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ON THE CONTINUOUS RESONANT EQUATION FOR NLS II: STATISTICAL STUDY

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We consider the continuous resonant (CR) system of the 2-dimensional cubic nonlinear Schrödinger (NLS) equation. This system arises in numerous instances as an effective equation for the long-time dynamics of NLS in confined regimes (e.g., on a compact domain or with a trapping potential). The system was derived and studied from a deterministic viewpoint in several earlier works, which uncovered many of its striking properties. This manuscript is devoted to a probabilistic study of this system. Most notably, we construct global solutions in negative Sobolev spaces, which leave Gibbs and white noise measures invariant. Invariance of white noise measure seems particularly interesting in view of the absence of similar results for NLS.

1. Introduction

Presentation of the equation. The purpose of this manuscript is to construct some invariant measures for the so-called continuous resonant (CR) system of the cubic nonlinear Schrödinger equation. This system can be written as

$$\begin{cases} i\,\partial_t u = \mathcal{T}(u,u,u), & (t,x)\in\mathbb{R}\times\mathbb{R}^2, \\ u(0,x) = f(x), \end{cases} \tag{CR}$$

where the operator $\mathcal{T}$ defining the nonlinearity has several equivalent formulations corresponding to different interpretations/origins of this system. In its original formulation [Faou et al. 2013] as the large-box limit[1] of the resonant cubic NLS,[2] $\mathcal{T}$ can be written as follows: for $z\in\mathbb{R}^2$ and $x=(x_1,x_2)\in\mathbb{R}^2$, letting $x^\perp=(-x_2,x_1)$, we have

$$\mathcal{T}(f_1,f_2,f_3)(z) := \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_1(x+z)\, f_2(\lambda x^\perp + z)\, \overline{f_3(x+\lambda x^\perp + z)}\, dx\, d\lambda.$$

This integral can be understood as an integral over all rectangles having z as a vertex. It has the equivalent formulation [Germain et al. 2015]

$$\mathcal{T}(f_1,f_2,f_3) = 2\pi \int_{\mathbb{R}} \mathrm{e}^{-i\tau\Delta}\big[(\mathrm{e}^{i\tau\Delta} f_1)(\mathrm{e}^{i\tau\Delta} f_2)(\overline{\mathrm{e}^{i\tau\Delta} f_3})\big]\, d\tau.$$

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[1] Starting with the equation on a torus of size L and letting $L\to\infty$.

[2] This is NLS with only the resonant interactions retained (also known as the first Birkhoff normal form). It gives an approximation of NLS for sufficiently small initial data.

It was shown in [Faou et al. 2013] that the dynamics of (CR) approximate that of the cubic NLS equation on a torus of size L (with L large enough) over time scales $\sim L^2/\varepsilon^2$ (up to logarithmic loss in L), where ε denotes the size of the initial data.

Another formulation of (CR) comes from the fact that it is also the resonant system for the cubic nonlinear Schrödinger equation with harmonic potential given by

$$i\partial_t u - \Delta u + |x|^2 u = \mu |u|^2 u, \quad \mu \in \mathbb{R} \text{ constant}. \tag{1-1}$$

In this picture, $\mathcal{T}$ can be written as follows: denoting by $H := -\Delta + |x|^2 = -\partial_{x_1}^2 - \partial_{x_2}^2 + x_1^2 + x_2^2$ the harmonic oscillator on $\mathbb{R}^2$,

$$\mathcal{T}(f_1, f_2, f_3) = 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{e}^{i\tau H}[(\mathrm{e}^{-i\tau H} f_1)(\mathrm{e}^{-i\tau H} f_2)(\overline{\mathrm{e}^{-i\tau H} f_3})]\, d\tau.$$

As a result, the dynamics of (CR) approximate the dynamics of (1-1) over long nonlinear time scales for small enough initial data.

The equation (CR) is Hamiltonian. Indeed, introducing the functional

$$\begin{aligned}\mathcal{E}(u_1, u_2, u_3, u_4) &:= \langle \mathcal{T}(u_1, u_2, u_3)\, ,\, u_4 \rangle_{L^2} \\ &= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\mathbb{R}^2} (\mathrm{e}^{-itH} u_1)(\mathrm{e}^{-itH} u_2)(\overline{\mathrm{e}^{-itH} u_3})(\overline{\mathrm{e}^{-itH} u_4})\, dx\, dt\end{aligned}$$

and setting

$$\mathcal{E}(u) := \mathcal{E}(u, u, u, u),$$

(CR) derives from the Hamiltonian $\mathcal{E}$ given the symplectic form $\omega(f, g) = -4\mathfrak{Im}\langle f, g\rangle_{L^2(\mathbb{R}^2)}$ on $L^2(\mathbb{R}^2)$, so that (CR) is equivalent to

$$i\partial_t f = \frac{1}{2}\frac{\partial \mathcal{E}(f)}{\partial \bar{f}}.$$

In addition to the two instances mentioned above in which (CR) appears to describe the long-time dynamics of the cubic NLS equation — with or without potential — we mention the following:

- The equation (CR) appears as a modified scattering limit of the cubic NLS on $\mathbb{R}^3$ with harmonic tapping in two directions. Here, (CR) appears as an asymptotic system and any information on the asymptotic dynamics of (CR) directly gives the corresponding behavior for NLS with partial harmonic trapping. We refer to [Hani and Thomann 2015] for more details.

- When restricted to the Bargmann–Fock space (see below), the equation (CR) turns out to be the lowest-Landau-level equation, which describes fast-rotating Bose–Einstein condensates (see [Aftalion et al. 2006; Nier 2007; Gérard et al. $\geq$ 2015]).

- The equation (CR) can also be interpreted as describing the effective dynamics of high-frequency envelopes for NLS on the unit torus $\mathbb{T}^2$. This means that, if the initial data $\varphi(0)$ for NLS has its Fourier transform given by[3] $\{\hat{\varphi}(0, k) \sim g_0(k/N)\}_{k\in\mathbb{Z}^2}$ and if $g(t)$ evolves according to (CR) with initial data g_0

[3] Up to a normalizing factor in H^s, $s > 1$.

and $\varphi(t)$ evolves according to NLS with initial data $\varphi(0)$, then $g(t,k/N)$ approximates the dynamics of $\hat{\varphi}(t,k)$ in the limit of large N (see [Faou et al. 2013, Theorem 2.6]).

Some properties and invariant spaces. We review some of the properties of the (CR) equation that will be useful in this paper. For a more detailed study of the equation we refer to [Faou et al. 2013; Germain et al. 2015].

First, (CR) is globally well-posed in $L^2(\mathbb{R}^2)$. Amongst its conserved quantities, we note

$$\int_{\mathbb{R}^2} |u|^2\,dx \quad \text{and} \quad \int_{\mathbb{R}^2} (|x|^2|u|^2 + |\nabla u|^2)\,dx = \int_{\mathbb{R}^2} \bar{u} H u\,dx,$$

(recall that H denotes the harmonic oscillator $H = -\Delta + |x|^2$). This equation also enjoys many invariant spaces, in particular:

- The eigenspaces $(E_N)_{N\geq 0}$ of the harmonic oscillator are stable (see [Faou et al. 2013; Germain et al. 2015]). This is a manifestation of the fact that (CR) is the resonant equation associated to (1-1). Recall that H admits a complete basis of eigenvectors for $L^2(\mathbb{R}^2)$; each eigenspace E_N ($N = 0,1,2,\dots$) has dimension $N+1$.

- The set of radial functions is stable, as follows from the invariance of H under rotations (see [Germain et al. 2015]). Global dynamics on $L^2_{\text{rad}}(\mathbb{R}^2)$, the radial functions of $L^2(\mathbb{R}^2)$, can be defined. A basis of normalized eigenfunctions of H for $L^2_{\text{rad}}(\mathbb{R}^2)$ is given by

$$\varphi_n^{\text{rad}}(x) = \frac{1}{\sqrt{\pi}} L_n^{(0)}(|x|^2) e^{-|x|^2/2} \quad \text{with} \quad L_n^{(0)}(x) = e^x \frac{1}{n!}\left(\frac{d}{dx}\right)^n (e^{-x}x^n) \quad \text{for } n \in \mathbb{N}.$$

We record that $H\varphi_n^{\text{rad}} = (4n+2)\varphi_n^{\text{rad}}$.

- If $\mathcal{O}(\mathbb{C})$ stands for the set of entire functions on $\mathbb{C}$ (with the identification $z = x_1 + ix_2$), the Bargmann–Fock space $L^2_{\text{hol}}(\mathbb{R}^2) = L^2(\mathbb{R}^2) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2})$ is invariant under the flow of (CR). Global dynamics on $L^2_{\text{hol}}(\mathbb{R}^2)$ can be defined. A basis of normalized eigenfunctions of H for $L^2_{\text{hol}}(\mathbb{R}^2)$ is given by the "holomorphic" Hermite functions, also known as the "special Hermite functions", namely

$$\varphi_n^{\text{hol}}(x) = \frac{1}{\sqrt{\pi n!}} (x_1 + ix_2)^n \mathrm{e}^{-|x|^2/2} \quad \text{for } n \in \mathbb{N}.$$

Notice that $H\varphi_n^{\text{hol}} = 2(n+1)\varphi_n^{\text{hol}}$. It is proved in [Germain et al. 2015] that

$$\mathcal{T}(\varphi_{n_1}^{\text{hol}}, \varphi_{n_2}^{\text{hol}}, \varphi_{n_3}^{\text{hol}}) = \alpha_{n_1,n_2,n_3,n_4} \varphi_{n_4}^{\text{hol}}, \quad n_4 = n_1 + n_2 - n_3, \tag{1-2}$$

with

$$\alpha_{n_1,n_2,n_3,n_4} = \mathcal{H}(\varphi_{n_1}^{\text{hol}}, \varphi_{n_2}^{\text{hol}}, \varphi_{n_3}^{\text{hol}}, \varphi_{n_4}^{\text{hol}}) = \frac{\pi}{8} \frac{(n_1+n_2)!}{2^{n_1+n_2}\sqrt{n_1!n_2!n_3!n_4!}} \mathbf{1}_{n_1+n_2=n_3+n_4}.$$

As a result, the (CR) system reduces to the following infinite-dimensional system of ODEs when restricted to $\text{Span}\{\varphi_n\}_{n\in\mathbb{N}}$:

$$i\partial_t c_n(t) = \sum_{\substack{n_1,n_2,n_3 \in \mathbb{N} \\ n_1+n_2-n_3=n}} \alpha_{n_1,n_2,n_3,n} c_{n_1}(t) c_{n_2}(t) \bar{c}_{n_3}(t).$$

Statistical solutions. In this paper we construct global probabilistic solutions on each of the above-mentioned spaces which leave invariant either Gibbs or white noise measures. More precisely, our main results can be summarized as follows:

- We construct global strong flows on
$$X^0_{\mathrm{rad}}(\mathbb{R}^2) = \bigcap_{\sigma>0} \mathcal{H}^{-\sigma}_{\mathrm{rad}}(\mathbb{R}^2)$$
and on
$$X^0_{\mathrm{hol}}(\mathbb{R}^2) := \Big(\bigcap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R}^2)\Big) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}),$$
which leave the Gibbs measures invariant (see Theorem 2.5).
- We construct global weak probabilistic solutions on
$$X^{-1}_{\mathrm{hol}}(\mathbb{R}^2) := \Big(\bigcap_{\sigma>1} \mathcal{H}^{-\sigma}(\mathbb{R}^2)\Big) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}),$$
and this dynamics leaves the white noise measure invariant (see Theorem 2.6).

