Analysis & PDE

Volume 8 No. 7 2015

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ANALYSIS AND PDE
Vol. 8, No. 7, 2015
dx.doi.org/10.2140/apde.2015.8.1675

A MODEL FOR STUDYING DOUBLE EXPONENTIAL GROWTH IN THE TWO-DIMENSIONAL EULER EQUATIONS

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We introduce a model for the two-dimensional Euler equations which is designed to study whether or not double exponential growth can be achieved for a short time at an interior point of the flow.

1. Background

The two-dimensional Euler equations for incompressible fluid flow are given by

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p,$$

together with

$$\nabla \cdot u = 0.$$

Here, $u : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$ is a time-varying vector field on $\mathbb{R}^2$ representing the velocity and $p : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ is a scalar representing the pressure.

The equation is solved with a given initial divergence-free velocity field u_0:

$$u(x, 0) = u_0(x).$$

When u_0 is chosen to be, for instance, smooth with compact support, a smooth solution to the Euler equation exists for all time. Moreover, a result of Beale, Kato, and Majda [Beale et al. 1984] shows that Sobolev norms grow at most double-exponentially in time.

Considerable work has been done recently to establish that such growth actually occurs. Denisov [2015] demonstrates growth similar to double exponential in an example that consists of a slightly smoothed, singular steady state solution together with a bump. For some time, the singular solution stretches the bump at a double exponential rate. Kiselev and Šverák [2014] do Denisov one better by creating a sustained double exponential growth near a boundary. This is a very similar idea to Denisov's. We may imagine that something quite similar to Denisov's singular steady state lives right at the boundary and is drawing bumps towards it. Another recent result on rapid growth in the Euler equations is [Zlatoš 2015].

Both authors were partially supported by NSF grant DMS 1266104.
MSC2010: 35Q31.
Keywords: fluid mechanics, Euler equations, two-dimensional Euler equations.

2. New results

Summary. The purpose of this paper is to create a tool for studying the question of whether double exponential growth can begin spontaneously at an interior point. We borrow from Pavlović's [2002] thesis the idea that the allowed fast growth in Euler is coming from low frequency to high frequency interactions. We model the impact of each scale on the vicinity of a given particle as a linear area-preserving map.

In a double exponential growth, energy has to cascade from low frequencies to high frequencies. We try to model this phenomenon. We start at a point in time where, heuristically, N dyadic scales are active. More precisely, we use the parameter N as a bound for short time on the sum of the L^∞ norms of Littlewood–Paley projections of ∇u, the gradient of the velocity field. When double exponential growth is taking place, this Besov norm should be growing exponentially, so its size stays stable (to within a constant) for time $O(1)$. This is assumption (1).

As each scale evolves, it alters the effects of the smaller scales. We can model this as a system of differential equations with one SL(2)-valued unknown for each scale. The main result in the paper is Theorem 3, below. It says that, during a time period of order $(\log N)/N$, this autonomous system of differential equations closely approximates the actual behavior of the Euler equation. This is a time period during which growth by a factor of a power of N can occur in the Sobolev norms of the velocity and during which our hypothesis stays stable. Indeed, such growth must occur during some such time period if double exponential growth is to take place. Thus, our simplified model can be used to study the possibility and likelihood of growth occurring spontaneously at an interior point. This is especially noteworthy because the previous examples of rapid growth in the two-dimensional Euler equations (such as [Denisov 2015; Kiselev and Šverák 2014]) do not occur spontaneously from energy cascading from low to high frequencies. Thus, we believe this phenomenon is definitely worthy of more study.

We comment briefly on some of the properties of the model. Clearly, the system is simpler than studying the Euler equations. This is because many of the parameters of the Euler equations lie in the initial condition ω_0 of the SL(2) system. Indeed, once a point in $\mathbb{R}^2$ is chosen (to study the accumulation of vorticity at that point as it moves through the flow) and the parameter N (the number of active scales) is fixed, the system has only $3N$ parameters. If the initial condition ω_0 can be designed so that the SL(2) system grows exponentially with rate N, this would indicate double exponential growth in the Euler equations. However, it is critical that such growth be sustained for a time period of order $(\log N)/N$, as the proof below makes it reasonably obvious that it is possible to do so for a time period of order $1/N$ (see Lemma 5). Currently, we do not have a strategy for designing such initial data. The purpose of this work is to establish a rigorous connection between the Euler equations and the model.

Admittedly, our model works for only a very short period of time. We cannot use the model to follow the equation for a longer period of time, because nonlinearities are breaking down its connection to the equation. The fact that it runs long enough to give some insight into the double exponential growth question is a consequence of the criticality of the equation for this problem. In supercritical problems like blow-up for surface quasigeostrophic equations or blow-up for the three-dimensional Euler equations, the same kind of model cannot work.

Notation and Prerequisites. By $a \lesssim b$ we mean that $a \leq kb$ for some constant k that does not depend on anything important. The notation $a \sim b$ means $a \lesssim b$ and $b \lesssim a$ simultaneously. The norm $|\cdot|$ is the usual Euclidean norm when applied to vectors in $\mathbb{R}^2$ and can be thought of as the maximum norm when applied to a matrix.

Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that

$$\psi(\xi) = \begin{cases} 1 & \text{for } 0 < |\xi| < 1, \\ 0 & \text{for } |\xi| > 2, \end{cases}$$

and define the operator P_0 to be the Fourier multiplier with symbol ψ. Let $\psi_1(\xi) = \psi\left(\frac{1}{2}\xi\right) - \psi(\xi)$ and, for $j > 1$, define P_j to be the Fourier multiplier with symbol $\psi_j(\xi) := \psi_1(2^{1-j}\xi)$. For convenience of notation, define $P_j = 0$ for $j < 0$. Thus, P_j acts like a projection onto the frequency annulus $\{\xi : |\xi| \sim 2^j\}$ for $j > 1$, and $\sum_j P_j$ is the identity because the sum telescopes. These P_j are commonly known as the Littlewood–Paley operators. Further, let $\widetilde{P}_j = \sum_{\alpha=-2}^{2} P_{j+\alpha}$ and $E_j = \sum_{k<j} P_k$. Note that

$$E_j f(x) = f * (2^{2k}\widehat{\psi}(2^k\cdot))(x) = \int f(x + 2^{-j}s)\widehat{\psi}(s)\,ds$$

and $\widehat{\psi}$ is a radial Schwartz function such that $\int \widehat{\psi} = \psi(0) = 1$. Hence, E_j acts like, and will be referred to as, an averaging operator on scale $\sim 2^{-j}$. All Littlewood–Paley operators in this work take their arguments in the spatial variable $x \in \mathbb{R}^2$ (and not in time).

