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SEMILINEAR WAVE EQUATIONS ON ASYMPTOTICALLY DE SITTER, KERR–DE SITTER AND MINKOWSKI SPACETIMES

PETER HINTZ AND ANDRÁS VASY

We show the small data solvability of suitable semilinear wave and Klein–Gordon equations on geometric classes of spaces, which include so-called asymptotically de Sitter and Kerr–de Sitter spaces as well as asymptotically Minkowski spaces. These spaces allow general infinities, called conformal infinity in the asymptotically de Sitter setting; the Minkowski-type setting is that of nontrapping Lorentzian scattering metrics introduced by Baskin, Vasy and Wunsch. Our results are obtained by showing the *global* Fredholm property, and indeed invertibility, of the underlying linear operator on suitable L^2-based function spaces, which also possess appropriate algebra or more complicated multiplicative properties. The linear framework is based on the b-analysis, in the sense of Melrose, introduced in this context by Vasy to describe the asymptotic behavior of solutions of linear equations. An interesting feature of the analysis is that *resonances*, namely poles of the inverse of the Mellin-transformed b-normal operator, which are "quantum" (not purely symbolic) objects, play an important role.

1. Introduction

In this paper we consider semilinear wave equations in contexts such as asymptotically de Sitter and Kerr–de Sitter spaces as well as asymptotically Minkowski spaces. The word "asymptotically" here does *not* mean that the asymptotic behavior has to be that of exact de Sitter, etc., spaces, or even a perturbation of these at infinity; much more general infinities, which nonetheless possess a similar structure as far as the underlying analysis is concerned, are allowed. Recent progress [Vasy 2013a; Baskin et al. 2014] allows one to set up the analysis of the associated linear problem *globally* as a Fredholm problem, concretely using

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the framework of Melrose's [1993] b-pseudodifferential operators on appropriate compactifications M of these spaces. (The b-analysis itself originates in Melrose's work on the propagation of singularities for the wave equation on manifolds with smooth boundary, and Melrose described a systematic framework for elliptic b-equations. Here "b" refers to analysis based on vector fields tangent to the boundary of the space; we give some details later in the introduction and further details in Section 2A, where we recall the setting of [Vasy 2013a].) This allows one to use the contraction mapping theorem to solve semilinear equations with small data in many cases, since typically the semilinear terms can be considered perturbations of the linear problem. That is, as opposed to solving an evolution equation on time intervals of some length, possibly controlling this length in some manner, and iterating the solution using (almost) conservation laws, we solve the equation globally in one step.

As Fredholm analysis means that one has to control the linear operator L modulo compact errors, which in these settings means modulo terms which are *both* smoother and more decaying, the underlying linear analysis involves both arguments based on the principal symbol of the wave operator and on its so-called (b-)normal operator family, which is a holomorphic family $\widehat{N}(L)(\sigma)$ of operators on ∂M. In settings in which there is an $\mathbb{R}^+$-action in the normal variable and the operator is dilation invariant, this simply means Mellin-transforming in the normal variable. Replacing the normal variable by its logarithm, this is equivalent to a Fourier transform.

At the principal symbol level, one encounters real-principal-type phenomena as well as radial points of the Hamilton flow at the boundary of the compactified underlying space M; these allow for the usual (for wave equations) loss of one (b-)derivative relative to elliptic problems. Physically, in the de Sitter and Kerr–de Sitter-type settings, radial points correspond to a red shift effect. In Kerr–de Sitter spaces there is an additional loss of derivatives due to trapping. On the other hand, the b-normal operator family enters via the poles σ_j of the meromorphic inverse $\widehat{N}(L)(\sigma)^{-1}$; these poles, called *resonances*, determine the decay and growth rates of solutions of the linear problem at ∂M, namely $\Im\sigma_j > 0$ means growing while $\Im\sigma_j < 0$ means decaying solutions. Translated into the nonlinear setting, taking powers of solutions of the linear equation means that growing linear solutions become even more growing, thus the nonlinear problem is uncontrollable; while decaying linear solutions become even more decaying, thus the nonlinear effects become negligible at infinity. Correspondingly, the location of these resonances becomes crucial for nonlinear problems. We note that, in addition to providing solvability of semilinear problems, our results can also be used to obtain the *asymptotic expansion* of the solution.

In short, we present a *systematic approach* to the analysis of semilinear wave and Klein–Gordon equations: Given an appropriate structure of the space at infinity and given that the location of the resonances fits well with the nonlinear terms — see the discussion below — one can solve (suitable) semilinear equations. Thus, the main purpose of this paper is to present the first step towards a general theory for the global study of linear and nonlinear wave-type equations; the semilinear applications we give are meant to show how far we can get in the nonlinear regime using relatively simple means and lend themselves to meaningful comparisons with existing literature; see the discussion below. In particular, our approach readily generalizes to the analysis of *quasilinear* equations, provided one understands the necessary (b-)analysis for nonsmooth metrics. Since the first version of this paper, we described such generalizations in detail in the context of asymptotically de Sitter [Hintz 2013] and asymptotically Kerr–de Sitter [Hintz and Vasy 2014a] spaces.

We now describe our setting in more detail. We consider semilinear wave equations of the form

$$(\Box_g - \lambda)u = f + q(u, du)$$

on a manifold M, where q is (typically, though more general functions are also considered) a polynomial vanishing at least quadratically at $(0, 0)$ (so contains no constant or linear terms, which should be included either in f or in the operator on the left-hand side). The derivative du is measured relative to the metric structure (e.g., when constructing polynomials in it). Here, g and λ fit in one of the following scenarios, which we state slightly informally, with references to the precise theorems. We discuss the terminology afterwards in more detail, but the reader unfamiliar with the terms could drop the word "asymptotically" and "even" to obtain specific examples.

(1) *A neighborhood of the backward light cone from future infinity in an asymptotically de Sitter space*: (This may be called a static region or patch of an asymptotically de Sitter space, even when there is no timelike Killing vector field.) In order to solve the semilinear equation, if $\lambda > 0$ one can let q be an arbitrary polynomial with quadratic vanishing at the origin, or indeed a more general function. If $\lambda = 0$ and q depends on du only, the same conclusion holds. Further, in either case, one obtains an expansion of the solution at infinity. See Theorems 2.25 and 2.37 and Corollary 2.28.

(2) *Kerr–de Sitter space, including a neighborhood of the event horizon, or more general spaces with normally hyperbolic trapping, discussed below*: In the main part of the section we assume $\lambda > 0$ and allow $q = q(u)$ with quadratic vanishing at the origin. We also obtain an expansion at infinity. See Theorems 3.7 and 3.11 and Corollary 3.10. However, in Section 3C we briefly discuss nonlinearities involving derivatives which are appropriately behaved at the trapped set.

(3) *Global* even *asymptotically de Sitter spaces*: These are in some sense the easiest examples as they correspond, via extension across the conformal boundary, to working on a manifold without boundary. Here, $\lambda = \frac{1}{4}(n-1)^2 + \sigma^2$. While the equation is unchanged if one replaces σ by $-\sigma$, the process of extending across the boundary breaks this symmetry, and in Section 4 we mostly consider $\Im\sigma \leq 0$. If $\Im\sigma < 0$ is sufficiently small and the dimension satisfies $n \geq 6$, quadratic vanishing of q suffices; if $n \geq 4$ then cubic vanishing is sufficient. If q does not involve derivatives, then $\Im\sigma \geq 0$ small also works, and if $\Im\sigma > 0$ and $n \geq 5$, or $\Im\sigma = 0$ and $n \geq 6$, then quadratic vanishing of q is sufficient. See Theorems 4.10, 4.12 and 4.15. Using the results from "static" asymptotically de Sitter spaces, quadratic vanishing of q in fact suffices for all $\lambda > 0$, and indeed $\lambda \geq 0$ if $q = q(du)$, but the decay estimates for solutions are lossy relative to the *decay* of the forcing. See Theorem 4.17.

(4) *Nontrapping Lorentzian scattering (generalized asymptotically Minkowski) spaces, $\lambda = 0$*: If $q = q(du)$, we allow q with quadratic vanishing at 0 if $n \geq 5$; and cubic if $n \geq 4$. If $q = q(u)$, we allow q with quadratic vanishing if $n \geq 6$; and cubic if $n \geq 4$. Further, for $q = q(du)$ quadratic satisfying a null condition, $n = 4$ also works. See Theorems 5.12, 5.14 and 5.20.

We now recall these settings in more detail. First — see [Vasy 2010] — an asymptotically de Sitter space is an appropriate generalization of the Riemannian conformally compact spaces of Mazzeo and Melrose [1987], namely a smooth manifold with boundary, $\widetilde{M}$, with interior $\widetilde{M}^\circ$ equipped with a Lorentzian metric $\tilde{g}$, which we take to be of signature $(1, n-1)$ for the sake of definiteness, and with a boundary defining function ρ such that $\hat{g} = \rho^2 \tilde{g}$ is a smooth, symmetric 2-cotensor of signature $(1, n-1)$ up to the boundary of $\widetilde{M}$ and $\hat{g}(d\rho, d\rho) = 1$ (thus, the boundary defining function is timelike, and thus the boundary is spacelike; the last equality makes the curvature asymptotically constant). In addition, $\partial\widetilde{M}$ has two components, $\widetilde{X}_\pm$ (each of which may be a union of connected components), with all null-geodesics $c = c(s)$ of $\tilde{g}$ tending to $\widetilde{X}_+$ as $s \to +\infty$ and to $\widetilde{X}_-$ as $s \to -\infty$, or vice versa. Notice that in the interior of $\widetilde{M}$ the conformal factor ρ^{-2} simply reparameterizes the null-geodesics, so equivalently one can require that null-geodesics of $\hat{g}$ reach $\widetilde{X}_\pm$ at finite parameter values. Analogously to asymptotically hyperbolic spaces, where this was shown by Graham and Lee [1991], on such a space one can always introduce a product decomposition $(\partial\widetilde{M})_z \times [0, \delta)_\rho$ near $\partial\widetilde{M}$ (possibly changing ρ) such that the metric has a warped product structure $\hat{g} = d\rho^2 - h(\rho, z, dz)$, $\tilde{g} = \rho^{-2}\hat{g}$; the metric is called even if h can be taken even in ρ, i.e., a smooth function of ρ^2. We refer to [Guillarmou 2005] for the introduction of even metrics in the asymptotically hyperbolic context and to [Vasy 2010; 2013a; 2014] for further discussion.

Blowing up a point p at $\widetilde{X}_+$, which essentially means introducing spherical coordinates around it, we obtain a manifold with corners $[\widetilde{M}; p]$ with a blow-down map $\beta : [\widetilde{M}; p] \to \widetilde{M}$ that is a diffeomorphism away from the *front face*, which gets mapped to p by β. Just like blowing up the origin in Minkowski space desingularizes the future (or past) light cone, this blow-up desingularizes the backward light cone from p on $\widetilde{M}$, which lifts to a smooth submanifold transversal to the front face on $[\widetilde{M}; p]$ which intersects the front face in a sphere Y. The interior of this lifted backward light cone, at least near the front face, is a generalization of the static patch in de Sitter space, and we refer to a neighborhood M_δ, $\delta > 0$, of the closure of the interior M_+ of the lifted backward light cone in $[\widetilde{M}; p]$ which only intersects the boundary of $[\widetilde{M}; p]$ in the interior of the front face (so M_δ is a noncompact manifold with boundary X_δ and, say, boundary defining function τ) as the "static" asymptotically de Sitter problem. See Figure 1. Via a doubling process, X_δ can be replaced by a compact manifold without boundary, X, and M_δ by $M = X \times [0, \tau_0)_\tau$, an approach taken in [Vasy 2013a], where complex absorption was used; or, indeed, one can instead work

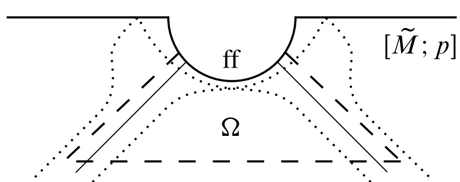

Figure 1. Setup of the "static" asymptotically de Sitter problem. Indicated are the blow-up of $\widetilde{M}$ at p and the front face, the lift of the backward light cone to $[\widetilde{M}; p]$ (solid), and lifts of backward light cones from points near to p (dotted); moreover, $\Omega \subset M$ (dashed boundary) is a submanifold with corners within M (which is not drawn here; see [Vasy 2013a] for a description of M using a doubling procedure in a similar context). The role of Ω is explained in Section 2A.

in a compact region $\Omega \subset M_\delta$ by adding artificial, spacelike boundaries, as we do here in Section 2A. With such an Ω, the distinction between M and M_δ is irrelevant, and we simply write M below.

See [Vasy 2010; 2013a] for relating the "global" and the "static" problems. We note that the lift of $\tilde{g}$ to M in the static region is a Lorentzian b-metric, that is, a smooth symmetric section of signature $(1, n-1)$ of the second tensor power of the b-cotangent bundle, ${}^{\mathrm{b}}T^*M$. The latter is the dual of ${}^{\mathrm{b}}TM$, whose smooth sections are smooth vector fields on M tangent to ∂M; sections of ${}^{\mathrm{b}}T^*M$ are smooth combinations of $d\tau/\tau$ and smooth one-forms on X, relative to a product decomposition $X \times [0, \delta)_\tau$ near $X = \partial M$. See also Section 2A.

As mentioned earlier, the methods of [Vasy 2013a] work in a rather general b-setting, including generalizations of "static" asymptotically de Sitter spaces. Kerr–de Sitter space, described from this perspective in [Vasy 2013a, §6], can be thought of as such a generalization. In particular, it still carries a Lorentzian b-metric, but with a somewhat more complicated structure, of which the only important part for us is that it has trapped rays. More concretely, it is best to consider the bicharacteristic flow in the b-cosphere bundle, ${}^{\mathrm{b}}S^*M$ (projections of null-bicharacteristics being just the null-geodesics), quotienting out by the $\mathbb{R}^+$-action on the fibers of ${}^{\mathrm{b}}T^*M \setminus o$. On the "static" asymptotically de Sitter space, each half of the spherical b-conormal bundle ${}^{\mathrm{b}}SN^*Y$ consists of (a family of) saddle points of the null-bicharacteristic flow (these are called *radial sets*); the stable and unstable directions are normal to ${}^{\mathrm{b}}SN^*Y$ itself, with one of the stable or unstable manifolds being the conormal bundle of the lifted light cone (which plays the role of the *event horizon* in black hole settings), and the other being the characteristic set within the boundary X (so, within the boundary, the radial sets ${}^{\mathrm{b}}SN^*Y$ are actually sources or sinks). Then, on asymptotically de Sitter spaces, all null-bicharacteristics over $\overline{M_+} \setminus X$ either leave Ω in finite time or (if they lie on the conormal bundle of the event horizon) tend to ${}^{\mathrm{b}}SN^*Y$ as the parameter goes to $\pm\infty$, with each bicharacteristic tending to ${}^{\mathrm{b}}SN^*Y$ in at most one direction. The main difference for Kerr–de Sitter space is that there are null-bicharacteristics which do not leave $\overline{M_+} \setminus X$ and do not tend to ${}^{\mathrm{b}}SN^*Y$. On de Sitter–Schwarzschild space (nonrotating black holes) these future-trapped rays project to a sphere, called the photon sphere, times $[0, \delta)_\tau$; on general Kerr–de Sitter space the trapped set deforms, but is still *normally hyperbolic*, a setting studied by Wunsch and Zworski [2011] and Dyatlov [2015].

We refer to [Baskin et al. 2014, §3] and to Section 5A here for a definition of asymptotically Minkowski spaces, but roughly they are manifolds with boundary M with Lorentzian metrics g on the interior M° conformal to a b-metric $\hat{g}$ as $g = \tau^{-2}\hat{g}$, with τ a boundary defining function[1] (so these are Lorentzian *scattering* metrics in the sense of [Melrose 1994], i.e., symmetric cotensors in the second power of the scattering cotangent bundle, and of signature $(1, n-1)$), with a real C^∞ function v defined on M with dv and $d\tau$ linearly independent at $S = \{v = 0, \tau = 0\}$, and with a specific behavior of the metric at S which reflects that of Minkowski space on its radial compactification near the boundary of the light cone at infinity (so S is the light cone at infinity in this greater generality). Concretely, the specific form is

$$\tau^2 g = \hat{g} = v\,\frac{d\tau^2}{\tau^2} - \left(\frac{d\tau}{\tau} \otimes \alpha + \alpha \otimes \frac{d\tau}{\tau}\right) - \tilde{h},$$

[1] In Section 5 we switch to ρ as the boundary defining function for consistency with [Baskin et al. 2014].

where α is a smooth one-form on M, equal to $\frac{1}{2}\,dv$ at S, and $\tilde{h}$ is a smooth 2-cotensor on M that is positive definite on the annihilator of $d\tau$ and dv (which is a codimension 2 space).[2] The difference between the de Sitter-type and Minkowski settings is in part this conformal factor, τ^{-2}, but more importantly, as this conformal factor again does not affect the behavior of the null-bicharacteristics, so one can consider those of $\hat{g}$ on ${}^{\mathrm{b}}S^*M$, at the spherical conormal bundle ${}^{\mathrm{b}}SN^*S$ of S (see Section 2) the nature of the radial points is source or sink rather than a saddle point of the flow. (One also makes a nontrapping assumption in the asymptotically Minkowski setting.)

Now we comment on the specific way these settings fit into the b-framework, and the way the various restrictions described above arise:

(1) *Asymptotically "static" de Sitter*: Due to a zero resonance for the linear problem when $\lambda = 0$, which moves to the lower half plane for $\lambda > 0$, in this setting $\lambda > 0$ works in general; $\lambda = 0$ works if q depends on du but not on u. The relevant function spaces are L^2-based b-Sobolev spaces (see Section 2) on the bordification (partial compactification) of the space, or analogous spaces plus a finite expansion. Further, the semilinear terms involving du have coefficients corresponding to the b-structure, i.e., b-objects are used to create functions from the differential forms or, equivalently, b-derivatives of u are used.

(2) *Kerr–de Sitter space*: This is an extension of (1); the framework is essentially the same, with the difference being that there is now trapping corresponding to the "photon sphere". This makes first-order terms in the nonlinearity nonperturbative, unless they are well adapted to the trapping. Thus, we assume $\lambda > 0$. The relevant function spaces are as in the asymptotically de Sitter setting.

(3) *Global* even *asymptotically de Sitter spaces*: In order to get reasonable results, one needs to measure regularity relatively finely, using the module of vector fields tangent to what used to be the conformal boundary in the extension. The relevant function spaces are thus Sobolev spaces with additional (finite) conormal regularity. Further, du has coefficients corresponding to the 0-structure of Mazzeo and Melrose, in the same sense the b-structure was used in (1). The range of λ here is limited by the process of extension across the boundary; for nonlinearities involving u only, the restriction amounts to (at least very slowly) decaying solutions for the linear problem (without extension across the conformal boundary).

Another possibility is to view global de Sitter space as a union of static patches. Here, the b-Sobolev spaces on the static parts translate into 0-Sobolev spaces on the global space, which have weights that are shifted by a dimension-dependent amount relative to the weights of the b-spaces. This approach allows many of the nonlinearities that we can deal with on static parts; however, the resulting decay estimates on u are quite lossy relative to the decay of the forcing term f.

(4) *Nontrapping Lorentzian scattering spaces (generalized asymptotically Minkowski spaces), $\lambda = 0$*: Note that if $\lambda > 0$, the type of the equation changes drastically; it naturally fits into Melrose's scattering algebra[3]

[2] More general, "long-range" scattering metrics also work for the purposes of this paper without any significant changes; the analysis of these is currently being completed by Baskin, Vasy and Wunsch. The difference is the presence of smooth multiples of $\tau\, d\tau^2/\tau^2$ in the metric near $\tau = 0$, $v = 0$. These do not affect the normal operator, but slightly change the dynamics in ${}^{\mathrm{b}}S^*M$. This, however, does not affect the function spaces to be used for our semilinear problem.

[3] In many ways the scattering algebra is actually much better behaved than the b-algebra, in particular it is symbolic in the sense of weights/decay. Thus, with numerical modifications, our methods should extend directly.

rather than the b-algebra which can be used for $\lambda = 0$. While the results here are quite robust and there are no issues with trapping, they are more involved as one needs to keep track of regularity relative to the module of vector fields tangent to the light cone at infinity. The relevant function spaces are b-Sobolev spaces with additional b-conormal regularity corresponding to the aforementioned module. Further, du has coefficients corresponding to Melrose's scattering structure. These spaces, in the special case of Minkowski space, are related to the spaces used by Klainerman [1985], using the infinitesimal generators of the Lorentz group, but, while Klainerman works in an $L^\infty L^2$ setting, we remain purely in a (weighted) L^2-based setting, as the latter is more amenable to the tools of microlocal analysis.

We reiterate that, while the way the four types of spaces fit into it differs somewhat, the underlying linear framework is that of L^2-based b-analysis on manifolds with boundary, except that in the global view of asymptotically de Sitter spaces one can eliminate the boundary altogether.

In order to underline the generality of the method, we emphasize that, corresponding to cases (1) and (2), in b-settings in which one can work on standard b-Sobolev spaces the restrictions on the solvability of the semilinear equations are simply given by the presence of resonances for the Mellin-transformed normal operator in $\Im\sigma \geq 0$, which would allow growing solutions to the equation (with the exception of $\Im\sigma = 0$, in which case the nonlinear iterative arguments produce growth unless the nonlinearity has a special structure), making the nonlinearity nonperturbative and the losses at high energy estimates for this Mellin-transformed operator and the closely related b-principal symbol estimates when one has trapping. (It is these losses that cause the difference in the trapping setting between nonlinearities with or without derivatives.) In particular, the results are necessarily optimal in the nontrapping setting of (1), as shown even by an ODE; see Remark 2.31. In the trapping setting it is not clear precisely what improvements are possible for nonlinearities with derivatives, though, when there are no derivatives in the nonlinearity, we already have no restrictions on the nonlinearity and to this extent the result is optimal.

On Lorentzian scattering spaces, more general function spaces are used and it is not in principle clear whether the results are optimal, but at least comparison with the work of Klainerman [1985; 1986] and Christodoulou [1986] for perturbations of Minkowski space gives consistent results; see the comments below. On global asymptotically de Sitter spaces, the framework of [Vasy 2013a; 2013b] is very convenient for the linear analysis, but it is not clear to what extent it gives optimal results in the nonlinear setting. The reason why more precise function spaces become necessary is the following: There are two basic properties of spaces of functions on manifolds with boundaries, namely differentiability and decay. Whether one can have both at the same time for the linear analysis depends on the (Hamiltonian) dynamical nature of radial points: when defining functions of the corresponding boundaries of the compactified cotangent bundle have opposite character (stable vs. unstable) one can have both at the same time, otherwise not; see Propositions 2.1 and 5.2 for details. For nonlinear purposes, the most convenient setting, in which we are in (1), is if one can work with spaces of arbitrarily high regularity and fast decay, and corresponds to saddle points of the flow in the above sense. In (4), however, working in higher regularity spaces, which is necessary in order to be able to make sense of the nonlinearity, requires using faster-growing (or at least less decaying) weights, which is problematic when dealing with nonlinearities (e.g., polynomials) since multiplication gives even worse growth properties then. Thus, to make the nonlinear analysis work, the function spaces we use need to have more structure; it is a module regularity that is used to capture some weaker regularity in order to enable work in spaces with acceptable weights.

While all results are stated for the scalar equation, analogous results hold in many cases for operators on natural vector bundles, such as the d'Alembertian (or Klein–Gordon operator) on differential forms, since the linear arguments work in general for operators with scalar principal symbol whose subprincipal symbol satisfies appropriate estimates at radial sets — see [Vasy 2013a, Remark 2.1] — though of course for semilinear applications the presence of resonances in the closed upper half plane has to be checked. This already suffices to obtain the well-posedness of the semilinear equations on asymptotically de Sitter spaces that we consider in this paper; for this purpose one needs to know the poles of the resolvent of the Laplacian on forms on *exact* hyperbolic space only. On asymptotically Minkowski spaces, the absence of poles of an asymptotically hyperbolic resolvent in a region has to be checked in addition — see Theorem 5.3 — and the situation depends crucially on the delicate balance of weights and regularity, as alluded to above. Note that, on *perturbations* of Minkowski space, this absence of poles follows from the appropriate behavior of the poles of the resolvent of the Laplacian on forms on *exact* hyperbolic space.

The degree to which these nonlinear problems have been studied differs, with the Minkowski problem (on perturbations of Minkowski space, as opposed to our more general setting) being the most studied. There semilinear and indeed even quasilinear equations are well understood due to the work of Christodoulou [1986] and Klainerman [1985; 1986], with their book [1993] on the global stability of Einstein's equation being one of the main achievements. (We also refer to the work of Lindblad and Rodnianski [2005; 2010] simplifying some of the arguments, of Bieri [2009] relaxing some of the decay conditions, of Wang [2010] obtaining asymptotic expansions, and of Lindblad [2008] for results on a class of quasilinear equations. Hörmander's [1997] book provides further references in the general area. There are numerous works on the *linear* problem, and estimates this yields for the nonlinear problems, such as Strichartz estimates; here we refer to the recent work of Metcalfe and Tataru [2012] for a parametrix construction in low regularity, and references therein.) We obtain results comparable to these (when restricted to the semilinear setting), on a larger class of manifolds; see Remark 5.17. For nonlinearities which do not involve derivatives, slightly stronger results have been obtained, in a slightly different setting, in [Chruściel and Łęski 2006]; see Remark 5.18.

On the other hand, there is little (nonlinear) work on the asymptotically de Sitter and Kerr–de Sitter settings; indeed the only paper the authors are aware of is [Baskin 2013] in roughly comparable generality in terms of the setting, though in *exact* de Sitter space Yagdjian [2009; 2012] has studied a large class of semilinear equations with no derivatives. Baskin's result is for a semilinear equation with no derivatives and a single exponent, using his [2010] parametrix construction, namely u^p with[4] $p = 1 + 4/(n-2)$, and for $\lambda > \frac{1}{4}(n-1)^2$. In the same setting, $p > 1 + 4/(n-1)$ works for us, and thus Baskin's setting is in particular included. Yagdjian works with the explicit solution operator (derived using special functions) in exact de Sitter space, again with no derivatives in the nonlinearity. While there are some exponents that his results cover (for $\lambda > \frac{1}{4}(n-1)^2$, all $p > 1$ work for him) that ours do not directly (but indirectly, via the static model, we in fact obtain such results), the range $\left(\frac{1}{4}(n-1)^2 - \frac{1}{4}, \frac{1}{4}(n-1)^2\right)$ is excluded by

[4]The dimension of the spacetime in Baskin's paper is $n+1$; we continue using our notation above.

him while covered by our work for sufficiently large p. In the (asymptotically) Kerr–de Sitter setting, to our knowledge, there has been no similar semilinear work, however Luk [2013] and Tohaneanu [2012] studied semilinear waves on Kerr spacetimes. We recall finally that there is more work on the linear problem in de Sitter, de Sitter–Schwarzschild and Kerr–de Sitter spaces. We refer to [Vasy 2013a] for more detail; some references are [Polarski 1989; Yagdjian and Galstian 2009; Sá Barreto and Zworski 1997; Bony and Häfner 2008; Vasy 2010; Baskin 2010; Dafermos and Rodnianski 2007; Dyatlov 2011a; 2011b] and Melrose, Sá Barreto and Vasy [Melrose et al. 2014]. Also, while it received more attention, the linear problem on Kerr space does not fit directly into our setting; see the introduction of [Vasy 2013a] for an explanation and for further references, and [Dafermos and Rodnianski 2013] for more background and additional references.

While the basic ingredients of the necessary linear b-analysis were analyzed in [Vasy 2013a], the solvability framework was only discussed in the dilation-invariant setting, and in general the asymptotic expansion results were slightly lossy in terms of derivatives in the non-dilation-invariant case. We remedy these issues in this paper, providing a full Fredholm framework. The key technical tools are the propagation of b-singularities at b-radial points which are saddle points of the flow in ${}^{\mathrm{b}}S^*M$ — see Proposition 2.1 — as well as the b-normally hyperbolic versions, proved in [Hintz and Vasy 2014b], of the semiclassical normally hyperbolic trapping estimates of Wunsch and Zworski [2011]; the rest of the Fredholm setup is discussed in Section 2A in the nontrapping and Section 3A in the normally hyperbolic trapping setting. The analogue of Proposition 2.1 for sources and sinks was already proved in [Baskin et al. 2014, §4]; our Lorentzian scattering metric Fredholm discussion, which relies on this, is in Section 5A.

We emphasize that our analysis would be significantly less cumbersome in terms of technicalities if we were not including Cauchy hypersurfaces and solved a globally well-behaved problem by imposing sufficiently rapid decay at past infinity instead (it is standard to convert a Cauchy problem into a forward solution problem). Cauchy hypersurfaces are only necessary for us if we deal with a problem ill-behaved in the past because complex absorption does not force appropriate forward supports even though it does so at the level of singularities; otherwise we can work with appropriate (weighted) Sobolev spaces. The latter is the case with Lorentzian scattering spaces, which thus provide an ideal example for our setting. It can also be done in the global setting of asymptotically de Sitter spaces, as in setting (3) above, essentially by realizing these as the boundary of the appropriate compactification of a Lorentzian scattering space; see [Vasy 2014]. In the case of Kerr–de Sitter black holes, in the presence of dilation invariance, one has access to a similar luxury: complex absorption does the job, as in [Vasy 2013a]; the key aspect is that it needs to be imposed *outside* the static region we consider. For a general Lorentzian b-metric with a normally hyperbolic trapped set, this may not be easy to arrange, and we do work by adding Cauchy hypersurfaces, even at the cost of the resulting technical complications, which are rather artificial in terms of PDE theory. For perturbations of Kerr–de Sitter space, however, it is possible to forego the latter for well-posedness by an appropriate gluing to complete the space with actual Kerr–de Sitter space in the past for the purposes of functional analysis. We remark that Cauchy hypersurfaces are somewhat ill-behaved for L^2-based estimates, which we use, but match $L^\infty L^2$ estimates quite well, which explains the large role they play in existing hyperbolic theory, such as [Klainerman 1985] or [Hörmander 1985a, Chapter 23.2]. We hope that adopting this more commonly used form of "truncation" of hyperbolic problems will aid the readability of the paper.

We also explain the role that the energy estimates (as opposed to microlocal energy estimates) play. These mostly arise to deal with the artificially introduced boundaries; if other methods are used to truncate the flow, their role reduces to checking that, in certain cases, when the microlocal machinery only guarantees Fredholm properties of the underlying linear operators, the potential finite-dimensional kernel and cokernel are indeed trivial. Asymptotically Minkowski spaces illustrate this best, as the Hamilton flow is globally well behaved there; see Section 5A.

The other key technical tool is the algebra property of b-Sobolev spaces and other spaces with additional conormal regularity. These are stated in the respective sections; the case of the standard b-Sobolev spaces reduces to the algebra property of the standard Sobolev spaces on $\mathbb{R}^n$. Given the algebra properties, the results are proved by applying the contraction mapping theorem to the linear operator.

In summary, the plan of this paper is the following. In each of the sections below we consider one of these settings, and first describe the Sobolev spaces on which one has invertibility for the linear problems of interest, then analyze the algebra properties of these Sobolev spaces, finally proving the solvability of the semilinear equations by checking that the hypotheses of the contraction mapping theorem are satisfied.

2. Asymptotically de Sitter spaces: generalized static model

In this section we discuss solving semilinear wave equations on asymptotically de Sitter spaces from the "static perspective", i.e., in neighborhoods (in a blown-up space) of the backward light cone from a fixed point at future conformal infinity; see Figure 1. The main ingredient is extending the linear theory from that of [Vasy 2013a] in various ways, which is the subject of Section 2A. In the following parts of this section we use this extension to solve semilinear equations and to obtain their asymptotic behavior.

First, however, we recall some of the basics of b-analysis. As a general reference, we refer the reader to [Melrose 1993]. Thus, let M be an n-dimensional manifold with boundary X and denote by $\mathcal{V}_{\mathrm{b}}(M)$ the space of *b-vector fields*, which consists of all vector fields on M which are tangent to X. Elements of $\mathcal{V}_{\mathrm{b}}(M)$ are sections of a natural vector bundle over M, the *b-tangent bundle* ${}^{\mathrm{b}}TM$. Its dual, the *b-cotangent bundle*, is denoted ${}^{\mathrm{b}}T^*M$. In local coordinates, $(\tau, z) \in [0,\infty) \times \mathbb{R}^{n-1}$ near the boundary, the fibers of ${}^{\mathrm{b}}TM$ are spanned by $\tau\partial_\tau$, $\partial_{z_1}, \dots, \partial_{z_{n-1}}$, with $\tau\partial_\tau$ being a nontrivial b-vector field up to and including $\tau = 0$ (even though it degenerates as an ordinary vector field), while the fibers of ${}^{\mathrm{b}}T^*M$ are spanned by $d\tau/\tau$, $dz_1, \dots, dz_{n-1}$. A *b-metric* g on M is then simply a nondegenerate section of the second symmetric tensor power of ${}^{\mathrm{b}}T^*M$, that is, of the form

$$g = g_{00}(\tau,z)\,\frac{d\tau^2}{\tau^2} + \sum_{i=1}^{n-1} g_{0i}(\tau,z)\left(\frac{d\tau}{\tau} \otimes dz_i + dz_i \otimes \frac{d\tau}{\tau}\right) + \sum_{i,j=1}^{n-1} g_{ij}(\tau,z)\,dz_i \otimes dz_j, \quad g_{ij} = g_{ji},$$

with smooth coefficients $g_{k\ell}$. In terms of the coordinate $t = -\log\tau \in \mathbb{R}$ — thus $d\tau/\tau = -dt$ — the b-metric g therefore approaches a stationary (t-independent in the local coordinate system) metric exponentially fast as $\tau = e^{-t}$.

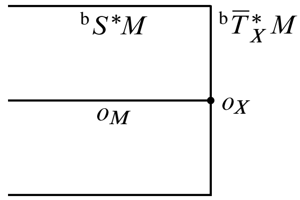

Figure 2. The radially compactified cotangent bundle ${}^{\mathrm{b}}\overline{T}^*M$ near ${}^{\mathrm{b}}\overline{T}^*_X M$; the cosphere bundle ${}^{\mathrm{b}}S^*M$, viewed as the boundary at fiber infinity of ${}^{\mathrm{b}}\overline{T}^*M$, is also shown, as well as the zero section $o_M \subset {}^{\mathrm{b}}\overline{T}^*M$ and the zero section over the boundary $o_X \subset {}^{\mathrm{b}}\overline{T}^*_X M$.

The *b-conormal bundle* ${}^{\mathrm{b}}N^*Y$ of a boundary submanifold $Y \subset X$ of M is the subbundle of ${}^{\mathrm{b}}T^*_Y M$ whose fiber over $p \in Y$ is the annihilator of vector fields on M tangent to Y and X. In local coordinates (τ, z', z''), where Y is defined by $z' = 0$ in X, these vector fields are smooth linear combinations of $\tau\partial_\tau$, $\partial_{z''_j}$, $z'_i \partial_{z'_j}$ and $\tau\partial_{z'_k}$, whose span in ${}^{\mathrm{b}}T_p M$ is that of $\tau\partial_\tau$ and $\partial_{z''_j}$, and thus the fiber of the b-conormal bundle is spanned by the dz'_j, i.e., has the same dimension as the codimension of Y in X (and *not* that in M, corresponding to $d\tau/\tau$ not annihilating $\tau\partial_\tau$).

We define the *b-cosphere bundle* ${}^{\mathrm{b}}S^*M$ to be the quotient of ${}^{\mathrm{b}}T^*M \setminus o$ by the $\mathbb{R}^+$-action; here o is the zero section. Likewise, we define the spherical b-conormal bundle of a boundary submanifold $Y \subset X$ as the quotient of ${}^{\mathrm{b}}N^*Y \setminus o$ by the $\mathbb{R}^+$-action; it is a submanifold of ${}^{\mathrm{b}}S^*M$. A better way to view ${}^{\mathrm{b}}S^*M$ is as the boundary at fiber infinity of the fiber-radial compactification ${}^{\mathrm{b}}\overline{T}^*M$ of ${}^{\mathrm{b}}T^*M$, where the fibers are replaced by their radial compactification; see [Vasy 2013a, §2] and also Section 5A. The b-cosphere bundle ${}^{\mathrm{b}}S^*M \subset {}^{\mathrm{b}}\overline{T}^*M$ still contains the boundary of the compactification of the "old" boundary ${}^{\mathrm{b}}\overline{T}^*_X M$; see Figure 2.

Next, the algebra $\mathrm{Diff}_{\mathrm{b}}(M)$ of *b-differential operators* generated by $\mathcal{V}_{\mathrm{b}}(M)$ consists of operators of the form

$$\mathcal{P} = \sum_{|\alpha|+j \le m} a_\alpha(\tau, z)(\tau D_\tau)^j D_z^\alpha$$

with $a_\alpha \in C^\infty(M)$, writing $D = \frac{1}{i}\partial$ as usual. (With $t = -\log\tau$ as above, the coefficients of $\mathcal{P}$ are thus constant up to exponentially decaying remainders as $t \to \infty$.) Writing elements of ${}^{\mathrm{b}}T^*M$ as

$$\sigma \frac{d\tau}{\tau} + \sum_j \zeta_j \, dz_j, \tag{2-1}$$

we have the principal symbol

$$\sigma_{\mathrm{b},m}(\mathcal{P}) = \sum_{|\alpha|+j=m} a_\alpha(\tau, z)\sigma^j \zeta^\alpha,$$

which is a homogeneous degree-m function in ${}^{\mathrm{b}}T^*M \setminus o$. Principal symbols are multiplicative, i.e., $\sigma_{\mathrm{b},m+m'}(\mathcal{P} \circ \mathcal{P}') = \sigma_{\mathrm{b},m}(\mathcal{P})\sigma_{\mathrm{b},m'}(\mathcal{P}')$, and one has a connection between operator commutators and Poisson brackets, to wit

$$\sigma_{\mathrm{b},m+m'-1}(i[\mathcal{P}, \mathcal{P}']) = \mathsf{H}_p p', \quad p = \sigma_{\mathrm{b},m}(\mathcal{P}), \; p' = \sigma_{\mathrm{b},m'}(\mathcal{P}'),$$

where H_p is the extension of the Hamilton vector field from $T^*M^\circ \setminus o$ to ${}^{\mathrm{b}}T^*M \setminus o$, which is thus a homogeneous degree-$(m-1)$ vector field on ${}^{\mathrm{b}}T^*M \setminus o$ tangent to the boundary ${}^{\mathrm{b}}T^*_X M$. In local coordinates (τ, z) on M near X, with b-dual coordinates (σ, ζ) as in (2-1), this has the form

$$\mathsf{H}_p = (\partial_\sigma p)(\tau\partial_\tau) - (\tau\partial_\tau p)\partial_\sigma + \sum_j ((\partial_{\zeta_j} p)\partial_{z_j} - (\partial_{z_j} p)\partial_{\zeta_j}); \tag{2-2}$$

see [Baskin et al. 2014, Equation (3.20)], where a somewhat different notation is used, given by [Baskin et al. 2014, Equation (3.19)].

While elements of $\mathrm{Diff}_{\mathrm{b}}(M)$ commute to leading order in the symbolic sense, they do not commute in the sense of the order of decay of their coefficients. (This is in contrast to the scattering algebra; see [Melrose 1994].) The *normal operator* captures the leading-order part of $\mathcal{P} \in \mathrm{Diff}^m_{\mathrm{b}}(M)$ in the latter sense, namely

$$N(\mathcal{P}) = \sum_{j+|\alpha|\le m} a_\alpha(0, z)(\tau D_\tau)^j D_z^\alpha.$$

One can define $N(\mathcal{P})$ invariantly as an operator on the model space $M_I := [0, \infty)_\tau \times X$ by fixing a boundary defining function of M; see [Vasy 2013a, §3]. Identifying a collar neighborhood of $X \subset M$ with a neighborhood of $\{0\} \times X$ in M_I, we then have $\mathcal{P} - N(\mathcal{P}) \in \tau\,\mathrm{Diff}^m_{\mathrm{b}}(M)$ (near ∂M). Since $N(\mathcal{P})$ is dilation-invariant (equivalently, translation-invariant in $t = -\log\tau$), it is naturally studied via the Mellin transform in τ (equivalently, Fourier transform in $-t$), which leads to the *(Mellin-transformed) normal operator family*

$$\widehat{N}(\mathcal{P})(\sigma) \equiv \widehat{\mathcal{P}}(\sigma) = \sum_{j+|\alpha|\le m} a_\alpha(0, z)\sigma^j D_z^\alpha,$$

which is a holomorphic family of operators $\widehat{\mathcal{P}}(\sigma) \in \mathrm{Diff}^m(X)$.

Passing from $\mathrm{Diff}_{\mathrm{b}}(M)$ to the algebra of *b-pseudodifferential operators* $\Psi_{\mathrm{b}}(M)$ amounts to allowing symbols to be more general functions than polynomials; apart from symbols being smooth functions on ${}^{\mathrm{b}}T^*M$ rather than on T^*M if M was boundaryless, this is entirely analogous to the way one passes from differential to pseudodifferential operators, with the technical details being a bit more involved. One can have a rather accurate picture of b-pseudodifferential operators, however, by considering the following: For $a \in C^\infty({}^{\mathrm{b}}T^*M)$, we say $a \in S^m({}^{\mathrm{b}}T^*M)$ if a satisfies

$$|\partial_w^\alpha \partial_\xi^\beta a(w, \xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|} \quad \text{for all multiindices } \alpha,\ \beta$$

in any coordinate chart, where w are coordinates in the base and ξ coordinates in the fiber; more precisely, in local coordinates (τ, z) near X, we take $\xi = (\sigma, \zeta)$ as above. We define the quantization $\mathrm{Op}(a)$ of a, acting on smooth functions u supported in a coordinate chart, by

$$\mathrm{Op}(a)u(\tau, z) = (2\pi)^{-n}\int e^{i(\tau-\tau')\tilde\sigma + i(z-z')\zeta}\phi\left(\frac{\tau-\tau'}{\tau}\right)a(\tau, z, \tau\tilde\sigma, \zeta)u(\tau', z')\,d\tau'\,dz'\,d\tilde\sigma\,d\zeta,$$

where the τ'-integral is over $[0, \infty)$, and $\phi \in C_c^\infty\big((-\tfrac12, \tfrac12)\big)$ is identically 1 near 0. The cutoff ϕ ensures that these operators lie in the "small b-calculus" of Melrose, in particular that such quantizations act on weighted b-Sobolev spaces, defined below. For general u, define $\mathrm{Op}(a)u$ using a partition of unity. We write $\mathrm{Op}(a) \in \Psi_{\mathrm{b}}^m(M)$; every element of $\Psi_{\mathrm{b}}^m(M)$ is of the form $\mathrm{Op}(a)$ for some $a \in S^m({}^{\mathrm{b}}T^*M)$ modulo the set $\Psi_{\mathrm{b}}^{-\infty}(M)$ of smoothing operators. We say that a is a *symbol* of $\mathrm{Op}(a)$. The equivalence class of a in $S^m({}^{\mathrm{b}}T^*M)/S^{m-1}({}^{\mathrm{b}}T^*M)$ is invariantly defined on ${}^{\mathrm{b}}T^*M$ and is called the *principal symbol* of $\mathrm{Op}(a)$.

If $A \in \Psi_{\mathrm{b}}^{m_1}(M)$ and $B \in \Psi_{\mathrm{b}}^{m_2}(M)$, then AB, $BA \in \Psi_{\mathrm{b}}^{m_1+m_2}(M)$, while $[A, B] \in \Psi_{\mathrm{b}}^{m_1+m_2-1}(M)$, and its principal symbol is $\frac{1}{i}\mathsf{H}_a b \equiv \frac{1}{i}\{a, b\}$, with H_a as above.

Lastly, we recall the notion of *b-Sobolev spaces*: Fixing a volume b-density ν on M, which locally is a positive multiple of $|(1/\tau)\,d\tau\,dz|$, we define, for $s \in \mathbb{N}$,

$$H_{\mathrm{b}}^s(M) = \{u \in L^2(M, \nu) : V_1 \cdots V_j u \in L^2(M, \nu),\, V_i \in \mathcal{V}_{\mathrm{b}}(M),\, 1 \le i \le j \le s\},$$

which one can extend to $s \in \mathbb{R}$ by duality and interpolation. *Weighted b-Sobolev spaces* are denoted $H_{\mathrm{b}}^{s,\alpha}(M) = \tau^\alpha H_{\mathrm{b}}^s(M)$, that is, their elements are of the form $\tau^\alpha u$ with $u \in H_{\mathrm{b}}^s(M)$. Any b-pseudodifferential operator $\mathcal{P} \in \Psi_{\mathrm{b}}^m(M)$ defines a bounded linear map $\mathcal{P} : H_{\mathrm{b}}^{s,\alpha}(M) \to H_{\mathrm{b}}^{s-m,\alpha}(M)$ for all $s, \alpha \in \mathbb{R}$. Correspondingly, there is a notion of wave front set $\mathrm{WF}_{\mathrm{b}}^{s,\alpha}(u) \subset {}^{\mathrm{b}}S^*M$ for a distribution $u \in H_{\mathrm{b}}^{-\infty,\alpha}(M)$, defined analogously to the wave front set of distributions on $\mathbb{R}^n$ or closed manifolds. A point $\varpi \in {}^{\mathrm{b}}S^*M$ is *not* in $\mathrm{WF}_{\mathrm{b}}^{s,\alpha}(u)$ if and only if there exists $\mathcal{P} \in \Psi_{\mathrm{b}}^0(M)$, elliptic at ϖ (i.e., with principal symbol nonvanishing on the ray corresponding to ϖ) such that $\mathcal{P}u \in H_{\mathrm{b}}^{s,\alpha}(M)$. Notice however that we *do* need to have a priori control on the weight α (we are *assuming* $u \in H_{\mathrm{b}}^{-\infty,\alpha}(M)$), which again reflects the lack of commutativity of $\Psi_{\mathrm{b}}(M)$ even to leading order in the sense of decay of coefficients at ∂M.