Since the ’90s, there have been many works devoted to the construction of Gibbs measures for dispersive equations and, more recently, much attention has been paid to the well-posedness of these equations with random initial conditions. We refer to the introduction of [Poiret et al. 2014] for references on the subject. In particular, concerning the construction of strong solutions for the nonlinear harmonic oscillator (which is related to (CR)), we refer to [Thomann 2009; Burq et al. 2013; Deng 2012; Poiret 2012a; 2012b; Poiret et al. 2014].

Let us define what we mean by white noise measure in our context. Denote by $(e_n)_{n\geq 0}$ a Hilbert basis of $L^2(0,1)$ and consider independent standard Gaussians $(g_n)_{n\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \boldsymbol{p})$. Then it is well known (see, e.g., [Hida 1980, Chapter 2]) that the random series
$$B_t = \sum_{n=0}^{+\infty} g_n \int_0^t e_n(s)\,ds$$
converges in $L^2(\Omega, \mathcal{F}, \boldsymbol{p})$ and defines a Brownian motion. The white noise measure is then defined by the map
$$\omega \mapsto W(t,\omega) = \frac{dB_t}{dt}(\omega) = \sum_{n=0}^{+\infty} g_n(\omega) e_n(t). \tag{1-3}$$
Now consider a Hilbert space $\mathcal{H}$ which is a space of functions on a manifold M and consider a Hilbert basis $(e_n)_{n\geq 0}$ of $\mathcal{H}$. We define the mean-zero Gaussian white noise (measure) on $\mathcal{H}$ as $\mu = \boldsymbol{p} \circ W^{-1}$, where
$$W(x,\omega) = \sum_{n=0}^{+\infty} g_n(\omega) e_n(x).$$

Notice that this measure is independent of the choice of the Hilbert basis of $\mathscr{H}$. It is clear that, for all $x \in M$, $\mathbb{E}_{\boldsymbol{p}}[W(x,\cdot)] = 0$. Moreover, for all $x,\, y \in M$ we have

$$\mathbb{E}_{\boldsymbol{p}}[W(x,\cdot)\overline{W(y,\cdot)}] = \sum_{n=0}^{+\infty} e_n(x)\overline{e_n(y)} = \delta(x-y),$$

since the sum in the previous line is the kernel of the identity projector on $\mathscr{H}$. For more details on Gaussian measures on Hilbert spaces, we refer to [Janson 1997].

Construction of flows invariant under white noise measures is much trickier due to the low regularity of the support of such measures, and there seem to be no results of this sort for NLS equations. We construct weak solutions on the support of the white noise measure on $X^{-1}_{\mathrm{hol}}(\mathbb{R}^2)$ using a method based on a compactness argument in the space of measures (the Prokhorov theorem) combined with a representation theorem of random variables (the Skorohod theorem). This approach has been first applied to the Navier–Stokes and Euler equations in [Albeverio and Cruzeiro 1990; Da Prato and Debussche 2002] and extended to dispersive equations by Burq, Thomann and Tzvetkov [Burq et al. 2014], who give a self-contained presentation of the method.

Notations. Define the harmonic Sobolev spaces for $s \in \mathbb{R}$ and $p \geq 1$ by

$$\mathscr{W}^{s,p} = \mathscr{W}^{s,p}(\mathbb{R}^2) = \{u \in L^p(\mathbb{R}^2) : H^{s/2}u \in L^p(\mathbb{R}^2)\}, \quad \mathscr{H}^s = \mathscr{W}^{s,2}.$$

They are endowed with the natural norms $\|u\|_{\mathscr{W}^{s,p}}$. Up to equivalence of norms we have, for $s \geq 0$ and $1 < p < +\infty$ (see [Yajima and Zhang 2004, Lemma 2.4]),

$$\|u\|_{\mathscr{W}^{s,p}} = \|H^{s/2}u\|_{L^p} \equiv \|(-\Delta)^{s/2}u\|_{L^p} + \|\langle x\rangle^s u\|_{L^p}. \tag{1-4}$$

Consider a probability space $(\Omega, \mathscr{F}, \boldsymbol{p})$. Throughout the paper, $\{g_n : n \geq 0\}$ and $\{g_{n,k} : n \geq 0,\ 0 \leq k \leq n\}$ are independent standard complex Gaussians $\mathscr{N}_{\mathbb{C}}(0,1)$ (their probability density function is $(1/\pi)e^{-|z|^2}\,dz$, dz being Lebesgue measure on $\mathbb{C}$). If X is a random variable, we denote by $\mathscr{L}(X)$ its law (or distribution).

We will sometimes use the notation $L^p_T = L^p(-T,T)$ for $T > 0$. If E is a Banach space and μ is a measure on E, we write $L^p_\mu = L^p(d\mu)$ and $\|u\|_{L^p_\mu E} = \big\| \|u\|_E \big\|_{L^p_\mu}$. We define $X^\sigma(\mathbb{R}^2) = \bigcap_{\tau<\sigma} \mathscr{H}^\tau(\mathbb{R}^2)$ and, if $I \subset \mathbb{R}$ is an interval, with an abuse of notation we write $\mathscr{C}(I; X^\sigma(\mathbb{R}^2)) = \bigcap_{\tau<\sigma} \mathscr{C}(I; \mathscr{H}^\tau(\mathbb{R}^2))$.

Finally, $\mathbb{N}$ denotes the set of natural integers including 0; $c, C > 0$ denote constants, the value of which may change from line to line. These constants will always be universal or uniformly bounded with respect to the other parameters. For two quantities A and B, we write $A \lesssim B$ if $A \leq CB$ and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

2. Statement of the results

As mentioned above, we will construct strong solutions on the support of Gibbs measures and prove the invariance of such measures. For white noise measures, solutions are weak and belong to the space $C_T X^{-1}$. We start by discussing the former case.

Global strong solutions invariant under Gibbs measure.

Measures and dynamics on the space E_N. The operator H is self-adjoint on $L^2(\mathbb{R}^2)$ and has the discrete spectrum $\{2N+2 : N \in \mathbb{N}\}$. For $N \geq 0$, denote by E_N the eigenspace associated to the eigenvalue $2N+2$. This space has dimension $N+1$. Consider any orthonormal basis $(\varphi_{N,k})_{0\leq k\leq N}$ of E_N. Define $\gamma_N \in L^2(\Omega; E_N)$ by

$$\gamma_N(\omega, x) = \frac{1}{\sqrt{N+1}} \sum_{k=0}^{N} g_{N,k}(\omega)\varphi_{N,k}(x).$$

The distribution of the random variable γ_N does not depend on the choice of the basis, and observe that the law of large numbers gives

$$\|\gamma_N\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{N+1} \sum_{k=0}^{N} |g_{N,k}(\omega)|^2 \to 1 \quad \text{a.s. when } N \to +\infty.$$

Then we define the probability measure $\mu_N = \gamma_{\#}\boldsymbol{p} := \boldsymbol{p} \circ \gamma_N^{-1}$ on E_N.

The L^p properties of the measures μ_N have been studied in [Poiret et al. 2015] with an improvement in [Robert and Thomann 2015]. We mention in particular the following result:

Theorem 2.1 [Poiret et al. 2015; Robert and Thomann 2015]. *There exist $c, C_1, C_2 > 0$ such that, for all $N \geq N_0$,*

$$\mu_N\big[u \in E_N : C_1 N^{-1/2}(\log N)^{1/2}\|u\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^\infty(\mathbb{R}^2)} \leq C_2 N^{-1/2}(\log N)^{1/2}\|u\|_{L^2(\mathbb{R}^2)}\big] \geq 1 - N^{-c}.$$

This proposition is a direct application of [Robert and Thomann 2015, Theorem 3.8] with $h = N^{-1}$ and $d = 2$. Notice that, for all $u \in E_N$, we have $\|u\|_{\mathcal{H}^s} = (2N+2)^{s/2}\|u\|_{L^2}$. The best (deterministic) L^∞ bound for an eigenfunction $u \in E_N$ is given by [Koch and Tataru 2005]:

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq C\|u\|_{L^2(\mathbb{R}^2)}, \tag{2-1}$$

and this estimate is optimal, since it is saturated by the radial Hermite functions. Therefore, the result of Theorem 2.1 shows that there is almost a gain of one derivative compared to the deterministic estimate (2-1).

It turns out that the measures μ_N are invariant under the flow of (CR), and we have the following:

Theorem 2.2. *For all $N \geq 1$, the measure μ_N is invariant under the flow Φ of* (CR) *restricted to E_N. Therefore, by the Poincaré theorem, μ_N-almost all $u \in E_N$ are recurrent in the following sense: for μ_N-almost all $u_0 \in E_N$ there exists a sequence of times $t_n \to +\infty$ such that*

$$\lim_{n\to+\infty} \|\Phi(t_n)u_0 - u_0\|_{L^2(\mathbb{R}^2)} = 0.$$

In the previous result, one only uses the invariance of the probability measure μ_N under the flow and no additional property of the equation (CR).

Gibbs measure on the space $X^0_\star(\mathbb{R}^2)$ and a well-posedness result. In the sequel we either consider the family $(\varphi_n^{\mathrm{rad}})_{n\geq 0}$ of the radial Hermite functions which are eigenfunctions of H associated to the eigenvalue $\lambda_n^{\mathrm{rad}} = 4n+2$, or the family $(\varphi_n^{\mathrm{hol}})_{n\geq 0}$ of the holomorphic Hermite functions which are eigenvalues of H associated to the eigenvalue $\lambda_n^{\mathrm{hol}} = 2n+2$. Set

$$X^0_{\mathrm{rad}}(\mathbb{R}^2) = \bigcap_{\sigma>0} \mathcal{H}^{-\sigma}_{\mathrm{rad}}(\mathbb{R}^2),$$

$$X^0_{\mathrm{hol}}(\mathbb{R}^2) := \left(\bigcap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R}^2) \right) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}).$$

In the following, we write $X^0_\star(\mathbb{R}^2)$ for $X^0_{\mathrm{rad}}(\mathbb{R}^2)$ or $X^0_{\mathrm{hol}}(\mathbb{R}^2)$, $\varphi_n^\star$ for φ_n^{rad} or φ_n^{hol}, etc.

Now define $\gamma_\star \in L^2(\Omega; X^0_\star(\mathbb{R}^2))$ by

$$\gamma_\star(\omega, x) = \sum_{n=0}^{+\infty} \frac{g_n(\omega)}{\sqrt{\lambda_n^\star}} \varphi_n^\star(x)$$

and consider the Gaussian probability measure $\mu_\star = (\gamma_\star)_\# \boldsymbol{p} := \boldsymbol{p} \circ \gamma_\star^{-1}$.