Let $u : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$ be the velocity field of a two-dimensional, inviscid, incompressible fluid flow and $\omega = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ the associated vorticity. We make some assumptions about u over the time period we will be considering, which is of order $(\log N)/N$. We will assume that

$$\sum_{j=0}^{\infty} \|P_j \nabla u\|_{L^\infty} \lesssim N \tag{1}$$

$$\text{and} \qquad \|P_j \nabla u\|_{L^\infty} \lesssim 1 \quad \text{for all } j \geq 0. \tag{2}$$

Note that (2) is automatic in the case that $\omega_0 \in L^\infty$. Above, and throughout this work, $L^p = L^p(\mathbb{R}^2)$, that is, all L^p norms are taken in the spatial variable $x \in \mathbb{R}^2$. Also as above, explicit mention of the dependence on time (t) will often be omitted for brevity. We define the flow maps $\phi(x, t)$ to be solutions of the differential equations

$$\begin{aligned} \frac{\partial}{\partial t}\phi(x,t) &= u(\phi(x,t),t), \\ \phi(x,0) &= x, \end{aligned} \tag{3}$$

so the point $\phi(x, t)$ is the image of the point x under the flow with velocity field u at time t. Thus, the Jacobian matrix of ϕ, which we denote by $D\phi$, satisfies the differential equation

$$\begin{aligned} \frac{\partial}{\partial t} D\phi(x,t) &= ((\nabla u)\circ\phi)(x,t) \cdot D\phi(x,t), \\ D\phi(x,0) &= I, \end{aligned} \tag{4}$$

for each $x \in \mathbb{R}^2$. By both D and ∇ we mean the Jacobian derivative in the spatial variable x and not in the coordinates of the particle trajectories $\phi(x,t)$. Indeed, it should be noted that the equations (3) and (4) invite a change of coordinates via the map $x \mapsto \phi(x,t)$. This change of coordinates is especially convenient because incompressibility, $\nabla_x \cdot u = 0$, gives $\det(D_x\phi(x,t)) \equiv 1$. This will make it useful for our purposes to use the Lagrangian reference frame; that is, spatial variables will be evaluated along the particle trajectories $\phi(x,t)$. A thorough discussion of particle trajectory maps and the Lagrangian reference frame can be found in [Majda and Bertozzi 2002]. Other recent results use the Lagrangian reference frame; see [Bourgain and Li 2015a; 2015b].

Proceeding formally, if we define $R_i := \Delta^{-\frac{1}{2}}\partial/\partial x_i$, we have the so-called Biot–Savart law,

$$\nabla u = \begin{pmatrix} -R_1R_2\omega & -R_2^2\omega \\ R_1^2\omega & R_1R_2\omega \end{pmatrix}.$$

Using the Green's function for the Laplace operator, we can calculate the nonlocal parts of the composed Riesz operators by giving the nonsingular part of their kernels. (The local part, of course, lives in the singular part of the kernel located on the diagonal.) These are

$$R_1R_2\omega = K_{12} * \omega(\cdot,t), \quad R_1^2\omega = K_{11} * \omega(\cdot,t), \quad \text{and} \quad R_2^2 = -K_{11} * \omega(\cdot,t),$$

where

$$K_{12}(x_1,x_2) = \frac{x_1x_2}{\pi(x_1^2+x_2^2)^2} \quad \text{and} \quad K_{11}(x_1,x_2) = \frac{x_2^2-x_1^2}{2\pi(x_1^2+x_2^2)^2}.$$

The main result. The following definition is the one of the main fixtures of this paper. We will define an approximation of $\nabla u(\phi(0,t),t)$ so that, for a short time of order $(\log N)/N$, the flow is given by a linear area-preserving map at each physical scale around the point $\phi(0,t)$. That is, the contribution to $\nabla u(\phi(0,t),t)$ from the part of the vorticity which at time 0 was at an annulus at scale 2^{-j} around 0 is calculated as though the flow on the annulus were linear and given by some $h_j \in \mathrm{SL}(2)$. This is inspired by the following version of the Biot–Savart law:

$$\begin{aligned}\nabla u(\phi(0,t),t) &= \int \omega(s,t)K(s-\phi(0,t))\,ds \\ &= \int \omega(\phi(s,t),t)K(\phi(s,t)-\phi(0,t))\,ds \\ &= \sum_{j\in\mathbb{Z}} \int_{A_j} \omega_0(s)K(\phi(s,t)-\phi(0,t))\,ds, \end{aligned} \tag{5}$$

where $A_j = \{x : 2^{-j} \le |x| < 2^{1-j}\}$ and by dropping the index of K we mean a generic entry in the matrix $\nabla u(\phi(x,t),t)$. We have used the aforementioned change of coordinates $s \mapsto \phi(s,t)$. This change of coordinates is especially convenient because, in two space dimensions, the vorticity is purely transported by the flow map. That is,

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = 0 \quad \text{and so} \quad \omega_0(s) = \omega(\phi(s,t),t)$$

for all $t \geq 0$. In (5), we are focusing on $\phi(0,t)$, which we think of as a generic interior point of a fluid flow, in order to study whether double exponential growth is in the making at that point as it moves along the flow.

Definition 1. Let $h(t)$ be an element of SL(2). We define

$$(\nabla u)_{j,h}(t) = \begin{pmatrix} -(\nabla u)_{j,h,2} & (\nabla u)_{j,h,1} \\ (\nabla u)_{j,h,1} & (\nabla u)_{j,h,2} \end{pmatrix},$$

where

$$(\nabla u)_{j,h,i}(t) = \int_{A_j} \omega_{0,j}(s) K_{1i}(h(t)\cdot s)\, ds, \tag{6}$$

$$\omega_j = \chi_{A_j}(E_{j+\log N} - E_{j-\log N})\omega,$$

$$\text{and} \quad \omega_{0,j}(x) = \omega_j(x,0).$$

Remark 2. Two things regarding the above definition are worth emphasizing. First, a heuristic note: the object $\sum_j (\nabla u)_{j,h_j}(t)$ should be thought of as an approximation of $\nabla u(\phi(0,t),t)$. This is, of course, in the event that each matrix $h_j(t)$ is a linear approximation of the movement of the fluid particles roughly distance 2^{-j} from $\phi(0,t)$. Indeed, if in (6) we replaced $\omega_{0,j}$ with ω_0 and $h(t)\cdot s$ with $\phi(x,t)-\phi(0,t)$, we would have $\sum_j (\nabla u)_{j,h_j}(t) = \nabla u(\phi(0,t),t)$ (at least formally).

Second, despite the notation, $(\nabla u)_{j,h,i}(t)$ does not explicitly depend on the velocity field u at time t.

We now state the main result: for a short time, we can approximate the average of the Jacobian of the flow map at the scale 2^{-j} by a linear map for each j and these linear maps satisfy an autonomous system of differential equations not depending on the solution to the Euler equations. The behavior of this system can be a test for whether double exponential growth can occur and what it should look like.