2A. ***The linear Fredholm framework.*** The goal of this section is to fully extend the results of [Vasy 2013a] on linear estimates for wave equations for b-metrics to non-dilation-invariant settings, and to explicitly discuss Cauchy hypersurfaces, since that work concentrated on complex absorption. Namely, while the results there on linear estimates for wave equations for b-metrics are optimally stated when the metrics and thus the corresponding operators are dilation-invariant, that is, when near $\tau = 0$ the normal operator can be identified with the operator itself — see Vasy's Lemma 3.1 — the estimates for Sobolev derivatives are lossy for general b-metrics in [Vasy 2013a, Proposition 3.5], essentially because one should not treat the difference between the normal operator and the actual operator purely as a perturbation. Therefore, we first strengthen the linear results of Vasy in the non-dilation-invariant setting by analyzing b-radial points which are saddle points of the Hamilton flow. This is similar to [Baskin et al. 2014, §4], where the analogous result was proved when the b-radial points are sources or sinks. This is then used to set up a Fredholm framework for the linear problem. If one is mainly interested in the dilation-invariant case, one can use [Vasy 2013a, Lemma 3.1] in place of Theorems 2.18–2.21 below, either adding the boundary corresponding to H_2 below, or still using complex absorption as was done in [Vasy 2013a].

So suppose $\mathcal{P} \in \Psi_{\mathrm{b}}^m(M)$ with M a manifold with boundary. (The dilation-invariant analysis of [Vasy 2013a, §2] applies to the Mellin-transformed normal operator $\widehat{\mathcal{P}}(\sigma)$.) Let p be the principal symbol of $\mathcal{P}$, which we assume to be *real-valued*, and let H_p be the Hamilton vector field of p. Let $\tilde{\rho}$ denote a homogeneous defining function of ${}^{\mathrm{b}}S^*M$ of degree -1. Then the rescaled Hamilton vector field

$$V = \tilde{\rho}^{m-1}\mathsf{H}_p$$

is a C^∞ vector field on ${}^{\mathrm{b}}\overline{T}^*M$ away from the 0-section, and it is tangent to all boundary faces. The characteristic set Σ is the zero-set of the smooth function $\tilde{\rho}^m p$ in ${}^{\mathrm{b}}S^*M$. We refer to the flow of V in $\Sigma \subset {}^{\mathrm{b}}S^*M$ as the Hamilton, or (null-)bicharacteristic flow; its integral curves, the (null-)bicharacteristics, are reparameterizations of those of the Hamilton vector field H_p, projected by the quotient map ${}^{\mathrm{b}}T^*M \setminus o \to {}^{\mathrm{b}}S^*M$.

2A1. *Generalized b-radial sets.* The standard propagation of singularities theorem in the characteristic set Σ in the b-setting is that, for $u \in H_{\mathrm{b}}^{-\infty,r}(M)$, within Σ, $\mathrm{WF}_{\mathrm{b}}^{s,r}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{s-m+1,r}(\mathcal{P}u)$ is a union of maximally extended integral curves (i.e., null-bicharacteristics) of $\mathcal{P}$. This is vacuous at points where V vanishes (as a smooth vector field); these points are called *radial points*, since, at such a point, H_p itself (on ${}^{\mathrm{b}}T^*M \setminus o$) is radial, that is, is a multiple of the generator of the dilations of the fiber of the b-cotangent bundle. At a radial point α, V acts on the ideal $\mathcal{I}$ of C^∞ functions vanishing at α, and thus on $T^*_\alpha {}^{\mathrm{b}}\overline{T}^*M$, which can be identified with $\mathcal{I}/\mathcal{I}^2$. Since V is tangent to both boundary hypersurfaces, given by $\tau = 0$ and $\tilde{\rho} = 0$, $d\tau$ and $d\tilde{\rho}$ are automatically eigenvectors of the linearization of V. We are interested in a generalization of the situation, in which we have a smooth submanifold L of ${}^{\mathrm{b}}S^*_X M$ consisting of radial points which are a source or sink for V within ${}^{\mathrm{b}}T^*_X M$ but, if a source — so in particular $d\tilde{\rho}$ is in an unstable eigenspace — then $d\tau$ is in the (necessarily one-dimensional) stable eigenspace, and vice versa. Thus, L is a saddle point of the Hamilton flow.

In view of the bicharacteristic flow on Kerr–de Sitter space (which, unlike the nonrotating de Sitter–Schwarzschild black holes, does not have this precise radial point structure), it is important to be slightly more general, as in [Vasy 2013a, §2.2]. Thus, we assume that dp does not vanish where p does, namely, at Σ, and is linearly independent of $d\tau$ at $\{\tau = 0, p = 0\} = \Sigma \cap {}^{\mathrm{b}}S^*_X M$, so Σ is a smooth submanifold of ${}^{\mathrm{b}}S^*M$ transversal to ${}^{\mathrm{b}}S^*_X M$. For L, assume simply that $L = L_+ \cup L_-$, where $L_\pm = \mathcal{L}_\pm \cap {}^{\mathrm{b}}S^*_X M$ are smooth disjoint submanifolds of ${}^{\mathrm{b}}S^*_X M$ and $\mathcal{L}_\pm$ are smooth disjoint submanifolds of Σ transversal to ${}^{\mathrm{b}}S^*_X M$ (these play the role of the two halves of the conormal bundles of event horizons), defined locally near ${}^{\mathrm{b}}S^*_X M$, with $\tilde{\rho}^{m-1}\mathsf{H}_p$ tangent to $\mathcal{L}_\pm$, with a homogeneous degree-zero quadratic defining function ρ_0 (explained below) of $\mathcal{L}$ within Σ such that

$$\tilde{\rho}^{m-2}\mathsf{H}_p\tilde{\rho}|_{L_\pm} = \mp\beta_0 \quad \text{and} \quad -\tilde{\rho}^{m-1}\tau^{-1}\mathsf{H}_p\tau|_{L_\pm} = \mp\tilde{\beta}\beta_0, \quad \beta_0, \tilde{\beta} \in C^\infty(L_\pm) \text{ with } \beta_0, \tilde{\beta} > 0, \tag{2-3}$$

and, with $\beta_1 > 0$,

$$\mp\tilde{\rho}^{m-1}\mathsf{H}_p\rho_0 - \beta_1\rho_0 \tag{2-4}$$

is nonnegative modulo cubic vanishing terms at $L_\pm$. Here, the phrase "quadratic defining function ρ_0" means that ρ_0 vanishes quadratically at $\mathcal{L}$ (and vanishes only at $\mathcal{L}$), with the vanishing nondegenerate, in the sense that the Hessian is positive definite, corresponding to ρ_0 being a sum of squares of linear defining functions whose differentials span the conormal bundle of $\mathcal{L}$ within Σ.

Under these assumptions, L_- is a source and L_+ is a sink within ${}^{\mathrm{b}}S^*_X M$, in the sense that nearby bicharacteristics *within* ${}^{\mathrm{b}}S^*_X M$ all tend to $L_\pm$ as the parameter along them goes to $\pm\infty$, but at L_- there is also a stable, and at L_+ an unstable, manifold, namely $\mathscr{L}_-$ and $\mathscr{L}_+$. Indeed, bicharacteristics in $\mathscr{L}_\pm$ remain there by the tangency of $\tilde\rho^{m-1}\mathsf{H}_p$ to $\mathscr{L}_\pm$; further, $\tau \to 0$ along them as the parameter goes to $\mp\infty$ by (2-3), at least sufficiently close to $\tau = 0$, since $L_\pm$ are defined in $\mathscr{L}_\pm$ by $\tau = 0$.

In order to simplify the statements, we assume that

$$\tilde\beta \text{ is constant on } L_\pm, \quad \tilde\beta = \beta > 0;$$

we refer the reader to [Vasy 2013a, Equations (2.5)–(2.6)] and the discussion throughout that paper, where a general $\tilde\beta$ is allowed, at the cost of either $\sup\tilde\beta$ or $\inf\tilde\beta$ playing a role in various statements depending on signs. Finally, we assume that $\mathcal{P} - \mathcal{P}^* \in \Psi_{\mathrm{b}}^{m-2}(M)$ for convenience (with respect to some b-metric), as this is the case for the Klein–Gordon equation.[5]

Proposition 2.1. *Suppose $\mathcal{P}$ is as above.*

If $s \ge s'$, $s' - \frac{1}{2}(m-1) > \beta r$ and $u \in H_{\mathrm{b}}^{-\infty,r}(M)$, then $L_\pm$ (and thus a neighborhood of $L_\pm$) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$ provided $L_\pm \cap \mathrm{WF}_{\mathrm{b}}^{s-m+1,r}(\mathcal{P}u) = \varnothing$ and $L_\pm \cap \mathrm{WF}_{\mathrm{b}}^{s',r}(u) = \varnothing$, and, in a neighborhood of $L_\pm$, $\mathscr{L}_\pm \cap \{\tau > 0\}$ are disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$.

On the other hand, if $s - \frac{1}{2}(m-1) < \beta r$ and $u \in H_{\mathrm{b}}^{-\infty,r}(M)$, then $L_\pm$ (and thus a neighborhood of $L_\pm$) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$ provided $L_\pm \cap \mathrm{WF}_{\mathrm{b}}^{s-m+1,r}(\mathcal{P}u) = \varnothing$ and a punctured neighborhood of $L_\pm$ in $\Sigma \cap {}^{\mathrm{b}}S_X^*M$, with $L_\pm$ removed, is disjoint from $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$.

Remark 2.2. The decay order r plays the role of $-\Im\sigma$ in [Vasy 2013a] in view of the Mellin transform in the dilation-invariant setting identifying weighted b-Sobolev spaces of weight r with semiclassical Sobolev spaces on the boundary on the line $\Im\sigma = -r$; see [ibid., Equation (3.8)–(3.9)]. Thus, the threshold regularity in this proposition is a direct translation of that in Vasy's Propositions 2.3–2.4.

Proof. We remark first that $\tilde\rho^{m-1}\mathsf{H}_p\rho_0$ vanishes quadratically on $\mathscr{L}_\pm$, since $\tilde\rho^{m-1}\mathsf{H}_p$ is tangent to $\mathscr{L}_\pm$ and ρ_0 itself vanishes there quadratically. Further, this quadratic expression is positive definite near $\tau = 0$ since it is so at $\tau = 0$. Correspondingly, we can strengthen (2-4) to

$$\mp\tilde\rho^{m-1}\mathsf{H}_p\rho_0 - \tfrac{1}{2}\beta_1\rho_0 \tag{2-5}$$

being nonnegative modulo cubic terms vanishing at $\mathscr{L}_\pm$ in a neighborhood of $\tau = 0$.

Notice next that, using (2-5) in the first case and (2-3) in the second, and that $L_\pm$ is defined in Σ by $\tau = 0$ and $\rho_0 = 0$, there exist $\delta_0 > 0$ and $\delta_1 > 0$ such that

$$\alpha \in \Sigma, \quad \rho_0(\alpha) < \delta_0, \quad \tau(\alpha) < \delta_1 \quad \text{and} \quad \rho_0(\alpha) \neq 0 \quad \Longrightarrow \quad (\mp\tilde\rho^{m-1}\mathsf{H}_p\rho_0)(\alpha) > 0$$

and

$$\alpha \in \Sigma, \quad \rho_0(\alpha) < \delta_0 \quad \text{and} \quad \tau(\alpha) < \delta_1 \quad \Longrightarrow \quad (\pm\tilde\rho^{m-1}\tau^{-1}\mathsf{H}_p\tau)(\alpha) > 0.$$

[5] The natural assumption is that the principal symbol of $\frac{1}{2i}(\mathcal{P} - \mathcal{P}^*) \in \Psi_{\mathrm{b}}^{m-1}(M)$ at $L_\pm$ is

$$\pm\hat\beta\beta_0\tilde\rho^{-m+1}, \quad \hat\beta \in C^\infty(L_\pm).$$

If $\hat\beta$ vanishes, Proposition 2.1 is valid without a change; otherwise, it shifts the threshold quantity $s - \frac{1}{2}(m-1) - \beta r$ below in Proposition 2.1 to $s - \frac{1}{2}(m-1) - \beta r + \hat\beta$ if $\hat\beta$ is constant, with modifications as in [Vasy 2013a, Proof of Propositions 2.3–2.4] otherwise.

Similarly to [Vasy 2013a, Proof of Propositions 2.3–2.4], which is not in the b-setting, and [Baskin et al. 2014, Proof of Proposition 4.4], which is, but concerns only sources and sinks (corresponding to Minkowski-type spaces), we consider commutants

$$C \in \tau^{-r}\Psi_{\mathrm{b}}^{s-(m-1)/2}(M) = \Psi_{\mathrm{b}}^{s-(m-1)/2,-r}(M)$$

with principal symbol

$$c = \phi(\rho_0)\phi_0(p_0)\phi_1(\tau)\tilde{\rho}^{-s+(m-1)/2}\tau^{-r}, \quad p_0 = \tilde{\rho}^m p,$$

where $\phi_0 \in C_c^\infty(\mathbb{R})$ is identically 1 near 0, $\phi \in C_c^\infty(\mathbb{R})$ is identically 1 near 0 with $\phi' \le 0$ in $[0,\infty)$ and ϕ supported in $(-\delta_0, \delta_0)$, while $\phi_1 \in C_c^\infty(\mathbb{R})$ is identically 1 near 0 with $\phi_1' \le 0$ in $[0,\infty)$ and ϕ_1 supported in $(-\delta_1, \delta_1)$, so that

$$\alpha \in \operatorname{supp} d(\phi\circ\rho_0) \cap \operatorname{supp}(\phi_1\circ\tau) \cap \Sigma \quad \Longrightarrow \quad \mp(\tilde{\rho}^{m-1}\mathsf{H}_p\rho_0)(\alpha) > 0$$

and

$$\pm\tilde{\rho}^{m-1}\tau^{-1}\mathsf{H}_p\tau$$

remains positive on $\operatorname{supp}(\phi_1\circ\tau)\cap\operatorname{supp}(\phi\circ\rho_0)$.

The main contribution then comes from the weights, which give

$$\tilde{\rho}^{m-1}\mathsf{H}_p(\tilde{\rho}^{-s+(m-1)/2}\tau^{-r}) = \mp\bigl(-s+\tfrac{1}{2}(m-1)+\beta r\bigr)\beta_0\tilde{\rho}^{-s+(m-1)/2}\tau^{-r},$$

where the sign of the factor in parentheses on the right-hand side being negative (resp. positive) gives the first (resp. second) case of the statement of the proposition. Further, the sign of the term in which $\phi_1(\tau)$ (resp. $\phi(\rho_0)$) gets differentiated, yielding $\pm\tau\tilde{\beta}\beta_0\phi_1'(\tau)$ (resp. $\phi'(\rho_0)\tilde{\rho}^{m-1}\mathsf{H}_p\rho_0$) is, when $s-\frac{1}{2}(m-1)-\beta r > 0$, the opposite of (resp. same as) these terms, while when $s-\frac{1}{2}(m-1)-\beta r < 0$, it is the same as (resp. opposite of) these terms. Correspondingly,

$$\begin{aligned}\sigma_{2s}(i[\mathcal{P}, C^*C]) = \mp 2\bigl(&-\beta_0\bigl(s-\tfrac{1}{2}(m-1)-\beta r\bigr)\phi\phi_0\phi_1 - \beta_0\tilde{\beta}\tau\phi\phi_0\phi_1' \\ &\mp(\tilde{\rho}^{m-1}\mathsf{H}_p\rho_0)\phi'\phi_0\phi_1 + m\beta_0 p_0\phi\phi_0'\phi_1\bigr)\phi\phi_0\phi_1\tilde{\rho}^{-2s}\tau^{-2r}.\end{aligned}$$

We can regularize using $S_\epsilon \in \Psi_{\mathrm{b}}^{-\delta}(M)$ for $\epsilon > 0$, uniformly bounded in $\Psi_{\mathrm{b}}^0(M)$, converging to Id in $\Psi_{\mathrm{b}}^{\delta'}(M)$ for $\delta' > 0$, with principal symbol $(1+\epsilon\tilde{\rho}^{-1})^{-\delta}$, as in [Vasy 2013a, Proof of Propositions 2.3–2.4], where the only difference was that the calculation was on $X = \partial M$, and thus the pseudodifferential operators were standard ones, rather than b-pseudodifferential operators. The a priori regularity assumption on $\mathrm{WF}_{\mathrm{b}}^{s',r}(u)$ arises as the regularizer has the opposite sign as compared to the contribution of the weights, thus the amount of regularization one can do is limited. The positive commutator argument then proceeds completely analogously to [Vasy 2013a, Proof of Propositions 2.3–2.4], except that, as in that reference, one has to assume a priori bounds on the term with the sign opposite to that of $s-\frac{1}{2}(m-1)-\beta r$, of which there is exactly one for either sign (unlike in [Vasy 2013a], in which only $s-\frac{1}{2}(m-1)+\beta\Im\sigma < 0$ has such a term), thus on $\Sigma\cap\operatorname{supp}(\phi_1'\circ\tau)\cap\operatorname{supp}(\phi\circ\rho_0)$ when $s-\frac{1}{2}(m-1)-\beta r > 0$ and on $\Sigma\cap\operatorname{supp}(\phi_1\circ\tau)\cap\operatorname{supp}(\phi'\circ\rho_0)$ when $s-\frac{1}{2}(m-1)-\beta r < 0$.

Using the openness of the complement of the wave front set, we can finally choose ϕ and ϕ_1 (satisfying the support conditions, among others) so that the a priori assumptions are satisfied, choosing ϕ_1 first and then shrinking the support of ϕ in the first case, with the choice being made in the opposite order in the second case, completing the proof of the proposition. $\square$

2A2. *Complex absorption.* In order to have good Fredholm properties we either need a complete Hamilton flow, or need to "stop it" in a manner that gives suitable estimates; one may want to do the latter to avoid global assumptions on the flow on the ambient space. The microlocally best-behaved version is given by complex absorption; it is microlocal, works easily with Sobolev spaces of arbitrary order, and makes the operator elliptic in the absorbing region, giving rise to very convenient analysis. The main downside of complex absorption is that it does not automatically give forward mapping properties for the support of solutions in settings like the wave equation, even though at the level of singularities it does have the desired forward property. It was used extensively in [Vasy 2013a] — in the dilation-invariant setting, the bicharacteristics on $X \times (0,\infty)_\tau$ are controlled (by the invariance) as $\tau \to \infty$ as well as when $\tau \to 0$, and thus one need not use complex absorption there but, instead, decay as $\tau \to \infty$ (corresponding to growth as $\tau \to 0$ on these dilation-invariant spaces) gives the desired forward property; complex absorption was only used to cut off the flow within X. Here we want to localize in τ as well and, while complex absorption can achieve this, it loses the forward *support* character of the problem. Thus, complex absorption will not be of use to us when solving semilinear forward problems later on; however, as it is conceptually much cleaner, we discuss Fredholm properties using it first before turning to adding artificial (spacelike) boundary hypersurfaces in the next section, which allow for the solution of forward problems but require additional technicalities.

Thus, we now consider $\mathcal{P} - i\mathcal{Q} \in \Psi_{\mathrm{b}}^m(M)$ and $\mathcal{Q} \in \Psi_{\mathrm{b}}^m(M)$, with real principal symbol q, being the complex absorption similar to [Vasy 2013a, §§2.2 and 2.8]; we assume that $\mathrm{WF}_{\mathrm{b}}'(\mathcal{Q}) \cap L = \varnothing$. Here the semiclassical version, discussed in the above work with further references there, is a close parallel to our b-setting; it is essentially equivalent to the b-setting in the special case that $\mathcal{P}$ and $\mathcal{Q}$ are dilation-invariant, for then the Mellin transform gives rise exactly to the semiclassical problem as the Mellin-dual parameter goes to infinity. Thus, we assume that the characteristic set Σ of $\mathcal{P}$ has the form

$$\Sigma = \Sigma_+ \cup \Sigma_-,$$

with each of $\Sigma_\pm$ being a union of connected components and

$$\mp q \geq 0 \quad \text{near} \quad \Sigma_\pm.$$

Recall from [Vasy 2013a, §2.5], which in turn is a simple modification of the semiclassical results of Nonnenmacher and Zworski [2009], and Datchev and Vasy [2012], that, under these sign conditions on q, estimates can be propagated in the backward direction along the Hamilton flow on Σ_+ and in the forward direction for Σ_-, or, phrased as a wave front set statement (the property of being singular propagates in the opposite direction as the property of being regular!), $\mathrm{WF}^s(u)$ is invariant in $(\Sigma_+ \setminus {}^{\mathrm{b}}S_X^*M) \setminus \mathrm{WF}^{s-m+1}((\mathcal{P} - i\mathcal{Q})u)$ under the forward Hamilton flow, and is invariant in $(\Sigma_- \setminus {}^{\mathrm{b}}S_X^*M) \setminus \mathrm{WF}^{s-m+1}((\mathcal{P} - i\mathcal{Q})u)$ under the backward flow. (That is, the invariance is away from the boundary X; we address the behavior at the boundary in the rest of the paragraph.) Since this is a principal symbol argument, given in [Vasy 2013a, §2.5; Datchev and Vasy 2012, Lemma 5.1], its extension to the b-setting only requires minimal changes. Namely, assuming one is away from radial points, as one may (since at these the statement is vacuous), one constructs the principal symbol c of the commutant on ${}^{\mathrm{b}}T^*M \setminus o$ as a C^∞ function c_0 on ${}^{\mathrm{b}}S^*M$ with derivative of a fixed sign along the Hamilton flow in the region where one wants to obtain the estimate (exactly the same way as for real-principal-type proofs) multiplied by weights in τ and $\tilde{\rho}$, making the Hamilton derivative of c_0 large relative to c_0 to control the error terms from the weights, and computes $\langle u, -i[C^*C, \widetilde{\mathcal{P}}]u\rangle$, where $\widetilde{\mathcal{P}}$ is the symmetric part of $\mathcal{P} - i\mathcal{Q}$ (so has principal symbol p) and $\widetilde{\mathcal{Q}}$ is the antisymmetric part. This gives

$$-2\Re\langle u, iC^*C(\mathcal{P} - i\mathcal{Q})u\rangle - 2\Re\langle u, C^*C\widetilde{\mathcal{Q}}u\rangle.$$

The issue here is that the second term on the right-hand side involves $C^*C\widetilde{\mathcal{Q}}$, which is one order higher than $[C^*C, \widetilde{\mathcal{P}}]$, so, while it itself has a desirable sign, one needs to be concerned about subprincipal terms.[6] However, one rewrites

$$2\Re\langle u, C^*C\widetilde{\mathcal{Q}}u\rangle = 2\Re\langle u, C^*\widetilde{\mathcal{Q}}Cu\rangle + 2\Re\langle u, C^*[C, \widetilde{\mathcal{Q}}]u\rangle.$$

Now, the first term is positive modulo a controllable error by the sharp Gårding inequality or if one arranges that q is the square of a symbol. This controllability claim uses the derivative of c, arising in the symbol of the commutator with $\widetilde{\mathcal{P}}$, to provide the control: since $\widetilde{\mathcal{Q}}$ is positive modulo an operator one order lower and in the term involving this operator, the principal symbol c of C is not differentiated, writing c as c_0 times a weight, where c_0 is homogeneous of degree zero, and taking the derivative of c_0 large relative to c_0, as is already used to control weights, etc., controls this error term (modulo which we have positivity) as well. On the other hand, the second can be rewritten in terms of $[C, [C, \widetilde{\mathcal{Q}}]]$, $(C^* - C)[C, \widetilde{\mathcal{Q}}]$, etc., which are all controllable as they drop *two* orders relative to the product $C^*C\widetilde{\mathcal{Q}}$. This gives rise to the result, namely that, for $u \in H_{\mathrm{b}}^{-\infty,r}$, $\mathrm{WF}_{\mathrm{b}}^{s,r}(u)$ is invariant in $\Sigma_+ \setminus \mathrm{WF}^{s-m+1,r}((\mathcal{P} - i\mathcal{Q})u)$ under the forward Hamilton flow and in $\Sigma_- \setminus \mathrm{WF}^{s-m+1,r}((\mathcal{P} - i\mathcal{Q})u)$ under the backward flow.

In analogy with [Vasy 2013a, Definition 2.12], we say that $\mathcal{P} - i\mathcal{Q}$ is *nontrapping* if all bicharacteristics in Σ from any point in $\Sigma \setminus (L_+ \cup L_-)$ flow to $\mathrm{Ell}(q) \cup L_+ \cup L_-$ in both the forward and backward directions (i.e., either enter $\mathrm{Ell}(q)$ in finite time or tend to $L_+ \cup L_-$). Notice that, as $\Sigma_\pm$ are closed under the Hamilton flow, bicharacteristics in $\mathcal{L}_\pm \setminus (L_+ \cup L_-)$ necessarily enter the elliptic set of $\mathcal{Q}$ in the forward, in Σ_+ (resp. backward, in Σ_-), direction. Indeed, by the nontrapping hypothesis, these bicharacteristics have to reach the elliptic set of $\mathcal{Q}$ as they cannot tend to L_+ (resp. L_-): $\mathcal{L}_+$ and $\mathcal{L}_-$ are unstable (resp. stable) manifolds and these bicharacteristics cannot enter the boundary — which is preserved by the flow — so cannot lie in the stable (resp. unstable) manifolds of $L_+ \cup L_-$, which are within ${}^{\mathrm{b}}S^*_X M$. Similarly, bicharacteristics in $(\Sigma \cap {}^{\mathrm{b}}S^*_X M) \setminus (L_+ \cup L_-)$ necessarily reach the elliptic set

[6]In fact, as the principal symbol of $C^*C\widetilde{\mathcal{Q}}$ is real, the real part of its subprincipal symbol is well defined and is the real part of c^2q, where c and q include the real parts of their subprincipal terms, and is all that matters for this argument, so one could proceed symbolically.

of $\mathcal{Q}$ in the backward, in Σ_+ (resp. forward, in Σ_-), direction. Then, for s and r satisfying

$$s - \tfrac{1}{2}(m-1) > \beta r,$$

one has an estimate

$$\|u\|_{H_b^{s,r}} \leq C\|(\mathcal{P} - i\mathcal{Q})u\|_{H_b^{s-m+1,r}} + C\|u\|_{H_b^{s',r}} \tag{2-6}$$

provided one assumes $s' < s$ and

$$s' - \tfrac{1}{2}(m-1) > \beta r, \quad u \in H_b^{s',r}.$$

Indeed, this is a simple consequence of $u \in H_b^{s',r}$ and $(\mathcal{P} - i\mathcal{Q})u \in H_b^{s-m+1,r}$ implying $u \in H_b^{s,r}$ via the closed graph theorem; see [Hörmander 1985b, Proof of Theorem 26.1.7; Vasy 2013b, §4.3]. This implication in turn holds as, on the elliptic set of $\mathcal{Q}$, one has the stronger statement $u \in H_b^{s+1,r}$ under these conditions, and then, using real-principal-type propagation of regularity in the *backward* direction on Σ_+ and the *forward* direction on Σ_-, one can propagate the microlocal membership of $H_b^{s,r}$ (i.e., the absence of the corresponding wave front set) in the backward (resp. forward) direction on Σ_+ (resp. Σ_-). Since bicharacteristics in $\mathcal{L}_\pm \setminus (L_+ \cup L_-)$ necessarily enter the elliptic set of $\mathcal{Q}$ in the forward (resp. backward) direction, and thus one has $H_b^{s,r}$ membership along them by what we have shown, Proposition 2.1 extends this membership to $L_\pm$, and hence to a neighborhood of these, and by our nontrapping assumption every bicharacteristic enters either this neighborhood of $L_\pm$ or the elliptic set of $\mathcal{Q}$ in finite time in the backward (resp. forward) direction, so by the real-principal-type propagation of singularities we have the claimed microlocal membership everywhere.

Reversing the direction in which one propagates estimates, one also has a similar estimate for the adjoint $\mathcal{P}^* + i\mathcal{Q}^*$, except now one needs to have

$$s - \tfrac{1}{2}(m-1) < \beta r$$

in order to propagate through the saddle points in the opposite direction, that is, from within ${}^{\mathrm{b}}S^*_X M$ to $\mathcal{L}_\pm$. Then, for $s' < s$,

$$\|u\|_{H_b^{s,r}} \leq C\|(\mathcal{P}^* + i\mathcal{Q}^*)u\|_{H_b^{s-m+1,r}} + C\|u\|_{H_b^{s',r}}. \tag{2-7}$$

The issue with these estimates is that $H_b^{s,r}$ does not include compactly into the error term $H_b^{s',r}$ on the right-hand side, due to the lack of additional decay. Thus, these estimates are insufficient to show Fredholm properties, which in fact do not hold in general.

We thus further assume that there are no poles of the inverse of the Mellin conjugate $\widehat{(\mathcal{P} - i\mathcal{Q})}(\sigma)$ of the normal operator $N(\mathcal{P} - i\mathcal{Q})$ on the line $\Im\sigma = -r$. Here we refer to [Vasy 2013a, §3.1] for a brief discussion of the normal operator and the Mellin transform; this cited section also contains more detailed references to [Melrose 1993]. Then, using the Mellin transform, which is an isomorphism between weighted b-Sobolev spaces and semiclassical Sobolev spaces (see Equations (3.8)–(3.9) in [Vasy 2013a]) and the estimates for $(\mathcal{P}-i\mathcal{Q})\widehat{\ }(\sigma)$ (including the high-energy, i.e., semiclassical, estimates,[7] all of which is discussed in detail in [Vasy 2013a, §2] — the high energy assumptions of [Vasy 2013a, §2] hold by our assumptions on the b-flow at ${}^{\mathrm{b}}S^*_X M$ — and which imply that, for all but a discrete set of r, the aforementioned lines do not contain such poles), we obtain that, on $\mathbb{R}^+_\rho \times \partial M$,

$$\|v\|_{H_{\mathrm{b}}^{s,r}} \le C\|N(\mathcal{P}-i\mathcal{Q})v\|_{H_{\mathrm{b}}^{s-m+1,r}} \tag{2-8}$$

when

$$s-\tfrac{1}{2}(m-1) > \beta r.$$

Again, we have an analogous estimate for $N(\mathcal{P}^*+i\mathcal{Q}^*)$:

$$\|v\|_{H_{\mathrm{b}}^{s,r}} \le C\|N(\mathcal{P}^*+i\mathcal{Q}^*)v\|_{H_{\mathrm{b}}^{s-m+1,r}} \tag{2-9}$$

provided $-r$ is not the imaginary part of a pole of the inverse of $(\mathcal{P}^*+i\mathcal{Q}^*)\widehat{\ }$ and provided

$$s-\tfrac{1}{2}(m-1) < \beta r.$$

As $(\mathcal{P}^*+i\mathcal{Q}^*)\widehat{\ }(\sigma) = (\widehat{\mathcal{P}}-i\widehat{\mathcal{Q}})^*(\bar\sigma)$ — see the discussion in [Vasy 2013a] preceding Equation (3.25) — the requirement on $-r$ is the same as r not being the imaginary part of a pole of the inverse of $\widehat{\mathcal{P}}-i\widehat{\mathcal{Q}}$.

We apply these results by first letting $\chi \in C_c^\infty(M)$ be identically 1 near ∂M supported in a collar neighborhood of ∂M, which we identify with $(0,\epsilon)_\tau \times \partial M$ of the normal operator space. Then, *assuming* $s'-\frac{1}{2}(m-1) > \beta r$,

$$\|u\|_{H_{\mathrm{b}}^{s',r}} \le \|\chi u\|_{H_{\mathrm{b}}^{s',r}} + \|(1-\chi)u\|_{H_{\mathrm{b}}^{s',r}} \le C\|N(\mathcal{P}-i\mathcal{Q})\chi u\|_{H_{\mathrm{b}}^{s'-m+1,r}} + \|(1-\chi)u\|_{H_{\mathrm{b}}^{s',r}}. \tag{2-10}$$

Now, if $K=\operatorname{supp}(1-\chi)$, then

$$\|(1-\chi)u\|_{H_{\mathrm{b}}^{s',r}} \le C\|u\|_{H^{s'}(K)} \le C'\|u\|_{H_{\mathrm{b}}^{s',\tilde r}} \le C''\|u\|_{H_{\mathrm{b}}^{s'+1,\tilde r}}$$

for any $\tilde r$. On the other hand, $N(\mathcal{P}-i\mathcal{Q})-(\mathcal{P}-i\mathcal{Q}) \in \tau\Psi_{\mathrm{b}}^m([0,\epsilon)\times\partial M)$, so

$$\begin{aligned} N(\mathcal{P}-i\mathcal{Q})\chi u &= (\mathcal{P}-i\mathcal{Q})\chi u + (N(\mathcal{P}-i\mathcal{Q})-(\mathcal{P}-i\mathcal{Q}))\chi u \\ &= \chi(\mathcal{P}-i\mathcal{Q})u + [\mathcal{P}-i\mathcal{Q},\chi]u + (N(\mathcal{P}-i\mathcal{Q})-(\mathcal{P}-i\mathcal{Q}))\chi u \end{aligned}$$

plus the fact that $[\mathcal{P}-i\mathcal{Q},\chi]$ is supported in K and $\|\chi(\mathcal{P}-i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s'-m+1,r}} \le \|(\mathcal{P}-i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s'-m+1,r}}$ show that, for all $\tilde r$,

$$\|N(\mathcal{P}-i\mathcal{Q})\chi u\|_{H_{\mathrm{b}}^{s'-m+1,r}} \le \|(\mathcal{P}-i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s'-m+1,r}} + C\|u\|_{H_{\mathrm{b}}^{s'+1,\tilde r}} + C\|u\|_{H_{\mathrm{b}}^{s'+1,r-1}}. \tag{2-11}$$

Combining (2-6), (2-10) and (2-11), we deduce that (with new constants, and taking s' sufficiently small and $\tilde r = r-1$)

$$\|u\|_{H_{\mathrm{b}}^{s,r}} \le C\|(\mathcal{P}-i\mathcal{Q})u\|_{H_{\mathrm{b}}^{s-m+1,r}} + C\|u\|_{H_{\mathrm{b}}^{s'+1,r-1}}, \tag{2-12}$$

[7] The high-energy estimates are actually implied by b-principal symbol-based estimates on the normal operator space $M_\infty = X\times\overline{\mathbb{R}^+}$, $X=\partial M$, on spaces $\tau^r H_{\mathrm{b}}^s(M_\infty)$ corresponding to $\Im\sigma=-r$, but we do not explicitly discuss this here.

where now the inclusion $H_{\mathrm{b}}^{s,r} \to H_{\mathrm{b}}^{s'+1,r-1}$ is compact when we choose, as we may, $s' < s-1$, requiring, however, $s' - \frac{1}{2}(m-1) > \beta r$. Recall that this argument required that s, r and s' satisfied the requirements preceding (2-6) and that $-r$ was not the imaginary part of any pole of $(\mathcal{P} - i\mathcal{Q})\widehat{\ }$.

Analogous estimates hold for $(\mathcal{P} - i\mathcal{Q})^*$, where now we write $\tilde{s}$, $\tilde{r}$ and $\tilde{s}'$ for the Sobolev orders for the eventual application:

$$\|u\|_{H_{\mathrm{b}}^{\tilde{s},\tilde{r}}} \leq C\|(\mathcal{P} - i\mathcal{Q})^* u\|_{H_{\mathrm{b}}^{\tilde{s}-m+1,\tilde{r}}} + C\|u\|_{H_{\mathrm{b}}^{\tilde{s}'+1,\tilde{r}-1}} \tag{2-13}$$

provided $\tilde{s}$ and $\tilde{r}$ in place of s and r satisfy the requirements stated before (2-7), and provided $-\tilde{r}$ is not the imaginary part of a pole of $(\mathcal{P}^* + i\mathcal{Q}^*)\widehat{\ }$ (i.e., $\tilde{r}$ of $\widehat{\mathcal{P}} - i\widehat{\mathcal{Q}}$). Note that we *do not* have a stronger requirement for $\tilde{s}'$, unlike for s' above, since upper bounds for s imply those for $s' \leq s$.

Via a standard functional analytic argument — see [Hörmander 1985b, Proof of Theorem 26.1.7] and also [Vasy 2013a, §2.6] in the present context — we thus obtain Fredholm properties of $\mathcal{P} - i\mathcal{Q}$, in particular solvability, modulo a (possible) finite-dimensional obstruction in $H_{\mathrm{b}}^{s,r}$ if

$$s - \tfrac{1}{2}(m-1) - 1 > \beta r. \tag{2-14}$$

Concretely, we take $\tilde{s} = m-1-s$, $\tilde{r} = -r$, and $s' < s-1$ sufficiently close to $s-1$ that $s' - \frac{1}{2}(m-1) > \beta r$ (which is possible by (2-14)). Thus, $s - \frac{1}{2}(m-1) > \beta r$ means $\tilde{s} - \frac{1}{2}(m-1) = \frac{1}{2}(m-1) - s < -\beta r = \beta\tilde{r}$, so the space on the left-hand side of (2-12) is dual to that in the first term on the right-hand side of (2-13), and the same for the equations interchanged, and notice that the condition on the poles of the inverse of the Mellin-transformed normal operators is the same for both $\mathcal{P} - i\mathcal{Q}$ and $\mathcal{P}^* + i\mathcal{Q}^*$: $-r$ is not the imaginary part of a pole of $(\mathcal{P} - i\mathcal{Q})\widehat{\ }$. Let

$$\mathcal{Y}^{s,r} = H_{\mathrm{b}}^{s,r}(M), \quad \mathcal{X}^{s,r} = \{u \in H_{\mathrm{b}}^{s,r}(M) : (\mathcal{P} - i\mathcal{Q})u \in H_{\mathrm{b}}^{s-1,r}(M)\},$$

and note that $\mathcal{Y}^{s,r}$ and $\mathcal{X}^{s,r}$ are complete, where, in the case of $\mathcal{X}^{s,r}$, the natural norm is

$$\|u\|^2_{\mathcal{X}^{s,r}} = \|u\|^2_{H_{\mathrm{b}}^{s,r}(M)} + \|(\mathcal{P} - i\mathcal{Q})u\|^2_{H_{\mathrm{b}}^{s-1,r}(M)};$$

see Remark 2.19. Our discussion thus far yields:

Proposition 2.3. *Suppose that $\mathcal{P}$ is nontrapping. Suppose $s, r \in \mathbb{R}$, $s - \frac{1}{2}(m-1) - 1 > \beta r$, and $-r$ is not the imaginary part of a pole of $(\mathcal{P} - i\mathcal{Q})\widehat{\ }$, where $\mathcal{P} - i\mathcal{Q}$ is a priori a map*

$$\mathcal{P} - i\mathcal{Q} : H_{\mathrm{b}}^{s,r}(M) \to H_{\mathrm{b}}^{s-2,r}(M).$$

Then

$$\mathcal{P} - i\mathcal{Q} : \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r}$$

is Fredholm.

2A3. *Initial value problems.* As already mentioned, the main issue with the argument using complex absorption that it does not guarantee the forward nature (in terms of supports) of the solution for a wave-like equation, although it does guarantee the correct microlocal structure. So now we assume that $\mathcal{P} \in \mathrm{Diff}_{\mathrm{b}}^2(M)$ and that there is a Lorentzian b-metric g such that

$$\mathcal{P} - \Box_g \in \mathrm{Diff}_{\mathrm{b}}^1(M), \quad \mathcal{P} - \mathcal{P}^* \in \mathrm{Diff}_{\mathrm{b}}^0(M). \tag{2-15}$$

Then one can run a completely analogous argument using energy-type estimates by restricting the domain we consider to be a manifold with corners, where the new boundary hypersurfaces are spacelike with respect to g, i.e., given by level sets of timelike functions. Such a possibility was mentioned in [Vasy 2013a, Remark 2.6], though it was not described in detail as it was not needed there, essentially because the existence and uniqueness argument for forward solutions was given only for dilation-invariant operators. The main difference between using complex absorption and adding boundary hypersurfaces is that the latter limit the Sobolev regularity one can use, with the most natural choice coming from energy estimates. However, a posteriori one can improve the result to better Sobolev spaces using propagation of singularities results.

So assume now that $U \subset M$ is open and we have two functions $\mathfrak{t}_1$ and $\mathfrak{t}_2$ in $C^\infty(M)$, both of which, restricted to U, are timelike (in particular have nonzero differential) near their respective 0-level sets H_1 and H_2, and

$$\Omega = \mathfrak{t}_1^{-1}([0,\infty)) \cap \mathfrak{t}_2^{-1}([0,\infty)) \subset U.$$

Notice that the timelike assumption forces $d\mathfrak{t}_j$ to not lie in $N^*X = N^*\partial M$ (for its image in the b-cosphere bundle would be zero) and thus, if the H_j intersect X, they do so transversally. We assume that the H_j intersect only away from X and that they do so transversally, that is, the differentials of $\mathfrak{t}_j$ are independent at the intersection. Then Ω is a manifold with corners with boundary hypersurfaces H_1, H_2 and X (all intersected with Ω). We, however, keep thinking of Ω as a domain in M. The role of the elliptic set of $\mathcal{Q}$ is now played by ${}^{\mathrm{b}}S^*_{H_j}M$, $j = 1, 2$. The *nontrapping* assumption becomes (see Figure 3) that:

(1) All bicharacteristics in $\Sigma_\Omega = \Sigma \cap {}^{\mathrm{b}}S^*_\Omega M$ from any point in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ flow (within Σ_Ω) to ${}^{\mathrm{b}}S^*_{H_1}M \cup L_+$ in the forward direction (i.e., either enter ${}^{\mathrm{b}}S^*_{H_1}M$ in finite time or tend to L_+) and to ${}^{\mathrm{b}}S^*_{H_2}M \cup L_+$ in the backward direction.

(2) From any point in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ the bicharacteristics flow to ${}^{\mathrm{b}}S^*_{H_2}M \cup L_-$ in the forward direction and to ${}^{\mathrm{b}}S^*_{H_1}M \cup L_-$ in the backward direction.

In particular, orienting the characteristic set by letting Σ_- be the *future-oriented* and Σ_+ the *past-oriented* part, $d\mathfrak{t}_1$ is future-oriented, while $d\mathfrak{t}_2$ is past-oriented.

On a manifold with corners, such as Ω, one can consider supported and extendible distributions; see [Hörmander 1985a, Appendix B.2] for the smooth boundary setting, with simple changes needed only for the corners setting, which is discussed in [Vasy 2008, §3], for example. Here we consider Ω as a domain in M, and thus its boundary face $X \cap \Omega$ is regarded as having a different character from the $H_j \cap \Omega$, that is, the support and extendibility considerations do not arise at X — all distributions are regarded as acting on a subspace of C^∞ functions on Ω vanishing at X to infinite order, i.e., they are automatically extendible distributions at X. On the other hand, at H_j we consider both extendible distributions, acting on C^∞ functions vanishing to infinite order at H_j, and supported distributions, which act on all C^∞

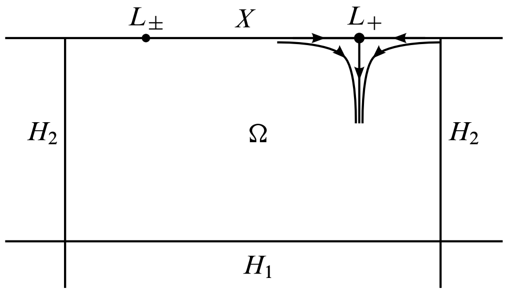

Figure 3. Setup for the discussion of the forward problem. Near the spacelike hypersurfaces H_1 and H_2, which are the replacement for the complex absorbing operator $\mathcal{Q}$, we use standard (nonmicrolocal) energy estimates, and away from them, we use b-microlocal propagation results, including at the radial sets $L_\pm$. The bicharacteristic flow — in fact, its projection to the base — is only indicated near L_+; near L_-, the directions of the flowlines are reversed.

functions (as far as conditions at H_j are concerned). For example, the space of supported distributions at H_1 extendible at H_2 (and at X, as we always tacitly assume) is the dual space of the subspace of $C^\infty(\Omega)$ consisting of functions vanishing to infinite order at H_2 and X (but not necessarily at H_1). An equivalent way of characterizing this space of distributions is that they are restrictions of elements of the dual of $\dot{C}^\infty(M)$ (consisting of C^∞ functions on M vanishing to infinite order at X) with support in $\mathfrak{t}_1 \geq 0$ to C^∞ functions on Ω which vanish to infinite order at X and H_2, thus, in the terminology of [Hörmander 1985a], restrictions to $\Omega \setminus (H_2 \cup X)$.

The main interest is in spaces induced by the Sobolev spaces $H_{\mathrm{b}}^{s,r}(M)$. Notice that the Sobolev norm is of a completely different nature at X than at the H_j, namely the derivatives are based on complete, rather than incomplete, vector fields: $\mathcal{V}_{\mathrm{b}}(M)$ is being restricted to Ω, so one obtains vector fields tangent to X but not to the H_j. As for supported and extendible distributions corresponding to $H_{\mathrm{b}}^{s,r}(M)$, we have, for instance,

$$H_{\mathrm{b}}^{s,r}(M)^{\bullet,-},$$

with the first superscript on the right denoting whether supported ($\bullet$) or extendible ($-$) distributions are discussed at H_1, and the second the analogous property at H_2, which consists of restrictions of elements of $H_{\mathrm{b}}^{s,r}(M)$ with support in $\mathfrak{t}_1 \geq 0$ to $\Omega \setminus (H_2 \cup X)$. Then elements of $C^\infty(\Omega)$ with the analogous vanishing conditions, so in the example vanishing to infinite order at H_1 and X, are dense in $H_{\mathrm{b}}^{s,r}(M)^{\bullet,-}$; further, the dual of $H_{\mathrm{b}}^{s,r}(M)^{\bullet,-}$ is $H_{\mathrm{b}}^{-s,-r}(M)^{-,\bullet}$ with respect to the L^2 (sesquilinear) pairing.

First we work locally. For this purpose it is convenient to introduce another timelike function $\tilde{\mathfrak{t}}_j$, not necessarily timelike, and consider

$$\Omega_{[t_0,t_1]} = \mathfrak{t}_j^{-1}([t_0,\infty)) \cap \tilde{\mathfrak{t}}_j^{-1}((-\infty,t_1]) \quad \text{and} \quad \Omega_{(t_0,t_1)} = \mathfrak{t}_j^{-1}((t_0,\infty)) \cap \tilde{\mathfrak{t}}_j^{-1}((-\infty,t_1)),$$

and similarly on half-open, half-closed intervals. Thus, $\Omega_{[t_0,t_1]}$ becomes smaller as t_0 becomes larger or t_1 becomes smaller.

