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A POINTWISE INEQUALITY FOR THE FOURTH-ORDER LANE–EMDEN EQUATION

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We prove the pointwise inequality

$$-\Delta u \geq \left(\frac{2}{(p+1)-c_n}\right)^{\frac{1}{2}} |x|^{a/2} u^{(p+1)/2} + \frac{2}{n-4}\frac{|\nabla u|^2}{u} \quad \text{in } \mathbb{R}^n,$$

where $c_n := 8/(n(n-4))$, for positive bounded solutions of the fourth-order Hénon equation, that is,

$$\Delta^2 u = |x|^a u^p \quad \text{in } \mathbb{R}^n$$

for some $a \geq 0$ and $p > 1$. Motivated by Moser's proof of Harnack's inequality as well as Moser iteration-type arguments in the regularity theory, we develop an iteration argument to prove the above pointwise inequality. As far as we know this is the first time that such an argument is applied towards constructing pointwise inequalities for partial differential equations. An interesting point is that the coefficient $2/(n-4)$ also appears in the fourth-order Q-curvature and the Paneitz operator. This, in particular, implies that the scalar curvature of the conformal metric with conformal factor $u^{4/(n-4)}$ is positive.

1. Introduction

We are interested in proving an a priori pointwise estimate for positive solutions of the fourth-order Hénon equation

$$\Delta^2 u = |x|^a u^p \quad \text{in } \mathbb{R}^n, \tag{1-1}$$

where $p > 1$ and $a \geq 0$. Let us first mention that, for the case $a = 0$, it is known that (1-1) only admits $u = 0$ as a nonnegative solution when p is a subcritical exponent, that is, $1 < p < (n+4)/(n-4)$ when $n \geq 5$, and $1 < p$ when $n \leq 4$. Moreover, for the critical case $p = (n+4)/(n-4)$, all entire positive solutions are classified. See [Lin 1998; Wei and Xu 1999]. This is a counterpart of the standard Liouville theorem of Gidas and Spruck [1981a; 1981b] for the second-order Lane–Emden equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^n, \tag{1-2}$$

stating that $u = 0$ is the only nonnegative solution for (1-2) when p is a subcritical exponent, that is, $1 < p < (n+2)/(n-2)$ when $n \geq 3$. Note also that, for the fourth-order Hénon equation, it is conjectured that $u = 0$ is the only nonnegative solution of (1-1) when p is a subcritical exponent, that is, when

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$1 < p < (n+4+2a)/(n-4)$ and $n \geq 5$; see [Fazly and Ghoussoub 2014]. Therefore, throughout this note, when we are dealing with (1-1) we assume that $p > (n+4+2a)/(n-4)$ and $n \geq 5$. For more information, see [Fazly and Ghoussoub 2014; Souplet 2009] and references therein.

Pointwise estimates have had tremendous impact on the theory of elliptic partial differential equations. In what follows, we list some of the celebrated pointwise inequalities for certain semilinear elliptic equations and systems. These inequalities have been used to tackle well-known conjectures and open problems. The following inequality has been one of the main techniques to solve De Giorgi's conjecture (1978) for the Allen–Cahn equation and to analyze various semilinear equations and problems.

Theorem 1.1 [Modica 1985]. *Let $F \in C^2(\mathbb{R})$ be a nonnegative function and u be a bounded entire solution of*

$$\Delta u = F'(u) \quad in\ \mathbb{R}^n. \tag{1-3}$$

Then

$$|\nabla u|^2 \leq 2F(u) \quad in\ \mathbb{R}^n. \tag{1-4}$$

For the specific case $F(u) = \frac{1}{4}(1-u^2)^2$, equation (1-3) is known as the Allen–Cahn equation. Note also that [Caffarelli et al. 1994] extended this inequality to quasilinear equations. We refer interested readers to [Farina and Valdinoci 2010; 2011; 2013; 2014; Castellaneta et al. 2012; Farina et al. 2008] regarding pointwise gradient estimates and certain improvements of (1-4). For the fourth-order counterpart of (1-3) with an arbitrary nonlinearity, a general inequality of the form (1-4) is not known. However, for a particular nonlinearity known as the fourth-order Lane–Emden equation, i.e.,

$$\Delta^2 u = u^p \quad \text{in } \mathbb{R}^n \tag{1-5}$$

it was shown by Wei and Xu [1999, Theorem 3.1] that the negative Laplacian of the positive solutions is nonnegative, that is, $-\Delta u \geq 0$ in $\mathbb{R}^n$. Set $v = -\Delta u$ and, from the fact that $-\Delta u \geq 0$, we can consider (1-5) as a special case (when $q = 1$) of the Lane–Emden system

$$\begin{cases} -\Delta u = v^q & \text{in } \mathbb{R}^n, \\ -\Delta v = u^p & \text{in } \mathbb{R}^n, \end{cases} \tag{1-6}$$

where $p \geq q \geq 1$. Note that there is a significance difference between system (1-6) and equation (1-5), in the sense that this system has Hamiltonian structure while the equation has gradient structure. This system has been of great interest, at least in the past two decades. In particular, the Lane–Emden conjecture, stating that $u = v = 0$ is the only nonnegative solution for this system when $1/(p+1) + 1/(q+1) > (n-2)/n$ has been studied extensively and various methods and techniques have been developed to tackle this conjecture. Among these methods, Souplet [2009] proved the following pointwise inequality for solutions of (1-6) and then used it to prove the Lane–Emden conjecture in four dimensions. Note that the particular case $1 < p < 2$ was done by Phan [2012].

Theorem 1.2 [Souplet 2009]. *Let u and v be nonnegative solutions of (1-6). Then*

$$\frac{u^{p+1}}{p+1} \leq \frac{v^{q+1}}{q+1} \quad in\ \mathbb{R}^n. \tag{1-7}$$

Applying this theorem, the following pointwise inequality holds for nonnegative solutions of (1-5):

$$-\Delta u \geq \sqrt{\frac{2}{p+1}}\,u^{(p+1)/2} \quad \text{in } \mathbb{R}^n. \tag{1-8}$$

Note also that Phan [2012], with similar methods to those [Souplet 2009], extended the pointwise inequality (1-7) to nonnegative solutions of the Hénon–Lane–Emden system

$$\begin{cases} -\Delta u = |x|^b v^q & \text{in } \mathbb{R}^n, \\ -\Delta v = |x|^a u^p & \text{in } \mathbb{R}^n, \end{cases} \tag{1-9}$$

where $p \geq q \geq 1$. Suppose that $0 \leq a-b \leq (n-2)(p-q)$; then

$$|x|^a \frac{u^{p+1}}{p+1} \leq |x|^b \frac{v^{q+1}}{q+1} \quad \text{in } \mathbb{R}^n. \tag{1-10}$$

The standard method to prove a pointwise inequality, as is used to prove (1-7) and (1-4), is to derive an appropriate equation — call it an auxiliary equation — for the function that is the difference between the right-hand and left-hand sides of the inequality. Then, whenever we have enough decay estimates on solutions of the auxiliary equation, maximum principles can be applied to prove that the difference function has a fixed sign. So, the key point here is to manipulate a suitable auxiliary equation.

In a more technical framework, to construct an auxiliary equation to prove (1-7) and (1-8), a few positive terms, including a gradient term of the form $|\nabla u|^2 u^{t-2}$ for some number t, are not considered in [Souplet 2009]. To be more explicit, in order to prove (1-8), which is a particular case of (1-7), the difference function $w(x) := \Delta u + \sqrt{2/(p+1)}\,u^{(p+1)/2}$ is considered. Straightforward calculations show that the following auxiliary equation holds:

$$\left(\sqrt{\frac{2}{p+1}}\,u^{(1-p)/2}\right)\Delta w = \Delta u + \sqrt{\frac{2}{p+1}}\,u^{(p+1)/2} + \frac{p-1}{2}\frac{|\nabla u|^2}{u}. \tag{1-11}$$

In order to show that Δw is nonnegative when w is nonnegative, via maximum principles for the above equation, the gradient term $|\nabla u|^2/u$ is not considered in [Souplet 2009]. Note that (1-11) implies, in spirit, that the gradient term $|\nabla u|^2/u$ should have an impact on the inequality, just like the Laplacian operator and the power term $u^{(p+1)/2}$. This is our motivation to attempt to include the gradient term in the inequality (1-8) that gives a lower bound on the Laplacian operator. Let us briefly mention that Modica, in his proof of (1-4), took advantage of similar gradient terms to construct an auxiliary equation. Following ideas provided by Modica [1985] and Souplet [2009], as we shall see in the proof of Proposition 3.1, we manage to keep most of the positive terms when looking for an auxiliary equation.

In this paper, we develop a Moser iteration-type argument to prove a lower bound for the negative Laplacian of positive bounded solutions of (1-1) that involves powers of u and the new term $|\nabla u|^2/u$ with $2/(n-4)$ as the coefficient. The remarkable point is that the coefficient $2/(n-4)$ is exactly what we need in the estimate of the scalar curvature for the conformal metric $g = u^{2/(n-4)} g_0$.

Here is our main result:

Theorem 1.3. *Let u be a bounded positive solution of (1-1). Then the following pointwise inequality holds*:

$$-\Delta u \geq \sqrt{\frac{2}{(p+1)-c_n}}|x|^{a/2}u^{(p+1)/2} + \frac{2}{n-4}\frac{|\nabla u|^2}{u} \quad in\ \mathbb{R}^n, \tag{1-12}$$

where $c_n := 8/(n(n-4))$ *and* $0 \leq a \leq \inf_{k\geq 0} A_k$ (A_k *is defined in* (4-28)).

Remark 1.4. A natural question here is: what are the best constants in the inequality (1-12)?

Let us now put the inequality (1-12) in a more geometric text. By the conformal change $g = u^{4/(n-4)}g_0$, where g_0 is the usual Euclidean metric, the new scalar curvature becomes

$$S_g = -\frac{4(n-1)}{n-2}u^{-(n+2)/(n-4)}\Delta(u^{(n-2)/(n-4)}).$$

An immediate consequence of (1-12) is that the conformal scalar curvature is positive. Note that this cannot be deduced from the inequality (1-8).

The idea of proving a lower bound for the negative of the Laplacian operator is also used in the context of nonlinear eigenvalue problems to prove certain regularity results; see, e.g., [Cowan et al. 2010]. Similar pointwise inequalities are used to prove Liouville theorems in the notion of stability in [Wei et al. 2013; Wei and Ye 2013] and references therein as well. We would like to mention that Gui [2008] proved a very interesting Hamiltonian identity for elliptic systems that may be regarded as a generalization of Modica's inequality. He used this identity to rigorously analyze the structure of level curves of saddle solutions of the Allen–Cahn equation as well as Young's law for the contact angles in triple junction formation. Note also that, as is shown by Farina [2004] for the Ginzburg–Landau system, the analog of Modica's estimate is false for systems in general. We refer interested readers to [Alikakos 2013] for a review of this topic and to [Fazly and Ghoussoub 2013] for De Giorgi-type results for systems.

Here is the organization of the paper. In Section 2, we provide certain standard elliptic estimates that are consequences of Sobolev embeddings and the regularity theory. Then, in Section 3 we develop a Moser iteration-type argument, following ideas provided by Modica [1985] and Souplet [2009]. Finally, in Section 4, we first give a certain maximum principle argument for a quasilinear equation that arises in the Moser iteration process. Then we apply the estimates and methods developed in the earlier sections. We suggest the reader ignores the weight function $|x|^a$ in (1-1) when reading the paper for the first time.

2. Technical elliptic estimates

In this section, we provide some elliptic decay estimates that we use frequently later in the proofs. Deriving the right decay estimates for solutions of (1-1) plays a fundamental role in our proofs. Similar estimates have been also used in the literature to construct Liouville theorems and regularity results. We refer interested readers to [Fazly 2014; Fazly and Ghoussoub 2014; Phan 2012; Souplet 2009; Phan and Souplet 2012]. We start with the following standard estimate:

Lemma 2.1 (L^p-estimate on B_R). *Suppose that u is a nonnegative solution of (1-1); then, for any $R > 1$ we have*

$$\int_{B_R} |x|^a u^p \le C R^{n-(4p+a)/(p-1)},$$

where $C = C(n, p, a) > 0$ is independent of R.

Proof. Consider the following test function $\phi_R \in C_c^4(\mathbb{R}^n)$ with $0 \le \phi_R \le 1$:

$$\phi_R(x) = \begin{cases} 1 & \text{if } |x| < R, \\ 0 & \text{if } |x| > 2R, \end{cases}$$

where $\|D^i \phi_R\|_\infty \le C/R^i$ for $1 \le i \le 4$. For fixed $m \ge 2$, we have

$$|\Delta^2 \phi_R^m(x)| \le \begin{cases} 0 & \text{if } |x| < R \text{ or } |x| > 2R, \\ C R^{-4} \phi_R^{m-4} & \text{if } R < |x| < 2R, \end{cases}$$

where $C > 0$ is independent of R. For $m \ge 2$, multiply the equation by ϕ_R^m and integrate to get

$$\int_{B_{2R}} |x|^a u^p \phi_R^m = \int_{B_{2R}} \Delta^2 u \phi_R^m = \int_{B_{2R}} u \Delta^2 \phi_R^m \le C R^{-4} \int_{B_{2R} \setminus B_R} u \phi_R^{m-4}.$$

Applying Hölder's inequality, we get

$$\begin{aligned} \int_{B_{2R}} |x|^a u^p \phi_R^m &\le C R^{-4} \left(\int_{B_{2R} \setminus B_R} |x|^{(-a/p)p'} \right)^{\frac{1}{p'}} \left(\int_{B_{2R} \setminus B_R} |x|^a u^p \phi_R^{(m-4)p} \right)^{\frac{1}{p}} \\ &\le C R^{(n-(a/p)p')/p'-4} \left(\int_{B_{2R} \setminus B_R} |x|^a u^p \phi_R^{(m-4)p} \right)^{\frac{1}{p}}, \end{aligned}$$

where $p' = p/(p-1)$. Set $m = (m-4)p$, so that $m = 4p/(p-1)$, to get

$$\int_{B_{2R}} |x|^a u^p \phi_R^m \le C R^{(n-(a/p)p')/p'-4} \left(\int_{B_{2R}} |x|^a u^p \phi_R^m \right)^{\frac{1}{p}}.$$

Therefore,

$$\int_{B_{2R}} |x|^a u^p \phi_R^m \le C R^{(n-(a/p)p')-4p'}.$$

This finishes the proof. □

From Hölder's inequality we get the following:

Corollary 2.2. *Under the same assumptions as Lemma 2.1,*

$$\int_{B_R \setminus B_{R/2}} u \le C R^{n-(a+4)/(p-1)},$$

where $C = C(n, p, a) > 0$ is independent of R.

We now show that the operator $-\Delta u$ has a sign. Then, we apply this to provide various elliptic estimates for derivatives of u. In addition, later on this helps us to start an iteration argument.

Proposition 2.3. *Let u be a positive solution of (1-1). Then, $-\Delta u \ge 0$ in $\mathbb{R}^n$.*

Proof. Let $v = -\Delta u$. Ideas and methods applied in this proof are strongly motivated by the ones given in [Wei and Xu 1999]. Suppose that there is $x_0 \in \mathbb{R}^n$ such that $v(x_0) < 0$. Without loss of generality we take $x_0 = 0$, i.e., if $x_0 \neq 0$ set $\omega(x) = v(x + x_0)$ and apply the same argument. We use the notation $\bar{f}(r) = (1/|\partial B_r|) \int_{\partial B_r} f \, dS$ for the average of a function $f(x)$ on the boundary of B_r. We refer interested readers to [Ni 1982] regarding the average function. Applying Hölder's inequality,

$$\begin{cases} -\Delta_r \bar{u}(r) = \bar{v}(r) & \text{in } \mathbb{R}, \\ -\Delta_r \bar{v}(r) \geq r^a (\bar{u})^p & \text{in } \mathbb{R}, \end{cases} \tag{2-1}$$

where Δ_r is the Laplacian operator in polar coordinates, i.e.,

$$\Delta_r \bar{f}(r) = r^{1-n} (r^{n-1} \bar{f}'(r))'.$$

It is straightforward to see that

$$\bar{v}'(r) = \frac{1}{|\partial B_r|} \int_{B_r} \Delta v = -\frac{1}{|\partial B_r|} \int_{B_r} |x|^a u^p \leq 0.$$

Therefore, $\bar{v}(r) \leq \bar{v}(0) < 0$ for $r > 0$. Similarly, for $\bar{u}'(r)$ we have

$$\bar{u}'(r) = -\frac{1}{|\partial B_r|} \int_{B_r} v = -r^{1-n} \int_0^r s^{n-1} \bar{v}(s) \, ds \geq -\bar{v}(0) r^{1-n} \int_0^r s^{n-1} \, ds = -\frac{\bar{v}(0)}{n} r.$$

From this, for any $r \geq r_0$ we get

$$\bar{u}(r) \geq \alpha r^2, \tag{2-2}$$

where $\alpha = -\bar{v}(0)/(2n) > 0$. We now have a lower bound on $\bar{u}(r)$. Suppose instead that the following more general lower bound holds on $\bar{u}(r)$:

$$\bar{u}(r) \geq \frac{\alpha^{p^k}}{\beta^{s_k}} r^{t_k} \quad \text{for } r \geq r_k, \tag{2-3}$$

where $s_0 := 0$, $t_0 := 2$, $\alpha := -\bar{v}(0)/(2n) > 0$ and $\beta := 2p + a + n + 4 > 0$. Note that (2-1) gives a relation between the two functions $\bar{u}(r)$ and $\bar{v}(r)$. Therefore, the lower bound on $\bar{u}(r)$ forces an upper bound on $\bar{v}(r)$ and vice versa. In the light of this fact, we can construct an iteration argument to improve the bound (2-3). Integrating the second equation of (2-1) over $[r_k, r]$ when $r \geq r_k$, we get

$$\begin{aligned} r^{n-1} \bar{v}'(r) &\leq r_k^{n-1} \bar{v}'(r_k) - \frac{\alpha^{p^{k+1}}}{\beta^{p s_k}} \int_{r_k}^r s^{n-1+a+p t_k} \, ds \\ &\leq -\frac{\alpha^{p^{k+1}}}{\beta^{p s_k} (p t_k + n + a)} (r^{p t_k + n + a} - r_k^{p t_k + n + a}) \quad \text{since } \bar{v}' < 0. \end{aligned}$$

Therefore, $\bar{v}'(r) \leq -\big(\alpha^{p^{k+1}} / (\beta^{p s_k} (p t_k + n + a))\big)(r^{p t_k + a + 1} - r_k^{p t_k + a + 1})$ for all $r \geq r_k$, that is,

$$\bar{v}'(r) \leq -\frac{\alpha^{p^{k+1}}}{2 \beta^{p s_k} (p t_k + n + a)} r^{p t_k + a + 1} \quad \text{for all } r \geq 2^{1/(p t_k + a + 1)} r_k.$$

Integrating the last inequality over $[2^{1/(pt_k+a+1)}r_k, r]$ when $r \geq 2^{1/(pt_k+a+1)}r_k = \tilde{r}_k$, we obtain

$$\bar{v}(r) \leq \bar{v}(\tilde{r}_k) - \frac{\alpha^{p^{k+1}}}{2\beta^{ps_k} T_{k,n,a,p}}(r^{pt_k+a+2} - \tilde{r}_k^{pt_k+a+2}),$$

where $T_{k,n,a,p} := (pt_k + n + a)(pt_k + 2 + a)$. By similar discussions and by taking r large enough, that is, $r \geq 2^{1/(pt_k+a+1)} 2^{1/(pt_k+a+2)} r_k = \tilde{\tilde{r}}_k$, we end up with

$$\bar{v}(r) \leq -\frac{\alpha^{p^{k+1}}}{4\beta^{ps_k} T_{k,n,a,p}} r^{pt_k+a+2}. \tag{2-4}$$

Applying (2-4) and integrating (2-1) again over $[\tilde{\tilde{r}}_k, r]$ when $r \geq \tilde{\tilde{r}}_k$, we have

$$r^{n-1}\bar{u}'(r) = \tilde{r}_k^{n-1}\bar{u}'(\tilde{r}_k) - \int_{\tilde{r}_k}^{r} s^{n-1}\bar{v}(s)\, ds \geq \frac{\alpha^{p^{k+1}}}{4\beta^{ps_k} T_{k,n,a,p}} \int_{\tilde{r}_k}^{r} s^{pt_k+a+n+1}\, ds.$$

Therefore, the following new lower bound on $\bar{u}(r)$ holds:

$$\bar{u}(r) \geq \frac{\alpha^{p^{k+1}}}{2^4 \beta^{ps_k} \tilde{T}_{k,n,a,p}} r^{pt_k+a+n+4},$$

where

$$r \geq 2^{1/(pt_k+a+3)} 2^{1/(pt_k+a+4)} \tilde{\tilde{r}}_k = 2^{\sum_{i=1}^{4} 1/(pt_k+a+i)} r_k$$

and

$$\begin{aligned}\tilde{T}_{k,n,a,p} &= (pt_k + n + a + 2)(pt_k + 4 + a) T_{k,n,a,p} \\ &= (pt_k + n + a)(pt_k + 2 + a)(pt_k + n + a + 2)(pt_k + 4 + a) \\ &\leq (pt_k + n + a + 4)^4.\end{aligned}$$

We now modify this estimate to make the coefficients similar to (2-3). After simplifying, we get

$$\bar{u}(r) \geq \frac{\alpha^{p^{k+1}}}{\beta^{ps_k} M_k} r^{pt_k+a+4} \quad \text{for } r \geq 2^{4/(pt_k+a+1)} r_k, \tag{2-5}$$

where $M_k := 2^4(pt_k + n + a + 4)^4$. In what follows, we put an upper bound on M_k that is expressed as a power of β. Note that

$$\tfrac{1}{2}\sqrt[4]{M_{k+1}} = pt_{k+1} + n + a + 4 = p(pt_k + n + a + 4) + + n + a + 4 \leq (pt_k + n + a + 4)(p+1) = \tfrac{1}{2}(p+1)\sqrt[4]{M_k}.$$

From this we have $M_{k+1} \leq (p+1)^4 M_k$ and therefore $M_k \leq (p+1)^{4k} M_0$, where $M_0 = 2^4(2p+n+a+4)^4$ because $t_0 = 2$. Since the constant β is defined as $\beta = 2p + n + a + 4$, we get the bound

$$M_k \leq \beta^{4k+4}. \tag{2-6}$$

From this, (2-3) and (2-5), and to complete the iteration process, we set

$$t_{k+1} := pt_k + a + 4 \qquad \text{for } t_0 = 2, \tag{2-7}$$

$$s_{k+1} := ps_k + 4k + 4 \qquad \text{for } s_0 = 0, \tag{2-8}$$

and, therefore,

$$\bar{u}(r) \geq \frac{\alpha^{p^{k+1}}}{\beta^{s_{k+1}}} r^{t_{k+1}} \quad \text{for } r \geq r_{k+1}, \tag{2-9}$$

where $r_{k+1} := 2^{4/(pt_k+a+1)} r_k \geq 2^{\sum_{i=1}^{4} 1/(pt_k+a+i)} r_k$. By direct calculations on these recursive sequences, we get the explicit sequences

$$\begin{aligned} t_k &= \frac{2p^{k+1} + (a+2)p^k - (a+4)}{p-1}, \\ s_k &= \frac{4p^{k+1} - 4p(k+1) + 4k}{(p-1)^2}, \\ r_k &= 2^{\sum_{i=0}^{k-1} 4/(pt_i+a+1)} r_0 \leq 2^{\sum_{i=0}^{\infty} 4/(pt_i+a+1)} r_0 =: r^* < \infty. \end{aligned}$$

Set $R := \beta^{2/(p-1)} M$, where $M = \max\{\alpha^{-1}, m\}$ when $m > 1$ is large enough to ensure $m\beta^{2/(p-1)} \geq r^*$. Therefore, $R \geq r^* \geq r_k$ for any k and we have

$$\bar{u}(R) \geq M^{t_k - p^k} \beta^{2t_k/(p-1) - s_k}.$$

If we take k large enough, e.g., $k \geq (\ln(a+4) - \ln(a+2))/\ln p$, then $t_k > p^k$. The fact that $M > 1$ gives us

$$\bar{u}(R) \geq \beta^{2t_k/(p-1) - s_k} = \beta^{(2(a+2)p^k + 4k(p-1) + 4p - 2(a+4))/(p-1)^2}.$$

Since we have assumed that $a+2 > 0$ and $\beta > 1$, we get $\bar{u}(R) \to \infty$ as $k \to \infty$. Note that $0 < R < \infty$ is independent of k. This finishes the proof. □

We now apply Proposition 2.3 to conclude that $-\Delta u \geq 0$ and therefore we can consider (1-1) as a special case of the Hénon–Lane–Emden equation.

Lemma 2.4 (L^1-estimates on B_R). *Suppose that u is a nonnegative solution of (1-1); then, for any $R > 1$ we have*

$$\int_{B_R} |\Delta u| \leq C R^{n-(2p+2+a)/(p-1)},$$

where $C = C(n, p, a) > 0$ is independent of R.

Proof. Set $v = -\Delta u$. From Proposition 2.3 we know that $v \geq 0$. Therefore, the pair (u, v) satisfies the system

$$\begin{cases} -\Delta u = v & \text{in } \mathbb{R}^n, \\ -\Delta v = |x|^a u^p \text{in } \mathbb{R}^n, \end{cases} \tag{2-10}$$

which is a particular case of the Hénon–Lane–Emden system. From the estimates provided in [Fazly and Ghoussoub 2014, Lemma 2.1], we get the desired result. □

Lemma 2.5 (an interpolation inequality on B_R). *Let $R > 1$ and $z \in W^{2,1}(B_{2R})$. Then*

$$\int_{B_R \setminus B_{R/2}} |Dz| \leq CR \int_{B_{2R} \setminus B_{R/4}} |\Delta z| + CR^{-1} \int_{B_{2R} \setminus B_{R/4}} |z|,$$

where $C = C(n) > 0$ is independent of R.

Corollary 2.6. *Under the same assumptions as Lemma 2.1. The following estimate holds*:

$$\int_{B_R\setminus B_{R/2}} |Du| \le C R^{n-(p+3+a)/(p-1)},$$

where $C = C(n, p, a) > 0$ *is independent of* R.

Lemma 2.7 (L^τ-estimate on B_R). *Let* $1 < \tau < \infty$ *and* $z \in W^{2,\tau}(B_{2R})$. *Then*

$$\int_{B_R\setminus B_{R/2}} |D^2 z|^\tau \le C \int_{B_{2R}\setminus B_{R/4}} |\Delta z|^\tau + C R^{-2\tau} \int_{B_{2R}\setminus B_{R/4}} |z|^\tau,$$

where $C = C(n, \tau) > 0$ *does not depend on* R.

Lemma 2.8 (L^2-estimates on B_R). *Suppose that* u *is a bounded nonnegative solution of* (1-1); *then, for any* $R > 1$ *we have*

$$\int_{B_R} |\Delta u|^2 \le C \int_{B_{2R}} |x|^a u^{p+1} + C R^{-2} \int_{B_{2R}} |\Delta u| + C R^{-4} \int_{B_{2R}\setminus B_R} u, \tag{2-11}$$

where $C = C(n, p, a) > 0$ *does not depend on* R.

Proof. We proceed in two steps.

Step 1: Multiply both sides of (1-1) by $u\phi^2$, where $\phi \in C_c^\infty(\mathbb{R}^n)\cap[0, 1]$ is a test function. Then, integrating by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta u|^2\phi^2 &= \int_{\mathbb{R}^n} |x|^a u^{p+1}\phi^2 - 4\int_{\mathbb{R}^n} \Delta u \nabla u\cdot\nabla\phi\phi - \int_{\mathbb{R}^n} u\Delta u(2|\nabla\phi|^2 + 2\phi\Delta\phi)\\ &\le \int_{\mathbb{R}^n} |x|^a u^{p+1}\phi^2 + \delta\int_{\mathbb{R}^n} |\Delta u|^2\phi^2 + C(\delta)\int_{\mathbb{R}^n} |\nabla u|^2|\nabla\phi|^2 + C\int_{\mathbb{R}^n} |\Delta u|(|\nabla\phi|^2 + |\Delta\phi|)\end{aligned}$$

for some constant $C > 0$. Here, we have used Cauchy's inequality for $0 < \delta < 1$. Therefore, if we set ϕ to be the standard test function, that is, $\phi = 1$ in B_R and $\phi = 0$ in $\mathbb{R}^n \setminus B_{2R}$ with $\|D_x^i\phi\|_{L^\infty(B_{2R}\setminus B_R)} \le C R^{-i}$ for $i = 1, 2$, then we get

$$\int_{B_R} |\Delta u|^2 \le \int_{B_{2R}} |x|^a u^{p+1} + C R^{-2}\int_{B_{2R}\setminus B_R} |\nabla u|^2 + C R^{-2}\int_{B_{2R}\setminus B_R} |\Delta u|, \tag{2-12}$$

where $C = C(n, p, a) > 0$ does not depend on R.

Step 2: Multiply both sides of $-\Delta u = v$ by $u\phi^2$, where ϕ is the same test function as in Step 1. Integrating by parts again, we get

$$\int_{\mathbb{R}^n} |\nabla u|^2\phi^2 = \int_{\mathbb{R}^n} uv\phi^2 - 2\int_{\mathbb{R}^n} u\nabla u\cdot\nabla\phi\phi \le \int_{\mathbb{R}^n} uv\phi^2 + \delta\int_{\mathbb{R}^n} |\nabla u|^2\phi^2 + C(\delta)\int_{\mathbb{R}^n} |\nabla\phi|^2 u^2,$$

where we have also used Cauchy's inequality for $0 < \delta < 1$. So,

$$\int_{B_R} |\nabla u|^2 \le C\int_{B_{2R}} |\Delta u| + C R^{-2}\int_{B_{2R}\setminus B_R} u, \tag{2-13}$$

where we have used the boundedness of u. From (2-12) and (2-13) we get

$$\int_{B_R} |\Delta u|^2 \le \int_{B_{2R}} |x|^a u^{p+1} + CR^{-2}\int_{B_{2R}} |\Delta u| + CR^{-4}\int_{B_{2R}\setminus B_R} u. \tag{2-14}$$

This completes the proof. □

We now apply Lemma 2.1, Lemma 2.8 and Corollary 2.2 to get the following:

Corollary 2.9. *Suppose that the assumptions of Lemma 2.1 hold. Moreover, let u be bounded; then*

$$\int_{B_R} |\Delta u|^2 \le CR^{n-(4p+a)/(p-1)}, \tag{2-15}$$

where $C = C(n, p, a) > 0$ is independent of R.

Lemma 2.10 (Sobolev inequalities on the sphere S^{n-1}). *Fix $n \ge 2$, a positive integer i and $1 < t < \tau \le \infty$. For $z \in W^{i,t}(S^{n-1})$,*

$$\|z\|_{L^\tau(S^{n-1})} \le C\|D^i_\theta z\|_{L^t(S^{n-1})} + C\|z\|_{L^1(S^{n-1})},$$

where

$$\begin{cases} \dfrac{1}{\tau} = \dfrac{1}{t} - \dfrac{i}{n-1} & \text{if } it+1 < n, \\ \tau = \infty & \text{if } it+1 > n, \end{cases}$$

and $C = C(i, t, n, \tau) > 0$.

3. Developing the iteration argument

In this section, we develop a counterpart of the Moser iteration argument [1961] for solutions of (1-1). We define a sequence of functions $(w_k)_{k=-1}$ of the form

$$w_k := \Delta u + \alpha_k |\nabla u|^2 (u+\epsilon)^{-1} + \beta_k |x|^{a/2} u^{(p+1)/2},$$

where α_k and β_k are certain nondecreasing sequences of nonnegative numbers with $\alpha_{-1} = \beta_{-1} = 0$.

Assuming that $w_k \le 0$, that is, essentially, a lower bound on the negative Laplacian operator holds, we construct a differential inequality for w_{k+1} with $\alpha_{k+1} \ge \alpha_k$ and $\beta_{k+1} \ge \beta_k$. Then, applying certain maximum principle arguments, we show that $w_{k+1} \le 0$. Note that $w_{k+1} \le 0$ is stronger than $w_k \le 0$, because it forces a stronger lower bound on the negative of the Laplacian operator.

We start by proving that w_{-1}, which is the Laplacian operator of u, is nonpositive; see Proposition 2.3. Then, using this fact and applying (1-9) and (1-10) when $q = 1$ and $b = 0$, we get the following inequality for nonnegative solutions of the fourth-order Hénon equation (1-1):

$$-\Delta u \ge \sqrt{\frac{2}{p+1}} |x|^{a/2} u^{(p+1)/2} \quad \text{in } \mathbb{R}^n, \tag{3-1}$$

where $0 \le a \le (n-2)(p-1)$. Inequality (3-1) is the first step of the iteration argument, meaning that $w_0 \le 0$ for $\alpha_0 = 0$ and $\beta_0 = \sqrt{2/(p+1)}$.

We now perform the iteration argument:

Proposition 3.1. *Let u be a positive classical solution of (1-1). Suppose that $(\alpha_k)_{k=0}$ and $(\beta_k)_{k=0}$ are sequences of numbers. Define the sequence of functions*

$$w_k := \Delta u + \alpha_k |\nabla u|^2 (u+\epsilon)^{-1} + \beta_k |x|^{a/2} u^{(p+1)/2}, \tag{3-2}$$

where $\epsilon = \epsilon(k)$ is a positive constant. Suppose that $w_k \le 0$; then w_{k+1} satisfies the differential inequality

$$\begin{aligned}\Delta w_{k+1} - 2\alpha_{k+1}(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} + \alpha_{k+1} w_{k+1}(u+\epsilon)^{-2}|\nabla u|^2 - \tfrac{1}{2}\beta_{k+1}(p+1)u^{(p-1)/2}|x|^{a/2} w_{k+1} \\ \ge I^{(1)}_{\epsilon,\beta_k}|x|^a u^p + \alpha_{k+1} I^{(2)}_{\alpha_k}|\nabla u|^4 (u+\epsilon)^{-3} + I^{(4)}_{a,\alpha_k,\beta_k}|x|^{a-2}u^{(p+1)/2} \\ + I^{(3)}_{\epsilon,\alpha_k,\beta_k}|x|^a u^{(p+1)/2}\left|\frac{\nabla u}{u} + \frac{a\beta_{k+1}\bigl(\frac{1}{2}(p+1) - \alpha_{k+1}u/(u+\epsilon)\bigr)}{2I^{(3)}_{\epsilon,\alpha_k,\beta_k}}\frac{x}{|x|^2}\right|^2,\end{aligned} \tag{3-3}$$

where

$$\begin{aligned} I^{(1)}_{\epsilon,\alpha_k,\beta_k} &:= 1 - \frac{p+1}{2}\beta_{k+1}^2 + \frac{2}{n}\alpha_{k+1}\beta_k^2\frac{u}{u+\epsilon}, \\ I^{(2)}_{\alpha_k} &:= \frac{2}{n}(\alpha_{k+1}+\alpha_k+1)^2 - 2\alpha_{k+1}(\alpha_{k+1}+1) + \alpha_{k+1}, \\ I^{(3)}_{\epsilon,\alpha_k,\beta_k} &:= \frac{4}{n}\alpha_{k+1}\beta_k(\alpha_{k+1}+\alpha_k+1)\frac{u^2}{(u+\epsilon)^2} + \beta_{k+1}\alpha_{k+1}\frac{u^2}{(u+\epsilon)^2} \\ &\qquad - (p+1)\beta_{k+1}\alpha_{k+1}\frac{u}{u+\epsilon} + \frac{p+1}{2}\Bigl(\frac{p-1}{2} - \alpha_{k+1}\frac{u}{u+\epsilon}\Bigr)\beta_{k+1}, \\ I^{(4)}_{a,\epsilon,\alpha_k,\beta_k} &:= \frac{a}{2}\beta_{k+1}\Bigl(n + \frac{a}{2} - 2\Bigr) - \frac{a^2\beta_{k+1}^2\bigl(\frac{1}{2}(p+1) - \alpha_{k+1}u/(u+\epsilon)\bigr)^2}{4I^{(3)}_{\epsilon,\alpha_k,\beta_k}}. \end{aligned}$$

Proof. For the sake of simplicity in calculations, set $b := \frac{1}{2}a$ and $q := \frac{1}{2}(p+1)$. From (3-2), the function w_{k+1} is defined as

$$w_{k+1} := \Delta u + \alpha_{k+1}|\nabla u|^2(u+\epsilon)^{-1} + \beta_{k+1}|x|^b u^q.$$

Taking Laplacian of w_{k+1} and using (1-1), we get

$$\Delta w_{k+1} = \Delta^2 u + \alpha_{k+1}\Delta(|\nabla u|^2(u+\epsilon)^{-1}) + \beta_{k+1}\Delta(|x|^b u^q) = |x|^a u^p + I + J, \tag{3-4}$$

where $I := \alpha_{k+1}\Delta(|\nabla u|^2(u+\epsilon)^{-1})$ and $J := \beta_{k+1}\Delta(|x|^b u^q)$. In what follows, we simplify I and J as well as finding lower bounds for these terms. We start with J:

$$\begin{aligned}\frac{J}{\beta_{k+1}} &= \Delta(|x|^b u^q) = \Delta|x|^b u^q + \Delta u^q |x|^b + 2\nabla|x|^b\cdot\nabla u^q \\ &= b(n+b-2)|x|^{b-2}u^q + q(q-1)|x|^b u^{q-2}|\nabla u|^2 + q|x|^b u^{q-1}\Delta u + 2bq|x|^{b-2}u^{q-1}\nabla u\cdot x.\end{aligned}$$

From the definition of w_{k+1}, we have

$$\Delta u = w_{k+1} - \alpha_{k+1}|\nabla u|^2(u+\epsilon)^{-1} - \beta_{k+1}|x|^b u^q. \tag{3-5}$$

Substitute this into the previous equation to simplify J as

$$\frac{J}{\beta_{k+1}} = qu^{q-1}|x|^b w_{k+1} - q\beta_{k+1}u^{2q-1}|x|^{2b} + \Big(q(q-1) - q\alpha_{k+1}\frac{u}{u+\epsilon}\Big)|x|^b u^{q-2}|\nabla u|^2 + b(n+b-2)|x|^{b-2}u^q + 2bq|x|^{b-2}u^{q-1}\nabla u\cdot x. \quad (3\text{-}6)$$

We now simplify I:

$$\begin{aligned}\frac{I}{\alpha_{k+1}} &= \Delta(|\nabla u|^2(u+\epsilon)^{-1}) = \sum_{i,j}\partial_{jj}(u_i^2(u+\epsilon)^{-1})\\ &= 2(u+\epsilon)^{-1}\sum_{i,j}(\partial_{ij}u)^2 + 2(u+\epsilon)^{-1}\nabla u\cdot\nabla\Delta u - 4(u+\epsilon)^{-2}\sum_{i,j}\partial_i u\partial_j u\partial_{ij}u\\ &\qquad - |\nabla u|^2(u+\epsilon)^{-2}\Delta u + 2|\nabla u|^4(u+\epsilon)^{-3}.\end{aligned}$$

Again substituting (3-5) into the term $2(u+\epsilon)^{-1}\nabla u\cdot\nabla\Delta u$ that appears above, we get

$$\begin{aligned}\frac{I}{\alpha_{k+1}} &= 2(u+\epsilon)^{-1}\sum_{i,j}(\partial_{ij}u)^2 - 4(u+\epsilon)^{-2}\sum_{i,j}\partial_i u\partial_j u\partial_{ij}u + 2|\nabla u|^4(u+\epsilon)^{-3} - |\nabla u|^2(u+\epsilon)^{-3}\Delta u\\ &\quad + 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} - 2\alpha_{k+1}(u+\epsilon)^{-1}\nabla u\cdot(|\nabla u|^2(u+\epsilon)^{-1})\\ &\quad - 2\beta_{k+1}(u+\epsilon)^{-1}\nabla u\cdot\nabla(|x|^b u^q).\end{aligned}$$

Then, collecting similar terms, we obtain

$$\begin{aligned}&\frac{I}{\alpha_{k+1}} - 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1}\\ &\quad = 2(u+\epsilon)^{-1}\sum_{i,j}(\partial_{ij}u)^2 - 4(\alpha_{k+1}+1)(u+\epsilon)^{-2}\sum_{i,j}\partial_i u\partial_j u\partial_{ij}u + 2(\alpha_{k+1}+1)|\nabla u|^4(u+\epsilon)^{-3}\\ &\qquad - |\nabla u|^2(u+\epsilon)^{-2}\Delta u - 2\beta_{k+1}b|x|^{b-2}(u+\epsilon)^{-1}u^q\nabla u\cdot x - 2\beta_{k+1}q|x|^b u^{q-1}(u+\epsilon)^{-1}|\nabla u|^2.\end{aligned}$$

Completing the square, we get

$$\begin{aligned}&\frac{I}{\alpha_{k+1}} - 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1}\\ &\quad = 2(u+\epsilon)^{-1}\sum_{i,j}(\partial_{ij}u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}\partial_i u\partial_j u)^2 - 2\alpha_{k+1}(\alpha_{k+1}+1)|\nabla u|^4(u+\epsilon)^{-3}\\ &\qquad - |\nabla u|^2(u+\epsilon)^{-2}\Delta u - 2\beta_{k+1}b|x|^{b-2}(u+\epsilon)^{-1}u^q\nabla u\cdot x - 2\beta_{k+1}q|x|^b u^{q-1}(u+\epsilon)^{-1}|\nabla u|^2. \quad (3\text{-}7)\end{aligned}$$

Note that, for any $n\times n$ matrix $A = (a_{i,j})$, the Hilbert–Schmidt norm is defined by $\|A\|_2 = \sqrt{\sum_{i,j}|a_{i,j}|^2} = \sqrt{\text{trace}(AA^*)}$, where A^* denotes the conjugate transpose of A. From the Cauchy–Schwarz inequality, the following inequality holds:

$$|\text{trace } A|^2 = |(A, I)|^2 \le \|A\|_2^2\|I\|_2^2 = n\sum_{i,j}|a_{i,j}|^2. \quad (3\text{-}8)$$

Set $a_{i,j} := \partial_{ij}u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}\partial_i u \partial_j u$ in (3-8) to get

$$\sum_{i,j}^{n}(\partial_{ij}u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}\partial_i u\partial_j u)^2 \geq \frac{1}{n}(\Delta u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}|\nabla u|^2)^2.$$

From this lower bound for the Hessian and (3-7), we get

$$\frac{I}{\alpha_{k+1}} - 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} \geq \frac{2}{n}(u+\epsilon)^{-1}(\Delta u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}|\nabla u|^2)^2 \\ -2\alpha_{k+1}(\alpha_{k+1}+1)|\nabla u|^4(u+\epsilon)^{-3} - |\nabla u|^2(u+\epsilon)^{-2}\Delta u + T_k, \quad (3\text{-}9)$$

where

$$T_k := -2\beta_{k+1}b|x|^{b-2}(u+\epsilon)^{-1}u^q\nabla u\cdot x - 2\beta_{k+1}q|x|^b u^{q-1}(u+\epsilon)^{-1}|\nabla u|^2.$$

Note also that, from the assumption $w_k \leq 0$, we have the upper bound on the Laplacian operator $\Delta u \leq -\alpha_k|\nabla u|^2(u+\epsilon)^{-1} - \beta_k|x|^b u^q$. Elementary calculations show that, if $t \leq t_* \leq 0$ and $s \geq 0$, then $(t-s)^2 \geq t_*^2 - 2t_*s + s^2$. Set the parameters as $t = \Delta u$, $t_* = -\alpha_k|\nabla u|^2(u+\epsilon)^{-1} - \beta_k|x|^b u^q$ and $s = (\alpha_{k+1}+1)(u+\epsilon)^{-1}|\nabla u|^2$ to get the following lower bound on the square term that appears in (3-9):

$$(\Delta u - (\alpha_{k+1}+1)(u+\epsilon)^{-1}|\nabla u|^2)^2 \\ \geq (\alpha_k|\nabla u|^2(u+\epsilon)^{-1}+\beta_k|x|^b u^q)^2 + 2(\alpha_k|\nabla u|^2(u+\epsilon)^{-1}+\beta_k|x|^b u^q)(\alpha_{k+1}+1)(u+\epsilon)^{-1}|\nabla u|^2 \\ + (\alpha_{k+1}+1)^2(u+\epsilon)^{-2}|\nabla u|^4. \quad (3\text{-}10)$$

Substitute (3-5) into the term $-|\nabla u|^2(u+\epsilon)^{-2}\Delta u$ that appears in (3-9) to eliminate the Laplacian operator. Then, apply inequality (3-10) to simplify (3-9) as

$$\frac{I}{\alpha_{k+1}} - 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} \\ \geq \frac{2}{n}(u+\epsilon)^{-1}\big((\alpha_{k+1}+\alpha_k+1)^2|\nabla u|^4(u+\epsilon)^{-2} + \beta_k^2|x|^{2b}u^{2q} + 2\beta_k(\alpha_{k+1}+\alpha_k+1)|x|^b u^q(u+\epsilon)^{-1}|\nabla u|^2\big) \\ - w_{k+1}(u+\epsilon)^{-2}|\nabla u|^2 - \alpha_{k+1}(2\alpha_{k+1}+1)|\nabla u|^4(u+\epsilon)^{-3} + \beta_{k+1}|x|^b u^q(u+\epsilon)^{-2}|\nabla u|^2 + T_k.$$

Collecting similar terms and using the value of T_k, we end up with

$$\frac{I}{\alpha_{k+1}} - 2(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} + w_{k+1}(u+\epsilon)^{-2}|\nabla u|^2 \\ \geq \frac{2}{n}\beta_k^2|x|^{2b}u^{2q}(u+\epsilon)^{-1} + I_{\alpha_k}^{(2)}|\nabla u|^4(u+\epsilon)^{-3} + S_{\epsilon,\alpha_k,\beta_k}|\nabla u|^2u^{q-2}|x|^b - 2\beta_{k+1}b|x|^{b-2}(u+\epsilon)^{-1}u^q\nabla u\cdot x,$$

where

$$I_{\alpha_k}^{(2)} := \frac{2}{n}(\alpha_{k+1}+\alpha_k+1)^2 - 2\alpha_{k+1}(\alpha_{k+1}+1) + \alpha_{k+1},$$

$$S_{\epsilon,\alpha_k,\beta_k} := \frac{4}{n}\beta_k(\alpha_{k+1}+\alpha_k+1)\frac{u^2}{(u+\epsilon)^2} + \beta_{k+1}\frac{u^2}{(u+\epsilon)^2} - 2\beta_{k+1}q\frac{u}{u+\epsilon}.$$

Therefore, the following lower bound for I holds:

$$I \geq 2\alpha_{k+1}(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} - \alpha_{k+1}w_{k+1}(u+\epsilon)^{-2}|\nabla u|^2 + \frac{2}{n}\alpha_{k+1}\beta_k^2|x|^{2b}u^{2q}(u+\epsilon)^{-1} + I_{\alpha_k}|\nabla u|^4(u+\epsilon)^{-3} + S_{\epsilon,\alpha_k,\beta_k}|\nabla u|^2u^{q-2}|x|^b - 2\beta_{k+1}b|x|^{b-2}(u+\epsilon)^{-1}u^q\nabla u\cdot x. \quad (3\text{-}11)$$

Finally, applying this lower bound for I and the lower bound given for J in (3-6), from (3-3) we get

$$\begin{aligned}\Delta w_{k+1} - 2\alpha_{k+1}(u+\epsilon)^{-1}\nabla u\cdot\nabla w_{k+1} + \alpha_{k+1}(u+\epsilon)^{-2}|\nabla u|^2 w_{k+1} - \beta_{k+1}qu^{q-1}|x|^b w_{k+1} \\ \geq |x|^a u^p\Big(1 - q\beta_{k+1}^2 + \frac{2}{n}\alpha_{k+1}\beta_k^2\frac{u}{u+\epsilon}\Big) + \alpha_{k+1}I_{\alpha_k}^{(2)}|\nabla u|^4(u+\epsilon)^{-3} \\ + \Big(\alpha_{k+1}S_{\epsilon,\alpha_k,\beta_k} + \Big(q(q-1) - \alpha_{k+1}q\frac{u}{u+\epsilon}\Big)\beta_{k+1}\Big)|\nabla u|^2u^{q-2}|x|^b \\ + 2b\beta_{k+1}\Big(q - \alpha_{k+1}\frac{u}{u+\epsilon}\Big)|x|^{b-2}u^{q-1}\nabla u\cdot x + b\beta_{k+1}(n+b-2)|x|^{b-2}u^q.\end{aligned}$$

Completing the square finishes the proof. □

4. Proof of Theorem 1.3 via iteration arguments

To apply the iteration argument, we need to develop a maximum principle argument for the equation

$$\Delta w - 2\alpha(u+\epsilon)^{-1}\nabla u\cdot\nabla w + \alpha w(u+\epsilon)^{-2}|\nabla u|^2 - \tfrac{1}{2}\beta(p+1)|x|^{a/2}u^{(p-1)/2}w = f(x) \geq 0 \quad \text{in } \mathbb{R}^n \quad (4\text{-}1)$$

that appears in Proposition 3.1, where α and β are positive constants, u is a solution of (1-1) and $w,\ f \in C^\infty(\mathbb{R}^n)$.

Lemma 4.1. *Suppose that w is a solution of the differential inequality (4-1), where u is a solution of (1-1) and*

$$w = \Delta u + \alpha(u+\epsilon)^{-1}|\nabla u|^2 + \beta|x|^{a/2}u^{(p+1)/2} \quad (4\text{-}2)$$

for positive constants ϵ, α and β. Then, assuming that $p+1 > 2\alpha$,

$$\Delta\tilde{w} \geq 0 \quad \text{on } \{w \geq 0\} \subset \mathbb{R}^n, \quad (4\text{-}3)$$

where $\tilde{w} = (u+\epsilon)^t w$ for $t = -\alpha$.

Proof. Straightforward calculations show that

$$\Delta\tilde{w} = (u+\epsilon)^t\Delta w + 2t(u+\epsilon)^{t-1}\nabla u\cdot\nabla w + t(u+\epsilon)^{t-1}w\Delta u + t(t-1)w(u+\epsilon)^{t-2}|\nabla u|^2.$$

We now add and subtract two terms, $\frac{1}{2}\beta(p+1)|x|^{a/2}u^{(p-1)/2}(u+\epsilon)^t w$ and $tw(u+\epsilon)^{t-2}|\nabla u|^2$, to the above identity and collect similar terms to get

$$\begin{aligned}\Delta\tilde{w} = (u+\epsilon)^t\big(\Delta w + 2t(u+\epsilon)^{-1}\nabla u\cdot\nabla w - tw(u+\epsilon)^{-2}|\nabla u|^2 - \tfrac{1}{2}\beta(p+1)|x|^{a/2}u^{(p-1)/2}w\big) \\ + \tfrac{1}{2}\beta(p+1)|x|^{a/2}u^{(p-1)/2}(u+\epsilon)^t w + tw(u+\epsilon)^{t-2}|\nabla u|^2 + t(u+\epsilon)^{t-1}w\Delta u \\ + t(t-1)w(u+\epsilon)^{t-2}|\nabla u|^2.\end{aligned}$$

From the fact that $t=-\alpha$ and w satisfies (4-1), we get

$$\Delta\tilde{w}\geq\tfrac{1}{2}\beta(p+1)|x|^{a/2}u^{(p-1)/2}(u+\epsilon)^{t}w+t(u+\epsilon)^{t-1}w\Delta u+t^{2}w(u+\epsilon)^{t-1}\frac{|\nabla u|^{2}}{u+\epsilon}.$$

Note that we can eliminate the gradient term using (4-2), that is,

$$\alpha(u+\epsilon)^{-1}|\nabla u|^{2}=w-\Delta u-\beta|x|^{a/2}u^{(p+1)/2}.$$

Therefore, after collecting similar terms we get

$$\begin{aligned}\Delta\tilde{w}&\geq\frac{t^{2}}{\alpha}w^{2}(u+\epsilon)^{t-1}+(u+\epsilon)^{t-1}wt\Big(1-\frac{t}{\alpha}\Big)\Delta u\\&\qquad+\beta(u+\epsilon)^{t-1}|x|^{a/2}u^{(p-1)/2}w\Big(\frac{(p+1)\epsilon}{2}+u\Big(\frac{p+1}{2}-\frac{t^{2}}{\alpha}\Big)\Big)\\&=:R_{1}+R_{2}+R_{3}.\end{aligned}$$

We claim that the above three terms, R_1, R_2 and R_3, are nonnegative when $w\geq 0$. From the fact that $\alpha>0$ one can see that R_1 is nonnegative. From the definition of $t=-\alpha<0$, we have $t(1-t/\alpha)=-2\alpha<0$. This together with Proposition 2.3, that is, $\Delta u\leq 0$, confirms that R_2 is nonnegative. Positivity of R_3 is an immediate consequence of the assumptions: β is positive and $\frac{1}{2}(p+1)-t^2/\alpha=\frac{1}{2}(p+1)-\alpha$ is also positive. This finishes the proof. □

We now apply Lemma 4.1 to show that any solution w of (4-1) is negative.

Lemma 4.2. *Suppose that $\tilde{w}$ and w as in Lemma 4.1. Let u be a bounded solution of (1-1); then $w\leq 0$.*

Proof. The methods and ideas that we apply in the proof are motivated by Souplet [2009]. Multiply (4-3) by $\tilde{w}_+^s$, where $s>0$ is a parameter that will be determined later. Then, integration by parts over B_R gives us

$$0\leq\int_{B_R}\Delta\tilde{w}\tilde{w}_+^s=-s\int_{B_R}|\nabla\tilde{w}_+|^2\tilde{w}_+^{s-1}+R^{n-1}\int_{S^{n-1}}\tilde{w}_r\tilde{w}_+^s.\tag{4-4}$$

Therefore,

$$\int_{B_R}|\nabla\tilde{w}_+|^2\tilde{w}_+^{s-1}\leq\frac{1}{s(s+1)}R^{n-1}\int_{S^{n-1}}(\tilde{w}_+^{s+1})_r=C(s)R^{n-1}I'(R),\tag{4-5}$$

where

$$I(R):=\int_{S^{n-1}}\tilde{w}_+^{s+1}=\int_{S^{n-1}}(u+\epsilon)^{-(s+1)\alpha}w_+^{s+1}$$

and $C(s)$ is a constant independent of R. Note that w, given as $w=\Delta u+\alpha|\nabla u|^2(u+\epsilon)^{-1}+\beta|x|^{a/2}u^{(p+1)/2}$, satisfies $w\geq 0$ if and only if $-\Delta u\leq\alpha|\nabla u|^2(u+\epsilon)^{-1}+\beta|x|^{a/2}u^{(p+1)/2}$. Therefore,

$$w_+^{s+1}\leq C|\nabla u|^{2(s+1)}(u+\epsilon)^{-(s+1)}+C|x|^{(s+1)a/2}u^{(s+1)(p+1)/2},\tag{4-6}$$

where $C = C(\alpha, \beta, s)$. Applying this upper bound for w_+, we can get an upper bound for $I(R)$:

$$\begin{aligned} I(R) &\le C \int_{S^{n-1}} (u+\epsilon)^{-(s+1)(\alpha+1)} |\nabla u|^{2(s+1)} + C R^{(s+1)a/2} \int_{S^{n-1}} (u+\epsilon)^{-\alpha(s+1)} u^{(s+1)(p+1)/2} \\ &\le C(\epsilon) \int_{S^{n-1}} |\nabla u|^{2(s+1)} + C(\epsilon) R^{(s+1)a/2} \int_{S^{n-1}} u^{(s+1)(p+1)/2} \\ &=: C(\epsilon)(I_1(R) + I_2(R)). \end{aligned} \tag{4-7}$$

In what follows, we show that there is a sequence R such that the two terms $I_1(R)$ and $I_2(R)$ decay to zero for a fixed ϵ. We start with $I_2(R)$, which includes an integral of a positive power of u over the sphere. Due to the boundedness assumption on u, it is straightforward to relate this term to L^p estimates of u over the sphere. As a matter of fact, if $(s+1)(p+1) > 2p$ then, from the boundedness of u, we have

$$\int_{S^{n-1}} u^{(s+1)(p+1)/2} \le C(n) \|u\|^p_{L^p(S^{n-1})} \tag{4-8}$$

and for the case $(s+1)(p+1) \le 2p$ we can use Hölder's inequality to get

$$\int_{S^{n-1}} u^{(s+1)(p+1)/2} \le C(n, p) \|u\|^{(p+1)(s+1)/2}_{L^p(S^{n-1})}. \tag{4-9}$$

So, to prove a decay estimate for $I_2(R)$ we need to construct a decay estimate for $\|u\|_{L^p(S^{n-1})}$. On the other hand, we apply Lemma 2.10 to get an upper bound for the first term in (4-7), $I_1(R)$. In fact, from Lemma 2.10 with $i = 1$, $\tau = 2(s+1)$ and $t = 2$, we have

$$\begin{aligned} \|D_x u\|_{L^{2(s+1)}(S^{n-1})} &\le C \|D_\theta D_x u\|_{L^2(S^{n-1})} + C \|D_x u\|_{L^1(S^{n-1})} \\ &\le C R \|D_x^2 u\|_{L^2(S^{n-1})} + C \|D_x u\|_{L^1(S^{n-1})} \end{aligned} \tag{4-10}$$

for $s = 2/(n-3)$. In order to get a decay estimate for $I_1(R)$, we need decay estimates for the two terms in the right-hand side of (4-10), $\|D_x^2 u\|_{L^2(S^{n-1})}$ and $\|D_x u\|_{L^1(S^{n-1})}$.

We now apply the elliptic estimates given in Section 2 to provide decay estimates for $\|u\|_{L^p(S^{n-1})}$, $\|D_x u\|_{L^1(S^{n-1})}$ and $\|D_x^2 u\|_{L^2(S^{n-1})}$. To do so we first find appropriate upper bounds for these terms on the ball of radius R. Then we use certain measure-comparison arguments to construct decay estimates over the sphere. So, from Lemma 2.7 with $\tau = 2$, we get

$$\int_{R/2}^{R} \|D_x^2 u\|^2_{L^2(S^{n-1})} r^{n-1}\, dr \le C \int_{B_{2R}\setminus B_{R/4}} |\Delta u|^2 + C R^{-4} \int_{B_{2R}\setminus B_{R/4}} u^2. \tag{4-11}$$

We now apply Corollary 2.9 and Corollary 2.2 to get a decay estimate for the right-hand side of (4-11), namely,

$$\begin{aligned} &R^{-4} \int_{B_{2R}\setminus B_{R/4}} u^2 \le C R^{-4} \int_{B_{2R}\setminus B_{R/4}} u \le C R^{-4} R^{n-(a+4)/(p-1)} = C R^{n-(a+4p)/(p-1)}, \\ &\int_{B_{2R}\setminus B_{R/4}} |\Delta u|^2 \le C R^{n-(a+4p)/(p-1)}, \end{aligned}$$

where C is independent from R. From this and (4-11), we obtain the desired decay estimate on the Hessian operator of u,

$$\int_{R/2}^{R} \|D_x^2 u\|^2_{L^2(S^{n-1})} r^{n-1}\, dr \le C R^{n-(4p+a)/(p-1)}. \tag{4-12}$$

Similarly, from Corollary 2.6 and Lemma 2.1, we have

$$\int_{R/2}^{R} \|D_x u\|_{L^1(S^{n-1})} r^{n-1}\, dr \le C R^{n-(p+3+a)/(p-1)}, \tag{4-13}$$

$$\int_{R/2}^{R} \|u\|^p_{L^p(S^{n-1})} r^{n-1}\, dr \le C R^{n-(a+4)p/(p-1)}. \tag{4-14}$$

Now let's define the following sets. These sets are meant to facilitate our arguments towards construction of decay estimates for $\|u\|_{L^p(S^{n-1})}$, $\|D_x u\|_{L^1(S^{n-1})}$ and $\|D_x^2 u\|_{L^2(S^{n-1})}$. For a large number M, which will be determined later, define

$$\begin{aligned}
\Gamma_1(R) &:= \{r \in (R/2, R) : \|u\|^p_{L^p(S^{n-1})} > M R^{-(a+4)p/(p-1)}\},\\
\Gamma_2(R) &:= \{r \in (R/2, R) : \|D_x u\|_{L^1(S^{n-1})} > M R^{-(p+3+a)/(p-1)}\},\\
\Gamma_3(R) &:= \{r \in (R/2, R) : \|D_x^2 u\|^2_{L^2(S^{n-1})} > M R^{-(a+4p)/(p-1)}\}.
\end{aligned}$$

We claim that $|\Gamma_i(R)| \le R/4$ for $1 \le i \le 3$: Using (4-12), we get

$$\begin{aligned}
C &\ge R^{-n+(a+4p)/(p-1)} \int_{R/2}^{R} \|D_x^2 u\|^2_{L^2(S^{n-1})} r^{n-1}\, dr\\
&\ge N R^{-n+(a+4p)/(p-1)} R^{n-1} \int_{R/2}^{R} \|D_x^2 u\|^2_{L^2(S^{n-1})}\, dr\\
&\ge N M R^{-n+(a+4p)/(p-1)} R^{n-1} \int_{|\Gamma_3(R)|} R^{-(a+4p)/(p-1)}\, dr\\
&\ge N M R^{-n+(a+4p)/(p-1)} R^{n-1} |\Gamma_3(R)| R^{-(a+4p)/(p-1)} = N M |\Gamma_3(R)| R^{-1},
\end{aligned}$$

where $N = \left(\frac{1}{2}\right)^{n-1}$. Therefore, $|\Gamma_3(R)| \le CR/NM$. Now, choosing M to be large enough, that is, $M > 4C/N$, we get $|\Gamma_3(R)| \le R/4$. Similarly, applying (4-13) and (4-14), one can show that $|\Gamma_i(R)| \le R/4$ for $i = 1, 2$. Hence, $|\Gamma_i(R)| \le R/4$ for $1 \le i \le 3$ while $\Gamma_i(R) \subset (R/2, R)$. So, we can find a sequence of $\tilde{R}$ such that

$$\tilde{R} \in (R/2, R) \setminus \bigcup_{i=1}^{i=3} \Gamma_i(R) \ne \varnothing. \tag{4-15}$$

Therefore, for the sequence $\tilde{R}$, we obtain

$$\|u\|^p_{L^p(S^{n-1})} \le M R^{-(a+4)p/(p-1)}, \tag{4-16}$$

$$\|D_x u\|_{L^1(S^{n-1})} \le M R^{-(p+3+a)/(p-1)}, \tag{4-17}$$

$$\|D_x^2 u\|^2_{L^2(S^{n-1})} \le M R^{-(a+4p)/(p-1)}. \tag{4-18}$$

Substituting (4-16) into (4-8) and (4-9), we get the decay estimate on $I_2(R)$

$$\begin{aligned} I_2(R) &\le C\chi\{(s+1)(p+1) > 2p\}R^{(s+1)/2a-(a+4)p/(p-1)} \\ &\quad + C\chi\{(s+1)(p+1) \le 2p\}R^{(s+1)a/2-(a+4)(p+1)(s+1)/(2(p-1))} \\ &= C\chi\{(s+1)(p+1) > 2p\}R^{-\eta_1} + C\chi\{(s+1)(p+1) > 2p\}R^{-\eta_2}, \end{aligned} \tag{4-19}$$

where χ is the characteristic function, $\eta_1 := a\bigl(p/(p-1) - \frac{1}{2}(s+1)\bigr) + 4p/(p-1) > 0$ and $\eta_2 := (s+1)(ap+2(p+1))/(p+1) > 0$. Note that we have used the fact that $p/(p-1) - \frac{1}{2}(s+1) > 0$ because $0 < s = 2/(n-3) \le 1$ when $n \ge 5$. On the other hand, substituting (4-17) and (4-18) into the Sobolev embedding (4-10), we get

$$\|D_x u\|_{L^{2(s+1)}(S^{n-1})} \le CR^{1-(a+4p)/(p-1)} + CR^{-(p+3+a)/(p-1)} = 2CR^{-(p+3+a)/(p-1)}. \tag{4-20}$$

From this and the definition of $I_1(R)$, we end up with the decay estimate on $I_1(R)$

$$I_1(R) = \int_{S^{n-1}} |\nabla u|^{2(s+1)} \le CR^{-2(p+3+a)(s+1)/(p-1)} = CR^{-\eta_3}, \tag{4-21}$$

where $\eta_3 := 2(p+3+a)(s+1)/(p-1) > 0$. Finally, from (4-21) and (4-19), we observe that

$$I(R) \le CR^{-\eta} \quad \text{for all } R > 1,$$

where $\eta := \min\{\eta_1, \eta_2, \eta_3\} > 0$. So, $I(R) \to 0$ as $R \to \infty$. Note that $\tilde{R} \to \infty$ as $R \to \infty$. Since $I(R)$ is a positive function and converges to zero, there is a sequence such that the functional $I'(R)$ is nonpositive. Therefore, (4-5) yields

$$\int_{B_R} |\nabla \tilde{w}_+|^2 \tilde{w}_+^{s-1} \le 0. \tag{4-22}$$

Hence, $\tilde{w}_+$ has to be a constant. From the continuity of $\tilde{w}$, we have $\tilde{w} \equiv C$. Note that the constant C cannot be strictly positive. So, $\tilde{w}_+ = 0$ and therefore $w_+ = 0$. This finishes the proof. □

Note that Lemma 4.1 and Lemma 4.2 imply an iteration argument for the sequence of functions, for $k \ge -1$,

$$w_k = \Delta u + \alpha_k (u+\epsilon)^{-1}|\nabla u|^2 + \beta_k |x|^{a/2} u^{(p+1)/2} \tag{4-23}$$

as long as the right-hand side of (3-3) stays nonnegative. For the rest of this section, we construct sequences $\{\alpha_k\}_{k=-1}$ and $\{\beta_k\}_{k=-1}$ such that the right-hand side of (3-3) is nonnegative.

Constructing sequences α_k and β_k. In this part, we define sequences α_k and β_k needed for the iteration argument.

Lemma 4.3. *Suppose $\alpha_0 = 0$ and define*

$$\alpha_{k+1} := \frac{4(\alpha_k+1) - n + \sqrt{n(16\alpha_k^2 + 24\alpha_k + n + 8)}}{4(n-1)}. \tag{4-24}$$

Then $(\alpha_k)_k$ is a positive, bounded and increasing sequence that converges to $\alpha := 2/(n-4)$ provided $n > 4$ and $p > 1$. Moreover, for this choice of $(\alpha_k)_k$, the sequence $I^{(2)}_{\alpha_k}$ of coefficients defined in Proposition 3.1 equals zero.

Proof. It is straightforward to show that $\alpha_k > 0$ for any $k \geq 0$. Also, direct calculations show that $\alpha_k \to \alpha := 2/(n-4)$ provided α_k is convergent. Note that $\alpha_1 = (4-n+\sqrt{n^2+8n})/(4n-4) < 2/(n-4)$ and, by induction, one can see that $\alpha_k \leq \alpha$ for all $k \geq 0$. Lastly, we show that α_k is an increasing sequence: For any k,

$$\begin{aligned}\alpha_{k+1} - \alpha_k &= \frac{\sqrt{n(16\alpha_k^2 + 24\alpha_k + n + 8)} - ((n-4) + 4a_k(n-2))}{4(n-1)} \\ &= \frac{8(n-1)(n-4)(2\alpha_k + 1)}{S_{n,k}}\left(\frac{2}{n-4} - \alpha_k\right),\end{aligned}$$

where $S_{n,k} = \sqrt{n(16\alpha_k^2 + 24\alpha_k + n + 8)} + (n-4) + 4a_k(n-2) > 0$. Therefore, from the fact that $\alpha_k \leq \alpha = 2/(n-4)$, we get the desired result. □

Similarly, we provide an explicit formula for the sequence β_k:

Lemma 4.4. *Suppose $\beta_0 = \sqrt{2/(p+1)}$ and define*

$$\beta_{k+1} := \sqrt{\frac{2}{p+1} + \frac{4}{(p+1)n}\alpha_k \beta_k^2}, \tag{4-25}$$

where $(\alpha_k)_k$ is as in Lemma 4.3. Then $(\beta_k)_k$ is a positive, bounded and increasing sequence that converges to $\beta := \sqrt{2/((p+1)-c_n)}$, where $c_n = 8/(n(n-4))$ provided that $n > 4$ and $p > 1$. Moreover, for this choice of $(\alpha_k)_k$ and $(\beta_k)_k$, the sequence $I^{(1)}_{0,\alpha_k,\beta_k}$ of coefficients defined in Proposition 3.1 is strictly positive.

Proof. The sequence $(\beta_k)_k$ for all $k \geq 0$ is positive. Note that boundedness of the sequence $(\alpha_k)_k$ forces the boundedness of the $(\beta_k)_k$, meaning that $\beta_{k+1} \leq \sqrt{2/(p+1) + (4\alpha/((p+1)n))\beta_k^2}$ for any k. By straightforward calculations we get

$$\beta_{k+1}^2 \leq \frac{2}{p+1}\sum_{i=0}^{k+1}\left(\frac{4\alpha}{n(p+1)}\right)^i.$$

Note that $4\alpha/(n(p+1)) = 8/(n(n-4)(p+1)) < 1$ provided that $n > 4$ and $p > 1$. Therefore, $\sum_{i=0}^{\infty}\bigl(4\alpha/(n(p+1))\bigr)^i < \infty$. This proves the boundedness of $(\beta_k)_k$.

Since $(\alpha_k)_{k=0}$ is an increasing sequence, the sequence $(\beta_k)_{k=0}$ will be nondecreasing by induction. Note that

$$\beta_1 = \beta_0 \quad \text{and} \quad \beta_2 = \sqrt{\frac{2}{p+1} + \frac{8}{(p+1)^2 n}\frac{4-n+\sqrt{n^2+8n}}{4n-4}} > \beta_1 = \sqrt{\frac{2}{p+1}}.$$

Suppose that $\beta_{k-1} \leq \beta_k$ for a certain index $k \geq 2$; then we apply the fact that $\alpha_k \geq \alpha_{k-1}$ to show $\beta_k \leq \beta_{k+1}$. This can be found as a consequence of

$$\beta_{k+1} - \beta_k = \frac{\beta_{k+1}^2 - \beta_k^2}{\beta_{k+1} + \beta_k} = \frac{4}{(p+1)n(\beta_{k+1} + \beta_k)}(\beta_k^2\alpha_k - \beta_{k-1}^2\alpha_{k-1}) \geq \frac{4\alpha_{k-1}(\beta_k + \beta_{k-1})}{(p+1)n(\beta_{k+1} + \beta_k)}(\beta_k - \beta_{k-1}).$$

So, $(\beta_k)_k$ is convergent and converges to $\beta := \sqrt{2n(n-4)/((p+1)(n-4)n-8)}$. Since $(p+1)n(n-4) > 8$ for $p > 1$ and $n > 4$, β is well-defined. □

Note that, based on the definition of the sequences $\{\alpha_k\}_{k=-1}$ and $\{\beta_k\}_{k=-1}$, we concluded that $I^{(1)}_{0,\alpha_k,\beta_k} > 0$ and $I^{(2)}_{\alpha_k} = 0$. In the next two lemmata we investigate the positivity of $I^{(3)}_{\epsilon,\alpha_k,\beta_k}$ and $I^{(4)}_{a,\epsilon,\alpha_k,\beta_k}$, the sequences that appeared in (3-3) in Proposition 3.1.

Lemma 4.5. *Set $\epsilon = 0$ in $I^{(3)}_{\epsilon,\alpha_k,\beta_k}$, which is defined in Proposition 3.1. Then*

$$I^{(3)}_{0,\alpha_k,\beta_k} \to I^{(3)}_{0,\alpha,\beta} := \frac{4}{n}\alpha\beta(2\alpha+1) + \alpha\beta + \beta q(q-3\alpha-1) \tag{4-26}$$

as $k \to \infty$. The constant $I^{(3)}_{0,\alpha,\beta}$ is positive provided $p > (n+4)/(n-4)$ and $n > 4$.

Proof. Note that when $p > (n+4)/(n-4)$ and $n > 4$, we have $\frac{1}{2}(p+1) > n/(n-4)$. As $k \to \infty$, from Lemma 4.3 and Lemma 4.4, the sequences $\alpha_k \to \alpha := 2/(n-4)$ and $\beta_k \to \beta := \sqrt{2/((p+1)-c_n)}$. Therefore,

$$\begin{aligned}
\frac{I^{(3)}_{0,\alpha,\beta}}{\beta} &= \frac{4}{n}\Big(\frac{2}{n-4}\Big)\Big(\frac{4}{n-4}+1\Big) + \frac{2}{n-4} + \frac{p+1}{2}\Big(\frac{p-1}{2} - \frac{6}{n-4}\Big) \\
&= \Big(\frac{p+1}{2}\Big)^2 - \Big(\frac{p+1}{2}\Big)\Big(\frac{n+2}{n-4}\Big) + \frac{2n}{(n-4)^2} \\
&= \Big(\frac{p+1}{2} - \frac{n}{n-4}\Big)\Big(\frac{p+1}{2} - \frac{2}{n-4}\Big) > 0.
\end{aligned}$$

□

Note that $I^{(4)}_{a,\epsilon,\alpha_k,\beta_k}$ appears in (3-3) mainly because of the weight function $|x|^a$. In other words, we have $I^{(4)}_{0,\epsilon,\alpha_k,\beta_k} = 0$ in the case of $a = 0$.

Lemma 4.6. *For any $k \ge 0$,*

$$I^{(3)}_{0,\alpha_k,\beta_k} < \beta_{k+1}\big(\tfrac{1}{2}(p+1) - \alpha_{k+1}\big)^2 \tag{4-27}$$

provided $p > (n+4)/(n-4)$ and $n > 4$. Therefore, for any $a \ge 0$ that satisfies the upper bound

$$a \le A_k := \frac{2(n-2)I^{(3)}_{0,\alpha_k,\beta_k}}{\beta_{k+1}\big(\tfrac{1}{2}(p+1) - \alpha_{k+1}\big)^2 - I^{(3)}_{0,\alpha_k,\beta_k}}, \tag{4-28}$$

the sequence $I^{(4)}_{a,0,\alpha_k,\beta_k}$ is positive for any k.

Proof. Basic calculations show that

$$\begin{aligned}
&\beta_{k+1}\Big(\frac{p+1}{2} - \alpha_{k+1}\Big)^2 - I^{(3)}_{0,\alpha_k,\beta_k} \\
&\quad= \beta_{k+1}\Big(\frac{p+1}{2} - \alpha_{k+1}\Big)^2 - \frac{4}{n}\alpha_{k+1}\beta_k(\alpha_{k+1}+\alpha_k+1) - \alpha_{k+1}\beta_{k+1} - \beta_{k+1}\frac{p+1}{2}\Big(\frac{p+1}{2} - 3\alpha_{k+1} - 1\Big) \\
&\quad\ge \beta_{k+1}\Big(\Big(\frac{p+1}{2} - \alpha_{k+1}\Big)^2 - \frac{4}{n}\alpha_{k+1}(\alpha_{k+1}+\alpha_k+1) - \alpha_{k+1} - \frac{p+1}{2}\Big(\frac{p+1}{2} - 3\alpha_{k+1} - 1\Big)\Big) \\
&\quad= \beta_{k+1}\Big(\frac{n-4}{n}\alpha^2_{k+1} - \frac{4}{n}\alpha^2_{k+1} - \frac{4}{n}\alpha_{k+1} + \frac{p-1}{2}\alpha_{k+1} + \frac{p+1}{2}\Big),
\end{aligned}$$

where we have used the fact that β_k and α_k are increasing sequences in the first and the second inequality, respectively. Therefore,

$$\begin{aligned}\beta_{k+1}\Big(\frac{p+1}{2}-\alpha_{k+1}\Big)^2 - I^{(3)}_{0,\alpha_k,\beta_k} &\geq \beta_{k+1}\Big(\frac{n-4}{n}\alpha_{k+1}^2+\alpha_{k+1}\Big(\frac{p-1}{2}-\frac{4}{n}\alpha_{k+1}\Big)+\frac{p+1}{2}-\frac{4}{n}\alpha_{k+1}\Big)\\ &\geq \beta_{k+1}\Big(\frac{n-4}{n}\alpha_{k+1}^2+(\alpha_{k+1}+1)\Big(\frac{p-1}{2}-\frac{4}{n}\alpha\Big)\Big)\\ &> 0.\end{aligned}$$

Note that in the last inequality we have used the fact that

$$\frac{p-1}{2}-\frac{4}{n}\alpha = \frac{p-1}{2}-\frac{4}{n}\frac{2}{n-4} > \frac{4}{(n-4)n}(n-2) > 0,$$

since $p > (n+4)/(n-4)$ and $n > 4$. □

Remark 4.7. It would be interesting if a counterpart of (1-12) could be proved for bounded solutions of the fourth-order semilinear equation $\Delta^2 u = f(u)$ under certain assumptions on the arbitrary nonlinearity $f \in C^1(\mathbb{R})$. We expect that such an inequality could be established for some convex nonlinearity f.

Appendix

We would like to mention that given the estimates in Lemma 2.1 and Lemma 2.4, one can provide a somewhat simpler proof of Proposition 2.3, as follows.

Second proof of Proposition 2.3. From Lemma 2.1, we have $\int_{\mathbb{R}^n} |x|^{2-n+a}u^p\,dx < \infty$. Hence, we define the function

$$w(x) = \frac{1}{n(n-2)\omega_n}\int_{\mathbb{R}^n}\frac{|y|^a u^p(y)}{|x-y|^{n-2}}\,dy.$$

It is clear that $w(x) \geq 0$ and $\Delta w = -|x|^a u^p$. This implies that, for a solution u of (1-1), the function $h(x) := w(x) + \Delta u(x)$ is a well-defined harmonic function on $\mathbb{R}^n$. Thus, for any $x_0 \in \mathbb{R}^n$ and any $R > 0$, by the mean value theorem for harmonic functions we will have

$$h(x_0) := \int_{\partial B_R(x_0)} h\,d\sigma = \int_{\partial B_R(x_0)} (w+\Delta u)\,d\sigma \leq \int_{\partial B_R(x_0)} w\,d\sigma + \int_{\partial B_R(x_0)} |\Delta u|\,d\sigma. \tag{A-1}$$

Since $w(x_0) < \infty$, through Tonelli's theorem, we can change the order of the integrations to see that the first integral on the right-hand side of (A-1) tends to zero as $R \to \infty$ for all R. To be more precise, notice that, up to a constant multiple, the first integral can be written as

$$\int_{\mathbb{R}^n}\int_{\partial B_R(x_0)}\frac{d\sigma_x}{|x-y|^{n-2}}|y|^a u^p(y)\,dy.$$

Then we use the fact that $\int_{\partial B_R(x_0)} 1/|x-y|^{n-2}\,d\sigma_x = |y-x_0|^{2-n}$ if $|y-x_0| > R$ and equals R^{2-n} if $|y-x_0| < R$. Thus the integral will split into two parts. The outside part tends to zero as $R \to \infty$ due to the fact that $w(x_0) < \infty$, while the inside part tends to zero due to the fact that, by Lemma 2.1,

$$R^{2-n}\int_{B_R(x_0)} |y|^a u^p(y)\,dy \leq R^{2-n}\int_{B_{R+|x_0|}(0)} |y|^a u^p\,dy \leq CR^{2-n}(R+|x_0|)^{n-(4p+a)/(p-1)}$$

tends to zero as $R \to \infty$. The second integral will tend to zero for some sequence of R by Lemma 2.4 again. Apply the above inequality to this sequence to see that $h(x_0) \leq 0$. Since x_0 is arbitrary, we have $-\Delta u \geq 0$. □

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CONVERGENCE RATES AND HÖLDER ESTIMATES IN ALMOST-PERIODIC HOMOGENIZATION OF ELLIPTIC SYSTEMS

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For a family of second-order elliptic systems in divergence form with rapidly oscillating, almost-periodic coefficients, we obtain estimates for approximate correctors in terms of a function that quantifies the almost periodicity of the coefficients. The results are used to investigate the problem of convergence rates. We also establish uniform Hölder estimates for the Dirichlet problem in a bounded $C^{1,\alpha}$ domain.

1. Introduction and statement of main results

In this paper we consider a family of second-order elliptic operators in divergence form with rapidly oscillating, *almost-periodic* coefficients,

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial}{\partial x_j}\right), \quad \varepsilon > 0. \tag{1-1}$$

We will assume that $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \le i, j \le d$ and $1 \le \alpha, \beta \le m$ is real and satisfies the ellipticity condition

$$\mu|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \le \frac{1}{\mu}|\xi|^2 \quad \text{for } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times m}, \tag{1-2}$$

where $\mu > 0$ (the summation convention is used throughout the paper). We further assume that $A = A(y)$ is uniformly almost-periodic in $\mathbb{R}^d$; i.e., A is the uniform limit of a sequence of trigonometric polynomials in $\mathbb{R}^d$.

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Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be the weak solution of the Dirichlet problem

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \qquad \text{and} \qquad u_\varepsilon = g \quad \text{on } \partial\Omega, \tag{1-3}$$

where $F \in H^{-1}(\Omega; \mathbb{R}^m)$ and $g \in H^{1/2}(\partial\Omega; \mathbb{R}^m)$. Under the ellipticity condition (1-2) and the almost periodicity condition on A, it is known that $u_\varepsilon \to u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$ and thus strongly in $L^2(\Omega; \mathbb{R}^m)$ as $\varepsilon \to 0$. Furthermore, the function u_0 is the solution of

$$\mathcal{L}_0(u_0) = F \quad \text{in } \Omega \qquad \text{and} \qquad u_0 = g \quad \text{on } \partial\Omega, \tag{1-4}$$

where $\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla)$ is a second-order elliptic operator with constant coefficients, uniquely determined by $A(y)$. As in the periodic case (see, e.g., [Bensoussan et al. 1978]), the constant matrix $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$ is called the homogenized matrix for A and $\mathcal{L}_0$ the homogenized operator for $\mathcal{L}_\varepsilon$. In this paper we shall be interested in quantitative homogenization results as well as uniform estimates for solutions of (1-3).

Homogenization of elliptic equations with rapidly oscillating, almost-periodic or random coefficients was studied first by S. M. Kozlov [1978; 1979] and by G. C. Papanicolaou and S. R. S. Varadhan [1981]. In particular, the $o(1)$ convergence rate of $u_\varepsilon - u_0$ in $C^\sigma(\overline{\Omega})$ for some $\sigma > 0$ was obtained in [Kozlov 1978] for a scalar second-order elliptic equation in divergence form with almost-periodic coefficients. Under some additional conditions on the frequencies in the spectrum of $A(y)$, the sharp $O(\varepsilon)$ rate in $C(\overline{\Omega})$ was proved in [Kozlov 1978] for operators with sufficiently smooth quasiperiodic coefficients. It is known that, without additional structure conditions on $A(y)$, the $O(\varepsilon)$ rate cannot be expected in general (see [Bondarenko et al. 2005] for some interesting results in the 1-dimensional case).

In contrast to the periodic case, the equation for the exact correctors $\chi(y)$,

$$-\operatorname{div}(A(y)\nabla\chi(y)) = \operatorname{div}(A(y)\nabla P(y)) \quad \text{in } \mathbb{R}^d, \tag{1-5}$$

may not be solvable in the almost-periodic (or random) setting for linear functions $P(y)$. In [Kozlov 1978], solutions $\chi(y)$ of (1-5) with sublinear growth and almost-periodic gradient were constructed and, as a result, homogenization was obtained for operators with trigonometric polynomial coefficients, by a lifting method. The homogenization result for the general case follows by an approximation argument. A different approach, which also gives the homogenization of the second-order elliptic equations with random coefficients, is to formulate and solve an abstract auxiliary equation in a Hilbert space for $\psi(y) = \nabla\chi(y)$. We outline this approach in Section 2 and refer the reader to [Jikov et al. 1994] for a detailed presentation and references.

Another approach to homogenization involves the use of the so-called approximate correctors [Papanicolaou and Varadhan 1981; Kozlov 1979]. Under certain mixing conditions, the approach has been employed successfully to establish quantitative homogenization results for second-order linear elliptic equations and systems in divergence form with random coefficients [Yurinskiĭ 1986; Pozhidaev and Yurinskiĭ 1989; Bourgeat and Piatnitski 2004]. For nonlinear second-order elliptic equations and Hamilton–Jacobi equations, we refer the reader to [Caffarelli and Souganidis 2010; Armstrong et al. 2014; Armstrong and Smart 2014] for recent advances and references on quantitative homogenization results. We point out that the almost-periodic case, which does not satisfy the mixing conditions generally imposed in the random case, is studied in [Caffarelli and Souganidis 2010; Armstrong et al. 2014]. We also mention that sharp quantitative results were obtained recently in [Gloria and Otto 2011; 2012; Gloria et al. 2014] for stochastic homogenization of discrete linear elliptic equations in divergence form.

In this paper we carry out a quantitative study of the approximate correctors $\chi_T = (\chi_{T,j}^\beta)$ for $\mathscr{L}_\varepsilon$ in (1-1), where, for $1 \le j \le d$ and $1 \le \beta \le m$, $u = \chi_{T,j}^\beta$ is defined by

$$-\operatorname{div}(A(y)\nabla u) + T^{-2} u = \operatorname{div}(A(y)\nabla P_j^\beta(y)) \quad \text{in } \mathbb{R}^d \tag{1-6}$$

and $P_j^\beta(y) = y_j(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the β-th position. Among other things, we will prove that, for $T \ge 1$ and $\sigma \in (0, 1)$,

$$T^{-1} \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \le C_\sigma \Theta_\sigma(T), \tag{1-7}$$

$$|\chi_T(x) - \chi_T(y)| \le C_\sigma T^{1-\sigma} |x-y|^\sigma \quad \text{for any } x, y \in \mathbb{R}^d, \tag{1-8}$$

and, for $0 < r \le T$,

$$\sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,r)} |\nabla \chi_T|^2 \right)^{\frac{1}{2}} \le C_\sigma \left(\frac{T}{r}\right)^\sigma, \tag{1-9}$$

where C_σ depends only on d, m, σ and A. The continuous function $\Theta_\sigma(T)$, which is decreasing and converges to zero as $T \to \infty$, is defined by

$$\Theta_\sigma(T) = \inf_{0<R\le T} \left(\rho(R) + \left(\frac{R}{T}\right)^\sigma \right), \tag{1-10}$$

where

$$\rho(R) = \sup_{y\in\mathbb{R}^d} \inf_{\substack{z\in\mathbb{R}^d \\ |z|\le R}} \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)} \tag{1-11}$$

is a decreasing and continuous function that quantifies the almost periodicity of A. Indeed, a bounded continuous function A in $\mathbb{R}^d$ is uniformly almost-periodic if and only if $\rho(R) \to 0$ as $R \to \infty$.

With the estimates (1-7), (1-8) and (1-9) at our disposal, we obtain the following theorems on the convergence rates. Our results in Theorems 1.2 and 1.4 are new even in the scalar case $m = 1$.

Theorem 1.1. *Suppose that $A(y) = (a_{ij}^{\alpha\beta}(y))$ satisfies the ellipticity condition (1-2) and is uniformly almost-periodic in $\mathbb{R}^d$. Let $p > d$, $\sigma \in (0, 1)$, and Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$ for some $\alpha > 0$. Then there exists a modulus $\eta : (0, 1] \to [0, \infty)$, which depends only on A and σ, such that $\lim_{t\to 0} \eta(t) = 0$ and*

$$\|u_\varepsilon - u_0\|_{C^\sigma(\bar{\Omega})} \le C\eta(\varepsilon) \|u_0\|_{W^{2,p}(\Omega)} \tag{1-12}$$

for $\varepsilon \in (0, 1)$ whenever $u_\varepsilon \in H^1(\Omega)$ is the weak solution of (1-3) and $u_0 \in W^{2,p}(\Omega)$ is the solution of (1-4). Furthermore, we have

$$\|u_\varepsilon - u_0 - \varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)} \le C\eta(\varepsilon)\|u_0\|_{W^{2,p}(\Omega)}, \tag{1-13}$$

where $T = \varepsilon^{-1}$ and $\chi_T(y)$ denotes the approximate corrector defined by (1-6). The constants C in (1-12) and (1-13) depend only on Ω, p, σ and A.

The next theorem gives more precise rates of convergence, provided $\rho(R)$ decays fast enough that $\int_1^\infty (\rho(r)/r)\,dr < \infty$.

Theorem 1.2. *Under the same assumptions as in Theorem 1.2,*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\|u_0\|_{W^{2,p}(\Omega)} \left(\int_{1/(2\varepsilon)}^\infty \frac{\Theta_\sigma(r)}{r}\,dr + [\Theta_1(\varepsilon^{-1})]^\sigma \right) \tag{1-14}$$

and

$$\|u_\varepsilon - u_0 - \varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)} \le C\|u_0\|_{W^{2,p}(\Omega)} \left(\int_{1/(2\varepsilon)}^\infty \frac{\Theta_\sigma(r)}{r}\,dr + [\Theta_1(\varepsilon^{-1})]^{\sigma/2} \right) \tag{1-15}$$

for any $\sigma \in (0, 1)$, where $T = \varepsilon^{-1}$ and C depends only on Ω, A, p and σ.

Remark 1.3. By taking $R = \sqrt{T}$ in (1-10), we obtain $\Theta_\sigma(T) \le \rho(\sqrt{T}) + T^{-\sigma/2}$ for $T \ge 1$. It follows that

$$\int_1^\infty \frac{\rho(r)}{r}\,dr < \infty \implies \int_1^\infty \frac{\Theta_\sigma(r)}{r}\,dr < \infty \tag{1-16}$$

for any $\sigma \in (0, 1]$. It is not clear whether estimates (1-14) and (1-15) are sharp. However, let us suppose that there exist $\tau > 0$ and $C > 0$ such that

$$\rho(R) \le CR^{-\tau} \quad \text{for all } R \ge 1. \tag{1-17}$$

Then, for $T \ge 1$,

$$\Theta_\sigma(T) \le CT^{-\sigma\tau/(\sigma+\tau)}.$$

It follows from (1-14) that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon^{\sigma\tau/(\sigma+\tau)}\|u_0\|_{W^{2,p}(\Omega)}.$$

Since $\sigma \in (0, 1)$ is arbitrary, this gives

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C_\gamma \varepsilon^\gamma \|u_0\|_{W^{2,p}(\Omega)} \quad \text{for any } 0 < \gamma < \frac{\tau}{\tau+1}. \tag{1-18}$$

Similarly, one may deduce from (1-15) that

$$\|u_\varepsilon - u_0 - \varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)} \le C_\gamma \varepsilon^\gamma \|u_0\|_{W^{2,p}(\Omega)} \tag{1-19}$$

for any $0 < \gamma < \tau/(2(\tau+1))$. It is interesting to note that if A is periodic then $\rho(R) = 0$ for R large and thus the condition (1-17) holds for any $\tau > 1$. Consequently, estimates (1-18) and (1-19) yield convergence rates $O(\varepsilon^{1-\delta})$ and $O(\varepsilon^{1/2-\delta})$ for any $\delta > 0$ in $L^2(\Omega)$ and $H^1(\Omega)$, respectively, which are near optimal. Also note that, under the condition (1-17), our estimate (1-7) gives

$$\|\chi_T\|_{L^\infty} \le C_\delta T^{1/(\tau+1)+\delta} \tag{1-20}$$

for any $\delta > 0$, while one has $\|\chi_T\|_{L^\infty} \le C$ if A is periodic. Section 8 contains some examples of quasiperiodic functions for which condition (1-17) is satisfied.

In this paper we also establish the uniform Hölder estimates for the Dirichlet problem (1-3).

Theorem 1.4. *Suppose that $A(y) = (a_{ij}^{\alpha\beta}(y))$ satisfies the ellipticity condition (1-2) and is uniformly almost-periodic in $\mathbb{R}^d$. Let Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$ for some $\alpha > 0$. Let u_ε be a weak solution of*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F + \operatorname{div}(f) \quad \text{in } \Omega \qquad \text{and} \qquad u_\varepsilon = g \quad \text{on } \partial\Omega. \tag{1-21}$$

Then, for any $\sigma \in (0, 1)$,

$$\|u_\varepsilon\|_{C^\sigma(\bar{\Omega})} \le C\biggl(\|g\|_{C^\sigma(\partial\Omega)} + \sup_{\substack{x\in\Omega\\ 0<r<r_0}} r^{2-\sigma} \fint_{B(x,r)\cap\Omega} |F| + \sup_{\substack{x\in\Omega\\ 0<r<r_0}} r^{1-\sigma}\biggl(\fint_{B(x,r)\cap\Omega} |f|^2\biggr)^{\frac{1}{2}}\biggr), \tag{1-22}$$

where $r_0 = \operatorname{diam}(\Omega)$ and C depends only on σ, A and Ω.

We now describe the outline of this paper as well as some of key ideas used in the proof of its main results. In Section 2 we give a brief review of the homogenization of second-order elliptic systems with almost-periodic coefficients, based on an auxiliary equation in $B^2(\mathbb{R}^d)$, the Besicovich space of almost-periodic functions. We also prove a homogenization theorem (Theorem 2.2) for a sequence of operators $\{-\operatorname{div}(B_\ell(x/\varepsilon_\ell)\nabla)\}$, where $\varepsilon_\ell \to 0$ and $\{B_\ell(y)\}$ are obtained from $A(y)$ through rotations and translations. With this theorem, a compactness argument is used in Sections 3 and 4 to establish the uniform interior and boundary Hölder estimates for local solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = F + \operatorname{div}(f)$. The proof of Theorem 1.4 is given in Section 4. We mention that the compactness argument, which originated from the regularity theory in the calculus of variations and minimal surfaces, was introduced to the study of homogenization problems by M. Avellaneda and F. Lin [1987; 1989]. It was used recently in [Kenig et al. 2013] to establish the Lipschitz estimates for the Neumann problem in periodic homogenization. Also see related work in [Shen 2008; Geng et al. 2012; Shen and Geng 2015]. In the almost-periodic setting, the compactness argument was used in [Dungey et al. 2001] to obtain the interior Hölder estimate for operators with complex coefficients. However, we point out that some version of Theorem 2.2 seems to be necessary to ensure that the constants are independent of the centers of balls.

The approximate correctors χ_T are constructed in Section 5, while estimates (1-7), (1-8) and (1-9) are established in Section 6. The proof of (1-8) and (1-9) relies on the uniform Hölder estimates for $\mathcal{L}_\varepsilon$. We will also show that

$$|\chi_T(x) - \chi_T(y)| \le CT\|A(\cdot + x) - A(\cdot + y)\|_{L^\infty} \quad \text{for any } x, y \in \mathbb{R}^d. \tag{1-23}$$

The estimate (1-7) follows from (1-23) and (1-8) in a manner somewhat similar to the case of Hamilton–Jacobi equations in the almost-periodic setting [Ishii 2000; Lions and Souganidis 2005; Armstrong et al. 2014].

Theorems 1.1 and 1.2 are proved in Section 7. Here we follow an approach for the periodic case by considering

$$w_\varepsilon = u_\varepsilon(x) - u_0(x) - \varepsilon\chi_T(x/\varepsilon)\nabla u_0(x) + v_\varepsilon(x),$$

where $T=\varepsilon^{-1}$ and v_ε is the weak solution of the problem $\mathscr{L}_\varepsilon(v_\varepsilon)=0$ in Ω and $v_\varepsilon=\varepsilon\chi_T(x/\varepsilon)\nabla u_0(x)$ on $\partial\Omega$. We are able to show that

$$\|w_\varepsilon\|_{H^1(\Omega)} \le C_\sigma\big(\Theta_\sigma(T)+\langle|\psi-\nabla\chi_T|\rangle\big)\|u_0\|_{W^{2,2}(\Omega)} \tag{1-24}$$

for any $\sigma\in(0,1)$, where ψ is the limit of $\nabla\chi_T$ in $B^2(\mathbb{R}^d)$ as $T\to\infty$. In the periodic case, one of the key steps is to write $\hat{A}-A(y)-A(y)\nabla\chi(y)$ as a divergence of some bounded periodic function. In the almost-periodic setting, this will be replaced by solving the equation

$$-\Delta u+T^{-2}u=B_T-\langle B_T\rangle \quad \text{in } \mathbb{R}^d, \tag{1-25}$$

where $B_T(y)=\hat{A}-A(y)-A(y)\nabla\chi_T(y)$. The same ideas for proving (1-7)–(1-9) are used to obtain the desired estimates for $\|u\|_{L^\infty}$ and $\|\nabla u\|_{L^\infty}$ in terms of the function $\Theta_\sigma(T)$. Finally, in Section 8 we consider the case of quasiperiodic coefficients and provide some sufficient conditions on the frequencies of $A(y)$ for the estimate (1-17) on $\rho(R)$.

Throughout this paper, unless indicated otherwise, we always assume that $A=(a_{ij}^{\alpha\beta})$ satisfies the ellipticity condition (1-2) and is uniformly almost-periodic in $\mathbb{R}^d$. We will use $\fint_E f=(1/|E|)\int_E f$ to denote the L^1 average of f over E, and C to denote constants that depend on $A(y)$, Ω and other relevant parameters, but never on ε or T.

2. Homogenization and compactness

This section contains a brief review of homogenization theory of elliptic systems with almost-periodic coefficients. We refer the reader to [Jikov et al. 1994, pp. 238–242] for a detailed presentation. We also prove a homogenization theorem for a sequence of operators obtained from $\mathscr{L}_\varepsilon$ through translations and rotations.

Let $\mathrm{Trig}(\mathbb{R}^d)$ denote the set of (real) trigonometric polynomials in $\mathbb{R}^d$. A bounded continuous function f in $\mathbb{R}^d$ is said to be uniformly almost-periodic (or almost-periodic in the sense of Bohr) if f is a limit of a sequence of functions in $\mathrm{Trig}(\mathbb{R}^d)$ with respect to the norm $\|f\|_{L^\infty}$. A function f in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ is said to belong to $B^2(\mathbb{R}^d)$ if f is a limit of a sequence of functions in $\mathrm{Trig}(\mathbb{R}^d)$ with respect to the seminorm

$$\|f\|_{B^2}=\limsup_{R\to\infty}\left(\fint_{B(0,R)}|f|^2\right)^{\frac{1}{2}}. \tag{2-1}$$

Functions in $B^2(\mathbb{R}^d)$ are said to be almost-periodic in the sense of Besicovich. It is not hard to see that, if $f\in B^2(\mathbb{R}^d)$ and g is uniformly almost-periodic, then $fg\in B^2(\mathbb{R}^d)$.

Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. A number $\langle f\rangle$ is called the mean value of f if

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^d} f(x/\varepsilon)\varphi(x)\,dx=\langle f\rangle\int_{\mathbb{R}^d}\varphi \tag{2-2}$$

for any $\varphi\in C_0^\infty(\mathbb{R}^d)$. If $f\in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $\|f\|_{B^2}<\infty$, the existence of $\langle f\rangle$ is equivalent to the condition that, as $\varepsilon\to 0$, $f(x/\varepsilon)\rightharpoonup\langle f\rangle$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, i.e., $f(x/\varepsilon)\rightharpoonup\langle f\rangle$ weakly in $L^2(B(0,R))$ for any $R > 1$. In this case, one has

$$\langle f \rangle = \lim_{L\to\infty} \fint_{B(0,L)} f.$$

It is known that if $f, g \in B^2(\mathbb{R}^d)$ then fg has the mean value. Furthermore, under the equivalent relation that $f \sim g$ if $\|f - g\|_{B^2} = 0$, the set $B^2(\mathbb{R}^d)/\!\sim$ is a Hilbert space with the inner product defined by $(f, g) = \langle fg \rangle$.

A function $f = (f_i^\alpha)$ in $\mathrm{Trig}(\mathbb{R}^d; \mathbb{R}^{d\times m})$ is called potential if there exists $g = (g^\alpha) \in \mathrm{Trig}(\mathbb{R}^d; \mathbb{R}^m)$ such that $f_i^\alpha = \partial g^\alpha/\partial x_i$. A function $f = (f_i^\alpha)$ in $\mathrm{Trig}(\mathbb{R}^d; \mathbb{R}^{d\times m})$ is called solenoidal if $\partial f_i^\alpha/\partial x_i = 0$ for $1 \le \alpha \le m$. Let V^2_{pot} (resp. V^2_{sol}) denote the closure of potential (resp. solenoidal) trigonometric polynomials with mean value zero in $B^2(\mathbb{R}^d; \mathbb{R}^{d\times m})$. Then

$$B^2(\mathbb{R}^d; \mathbb{R}^{d\times m}) = V^2_{\mathrm{pot}} \oplus V^2_{\mathrm{sol}} \oplus \mathbb{R}^{d\times m}. \tag{2-3}$$

By the Lax–Milgram theorem and the ellipticity condition (1-2), for any $1 \le j \le d$ and $1 \le \beta \le m$ there exists a unique $\psi_j^\beta = (\psi_{ij}^{\alpha\beta}) \in V^2_{\mathrm{pot}}$ such that

$$\langle a_{ik}^{\alpha\gamma} \psi_{kj}^{\gamma\beta} \phi_i^\alpha \rangle = -\langle a_{ij}^{\alpha\beta} \phi_i^\alpha \rangle \quad \text{for any } \phi = (\phi_i^\alpha) \in V^2_{\mathrm{pot}}. \tag{2-4}$$

Let

$$\hat{a}_{ij}^{\alpha\beta} = \langle a_{ij}^{\alpha\beta} \rangle + \langle a_{ik}^{\alpha\gamma} \psi_{kj}^{\gamma\beta} \rangle \tag{2-5}$$

and $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$. Then

$$\mu|\xi|^2 \le \hat{a}_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \le \mu_1 |\xi|^2 \tag{2-6}$$

for any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times m}$, where μ_1 depends only on d, m and μ. It is also known that $\widehat{A^*} = (\hat{A})^*$, where A^* denotes the adjoint of A, i.e., $A^* = (b_{ij}^{\alpha\beta})$ with $b_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$.

As the following theorem shows, the homogenized operator for $\mathcal{L}_\varepsilon$ is given by $\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla)$.

Theorem 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $F \in H^{-1}(\Omega; \mathbb{R}^m)$. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω. Suppose $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$. Then $A(x/\varepsilon)\nabla u_\varepsilon \rightharpoonup \hat{A}\nabla u_0$ weakly in $L^2(\Omega; \mathbb{R}^{dm})$. Consequently, if $f \in H^{1/2}(\partial\Omega; \mathbb{R}^m)$ and u_ε is the unique weak solution in $H^1(\Omega; \mathbb{R}^m)$ of the Dirichlet problem $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω and $u_\varepsilon = f$ on $\partial\Omega$, then, as $\varepsilon \to 0$, $u_\varepsilon \to u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$ and strongly in $L^2(\Omega; \mathbb{R}^m)$, where u_0 is the unique weak solution in $H^1(\Omega; \mathbb{R}^m)$ of the Dirichlet problem $\mathcal{L}_0(u_0) = F$ in Ω and $u_0 = f$ on $\partial\Omega$.*

Proof. See [Jikov et al. 1994] for the single equation case ($m = 1$). The proof for the case $m > 1$ is exactly the same. □

In Sections 3 and 4 we will use a compactness argument to establish the uniform Hölder estimates for local solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f) + F$. This requires us to work with a class of operators that are obtained from $\mathcal{L}^A = -\operatorname{div}(A(x)\nabla)$ through translations and rotations of coordinates in $\mathbb{R}^d$. Observe that, if $\mathcal{L}^A(u) = F$ and $x = Oy + z$ for some rotation $O = (O_{ij})$ and $z \in \mathbb{R}^d$, then $\mathcal{L}^B(v) = G$, where

$v(y)=u(Oy+z)$, $B=(b_{ij}^{\alpha\beta}(y))$ with $b_{ij}^{\alpha\beta}(y)=a_{\ell k}^{\alpha\beta}(Oy+z)O_{\ell i}O_{kj}$, and $G(y)=F(Oy+z)$. Thus, for each $A=(a_{ij}^{\alpha\beta})$ fixed, we shall consider the set of matrices

$$\mathcal{A}=\left\{B=(b_{ij}^{\alpha\beta}(y)):b_{ij}^{\alpha\beta}(y)=a_{\ell k}^{\alpha\beta}(Oy+z)O_{\ell i}O_{kj}\ \text{ for some rotation }\ O=(O_{ij})\ \text{ and }\ z\in\mathbb{R}^d\right\}. \tag{2-7}$$

Note that, if $B(y)=O^tA(Oy+z)O\in\mathcal{A}$, where O^t denotes the transpose of O, then the homogenized matrix $\hat{B}$ equals $O^t\hat{A}O$.

The proof of Theorems 3.1 and 4.1 relies on the following extension of Theorem 2.1:

Theorem 2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $F\in H^{-1}(\Omega;\mathbb{R}^m)$. Let $u_\ell\in H^1(\Omega;\mathbb{R}^m)$ be a weak solution of $-\operatorname{div}(A_\ell(x/\varepsilon_\ell)\nabla u_\ell)=F$ in Ω, where $\varepsilon_\ell\to 0$ and $A_\ell\in\mathcal{A}$. Suppose that $u_\ell\rightharpoonup u$ weakly in $H^1(\Omega;\mathbb{R}^m)$. Then u is a weak solution of $-\operatorname{div}(\tilde{A}\nabla u)=F$ in Ω, where $\tilde{A}=O^t\hat{A}O$ for some rotation O in $\mathbb{R}^d$.*

Proof. Suppose that $A_\ell(y)=O_\ell^tA(O_\ell y+z_\ell)O_\ell$ for some rotations O_ℓ and $z_\ell\in\mathbb{R}^d$. By passing to a subsequence we may assume that $O_\ell\to O$ as $\ell\to\infty$. Since $A(y)$ is uniformly almost-periodic, $\{A(y+z_\ell)\}_{\ell=1}^\infty$ is precompact in $C_b(\mathbb{R}^d)$, the set of bounded continuous functions in $\mathbb{R}^d$. Thus, by passing to a subsequence, we may also assume that $A(y+z_\ell)$ converges uniformly in $\mathbb{R}^d$ to an almost-periodic matrix $B(y)$. Consequently, we obtain $A_\ell(y)\to\tilde{B}(y)=O^tB(Oy)O$ uniformly in $\mathbb{R}^d$. Note that $\hat{\tilde{B}}=O^t\hat{B}O=O^t\hat{A}O$.

Now, let $v_\ell\in H^1(\Omega;\mathbb{R}^m)$ be the weak solution of the Dirichlet problem

$$-\operatorname{div}(\tilde{B}(x/\varepsilon_\ell)\nabla v_\ell)=F\quad\text{in }\Omega\qquad\text{and}\qquad v_\ell=u_\ell\quad\text{on }\partial\Omega.$$

Using $-\operatorname{div}\big(A_\ell(x/\varepsilon_\ell)\nabla(u_\ell-v_\ell)\big)=\operatorname{div}\big((A_\ell(x/\varepsilon_\ell)-\tilde{B}(x/\varepsilon_\ell))\nabla v_\ell\big)$ in Ω and $u_\ell-v_\ell=0$ on $\partial\Omega$, we may use the energy estimates to deduce that

$$\|u_\ell-v_\ell\|_{H^1(\Omega)}\le C\|A_\ell-\tilde{B}\|_{L^\infty}\|\nabla v_\ell\|_{L^2(\Omega)}\le C\|A_\ell-\tilde{B}\|_{L^\infty}\{\|u_\ell\|_{H^1(\Omega)}+\|F\|_{H^{-1}(\Omega)}\}.$$

It follows that $u_\ell-v_\ell\to 0$ in $H^1(\Omega;\mathbb{R}^m)$ as $\ell\to\infty$.

Finally, since $v_\ell=v_\ell-u_\ell+u_\ell\rightharpoonup u$ weakly in $H^1(\Omega;\mathbb{R}^m)$, it follows from Theorem 2.1 that $\tilde{B}(x/\varepsilon_\ell)\nabla v_\ell\rightharpoonup\tilde{A}\nabla u$ weakly in $H^1(\Omega;\mathbb{R}^{d\times m})$, where $\tilde{A}=\hat{\tilde{B}}=O^t\hat{A}O$. As a result, we obtain $-\operatorname{div}(\tilde{A}\nabla u)=F$ in Ω. This completes the proof. □

3. Uniform interior Hölder estimates

The goal of this and the next section is to establish uniform interior and boundary Hölder estimates for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon)=f+\operatorname{div}(g)$. We will first use a compactness method to deal with the special case $\mathcal{L}_\varepsilon(u_\varepsilon)=0$. The results are then used to establish size and Hölder estimates for fundamental solutions and Green functions for $\mathcal{L}_\varepsilon$. The general case follows from the estimates for fundamental solutions and Green functions.

Theorem 3.1. *Let $u_\varepsilon \in H^1(B(x_0, 2r); \mathbb{R}^m)$ be a weak solution of* $\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ *in* $B(x_0, 2r)$ *for some* $x_0 \in \mathbb{R}^d$ *and* $r > 0$. *Let* $\sigma \in (0, 1)$. *Then*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C_\sigma \left(\frac{|x-y|}{r}\right)^\sigma \left(\fint_{B(x_0,2r)} |u_\varepsilon|^2\right)^{\frac{1}{2}} \tag{3-1}$$

for any $x, y \in B(x_0, r)$, *where* C_σ *depends only on* d, m, σ *and* A (*not on* ε, x_0 *or* r).

Theorem 3.1 follows from Theorem 2.2 by a three-step compactness argument, similar to the periodic case in [Avellaneda and Lin 1987].

Lemma 3.2. *Let* $0 < \sigma < 1$. *Then there exist constants* $\varepsilon_0 > 0$ *and* $\theta \in \left(0, \frac{1}{4}\right)$, *depending only on* σ *and* A, *such that*

$$\fint_{B(y,\theta)} \left| u_\varepsilon - \fint_{B(y,\theta)} u_\varepsilon \right|^2 \le \theta^{2\sigma} \quad \textit{for any } 0 < \varepsilon < \varepsilon_0 \tag{3-2}$$

whenever $u_\varepsilon \in H^1(B(y, 1); \mathbb{R}^m)$ *is a weak solution of* $\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ *in* $B(y, 1)$ *for some* $y \in \mathbb{R}^d$ *and*

$$\fint_{B(y,1)} |u_\varepsilon|^2 \le 1.$$

Proof. If $\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y, 1)$ and $v(x) = u_\varepsilon(x + y)$, then $\operatorname{div}(B(x/\varepsilon)\nabla v) = 0$ in $B(0, 1)$, where $B(x) = A(x + \varepsilon^{-1} y) \in \mathscr{A}$. As a result, it suffices to establish estimate (3-2) for $y = 0$ and for solutions u_ε of $\operatorname{div}(B(x/\varepsilon)\nabla u_{)} = 0$ in $B(0, 1)$, where $B \in \mathscr{A}$.

To this end, we first note that, if w is a solution of a second-order elliptic system in $B\left(0, \frac{1}{2}\right)$ with constant coefficients satisfying the ellipticity condition (2-6), then

$$\fint_{B(0,\theta)} \left| w - \fint_{B(0,\theta)} w \right|^2 \le C_0 \theta^2 \fint_{B(0,1/2)} |w|^2 \quad \text{for any } 0 < \theta < \tfrac{1}{4}, \tag{3-3}$$

where C_0 depends only on d, m and μ. We now choose $\theta \in \left(0, \frac{1}{4}\right)$ so small that

$$2^d C_0 \theta^2 < \theta^{2\sigma}. \tag{3-4}$$

We claim that the estimate (3-2) with $y = 0$ holds for this θ and for some $\varepsilon_0 > 0$, which depends only on A, whenever u_ε is a weak solution of $\operatorname{div}(B(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(0, 1)$ for some $B \in \mathscr{A}$.

Suppose this is not the case. Then there exist $\{\varepsilon_\ell\} \subset \mathbb{R}_+$, $\{B_\ell\} \subset \mathscr{A}$ and $\{u_\ell\} \subset H^1(B(0, 1); \mathbb{R}^m)$ such that $\varepsilon_\ell \to 0$,

$$\begin{cases} \operatorname{div}(B_\ell(x/\varepsilon_\ell)\nabla u_\ell) = 0 & \text{in } B(0, 1), \\ \displaystyle\fint_{B(0,1)} |u_\ell|^2 \le 1, \end{cases} \tag{3-5}$$

and

$$\fint_{B(0,\theta)} \left| u_\ell - \fint_{B(0,\theta)} u_\ell \right|^2 > \theta^{2\sigma}. \tag{3-6}$$

Since $\{u_\ell\}$ is bounded in $L^2(B(0, 1); \mathbb{R}^m)$, by Cacciopoli's inequality, $\{u_\ell\}$ is bounded in $H^1\left(B\left(0, \frac{1}{2}\right); \mathbb{R}^m\right)$. By passing to a subsequence, we may assume $u_\ell \rightharpoonup u$ weakly in $H^1\left(B\left(0, \frac{1}{2}\right); \mathbb{R}^m\right)$ and in $L^2(B(0, 1); \mathbb{R}^m)$.

It follows from Theorem 2.2 that u is a solution of $\operatorname{div}(\tilde{A}u) = 0$ in $B\big(0, \frac{1}{2}\big)$, where $\tilde{A} = O^t \hat{A} O$ for some rotation O in $\mathbb{R}^d$. Since the matrix $O^t \hat{A} O$ satisfies the ellipticity condition (2-6), estimate (3-3) holds for $w = u$. However, since $u_\ell \to u$ strongly in $L^2\big(B\big(0, \frac{1}{2}\big); \mathbb{R}^m\big)$, we may deduce from (3-6) that

$$\theta^{2\sigma} \le \fint_{B(0,\theta)} \Big| u - \fint_{B(0,\theta)} u \Big|^2 \le C_0 \theta^2 \fint_{B(0,1/2)} |u|^2 \le 2^d C_0 \theta^2 \fint_{B(0,1)} |u|^2, \tag{3-7}$$

where we have used (3-3) for the second inequality.

Finally, we note that the weak convergence of u_ℓ in $L^2(B(0, 1); \mathbb{R}^m)$ and the inequality in (3-5) give

$$\fint_{B(0,1)} |u|^2 \le 1.$$

In view of (3-7), we obtain $\theta^{2\sigma} \le 2^d C_0 \theta^2$, which contradicts (3-4). This completes the proof. □

Lemma 3.3. *Fix $0 < \sigma < 1$. Let ε_0 and θ be the constants given by Lemma 3.2. Let $u_\varepsilon \in H^1(B(y, 1); \mathbb{R}^m)$ be a weak solution of $\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y, 1)$ for some $y \in \mathbb{R}^d$. Then, if $0 < \varepsilon < \varepsilon_0 \theta^{k-1}$ for some $k \ge 1$,*

$$\fint_{B(y,\theta^k)} \Big| u_\varepsilon - \fint_{B(y,\theta^k)} u_\varepsilon \Big|^2 \le \theta^{2k\sigma} \fint_{B(y,1)} |u_\varepsilon|^2. \tag{3-8}$$

Proof. The lemma is proved by an induction argument on k, using Lemma 3.2 and the rescaling property that, if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(y, 1)$ and $v(x) = u_\varepsilon(\theta^k x)$, then

$$\mathcal{L}_{\varepsilon/\theta^k}(v) = 0 \quad \text{in } B(\theta^{-k} y, \theta^{-k}).$$

See [Avellaneda and Lin 1987] for the periodic case. □

Proof of Theorem 3.1. By rescaling we may assume that $r = 1$. Suppose that $u_\varepsilon \in H^1(B(y, 2); \mathbb{R}^m)$ and $\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y, 2)$ for some $y \in \mathbb{R}^d$. We show that

$$\fint_{B(z,t)} \Big| u_\varepsilon - \fint_{B(z,t)} u_\varepsilon \Big|^2 \le C t^{2\sigma} \fint_{B(z,1)} |u_\varepsilon|^2 \tag{3-9}$$

for any $0 < t < \theta$ and $z \in B(y, 1)$, where $\theta \in \big(0, \frac{1}{4}\big)$ is given by Lemma 3.2. The estimate (3-1) follows from (3-9) by Campanato's characterization of Hölder spaces.