Theorem 3. *Assume that*

$$\sum_{j=0}^{\infty} \|P_j \nabla u\|_{L^\infty} \lesssim N$$

$$\textit{and} \qquad \|P_j \nabla u\|_{L^\infty} \lesssim 1,$$

and let $h_j \in \mathrm{SL}(2)$ be defined as the solution to the ODE

$$\frac{dh_j}{dt} = \Big(\sum_{k<j} (\nabla u)_{k,h_k}\Big) h_j,$$

$$h_j(0) = I.$$

Then there is a (small) universal constant $C > 0$ such that, for all times $0 \leq t \leq C(\log N)/N$, we have

$$|h_j(t) - E_j D\phi(0,t)| = O(N^{-\frac{7}{10}})$$

for all $j > 0$.

Remark 4. The purpose of using $\omega_{0,j}$ instead of just ω_0 is a technical advantage: $\omega_{0,j}$ is a projection onto the frequencies of ω that make a significant contribution to $\nabla u(\phi(0,t),t)$ coming from the annulus A_j. Indeed, if, in light of (5), we define

$$\widetilde{\nabla} u(\phi(0,t),t) := \sum_{j\in\mathbb{Z}} \int_{A_j} \omega_j(\phi(s,t),t) K(\phi(s,t)-\phi(0,t))\, ds,$$

whereas (at least formally)

$$\nabla u(\phi(0,t),t) = \sum_{j\in\mathbb{Z}} \int_{A_j} \omega(\phi(s,t),t) K(\phi(s,t)-\phi(0,t))\, ds,$$

the difference is

$$\begin{aligned}
&\sum_j \sum_{|k-j|>\log N} \int_{A_j} P_k(\omega(\phi(s,t),t)) K(\phi(s,t)-\phi(0,t))\, ds \\
&\quad = \sum_{\{(j,k):|k-j|>\log N\}} \int_{A_j} \biggl(\int \omega(y,t) \check{\psi}_k(y-\phi(s,t))\, dy \biggr) K(\phi(s,t)-\phi(0,t))\, ds \\
&\quad = \sum_{\{(j,k):|k-j|>\log N\}} \int_{A_j} \biggl(\int \omega(y,t) \biggl(\int e^{2\pi i (y-\phi(s,t))\cdot\xi} \psi_k(\xi)\, d\xi \biggr) dy \biggr) K(\phi(s,t)-\phi(0,t))\, ds. \quad (7)
\end{aligned}$$

Note that, by (3), the fundamental theorem of calculus, and (1), over a time period of order $(\log N)/N$ we have

$$\frac{|s|}{\log N} \lesssim |\phi(s,t)| \lesssim |s| \log N. \quad (8)$$

In the right-hand side of (7), we integrate by parts in $\int e^{2\pi i (y-\phi(s,t))\cdot\xi} \psi_k(\xi)\, d\xi$, moving a derivative ∇_ξ from the exponential onto ψ_k for terms in which $k > j + \log N$, and the opposite way for terms where $k < j + \log N$. Since $|s| \sim 2^{-j}$ in each A_j, this gives us a factor of $(\log N) 2^{\pm j}$ from the exponential term (because of (8)) and $2^{\mp k}$ from the dilation of ψ_1, and this gives the estimate

$$(7) \lesssim \log N \sum_{\{(j,k):|k-j|>\log N\}} 2^{-|k-j|} \| P_k \omega \|_{L^\infty} \int_{A_j} K(\phi(s,t)-\phi(0,t))\, ds.$$

Since, by definition, $|K(x)| \sim |x|^{-2}$, we have $\int_{A_j} K(\phi(s,t)-\phi(0,t))\, ds \lesssim (\log N)^2$ for all j. Hence, we now have the bound

$$\begin{aligned}
(7) &\lesssim (\log N)^3 \sum_{\{(j,k):|k-j|>\log N\}} \| P_k \omega \|_{L^\infty} 2^{-|k-j|} \\
&\lesssim \frac{(\log N)^3}{N} \sum_k \| P_k \omega \|_{L^\infty} \\
&\lesssim (\log N)^3. \quad (9)
\end{aligned}$$

We also used (1) and the fact that $\| P_k \omega \|_{L^\infty} \sim \| P_k \nabla u \|_{L^\infty}$.

The technical advantage of using $\omega_{0,j}$ is that we have, similarly to (1),

$$\sum_{j=0}^{\infty} \|\omega_j\|_{L^\infty} \lesssim \sum_{j=0}^{\infty} \sum_{\{(k,j):|k-j|<\log N} \|P_k \omega\|_{L^\infty} \lesssim N \log N. \tag{10}$$

The reader might ask why we chose to have the error estimate in (9) come to $(\log N)^3$. It is entirely arbitrary. By replacing the range of $\log N$ scales by a range of $C \log N$ scales, which would only cost us a constant in the estimate (10), we could reduce the estimate to an arbitrary negative power of N, but the point is that, because of the brevity of our time period, any estimate for the error which has a power of N lower than 1 will work. The error is smaller than the worst case we have for $\|\nabla u\|_{L^\infty}$. The important part of these estimates is that we lose (at most) a factor of a power of $\log N$ in (10), which is enough for our purposes, mainly because of the assumption (1).

3. Methods of the proof

Here is a brief outline of the proof. The proofs of the lemmas and the main theorem will follow in the next section.

Many of the estimates will be based on the following Gronwall-type lemma, which says that solutions to similar ODEs remain similar for a short time.

Lemma 5. *Suppose that F, G_1, G_2, w and v are real-valued functions of time with domain $[0,\infty)$ such that $F(t) = O(N)$, and that*

$$\begin{aligned}
\frac{dw}{dt}(t) &= F(t)w(t) + G_1(t),\\
\frac{dv}{dt}(t) &= F(t)v(t) + G_2(t),\\
w(0) &= v(0).
\end{aligned}$$

Assume further that, for some constant E,

$$|G_1(t) - G_2(t)| \lesssim \begin{cases} E & \text{for } 0 < t \lesssim 1/N,\\ |F(t)(w(t)-v(t))| & \text{for } t \gtrsim 1/N.\end{cases}$$

Then, there is a (small) universal constant C, independent of N, such that $|(w-v)(t)| \lesssim EN^{-\frac{9}{10}-\frac{1}{100}}$ for all times $t \le C(\log N)/N$.

The idea behind Lemma 5 is that, since the difference starts out at 0, the "error" term $G_1 - G_2$ dominates for times $t \lesssim 1/N$. At that time, the main term, $F(t)(w(t)-v(t))$, becomes the dominant term but $|(w-v)(t)|$ remains relatively small for an additional time $\lesssim \log N$. Most of the time, we will not need the extra factor of $N^{-\frac{1}{100}}$. It will be used to eliminate factors of $\log N$ that show up in the error term E.