We then consider energy estimates on $\Omega_{[T_0,T_1]}$. In order to set up the following arguments, choose

$$T_- < T'_- < T_0 \quad \text{and} \quad T_1 < T'_+ < T_+,$$

and assume that $\Omega_{[T_-,T_+]}$ is compact, $\Omega_{[T_0,T_1]}$ is nonempty, and $\mathfrak{t}_j$ is timelike on $\Omega_{[T_-,T_+]}$. The energy estimates propagate estimates in the direction of either increasing or decreasing $\mathfrak{t}_j$. With the extendible or supported character of distributions at $\tilde{\mathfrak{t}}_j = T_+$ being irrelevant for this matter in the case being considered and thus dropped from the notation (so $(-)$ refers to extendibility at $\mathfrak{t}_j = T_0$), consider

$$\mathcal{P} : H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_+]})^- \to H_{\mathrm{b}}^{s-2,r}(\Omega_{[T_0,T_+]})^-, \quad s,\ r \in \mathbb{R}.$$

The energy estimate, with backward propagation in $\mathfrak{t}_j$, from $\tilde{\mathfrak{t}}_j^{-1}([T'_+, T_+])$, in this setting takes the form:

Lemma 2.4. *Let $r \in \mathbb{R}$. There is $C > 0$ such that, for $u \in H_{\mathrm{b}}^{2,r}(\Omega_{[T_0,T_+]})^-$,*

$$\|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_1]})^-} \le C\big(\|\mathcal{P}u\|_{H_{\mathrm{b}}^{0,r}(\Omega_{[T_0,T_+]})^-} + \|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_+]}\cap\tilde{\mathfrak{t}}_j^{-1}([T'_+,T_+]))^-}\big). \tag{2-16}$$

This also holds with $\mathcal{P}$ replaced by $\mathcal{P}^*$, acting on the same spaces.

Remark 2.5. The lemma is also valid if one has several boundary hypersurfaces, that is, if one replaces $\mathfrak{t}_j^{-1}([t_0,\infty))$ by $\mathfrak{t}_j^{-1}([t_{j,0},\infty)) \cap \mathfrak{t}_k^{-1}([t_{k,0},\infty))$ in the definition of $\Omega_{[t_0,t_1]}$, and/or $\tilde{\mathfrak{t}}_j^{-1}((-\infty,t_1])$ by $\tilde{\mathfrak{t}}_j^{-1}((-\infty,t_{j,1}])\cap\tilde{\mathfrak{t}}_k^{-1}((-\infty,t_{k,1}])$, i.e., regarding $\mathfrak{t}_j$ and/or $\tilde{\mathfrak{t}}_j$ as vector-valued, and propagating backwards in $\mathfrak{t}_{j_0}$ for some fixed j_0, under the additional hypothesis that $\mathfrak{t}_{j_0}$ is timelike in $\Omega_{[t_0,t_1]}$, and all $\mathfrak{t}_j$, $j \neq j_0$, are timelike near their respective zero sets, with the same timelike character at $\mathfrak{t}_{j_0}$. (One can also have more than two such functions.) To see this, replace $\chi(\mathfrak{t}_j)$ by $\chi_{j_0}(\mathfrak{t}_{j_0})\chi_k(\mathfrak{t}_k)$ and analogously for $\tilde{\chi}$ in the definition of V in (2-17), where χ_k is the characteristic function of $[t_{k,0},\infty)$, while letting $W = G({}^{\mathrm{b}}d\mathfrak{t}_{j_0},\cdot)$. Then $\chi'\tilde{\chi}\tau^\alpha A^\sharp$ is replaced by $\chi'_j\chi_k\tilde{\chi}_j\tilde{\chi}_k\tau^\alpha A^\sharp + \chi_j\chi'_k\tilde{\chi}_j\tilde{\chi}_k\tau^\alpha A^\sharp$, etc., and our additional hypothesis guarantees that the matrix $A^\sharp$ is indeed positive definite: The contribution from differentiating χ_{j_0} is positive definite by the timelike nature of $d\mathfrak{t}_{j_0}$, while the contribution from differentiating χ_j, $j \neq j_0$, giving δ-distributions at the hypersurfaces $\mathfrak{t}_j^{-1}(t_{j,0})$, is positive definite by the second part of the above additional hypothesis and can therefore be dropped as in the proof of Lemma 2.4 below. Thus χ'_{j_0} can still be used to dominate χ_{j_0}; the terms in which $\tilde{\chi}_j$ is differentiated have support where $\tilde{\mathfrak{t}}_j$ is in $(T'_{+,j}, T_{+,j})$, so the control region on the right-hand side of (2-16) is the union of these sets.

In our application this situation arises as we need the estimates on $\mathfrak{t}_1^{-1}([T_0,T_1]) \cap \mathfrak{t}_2^{-1}([0,\infty))$ and $\mathfrak{t}_1^{-1}([0,\infty)) \cap \mathfrak{t}_2^{-1}([T_0,T_1])$, with $T_0 = 0$ and $T_1 > 0$ small. For instance, in the latter case $\mathfrak{t}_2$ plays the role of $\mathfrak{t}_j$ above, while $-\mathfrak{t}_1$ and $\mathfrak{t}_2$ play the role of $\tilde{\mathfrak{t}}_j$ and $\tilde{\mathfrak{t}}_k$; see Figure 4.

Proof of Lemma 2.4. To see (2-16), one proceeds as in [Vasy 2013a, §3.3] and considers

$$V = -i\chi(\mathfrak{t}_j)\tilde{\chi}(\tilde{\mathfrak{t}}_j)\tau^\alpha W \tag{2-17}$$

with $W = G(d\mathfrak{t}_j,\cdot)$ a timelike vector field and with χ, $\tilde{\chi} \in C^\infty(\mathbb{R})$, both nonnegative, to be specified. Then, choosing a *Riemannian* b-metric $\tilde{g}$,

$$-i(V^*\Box_g - \Box_g^* V) = {}^{\mathrm{b}}d_{\tilde{g}}^* C^{\mathrm{b}}\,{}^{\mathrm{b}}d,$$

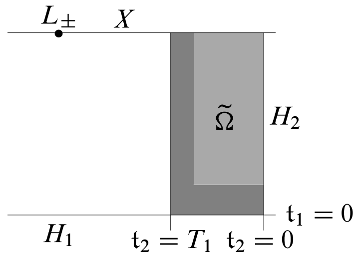

Figure 4. A domain $\tilde\Omega = \mathfrak{t}_2^{-1}([0,\infty)) \cap \big((-\mathfrak{t}_1)^{-1}((-\infty,0]) \cap \mathfrak{t}_2^{-1}((-\infty,T_1])\big)$ on which we will apply the energy estimate (2-16). The a priori control region is indicated in dark gray.

with the subscript on the adjoint on the right-hand side denoting the metric with respect to which it is taken, ${}^{\mathrm{b}}d : C^\infty(M) \to C^\infty(M; {}^{\mathrm{b}}T^*M)$ being the b-differential, and with

$$C^{\mathrm{b}} = \chi'\tilde\chi\tau^\alpha A^\sharp + \chi\tilde\chi'\tau^\alpha \tilde A^\sharp + \chi\tilde\chi\tau^\alpha R^{\mathrm{b}},$$

where $A^\sharp$, $\tilde A^\sharp$ and R^{b} are bundle endomorphisms of ${}^{\mathbb{C}\mathrm{b}}T^*M$, and $A^\sharp$ and $\tilde A^\sharp$ are positive definite. Proceeding further, replacing $\Box_g$ by $\mathcal{P}$ one has

$$\begin{aligned} -i(V^*\mathcal{P} - \mathcal{P}^*V) &= {}^{\mathrm{b}}d_{\tilde g}^* C^\sharp\, {}^{\mathrm{b}}d + (\tilde E_1)_{\tilde g}^*\tau^\alpha\chi\tilde\chi\, {}^{\mathrm{b}}d + {}^{\mathrm{b}}d_{\tilde g}^*\tau^\alpha\chi\tilde\chi\tilde E_2, \\ C^\sharp &= \chi'\tilde\chi\tau^\alpha A^\sharp + \chi\tilde\chi'\tau^\alpha\tilde A^\sharp + \chi\tilde\chi\tau^\alpha\tilde R^\sharp \end{aligned} \tag{2-18}$$

with $\tilde E_j$ bundle maps from the trivial bundle over M to ${}^{\mathbb{C}\mathrm{b}}T^*M$, $A^\sharp$ and $\tilde A^\sharp$ as before, and $\tilde R^\sharp$ a bundle endomorphism of ${}^{\mathbb{C}\mathrm{b}}T^*M$, as follows by expanding

$$-i(V^*(\mathcal{P} - \Box_g) - (\mathcal{P} - \Box_g)^*V),$$

using that $\mathcal{P} - \Box_g \in \mathrm{Diff}_{\mathrm{b}}^1(M)$. We regard the second term on the right-hand side of (2-18) as the one requiring a priori control by $\|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_+]}\cap\mathfrak{t}_j^{-1}([T'_+,T_+]))^-}$; we achieve this by making $\tilde\chi$ supported in $(-\infty, T_+)$, identically 1 near $(-\infty, T'_+]$, so $d\tilde\chi$ is supported in (T'_+, T_+). Now, making $\chi' \geq 0$ large relative to χ on $\operatorname{supp}(\chi\tilde\chi)$, as in[8] [Vasy 2013a, Equation (3.27)], allows one to dominate all terms without derivatives of χ. In order to obtain a nondegenerate estimate up to $\mathfrak{t}_j = T_0$, one cuts off χ at $\mathfrak{t}_j = T_0$ using the Heaviside function, so χ' gives a (positive!) δ-distribution there. Applying (2-18) to v, pairing with v and integrating by parts, the δ-distributions have the same sign as $\chi' A^\sharp$ and can thus be dropped. Put differently, without the sharp cutoff, one again computes the same pairing, but this time on the domain $\Omega_{[T_0,T_+]}$, thus picking up boundary terms with the correct sign in the integration by parts, so these terms can be dropped. This proves the energy estimate (2-16) when one takes $\alpha = -2r$. ☐

Propagating in the forward direction, from $\mathfrak{t}_j^{-1}([T_-, T'_-])$, where now $-$ denotes the character of the space at T_1 (so $-$ refers to extendibility at $\mathfrak{t}_j = T_1$),

$$\|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_1]})^-} \leq C\big(\|\mathcal{P}u\|_{H_{\mathrm{b}}^{0,r}(\Omega_{[T_-,T_1]})^-} + \|u\|_{H_{\mathrm{b}}^{1,r}(\Omega_{[T_-,T_1]}\cap\mathfrak{t}_j^{-1}([T_-,T'_-]))^-}\big). \tag{2-19}$$

[8]Though, there, the sign of χ' is opposite, as the estimate is propagated in the opposite direction.

In particular, for u supported in $\mathfrak{t}_j \geq T_0$, the last estimate becomes, with the first superscript on the right denoting whether supported ($\bullet$) or extendible ($-$) distributions are discussed at $\mathfrak{t} = T_0$ and the second superscript the same at $\mathfrak{t} = T_1$,

$$\|u\|_{H_b^{1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_b^{0,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \tag{2-20}$$

when

$$\mathcal{P} : H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-} \to H_b^{s-2,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$$

and $u \in H_b^{2,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. To summarize, we state both this and (2-16) in terms of these supported spaces:

Corollary 2.6. *Let $r, \tilde{r} \in \mathbb{R}$. For $u \in H_b^{2,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, one has*

$$\|u\|_{H_b^{1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_b^{0,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \tag{2-21}$$

while, for $v \in H_b^{2,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}$, the estimate

$$\|v\|_{H_b^{1,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}} \leq C\|\mathcal{P}^* v\|_{H_b^{0,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}} \tag{2-22}$$

holds.

A duality argument, combined with propagation of singularities, thus gives:

Lemma 2.7. *Let $s \geq 0$, $r \in \mathbb{R}$. Then there is $C > 0$ with the following property: If $f \in H_b^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, then there exists $u \in H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ such that $\mathcal{P}u = f$ and*

$$\|u\|_{H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|f\|_{H_b^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}}.$$

Remark 2.8. As in Remark 2.5, the lemma remains valid in more generality, namely, if one replaces $\mathfrak{t}_j^{-1}([t_0,\infty))$ by $\mathfrak{t}_j^{-1}([t_{j,0},\infty)) \cap \mathfrak{t}_k^{-1}([t_{k,0},\infty))$ and/or $\tilde{\mathfrak{t}}_j^{-1}((-\infty,t_1])$ by $\tilde{\mathfrak{t}}_j^{-1}((-\infty,t_{j,1}]) \cap \tilde{\mathfrak{t}}_j^{-1}((-\infty,t_{k,1}])$ in the definition of $\Omega_{[t_0,t_1]}$, provided that the $\mathfrak{t}_j$ have linearly independent differentials on their joint zero set, and similarly for the $\tilde{\mathfrak{t}}_j$. The place where this linear independence is used (the energy estimate above does not need this) is for the continuous Sobolev extension map, valid on manifolds with corners; see [Vasy 2008, §3].

Proof. We work on the slightly bigger region $\Omega_{[T'_-,T'_+]}$, applying the energy estimates with T_0 replaced by T'_-, T_1 replaced by T'_+. First, by the supported property at $\mathfrak{t}_j = T_0$, one can regard f as an element of $H_b^{s-1,r}(\Omega_{[T'_-,T_1]})^{\bullet,-}$ with support in $\Omega_{[T_0,T_1]}$. Let

$$\tilde{f} \in H_b^{s-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-} \subset H_b^{-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}$$

be an extension of f, so $\tilde{f}$ is supported in $\Omega_{[T_0,T'_+]}$ and restricts to f; by the definition of spaces of extendible distributions as quotients of spaces of distributions on a larger space — see [Hörmander 1985a, Appendix B.2] — we can assume

$$\|\tilde{f}\|_{H_b^{s-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}} \leq 2\|f\|_{H_b^{s-1,r}(\Omega_{[T'_-,T_1]})^{\bullet,-}}. \tag{2-23}$$

By (2-16) applied with $\mathcal{P}$ replaced by $\mathcal{P}^*$ and $\tilde{r} = -r$,

$$\|\phi\|_{H_b^{1,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}} \le C\|\mathcal{P}^*\phi\|_{H_b^{0,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}} \tag{2-24}$$

for $\phi \in H_b^{2,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}$. Correspondingly, by the Hahn–Banach theorem, there exists

$$\tilde{u} \in (H_b^{0,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet})^* = H_b^{0,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}$$

such that

$$\langle \mathcal{P}\tilde{u}, \phi\rangle = \langle \tilde{u}, \mathcal{P}^*\phi\rangle = \langle \tilde{f}, \phi\rangle, \quad \phi \in H_b^{2,\tilde{r}}(\Omega_{[T'_-,T'_+]})^{-,\bullet}$$

and

$$\|\tilde{u}\|_{H_b^{0,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}} \le C\|\tilde{f}\|_{H_b^{-1,r}(\Omega_{[T'_-,T'_+]})^{\bullet,-}}. \tag{2-25}$$

One can regard $\tilde{u}$ as an element of $H_b^{0,r}(\Omega_{[T_-,T'_+]})^{\bullet,-}$ with support in $\Omega_{[T'_-,T'_+]}$, with $\tilde{f}$ similarly extended; then $\langle \mathcal{P}\tilde{u}, \phi\rangle = \langle \tilde{f}, \phi\rangle$ for $\phi \in \dot{C}_c^\infty(\Omega_{(T_-,T'_+)})$ (here the dot over C^∞ refers to infinite-order vanishing at $X = \partial M$!), so $\mathcal{P}\tilde{u} = \tilde{f}$ in distributions. Since $\tilde{u}$ vanishes on $\Omega_{(T_-,T'_-)}$ and

$$\tilde{f} \in H_b^{s-1,r}(\Omega_{[T_-,T'_+]})^{\bullet,-},$$

propagation of singularities applied on $\Omega_{(T_-,T'_+)}$ (which has only the boundary $\partial M = X$) gives that $\tilde{u} \in H_{b,\mathrm{loc}}^{s,r}(\Omega_{(T_-,T'_+)})$ (here we are ignoring the two boundaries, $\mathfrak{t}_j = T_-, T'_+$, not making a uniform statement there, but we are not ignoring $\partial M = X$). In addition, for $\chi, \tilde{\chi} \in C_c^\infty(\Omega_{(T_-,T'_+)})$ with $\tilde{\chi} \equiv 1$ on $\operatorname{supp}\chi$, we have the estimate

$$\|\chi\tilde{u}\|_{H_b^{s,r}(\Omega_{[T_-,T'_+]})} \le C\big(\|\tilde{\chi}\mathcal{P}\tilde{u}\|_{H_b^{s-1,r}(\Omega_{[T_-,T'_+]})} + \|\tilde{\chi}\tilde{u}\|_{H_b^{0,r}(\Omega_{[T_-,T'_+]})}\big). \tag{2-26}$$

In view of the support property of $\tilde{u}$, this gives that, restricting to $\Omega_{(T_-,T_1]}$, we obtain an element of $H_b^{s,r}(\Omega_{(T_-,T_1]})^-$ with support in $\Omega_{[T_0,T_1]}$, i.e., an element of $H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. The desired estimate follows from (2-25), controlling the second term of the right-hand side of (2-26), and (2-23) as well as using $\mathcal{P}\tilde{u} = \tilde{f}$. □

At this point, u given by Lemma 2.7 is not necessarily unique. However:

Lemma 2.9. *Let $s, r \in \mathbb{R}$. If $u \in H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ is such that $\mathcal{P}u = 0$, then $u = 0$.*

Proof. Propagation of singularities, as in the proof of Lemma 2.7, regarding u as a distribution on (T_-, T_1) with support in $[T_0, T_1)$ gives that $u \in H_{b,\mathrm{loc}}^{\infty,r}(\Omega_{(T_-,T_1)})$. Taking $T_0 < T'_1 < T_1$, letting $u' = u|_{[T_0,T'_1]}$, (2-21) shows that $u' = 0$. Since T'_1 is arbitrary, this shows $u = 0$. □

Corollary 2.10. *Let $s \ge 0$ and $r \in \mathbb{R}$. Then there is $C > 0$ with the following property*:

If $f \in H_b^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, then there exists a unique $u \in H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ such that $\mathcal{P}u = f$. Further, this unique u satisfies

$$\|u\|_{H_b^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \le C\|f\|_{H_b^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}}.$$

Proof. Existence is Lemma 2.7; uniqueness is linearity plus Lemma 2.9, which, together with the estimate in Lemma 2.7, prove the corollary. □

Corollary 2.11. *Let $s \geq 0$ and $r, \tilde{r} \in \mathbb{R}$. For $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ with $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$,*

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}} \tag{2-27}$$

while, for $v \in H_{\mathrm{b}}^{s,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}$ with $\mathcal{P}^*v \in H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}$,

$$\|v\|_{H_{\mathrm{b}}^{s,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}} \leq C\|\mathcal{P}^*v\|_{H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega_{[T_0,T_1]})^{-,\bullet}}. \tag{2-28}$$

Remark 2.12. Again, this estimate remains valid for vector-valued $\mathfrak{t}_j$ and $\tilde{\mathfrak{t}}_j$ — see Remarks 2.5 and 2.8 — under the linear independence condition of the latter.

Proof. It suffices to consider (2-27). Let $f = \mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ and let $u' \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ be given by Corollary 2.10. In view of the uniqueness statement of Corollary 2.10, $u = u'$. Then the estimate of Corollary 2.10 proves the claim. □

This yields the following kind of propagation of singularities result:

Proposition 2.13. *Let $s \geq 0$ and $r \in \mathbb{R}$. If $u \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega_{[T_0,T_1]})^{\bullet,-}$ with $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, then $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$.*

If instead $u \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega_{[T_0,T_1]})^{-,-}$ with $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{-,-}$ and, for some $\widetilde{T}_0 > T_0$, $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]} \setminus \Omega_{(\widetilde{T}_0,T_1]})^{-,-}$, then $u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{-,-}$.

Remark 2.14. One can "mix and match" the two parts of the proposition in the setting of Remark 2.5, with, say, a supportedness condition at $\tilde{\mathfrak{t}}_j$ and only an extendibility assumption at $\tilde{\mathfrak{t}}_k$, but with an $H_{\mathrm{b}}^{s,r}$ membership assumption on u in $\Omega_{[T_0,T_1]} \setminus \tilde{\mathfrak{t}}_k^{-1}((-\infty, \widetilde{T}_1))$, $\widetilde{T}_1 < T_1$, with a completely analogous argument. For instance, in the setting of Figure 4, one gets the regularity under supportedness assumptions at H_1, just extendibility at $\mathfrak{t}_2 = T_1$, but a priori regularity for $\mathfrak{t}_2 \in (\widetilde{T}_1, T_1)$.

Proof. Let $u' \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ be the unique solution in $H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ of $\mathcal{P}u' = f$ where $f = \mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$; we obtain u' by applying the existence part of Corollary 2.10. Then $u, u' \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega_{[T_0,T_1]})^{\bullet,-}$ and $\mathcal{P}(u-u') = 0$. Applying Lemma 2.9, we conclude that $u = u'$, which completes the proof of the first part.

For the second part, let $\chi \in C^\infty(\mathbb{R})$ be supported in (T_0, ∞), identically 1 near $[\widetilde{T}_0, \infty)$, and consider $u' = (\chi \circ \mathfrak{t}_j)u \in H_{\mathrm{b}}^{1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$, with the support property arising from the vanishing of χ near T_0. Then $\mathcal{P}u' = [\mathcal{P}, (\chi \circ \mathfrak{t}_j)]u + (\chi \circ \mathfrak{t}_j)\mathcal{P}u$. Now the first term on the right-hand side is in $H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$ because, on the support of $d\chi$, which is in $\Omega_{[T_0,T_1]} \setminus \Omega_{(\widetilde{T}_0,T_1]}$, u is in $H_{\mathrm{b}}^{s,r}$ and the commutator is first order, while the second term is in the desired space since $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega_{[T_0,T_1]})^{-,-}$, and, as for u itself, the cutoff improves the support property. Thus, the first part of the lemma is applicable, giving that $\chi u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{\bullet,-}$. Since $(1-\chi)u \in H_{\mathrm{b}}^{s,r}(\Omega_{[T_0,T_1]})^{-,-}$ by the a priori assumption, the conclusion follows. □

We take $T_0 = 0$ and thus consider, for $s \geq 0$,

$$\mathcal{P} : H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{\bullet,-} \tag{2-29}$$

$$\text{and} \quad \mathcal{P}^* : H_{\mathrm{b}}^{s,r}(\Omega)^{-,\bullet} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{-,\bullet}. \tag{2-30}$$

In combination with the real-principal-type propagation results and Proposition 2.1, this yields, under the nontrapping assumptions, much as in the complex absorbing case, that[9]

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}} + C\|u\|_{H_{\mathrm{b}}^{0,r}(\Omega)^{\bullet,-}}, \quad \beta r < -\tfrac{1}{2},\ s > 0, \tag{2-31}$$

and

$$\|u\|_{H_{\mathrm{b}}^{s,\tilde{r}}(\Omega)^{-,\bullet}} \leq C\|\mathcal{P}^*u\|_{H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega)^{-,\bullet}} + C\|u\|_{H_{\mathrm{b}}^{0,\tilde{r}}(\Omega)^{-,\bullet}}, \quad \beta\tilde{r} > s - \tfrac{1}{2},\ s > 0. \tag{2-32}$$

We could proceed as in the complex absorption case to make the space on the left-hand side include compactly into the "error term" on the right using the normal operators. As this imposes some constraints — see (2-14) — which, together with the requirements of the energy estimates, namely that the Sobolev order is nonnegative, mean that we would get slightly too strong restrictions on s — see Remark 2.20 — we proceed instead with a direct energy estimate. We thus assume that Ω is sufficiently small that there is a boundary defining function τ of M with $d\tau/\tau$ timelike on Ω, of the same timelike character as $\mathfrak{t}_2$, opposite to $\mathfrak{t}_1$. As explained in [Vasy 2013a, §7], in this case there is $C > 0$ such that, for $\Im\sigma > C$, $\widehat{P}(\sigma)$ is necessarily invertible.

The energy estimate is:

Lemma 2.15. *There exists $r_0 < 0$ such that, for $r \leq r_0$ and $-\tilde{r} \leq r_0$, there is $C > 0$ such that, for $u \in H_{\mathrm{b}}^{2,r}(\Omega)^{\bullet,-}$ and $v \in H_{\mathrm{b}}^{2,\tilde{r}}(\Omega)^{-,\bullet}$, one has*

$$\begin{aligned} \|u\|_{H_{\mathrm{b}}^{1,r}(\Omega)^{\bullet,-}} &\leq C\|\mathcal{P}u\|_{H_{\mathrm{b}}^{0,r}(\Omega)^{\bullet,-}}, \\ \|v\|_{H_{\mathrm{b}}^{1,\tilde{r}}(\Omega)^{-,\bullet}} &\leq C\|\mathcal{P}^*v\|_{H_{\mathrm{b}}^{0,\tilde{r}}(\Omega)^{-,\bullet}}. \end{aligned} \tag{2-33}$$

Proof. We run the argument of Lemma 2.4 globally on Ω using a timelike vector field (e.g., starting with $W = G(d\tau/\tau, \cdot)$) that has, as a multiplier, a sufficiently large positive power $\alpha = -2r$ of τ, that is, replacing (2-17) by

$$V = -i\tau^\alpha W.$$

Then the term with τ^α differentiated (which in (2-18) is included in the $\widetilde{R}^\sharp$ term), and thus possessing a factor of α, is used to dominate the other, "error", terms in (2-18), completing the proof of the lemma as in Lemma 2.4. □

This can be used as in Lemma 2.7 to provide solvability and, using the propagation of singularities — which in this case includes the use of Proposition 2.1, noting that $s - \frac{1}{2} > \beta r$ is automatically satisfied — improved regularity. In particular, we obtain the following analogues of Corollaries 2.10–2.11:

Corollary 2.16. *There is $r_0 < 0$ such that, for $r \leq r_0$ and $s \geq 0$, there is $C > 0$ with the following property: If $f \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$, then there exists a unique $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ such that $\mathcal{P}u = f$.*

Further, this unique u satisfies

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} \leq C\|f\|_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}}.$$

[9] In fact, the error term on the right-hand side can be taken to be supported in a smaller region, since, at H_1 in the first case and at H_2 in the second, there are no error terms due to the energy estimates (2-21), applied with $\mathcal{P}^*$ in place of $\mathcal{P}$ in the second case.

Corollary 2.17. *There is $r_0 < 0$ such that, if $r < r_0$, $-\tilde{r} < r_0$ and $s \geq 0$, then there is $C > 0$ such that the following holds*:

For $u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$ with $\mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}$, one has

$$\|u\|_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} \leq C\|\mathcal{P}u\|_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}} \tag{2-34}$$

while, for $v \in H_{\mathrm{b}}^{s,\tilde{r}}(\Omega)^{-,\bullet}$ with $\mathcal{P}^*v \in H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega)^{-,\bullet}$, one has

$$\|v\|_{H_{\mathrm{b}}^{s,\tilde{r}}(\Omega)^{-,\bullet}} \leq C\|\mathcal{P}^*v\|_{H_{\mathrm{b}}^{s-1,\tilde{r}}(\Omega)^{-,\bullet}}. \tag{2-35}$$

We restate Corollary 2.16 as an invertibility statement.

Theorem 2.18. *There is $r_0 < 0$ with the following property. Suppose $s \geq 0$, $r \leq r_0$, and let*

$$\mathcal{Y}^{s,r} = H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}, \quad \mathcal{X}^{s,r} = \{u \in H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} : \mathcal{P}u \in H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}\},$$

where $\mathcal{P}$ is a priori a map $\mathcal{P} : H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{\bullet,-}$. Then

$$\mathcal{P} : \mathcal{X}^{s,r} \to \mathcal{Y}^{s-1,r}$$

is a continuous, invertible map, with continuous inverse.

Remark 2.19. Both $\mathcal{Y}^{s,r}$ and $\mathcal{X}^{s,r}$ are complete, in the case of $\mathcal{X}^{s,r}$ with the natural norm being

$$\|u\|^2_{\mathcal{X}^{s,r}} = \|u\|^2_{H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}} + \|\mathcal{P}u\|^2_{H_{\mathrm{b}}^{s-1,r}(\Omega)^{\bullet,-}},$$

as follows by the continuity of $\mathcal{P}$ as a map $H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-2,r}(\Omega)^{\bullet,-}$ and the completeness of the b-Sobolev spaces $H_{\mathrm{b}}^{s,r}(\Omega)^{\bullet,-}$.

Remark 2.20. Using normal operators as in the discussion leading to Proposition 2.3, one would get the following statement: Suppose $s > 1$ and $s - \frac{3}{2} > \beta r$. Then, with $\mathcal{X}^{s,r}$ and $\mathcal{Y}^{s,r}$ as above, $\mathcal{P} : \mathcal{X}^{s,r} \to \mathcal{Y}^{s,r}$ is Fredholm. Here the main loss is that one needs to assume $s > 1$; this is done since, in the argument, one needs to take s' with $s' + 1 < s$ in order to transition the normal operator estimates from $N(\mathcal{P})u$ to $\mathcal{P}u$ and still have a compact inclusion, but the normal operator estimates need $s' \geq 0$ as, due to the boundary H_2, they are again based on energy estimates. Using the direct global energy estimate eliminates this loss, which is an artifact of combining local energy estimates with the b-theory. In particular, in the complex absorption setting, this problem does not arise, but, on the other hand, one need not have the forward support property of the solution.

The results of [Vasy 2013a] then are immediately applicable to obtain an expansion of the solutions; the main point of the following theorem being the elimination of the losses in differentiability in Vasy's Proposition 3.5 due to Proposition 2.1.

Theorem 2.21 (strengthened version of [Vasy 2013a, Proposition 3.5]). *Let M be a manifold with a nontrapping b-metric g as above, with boundary X and let τ be a boundary defining function, $\mathcal{P}$ as in* (2-15). *Suppose the domain Ω is as defined above and $d\tau/\tau$ is timelike.*

Let σ_j be the poles of $\widehat{\mathcal{P}}^{-1}$ and let ℓ be such that $\Im\sigma_j + \ell \notin \mathbb{N}$ for all j. Let $\phi \in C^\infty(\mathbb{R})$ be such that $\operatorname{supp}\phi \subset (0,\infty)$ and $\phi \circ \mathfrak{t}_1 \equiv 1$ near $X \cap \Omega$. Then, for $s > \frac{3}{2} + \beta\ell$, there are $m_{jl} \in \mathbb{N}$ such that solutions of $\mathcal{P}u = f$ with $f \in H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$ and $u \in H_{\mathrm{b}}^{s_0,r_0}(\Omega)^{\bullet,-}$, $s \geq s_0 \geq 1$, $s_0 - \frac{1}{2} > \beta r_0$, satisfy that, for some $a_{jl\kappa} \in C^\infty(X \cap \Omega)$,

$$u' = u - \sum_j \sum_{l \in \mathbb{N}} \sum_{\kappa \leq m_{jl}} \tau^{i\sigma_j + l} (\log\tau)^\kappa (\phi \circ \mathfrak{t}_1) a_{jl\kappa} \in H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}, \tag{2-36}$$

where the sum is understood to be over a finite set with $-\Im\sigma_j + l < \ell$.

Here the (semi)norms of both $a_{jl\kappa}$ in $C^\infty(X \cap \Omega)$ and u' in $H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}$ are bounded by a constant times that of f in $H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$.

The analogous result also holds if f possesses an expansion modulo $H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$, namely

$$f = f' + \sum_j \sum_{\kappa \leq m'_j} \tau^{\alpha_j} (\log\tau)^\kappa (\phi \circ \mathfrak{t}_1) a_{j\kappa}$$

with $f' \in H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$ and $a_{j\kappa} \in C^\infty(X \cap \Omega)$, where terms corresponding to the expansion of f are added to (2-36) *in the sense of the extended union of index sets* [Melrose 1993, §5.18], *recalled below in Definition 2.32.*

Remark 2.22. Here the factor $\phi \circ \mathfrak{t}_1$ is added to cut off the expansion away from H_1, thus assuring that u' is in the indicated space (a supported distribution).

Also, the sum over l is generated by the lack of dilation invariance of $\mathcal{P}$. If we take ℓ such that $-\Im\sigma_j > \ell - 1$ for all j then all the terms in the expansion arise directly from the resonances, thus $l = 0$ and $m_{j0} + 1$ is the order of the pole of $\widehat{\mathcal{P}}^{-1}$ at σ_j, with the $a_{j0\kappa}$ being resonant states.

Proof. First assume that $-\Im\sigma_j > \ell$ for every j; thus there are no terms subtracted from u in (2-36). We proceed as in [Vasy 2013a, Proposition 3.5], but use the propagation of singularities, in particular Propositions 2.1 and 2.13, to eliminate the losses. First, by the propagation of singularities, using $s_0 - \frac{1}{2} > \beta r_0$ and $s \geq s_0$, $s \geq 0$,

$$u \in H_{\mathrm{b}}^{s,r_0}(\Omega)^{\bullet,-}.$$

Thus, as $\mathcal{P} - N(\mathcal{P}) \in \tau\,\mathrm{Diff}_{\mathrm{b}}^2(M)$,

$$N(\mathcal{P})u = f - \tilde{f}, \quad \text{where} \quad \tilde{f} = (\mathcal{P} - N(\mathcal{P}))u \in H_{\mathrm{b}}^{s-2,r_0+1}(\Omega)^{\bullet,-}. \tag{2-37}$$

We apply [Vasy 2013a, Lemma 3.1] (using $s \geq s_0 \geq 1$), which is the lossless version of Vasy's Proposition 3.5 in the dilation-invariant case. Note that the lemma is stated on the normal operator space M_∞, which does not have a boundary face corresponding to H_2, i.e., $S_2 \times (0,\infty)$, with complex absorption being used instead. However, given the analysis on $X \cap \Omega$ discussed above, all the arguments go through essentially unchanged: this is a Mellin transform and contour deformation argument.

One thus obtains (2-36) with ℓ replaced by $\ell' = \min(\ell, r_0 + 1)$, except that $u = u' \in H_{\mathrm{b}}^{s-1,\ell'}(\Omega)^{\bullet,-}$, corresponding to the $\tilde{f}$ term in $N(\mathcal{P})u$ rather than $u = u' \in H_{\mathrm{b}}^{s,\ell'}(\Omega)^{\bullet,-}$, as desired. However, using $\mathcal{P}u = f \in H_{\mathrm{b}}^{s-1,\ell'}(\Omega)^{\bullet,-}$, we deduce by the propagation of singularities, using $s - 1 > \beta\ell' + \frac{1}{2}$, $s \geq 0$, that $u = u' \in H_{\mathrm{b}}^{s,\ell'}(\Omega)^{\bullet,-}$. If $\ell \le r_0 + 1$, we have proved (2-36). Otherwise we iterate, replacing r_0 by $r_0 + 1$. We thus reach the conclusion, (2-36), in finitely many steps.

If there are j such that $-\Im\sigma_j < \ell$ then, in the first step, when using [Vasy 2013a, Lemma 3.1], we obtain the partial expansion u_1 corresponding to $\ell' = \min(\ell, r_0 + 1)$ in place of ℓ; here we may need to decrease ℓ' by an arbitrarily small amount to make sure that ℓ' is not $-\Im\sigma_j$ for any j. Further, the terms of the partial expansion are annihilated by $N(\mathcal{P})$, so u' satisfies

$$\mathcal{P}u' = \mathcal{P}u - N(\mathcal{P})u_1 - (\mathcal{P} - N(\mathcal{P}))u_1 \in H_{\mathrm{b}}^{s-1,\ell'}(\Omega)^{\bullet,-}$$

as $(\mathcal{P} - N(\mathcal{P}))u_1 \in H_{\mathrm{b}}^{\infty,r_0+1}(\Omega)^{\bullet,-}$ in fact, due to the conormality of u_1 and $\mathcal{P} - N(\mathcal{P}) \in \tau\,\mathrm{Diff}_{\mathrm{b}}^2(M)$. Correspondingly, the propagation of singularities result is applicable as above to conclude that $u' \in H_{\mathrm{b}}^{s,\ell'}(\Omega)^{\bullet,-}$. If $\ell \le r_0 + 1$ we are done. Otherwise, we have better information on $\tilde{f}$ in the next step, namely

$$\tilde{f} = (\mathcal{P} - N(\mathcal{P}))u = (\mathcal{P} - N(\mathcal{P}))u' + (\mathcal{P} - N(\mathcal{P}))u_1$$

with the first term in $H_{\mathrm{b}}^{s-2,r_0+1}(\Omega)^{\bullet,-}$ (same as in the case first considered above, without relevant resonances), while the expansion of u_1 shows that $(\mathcal{P} - N(\mathcal{P}))u_1$ has a similar expansion, but with an extra power of τ (i.e., $\tau^{i\sigma_j}$ is shifted to $\tau^{i\sigma_j+1}$). We can now apply Vasy's Lemma 3.1 again; in the case of the terms arising from the partial expansion, u_1, there are now new terms corresponding to shifting the powers $\tau^{i\sigma_j}$ to $\tau^{i\sigma_j+1}$, as stated in the referred lemma, and possibly causing logarithmic terms if $\sigma_j - i$ is also a pole of $\widehat{\mathcal{P}}^{-1}$. Iterating in the same manner proves the theorem when $f \in H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$. When f has an expansion modulo $H_{\mathrm{b}}^{s-1,\ell}(\Omega)^{\bullet,-}$, the same argument works; [Vasy 2013a, Lemma 3.1] gives the terms with the extended union, which then further generate additional terms due to $\mathcal{P} - N(\mathcal{P})$, just as the resonance terms did. □

There is one problem with this theorem for the purposes of semilinear equations: the resonant terms with $\Im\sigma_j \ge 0$ which give rise to unbounded, or at most just bounded, terms in the expansion which become larger when one takes powers of these, or when one iteratively applies $\mathcal{P}^{-1}$ (with the latter being the only issue if $\Im\sigma_j = 0$ and the pole is simple).

Concretely, we now consider an asymptotically de Sitter space $(\widetilde{M}, \tilde{g})$. We then blow up a point p at the future boundary $\widetilde{X}_+$, as discussed in the introduction (see p. 1810), to obtain the analogue of the static model of de Sitter space $M = [\widetilde{M}; p]$ with the pulled back metric g, which is a b-metric near the front face (but away from the side face); let $\mathcal{P} = \Box_g - \lambda$. If $\widetilde{M}$ is actual de Sitter space, then M is the actual static model; otherwise, the metric of the asymptotically de Sitter space, frozen at p, induces a de Sitter metric, g_0, which is well defined at the front face of the blow-up M (but away from its side faces) as a b-metric. In particular, the resonances in the "static region" of any asymptotically de Sitter space are the same as in the static model of actual de Sitter space.

On actual de Sitter space, the poles of $\widehat{\mathcal{P}}^{-1}$ are those on the hyperbolic space in the interior of the light cone equipped with a potential, as described in [Vasy 2010, Lemma 7.11], or indeed in [Vasy 2013a, Proposition 4.2], where essentially the present notation is used.[10] As shown in [Vasy 2010,

[10] In [Vasy 2010, Lemma 7.11] $-\sigma^2$ plays the same role as σ^2 here or in [Vasy 2013a, Proposition 4.2].

Corollary 7.18], converted to our notation, the only possible poles are at

$$i\hat{s}_\pm(\lambda) - i\mathbb{N}, \quad \hat{s}_\pm(\lambda) = -\tfrac{1}{2}(n-1) \pm \sqrt{\tfrac{1}{4}(n-1)^2 - \lambda}. \tag{2-38}$$

In particular, when $\lambda = m^2$, $m > 0$, we conclude:

Lemma 2.23. *For $m > 0$, $\mathcal{P} = \Box_g - m^2$, with g induced by an asymptotically de Sitter metric as above, all poles of $\widehat{\mathcal{P}}^{-1}$ have strictly negative imaginary part.*

In other words, for small mass $m > 0$, there are no resonances σ of the Klein–Gordon operator with $\Im\sigma \geq -\epsilon_0$ for some $\epsilon_0 > 0$. Therefore, the expansion of u as in (2-36) no longer has a constant term. Let us fix such $m > 0$ and $\epsilon_0 > 0$, which ensures that, for $0 < \epsilon < \epsilon_0$, the only term in the asymptotic expansion (2-36), when $s > \frac{1}{2} + \epsilon$ and $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$, is the "remainder" term $u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$. Here we use that $\beta = 1$ in de Sitter space, hence also on an asymptotically de Sitter space; see [Vasy 2013a, §4.4], in particular the second displayed equation after Equation (4.16) there, which computes β in accordance with Remark 2.2.

Being interested in finding *forward solutions* to (nonlinear) wave equations on *asymptotically de Sitter spaces*, we now define the forward solution operator

$$S_{\mathrm{KG}} : H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \tag{2-39}$$

using Theorems 2.18 and 2.21.

Remark 2.24. If $\widetilde{M} \subset M$ is an asymptotically de Sitter space with global time function t, $\tau = e^{-t}$ is the defining function for future infinity, and the domain Ω is such that $\Omega \cap \widetilde{M} = \{\tau < \tau_0\}$, then S_{KG} in fact restricts to a forward solution operator on $\widetilde{M}$ itself; indeed, if $E : H_{\mathrm{b}}^{s-1,\epsilon}(\{\tau < \tau_0\}) \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ is an extension operator, then the forward solution operator on $\{\tau < \tau_0\}$ is given by extending $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\{\tau < \tau_0\})$ using E, finding the forward solution on Ω using S_{KG}, and restricting back to $\{\tau < \tau_0\}$. The result is independent of the extension operator, as is easily seen from standard energy estimates; see in particular [Vasy 2013a, Proposition 3.9].

2B. ***A class of semilinear equations.*** Let us fix $m > 0$ and $\epsilon_0 > 0$ as above for statements about semilinear equations involving the Klein–Gordon operator; for equations involving the wave operator only, let $-\epsilon_0$ be equal to the largest imaginary part of all nonzero resonances of $\Box_g$. In Theorem 2.25 and further in the subsequent sections, bundles like ${}^{\mathrm{b}}T^*\Omega$ refer to ${}^{\mathrm{b}}T^*_\Omega M$; the boundaries H_j of Ω are regarded as artificial and do not affect the cotangent bundle or the corresponding vector fields.

Theorem 2.25. *Let $0 \leq \epsilon < \epsilon_0$ and $s > \frac{3}{2} + \epsilon$. Moreover, let*

$$q : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

be a continuous function with $q(0,0) = 0$ such that there exists a continuous nondecreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying

$$\|q(u, {}^{\mathrm{b}}du) - q(v, {}^{\mathrm{b}}dv)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R,$$

where we use the norms corresponding to the map q. Then there is a constant $C_L > 0$ such that the following holds: If $L(0) < C_L$, then for small $R > 0$ there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \le C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du) \tag{2-40}$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\| \le R$, that depends continuously on f.

More generally, suppose

$$q : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

satisfies $q(0,0,0) = 0$ and

$$\|q(u, {}^{\mathrm{b}}du, w) - q(u', {}^{\mathrm{b}}du', w')\| \le L(R)(\|u-u'\| + \|w-w'\|)$$

provided $\|u\| + \|w\|$, $\|u'\| + \|w'\| \le R$, where we use the norms corresponding to the map q, for a continuous nondecreasing function $L : \mathbb{R}_{\ge 0} \to \mathbb{R}$. Then there is a constant $C_L > 0$ such that the following holds: If $L(0) < C_L$, then for small $R > 0$ there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \le C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du, \Box_g u) \tag{2-41}$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|\Box_g u\|_{H_{\mathrm{b}}^{s-1,\epsilon}} \le R$, that depends continuously on f.

Further, if $\epsilon > 0$ and the nonlinearity is of the form $q({}^{\mathrm{b}}du)$, with

$$q : H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

having a small Lipschitz constant near 0, then for small $R > 0$ there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \le C$, the equation

$$\Box_g u = f + q({}^{\mathrm{b}}du)$$

has a unique solution u with $u - (\phi \circ \mathfrak{t}_1)c = u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, where $c \in \mathbb{C}$, that depends continuously on f, i.e., $c \in \mathbb{C}$ and $u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ depend continuously on f. Here, $\phi \in C^\infty(\mathbb{R})$ with support in $(0,\infty)$ and $\mathfrak{t}_1$ are as in Theorem 2.21. In fact, the statement even holds for nonlinearities $q(u, {}^{\mathrm{b}}du)$ provided

$$q : (\mathbb{C}(\phi \circ \mathfrak{t}_1) \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)) \times H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

has a small Lipschitz constant near 0.

Proof. To prove the first part, let S_{KG} be the forward solution operator for $\Box_g - m^2$ as in (2-39). We want to apply the Banach fixed point theorem to the operator $T_{\mathrm{KG}} : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $T_{\mathrm{KG}}u = S_{\mathrm{KG}}(f + q(u, {}^{\mathrm{b}}du))$.

Let $C_L = \|S_{\mathrm{KG}}\|^{-1}$; then we have the estimate

$$\|T_{\mathrm{KG}}u - T_{\mathrm{KG}}v\| \le \|S_{\mathrm{KG}}\| L(R')\|u - v\| \le C_0\|u - v\| \tag{2-42}$$

for $\|u\|$, $\|v\| \leq R$ and a constant $C_0 < 1$, granted that $L(R) \leq C_0 \|S_{\mathrm{KG}}\|^{-1}$, which holds for small $R > 0$ by assumption on L. Then, T_{KG} maps the R-ball in $H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ into itself if $\|S_{\mathrm{KG}}\|(\|f\| + L(R)R) \leq R$, i.e., if $\|f\| \leq R(\|S_{\mathrm{KG}}\|^{-1} - L(R))$. Put

$$C = R(\|S_{\mathrm{KG}}\|^{-1} - L(R)).$$

Then the existence of a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ with $\|u\| \leq R$ to the PDE (2-40) with $\|f\|_{H_{\mathrm{b}}^{s-1,\epsilon}} \leq C$ follows from the Banach fixed point theorem.