Lemma 2.3. *In each of the previous cases, the measure $\mu_\star$ is a probability measure on $X^0_\star(\mathbb{R}^2)$.*

Notice that, since (CR) conserves the $\mathcal{H}^1$ norm, $\mu_\star$ is formally invariant under its flow. More generally, we can define a family $(\rho_{\star,\beta})_{\beta\geq 0}$ of probability measures on $X^0_\star(\mathbb{R}^2)$ which are formally invariant under (CR) in the following way: define, for $\beta \geq 0$, the measure $\rho_\star = \rho_{\star,\beta}$ by

$$d\rho_\star(u) = C_\beta \mathrm{e}^{-\beta \mathcal{E}(u)}\, d\mu_\star(u), \tag{2-2}$$

where $C_\beta > 0$ is a normalizing constant. In Lemma 3.2, we will show that $\mathcal{E}(u) < +\infty$ $\mu_\star$-a.s., which enables us to define this probability measure.

For all $\beta \geq 0$, $\rho_\star(X^0_\star(\mathbb{R}^2)) = 1$ and $\rho_\star(L^2_\star(\mathbb{R}^2)) = 0$.

Remark 2.4. We could also give sense to a generalized version of (2-2) when $\beta < 0$ using the renormalizing method of Lebowitz, Rose and Speer. We do not give the details and refer to [Burq et al. 2013] for such a construction.

We are now able to state the following global existence result:

Theorem 2.5. *Let $\beta \geq 0$. There exists a set $\Sigma \subset X^0_\star(\mathbb{R}^2)$ of full $\rho_\star$ measure such that, for every $f \in \Sigma$, the equation* (CR) *with initial condition $u(0) = f$ has a unique global solution $u(t) = \Phi(t) f$ such that, for any $0 < s < \frac{1}{2}$,*

$$u(t) - f \in \mathcal{C}(\mathbb{R}; \mathcal{H}^s(\mathbb{R}^2)).$$

Moreover, for all $\sigma > 0$ and $t \in \mathbb{R}$,

$$\|u(t)\|_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)} \leq C(\Lambda(f,\sigma) + \ln^{1/2}(1+|t|))$$

and the constant $\Lambda(f,\sigma)$ satisfies the bound $\mu_\star(f : \Lambda(f,\sigma) > \lambda) \leq Ce^{-c\lambda^2}$.

Furthermore, the measure $\rho_\star$ is invariant under Φ: for any $\rho_\star$-measurable set $A \subset \Sigma$ and any $t \in \mathbb{R}$, $\rho_\star(A) = \rho_\star(\Phi(t)(A))$.

White noise measure on the space $X^{-1}_{\mathrm{hol}}(\mathbb{R}^2)$ and weak solutions. Our aim is now to construct weak solutions on the support of the white noise measure. Consider the Gaussian random variable

$$\gamma(\omega, x) = \sum_{n=0}^{+\infty} g_n(\omega)\varphi_n^{\mathrm{hol}}(x) = \frac{1}{\sqrt{\pi}}\left(\sum_{n=0}^{+\infty} \frac{(x_1 + i x_2)^n g_n(\omega)}{\sqrt{n!}}\right)\mathrm{e}^{-|x|^2/2} \tag{2-3}$$

and the measure $\mu = \boldsymbol{p} \circ \gamma^{-1}$. As in Lemma 2.3, we can show that the measure μ is a probability measure on

$$X^{-1}_{\mathrm{hol}}(\mathbb{R}^2) := \left(\bigcap_{\sigma>1} \mathscr{H}^{-\sigma}(\mathbb{R}^2)\right) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}).$$

Since $\|u\|_{L^2(\mathbb{R}^2)}$ is preserved by (CR), μ is formally invariant under (CR). We are not able to define a flow at this level of regularity; however, using compactness arguments combined with probabilistic methods, we will construct weak solutions.

Theorem 2.6. *There exists a set $\Sigma \subset X^{-1}_{\mathrm{hol}}(\mathbb{R}^2)$ of full μ measure such that, for every $f \in \Sigma$, the equation* (CR) *with initial condition $u(0) = f$ has a solution*

$$u \in \bigcap_{\sigma>1} \mathscr{C}(\mathbb{R}; \mathscr{H}^{-\sigma}(\mathbb{R}^2)).$$

The distribution of the random variable $u(t)$ is equal to μ (and thus independent of $t \in \mathbb{R}$):

$$\mathscr{L}_{X^{-1}(\mathbb{R}^2)}(u(t)) = \mathscr{L}_{X^{-1}(\mathbb{R}^2)}(u(0)) = \mu \quad \textit{for all } t \in \mathbb{R}.$$

Remark 2.7. One can also define the Gaussian measure $\mu = \boldsymbol{p} \circ \gamma^{-1}$ on $X^{-1}(\mathbb{R}^2) = \bigcap_{\sigma>1} \mathscr{H}^{-\sigma}(\mathbb{R}^2)$ by

$$\gamma(\omega, x) = \sum_{n=0}^{+\infty} \frac{1}{\sqrt{\lambda_n}} \sum_{k=-n}^{n} g_{n,k}(\omega)\varphi_{n,k}(x), \quad \lambda_n = 2n + 2,$$

(where the $\varphi_{n,k}$ are an orthonormal basis of eigenfunctions of the harmonic oscillator and the angular momentum operator). Since $\|u\|_{\mathscr{H}^1(\mathbb{R}^2)}$ is preserved by (CR), μ is formally invariant under (CR), but we are not able to obtain an analogous result in this case.

The same comment holds for the white noise measure $\mu = \boldsymbol{p} \circ \gamma^{-1}$ on $X^{-1}_{\mathrm{rad}}(\mathbb{R}^2) = \bigcap_{\sigma>1} \mathscr{H}^{-\sigma}_{\mathrm{rad}}(\mathbb{R}^2)$ with

$$\gamma(\omega, x) = \sum_{n=0}^{+\infty} g_n(\omega)\varphi_n^{\mathrm{rad}}(x),$$

which is also formally invariant under (CR).

Plan of the paper. The rest of the paper is organized as follows. In Section 3 we prove the results concerning the strong solutions and in Section 4 we construct the weak solutions.

3. Strong solutions

Proof of Theorem 2.2. The proof of Theorem 2.2 is an application of the Liouville theorem. Indeed, write $u_N = \sum_{k=0}^{N} c_{N,k}\varphi_{N,k} \in E_N$; then

$$d\mu_N = \frac{(N+1)^{N+1}}{\pi^{N+1}} \exp\biggl(-(N+1)\sum_{k=0}^{N}|c_{N,k}|^2\biggr)\prod_{k=0}^{N} da_{N,k}\,db_{N,k},$$

where $c_{N,k} = a_{N,k} + i b_{N,k}$.

The Lebesgue measure $\prod_{k=0}^{N} da_{N,k}\,db_{N,k}$ is preserved since (CR) is Hamiltonian and $\sum_{k=0}^{N}|c_{N,k}|^2 = \|u_N\|_{L^2}^2$ is a constant of motion.

Proof of Theorem 2.5. We start with the proof of Lemma 2.3.

Proof of Lemma 2.3. We only consider the case $X^0_\star(\mathbb{R}^2) = X^0_{\mathrm{hol}}(\mathbb{R}^2)$. It is enough to show that $\gamma_{\mathrm{hol}} \in X^0_{\mathrm{hol}}(\mathbb{R}^2)$ $\boldsymbol{p}$-a.s. First, for all $\sigma > 0$, we have

$$\int_\Omega \|\gamma_{\mathrm{hol}}\|^2_{\mathscr{H}^{-\sigma}(\mathbb{R}^2)}\,d\boldsymbol{p}(\omega) = \int_\Omega \sum_{n=0}^{+\infty}\frac{|g_n|^2}{(\lambda_n^{\mathrm{hol}})^{\sigma+1}}\,d\boldsymbol{p}(\omega) = C\sum_{n=0}^{+\infty}\frac{1}{(n+1)^{\sigma+1}} < +\infty, \tag{3-1}$$

therefore $\gamma_{\mathrm{hol}} \in \bigcap_{\sigma>0} L^2(\Omega;\mathscr{H}^{-\sigma}(\mathbb{R}^2))$. Next, by [Colliander and Oh 2012, Lemma 3.4], for all $A \ge 1$ there exists a set $\Omega_A \subset \Omega$ such that $\boldsymbol{p}(\Omega_A^c) \le \exp(-A^\delta)$ and, for all $\omega \in \Omega_A$, $\varepsilon > 0$ and $n \ge 0$,

$$|g_n(\omega)| \le CA(n+1)^\varepsilon.$$

Then, for $\omega \in \bigcup_{A\ge 1}\Omega_A$, we have $\sum_{n=0}^{+\infty} z^n g_n(\omega)/\sqrt{\lambda_n^{\mathrm{hol}} n!} \in \mathbb{O}(\mathbb{C})$. □

We first define a smooth version of the usual spectral projector. Choose $\chi \in \mathscr{C}_0^\infty(-1,1)$ so that $0 \le \chi \le 1$ with $\chi = 1$ on $\bigl[-\frac{1}{2},\frac{1}{2}\bigr]$. We define the operators $S_N = \chi(H/\lambda_N)$ as

$$S_N\biggl(\sum_{n=0}^{\infty} c_n\varphi_n^\star\biggr) = \sum_{n=0}^{\infty}\chi\biggl(\frac{\lambda_n^\star}{\lambda_N^\star}\biggr)c_n\varphi_n^\star.$$

Then, for all $1 < p < +\infty$, the operator S_N is bounded in $L^p(\mathbb{R}^2)$ (see [Deng 2012, Proposition 2.1] for a proof).

Local existence. It will be useful to work with an approximation of (CR). We consider the dynamical system given by the Hamiltonian $\mathscr{H}_N(u) := \mathscr{H}(S_N u)$. This system reads

$$\begin{cases} i\partial_t u_N = \mathscr{T}_N(u_N), & (t,x) \in \mathbb{R}\times\mathbb{R}^2,\\ u_N(0,x) = f, \end{cases} \tag{3-2}$$

with $\mathscr{T}_N(u_N) := S_N\mathscr{T}(S_N u, S_N u, S_N u)$. Observe that (3-2) is a finite-dimensional dynamical system on $\bigoplus_{k=0}^{N} E_k$ and that the projection of $u_N(t)$ on its complement is constant. For $\beta \ge 0$ and $N \ge 0$, we define the measures $\rho_\star^N$ by

$$d\rho_\star^N(u) = C_\beta^N \mathrm{e}^{-\beta\mathscr{H}_N(u)}\,d\mu_\star(u),$$

where $C_\beta^N > 0$ is a normalizing constant. We have the following result:

Lemma 3.1. *The system* (3-2) *is globally well-posed in* $L^2(\mathbb{R}^2)$. *Moreover, the measures* $\rho_\star^N$ *are invariant under its flow, denoted by* Φ_N.