We will often use the following estimate, which says the individual Littlewood–Paley pieces of $D\phi$ stay small:

Lemma 6. *Under assumptions (1)–(2), $\sup_{j>0} \|P_j D\phi(t)\|_{L^\infty} \lesssim N^{-\frac{9}{10}-\frac{1}{100}}$ for times $t \le C(\log N)/N$.*

In order to prove Theorem 3, we will show that, for the time period we are considering, the flow maps are essentially linear on a given dyadic annulus. That is, we will estimate the difference between the linear map $(E_j D\phi(0,t))\cdot x$ and the difference $\phi(x,t)-\phi(0,t)$ for $|x|\sim 2^{-j}$. To do so, we first show that the averages of the Jacobians of the flow maps are close to the averages of the differences in the flow maps, that is,

$$|E_j D\phi(0,t)\cdot x-(E_j\phi(x,t)-E_j\phi(0,t))|\lesssim 2^{-j}N^{-\frac{9}{10}},$$

a sort of approximate mean value theorem. We do this by using (3) to examine the time derivative of the difference of the flow maps and (4) to examine the average of $D\phi$ at the appropriate scale. With the fundamental theorem of calculus, and on frequency support grounds, we have that the time derivative of the difference is essentially

$$\left(\int_0^1 E_j\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(E_j\phi(x,t)-E_j\phi(0,t)).$$

If we throw away $\log N$ many frequencies from the integrand, it is almost constant on its domain. The error from doing so is acceptable, so we have now, essentially,

$$E_j((\nabla u)\circ\phi)(0,t)\cdot(E_j\phi(x,t)-E_j\phi(0,t))+O(2^{-j}\log N)$$

and we apply Lemma 5. We will still have to show that the difference of averages is close to the actual difference for x at the appropriate scale. Since $\sum P_k=1$, this is entirely a matter of controlling the frequency bands bigger than 2^j. We do this by first using a trivial bound for the high (at least $j+\log N$) frequencies, which comes from the fundamental theorem of calculus. For $j\le k\le j+\log N$, we can again exploit the fact that averages at scale 2^{-j} are essentially constant at scale $2^{-j-\log N}$.

Putting all of this together, we have:

Lemma 7. *For times $t\le C(\log N)/N$ and $|x|\sim 2^{-j}$, we have*

$$\left|(E_j D\phi(0,t))\cdot x-(\phi(x,t)-\phi(0,t))\right|=O(2^{-j}N^{-\frac{9}{10}}).$$

Finally, we will prove Theorem 3 by using Lemma 7 to substitute the linear map $(E_j D\phi(0,t))\cdot x$ for the difference $\phi(x,t)-\phi(0,t)$ in each piece of the convolution used to calculate ∇u by the Biot–Savart law.

4. The proof

Note that the constant C may change from line to line. It will only change finitely many times and, in the end, it will be a universal constant which is independent of N.

Proof of Lemma 5. Observe that

$$\begin{aligned}\frac{d(w-v)}{dt}(t)&=F(t)(w-v)(t)-(G_1(t)-G_2(t)),\\(w-v)(0)&=0,\end{aligned}$$

and suppose that T is the first time that $|(w-v)(T)| = E/N$. Then, for times $t \le \min\{1/N, T\}$, we have, by assumption,

$$F(t)(w-v)(t) = O(E) \implies \left|\frac{d(w-v)}{dt}(t)\right| \lesssim E.$$

Therefore, since the growth of the difference is at most linear of rate E, it follows that $T = O(1/N)$. For $T \le t \le C(\log N)/N$, we have

$$\left|\frac{d(w-v)}{dt}(t)\right| \lesssim |F(t)(w-v)(t)| = O(N)|(w-v)(t)|$$

and so, by Gronwall's lemma, we have

$$|(w-v)(t)| \lesssim \frac{E}{N} e^{Nt} \lesssim E N^{-\frac{9}{10}-\frac{1}{100}},$$

where we get the last inequality by choosing C such that $t \lesssim C(\log N)/N \le (\log(N^{\frac{1}{10}-\frac{1}{100}}))/N$. $\square$

Proof of Lemma 6. Taking P_j of both sides of (4), we have, on frequency support grounds

$$\begin{aligned}\frac{\partial}{\partial t} P_j D\phi = P_j(E_{j+3}((\nabla u)\circ\phi)\cdot \widetilde{P}_j D\phi) + P_j(\widetilde{P}_j((\nabla u)\circ\phi)\cdot E_{j-2}D\phi)\\ + P_j\biggl(\sum_{k=j}^{\infty} \widetilde{P}_{k+1}((\nabla u)\circ\phi)\cdot P_{k+3}D\phi + P_{k+3}((\nabla u)\circ\phi)\cdot \widetilde{P}_k D\phi\biggr). \quad (11)\end{aligned}$$

(Notice that explicit dependence on x and t has been omitted for convenience of notation. This will continue throughout this work.) We will make frequent use of the following versions of the cheap Littlewood–Paley inequality:

$$\sup_j \|P_j f\|_{L^\infty} \lesssim \|f\|_{L^\infty} \quad (12)$$

$$\text{and} \quad \sup_j \|E_j f\|_{L^\infty} \lesssim \|f\|_{L^\infty}. \quad (13)$$

To prove (13), observe that, for any x,

$$|E_j f(x)| = \left|\int f(x+2^{-j}s)\widehat{\psi}(s)\,ds\right| \le \|f\|_{L^\infty}\left|\int \widehat{\psi}(s)\,ds\right| = \psi(0)\cdot\|f\|_{L^\infty} = \|f\|_{L^\infty},$$

by the definition of ψ. The estimate (12) is proven analogously.

Let $S(t) = \sup_{j>0}\|P_j D\phi(t)\|_{L^\infty}$. For the first term of (11) we then have

$$\bigl|P_j\bigl(E_{j+3}((\nabla u)\circ\phi)\cdot\widetilde{P}_j D\phi\bigr)\bigr| \lesssim \|E_{j+3}((\nabla u)\circ\phi)\|_{L^\infty}\|\widetilde{P}_j D\phi\|_{L^\infty} \lesssim \sum_{j=0}^{\infty}\|P_j\nabla u\|_{L^\infty} S(t) \lesssim O(N)S(t).$$