To prove the continuous dependence of u on f, suppose we are given $u_j \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $j = 1, 2$, with $\|u_j\| \leq R$, and $f_j \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f_j\| \leq C$, such that

$$(\Box_g - m^2)u_j = f_j + q(u_j, {}^{\mathrm{b}}du_j), \quad j = 1, 2.$$

Then

$$(\Box_g - m^2)(u_1 - u_2) = f_1 - f_2 + q(u_1, {}^{\mathrm{b}}du_1) - q(u_2, {}^{\mathrm{b}}du_2),$$

hence

$$\|u_1 - u_2\| \leq \|S_{\mathrm{KG}}\|\big(\|f_1 - f_2\| + L(R)\|u_1 - u_2\|\big),$$

which in turn gives

$$\|u_1 - u_2\| \leq \frac{\|f_1 - f_2\|}{1 - C_0}.$$

This completes the proof of the first part.

For the more general statement, we use the fact that one can think of $\Box_g$ in the nonlinearity as a first-order operator. Concretely, we work on the coisotropic space

$$\mathscr{X} = \{u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} : \Box_g u \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}\}$$

with norm

$$\|u\|_{\mathscr{X}} = \|u\|_{H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}} + \|\Box_g u\|_{H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}}.$$

This is a Banach space: if (u_k) is a Cauchy sequence in $\mathscr{X}$, then $u_k \to u$ in $H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ and $\Box_g u_k \to v$ in $H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$; in particular,

$$\Box_g u_k \to \Box_g u \quad \text{and} \quad \Box_g u_k \to v \quad \text{in } \tau^\epsilon H_{\mathrm{b}}^{s-2}(\Omega)^{\bullet,-},$$

thus $\Box_g u = v \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$, which was to be shown. We then define $T_{\mathrm{KG}} : \mathscr{X} \to \mathscr{X}$ by $T_{\mathrm{KG}} u = S_{\mathrm{KG}}(f + q(u, {}^{\mathrm{b}}du, \Box_g u))$ and obtain the estimate

$$\begin{aligned}
\|T_{\mathrm{KG}} u - T_{\mathrm{KG}} v\|_{\mathscr{X}} &= \|T_{\mathrm{KG}} u - T_{\mathrm{KG}} v\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|q(u, {}^{\mathrm{b}}du, \Box_g u) - q(v, {}^{\mathrm{b}}dv, \Box_g v)\|_{H_{\mathrm{b}}^{s-1,\epsilon}} \\
&\leq (\|S_{\mathrm{KG}}\| + 1)L(R)(\|u - v\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|\Box_g u - \Box_g v\|_{H_{\mathrm{b}}^{s-1,\epsilon}}) \\
&= (\|S_{\mathrm{KG}}\| + 1)L(R)\|u - v\|_{\mathscr{X}} \leq C_0 \|u - v\|_{\mathscr{X}}
\end{aligned}$$

for u, $v \in \mathscr{X}$ with norms bounded by R, with $C_0 < 1$ if $R > 0$ is small enough, provided we require $L(0) < C_L := (\|S_{\mathrm{KG}}\| + 1)^{-1}$. Then, for $u \in \mathscr{X}$ with $\|u\| \leq R$,

$$\|T_{\mathrm{KG}} u\|_{\mathscr{X}} \leq (\|S_{\mathrm{KG}}\| + 1)(\|f\|_{H_{\mathrm{b}}^{s-1,\epsilon}} + L(R)R) \leq R$$

if $\|f\| \leq C$ with $C > 0$ small. Thus, T_{KG} is a contraction on $\mathscr{X}$, and we obtain the solvability of (2-41). The continuous dependence of the solution on the forcing term f is proved as above.

For the third part, we use the forward solution operator $S : H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-} \to \mathscr{Y} := \mathbb{C} \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ for $\Box_g$; note that $\mathscr{Y}$ is a Banach space with norm $\|(c,u')\|_{\mathscr{Y}} = |c| + \|u'\|_{H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}}$. (See Section 2C for related, more general statements.) We will apply the Banach fixed point theorem to the operator $T : \mathscr{Y} \to \mathscr{Y}$, $Tu = S(f + q(u, {}^{\mathrm{b}}du))$: we again have an estimate like (2-42), since ${}^{\mathrm{b}}du \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-}$ for $u \in \mathscr{Y}$ and, for small $R > 0$, T maps the R-ball around 0 in $\mathscr{Y}$ into itself if the norm of f in $H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ is small, as above. The continuous dependence of the solution on the forcing term is proved as above. $\square$

The following basic statement ensures that there are interesting nonlinearities q that satisfy the requirements of the theorem; see also Section 2C.

Lemma 2.26. *Let $s > \frac{1}{2}n$; then $H_{\mathrm{b}}^s(\mathbb{R}_+^n)$ is an algebra. In particular, $H_{\mathrm{b}}^s(N)$ is an algebra on any compact n-dimensional manifold N with boundary which is equipped with a b-metric.*

Proof. The first statement is the special case $k = 0$ of Lemma 4.4 after a logarithmic change of coordinates, which gives an isomorphism $H_{\mathrm{b}}^s(\mathbb{R}_+^n) \cong H^s(\mathbb{R}^n)$; the lemma is well known in this case (see, e.g., [Taylor 1997, Chapter 13.3]). The second statement follows by localization and from the coordinate invariance of H_{b}^s. $\square$

More, related statements will be given in Section 4B.

Remark 2.27. The algebra property of $H_{\mathrm{b}}^s(N)$ for $s > \frac{1}{2}\dim(N)$ is a special case of the fact that, for any $F \in C^\infty(\mathbb{R})$ (for real-valued u) or $F \in C^\infty(\mathbb{C})$ (for complex-valued u) with $F(0) = 0$, the composition map $H_{\mathrm{b}}^s(N) \to H_{\mathrm{b}}^s(N)$, $u \mapsto F \circ u$, is well defined and continuous; see, for example, [Taylor 1997, Chapter 13.10]. In the real-valued u case, if $F(0) \neq 0$ then writing $F(t) = F(0) + tF_1(t)$ shows that $F \circ u \in \mathbb{C} + H_{\mathrm{b}}^s(N)$. If $r > 0$, then $H_{\mathrm{b}}^{s,r}(N) \subset H_{\mathrm{b}}^s(N)$ shows that $F_1(u) \in H_{\mathrm{b}}^s(N)$, thus $F \circ u = F(0) + uF_1(u) \in \mathbb{C} + H_{\mathrm{b}}^{s,r}(N)$; and, if F vanishes to order k at 0, then $F(t) = t^k F_k(t)$, so $F \circ u = u^k(F_k \circ u)$, and the multiplicative properties of $H_{\mathrm{b}}^{s,r}(N)$ show that $F \circ u \in H_{\mathrm{b}}^{s,kr}(N)$. The argument is analogous for complex-valued u, indeed for $\mathbb{R}^L$-valued u, using Taylor's theorem on F at the origin.

Corollary 2.28. *If $s > \frac{1}{2}n$, the hypotheses of Theorem 2.25 hold for nonlinearities $q(u) = cu^p$, $p \geq 2$ an integer, $c \in \mathbb{C}$, as well as $q(u) = q_0 u^p$, $q_0 \in H_{\mathrm{b}}^s(M)$.*

If $s - 1 > \frac{1}{2}n$, the hypotheses of Theorem 2.25 hold for nonlinearities

$$q(u, {}^{\mathrm{b}}du) = \sum_{2 \leq j + |\alpha| \leq d} q_{j\alpha} u^j \prod_{k \leq |\alpha|} X_{\alpha,k} u, \tag{2-43}$$

where $q_{j,\alpha} \in \mathbb{C} + H_{\mathrm{b}}^s(M)$, $X_{\alpha,k} \in \mathcal{V}_{\mathrm{b}}(M)$.

Thus, in either case, for $m > 0$, $0 \leq \epsilon < \epsilon_0$, $s > \frac{3}{2} + \epsilon$ and for small $R > 0$, there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du) \tag{2-44}$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

The analogous conclusion also holds for $\Box_g u = f + q(u, {}^{\mathrm{b}}du)$ provided $\epsilon > 0$ and

$$q(u, {}^{\mathrm{b}}du) = \sum_{\substack{2 \leq j + |\alpha| \leq d \\ |\alpha| \geq 1}} q_{j\alpha} u^j \prod_{k \leq |\alpha|} X_{\alpha,k} u, \tag{2-45}$$

with the solution being in $\mathbb{C}(\phi \circ \mathfrak{t}_1) \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $\phi \circ \mathfrak{t}_1$ identically 1 near $X \cap \Omega$ and vanishing near H_1.

Remark 2.29. For such polynomial nonlinearities, the Lipschitz constant $L(R)$ in the statement of Theorem 2.25 satisfies $L(0) = 0$.

Remark 2.30. In this paper, we do not prove that one obtains smooth (i.e., conormal) solutions if the forcing term is smooth (conormal); see [Hintz 2013] for such a result in the quasilinear setting.

Since in Theorem 2.25 we allow q to depend on $\Box_g u$, we can in particular solve certain quasilinear equations if $s > \max\left(\frac{1}{2} + \epsilon, \frac{1}{2}n + 1\right)$: Suppose for example that $q' : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1}(\Omega)^{\bullet,-}$ is continuous with $\|q'(u) - q'(v)\| \leq L'(R)\|u - v\|$ for $u, v \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ with norms bounded by R, where $L' : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is locally bounded; then we can solve the equation

$$(1 + q'(u))(\Box_g - m^2)u = f \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$$

provided the norm of f is small. Indeed, if we put $q(u, w) = -q'(u)(w - m^2 u)$, then $q(u, \Box_g u) = -q'(u)(\Box_g - m^2)u$ and the PDE becomes

$$(\Box_g - m^2)u = f + q(u, \Box_g u),$$

which is solvable by Theorem 2.25, since, with $\|\cdot\| = \|\cdot\|_{H_{\mathrm{b}}^{s-1,\epsilon}}$, for u, $u' \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ and $w, w' \in H_{\mathrm{b}}^{s-1,\epsilon}(\Omega)^{\bullet,-}$ with $\|u\| + \|w\|, \|u'\| + \|w'\| \leq R$, we have

$$\begin{aligned}\|q(u, w) - q(u', w')\| &\leq \|q'(u) - q'(u')\| \, \|w - m^2 u\| + \|q'(u')\| \, \|w - w' - m^2(u - u')\| \\ &\leq L'(R)((1 + m^2)R + m^2 R)\|u - u'\| + L'(R)R\|w - w'\| \\ &\leq L(R)(\|u - u'\| + \|w - w'\|)\end{aligned}$$

with $L(R) \to 0$ as $R \to 0$.

By a similar argument, one can also allow q' to depend on ${}^{\mathrm{b}}du$ and $\Box_g u$.

Remark 2.31. Recalling the discussion following Theorem 2.21, let us emphasize the importance of $\widehat{P}(\sigma)^{-1}$ having no poles in the closed upper half plane by looking at the explicit example of the operator $\mathcal{P} = \partial_x$ in 1 dimension. In terms of $\tau = e^{-x}$, we have $\mathcal{P} = -\tau\partial_\tau$, thus $\widehat{P}(\sigma) = -i\sigma$, considered as an operator on the boundary (which is a single point) at $+\infty$ of the radial compactification of $\mathbb{R}$; hence $\widehat{P}(\sigma)^{-1}$ has a simple pole at $\sigma = 0$, corresponding to constants being annihilated by $\mathcal{P}$. Now suppose we want to find a forward solution of $u' = u^2 + f$, where $f \in C_c^\infty(\mathbb{R})$. In the first step of the iterative procedure described above, we will obtain a constant term; the next step gives a term that is linear in x (x being the antiderivative of 1), i.e., in $\log\tau$, then we get quadratic terms and so on, therefore the iteration does not converge (for general f), which is of course to be expected, since solutions to $u' = u^2 + f$ in general blow up in finite time. On the other hand, if $\mathcal{P} = \partial_x + 1$ then $\widehat{P}(\sigma)^{-1} = (1 - i\sigma)^{-1}$, which has a simple pole at $\sigma = -i$, which means that forward solutions u of $u' + u = u^2 + f$ with f as above can be constructed iteratively and the first term of the expansion of u at $+\infty$ is $c\tau^{i(-i)} = ce^{-x}$, $c \in \mathbb{C}$.

2C. ***Semilinear equations with polynomial nonlinearity.*** With polynomial nonlinearities as in (2-43), we can use the second part of Theorem 2.21 to obtain an asymptotic expansion for the solution; see Remark 2.38 and, in a slightly different setting, Section 3B for details on this. Here, we instead define a space that encodes asymptotic expansions directly in such a way that we can run a fixed point argument directly.

To describe the exponents appearing in the expansion, we use index sets, as introduced by Melrose [1993].

Definition 2.32. (1) An *index set* is a discrete subset $\mathcal{E}$ of $\mathbb{C} \times \mathbb{N}_0$ satisfying the conditions

(a) if $(z, k) \in \mathcal{E}$ then $(z, j) \in \mathcal{E}$ for $0 \le j \le k$, and

(b) if (z_j, k_j) is a sequence of elements of $\mathcal{E}$ with $|z_j| + k_j \to \infty$, then $\Re z_j \to \infty$.

(2) For any index set $\mathcal{E}$, define

$$w_{\mathcal{E}}(z) = \begin{cases} \max\{k \in \mathbb{N}_0 : (z,k) \in \mathcal{E}\} & \text{if } (z,0) \in \mathcal{E}, \\ -\infty & \text{otherwise.} \end{cases}$$

(3) For two index sets $\mathcal{E}$ and $\mathcal{E}'$, define their extended union by

$$\mathcal{E} \mathbin{\overline{\cup}} \mathcal{E}' = \mathcal{E} \cup \mathcal{E}' \cup \{(z, l + l' + 1) : (z,l) \in \mathcal{E},\ (z,l') \in \mathcal{E}'\}$$

and their product by $\mathcal{E}\mathcal{E}' = \{(z + z', l + l') : (z,l) \in \mathcal{E},\ (z', l') \in \mathcal{E}'\}$. We shall write $\mathcal{E}^k$ for the k-fold product of $\mathcal{E}$ with itself.

(4) A *positive index set* is an index set $\mathcal{E}$ with the property that $\Re z > 0$ for all $z \in \mathbb{C}$ with $(z, 0) \in \mathcal{E}$.

Remark 2.33. To ensure that the class of polyhomogeneous conormal distributions with a given index set $\mathcal{E}$ is invariantly defined, [Melrose 1993] in addition requires that $(z, k) \in \mathcal{E}$ implies $(z + j, k) \in \mathcal{E}$ for all $j \in \mathbb{N}_0$. In particular, this is a natural condition in non-dilation-invariant settings, as in Theorem 2.21. A convenient way to enforce this condition in all relevant situations is to enlarge the index set corresponding to the poles of the inverse of the normal operator accordingly; see the statement of Theorem 2.37.

Observe though that this condition is not needed in the dilation-invariant cases of the solvability statements below.

Since we want to capture the asymptotic behavior of solutions near $X \cap \Omega$, we fix a cutoff $\phi \in C^\infty(\mathbb{R})$ with support in $(0, \infty)$ such that $\phi \circ \mathfrak{t}_1 \equiv 1$ near $X \cap \Omega$ (we already used such a cutoff in Theorem 2.21), and make the following definition:

Definition 2.34. Let $\mathcal{E}$ be an index set, and let $s, r \in \mathbb{R}$. For $\epsilon > 0$ with the property that there is no $(z, 0) \in \mathcal{E}$ with $\Re z = \epsilon$, define the space $\mathcal{X}^{s,r,\epsilon}_{\mathcal{E}}$ to consist of all tempered distributions v on M with support in $\overline{\Omega}$ such that

$$v' = v - \sum_{\substack{(z,k)\in\mathcal{E} \\ \Re z<\epsilon}} \tau^z (\log\tau)^k (\phi\circ \mathfrak{t}_1) v_{z,k} \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \tag{2-46}$$

for certain $v_{z,k} \in H^r(X\cap\Omega)$.

Observe that the terms $v_{z,k}$ in the expansion (2-46) are uniquely determined by v, since $\epsilon > \Re z$ for all $z\in\mathbb{C}$ for which $(z,0)$ appears in the sum (2-46); then also v' is uniquely determined by v. Therefore, we can use the isomorphism

$$\mathcal{X}_{\mathcal{E}}^{s,r,\epsilon} \cong \biggl(\bigoplus_{\substack{(z,k)\in\mathcal{E} \\ \Re z<\epsilon}} H^r(X\cap\Omega)\biggr) \oplus H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$$

to give $\mathcal{X}_{\mathcal{E}}^{s,r,\epsilon}$ the structure of a Banach space.

Lemma 2.35. *Let $\mathcal{P}$ and $\mathcal{F}$ be positive index sets, and let $\epsilon > 0$. Define $\mathcal{E}_0' = \mathcal{P}\mathbin{\overline{\cup}}\mathcal{F}$ and, recursively, $\mathcal{E}_{N+1}' = \mathcal{P}\mathbin{\overline{\cup}}\bigl(\mathcal{F}\cup\bigcup_{k\geq 2}(\mathcal{E}_N')^k\bigr)$; put $\mathcal{E}_N = \{(z,k)\in\mathcal{E}_N' : 0<\Re z\leq\epsilon\}$. Then there exists $N_0\in\mathbb{N}$ such that $\mathcal{E}_N = \mathcal{E}_{N_0}$ for all $N\geq N_0$; moreover, the limiting index set $\mathcal{E}_\infty(\mathcal{P},\mathcal{F},\epsilon) := \mathcal{E}_{N_0}$ is finite.*

Proof. Writing $\pi_1 : \mathbb{C}\times\mathbb{N}_0\to\mathbb{C}$ for the projection, one has

$$\pi_1\mathcal{E}_1 = \biggl\{z = \sum_{j=1}^k z_j \ :\ 0<\Re z\leq\epsilon,\ k\geq 1,\ z_j\in\pi_1\mathcal{E}_0\biggr\},$$

and it is then clear that $\pi_1\mathcal{E}_N = \pi_1\mathcal{E}_1$ for all $N\geq 1$. Since $\mathcal{E}_0$ is a positive index set, there exists $\delta>0$ such that $\Re z\geq\delta$ for all $z\in\mathcal{E}_0$; hence, $\pi_1\mathcal{E}_\infty = \pi_1\mathcal{E}_1$ is finite.

To finish the proof, we need to show that, for all $z\in\mathbb{C}$, the number $w_{\mathcal{E}_N}(z)$ stabilizes. Defining $p(z) = w_{\mathcal{P}}(z)+1$ for $z\in\pi_1\mathcal{P}$ and $p(z)=0$ otherwise, we have a recursion relation

$$w_{\mathcal{E}_N}(z) = p(z) + \max\biggl\{w_{\mathcal{F}}(z),\ \max_{\substack{z=z_1+\cdots+z_k \\ k\geq 2,\, z_j\in\pi_1\mathcal{E}_\infty}} \biggl\{\sum_{j=1}^k w_{\mathcal{E}_{N-1}}(z_j)\biggr\}\biggr\},\quad N\geq 1. \tag{2-47}$$

For each z_j appearing in the sum, we have $\Im z_j\leq\Im z-\delta$. Thus, we can use (2-47) with z replaced by such z_j and N replaced by $N-1$ to express $w_{\mathcal{E}_N}(z)$ in terms of a finite number of $p(z_\alpha)$ and $w_{\mathcal{F}}(z_\alpha)$, $\Im z_\alpha\leq\Im z$, and a finite number of $w_{\mathcal{E}_{N-2}}(z_\beta)$, $z_\beta\leq\Im z-2\delta$. Continuing in this way, after $N_0 = \lfloor(\Im z)/\delta\rfloor+1$ steps we have expressed $w_{\mathcal{E}_N(z)}$ in terms of a finite number of $p(z_\gamma)$ and $w_{\mathcal{F}}(z_\gamma)$, $\Im z_\gamma\leq\Im z$, only, and this expression is independent of N as long as $N\geq N_0$. □

Definition 2.36. Let $\mathcal{P}$ and $\mathcal{F}$ be positive index sets and let $\epsilon>0$ be such that there is no $(z,0)$ in $\mathcal{E}_\infty(\mathcal{P},\mathcal{F},\epsilon)$ with $\Re z=\epsilon$, with $\mathcal{E}_\infty(\mathcal{P},\mathcal{F},\epsilon)$ as defined in the statement of Lemma 2.35. Then, for s, $r\in\mathbb{R}$, define the Banach spaces

$$\mathcal{X}_{\mathcal{P},\mathcal{F}}^{s,r,\epsilon} := \mathcal{X}_{\mathcal{E}_\infty(\mathcal{P},\mathcal{F},\epsilon)}^{s,r,\epsilon},$$
$${}^0\mathcal{X}_{\mathcal{P},\mathcal{F}}^{s,r,\epsilon} := \mathcal{X}_{\mathcal{E}_\infty(\mathcal{P},\mathcal{F},\epsilon)\cup\{(0,0)\}}^{s,r,\epsilon}.$$

Note that the spaces ${}^{(0)}\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$ are Banach algebras for $s > \frac{1}{2}n$ in the sense that there is a constant $C > 0$ such that $\|uv\| \le C\|u\|\,\|v\|$ for all $u,\ v \in {}^{(0)}\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$. Moreover, $\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$ interacts well with the forward solution operator S_{KG} of $\Box_g - m^2$ in the sense that $u \in \mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$ and $k \ge 2$ — with $\mathscr{P}$ being related to the poles of $\widehat{\mathcal{P}}(\sigma)^{-1}$, where $\mathcal{P} = \Box_g - m^2$, as will be made precise in the statement of Theorem 2.37 below — implies $S_{\mathrm{KG}}(u^k) \in \mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$.

We can now state the result giving an asymptotic expansion of the solution of $(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du)$ for polynomial nonlinearities q.

Theorem 2.37. *Let $\epsilon > 0$, $s > \max\big(\frac{3}{2} + \epsilon, \frac{1}{2}n + 1\big)$, and q as in* (2-43). *Moreover, if $\sigma_j \in \mathbb{C}$ are the poles of the inverse family $\widehat{\mathcal{P}}(\sigma)^{-1}$, where $\mathcal{P} = \Box_g - m^2$, and $m_j + 1$ is the order of the pole of $\widehat{\mathcal{P}}(\sigma)^{-1}$ at σ_j, let $\mathscr{P} = \{(i\sigma_j + k, \ell) : 0 \le \ell \le m_j,\ k \in \mathbb{N}_0\}$. Assume that $\epsilon \ne \Re(i\sigma_j)$ for all j and that, moreover, $m > 0$, which implies that $\mathscr{P}$ is a positive index set; see Lemma 2.23. Finally, let $\mathscr{F}$ be a positive index set.*

Then, for small enough $R > 0$, there exists $C > 0$ such that, for all $f \in \mathscr{X}^{s-1,s-1,\epsilon}_{\mathscr{F}}$ with $\|f\| \le C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^{\mathrm{b}}du)$$

has a unique solution $u \in \mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, with $\|u\| \le R$, that depends continuously on f; in particular, u has an asymptotic expansion with remainder term in $H^{s,\epsilon}_{\mathrm{b}}(\Omega)^{\bullet,-}$.

Further, if the polynomial nonlinearity is of the form $q({}^{\mathrm{b}}du)$ then, for small $R > 0$, there exists $C > 0$ such that, for all $f \in \mathscr{X}^{s-1,s-1,\epsilon}_{\mathscr{F}}$ with $\|f\| \le C$, the equation

$$\Box_g u = f + q({}^{\mathrm{b}}du)$$

has a unique solution $u \in {}^{0}\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, with $\|u\| \le R$, that depends continuously on f.

Proof. By Theorem 2.21 and the definition of the space $\mathscr{X} = \mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, we have a forward solution operator $S_{\mathrm{KG}} : \mathscr{X} \to \mathscr{X}$ of $\Box_g - m^2$. Thus, we can apply the Banach fixed point theorem to the operator $T : \mathscr{X} \to \mathscr{X}$, $Tu = S_{\mathrm{KG}}(f + q(u, {}^{\mathrm{b}}du))$, where we note that $q : \mathscr{X} \to \mathscr{X}$, which follows from the definition of $\mathscr{X}$ and the fact that q is a polynomial only involving terms of the form $u^j \prod_{k \le |\alpha|} X_{\alpha,k} u$ for $j + |\alpha| \ge 2$. This condition on q also ensures that T is a contraction on a sufficiently small ball in $\mathscr{X}_+$.

For the second part, writing ${}^{0}\mathscr{X} = {}^{0}\mathscr{X}^{s,s,\epsilon}_{\mathscr{P},\mathscr{F}}$, we have a forward solution operator $S : \mathscr{X} \to {}^{0}\mathscr{X}$. But $q({}^{\mathrm{b}}du) : {}^{0}\mathscr{X} \to \mathscr{X}$, since ${}^{\mathrm{b}}d$ annihilates constants, and we can thus finish the proof as above.

The continuous dependence of the solution on the right-hand side is proved as in Theorem 2.25. ☐

Note that $\epsilon > 0$ is (almost) unrestricted here, and thus we can get arbitrarily many terms in the asymptotic expansion if we work with arbitrarily high Sobolev spaces.

The condition that the polynomial $q(u, {}^{\mathrm{b}}du)$ does not involve a linear term is very important as it prevents logarithmic terms from stacking up in the iterative process used to solve the equation. Also, adding a term νu to $q(u, {}^{\mathrm{b}}du)$ effectively changes the Klein–Gordon parameter from $-m^2$ to $\nu - m^2$, which changes the location of the poles of $\widehat{P}(\sigma)^{-1}$; in the worst case, if $\nu > m^2$, this would even cause a pole to move to $\Im\sigma > 0$, corresponding to a resonant state that blows up exponentially in time. Lastly, let us remark that the form (2-45) of the nonlinearity is not sufficient to obtain an expansion beyond leading order, since, in the iterative procedure, logarithmic terms would stack up in the next-to-leading-order term of the expansion.

Remark 2.38. Instead of working with the spaces ${}^{(0)}\mathcal{X}^{s,s,\epsilon}_{\mathcal{P},\mathcal{F}}$, which have the expansion built in, one could alternatively first prove the existence of a solution u in a (slightly) decaying b-Sobolev space, which then allows one to regard the polynomial nonlinearity as a perturbation of the linear operator $\Box_g - m^2$; then an iterative application of the dilation-invariant result [Vasy 2013a, Lemma 3.1] gives an expansion of the solution to the nonlinear equation. We will follow this idea in the discussion of polynomial nonlinearities on asymptotically Kerr-de Sitter spaces in the next section.

3. Kerr–de Sitter space

In this section we analyze semilinear waves on Kerr–de Sitter space and, more generally, on spaces with normally hyperbolic trapping, discussed below. The effect of the latter is a loss of derivatives for the linear estimates in general, but we show that at least derivatives with principal symbol vanishing on the trapped set are well behaved. We then use these results to solve semilinear equations in the rest of the section.

3A. ***Linear Fredholm theory.*** The linear theorem in the case of normally hyperbolic trapping for dilation-invariant operators $\mathcal{P} = \Box_g - \lambda$ is the following:

Theorem 3.1 (see [Vasy 2013a, Theorem 1.4]). *Let M be a manifold with a b-metric g as above, with boundary X, and let τ be the boundary defining function with $\mathcal{P}$ as in* (2-15). *If g has normally hyperbolic trapping, $\mathfrak{t}_1$ and Ω are as above, and $\phi \in C^\infty(\mathbb{R})$ is as in Theorem 2.21, then there exist $C' > 0$, $\varkappa > 0$ and $\beta \in \mathbb{R}$ such that, for $0 \le \ell < C'$ and $s > \frac{1}{2} + \beta\ell$, $s \ge 0$, solutions $u \in H_{\mathrm{b}}^{-\infty,-\infty}(\Omega)^{\bullet,-}$ of $(\Box_g - \lambda)u = f$ with $f \in H_{\mathrm{b}}^{s-1+\varkappa,\ell}(\Omega)^{\bullet,-}$ satisfy that, for some $a_{j\kappa} \in C^\infty(\Omega \cap X)$* (*which are the resonant states*) *and $\sigma_j \in \mathbb{C}$* (*which are the resonances*),

$$u' = u - \sum_j \sum_{\kappa \le m_j} \tau^{i\sigma_j} (\log \tau)^\kappa (\phi \circ \mathfrak{t}_1) a_{j\kappa} \in H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}. \tag{3-1}$$

Here the (*semi*)*norms of both $a_{j\kappa}$ in $C^\infty(\Omega \cap X)$ and u' in $H_{\mathrm{b}}^{s,\ell}(\Omega)^{\bullet,-}$ are bounded by a constant times that of f in $H_{\mathrm{b}}^{s-1+\varkappa,\ell}(\Omega)^{\bullet,-}$. The same conclusion holds for sufficiently small perturbations of the metric as a symmetric bilinear form on ${}^{\mathrm{b}}TM$ provided the trapping is normally hyperbolic.*

In order to state the analogue of Theorems 2.18 and 2.21 when one has *normally hyperbolic trapping* at $\Gamma \subset {}^{\mathrm{b}}S^*_X M$, we will employ nontrapping estimates in certain so-called normally isotropic functions spaces, established in [Hintz and Vasy 2014b]. To put our problem into the context of [Hintz and Vasy 2014b], we need some notation in addition to that in Section 2; in the setting of Section 2, as leading up to Theorem 2.18 — see the discussion above Figure 3 — we define

(1) the *forward trapped set in* Σ_+ as the set of points in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ through which bicharacteristics do not flow (within Σ_Ω) to ${}^{\mathrm{b}}S^*_{H_1} M \cup L_+$ in the forward direction (i.e., they do not reach ${}^{\mathrm{b}}S^*_{H_1} M$ in finite time and they do not tend to L_+),

(2) the *backward trapped set in* Σ_+ as the set of points in $\Sigma_\Omega \cap (\Sigma_+ \setminus L_+)$ through which bicharacteristics do not flow to ${}^{\mathrm{b}}S^*_{H_2} M \cup L_+$ in the backward direction,

(3) the *forward trapped set in* Σ_- as the set of points in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ through which bicharacteristics do not flow to ${}^{\mathrm{b}}S^*_{H_2}M \cup L_-$ in the forward direction, and

(4) the *backward trapped set in* Σ_- as the set of points in $\Sigma_\Omega \cap (\Sigma_- \setminus L_-)$ through which bicharacteristics do not flow to ${}^{\mathrm{b}}S^*_{H_1}M \cup L_-$ in the backward direction.

The *forward trapped set* Γ_- is the union of the forward trapped sets in $\Sigma_\pm$, and analogously for the *backward trapped set* Γ_+. The *trapped set* Γ is the intersection of the forward and backward trapped sets. We say that $\mathcal{P}$ is *normally hyperbolically trapping*, or has normally hyperbolic trapping, if $\Gamma \subset {}^{\mathrm{b}}S^*_X M$ is b-normally hyperbolic in the sense discussed in [Hintz and Vasy 2014b, §3.2].

Following [Hintz and Vasy 2014b], we introduce replacements for the b-Sobolev spaces used in Section 2, which are called *normally isotropic at* Γ; these spaces $\mathcal{H}^s_{\mathrm{b},\Gamma}$ — see also (3-2) — and dual spaces $\mathcal{H}^{*,-s}_{\mathrm{b},\Gamma}$ are just the standard b-Sobolev spaces $H^s_{\mathrm{b}}(M)$ and $H^{-s}_{\mathrm{b}}(M)$, respectively, microlocally away from Γ.

Concretely, suppose Γ is locally (in a neighborhood U_0 of Γ) defined by $\tau = 0$, $\phi_+ = \phi_- = 0$, $\hat{p} = 0$ in ${}^{\mathrm{b}}S^*M$, with $d\tau$, $d\phi_+$, $d\phi_-$, $d\hat{p}$ and $\hat{p} = \tilde{\rho}^m p$, linearly independent at Γ. Here, one should think of ϕ_- as being a defining function of $\Gamma_+ \cap \Sigma_+$ or $\Gamma_- \cap \Sigma_-$ within ${}^{\mathrm{b}}S^*M$, and ϕ_+ of $\Gamma_\pm \cap \Sigma_\mp$ within ${}^{\mathrm{b}}S^*_X M$. Then, taking any $Q_\pm \in \Psi^0_{\mathrm{b}}(M)$ with principal symbol $\phi_\pm$, $\widehat{P} \in \Psi^0_{\mathrm{b}}(M)$ with principal symbol $\hat{p}$, and $Q_0 \in \Psi^0_{\mathrm{b}}(M)$ elliptic on U_0^c with $\mathrm{WF}'_{\mathrm{b}}(Q_0) \cap \Gamma = \varnothing$, we define the (global) b-normally isotropic spaces at Γ of order s, $\mathcal{H}^s_{\mathrm{b},\Gamma} = \mathcal{H}^s_{\mathrm{b},\Gamma}(M)$, by the norm

$$\|u\|^2_{\mathcal{H}^s_{\mathrm{b},\Gamma}} = \|Q_0 u\|^2_{H^s_{\mathrm{b}}} + \|Q_+ u\|^2_{H^s_{\mathrm{b}}} + \|Q_- u\|^2_{H^s_{\mathrm{b}}} + \|\tau^{1/2} u\|^2_{H^s_{\mathrm{b}}} + \|\widehat{P} u\|^2_{H^s_{\mathrm{b}}} + \|u\|^2_{H^{s-1/2}_{\mathrm{b}}}, \tag{3-2}$$

and let $\mathcal{H}^{*,-s}_{\mathrm{b},\Gamma}$ be the dual space relative to L^2, which is

$$Q_0 H^{-s}_{\mathrm{b}} + Q_+ H^{-s}_{\mathrm{b}} + Q_- H^{-s}_{\mathrm{b}} + \tau^{1/2} H^{-s}_{\mathrm{b}} + \widehat{P} H^{-s}_{\mathrm{b}} + H^{-s+1/2}_{\mathrm{b}}.$$

In particular,

$$\begin{aligned} H^s_{\mathrm{b}}(M) \subset \mathcal{H}^s_{\mathrm{b},\Gamma}(M) &\subset H^{s-1/2}_{\mathrm{b}}(M) \cap H^{s,-1/2}_{\mathrm{b}}(M), \\ H^{s+1/2}_{\mathrm{b}}(M) + H^{s,1/2}_{\mathrm{b}}(M) \subset \mathcal{H}^{*,s}_{\mathrm{b},\Gamma}(M) &\subset H^s_{\mathrm{b}}(M). \end{aligned} \tag{3-3}$$

Microlocally away from Γ, $\mathcal{H}^s_{\mathrm{b},\Gamma}(M)$ is indeed just the standard H^s_{b} space, while $\mathcal{H}^{*,-s}_{\mathrm{b},\Gamma}$ is H^{-s}_{b}, since at least one of Q_0, $Q_\pm$, τ and $\widehat{P}$ is elliptic; the space is independent of the choice of Q_0 satisfying the criteria, since at least one of $Q_\pm$, τ and $\widehat{P}$ is elliptic on $U_0 \setminus \Gamma$. Moreover, every operator in $\Psi^k_{\mathrm{b}}(M)$ defines a continuous map $\mathcal{H}^s_{\mathrm{b},\Gamma}(M) \to \mathcal{H}^{s-k}_{\mathrm{b},\Gamma}(M)$ because, for $A \in \Psi^k_{\mathrm{b}}(M)$, $Q_+ A u = A Q_+ u + [Q_+, A]u$ and $[Q_+, A] \in \Psi^{k-1}_{\mathrm{b}}(M)$; the analogous statement also holds for the dual spaces.

The nontrapping estimates then are:

Proposition 3.2 (see [Hintz and Vasy 2014b, Theorem 3]). *With $\mathcal{P}$, $\mathcal{H}^s_{\mathrm{b},\Gamma}$ and $\mathcal{H}^{*,s}_{\mathrm{b},\Gamma}$ as above, for any neighborhood U of Γ and any N, there exist $B_0 \in \Psi^0_{\mathrm{b}}(M)$ elliptic at Γ and B_1, $B_2 \in \Psi^0_{\mathrm{b}}(M)$ with $\mathrm{WF}'_{\mathrm{b}}(B_j) \subset U$, $j = 0, 1, 2$, $\mathrm{WF}'_{\mathrm{b}}(B_2) \cap \Gamma_+ = \varnothing$, and $C > 0$, such that*

$$\|B_0 u\|_{\mathcal{H}^s_{\mathrm{b},\Gamma}} \leq \|B_1 \mathcal{P} u\|_{\mathcal{H}^{*,s-m+1}_{\mathrm{b},\Gamma}} + \|B_2 u\|_{H^s_{\mathrm{b}}} + C\|u\|_{H^{-N}_{\mathrm{b}}}, \tag{3-4}$$

i.e., if all the functions on the right-hand side are in the indicated spaces ($B_1\mathcal{P}u \in \mathcal{H}^{*,s-m+1}_{\mathrm{b},\Gamma}$, *etc.*) *then* $B_0 u \in \mathcal{H}^s_{\mathrm{b},\Gamma}$, *and the inequality holds.*

The same conclusion also holds if we assume $\mathrm{WF}'_{\mathrm{b}}(B_2)\cap\Gamma_- = \varnothing$ *instead of* $\mathrm{WF}'_{\mathrm{b}}(B_2)\cap\Gamma_+ = \varnothing$.

Finally, if $r<0$ *then, with* $\mathrm{WF}'_{\mathrm{b}}(B_2)\cap\Gamma_+ = \varnothing$, (3-4) *becomes*

$$\|B_0u\|_{H^{s,r}_{\mathrm{b}}} \le \|B_1\mathcal{P}u\|_{H^{s-m+1,r}_{\mathrm{b}}} + \|B_2u\|_{H^{s,r}_{\mathrm{b}}} + C\|u\|_{H^{-N,r}_{\mathrm{b}}} \tag{3-5}$$

while, if $r>0$ *then, with* $\mathrm{WF}'_{\mathrm{b}}(B_2)\cap\Gamma_- = \varnothing$,

$$\|B_0u\|_{H^{s,r}_{\mathrm{b}}} \le \|B_1\mathcal{P}u\|_{H^{s-m+1,r}_{\mathrm{b}}} + \|B_2u\|_{H^{s,r}_{\mathrm{b}}} + C\|u\|_{H^{-N,r}_{\mathrm{b}}}. \tag{3-6}$$

Remark 3.3. Note that the weighted versions (3-5)–(3-6) use *standard* weighted b-Sobolev spaces.

Next, if $\Omega\subset M$, as in Section 2, is such that ${}^{\mathrm{b}}S^*_{H_j}\Omega\cap\Gamma=\varnothing$, $j=1,2$, then spaces such as

$$\mathcal{H}^{*,s}_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-}$$

are not only well defined but are standard H^s_{b}-spaces near the H_j. The inclusions analogous to (3-3) also hold for the corresponding spaces over Ω.

Notice that elements of $\Psi^p_{\mathrm{b}}(M)$ only map $\mathcal{H}^s_{\mathrm{b},\Gamma}(M)$ to $\mathcal{H}^{*,s-p-1}_{\mathrm{b},\Gamma}(M)$, with the issues being at Γ corresponding to (3-3) (thus there is no distinction between the behavior on the Ω vs. the M-based spaces). However, if $A\in\Psi^p_{\mathrm{b}}(M)$ has principal symbol vanishing on Γ then

$$A:\mathcal{H}^s_{\mathrm{b},\Gamma}(M)\to H^{s-p}_{\mathrm{b}}(M) \quad\text{and}\quad A:H^s_{\mathrm{b}}(M)\to\mathcal{H}^{*,s-p}_{\mathrm{b},\Gamma}(M), \tag{3-7}$$

as A can be expressed as $A_+Q_+ + A_-Q_- + A_\partial\tau + \hat{A}\hat{P} + A_0Q_0 + R$ with $A_\pm, A_0, A_\partial, \hat{A}\in\Psi^0_{\mathrm{b}}(M)$ and $R\in\Psi^{-1}_{\mathrm{b}}(M)$, with the second mapping property following by duality as $\Psi^p_{\mathrm{b}}(M)$ is closed under adjoints and the principal symbol of the adjoint vanishes wherever that of the original operator does. Correspondingly, if $A_j\in\Psi^{m_j}_{\mathrm{b}}(M)$, $j=1,2$, have principal symbol vanishing at Γ then $A_1A_2u:\mathcal{H}^s_{\mathrm{b},\Gamma}(M)\to\mathcal{H}^{*,s-m_1-m_2}_{\mathrm{b},\Gamma}(M)$.

We consider $\mathcal{P}$ as a map

$$\mathcal{P}:\mathcal{H}^s_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-}\to\mathcal{H}^{s-2}_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-}$$

and let

$$\mathcal{Y}^s_\Gamma=\mathcal{H}^{*,s}_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-},\quad \mathcal{X}^s_\Gamma=\{u\in\mathcal{H}^s_{\mathrm{b},\Gamma}(\Omega)^{\bullet,-}:\mathcal{P}u\in\mathcal{Y}^{s-1}_\Gamma\}.$$

While $\mathcal{X}^s_\Gamma$ is complete,[11] it is a slightly exotic space, unlike $\mathcal{X}^s$ in Theorem 2.18, which is a coisotropic space depending on Σ (and thus the principal symbol of $\mathcal{P}$) only, since elements of $\Psi^p_{\mathrm{b}}(M)$ only map $\mathcal{H}^s_{\mathrm{b},\Gamma}(M)$ to $\mathcal{H}^{*,s-p-1}_{\mathrm{b},\Gamma}(M)$, as remarked earlier. In fact, $\mathcal{X}^s_\Gamma$ actually depends on $\mathcal{P}$ modulo $\Psi^0_{\mathrm{b}}(M)$ plus

[11] Also, elements of $C^\infty(\Omega)$ vanishing to infinite order at H_1 and $X\cap\Omega$ are dense in $\mathcal{X}^s_\Gamma$. Indeed, in view of [Melrose et al. 2013, Lemma A.3] the only possible issue is at Γ, thus the distinction between Ω and M may be dropped. To complete the argument, one proceeds as in the quoted lemma, using the ellipticity of σ at Γ, letting $\Lambda_n\in\Psi^{-\infty}_{\mathrm{b}}(M)$, $n\in\mathbb{N}$, be a quantization of $\phi(\sigma/n)a$ with $a\in C^\infty({}^{\mathrm{b}}S^*M)$ supported in a neighborhood of Γ and identically 1 near Γ, and $\phi\in C^\infty_c(\mathbb{R})$, noting that $[\Lambda_n,\mathcal{P}]\in\Psi^{-\infty}_{\mathrm{b}}(M)$ is uniformly bounded in $\Psi^0_{\mathrm{b}}(M)+\tau\Psi^1_{\mathrm{b}}(M)$ in view of (2-2), and thus, for $u\in\mathcal{X}^s_\Gamma$, $\mathcal{P}\Lambda_nu=\Lambda_n\mathcal{P}u+[\mathcal{P},\Lambda_n]u\to\mathcal{P}u$ in $\mathcal{H}^{*,s-1}_{\mathrm{b},\Gamma}$ since $[\mathcal{P},\Lambda_n]$ is uniformly bounded, so $H^{s-1/2}_{\mathrm{b}}\cap H^{s,-1/2}_{\mathrm{b}}\to H^{s-1/2}_{\mathrm{b}}\cap H^{s-1,1/2}_{\mathrm{b}}$, and thus $\mathcal{H}^s_{\mathrm{b},\Gamma}\to\mathcal{H}^{*,s-1}_{\mathrm{b},\Gamma}$ by (3-3).

first-order pseudodifferential operators of the form $A_1 A_2$, $A_1 \in \Psi_b^0(M)$ with $A_2 \in \Psi_b^1(M)$, both with principal symbol vanishing at Γ. Here, the operators should have Schwartz kernels supported away from the H_j; near H_j (but away from Γ), one should say $\mathcal{P}$ matters modulo $\mathrm{Diff}_b^1(M)$, i.e., only the principal symbol of $\mathcal{P}$ matters.

We then have:

Theorem 3.4. *Suppose $s \geq \frac{3}{2}$ and that the inverse of the Mellin-transformed normal operator $\widehat{\mathcal{P}}(\sigma)^{-1}$ has no poles with $\Im\sigma \geq 0$. Then*

$$\mathcal{P} : \mathcal{X}_\Gamma^s \to \mathcal{Y}_\Gamma^{s-1}$$

is invertible, giving the forward solution operator.

Proof. First, with $r < -\frac{1}{2}$, so with dual spaces having weight $\tilde{r} > \frac{1}{2}$, Theorem 2.18 holds without changes, as Proposition 3.2 gives nontrapping estimates in this case on the standard b-Sobolev spaces. In particular, if $r \ll 0$, $\operatorname{Ker}\mathcal{P}$ is trivial even on $H_b^{s-1/2,r}(\Omega)^{\bullet,-}$, hence certainly on its subspace $\mathcal{H}_{b,\Gamma}^s(\Omega)^{\bullet,-}$. Similarly, $\operatorname{Ker}\mathcal{P}^*$ is trivial on $H_b^{s,\tilde{r}}(\Omega)^{-,\bullet}$ for $\tilde{r} \gg 0$, and thus, with $r < -\frac{1}{2}$, for $f \in H_b^{-1,r}(\Omega)^{\bullet,-}$ there exists $u \in H_b^{0,r}(\Omega)^{\bullet,-}$ with $\mathcal{P}u = f$. Further, making use of the nontrapping estimates in Proposition 3.2, if $r < 0$ and $f \in H_b^{s-1,r}(\Omega)^{\bullet,-}$ then the argument of Theorem 2.21 improves this statement to $u \in H_b^{s,r}(\Omega)^{\bullet,-}$.