With Lemma 3.3 at our disposal, the proof of (3-9) follows the same line of argument as in the periodic case. We refer the reader to [Avellaneda and Lin 1987] for details. We point out that the classical local Hölder estimates for solutions of elliptic systems in divergence form with continuous coefficients are needed to handle the case $\varepsilon \ge \theta\varepsilon_0$ and $0 < t < \theta$, as well as the case $0 < \varepsilon < \theta\varepsilon_0$ and $0 < t < \varepsilon/\varepsilon_0$. □

It follows from (3-1) and Cacciopoli's inequality that

$$\fint_{B(y,t)} |\nabla u_\varepsilon|^2 \le C_\sigma \Big(\frac{t}{r}\Big)^\sigma \fint_{B(y,r)} |\nabla u_\varepsilon|^2 \quad \text{for any } 0 < t < r \tag{3-10}$$

if $\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y, r)$. Since A^* satisfies the same ellipticity and almost periodicity conditions as A, estimate (3-16) also holds for solutions of $\operatorname{div}(A^*(x/\varepsilon)\nabla u_\varepsilon) = 0$ in $B(y, r)$. As a result, one may construct an $m \times m$ matrix of fundamental solutions $\Gamma_\varepsilon(x, y) = (\Gamma_\varepsilon^{\alpha\beta}(x, y))$ such that, for each $y \in \mathbb{R}^d$, $\nabla_x \Gamma_\varepsilon(x, y)$ is locally integrable and

$$\phi^\gamma(y) = \int_{\mathbb{R}^d} a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_j} (\Gamma_\varepsilon^{\beta\gamma}(x, y)) \frac{\partial \phi^\alpha}{\partial x_i} \, dx \tag{3-11}$$

for any $\phi = (\phi^\alpha) \in C_0^1(\mathbb{R}^d, \mathbb{R}^m)$ (see, e.g., [Hofmann and Kim 2007]). Moreover, if $d \geq 3$, the matrix $\Gamma_\varepsilon(x, y)$ satisfies

$$|\Gamma_\varepsilon(x, y)| \leq C|x - y|^{2-d} \tag{3-12}$$

for any $x, y \in \mathbb{R}^d$ with $x \neq y$, and

$$\begin{aligned} |\Gamma_\varepsilon(x + h, y) - \Gamma_\varepsilon(x, y)| &\leq \frac{C_\sigma |h|^\sigma}{|x - y|^{d-2+\sigma}}, \\ |\Gamma_\varepsilon(x, y + h) - \Gamma_\varepsilon(x, y)| &\leq \frac{C_\sigma |h|^\sigma}{|x - y|^{d-2+\sigma}}, \end{aligned} \tag{3-13}$$

where $x, y, h \in \mathbb{R}^d$ and $0 < |h| \leq \frac{1}{2}|x - y|$. Since $\mathcal{L}_\varepsilon^*(\Gamma_\varepsilon(x, \cdot)) = 0$ in $\mathbb{R}^d \setminus \{x\}$, using Caccioppoli's inequality and (3-12)–(3-13) we obtain

$$\left(\fint_{R \leq |y - x| \leq 2R} |\nabla_y \Gamma_\varepsilon(x, y)|^2 \, dy \right)^{\frac{1}{2}} \leq \frac{C}{R^{d-1}} \tag{3-14}$$

and

$$\left(\fint_{R \leq |y - x_0| \leq 2R} |\nabla_y \{\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)\}|^2 \, dy \right)^{\frac{1}{2}} \leq \frac{C|x - z|^\sigma}{R^{d-1+\sigma}}, \tag{3-15}$$

where $x, z \in B(x_0, r)$ and $R \geq 2r$.

Theorem 3.4. *Let $u_\varepsilon \in H^1(B(x_0, 2r); \mathbb{R}^m)$ be a weak solution of*

$$-\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = f + \operatorname{div}(g) \quad \text{in } 2B = B(x_0, 2r).$$

Let $0 < \sigma < 1$. Then, for any $x, z \in B = B(x_0, r)$,

$$|u_\varepsilon(x) - u_\varepsilon(z)| \leq C|x - z|^\sigma \left(r^{-\sigma} \left(\fint_{2B} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + \sup_{\substack{y \in B \\ 0 < t < r}} t^{2-\sigma} \left(\fint_{B(y,t)} |f|^2 \right)^{\frac{1}{2}} + \sup_{\substack{y \in B \\ 0 < t < r}} t^{1-\sigma} \left(\fint_{B(y,t)} |g|^2 \right)^{\frac{1}{2}} \right), \tag{3-16}$$

where C depends only on p, σ and A. In particular,

$$\|u_\varepsilon\|_{L^\infty(B)} \leq C \left(\fint_{2B} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + Cr^\sigma \sup_{\substack{y \in B \\ 0 < t < r}} t^{2-\sigma} \left(\fint_{B(y,t)} |f|^2 \right)^{\frac{1}{2}} + Cr^\sigma \sup_{\substack{y \in B \\ 0 < t < r}} t^{1-\sigma} \left(\fint_{B(y,t)} |g|^2 \right)^{\frac{1}{2}}, \tag{3-17}$$

where C depends only on p, σ and A.

Proof. We first note that the L^∞ estimate (3-17) follows easily from (3-16). To see (3-16), we assume $d \geq 3$; the case $d = 2$ follows from the case $d = 3$ by adding a dummy variable (the method of ascending). We choose a cut-off function $\varphi \in C_0^\infty\big(B\big(x_0, \frac{7}{4}r\big)\big)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ in $B\big(x_0, \frac{3}{2}r\big)$, and $|\nabla\varphi| \le Cr^{-1}$. Since

$$\mathcal{L}_\varepsilon(u_\varepsilon) = f\varphi + \operatorname{div}(g\varphi) - g\nabla\varphi - A(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi - \nabla\{A(x/\varepsilon)u_\varepsilon \cdot \nabla\varphi\},$$

we obtain that, for $x \in B(x_0, r)$,

$$\begin{aligned} u_\varepsilon(x) = \int_{\mathbb{R}^d} \Gamma_\varepsilon(x, y) f(y)\varphi(y)\, dy - \int_{\mathbb{R}^d} \nabla_y \Gamma_\varepsilon(x, y) g(y)\varphi(y)\, dy \\ - \int_{\mathbb{R}^d} \Gamma_\varepsilon(x, y) g(y)\nabla\varphi(y)\, dy - \int_{\mathbb{R}^d} \Gamma_\varepsilon(x, y) A(y/\varepsilon)\nabla u_\varepsilon(y) \cdot \nabla\varphi(y)\, dy \\ + \int_{\mathbb{R}^d} \nabla_y \Gamma_\varepsilon(x, y) A(y/\varepsilon) u_\varepsilon(y)\nabla\varphi(y)\, dy. \end{aligned} \tag{3-18}$$

It follows that, for any $x, z \in B(x_0, r)$,

$$\begin{aligned} |u_\varepsilon(x) - u_\varepsilon(z)| \le C \int_{2B} |\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)|\,|f(y)|\, dy \\ + C \int_{2B} |\nabla_y\{\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)\}|\,|g(y)|\, dy \\ + C \int_{2B} |\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)|\,|g(y)|\,|\nabla\varphi(y)|\, dy \\ + C \int_{2B} |\Gamma_\varepsilon(x, y) - \Gamma_\varepsilon(z, y)|\,|\nabla u_\varepsilon(y)|\,|\nabla\varphi(y)|\, dy \\ + C \int_{2B} |\nabla_y \Gamma_\varepsilon(x, y) - \nabla_y \Gamma_\varepsilon(z, y)|\,|u_\varepsilon(y)|\,|\nabla\varphi(y)|\, dy, \end{aligned} \tag{3-19}$$

where $2B = B(x_0, 2r)$. Since $|\nabla\varphi| = 0$ in $B\big(x_0, \frac{3}{2}r\big)$ and $x, z \in B(x_0, r)$, the last three terms in the right-hand side of (3-19) may be handled easily, using estimate (3-13), Cacciopoli's inequality and (3-15). They are bounded by

$$C_\sigma \left(\frac{|y - z|}{r}\right)^\sigma \left(\left(\fint_{2B} |u_\varepsilon|^2\right)^{\frac{1}{2}} + r^2\left(\fint_{2B} |f|^2\right)^{\frac{1}{2}} + r\left(\fint_{2B} |g|^2\right)^{\frac{1}{2}}\right)$$

for any $\sigma \in (0, 1)$.

Next, we use (3-12) and (3-13) to bound the first term in the right-hand side of (3-19) by

$$C \int_{B(x,4s)} \frac{|f(y)|\, dy}{|x - y|^{d-2}} + C \int_{B(z,5s)} \frac{|f(y)|\, dy}{|z - y|^{d-2}} + Cs^{\sigma_1} \int_{2B\setminus B(x,4s)} \frac{|f(y)|\, dy}{|x - y|^{d-2+\sigma_1}}, \tag{3-20}$$

where $s = |x - z|$ and $\sigma_1 \in (\sigma, 1)$. By decomposing $B(x, 4s)$ as a union of sets $\{y : |y - x| \sim 2^j s\}$, it is not hard to verify that the first term in (3-20) is bounded by

$$Cs^\sigma \sup_{\substack{y\in B \\ 0<t<r}} t^{2-\sigma}\left(\fint_{B(y,t)} |f|^2\right)^{\frac{1}{2}}.$$

The other two terms in (3-20) may be handled in a similar manner.

Finally, the second term in the right-hand side of (3-19) is bounded by

$$\int_{B(x,4s)} |\nabla_y \Gamma_\varepsilon(x,y)||g(y)|\,dy + \int_{B(z,5s)} |\nabla_y \Gamma_\varepsilon(z,y)||g(y)|\,dy + \int_{2B\setminus B(x,4s)} |\nabla_y\{\Gamma_\varepsilon(x,y)-\Gamma_\varepsilon(z,y)\}||g(y)|\,dy. \tag{3-21}$$

By decomposing $2B \setminus B(x,4s)$ as a union of sets $\{y : |y-x| \sim 2^j s\}$ and using Hölder's inequality and (3-15) (with σ replaced by some $\sigma_1 \in (\sigma,1)$), we may bound the third term in (3-21) by

$$Cs^\sigma \sup_{\substack{y\in B\\ 0<t<r}} t^{1-\sigma}\left(\fint_{B(y,t)} |g|^2\right)^{\frac{1}{2}}.$$

The other two terms in (3-21) may be handled in a similar manner. This completes the proof. □

Remark 3.5. Suppose that $-\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = f$ in $2B$ and $f \in L^p(2B;\mathbb{R}^m)$ for some $p \ge 2$, where $2B = B(x_0,2r)$. Assume $d \ge 3$. Using (3-18) and Cacciopoli's inequality, we may obtain that

$$|u_\varepsilon(x)| \le C\int_{2B} \frac{|f(y)|}{|x-y|^{d-2}}\,dy + C\left(\fint_{2B} |u_\varepsilon|^2\right)^{\frac{1}{2}} + Cr^2\left(\fint_{2B} |f|^2\right)^{\frac{1}{2}} \tag{3-22}$$

for any $x \in B = B(x_0,r)$. By the fractional integral estimates, this gives

$$\left(\fint_B |u_\varepsilon|^q\right)^{\frac{1}{q}} \le C\left(\fint_{2B} |u_\varepsilon|^2\right)^{\frac{1}{2}} + Cr^2\left(\fint_{2B} |f|^p\right)^{\frac{1}{p}}, \tag{3-23}$$

where $0 < 1/p - 1/q \le 2/d$.

4. Uniform boundary Hölder estimates and proof of Theorem 1.4

For $x_0 \in \partial\Omega$ and $0 < r < r_0 = \operatorname{diam}(\Omega)$, define

$$\Omega_r(x_0) = B(x_0,r)\cap\Omega \quad \text{and} \quad \Delta_r(x_0) = B(x_0,r)\cap\partial\Omega. \tag{4-1}$$

Theorem 4.1. *Let Ω be a bounded $C^{1,\eta}$ domain in $\mathbb{R}^d$ for some $\eta > 0$. Let $u_\varepsilon \in H^1(\Omega_r(x_0);\mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $\Omega_r(x_0)$ and $u_\varepsilon = 0$ on $\Delta_r(x_0)$ for some $x_0 \in \partial\Omega$ and $0 < r < r_0$. Then, for any $0 < \sigma < 1$ and $x, y \in \Omega_{r/2}(x_0)$,*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\left(\frac{|x-y|}{r}\right)^\sigma \left(\fint_{\Omega_r(x_0)} |u_\varepsilon|^2\right)^{\frac{1}{2}}, \tag{4-2}$$

where C depends only on σ, A and Ω.

Let $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a $C^{1,\eta}$ function such that

$$\phi(0) = 0, \quad \nabla\phi(0) = 0 \quad \text{and} \quad \|\nabla\phi\|_{C^{0,\eta}(\mathbb{R}^{d-1})} \le M_0. \tag{4-3}$$

Let

$$\begin{aligned} D(r) = D(r,\phi) &= \big\{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \phi(x') < x_d < \phi(x') + 10(M_0+1)r\big\}, \\ I(r) = I(r,\phi) &= \big\{(x', \phi(x')) \in \mathbb{R}^d : |x'| < r\big\}. \end{aligned} \tag{4-4}$$

By translation and rotation, Theorem 4.1 may be reduced to the following:

Theorem 4.2. *Let $u_\varepsilon \in H^1(D(r); \mathbb{R}^m)$ be a weak solution of* $\operatorname{div}(B(x/\varepsilon)\nabla u_\varepsilon) = 0$ *in $D(r)$ and $u_\varepsilon = 0$ on $I(r)$ for some $r > 0$ and $B \in \mathscr{A}$. Then, for any $0 < \sigma < 1$ and $x, y \in D(r/2)$,*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\left(\frac{|x-y|}{r}\right)^\sigma \left(\fint_{D_r} |u_\varepsilon|^2\right)^{\frac{1}{2}}, \tag{4-5}$$

where C depends only on σ, A and (η, M_0) in (4-3).

To prove Theorem 4.2, we need a homogenization result for a sequence of matrices in the class $\mathscr{A}$ on a sequence of domains.

Lemma 4.3. *Let $\{B_\ell\}$ be a sequence of matrices in $\mathscr{A}$. Let $\{\phi_\ell\}$ be a sequence of $C^{1,\eta}$ functions satisfying (4-3). Suppose that* $\operatorname{div}(B_\ell(x/\varepsilon_\ell)\nabla u_\ell) = 0$ *in $D(r, \phi_\ell)$ and $u_\ell = 0$ on $I(r, \phi_\ell)$ for some $r > 0$, where $\varepsilon_\ell \to 0$ and $\|u_\ell\|_{H^1(D(r,\phi_\ell))} \le C$. Then there exist subsequences of $\{\phi_\ell\}$ and $\{u_\ell\}$, which we still denote by $\{\phi_\ell\}$ and $\{u_\ell\}$, respectively, a function ϕ satisfying (4-3) with $u \in H^1(D(r, \phi); \mathbb{R}^m)$, and a constant matrix $\tilde{B}$, such that*

$$\begin{cases} \phi_\ell \to \phi & \text{in } C^1(|x'| < r), \\ u_\ell(x', x_d - \phi_\ell(x')) \rightharpoonup u(x', x_d - \phi(x')) & \text{weakly in } H^1(D(r, 0); \mathbb{R}^m), \end{cases} \tag{4-6}$$

and

$$\operatorname{div}(\tilde{B}\nabla u) = 0 \quad \text{in } D(r,\phi) \qquad \text{and} \qquad u = 0 \quad \text{on } I(r,\phi). \tag{4-7}$$

Moreover, the matrix $\tilde{B}$, which is given by $O^t \hat{A} O$ for some rotation O in $\mathbb{R}^d$, satisfies the ellipticity condition (2-6).

Proof. Since $\|\nabla\phi_\ell\|_{C^{0,\eta}(\mathbb{R}^{d-1})} \le M_0$ and $\|u_\ell\|_{H^1(D(r,\phi_\ell))} \le C$, (4-6) follows by passing to subsequences. Suppose that $B_\ell(y) = O_\ell^t A(O_\ell y + z_\ell) O_\ell$ for some rotation O_ℓ and $z_\ell \in \mathbb{R}^d$. By passing to a subsequence, we may assume that $O_\ell \to O$. Since $u_\ell \to u$ weakly in $H^1(\Omega; \mathbb{R}^m)$ for any $\Omega \Subset D(r, \phi)$, it follows from Theorem 2.2 that $\operatorname{div}(\tilde{B} u) = 0$ in $D(r, \phi)$, where $\tilde{B} = O^t \hat{A} O$. Finally, since $v_\ell(x', x_d) = u_\ell(x', x_d + \phi_\ell(x')) \rightharpoonup v(x', x_d + \phi(x'))$ weakly in $H^1(D(r, 0))$ and $v_\ell = 0$ on $I(r, 0)$, we may conclude that $v = 0$ on $I(r, 0)$. Hence, $u = 0$ on $I(r, \phi)$. □

Proof of Theorem 4.2. With Lemma 4.3 at our disposal, Theorem 4.2 follows by the three-step compactness argument, as in the periodic case. We refer the reader to [Avellaneda and Lin 1987] for details. □

With interior and boundary Hölder estimates in Theorems 3.1 and 4.1, one may construct an $m \times m$ matrix $G_\varepsilon(x, y) = (G_\varepsilon^{\alpha\beta}(x, y))$ of Green functions for $\mathscr{L}_\varepsilon$ for a bounded $C^{1,\eta}$ domain Ω. Moreover, if $d \ge 3$,

$$|G_\varepsilon(x, y)| \le C|x - y|^{2-d} \tag{4-8}$$

for any $x, y \in \Omega$ and

$$|G_\varepsilon(x, y) - G_\varepsilon(z, y)| \le \frac{C_\sigma |x - z|^\sigma}{|x - y|^{d-2+\sigma}} \tag{4-9}$$

for any $x, y, z \in \Omega$ with $|x - z| < \frac{1}{2}|x - y|$ and any $0 < \sigma < 1$. Since $G_\varepsilon(\cdot, y) = 0$ and $G_\varepsilon(y, \cdot) = 0$ on $\partial\Omega$, one also has

$$|G_\varepsilon(x, y)| \le \frac{C(\delta(x))^{\sigma_1}(\delta(y))^{\sigma_2}}{|x - y|^{d-2+\sigma_1+\sigma_2}} \tag{4-10}$$

for any $x, y \in \Omega$ and any $0 \le \sigma_1, \sigma_2 < 1$, where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ and C depends only on A, Ω, σ_1 and σ_2.

Theorem 4.4. *Let Ω be a bounded $C^{1,\eta}$ domain in $\mathbb{R}^d$ for some $\eta > 0$. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω and $u_\varepsilon = 0$ on $\partial\Omega$. Then*

$$\|u_\varepsilon\|_{C^\alpha(\bar{\Omega})} \le C_\alpha \sup_{\substack{x\in\Omega \\ 0<r<r_0}} r^{2-\alpha} \fint_{\Omega(x,r)} |F| \tag{4-11}$$

for any $0 < \alpha < 1$, where $r_0 = \operatorname{diam}(\Omega)$ and C_α depends only on A, Ω and α.

Proof. Since

$$u_\varepsilon(x) = \int_\Omega G_\varepsilon(x, y) F(y)\, dy,$$

it follows that, for any $x, z \in \Omega$,

$$|u_\varepsilon(x) - u_\varepsilon(z)| \le \int_\Omega |G_\varepsilon(x, y) - G_\varepsilon(z, y)|\,|F(y)|\, dy.$$

Let $t = |x - z|$ and write $\Omega = [\Omega \setminus B(x, 4t)] \cup \Omega(x, 4t)$. We use (4-8) to estimate the integral of $|G_\varepsilon(x, y) - G_\varepsilon(z, y)|\,|F(y)|$ over $\Omega(x, 4t)$. This gives

$$\begin{aligned}\int_{\Omega(x,4t)} |G_\varepsilon(x, y) - G_\varepsilon(z, y)|\,|F(y)|\, dy &\le C \int_{\Omega(x,4t)} \frac{|F(y)|\, dy}{|x - y|^{d-2}} + C \int_{\Omega(z,5t)} \frac{|F(y)|\, dy}{|z - y|^{d-2}} \\ &\le C t^\alpha \sup_{\substack{x\in\Omega \\ 0<r<r_0}} r^{2-\alpha} \fint_{\Omega(x,r)} |F|.\end{aligned}$$

For the integral over $\Omega \setminus B(x, 4t)$, we choose $\beta \in (\alpha, 1)$ and use (4-9) to obtain

$$\int_{\Omega\setminus B(x,4t)} |G_\varepsilon(x, y) - G_\varepsilon(z, y)|\,|F(y)|\, dy \le C t^\beta \int_{\Omega\setminus B(x,4t)} \frac{|F(y)|\, dy}{|x - y|^{d-2+\beta}} \le C t^\alpha \sup_{\substack{x\in\Omega \\ 0<r<r_0}} r^{2-\alpha} \fint_{\Omega(x,r)} |F|.$$

Thus we have proved that $|u(x) - u(z)|/|x - z|^\alpha$ is bounded by the right-hand side of (4-11). The remaining estimate for $\|u_\varepsilon\|_{L^\infty(\Omega)}$ is similar. □

Theorem 4.5. *Let Ω be a bounded $C^{1,\eta}$ domain in $\mathbb{R}^d$ for some $\eta > 0$. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f)$ in Ω and $u_\varepsilon = 0$ on $\partial\Omega$. Then*

$$\|u_\varepsilon\|_{C^\alpha(\bar{\Omega})} \le C_\alpha \sup_{\substack{x\in\Omega \\ 0<r<r_0}} r^{1-\alpha} \left(\fint_{\Omega(x,r)} |f|^2 \right)^{\frac{1}{2}} \tag{4-12}$$

for any $0 < \alpha < 1$, *where* $r_0 = \operatorname{diam}(\Omega)$ *and* C_α *depends only on* A, Ω *and* α.

Proof. The proof is similar to that of Theorem 4.4, using

$$|u_\varepsilon(x) - u_\varepsilon(z)| \le \int_\Omega \big|\nabla_y (G_\varepsilon(x, y) - G_\varepsilon(z, y))\big|\,|f(y)|\,dy.$$

The lack of pointwise estimates for $\nabla_y G_\varepsilon(x, y)$ is overcome by using the following estimates:

$$\begin{aligned} \int_{r\le|y-x|\le 2r} |\nabla_y G_\varepsilon(x, y)|^2\,dy &\le \frac{C}{r^2}\int_{r/2\le|y-x|\le 3r} |G_\varepsilon(x, y)|^2\,dy,\\ \int_{R\le|y-x|\le 2R} |\nabla_y (G_\varepsilon(x, y) - G_\varepsilon(z, y))|^2\,dy &\le \frac{C}{R^2}\int_{R/2\le|y-x|\le 3R} |G_\varepsilon(x, y) - G_\varepsilon(z, y)|^2\,dy, \end{aligned} \tag{4-13}$$

where $|x - z| < \frac{1}{4}|x - y|$. Estimate (4-13) follows from Caccioppoli's inequality. We omit the rest of the proof. □

Theorem 4.6. *Let* Ω *be a bounded* $C^{1,\eta}$ *domain in* $\mathbb{R}^d$ *for some* $\eta > 0$. *Suppose that* $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ *in* Ω *and* $u_\varepsilon = g$ *on* $\partial\Omega$. *Then*

$$\|u_\varepsilon\|_{C^\alpha(\overline{\Omega})} \le C_\alpha \|g\|_{C^\alpha(\partial\Omega)} \tag{4-14}$$

for any $0 < \alpha < 1$, *where* C_α *depends only on* A, Ω *and* α.

Proof. Without loss of generality we may assume that $\|g\|_{C^\alpha(\partial\Omega)} = 1$. Let v be the harmonic function in Ω such that $v \in C(\overline{\Omega})$ and $v = g$ on $\partial\Omega$. It is well known that $\|v\|_{C^\alpha(\overline{\Omega})} \le C_\alpha \|g\|_{C^\alpha(\partial\Omega)} = C_\alpha$, where C_α depends only on α and Ω. By interior estimates for harmonic functions, one also has

$$|\nabla v(x)| \le C_\alpha (\delta(x))^{\alpha-1} \tag{4-15}$$

for any $x \in \Omega$. Since $\mathcal{L}_\varepsilon(u_\varepsilon - v) = -\mathcal{L}_\varepsilon(v)$ in Ω and $u_\varepsilon - v = 0$ on $\partial\Omega$, it follows that

$$u_\varepsilon(x) - v(x) = -\int_\Omega \nabla_y G_\varepsilon(x, y) A(y/\varepsilon) \nabla v(y)\,dy.$$

This, together with (4-15), gives

$$|u_\varepsilon(x) - v(x)| \le C_\alpha \int_\Omega |\nabla_y G_\varepsilon(x, y)|\,(\delta(y))^{\alpha-1}\,dy. \tag{4-16}$$

We will show that

$$\int_\Omega |\nabla_y G_\varepsilon(x, y)|\,(\delta(y))^{\alpha-1}\,dy \le C_\alpha (\delta(x))^\alpha \quad \text{for any } x \in \Omega. \tag{4-17}$$

Assume (4-17) for a moment. Then

$$|u_\varepsilon(x) - v(x)| \le C_\alpha (\delta(x))^\alpha \quad \text{for any } x \in \Omega. \tag{4-18}$$

It follows that $\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega)} + C \le C$. Let $x, y \in \Omega$. To show $|u_\varepsilon(x) - u_\varepsilon(y)| \le C|x - y|^\alpha$, we consider three cases: (1) $|x - y| < \frac{1}{4}\delta(x)$; (2) $|x - y| < \frac{1}{4}\delta(y)$; (3) $|x - y| \ge \max\big(\frac{1}{4}\delta(x), \frac{1}{4}\delta(y)\big)$. In the first case, since $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω, we may use the interior Hölder estimates in Theorem 3.1 to obtain

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C_\alpha |x - y|^\alpha \|u_\varepsilon\|_{L^\infty(B(x,\delta(x)/2))} \le C_\alpha |x - y|^\alpha.$$

The second case is handled in the same manner. For the third case we use (4-18) and Hölder estimates for v to see that

$$\begin{aligned}|u_\varepsilon(x)-u_\varepsilon(y)| &\le |u_\varepsilon(x)-v(x)|+|v(x)-v(y)|+|v(y)-u_\varepsilon(y)|\\ &\le C(\delta(x))^\alpha + C|x-y|^\alpha + C(\delta(y))^\alpha\\ &\le C_\alpha |x-y|^\alpha.\end{aligned}$$

It remains to prove (4-17). To this end we fix $x\in\Omega$ and let $r=\delta(x)/2$. We first note that

$$\int_{B(x,r)} |\nabla_y G_\varepsilon(x,y)|(\delta(y))^{\alpha-1}\,dy \le Cr^{\alpha-1}\int_{B(x,r)} |\nabla_y G_\varepsilon(x,y)|\,dy \le Cr^\alpha, \tag{4-19}$$

where the last inequality follows from the first estimate in (4-13) by decomposing $B(x,r)\setminus\{0\}$ as $\bigcup_{j=0}^\infty\{B(x,2^{-j}r)\setminus B(x,2^{-j-1}r)\}$. To estimate the integral on $\Omega\setminus B(x,r)$, we observe that, if Q is a cube in $\mathbb{R}^d$ with the property that $3Q\subset\Omega\setminus\{x\}$ and $\ell(Q)\sim \operatorname{dist}(Q,\partial\Omega)$, then

$$\begin{aligned}\int_Q |\nabla_y G_\varepsilon(x,y)|(\delta(y))^{\alpha-1}\,dy &\le C(\ell(Q))^{\alpha-1}|Q|\left(⨍_Q |\nabla_y G_\varepsilon(x,y)|^2\,dy\right)^{\frac12}\\ &\le C(\ell(Q))^{\alpha-2}|Q|\left(⨍_{2Q} |G_\varepsilon(x,y)|^2\,dy\right)^{\frac12}\\ &\le Cr^{\alpha_1}(\ell(Q))^{\alpha+\alpha_2-2}|Q|\left(⨍_{2Q} \frac{dy}{|x-y|^{2(d-2+\alpha_1+\alpha_2)}}\right)^{\frac12},\end{aligned} \tag{4-20}$$

where $\alpha_1,\alpha_2\in(0,1)$. We remark that Cacciopoli's inequality was used for the second inequality above, while the estimate (4-10) was used for the third. Since $3Q\subset\Omega\setminus\{x\}$, we see that $|x-y|\sim|x-z|$ for any $y,z\in 2Q$. As a result, it follows from (4-20) that

$$\int_Q |\nabla_y G_\varepsilon(x,y)|(\delta(y))^{\alpha-1}\,dy \le Cr^{\alpha_1}\int_Q \frac{(\delta(y))^{\alpha+\alpha_2-2}}{|x-y|^{d-2+\alpha_1+\alpha_2}}\,dy. \tag{4-21}$$

By decomposing $\Omega\setminus B(x,r)$ as a nonoverlapping union of cubes Q with the said property (a Whitney-type decomposition of Ω), we obtain

$$\begin{aligned}\int_{\Omega\setminus B(x,r)} |\nabla_y G_\varepsilon(x,y)|(\delta(y))^{\alpha-1}\,dy &\le Cr^{\alpha_1}\int_\Omega \frac{(\delta(y))^{\alpha+\alpha_2-2}}{(|x-y|+r)^{d-2+\alpha_1+\alpha_2}}\,dy\\ &\le Cr^{\alpha_1}\int_{\mathbb{R}^d_+} \frac{y_d^{\alpha+\alpha_2-2}dy}{(|r-y_d|+r+|y'|)^{d-2+\alpha_1+\alpha_2}}.\end{aligned} \tag{4-22}$$

Finally, a direct computation shows that the integral on the right-hand side of (4-22) is bounded by $Cr^{\alpha-\alpha_1}$ provided that $\alpha_1>\alpha$ and $\alpha_2>1-\alpha$. This completes the proof. □

Proof of Theorem 1.4. This follows from Theorems 4.4, 4.5 and 4.6 by writing $u_\varepsilon = u_\varepsilon^{(1)}+u_\varepsilon^{(2)}+u_\varepsilon^{(3)}$, where $u_\varepsilon^{(1)}$, $u_\varepsilon^{(2)}$ and $u_\varepsilon^{(3)}$ satisfy the conditions in Theorems 4.4, 4.5 and 4.6, respectively. □

5. Construction of approximate correctors

In this section we construct the approximate correctors $\chi_T = (\chi_{T,j}^{\beta}) = (\chi_{T,j}^{\alpha\beta})$ and obtain some preliminary estimates.

Proposition 5.1. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ and $g = (g_1, \ldots, g_d) \in L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^{d\times m})$. Assume that*

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} (|f|^2 + |g|^2) < \infty.$$

Then, for $T > 0$, there exists a unique $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ such that

$$-\operatorname{div}(A(x)\nabla u) + T^{-2} u = f + \operatorname{div}(g) \quad \text{in } \mathbb{R}^d \tag{5-1}$$

and

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} (|\nabla u|^2 + |u|^2) < \infty. \tag{5-2}$$

Moreover, the solution u satisfies the estimate

$$\sup_{x\in\mathbb{R}^d} \fint_{B(x,T)} (|\nabla u|^2 + T^{-2}|u|^2) \le C \sup_{x\in\mathbb{R}^d} \fint_{B(x,T)} (|g|^2 + T^2|f|^2), \tag{5-3}$$

where C depends only on d, m and μ.

Proof. By rescaling we may assume that $T = 1$. The proof of the existence and estimate (5-3) may be found in [Pozhidaev and Yurinskiĭ 1989]. It uses the fact that, for $f \in L^2(\mathbb{R}^d; \mathbb{R}^m)$ and $g = (g_1, \ldots, g_d) \in L^2(\mathbb{R}^d; \mathbb{R}^{d\times m})$ with compact support, there exists a constant $\lambda > 0$, depending only on d, m and μ, such that the solution of (5-1) in $H^1(\mathbb{R}^d; \mathbb{R}^m)$ satisfies

$$\int_{\mathbb{R}^d} e^{\lambda|x|}(|\nabla u|^2 + |u|^2)\, dx \le C \int_{\mathbb{R}^d} e^{\lambda|x|}(|f|^2 + |g|^2)\, dx.$$

For the uniqueness, assume that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ satisfies (5-2) and $-\operatorname{div}(A(x)\nabla u) + u = 0$ in $\mathbb{R}^d$. By Cacciopoli's inequality,

$$\int_{B(0,R)} |\nabla u|^2 + \int_{B(0,R)} |u|^2 \le \frac{C}{R^2} \int_{B(0,2R)} |u|^2$$

for any $R \ge 1$. It follows that

$$\int_{B(0,R)} |u|^2 \le \frac{C}{R^{2d}} \int_{B(0,2^d R)} |u|^2$$

for any $R \ge 1$. However, the condition (5-2) implies that $\int_{B(0,2^d R)} |u|^2 \le C_u R^d$. Consequently, we obtain $\int_{B(0,R)} |u|^2 \le C_u R^{-d}$ for any $R \ge 1$ and thus $u \equiv 0$ in $\mathbb{R}^d$. ☐

Remark 5.2. The solution u of (5-1), given by Proposition 5.1, in fact satisfies

$$\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}|\nabla u|^p\right)^{\frac{1}{p}} \le C\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}|g|^p\right)^{\frac{1}{p}} + C\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}T^2|f|^2\right)^{\frac{1}{2}}, \tag{5-4}$$

$$\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}T^{-q}|u|^q\right)^{\frac{1}{q}} \le C\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}|g|^p\right)^{\frac{1}{p}} + C\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}T^2|f|^2\right)^{\frac{1}{2}} \tag{5-5}$$

for some $p>2$, depending only on d, m and μ, where $1/q = 1/p - 1/d$ for $d\ge 3$. If $d=2$, the left-hand side of (5-5) should be replaced by $T^{-1}\|u\|_{L^\infty}$.

To see (5-4), one uses the weak reverse Hölder estimate: if u is a weak solution of $-\operatorname{div}(A(x)u) = f + \operatorname{div}(g)$ in $B_r = B(x_0,r)$, then

$$\left(\fint_{B_{r/2}}|\nabla u|^p\right)^{\frac{1}{p}} \le \frac{C}{r}\left(\fint_{B_r}|u|^2\right)^{\frac{1}{2}} + C\left(\fint_{B_r}|g|^p\right)^{\frac{1}{p}} + Cr\left(\fint_{B_r}|f|^2\right)^{\frac{1}{2}}$$

for some $p>2$, depending only on d, m and μ (see, e.g., [Giaquinta 1983]). Estimate (5-5) follows from (5-4) by Sobolev imbedding.

Let $P_j^\beta(x) = x_j e^\beta$, where $1\le j\le d$, $1\le\beta\le m$, and $e^\beta = (0,\dots,0,1,0,\dots,0)$ with 1 in the β-th position. For $T>0$, the approximate corrector is defined as $\chi_T = (\chi_{T,j}^{\alpha\beta})$, where, for each $1\le j\le d$ and $1\le\beta\le m$, $u = \chi_{T,j}^\beta = (\chi_{T,j}^{1\beta},\dots,\chi_{T,j}^{m\beta})$ is the weak solution of

$$-\operatorname{div}(A(x)\nabla u) + T^{-2}u = \operatorname{div}(A(x)\nabla P_j^\beta) \quad \text{in } \mathbb{R}^d, \tag{5-6}$$

given by Proposition 5.1. It follows from (5-3) that

$$\sup_{x\in\mathbb{R}^d}\fint_{B(x,T)}(|\nabla\chi_T|^2 + T^{-2}|\chi_T|^2) \le C, \tag{5-7}$$

where C depends only on d, m and μ. Clearly, this gives

$$\sup_{\substack{x\in\mathbb{R}^d\\ L\ge T}}\fint_{B(x,L)}(|\nabla\chi_T|^2 + T^{-2}|\chi_T|^2) \le C, \tag{5-8}$$

where C depends only on d, m and μ.

Lemma 5.3. *Let $x,y,z\in\mathbb{R}^d$. Then*

$$\begin{aligned}\left(\fint_{B(x,T)}|\nabla\chi_T(t+y) - \nabla\chi_T(t+z)|^2\,dt\right)^{\frac{1}{2}} &\le C\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty(\mathbb{R}^d)},\\ T^{-1}\left(\fint_{B(x,T)}|\chi_T(t+y) - \chi_T(t+z)|^2\,dt\right)^{\frac{1}{2}} &\le C\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty(\mathbb{R}^d)},\end{aligned} \tag{5-9}$$

where C depends only on d, m and μ.

Proof. Fix $y, z \in \mathbb{R}^d$ and $1 \le j \le d$, $1 \le \beta \le m$. Let $u(t) = \chi_{T,j}^{\beta}(t+y)$ and $v(t) = \chi_{T,j}^{\beta}(t+z)$. Then $w = u - v$ is a solution of

$$-\operatorname{div}(A(t+y)\nabla w) + T^{-2} w = \operatorname{div}\big([A(t+y) - A(t+z)]\nabla P_j^{\beta}\big) + \operatorname{div}\big([A(t+y) - A(t+z)]\nabla v\big).$$

In view of Proposition 5.1, we obtain

$$\begin{aligned}
&\fint_{B(x,T)} (|\nabla w|^2 + T^{-2}|w|^2) \\
&\qquad \le C \sup_{x\in\mathbb{R}^d} \fint_{B(x,T)} |A(t+y) - A(t+z)|^2\, dt + C \sup_{x\in\mathbb{R}^d} \fint_{B(x,T)} |A(t+y) - A(t+z)|^2 |\nabla v|^2\, dt \\
&\qquad \le C \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty}^2 + C\|A(\cdot + y) - A(\cdot + z)\|_{L^\infty}^2 \sup_{x\in\mathbb{R}^d} \fint_{B(x,T)} |\nabla v|^2 \\
&\qquad \le C\|A(\cdot + y) - A(\cdot + z)\|_{L^\infty}^2,
\end{aligned}$$

where we have used (5-7) in the last inequality. This completes the proof. □

Remark 5.4. For $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, define

$$\|f\|_{W^2} = \limsup_{L\to\infty} \sup_{x\in\mathbb{R}^d} \left(\fint_{B(x,L)} |f|^2 \right)^{\frac{1}{2}}. \tag{5-10}$$

Note that, by (5-7),

$$\|\nabla \chi_T\|_{W^2} + T^{-1}\|\chi_T\|_{W^2} \le C, \tag{5-11}$$

where C depends only on d, m and μ. Moreover, by Lemma 5.3, for any $\tau \in \mathbb{R}^d$,

$$\|\nabla \chi_T(\cdot + \tau) - \nabla \chi_T(\cdot)\|_{W^2} + T^{-1}\|\chi_T(\cdot + \tau) - \chi_T(\cdot)\|_{W^2} \le C\|A(\cdot + \tau) - A(\cdot)\|_{L^\infty}. \tag{5-12}$$

Since A is uniformly almost-periodic, for any $\varepsilon > 0$, the set

$$\{\tau \in \mathbb{R}^d : \|A(\cdot + \tau) - A(\cdot)\|_{L^\infty(\mathbb{R}^d)} < \varepsilon\}$$

is relatively dense in $\mathbb{R}^d$. It follows that, for any $\varepsilon > 0$, the set of τ for which the left-hand side of (5-12) is less than ε is also relatively dense in $\mathbb{R}^d$. By [Besicovitch and Bohr 1931], this implies that $\nabla \chi_T$ and χ_T are limits of sequences of trigonometric polynomials with respect to the seminorm $\|\cdot\|_{W^2}$ in (5-10). In particular, $\nabla\chi_T, \chi_T \in B^2(\mathbb{R}^d)$ for any $T > 0$.

Lemma 5.5. *Let $u_T = \chi_{T,j}^{\beta}$ for some $T > 0$, $1 \le j \le d$ and $1 \le \beta \le m$. Then*

$$\left\langle a_{ik}^{\alpha\gamma} \frac{\partial u_T^{\gamma}}{\partial x_k} \frac{\partial v^{\alpha}}{\partial x_i} \right\rangle + T^{-2}\langle u_T \cdot v\rangle = -\left\langle a_{ij}^{\alpha\beta} \frac{\partial v^{\alpha}}{\partial x_i} \right\rangle, \tag{5-13}$$

where $v = (v^\alpha) \in H^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ and $v^\alpha, \nabla v^\alpha \in B^2(\mathbb{R}^d)$.

Proof. For any $\phi = (\phi^\alpha) \in H^1(\mathbb{R}^d; \mathbb{R}^m)$ with compact support, we have

$$\int_{\mathbb{R}^d} a_{ik}^{\alpha\gamma} \frac{\partial u_T^{\gamma}}{\partial x_k} \cdot \frac{\partial \phi^{\alpha}}{\partial x_i} + \frac{1}{T^2} \int_{\mathbb{R}^d} u_T \cdot \phi = -\int_{\mathbb{R}^d} a_{ij}^{\alpha\beta} \frac{\partial \phi^{\alpha}}{\partial x_i}. \tag{5-14}$$

Let $v = (v^\alpha) \in H^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$. Suppose that $v^\alpha \in B^2(\mathbb{R}^d)$ and $\nabla v^\alpha \in B^2(\mathbb{R}^d)$. Choose $\phi(x) = \varphi(\varepsilon x) v(x)$ in (5-14), where $\varphi \in C_0^\infty(\mathbb{R}^d)$. The desired result follows by a simple change of variables $x \mapsto x/\varepsilon$ in (5-14), multiplying both sides of the equation by ε^d, and finally letting $\varepsilon \to 0$. □

Letting v be a constant in (5-13), we see that

$$\langle \chi^\beta_{T,j} \rangle = 0. \tag{5-15}$$

By taking $v = \chi^\beta_{T,j}$, we obtain

$$\langle A \nabla \chi^\beta_{T,j} \cdot \nabla \chi^\beta_{T,j} \rangle + T^{-2} \langle |\chi^\beta_{T,j}|^2 \rangle = -\langle A^* \nabla \chi^\beta_{j,T} \rangle, \tag{5-16}$$

where A^* denotes the adjoint of A. This, in particular, implies that

$$\langle |\nabla \chi_T|^2 \rangle + T^{-2} \langle |\chi_T|^2 \rangle \le C,$$

where C depends only on d, m and μ.

Lemma 5.6. *Let $\psi = (\psi^{\alpha\beta}_{ij})$ be defined by (2-4). Then, as $T \to \infty$,*

$$\frac{\partial}{\partial x_i} (\chi^{\alpha\beta}_{T,j}) \rightharpoonup \psi^{\alpha\beta}_{ij} \quad \text{weakly in } B^2(\mathbb{R}^d). \tag{5-17}$$

Proof. Fix $1 \le j \le d$ and $1 \le \beta \le m$. Let $\tilde{\psi}^\beta_j = (\tilde{\psi}^{\alpha\beta}_{ij}) \in B^2(\mathbb{R}^d; \mathbb{R}^{dm})$ be the weak limit in $B^2(\mathbb{R}^d)$ of a subsequence $\nabla \chi^\beta_{T_\ell, j}$, where $T_\ell \to \infty$. Since $\nabla \chi^\beta_{T,j} \in V^2_{\mathrm{pot}}$, we see that $\tilde{\psi}^\beta_j \in V^2_{\mathrm{pot}}$. Moreover, since $T^{-2} \langle |\chi_T|^2 \rangle \le C$, it follows by letting $T \to \infty$ in (5-13) that

$$\left\langle a^{\alpha\gamma}_{ik} \tilde{\psi}^{\gamma\beta}_{kj} \frac{\partial v^\alpha}{\partial x_i} \right\rangle = -\left\langle a^{\alpha\beta}_{ij} \frac{\partial v^\alpha}{\partial x_i} \right\rangle$$

for any $v = (v^\alpha) \in \mathrm{Trig}(\mathbb{R}^d; \mathbb{R}^m)$. This implies that $\tilde{\psi}^\beta_j$ is a solution of (2-4). By the uniqueness of the solution, we obtain $\tilde{\psi}^\beta_j = \psi^\beta_j$ and hence (5-17). □

Theorem 5.7. *As $T \to \infty$, $T^{-2} \langle |\chi_T|^2 \rangle \to 0$.*

Proof. Note that

$$\begin{aligned} \mu \langle |\psi - \nabla \chi_T|^2 \rangle &\le \left\langle a^{\alpha\gamma}_{ik} \left(\psi^{\gamma\beta}_{kj} - \frac{\partial}{\partial x_k} (\chi^{\gamma\beta}_{T,j}) \right) \left(\psi^{\alpha\beta}_{ij} - \frac{\partial}{\partial x_i} (\chi^{\alpha\beta}_{T,j}) \right) \right\rangle \\ &= \langle a^{\alpha\beta}_{ik} \psi^{\gamma\beta}_{kj} \psi^{\alpha\beta}_{ij} \rangle - \left\langle a^{\alpha\gamma}_{ik} \frac{\partial}{\partial x_k} (\chi^{\gamma\beta}_{T,j}) \psi^{\alpha\beta}_{ij} \right\rangle - T^{-2} \langle |\chi_T|^2 \rangle, \end{aligned}$$

where we have used equations (2-4) and (5-13). In view of Lemma 5.6, this implies that, as $T \to \infty$, $T^{-2} \langle |\chi_T|^2 \rangle \to 0$ and

$$\|\psi - \nabla \chi_T\|_{B^2} \to 0. \tag{5-18}$$

This concludes the proof. □

Remark 5.8. For $T > 0$, let

$$\hat{a}^{\alpha\beta}_{T,ij} = \langle a^{\alpha\beta}_{ij} \rangle + \left\langle a^{\alpha\gamma}_{ik} \frac{\partial}{\partial x_k} (\chi^{\gamma\beta}_{T,j}) \right\rangle \tag{5-19}$$

be the approximate homogenized coefficients. Then

$$|\hat{a}_{ij}^{\alpha\beta} - \hat{a}_{T,ij}^{\alpha\beta}| = \left| \left\langle a_{ik}^{\alpha\gamma} \left(\psi_{kj}^{\gamma\beta} - \frac{\partial}{\partial x_k} (\chi_{T,j}^{\gamma\beta}) \right) \right\rangle \right| \le C \|\psi - \nabla \chi_T\|_{B^2}. \tag{5-20}$$

6. Estimates of approximate correctors

In this section we will establish sharp estimates for approximate correctors χ_T. The proof relies on the uniform L^∞ and Hölder estimates obtained in Section 3 for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = f + \operatorname{div}(g)$.

Lemma 6.1. *For* $T \ge 1$,

$$\|\chi_T\|_{L^\infty(\mathbb{R}^d)} \le CT, \tag{6-1}$$

where C *is independent of* T. *Moreover, for any* $0 < \sigma < 1$ *and* $|x-y| \le T$,

$$|\chi_T(x) - \chi_T(y)| \le C_\sigma T^{1-\sigma} |x-y|^\sigma, \tag{6-2}$$

where C_σ *depends only on* σ *and* A.

Proof. We consider the case $d \ge 3$. The 2-dimensional case follows by the method of ascending.

Let $1 \le j \le d$ and $1 \le \beta \le m$. Fix $z \in \mathbb{R}^d$ and consider the function

$$u(x) = \chi_{T,j}^\beta(x) + P_j^\beta(x-z). \tag{6-3}$$

It follows from (5-7) that

$$\left(\fint_{B(z,4T)} |u|^2 \right)^{\frac{1}{2}} \le CT. \tag{6-4}$$

Since

$$\operatorname{div}(A(x)\nabla u) = T^{-2} \chi_{T,j}^\beta \quad \text{in } \mathbb{R}^d, \tag{6-5}$$

we may apply the estimate (3-23) repeatedly to show that

$$\left(\fint_{B(z,2T)} |u|^p \right)^{\frac{1}{p}} \le C_p T \tag{6-6}$$

for any $2 < p < \infty$, where C_p depends only on p and A. This, together with (3-17), gives

$$\|u\|_{L^\infty(B(z,T))} \le CT.$$

Hence, $|\chi_{T,j}^\beta(z)| \le CT$ for any $z \in \mathbb{R}^d$. Finally, (6-2) follows from (6-1) and the Hölder estimate (3-16). □

Lemma 6.2. *Let* $\sigma_1, \sigma_2 \in (0,1)$ *and* $2 < p < \infty$. *Then, for any* $1 \le r \le T$,

$$\sup_{x \in \mathbb{R}^d} \left(\fint_{B(x,r)} |\nabla \chi_T|^p \right)^{\frac{1}{p}} \le C T^{\sigma_1} \left(\frac{T}{r} \right)^{\sigma_2}, \tag{6-7}$$

where C *depends only on* p, σ_1, σ_2 *and* A.

Proof. Let u be the same as in the proof of Lemma 6.1. By Cacciopoli's inequality,

$$\fint_{B(z,r)} |\nabla u|^2 \le Cr^{-2} \fint_{B(z,2r)} |u-u(z)|^2 + Cr^2 \|T^{-2}\chi_T\|_{L^\infty}^2,$$

where $0 < r \le T$. In view of (6-1) and (6-2), this gives

$$\sup_{z\in\mathbb{R}^d} \left(\fint_{B(z,r)} |\nabla \chi_T|^2 \right)^{\frac{1}{2}} \le C_\sigma \left(\frac{T}{r} \right)^\sigma \tag{6-8}$$

for any $\sigma \in (0, 1)$ and $0 < r \le T$. Since A is uniformly continuous in $\mathbb{R}^d$, by the local $W^{1,p}$ estimates for elliptic systems in divergence form, it follows from (6-5) that

$$\left(\fint_{B(z,1)} |\nabla u|^p \right)^{\frac{1}{p}} \le C_p \left(\fint_{B(z,2)} |\nabla u|^2 \right)^{\frac{1}{2}} + CT^{-2} \|\chi_T\|_{L^\infty}$$

for any $z \in \mathbb{R}^d$ and $2 < p < \infty$, where C_p depends only on p and A. This, together with (6-8), yields

$$\sup_{z\in\mathbb{R}^d} \left(\fint_{B(z,1)} |\nabla \chi_T|^p \right)^{\frac{1}{p}} \le C_{p,\sigma} T^\sigma$$

for any $\sigma \in (0, 1)$ and $p \in (2, \infty)$. Consequently, for any $1 \le r \le T$ and $\sigma \in (0, 1)$,

$$\sup_{z\in\mathbb{R}^d} \left(\fint_{B(z,r)} |\nabla \chi_T|^p \right)^{\frac{1}{p}} \le C_{p,\sigma} T^\sigma. \tag{6-9}$$

The desired estimate (6-7) now follows from (6-8) and (6-9) by a simple interpolation of L^p norms. □

Theorem 6.3. *Let $T \ge 1$. The approximate corrector χ_T is uniformly almost-periodic in $\mathbb{R}^d$. Moreover, for any $y, z \in \mathbb{R}^d$,*

$$\|\chi_T(\cdot + y) - \chi_T(\cdot + z)\|_{L^\infty(\mathbb{R}^d)} \le CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(\mathbb{R}^d)}, \tag{6-10}$$

where C is independent of T, y and z.

Proof. We assume $d \ge 3$. The case $d = 2$ follows from the case $d = 3$ by the method of ascending. Fix $y, z \in \mathbb{R}^d$ and $1 \le j \le d$, $1 \le \beta \le m$. Let

$$u(x) = \chi_{T,j}^\beta(x + y) - \chi_{T,j}^\beta(x + z).$$

Note that

$$-\operatorname{div}(A(x+y)\nabla u) = -T^{-2}u + \operatorname{div}\big((A(x+y)-A(x+z))\nabla P_j^\beta\big) + \operatorname{div}\big((A(x+y)-A(x+z))\nabla v\big), \tag{6-11}$$

where $v(x) = \chi_{T,j}^\beta(x + z)$. Let $B = B(x_0, T)$. As in the proof of Theorem 3.4, we choose a cut-off function $\varphi \in C_0^\infty\big(B\big(x_0, \tfrac{7}{4}T\big)\big)$ such that $\varphi = 1$ in $B\big(x_0, \tfrac{3}{2}T\big)$ and $|\nabla\varphi| \le CT^{-1}$. Using the representation formula by fundamental solutions and (6-11), we obtain, for any $x \in B$,

$$
\begin{aligned}
|u(x)| \le CT^{-2} \int_{2B} |\Gamma^y(x,t)|\,|u(t)|\,dt + C\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty} \int_{2B} |\nabla_t(\Gamma^y(x,t)\varphi(t))|\,dt \\
+ C\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty} \int_{2B} |\nabla v(t)|\,|\nabla_t(\Gamma^y(x,t)\varphi(t))|\,dt \\
+ CT\biggl(\fint_{2B} |\nabla u|^2 \biggr)^{\frac{1}{2}} + C\biggl(\fint_{2B} |u|^2 \biggr)^{\frac{1}{2}},
\end{aligned}
\tag{6-12}
$$

where we have used $\Gamma^y(x,t) = \Gamma(x+y, t+y)$ to denote the matrix of fundamental solutions for the operator $-\operatorname{div}(A(\cdot+y)\nabla)$ in $\mathbb{R}^d$. By Lemma 5.3, the last two terms in the right-hand side of (6-12) are bounded by the right-hand side of (6-10). Using the size estimate (3-12) and Cacciopoli's inequality, it is also not hard to see that the second term in the right-hand side of (6-12) is bounded by the right-hand side of (6-10).

To treat the third term in the right-hand side of (6-12), we note that

$$
\begin{aligned}
&\int_{2B} |\nabla v(t)|\,|\nabla_t(\Gamma^y(x,t)\varphi(t))|\,dt \\
&\qquad \le C \sum_{\ell=0}^{\infty} \biggl(\fint_{|t-x|\sim 2^{-\ell}T} |\nabla v(t)|^2\,dt \biggr)^{\frac{1}{2}} \biggl(\fint_{|t-x|\sim 2^{-\ell}T} |\nabla_t(\Gamma^y(x,t)\varphi)|^2\,dt \biggr)^{\frac{1}{2}} (2^{-\ell}T)^d \\
&\qquad \le C \sum_{\ell=0}^{\infty} (2^{\ell})^{\sigma} \cdot (2^{-\ell}T)^{1-d} \cdot (2^{-\ell}T)^d \\
&\qquad \le CT,
\end{aligned}
$$

where $\sigma \in (0,1)$ and we have used (6-8) to estimate the integral involving $|\nabla v(t)|^2$ for the second inequality. As a result, we have proved that, for any $x \in B$,

$$
|u(x)| \le CT^{-2} \int_{2B} \frac{|u(t)|}{|x-t|^{d-2}}\,dt + CT\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty}. \tag{6-13}
$$

By the fractional integral estimates, this implies that

$$
\biggl(\fint_{B} |u|^q \biggr)^{\frac{1}{q}} \le C\biggl(\fint_{2B} |u|^p \biggr)^{\frac{1}{p}} + CT\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty},
$$

where $1 < p < q \le \infty$ and $1/p - 1/q < 2/d$. Since

$$
\biggl(\fint_{2B} |u|^2 \biggr)^{\frac{1}{2}} \le CT\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty}
$$

by Lemma 5.3, a simple iteration argument shows that

$$
\|u\|_{L^\infty(B)} \le CT\|A(\cdot+y) - A(\cdot+z)\|_{L^\infty}.
$$

This completes the proof. □

Remark 6.4. Let $u(x)=\chi_T(x+y)-\chi_T(x+z)$, as in the proof of Theorem 6.3. Then

$$|u(t)-u(s)|\le C_\sigma\left(\frac{|t-s|}{T}\right)^\sigma T\|A(\cdot+y)-A(\cdot+z)\|_{L^\infty} \tag{6-14}$$

for any $\sigma\in(0,1)$ and $t,s\in\mathbb{R}^d$, where C_σ depends only on σ and A. This follows from (6-11), (6-10) and (3-16). By Cacciopoli's inequality and (6-14), we may deduce that

$$\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,r)}|\nabla u|^2\right)^{\frac{1}{2}}\le C_\sigma\left(\frac{T}{r}\right)^\sigma\|A(\cdot+y)-A(\cdot+z)\|_{L^\infty} \tag{6-15}$$

for any $\sigma\in(0,1)$.

Theorem 6.5. *Let $T\ge 1$. Then*

$$T^{-1}\|\chi_T\|_{L^\infty(\mathbb{R}^d)}\le C_\sigma\left(\rho(R)+\left(\frac{R}{T}\right)^\sigma\right) \tag{6-16}$$

for any $R>0$ and $\sigma\in(0,1)$, where C_σ depends only on σ and A. In particular, $T^{-1}\|\chi_T\|_{L^\infty(\mathbb{R}^d)}\to 0$ as $T\to\infty$.

Proof. Let $y,z\in\mathbb{R}^d$. Suppose $|z|\le R$. Then

$$|\chi_T(y)-\chi_T(0)|\le|\chi_T(y)-\chi_T(z)|+|\chi_T(z)-\chi_T(0)|\le CT\|A(\cdot+y)-A(\cdot+z)\|_{L^\infty(\mathbb{R}^d)}+C_\sigma T^{1-\sigma}R^\sigma,$$

where we have used Theorem 6.3 and Lemma 6.1. It follows that

$$\sup_{y\in\mathbb{R}^d}T^{-1}|\chi_T(y)-\chi_T(0)|\le C\rho(R)+C_\sigma\left(\frac{R}{T}\right)^\sigma \tag{6-17}$$

for any $R>0$.

Finally, we observe that

$$|\chi_T(0)|\le\left|\fint_{B(0,L)}(\chi_T(y)-\chi_T(0))\,dy\right|+\left|\fint_{B(0,L)}\chi_T(y)\,dy\right|\le\sup_{y\in\mathbb{R}^d}|\chi_T(y)-\chi_T(0)|+\left|\fint_{B(0,L)}\chi_T(y)\,dy\right|.$$

Since $\langle\chi_T\rangle=0$, we may let $L\to\infty$ in the estimate above to obtain

$$|\chi_T(0)|\le\sup_{y\in\mathbb{R}^d}|\chi_T(y)-\chi_T(0)|.$$

This, together with (6-17), yields the estimate (6-16). □

For $T\ge 1$ and $\sigma>0$, define

$$\Theta_\sigma(T)=\inf_{0<R\le T}\left(\rho(R)+\left(\frac{R}{T}\right)^\sigma\right). \tag{6-18}$$

Note that $\Theta_\sigma(T)$ is a decreasing and continuous function of T and $\Theta_\sigma(T)\to 0$ as $T\to\infty$. It follows from Theorem 6.5 that

$$T^{-1}\|\chi_T\|_{L^\infty(\mathbb{R}^d)}\le C_\sigma\Theta_\sigma(T)\quad\text{for any }\ T\ge 1, \tag{6-19}$$

where $\sigma \in (0,1)$. By taking $R = T^\alpha$ for some $\alpha \in (0,1)$ in (6-18), we see that

$$\Theta_\sigma(T) \le \rho(T^\alpha) + T^{-\sigma(1-\alpha)}. \tag{6-20}$$

This, in particular, implies that

$$\int_1^\infty \frac{\rho(r)}{r}\,dr < \infty \implies \int_1^\infty \frac{\Theta_\sigma(r)}{r}\,dr < \infty.$$

Theorem 6.6. *Let $T \ge 1$. Then*

$$\langle |\psi - \nabla \chi_T|^2\rangle^{1/2} \le C_\sigma \int_{T/2}^\infty \frac{\Theta_\sigma(r)}{r}\,dr \tag{6-21}$$

for $\sigma \in (0,1)$, where C_σ depends only on σ and A.

Proof. Fix $1 \le j \le d$ and $1 \le \beta \le m$. Let $u = \chi^\beta_{T,j}$, $v = \chi^\beta_{2T,j}$ and $w = u - v$. It follows from Lemma 5.5 that

$$\langle A\nabla w \cdot \nabla \varphi\rangle = \frac{1}{4T^2}\langle v \cdot \varphi\rangle - \frac{1}{T^2}\langle u \cdot \varphi\rangle$$

for any $\varphi \in H^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m)$ with $\varphi, \nabla\varphi \in B^2(\mathbb{R}^d)$. By taking $\varphi = w$, we obtain

$$\langle |\nabla w|^2\rangle \le C T^{-2}\big(\langle |u|^2\rangle + \langle |v|^2\rangle\big) \le C_\sigma (\Theta_\sigma(T) + \Theta_\sigma(2T))^2, \tag{6-22}$$

where we have used (6-19) for the second inequality. Hence, we have proved that

$$\langle |\nabla \chi_T - \nabla \chi_{2T}|^2\rangle^{1/2} \le C_\sigma \int_{T/2}^T \frac{\Theta_\sigma(r)}{r}\,dr,$$

where we have used the fact that $\Theta_\sigma(r)$ is decreasing. Consequently,

$$\sum_{\ell=0}^\infty \langle |\nabla \chi_{2^\ell T} - \nabla \chi_{2^{\ell+1}T}|^2\rangle^{1/2} \le C_\sigma \int_{T/2}^\infty \frac{\Theta_\sigma(r)}{r}\,dr. \tag{6-23}$$

Recall that, by (5-18), $\langle |\psi - \nabla\chi_T|^2\rangle \to 0$ as $T \to \infty$. The estimate (6-21) now follows from (6-23). □

Remark 6.7. Suppose that there exist $C > 0$ and $\tau > 0$ such that

$$\rho(R) \le \frac{C}{R^\tau} \quad \text{for } R \ge 1. \tag{6-24}$$

By taking $R = T^{\sigma/(\tau+\sigma)}$ in (6-16), we obtain

$$T^{-1}\|\chi_T\|_{L^\infty} \le C\Theta_\sigma(T) \le C T^{-\tau\sigma/(\tau+\sigma)}.$$

Since $\sigma \in (0,1)$ is arbitrary, this shows that

$$T^{-1}\|\chi_T\|_{L^\infty} \le C_\delta T^{-\tau/(\tau+1)+\delta} \tag{6-25}$$

for any $\delta \in (0,1)$, where C_δ depends only on δ and A. Under the condition (6-24), by Theorem 6.6, we also obtain

$$\langle |\psi - \nabla\chi_T|^2\rangle^{1/2} \le C_\delta T^{-\tau/(\tau+1)+\delta} \quad \text{for any } \delta \in (0,1). \tag{6-26}$$

7. Convergence rates

In this section we give the proof of Theorems 1.1 and 1.2.

Lemma 7.1. *Let $h \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $T > 0$. Suppose that there exists $\sigma \in (0,1)$ such that*

$$\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,r)}|h|^2\right)^{\frac{1}{2}} \le \left(\frac{T}{r}\right)^{1-\sigma} \quad \text{for any } 0<r\le T. \tag{7-1}$$

Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ be the solution of

$$-\Delta u + T^{-2}u = h \quad \text{in } \mathbb{R}^d \tag{7-2}$$

given by Proposition 5.1. Then

$$\|u\|_{L^\infty} \le CT^2, \qquad \|\nabla u\|_{L^\infty} \le CT, \tag{7-3}$$

and

$$|\nabla u(x) - \nabla u(y)| \le CT^{1-\sigma}|x-y|^\sigma \quad \text{for any } x, y \in \mathbb{R}^d, \tag{7-4}$$

where C depends only on d and σ. Furthermore, $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,T)}|\nabla^2 u|^2\right)^{\frac{1}{2}} \le C. \tag{7-5}$$

Proof. By rescaling we may assume $T = 1$. It follows from Proposition 5.1 and (7-1) that

$$\sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,1)}|u|^2\right)^{\frac{1}{2}} \le C \quad \text{and} \quad \sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,1)}|\nabla u|^2\right)^{\frac{1}{2}} \le C, \tag{7-6}$$

where C depends only on d. Fix $x_0 \in \mathbb{R}^d$ and let $\phi \in C_0^\infty(B(x_0,2))$ be a cut-off function such that $\phi = 1$ in $B(x_0,1)$. By representing $u\phi$ as an integral and using the fundamental solution for $-\Delta$, the desired estimates follow from (7-1) by a standard procedure. We leave the details to the reader. □

Under additional almost periodicity conditions on h, the next lemma gives much sharper estimates for the solution u of (7-2).

Lemma 7.2. *Let $h \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $T > 0$. Suppose that there exists $\sigma \in (0,1)$ such that*

$$\begin{aligned} \sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,r)}|h|^2\right)^{\frac{1}{2}} &\le C_0\left(\frac{T}{r}\right)^{1-\sigma}, \\ \sup_{x\in\mathbb{R}^d}\left(\fint_{B(x,r)}|h(t+y)-h(t+z)|^2\,dt\right)^{\frac{1}{2}} &\le C_0\left(\frac{T}{r}\right)^{1-\sigma}\|A(\cdot+y)-A(\cdot+z)\|_{L^\infty} \end{aligned} \tag{7-7}$$

for any $0<r\le T$ and $y, z \in \mathbb{R}^d$. Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ be the solution of (7-2), given by Proposition 5.1. Then

$$\begin{aligned} T^{-2}\|u\|_{L^\infty} &\le C\Theta_1(T) + |\langle h\rangle|, \\ T^{-1}\|\nabla u\|_{L^\infty} &\le C\Theta_\sigma(T), \end{aligned} \tag{7-8}$$

where $\Theta_\sigma(T)$ is defined by (6-18) and C depends at most on d, σ and C_0.

Proof. By applying Lemma 7.1 to the function

$$\frac{u(x+y)-u(x+z)}{C_0\|A(\cdot+y)-A(\cdot+z)\|_{L^\infty}}$$

with y and z fixed, we obtain

$$\begin{aligned}\|u(\cdot+y)-u(\cdot+z)\|_{L^\infty} &\le CT^2\|A(\cdot+y)-A(\cdot+z)\|_{L^\infty},\\ \|\nabla u(\cdot+y)-\nabla u(\cdot+z)\|_{L^\infty} &\le CT\|A(\cdot+y)-A(\cdot+z)\|_{L^\infty},\end{aligned} \tag{7-9}$$

where C depends only on d, C_0 and σ. This shows that u and ∇u are uniformly almost periodic. In particular, u and ∇u have mean values and $\langle\nabla u\rangle = 0$. Also, note that condition (7-7) implies that $h \in B^2(\mathbb{R}^d)$ and hence has the mean value $\langle h\rangle$. It is easy to deduce from (7-2) that $\langle u\rangle = T^2\langle h\rangle$.

Note that, for any $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^d$ with $|z| \le R \le T$,

$$T^{-2}|u(y)-u(0)| \le T^{-2}|u(y)-u(z)| + T^{-2}|u(z)-u(0)| \le C\|A(\cdot+y)-A(\cdot+z)\|_{L^\infty} + CT^{-1}R,$$

where we have used (7-9) and $\|\nabla u\|_{L^\infty} \le CT$ for the second inequality. It follows from the definition of $\rho(R)$ that

$$\sup_{y\in\mathbb{R}^d} T^{-2}|u(y)-u(0)| \le C(\rho(R)+T^{-1}R) \quad \text{for any } 0 < R \le T.$$

By the definition of Θ_1, this gives

$$\sup_{y\in\mathbb{R}^d} T^{-2}|u(y)-u(0)| \le C\Theta_1(T). \tag{7-10}$$

Using

$$|T^{-2}u(0)| \le T^{-2}\left|\fint_{B(0,L)} (u(y)-u(0))\,dy\right| + \left|\fint_{B(0,L)} u(x)\right|$$

for any $L > 0$ and (7-10), we see that, by letting $L \to \infty$,

$$|T^{-2}u(0)| \le C\Theta_1(T) + T^{-2}|\langle u\rangle| = C\Theta_1(T) + |\langle h\rangle|. \tag{7-11}$$

The first inequality in (7-8) now follows from (7-10) and (7-11).

Finally, we point out that the second inequality in (7-8) follows in the same manner, using (7-9) and (7-4) as well as the fact that the mean value of ∇u is zero. □

We are now ready to estimate the rates of convergence of u_ε to u_0.

Theorem 7.3. *Let u_ε ($\varepsilon \ge 0$) be the weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω and $u_\varepsilon = g$ on $\partial\Omega$. Suppose that $u_0 \in W^{2,2}(\Omega)$. Let*

$$w_\varepsilon(x) = u_\varepsilon(x) - u_0(x) - \varepsilon\chi_{T,j}(x/\varepsilon)\frac{\partial u_0}{\partial x_j} + v_\varepsilon, \tag{7-12}$$

where $T = \varepsilon^{-1}$ and $v_\varepsilon \in H^1(\Omega;\mathbb{R}^m)$ is the weak solution of the Dirichlet problem

$$\mathcal{L}_\varepsilon(v_\varepsilon) = 0 \quad \text{in } \Omega \qquad \text{and} \qquad v_\varepsilon = \varepsilon\chi_{T,j}(x/\varepsilon)\frac{\partial u_0}{\partial x_j} \quad \text{on } \partial\Omega. \tag{7-13}$$

Then

$$\|w_\varepsilon\|_{H^1(\Omega)} \le C_\sigma\big(\Theta_\sigma(T) + \langle|\psi - \nabla\chi_T|\rangle\big)\|u_0\|_{W^{2,2}(\Omega)} \tag{7-14}$$

for any $\sigma \in (0, 1)$, *where* C_σ *depends only on* σ, A *and* Ω.

Proof. With loss of generality we may assume that

$$\|u_0\|_{W^{2,2}(\Omega)} = 1. \tag{7-15}$$

A direct computation shows that

$$\mathcal{L}_\varepsilon(w_\varepsilon) = -\operatorname{div}(B_T(x/\varepsilon)\nabla u_0) + \varepsilon \operatorname{div}(A(x/\varepsilon)\chi_T(x/\varepsilon)\nabla^2 u_0), \tag{7-16}$$

where $B_T(y) = (b_{T,ij}^{\alpha\beta}(y))$ is given by

$$b_{T,ij}^{\alpha\beta}(y) = \hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(y) - a_{ik}^{\alpha\gamma}(y)\frac{\partial}{\partial y_k}(\chi_{T,j}^{\gamma\beta}(y)). \tag{7-17}$$

Since $w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^m)$, it follows from (7-16) that

$$c\int_\Omega |\nabla w_\varepsilon|^2\,dx \le \left|\int_\Omega \operatorname{div}(B_T(x/\varepsilon)\nabla u_0)\cdot w_\varepsilon\,dx\right| + \int_\Omega |\varepsilon\chi_T(x/\varepsilon)||\nabla^2 u_0||\nabla w_\varepsilon|\,dx = I_1 + I_2. \tag{7-18}$$

It suffices to show that

$$I_1 + I_2 \le C_\sigma\big(\Theta_\sigma(T) + \langle|\psi - \nabla\chi_T|\rangle\big)\|w_\varepsilon\|_{H^1(\Omega)} \tag{7-19}$$

for any $\sigma \in (0, 1)$.

First, it is easy to see that

$$I_2 \le C\varepsilon\|\chi_T\|_{L^\infty}\|\nabla w_\varepsilon\|_{L^2(\Omega)} \le C\Theta_\sigma(T)\|\nabla w_\varepsilon\|_{L^2(\Omega)} \tag{7-20}$$

for any $\sigma \in (0, 1)$, where we have used (7-15) and (6-19).

Next, to estimate I_1, we let $h(y) = h_T(y) = B_T(y) - \langle B_T\rangle$ and solve (7-2). More precisely, let $h = (h_{ij}^{\alpha\beta})$ and $f = (f_{ij}^{\alpha\beta})$, where $f_{ij}^{\alpha\beta} \in H^2_{\text{loc}}(\mathbb{R}^d)$ solves

$$-\Delta f_{ij}^{\alpha\beta} + T^{-2} f_{ij}^{\alpha\beta} = h_{ij}^{\alpha\beta} \quad \text{in } \mathbb{R}^d. \tag{7-21}$$

By (6-8) and (6-15), the function h satisfies the condition (7-7) for any $\sigma \in (0, 1)$. Since $\langle h\rangle = 0$, it follows from Lemma 7.2 that

$$\begin{aligned} T^{-2}\|f\|_{L^\infty} &\le C\Theta_1(T),\\ T^{-1}\|\nabla f\|_{L^\infty} &\le C\Theta_\sigma(T) \end{aligned} \tag{7-22}$$

for any $\sigma \in (0, 1)$. Using (7-21) and integration by parts, we may bound I_1 in (7-18) by

$$\left|\int_\Omega \operatorname{div}(\Delta f(x/\varepsilon)\nabla u_0)\cdot w_\varepsilon\,dx\right| + T^{-2}\int_\Omega |f(x/\varepsilon)||\nabla u_0||\nabla w_\varepsilon|\,dx + C\langle|\psi - \nabla\chi_T|\rangle\|w_\varepsilon\|_{L^2(\Omega)}, \tag{7-23}$$

where we have used the fact that $|\langle B_T\rangle| \le C\langle|\psi - \nabla\chi_T|\rangle$. Note that, by (7-22), the second term in (7-23) is bounded by $C\Theta_1(T)\|\nabla w_\varepsilon\|_{L^2(\Omega)}$.

It remains to estimate the first term in (7-23), which we denote by I_{11}. To this end we write

$$\begin{aligned}\operatorname{div}(\Delta f(x/\varepsilon)\nabla u_0)\cdot w_\varepsilon &= \frac{\partial}{\partial x_i}\left(\Delta f_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\right)\cdot w_\varepsilon^\alpha \\ &= \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_k}\left(\frac{\partial f_{ij}^{\alpha\beta}}{\partial x_k}-\frac{\partial f_{kj}^{\alpha\beta}}{\partial x_i}\right)(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\right)\cdot w_\varepsilon^\alpha + \frac{\partial}{\partial x_i}\left(\frac{\partial^2 f_{kj}^{\alpha\beta}}{\partial x_k\partial x_i}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\right)\cdot w_\varepsilon^\alpha \\ &= -\frac{\partial}{\partial x_i}\left(\varepsilon\left(\frac{\partial f_{ij}^{\alpha\beta}}{\partial x_k}-\frac{\partial f_{kj}^{\alpha\beta}}{\partial x_i}\right)(x/\varepsilon)\frac{\partial^2 u_0^\beta}{\partial x_k\partial x_j}\right)\cdot w_\varepsilon^\alpha + \frac{\partial}{\partial x_i}\left(\frac{\partial^2 f_{kj}^{\alpha\beta}}{\partial x_k\partial x_i}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\right)\cdot w_\varepsilon^\alpha,\end{aligned}$$

where we have used the product rule and the fact that

$$\frac{\partial^2}{\partial x_i\partial x_k}\left(\left(\frac{\partial f_{ij}^{\alpha\beta}}{\partial x_k}-\frac{\partial f_{kj}^{\alpha\beta}}{\partial x_i}\right)(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\right)=0.$$

It then follows from an integration by parts that

$$I_{11}\le C\varepsilon\int_\Omega|\nabla f(x/\varepsilon)||\nabla^2 u_0||\nabla w_\varepsilon|\,dx + C\sum_{j,\alpha,\beta}\int_\Omega\left|\nabla\frac{\partial f_{kj}^{\alpha\beta}}{\partial x_k}(x/\varepsilon)\right||\nabla u_0||\nabla w_\varepsilon|\,dx = I_{11}^{(1)}+I_{11}^{(2)}. \tag{7-24}$$

In view of (7-22), we have

$$I_{11}^{(1)}\le C\varepsilon\|\nabla f\|_{L^\infty}\|\nabla w_\varepsilon\|_{L^2(\Omega)}\le C\Theta_\sigma(T)\|\nabla w_\varepsilon\|_{L^2(\Omega)} \tag{7-25}$$

for any $\sigma\in(0,1)$.

Finally, to estimate $I_{11}^{(2)}$, we note that, by the definition of χ_T,

$$\frac{\partial h_{ij}^{\alpha\beta}}{\partial y_i}=\frac{\partial}{\partial y_i}(b_{T,ij}^{\alpha\beta})=-\frac{1}{T^2}\chi_{T,j}^{\alpha\beta}.$$

It follows that

$$-\Delta\frac{\partial f_{ij}^{\alpha\beta}}{\partial y_i}+\frac{1}{T^2}\frac{\partial f_{ij}^{\alpha\beta}}{\partial y_i}=-\frac{1}{T^2}\chi_{T,j}^{\alpha\beta}.$$

Observe that the function $T^{-1}\chi_T$ satisfies the assumption on h in Lemma 7.2 with $\sigma=1$. As a result, we obtain

$$\left\|\nabla\frac{\partial f_{ij}^{\alpha\beta}}{\partial x_i}\right\|_{L^\infty}\le C_\sigma\Theta_\sigma(T)$$

for any $\sigma\in(0,1)$. This allows us to bound $I_{11}^{(2)}$ by $C_\sigma\Theta_\sigma(T)\|\nabla w_\varepsilon\|_{L^2(\Omega)}$ and completes the proof. □

The next lemma gives an estimate for the norm of v_ε in $H^1(\Omega)$.

Lemma 7.4. *Let v_ε be the weak solution of (7-13) with $T=\varepsilon^{-1}$. Then*

$$\|v_\varepsilon\|_{H^1(\Omega)}\le C_\sigma(T^{-1}\|\chi_T\|_{L^\infty})^{1/2-\sigma}(\|\nabla u_0\|_{L^\infty(\Omega)}+\|\nabla^2 u_0\|_{L^2(\Omega)}) \tag{7-26}$$

for any $\sigma\in\left(0,\frac{1}{2}\right)$, where C_σ depends only on A, Ω and σ.

Proof. We may assume that $\|\nabla u_0\|_{L^\infty(\Omega)}+\|\nabla^2 u_0\|_{L^2(\Omega)}=1$. We may also assume that $\delta=T^{-1}\|\chi_T\|_{L^\infty}>0$ is small, since $\delta\to 0$ as $T\to\infty$. Choose a cut-off function $\eta_\delta\in C_0^\infty(\mathbb{R}^d)$ so that $0\le\eta_\delta\le 1$, $\eta_\delta(x)=1$ if $\operatorname{dist}(x,\partial\Omega)<\delta$, $\eta_\delta(x)=0$ if $\operatorname{dist}(x,\partial\Omega)\ge 2\delta$, and $|\nabla\eta_\delta|\le C\delta^{-1}$. Note that

$$\begin{aligned}\|v_\varepsilon\|_{H^1(\Omega)}&\le C\varepsilon\|\chi_T(x/\varepsilon)\nabla u_0\|_{H^{1/2}(\partial\Omega)}\\&\le C\varepsilon\|\eta_\delta\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)}\\&\le C\biggl(\|\chi_T\|_{L^\infty}\delta^{-1/2}\varepsilon+\biggl(\int_{\Omega_\delta}|\nabla\chi_T(x/\varepsilon)|^2\,dx\biggr)^{\frac12}\biggr),\end{aligned}\tag{7-27}$$

where $\Omega_\delta=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)\le 2\delta\}$. Since $\|\chi_T\|_{L^\infty}\delta^{-1/2}\varepsilon=\delta^{1/2}$, we only need to estimate the integral of $|\nabla\chi_T(x/\varepsilon)|^2$ over Ω_δ.

To this end, we cover Ω_δ with cubes Q_j of side length δ such that $\sum_j|Q_j|\le C\delta$. It follows that

$$\begin{aligned}\int_{\Omega_\delta}|\nabla\chi_T(x/\varepsilon)|^2\,dx\le\sum_j\int_{Q_j}|\nabla\chi_T(x/\varepsilon)|^2\,dx&\le\sum_j|Q_j|\fint_{(1/\varepsilon)Q_j}|\nabla\chi_T|^2\\&\le C\delta\sup_{\ell(Q)=\delta T}\fint_Q|\nabla\chi_T|^2\le C_\sigma\delta^{1-\sigma}\end{aligned}\tag{7-28}$$

for any $\sigma\in(0,1)$, where we have used the estimate (6-8) in the last inequality. This, together with (7-27), gives (7-26). □

We are now in a position to give the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. It follows from Theorem 7.3 and Lemma 7.4 that, for any $\sigma\in(0,1)$ and $\delta\in\bigl(0,\tfrac12\bigr)$,

$$\begin{aligned}&\|u_\varepsilon-u_0-\varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)}\\&\qquad\le C\bigl(\Theta_\sigma(T)+\langle|\psi-\nabla\chi_T|\rangle\bigr)\|u_0\|_{W^{2,2}(\Omega)}+C(\Theta_\sigma(T))^{1/2-\delta}(\|\nabla u_0\|_{L^\infty(\Omega)}+\|\nabla^2u_0\|_{L^2(\Omega)})\\&\qquad\le C\bigl(\langle|\psi-\nabla\chi_T|\rangle+(\Theta_\sigma(T))^{1/2-\delta}\bigr)\|u_0\|_{W^{2,p}(\Omega)}\\&\qquad\le C\bigl(\langle|\psi-\nabla\chi_T|\rangle+(\Theta_1(T))^{\sigma(1/2-\delta)}\bigr)\|u_0\|_{W^{2,p}(\Omega)},\end{aligned}\tag{7-29}$$

where $T=\varepsilon^{-1}$ and we have used the Sobolev imbedding $\|\nabla u_0\|_{L^\infty(\Omega)}\le C\|u_0\|_{W^{2,p}(\Omega)}$ for $p>d$. This implies that

$$\begin{aligned}\|u_\varepsilon-u_0\|_{L^2(\Omega)}&\le\|\varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{L^2(\Omega)}+C\bigl(\langle|\psi-\nabla\chi_T|\rangle+(\Theta_1(T))^{1/4}\bigr)\|u_0\|_{W^{2,p}(\Omega)}\\&\le C\bigl(\langle|\psi-\nabla\chi_T|\rangle+(\Theta_1(T))^{1/4}\bigr)\|u_0\|_{W^{2,p}(\Omega)},\end{aligned}$$

where C depends only on A and Ω. Since $\langle|\psi-\nabla\chi_T|\rangle+(\Theta_1(T))^{1/4}\to 0$ as $T\to\infty$, one may find a modulus η on $(0,1]$, depending only on A, such that $\eta(0+)=0$ and

$$\langle|\psi-\nabla\chi_T|\rangle+(\Theta_1(T))^{1/4}\le\eta(T^{-1})$$

for $T\ge 1$. As a result, we obtain

$$\begin{aligned}\|u_\varepsilon-u_0-\varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{H^1(\Omega)}&\le C\eta(\varepsilon)\|u_0\|_{W^{2,p}(\Omega)},\\\|u_\varepsilon-u_0\|_{L^2(\Omega)}&\le C\eta(\varepsilon)\|u_0\|_{W^{2,p}(\Omega)}.\end{aligned}$$

Finally, we observe that, by Theorem 1.4, for any $\sigma \in (0, 1)$,

$$\|u_\varepsilon\|_{C^\sigma(\bar{\Omega})} \le C(\|g\|_{C^\sigma(\partial\Omega)} + \|F\|_{L^d(\Omega)}) \le C(\|u_0\|_{C^\sigma(\bar{\Omega})} + \|\nabla^2 u_0\|_{L^d(\Omega)}) \le C\|u_0\|_{W^{2,d}(\Omega)}.$$

It follows by interpolation that, for any $\sigma \in (0, 1)$,

$$\|u_\varepsilon - u_0\|_{C^\sigma(\bar{\Omega})} \le C\tilde{\eta}(\varepsilon)\|u_0\|_{W^{2,p}(\Omega)},$$

where $\tilde{\eta}$ is a modulus function depending only on A and σ, and $\tilde{\eta}(0+) = 0$. This completes the proof. □

Proof of Theorem 1.2. Estimate (1-15) follows directly from (7-29) and Theorem 6.6. To see (1-14), we use

$$\begin{aligned}\|u_\varepsilon - u_0\|_{L^2(\Omega)} &\le \|u_\varepsilon - u_0 - \varepsilon\chi_T(x/\varepsilon)\nabla u_0 + v_\varepsilon\|_{L^2(\Omega)} + \|v_\varepsilon\|_{L^2(\Omega)}\\ &\le C_\sigma\big(\Theta_\sigma(T) + \langle|\psi - \nabla\psi_T|\rangle\big)\|u_0\|_{W^{2,2}(\Omega)} + \|v_\varepsilon\|_{L^2(\Omega)},\end{aligned} \tag{7-30}$$

where v_ε is as defined in Theorem 7.3. By Theorem 1.4 we obtain

$$\begin{aligned}\|v_\varepsilon\|_{L^2(\Omega)} &\le C\|v_\varepsilon\|_{L^\infty(\Omega)}\\ &\le C\|\varepsilon\chi_T(x/\varepsilon)\nabla u_0\|_{C^{\sigma_1}(\partial\Omega)}\\ &\le C(\varepsilon^{1-\sigma_1}\|\chi_T\|_{C^{0,\sigma_1}} + \Theta_\sigma(T))\|\nabla u_0\|_{C^{\sigma_1}(\partial\Omega)}\\ &\le C(T^{\sigma_1-1}\|\chi_T\|_{C^{0,\sigma_1}} + \Theta_\sigma(T))\|u_0\|_{W^{2,p}(\Omega)},\end{aligned}$$

where $p > d$, $\sigma \in (0, 1)$ and $0 < \sigma_1 < 1 - d/p$. Since $T^{-1}\|\chi_T\|_{L^\infty} \le C_\sigma\Theta_\sigma(T)$ and $|\chi_T(x) - \chi_T(y)| \le C_\alpha T^{1-\alpha}|x-y|^\alpha$ for any $\alpha \in (0, 1)$, it follows by interpolation that

$$T^{\sigma_1-1}\|\chi_T\|_{C^{0,\sigma_1}} \le C(\Theta_\sigma(T))^{1-\sigma_2}$$

for any $\sigma_2 > \sigma_1$. Hence,

$$\|v_\varepsilon\|_{L^2(\Omega)} \le C(\Theta_\sigma(T))^{1-\delta}\|u_0\|_{W^{2,p}(\Omega)} \le C(\Theta_1(T))^{\sigma(1-\delta)}\|u_0\|_{W^{2,p}(\Omega)}$$

for any $\delta, \sigma \in (0, 1)$ and $p > d$, where C depends only on δ, p, σ, A and Ω. This, together with (7-30) and Theorem 6.6, gives

$$\begin{aligned}\|u_\varepsilon - u_0\|_{L^2(\Omega)} &\le C\big(\langle|\psi - \nabla\chi_T|\rangle + (\Theta_1(T))^\sigma\big)\|u_0\|_{W^{2,p}(\Omega)}\\ &\le C\bigg(\int_{1/(2\varepsilon)}^\infty \frac{\Theta_\sigma(r)}{r}\,dr + (\Theta_1(\varepsilon^{-1}))^\sigma\bigg)\|u_0\|_{W^{2,p}(\Omega)}\end{aligned}$$

for any $\sigma \in (0, 1)$, and completes the proof. □

8. Quasiperiodic coefficients

In this section we consider the case where $A(x)$ is quasiperiodic and continuous. More precisely, without loss of generality, we will assume that

$$\begin{cases} A(x) = B(j_\lambda(x)),\\ B \text{ is 1-periodic and continuous in } \mathbb{R}^M,\end{cases} \tag{8-1}$$

where $M = m_1 + m_2 + \cdots + m_d$ and, for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$,

$$j_\lambda(x) = (\lambda_1^1 x_1, \lambda_1^2 x_1, \ldots, \lambda_1^{m_1} x_1, \lambda_2^1 x_2, \ldots, \lambda_2^{m_2} x_2, \ldots, \lambda_d^1 x_d, \ldots, \lambda_d^{m_d} x_d) \in \mathbb{R}^M.$$

Also, for each $i = 1, 2, \ldots, d$, the set $\{\lambda_i^1, \ldots, \lambda_i^{m_i}\}$ is assumed to be linearly independent over $\mathbb{Z}$. Under these conditions, it is known that $A(x)$ is uniformly almost periodic. We shall be interested in conditions on $\lambda = (\lambda_i^j)$ that imply the power decay of $\rho(R)$ as $R \to \infty$. For convenience we consider

$$\rho_1(R) = \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ \|z\|_\infty \le R}} \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty}, \tag{8-2}$$

where $\|z\|_\infty = \max(|z_1|, \ldots, |z_d|)$ for $z = (z_1, \ldots, z_d)$. It is easy to see that $\rho_1(\sqrt{d}R) \le \rho(R) \le \rho_1(R)$.

Let

$$\omega(\delta) = \sup\{|B(x) - B(y)| : \|x - y\|_\infty \le \delta\}, \quad \delta > 0,$$

denote the modulus of continuity of $B(x)$. For $x \in \mathbb{R}$, write $x = [x] + <x>$, where $[x] \in \mathbb{Z}$ and $<x> \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. If $x = (x_1, \ldots, x_M) \in \mathbb{R}^M$, define $[x] = ([x_1], \ldots, [x_N])$ and $<x> = (<x_1>, \ldots, <x_M>)$. It is easy to see that $\|<x>\|_\infty$ gives the distance from x to $\mathbb{Z}^M$ with respect to the norm $\|\cdot\|_\infty$.

Lemma 8.1. *Let $\rho_1(R)$ be defined by (8-2). Then, for any $R > 0$, $\rho_1(R) \le \omega(\theta_\lambda(R))$, where*

$$\theta_\lambda(R) = \sup_{x \in [-1/2, 1/2]^M} \inf_{\substack{z \in \mathbb{R}^d \\ \|z\|_\infty \le R}} \|x - <j_\lambda(z)>\|_\infty. \tag{8-3}$$

Proof. Note that, since B is 1-periodic,

$$|B(x) - B(y)| = |B(y + [x - y] + <x - y>) - B(y)| = |B(y + <x - y>) - B(y)| \le \omega(\|<x - y>\|_\infty)$$

for any $x, y \in \mathbb{R}^M$. It follows that

$$|A(x + y) - A(x + z)| \le \omega(\|<j_\lambda(y) - j_\lambda(z)>\|_\infty)$$

for any $x, y, z \in \mathbb{R}^d$. This implies that

$$\rho_1(R) \le \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ \|z\|_\infty \le R}} \omega(\|<j_\lambda(y) - j_\lambda(z)>\|_\infty).$$

Using

$$\|<j_\lambda(y) - j_\lambda(z)>\|_\infty = \|<<j_\lambda(y)> - <j_\lambda(z)>>\|_\infty \le \|<j_\lambda(y)> - <j_\lambda(z)>\|_\infty,$$

we obtain

$$\rho_1(R) \le \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ \|z\|_\infty \le R}} \omega(\|<j_\lambda(y)> - <j_\lambda(z)>\|_\infty) \le \omega(\theta_\lambda(R)),$$

where we have used the continuity of $\omega(\delta)$ for the second inequality. □

Let $\lambda_i = (\lambda_i^1, \lambda_i^2, \ldots, \lambda_i^{m_i}) \in \mathbb{R}^{m_i}$ for each $1 \le i \le d$ and

$$j_{\lambda_i}(t) = (\lambda_i^1 t, \lambda_i^2 t, \ldots, \lambda_i^{m_i} t) \in \mathbb{R}^{m_i} \quad \text{for } t \in \mathbb{R}.$$

Thus, for $z = (z_1, z_2, \dots, z_d) \in \mathbb{R}^d$,

$$j_\lambda(z) = (j_{\lambda_1}(z_1), j_{\lambda_2}(z_2), \dots, j_{\lambda_d}(z_d)).$$

It follows that

$$\|x - \langle j_\lambda(z)\rangle\|_\infty = \max_{1\le i\le d} \|x_i - \langle j_{\lambda_i}(z_i)\rangle\|_\infty,$$

where $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^M$ and $x_i \in \mathbb{R}^{m_i}$. This implies that

$$\theta_\lambda(R) = \max_{1\le i\le d} \theta_{\lambda_i}(R), \tag{8-4}$$

where

$$\theta_{\lambda_i}(R) = \sup_{x\in[-1/2,1/2]^{m_i}} \inf_{\substack{t\in\mathbb{R}\\ |t|\le R}} \|x - \langle j_{\lambda_i}(t)\rangle\|_\infty. \tag{8-5}$$

Note that if $m_i = 1$ then $\theta_{\lambda_i}(R) = 0$ for R large. We will use the Erdős–Turán–Koksma inequality in the discrepancy theory to estimate the function $\theta_{\lambda_i}(R)$, defined by (8-5), for $m_i \ge 2$.

Let $P = P_N = \{x_1, x_2, \dots, x_N\}$ be a finite subset of $\left[-\frac{1}{2}, \frac{1}{2}\right]^m$. The discrepancy of P is defined as

$$D_N(P) = \sup_B \left| \frac{A(B; P)}{N} - |B| \right|,$$

where the supremum is taken over all rectangular boxes $B = [a_1, b_1] \times \cdots \times [a_m, b_m] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^m$ and $A(B; P)$ denotes the number of elements of P in B. It follows from the Erdős–Turán–Koksma inequality that

$$D_N(P) \le C\left\{ \frac{1}{H} + \sum_{\substack{n\in\mathbb{Z}^m\\ 0<\|n\|_\infty\le H}} \frac{1}{(1+|n_1|)\cdots(1+|n_m|)} \left| \frac{1}{N} \sum_{x\in P} e^{2\pi i(n\cdot x)} \right| \right\} \tag{8-6}$$

for any $H \ge 1$, where C depends only on m (see, e.g., [Drmota and Tichy 1997, p. 15]). It is not hard to see that

$$\max_{y\in[-1/2,1/2]^m} \min_{z\in P_N} \|y - z\|_\infty \le \tfrac{1}{2}[D_N(P_N)]^{1/m}. \tag{8-7}$$

Lemma 8.2. *Let $R \ge 2$ and $\ell \ge 2$ be two positive integers. We divide the interval $[-R, R]$ into $2R\ell$ subintervals of length $1/\ell$. Let $N = 2R\ell$ and*

$$P_N = \left\{ x = \langle j_\lambda(t)\rangle \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^m : t = j + \frac{k}{\ell},\ -R \le j \le R-1 \ \text{and}\ 0 \le k \le \ell - 1 \right\},$$

where $\lambda = (\lambda^1, \dots, \lambda^m) \in \mathbb{R}^m$ and $m \ge 2$. Suppose that there exist $c_0 > 0$ and $\tau > 0$ such that

$$|n\cdot\lambda| \ge c_0 |n|^{-\tau} \quad \text{for any } n \in \mathbb{Z}^m \setminus \{0\}. \tag{8-8}$$

Then

$$D_N(P_N) \le C(R^{-1/(\tau+1)}(\log R)^{m-1} + N^{-1} R^{1+1/(\tau+1)}(\log R)^{m-1}), \tag{8-9}$$

where C depends only on m, c_0, $|\lambda|$ and τ.

Proof. Let $f(t)=e^{2\pi i (n\cdot\lambda)t}$ and

$$I_n=\frac{1}{N}\sum_{x\in P_N} e^{2\pi i(n\cdot x)}=\frac{1}{N}\sum_{j,k} f(t_{jk}), \tag{8-10}$$

where $n\in\mathbb{Z}^m\setminus\{0\}$, $j=-R,\ldots,R-1$, $k=0,\ldots,\ell-1$ and $t_{jk}=j+k/\ell$. Using

$$\left|\frac{1}{2R}\int_{-R}^{R} f(t)\,dt-\frac{1}{N}\sum_{j,k} f(t_{jk})\right|\le C\ell^{-1}\|f'\|_\infty,$$

we obtain

$$|I_n|\le C\ell^{-1}\|f'\|_\infty+\left|\frac{1}{2R}\int_{-R}^{R} f(t)\,dt\right|\le C\ell^{-1}|n\cdot\lambda|+\frac{C}{R|n\cdot\lambda|}\le C(\ell^{-1}|n|+R^{-1}|n|^\tau),$$

where we have used the assumption (8-8). In view of (8-6), we obtain

$$\begin{aligned}
D_N(P_N)&\le C\biggl(\frac{1}{H}+\sum_{\substack{n\in\mathbb{Z}^m\\ 0<\|z\|_\infty\le H}}\frac{\ell^{-1}|n|+R^{-1}|n|^\tau}{(1+|n_1|)\cdots(1+|n_m|)}\biggr)\\
&\le C\biggl(\frac{1}{H}+\int_{|x|\le CH}\frac{\ell^{-1}|x|+R^{-1}|x|^\tau}{(1+|x_1|)\cdots(1+|x_m|)}\,dx\biggr)\\
&\le C\Bigl(\frac{1}{H}+RN^{-1}H(\log H)^{m-1}+R^{-1}H^\tau(\log H)^{m-1}\Bigr)
\end{aligned}$$

for any $H\ge 2$. By taking $H=R^{1/(\tau+1)}$, we obtain the estimate (8-9). □

Theorem 8.3. *Let $\lambda=(\lambda_1,\ldots,\lambda_d)$ with $\lambda_i=(\lambda_i^1,\ldots,\lambda_i^{m_i})\in\mathbb{R}^{m_i}$ for $1\le i\le d$. Suppose that there exist $c_0>0$ and $\tau>0$ such that, for each $1\le i\le d$ with $m_i\ge 2$,*

$$|n\cdot\lambda_i|\ge c_0|n|^{-\tau}\quad\text{for any } n\in\mathbb{Z}^{m_i}\setminus\{0\}. \tag{8-11}$$

Then, for any $R\ge 2$,

$$\theta_\lambda(R)\le CR^{-1/(\tilde m(\tau+1))}(\log R)^{1-1/\tilde m}, \tag{8-12}$$

where $\tilde m=\max\{m_1,\ldots,m_d\}$ and C depends only on d, $\tilde m$, c_0 and τ.

Proof. Suppose $m_i\ge 2$. Let $P=P_N$ be same as in Lemma 8.2. It follows from (8-7) and Lemma 8.2 that

$$\theta_{\lambda_i}(R)\le C\bigl(R^{-1/(\tau+1)}(\log R)^{m_i-1}+N^{-1}R^{1+1/(\tau+1)}(\log R)^{m_i-1}\bigr)^{1/m_i}\le CR^{-1/(m_i(\tau+1))}(\log R)^{1-1/m_i},$$

where we have taken $N=CR^{1+2/(\tau+1)}$. This, together with (8-4), gives (8-12). □

Remark 8.4. Suppose that $A(x)=B(j_\lambda(x))$ and $B(y)$ is 1-periodic. Also assume that λ satisfies the condition (8-11) and $B(y)$ is Hölder continuous of order α for some $\alpha\in(0,1]$. It follows from Lemma 8.1 and Theorem 8.3 that

$$\rho(R)\le CR^{-\alpha/(\tilde m(\tau+1))}(\log R)^{\alpha(1-1/\tilde m)} \tag{8-13}$$

for $R\ge 1$. In view of Remark 1.3, this leads to

$$\|u_\varepsilon-u_0\|_{L^2(\Omega)}\le C_\gamma\varepsilon^\gamma\|u_0\|_{W^{2,p}(\Omega)}$$

for any $0 < \gamma < \alpha/(\alpha + \tilde{m}(\tau + 1))$. We point out that, for $A(y)$ that satisfies the condition (8-11) and is sufficiently smooth, the sharp estimate $\|u_\varepsilon - u_0\|_{L^2(\Omega)} = O(\varepsilon)$ was obtained in [Kozlov 1978].

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QUANTITATIVE DECAY RATES FOR DISPERSIVE SOLUTIONS TO THE EINSTEIN-SCALAR FIELD SYSTEM IN SPHERICAL SYMMETRY

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We study the future causally geodesically complete solutions of the spherically symmetric Einstein-scalar field system. Under the *a priori assumption* that the scalar field ϕ scatters locally in the scale-invariant bounded-variation (BV) norm, we prove that ϕ and its derivatives decay polynomially. Moreover, we show that the decay rates are sharp. In particular, we obtain sharp quantitative decay for the class of global solutions with small BV norms constructed by Christodoulou. As a consequence of our results, for every future causally geodesically complete solution with sufficiently regular initial data, we show the dichotomy that either the sharp power law tail holds or that the spacetime blows up at infinity in the sense that some scale invariant spacetime norms blow up.

1. Introduction

In this paper, we study the quantitative long time dynamics for the spherically symmetric dispersive spacetimes satisfying the Einstein-scalar field equations. More precisely, these are spherically symmetric solutions $(\mathcal{M}, \boldsymbol{g}, \phi)$ to the Einstein-scalar field system, where $\boldsymbol{g}$ is a Lorentzian metric and ϕ is a real-valued function on a 3+1-dimensional manifold $\mathcal{M}$, such that $(\mathcal{M}, \boldsymbol{g})$ is future causally geodesically complete and ϕ scatters locally in the scale-invariant bounded-variation (BV) norm. For these spacetimes, we establish a Price-law-type decay for the scalar field ϕ, the Christoffel symbols associated to $\boldsymbol{g}$ and all of their derivatives. To obtain the decay results, we do not need to assume any smallness of the initial data. Moreover, we show that the decay rates in this paper are sharp.

MSC2010: 35Q76.
Keywords: Einstein-scalar field system, spherical symmetry, quantitative decay rate.

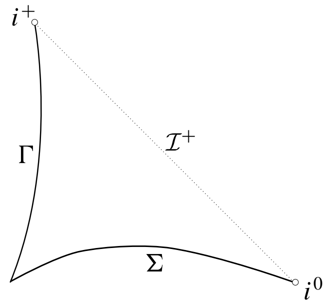

Figure 1. The dispersive case.

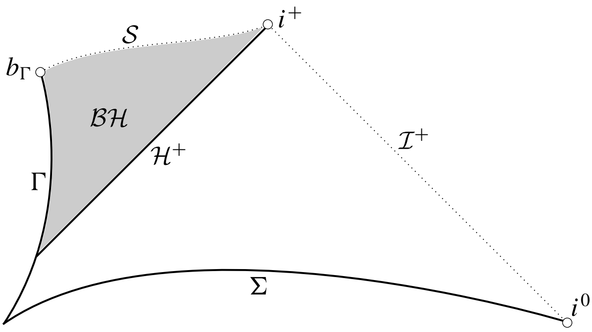

Figure 2. The black hole case.

The spherically symmetric Einstein-scalar field system, being one of the simplest models of self-gravitating matter in this symmetry class, has been studied extensively both numerically and mathematically. In a seminal series of papers by Christodoulou [1987; 1991; 1993; 1994; 1999], he achieved a complete understanding of the singularity structure of spherically symmetric spacetime solutions to this system. The culmination of the results shows that generic[1] spherically symmetric initial data with one asymptotically flat end give rise to a spacetime whose global geometry is either dispersive (with a Penrose diagram represented by Figure 1) or contains a black hole region $\mathcal{BH}$ which terminates in a spacelike curvature singularity $\mathcal{S}$ (with a Penrose diagram represented by Figure 2). In particular, in either of these generic scenarios, the spacetime possesses a complete null infinity $\mathcal{I}^+$ and thus obeys the weak cosmic censorship conjecture. Moreover, in either case, the maximal Cauchy development of the data is inextendible with a C^2 Lorentzian metric and therefore also verifies the strong cosmic censorship conjecture. We refer the readers to [Kommemi 2013] for a comprehensive discussion of general singularity structures for spherically symmetric spacetimes.

The remarkable resolution of the cosmic censorship conjectures, however, gives very little information on the long time dynamics for these spacetimes except for the small data[2] case [Christodoulou 1993]. In particular, not much is known about the asymptotic decay of the scalar field as measured by a far-away

[1] In the BV class, i.e., the initial data for $\partial_v(r\phi)$ has bounded variation. More precisely, Christodoulou showed that the nongeneric set of initial data has codimension at least two in the BV topology.

[2] That is, when the initial data is close to that of Minkowski space.

observer at null infinity. In the dispersive case, Christodoulou showed that the Bondi mass decays to zero along null infinity without an explicit decay rate. In the black hole case, he showed that the Bondi mass approaches the mass of the black hole, from which one can infer the nonquantitative decay for the scalar field along null infinity [Christodoulou 1993].

The long time dynamics in the case where the spacetime settles to a black hole was subsequently studied[3] in the seminal work of Dafermos and Rodnianski [2005]. They proved a quantitative decay rate for the scalar field (and its derivatives) in the spacetime including along null infinity $\mathcal{I}^+$ and the event horizon $\mathcal{H}^+$. The proof is based on the local conservation of energy, which is subcritical, together with techniques exploiting the conformal geometry of the spacetime and the celebrated red-shift effect along the event horizon. The result in particular justified, in a nonlinear setting, the heuristics of [Price 1972]. It turns out that the quantitative decay rates, when combined with the results of [Dafermos 2005], also have interesting consequences for the strong cosmic censorship conjecture in the context of the spherically symmetric Einstein–Maxwell-scalar field system.

In this paper, we study the other generic scenario, spherically symmetric dispersive spacetime solutions to the Einstein-scalar field system. Unlike in the black hole case, the monotonic Hawking mass is *supercritical* and provides no control over the dynamics of the solution. We thus do not expect to be able to obtain quantitative decay rates for large solutions without imposing extra assumptions. Instead, we assume a priori the nonquantitative decay of a *critical* quantity — the BV norm[4] — but only locally in a region where the area of the orbit of the symmetry group SO(3) remains uniformly bounded. Under this assumption of local BV scattering, we show that the scalar field and all its derivatives decay with a quantitative rate, reminiscent of the Price law decay rates in the black hole case. (We refer the readers to the statement of the main theorems in Section 3 for the precise rates that we obtain.) We prove, in particular, a quantitative decay rate for the scalar field along null infinity.

Our results apply in particular to the class of solutions arising from initial data with small BV norm. Christodoulou [1993] showed that these spacetimes are future causally geodesically complete. One can easily deduce from [Christodoulou 1993] that in fact these spacetimes satisfy the BV scattering assumption and therefore the solutions obey the quantitative decay estimates of our main theorem (see Theorem 3.15). On the other hand, our results do not require any smallness assumptions on the initial data. We conjecture that indeed our class of spacetimes contains those arising from large data:

Conjecture 1. There exists initial data of arbitrarily large BV norm whose maximal global development scatters locally in the BV norm.

In addition to the upper bounds that we obtain in our main theorem, we also construct examples where we prove lower bounds for the solutions with the same rates as the upper bounds. In particular, there exists a class of initial data with compactly supported scalar field whose future development saturates the decay estimates in the main theorem. This shows that the decay rates are sharp. We note that the decay rate is also consistent with the numerical study of Bizón, Chmaj and Rostworowski [Bizoń et al. 2009].

[3] In fact, they considered the more general Einstein–Maxwell-scalar field equations.

[4] Solutions of bounded variation were first studied by Christodoulou [1993] and play an important role in the proof of the cosmic censorship conjectures [Christodoulou 1999].

As a corollary of the main result on decay, we show the following dichotomy: either the quantitative decay rates are satisfied or the solution blows up at infinity. The latter are solutions such that some scale-invariant spacetime norms become infinite (see the precise definition in Definition 3.12).

The decay result in this paper easily implies that the locally BV scattering solutions that we consider are stable against small, regular, *spherically symmetric* perturbations. More ambitiously, one may conjecture:

Conjecture 2. Spherically symmetric, locally BV scattering dispersive solutions to the Einstein-scalar field equations are stable against *nonspherically symmetric* perturbations.

Conjecture 2, if true, will generalize the monumental theorem on the nonlinear stability of Minkowski spacetime of [Christodoulou and Klainerman 1993] (see also a simpler proof in [Lindblad and Rodnianski 2010]). For nonlinear wave equations satisfying the null condition, it is known [Alinhac 2009; Yang 2015] that *large* solutions decaying sufficiently fast are globally stable against small perturbations. On the other hand, our main theorem shows a quantitative decay rate for spherically symmetric, locally BV scattering dispersive spacetimes. Conjecture 2 can therefore be viewed as an attempt to generalize the results in [Alinhac 2009; Yang 2015] to the Einstein-scalar field system. We will address both Conjectures 1 and 2 in future works.

1A. ***Outline of the paper.*** In Section 2, we discuss the set-up of the problem and in particular define the class of solutions considered in the main theorem, i.e., the locally BV scattering solutions. In Section 3, we state the main theorems in the paper (Theorems 3.1 and 3.2), their consequences and additional theorems on the optimality of the decay rates. In the same section, we outline the main ideas of the proof. In Sections 4 and 5, we explain the main analytic features of the equations and the geometry of the class of spacetimes that we consider.

Sections 6 and 7 consist of the main content of this paper. In Section 6, we prove the decay estimates for ϕ, $\partial_v(r\phi)$ and $\partial_u(r\phi)$, that is, the first main theorem (Theorem 3.1). In Section 7, using in particular the results in Section 6, we derive the decay bounds for the second derivatives for $r\phi$ and the metric components, that is, the second main theorem (Theorem 3.2).

In the remaining sections of the paper, we turn to other theorems stated in Section 3. In Section 8, we give a proof of the dichotomy alluded to above, that either the quantitative decay holds or the spacetime blows up at infinity. In Section 9, we sketch a proof of a refinement of the conclusions of the main theorems in the small data case. Finally, in Section 10, we prove optimality of the decay rates asserted by the main theorems.

2. Set-up

In this section, we define the set-up, formulate the equations in a double null coordinate system and explain the characteristic initial value problem. This will allow us to state the main theorem in the next section.

2A. ***Spherically symmetric Einstein-scalar-field system (SSESF).*** We begin with a brief discussion on the derivation of the spherically symmetric Einstein-scalar-field system (SSESF) from the 3+1-dimensional Einstein-scalar-field system.

Solutions to the Einstein-scalar field equations can be represented by a triplet $(\mathcal{M}, \boldsymbol{g}_{\mu\nu}, \phi)$, where $(\mathcal{M}, \boldsymbol{g}_{\mu\nu})$ is a 3+1-dimensional Lorentzian manifold and ϕ a real-valued function on $\mathcal{M}$. The spacetime metric $\boldsymbol{g}_{\mu\nu}$ and the scalar field ϕ satisfy the Einstein-scalar-field system

$$\begin{cases} \boldsymbol{R}_{\mu\nu} - \frac{1}{2}\boldsymbol{g}_{\mu\nu} R = 2\boldsymbol{T}_{\mu\nu}, \\ \nabla^{\mu}\partial_{\mu}\phi = 0, \end{cases} \tag{2-1}$$

where $\boldsymbol{R}_{\mu\nu}$ is the Ricci curvature of $\boldsymbol{g}_{\mu\nu}$, R is the scalar curvature and ∇_{μ} is the covariant derivative given by the Levi-Civita connection on $(\mathcal{M}, \boldsymbol{g})$. The energy–momentum tensor $\boldsymbol{T}_{\mu\nu}$ is given by the scalar field ϕ:

$$\boldsymbol{T}_{\mu\nu} = \partial_{\mu}\phi\,\partial_{\nu}\phi - \tfrac{1}{2}\boldsymbol{g}_{\mu\nu}\,\partial^{\lambda}\phi\,\partial_{\lambda}\phi. \tag{2-2}$$

Assume that the solution $(\mathcal{M}, \boldsymbol{g}_{\mu\nu}, \phi)$ is spherically symmetric, that is, the group SO(3) of three-dimensional rotations acts smoothly and isometrically on $(\mathcal{M}, \boldsymbol{g})$, where each orbit is either a point or is isometric to $\mathbb{S}^2$ with a round metric. The scalar field ϕ is required to be constant on each of the orbits. These assumptions are propagated by (2-1); hence, if $(\mathcal{M}, \boldsymbol{g}_{\mu\nu}, \phi)$ is a Cauchy development, then it suffices to assume spherical symmetry only on the initial data.

The quotient $\mathcal{M}/\,\mathrm{SO}(3)$ gives rise to a 1+1-dimensional Lorentzian manifold with boundary, which we denote by $(\mathcal{Q}, g_{ab})$. The boundary Γ consists of fixed points of the group action. We define the *area radius function* r on $\mathcal{Q}$ to be

$$r := \sqrt{\frac{\text{Area of symmetry sphere}}{4\pi}}$$

and $r = 0$ at Γ. Note that each component of Γ is a timelike geodesic.

We assume that Γ is nonempty and connected, and moreover that there exist *global double null coordinates* (u, v), i.e., a coordinate system (u, v) covering $\mathcal{Q}$ in which the metric takes the form

$$g_{ab}\,\mathrm{d}x^a \cdot \mathrm{d}x^b = -\Omega^2\,\mathrm{d}u \cdot \mathrm{d}v \tag{2-3}$$

for some $\Omega > 0$. We remark that both assumptions are easily justified if $(\mathcal{M}, \boldsymbol{g})$ is a Cauchy development of a spacelike hypersurface homeomorphic to $\mathbb{R}^3$.

The metric $\boldsymbol{g}_{\mu\nu}$ of $\mathcal{M}$ is characterized by Ω and r and takes the form

$$\boldsymbol{g}_{\mu\nu}\,\mathrm{d}x^{\mu} \cdot \mathrm{d}x^{\nu} = -\Omega^2\,\mathrm{d}u \cdot \mathrm{d}v + r^2\,\mathrm{d}s^2_{\mathbb{S}^2}, \tag{2-4}$$

where $\mathrm{d}s^2_{\mathbb{S}^2}$ is the standard line element on the unit sphere $\mathbb{S}^2$. Therefore, we may reformulate the *spherically symmetric Einstein-scalar field system* (SSESF) in terms of the triplet (ϕ, r, Ω) as

$$\begin{cases} r\,\partial_u\partial_v r = -\partial_u r\,\partial_v r - \frac{1}{4}\Omega^2, \\ r^2\,\partial_u\partial_v \log\Omega = \partial_u r\,\partial_v r + \frac{1}{4}\Omega^2 - r^2\,\partial_u\phi\,\partial_v\phi, \\ r\,\partial_u\partial_v\phi = -\partial_u r\,\partial_v\phi - \partial_v r\,\partial_u\phi, \\ 2\Omega^{-1}\,\partial_u r\,\partial_u\Omega = \partial_u^2 r + r(\partial_u\phi)^2, \\ 2\Omega^{-1}\,\partial_v r\,\partial_v\Omega = \partial_v^2 r + r(\partial_v\phi)^2, \end{cases} \tag{SSESF}$$

with the boundary condition $r = 0$ along Γ.

2B. ***Basic assumptions, notations and conventions.*** In this subsection, we introduce the basic assumptions on the base manifold $\mathcal{Q}$, as well as some notations and conventions that will be used in the rest of the paper.

Definition of $\mathcal{Q}$ and $\mathcal{M}$. Denote by $\mathbb{R}^{1+1}$ the 1+1-dimensional Minkowski space, with the standard double null coordinates (u, v). Let $\mathcal{Q}$ be a 1+1-dimensional Lorentzian manifold which is conformally embedded into $\mathbb{R}^{1+1}$ with $\mathrm{d}s^2_{\mathcal{Q}} = -\Omega^2 \, \mathrm{d}u \cdot \mathrm{d}v$. Given a nonnegative function r on $\mathcal{Q}$, we define the set $\Gamma := \{(u, v) \in \mathcal{Q} : r(u, v) = 0\}$, called the *axis of symmetry*. We also define $(\mathcal{M}, \boldsymbol{g}_{\mu\nu})$ to be the 1+3-dimensional Lorentzian manifold with $\mathcal{M} = \mathcal{Q} \times \mathbb{S}^2$ and $\boldsymbol{g}_{\mu\nu}$ given by (2-3); this is to be thought of as the full spacetime before the symmetry reduction. (We refer to Section 2A for the full interpretation.)

Assumptions on the conformal geometry of $\mathcal{Q}$. We assume that $\Gamma \subset \mathcal{Q}$ is a connected set which is the image of a future-directed timelike curve emanating from the point $(1, 1)$. We also assume that $C_1 \subset \mathcal{Q}$, where

$$C_1 = \{(u, v) \in \mathbb{R}^{1+1} : u = 1, 1 \leq v < \infty\}.$$

Furthermore, $\mathcal{Q}$ is assumed to be the domain of dependence of Γ and C_1 to the future, in the sense that every causal curve in $\mathcal{Q}$ has its past endpoint on either Γ or C_1.

Notations for the conformal geometry of $\mathcal{Q}$. Denote by C_u (resp. $\underline{C}_v$) the constant-u (resp. constant-v) curve in $\mathcal{Q}$. Note that these are null curves in $\mathcal{Q}$.

Given $(u_0, v_0) \in \mathcal{Q}$, we define the *domain of dependence* of the line segment $C_{u_0} \cap \{v \leq v_0\}$, denoted by $\mathcal{D}(u_0, v_0)$, to be the set of all points $p \in \mathcal{Q}$ such that all past-directed causal curves passing p intersect $\Gamma \cup (C_{u_0} \cap \{v \leq v_0\})$ plus the line segment $(C_{u_0} \cap \{v \leq v_0\})$ itself.

Also, we define the *future null infinity* $\mathcal{I}^+$ to be the set of ideal points $(u, +\infty)$ such that $\sup_{C_u} r = \infty$.