The first inequality follows from (12), the second follows from the triangle inequality and from adding nonnegative terms, and the definitions of $\widetilde{P}_j$ and $S(t)$, and the last follows from assumption (1). Along similar lines, for the second term of (11) we have

$$\begin{aligned}\left|P_j\big(\widetilde{P}_j((\nabla u)\circ\phi)\cdot E_{j-2}D\phi\big)\right| &\lesssim \|\widetilde{P}_j((\nabla u)\circ\phi)\|_{L^\infty}\|E_{j-2}D\phi\|_{L^\infty}\\ &\lesssim \sup_j\|P_j\nabla u\|_{L^\infty}\|D\phi\|_{L^\infty}\\ &\lesssim O(\|D\phi\|_{L^\infty}),\end{aligned}$$

which follows from (12), the definition of $\widetilde{P}_j$ and (13), and assumption (2). Finally, for the last term in (11),

$$\begin{aligned}&\left|P_j\left(\sum_{k=j}^{\infty}\widetilde{P}_{k+1}((\nabla u)\circ\phi)\cdot P_{k+3}D\phi + P_{k+3}((\nabla u)\circ\phi)\cdot\widetilde{P}_k D\phi\right)\right|\\ &\qquad\lesssim \sum_{k=j}^{\infty}\|\widetilde{P}_{k+1}\nabla u\|_{L^\infty}\|P_{k+3}D\phi\|_{L^\infty}+\|P_{k+3}\nabla u\|_{L^\infty}\|\widetilde{P}_k D\phi\|_{L^\infty}\\ &\qquad\lesssim S(t)\sum_{k=j}^{\infty}\|P_k\nabla u\|_{L^\infty}\\ &\qquad\lesssim O(N)S(t),\end{aligned}$$

which we justify with (12), the definitions of $\widetilde{P}_k$ and $S(t)$, and assumption (1). Putting together these three estimates, we have

$$\frac{d}{dt}S(t)=O(N)S(t)+O(\|D\phi\|_{L^\infty}).$$

Further, using (4), (1) and Gronwall's lemma, we see that

$$\|D\phi\|_{L^\infty}\lesssim e^{Nt} \tag{14}$$

and so

$$\frac{d}{dt}S(t)=O(N)S(t)+O(e^{Nt}).$$

From here, we can apply a traditional, inhomogeneous version of Gronwall's lemma to achieve the desired bound. □

The proof of Lemma 7 is achieved in two parts. First, we show that the average of the Jacobian of a flow map is closely approximated by the average difference of a flow map at a fixed scale. That is, for $|x|\sim 2^{-j}$, we have

$$\left|(E_jD\phi(0,t))\cdot x-(E_j\phi(x,t)-E_j\phi(0,t))\right|=O(2^{-j}N^{-\frac{9}{10}}).$$

We do this by comparing the time derivatives of each expression and using Lemma 5. Then we show that the differences of the flow maps themselves at scale $\sim 2^{-j}$ are closely approximated by their averages at the same scale, that is,

$$\left|E_j\phi(x,t)-E_j\phi(0,t)-(\phi(x,t)-\phi(0,t))\right|\lesssim 2^{-j}N^{-\frac{9}{10}},$$

hence proving Lemma 7 by the triangle inequality.

Proof of Lemma 7. First, we claim that, for $|x| \sim 2^{-j}$,

$$\big|(E_j D\phi(0,t)) \cdot x - (E_j \phi(x,t) - E_j \phi(0,t))\big| = O(2^{-j} N^{-\frac{9}{10}}). \tag{15}$$

We examine $\partial\big((E_j D\phi(0,t)) \cdot x\big)\big/\partial t$ using (4). The goal is to use Lemma 5. In this case, we want to show that $\partial\big((E_j D\phi(0,t)) \cdot x\big)\big/\partial t = E_j((\nabla u) \circ \phi)(0,t) \cdot \big((E_j D\phi(0,t)) \cdot x\big)$ plus an error term which obeys acceptable bounds. Taking E_j and then the product with x of both sides of (4), we have, purely on frequency support grounds,

$$\begin{aligned}\frac{\partial}{\partial t}\big((E_j D\phi(0,t)) \cdot x\big) &= E_j\big(E_{j+3}((\nabla u) \circ \phi)(0,t) \cdot E_{j+3} D\phi(0,t)\big) \cdot x \\ &\quad + E_j\bigg(\sum_{k=j}^{\infty}\big(\widetilde{P}_{k+1}((\nabla u) \circ \phi) \cdot P_{k+3} D\phi + P_{k+3}((\nabla u) \circ \phi) \cdot \widetilde{P}_k D\phi\big)\bigg) \cdot x. \end{aligned} \tag{16}$$

(Note that, in the second line, we have again omitted the arguments of $(\nabla u) \circ \phi$ and $D\phi$ for brevity.) The second term is entirely an error term. Observe that, by (2), Lemma 6, and frequency support, the second term of (16) is

$$O\big(\sup_j \|P_j \nabla u\|_{L^\infty}\big) E_j\bigg(\sum_{k=j}^{\infty} P_k D\phi(0,t)\bigg) \cdot x = O(2^{-j}), \tag{17}$$

which will prove to be a tolerable error. For the first term of (16), since E_j is not actually a projection, we have to separate some of the frequencies. We use the fact that $E_j E_{j-2} = E_{j-2}$, giving

$$\begin{aligned} E_j\big(E_{j+3}((\nabla u) \circ \phi)(0,t) \cdot E_{j+3} D\phi(0,t)\big) \cdot x &= E_{j-2}((\nabla u) \circ \phi)(0,t) \cdot \big((E_{j-2} D\phi(0,t)) \cdot x\big) \\ &\quad + E_j\bigg(\sum_{k,l=j-2}^{j+2} P_k((\nabla u) \circ \phi)(0,t) \cdot P_l D\phi(0,t)\bigg) \cdot x. \end{aligned}$$

We now add and subtract $\big(\sum_{k,l=j-2}^{j+2} P_k((\nabla u) \circ \phi)(0,t) \cdot P_l D\phi(0,t)\big) \cdot x$. This gives us, from (16) and (17),

$$\begin{aligned} &\frac{\partial}{\partial t}((E_j D\phi(0,t)) \cdot x) \\ &= E_j((\nabla u) \circ \phi)(0,t) \cdot ((E_j D\phi(0,t)) \cdot x) + O(2^{-j}) \\ &\qquad + \sum_{k,l=j}^{j+2} \big(P_k((\nabla u) \circ \phi)(0,t) \cdot P_l D\phi(0,t)\big) \cdot x \\ &\qquad\quad + \sum_{k,l=j-2}^{j+2} E_j\big(P_k((\nabla u) \circ \phi)(0,t) \cdot P_l D\phi(0,t)\big) \cdot x - P_k((\nabla u) \circ \phi)(0,t) \cdot P_l D\phi(0,t) \cdot x, \end{aligned} \tag{18}$$

where the last two lines are error terms which we denote by Ψ. Notice that, for a typical term in the last sum, we have

$$\begin{aligned}
&\big|E_j\big(P_k((\nabla u)\circ\phi)(0,t)\cdot P_l D\phi(0,t)\big)\cdot x - P_k((\nabla u)\circ\phi)(0,t)\cdot P_l D\phi(0,t)\cdot x\big| \\
&\qquad \le \big|E_j\big(P_k((\nabla u)\circ\phi)(0,t)\cdot P_l D\phi(0,t)\big)\cdot x\big| + \big|P_k((\nabla u)\circ\phi)(0,t)\cdot P_l D\phi(0,t)\cdot x\big| \\
&\qquad \lesssim 2^{-j}\|P_k((\nabla u)\circ\phi)\cdot P_l D\phi\|_{L^\infty} \\
&\qquad \lesssim 2^{-j}\|P_k\nabla u\|_{L^\infty}\|P_l D\phi\|_{L^\infty} \\
&\qquad = O(2^{-j}),
\end{aligned} \tag{19}$$

where we have used the triangle inequality, (13), the hypothesis that $|x|\sim 2^{-j}$, (2) and Lemma 6. A similar (simpler) argument can be used to achieve the same estimate for a typical term in the first sum and, since both sums have only $O(1)$ many terms, we now have the estimate

$$|\Psi| = O(2^{-j}).$$

We have now achieved the goal,

$$\frac{\partial}{\partial t}\big((E_j D\phi(0,t))\cdot x\big) = E_j((\nabla u)\circ\phi)(0,t)\cdot\big((E_j D\phi(0,t))\cdot x\big) + O(2^{-j}). \tag{20}$$