In particular, if $f \in \mathcal{H}_{b,\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \subset H_b^{s-1,0}(\Omega)^{\bullet,-}$, then $u \in H_b^{s,r}(\Omega)^{\bullet,-}$ for $r < 0$. This can be improved using the argument of Theorem 2.21. Indeed, with $-1 \leq r < 0$ arbitrary, $\mathcal{P} - N(\mathcal{P}) \in \tau\,\mathrm{Diff}_b^2(M)$ implies, as in (2-37), that

$$N(\mathcal{P})u = f - \tilde{f}, \quad \text{where} \quad \tilde{f} = (\mathcal{P} - N(\mathcal{P}))u \in H_b^{s-2,r+1}(\Omega)^{\bullet,-}. \tag{3-8}$$

But $f \in \mathcal{H}_{b,\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \subset H_b^{s-1,0}(\Omega)^{\bullet,-}$, hence the right-hand side is in $H_b^{s-2,0}(\Omega)^{\bullet,-}$; thus the dilation-invariant result, [Vasy 2013a, Lemma 3.1], gives $u \in H_b^{s-1,0}(\Omega)^{\bullet,-}$. This can then be improved further since, in view of $\mathcal{P}u = f \in \mathcal{H}_{b,\Gamma}^{*,s-1}(\Omega)^{\bullet,-}$, propagation of singularities, most crucially Proposition 3.2, yields $u \in \mathcal{H}_{b,\Gamma}^s(\Omega)^{\bullet,-}$. This completes the proof of the theorem. □

This result shows the importance of controlling the resonances in $\Im\sigma \geq 0$. For the wave operator on exact Kerr–de Sitter space, Dyatlov's [2011a; 2011b] analysis shows that the zero resonance of $\Box_g$ is the only one in $\Im\sigma \geq 0$, the residue at 0 having constant functions as its range. For the Klein–Gordon operator $\Box_g - m^2$, the statement is even better from our perspective as there are no resonances in $\Im\sigma \geq 0$ for $m > 0$ small. This is pointed out in [Dyatlov 2011a]; we give a direct proof based on perturbation theory.

Lemma 3.5. *Let $\mathcal{P} = \Box_g$ on exact Kerr–de Sitter space. Then, for small $m > 0$, all poles of $(\widehat{\mathcal{P}}(\sigma) - m^2)^{-1}$ have strictly negative imaginary part.*

Proof. By perturbation theory, the inverse family of $\widehat{\mathcal{P}}(\sigma) - \lambda$ has a simple pole at $\sigma(\lambda)$ coming with a single resonant state $\phi(\lambda)$ and a dual state $\psi(\lambda)$, with analytic dependence on λ, where $\sigma(0) = 0$, $\phi(0) \equiv 1$, and $\psi(0) = 1_{\{\mu > 0\}}$, where we use the notation of [Vasy 2013a, §6]. Differentiating $\widehat{\mathcal{P}}(\sigma(\lambda))\phi(\lambda) = \lambda\phi(\lambda)$ with respect to λ and evaluating at $\lambda = 0$ gives

$$\sigma'(0)\widehat{\mathcal{P}}'(0)\phi(0) + \widehat{\mathcal{P}}(0)\phi'(0) = \phi(0).$$

Pairing this with $\psi(0)$, which is orthogonal to $\operatorname{Ran}\widehat{\mathcal{P}}(0)$, yields

$$\sigma'(0) = \frac{\langle\psi(0), \phi(0)\rangle}{\langle\psi(0), \widehat{\mathcal{P}}'(0)\phi(0)\rangle}.$$

Since $\phi(0) = 1$ and $\psi(0) = 1_{\{\mu>0\}}$, this implies

$$\operatorname{sgn}\Im\sigma'(0) = -\operatorname{sgn}\Im\langle\psi(0), \widehat{\mathcal{P}}'(0)\phi(0)\rangle. \tag{3-9}$$

To find the latter quantity, we note that the only terms in the general form of the d'Alembertian that could possibly yield a nonzero contribution here are terms involving τD_τ and either D_r, D_ϕ or D_θ. Concretely, using the explicit form of the dual metric G — see Equation (6.1) in [Vasy 2013a] — in the new coordinates $t = \tilde{t} + h(r)$, $\phi = \tilde{\phi} + P(r)$ and $\tau = e^{-t}$, with $h(r)$ and $P(r)$ as in Vasy's Equation (6.5),

$$G = -\rho^{-2}\biggl(\tilde{\mu}(\partial_r - h'(r)\tau\partial_\tau + P'(r)\partial_\phi)^2 + \frac{(1+\gamma)^2}{\varkappa\sin^2\theta}(-a\sin^2\theta\,\tau\partial_\tau + \partial_\phi)^2 + \varkappa\partial_\theta^2 \\ - \frac{(1+\gamma)^2}{\tilde{\mu}}(-(r^2+a^2)\tau\partial_\tau + a\partial_\phi)^2\biggr),$$

and its determinant is $|\det G|^{1/2} = (1+\gamma)^2\rho^{-2}(\sin\theta)^{-1}$, so we see that the only nonzero contribution to the right-hand side of (3-9) comes from the term

$$(1+\gamma)^2\rho^{-2}(\sin\theta)^{-1}D_r\bigl((1+\gamma)^{-2}\rho^2\sin\theta\,\rho^{-2}\tilde{\mu}h'(r)\bigr)\tau D_\tau = -i\rho^{-2}\partial_r(\tilde{\mu}h'(r))\tau D_\tau$$

of the d'Alembertian. Mellin-transforming this amounts to replacing τD_τ by σ; then differentiating the result with respect to σ gives

$$\begin{aligned}\langle\psi(0), \widehat{\mathcal{P}}'(0)\phi(0)\rangle &= -i\int_{\tilde{\mu}>0}\rho^{-2}\partial_r(\tilde{\mu}h'(r))\,d\mathrm{vol} \\ &= -i\int_0^\pi\int_0^{2\pi}\int_{r_-}^{r_+}(1+\gamma)^{-2}\sin\theta\,\partial_r(\tilde{\mu}h'(r))\,dr\,d\phi\,d\theta \\ &= -\frac{4\pi i}{(1+\gamma)^2}\bigl[(\tilde{\mu}h'(r))\big|_{r_+} - (\tilde{\mu}h'(r))\big|_{r_-}\bigr].\end{aligned} \tag{3-10}$$

Since the singular part of $h'(r)$ at $r_\pm$ (which are the roots of $\tilde{\mu}$) is $h'(r) = \mp(1+\gamma)(r^2+a^2)/\tilde{\mu}$, the right-hand side of (3-10) is positive up to a factor of i; thus $\Im\sigma'(0) < 0$, as claimed. □

In other words, for small mass $m > 0$, there are no resonances σ of the Klein–Gordon operator with $\Im\sigma \geq -\epsilon_0$ for some $\epsilon_0 > 0$. Therefore, the expansion of u as in (3-1) no longer has a constant term. Correspondingly, for $\epsilon \in \mathbb{R}$, $\epsilon \leq \epsilon_0$, Theorem 3.1 gives the forward solution operator

$$S_{\mathrm{KG,I}} : H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \tag{3-11}$$

in the dilation-invariant case.

Further, Theorem 3.4 is applicable and gives the forward solution operator

$$S_{\mathrm{KG}} : \mathcal{H}_{\mathrm{b},\Gamma}^{*,s-1}(\Omega)^{\bullet,-} \to \mathcal{H}_{\mathrm{b},\Gamma}^{s}(\Omega)^{\bullet,-} \tag{3-12}$$

on the normally isotropic spaces.

For the semilinear application, for nonlinearities without derivatives, it is important that the loss of derivatives $\varkappa$ in the space $H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}$ is at most 1. This is not explicitly specified in [Wunsch and Zworski 2011], though their proof directly gives (see especially the part before their Section 4.4) that, for small $\epsilon > 0$, $\varkappa$ can be taken proportional to ϵ and there is $\epsilon_0' > 0$ such that $\varkappa \in (0,1]$ for $\epsilon < \epsilon_0'$. We reduce $\epsilon_0 > 0$ above if needed so that $\epsilon_0 \le \epsilon_0'$; then (3-11) holds with $\varkappa = c\epsilon \in (0,1]$ if $\epsilon < \epsilon_0$, where $c > 0$.

In fact, one does not need to go through Wunsch and Zworski's proof, as the Phragmén–Lindelöf theorem allows one to obtain the same conclusion from their final result:

Lemma 3.6. *Let $h : U \to E$ be a holomorphic function on the half strip $U = \{z \in \mathbb{C} : 0 \le \Im z \le c,\ \Re z \ge 1\}$ that is continuous on $\overline{U}$ with values in a Banach space E and suppose, moreover, that there are constants $A, C > 0$ such that*

$$\|h(z)\| \le \begin{cases} C|z|^{k_1} & \text{if } \Im z = 0, \\ C|z|^{k_2} & \text{if } \Im z = c, \\ C\exp(A|z|) & \text{if } z \in \overline{U}. \end{cases}$$

Then there is a constant $C' > 0$ such that

$$\|h(z)\| \le C'|z|^{k_1(1-(\Im z)/c)+k_2(\Im z)/c}$$

for all $z \in \overline{U}$.

Proof. Consider the function $f(z) = z^{k_1 - i(k_2-k_1)z/c}$, which is holomorphic on a neighborhood of $\overline{U}$. Writing $z \in \overline{U}$ as $z = x + iy$ with $x, y \in \mathbb{R}$, one has

$$|f(z)| = |z|^{k_1}\exp\left(\Im\left(\frac{k_2-k_1}{c} z\log z\right)\right) = |z|^{k_1}|z|^{(k_2-k_1)\Im z/c}\exp\left(\frac{k_2-k_1}{c} x\arctan\left(\frac{y}{x}\right)\right).$$

Noting that $|x\arctan(y/x)| = y|(x/y)\arctan(y/x)|$ is bounded by c for all $x + iy \in \overline{U}$, we conclude that

$$e^{-|k_2-k_1|}|z|^{k_1(1-\Im z/c)+k_2\Im z/c} \le |f(z)| \le e^{|k_2-k_1|}|z|^{k_1(1-\Im z/c)+k_2\Im z/c}.$$

Therefore, $f(z)^{-1}h(z)$ is bounded by a constant C' on $\partial\overline{U}$, and satisfies an exponential bound for $z \in U$. By the Phragmén–Lindelöf theorem, $\|f(z)^{-1}h(z)\|_E \le C'$, and the claim follows. ☐

Since, for any $\delta > 0$, we can bound $|\log z| \le C_\delta |z|^\delta$ for $|\Re z| \ge 1$, we obtain that the inverse family $R(\sigma) = \widehat{\mathcal{P}}(\sigma)^{-1}$ of the normal operator of $\Box_g$ on (asymptotically) Kerr–de Sitter spaces — as in [Vasy 2013a] but here in the setting of artificial boundaries, as opposed to complex absorption — satisfies a bound

$$\|R(\sigma)\|_{|\sigma|^{-(s-1)}H^{s-1}_{|\sigma|^{-1}}(X\cap\Omega)\to|\sigma|^{-s}H^{s}_{|\sigma|^{-1}}(X\cap\Omega)} \le C_\delta|\sigma|^{-1+\varkappa'+\delta} \tag{3-13}$$

for any $\delta > 0$, $\Im\sigma \ge -c\varkappa'$ and $|\Re\sigma|$ large. Therefore, as mentioned above, by the proof of Theorem 3.1, in particular using [Vasy 2013a, Lemma 3.1], we can assume $\varkappa \in (0,1]$ in the dilation-invariant result, Theorem 3.1, if we take $C' > 0$ small enough, i.e., if we do not go too far into the lower half plane $\Im\sigma < 0$, which amounts to only taking terms in the expansion (3-1) which decay to at most some fixed order, which we may assume to be less than $-\Im\sigma_j$ for all resonances σ_j.

3B. ***A class of semilinear equations; equations with polynomial nonlinearity.*** In the following semilinear applications, let us fix $\varkappa \in (0, 1]$ and ϵ_0 as explained before Lemma 3.6, so that we have the forward solution operator $S_{\mathrm{KG,I}}$ as in (3-11).

We then have statements paralleling Theorems 2.25 and 2.37 and Corollary 2.28, namely Theorems 3.7 and 3.11 and Corollary 3.10, respectively.

Theorem 3.7. *Suppose (M, g) is dilation invariant. Let $-\infty < \epsilon < \epsilon_0$, $s > \frac{1}{2} + \beta\epsilon$, $s \geq 1$, and let $q : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$ be a continuous function with $q(0) = 0$ such that there exists a continuous nondecreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying*

$$\|q(u) - q(v)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ such that the following holds: if $L(0) < C_L$ then, for small $R > 0$, there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

More generally, suppose

$$q : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$$

satisfies $q(0, 0) = 0$ and

$$\|q(u, w) - q(u', w')\| \leq L(R)(\|u - u'\| + \|w - w'\|)$$

provided $\|u\| + \|w\|$, $\|u'\| + \|w'\| \leq R$, where we use the norms corresponding to the map q, for a continuous nondecreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Then there is a constant $C_L > 0$ such that the following holds: if $L(0) < C_L$ then, for small $R > 0$, there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u, \Box_g u)$$

has a unique solution $u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, with $\|u\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|\Box_g u\|_{H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}} \leq R$, that depends continuously on f.

Proof. We use the proof of the first part of Theorem 2.25, where, in the current setting, the solution operator $S_{\mathrm{KG,I}}$ maps $H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$ and the contraction map is $T : H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-}$, $Tu = S_{\mathrm{KG,I}}(f + q(u))$.

For the general statement, we follow the proof of the second part of Theorem 2.25, where we now instead use the Banach space

$$\mathscr{X} = \{u \in H_{\mathrm{b}}^{s,\epsilon}(\Omega)^{\bullet,-} : \Box_g u \in H_{\mathrm{b}}^{s-1+\varkappa,\epsilon}(\Omega)^{\bullet,-}\}$$

with norm

$$\|u\|_{\mathcal{X}} = \|u\|_{H_{\mathrm{b}}^{s,\epsilon}} + \|\Box_g u\|_{\tau^\epsilon H_{\mathrm{b}}^{s-1+\varkappa}},$$

which is a Banach space by the same argument as in the proof of Theorem 2.25. $\square$

We have a weaker statement in the general, non-dilation-invariant case, where we work in unweighted spaces.

Theorem 3.8. *Let $s \geq 1$ and suppose $q : H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$ is a continuous function with $q(0) = 0$ such that there exists a continuous nondecreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying*

$$\|q(u) - q(v)\| \leq L(R)\|u - v\|, \quad \|u\|, \|v\| \leq R.$$

Then there is a constant $C_L > 0$ such that the following holds: if $L(0) < C_L$ then, for small $R > 0$, there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

An analogous statement holds for nonlinearities $q = q(u, \Box_g u)$ which are continuous maps

$$q : H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \times H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \to H_{\mathrm{b}}^s(\Omega)^{\bullet,-},$$

vanish at $(0, 0)$, and have a small Lipschitz constant near 0.

Proof. Since

$$S_{\mathrm{KG}} : H_{\mathrm{b}}^s(\Omega)^{\bullet,-} \subset \mathcal{H}_{\mathrm{b},\Gamma}^{*,s-1/2}(\Omega)^{\bullet,-} \to \mathcal{H}_{\mathrm{b},\Gamma}^{s+1/2}(\Omega)^{\bullet,-} \subset H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$$

by (3-3) and (3-12), this follows again from the Banach fixed point theorem. $\square$

Remark 3.9. The proof of Theorem 3.4 shows that equations on function spaces with negative weights (i.e., growing near infinity) behave as nicely as equations on the static part of asymptotically de Sitter spaces, discussed in Section 2. However, naturally occurring nonlinearities (e.g., polynomials) will not be continuous nonlinear operators on such growing spaces.

Corollary 3.10. *If $s > \frac{1}{2}n$, the hypotheses of Theorem 3.8 hold for nonlinearities $q(u) = cu^p$, $p \geq 2$ an integer, $c \in \mathbb{C}$, as well as $q(u) = q_0 u^p$, $q_0 \in H_{\mathrm{b}}^s(M)$.*

Thus, for small $m > 0$ and $R > 0$, there exists $C > 0$ such that, for all $f \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u)$$

has a unique solution $u \in H_{\mathrm{b}}^s(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

If f satisfies stronger decay assumptions, then u does as well. More precisely, denoting the inverse family of the normal operator of the Klein–Gordon operator with (small) mass m by $R_m(\sigma) = (\widehat{\mathcal{P}}(\sigma) - m^2)^{-1}$, which has poles only in $\Im\sigma < 0$ (see Lemma 3.5 and [Dyatlov 2011a; Vasy 2013a]) and, moreover, defining the spaces $\mathcal{X}_{\mathcal{F}}^{s,r,\epsilon}$ and $\mathcal{X}_{\mathcal{P},\mathcal{F}}^{s,r,\epsilon}$ analogously to the corresponding spaces in Section 2C, we have the following result:

Theorem 3.11. *Fix $0 < \epsilon < \min\{C', \frac{1}{2}\}$ and let $s \gg s' \geq \max(\frac{1}{2} + \beta\epsilon, \frac{1}{2}n, 1 + \varkappa)$. (A concrete bound for s will be given in the course of the proof; see Equation* (3-15).) *Let*

$$q(u) = \sum_{p=2}^{d} q_p u^p, \quad q_p \in H_{\mathrm{b}}^s(M).$$

Moreover, if $\sigma_j \in \mathbb{C}$ are the poles of the inverse family $R_m(\sigma)$, and $m_j + 1$ is the order of the pole of $R_m(\sigma)$ at σ_j, let $\mathscr{P} = \{(i\sigma_j + k, \ell) : 0 \leq \ell \leq m_j,\ k \in \mathbb{N}_0\}$. Assume that $\epsilon \neq \Re(i\sigma_j)$ for all j, and that $m > 0$ is so small that $\mathscr{P}$ is a positive index set. Finally, let $\mathscr{F}$ be a positive index set.

Then, for small enough $R > 0$, there exists $C > 0$ such that, for all $f \in \mathscr{X}_{\mathscr{F}}^{s,s,\epsilon}$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u) \tag{3-14}$$

has a unique solution $u \in \mathscr{X}_{\mathscr{P},\mathscr{F}}^{s',s',\epsilon}$, with $\|u\| \leq R$, that depends continuously on f; in particular, u has an asymptotic expansion with remainder in $H_{\mathrm{b}}^{s',\epsilon}(\Omega)^{\bullet,-}$.

Proof. Let us write $\mathscr{P} = \Box_g - m^2$. Let $\delta < \frac{1}{2}$ be such that $0 < 2\delta < \Re z$ for all $(z, 0) \in \mathscr{F}$; then $f \in H_{\mathrm{b}}^{s,2\delta}(\Omega)^{\bullet,-}$. Now, for $u \in H_{\mathrm{b}}^{s,\delta}(\Omega)^{\bullet,-}$, consider $Tu := S_{\mathrm{KG}}(f + q(u))$. First of all, $f + q(u) \in H_{\mathrm{b}}^{s,2\delta}(\Omega)^{\bullet,-} \subset H_{\mathrm{b}}^{s}(\Omega)^{\bullet,-}$, thus the proof of Theorem 3.4 shows that $Tu \in H_{\mathrm{b}}^{s+1,r}(\Omega)^{\bullet,-}$ with $r < 0$ arbitrary. Therefore,

$$N(\mathscr{P})u = f + q(u) + (N(\mathscr{P}) - \mathscr{P})u \in H_{\mathrm{b}}^{s,2\delta}(\Omega)^{\bullet,-} + H_{\mathrm{b}}^{s-1,r+1}(\Omega)^{\bullet,-} \subset H_{\mathrm{b}}^{s-1,2\delta}(\Omega)^{\bullet,-},$$

and thus, if $\delta > 0$ is sufficiently small, namely, $\delta < \frac{1}{2}\inf\{-\Im\sigma_j\}$, Theorem 3.1 implies $u \in H_{\mathrm{b}}^{s-\varkappa,2\delta}(\Omega)^{\bullet,-}$. Since we can choose $\varkappa = c\delta$ for some constant $c > 0$, we obtain

$$Tu \in \bigcap_{r>0} H_{\mathrm{b}}^{s+1,r}(\Omega)^{\bullet,-} \cap H_{\mathrm{b}}^{s-c\delta,2\delta}(\Omega)^{\bullet,-} \subset \bigcap_{r'>0} H_{\mathrm{b}}^{s,2\delta-2c\delta^2/(1+c\delta)-r'}(\Omega)^{\bullet,-}$$

by interpolation. In particular, choosing $\delta > 0$ even smaller if necessary, we obtain $Tu \in H_{\mathrm{b}}^{s,\delta}(\Omega)^{\bullet,-}$. Applying the Banach fixed point theorem to the map T thus gives a solution $u \in H_{\mathrm{b}}^{s,\delta}(\Omega)^{\bullet,-}$ to (3-14).

For this solution u, we obtain

$$N(\mathscr{P})u = \mathscr{P}u + (N(\mathscr{P}) - \mathscr{P})u \in H_{\mathrm{b}}^{s,2\delta} + H_{\mathrm{b}}^{s-2,\delta+1} \subset H_{\mathrm{b}}^{s-2,2\delta}$$

since q only has quadratic and higher terms. Hence Theorem 3.1 implies that $u = u_1 + u'$, where u_1 is an expansion with terms coming from poles of $\widehat{\mathscr{P}}^{-1}$ whose decay order lies between δ and 2δ, and $u' \in H_{\mathrm{b}}^{s-1-\varkappa,2\delta}(\Omega)^{\bullet,-}$. This in turn implies that $f + q(u)$ has an expansion with remainder term in $H_{\mathrm{b}}^{s-1-\varkappa,\min\{4\delta,\epsilon\}}(\Omega)^{\bullet,-}$; thus

$$N(\mathscr{P})u \in H_{\mathrm{b}}^{s-3-\varkappa,\min\{4\delta,\epsilon\}}(\Omega)^{\bullet,-} \text{ plus an expansion,}$$

and we proceed iteratively, until, after k more steps, we have $4 \cdot 2^k\delta \geq \epsilon$, and then u has an expansion with remainder term $H_{\mathrm{b}}^{s-3-2k-\varkappa,\epsilon}(\Omega)^{\bullet,-}$ provided we can apply Theorem 3.1 in the iterative procedure, i.e., provided $s-3-2k-\varkappa =: s' > \max\bigl(\frac{1}{2}+\beta\epsilon, \frac{1}{2}n, 1+\varkappa\bigr)$. This is satisfied if

$$s > \max\bigl(\tfrac{1}{2}+\beta\epsilon, \tfrac{1}{2}n, 1+\varkappa\bigr) + 2\lceil \log_2(\epsilon/\delta)\rceil + \varkappa - 1. \tag{3-15}$$

This concludes the proof. □

3C. ***Semilinear equations with derivatives in the nonlinearities.*** Theorem 3.4 allows one to solve even semilinear equations with derivatives in some cases. For instance, in the case of de Sitter–Schwarzschild space, within $\Sigma \cap {}^{\mathrm{b}}S^*_X M$, Γ is given by $r = r_c$, $\sigma_1(D_r) = 0$, where $r_c = \frac{3}{2}r_s$ is the radius of the photon sphere; see, e.g., [Vasy 2013a, §6.4]. Thus, nonlinear terms such as $(r-r_c)(\partial_r u)^2$ are allowed for $s > \frac{1}{2}n+1$ since $\partial_r : \mathscr{H}^s_{\mathrm{b},\Gamma}(M) \to H^{s-1}_{\mathrm{b}}(M)$, with the latter space being an algebra, while multiplication by $r-r_c$ maps this space to $\mathscr{H}^{*,s-1}_{\mathrm{b},\Gamma}$, by (3-7). Thus, a straightforward modification of Theorem 3.8, applying the fixed point theorem on the normally isotropic spaces directly, gives well-posedness.

4. Asymptotically de Sitter spaces: global approach

We can approach the problem of solving nonlinear wave equations on global asymptotically de Sitter spaces in two ways: either we proceed as in the previous two sections, first showing invertibility of the linear operator on suitable spaces and then applying the contraction mapping principle to solve the nonlinear problem; or we use the solvability results from Section 2 for backward light cones from points at future conformal infinity and glue the solutions on all these “static” parts together to obtain a global solution. The first approach, which we will follow in Section 4A–4D, has the disadvantage that the conditions on the nonlinearity that guarantee the existence of solutions are quite restrictive, however, if the conditions are met, one has good decay estimates for solutions. The second approach, on the other hand, detailed in Section 4E, allows many of the nonlinearities, suitably reinterpreted, that work on “static parts” of asymptotically de Sitter spaces (i.e., backward light cones), but the decay estimates for solutions are quite weak relative to the decay of the forcing term because of the gluing process.

4A. ***The linear framework.*** Let g be the metric on an n-dimensional asymptotically de Sitter space X with global time function t [Vasy 2010]. Then, following [Vasy 2013a, Section 4], the operator[12]

$$P_\sigma = \mu^{-1/2}\mu^{i\sigma/2-(n+1)/4}\bigl(\Box_g - \bigl(\tfrac{1}{2}(n-1)\bigr)^2 - \sigma^2\bigr)\mu^{-i\sigma/2+(n+1)/4}\mu^{-1/2} \tag{4-1}$$

extends nondegenerately to an operator on a closed manifold $\widetilde{X}$ which contains the compactification $\overline{X}$ of the asymptotically de Sitter space as a submanifold with boundary Y, where $Y = Y_- \cup Y_+$ has two connected components, which we call the boundary of X at past and future infinity, respectively. The expression “nondegenerately” here means that, near $Y_\pm$, P_σ fits into the framework of [Vasy 2013a]. Here, $\mu = 0$ is the defining function of Y and $\mu > 0$ is the interior of the asymptotically de Sitter space. Moreover, null-bicharacteristics of P_σ tend to $Y_\pm$ as $t \to \pm\infty$.

Following [Vasy 2014], let us in fact assume that $\widetilde{X} = \overline{C_-} \cup \overline{X} \cup \overline{C_+}$ is the union of the compactifications of asymptotically de Sitter space X and two asymptotically hyperbolic caps $C_\pm$; as Vasy explains, one

[12] P_σ in our notation corresponds to $P^*_{\bar\sigma}$ in [Vasy 2013a], the latter operator being the one for which one solves the forward problem.

might need to take two copies of X to construct $\widetilde{X}$. For the purposes of the next statement, we recall that variable-order Sobolev spaces $H^s(\widetilde{X})$ were discussed in [Baskin et al. 2014, Appendix A]. Then P_σ is the restriction to X of an operator $\widetilde{P}_\sigma \in \mathrm{Diff}^2(\widetilde{X})$, which is Fredholm as a map

$$\widetilde{P}_\sigma : \widetilde{\mathscr{X}}^s \to \widetilde{\mathscr{Y}}^{s-1}, \quad \widetilde{\mathscr{X}}^s = \{u \in H^s : \widetilde{P}_\sigma u \in H^{s-1}\}, \quad \widetilde{\mathscr{Y}}^{s-1} = H^{s-1},$$

where $s \in C^\infty(S^*\widetilde{X})$, monotone along the bicharacteristic flow, is such that $s|_{N^*Y_-} > \frac{1}{2} - \Im\sigma$, $s|_{N^*Y_+} < \frac{1}{2} - \Im\sigma$, and s is constant near $S^*Y_\pm$. Note that the choice of signs here is opposite to the one in [Vasy 2014], since here we are going to construct the forward solution operator on X.

Restricting our attention to X, we define the space $H^s(X)^{\bullet,-}$ to be the completion in $H^s(X)$ of the space of C^∞ functions that vanish to infinite order at Y_-; thus, the superscripts indicate that distributions in $H^s(X)^{\bullet,-}$ are supported distributions near Y_- and extendible distributions near Y_+. Then, define the spaces

$$\mathscr{X}^s = \{u \in H^s(X)^{\bullet,-} : P_\sigma u \in H^{s-1}(X)^{\bullet,-}\}, \quad \mathscr{Y}^{s-1} = H^{s-1}(X)^{\bullet,-}.$$

Theorem 4.1. *Fix $\sigma \in \mathbb{C}$ and $s \in C^\infty(S^*\overline{X})$ as above. Then $P_\sigma : \mathscr{X}^s \to \mathscr{Y}^{s-1}$ is invertible and $P_\sigma^{-1} : H^{s-1}(X)^{\bullet,-} \to H^s(X)^{\bullet,-}$ is the forward solution operator of P_σ.*

Proof. First, let us assume $\Re\sigma \gg 0$, so semiclassical and large parameter estimates are applicable to $\widetilde{P}_\sigma$, and let $T_0 \in \mathbb{R}$ be such that s is constant in $\{t \le T_0\}$. Then, for any $T_1 \le T_0$, we can paste together microlocal energy estimates for $\widetilde{P}_\sigma$ near $\overline{C_-}$ and standard energy estimates for the wave equation in $\{t \le T_1\}$ away from Y_-, as in the derivation of Equation (3.29) of [Vasy 2013a], and thereby obtain

$$\|u\|_{H^1(\{t \le T_1\})} \lesssim \|\widetilde{P}_\sigma u\|_{H^0(\{t \le T_1\})}; \tag{4-2}$$

thus, for $f \in C^\infty(\widetilde{X})$, $\operatorname{supp} f \subset \{t \ge T_1\}$ implies $\operatorname{supp} \widetilde{P}_\sigma^{-1} f \subset \{t \ge T_1\}$. Choosing $\phi \in C_c^\infty(X)$ with support in $\{t \ge T_1\}$ and $\psi \in C^\infty(\widetilde{X})$ with support in $\{t \le T_1\}$, we therefore obtain $\psi \widetilde{P}_\sigma^{-1} \phi = 0$. Since $\widetilde{P}_\sigma^{-1}$ is meromorphic, this continues to hold for all $\sigma \in \mathbb{C}$ such that $\Im\sigma > \frac{1}{2} - s$. Since $T_1 \le T_0$ is arbitrary, this, together with standard energy estimates on the asymptotically de Sitter space X, proves that P_σ^{-1} propagates supports forward, provided P_σ is invertible. Moreover, elements of $\ker \widetilde{P}_\sigma$ are supported in $\overline{C_+}$.

The invertibility of P_σ is a consequence of [Baskin et al. 2014, Lemma 8.3] (also see Footnote 15 there): let $E : H^{s-1}(X)^{\bullet,-} \to H^{s-1}(\widetilde{X})$ be a continuous extension operator that extends by 0 in $\overline{C_-}$ and $R : H^s(\widetilde{X}) \to H^s(X)^{-,-}$ the restriction; then $R \circ \widetilde{P}_\sigma^{-1} \circ E$ does not have poles, and, since

$$\bigcup_{T_1 \le T_0} H^s(\{t > T_1\})^{\bullet,-} \subset H^s(X)^{\bullet,-}$$

(where $\bullet$ denotes supported distributions at $\{t = T_1\}$ and Y_-, respectively) is dense, $R \circ \widetilde{P}_\sigma^{-1} \circ E$ in fact maps into $H^s(X)^{\bullet,-}$; thus $P_\sigma^{-1} = R \circ \widetilde{P}_\sigma^{-1} \circ E$ indeed exists and has the claimed properties. $\square$

In our quest for forward solutions of semilinear equations, we restrict ourselves to a submanifold with boundary $\Omega \subset \overline{X}$ containing and localized near future infinity, so that we can work in fixed-order Sobolev spaces; moreover, it will be useful to measure the conormal regularity of solutions to the linear equation at the conormal bundle of the boundary of X at future infinity more precisely. So let $H^{s,k}(\widetilde{X}, Y_+)$ be the subspace of $H^s(\widetilde{X})$ with k-fold regularity with respect to the $\Psi^0(\widetilde{X})$-module $\mathcal{M}$ of first-order pseudodifferential operators with principal symbol vanishing on N^*Y_+. A result of Haber and Vasy [2013, Theorem 6.3], with $s_0 = \frac{1}{2} - \Im\sigma$ in our case, shows that $f \in H^{s-1,k}(\widetilde{X}, Y_+)$, $\widetilde{P}_\sigma u = f$ with u a distribution, in fact imply that $u \in H^{s,k}(\widetilde{X}, Y_+)$. So, if we let $H^{s,k}(\Omega)^{\bullet,-}$ denote the space of all $u \in H^s(X)^{\bullet,-}$ which are restrictions to Ω of functions in $H^{s,k}(\widetilde{X}, Y_+)$, supported in $\Omega \cup \overline{C_+}$, the argument of Theorem 4.1 shows that we have a forward solution operator $S_\sigma : H^{s-1,k}(\Omega)^{\bullet,-} \to H^{s,k}(\Omega)^{\bullet,-}$ provided

$$s < \tfrac{1}{2} - \Im\sigma. \tag{4-3}$$

4A1. *The backward problem.* Another problem that we will briefly consider below is the backward problem, i.e., where one solves the equation on X backward from Y_+, which is the same, up to relabelling, as solving the equation forward from Y_-. Thus, we have a backward solution operator $S_\sigma^- : H^{s-1,k}(\Omega)^{-,\bullet} \to H^{s,k}(\Omega)^{-,\bullet}$ (where Ω is chosen as above so that we can use fixed-order Sobolev spaces) provided $s > \frac{1}{2} - \Im\sigma$. Similarly to the above, $-$ denotes extendible distributions at $\partial\Omega \cap X^\circ$ and $\bullet$ denotes supported distributions at Y_+; the module regularity is measured at Y_+.

4B. ***Algebra properties of $H^{s,k}(\Omega)^{\bullet,-}$.*** Let us call a polynomially bounded, measurable function $w : \mathbb{R}^n \to (0, \infty)$ a *weight function*. For such a weight function w, we define

$$H^{(w)}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : w\hat{u} \in L^2(\mathbb{R}^n)\}.$$

The following lemma is similar in spirit to, but different from, Strichartz's [1971] result on Sobolev algebras; it is the basis for the multiplicative properties of the more delicate spaces considered below.

Lemma 4.2. *Let w_1, w_2 and w be weight functions such that one of the quantities*

$$\begin{aligned} M_+ &:= \sup_{\xi\in\mathbb{R}^n} \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\eta, \\ M_- &:= \sup_{\eta\in\mathbb{R}^n} \int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\right)^2 d\xi \end{aligned} \tag{4-4}$$

is finite. Then $H^{(w_1)}(\mathbb{R}^n) \cdot H^{(w_2)}(\mathbb{R}^n) \subset H^{(w)}(\mathbb{R}^n)$.

Proof. For $u, v \in S(\mathbb{R}^n)$, we use Cauchy–Schwarz to estimate

$$\begin{aligned} \|uv\|^2_{H^{(w)}} &= \int w(\xi)^2 |\widehat{uv}(\xi)|^2 \, d\xi \\ &= \int w(\xi)^2 \left(\int w_1(\eta)|\hat{u}(\eta)| w_2(\xi-\eta)|\hat{v}(\xi-\eta)| w_1(\eta)^{-1} w_2(\xi-\eta)^{-1} \, d\eta \right)^2 d\xi \\ &\le \int \left(\int \left(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)} \right)^2 d\eta \right) \left(\int w_1(\eta)^2 |\hat{u}(\eta)|^2 w_2(\xi-\eta)^2 |\hat{v}(\xi-\eta)|^2 \, d\eta \right) d\xi \\ &\le M_+ \|u\|^2_{H^{(w_1)}} \|v\|^2_{H^{(w_2)}} \end{aligned}$$

as well as

$$\begin{aligned}\|uv\|_{H^{(w)}}^2 &\le \int\Big(\int w_2(\xi-\eta)^2|\hat{v}(\xi-\eta)|^2\,d\eta\Big)\Big(\int\Big(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\Big)^2 w_1(\eta)^2|\hat{u}(\eta)|^2\,d\eta\Big)\,d\xi\\ &= \|v\|_{H^{(w_2)}}^2\int w_1(\eta)^2|\hat{u}(\eta)|^2\Big(\int\Big(\frac{w(\xi)}{w_1(\eta)w_2(\xi-\eta)}\Big)^2\,d\xi\Big)\,d\eta\\ &\le M_-\|u\|_{H^{(w_1)}}^2\|v\|_{H^{(w_2)}}^2.\end{aligned}$$

Since $S(\mathbb{R}^n)$ is dense in $H^{(w_1)}(\mathbb{R}^n)$ and $H^{(w_2)}(\mathbb{R}^n)$, the lemma follows. □

In particular, if

$$\Big\|\frac{w(\xi)}{w(\eta)w(\xi-\eta)}\Big\|_{L^\infty_\xi L^2_\eta} < \infty, \tag{4-5}$$

then $H^{(w)}$ is an algebra.

For example, the weight function $w(\xi)=\langle\xi\rangle^s$ for $s>\frac{1}{2}n$ satisfies (4-5), as we will check below, which implies that $H^s(\mathbb{R}^n)$ is an algebra for $s>\frac{1}{2}n$; this is the special case $k=0$ of Lemma 4.4 below and is well known; see, e.g., [Taylor 1997, Chapter 13.3]. Also, product-type weight functions $w_d(\xi)=\langle\xi'\rangle^s\langle\xi''\rangle^k$ (where $\xi=(\xi',\xi'')\in\mathbb{R}^{d+(n-d)}$) for $s>\frac{1}{2}d$ and $k>\frac{1}{2}(n-d)$ satisfy (4-5).

The following lemma, together with the triangle inequality $\langle\xi\rangle^\alpha\lesssim\langle\eta\rangle^\alpha+\langle\xi-\eta\rangle^\alpha$ for $\alpha\ge 0$, will often be used to check conditions like (4-4).

Lemma 4.3. *Suppose $\alpha,\beta\ge 0$ are such that $\alpha+\beta>n$. Then*

$$\int_{\mathbb{R}^n}\frac{d\eta}{\langle\eta\rangle^\alpha\langle\xi-\eta\rangle^\beta}\in L^\infty(\mathbb{R}^n_\xi).$$

Proof. Splitting the domain of integration into the two regions $\{\langle\eta\rangle<\langle\xi-\eta\rangle\}$ and $\{\langle\eta\rangle\ge\langle\xi-\eta\rangle\}$, we obtain the bound

$$\int_{\mathbb{R}^n}\frac{d\eta}{\langle\eta\rangle^\alpha\langle\xi-\eta\rangle^\beta}\le 2\int_{\mathbb{R}^n}\frac{d\eta}{\langle\eta\rangle^{\alpha+\beta}},$$

which is finite in view of $\alpha+\beta>n$. □

Another important consequence of Lemma 4.2 is that $H^{s'}(\mathbb{R}^n)$ is an $H^s(\mathbb{R}^n)$-module provided $|s'|\le s$ and $s>\frac{1}{2}n$, which follows for $s'\ge 0$ from $M_+<\infty$, and for $s'<0$ either by duality or from $M_-<\infty$ (with $M_\pm$ as in the statement of the lemma, with the corresponding weight functions).

Lemma 4.4. *Write $x\in\mathbb{R}^n$ as $x=(x',x'')\in\mathbb{R}^{d+(n-d)}$. For $s\in\mathbb{R}$ and $k\in\mathbb{N}_0$, let*

$$\mathcal{Y}_d^{s,k}(\mathbb{R}^n)=\{u\in H^s(\mathbb{R}^n): D^k_{x''}u\in H^s(\mathbb{R}^n)\}.$$

Then, for $s>\frac{1}{2}d$ and $s+k>\frac{1}{2}n$, $\mathcal{Y}_d^{s,k}(\mathbb{R}^n)$ is an algebra.

Proof. Using the Leibniz rule, we see that it suffices to show that if $u,v\in\mathcal{Y}_d^{s,k}$ then $D^\alpha_{x''}uD^\beta_{x''}v\in H^s$, provided $|\alpha|+|\beta|\le k$. Since $D^\alpha_{x''}u\in\mathcal{Y}_d^{s,k-|\alpha|}$ and $D^\beta_{x''}v\in\mathcal{Y}_d^{s,k-|\beta|}$, this amounts to showing that

$$\mathcal{Y}_d^{s,a}\cdot\mathcal{Y}_d^{s,b}\subset H^s\quad\text{if } a+b\ge k. \tag{4-6}$$

Using the characterization $\mathcal{Y}_d^{s,a} = H^{(w)}$ for $w(\xi) = \langle\xi\rangle^s \langle\xi''\rangle^k$, Lemma 4.2 in turn reduces this to the estimate

$$\begin{aligned}\int \frac{\langle\xi\rangle^{2s}}{\langle\eta\rangle^{2s}\langle\eta''\rangle^{2a}\langle\xi-\eta\rangle^{2s}\langle\xi''-\eta''\rangle^{2b}}\,d\eta \\ &\lesssim \int \frac{d\eta}{\langle\eta''\rangle^{2a}\langle\xi-\eta\rangle^{2s}\langle\xi''-\eta''\rangle^{2b}} + \int \frac{d\eta}{\langle\eta\rangle^{2s}\langle\eta''\rangle^{2a}\langle\xi''-\eta''\rangle^{2b}} \\ &\leq \int \frac{d\eta'}{\langle\xi'-\eta'\rangle^{2s'}} \int \frac{d\eta''}{\langle\eta''\rangle^{2a}\langle\xi''-\eta''\rangle^{2b+2(s-s')}} + \int \frac{d\eta'}{\langle\eta'\rangle^{2s'}} \int \frac{d\eta''}{\langle\eta''\rangle^{2a+2(s-s')}\langle\xi''-\eta''\rangle^{2b}},\end{aligned}$$

where we choose $\frac{1}{2}d < s' < s$ such that $a+b+s-s' > \frac{1}{2}(n-d)$, which holds if $k+s > \frac{1}{2}(n-d)+s'$, which is possible by our assumptions on s and k. The integrals are uniformly bounded in ξ: for the η'-integrals, this follows from $s' > \frac{1}{2}d$; for the η''-integrals, we use Lemma 4.3. □

We shall now use this (noninvariant) result to prove algebra properties of spaces with iterated module regularity: Consider a compact manifold without boundary X and a submanifold Y. Let $\mathcal{M} \supset \Psi^0(X)$ be the $\Psi^0(X)$-module of first-order pseudodifferential operators whose principal symbol vanishes on N^*Y. For $s \in \mathbb{R}$ and $k \in \mathbb{N}_0$, define

$$H^{s,k}(X,Y) = \{u \in H^s(X) : \mathcal{M}^k u \in H^s(X)\}.$$

Proposition 4.5. *Suppose* $\dim(X) = n$ *and* $\operatorname{codim}(Y) = d$. *Assume that* $s > \frac{1}{2}d$ *and* $s+k > \frac{1}{2}n$. *Then* $H^{s,k}(X,Y)$ *is an algebra.*

Proof. Away from Y, $H^{s,k}(X,Y)$ is just $H^{s+k}(X)$, which is an algebra since $s+k > \frac{1}{2}\dim(X)$. Thus, since the statement is local, we may assume that we have a product decomposition near Y, namely $X = \mathbb{R}^d_{x'} \times \mathbb{R}^{n-d}_{x''}$, $Y = \{x' = 0\}$, and that we are given arbitrary $u, v \in H^{s,k}(X,Y)$ with compact support close to $(0,0)$ for which we have to show $uv \in H^{s,k}(X,Y)$. Notice that, for $f \in H^s(X)$ with such small support, $f \in H^{s,k}(X,Y)$ is equivalent to $\mathcal{M}'^k f \in H^s(X)$, where $\mathcal{M}'$ is the $C^\infty(M)$-module of differential operators generated by Id, $\partial_{x''_i}$ and $x'_j\partial_{x'_k}$, where $1 \leq i \leq n-d$ and $1 \leq j,k \leq d$.

Thus the proposition follows from the following statement: for s and k as in the statement of the proposition,

$$H^{s,k}(\mathbb{R}^n, \mathbb{R}^{n-d}) := \{u \in H^s(\mathbb{R}^n) : (x')^{\tilde\alpha} D^\alpha_{x'} D^\beta_{x''} u \in H^s(\mathbb{R}^n),\ |\tilde\alpha| = |\alpha|,\ |\alpha|+|\beta| \leq k\}$$

is an algebra. Using the Leibniz rule, we thus have to show that

$$((x')^{\tilde\alpha} D^\alpha_{x'} D^\beta_{x''} u)((x')^{\tilde\gamma} D^\gamma_{x'} D^\delta_{x''} v) \in H^s \tag{4-7}$$

provided $|\tilde\alpha| = |\alpha|$, $|\tilde\gamma| = |\gamma|$ and $|\alpha|+|\beta|+|\gamma|+|\delta| \leq k$. Since the two factors in (4-7) lie in $H^{s,k-|\alpha|-|\beta|}$ and $H^{s,k-|\gamma|-|\delta|}$, respectively, this amounts to showing that $H^{s,a} \cdot H^{s,b} \subset H^s$ for $a+b \geq k$. This, however, is easy to see, since $H^{s,c} \subset \mathcal{Y}_d^{s,c}$ for all $c \in \mathbb{N}_0$ and $\mathcal{Y}_d^{s,a} \cdot \mathcal{Y}_d^{s,b} \subset H^s$ was proved in (4-6). □

In order to be able to obtain sharper results for particular nonlinear equations in Section 4C, we will now prove further results in the case $\operatorname{codim}(Y) = 1$, which we will assume to hold from now on; also, we fix $n = \dim(X)$.

Proposition 4.6. *Assume that $s > \frac{1}{2}$ and $k > \frac{1}{2}(n-1)$. Then $H^{s,k}(X,Y) \cdot H^{s-1,k}(X,Y) \subset H^{s-1,k}(X,Y)$.*

Proof. Using the Leibniz rule, this follows from $\mathcal{Y}_1^{s,a} \cdot \mathcal{Y}_1^{s-1,b} \subset H^{s-1}$ for $a+b \geq k$. This, as before, can be reduced to the local statement on $\mathbb{R}^n = \mathbb{R}_{x_1} \times \mathbb{R}^{n-1}_{x'}$ with $Y = \{x_1 = 0\}$. We write $\xi = (\xi_1, \xi') \in \mathbb{R}^{1+(n-1)}$ and $\eta = (\eta_1, \eta') \in \mathbb{R}^{1+(n-1)}$. By Lemma 4.2, the case $s \geq 1$ follows from the estimate

$$\begin{aligned}\int \frac{\langle \xi \rangle^{2(s-1)}}{\langle \eta \rangle^{2s} \langle \eta' \rangle^{2a} \langle \xi - \eta \rangle^{2(s-1)} \langle \xi' - \eta' \rangle^{2b}} \, d\eta& \\ \lesssim \int \frac{d\eta}{\langle \eta \rangle^{2} \langle \eta' \rangle^{2a} \langle \xi - \eta \rangle^{2(s-1)} \langle \xi' - \eta' \rangle^{2b}} &+ \int \frac{d\eta}{\langle \eta \rangle^{2s} \langle \eta' \rangle^{2a} \langle \xi' - \eta' \rangle^{2b}} \\ \leq 2 \int \frac{d\eta_1}{\langle \eta_1 \rangle^{2s}} \int \frac{d\eta'}{\langle \eta' \rangle^{2a} \langle \xi' - \eta' \rangle^{2b}} &\in L^\infty_\xi\end{aligned}$$

by Lemma 4.3.