Integration over null curves. Whenever we integrate over a subset of C_u or $\underline{C}_v$, we will use the standard line element $\mathrm{d}v$ or $\mathrm{d}u$ for the integrals, i.e.,

$$\int_{\underline{C}_v \cap \{u_1 \leq u \leq u_2\}} f = \int_{u_1}^{u_2} f(u', v) \, \mathrm{d}u',$$
$$\int_{C_u \cap \{v_1 \leq v \leq v_2\}} f = \int_{v_1}^{v_2} f(u, v') \, \mathrm{d}v'.$$

Functions of bounded variation. Unless otherwise specified, functions of bounded variation (BV) considered in this paper will be assumed to be right-continuous. By convention,

$$\partial_v f \, \mathrm{d}v \quad \text{or} \quad \partial_v f$$

will refer to the distributional derivative of f, which is a finite signed measure, and

$$|\partial_v f| \, \mathrm{d}v \quad \text{or} \quad |\partial_v f|$$

will denote the total variation measure. Unless otherwise specified, these measures will be evaluated on intervals of the form $(v_1, v_2]$. Thus, according to our conventions,

$$\int_{v_1}^{v_2} \partial_v f(v)\, \mathrm{d}v = f(v_2) - f(v_1),$$

$$\int_{v_1}^{v_2} |\partial_v f(v)|\, \mathrm{d}v = \mathrm{TV}_{(v_1,v_2]}[f].$$

New variables. We introduce the following notation for the directional derivatives of r:

$$\lambda := \frac{\partial r}{\partial v}, \qquad \nu := \frac{\partial r}{\partial u},$$

The *Hawking mass* $m(u, v)$ is defined by the relation

$$1 - \frac{2m}{r} = \partial^a r \partial_a r = -4\Omega^{-2}\partial_u r\, \partial_v r. \tag{2-5}$$

For a solution to (SSESF), the quantity m possesses useful monotonicity properties (see Lemma 4.3), which will be one of the key ingredients of our analysis. We define the *mass ratio* to be

$$\mu := \frac{2m}{r}.$$

We also define the *Bondi mass* on C_u by $M(u) := \lim_{v\to\infty} m(u, v)$, provided the limit exists. The Bondi mass $M_i := M(1) = \lim_{v\to\infty} m(1, v)$ on the initial curve C_1 is called the *initial Bondi mass*.

2C. ***Reformulation in terms of the Hawking mass.*** The Hawking mass as defined in (2-5) turns out to obey useful monotonicity (see Section 4B). We therefore reformulate (SSESF) in terms of m and eliminate Ω. Notice that, by (2-4) and (2-5), the metric is determined by r and m.

We say that (ϕ, r, m) *on* $\mathcal{Q}$ *is a solution to* (SSESF) if the following equations hold:

$$\partial_u \lambda = \frac{\mu}{(1-\mu)r}\lambda\nu, \quad \text{and} \quad \partial_v \nu = \frac{\mu}{(1-\mu)r}\lambda\nu, \tag{2-6}$$

$$2\nu\, \partial_u m = (1-\mu) r^2 (\partial_u \phi)^2 \quad \text{and} \quad 2\lambda\, \partial_v m = (1-\mu) r^2 (\partial_v \phi)^2, \tag{2-7}$$

$$\partial_u \partial_v (r\phi) = \frac{\mu\lambda\nu}{(1-\mu)r}\phi, \tag{2-8}$$

and, moreover, the following boundary conditions hold:

$$r = 0 \quad \text{and} \quad m = 0 \quad \text{along } \Gamma.$$

We remark that, using (2-6), the wave equation (2-8) for ϕ may be rewritten in either of the following two equivalent forms:

$$\partial_u(\partial_v(r\phi)) = (\partial_u \lambda)\phi, \tag{2-8$'$}$$

$$\partial_v(\partial_u(r\phi)) = (\partial_v \nu)\phi. \tag{2-8$''$}$$

2D. ***Choice of coordinates.*** Note that $\mathcal{Q}$ is ruled by the family of null curves C_u. Since a null curve C_u with $u \neq 1$ cannot intersect C_1, its past endpoint must be on Γ. Therefore, our assumptions so far impose the following conditions on the double null coordinates (u, v) on $\mathcal{Q}$: u is constant on each future-directed null curve emanating from Γ and v is constant on each conjugate null curve. However, these conditions are insufficient to give a unique choice of a coordinate system, as the system (SSESF) and assumptions so far are invariant under the change of coordinates

$$u \mapsto U(u), \quad v \mapsto V(v), \quad U(1) = V(1) = 1$$

for any strictly increasing functions U and V. To remove this ambiguity, we fix the choice of the coordinate system, once and for all, as follows.

We first fix v on C_1, relating it to the function r. Specifically, we will require that $v = 2r + 1$ on C_1, which in particular implies that

$$\lambda(1, v) = \tfrac{1}{2}. \tag{2-9}$$

Next, in order to fix u, we prescribe u such that $\Gamma = \{(u, v) : u = v\}$. To do so, for every outgoing null curve C in $\mathcal{Q}$, follow the incoming null curve to the past starting from $C \cap \Gamma$ until the point p_* where it intersects the initial curve C_1. We then define the u-coordinate value for C to be the v-coordinate value for p_*.

Under this coordinate choice, $\mathcal{D}(u_0, v_0)$ may be expressed as

$$\mathcal{D}(u_0, v_0) = \{(u, v) \in \mathcal{Q} : u \in [u_0, v_0],\, v \in [u, v_0]\}.$$

Moreover, if r and ϕ are sufficiently regular functions on $\mathcal{Q}$, then our coordinate choice leads to

$$\lim_{v \to u+} (\lambda + \nu)(u, v) = \lim_{u \to v-} (\lambda + \nu)(u, v) = 0,$$
$$\lim_{v \to u+} (\partial_v + \partial_u)(r\phi)(u, v) = \lim_{u \to v-} (\partial_v + \partial_u)(r\phi)(u, v) = 0.$$

These conditions will be incorporated into precise formulations of solutions to (SSESF) with limited regularity in the following subsection.

2E. ***Characteristic initial value problem.*** We will study the characteristic initial value problem for (SSESF) with data prescribed on C_1 under quite general assumptions on the regularity. In this subsection, we give precise formulations of initial data and solutions to (SSESF) to be considered.

We begin with a discussion on the constraint imposed by (SSESF) (more precisely, (2-6)–(2-8)) on initial data for (ϕ, r, m). In fact, the constraint is very simple, thanks to the fact that they are prescribed on a characteristic (i.e., null) curve C_1. Once we prescribe ϕ on C_1, the coordinate condition (2-9) dictates the initial values of r, and the initial values of m are then determined by the constraint (2-7) along the v-direction as well as the boundary condition $m(1, 1) = 0$. In other words, initial data for (ϕ, r, m) possess only one degree of freedom, namely the prescription of a single real-valued function $\phi(1, v)$ or, equivalently, $\partial_v(r\phi)(1, v)$.

Following [Christodoulou 1993], we say that an initial data set for (ϕ, r, m) is of *bounded variation* (BV) if $\partial_v(r\phi)(1, \cdot)$ is a (right-continuous) BV function on $[1, \infty)$ with finite total variation on $(1, \infty)$. We also define the notion of *solution of bounded variation* to (SSESF) as follows:

Definition 2.1. A solution (ϕ, r, m) to (SSESF) is called a *solution of bounded variation* on $\mathcal{Q}$ if, on every compact domain of dependence $\mathcal{D}(u_0, v_0)$, the following conditions hold:

(1) $\sup_{\mathcal{D}(u_0,v_0)}(-\nu) < \infty$ and $\sup_{\mathcal{D}(u_0,v_0)} \lambda^{-1} < \infty$.

(2) λ is BV on each $C_u \cap \mathcal{D}(u_0, v_0)$ uniformly in u, and ν is BV on each $\underline{C}_v \cap \mathcal{D}(u_0, v_0)$ uniformly in v.

(3) For each a with $(a, a) \in \Gamma$,

$$\lim_{\epsilon\to 0+} (\nu + \lambda)(a, a + \epsilon) = 0.$$

(4) ϕ is an absolutely continuous function on each $C_u \cap \mathcal{D}(u_0, v_0)$ with total variation bounded uniformly in u, and also an absolutely continuous function on each $\underline{C}_v \cap \mathcal{D}(u_0, v_0)$ with total variation bounded uniformly in v.

(5) For each a with $(a, a) \in \Gamma$,

$$\lim_{\epsilon\to 0} \sup_{0<\delta\le\epsilon} \mathrm{TV}_{\{a-\delta\}\times(a-\delta,a)}[\phi] = 0, \qquad \lim_{\epsilon\to 0} \sup_{0<\delta\le\epsilon} \mathrm{TV}_{(a-\epsilon,a-\delta)\times\{a-\delta\}}[\phi] = 0,$$
$$\lim_{\epsilon\to 0} \sup_{0<\delta\le\epsilon} \mathrm{TV}_{(a,a+\delta)\times\{a+\delta\}}[\phi] = 0, \qquad \lim_{\epsilon\to 0} \sup_{0<\delta\le\epsilon} \mathrm{TV}_{\{a+\delta\}\times(a+\delta,a+\epsilon)}[\phi] = 0.$$

(6) $\partial_v(r\phi)$ is BV on each $C_u \cap \mathcal{D}(u_0, v_0)$ uniformly in u, and $\partial_u(r\phi)$ is BV on each $\underline{C}_v \cap \mathcal{D}(u_0, v_0)$ uniformly in v.

(7) For each a with $(a, a) \in \Gamma$,

$$\lim_{\epsilon\to 0+} (\partial_v(r\phi) + \partial_u(r\phi))(a, a + \epsilon) = 0.$$

We also consider more regular data and solutions, as follows. We say that an initial data set for (ϕ, r, m) is C^1 if $\partial_v(r\phi)(1, \cdot)$ is C^1 on $[1, \infty)$ with $\sup_{C_1} |\partial_v^2(r\phi)| < \infty$. In the following definition, we define the corresponding notion of a C^1 *solution* to (SSESF).

Definition 2.2. A solution (ϕ, r, m) to (SSESF) is called a C^1 *solution* on $\mathcal{Q}$ if the following conditions hold on every compact domain of dependence $\mathcal{D}(u_0, v_0)$:

(1) $\sup_{\mathcal{D}(u_0,v_0)}(-\nu) < \infty$ and $\sup_{\mathcal{D}(u_0,v_0)} \lambda^{-1} < \infty$.

(2) λ and ν are C^1 on $\mathcal{D}(u_0, v_0)$.

(3) For each a with $(a, a) \in \Gamma$,

$$\lim_{\epsilon\to 0+} (\nu + \lambda)(a, a + \epsilon) = \lim_{\epsilon\to 0+} (\nu + \lambda)(a - \epsilon, a) = 0.$$

(4) $\partial_v(r\phi)$ and $\partial_u(r\phi)$ are C^1 on $\mathcal{D}(u_0, v_0)$.

(5) For each a with $(a, a) \in \Gamma$,

$$\lim_{\epsilon \to 0+} (\partial_v(r\phi) + \partial_u(r\phi))(a, a+\epsilon) = \lim_{\epsilon \to 0+} (\partial_v(r\phi) + \partial_v(r\phi))(a-\epsilon, a) = 0.$$

Remark 2.3. By [Christodoulou 1993, Theorem 6.3], a BV initial data set leads to a unique BV solution to (SSESF) on $\{(u, v) : 1 \le u \le 1+\delta, v \ge u\}$ for some $\delta > 0$. If the initial data set is furthermore C^1, then it is not difficult to see that the corresponding solution is also C^1 (persistence of regularity). In fact, this statement follows from the arguments in Section 7; see, in particular, the proof of Lemma 7.7.

2F. *Local scattering in BV and asymptotic flatness.* We are now ready to formulate the precise notion of *locally BV scattering solutions* to (SSESF), which is the class of solutions that we consider. In particular, for this class of solutions, we make a priori assumptions on its global geometry.

Definition 2.4. We say that a BV solution (ϕ, r, m) to (SSESF) is *locally scattering in the bounded variation norm (BV)*, or a *locally BV scattering solution*, if the following conditions hold:

(1) *Future completeness of radial null geodesics*: Every incoming null geodesic in $\mathcal{Q}$ has its future endpoint on Γ, and every outgoing null geodesic in $\mathcal{Q}$ is infinite towards the future in the affine parameter. Moreover, there exists a global system of null coordinates (u, v) and $\mathcal{Q}$ is given by

$$\mathcal{Q} = \{(u, v) : u \in [1, \infty), v \in [u, \infty)\}. \tag{2-10}$$

(2) *Vanishing final Bondi mass*:

$$M_f := \lim_{u \to \infty} M(u) = 0. \tag{2-11}$$

(3) *Scattering in BV in a compact r-region*: There exists $R > 0$ such that, for the region $\mathcal{Q}_{\mathrm{cpt}} := \{(u, v) \in \mathcal{Q} : r(u, v) \le R\}$, we have

$$\int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v^2(r\phi)| \to 0 \quad \text{and} \quad \int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v \log \lambda| \to 0 \tag{2-12}$$

as $u \to \infty$.

Several remarks concerning Definition 2.4 are in order.

Remark 2.5. In fact, the condition (2-10) is a consequence of future completeness of radial null geodesics and the preceding assumptions. To see this, first recall our assumption that $C_1 = \{(u, v) : u = 1, v \in [1, \infty)\}$. Hence from our choice of the coordinate u and future completeness of incoming radial null geodesics, it follows that the range of u must be $[1, \infty)$. Furthermore, for each $u \in [1, \infty)$, the range of v on C_u is $[u, \infty)$ by the future completeness of outgoing radial null geodesics and Definition 2.1. More precisely, future completeness of C_u implies that it can be continued past $\{u\} \times [u, v_0]$ as long as $\int_u^{v_0} \Omega^2 \, dv < \infty$, and Definition 2.1 implies[5] that $\Omega^2 = -4\lambda\nu/(1-\mu)$ indeed remains bounded on $\{u\} \times [u, v_0]$ for every finite v_0.

[5] We refer to the proof of Proposition 5.3 below for details of estimating $-\nu/(1-\mu)$ in terms of assumptions on ϕ, $\partial_v(r\phi)$ and λ.

Remark 2.6. For more regular (e.g., C^1) asymptotically flat solutions, (1) and (2) in Definition 2.4 may be replaced by a single equivalent condition, namely requiring the full spacetime $(\mathcal{M}, \boldsymbol{g})$ to be *future casually geodesically complete* as a 1+3-dimensional Lorentzian manifold. In particular, (2) follows from the deep work [Christodoulou 1987], in which it was proved that if $M_f > 0$ then the space-time necessarily contains a black hole and thus is not future causally geodesically complete.

Remark 2.7. As we will see in the proof, there exists a universal $\tilde{\epsilon}_0$ such that (3) in Definition 2.4 can be replaced by the weaker requirement that there exists $R > 0$ and $U > 0$ such that

$$\int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v^2(r\phi)| \le \tilde{\epsilon}_0 \quad \text{and} \quad \int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v \log \lambda| \le \tilde{\epsilon}_0$$

for $u \ge U$. To simplify the exposition, we will omit the proof of this improvement.

Remark 2.8. For a sufficiently regular, asymptotically flat solution to (SSESF), Definition 2.4(1) is equivalent to requiring that the conformal compactification of $\mathcal{Q}$ is depicted by a Penrose diagram as in Figure 1 (in the introduction). For more discussion on Penrose diagrams, we refer the reader to [Dafermos and Rodnianski 2005, Appendix C; Kommemi 2013]. In fact, this equivalence follows easily from the classification of all possible Penrose diagrams for the system (SSESF) given in the latter reference.

We also define the precise notion of *asymptotic flatness* for initial data with BV or C^1 regularity. As we shall see soon, in the main theorems, the rate of decay for the initial data, measured in r, is directly related to the rate of decay of the corresponding solution in both u and r.

Definition 2.9 (asymptotic flatness of order ω' in BV or C^1). For $\omega' > 1$, we make the following definition:

(1) We say that an initial data set is *asymptotically flat of order ω' in BV* if $\partial_v(r\phi)(1, \cdot) \in \mathrm{BV}[1, \infty)$ and there exists $\mathcal{I}_1 > 0$ such that

$$\sup_{C_1} (1+r)^{\omega'} |\partial_v(r\phi)| \le \mathcal{I}_1 < \infty. \tag{2-13}$$

(2) We say that an initial data set is *asymptotically flat of order ω' in C^1* if $\partial_v(r\phi)(1, \cdot) \in C^1[1, \infty)$ and there exists $\mathcal{I}_2 > 0$ such that

$$\sup_{C_1} (1+r)^{\omega'} |\partial_v(r\phi)| + \sup_{C_1} (1+r)^{\omega'+1} |\partial_v^2(r\phi)| \le \mathcal{I}_2 < \infty. \tag{2-14}$$

Remark 2.10. The initial Bondi mass $M_i := \lim_{v\to\infty} m(1, v)$ can be easily bounded by $C\mathcal{I}_1^2$; see Lemma 4.5.

Remark 2.11. Observe that both conditions imply that $(r\phi)(1, v)$ tends to a finite limit as $v \to \infty$; in particular, $\lim_{v\to\infty} \phi(1, v) = 0$. This serves to fix the gauge freedom $(\phi, r, m) \mapsto (\phi + c, r, m)$ for solutions to (SSESF).

3. Main results

3A. *Main theorems.* With the definitions of locally BV scattering solutions and asymptotically flat initial data, we now have the necessary means to state the main theorems of this paper. Roughly speaking, these theorems say that locally BV scattering solutions with asymptotically flat initial data exhibit quantitative decay rates, which can be read off from the rate ω' in Definition 2.9. The first theorem is for initial data and solutions in BV.

Theorem 3.1 (main theorem in BV). *Let (ϕ, r, m) be a locally BV scattering solution to (SSESF) with asymptotically flat initial data of order ω' in BV. Then, for $\omega := \min\{\omega', 3\}$, there exists a constant $A_1 > 0$ such that*

$$|\phi| \le A_1 \min\{u^{-\omega}, r^{-1}u^{-(\omega-1)}\}, \tag{3-1}$$

$$|\partial_v(r\phi)| \le A_1 \min\{u^{-\omega}, r^{-\omega}\}, \tag{3-2}$$

$$|\partial_u(r\phi)| \le A_1 u^{-\omega}. \tag{3-3}$$

The second theorem is for initial data and solutions in C^1.

Theorem 3.2 (main theorem in C^1). *Let (ϕ, r, m) be a locally BV scattering solution to (SSESF) with asymptotically flat initial data of order ω' in C^1. Then, in addition to the bounds (3-1)–(3-3), there exists a constant $A_2 > 0$ such that, for $\omega := \min\{\omega', 3\}$,*

$$|\partial_v^2(r\phi)| \le A_2 \min\{u^{-(\omega+1)}, r^{-(\omega+1)}\}, \tag{3-4}$$

$$|\partial_u^2(r\phi)| \le A_2 u^{-(\omega+1)}, \tag{3-5}$$

$$|\partial_v \lambda| \le A_2 \min\{u^{-3}, r^{-3}\}, \tag{3-6}$$

$$|\partial_u \nu| \le A_2 u^{-3}. \tag{3-7}$$

Some remarks regarding the main theorems are in order.

Remark 3.3. Notice that in Theorem 3.2 the decay rates for $\partial_v\lambda$ and $\partial_u\nu$ are independent of the order ω' of asymptotic flatness of the initial data. This is because the scalar field terms enter the equations for $\partial_u\partial_v \log\lambda$ and $\partial_v\partial_u \log\nu$ quadratically (see equations (4-6) and (4-7)) and thus, as long as $\omega' > 1$, their contributions to the decay rates of $\partial_v\lambda$ and $\partial_u\nu$ are of lower order than the term involving the Hawking mass.

Remark 3.4. By Remark 2.3, a C^1 initial data set gives rise to a C^1 solution. Hence Remark 2.6 applies, and conditions (1)–(2) of Definition 2.4 may be replaced by a single equivalent condition of *future causal geodesic completeness* of $(\mathcal{M}, \boldsymbol{g})$ in the case of Theorem 3.2.

Remark 3.5. In general, the constants A_1 and A_2 depend not only on the size of the initial data (e.g., $\mathcal{I}_1$ and $\mathcal{I}_2$), but rather on the full profile of the solution. Nevertheless, for the special case of small initial total variation of $\partial_v(r\phi)$, A_1 and A_2 *do* depend only on the size of the initial data; see Section 3C.

Remark 3.6. If the initial data also verify higher-derivative estimates, then the techniques in proving Theorems 3.1 and 3.2 also allow us to derive decay bounds for higher-order derivatives. The proof of the higher-derivative decay estimates is in fact easier than the proofs of the first- and second-derivative decay bounds, since we have already obtained sufficiently strong control of the scalar field and the geometry of the spacetime. We will omit the details.

Remark 3.7. The decay rates that we obtain in these variables imply, as immediate corollaries, decay rates for $\partial_v \phi$, $\partial_u \phi$, etc. See Corollaries 6.9 and 7.13.

Remark 3.8. The decay rates in the main theorems are measured with respect to the double null coordinates (u, v) normalized at the initial curve and the axis Γ as in Section 2D. To measure the decay rate along null infinity, one can alternatively normalize the u-coordinate[6] by requiring $\partial_u r = -\frac{1}{2}$ at future null infinity. As we will show in Section 5C, for the class of spacetimes considered in this paper, the decay rates with respect to this new system of null coordinates are the same up to a constant multiplicative factor.

Remark 3.9. In view of Remark 2.7, the assumption of local BV scattering can be replaced by the *boundedness* of the *subcritical* quantities:

$$\int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v^2 (r\phi)|^p \leq C \quad \text{and} \quad \int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v \log \lambda|^p \leq C \quad \text{for } p > 1.$$

This is because, for fixed $\tilde{\epsilon}_0$, one can choose R to be sufficiently small (depending on C) and apply Hölder's inequality to ensure that

$$\int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v^2 (r\phi)| \leq \tilde{\epsilon}_0 \quad \text{and} \quad \int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v \log \lambda| \leq \tilde{\epsilon}_0.$$

Remark 3.10. We also notice that the proof of our main theorem can be carried out in an identical manner for locally BV scattering solutions arising from asymptotically flat *Cauchy data*. More precisely, we can consider initial data given on a Cauchy hypersurface,

$$v = f(u) \quad \text{with} \quad C^{-1} \leq -f'(u) \leq C,$$

and satisfying the constraint equation together with the following bounds on the initial data:

$$(1+r)|\phi| + (1+r)^{\omega'} \big(|\partial_v (r\phi)| + |\partial_u (r\phi)| + \big|\lambda - \tfrac{1}{2}\big| + \big|\nu + \tfrac{1}{2}\big| \big) \leq \tilde{\mathcal{I}}_1$$

and

$$(1+r)^{\omega'+1} (|\partial_v^2 (r\phi)| + |\partial_u^2 (r\phi)| + |\partial_v \log \lambda| + |\partial_u \log \nu|) \leq \tilde{\mathcal{I}}_2.$$

Then, if we assume in addition that the spacetime is locally BV scattering to the future, the conclusions of Theorems 3.1 and 3.2 hold.

Remark 3.11. Our main theorems can also be viewed as results on upgrading qualitative decay to quantitative decay estimates. Such problems have been widely studied in the *linear* setting (without the assumption on spherical symmetry) on nontrapping asymptotically flat Lorentzian manifolds [Dafermos and Rodnianski 2010; Tataru 2013; Metcalfe et al. 2012], as well as for the obstacle problem on Minkowski

[6]In particular, this normalization is used in [Dafermos and Rodnianski 2005] for the black hole case. By changing the null coordinates, we can thus more easily compare the decay rates in our setting with those in [Dafermos and Rodnianski 2005].

space [Morawetz 1975; Strauss 1975]. In the *nonlinear* setting, we mention the work of Christodoulou and Tahvildar-Zadeh [1993], who studied the energy critical, 2-dimensional, spherically symmetric wave map system and proved asymptotic decay for the solution and its derivatives.

3B. *BV scattering and the blow-up at infinity scenario.* The condition of local BV scattering in the main theorems follows if one rules out, a priori, a blow-up at infinity scenario. More precisely, we say that a solution blows up at infinity if some scale-invariant spacetime norms are infinite, as follows:

Definition 3.12. Let (ϕ, r, m) be a BV solution to (SSESF) such that condition (1) of Definition 2.4 holds. We say that the solution *blows up at infinity* if at least one of the following holds:

(1) $\sup \lambda_\Gamma^{-1} = \infty$, where $\lambda_\Gamma(u) := \lim_{v\to u+} \lambda(u, v)$.

(2) $\int_1^\infty \int_u^\infty |\partial_v\lambda\, \partial_u\phi - \partial_u\lambda\, \partial_v\phi| \,\mathrm{d}v\, \mathrm{d}u = \infty$.

(3) $\int_1^\infty \int_u^\infty |\partial_u\phi\, \partial_v(\nu^{-1}\partial_u(r\phi)) - \partial_v\phi\, \partial_u(\nu^{-1}\partial_u(r\phi))| \,\mathrm{d}v\, \mathrm{d}u = \infty$.

Remark 3.13. We do not prove in the paper the existence or nonexistence of solutions that blow up at infinity. This is analogous to the blow-up at infinity scenarios which have recently been constructed in some simpler *semilinear*, *critical* wave equations [Donninger and Krieger 2013].

It follows from our main theorem that, if a solution does not blow up at infinity, it obeys quantitative decay estimates. More precisely, we have:

Theorem 3.14 (dichotomy between blow-up at infinity and BV scattering). *Let (ϕ, r, m) be a BV solution to (SSESF) such that condition* (1) *of Definition 2.4 holds. Assume furthermore that the initial data for (ϕ, r, m) obey the condition*[7] $\lim_{v\to\infty} \phi(1, v) = 0$ *and*

$$\int_{C_1} |\partial_v^2(r\phi)| \,\mathrm{d}v + \sup_{C_1} |\partial_v(r\phi)| < \infty. \tag{3-8}$$

Then, either:

(1) *the solution blows up at infinity*; *or*

(2) *the solution is globally BV scattering*, *in the sense that conditions* (2) *and* (3) *of Definition 2.4 hold with $R = \infty$.*

This theorem is established in Section 8. It then follows from our main theorems (Theorems 3.1 and 3.2) that, if a BV solution does not blow up at infinity and possesses asymptotically flat initial data, then it obeys quantitative decay estimates.

3C. *Refinement in the small data in BV case.* By a theorem of Christodoulou [1993], the maximal development of data with small BV norms does not blow up at infinity. The previous theorem applies, and thus the corresponding solution is globally BV scattering, in the sense described in Theorem 3.14. Moreover, a closer inspection of the proof of the main theorems reveals that the following stronger conclusion holds in this case:

[7] By Remark 2.11, this is the only condition on $\lim_{v\to\infty} \phi(1, v)$ which is consistent with asymptotic flatness.

Theorem 3.15 (sharp decay for data with small BV norm). *There exists a universal $\epsilon_0 > 0$ such that, for $0 < \epsilon \le \epsilon_0$, the following statements hold*:

(1) *If the initial data set is asymptotically flat of order ω' in BV and*

$$\int_{C_1} |\partial_v^2(r\phi)| < \epsilon,$$

then the maximal development (ϕ, r, m) is globally BV scattering, in the sense that Definition 2.4 holds with arbitrarily large $R > 0$. Moreover, it satisfies estimates (3-1)–(3-3) with $A_1 \le C_{\mathcal{I}_1}(\mathcal{I}_1 + \epsilon)$.

Here (and similarly in (2)), we use the convention that $C_{\mathcal{I}_1}$ depends on $\mathcal{I}_1$ in a nondecreasing fashion.[8]

(2) *If, in addition, the initial data set is asymptotically flat of order ω' in C^1, then the maximal development also satisfies (3-4)–(3-7) with $A_2 \le C_{\mathcal{I}_2}(\mathcal{I}_2 + \epsilon)$.*

The point of this theorem is that we only need the initial total variation to be small in order to conclude pointwise decay rates; in particular, $\mathcal{I}_1$ and $\mathcal{I}_2$ can be arbitrarily large. In this sense, Theorem 3.15 generalizes both the small BV global well-posedness theorem [Christodoulou 1993, Theorem 6.2] and the earlier small data scattering theorem [Christodoulou 1986] for data that are small in a weighted C^1 norm. A proof of this theorem will be sketched in Section 9.

3D. *Optimality of the decay rates.* Our main theorems show upper bounds for the decay rates of the scalar field ϕ and its derivatives both towards null infinity (i.e., in r) and along null infinity (in u). For $\omega' = \omega < 3$, if the decay rate of the initial data towards null infinity also satisfies a lower bound, then we can show that both the r and u decay rates in Theorem 3.1 are saturated. More precisely:

Theorem 3.16 (sharpness of $t^{-\omega}$ tail for $1 < \omega < 3$). *Let $1 < \omega < 3$. Suppose, in addition to the assumptions of Theorem 3.1, that there exists $V \ge 1$ such that the initial data set satisfies the lower bound*

$$r^{\omega}\, \partial_v(r\phi)(1, v) \ge L > 0$$

for $v \ge V$. Then there exists a constant $L_\omega > 0$ such that

$$\partial_v(r\phi)(u, v) \ge L_\omega \min\{r^{-\omega}, u^{-\omega}\},$$
$$-\partial_u(r\phi)(u, v) \ge L_\omega u^{-\omega},$$

for u sufficiently large.

Remark 3.17. One can also infer the sharpness of the decay of ϕ from that of its derivatives. We will omit the details.

This theorem will be proved in Section 10A. In fact, the proof of this theorem is similar to the proof of the upper bounds in the first main theorem (Theorem 3.1). We will show that after restricting to u sufficiently large, the initial lower bound propagates and the nonlinear terms only give lower-order contributions. Notice also that the analogous statement is false for $\omega' \ge 3$, since the nonlinear terms may dominate the contribution of the initial data.

[8] In particular, for $\mathcal{I}_1$ sufficiently small, we have the estimate $A_1 \le C(\mathcal{I}_1 + \epsilon)$ for some absolute constant C.

For $\omega' \geq 3$, we can show that the decay rates in Theorem 3.1 are sharp, in the following sense:

Theorem 3.18 (sharpness of t^{-3} tail). *For arbitrarily small $\epsilon > 0$, there exists a locally BV scattering solution (ϕ, r, m) to (SSESF) which satisfies the following properties*:

(1) $\partial_v(r\phi)(1, v)$ *is smooth, compactly supported in the v-variable and has total variation less than ϵ*:

$$\int_{C_1} |\partial_v^2(r\phi)| < \epsilon.$$

(2) *There exists a constant $L_3 > 0$ such that*

$$\partial_v(r\phi)(u, v) \geq L_3 \min\{r^{-3}, u^{-3}\},$$
$$-\partial_u(r\phi)(u, v) \geq L_3 u^{-3},$$

for u sufficiently large.

To prove Theorem 3.18, we will first establish a sufficient condition for the desired lower bounds in terms of (nonvanishing of) a single real number $\mathfrak{L}$, which is computed from information at the null infinity. This result (Lemma 10.1) is proved using the decay rates proved in the main theorems, and we believe it might be of independent interest. In Section 10C, we will complete the proof of Theorem 3.18, by constructing an initial data set for which $\mathfrak{L}$ can be bounded away from zero. This can be achieved by showing that the solution is close to that of a corresponding linear problem, controlling the error terms after taking $\epsilon > 0$ to be sufficiently small and using Theorem 3.15.

3E. ***Strategy of the proof of the main theorems.*** Roughly speaking, the proof of decay of ϕ and its derivatives can be split into three steps. In the first two steps, we control the incoming part[9] of the derivatives of the scalar field and metric components, that is, $\partial_v(r\phi)$, $\partial_v^2(r\phi)$ and $\partial_v\lambda$. To this end, we split the spacetime into the exterior region $\mathcal{Q}_{\text{ext}} := \{(u, v) \in \mathcal{Q} : v \geq 3u\}$ and the interior region $\mathcal{Q}_{\text{int}} := \{(u, v) \in \mathcal{Q} : v \leq 3u\}$. In the first step, we control the incoming part of the solution in the exterior region. In this region, we have $r \gtrsim v, u$, thus the negative r-weights in the equations give the required decay of ϕ and its derivatives. We then prove bounds in the interior region in the second step. Here, we exploit certain (nonquantitative) smallness in the spacetimes quantities as $u \to \infty$, given by the assumption of local BV scattering, to propagate the decay estimates from the exterior region to the interior region all the way up to the axis. Finally, in the third step, we control the outgoing part of the solution, that is, $\partial_u(r\phi)$, $\partial_u^2(r\phi)$ and $\partial_u\nu$, by showing that the decay bounds that we have proved along the axis can be propagated in the outgoing direction.

We remind the readers that the above sketch is only a heuristic argument and is not true if taken literally. In particular, in order to carry out this procedure we need to first show that the local BV scattering assumption provides some control over the spacetime geometry. As we will show below, the estimates are derived in slightly different fashions for the first and second derivatives of $r\phi$. We note in particular that carrying out this general scheme relies heavily on the analytic structure of the Einstein-scalar field equations, including the monotonicity properties as well as the null structure of the (renormalized) equations.

[9] We call these variables "incoming" because they obey a transport equation in the ∂_u-direction.

3E1. *Estimates for first derivatives of $r\phi$.* To obtain decay bounds for the first derivatives of $r\phi$, we will rely on the wave equation

$$\partial_u \partial_v (r\phi) = \frac{2m\lambda\nu}{(1-\mu)r^2}\phi.$$

Notice that, when we solve for the incoming radiation $\partial_v(r\phi)$ using this as a transport equation in u, the right-hand side does not depend explicitly on the outgoing radiation $\partial_u(r\phi)$. Instead, the right-hand side consists of terms that are either lower order (in terms of derivatives) or satisfy a certain monotonicity property.

In particular, this equation shows that, as long as ϕ can be controlled, we can estimate $\partial_v(r\phi)$ by integrating along the incoming u direction. On the other hand, we can also control ϕ once a bound on $\partial_v(r\phi)$ is known, by integrating along the outgoing v-direction.

To achieve the desired decay rates for ϕ, $\partial_v(r\phi)$ and $\partial_u(r\phi)$, we follow the three steps outlined above:

(1) Bounds[10] for $\partial_v(r\phi)$ and ϕ in $v \geq 3u$: In the exterior region, we have $r \gtrsim u, v$; it is therefore sufficient to prove the decay in r. First, we prove that $\sup_{C_u}(1+r)\phi$ is bounded. This is achieved in a compact region by continuity of the solution[11] and in the region of large r by integrating $\partial_v(r\phi)$ in the outgoing direction from the compact region. Since $\partial_v(r\phi)$ can in turn be controlled by ϕ, we get the desired bound. To improve over this bound we define

$$\mathcal{B}_1(U) := \sup_{u\in[1,U]} \sup_{C_u} (u^{\omega}|\phi| + r u^{\omega-1}|\phi|)$$

and show via the wave equation that

$$r^{\omega}|\partial_v(r\phi)| \leq C(u_1) + \epsilon(u_1)\mathcal{B}_1(U),$$

where $\epsilon \to 0$ as $u_1 \to \infty$. This gives the optimal decay rate for $\partial_v(r\phi)$ in the exterior region, up to an arbitrarily small loss, which can be estimated once $\mathcal{B}_1(U)$ can be controlled.

(2) Bounds for $\partial_v(r\phi)$ and ϕ in $v \leq 3u$: For the decay of the first derivatives, the interior region $\{v \leq 3u\}$ is further divided into the intermediate region $\{r \geq R\}$ and the compact region $\{r \leq R\}$. In these two regions, the r-weight in the equation is not sufficient to give the sharp decay rate. Instead, we start from the decay rate $\partial_v(r\phi)$ obtained in the first step in the exterior region and propagate this decay estimate inwards. To achieve this, we need to show that $\int 2m\lambda\nu/((1-\mu)r^2)$ is small when u is sufficiently large.

(2a) $r \geq R$ and $v \leq 3u$: In the intermediate region, where we still have a lower bound on r, the required smallness is given by the *qualitative* information that the Hawking mass approaches 0. Thus, from some large time onwards, $\int 2m\lambda\nu/((1-\mu)r^2)$ becomes sufficiently small and we can integrate the wave equation directly to obtain the desired decay bounds.

[10] The estimates in this region are similar to the corresponding bounds for the black hole case in [Dafermos and Rodnianski 2005]. There, it was observed that the quantity $\partial_v(r\phi)$, which Dafermos and Rodnianski called an almost Riemann invariant, verifies an equation such that the right-hand side has useful weights in r and give the desired decay rates.

[11] In particular, since we are simply using compactness, the constants in Theorem 3.1 depend not only on the size of the initial data.

(2b) $r \leq R$ and $v \leq 3u$: In this region, we use the local BV scattering assumption to show that $\int_{\{r\leq R\}} 2m\lambda\nu/((1-\mu)r^2) \to 0$ as $u\to\infty$. This smallness allows us to propagate the decay estimates from the curve $r = R$ to the region $r < R$. At this point, we can also recover the control for $\mathcal{B}_1(U)$ and close the estimates in step (1). This allows us to derive all the optimal decay rates for ϕ and $\partial_v(r\phi)$.

(3) Bounds for $\partial_u(r\phi)$: To achieve the bounds for $\partial_u(r\phi)$, first note that along the axis we have $\partial_u(r\phi) = -\partial_v(r\phi)$. Thus, by the previous derived control for $\partial_v(r\phi)$, we also have the decay of $\partial_u(r\phi)$ along the axis. We then consider the wave equation as a transport equation in the outgoing direction for $\partial_u(r\phi)$ to obtain the sharp decay for $\partial_u(r\phi)$ in the whole spacetime.

3E2. *Estimates for the second derivatives of $r\phi$.* As for the first derivatives, we control the second derivatives by first integrating the equation in the exterior region up to a curve $v = 3u$. We then propagate the decay bounds from the exterior region to the interior region using the estimates already derived for the first derivative of ϕ, as well as the local BV scattering assumption. However, at this level of derivatives, some new difficulties arise, as we now describe.

Renormalization and the null structure. The assumption of local BV scattering implies that

$$\int_{C_u\cap\{r\leq R\}} (|\partial_v\phi| + |\partial_v^2(r\phi)|) \to 0 \tag{3-9}$$

as $u\to\infty$. When combined with Christodoulou's BV theory, this also implies that, as $v\to\infty$, we have

$$\int_{\underline{C}_v\cap\{r\leq R\}} (|\partial_u\phi| + |\partial_u^2(r\phi)|) \to 0. \tag{3-10}$$

Notice that on C_u (resp. $\underline{C}_v$), we only control the integral of $\partial_v^2(r\phi)$ and $\partial_v\phi$ (resp. $\partial_u^2(r\phi)$ and $\partial_u\phi$).

Suppose, when integrating along the incoming direction to control $\partial_v^2(r\phi)$ and $\partial_v\lambda$, we need to estimate terms of the form

$$\int_{\underline{C}_v\cap\{r\leq R\}} |\partial_u\phi\, \partial_v\phi|.$$

We can apply the BV theory to show that, for v sufficiently large,

$$\int_{\underline{C}_v\cap\{r\leq R\}} |\partial_u\phi| \leq \epsilon.$$

On the other hand, one can show that

$$\sup_{\underline{C}_v\cap\{r\leq R\}} |\partial_v\phi| \leq C \sup_{J^-(\underline{C}_v\cap\mathcal{Q}_{\mathrm{cpt}})} |\partial_v^2(r\phi)|,$$

which can be controlled by the quantity that we are estimating.

However, in (4-2) for $\partial_v^2(r\phi)$, derived by differentiating (2-8), there are terms of the form

$$\partial_v\phi\, \partial_v\phi$$

such that neither of the factors can be controlled a priori in L^1 by the local BV scattering assumption. In other words, the equation does not obey any null condition.

To deal with this problem, we follow [Christodoulou 1993] and introduce the renormalized variables

$$\partial_v^2(r\phi) - (\partial_v\lambda)\phi, \qquad \partial_v \log\lambda - \frac{\lambda}{1-\mu}\frac{\mu}{r} + \partial_v\phi(\lambda^{-1}\partial_v(r\phi) - \nu^{-1}\partial_u(r\phi)),$$
$$\partial_u^2(r\phi) - (\partial_u\nu)\phi, \qquad \partial_u \log(-\nu) - \frac{\nu}{1-\mu}\frac{\mu}{r} + \partial_u\phi(\lambda^{-1}\partial_v(r\phi) - \nu^{-1}\partial_u(r\phi)),$$

which have the property that the nonlinear terms arising in the equations for these variables in fact have a null structure. In particular, we can apply the above heuristic procedure to obtain decay estimates in the compact region $r \le R$.

Nonrenormalized variables and decay towards null infinity. While the renormalization allows us to apply the BV theory in the interior region, it does not give the optimal r decay rates in the exterior region. For example, the renormalized quantity

$$\partial_v \log\lambda - \frac{\mu}{1-\mu}\frac{\lambda}{r} + \partial_v\phi(\lambda^{-1}\partial_v(r\phi) - \nu^{-1}\partial_u(r\phi))$$

decays only as r^{-2} towards null infinity due to the contribution of $(\mu/(1-\mu))\lambda/r$, which is weaker than the desired r^{-3} decay for $\partial_v \log\lambda$. Therefore, in order to obtain the optimal estimates everywhere in the spacetime, we need to use the variables $\partial_v^2(r\phi)$, $\partial_u^2(r\phi)$, $\partial_v\lambda$ and $\partial_u\nu$ together with their renormalized versions.

Coupling of the incoming and outgoing parts. Finally, an additional challenge is that, unlike the estimates for the first derivatives of the scalar field, the bounds for the incoming part of the solution, $\partial_v^2(r\phi)$ and $\partial_v\lambda$ are coupled to that for the outgoing part, $\partial_u^2(r\phi)$ and $\partial_u\nu$. Likewise, to control $\partial_u^2(r\phi)$, we need estimates for $\partial_v^2(r\phi)$ and $\partial_v\lambda$. For example, in the equation for $\partial_v \log\lambda - (\mu/(1-\mu))\lambda/r + \partial_v\phi(\lambda^{-1}\partial_v(r\phi) - \nu^{-1}\partial_u(r\phi))$, there is a term involving $\partial_u^2(r\phi)$ on the right-hand side. In particular, in order to obtain the desired decay for $\partial_v\lambda$, we need to at the same time prove the decay for $\partial_u^2(r\phi)$.

Strategy for obtaining the decay estimates. With the above difficulties in mind, we can now give a very rough sketch of the strategy of the proof:

(1) Bounds for $\partial_v^2(r\phi)$ and $\partial_v\lambda$ for large r: As in the case for the first derivatives, we first prove the optimal r decay for $\partial_v^2(r\phi)$ and $\partial_v\lambda$ in the exterior region. To this end, we integrate the equations satisfied by the *nonrenormalized* variables. We note that the error terms can all be bounded using the local BV scattering assumption and the decay estimates already proved for the first derivatives.

(2) Bounds for all second derivatives: Steps (2) and (3) for the decay bounds for the first derivatives are now coupled. Define

$$\mathcal{B}_2(U) := \sup_{u\in[1,U]} \sup_{C_u} \big(u^{\omega}|\partial_v^2(r\phi)| + u^{\omega}|\partial_u^2(r\phi)| + u^{\omega}|\partial_v\lambda| + u^{\omega}|\partial_u\nu|\big).$$

We then show that $\mathcal{B}_2(U)$ can control the error terms arising from integrating the *renormalized* equations in the sense that we can obtain an inequality of the form

$$|\text{weighted renormalized variables}| \le C(u_2) + \epsilon(u_2)\mathcal{B}_2(U),$$

where $\epsilon(u_2) \to 0$ as $u_2 \to \infty$. We then prove that the renormalized variables in fact control all the weighted second derivatives in $\mathcal{B}_2$. After choosing u_2 to be sufficiently large, we show that $\mathcal{B}_2(U)$ is bounded independently of U and thus all the second derivatives have $u^{-\omega}$ decay.

(3) Optimal bounds in terms of u decay: While we have obtained $u^{-\omega}$ decay for the second derivatives, the decay rates are not the sharp rates claimed in the main theorem. To finally obtain the desired bounds, we integrate the equations of the nonrenormalized variables and use the preliminary estimates obtained in (1) and (2) above. Here, we make use of the fact that the estimates obtained in step (2) are sufficiently strong (both in terms of regularity and decay) to control the error terms in the nonrenormalized equations.

4. Analytic properties of (SSESF)

In this section, we discuss the analytic properties of (SSESF). These include scaling, monotonicity and the null structure of the system. All these features will play crucial roles in the analysis.

4A. ***Scaling.*** For $a > 0$, (SSESF) is invariant under the scaling of the coordinate system

$$u \mapsto au, \quad v \mapsto av$$

together with the scaling of the functions

$$r \mapsto ar, \quad m \mapsto am, \quad \Omega \mapsto \Omega, \quad \phi \mapsto \phi.$$

This in particular implies that the BV norms

$$\int_u^\infty |\partial_v^2(r\phi)(u, v')| \, \mathrm{d}v' \quad \text{and} \quad \int_u^\infty |\partial_v \lambda(u, v')| \, \mathrm{d}v'$$

are scale invariant. Thus the a priori assumptions (2-12) are taken with respect to localized versions of scale-invariant norms.

4B. ***Monotonicity properties.*** We first begin with basic monotonicity properties of r.

Lemma 4.1 (monotonicity of r). *Let (ϕ, r, m) be a BV solution to (SSESF). Then we have*

$$\nu < 0 \quad \textit{in } \mathcal{Q}$$

and

$$\begin{cases} \lambda > 0 & \textit{when } 1-\mu > 0, \\ \lambda = 0 & \textit{when } 1-\mu = 0, \\ \lambda < 0 & \textit{when } 1-\mu < 0. \end{cases}$$

Proof. This was proved in [Christodoulou 1993, Propositions 1.1 and 1.2]; we reproduce the proof for the reader's convenience. Note the equation

$$\partial_u \partial_v (r^2) = -\tfrac{1}{2}\Omega^2,$$

which easily follows from (SSESF). As $\partial_u r^2 = 2r\partial_u r = 0$ on Γ and $r > 0$ on $\mathcal{Q}$, we easily see that $\nu < 0$. Then, from the definition of $1-\mu$, the second conclusion also follows. □

According to the sign of λ, a general Penrose diagram $\mathcal{Q}$ is divided into three subregions:

$$\mathcal{T} := \{(u, v) \in \mathcal{Q} : \lambda < 0\}, \quad \mathcal{A} := \{(u, v) \in \mathcal{Q} : \lambda = 0\} \quad \text{and} \quad \mathcal{R} := \{(u, v) \in \mathcal{Q} : \lambda > 0\}.$$

These are called the *trapped region*, *apparent horizon* and *regular region*, respectively. The next lemma, which we borrow from [Christodoulou 1993], shows that the solutions to (SSESF) considered in this paper consist only of the regular region $\mathcal{R}$. Therefore, extensive discussion of $\mathcal{T}$ and $\mathcal{A}$ will be suppressed.

Lemma 4.2 [Christodoulou 1993, Proposition 1.4]. *Let (ϕ, r, m) be a BV solution to (SSESF). Then the causal past of Γ in $\mathcal{Q}$ is contained in $\mathcal{R}$. In particular, $\mathcal{Q} = \mathcal{R}$ if (ϕ, r, m) satisfies condition* (1) *in Definition 2.4.*

Next, we turn to monotonicity properties of the Hawking mass m, which will play an important role in our paper. The following lemma is an obvious consequence of (2-7):

Lemma 4.3 (monotonicity of m). *For a BV solution (ϕ, r, m) to (SSESF), we have*

$$\partial_v m \geq 0 \quad \textit{and} \quad \partial_u m \leq 0 \quad \textit{in } \mathcal{R}.$$

By the monotonicity $\partial_v m \geq 0$, the limit $M(u) := \lim_{v\to\infty} m(u, v)$ exists (possibly $+\infty$ at this point) for each u. This is called the *Bondi mass* at retarded time u. The following statement is an easy corollary of the preceding lemma:

Corollary 4.4 (monotonicity of the Bondi mass). *Let (ϕ, r, m) be a BV solution to (SSESF) and suppose that $C_u \subset \mathcal{R}$ for $u \in [u_1, u_2]$. Then the Bondi mass $M(u)$ is a nonincreasing function on $[u_1, u_2]$.*

The following lemma shows that $M_i < \infty$ for initial data sets considered in this paper:

Lemma 4.5. *Suppose that $\partial_v(r\phi)(1, \cdot)$ is asymptotically flat of order $\omega' > 1$ in the sense of Definition 2.9. Then we have*

$$M_i := \lim_{v\to\infty} m(1, v) \leq C\mathcal{I}_1^2. \tag{4-1}$$

This is an easy consequence of (2-7) and Lemma 4.1; we omit its proof. By the preceding corollary, it follows that $M(u) < \infty$ for each u.

We conclude this subsection with additional monotonicity properties of solutions to (SSESF), useful for controlling the geometry of locally BV scattering solutions to (SSESF).

Lemma 4.6. *Let (ϕ, r, m) be a BV solution to (SSESF). For $(u, v) \in \mathcal{R}$, we have*

$$\frac{\lambda}{1-\mu}(u, v) \leq \frac{\lambda}{1-\mu}(1, v) \quad \textit{and} \quad \partial_u \lambda = \partial_v \nu \leq 0.$$

Proof. The lemma follows from the formula

$$\partial_u \log\left|\frac{\lambda}{1-\mu}\right| = -(-\nu)^{-1} r (\partial_u \phi)^2$$

and (2-6). □

4C. ***Null structure of the evolution equations.*** In this subsection, we follow [Christodoulou 1993] and demonstrate that the evolution equations verify a form of null structure. In particular, the null structure occurs in the equations for the second derivatives of the scalar field and the metric. However, it is not apparent if we simply take the derivatives of the equations (2-6) and (2-8). Instead, we rewrite the equations in renormalized variables for which the null structure is manifest. We will perform this renormalization separately for the wave equations for ϕ and for the equations for λ and ν.

The wave equation for ϕ. Taking ∂_v of (2-8), we obtain

$$\partial_u(\partial_v^2(r\phi)) = \partial_v(\partial_u\lambda\,\phi) = \partial_u\lambda\,\partial_v\phi + (\partial_v\partial_u\lambda)\phi,$$

or equivalently, after substituting in the first equation in (2-6),

$$\partial_u(\partial_v^2(r\phi)) = \frac{2m\lambda\nu}{(1-\mu)r^2}\,\partial_v\phi + \frac{\nu}{1-\mu}(\partial_v\phi)^2\phi + \frac{2m\nu}{(1-\mu)r^2}(\partial_v\lambda)\phi - \frac{4m}{(1-\mu)r^3}\lambda^2\nu\phi. \tag{4-2}$$

Some terms on the right-hand side, such as $(1-\mu)^{-1}\nu(\partial_v\phi)^2\phi$, do not exhibit null structure and are dangerous near Γ. To tackle this, we rewrite

$$(\partial_v\partial_u\lambda)\phi = \partial_u[(\partial_v\lambda)\phi] - \partial_v\lambda\,\partial_u\phi.$$

Thus, from the first equation, we derive

$$\partial_u[\partial_v^2(r\phi) - (\partial_v\lambda)\phi] = \partial_u\lambda\,\partial_v\phi - \partial_v\lambda\,\partial_u\phi. \tag{4-3}$$

By switching u and v, we obtain the following analogous equations in the conjugate direction:

$$\partial_v(\partial_u^2(r\phi)) = \frac{2m\lambda\nu}{(1-\mu)r^2}\,\partial_u\phi + \frac{\lambda}{1-\mu}(\partial_u\phi)^2\phi + \frac{2m\lambda}{(1-\mu)r^2}(\partial_u\nu)\phi - \frac{4m}{(1-\mu)r^3}\lambda\nu^2\phi, \tag{4-4}$$

$$\partial_v[\partial_u^2(r\phi) - (\partial_u\nu)\phi] = \partial_v\nu\,\partial_u\phi - \partial_u\nu\,\partial_v\phi. \tag{4-5}$$

The equations for λ *and* ν. From (2-6), we have

$$\partial_u \log\lambda = \frac{\mu}{(1-\mu)r}\nu, \qquad \partial_v \log(-\nu) = \frac{\mu}{(1-\mu)r}\lambda.$$

Take ∂_v and ∂_u of the first and second equations, respectively. Using (2-6), it is not difficult to verify that

$$\partial_u\partial_v \log\lambda = \frac{1}{1-\mu}\lambda^{-1}\nu(\partial_v\phi)^2 - \frac{4m}{(1-\mu)r^3}\lambda\nu, \tag{4-6}$$

$$\partial_v\partial_u \log(-\nu) = \frac{1}{1-\mu}\nu^{-1}\lambda(\partial_u\phi)^2 - \frac{4m}{(1-\mu)r^3}\lambda\nu. \tag{4-7}$$

To reveal the null structure, we must carry out the renormalization as we have done for (4-3) and (4-5). Following [Christodoulou 1993], it is easy to check that the above two equations are equivalent to

$$\partial_u\left[\partial_v \log\lambda - \frac{\mu}{1-\mu}\frac{\lambda}{r} + \partial_v\phi(\lambda^{-1}\partial_v(r\phi) - \nu^{-1}\partial_u(r\phi))\right] = \partial_u\phi\,\partial_v(\nu^{-1}\partial_u(r\phi)) - \partial_v\phi\,\partial_u(\nu^{-1}\partial_u(r\phi)) \tag{4-8}$$

and the conjugate equation

$$\partial_v\left[\partial_u \log(-\nu)-\frac{\mu}{1-\mu}\frac{\nu}{r}+\partial_u\phi(\lambda^{-1}\partial_v(r\phi)-\nu^{-1}\partial_u(r\phi))\right]=-\partial_u\phi\,\partial_v(\lambda^{-1}\partial_v(r\phi))+\partial_v\phi\,\partial_u(\lambda^{-1}\partial_v(r\phi)). \tag{4-9}$$

5. Basic estimates for locally BV scattering solutions

In this section, we gather some basic estimates concerning locally BV scattering solutions. These estimates will apply, in particular, to solutions satisfying the hypotheses of Theorem 3.1.

5A. *Integration lemmas for ϕ.* We first derive some basic inequalities for ϕ, $\lambda^{-1}\partial_v(r\phi)$ and $\partial_v\phi$. We remark that these are functional inequalities which hold under very general assumptions and in particular do not rely on the locally BV scattering assumption.

Lemma 5.1. *Let $\phi(u,\cdot)$ and $r(u,\cdot)$ be Lipschitz functions on $[u,v]$ with $\lambda>0$ and $r(u,u)=0$. Then the following inequality holds*:

$$|\phi(u,v)|\le \sup_{v'\in[u,v]}\left|\frac{\partial_v(r\phi)}{\lambda}(u,v')\right|. \tag{5-1}$$

More generally, for $u\le v_1\le v_2$, we have

$$|r\phi(u,v_1)-r\phi(u,v_2)|\le (r(u,v_2)-r(u,v_1))\sup_{v'\in[v_1,v_2]}\left|\frac{\partial_v(r\phi)}{\lambda}(u,v')\right|. \tag{5-2}$$

Proof. We shall prove (5-2), since (5-1) then follows as a special case. Integrating $\partial_v(r\phi)(u,v')$ over $v'\in[v_1,v_2]$, we get

$$\begin{aligned}|r\phi(u,v_1)-r\phi(u,v_2)|&\le\int_{v_1}^{v_2}|\partial_v(r\phi)(u,v')|\,dv'\\&\le\sup_{v'\in[v_1,v_2]}\left|\frac{\partial_v(r\phi)}{\lambda}(u,v')\right|\times\int_{v_1}^{v_2}\lambda(u,v')\,dv'\\&=(r(u,v_2)-r(u,v_1))\sup_{v'\in[v_1,v_2]}\left|\frac{\partial_v(r\phi)}{\lambda}(u,v')\right|.\end{aligned}$$

□

Lemma 5.2. *Let $\phi(u,\cdot)$ and $r(u,\cdot)$ be functions on $[u,v]$ such that $\partial_v\phi$ is integrable, r is Lipschitz with $\lambda>0$ and $r(u,u)=0$. Suppose furthermore that $\lambda^{-1}\partial_v(r\phi)(u,\cdot)$ is BV on $[u,v]$. Then the following statements hold*:

(1) *We have*

$$\int_u^v|\partial_v\phi(u,v')|\,dv'\le\int_u^v|\partial_v(\lambda^{-1}\partial_v(r\phi))(u,v')|\,dv'. \tag{5-3}$$

(2) *Suppose, in addition, that $\lambda^{-1}\partial_v(r\phi)(u,\cdot)$ is Lipschitz on $[u,v]$. Then we have*

$$|\partial_v\phi(u,v)|\le\frac{1}{2}\frac{\sup_{v'\in[u,v]}\lambda(u,v')}{\inf_{v'\in[u,v]}\lambda(u,v')}\sup_{v'\in[u,v]}|\partial_v(\lambda^{-1}\partial_v(r\phi))(u,v')|. \tag{5-4}$$

Proof. We proceed formally to compute

$$\begin{aligned}\partial_v\phi(u,v) &= \frac{\lambda}{r}(\lambda^{-1}\partial_v(r\phi)-\phi)(u,v)\\ &= \frac{\lambda}{r^2}(u,v)\int_u^v\biggl(\int_{v'}^v \partial_v(\lambda^{-1}\partial_v(r\phi))(u,v'')\,\mathrm{d}v''\biggr)\lambda(u,v')\,\mathrm{d}v'\\ &= \frac{\lambda}{r^2}(u,v)\int_u^v r(u,v'')\partial_v(\lambda^{-1}\partial_v(r\phi))(u,v'')\,\mathrm{d}v''.\end{aligned}$$

The above computation is justified thanks to the hypotheses, where we interpret

$$\partial_v(\lambda^{-1}\partial_v(r\phi))(u,v'')\,\mathrm{d}v''$$

to be the weak derivative of $\lambda^{-1}\partial_v(r\phi)$, which is a finite signed measure. For a fixed (u,v), observe that

$$\sup_{v''\in[u,v]} r(u,v'')\int_{v''}^v \frac{\lambda(u,v')}{r^2(u,v')}\,\mathrm{d}v' \le 1.$$

This proves (5-3). For (5-4), note that the function $\lambda^{-1}\partial_v(r\phi)$ is absolutely continuous on $[u,v]$, so $\partial_v(\lambda^{-1}\partial_v(r\phi)(u,\cdot))$ exists almost everywhere on $[u,v]$; moreover, it belongs to L^∞ by the Lipschitz assumption. Noting that

$$\sup_{v'\in[u,v]} \frac{\lambda(u,v')}{r^2(u,v')}\int_u^{v'} r(u,v'')\,\mathrm{d}v'' \le \frac{1}{2}\frac{\sup_{v'\in[u,v]}\lambda(u,v')}{\inf_{v'\in[u,v]}\lambda(u,v')},$$

we obtain (5-4). □

5B. *Geometry of locally BV scattering solutions.* The goal of this subsection is to prove the following proposition:

Proposition 5.3. *Let (ϕ, r, m) be a locally BV scattering solution to* (SSESF) *as in Definition 2.4. Assume furthermore that, on the initial slice C_1, we have $\lambda(1,\cdot)=\frac{1}{2}$ and*

$$\sup_{C_1}|\partial_v(r\phi)| + M_i < \infty.$$

Then there exist finite constants $K, \Lambda > 0$ such that the following bounds hold for all $(u,v)\in\mathcal{Q}$:

$$\Lambda^{-1} \le \lambda(u,v) \le \tfrac{1}{2}, \tag{5-5}$$

$$\Lambda^{-1} \le -\nu(u,v) \le K, \tag{5-6}$$

$$1 \le (1-\mu(u,v))^{-1} \le K\Lambda, \tag{5-7}$$

$$0 < \frac{-\nu}{1-\mu(u,v)} \le K. \tag{5-8}$$

Moreover, there exists a finite constant $\Psi > 0$ such that, for all $(u,v)\in\mathcal{Q}$, we have

$$|\partial_v(r\phi)(u,v)| \le \Psi, \tag{5-9}$$

$$|\phi(u,v)| \le \Lambda\Psi. \tag{5-10}$$

Once we have this proposition, we will denote by Λ, K and Ψ the best constants such that (5-5)–(5-10) hold.

By Lemma 4.2, we already know that $\lambda > 0$, $-\nu > 0$ and $(1-\mu)^{-1} < \infty$. The first three bounds, namely (5-5)–(5-7), ensure that these bounds concerning the geometry of the spacetime do not degenerate anywhere, in particular along the axis Γ. They will be very useful in the analysis that follows.

The proof of Proposition 5.3 will consist of several steps. We begin with elementary bounds for λ and ν.

Lemma 5.4. *Let (ϕ, r, m) be a BV solution to (SSESF) with $\mathcal{Q} = \mathcal{R}$. Then, for every $(u, v) \in \mathcal{Q}$, we have*

$$\lambda(u, v) \le \lambda(1, v), \tag{5-11}$$

$$\lambda^{-1}(u, v) \le \lim_{u' \to v-} \lambda^{-1}(u', v), \tag{5-12}$$

$$\nu(u, v) \le -\lim_{v' \to u+} \lambda(u, v'). \tag{5-13}$$

Proof. By (2-6), we have

$$\lambda(u, v) = \lambda(1, v) \exp\left(\int_1^u \left(\frac{2m}{(1-\mu)r^2} \nu \right)(u', v) \, du' \right),$$

$$\lambda^{-1}(u, v) = \lim_{u' \to v-} \lambda(u', v)^{-1} \exp\left(\int_u^v \left(\frac{2m}{(1-\mu)r^2} \nu \right)(u', v) \, du' \right),$$

$$\nu(u, v) = \lim_{v' \to u+} \nu(u, v') \exp\left(\int_u^v \left(\frac{2m}{(1-\mu)r^2} \lambda \right)(u, v') \, dv' \right).$$

Since $-\nu$, $(1-\mu) > 0$ everywhere, (5-11) and (5-12) follow. Moreover, since

$$\lim_{v' \to u+} \nu(u, v') = -\lim_{v' \to u+} \lambda(u, v'),$$

and $\lambda > 0$ on $\mathcal{Q}$, (5-13) follows as well. □

By Lemma 4.2, $\mathcal{Q} = \mathcal{R}$ holds for a solution (SSESF) satisfying the hypotheses of Proposition 5.3. As an immediate corollary, we have the following easy upper bound for λ:

Corollary 5.5. *Let (ϕ, r, m) be a solution to (SSESF) satisfying the hypotheses of Proposition 5.3. Then, by the coordinate condition $\lambda(1, v) = \frac{1}{2}$ and (5-11), we have*

$$\sup_{\mathcal{Q}} \lambda \le \tfrac{1}{2}.$$

Next, we proceed to prove the lower bounds of (5-5) and (5-6). We begin with a technical lemma concerning a large-r region, which will also be useful in our proof of (5-9) and (5-10).

Lemma 5.6. *Let (ϕ, r, m) be a solution to (SSESF) satisfying the hypotheses of Proposition 5.3. Then, for arbitrarily small $\epsilon > 0$, there exists $r_0 > 1$ such that*

$$\sup_{(u,v) \in \{r \ge r_0\}} \int_1^u \left| \frac{\mu}{1-\mu} \frac{\nu}{r}(u', v) \right| du' < \epsilon. \tag{5-14}$$

Proof. For $(u, v) \in \{r \geq r_0\}$, we begin by simply estimating as follows:

$$\left| \frac{\mu}{1-\mu} \frac{\nu}{r} \right| \leq \frac{2M_i}{(1-2M_i/r_0)} \frac{(-\nu)}{r^2}$$

The above inequality holds as long as[12] we choose $r_0 > \max\{2M_i, R\}$. Note that if $(u, v) \in \{r \geq r_0\}$, then the null curve $\{(u', v) : u' \in [1, u]\}$ from the initial slice C_1 to (u, v) lies entirely in $\{r \geq r_0\}$. Integrating along this curve, we obtain for $(u, v) \in \{r \geq r_0\}$

$$\int_1^u \left| \frac{\mu}{1-\mu} \frac{\nu}{r}(u', v) \right| \mathrm{d}u' < \frac{2M_i}{(1-2M_i/r_0)} \frac{1}{r_0}$$

Taking r_0 sufficiently large, (5-14) follows. □

Next, we prove an analogous result in a large-u region. Key to its proof will be the identity (5-16) below, which will also be used to relate (5-14) and (5-15) to the desired lower bounds of λ and $-\nu$.

Lemma 5.7. *Let (ϕ, r, m) be a solution to (SSESF) satisfying the hypotheses of Proposition 5.3. Then, for arbitrarily small $\epsilon > 0$, there exists $U > 1$ such that*

$$\sup_{v \geq U} \int_U^v \left| \frac{\mu}{1-\mu} \frac{\nu}{r}(u', v) \right| \mathrm{d}u' < \epsilon. \tag{5-15}$$

Proof. Let $\epsilon > 0$ be an arbitrary positive number. Using (2-6) and the fact that $1 - \mu > 0$ and $-\nu > 0$ on $\mathcal{Q}$, we have, for any $1 \leq u_1 \leq u_2 < v$,

$$\int_{u_1}^{u_2} \left| \frac{\mu}{1-\mu} \frac{\nu}{r}(u', v) \right| \mathrm{d}u' = \log \lambda(u_1, v) - \log \lambda(u_2, v). \tag{5-16}$$

In order to prove (5-15), it therefore suffices to exhibit $U > 1$ such that

$$\sup_{(u,v),(u',v') \in \{u \geq U\}} |\log \lambda(u, v) - \log \lambda(u', v')| < \epsilon. \tag{5-17}$$

In order to proceed, we divide $\mathcal{Q}$ into three regions: $\mathcal{Q}_{\mathrm{cpt}} := \{r \leq R\}$, $\mathcal{Q}_{[R,r_0]} := \{R \leq r \leq r_0\}$ and $\mathcal{Q}_{[r_0,\infty)} := \{r \geq r_0\}$, where $r_0 > \max\{2M_i, R\}$ is chosen via Lemma 5.6 so that

$$\sup_{(u,v) \in \mathcal{Q}_{[r_0,\infty)}} \int_1^u \left| \frac{\mu}{1-\mu} \frac{\nu}{r}(u', v) \right| \mathrm{d}u' < \frac{\epsilon}{8}.$$

Using (5-16) and the fact that $\log \lambda(1, v) = \frac{1}{2}$, the preceding inequality is equivalent to

$$\sup_{(u,v) \in \mathcal{Q}_{[r_0,\infty)}} \left| \log \lambda(u, v) - \tfrac{1}{2} \right| < \frac{\epsilon}{8}. \tag{5-18}$$

Next, we turn to the region $\mathcal{Q}_{[R,r_0]}$; here we exploit the vanishing of the final Bondi mass. Indeed, taking U_1 large enough so that $2M(U_1) < R$, we may estimate

$$\left| \frac{\mu}{1-\mu} \frac{\nu}{r} \right| \leq \frac{2M(U_1)}{(1-2M(U_1)/R)R^2}(-\nu) \quad \text{for } u \geq U_1.$$

[12] Indeed, it suffices to choose $r_0 > 2M_i$ here. The condition $r_0 > R$ will be used in the proof of Lemma 5.7.

Consider now the timelike curve given by $\gamma_0 := \{(u', v') : r(u', v') = r_0\}$. On $\gamma_0 \cap \{(u, v) : u \geq U_1\}$, note that (5-18) holds. Integrating the preceding inequality along incoming null curves emanating from $\gamma_0 \cap \{(u, v) : u \geq U_1\}$, we obtain, for $(u, v) \in \mathcal{Q}_{[R,r_0]} \cap \{(u, v) : u \geq U_2\}$,

$$\left|\log \lambda(u, v) - \tfrac{1}{2}\right| < \frac{\epsilon}{8} + \frac{2M(U_1)(r_0 - R)}{(1 - 2M(U_1)/R)R^2},$$

where $U_2 = U_2(U_1, r_0)$ is the future endpoint of the incoming null curve in $\mathcal{Q}_{[R,r_0]}$ from the past endpoint of $\gamma_0 \cap \{(u, v) : u \geq U_1\}$; more precisely, $U_2 = \sup\{u : r(u, V_1) \geq R\}$, where V_1 is defined by $r(U_1, V_1) = r_0$. Choosing U_1 sufficiently large, we then obtain

$$\sup_{(u,v)\in\mathcal{Q}_{[R,r_0]}\cap\{u\geq U_2\}} \left|\log \lambda(u, v) - \tfrac{1}{2}\right| < \frac{\epsilon}{4}. \tag{5-19}$$

Finally, in $\mathcal{Q}_{\text{cpt}}$, we use the local BV scattering condition (2-12) to choose $U \geq U_2$ large enough so that we have

$$\sup_{(u,v),(u,v')\in\mathcal{Q}_{\text{cpt}}\cap\{u\geq U\}} |\log \lambda(u, v) - \log \lambda(u, v')| < \frac{\epsilon}{4}. \tag{5-20}$$

To compare $\log \lambda(u, v)$ and $\log \lambda(u', v')$ with $u \neq u'$, we use (5-19), (5-20) and the triangle inequality. Thus, the desired conclusion (5-17) follows. □

As a corollary of the preceding lemmas and (5-16) (or, more directly, (5-17) and (5-18)), we immediately see that λ and $-\nu$ are uniformly bounded away from zero.

Corollary 5.8. *Let (ϕ, r, m) be a solution to (SSESF) satisfying the hypotheses of Proposition 5.3. Then there exists $0 < \Lambda < \infty$ such that, for all $(u, v) \in \mathcal{Q}$, we have*

$$\Lambda^{-1} \leq \lambda(u, v) \quad \textit{and} \quad \Lambda^{-1} \leq -\nu(u, v).$$

Together with Corollary 5.5, this concludes the proof of (5-5). Next, using Lemmas 5.1, 5.6 and 5.7 and the wave equation (2-8) for ϕ, we prove (5-9) and (5-10) in the following lemma:

Lemma 5.9. *Let (ϕ, r, m) be a solution to (SSESF) satisfying the hypotheses of Proposition 5.3. Then there exists a constant $0 < \Psi < \infty$ such that*

$$\sup_{\mathcal{Q}} |\partial_v(r\phi)| \leq \Psi \quad \textit{and} \quad \sup_{\mathcal{Q}} |\phi| \leq \Lambda\Psi, \tag{5-21}$$

where Λ is the best constant such that Corollary 5.8 holds.

Proof. Note that the second inequality of (5-21) is an immediate consequence of the first inequality, Lemma 5.1 and Corollary 5.8. The proof of the first inequality will proceed in two steps: First, we shall show that $\partial_v(r\phi)$ is uniformly bounded on the large-r region, essentially via Lemma 5.6. By compactness, it immediately follows that $\partial_v(r\phi)$ is uniformly bounded on the finite-u region. Then in the second step, we shall show that $\partial_v(r\phi)$ is uniformly bounded on a large-u region as well using Lemma 5.7.

By Lemma 5.6, choose $r_0 > 0$ so that

$$\sup_{(u,v)\in\{r\geq r_0\}} \int_1^u \left|\frac{\mu}{1-\mu}\frac{\nu}{r}(u', v)\right| \mathrm{d}u' < \frac{1}{10\Lambda}. \tag{5-22}$$

We also borrow the notation $\mathcal{Q}_{[r_0,\infty)} := \{(u,v) : r(u,v) \geq r_0\}$ from the proof of Lemma 5.7. Given $U \geq 1$, define $\Psi_{[r_0,\infty)}(U)$ to be

$$\Psi_{[r_0,\infty)}(U) := \sup_{(u,v)\in \mathcal{Q}_{[r_0,\infty)}\cap\{1\leq u\leq U\}} |\partial_v(r\phi)(u,v)|.$$

Let $(u,v) \in \mathcal{Q}_{[r_0,\infty)}$. Using (2-8), we then write

$$\partial_u\partial_v(r\phi) = \frac{\mu}{1-\mu}\frac{\nu}{r}\left(\frac{\lambda}{r}(r\phi - r_0\phi_{r_0}) + \frac{\lambda}{r}r_0\phi_{r_0}\right).$$

Here, $\phi_{r_0}(u,v) := \phi(u, v_0^\star(u))$, where $v_0^\star(u)$ is the unique v-value for which $r(u, v_0^\star(u)) = r_0$. Note that the outgoing null curve from $(u, v_0^\star(u))$ to $(u,v) \in \mathcal{Q}_{[r_0,\infty)}$ lies entirely in $\mathcal{Q}_{[r_0,\infty)}$. Thus, by Lemma 5.1 and (5-5), we see that, for $(u,v) \in \mathcal{Q}_{[r_0,\infty)}$ with $1 \leq u \leq U$,

$$|\partial_u\partial_v(r\phi)| \leq \left|\frac{\mu}{1-\mu}\frac{\nu}{r}\right|\left(\frac{r-r_0}{2r}\Lambda\Psi_{[r_0,\infty)}(U) + \frac{r_0}{2r}|\phi_{r_0}|\right) \leq \left|\frac{\mu}{1-\mu}\frac{\nu}{r}\right|(\Lambda\Psi_{[r_0,\infty)}(U) + |\phi_{r_0}|).$$

Integrating this over the incoming null curve from $(1,v)$ to (u,v) (which lies in $\mathcal{Q}_{[r_0,\infty)} \cap \{1 \leq u \leq U\}$) and using Lemma 5.6, we then obtain

$$\Psi_{[r_0,\infty)}(U) \leq \sup_{C_1\cap\mathcal{Q}_{[r_0,\infty)}} |\partial_v(r\phi)| + \frac{1}{10}\Psi_{[r_0,\infty)}(U) + \frac{1}{10\Lambda}\sup_{\gamma_0\cap\{1\leq u\leq U\}}|\phi|,$$

where γ_0 is the timelike curve $\{(u,v) : r(u,v) = r_0\}$. Note that the first term on the right-hand side is finite by the assumptions on the initial data, whereas the last term is finite for every $1 \leq U < \infty$ by compactness of $\gamma_0 \cap \{(u,v) : 1 \leq u \leq U\}$ and continuity of ϕ. Then, by a simple continuity argument, it follows that $\Psi_{[r_0,\infty)}(U) < \infty$ for every $1 \leq U < \infty$. Moreover, by compactness of $\{(u,v) : r(u,v) \leq r_0, 1 \leq u \leq U\}$, as well as the uniform BV assumption on $\partial_v(r\phi)$, we also have

$$\Psi_{[0,\infty)}(U) := \sup_{(u,v)\in\{1\leq u\leq U\}} |\partial_v(r\phi)(u,v)| < \infty.$$

We now proceed to deal with the large-u region, namely $\{(u,v) : u \geq U\}$. Using Lemma 5.7, we choose $U_0 \geq 1$ sufficiently large that

$$\sup_{v\geq U_0}\int_{U_0}^{v}\left|\frac{\mu}{1-\mu}\frac{\nu}{r}(u',v)\right|du' < \frac{1}{10\Lambda}. \tag{5-23}$$

Proceeding, as before, via Lemma 5.1, we estimate, for $(u,v) \in \{(u,v) : u \geq U_0\}$,

$$|\partial_u\partial_v(r\phi)(u,v)| \leq \left|\frac{\mu}{1-\mu}\frac{\nu}{r}\right|\Lambda\sup_{v'\in[u,v]}|\partial_v(r\phi)(u,v')|.$$

Integrating along incoming null curves from C_{U_0}, we see that

$$\Psi_{[0,\infty)}(U) \leq \Psi_{[0,\infty)}(U_0) + \tfrac{1}{10}\Psi_{[0,\infty)}(U)$$

for any $U \geq U_0$. Absorbing the second term on the right-hand side into the left-hand side and taking $U \to \infty$, we obtain (5-21) with $\Psi \leq \frac{10}{9}\Psi_{[0,\infty)}(U_0) < \infty$. □

We are finally ready to conclude the proof of Proposition 5.3, by proving (5-8). Indeed, the upper bounds in (5-6) and (5-7) would then follow immediately. Moreover, the lower bound in (5-7) is trivial, as $\mu = 2m/r \geq 0$.

Lemma 5.10. *Let (ϕ, r, m) be a solution to (SSESF) satisfying the hypotheses of Proposition 5.3. Then there exists a finite constant $K > 0$ such that, for all $(u, v) \in \mathcal{Q}$,*

$$\frac{-\nu}{1-\mu}(u, v) \leq K. \tag{5-24}$$

Proof. To prove (5-24), we shall rely on the equation

$$\partial_v \log\left(\frac{-\nu}{1-\mu}\right) = \lambda^{-1} r (\partial_v \phi)^2, \tag{5-25}$$

which may be easily derived from (2-6) and (2-7).

For $(u, v) \in \mathcal{Q}$, we begin by integrating (5-25) on the outgoing null curve from $(u, u) \in \Gamma$ to (u, v), which gives

$$\left(\frac{-\nu}{1-\mu}\right)(u, v) \leq \left(\lim_{v' \to u+} \left(\frac{-\nu}{1-\mu}\right)(u, v')\right) \exp\left(\int_u^v \lambda^{-1} r (\partial_v \phi)^2 (u, v') \, dv'\right).$$

We claim that $\lim_{v' \to u+} (-\nu)(u, v') = \lim_{v' \to u+} \lambda(u, v') \leq \frac{1}{2}$ and $\lim_{v' \to u+} \mu(u, v') = 0$. The first assertion is obvious. To prove the second one, we first use (2-7) to write

$$m(u, v) \leq \tfrac{1}{2}\Big(\sup_{v' \in [u, v]} |r^2 \partial_v \phi|(u, v')\Big) \int_u^v |\partial_v \phi(u, v')| \, dv'.$$

Now observe that $\sup_{v' \in [u,v]} |r^2 \partial_v \phi|(u, v') \leq C r(u, v) \sup_{v' \in [u,v]} |\partial_v (r\phi)|$ and the remaining integral goes to 0 as $v \to u+$, since ϕ is assumed to be absolutely continuous on C_u near the axis by Definition 2.1.

By the above claim, we have

$$\left(\frac{-\nu}{1-\mu}\right)(u, v) \leq \tfrac{1}{2} \exp\left(\int_u^v \lambda^{-1} r (\partial_v \phi)^2 (u, v') \, dv'\right).$$

The lemma would therefore follow if we could prove

$$\sup_{(u,v) \in \mathcal{Q}} \int_u^v \lambda^{-1} r (\partial_v \phi)^2 (u, v') \, dv' < \infty.$$

To achieve this, we shall divide the integral into two parts, one in $\mathcal{Q}_{\mathrm{cpt}}$ and the other in its complement $\mathcal{Q}^c_{\mathrm{cpt}}$. Indeed, defining $v^\star(u)$ to be the unique v-value such that $r(u, v^\star(u)) = R$, we split the integral into $\int_u^{v^\star(u)}$ and $\int_{v^\star(u)}^v$. If $v < v^\star(u)$, the latter integral will be taken to be zero.

For the first integral, let us begin by pulling out $\lambda^{-1} r \partial_v \phi$ from the integral. Using the identity $\lambda^{-1} r \partial_v \phi = \lambda^{-1} \partial_v (r\phi) - \phi$, we have

$$\int_u^{v^\star(u)} \lambda^{-1} r (\partial_v \phi)^2 (u, v') \, dv' \leq \sup_{v' \in [u, v^\star(u)]} \big(\lambda^{-1} |\partial_v (r\phi)|(u, v') + |\phi|(u, v')\big) \int_u^{v^\star(u)} |\partial_v \phi(u, v')| \, dv'.$$

Then, by Lemmas 5.2 and 5.9 and the local BV scattering assumption, the right-hand side is uniformly bounded in u from above, as desired. For the second integral, note that, by Lemma 4.6 and Corollary 5.8, we have

$$(1-\mu)^{-1}(u,v) \le \Lambda \frac{\lambda}{1-\mu}(u,v) \le \tfrac{1}{2}\Lambda \sup_{C_1}(1-\mu)^{-1}.$$

Notice that the quantity $\sup_{C_1}(1-\mu)^{-1}$ for the initial data is finite, since $1-\mu > 0$ everywhere and $1-\mu(1,v) \to 1$ as $v \to \infty$. Moreover, for $v \ge v^\star(u)$, we have $r(u,v) \ge R$. Therefore, in view of (2-7), we may estimate

$$\begin{aligned}\int_{v^\star(u)}^{v} \lambda^{-1} r(\partial_v\phi)^2 \,\mathrm{d}v' &\le \frac{\Lambda}{R}\sup_{C_1}(1-\mu)^{-1}\int_{v^\star(u)}^{v} \tfrac{1}{2}\lambda^{-1}(1-\mu)r^2(\partial_v\phi)^2(u,v')\,\mathrm{d}v' \\ &\le \frac{\Lambda}{R}\sup_{C_1}(1-\mu)^{-1}\big(m(u,v)-m(u,v^\star(u))\big) \\ &\le C_{\Lambda,R,M_i,\sup_{C_1}(1-\mu)^{-1}} < \infty,\end{aligned}$$

from which the lemma follows. □

We conclude this subsection with a pair of identities which are useful for estimating $\int |\partial_u\lambda|\,\mathrm{d}u$ and $\int |\partial_v \nu|\,\mathrm{d}v$ in terms of information on ϕ.

Lemma 5.11. *From* (SSESF), *the following identities hold*:

$$\int_u^v \frac{\mu}{1-\mu}\frac{\lambda}{r}(u,v')\,\mathrm{d}v' = \log(1-\mu)(u,v) + \int_u^v \lambda^{-1} r(\partial_v\phi)^2(u,v')\,\mathrm{d}v', \tag{5-26}$$

$$\int_u^v \frac{\mu}{1-\mu}\frac{(-\nu)}{r}(u',v)\,\mathrm{d}u' = \log(1-\mu)(u,v) + \int_u^v (-\nu)^{-1} r(\partial_u\phi)^2(u',v)\,\mathrm{d}u'. \tag{5-27}$$

Proof. We shall prove (5-26), leaving the similar proof of (5-27) to the reader. From the proof of Lemma 5.4, we have

$$\int_u^v \frac{\mu}{1-\mu}\frac{\lambda}{r}(u,v') = \log \frac{\nu(u,v)}{\lim_{v'\to u+}\nu(u,v')}.$$

Comparing with the integral of (5-25), along with the fact that $\lim_{v'\to u+}(1-\mu)(u,v') = 1$, we arrive at (5-26). □

5C. *Normalization of the coordinate system.* In Section 2D, the coordinates are normalized so that λ is constant on the initial hypersurface $\{u=1\}$. Alternatively, one can introduce a new coordinate system (u_∞, v_∞) which is normalized at future null infinity by requiring that $\nu_\infty \to -\frac{1}{2}$ along each outgoing null curve towards null infinity and require, as before, that $\Gamma = \{(u,v) : u = v\}$. We will show that the coordinate functions u and u_∞ are comparable and thus the main theorem on the decay rates can also be stated in this alternatively normalized coordinate system.

We can explicitly compute the coordinate change, which is given by

$$\frac{du_\infty}{du}(u) = -2\lim_{v\to\infty}\nu(u,v), \quad u_\infty(1) = 1 \quad \text{and} \quad v_\infty(v) = u_\infty(v).$$

Notice that the limit $\lim_{v\to\infty} \nu(u, v)$ is well-defined due to the monotonicity of ν, and

$$u_\infty(u) = -2\int_1^u \big(\lim_{v\to\infty} \nu(u', v)\big)\, du' + 1.$$

By Proposition 5.3, the following estimate holds:

$$2(\Lambda)^{-1}(u-1) \le u_\infty - 1 \le 2K(u-1).$$

5D. *Consequence of local BV scattering.* In this subsection, we give some estimates for $\partial_u^2(r\phi)$, $\partial_u\phi$ and $\partial_u\nu$ that follow from the local BV scattering assumption. To this end, we will need the analysis for solutions to (SSESF) with small bounded variation norm in [Christodoulou 1993], in particular:

Theorem 5.12 [Christodoulou 1993, Theorem 6.2]. *There exist universal constants ϵ_0 and C_0 such that, for $\epsilon < \epsilon_0$, if $\lambda(1, \cdot) = \frac{1}{2}$ and $\partial_v(r\phi)(1, \cdot)$ is of bounded variation with*

$$\int_{C_1} |\partial_v^2(r\phi)| < \epsilon, \tag{5-28}$$

then its maximal development (ϕ, r, m) satisfies condition (1) in Definition 2.4 and obeys

$$\tfrac{1}{3} \le \lambda \le \tfrac{1}{2}, \quad \tfrac{1}{3} \le -\nu \le \tfrac{2}{3}, \quad \tfrac{2}{3} \le 1-\mu \le 1, \tag{5-29}$$

$$\sup_{u\ge 1} \int_{C_u} \big(|\partial_v(\lambda^{-1}\partial_v(r\phi))| + |\partial_v\phi| + |\partial_v \log\lambda|\big) < C_0\epsilon, \tag{5-30}$$

$$\sup_{v\ge 1} \int_{\underline{C}_v} \big(|\partial_u(\nu^{-1}\partial_u(r\phi))| + |\partial_u\phi| + |\partial_u \log\nu|\big) < C_0\epsilon. \tag{5-31}$$

Remark 5.13. In [Christodoulou 1993], it is implicitly assumed[13] that $\phi(1, 1) = 0$. Note, however, that the bounds in the above theorem are stated in such a way that they are invariant under the transform $(\phi, r, m) \mapsto (\phi+c, r, m)$, under which (SSESF) is also invariant. Any solution may then be transformed to satisfy $\phi(1, 1) = 0$. As a consequence, we do not need to check $\phi(1, 1) = 0$ in order to apply the theorem.

Using Theorem 5.12, we prove the following bound for locally BV scattering solutions to (SSESF):

Theorem 5.14. *Let (ϕ, r, m) be a locally BV scattering solution to (SSESF). For every $\epsilon > 0$, there exists $u_0 > 1$ such that the following estimate holds:*

$$\sup_{v\in[u_0,\infty)} \bigg(\int_{\underline{C}_v\cap\{u\ge u_0\}\cap\mathcal{Q}_{\mathrm{cpt}}} |\partial_u^2(r\phi)| + \int_{\underline{C}_v\cap\{u\ge u_0\}\cap\mathcal{Q}_{\mathrm{cpt}}} |\partial_u\phi| + \int_{\underline{C}_v\cap\{u\ge u_0\}\cap\mathcal{Q}_{\mathrm{cpt}}} |\partial_u \log\nu|\bigg) < \epsilon.$$

Moreover, we have

$$\sup_{\mathcal{Q}} |\partial_u(r\phi)| \le C_{K,\Lambda}\Psi. \tag{5-32}$$

[13] In [Christodoulou 1993], the intersection between Γ and the initial outgoing null curve is called $(0, 0)$, as opposed to our convention $(1, 1)$.

Proof. We first show that, for a locally BV scattering solution to (SSESF),

$$\int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v(\lambda^{-1}\partial_v(r\phi))| \to 0 \quad \text{as } u \to \infty.$$

Expanding this expression, we have

$$\int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v(\lambda^{-1}\partial_v(r\phi))| \leq \int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} \lambda^{-1}\big(|\partial_v^2(r\phi)| + |(\partial_v \log\lambda)\partial_v(r\phi)|\big).$$

By (5-5) and (5-9), we have

$$\int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v(\lambda^{-1}\partial_v(r\phi))| \leq C_{\Lambda,\Psi} \int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v^2(r\phi)| + |\partial_v \log\lambda|,$$

which, by (2-12) in Definition 2.4, tends to 0 as $u \to \infty$. Notice that the quantity $\int_{C_u \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_v(\lambda^{-1}\partial_v(r\phi))|$ which we have controlled is invariant under any rescaling of the coordinate v and also under the transform $(\phi, r, m) \mapsto (\phi + c, r, m)$.

We now proceed to the proof of the theorem. Let v_0 be sufficiently large and $u^\star(v_0)$ be the unique $r(u^\star(v_0), v_0) = R$. By the finite speed of propagation of the equations, the solution on $\underline{C}_{v_0} \cap \mathcal{Q}_{\mathrm{cpt}}$ depends only on the data on $C_{u^\star(v_0)} \cap \mathcal{Q}_{\mathrm{cpt}}$.

In order to apply Theorem 5.12, we change coordinates $(u, v) \mapsto (U(u), V(v))$ in the region bounded by $C_{u^\star(v_0)}$ and $\underline{C}_{v_0}$ to a new double null coordinate (U, V) such that, for $U^\star = U(u^\star(v_0))$, we have $\lambda(U^\star, V) = \frac{1}{2}$. To this end, define $V(v)$ by

$$\frac{dV}{dv} = 2\lambda(u^\star(v_0), v), \qquad V(v_0) = v_0.$$

Notice that this is acceptable since $\lambda > 0$. In order for the condition $U = V$ to hold on Γ, we require $U(u) = V(u)$. Then, with respect to the coordinate V,

$$\partial_V r(U^\star, V) = \frac{1}{2}.$$

By (5-5), we have

$$\Lambda^{-1} \leq \frac{dV}{dv}, \frac{dU}{du} \leq \frac{1}{2}.$$

Moreover,

$$\int_{u^\star(v_0)}^{v_0} \left|\frac{d^2V}{dv^2}(v')\right| dv' \leq 2\int_{u^\star(v_0)}^{v_0} |\partial_v\lambda(u^\star(v_0), v')|\, dv',$$

which tends to 0 as $v_0 \to \infty$ by the assumption of local BV scattering. For v_0 sufficiently large, in the (U, V) coordinate system, $\int_{C_{u^\star(v_0)} \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_V((\partial_V r)^{-1}\partial_V(r\phi))|\, dV$ is small and $\partial_V r = \frac{1}{2}$. The data satisfy the assumptions of Theorem 5.12 and therefore[14]

$$\int_{\underline{C}_{v_0}\cap\mathcal{Q}_{\mathrm{cpt}}} \big(|\partial_U((\partial_U r)^{-1}\partial_U(r\phi))| + |\partial_U\phi| + |\partial_U \log \partial_U r|\big)\, dU \to 0$$

as $v_0 \to \infty$. Returning to the original coordinate system (u, v), the first statement easily follows.

Finally, for the L^∞ estimate for $\partial_u(r\phi)$, notice that $|\partial_u(r\phi)| \le \Psi$ at the axis by (5-9) and (7) of Definition 2.1. Using (2-8″), (2-6) (in particular, the fact that $\partial_v \nu \le 0$), (5-10) and (5-6), we have

$$|\partial_u(r\phi)(u, v)| \le \Psi + \Lambda\Psi \int_u^v (-\partial_v \nu)\, dv' \le C_{K,\Lambda}\Psi. \qquad \square$$

6. Decay of ϕ and its first derivatives

In this section, we prove the first main theorem (Theorem 3.1). Throughout this section, we assume that (ϕ, r, m) is a locally BV scattering solution to (SSESF) with asymptotically flat initial data of order ω' in BV, as in Definitions 2.4 and 2.9. Let $\omega = \min\{\omega', 3\}$.

6A. *Preparatory lemmas.* The following lemma will play a key role in the proof of both Theorems 3.1 and 3.2. It is a consequence of the scattering assumption (2-12) and vanishing of the final Bondi mass.

Lemma 6.1. *Let $\epsilon > 0$ be an arbitrary positive number. For $u_1 > 1$ sufficiently large, we have*

$$\sup_{v\in[u_1,\infty)} \int_{\underline{C}_v\cap\{u\ge u_1\}} \left|\frac{2m\nu}{(1-\mu)r^2}\right| < \epsilon, \tag{6-1}$$

$$\sup_{u\in[u_1,\infty)} \int_{C_u} \left|\frac{2m\lambda}{(1-\mu)r^2}\right| < \epsilon. \tag{6-2}$$

Proof. The first statement, (6-1), was proved in Lemma 5.7; thus it only remains to prove (6-2).

Divide $\mathcal{Q}$ into $\mathcal{Q}_{\mathrm{cpt}} = \mathcal{Q}\cap\{r \le R\}$ and $\mathcal{Q}^c_{\mathrm{cpt}} := \mathcal{Q}\setminus\mathcal{Q}_{\mathrm{cpt}}$. First, note that by (2-12) we have

$$\sup_{u\in[u_1,\infty)} \int_{C_u\cap\mathcal{Q}_{\mathrm{cpt}}} \left|\frac{2m\lambda}{(1-\mu)r^2}\right| < \frac{\epsilon}{2}$$

for u_1 sufficiently large. Next, we consider $\mathcal{Q}^c_{\mathrm{cpt}}$. Define $v^\star(u) := \sup\{v \in [u,\infty) : r(u, v) \ge R\}$; note that $r(u, v^\star(u)) = R$ by continuity. We now compute

$$\int_{C_u\cap\mathcal{Q}^c_{\mathrm{cpt}}} \left|\frac{2m\lambda}{(1-\mu)r^2}\right| = \int_{v^\star(u)}^\infty \left|\frac{2m\lambda}{(1-\mu)r^2}(u, v')\right| dv' \le 2K\Lambda M(u_1) \int_{v^\star(u)}^\infty \frac{\lambda}{r^2}(u, v')\, dv' \le 2R^{-1}K\Lambda M(u_1)$$

uniformly in $u \ge u_1$. As $\lim_{u_1\to\infty} M(u_1) = 0$ by (2-11), the last line can be made arbitrarily small by taking u_1 sufficiently large. This proves (6-2). $\square$

The following lemma allows us to estimate ϕ in terms of $|\partial_v(r\phi)|$.

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[14] More precisely, we apply Theorem 5.12 to the truncated initial data

$$\partial_V(r\tilde{\phi})(U^\star, V) = \begin{cases} \partial_V(r\phi)(U^\star, V) & \text{for } V < v_0, \\ \partial_V(r\phi)(U^\star, v_0) & \text{for } V \ge v_0. \end{cases}$$

Lemma 6.2. *The following estimates hold*:

$$|\phi|(u,v) \leq \Lambda \sup_{C_u} |\partial_v(r\phi)| \quad \textit{and} \quad ru^{\omega-1}|\phi|(u,v) \leq C\Lambda\big(\sup_{C_u} u^{\omega}|\partial_v(r\phi)| + \sup_{C_u} r^{\omega}|\partial_v(r\phi)|\big).$$

Proof. The first estimate follows from Lemma 5.1 and Proposition 5.3. The second estimate is a consequence of the first when $r(u,v) \leq u$, so it suffices to assume $r(u,v) \geq u$. Introducing a parameter $v_1 \in [u,v]$, we estimate

$$\begin{aligned} ru^{\omega-1}|\phi|(u,v) &\leq u^{\omega-1} \int_u^v |\partial_v(r\phi)(u,v')|\,\mathrm{d}v' \\ &\leq \Lambda u^{\omega-1}\big(\sup_{C_u}|\partial_v(r\phi)|\big)\int_u^{v_1} \lambda(u,v')\,\mathrm{d}v' + \Lambda u^{\omega-1}\big(\sup_{C_u} r^{\omega}|\partial_v(r\phi)|\big)\int_{v_1}^v \frac{\lambda}{r^{\omega}}(u,v')\,\mathrm{d}v' \\ &\leq \Lambda\left(\frac{r(u,v_1)}{u}\right)\sup_{C_u} u^{\omega}|\partial_v(r\phi)| + \frac{\Lambda}{\omega-1}\frac{u^{\omega-1}}{r(u,v_1)^{\omega-1}}\sup_{C_u} r^{\omega}|\partial_v(r\phi)|. \end{aligned}$$

Choosing v_1 so that $r(u,v_1) = u$ (which is possible since $r(u,v) \geq u$), the desired estimate follows. □

6B. *Preliminary r-decay for ϕ.* In this subsection, we derive bounds for ϕ which are sharp in terms of r-weights. As a consequence, they give sharp decay rates towards null infinity.

Lemma 6.3. *There exists a constant* $0 < H_1 < \infty$ *such that*

$$\sup_{\mathcal{Q}} (1+r)|\phi| \leq H_1. \tag{6-3}$$

Proof. Let $r_1 > 0$ be a large number, to be chosen below. Different arguments will be used in $\{r \geq r_1\}$ and $\{r \leq r_1\}$. For each $u \geq 1$, let $v_1^{\star}(u)$ be the unique v-value for which $r(u, v_1^{\star}(u)) = r_1$. By the fundamental theorem of calculus, we have

$$r\phi = r_1\phi(u, v_1^{\star}(u)) + \int_{v_1^{\star}(u)}^v \partial_v(r\phi)(u,v')\,\mathrm{d}v'. \tag{6-4}$$

Integrate (2-8) along the incoming direction from $(1,v)$ to (u,v). By Corollary 4.4 and Proposition 5.3, we have

$$\begin{aligned} |\partial_v(r\phi)(u,v)| &\leq |\partial_v(r\phi)(1,v)| + \left|\int_1^u \frac{2m\lambda\nu}{(1-\mu)r^3}(r\phi)(u',v)\,\mathrm{d}u'\right| \\ &\leq |\partial_v(r\phi)(1,v)| + \frac{K\Lambda M_i}{2}\frac{1}{r^2(u,v)}\sup_{u'\in[1,u]} |r\phi(u',v)|. \end{aligned}$$

Substituting the preceding bound into (6-4), we obtain

$$\sup_{C_u \cap \{r \geq r_1\}} |r\phi| \leq |r_1\phi(u, v_1^{\star}(u))| + \int_{v_1^{\star}(u)}^v |\partial_v(r\phi)(1,v')|\,\mathrm{d}v' + \frac{K\Lambda^2 M_i}{2r_1}\sup_{u'\in[1,u]} sup_{C_{u'}\cap\{r\geq r_1\}}|r\phi|. \tag{6-5}$$

The first term on the right-hand side is bounded by $r_1\Lambda\Psi$, by (5-10), whereas the second term depends only on the initial data and can be estimated in terms of $\mathcal{I}_1$ as follows:

$$\int_{v_1^{\star}(u)}^v |\partial_v(r\phi)(1,v')|\,\mathrm{d}v' \leq \Lambda\mathcal{I}_1 \int_1^{\infty} (1+r(1,v'))^{-\omega'}\lambda(1,v')\,\mathrm{d}v' \leq \frac{\Lambda}{\omega'-1}\mathcal{I}_1.$$

Moreover, choosing r_1 to be large enough that

$$\frac{K\Lambda^2 M_i}{2r_1} \leq \frac{1}{2},$$

the last term of (6-5) can be absorbed in to the left-hand side and we conclude

$$\sup_{\{r\geq r_1\}} |r\phi| \leq 2r_1\Lambda\Psi + \frac{2}{\omega'-1}\Lambda\mathcal{I}_1.$$

On the other hand, in $\{r \leq r_1\}$ we have

$$\sup_{\{r\leq r_1\}} |r\phi| \leq r_1\Lambda\Psi$$

by (5-10). Combining the bounds in $\{r \geq r_1\}$ and $\{r \leq r_1\}$, the lemma follows. □

Remark 6.4. The preceding argument shows that Lemma 6.3 holds with[15]

$$H_1 \leq C_{\mathcal{I}_1,K,\Lambda}(\mathcal{I}_1 + \Psi). \tag{6-6}$$

6C. *Propagation of u-decay for $\partial_u(r\phi)$.* Here, we show that u-decay estimates proved for $\partial_v(r\phi)$ and ϕ may be "transferred" to $\partial_u(r\phi)$; this reduces the proof of Theorem 3.1 to showing only (3-1) and (3-2). To this end, we integrate $\partial_v\partial_u(r\phi)$ from the axis Γ, along which $\partial_u(r\phi) = -\partial_v(r\phi)$.

Lemma 6.5. *Suppose that there exists a finite positive constant A such that*

$$\sup_{\mathcal{Q}}|\phi| \leq Au^{-\omega} \quad \textit{and} \quad \sup_{\mathcal{Q}}|\partial_v(r\phi)| \leq Au^{-\omega}.$$

Then

$$\sup_{\mathcal{Q}}|\partial_u(r\phi)| \leq (1+K)Au^{-\omega}.$$

Proof. Fix $u \geq 1$ and $v \geq u$. Integrate (2-8″) along the outgoing direction from (u, u) to (u, v) and take the absolute value. Using (7) of Definition 2.1, (2-6) (in particular, $\partial_v\nu \leq 0$), (5-6) and the hypotheses, we have

$$|\partial_u(r\phi)(u,v)| \leq \lim_{v'\to u+}|\partial_v(r\phi)(u,v')| + \sup_{u\leq v'\leq v}|\phi(u,v')|\int_u^v(-\partial_v\nu)\,dv' \leq Au^{-\omega} + KAu^{-\omega}.$$ □

6D. *Full decay for ϕ and $\partial_v(r\phi)$.* In this subsection, we finish the proof of Theorem 3.1. By Lemma 6.5, it suffices to establish the full decay of ϕ and $\partial_v(r\phi)$, i.e., (3-1) and (3-2). For the convenience of the reader, we recall these estimates:

$$|\phi| \leq A\min\{u^{-\omega}, r^{-1}u^{-(\omega-1)}\} \quad \text{and} \quad |\partial_v(r\phi)| \leq A\min\{u^{-\omega}, r^{-\omega}\}.$$

For $U > 1$, let

$$\mathcal{B}_1(U) := \sup_{u\in[1,U]}\sup_{C_u}(u^{\omega}|\phi| + ru^{\omega-1}|\phi|).$$

[15] Notice that, while the constant $C_{\mathcal{I}_1,K,\Lambda}$ depends on $\mathcal{I}_1$, the preceding argument moreover allows us to choose $C_{\mathcal{I}_1,K,\Lambda}$ to be nondecreasing in $\mathcal{I}_1$. In particular, *for $\mathcal{I}_1$ sufficiently small*, we have $H_1 \leq C_{K,\Lambda}(\mathcal{I}_1+\Psi)$. It is for this reason that we prefer to write the expression $C_{\mathcal{I}_1,K,\Lambda}(\mathcal{I}_1+\Psi)$ instead of the more general $C_{\mathcal{I}_1,K,\Lambda,\Psi}$.

Notice that this is finite for every fixed U, by Lemma 6.3. To establish the decay estimate (3-1), it suffices to prove that $\mathcal{B}_1(U)$ is bounded by a finite constant which is *independent of* U. We will show that this also implies (3-2). Divide $\mathcal{Q}$ into $\mathcal{Q}_{\text{ext}} \cup \mathcal{Q}_{\text{int}}$, defined by

$$\mathcal{Q}_{\text{ext}} := \{(u, v) \in \mathcal{Q} : v \geq 3u\}, \qquad \mathcal{Q}_{\text{int}} := \{(u, v) \in \mathcal{Q} : v \leq 3u\}.$$

We first establish a bound for $\partial_v(r\phi)$ with the sharp r-weight, which thus gives the sharp decay rate in $\mathcal{Q}_{\text{ext}}$.

Lemma 6.6. *Let $u_1 > 1$. Then, for $u_1 \leq u \leq U$,*

$$\sup_{C_u} r^{\omega} |\partial_v(r\phi)| \leq \mathcal{I}_1 + C_{K,M_i}\, u_1 H_1 + C u_1^{-1} K M_i\, \mathcal{B}_1(U). \tag{6-7}$$

Proof. We separate the proof into cases $\omega \geq 2$ and $1 < \omega \leq 2$.

Case 1: $\omega \geq 2$. First, notice that

$$|\phi| \leq \mathcal{B}_1(U)(r^{-1}u^{-(\omega-1)})^{\omega-2}(u^{-\omega})^{1-(\omega-2)} \leq \mathcal{B}_1(U) r^{-(\omega-2)} u^{-2}.$$

Applying Lemma 6.3, we also have

$$|\phi| \leq (1+r)^{-1} H_1.$$

By Corollary 4.4 and Proposition 5.3, we have the following pointwise bounds:

$$\sup_{u' \in [1,u_1]} \left| \frac{m\lambda\nu}{1-\mu} \right| \leq \frac{K M_i}{2}, \qquad \sup_{u' \in [u_1,\infty)} \left| \frac{m\lambda\nu}{1-\mu} \right| \leq \frac{K M(u_1)}{2}.$$

Therefore, integrating (2-8) along the incoming direction from $(1, v)$ to (u, v), we have

$$\begin{aligned} |\partial_v(r\phi)(u,v)| &\leq |\partial_v(r\phi)(1,v)| + \left| \int_1^u \frac{2m\lambda\nu\phi}{(1-\mu)r^2}(u', v)\, du' \right| \\ &\leq |\partial_v(r\phi)(1,v)| + \frac{K M_i}{r^2(u,v)(1+r(u,v))} H_1 \int_1^{u_1} du' + \frac{K M(u_1)}{r^{\omega}(u,v)} \mathcal{B}_1(U) \int_{u_1}^{u} (u')^{-2}\, du' \\ &\leq |\partial_v(r\phi)(1,v)| + \frac{u_1 K M_i}{r^2(u,v)(1+r(u,v))} H_1 + \frac{K M(u_1)}{u_1 r^{\omega}(u,v)} \mathcal{B}_1(U). \end{aligned}$$

Multiplying both sides by $r^{\omega}(u, v)$ and using the fact that $r(u, v) \leq r(1, v)$, we conclude

$$\begin{aligned} r^{\omega}|\partial_v(r\phi)|(u,v) &\leq r^{\omega}|\partial_v(r\phi)|(1,v) + u_1 \frac{r^{\omega-2}}{1+r} K M_i\, H_1 + u_1^{-1} K M(u_1)\, \mathcal{B}_1(U) \\ &\leq \mathcal{I}_1 + C_{u_1,K,M_i} H_1 + u_1^{-1} K M_i\, \mathcal{B}_1(U). \end{aligned}$$

Case 2: $1 < \omega \leq 2$. We will use the following bounds for ϕ. First,

$$|\phi| \leq \mathcal{B}_1(U)(r^{-1}u^{-(\omega-1)})^{\omega-1}(u^{-\omega})^{(2-\omega)} \leq \mathcal{B}_1(U) r^{-(\omega-1)} u^{-1}.$$

Also, Lemma 6.3 implies

$$|\phi| \leq (1+r)^{-1} H_1.$$

As in Case 1 we integrate (2-8) along the incoming direction from $(1, v)$ to (u, v):

$$\begin{aligned}|\partial_v(r\phi)(u,v)| &\le |\partial_v(r\phi)(1,v)| + \left|\int_1^u \frac{2m\lambda\nu\phi}{(1-\mu)r^2}(u',v)\,du'\right| \\ &\le |\partial_v(r\phi)(1,v)| + \frac{K\Lambda M_i H_1}{1+r}\int_1^{u_1}\frac{-\nu}{r^2}\,du' + \frac{K\Lambda M(u_1)}{u_1}\mathcal{B}_1(U)\int_{u_1}^u \frac{-\nu}{r^{\omega+1}}\,du' \\ &\le |\partial_v(r\phi)(1,v)| + \frac{\omega K\Lambda M_i}{r(u,v)(1+r(u,v))}H_1 + \frac{\omega K\Lambda M(u_1)}{u_1 r^\omega(u,v)}\mathcal{B}_1(U).\end{aligned}$$

Multiply both sides by r^ω to arrive at the conclusion, as in Case 1. In this case, note that the second term is a bit better than what is claimed, as there is no dependence on $u_1 \ge 1$. □

Remark 6.7. The proof of this lemma requires $\omega \le 3$. More precisely, this limitation comes from the contribution of the right-hand side of (2-8)

We are now ready to prove the bounds (3-1) and (3-2). The idea is to "propagate" the exterior decay estimate (6-7) into $\mathcal{Q}_{\text{int}}$ to obtain decay in u, using the smallness coming from Lemma 6.1 in the region where u is sufficiently large. On the other hand, the preliminary r-decay estimates proved in Section 6B will give the desired r-decay rates in rest of the space-time.

Proof of (3-1) *and* (3-2). Let $1 \le u_1 \le U$. For $(u, v) \in \mathcal{Q}$ with $u \in [3u_1, U]$, integrate (2-8) along the incoming direction from $(u/3, v)$ to (u, v). Then

$$|\partial_v(r\phi)(u,v)| \le |\partial_v(r\phi)(u/3,v)| + \tfrac{1}{2}\Big(\sup_{u'\in[u/3,u]}\sup_{C_{u'}}|\phi|\Big)\int_{u/3}^u\left|\frac{2m\nu}{(1-\mu)r^2}(u',v)\right|du'. \tag{6-8}$$

Multiply both sides by u^ω and estimate each term on the right-hand side. For the first term, the key observation is the following: for $v \ge u$, the point $\big(\tfrac{1}{3}u, v\big)$ lies in $\mathcal{Q}_{\text{ext}}$, where (6-7) is effective. Indeed, note that

$$\big(\tfrac{2}{3}\Lambda\big)u \le \Lambda^{-1}\big(v - \big(\tfrac{1}{3}u\big)\big) \le r\big(\tfrac{1}{3}u, v\big).$$

Thus, by (6-7),

$$\begin{aligned}u^\omega\big|\partial_v(r\phi)\big(\tfrac{1}{3}u,v\big)\big| &\le \big(\tfrac{3}{2}\Lambda\big)^\omega\big(r^\omega\big(\tfrac{1}{3}u,v\big)\big|\partial_v(r\phi)\big(\tfrac{1}{3}u,v\big)\big|\big) \\ &\le \big(\tfrac{3}{2}\Lambda\big)^\omega\big(\mathcal{I}_1 + C_{u_1,K,M_i}H_1 + Cu_1^{-1}KM_i\mathcal{B}_1(U)\big) \\ &\le C_{u_1,K,\Lambda,M_i}(\mathcal{I}_1 + H_1) + C_{K,\Lambda}M_i u_1^{-1}\mathcal{B}_1(U).\end{aligned}$$

For the second term on the right-hand side of (6-8), we have

$$\frac{u^\omega}{2}\Big(\sup_{u'\in[u/3,u]}\sup_{C_{u'}}|\phi|\Big)\int_{u/3}^u\left|\frac{2m\nu}{(1-\mu)r^2}(u',v)\right|du' \le \frac{3^\omega}{2}\left(\int_{u/3}^u\left|\frac{2m\nu}{(1-\mu)r^2}(u',v)\right|du'\right)\mathcal{B}_1(U).$$

Combining these estimates, we deduce

$$\begin{aligned}&\sup_{C_u} u^\omega|\partial_v(r\phi)(u,v)| \\ &\qquad\le C_{u_1,K,\Lambda,M_i}(\mathcal{I}_1 + H_1) + \left(C_{K,\Lambda}M_i u_1^{-1} + C\int_{u/3}^u\left|\frac{2m\nu}{(1-\mu)r^2}(u',v)\right|du'\right)\mathcal{B}_1(U).\end{aligned} \tag{6-9}$$

Recalling the bounds of ϕ in terms of $\partial_v(r\phi)$ in Lemma 6.2, we have

$$\mathcal{B}_1(U) \le (1+2\Lambda) \sup_{u\in[1,U]} \sup_{C_u} \big(u^{\omega}|\partial_v(r\phi)| + r^{\omega}|\partial_v(r\phi)|\big).$$

The right-hand side can be controlled by (6-9) and (6-7), from which we conclude

$$\mathcal{B}_1(U) \le C_{u_1,K,\Lambda,M_i}(\mathcal{I}_1 + H_1) + \left(C_{K,\Lambda} M_i u_1^{-1} + C \int_{u/3}^{u} \left| \frac{2m\nu}{(1-\mu)r^2}(u',v)\right| \mathrm{d}u' \right) \mathcal{B}_1(U). \tag{6-10}$$

As a consequence of Lemma 6.1, the entire coefficient in front of $\mathcal{B}_1(U)$ can made to be smaller than (say) $\frac{1}{2}$ by taking u_1 sufficiently large. Since $\mathcal{B}_1(U) < \infty$, we can then absorb this term into the left-hand side. Observing that this bound is independent of $U > 1$, we have thus obtained (3-1).

To prove (3-2), simply apply (6-9) and (6-7), which shows that

$$\sup_{u\in[1,U]} \sup_{C_u} \big(u^{\omega}|\partial_v(r\phi)| + r^{\omega}|\partial_v(r\phi)|\big) \le C_{u_1,K,\Lambda,M_i}(\mathcal{I}_1 + H_1) + \left(C_{K,\Lambda} M_i u_1^{-1} + C \int_{u/3}^{u} \left| \frac{2m\nu}{(1-\mu)r^2}(u',v)\right| \mathrm{d}u' \right) \mathcal{B}_1(U).$$

This boundedness of $\mathcal{B}_1(U)$ that we just proved thus implies (3-2). □

Remark 6.8. According to the proof that we have just given, the constant $A_1 > 0$ depends on our choice of $u_1 > 1$, which in turn depends on how fast the coefficient in front of $\mathcal{B}_1(U)$ in (6-10) vanishes as $u_1 \to \infty$. This explains why $A_1 > 0$ does not depend only on the size of the initial data, as remarked in Section 3. Controlling the size of $u_1 > 1$ under an additional small data assumption will be key to proving (1) of Theorem 3.15 in Section 9.

6E. *Additional decay estimates.* We end this section with the following decay estimates for $\partial_v\phi$, $\partial_u\phi$ and m.

Corollary 6.9. *Let (ϕ, r, m) be a locally BV scattering solution to (SSESF) with asymptotically flat initial data of order ω' in BV, and define $\omega = \min\{\omega', 3\}$. Let A_1 be the constant in Theorem 3.1. Then the following decay estimates hold*:

$$|\partial_v\phi| \le C A_1 \min\{r^{-1}u^{-\omega}, r^{-2}u^{-(\omega-1)}\}, \tag{6-11}$$

$$|\partial_u\phi| \le C_K A_1 r^{-1}u^{-\omega}, \tag{6-12}$$

$$m \le C_\Lambda A_1^2 \min\{r u^{-2\omega}, u^{-(2\omega-1)}\}. \tag{6-13}$$

Proof. Let $u \ge 1$ and $v \ge u$. Since

$$r\partial_v\phi = \partial_v(r\phi) - \lambda\phi, \quad r\partial_u\phi = \partial_u(r\phi) - \nu\phi,$$

the estimates (6-11) and (6-12) follow from (3-1)–(3-3) and the fact that $\sup_{\mathcal{Q}}|\lambda| \le \frac{1}{2}$, $\sup_{\mathcal{Q}}|\nu| \le K$.

On the other hand, by (2-7), we have

$$m(u,v) = \frac{1}{2}\int_u^v \lambda^{-1}(1-\mu)r^2(\partial_v\phi)^2(u,v')\,\mathrm{d}v'. \tag{6-14}$$

Using $|\partial_v \phi(u,v)| \le C A_1 r^{-1} u^{-\omega}$ (which has just been established), we obtain

$$m(u,v) \le C_\Lambda A_1^2 r u^{-2\omega},$$

which proves a "half" of (6-13). To prove the other "half", let us introduce a parameter $r_1 > 0$ (to be determined later) and define $v_1^\star(u)$ to be the unique v-value such that $r(u, v_1^\star(u)) = r_1$. For $v \ge v_1^\star(u)$, split the v'-integral in (6-14) into $\int_u^{v_1^\star(u)} + \int_{v_1^\star(u)}^{v}$ and use $|\partial_v \phi(u,v)| \le C A_1 r^{-1} u^{-\omega}$ for the former and $|\partial_v \phi(u,v)| \le C A_1 r^{-2} u^{-(\omega-1)}$ for the latter. As $m(u,v)$ is nondecreasing in v, we then arrive at the estimate

$$\sup_{C_u} m \le C_\Lambda A_1^2 r_1 u^{-2\omega} + C_\Lambda A_1^2 r_1^{-1} u^{-2(\omega-1)}.$$

Choosing $r_1 = u$, we obtain (6-13). □

7. Decay of second derivatives

In this section, we establish our second main theorem (Theorem 3.2). Throughout the section, we assume that (ϕ, r, m) is a locally BV scattering solution to (SSESF) with asymptotically flat initial data of order ω' in C^1, as in Definitions 2.4 and 2.9. As discussed in Remark 2.3, (ϕ, r, m) is then a C^1 solution to (SSESF). As before, let $\omega = \min\{\omega', 3\}$.

7A. *Preparatory lemmas.* The following lemma, along with Lemma 6.1, provides the crucial smallness for our proof of Theorem 3.2.