Proof. The global existence follows from the conservation of $\|u_N\|_{L^2(\mathbb{R}^2)}$. The invariance of the measures is a consequence of the Liouville theorem and the conservation of $\sum_{k=0}^\infty \lambda_k |c_k|^2$ by the flow of (CR) (see [Faou et al. 2013]). We refer to [Burq et al. 2013, Lemma 8.1 and Proposition 8.2] for the details. ☐

We now state a result concerning dispersive bounds of Hermite functions.

Lemma 3.2. *For all* $2 \le p \le +\infty$,

$$\|\varphi_n^{\mathrm{hol}}\|_{L^p(\mathbb{R}^d)} \le C n^{\frac{1}{2p}-\frac{1}{4}}, \tag{3-3}$$

$$\|\varphi_n^{\mathrm{rad}}\|_{L^4(\mathbb{R}^d)} \le C n^{-\frac{1}{4}} (\ln n)^{\frac{1}{4}}. \tag{3-4}$$

Proof. By Stirling, we easily get that $\|\varphi_n^{\mathrm{hol}}\|_{L^\infty(\mathbb{R}^d)} \le C n^{-1/4}$, which is (3-3) for $p = \infty$; the estimate for $2 \le p \le \infty$ follows by interpolation. For the proof of (3-4), we refer to [Imekraz et al. 2015, Proposition 2.4]. ☐

Lemma 3.3. *(i) We have*

$\exists C > 0\ \exists c > 0\ \forall \lambda \ge 1\ \forall N \ge 1$

$$\mu_\star\big(u \in X_\star^0(\mathbb{R}^2) : \|e^{-itH} S_N u\|_{L^4([-\pi/4,\pi/4]\times\mathbb{R}^2)} > \lambda\big) \le C e^{-c\lambda^2}. \tag{3-5}$$

(ii) There exists $\beta > 0$ *such that*

$\exists C > 0\ \exists c > 0\ \forall \lambda \ge 1\ \forall N \ge N_0 \ge 1$

$$\mu_\star\big(u \in X_\star^0(\mathbb{R}^2) : \|e^{-itH}(S_N - S_{N_0}) u\|_{L^4([-\pi/4,\pi/4]\times\mathbb{R}^2)} > \lambda\big) \le C e^{-c N_0^\beta \lambda^2}. \tag{3-6}$$

(iii) In the holomorphic case, for all $2 \le p < +\infty$ *and* $s < \frac{1}{2} - \frac{1}{p}$,

$\exists C > 0\ \exists c > 0\ \forall \lambda \ge 1\ \forall N \ge 1$

$$\begin{aligned} \mu_{\mathrm{hol}}\big(u \in X_{\mathrm{hol}}^0(\mathbb{R}^2) : \|e^{-itH} u\|_{L^p([-\pi/4,\pi/4])\mathcal{W}^{s,p}(\mathbb{R}^2)} > \lambda\big) \le C e^{-c\lambda^2}, \\ \mu_{\mathrm{hol}}\big(u \in X_{\mathrm{hol}}^0(\mathbb{R}^2) : \|e^{-itH} u\|_{L^{8/3}([-\pi/4,\pi/4]\times\mathbb{R}^2)} > \lambda\big) \le C e^{-c\lambda^2}. \end{aligned} \tag{3-7}$$

(iv) In the radial case, for all $s < \frac{1}{2}$,

$\exists C > 0\ \exists c > 0\ \forall \lambda \ge 1\ \forall N \ge 1$

$$\mu_{\mathrm{rad}}\big(u \in X_{\mathrm{rad}}^0(\mathbb{R}^2) : \|e^{-itH} u\|_{L^4([-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)} > \lambda\big) \le C e^{-c\lambda^2}. \tag{3-8}$$

Proof. We have that

$$\mu_\star\big(u \in X_\star^0(\mathbb{R}^2) : \|e^{-itH} S_N u\|_{L^4([-\pi/4,\pi/4]\times\mathbb{R}^2)} > \lambda\big) = \boldsymbol{p}\bigg(\bigg\|\sum_{n=0}^\infty e^{-it\lambda_n} \chi\Big(\frac{\lambda_n}{\lambda_N}\Big) \frac{g_n(\omega)}{\sqrt{\lambda_n}} \varphi_n^\star(x)\bigg\|_{L^4([-\pi/4,\pi/4]\times\mathbb{R}^2)} > \lambda\bigg).$$

Set

$$F(\omega,t,x)\equiv\sum_{n=0}^{\infty}e^{-it\lambda_n^\star}\chi\Big(\frac{\lambda_n^\star}{\lambda_N^\star}\Big)\frac{g_n(\omega)}{\sqrt{\lambda_n^\star}}\varphi_n^\star(x).$$

Let $q\geq p\geq 2$ and $s\geq 0$. Recall here the Khintchine inequality (see, e.g., [Burq and Tzvetkov 2008, Lemma 3.1] for a proof): there exists $C>0$ such that, for all real $k\geq 2$ and $(a_n)\in\ell^2(\mathbb{N})$,

$$\Big\|\sum_{n\geq 0}g_n(\omega)\,a_n\Big\|_{L^k_p}\leq C\sqrt{k}\Big(\sum_{n\geq 0}|a_n|^2\Big)^{\frac{1}{2}}\tag{3-9}$$

if the g_n are i.i.d. normalized Gaussians. Applying it to (3-9) we get

$$\|H^{s/2}F(\omega,t,x)\|_{L^q_\omega}\leq C\sqrt{q}\Big(\sum_{n=0}^{\infty}\chi^2\Big(\frac{\lambda_n^\star}{\lambda_N^\star}\Big)\frac{|\varphi_n^\star(x)|^2}{\lambda_n^{\star\,1-s}}\Big)^{\frac{1}{2}}\leq C\sqrt{q}\Big(\sum_{n=0}^{\infty}\frac{|\varphi_n^\star(x)|^2}{\langle n\rangle^{1-s}}\Big)^{\frac{1}{2}}$$

and using the Minkowski inequality for $q\geq p$ twice gives

$$\|H^{s/2}F(\omega,t,x)\|_{L^q_\omega L^p_{t,x}}\leq\|H^{s/2}F(\omega,t,x)\|_{L^p_{t,x}L^q_\omega}\leq C\sqrt{q}\Big(\sum_{n=0}^{\infty}\frac{\|\varphi_n^\star(x)\|^2_{L^p(\mathbb{R}^2)}}{\langle n\rangle^{1-s}}\Big)^{\frac{1}{2}}.\tag{3-10}$$

We are now ready to prove (3-5). Set $p=4$ and $s=0$. By Lemma 3.2 we have $\|\varphi_n^\star\|_{L^4(\mathbb{R}^2)}\leq Cn^{-1/8}$, so we get, from (3-10),

$$\|F(\omega,t,x)\|_{L^q_\omega L^4_{t,x}}\leq C\sqrt{q}.$$

The Bienaymé–Chebyshev inequality then gives

$$\boldsymbol{p}\big(\|F(\omega,t,x)\|_{L^4_{t,x}}>\lambda\big)\leq(\lambda^{-1}\|F(\omega,t,x)\|_{L^q_\omega L^4_{t,x}})^q\leq(C\lambda^{-1}\sqrt{q})^q.$$

Thus, by choosing $q=\delta\lambda^2\geq 4$, for δ small enough we get the bound

$$\boldsymbol{p}\big(\|F(\omega,t,x)\|_{L^4_{t,x}}>\lambda\big)\leq Ce^{-c\lambda^2},$$

which is (3-5).

For the proof of (3-6), we analyze the function

$$G(\omega,t,x)\equiv\sum_{n=0}^{\infty}e^{-it\lambda_n^\star}\Big(\chi\Big(\frac{\lambda_n^\star}{\lambda_N^\star}\Big)-\chi\Big(\frac{\lambda_n^\star}{\lambda_{N_0}^\star}\Big)\Big)\frac{g_n(\omega)}{\sqrt{\lambda_n^\star}}\varphi_n^\star(x)$$

and we use that a negative power of N_0 can be gained in the estimate. Namely, there is $\gamma>0$ such that

$$\|G(\omega,t,x)\|_{L^q_\omega L^4_{t,x}}\leq C\sqrt{q}N_0^{-\gamma},$$

which implies (3-6).

To prove (3-7)–(3-8), we come back to (3-10) and argue similarly. This completes the proof of Lemma 3.3. ◻

Lemma 3.4. *Let $\beta \geq 0$. Let $p \in [1, \infty[$; then, when $N \to +\infty$,*

$$C_\beta^N e^{-\beta \mathcal{H}_N(u)} \to C_\beta e^{-\beta \mathcal{H}(u)} \quad \text{in } L^p(d\mu_\star(u)).$$

In particular, for all measurable sets $A \subset X_\star^0(\mathbb{R}^2)$,

$$\rho_\star^N(A) \to \rho_\star(A).$$

Proof. Let $G_\beta^N(u) = \mathrm{e}^{-\beta \mathcal{H}_N(u)}$ and $G_\beta(u) = \mathrm{e}^{-\beta \mathcal{H}(u)}$. By (3-6), we deduce that $\mathcal{H}_N(u) \to \mathcal{H}(u)$ in measure with respect to $\mu_\star$. In other words, for $\varepsilon > 0$ and $N \geq 1$, we let

$$A_{N,\varepsilon} = \{u \in X_\star^0(\mathbb{R}^2) : |G_\beta^N(u) - G_\beta(u)| \leq \varepsilon\},$$

then $\mu_\star(A_{N,\varepsilon}^c) \to 0$ when $N \to +\infty$. Since $0 \leq G, G_N \leq 1$,

$$\begin{aligned} \|G_\beta - G_\beta^N\|_{L^p_{\mu_\star}} &\leq \|(G_\beta - G_\beta^N)\mathbf{1}_{A_{N,\varepsilon}}\|_{L^p_{\mu_\star}} + \|(G_\beta - G_\beta^N)\mathbf{1}_{A^c_{N,\varepsilon}}\|_{L^p_{\mu_\star}} \\ &\leq \varepsilon(\mu_\star(A_{N,\varepsilon}))^{1/p} + 2(\mu_\star(A^c_{N,\varepsilon}))^{1/p} \\ &\leq C\varepsilon \end{aligned}$$

for N large enough. Finally, we have, when $N \to +\infty$,

$$C_\beta^N = \left(\int \mathrm{e}^{-\beta \mathcal{H}_N(u)}\, d\mu_\star(u)\right)^{-1} \to \left(\int \mathrm{e}^{-\beta \mathcal{H}(u)}\, d\mu_\star(u)\right)^{-1} = C_\beta,$$

and this ends the proof. ☐

We look for a solution to (CR) of the form $u = f + v$; thus v has to satisfy

$$\begin{cases} i\partial_t v = \mathcal{T}(f + v), & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ v(0, x) = 0, \end{cases} \tag{3-11}$$

with $\mathcal{T}(u) = \mathcal{T}(u, u, u)$. Similarly, we introduce

$$\begin{cases} i\partial_t v_N = \mathcal{T}_N(f + v_N), & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ v(0, x) = 0. \end{cases} \tag{3-12}$$