If $\frac{1}{2} < s \leq 1$, then ξ_1 and ξ' play different roles. Indeed, the background regularity to be proved is H^{s-1}, $s-1 \leq 0$, thus the continuity of multiplication in the conormal direction to Y is proved by "duality" (using Lemma 4.2 with $M_- < \infty$), whereas the continuity in the tangential (to Y) directions, where both factors have $k > \frac{1}{2}(n-1)$ derivatives, is proved directly (using Lemma 4.2 with $M_+ < \infty$). So, let $u \in \mathcal{Y}_1^{s,a}$ and $v \in \mathcal{Y}_1^{s-1,b}$, and put

$$u_0(\xi) = \langle \xi \rangle^s \langle \xi' \rangle^a u(\xi) \in L^2(\mathbb{R}^n), \quad v_0(\xi) = \langle \xi \rangle^{s-1} \langle \xi' \rangle^b v(\xi) \in L^2(\mathbb{R}^n).$$

Then

$$\langle \xi \rangle^{s-1} \widehat{uv}(\xi) = \int \frac{\langle \eta \rangle^{1-s}}{\langle \xi \rangle^{1-s} \langle \eta' \rangle^b \langle \xi - \eta \rangle^s \langle \xi' - \eta' \rangle^a} u_0(\xi - \eta) v_0(\eta) \, d\eta,$$

hence, by Cauchy–Schwarz and Lemma 4.3,

$$\begin{aligned}&\int \langle \xi \rangle^{2(s-1)} |\widehat{uv}(\xi)|^2 \, d\xi \\ &\quad \leq \int \Big(\int \frac{d\eta'}{\langle \eta' \rangle^{2b} \langle \xi' - \eta' \rangle^{2a}} \Big) \Big(\int \Big| \int \frac{\langle \eta \rangle^{1-s}}{\langle \xi \rangle^{1-s} \langle \xi - \eta \rangle^s} u_0(\xi - \eta) v_0(\eta) \, d\eta_1 \Big|^2 \, d\eta' \Big) \, d\xi \\ &\quad \lesssim \iint \Big(\int |u_0(\xi - \eta)|^2 \, d\eta_1 \Big) \Big(\int \frac{\langle \eta \rangle^{2(1-s)}}{\langle \xi \rangle^{2(1-s)} \langle \xi - \eta \rangle^{2s}} |v_0(\eta)|^2 \, d\eta_1 \Big) \, d\eta' \, d\xi \\ &\quad \lesssim \iint \|u_0(\cdot, \xi' - \eta')\|^2_{L^2} |v_0(\eta)|^2 \Big(\int \frac{1}{\langle \xi - \eta \rangle^{2s}} + \frac{1}{\langle \xi \rangle^{2(1-s)} \langle \xi - \eta \rangle^{2(2s-1)}} \, d\xi_1 \Big) \, d\xi' \, d\eta \\ &\quad \lesssim \|u\|^2_{\mathcal{Y}_1^{s,a}} \|v\|^2_{\mathcal{Y}_1^{s-1,b}},\end{aligned}$$

since $\frac{1}{2} < s \leq 1$, so $1 - s \geq 0$ and $2s - 1 > 0$, and the ξ_1-integral is thus bounded from above by

$$\int \frac{1}{\langle \xi_1 - \eta_1 \rangle^{2s}} + \frac{1}{\langle \xi_1 \rangle^{2(1-s)} \langle \xi_1 - \eta_1 \rangle^{2(2s-1)}} \, d\xi_1 \in L^\infty_{\eta_1}.$$

The proof is complete. ☐

For semilinear equations whose nonlinearity does not involve any derivatives, one can afford to lose derivatives in multiplication statements. We give two useful results in this context, the first being a consequence of Proposition 4.6.

Corollary 4.7. *Let $\mu \in C^\infty(X)$ be a defining function for Y, i.e., $\mu|_Y \equiv 0$, $d\mu \neq 0$ on Y, and μ vanishes on Y only. Suppose $s > \frac{1}{2}$ and $\ell \in \mathbb{C}$ are such that $\Re\ell + \frac{3}{2} > s$. Then multiplication by μ_+^ℓ defines a continuous map $H^{s,k}(X,Y) \to H^{s-1,k}(X,Y)$ for all $k \in \mathbb{N}_0$.*

Proof. By the Leibniz rule it suffices to prove the statement for $k = 0$. We have $\mu_+^\ell \in H^{\Re\ell+1/2-\epsilon;\infty}(X,Y)$ for all $\epsilon > 0$: indeed, the Fourier transform of $\chi(x)x_+^\ell$ on $\mathbb{R}$, with $\chi \in C_c^\infty(\mathbb{R})$, is bounded by a constant multiple of $\langle\xi\rangle^{-\Re\ell-1}$, which is an element of $\langle\xi\rangle^{-r}L^2_\xi$ if and only if $r - \Re\ell - 1 < -\frac{1}{2}$, that is, if $\Re\ell + \frac{1}{2} > r$. Hence, the corollary follows from Proposition 4.6, since one has $\Re\ell + \frac{1}{2} - \epsilon \geq s - 1$ for some $\epsilon > 0$ provided $\Re\ell + \frac{3}{2} > s$. □

Proposition 4.8. *Let $0 \leq s', s_1, s_2 < \frac{1}{2}$ be such that $s' < s_1 + s_2 - \frac{1}{2}$, and let $k > \frac{1}{2}(n-1)$. Then $H^{s_1,k}(X,Y) \cdot H^{s_2,k}(X,Y) \subset H^{s',k}(X,Y)$.*

Proof. Using the Leibniz rule, this reduces to the statement that $\mathcal{Y}_1^{s_1,a} \cdot \mathcal{Y}_1^{s_2,b} \subset H^{s'}$ if $a + b \geq k$. Splitting variables $\xi = (\xi_1, \xi')$, $\eta = (\eta_1, \eta')$, Lemma 4.2 in turn reduces this to the observation that

$$\int \frac{\langle\xi\rangle^{2s'}}{\langle\eta\rangle^{2s_1}\langle\eta'\rangle^{2a}\langle\xi-\eta\rangle^{2s_2}\langle\xi'-\eta'\rangle^{2b}}\,d\eta \lesssim \left(\int \frac{d\eta_1}{\langle\eta_1\rangle^{2(s_1-s')}\langle\xi_1-\eta_1\rangle^{2s_2}} + \int \frac{d\eta_1}{\langle\eta_1\rangle^{2s_1}\langle\xi_1-\eta_1\rangle^{2(s_2-s')}}\right)\int \frac{d\eta'}{\langle\eta'\rangle^{2a}\langle\xi'-\eta'\rangle^{2b}}$$

is uniformly bounded in ξ by Lemma 4.3, in view of $s' < s_1 + s_2 - \frac{1}{2} < \min\{s_1, s_2\}$, thus $s_1 - s' > 0$ and $s_2 - s' > 0$, and $s_1 + s_2 - s' > \frac{1}{2}$, as well as $a + b > \frac{1}{2}(n-1)$. □

Corollary 4.9. *Let $p \in \mathbb{N}$ and $s = \frac{1}{2} - \epsilon$ with $0 \leq \epsilon < 1/(2p)$, and let $k > \frac{1}{2}(n-1)$. Then $u \in H^{s,k}(X,Y)$ implies $u^p \in H^{0,k}(X,Y)$.*

Proof. Proposition 4.8 gives $u^2 \in H^{1/2-2\epsilon-\epsilon_2',k}$ for all $\epsilon_2' > 0$, thus $u^3 \in H^{1/2-3\epsilon-\epsilon_3',k}$ for all $\epsilon_3' > 0$, since $\epsilon_2' > 0$ is arbitrary; continuing in this way gives $u^p \in H^{1/2-p\epsilon-\epsilon_p',k}$ for all $\epsilon_p' > 0$, and the claim follows. □

4C. ***A class of semilinear equations.*** Recall that, provided $s < \frac{1}{2} - \Im\sigma$, we have a forward solution operator $S_\sigma : H^{s-1,k}(\Omega)^{\bullet,-} \to H^{s,k}(\Omega)^{\bullet,-}$ of P_σ, defined in (4-1). Let us fix such $s \in \mathbb{R}$ and $\sigma \in \mathbb{C}$. Undoing the conjugation, we obtain a forward solution operator

$$S = \mu^{-1/2}\mu^{-i\sigma/2+(n+1)/4}S_\sigma\mu^{i\sigma/2-(n+1)/4}\mu^{-1/2},$$
$$S : \mu^{(n+3)/4+\Im\sigma/2}H^{s-1,k}(\Omega)^{\bullet,-} \to \mu^{(n-1)/4+\Im\sigma/2}H^{s,k}(\Omega)^{\bullet,-},$$

of $\Box_g - \frac{1}{4}(n-1)^2 - \sigma^2$. Since g is a 0-metric, the natural vector fields to appear in a nonlinear equation are 0-vector fields; see Section 4E for a brief discussion of these concepts. However, since the analysis is based on ordinary Sobolev spaces relative to which one has b-regularity (regularity with respect to the module $\mathcal{M}$), we consider b-vector fields in the nonlinearities. In case one does use 0-vector fields, the solvability conditions can be relaxed; see Section 4D.

Theorem 4.10. *Suppose $s < \frac{1}{2} - \Im\sigma$. Let*

$$q : \mu^{(n-1)/4+\Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-} \times \mu^{(n-1)/4+\Im\sigma/2} H^{s,k-1}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \to \mu^{(n+3)/4+\Im\sigma/2} H^{s-1,k}(\Omega)^{\bullet,-}$$

be a continuous function with $q(0,0) = 0$ such that there exists a continuous nondecreasing function $L : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying

$$\|q(u, {}^{\mathrm{b}}du) - q(v, {}^{\mathrm{b}}dv)\| \leq L(R)\|u - v\|, \quad \|u\|,\ \|v\| \leq R.$$

Then there is a constant $C_L > 0$ such that the following holds: if $L(0) < C_L$ then, for small $R > 0$, there exists $C > 0$ such that, for all $f \in \mu^{(n+3)/4+\Im\sigma/2} H^{s-1,k}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the equation

$$\left(\Box_g - \left(\tfrac{1}{2}(n-1)\right)^2 - \sigma^2\right)u = f + q(u, {}^{\mathrm{b}}du)$$

has a unique solution $u \in \mu^{(n-1)/4+\Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

Proof. Use the Banach fixed point theorem as in the proof of Theorem 2.25. □

Remark 4.11. As in Theorem 2.25, we can also allow nonlinearities $q(u, {}^{\mathrm{b}}du, \Box_g u)$, provided

$$q : \mu^{(n-1)/4+\Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-} \times \mu^{(n-1)/4+\Im\sigma/2} H^{s-1,k}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet,-} \times \mu^{(n+3)/4+\Im\sigma/2} H^{s-1,k}(\Omega)^{\bullet,-} \\ \to \mu^{(n+3)/4+\Im\sigma/2} H^{s-1,k}(\Omega)^{\bullet,-}$$

is continuous, $q(0,0,0) = 0$ and q has a small Lipschitz constant near 0.

4D. *Semilinear equations with polynomial nonlinearity.*

Next, we want to find a forward solution of the semilinear PDE

$$\left(\Box_g - \left(\tfrac{1}{2}(n-1)\right)^2 - \sigma^2\right)u = f + c\mu^A u^p X(u), \tag{4-8}$$

where $c \in C^\infty(\widetilde{X})$ and $X(u) = \prod_{j=1}^q X_j u$ is a q-fold product of derivatives of u along vector fields $X_j \in \mathcal{M}$. The purpose of the following computations is to obtain conditions on A, p and q which guarantee that the map $u \mapsto c\mu^A u^p X(u)$ satisfies the conditions of the map q in Theorem 4.10. Note that the derivatives in the nonlinearity lie in the module $\mathcal{M}$ (in coordinates: $\mu\partial_\mu$ and ∂_y), whereas, as mentioned above, the natural vector fields are 0-derivatives (in coordinates: $x\partial_x = 2\mu\partial_\mu$ and $x\partial_y = \mu^{1/2}\partial_y$) but, since it does not make the computation more difficult, we consider module instead of 0-derivatives and compensate this by allowing any weight μ^A in front of the nonlinearity.

Rephrasing the PDE in terms of P_σ using $\tilde{u} = \mu^{i\sigma/2-(n+1)/4+1/2}u$ and $\tilde{f} = \mu^{-1/2+i\sigma/2-(n+1)/4} f$, we obtain

$$\begin{aligned} P_\sigma \tilde{u} &= \tilde{f} + c\mu^A \mu^{-1/2+i\sigma/2-(n+1)/4} \mu^{(p+q)(-i\sigma/2+(n-1)/4)} \tilde{u}^p \prod_{j=1}^q (f_j + X_j \tilde{u}) \\ &= \tilde{f} + c\mu^\ell \tilde{u}^p \prod_{j=1}^q (f_j + X_j \tilde{u}), \end{aligned}$$

where $f_j \in C^\infty(\widetilde{X})$ and

$$\ell = A + (p+q-1)\big(-\tfrac{1}{2}i\sigma + \tfrac{1}{4}(n-1)\big) - 1. \tag{4-9}$$

Therefore, if $\tilde{u} \in H^{s,k}(\Omega)^{\bullet,-}$, we obtain that the right-hand side of the equation lies in $H^{s,k-1}(\Omega)^{\bullet,-}$ if $\tilde{f} \in H^{s,k-1}(\Omega)^{\bullet,-}$, $s > \frac{1}{2}$, $k > \frac{1}{2}(n+1)$ — which, by Proposition 4.5, implies that $H^{s,k-1}(\Omega)^{\bullet,-}$ is an algebra — and

$$\Re\ell + \tfrac{1}{2} = A + (p+q-1)\big(\tfrac{1}{2}\Im\sigma + \tfrac{1}{4}(n-1)\big) - \tfrac{1}{2} > s, \tag{4-10}$$

since this condition ensures that $\mu^\ell \in H^{s,\infty}(X)$, which implies that multiplication by μ^ℓ is a bounded map $H^{s,k-1}(\Omega)^{\bullet,-} \to H^{s,k-1}(\Omega)^{\bullet,-}$.[13] Given the restriction (4-3) on s and $\Im\sigma$, we see that, by choosing $s > \frac{1}{2}$ close to $\frac{1}{2}$ and $\Im\sigma < 0$ close to 0, we obtain the condition

$$p + q > 1 + \frac{4(1-A)}{n-1}. \tag{4-11}$$

If these conditions are satisfied, the right side of the rewritten PDE lies in $H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-}$, so Theorem 4.10 is applicable, and thus (4-8) is well posed in these spaces.

From (4-11) with $A = 0$, we see that quadratic nonlinearities are fine for $n \geq 6$, and cubic ones for $n \geq 4$.

To sum this up, we revert back to $u = \mu^{(n-1)/4 - i\sigma/2}\tilde{u}$ and $f = \mu^{(n+3)/4 - i\sigma/2}\tilde{f}$:

Theorem 4.12. *Let $s > \frac{1}{2}$ and $k > \frac{1}{2}(n+1)$, and assume $A \in \mathbb{R}$ and $p, q \in \mathbb{N}_0$, $p + q \geq 2$, satisfy condition* (4-10). *Moreover, suppose $\sigma \in \mathbb{C}$ satisfies* (4-3), *i.e., $\Im\sigma < \frac{1}{2} - s$. Finally, let $c \in C^\infty(\widetilde{M})$ and $X(u) = \prod_{j=1}^q X_j u$, where X_j are vector fields in $\mathcal{M}$. Then, for small enough $R > 0$, there exists a constant $C > 0$ such that, for all $f \in \mu^{(n+3)/4 + \Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the PDE*

$$\big(\Box_g - \big(\tfrac{1}{2}(n-1)\big)^2 - \sigma^2\big)u = f + c\mu^A u^p X(u)$$

has a unique solution $u \in \mu^{(n-1)/4 + \Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

The same conclusion holds if the nonlinearity is a finite sum of terms of the form $c\mu^A u^p X(u)$ provided each such term separately satisfies (4-3).

Proof. Reformulating the PDE in terms of $\tilde{u}$ and $\tilde{f}$ as above, this follows from an application of the Banach fixed point theorem to the map

$$H^{s,k}(\Omega)^{\bullet,-} \to H^{s,k}(\Omega)^{\bullet,-}, \quad \tilde{u} \mapsto S_\sigma\bigg(\tilde{f} + \mu^\ell \tilde{u}^p \prod_{j=1}^q (f_j + X_j\tilde{u})\bigg),$$

with ℓ given by (4-9) and $f_j \in C^\infty(\widetilde{X})$. Here, $p + q \geq 2$ and the smallness of R ensure that this map is a contraction on the ball of radius R in $H^{s,k}(\Omega)^{\bullet,-}$. $\square$

[13] If one works in higher regularity spaces, $s \geq \frac{3}{2}$, we in fact only need $\Re\ell + \frac{3}{2} > s$, since then multiplication by μ^ℓ is a bounded map $H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-} \to H^{s-1,k}(\Omega)^{\bullet,-}$. However, the solvability criterion (4-11) would be weaker, namely the role of the dimension n shifts by 2, since in order to use $s \geq \frac{3}{2}$ we need $\Im\sigma < -1$.

Remark 4.13. Even though the above conditions force $\Im\sigma < 0$, let us remark that the conditions of the theorem, most importantly (4-10), can be satisfied if $m^2 = \frac{1}{4}(n-1)^2 + \sigma^2 > 0$ is real, which thus means that we are in fact considering a nonlinear equation involving the Klein–Gordon operator $\Box_g - m^2$. Indeed, let $\sigma = i\tilde{\sigma}$ with $\tilde{\sigma} < 0$; then condition (4-10) with $A = 0$ and $p+q = 2$ becomes $\tilde{\sigma} > 2 - \frac{1}{2}(n-1)$ (where we accordingly have to choose $s > \frac{1}{2}$ close to $\frac{1}{2}$, depending on $\tilde{\sigma}$), and the requirement $\tilde{\sigma} < 0$ forces $n \geq 6$. On the other hand, we want $\frac{1}{4}(n-1)^2 - \tilde{\sigma}^2 = m^2 > 0$; we thus obtain the condition

$$0 < m^2 < \bigl(\tfrac{1}{2}(n-1)\bigr)^2 - \bigl(2 - \tfrac{1}{2}(n-1)\bigr)^2$$

for masses m that Theorem 4.12 can handle, which does give a nontrivial range of allowed m for $n \geq 6$.

Remark 4.14. Let us compare the situation in Theorem 4.12 with the situation for the static model of an asymptotically de Sitter space in Section 2. First, we can solve fewer equations globally on asymptotically de Sitter spaces and, second, we need stronger regularity assumptions in order to make an iterative argument work: In the static model, we needed to be in a b-Sobolev space of order greater than $\frac{1}{2}(n+2)$, which in the non-blown-up picture corresponds to 0-regularity of order greater than $\frac{1}{2}(n+2)$, whereas, in the global version, we need a background Sobolev regularity greater than $\frac{1}{2}$, relative to which we have "b-regularity" (i.e., regularity with respect to the module $\mathcal{M}$) of order greater than $\frac{1}{2}(n+1)$. This comparison is of course only a qualitative one, though, since the underlying geometries in the two cases are different.

Using Proposition 4.6 and Corollary 4.7, one can often improve this result. Thus, let us consider the most natural case of (4-8), in which we use 0-derivatives X_j, corresponding to the 0-structure on the *not* even-ified manifold X, and no additional weight. The only difference this makes is if there are tangential 0-derivatives (in coordinates: $\mu^{1/2}\partial_y$). For simplicity of notation, let us therefore assume that $X_j = \mu^{1/2}\widetilde{X}_j$, $1 \leq j \leq \alpha$, and $X_j = \widetilde{X}_j$, $\alpha < j \leq q$, where the $\widetilde{X}_j$ are vector fields in $\mathcal{M}$. Then the PDE (4-8), rewritten in terms of P_σ, $\tilde{u}$ and $\tilde{f}$, becomes

$$P_\sigma \tilde{u} = \tilde{f} + c\mu^\ell \tilde{u}^p \prod_{j=1}^{q} (\tilde{f}_j + \widetilde{X}_j \tilde{u}) \tag{4-12}$$

with $\tilde{f}_j \in C^\infty(\widetilde{X})$, where

$$\ell = \tfrac{1}{2}\alpha + (p+q-1)\bigl(-\tfrac{1}{2}(i\sigma) + \tfrac{1}{4}(n-1)\bigr) - 1.$$

First, suppose that there are no derivatives in the nonlinearity, so that $p \geq 2$ and $q = \alpha = 0$. Then $\mu^\ell \tilde{u}^p \in H^{s-1,k}(\Omega)^{\bullet,-}$ provided $\Re\ell + \frac{3}{2} > s > \frac{1}{2}$ by Corollary 4.7; choosing s arbitrarily close to $\frac{1}{2}$, this is equivalent to

$$\tfrac{1}{2}\Im\sigma + \tfrac{1}{4}(n-1) > 0. \tag{4-13}$$

This is a very natural condition: the solution operator for the linear wave equation produces solutions with asymptotics $\mu^{(n-1)/4 \pm i\sigma/2}$; see (2-38), and recall that we are working with the even-ified manifold with boundary defining function $\mu = x^2$. The nonlinear equation (4-8) should therefore only be well behaved if solutions to the linear equation decay at infinity, i.e., if $\pm\Im\sigma + \frac{1}{4}(n-1) \geq 0$. Since we need

$\Im\sigma < 0$ to be allowed to take $s > \frac{1}{2}$, condition (4-13) is equivalent to the (small) decay of solutions to the linear equation at infinity (where $\mu = 0$).

Next, let us assume that $q > 0$. Then the nonlinear term in (4-12) is an element of

$$\mu^\ell H^{s,k}(\Omega)^{\bullet,-} \cdot H^{s,k-1}(\Omega)^{\bullet,-} \subset H^{s,k-1}(\Omega)^{\bullet,-}$$

by Proposition 4.6, provided $\Re\ell + \frac{1}{2} > s > \frac{1}{2}$, which gives the condition

$$\tfrac{1}{2}\Im\sigma + \tfrac{1}{4}(n-1) > 1 - \tfrac{1}{2}\alpha,$$

where we again choose $s > \frac{1}{2}$ arbitrarily close to $\frac{1}{2}$, so for $\alpha = 2$ we again get condition (4-13) and for $\alpha > 2$ we get an even weaker one.

Finally, let us discuss a nonlinear term of the form $c\mu^A u^p$, $p \geq 2$, in the setting of even lower regularity $0 \leq s < \frac{1}{2}$, the technical tool here being Corollary 4.9: Rewriting the PDE (4-8) with this nonlinearity in terms of P_σ, $\tilde{u}$ and $\tilde{f}$, we get

$$P_\sigma \tilde{u} = \tilde{f} + c\mu^\ell \tilde{u}^p, \quad \ell = A + (p-1)\big(-\tfrac{1}{2}i\sigma + \tfrac{1}{4}(n-1)\big) - 1.$$

Let $s = \frac{1}{2} - \epsilon$ with $0 \leq \epsilon < 1/(2p)$. Then, if $\tilde{u} \in H^{1/2-\epsilon,k}(\Omega)^{\bullet,-}$ with $k > \frac{1}{2}(n-1)$, Corollary 4.9 yields $\tilde{u}^p \in H^{0,k}(\Omega)^{\bullet,-}$; thus

$$\mu^\ell \tilde{u}^p \in H^{0,k}(\Omega)^{\bullet,-} \subset H^{s-1,k}(\Omega)^{\bullet,-}$$

provided $\Re\ell \geq 0$, that is,

$$n > 1 + \frac{4(1-A)}{p-1} - 2\Im\sigma, \tag{4-14}$$

where we still require $\Im\sigma < \frac{1}{2} - s = \epsilon$, which in particular allows σ to be real if $\epsilon > 0$.

In summary:

Theorem 4.15. *Let $p \geq 2$ be an integer, $\frac{1}{2} - 1/(2p) < s \leq \frac{1}{2}$, $k > \frac{1}{2}(n-1)$, and suppose $\sigma \in \mathbb{C}$ is such that $\Im\sigma < \frac{1}{2} - s$. Moreover, assume $A \in \mathbb{R}$ and the dimension n satisfy condition* (4-14). *Then, for small enough $R > 0$, there exists a constant $C > 0$ such that, for all $f \in \mu^{(n+3)/4+\Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$ with $\|f\| \leq C$, the PDE*

$$\big(\Box_g - \big(\tfrac{1}{2}(n-1)\big)^2 - \sigma^2\big)u = f + c\mu^A u^p$$

has a unique solution $u \in \mu^{(n-1)/4+\Im\sigma/2} H^{s,k}(\Omega)^{\bullet,-}$, with $\|u\| \leq R$, that depends continuously on f.

In particular, if $\frac{1}{4} < s < \frac{1}{2}$, $0 < \Im\sigma < \frac{1}{2} - s$ and $A = 0$, then quadratic nonlinearities are fine for $n \geq 5$; if $\Im\sigma = 0$ and $A = 0$, then they work for $n \geq 6$.

4D1. *Backward solutions to semilinear equations with polynomial nonlinearity.* Recalling the setting of Section 4A1, let us briefly turn to the backward problem for (4-8), which we rephrase in terms of P_σ as above. For simplicity, let us only consider the "least sophisticated" conditions, namely $s > \frac{1}{2}$, $k > \frac{1}{2}(n+1)$,

$$A + (p+q-1)\big(\tfrac{1}{2}\Im\sigma + \tfrac{1}{4}(n-1)\big) - \tfrac{1}{2} > s, \tag{4-15}$$

and — this is the important change compared to the forward problem — $s > \frac{1}{2} - \Im\sigma$, where the latter guarantees the existence of the backward solution operator S_σ^-. Thus, if $\Im\sigma > 0$ is large enough and $s > \frac{1}{2}$ satisfies (4-15), then (4-8) is solvable in any dimension.

In the special case that we only consider 0-derivatives and no extra weight, which corresponds to putting $A = q + \frac{1}{2}\alpha$, we obtain the condition

$$\Im\sigma > \frac{4\left(1 - q - \frac{1}{2}\alpha\right) - (p+q-1)(n-1)}{2(p+q+1)}$$

if we choose $s > \frac{1}{2} - \Im\sigma$ close to $\frac{1}{2}$, which in particular allows $\Im\sigma \geq 0$, and thus σ^2 *arbitrary*, if $p > 1 + 4/(n-1)$ (so $p \geq 2$ is acceptable if $n \geq 6$) or $q + \frac{1}{2}\alpha \geq 1$.

4E. ***From static parts to global asymptotically de Sitter spaces.*** Let us consider the equation

$$(\Box_g - m^2)u = f + q(u, {}^0du), \tag{4-16}$$

where the reason for using the 0-differential ${}^0d$ (see below) will be given momentarily. The idea is that every point in X lies in the interior of the backward light cone from some point p at future infinity Y_+, denoted S_p; that is, the blow-up of $\overline{X}$ at p contains the static part S_p of an asymptotically de Sitter space, where the solvability statements have been explained in Section 2. Consider a suitable neighborhood $\Omega_p \subset [\overline{X}; p]$ of the static patch as in Section 2, so the boundary of Ω_p is the union of ∂S_p and an "artificial" spacelike boundary, which on the non-blown-up space $\overline{X}$ all meet at the point p, and a Cauchy surface. In fact, we may choose the Ω_p in a fashion that is uniform in p. We then solve (4-16) on Ω_p, thereby obtaining a forward solution u_p and, by local uniqueness for $\Box_g - m^2$ in X, all such solutions agree on their overlap, i.e., $u_p \equiv u_q$ on $\Omega_p \cap \Omega_q$. Therefore, we can define a function u by setting $u = u_p$ on Ω_p, $p \in Y_+$, which then is a solution of (4-16) on X. To make this precise, we need to analyze the relationships between the function spaces on the Ω_p, $p \in Y_+$, and X. As we will see in Lemma 4.16, b-Sobolev spaces on the blow-ups Ω_p of $\overline{X}$ at boundary points are closely related to 0-Sobolev spaces on X.

Recall the definition of 0-Sobolev spaces on a manifold with boundary M (for us, $M = \overline{X}$) with a 0-metric, that is, a metric of the form $x^{-2}\hat{g}$ with x a boundary defining function, where $\hat{g}$ extends nondegenerately to the boundary: If $\mathcal{V}_0(M) = x\mathcal{V}(M)$ denotes the Lie algebra of 0-vector fields, where $\mathcal{V}(M)$ are smooth vector fields on M, and $\mathrm{Diff}_0^*(M)$ the enveloping algebra of 0-differential operators, then

$$H_0^k(M) = \{u \in L^2(M, d\mathrm{vol}) : Pu \in L^2(M, d\mathrm{vol}),\ P \in \mathrm{Diff}_0^k(M)\}$$

and $H_0^{k,\ell}(M) = x^\ell H_0^k(M)$. For clarity, we shall write $L_0^2(M) = L^2(M, d\mathrm{vol})$. We also recall the definition of the 0-(co)tangent spaces: if $\mathcal{I}_p$ denotes the ideal of $C^\infty(M)$ functions vanishing at $p \in M$, then the 0-tangent space at p is defined as ${}^0T_pM = \mathcal{V}_0(M)/\mathcal{I}_p \cdot \mathcal{V}_0(M)$, and the 0-cotangent space at p, ${}^0T_p^*M$, as the dual of ${}^0T_pM$. In local coordinates $(x, y) \in \mathbb{R}_x \times \mathbb{R}_y^{n-1}$ near the boundary of M, we have $d\mathrm{vol} = f(x, y)(dx/x)(dy/x^{n-1})$ with f smooth and nonvanishing, and $\mathcal{V}_0(M)$ is spanned by $x\partial_x$ and $x\partial_y$; also, $x\partial_x$ and $x\partial_{y_j}$, $j = 2, \dots, n$, form a basis of ${}^0T_pM$ (for $p \in \partial M$, which is the only place where 0-spaces differ from the standard spaces), and dx/x and dy_j/x, $j = 2, \dots, n$, form a basis of ${}^0T^*_pM$. The exterior derivative d induces the first-order 0-differential operator ${}^0d$ on sections of Λ^0TM; this follows from

$$df = (\partial_x f)\,dx + (\partial_y f)\,dy = (x\partial_x f)\,\frac{dx}{x} + (x\partial_y f)\,\frac{dy}{x}.$$

Now, let $\Omega \subset \overline{X}$ be a domain as in Section 4A. Moreover, let $\beta_p : \Omega_p \to X$ be the blow-down map. We then have:

Lemma 4.16. *Let $k \in \mathbb{N}_0$ and $\ell \in \mathbb{R}$. Then there are constants $C > 0$ and $C_\delta > 0$ such that, for all $\delta > 0$,*

$$\|f\|_{H_0^{k,\ell-(n-1)/2-\delta}(\Omega)^\bullet} \leq C_\delta \sup_{p \in Y_+} \|\beta_p^* f\|_{H_b^{k,\ell}(\Omega_p)^{\bullet,-}} \leq CC_\delta \|f\|_{H_0^{k,\ell}(\Omega)^\bullet}. \tag{4-17}$$

Here, $\bullet$ indicates supported distributions at the "artificial" boundary and $-$ extendible distributions at all other boundary hypersurfaces.

Proof. Let us work locally near a point $p \in Y_+$; since $Y_+ \cong \mathbb{S}^{n-1}$ is compact, all constructions below can be made uniformly in p. The only possible issues are near the boundary $Y_+ = \{x = 0\}$, with x a boundary defining function; hence, let us work in a product neighborhood $Y_+ \times [0, 2\epsilon)_x$, $\epsilon > 0$, of Y_+, and let us assume u is supported is $Y_+ \times [0, \epsilon]$.

We use coordinates $x, y_2, \dots, y_n$ such that $y_j = 0$ at p. Coordinates on S_p are then $x, z_2, \dots, z_n$ with $z_j = y_j/x$, that is, $\beta_p(x, z) = (x, xz)$, with the restriction $\sum_{j=2}^n |z_j|^2 \leq 1$. Therefore,

$$\|\beta_p^* f\|^2_{L^2_b} \approx \int_{S_p} |\beta_p^* f(x,z)|^2\,\frac{dx}{x}\,dz = \int_{\beta_p(S_p)} |f(x,xz)|^2\,\frac{dx}{x}\,dz \leq \int |f(x,y)|^2\,\frac{dx}{x}\,\frac{dy}{x^{n-1}} \approx \|f\|^2_{L^2_0}.$$

Adding weights to this estimate is straightforward. Next, we observe

$$\begin{aligned} x\partial_x(\beta_p^* f)(x,z) &= x\partial_x f(x,xz) + zx\partial_y f(x,xz), \\ \partial_z(\beta_p^* f)(x,z) &= x\partial_y f(x,xz) \end{aligned} \tag{4-18}$$

and, since $|z| \leq 1$, we conclude that $\beta_p^* f \in H_b^1(S_p)$ is equivalent to $f, x\partial_x f, x\partial_y f \in L_0^2(\beta_p(S_p))$, which proves the second inequality in (4-17) in the case $k = 1$; the general case is similar.

For the first inequality in (4-17), we first note that the additional weight comes from the number of static parts, i.e., interiors of backward light cones from points in Y_+, that one needs to cover any fixed half space $\{x \geq x_0\}$. Namely, for $0 < x_0 \leq \epsilon$, let $\mathscr{B}(x_0) \subset Y_+$ be a set of points such that every point in $\{x \geq x_0\}$ lies in S_p for some $p \in \mathscr{B}(x_0)$; then we can choose $\mathscr{B}(x_0)$ such that $|\mathscr{B}(x_0)| \leq Cx_0^{-(n-1)}$, where $|\cdot|$ denotes the number of elements in a set. This follows from the observation that the area of the slice $x = x_0$ of S_p within $Y_+ \cong \mathbb{S}^{n-1}$ (keeping in mind that we are working in a product neighborhood of Y_+) is bounded from below by cx_0^{n-1} for some p-independent constant $c > 0$. Indeed, note that null-geodesics of the 0-metric g are, up to reparametrization, the same as null-geodesics of the conformally related metric $x^2 g$, which is a nondegenerate Lorentzian metric up to Y_+. See also Figure 5 below.

Thus, putting $\alpha = \frac{1}{2}(n-1)+\delta$, $\delta > 0$, we estimate

$$\begin{aligned}\int_{x\leq\epsilon} |x^\alpha f(x,y)| \frac{dx}{x}\frac{dy}{x^{n-1}} &= \sum_{j=0}^\infty \int_{2^{-j-1}\epsilon<x\leq 2^{-j}\epsilon} |x^\alpha f(x,y)|^2 \frac{dx}{x}\frac{dy}{x^{n-1}} \\ &\lesssim \sum_{j=0}^\infty 2^{-2\alpha j} \sum_{p\in\mathscr{B}(2^{-j-1}\epsilon)} \|\beta_p^* f\|^2_{L^2_{\mathrm{b}}} \\ &\lesssim \sum_{j=0}^\infty 2^{-2\alpha j}(2^{-j-1}\epsilon)^{-n+1} \sup_{p\in Y_+} \|\beta_p^* f\|^2_{L^2_{\mathrm{b}}} \\ &\lesssim \sum_{j=0}^\infty 2^{-j(2\alpha-n+1)} \sup_{p\in Y_+} \|\beta_p^* f\|^2_{L^2_{\mathrm{b}}},\end{aligned}$$

with the sum converging since $2\alpha - n + 1 = 2\delta > 0$. Weights and higher-order Sobolev spaces are handled similarly, using (4-18). ☐

In particular, this explains why in (4-16) we take $d = {}^0 d : H_0^{k,\ell}(X) \to H_0^{k-1,\ell}(X; {}^0T^*X)$, namely this is necessary in order to make the global equation interact well with the static patches.

Since we want to consider local problems to solve the global one, the nonlinearity q must be local in the sense that $q(u, {}^0du)(p)$ for $p \in X$ only depends on p and its arguments evaluated at p; let us, for simplicity, assume that q is in fact a polynomial, as in (2-43).

Using Corollary 2.28, we then obtain:

Theorem 4.17. *Let $0 \leq \epsilon < \epsilon_0$ with ϵ_0 as in Section 2B, and $s > \max\bigl(\frac{3}{2}+\epsilon, \frac{1}{2}n+1\bigr)$, $s \in \mathbb{N}$. Let*

$$q(u, {}^0du) = \sum_{2\leq j+|\alpha|\leq d} q_{j\alpha} u^j \prod_{k\leq|\alpha|} X_{\alpha,k} u,$$

where $q_{j,\alpha} \in \mathbb{C} + H_0^s(\overline{X})$, $X_{\alpha,k} \in \mathcal{V}_0(M)$. Then there exists $C > 0$ such that, for all $f \in H_0^{s-1,\epsilon}(\Omega)^\bullet$ with $\|f\| \leq C$, the equation

$$(\Box_g - m^2)u = f + q(u, {}^0du)$$

has a unique solution $u \in \bigcap_{\delta>0} H_0^{s,\epsilon-(n-1)/2-\delta}(\Omega)^\bullet$ that depends continuously on f. Here, we allow $m = 0$ if every summand of q contains at least one 0-derivative, and require $m > 0$ if this is not the case, e.g., if $q = q(u)$ is simply the sum of (multiples of) powers of u.

The analogous conclusion also holds for $\Box_g u = f + q({}^0du)$ provided $\epsilon > 0$, with the solution u being in $\bigcap_{\delta>0} H_0^{s,-(n-1)/2-\delta}(\Omega)^\bullet$. Moreover, for all $p \in Y_+$, the limit $u_\partial(p) := \lim_{p'\to p, p'\in X} u(p')$ exists, $u_\partial \in C^{0,\epsilon}(Y_+)$, and $u - u_\partial(\phi\circ \mathfrak{t}_1) \in x^\epsilon C^0(\overline{X})$, where $\phi \circ \mathfrak{t}_1$ is identically 1 near Y_+ and vanishes near the "artificial" boundary of Ω.

Proof. We start by proving the first part: If $f \in H_0^{s-1,\epsilon}(\Omega)^\bullet$ then $f_p = \beta_p^* f \in H_{\mathrm{b}}^{s-1,\epsilon}(S_p)$ is a uniformly bounded family in the respective norms, by Lemma 4.16. We can then use Corollary 2.28 to solve

$$(\Box_g - m^2)u_p = f_p + q(u_p, {}^{\mathrm{b}}du_p)$$

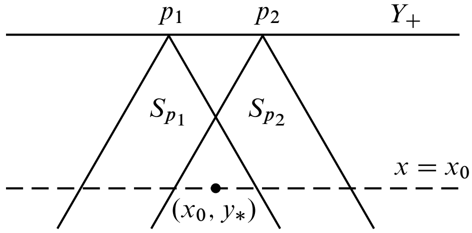

Figure 5. Setup for the proof of $u_\partial \in C^{0,\epsilon}(Y_+)$; shown are the backward light cones from two nearby points $p_1,\ p_2 \in Y_+$ that intersect within the slice $\{x = x_0\}$ at a point (x_0, y_*).

in the static part S_p, where we use that q is a polynomial and the fact that ${}^{\mathrm{b}}T^*_{p'}S_p$ naturally injects into ${}^0T^*_{\beta_p(p')}\Omega$ for $p' \in S_p$ to make sense of the nonlinearity; we thus obtain a uniformly bounded family $u_p = \tilde{u}_p|_{S_p} \in H_{\mathrm{b}}^{s,\epsilon}(S_p)^{\bullet,-}$. By local uniqueness and since f vanishes near Y_-, we see that the function u, defined by $u(\beta_p(p')) = u_p(p')$ for $p \in Y_+$ and $p' \in S_p$, is well defined and, by Lemma 4.16, we indeed have $u \in H_0^{s,\epsilon-(n-1)/2-\delta}(\Omega)^\bullet$ for all $\delta > 0$.

For the second part, we follow the same strategy, obtaining solutions $u_p = c_p(\phi \circ \mathrm{t}_1) + u_p'$ of

$$\Box_g u_p = f_p + q({}^{\mathrm{b}}du_p),$$

where $c_p \in \mathbb{C}$ and $u_p' \in H_{\mathrm{b}}^{s,\epsilon}(S_p)^{\bullet,-}$ are uniformly bounded, thus u_p is uniformly bounded in $H_{\mathrm{b}}^{s,-\delta}(\Omega)^\bullet$ for every fixed $\delta > 0$ and, therefore, the existence of a unique solution u follows as before. Put $u_\partial(p) := c_p$; then $u_\partial(p) = \lim_{p' \to p, p' \in S_p} u(p')$, since $u_p' \in x^\epsilon C^0(S_p)$ by the Sobolev embedding theorem. We first prove that u_∂ so defined is ϵ-Hölder continuous. Let us work in local coordinates (x, y) near a point $(0, y_0)$ in Y_+. Now, u_p' is uniformly bounded in $x^\epsilon C^0(S_p)$ and since, for $x_0 > 0$ arbitrary, we have $c_{p_1} + u'_{p_1}(x_0, y_*) = c_{p_2} + u'_{p_2}(x_0, y_*)$ for all $p_1,\ p_2 \in Y_+$ provided $|p_1 - p_2| \le c x_0$ for some constant $c > 0$, which ensures that $S_{p_1} \cap S_{p_2} \cap \{x = x_0\}$ is nonempty and thus contains a point (x_0, y_*) (see Figure 5), we obtain

$$|c_{p_1} - c_{p_2}| = |u'_{p_1}(x_0, y_*) - u'_{p_2}(x_0, y_*)| \le C x_0^\epsilon \quad \text{when} \quad |p_1 - p_2| \le c x_0$$

for all x_0, thus

$$\frac{|u_\partial(p_1) - u_\partial(p_2)|}{|p_1 - p_2|^\epsilon} \le C, \qquad p_1,\ p_2 \in Y_+.$$

This in particular implies that

$$\begin{aligned} |u(x, y) - u_\partial(0, y_0)| &\le |u(x, y) - u_\partial(0, y)| + |u_\partial(0, y) - u_\partial(0, y_0)| \\ &\le C(|y - y_0|^\epsilon + x^\epsilon) \to 0 \quad \text{as } x \to 0,\ y \to y_0, \end{aligned} \tag{4-19}$$

hence we in fact have $u_\partial(p) = \lim_{p' \to p,\, p' \in X} u(p')$. Finally, putting $y = y_0$ in (4-19) proves that $u - u_\partial(\phi \circ \mathrm{t}_1) \in x^\epsilon C^0(\overline{X})$. $\square$

The major lossy part of the argument is the conversion from f to the family $\beta_p^* f$: even though the second inequality in Lemma 4.16 is optimal (e.g., for functions which are supported in a single static patch), one loses $\frac{1}{2}(n-1)$ orders of decay relative to the gluing estimate, i.e., the first inequality in Lemma 4.16, which is used to pass from the family u_p to u.

Observe on the other hand that the decay properties of u, without regard to those of f, in the first part of the theorem are very natural, since the constant function 1 is an element of $\bigcap_{\delta>0} H_0^{\infty,-(n-1)/2-\delta}(X)$, thus u has an additional decay of ϵ relative to constants.

Remark 4.18. For the proof of Theorem 4.17 it is irrelevant whether certain 0-Sobolev spaces are algebras, since the main analysis, Corollary 2.28, is carried out on b-Sobolev spaces.

5. Lorentzian scattering spaces

5A. ***The linear Fredholm framework.*** We now consider n-dimensional nontrapping asymptotically Minkowski spacetimes (M, g), a notion which includes the radial compactification of Minkowski spacetime. This notion was briefly recalled in the introduction (see p. 1811); here we restate this in the notation of [Baskin et al. 2014, §3], where this notion was introduced.

Thus, M is compact with smooth boundary, with a boundary defining function ρ (we switch the notation from τ mainly to emphasize that ρ is not everywhere timelike) and *scattering vector fields* $V \in \mathcal{V}_{\mathrm{sc}}(M)$, introduced by Melrose [1994], are smooth vector fields of the form $\rho V'$, $V' \in \mathcal{V}_{\mathrm{b}}(M)$. Hence, if the z_j are local coordinates on ∂M extended to a neighborhood in M, then a local basis of these vector fields over $C^\infty(M)$ is $\rho^2\partial_\rho$, $\rho\partial_{z_j}$. Correspondingly, $\mathcal{V}_{\mathrm{sc}}(M)$ is the set of smooth sections of a vector bundle ${}^{\mathrm{sc}}TM$, which is therefore, roughly speaking, $\rho\,{}^{\mathrm{b}}TM$. The vector field $\rho^2\partial_\rho$ is well defined up to a positive factor at $\rho = 0$ and is called the *scattering normal vector field* of ∂M. The dual bundle of ${}^{\mathrm{sc}}TM$, called the *scattering cotangent bundle*, is denoted by ${}^{\mathrm{sc}}T^*M$. If M is the radial compactification of $\mathbb{R}^n$, obtained by gluing a sphere at infinity via the reciprocal polar coordinate map $(r, \omega) \mapsto (r^{-1}, \omega) \in (0, 1)_\rho \times \mathbb{S}^{n-1}_\omega$, that is, adding $\rho = 0$ to the right-hand side (corresponding to "$r = \infty$"), then $\mathcal{V}_{\mathrm{sc}}(M)$ is spanned by (the lifts of) the translation-invariant vector fields over $C^\infty(M)$.

A *Lorentzian scattering metric* g is a Lorentzian signature, taken to be $(1, n-1)$, metric on ${}^{\mathrm{sc}}TM$, i.e., a smooth symmetric section of ${}^{\mathrm{sc}}T^*M \otimes {}^{\mathrm{sc}}T^*M$ with this signature with the following additional properties:

(1) There is a real C^∞ function v defined on M with dv and $d\rho$ linearly independent at "the light cone at infinity", $S = \{v = 0,\ \rho = 0\}$.

(2) $g(\rho^2\partial_\rho, \rho^2\partial_\rho)$ has the same sign as v at $\rho = 0$, i.e., $\rho^2\partial_\rho$ is timelike in $v > 0$ and spacelike in $v < 0$.

(3) Near S,
$$g = v\frac{d\rho^2}{\rho^4} - \left(\frac{d\rho}{\rho^2} \otimes \frac{\alpha}{\rho} + \frac{\alpha}{\rho} \otimes \frac{d\rho}{\rho^2}\right) - \frac{\tilde{h}}{\rho^2},$$
where α is a smooth one-form on M,
$$\alpha = \tfrac{1}{2}\,dv + \mathcal{O}(v) + \mathcal{O}(\rho),$$
and $\tilde{h}$ is a smooth 2-cotensor on M, which is positive definite on the (codimension-two) annihilator of $d\rho$ and dv.