Lemma 7.1. *For every $\epsilon > 0$, there exists $u_2 > 1$ such that*

$$\sup_{v \in [u_2, \infty)} \int_{\underline{C}_v \cap \{u \ge u_2\}} |\partial_u \phi| < \epsilon, \tag{7-1}$$

$$\sup_{u \in [u_2, \infty)} \int_{C_u} |\partial_v \phi| < \epsilon. \tag{7-2}$$

Proof. We will only prove (7-1), leaving the similar proof of (7-2) to the reader. As in the proof of Lemma 6.1, we divide $\mathcal{Q}$ into $\mathcal{Q}_{\mathrm{cpt}} := \mathcal{Q} \cap \{r \le R\}$ and $\mathcal{Q}_{\mathrm{cpt}}^c := \mathcal{Q} \setminus \mathcal{Q}_{\mathrm{cpt}}$, and argue separately. First, by Theorem 5.14, we have

$$\sup_{v \in [u_2, \infty)} \int_{\underline{C}_v \cap \{u \ge u_2\} \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_u \phi| < \frac{\epsilon}{2}$$

for u_2 sufficiently large. Next, to derive (7-1) in $\mathcal{Q}_{\mathrm{cpt}}^c$, we define $u^\star(v) := \sup\{u \in [u_2, v] : r(u,v) \ge R\}$, where we use the convention $u^\star(v) = u_2$ when the set is empty. Then, using Proposition 5.3 and Schwarz, we compute

$$\begin{aligned}\int_{\underline{C}_v \cap \{u \ge u_2\} \cap \mathcal{Q}_{\mathrm{cpt}}^c} |\partial_u \phi| = \int_{u_2}^{u^\star(v)} |\partial_u \phi(u', v)| \, du' &\le \sqrt{\frac{2K\Lambda}{R}} \sqrt{\int_{u_2}^{u^\star(v)} \tfrac{1}{2}(-\nu)^{-1}(1-\mu) r^2 (\partial_u \phi)^2(u', v) \, du'} \\ &\le \sqrt{\frac{2K\Lambda}{R} m(u_2, v)} \le \sqrt{\frac{2K\Lambda}{R} M(u_2)}.\end{aligned}$$

By (2-11), $\lim_{u_2\to\infty} M(u_2) = 0$, and (7-1) thus follows. □

The next lemma allows us to estimate the first derivative of ϕ at (u, v) in terms of information on $C_u \cap \{(u, v') : u \le v' \le v\}$.

Lemma 7.2. *For every* $(u, v) \in \mathcal{Q}$, *the following inequalities hold*:

$$|\partial_v \phi(u, v)| \le \tfrac{1}{4}\Lambda^2 \sup_{u\le v'\le v} |\partial_v^2(r\phi)(u, v')| + \tfrac{1}{4}\Lambda^3 \sup_{u\le v'\le v} |\partial_v(r\phi)(u, v')| \sup_{u\le v'\le v} |\partial_v \lambda(u, v')|,$$
$$|\partial_u \phi(u, v)| \le \Lambda \sup_{u\le v'\le v} (-\nu)(u, v')|\partial_v \phi(u, v')|.$$

Proof. The first is an easy consequence of (5-4) in Section 5A. To prove the second inequality, we start from the equation

$$\partial_v(r\partial_u \phi) = -\nu \partial_v \phi,$$

which follows from (2-6) and (2-8). Therefore, we have

$$|\partial_u \phi(u, v)| \le \frac{1}{r(u, v)} \int_u^v (-\nu)|\partial_v \phi|(u, v')\, dv',$$

from which the second inequality easily follows. □

In the next lemma, we show that improved estimates for m near Γ hold if we assume an L^∞ control of $\partial_v \phi$.

Lemma 7.3. *For every* $(u, v) \in \mathcal{Q}$, *the following inequalities hold*:

$$\frac{\mu}{r}(u, v) \le \Lambda^2 \sup_{u\le v'\le v} |\partial_v(r\phi)(u, v')| \sup_{u\le v'\le v} |\partial_v \phi(u, v')|, \tag{7-3}$$
$$\frac{\mu}{r^2}(u, v) \le \tfrac{1}{3}\Lambda^2 \sup_{u\le v'\le v} |\partial_v \phi(u, v')|^2. \tag{7-4}$$

Proof. Recall $\mu = 2m/r$. By (2-7), we have

$$2m(u, v) = \int_u^v (1-\mu)\lambda^{-1} r^2 (\partial_v \phi)^2(u, v')\, dv'.$$

Pulling everything except $r^2\lambda$ outside the integral and using $\int_u^v r^2\lambda(u, v')\, dv' = \frac{1}{3}r^3(u, v)$, we obtain (7-4). On the other hand, using $\lambda^{-1} r \partial_v \phi = \lambda^{-1}\partial_v(r\phi) - \phi$ and $\int_u^v r\lambda(u, v')\, dv' = \frac{1}{2}r^2(u, v)$, we easily deduce

$$\frac{\mu}{r}(u, v) \le \tfrac{1}{2} \sup_{u\le v'\le v} \big(\Lambda^2|\partial_v(r\phi)(u, v')| + \Lambda|\phi(u, v')|\big)|\partial_v \phi(u, v')|.$$

From the fact that $|\phi(u, v)| \le \Lambda \sup_{u\le v'\le v}|\partial_v(r\phi)(u, v')|$, (7-3) easily follows. □

7B. *Preliminary r-decay for $\partial_v^2(r\phi)$ and $\partial_v\lambda$.* In this subsection, we establish decay estimates for $\partial_v^2(r\phi)$ and $\partial_v\lambda$ which are sharp in terms of r-weights in the region $\mathcal{Q}_{\text{ext}}$. We remind the reader the decomposition $\mathcal{Q} = \mathcal{Q}_{\text{ext}} \cup \mathcal{Q}_{\text{int}}$, where

$$\mathcal{Q}_{\text{ext}} = \{(u, v) \in \mathcal{Q} : v \geq 3u\}, \qquad \mathcal{Q}_{\text{int}} = \{(u, v) \in \mathcal{Q} : v \leq 3u\}.$$

In particular, note that $r \geq 2\Lambda^{-1}u > 0$ in $\mathcal{Q}_{\text{ext}}$.

Lemma 7.4. *The following estimates hold*:

$$\sup_{\mathcal{Q}_{\text{ext}}} r^3 |\partial_v \lambda| \leq C_{K,\Lambda} A_1^2, \tag{7-5}$$

$$\sup_{\mathcal{Q}_{\text{ext}}} r^{\omega+1} |\partial_v^2(r\phi)| \leq C\mathcal{I}_2 + C_{K,\Lambda,M_i} A_1^3. \tag{7-6}$$

Proof. We begin by proving (7-5). Recall (4-6):

$$\partial_u \partial_v \log\lambda = \frac{1}{1-\mu}\lambda^{-1}\nu(\partial_v\phi)^2 - \frac{4m}{(1-\mu)r^3}\lambda\nu.$$

Note that $\partial_v \log\lambda = 0$ on C_1 by our choice of coordinates. Therefore, integrating the preceding equation along the incoming direction from $(1, v)$ to (u, v), we have

$$|\partial_v \log\lambda(u, v)| \leq \int_1^u \left| \frac{1}{1-\mu}\lambda^{-1}\nu(\partial_v\phi)^2(u', v) \right| du' + \int_1^u \left| \frac{4m}{(1-\mu)r^3}\lambda\nu(u', v) \right| du'.$$

Then (7-5) follows using Proposition 5.3, (6-11) and (6-13). We remark that the power of r is dictated by the second integral.

The proof of (7-6) is very similar. We start by recalling (4-2):

$$\partial_u(\partial_v^2(r\phi)) = \frac{2m\lambda\nu}{(1-\mu)r^2}\,\partial_v\phi + \frac{\nu}{1-\mu}(\partial_v\phi)^2\phi + \frac{2m\nu}{(1-\mu)r^2}(\partial_v\lambda)\phi - \frac{4m}{(1-\mu)r^3}\lambda^2\nu\phi.$$

For $u \geq 1$, we have $r(u, v) \leq r(1, v)$; moreover, by hypothesis, we have the estimate for the initial data term

$$(1 + r(1, v))^{\omega'+1} |\partial_v^2(r\phi)(1, v)| \leq \mathcal{I}_2.$$

Therefore, by the fundamental theorem of calculus, it suffices to bound

$$\int_1^u \left| \frac{2m\lambda\nu}{(1-\mu)r^2}\,\partial_v\phi(u', v) \right| du' + \int_1^u \left| \frac{\nu}{1-\mu}(\partial_v\phi)^2\phi(u', v) \right| du' \\ + \int_1^u \left| \frac{2m\nu}{(1-\mu)r^2}(\partial_v\lambda)\phi(u', v) \right| du' + \int_1^u \left| \frac{4m}{(1-\mu)r^3}\lambda^2\nu\phi(u', v) \right| du'$$

by $C_{K,\Lambda,M_i} A_1^3 r^{-(\omega+1)}$. This is an easy consequence of Proposition 5.3, (3-1), (6-11), (6-13) and also (7-5), which has just been established. Note that the last term is what limits $\omega \leq 3$. □

7C. ***Propagation of u-decay for $\partial_u^2(r\phi)$ and $\partial_u\nu$.*** Here, we show that certain u-decay for $\partial_u^2(r\phi)$ and $\partial_u\nu$ proved in $\mathcal{Q}_{\mathrm{int}}$ can be propagated to $\mathcal{Q}$. The technique employed is very similar to that in the previous subsection.

Lemma 7.5. *For $U \geq 1$, suppose that there exist finite positive constants A, k_1 and k_2 such that*

$$0 \leq k_1 \leq 2\omega + 1, \quad 0 \leq k_2 \leq 3\omega + 1$$

and, for $u \in [1, U]$, we have

$$\sup_{C_u \cap \mathcal{Q}_{\mathrm{int}}} u^{k_1}|\partial_u\nu| \leq A \quad \text{and} \quad \sup_{C_u \cap \mathcal{Q}_{\mathrm{int}}} u^{k_2}|\partial_u^2(r\phi)| \leq A.$$

Then for $u \in [1, U]$, the following estimates hold:

$$\sup_{C_u} u^{k_1}|\partial_u\nu| \leq C_{K,\Lambda}A + C_{K,\Lambda}A_1^2, \tag{7-7}$$

$$\sup_{C_u} u^{k_2}|\partial_u^2(r\phi)| \leq A + C_{K,\Lambda}A_1^3 + C_{K,\Lambda}A_1^3 \sup_{C_u} u|\partial_u\nu|. \tag{7-8}$$

Furthermore, the following alternative to (7-8) also holds:

$$\sup_{C_u} u^{k_2}|\partial_u^2(r\phi)| \leq A + C_{K,\Lambda}A_1^3 + C_{K,\Lambda}\Psi \int_{3u}^{\infty} \left|\frac{2m\lambda}{(1-\mu)r^2}\right|(u, v')\, dv' \cdot \sup_{C_u} u^{k_2}|\partial_u\nu|. \tag{7-9}$$

Proof. Let us begin with (7-7). Recall (4-7):

$$\partial_v\partial_u \log\nu = \frac{1}{1-\mu}\lambda\nu^{-1}(\partial_u\phi)^2 - \frac{4m}{(1-\mu)r^3}\lambda\nu.$$

Given $(u, v) \in \mathcal{Q}_{\mathrm{ext}}$ (with $u \in [1, U]$), let us integrate this equation along the outgoing direction from $(u, 3u)$ to (u, v), take the absolute value and multiply by u^{k_1}. Using the hypothesis

$$\sup_{\mathcal{Q}_{\mathrm{int}} \cap \{(u,v) \in \mathcal{Q}: u \in [1,U]\}} u^{k_1}|\partial_u\nu| \leq A,$$

(7-7) is reduced to showing

$$u^{k_1}\int_{3u}^{\infty}\left|\frac{1}{1-\mu}\lambda\nu^{-1}(\partial_u\phi)^2(u, v)\right| dv \leq C_{K,\Lambda}A_1^2, \tag{7-10}$$

$$u^{k_1}\int_{3u}^{\infty}\left|\frac{4m}{(1-\mu)r^3}\lambda\nu(u, v)\right| dv \leq C_{K,\Lambda}A_1^2, \tag{7-11}$$

for $u \in [1, U]$.

Using Proposition 5.3 and (6-12), the left-hand side of (7-10) is bounded by

$$C_{K,\Lambda}A_1^2 u^{k_1-2\omega}\int_{3u}^{\infty}\frac{1}{r^2}\lambda\, dv \leq C_{K,\Lambda}A_1^2 u^{k_1-2\omega}r^{-1}(u, 3u).$$

As $u \geq 1$ and $r(u, 3u) \geq 2\Lambda^{-1}u$, (7-10) follows. Similarly, by (5-7) and (6-13), the left-hand side of (7-11) is also bounded by‘$C_{K,\Lambda}A_1^2 u^{k_1-2\omega}r^{-1}(u, 3u)$, from which (7-11) immediately follows.

Next, we turn to (7-8) and (7-9); as they are proved similarly to before, we will only outline the main points. Recall (4-4):

$$\partial_v(\partial_u^2(r\phi)) = \frac{2m\lambda\nu}{(1-\mu)r^2}\,\partial_u\phi + \frac{\lambda}{1-\mu}(\partial_u\phi)^2\phi + \frac{2m\lambda}{(1-\mu)r^2}(\partial_u\nu)\phi - \frac{4m}{(1-\mu)r^3}\lambda\nu^2\phi.$$

Fix $(u, v) \in \mathcal{Q}_{\text{ext}}$ with $u \in [1, U]$. We then integrate the preceding equation along the outgoing direction from $(u, 3u)$ to (u, v), take the absolute value and multiply by u^{k_2}. In order to prove (7-8), in view of the hypothesis

$$\sup_{\mathcal{Q}_{\text{int}}\cap\{(u,v)\in\mathcal{Q}:u\in[1,U]\}} u^{k_2}|\partial_u^2(r\phi)| \le A,$$

it suffices to establish the following estimates for $u \in [1, U]$:

$$u^{k_2}\int_{3u}^{\infty}\left|\frac{2m\lambda\nu}{(1-\mu)r^2}\,\partial_u\phi(u,v)\right| dv \le C_{K,\Lambda}A_1^3,$$

$$u^{k_2}\int_{3u}^{\infty}\left|\frac{\lambda}{1-\mu}(\partial_u\phi)^2\phi(u,v)\right| dv \le C_{K,\Lambda}A_1^3,$$

$$u^{k_2}\int_{3u}^{\infty}\left|\frac{2m\lambda}{(1-\mu)r^2}(\partial_u\nu)\phi(u,v)\right| dv \le C_{K,\Lambda}A_1^3\,\sup_{\mathcal{Q}} u|\partial_u\nu|,$$

$$u^{k_2}\int_{3u}^{\infty}\left|\frac{4m}{(1-\mu)r^3}\lambda\nu^2\phi(u,v)\right| dv \le C_{K,\Lambda}A_1^3.$$

The proof of these estimates is similar to that of (7-10) and (7-11); we omit the details. To prove (7-9), we replace the third estimate by

$$u^{k_2}\int_{3u}^{\infty}\left|\frac{2m\lambda}{(1-\mu)r^2}(\partial_u\nu)\phi(u,v)\right| dv \le C_{K,\Lambda}\Psi\int_{3u}^{\infty}\left|\frac{2m\lambda}{(1-\mu)r^2}\right|(u,v')\,dv'\cdot\sup_{C_u} u^{k_2}|\partial_u\nu|,$$

which is an easy consequence of Proposition 5.3. □

7D. ***Full decay for $\partial_v^2(r\phi)$, $\partial_u^2(r\phi)$, $\partial_v\lambda$ and $\partial_u\nu$.*** With all the preparations so far, we are finally ready to prove Theorem 3.2. Our proof consists of two steps. The first step is to use the local BV scattering assumption to prove a preliminary decay rate of $u^{-\omega}$ for $\partial_v^2(r\phi)$, $\partial_u^2(r\phi)$, $\partial_v\lambda$ and $\partial_u\nu$. In this step, it is crucial to pass to the *renormalized variables* and exploit the null structure of (SSESF), in order to utilize the a priori bounds in the local BV scattering assumption. The second step to upgrade these decay rates to those that are claimed in Theorem 3.2. Thanks to the preliminary $u^{-\omega}$ decay from the first step, the null structure is not necessary at this point.

We now begin with the first step. The null structure of (SSESF), as demonstrated in Section 4C, is used in an essential way.

Proposition 7.6. *There exists a finite constant $A_2' > 0$ such that*

$$|\partial_v^2(r\phi)| \le A_2' u^{-\omega}, \qquad |\partial_u^2(r\phi)| \le A_2' u^{-\omega},$$
$$|\partial_v\lambda| \le A_2' u^{-\omega}, \qquad |\partial_u\nu| \le A_2' u^{-\omega}.$$

Proof. For $U > 1$, we define

$$\mathcal{B}_2(U) := \sup_{u\in[1,U]} \sup_{C_u} \big(u^\omega|\partial_v^2(r\phi)| + u^\omega|\partial_u^2(r\phi)| + u^\omega|\partial_v\lambda| + u^\omega|\partial_u\nu|\big). \tag{7-12}$$

Notice that the above is finite for every fixed U due to Lemmas 7.4 and 7.5. As indicated earlier, we need to use the null structure of the system (SSESF) as in Section 4C. For convenience, we define the shorthands

$$F_1 := \partial_v^2(r\phi) - (\partial_v\lambda)\phi,$$
$$G_1 := \partial_u^2(r\phi) - (\partial_u\nu)\phi,$$

and

$$F_2 := \partial_v \log\lambda - \frac{\lambda}{1-\mu}\frac{\mu}{r} + \partial_v\phi(\lambda^{-1}\partial_v(r\phi) - \nu^{-1}\partial_u(r\phi)),$$
$$G_2 := \partial_u \log(-\nu) - \frac{\nu}{1-\mu}\frac{\mu}{r} + \partial_u\phi(\lambda^{-1}\partial_v(r\phi) - \nu^{-1}\partial_u(r\phi)).$$

Then (4-3), (4-5), (4-8) and (4-9) may be rewritten in the following fashion:

$$\partial_u F_1 = \partial_u\lambda\,\partial_v\phi - \partial_v\lambda\,\partial_u\phi, \tag{7-13}$$
$$\partial_u F_2 = \partial_u\phi\,\partial_v(\nu^{-1}\partial_u(r\phi)) - \partial_v\phi\,\partial_u(\nu^{-1}\partial_u(r\phi)), \tag{7-14}$$
$$\partial_v G_1 = \partial_v\nu\,\partial_u\phi - \partial_u\nu\,\partial_v\phi, \tag{7-15}$$
$$\partial_v G_2 = -\partial_u\phi\,\partial_v(\lambda^{-1}\partial_v(r\phi)) + \partial_v\phi\,\partial_u(\lambda^{-1}\partial_v(r\phi)). \tag{7-16}$$

The following lemma is the key technical component of the proof:

Lemma 7.7. *There exists a finite positive constant $C = C_{A_1,\mathcal{I}_2,K,\Lambda}$ and positive function $\epsilon(u)$ satisfying*

$$\epsilon(u) \to 0 \quad \textit{as } u \to \infty$$

such that the following inequalities holds for $1 \le u_2 \le U$:

$$\sup_{\mathcal{Q}_{\text{int}}\cap\{(u,v)\in\mathcal{Q}:u\in[3u_2,U]\}} (u^\omega|F_1| + u^\omega|G_1|) \le C_\Lambda\mathcal{I}_2 + C_{K,\Lambda,M_i}A_1^3 + \epsilon(u_2)\mathcal{B}_2(U), \tag{7-17}$$

$$\sup_{\mathcal{Q}_{\text{int}}\cap\{(u,v)\in\mathcal{Q}:u\in[3u_2,U]\}} (u^\omega|F_2| + u^\omega|G_2|) \le C_{K,\Lambda}A_1^2 + \epsilon(u_2)\mathcal{B}_2(U). \tag{7-18}$$

We defer the proof of this lemma. Instead, we first finish the proof of Proposition 7.6, assuming Lemma 7.7.

First, we claim that (7-17) and (7-18) imply

$$\sup_{\mathcal{Q}_{\text{int}}\cap\{(u,v)\in\mathcal{Q}:u\in[3u_2,U]\}} u^\omega\big(|\partial_v^2(r\phi)| + |\partial_u^2(r\phi)| + |\partial_v\lambda| + |\partial_u\nu|\big) \le H_2 + (\epsilon+\epsilon')(u_2)\mathcal{B}_2(U) \tag{7-19}$$

for some constant $0 < H_2 < \infty$ and some positive function $\epsilon'(u_2)$ which tends to zero as $u_2 \to \infty$.

The point is that F_1, F_2, G_1 and G_2 control $\partial_v^2(r\phi)$, $\partial_u^2(r\phi)$, $\partial_v\lambda$ and $\partial_u\nu$, respectively, up to higher-order terms, which may be absorbed into the second term on the right-hand side. Indeed, consider $u \in [3u_2, U]$.

For F_1 and G_1, we estimate

$$u^\omega|\partial_v^2(r\phi)(u,v)| = u^\omega|F_1 + (\partial_v\lambda)\phi|(u,v) \le u^\omega|F_1(u,v)| + \sup_{C_u}|\phi|\cdot\mathcal{B}_2(U),$$
$$u^\omega|\partial_u^2(r\phi)(u,v)| = u^\omega|G_1 + (\partial_u\nu)\phi|(u,v) \le u^\omega|G_1(u,v)| + \sup_{C_u}|\phi|\cdot\mathcal{B}_2(U),$$

which are acceptable, because $\sup_{C_u}|\phi| \to 0$ as $u \ge 3u_2 \to \infty$, by Theorem 3.1. For F_2, we use Proposition 5.3 to estimate

$$\begin{aligned} u^\omega|\partial_v\lambda| &= u^\omega\lambda\left|F_2 + \frac{\lambda}{1-\mu}\frac{\mu}{r} + \partial_v\phi(\lambda^{-1}\partial_v(r\phi) - \nu^{-1}\partial_u(r\phi))\right| \\ &\le \tfrac{1}{2}u^\omega|F_2| + \tfrac{1}{4}K\Lambda u^\omega\left|\frac{\mu}{r}\right| + \tfrac{1}{2}\Lambda u^\omega|\partial_v\phi|\big(|\partial_v(r\phi)| + |\partial_u(r\phi)|\big). \end{aligned}$$

Applying (7-3) (from Lemma 7.3) to the second term on the last line, and then using Lemma 7.2 to control $u^\omega|\partial_v\phi|$, we arrive at

$$u^\omega|\partial_v\lambda(u,v)| \le \tfrac{1}{2}u^\omega|F_2(u,v)| + C_{K,\Lambda}\,\Psi\sup_{C_u}\big(|\partial_v(r\phi)| + |\partial_u(r\phi)|\big)\cdot\mathcal{B}_2(U),$$

which is acceptable in view of Theorem 3.1. Proceeding similarly, but also using the second inequality of Lemma 7.2 to control $|\partial_u\phi|$, we obtain

$$u^\omega|\partial_u\nu(u,v)| \le Ku^\omega|G_2(u,v)| + C_{K,\Lambda}\,\Psi\sup_{C_u}\big(|\partial_v(r\phi)| + |\partial_u(r\phi)|\big)\cdot\mathcal{B}_2(U).$$

Combining these estimates with (7-17) and (7-18), we conclude (7-19) with

$$H_2 = C_\Lambda\mathcal{I}_2 + C_{K,\Lambda,M_i}A_1^3 + C_{K,\Lambda}A_1^2, \tag{7-20}$$
$$\epsilon'(u_2) = C\sup_{u\ge 3u_2}|\phi| + C_{K,\Lambda}\Psi\sup_{u\ge 3u_2}\big(|\partial_v(r\phi)| + |\partial_u(r\phi)|\big). \tag{7-21}$$

Next, note that the (nondecreasing) function

$$H_2'(u_2) := \sup_{\mathcal{Q}_{\mathrm{int}}\cap\{(u,v)\in\mathcal{Q}:u\in[1,3u_2]\}} u^\omega\big(|\partial_v^2(r\phi)| + |\partial_u^2(r\phi)| + |\partial_v\lambda| + |\partial_u\nu|\big) \ge 0 \tag{7-22}$$

is always *finite* for any fixed $u_2 \ge 1$, as the region $\mathcal{Q}_{\mathrm{int}}\cap\{(u,v)\in\mathcal{Q}: u\in[1,3u_2]\}$ is compact and each of these terms is a continuous function, since (ϕ, r, m) is a C^1 solution (see Definition 2.2). Combining with (7-19), we obtain

$$\sup_{\mathcal{Q}_{\mathrm{int}}\cap\{(u,v)\in\mathcal{Q}:u\in[1,U]\}} u^\omega\big(|\partial_v^2(r\phi)| + |\partial_u^2(r\phi)| + |\partial_v\lambda| + |\partial_u\nu|\big) \le H_2 + H_2'(u_2) + (\epsilon+\epsilon')(u_2)\mathcal{B}_2(U)$$

for every $u_2 \in [1, U]$.

Now apply (7-7) and (7-9) in Lemma 7.5 to $\partial_u^2(r\phi)$, $\partial_u\nu$. Also apply Lemma 7.4 (along with the fact that $r(u,v) \ge 2\Lambda^{-1}u$ in $\mathcal{Q}_{\mathrm{ext}}$ and $\omega \le 3$) to $\partial_v^2(r\phi)$, $\partial_v\lambda$ in $\mathcal{Q}_{\mathrm{ext}}$. Then we see that there exist a nonnegative and nondecreasing function $H_2''(u_2)$ and a positive function $\epsilon''(u_2)$ such that

$$\mathcal{B}_2(U) \le H_2''(u_2) + \epsilon''(u_2)\mathcal{B}_2(U)$$

and $\epsilon''(u_2) \to 0$ as $u_2 \to \infty$. Taking u_2 sufficiently large, the second term on the right-hand side can be absorbed into the left-hand side; then we conclude that $\mathcal{B}_2(U) \le C_{A_1,K,\Lambda} H_2''(u_2)$. As this bound is independent of U, Proposition 7.6 then follows. □

Remark 7.8. Using (7-7) and (7-9) in Lemma 7.5 and (7-5)–(7-6) in Lemma 7.4, the functions $H_2''(u_2)$ and $\epsilon''(u_2)$ can be explicitly bounded from above as follows:

$$H_2''(u_2) \le C_{K,\Lambda}\left(1+\Psi\int_3^\infty \left|\frac{2m\lambda}{(1-\mu)r^2}\right|(u,v')\,dv'\right)\cdot(H_2+H'(u_2)+A_1^2+A_1^3) \\ +C\mathcal{I}_2+C_{K,\Lambda}A_1^2+C_{K,\Lambda,M_i}A_1^3, \tag{7-23}$$

$$\epsilon''(u_2) \le C_{K,\Lambda}\left(1+\Psi\int_3^\infty \left|\frac{2m\lambda}{(1-\mu)r^2}\right|(u,v')\,dv'\right)\cdot(\epsilon+\epsilon')(u_2). \tag{7-24}$$

These bounds will be useful in our proof of Theorem 3.15 in Section 9.

At this point, in order to complete the proof of Proposition 7.6, we are only left to prove Lemma 7.7.

Proof of Lemma 7.7. Let $(u,v)\in\mathcal{Q}_{\text{int}}$ (i.e., $v\in[u,3u]$) with $u\in[3u_2,U]$. In this proof, we will use the notation $\epsilon(u_2)$ to refer to a positive quantity which may be made arbitrarily small by choosing u_2 large enough, which may vary from line to line.

We first estimate F_1 and F_2. Integrating (7-13) and (7-14) along the incoming direction from $\left(\frac{1}{3}u,v\right)$ to (u,v), we obtain

$$|F_1(u,v)| \le \left|F_1\left(\tfrac{1}{3}u,v\right)\right| + \int_{u/3}^u |\partial_u\lambda\,\partial_v\phi(u',v)| + |\partial_v\lambda\,\partial_u\phi(u',v)|\,du',$$
$$|F_2(u,v)| \le \left|F_2\left(\tfrac{1}{3}u,v\right)\right| + \int_{u/3}^u |\partial_u\phi\,\partial_v(\nu^{-1}\partial_u(r\phi))(u',v)| + |\partial_v\phi\,\partial_u(\nu^{-1}\partial_u(r\phi))(u',v)|\,du'.$$

Multiply both sides of these inequalities by u^ω. For $v\in[u,3u]$, note that $\left(\frac{1}{3}u,v\right)\in\mathcal{Q}_{\text{ext}}$ and $u\le\frac{3}{2}\Lambda r\left(\frac{1}{3}u,v\right)$. Therefore, using Theorem 3.1 for ϕ and $\partial_v(r\phi)$, Corollary 6.9 for $\partial_v\phi$, Lemma 7.3 for μ/r, and Lemma 7.4 for $\partial_v^2(r\phi)$ and $\partial_v\lambda$, we have

$$\begin{aligned} u^\omega\left|F_1\left(\tfrac{1}{3}u,v\right)\right| &\le C_\Lambda r^\omega\left(|\partial_v^2(r\phi)|+|(\partial_v\lambda)\phi|\right)\left(\tfrac{1}{3}u,v\right) \\ &\le C_\Lambda\mathcal{I}_2 + C_{K,\Lambda,M_i}A_1^3, \\ u^\omega\left|F_2\left(\tfrac{1}{3}u,v\right)\right| &\le C_\Lambda r^\omega\left(|\lambda^{-1}\partial_v\lambda| + \frac{\mu}{1-\mu}\frac{\lambda}{r} + \left|\partial_v\phi(\lambda^{-1}\partial_v(r\phi)-\nu^{-1}\partial_u(r\phi))\right|\right)\left(\tfrac{1}{3}u,v\right) \\ &\le C_{K,\Lambda}A_1^2. \end{aligned}$$

Therefore, we only need to deal with the u'-integrals. For $u\in[3u_2,U]$, we claim that

$$u^\omega\int_{u/3}^u |\partial_u\lambda(u',v)|\,|\partial_v\phi(u',v)|\,du' \le \epsilon(u_2)\mathcal{B}_2(U), \tag{7-25}$$

$$u^\omega\int_{u/3}^u |\partial_v\lambda(u',v)|\,|\partial_u\phi(u',v)|\,du' \le \epsilon(u_2)\mathcal{B}_2(U), \tag{7-26}$$

$$u^{\omega}\int_{u/3}^{u}|\partial_u\phi(u',v)||\partial_v(\nu^{-1}\partial_u(r\phi))(u',v)|\,\mathrm{d}u' \le \epsilon(u_2)\mathcal{B}_2(U), \tag{7-27}$$

$$u^{\omega}\int_{u/3}^{u}|\partial_v\phi(u',v)||\partial_u(\nu^{-1}\partial_u(r\phi))(u',v)|\,\mathrm{d}u' \le \epsilon(u_2)\mathcal{B}_2(U). \tag{7-28}$$

Proof of (7-25). We proceed similarly as in the proof of Theorem 3.1. By (2-6), (5-9) and Lemma 7.2, we estimate

$$\begin{aligned}u^{\omega}&\int_{u/3}^{u}|\partial_u\lambda(u',v)||\partial_v\phi(u',v)|\,\mathrm{d}u'\\ &\le C_{\Lambda}\bigg(\int_{u_2}^{v}\bigg|\frac{2m\nu}{(1-\mu)r^2}(u',v)\bigg|\,\mathrm{d}u'\bigg)\sup_{u'\in[u/3,u]}\sup_{C_{u'}}(u')^{\omega}\big(|\partial_v^2(r\phi)|+\Psi|\partial_v\lambda|\big)\\ &\le C_{\Lambda,\Psi}\bigg(\int_{u_2}^{v}\bigg|\frac{2m\nu}{(1-\mu)r^2}(u',v)\bigg|\,\mathrm{d}u'\bigg)\mathcal{B}_2(U).\end{aligned}$$

Thus (7-25) follows by Lemma 6.1.

Proof of (7-26). We have

$$\begin{aligned}u^{\omega}\int_{u/3}^{u}|\partial_v\lambda(u',v)||\partial_u\phi(u',v)|\,\mathrm{d}u' &\le C\bigg(\int_{u_2}^{v}|\partial_u\phi(u',v)|\,\mathrm{d}u'\bigg)\sup_{u'\in[u/3,u]}\sup_{C_{u'}}(u')^{\omega}|\partial_v\lambda|\\ &\le C\bigg(\int_{u_2}^{v}|\partial_u\phi(u',v)|\,\mathrm{d}u'\bigg)\mathcal{B}_2(U).\end{aligned}$$

Thus (7-26) follows by Lemma 7.1.

Proof of (7-27). We start with the identity

$$\partial_v(\nu^{-1}\partial_u(r\phi)) = -\frac{2m}{(1-\mu)r^2}\lambda(\nu^{-1}\partial_u(r\phi)-\phi),$$

which is readily verifiable using (2-6) and (2-8). By (5-10) and (5-32), we estimate

$$\begin{aligned}u^{\omega}&\int_{u/3}^{u}|\partial_u\phi(u',v)||\partial_v(\nu^{-1}\partial_u(r\phi))(u',v)|\,\mathrm{d}u'\\ &\le C_{K,\Lambda}\Psi\bigg(\int_{u_2}^{v}\bigg|\frac{2m\nu}{(1-\mu)r^2}(u',v)\bigg|\,\mathrm{d}u'\bigg)\sup_{u'\in[u/3,u]}\sup_{C_{u'}}(u')^{\omega}|\partial_u\phi|.\end{aligned}$$

The u'-integral vanishes as $u_2\to\infty$ by Lemma 6.1. On the other hand, by Lemma 7.2 and Proposition 5.3, we have

$$\sup_{C_{u'}}(u')^{\omega}|\partial_u\phi| \le C_{K,\Lambda}\sup_{C_{u'}}(u')^{\omega}|\partial_v\phi| \le C_{K,\Lambda,\Psi}\mathcal{B}_2(U) \tag{7-29}$$

for any $u'\in[1,U]$. Therefore, (7-27) follows.

Proof of (7-28). Here we divide the integral into two, one in $\mathcal{Q}_{\mathrm{cpt}}$ and the other outside. Recall the notation $u^\star(v) = \sup\{u \in [1, v] : r(u, v) \geq R\}$. Below, we will consider the case $u^\star(v) \in \left[\frac{1}{3}u, u\right]$, i.e., when the line segment $\left\{(u', v) \in \mathcal{Q} : u' \in \left[\frac{1}{3}u, u\right]\right\}$ crosses $\{r = R\}$; the other case is easier and can be handled with a minor modification.

We first deal with the integral over the portion in $\mathcal{Q}_{\mathrm{cpt}}$. We claim that

$$u^\omega \int_{u^\star(v)}^{u} |\partial_v \phi(u', v)|\,|\partial_u(\nu^{-1}\partial_u(r\phi))(u', v)|\,\mathrm{d}u' \leq \epsilon(u_2)\mathcal{B}_2(U).$$

This is an easy consequence of the bound for $|\partial_v \phi|$ in Lemma 7.2, the fact that u and u' are comparable over the domain of integration, and

$$\sup_{v \in [u_2, \infty)} \int_{C_v \cap \{u \geq u_2\} \cap \mathcal{Q}_{\mathrm{cpt}}} |\partial_u(\nu^{-1}\partial_u(r\phi))| \to 0 \quad \text{as } u_2 \to \infty,$$

which follows from (3-3), (5-6) and Theorem 5.14.

We now consider the remaining contribution to the integral. We begin as follows:

$$\begin{aligned}
u^\omega \int_{u/3}^{u^\star(v)} & |\partial_v \phi(u', v)|\,|\partial_u(\nu^{-1}\partial_u(r\phi))(u', v)|\,\mathrm{d}u' \\
&\leq C_{K,\Lambda}\left(\int_{u/3}^{u^\star(v)} |\partial_v \phi(u', v)|\,\mathrm{d}u'\right) \sup_{u' \in [u/3, u^\star(v)]} \sup_{C_{u'}} (u')^\omega\left(|\partial_u^2(r\phi)| + \Psi|\partial_u \nu|\right) \\
&\leq C_{K,\Lambda,\Psi}\left(\int_{u/3}^{u^\star(v)} |\partial_v \phi(u', v)|\,\mathrm{d}u'\right)\mathcal{B}_2(U).
\end{aligned}$$

For $u' \in \left[\frac{1}{3}u, u^\star(v)\right]$, we have $r(u', v) \geq R$. Thus, by (6-11), we have

$$\int_{u/3}^{u^\star(v)} |\partial_v \phi(u', v)|\,\mathrm{d}u' \leq \frac{C_K A_1}{R}\int_{u_2}^{\infty} (u')^{-\omega}\,\mathrm{d}u' \leq \frac{C_K A_1}{R} u_2^{-(\omega-1)},$$

which vanishes as $u_2 \to \infty$. Therefore, in the case under consideration, (7-28) follows.

We have therefore obtained the desired bounds for F_1 and F_2. Next, we estimate G_1 and G_2. Let us integrate (7-15) and (7-16) along the outgoing direction from (u, u) on the axis to (u, v). Then we obtain

$$|G_1(u, v)| \leq \lim_{v' \to u+} |G_1(u, v')| + \int_u^v |\partial_v \nu\, \partial_u \phi(u, v')| + |\partial_u \nu\, \partial_v \phi(u, v')|\,\mathrm{d}v',$$

$$|G_2(u, v)| \leq \lim_{v' \to u+} |G_2(u, v')| + \int_u^v |\partial_v \phi\, \partial_u(\lambda^{-1}\partial_v(r\phi))(u, v')| + |\partial_u \phi\, \partial_v(\lambda^{-1}\partial_v(r\phi))(u, v')|\,\mathrm{d}v'.$$

Note that

$$\lim_{v \to u+} \frac{\mu}{r}(u, v) = 0 \quad \text{and} \quad \lim_{v \to u+} (\lambda^{-1}\partial_v(r\phi)(u, v) - \nu^{-1}\partial_u(r\phi)(u, v)) = 0,$$

since (ϕ, r, m) is a C^1 solution. It follows that $\lim_{v \to u+} \partial_u \partial_v(r\phi)(u, v) = 0$ and $\lim_{v \to u+} \partial_u \partial_v r(u, v) = 0$. Moreover, we have

$$\lim_{v \to u+} \partial_v^2(r\phi)(u, v) = -\lim_{v \to u+} \partial_u^2(r\phi)(u, v) \quad \text{and} \quad \lim_{v \to u+} \partial_v \lambda(u, v) = -\lim_{v \to u+} \partial_u \nu(u, v).$$

As a consequence,

$$\lim_{v'\to u+} G_1(u,v') = -\lim_{v'\to u+} F_1(u,v') \quad \text{and} \quad \lim_{v'\to u+} G_2(u,v') = \lim_{v'\to u+} F_2(u,v').$$

Therefore, by the previous estimates for F_1 and F_2, we have

$$u^\omega \lim_{v'\to u+} |G_1(u,v')| \le C_\Lambda \mathcal{I}_2 + C_{K,\Lambda,M_i} A_1^3 + \epsilon(u_2)\mathcal{B}_2(U),$$
$$u^\omega \lim_{v'\to u+} |G_2(u,v')| \le C_{K,\Lambda} A_1^2 + \epsilon(u_2)\mathcal{B}_2(U),$$

which are acceptable. Recalling that we are considering $(u,v) \in \mathcal{Q}_{\text{int}}$, hence $v \in [u,3u]$, we are now left to establish the following estimates:

$$u^\omega \int_u^{3u} |\partial_v \nu(u,v')||\partial_u \phi(u,v')| \, dv' \le \epsilon(u_2)\mathcal{B}_2(U), \tag{7-30}$$

$$u^\omega \int_u^{3u} |\partial_u \nu(u,v')||\partial_v \phi(u,v')| \, dv' \le \epsilon(u_2)\mathcal{B}_2(U), \tag{7-31}$$

$$u^\omega \int_u^{3u} |\partial_v \phi(u,v')||\partial_u (\lambda^{-1}\partial_v(r\phi))(u,v')| \, dv' \le \epsilon(u_2)\mathcal{B}_2(U), \tag{7-32}$$

$$u^\omega \int_u^{3u} |\partial_u \phi(u,v')||\partial_v (\lambda^{-1}\partial_v(r\phi))(u,v')| \, dv' \le \epsilon(u_2)\mathcal{B}_2(U). \tag{7-33}$$

Proof of (7-30). Substituting $\partial_v \nu$ by (2-6) and using (7-29), we have

$$\begin{aligned} u^\omega \int_u^{3u} |\partial_v \nu(u,v')||\partial_u \phi(u,v')| \, dv' &\le K \left(\int_u^\infty \left| \frac{2m\lambda}{(1-\mu)r^2}(u,v') \right| dv' \right) \sup_{v'\in[u,3u]} u^\omega |\partial_u \phi(u,v')| \\ &\le C_{K,\Lambda,\Psi} \left(\sup_{u\ge 3u_2} \int_u^\infty \left| \frac{2m\lambda}{(1-\mu)r^2}(u,v') \right| dv' \right) \mathcal{B}_2(U). \end{aligned}$$

Thus (7-30) follows by Lemma 6.1.

Proof of (7-31). We have

$$\begin{aligned} u^\omega \int_u^{3u} |\partial_u \nu(u,v')||\partial_v \phi(u,v')| \, dv' &\le \int_u^\infty |\partial_v \phi(u,v')| \, dv' \sup_{v'\in[u,3u]} u^\omega |\partial_u \nu(u,v')| \\ &\le \left(\sup_{u\ge 3u_2} \int_u^\infty |\partial_v \phi(u,v')| \, dv' \right) \mathcal{B}_2(U). \end{aligned}$$

Thus (7-31) follows by Lemma 7.1.

Proof of (7-32). By (2-6) and (2-8), we have the identity

$$\partial_u(\lambda^{-1}\partial_v(r\phi)) = -\frac{2m}{(1-\mu)r^2}\nu(\lambda^{-1}\partial_v(r\phi) - \phi).$$

Then, by Proposition 5.3, we have

$$u^{\omega}\int_u^{3u} |\partial_v\phi(u,v')||\partial_u(\lambda^{-1}\partial_v(r\phi))(u,v')|\,\mathrm{d}v' \le C_{K,\Lambda}\Psi\biggl(\int_u^{\infty}\biggl|\frac{2m\lambda}{(1-\mu)r^2}(u,v')\biggr|\,\mathrm{d}v'\biggr)\sup_{v'\in[u,3u]} u^{\omega}|\partial_v\phi(u,v')|.$$

Now (7-32) follows by Lemmas 6.1 and 7.2 and (5-9).

Proof of (7-33). As in the proof of (7-28), we will divide the integral into two pieces. More precisely, let us define $v^{\star}(u)$ to be the unique v-value such that $r(u,v^{\star}(u))=R$. Assuming $v^{\star}(u)\in[u,3u]$, the integral $\int_u^{3u}$ will be divided into $\int_u^{v^{\star}(u)}$ and $\int_{v^{\star}(u)}^{3u}$. The remaining case $v^{\star}(u)>3u$ can be dealt with by adapting the argument for the first integral.

For the first integral, we claim that

$$u^{\omega}\int_u^{v^{\star}(u)} |\partial_u\phi(u,v')||\partial_v(\lambda^{-1}\partial_v(r\phi))(u,v')|\,\mathrm{d}v' \le \epsilon(u_2)\mathcal{B}_2(U).$$

From the locally BV scattering assumption (2-12), we have

$$\sup_{u\in[3u_2,\infty)}\int_{C_u\cap\mathcal{Q}_{\mathrm{cpt}}} |\partial_v(\lambda^{-1}\partial_v(r\phi))| \to 0 \quad \text{as } u_2\to\infty.$$

Combined with (7-29), the claim follows.

Next, we turn to the second integral. By (5-5) and (5-9), we estimate

$$\begin{aligned} u^{\omega}\int_{v^{\star}(u)}^{3u} |\partial_u\phi(u,v')||\partial_v(\lambda^{-1}\partial_v(r\phi))(u,v')|\,\mathrm{d}v' &\le \sup_{v'\in[v^{\star}(u),3u]} u^{\omega}\bigl|\partial_v(\lambda^{-1}\partial_v(r\phi))(u,v')\bigr| \int_{v^{\star}(u)}^{3u}|\partial_u\phi(u,v')|\,\mathrm{d}v' \\ &\le C_{\Lambda,\Psi}\mathcal{B}_2(U)\int_{v^{\star}(u)}^{3u}|\partial_u\phi(u,v')|\,\mathrm{d}v'. \end{aligned}$$

For $v'\in[v^{\star}(u),3u]$, we have $r(u,v)\ge R$. Thus, by (6-12), we have

$$\int_{v^{\star}(u)}^{3u}|\partial_u\phi(u,v')|\,\mathrm{d}v' \le \frac{C_K A_1}{R}\int_u^{3u} u^{-\omega}\,\mathrm{d}v' \le \frac{C_K A_1}{R}u_2^{-(\omega-1)},$$

which vanishes as $u_2\to\infty$, and therefore finishes the proof of (7-33). We remark that the fact that we are in $\mathcal{Q}_{\mathrm{int}}$ is used crucially here, as otherwise the integral would not be convergent. □

Remark 7.9. In the case where we have global BV scattering (i.e., conditions (2) and (3) of Definition 2.4 are satisfied with $R=\infty$), we can take $R=\infty$ in the preceding argument to obtain the following explicit upper bound on $\epsilon(u_2)$:

$$\begin{aligned}\epsilon(u_2) \le C_{K,\Lambda,\Psi} \sup_{v\in[u_2,\infty)} \int_{\underline{C}_v\cap\{u\ge u_2\}} \left|\frac{2m\nu}{(1-\mu)r^2}\right| + C \sup_{v\in[u_2,\infty)} \int_{\underline{C}_v\cap\{u\ge u_2\}} |\partial_u\phi| \\ + C_{K,\Lambda,\Psi} \sup_{v\in[u_2,\infty)} \int_{\underline{C}_v\cap\{u\ge u_2\}} \left|\partial_u(\nu^{-1}\partial_u(r\phi))\right| \\ + C_{K,\Lambda,\Psi} \sup_{u\ge 3u_2} \int_{C_u} \left|\frac{2m\lambda}{(1-\mu)r^2}\right| + C \sup_{u\ge 3u_2} \int_{C_u} |\partial_v\phi| \\ + C_{K,\Lambda,\Psi} \sup_{u\in[3u_2,\infty)} \int_{C_u} \left|\partial_v(\lambda^{-1}\partial_v(r\phi))\right|. \end{aligned} \tag{7-34}$$

This will be useful in our proof of the sharp decay rate in the case of small BV norm (Theorem 3.15) in Section 9.

In the second step of our proof of Theorem 3.2, we use the preliminary $u^{-\omega}$ decay proved in Proposition 7.6 to obtain the optimal the u-decay. Key to this step is the following proposition, which claims optimal u-decay in $\mathcal{Q}_{\text{int}}$:

Proposition 7.10. *There exists a constant* $0 < A_2'' < \infty$ *such that*

$$\sup_{\mathcal{Q}_{\text{int}}} u^{\omega+1}|\partial_v^2(r\phi)| \le A_2'', \tag{7-35}$$

$$\sup_{\mathcal{Q}_{\text{int}}} u^{\omega+1}|\partial_u^2(r\phi)| \le A_2'', \tag{7-36}$$

$$\sup_{\mathcal{Q}_{\text{int}}} u^{3}|\partial_v\lambda| \le A_2'', \tag{7-37}$$

$$\sup_{\mathcal{Q}_{\text{int}}} u^{3}|\partial_u\nu| \le A_2''. \tag{7-38}$$

Once we establish Proposition 7.10, the desired decay for $\partial_v^2(r\phi)$ and $\partial_v\lambda$ follow from Lemma 7.4 and the fact that $r \ge 2\Lambda^{-1}u$ in $\mathcal{Q}_{\text{ext}}$. Furthermore, the desired decay for $\partial_u^2(r\phi)$ and $\partial_u\nu$ follow from Lemma 7.5.

Proof. Thanks to the fact that we have pointwise bounds for sufficient number of derivatives (albeit with suboptimal decay) near Γ at this point, it suffices to work with the "nonrenormalized" equations (4-2), (4-4), (4-6) and (4-7). In particular, we need not utilize the null structure of (SSESF).

Let $(u, v) \in \mathcal{Q}_{\text{int}}$ (i.e., $v \in [u, 3u]$) with $u \ge 3$. We begin with (7-35). Integrating $\partial_u\partial_v^2(r\phi)$ in the u-direction from $\frac{1}{3}u$ to u, multiplying by $u^{\omega+1}$ and using $r\left(\frac{1}{3}u, v\right) \ge \frac{2}{3}\Lambda^{-1}u$, we obtain

$$u^{\omega+1}|\partial_v^2(r\phi)|(u,v) \le C_\Lambda r^{\omega+1}|\partial_v^2(r\phi)|\left(\tfrac{1}{3}u, v\right) + u^{\omega+1}\int_{u/3}^{u} |\partial_u\partial_v^2(r\phi)|(u',v)\,du'. \tag{7-39}$$

Since $\left(\frac{1}{3}u, v\right) \in \mathcal{Q}_{\text{ext}}$, the first term on the right-hand side is bounded by $C_\Lambda\mathcal{I}_2 + C_{K,\Lambda,M_i}A_1^3$, thanks to Lemma 7.4. To estimate the u'-integral, we substitute $\partial_u\partial_v^2(r\phi)$ by (4-2). Then, applying Proposition 5.3, Lemma 7.2, Lemma 7.3, Theorem 3.1 and Proposition 7.6, we obtain

$$|\partial_u \partial_v^2 (r\phi)|(u', v) \le C_{A_1,K,\Lambda} (u')^{-3\omega} A_1 (A_2')^2.$$

Thus we have

$$u^{\omega+1} |\partial_v^2 (r\phi)|(u, v) \le C_\Lambda \mathcal{I}_2 + C_{K,\Lambda,M_i} A_1^3 + C_{A_1,K,\Lambda} A_1 (A_2')^2, \tag{7-40}$$

where we have used the fact that $\omega > 1$, and thus $3\omega - 1 > \omega + 1$, to throw away the u-weight in the last term. This proves (7-35).

Next, we prove (7-36). Integrating $\partial_v \partial_u^2 (r\phi)$ in the v-direction from $u+$ to v and multiplying by $u^{\omega+1}$, we have

$$u^{\omega+1} |\partial_u^2 (r\phi)|(u, v) \le \lim_{v' \to u+} u^{\omega+1} |\partial_u^2 (r\phi)|(u, v') + u^{\omega+1} \int_u^{3u} |\partial_v \partial_u^2 (r\phi)|(u, v')\, dv'. \tag{7-41}$$

Recall that $\lim_{v' \to u+} \partial_u^2 (r\phi)(u, v') = \lim_{v' \to u+} \partial_v^2 (r\phi)(u, v')$, as (ϕ, r, m) is a C^1 solution. Thus the first term on the right-hand side can be estimated via (7-40). Substitute $\partial_v \partial_u^2 (r\phi)$ by (4-4) and apply, as before, Proposition 5.3, Lemma 7.2, Lemma 7.3, Theorem 3.1 and Proposition 7.6. Then we have

$$|\partial_v \partial_u^2 (r\phi)|(u, v') \le C_{A_1,K,\Lambda} u^{-3\omega} A_1 (A_2')^2.$$

It now follows that

$$u^{\omega+1} |\partial_u^2 (r\phi)|(u, v) \le C_\Lambda \mathcal{I}_2 + C_{K,\Lambda,M_i} A_1^3 + C_{A_1,K,\Lambda} A_1 (A_2')^2, \tag{7-42}$$

which proves (7-36).

At this point, combining Lemma 7.2, Theorem 3.1, Lemma 7.4 and (7-40), note that we have the following improved u-decay for $\partial_v \phi$:

$$\sup_{\mathcal{Q}} u^{\omega+1} |\partial_v \phi| \le C_\Lambda \sup_{\mathcal{Q}} \big(u^{\omega+1} |\partial_v^2 (r\phi)| + u A_1 |\partial_v \lambda| \big) \le B, \tag{7-43}$$

where

$$B := C_\Lambda \mathcal{I}_2 + C_{K,\Lambda,M_i} A_1^3 + C_{A_1,K,\Lambda} A_1 (A_2')^2 + C_\Lambda A_1 A_2'. \tag{7-44}$$

We now turn to (7-37). Integrating $\partial_u \partial_v \log \lambda$ in the u-direction from $\frac{1}{3} u$ to u, multiplying by u^3 and using $r\big(\frac{1}{3} u, v\big) \ge \frac{2}{3} \Lambda^{-1} u$, we obtain

$$u^3 |\partial_v \log \lambda|(u, v) \le C r^3 |\partial_v \log \lambda|\big(\tfrac{1}{3} u, v\big) + u^3 \int_{u/3}^{u} |\partial_u \partial_v \log \lambda|(u', v)\, du'. \tag{7-45}$$

Since $\big(\frac{1}{3} u, v\big) \in \mathcal{Q}_{\mathrm{ext}}$, the first term on the right-hand side is bounded by $C_{K,\Lambda} A_1^2$, by Lemma 7.4 and the fact that $\lambda^{-1} \le \Lambda$. Next, substituting $\partial_u \log \lambda$ by (4-6), applying Proposition 5.3, Lemma 7.3 and Lemma 7.2 and using the improved bound (7-43), we have

$$|\partial_u \partial_v \log \lambda|(u', v) \le C_{K,\Lambda} B^2 (u')^{-2(\omega+1)}.$$

Therefore

$$u^3 |\partial_v \lambda|(u, v) \le C_{K,\Lambda} A_1^2 + C_{K,\Lambda} B^2, \tag{7-46}$$

where we used $2(\omega+1)-1>3$ to throw away the u-weight in the last term. This proves (7-37).

Finally, we prove (7-38). Integrating $\partial_v\partial_u \log\nu$ in the v-direction from $u+$ to v and multiplying by u^3, we have

$$u^3|\partial_u \log\nu|(u,v) \le \lim_{v'\to u+} u^3|\partial_u \log\nu|(u,v') + u^3\int_u^{3u} |\partial_v\partial_u \log\nu|(u,v')\,dv'. \tag{7-47}$$

Since $\lim_{v'\to u+}\partial_u\nu(u,v') = -\lim_{v'\to u+}\partial_v\lambda(u,v')$, the first term is bounded by (7-46). Furthermore, substituting $\partial_v\partial_u\log\nu$ by (4-7) and applying Proposition 5.3, Lemma 7.3 and Lemma 7.2 and using the improved bound (7-43), we have

$$|\partial_v\partial_u\log\nu|(u,v') \le C_{K,\Lambda}B^2u^{-2(\omega+1)}.$$

As before, it follows that

$$u^3|\partial_u\nu|(u,v) \le C_{K,\Lambda}A_1^2 + C_{K,\Lambda}B^2, \tag{7-48}$$

which proves (7-38). □

Remark 7.11. Combining (7-40), (7-42), (7-46) and (7-48), we see that Proposition 7.10 holds with

$$A_2'' \le C_\Lambda\mathcal{I}_2 + C_{K,\Lambda,M_i}A_1^3 + C_{A_1,K,\Lambda}A_1(A_2')^2 + C_{K,\Lambda}A_1^2 + C_{K,\Lambda}B^2, \tag{7-49}$$

where B is as in (7-44).

Remark 7.12. According to the argument of this subsection, note that the size of A_2' in Proposition 7.6 depends on the choice of u_2 through the term $H_2'(u_2)$, where the size of u_2 depends on the rate of convergence of $\epsilon''(u_2)\to 0$ as $u_2\to\infty$. This explains why A_2 does not depend only on the size of the initial data, as remarked in Section 3. On the other hand, as stated in Theorem 3.15(2), we shall show that, in the case of small BV initial data, A_2 depends only on the size of the initial data. To achieve this, we show in Section 9 that we may take $u_2=1$ under this small data assumption.

7E. *Additional decay estimates.* As in the previous section, we conclude this section by providing additional decay rates concerning second derivatives of ϕ and r and improved decay for m near Γ.

Corollary 7.13. *Let (ϕ,r,m) be a locally BV scattering solution to (SSESF) with asymptotically flat C^1 initial data of order ω'. Let A_1 and A_2 be the constants in Theorems 3.1 and 3.2, respectively. Then the following bounds hold:*

$$|\partial_v\phi| \le C_\Lambda(A_1+A_2+A_1A_2)\min\{u^{-(\omega+1)}, r^{-2}u^{-(\omega-1)}\} \tag{7-50}$$

$$|\partial_u\phi| \le C_{K,\Lambda}(A_1+A_2+A_1A_2)\min\{u^{-(\omega+1)}, r^{-1}u^{-\omega}\} \tag{7-51}$$

$$|\partial_v^2\phi| \le C_\Lambda(A_1+A_2+A_1A_2)\min\{r^{-1}u^{-(\omega+1)}, r^{-3}u^{-(\omega-1)}\}, \tag{7-52}$$

$$|\partial_u\partial_v\phi| \le C_{K,\Lambda}(A_1+A_2+A_1A_2)\min\{r^{-1}u^{-(\omega+1)}, r^{-2}u^{-\omega}\}, \tag{7-53}$$

$$|\partial_u^2\phi| \le C_{K,\Lambda}(A_1+A_2+A_1A_2)r^{-1}u^{-(\omega+1)}, \tag{7-54}$$

$$|\partial_u\partial_v r| \le C_{K,\Lambda}(A_1+A_2+A_1A_2)^2\min\{ru^{-(2\omega+2)}, r^{-2}u^{-(2\omega-1)}\}, \tag{7-55}$$

$$m \le C_{K,\Lambda}(A_1+A_2+A_1A_2)^2\min\{r^3u^{-(2\omega+2)}, u^{-(2\omega-1)}\}. \tag{7-56}$$

This corollary follows immediately from the estimates derived in Theorem 3.2. We sketch the proof:

Proof. First, note that (7-50) and (7-51) follows from Corollary 6.9, Theorem 3.2 and Lemma 7.2. Next, (7-52) and (7-54) are easy consequences of the preceding estimates, Theorems 3.1 and 3.2, and the identities

$$r\,\partial_v^2\phi = \partial_v^2(r\phi) - (\partial_v\lambda)\phi - 2\lambda\,\partial_v\phi, \quad r\,\partial_u^2\phi = \partial_u^2(r\phi) - (\partial_u\nu)\phi - 2\nu\,\partial_u\phi.$$

On the other hand, for (7-53), we use the identity

$$r\partial_u\partial_v\phi = -\lambda\,\partial_u\phi - \nu\,\partial_v\phi,$$

which may be verified from (2-6) and (2-8).

Next, (7-56) follows from Corollary 6.9, Lemma 7.3 and (7-50). Finally, using Corollary 5.5, Lemma 5.10, (7-56) and (2-6), we conclude (7-55). □

8. Decay and blow-up at infinity

In this section, we prove Theorem 3.14, that is, unless the solution blows up at infinity, a "future causally geodesically complete" solution scatters in BV.

Take a BV solution to (SSESF) satisfying the hypotheses of Theorem 3.14 which does not blow up at infinity. Note, in particular, that $\mathcal{Q} = \mathcal{R}$ by (1) of Definition 2.4 and Lemma 4.2. In order to prove Theorem 3.14, our goal is to show that such a spacetime is in fact BV scattering, i.e., (1), (2) and (3) in Definition 2.4 hold and, moreover, (3) holds with $R = \infty$.

The main step will be to show that there exists a constant C_Λ such that, for every $\epsilon > 0$, there exists U such that, for every $u \geq U$, we have

$$\int_{C_u} |\partial_v^2(r\phi)| + \int_{C_u} |\partial_v\lambda| \leq C_\Lambda\epsilon. \tag{8-1}$$

This will be achieved in a sequence of lemmas and propositions below.

Before we proceed, we first prove a preliminary bound on λ:

Proposition 8.1. *There exists* $0 < \Lambda < \infty$ *such that*

$$\Lambda^{-1} \leq \lambda(u, v) \leq \tfrac{1}{2}. \tag{8-2}$$

Proof. By (1) in Definition 3.12, there exists $0 < \Lambda < \infty$ such that $\sup \lambda_\Gamma^{-1} \leq \Lambda$. As $\lim_{u\to v-}\lambda_\Gamma(u) = \lim_{u\to v-}\lambda(u', v)$ (see [Christodoulou 1993, Section 7]), it follows from Lemma 5.4 that, for every $(u, v) \in \mathcal{Q}$, we have the estimate (8-2). □

We now proceed to show (8-1). The first step is to show that, for u sufficiently large, the integrals along C_u of $|F_1|$ and $|F_2|$ are small. Here, we recall the notation in the proof of Proposition 7.6,

$$F_1 := \partial_v^2(r\phi) - (\partial_v\lambda)\phi,$$
$$F_2 := \partial_v \log\lambda - \frac{\lambda}{1-\mu}\frac{\mu}{r} + \partial_v\phi(\lambda^{-1}\partial_v(r\phi) - \nu^{-1}\partial_u(r\phi)).$$

Once we obtain the desired bounds for F_1 and F_2, we then derive (8-1) from these bounds. This will be the most technical part (see discussions in Remark 8.3).

First, we bound the integrals of F_1 and F_2:

Proposition 8.2. *For every $\epsilon > 0$, there exists V sufficiently large such that the following bound holds for $u \geq V$:*

$$\int_{C_u} (|F_1| + |F_2|)(u, v) \leq 3\epsilon. \tag{8-3}$$

Proof. By (2) and (3) in Definition 3.12, we have

$$\int_1^\infty \int_u^\infty |\partial_v \lambda \, \partial_u \phi - \partial_u \lambda \, \partial_v \phi| \, dv \, du < \infty$$

and

$$\int_1^\infty \int_u^\infty \left| \partial_u \phi \, \partial_v (\nu^{-1} \partial_u (r\phi)) - \partial_v \phi \, \partial_u (\nu^{-1} \partial_u (r\phi)) \right| dv \, du < \infty.$$

Thus, by choosing V sufficiently large, we have

$$\int_V^\infty \int_u^\infty |\partial_v \lambda \, \partial_u \phi - \partial_v \lambda \, \partial_v \phi| \, dv \, du < \epsilon \tag{8-4}$$

and

$$\int_V^\infty \int_u^\infty \left| \partial_u \phi \, \partial_v (\nu^{-1} \partial_u (r\phi)) - \partial_v \phi \, \partial_u (\nu^{-1} \partial_u (r\phi)) \right| dv \, du < \epsilon. \tag{8-5}$$

From the initial conditions, we easily see that $F_1(1, \cdot)$ and $F_2(1, \cdot)$ obey $\int_{C_1} |F_1| + |F_2| < \infty$. Thus, by choosing V larger if necessary, we have

$$\int_V^\infty (|F_1| + |F_2|)(1, v) \, dv \leq \epsilon. \tag{8-6}$$

Notice that, by equations (7-13) and (7-14), the estimates (8-4) and (8-5) control $\iint |\partial_u F_1| \, du \, dv$ and $\iint |\partial_u F_2| \, du \, dv$. Thus, we have

$$\int_{\max\{u, V\}}^\infty (|F_1| + |F_2|)(u, v) \, dv \leq 3\epsilon$$

for every $u \geq 1$. In particular, for $u \geq V$, we have

$$\int_{C_u} (|F_1| + |F_2|)(u, v) \leq 3\epsilon,$$

as desired. □

The inequality (8-3) is the starting point for our proof of (8-1). More precisely, our basic strategy is to use a continuous induction on v, beginning from the axis, to remove the quadratic and higher terms from (8-3) and infer (8-1).

Remark 8.3. Before beginning the proof in earnest, we would like to point out two technical nuisances that we confront: First, in order to estimate the scalar field ϕ itself from F_1 and F_2, we need to integrate essentially from null infinity,[16] which is opposite to the direction of our method of continuity. Second, as $\partial_v(r\phi)$ is only assumed to be BV, the left-hand side of (8-1) is not continuous in v in general. To overcome the first, we make use of the invariance of (SSESF) and of F_1 and F_2 under the change $\phi \mapsto \phi + c$. To take care of the second, we carefully keep track of the evolution of discontinuities of $\partial_v(r\phi)$.

Notice that, in order to obtain (8-1) from (8-3), we only need to integrate on a *fixed* hypersurface C_u. We now fix $u_0 \geq V$ and define a new function $\bar{\phi}_{u_0}$ by

$$\bar{\phi}_{u_0}(u, v) := \phi(u, v) - \lim_{v' \to u_0+} \phi(u_0, v'). \tag{8-7}$$

As remarked before, note that (SSESF) is invariant under the change $(\phi, r, m) \mapsto (\bar{\phi}_{u_0}, r, m)$, that is, $(\bar{\phi}_{u_0}, r, m)$ is still a solution to (SSESF). Moreover, it is easy to check that F_1 and F_2 are also invariant under this change, i.e.,

$$\begin{aligned} F_1 &= \partial_v^2(r\bar{\phi}_{u_0}) - (\partial_v\lambda)\bar{\phi}_{u_0}, \\ F_2 &= \partial_v \log\lambda - \frac{\lambda}{1-\mu}\frac{\mu}{r} + \partial_v\bar{\phi}_{u_0}(\lambda^{-1}\partial_v(r\bar{\phi}_{u_0}) - \nu^{-1}\partial_u(r\bar{\phi}_{u_0})). \end{aligned} \tag{8-8}$$

The new scalar field has been chosen so that $\bar{\phi}_{u_0}(u_0, \cdot\,)$ and $\partial_v(r\bar{\phi}_{u_0})(u_0, \cdot\,)$ vanish at the axis, that is,

$$\lim_{v \to u_0+} \bar{\phi}_{u_0}(u_0, v) = \lim_{v \to u_0+} \partial_v(r\bar{\phi}_{u_0})(u_0, v) = \lim_{v \to u_0+} \partial_u(r\bar{\phi}_{u_0})(u_0, v) = 0. \tag{8-9}$$

We claim that the original scalar field $\phi(u, v)$ obeys the condition

$$\lim_{v \to \infty} \phi(u_0, v) = 0 \tag{8-10}$$

for every $u_0 \geq 1$. Therefore, by the definition given in (8-7), we see that ϕ and $\bar{\phi}_{u_0}$ are also related by

$$\phi(u, v) = \bar{\phi}_{u_0}(u, v) - \lim_{v' \to \infty} \bar{\phi}_{u_0}(u, v'). \tag{8-11}$$

To establish the claim (8-10), we proceed as in the proof of Lemma 6.3, but work with ϕ rather than $r\phi$. Fix $u_0 > 1$ and let $r_1 > 0$ be a large number, to be determined. For each $u \geq 1$, let $v_1^\star(u)$ be the unique v-value such that $r(u, v_1^\star(u)) = r_1$. Consider $(u, v) \in \{1 \leq u \leq u_0\} \cap \{r \geq r_1\}$. Using the uniform bound of m and $\lambda/(1-\mu)$ in terms of the data at $u = 1$ (which holds thanks to monotonicity), we may integrate (2-8) along the incoming direction to estimate

$$|\partial_v(r\phi)(u, v) - \partial_v(r\phi)(1, v)| \leq \frac{C_0}{r(u, v)} \sup_{u' \in [1, u]} |\phi(u', v)|,$$

[16] More precisely, ϕ is determined from $\partial_v(r\phi)$, which in turn can be determined from $\int |\partial_v^2(r\phi)|$ by integrating from $v = \infty$. Another conceptual reason why information near $v = \infty$ is relevant for estimating ϕ is that the initial condition $\lim_{v\to\infty} \phi(1, v) = 0$ implies that $\lim_{v\to\infty} \phi(u, v) = 0$ for every $u \geq 1$. See the discussion before (8-11).

where C_0 depends only on the data at $u = 1$. Integrating both sides in the outgoing direction from $v_1^\star(u)$ to v (using (8-2) for the right-hand side) and dividing by $r = r(u, v)$, we obtain

$$|\phi(u, v)| \leq \frac{r_1}{r}|\phi(u, v_1^\star(u))| + \frac{r(1, v_1^\star(u))}{r}|\phi(1, v_1^\star(u))| + \frac{r(1, v)}{r}|\phi(1, v)| + \frac{C_0\Lambda}{r}\log\frac{r}{r_1} \sup_{1\leq u'\leq u,\, v\geq v_1^\star(u)} |\phi|. \quad (8\text{-}12)$$

Now the idea is to use (8-12) to first show that ϕ is bounded on the region $\{1 \leq u \leq u_0\}$ and then use (8-12) again with the additional boundedness of ϕ to conclude that (8-10) holds. To begin with, observe that ϕ is bounded on each set compact subset of $\mathcal{Q}$, since it is a BV solution in the sense of Definition 2.1. Combined with the hypothesis that $\phi(1, v) \to 0$ as $v \to \infty$, we see that the first three terms are bounded by a constant that depends on r_1. On the other hand, by taking r_1 sufficiently large, the coefficient $(C_0\Lambda/r)\log(r/r_1)$ of the last term can be made arbitrarily small for $r \geq r_1$. This smallness allows us to absorb the last term to the left-hand side, and conclude the desired boundedness of ϕ on the region $\{1 \leq u \leq u_0\}$. Then, plugging in $u = u_0$ and the uniform bound for ϕ into (8-12), the claim (8-10) follows from the hypothesis $\lim_{v\to\infty}\phi(1, v) = 0$.

Let

$$I_1(u, v) := \int_u^v |\partial_v^2(r\bar{\phi}_{u_0})|(u, v')\,\mathrm{d}v' \quad \text{and} \quad I_2(u, v) := \int_u^v |\partial_v\lambda|(u, v')\,\mathrm{d}v'.$$

In the following two lemmas, we will show that

$$I_1(u_0, v) \leq 3\epsilon + C_\Lambda I_1(u_0, v) I_2(u_0, v), \quad (8\text{-}13)$$

$$I_2(u_0, v) \leq 3\epsilon + C_\Lambda I_1(u_0, v)^2(1 + I_1(u_0, v))^2(1 + I_2(u_0, v))^2 e^{C_\Lambda I_1(u_0,v)^2(1+I_2(u_0,v))}, \quad (8\text{-}14)$$

for every $V \leq u_0 \leq v$, with C_Λ independent of u_0 and v.

Lemma 8.4. *There exists a constant $C_\Lambda > 0$ such that, for every $V \leq u_0 \leq v$,*

$$I_1(u_0, v) \leq 3\epsilon + C_\Lambda I_1(u_0, v) I_2(u_0, v).$$

Proof. In this proof, we fix $u_0 \geq V$ and use the abbreviations

$$\bar{\phi} := \bar{\phi}_{u_0}, \quad \partial_v(r\bar{\phi}) := \partial_v(r\bar{\phi}_{u_0}) \quad \text{and} \quad \partial_v^2(r\bar{\phi}) := \partial_v^2(r\bar{\phi}_{u_0}). \quad (8\text{-}15)$$

By Lemma 5.1, we have

$$|\bar{\phi}(u_0, v)| \leq \frac{1}{r}\int_{u_0}^v \partial_v(r\bar{\phi})(u_0, v')\,\mathrm{d}v' \leq \Lambda \sup_{u_0\leq v'\leq v} |\partial_v(r\bar{\phi})(u_0, v')|. \quad (8\text{-}16)$$

By the fundamental theorem of calculus and (8-9), note that

$$\sup_{u_0\leq v'\leq v} |\partial_v(r\bar{\phi})(u_0, v')| \leq I_1(u_0, v). \quad (8\text{-}17)$$

Thus, recalling the definition of F_1 in (8-8), we have

$$\begin{aligned} I_1(u_0, v) &\le \int_{u_0}^{v} |F_1(u_0, v')|\, dv' + \int_{u_0}^{v} |\partial_v \lambda||\bar{\phi}|(u_0, v')\, dv' \\ &\le \int_{u_0}^{v} |F_1(u_0, v')|\, dv' + \Lambda\, I_1(u_0, v) I_2(u_0, v) \\ &\le 3\epsilon + C_\Lambda I_1(u_0, v) I_2(u_0, v). \end{aligned}$$

□

We now move on to estimate $I_2(u_0, v)$.

Lemma 8.5. *There exists a constant $C_\Lambda > 0$ such that, for every $V \le u_0 \le v$,*

$$I_2(u_0, v) \le 3\epsilon + C_\Lambda I_1(u_0, v)^2 (1 + I_1(u_0, v))^2 (1 + I_2(u_0, v))^2 e^{C_\Lambda I_1(u_0,v)^2(1+I_2(u_0,v))}.$$

Proof. Again, we fix $u_0 \ge V$ and use the abbreviation (8-15), as well as

$$\partial_u(r\bar{\phi})(u, v) := \partial_u(r\bar{\phi}_{u_0})(u, v). \tag{8-18}$$

Recalling the equation for F_2 in (8-8), in order to control $I_2(u_0, v)$ from F_2, we need to estimate

$$\int_{u_0}^{v} \left(\frac{\lambda}{1-\mu}\frac{\mu}{r}\right)(u_0, v')\, dv' \quad \text{and} \quad \int_{u_0}^{v} \partial_v\bar{\phi}(\lambda^{-1}\partial_v(r\bar{\phi}) - \nu^{-1}\partial_u(r\bar{\phi}))(u_0, v')\, dv'.$$

By Lemma 5.11,

$$\int_{u_0}^{v} \left(\frac{\lambda}{1-\mu}\frac{\mu}{r}\right)(u_0, v')\, dv' = \log(1 - \mu(u_0, v)) + \int_{u_0}^{v} \frac{r(\partial_v\bar{\phi})^2}{\lambda}(u_0, v')\, dv'.$$

Since $\mathcal{Q} = \mathcal{R}$, the integrand on the left-hand side is nonnegative. Notice furthermore that, since $\mu \ge 0$, $\log(1 - \mu(u_0, v)) < 0$. Thus,

$$\begin{aligned} \int_{u_0}^{v} \left(\frac{\lambda}{1-\mu}\frac{\mu}{r}\right)(u_0, v')\, dv' &\le \left| \int_{u_0}^{v} \frac{r(\partial_v\bar{\phi})^2}{\lambda}(u_0, v')\, dv' \right| \\ &\le \int_{u_0}^{v} |\partial_v\bar{\phi}(u_0, v')| \left| \left(\frac{\partial_v(r\bar{\phi})}{\lambda} - \bar{\phi}\right)(u_0, v') \right| dv' \\ &\le 2\Lambda I_1(u_0, v) \int_{u_0}^{v} |\partial_v\bar{\phi}(u_0, v')|\, dv', \end{aligned}$$

where we have used (8-16) and (8-17) on the last line. Using Lemma 5.2, we estimate the integral on the last line by

$$\int_{u_0}^{v} |\partial_v\bar{\phi}(u_0, v')|\, dv' \le \int_{u_0}^{v} |\partial_v(\lambda^{-1}\partial_v(r\bar{\phi}))(u_0, v')|\, dv', \tag{8-19}$$

and the right-hand side can in turn be estimated, using (8-17), by

$$\begin{aligned} \int_{u_0}^{v} |\partial_v(\lambda^{-1}\partial_v(r\bar{\phi}))(u_0, v')|\, dv' &\le \int_{u_0}^{v} \lambda^{-1}|\partial_v^2(r\bar{\phi})(u_0, v')|\, dv' + \int_{u_0}^{v} \lambda^{-2}|\partial_v\lambda\, \partial_v(r\bar{\phi})(u_0, v')|\, dv' \\ &\le \Lambda I_1(u_0, v) + \Lambda^2 I_1(u_0, v) I_2(u_0, v). \end{aligned}$$

Therefore, we have

$$\int_{u_0}^{v}\Big(\frac{\lambda}{1-\mu}\frac{\mu}{r}\Big)(u_0,v')\,dv' \le C_\Lambda I_1(u_0,v)^2(1+I_2(u_0,v)). \tag{8-20}$$

We now move on to bound $\int_{u_0}^{v}\partial_v\bar{\phi}\,\lambda^{-1}\partial_v(r\bar{\phi})(u_0,v')\,dv'$. Using (8-17) and (8-19), we easily estimate

$$\begin{aligned}\int_{u_0}^{v}|\partial_v\bar{\phi}\,\lambda^{-1}\partial_v(r\bar{\phi})(u_0,v')|dv' &\le \Lambda\int_{u_0}^{v}|\partial_v\bar{\phi}|(u_0,v')\,dv'\sup_{u_0\le v'\le v}|\partial_v(r\bar{\phi})(u_0,v')|\\ &\le C_\Lambda I_1(u_0,v)^2(1+I_2(u_0,v)).\end{aligned} \tag{8-21}$$

Finally, we are only left to bound $-\int_{u_0}^{v}\partial_v\bar{\phi}\,\nu^{-1}\partial_u(r\bar{\phi})(u_0,v')dv'$. As before, we begin by estimating

$$\begin{aligned}\int_{u_0}^{v}|\partial_v\bar{\phi}\,\nu^{-1}\partial_u(r\bar{\phi})(u_0,v')|\,dv' &\le \int_{u_0}^{v}|\partial_v\bar{\phi}|(u_0,v')\,dv'\sup_{u_0\le v'\le v}|\nu^{-1}\partial_u(r\bar{\phi})(u_0,v')|\\ &\le C_\Lambda I_1(u_0,v)(1+I_2(u_0,v))\sup_{u_0\le v'\le v}|\nu^{-1}\partial_u(r\bar{\phi})(u_0,v')|.\end{aligned} \tag{8-22}$$

In this case, we do not wish to pull out ν as we have not assumed any bound on it. Instead, we consider $\nu^{-1}\partial_u(r\bar{\phi})$ as a whole and note that

$$\partial_v(\nu^{-1}\partial_u(r\bar{\phi})) = -\Big(\frac{\lambda}{1-\mu}\frac{\mu}{r}\Big)\nu^{-1}\partial_u(r\bar{\phi}) + \Big(\frac{\lambda}{1-\mu}\frac{\mu}{r}\Big)\bar{\phi}. \tag{8-23}$$

Then (8-23) holds since, by (2-6) and (2-8), we have

$$\partial_v(\nu^{-1}\partial_u(r\phi)) = -\Big(\frac{\lambda}{1-\mu}\frac{\mu}{r}\Big)\nu^{-1}\partial_u(r\phi) + \Big(\frac{\lambda}{1-\mu}\frac{\mu}{r}\Big)\phi$$

and, moreover, both the left-hand side and the right-hand side of the equation are invariant under the transformation $\phi\mapsto\phi+c$.

Therefore, by the variation of constants formula and (8-9), we have

$$\nu^{-1}\partial_u(r\bar{\phi})(u_0,v) = e^{-J(u_0,v)}\int_{u_0}^{v}e^{J(u_0,v')}\frac{\lambda}{1-\mu}\frac{\mu}{r}\bar{\phi}(u_0,v')\,dv',$$

where

$$J(u_0,v) := \int_{u_0}^{v}\frac{\lambda}{1-\mu}\frac{\mu}{r}(u_0,v')\,dv'.$$

By (8-17) and (8-20), we have

$$\sup_{u_0\le v'\le v}|\nu^{-1}\partial_u(r\bar{\phi})(u_0,v')| \le C_\Lambda I_1(u_0,v)^3(1+I_2(u_0,v))e^{C_\Lambda I_1(u_0,v)^2(1+I_2(u_0,v))}.$$

Then, by (8-22), we conclude that

$$\int_{u_0}^{v}|\partial_v\bar{\phi}\,\nu^{-1}\partial_u(r\bar{\phi})(u_0,v')|\,dv' \le C_\Lambda I_1(u_0,v)^4(1+I_2(u_0,v))^2e^{C_\Lambda I_1(u_0,v)^2(1+I_2(u_0,v))}. \tag{8-24}$$

Combining (8-20), (8-21) and (8-24), we conclude that (8-14) holds. □

Next, we apply (8-13) and (8-14) to show:

Proposition 8.6. *For u_0 sufficiently large and $v \geq u_0$, we have*

$$I_1(u_0, v) + I_2(u_0, v) \leq C_\Lambda \epsilon.$$

Remark 8.7. If it is the case that $\int_{u_0}^{v} \big(|\partial_v^2(r\bar{\phi})| + |\partial_v \lambda|\big)(u_0, v')\, dv'$ is continuous in v for each fixed u_0, then the desired conclusion follows from (8-13) and (8-14) via a simple continuity argument in v. In particular, the conclusion follows in the case where the initial data of $\partial_v(r\phi)$ are in $W^{1,1}$ or C^1. The only remaining difficulty is, therefore, to control the size of the delta function singularities in $\partial_v^2(r\bar{\phi})$ in the general case where we only have a BV solution.

Proof. We begin by studying the propagation of discontinuities for a BV solution to (SSESF). In the general case where $\partial_v(r\phi)(1, \cdot)$ is only in BV and contains jump discontinuities (at which $\partial_v(r\phi)(1, \cdot)$ is assumed to be right-continuous), notice that the jump discontinuities for a BV function are discrete, i.e., they occur only at a (possibly infinite) sequence of points $V < v_1 < v_2 < v_3 < \cdots$. On the other hand, note that, by the initial condition $r = 2v$ on C_1, we have $\lambda(1, v) = \frac{1}{2}$; in particular, λ is continuous initially.

Thanks to the initial condition, it follows that λ does not possess any discontinuities outside Γ. Indeed, from the definition of a BV solution, m and r are continuous. Then, by (2-6), we see that ν is Lipschitz in the v-direction outside of Γ, with bounded Lipschitz constant on each compact interval of u. Looking back at (2-6) and recalling that $\lambda(1, v) = \frac{1}{2}$, we then see that λ does not possess any discontinuities outside Γ, as desired. Since λ is a priori in BV, it follows that $\int_{u_0}^{v} |\partial_v \lambda(u_0, v')|\, dv'$ is continuous in $v \in (u_0, \infty)$ with $\int_{u_0}^{v} |\partial_v \lambda(u_0, v')|\, dv' \to 0$ as $v \to u_0+$.

By the above regularity statements and (2-8), as well as the fact that ϕ is continuous outside Γ by the definition of a BV solution, it now follows that the jump discontinuities of $\partial_v(r\phi)$ are propagated along constant-v_i curves. Therefore, for $u_0 \geq V$, we see that $\partial_v(r\phi)(u_0, v)$ is a right-continuous BV function on (u_0, ∞) with jump discontinuities at $u_0 < v_1 < v_2 < v_3 < \cdots$ with the same sizes as $\partial_v(r\phi)(1, v)$. From (8-7), notice that, using the abbreviations in (8-15) and (8-18),

$$\int_{u_0}^{v} |\partial_v^2(r\bar{\phi})(u_0, v')|\, dv' = \int_{u_0}^{v} |(\partial_v^2(r\phi) - c\partial_v \lambda)(u_0, v')|\, dv'$$

for the constant $c = \lim_{v \to u_0+} \phi(u_0, v)$, which is independent of v. Thanks to the continuity property of $\lambda(u_0, \cdot)$, we see that the integral of $|\partial_v^2(r\bar{\phi})(u_0, \cdot)|$ has the same jump discontinuities as $|\partial_v^2(r\phi)(u_0, \cdot)|$. In particular, by (8-6), each jump of $I_1(u_0, v)$ is at most of size ϵ.

Fix $C_\Lambda > 1$ to be larger than the maximum of the constants from (8-13) and (8-14). First, a standard continuity argument using (8-13) and (8-14) implies that, if

$$\lim_{v \to v_i+} \int_{u_0}^{v} |\partial_v^2(r\bar{\phi})(u_0, v')|\, dv' \leq 5C_\Lambda \epsilon \quad \text{and} \quad \lim_{v \to v_i+} \int_{u_0}^{v} |\partial_v \lambda(u_0, v')|\, dv' \leq 5\epsilon$$

(with the convention $v_0 := u_0$), then

$$\int_{u_0}^{v} |\partial_v^2(r\bar{\phi})(u_0, v')|\, dv' \leq 4C_\Lambda \epsilon \quad \text{and} \quad \int_{u_0}^{v} |\partial_v \lambda(u_0, v')|\, dv' \leq 4\epsilon$$

for $v_i < v < v_{i+1}$.

Assume, for the sake of contradiction, that the conclusion of the proposition is not satisfied. Recall that the integral of $|\partial_v \lambda|$ is continuous. Thus, we have that, for some v_i with $i > 0$,

$$\lim_{v \to v_i -} \int_{u_0}^{v} |\partial_v^2 (r\bar{\phi})(u_0, v')| \, dv' \leq 4 C_\Lambda \epsilon$$

holds, but at the same time

$$\lim_{v \to v_i +} \int_{u_0}^{v} |\partial_v^2 (r\bar{\phi})(u_0, v')| \, dv' > 5 C_\Lambda \epsilon.$$

However, we have seen that the size of the jump in $I_1(u_0, v)$ is bounded by ϵ, which is smaller than $C_\Lambda \epsilon$ if $C_\Lambda > 1$. This leads to a contradiction and thus the conclusion of the proposition holds. □

We are now ready to conclude the proof of Theorem 3.14.

Proof of Theorem 3.14. We first establish (8-1). In what follows, we use the abbreviations in (8-15) and (8-18), such as $\bar{\phi} = \bar{\phi}_{u_0}$. The idea is to transform back, $(\bar{\phi}, r, m) \mapsto (\phi, r, m)$, using (8-11). Note that $|\partial_v \lambda|$ remains the same under this change, so it suffices to estimate $|\partial_v^2 (r\phi)|$. By (8-19) and Proposition 8.6, for sufficiently large u_0 the limit $\bar{\phi}(u_0, \infty) := \lim_{v \to \infty} \bar{\phi}(u_0, v)$ exists and satisfies

$$|\bar{\phi}(u_0, \infty)| \leq C_\Lambda \epsilon,$$

where we note that C_Λ is independent of u_0.

By (8-11), we have $\phi(u, v) = \bar{\phi}(u, v) - \bar{\phi}(u, \infty)$ for all u. Thus, using Proposition 8.6, we estimate

$$\begin{aligned} \int_{u_0}^{\infty} |\partial_v^2 (r\phi)(u_0, v)| \, dv &= \int_{u_0}^{\infty} \big|\partial_v^2 (r\bar{\phi}(u_0, v) - r\bar{\phi}(u_0, \infty))\big| \, dv \\ &\leq \int_{u_0}^{\infty} |\partial_v^2 (r\bar{\phi})(u_0, v)| \, dv + |\bar{\phi}(u_0, \infty)| \int_{u_0}^{\infty} |\partial_v \lambda(u_0, v)| \, dv \\ &\leq C_\Lambda (\epsilon + \epsilon^2). \end{aligned}$$

Since $u_0 \geq V$ is arbitrary, this proves (8-1).

Finally, we prove that conditions (2) and (3) of Definition 2.4 hold. Indeed, since $\partial_v \log \lambda = \lambda^{-1} \partial_v \lambda$, (3) in Definition 2.4 follows from (8-1) and (8-2); in fact, it holds with arbitrarily large $R > 0$. Next, by (2-7), nonnegativity of $1 - \mu$ and μ (by Lemma 4.1) and the fact that m is invariant under $\phi \mapsto \bar{\phi}$,

$$m(u_0, v) \leq \tfrac{1}{2} \sup_{u_0 \leq v' \leq v} |(\lambda^{-1} \partial_v (r\bar{\phi}) - \bar{\phi})(u_0, v')| \int_{u_0}^{v} |\partial_v \bar{\phi}(u_0, v')| \, dv',$$

where the right-hand side is bounded by $C_{\epsilon,\Lambda} \epsilon$ (with $C_{\epsilon,\Lambda}$ nondecreasing in ϵ) by the estimates proved so far. Therefore, (2) of Definition 2.4 follows. This concludes the proof of Theorem 3.14. □

9. Refinement in the small data case

In this section, we sketch a proof of Theorem 3.15. The idea is to revisit the proofs of the main theorems (Theorems 3.1 and 3.2), and notice that all the required smallness can be obtained by taking initial total variation of $\partial_v (r\phi)$ small. Key to this idea is the following lemma:

Lemma 9.1. *There exist universal constants ϵ_0 and C_0 such that, for $\epsilon < \epsilon_0$, the following holds: Suppose that $\lambda(1,\cdot) = \frac{1}{2}$ and $\partial_v(r\phi)(1,\cdot)$ is of bounded variation with*

$$\int_{C_1} |\partial_v^2(r\phi)| < \epsilon. \tag{9-1}$$

Suppose furthermore that $\lim_{v\to\infty} \phi(1,v) = 0$. *Then the maximal development (ϕ, r, m) satisfies condition* (1) *of Definition 2.4 and obeys*

$$\sup_{v\in[1,\infty)} \int_{\underline{C}_v} \left|\frac{\mu}{1-\mu}\frac{\nu}{r}\right| + \sup_{u\in[1,\infty)} \int_{C_u} \left|\frac{\mu}{1-\mu}\frac{\lambda}{r}\right| \le C_0\epsilon^2, \tag{9-2}$$

$$\sup_{v\in[1,\infty)} \int_{\underline{C}_v} |\partial_u \phi| + \sup_{u\in[1,\infty)} \int_{C_u} |\partial_v \phi| \le C_0\epsilon, \tag{9-3}$$

$$\sup_{v\in[1,\infty)} \int_{\underline{C}_v} (|\partial_u^2(r\phi)| + \partial_u \log \nu) + \sup_{u\in[1,\infty)} \int_{C_u} (|\partial_v^2(r\phi)| + \partial_v \log \lambda) \le C_0\epsilon. \tag{9-4}$$

Moreover, the bounds in Proposition 5.3 hold with

$$K + \Lambda \le C_0, \quad \Psi \le C_0\epsilon. \tag{9-5}$$

Proof. This lemma is an easy consequence of Theorem 5.12 and Lemma 5.11 once we show

$$\sup_{\mathcal{Q}} |\partial_v(r\phi)| \le C_0\epsilon,$$

using the additional condition $\lim_{v\to\infty} \phi(1,v) = 0$. By Lemma 5.2, note that $\int_{C_1} |\partial_v \phi| \le C\epsilon$; therefore, integrating from $v = \infty$, we have $\lim_{v\to 1+} |\phi(1,v)| \le C\epsilon$. Then, using (9-1) to integrate from $v = 1$, where we note that $\lim_{v\to 1+} \phi(1,v) = \lim_{v\to 1+} \partial_v(r\phi)(1,v)$, we obtain

$$\sup_{C_1} |\partial_v(r\phi)| \le C\epsilon.$$

Using (2-8′), $\partial_u \lambda \le 0$, Lemma 5.1 (to control $|\phi|$ from $|\partial_v(r\phi)|$) and $\frac{1}{3} \le \lambda \le \frac{1}{2}$ (by Theorem 5.12), it follows that

$$\begin{aligned} \sup_{\mathcal{D}(1,v)} |\partial_v(r\phi)| &\le \sup_{1\le v'\le v} |\partial_v(r\phi)(1,v')| + \sup_{(u,v)\in\mathcal{D}(1,v)} \sup_{1\le u'\le u} |\phi(u',v)| \int_1^u (-\partial_u\lambda)(u',v)\, du' \\ &\le C\epsilon + \tfrac{1}{2} \sup_{\mathcal{D}(1,v)} |\partial_v(r\phi)|, \end{aligned}$$

which proves $\sup_{\mathcal{Q}} |\partial_v(r\phi)| \le C_0\epsilon$, as desired. □

Equipped with Lemma 9.1, we now proceed to outline the proof of Theorem 3.15.

Proof of (1) *in Theorem 3.15.* That (ϕ, r, m) is globally BV scattering follows from Theorem 3.14 and the fact that initial data with small total variation cannot lead to a development which blows up at infinity; the latter fact follows from Theorem 6.2 in [Christodoulou 1993], as well as estimates proved in [Christodoulou 1993, Section 4].

It remains to prove that (3-1)–(3-3) hold with $A_1 \le C_{\mathcal{I}_1}(\mathcal{I}_1+\epsilon)$ if $\epsilon > 0$ is sufficiently small. By (6-6), it follows that Lemma 6.3 holds with $H_1 \le C_{\mathcal{I}_1}(\mathcal{I}_1+\epsilon)$, and (6-7) in Lemma 6.6 becomes

$$\sup_{C_u} r^{\omega}|\partial_v(r\phi)| \le C_{\mathcal{I}_1} u_1(\mathcal{I}_1+\epsilon) + CM_i u_1^{-1}\mathcal{B}_1(U). \tag{(6-7)$'$}$$

Note that $M_i \le C\mathcal{I}_1^2$. Then, repeating the arguments in Section 6D, we see that (6-10) becomes

$$\mathcal{B}_1(U) \le C_{\mathcal{I}_1} u_1(\mathcal{I}_1+\epsilon) + C(\mathcal{I}_1^2 u_1^{-1} + \epsilon^2)\mathcal{B}_1(U). \tag{(6-10)$'$}$$

It is important to note that the constant C in the last term does not depend on $\mathcal{I}_1$. Take $u_1 = 1000C(1+\mathcal{I}_{15})^2$. Then, for $\epsilon > 0$ sufficiently small (independent of $\mathcal{I}_1$), we derive

$$\mathcal{B}_1(U) \le C_{\mathcal{I}_1}(\mathcal{I}_1+\epsilon).$$

It then follows that (3-1) and (3-2) hold with $A_1 \le C_{\mathcal{I}_1}(\mathcal{I}_1+\epsilon)$. Applying Lemma 6.5, we conclude that (3-3) holds with $A_1 \le C_{\mathcal{I}_1}(\mathcal{I}_1+\epsilon)$ as well. □

Proof of (2) *in Theorem 3.15.* We need to prove that (3-4)–(3-7) hold with $A_2 \le C_{\mathcal{I}_2}(\mathcal{I}_2+\epsilon)$. The key is to show that Proposition 7.6 holds with

$$A_2' \le C_{\mathcal{I}_2}(\mathcal{I}_2+\epsilon). \tag{9-6}$$

Indeed, by the explicit bounds on the constants (in particular, (7-5), (7-6), (7-7), (7-8), (7-44) and (7-49)), the desired conclusion easily follows once (9-6) is established.

Note that $\mathcal{I}_1 \le \mathcal{I}_2$ by definition, and thus $A_1 \le C_{\mathcal{I}_2}(\mathcal{I}_2+\epsilon)$ by the preceding proof. We furthermore claim that the following statements hold:

- Lemma 7.7 holds with
$$\epsilon(u_2) \le C\epsilon, \tag{9-7}$$
for every $u_2 \ge 1$.
- We have
$$H_2'(1) \le C_{\mathcal{I}_2}(\mathcal{I}_2+\epsilon), \tag{9-8}$$
where we remind the reader that
$$H_2'(1) = \sup_{\{(u,v):u\in[1,3],v\in[u,3u]\}} u^{\omega}\big(|\partial_v^2(r\phi)| + |\partial_u^2(r\phi)| + |\partial_v\lambda| + |\partial_u\nu|\big),$$
according to (7-22).

The first claim follows easily from Lemma 9.1 and (7-34). For the second claim, since $1 \le u \le 3$, it suffices to prove[17]

$$\sup_{\mathcal{D}(1,9)} \big(|\partial_v^2(r\phi)| + |\partial_u^2(r\phi)| + |\partial_v\lambda| + |\partial_u\nu|\big) \le C(\mathcal{I}_2+\epsilon),$$

which follows from a persistence-of-regularity argument, similar to our proof of Lemma 7.7.

[17] Recall that $\mathcal{D}(1,9) = \{(u,v) : u \in [1,3],\ v \in [u,3u]\}$ is the domain of dependence of $C_1 \cap \{1 \le v \le 9\}$.

To conclude the proof, recall that we had

$$\mathcal{B}_2(U) \leq H_2''(u_2) + \epsilon''(u_2)\mathcal{B}_2(U),$$

where $\mathcal{B}_2(U)$ was defined in (7-12), and $H_2''(u_2)$ and $\epsilon''(u_2)$ obey the bounds in (7-23) and (7-24), respectively. Thanks to (9-7), it follows that we may take $u_2 = 1$ and $\epsilon''(1) \leq C\epsilon$, where C does not depend on $\mathcal{I}_2$. Next, since $u_2 = 1$, we see that $H_2''(1) \leq C_{\mathcal{I}_2}(\mathcal{I}_2 + \epsilon)$, by (9-8). Therefore, for $\epsilon > 0$ sufficiently small (independent of $\mathcal{I}_2$), we conclude that

$$\mathcal{B}_2(U) \leq C_{\mathcal{I}_2}(\mathcal{I}_2 + \epsilon),$$

which proves that Proposition 7.6 holds with (9-6), as desired. □

10. Optimality of the decay rates

In this section, we show the optimality of the decay rates obtained above, i.e., we prove Theorems 3.16 and 3.18.

10A. ***Optimality of the decay rates in the case $1 < \omega' < 3$.*** In this subsection, we prove Theorem 3.16. More precisely, we will demonstrate that the proof of the upper bounds for ϕ and its derivatives can in fact be sharpened to give also lower bounds for $\partial_v(r\phi)$ and $\partial_u(r\phi)$ if the initial data satisfy appropriate lower bounds for $\omega < 3$.

Proof of Theorem 3.16. We first prove the lower bound for $\partial_v(r\phi)$. We split the spacetime into the exterior region $\mathcal{Q}_{\text{ext}}$ and interior region $\mathcal{Q}_{\text{int}}$, as before. Notice that, in the exterior region, $u \lesssim r$ and it suffices to prove a lower bound for $r^\omega \partial_v(r\phi)$. Similarly, in the interior region, $r \lesssim u$ and it suffices to prove a lower bound for $u^\omega \partial_v(r\phi)$.

Revisiting the proof of Lemma 6.6, we note that, instead of controlling $\partial_v(r\phi)$ by the initial data and error terms, we can bound the difference between $\partial_v(r\phi)(u, v)$ and the corresponding initial value of $\partial_v(r\phi)(1, v)$. More precisely, from the proof of Lemma 6.6, we have

$$|\partial_v(r\phi)(u, v) - \partial_v(r\phi)(1, v)| \leq \frac{u_1 K M_i}{r^2(u, v)(1 + r(u, v))} H_1 + \frac{K M(u_1)}{u_1 r^\omega(u, v)} \mathcal{B}_1(U)$$

in the case $2 < \omega < 3$ and

$$|\partial_v(r\phi)(u, v) - \partial_v(r\phi)(1, v)| \leq \frac{\omega K M_i}{r(u, v)(1 + r(u, v))} H_1 + \frac{\omega K M(u_1)}{u_1 r^\omega(u, v)} \mathcal{B}_1(U)$$

in the case $1 < \omega \leq 2$. By the decay results proved in Section 6D, we have

$$\sup_u (H_1 + \mathcal{B}_1(u)) \leq A$$

for some constant A. Therefore, by choosing u_1 sufficiently large, we have, in the region $3u \leq v$,

$$r^\omega |\partial_v(r\phi)(u, v) - \partial_v(r\phi)(1, v)| \leq \tfrac{1}{4} L,$$

as long as $u \geq u_1$. We now apply the assumption on the lower bound for the initial data $r^{\omega}\partial_v(r\phi)(1, v) \geq L$ for $v \geq V$. Choosing u larger if necessary, we can assume that $u \geq V$. Then, we derive that, in $3u \leq v$,

$$r^{\omega}\partial_v(r\phi)(u, v) \geq \tfrac{1}{2}L.$$

We now move to the interior region, where $3u \geq v$. To this end, we improve the bounds in (6-8). First, notice that the lower bound in the exterior region implies that there exists L' such that

$$u^{\omega}\partial_v(r\phi)(u, v) \geq L' \tag{10-1}$$

for $3u \leq v$. Then, integrating (2-8) along the incoming direction from $\left(\frac{1}{3}u, v\right)$ to (u, v), we get

$$|\partial_v(r\phi)(u, v) - \partial_v(r\phi)\left(\tfrac{1}{3}u, v\right)| \leq \tfrac{1}{2}\Big(\sup_{u' \in [u/3, u]} \sup_{C_{u'}} |\phi|\Big) \int_{u/3}^{u} \left| \frac{2m\nu}{(1-\mu)r^2}(u', v) \right| du'.$$

By Theorem 3.1, we have

$$\sup_{C_u} |\phi| \leq A_1 u^{-\omega}$$

for some $A_1 > 0$. Lemma 6.1 implies that

$$\int_{u/3}^{u} \left| \frac{2m\nu}{(1-\mu)r^2}(u', v) \right| du' \to 0$$

as $u \to \infty$. Thus the right-hand side can be bounded by $\frac{1}{2}L'u^{\omega}$ after choosing u to be sufficiently large. Combining this with the lower bound (10-1), we have

$$u^{\omega}\partial_v(r\phi)(u, v) \geq \tfrac{1}{2}L'$$

for $3u \leq v$ and u sufficiently large.

We now proceed to obtain the lower bound for $\partial_u(r\phi)$ by revisiting the proof of Lemma 6.5. Integrating (2-8) along the outgoing direction from (u, u) to (u, v), we have

$$\left|\partial_u(r\phi)(u, v) - \lim_{v' \to u+} \partial_u(r\phi)(u, v')\right| \leq \int_{C_u} \left| \frac{\mu\lambda\nu}{(1-\mu)r}\phi \right|. \tag{10-2}$$

As before, we use Theorem 3.1, i.e.,

$$\sup_{C_u} |\phi| \leq A_1 u^{-\omega}$$

for some $A_1 > 0$. By Lemma 6.1 and the upper bound (5-6) for $|\nu|$, we have

$$\int_{C_u} \left| \frac{\mu\lambda\nu}{(1-\mu)r} \right| \to 0 \quad \text{as } u \to \infty.$$

Therefore, we can choose u sufficiently large such that

$$u^{\omega} \int_{C_u} \left| \frac{\mu\lambda\nu}{(1-\mu)r}\phi \right| \leq \tfrac{1}{4}L'.$$

Returning to (10-2) and recalling that, for u large,

$$-\lim_{v'\to u+} \partial_u(r\phi)(u, v') = \lim_{v'\to u+} \partial_v(r\phi)(u, v') \geq \tfrac{1}{2}L' u^{-\omega},$$

we have

$$-\partial_u(r\phi)(u, v) \geq \tfrac{1}{4}L' u^{-\omega}$$

for u sufficiently large, as desired. □

10B. *Key lower bound lemma.* The goal of the remainder of this section is to prove Theorem 3.18. In this subsection we establish a sufficient condition for the desired lower bounds on the decay of ϕ in terms of a number (called $\mathfrak{L}$) computed on $\mathcal{I}^+$. This will be an important ingredient for our proof of Theorem 3.18 in the next subsection.

Lemma 10.1 (key lower bound lemma). *Let (ϕ, r, m) be a C^1 solution to (SSESF) which is locally BV scattering and asymptotically flat initial data of order $\omega = 3$ in C^1. Suppose, furthermore, that*

$$\mathfrak{L} := \lim_{v\to\infty} r^3 \partial_v(r\phi)(1, v) + \int_1^\infty (M\nu_\infty \Phi)(u)\, \mathrm{d}u \neq 0,$$

where $M(u) := \lim_{v\to\infty} m(u, v)$, $\nu_\infty(u) := \lim_{v\to\infty} \nu(u, v)$ and $\Phi(u) := \lim_{v\to\infty} r\phi(u, v)$. Then there exist constants $U, L_3 > 0$ such that the following lower bounds for the decay of $\partial_v(r\phi)$, $\partial_u(r\phi)$ hold on $\{(u, v) : u \geq U\}$:

$$|\partial_v(r\phi)(u, v)| \geq L_3 \min\{r(u, v)^{-3}, u^{-3}\}, \tag{10-3}$$

$$|\partial_u(r\phi)(u, v)| \geq L_3 u^{-3}. \tag{10-4}$$

Remark 10.2. By (10-3) and (10-4), $\partial_v(r\phi)$ and $\partial_u(r\phi)$ have definite signs. In fact, the proof below shows that the signs of $\partial_v(r\phi)$ and $-\partial_u(r\phi)$ agree with that of $\mathfrak{L}$.

Proof. Without loss of generality, assume that $\mathfrak{L} > 0$. For $0 < \eta \leq 1$, define the *η-exterior region* by

$$\mathcal{Q}^\eta_{\mathrm{ext}} := \{(u, v) \in \mathcal{Q} : u \leq \eta v\}.$$

Step 1. In the first step, we make precise the relation between r and v in $\mathcal{Q}^\eta_{\mathrm{ext}}$ for small η. We claim that $r \sim \frac{1}{2}v$ in this region; more precisely,

$$\left|\frac{r(u, v)}{v} - \frac{1}{2}\right| \leq \eta C_{A_1, A_2, K, \Lambda}. \tag{10-5}$$

Integrating by parts, we have

$$r(u, v) = \int_u^v \lambda(u, v')\, \mathrm{d}v' = -\int_u^v \partial_v\lambda(u, v') v'\, \mathrm{d}v' + v\lambda(u, v) - u\lambda(u, u).$$

To make the leading term $v\lambda(u, v)$ and small number u/v explicit, we rewrite the last expression as follows:

$$r(u, v) = v\left[\lambda(u, v) - \frac{u}{v}\left(\lambda(u, u) + \int_u^v \partial_v\lambda(u, v')\frac{v'}{u}\, \mathrm{d}v'\right)\right].$$

Recall that λ is uniformly bounded from above and below on $\mathcal{Q}$, namely, $\Lambda^{-1} \le \lambda \le \frac{1}{2}$. Moreover, by the decay estimates for $\partial_v \lambda$ proved in Theorem 3.2, we have

$$\sup_{(u,v)\in\mathcal{Q}} \int_u^v |\partial_v \lambda(u, v')| \frac{v'}{u} \, dv' \le C_{A_2}.$$

As a consequence,

$$\left| \frac{r(u,v)}{v} - \lambda(u,v) \right| \le \eta C_{A_2,\Lambda}.$$

Thus (10-5) will follow once we establish

$$\left|\lambda(u,v) - \tfrac{1}{2}\right| \le \eta^2 C_{A_1,A_2,K,\Lambda}. \tag{10-6}$$

This inequality is proved by integrating the decay estimate (7-55) for $\partial_u \lambda = \partial_u \partial_v r$ along the incoming direction, starting from the normalization $\lambda(1,v) = \frac{1}{2}$. Here, we use the easy geometric fact that, if (u,v) lies in $\mathcal{Q}^\eta_{\text{ext}}$, then so does the incoming null curve from $(1,v)$ to (u,v).

Step 2. We claim that, for $U_1 \ge 1$ sufficiently large and $0 < \eta \le 1$ suitably small, we have

$$\partial_v(r\phi)(u,v) \ge \tfrac{1}{2}\mathfrak{L}\left(\tfrac{1}{2}v\right)^{-3} \tag{10-7}$$

for $(u,v) \in \mathcal{Q}^\eta_{\text{ext}} \cap \{u \ge U_1\}$.

We begin with

$$\left(\tfrac{1}{2}v\right)^3 \partial_v(r\phi)(u,v) = \left(\tfrac{1}{2}v\right)^3 \partial_v(r\phi)(1,v) + \left(\tfrac{1}{2}v\right)^3 \int_1^u \frac{2m\lambda\nu}{(1-\mu)r^3} r\phi(u',v)\, du', \tag{10-8}$$

obtained by integrating the $\partial_u \partial_v(r\phi)$ equation and multiplying by $\left(\frac{1}{2}v\right)^3$. To prove (10-7), it suffices to show that the right-hand side of (10-8) is bounded from below by $\frac{1}{2}\mathfrak{L}$ for $(u,v) \in \mathcal{Q}^\eta_{\text{ext}} \cap \{u \ge U_1\}$ with sufficiently large $U_1 \ge 1$ and small $0 < \eta \le 1$.

Note that $r = \frac{1}{2}(v-1)$ on C_1, and $v \ge \eta^{-1}$ if $(u,v) \in \mathcal{Q}^\eta_{\text{ext}}$. Thus, for $(u,v) \in \mathcal{Q}^\eta_{\text{ext}}$ and $0 < \eta \le 1$ sufficiently small, we have

$$\left|\left(\tfrac{1}{2}v\right)^3 \partial_v(r\phi)(1,v) - \lim_{v\to\infty} r^3 \partial_v(r\phi)(1,v)\right| < \tfrac{1}{8}\mathfrak{L}.$$

In order to proceed, it is useful to keep in mind the following technical point: for $U_1 \ge 1$, by the decay estimates (3-1) and (6-13), we have

$$\sup_{v \ge U_1} \int_{U_1}^v \left| \frac{2m\lambda\nu}{1-\mu} r\phi(u',v) \right| du' \le U_1^{-6} C_{A_1,\Lambda}. \tag{10-9}$$

In what follows, let $(u,v) \in \mathcal{Q}^\eta_{\text{ext}} \cap \{u \ge U_1\}$. Using (10-5), (10-9) and the fact that the null segment from $(1,v)$ to (u,v) lies in $\mathcal{Q}^\eta_{\text{ext}}$, we get

$$\left| \left(\tfrac{1}{2}v\right)^3 \int_1^u \frac{2m\lambda\nu}{(1-\mu)r^3} r\phi(u',v)\, du' - \int_1^u \frac{2m\lambda\nu}{1-\mu} r\phi(u',v)\, du' \right| \le \eta C_{A_1,A_2,K,\Lambda}.$$

Taking $U_1 \geq 1$ large enough and using (10-9), we may arrange

$$\sup_{v \geq U_1} \int_{U_1}^{v} \left| \frac{2m\lambda\nu}{1-\mu} r\phi(u', v) \right| \mathrm{d}u' + \int_{U_1}^{\infty} |M\nu_\infty \Phi(u')| \, \mathrm{d}u' < \tfrac{1}{8}\mathfrak{L}.$$

On the other hand, note that $2m\lambda\nu(1-\mu)^{-1} r\phi(u, v) \to M\nu_\infty \Phi(u)$ for each $u \geq 1$ as $v \to \infty$. Therefore, by the dominated convergence theorem, for $0 < \eta \leq 1$ sufficiently small (so that v is large), we have

$$\left| \int_{1}^{U_1} \frac{2m\lambda\nu}{1-\mu} r\phi(u', v) \, \mathrm{d}u' - \int_{1}^{U_1} M\nu_\infty \Phi(u') \, \mathrm{d}u' \right| < \tfrac{1}{8}\mathfrak{L}.$$

Putting these together and taking $0 < \eta \leq 1$ sufficiently small, we conclude (10-7).

Step 3. Next, we claim that there exists $U_2 = U_2(U_1, A_2, \Lambda, K, \eta) \geq 1$ such that $U_2 \geq U_1$ and, for $(u, v) \in (\mathcal{Q} \setminus \mathcal{Q}^{\eta}_{\mathrm{ext}}) \cap \{u \geq U_2\}$, we have

$$\partial_v (r\phi)(u, v) \geq 2\eta^3 \mathfrak{L} u^{-3}. \tag{10-10}$$

Combined with (10-7) (keeping in mind that $r \sim \frac{1}{2}v$ in $\mathcal{Q}^{\eta}_{\mathrm{ext}}$ by (10-5)), this would establish (10-3).

Take $U_2 \geq \eta^{-1} U_1$, and consider $(u, v) \in (\mathcal{Q} \setminus \mathcal{Q}^{\eta}_{\mathrm{ext}}) \cap \{u \geq U_2\}$. Integrating (2-8), we have

$$\partial_v (r\phi)(u, v) = \partial_v (r\phi)(\eta u, v) + \int_{\eta u}^{u} \frac{2m\lambda\nu}{r^2} \phi(u', v) \, \mathrm{d}u'.$$

Note that $(\eta u, v) \in \mathcal{Q}^{\eta}_{\mathrm{ext}} \cap \{u \geq U_1\}$ since $v \geq u$ and $\eta u \geq \eta U_2 \geq U_1$. Therefore, by (10-7) and the fact that $\eta^{-1} u > v$ (as $(u, v) \in \mathcal{Q} \setminus \mathcal{Q}^{\eta}_{\mathrm{ext}}$), the first term on the right-hand side obeys the lower bound

$$\partial_v (r\phi)(\eta u, v) \geq \tfrac{1}{2}\mathfrak{L}\big(\tfrac{1}{2}v\big)^{-3} > 4\eta^3 \mathfrak{L} u^{-3}.$$

On the other hand, using (3-1) and (7-56), we have

$$\left| \int_{\eta u}^{u} \frac{2m\lambda\nu}{r^2} \phi(u', v) \, \mathrm{d}u' \right| \leq C_{A_1, A_2, \Lambda, K} \int_{\eta u}^{u} \frac{1}{(u')^{10}} \, \mathrm{d}u' \leq C_{A_1, A_2, \Lambda, K, \eta} \, u^{-9}.$$

Taking U_2 large enough, we conclude that (10-10) holds.

Step 4. Finally, we claim that there exists $U = U(U_2, A_2, \Lambda, K, \eta) \geq 1$ such that $U \geq U_2 \geq U_1$ and, for $(u, v) \in \{u \geq U\}$, we have

$$-\partial_u (r\phi)(u, v) \geq \eta^3 \mathfrak{L} u^{-3}. \tag{10-11}$$

This would prove (10-4), thereby completing the proof of Lemma 10.1.

Our argument will be very similar to the previous step. Take $U \geq U_2$ and consider $(u, v) \in \{u \geq U\}$. Integrating (2-8) along the outgoing direction, we have

$$-\partial_u (r\phi)(u, v) = -\partial_u (r\phi)(u, u) - \int_{u}^{v} \frac{2m\lambda\nu}{r^2} \phi(u, v') \, \mathrm{d}v'.$$

Recall that $\lim_{v \to u+} \partial_u (r\phi)(u, v) = -\lim_{v \to u+} \partial_v (r\phi)(u, v)$. By (10-10) and the fact that $u \geq U \geq U_2$, we see that the first term on the right-hand side obeys the lower bound

$$-\partial_u (r\phi)(u, u) \geq 2\eta^3 \mathfrak{L} u^{-3}.$$

On the other hand, using (3-1) and (7-56), we have

$$\left| \int_u^v \frac{2m\lambda\nu}{r^2} \phi(u, v') \, \mathrm{d}v' \right| \le C_{A_1,A_2,K} \int_u^v \min\{u^{-10}, r^{-2}u^{-8}\} \, \lambda \, \mathrm{d}v' \le C_{A_1,A_2,K} \, u^{-9}.$$

Taking U sufficiently large, we conclude that (10-11) holds. □

10C. ***Optimality of the decay rates, in the case $\omega' \ge 3$.*** In this subsection, we prove Theorem 3.18 by studying the solution to (SSESF) arising from the initial value

$$\partial_v(r\phi)(1, v) = \epsilon \tilde{\chi}\left(\frac{v - v_0}{N}\right),$$

where $\tilde{\chi} : (-\infty, \infty) \to [0, \infty)$ is a smooth function such that

$$\operatorname{supp} \tilde{\chi} \subset \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \qquad \int_{\mathbb{R}} \tilde{\chi} = 1.$$

We also require that $v_0 \ge 2$ and $N \le v_0$. With such data, the initial total variation is of size at most $C\epsilon$:

$$\int_1^\infty |\partial_v^2(r\phi)(1, v)| \, \mathrm{d}v \le \epsilon \int_{-\infty}^\infty \left| \tilde{\chi}'\left(\frac{v - v_0}{N}\right) \right| \frac{\mathrm{d}v}{N} \le C\epsilon.$$

We also see that $\mathcal{I}_1 \le C\epsilon v_0^3$ and $\mathcal{I}_2 \le C\epsilon v_0^4/N$ with $\omega' = 3$, as

$$\sup_{v \in [1,\infty)} (1+r)^3 |\partial_v(r\phi)|(1, v) \le C\epsilon v_0^3 \quad \text{and} \quad \sup_{v \in [1,\infty)} (1+r)^4 |\partial_v^2(r\phi)|(1, v) \le C\epsilon \frac{v_0^4}{N}.$$

We are now ready to give a proof of Theorem 3.18. The idea is to compute $\mathfrak{L}$ to the leading order (which turns out to be $-c\epsilon^3$ for some $c > 0$) and then control the lower order terms by taking $\epsilon > 0$ sufficiently small and applying Theorem 3.15.

Proof of Theorem 3.18. For this proof, we fix $v_0 = 4$ and $N = 1$. We use the shorthand

$$\chi(v) := \tilde{\chi}(v - 4).$$

By the preceding discussion on the size of initial data, we see that Theorem 3.15 applies when $\epsilon > 0$ is sufficiently small. Therefore, there exists a constant $C > 0$ independent of $\epsilon > 0$ such that Theorems 3.1 and 3.2 and Proposition 5.3 hold with

$$A_1, A_2 \le C\epsilon, \qquad K, \Lambda \le C. \tag{10-12}$$

We begin by showing

$$\partial_v(r\phi)(u, v) = \epsilon\chi(v) + \mathrm{Err}_1(u, v), \tag{10-13}$$

where

$$|\mathrm{Err}_1(u, v)| \le C\epsilon^3 \min\{u^{-3}, r(u, v)^{-3}\}. \tag{10-14}$$

The argument is similar to the proof of Theorem 3.16, but this time we rely on Theorem 3.15 to make the dependence of Err_1 on ϵ explicit. Indeed, by (2-8), we have

$$|\mathrm{Err}_1(u,v)| \le \int_1^u \left| \frac{\mu\lambda\nu}{(1-\mu)r}\phi \right| (u',v)\,\mathrm{d}u'.$$

Then, estimating the right-hand side using Theorem 3.1, Proposition 5.3 and Corollary 7.13, and using (10-12) to make the ϵ-dependence explicit, (10-14) follows.