Recall that $X_\star^0(\mathbb{R}^2)$ equals $X_{\mathrm{hol}}^0(\mathbb{R}^2)$ or $X_{\mathrm{rad}}^0(\mathbb{R}^2)$. Define the sets, for $s < \frac{1}{2}$,

$$A_{\mathrm{rad}}^s(D) = \{f \in X_{\mathrm{rad}}^0(\mathbb{R}^2) : \|\mathrm{e}^{-itH} f\|_{L^4([-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)} \leq D\};$$

choosing $p(s) = 4/(1-2s)$, so that $s < \frac{1}{2} - \frac{1}{p}$,

$$\begin{aligned} &A_{\mathrm{hol}}^s(D) \\ &\quad = \{f \in X_{\mathrm{hol}}^0(\mathbb{R}^2) : \|\mathrm{e}^{-itH} f\|_{L^{8/3}([-\pi/4,\pi/4])L^{8/3}(\mathbb{R}^2)} + \|\mathrm{e}^{-itH} f\|_{L^{p(s)}([-\pi/4,\pi/4])\mathcal{W}^{s,p(s)}(\mathbb{R}^2)} \leq D\}. \end{aligned}$$

In the sequel, we write $A_\star^s(D)$ for $A_{\mathrm{hol}}^s(D)$ or $A_{\mathrm{rad}}^s(D)$. Then we have the following result:

Lemma 3.5. *Let $\beta \geq 0$. There exist $c, C > 0$ such that, for all $N \geq 0$,*

$$\rho_\star^N(A_\star^s(D)^c) \leq Ce^{-cD^2}, \quad \rho_\star(A_\star^s(D)^c) \leq Ce^{-cD^2} \quad \text{and} \quad \mu_\star(A_\star^s(D)^c) \leq Ce^{-cD^2}.$$

Proof. Since $\beta \geq 0$, we have $\rho_\star^N(A_\star^s(D)^c), \rho_\star(A_\star^s(D)^c) \leq C\mu_\star(A_\star^s(D)^c)$. The result is therefore given by (3-7) and (3-8). □

Proposition 3.6. *Let $s < \frac{1}{2}$. There exists $c > 0$ such that, for any $D \geq 0$, setting $\tau(D) = cD^{-2}$, for any $f \in A_\star^s(D)$ there exists a unique solution $v \in L^\infty([-\tau,\tau]; L^2(\mathbb{R}^2))$ to the equation* (3-11) *and a unique solution $v_N \in L^\infty([-\tau,\tau]; L^2(\mathbb{R}^2))$ to the equation* (3-12), *which furthermore satisfy*

$$\|v\|_{L^\infty([-\tau,\tau];\mathcal{H}^s(\mathbb{R}^2))},\ \|v_N\|_{L^\infty([-\tau,\tau];\mathcal{H}^s(\mathbb{R}^2))} \leq D.$$

The key ingredient in the proof of this result is the following trilinear estimate:

Lemma 3.7. *Assume that, for $1 \leq j \leq 3$ and $1 \leq k \leq 4$, $(p_{jk}, q_{jk}) \in [2,+\infty[^2$ are Strichartz admissible pairs, that is, they satisfy*

$$\frac{1}{q_{jk}} + \frac{1}{p_{jk}} = \frac{1}{2},$$

and they are such that, for $1 \leq j \leq 4$,

$$\frac{1}{p_{j1}} + \frac{1}{p_{j2}} + \frac{1}{p_{j3}} + \frac{1}{p_{j4}} = \frac{1}{q_{j1}} + \frac{1}{q_{j2}} + \frac{1}{q_{j3}} + \frac{1}{q_{j4}} = 1.$$

Then, for all $s \geq 0$, there exists $C > 0$ such that

$$\begin{aligned}\|\mathcal{T}(u_1,u_2,u_3)\|_{\mathcal{H}^s(\mathbb{R}^2)} \leq C&\|e^{-itH}u_1\|_{L^{p_{11}}\mathcal{W}^{s,q_{11}}}\|e^{-itH}u_2\|_{L^{p_{12}}L^{q_{12}}}\|e^{-itH}u_3\|_{L^{p_{13}}L^{q_{13}}}\\ &+ C\|e^{-itH}u_1\|_{L^{p_{21}}L^{q_{21}}}\|e^{-itH}u_2\|_{L^{p_{22}}\mathcal{W}^{s,q_{22}}}\|e^{-itH}u_3\|_{L^{p_{23}}L^{q_{23}}}\\ &+ C\|e^{-itH}u_1\|_{L^{p_{31}}L^{q_{31}}}\|e^{-itH}u_2\|_{L^{p_{32}}L^{q_{32}}}\|e^{-itH}u_3\|_{L^{p_{33}}\mathcal{W}^{s,q_{33}}},\end{aligned}$$

with the notation $L^p\mathcal{W}^{s,q} = L^p([-\pi/4,\pi/4]; \mathcal{W}^{s,q}(\mathbb{R}^2))$.

Proof. By duality,

$$\begin{aligned}&\|\mathcal{T}(u_1,u_2,u_3)\|_{\mathcal{H}^s(\mathbb{R}^2)}\\ &\qquad= \sup_{\|u\|_{L^2(\mathbb{R}^2)}=1} \langle H^{s/2}\mathcal{T}(u_1,u_2,u_3), u\rangle_{L^2(\mathbb{R}^2)}\\ &\qquad= 2\pi \sup_{\|u\|_{L^2(\mathbb{R}^2)}=1} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{\mathbb{R}^2} H^{s/2}\big((\mathrm{e}^{-itH}u_1)(\mathrm{e}^{-itH}u_2)(\overline{\mathrm{e}^{-itH}u_3})\big)(\overline{\mathrm{e}^{-itH}u})\,dx\,dt.\end{aligned}$$

Then, by Strichartz, for all u of unit norm in L^2 and for any admissible pair (p_4, q_4),

$$\begin{aligned}\|\mathcal{T}(u_1,u_2,u_3)\|_{\mathcal{H}^s(\mathbb{R}^2)} &\leq C\|(\mathrm{e}^{-itH}u_1)(\mathrm{e}^{-itH}u_2)(\overline{\mathrm{e}^{-itH}u_3})\|_{L^{p_4'}\mathcal{W}^{s,q_4'}}\|\mathrm{e}^{-itH}u\|_{L^{p_4}L^{q_4}}\\ &\leq C\|(\mathrm{e}^{-itH}u_1)(\mathrm{e}^{-itH}u_2)(\overline{\mathrm{e}^{-itH}u_3})\|_{L^{p_4'}\mathcal{W}^{s,q_4'}}.\end{aligned}$$

We then conclude using (1-4) and applying the following lemma twice. □

We have the following product rule:

Lemma 3.8. *Let $s \geq 0$, then*

$$\|u\,v\|_{\mathcal{W}^{s,q}} \leq C\|u\|_{L^{q_1}}\|v\|_{\mathcal{W}^{s,q_1'}} + C\|v\|_{L^{q_2}}\|u\|_{\mathcal{W}^{s,q_2'}}$$

for $1 < q < \infty$, $1 < q_1, q_2 < \infty$ *and* $1 \leq q_1', q_2' < \infty$ *such that*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_1'} = \frac{1}{q_2} + \frac{1}{q_2'}.$$

For the proof with the usual Sobolev spaces, we refer to [Taylor 2000, Proposition 1.1, p. 105]. The result in our context follows by using (1-4).

Proof of Proposition 3.6. We only consider (3-11), the other case being similar by the boundedness of S_N on $L^p(\mathbb{R}^2)$. For $s < \frac{1}{2}$, we define the space

$$Z^s(\tau) = \big\{v \in \mathcal{C}([-\tau,\tau];\mathcal{H}^s(\mathbb{R}^2)) : v(0) = 0 \text{ and } \|v\|_{Z^s(\tau)} \leq D\big\},$$

with $\|v\|_{Z^s(\tau)} = \|v\|_{L^\infty_{[-\tau,\tau]}\mathcal{H}^s(\mathbb{R}^2)}$ and, for $f \in A^s_\star(D)$, we define the operator

$$K(v) = -i\int_0^t \mathcal{T}(f+v)\,ds.$$

We will show that K has a unique fixed point $v \in Z^s(\tau)$.

<u>The case of radial Hermite functions</u>: By Lemma 3.7 with $(p_{jk}, q_{jk}) = (4,4)$, we have, for all $v \in Z^s(\tau)$,

$$\begin{aligned}\|K(v)\|_{Z^s(\tau)} &\leq \tau\|\mathcal{T}(f+v)\|_{Z^s(\tau)}\\ &\leq C\tau\big\|\|e^{-isH}(f+v)(t)\|^3_{L^4(s\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)}\big\|_{L^\infty_{t\in[-\tau,\tau]}}.\end{aligned} \tag{3-13}$$

Next, by Strichartz and since $v \in Z^s(\tau)$,

$$\begin{aligned}&\|e^{-isH}(f+v)(t)\|_{L^4(s\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)}\\ &\qquad\leq \|e^{-isH}f\|_{L^4(s\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)} + \|e^{-isH}v(t)\|_{L^4(s\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)}\\ &\qquad\leq C(D + \|v(t)\|_{\mathcal{H}^s(\mathbb{R}^2)})\\ &\qquad\leq 2CD.\end{aligned}$$

Therefore, from (3-13) we deduce

$$\|K(v)\|_{Z^s(\tau)} \leq C\tau D^3,$$

which implies that K maps $Z^s(\tau)$ into itself when $\tau \leq cD^{-2}$ for $c > 0$ small enough.

Similarly, for $v_1, v_2 \in Z^s(\tau)$, we have the bound

$$\|K(v_2) - K(v_1)\|_{Z^s(\tau)} \leq C\tau D^2\|v_2 - v_1\|_{Z^s(\tau)}, \tag{3-14}$$

which shows that if $\tau \leq cD^{-2}$ then K is a contraction of $Z^s(\tau)$. The Picard fixed point theorem gives the desired result.

The case of holomorphic Hermite functions: For $s < \frac{1}{2}$, recall that we set $p = p(s) = 4/(1-2s)$, so that $s < \frac{1}{2} - \frac{1}{p}$. We have

$$\begin{aligned}\|K(v)\|_{Z^s(\tau)} &\le \tau \|\mathcal{T}(f+v)\|_{Z^s(\tau)} \\ &\le C\tau\big(\|\mathcal{T}(f,f,f)\|_{Z^s} + \|\mathcal{T}(f,f,v)\|_{Z^s} + \|\mathcal{T}(f,v,v)\|_{Z^s} + \|\mathcal{T}(v,v,v)\|_{Z^s}\big).\end{aligned}$$

We estimate each term, thanks to Lemma 3.7 and Strichartz. The conjugation plays no role, so we forget it.