To use Lemma 5, we need an analogous statement for $\partial(E_j\phi(x,t)-E_j\phi(0,t))/\partial t$. We begin by using (3). Since $\partial/\partial t$ commutes with E_j, and by (3) and the fundamental theorem of calculus, we have

$$\begin{aligned}
\frac{\partial}{\partial t}(E_j\phi(x,t)-E_j\phi(0,t)) &= E_j\frac{\partial}{\partial t}(\phi(x,t)-\phi(0,t)) \\
&= E_j\big(u(\phi(x,t),t)-u(\phi(0,t),t)\big) \\
&= E_j\bigg(\bigg(\int_0^1 \nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\bigg)\cdot(\phi(x,t)-\phi(0,t))\bigg).
\end{aligned}$$

We now take E_j of the product, move E_j inside the integral, and the above expression gives

$$\begin{aligned}
&E_j\bigg(\bigg(\int_0^1 E_{j+3}\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\bigg)\cdot(E_{j+3}\phi(x,t)-E_{j+3}\phi(0,t))\bigg) \\
&+E_j\bigg(\sum_{k=j}^{\infty}\widetilde{P}_{k+1}((\nabla u)\circ\phi)\cdot P_{k+3}(\phi(x,t)-\phi(0,t)) + P_{k+3}((\nabla u)\circ\phi)\cdot\widetilde{P}_k(\phi(x,t)-\phi(0,t))\bigg),
\end{aligned} \tag{21}$$

which we justify on frequency support grounds. We use the same technique on the first term as we used to achieve (20). That is, we add and subtract

$$\sum_{k,l=j-2}^{j+2}\bigg(\int_0^1 P_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\bigg)\cdot(P_l\phi(x,t)-P_l\phi(0,t))$$

to exploit the fact that $E_j E_{j-2} = E_{j-2}$. This gives us that (21) equals

$$\begin{aligned}
&\left(\int_0^1 E_j \nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(E_j\phi(x,t)-E_j\phi(0,t))\\
&\quad+\sum_{k,l=j-2}^{j+2} E_j\left(\left(\int_0^1 P_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(P_l\phi(x,t)-P_l\phi(0,t))\right)\\
&\quad-\sum_{k,l=j-2}^{j+2}\left(\int_0^1 P_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(P_l\phi(x,t)-P_l\phi(0,t))\\
&\quad+E_j\left(\sum_{k=j}^{\infty}\widetilde{P}_{k+1}((\nabla u)\circ\phi)\cdot P_{k+3}(\phi(x,t)-\phi(0,t))+P_{k+3}((\nabla u)\circ\phi)\cdot\widetilde{P}_k(\phi(x,t)-\phi(0,t))\right). \qquad (22)
\end{aligned}$$

The term in the first line is good and the remaining terms are error terms. Denote the difference of the middle two sums by Φ; the indices in these sums match and we can use (13) on each term in the first sum and the fact that there are only $O(1)$ many terms in the sum to estimate

$$|\Phi| = O\left(\left\|\left(\int_0^1 P_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(P_l\phi(x,t)-P_l\phi(0,t))\right\|_{L^\infty}\right). \qquad (23)$$

We use assumption (2) to estimate the integral, giving

$$\begin{aligned}
&\left\|\left(\int_0^1 P_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(P_l\phi(x,t)-P_l\phi(0,t))\right\|_{L^\infty}\\
&\qquad\qquad = O\big(\|P_l\phi(x,t)-P_l\phi(0,t)\|_{L^\infty}\big). \qquad (24)
\end{aligned}$$

Using (12), (8) and the hypothesis that $|x|\sim 2^{-j}$, we now have the estimate

$$|\Phi| = O(2^{-j}\log N).$$

We now estimate the last error term of (21), which we denote by Ξ. Using assumption (2), we have

$$\begin{aligned}
|\Xi| &\lesssim \sup_k \|P_k\nabla u\|_{L^\infty}\left|E_j\left(\sum_{k=j}^{\infty}P_{k+3}(\phi(x,t)-\phi(0,t))+\widetilde{P}_k(\phi(x,t)-\phi(0,t))\right)\right|\\
&\lesssim \left|E_j\left(\sum_{k=j}^{\infty}P_{k+3}(\phi(x,t)-\phi(0,t))+\widetilde{P}_k(\phi(x,t)-\phi(0,t))\right)\right|. \qquad (25)
\end{aligned}$$

Because of the operator E_j, by frequency support, there are only $O(1)$ many terms left in the sum. Therefore, it suffices to estimate a typical term in the sum, such as

$$P_k(\phi(x,t)-\phi(0,t)),$$

where $k\sim j$. Using (12), (8), and the fact that $|x|\sim 2^{-j}$, we have

$$|P_k(\phi(x,t)-\phi(0,t))| \lesssim 2^{-j}\log N$$

and hence

$$|\Xi| = O(2^{-j}\log N).$$

Using these estimates on $|\Phi|$ and $|\Xi|$, we now have that (21) equals

$$\left(\int_0^1 E_j\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(E_j\phi(x,t)-E_j\phi(0,t))+O(2^{-j}\log N). \tag{26}$$

At this point, we reiterate that the goal is to show that the above expression is equal to

$$E_j((\nabla u)\circ\phi)(0,t)(E_j\phi(x,t)-E_j\phi(0,t))$$

plus an acceptable error term, and that the error so far, $O(2^{-j}\log N)$, is acceptable. For convenience, we adopt the following notation for the integral term in (26):

$$\begin{aligned}\mathcal{I}(t) &:= \int_0^1 E_j\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\\ &= \int_0^1\left(E_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)+\sum_{l=k}^{j-1}P_l\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\right)ds,\end{aligned}$$

where we chose $k = j-\log N$ so that the first part of the integral is essentially constant. Indeed, if $\|f\|_{L^\infty}\lesssim N$ and $|x-y|\le 2^{-j}\log N$, with this choice of k we have

$$\begin{aligned}E_kf(x)-E_kf(y) &= \int_{\mathbb{R}^2} f(s)2^{2k}\big(\hat{\psi}(2^k(x+s))-\hat{\psi}(2^k(y+s))\big)\,ds\\ &\lesssim 2^{2k}\|f\|_{L^\infty}\|\nabla\hat{\psi}\|_{L^\infty}2^k|x-y|\,|B_{2^{-j}}(0)|\\ &\lesssim \|f\|_{L^\infty}2^{k-j}\log N\\ &\lesssim N2^{-\log N}\log N\\ &\lesssim \log N,\end{aligned}$$

wherein we can move from the first line to the second line by the definition of ψ. Since the first part of the integrand is essentially constant, we can choose any point in the domain we want for its argument (we choose $\phi(0,t)$). We then add and subtract the extra frequencies (that is, those between k and j) and we have

$$\mathcal{I}(t) = E_j\nabla u(\phi(0,t),t)+\int_0^1\left(\sum_{l=k}^{j}P_l\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)-P_l\nabla u(\phi(0,t),t)\right)ds+\log N.$$