Integrating (10-13), we also have

$$\begin{aligned} r\phi(u,v) &= \int_u^v \partial_v(r\phi)(u,v')\,\mathrm{d}v' \\ &= \epsilon \int_u^v \chi(v')\,\mathrm{d}v' + \int_u^v \mathrm{Err}_1(u,v')\,\mathrm{d}v' \\ &= \epsilon X(u,v) + \mathrm{Err}_2(u,v), \end{aligned}$$

where $X(u,v) := \int_u^v \chi(v')\,\mathrm{d}v'$ and $\mathrm{Err}_2(u,v) := \int_u^v \mathrm{Err}_1(u,v')\,\mathrm{d}v'$. Integrating (10-14) and using the bound $C^{-1} \le \lambda \le \frac{1}{2}$, we easily obtain

$$|\mathrm{Err}_2(u,v)| \le C\epsilon^3 \min\{ru^{-3}, u^{-2}\}. \tag{10-15}$$

In particular, taking $v \to \infty$, we see that

$$|\Phi(u) - \epsilon X(u,\infty)| \le C\epsilon^3 u^{-2}. \tag{10-16}$$

We now proceed to estimate $M(u)$. We begin with the easy observation

$$M(u) \le C\epsilon^2 u^{-5}, \tag{10-17}$$

which follows from Corollary 7.13 and (10-12). On the other hand, recalling the definition of $M(u)$ from (2-7) and using the elementary inequality $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$,

$$\begin{aligned} M(u) &= \frac{1}{2}\int_u^\infty \frac{1-\mu}{\lambda}\left[\partial_v(r\phi) - \frac{\lambda}{r}(r\phi)\right]^2 (u,v)\,\mathrm{d}v \\ &\ge \frac{\epsilon^2}{4}\int_u^\infty \frac{1-\mu}{\lambda}(u,v)\left[\chi(v) - \frac{\lambda}{r}X(u,v)\right]^2 \mathrm{d}v - \frac{1}{2}\int_u^\infty \frac{1-\mu}{\lambda}\left[\mathrm{Err}_1 - \frac{\lambda}{r}\mathrm{Err}_2\right]^2 (u,v)\,\mathrm{d}v. \end{aligned}$$

By (10-12), (10-14) and (10-15), we have

$$\left| \frac{1}{2}\int_u^\infty \frac{1-\mu}{\lambda}\left[\mathrm{Err}_1 - \frac{\lambda}{r}\mathrm{Err}_2\right]^2 (u,v)\,\mathrm{d}v \right| \le C\epsilon^6.$$

Furthermore, note that $(1-\mu) \ge (K\Lambda)^{-1} \ge C^{-1} > 0$, by Proposition 5.3 and (10-12). Also, for $(u,v) \in [1,2] \times [8,\infty)$, note that $\chi(v) = 0$ and $X(u,v) = 1$. Therefore, for $1 \le u \le 2$, there exists $c > 0$

(independent of $\epsilon > 0$) such that

$$\begin{aligned}\frac{1}{4}\int_u^\infty \frac{1-\mu}{\lambda}(u,v)\Big[\chi - \frac{\lambda}{r}X\Big]^2(u,v)\,\mathrm{d}v &\geq (4C)^{-1}\int_u^\infty \Big[\chi - \frac{\lambda}{r}X\Big]^2(u,v)\,\lambda^{-1}(u,v)\,\mathrm{d}v \\ &\geq (4C)^{-1}\int_8^\infty \frac{\lambda}{r^2}(u,v)\,\mathrm{d}v \\ &\geq c.\end{aligned}$$

Therefore, we conclude that

$$M(u) \geq c\epsilon^2 - C\epsilon^6 \quad \text{for } 1 \leq u \leq 2. \tag{10-18}$$

We are now ready to compute $\mathfrak{L}$ and complete the proof. We begin by observing that

$$\lim_{v\to\infty} r^3|\partial_v(r\phi)(1,v)| = 0$$

by our choice of data. Therefore,

$$-\mathfrak{L} = \int_1^\infty M(-\nu_\infty)\Phi(u)\,\mathrm{d}u = \epsilon\int_1^\infty M(u)(-\nu_\infty)(u)X(u,\infty)\,\mathrm{d}u + \int_1^\infty M(u)(-\nu_\infty)(u)\,\mathrm{Err}_2(u,\infty)\,\mathrm{d}u.$$

By Proposition 5.3, (10-12), (10-15) and (10-17), we have

$$\left|\int_1^\infty M(u)(-\nu_\infty)(u)\,\mathrm{Err}_2(u,\infty)\,\mathrm{d}u\right| \leq C\epsilon^5.$$

On the other hand, by Proposition 5.3, (10-12) and (10-18), we have (taking $c > 0$ smaller if necessary)

$$\begin{aligned}\epsilon\int_1^\infty M(u)(-\nu_\infty)(u)X(u,\infty)\,\mathrm{d}u &\geq \epsilon\int_1^2 M(u)(-\nu_\infty)(u)X(u,\infty)\,\mathrm{d}u \\ &\geq \Lambda^{-1}\epsilon\int_1^2 M(u)\,\mathrm{d}u \geq c\epsilon^3 - C\epsilon^7.\end{aligned}$$

Therefore, taking $\epsilon > 0$ sufficiently small, we see that $-\mathfrak{L} > \frac{1}{2}c\epsilon^3 > 0$. □

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A MODEL FOR STUDYING DOUBLE EXPONENTIAL GROWTH IN THE TWO-DIMENSIONAL EULER EQUATIONS

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We introduce a model for the two-dimensional Euler equations which is designed to study whether or not double exponential growth can be achieved for a short time at an interior point of the flow.

1. Background

The two-dimensional Euler equations for incompressible fluid flow are given by

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p,$$

together with

$$\nabla \cdot u = 0.$$

Here, $u : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$ is a time-varying vector field on $\mathbb{R}^2$ representing the velocity and $p : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ is a scalar representing the pressure.

The equation is solved with a given initial divergence-free velocity field u_0:

$$u(x, 0) = u_0(x).$$

When u_0 is chosen to be, for instance, smooth with compact support, a smooth solution to the Euler equation exists for all time. Moreover, a result of Beale, Kato, and Majda [Beale et al. 1984] shows that Sobolev norms grow at most double-exponentially in time.

Considerable work has been done recently to establish that such growth actually occurs. Denisov [2015] demonstrates growth similar to double exponential in an example that consists of a slightly smoothed, singular steady state solution together with a bump. For some time, the singular solution stretches the bump at a double exponential rate. Kiselev and Šverák [2014] do Denisov one better by creating a sustained double exponential growth near a boundary. This is a very similar idea to Denisov's. We may imagine that something quite similar to Denisov's singular steady state lives right at the boundary and is drawing bumps towards it. Another recent result on rapid growth in the Euler equations is [Zlatoš 2015].

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2. New results

Summary. The purpose of this paper is to create a tool for studying the question of whether double exponential growth can begin spontaneously at an interior point. We borrow from Pavlović's [2002] thesis the idea that the allowed fast growth in Euler is coming from low frequency to high frequency interactions. We model the impact of each scale on the vicinity of a given particle as a linear area-preserving map.

In a double exponential growth, energy has to cascade from low frequencies to high frequencies. We try to model this phenomenon. We start at a point in time where, heuristically, N dyadic scales are active. More precisely, we use the parameter N as a bound for short time on the sum of the L^∞ norms of Littlewood–Paley projections of ∇u, the gradient of the velocity field. When double exponential growth is taking place, this Besov norm should be growing exponentially, so its size stays stable (to within a constant) for time $O(1)$. This is assumption (1).

As each scale evolves, it alters the effects of the smaller scales. We can model this as a system of differential equations with one SL(2)-valued unknown for each scale. The main result in the paper is Theorem 3, below. It says that, during a time period of order $(\log N)/N$, this autonomous system of differential equations closely approximates the actual behavior of the Euler equation. This is a time period during which growth by a factor of a power of N can occur in the Sobolev norms of the velocity and during which our hypothesis stays stable. Indeed, such growth must occur during some such time period if double exponential growth is to take place. Thus, our simplified model can be used to study the possibility and likelihood of growth occurring spontaneously at an interior point. This is especially noteworthy because the previous examples of rapid growth in the two-dimensional Euler equations (such as [Denisov 2015; Kiselev and Šverák 2014]) do not occur spontaneously from energy cascading from low to high frequencies. Thus, we believe this phenomenon is definitely worthy of more study.

We comment briefly on some of the properties of the model. Clearly, the system is simpler than studying the Euler equations. This is because many of the parameters of the Euler equations lie in the initial condition ω_0 of the SL(2) system. Indeed, once a point in $\mathbb{R}^2$ is chosen (to study the accumulation of vorticity at that point as it moves through the flow) and the parameter N (the number of active scales) is fixed, the system has only $3N$ parameters. If the initial condition ω_0 can be designed so that the SL(2) system grows exponentially with rate N, this would indicate double exponential growth in the Euler equations. However, it is critical that such growth be sustained for a time period of order $(\log N)/N$, as the proof below makes it reasonably obvious that it is possible to do so for a time period of order $1/N$ (see Lemma 5). Currently, we do not have a strategy for designing such initial data. The purpose of this work is to establish a rigorous connection between the Euler equations and the model.

Admittedly, our model works for only a very short period of time. We cannot use the model to follow the equation for a longer period of time, because nonlinearities are breaking down its connection to the equation. The fact that it runs long enough to give some insight into the double exponential growth question is a consequence of the criticality of the equation for this problem. In supercritical problems like blow-up for surface quasigeostrophic equations or blow-up for the three-dimensional Euler equations, the same kind of model cannot work.

Notation and Prerequisites. By $a \lesssim b$ we mean that $a \le kb$ for some constant k that does not depend on anything important. The notation $a \sim b$ means $a \lesssim b$ and $b \lesssim a$ simultaneously. The norm $|\cdot|$ is the usual Euclidean norm when applied to vectors in $\mathbb{R}^2$ and can be thought of as the maximum norm when applied to a matrix.

Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that

$$\psi(\xi) = \begin{cases} 1 & \text{for } 0 < |\xi| < 1, \\ 0 & \text{for } |\xi| > 2, \end{cases}$$

and define the operator P_0 to be the Fourier multiplier with symbol ψ. Let $\psi_1(\xi) = \psi\left(\frac{1}{2}\xi\right) - \psi(\xi)$ and, for $j > 1$, define P_j to be the Fourier multiplier with symbol $\psi_j(\xi) := \psi_1(2^{1-j}\xi)$. For convenience of notation, define $P_j = 0$ for $j < 0$. Thus, P_j acts like a projection onto the frequency annulus $\{\xi : |\xi| \sim 2^j\}$ for $j > 1$, and $\sum_j P_j$ is the identity because the sum telescopes. These P_j are commonly known as the Littlewood–Paley operators. Further, let $\widetilde{P}_j = \sum_{\alpha=-2}^{2} P_{j+\alpha}$ and $E_j = \sum_{k<j} P_k$. Note that

$$E_j f(x) = f * (2^{2k}\widehat{\psi}(2^k \cdot))(x) = \int f(x + 2^{-j}s)\widehat{\psi}(s)\, ds$$

and $\widehat{\psi}$ is a radial Schwartz function such that $\int \widehat{\psi} = \psi(0) = 1$. Hence, E_j acts like, and will be referred to as, an averaging operator on scale $\sim 2^{-j}$. All Littlewood–Paley operators in this work take their arguments in the spatial variable $x \in \mathbb{R}^2$ (and not in time).

Let $u : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$ be the velocity field of a two-dimensional, inviscid, incompressible fluid flow and $\omega = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ the associated vorticity. We make some assumptions about u over the time period we will be considering, which is of order $(\log N)/N$. We will assume that

$$\sum_{j=0}^{\infty} \|P_j \nabla u\|_{L^\infty} \lesssim N \tag{1}$$

$$\text{and} \qquad \|P_j \nabla u\|_{L^\infty} \lesssim 1 \quad \text{for all } j \ge 0. \tag{2}$$

Note that (2) is automatic in the case that $\omega_0 \in L^\infty$. Above, and throughout this work, $L^p = L^p(\mathbb{R}^2)$, that is, all L^p norms are taken in the spatial variable $x \in \mathbb{R}^2$. Also as above, explicit mention of the dependence on time (t) will often be omitted for brevity. We define the flow maps $\phi(x, t)$ to be solutions of the differential equations

$$\begin{aligned} \frac{\partial}{\partial t}\phi(x,t) &= u(\phi(x,t), t), \\ \phi(x, 0) &= x, \end{aligned} \tag{3}$$

so the point $\phi(x, t)$ is the image of the point x under the flow with velocity field u at time t. Thus, the Jacobian matrix of ϕ, which we denote by $D\phi$, satisfies the differential equation

$$\begin{aligned} \frac{\partial}{\partial t} D\phi(x,t) &= ((\nabla u) \circ \phi)(x,t) \cdot D\phi(x,t), \\ D\phi(x, 0) &= I, \end{aligned} \tag{4}$$

for each $x \in \mathbb{R}^2$. By both D and ∇ we mean the Jacobian derivative in the spatial variable x and not in the coordinates of the particle trajectories $\phi(x,t)$. Indeed, it should be noted that the equations (3) and (4) invite a change of coordinates via the map $x \mapsto \phi(x,t)$. This change of coordinates is especially convenient because incompressibility, $\nabla_x \cdot u = 0$, gives $\det(D_x\phi(x,t)) \equiv 1$. This will make it useful for our purposes to use the Lagrangian reference frame; that is, spatial variables will be evaluated along the particle trajectories $\phi(x,t)$. A thorough discussion of particle trajectory maps and the Lagrangian reference frame can be found in [Majda and Bertozzi 2002]. Other recent results use the Lagrangian reference frame; see [Bourgain and Li 2015a; 2015b].

Proceeding formally, if we define $R_i := \Delta^{-\frac{1}{2}}\partial/\partial x_i$, we have the so-called Biot–Savart law,

$$\nabla u = \begin{pmatrix} -R_1R_2\omega & -R_2^2\omega \\ R_1^2\omega & R_1R_2\omega \end{pmatrix}.$$

Using the Green's function for the Laplace operator, we can calculate the nonlocal parts of the composed Riesz operators by giving the nonsingular part of their kernels. (The local part, of course, lives in the singular part of the kernel located on the diagonal.) These are

$$R_1R_2\omega = K_{12} * \omega(\cdot,t), \quad R_1^2\omega = K_{11} * \omega(\cdot,t), \quad \text{and} \quad R_2^2 = -K_{11} * \omega(\cdot,t),$$

where

$$K_{12}(x_1,x_2) = \frac{x_1x_2}{\pi(x_1^2+x_2^2)^2} \quad \text{and} \quad K_{11}(x_1,x_2) = \frac{x_2^2-x_1^2}{2\pi(x_1^2+x_2^2)^2}.$$

The main result. The following definition is the one of the main fixtures of this paper. We will define an approximation of $\nabla u(\phi(0,t),t)$ so that, for a short time of order $(\log N)/N$, the flow is given by a linear area-preserving map at each physical scale around the point $\phi(0,t)$. That is, the contribution to $\nabla u(\phi(0,t),t)$ from the part of the vorticity which at time 0 was at an annulus at scale 2^{-j} around 0 is calculated as though the flow on the annulus were linear and given by some $h_j \in \mathrm{SL}(2)$. This is inspired by the following version of the Biot–Savart law:

$$\begin{aligned} \nabla u(\phi(0,t),t) &= \int \omega(s,t)K(s-\phi(0,t))\,ds \\ &= \int \omega(\phi(s,t),t)K(\phi(s,t)-\phi(0,t))\,ds \\ &= \sum_{j\in\mathbb{Z}} \int_{A_j} \omega_0(s)K(\phi(s,t)-\phi(0,t))\,ds, \end{aligned} \tag{5}$$

where $A_j = \{x : 2^{-j} \le |x| < 2^{1-j}\}$ and by dropping the index of K we mean a generic entry in the matrix $\nabla u(\phi(x,t),t)$. We have used the aforementioned change of coordinates $s \mapsto \phi(s,t)$. This change of coordinates is especially convenient because, in two space dimensions, the vorticity is purely transported by the flow map. That is,

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = 0 \quad \text{and so} \quad \omega_0(s) = \omega(\phi(s,t),t)$$

for all $t \geq 0$. In (5), we are focusing on $\phi(0,t)$, which we think of as a generic interior point of a fluid flow, in order to study whether double exponential growth is in the making at that point as it moves along the flow.

Definition 1. Let $h(t)$ be an element of SL(2). We define

$$(\nabla u)_{j,h}(t) = \begin{pmatrix} -(\nabla u)_{j,h,2} & (\nabla u)_{j,h,1} \\ (\nabla u)_{j,h,1} & (\nabla u)_{j,h,2} \end{pmatrix},$$

where

$$(\nabla u)_{j,h,i}(t) = \int_{A_j} \omega_{0,j}(s) K_{1i}(h(t)\cdot s)\, ds, \tag{6}$$

$$\omega_j = \chi_{A_j}(E_{j+\log N} - E_{j-\log N})\omega,$$

$$\text{and} \quad \omega_{0,j}(x) = \omega_j(x,0).$$

Remark 2. Two things regarding the above definition are worth emphasizing. First, a heuristic note: the object $\sum_j (\nabla u)_{j,h_j}(t)$ should be thought of as an approximation of $\nabla u(\phi(0,t),t)$. This is, of course, in the event that each matrix $h_j(t)$ is a linear approximation of the movement of the fluid particles roughly distance 2^{-j} from $\phi(0,t)$. Indeed, if in (6) we replaced $\omega_{0,j}$ with ω_0 and $h(t)\cdot s$ with $\phi(x,t)-\phi(0,t)$, we would have $\sum_j (\nabla u)_{j,h_j}(t) = \nabla u(\phi(0,t),t)$ (at least formally).

Second, despite the notation, $(\nabla u)_{j,h,i}(t)$ does not explicitly depend on the velocity field u at time t.

We now state the main result: for a short time, we can approximate the average of the Jacobian of the flow map at the scale 2^{-j} by a linear map for each j and these linear maps satisfy an autonomous system of differential equations not depending on the solution to the Euler equations. The behavior of this system can be a test for whether double exponential growth can occur and what it should look like.

Theorem 3. *Assume that*

$$\sum_{j=0}^{\infty} \|P_j \nabla u\|_{L^\infty} \lesssim N$$

$$\textit{and} \qquad \|P_j \nabla u\|_{L^\infty} \lesssim 1,$$

and let $h_j \in$ SL(2) be defined as the solution to the ODE

$$\frac{dh_j}{dt} = \Big(\sum_{k<j} (\nabla u)_{k,h_k}\Big) h_j,$$

$$h_j(0) = I.$$

Then there is a (small) universal constant $C > 0$ such that, for all times $0 \leq t \leq C(\log N)/N$, we have

$$|h_j(t) - E_j D\phi(0,t)| = O(N^{-\frac{7}{10}})$$

for all $j > 0$.

Remark 4. The purpose of using $\omega_{0,j}$ instead of just ω_0 is a technical advantage: $\omega_{0,j}$ is a projection onto the frequencies of ω that make a significant contribution to $\nabla u(\phi(0,t),t)$ coming from the annulus A_j. Indeed, if, in light of (5), we define

$$\widetilde{\nabla} u(\phi(0,t),t) := \sum_{j\in\mathbb{Z}} \int_{A_j} \omega_j(\phi(s,t),t) K(\phi(s,t)-\phi(0,t))\, ds,$$

whereas (at least formally)

$$\nabla u(\phi(0,t),t) = \sum_{j\in\mathbb{Z}} \int_{A_j} \omega(\phi(s,t),t) K(\phi(s,t)-\phi(0,t))\, ds,$$

the difference is

$$\begin{aligned}
&\sum_j \sum_{|k-j|>\log N} \int_{A_j} P_k(\omega(\phi(s,t),t)) K(\phi(s,t)-\phi(0,t))\, ds \\
&\quad = \sum_{\{(j,k):|k-j|>\log N\}} \int_{A_j} \Big(\int \omega(y,t) \check{\psi}_k(y-\phi(s,t))\, dy \Big) K(\phi(s,t)-\phi(0,t))\, ds \\
&\quad = \sum_{\{(j,k):|k-j|>\log N\}} \int_{A_j} \Big(\int \omega(y,t) \Big(\int e^{2\pi i (y-\phi(s,t))\cdot\xi} \psi_k(\xi)\, d\xi \Big) dy \Big) K(\phi(s,t)-\phi(0,t))\, ds. \qquad (7)
\end{aligned}$$

Note that, by (3), the fundamental theorem of calculus, and (1), over a time period of order $(\log N)/N$ we have

$$\frac{|s|}{\log N} \lesssim |\phi(s,t)| \lesssim |s| \log N. \qquad (8)$$

In the right-hand side of (7), we integrate by parts in $\int e^{2\pi i (y-\phi(s,t))\cdot\xi} \psi_k(\xi)\, d\xi$, moving a derivative ∇_ξ from the exponential onto ψ_k for terms in which $k > j + \log N$, and the opposite way for terms where $k < j + \log N$. Since $|s| \sim 2^{-j}$ in each A_j, this gives us a factor of $(\log N)2^{\pm j}$ from the exponential term (because of (8)) and $2^{\mp k}$ from the dilation of ψ_1, and this gives the estimate

$$(7) \lesssim \log N \sum_{\{(j,k):|k-j|>\log N\}} 2^{-|k-j|} \|P_k \omega\|_{L^\infty} \int_{A_j} K(\phi(s,t)-\phi(0,t))\, ds.$$

Since, by definition, $|K(x)| \sim |x|^{-2}$, we have $\int_{A_j} K(\phi(s,t)-\phi(0,t))\, ds \lesssim (\log N)^2$ for all j. Hence, we now have the bound

$$\begin{aligned}
(7) &\lesssim (\log N)^3 \sum_{\{(j,k):|k-j|>\log N\}} \|P_k \omega\|_{L^\infty} 2^{-|k-j|} \\
&\lesssim \frac{(\log N)^3}{N} \sum_k \|P_k \omega\|_{L^\infty} \\
&\lesssim (\log N)^3. \qquad (9)
\end{aligned}$$

We also used (1) and the fact that $\|P_k \omega\|_{L^\infty} \sim \|P_k \nabla u\|_{L^\infty}$.

The technical advantage of using $\omega_{0,j}$ is that we have, similarly to (1),

$$\sum_{j=0}^{\infty}\|\omega_j\|_{L^\infty} \lesssim \sum_{j=0}^{\infty}\sum_{\{(k,j):|k-j|<\log N\}}\|P_k\omega\|_{L^\infty} \lesssim N\log N. \tag{10}$$

The reader might ask why we chose to have the error estimate in (9) come to $(\log N)^3$. It is entirely arbitrary. By replacing the range of $\log N$ scales by a range of $C\log N$ scales, which would only cost us a constant in the estimate (10), we could reduce the estimate to an arbitrary negative power of N, but the point is that, because of the brevity of our time period, any estimate for the error which has a power of N lower than 1 will work. The error is smaller than the worst case we have for $\|\nabla u\|_{L^\infty}$. The important part of these estimates is that we lose (at most) a factor of a power of $\log N$ in (10), which is enough for our purposes, mainly because of the assumption (1).

3. Methods of the proof

Here is a brief outline of the proof. The proofs of the lemmas and the main theorem will follow in the next section.

Many of the estimates will be based on the following Gronwall-type lemma, which says that solutions to similar ODEs remain similar for a short time.

Lemma 5. *Suppose that F, G_1, G_2, w and v are real-valued functions of time with domain $[0,\infty)$ such that $F(t)=O(N)$, and that*

$$\frac{dw}{dt}(t) = F(t)w(t)+G_1(t),$$
$$\frac{dv}{dt}(t) = F(t)v(t)+G_2(t),$$
$$w(0)=v(0).$$

Assume further that, for some constant E,

$$|G_1(t)-G_2(t)| \lesssim \begin{cases} E & \text{for } 0<t\lesssim 1/N, \\ |F(t)(w(t)-v(t))| & \text{for } t\gtrsim 1/N. \end{cases}$$

Then, there is a (small) universal constant C, independent of N, such that $|(w-v)(t)| \lesssim EN^{-\frac{9}{10}-\frac{1}{100}}$ for all times $t\le C(\log N)/N$.

The idea behind Lemma 5 is that, since the difference starts out at 0, the "error" term G_1-G_2 dominates for times $t\lesssim 1/N$. At that time, the main term, $F(t)(w(t)-v(t))$, becomes the dominant term but $|(w-v)(t)|$ remains relatively small for an additional time $\lesssim \log N$. Most of the time, we will not need the extra factor of $N^{-\frac{1}{100}}$. It will be used to eliminate factors of $\log N$ that show up in the error term E.

We will often use the following estimate, which says the individual Littlewood–Paley pieces of $D\phi$ stay small:

Lemma 6. *Under assumptions (1)–(2), $\sup_{j>0}\|P_jD\phi(t)\|_{L^\infty} \lesssim N^{-\frac{9}{10}-\frac{1}{100}}$ for times $t\le C(\log N)/N$.*

In order to prove Theorem 3, we will show that, for the time period we are considering, the flow maps are essentially linear on a given dyadic annulus. That is, we will estimate the difference between the linear map $(E_j D\phi(0,t)) \cdot x$ and the difference $\phi(x,t) - \phi(0,t)$ for $|x| \sim 2^{-j}$. To do so, we first show that the averages of the Jacobians of the flow maps are close to the averages of the differences in the flow maps, that is,

$$|E_j D\phi(0,t) \cdot x - (E_j\phi(x,t) - E_j\phi(0,t))| \lesssim 2^{-j} N^{-\frac{9}{10}},$$

a sort of approximate mean value theorem. We do this by using (3) to examine the time derivative of the difference of the flow maps and (4) to examine the average of $D\phi$ at the appropriate scale. With the fundamental theorem of calculus, and on frequency support grounds, we have that the time derivative of the difference is essentially

$$\left(\int_0^1 E_j \nabla u(s\phi(x,t) + (1-s)\phi(0,t), t)\, ds \right) \cdot (E_j\phi(x,t) - E_j\phi(0,t)).$$

If we throw away $\log N$ many frequencies from the integrand, it is almost constant on its domain. The error from doing so is acceptable, so we have now, essentially,

$$E_j((\nabla u) \circ \phi)(0,t) \cdot (E_j\phi(x,t) - E_j\phi(0,t)) + O(2^{-j} \log N)$$

and we apply Lemma 5. We will still have to show that the difference of averages is close to the actual difference for x at the appropriate scale. Since $\sum P_k = 1$, this is entirely a matter of controlling the frequency bands bigger than 2^j. We do this by first using a trivial bound for the high (at least $j + \log N$) frequencies, which comes from the fundamental theorem of calculus. For $j \le k \le j + \log N$, we can again exploit the fact that averages at scale 2^{-j} are essentially constant at scale $2^{-j-\log N}$.

Putting all of this together, we have:

Lemma 7. *For times $t \le C(\log N)/N$ and $|x| \sim 2^{-j}$, we have*

$$\left|(E_j D\phi(0,t)) \cdot x - (\phi(x,t) - \phi(0,t))\right| = O(2^{-j} N^{-\frac{9}{10}}).$$

Finally, we will prove Theorem 3 by using Lemma 7 to substitute the linear map $(E_j D\phi(0,t)) \cdot x$ for the difference $\phi(x,t) - \phi(0,t)$ in each piece of the convolution used to calculate ∇u by the Biot–Savart law.

4. The proof

Note that the constant C may change from line to line. It will only change finitely many times and, in the end, it will be a universal constant which is independent of N.

Proof of Lemma 5. Observe that

$$\frac{d(w-v)}{dt}(t) = F(t)(w-v)(t) - (G_1(t) - G_2(t)),$$
$$(w-v)(0) = 0,$$

and suppose that T is the first time that $|(w-v)(T)| = E/N$. Then, for times $t \le \min\{1/N, T\}$, we have, by assumption,

$$F(t)(w-v)(t) = O(E) \implies \left|\frac{d(w-v)}{dt}(t)\right| \lesssim E.$$

Therefore, since the growth of the difference is at most linear of rate E, it follows that $T = O(1/N)$. For $T \le t \le C(\log N)/N$, we have

$$\left|\frac{d(w-v)}{dt}(t)\right| \lesssim |F(t)(w-v)(t)| = O(N)|(w-v)(t)|$$

and so, by Gronwall's lemma, we have

$$|(w-v)(t)| \lesssim \frac{E}{N}e^{Nt} \lesssim EN^{-\frac{9}{10}-\frac{1}{100}},$$

where we get the last inequality by choosing C such that $t \lesssim C(\log N)/N \le (\log(N^{\frac{1}{10}-\frac{1}{100}}))/N$. □

Proof of Lemma 6. Taking P_j of both sides of (4), we have, on frequency support grounds

$$\frac{\partial}{\partial t} P_j D\phi = P_j(E_{j+3}((\nabla u)\circ\phi)\cdot \widetilde{P}_j D\phi) + P_j(\widetilde{P}_j((\nabla u)\circ\phi)\cdot E_{j-2}D\phi) \\ + P_j\biggl(\sum_{k=j}^{\infty} \widetilde{P}_{k+1}((\nabla u)\circ\phi)\cdot P_{k+3}D\phi + P_{k+3}((\nabla u)\circ\phi)\cdot \widetilde{P}_k D\phi\biggr). \quad (11)$$

(Notice that explicit dependence on x and t has been omitted for convenience of notation. This will continue throughout this work.) We will make frequent use of the following versions of the cheap Littlewood–Paley inequality:

$$\sup_j \|P_j f\|_{L^\infty} \lesssim \|f\|_{L^\infty} \quad (12)$$

$$\text{and} \quad \sup_j \|E_j f\|_{L^\infty} \lesssim \|f\|_{L^\infty}. \quad (13)$$

To prove (13), observe that, for any x,

$$|E_j f(x)| = \left|\int f(x+2^{-j}s)\hat{\psi}(s)\,ds\right| \le \|f\|_{L^\infty}\left|\int \hat{\psi}(s)\,ds\right| = \psi(0)\cdot\|f\|_{L^\infty} = \|f\|_{L^\infty},$$

by the definition of ψ. The estimate (12) is proven analogously.

Let $S(t) = \sup_{j>0}\|P_j D\phi(t)\|_{L^\infty}$. For the first term of (11) we then have

$$\bigl|P_j\bigl(E_{j+3}((\nabla u)\circ\phi)\cdot \widetilde{P}_j D\phi\bigr)\bigr| \lesssim \|E_{j+3}((\nabla u)\circ\phi)\|_{L^\infty}\|\widetilde{P}_j D\phi\|_{L^\infty} \lesssim \sum_{j=0}^{\infty}\|P_j \nabla u\|_{L^\infty} S(t) \lesssim O(N)S(t).$$

The first inequality follows from (12), the second follows from the triangle inequality and from adding nonnegative terms, and the definitions of $\widetilde{P}_j$ and $S(t)$, and the last follows from assumption (1). Along similar lines, for the second term of (11) we have

$$\begin{aligned}\left|P_j\big(\widetilde{P}_j((\nabla u)\circ\phi)\cdot E_{j-2}D\phi\big)\right| &\lesssim \|\widetilde{P}_j((\nabla u)\circ\phi)\|_{L^\infty}\|E_{j-2}D\phi\|_{L^\infty}\\ &\lesssim \sup_j\|P_j\nabla u\|_{L^\infty}\|D\phi\|_{L^\infty}\\ &\lesssim O(\|D\phi\|_{L^\infty}),\end{aligned}$$

which follows from (12), the definition of $\widetilde{P}_j$ and (13), and assumption (2). Finally, for the last term in (11),

$$\begin{aligned}&\left|P_j\left(\sum_{k=j}^{\infty}\widetilde{P}_{k+1}((\nabla u)\circ\phi)\cdot P_{k+3}D\phi+P_{k+3}((\nabla u)\circ\phi)\cdot\widetilde{P}_kD\phi\right)\right|\\ &\qquad\lesssim\sum_{k=j}^{\infty}\|\widetilde{P}_{k+1}\nabla u\|_{L^\infty}\|P_{k+3}D\phi\|_{L^\infty}+\|P_{k+3}\nabla u\|_{L^\infty}\|\widetilde{P}_kD\phi\|_{L^\infty}\\ &\qquad\lesssim S(t)\sum_{k=j}^{\infty}\|P_k\nabla u\|_{L^\infty}\\ &\qquad\lesssim O(N)S(t),\end{aligned}$$

which we justify with (12), the definitions of $\widetilde{P}_k$ and $S(t)$, and assumption (1). Putting together these three estimates, we have

$$\frac{d}{dt}S(t)=O(N)S(t)+O(\|D\phi\|_{L^\infty}).$$

Further, using (4), (1) and Gronwall's lemma, we see that

$$\|D\phi\|_{L^\infty}\lesssim e^{Nt} \tag{14}$$

and so

$$\frac{d}{dt}S(t)=O(N)S(t)+O(e^{Nt}).$$

From here, we can apply a traditional, inhomogeneous version of Gronwall's lemma to achieve the desired bound. □

The proof of Lemma 7 is achieved in two parts. First, we show that the average of the Jacobian of a flow map is closely approximated by the average difference of a flow map at a fixed scale. That is, for $|x|\sim 2^{-j}$, we have

$$\left|(E_jD\phi(0,t))\cdot x-(E_j\phi(x,t)-E_j\phi(0,t))\right|=O(2^{-j}N^{-\frac{9}{10}}).$$

We do this by comparing the time derivatives of each expression and using Lemma 5. Then we show that the differences of the flow maps themselves at scale $\sim 2^{-j}$ are closely approximated by their averages at the same scale, that is,

$$\left|E_j\phi(x,t)-E_j\phi(0,t)-(\phi(x,t)-\phi(0,t))\right|\lesssim 2^{-j}N^{-\frac{9}{10}},$$

hence proving Lemma 7 by the triangle inequality.

Proof of Lemma 7. First, we claim that, for $|x| \sim 2^{-j}$,

$$\left|(E_j D\phi(0,t)) \cdot x - (E_j \phi(x,t) - E_j \phi(0,t))\right| = O(2^{-j} N^{-\frac{9}{10}}). \tag{15}$$

We examine $\partial\big((E_j D\phi(0,t)) \cdot x\big)/\partial t$ using (4). The goal is to use Lemma 5. In this case, we want to show that $\partial\big((E_j D\phi(0,t)) \cdot x\big)/\partial t = E_j((\nabla u) \circ \phi)(0,t) \cdot \big((E_j D\phi(0,t)) \cdot x\big)$ plus an error term which obeys acceptable bounds. Taking E_j and then the product with x of both sides of (4), we have, purely on frequency support grounds,

$$\begin{aligned}\frac{\partial}{\partial t}\big((E_j D\phi(0,t)) \cdot x\big) &= E_j\big(E_{j+3}((\nabla u) \circ \phi)(0,t) \cdot E_{j+3} D\phi(0,t)\big) \cdot x \\ &\quad + E_j\left(\sum_{k=j}^{\infty}\big(\widetilde{P}_{k+1}((\nabla u) \circ \phi) \cdot P_{k+3} D\phi + P_{k+3}((\nabla u) \circ \phi) \cdot \widetilde{P}_k D\phi\big)\right) \cdot x. \end{aligned} \tag{16}$$

(Note that, in the second line, we have again omitted the arguments of $(\nabla u) \circ \phi$ and $D\phi$ for brevity.) The second term is entirely an error term. Observe that, by (2), Lemma 6, and frequency support, the second term of (16) is

$$O\big(\sup_j \|P_j \nabla u\|_{L^\infty}\big) E_j\left(\sum_{k=j}^{\infty} P_k D\phi(0,t)\right) \cdot x = O(2^{-j}), \tag{17}$$

which will prove to be a tolerable error. For the first term of (16), since E_j is not actually a projection, we have to separate some of the frequencies. We use the fact that $E_j E_{j-2} = E_{j-2}$, giving

$$\begin{aligned} E_j\big(E_{j+3}((\nabla u) \circ \phi)(0,t) \cdot E_{j+3} D\phi(0,t)\big) \cdot x &= E_{j-2}((\nabla u) \circ \phi)(0,t) \cdot \big((E_{j-2} D\phi(0,t)) \cdot x\big) \\ &\quad + E_j\left(\sum_{k,l=j-2}^{j+2} P_k((\nabla u) \circ \phi)(0,t) \cdot P_l D\phi(0,t)\right) \cdot x. \end{aligned}$$

We now add and subtract $\big(\sum_{k,l=j-2}^{j+2} P_k((\nabla u) \circ \phi)(0,t) \cdot P_l D\phi(0,t)\big) \cdot x$. This gives us, from (16) and (17),

$$\begin{aligned} &\frac{\partial}{\partial t}((E_j D\phi(0,t)) \cdot x) \\ &= E_j((\nabla u) \circ \phi)(0,t) \cdot ((E_j D\phi(0,t)) \cdot x) + O(2^{-j}) \\ &\quad + \sum_{k,l=j}^{j+2} \big(P_k((\nabla u) \circ \phi)(0,t) \cdot P_l D\phi(0,t)\big) \cdot x \\ &\quad + \sum_{k,l=j-2}^{j+2} E_j\big(P_k((\nabla u) \circ \phi)(0,t) \cdot P_l D\phi(0,t)\big) \cdot x - P_k((\nabla u) \circ \phi)(0,t) \cdot P_l D\phi(0,t) \cdot x, \end{aligned} \tag{18}$$

where the last two lines are error terms which we denote by Ψ. Notice that, for a typical term in the last sum, we have

$$\begin{aligned}
&\big|E_j\big(P_k((\nabla u)\circ\phi)(0,t)\cdot P_l D\phi(0,t)\big)\cdot x - P_k((\nabla u)\circ\phi)(0,t)\cdot P_l D\phi(0,t)\cdot x\big| \\
&\qquad \le \big|E_j\big(P_k((\nabla u)\circ\phi)(0,t)\cdot P_l D\phi(0,t)\big)\cdot x\big| + \big|P_k((\nabla u)\circ\phi)(0,t)\cdot P_l D\phi(0,t)\cdot x\big| \\
&\qquad \lesssim 2^{-j}\|P_k((\nabla u)\circ\phi)\cdot P_l D\phi\|_{L^\infty} \\
&\qquad \lesssim 2^{-j}\|P_k\nabla u\|_{L^\infty}\|P_l D\phi\|_{L^\infty} \\
&\qquad = O(2^{-j}),
\end{aligned} \tag{19}$$

where we have used the triangle inequality, (13), the hypothesis that $|x|\sim 2^{-j}$, (2) and Lemma 6. A similar (simpler) argument can be used to achieve the same estimate for a typical term in the first sum and, since both sums have only $O(1)$ many terms, we now have the estimate

$$|\Psi| = O(2^{-j}).$$

We have now achieved the goal,

$$\frac{\partial}{\partial t}\big((E_j D\phi(0,t))\cdot x\big) = E_j((\nabla u)\circ\phi)(0,t)\cdot\big((E_j D\phi(0,t))\cdot x\big) + O(2^{-j}). \tag{20}$$

To use Lemma 5, we need an analogous statement for $\partial(E_j\phi(x,t)-E_j\phi(0,t))/\partial t$. We begin by using (3). Since $\partial/\partial t$ commutes with E_j, and by (3) and the fundamental theorem of calculus, we have

$$\begin{aligned}
\frac{\partial}{\partial t}(E_j\phi(x,t)-E_j\phi(0,t)) &= E_j\frac{\partial}{\partial t}(\phi(x,t)-\phi(0,t)) \\
&= E_j\big(u(\phi(x,t),t)-u(\phi(0,t),t)\big) \\
&= E_j\bigg(\bigg(\int_0^1 \nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\bigg)\cdot(\phi(x,t)-\phi(0,t))\bigg).
\end{aligned}$$

We now take E_j of the product, move E_j inside the integral, and the above expression gives

$$\begin{aligned}
&E_j\bigg(\bigg(\int_0^1 E_{j+3}\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\bigg)\cdot(E_{j+3}\phi(x,t)-E_{j+3}\phi(0,t))\bigg) \\
&+E_j\bigg(\sum_{k=j}^{\infty}\widetilde{P}_{k+1}((\nabla u)\circ\phi)\cdot P_{k+3}(\phi(x,t)-\phi(0,t)) + P_{k+3}((\nabla u)\circ\phi)\cdot\widetilde{P}_k(\phi(x,t)-\phi(0,t))\bigg),
\end{aligned} \tag{21}$$

which we justify on frequency support grounds. We use the same technique on the first term as we used to achieve (20). That is, we add and subtract

$$\sum_{k,l=j-2}^{j+2}\bigg(\int_0^1 P_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\bigg)\cdot(P_l\phi(x,t)-P_l\phi(0,t))$$

to exploit the fact that $E_j E_{j-2} = E_{j-2}$. This gives us that (21) equals

$$\begin{aligned}
&\left(\int_0^1 E_j \nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(E_j\phi(x,t)-E_j\phi(0,t))\\
&\quad+\sum_{k,l=j-2}^{j+2} E_j\left(\left(\int_0^1 P_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(P_l\phi(x,t)-P_l\phi(0,t))\right)\\
&\quad-\sum_{k,l=j-2}^{j+2}\left(\int_0^1 P_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(P_l\phi(x,t)-P_l\phi(0,t))\\
&\quad+E_j\left(\sum_{k=j}^{\infty}\widetilde{P}_{k+1}((\nabla u)\circ\phi)\cdot P_{k+3}(\phi(x,t)-\phi(0,t))+P_{k+3}((\nabla u)\circ\phi)\cdot\widetilde{P}_k(\phi(x,t)-\phi(0,t))\right).
\end{aligned}\tag{22}$$

The term in the first line is good and the remaining terms are error terms. Denote the difference of the middle two sums by Φ; the indices in these sums match and we can use (13) on each term in the first sum and the fact that there are only $O(1)$ many terms in the sum to estimate

$$|\Phi| = O\left(\left\|\left(\int_0^1 P_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(P_l\phi(x,t)-P_l\phi(0,t))\right\|_{L^\infty}\right).\tag{23}$$

We use assumption (2) to estimate the integral, giving

$$\begin{aligned}
&\left\|\left(\int_0^1 P_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(P_l\phi(x,t)-P_l\phi(0,t))\right\|_{L^\infty}\\
&\qquad\qquad= O\big(\|P_l\phi(x,t)-P_l\phi(0,t)\|_{L^\infty}\big).
\end{aligned}\tag{24}$$

Using (12), (8) and the hypothesis that $|x|\sim 2^{-j}$, we now have the estimate

$$|\Phi| = O(2^{-j}\log N).$$

We now estimate the last error term of (21), which we denote by Ξ. Using assumption (2), we have

$$\begin{aligned}
|\Xi| &\lesssim \sup_k\|P_k\nabla u\|_{L^\infty}\left|E_j\left(\sum_{k=j}^{\infty}P_{k+3}(\phi(x,t)-\phi(0,t))+\widetilde{P}_k(\phi(x,t)-\phi(0,t))\right)\right|\\
&\lesssim \left|E_j\left(\sum_{k=j}^{\infty}P_{k+3}(\phi(x,t)-\phi(0,t))+\widetilde{P}_k(\phi(x,t)-\phi(0,t))\right)\right|.
\end{aligned}\tag{25}$$

Because of the operator E_j, by frequency support, there are only $O(1)$ many terms left in the sum. Therefore, it suffices to estimate a typical term in the sum, such as

$$P_k(\phi(x,t)-\phi(0,t)),$$

where $k\sim j$. Using (12), (8), and the fact that $|x|\sim 2^{-j}$, we have

$$|P_k(\phi(x,t)-\phi(0,t))|\lesssim 2^{-j}\log N$$

and hence

$$|\Xi| = O(2^{-j}\log N).$$

Using these estimates on $|\Phi|$ and $|\Xi|$, we now have that (21) equals

$$\left(\int_0^1 E_j \nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\right)\cdot(E_j\phi(x,t)-E_j\phi(0,t))+O(2^{-j}\log N). \tag{26}$$

At this point, we reiterate that the goal is to show that the above expression is equal to

$$E_j((\nabla u)\circ\phi)(0,t)(E_j\phi(x,t)-E_j\phi(0,t))$$

plus an acceptable error term, and that the error so far, $O(2^{-j}\log N)$, is acceptable. For convenience, we adopt the following notation for the integral term in (26):

$$\begin{aligned}\mathcal{I}(t) &:= \int_0^1 E_j\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\,ds\\ &= \int_0^1\left(E_k\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)+\sum_{l=k}^{j-1}P_l\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)\right)ds,\end{aligned}$$

where we chose $k = j-\log N$ so that the first part of the integral is essentially constant. Indeed, if $\|f\|_{L^\infty}\lesssim N$ and $|x-y|\le 2^{-j}\log N$, with this choice of k we have

$$\begin{aligned}E_kf(x)-E_kf(y) &= \int_{\mathbb{R}^2} f(s)2^{2k}\big(\hat{\psi}(2^k(x+s))-\hat{\psi}(2^k(y+s))\big)\,ds\\ &\lesssim 2^{2k}\|f\|_{L^\infty}\|\nabla\hat{\psi}\|_{L^\infty}2^k|x-y||B_{2^{-j}}(0)|\\ &\lesssim \|f\|_{L^\infty}2^{k-j}\log N\\ &\lesssim N2^{-\log N}\log N\\ &\lesssim \log N,\end{aligned}$$

wherein we can move from the first line to the second line by the definition of ψ. Since the first part of the integrand is essentially constant, we can choose any point in the domain we want for its argument (we choose $\phi(0,t)$). We then add and subtract the extra frequencies (that is, those between k and j) and we have

$$\mathcal{I}(t)=E_j\nabla u(\phi(0,t),t)+\int_0^1\left(\sum_{l=k}^{j}P_l\nabla u(s\phi(x,t)+(1-s)\phi(0,t),t)-P_l\nabla u(\phi(0,t),t)\right)ds+\log N.$$

The integral of the sum is clearly $\lesssim \log N$ because of (2) and the choice of k. Substituting this into (26) and using (8), we have (finally)

$$\frac{\partial}{\partial t}(E_j\phi(x,t)-E_j\phi(0,t)) = E_j\nabla u(\phi(0,t),t)\cdot(E_j\phi(x,t)-E_j\phi(0,t))+O(2^{-j}(\log N)^2).$$

Using this, together with (20), we can apply Lemma 5 with

$$w=(E_jD\phi(0,t))\cdot x,\quad v=E_j\phi(x,t)-E_j\phi(0,t),\quad\text{and}\quad E=2^{-j}(\log N)^2,$$

which proves the claim that

$$|E_j D\phi(0,t)\cdot x - (E_j\phi(x,t) - E_j\phi(0,t))| = O(2^{-j}N^{-\frac{9}{10}})$$

for $|x| \sim 2^{-j}$.

Now, to prove the lemma, it suffices to show that

$$|E_j\phi(x,t) - E_j\phi(0,t) - (\phi(x,t) - \phi(0,t))| \lesssim 2^{-j}N^{-\frac{9}{10}}.$$

By the definition of the Littlewood–Paley operators, we have

$$E_j\phi(x,t) - E_j\phi(0,t) - (\phi(x,t) - \phi(0,t)) = \sum_{k\geq j} P_k\phi(x,t) - P_k\phi(0,t),$$

which we now estimate in two parts. First, for the large frequencies, we have (for arbitrary y)

$$\begin{aligned}
\sum_{k=j+\log N}^{\infty} P_k\phi(y,t) &= \sum_{k=j+\log N}^{\infty} E_{k+1}\phi(y,t) - E_k\phi(y,t) \\
&= \sum_{k=j+\log N}^{\infty} \int_{\mathbb{R}^2} \big(\phi(y + 2^{-(k+1)}s, t) - \phi(y + 2^{-k}s, t)\big)\hat{\psi}(s)\, ds \\
&\lesssim \|D\phi\|_{L^\infty} \sum_{k=j+\log N}^{\infty} 2^{-k} \\
&\lesssim N^{\frac{1}{10}} 2^{-j-\log N} \\
&\lesssim 2^{-j}N^{-\frac{9}{10}}, \qquad (27)
\end{aligned}$$

where we have used the definition of E_k, (14) and our choice of C (as in the proof of Lemma 5). For the smaller frequencies, we have left

$$\sum_{k=j}^{l} (P_k\phi(x,t) - P_k\phi(0,t)), \qquad (28)$$

where $l = j + \log N - 1$. We will estimate an arbitrary frequency band $P_k\phi(x,t) - P_k\phi(0,t)$ in this range. Take x_i to be points on the line segment from 0 to x such that $|x_{i+1} - x_i| \sim 2^{-l}$; thus we have $\sim 2^{l-j} \sim N$ points x_i. For convenience of notation, take $x_0 = 0$ and $x_N = x$. By adding and subtracting $P_k\phi(x_i,t)$ for each i, we have

$$|P_k\phi(x,t) - P_k\phi(0,t)| \lesssim 2^{l-j} \max_i |P_k\phi(x_{i+1},t) - P_k\phi(x_i,t)|. \qquad (29)$$

For each i, we have from Lemma 6 that

$$P_k(\phi(x_{i+1},t) - \phi(x_i,t)) \lesssim 2^{-l}\|P_k D\phi\|_{L^\infty} \lesssim 2^{-l}N^{-\frac{9}{10}-\frac{1}{100}}.$$

Plugging this into (29) and, in turn, plugging the result into (28), we can use the factor of $N^{-\frac{1}{100}}$ and the fact that there are only $\sim \log N$ terms in the sum to obtain

$$\sum_{k=j}^{l} P_k\phi(x,t) - P_k\phi(0,t) \lesssim 2^{-j} N^{-\frac{9}{10}}.$$

This, together with (27), proves the claim that

$$|E_j\phi(x,t) - E_j\phi(0,t) - (\phi(x,t) - \phi(0,t))| \lesssim 2^{-j} N^{-\frac{9}{10}}$$

and we have already shown that

$$|E_j D\phi(0,t)\cdot x - (\phi(x,t) - \phi(0,t))| = O(2^{-j} N^{-\frac{9}{10}});$$

applying the triangle inequality, we complete the proof of Lemma 7. □

It only remains to prove the main theorem.

Proof of Theorem 3. Our goal is to show that

$$\frac{d(E_j D\phi - h_j)}{dt} = \Bigl(\sum_{k<j} (\nabla u)_{k,E_k D\phi,i}\Bigr) E_j D\phi - \Bigl(\sum_{k<j} (\nabla u)_{k,h_k,i}\Bigr) h_j + O(N^{\frac{1}{5}}) E_j D\phi \tag{30}$$

and apply a version of Lemma 5. (We remove explicit dependence on 0 and t in order to simplify notation.) We use the definition of h_j and (20) (from which we may omit the product with x) and, with some adding and subtracting, we have

$$\begin{aligned}\frac{d(E_j D\phi - h_j)}{dt} &= \bigl(E_j((\nabla u)\circ\phi) - E_j((\tilde{\nabla} u)\circ\phi)\bigr)\cdot E_j D\phi + O(1)\\ &\quad + \Bigl(E_j((\tilde{\nabla} u\circ\phi)) - \Bigl(\sum_{k<j} (\nabla u)_{k,E_k D\phi}\Bigr)\Bigr)\cdot E_j D\phi\\ &\quad + \Bigl(\sum_{k<j} (\nabla u)_{k,E_k D\phi}\Bigr) E_j D\phi - \Bigl(\sum_{k<j} (\nabla u)_{k,h_k}\Bigr) h_j.\end{aligned} \tag{31}$$

(We are also omitting the explicit dependence on i, meaning that we are referring to a generic entry in the matrix.) We want the last line of (31) to achieve (30). The other terms are error terms, which we require to be controlled by $O(N^{\frac{1}{5}})E_j D\phi$. We can easily estimate the coefficient of $E_j D\phi$ in the first line using (13) and (9):

$$\bigl|\bigl(E_j((\nabla u)\circ\phi) - E_j((\tilde{\nabla} u)\circ\phi)\bigr)\bigr| \lesssim \|(\nabla u)\circ\phi - (\tilde{\nabla} u)\circ\phi\|_{L^\infty} \lesssim (\log N)^3, \tag{32}$$

which gives that the first term in (31) is

$$O(N^{\frac{1}{5}}) E_j D\phi \tag{33}$$

and so, in order to have (30), we only have to control the coefficient of $E_j D\phi$ in the middle term. By definition and using the Biot–Savart law, this is equal to

$$E_j\Bigg(\sum_{k\in\mathbb{Z}}\int_{A_k}\omega_{0,k}(s)K(\phi(s,t)-\phi(0,t))\,ds\Bigg)-\sum_{k<j}\int_{A_k}\omega_{0,k}(s)K(E_k D\phi(0,t)\cdot s)\,ds. \tag{34}$$

We split the sum on the left into two parts, $k \geq j$ and $k < j$. For $k \geq j$, the sum is equal to

$$\begin{aligned} E_j\Bigg(\sum_{k\geq j}\int_{A_k}\omega_{0,k}(s)K(\phi(s,t)-\phi(0,t))\,ds\Bigg) &\lesssim \sum_{k\geq j}\|K\|_{L^\infty(\phi(A_k,t))}\int_{A_k}E_j\omega_{0,k}(s)\,ds \\ &\lesssim (\log N)^2\sum_{k\geq j}\|E_j\omega_{0,k}\|_{L^\infty}. \end{aligned} \tag{35}$$

Above, we get a factor of $2^{2k}(\log N)^2$ from integrating K and using (8), and a factor of 2^{-2k} comes from integrating $E_j\omega_{0,k}$ on A_j. For each k, $\|P_k\omega\|_{L^\infty} \lesssim 1$ and, by frequency support (after using the triangle inequality), there are fewer than $(\log N)^2$ many terms in the sum. Hence the error contributed by (35) is only $O((\log N)^4) \lesssim N^{\frac{1}{5}}$.

The rest of the error term, (34), where the first sum is over $k < j$, is

$$\sum_{k<j}\int_{A_k}\omega_{0,k}(s)\big(K(\phi(s,t)-\phi(0,t))-K(E_k D\phi(0,t)\cdot s)\big)\,ds. \tag{36}$$

By Lemma 7, we have $|\phi(s,t)-\phi(0,t)-E_k D\phi(0,t)\cdot s| \lesssim 2^{-k}N^{-\frac{9}{10}}$ when $|s| \sim 2^{-k}$. Further, by (14) and (8), we may choose C so that, if $x = \phi(s,t)-\phi(0,t)$, $y = E_k D\phi(0,t)\cdot s$ and $\epsilon = \frac{1}{50}-\frac{1}{500}$, we have

$$2^{-k}N^{-\epsilon} \lesssim |x|, |y| \lesssim 2^{-k}N^{\epsilon}$$

for times $0 \leq t \leq C(\log N)/N$. Then we have the bound

$$\begin{aligned} K_{12}(x)-K_{12}(y) &\lesssim 2^{4k}N^{4\epsilon}(x_1x_2-y_1y_2) \\ &= 2^{4k}N^{4\epsilon}(x_1(x_2-y_2)+y_2(x_1-y_1)) \\ &\lesssim N^{5\epsilon}2^{3k}\max_i\{|x_i-y_i|\} \\ &\lesssim 2^{2k}N^{5\epsilon-\frac{9}{10}} \\ &\lesssim 2^{2k}N^{-\frac{4}{5}-\frac{1}{100}} \end{aligned}$$

and similarly for K_{11}. We can then estimate the sum (36) by

$$N^{-\frac{4}{5}-\frac{1}{100}}\sum_{k<j}\|\omega_{0,k}\|_{L^\infty}ds \lesssim N^{\frac{1}{5}-\frac{1}{100}}\log N \lesssim N^{\frac{1}{5}}$$

and with this we have the estimate that the middle term in (31) is $O(N^{\frac{1}{5}})E_j D\phi$ and, therefore, we have (30).

We will now apply a version of Lemma 5 using (30). Assume for contradiction that the estimate $|h_k(t) - E_k D\phi(0,t)| = O(N^{-\frac{7}{10}})$ fails for the first time at time $t_0 < C(\log N)/N$ and at scale j. So, for $k < j$ and times $t < t_0$, the estimate holds. Therefore, we have, for $t < t_0$,

$$\begin{aligned}
&\frac{d(E_j D\phi - h_j)}{dt} \\
&\quad = \Big(\sum_{k<j} (\nabla u)_{k,E_k D\phi}\Big) E_k D\phi - \Big(\sum_{k<j} (\nabla u)_{k,h_k}\Big) h_j + O(N^{\frac{1}{5}}) E_j D\phi \\
&\quad = \Big(\sum_{k<j} (\nabla u)_{k,h_k}\Big)(E_k D\phi - h_j) + \Big(\sum_{k<j} (\nabla u)_{k,E_k D\phi} - (\nabla u)_{k,h_k}\Big) E_j D\phi + O(N^{\frac{1}{5}}) E_j D\phi \\
&\quad \lesssim \Big(\sum_{k<j} (\nabla u)_{k,h_k}\Big)(E_j D\phi - h_j) + O(N^{\frac{1}{5}}) E_j D\phi,
\end{aligned}$$

where, for the last line, we used our assumption that the estimate holds on scales $k < j$ and the estimates on the Biot–Savart kernels K_{1i}. Note that, at time $t = 0$, the difference $E_j D\phi - h_j$ equals 0. Suppose that T is the first time such that $E_j D\phi - h_j = N^{-\frac{4}{5}}$. If $t \le \min\{1/N, T\}$, we have

$$\frac{d(E_j D\phi - h_j)}{dt} \lesssim N^{\frac{1}{5}} \qquad \text{since} \qquad N^{\frac{1}{5}}(E_j D\phi - h_j) = O(1)$$

and it follows that $T = O(1/N)$. For times t such that $T \le t \le t_0 < C(\log N)/N$, the first term dominates and

$$E_j D\phi - h_j = O\big(N^{-\frac{4}{5}} \exp(t O(N))\big) = O(N^{-\frac{7}{10}}),$$

where the last equality comes from our choice of C, since $t_0 < C(\log N)/N \le (\log N^{\frac{1}{10}})/N$. Thus, the assumption that the estimate breaks down at scale j and at time $t_0 < C(\log N)/N$ was false, and hence it holds for all j and $t \le C(\log N)/N$, proving the theorem. □

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ASYMPTOTICS OF HADAMARD TYPE FOR EIGENVALUES OF THE NEUMANN PROBLEM ON C^1-DOMAINS FOR ELLIPTIC OPERATORS

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This article investigates how the eigenvalues of the Neumann problem for an elliptic operator depend on the domain in the case when the domains involved are of class C^1. We consider the Laplacian and use results developed previously for the corresponding Lipschitz case. In contrast with the Lipschitz case, however, in the C^1-case we derive an asymptotic formula for the eigenvalues when the domains are of class C^1. Moreover, as an application we consider the case of a C^1-perturbation when the reference domain is of class $C^{1,\alpha}$.

1. Introduction

The results presented in this article are based on an abstract framework for eigenvalues of the Neumann problem previously developed by Kozlov and Thim [2014], where we considered applications to Lipschitz- and $C^{1,\alpha}$-domains. However, the corresponding result for C^1-domains was omitted. In this study we present an asymptotic formula of Hadamard type for perturbations in the case when the domains are of class C^1. We also apply this theorem to the case when the reference domain is $C^{1,\alpha}$, which simplifies the expressions involved.

Partial differential equations are typically not easily solvable when the domain is merely C^1. Indeed, the existence result for Laplace's equation on a general C^1-domain with L^p-data on the boundary was only finally resolved by [Fabes et al. 1978]. This problem was difficult due to the fact that proving that the layer potentials define compact operators (so Fredholm theory is applicable, similar to the $C^{1,\alpha}$-case) was rather technical. The results are based on estimates for the Cauchy integral on Lipschitz curves and we only obtain L^p-estimates for the gradient. As a consequence, the problem of eigenvalue dependence on a C^1-domain becomes difficult.

Hadamard [1908] — see also [Maz′ya and Shaposhnikova 1998] — studied a special type of perturbations of domains with smooth boundary in the early twentieth century, where the perturbed domain Ω_ε is represented by $x_\nu = h(x')$ with $x' \in \partial\Omega_0$, x_ν the signed distance to the boundary ($x_\nu < 0$ for $x \in \Omega_0$), and h a smooth function bounded by a small parameter ε. Hadamard considered the Dirichlet problem, but a formula of Hadamard type for the first nonzero eigenvalue of the Neumann Laplacian is given by

$$\Lambda(\Omega_\varepsilon) = \Lambda(\Omega_0) + \int_{\partial\Omega_0} h(|\nabla\varphi|^2 - \Lambda(\Omega_0)\varphi^2)\, dS + o(\varepsilon),$$

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where dS is the surface measure on $\partial\Omega_0$ and φ is an eigenfunction corresponding to $\Lambda(\Omega_0)$ such that $\|\varphi\|_{L^2(\Omega_0)} = 1$; compare with [Grinfeld 2010]. In more general terms, eigenvalue dependence on domain perturbations is a classical and important problem going far back. Moreover, these problems are closely related to shape optimization; see, e.g., [Henrot 2006; Sokołowski and Zolésio 1992], and references found therein.

Specifically, let Ω_1 and Ω_2 be domains in $\mathbb{R}^n$, $n \geq 2$, and consider the spectral problems

$$\begin{cases} -\Delta u = \Lambda(\Omega_1)u & \text{in } \Omega_1, \\ \partial_\nu u = 0 & \text{on } \partial\Omega_1 \end{cases} \tag{1-1}$$

and

$$\begin{cases} -\Delta v = \Lambda(\Omega_2)v & \text{in } \Omega_2, \\ \partial_\nu v = 0 & \text{on } \partial\Omega_2, \end{cases} \tag{1-2}$$

where ∂_ν is the normal derivative with respect to the outward normal. In the case of nonsmooth boundary, we consider the corresponding weak formulations. The analogous Dirichlet problems have previously been considered [Kozlov 2006; 2013; Kozlov and Nazarov 2010; 2012], however the Neumann problem requires a different approach as regards what one can use as a proximity quantity between the two domains and the operators involved.

We will require that the domains are close, in the sense that the Hausdorff distance between the sets Ω_1 and Ω_2, namely

$$d = \max\Big\{ \sup_{x\in\Omega_1} \inf_{y\in\Omega_2} |x-y|, \ \sup_{y\in\Omega_2} \inf_{x\in\Omega_1} |x-y| \Big\}, \tag{1-3}$$

is small. For example, if the problem in (1-1) has a discrete spectrum and the two domains Ω_1 and Ω_2 are close, then the problem in (1-2) has precisely J_m eigenvalues $\Lambda_k(\Omega_2)$ close to $\Lambda_m(\Omega_1)$; see, for instance, Lemma 3.1 in [Kozlov and Thim 2014]. Here, J_m is the dimension of the eigenspace X_m corresponding to $\Lambda_m(\Omega_1)$. The aim is to characterize the difference $\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1)$ for $k = 1, 2, \ldots, J_m$.

In a previous study [Kozlov and Thim 2014], we considered the cases when the domains are Lipschitz or $C^{1,\alpha}$, with $0 < \alpha < 1$, as applications of an abstract framework. The main result is an asymptotic result for $C^{1,\alpha}$-domains, where Ω_1 is a $C^{1,\alpha}$-domain and Ω_2 is a Lipschitz perturbation of Ω_1, in the sense that the perturbed domain Ω_2 can be characterized by a function h defined on the boundary $\partial\Omega_1$ such that every point $(x', x_\nu) \in \partial\Omega_2$ is represented by $x_\nu = h(x')$, where $(x', 0) \in \partial\Omega_1$ and x_ν is the signed distance to $\partial\Omega_1$ as defined above. Moreover, the function h is assumed to be Lipschitz continuous and satisfy $|\nabla h| \leq C d^\alpha$. We proved that, if the problem in (1-1) has a discrete spectrum and m is fixed, then there exists a constant $d_0 > 0$ such that, if $d \leq d_0$, then

$$\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1) = \kappa_k + O(d^{1+\alpha}) \tag{1-4}$$

for every $k = 1, 2, \ldots, J_m$. Here, $\kappa = \kappa_k$ is an eigenvalue of the problem

$$\kappa(\varphi, \psi) = \int_{\partial\Omega_1} h(x')(\nabla\varphi\cdot\nabla\psi - \Lambda_m(\Omega_1)\varphi\psi)\, dS(x') \quad \text{for all } \psi \in X_m, \tag{1-5}$$

where $\varphi \in X_m$. Moreover, $\kappa_1, \kappa_2, \ldots, \kappa_{J_m}$ in (1-4) run through all eigenvalues of (1-5), counting their multiplicities; see Theorem 1.1 in [Kozlov and Thim 2014].

In the case when the domains are merely Lipschitz, we only obtain that there exists a constant C, independent of d, such that $|\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1)| \le Cd$ for every $k = 1, 2, \ldots, J_m$; see Corollary 6.11 in [Kozlov and Thim 2014]. Furthermore, in Section 6.7 there, we provide an example which shows that we can not get an asymptotic result of the type above for the Lipschitz case.

1A. *New results.* The main result of this article is proved in Section 4B, where an asymptotic formula for $\Lambda_m(\Omega_2) - \Lambda_k(\Omega_1)$ in the case of C^1-domains is derived. The main term consists of extensions of eigenfunctions to (1-1) and the remainder is of order $o(d)$; see Theorem 4.4. We suppose that Ω_2 is a Lipschitz perturbation of a C^1-domain Ω_1 such that the Hausdorff distance d between Ω_1 and Ω_2 is small and the outward normals n_1 and n_2 — taken at the corresponding points of Ω_1 and Ω_2, respectively — are comparable in the sense that $n_1 - n_2 = o(1)$ as $d \to 0$ (uniformly). If we also require that $\Omega_2 \subset \Omega_1$ to avoid the need for extension theorems, we obtain the following result:

Theorem 1.1. *Suppose that Ω_1 is a C^1-domain, that Ω_2 is as described above, and that $\Omega_2 \subset \Omega_1$. In addition, assume that the problem in (1-1) has a discrete spectrum and that m is fixed. Then there exists a constant $d_0 > 0$ such that, if $d \le d_0$, then*

$$\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1) = \tau_k + o(d) \quad \text{for } k = 1, 2, \ldots, J_m. \tag{1-6}$$

Here, $\tau = \tau_k$ is an eigenvalue of

$$\tau(\varphi, \psi) = \int_{\Omega_1 \setminus \Omega_2} (\nabla\varphi \cdot \nabla\psi - \Lambda_m(\Omega_1)\varphi\psi)\, dx \quad \text{for all } \psi \in X_m, \tag{1-7}$$

where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \ldots, \tau_{J_m}$ in (1-6) run through all eigenvalues of (1-7), counting their multiplicities.

Note that the main term is of order d and that the remainder is strictly smaller as $d \to 0$.

As an application, in Section 5 we consider the case when the perturbation is of Hadamard type and we assume that the reference domain Ω_1 is a $C^{1,\alpha}$-domain. Indeed, if Ω_2 is a perturbation of Ω_1 in the sense that the perturbed domain Ω_2 can be characterized by a Lipschitz function h defined on the boundary $\partial\Omega_1$ such that $(x', x_\nu) \in \partial\Omega_2$ is represented by $x_\nu = h(x')$, where $(x', 0) \in \partial\Omega_1$, x_ν is the signed distance to $\partial\Omega_1$ as defined above, and $\nabla h = o(1)$ as $d \to 0$ (uniformly), we obtain the following result; see Theorem 5.1.

Theorem 1.2. *Suppose that Ω_1 is a $C^{1,\alpha}$-domain, that Ω_2 is a perturbation as described above, that the problem in (1-1) has a discrete spectrum, and that m is fixed. Then there exists a constant $d_0 > 0$ such that, if $d \le d_0$, then*

$$\Lambda_k(\Omega_2) - \Lambda_m(\Omega_1) = \kappa_k + o(d) \tag{1-8}$$

for every $k = 1, 2, \ldots, J_m$. Here, $\kappa = \kappa_k$ is an eigenvalue of the problem

$$\kappa(\varphi, \psi) = \int_{\partial\Omega_1} h(x')(\nabla\varphi \cdot \nabla\psi - \Lambda_m(\Omega_1)\varphi\psi)\, dS(x') \quad \text{for all } \psi \in X_m, \tag{1-9}$$

where $\varphi \in X_m$. Moreover, $\kappa_1, \kappa_2, \ldots, \kappa_{J_m}$ in (1-8) run through all eigenvalues of (1-9) counting their multiplicities.

We also note here that Theorem 1.2 is sharp. Indeed, the main term in (1-9) is of order d and the example given in Section 6.7 in [Kozlov and Thim 2014] shows that this cannot be improved.

2. Notation and definitions

We will use the same abstract setting and notation that was used in [Kozlov and Thim 2014]. Let us summarize the notation. We consider the operator $1-\Delta$; a number λ is an eigenvalue of the operator $1-\Delta$ if and only if $\lambda-1$ is an eigenvalue of $-\Delta$. The reason for considering $1-\Delta$ is to avoid technical difficulties due to the eigenvalue zero. Enumerate the eigenvalues $\Lambda_k(\Omega_1)=\lambda_k-1$ for $k=1,2,\dots$ of (1-1) according to $0<\lambda_1<\lambda_2<\cdots$. Similarly, we let $\Lambda_k(\Omega_2)=\mu-1$ be the eigenvalues of (1-2). Suppose that H_1 and H_2 are infinite-dimensional subspaces of a Hilbert space H. We denote the inner product on H by $(\cdot,\cdot)$. Let the operators $K_j\colon H_j\to H_j$ be positive definite and self-adjoint for $j=1,2$. Furthermore, let K_1 be compact. We consider the spectral problems

$$K_1\varphi=\lambda^{-1}\varphi,\quad \varphi\in H_1, \tag{2-1}$$

and

$$K_2U=\mu^{-1}U,\quad U\in H_2, \tag{2-2}$$

and denote by λ_k^{-1} for $k=1,2,\dots$ the eigenvalues of K_1. Let $X_k\subset H_1$ be the eigenspace corresponding to the eigenvalue λ_k^{-1}. Moreover, we denote the dimension of X_k by J_k and define $\mathscr{X}_m=X_1+X_2+\cdots X_m$, where $m\geq 1$ is any integer. In this article we study eigenvalues of (2-2) located in a neighborhood of λ_m^{-1}, where m is fixed. Note that it is known that there are precisely J_m eigenvalues of (1-2) near λ_m^{-1}; see, e.g., Lemma 3.1 in [Kozlov and Thim 2014]. We wish to describe how close they are in the case of C^1-domains.

Let $S_1\colon H\to H_1$ and $S_2\colon H\to H_2$ be orthogonal projectors and define S as the restriction of S_2 to H_1. To compare K_1 and K_2, we define the operator $B\colon H_1\to H_2$ as $B=K_2S-SK_1$. For $\varphi\in\mathscr{X}_m$, $B\varphi$ is typically small in applications. Furthermore, we use the convention that C is a generic constant that can change from line to line, but always depend only on the parameters. We also use the notation κ for a generic function $\kappa\colon[0,\infty)\mapsto[0,\infty)$ such that $\kappa(\delta)=o(1)$ as $\delta\to 0$.

2A. ***Domains in*** $\mathbb{R}^n$**.** Let Ω_1 be the reference domain, which will be fixed throughout. We will assume that Ω_1 and Ω_2 are at least Lipschitz domains. Then there exists a positive constant M such that the boundary $\partial\Omega_1$ can be covered by a finite number of balls B_k, $k=1,2,\dots,N$, where there exist orthogonal coordinate systems in which

$$\Omega_1\cap B_k=\{y=(y',y_n):y_n>h_k^{(1)}(y')\}\cap B_k,$$

where the center of B_k is at the origin and $h_k^{(1)}$ are Lipschitz functions, i.e.,

$$|h_k^{(1)}(y')-h_k^{(1)}(x')|\leq M|y'-x'|,$$

such that $h_k^{(1)}(0) = 0$. We assume that Ω_2 belongs to the class of domains where Ω_2 is close to Ω_1 in the sense that Ω_2 can be described by

$$\Omega_2 \cap B_k = \{y = (y', y_n) : y_n > h_k^{(2)}(y')\} \cap B_k,$$

where $h_k^{(2)}$ are also Lipschitz continuous with Lipschitz constant M.

The case when Ω_1 is a C^1- or $C^{1,\alpha}$-domain is defined analogously, with the addition that $h_k^{(1)} \in C^1(\mathbb{R}^{n-1})$ (or $C^{1,\alpha}(\mathbb{R}^{n-1})$) such that

$$h_k^{(1)}(0) = \partial_{x_i} h_k^{(1)}(0) = 0, \quad i = 1, 2, \ldots, n-1.$$

Note that when Ω_1 is a C^1-domain we obtain that, for $P, Q \in \partial\Omega_1$, the outward normal n_1 of Ω_1 satisfies

$$n_1(P) - n_1(Q) = o(1) \quad \text{as } |P - Q| \to 0$$

uniformly.

2B. ***Perturbations of C^1-domains.*** The situation we consider is the case when the reference domain Ω_1 is a C^1-domain and the perturbed domain Ω_2 is close in the sense of Section 2A. We require that Ω_2 is a Lipschitz domain such that

$$|\nabla(h_k^{(1)} - h_k^{(2)})| = o(1) \quad \text{as } d \to 0 \tag{2-3}$$

uniformly. This condition can be compared to the one we used in [Kozlov and Thim 2014] for perturbations of $C^{1,\alpha}$-domains:

$$|\nabla(h_k^{(1)} - h_k^{(2)})| \le C d^{\alpha}. \tag{2-4}$$

Note that $h_k^{(2)}$ are only assumed to be Lipschitz continuous and satisfy (2-3) and (2-4), respectively.

3. Definition of the operators K_j

Let Ω_1 and Ω_2 be two domains in $\mathbb{R}^n$ ($\Omega_1 \cap \Omega_2 \neq \varnothing$) and put $H = L^2(\mathbb{R}^n)$ and $H_j = L^2(\Omega_j)$ for $j = 1, 2$, where functions in H_j are extended to $\mathbb{R}^n$ by zero outside of Ω_j if necessary. For $f \in L^2(\Omega_j)$, the weak solution to the Neumann problem $(1 - \Delta)W_j = f$ in Ω_j and $\partial_\nu W_j = 0$ on $\partial\Omega_j$ for $j = 1, 2$ satisfies

$$\int_{\Omega_j} (\nabla W_j \cdot \nabla v + W_j v)\, dx = \int_{\Omega_j} f v\, dx \quad \text{for every } v \in H^1(\Omega_j),$$

and the Cauchy–Schwarz inequality implies that

$$\|\nabla W_j\|_{L^2(\Omega_j)} + \|W_j\|_{L^2(\Omega_j)} \le \|f\|_{L^2(\Omega_j)} \quad \text{for all } f \in L^2(\Omega_j).$$

We define the operators K_j on $L^2(\Omega_j)$, $j = 1, 2$, as the solution operators corresponding to the domains Ω_j, i.e., $K_j f = W_j$. The operators K_j are self-adjoint and positive definite and, if Ω_j are, e.g., Lipschitz, also compact.

3A. *Results for Lipschitz domains.* We will work with results for Lipschitz domains and then refine estimates using the additional smoothness of the C^1-case. Let Ω be a Lipschitz domain. The truncated cones $\Gamma(x')$ at $x' \in \partial\Omega$ are given by, e.g.,

$$\Gamma(x') = \{x \in \Omega : |x - x'| < 2\,\mathrm{dist}(x, \partial\Omega)\}$$

and the nontangential maximal function is defined on the boundary $\partial\Omega$ by

$$N(u)(x') = \max_{k=1,2,\dots,N} \sup\{|u(x)| : x \in \Gamma(x') \cap B_k\}.$$

For the case when Ω_1 and Ω_2 are Lipschitz, one can show that

$$\|N(K_j u)\|_{L^2(\partial\Omega_j)} + \|N(\nabla K_j u)\|_{L^2(\partial\Omega_j)} \le C\|u\|_{L^2(\Omega_j)}, \qquad j = 1, 2, \tag{3-1}$$

where the constant C depends only on the Lipschitz constant M and $B_1, B_2, \dots, B_N$. We interpret $\partial_\nu K_j u = 0$ on $\partial\Omega_j$ in the sense that $n \cdot \nabla K_j u \to 0$ nontangentially (with limits taken inside cones $\Gamma(x')$) at almost every point on $\partial\Omega$, where n is the outward normal. These results are discussed further in Section 6.2 of [Kozlov and Thim 2014]. Let us summarize that reference's Lemmas 6.2 and 6.3 for convenience.

Lemma 3.1. *Let Ω be a Lipschitz domain. Then*:

(i) *If $g \in L^2(\partial\Omega)$, then there exists a unique (up to constants) function u in $H^1(\Omega)$ such that $(1-\Delta)u = 0$ in Ω and $\partial_\nu u = g$ a.e. on $\partial\Omega$ in the nontangential sense and, moreover,*

$$\|N(u)\|_{L^2(\partial\Omega)} + \|N(\nabla u)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}.$$

(ii) *If $f \in L^2(\Omega)$, then there exists a unique function u in $H^1(\Omega)$ such that $(1-\Delta)u = f$ in Ω and $\partial_\nu u = 0$ on $\partial\Omega$ in the nontangential sense, and*

$$\|N(u)\|_{L^2(\partial\Omega)} + \|N(\nabla u)\|_{L^2(\partial\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

Here, the constant C depends only on M and $B_1, B_2, \dots, B_N$.

The corresponding lemma for the Dirichlet case is also known and one can prove it using an argument similar to the one used to prove Lemmas 6.2 and 6.3 in [Kozlov and Thim 2014].

Lemma 3.2. *Let Ω be a Lipschitz domain. Then*:

(i) *If $g \in L^2(\partial\Omega)$, then there exists a unique function $u \in H^1(\Omega)$ such that $(1-\Delta)u = 0$ in Ω, $u = g$ on $\partial\Omega$ in the nontangential sense, and*

$$\|N(u)\|_{L^2(\partial\Omega)} \le C\|g\|_{L^2(\partial\Omega)}.$$

(ii) *If $f \in L^2(\Omega)$, then there exists a unique function $u \in H^1(\Omega)$ such that $(1-\Delta)u = f$ in Ω, $u = 0$ on $\partial\Omega$ in the nontangential sense, and*

$$\|N(u)\|_{L^2(\partial\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

Here, the constant C depends only on M and $B_1, B_2, \dots, B_N$.

We conclude with an extension result for Lipschitz domains; see, e.g., [Kozlov and Thim 2014, Lemma 6.4(i)] for a proof.

Lemma 3.3. *Suppose that $f \in H^1(\partial\Omega)$ and $g \in L^2(\partial\Omega)$, where Ω is a Lipschitz domain. Then there exists a function $u \in H^1(\Omega^c)$ such that $u \to f$ and $n \cdot \nabla u \to g$ nontangentially at almost every point on $\partial\Omega$, where n is the outward normal of Ω, and there exists a constant C such that*

$$\|N(u)\|_{L^2(\partial\Omega)} + \|N(\nabla u)\|_{L^2(\partial\Omega)} \le C(\|f\|_{H^1(\partial\Omega)} + \|g\|_{L^2(\partial\Omega)}),$$

where C depends only on M and $B_1, B_2, \dots, B_N$.

4. Main results

Let us proceed to prove the main results. In Section 4A, we prove a key lemma concerning an estimate for $\partial_\nu K_j S_j \varphi$ on $\partial(\Omega_1 \cap \Omega_2)$. Using this estimate, we can refine results for Lipschitz domains that were previously developed in [Kozlov and Thim 2014] and, as a result, obtain an asymptotic formula describing the difference between λ_m^{-1} and μ_m^{-1} in terms of eigenfunctions of K_1.

4A. *Boundary estimates for C^1-domains.* Since $\partial_\nu \varphi = 0$ on $\partial\Omega_1$, we would expect that $\partial_\nu \varphi$ is small also on Ω_2 if the domains are close. However, since in the C^1-case we only obtain solutions with derivatives in L^p, this problem becomes more difficult than the corresponding issue in the $C^{1,\alpha}$-case (which was solved in [Kozlov and Thim 2014]). To this end, we will exploit that, locally on the boundaries $\partial\Omega_j$, the normal vectors can be approximated by constant unit vectors e_n (with respect to the local coordinate system). That is, we approximate the surface by its tangent plane at a specific point. We obtain the following result:

Lemma 4.1. *Let $P \in \partial(\Omega_1 \cap \Omega_2)$ and $\delta > 0$ such that $B(P, 2\delta) \subset B_k$ for some k, where B_k are the balls covering $\Omega_1 \cap \Omega_2$ given in Section 2A. Then there exists a function $\kappa(\delta)$ such that*

$$\int_{\partial(\Omega_1\cap\Omega_2)\cap B(P,\delta)} |\partial_\nu K_j S_j \varphi|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx, \qquad j = 1, 2, \tag{4-1}$$

for every $\varphi \in \mathscr{X}_m$, where $\kappa(\delta) = o(1)$ as $\delta \to 0$.

Proof. Let $B = B(P, 2\delta)$. We wish to consider $\partial_\nu K_j S_j \varphi$ on $\partial(\Omega_1 \cap \Omega_2)$. However, since $\nabla K_j S_j \varphi$ only exist in the sense of L^2, it is nontrivial to exploit the fact that $\partial_\nu K_j S_j \varphi$ is zero on $\partial\Omega_j$. Therefore, let us instead consider $\partial_{x_n} K_j S_j \varphi$ (with respect to the coordinate system in B_k). The outward normal of Ω_j is comparable to e_n in B_k and $\partial_\nu K_j S_j \varphi = 0$ on $\partial\Omega_j$, so we expect $\partial_{x_n} K_j S_j \varphi$ to be small on $\partial\Omega_j \cap B_k$. Indeed, since $\nabla K_j S_j \varphi \cdot n_j \to 0$ nontangentially on $\partial\Omega_j$ and $n_j = e_n + o(1)$ as $\delta \to 0$, we obtain that

$$\int_{\partial\Omega_j \cap B} |\partial_{x_n} K_j S_j \varphi|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx. \tag{4-2}$$

However, we cannot expect $\partial_{x_n} K_j S_j \varphi$ to be small on all of Ω_j. The idea is to use the fact that ∂_{x_n} commutes with $(1 - \lambda_m - \Delta)$. Indeed, we see that if $\Phi = \partial_{x_n} K_1 S_1 \varphi$, then $(1 - \lambda_m - \Delta)\Phi = 0$ in Ω_1 and $\Phi = \partial_{x_n} K_1 S_1 \varphi$ on $\partial\Omega_1$. The case when $j = 2$ will be treated similarly but requires some additional steps. Let us consider the equation $(1-\lambda_m-\Delta)\Phi = 0$ in Ω_1 and $\Phi = \partial_{x_n} K_1 S_1 \varphi$ on $\partial\Omega_1$. We split this equation in two separate parts.

Part 1. Let Φ_p be the solution to $(1-\lambda_m-\Delta)\Phi_p = 0$ in Ω_1, $\Phi_p = \partial_{x_n} K_1 S_1 \varphi$ on $\partial\Omega_1 \cap B$ and, on $\partial\Omega_1 \cap B^c$, we let $\Phi_p = 0$. Lemma 3.2 implies that Φ_p satisfies

$$\int_{\partial\Omega_1} |N(\Phi_p)|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx. \tag{4-3}$$

Then it follows that

$$\int_{\Omega_1 \cap \partial\Omega_2 \cap B} |\Phi_p|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2. \tag{4-4}$$

Part 2. Let Φ_h be the solution to $(1-\lambda_m-\Delta)\Phi_h = 0$ in Ω_1, $\Phi_h = 0$ on $\partial\Omega_1 \cap B$, and $\Phi_h = \partial_{x_n} K_1 S_1 \varphi$ on $\partial\Omega_1 \cap B^c$. To prove an estimate for Φ_h on $\partial\Omega_1 \cap B$ similar to the one given for Φ_p in (4-4), we use a local estimate for solutions to the Dirichlet problem, where we exploit that the boundary data is zero on $\Omega_1 \cap B$. Indeed, let $\frac{1}{2}B$ be the ball with the same center as B but half the radius. Then Theorem 5.24 in [Kenig and Pipher 1993] (for example) implies that

$$\int_{\partial\Omega_1 \cap \frac{1}{2}B} |N(\nabla\Phi_h)|^2 \, dS(x') \le C \int_{\Omega_1 \cap B} |\nabla\Phi_h|^2 \, dx \tag{4-5}$$

since the tangential gradient of Φ_h is zero on the boundary. This, in turn, implies that the left-hand side in (4-5) is finite and, furthermore, since also $\Phi_h = 0$ on $\Omega_1 \cap B$, it follows that

$$\int_{\Omega_1 \cap \partial\Omega_2 \cap \frac{1}{2}B} |\Phi_h|^2 \, dS(x') \le Cd \int_{\Omega_1} |\varphi|^2 \, dx, \tag{4-6}$$

where d is the Hausdorff distance between Ω_1 and Ω_2.

Equations (4-4) and (4-6) are sufficient to obtain that

$$\int_{\partial(\Omega_1 \cap \Omega_2) \cap \frac{1}{2}B} |N(\partial_{x_n} K_1 S_1 \varphi)|^2 \, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2 \, dx$$

since $\Phi = \Phi_p + \Phi_h$.

Turning our attention to when $j = 2$, we see that $(1-\Delta)K_2 S_2 \varphi = S_2 \varphi$ and that this equation is not homogeneous. Moreover, the right-hand side is not necessarily small. However, since $S\varphi = \lambda_m K_2 S \varphi - \lambda_m B\varphi$ and $B\varphi$ is small, we can consider

$$(1-\lambda_m-\Delta)K_2 S_2 \varphi = -\lambda_m B\varphi. \tag{4-7}$$

Let Ψ be the weak solution to $(1-\lambda_m-\Delta)\Psi = -\lambda_m B\varphi$ in Ω_2 and $\Psi = 0$ on $\partial\Omega_2$. Then

$$\|\Psi\|_{H^1(\Omega_2)} \le C\|B\varphi\|_{L^2(\Omega_2)}$$

and the trace of Ψ is defined on $\partial\Omega$. Moreover, from Lemma 3.2 we obtain that

$$\|N(\Psi)\|_{L^2(\partial\Omega_2)} \le C\|B\varphi\|_{L^2(\Omega_2)}. \tag{4-8}$$

Now, put $\Phi = \Psi + W$. Then $(1-\lambda_m-\Delta)W = 0$ and $W = \partial_{x_n} K_2 S_2\varphi$ on $\partial\Omega_2$. It is now possible to carry out steps 1 and 2 for W in Ω_2 analogously to Φ in Ω_1, exchanging the roles of Ω_1 and Ω_2. Thus, using the same notation, we obtain that

$$\int_{\partial(\Omega_1\cap\Omega_2)\cap\frac{1}{2}B} |N(W)|^2\, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2\, dx. \tag{4-9}$$

Finally, Lemma 6.6 in [Kozlov and Thim 2014] states that $\|B\varphi\|^2_{L^2(\Omega_2)} \le Cd\|\varphi\|^2_{L^2(\Omega_1)}$, so this fact and equations (4-8) and (4-9) prove that

$$\int_{\partial(\Omega_1\cap\Omega_2)\cap\frac{1}{2}B} |N(\partial_{x_n} K_2 S_2\varphi)|^2\, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2\, dx. \tag{4-10}$$

We can now conclude the proof by observing that the outward normal on $\partial(\Omega_1\cap\Omega_2)$ is given by n_1 or n_2 at almost every point, and $n_j = e_n + r_j$ with $r_j = \kappa(\delta)$, $j = 1, 2$, so we obtain that

$$\int_{\partial(\Omega_1\cap\Omega_2)\cap\frac{1}{2}B} |\partial_\nu K_j S_j\varphi|^2\, dS(x') \le \kappa(\delta) \int_{\Omega_1} |\varphi|^2\, dx. \qquad \square$$

The previous lemma is local in nature, but due to compactness we can prove the following corollary:

Corollary 4.2. *There exists a constant $d_0 > 0$ such that, if $d \le d_0$, then*

$$\int_{\partial(\Omega_1\cap\Omega_2)} |\partial_\nu K_j S_j\varphi|^2\, dS(x') \le \kappa(d) \int_{\Omega_1} |\varphi|^2\, dx, \quad j = 1, 2, \tag{4-11}$$

for every $\varphi \in \mathscr{X}_m$, where $\kappa(d) = o(1)$ as $d \to 0$.

Proof. By compactness, if d is small we can cover $\partial(\Omega_1\cap\Omega_2)$ by a finite number of balls $B(P, d)$ such that $B(P, 2d) \subset B_k$ for some k, where B_k are the covering balls from Section 2A. By choosing d_0 small enough and letting $\delta = d$ in the previous lemma, the result in the corollary now follows. $\square$

4B. *Proof of Theorem 1.1.* The following proposition is a reformulation of Proposition 6.10 in [Kozlov and Thim 2014], where the proof can also be found. The expressions with tildes are the extensions of the corresponding functions provided by Lemma 3.3. We will use this result and Corollary 4.2 to prove Theorem 1.1.

Proposition 4.3. *Suppose that Ω_1 and Ω_2 are Lipschitz domains in the sense of Section 2A. Then*

$$\lambda_m^{-1} - \mu_k^{-1} = \tau_k + O(d^{3/2}) \quad \textit{for } k = 1, 2, \ldots, J_m. \tag{4-12}$$

Here, $\tau = \tau_k$ is an eigenvalue of

$$\tau(\varphi, \psi) = \lambda_m^{-1} \int_{\Omega_1\setminus\Omega_2} ((1-\lambda_m)\widetilde{K_2 S\varphi}\psi + \nabla\widetilde{K_2 S\varphi}\cdot\nabla\psi)\, dx - \lambda_m^{-1} \int_{\Omega_2\setminus\Omega_1} ((1-\lambda_m)(K_2 S\varphi)\widetilde{\psi} + \nabla K_2 S\varphi\cdot\nabla\widetilde{\psi})\, dx \tag{4-13}$$

for all $\psi \in X_m$, where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \ldots, \tau_{J_m}$ in (4-12) run through all eigenvalues of (4-13), counting their multiplicities.

Let us now prove a version of this proposition that holds specifically for C^1-domains. We will show the following result:

Theorem 4.4. *Suppose that Ω_1 is a C^1-domain and that Ω_2 is a perturbation in the sense of Section 2B satisfying (2-3). Then*

$$\lambda_m^{-1} - \mu_k^{-1} = \tau_k + o(d) \quad \text{for } k = 1, 2, \dots, J_m. \tag{4-14}$$

Here, $\tau = \tau_k$ is an eigenvalue of

$$\tau(\varphi,\ \psi) = \lambda_m^{-1} \int_{\Omega_1 \setminus \Omega_2} ((1-\lambda_m)\varphi\psi + \nabla\varphi \cdot \nabla\psi)\, dx - \lambda_m^{-1} \int_{\Omega_2 \setminus \Omega_1} ((1-\lambda_m)\widetilde{\varphi}\widetilde{\psi} + \nabla\widetilde{\varphi} \cdot \nabla\widetilde{\psi})\, dx \tag{4-15}$$

for all $\psi \in X_m$, where $\varphi \in X_m$. Moreover, $\tau_1, \tau_2, \dots, \tau_{J_m}$ in (4-14) run through all eigenvalues of (4-15), counting their multiplicities.

Proof. We need to prove that (4-13) can be expressed as (4-15) up to a term of order $o(d)$. Since $K_2 S\varphi = B\varphi + \lambda_m^{-1} S\varphi$, we let

$$\widetilde{K_2 S\varphi} = \widetilde{B\varphi} + \lambda_m^{-1}\widetilde{\varphi},$$

where $\widetilde{B\varphi}$ is the extension of $B\varphi$ from $\Omega_1 \cap \Omega_2$ and $\widetilde{\varphi}$ is the extension of φ from Ω_1, both provided by Lemma 3.3. We show that $\widetilde{B\varphi}$ is small and that $\lambda_m^{-1}\widetilde{\varphi}$ gives the main term. To this end, let $V = B\varphi$ in $\Omega_1 \cap \Omega_2$. Then $(1-\Delta)V = 0$ in $\Omega_1 \cap \Omega_2$, $\partial_\nu V = \partial_\nu K_2 S\varphi$ on $\partial\Omega_1 \cap \Omega_2$, and $\partial_\nu V = -\partial_\nu K_1 \varphi$ on $\Omega_1 \cap \partial\Omega_2$. Using Corollary 4.2 and Lemma 3.1, we then obtain that

$$\|N(V)\|_{L^2(\partial(\Omega_1\cap\Omega_2))} + \|N(\nabla V)\|_{L^2(\partial(\Omega_1\cap\Omega_2))} \le \kappa(d)\|\varphi\|_{L^2(\Omega_1)},$$

where $\kappa(d) = o(1)$ as $d \to 0$, and thus

$$\|N(\widetilde{B\varphi})\|_{L^2(\partial(\Omega_1\cap\Omega_2))} + \|N(\nabla\widetilde{B\varphi})\|_{L^2(\partial(\Omega_1\cap\Omega_2))} \le \kappa(d)\|\varphi\|^2_{L^2(\Omega_1)}.$$

Now, the Cauchy–Schwarz inequality implies that

$$\begin{aligned}
\int_{\Omega_1\setminus\Omega_2} |\nabla\widetilde{B\varphi} \cdot \nabla\psi|\, dx &\le \left(\int_{\Omega_1\setminus\Omega_2} |\nabla\widetilde{B\varphi}|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\Omega_1\setminus\Omega_2} |\nabla\psi|^2\, dx\right)^{\frac{1}{2}} \\
&\le Cd\left(\int_{\partial(\Omega_1\cap\Omega_2)} N(\nabla\widetilde{B\varphi})^2\, dS(x')\right)^{\frac{1}{2}} \left(\int_{\Omega_1\setminus\Omega_2} |\nabla\psi|^2\, dx\right)^{\frac{1}{2}} \\
&= o(d)
\end{aligned}$$

and, similarly,

$$\int_{\Omega_1\setminus\Omega_2} |\widetilde{B\varphi}\psi|\, dx \le Cd\left(\int_{\partial(\Omega_1\cap\Omega_2)} N(\widetilde{B\varphi})^2\, dS(x')\right)^{\frac{1}{2}} \left(\int_{\Omega_1\setminus\Omega_2} |\psi|^2\, dx\right)^{\frac{1}{2}} = o(d).$$

Analogously, one can show that the corresponding expressions involving $B\varphi$ on $\Omega_2 \setminus \Omega_1$ are also of order $o(d)$. □

To pass from $\lambda_m^{-1}-\mu_m^{-1}$ to $\Lambda_k(\Omega_2)-\Lambda_m(\Omega_1)$, observe that

$$\lambda_m^{-1}-\mu_k^{-1}=\lambda_m^{-2}\left(\frac{\lambda_m}{\mu_k}(\mu_k-\lambda_m)\right)=\lambda_m^{-2}\left(\mu_k-\lambda_m-\frac{(\mu_k-\lambda_m)^2}{\mu_k}\right),$$

where $(\mu_k-\lambda_m)^2=O(d^2)$ since Ω_1 and Ω_2 are at least Lipschitz; see Corollary 6.11 in [Kozlov and Thim 2014]. Note also that, if it is the case that $\Omega_2\subset\Omega_1$, we can simplify the previous theorem by removing the second integral in (4-15) and avoid the use of extensions of eigenfunctions; compare with the statement of Theorem 1.1 in the introduction.

5. C^1-perturbations of $C^{1,\alpha}$-domains

Suppose that Ω_1 is a $C^{1,\alpha}$-domain and that it is possible to characterize the perturbed domain Ω_2 by a Lipschitz function h defined on the boundary $\partial\Omega_1$ such that $(x',x_\nu)\in\partial\Omega_2$ is represented by $x_\nu=h(x')$, where $(x',0)\in\partial\Omega_1$ and x_ν is the signed distance to the boundary $\partial\Omega_1$ (with $x_\nu<0$ when $x\in\Omega_1$). We assume that $\nabla h=o(1)$ as $d\to 0$ (uniformly). In this case, we can simplify the expression given in Theorem 4.4 and avoid the use of extensions by stating the formula (4-14) as a boundary integral.

Theorem 5.1. *Suppose that Ω_1 is a $C^{1,\alpha}$-domain and that Ω_2 is as described above. Then*

$$\lambda_m^{-1}-\mu_k^{-1}=\tau_k+o(d) \tag{5-1}$$

for $k=1,2,\dots,J_m$. Here, $\tau=\tau_k$ is an eigenvalue of

$$\tau(\varphi,\ \psi)=\lambda_m^{-2}\int_{\partial\Omega_1}h(x')((1-\lambda_m)\varphi\psi+\nabla\varphi\cdot\nabla\psi)\,dS(x')\quad\text{for all }\ \psi\in X_m, \tag{5-2}$$

where $\varphi\in X_m$. Moreover, $\tau_1,\tau_2,\dots,\tau_{J_m}$ in (5-1) run through all eigenvalues of (5-2), counting their multiplicities.

Proof. Since Ω_1 is a $C^{1,\alpha}$-domain, we can use results from the proof of Corollary 6.17 in [Kozlov and Thim 2014]. In that proof, we showed that $\varphi\in C^{1,\alpha}(\Omega_1)$ and also that φ can be extended to a function $\widetilde{\varphi}\in C^{1,\alpha}(\mathbb{R}^n)$ such that

$$\int_{\Omega_1\setminus\Omega_2}\left(|\varphi(x)-\varphi(x',0)|^2+|\nabla\varphi(x)-\nabla\varphi(x',0)|^2\right)dx\le Cd^{1+\alpha}\,\|\varphi\|^2_{L^2(\Omega_1)},$$

with the corresponding estimate holding for $\widetilde{\varphi}$ on $\Omega_2\setminus\Omega_1$. Hence, Theorem 4.4 implies that $\lambda_m^{-1}-\mu_k^{-1}$ is given by

$$\lambda_m^{-2}\biggl(\int_{\partial\Omega_1\cap\Omega_2^c}\int_0^{h(x')}\bigl((1-\lambda_m)\varphi(x',0)\psi(x',0)+\nabla\varphi(x',0)\cdot\nabla\psi(x',0)\bigr)\,dx_\nu\,dS(x')$$
$$-\int_{\partial\Omega_1\cap\Omega_2}\int_0^{-h(x')}\bigl((1-\lambda_m)\widetilde{\varphi}(x',0)\widetilde{\psi}(x',0)+\nabla\widetilde{\varphi}(x',0)\cdot\nabla\widetilde{\psi}(x',0)\bigr)\,dx_\nu\,dS(x')\biggr)+o(d).$$

The desired conclusion follows from this statement. □

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SCALING LIMIT FOR THE KERNEL OF THE SPECTRAL PROJECTOR AND REMAINDER ESTIMATES IN THE POINTWISE WEYL LAW

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Let (M, g) be a compact, smooth, Riemannian manifold. We obtain new off-diagonal estimates as $\lambda \to \infty$ for the remainder in the pointwise Weyl law for the kernel of the spectral projector of the Laplacian onto functions with frequency at most λ. A corollary is that, when rescaled around a non-self-focal point, the kernel of the spectral projector onto the frequency interval $(\lambda, \lambda+1]$ has a universal scaling limit as $\lambda \to \infty$ (depending only on the dimension of M). Our results also imply that, if M has no conjugate points, then immersions of M into Euclidean space by an orthonormal basis of eigenfunctions with frequencies in $(\lambda, \lambda+1]$ are embeddings for all λ sufficiently large.

1. Introduction

Suppose that (M, g) is a smooth, compact, Riemannian manifold without boundary of dimension $n \geq 2$. Let Δ_g be the nonnegative Laplacian acting on $L^2(M, g, \mathbb{R})$ and let $\{\varphi_j\}_j$ be an orthonormal basis of eigenfunctions:

$$\Delta_g \varphi_j = \lambda_j^2 \varphi_j, \tag{1}$$

with $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots$. This article concerns the $\lambda \to \infty$ asymptotics of the Schwartz kernel

$$E_\lambda(x, y) = \sum_{\lambda_j \leq \lambda} \varphi_j(x)\varphi_j(y) \tag{2}$$

of the spectral projection

$$E_\lambda : L^2(M, g) \to \bigoplus_{\mu \in (0,\lambda]} \ker(\Delta_g - \mu^2)$$

onto functions with frequency at most λ. We are primarily concerned with the behavior of $E_\lambda(x, y)$ at points $x, y \in M$ for which the Riemannian distance $\mathrm{dist}_g(x, y)$ is less than the injectivity radius $\mathrm{inj}(M, g)$, so that the inverse of the exponential map $\exp_y^{-1}(x)$ is well defined. We write

$$E_\lambda(x, y) = \frac{\lambda^n}{(2\pi)^n} \int_{|\xi|_{g_y} < 1} e^{i\lambda \langle \exp_y^{-1}(x), \xi \rangle_{g_y}} \frac{d\xi}{\sqrt{|g_y|}} + R(x, y, \lambda), \tag{3}$$

where the remainder $R(x, y, \lambda)$ is a smooth function of x and y. The integral in (3) is over the cotangent fiber $T_y^* M$ and it is coordinate-independent because the integration measure $d\xi/\sqrt{|g_y|}$ is the quotient

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of the natural symplectic form $d\xi\, dy$ on T^*M by the Riemannian volume form $\sqrt{|g_y|}\, dy$. The integral is also symmetric in x and y, which can be seen by changing variables from T_y^*M to T_x^*M using the parallel transport operator (see (28)).

Our main result, Theorem 2, fits into a long history of estimates on $R(x, y, \lambda)$ as $\lambda \to +\infty$ (see Section 1.2 for some background). To state it, we need a definition from [Safarov 1988; Sogge and Zelditch 2013]:

Definition 1. A point $x \in M$ is said to be *non-self-focal* if the set of unit covectors

$$\mathscr{L}_x = \{\xi \in S_x^* M \mid \exp_x(t\xi) = x \ \text{ for some } \ t > 0\} \tag{4}$$

has zero measure with respect to the surface measure induced by g on $S_x^* M$.

Theorem 2. *Let (M, g) be a compact, smooth, Riemannian manifold of dimension $n \geq 2$ with no boundary. Suppose $x_0 \in M$ is a non-self-focal point and let r_λ be a nonnegative function with $\lim_{\lambda\to\infty} r_\lambda = 0$. Then*

$$\sup_{x,y\in B(x_0,r_\lambda)} |R(x, y, \lambda)| = o(\lambda^{n-1}) \tag{5}$$

as $\lambda \to \infty$. Here, $B(x_0, r_\lambda)$ denotes the geodesic ball of radius r_λ centered at x_0 and the rate of convergence depends on x_0 and r_λ.

The little-o estimate (5) is not new for $x = y$ (i.e., $r_\lambda = 0$). Both Safarov [1988] and Sogge and Zelditch [2002] show that $R(x, x, \lambda) = o(\lambda^{n-1})$ when x belongs to a compact subset of the diagonal in $M \times M$ consisting only of non-self-focal points (see also [Safarov and Vassiliev 1997]). Safarov [1988] also obtained $o(\lambda^{n-1})$ estimates on $R(x, y, \lambda)$ for (x, y) in a compact subset of $M \times M$ that does not intersect the diagonal (under the assumptions of Theorem 6). Theorem 2 simultaneously allows $x \neq y$ and $\mathrm{dist}_g(x, y) \to 0$ as $\lambda \to \infty$, closing the gap between the two already-known regimes. We refer the reader to Section 1.2 for further discussion and motivation for Theorem 2 and to Section 2 for an outline of the proof.

An elementary corollary of Theorem 2 is Theorem 3, which gives scaling asymptotics for the Schwartz kernel

$$E_{(\lambda,\lambda+1]}(x, y) := \sum_{\lambda<\lambda_j\leq\lambda+1} \varphi_j(x)\varphi_j(y) \tag{6}$$

of the orthogonal projection

$$E_{(\lambda,\lambda+1]} = E_{\lambda+1} - E_\lambda : L^2(M, g) \to \bigoplus_{\mu\in(\lambda,\lambda+1]} \ker(\Delta_g - \mu^2).$$

Passing to polar coordinates in (3) and using that

$$\int_{S^{n-1}} e^{i\langle v,\omega\rangle}\, d\omega = (2\pi)^{n/2} \frac{J_{(n-2)/2}(|v|)}{|v|^{(n-2)/2}}, \tag{7}$$

it is straightforward to obtain the following result:

Theorem 3. *Let (M, g) be a compact, smooth, Riemannian manifold of dimension $n \geq 2$ with no boundary. Let $x_0 \in M$ be a non-self-focal point. Consider any nonnegative function r_λ satisfying $r_\lambda \to 0$ as $\lambda \to \infty$. Then*

$$\sup_{x,y \in B(x_0, r_\lambda)} \left| E_{(\lambda,\lambda+1]}(x, y) - \frac{\lambda^{n-1}}{(2\pi)^{n/2}} \frac{J_{(n-2)/2}(\lambda \operatorname{dist}_g(x, y))}{(\lambda \operatorname{dist}_g(x, y))^{(n-2)/2}} \right| = o(\lambda^{n-1}), \tag{8}$$

where J_ν is the Bessel function of the first kind with index ν, $B(x_0, r_\lambda)$ denotes the geodesic ball of radius r_λ centered at x_0, and dist_g is the Riemannian distance.

Remark 4. Under the assumptions of Theorem 3, relation (8) holds for $E_{(\lambda,\lambda+\delta]}$ with any $\delta > 0$. The difference is that the Bessel function term is multiplied by δ and that the rate of convergence depends on δ. Our proof of Theorem 3 is insensitive to the choice of δ.

In normal coordinates at x_0, (8) therefore implies

$$\sup_{|u|,|v|<r_0} \left| E_{(\lambda,\lambda+1]}\left(x_0 + \frac{u}{\lambda}, x_0 + \frac{v}{\lambda}\right) - \frac{\lambda^{n-1}}{(2\pi)^n} \int_{S^{n-1}} e^{i\langle u-v,w\rangle} \, d\omega \right| = o(\lambda^{n-1}) \tag{9}$$

as $\lambda \to \infty$. The measure $d\omega$ is the Euclidean surface measure on the unit sphere S^{n-1} and the rate of convergence of the error term depends on r_0 and the point x_0. The integral over S^{n-1} in (9) is the kernel of the spectral projector onto the generalized eigenspace of eigenvalue 1 for the flat Laplacian on $\mathbb{R}^n$ (see [Helgason 1981; Zelditch 2008, §2.1]).

We believe (5) holds for any number of covariant derivatives $\nabla_x^j \nabla_y^k$ of the remainder $R(x, y, \lambda)$ with $o(\lambda^{n-1})$ replaced by $o(\lambda^{n-1+j+k})$. This would immediately imply that the C^0 convergence in (8) can be upgraded to C^k convergence for all k. Proving this is work in progress by the authors. Since $E_{(\lambda,\lambda+1]}$ is the covariance kernel for asymptotically fixed frequency random waves on M (see [Sarnak and Wigman 2014; Sodin 2012; Zelditch 2009]), this C^∞ convergence would show that the integral statistics of monochromatic random waves near a non-self-focal point depend only on the dimension of M. We refer the reader to Section 1.3 for further discussion and motivation for Theorem 3.

1.1. ***Applications.*** Combining Theorem 2 with prior results of Safarov [1988], we obtain little-o estimates on $R(x, y, \lambda)$ without requiring x or y to be in a shrinking neighborhood of a single nonfocal point. We recall the following definition from [Safarov 1988; Sogge and Zelditch 2013]:

Definition 5. Let (M, g) be a Riemannian manifold. We say that $x, y \in M$ are *mutually nonfocal* if the set of unit covectors

$$\mathcal{L}(x, y) = \{\xi \in S_x^* M \mid \exp_x(t\xi) = y \text{ for some } t > 0\} \tag{10}$$

has zero measure with respect to the Euclidean surface measure induced by g on $S_x^* M$.

Theorem 6. *Let (M, g) be a compact, smooth, Riemannian manifold of dimension $n \geq 2$ with no boundary. Consider any compact set $K \subseteq M \times M$ such that, if $(x, y) \in K$, then x and y are mutually nonfocal and either x or y is a non-self-focal point. Then, as $\lambda \to \infty$, we have*

$$\sup_{(x,y) \in K} |R(x, y, \lambda)| = o(\lambda^{n-1}). \tag{11}$$

Remark 7. Theorem 6 applies with $K = M \times M$ if (M, g) has no conjugate points.

Theorem 6 — proved in Section 7 — can be applied to studying immersions of (M, g) into Euclidean space by arrays of high-frequency eigenfunctions. Let $\{\varphi_{j_1}, \dots, \varphi_{j_{m_\lambda}}\}$ be an orthonormal basis for $\bigoplus_{\lambda<\mu\le\lambda+1} \ker(\Delta_g - \mu^2)$ and consider the maps

$$\Psi_{(\lambda,\lambda+1]} : M \to \mathbb{R}^{m_\lambda}, \quad \Psi_{(\lambda,\lambda+1]}(x) = \sqrt{\frac{(2\pi)^n}{2\lambda^{n-1}}}(\varphi_{j_1}(x), \dots, \varphi_{j_{m_\lambda}}(x)). \tag{12}$$

The $\lambda^{-(n-1)/2}$ normalization is chosen so that the diameter of $\Psi_{(\lambda,\lambda+1]}(M)$ in $\mathbb{R}^{m_\lambda}$ is bounded above and below as $\lambda \to \infty$. Maps related to Ψ_λ are studied in [Bérard et al. 1994; Jones et al. 2008; Potash 2014; Zelditch 2009]. In particular, Zelditch [2009, Proposition 2.3] showed that the maps $\Psi_{(\lambda,\lambda+1]}$ are almost-isometric immersions for large λ, in the sense that a certain rescaling of the pullback $\Psi_\lambda^*(g_{\text{euc}})$ of the Euclidean metric on $\mathbb{R}^{m_\lambda}$ converges pointwise to g. A consequence of Theorem 6 is that these maps are actually embeddings for λ sufficiently large:

Theorem 8. *Let (M, g) be a compact, smooth, Riemannian manifold of dimension $n \ge 2$ with no boundary. If every point $x \in M$ is non-self-focal and all pairs $x, y \in M$ are mutually nonfocal, then there exists $\lambda_0 > 0$ such that the maps $\Psi_{(\lambda,\lambda+1]} : M \to \mathbb{R}^{m_\lambda}$ are embeddings for all $\lambda \ge \lambda_0$.*

We prove Theorem 8 in Section 7. Note that this result does not hold on the round spheres $S^n \subseteq \mathbb{R}^{n+1}$, since even spherical harmonics take on equal values at antipodal points. Since $\Psi_{(\lambda,\lambda+1]}$ are embeddings for λ large, it is natural to study $\Psi_{(\lambda,\lambda+1]}(M)$ as a metric space equipped with the distance, dist_λ, induced by the embedding:

$$\begin{aligned}\text{dist}_\lambda^2(x, y) &:= \|\Psi_{(\lambda,\lambda+1]}(x) - \Psi_{(\lambda,\lambda+1]}(y)\|_{l^2(\mathbb{R}^{m_\lambda})}^2 \\ &= \frac{(2\pi)^n}{2\lambda^{n-1}}(E_{(\lambda,\lambda+1]}(x, x) + E_{(\lambda,\lambda+1]}(y, y) - 2E_{(\lambda,\lambda+1]}(x, y)).\end{aligned} \tag{13}$$

Theorem 9, also proved in Section 7, gives precise asymptotics for $\text{dist}_\lambda(x, y)$ in terms of $\text{dist}_g(x, y)$:

Theorem 9. *Let (M, g) be a compact, smooth, Riemannian manifold of dimension $n \ge 2$ with no boundary. Suppose further that every $x \in M$ is non-self-focal and all pairs $x, y \in M$ are mutually nonfocal. As $\lambda \to \infty$, we have*

$$\sup_{x,y\in M} \left| \frac{1}{\lambda^2 \,\text{dist}_g^2(x, y)} \left[\text{dist}_\lambda^2(x, y) - \left(\text{vol}(S^{n-1}) - (2\pi)^{n/2} \frac{J_{(n-2)/2}(\lambda \,\text{dist}_g(x, y))}{(\lambda \,\text{dist}_g(x, y))^{(n-2)/2}} \right) \right] \right| = o(1). \tag{14}$$

1.2. ***Discussion of Theorem 2.*** Theorem 2 is an extension of Hörmander's pointwise Weyl law [1968, Theorem 4.4]. Hörmander proved that there exists $\varepsilon > 0$ such that, if the Riemannian distance $\text{dist}_g(x, y)$ between x and y is less than ε, then

$$E_\lambda(x, y) = \frac{\lambda^n}{(2\pi)^n} \int_{|\xi|_{g_y}<1} e^{i\lambda\psi(x,y,\xi)} \frac{d\xi}{\sqrt{|g_y|}} + O(\lambda^{n-1}), \tag{15}$$

where, in Hörmander's terminology, the phase function ψ is adapted to the principal symbol $|\xi|_{g_y}$ of $\sqrt{\Delta_g}$. After his Theorem 4.4, Hörmander [1968] remarks that the choice of ψ is not unique. However, every adapted phase function satisfies

$$\psi(x, y, \xi) = \langle x - y, \xi \rangle + O(|x - y|^2 |\xi|).$$

In particular, since $\langle \exp_y^{-1}(x), \xi \rangle_{g_y} = \langle x - y, \xi \rangle + O(|x - y|^2 |\xi|)$, Taylor-expanding (15) yields, for any $r_0 > 0$,

$$\sup_{\mathrm{dist}_g(x,y) < r_0/\lambda} \left| E_\lambda(x, y) - \frac{\lambda^n}{(2\pi)^n} \int_{|\xi|_{g_y} < 1} e^{i\lambda \langle \exp_y^{-1}(x), \xi \rangle_{g_y}} \frac{d\xi}{\sqrt{|g_y|}} \right| = O(\lambda^{n-1}).$$

Changing from one adapted phase to another produces, a priori, an error of $O(\lambda^{n-1})$ in (15). With the additional assumption that x and y are near a non-self-focal point, Theorem 2 therefore extends Hörmander's result in two ways. First, our careful choice of phase function $\langle \exp_y^{-1}(x), \xi \rangle_{g_y}$ allows us to obtain a $o(\lambda^{n-1})$ estimate on R while keeping the amplitude equal to 1. Second, we allow $\mathrm{dist}_g(x, y)$ to shrink arbitrarily slowly with λ.

Hörmander's phase functions $\psi(x, y, \xi)$ are difficult to analyze directly when $x \neq y$, since they are the solutions to certain Hamilton–Jacobi equations (see [Hörmander 1968, Definition 3.1; 1985b, (29.1.7)]) which we cannot describe explicitly. Instead, in proving Theorem 2, we use a parametrix for the half-wave operator $U(t) = e^{-it\sqrt{\Delta_g}}$ with the geometric phase function $\phi : \mathbb{R} \times M \times T^*M :\to \mathbb{R}$ given by $\phi(t, x, y, \xi) = \langle \exp_y^{-1}(x), \xi \rangle - t|\xi|_{g_y}$. Such a parametrix was previously used by Zelditch [2009], where a construction for the amplitude was omitted. Our construction, given in Section 3, makes clear the off-diagonal behavior of $E_\lambda(x, y)$ and uses the results of Laptev, Safarov and Vassiliev [Laptev et al. 1994], who treat Fourier integral operators (FIOs) with global phase functions.

Using the phase function ϕ simplifies our computations considerably, since the half-density factor $\sqrt{\det \phi_{x,\xi}(t, x, y, \xi)}$, which comes up in the usual parametrix construction for $U(t)$ acting on half-densities, is independent of t and ξ. This makes it easy to obtain the amplitude in a parametrix for $U(t)$ acting on functions from that of $U(t)$ acting on half-densities. For more details, see the outline of the proof of Theorem 2 given in Section 2, as well as Section 3, especially (37).

The error estimate in (15) is sharp on Zoll manifolds (see [Zelditch 1997]), such as the round sphere. The majority of the prior estimates on $R(x, y, \lambda)$ actually treat the case $x = y$. Notably, Bérard [1977] showed that on all compact manifolds of dimension $n \geq 3$ with nonpositive sectional curvatures and on all Riemannian surfaces without conjugate points we have $R(x, x, \lambda) = O(\lambda^n / \log \lambda)$. The $O(\lambda^{n-1})$ error in the Weyl asymptotics for the spectral counting function

$$\#\{j : \lambda_j \in [0, \lambda]\} = \int_M E_\lambda(x, x)\, dv_g(x) = \left(\frac{\lambda}{2\pi}\right)^n \mathrm{vol}_g(M) \cdot \mathrm{vol}_{\mathbb{R}^n}(B_1) + \int_M R(x, x, \lambda)\, dv_g(x)$$

has also been improved under various assumptions on the structure of closed geodesics on (M, g) (see [Bérard 1977; Colin de Verdière 1980; Duistermaat and Guillemin 1975; Ivriĭ 1984; Nicolaescu 2012; Petridis and Toth 2002; Randol 1981; Safarov and Vassiliev 1997]). For instance, [Duistermaat and Guillemin 1975; Ivriĭ 1984] prove that $\int_M R(x, x, \lambda)\, dv_g(x) = o(\lambda^{n-1})$ if (M, g) is aperiodic (i.e., the set of all closed geodesics has measure zero in S^*M).