For the trilinear term in v,

$$\|\mathcal{T}(v,v,v)\|_{\mathcal{H}^s} \le C\|\mathrm{e}^{-it'H}v\|^3_{L^4_{t'\in[-\pi/4,\pi/4]}\mathcal{W}^{s,4}(\mathbb{R}^2)} \le C\|v\|^3_{\mathcal{H}^s(\mathbb{R}^2)}.$$

For the quadratic term in v, for $\delta > 0$ such that $2\big/\big(\frac{8}{3}+\delta\big) + \frac{1}{p} + \frac{1}{4} = 1$,

$$\begin{aligned}\|\mathcal{T}(v,v,f)\|_{\mathcal{H}^s} &\le C\|\mathrm{e}^{-it'H}v\|^2_{L^{8/3+\delta}(t'\in[-\pi/4,\pi/4])L^{8/3+\delta}(\mathbb{R}^2)}\|\mathrm{e}^{-it'H}f\|_{L^p(t'\in[-\pi/4,\pi/4])\mathcal{W}^{s,p}(\mathbb{R}^2)} \\ &\qquad + \|\mathrm{e}^{-it'H}v\|_{L^4(t'\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)}\|\mathrm{e}^{-it'H}v\|_{L^4(t'\in[-\pi/4,\pi/4])L^4(\mathbb{R}^2)} \\ &\qquad\qquad \times \|\mathrm{e}^{-it'H}f\|_{L^4(t'\in[-\pi/4,\pi/4])L^4(\mathbb{R}^2)} \\ &\le CD\|v\|^2_{\mathcal{H}^s(\mathbb{R}^2)}.\end{aligned}$$

For the linear term in v, with the same δ as above,

$$\begin{aligned}\|\mathcal{T}(v,f,f)\|_{\mathcal{H}^s} &\le C\|\mathrm{e}^{-it'H}v\|_{L^{8/3+\delta}(t'\in[-\pi/4,\pi/4])L^{8/3+\delta}(\mathbb{R}^2)}\|\mathrm{e}^{-it'H}f\|_{L^{8/3+\delta}(t'\in[-\pi/4,\pi/4])L^{8/3+\delta}(\mathbb{R}^2)} \\ &\qquad\qquad \times \|\mathrm{e}^{-it'H}f\|_{L^p(t'\in[-\pi/4,\pi/4])\mathcal{W}^{s,p}(\mathbb{R}^2)} \\ &\qquad + \|\mathrm{e}^{-it'H}v\|_{L^4(t'\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)}\|\mathrm{e}^{-it'H}f\|^2_{L^4(t'\in[-\pi/4,\pi/4])L^4(\mathbb{R}^2)} \\ &\le CD^2\|v\|_{\mathcal{H}^s(\mathbb{R}^2)}.\end{aligned}$$

For the constant term in v,

$$\begin{aligned}\|\mathcal{T}(f,f,f)\|_{\mathcal{H}^s} &\le C\|\mathrm{e}^{-it'H}f\|^2_{L^{8/3+\delta}(t'\in[-\pi/4,\pi/4])L^{8/3+\delta}(\mathbb{R}^2)}\|\mathrm{e}^{-it'H}f\|_{L^p(t'\in[-\pi/4,\pi/4])\mathcal{W}^{s,p}(\mathbb{R}^2)} \\ &\le CD^3.\end{aligned}$$

With these estimates at hand, the result follows by the Picard fixed point theorem. □

Approximation and invariance of the measure.

Lemma 3.9. *Fix $D \ge 0$ and $s < \frac{1}{2}$. Then, for all $\varepsilon > 0$, there exists $N_0 \ge 0$ such that, for all $f \in A^s_\star(D)$ and $N \ge N_0$,*

$$\|\Phi(t)f - \Phi_N(t)f\|_{L^\infty([-\tau_1,\tau_1];\mathcal{H}^s(\mathbb{R}^2))} \le \varepsilon,$$

where $\tau_1 = cD^{-2}$ for some $c > 0$.

Proof. Denoting for simplicity $\mathcal{T}(f) = \mathcal{T}(f,f,f)$,

$$v - v_N = -i\int_0^t \big[S_N(\mathcal{T}(f+v) - \mathcal{T}(f+v_N)) + (1-S_N)\mathcal{T}(f+v)\big]\,ds.$$

As in (3-14), we get

$$\|v-v_N\|_{Z^s(\tau)} \le C\tau D^2\|v-v_N\|_{Z^s(\tau)} + \int_{-\tau}^{\tau} \|(1-S_N)\mathcal{T}(f+v)\|_{\mathcal{H}^s(\mathbb{R}^2)}\, ds,$$

which in turn implies, when $C\tau D^2 \le \frac{1}{2}$,

$$\|v-v_N\|_{Z^s(\tau)} \le 2\int_{-\tau}^{\tau} \|(1-S_N)\mathcal{T}(f+v)\|_{\mathcal{H}^s(\mathbb{R}^2)}\, ds.$$

Choose $\eta>0$ so that $s+\eta<\frac{1}{2}$. Then, by the proof of Proposition 3.6, $\|\mathcal{T}(f+v)\|_{L^\infty_{[-\tau,\tau]}\mathcal{H}^{s+\eta}(\mathbb{R}^2)} \le CD^3$ if $\tau \le c_0 D^{-2}$ and, therefore, there exists $N_0 = N_0(\varepsilon, D)$ which satisfies the claim. □

In the next result, we summarize the results obtained by de Suzzoni [2011, Sections 3.3 and 4]. Since the proofs are very similar in our context, we skip them.

Let $D_{i,j} = (i + j^{1/2})^{1/2}$, with $i, j \in \mathbb{N}$, and set $T_{i,j} = \sum_{\ell=1}^{j} \tau_1(D_{i,\ell})$. Let

$$\Sigma_{N,i} := \{f : \Phi_N(\pm T_{i,j})f \in A^s_\star(D_{i,j+1}) \text{ for all } j \in \mathbb{N}\}$$

and

$$\Sigma_i := \limsup_{N\to+\infty} \Sigma_{N,i}, \qquad \Sigma := \bigcup_{i\in\mathbb{N}} \Sigma_i.$$

Proposition 3.10. *Let $\beta \ge 0$; then:*

(i) The set Σ is of full $\rho_\star$ measure.

(ii) For all $f \in \Sigma$, there exists a unique global solution $u = f + v$ to (CR). *This define a global flow Φ on Σ*

(iii) For all measurable set $A \subset \Sigma$ and all $t \in \mathbb{R}$,

$$\rho_\star(A) = \rho_\star(\Phi(t)(A)).$$

4. Weak solutions: proof of Theorem 2.6

Definition of $\mathcal{T}(u,u,u)$ on the support of μ. For $N \ge 0$, denote by Π_N the orthogonal projector on the space $\bigoplus_{k=0}^{N} E_k$ (in this section, we do not need the smooth cut-offs S_N). In the sequel, we write $\mathcal{T}(u) = \mathcal{T}(u,u,u)$ and $\mathcal{T}_N(u) = \Pi_N\mathcal{T}(\Pi_N u, \Pi_N u, \Pi_N u)$.

Proposition 4.1. *For all $p \ge 2$ and all $\sigma > 1$, the sequence $(\mathcal{T}_N(u))_{N\ge1}$ is a Cauchy sequence in $L^p(X^{-1}(\mathbb{R}^2), \mathcal{B}, d\mu; \mathcal{H}^{-\sigma}(\mathbb{R}^2))$. Namely, for all $p \ge 2$, there exist $\delta > 0$ and $C > 0$ such that, for all $1 \le M < N$,*

$$\int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(u) - \mathcal{T}_M(u)\|^p_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}\, d\mu(u) \le CM^{-\delta}.$$

We denote by $\mathcal{T}(u) = \mathcal{T}(u,u,u)$ the limit of this sequence and we have, for all $p \ge 2$,

$$\|\mathcal{T}(u)\|_{L^p_\mu \mathcal{H}^{-\sigma}(\mathbb{R}^2)} \le C_p. \tag{4-1}$$

Before we turn to the proof of Proposition 4.1, let us state two elementary results which will be needed in the sequel.

Lemma 4.2. *For all* $n \in \mathbb{N}$,

$$\sum_{k=n}^{+\infty} \frac{1}{2^k}\binom{k}{n} = \sum_{k=n}^{+\infty} \frac{k!}{2^k n!(k-n)!} = 2.$$

Proof. For $|z| < 1$ we have $1/(1-z) = \sum_{k=0}^{+\infty} z^k$. If we differentiate this formula n times we get

$$\frac{n!}{(1-z)^{n+1}} = \sum_{k=n}^{+\infty} \frac{k!}{(k-n)!} z^{k-n},$$

which implies the result, taking $z = \frac{1}{2}$. □

Lemma 4.3. *Choose* $0 < \varepsilon < 1$ *and* $p, L \geq 1$ *so that* $p \leq L^{\varepsilon}$. *Then*

$$\frac{L!}{2^L (L-p)!} \leq C 2^{-L/2}.$$

Proof. The proof is straightforward. By the assumption $p \leq L^{\varepsilon}$,

$$\frac{L!}{(L-p)!} \leq L^p \leq C 2^{L/2},$$

which was the claim. □

Proof of Proposition 4.1. By the result [Thomann and Tzvetkov 2010, Proposition 2.4] on the Wiener chaos, we only have to prove the statement for $p = 2$.

Firstly, by definition of the measure μ,

$$\int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(u) - \mathcal{T}_M(u)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}\, d\mu(u) = \int_{\Omega} \|\mathcal{T}_N(\gamma(\omega)) - \mathcal{T}_M(\gamma(\omega))\|^2_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}\, d\boldsymbol{p}(\omega).$$

Therefore, it is enough to prove that $(\mathcal{T}_N(\gamma))_{N \geq 1}$ is a Cauchy sequence in $L^2(\Omega; \mathcal{H}^{-\sigma}(\mathbb{R}^2))$. Let $1 \leq M < N$ and fix $\alpha > \frac{1}{2}$. By (1-2), we get

$$\begin{aligned}
H^{-\alpha}\mathcal{T}_N(\gamma) &= \frac{1}{2^\alpha} \sum_{A_N} \frac{g_{n_1} g_{n_2} \bar{g}_{n_3}}{(n_1+n_2-n_3+1)^\alpha} \mathcal{T}(\varphi^{\mathrm{hol}}_{n_1}, \varphi^{\mathrm{hol}}_{n_2}, \varphi^{\mathrm{hol}}_{n_3}) \\
&= \frac{\pi}{8 \cdot 2^\alpha} \sum_{A_N} \frac{(n_1+n_2)!}{2^{n_1+n_2}\sqrt{n_1! n_2! n_3! (n_1+n_2-n_3)!}} \frac{g_{n_1} g_{n_2} \bar{g}_{n_3}}{(n_1+n_2-n_3+1)^\alpha} \varphi^{\mathrm{hol}}_{n_1+n_2-n_3} \\
&= \frac{\pi}{8 \cdot 2^\alpha} \sum_{p=0}^{N} \frac{1}{(p+1)^\alpha} \Bigg(\sum_{A_N^{(p)}} \frac{(n_1+n_2)!}{2^{n_1+n_2}\sqrt{n_1! n_2! n_3! p!}} g_{n_1} g_{n_2} \bar{g}_{n_3} \Bigg) \varphi^{\mathrm{hol}}_p
\end{aligned}$$

with

$$\begin{aligned}
A_N &= \{n \in \mathbb{N}^3 : 0 \leq n_j \leq N,\ 0 \leq n_1+n_2-n_3 \leq N\}, \\
A_N^{(p)} &= \{n \in \mathbb{N}^3 : 0 \leq n_j \leq N,\ n_1+n_2-n_3 = p\} \quad \text{if } 0 \leq p \leq N.
\end{aligned}$$