A Lorentzian scattering metric is *nontrapping* if:

(1) $S = S_+ \cup S_-$ (each a disjoint union of connected components) and in $X = \partial M$ the open set $\{v > 0\} \cap X$ decomposes as $C_+ \cup C_-$ (disjoint union), with $\partial C_+ = S_+$ and $\partial C_- = S_-$; we write $C_0 = \{v < 0\} \cap X$.

(2) The projections of all null-bicharacteristics in ${}^{\mathrm{sc}}T^*M \setminus o$ to M tend to $S_\pm$ as their parameter tends to $\pm\infty$ or vice versa.

Since a conformal factor only reparameterizes bicharacteristics, this means that, with $\hat{g} = \rho^2 g$, which is a b-metric on M, the projections of all null-bicharacteristics of $\hat{g}$ in ${}^{\mathrm{b}}T^*M \setminus o$ tend to $S_\pm$. As already pointed out in the introduction (see p. 1812), the difference between the de Sitter-type and Minkowski settings is that at the spherical conormal bundle ${}^{\mathrm{b}}SN^*S$ of S the nature of the radial points is source or sink rather than a saddle point of the flow at $L_\pm$ discussed in Section 2A.

We first state solvability properties, namely we show that, under the assumptions of [Baskin et al. 2014, §3], the problem of finding a tempered solution to $\Box_g w = f$ is a Fredholm problem in suitable weighted Sobolev spaces. In particular, there is only a finite-dimensional obstruction to existence. Then we strengthen the assumptions somewhat and show actual solvability in the strong sense that, in these spaces, the solution w satisfies that, if f is vanishing to infinite order near $\overline{C_-}$, then so is w.

Let

$$L = \rho^{-(n-2)/2}\rho^{-2}\Box_g \rho^{(n-2)/2} \in \mathrm{Diff}_{\mathrm{b}}^2(M)$$

be the "conjugated" b-wave operator (as in [Baskin et al. 2014, §4]), which is formally self-adjoint with respect to the density of the Lorentzian b-metric $\hat{g} = \rho^2 g$; further, $L = \Box_{\hat{g}} - \gamma$, where $\gamma \in C^\infty(M)$ is real-valued. Choose

$$m \in C^\infty({}^{\mathrm{b}}S^*M) \text{ a variable (Sobolev) order function, decreasing along the direction of the Hamilton flow oriented to the future, i.e., towards } S_+. \tag{5-1}$$

Remark 5.1. In the actual application of asymptotically Minkowski spaces, one can take m to be a function on M rather than ${}^{\mathrm{b}}S^*M$ by making it take constant values near $\overline{C_+}$ (resp. $\overline{C_-}$) corresponding to the requirements at $\mathcal{R}_+$ (resp. $\mathcal{R}_-$) below, and transitioning in between using a time function as in the discussion preceding Theorem 5.3, i.e., making m of the form $F \circ \tilde{\mathfrak{t}}$ for appropriate F. Since this simplifies some arguments below, we assume this whenever it is convenient.

With

$$\mathcal{R}_+ = {}^{\mathrm{b}}SN^*S_+ \quad (\text{resp. } \mathcal{R}_- = {}^{\mathrm{b}}SN^*S_-)$$

the future (resp. past) radial sets in ${}^{\mathrm{b}}S^*M$ — see [Baskin et al. 2014, §3.6] — and with

$$m + l < \tfrac{1}{2} \quad \text{at } \mathcal{R}_+, \qquad m + l > \tfrac{1}{2} \quad \text{at } \mathcal{R}_-,$$

and m constant near $\mathcal{R}_+ \cup \mathcal{R}_-$, one has an estimate

$$\|u\|_{H_{\mathrm{b}}^{m,l}} \le C\|Lu\|_{H_{\mathrm{b}}^{m-1,l}} + C\|u\|_{H_{\mathrm{b}}^{m',l}} \tag{5-2}$$

provided one assumes $m' < m$,

$$m' + l > \tfrac{1}{2} \quad \text{at } \mathcal{R}_- \qquad \text{and} \qquad u \in H_{\mathrm{b}}^{m',l}.$$

To see this, we recall and record a slight improvement of [Baskin et al. 2014, Proposition 4.4]:

Proposition 5.2. *Suppose L is as above.*

If $m+l < \frac{1}{2}$ and $u \in H_{\mathrm{b}}^{-\infty,l}(M)$, then $\mathcal{R}_\pm$ (and thus a neighborhood of $\mathcal{R}_\pm$) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{m,l}(u)$ provided $\mathcal{R}_\pm \cap \mathrm{WF}_{\mathrm{b}}^{m-1,l}(Lu) = \varnothing$ and a punctured neighborhood of $\mathcal{R}_\pm$, with $\mathcal{R}_\pm$ removed, in $\Sigma \cap {}^{\mathrm{b}}S^*M$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{m,l}(u)$.

On the other hand, if $m' + l > \frac{1}{2}$, $m \geq m'$, $u \in H_{\mathrm{b}}^{-\infty,l}(M)$ and $\mathrm{WF}_{\mathrm{b}}^{m',l}(u) \cap \mathcal{R}_\pm = \varnothing$, then $\mathcal{R}_\pm$ (and thus a neighborhood of $\mathcal{R}_\pm$) is disjoint from $\mathrm{WF}_{\mathrm{b}}^{m,l}(u)$ provided $\mathcal{R}_\pm \cap \mathrm{WF}_{\mathrm{b}}^{m-1,l}(Lu) = \varnothing$.

Proof. The first statement is proved in [Baskin et al. 2014, Proposition 4.4]. The second statement follows the same way, but in that case the product of the required powers of the boundary defining functions, $\rho^{-2l}\tilde{\rho}^{-2m+1}$, with $\tilde{\rho}$ the defining function of fiber infinity[14] as in Section 2A, in the commutant of [Baskin et al. 2014, Proposition 4.4] provides a favorable sign, thus [Baskin et al. 2014, Equation (4.1)] holds without the E term. However, when regularizing, the regularizer contributes a term with the opposite sign, exactly as in [Vasy 2013a, Proof of Propositions 2.3–2.4]; this forces the requirement on the a priori regularity, namely $\mathrm{WF}_{\mathrm{b}}^{m',l}(u) \cap \mathcal{R}_\pm = \varnothing$, exactly as in those propositions; see also Proposition 2.1 above. ☐

Indeed, due to the closed graph theorem, (5-2) follows immediately from the b-radial point regularity statements of Proposition 5.2 for sources and sinks, and the propagation of b-singularities for variable-order Sobolev spaces, which is not proved in [Baskin et al. 2014], but whose analogue in standard Sobolev spaces is proved there in [Baskin et al. 2014, Proposition A.1] (with additional references given to related results in the literature) and, as it is a purely symbolic argument, the extension to the b-setting is straightforward. (We refer to Proposition 2.1 here and [Baskin et al. 2014, Proposition 4.4] extending the radial point results, Propositions 2.3–2.4, of [Vasy 2013a], from the boundaryless setting to the b-setting.)

One also has a similar estimate for L when one replaces m by a weight $\widetilde{m}$ which is increasing along the direction of the Hamilton flow oriented towards the past,

$$\widetilde{m} + \tilde{l} > \tfrac{1}{2} \quad \text{at } \mathcal{R}_+, \qquad \widetilde{m} + \tilde{l} < \tfrac{1}{2} \quad \text{at } \mathcal{R}_-,$$

provided one assumes $\widetilde{m}' < \widetilde{m}$,

$$\widetilde{m}' + \tilde{l} > \tfrac{1}{2} \quad \text{at } \mathcal{R}_+, \qquad u \in H_{\mathrm{b}}^{\widetilde{m}',\tilde{l}}.$$

Further, L can be replaced by L^*. Thus,

$$\|u\|_{H_{\mathrm{b}}^{\widetilde{m},\tilde{l}}} \leq C\|L^*u\|_{H_{\mathrm{b}}^{\widetilde{m}-1,\tilde{l}}} + C\|u\|_{H_{\mathrm{b}}^{\widetilde{m}',\tilde{l}}}. \tag{5-3}$$

Just as in the asymptotically de Sitter and Kerr–de Sitter settings, one wants to improve these estimates so that the space $H_{\mathrm{b}}^{m,l}$ and, respectively, $H_{\mathrm{b}}^{\widetilde{m},\tilde{l}}$ on the left-hand side includes compactly into the error term

[14] This defining function is denoted by ν in [Baskin et al. 2014].

on the right-hand side. This argument is completely analogous to Section 2A using the Mellin-transformed normal operator estimates obtained in [Baskin et al. 2014, §5]. We thus further assume that there are no poles of the Mellin conjugate $\hat{L}(\sigma)$ on the line $\Im\sigma = -l$. Then, using the Mellin transform and the estimates for $\hat{L}(\sigma)$ (including the high-energy estimates, which imply that for all but a discrete set of l the aforementioned lines do not contain such poles), as in Section 2A we obtain that, on $\mathbb{R}^+_\rho \times \partial M$,

$$\|v\|_{H_{\mathrm{b}}^{\widehat{m},l}} \leq C\|N(L)v\|_{H_{\mathrm{b}}^{\widehat{m}-1,l}} \tag{5-4}$$

when $\widehat{m} \in C^\infty(S^*\partial M)$ is a variable-order function decreasing along the direction of the Hamilton flow oriented to the future, Λ_+ (resp. Λ_-) is the future (resp. past) radial set in $S^*\partial M$, and with

$$\widehat{m} + l < \tfrac{1}{2} \quad \text{at } \Lambda_+, \qquad \widehat{m} + l > \tfrac{1}{2} \quad \text{at } \Lambda_-.$$

One can take

$$\widehat{m} = m|_{T^*\partial M},$$

for instance, under the identification of $T^*\partial M$ as a subspace of ${}^{\mathrm{b}}T^*_{\partial M}M$, taking into account that homogeneous degree-zero functions on $T^*\partial M \setminus o$ are exactly functions on $S^*\partial M$, and analogously on ${}^{\mathrm{b}}T^*_{\partial M}M$. However, in the limit $\sigma \to \infty$, one should use norms depending on σ, reflecting the dependence of the semiclassical norm on h. We recall from Remark 5.1 that in the main case of interest one can take m to be a pullback from M and thus the Mellin-transformed operator norms are independent of σ. In either case, we simply write m in place of $\widehat{m}$.

Again, we have an analogous estimate for $N(L^*)$:

$$\|v\|_{H_{\mathrm{b}}^{\tilde{m},\tilde{l}}} \leq C\|N(L^*)v\|_{H_{\mathrm{b}}^{\tilde{m}-1,\tilde{l}}} \tag{5-5}$$

provided $-\tilde{l}$ is not the imaginary part of a pole of $\widehat{L^*}$, and provided $\tilde{m}$ satisfies the requirements above. As $\widehat{L^*}(\sigma) = (\hat{L})^*(\bar{\sigma})$, the requirement on $-\tilde{l}$ is the same as $\tilde{l}$ not being the imaginary part of a pole of $\hat{L}$.

At this point, the argument of the paragraph of (2-10) in Section 2A can be repeated verbatim to yield that, for m with $m+l > \frac{3}{2}$ at $\mathcal{R}_-$ (with the stronger restriction coming from the requirements on m' at $\mathcal{R}_-$, $\tilde{m}'$ at $\mathcal{R}_+$, and $m' < m-1$, $\tilde{m}' < \tilde{m}-1$; recall that one needs to estimate the normal operator on these primed spaces) and $m+l < \frac{1}{2}$ at $\mathcal{R}_+$,

$$\|u\|_{H_{\mathrm{b}}^{m,l}} \leq C\|Lu\|_{H_{\mathrm{b}}^{m-1,l}} + C\|u\|_{H_{\mathrm{b}}^{m'+1,l-1}}, \tag{5-6}$$

where now the inclusion $H_{\mathrm{b}}^{m,l} \to H_{\mathrm{b}}^{m'+1,l-1}$ is compact (as we choose $m' < m-1$); this argument required m, l and m' satisfied the requirements preceding (5-2), and that $-l$ is not the imaginary part of any pole of $\hat{L}$.

Analogous estimates hold for L^*:

$$\|u\|_{H_{\mathrm{b}}^{\tilde{m},\tilde{l}}} \leq C\|L^*u\|_{H_{\mathrm{b}}^{\tilde{m}-1,\tilde{l}}} + C\|u\|_{H_{\mathrm{b}}^{m'+1,\tilde{l}-1}} \tag{5-7}$$

provided $\tilde{m}$, $\tilde{l}$ and $\tilde{m}'$ satisfy the requirements stated before (5-3), $\tilde{m}' < \tilde{m}-1$, and $-\tilde{l}$ is not the imaginary part of a pole of $\widehat{L^*}$ (i.e., $\tilde{l}$ of $\hat{L}$).

Via the same functional analytic argument as in Section 2A, we thus obtain Fredholm properties of L, in particular solvability, modulo a (possible) finite-dimensional obstruction, in $H_{\mathrm{b}}^{m,l}$ if

$$m+l>\tfrac{3}{2}\quad\text{at }\mathcal{R}_-,\qquad m+l<-\tfrac{1}{2}\quad\text{at }\mathcal{R}_+.$$

More precisely, we take $\tilde{m}=1-m$ and $\tilde{l}=-l$, so $m+l<-\frac{1}{2}$ at $\mathcal{R}_+$ means $\tilde{m}+\tilde{l}=1-(m+l)>\frac{3}{2}$, so the space on the left-hand side of (5-6) is dual to that in the first term on the right-hand side of (5-7), and the same for the equations interchanged. Then the Fredholm statement is for

$$L:\mathcal{X}^{m,l}\to\mathcal{Y}^{m-1,l},$$

with

$$\mathcal{Y}^{s,r}=H_{\mathrm{b}}^{s,r},\quad \mathcal{X}^{s,r}=\{u\in H_{\mathrm{b}}^{s,r}:Lu\in H_{\mathrm{b}}^{s-1,r}\}.$$

Note that, by propagation of singularities, i.e., most importantly using Proposition 5.2, with $\operatorname{Ker}L\subset H_{\mathrm{b}}^{m,l}$ and $\operatorname{Ker}L^*\subset H_{\mathrm{b}}^{1-m,-l}$ a priori,

$$\operatorname{Ker}L\subset H_{\mathrm{b}}^{m^\flat,l}\ \text{ and }\ \operatorname{Ker}L^*\subset H_{\mathrm{b}}^{1-m^\flat,-l}\quad\text{if}\quad m^\flat+l>\tfrac{1}{2}\text{ at }\mathcal{R}_-\text{ and }m^\flat+l<\tfrac{1}{2}\text{ at }\mathcal{R}_+.\tag{5-8}$$

We can improve this further using the propagation of singularities. Namely, suppose one merely has

$$m+l>\tfrac{3}{2}\quad\text{at }\mathcal{R}_-,\qquad m+l<\tfrac{1}{2}\quad\text{at }\mathcal{R}_+,\tag{5-9}$$

so the requirement at $\mathcal{R}_+$ is weakened. Then let $m^\sharp=m-1$ near $\mathcal{R}_+$ and $m^\sharp\le m$ everywhere, but still satisfying the requirements for the order function along the Hamilton flow, so the Fredholm result is applicable with $m^\sharp$ in place of m. Now, if $u\in\mathcal{X}^{m^\sharp,l}$, $Lu=f$ and $f\in\mathcal{Y}^{m-1,l}\subset\mathcal{Y}^{m^\sharp-1,l}$, then Proposition 5.2 gives $u\in\mathcal{X}^{m,l}$. Further, if $\operatorname{Ker}L$ and $\operatorname{Ker}L^*$ are trivial, this gives that, for m and l as in (5-9) satisfying also the conditions along the Hamilton flow, $L:\mathcal{X}^{m,l}\to\mathcal{Y}^{m-1,l}$ is invertible.

Now, as invertibility (the absence of kernel and cokernel) is preserved under sufficiently small perturbations, it holds in particular for perturbations of the Minkowski metric which are Lorentzian scattering metrics in our sense, with closeness measured in smooth sections of the second symmetric power of ${}^{\mathrm{b}}T^*M$. (Note that nontrapping is also preserved under such perturbations.)

For more general asymptotically Minkowski metrics we note that, due to Theorem 2.21 (which does not have any requirements for the timelike nature of the boundary defining function, and which works locally near $\overline{C_-}$ either by working on (extendible) function spaces or by using the localization given by wave propagation as in §3.3 of [Vasy 2013a] or Section 4A here), elements of $\operatorname{Ker}L$ on $H_{\mathrm{b}}^{m,l}$, with m and l as above, lie in $\dot{C}^\infty(M)$ locally near $\overline{C_-}$ provided all resonances, i.e., poles of $\hat{L}(\sigma)$, in $\Im\sigma<-l$ have polar parts (coefficients of the Laurent series) that map into distributions supported on $\overline{C_+}$. As shown in [Vasy 2014, Remark 4.17], when $\hat{L}(\sigma)$ arises from a Lorentzian conic metric as in[15] [Vasy 2014, Equation (3.5)], but with the arguments applicable without significant changes in our more general

[15]In [Vasy 2014], the boundary defining function used to define the Mellin transform is replaced by its reciprocal, which effectively switches the sign of σ in the operator, but also the backward propagator is considered (propagating toward the past light cone), which reverses the role of σ and $-\sigma$ again, so in fact, the signs in [Vasy 2014] and [Baskin et al. 2014] agree for the formulae connecting the asymptotically hyperbolic resolvents and the global operator, $\hat{L}(\sigma)$.

case, see also [Baskin et al. 2014, §7] for our general setting, and [Vasy 2013a, Remark 4.6] for a related discussion with complex absorption, the resonances of $\hat{L}(\sigma)$ consist of the resonances of the asymptotically hyperbolic resolvents on the caps, namely $\mathcal{R}_{C_+}(\sigma)$ and $\mathcal{R}_{C_-}(-\sigma)$, as well as possibly imaginary integers $\sigma \in i\mathbb{Z} \setminus \{0\}$, with resonant states when $\Im\sigma < 0$ being differentiated delta distributions at $S_+ = \partial C_+$ while the dual states are differentiated delta distributions at $S_- = \partial C_-$ when $\Im\sigma > 0$; the latter arise, e.g., as poles on even-dimensional Minkowski space. More generally, when composed with extension of $C_c^\infty(\overline{C_-} \cup C_0)$ by zero to $C^\infty(X)$ from the right and with restriction to $\overline{C_-} \cup C_0$ from the left, the only poles of $\hat{L}(\sigma)$ are those of $\mathcal{R}_{C_-}(-\sigma)$ as well as the possible $\sigma \in i\mathbb{N}_+$. Thus, fixing $l > -1$, one can conclude that elements of $\operatorname{Ker} L$ are in $\dot{C}^\infty(M)$ locally near $\overline{C_-}$ provided $\mathcal{R}_{C_-}(\tilde{\sigma})$ has no poles in $\Im\tilde{\sigma} > l$. (The only change for $l \le -1$ is that one needs to exclude the potential pure imaginary integer poles as well.) The analogous statement for $\operatorname{Ker} L^*$ on $H_{\mathrm{b}}^{\tilde{m},\tilde{l}}$ is that, fixing $\tilde{l} > -1$, elements are in $\dot{C}^\infty(M)$ near $\overline{C_+}$ provided $\mathcal{R}_{C_+}(\tilde{\sigma})$ has no poles in $\Im\tilde{\sigma} > \tilde{l}$. As $\tilde{l} = -l$ for our duality arguments, the weakest symmetric assumption (in terms of strength at C_+ and C_-) is that $\mathcal{R}_{C_\pm}$ do not have any poles in the closed upper half plane; here the closure is added to make sure L is actually Fredholm on $H_{\mathrm{b}}^{m,l}$ with $l = 0$. In general, if one wants to use other values of l, one needs to assume the absence of poles in $\Im\sigma \ge -|l|$ (if one wants to keep the hypotheses symmetric).

Note that, assuming $d\rho/\rho$ is timelike (with respect to $\hat{g}$) near $\overline{C_-}$, one *automatically* has the absence of poles of $\mathcal{R}_{C_-}$ in an upper half plane, and the finiteness (with multiplicity) of the number of poles in any upper half plane, by the semiclassical estimates of [Vasy 2013a, §§3.2 and 7.2] (one can ignore the complex absorption discussion there), so in this case the issue is that of a possible finite number of resonances. There is an analogous statement if $d\rho/\rho$ is timelike near $\overline{C_+}$ for $\mathcal{R}_{C_+}$.

Now, assuming still that $d\rho/\rho$ is timelike at, and hence near, $\overline{C_-}$, it is easy to construct a function $\mathfrak{t}$ which has a timelike differential near $\overline{C_-}$, and appropriate sublevel sets are small neighborhoods of $\overline{C_-}$. Once one has such a function $\mathfrak{t}$, energy estimates can be used to conclude that, in such a neighborhood, rapidly vanishing solutions of $Lu = 0$ actually vanish in this neighborhood, so elements of $\operatorname{Ker} L$ have support disjoint from $\overline{C_-}$; similarly, elements of $\operatorname{Ker} L^*$ have support disjoint from $\overline{C_+}$.

Concretely, with $\widehat{G}$ the dual b-metric of $\hat{g}$, let U_- be a neighborhood of $\overline{C_-}$ and let $0 < \epsilon_0 < \epsilon_1$, $\tilde{\epsilon} > 0$ and $\delta > 0$ be such that $\{\rho \le \tilde{\epsilon},\ v \ge -\epsilon_1\} \cap U_-$ is a compact subset of U_- and, on U_-,

$$\rho < \tilde{\epsilon} \quad \text{and} \quad v > -\epsilon_1 \quad \Longrightarrow \quad \widehat{G}\left(\frac{d\rho}{\rho}, \frac{d\rho}{\rho}\right) > \delta,$$

$$\rho < \tilde{\epsilon}, \quad \text{and} \quad -\epsilon_1 < v < -\epsilon_0 \quad \Longrightarrow \quad \widehat{G}\left(\frac{d\rho}{\rho}, dv\right) < 0 \quad \text{and} \quad \widehat{G}(dv, dv) > 0.$$

Such U_- and constants indeed exist. First, there is U_- and $\tilde{\epsilon}' > 0$, $\epsilon_1' > 0$ such that $\{\rho \le \tilde{\epsilon}',\ v \ge -\epsilon_1'\} \cap U_-$ is a compact subset of U_- since $\overline{C_-}$ is defined by $\{\rho = 0,\ v \ge 0\}$ in a neighborhood of $\overline{C_-}$ with $d\rho \ne 0$ there and $dv \ne 0$ near $v = 0$; we then consider $\tilde{\epsilon} < \tilde{\epsilon}'$ and $\epsilon_1 < \epsilon_1'$ below. Next, since $\widehat{G}(d\rho/\rho, d\rho/\rho)$ is positive on a neighborhood of $\overline{C_-}$ by assumption (thus, for any sufficiently small ϵ_1 and $\tilde{\epsilon}$ there is a desired δ such that the first inequality is satisfied) and $\widehat{G}(d\rho/\rho, dv)|_{S_-} = -2$, any sufficiently small ϵ_1 and $\tilde{\epsilon}$ give $\widehat{G}(d\rho/\rho, dv) < 0$ in the desired region, and finally $\widehat{G}(dv, dv) > 0$ on C_0 near S_- (as

$\widehat{G}(dv, dv) = -4v + \mathcal{O}(v^2)$ there), so, choosing ϵ_1 sufficiently small, $\epsilon_0 < \epsilon_1$, and then $\tilde{\epsilon}$ sufficiently small we satisfy all criteria.

Now let ϵ_- and ϵ_+ be such that $0 < \epsilon_- < \epsilon_+ < \tilde{\epsilon}$, and let $\phi \in C^\infty(\mathbb{R})$ have $\phi' \le 0$, $\phi = 0$ near $[-\epsilon_0, \infty)$, $\phi > \tilde{\epsilon}$ near $(-\infty, -\epsilon_1]$ and $\phi' < 0$ when ϕ takes values in $[\epsilon_-, \epsilon_+]$. Then $\mathfrak{t} = \rho + \phi(v)$ has the property that, on U_-,

$$\mathfrak{t} \le \epsilon_+ \quad \Longrightarrow \quad \rho,\, \phi(v) \le \epsilon_+ \quad \Longrightarrow \quad \rho < \tilde{\epsilon} \quad \text{and} \quad v > -\epsilon_1,$$

and

$$v \ge -\epsilon_0 \quad \Longrightarrow \quad \mathfrak{t} = \rho.$$

Thus, on U_-, if $v \ge -\epsilon_0$ and $\mathfrak{t} \le \epsilon_+$ then $d\mathfrak{t}$ is timelike as $d\rho$ is such, while if $v < -\epsilon_0$ and $\mathfrak{t} \le \epsilon_+$ then

$$\widehat{G}(d\mathfrak{t}, d\mathfrak{t}) = \rho^2 \widehat{G}\left(\frac{d\rho}{\rho}, \frac{d\rho}{\rho}\right) + 2\phi'(v)\rho \widehat{G}\left(\frac{d\rho}{\rho}, dv\right) + (\phi'(v))^2 \widehat{G}(dv, dv)$$

and all terms are nonnegative in view of $-\epsilon_1 < v < -\epsilon_0$ and $\rho \le \tilde{\epsilon}$, with the inequality being strict when $\mathfrak{t} \in [\epsilon_-, \epsilon_+]$ (as well as in $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+])$). Thus, near $\mathfrak{t}^{-1}([\epsilon_-, \epsilon_+]) \cap U_-$, $\mathfrak{t}$ is a timelike function; the same is true on $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+]) \cap U_-$. Choose $\chi \in C^\infty(\mathbb{R})$ with $\chi' \le 0$, $\chi = 1$ near $(-\infty, \epsilon_-]$ and $\chi = 0$ near $[\epsilon_+, \infty)$, and let $\chi \circ \mathfrak{t}$, defined by this formula in U_-, be extended to M as 0 outside U_-; since $\mathfrak{t}^{-1}((-\infty, \epsilon_+]) \cap U_-$ is a compact subset of U_-, this gives a C^∞ function. Further, ρ is also timelike, with $d\rho/\rho$ and $d\mathfrak{t}$ in the same component of the timelike cone; see Figure 6. Correspondingly, one can apply energy estimates using the timelike vector field $V = (\chi \circ \mathfrak{t})\rho^{-\ell} \widehat{G}(d\rho/\rho, \cdot)$; see [Vasy 2013a, §3.3] leading up to Equation (3.24) and the subsequent discussion, which in turn is based on [Vasy 2012, §§3–4]. Here one needs to make both $-\chi'$ large relative to χ and $\ell > 0$ large (making the b-derivative of $\rho^{-\ell}$ large relative to $\rho^{-\ell}$), as discussed in the Mellin-transformed setting in [Vasy 2013a, §3.3], in [Vasy 2012, §§3–4], as well as in Section 2A here (with τ in place of ρ, but with the sign of ℓ reversed due to the difference between b-saddle points and b-sinks/sources). Notice that taking ℓ large is exactly where the rapid decay near $\overline{C_-}$ is used.

We have seen that the existence of appropriate timelike functions, such as $\mathfrak{t}$, in a neighborhood of $\overline{C_+}$ and $\overline{C_-}$ is automatic (in a slightly degenerate sense at $\overline{C_\pm}$ themselves) when $d\rho/\rho$ is timelike in these regions; indeed these functions could be extended to a neighborhood of C_0 if v is appropriately chosen.

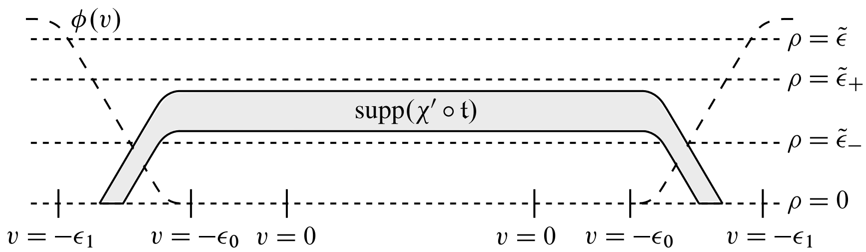

Figure 6. Setup for energy estimates near $\overline{C_-}$; the shaded region is the support of $\chi' \circ \mathfrak{t}$, where $-\chi'$ is used to dominate χ to give positivity in the energy estimate; near $\rho = 0$ and on supp$(\chi \circ \mathfrak{t})$, i.e., in the region between $\rho = 0$ and the shaded region, a sufficiently large weight $\rho^{-\ell}$ gives positivity.

In order to conclude that elements of Ker L and Ker L^* vanish globally, however, we need to control *all* of the interior of M. This can be accomplished by showing global hyperbolicity of M°, which in turn can be seen by applying a result due to Geroch.[16] Namely, by [Geroch 1970, Theorem 11] it suffices to show that a suitable $\mathcal{S}$ is a Cauchy surface, which, by [ibid., Property 6], follows if we show that $\mathcal{S}$ is achronal, closed, and every null-geodesic intersects and then reemerges from $\mathcal{S}$. In order to define $\mathcal{S}$, it is useful to define $\hat{\mathfrak{t}} = \psi \circ \mathfrak{t}$ in U_-, where $\psi \in C^\infty(\mathbb{R})$, $\psi' \geq 0$, $\psi(t) = t$ near $t \leq \epsilon_-$, $\psi'(t) > 0$ for $t < \epsilon_+$ and $\psi'(t) = 0$ for $t \geq \epsilon_+$; let $T = \psi(\epsilon_+) > \epsilon_-$. Further, extend $\hat{\mathfrak{t}}$ to M as equal to T outside U_-; since $U_- \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+])$ is compact, this gives a C^∞ function on M. Thus, $\hat{\mathfrak{t}} \in C^\infty(M)$ is a globally weakly timelike function, in that $\widehat{G}(d\hat{\mathfrak{t}}, d\hat{\mathfrak{t}}) \geq 0$, and it is strictly timelike in $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+))$. In particular, it is monotone along all null-geodesics. Further, $\hat{\mathfrak{t}} = 0$ at S_- and $\hat{\mathfrak{t}} = T > 0$ at S_+, and indeed near S_+. Then we claim that $\mathcal{S} = \hat{\mathfrak{t}}^{-1}(\epsilon_-) \cap M^\circ$ is a Cauchy surface.

Now, $\mathcal{S}$ is closed in M° since $\overline{\mathcal{S}}$ is closed in M; indeed, it is a closed embedded submanifold. By our nontrapping assumption, every null-geodesic in M° tends to S_+ in one direction and S_- in the other direction, so on future-oriented null-geodesics (ones tending to S_+), $\hat{\mathfrak{t}}$ is monotone increasing, attaining all values in $(0, T]$. Since at the ϵ_- level set of $\mathfrak{t}$, and hence of $\hat{\mathfrak{t}}$, $d\hat{\mathfrak{t}}$ is strictly timelike, the value ϵ_- is attained exactly once for $\hat{\mathfrak{t}}$ along null-geodesics. Thus, every null-geodesic intersects $\mathcal{S}$ and then reemerges from it. Finally, $\mathcal{S}$ is achronal, i.e., there exist no timelike curves connecting two points on $\mathcal{S}$: any future-oriented timelike curve (meaning with tangent vector in the timelike cone whose boundary is the future light cone) in $M^\circ \cap \mathfrak{t}^{-1}((-\infty, \epsilon_+))$ has $\hat{\mathfrak{t}}$ monotone increasing, with the increase being strict near $\mathcal{S}$, so again the value ϵ_- can be attained at most once on such a curve. In summary, this proves that M° is globally hyperbolic, so every solution of $Lu = 0$ with vanishing Cauchy data on $\mathcal{S}$ vanishes identically; in particular, by what we have observed, Ker L and Ker L^* are trivial on the indicated spaces.

In summary:

Theorem 5.3. *If* (M, g) *is a nontrapping Lorentzian scattering metric in the sense of* [Baskin et al. 2014], $|l| < 1$, *and*

(1) *the induced asymptotically hyperbolic resolvents* $\mathcal{R}_{C_\pm}$ *have no poles in* $\Im\sigma \geq -|l|$, *and*

(2) $d\rho/\rho$ *is timelike near* $\overline{C_+} \cup \overline{C_-}$,

then, for order functions $m \in C^\infty({}^{\mathrm{b}}S^*M)$ *satisfying* (5-1) *and* (5-9), *the forward problem for the conjugated wave operator* L, *that is, with* L *considered as a map*

$$L : \mathcal{X}^{m,l} \to \mathcal{Y}^{m-1,l},$$

is invertible.

Extending the notation of [Baskin et al. 2014], especially §4, for $m, l \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we denote by $H_{\mathrm{b}}^{m,l,k}(M)$ the space of all $u \in H_{\mathrm{b}}^{m,l}(M)$ (i.e., $u \in \rho^l H_{\mathrm{b}}^m(M)$, where ρ is the boundary defining function of M) such that $\mathcal{M}^j u \in H_{\mathrm{b}}^{m,l}(M)$ for all $0 \leq j \leq k$. Here, $\mathcal{M} \subset \Psi_{\mathrm{b}}^1(M)$ is the $\Psi_{\mathrm{b}}^0(M)$-module of pseudodifferential operators with principal symbol vanishing on the radial set $\mathcal{R}_+$ of the operator $L = \rho^{-(n-2)/2}\rho^{-2}\Box_g \rho^{(n-2)/2}$; in the coordinates ρ, v, y as in [Baskin et al. 2014] (ρ being as above, v

[16] In Geroch's notation, our M° is M.

a defining function of the light cone at infinity within ∂M, and y coordinates within in the light cone at infinity), $\mathcal{M}$ has local generators $\rho\partial_\rho$, $\rho\partial_v$, $v\partial_v$, ∂_y. Then Baskin's results extend our theorem to the spaces with module regularity.

Namely, [Baskin et al. 2014, Proposition 4.4], guarantees the module regularity $u \in H_{\mathrm{b}}^{m,l,k}(M)$ of a solution u of $Lu = f$ if f has matching module regularity $f \in H_{\mathrm{b}}^{m-1,l,k}(M)$ and if u is in $H_{\mathrm{b}}^{m+k,l}(M)$ near $\overline{C_-}$. To be precise, that proposition is stated making the stronger assumption, $f \in H_{\mathrm{b}}^{m-1+k,l}(M)$. However, the proof goes through for just $f \in H_{\mathrm{b}}^{m-1,l,k}(M)$ in a completely analogous manner to the result of Haber and Vasy [2013, Theorem 6.3], where (in the boundaryless setting, for a Lagrangian radial set) the result is stated in this generality.

If $f \in H_{\mathrm{b}}^{m-1,l,k}(M)$ then, in particular, f is locally in $H_{\mathrm{b}}^{m+k-1,l}$ near $\overline{C_-}$, thus, taking into account that $m + l > \frac{1}{2}$ already there, u is in $H_{\mathrm{b}}^{m+k,l}$ in that region by Proposition 5.2 (by the first case there, that is, in the high-regularity regime). Thus, an application of the closed graph theorem gives the following boundedness result:

Theorem 5.4. *Under the assumptions of Theorem 5.3, L^{-1} has the property that it restricts to*

$$L^{-1} : H_{\mathrm{b}}^{m-1,l,k} \to H_{\mathrm{b}}^{m,l,k}, \quad k \geq 0,$$

as a bounded map.

In particular, letting $\Omega = \{\tilde{\mathfrak{t}} \geq 0\}$, where $\tilde{\mathfrak{t}} = \hat{\mathfrak{t}} - \epsilon_-$ so that it attains the value 0 within $M \setminus (\overline{C_+} \cup \overline{C_-})$, we have a forward solution operator S of L which maps $H_{\mathrm{b}}^{m-1,l,k}(\Omega)^\bullet$ into $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$, given that $m + l < \frac{1}{2}$; let us assume that m is constant in Ω. Here, $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ consists of supported distributions at $\partial\Omega \cap C_0^\circ = \{\tilde{\mathfrak{t}} = 0\}$.

Remark 5.5. Using the arguments leading to Theorem 5.3 in the current, forward problem, setting, but now also using standard energy estimates near the artificial boundary $\tilde{\mathfrak{t}} = 0$ of Ω, we see that it suffices to control the resonances of the asymptotically hyperbolic resolvent in the upper cap C_+ in order to ensure the invertibility of the forward problem.

5B. ***Algebra properties of $H_{\mathrm{b}}^{m,-\infty,k}$.*** In order to discuss nonlinear wave equations on an asymptotically Minkowski space, we need to discuss the algebra properties of $H_{\mathrm{b}}^{m,-\infty,k} = \bigcup_{l \in \mathbb{R}} H_{\mathrm{b}}^{m,l,k}$. Even though we are only interested in the space $H_{\mathrm{b}}^{m,-\infty,k}(\Omega)^\bullet$, we consider $H_{\mathrm{b}}^{m,-\infty,k}(M)$, where m is constant on M for notational simplicity, and the results we prove below are valid for $H_{\mathrm{b}}^{m,-\infty,k}(\Omega)^\bullet$ by the same proofs.

We start with the following lemma:

Lemma 5.6. *Let $l_1, l_2 \in \mathbb{R}$ and $k > \frac{1}{2}n$. Then $H_{\mathrm{b}}^{0,l_1,k} \cdot H_{\mathrm{b}}^{0,l_2,k} \subset H_{\mathrm{b}}^{0,l_1+l_2-1/2,k}$.*

Proof. The generators $\rho\partial_\rho$, $\rho\partial_v$, $v\partial_v$, ∂_y of $\mathcal{M}$ take on a simpler form if we blow up the point $(\rho, v) = (0, 0)$. It is most convenient to use projective coordinates on the blown-up space, namely:

(1) Near the interior of the front face, we use the coordinates $\tilde{\rho} = \rho \geq 0$ and $s = v/\rho \in \mathbb{R}$. We compute $\rho\partial_\rho = \tilde{\rho}\partial_{\tilde{\rho}} - s\partial_s$, $v\partial_v = s\partial_s$ and $\rho\partial_v = \partial_s$; since $(d\rho/\rho)\, dv\, dy = d\tilde{\rho}\, ds\, dy$ (this is the b-density

from $H_{\mathrm{b}}^{0,l,k}$), the space $H_{\mathrm{b}}^{0,l,k}$ becomes

$$A^{l,k} := \{u \in \tilde\rho^l L^2(d\tilde\rho\, ds\, dy) : \mathscr{A}^j u \in \tilde\rho^l L^2(d\tilde\rho\, ds\, dy),\ 0 \le j \le k\},$$

where $\mathscr{A}$ is the C^∞-module of differential operators generated by ∂_s, $\tilde\rho\partial_{\tilde\rho}$ and ∂_y.

Now, observe that $\tilde\rho^l L^2(d\tilde\rho\, ds\, dy) = \tilde\rho^{l-1/2} L^2((d\tilde\rho/\rho)\, ds\, dy)$; therefore, we can rewrite

$$A^{l,k} = \Big\{u \in \tilde\rho^{l-1/2} L^2\Big(\frac{d\tilde\rho}{\rho}\, ds\, dy\Big) : \mathscr{A}^j u \in \tilde\rho^{l-1/2} L^2\Big(\frac{d\tilde\rho}{\rho}\, ds\, dy\Big),\ 0 \le j \le k\Big\} = \tilde\rho^{l-1/2} H_{\mathrm{b}}^k\Big(\frac{d\tilde\rho}{\rho}\, ds\, dy\Big).$$

In particular, by the Sobolev algebra property, Lemma 2.26, and the locality of the multiplication, choosing $k > \frac{1}{2}n$ ensures that $\tilde\rho^{l_1-1/2} H_{\mathrm{b}}^k \cdot \tilde\rho^{l_2-1/2} H_{\mathrm{b}}^k \subset \tilde\rho^{l_1+l_2-1} H_{\mathrm{b}}^k$, which is to say $A^{l_1,k} \cdot A^{l_2,k} \subset A^{l_1+l_2-1/2,k}$.

(2) Near either corner of the blown-up space, we use $\tilde v = v$ and $t = \rho/v$ (say, $\tilde v \ge 0$, $t \ge 0$). We compute $\rho\partial_\rho = t\partial_t$, $v\partial_v = \tilde v\partial_{\tilde v} - t\partial_t$, $\rho\partial_v = t\tilde v\partial_{\tilde v} - t^2\partial_t$; and, since $(d\rho/\rho)\, dv\, dy = (dt/t)\, d\tilde v\, dy$, the space $H_{\mathrm{b}}^{0,l,k}$ becomes

$$B^{l,k} := \Big\{u \in (t\tilde v)^l L^2\Big(\frac{dt}{t}\, d\tilde v\, dy\Big) : \mathscr{B}^j u \in (t\tilde v)^l L^2\Big(\frac{dt}{t}\, d\tilde v\, dy\Big),\ 0 \le j \le k\Big\},$$

where $\mathscr{B}$ is the C^∞-module of differential operators generated by $t\partial t$, $\tilde v\partial_{\tilde v}$ and ∂_y. Again, we can rewrite this as

$$B^{l,k} = t^l \tilde v^{l-1/2} H_{\mathrm{b}}^k\Big(\frac{dt}{t}\, \frac{d\tilde v}{\tilde v}\, dy\Big),$$

which implies that, for $k > \frac{1}{2}n$,

$$B^{l_1,k} \cdot B^{l_2,k} \subset t^{l_1+l_2} v^{l_1+l_2-1} H_{\mathrm{b}}^k\Big(\frac{dt}{t}\, \frac{d\tilde v}{\tilde v}\, dy\Big) \subset B^{l_1+l_2-1/2,k}.$$

To relate these two statements to the statement of the lemma, we use cutoff functions χ_A and χ_B to localize within the two coordinate systems. More precisely, choose a cutoff function $\chi \in C_c^\infty(\mathbb{R}_s)$ such that $\chi(s) \equiv 1$ near $s = 0$, $\chi(s) = 0$ for $|s| \ge 2$, and $\chi^{1/2} \in C_c^\infty(\mathbb{R}_s)$. Then multiplication with $\chi_A(\rho, v) := \chi(v/\rho)$ is a continuous map $H_{\mathrm{b}}^{0,l,k} \to A^{l,k}$. Indeed, to check this, one simply observes that $\mathscr{M}^j \chi_A \in L^\infty$ for all $j \in \mathbb{N}_0$. Similarly, letting $\chi_B(\rho, v) := 1 - \chi_A(\rho, v)$, multiplication with χ_B is a continuous map $H_{\mathrm{b}}^{0,l,k} \to B^{l,k}$. Finally, note that we have $A^{l,k}$, $B^{l,k} \subset H_{\mathrm{b}}^{0,l,k}$.

To put everything together, take $u_j \in H_{\mathrm{b}}^{0,l_j,k}$ ($j = 1, 2$); then

$$u_1u_2 = (\chi_A u_1)(\chi_A u_2) + (\chi_B u_1)(\chi_B u_2) + (\chi_A u_1)(\chi_B u_2) + (\chi_B u_1)(\chi_A u_2).$$

The first two terms then lie in $H_{\mathrm{b}}^{0,l_1+l_2-1/2,k}$. To deal with the third term, write

$$(\chi_A u_1)(\chi_B u_2) = (\chi_A^{1/2} u_1)(\chi_A^{1/2} \chi_B u_2) \in A^{l_1,k} \cdot A^{l_2,k} \subset H_{\mathrm{b}}^{0,l_1+l_2-1/2,k},$$

and likewise for the fourth term. Thus, $u_1u_2 \in H_{\mathrm{b}}^{0,l_1+l_2-1/2,k}$, as claimed. □

Remark 5.7. The proof actually shows more, namely that

$$H_{\mathrm{b}}^{0,l,k} H_{\mathrm{b}}^{0,l',k} \subset \rho_{\mathrm{ff}}^{-1/2} H_{\mathrm{b}}^{0,l+l',k}, \tag{5-10}$$

where ρ_{ff} is the defining function of the front face $\rho = v = 0$, e.g., $\rho_{\mathrm{ff}} = (\rho^2 + v^2)^{1/2}$. The reason that (5-10) is a natural statement is that module- and b-derivatives are the same away from $\rho = v = 0$; hence, regularity with respect to the module $\mathcal{M}$ is, up to a weight that is a power of ρ_{ff}, the same as b-regularity.

More abstractly speaking, the above proof shows the following: if ρ_b denotes a boundary defining function of the other boundary hypersurface $\partial[M; S_+] \setminus \mathrm{ff}$ of $[M; S_+]$, then

$$H_{\mathrm{b}}^{0,l,k} \cong \rho_{\mathrm{ff}}^{-1/2} (\rho_{\mathrm{ff}} \rho_b)^l H_{\mathrm{b}}^k([M; S_+]).$$

Note that one can also show this in one step, introducing the coordinates $\rho_{\mathrm{ff}} \geq 0$ and $s = v/(\rho + \rho_{\mathrm{ff}}) \in [-1, 1]$ on $[M; S_+]$ in a neighborhood of ff, and mimicking the above proof, which, however, is computationally less convenient.

Remark 5.8. We can extend the lemma to $H_{\mathrm{b}}^{m,l,k} H_{\mathrm{b}}^{m,l',k} \subset H_{\mathrm{b}}^{m,l+l'-1/2,k}$ for $m \in \mathbb{N}_0$ using the Leibniz rule to distribute the m b-derivatives among the two factors and then using the lemma for the case $m = 0$.

The following corollary, which will play an important role in Section 5E, improves Lemma 5.6 if we have higher b-regularity.

Corollary 5.9. *Let $k > \frac{1}{2}n$, $0 \leq \delta < 1/n$ and $l, l' \in \mathbb{R}$. Then*:

(1) $H_{\mathrm{b}}^{1,l,k} H_{\mathrm{b}}^{0,l',k} \subset H_{\mathrm{b}}^{0,l+l'-1/2+\delta,k}$.

(2) $H_{\mathrm{b}}^{1,l,k} H_{\mathrm{b}}^{1,l',k} \subset H_{\mathrm{b}}^{1,l+l'-1/2+\delta,k}$.