The integral of the sum is clearly $\lesssim\log N$ because of (2) and the choice of k. Substituting this into (26) and using (8), we have (finally)

$$\frac{\partial}{\partial t}(E_j\phi(x,t)-E_j\phi(0,t)) = E_j\nabla u(\phi(0,t),t)\cdot(E_j\phi(x,t)-E_j\phi(0,t))+O(2^{-j}(\log N)^2).$$

Using this, together with (20), we can apply Lemma 5 with

$$w=(E_jD\phi(0,t))\cdot x,\quad v=E_j\phi(x,t)-E_j\phi(0,t),\quad\text{and}\quad E=2^{-j}(\log N)^2,$$

which proves the claim that

$$|E_j D\phi(0,t)\cdot x - (E_j\phi(x,t) - E_j\phi(0,t))| = O(2^{-j} N^{-\frac{9}{10}})$$

for $|x| \sim 2^{-j}$.

Now, to prove the lemma, it suffices to show that

$$|E_j\phi(x,t) - E_j\phi(0,t) - (\phi(x,t) - \phi(0,t))| \lesssim 2^{-j} N^{-\frac{9}{10}}.$$

By the definition of the Littlewood–Paley operators, we have

$$E_j\phi(x,t) - E_j\phi(0,t) - (\phi(x,t) - \phi(0,t)) = \sum_{k\geq j} P_k\phi(x,t) - P_k\phi(0,t),$$

which we now estimate in two parts. First, for the large frequencies, we have (for arbitrary y)

$$\begin{aligned}\sum_{k=j+\log N}^{\infty} P_k\phi(y,t) &= \sum_{k=j+\log N}^{\infty} E_{k+1}\phi(y,t) - E_k\phi(y,t)\\ &= \sum_{k=j+\log N}^{\infty} \int_{\mathbb{R}^2} \big(\phi(y+2^{-(k+1)}s,t) - \phi(y+2^{-k}s,t)\big)\widehat{\psi}(s)\,ds\\ &\lesssim \|D\phi\|_{L^\infty} \sum_{k=j+\log N}^{\infty} 2^{-k}\\ &\lesssim N^{\frac{1}{10}} 2^{-j-\log N}\\ &\lesssim 2^{-j} N^{-\frac{9}{10}}, \end{aligned} \tag{27}$$

where we have used the definition of E_k, (14) and our choice of C (as in the proof of Lemma 5). For the smaller frequencies, we have left

$$\sum_{k=j}^{l} (P_k\phi(x,t) - P_k\phi(0,t)), \tag{28}$$

where $l = j + \log N - 1$. We will estimate an arbitrary frequency band $P_k\phi(x,t) - P_k\phi(0,t)$ in this range. Take x_i to be points on the line segment from 0 to x such that $|x_{i+1} - x_i| \sim 2^{-l}$; thus we have $\sim 2^{l-j} \sim N$ points x_i. For convenience of notation, take $x_0 = 0$ and $x_N = x$. By adding and subtracting $P_k\phi(x_i,t)$ for each i, we have

$$|P_k\phi(x,t) - P_k\phi(0,t)| \lesssim 2^{l-j} \max_i |P_k\phi(x_{i+1},t) - P_k\phi(x_i,t)|. \tag{29}$$

For each i, we have from Lemma 6 that

$$P_k(\phi(x_{i+1},t) - \phi(x_i,t)) \lesssim 2^{-l} \|P_k D\phi\|_{L^\infty} \lesssim 2^{-l} N^{-\frac{9}{10}-\frac{1}{100}}.$$

Plugging this into (29) and, in turn, plugging the result into (28), we can use the factor of $N^{-\frac{1}{100}}$ and the fact that there are only $\sim \log N$ terms in the sum to obtain

$$\sum_{k=j}^{l} P_k\phi(x,t) - P_k\phi(0,t) \lesssim 2^{-j} N^{-\frac{9}{10}}.$$

This, together with (27), proves the claim that

$$|E_j\phi(x,t) - E_j\phi(0,t) - (\phi(x,t) - \phi(0,t))| \lesssim 2^{-j} N^{-\frac{9}{10}}$$

and we have already shown that

$$|E_j D\phi(0,t)\cdot x - (\phi(x,t) - \phi(0,t))| = O(2^{-j} N^{-\frac{9}{10}});$$

applying the triangle inequality, we complete the proof of Lemma 7. □

It only remains to prove the main theorem.

Proof of Theorem 3. Our goal is to show that

$$\frac{d(E_j D\phi - h_j)}{dt} = \Big(\sum_{k<j} (\nabla u)_{k,E_k D\phi,i}\Big) E_j D\phi - \Big(\sum_{k<j} (\nabla u)_{k,h_k,i}\Big) h_j + O(N^{\frac{1}{5}}) E_j D\phi \tag{30}$$

and apply a version of Lemma 5. (We remove explicit dependence on 0 and t in order to simplify notation.) We use the definition of h_j and (20) (from which we may omit the product with x) and, with some adding and subtracting, we have

$$\begin{aligned}\frac{d(E_j D\phi - h_j)}{dt} = {} & \big(E_j((\nabla u)\circ\phi) - E_j((\tilde{\nabla} u)\circ\phi)\big)\cdot E_j D\phi + O(1) \\ & + \Big(E_j((\tilde{\nabla} u\circ\phi)) - \Big(\sum_{k<j} (\nabla u)_{k,E_k D\phi}\Big)\Big)\cdot E_j D\phi \\ & + \Big(\sum_{k<j} (\nabla u)_{k,E_k D\phi}\Big) E_j D\phi - \Big(\sum_{k<j} (\nabla u)_{k,h_k}\Big) h_j.\end{aligned} \tag{31}$$