Also related to this article are lower bounds for $R(x, y, \lambda)$ obtained by Jakobson and Polterovich [2007] as well as estimates on averages of $R(x, y, \lambda)$ with respect to either $y \in M$ or $\lambda \in \mathbb{R}_{>0}$ studied by Lapointe, Polterovich and Safarov [Lapointe et al. 2009].

1.3. ***Discussion of Theorem 3.*** The scaling asymptotics (9) were first stated — without proof and without any assumptions on $\mathcal{L}_{x_0}$ — by Zelditch [2001, Theorem 2.1]. When $(M, g) = (S^2, g_{\text{round}})$ is the standard 2-sphere, the square roots of the Laplace eigenvalues are $\lambda_k = k \cdot \sqrt{1 + 1/k}$ for $k \in \mathbb{Z}_+$, and $\mathcal{L}_{x_0} = S^*_{x_0} M$, since the geodesic flow is 2π-periodic. There is therefore no $x_0 \in S^2$ satisfying the assumptions of Theorem 3. Nonetheless, (8) holds with E_λ replaced by the kernel of the spectral projection onto the λ_k^2 eigenspace and is known as Mehler–Heine asymptotics (see §8.1 in [Szegő 1975]). More generally, on any Zoll manifold, the square roots of Laplace eigenvalues come in clusters that concentrate along an arithmetic progression. The width of the k-th cluster is on the order of k^{-1} and we conjecture that the scaling asymptotics (8) hold for the spectral projectors onto these clusters (see [Zelditch 1997] for background on the spectrum of Zoll manifolds).

If one perturbs the standard metric on S^2 or on a Zoll surface, one can create smooth metrics possessing self-focal points x_0 where only a fraction of the measure of initial directions at x_0 give geodesics that return to x_0. These points complicate the remainder estimate for the general case. Indeed, it was pointed out to the authors by Safarov that even on the diagonal there is a two-term asymptotic formula with the second term of the form $Q(x, \lambda)\lambda^{n-1}$, where Q is a bounded function. The function Q is identically zero if x_0 is non-self-focal or if a full measure of geodesics emanating from x_0 return to x_0 at the same time. In general, however, Q will contribute an extra term on the order of λ^{n-1} to the asymptotics in (8). We refer the interested reader to §1.8 in [Safarov and Vassiliev 1997].

1.4. ***Notation.*** Given a Riemannian manifold (M, g), let $\text{vol}_g(M)$ be its volume, $\text{dist}_g : M \times M \to \mathbb{R}$ be the induced distance function, and $\text{inj}(M, g)$ be its injectivity radius. For $x \in M$ we write $S^*_x M$ for the unit sphere in the cotangent fiber $T^*_x M$. We denote by $\langle \cdot, \cdot \rangle_{g_x} : T^*_x M \times T^*_x M \to \mathbb{R}$ the Riemannian inner product on $T^*_x M$ and by $|\cdot|_{g_x}$ the corresponding norm. When $M = \mathbb{R}^n$ we simply write $\langle \cdot, \cdot \rangle$ and $|\cdot|$. In addition, for $(x, \xi) \in T^*M$, we will sometimes write $g_x^{1/2}(\xi)$ for the square root of the matrix g_x applied to the covector ξ and we write $|g_x|$ for the determinant of g_x.

We denote by S^k the space of classical symbols of degree k, and we will write $S^k_{\text{hom}} \subseteq S^k$ for those symbols that are homogeneous of degree k. We also denote by $\Psi^k(M)$ the class of pseudodifferential operators of order k on M.

2. Outline for the proof of Theorem 2

Fix (M, g) and a non-self-focal point $x_0 \in M$. Theorem 2 follows from the existence of a constant $c > 0$ such that, for all $\varepsilon > 0$, there exist $\tilde{\lambda}_\varepsilon > 0$, an open neighborhood $\mathcal{U}_\varepsilon$ of x_0 and a positive constant c_ε such that

$$\sup_{x, y \in \mathcal{U}_\varepsilon} |R(x, y, \lambda)| \leq c\varepsilon\lambda^{n-1} + c_\varepsilon \lambda^{n-2} \tag{16}$$

for all $\lambda \geq \tilde{\lambda}_\varepsilon$. Indeed, if r_λ is a positive function with $\lim_{\lambda\to\infty} r_\lambda = 0$, then it suffices to choose $\lambda_\varepsilon := \max\{\tilde{\lambda}_\varepsilon, \inf\{\lambda : B(x_0, r_\lambda) \subset \mathcal{U}_\varepsilon\}\}$ to get

$$\sup_{x,y\in B(x_0,r_\lambda)} |R(x,y,\lambda)| \leq c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2} \quad \text{for all } \lambda \geq \lambda_\varepsilon.$$

By the definition of R in (3) and the definition of E_λ, (2), we seek to find a constant $c > 0$ such that, for all $\varepsilon > 0$, there exist $\tilde{\lambda}_\varepsilon > 0$, an open neighborhood $\mathcal{U}_\varepsilon$ of x_0 and a positive constant c_ε satisfying

$$\sup_{x,y\in\mathcal{U}_\varepsilon} \left| E_\lambda(x,y) - \frac{\lambda^n}{(2\pi)^n} \int_{|\xi|_{g_y}<1} e^{i\lambda\langle \exp_y^{-1}(x),\xi\rangle_{g_y}} \frac{d\xi}{\sqrt{|g_y|}} \right| \leq c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2} \tag{17}$$

for all $\lambda \geq \tilde{\lambda}_\varepsilon$. We prove (17) using the so-called wave kernel method. That is, we use that the derivative of the spectral function is the inverse Fourier transform of the fundamental solution of the half-wave equation on (M, g):

$$E_\lambda(x,y) = \int_0^\lambda \sum_j \delta(\mu-\lambda_j)\varphi_j(x)\varphi_j(y)\,d\mu = \int_0^\lambda \mathcal{F}^{-1}_{t\to\mu}(U(t,x,y))(\mu)\,d\mu, \tag{18}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and $U(t,x,y)$ is the Schwartz kernel of $e^{-it\sqrt{\Delta_g}}$. The singularities of $U(t,x,y)$ control the $\lambda \to \infty$ behavior of E_λ. We first study the contribution of the singularity of $U(t,x,y)$ coming at $t = \operatorname{dist}_g(x,y)$ by taking a Schwartz function $\rho \in \mathcal{S}(\mathbb{R})$ that satisfies $\operatorname{supp}(\hat{\rho}) \subseteq (-\operatorname{inj}(M,g), \operatorname{inj}(M,g))$ and

$$\hat{\rho}(t) = 1 \quad \text{for all } |t| < \tfrac{1}{2}\operatorname{inj}(M,g). \tag{19}$$

We prove in Section 5 the following proposition, which shows that (17) holds with E_λ replaced by $\rho * E_\lambda$.

Proposition 10 (smoothed projector). *Let (M, g) be a compact, smooth, Riemannian manifold of dimension $n \geq 2$ with no boundary. Then there exist constants $c, C > 0$ such that*

$$\left| \rho * E_\lambda(x,y) - \frac{1}{(2\pi)^n} \int_{|\xi|_{g_y}<\lambda} e^{i\langle \exp_y^{-1}(x),\xi\rangle_{g_y}} \frac{d\xi}{\sqrt{|g_y|}} \right| \leq c\operatorname{dist}_g(x,y)\lambda^{n-1} + C\lambda^{n-2} \tag{20}$$

for all $x, y \in M$ with $\operatorname{dist}_g(x,y) \leq \frac{1}{2}\operatorname{inj}(M,g)$ and all $\lambda > 0$.

Note that Proposition 10 does not assume that x and y are near a non-self-focal point. The reason is that convolving E_λ with ρ multiplies the half-wave kernel $U(t,x,y)$ in (18) by the Fourier transform $\hat{\rho}(t)$, which cuts out all but the singularity at $t = \operatorname{dist}_g(x,y)$. The proof of (20) relies on the construction in Section 3 of a short-time parametrix for $U(t)$, which differs from the celebrated Hörmander parametrix because it uses the coordinate-independent phase function

$$\phi(t,x,y,\xi) := \langle \exp_y^{-1}(x), \xi\rangle_{g_y} - t|\xi|_{g_y}, \quad (t,x,y,\xi) \in \mathbb{R}\times M\times T^*M. \tag{21}$$

It remains to estimate the difference $|E_\lambda(x,y) - \rho * E_\lambda(x,y)|$, which is the content of the following result:

Proposition 11 (smooth vs. rough projector). *Let (M, g) be a compact, smooth, Riemannian manifold of dimension $n \geq 2$ with no boundary. Let $x_0 \in M$ be a non-self-focal point. Then there exists $c > 0$ such that, for all $\varepsilon > 0$, there exist an open neighborhood $\mathcal{U}_\varepsilon$ of x_0 and a positive constant c_ε with*

$$\sup_{x,y\in\mathcal{U}_\varepsilon} |E_\lambda(x, y) - \rho * E_\lambda(x, y)| \leq c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2} \tag{22}$$

for all $\lambda \geq 1$.

The assumption that x and y are near a non-self-focal point x_0 guarantees that the dominant contribution to $E_\lambda(x, y)$ comes from the singularity of $U(t, x, y)$ at $t = \operatorname{dist}_g(x, y)$. Following the technique in [Sogge and Zelditch 2002], we prove Proposition 11 in Section 6 by microlocalizing $U(t)$ near x_0 (see Section 4) and applying two Tauberian-type theorems (presented in Section 6.1). Relation (17), and consequently Theorem 2, are a direct consequence of combining Proposition 10 with Proposition 11.

3. Parametrix for the half-wave group

The half-wave group is the one-parameter family of unitary operators $U(t) = e^{-it\sqrt{\Delta_g}}$ acting on $L^2(M, g)$. It solves the initial value problem

$$\Big(\frac{1}{i}\partial_t + \sqrt{\Delta_g}\Big)U(t) = 0, \quad U(0) = \mathrm{Id},$$

and its Schwartz kernel $U(t, x, y)$ is related to the kernel of the spectral projector $E_\lambda(x, y)$ via (18). It is well known (see [Duistermaat and Guillemin 1975; Hörmander 1985b]) that U is a FIO in $I^{-1/4}(\mathbb{R} \times M, M; \Gamma)$ associated to the canonical relation

$$\Gamma = \big\{(t, \tau, x, \eta, y, \xi) \in T^*(\mathbb{R} \times M \times M) \mid \tau = -|\xi|_{g_y}, G^t(y, \xi) = (x, \eta)\big\}, \tag{23}$$

where G^t denotes geodesic flow.

Our goal in this section is to construct a short-time parametrix for $U(t)$ that is similar to Hörmander's parametrix [1968; 1985b, §29] but uses the coordinate-independent phase function $\phi : \mathbb{R} \times M \times T^*M \to \mathbb{R}$ defined in (21). Such a parametrix was used by Zelditch [2009], where a detailed construction was omitted. To construct the amplitude we follow [Laptev et al. 1994], who give a detailed treatment of FIOs that are built using global phase functions such as ϕ. Denote by $\chi \in C^\infty([0, +\infty), [0, 1])$ a compactly supported, smooth cut-off function with

$$\operatorname{supp} \chi \subset [0, \operatorname{inj}(M, g)) \qquad \text{and} \qquad \chi(s) = 1 \quad \text{for } s \in \big[0, \tfrac{1}{2} \operatorname{inj}(M, g)\big).$$

Further, following [Bérard et al. 1994; Berger et al. 1971, Proposition C.III.2], define

$$\Theta(x, y) := |\det_g D_{\exp_x^{-1}(y)} \exp_x|. \tag{24}$$

The subscript g means that we use the inner products on $T_{\exp_x^{-1}(y)}(T_x M)$ and $T_y^* M$ induced from g and, as explained in [Berger et al. 1971], $\Theta(x, y) = \sqrt{|g_x|}$ in normal coordinates at y. The main result of this section is the following:

Proposition 12. *For $|t| < \operatorname{inj}(M, g)$ we have*

$$U(t, x, y) = \frac{\chi(\operatorname{dist}_g(x, y))}{(2\pi)^n \Theta(x, y)^{1/2}} \int_{T_y^* M} e^{i\phi(t,x,y,\xi)} A(t, y, \xi) \frac{d\xi}{\sqrt{|g_y|}}, \tag{25}$$

where the equality is modulo smoothing kernels. The amplitude A, which is an order-0 polyhomogeneous symbol, is uniquely determined by ϕ modulo $S^{-\infty}$ and satisfies:

- *For all $y \in M$ and $\xi \in T_y^* M$,*
$$A(0, y, \xi) = 1. \tag{26}$$
- *For $|t| < \operatorname{inj}(M, g)$ and all $(y, \xi) \in T_y^* M$, we have*
$$A(t, y, \xi) - 1 \in S^{-1}. \tag{27}$$

There are many choices of amplitude functions in (25) that depend on t, x, y and ξ. When we write that A is uniquely determined modulo $S^{-\infty}$, we mean that it is unique among amplitudes that are independent of x. The proof of Proposition 12 is divided into two steps. First, we prove in Section 3.1 that ϕ parametrizes Γ. Then, in Section 3.2, we construct the amplitude A.

3.1. *Properties of the phase function.* Throughout this section, we will denote by $\mathcal{T}_{y \to x} : T_y^* M \to T_x^* M$ the parallel transport operator (along the unique shortest geodesic from x to y) for all x and y sufficiently close. We will use that

$$\mathcal{T}_{y \to x} \exp_y^{-1}(x) = -\exp_x^{-1}(y) \quad \text{and} \quad \mathcal{T}_{y \to x} = \mathcal{T}_{x \to y}^*. \tag{28}$$

Lemma 13. *The phase function $\phi(t, x, y, \xi)$ parametrizes the canonical relation Γ for $|t| < \operatorname{inj}(M, g)$ and $\operatorname{dist}_g(x, y) < \frac{1}{2} \operatorname{inj}(M, g)$, in the sense that*

$$\Gamma = i_\phi(C_\phi) \tag{29}$$

is the image of the critical set

$$C_\phi = \left\{ (t, x, y, \xi) \in \mathbb{R} \times M \times T^* M \;\middle|\; x = \exp_y\left(\frac{t\xi}{|\xi|_{g_y}} \right) \right\}$$

under the immersion $i_\phi(t, x, y, \xi) = (t, d_t \phi, x, d_x \phi, y, -d_y \phi)$.

Proof. When $|t| < \operatorname{inj}(M, g)$, we have that $(t, x, y, \xi) \in C_\phi$ if and only if $t = 0$ and $x = y$, or

$$t = \operatorname{dist}_g(x, y) \neq 0 \quad \text{and} \quad \frac{\xi}{|\xi|_{g_y}} = \frac{\exp_y^{-1}(x)}{\operatorname{dist}_g(x, y)}.$$

To prove (29) when $t = 0$, we must show that

$$i_\phi(0, x, x, \xi) = \{(0, -|\xi|_{g_x}, x, \xi, x, \xi) \mid \xi \in T_x^* M\} = \Gamma|_{t=0}. \tag{30}$$

Since $d_x|_{x=y} \exp_y^{-1}(x)$ is the identity on $T_y^* M$,

$$d_x|_{x=y} \phi(0, x, y, \xi) = \xi.$$

Next, using (28), we have

$$\phi(0,x,y,\xi) = \langle -\exp_x^{-1}(y), \mathcal{T}_{y\to x}\xi\rangle_{g_x}.$$

Therefore,

$$d_y|_{y=x}\phi(0,x,y,\xi) = -\xi,$$

which proves (30). To establish (29) when $t \neq 0$, we write

$$\partial_{x_k}\phi(t,x,y,\xi) = \sum_{i,j} g^{ij}(y)\partial_{x_k}[\exp_y^{-1}(x)]_i\xi_j, \quad k = 1,\dots,n. \tag{31}$$

Since $d_x \operatorname{dist}_g(x,y) = -\exp_x^{-1}(y)/\operatorname{dist}_g(x,y)$, evaluating (31) at

$$\xi = |\xi|_{g_y}\frac{\exp_y^{-1}(x)}{\operatorname{dist}_g(x,y)},$$

we obtain

$$d_x\phi(t,x,y,\xi) = \frac{|\xi|_{g_y}}{2\operatorname{dist}_g(x,y)}d_x[\operatorname{dist}_g(x,y)^2] = |\xi|_{g_y} d_x \operatorname{dist}_g(x,y) = -|\xi|_{g_y}\frac{\exp_x^{-1}(y)}{\operatorname{dist}_g(x,y)}. \tag{32}$$

Since $G^t(y,\exp_y^{-1}(x)) = (x, -\exp_x^{-1}(y))$, it remains to check that

$$-d_y\phi(t,x,y,\xi) = |\xi|_{g_y}\frac{\exp_y^{-1}(x)}{\operatorname{dist}_g(x,y)},$$

which we verify in normal coordinates at y. We have that

$$d_z|_{z=y}|\xi|_z = 0 \quad \text{and} \quad \partial_{z_k}|_{z=y}(\exp_z^{-1}(x))_j = -\delta_{kj}.$$

Thus,

$$\partial_{z_k}|_{z=y}\phi(t,x,z,\xi) = -\xi_k.$$

Evaluating at $\xi = |\xi|\cdot x/|x|$, we find that

$$-d_y\phi(t,x,y,\xi) = |\xi|\cdot\frac{x}{|x|} = |\xi|_{g_y}\frac{\exp_y^{-1}(x)}{\operatorname{dist}_g(x,y)},$$

as desired. □

We need one more lemma before constructing the amplitude A in Proposition 12.

Lemma 14. *Let $\beta : M \times M \to \mathbb{R}$ be any smooth function such that $\beta(x,x) = 1$. The kernel of the identity operator acting on functions relative to the Riemannian volume form $\sqrt{|g_y|}\,dy$ admits the following representation as an oscillatory integral*:

$$\begin{aligned}\delta(x,y) &= \frac{\chi(\operatorname{dist}_g(x,y))}{(2\pi)^n}\beta(x,y)\int_{T_x^*M} e^{-i\langle \exp_x^{-1}(y),\eta\rangle_{g_x}}\frac{d\eta}{\sqrt{|g_x|}}\\ &= \frac{\chi(\operatorname{dist}_g(x,y))}{(2\pi)^n}\beta(x,y)\int_{T_y^*M} e^{i\langle \exp_y^{-1}(x),\xi\rangle_{g_y}}\frac{d\xi}{\sqrt{|g_y|}}.\end{aligned} \tag{33}$$

Proof. Fix $x \in M$ and let $f \in C^\infty(M)$. Without loss of generality, assume that f is supported in an open set $U \subset B(x, \mathrm{inj}(M, g))$ that contains the point x. Set $V = \exp_x^{-1}(U) \subset \mathbb{R}^n$ and consider normal coordinates at x:

$$h : V \to U, \quad h(z) = \exp_x(z). \tag{34}$$

The pairing of the right-hand side of (33) with f is then

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\langle z,\eta\rangle} \chi(|z|) f(h(z)) \beta(0, z) \sqrt{|g_{h(z)}|}\, dz\, d\eta = \chi(|0|) f(h(0)) \sqrt{|g_{h(0)}|}\beta(0, 0) = f(x).$$

This proves (33). To explain why the two oscillatory integrals in the statement of the present lemma define the same distribution, we will use the parallel transport operator (see (28)). We write (33) as

$$\frac{\chi(\mathrm{dist}_g(x, y))}{(2\pi)^n} \beta(x, y) \int_{T_x^*M} e^{i\langle \exp_y^{-1}(x), \mathcal{T}_{y\to x}\eta\rangle_{g_y}} \frac{d\eta}{\sqrt{|g_x|}}. \tag{35}$$

Let $(y^1, \dots, y^n)$ be any local coordinates near x. We note that, for every y, the collection of covectors $\{g_y^{1/2}\, dy^j|_y\}_{j=1}^n$ is an orthonormal basis for T_y^*M. Hence, the Lebesgue measure on T_y^*M in our coordinates is $|g_y|^{1/2}\, dy^1|_y \wedge \dots \wedge dy^n|_y$ and, since $\mathcal{T}_{y\to x}$ is an isometry,

$$\xi = \mathcal{T}_{y\to x}\eta \quad \Longrightarrow \quad d\xi = \frac{|g_y|^{1/2}}{|g_x|^{1/2}} d\eta.$$

This allows us to change variables in (35) to obtain the integral over T_y^*M in the statement of the lemma. □

3.2. ***Construction of the amplitude.*** To construct the amplitude A in Proposition 12, let us write $\widetilde{U}(t)$ for the wave operator acting on sections of the half-density bundle $\Omega^{1/2}(M)$. Lemma 13 combined with Theorem 3.4 in [Laptev et al. 1994] (or Proposition 25.1.5 in [Hörmander 1985b]) shows that there exists a polyhomogeneous symbol A of order 0 that is supported in a neighborhood of C_ϕ for which

$$\widetilde{U}(t, x, y) = \frac{\chi(\mathrm{dist}_g(x, y))}{(2\pi)^n} \int_{T_y^*M} e^{i\phi(t,x,y,\xi)} A(t, y, \xi)\, d_\phi(t, x, y, \xi)\, d\xi \quad (\mathrm{mod}\ C^\infty), \tag{36}$$

where

$$d_\phi = \sqrt{|\det d_{x,\xi}\phi|} \in \Omega_x^{1/2}(M) \otimes \Omega_y^{-1/2}(M)$$

is a $\frac{1}{2}$-density in x and a $\left(-\frac{1}{2}\right)$-density in y. Since $d\xi$ behaves like a 1-density in y, $\widetilde{U}(t, x, y)$ is in $\Omega_x^{1/2}(M) \otimes \Omega_y^{1/2}(M)$. The square root of the Riemannian volume form,

$$g_y^{1/4} = |g_y|^{1/4}\, |dy|^{1/2} \in \Omega_y^{1/2}(M),$$

identifies L^2 global sections $\Gamma(\Omega^{1/2}(M))$ with $L^2(M)$ via

$$L^2(M) \to \Gamma(\Omega^{1/2}(M)), \quad f(y) \mapsto f(y) \cdot g_y^{1/4}.$$

Then, computing in normal coordinates at y, we have

$$d_\phi(t,x,y,\xi)g_y^{1/4}g_x^{-1/4} = \frac{1}{|g_x|^{1/4}} = \frac{1}{\Theta(x,y)^{1/2}}. \tag{37}$$

In addition, since $U(t,x,y) = \widetilde{U}(t,x,y)g_x^{-1/4}g_y^{-1/4}$, relation (37) gives

$$U(t,x,y) = \frac{\chi(\mathrm{dist}_g(x,y))}{(2\pi)^n\Theta(x,y)^{1/2}} \int_{T_y^*M} e^{i\phi(t,x,y,\xi)} A(t,y,\xi)\, \frac{d\xi}{\sqrt{|g_y|}} \qquad (\mathrm{mod}\ C^\infty). \tag{38}$$

Write $A \sim \sum_{j\geq 0} A_{-j}$ for the polyhomogeneous expansion of A. Note that

$$A_0(t,y,\xi) = 1 \quad \text{for all } t,$$

because the principal symbol $\widetilde{U}(t)$ is independent of t and equals 1 at $t = 0$ [Laptev et al. 1994, Theorem 4.1]. Next, since

$$\widetilde{U}(0,x,y) = \frac{\chi(\mathrm{dist}_g(x,y))}{(2\pi)^n} \int_{T_y^*M} e^{i\phi(t,x,y,\xi)} A(0,y,\xi)\, d_\phi(t,x,y,\xi)\, \frac{d\xi}{\sqrt{|g_y|}}$$

is a kernel for the identity modulo C^∞ and $A(0,y,\xi)$ is uniquely determined by ϕ mod $S^{-\infty}$ (Theorem 3.4 in [Laptev et al. 1994]), it follows from Lemma 14 and (37), with $\beta(x,y) = \Theta(x,y)^{-1/2}$, that

$$A_{-j}(0,y,\xi) = 0 \quad \text{for all } j \geq 1,$$

as desired.

4. Microlocalizing the identity operator at non-self-focal points

In this section we microlocalize the identity operator near a non-self-focal point x_0. For every $\varepsilon > 0$ we make a microlocal decomposition of the identity, $\mathrm{Id} = B_\varepsilon + C_\varepsilon$ near x_0, where the operator B_ε is supported on the set of "bad" loopset directions and is built so that its support has measure smaller than ε. This construction follows closely that of Sogge and Zelditch [2002].

Lemma 15. *There exists a constant $\gamma > 0$ such that, for every $\varepsilon > 0$, there is a neighborhood $\mathcal{O}_\varepsilon$ of x_0, a function $\psi_\varepsilon \in C_c^\infty(M)$ and real-valued operators B_ε, $C_\varepsilon \in \Psi^0(M)$ supported in $\mathcal{O}_\varepsilon$ satisfying the following properties*:

(1) *For every ε, $\mathrm{supp}(\psi_\varepsilon) \subset \mathcal{O}_\varepsilon$ and $\psi_\varepsilon = 1$ on a neighborhood of x_0.*

(2) *For every ε,*

$$B_\varepsilon + C_\varepsilon = \psi_\varepsilon^2. \tag{39}$$

(3) *$U(t)C_\varepsilon^*$ is a smoothing operator for $\frac{1}{2}\mathrm{inj}(M,g) < |t| < \frac{1}{\varepsilon}$.*

(4) *Denote by b_0 and c_0 the principal symbols of B_ε and C_ε respectively. Then, for all $x \in M$, we have*

$$\frac{1}{\varepsilon}\int_{|\xi|_{g_x}\leq 1} |b_0(x,\xi)|^2\, d\xi + \int_{|\xi|_{g_x}\leq 1} |c_0(x,\xi)|^2\, d\xi \leq \gamma \tag{40}$$

and both b_0 and c_0 are constant in an open neighborhood of x_0.

Proof. For every $x, y \in M$ and $\xi \in S^*_x M$, define the loopset function

$$\mathcal{L}^*(x, y, \xi) = \inf\{t > 0 \mid \exp_x(t\xi) = y\}$$

with $\mathcal{L}^*(x, y, \xi) = +\infty$ if the infimum is taken over the empty set. Unlike the loopset function studied in [Sogge and Zelditch 2002], we are interested in $x \neq y$ (but with $\mathrm{dist}_g(x, y) < \frac{1}{2}\mathrm{inj}(M, g)$).

Fix a coordinate chart $(\kappa_{x_0}, \mathcal{V}_{x_0})$ containing x_0 with $\kappa_{x_0} : \mathcal{V}_{x_0} \subset \mathbb{R}^n \to M$. We first note that the function $f : \mathcal{V}_{x_0} \times \mathcal{V}_{x_0} \times S^{n-1} \to \mathbb{R}$ defined as $f(x, y, \xi) = 1/\mathcal{L}^*(x, y, \xi)$ is upper semicontinuous and so, by the proof of [Sogge and Zelditch 2002, Lemma 3.1], there exist a neighborhood $\mathcal{N}_\varepsilon \subset \mathcal{V}_{x_0}$ of x_0 and an open set $\Omega_\varepsilon \subset S^{n-1}$ for which

$$\mathcal{L}^*(x, y, \xi) > \tfrac{1}{\varepsilon} \quad \text{in } \mathcal{N}_\varepsilon \times \mathcal{N}_\varepsilon \times \Omega_\varepsilon^c, \tag{41}$$

$$|\Omega_\varepsilon| \leq \varepsilon. \tag{42}$$

In addition, there exists a function $\varrho_\varepsilon \in C^\infty(S^{n-1}, [0, 1])$ satisfying that $\varrho_\varepsilon \equiv 1$ on Ω_ε, $\varrho_\varepsilon(\xi) = \varrho(-\xi)$ for all $\xi \in S^{n-1}$ and $|\operatorname{supp}(\varrho_\varepsilon)| < 2\varepsilon$. In particular,

$$\mathcal{L}^*(x, y, \xi) > \frac{1}{\varepsilon} \quad \text{on } \mathcal{N}_\varepsilon \times \mathcal{N}_\varepsilon \times \operatorname{supp}(1 - \varrho_\varepsilon).$$

As in [Sogge and Zelditch 2002], we choose a real-valued function $\tilde{\psi}_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\tilde{\psi}_\varepsilon) \subset \mathcal{N}_\varepsilon$ that is equal to 1 in a neighborhood of $\kappa_{x_0}^{-1}(x_0)$. Define symbols on $\mathbb{R}^{3n}$ by

$$\tilde{b}_\varepsilon(x, y, \xi) = \tilde{\psi}_\varepsilon(x)\tilde{\psi}_\varepsilon(y)\varrho_\varepsilon\Big(\frac{\xi}{|\xi|}\Big) \quad \text{and} \quad \tilde{c}_\varepsilon(x, y, \xi) = \tilde{\psi}_\varepsilon(x)\tilde{\psi}_\varepsilon(y)\Big(1 - \varrho_\varepsilon\Big(\frac{\xi}{|\xi|}\Big)\Big),$$

and consider their respective quantizations $\mathrm{Op}(\tilde{b}_\varepsilon), \mathrm{Op}(\tilde{c}_\varepsilon) \in \Psi^0(\mathbb{R}^n)$. Properties (1) and (2) follow from setting

$$B_\varepsilon := (\kappa_{x_0}^{-1})^* \,\mathrm{Op}(\tilde{b}_\varepsilon), \qquad C_\varepsilon := (\kappa_{x_0}^{-1})^* \,\mathrm{Op}(\tilde{c}_\varepsilon)$$

and

$$\mathcal{O}_\varepsilon = \kappa_{x_0}(\mathcal{N}_\varepsilon), \qquad \psi_\varepsilon := (\kappa_{x_0}^{-1})^* \tilde{\psi}_\varepsilon.$$

Note that if, for some time, $\frac{1}{2}\,\mathrm{inj}(M, g) < t < \frac{1}{\varepsilon}$, we have $\exp_x(t\xi/|\xi|) = y$ for some $x, y \in M$ and $\xi \in T^*_x M$, then $\mathcal{L}^*(x, y, \xi/|\xi|) \leq \frac{1}{\varepsilon}$, and the latter implies $\tilde{c}_\varepsilon(x, y, \xi) = 0$. Therefore, we see that, if we write c_ε for the symbol of C_ε, then

$$c_\varepsilon(x, y, \xi) = 0 \qquad \text{if} \qquad (t, x, y; \tau, \xi, \eta) \in \Gamma \quad \text{with} \quad \tfrac{1}{2}\,\mathrm{inj}(M, g) < t < \tfrac{1}{\varepsilon},$$

where Γ is the canonical relation underlying $U(t)$ (see (23)). Thus, the kernel of $U(t)C_\varepsilon^*$ is a smooth function for $\frac{1}{2}\,\mathrm{inj}(M, g) < t < \frac{1}{\varepsilon}$ and for (x, y) in $\mathcal{O}_\varepsilon \times \mathcal{O}_\varepsilon$, which is precisely statement (3). For all $x \in \mathcal{N}_\varepsilon$, we have that the principal symbols b_0 and c_0 satisfy the inequality (40), since $|\operatorname{supp} \varrho_\varepsilon| < 2\varepsilon$. Also, since b_ε and c_ε are real valued and invariant under $\xi \mapsto -\xi$, we have that B_ε and C_ε are real valued as well. ☐

Remark 16. By construction, the subprincipal symbols of B_ε and C_ε (acting on half-densities) are zero in a neighborhood of x_0. Indeed, the principal symbols are constant as functions of x in a neighborhood of x_0 and, in the coordinates κ_{x_0} used in Lemma 15, the total symbols of B_ε and C_ε are homogeneous functions of order zero. Thus, in any coordinates, the parts of order -1 of the polyhomogeneous expansions of the total symbols of B_ε and C_ε vanish in a neighborhood of x_0.

Remark 17. We record precise asymptotics for the on-diagonal behavior of $QEQ^*(x,x,\mu)$ for all $x \in \mathbb{O}_\varepsilon$ and $Q \in \{\mathrm{Id}, B_\varepsilon, C_\varepsilon\}$. Write q_0 for the principal symbol of Q. Using that the subprincipal symbols of both Q and QQ^* (acting on half-densities) vanish identically in a neighborhood $\widetilde{\mathbb{O}}_\varepsilon$ of x_0, Lemmas 3.2 and 3.3 in [Sogge and Zelditch 2002] show that there exist constants $c, c_\varepsilon > 0$ such that, for all $x \in \widetilde{\mathbb{O}}_\varepsilon$,

$$QEQ^*(x,x,\lambda) = \frac{1}{(2\pi)^n} \int_{|\xi|_{g_x}<\lambda} |q_0(x,\xi)|^2 \, d\xi + R_Q(x,x,\lambda)$$

with

$$|R_Q(x,x,\lambda)| \le c\varepsilon\lambda^{n-1} + c_\varepsilon \lambda^{n-2} \tag{43}$$

for all $\lambda \ge 1$. We note that a similar result is obtained in [Safarov and Vassiliev 1997, Theorem 1.8.7], with the difference that the latter is proved for points x that are nonfocal.

5. Smoothed projector: proof of Proposition 10

Proposition 18 below is our main technical estimate on $E_\lambda(x,y)$. We use it to prove Propositions 10 and 11 in Sections 5 and 6, respectively.

Proposition 18. *Let (M,g) be a compact, smooth, Riemannian manifold of dimension $n \ge 2$ with no boundary. Let $\varepsilon > 0$ and $Q \in \{\mathrm{Id}, B_\varepsilon, C_\varepsilon\}$ for B_ε and C_ε, as introduced in Lemma 15. Let q_0 be the principal symbol of Q. Then, for all $x, y \in \mathbb{O}_\varepsilon$ with $\mathrm{dist}_g(x,y) \le \frac{1}{2}\mathrm{inj}(M,g)$ and all $\mu \ge 1$, we have*

$$\begin{aligned}\partial_\mu(\rho * EQ^*)(x,y,\mu) &= \frac{\mu^{n-1}}{(2\pi)^n \Theta(x,y)^{1/2}} \biggl[\int_{S_y^*M} e^{i\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} q_0(y,\omega) \frac{d\omega}{\sqrt{|g_y|}} \\ &\qquad + \int_{S_y^*M} e^{i\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} D_{-1}^Q(y,\omega) \frac{d\omega}{\sqrt{|g_y|}} \biggr] + W(x,y,\mu).\end{aligned} \tag{44}$$

Here, $d\omega$ is the Euclidean surface measure on S_y^*M and the function Θ is as defined in (24). The function D_{-1}^Q belongs to S^{-1} and there exists $C > 0$ such that, for every $\varepsilon > 0$,

$$D_{-1}^{B_\varepsilon}(y,\xi) + D_{-1}^{C_\varepsilon}(y,\xi) = 0 \quad \textit{for all } \ y \in \mathbb{O}_\varepsilon, \tag{45}$$

$$\sup_{x,y\in\mathbb{O}_\varepsilon} \left| \int_{S_y^*M} e^{i\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} D_{-1}^Q(y,\omega) \frac{d\omega}{\sqrt{|g_y|}} \right| \le C\varepsilon. \tag{46}$$

In addition, W is a smooth function in (x,y) for which there exists $C > 0$ such that, for all $\mu > 0$,

$$\sup_{\mathrm{dist}_g(x,y)\le\frac{1}{2}\mathrm{inj}(M,g)} |W(x,y,\mu)| \le C(\mu^{n-2}\,\mathrm{dist}_g(x,y) + (1+\mu)^{n-3}). \tag{47}$$

Proof. Let $x, y \in M$ with $\operatorname{dist}_g(x, y) \le \frac{1}{2}\operatorname{inj}(M, g)$. Note that

$$\partial_\mu(\rho * EQ^*)(x, y, \mu) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it\lambda}\hat{\rho}(t)U(t)Q^*(x, y)\,dt. \tag{48}$$

We start by rewriting $U(t)Q^*(x, y)$ using the parametrix (25) for $U(t)$. We have

$$U(t)Q^*(x, y) = \frac{\chi(d_g(x, y))}{(2\pi)^n\Theta(x, y)^{1/2}}\int_{T_y^*M} e^{i\langle \exp_y^{-1}(x),\xi\rangle_{g_y} - t|\xi|_{g_y}} D^Q(t, y, \xi)\,\frac{d\xi}{\sqrt{|g_y|}} \tag{49}$$

for some $D^Q \in S^0$ with polyhomogeneous expansion $D^Q \sim \sum_{j\ge 0} D^Q_{-j}$. We claim that

$$D^Q_0(0, y, \xi) = q_0(y, \xi) \tag{50}$$

and that, for all $\varepsilon > 0$,

$$D^{B^\varepsilon}_{-1}(0, y, \xi) + D^{C^\varepsilon}_{-1}(0, y, \xi) = 0, \tag{51}$$

$$\sup_{x,y\in\mathbb{O}_\varepsilon}\left|\int_{S_y^*M} e^{i\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} D^Q_{-1}(0, y, \xi)\,\frac{d\omega}{\sqrt{|g_y|}}\right| \le C\varepsilon, \tag{52}$$

where C is a constant independent of ε. Indeed, let $\widetilde{U}(t)\widetilde{Q}^*$ denote the operator $U(t)Q^*$ when regarded as acting on half-densities and note that, by the same computations that deduce (38) from (36), we have

$$\widetilde{U}(t)\widetilde{Q}^*(x, y) = \frac{\chi(d_g(x, y))}{(2\pi)^n}\int_{T_y^*M} e^{i\langle \exp_y^{-1}(x),\xi\rangle_{g_y} - t|\xi|_{g_y}} D^Q(t, y, \xi)\,d_\phi(t, x, y, \xi)\,d\xi.$$

Since the principal symbols of both $\widetilde{U}$ and $\widetilde{Q}$ are independent of t, and $\widetilde{U}(0) = \mathrm{Id}$, we know

$$D^Q_0(t, y, \xi) = q_0(y, \xi).$$

Moreover, note that $D^{\mathrm{Id}}_{-1}(0, y, \xi) = 0$ by Proposition 12 and that D^{Id} is uniquely determined modulo $S^{-\infty}$ by the phase function ϕ (see [Laptev et al. 1994]). This proves (51), since on $\mathbb{O}_\varepsilon$ we have $\mathrm{Id} = B_\varepsilon + C_\varepsilon$. Finally, by the construction of B_ε, we see that the size of the support of $D^{B_\varepsilon}_{-1}(0, y, \xi)$ is smaller than a constant times ε. This proves (52) for $Q = B_\varepsilon$ and hence for $Q = C_\varepsilon$, since $D^{B_\varepsilon}_{-1} = -D^{C_\varepsilon}_{-1}$.

Combining (48) and (49) and changing coordinates $\xi \mapsto \mu r\omega$, where $(r, \omega) \in [0, +\infty) \times S_y^*M$, we obtain up to an $O(\mu^{-\infty})$ error that

$$\begin{aligned}&\Theta(x, y)^{1/2}\cdot\partial_\mu(\rho * EQ^*)(x, y, \mu)\\ &\quad= \frac{\mu^n}{(2\pi)^{n+1}}\int_{\mathbb{R}}\int_0^\infty \hat{\rho}(t)e^{i\mu t(1-r)}\chi(r)r^{n-1}\left(\int_{S_y^*M} e^{i\mu r\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} D^Q(t, y, r\mu\omega)\,d\omega\right)dr\,dt,\end{aligned} \tag{53}$$

where $\chi \in C_c^\infty(\mathbb{R})$ is a cut-off function that is identically 1 near $r = 1$ and vanishes for $r \notin \left[\frac{1}{2}, \frac{3}{2}\right]$. Indeed, on the support of $1-\chi$, the operator $L = (1/i\mu(1-r))\,\partial_t$ is well defined, preserves $e^{i\mu t(1-r)}$, and its adjoint L^* satisfies that, for all $k \in \mathbb{Z}^+$,

$$\left|(L^*)^k\left(r^{n-1}(1-\chi(r))\hat{\rho}(t)\int_{S_y^*M} e^{i\mu r\langle \exp_y^{-1}(x),\omega\rangle_{g_y}} D^Q(t, y, r\mu\omega)\,d\omega\right)\right| \le (1+\mu)^{-k}\cdot c_k$$

for some $c_k > 0$. Define

$$S^Q(t, y, \xi) := q_0(y, \xi) + D^Q_{-1}(t, y, \xi)$$

to be the two leading terms of D^Q. Since $D^Q - S^Q \in S^{-2}$, up to a $O(\mu^{n-3})$ error we have

$$\Theta(x, y)^{1/2} \cdot \partial_\mu(\rho * EQ^*)(x, y, \mu)$$
$$= \frac{\mu^n}{(2\pi)^{n+1}} \int_{\mathbb{R}} \int_0^\infty \hat{\rho}(t) e^{i\mu t(1-r)} \chi(r) r^{n-1} \left(\int_{S_y^* M} e^{i\mu r \langle \exp_y^{-1}(x), \omega \rangle_{g_y}} S^Q(t, y, r\mu\omega) d\omega \right) dr\, dt. \quad (54)$$

According to [Sogge 1993, Theorem 1.2.1], there exist smooth functions $a_\pm, b_\pm \in C^\infty(M \times \mathbb{R}^n)$ such that, for all $(y, \eta) \in M \times T_y^* M$,

$$\int_{S_y^* M} e^{i\langle \eta, \omega \rangle_{g_y}} S^Q(t, y, \mu r \omega) \frac{d\omega}{\sqrt{|g_y|}} = \sum_\pm e^{\pm i|\eta|_{g_y}} (a_\pm(y, \eta) + r^{-1}\mu^{-1} \cdot b_\pm(t, y, \eta)) \quad (55)$$

and

$$|\partial_\eta^\alpha a_\pm(y, \eta)| \le C_\alpha (1 + |\eta|_{g_y})^{-(n-1)/2 - |\alpha|}, \quad (56)$$
$$|\partial_t^\beta \partial_\eta^\alpha b_\pm(t, y, \eta)| \le C_{\alpha,\beta} (1 + |\eta|_{g_y})^{-(n-1)/2 - |\alpha| - 1}, \quad (57)$$

for all multi-indices $\alpha \ge 0$ and $\beta \ge 0$ and for some $C_\alpha, C_{\alpha,\beta} > 0$ independent of t, y and η. Hence, (54) equals

$$\frac{\mu^n}{(2\pi)^{n+1}} \sum_\pm \int_{\mathbb{R}} \int_0^\infty e^{i\mu\psi_\pm(t,r,x,y)} g_\pm(t, r, x, y, \mu)\, dr\, dt, \quad (58)$$

where $\psi_\pm(t, r, x, y) = t(1-r) \pm r \operatorname{dist}_g(x, y)$ and

$$g_\pm(t, r, x, y, \mu) = \frac{1}{(2\pi)^n} r^{n-1} \chi(r) \hat{\rho}(t) \big(a_\pm(y, r\mu \exp_y^{-1}(x)) + r^{-1}\mu^{-1} b_\pm(t, y, r\mu \exp_y^{-1}(x)) \big). \quad (59)$$

Note that the critical points of $\psi_\pm$ are $(t_c^\pm, r_c^\pm) = (\pm \operatorname{dist}_g(x, y), 1)$ and that

$$\det(\operatorname{Hess} \psi_\pm(t_c^\pm, r_c^\pm, x, y)) = 1.$$

Hence, we apply the method of stationary phase to get that (58) is

$$\mu^{n-1} e^{\pm i\mu \operatorname{dist}_g(x,y)} \sum_\pm (g_\pm(t_c^\pm, r_c^\pm, x, y, \mu) - i\mu^{-1} \partial_r \partial_t g_\pm(t_c^\pm, r_c^\pm, x, y, \mu))$$
$$+ O\big(\mu^{n-3} \sup_{(t,r) \in \operatorname{supp}(g_\pm)} \sup_{\alpha+\beta \le 7} |\partial_t^\alpha \partial_r^\beta g_\pm(t, r, x, y, \mu)|\big). \quad (60)$$

We take 7 derivatives in the last term, since, in stationary phase with a quadratic phase over $\mathbb{R}^k$, the remainder after the first N terms is bounded by $k + 1 + 2N$ derivatives of the amplitude. Note that $\partial_t \hat{\rho}(t) = 0$ for $t = \pm \operatorname{dist}_g(x, y)$. Hence, since $a_\pm$ are independent of t, we have

$$i\mu^{-1} \partial_r \partial_t g_\pm(t_c^\pm, r_c^\pm, x, y, \mu) = O(\mu^{-2}). \quad (61)$$

Moreover, by (56) and (57), the derivatives of g in t and r are uniformly bounded. Hence,

$$\frac{\mu^{n-1}}{(2\pi)^n}\int_{S_y^*M} e^{i\langle \exp_y^{-1}(x),\omega\rangle_{g_y}}(q_0(y,\omega)+\mu^{-1}D_{-1}^{Q}(\operatorname{dist}_g(x,y),y,\omega))\frac{d\omega}{\sqrt{|g_y|}}+O(\mu^{n-3}). \tag{62}$$

Taylor-expanding $D_{-1}^{Q}(\operatorname{dist}_g(x,y),y,\omega)=D_{-1}^{Q}(0,y,\omega)+O(\operatorname{dist}_g(x,y))$ and recalling (51) and (52) completes the proof. □

Proof of Proposition 10. Proposition 10 follows by integrating (44) with respect to μ from 0 to λ applied to $Q=\mathrm{Id}$. We have

$$\rho*E(x,y,\lambda)=\int_0^\lambda \frac{\mu^{n-1}}{(2\pi)^n\Theta(x,y)^{1/2}}\left(\int_{S_y^*M} e^{i\mu\langle \exp_y^{-1}(x),\omega\rangle_{g_y}}\frac{d\omega}{\sqrt{|g_y|}}\right)d\mu+\int_0^\lambda W(x,y,\mu)\,d\mu. \tag{63}$$

Changing coordinates to $\xi=\mu\omega$, we find

$$\rho*E(x,y,\lambda)=\frac{\lambda^n}{(2\pi)^n\Theta(x,y)^{1/2}}\int_{|\xi|_{g_y}<1} e^{i\lambda\langle \exp_y^{-1}(x),\xi\rangle_{g_y}}\frac{d\xi}{\sqrt{|g_y|}}+\int_0^\lambda W(x,y,\mu)\,d\mu. \tag{64}$$

Note that

$$\Theta(x,y)^{-1/2}=1+O(\operatorname{dist}_g(x,y)^2)$$

and

$$\frac{\exp_y^{-1}(x)}{i\lambda\operatorname{dist}_g(x,y)^2}\nabla_\xi e^{i\lambda\langle \exp_y^{-1}(x),\xi\rangle_{g_y}}=e^{i\lambda\langle \exp_y^{-1}(x),\xi\rangle_{g_y}}.$$

Therefore, we may integrate by parts once in (64) to obtain

$$\begin{aligned}\rho*E(x,y,\lambda)=\frac{\lambda^n}{(2\pi)^n}\int_{|\xi|_{g_y}<1} e^{i\lambda\langle \exp_y^{-1}(x),\xi\rangle_{g_y}}\frac{d\xi}{\sqrt{|g_y|}}+\int_0^\lambda W(x,y,\mu)\,d\mu\\ +O\left(\operatorname{dist}_g(x,y)\lambda^{n-1}\int_{|\xi|_{g_y}=1} e^{i\lambda\langle \exp_y^{-1}(x),\omega\rangle_{g_y}}\,d\omega\right).\end{aligned}$$

Since

$$\sup_{\operatorname{dist}_g(x,y)<\operatorname{inj}(M,g)}\left|\operatorname{dist}_g(x,y)\int_{|\xi|_{g_y}=1} e^{i\lambda\langle \exp_y^{-1}(x),\omega\rangle_{g_y}}\,d\omega\right|=o(1)$$

as $\lambda\to\infty$, we find that

$$\rho*E(x,y,\lambda)=\frac{\lambda^n}{(2\pi)^n}\int_{|\xi|_{g_y}<1} e^{i\lambda\langle \exp_y^{-1}(x),\xi\rangle_{g_y}}\frac{d\xi}{\sqrt{|g_y|}}+\int_0^\lambda W(x,y,\mu)\,d\mu+o(\lambda^{n-1}).$$

By (47), we have

$$\sup_{x,y\in B(x_0,\operatorname{inj}(M,g)/2)}\left|\int_0^\lambda W(x,y,\mu)\,d\mu\right|\le c\operatorname{dist}_g(x,y)\lambda^{n-1}+C\lambda^{n-2}$$

for some c, $C>0$ as claimed. □

6. Smooth vs. rough projector: proof of Proposition 11

Let $x_0 \in M$ be a non-self-focal point and fix $\varepsilon > 0$. The proof of Proposition 11 amounts to showing that there exists $c > 0$ such that, for all $\varepsilon > 0$, there is an open neighborhood $\mathcal{U}_\varepsilon$ of x_0 and a positive constant c_ε with

$$\sup_{x,y\in\mathcal{U}_\varepsilon} |E_\lambda(x, y) - \rho * E_\lambda(x, y)| \le c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2} \tag{65}$$

for all $\lambda \ge 1$. It is at this point that the assumption that x_0 is a non-self-focal point is needed. In Section 4 we construct a partition of the identity operator localized to x_0. We use this partition to split $|E_\lambda(x, y) - \rho * E_\lambda(x, y)|$ into different pieces, each of which we shall control using two types of Tauberian theorems, described in Section 6.1. We conclude this section by presenting the proof of Proposition 11 in Section 6.2.

To ease the notation, we will write

$$E(x, y, \lambda) := E_\lambda(x, y).$$

To prove (65), we use the operators B_ε and C_ε and the function ψ_ε constructed in Lemma 15. We set

$$\alpha_\varepsilon(x, y, \lambda) := EC_\varepsilon^*(x, y, \lambda) + \tfrac{1}{2}(E(x, x, \lambda) + C_\varepsilon EC_\varepsilon^*(y, y, \lambda)), \tag{66}$$

$$\beta_\varepsilon(x, y, \lambda) := \rho * EC_\varepsilon^*(x, y, \lambda) + \tfrac{1}{2}(E(x, x, \lambda) + C_\varepsilon EC_\varepsilon^*(y, y, \lambda)), \tag{67}$$

where x and y are any two points in M. Note that

$$|\alpha_\varepsilon(x, y, \lambda) - \beta_\varepsilon(x, y, \lambda)| = |EC_\varepsilon^*(x, y, \lambda) - \rho * EC_\varepsilon^*(x, y, \lambda)|.$$

In addition, observe that

$$\alpha_\varepsilon(x, y, \lambda) := \tfrac{1}{2} \sum_{\lambda_j \le \lambda} [\varphi_j(x) + (C_\varepsilon\varphi_j)(y)]^2$$

and so $\alpha_\varepsilon(x, y, \lambda)$ is an increasing function of λ for any fixed x and y. We also set

$$g_\varepsilon(x, y, \lambda) := EB_\varepsilon^*(x, y, \lambda) - \rho * EB_\varepsilon^*(x, y, \lambda). \tag{68}$$

Since $B_\varepsilon + C_\varepsilon = \psi_\varepsilon^2$ and $\psi_\varepsilon = 1$ in a neighborhood of x_0, relation (65) would hold if we proved that there exist positive constants c and c_ε with c independent of ε, and a neighborhood $\mathcal{U}_\varepsilon$ of x_0, such that, for all $\lambda \ge 1$,

$$\sup_{x,y\in\mathcal{U}_\varepsilon} |\alpha_\varepsilon(x, y, \lambda) - \beta_\varepsilon(x, y, \lambda)| \le c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2}, \tag{69}$$

$$\sup_{x,y\in\mathcal{U}_\varepsilon} |g_\varepsilon(x, y, \lambda)| \le c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2}. \tag{70}$$

6.1. ***Tauberian theorems.*** To control $|\alpha_\varepsilon(x, y, \lambda) - \beta_\varepsilon(x, y, \lambda)|$ and $|g_\varepsilon(x, y, \lambda)|$ we use two different Tauberian-type theorems. To state the first one, fix a positive function $\phi \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp} \hat{\phi} \subseteq (-1, 1)$ and $\hat{\phi}(0) = 1$. We have written $\hat{f}$ for the Fourier transform of f. Define, for each $a > 0$,

$$\phi_a(\lambda) := \frac{1}{a}\phi\Big(\frac{\lambda}{a}\Big), \tag{71}$$

so that $\hat{\phi}_a(t) = \hat{\phi}(at)$.

Lemma 19 (Tauberian theorem for monotone functions). *Let α be an increasing temperate function with $\alpha(0) = 0$ and let β be a function of locally bounded variation with $\beta(0) = 0$. Suppose further that there exist $M_0 > 0$, $a > 0$ and a constant c_a such that*:

(a) *There exists $m \in \mathbb{N}$ such that*

$$\int_{\mu-a}^{\mu+a} |d\beta| \le a M_0 (1+|\mu|)^{m-1} + c_a |\mu|^{m-2} \quad \textit{for all } \mu \ge 0.$$

(b) *There exist $\kappa \in \mathbb{Z} \setminus \{-1\}$ with $\kappa \le m-1$, and $M_a > 0$, such that*

$$|(d\alpha - d\beta) * \phi_a(\mu)| \le M_a (1+|\mu|)^{\kappa} \quad \textit{for all } \mu \ge 0.$$

Then there exists $c > 0$ depending only on ϕ such that

$$|\alpha(\mu) - \beta(\mu)| \le c\big(a M_0 |\mu|^{m-1} + c_a |\mu|^{m-2} + M_a (1+|\mu|)^{\kappa+1}\big) \tag{72}$$

for all $\mu \ge 0$.

Proof. The proof is identical to argument for Lemma 17.5.6 in [Hörmander 1985a]. □

We will also need the following result:

Lemma 20 (Tauberian theorem for nonmonotone functions [Hörmander 1968]). *Let g be a piecewise continuous function such that there exists $a > 0$ with $\hat{g}(t) \equiv 0$ for $|t| \le a$. Suppose further that, for all $\mu \in \mathbb{R}$, there exist constants $m \in \mathbb{N}$ and $c_1, c_2 > 0$ such that*

$$|g(\mu+s) - g(\mu)| \le c_1 (1+|\mu|)^m + c_2 (1+|\mu|)^{m-1} \quad \textit{for all } s \in [0,1]. \tag{73}$$

Then there exists a positive constant $c_{m,a}$, depending only on m and a, such that, for all μ,

$$|g(\mu)| \le c_{m,a}\big(c_1 (1+|\mu|)^m + c_2 (1+|\mu|)^{m-1}\big).$$

6.2. *Proof of Proposition 11.* As explained above, the proof of Proposition 11 reduces to establishing relations (69) and (70).

Proof of (69). We seek to apply Lemma 19 to α_ε and β_ε. Let $a = \varepsilon$, $m = n$ and $\kappa = -2$. We first verify condition (a). From Remark 17, it follows that there exist an open neighborhood $\mathcal{U}_\varepsilon$ of x_0 and constants $c_1, c_\varepsilon > 0$ such that, for all $x, y \in \mathcal{U}_\varepsilon$ and all $\lambda \ge 1$,

$$\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \big(|\partial_\nu E(x,x,\nu)| + |\partial_\nu (C_\varepsilon E C_\varepsilon^*)(y,y,\nu)|\big)\, d\nu = \sum_{|\lambda_j - \lambda| \le \varepsilon} (\varphi_j(x))^2 + (C_\varepsilon \varphi_j(y))^2$$
$$\le c_1 \varepsilon \lambda^{n-1} + c_\varepsilon \lambda^{n-2}. \tag{74}$$

Combining (74) with the estimate in Proposition 18 applied to $Q = C_\varepsilon$, we see that there exist positive constants M_0 and c_ε for which

$$\sup_{x,y \in \mathcal{U}_\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |\partial_\nu \beta_\varepsilon(x,y,\nu)|\, d\nu \le M_0 \varepsilon \lambda^{n-1} + c_\varepsilon \lambda^{n-2}$$

for all $\lambda \geq 1$. It remains to verify condition (b). Note that

$$\partial_\lambda (\alpha_\varepsilon(x, y, \cdot) - \beta_\varepsilon(x, y, \cdot)) * \phi_\varepsilon(\lambda) = \mathscr{F}^{-1}_{t \to \lambda}\big((1 - \hat{\rho}(t))\hat{\phi}_\varepsilon(t)(U(t)C_\varepsilon^*)(x, y)\big)(\lambda),$$

where $\mathscr{F}$ is the Fourier transform and ϕ_ε is defined in (71). According to Lemma 15, $U(t)C_\varepsilon^*$ is a smoothing operator for $\frac{1}{2}\operatorname{inj}(M, g) < |t| < \frac{1}{\varepsilon}$. Hence, since

$$\operatorname{supp} \hat{\phi}_\varepsilon \subset \big\{t : |t| < \tfrac{1}{\varepsilon}\big\} \quad \text{and} \quad \operatorname{supp}(1 - \hat{\rho}) \subset \big\{t : |t| > \tfrac{1}{2}\operatorname{inj}(M, g)\big\},$$

we find that, for each N, there are constants $c_{N,\varepsilon}$ depending on N and ε that satisfy

$$\sup_{x,y \in M} \big|\partial_\lambda (\alpha_\varepsilon(x, y, \cdot) - \beta_\varepsilon(x, y, \cdot)) * \phi_\varepsilon(\lambda)\big| \leq c_{N,\varepsilon}(1 + |\lambda|)^{-N}$$

for all $\lambda > 0$. □

Proof of (70). We seek to apply Lemma 20 to g_ε. First, note that, since

$$g_\varepsilon(x, y, \lambda) = EB_\varepsilon^*(x, y, \lambda) - \rho * EB_\varepsilon^*(x, y, \lambda),$$

the function $g_\varepsilon(x, y, \cdot)$ is piecewise continuous in the λ variable. Next, we check that $\hat{g_\varepsilon}(t) \equiv 0$ in a neighborhood of $t = 0$. We have

$$\partial_\lambda g_\varepsilon(x, y, \lambda) = \mathscr{F}^{-1}_{t \to \lambda}\big((1 - \hat{\rho}(t))(U(t)B_\varepsilon^*)(x, y)\big)(\lambda).$$

Since $\hat{\rho} \equiv 1$ on $\big(-\frac{1}{2}\operatorname{inj}(M, g), \frac{1}{2}\operatorname{inj}(M, g)\big)$, we have $\mathscr{F}_{\lambda \to t}(\partial_\lambda g_\varepsilon(x, y, \cdot))(t) = 0$ for $|t| \leq \frac{1}{2}\operatorname{inj}(M, g)$. Equivalently,

$$t \cdot \mathscr{F}_{\lambda \to t}(g_\varepsilon(x, y, \cdot))(t) = 0, \quad |t| \leq \tfrac{1}{2}\operatorname{inj}(M, g).$$

In addition, we must have $\mathscr{F}_{\lambda \to t}(g_\varepsilon(x, y, \cdot))(0) = 0$, for otherwise $g_\varepsilon(x, y, \cdot)$ would include a sum of derivatives of delta functions but this is not possible, since $g_\varepsilon(x, y, \cdot)$ is piecewise continuous. It follows that

$$\mathscr{F}_{\lambda \to t}(g_\varepsilon(x, y, \cdot))(t) = 0, \quad |t| \leq \tfrac{1}{2}\operatorname{inj}(M, g),$$

as desired. It therefore remains to check that g_ε satisfies (73). Let $s \in [0, 1]$, $\lambda \in \mathbb{R}$ and write

$$\begin{aligned} &g_\varepsilon(x, y, \lambda + s) - g_\varepsilon(x, y, \lambda) \\ &\quad = EB_\varepsilon^*(x, y, \lambda + s) - EB_\varepsilon^*(x, y, \lambda) + \rho * EB_\varepsilon^*(x, y, \lambda + s) - \rho * EB_\varepsilon^*(x, y, \lambda). \end{aligned} \tag{75}$$

To estimate $EB_\varepsilon^*(x, y, \lambda + s) - EB_\varepsilon^*(x, y, \lambda)$ we apply the Cauchy–Schwarz inequality:

$$\begin{aligned} EB_\varepsilon^*(x, y, \lambda + s) - EB_\varepsilon^*(x, y, \lambda) &= \sum_{\lambda \leq \lambda_j \leq \lambda + s} \varphi_j(x) B_\varepsilon \varphi_j(y) \\ &\leq \Big(\sum_{\lambda \leq \lambda_j \leq \lambda + s} (\varphi_j(x))^2\Big)^{\frac{1}{2}} \Big(\sum_{\lambda \leq \lambda_j \leq \lambda + s} (B_\varepsilon \varphi_j(y))^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Applying Remark 17 to $Q = \mathrm{Id}$ and $Q = B_\varepsilon$, there exist an open neighborhood $\mathcal{U}_\varepsilon$ of x_0 and constants $c, c_\varepsilon > 0$ such that

$$|EB_\varepsilon^*(x, y, \lambda + s) - EB_\varepsilon^*(x, y, \lambda)| \le c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2} \tag{76}$$

for all $\lambda \ge 1$, $s \in [0, 1]$ and $x, y \in \mathcal{U}_\varepsilon$. The ε factor is due to the fact that $\|b_0\|_1 < \varepsilon$.

To estimate $\rho * EB_\varepsilon^*(x, y, \lambda + s) - \rho * EB_\varepsilon^*(x, y, \lambda)$ we apply Proposition 18 to the operator $Q = B_\varepsilon$. Since there exists $\tilde{c} > 0$ with

$$|\partial_\lambda \rho * EB_\varepsilon^*(x, y, \lambda)| \le \tilde{c}(\|b_0\|_1\lambda^{n-1} + \lambda^{n-2}) \quad \text{for all } \lambda \ge 1$$

and $\|b_0\|_1 \le \varepsilon$, we get (after possibly enlarging c and c_ε) that

$$|\rho * EB_\varepsilon^*(x, y, \lambda + s) - \rho * EB_\varepsilon^*(x, y, \lambda)| \le c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2} \quad \text{for all } \lambda \ge 1. \tag{77}$$

Combining (76) and (77) into (75), we conclude the existence of positive constants c and c_ε such that

$$|g_\varepsilon(x, y, \lambda + s) - g_\varepsilon(x, y, \lambda)| \le c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2} \quad \text{for all } \lambda \ge 1$$

and $s \in [0, 1]$, as desired. Applying Lemma 20 with $m = n$ and $a = \frac{1}{2}\,\mathrm{inj}(M, g)$ proves (70). □

7. Proof of Theorems 6–9

Proof of Theorem 6. Suppose that (M, g) is a smooth, compact, Riemannian manifold with no boundary. Let $K \subseteq M \times M$ be a compact set satisfying that any pair of points in it are mutually nonfocal. We aim to show that there exists $c > 0$ such that, for every $\varepsilon > 0$, there are constants $\lambda_\varepsilon > 0$ and $c_\varepsilon > 0$ such that

$$\sup_{(x,y)\in K} |R(x, y, \lambda)| \le c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2}$$

for all $\lambda > \lambda_\varepsilon$. Fix $\varepsilon > 0$ and write $\Delta \subseteq M \times M$ for the diagonal. Define

$$\widetilde{K} = K \cap \Delta.$$

By (16), there exists $\lambda_\varepsilon > 0$, a finite collection $\{x_j : j = 1, \dots, N_\varepsilon\}$ and open neighborhoods $\mathcal{U}_\varepsilon^{x_j}$ of x_j such that

$$\widetilde{K} \subseteq \bigcup_j \mathcal{U}_\varepsilon^{x_j} \times \mathcal{U}_\varepsilon^{x_j}$$

and

$$\sup_{x,y\in\mathcal{U}_\varepsilon^{x_j}} |R(x, y, \lambda)| \le c\varepsilon\lambda^{n-1} + c_\varepsilon\lambda^{n-2} \tag{78}$$

for all $\lambda > \lambda_\varepsilon$. Define

$$K_\varepsilon := K \setminus \bigcup_j \mathcal{U}_\varepsilon^{x_j} \times \mathcal{U}_\varepsilon^{x_j}.$$

Safarov [1988, Theorem 3.3] proved under the mutually nonfocal assumption that

$$\sup_{(x,y)\in K_\varepsilon} |R(x, y, \lambda)| = o_\varepsilon(\lambda^{n-1}). \tag{79}$$

Combining (78) and (79) completes the proof. □

Proof of Theorem 8. The injectivity of the maps $\Psi_{(\lambda,\lambda+1]} : M \to \mathbb{R}^{m_\lambda}$ for λ large enough is implied by the existence of positive constants c_1, c_2, r_0 and λ_{r_0} such that, if $\lambda > \lambda_{r_0}$, then

$$\inf_{x,y:\lambda\,\mathrm{dist}_g(x,y)\geq r_0} \mathrm{dist}^2_\lambda(x,y) > c_1 \tag{80}$$

and

$$\inf_{x,y:\lambda\,\mathrm{dist}_g(x,y)< r_0} \frac{\mathrm{dist}^2_\lambda(x,y)}{\lambda^2\,\mathrm{dist}_g(x,y)^2} > c_2. \tag{81}$$

We first prove (80). By Theorem 6, for all $x, y \in M$,

$$\mathrm{dist}^2_\lambda(x,y) = f(\lambda\,\mathrm{dist}_g(x,y)) + \widetilde{R}(x,y,\lambda), \tag{82}$$

where $\sup_{x,y\in M} |\widetilde{R}(x,y,\lambda)| = o(1)$ and $f : [0,+\infty) \to \mathbb{R}$ is the function

$$f(r) := \int_{S^{n-1}} (1 - e^{ir\omega_1})\,d\omega.$$

Observe that $f(r) \geq 0$, with $f(r) = 0$ only if $r = 0$. Moreover,

$$f(r) = \sigma_n + O(r^{-(n-1)/2}) \quad \text{as } r \to \infty \qquad \text{and} \qquad f(r) = r^2 \cdot \tilde{f}(r) \tag{83}$$

for some smooth and positive function $\tilde{f}$, where σ_n is the volume of S^{n-1}. According to the first relation in (83), we may choose $r_0 > 0$ so that

$$\lambda\,\mathrm{dist}_g(x,y) \geq r_0 \implies |f(\lambda\,\mathrm{dist}_g(x,y)) - \sigma_n| \leq \tfrac{1}{4}\sigma_n. \tag{84}$$

Moreover, by Theorem 6 we may choose λ_{r_0} so that, if $\lambda > \lambda_{r_0}$, then

$$\sup_{x,y\in M} |\widetilde{R}(x,y,\lambda)| \leq \tfrac{1}{4}\sigma_n. \tag{85}$$

Combining (82) , (84) and (85), we find that, for all $\lambda > \lambda_{r_0}$ and all $x, y \in M$ with $\lambda\,\mathrm{dist}_g(x,y) \geq r_0$,

$$\mathrm{dist}^2_\lambda(x,y) \geq \tfrac{1}{2}\sigma_n,$$

as desired. To verify (81), write, as above,

$$\mathrm{dist}^2_\lambda(x,y) = \frac{(2\pi)^n}{2\lambda^{n-1}}\big(E_{(\lambda,\lambda+1]}(x,x) + E_{(\lambda,\lambda+1]}(y,y) - 2E_{(\lambda,\lambda+1]}(x,y)\big)$$

and note that the first derivatives of $\mathrm{dist}^2_\lambda(x,y)$ in x and y all vanish when $x = y$. Moreover, by [Zelditch 2009, Proposition 2.3], we have that the Hessian of $E_{(\lambda,\lambda+1]}$ may be written as

$$d_x \otimes d_y\big|_{x=y} E_{(\lambda,\lambda+1]}(x,y) = C_n\lambda^{n+1} g_x + o(\lambda^{n+1}),$$

where g_x is the metric g on T_xM, and Equation (1.2) in [Potash 2014] shows that

$$C_n = \frac{\sigma_n}{n(2\pi)^n}.$$

Therefore, applying Taylor's theorem, there exists $C_0 > 0$ for which

$$\left|\frac{\text{dist}_\lambda^2(x,y)}{\lambda^2\,\text{dist}_g^2(x,y)} - \frac{\sigma_n}{2n}\right| \le C_0 \cdot \lambda\,\text{dist}_g(x,y). \tag{86}$$

The extra factor of λ on the right-hand side of (86) comes from the fact that

$$\sup_{|\alpha|=3} \left|\partial_x^\alpha|_{x=y} E_{(\lambda,\lambda+1]}(x,y)\right| = O(m_\lambda \lambda^3),$$

which is proved, for example, in [Xu 2006, Equation (2.7)]. Equation (86) shows that

$$\inf_{\lambda\,\text{dist}_g(x,y)<\sigma_n/(4nC_0)} \frac{\text{dist}_\lambda^2(x,y)}{\lambda^2 d_g^2(x,y)} \ge \frac{\sigma_n}{2n} > 0.$$

If $r_0 \le \sigma_n/(4nC_0)$, then the claim (81) follows. Otherwise, it remains to show that there exists $c_2 > 0$ with

$$\inf_{\sigma_n/(4nC_0)\le\lambda\,\text{dist}_g(x,y)<r_0} \frac{\text{dist}_\lambda^2(x,y)}{\lambda^2 d_g^2(x,y)} > c_2 \tag{87}$$

for all λ sufficiently large. Theorem 6 shows that, after possibly enlarging λ_{r_0}, we have

$$\sup_{x,y\in M} |\widetilde{R}(x,y,\lambda)| \le \left(\frac{\sigma_n}{4nC_0}\right)^2 \inf_{r<r_0} \tilde{f}(r)$$

for all $\lambda > \lambda_{r_0}$. Then the second relation in (83) combined with (82) yields that, for all $\lambda > \lambda_{r_0}$,

$$\inf_{\sigma_n/(4nC_0)\le\lambda\,\text{dist}_g(x,y)<r_0} \text{dist}_\lambda^2(x,y) \ge \left(\frac{\sigma_n}{4nC_0}\right)^2 \inf_{r<r_0} \tilde{f}(r) > 0.$$

This completes the proof of (81). □

Proof of Theorem 9. By (13) and Theorem 6 we have that

$$\sup_{x,y\in M}\left|\text{dist}_\lambda^2(x,y) - \int_{S^{n-1}} (1 - e^{i\lambda\,\text{dist}_g(x,y)\omega_1})\,d\omega\right| = o(1)$$

as $\lambda \to \infty$. Combining this with

$$\frac{1}{\lambda^2\,\text{dist}_g(x,y)^2} \int_{S^{n-1}} (1 - e^{i\lambda\,\text{dist}_g(x,y)\omega_1})\,d\omega = \frac{\sigma_n}{2n} + O(\lambda^2\,\text{dist}_g^2(x,y))$$

and (86) completes the proof. □

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ON THE CONTINUOUS RESONANT EQUATION FOR NLS II: STATISTICAL STUDY

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We consider the continuous resonant (CR) system of the 2-dimensional cubic nonlinear Schrödinger (NLS) equation. This system arises in numerous instances as an effective equation for the long-time dynamics of NLS in confined regimes (e.g., on a compact domain or with a trapping potential). The system was derived and studied from a deterministic viewpoint in several earlier works, which uncovered many of its striking properties. This manuscript is devoted to a probabilistic study of this system. Most notably, we construct global solutions in negative Sobolev spaces, which leave Gibbs and white noise measures invariant. Invariance of white noise measure seems particularly interesting in view of the absence of similar results for NLS.

1. Introduction

Presentation of the equation. The purpose of this manuscript is to construct some invariant measures for the so-called continuous resonant (CR) system of the cubic nonlinear Schrödinger equation. This system can be written as

$$\begin{cases} i\partial_t u = \mathcal{T}(u,u,u), & (t,x)\in\mathbb{R}\times\mathbb{R}^2, \\ u(0,x) = f(x), \end{cases} \tag{CR}$$

where the operator $\mathcal{T}$ defining the nonlinearity has several equivalent formulations corresponding to different interpretations/origins of this system. In its original formulation [Faou et al. 2013] as the large-box limit[1] of the resonant cubic NLS,[2] $\mathcal{T}$ can be written as follows: for $z\in\mathbb{R}^2$ and $x=(x_1,x_2)\in\mathbb{R}^2$, letting $x^\perp=(-x_2,x_1)$, we have

$$\mathcal{T}(f_1,f_2,f_3)(z) := \int_{\mathbb{R}}\int_{\mathbb{R}^2} f_1(x+z)\,f_2(\lambda x^\perp+z)\,\overline{f_3(x+\lambda x^\perp+z)}\,dx\,d\lambda.$$

This integral can be understood as an integral over all rectangles having z as a vertex. It has the equivalent formulation [Germain et al. 2015]

$$\mathcal{T}(f_1,f_2,f_3) = 2\pi\int_{\mathbb{R}} \mathrm{e}^{-i\tau\Delta}[(\mathrm{e}^{i\tau\Delta}f_1)(\mathrm{e}^{i\tau\Delta}f_2)(\overline{\mathrm{e}^{i\tau\Delta}f_3})]\,d\tau.$$

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[1] Starting with the equation on a torus of size L and letting $L\to\infty$.

[2] This is NLS with only the resonant interactions retained (also known as the first Birkhoff normal form). It gives an approximation of NLS for sufficiently small initial data.

It was shown in [Faou et al. 2013] that the dynamics of (CR) approximate that of the cubic NLS equation on a torus of size L (with L large enough) over time scales $\sim L^2/\varepsilon^2$ (up to logarithmic loss in L), where ε denotes the size of the initial data.

Another formulation of (CR) comes from the fact that it is also the resonant system for the cubic nonlinear Schrödinger equation with harmonic potential given by

$$i\partial_t u - \Delta u + |x|^2 u = \mu |u|^2 u, \quad \mu \in \mathbb{R} \text{ constant.} \tag{1-1}$$

In this picture, $\mathcal{T}$ can be written as follows: denoting by $H := -\Delta + |x|^2 = -\partial_{x_1}^2 - \partial_{x_2}^2 + x_1^2 + x_2^2$ the harmonic oscillator on $\mathbb{R}^2$,

$$\mathcal{T}(f_1, f_2, f_3) = 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{e}^{i\tau H}[(\mathrm{e}^{-i\tau H} f_1)(\mathrm{e}^{-i\tau H} f_2)(\overline{\mathrm{e}^{-i\tau H} f_3})]\, d\tau.$$

As a result, the dynamics of (CR) approximate the dynamics of (1-1) over long nonlinear time scales for small enough initial data.

The equation (CR) is Hamiltonian. Indeed, introducing the functional

$$\begin{aligned}\mathscr{E}(u_1, u_2, u_3, u_4) &:= \langle \mathcal{T}(u_1, u_2, u_3)\,,\, u_4 \rangle_{L^2} \\ &= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\mathbb{R}^2} (\mathrm{e}^{-itH} u_1)(\mathrm{e}^{-itH} u_2)(\overline{\mathrm{e}^{-itH} u_3})(\overline{\mathrm{e}^{-itH} u_4})\, dx\, dt\end{aligned}$$

and setting

$$\mathscr{E}(u) := \mathscr{E}(u, u, u, u),$$

(CR) derives from the Hamiltonian $\mathscr{E}$ given the symplectic form $\omega(f, g) = -4\mathfrak{Im}\langle f, g\rangle_{L^2(\mathbb{R}^2)}$ on $L^2(\mathbb{R}^2)$, so that (CR) is equivalent to

$$i\partial_t f = \frac{1}{2} \frac{\partial \mathscr{E}(f)}{\partial \bar{f}}.$$

In addition to the two instances mentioned above in which (CR) appears to describe the long-time dynamics of the cubic NLS equation — with or without potential — we mention the following:

- The equation (CR) appears as a modified scattering limit of the cubic NLS on $\mathbb{R}^3$ with harmonic tapping in two directions. Here, (CR) appears as an asymptotic system and any information on the asymptotic dynamics of (CR) directly gives the corresponding behavior for NLS with partial harmonic trapping. We refer to [Hani and Thomann 2015] for more details.

- When restricted to the Bargmann–Fock space (see below), the equation (CR) turns out to be the lowest-Landau-level equation, which describes fast-rotating Bose–Einstein condensates (see [Aftalion et al. 2006; Nier 2007; Gérard et al. ≥ 2015]).

- The equation (CR) can also be interpreted as describing the effective dynamics of high-frequency envelopes for NLS on the unit torus $\mathbb{T}^2$. This means that, if the initial data $\varphi(0)$ for NLS has its Fourier transform given by[3] $\{\hat{\varphi}(0, k) \sim g_0(k/N)\}_{k \in \mathbb{Z}^2}$ and if $g(t)$ evolves according to (CR) with initial data g_0

[3] Up to a normalizing factor in H^s, $s > 1$.

and $\varphi(t)$ evolves according to NLS with initial data $\varphi(0)$, then $g(t, k/N)$ approximates the dynamics of $\hat{\varphi}(t,k)$ in the limit of large N (see [Faou et al. 2013, Theorem 2.6]).

Some properties and invariant spaces. We review some of the properties of the (CR) equation that will be useful in this paper. For a more detailed study of the equation we refer to [Faou et al. 2013; Germain et al. 2015].

First, (CR) is globally well-posed in $L^2(\mathbb{R}^2)$. Amongst its conserved quantities, we note

$$\int_{\mathbb{R}^2} |u|^2\,dx \quad \text{and} \quad \int_{\mathbb{R}^2} (|x|^2|u|^2 + |\nabla u|^2)\,dx = \int_{\mathbb{R}^2} \bar{u} H u\,dx,$$

(recall that H denotes the harmonic oscillator $H = -\Delta + |x|^2$). This equation also enjoys many invariant spaces, in particular:

- The eigenspaces $(E_N)_{N\geq 0}$ of the harmonic oscillator are stable (see [Faou et al. 2013; Germain et al. 2015]). This is a manifestation of the fact that (CR) is the resonant equation associated to (1-1). Recall that H admits a complete basis of eigenvectors for $L^2(\mathbb{R}^2)$; each eigenspace E_N ($N = 0, 1, 2, \dots$) has dimension $N + 1$.

- The set of radial functions is stable, as follows from the invariance of H under rotations (see [Germain et al. 2015]). Global dynamics on $L^2_{\text{rad}}(\mathbb{R}^2)$, the radial functions of $L^2(\mathbb{R}^2)$, can be defined. A basis of normalized eigenfunctions of H for $L^2_{\text{rad}}(\mathbb{R}^2)$ is given by

$$\varphi_n^{\text{rad}}(x) = \frac{1}{\sqrt{\pi}} L_n^{(0)}(|x|^2) e^{-|x|^2/2} \quad \text{with} \quad L_n^{(0)}(x) = e^x \frac{1}{n!}\left(\frac{d}{dx}\right)^n (e^{-x}x^n) \quad \text{for } n \in \mathbb{N}.$$

We record that $H\varphi_n^{\text{rad}} = (4n+2)\varphi_n^{\text{rad}}$.

- If $\mathcal{O}(\mathbb{C})$ stands for the set of entire functions on $\mathbb{C}$ (with the identification $z = x_1 + ix_2$), the Bargmann–Fock space $L^2_{\text{hol}}(\mathbb{R}^2) = L^2(\mathbb{R}^2) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2})$ is invariant under the flow of (CR). Global dynamics on $L^2_{\text{hol}}(\mathbb{R}^2)$ can be defined. A basis of normalized eigenfunctions of H for $L^2_{\text{hol}}(\mathbb{R}^2)$ is given by the "holomorphic" Hermite functions, also known as the "special Hermite functions", namely

$$\varphi_n^{\text{hol}}(x) = \frac{1}{\sqrt{\pi n!}}(x_1 + ix_2)^n \mathrm{e}^{-|x|^2/2} \quad \text{for } n \in \mathbb{N}.$$

Notice that $H\varphi_n^{\text{hol}} = 2(n+1)\varphi_n^{\text{hol}}$. It is proved in [Germain et al. 2015] that

$$\mathcal{T}(\varphi_{n_1}^{\text{hol}}, \varphi_{n_2}^{\text{hol}}, \varphi_{n_3}^{\text{hol}}) = \alpha_{n_1,n_2,n_3,n_4}\varphi_{n_4}^{\text{hol}}, \quad n_4 = n_1 + n_2 - n_3, \tag{1-2}$$

with

$$\alpha_{n_1,n_2,n_3,n_4} = \mathcal{H}(\varphi_{n_1}^{\text{hol}}, \varphi_{n_2}^{\text{hol}}, \varphi_{n_3}^{\text{hol}}, \varphi_{n_4}^{\text{hol}}) = \frac{\pi}{8}\frac{(n_1+n_2)!}{2^{n_1+n_2}\sqrt{n_1!n_2!n_3!n_4!}}\mathbf{1}_{n_1+n_2=n_3+n_4}.$$

As a result, the (CR) system reduces to the following infinite-dimensional system of ODEs when restricted to $\operatorname{Span}\{\varphi_n\}_{n\in\mathbb{N}}$:

$$i\partial_t c_n(t) = \sum_{\substack{n_1,n_2,n_3\in\mathbb{N} \\ n_1+n_2-n_3=n}} \alpha_{n_1,n_2,n_3,n} c_{n_1}(t)c_{n_2}(t)\bar{c}_{n_3}(t).$$

Statistical solutions. In this paper we construct global probabilistic solutions on each of the above-mentioned spaces which leave invariant either Gibbs or white noise measures. More precisely, our main results can be summarized as follows:

- We construct global strong flows on
$$X^0_{\mathrm{rad}}(\mathbb{R}^2) = \bigcap_{\sigma>0} \mathcal{H}^{-\sigma}_{\mathrm{rad}}(\mathbb{R}^2)$$
and on
$$X^0_{\mathrm{hol}}(\mathbb{R}^2) := \Big(\bigcap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R}^2)\Big) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}),$$
which leave the Gibbs measures invariant (see Theorem 2.5).

- We construct global weak probabilistic solutions on
$$X^{-1}_{\mathrm{hol}}(\mathbb{R}^2) := \Big(\bigcap_{\sigma>1} \mathcal{H}^{-\sigma}(\mathbb{R}^2)\Big) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}),$$
and this dynamics leaves the white noise measure invariant (see Theorem 2.6).

Since the '90s, there have been many works devoted to the construction of Gibbs measures for dispersive equations and, more recently, much attention has been paid to the well-posedness of these equations with random initial conditions. We refer to the introduction of [Poiret et al. 2014] for references on the subject. In particular, concerning the construction of strong solutions for the nonlinear harmonic oscillator (which is related to (CR)), we refer to [Thomann 2009; Burq et al. 2013; Deng 2012; Poiret 2012a; 2012b; Poiret et al. 2014].

Let us define what we mean by white noise measure in our context. Denote by $(e_n)_{n\geq 0}$ a Hilbert basis of $L^2(0,1)$ and consider independent standard Gaussians $(g_n)_{n\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \boldsymbol{p})$. Then it is well known (see, e.g., [Hida 1980, Chapter 2]) that the random series

$$B_t = \sum_{n=0}^{+\infty} g_n \int_0^t e_n(s)\, ds$$

converges in $L^2(\Omega, \mathcal{F}, \boldsymbol{p})$ and defines a Brownian motion. The white noise measure is then defined by the map

$$\omega \mapsto W(t,\omega) = \frac{dB_t}{dt}(\omega) = \sum_{n=0}^{+\infty} g_n(\omega) e_n(t). \tag{1-3}$$

Now consider a Hilbert space $\mathcal{H}$ which is a space of functions on a manifold M and consider a Hilbert basis $(e_n)_{n\geq 0}$ of $\mathcal{H}$. We define the mean-zero Gaussian white noise (measure) on $\mathcal{H}$ as $\mu = \boldsymbol{p} \circ W^{-1}$, where

$$W(x,\omega) = \sum_{n=0}^{+\infty} g_n(\omega) e_n(x).$$

Notice that this measure is independent of the choice of the Hilbert basis of $\mathcal{K}$. It is clear that, for all $x \in M$, $\mathbb{E}_{\boldsymbol{p}}[W(x,\cdot)] = 0$. Moreover, for all $x, y \in M$ we have

$$\mathbb{E}_{\boldsymbol{p}}[W(x,\cdot)\overline{W(y,\cdot)}] = \sum_{n=0}^{+\infty} e_n(x)\overline{e_n(y)} = \delta(x-y),$$

since the sum in the previous line is the kernel of the identity projector on $\mathcal{K}$. For more details on Gaussian measures on Hilbert spaces, we refer to [Janson 1997].

Construction of flows invariant under white noise measures is much trickier due to the low regularity of the support of such measures, and there seem to be no results of this sort for NLS equations. We construct weak solutions on the support of the white noise measure on $X^{-1}_{\mathrm{hol}}(\mathbb{R}^2)$ using a method based on a compactness argument in the space of measures (the Prokhorov theorem) combined with a representation theorem of random variables (the Skorohod theorem). This approach has been first applied to the Navier–Stokes and Euler equations in [Albeverio and Cruzeiro 1990; Da Prato and Debussche 2002] and extended to dispersive equations by Burq, Thomann and Tzvetkov [Burq et al. 2014], who give a self-contained presentation of the method.

Notations. Define the harmonic Sobolev spaces for $s \in \mathbb{R}$ and $p \geq 1$ by

$$\mathcal{W}^{s,p} = \mathcal{W}^{s,p}(\mathbb{R}^2) = \{u \in L^p(\mathbb{R}^2) : H^{s/2}u \in L^p(\mathbb{R}^2)\}, \quad \mathcal{H}^s = \mathcal{W}^{s,2}.$$

They are endowed with the natural norms $\|u\|_{\mathcal{W}^{s,p}}$. Up to equivalence of norms we have, for $s \geq 0$ and $1 < p < +\infty$ (see [Yajima and Zhang 2004, Lemma 2.4]),

$$\|u\|_{\mathcal{W}^{s,p}} = \|H^{s/2}u\|_{L^p} \equiv \|(-\Delta)^{s/2}u\|_{L^p} + \|\langle x\rangle^s u\|_{L^p}. \tag{1-4}$$

Consider a probability space $(\Omega, \mathcal{F}, \boldsymbol{p})$. Throughout the paper, $\{g_n : n \geq 0\}$ and $\{g_{n,k} : n \geq 0,\ 0 \leq k \leq n\}$ are independent standard complex Gaussians $\mathcal{N}_{\mathbb{C}}(0,1)$ (their probability density function is $(1/\pi)e^{-|z|^2}\,dz$, dz being Lebesgue measure on $\mathbb{C}$). If X is a random variable, we denote by $\mathscr{L}(X)$ its law (or distribution).

We will sometimes use the notation $L^p_T = L^p(-T,T)$ for $T > 0$. If E is a Banach space and μ is a measure on E, we write $L^p_\mu = L^p(d\mu)$ and $\|u\|_{L^p_\mu E} = \big\| \|u\|_E \big\|_{L^p_\mu}$. We define $X^\sigma(\mathbb{R}^2) = \bigcap_{\tau<\sigma} \mathcal{H}^\tau(\mathbb{R}^2)$ and, if $I \subset \mathbb{R}$ is an interval, with an abuse of notation we write $\mathscr{C}(I; X^\sigma(\mathbb{R}^2)) = \bigcap_{\tau<\sigma} \mathscr{C}(I; \mathcal{H}^\tau(\mathbb{R}^2))$.

Finally, $\mathbb{N}$ denotes the set of natural integers including 0; $c, C > 0$ denote constants, the value of which may change from line to line. These constants will always be universal or uniformly bounded with respect to the other parameters. For two quantities A and B, we write $A \lesssim B$ if $A \leq CB$ and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

2. Statement of the results

As mentioned above, we will construct strong solutions on the support of Gibbs measures and prove the invariance of such measures. For white noise measures, solutions are weak and belong to the space $C_T X^{-1}$. We start by discussing the former case.

Global strong solutions invariant under Gibbs measure.

Measures and dynamics on the space E_N. The operator H is self-adjoint on $L^2(\mathbb{R}^2)$ and has the discrete spectrum $\{2N+2 : N \in \mathbb{N}\}$. For $N \geq 0$, denote by E_N the eigenspace associated to the eigenvalue $2N+2$. This space has dimension $N+1$. Consider any orthonormal basis $(\varphi_{N,k})_{0\leq k\leq N}$ of E_N. Define $\gamma_N \in L^2(\Omega; E_N)$ by

$$\gamma_N(\omega, x) = \frac{1}{\sqrt{N+1}} \sum_{k=0}^{N} g_{N,k}(\omega)\varphi_{N,k}(x).$$

The distribution of the random variable γ_N does not depend on the choice of the basis, and observe that the law of large numbers gives

$$\|\gamma_N\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{N+1} \sum_{k=0}^{N} |g_{N,k}(\omega)|^2 \to 1 \quad \text{a.s. when } N \to +\infty.$$

Then we define the probability measure $\mu_N = \gamma_{\#}\boldsymbol{p} := \boldsymbol{p} \circ \gamma_N^{-1}$ on E_N.

The L^p properties of the measures μ_N have been studied in [Poiret et al. 2015] with an improvement in [Robert and Thomann 2015]. We mention in particular the following result:

Theorem 2.1 [Poiret et al. 2015; Robert and Thomann 2015]. *There exist $c, C_1, C_2 > 0$ such that, for all $N \geq N_0$,*

$$\mu_N\big[u \in E_N : C_1 N^{-1/2}(\log N)^{1/2}\|u\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^\infty(\mathbb{R}^2)} \leq C_2 N^{-1/2}(\log N)^{1/2}\|u\|_{L^2(\mathbb{R}^2)}\big] \geq 1 - N^{-c}.$$

This proposition is a direct application of [Robert and Thomann 2015, Theorem 3.8] with $h = N^{-1}$ and $d = 2$. Notice that, for all $u \in E_N$, we have $\|u\|_{\mathcal{H}^s} = (2N+2)^{s/2}\|u\|_{L^2}$. The best (deterministic) L^∞ bound for an eigenfunction $u \in E_N$ is given by [Koch and Tataru 2005]:

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq C\|u\|_{L^2(\mathbb{R}^2)}, \tag{2-1}$$

and this estimate is optimal, since it is saturated by the radial Hermite functions. Therefore, the result of Theorem 2.1 shows that there is almost a gain of one derivative compared to the deterministic estimate (2-1).

It turns out that the measures μ_N are invariant under the flow of (CR), and we have the following:

Theorem 2.2. *For all $N \geq 1$, the measure μ_N is invariant under the flow Φ of (CR) restricted to E_N. Therefore, by the Poincaré theorem, μ_N-almost all $u \in E_N$ are recurrent in the following sense: for μ_N-almost all $u_0 \in E_N$ there exists a sequence of times $t_n \to +\infty$ such that*

$$\lim_{n\to+\infty} \|\Phi(t_n)u_0 - u_0\|_{L^2(\mathbb{R}^2)} = 0.$$

In the previous result, one only uses the invariance of the probability measure μ_N under the flow and no additional property of the equation (CR).

Gibbs measure on the space $X^0_\star(\mathbb{R}^2)$ *and a well-posedness result.* In the sequel we either consider the family $(\varphi_n^{\mathrm{rad}})_{n\geq 0}$ of the radial Hermite functions which are eigenfunctions of H associated to the eigenvalue $\lambda_n^{\mathrm{rad}} = 4n+2$, or the family $(\varphi_n^{\mathrm{hol}})_{n\geq 0}$ of the holomorphic Hermite functions which are eigenvalues of H associated to the eigenvalue $\lambda_n^{\mathrm{hol}} = 2n+2$. Set

$$X^0_{\mathrm{rad}}(\mathbb{R}^2) = \bigcap_{\sigma>0} \mathcal{H}^{-\sigma}_{\mathrm{rad}}(\mathbb{R}^2),$$

$$X^0_{\mathrm{hol}}(\mathbb{R}^2) := \Big(\bigcap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R}^2)\Big) \cap (\mathcal{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}).$$

In the following, we write $X^0_\star(\mathbb{R}^2)$ for $X^0_{\mathrm{rad}}(\mathbb{R}^2)$ or $X^0_{\mathrm{hol}}(\mathbb{R}^2)$, $\varphi_n^\star$ for φ_n^{rad} or φ_n^{hol}, etc.

Now define $\gamma_\star \in L^2(\Omega; X^0_\star(\mathbb{R}^2))$ by

$$\gamma_\star(\omega, x) = \sum_{n=0}^{+\infty} \frac{g_n(\omega)}{\sqrt{\lambda_n^\star}} \varphi_n^\star(x)$$

and consider the Gaussian probability measure $\mu_\star = (\gamma_\star)_\# \boldsymbol{p} := \boldsymbol{p} \circ \gamma_\star^{-1}$.

Lemma 2.3. *In each of the previous cases, the measure* $\mu_\star$ *is a probability measure on* $X^0_\star(\mathbb{R}^2)$.

Notice that, since (CR) conserves the $\mathcal{H}^1$ norm, $\mu_\star$ is formally invariant under its flow. More generally, we can define a family $(\rho_{\star,\beta})_{\beta\geq 0}$ of probability measures on $X^0_\star(\mathbb{R}^2)$ which are formally invariant under (CR) in the following way: define, for $\beta \geq 0$, the measure $\rho_\star = \rho_{\star,\beta}$ by

$$d\rho_\star(u) = C_\beta \mathrm{e}^{-\beta\mathcal{E}(u)}\, d\mu_\star(u), \tag{2-2}$$

where $C_\beta > 0$ is a normalizing constant. In Lemma 3.2, we will show that $\mathcal{E}(u) < +\infty$ $\mu_\star$-a.s., which enables us to define this probability measure.

For all $\beta \geq 0$, $\rho_\star(X^0_\star(\mathbb{R}^2)) = 1$ and $\rho_\star(L^2_\star(\mathbb{R}^2)) = 0$.

Remark 2.4. We could also give sense to a generalized version of (2-2) when $\beta < 0$ using the renormalizing method of Lebowitz, Rose and Speer. We do not give the details and refer to [Burq et al. 2013] for such a construction.

We are now able to state the following global existence result:

Theorem 2.5. *Let* $\beta \geq 0$. *There exists a set* $\Sigma \subset X^0_\star(\mathbb{R}^2)$ *of full* $\rho_\star$ *measure such that, for every* $f \in \Sigma$, *the equation* (CR) *with initial condition* $u(0) = f$ *has a unique global solution* $u(t) = \Phi(t) f$ *such that, for any* $0 < s < \frac{1}{2}$,

$$u(t) - f \in \mathcal{C}(\mathbb{R}; \mathcal{H}^s(\mathbb{R}^2)).$$

Moreover, for all $\sigma > 0$ *and* $t \in \mathbb{R}$,

$$\|u(t)\|_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)} \leq C(\Lambda(f,\sigma) + \ln^{1/2}(1+|t|))$$

and the constant $\Lambda(f,\sigma)$ *satisfies the bound* $\mu_\star(f : \Lambda(f,\sigma) > \lambda) \leq Ce^{-c\lambda^2}$.

Furthermore, the measure $\rho_\star$ is invariant under Φ: for any $\rho_\star$-measurable set $A \subset \Sigma$ and any $t \in \mathbb{R}$, $\rho_\star(A) = \rho_\star(\Phi(t)(A))$.

White noise measure on the space $X^{-1}_{\mathrm{hol}}(\mathbb{R}^2)$ and weak solutions. Our aim is now to construct weak solutions on the support of the white noise measure. Consider the Gaussian random variable

$$\gamma(\omega, x) = \sum_{n=0}^{+\infty} g_n(\omega)\varphi_n^{\mathrm{hol}}(x) = \frac{1}{\sqrt{\pi}}\bigg(\sum_{n=0}^{+\infty} \frac{(x_1 + i x_2)^n g_n(\omega)}{\sqrt{n!}}\bigg)\mathrm{e}^{-|x|^2/2} \tag{2-3}$$

and the measure $\mu = \boldsymbol{p} \circ \gamma^{-1}$. As in Lemma 2.3, we can show that the measure μ is a probability measure on

$$X^{-1}_{\mathrm{hol}}(\mathbb{R}^2) := \bigg(\bigcap_{\sigma>1} \mathcal{H}^{-\sigma}(\mathbb{R}^2)\bigg) \cap (\mathbb{O}(\mathbb{C})\mathrm{e}^{-|z|^2/2}).$$

Since $\|u\|_{L^2(\mathbb{R}^2)}$ is preserved by (CR), μ is formally invariant under (CR). We are not able to define a flow at this level of regularity; however, using compactness arguments combined with probabilistic methods, we will construct weak solutions.

Theorem 2.6. *There exists a set $\Sigma \subset X^{-1}_{\mathrm{hol}}(\mathbb{R}^2)$ of full μ measure such that, for every $f \in \Sigma$, the equation (CR) with initial condition $u(0) = f$ has a solution*

$$u \in \bigcap_{\sigma>1} \mathscr{C}(\mathbb{R}; \mathcal{H}^{-\sigma}(\mathbb{R}^2)).$$

The distribution of the random variable $u(t)$ is equal to μ (and thus independent of $t \in \mathbb{R}$):

$$\mathscr{L}_{X^{-1}(\mathbb{R}^2)}(u(t)) = \mathscr{L}_{X^{-1}(\mathbb{R}^2)}(u(0)) = \mu \quad \textit{for all } t \in \mathbb{R}.$$

Remark 2.7. One can also define the Gaussian measure $\mu = \boldsymbol{p} \circ \gamma^{-1}$ on $X^{-1}(\mathbb{R}^2) = \bigcap_{\sigma>1} \mathcal{H}^{-\sigma}(\mathbb{R}^2)$ by

$$\gamma(\omega, x) = \sum_{n=0}^{+\infty} \frac{1}{\sqrt{\lambda_n}} \sum_{k=-n}^{n} g_{n,k}(\omega)\varphi_{n,k}(x), \quad \lambda_n = 2n+2,$$

(where the $\varphi_{n,k}$ are an orthonormal basis of eigenfunctions of the harmonic oscillator and the angular momentum operator). Since $\|u\|_{\mathcal{H}^1(\mathbb{R}^2)}$ is preserved by (CR), μ is formally invariant under (CR), but we are not able to obtain an analogous result in this case.

The same comment holds for the white noise measure $\mu = \boldsymbol{p} \circ \gamma^{-1}$ on $X^{-1}_{\mathrm{rad}}(\mathbb{R}^2) = \bigcap_{\sigma>1} \mathcal{H}^{-\sigma}_{\mathrm{rad}}(\mathbb{R}^2)$ with

$$\gamma(\omega, x) = \sum_{n=0}^{+\infty} g_n(\omega)\varphi_n^{\mathrm{rad}}(x),$$

which is also formally invariant under (CR).

Plan of the paper. The rest of the paper is organized as follows. In Section 3 we prove the results concerning the strong solutions and in Section 4 we construct the weak solutions.

3. Strong solutions

Proof of Theorem 2.2. The proof of Theorem 2.2 is an application of the Liouville theorem. Indeed, write $u_N = \sum_{k=0}^{N} c_{N,k}\varphi_{N,k} \in E_N$; then

$$d\mu_N = \frac{(N+1)^{N+1}}{\pi^{N+1}} \exp\biggl(-(N+1)\sum_{k=0}^{N}|c_{N,k}|^2\biggr)\prod_{k=0}^{N} da_{N,k}\, db_{N,k},$$

where $c_{N,k} = a_{N,k} + i b_{N,k}$.

The Lebesgue measure $\prod_{k=0}^{N} da_{N,k}\, db_{N,k}$ is preserved since (CR) is Hamiltonian and $\sum_{k=0}^{N}|c_{N,k}|^2 = \|u_N\|_{L^2}^2$ is a constant of motion.

Proof of Theorem 2.5. We start with the proof of Lemma 2.3.

Proof of Lemma 2.3. We only consider the case $X_\star^0(\mathbb{R}^2) = X_{\mathrm{hol}}^0(\mathbb{R}^2)$. It is enough to show that $\gamma_{\mathrm{hol}} \in X_{\mathrm{hol}}^0(\mathbb{R}^2)$ $\boldsymbol{p}$-a.s. First, for all $\sigma > 0$, we have

$$\int_\Omega \|\gamma_{\mathrm{hol}}\|_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}^2\, d\boldsymbol{p}(\omega) = \int_\Omega \sum_{n=0}^{+\infty} \frac{|g_n|^2}{(\lambda_n^{\mathrm{hol}})^{\sigma+1}}\, d\boldsymbol{p}(\omega) = C\sum_{n=0}^{+\infty}\frac{1}{(n+1)^{\sigma+1}} < +\infty, \tag{3-1}$$

therefore $\gamma_{\mathrm{hol}} \in \bigcap_{\sigma>0} L^2(\Omega; \mathcal{H}^{-\sigma}(\mathbb{R}^2))$. Next, by [Colliander and Oh 2012, Lemma 3.4], for all $A \geq 1$ there exists a set $\Omega_A \subset \Omega$ such that $\boldsymbol{p}(\Omega_A^c) \leq \exp(-A^\delta)$ and, for all $\omega \in \Omega_A$, $\varepsilon > 0$ and $n \geq 0$,

$$|g_n(\omega)| \leq CA(n+1)^\varepsilon.$$

Then, for $\omega \in \bigcup_{A\geq 1}\Omega_A$, we have $\sum_{n=0}^{+\infty} z^n g_n(\omega)/\sqrt{\lambda_n^{\mathrm{hol}} n!} \in \mathcal{O}(\mathbb{C})$. □

We first define a smooth version of the usual spectral projector. Choose $\chi \in \mathcal{C}_0^\infty(-1,1)$ so that $0 \leq \chi \leq 1$ with $\chi = 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. We define the operators $S_N = \chi(H/\lambda_N)$ as

$$S_N\biggl(\sum_{n=0}^{\infty} c_n\varphi_n^\star\biggr) = \sum_{n=0}^{\infty}\chi\biggl(\frac{\lambda_n^\star}{\lambda_N^\star}\biggr)c_n\varphi_n^\star.$$

Then, for all $1 < p < +\infty$, the operator S_N is bounded in $L^p(\mathbb{R}^2)$ (see [Deng 2012, Proposition 2.1] for a proof).

Local existence. It will be useful to work with an approximation of (CR). We consider the dynamical system given by the Hamiltonian $\mathcal{H}_N(u) := \mathcal{H}(S_N u)$. This system reads

$$\begin{cases} i\partial_t u_N = \mathcal{T}_N(u_N), & (t,x) \in \mathbb{R}\times\mathbb{R}^2, \\ u_N(0,x) = f, \end{cases} \tag{3-2}$$

with $\mathcal{T}_N(u_N) := S_N\mathcal{T}(S_N u, S_N u, S_N u)$. Observe that (3-2) is a finite-dimensional dynamical system on $\bigoplus_{k=0}^{N} E_k$ and that the projection of $u_N(t)$ on its complement is constant. For $\beta \geq 0$ and $N \geq 0$, we define the measures $\rho_\star^N$ by

$$d\rho_\star^N(u) = C_\beta^N \mathrm{e}^{-\beta\mathcal{H}_N(u)}\, d\mu_\star(u),$$

where $C_\beta^N > 0$ is a normalizing constant. We have the following result:

Lemma 3.1. *The system (3-2) is globally well-posed in $L^2(\mathbb{R}^2)$. Moreover, the measures $\rho_\star^N$ are invariant under its flow, denoted by Φ_N.*

Proof. The global existence follows from the conservation of $\|u_N\|_{L^2(\mathbb{R}^2)}$. The invariance of the measures is a consequence of the Liouville theorem and the conservation of $\sum_{k=0}^{\infty} \lambda_k |c_k|^2$ by the flow of (CR) (see [Faou et al. 2013]). We refer to [Burq et al. 2013, Lemma 8.1 and Proposition 8.2] for the details. □

We now state a result concerning dispersive bounds of Hermite functions.

Lemma 3.2. *For all $2 \le p \le +\infty$,*

$$\|\varphi_n^{\text{hol}}\|_{L^p(\mathbb{R}^d)} \le C n^{\frac{1}{2p}-\frac{1}{4}}, \tag{3-3}$$

$$\|\varphi_n^{\text{rad}}\|_{L^4(\mathbb{R}^d)} \le C n^{-\frac{1}{4}} (\ln n)^{\frac{1}{4}}. \tag{3-4}$$

Proof. By Stirling, we easily get that $\|\varphi_n^{\text{hol}}\|_{L^\infty(\mathbb{R}^d)} \le C n^{-1/4}$, which is (3-3) for $p = \infty$; the estimate for $2 \le p \le \infty$ follows by interpolation. For the proof of (3-4), we refer to [Imekraz et al. 2015, Proposition 2.4]. □

Lemma 3.3. *(i) We have*

$\exists C > 0 \ \exists c > 0 \ \forall \lambda \ge 1 \ \forall N \ge 1$

$$\mu_\star\big(u \in X_\star^0(\mathbb{R}^2) : \|e^{-itH} S_N u\|_{L^4([-\pi/4,\pi/4]\times\mathbb{R}^2)} > \lambda\big) \le C e^{-c\lambda^2}. \tag{3-5}$$

(ii) There exists $\beta > 0$ such that

$\exists C > 0 \ \exists c > 0 \ \forall \lambda \ge 1 \ \forall N \ge N_0 \ge 1$

$$\mu_\star\big(u \in X_\star^0(\mathbb{R}^2) : \|e^{-itH}(S_N - S_{N_0})u\|_{L^4([-\pi/4,\pi/4]\times\mathbb{R}^2)} > \lambda\big) \le C e^{-cN_0^\beta \lambda^2}. \tag{3-6}$$

(iii) In the holomorphic case, for all $2 \le p < +\infty$ and $s < \frac{1}{2} - \frac{1}{p}$,

$\exists C > 0 \ \exists c > 0 \ \forall \lambda \ge 1 \ \forall N \ge 1$

$$\begin{aligned} \mu_{\text{hol}}\big(u \in X_{\text{hol}}^0(\mathbb{R}^2) : \|e^{-itH} u\|_{L^p([-\pi/4,\pi/4])\mathcal{W}^{s,p}(\mathbb{R}^2)} > \lambda\big) &\le C e^{-c\lambda^2}, \\ \mu_{\text{hol}}\big(u \in X_{\text{hol}}^0(\mathbb{R}^2) : \|e^{-itH} u\|_{L^{8/3}([-\pi/4,\pi/4]\times\mathbb{R}^2)} > \lambda\big) &\le C e^{-c\lambda^2}. \end{aligned} \tag{3-7}$$

(iv) In the radial case, for all $s < \frac{1}{2}$,

$\exists C > 0 \ \exists c > 0 \ \forall \lambda \ge 1 \ \forall N \ge 1$

$$\mu_{\text{rad}}\big(u \in X_{\text{rad}}^0(\mathbb{R}^2) : \|e^{-itH} u\|_{L^4([-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)} > \lambda\big) \le C e^{-c\lambda^2}. \tag{3-8}$$

Proof. We have that

$$\mu_\star\big(u \in X_\star^0(\mathbb{R}^2) : \|e^{-itH} S_N u\|_{L^4([-\pi/4,\pi/4]\times\mathbb{R}^2)} > \lambda\big) = \boldsymbol{p}\left(\left\|\sum_{n=0}^{\infty} e^{-it\lambda_n} \chi\left(\frac{\lambda_n}{\lambda_N}\right) \frac{g_n(\omega)}{\sqrt{\lambda_n}} \varphi_n^\star(x)\right\|_{L^4([-\pi/4,\pi/4]\times\mathbb{R}^2)} > \lambda\right).$$

Set

$$F(\omega,t,x)\equiv\sum_{n=0}^{\infty}e^{-it\lambda_n^\star}\chi\Big(\frac{\lambda_n^\star}{\lambda_N^\star}\Big)\frac{g_n(\omega)}{\sqrt{\lambda_n^\star}}\varphi_n^\star(x).$$

Let $q\geq p\geq 2$ and $s\geq 0$. Recall here the Khintchine inequality (see, e.g., [Burq and Tzvetkov 2008, Lemma 3.1] for a proof): there exists $C>0$ such that, for all real $k\geq 2$ and $(a_n)\in\ell^2(\mathbb{N})$,

$$\Big\|\sum_{n\geq 0}g_n(\omega)\,a_n\Big\|_{L^k_p}\leq C\sqrt{k}\Big(\sum_{n\geq 0}|a_n|^2\Big)^{\frac{1}{2}} \tag{3-9}$$

if the g_n are i.i.d. normalized Gaussians. Applying it to (3-9) we get

$$\|H^{s/2}F(\omega,t,x)\|_{L^q_\omega}\leq C\sqrt{q}\Big(\sum_{n=0}^{\infty}\chi^2\Big(\frac{\lambda_n^\star}{\lambda_N^\star}\Big)\frac{|\varphi_n^\star(x)|^2}{\lambda_n^{\star\,1-s}}\Big)^{\frac{1}{2}}\leq C\sqrt{q}\Big(\sum_{n=0}^{\infty}\frac{|\varphi_n^\star(x)|^2}{\langle n\rangle^{1-s}}\Big)^{\frac{1}{2}}$$

and using the Minkowski inequality for $q\geq p$ twice gives

$$\|H^{s/2}F(\omega,t,x)\|_{L^q_\omega L^p_{t,x}}\leq\|H^{s/2}F(\omega,t,x)\|_{L^p_{t,x}L^q_\omega}\leq C\sqrt{q}\Big(\sum_{n=0}^{\infty}\frac{\|\varphi_n^\star(x)\|^2_{L^p(\mathbb{R}^2)}}{\langle n\rangle^{1-s}}\Big)^{\frac{1}{2}}. \tag{3-10}$$

We are now ready to prove (3-5). Set $p=4$ and $s=0$. By Lemma 3.2 we have $\|\varphi_n^\star\|_{L^4(\mathbb{R}^2)}\leq Cn^{-1/8}$, so we get, from (3-10),

$$\|F(\omega,t,x)\|_{L^q_\omega L^4_{t,x}}\leq C\sqrt{q}.$$

The Bienaymé–Chebyshev inequality then gives

$$\boldsymbol{p}\big(\|F(\omega,t,x)\|_{L^4_{t,x}}>\lambda\big)\leq(\lambda^{-1}\|F(\omega,t,x)\|_{L^q_\omega L^4_{t,x}})^q\leq(C\lambda^{-1}\sqrt{q})^q.$$

Thus, by choosing $q=\delta\lambda^2\geq 4$, for δ small enough we get the bound

$$\boldsymbol{p}\big(\|F(\omega,t,x)\|_{L^4_{t,x}}>\lambda\big)\leq Ce^{-c\lambda^2},$$

which is (3-5).

For the proof of (3-6), we analyze the function

$$G(\omega,t,x)\equiv\sum_{n=0}^{\infty}e^{-it\lambda_n^\star}\Big(\chi\Big(\frac{\lambda_n^\star}{\lambda_N^\star}\Big)-\chi\Big(\frac{\lambda_n^\star}{\lambda_{N_0}^\star}\Big)\Big)\frac{g_n(\omega)}{\sqrt{\lambda_n^\star}}\varphi_n^\star(x)$$

and we use that a negative power of N_0 can be gained in the estimate. Namely, there is $\gamma>0$ such that

$$\|G(\omega,t,x)\|_{L^q_\omega L^4_{t,x}}\leq C\sqrt{q}N_0^{-\gamma},$$

which implies (3-6).

To prove (3-7)–(3-8), we come back to (3-10) and argue similarly. This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let $\beta \geq 0$. Let $p \in [1, \infty[$; then, when $N \to +\infty$,*

$$C_\beta^N e^{-\beta \mathcal{H}_N(u)} \to C_\beta e^{-\beta \mathcal{H}(u)} \quad \text{in } L^p(d\mu_\star(u)).$$

In particular, for all measurable sets $A \subset X_\star^0(\mathbb{R}^2)$,

$$\rho_\star^N(A) \to \rho_\star(A).$$

Proof. Let $G_\beta^N(u) = \mathrm{e}^{-\beta \mathcal{H}_N(u)}$ and $G_\beta(u) = \mathrm{e}^{-\beta \mathcal{H}(u)}$. By (3-6), we deduce that $\mathcal{H}_N(u) \to \mathcal{H}(u)$ in measure with respect to $\mu_\star$. In other words, for $\varepsilon > 0$ and $N \geq 1$, we let

$$A_{N,\varepsilon} = \{u \in X_\star^0(\mathbb{R}^2) : |G_\beta^N(u) - G_\beta(u)| \leq \varepsilon\},$$

then $\mu_\star(A_{N,\varepsilon}^c) \to 0$ when $N \to +\infty$. Since $0 \leq G, G_N \leq 1$,

$$\begin{aligned} \|G_\beta - G_\beta^N\|_{L^p_{\mu_\star}} &\leq \|(G_\beta - G_\beta^N)\mathbf{1}_{A_{N,\varepsilon}}\|_{L^p_{\mu_\star}} + \|(G_\beta - G_\beta^N)\mathbf{1}_{A_{N,\varepsilon}^c}\|_{L^p_{\mu_\star}} \\ &\leq \varepsilon(\mu_\star(A_{N,\varepsilon}))^{1/p} + 2(\mu_\star(A_{N,\varepsilon}^c))^{1/p} \\ &\leq C\varepsilon \end{aligned}$$

for N large enough. Finally, we have, when $N \to +\infty$,

$$C_\beta^N = \left(\int \mathrm{e}^{-\beta \mathcal{H}_N(u)}\, d\mu_\star(u)\right)^{-1} \to \left(\int \mathrm{e}^{-\beta \mathcal{H}(u)}\, d\mu_\star(u)\right)^{-1} = C_\beta,$$

and this ends the proof. □

We look for a solution to (CR) of the form $u = f + v$; thus v has to satisfy

$$\begin{cases} i\partial_t v = \mathcal{T}(f + v), & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ v(0, x) = 0, \end{cases} \tag{3-11}$$

with $\mathcal{T}(u) = \mathcal{T}(u, u, u)$. Similarly, we introduce

$$\begin{cases} i\partial_t v_N = \mathcal{T}_N(f + v_N), & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ v(0, x) = 0. \end{cases} \tag{3-12}$$

Recall that $X_\star^0(\mathbb{R}^2)$ equals $X_{\text{hol}}^0(\mathbb{R}^2)$ or $X_{\text{rad}}^0(\mathbb{R}^2)$. Define the sets, for $s < \frac{1}{2}$,

$$A_{\text{rad}}^s(D) = \{f \in X_{\text{rad}}^0(\mathbb{R}^2) : \|\mathrm{e}^{-itH} f\|_{L^4([-\pi/4, \pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)} \leq D\};$$

choosing $p(s) = 4/(1 - 2s)$, so that $s < \frac{1}{2} - \frac{1}{p}$,

$$\begin{aligned} &A_{\text{hol}}^s(D) \\ &\quad = \{f \in X_{\text{hol}}^0(\mathbb{R}^2) : \|\mathrm{e}^{-itH} f\|_{L^{8/3}([-\pi/4, \pi/4])L^{8/3}(\mathbb{R}^2)} + \|\mathrm{e}^{-itH} f\|_{L^{p(s)}([-\pi/4, \pi/4])\mathcal{W}^{s,p(s)}(\mathbb{R}^2)} \leq D\}. \end{aligned}$$

In the sequel, we write $A_\star^s(D)$ for $A_{\text{hol}}^s(D)$ or $A_{\text{rad}}^s(D)$. Then we have the following result:

Lemma 3.5. *Let $\beta \geq 0$. There exist $c, C > 0$ such that, for all $N \geq 0$,*

$$\rho_\star^N(A_\star^s(D)^c) \leq Ce^{-cD^2}, \quad \rho_\star(A_\star^s(D)^c) \leq Ce^{-cD^2} \quad \text{and} \quad \mu_\star(A_\star^s(D)^c) \leq Ce^{-cD^2}.$$

Proof. Since $\beta \geq 0$, we have $\rho_\star^N(A_\star^s(D)^c), \rho_\star(A_\star^s(D)^c) \leq C\mu_\star(A_\star^s(D)^c)$. The result is therefore given by (3-7) and (3-8). □

Proposition 3.6. *Let $s < \frac{1}{2}$. There exists $c > 0$ such that, for any $D \geq 0$, setting $\tau(D) = cD^{-2}$, for any $f \in A_\star^s(D)$ there exists a unique solution $v \in L^\infty([-\tau,\tau]; L^2(\mathbb{R}^2))$ to the equation (3-11) and a unique solution $v_N \in L^\infty([-\tau,\tau]; L^2(\mathbb{R}^2))$ to the equation (3-12), which furthermore satisfy*

$$\|v\|_{L^\infty([-\tau,\tau];\mathcal{H}^s(\mathbb{R}^2))}, \ \|v_N\|_{L^\infty([-\tau,\tau];\mathcal{H}^s(\mathbb{R}^2))} \leq D.$$

The key ingredient in the proof of this result is the following trilinear estimate:

Lemma 3.7. *Assume that, for $1 \leq j \leq 3$ and $1 \leq k \leq 4$, $(p_{jk}, q_{jk}) \in [2,+\infty[^2$ are Strichartz admissible pairs, that is, they satisfy*

$$\frac{1}{q_{jk}} + \frac{1}{p_{jk}} = \frac{1}{2},$$

and they are such that, for $1 \leq j \leq 4$,

$$\frac{1}{p_{j1}} + \frac{1}{p_{j2}} + \frac{1}{p_{j3}} + \frac{1}{p_{j4}} = \frac{1}{q_{j1}} + \frac{1}{q_{j2}} + \frac{1}{q_{j3}} + \frac{1}{q_{j4}} = 1.$$

Then, for all $s \geq 0$, there exists $C > 0$ such that

$$\begin{aligned}\|\mathcal{T}(u_1,u_2,u_3)\|_{\mathcal{H}^s(\mathbb{R}^2)} \leq{}& C\|e^{-itH}u_1\|_{L^{p_{11}}\mathcal{W}^{s,q_{11}}}\|e^{-itH}u_2\|_{L^{p_{12}}L^{q_{12}}}\|e^{-itH}u_3\|_{L^{p_{13}}L^{q_{13}}}\\ &+ C\|e^{-itH}u_1\|_{L^{p_{21}}L^{q_{21}}}\|e^{-itH}u_2\|_{L^{p_{22}}\mathcal{W}^{s,q_{22}}}\|e^{-itH}u_3\|_{L^{p_{23}}L^{q_{23}}}\\ &+ C\|e^{-itH}u_1\|_{L^{p_{31}}L^{q_{31}}}\|e^{-itH}u_2\|_{L^{p_{32}}L^{q_{32}}}\|e^{-itH}u_3\|_{L^{p_{33}}\mathcal{W}^{s,q_{33}}},\end{aligned}$$

with the notation $L^p\mathcal{W}^{s,q} = L^p([-\pi/4,\pi/4]; \mathcal{W}^{s,q}(\mathbb{R}^2))$.