Therefore,

$$\|\mathcal{T}_N(\gamma)-\mathcal{T}_M(\gamma)\|^2_{\mathcal{H}^{-\alpha}(\mathbb{R}^2)}$$
$$=\frac{\pi^2}{64\cdot 2^{2\alpha}}\sum_{p=0}^{N}\frac{1}{(p+1)^{2\alpha}}\sum_{(n,m)\in A^{(p)}_{M,N}\times A^{(p)}_{M,N}}\frac{(n_1+n_2)!\,(m_1+m_2)!\,g_{n_1}g_{n_2}\bar{g}_{n_3}\bar{g}_{m_1}g_{m_2}g_{m_3}}{2^{n_1+n_2}2^{m_1+m_2}\,p!\sqrt{n_1!n_2!n_3!}\sqrt{m_1!m_2!m_3!}},$$

where $A^{(p)}_{M,N}$ is the set defined by

$$A^{(p)}_{M,N}=\big\{n\in\mathbb{N}^3:0\le n_j\le N,\ n_1+n_2-n_3=p\in\{0,\dots,N\}\ \text{ and }\ \max\{n_1,n_2,n_3,p\}>M\big\}.$$

Now we take the integral over Ω. Since $(g_n)_{n\ge 0}$ are independent and centered Gaussians, we deduce that each term in the right-hand side vanishes unless one of two cases holds:

Case 1: $(n_1,n_2,n_3)=(m_1,m_2,m_3)$ or $(n_1,n_2,n_3)=(m_2,m_1,m_3)$.

Case 2: We have one of

$$(n_1,n_2,m_1)=(n_3,m_2,m_3),\quad (n_1,n_2,m_2)=(n_3,m_1,m_3),$$
$$(n_1,n_2,m_3)=(m_1,n_3,m_2),\quad (n_1,n_2,m_3)=(m_2,n_3,m_1).$$

We write

$$\int_\Omega \|\mathcal{T}_N(\gamma)-\mathcal{T}_M(\gamma)\|^2_{\mathcal{H}^{-2\alpha}(\mathbb{R}^2)}\,d\boldsymbol{p}=J_1+J_2,$$

where J_1 and J_2 correspond to the contribution in the sum of each of cases 1 and 2, respectively.

Contribution in case 1: By symmetry, we can assume that $(n_1,n_2,n_3)=(m_1,m_2,m_3)$. Define

$$B^{(p)}_{M,N}=\big\{n\in\mathbb{N}^2:0\le n_j\le N\ \text{ and }\ \max\{n_1,n_2,n_1+n_2-p,p\}>M\big\}.$$

Then

$$J_1\le C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\sum_{B^{(p)}_{M,N}}\frac{((n_1+n_2)!)^2}{2^{2(n_1+n_2)}\,p!\,n_1!\,n_2!\,(n_1+n_2-p)!}.$$

In the previous sum, we make the change of variables $L=n_1+n_2$ and we observe that on $B^{(p)}_{M,N}$ we have $L\ge M$; then

$$J_1\le C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\sum_{L\ge p+M}\sum_{n_1=0}^{L}\frac{(L!)^2}{2^{2L}\,p!\,n_1!\,(L-n_1)!\,(L-p)!}$$
$$=C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\sum_{L\ge p+M}\frac{L!}{2^L\,p!\,(L-p)!},$$

where we used the fact that $\sum_{n_1=0}^{L}\binom{L}{n_1}=2^L$. Let $\varepsilon>0$ and split the previous sum into two pieces:

$$\begin{aligned}
J_1 &\le C\sum_{p=0}^{M^\varepsilon}\frac{1}{(1+p)^{2\alpha}}\sum_{L=M}^{+\infty}\frac{L!}{2^L p!(L-p)!}+C\sum_{p=M^\varepsilon+1}^{+\infty}\frac{1}{(1+p)^{2\alpha}}\sum_{L=p}^{+\infty}\frac{L!}{2^L p!(L-p)!}\\
&\le C\sum_{p=0}^{M^\varepsilon}\frac{1}{(1+p)^{2\alpha}}\sum_{L=M}^{+\infty}\frac{L!}{2^L p!(L-p)!}+2C\sum_{p=M^\varepsilon+1}^{+\infty}\frac{1}{(1+p)^{2\alpha}}=:J_{11}+J_{12},
\end{aligned}$$

by Lemma 4.2. For the first sum, we can use Lemma 4.3, since $p\le M^\varepsilon\le L^\varepsilon$; thus

$$J_{11}\le C\sum_{p=0}^{M^\varepsilon}\frac{1}{(1+p)^{2\alpha}p!}\sum_{L=M}^{+\infty}\frac{1}{2^{L/2}}\le C\sum_{L=M}^{+\infty}\frac{1}{2^{L/2}}\le CM^{-\delta}.$$

Next, clearly, $J_{12}\le CM^{-\delta}$ because $\alpha>\frac{1}{2}$, and this gives $J_1\le CM^{-\delta}$.

<u>Contribution in case 2</u>: We can assume that $(n_1,n_2,m_1)=(n_3,m_2,m_3)$. Then, for $n,m\in A_{M,N}^{(p)}$, we have $n_2=m_2=p$. Moreover, by symmetry, we can assume that $n_1>M$ or $p>M$. Thus,

$$\begin{aligned}
J_2 \le C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\sum_{n_1=M+1}^{+\infty}\sum_{m_1=0}^{+\infty}\frac{(n_1+p)!(m_1+p)!}{2^{n_1+p}2^{m_1+p}n_1!m_1!(p!)^2}\\
+C\sum_{p\ge M+1}\frac{1}{(1+p)^{2\alpha}}\sum_{n_1=0}^{+\infty}\sum_{m_1=0}^{+\infty}\frac{(n_1+p)!(m_1+p)!}{2^{n_1+p}2^{m_1+p}n_1!m_1!(p!)^2}=:J_{21}+J_{22}.
\end{aligned}$$

To begin with, by Lemma 4.2, we have

$$\begin{aligned}
J_{22}&=C\sum_{p\ge M+1}\frac{1}{(1+p)^{2\alpha}}\Big(\sum_{n_1=0}^{+\infty}\frac{(n_1+p)!}{2^{n_1+p}n_1!\,p!}\Big)\Big(\sum_{m_1=0}^{+\infty}\frac{(m_1+p)!}{2^{m_1+p}m_1!\,p!}\Big)\\
&=4C\sum_{p\ge M+1}\frac{1}{(1+p)^{2\alpha}}\le cM^{-\delta}.
\end{aligned}$$

Then, by Lemma 4.2 again,

$$\begin{aligned}
J_{21}&=C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\Big(\sum_{n_1=M+1}^{+\infty}\frac{(n_1+p)!}{2^{n_1+p}n_1!\,p!}\Big)\Big(\sum_{m_1=0}^{+\infty}\frac{(m_1+p)!}{2^{m_1+p}m_1!\,p!}\Big)\\
&=2C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\Big(\sum_{n_1=M+1}^{+\infty}\frac{(n_1+p)!}{2^{n_1+p}n_1!\,p!}\Big)\\
&=2C\sum_{p=0}^{M^\varepsilon}\frac{1}{(1+p)^{2\alpha}}\Big(\sum_{n_1=M+1}^{+\infty}\frac{(n_1+p)!}{2^{n_1+p}n_1!\,p!}\Big)+2C\sum_{p=M^\varepsilon+1}^{+\infty}\frac{1}{(1+p)^{2\alpha}}\Big(\sum_{n_1=M+1}^{+\infty}\frac{(n_1+p)!}{2^{n_1+p}n_1!\,p!}\Big)\\
&=:K_1+K_2.
\end{aligned}$$

On the one hand, by Lemma 4.3,

$$K_1 \le C\Big(\sum_{p=0}^{M^\varepsilon} \frac{1}{(1+p)^{2\alpha}p!}\Big)\Big(\sum_{n_1=M+1}^{+\infty} 2^{-n_1/2}\Big) \le CM^{-\delta}$$

and, on the other hand, by Lemma 4.2, since $\alpha > \frac{1}{2}$,

$$K_2 \le C \sum_{p=M^\varepsilon+1}^{+\infty} \frac{1}{(1+p)^{2\alpha}} \le CM^{-\delta}.$$

Putting all the estimates together, we get $J_2 \le CM^{-\delta}$, which concludes the proof. □

Study of the measure ν_N. Let $N \ge 1$. We then consider the following approximation of (CR):

$$\begin{cases} i\partial_t u = \mathcal{T}_N(u), & (t,x) \in \mathbb{R}\times\mathbb{R}^2, \\ u(0,x) = f(x) \in X^{-1}(\mathbb{R}^2). \end{cases} \tag{4-2}$$

The equation (4-2) is an ODE in the frequencies less than N and $(1-\Pi_N)u(t) = (1-\Pi_N)f$ for all $t \in \mathbb{R}$.

The main reason to introduce this system is the following proposition, whose proof we omit.

Proposition 4.4. *The equation* (4-2) *has a global flow Φ_N. Moreover, the measure μ is invariant under Φ_N: for any Borel set $A \subset X^{-1}(\mathbb{R}^2)$ and for all $t \in \mathbb{R}$, $\mu(\Phi_N(t)(A)) = \mu(A)$.*

In particular, if $\mathscr{L}_{X^{-1}}(v) = \mu$ then, for all $t \in \mathbb{R}$, $\mathscr{L}_{X^{-1}}(\Phi_N(t)v) = \mu$.

We denote by ν_N the measure on $\mathscr{C}([-T,T]; X^{-1}(\mathbb{R}^2))$, defined as the image measure of μ under the map

$$X^{-1}(\mathbb{R}^2) \to \mathscr{C}([-T,T]; X^{-1}(\mathbb{R}^2)),$$
$$v \mapsto \Phi_N(t)(v).$$

Lemma 4.5. *Let $\sigma > 1$ and $p \ge 2$. Then there exists $C > 0$ such that, for all $N \ge 1$,*

$$\big\| \|u\|_{W_T^{1,p}\mathscr{H}_x^{-\sigma}} \big\|_{L^p_{\nu_N}} \le C.$$

Proof. Firstly, we have that, for $\sigma > 1$, $p \ge 2$ and $N \ge 1$,

$$\big\| \|u\|_{L_T^p\mathscr{H}_x^{-\sigma}} \big\|_{L^p_{\nu_N}} \le C.$$

Indeed, by the definition of ν_N and the invariance of μ under Φ_N, we have

$$\|u\|_{L^p_{\nu_N}L^p_T\mathscr{H}_x^{-\sigma}} = (2T)^{1/p}\|v\|_{L^p_\mu\mathscr{H}_x^{-\sigma}} = (2T)^{1/p}\|\gamma\|_{L^p_{\boldsymbol{P}}\mathscr{H}_x^{-\sigma}}.$$

Then, by the Khintchine inequality (3-9) and (3-1), for all $p \ge 2$,

$$\|\gamma\|_{L^p_{\boldsymbol{P}}\mathscr{H}_x^{-\sigma}} \le C\sqrt{p}\|\gamma\|_{L^2_{\boldsymbol{P}}\mathscr{H}_x^{-\sigma}} \le C.$$

We refer to [Burq et al. 2014, Proposition 3.1] for the details.