(We are also omitting the explicit dependence on i, meaning that we are referring to a generic entry in the matrix.) We want the last line of (31) to achieve (30). The other terms are error terms, which we require to be controlled by $O(N^{\frac{1}{5}}) E_j D\phi$. We can easily estimate the coefficient of $E_j D\phi$ in the first line using (13) and (9):

$$\big|\big(E_j((\nabla u)\circ\phi) - E_j((\tilde{\nabla} u)\circ\phi)\big)\big| \lesssim \|(\nabla u)\circ\phi - (\tilde{\nabla} u)\circ\phi\|_{L^\infty} \lesssim (\log N)^3, \tag{32}$$

which gives that the first term in (31) is

$$O(N^{\frac{1}{5}}) E_j D\phi \tag{33}$$

and so, in order to have (30), we only have to control the coefficient of $E_j D\phi$ in the middle term. By definition and using the Biot–Savart law, this is equal to

$$E_j\left(\sum_{k\in\mathbb{Z}}\int_{A_k}\omega_{0,k}(s)K(\phi(s,t)-\phi(0,t))\,ds\right)-\sum_{k<j}\int_{A_k}\omega_{0,k}(s)K(E_k D\phi(0,t)\cdot s)\,ds. \tag{34}$$

We split the sum on the left into two parts, $k \geq j$ and $k < j$. For $k \geq j$, the sum is equal to

$$\begin{aligned}E_j\left(\sum_{k\geq j}\int_{A_k}\omega_{0,k}(s)K(\phi(s,t)-\phi(0,t))\,ds\right) &\lesssim \sum_{k\geq j}\|K\|_{L^\infty(\phi(A_k,t))}\int_{A_k}E_j\omega_{0,k}(s)\,ds\\ &\lesssim (\log N)^2\sum_{k\geq j}\|E_j\omega_{0,k}\|_{L^\infty}.\end{aligned} \tag{35}$$

Above, we get a factor of $2^{2k}(\log N)^2$ from integrating K and using (8), and a factor of 2^{-2k} comes from integrating $E_j\omega_{0,k}$ on A_j. For each k, $\|P_k\omega\|_{L^\infty} \lesssim 1$ and, by frequency support (after using the triangle inequality), there are fewer than $(\log N)^2$ many terms in the sum. Hence the error contributed by (35) is only $O((\log N)^4) \lesssim N^{\frac{1}{5}}$.

The rest of the error term, (34), where the first sum is over $k < j$, is

$$\sum_{k<j}\int_{A_k}\omega_{0,k}(s)\big(K(\phi(s,t)-\phi(0,t))-K(E_k D\phi(0,t)\cdot s)\big)\,ds. \tag{36}$$

By Lemma 7, we have $|\phi(s,t)-\phi(0,t)-E_k D\phi(0,t)\cdot s| \lesssim 2^{-k}N^{-\frac{9}{10}}$ when $|s| \sim 2^{-k}$. Further, by (14) and (8), we may choose C so that, if $x = \phi(s,t)-\phi(0,t)$, $y = E_k D\phi(0,t)\cdot s$ and $\epsilon = \frac{1}{50}-\frac{1}{500}$, we have

$$2^{-k}N^{-\epsilon} \lesssim |x|, |y| \lesssim 2^{-k}N^{\epsilon}$$

for times $0 \leq t \leq C(\log N)/N$. Then we have the bound

$$\begin{aligned}K_{12}(x)-K_{12}(y) &\lesssim 2^{4k}N^{4\epsilon}(x_1x_2-y_1y_2)\\ &= 2^{4k}N^{4\epsilon}(x_1(x_2-y_2)+y_2(x_1-y_1))\\ &\lesssim N^{5\epsilon}2^{3k}\max_i\{|x_i-y_i|\}\\ &\lesssim 2^{2k}N^{5\epsilon-\frac{9}{10}}\\ &\lesssim 2^{2k}N^{-\frac{4}{5}-\frac{1}{100}}\end{aligned}$$

and similarly for K_{11}. We can then estimate the sum (36) by

$$N^{-\frac{4}{5}-\frac{1}{100}}\sum_{k<j}\|\omega_{0,k}\|_{L^\infty}ds \lesssim N^{\frac{1}{5}-\frac{1}{100}}\log N \lesssim N^{\frac{1}{5}}$$

and with this we have the estimate that the middle term in (31) is $O(N^{\frac{1}{5}})E_j D\phi$ and, therefore, we have (30).

We will now apply a version of Lemma 5 using (30). Assume for contradiction that the estimate $|h_k(t) - E_k D\phi(0,t)| = O(N^{-\frac{7}{10}})$ fails for the first time at time $t_0 < C(\log N)/N$ and at scale j. So, for $k < j$ and times $t < t_0$, the estimate holds. Therefore, we have, for $t < t_0$,

$$\begin{aligned}
&\frac{d(E_j D\phi - h_j)}{dt} \\
&\qquad = \Bigl(\sum_{k<j} (\nabla u)_{k,E_k D\phi}\Bigr) E_k D\phi - \Bigl(\sum_{k<j} (\nabla u)_{k,h_k}\Bigr) h_j + O(N^{\frac{1}{5}}) E_j D\phi \\
&\qquad = \Bigl(\sum_{k<j} (\nabla u)_{k,h_k}\Bigr)(E_k D\phi - h_j) + \Bigl(\sum_{k<j} (\nabla u)_{k,E_k D\phi} - (\nabla u)_{k,h_k}\Bigr) E_j D\phi + O(N^{\frac{1}{5}}) E_j D\phi \\
&\qquad \lesssim \Bigl(\sum_{k<j} (\nabla u)_{k,h_k}\Bigr)(E_j D\phi - h_j) + O(N^{\frac{1}{5}}) E_j D\phi,
\end{aligned}$$

where, for the last line, we used our assumption that the estimate holds on scales $k < j$ and the estimates on the Biot–Savart kernels K_{1i}. Note that, at time $t = 0$, the difference $E_j D\phi - h_j$ equals 0. Suppose that T is the first time such that $E_j D\phi - h_j = N^{-\frac{4}{5}}$. If $t \le \min\{1/N, T\}$, we have

$$\frac{d(E_j D\phi - h_j)}{dt} \lesssim N^{\frac{1}{5}} \quad \text{since} \quad N^{\frac{1}{5}}(E_j D\phi - h_j) = O(1)$$

and it follows that $T = O(1/N)$. For times t such that $T \le t \le t_0 < C(\log N)/N$, the first term dominates and

$$E_j D\phi - h_j = O\bigl(N^{-\frac{4}{5}} \exp(tO(N))\bigr) = O(N^{-\frac{7}{10}}),$$

where the last equality comes from our choice of C, since $t_0 < C(\log N)/N \le (\log N^{\frac{1}{10}})/N$. Thus, the assumption that the estimate breaks down at scale j and at time $t_0 < C(\log N)/N$ was false, and hence it holds for all j and $t \le C(\log N)/N$, proving the theorem. □

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Received 16 Oct 2014. Revised 8 May 2015. Accepted 24 Jun 2015.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from MSP.

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ANALYSIS & PDE

Volume 8 No. 7 2015

2157-5045(2015)8:7;1-7