Proof. By duality,

$$\begin{aligned}&\|\mathcal{T}(u_1,u_2,u_3)\|_{\mathcal{H}^s(\mathbb{R}^2)}\\ &\quad= \sup_{\|u\|_{L^2(\mathbb{R}^2)}=1} \langle H^{s/2}\mathcal{T}(u_1,u_2,u_3), u\rangle_{L^2(\mathbb{R}^2)}\\ &\quad= 2\pi \sup_{\|u\|_{L^2(\mathbb{R}^2)}=1} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{\mathbb{R}^2} H^{s/2}\big((\mathrm{e}^{-itH}u_1)(\mathrm{e}^{-itH}u_2)(\overline{\mathrm{e}^{-itH}u_3})\big)(\overline{\mathrm{e}^{-itH}u})\,dx\,dt.\end{aligned}$$

Then, by Strichartz, for all u of unit norm in L^2 and for any admissible pair (p_4, q_4),

$$\begin{aligned}\|\mathcal{T}(u_1,u_2,u_3)\|_{\mathcal{H}^s(\mathbb{R}^2)} &\leq C\|(\mathrm{e}^{-itH}u_1)(\mathrm{e}^{-itH}u_2)(\overline{\mathrm{e}^{-itH}u_3})\|_{L^{p_4'}\mathcal{W}^{s,q_4'}}\|\mathrm{e}^{-itH}u\|_{L^{p_4}L^{q_4}}\\ &\leq C\|(\mathrm{e}^{-itH}u_1)(\mathrm{e}^{-itH}u_2)(\overline{\mathrm{e}^{-itH}u_3})\|_{L^{p_4'}\mathcal{W}^{s,q_4'}}.\end{aligned}$$

We then conclude using (1-4) and applying the following lemma twice. □

We have the following product rule:

Lemma 3.8. *Let $s \geq 0$, then*

$$\|u\, v\|_{\mathcal{W}^{s,q}} \leq C\|u\|_{L^{q_1}}\|v\|_{\mathcal{W}^{s,q_1'}} + C\|v\|_{L^{q_2}}\|u\|_{\mathcal{W}^{s,q_2'}}$$

for $1 < q < \infty$, $1 < q_1, q_2 < \infty$ and $1 \leq q_1', q_2' < \infty$ such that

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_1'} = \frac{1}{q_2} + \frac{1}{q_2'}.$$

For the proof with the usual Sobolev spaces, we refer to [Taylor 2000, Proposition 1.1, p. 105]. The result in our context follows by using (1-4).

Proof of Proposition 3.6. We only consider (3-11), the other case being similar by the boundedness of S_N on $L^p(\mathbb{R}^2)$. For $s < \frac{1}{2}$, we define the space

$$Z^s(\tau) = \bigl\{v \in \mathscr{C}([-\tau,\tau]; \mathscr{H}^s(\mathbb{R}^2)) : v(0) = 0 \text{ and } \|v\|_{Z^s(\tau)} \leq D\bigr\},$$

with $\|v\|_{Z^s(\tau)} = \|v\|_{L^\infty_{[-\tau,\tau]}\mathscr{H}^s(\mathbb{R}^2)}$ and, for $f \in A^s_\star(D)$, we define the operator

$$K(v) = -i\int_0^t \mathcal{T}(f+v)\,ds.$$

We will show that K has a unique fixed point $v \in Z^s(\tau)$.

<u>The case of radial Hermite functions</u>: By Lemma 3.7 with $(p_{jk}, q_{jk}) = (4,4)$, we have, for all $v \in Z^s(\tau)$,

$$\begin{aligned}\|K(v)\|_{Z^s(\tau)} &\leq \tau\|\mathcal{T}(f+v)\|_{Z^s(\tau)} \\ &\leq C\tau\Bigl\|\|e^{-isH}(f+v)(t)\|^3_{L^4(s\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)}\Bigr\|_{L^\infty_{t\in[-\tau,\tau]}}.\end{aligned} \tag{3-13}$$

Next, by Strichartz and since $v \in Z^s(\tau)$,

$$\begin{aligned}&\|e^{-isH}(f+v)(t)\|_{L^4(s\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)} \\ &\qquad\leq \|e^{-isH}f\|_{L^4(s\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)} + \|e^{-isH}v(t)\|_{L^4(s\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)} \\ &\qquad\leq C(D + \|v(t)\|_{\mathscr{H}^s(\mathbb{R}^2)}) \\ &\qquad\leq 2CD.\end{aligned}$$

Therefore, from (3-13) we deduce

$$\|K(v)\|_{Z^s(\tau)} \leq C\tau D^3,$$

which implies that K maps $Z^s(\tau)$ into itself when $\tau \leq cD^{-2}$ for $c > 0$ small enough.

Similarly, for $v_1, v_2 \in Z^s(\tau)$, we have the bound

$$\|K(v_2) - K(v_1)\|_{Z^s(\tau)} \leq C\tau D^2\|v_2 - v_1\|_{Z^s(\tau)}, \tag{3-14}$$

which shows that if $\tau \leq cD^{-2}$ then K is a contraction of $Z^s(\tau)$. The Picard fixed point theorem gives the desired result.

The case of holomorphic Hermite functions: For $s < \frac{1}{2}$, recall that we set $p = p(s) = 4/(1-2s)$, so that $s < \frac{1}{2} - \frac{1}{p}$. We have

$$\begin{aligned}\|K(v)\|_{Z^s(\tau)} &\le \tau \|\mathcal{T}(f+v)\|_{Z^s(\tau)} \\ &\le C\tau\big(\|\mathcal{T}(f,f,f)\|_{Z^s} + \|\mathcal{T}(f,f,v)\|_{Z^s} + \|\mathcal{T}(f,v,v)\|_{Z^s} + \|\mathcal{T}(v,v,v)\|_{Z^s}\big).\end{aligned}$$

We estimate each term, thanks to Lemma 3.7 and Strichartz. The conjugation plays no role, so we forget it.

For the trilinear term in v,

$$\|\mathcal{T}(v,v,v)\|_{\mathcal{H}^s} \le C\|e^{-it'H}v\|^3_{L^4_{t'\in[-\pi/4,\pi/4]}\mathcal{W}^{s,4}(\mathbb{R}^2)} \le C\|v\|^3_{\mathcal{H}^s(\mathbb{R}^2)}.$$

For the quadratic term in v, for $\delta > 0$ such that $2/\big(\frac{8}{3}+\delta\big) + \frac{1}{p} + \frac{1}{4} = 1$,

$$\begin{aligned}\|\mathcal{T}(v,v,f)\|_{\mathcal{H}^s} &\le C\|e^{-it'H}v\|^2_{L^{8/3+\delta}(t'\in[-\pi/4,\pi/4])L^{8/3+\delta}(\mathbb{R}^2)}\|e^{-it'H}f\|_{L^p(t'\in[-\pi/4,\pi/4])\mathcal{W}^{s,p}(\mathbb{R}^2)} \\ &\quad + \|e^{-it'H}v\|_{L^4(t'\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)}\|e^{-it'H}v\|_{L^4(t'\in[-\pi/4,\pi/4])L^4(\mathbb{R}^2)} \\ &\qquad \times \|e^{-it'H}f\|_{L^4(t'\in[-\pi/4,\pi/4])L^4(\mathbb{R}^2)} \\ &\le CD\|v\|^2_{\mathcal{H}^s(\mathbb{R}^2)}.\end{aligned}$$

For the linear term in v, with the same δ as above,

$$\begin{aligned}\|\mathcal{T}(v,f,f)\|_{\mathcal{H}^s} &\le C\|e^{-it'H}v\|_{L^{8/3+\delta}(t'\in[-\pi/4,\pi/4])L^{8/3+\delta}(\mathbb{R}^2)}\|e^{-it'H}f\|_{L^{8/3+\delta}(t'\in[-\pi/4,\pi/4])L^{8/3+\delta}(\mathbb{R}^2)} \\ &\qquad \times \|e^{-it'H}f\|_{L^p(t'\in[-\pi/4,\pi/4])\mathcal{W}^{s,p}(\mathbb{R}^2)} \\ &\quad + \|e^{-it'H}v\|_{L^4(t'\in[-\pi/4,\pi/4])\mathcal{W}^{s,4}(\mathbb{R}^2)}\|e^{-it'H}f\|^2_{L^4(t'\in[-\pi/4,\pi/4])L^4(\mathbb{R}^2)} \\ &\le CD^2\|v\|_{\mathcal{H}^s(\mathbb{R}^2)}.\end{aligned}$$

For the constant term in v,

$$\begin{aligned}\|\mathcal{T}(f,f,f)\|_{\mathcal{H}^s} &\le C\|e^{-it'H}f\|^2_{L^{8/3+\delta}(t'\in[-\pi/4,\pi/4])L^{8/3+\delta}(\mathbb{R}^2)}\|e^{-it'H}f\|_{L^p(t'\in[-\pi/4,\pi/4])\mathcal{W}^{s,p}(\mathbb{R}^2)} \\ &\le CD^3.\end{aligned}$$

With these estimates at hand, the result follows by the Picard fixed point theorem. □

Approximation and invariance of the measure.

Lemma 3.9. *Fix $D \ge 0$ and $s < \frac{1}{2}$. Then, for all $\varepsilon > 0$, there exists $N_0 \ge 0$ such that, for all $f \in A^s_\star(D)$ and $N \ge N_0$,*

$$\|\Phi(t)f - \Phi_N(t)f\|_{L^\infty([-\tau_1,\tau_1];\mathcal{H}^s(\mathbb{R}^2))} \le \varepsilon,$$

where $\tau_1 = cD^{-2}$ for some $c > 0$.

Proof. Denoting for simplicity $\mathcal{T}(f) = \mathcal{T}(f,f,f)$,

$$v - v_N = -i\int_0^t \big[S_N(\mathcal{T}(f+v) - \mathcal{T}(f+v_N)) + (1-S_N)\mathcal{T}(f+v)\big]\,ds.$$

As in (3-14), we get

$$\|v - v_N\|_{Z^s(\tau)} \le C\tau D^2 \|v - v_N\|_{Z^s(\tau)} + \int_{-\tau}^{\tau} \|(1 - S_N)\mathcal{T}(f + v)\|_{\mathcal{H}^s(\mathbb{R}^2)}\, ds,$$

which in turn implies, when $C\tau D^2 \le \frac{1}{2}$,

$$\|v - v_N\|_{Z^s(\tau)} \le 2 \int_{-\tau}^{\tau} \|(1 - S_N)\mathcal{T}(f + v)\|_{\mathcal{H}^s(\mathbb{R}^2)}\, ds.$$

Choose $\eta > 0$ so that $s + \eta < \frac{1}{2}$. Then, by the proof of Proposition 3.6, $\|\mathcal{T}(f + v)\|_{L^\infty_{[-\tau,\tau]}\mathcal{H}^{s+\eta}(\mathbb{R}^2)} \le CD^3$ if $\tau \le c_0 D^{-2}$ and, therefore, there exists $N_0 = N_0(\varepsilon, D)$ which satisfies the claim. □

In the next result, we summarize the results obtained by de Suzzoni [2011, Sections 3.3 and 4]. Since the proofs are very similar in our context, we skip them.

Let $D_{i,j} = (i + j^{1/2})^{1/2}$, with $i, j \in \mathbb{N}$, and set $T_{i,j} = \sum_{\ell=1}^{j} \tau_1(D_{i,\ell})$. Let

$$\Sigma_{N,i} := \{f : \Phi_N(\pm T_{i,j}) f \in A^s_\star(D_{i,j+1}) \text{ for all } j \in \mathbb{N}\}$$

and

$$\Sigma_i := \limsup_{N \to +\infty} \Sigma_{N,i}, \qquad \Sigma := \bigcup_{i \in \mathbb{N}} \Sigma_i.$$

Proposition 3.10. *Let* $\beta \ge 0$; *then*:

(i) The set Σ *is of full* $\rho_\star$ *measure.*

(ii) For all $f \in \Sigma$, *there exists a unique global solution* $u = f + v$ *to* (CR). *This define a global flow* Φ *on* Σ

(iii) For all measurable set $A \subset \Sigma$ *and all* $t \in \mathbb{R}$,

$$\rho_\star(A) = \rho_\star(\Phi(t)(A)).$$

4. Weak solutions: proof of Theorem 2.6

Definition of $\mathcal{T}(u,u,u)$ ***on the support of*** μ. For $N \ge 0$, denote by Π_N the orthogonal projector on the space $\bigoplus_{k=0}^{N} E_k$ (in this section, we do not need the smooth cut-offs S_N). In the sequel, we write $\mathcal{T}(u) = \mathcal{T}(u,u,u)$ and $\mathcal{T}_N(u) = \Pi_N \mathcal{T}(\Pi_N u, \Pi_N u, \Pi_N u)$.

Proposition 4.1. *For all* $p \ge 2$ *and all* $\sigma > 1$, *the sequence* $(\mathcal{T}_N(u))_{N \ge 1}$ *is a Cauchy sequence in* $L^p(X^{-1}(\mathbb{R}^2), \mathcal{B}, d\mu; \mathcal{H}^{-\sigma}(\mathbb{R}^2))$. *Namely, for all* $p \ge 2$, *there exist* $\delta > 0$ *and* $C > 0$ *such that, for all* $1 \le M < N$,

$$\int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(u) - \mathcal{T}_M(u)\|^p_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}\, d\mu(u) \le C M^{-\delta}.$$

We denote by $\mathcal{T}(u) = \mathcal{T}(u,u,u)$ *the limit of this sequence and we have, for all* $p \ge 2$,

$$\|\mathcal{T}(u)\|_{L^p_\mu \mathcal{H}^{-\sigma}(\mathbb{R}^2)} \le C_p. \tag{4-1}$$

Before we turn to the proof of Proposition 4.1, let us state two elementary results which will be needed in the sequel.

Lemma 4.2. *For all $n \in \mathbb{N}$,*

$$\sum_{k=n}^{+\infty} \frac{1}{2^k}\binom{k}{n} = \sum_{k=n}^{+\infty} \frac{k!}{2^k n!(k-n)!} = 2.$$

Proof. For $|z| < 1$ we have $1/(1-z) = \sum_{k=0}^{+\infty} z^k$. If we differentiate this formula n times we get

$$\frac{n!}{(1-z)^{n+1}} = \sum_{k=n}^{+\infty} \frac{k!}{(k-n)!} z^{k-n},$$

which implies the result, taking $z = \frac{1}{2}$. □

Lemma 4.3. *Choose $0 < \varepsilon < 1$ and $p, L \geq 1$ so that $p \leq L^\varepsilon$. Then*

$$\frac{L!}{2^L (L-p)!} \leq C 2^{-L/2}.$$

Proof. The proof is straightforward. By the assumption $p \leq L^\varepsilon$,

$$\frac{L!}{(L-p)!} \leq L^p \leq C 2^{L/2},$$

which was the claim. □

Proof of Proposition 4.1. By the result [Thomann and Tzvetkov 2010, Proposition 2.4] on the Wiener chaos, we only have to prove the statement for $p = 2$.

Firstly, by definition of the measure μ,

$$\int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(u) - \mathcal{T}_M(u)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}\, d\mu(u) = \int_\Omega \|\mathcal{T}_N(\gamma(\omega)) - \mathcal{T}_M(\gamma(\omega))\|^2_{\mathcal{H}^{-\sigma}(\mathbb{R}^2)}\, d\boldsymbol{p}(\omega).$$

Therefore, it is enough to prove that $(\mathcal{T}_N(\gamma))_{N\geq 1}$ is a Cauchy sequence in $L^2(\Omega; \mathcal{H}^{-\sigma}(\mathbb{R}^2))$. Let $1 \leq M < N$ and fix $\alpha > \frac{1}{2}$. By (1-2), we get

$$\begin{aligned}
H^{-\alpha}\mathcal{T}_N(\gamma) &= \frac{1}{2^\alpha} \sum_{A_N} \frac{g_{n_1} g_{n_2} \bar{g}_{n_3}}{(n_1+n_2-n_3+1)^\alpha} \mathcal{T}(\varphi^{\text{hol}}_{n_1}, \varphi^{\text{hol}}_{n_2}, \varphi^{\text{hol}}_{n_3}) \\
&= \frac{\pi}{8\cdot 2^\alpha} \sum_{A_N} \frac{(n_1+n_2)!}{2^{n_1+n_2}\sqrt{n_1!n_2!n_3!(n_1+n_2-n_3)!}} \frac{g_{n_1} g_{n_2} \bar{g}_{n_3}}{(n_1+n_2-n_3+1)^\alpha} \varphi^{\text{hol}}_{n_1+n_2-n_3} \\
&= \frac{\pi}{8\cdot 2^\alpha} \sum_{p=0}^{N} \frac{1}{(p+1)^\alpha} \Bigg(\sum_{A_N^{(p)}} \frac{(n_1+n_2)!}{2^{n_1+n_2}\sqrt{n_1!n_2!n_3!\,p!}} g_{n_1} g_{n_2} \bar{g}_{n_3} \Bigg) \varphi^{\text{hol}}_p
\end{aligned}$$

with

$$\begin{aligned}
A_N &= \{n \in \mathbb{N}^3 : 0 \leq n_j \leq N,\ 0 \leq n_1+n_2-n_3 \leq N\}, \\
A_N^{(p)} &= \{n \in \mathbb{N}^3 : 0 \leq n_j \leq N,\ n_1+n_2-n_3 = p\} \quad \text{if } 0 \leq p \leq N.
\end{aligned}$$

Therefore,

$$\|\mathcal{T}_N(\gamma)-\mathcal{T}_M(\gamma)\|^2_{\mathcal{H}^{-\alpha}(\mathbb{R}^2)} = \frac{\pi^2}{64\cdot 2^{2\alpha}}\sum_{p=0}^{N}\frac{1}{(p+1)^{2\alpha}}\sum_{(n,m)\in A^{(p)}_{M,N}\times A^{(p)}_{M,N}}\frac{(n_1+n_2)!(m_1+m_2)!g_{n_1}g_{n_2}\bar g_{n_3}\bar g_{m_1}g_{m_2}g_{m_3}}{2^{n_1+n_2}2^{m_1+m_2}p!\sqrt{n_1!n_2!n_3!}\sqrt{m_1!m_2!m_3!}},$$

where $A^{(p)}_{M,N}$ is the set defined by

$$A^{(p)}_{M,N}=\big\{n\in\mathbb{N}^3: 0\le n_j\le N,\ n_1+n_2-n_3=p\in\{0,\dots,N\}\ \text{and}\ \max\{n_1,n_2,n_3,p\}>M\big\}.$$

Now we take the integral over Ω. Since $(g_n)_{n\ge 0}$ are independent and centered Gaussians, we deduce that each term in the right-hand side vanishes unless one of two cases holds:

Case 1: $(n_1,n_2,n_3)=(m_1,m_2,m_3)$ or $(n_1,n_2,n_3)=(m_2,m_1,m_3)$.

Case 2: We have one of

$$(n_1,n_2,m_1)=(n_3,m_2,m_3),\quad (n_1,n_2,m_2)=(n_3,m_1,m_3),$$
$$(n_1,n_2,m_3)=(m_1,n_3,m_2),\quad (n_1,n_2,m_3)=(m_2,n_3,m_1).$$

We write

$$\int_\Omega \|\mathcal{T}_N(\gamma)-\mathcal{T}_M(\gamma)\|^2_{\mathcal{H}^{-2\alpha}(\mathbb{R}^2)}\,d\boldsymbol{p}=J_1+J_2,$$

where J_1 and J_2 correspond to the contribution in the sum of each of cases 1 and 2, respectively.

Contribution in case 1: By symmetry, we can assume that $(n_1,n_2,n_3)=(m_1,m_2,m_3)$. Define

$$B^{(p)}_{M,N}=\big\{n\in\mathbb{N}^2: 0\le n_j\le N\ \text{and}\ \max\{n_1,n_2,n_1+n_2-p,p\}>M\big\}.$$

Then

$$J_1\le C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\sum_{B^{(p)}_{M,N}}\frac{((n_1+n_2)!)^2}{2^{2(n_1+n_2)}p!n_1!n_2!(n_1+n_2-p)!}.$$

In the previous sum, we make the change of variables $L=n_1+n_2$ and we observe that on $B^{(p)}_{M,N}$ we have $L\ge M$; then

$$J_1\le C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\sum_{L\ge p+M}\sum_{n_1=0}^{L}\frac{(L!)^2}{2^{2L}p!n_1!(L-n_1)!(L-p)!}$$
$$=C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\sum_{L\ge p+M}\frac{L!}{2^L p!(L-p)!},$$

where we used the fact that $\sum_{n_1=0}^{L}\binom{L}{n_1}=2^L$. Let $\varepsilon>0$ and split the previous sum into two pieces:

$$\begin{aligned}J_1 &\le C\sum_{p=0}^{M^\varepsilon}\frac{1}{(1+p)^{2\alpha}}\sum_{L=M}^{+\infty}\frac{L!}{2^L p!(L-p)!}+C\sum_{p=M^\varepsilon+1}^{+\infty}\frac{1}{(1+p)^{2\alpha}}\sum_{L=p}^{+\infty}\frac{L!}{2^L p!(L-p)!}\\ &\le C\sum_{p=0}^{M^\varepsilon}\frac{1}{(1+p)^{2\alpha}}\sum_{L=M}^{+\infty}\frac{L!}{2^L p!(L-p)!}+2C\sum_{p=M^\varepsilon+1}^{+\infty}\frac{1}{(1+p)^{2\alpha}}=:J_{11}+J_{12},\end{aligned}$$

by Lemma 4.2. For the first sum, we can use Lemma 4.3, since $p\le M^\varepsilon\le L^\varepsilon$; thus

$$J_{11}\le C\sum_{p=0}^{M^\varepsilon}\frac{1}{(1+p)^{2\alpha}p!}\sum_{L=M}^{+\infty}\frac{1}{2^{L/2}}\le C\sum_{L=M}^{+\infty}\frac{1}{2^{L/2}}\le CM^{-\delta}.$$

Next, clearly, $J_{12}\le CM^{-\delta}$ because $\alpha>\frac{1}{2}$, and this gives $J_1\le CM^{-\delta}$.

Contribution in case 2: We can assume that $(n_1,n_2,m_1)=(n_3,m_2,m_3)$. Then, for $n,m\in A_{M,N}^{(p)}$, we have $n_2=m_2=p$. Moreover, by symmetry, we can assume that $n_1>M$ or $p>M$. Thus,

$$\begin{aligned}J_2\le C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}&\sum_{n_1=M+1}^{+\infty}\sum_{m_1=0}^{+\infty}\frac{(n_1+p)!(m_1+p)!}{2^{n_1+p}2^{m_1+p}n_1!m_1!(p!)^2}\\ &+C\sum_{p\ge M+1}\frac{1}{(1+p)^{2\alpha}}\sum_{n_1=0}^{+\infty}\sum_{m_1=0}^{+\infty}\frac{(n_1+p)!(m_1+p)!}{2^{n_1+p}2^{m_1+p}n_1!m_1!(p!)^2}=:J_{21}+J_{22}.\end{aligned}$$

To begin with, by Lemma 4.2, we have

$$\begin{aligned}J_{22}&=C\sum_{p\ge M+1}\frac{1}{(1+p)^{2\alpha}}\Big(\sum_{n_1=0}^{+\infty}\frac{(n_1+p)!}{2^{n_1+p}n_1!\,p!}\Big)\Big(\sum_{m_1=0}^{+\infty}\frac{(m_1+p)!}{2^{m_1+p}m_1!\,p!}\Big)\\ &=4C\sum_{p\ge M+1}\frac{1}{(1+p)^{2\alpha}}\le cM^{-\delta}.\end{aligned}$$

Then, by Lemma 4.2 again,

$$\begin{aligned}J_{21}&=C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\Big(\sum_{n_1=M+1}^{+\infty}\frac{(n_1+p)!}{2^{n_1+p}n_1!\,p!}\Big)\Big(\sum_{m_1=0}^{+\infty}\frac{(m_1+p)!}{2^{m_1+p}m_1!\,p!}\Big)\\ &=2C\sum_{p\ge 0}\frac{1}{(1+p)^{2\alpha}}\Big(\sum_{n_1=M+1}^{+\infty}\frac{(n_1+p)!}{2^{n_1+p}n_1!\,p!}\Big)\\ &=2C\sum_{p=0}^{M^\varepsilon}\frac{1}{(1+p)^{2\alpha}}\Big(\sum_{n_1=M+1}^{+\infty}\frac{(n_1+p)!}{2^{n_1+p}n_1!\,p!}\Big)+2C\sum_{p=M^\varepsilon+1}^{+\infty}\frac{1}{(1+p)^{2\alpha}}\Big(\sum_{n_1=M+1}^{+\infty}\frac{(n_1+p)!}{2^{n_1+p}n_1!\,p!}\Big)\\ &=:K_1+K_2.\end{aligned}$$

On the one hand, by Lemma 4.3,

$$K_1 \leq C\Big(\sum_{p=0}^{M^\varepsilon} \frac{1}{(1+p)^{2\alpha}p!}\Big)\Big(\sum_{n_1=M+1}^{+\infty} 2^{-n_1/2}\Big) \leq CM^{-\delta}$$

and, on the other hand, by Lemma 4.2, since $\alpha > \frac{1}{2}$,

$$K_2 \leq C \sum_{p=M^\varepsilon+1}^{+\infty} \frac{1}{(1+p)^{2\alpha}} \leq CM^{-\delta}.$$

Putting all the estimates together, we get $J_2 \leq CM^{-\delta}$, which concludes the proof. □

Study of the measure ν_N. Let $N \geq 1$. We then consider the following approximation of (CR):

$$\begin{cases} i\partial_t u = \mathcal{T}_N(u), & (t,x) \in \mathbb{R}\times\mathbb{R}^2, \\ u(0,x) = f(x) \in X^{-1}(\mathbb{R}^2). \end{cases} \tag{4-2}$$

The equation (4-2) is an ODE in the frequencies less than N and $(1-\Pi_N)u(t) = (1-\Pi_N)f$ for all $t \in \mathbb{R}$.

The main reason to introduce this system is the following proposition, whose proof we omit.

Proposition 4.4. *The equation (4-2) has a global flow Φ_N. Moreover, the measure μ is invariant under Φ_N: for any Borel set $A \subset X^{-1}(\mathbb{R}^2)$ and for all $t \in \mathbb{R}$, $\mu(\Phi_N(t)(A)) = \mu(A)$.*

In particular, if $\mathscr{L}_{X^{-1}}(v) = \mu$ then, for all $t \in \mathbb{R}$, $\mathscr{L}_{X^{-1}}(\Phi_N(t)v) = \mu$.

We denote by ν_N the measure on $\mathscr{C}([-T,T]; X^{-1}(\mathbb{R}^2))$, defined as the image measure of μ under the map

$$X^{-1}(\mathbb{R}^2) \to \mathscr{C}([-T,T]; X^{-1}(\mathbb{R}^2)),$$
$$v \mapsto \Phi_N(t)(v).$$

Lemma 4.5. *Let $\sigma > 1$ and $p \geq 2$. Then there exists $C > 0$ such that, for all $N \geq 1$,*

$$\big\| \|u\|_{W_T^{1,p}\mathscr{H}_x^{-\sigma}} \big\|_{L^p_{\nu_N}} \leq C.$$

Proof. Firstly, we have that, for $\sigma > 1$, $p \geq 2$ and $N \geq 1$,

$$\big\| \|u\|_{L_T^{p}\mathscr{H}_x^{-\sigma}} \big\|_{L^p_{\nu_N}} \leq C.$$

Indeed, by the definition of ν_N and the invariance of μ under Φ_N, we have

$$\|u\|_{L^p_{\nu_N} L^p_T \mathscr{H}_x^{-\sigma}} = (2T)^{1/p}\|v\|_{L^p_\mu \mathscr{H}_x^{-\sigma}} = (2T)^{1/p}\|\gamma\|_{L^p_{\boldsymbol{p}}\mathscr{H}_x^{-\sigma}}.$$

Then, by the Khintchine inequality (3-9) and (3-1), for all $p \geq 2$,

$$\|\gamma\|_{L^p_{\boldsymbol{p}}\mathscr{H}_x^{-\sigma}} \leq C\sqrt{p}\|\gamma\|_{L^2_{\boldsymbol{p}}\mathscr{H}_x^{-\sigma}} \leq C.$$

We refer to [Burq et al. 2014, Proposition 3.1] for the details.

Next, we show that $\big\| \|\partial_t u\|_{L^p_T \mathcal{H}^{-\sigma}_x} \big\|_{L^p_{\nu_N}} \le C$. By definition of ν_N,

$$\|\partial_t u\|^p_{L^p_{\nu_N} L^p_T \mathcal{H}^{-\sigma}_x} = \int_{\mathscr{C}([-T,T];X^{-1}(\mathbb{R}^2))} \|\partial_t u\|^p_{L^p_T \mathcal{H}^{-\sigma}_x} \, d\nu_N(u) = \int_{X^{-1}(\mathbb{R}^2)} \|\partial_t \Phi_N(t)(v)\|^p_{L^p_T \mathcal{H}^{-\sigma}_x} \, d\mu(v).$$

Now, since $\Phi_N(t)(v)$ satisfies (4-2) and by the invariance of μ, we have

$$\|\partial_t u\|^p_{L^p_{\nu_N} L^p_T \mathcal{H}^{-\sigma}_x} = \int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(\Phi_N(t)(v))\|^p_{L^p_T \mathcal{H}^{-\sigma}_x} \, d\mu(v) = 2T \int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_N(v)\|^p_{\mathcal{H}^{-\sigma}_x} \, d\mu(v)$$

and we conclude with (4-1) and Proposition 4.1. □

The convergence argument. The importance of Lemma 4.5 above comes from the fact that it allows us to establish the following tightness result for the measures ν_N. We refer to [Burq et al. 2014, Proposition 4.11] for the proof.

Proposition 4.6. *Let $T > 0$ and $\sigma > 1$. Then the family of measures*

$$(\nu_N)_{N \ge 1} \quad \textit{with} \quad \nu_N = \mathscr{L}_{\mathscr{C}_T \mathcal{H}^{-\sigma}}(u_N(t); t \in [-T, T])$$

is tight in $\mathscr{C}([-T, T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2))$.

The result of Proposition 4.6 enables us to use the Prokhorov theorem: for each $T > 0$ there exists a subsequence ν_{N_k} and a measure ν on the space $\mathscr{C}([-T, T]; X^{-1}(\mathbb{R}^2))$ such that, for all $\tau > 1$ and all bounded continuous functions $F : \mathscr{C}([-T, T]; \mathcal{H}^{-\tau}(\mathbb{R}^2)) \to \mathbb{R}$,

$$\int_{\mathscr{C}([-T,T];\mathcal{H}^{-\tau}(\mathbb{R}^2))} F(u) \, d\nu_{N_k}(u) \to \int_{\mathscr{C}([-T,T];\mathcal{H}^{-\tau}(\mathbb{R}^2))} F(u) \, d\nu(u).$$

By the Skorohod theorem, there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \tilde{\boldsymbol{p}})$, a sequence of random variables $(\tilde{u}_{N_k})$ and a random variable $\tilde{u}$ with values in $\mathscr{C}([-T, T]; X^{-1}(\mathbb{R}^2))$ such that

$$\mathscr{L}(\tilde{u}_{N_k}; t \in [-T, T]) = \mathscr{L}(u_{N_k}; t \in [-T, T]) = \nu_{N_k}, \quad \mathscr{L}(\tilde{u}; t \in [-T, T]) = \nu, \tag{4-3}$$

and, for all $\tau > 1$,

$$\tilde{u}_{N_k} \to \tilde{u} \quad \tilde{\boldsymbol{p}}\text{-a.s. in } \mathscr{C}([-T, T]; \mathcal{H}^{-\tau}(\mathbb{R}^2)). \tag{4-4}$$

We now claim that $\mathscr{L}_{X^{-1}}(u_{N_k}(t)) = \mathscr{L}_{X^{-1}}(\tilde{u}_{N_k}(t)) = \mu$ for all $t \in [-T, T]$ and $k \ge 1$. Indeed, for all $t \in [-T, T]$, the evaluation map

$$\begin{aligned} R_t : \mathscr{C}([-T, T]; X^{-1}(\mathbb{R}^2)) &\to X^{-1}(\mathbb{R}^2), \\ u &\mapsto u(t, \cdot), \end{aligned}$$

is well-defined and continuous.

Thus, for all $t \in [-T, T]$, $u_{N_k}(t)$ and $\tilde{u}_{N_k}(t)$ have same distribution $(R_t)_{\#}\nu_{N_k}$. By Proposition 4.4, we obtain that this distribution is μ.

Thus, from (4-4) we deduce that

$$\mathscr{L}_{X^{-1}}(\tilde{u}(t)) = \mu \quad \text{for all } t \in [-T, T]. \tag{4-5}$$

Let $k \geq 1$ and $t \in \mathbb{R}$ and consider the random variable X_k given by

$$X_k = u_{N_k}(t) - R_0(u_{N_k}(t)) + i \int_0^t \mathcal{T}_{N_k}(u_{N_k})\, ds.$$

Define $\widetilde{X}_k$ similarly to X_k, with u_{N_k} replaced by $\tilde{u}_{N_k}$. Then, by (4-3),

$$\mathcal{L}_{\mathscr{C}_T X^{-1}}(\widetilde{X}_{N_k}) = \mathcal{L}_{\mathscr{C}_T X^{-1}}(X_{N_k}) = \delta_0.$$

In other words, $\widetilde{X}_k = 0$ $\tilde{\boldsymbol{p}}$-a.s. and $\tilde{u}_{N_k}$ satisfies the following equation $\tilde{\boldsymbol{p}}$-a.s.:

$$\tilde{u}_{N_k}(t) = R_0(\tilde{u}_{N_k}(t)) - i \int_0^t \mathcal{T}_{N_k}(\tilde{u}_{N_k})\, ds. \tag{4-6}$$

We now show that we can pass to the limit $k \to +\infty$ in (4-6) in order to show that $\tilde{u}$ is $\tilde{\boldsymbol{p}}$-a.s. a solution to (CR), written in integral form as

$$\tilde{u}(t) = R_0(\tilde{u}(t)) - i \int_0^t \mathcal{T}(\tilde{u})\, ds. \tag{4-7}$$

Firstly, from (4-4) we deduce the convergence of the linear terms in (4-6) to those in (4-7). The following lemma gives the convergence of the nonlinear term:

Lemma 4.7. *Up to a subsequence,*

$$\mathcal{T}_{N_k}(\tilde{u}_{N_k}) \to \mathcal{T}(\tilde{u}) \quad \tilde{\boldsymbol{p}}\text{-a.s. in } L^2([-T,T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2)).$$

Proof. In order to simplify the notations, in this proof we drop the tildes and write $N_k = k$. Let $M \geq 1$ and write

$$\mathcal{T}_k(u_k) - \mathcal{T}(u) = (\mathcal{T}_k(u_k) - \mathcal{T}(u_k)) + (\mathcal{T}(u_k) - \mathcal{T}_M(u_k)) + (\mathcal{T}_M(u_k) - \mathcal{T}_M(u)) + (\mathcal{T}_M(u) - \mathcal{T}(u)).$$

To begin with, by continuity of the product in finite dimensions, when $k \to +\infty$,

$$\mathcal{T}_M(u_k) \to \mathcal{T}_M(u) \quad \tilde{\boldsymbol{p}}\text{-a.s. in } L^2([-T,T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2)).$$

We now deal with the other terms. It is sufficient to show the convergence in the space $X := L^2(\Omega \times [-T,T]; \mathcal{H}^{-\sigma}(\mathbb{R}^2))$, since the almost sure convergence follows after extraction of a subsequence. By definition and the invariance of μ, we obtain

$$\begin{aligned}
\|\mathcal{T}_M(u_k) - \mathcal{T}(u_k)\|_X^2 &= \int_{\mathscr{C}([-T,T];X^{-1})} \|\mathcal{T}_M(v) - \mathcal{T}(v)\|^2_{L^2_T \mathcal{H}^{-\sigma}_x}\, d\nu_k(v) \\
&= \int_{X^{-1}(\mathbb{R}^2)} \big\|\mathcal{T}_M(\Phi_k(t)(f)) - \mathcal{T}(\Phi_k(t)(f))\big\|^2_{L^2_T \mathcal{H}^{-\sigma}_x}\, d\mu(f) \\
&= \int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_M(f) - \mathcal{T}(f)\|^2_{L^2_T \mathcal{H}^{-\sigma}_x}\, d\mu(f) \\
&= 2T \int_{X^{-1}(\mathbb{R}^2)} \|\mathcal{T}_M(f) - \mathcal{T}(f)\|^2_{\mathcal{H}^{-\sigma}_x}\, d\mu(f),
\end{aligned}$$

which tends to 0 uniformly in $k \geq 1$ when $M \to +\infty$, according to Proposition 4.1.

The term $\|\mathcal{T}_M(u) - \mathcal{T}(u)\|_X$ is treated similarly. Finally, with the same argument, we show

$$\|\mathcal{T}_k(u_k) - \mathcal{T}(u_k)\|_X \le C\|\mathcal{T}_k(f) - \mathcal{T}(f)\|_{L^2_\mu \mathcal{H}_x^{-\sigma}},$$

which tends to 0 when $k \to +\infty$. This completes the proof. □

Conclusion of the proof of Theorem 2.6. Define $\tilde{f} = \tilde{u}(0) := R_0(\tilde{u})$. Then, by (4-5), $\mathscr{L}_{X^{-1}}(\tilde{f}) = \mu$ and, by the previous arguments, there exists $\widetilde{\Omega}' \subset \widetilde{\Omega}$ such that $\tilde{\boldsymbol{p}}(\widetilde{\Omega}') = 1$ and, for each $\omega' \in \widetilde{\Omega}'$, the random variable $\tilde{u}$ satisfies the equation

$$\tilde{u} = \tilde{f} - i\int_0^t \mathcal{T}(\tilde{u})\,dt, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^2. \tag{4-8}$$

Set $\Sigma = \tilde{f}(\Omega')$; then $\mu(\Sigma) = \tilde{\boldsymbol{p}}(\widetilde{\Omega}') = 1$. It remains to check that we can construct a global dynamics. Take a sequence $T_N \to +\infty$ and perform the previous argument for $T = T_N$. For all $N \ge 1$, let Σ_N be the corresponding set of initial conditions and set $\Sigma = \bigcap_{N\in\mathbb{N}} \Sigma_N$. Then $\mu(\Sigma) = 1$ and, for all $\tilde{f} \in \Sigma$, there exists

$$\tilde{u} \in \mathscr{C}(\mathbb{R}; X^{-1}(\mathbb{R}^2))$$

which solves (4-8). This completes the proof of Theorem 2.6.

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BLOW-UP ANALYSIS OF A NONLOCAL LIOUVILLE-TYPE EQUATION

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We establish an equivalence between the *Nirenberg problem* on the circle and the boundary of holomorphic immersions of the disk into the plane. More precisely we study the nonlocal Liouville-type equation

$$(-\Delta)^{\frac{1}{2}} u = \kappa e^u - 1 \quad \text{in } S^1, \tag{1}$$

where $(-\Delta)^{\frac{1}{2}}$ stands for the fractional Laplacian and κ is a bounded function. The equation (1) can actually be interpreted as the prescribed curvature equation for a curve in conformal parametrization. Thanks to this geometric interpretation we perform a subtle blow-up and quantization analysis of (1). We also show a relation between (1) and the analogous equation in $\mathbb{R}$,

$$(-\Delta)^{\frac{1}{2}} u = K e^u \quad \text{in } \mathbb{R} \tag{2}$$

with K bounded on $\mathbb{R}$.

1. Introduction

A famous problem posed by Louis Nirenberg is the question of for which positive functions K on the standard sphere (S^n, g_{S^n}) there exists a function u on S^n such that the scalar curvature (Gauss curvature in dimension $n = 2$) of the conformal metric $g = e^{2u} g_{S^n}$ is equal to K. This problem, prescribing the scalar curvature within a conformal class of manifolds, has stimulated a lot of works in geometry and analysis. In dimension $n = 2$ it consists in solving the so-called Liouville equation. More precisely, if (Σ, g_0) is a smooth, closed Riemann surface with Gauss curvature K_{g_0}, an easy computation shows that a function $K(x)$ is the Gauss curvature for some metric $g = e^{2u} g_0$ conformally equivalent to the metric g_0 with $u : \Sigma \to \mathbb{R}$ if and only if there exists a solution $u = u(x)$ of

$$-\Delta_{g_0} u = K e^{2u} - K_{g_0} \quad \text{on } \Sigma, \tag{3}$$

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where Δ_{g_0} is the Laplace–Beltrami operator on (Σ, g_0) (see, e.g., [Chang 2005] for more details).

In particular, when $\Sigma = \mathbb{R}^2$ or $\Sigma = S^2$, (3) reads, respectively,

$$-\Delta u = K e^{2u} \quad \text{on } \mathbb{R}^2 \tag{4}$$

and

$$-\Delta_{S^2} u = K e^{2u} - 1 \quad \text{on } S^2. \tag{5}$$

Singular Liouville equations of the form

$$-\Delta_{g_0} u = K e^{2u} - K_{g_0} - 2\pi \sum_{i=1}^{m} \alpha_i \delta_{p_i} \quad \text{on } \Sigma \tag{6}$$

have a role in fluid dynamics — see [Tur and Yanovsky 2004] — as well as in the study of electroweak theory or abelian Chern–Simons vortices; see, e.g., [Tarantello 2008]. For the latter cases, singular points represent zeroes of the scalar wave function involved in the model.

Equations (4), (5) and also (6) have been largely studied in the literature. Here we would like to recall the famous blow-up result of Brezis and Merle [1991] concerning (4):

Theorem 1.1 [Brezis and Merle 1991, Theorem 3]. *Assume that $(u_k) \subset L^1(\Omega)$, Ω an open subset of $\mathbb{R}^2$, is a sequence of solutions to (4) satisfying $K_k \geq 0$, $\|K_k\|_{L^p} \leq C_1$, and $\|e^{u_k}\|_{L^{p'}} \leq C_2$ for some $1 < p \leq \infty$. Then, up to subsequences, one of the following alternatives holds: either (u_k) is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$, or $u_k(x) \to -\infty$ uniformly on compact subsets of Ω, or there is a finite nonempty (blow-up) set $B = \{a_1, \ldots, a_N\} \subset \Omega$ such that $u_k(x) \to -\infty$ on compact subsets of $\Omega \setminus B$. In addition, in this last case, $K_k e^{2u_k}$ converges in the sense of measure on Ω to $\sum_{i=1}^N \alpha_i \delta_{a_i}$, with $\alpha_i \geq 2\pi/p'$.*

The purpose of this work is to investigate an analogous prescribed curvature problem in dimension 1. Even if this is a classical problem, it has never been studied so far (to our knowledge) from the point of view of conformal geometry. In the case, for instance, of a planar Jordan curve (namely, a continuous closed and simple curve) there is the possibility to parametrize it through the trace of the Riemann mapping between the disk D^2 and the simply connected domain enclosed by the curve. The equation corresponding to such a parametrization is

$$(-\Delta)^{\frac{1}{2}} \lambda = \kappa e^{\lambda} - 1 \quad \text{in } S^1, \tag{7}$$

where $e^{\lambda} d\theta$ and $\kappa e^{\lambda} d\theta$ are the length form and the curvature density, respectively, of the curve in this parametrization. The definition and relevant properties of the operator $(-\Delta)^{\frac{1}{2}}$ will be given in Appendix A.

One of the main results of this paper is the one-to-one correspondence between the solutions to the Nirenberg problem (7) in S^1 and the space of holomorphic immersions of the disk D^2 (see Theorem 1.4 below). This correspondence can be seen as a sort of generalized Riemann mapping theorem.

This permits us to perform a complete blow-up analysis of (7) in the spirit of Theorem 1.1, even if we do not get exactly the same dichotomy. More precisely, our first main result is the following theorem:

Theorem 1.2. *Let $(\lambda_k) \subset L^1(S^1, \mathbb{R})$ be a sequence with*

$$L_k := \|e^{\lambda_k}\|_{L^1(S^1)} \leq \bar{L} \tag{8}$$

satisfying

$$(-\Delta)^{\frac{1}{2}}\lambda_k = \kappa_k e^{\lambda_k} - 1 \quad \text{in } S^1, \tag{9}$$

where $\kappa_k \in L^\infty(S^1, \mathbb{R})$ *satisfies*

$$\|\kappa_k\|_{L^\infty(S^1)} \leq \bar{\kappa}. \tag{10}$$

Then up to subsequence we have $\kappa_k e^{\lambda_k} \rightharpoonup \mu$ *weakly in* $W^{1,p}_{\mathrm{loc}}(S^1 \setminus B)$ *for every* $p < \infty$, *where* μ *is a Radon measure,* $B := \{a_1, \ldots, a_N\}$ *is a (possibly empty) subset of* S^1 *and* $\kappa_k \overset{*}{\rightharpoonup} \kappa_\infty$ *in* $L^\infty(S^1)$. *Set* $\bar{\lambda}_k := (1/2\pi)\int_{S^1}\lambda_k\, d\theta$. *Then one of the following alternatives holds*:

(i) $\bar{\lambda}_k \to -\infty$ *as* $k \to \infty$, $N = 1$ *and* $\mu = 2\pi\delta_{a_1}$. *In this case,*

$$v_k := \lambda_k - \bar{\lambda}_k \rightharpoonup v_\infty \quad \text{in } W^{1,p}_{\mathrm{loc}}(S^1 \setminus \{a_1\}) \text{ for every } p < \infty,$$

where $v_\infty(e^{i\theta}) = -\log\bigl(2(1-\cos(\theta-\theta_1))\bigr)$ *for* $a_1 = e^{i\theta_1}$, *solving*

$$(-\Delta)^{\frac{1}{2}} v_\infty = -1 + 2\pi\delta_{a_1} \quad \text{in } S^1. \tag{11}$$

(ii) $\bar{\lambda}_k \to -\infty$ *as* $k \to \infty$, $N = 2$ *and* $\mu = \pi(\delta_{a_1} + \delta_{a_2})$. *In this case,*

$$v_k := \lambda_k - \bar{\lambda}_k \rightharpoonup v_\infty \quad \text{in } W^{1,p}_{\mathrm{loc}}(S^1 \setminus \{a_1, a_2\}) \text{ for every } p < \infty,$$

where

$$v_\infty(e^{i\theta}) = -\tfrac{1}{2}\log\bigl(2(1-\cos(\theta-\theta_1))\bigr) - \tfrac{1}{2}\log\bigl(2(1-\cos(\theta-\theta_2))\bigr), \quad a_1 = e^{i\theta_1},\ a_2 = e^{i\theta_2},$$

solves

$$(-\Delta)^{\frac{1}{2}} v_\infty = -1 + \pi\delta_{a_1} + \pi\delta_{a_2} \quad \text{in } S^1. \tag{12}$$

(iii) $|\bar{\lambda}_k| \leq C$ *and* $\mu = \kappa_\infty e^{\lambda_\infty} + \pi(\delta_{a_1} + \cdots + \delta_{a_N})$ *for some* $\lambda_\infty \in W^{1,p}_{\mathrm{loc}}(S^1 \setminus B)$, *with* $\lambda_\infty, e^{\lambda_\infty} \in L^1(S^1)$ *and*

$$(-\Delta)^{\frac{1}{2}}\lambda_\infty = \kappa_\infty e^{\lambda_\infty} - 1 + \sum_{i=1}^{N}\pi\delta_{a_i} \quad \text{in } S^1. \tag{13}$$

We would like to stress that we obtain a *quantization-type* result, namely the curvature concentrating at each blow-up point is precisely π, without any assumption on the sign of the curvature (this hypothesis is crucial in [Brezis and Merle 1991]) and on the convergence of the κ_k. Actually, several works on equations (4) and (5) have extended the result of Brezis and Merle, showing that, under the crucial assumption that the prescribed curvatures K_k converge in C^0, the amount of curvature concentrating at each point is a multiple of 4π, i.e., a multiple of the total Gaussian curvature of S^2; see, e.g., [Li and Shafrir 1994]. (Also, higher-dimensional extensions were studied under the same strong assumptions of convergence of K_k in C^0 or even C^1; see, e.g., [Druet and Robert 2006; Malchiodi 2006; Martinazzi 2009b].) In [Brezis and Merle 1991] the functions K_k can belong to $L^p(\mathbb{R})$, with $1 < p \leq +\infty$. We believe that in the case of the nonlocal Liouville equation (7) the quantization result by π does not hold once we replace $\kappa \in L^\infty$ by $\kappa \in L^p$ with $1 < p < +\infty$.

The fact that we are able to get a quantization result only under the minimal (and geometrically meaningful) bounds (8) and (10) is better understood through the above-mentioned one-to-one correspondence between the solutions to (7) and the space of holomorphic immersions of the disk D^2. Precisely, given a solution λ to (7) with $\kappa \in L^\infty(S^1)$, the function e^λ provides a “conformal” parametrization of a closed curve $\gamma : S^1 \to \mathbb{C}$ in normal parametrization whose curvature at the point $\gamma(z)$ is exactly $\kappa(z)$.

Definition 1.3. A function $\Phi \in C^1(\bar{D}^2, \mathbb{C})$ is called a holomorphic immersion if Φ is holomorphic in D^2 and $\Phi'(z) := \partial_z \Phi(z) \neq 0$ for every $z \in \bar{D}^2$.

A curve $\gamma \in C^1(S^1, \mathbb{C})$ is said to be in normal parametrization if $|\dot{\gamma}|$ is constant, and is in conformal parametrization if there exists a holomorphic immersion $\Phi \in C^1(\bar{D}^2, \mathbb{C})$ with $\Phi|_{S^1} = \gamma$.

Then we have the following characterization:

Theorem 1.4. *A function $\lambda \in L^1(S^1, \mathbb{C})$ with $L := \|e^\lambda\|_{L^1(S^1)} < \infty$ satisfies*

$$(-\Delta)^{\frac{1}{2}} \lambda = \kappa e^\lambda - 1 \quad in\ S^1 \tag{14}$$

for some function $\kappa : S^1 \to \mathbb{R}$, $\kappa \in L^\infty(S^1)$, if and only if there exists a closed curve $\gamma \in W^{2,\infty}(S^1, \mathbb{C})$ with $|\dot{\gamma}| \equiv L/(2\pi)$, a holomorphic immersion $\Phi : \bar{D}^2 \to \mathbb{C}$ and a diffeomorphism $\sigma : S^1 \to S^1$ such that, for all $z \in S^1$, we have $\Phi \circ \sigma(z) = \gamma(z)$,

$$|\Phi'(z)| = e^{\lambda(z)} \tag{15}$$

and the curvature of $\Phi(S^1)$ is κ. While Φ uniquely determines λ via (15), λ determines Φ up to a rotation and a translation. Moreover,

$$|\Phi'(z)| = e^{\tilde{\lambda}(z)}, \quad z \in \bar{D}^2, \tag{16}$$

where $\tilde{\lambda} : D^2 \to \mathbb{R}$ is the harmonic extension of λ.

Figures 1, 2 and 5 provide some examples of curves satisfying the assumptions of Theorem 1.4. Theorem 1.4 allows us to interpret and reformulate Theorem 1.2 from the point of view of the behavior

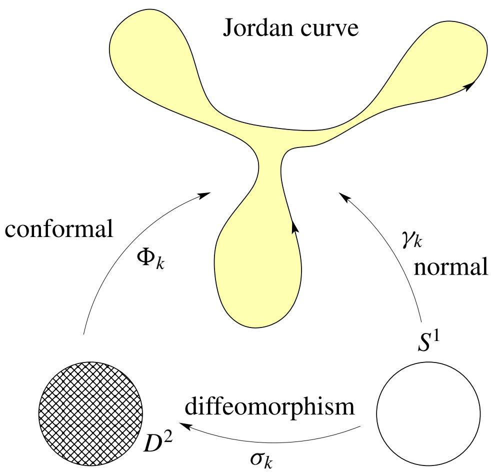

Figure 1. A domain bounded by a Jordan curve γ_k and biholomorphic to the unit disk D^2 via a map $\Phi_k : \bar{D}^2 \to \mathbb{C}$.

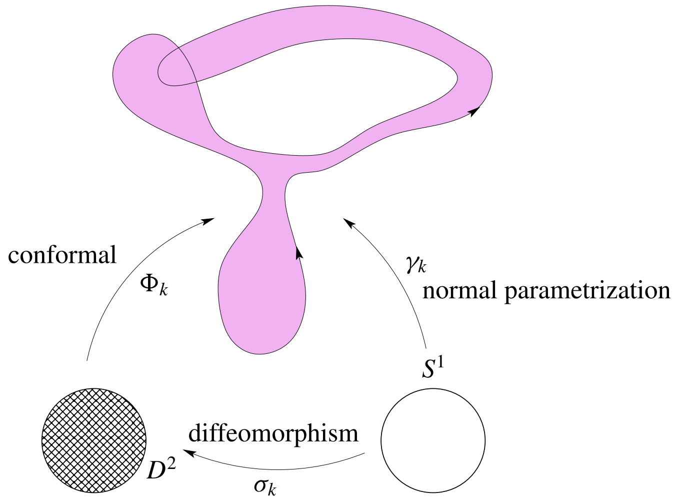

Figure 2. The curve γ_k can have self-intersections. In this case, $\Phi_k : \overline{D}^2 \to \mathbb{C}$ is a holomorphic immersion but it is not injective.

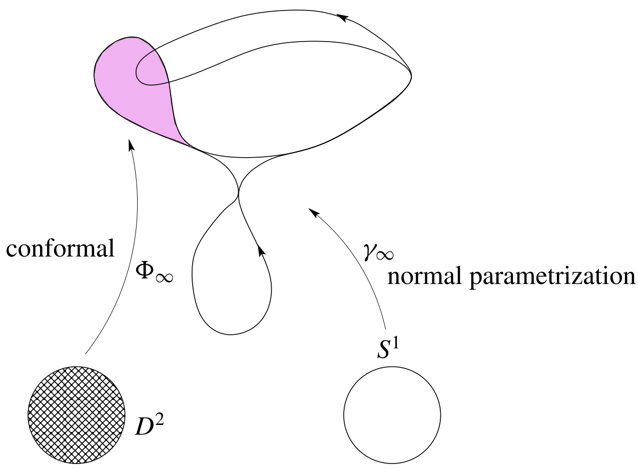

Figure 3. As $k \to \infty$ the curves γ_k can generate a pinching phenomenon. In this case, Φ_k can converge to a constant or, as in this figure, to a holomorphic immersion Φ_∞ (singular at finitely many points of ∂D^2) whose image "selects" one of the "components" bounded by γ_∞.

of the sequences of the curves γ_k (in normal parametrization) and of the immersions Φ_k corresponding to a sequence of solutions to (9); see Figures 3 and 4.

Theorem 1.5. *Let a sequence $(\lambda_k) \subset L^1(S^1, \mathbb{R})$ satisfy (8)–(10), let $\Phi_k : \overline{D}^2 \to \mathbb{C}$ be a holomorphic immersion satisfying (15), and let σ_k and γ_k with $\gamma_k = \Phi_k \circ \sigma_k$ be as given by Theorem 1.4. Then, up to extracting a subsequence, there exists an at most countable family J such that for every $j \in J$ there exist a sequence of Möbius transformations $f_k^j : \overline{D}^2 \to \overline{D}^2$ and a finite set of finitely many points*

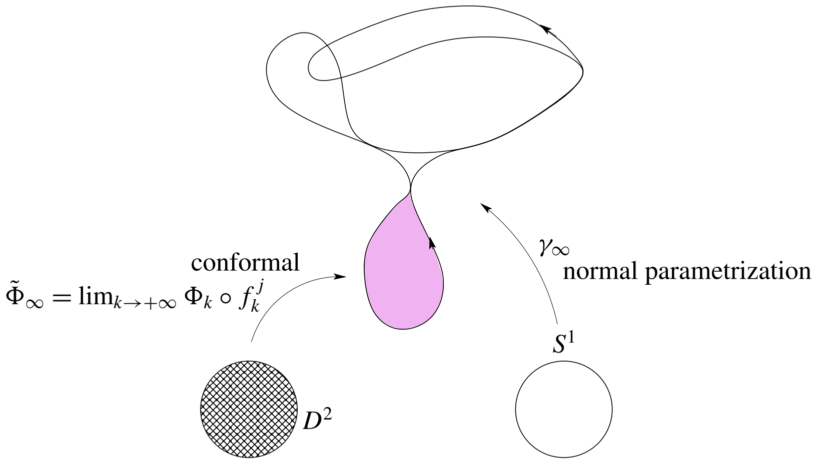

Figure 4. Composing Φ_k as in Figure 3 with suitable Möbius transformations, one can have Φ_∞ cover a different "component" bounded by γ_∞. In this figure one can choose among 4 different components, or choose Φ_∞ to be constant.

$B_j = \{a_1^j, \dots a_{N_j}^j\} \subset S^1$ *such that*

$$\gamma_k \rightharpoonup \gamma_\infty \quad in\ W^{2,p}(S^1) \qquad and \qquad \tilde{\Phi}_k^j := \Phi_k \circ f_k^j \rightharpoonup \tilde{\Phi}_\infty^j \quad in\ W^{2,p}_{\mathrm{loc}}(\overline{D}^2 \setminus B_j),$$

where $p < \infty$, *the* $\tilde{\Phi}_\infty^j : \overline{D}^2 \setminus B_j \to \mathbb{C}$ *are holomorphic immersions satisfying*

$$(\gamma_\infty)_*[S^1] = \sum_{j\in J} (\tilde{\Phi}_\infty^j)_*[S^1 \setminus B_j], \tag{17}$$

where, for any $\phi : S^1 \to \mathbb{C}$ *and differential form* ω *on* $\mathbb{C}$,

$$\langle \phi_*[S^1], \omega\rangle := \int_{S^1} \phi^*\omega.$$

If $\lambda_k^j := \log\big|(\tilde{\Phi}_k^j)'|_{S^1}\big|$ *then, up to a subsequence,* $\lambda_k^j \rightharpoonup \lambda_\infty^j$ *in* $W^{1,p}_{\mathrm{loc}}(S^1 \setminus B_j)$, *where*

$$(-\Delta)^{\frac{1}{2}} \lambda_\infty^j = \kappa_\infty^j e^{\lambda_\infty^j} - 1 - \sum_{i=1}^{N_j} \pi \delta_{a_i^j} \tag{18}$$

and $\kappa_k \circ f_k^j \overset{*}{\rightharpoonup} \kappa_\infty^j$ *in* $L^\infty(S^1, \mathbb{R})$ *as* $k \to +\infty$.

Theorem 1.5 says that it is always possible, up to the action of sequences of Möbius transformations, to recover all the connected components enclosed by the limiting curve γ_∞ (see in particular (17)). We will also see that these components are separated by what we call *pinched points* (see Definition 3.7), namely (roughly speaking) a pair of points $p \neq p' \in S^1$ such that $\gamma_\infty(p) = \gamma_\infty(p')$. The angle between the tangent vectors in these pairs of points is shown to necessarily be π. This also explains the coefficient π in front of each δ_{a_i} in (18).

It would be interesting to compare Theorems 1.2 and 1.5 to the blow-up analysis obtained recently by Mondino and Rivière [2014] in the case of sequences of weak conformal immersions from S^2 into $\mathbb{R}^m$; they study the possible limit of the Liouville equation

$$-\Delta_{g_0} u = K e^{2u} - 1 \quad \text{on } S^2 \tag{19}$$

satisfied by the conformal factor of the immersion Φ ($g_\Phi = e^{2u} g_0$) under the assumption that the second fundamental form is bounded in L^2. Also in their case, a sort of bubbling phenomenon occurs and the choice of different sequences of Möbius transformations of S^2 permits them to detect all the limiting enclosed currents. However, the 2-dimensional blow-up analysis differs substantially from the 1-dimensional case: in the 2-dimensional case the area is quantized, namely there is no production of area in the neck region between the different bubbles, whereas in the 1-dimensional case the quantization of the length does not hold. Precisely, Mondino and Rivière [2014] show that

$$\sum_{\text{“bubbles”}} \int_{S^2} e^{2u_\infty}\, dv = \liminf_{k\to+\infty} \int_{S^2} e^{2u_k}\, dv,$$

whereas in the present situation one can produce examples such that

$$\sum_{\text{“bubbles”}} \int_{S^1} e^{\lambda_\infty}\, d\theta < \liminf_{k\to+\infty} \int_{S^1} e^{\lambda_k}\, d\theta.$$

We insist on the fact that “conformal” parametrizations of planar curves are relevant in different applications. For instance, they should be one of the main tools of the Willmore plateau problem, of the analysis of the renormalizing area of surfaces in the hyperbolic space $\mathscr{H}^2$ and of the free boundaries problem. In particular, for the latter, Da Lio [2015] has observed that there is a one-to-one correspondence between free boundaries and $\frac{1}{2}$-harmonic maps and here we show that the holomorphic immersion ϕ for which $e^{\lambda(z)} = |\partial\phi/\partial\theta(z)|$, $z \in S^1$, is a $\frac{1}{2}$-harmonic map into $\phi(S^1)$.

In forthcoming work, we are going to investigate the topological and differential structure of the subspace of $C^{1,\alpha}(S^1) \times C^{0,\alpha}(S^1)$ made of solutions (u, κ) of the Nirenberg problem in S^1 (the Nirenberg moduli space). The present work should be interpreted as an attempt to describe the “boundary of the Nirenberg moduli space”. We mention that a nonlocal version of the Nirenberg problem in dimension $n \geq 2$ has recently been studied in [Jin et al. 2014; 2015a].

We finally prove a link between (7) and the analogous nonlocal equation in $\mathbb{R}$. Precisely, if $u \in L_{\frac{1}{2}}(\mathbb{R})$ (see (130)), $e^u \in L^1(\mathbb{R})$ and u satisfies

$$(-\Delta)^{\frac{1}{2}} u = K e^u \quad \text{in } \mathbb{R} \tag{20}$$

for some $K \in L^\infty(\mathbb{R})$, then $\lambda(z) := u(\Pi(z)) - \log(1+\sin z)$ (where $\Pi : S^1 \setminus \{-i\} \to \mathbb{R}$ is the stereographic projection) satisfies

$$(-\Delta)^{\frac{1}{2}} \lambda = K \circ \Pi e^\lambda - 1 + (2\pi - \|(-\Delta)^{\frac{1}{2}} u\|_{L^1})\delta_{-i} \quad \text{in } S^1. \tag{21}$$

Owing to this correspondence from Theorem 1.2, we can deduce the following compactness result in $\mathbb{R}$:

Theorem 1.6. *Let $u_k \in L_{\frac{1}{2}}(\mathbb{R})$ be a sequence of solutions to*

$$(-\Delta)^{\frac{1}{2}} u_k = K_k e^{u_k} \quad in \ \mathbb{R}$$

with $\|K_k\|_{L^\infty} \le C$ and $\|e^{u_k}\|_{L^1} \le C$. Then, up to subsequence, we have $K_k e^{u_k} \rightharpoonup \mu$ weakly in $W^{1,p}_{\mathrm{loc}}(\mathbb{R} \setminus B)$ for every $p < \infty$, where μ is a finite Radon measure in $\mathbb{R}$, $B := \{a_1, \ldots, a_N\}$ is a (possibly empty) subset of $\mathbb{R}$ and $K_k \overset{*}{\rightharpoonup} K_\infty$ in $L^\infty(\mathbb{R})$. Moreover, one of the following alternatives holds:

(i) $\mu|_{\mathbb{R}\setminus B} = K_\infty e^{u_\infty}$ *for some* $u_\infty \in W^{1,p}_{\mathrm{loc}}(\mathbb{R} \setminus B)$ *satisfying*

$$(-\Delta)^{\frac{1}{2}} u_\infty = K_\infty e^{u_\infty} + \sum_{i=1}^{N} \pi \delta_{a_i} \quad in \ \mathbb{R}. \tag{22}$$

(ii) $\mu|_{\mathbb{R}\setminus B} \equiv 0$, $N \le 2$ *and* $u_k \to -\infty$ *locally uniformly in* $\mathbb{R} \setminus B$.

In particular, we can deduce the following:

Corollary 1.7. *Under the hypotheses of Theorem 1.6, if $K_k \ge 0$ and*

$$\int_{\mathbb{R}} K_k e^{u_k}\, dx \le 2\pi,$$

then either $N = 1$ and $u_k \to -\infty$ locally uniformly $\mathbb{R} \setminus \{a_1\}$, or $N = 0$ and $u_k \rightharpoonup u_\infty$ in $W^{1,p}(\mathbb{R})$ as $k \to +\infty$, where u_∞ solves

$$(-\Delta)^{\frac{1}{2}} u_\infty = K_\infty e^{u_\infty}. \tag{23}$$

We will give the proof of Theorem 1.6 and Corollary 1.7 in a forthcoming paper.

An interesting consequence of Theorem 1.4 is a proof of the classification of the solutions to the nonlocal equation

$$(-\Delta)^{\frac{1}{2}} u = e^u \quad \text{in } \mathbb{R} \tag{24}$$

under the integrability condition

$$L := \int_{\mathbb{R}} e^u\, dx < \infty. \tag{25}$$

Equation (24) is a special case of the problem

$$(-\Delta)^{n/2} u = (n-1)!\, e^{nu} \quad \text{in } \mathbb{R}^n, \qquad V := \int_{\mathbb{R}^n} e^{nu}\, dx < \infty, \tag{26}$$

which has been studied by several authors in the last decades (see, e.g., [Chen and Li 1991; Chang and Yang 1997; Lin 1998; Jin et al. 2015b; Martinazzi 2009a]). Geometrically, if u solves (26) and $n \ge 2$, then the metric $e^{2u}|dx|^2$ on $\mathbb{R}^n$ has constant Q-curvature $(n-1)!$ and volume V; see, e.g., [Chang 2004]. All the above-mentioned works rely on the application of a moving-plane technique, in order to show that under certain growth conditions at infinity (needed only when $n \ge 3$) the solutions to (26) have the form

$$u_{\mu,x_0}(x) := \log \frac{2\mu}{1 + \mu^2 |x - x_0|^2}, \quad x \in \mathbb{R}^n, \tag{27}$$

for some $\mu > 0$ and $x_0 \in \mathbb{R}^n$. For the case $n = 1$, instead of using the moving-plane technique, we will use stereographic projection to transform (24) into (14), and use the geometric interpretation of the latter (Theorem 1.4) to compute all its solutions (Corollary 2.3 below). This will yield:

Theorem 1.8. *Every function $u \in L_{\frac{1}{2}}(\mathbb{R})$ solving (24)–(25) is of the form (27) for some $\mu > 0$ and $x_0 \in \mathbb{R}$.*

We also remark that, by changing the sign of the nonlinearity in (24), the problem has no solutions. More precisely:

Proposition 1.9. *Given a function $K \in L^\infty(\mathbb{R})$ with $K \leq 0$, the equation*

$$(-\Delta)^{\frac{1}{2}} u = K e^u \quad \text{in } \mathbb{R}$$

has no solution satisfying (25).

The proof of Proposition 1.9 is a simple application of the maximum principle for the operator $(-\Delta)^{\frac{1}{2}}$, but it is worth remarking that, for $n \geq 4$, even solutions to (26) with $(n-1)!$ replaced by $-(n-1)!$ (or any negative constant) do exist, as shown in [Martinazzi 2008].

The paper is organized as follows. In Section 2 we introduce the nonlocal Liouville equation (7) in S^1 and we explain its geometric interpretation. In Section 3 we perform the blow-up and quantization analysis of (7) and in particular we prove Theorems 1.2 and 1.5. Section 4 is devoted to the description of the relation between equations (7) and (20). Finally, in Section 5 we prove Theorem 1.8 and Proposition 1.9.

Notations. We denote by $\langle x, y\rangle$ the scalar product of $x, y \in \mathbb{R}^n$. Let $h : \Omega \subset \mathbb{C} \to \mathbb{R}$ and let $\gamma : S^1 \to \mathbb{C}$ be a curve. We denote by $\int_\gamma h(z)\,|dz|$ or $\int_\gamma h(z)\,d\theta$ the line integral of h along γ. Given $z \in \mathbb{C}$, we denote by $\Re(z)$ and $\Im(z)$ its real and imaginary part, respectively.

2. Nonlocal Liouville equation in S^1

In this section we study the nonlocal Liouville-type equation

$$(-\Delta)^{\frac{1}{2}} u = \kappa e^u - 1 \quad \text{in } S^1,$$

where $u \in L^1(S^1)$, $(-\Delta)^{\frac{1}{2}} u$ stands for the fractional Laplacian and $\kappa : S^1 \to \mathbb{R}$ is a bounded function. In Appendix A we recall the definition and some properties of the fractional Laplacian in S^1.

Geometric interpretation of the Liouville equation in S^1. The first key step in our analysis is the geometric interpretation of (7). Roughly speaking, such an equation prescribes the curvature of a closed curve in conformal parametrization.

It is easy to verify that for $\phi \in L^1(S^1)$ we have

$$(-\Delta)^{\frac{1}{2}} \phi(\theta) = \sum_{n \in \mathbb{Z}} |n| \hat{\phi}(n) e^{in\theta} = \mathcal{H}\left(\frac{\partial \phi}{\partial \theta}\right) = \frac{\partial \mathcal{H}(\phi)}{\partial \theta}, \tag{28}$$

where $\mathcal{H}$ is the Hilbert transform on S^1 defined by

$$\mathcal{H}(f)(\theta) := \sum_{n \in \mathbb{Z}} -i\,\text{sign}(n) \hat{f}(n) e^{in\theta}, \quad f \in \mathcal{D}'(S^1).$$

We recall that the Hilbert transform has the following property, a proof of which can be found, e.g., in [Katznelson 2004, Chapter III].

Lemma 2.1. *The Hilbert transform $\mathcal{H}$ is bounded from $L^p(S^1)$ into itself for $1 < p < +\infty$ and it is of weak type $(1, 1)$. A function $f := u + iv$ with $u, v \in L^1(S^1, \mathbb{R})$ can be extended to a holomorphic function in D^2 if and only if $v = \mathcal{H}(u) + a$ for some $a \in \mathbb{C}$.*

Proof of Theorem 1.4. (1) Let $\Phi \in C^1(\bar{D}^2, \mathbb{C})$ be a holomorphic immersion. Set $\lambda := (\log|\Phi'|)|_{S^1}$. Since $\Phi' : D^2 \to \mathbb{C} \setminus \{0\}$ is holomorphic, $\Phi'|_{S^1} = e^{\lambda + i\rho + i\theta_0}$ for some $\theta_0 \in [0, 2\pi)$, where $\rho := \mathcal{H}(\lambda)$ is the Hilbert transform of λ. Indeed, by Lemma 2.1, the function $f := \lambda + i\rho$ has a holomorphic extension $\tilde{f}$ to D^2; hence, $e^{\tilde{f}}$ is holomorphic in D^2 and $e^{\tilde{f}}|_{S^1} = e^f = e^{\lambda + i\rho}$. But $|e^f| = e^\lambda = (|\Phi'|)|_{S^1}$, so that by Lemma B.1 we have $\Phi'/e^{\tilde{f}} = e^{i\theta_0}$ for some constant θ_0. Up to a rotation of Φ we can assume that $\theta_0 = 0$. Up to such a rotation and a translation, Φ is determined by λ, and we have

$$\frac{\partial \Phi(z)}{\partial \theta}(z) = i e^{\lambda(z) + i\rho(z) + i\theta}. \tag{29}$$

Now let

$$s(\theta) := \int_0^\theta \left| \frac{\partial \Phi(e^{i\theta'})}{\partial \theta'} \right| d\theta'.$$

We have $s : [0, 2\pi] \to [0, L]$, where $L = \|\partial\Phi/\partial\theta\|_{L^1(S^1)}$ is the length of the curve $\Phi(S^1)$, and up to a scaling we will assume that $L = 2\pi$. Let $\theta := s^{-1} : [0, 2\pi] \to [0, 2\pi]$. One can also easily see that $\theta \in C^1([0, 2\pi], [0, 2\pi])$. Then, using (29) and that

$$\dot{s}(\theta) = |\Phi'(e^{i\theta})| = e^{\lambda(e^{i\theta})} > 0, \quad \dot{\theta}(s) = e^{-\lambda(e^{i\theta(s)})},$$

we compute

$$\tau(s) := \frac{d}{ds}\Phi(e^{i\theta(s)}) = \Phi'(e^{i\theta(s)}) i e^{i\theta(s)} \dot{\theta}(s) = \frac{\partial \Phi}{\partial \theta}(e^{i\theta(s)}) e^{-\lambda(e^{i\theta(s)})}.$$

Notice that $|\tau| \equiv 1$, i.e., the curve $\gamma : e^{is} \mapsto \Phi(e^{i\theta(s)})$ is parametrized by arc-length and τ is its unit tangent vector. Using (28), (29) and identifying s with e^{is}, the curvature of γ is given by

$$\begin{aligned} \kappa(s) = \langle i\tau(s), \dot{\tau}(s) \rangle &= \left\langle i\tau(s), \frac{d}{ds}\big(i e^{i\rho(e^{i\theta(s)}) + i\theta(s)}\big) \right\rangle \\ &= \left(\frac{d\rho(e^{i\theta(s)})}{d\theta} + 1 \right) \dot{\theta}(s) \\ &= ((-\Delta)^{\frac{1}{2}} \lambda(e^{i\theta(s)}) + 1) e^{-\lambda(e^{i\theta(s)})}. \end{aligned} \tag{30}$$

From (30) it follows that λ satisfies (14) with $\kappa(e^{is(\theta)}) := \langle i\tau(s(\theta)), \dot{\tau}(s(\theta)) \rangle$. Since $|\kappa(e^{is})| = |\ddot{\gamma}(e^{is})|$ is in $L^\infty(S^1)$, we also have $\gamma \in W^{2,\infty}(S^1, \mathbb{C})$.

(2) Conversely, let us assume that $\lambda \in L^1(S^1)$ with $e^\lambda \in L^1(S^1)$ weakly satisfies (14) for some $\kappa \in L^\infty(S^1)$. By regularity theory, $\lambda \in W^{1,p}(S^1)$ for any $p < \infty$. We set $\rho := \mathcal{H}(\lambda)$. Let $\phi \in W^{1,p}(\bar{D}^2, \mathbb{C})$ be the holomorphic extension of the function $e^{\lambda + i\rho} \in W^{1,p}(S^1)$ and set

$$\Phi(z) := \int_{\Sigma_{0,z}} \phi(w)\, dw, \quad z \in \bar{D}^2, \tag{31}$$

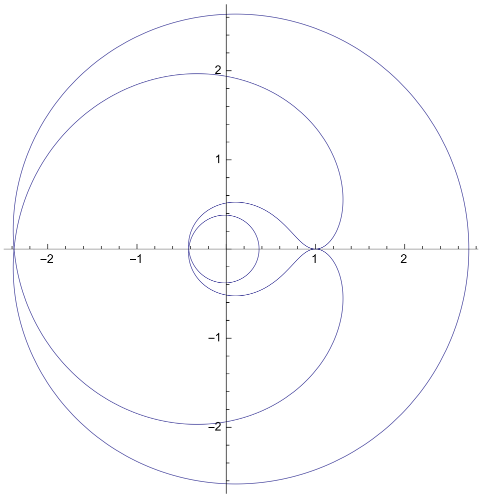

Figure 5. Plot of the curve $e^{\cos\theta}(\cos(2\pi\sin\theta)+i\sin(2\pi\sin\theta))$, $\theta\in[0,2\pi]$. It has the same kind of self-intersections as the curve $\Phi(e^{i\theta})=e^{2\pi e^{i\theta}}$, whose plot is difficult to inspect, since $|\Phi(z)|$ oscillates between $e^{2\pi}$ and $e^{-2\pi}$.

where $\Sigma_{0,z}$ is any path in $\bar{D}^2$ connecting 0 and z. Then $\Phi\in W^{2,p}(\bar{D}^2,\mathbb{C})$ satisfies (29). From part (1) we see that κ is the curvature of the curve $\Phi(S^1)$ in normal parametrization.

Let $\hat{\Phi}:\bar{D}^2\to\mathbb{C}$ be another holomorphic immersion such that $|\hat{\Phi}'(z)|=e^{\lambda(z)}$, $z\in S^1$. We claim that

$$\Phi=e^{i\theta_0}\hat{\Phi}+a \quad \text{in } \bar{D}^2 \text{ for some } \theta_0\in\mathbb{R},\ a\in\mathbb{C}. \tag{32}$$

Indeed, the function $h:=\Phi'/\hat{\Phi}'$ never vanishes in $\bar{D}^2$ and satisfies

$$|h(z)|=\frac{|\Phi'(z)|}{|\hat{\Phi}'(z)|}=\frac{e^{\lambda(z)}}{e^{\lambda(z)}}=1, \quad z\in S^1.$$

It follows from Lemma B.1 that h is a constant of modulus 1, say $h\equiv e^{i\theta_0}$, and (32) follows at once. □

Remark 2.2. In Theorem 1.4, we cannot expect that Φ is a biholomorphism from $\bar{D}^2$ onto $\Phi(\bar{D}^2)$. For instance, the function $\Phi(z):=e^{az}$ for any $a>0$ is an immersion and $\Phi(S^1)$ has self-intersections whenever $a\ge\pi$, as is easily seen by writing (see Figure 5)

$$\Phi(e^{i\theta})=e^{a\cos\theta}(\cos(a\sin\theta)+i\sin(a\sin\theta)).$$

Corollary 2.3. *All functions $\lambda\in L^1(S^1)$ with $e^\lambda\in L^1(S^1)$ that are solutions to*

$$(-\Delta)^{\frac{1}{2}}\lambda=C_0e^\lambda-1 \quad \text{on } S^1, \tag{33}$$

where C_0 is an arbitrary positive constant, are given by

$$\lambda(\theta)=\log\left|\frac{\partial}{\partial\theta}\frac{z-a_1}{1-\bar{a}_1z}\right|-\log C_0 \tag{34}$$

for some a_1 in D^2.

Proof. Up to the translation $\tilde{\lambda} = \lambda + \log C_0$ we can assume $C_0 = 1$. By Theorem 1.4, the function λ determines a holomorphic immersion $\Phi \in C^1(\bar{D}^2, \mathbb{C})$ such that $\Phi(S^1)$ is a curve of curvature 1; hence, up to a translation, $\Phi(S^1) \subseteq S^1$, and therefore it is a Möbius transformation of the disk. From (15) we infer that $\lambda = \log\big|\Phi'|_{S^1}\big|$, and we conclude. □

The following corollary is an easy consequence of Theorem 1.4 and Corollary 2.3:

Corollary 2.4. *Let Φ, λ and κ be as in Theorem 1.4 and let $f : \bar{D}^2 \to \bar{D}^2$ be a Möbius diffeomorphism. Set $\tilde{\Phi} := \Phi \circ f$, $\tilde{\lambda} := \log\big|\tilde{\Phi}'|_{S^1}\big|$ and $\tilde{\kappa} := \kappa \circ f|_{S^1}$. Then*

$$\tilde{\lambda} = \lambda \circ f|_{S^1} + \log\big|f'|_{S^1}\big| \quad \textit{and} \quad (-\Delta)^{\frac{1}{2}}\tilde{\lambda} = \tilde{\kappa} e^{\tilde{\lambda}} - 1.$$

Remark 2.5. One can also give an analogous geometric characterization for an equation of the type

$$(-\Delta)^{\frac{1}{2}}\lambda = \kappa e^{\lambda} - n \quad \text{in } S^1 \tag{35}$$

with $n > 1$. In this case there is a correspondence between the solutions of (35) and holomorphic functions $\Phi : D^2 \to \mathbb{C}$ of the form $\Phi'(z) = \Psi(z)h(z)$, where Ψ is the Blaschke product

$$\Psi(z) := \prod_{k=1}^{n-1} \frac{z - a_k}{1 - \bar{a}_k z}, \qquad a_1, \dots, a_{n-1} \in D^2,$$

and $h(z) \neq 0$ for every $z \in \bar{D}^2$. In this case, $n - 1 = i\Psi \cdot \partial\Psi/\partial\theta = \deg \Psi$.

Next, we show that the existence of a holomorphic immersion of the disk $\bar{D}^2$ is equivalent to the existence of a positive diffeomorphism of the disc $\bar{D}^2$. Such a result can be seen as a sort of generalized Riemann mapping theorem in the case of closed curves which are not necessarily injective. We start with the following lemma, giving better regularity up to the boundary of a holomorphic immersion $u : D^2 \to \mathbb{C}$ under the assumption that the curve $u|_{S^1}$ has a $W^{2,\infty}$-constant-speed parametrization.

Lemma 2.6. *Let $u \in C^0(\bar{D}^2, \mathbb{C})$ be holomorphic in D^2 with $\partial_z u \neq 0$ in D^2 and suppose there is $\gamma \in W^{2,\infty}(S^1, \mathbb{C})$ with $|\dot{\gamma}|$ constant and a homeomorphism $\sigma : S^1 \to S^1$ such that $\gamma = u \circ \sigma$. Then $u \in W^{2,p}(\bar{D}^2, \mathbb{C})$ for every $p < +\infty$ and $\partial_z u(z) \neq 0$ for all $z \in S^1$.*

Proof. Let $z_0 \in S^1$. Since $\dot{\gamma}(z_0) \neq 0$, we can find some $\rho > 0$ such that $\gamma(S^1 \cap B(z_0, \rho))$ coincides up to a rotation with a piece of the graph of a function $\varphi \in C^{1,\alpha}(\mathbb{R})$ that satisfies $\varphi'(u_1(x_0)) = 0$. We may also assume that $u = u_1 + iu_2$ takes values in the set $\{(\xi, \eta) \in \mathbb{R}^2 \mid \eta \geq \varphi(\xi)\}$. Define

$$\hat{u} = \hat{u}_1 + i\hat{u}_2 \quad \text{with} \quad \hat{u}_1 := u_1, \quad \hat{u}_2 := u_2 - \varphi(u_1).$$

Claim. *The function $\hat{u}_2$ satisfies*

$$\begin{cases} \partial_{x_i}(a_{ij}\partial_{x_j}\hat{u}_2) = 0 & \textit{in } B(x_0, \rho) \cap D^2, \\ \hat{u}_2 = 0 & \textit{in } B(x_0, \rho) \cap S^1, \end{cases} \tag{36}$$

where the matrix

$$(a_{ij}) = \begin{pmatrix} 1 - \dfrac{1}{1 + (\varphi')^2(u_1)} & \dfrac{\varphi'(u_1)}{1 + (\varphi')^2(u_1)} \\ -\dfrac{\varphi'(u_1)}{1 + (\varphi')^2(u_1)} & 1 - \dfrac{1}{1 + (\varphi')^2(u_1)} \end{pmatrix} \tag{37}$$

is in $L^\infty(\bar{D}^2)$ and uniformly elliptic.

Proof. We can write $u = \hat{u} + i\varphi(u_1)$. Since, by hypothesis, $\partial_{\bar{z}} u(z) = 0$ for all $z \in D^2$, the following estimates hold:

$$\begin{aligned} \partial_{\bar{z}} u_1 &= -i\,\partial_{\bar{z}} u_2, \\ \partial_{\bar{z}} \hat{u}(z) &= -i\varphi'(u_1)\partial_{\bar{z}} u_1 = -\varphi'(u_1)\partial_{\bar{z}} u_2, \\ \partial_{\bar{z}} u_1 + i\,\partial_{\bar{z}} \hat{u}_2(z) &= -i\varphi'(u_1)\partial_{\bar{z}} u_1, \\ \partial_{\bar{z}} u_1 &= -\frac{i}{1+i\varphi'(u_1)}\partial_{\bar{z}} \hat{u}_2(z), \\ \partial_{\bar{z}} \hat{u} &= -\frac{\varphi'(u_1)}{1+i\varphi'(u_1)}\partial_{\bar{z}} \hat{u}_2(z). \end{aligned}$$

Therefore,

$$\Delta \hat{u}_2 = 4\Im(\partial_z \partial_{\bar{z}} \hat{u}) = -4\Im\left[\partial_z\left[\frac{\varphi'(u_1)}{1+i\varphi'(u_1)}\partial_{\bar{z}} \hat{u}_2(z)\right]\right]. \tag{38}$$

Writing

$$\frac{\varphi'(u_1)}{1+i\varphi'(u_1)}\partial_{\bar{z}} \hat{u}_2(z) = \frac{\varphi'(u_1)}{1+(\varphi')^2(u_1)}\,\frac{\partial_{x_1}\hat{u}_2 + \varphi'(u_1)\partial_{x_2}\hat{u}_2 + i(\partial_{x_2}\hat{u}_2 - \varphi'(u_1)\partial_{x_1}\hat{u}_2)}{2},$$

we compute the right-hand side of (38) and get

$$\Delta \hat{u}_2 = -\Im\left[(\partial_{x_1} - i\,\partial_{x_2})\frac{\varphi'(u_1)}{1+(\varphi')^2(u_1)}[(\partial_{x_1}\hat{u}_2 + \varphi'(u_1)\partial_{x_2}\hat{u}_2) + i(\partial_{x_2}\hat{u}_2 - \varphi'(u_1)\partial_{x_1}\hat{u}_2)]\right].$$

Therefore $\hat{u}_2$ satisfies (36)–(37) and the claim is proven. □

Elliptic estimates imply that $\hat{u}_2 \in W^{2,p}(\bar{B}(z_0, r/4) \cap \bar{D}^2)$ for every $p < +\infty$; in particular, it is in $C^{1,\alpha}(\bar{B}(z_0, r/4) \cap \bar{D}^2)$ for every $\alpha \in (0, 1)$. Now, since $\hat{u}_2 \geq 0$ in $\bar{D}^2$ and $\hat{u}_2(z_0) = 0$, Hopf's lemma yields that $\partial_r \hat{u}_2(z_0) \neq 0$. Since $u = \hat{u} + i\varphi(u_1)$, it follows that

$$\partial_r u(z_0) = \partial_r \hat{u}_1(z_0) + i\,\partial_r \hat{u}_2(z_0) + i\,\underbrace{\varphi'(u_1(z_0))}_{=0}\,\partial_r \hat{u}_1(z_0) \neq 0$$

and, since $z_0 \in S^1$ was arbitrary, we conclude that $\partial_r u \neq 0$ everywhere on S^1. Then, since u is conformal up to the boundary, we also have $\partial_z u \neq 0$ on S^1. □

We introduce the set

$$\mathcal{T} := \big\{\gamma : S^1 \to \mathbb{C} \,\big|\, \gamma \in W^{2,\infty},\ |\dot{\gamma}| \text{ constant, and there is } \Psi \in C^1(\bar{D}^2, \mathbb{C}) \text{ with } \det \mathrm{Jac}(\Psi(z)) > 0,\ z \in D^2, \\ \text{and } (\Psi \circ \sigma)(z) = \gamma(z),\ z \in S^1, \text{ for some diffeomorphism } \sigma : S^1 \to S^1\big\}.$$

Theorem 2.7 (generalized Riemann mapping theorem). *A curve γ is in $\mathcal{T}$ if and only if there exists a holomorphic immersion $\Phi : \bar{D}^2 \to \mathbb{C}$ and a diffeomorphism $\sigma : S^1 \to S^1$ such that $\Phi \circ \sigma = \gamma$.*

Proof. (1) Suppose that there exists a holomorphic immersion $\Phi : \bar{D}^2 \to \mathbb{C}$ and a diffeomorphism $\sigma : S^1 \to S^1$ such that $\Phi \circ \sigma = \gamma$. Then one can take $\Psi = \Phi$. Therefore, $\gamma \in \mathcal{T}$.

(2) Conversely, let $\Psi \in C^1(\bar{D}^2, \mathbb{C})$ with $\Psi|_{S^1} = \gamma$ and $\det \mathrm{Jac}(\Psi) > 0$ in D^2.

(2i) Consider the pull-back of the Euclidean metric g on $\mathbb{R}^2$ by Ψ,

$$h_{ij} := \langle \partial_{x_i}\Psi, \partial_{x_j}\Psi\rangle.$$

Since $\det \mathrm{Jac}(\psi) > 0$, we have

$$c^{-1}\delta_{ij} \le (h_{ij}) \le c\delta_{ij}.$$

We can write

$$h = h_{11}\,dx^2 + 2h_{12}\,dx\,dy + h_{22}\,dy^2. \tag{39}$$

Setting $z = x + iy$, one can write h in the form

$$h = \nu\,|dz + \mu\,d\bar{z}|^2,$$

where ν is a positive continuous function on U and μ is a complex-valued continuous function with $\|\mu\|_{L^\infty(\bar{D}^2)} < 1$ on U. Actually, ν and μ are given by

$$\nu = \tfrac{1}{4}\big(h_{11} + h_{22} + 2\sqrt{h_{11}h_{22} - h_{12}^2}\big),$$
$$\mu = \frac{h_{11} - h_{22} + 2ih_{12}}{h_{11} + h_{22} + 2\sqrt{h_{11}h_{22} - h_{12}^2}}.$$

Moreover, Ψ solves the equation

$$\frac{\partial_{\bar{w}}\Psi(w)}{\partial_w\Psi(w)} = \mu(w) \quad \text{in } D^2. \tag{40}$$

The function μ is the so-called Beltrami coefficient associated to the metric h. Now we extend μ by 0 outside $\bar{D}^2$ (we still denote this extension by μ). Then there exists a unique homeomorphism $\xi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ (here $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \simeq S^2$) which satisfies, in a distributional sense,

$$\partial_{\bar{z}}\xi = \mu(z)\,\partial_z\xi \quad \text{in } \mathbb{C}$$

and the normalization conditions

$$\xi(0) = 0, \quad \xi(1) = 1, \quad \xi(\infty) = \infty.$$

Moreover, $\xi \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for some $p > 2$ and $\partial_z\xi \ne 0$ a.e. in $\mathbb{C}$. The function ξ is called a quasiconformal map with dilation coefficient μ (see, e.g., Theorem 4.30 in [Imayoshi and Taniguchi 1992]).

Since ξ is a homeomorphism, $\xi(S^1)$ is a Jordan curve.

(2ii) Consider now $\tilde{\Psi} := \Psi \circ \xi^{-1} : \xi(\bar{D}^2) \to \mathbb{C}$. From [Imayoshi and Taniguchi 1992, Proposition 4.13] it follows that the complex dilatation of $\tilde{\Psi}$ is 0 in $\xi(D^2)$; therefore, $\partial_{\bar{z}}\tilde{\Psi} = 0$ and $\tilde{\Psi}$ is holomorphic in $\xi(D^2)$; see [Imayoshi and Taniguchi 1992, Lemma 4.6].

(2iii) Now we apply the Riemann mapping theorem: there exists a biholomorphic map u from D^2 onto $\xi(D^2)$. In particular, $\partial_z u \ne 0$ in D^2. Take $\Phi := \Psi \circ \xi^{-1} \circ u$. We observe that $\det \mathrm{Jac}(\Psi) > 0$ implies $\partial_z\Psi \ne 0$ in $\bar{D}^2$. Therefore,

$$\partial_z\Phi = \partial_w(\Psi\circ\xi^{-1})\partial_z u + \partial_{\bar{w}}(\Psi\circ\xi^{-1})\partial_z\bar{u} = \partial_w(\Psi\circ\xi^{-1})\partial_z u + \partial_{\bar{w}}(\Psi\circ\xi^{-1})\overline{\partial_{\bar{z}}u} = \partial_w(\Psi\circ\xi^{-1})\partial_z u.$$

We observe that Φ is holomorphic in D^2 because it is the composition of two holomorphic maps and $\partial_z\Phi \neq 0$ in D^2. From Lemma 2.6, it follows that $\partial_z\Phi \neq 0$ in $\bar{D}^2$ and we conclude the proof of Theorem 2.7. □

From the next lemma we can deduce that if $\gamma \in \mathcal{T}$ then the winding number (or equivalently the degree) of γ is 1.

Lemma 2.8. *Let $\Phi \in W^{2,p}(\bar{D}^2, \mathbb{C})$ for some $1 < p \leq +\infty$ be a holomorphic function such that $\partial_z\Phi \neq 0$ in $\bar{D}^2$. Then*

$$\deg\Phi = \frac{1}{2\pi}\int_0^{2\pi} \frac{\langle i\partial_\theta\Phi, \partial_\theta^2\Phi\rangle}{|\partial_\theta\Phi|^2}\, d\theta = 1 + \frac{1}{2\pi i}\int_{S^1} \frac{f'(z)}{f(z)}\, dz = 1, \tag{41}$$

where $f(z) = \Phi'(z)$.

We note that Lemma 2.8 is a direct corollary of Theorem 1.4. Indeed, $\deg\Phi|_{S^1} = (1/2\pi)\int_{S^1} \kappa|\Phi'|\, d\theta = (1/2\pi)\int_{S^1} \kappa e^\lambda\, d\theta$ but, since $(-\Delta)^{\frac{1}{2}}\lambda = \kappa e^\lambda - 1$, integrating gives $\int_{S^1} \kappa e^\lambda\, d\theta = 2\pi$.

Anyway, we provide a direct proof for the reader's convenience:

Proof. We recall that

$$\Phi'(z) = \frac{1}{2}e^{-i\theta}\left(\frac{\partial\Phi}{\partial r} - \frac{i}{r}\frac{\partial\Phi}{\partial\theta}\right) =: f(z).$$

Since Φ is holomorphic, we have

$$\frac{\partial\Phi}{\partial r} = -\frac{i}{r}\frac{\partial\Phi}{\partial\theta}. \tag{42}$$

Hence,

$$\begin{aligned}
\int_{S^1}\frac{f'(z)}{f(z)}\,dz &= \int_{S^1} \frac{\frac{e^{-i\theta}}{2}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right)\frac{e^{-i\theta}}{2}\left(\frac{\partial\Phi}{\partial r} - \frac{i}{r}\frac{\partial\Phi}{\partial\theta}\right)}{\frac{e^{-i\theta}}{2}\left(\frac{\partial\Phi}{\partial r} - \frac{i}{r}\frac{\partial\Phi}{\partial\theta}\right)}\,dz \\
&= \int_{S^1} \frac{\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right)\left(-\frac{i}{r}e^{-i\theta}\frac{\partial\Phi}{\partial\theta}\right)}{\frac{\partial\Phi}{\partial r} - \frac{i}{r}\frac{\partial\Phi}{\partial\theta}}\,dz \qquad \text{(by (42))} \\
&= \int_{S^1} e^{-i\theta}\frac{\frac{2i}{r^2}\frac{\partial\Phi}{\partial\theta} - \frac{i}{r}\frac{\partial^2\Phi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial^2\Phi}{\partial^2\theta}}{\frac{-2i}{r}\frac{\partial\Phi}{\partial\theta}}\,dz \\
&= -\int_{S^1} e^{-i\theta}\,dz + \int_{S^1} e^{-i\theta}\frac{\frac{\partial^2\Phi}{\partial r\partial\theta}}{-2i\frac{\partial\Phi}{\partial\theta}}\,dz \int_{S^1} e^{-i\theta}\frac{\frac{\partial^2\Phi}{\partial^2\theta}}{-2i\frac{\partial\Phi}{\partial\theta}}\,dz \qquad (\text{since } r = 1 \text{ on } S^1) \\
&= -2\pi i - \frac{i}{2}\int_0^{2\pi}\frac{\frac{\partial^2\Phi}{\partial r\partial\theta}}{\frac{\partial\Phi}{\partial\theta}}\,d\theta - \frac{1}{2}\int_0^{2\pi}\frac{\frac{\partial^2\Phi}{\partial\theta\partial\theta}}{\frac{\partial\Phi}{\partial\theta}}\,d\theta \\
&= -2\pi i - \int_0^{2\pi}\frac{\frac{\partial^2\Phi}{\partial\theta\partial\theta}}{\frac{\partial\Phi}{\partial\theta}}\,d\theta \qquad \text{(by (42))}.
\end{aligned} \tag{43}$$

On the other hand, we have

$$\int_0^{2\pi} \frac{\langle i\partial_\theta\Phi, \partial_\theta^2\Phi\rangle}{|\partial_\theta\Phi|^2}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{-i\overline{\partial_\theta\Phi}\partial^2_{\theta^2}\Phi}{\overline{\partial_\theta\Phi}\partial_\theta\Phi}\,d\theta + \frac{1}{2}\int_0^{2\pi} \frac{i\partial_\theta\Phi\overline{\partial^2_{\theta^2}\Phi}}{\overline{\partial_\theta\Phi}\partial_\theta\Phi}\,d\theta. \tag{44}$$

We observe that

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi} \frac{i\partial_\theta\Phi\overline{\partial^2_{\theta^2}\Phi}}{\overline{\partial_\theta\Phi}\partial_\theta\Phi}\,d\theta &= -\frac{i}{2}\int_0^{2\pi} \overline{\partial_\theta\Phi}\frac{\partial^2_{\theta^2}\Phi}{|\partial_\theta\Phi|^2}\,d\theta - \frac{i}{2}\int_0^{2\pi} |\partial_\theta\Phi|^2\partial_\theta(|\partial_\theta\Phi|^{-2})\,d\theta \\ &= -\frac{i}{2}\int_0^{2\pi} \frac{\partial^2_{\theta^2}\Phi}{\partial_\theta\Phi}\,d\theta. \end{aligned}\tag{45}$$

It follows that

$$\int_0^{2\pi} \frac{\langle i\partial_\theta\Phi, \partial_\theta^2\Phi\rangle}{|\partial_\theta\Phi|^2}\,d\theta = -i\int_0^{2\pi} \frac{\partial^2_{\theta^2}\Phi}{\partial_\theta\Phi}\,d\theta. \tag{46}$$

By combining the estimates (43)–(46), we get

$$\int_{S^1} \frac{1}{2\pi i}\frac{f'(z)}{f(z)}\,dz = -1 - \frac{1}{2\pi i}\int_0^{2\pi} \frac{\partial^2_{\theta^2}\Phi}{\partial_\theta\Phi}\,d\theta = -1 + \frac{1}{2\pi}\int_0^{2\pi} \frac{\langle i\partial_\theta\Phi, \partial_\theta^2\Phi\rangle}{|\partial_\theta\Phi|^2}\,d\theta. \qquad \square$$

Connection with half-harmonic maps. In this subsection we show an interesting connection between the solutions of (7) and the half-harmonic maps into a given curve Γ.

Let $\tilde{\phi} = \Phi \in C^1(\overline{D}^2, \mathbb{C})$ be the map given by Theorem 2.7 and set $\phi := \Phi|_{S^1}$. Then Φ is conformal up to the boundary, i.e., $\partial\phi/\partial\theta \cdot \partial\tilde{\phi}/\partial r = 0$ on S^1. Since $\partial\tilde{\phi}/\partial r\big|_{r=1} = (-\Delta)^{\frac{1}{2}}\phi$, we deduce

$$(-\Delta)^{\frac{1}{2}}\phi \perp T_\phi\Gamma, \quad \text{i.e.,} \quad \frac{\partial\phi}{\partial\theta}\cdot(-\Delta)^{\frac{1}{2}}\phi = 0 \quad \text{on } \mathcal{D}'(S^1). \tag{47}$$

Equation (47) says that ϕ is a $\frac{1}{2}$-harmonic map into Γ (see [Da Lio and Rivière 2011]).

We would like to recall a characterization of $\frac{1}{2}$-harmonic maps of S^1 into submanifolds of $\mathbb{R}^n$, which has been already observed in [Da Lio 2015] and then in [Millot and Sire 2015].

Theorem 2.9 [Da Lio et al. $\geq$ 2015]. *Let $u \in H^{\frac{1}{2}}(S^1, \mathcal{N})$, where $\mathcal{N}$ is a k-dimensional smooth submanifold of $\mathbb{R}^m$ without boundary. Then u is a weak $\frac{1}{2}$-harmonic map, i.e., $(-\Delta)^{\frac{1}{2}}u \perp T_u\mathcal{N}$, if and only if its harmonic extension $\tilde{u} \in W^{1,2}(D^2, \mathbb{R}^m)$ is conformal, in which case*

$$\partial_r\tilde{u} \perp T_u\mathcal{N} \quad \text{in } \mathcal{D}'(S^1). \tag{48}$$

Proof. Let $u \in H^{\frac{1}{2}}(S^1, \mathcal{N})$ be a weak $\frac{1}{2}$-harmonic map and let $\tilde{u} \in W^{1,2}(D, \mathbb{R}^m)$ be the harmonic extension of u. Then

$$E(u) := \int_{S^1} |(-\Delta)^{\frac{1}{4}}u|^2\,|dz| = \int_{D^2} |\nabla\tilde{u}|^2\,|dz|.$$

Claim. *For every $\tilde{X} \in C^\infty(\overline{D}^2, \mathbb{R}^2)$ such that $\tilde{X}(z)\cdot z = 0$ for $z \in S^1$,*

$$\left(\frac{d}{dt}\int_{D^2}\big|\nabla\tilde{u}(z + t\tilde{X}(z))\big|^2\,|dz|\right)\bigg|_{t=0} = 0. \tag{49}$$

Proof of the claim. It has been proved in [Da Lio and Rivière 2011] that, if u is $\frac{1}{2}$-harmonic, then $u \in C^\infty(S^1)$; in particular, u satisfies

$$\left(\frac{d}{dt}\int_{S^1} \left|(-\Delta)^{\frac{1}{4}} u(z+tX(z))\right|^2 |dz|\right)\bigg|_{t=0} = 0 \tag{50}$$

for every $X \in C^\infty(S^1)$.

Let $\tilde{X} \in C^\infty(\bar{D}^2, \mathbb{R}^2)$ be such that $\tilde{X}(z)\cdot z = 0$ for $z \in S^1$. We observe that, for all $z \in S^1$, $Y := d\tilde{u}\cdot\tilde{X} = du\cdot\tilde{X} \in T_u\mathcal{N}$ and

$$\left(\frac{d}{dt}\int_{D^2} |\nabla\tilde{u}(z+t\tilde{X}(z))|^2 |dz|\right)\bigg|_{t=0} = \int_{D^2} \nabla\tilde{u}\cdot\nabla Y\,|dz| = \int_{S^1} \partial_r\tilde{u}\cdot Y\,|dz| = -\int_{S^1} (-\Delta)^{\frac{1}{2}} u\cdot Y\,|dz| = 0,$$

where the last equality follows from (50). □

From Proposition 2.10 below and the regularity of $\tilde{u}$ up to the boundary, it follows that $\tilde{u}$ is also conformal in $\bar{D}^2$, i.e.,

$$|\partial_{x_1}\tilde{u}| = |\partial_{x_2}\tilde{u}|, \qquad \partial_{x_1}\tilde{u}\cdot\partial_{x_2}\tilde{u} = 0.$$

Conversely, suppose the harmonic extension $\tilde{u}$ of u is conformal and satisfies (48). Since $\partial_r\tilde{u} = -(-\Delta)^{\frac{1}{2}}u$, we deduce that u is $\frac{1}{2}$-harmonic. □

Proposition 2.10 [Rivière 2012, Proposition II.2]. *Let $\tilde{u}$ be a map in $W^{1,2}(D^2, \mathbb{R}^m)$ satisfying*

$$\left(\frac{d}{dt}\int_{D^2} |\nabla\tilde{u}_t|^2 |dz|\right)\bigg|_{t=0} = 0, \qquad u_t(x) := u(x+tX(x)),$$

for every $X \in C^\infty(\bar{D}^2, \mathbb{R}^2)$ such that $\langle X(x), x\rangle = 0$ for $x \in S^1$. Then $\tilde{u}$ is conformal in D^2.

In the case of $\frac{1}{2}$-harmonic maps $u : S^1 \to S^1$, we deduce from Theorem 2.9 the following:

Corollary 2.11. *Let $u \in H^{\frac{1}{2}}(S^1, S^1)$ with* $\deg u = 1$. *Then u is a weak $\frac{1}{2}$-harmonic map if and only if its harmonic extension $\tilde{u} : \bar{D}^2 \to \bar{D}^2$ is a Möbius map, namely it has the form*

$$\tilde{u}(z) = e^{i\theta_0}\frac{z-a}{1-\bar{a}z}$$

for some $|a| < 1$ and $\theta_0 \in [0, 2\pi)$.

3. Compactness of the Liouville equation in S^1

In this section we analyze the asymptotics of solutions to (7).

The ε-regularity lemma and first compactness result. A key point in the proof of Theorem 1.2 is an *ε-regularity lemma*, asserting, roughly speaking, that if the L^1 norm in conformal parametrization of the curvature ($\kappa_k e^{\lambda_k}$) is small (less than π) in a neighborhood of a point, then $\lambda_k - C_k$ is uniformly bounded in the same neighborhood for some constant C_k. This result (Lemma 3.3) depends on Theorem 3.2 below.

Lemma 3.1 (fundamental solution of $(-\Delta)^{\frac{1}{2}}$ on S^1). *The function*

$$G(\theta) := -\frac{1}{2\pi}\log(2(1-\cos\theta))$$

belongs to $\mathrm{BMO}(S^1)$, *can be decomposed as*

$$G(\theta)=\frac{1}{\pi}\log\frac{\pi}{|\theta|}+H(\theta),\qquad \theta\in[-\pi,\pi]\sim S^1,\qquad \textit{with}\ H\in C^0(S^1), \tag{51}$$

and satisfies

$$(-\Delta)^{\frac{1}{2}}G=\delta_1-\frac{1}{2\pi}\quad \textit{in}\ S^1,\qquad \int_{S^1}G(\theta)\,d\theta=0, \tag{52}$$

and, for every function $u\in L^1(S^1)$ *with* $(-\Delta)^{\frac{1}{2}}u\in L^1(S^1)$, *one has*

$$u-\bar u=G*(-\Delta)^{\frac{1}{2}}u:=\int_{S^1}G(\,\cdot-\theta)(-\Delta)^{\frac{1}{2}}u(\theta)\,d\theta\quad \textit{for almost every}\ t\in S^1. \tag{53}$$

Proof. The identity (52) follows at once from Lemma 4.3. That $G\in\mathrm{BMO}(S^1)$ follows from parametrizing $S^1=[-\pi,\pi]/\{\pi\sim-\pi\}$, writing $1-\cos\theta=\frac{1}{2}\theta^2+O(\theta^4)$ as $\theta\to 0$ and therefore

$$G(\theta)=-\frac{1}{2\pi}\big(\log\big(\tfrac{1}{2}\theta^2\big)+\log(1+O(\theta^2))\big)$$

as $\theta\to 0$. Similarly, (51) follows from the explicit expression of G, since

$$H(\theta):=G(\theta)-\frac{1}{\pi}\log\frac{\pi}{|\theta|}=C+\log(1+O(\theta)^2)\to C\quad \text{as }\ \theta\to 0$$

and $H(\theta)\to-(\log 2)/(2\pi)$ as $|\theta|\to\pi$, so that $H\in C^0(S^1)$.

To prove (53) for $u\in C^\infty$, we write

$$u(0)-\bar u=\Big\langle\delta_1-\frac{1}{2\pi},u\Big\rangle=\langle(-\Delta)^{\frac{1}{2}}G,u\rangle:=\int_{S^1}G(\theta)(-\Delta)^{\frac{1}{2}}u(\theta)\,d\theta$$

and, translating, one gets (53) also for $t\neq 0$. For a general function $u\in H^{1,1}_{\Delta}(S^1)$, take a sequence $(u_k)\subset C^\infty(S^1)$ with

$$u_k\to u,\quad (-\Delta)^{\frac{1}{2}}u_k\to(-\Delta)^{\frac{1}{2}}u\quad \text{in } L^1(S^1),$$

which can be easily obtained by convolution. Then

$$u\xleftarrow{L^1(S^1)}u_k=\int_{S^1}G(\,\cdot-\theta)(-\Delta)^s u_k(\theta)\,dy\xrightarrow{L^1(S^1)}\int_{S^1}G(\,\cdot-\theta)(-\Delta)^s(\theta)\,d\theta,$$

the convergence on the right following from (51) and Fubini's theorem:

$$\int_{S^1}\left|\int_{S^1}G(t-\theta)[(-\Delta)^s u_k(\theta)-(-\Delta)^s u(\theta)]\,d\theta\right|dt\le\|G\|_{L^1(S^1)}\|(-\Delta)^s u_k-(-\Delta)^s u\|_{L^1(S^1)}\to 0$$

as $k\to\infty$. Since the convergence in L^1 implies a.e. convergence (up to a subsequence), (53) follows. The last claim follows at once from the explicit expression of G. □

The following theorem, which is a generalization of Theorem I in [Brezis and Merle 1991], is a sort of Moser–Trudinger inequality and it is crucial for proving Lemma 3.3.

Theorem 3.2. *There exist constants $C_1, C_2 > 0$ such that, for any $\varepsilon \in (0, \pi)$, one has*

$$C_1 \le \sup_{\substack{u = G * f \\ \|f\|_{L^1(S^1)} \le 1}} \varepsilon \int_{S^1} e^{(\pi-\varepsilon)|u|}\, d\theta \le C_2 \tag{54}$$

and, in particular,

$$C_1 \le \sup_{\substack{u \in L^1(S^1): \\ \|(-\Delta)^{\frac{1}{2}} u - \alpha\|_{L^1(S^1)} \le 1 \\ \text{for some } \alpha \in \mathbb{R}}} \varepsilon \int_{S^1} e^{(\pi-\varepsilon)|u-\bar{u}|}\, d\theta \le C_2. \tag{55}$$

Proof. Clearly the second inequality in (55) follows from the second inequality in (54) and (52). Let us now prove (54). Given f with $\|f\|_{L^1(S^1)} \le 1$ and setting $u = G * f$, we get

$$\begin{aligned} |u(t)| &= \left| \frac{1}{\pi} \int_{t-\pi}^{t+\pi} \log\left(\frac{\pi}{|\theta - t|}\right) f(\theta)\, d\theta + \int_{t-\pi}^{t+\pi} H(\theta - t) f(\theta)\, d\theta \right| \\ &\le \frac{1}{\pi} \int_{t-\pi}^{t+\pi} \log\left(\frac{\pi}{|\theta - t|}\right) |f(\theta)|\, d\theta + C. \end{aligned}$$

With Jensen's inequality and Fubini's theorem, and using that $\|f\|_{L^1(S^1)} \le 1$, it follows that

$$\begin{aligned} \int_{-\pi}^{\pi} e^{(\pi-\varepsilon)|u(t)-\bar{u}|}\, dt &\le C \int_{-\pi}^{\pi} \exp\left(\frac{\pi-\varepsilon}{\pi} \int_{t-\pi}^{t+\pi} \log\left(\frac{\pi}{|\theta - t|}\right) |f(\theta)|\, d\theta\right) dt \\ &\le C \int_{-\pi}^{\pi} \int_{t-\pi}^{t+\pi} \exp\left(\frac{\pi-\varepsilon}{\pi} \log \frac{\pi}{|\theta - t|}\right) |f(\theta)|\, d\theta\, dt \\ &= C \int_{t-\pi}^{t+\pi} |f(\theta)| \int_{-\pi}^{\pi} \left(\frac{\pi}{|\theta - t|}\right)^{1-\frac{\varepsilon}{\pi}} dt\, d\theta \le \frac{C_2}{\varepsilon}. \end{aligned} \tag{56}$$

This proves the second inequality in (54).

To prove the first inequalities in (54) and in (55), fix $\varepsilon \in (0, \pi)$, choose $(f_k) \subset C^\infty(S^1)$ nonnegative such that $f_k \to \delta_0$ weakly in the sense of measures with $\|f_k\|_{L^1(S^1)} = 1$, and let u_k solve

$$(-\Delta)^{\frac{1}{2}} u_k = f_k - \frac{1}{2\pi} \qquad \text{in } S^1, \quad \bar{u}_k = 0.$$

Such u_k can easily be constructed using the Fourier formula for $(-\Delta)^{\frac{1}{2}}$; see (123). Then, by Lemma 3.1,

$$|u_k(t)| \ge \int_{S^1} G(t-\theta) f_k(\theta)\, d\theta \ge \frac{1}{\pi} \int_{t-\pi}^{t+\pi} \log\left(\frac{\pi}{|\theta - t|}\right) f_k(\theta)\, d\theta - C.$$

Multiplying by $\pi - \varepsilon$, exponentiating, integrating on S^1 and taking the limit as $k \to \infty$, one gets

$$\begin{aligned} \lim_{k\to\infty} \int_{S^1} e^{(\pi-\varepsilon)|u_k(t)|}\, dt &\ge \lim_{k\to\infty} \frac{1}{C} \int_{-\pi}^{\pi} \exp\left(\frac{\pi-\varepsilon}{\pi} \int_{t-\pi}^{t+\pi} \log\left(\frac{\pi}{|\theta - t|}\right) f_k(\theta)\, d\theta\right) dt \\ &= \frac{1}{C} \int_{-\pi}^{\pi} \exp\left(\frac{\pi-\varepsilon}{\pi} \log \frac{\pi}{|t|}\right) dt = \frac{1}{C} \int_{-\pi}^{\pi} \left(\frac{\pi}{|t|}\right)^{1-\frac{\varepsilon}{\pi}} dt = \frac{C_1}{\varepsilon}, \end{aligned}$$

which proves (54) and also (55), since $\bar{u}_k = 0$. □

Lemma 3.3 (ε-regularity lemma). *Let $u \in L^1(S^1)$ be a solution of*

$$(-\Delta)^{\frac{1}{2}} u = \kappa e^u - 1 \tag{57}$$

with $\kappa \in L^\infty(S^1)$, $e^u \in L^1(S^1)$ and $\Lambda := \|\kappa e^u\|_{L^1}$. Assume that, for some arc $A \subset S^1$,

$$\int_A |\kappa| e^u \, d\theta \le \pi - \varepsilon \tag{58}$$

for some $\varepsilon > 0$. Then, for every arc $A' \Subset A$ with $\operatorname{dist}(A^c, A') = \delta$,

$$\|u - \bar{u}\|_{L^\infty(A')} \le C(\delta, \varepsilon, \Lambda). \tag{59}$$

Proof. Set $f := (-\Delta)^{\frac{1}{2}} u$. We split $f = f_1 + f_2$, where

$$f_1 = \kappa e^u \chi_A, \qquad f_2 = \kappa e^u \chi_{A^c}.$$

Let us now define

$$u_i(t) := G * f_i(t) = \int_{S^1} G(t-\theta) f_i(\theta)\, d\theta, \quad i = 1, 2,$$

where G is as in Lemma 3.1. From (52) and (53) it follows that

$$u - \bar{u} = G * (\kappa e^u - 1) = G * (\kappa e^u) = u_1 + u_2.$$

Choose an arc A'' with $A' \Subset A'' \Subset A$ and $\operatorname{dist}(A'', A^c) = \operatorname{dist}(A', (A'')^c) = \frac{1}{2}\delta$. With (51) we easily bound

$$\|u_2\|_{L^\infty(A'')} \le C_1 = C_1(\Lambda, \delta). \tag{60}$$

It follows from (58) and Theorem 3.2 that $\|e^{|u_1|}\|_{L^p(S^1)} \le C_{p,\varepsilon}$ for some $p > 1$ and, consequently, also $e^{\bar{u}} \le C$. Then, for $t \in A'$ we have

$$\begin{aligned} u_1(t) &\le \int_A G(t-\theta)(|\kappa| e^{u_1(\theta)} e^{u_2(\theta)+\bar{u}} - 1)\, d\theta \\ &\le \|\kappa\|_{L^\infty} \Bigg(e^{C_1+\bar{u}} \underbrace{\int_{A''} G(t-\theta) e^{u_1(\theta)}\, d\theta}_{(1)} + \underbrace{\int_{A\setminus A''} G(t-\theta) e^{u(\theta)}\, d\theta}_{(2)} + C \Bigg) \\ &\le C, \end{aligned}$$

where in (1) we use that $G \in L^q(S^1)$ for $q \in [1, \infty)$ and in (2) we use that $G \in L^\infty(A' \times (A \setminus A''))$. $\square$

Lemma 3.4. *Let $\lambda : S^1 \to S^1$ satisfy $(-\Delta)^{\frac{1}{2}}\lambda \in L^1(S^1)$ and let $\tilde{\lambda}$ be the harmonic extension of λ to D^2. Then*

$$\|\nabla \tilde{\lambda}\|_{L^{(2,\infty)}(D^2)} \le C \|(-\Delta)^{\frac{1}{2}}\lambda\|_{L^1(S^1)} \tag{61}$$

and, for any ball $B_r(x_0)$,

$$\frac{1}{r}\int_{B_r(x_0)\cap D^2} |\nabla \tilde{\lambda}|\, dx \le C \|\nabla \tilde{\lambda}\|_{L^{(2,\infty)}(B_r(x_0)\cap D^2)}. \tag{62}$$

Proof. Let $\lambda : S^1 \to S^1$ satisfy $(-\Delta)^{\frac{1}{2}}\lambda \in L^1(S^1)$ and let $\tilde{\lambda}$ be the harmonic extension of λ to D^2. Then we can write

$$\tilde{\lambda}(x) = \int_{S^1} G(x, y)\frac{\partial \tilde{\lambda}}{\partial \nu}(y)\,dy = \int_{S^1} G(x, y)(-\Delta)^{\frac{1}{2}}\lambda(y)\,dy, \tag{63}$$

where G is the Green function associated to the Neumann problem. It is known that $\nabla_x(G(x, y))$ is in $L^{(2,\infty)}(S^1)$ (see, e.g., [Kenig 1994]). Therefore, $\nabla\tilde{\lambda}(x) \in L^{(2,\infty)}(D^2)$ as well and (61) holds.