Next, we show that $\big\| \|\partial_t u\|_{L^p_T \mathcal{H}^{-\sigma}_x} \big\|_{L^p_{\nu_N}} \le C$. By definition of ν_N,

$$\|\partial_t u\|^p_{L^p_{\nu_N} L^p_T \mathcal{H}^{-\sigma}_x} = \int_{\mathscr{C}([-T,T];X^{-1}(\mathbb{R}^2))} \|\partial_t u\|^p_{L^p_T \mathcal{H}^{-\sigma}_x} \, d\nu_N(u) = \int_{X^{-1}(\mathbb{R}^2)} \|\partial_t \Phi_N(t)(v)\|^p_{L^p_T \mathcal{H}^{-\sigma}_x} \, d\mu(v).$$

Now, since $\Phi_N(t)(v)$ satisfies (4-2) and by the invariance of μ, we have

$$\|\partial_t u\|^p_{L^p_{\nu_N} L^p_T \mathcal{H}^{-\sigma}_x} = \int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(\Phi_N(t)(v))\|^p_{L^p_T \mathcal{H}^{-\sigma}_x} \, d\mu(v) = 2T \int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(v)\|^p_{\mathcal{H}^{-\sigma}_x} \, d\mu(v)$$

and we conclude with (4-1) and Proposition 4.1. □

The convergence argument. The importance of Lemma 4.5 above comes from the fact that it allows us to establish the following tightness result for the measures ν_N. We refer to [Burq et al. 2014, Proposition 4.11] for the proof.

Proposition 4.6. *Let $T > 0$ and $\sigma > 1$. Then the family of measures*

$$(\nu_N)_{N \ge 1} \quad \textit{with} \quad \nu_N = \mathscr{L}_{\mathscr{C}_T \mathcal{H}^{-\sigma}}(u_N(t); t \in [-T, T])$$

is tight in $\mathscr{C}([-T, T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2))$.

The result of Proposition 4.6 enables us to use the Prokhorov theorem: for each $T > 0$ there exists a subsequence ν_{N_k} and a measure ν on the space $\mathscr{C}([-T, T]; X^{-1}(\mathbb{R}^2))$ such that, for all $\tau > 1$ and all bounded continuous functions $F : \mathscr{C}([-T, T]; \mathcal{H}^{-\tau}(\mathbb{R}^2)) \to \mathbb{R}$,

$$\int_{\mathscr{C}([-T,T];\mathcal{H}^{-\tau}(\mathbb{R}^2))} F(u) \, d\nu_{N_k}(u) \to \int_{\mathscr{C}([-T,T];\mathcal{H}^{-\tau}(\mathbb{R}^2))} F(u) \, d\nu(u).$$

By the Skorohod theorem, there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \tilde{\boldsymbol{p}})$, a sequence of random variables $(\tilde{u}_{N_k})$ and a random variable $\tilde{u}$ with values in $\mathscr{C}([-T, T]; X^{-1}(\mathbb{R}^2))$ such that

$$\mathscr{L}(\tilde{u}_{N_k}; t \in [-T, T]) = \mathscr{L}(u_{N_k}; t \in [-T, T]) = \nu_{N_k}, \quad \mathscr{L}(\tilde{u}; t \in [-T, T]) = \nu, \tag{4-3}$$

and, for all $\tau > 1$,

$$\tilde{u}_{N_k} \to \tilde{u} \quad \tilde{\boldsymbol{p}}\text{-a.s. in } \mathscr{C}([-T, T]; \mathcal{H}^{-\tau}(\mathbb{R}^2)). \tag{4-4}$$

We now claim that $\mathscr{L}_{X^{-1}}(u_{N_k}(t)) = \mathscr{L}_{X^{-1}}(\tilde{u}_{N_k}(t)) = \mu$ for all $t \in [-T, T]$ and $k \ge 1$. Indeed, for all $t \in [-T, T]$, the evaluation map

$$\begin{aligned} R_t : \mathscr{C}([-T, T]; X^{-1}(\mathbb{R}^2)) &\to X^{-1}(\mathbb{R}^2), \\ u &\mapsto u(t, \cdot), \end{aligned}$$

is well-defined and continuous.

Thus, for all $t \in [-T, T]$, $u_{N_k}(t)$ and $\tilde{u}_{N_k}(t)$ have same distribution $(R_t)_\# \nu_{N_k}$. By Proposition 4.4, we obtain that this distribution is μ.

Thus, from (4-4) we deduce that

$$\mathscr{L}_{X^{-1}}(\tilde{u}(t)) = \mu \quad \text{for all } t \in [-T, T]. \tag{4-5}$$

Let $k \geq 1$ and $t \in \mathbb{R}$ and consider the random variable X_k given by

$$X_k = u_{N_k}(t) - R_0(u_{N_k}(t)) + i \int_0^t \mathcal{T}_{N_k}(u_{N_k})\, ds.$$

Define $\widetilde{X}_k$ similarly to X_k, with u_{N_k} replaced by $\tilde{u}_{N_k}$. Then, by (4-3),

$$\mathcal{L}_{\mathscr{C}_T X^{-1}}(\widetilde{X}_{N_k}) = \mathcal{L}_{\mathscr{C}_T X^{-1}}(X_{N_k}) = \delta_0.$$

In other words, $\widetilde{X}_k = 0$ $\tilde{\boldsymbol{p}}$-a.s. and $\tilde{u}_{N_k}$ satisfies the following equation $\tilde{\boldsymbol{p}}$-a.s.:

$$\tilde{u}_{N_k}(t) = R_0(\tilde{u}_{N_k}(t)) - i \int_0^t \mathcal{T}_{N_k}(\tilde{u}_{N_k})\, ds. \tag{4-6}$$

We now show that we can pass to the limit $k \to +\infty$ in (4-6) in order to show that $\tilde{u}$ is $\tilde{\boldsymbol{p}}$-a.s. a solution to (CR), written in integral form as

$$\tilde{u}(t) = R_0(\tilde{u}(t)) - i \int_0^t \mathcal{T}(\tilde{u})\, ds. \tag{4-7}$$

Firstly, from (4-4) we deduce the convergence of the linear terms in (4-6) to those in (4-7). The following lemma gives the convergence of the nonlinear term:

Lemma 4.7. *Up to a subsequence*,

$$\mathcal{T}_{N_k}(\tilde{u}_{N_k}) \to \mathcal{T}(\tilde{u}) \quad \tilde{\boldsymbol{p}}\text{-a.s. in } L^2([-T,T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2)).$$

Proof. In order to simplify the notations, in this proof we drop the tildes and write $N_k = k$. Let $M \geq 1$ and write

$$\mathcal{T}_k(u_k) - \mathcal{T}(u) = (\mathcal{T}_k(u_k) - \mathcal{T}(u_k)) + (\mathcal{T}(u_k) - \mathcal{T}_M(u_k)) + (\mathcal{T}_M(u_k) - \mathcal{T}_M(u)) + (\mathcal{T}_M(u) - \mathcal{T}(u)).$$

To begin with, by continuity of the product in finite dimensions, when $k \to +\infty$,

$$\mathcal{T}_M(u_k) \to \mathcal{T}_M(u) \quad \tilde{\boldsymbol{p}}\text{-a.s. in } L^2([-T,T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2)).$$

We now deal with the other terms. It is sufficient to show the convergence in the space $X := L^2(\Omega \times [-T,T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2))$, since the almost sure convergence follows after extraction of a subsequence. By definition and the invariance of μ, we obtain

$$\begin{aligned}
\|\mathcal{T}_M(u_k) - \mathcal{T}(u_k)\|_X^2 &= \int_{\mathscr{C}([-T,T];X^{-1})} \|\mathcal{T}_M(v) - \mathcal{T}(v)\|^2_{L^2_T \mathcal{H}^{-\sigma}_x}\, d\nu_k(v) \\
&= \int_{X^{-1}(\mathbb{R}^2)} \big\|\mathcal{T}_M(\Phi_k(t)(f)) - \mathcal{T}(\Phi_k(t)(f))\big\|^2_{L^2_T \mathcal{H}^{-\sigma}_x}\, d\mu(f) \\
&= \int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_M(f) - \mathcal{T}(f)\|^2_{L^2_T \mathcal{H}^{-\sigma}_x}\, d\mu(f) \\
&= 2T \int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_M(f) - \mathcal{T}(f)\|^2_{\mathcal{H}^{-\sigma}_x}\, d\mu(f),
\end{aligned}$$

which tends to 0 uniformly in $k \geq 1$ when $M \to +\infty$, according to Proposition 4.1.

The term $\|\mathcal{T}_M(u)-\mathcal{T}(u)\|_X$ is treated similarly. Finally, with the same argument, we show

$$\|\mathcal{T}_k(u_k)-\mathcal{T}(u_k)\|_X \le C\|\mathcal{T}_k(f)-\mathcal{T}(f)\|_{L^2_\mu \mathcal{H}_x^{-\sigma}},$$

which tends to 0 when $k \to +\infty$. This completes the proof. $\square$

Conclusion of the proof of Theorem 2.6. Define $\tilde{f} = \tilde{u}(0) := R_0(\tilde{u})$. Then, by (4-5), $\mathscr{L}_{X^{-1}}(\tilde{f}) = \mu$ and, by the previous arguments, there exists $\widetilde{\Omega}' \subset \widetilde{\Omega}$ such that $\tilde{\boldsymbol{p}}(\widetilde{\Omega}') = 1$ and, for each $\omega' \in \widetilde{\Omega}'$, the random variable $\tilde{u}$ satisfies the equation

$$\tilde{u} = \tilde{f} - i\int_0^t \mathcal{T}(\tilde{u})\,dt, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^2. \tag{4-8}$$

Set $\Sigma = \tilde{f}(\Omega')$; then $\mu(\Sigma) = \tilde{\boldsymbol{p}}(\widetilde{\Omega}') = 1$. It remains to check that we can construct a global dynamics. Take a sequence $T_N \to +\infty$ and perform the previous argument for $T = T_N$. For all $N \ge 1$, let Σ_N be the corresponding set of initial conditions and set $\Sigma = \bigcap_{N\in\mathbb{N}} \Sigma_N$. Then $\mu(\Sigma) = 1$ and, for all $\tilde{f} \in \Sigma$, there exists

$$\tilde{u} \in \mathscr{C}(\mathbb{R}; X^{-1}(\mathbb{R}^2))$$

which solves (4-8). This completes the proof of Theorem 2.6.

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