The proof of (62) follows from O'Neil's inequality [1963]

$$\int_A |\nabla\tilde{\lambda}|\,dx \le \|\chi_A\|_{L^{(2,1)}(A)}\|\nabla\tilde{\lambda}\|_{L^{(2,\infty)}(A)} = \sqrt{|A|}\,\|\nabla\tilde{\lambda}\|_{L^{(2,\infty)}(A)}$$

for any $A \subset D^2$. □

Theorem 3.5. *Let (λ_k) be a sequence as in Theorem 1.2 and let $(\Phi_k) \subset C^1(\bar{D}^2, \mathbb{C})$ be holomorphic immersions with $\lambda_k(z) = \log|\Phi_k'(z)|$ for $z \in S^1$ and $\Phi_k(1) = 0$ (compare to Theorem 1.4) Then, up to extracting a subsequence, the set*

$$B := \left\{ a \in S^1 \,\middle|\, \lim_{r\to 0^+}\limsup_{k\to\infty}\int_{B(a,r)\cap S^1} |\kappa_k| e^{\lambda_k}\,d\theta \ge \pi \right\} = \{a_1, \dots, a_N\} \tag{64}$$

is finite and, for functions $v_\infty \in L^1(S^1, \mathbb{R})$ and $\Phi_\infty \in W^{1,2}(D^2, \mathbb{C})$ we have, for $1 \le p < \infty$,

$$\lambda_k - \bar{\lambda}_k \rightharpoonup v_\infty \quad \text{in } W^{1,p}_{\text{loc}}(S^1 \setminus B), \qquad \bar{\lambda}_k := \frac{1}{2\pi}\int_{S^1} \lambda_k\,d\theta, \tag{65}$$

and

$$\Phi_k \rightharpoonup \Phi_\infty \quad \text{in } W^{2,p}_{\text{loc}}(\bar{D}^2 \setminus B, \mathbb{C}) \text{ and in } W^{1,2}(D^2, \mathbb{C}). \tag{66}$$

Moreover, one of the following alternatives holds:

(1) *The sequence $(\lambda_k) \subset \mathbb{R}$ is bounded and Φ_∞ is a holomorphic immersion of $\bar{D}^2 \setminus B$ (i.e., it is holomorphic in D^2 and $\partial_z\Phi_\infty \neq 0$ for $z \in \bar{D}^2 \setminus B$).*

(2) *$\lambda_k \to -\infty$ locally uniformly as $k \to +\infty$ and $\Phi_\infty \equiv Q$ for some constant $Q \in \mathbb{C}$.*

Proof. The sequence of measures $|\kappa_k|e^{\lambda_k}\,d\theta$ on S^1 is bounded (for the total variation norm); hence, up to extracting a subsequence, we have $|\kappa_k|e^{\lambda_k}\,dx \overset{*}{\rightharpoonup} \mu$ weakly in the sense of measures for a Radon measure $\mu \in \mathcal{M}(S^1)$. Let $B := \{a \in S^1 \mid \mu(\{a\}) \ge \pi\}$. Then B is clearly finite, say $B = \{a_1, \dots, a_N\}$, and is characterized by the first identity in (64). Indeed, if $\mu(\{a\}) \ge \pi$, then for every $r > 0$ and $\varphi \in C^0(S^1)$ supported in $B(a, r) \cap S^1$ such that $0 \le \varphi \le 1 = \varphi(a)$ one has

$$\limsup_{k\to\infty}\int_{B(a,r)\cap S^1} |\kappa_k|e^{\lambda_k}\,d\theta \ge \limsup_{k\to\infty}\int_{S^1} |\kappa_k|e^{\lambda_k}\varphi\,d\theta = \int_{S^1}\varphi\,d\mu \ge \pi\varphi(a) = \pi,$$

and, conversely, if $\mu(\{a\}) < \pi$, then $\mu(B(a, r_0) \cap S^1) < \pi$ for some $r_0 > 0$; hence, taking $\varphi \in C^0(S^1)$ supported in $B(a, r_0) \cap S^1$ with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $B(a, r_0/2) \cap S^1$, one gets

$$\limsup_{k\to\infty}\int_{B(a,r_0/2)\cap S^1} |\kappa_k|e^{\lambda_k}\,d\theta \le \limsup_{k\to\infty}\int_{S^1} |\kappa_k|e^{\lambda_k}\varphi\,d\theta = \int_{S^1}\varphi\,d\mu \le \mu(B(a, r_0)) < \pi.$$

We now show that for every compact $K \subset S^1 \setminus B$ there exists a constant c_K depending on $\bar{L}$ and $\bar{\kappa}$ in (8)–(10) such that

$$\|e^{\lambda_k}\|_{L^\infty(K)} \le c_K \tag{67}$$

and

$$\|\lambda_k - \bar{\lambda}_k\|_{L^\infty(K)} \le c_K. \tag{68}$$

Indeed, cover K with finitely many arcs $A_i \cap S_1$ such that

$$\int_{A_i \cap S^1} |\kappa_k| e^{\lambda_k}\, d\theta < \pi.$$

From Lemma 3.3 it follows that $\lambda_k - \bar{\lambda}_k$ is bounded in each A_i, and (68) follows. Moreover, considering that $\|e^{\lambda_k}\|_{L^1(S^1)} = L_k \le \bar{L}$, it follows that $\bar{\lambda}_k$ and λ_k are bounded above, and this proves (67). Now, writing $\lambda_k - \bar{\lambda}_k = G * (\kappa_k e^{\lambda_k} - 1)$ as in (53) of Lemma 3.1, we can bootstrap regularity and obtain that $\lambda_k - \bar{\lambda}_k$ is bounded in $W^{1,p}(K)$ for every $p < \infty$, and (65) follows from weak compactness.

Let $\tilde{\lambda}_k$ be the harmonic extension of λ_k. From (68), (61) and (62) we get

$$\|\tilde{\lambda}_k - \bar{\lambda}_k\|_{L^\infty(\partial(D^2 \setminus \cup_{i=1}^N B(a_i,\delta)))} \le C_\delta \quad \text{for every } \delta > 0;$$

hence,

$$(\tilde{\lambda}_k - \bar{\lambda}_k) \quad \text{is bounded in } W^{1,p}_{\text{loc}}(\bar{D}^2 \setminus B). \tag{69}$$

Since Φ_k is harmonic and conformal,

$$\int_{D^2} |\nabla \Phi_k(z)|^2 \le \tfrac{1}{2} L_k^2. \tag{70}$$

Since $\Phi_k(1) = 0$, it follows that the sequence (Φ_k) is bounded in $W^{1,2}(D^2)$ and, up to a subsequence, $\Phi_k \rightharpoonup \Phi_\infty$ weakly in $W^{1,2}(D^2)$, where Φ_∞ is holomorphic.

From (16) it follows that $|\nabla \Phi_k|$ is bounded in $W^{1,p}_{\text{loc}}(S^1 \setminus B)$, so Φ_k is bounded in $W^{2,p}_{\text{loc}}(S^1 \setminus B)$ and up to a subsequence one gets $\Phi_k \rightharpoonup \Phi_\infty$ in $W^{2,p}_{\text{loc}}(D^1 \setminus B)$, as desired.

Further, if $\bar{\lambda}_k \to -\infty$ then (69) yields $\nabla \Phi_k \to 0$ uniformly locally in $\bar{D}^2 \setminus B$; hence, Φ_∞ is constant. Similarly, if $\lambda_k \ge -C$ then $|\nabla \Phi_k|$ is locally uniformly lower bounded on $D^2 \setminus B$; hence, $\nabla \Phi_\infty \ne 0$ in $D^2 \setminus B$. □

Blow-up analysis. In this section we associate to a sequence (λ_k) satisfying (8)–(10) a sequence of curves $(\gamma_k) \subset W^{2,\infty}(S^1, \mathbb{C})$ with bounded lengths $L_k \le \bar{L}$, curvatures bounded by $\bar{\kappa}$, and $|\dot{\gamma}_k| \equiv L_k/(2\pi)$; a sequence $(\Phi_k) \subset C^1(\bar{D}^2, \mathbb{C})$ of holomorphic immersions such that $|(\Phi_k')|_{S^1}| = e^{\lambda_k}$; and a sequence of diffeomorphisms $\sigma_k : S^1 \to S^1$ such that $\Phi_k \circ \sigma_k = \gamma_k$. Up to a translation we can assume that $\Phi_k(1) = 0$ and, by the Arzelà–Ascoli theorem, $\gamma_k \to \gamma_\infty$ in $C^1(S^1, \mathbb{C})$ for a curve $\gamma_\infty \in W^{2,\infty}(S^1, \mathbb{C})$.

Notice that (Φ_k) and (λ_k) satisfy the hypothesis of Theorem 3.5 and, up to a subsequence, we can assume that (65) and (66) hold for a finite set $B = \{a_1, \dots, a_N\}$ and functions $v_\infty \in L^1(S^1, \mathbb{R})$ and $\Phi_\infty \in W^{1,2}(D^2, \mathbb{C})$. Moreover, either (1) or (2) in Theorem 3.5 holds.

We introduce the following distance function $D_k : S^1 \times S^1 \to \mathbb{R}^+$:

$$D_k(q,q') = \inf\left\{ \left(\int_0^1 |\Phi_k'(\Delta_k(t))|^2 |\Delta_k'(t)|^2\,dt\right)^{\frac{1}{2}} \,\middle|\, \Delta_k \in W^{1,2}([0,1],\overline{D}^2),\ \Delta_k(0)=\sigma_k(q),\ \Delta_k(1)=\sigma_k(q') \right\}, \tag{71}$$

It is well known that the infimum in (71) is attained by a path Δ_k such that $|\Phi_k'(\Delta_k(t))||\Delta_k'(t)|$ is constant. For such a path we then have

$$\left(\int_0^1 |\Phi_k'(\Delta_k(t))|^2 |\Delta'(t)|^2\,dt\right)^{\frac{1}{2}} = \int_0^1 |\Phi_k'(\Delta_k(t))|\,|\Delta_k'(t)|\,dt =: \int_{\Delta_k} |\Phi_k'(z)|\,|dz|.$$

In the sequel we sometimes identify the parametrization of a curve Δ with its image.

Proposition 3.6. (1) *The function D_k is Lipschitz continuous with $\|\nabla D_k\|_{L^\infty} \le 1$ and it converges uniformly.*

(2) *The infimum in (71) is attained by a curve Δ_k in normal parametrization such that the curvature of $\Phi_k \circ \Delta_k$ is bounded by $\|\kappa_k\|_{L^\infty}$.*

Proof. (1) Let $q, q', \tilde{q}, \tilde{q}' \in S^1$. The following estimate holds:

$$D_k(q,q') \le D_k(\tilde{q},\tilde{q}') + |\mathrm{arc}\,(\gamma_k(q),\gamma_k(\tilde{q}))| + |\mathrm{arc}\,(\gamma_k(q'),\gamma_k(\tilde{q}'))| \le D_k(\tilde{q},\tilde{q}') + |q-\tilde{q}| + |q'-\tilde{q}'|,$$

where $\mathrm{arc}(\cdot,\cdot)$ is the shortest arc between two points. By exchanging (q,q') and $(\tilde{q},\tilde{q}')$, we get that

$$|D_k(q,q') - D_k(\tilde{q},\tilde{q}')| \le |q-\tilde{q}| + |q'-\tilde{q}'|,$$

and we conclude.

(2) For a geodesic Δ with respect to D_k, the curve $\Phi_k \circ \Delta$ is a geodesic in $\mathbb{C}$ under the constraint that $\Phi_k \circ \Delta \subset \Phi_k(\overline{D}^2)$. This must be a union of segments (contained in $\Phi_k(D^2)$) and arcs of the curve γ_k, where the segments touch the curve γ_k tangentially. Hence the curvature of $\Phi_k \circ \Delta$ is bounded by $\|\kappa_k\|_{L^\infty}$.

This completes the proof of Proposition 3.6. □

We give next the definition of a pinched point for the curve γ_∞.

Definition 3.7. A point $p \in S^1$ is called a *pinched point* for the sequence (γ_k) if there exists $p' \in S^1$, $p \neq p'$, such that $\lim_{k\to+\infty} D_k(p,p') = 0$. We call p' the "dual" of p and we will show in Lemma 3.12 below that this dual is unique. We denote by $\mathscr{P}$ the set of the pinched points of γ_∞.

Remark 3.8. The definition of pinched point is independent of Φ_k and σ_k in the sense that if $\tilde{\Phi}_k = \Phi_k \circ f_k$, where $f_k : \overline{D}^2 \to \overline{D}^2$ is a Möbius transformation, and if $\tilde{\sigma}_k = f_k^{-1} \circ \sigma_k$, then

$$\lim_{k\to+\infty} \int_0^1 |\Phi_k'(\Delta(t))|\,|\Delta'(t)|\,dt = 0 \iff \lim_{k\to+\infty} \int_0^1 |\tilde{\Phi}_k'(\tilde{\Delta}(t))|\,|\tilde{\Delta}'(t)|\,dt = 0.$$

Proposition 3.9. *Assume that we are in case* (2) *of Theorem 3.5, i.e., $\Phi_k \to Q$ in $C^{1,\alpha}_{\mathrm{loc}}(\overline{D}^2 \setminus \{a_1,\dots,a_N\})$ for a constant $Q \in \mathbb{C}$. Then $N \in \{1,2\}$. If $N = 2$, let $\mathscr{C}_+$ and $\mathscr{C}_-$ be the connected components of $S^1 \setminus \{a_1,a_2\}$. Then $\sigma_k^{-1} \to p^\pm$ locally uniformly on $\mathscr{C}_\pm$, where $p^+, p^- \in \mathscr{P}$ are dual. Moreover,*

$Q=\gamma_\infty(p^+)=\gamma_\infty(p^-)$ and $\dot\gamma_\infty(p^+)=-\dot\gamma_\infty(p^-)$, and $\kappa_k e^{\lambda_k}\overset{*}{\rightharpoonup}\pi(\delta_{a_1}+\delta_{a_2})$ and $v_k:=\lambda_k-\bar\lambda_k\rightharpoonup v_\infty$ in $W^{1,p}_{\mathrm{loc}}(S^1\setminus\{a_1,a_2\})$, where v_∞ solves (12). If $N=1$ then $v_k\to v_\infty$ that solves (11).

Proof. By Theorem 3.5 we have $\bar\lambda_k\to-\infty$ and $\lambda_k\to-\infty$ uniformly locally in $S^1\setminus B=\{a_1,\dots,a_N\}$. In particular, since the signed Radon measures $\kappa_k e^{\lambda_k}\,dx$ are uniformly bounded, we have $\mu_k\overset{*}{\rightharpoonup}\mu$ for a Radon measure supported in B, which we can then write as $\mu=\sum_{i=1}^N\alpha_i\delta_{a_i}$. Moreover, since

$$\int_{S_1}\kappa_k e^{\lambda_k}\,d\theta=2\pi,$$

we infer that $\sum_{i=1}^N\alpha_i=2\pi$.

Let us assume that $N\ge 2$. We want to prove that $\alpha_i=\pi$ for every i, so necessarily $N=2$. In order to prove that $\alpha_i=\pi$, up to a rotation we can reduce to proving that $\alpha_1=\pi$ and assume that $a_1=i$. We can also assume that $N=2$ and $a_2=-i$. If this is not the case, it suffices to compose Φ_k with Möbius diffeomorphisms $f_k(z)=(z-it_k)/(1+it_kz)$ with $t_k\uparrow 1$ slowly enough that $\tilde\Phi_k:=\Phi_k\circ f_k$ is still as in case (2) of Theorem 3.5, with $B=\{a_1=i,a_2=-i\}$.

Then let Φ_k be as above, with $\Phi_k\rightharpoonup Q$ in $W^{2,p}_{\mathrm{loc}}(\bar D^2\setminus\{i,-i\})$. Set

$$V_k(z)=e^{-\bar\lambda_k}(\Phi_k(z)-\Phi_k(0)),\qquad v_k=\log|V_k'|_{S^1}|=\lambda_k-\bar\lambda_k.$$

By Theorem 3.5 we have

$$v_k\rightharpoonup v_\infty\quad\text{in }W^{1,p}_{\mathrm{loc}}(S^1\setminus\{i,-i\})\text{ and in }\mathcal{D}'(S^1),$$

where v_∞ solves

$$(-\Delta)^{\frac12}v_\infty=\alpha\delta_i+(2\pi-\alpha)\delta_{-i}-1 \tag{72}$$

for some $\alpha\in\mathbb{R}$. Similarly, $V_k\rightharpoonup V_\infty$ in $W^{2,p}_{\mathrm{loc}}(\bar D^2\setminus\{i,-i\})$. Solutions to (72) can be computed explicitly using Lemma 3.1, so that

$$v_\infty(e^{i\theta})=-\frac{\alpha}{2\pi}\log(2(1-\sin\theta))-\frac{2\pi-\alpha}{2\pi}\log(2(1+\sin\theta)).$$

Notice that, writing $z=x+iy$, for $z=e^{i\theta}\in S^1$ we have

$$2(1-\sin\theta)=x^2+y^2-2y+1=|z-i|^2$$

and, similarly, $2(1+\sin\theta)=|z+i|^2$. In particular, v_∞ can be extended to a holomorphic function

$$\tilde v_\infty(z):=-\frac{\alpha}{2\pi}\log(|z-i|^2)-\frac{2\pi-\alpha}{2\pi}\log(|z+i|^2),\quad z\in\bar D^2\setminus\{i,-i\}. \tag{73}$$

The estimate (69) together with (16) implies that

$$c_\delta^{-1}\le|V_k'|\le c_\delta\quad\text{on }\bar D^2\setminus(B(i,\delta)\cup B(-i,\delta))\text{ for every }\delta>0.$$

Therefore, $V_k\rightharpoonup V_\infty$ as $k\to+\infty$ in $W^{2,p}_{\mathrm{loc}}(\bar D^2\setminus\{i,-i\})$, where V_∞ is a conformal immersion of $\bar D^2\setminus\{i,-i\}$. Moreover, still using (16), from (73) we obtain

$$|V_\infty'(z)|=\frac{1}{|z-i|^{\alpha/\pi}|z+i|^{2-\alpha/\pi}}.$$

Since V'_∞ is holomorphic in D^2, up to a rotation (i.e., multiplication by a constant $e^{i\theta_0}$) we obtain

$$V'_\infty(z) = \frac{1}{(z-i)^{\alpha/\pi}(z+i)^{2-\alpha/\pi}}, \quad V_\infty(z) = \int_0^z \frac{dz}{(z-i)^{\alpha/\pi}(z+i)^{2-\alpha/\pi}}.$$

Up to possibly switching i with $-i$, we may assume that $\alpha \le \pi$. The function V_∞ is also known as the Schwarz–Christoffel mapping[1] and sends the two arcs $\mathscr{C}_+, \mathscr{C}_- \subset S^1$ joining i and $-i$ (chosen so that $\pm 1 \in \mathscr{C}_\pm$) into two parallel straight lines if $\alpha = \pi$ and into two half-lines meeting at $V_\infty(i)$, forming an angle of $\pi - \alpha$ there if $\alpha < \pi$.

Claim 1. *As $k \to +\infty$ we have $\sigma_k^{-1} \to p^\pm$ in $L^\infty_{\text{loc}}(\mathscr{C}_\pm)$, where $p^+, p^- \in S^1$ with $p^+ \neq p^-$.*

Proof. Notice that $\Phi_k \rightharpoonup Q$ in $W^{2,p}_{\text{loc}}(\bar{D}^2 \setminus \{i, -i\})$ implies that

$$\frac{\partial \sigma_k^{-1}}{\partial \theta} \to 0 \quad \text{uniformly locally in } S^1 \setminus \{i, -i\} \text{ as } k \to +\infty.$$

This proves the first part of the claim. Assume for contradiction that $p^+ = p^-$. Set $p_k^\pm = \sigma_k^{-1}(\pm 1) \to p^\pm$. By assumption, $|\text{arc}(p_k^+, p_k^-)| \to 0$ (here $\text{arc}(p_k^+, p_k^-)$ denotes the shortest arc connecting p_k^+ to p_k^-). Since σ_k is a diffeomorphism, $\sigma_k(\text{arc}(p_k^+, p_k^-))$ contains either $S^1 \cap B(i, \delta)$ or $S^1 \cap B(-i, \delta)$ for small $\delta > 0$. Suppose it contains $S^1 \cap B(i, \delta)$. Then

$$\int_{S^1 \cap B(i,\delta)} e^{\lambda_k}\, d\theta = \int_{S^1 \cap B(i,\delta)} |\Phi'_k(e^{i\theta})|\, d\theta \le \int_{\text{arc}(p_k^+, p_k^-)} |\dot{\gamma}_k|\, d\theta = \frac{L_k}{2\pi}|\text{arc}(p_k^+, p_k^-)| \to 0 \tag{74}$$

as $k \to \infty$. This contradicts that $i \in B$ and concludes the proof of Claim 1. ☐

Claim 2. *p^+ is a pinched point and p^- is dual to it.*

Proof. Let $p_k^\pm = \sigma_k^{-1}(\pm 1)$ be as above. Consider the path

$$\Delta_k = \text{arc}(\sigma_k(p^+), 1) \cup \text{arc}(\sigma_k(p^-), -1) \cup [-1, 1],$$

where $[-1, 1]$ is the segment in $\bar{D}^2$ joining -1 to 1. Since, as $k \to \infty$, we have

$$\int_{\text{arc}(\sigma_k(p^\pm), \pm 1)} |\Phi'_k(e^{i\theta})|\, d\theta = \int_{\text{arc}(p_k^\pm, p^\pm)} |\dot{\gamma}_k|\, d\theta = \frac{L_k |\text{arc}(p_k^\pm, p^\pm)|}{2\pi} \to 0 \tag{75}$$

and

$$\int_{[-1,1]} |\Phi'_k|\, |dz| \le 2 \sup_{[-1,1]} |\Phi'_k|\, |dz| \to 0,$$

we immediately infer that

$$\int_{\Delta_k} |\Phi'_k|\, |dz| \to 0;$$

hence, p^+ is dual to p^-. This proves Claim 2. ☐

[1] Up to composition with a conformal transformation, since Schwarz–Christoffel maps are usually defined on the half-plane $\{z \in \mathbb{C} : \Re z > 0\}$ instead of the unit disk.

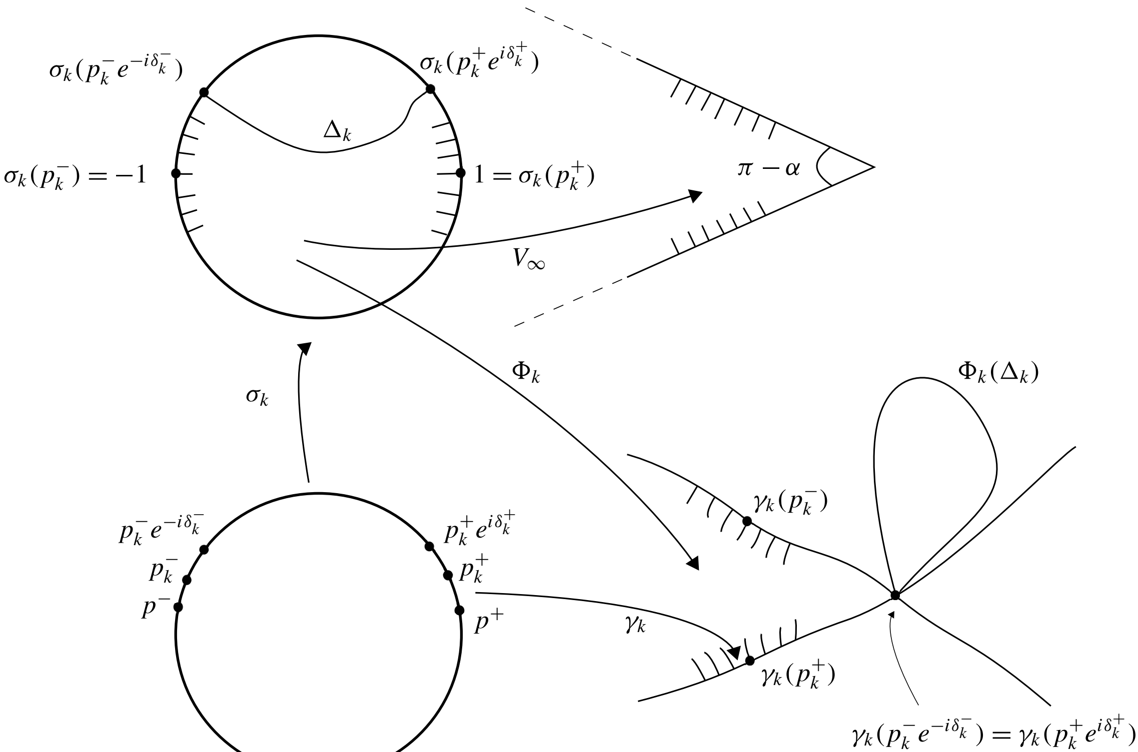

Figure 6. Case 1 in the proof of Proposition 3.9.

Now,

$$\frac{2\pi}{L_k}\dot{\gamma}_k(p_k^\pm) = \frac{\frac{\partial \Phi_k(\pm 1)}{\partial\theta}}{\left|\frac{\partial \Phi_k(\pm 1)}{\partial\theta}\right|} = \frac{\frac{\partial \Phi_k(\pm 1)}{\partial\theta}}{e^{\bar{\lambda}_k} e^{\lambda_k(\pm 1)-\bar{\lambda}_k}} = \frac{\partial V_\infty(\pm 1)}{\partial\theta} e^{\bar{\lambda}_k-\lambda_k(\pm 1)} + o(1) \quad \text{as } k\to\infty. \tag{76}$$

In particular, denoting by $(v, w)^\wedge$ the angle between two vectors, we have

$$(\dot{\gamma}_k(p_k^+), \dot{\gamma}_k(p_k^-))^\wedge \to \left(\frac{\partial V_\infty(1)}{\partial\theta}, \frac{\partial V_\infty(-1)}{\partial\theta}\right)^\wedge = \alpha. \tag{77}$$

We consider different cases:

Case 1: $0 < \alpha < \pi$. Since $p_k^\pm \to p^\pm$ and p^+ is pinched to p^-, and since

$$|\gamma_k(p_k^+)-\gamma_k(p_k^-)| \le D_k(p_k^+, p_k^-) \le D_k(p^+, p^-) + \frac{L_k}{2\pi}\big(|\mathrm{arc}(p^+, p_k^+)| + |\mathrm{arc}(p^-, p_k^-)|\big) \to 0 \quad \text{as } k\to\infty,$$

taking (77) and the bound $\bar{\kappa}$ on the curvature of γ_k into account we see that for positive numbers $\delta_k^\pm \to 0$ (as $k\to\infty$) we have

$$\gamma_k(p_k^+ e^{i\delta_k^+}) = \gamma_k(p_k^- e^{-i\delta_k^-}), \tag{78}$$

i.e., the two curves $t \mapsto \gamma_k(p_k^\pm e^{\pm it})$ cross in short time (see Figure 6).

Because $\delta_k^\pm \to 0$, we have

$$D_k(p_k^+ e^{i\delta_k^+}, p_k^- e^{-i\delta_k^-}) \le D_k(p_k^+, p_k^-) + \frac{L_k(\delta_k^+ + \delta_k^-)}{2\pi} \to 0 \quad \text{as } k \to \infty. \tag{79}$$

Now let $\Delta_k : [0, 1] \to \bar{D}^2$ be a geodesic realizing the distance on the left-hand side of (79). Then (78) implies that $\Phi_k \circ \Delta_k$ is a closed curve (nonconstant, since $p_k^+ e^{i\delta_k^+} \neq p_k^- e^{-i\delta_k^-}$ for k large), so that the integral of its curvature is at least π (see Lemma 3.10 below). On the other hand, Proposition 3.6 implies that the curvature of $\Phi_k \circ \Delta_k$ is bounded by $\bar{\kappa}$ and, since the length of this geodesic is going to 0 according to (79), we get a contradiction.

Case 2: $\alpha = 0$. Similarly to case 1, if the curves $\gamma_k(p_k^\pm e^{\pm it})$ cross for small times $\delta_k^\pm \to 0$, we conclude as before. If not, we can at least say that, up to a rotation of the axis,

$$V_\infty(D^2) = \{x + iy : y < 0\} \tag{80}$$

and that, for small times $\delta_k^\pm \to 0$,

$$\Re(\gamma_k(p_k^+ e^{i\delta_k^+})) = \Re(\gamma_k(p_k^- e^{-i\delta_k^-})) \tag{81}$$

and, without loss of generality,

$$\Im(\gamma_k(p_k^+ e^{i\delta_k^+})) > \Im(\gamma_k(p_k^- e^{-i\delta_k^-})), \tag{82}$$

where, for $x, y \in \mathbb{R}$, we use the notation $\Re(x + iy) = x$, $\Im(x + iy) = y$ (see Figure 7). Moreover, since the curvature of γ_k is uniformly bounded and $\delta_k^\pm \to 0$, using (76) and (80) we infer[2]

$$\frac{\dot{\gamma}_k(p_k^\pm e^{\pm i\delta_k^\pm})}{|\dot{\gamma}_k(p_k^\pm e^{\pm i\delta_k^\pm})|} = \frac{\dot{\gamma}_k(p_k^\pm)}{|\dot{\gamma}_k(p_k^\pm)|} + o(1) = -1 + o(1), \tag{83}$$

i.e., the curves $t \mapsto \gamma_k(p_k^\pm e^{\pm it})$ at the time $t = \delta_k^\pm$ are almost horizontal and pointing into opposite directions (notice the change of orientation between the curves $t \mapsto \gamma_k(e^{it})$ and $t \mapsto \gamma_k(p_k^- e^{-it})$). As before, (79) holds, so let $\Delta_k : [0, 1] \to \bar{D}^2$ be a geodesic realizing the distance in (79), with $\Delta_k(0) = \gamma_k(p_k^+ e^{i\delta_k^+})$ and $\Delta_k(1) = \gamma_k(p_k^- e^{-i\delta_k^-})$. Up to a reparametrization we can assume that $\tilde{\Delta}_k := \Phi_k \circ \Delta_k : [0, L] \to \mathbb{C}$ satisfies $|\dot{\tilde{\Delta}}_k(t)| \equiv 1$. Since the map Φ_k preserves the orientation, from (83) we infer

$$\Im(\dot{\tilde{\Delta}}_k(0)) \le 0 + o(1), \quad \Im(\dot{\tilde{\Delta}}_k(1)) \ge 0 + o(1),$$

i.e., up to $o(1) \to 0$ as $k \to \infty$ we have that $\dot{\tilde{\Delta}}_k(0)$ points downwards, while $\dot{\tilde{\Delta}}_k(1)$ points upwards. Now using (81) we see that the curve $\tilde{\Delta}_k$ has total curvature at least $\frac{1}{2}\pi - o(1)$ (see Lemma 3.11 below), again contradicting Proposition 3.6 and (79).

Case 3: $\alpha < 0$. Let Δ be the straight segment in $\bar{D}^2$ (seen as a smooth path) joining -1 to 1. Since $\Delta \subset \bar{D}^2 \setminus \{i, -i\}$ we have that $V_k \circ \Delta \to V_\infty \circ \Delta$ and, by the explicit form of V_∞, we deduce that the unit tangent vector of the curve $V_\infty \circ \Delta$ describes an arc in S^1 of length at least $|\alpha| + \pi$ (we are using

[2]The symbol $\dot{\gamma}_k(p_k^\pm e^{\pm i\delta_k^\pm})$ denotes the derivative of the curve $t \mapsto \gamma_k(e^{it})$ evaluated for $e^{it} = p_k^\pm e^{\pm i\delta_k^\pm}$ and *not* the derivative of the curve $t \mapsto \gamma_k(p_k^\pm e^{\pm it})$ evaluated for $t = \delta_k^\pm$.

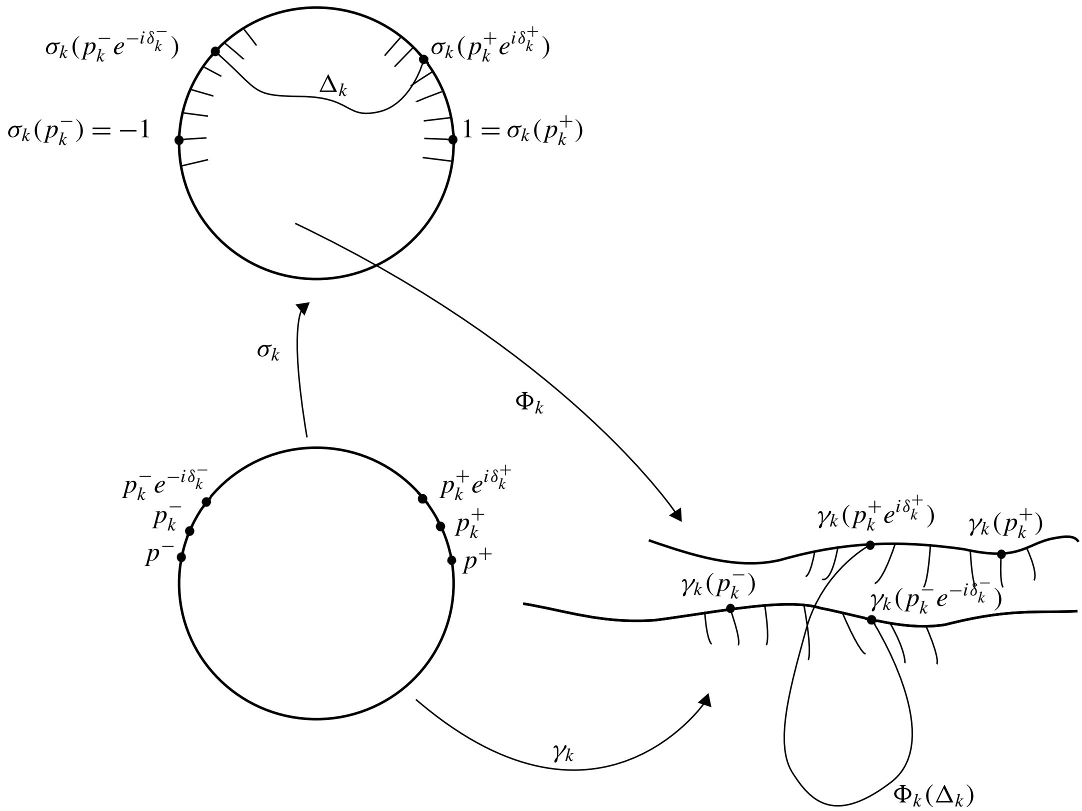

Figure 7. Case 2 in the proof of Proposition 3.9.

that Δ touches S^1 perpendicularly and V_∞ is conformal). This implies that, for k large enough, any C^1 curve of the form $\Phi_k \circ \tilde{\Delta}$ for a curve $\tilde{\Delta} \in C^1([0,1], \bar{D}^2)$ with $\tilde{\Delta}(0) = -1$ and $\tilde{\Delta}(1) = 1$ has a unit tangent vector describing an arc of length no less than $|\alpha| - o(1)$. If such a curve minimizes D_k, since, by Proposition 3.6, its curvature is bounded by $\bar{\kappa}$, its length cannot go to zero as $k \to \infty$. But this contradicts that p^+ and p^- are pinched points, since, if Δ_k is a geodesic minimizing $D_k(\sigma_k(p^+), \sigma_k(p^-))$ (with length going to 0 since p^+ and p^- are pinched), then joining Δ_k with the two arcs $\mathrm{arc}(\sigma_k(p^\pm), \pm 1)$ and using (75) one would obtain paths joining -1 to 1 of D_k-length going to 0.

The only case left is $\alpha = \pi$, which completes the proof of Proposition 3.9. □

In the proof of Proposition 3.9 we have used the following:

Lemma 3.10. *Let $\Delta \in W^{2,\infty}([0, L], \mathbb{C})$ be a curve satisfying $|\dot{\Delta}(t)| = 1$ for every $t \in [0, L]$ and $\Delta(0) = \Delta(L)$. Then*

$$\int_0^L |\kappa(t)|\, dt > \pi,$$

where κ is the curvature of Δ.

Proof. Let $\theta : [0, L] \to \mathbb{R}$ be a continuous function such that $\dot{\Delta}(t) = e^{i\theta(t)}$ for $t \in [0, L]$. Then it is easy to see that $\dot{\theta} = \kappa$. We have $\theta([0, L]) = [\theta_-, \theta_+] \subset \mathbb{R}$ for some $\theta_-, \theta_+ \in \mathbb{R}$. Assume now that

$$\theta_+ - \theta_- \le \pi \tag{84}$$

and set

$$\bar{\theta} := \tfrac{1}{2}(\theta_+ - \theta_-), \qquad v := e^{i\bar{\theta}}.$$

Then, since $|\theta(t) - \bar{\theta}| \le \frac{1}{2}\pi$ for every $t \in [0, L]$, we have

$$\frac{d}{dt}\langle \Delta(t), v\rangle = \langle \dot{\Delta}(t), v\rangle = \langle e^{i\theta(t)}, e^{i\bar{\theta}}\rangle \ge 0,$$

with equality possible only for a proper subset of $[0, L]$, where $|\theta(t) - \bar{\theta}| = \frac{1}{2}\pi$. But this contradicts that $\Delta(0) = \Delta(L)$. In particular, (84) cannot hold, and we get

$$\int_0^L |\kappa(t)|\, dt = \int_0^L |\dot{\theta}(t)|\, dt \ge \operatorname{osc}\theta = \theta_+ - \theta_- > \pi. \qquad \square$$

Lemma 3.11. *Let $\Delta \in W^{2,\infty}([0, L], \mathbb{C})$ be a curve satisfying $|\dot{\Delta}(t) = 1|$ for every $t \in [0, L]$. Assume that*

$$\Re(\Delta(0)) = \Re(\Delta(L)), \qquad \Im(\Delta(0)) < \Im(\Delta(L)), \tag{85}$$

and that for some (small) $\varepsilon > 0$ one has

$$\Im(\dot{\Delta}(0)) < \varepsilon \quad \text{and} \quad \Im(\dot{\Delta}(L)) > -\varepsilon. \tag{86}$$

Then

$$\int_0^L |\kappa(t)|\, dt > \frac{\pi}{2} - C\varepsilon,$$

where κ is the curvature of Δ and C is a universal constant.

Proof. Let $\theta \in W^{1,\infty}([0, L], \mathbb{R})$ be as in the proof of Lemma 3.10. Then (85) implies that for some $t_1, t_2 \in [0, L]$ one has $\Re(e^{i\theta(t_1)}) \le 0$ and $\Re(e^{i\theta(t_2)}) \ge 0$ (otherwise $\dot{\Delta}$ would always be pointing right, or always left). Condition (86) implies that $\Im(e^{i\theta(0)}) \le \varepsilon$ and $\Im(e^{i\theta(L)}) > -\varepsilon$. Then we immediately infer that the oscillation of θ is at least $\frac{1}{2}\pi - C\varepsilon$ and we conclude as in the proof of Lemma 3.10, using that $\kappa = \dot{\theta}$. $\square$

Next we prove some properties concerning the set $\mathcal{P}$:

Lemma 3.12. *Let p^+ and p^- be dual pinched points and assume that $\sigma_k(p^\pm) = \pm 1$. Then Φ_k is as in case* (2) *of Theorem 3.5, $B = \{a_1, a_2\}$ and $\pm 1 \notin B$. Moreover, every pinched point p has only one dual p' and $|\mathrm{arc}(p, p')| \ge C/\bar{\kappa}$.*

Proof. Let us start with the first claim. If Φ_k is as in case (1) of Theorem 3.5, then

$$\int_{\Delta_k} |\Phi_k'(z)|\, |dz| \ge C \quad \text{for every } \Delta_k \text{ with } \Delta_k(0) = -1,\ \Delta_k(1) = 1, \tag{87}$$

in contrast with the fact that p^+ and p^- are pinched. Thus we are in case (2) of Theorem 3.5 and, by Proposition 3.9, we have $N \in \{1, 2\}$. Assume now that $a_1 = 1 = \sigma_k(p^+)$ (the reasoning is similar if $a_1 = -1$). Then we compose Φ_k with the Möbius diffeomorphism $f_k(z) = (z - t_k)/(1 - t_k z)$, where $t_k \uparrow 1$ is chosen so that for a fixed small $\delta > 0$ we have, for k large enough,

$$\int_{S^1 \cap B_\delta(1)} |(\Phi_k \circ f_k)'(z)|\,|dz| = \frac{\pi}{2\bar{\kappa}}. \tag{88}$$

In other words, the effect of f_k is to stretch the disk to remove the concentration at the point $a_1 = 1$, concentrating the disk towards -1. Then $\tilde{\Phi}_k := \Phi_k \circ f_k$ is necessarily as in case (1) of Theorem 3.5. Moreover, the corresponding $\tilde{\sigma}_k := f_k^{-1} \circ \sigma_k$ still satisfies $\tilde{\sigma}_k(p^\pm) = \pm 1$, since f_k leaves ± 1 fixed. This, together with (88), contradicts that p^+ and p^- are pinched, since, by conformality and convergence of $\tilde{\Phi}_k$, in a neighborhood $B_{\delta/2}(1)$ we have $|\tilde{\Phi}_k'| \geq C$; hence, (87) holds with $\tilde{\Phi}_k$ instead of Φ_k. Therefore, going back to the original maps Φ_k, we have proven that $\pm 1 \notin B$.

To rule out the case $N = 1$ it suffices to observe that in this case $\sigma_k(p^+)$ and $\sigma_k(p^-)$ would belong to the same connected component of $S^1 \setminus B$; hence, since Φ_k is as in case (2) of Theorem 3.5, we would get $|\mathrm{arc}(\sigma_k^{-1}(1), \sigma_k^{-1}(1))| \to 0$, which is absurd, since $\sigma_k^{-1}(\pm 1) = p^\pm$ and $p^+ \neq p^-$.

Claim 1. *Every pinched point p has a unique dual p'.*

Proof. It suffices to prove that, given any pinched points p^+ and p^- dual to each other, $\dot{\gamma}_\infty(p^+) = -\dot{\gamma}_\infty(p^-)$ (since then a third point $\tilde{p}$ dual to p^+ would be also dual to p^-, whence $\dot{\gamma}_\infty(\tilde{p})$ would have to coincide both with $\dot{\gamma}_\infty(p^+)$ and its opposite, which is impossible). Let us therefore consider two pinched points p^+ and p^-, dual to each other. By considering $\tilde{\Phi}_k := \Phi_k \circ f_k$ and $\tilde{\sigma}_k = f_k^{-1} \circ \sigma_k$ for suitable Möbius transformations f_k, we can assume that $\tilde{\sigma}_k(p^\pm) = \pm 1$. Then, by the previous part of the lemma, $\tilde{\Phi}_k$ blows up at two points a_1 and a_2 different from ± 1. To such a $\tilde{\Phi}_k$ we can then apply Proposition 3.9 with $\mathscr{C}_\pm$ being the connected component of $S^1 \setminus \{a_1, a_2\}$ containing ± 1. We then infer that $\dot{\gamma}_\infty(p^+) = -\dot{\gamma}_\infty(p^-)$. ☐

Claim 2. *We have $|\mathrm{arc}(p, p')| \geq C/\bar{\kappa}$.*

Proof. This follows from the fact that both arcs $\mathscr{A}_1$ and $\mathscr{A}_1$ joining $\tilde{\sigma}_k(p^\pm) = \pm 1$ contain a blow-up point, a_1 or a_2, so that

$$\int_{\mathscr{A}_i} |\tilde{\kappa}_k| e^{\tilde{\lambda}_k}\,|dz| = \int_{f_k(\mathscr{A}_i)} |\kappa_k| e^{\lambda_k}\,|dz| \geq \pi - o(1).$$ ☐

This concludes the proof of Lemma 3.12. ☐

Lemma 3.13. *The set $\mathscr{P}$ is closed.*

Proof. Let $\{p_n\}$ and $\{p_n'\}$ be a sequence of pinched points and their duals, respectively, with $p_n \to p_\infty$ and $p_n' \to p_\infty'$ as $k \to +\infty$.

We first observe that $|p_n - p_n'| \geq C > 0$ for all $n \geq 0$, so $p_\infty \neq p_\infty'$.

For each p_n there exists a curve $\Delta_{n,k} \subseteq \bar{D}^2$ with $\partial \Delta_{n,k} = \{\sigma_k(p_n), \sigma(p_n')\}$ and

$$\lim_{k \to +\infty} \int_{\Delta_{n,k}} |\Phi_k'(z)|\,|dz| = 0.$$

Since $\gamma_k \to \gamma_\infty$ in $C^1(S^1)$ as $k \to +\infty$, we have

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\int_{\mathrm{arc}(p_n,p_\infty)}|\dot\gamma_k(t)|\,dt=0,\qquad \lim_{k\to+\infty}\lim_{n\to+\infty}\int_{\mathrm{arc}(p'_n,p'_\infty)}|\dot\gamma_k(t)|\,dt=0. \tag{89}$$

We set

$$\tilde\Delta_{n,k} := \Delta_{n,k}\cup \mathrm{arc}(\sigma_k(p_n),\sigma_k(p_\infty))\cup \mathrm{arc}(\sigma_k(p_n),\sigma_k(p_\infty)).$$

For all k, we have $\tilde\Delta_{n,k}\to\tilde\Delta_{\infty,k}$ as $n\to+\infty$ with $\partial\tilde\Delta_{k,\infty}=\{\sigma_k(p_\infty),\sigma_k(p'_\infty)\}$ and, since $\Phi_k\circ\sigma_k=\gamma_k$ on S^1 from (89), we have

$$\lim_{k\to+\infty}\int_{\tilde\Delta_{k,\infty}}|\Phi'_k(z)|\,|dz|=\lim_{k\to+\infty}\lim_{n\to+\infty}\int_{\tilde\Delta_{n,k}}|\Phi'_k(z)|\,|dz|=0.$$

Hence p_∞ is by definition a pinched point and p'_∞ is its dual. □

We now introduce the following equivalence relation on the set $S^1\setminus\{\mathcal{P}\}$:

Definition 3.14. Given $p, q\in S^1\setminus\{\mathcal{P}\}$, we say that $p\sim q$ if and only if there exists a sequence of paths $\Delta_k:[0,1]\to\bar D^2$ with $\Delta_k(0)=\sigma_k(p)$ and $\Delta_k(1)=\sigma_k(q)$ such that

$$\liminf_{k\to+\infty} d_k(\Delta_k,\sigma_k(\mathcal{P}))>0, \tag{90}$$

where $d_k:\bar D^2\times\bar D^2\to\mathbb{R}^+$ is the distance defined as

$$d_k(z,w)=\inf\left\{\left(\int_0^1|\Phi'_k(\Delta(t))|^2|\dot\Delta(t)|^2\,dt\right)^{\frac12}\ \middle|\ \Delta\in W^{1,2}([0,1],\bar D^2),\ \Delta(0)=z,\ \Delta(1)=w\right\}.$$

Proposition 3.15. *Let $q\in S^1\setminus\{\mathcal{P}\}$, and let $\mathcal{A}_q$ and $\mathcal{B}_q$ be the equivalence class and the connected component containing q, respectively. Then $\mathcal{B}_q\subseteq\mathcal{A}_q$.*

Proof. Let $q\in S^1\setminus\{\mathcal{P}\}$. We show that $\mathcal{A}_q\cap\mathcal{B}_q$ is open and closed in $\mathcal{B}_q$.

(1) $\mathcal{A}_q\cap\mathcal{B}_q$ is open in $\mathcal{B}_q$: Choose $\delta>0$ small enough so that $e^{it}q\in S^1\setminus\{\mathcal{P}\}$ for $t\in[-2\delta,2\delta]$ and

$$\int_{\sigma_k(\mathrm{arc}(e^{-2\delta i}q,e^{2\delta i}q))}|\Phi'_k(z)|\,|dz|<\frac{\pi}{2\bar\kappa}. \tag{91}$$

Now set $q_0=e^{-i\delta}q$, $q_1=q$ and $q_2=e^{i\delta}q$. Let f_k be the sequence of Möbius transformations of $\bar D^2$ such that $\tilde\sigma_k(q_0)=1$, $\tilde\sigma_k(q_1)=e^{2\pi i/3}$ and $\tilde\sigma_k(q_2)=e^{4\pi i/3}$. We apply Theorem 3.5 to $\tilde\Phi_k:=\Phi_k\circ f_k$ and notice that if we are in case (2) of Theorem 3.5, then there are one or two blow-up points. In the latter case, away from the blow-up points $\{a_1,a_2\}$, we have that σ_k^{-1} locally converges to two pinched points, which implies that one of the q_i lies in $\mathcal{P}$, a contradiction. In the former case, for one pair of points, say q_1 and q_2, one has

$$\int_{\mathrm{arc}(q_1,q_2)}|\dot\gamma(t)|\,dt=\int_{\mathrm{arc}(\tilde\sigma_k(q_1),\tilde\sigma_k(q_2))}|\tilde\Phi'_k(z)|\,|dz|\to0,$$

contradicting that $|\dot\gamma_k|$ is bounded away from 0 and $|\mathrm{arc}(q_1,q_2)|=\delta$.

Therefore we are in case (1) of Theorem 3.5 and $\tilde\Phi_k \rightharpoonup \tilde\Phi_\infty$ in $W^{1,2}(\bar D^2)$ and in $W^{2,p}_{\mathrm{loc}}(\bar D^2\setminus B)$, where $\tilde\Phi_\infty$ is a holomorphic immersion in $\bar D^2\setminus B$, $B=\{a_1,\dots,a_N\}$ and $e^{2j\pi i/3}\notin B$ for $j=0,1,2$. Since $|\tilde\Phi'_\infty|>C_\delta>0$ in $\bar D^2\setminus\bigcup_{i=1}^N B_\delta(a_i)$, for every $p\in\operatorname{arc}(q_0,q_2)$, choosing as Δ_k the segment joining $\sigma_k(p)$ to $\sigma_k(q)$ that satisfies (90) shows that $B_\delta(q)\cap S^1\subset\mathcal{A}_q$.

(2) $\mathcal{A}_q\cap\mathcal{B}_q$ is closed in $\mathcal{B}_q$: Let $q_n\in\mathcal{A}_q\cap\mathcal{B}_q$ be such that $q_n\to q_\infty\in\mathcal{B}_q$. For every n there exists Δ_n^k with $\Delta_n^k(0)=\sigma_k(q_n)$ and $\Delta_n^k(1)=\sigma_k(q)$, and

$$\liminf_{k\to+\infty} d_k(\Delta_n^k,\sigma_k(\mathcal{P}))>0. \tag{92}$$

Consider now the path $\Sigma_n^k=\operatorname{arc}(\sigma_k(q_\infty),\sigma_k(q_n))\cup\Delta_n^k$ joining $\sigma_k(q_\infty)$ to $\sigma_k(q)$. We claim that

$$\liminf_{k\to+\infty} d_k(\Sigma_n^k,\sigma_k(\mathcal{P}))>0.$$

Indeed, considering (92), it suffices to prove that, for n sufficiently large,

$$\liminf_{k\to+\infty} d_k(\operatorname{arc}(\sigma_k(q_\infty),\sigma_k(q_n)),\sigma_k(\mathcal{P}))>0. \tag{93}$$

Assume for contradiction that the lim inf in (93) is zero.

For every k and n, let $q_n^k\in\operatorname{arc}(q_\infty,q_n)$ and $p_n^k\in\mathcal{P}$ be such that

$$\liminf_{k\to+\infty} D_k(q_n^k,p_n^k)=0.$$

Up to a subsequence, $q_n^k\to q_\infty$ and $p_n^k\to p_\infty\in\mathcal{P}$ as $n,k\to\infty$, and

$$\lim_{k\to+\infty}\lim_{n\to+\infty} D_k(q_n^k,p_n^k)=\lim_{k\to+\infty} D_k(q_\infty,p_\infty)=0,$$

but this contradicts that $q_\infty\notin\mathcal{P}$. This contradiction proves that $q_\infty\in\mathcal{A}_q\cap\mathcal{B}_q$; hence, $\mathcal{A}_q\cap\mathcal{B}_q$ is closed in $\mathcal{B}_q$. □

Proposition 3.16. *Let $\mathcal{A}$ be an equivalence class in $S^1\setminus\{\mathcal{P}\}$. Then there exists a sequence $f_k:\bar D^2\to\bar D^2$ of Möbius transformations such that $\tilde\Phi_k:=\Phi_k\circ f_k\rightharpoonup\tilde\Phi_\infty$ in $W^{2,p}_{\mathrm{loc}}(\bar D^2\setminus B)$, $B=\{a_1,\dots,a_N\}$, and, as usual letting $\tilde\sigma_k$ be such that $\gamma_k=\tilde\Phi_k\circ\tilde\sigma_k$, one has $\tilde\sigma_k^{-1}\rightharpoonup\psi_\infty$ in $W^{2,p}_{\mathrm{loc}}(S^1\setminus B)$,*

$$\psi_\infty(S^1\setminus B)=\mathcal{A} \tag{94}$$

and $\gamma_\infty(\mathcal{A})=\tilde\Phi_\infty(S^1\setminus B)$. In fact, $(\gamma_\infty)_*[\mathcal{A}]=(\tilde\Phi_\infty)_*[S^1\setminus B]$.

Proof. Given $q\in\mathcal{A}$, take f_k as in the proof of Proposition 3.15 and set $\tilde\Phi_k:=\Phi_k\circ f_k$. We have shown that $\tilde\Phi_k\rightharpoonup\tilde\Phi_\infty$ in $W^{1,2}(\bar D^2)$ and in $W^{2,p}_{\mathrm{loc}}(\bar D^2\setminus B)$ for a finite set $B=\{a_1,\dots,a_N\}$, where $\tilde\Phi_\infty$ is a holomorphic immersion (Theorem 3.5, case (1)). In particular, this implies that $\psi_k:=\tilde\sigma_k^{-1}$ is bounded in $W^{2,p}_{\mathrm{loc}}(S^1\setminus B)$ and, up to a subsequence, $\psi_k\rightharpoonup\psi_\infty$ in $W^{2,p}_{\mathrm{loc}}(S^1\setminus B)$. Clearly,

$$\psi_\infty(S^1\setminus B)\subset\mathcal{A}.$$

Conversely, given $p\notin\psi_\infty(S^1\setminus B)$, we want to show that $p\notin\mathcal{A}$. Given such a p we have $\tilde\sigma_k(p)\to a_i$ for some $a_i\in B$, since otherwise we would have $p=\psi_k\circ\tilde\sigma_k(p)\to\psi_\infty(p_*)$ for $p_*\in S^1\setminus B$. Since

$\nabla\tilde{\Phi}_\infty \in L^2(D^2)$, from Fubini's theorem we can find a sequence $\delta_n^i \to 0$ such that

$$\lim_{n\to+\infty}\int_{\partial B(a_i,\delta_n^i)\cap\bar{D}^2} |\nabla\tilde{\Phi}_\infty(z)|^2\,|dz| = 0. \tag{95}$$

For every a_i, set $\{p_{k,n}^{i,-}, p_{k,n}^{i,+}\} = \tilde{\sigma}_k^{-1}(\partial B(a_i,\delta_n^i)\cap S^1)$. We have $|p_{k,n}^{i,-} - p_{k,n}^{i,+}| > C_0$ for any n and k large enough, since by definition of the blow-up points one has, for k large enough,

$$\int_{\operatorname{arc}(p_{k,n}^{i,-},p_{k,n}^{i,+})} |\dot{\gamma}_k(t)|\,dt = \int_{B(a_i,\delta_n^i)\cap S^1} e^{\lambda_k(z)}\,|dz| > \frac{\pi}{2}.$$

Therefore, up to a subsequence, $p_{k,n}^{i,-} \to p_\infty^{i,-}$ and $p_{k,n}^{i,+} \to p_\infty^{i,+}$ with $p_\infty^{i,+} \neq p_\infty^{i,-}$ and

$$\lim_{k\to\infty} D_k(\tilde{\sigma}_k(p_\infty^{i,-}), \tilde{\sigma}_k(p_\infty^{i,+})) = 0$$

In particular, $p_\infty^{i,-}$ and $p_\infty^{i,+}$ are pinched. Then condition (95) implies that any path Δ_k joining $\tilde{\sigma}_k(q)$ and $\tilde{\sigma}_k(p)$ for k large enough is close to $\tilde{\sigma}_k(p_\infty^{i,-}) \in \tilde{\sigma}_k(\mathcal{P})$, so $p \in S^1 \setminus \mathcal{A}$.

Finally,

$$\begin{aligned}
(\gamma_\infty)_*[\mathcal{A}] &= \lim_{\delta\to 0}(\gamma_\infty)_*\Big[\psi_\infty\Big(S^1 \setminus \bigcup_{a_i\in B} B(a_i,\delta)\Big)\Big],\\
&= \lim_{\delta\to 0}\lim_{k\to\infty}(\gamma_k)_*\Big[\tilde{\sigma}_k^{-1}\Big(S^1 \setminus \bigcup_{a_i\in B} B(a_i,\delta)\Big)\Big],\\
&= \lim_{\delta\to 0}\lim_{k\to\infty}(\tilde{\Phi}_k)_*\Big[S^1 \setminus \bigcup_{a_i\in B} B(a_i,\delta)\Big],\\
&= \lim_{\delta\to 0}(\tilde{\Phi}_\infty)_*\Big[S^1 \setminus \bigcup_{a_i\in B} B(a_i,\delta)\Big],\\
&= (\tilde{\Phi}_\infty)_*[S^1 \setminus B].
\end{aligned}$$

□

Quantization result: proof of Theorems 1.2 and 1.5. In this section we prove Theorems 1.2 and 1.5. In Theorem 1.2 we will show that, under the hypothesis of Theorem 3.5, $\kappa_k e^{\lambda_k} \rightharpoonup \mu$ weakly in the sense of Radon measures, where μ is a Radon measure which is the sum of a locally bounded (possibly vanishing) function and a (possibly empty) sum of Dirac masses. We also give precise estimates on the coefficients of the Dirac masses. In Theorem 1.5, we show that up to a suitable choice of Möbius transformations we can "detect" all the connected components arising in the limit.

Proof of Theorem 1.2. From Theorem 3.5 there is a (possibly empty) set $B = \{a_1, \dots, a_N\} \subset S^1$ such that (65) holds. Moreover, from (8) and (10) it follows that $\|(-\Delta)^{\frac{1}{2}}\lambda_k\|_{L^1(S^1)} \le C$. Therefore, (53) implies

$$\|\lambda_k - \bar{\lambda}_k\|_{L^q(S^1)} \le C \quad \text{for every } q < +\infty.$$

Up to extracting a further subsequence, we have $v_k := \lambda_k - \bar{\lambda}_k \rightharpoonup v_\infty$ in $L^q(S^1)$ and

$$\kappa_k e^{\lambda_k} \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad (-\Delta)^{\frac{1}{2}} v_k \overset{*}{\rightharpoonup} (-\Delta)^{\frac{1}{2}} v_\infty = \mu - 1 \quad \text{in } \mathcal{M}(S^1), \tag{96}$$

where $\mathcal{M}(S^1)$ denotes the space of finite signed measures on S^1. Up to a subsequence we also have $\kappa_k \overset{*}{\rightharpoonup} \kappa_\infty$ in $L^\infty(S^1)$. We now distinguish three cases.

Case 1: Suppose that we are in case (2) of Theorem 3.5 and $N = 1$, i.e., $\lambda_k \to -\infty$ locally uniformly in $S^1 \setminus \{a_1\}$. Then $\mu = c_1\delta_{a_1}$ and, since

$$\int_{S^1} \kappa_k e^{\lambda_k}\, d\theta = 2\pi,$$

it follows at once that $c_1 = 2\pi$. The explicit form of v_∞ follows from Lemma 3.1.

Case 2: Suppose that we are in case (2) of Theorem 3.5 and $N > 1$. Then we conclude by applying Proposition 3.9, which in particular implies that $N = 2$ and $\mu = \pi\delta_{a_1} + \pi\delta_{a_2}$. Again, the explicit form of v_∞ follows from Lemma 3.1.

Case 3: Suppose that we are in case (1) of Theorem 3.5, i.e., $\lambda_k \geq -C$. Then $\lambda_k \rightharpoonup \lambda_\infty$ weakly in $W^{1,p}_{\mathrm{loc}}(S^1 \setminus B)$ and for every $\varphi \in C^\infty_c(S^1 \setminus B)$ we have

$$0 = \lim_{k\to\infty} \int_{S^1} (\lambda_k(-\Delta)^{\frac{1}{2}}\varphi - (\kappa_k e^{\lambda_k} - 1)\varphi)\, d\theta = \lim_{k\to\infty} \int_{S^1} (\lambda_\infty(-\Delta)^{\frac{1}{2}}\varphi - (\mu - 1)\varphi)\, d\theta.$$

In particular, the distribution

$$T_\infty := (-\Delta)^{\frac{1}{2}}\lambda_\infty - \mu + 1$$

is supported in B and, since, by (96), $T_\infty \in \mathcal{M}(S^1)$, the order of T_∞ (as a distribution) is 0; hence,

$$T_\infty = \sum_{j=1}^{N} c_j\delta_{a_j}.$$

In order to compute the coefficients c_j, let $\chi_\delta : S^1 \to \mathbb{R}$ be 1 on $S^1 \cap \bigcup_{j=1}^n B(a_j, \delta)$ and 0 otherwise. We rewrite (9) as follows:

$$(-\Delta)^{\frac{1}{2}}\lambda_k = (1-\chi_\delta)\kappa_k e^{\lambda_k} + \chi_\delta\kappa_k e^{\lambda_k} - 1. \tag{97}$$

Since

$$\lim_{k\to\infty} (1-\chi_\delta)\kappa_k e^{\lambda_k} = (1-\chi_\delta)\kappa_\infty e^{\lambda_\infty} \quad \text{in } \mathcal{D}'(S^1),$$

testing (97) with $\varphi \in C^\infty(S^1)$ and letting $k \to \infty$ we get

$$\int_{S^1} (\lambda_\infty(-\Delta)^{\frac{1}{2}}\varphi - (1-\chi_\delta)\kappa_\infty e^{\lambda_\infty}\varphi + \varphi)\, d\theta = \lim_{k\to\infty} \int_{S^1} \chi_\delta\kappa_k e^{\lambda_k}\varphi\, d\theta$$

and, letting $\delta \to 0$, we infer

$$\langle T_\infty, \varphi\rangle = \lim_{\delta\to 0}\lim_{k\to\infty} \int_{S^1} \chi_\delta\kappa_k e^{\lambda_k}\varphi\, d\theta.$$

By choosing $\varphi = 1$ in a neighborhood of a_j for a fixed j and $\varphi = 0$ in a neighborhood of $B \setminus \{a_j\}$, we get

$$c_j = \lim_{\delta\to 0}\lim_{k\to\infty} \int_{S^1 \cap B(a_j,\delta)} \kappa_k e^{\lambda_k}\, d\theta.$$

We now want to compute c_j for a fixed $j \in \{1, \dots, N\}$. Consider the Möbius transformation $f_k(z) = (z - t_k a_j)/(1 - t_k \bar{a}_j z)$, and $\tilde{\Phi}_k := \Phi_k \circ f_k$, for a sequence $t_k \uparrow 1$ to be chosen. By Corollary 2.4 we have

$$\tilde{\lambda}_k := \log|\tilde{\Phi}'_k| = \lambda_k \circ f_k + \log|f'_k|, \quad \tilde{\kappa}_k := \kappa_k \circ f_k,$$

and

$$(-\Delta)^{\frac{1}{2}} \tilde{\lambda}_k = \tilde{\kappa}_k e^{\tilde{\lambda}_k} - 1.$$

Since $\log|f'_k| \to -\infty$ locally uniformly in $\bar{D}^2 \setminus \{a_j\}$ and $\log|f'_k(a_j)| \to \infty$, it is not difficult to see that, if $t_k \uparrow 1$ slowly enough, then $\tilde{\lambda}_k \to -\infty$ uniformly locally in $\bar{D}^2 \setminus \{a_j, -a_j\}$ and we can apply Proposition 3.9 to $\tilde{\Phi}_k$ and obtain that

$$\tilde{\kappa}_k e^{\tilde{\lambda}_k} \overset{*}{\rightharpoonup} \pi(\delta_{a_j} + \delta_{-a_j}).$$

With a change of variable we then get

$$\pi = \lim_{\delta \to 0} \lim_{k \to \infty} \int_{S^1 \cap B(a_j, \delta)} \tilde{\kappa}_k e^{\tilde{\lambda}_k} \, d\theta = \lim_{\delta \to 0} \lim_{k \to \infty} \int_{f_k(S^1 \cap B(a_j, \delta))} \kappa_k e^{\lambda_k} \, d\theta = c_j,$$

where the last identity holds up to having $t_k \uparrow 1$ slowly enough. □

Proof of Theorem 1.5. From Proposition 3.15 it follows that $S^1 \setminus \{\mathcal{P}\} = \bigcup_{j \in J} \mathcal{A}_i$, where J is an at most countable set and $\mathcal{A}_j$ is an equivalence class generated by the relation in Definition 3.14. From Proposition 3.16 it follows that for every class $\mathcal{A}_j$ there is a sequence of Möbius transformations $f_k^j(z)$ such that

$$\tilde{\Phi}_k^j := \Phi_k \circ f_k^j \rightharpoonup \tilde{\Phi}_\infty^j \quad \text{in } W^{2,p}_{\text{loc}}(\bar{D}^2 \setminus B_j), \qquad B_j = \{b_1^j, \dots b_{N_j}^j\},$$

where $\tilde{\Phi}_\infty^j : \bar{D}^2 \setminus B_j \to \mathbb{R}^2$ is a conformal immersion and $\gamma_\infty(\mathcal{A}_j) = \tilde{\Phi}_\infty^j(S^1 \setminus B_j)$. Moreover, we have

$$(\gamma_\infty)_*[S^1 \setminus \mathcal{P}] = \sum_{j \in J} (\tilde{\Phi}_\infty^j)_*[S^1 \setminus B_j].$$

We have

$$\sum_{j \in J} (\gamma_\infty)_*[\mathcal{A}_j] = \sum_{j \in J} (\tilde{\Phi}_\infty^j)_*[S^1 \setminus B_j]$$

and it remains to prove that

$$(\gamma_\infty)_*[\mathcal{P}] = 0.$$

In order to do that, let $\tau : \mathcal{P} \to \mathcal{P}$ be the bijection which, to a pinched point p, associates its dual. For a differential form $\phi : \mathbb{C} \to L(\mathbb{C}, \mathbb{C})$, we have

$$(\gamma_\infty)_*[\mathcal{P}](\phi) = \int_{\mathcal{P}} \phi(\gamma_\infty(t)) \dot{\gamma}_\infty(t) \, dt. \tag{98}$$

Now recall that

$$\gamma_\infty(t) = \gamma_\infty(\tau(t)), \quad \dot{\gamma}_\infty(t) = -\dot{\gamma}_\infty(\tau(t)). \tag{99}$$

For a sequence $t_n \in \mathcal{P}$ with $t_n \to t \in \mathcal{P}$ as $n \to \infty$, we have

$$\begin{aligned} \gamma_\infty(t_n) &= \gamma_\infty(t) + \dot\gamma_\infty(t)(t_n - t) + o(t_n - t), \\ \gamma_\infty(\tau(t_n)) &= \gamma_\infty(\tau(t)) + \dot\gamma_\infty(\tau(t))(\tau(t_n) - \tau(t)) + o(\tau(t_n) - \tau(t)), \end{aligned} \tag{100}$$

where for simplicity of notation we identified S^1 with the interval $[0, 2\pi]$, with zero corresponding to a point in $S^1 \setminus \mathcal{P}$. Using (99) and (100) we infer that

$$\lim_{n\to\infty} \frac{\tau(t_n) - \tau(t)}{t_n - t} = -1.$$

Then, at a density point of $\mathcal{P}$, we have $d\tau/dt = -1$ in the sense of approximate differentials (if the density of $\mathcal{P}$ is everywhere 0 then $|\mathcal{P}| = 0$ and we are done). Therefore,

$$\int_{\mathcal{P}} \phi(\gamma_\infty(t))\dot\gamma_\infty(t)\,dt = -\int_{\mathcal{P}} \phi\big(\gamma_\infty(\tau(t))\big)\dot\gamma_\infty(\tau(t))\,dt = -\int_{\tau(\mathcal{P})=\mathcal{P}} \phi(\gamma_\infty(t))\dot\gamma_\infty(t)\,dt,$$

where in the first identity we used (99) and in the second identity we made a change of variable. This proves that the integral in (98) vanishes for every differential form ϕ; hence, $(\gamma_\infty)_*[\mathcal{P}] = 0$.

Since, for every $j \in J$, the sequence $(\tilde\Phi_k^j)$ is as in case (1) of Theorem 3.5, i.e., setting $\lambda_k^j := \log\big|(\tilde\Phi_k^j)'|_{S^1}\big|$ we have $|\bar\lambda_k^j| \le C$, we can apply Theorem 1.2(iii) and it follows at once that the blow-up set of λ_k^j is B_j. □

4. Relation between the Liouville equations in $\mathbb{R}$ and S^1

Consider the conformal map $G : D^2 \to \mathbb{R}^2$ given by

$$G(z) = \frac{iz+1}{z+i} = \frac{z + \bar z + i(|z|^2 - 1)}{1 + |z|^2 + i(\bar z - z)}.$$

We will use on the domain D^2 the coordinate $z = \xi + i\eta$ and on the target $\mathbb{R}^2$ the coordinates (x, y) or $x + iy$. Writing G in components,

$$G^1(z) = \Re G(z) = \frac{2\xi}{(1+\eta)^2 + \xi^2}, \qquad G^2(z) = \Im G(z) = \frac{\xi^2 + \eta^2 - 1}{(1+\eta)^2 + \xi^2},$$

and using the polar coordinates (r, θ) on D^2 one easily verifies

$$\frac{\partial G^1}{\partial r}\bigg|_{r=1} = 0, \qquad \frac{\partial G^2}{\partial r}\bigg|_{r=1} = \frac{1}{1+\eta}, \qquad \frac{\partial G^1}{\partial \theta}\bigg|_{r=1} = -\frac{1}{1+\eta}, \qquad \frac{\partial G^2}{\partial \theta}\bigg|_{r=1} = 0.$$

Notice that $G|_{S^1}(\xi + i\eta) = \xi/(1+\eta)$, i.e., $\Pi := G^1|_{S^1}$ is the classical stereographic projection from $S^1 \setminus \{-i\}$ onto $\mathbb{R}$. Its inverse is

$$\Pi^{-1}(x) = \frac{2x}{1+x^2} + i\left(-1 + \frac{2}{1+x^2}\right). \tag{101}$$

If we write $\Pi^{-1}(x) = e^{i\theta(x)}$, we get the useful relation

$$1 + \sin(\theta(x)) = \frac{2}{1+x^2}, \qquad \frac{2}{1+\Pi(\theta)^2} = 1 + \sin\theta, \tag{102}$$

which follows easily from $\sin(\theta(x)) = \Im(\Pi^{-1}(x)) = (1-x^2)/(1+x^2)$.

Proposition 4.1. *Given $u:\mathbb{R}\to\mathbb{R}$, set $v := u\circ\Pi : S^1 \to \mathbb{R}$, where $\Pi := G^1|_{S^1}$. Then $u \in L_{\frac{1}{2}}(\mathbb{R})$ if and only if $v \in L^1(S^1)$. In this case,*

$$(-\Delta)^{\frac{1}{2}} v(e^{i\theta}) = \frac{((-\Delta)^{\frac{1}{2}}u)(\Pi(e^{i\theta}))}{1+\sin\theta} \quad \text{in } \mathscr{D}'(S^1\setminus\{-i\}), \tag{103}$$

that is,

$$\langle(-\Delta)^{\frac{1}{2}}v,\varphi\rangle = \langle(-\Delta)^{\frac{1}{2}}u,\varphi\circ\Pi^{-1}\rangle \quad \text{for every } \varphi\in C_0^\infty(S^1\setminus\{-i\}).$$

Further, if $(-\Delta)^{\frac{1}{2}}u \in L^1(\mathbb{R})$ or, equivalently, $(-\Delta)^{\frac{1}{2}}v|_{S^1\setminus\{-i\}} \in L^1(S^1)$, then

$$(-\Delta)^{\frac{1}{2}} v(e^{i\theta}) = \frac{((-\Delta)^{\frac{1}{2}}u)(\Pi(e^{i\theta}))}{1+\sin\theta} - \gamma\delta_{-i} \quad \text{in } \mathscr{D}'(S^1), \quad \gamma = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}}u\,dx. \tag{104}$$

Proof. Since

$$\int_{S^1} |v|\,d\theta = \int_{\mathbb{R}} \frac{2|v(\Pi^{-1}(x))|}{1+x^2}\,dx,$$

it is clear that $v\in L^1(S^1)$ if and only if $u\in L_{\frac{1}{2}}(\mathbb{R})$.

Given now $\varphi\in C_c^\infty(S^1\setminus\{-1\})$, set $\psi := \varphi\circ\Pi^{-1}\in C_c^\infty(\mathbb{R})$ and let $\tilde\varphi\in C^\infty(\bar{D}^2)$ and $\tilde\psi\in C^\infty\cap L^\infty(\bar{R}^2_+)$ be the harmonic extensions of φ and ψ given by the Poisson formulas (125) and (132), respectively. It is not difficult to see that, setting $\bar{G}=(G^1,-G^2)$, $\tilde\psi\circ\bar{G}|_{\bar{D}^2}$ is continuous, harmonic in D^2 and it coincides with $\tilde\varphi$ on S^1. Then, by the maximum principle, $\tilde\varphi = \tilde\psi\circ\bar{G}$ in $\bar{D}^2\setminus\{-i\}$.

Using polar coordinates we compute

$$\frac{\partial\tilde\varphi}{\partial r}\bigg|_{r=1}\circ\Pi^{-1} = \frac{\partial(\tilde\varphi\circ G^{-1})}{\partial x}\frac{\partial G^1}{\partial r}\bigg|_{r=1} + \frac{\partial(\tilde\varphi\circ G^{-1})}{\partial y}\frac{\partial G^2}{\partial r}\bigg|_{r=1} = -\frac{\partial\tilde\psi}{\partial y}\bigg|_{y=0}\frac{1+x^2}{2}.$$

Then, using Propositions A.1 and A.3, we get

$$\begin{aligned}\langle(-\Delta)^{\frac{1}{2}}v,\varphi\rangle &= \int_{S^1} v\frac{\partial\tilde\varphi}{\partial r}\bigg|_{r=1}d\theta\\ &= \int_{\mathbb{R}} (v\circ\Pi^{-1}(x))\bigg(\frac{\partial\tilde\varphi}{\partial r}\bigg|_{r=1}\circ\Pi^{-1}(x)\bigg)\frac{2}{1+x^2}\,dx\\ &= -\int_{\mathbb{R}} u\frac{\partial\tilde\psi}{\partial y}\bigg|_{y=0}dx\\ &= \langle(-\Delta)^{\frac{1}{2}}u,\psi\rangle,\end{aligned}$$

so that (103) is proven.

In order to prove (104), set $f := ((-\Delta)^{\frac{1}{2}}v)|_{S^1\setminus\{-i\}}\in\mathscr{D}'(S^1\setminus\{-i\})$ and notice that

$$\|f\|_{L^1(S^1)} = \|(-\Delta)^{\frac{1}{2}}u\|_{L^1(\mathbb{R})} = \gamma.$$

Since $f\in L^1(S^1)\subset\mathscr{D}'(S^1)$, we have

$$T := (-\Delta)^{\frac{1}{2}}v - f \in\mathscr{D}'(S^1) \tag{105}$$

and $\operatorname{supp}(T) \subset \{-i\}$. We claim that $T = c\delta_{-i}$ for some constant c. By a rotation of S^1, it is convenient to assume that T is supported at $\{1\}$. In this case, we can write

$$T = \sum_{k=0}^{N} c_k D^k \delta_0$$

for some $N \in \mathbb{N}$ and $c_0, \dots, c_N \in \mathbb{C}$, which leads to

$$\langle T, \varphi\rangle = \sum_{k=0}^{N} c_k (-1)^k D^k \varphi_0 = \sum_{k=0}^{N} c_k \sum_{n\in\mathbb{Z}} (-in)^k \overline{\hat{\varphi}(n)} \quad \text{for } \varphi \in \mathfrak{D}(S^1). \tag{106}$$

On the other hand, according to (124) we have, for $\varphi \in \mathfrak{D}(S^1)$,

$$\begin{aligned}\langle (-\Delta)^{\frac{1}{2}} v, \varphi\rangle &= \int_{S^1} v(\theta) \sum_{n\in\mathbb{N}} |n| \overline{\hat{\varphi}(n)} e^{-in\theta}\, d\theta \\ &= \sum_{n\in\mathbb{N}} |n| \overline{\hat{\varphi}(n)} \int_{S^1} v(\theta) e^{-in\theta}\, d\theta \\ &= 2\pi \sum_{n\in\mathbb{N}} |n| \hat{v}(n) \overline{\hat{\varphi}(n)},\end{aligned} \tag{107}$$

where the sum can be moved outside the integral because $\sum_{n\in\mathbb{N}} |n||\hat{\varphi}(n)| < \infty$. Similarly,

$$\langle f, \varphi\rangle = 2\pi \sum_{n\in\mathbb{N}} \hat{f}(n) \overline{\hat{\varphi}(n)} \quad \text{for } \varphi \in \mathfrak{D}(S^1). \tag{108}$$

Clearly (105), (106), (107) and (108) are compatible only if $c_k = 0$ for $k = 1, \dots, N$, hence proving (up to rotating back) that $T = c_0 \delta_{-i}$, as claimed. Finally, testing with $\varphi = 1$ we obtain

$$0 = \langle (-\Delta)^{\frac{1}{2}} v, 1\rangle = \langle f, 1\rangle + \langle T, 1\rangle = \|(-\Delta)^{\frac{1}{2}} u\|_{L^1} + c_0,$$

which implies that $c_0 = -\|(-\Delta)^{\frac{1}{2}} u\|_{L^1}$. □

Now, given $u \in L_{\frac{1}{2}}(\mathbb{R})$ we want to define a function $\lambda \in L^1(S^1)$ such that

$$\Pi^*(e^{2u}|dx|^2) = e^{2\lambda}|d\theta|^2,$$

where Π^* denotes the pull-back of the stereographic projection, while $|dx|^2$ and $|d\theta|^2$ are the standard metrics on $\mathbb{R}$ and S^1, respectively. Since

$$\Pi^*(e^{2u}|dx|^2) = \left(\frac{\partial \Pi}{\partial \theta}\right)^2 e^{2u(\Pi(\theta))} |d\theta|^2,$$

we find

$$\lambda(\theta) = u(\Pi(\theta)) + \log\left|\frac{\partial \Pi}{\partial \theta}\right| = u(\Pi(\theta)) - \log(1 + \sin\theta) \tag{109}$$

or equivalently, using (102),

$$u(x) = \lambda(\Pi^{-1}(x)) + \log \frac{2}{1 + x^2}. \tag{110}$$

Using Proposition 4.1 we can now easily relate $(-\Delta)^{\frac{1}{2}}u$ and $(-\Delta)^{\frac{1}{2}}\lambda$.

Proposition 4.2. *Given $u:\mathbb{R}\to\mathbb{R}$, set λ as in (109). Then $u\in L_{\frac{1}{2}}(\mathbb{R})$ if and only if $\lambda\in L^1(S^1)$, and $(-\Delta)^{\frac{1}{2}}u\in L^1(\mathbb{R})$ if and only if $(-\Delta)^{\frac{1}{2}}\lambda\in L^1(S^1\setminus\{-i\})$. In this case, u solves (20) if and only if λ solves*

$$(-\Delta)^{\frac{1}{2}}\lambda=\kappa e^{\lambda}-1+(2\pi-c)\delta_{-i}\quad in\ \ S^1 \tag{111}$$

with $\kappa=V\circ\Pi$ and $c=\|(-\Delta)^{\frac{1}{2}}u\|_{L^1(\mathbb{R})}$.

Proof. This follows at once from Proposition 4.2 and Lemma 4.3, below. □

Lemma 4.3. *We have*

$$(-\Delta)^{\frac{1}{2}}\log(1+\sin\theta)=1-2\pi\delta_{-i}.$$

Proof. Notice that by (102) we can write

$$\log(1+\sin\theta)=u_{1,0}(\Pi(\theta)),\qquad u_{1,0}(x)=\log\frac{2}{1+x^2}.$$

Then Propositions 5.1 and 4.1 imply

$$\begin{aligned}(-\Delta)^{\frac{1}{2}}\log(1+\sin\theta)=\frac{(-\Delta)^{\frac{1}{2}}u(\Pi(\theta))}{1+\sin\theta}-\|(-\Delta)^{\frac{1}{2}}u\|_{L^1}\delta_{-i}&=\frac{e^{u_{1,0}(\Pi(\theta))}}{1+\sin\theta}-\delta_i\int_{\mathbb{R}}e^{u_{1,0}(x)}\,dx\\&=1-2\pi\delta_{-i}.\end{aligned}$$ □

5. Proof of Theorem 1.8 and Proposition 1.9

Before proving Theorem 1.8, we show that the functions defined in (27) are indeed solutions of (24)–(25).

Proposition 5.1. *For every $\mu>0$ and $x_0\in\mathbb{R}$, the function u_{μ,x_0} defined in (27) belongs to $L_{\frac{1}{2}}(\mathbb{R})$, satisfies (25) with $L=2\pi$, and solves (24).*

Proof. That $u_{\lambda,x_0}\in L_{\frac{1}{2}}(\mathbb{R})$ and $\int_{\mathbb{R}}e^{u_{\lambda,x_0}}\,dx=2\pi$ is elementary. The equation is invariant under translations and dilations in the sense that, for all $x_0\in\mathbb{R}$ and $\lambda>0$, if u is a solution of (24) then $u(\lambda(x+x_0))+\log(\lambda)$ is a solution of (24) as well; hence, it suffices to prove that $u_{1,0}(x)=\log(2/(1+x^2))$ is a solution. From Proposition A.3 we get, integrating by parts,

$$\begin{aligned}\pi(-\Delta)^{\frac{1}{2}}u_{1,0}(x)&=\lim_{\varepsilon\to0}\int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]}\frac{\log\frac{1+y^2}{1+x^2}}{(x-y)^2}\,dy\\&=\lim_{\varepsilon\to0}\left\{-\frac{\log\frac{1+y^2}{1+x^2}}{y-x}\bigg|_{-\infty}^{x-\varepsilon}-\frac{\log\frac{1+y^2}{1+x^2}}{y-x}\bigg|_{x+\varepsilon}^{\infty}+\int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]}\frac{2y}{(y-x)(1+y^2)}\,dy\right\}\\&=\lim_{\varepsilon\to0}\left\{\frac{2\arctan y+x\log\frac{(y-x)^2}{1+y^2}}{1+x^2}\bigg|_{-\infty}^{x-\varepsilon}+\frac{2\arctan y+x\log\frac{(y-x)^2}{1+y^2}}{1+x^2}\bigg|_{x+\varepsilon}^{\infty}\right\}\\&=\frac{2\pi}{1+x^2}=\pi e^{u_{1,0}(x)}.\end{aligned}$$ □

Theorem 5.2. *There exist constants $C_1, C_2 > 0$ such that for any $\varepsilon \in (0, \pi)$ one has*

$$C_1 \le \sup_{\substack{u \in \tilde{H}^{1,1}_{\Delta}(I) \\ \|(-\Delta)^{\frac{1}{2}} u\|_{L^1(I)} \le 1}} \frac{\varepsilon}{|I|} \int_I e^{(\pi-\varepsilon)|u|}\, d\theta \le C_2, \tag{112}$$

where $\tilde{H}^{1,1}_{\Delta}(I) := \{u \in L^1(\mathbb{R}) : \operatorname{supp}(u) \subset \bar{I},\ (-\Delta)^{\frac{1}{2}} u \in L^1(\mathbb{R})\}$.

Lemma 5.3. *The Green function of $(-\Delta)^{\frac{1}{2}}$ on the interval $I = (-1, 1)$ can be decomposed as*

$$G_{\frac{1}{2}}(x, y) = F_{\frac{1}{2}}(|x-y|) + H_{\frac{1}{2}}(x, y),$$

where $F_{\frac{1}{2}}(x) := (1/\pi)\log(1/|x|)$ and $H_{\frac{1}{2}}$ is bounded above.

Proof. This follows from the explicit expression of $G(x, y)$ (see, e.g., [Blumenthal et al. 1961; Bucur 2015]), namely

$$G(x, y) = \frac{1}{2\pi}\int_0^{r_0(x,y)} \frac{1}{\sqrt{r(r+1)}}\, dr = \frac{1}{\pi}\log\big(\sqrt{r_0(x, y)} + \sqrt{r_0(x, y)+1}\big),$$

where

$$r_0(x, y) := \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}. \qquad \square$$

Proof of Theorem 5.2. Up to a translation and dilation we can assume that $I = (-1, 1)$. With Lemma 5.3 we write, for $u \in \tilde{H}^{1,1}_{\Delta}(I)$ and $f := (-\Delta)^{\frac{1}{2}} u$,

$$|u(x)| = \left|\int_I G(x, y) f(y)\, dy\right|,$$

and we bound

$$G(x, y) \le \frac{1}{\pi}\log\left(\frac{2}{|x-y|}\right) + C, \quad x, y, \in I,$$

hence

$$|u(x)| \le \frac{1}{\pi}\int_I \log\left(\frac{2}{|x-y|}\right)|f(y)|\, dy + C \tag{113}$$

and, exactly as in (56), one gets

$$\int_I e^{(\pi-\varepsilon)|u(x)|}\, dx \le C\int_I |f(y)| \int_I \left(\frac{2}{|x-y|}\right)^{1-\frac{\varepsilon}{\pi}} dx\, dy \le \frac{C}{\varepsilon}.$$

The rest of the proof is also similar to the proof of Theorem 3.2. $\square$

Remark 5.4. A slight modification of (112) is

$$C_1 \le \sup_{\substack{u = F_{1/2} * f \\ \operatorname{supp}(f) \subset \bar{I},\ \|f\|_{L^1(I)} \le 1}} \frac{\varepsilon}{|I|} \int_I e^{(\pi-\varepsilon)|u|}\, d\theta \le C_2, \tag{114}$$

where $F_{\frac{1}{2}}$ is as in Lemma 5.3. The proof of (114) is similar to the proof of (112), since $u = F_{\frac{1}{2}} * f$ obviously satisfies (113). An alternative proof of a nonsharp version of (114), namely

$$\sup_{\substack{u=F_{\frac{1}{2}}*f \\ \operatorname{supp}(f)\subset\bar{I},\ \|f\|_{L^1(I)}\le 1}} \int_I e^{\delta|u-\bar{u}|}\,d\theta \le C_2 \quad \text{for some } \delta>0 \text{ and } \bar{u} := \fint_I u\,dx,$$

can be obtained noticing that, for $u = F_{\frac{1}{2}} * f$, one has $[u]_{\mathrm{BMO}(I)} \le C[F_{\frac{1}{2}}]_{\mathrm{BMO}(\mathbb{R})}\|f\|_{L^1(I)}$, and one can apply the John–Nirenberg inequality.

Proposition 5.5. *Let $u \in L_{\frac{1}{2}}(\mathbb{R})$ satisfy (24)–(25). Then there is a constant $C_0 \in \mathbb{R}$ such that*

$$u(x) = \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{1+|y|}{|x-y|}\right) e^{u(y)}\,dy + C_0. \tag{115}$$

In the proof of Proposition 5.5 we use two lemmata.

Lemma 5.6. *For any $f \in L^1(\mathbb{R})$ the function*

$$w(x) := \mathcal{I}[f](x) := \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{1+|y|}{|x-y|}\right) f(y)\,dy \tag{116}$$

is well defined, belongs to $L_{\frac{1}{2}}(\mathbb{R})$ and satisfies

$$(-\Delta)^{\frac{1}{2}} w = f \quad \text{in } \mathcal{S}'. \tag{117}$$

Proof of Lemma 5.6. Let us first assume that f belongs to the Schwartz space $\mathcal{S}$. Remember that, for $F(x) := (1/\pi)\log(1/|x|)$, we have (see, e.g., [Vladimirov 1971, p. 132])

$$\hat{F}(\xi) = \mathcal{P}\frac{1}{|\xi|} + C\delta_0 \quad \text{in } \mathcal{S}', \tag{118}$$

where $\mathcal{P}(1/|\xi|) \in \mathcal{S}'$ is the tempered distribution defined by

$$\left\langle \mathcal{P}\frac{1}{|\xi|}, \varphi \right\rangle = \int_{|\xi|\le 1} \frac{\varphi(\xi)-\varphi(0)}{|\xi|}\,d\xi + \int_{|\xi|>1} \frac{\varphi(\xi)}{|\xi|}\,d\xi, \quad \varphi \in \mathcal{S}. \tag{119}$$

For every $f \in C_c^\infty(\mathbb{R})$ one easily sees that $F * f \in C^\infty(\mathbb{R})$ and $F * f \in L_{\frac{1}{2}}(\mathbb{R})$. Then

$$\begin{aligned}
\langle (-\Delta)^{\frac{1}{2}}(F*f), \varphi\rangle &:= \int_{\mathbb{R}} (F*f)\mathcal{F}^{-1}(|\xi|\hat{\varphi})\,dx \\
&= \int_{\mathbb{R}} F(\tilde{f} * \mathcal{F}^{-1}(|\xi|\hat{\varphi}))\,dx \\
&= \int_{\mathbb{R}} F\mathcal{F}\big(\mathcal{F}^{-1}(\tilde{f} * \mathcal{F}^{-1}(|\xi|^{2\sigma}\hat{\varphi}))\big)\,dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} F\mathcal{F}(\hat{f}|\xi|\hat{\varphi})\,dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}\hat{\hat{\varphi}}\,d\xi = \int_{\mathbb{R}} f\varphi\,dx,
\end{aligned} \tag{120}$$

where in order to apply (119) in the fifth identity can approximate the function $\psi(\xi) = \hat{f}|\xi|\hat{\varphi}$ by a sequence of functions $\psi_\varepsilon = \hat{f}\eta_\varepsilon\hat{\hat{\varphi}} \in \mathcal{S}(\mathbb{R})$ with $\eta_\varepsilon \in C^\infty(\mathbb{R})$ suitably chosen (see, for instance, [Jin et al. 2015b]). Hence, $(-\Delta)^{\frac{1}{2}}(F * f) = f$ in $\mathcal{D}'(\mathbb{R})$ and, since $f \in \mathcal{D}(\mathbb{R})$, the identity also holds in a strong sense. Moreover, since obviously

$$(-\Delta)^{\frac{1}{2}}\left(\frac{1}{\pi}\int_{\mathbb{R}} \log(1+|y|)f(y)\,dy\right) = 0,$$

we see that (117) is satisfied when $f \in \mathcal{D}(\mathbb{R})$.

For a general function $f \in L^1(\mathbb{R})$ we can find a sequence $(f_k) \subset \mathcal{D}(\mathbb{R})$ with $f_k \to f$ in $L^1(\mathbb{R})$ and take $\varphi \in \mathcal{S}(\mathbb{R})$. Then

$$(I)_k := \langle(-\Delta)^{\frac{1}{2}}\mathcal{I}[f_k], \varphi\rangle = \langle f_k, \varphi\rangle \to \langle f, \varphi\rangle$$

as $k \to \infty$, while

$$(I)_k = \langle \mathcal{I}[f_k], (-\Delta)^{\frac{1}{2}}\varphi\rangle = \int_{\mathbb{R}} \mathcal{I}[f_k](x)\psi(x)\,dx,$$

where $\psi := (-\Delta)^{\frac{1}{2}}\varphi$ satisfies

$$|\psi(x)| \le C(1+|x|^2). \tag{121}$$

It remains to show that

$$\int_{\mathbb{R}} \mathcal{I}[f_k - f](x)\psi(x)\,dx \to 0 \quad \text{as } k \to \infty.$$

Define $g_k := f_k - f \to 0$ in $L^1(\mathbb{R})$. Then, from $\|h_1 * h_2\|_{L^1} \le \|h_1\|_{L^1}\|h_2\|_{L^1}$, we get

$$\left|\int_{B(x,1)} \log\left(\frac{1+|y|}{|x-y|}\right) g_k(y)\,dy\right| \le \log(2+|x|)\|g_k\|_{L^1(\mathbb{R})} + C\|g_k\|_{L^1}$$

and, using that for $|x-y| \ge 1$ we have $\log((1+|y|)/|x-y|) \le C(1+\log(|x|))$,

$$\left|\int_{\mathbb{R}\setminus B(x,1)} \log\left(\frac{1+|y|}{|x-y|}\right) g_k(y)\,dy\right| \le C(1+\log|x|)\|g_k\|_{L^1}.$$

Therefore, taking (121) into account, we see that

$$(I)_k \to \langle \mathcal{I}[f], (-\Delta)^{\frac{1}{2}}\varphi\rangle \quad \text{as } k \to \infty;$$

hence, we conclude that $(-\Delta)^{\frac{1}{2}}w = f$ in $\mathcal{S}'(\mathbb{R})$. □

Lemma 5.7. *Let $f \in L_{\frac{1}{2}}(\mathbb{R})$ satisfy $(-\Delta)^{\frac{1}{2}}f = 0$. Then f is constant.*

Proof. This is identical to the proof of Lemma 14 in [Jin et al. 2015b]. □

Proof of Proposition 5.5. Set $w(x)$ as in (116) with $f(y) := e^{u(y)}$. Then $(-\Delta)^{\frac{1}{2}}(u-w) = 0$ by Lemma 5.6; hence, by Lemma 5.7, $u - w \equiv C_0$ for some $C_0 \in \mathbb{R}$. □

Proposition 5.8. *Let $u \in L_{\frac{1}{2}}(\mathbb{R})$ satisfy (24)–(25). Then $u \in C^\infty(\mathbb{R})$.*

Proof. Up to scaling, assume that

$$\int_{-1}^{1} e^{u(x)}\,dx < \varepsilon,$$

where ε will be fixed later.

Let us split $u = u_1 + u_2$, where

$$u_1(x) = \frac{1}{\pi}\int_{-1}^{1}\log\left(\frac{1+|y|}{|x-y|}\right)e^{u(y)}\,dy + C_0 = \frac{1}{\pi}\int_{-1}^{1}\log\left(\frac{1}{|x-y|}\right)e^{u(y)}\,dy + C_1. \tag{122}$$

Then (115) implies that u_2 is defined by the same formula, integrating over $\mathbb{R}\setminus[-1,1]$ instead of $\mathbb{R}$. It is easy to see that

$$\|u_2\|_{L^\infty([-\frac{1}{2},\frac{1}{2}])} \le C\int_{\mathbb{R}} e^{u(x)}\,dx < \infty.$$

From (114) if follows that, given $p < \infty$, choosing $\varepsilon > 0$ small enough (depending on p) we have $e^{|u_1|} \in L^p([-1,1])$, so $e^u \in L^p\left[-\frac{1}{2},\frac{1}{2}\right]$.

The same argument, together with translations and dilations, can be performed in a neighborhood of every point in $\mathbb{R}$, giving $e^u \in L^p_{\mathrm{loc}}(\mathbb{R})$ for $1 < p < \infty$. Going back to (115) it is easy to bootstrap regularity and prove that u is actually smooth. □

Corollary 5.9. *Every function $\lambda \in L^1(S^1)$ solving (33) with $(-\Delta)^{\frac{1}{2}}\lambda \in L^1(S^1)$ is smooth.*

Proof. By Proposition 4.2 the function $u:\mathbb{R}\to\mathbb{R}$ given by (110) is in $L_{\frac{1}{2}}(\mathbb{R})$ and it solves (24). Then, by Proposition 5.8, u is smooth; hence, $\lambda \in C^\infty(S^1\setminus\{-i\})$. Since (33) is invariant under rotations, we have that actually $\lambda \in C^\infty(S^1)$. □

Lemma 5.10. *For $u \in L_{\frac{1}{2}}(\mathbb{R})\cap C^1(\mathbb{R})$ solving (24)–(25), set*

$$\alpha := \int_{\mathbb{R}} e^{u(x)}\,dx.$$

Then $\alpha = 2\pi$.

Proof. This argument is taken from [Xu 2005] and is based on a Pohozaev-type identity. Differentiating (115) (for instance, by splitting the domain of integration into $[-a,a]$ and $\mathbb{R}\setminus[-a,a]$ for some $a > |x|$ and using elementary calculus) we obtain

$$x\frac{\partial u}{\partial x} = -\frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}}\frac{x}{x-y}e^{u(y)}\,dy.$$

Multiplying by $e^{u(x)}$ and integrating with respect to x on the interval $[-R,R]$, we get

$$(I) := \int_{-R}^{R} x\frac{\partial u}{\partial x}e^{u(x)}\,dx = -\frac{1}{\pi}\int_{-R}^{R}\mathrm{PV}\int_{\mathbb{R}}\frac{x}{x-y}e^{u(y)}\,dy\,e^{u(x)}\,dx =: (II).$$

Integrating by parts we find

$$(I) = \int_{-R}^{R} x\frac{\partial e^{u(x)}}{\partial x}\,dx = R(e^{u(R)}+e^{u(-R)}) - \int_{-R}^{R} e^{u(x)}\,dx \to -\alpha \quad \text{as } R\to\infty,$$

where we used that, at least on a subsequence, $R(e^{u(R)} - e^{u(-R)}) \to 0$ as $R \to \infty$, otherwise (25) would be violated. As for (II), we compute

$$(II) = -\frac{1}{2\pi}\int_{-R}^{R}\int_{\mathbb{R}} e^{u(y)}\,dy\, e^{u(x)}\,dx - \frac{1}{2\pi}\int_{-R}^{R} \mathrm{PV}\int_{\mathbb{R}} \frac{x+y}{x-y} e^{u(y)}\,dy\, e^{u(x)}\,dx \to -\frac{\alpha^2}{2\pi} + 0$$

as $R \to \infty$. Therefore, from $(I) = (II)$ we infer $\alpha = \alpha^2/(2\pi)$, i.e., $\alpha = 2\pi$. □

Proof of Theorem 1.8. Given $u \in L_{\frac{1}{2}}(\mathbb{R})$ satisfying (24)–(25), by Proposition 4.2 the function $\lambda(\theta) := u(\Pi(\theta)) - \log(1+\sin\theta)$ solves

$$(-\Delta)^{\frac{1}{2}}\lambda = e^{\lambda} - 1 + (2\pi - \alpha)\delta_{-i} \quad \text{in } S^1$$

and, by Lemma 5.10, $\alpha = 2\pi$; hence,

$$(-\Delta)^{\frac{1}{2}}\lambda = e^{\lambda} - 1 \quad \text{in } S^1.$$

By Corollary 2.3, λ is of the form given by (34) for some $a \in D^2$.

To complete the proof, write $a = \alpha e^{i\theta_0} = \alpha(t+is)$ with $\alpha, t, s \in \mathbb{R}$. We have

$$u(x) = \lambda \circ \Pi^{-1}(x) + \log\frac{2}{1+x^2} = \log\frac{2(1-\alpha^2)}{|1-\alpha(t+is)\Pi^{-1}(x)|^2(1+x^2)}.$$

The right-hand side can be computed using (101):

$$\begin{aligned} u(x) &= \log\frac{2(1-\alpha^2)}{\left|1+\alpha\frac{-2tx+s(1-x^2)}{1+x^2} - i\alpha\frac{2sx+t(1-x^2)}{1+x^2}\right|^2(1+x^2)} \\ &= \log\frac{2(1-\alpha^2)}{x^2(1-2\alpha s+\alpha^2) - 4\alpha t x + 1 + 2\alpha s + \alpha^2}. \end{aligned}$$

Completing the square in the denominator on the right-hand side, we get

$$u(x) = \log\frac{2(1-\alpha^2)}{(1-2\alpha s+\alpha^2)\left(x - \frac{2\alpha t}{1-2\alpha s+\alpha^2}\right)^2 + \frac{(1-\alpha^2)^2}{1-2\alpha s+\alpha^2}} = \log\frac{2\mu}{1+\mu^2(x-x_0)^2}$$

with

$$x_0 = \frac{2\alpha t}{1-2\alpha s+\alpha^2}, \qquad \mu = \frac{1-2\alpha s+\alpha^2}{1-\alpha^2}.$$ □

The following can been seen as a nonlocal version of the classical mean value property of harmonic functions. It appears in [Silvestre 2007, Proposition 2.2.6] in a slightly different case, but with a proof which readily extends to the following case.

Proposition 5.11. *There exists a positive function $\gamma_1 \in C^{1,1}(\mathbb{R})$ with $\int_{\mathbb{R}} \gamma_1\,dx = 1$ such that, setting $\gamma_\lambda(x) := (1/\lambda)\gamma_1(x/\lambda)$, we have*

$$u(x_0) \geq u * \gamma_\lambda(x_0)$$

for every $\lambda > 0$ and every $u \in L_{\frac{1}{2}}(\mathbb{R})$ satisfying $(-\Delta)^{\frac{1}{2}} u \geq 0$.

Proof of Proposition 1.9. Since $(-\Delta)^{\frac{1}{2}} u \le 0$, we have, by Proposition 5.11 below,

$$u(0) \le u * \gamma_\lambda(0) \quad \text{for every } \lambda > 0,$$

where γ_λ is as in Proposition 5.11. Since $d\mu_\lambda(x) := \gamma_\lambda(-x)\,dx$ satisfies $\int_{\mathbb{R}} d\mu_\lambda = 1$, from Jensen's inequality we get

$$\int_{\mathbb{R}} e^u \, d\mu_\lambda \ge \exp\biggl(\int_{\mathbb{R}} u \, d\mu_\lambda\biggr) = e^{u*\gamma_\lambda(0)} \ge e^{u(0)}.$$

On the other hand, since $d\mu_\lambda \le (C/\lambda)\,dx$, we estimate

$$\int_{\mathbb{R}} e^u \, dx \ge \frac{\lambda}{C} \int_{\mathbb{R}} e^u \, d\mu_\lambda \ge \frac{\lambda}{C} e^{u(0)} \to \infty \quad \text{as } \lambda \to \infty,$$

contradicting (25). □

Appendix A: The fractional Laplacian

The half-Laplacian on S^1. Given $u \in L^1(S^1)$, we define its Fourier coefficients as

$$\hat{u}(n) = \frac{1}{2\pi} \int_{S^1} u(\theta) e^{-in\theta} \, d\theta, \quad n \in \mathbb{Z}.$$

If u is smooth, we can define

$$(-\Delta)^{\frac{1}{2}} u(\theta) = \sum_{n \in \mathbb{Z}} |n| \hat{u}(n) e^{in\theta}. \tag{123}$$

For $u \in L^1(S^1)$, we can define $(-\Delta)^{\frac{1}{2}} u \in \mathscr{D}'(S^1)$ as a distribution as

$$\langle (-\Delta)^{\frac{1}{2}} u, \varphi \rangle := \int_{S^1} u (-\Delta)^{\frac{1}{2}} \varphi \, d\theta, \quad \varphi \in C^\infty(S^1). \tag{124}$$

Notice that $\varphi \in C^\infty(S^1)$ implies that $(-\Delta)^{\frac{1}{2}} \varphi \in C^\infty(S^1)$ (here, $(-\Delta)^{\frac{1}{2}} \varphi$ is defined as in (123)). In fact, given $\varphi \in L^1(S^1)$, we have $\varphi \in C^\infty(S^1)$ if and only if $\hat{\varphi}(n) = o(|n|^{-k})$ for every $k \ge 0$.

We can also give a definition of $(-\Delta)^{\frac{1}{2}} u$ in terms of harmonic extensions. If $u \in L^1(S^1)$, let $\tilde{u}(r, \theta)$ be its harmonic extension in D^2, explicitly given by the Poisson formula

$$\tilde{u}(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) u(t) \, dt, \quad P(r, \theta) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} \tag{125}$$

Then one can define (using polar coordinates)

$$(-\Delta)^{\frac{1}{2}} u = \frac{\partial \tilde{u}}{\partial r}\bigg|_{r=1} \quad \text{in } \mathscr{D}'(S^1), \tag{126}$$

where the distribution $\partial \tilde{u}/\partial r\big|_{r=1}$ is defined as

$$\biggl\langle \frac{\partial \tilde{u}}{\partial r}\bigg|_{r=1}, \varphi \biggr\rangle := \int_{S^1} u \frac{\partial \tilde{\varphi}}{\partial r}\bigg|_{r=1} d\theta,$$

where $\varphi \in C^\infty(S^1)$ and $\tilde{\varphi}$ is the harmonic extension of φ in D^2.

Notice that, if $u \in C^\infty(S^1)$, the equivalence of (123), (124) and, in fact, (126) is elementary, and (126) holds pointwise. For instance, the equivalence of (123) and (126) follows at once from

$$\tilde{u}(r,\theta) = \sum_{n\in\mathbb{Z}} \hat{u}(n) r^{|n|} e^{in\theta}.$$

Proposition A.1. *The definitions (124) and (126) are equivalent.*

Proof. Since (126) holds pointwise for smooth functions, one has, for $u \in L^1(S^1)$ and $\varphi \in C^\infty(S^1)$,

$$\langle (-\Delta)^{\frac{1}{2}} u, \varphi\rangle := \int_{S^1} u(-\Delta)^{\frac{1}{2}}\varphi\, dx = \int_{S^1} u \frac{\partial\tilde{\varphi}}{\partial\theta}\, d\theta =: \left\langle \frac{\partial \tilde{u}}{\partial r}\bigg|_{r=1}, \varphi \right\rangle. \qquad \square$$

For $u \in C^{1,\alpha}(S^1)$, there is also the following pointwise definition of $(-\Delta)^{\frac{1}{2}}u$:

Proposition A.2. *If $u \in C^{1,\alpha}(S^1)$ for some $\alpha \in (0,1]$, then $(-\Delta)^{\frac{1}{2}}u \in C^{0,\alpha}(S^1)$ and*

$$(-\Delta)^{\frac{1}{2}} u(e^{i\theta}) = \frac{1}{\pi}\,\mathrm{PV}\int_0^{2\pi} \frac{u(e^{i\theta}) - u(e^{it})}{2-2\cos(\theta-t)}\,dt, \tag{127}$$

where the principal value is well defined because $2 - 2r\cos(\theta - t) = (\theta - t)^2 + O((\theta-t)^4)$ as $t \to \theta$.

Proof. Considering Proposition A.1, it suffices to show the equivalence of (126) and (127). Set $\tilde{u}$ as in (125). Then

$$\begin{aligned}
\frac{\partial \tilde{u}(r,\theta)}{\partial r}\bigg|_{r=1} &= \lim_{r\uparrow 1} \frac{\tilde{u}(r,\theta) - u(e^{i\theta})}{r-1}\\
&= \lim_{r\uparrow 1} \frac{1}{2\pi(r-1)} \int_0^{2\pi} \frac{(1-r^2)(u(e^{i\theta}) - u(e^{it}))}{1-2r\cos(\theta-t)+r^2}\,dt\\
&= \lim_{r\uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{(1+r)(u(e^{i\theta}) - u(e^{it}))}{1-2r\cos(\theta-t)+r^2}\,dt\\
&= \frac{1}{\pi}\,\mathrm{PV}\int_0^{2\pi} \frac{u(e^{i\theta}) - u(e^{it})}{2-2r\cos(\theta-t)}\,dt. \qquad \square
\end{aligned}$$

The half-Laplacian on $\mathbb{R}$. For $u \in \mathcal{S}$ (the Schwarz space of rapidly decaying functions), we set

$$\widehat{(-\Delta)^{\frac{1}{2}} u}(\xi) = |\xi|\hat{u}(\xi), \qquad \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi}\,dx. \tag{128}$$

One can prove that

$$(-\Delta)^{\frac{1}{2}} u(x) = \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{u(x)-u(y)}{(x-y)^2}\,dy := \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus[-\varepsilon+x,x+\varepsilon]} \frac{u(x)-u(y)}{(x-y)^2}\,dy, \tag{129}$$

from which it follows that

$$\sup_{x\in\mathbb{R}} |(1+x^2)(-\Delta)^{\frac{1}{2}}\varphi(x)| < \infty \quad \text{for every } \varphi \in \mathcal{S}.$$

Then one can set

$$L_{\frac{1}{2}}(\mathbb{R}) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} \frac{|u(x)|}{1+x^2}\,dx < \infty \right\}, \tag{130}$$

and, for every $u \in L_{\frac{1}{2}}(\mathbb{R})$, one defines the tempered distribution $(-\Delta)^{\frac{1}{2}}u$ as

$$\langle(-\Delta)^{\frac{1}{2}}u, \varphi\rangle := \int_{\mathbb{R}} u(-\Delta)^{\frac{1}{2}}\varphi \, dx = \int_{\mathbb{R}} u\mathscr{F}^{-1}(|\xi|\hat{\varphi}(\xi)) \, dx \quad \text{for every } \varphi \in \mathscr{S}. \tag{131}$$

An alternative definition of $(-\Delta)^{\frac{1}{2}}$ can be given via the Poisson integral. For $u \in L_{\frac{1}{2}}(\mathbb{R})$, define the Poisson integral

$$\tilde{u}(x, y) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{yu(y)}{(y^2 + (x-\xi)^2)} \, d\xi, \qquad y > 0, \tag{132}$$

which is harmonic in $\mathbb{R} \times (0, \infty)$ and whose trace on $\mathbb{R} \times \{0\}$ is u. Then we have

$$(-\Delta)^{\frac{1}{2}}u = -\frac{\partial \tilde{u}}{\partial y}\bigg|_{y=0}, \tag{133}$$

where the identity is pointwise if u is regular enough (for instance, $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R})$), and has to be read in the sense of distributions in general, with

$$\left\langle -\frac{\partial \tilde{u}}{\partial y}\bigg|_{y=0}, \varphi \right\rangle := \left\langle u, -\frac{\partial \tilde{\varphi}}{\partial y}\bigg|_{y=0} \right\rangle, \qquad \varphi \in \mathscr{S}, \qquad \tilde{\varphi} \text{ as in (132)}. \tag{134}$$

More precisely:

Proposition A.3. *If $u \in L_{\frac{1}{2}}(\mathbb{R}) \cap C^{1,\alpha}_{\mathrm{loc}}((a, b))$ for some interval $(a, b) \subset \mathbb{R}$ and some $\alpha \in (0, 1)$, then $(-\Delta)^{\frac{1}{2}}u$, the tempered distribution defined in (131), coincides on the interval (a, b) with the functions given by (129) and (133). For general $u \in L_{\frac{1}{2}}(\mathbb{R})$, the definitions (131) and (133) are equivalent, where the right-hand side of (133) is defined by (134).*

Proof. Assume that $u \in L_{\frac{1}{2}}(\mathbb{R}) \cap C^{1,\alpha}_{\mathrm{loc}}((a, b))$. Following [Caffarelli and Silvestre 2007], we have, for $x \in (a, b)$,

$$\frac{\partial \tilde{u}(x, y)}{\partial y}\bigg|_{y=0} = \lim_{y\to 0} \frac{\tilde{u}(x, y) - \tilde{u}(x, 0)}{y} = \lim_{y\to 0} \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(\xi) - u(x)}{y^2 + (\xi - x)^2} \, d\xi = \frac{1}{\pi} \,\mathrm{PV} \int_{\mathbb{R}} \frac{u(\xi) - u(x)}{(\xi - x)^2} \, d\xi,$$

where the last convergence follows from dominated convergence outside $B_1(x)$ and by a Taylor expansion in a neighborhood of x. This proves the equivalence of (129) and (133). The equivalence between (129) and (131) amounts to showing that

$$\int_{\mathbb{R}} u\mathscr{F}^{-1}(|\xi|\hat{\varphi}(\xi)) \, dx = \frac{1}{\pi} \int_{\mathbb{R}} \mathrm{PV} \int_{\mathbb{R}} \frac{u(x) - u(y)}{(x - y)^2} \, dy \, \varphi(x) \, dx \tag{135}$$

whenever $\varphi \in \mathscr{S}$ is supported in (a, b). When $u \in \mathscr{S}$, the equivalence is shown, e.g., in [Caffarelli and Silvestre 2007] (passing through the definition given in (128)). In the general case, one approximates u with functions $u_k \in \mathscr{S}$ converging to u uniformly locally in (a, b) and in $L_{\frac{1}{2}}(\mathbb{R})$, as shown in Proposition 2.1.4 of [Silvestre 2007] (in order to have convergence in (135) as $u_k \to u$, it is convenient to consider φ compactly supported first, in case (a, b) is not bounded).

The last statement follows at once by noticing that, applying (133) to $\varphi \in \mathscr{S}$, one gets

$$\left\langle u, -\frac{\partial \tilde{\varphi}}{\partial y}\bigg|_{y=0} \right\rangle = \langle u, (-\Delta)^{\frac{1}{2}}\varphi\rangle. \qquad \square$$

Appendix B: Useful results from complex analysis

Lemma B.1. *Let $h \in C^0(\bar{D}^2, \mathbb{C})$ be holomorphic in D^2 with $h(S^1) \subset S^1$ and $0 \notin h(D^2)$. Then h is constant.*

Proof. Since h never vanishes, $\log |h|$ is well defined, harmonic and vanishes on S^1, hence everywhere. This implies that $|h| \equiv 1$ and, from the conformality of h, it follows that h is constant. □

The following is a generalization of Lemma B.1:

Lemma B.2 [Burckel 1979]. *If $h \in C^0(\bar{D}^2, \mathbb{C})$ be holomorphic in D^2 with $h(S^1) \subset S^1$ and $\deg h|_{S^1} = n \geq 0$, then h is a Blaschke product of degree n, i.e.,*

$$h(z) = e^{i\theta_0} \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}, \qquad a_1, \dots, a_n \in D^2,\ \theta_0 \in \mathbb{R}.$$

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