Proof. If $s = 1/(2\delta) > \frac{1}{2}n$, then

$$H_{\mathrm{b}}^{s,l,k} H_{\mathrm{b}}^{0,l',k} \subset H_{\mathrm{b}}^{0,l+l',k}; \tag{5-11}$$

indeed, using the Leibniz rule to distribute the k module-derivatives among the two factors and cancelling the weights, this amounts to showing that $H_{\mathrm{b}}^{s,0,k_1} H_{\mathrm{b}}^{0,0,k_2} \subset H_{\mathrm{b}}^{0,0,0}$ for $k_1 + k_2 \geq k$; but this is true even for $k_1 = k_2 = 0$, since H_{b}^s is a multiplier on H_{b}^0 provided $s > \frac{1}{2}n$.

On the other hand, the lemma gives

$$H_{\mathrm{b}}^{0,l,k} H_{\mathrm{b}}^{0,l',k} \subset \rho^{-1/2} H_{\mathrm{b}}^{0,l+l',k}. \tag{5-12}$$

Interpolating in the first factor between (5-11) and (5-12) thus gives the first statement.

For the second statement, use the Leibniz rule to distribute the one b-derivative to either factor; then one has to show $H_{\mathrm{b}}^{1,l,k} H_{\mathrm{b}}^{0,l',k} \subset H_{\mathrm{b}}^{0,l+l'-1/2+\delta,k}$ and the same inclusion with l and l' switched, which is what we just proved. ☐

Lemma 5.6 and Remark 5.7 imply that, for $u \in H_{\mathrm{b}}^{m,l,k}$, $p \geq 1$, with $m \geq 0$ and $k > \frac{1}{2}n$, we have $u^p \in H_{\mathrm{b}}^{m,pl-(p-1)/2,k}$; in fact, $u^p \in \rho_{\mathrm{ff}}^{-(p-1)/2} H_{\mathrm{b}}^{m,pl,k}$; see Remark 5.7. Using Corollary 5.9, we can improve this to the statement that $u \in H_{\mathrm{b}}^{m,l,k}$ implies $u^p \in H_{\mathrm{b}}^{m,pl-(p-1)/2+(p-1)\delta,k}$ for $m \geq 1$.

For nonlinearities that only involve powers u^p, we can afford to lose differentiability, as at the end of Section 4B, and gain decay in return, as the following lemma shows.

Lemma 5.10. *Let $\alpha > \frac{1}{2}$, $l \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Then $\rho_{\mathrm{ff}}^{-\alpha} H_{\mathrm{b}}^{0,l,k} \subset \rho^{1/2-\alpha} H_{\mathrm{b}}^{-1,l,k}$, where $\rho_{\mathrm{ff}} = (\rho^2 + v^2)^{1/2}$.*

Proof. We may assume $l = 0$ and that u is supported in $|v| < 1$, $\rho < 1$. First, consider the case $k = 0$. Let $u \in \rho_{\mathrm{ff}}^{-\alpha} H_{\mathrm{b}}^{0}$ and put

$$\tilde{u}(\rho, v, y) = \int_{-\infty}^{v} u(\rho, w, y)\, dw,$$

so $\partial_v \tilde{u} = u$. We have to prove $\chi\tilde{u} \in \rho^{1/2-\alpha} H_{\mathrm{b}}^{0}$ if $\chi \equiv 1$ near $\operatorname{supp} u$, which implies $u \in H_{\mathrm{b}}^{-1}$, as $\partial_v : H_{\mathrm{b}}^{0} \to H_{\mathrm{b}}^{-1}$ and the b-Sobolev space are local spaces. But

$$|\tilde{u}(\rho, v, y)|^2 \le \left(\int_{-1}^{1} \rho_{\mathrm{ff}}(\rho, w)^{2\alpha} |u(\rho, w, y)|^2\, dw \right) \int_{-1}^{1} \rho_{\mathrm{ff}}(\rho, w)^{-2\alpha}\, dw; \tag{5-13}$$

now,

$$\int_{-1}^{1} \rho_{\mathrm{ff}}^{-2\alpha}\, dw = \rho^{1-2\alpha} \int_{-1/\rho}^{1/\rho} \frac{dz}{(1+|z|^2)^{\alpha}} \lesssim \rho^{1-2\alpha}$$

for $\alpha > \frac{1}{2}$, so, with the v integral considered on a fixed interval, say $|v| < 2$ (notice that the right-hand side in (5-13) is independent of v!),

$$\iiint \rho^{2\alpha-1} |\tilde{u}(\rho, v, y)|^2 \frac{d\rho}{\rho}\, dv\, dy \lesssim \iiint \rho_{\mathrm{ff}}^{2\alpha} |u(\rho, w, y)|^2 \frac{d\rho}{\rho}\, dw\, dy,$$

proving the claim for $k = 0$. Now, $\rho\partial_\rho$ and ∂_y just commute with this calculation, so the corresponding derivatives are certainly well behaved. On the other hand, $\partial_v \tilde{u} = u$, so the estimates involving at least one v-derivative are just those for u itself. □

Corollary 5.11. *Let $k,\ p \in \mathbb{N}$ be such that $k > \frac{1}{2}n$ and $p \ge 2$. Let $l \in \mathbb{R}$ and $u \in H_{\mathrm{b}}^{0,l,k}$. Then $u^p \in H_{\mathrm{b}}^{-1, lp-(p-1)/2+1/2-\delta, k}$ with $\delta = 0$ if $p \ge 3$ and $\delta > 0$ if $p = 2$.*

Proof. This follows from $u^p \in \rho_{\mathrm{ff}}^{-(p-1)/2-\delta} H_{\mathrm{b}}^{0,lp,k}$ and the previous lemma, using that $\frac{1}{2}(p-1) + \delta > \frac{1}{2}$ with δ as stated. □

In other words, we gain the decay $\rho^{1/2-\delta}$ if we give up one derivative.

5C. ***A class of semilinear equations.*** We are now set to discuss solutions to nonlinear wave equations on an asymptotically Minkowski space. Under the assumptions of Theorem 5.3, we obtain a forward solution operator $S : H_{\mathrm{b}}^{m-1,l,k}(\Omega)^{\bullet} \to H_{\mathrm{b}}^{m,l,k}(\Omega)^{\bullet}$ of $P = \rho^{-(n-2)/2} \rho^{-2} \Box_g \rho^{(n-2)/2}$ provided $|l| < 1$, $m + l < \frac{1}{2}$ and $k \ge 0$.

Undoing the conjugation, we obtain a forward solution operator

$$\widetilde{S} = \rho^{(n-2)/2} S \rho^{-2} \rho^{-(n-2)/2}, \quad \widetilde{S} : H_{\mathrm{b}}^{m-1, l+(n-2)/2+2, k}(\Omega)^{\bullet} \to H_{\mathrm{b}}^{m, l+(n-2)/2, k}(\Omega)^{\bullet},$$

of $\Box_g$.

Since g is a Lorentzian scattering metric, the natural vector fields to appear in a nonlinear equation are scattering vector fields; more generally, since the analysis is carried out on b-spaces, we indeed allow b-vector fields in the following statement:

Theorem 5.12. *Let*

$$q : H_{\mathrm{b}}^{m, l+(n-2)/2, k}(\Omega)^{\bullet} \times H_{\mathrm{b}}^{m-1, l+(n-2)/2, k}(\Omega; {}^{\mathrm{b}}T^*\Omega)^{\bullet} \to H_{\mathrm{b}}^{m-1, l+(n-2)/2+2, k}(\Omega)^{\bullet}$$

be a continuous function with $q(0,0)=0$ such that there exists a continuous nondecreasing function $L:\mathbb{R}_{\geq 0}\to\mathbb{R}$ satisfying

$$\|q(u,{}^{\mathrm{b}}du)-q(v,{}^{\mathrm{b}}dv)\|\leq L(R)\|u-v\|,\quad \|u\|,\ \|v\|\leq R.$$

Then there is a constant $C_L>0$ such that the following holds: if $L(0)<C_L$ then, for small $R>0$, there exists $C>0$ such that, for all $f\in H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^{\bullet}$ with $\|f\|\leq C$, the equation

$$\Box_g u=f+q(u,{}^{\mathrm{b}}du)$$

has a unique solution $u\in H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^{\bullet}$, with $\|u\|\leq R$, that depends continuously on f.

Proof. Use the Banach fixed point theorem as in the proof of Theorem 2.25. ☐

Remark 5.13. Here, just as in Theorem 4.10, we can also allow q to depend on $\Box_g u$.

5D. ***Semilinear equations with polynomial nonlinearity.*** Next, we want to find a forward solution of the semilinear PDE

$$\Box_g u=f+cu^pX(u),$$

where $c\in C^\infty(M)$, $p\in\mathbb{N}_0$, and $X(u)=\prod_{j=1}^q\rho V_j(u)$ is a q-fold product of derivatives of u along scattering vector fields; here, V_j are b-vector fields. Let us assume $p+q\geq 2$ in order for the equation to be genuinely nonlinear. We rewrite the PDE as

$$L(\rho^{-(n-2)/2}u)=\rho^{-(n-2)/2-2}f+c\rho^{-2}\rho^{(p-1)(n-2)/2}(\rho^{-(n-2)/2}u)^p\prod_{j=1}^q\rho V_j(\rho^{(n-2)/2}\rho^{-(n-2)/2}u).$$

Introducing $\tilde{u}=\rho^{-(n-2)/2}u$ and $\tilde{f}=\rho^{-(n-2)/2-2}f$ yields the equation

$$\begin{aligned}L\tilde{u}&=\tilde{f}+c\rho^{(p-1)(n-2)/2-2}\tilde{u}^p\prod_{j=1}^q\rho^{n/2}(f_j\tilde{u}+V_j\tilde{u})\\&=\tilde{f}+c\rho^{(p-1)(n-2)/2+qn/2-2}\tilde{u}^p\prod_{j=1}^q(f_j\tilde{u}+V_j\tilde{u}),\end{aligned}\tag{5-14}$$

where the f_j are smooth functions. Now suppose that $\tilde{u}\in H_{\mathrm{b}}^{m,l,k}(\Omega)^{\bullet}$ with $m+l<\frac{1}{2}$, $m\geq 1$ and $k>\frac{1}{2}n$ (so that $H_{\mathrm{b}}^{m-1,-\infty,k}(\Omega)^{\bullet}$ is an algebra); then the second summand of the right-hand side of (5-14) lies in $H_{\mathrm{b}}^{m-1,\ell,k}(\Omega)^{\bullet}$, where

$$\ell=\tfrac{1}{2}(p-1)(n-2)+\tfrac{1}{2}qn-2+pl-\tfrac{1}{2}(p-1)+ql-\tfrac{1}{2}(q-1)-\tfrac{1}{2}.$$

For this space to lie in $H_{\mathrm{b}}^{m-1,l,k}(\Omega)^{\bullet}$ (which we want in order to be able to apply the solution operator S and land in $H_{\mathrm{b}}^{m,l,k}(\Omega)^{\bullet}$, so that a fixed point argument as in Section 2 can be applied), we thus need $\ell\geq l$, which can be rewritten as

$$\tfrac{1}{2}(p-1)(l+(n-3))+q\big(l+\tfrac{1}{2}(n-1)\big)\geq 2.\tag{5-15}$$

For $m = 1$ and $l < \frac{1}{2} - m$ less than, but close to, $-\frac{1}{2}$, we thus get the condition

$$(p-1)(n-4) + q(n-2) > 4.$$

If there are only nonlinearities involving derivatives of u, i.e., $p = 0$, we get the condition $q > 1+2/(n-2)$, that is, quadratic nonlinearities are fine for $n \geq 5$, and cubic ones for $n \geq 4$.

Note that, if $q = 0$, we can actually choose $m = 0$ and $l < \frac{1}{2}$ close to $\frac{1}{2}$, and we have Corollary 5.11 at hand. Thus we can improve (5-15) to $(p-1)\big(\frac{1}{2} + \frac{1}{2}(n-3)\big) > 2 - \frac{1}{2}$, i.e., $p > 1 + 3/(n-2)$, hence quadratic nonlinearities can be dealt with if $n \geq 6$, whereas cubic nonlinearities are fine as long as $n \geq 4$. Observe that this condition on p always implies $p > 1$, which is a natural condition, since $p = 1$ would amount to changing $\Box_g$ into $\Box_g - m^2$ (if one chooses the sign appropriately). But the Klein–Gordon operator naturally fits into a scattering framework, as mentioned in the introduction (see p. 1812), therefore requires a different analysis; we will not pursue this further in this paper.

To summarize the general case, note that $\tilde{u} \in H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$ is equivalent to $u \in H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^\bullet$, and $\tilde{f} \in H_{\mathrm{b}}^{m-1,l,k}(\Omega)^\bullet$ to $f \in H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$; thus:

Theorem 5.14. *Let $|l| < 1$, $m + l < \frac{1}{2}$, $k > \frac{1}{2}n$, and assume that $p, q \in \mathbb{N}_0$ with $p + q \geq 2$ satisfy condition* (5-15) *or the weaker conditions given above in the cases where $p = 0$ or $q = 0$; let $m \geq 0$ if $q = 0$, otherwise let $m \geq 1$. Moreover, let $c \in C^\infty(M)$ and $X(u) = \prod_{j=1}^{q} X_j u$, where X_j is a scattering vector field on M. Then, for small enough $R > 0$, there exists a constant $C > 0$ such that, for all $f \in H_{\mathrm{b}}^{m-1,l+(n-2)/2+2,k}(\Omega)^\bullet$ with $\|f\| \leq C$, the equation*

$$\Box_g u = f + cu^p X(u)$$

has a unique solution $u \in H_{\mathrm{b}}^{m,l+(n-2)/2,k}(\Omega)^\bullet$, with $\|u\| \leq R$, that depends continuously on f.

The same conclusion holds if the nonlinearity is a finite sum of terms of the form $cu^p X(u)$ provided each such term separately satisfies (5-15).

Proof. Reformulating the PDE in terms of $\tilde{u}$ and $\tilde{f}$ as above, this follows from an application of the Banach fixed point theorem to the map

$$H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet \to H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet, \quad \tilde{u} \mapsto S\bigg(\tilde{f} + c\rho^{(p-1)(n-2)/2+qn/2-2}\tilde{u}^p \prod_{j=1}^{q}(f_j\tilde{u} + V_j\tilde{u})\bigg)$$

with m, l and k as in the statement of the theorem. Here, $p + q \geq 2$ and the smallness of R ensure that this map is a contraction on the ball of radius R in $H_{\mathrm{b}}^{m,l,k}(\Omega)^\bullet$. ☐

Remark 5.15. If the derivatives in the nonlinearity only involve module-derivatives, we get a slightly better result, since we can work with $\tilde{u} \in H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet$. Indeed, a module-derivative falling on $\tilde{u}$ gives an element of $H_{\mathrm{b}}^{0,l,k-1}(\Omega)^\bullet$, applied to which the forward solution operator produces an element of $H_{\mathrm{b}}^{1,l,k-1}(\Omega)^\bullet \subset H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet$.

The numbers work out as follows: In condition (5-15), we now take $l < \frac{1}{2}$ close to $\frac{1}{2}$, thus obtaining

$$(p-1)(n-2) + qn > 4.$$

Thus, in the case that there are only derivatives in the nonlinearity, i.e., $p = 0$, we get $q > 1 + 2/n$, which allows for quadratic nonlinearities provided $n \geq 3$.

Remark 5.16. Observe that we can improve (5-15) in the case $p \geq 1$, $q \geq 1$ and $m \geq 1$ by using the δ-improvement from Corollary 5.9, namely, the right-hand side of (5-14) actually lies in $H_{\mathrm{b}}^{m-1,\ell,k}(\Omega)^\bullet$, where now

$$\ell = \tfrac{1}{2}(p-1)(n-2) + \tfrac{1}{2}qn - 2 + pl - \tfrac{1}{2}(p-1) + (p-1)\delta + ql - \tfrac{1}{2}(q-1) - \tfrac{1}{2} + \delta,$$

which satisfies $\ell \geq l$ if

$$\tfrac{1}{2}(p-1)(l + (n-3) + \delta) + q\bigl(l + \tfrac{1}{2}(n-1)\bigr) + \delta \geq 2,$$

which for $l < -\frac{1}{2}$ close to $-\frac{1}{2}$ means $(p-1)(n-4+2\delta) + q(n-2) + 2\delta > 4$, where $0 < \delta < 1/n$.

Remark 5.17. Let us compare the above result with Christodoulou's [1986]. A special case of his theorem states[17] that the Cauchy problem for the wave equation on Minkowski space with small initial data in $H_{k,k-1}(\mathbb{R}^{n-1})$ admits a global solution $u \in H^k_{\mathrm{loc}}(\mathbb{R}^n)$ with decay $|u(x)| \lesssim (1 + (v/\rho)^2)^{-(n-2)/2}$; here, $k = \frac{1}{2}n + 2$, and n is assumed to be at least 4 and even; when $n = 4$, the nonlinearity is moreover assumed to satisfy the null condition. The only polynomial nonlinearity that we cannot deal with using the above argument is thus the null-form nonlinearity in 4 dimensions.

To make a further comparison possible, we express $H_{k,\delta}(\mathbb{R}^{n-1})$ as a b-Sobolev space on the radial compactification of $\mathbb{R}^{n-1}$: Note that $u \in H_{k,\delta}(\mathbb{R}^{n-1})$ is equivalent to $(\langle x\rangle D_x)^\alpha u \in \langle x\rangle^{-\delta} L^2(\mathbb{R}^{n-1})$, $|\alpha| \leq k$. In terms of the boundary defining function ρ of $\partial\overline{\mathbb{R}^{n-1}}$ and the standard measure $d\omega$ on the unit sphere $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$, we have $L^2(\mathbb{R}^{n-1}) = L^2((d\rho/\rho^2)\,(dy/\rho^{n-2})) = \rho^{(n-1)/2} L^2((d\rho/\rho)\,dy)$, and thus $H_{k,\delta}(\mathbb{R}^{n-1}) = \rho^{(n-1)/2+\delta} H^k_{\mathrm{b}}(\tilde{\mathfrak{t}} = 0)$. Therefore, converting the Cauchy problem into a forward problem, the forcing lies in $H_{\mathrm{b}}^{k,(n-1)/2+k-1,0}(\Omega)^\bullet = H_{\mathrm{b}}^{n/2+2,n+1/2,0}(\Omega)^\bullet$. Comparing this with the space $H_{\mathrm{b}}^{0,l+(n-2)/2+2,n/2+1}$, with $l < \frac{1}{2}$, needed for our argument, we see that Christodoulou's result applies to a regime of fast decay which is disjoint from our slow decay (or even mild growth) regime.

Remark 5.18. In the case of nonlinearities u^p, the result of [Christodoulou 1986] implies the existence of global solutions to $\Box_g u = f + u^p$ if the spacetime dimension n is *even* and $n \geq 4$ if $p \geq 3$; in even dimensions $n \geq 6$, $p \geq 2$ suffices; the above result extends this to all dimensions satisfying the respective inequalities. In a somewhat similar context — see the work of Chruściel and Łęski [2006] — it has been proved that $p \geq 2$ in fact works in all dimensions $n \geq 5$.

5E. *Semilinear equations with null condition.* With g the Lorentzian scattering metric on an asymptotically Minkowski space satisfying the assumptions of Theorem 5.3 as before, define the null-form $Q({}^{\mathrm{sc}}du, {}^{\mathrm{sc}}dv) = g^{jk}\partial_j u \partial_k v$ and write $Q({}^{\mathrm{sc}}du)$ for $Q({}^{\mathrm{sc}}du, {}^{\mathrm{sc}}du)$. We are interested in solving the PDE

$$\Box_g u = Q({}^{\mathrm{sc}}du) + f.$$

[17] Note that n is the dimension of Minkowski space here, whereas Christodoulou uses $n + 1$.

The previous discussion solves this for $n \geq 5$; thus, let us from now on assume $n = 4$. To make the computations more transparent, we will keep the n in the notation and only substitute $n = 4$ when needed. Rewriting the PDE in terms of the operator $L = \rho^{-2}\rho^{-(n-2)/2}\Box_g \rho^{(n-2)/2}$ as above, we get

$$L\tilde{u} = \tilde{f} + \rho^{-(n-2)/2-2} Q({}^{\mathrm{sc}}d(\rho^{(n-2)/2}\tilde{u})),$$

where $\tilde{u} = \rho^{-(n-2)/2}u$ and $\tilde{f} = \rho^{-(n-2)/2-2}f$. We can write $Q({}^{\mathrm{sc}}du) = \frac{1}{2}\Box_g(u^2) - u\Box_g u$, so the PDE becomes

$$\begin{aligned} L\tilde{u} &= \tilde{f} + \rho^{-(n-2)/2-2}\bigl(\tfrac{1}{2}\Box_g(\rho^{n-2}\tilde{u}^2) - \rho^{(n-2)/2}\tilde{u}\Box_g(\rho^{(n-2)/2}\tilde{u})\bigr) \\ &= \tilde{f} + \tfrac{1}{2}L(\rho^{(n-2)/2}\tilde{u}^2) - \rho^{(n-2)/2}\tilde{u}L\tilde{u}. \end{aligned}$$

Since the results of Section 5B give small improvements on the decay of products of $H_{\mathrm{b}}^{1,*,*}$ functions with $H_{\mathrm{b}}^{m,*,*}$ functions ($m \geq 0$), one wants to solve this PDE on a function space that keeps track of these small improvements.

Definition 5.19. For $l \in \mathbb{R}$, $k \in \mathbb{N}_0$ and $\alpha \geq 0$, define the space

$$\mathscr{X}^{l,k,\alpha} := \{v \in H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^\bullet : Lv \in H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet\}$$

with norm

$$\|v\|_{\mathscr{X}^{l,k,\alpha}} = \|v\|_{H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^\bullet} + \|Lv\|_{H_{\mathrm{b}}^{0,l,k}(\Omega)^\bullet}. \tag{5-16}$$

By an argument similar to the one used in the proof of Theorem 2.25, we see that $\mathscr{X}^{l,k,\alpha}$ is a Banach space.

On $\mathscr{X}^{l,k,\alpha}$, with $\alpha > 0$ chosen below, we want to run an iteration argument: Start by defining the operator $T : \mathscr{X}^{l,k,\alpha} \to H_{\mathrm{b}}^{1,-\infty,k}(\Omega)^\bullet$ by

$$T : \tilde{u} \mapsto S(\tilde{f} - \rho^{(n-2)/2}\tilde{u}L\tilde{u}) + \tfrac{1}{2}\rho^{(n-2)/2}\tilde{u}^2.$$

Note that $\tilde{u} \in \mathscr{X}^{l,k,\alpha}$ implies, using Corollary 5.9 with $\delta < 1/n$,

$$\begin{aligned} \rho^{(n-2)/2}\tilde{u}^2 &\in \rho^{(n-2)/2}H_{\mathrm{b}}^{1,2(l+\alpha)-1/2+\delta,k}(\Omega)^\bullet = H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta+\alpha,k}(\Omega)^\bullet, \\ \rho^{(n-2)/2}\tilde{u}L\tilde{u} &\in H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet, \\ S(\rho^{(n-2)/2}\tilde{u}L\tilde{u}) &\in H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet, \end{aligned} \tag{5-17}$$

where in the last inclusion we need to require $1 + \bigl(2l + \alpha + \frac{1}{2}(n-3) + \delta\bigr) < \frac{1}{2}$, which for $n = 4$ means

$$l < -\tfrac{1}{2} - \tfrac{1}{2}(\alpha + \delta); \tag{5-18}$$

let us assume from now on that this condition holds. Furthermore, (5-17) implies that $T\tilde{u}$ is in $H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet$. Finally, we analyze

$$L(T\tilde{u}) \in H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^\bullet + \tfrac{1}{2}L(\rho^{(n-2)/2}\tilde{u}^2).$$

Using that L is a second-order b-differential operator, we have

$$\begin{aligned}\rho^{(n-2)/2}L(\tilde{u}^2) &\in 2\rho^{(n-2)/2}\tilde{u}L\tilde{u}+\rho^{(n-2)/2}H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^{\bullet}H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^{\bullet}\\ &\subset H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta,k}(\Omega)^{\bullet}+H_{\mathrm{b}}^{0,2(l+\alpha)+(n-3)/2,k}(\Omega)^{\bullet}\\ &= H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^{\bullet},\end{aligned}$$

which gives

$$\begin{aligned}L(\rho^{(n-2)/2}\tilde{u}^2) &\in L(\rho^{(n-2)/2})\tilde{u}^2+\rho^{(n-2)/2}L(\tilde{u}^2)+\rho^{(n-2)/2}H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^{\bullet}H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^{\bullet}\\ &\subset H_{\mathrm{b}}^{1,2l+\alpha+(n-3)/2+\delta+\alpha,k}(\Omega)^{\bullet}+H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^{\bullet}\\ &\qquad\qquad+H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\delta+\alpha}(\Omega)^{\bullet}\\ &= H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^{\bullet}.\end{aligned}$$

Hence, putting everything together,

$$L(T\tilde{u})\in H_{\mathrm{b}}^{0,2l+\alpha+(n-3)/2+\min\{\alpha,\delta\},k}(\Omega)^{\bullet}.$$

Therefore, we have $T\tilde{u}\in\mathscr{X}^{l,k,\alpha}$ provided

$$\begin{aligned}2l+\alpha+\tfrac{1}{2}(n-3)+\delta&\ge l+\alpha,\\ 2l+\alpha+\tfrac{1}{2}(n-3)+\min\{\alpha,\delta\}&\ge l,\end{aligned}$$

which for $0<\alpha<\delta$ and $n=4$ is equivalent to

$$l\ge-\tfrac{1}{2}-\delta,\quad l\ge-\tfrac{1}{2}-2\alpha. \tag{5-19}$$

This is consistent with condition (5-18) if $-\frac{1}{2}-\frac{1}{2}(\alpha+\delta)>-\frac{1}{2}-2\alpha$, that is, if $\alpha>\frac{1}{3}\delta$.

Finally, for the map T to be well defined, we need $S\tilde{f}\in\mathscr{X}^{l,k,\alpha}$, hence $\tilde{f}\in\operatorname{Ran}_{\mathscr{X}^{l,k,\alpha}}L$, which is in particular satisfied if $\tilde{f}\in H_{\mathrm{b}}^{0,l+\alpha,k}(\Omega)^{\bullet}$. Indeed, since $1+l+\alpha<1-\frac{1}{2}-\frac{1}{2}(\delta-\alpha)<\frac{1}{2}$ by condition (5-18), the element $S\tilde{f}\in H_{\mathrm{b}}^{1,l+\alpha,k}(\Omega)^{\bullet}$ is well defined.

We have proved:

Theorem 5.20. *Let $c\in\mathbb{C}$, $0<\delta<\frac{1}{4}$, $\frac{1}{3}\delta<\alpha<\delta$, and let $-\frac{1}{2}-2\alpha\le l<-\frac{1}{2}-\frac{1}{2}(\alpha+\delta)$. Then, for small enough $R>0$, there exists a constant $C>0$ such that, for all $f\in H_{\mathrm{b}}^{0,l+3+\alpha,k}(\Omega)^{\bullet}$ with $\|f\|\le C$, the equation*

$$\Box_g u=f+cQ({}^{\mathrm{sc}}du)$$

has a unique solution $u\in\mathscr{X}^{l+1,k,\alpha}$, with $\|u\|\le R$, that depends continuously on f.

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A JUNCTION CONDITION BY SPECIFIED HOMOGENIZATION AND APPLICATION TO TRAFFIC LIGHTS

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Given a coercive Hamiltonian which is quasiconvex with respect to the gradient variable and periodic with respect to time and space, at least "far away from the origin", we consider the solution of the Cauchy problem of the corresponding Hamilton–Jacobi equation posed on the real line. Compact perturbations of coercive periodic quasiconvex Hamiltonians enter into this framework, for example. We prove that the rescaled solution converges towards the solution of the expected effective Hamilton–Jacobi equation, but whose "flux" at the origin is "limited" in a sense made precise by Imbert and Monneau. In other words, the homogenization of such a Hamilton–Jacobi equation yields to supplement the expected homogenized Hamilton–Jacobi equation with a junction condition at the single discontinuous point of the effective Hamiltonian. We also illustrate possible applications of such a result by deriving, for a traffic flow problem, the effective flux limiter generated by the presence of a finite number of traffic lights on an ideal road. We also provide meaningful qualitative properties of the effective limiter.

1. Introduction

Setting of the general problem. This article is concerned with the study of the limit of the solution $u^\varepsilon(t,x)$ of the equation

$$u_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, u_x^\varepsilon\right) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathbb{R} \tag{1}$$

subject to the initial condition

$$u^\varepsilon(0,x) = u_0(x) \quad \text{for } x \in \mathbb{R} \tag{2}$$

for a Hamiltonian H satisfying the following assumptions:

MSC2010: 35B27, 49L25, 35F21.
Keywords: Hamilton–Jacobi equations, quasiconvex Hamiltonians, homogenization, junction condition, flux-limited solution, viscosity solution.

(A0) Continuity: $H : \mathbb{R}^3 \to \mathbb{R}$ is continuous.

(A1) Time periodicity: For all $k \in \mathbb{Z}$ and $(t, x, p) \in \mathbb{R}^3$,

$$H(t+k, x, p) = H(t, x, p).$$

(A2) Uniform modulus of continuity in time: There exists a modulus of continuity ω such that, for all $t, s, x, p \in \mathbb{R}$,

$$H(t, x, p) - H(s, x, p) \le \omega\big(|t-s|\big(1 + \max(H(s, x, p), 0)\big)\big).$$

(A3) Uniform coercivity: $$\lim_{|q|\to+\infty} H(t, x, q) = +\infty$$
uniformly with respect to (t, x).

(A4) Quasiconvexity of H for large x: There exists some $\rho_0 > 0$ such that, for all $x \in \mathbb{R} \setminus (-\rho_0, \rho_0)$, there exists a continuous map $t \mapsto p^0(t, x)$ such that

$$\begin{cases} H(t, x, \cdot) & \text{is nonincreasing in } (-\infty, p^0(t, x)), \\ H(t, x, \cdot) & \text{is nondecreasing in } (p^0(t, x), +\infty). \end{cases}$$

(A5) Left and right Hamiltonians: There exist two Hamiltonians $H_\alpha(t, x, p)$, $\alpha = L, R$, such that

$$\begin{cases} H(t, x+k, p) - H_L(t, x, p) \to 0 & \text{as } \mathbb{Z} \ni k \to -\infty, \\ H(t, x+k, p) - H_R(t, x, p) \to 0 & \text{as } \mathbb{Z} \ni k \to +\infty, \end{cases}$$

uniformly with respect to $(t, x, p) \in [0, 1]^2 \times \mathbb{R}$ and, for all $k, j \in \mathbb{Z}$, $(t, x, p) \in \mathbb{R}^3$ and $\alpha \in \{L, R\}$,

$$H_\alpha(t+k, x+j, p) = H_\alpha(t, x, p).$$

We have to impose some condition in order to ensure that effective Hamiltonians $\overline{H}_\alpha$ are quasiconvex; indeed, we will see that the effective equation should be solved with *flux-limited solutions*, recently introduced by Imbert and Monneau [2013]; such a theory relies on the quasiconvexity of the Hamiltonians.

(B-i) Quasiconvexity of the left and right Hamiltonians: H_α, $\alpha = L, R$, does not depend on time and there exists p^0_α (independent of (t, x)) such that

$$\begin{cases} H_\alpha(x, \cdot) & \text{is nonincreasing on } (-\infty, p^0_\alpha), \\ H_\alpha(x, \cdot) & \text{is nondecreasing on } (p^0_\alpha, +\infty). \end{cases}$$

(B-ii) Convexity of the left and right Hamiltonians: For each $\alpha = L, R$ and for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, the map $p \mapsto H_\alpha(t, x, p)$ is convex.

Example 1.1. A simple example of such a Hamiltonian is

$$H(t, x, p) = |p| - f(t, x)$$

with a continuous function f satisfying $f(t+1, x) = f(t, x)$ and $f(t, x) \to 0$ as $|x| \to +\infty$ uniformly with respect to $t \in \mathbb{R}$.

Main results. Our main result is concerned with the limit of the solution u^ε of (1)–(2). It joins part of the huge literature dealing with homogenization of Hamilton–Jacobi equation, starting with the pioneering work of Lions, Papanicolaou and Varadhan [Lions et al. 1986]. In particular, we need to use the perturbed test function introduced by Evans [1989]. As pointed out to us by the referee, there are few papers dealing with Hamiltonians that depend on time, which implies in particular that so-called correctors also depend on time. The reader is referred to [Barles and Souganidis 2000; Bernard and Roquejoffre 2004] for the large time behaviour and to [Forcadel et al. 2009a; 2009b; 2012] for homogenization results. This limit satisfies an effective Hamilton–Jacobi equation posed on the real line whose Hamiltonian is discontinuous. More precisely, the effective Hamiltonian equals the one which is expected (see (A5)) in $(-\infty; 0)$ and $(0; +\infty)$; in particular, it is discontinuous in the space variable (piecewise constant, in fact). In order to get a unique solution, a flux limiter should be identified [Imbert and Monneau 2013], henceforth abbreviated [IM].

Homogenized Hamiltonians and effective flux limiter. The homogenized left and right Hamiltonians are classically determined by the study of some "cell problems".

Proposition 1.2 (homogenized left and right Hamiltonians). *Assume* (A0)–(A5) *and either* (B-i) *or* (B-ii). *Then, for every $p \in \mathbb{R}$ and $\alpha = L, R$, there exists a unique $\lambda \in \mathbb{R}$ such that there exists a bounded solution v^α of*

$$\begin{cases} v_t^\alpha + H_\alpha(t, x, p + v_x^\alpha) = \lambda \quad \text{in } \mathbb{R} \times \mathbb{R}, \\ v^\alpha \text{ is } \mathbb{Z}^2\text{-periodic}. \end{cases} \tag{3}$$

If $\overline{H}_\alpha(p)$ denotes such a λ, then the map $p \mapsto \overline{H}_\alpha(p)$ is continuous, coercive and quasiconvex.

Remark 1.3. We recall that a function $\overline{H}_\alpha$ is quasiconvex if the sets $\{\overline{H}_\alpha \le \lambda\}$ are convex for all $\lambda \in \mathbb{R}$. If $\overline{H}_\alpha$ is also coercive, then $\bar{p}_\alpha^0$ denotes in proofs some $p \in \operatorname{argmin} \overline{H}_\alpha$.

The effective flux limiter $\bar{A}$ is the smallest $\lambda \in \mathbb{R}$ for which there exists a solution w of the global-in-time Hamilton–Jacobi equation

$$\begin{cases} w_t + H(t, x, w_x) = \lambda, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\ w \text{ is 1-periodic in } t. \end{cases} \tag{4}$$

Theorem 1.4 (effective flux limiter). *Assume* (A0)–(A5) *and either* (B-i) *or* (B-ii). *The set*

$$E = \{\lambda \in \mathbb{R} : \textit{there is a subsolution } w \textit{ of } (4)\}$$

is nonempty and bounded from below. Moreover, if $\bar{A}$ denotes the infimum of E, then

$$\bar{A} \ge A_0 := \max_{\alpha = L, R} (\min \overline{H}_\alpha). \tag{5}$$

Remark 1.5. We will see below (Theorem 4.6) that the infimum is in fact a minimum: there exists a global corrector which, in particular, can be rescaled properly.

We can now define the effective junction condition:

Definition 1.6. The *effective junction function* $F_{\bar A}$ is defined by

$$F_{\bar A}(p_L, p_R) := \max(\bar A, \overline{H}_L^+(p_L), \overline{H}_R^-(p_R)),$$

where

$$\overline{H}_\alpha^-(p) = \begin{cases} \overline{H}_\alpha(p) & \text{if } p < \bar p_\alpha^0, \\ \overline{H}_\alpha(\bar p_\alpha^0) & \text{if } p \geq \bar p_\alpha^0, \end{cases} \qquad \text{and} \qquad \overline{H}_\alpha^+(p) = \begin{cases} \overline{H}_\alpha(\bar p_\alpha^0) & \text{if } p \leq \bar p_\alpha^0, \\ \overline{H}_\alpha(p) & \text{if } p > \bar p_\alpha^0, \end{cases}$$

where $\bar p_\alpha^0 \in \operatorname{argmin} \overline{H}_\alpha$.

The convergence result. Our main result is the following theorem:

Theorem 1.7 (junction condition by homogenization). *Assume* (A0)–(A5) *and either* (B-i) *or* (B-ii). *Assume that the initial datum u_0 is Lipschitz continuous and, for $\varepsilon > 0$, let u^ε be the solution of* (1)–(2). *Then u^ε converges locally uniformly to the unique flux-limited solution u^0 of*

$$\begin{cases} u_t^0 + \overline{H}_L(u_x^0) = 0, & t > 0,\ x < 0, \\ u_t^0 + \overline{H}_R(u_x^0) = 0, & t > 0,\ x > 0, \\ u_t^0 + F_{\bar A}(u_x^0(t, 0^-), u_x^0(t, 0^+)) = 0, & t > 0,\ x = 0, \end{cases} \tag{6}$$

subject to the initial condition (2).

Remark 1.8. The notion of flux-limited solution for (6) was introduced in [IM].

This theorem asserts in particular that the slopes of the limit solution at the origin are characterized by the effective flux limiter $\bar A$. Its proof relies on the construction of a global "corrector", i.e., a solution of (4) which is close to an appropriate V-shaped function after rescaling. This latter condition is necessary so that the slopes at infinity of the corrector fit the expected slopes of the solution of the limit problem at the origin. Here is a precise statement:

Theorem 1.9 (existence of a global corrector for the junction). *Assume* (A0)–(A5) *and either* (B-i) *or* (B-ii). *There exists a solution w of* (4) *with $\lambda = \bar A$ such that the function*

$$w^\varepsilon(t, x) = \varepsilon w(\varepsilon^{-1} t, \varepsilon^{-1} x)$$

converges locally uniformly (*along a subsequence $\varepsilon_n \to 0$*) *towards a function $W = W(x)$ which satisfies $W(0) = 0$ and*

$$\hat p_R x 1_{\{x>0\}} + \hat p_L x 1_{\{x<0\}} \geq W(x) \geq \bar p_R x 1_{\{x>0\}} + \bar p_L x 1_{\{x<0\}}, \tag{7}$$

where

$$\begin{cases} \bar p_R = \min E_R, \\ \hat p_R = \max E_R, \end{cases} \qquad \textit{with} \quad E_R := \{p \in \mathbb{R} : \overline{H}_R^+(p) = \overline{H}_R(p) = \bar A\}, \tag{8}$$

$$\begin{cases} \bar p_L = \max E_L, \\ \hat p_L = \min E_L, \end{cases} \qquad \textit{with} \quad E_L := \{p \in \mathbb{R} : \overline{H}_L^-(p) = \overline{H}_L(p) = \bar A\}. \tag{9}$$

The construction of this global corrector is the reason why homogenization is referred to as being "specified"; see also related results on p. 1897. As a matter of fact, we will prove a stronger result; see Theorem 4.6.

Extension: application to traffic lights. The techniques developed to prove Theorem 1.7 allow us to deal with a different situation inspired by traffic flow problems. As explained in [Imbert et al. 2013], such problems are related to the study of some Hamilton–Jacobi equations. Theorem 1.12 below is motivated by aiming to figuring out how the traffic flow on an ideal (infinite, straight) road is modified by the presence of a finite number of traffic lights.

We can consider a Hamilton–Jacobi equation whose Hamiltonian does not depend on (t, x) for x outside a (small) interval of the form $N_\varepsilon = (b_1\varepsilon, b_N\varepsilon)$, and is piecewise constant with respect to x in $(b_1\varepsilon, b_N\varepsilon)$. At space discontinuities, junction conditions are imposed with ε-time-periodic flux limiters. The limit solution satisfies the equation after the "neighbourhood" N_ε disappears. We will see that the equation keeps memory of what happened there through a flux limiter at the origin $x = 0$.

Let us be more precise now. We are given, for $N \geq 1$ and $K \in \mathbb{N}$, a finite number of junction points $-\infty = b_0 < b_1 < b_2 < \cdots < b_N < b_{N+1} = +\infty$ and times $0 = \tau_0 < \tau_1 < \cdots < \tau_K < 1 = \tau_{K+1}$. For $\alpha \in \{0, \ldots, N\}$, ℓ_α denotes $b_{\alpha+1} - b_\alpha$. Note that $\ell_\alpha = +\infty$ for $\alpha = 0, N$.

We then consider the solution u^ε of (1) where the Hamiltonian H satisfies the following conditions:

(C1) The Hamiltonian is given by

$$H(t, x, p) = \begin{cases} \overline{H}_\alpha(p) & \text{if } b_\alpha < x < b_{\alpha+1}, \\ \max(\overline{H}^+_{\alpha-1}(p^-), \overline{H}^-_\alpha(p^+), a_\alpha(t)) & \text{if } x = b_\alpha, \ \alpha \neq 0. \end{cases}$$

(C2) The Hamiltonians $\overline{H}_\alpha$ for $\alpha = 0, \ldots, N$ are continuous, coercive and quasiconvex.

(C3) The flux limiters a_α for $\alpha = 1, \ldots, N$, and $i = 0, \ldots, K$, satisfy

$$a_\alpha(s + 1) = a_\alpha(s) \quad \text{with} \quad a_\alpha(s) = A^i_\alpha \quad \text{for all} \quad s \in [\tau_i, \tau_{i+1})$$

with $(A^i_\alpha)^{i=0,\ldots,K}_{\alpha=1,\ldots,N}$ satisfying $A^i_\alpha \geq \max_{\beta=\alpha-1,\alpha}(\min \overline{H}_\beta)$.

Remark 1.10. The Hamiltonians outside N_ε are denoted by $\overline{H}_\alpha$ instead of H_α in order to emphasize that they do not depend on time and space.

Remark 1.11. In view of the literature in traffic modelling, the Hamiltonians could be assumed to be convex. But we prefer to stick to the quasiconvex framework since it seems to us that it is the natural one (in view of [IM]).

The equation is supplemented with the initial condition

$$u^\varepsilon(0, x) = U^\varepsilon_0(x) \quad \text{for } x \in \mathbb{R} \tag{10}$$

with

$$U^\varepsilon_0 \text{ equi-Lipschitz continuous} \quad \text{and} \quad U^\varepsilon_0 \to u_0 \text{ locally uniformly.} \tag{11}$$

Then the following convergence result holds true:

Theorem 1.12 (time homogenization of traffic lights). *Assume* (C1)–(C3) *and* (11). *Let* u^ε *be the solution of* (1) *and* (10) *for all* $\varepsilon > 0$. *Then*:

(i) Homogenization: *There exists some* $\bar{A} \in \mathbb{R}$ *such that* u^ε *converges locally uniformly as* ε *tends to zero towards the unique viscosity solution* u^0 *of* (6) *and* (2) *with*

$$\overline{H}_L := \overline{H}_0, \quad \overline{H}_R := \overline{H}_N.$$

(ii) Qualitative properties of $\bar{A}$: *For* $\alpha = 1, \ldots, N$, $\langle a_\alpha \rangle$ *denotes* $\int_0^1 a_\alpha(s)\,ds$. *The effective limiter* $\bar{A}$ *satisfies the following properties*:

- *For all* α, $\bar{A}$ *is nonincreasing with respect to* ℓ_α.
- *For* $N = 1$,
$$\bar{A} = \langle a_1 \rangle. \tag{12}$$
- *For* $N \geq 1$,
$$\bar{A} \geq \max_{\alpha=1,\ldots,N} \langle a_\alpha \rangle. \tag{13}$$
- *For* $N \geq 2$, *there exists a critical distance* $d_0 \geq 0$ *such that*
$$\bar{A} = \max_{\alpha=1,\ldots,N} \langle a_\alpha \rangle \quad \textit{if} \quad \min_\alpha \ell_\alpha \geq d_0; \tag{14}$$
this distance d_0 *only depends on* $\max_{\alpha=1,\ldots,N} \|a_\alpha\|_\infty$, $\max_{\alpha=1,\ldots,N} \langle a_\alpha \rangle$ *and the* $\overline{H}_\alpha$.
- *We have*
$$\bar{A} \to \langle \bar{a} \rangle \quad \textit{as} \;\; (\ell_1, \ldots, \ell_{N-1}) \to (0, \ldots, 0), \tag{15}$$
where $\bar{a}(\tau) = \max_{\alpha=1,\ldots,N} a_\alpha(\tau)$.

Remark 1.13. Since the function $a(t)$ is piecewise constant, the way u^ε satisfies (1) has to be made precise. An L^1 theory in time (following for instance the approach of [Bourgoing 2008a; 2008b]) could probably be developed for such a problem, but we will use here a different, elementary approach. The Cauchy problem is understood as the solution of successive Cauchy problems. This is the reason why we will first prove a global Lipschitz bound on the solution, so that there indeed exists such a solution.

Remark 1.14. The result of Theorem 1.4 still holds for (1) under assumptions (C1)–(C3), with the set E defined for subsolutions which are moreover assumed to be globally Lipschitz (without fixed bound on the Lipschitz constant). The reader can check that the proof is unchanged.

Remark 1.15. It is somewhat easy to get (12) when the Hamiltonians $\overline{H}_\alpha$ are convex by using the optimal control interpretation of the problem. In the more general case of quasiconvex Hamiltonians, the result still holds true but the proof is more involved.

Remark 1.16. We may have $\bar{A} > \max_{\alpha=1,\ldots,N} \langle a_\alpha \rangle$. It is possible to deduce it from (15) in the case $N = 2$ by using the traffic light interpretation of the problem. If we have two traffic lights very close to each other (let us say that the distance in between is at most the space for only one car) and if the traffic lights have common period and are exactly in opposite phases (with, for instance, one minute for the green phase and one minute for the red phase), then the effect of the two traffic lights together gives a very low flux which is much lower than the effect of a single traffic light alone (i.e., here at most one car every two minutes will go through the two traffic lights).

Traffic flow interpretation of Theorem 1.12. We mentioned above that there are some connections between our problem and traffic flows.

Inequality (13) has a natural traffic interpretation, saying that the average limitation on the traffic flow created by several traffic lights on a single road is greater than or equal to the one created by the traffic light which creates the highest limitation. Moreover, this average limitation is smaller if the distances between traffic lights are bigger, as says the monotonicity of $\bar{A}$ with respect to the distances ℓ_α.

Property (14) says that the minimal limitation is reached if the distances between the traffic lights are bigger than a critical distance d_0. The proof of this result is quite involved and is reflected in the fact that the bounds that we have on d_0 are not continuous on the data (namely $\max_{\alpha=1,\dots,N} \|a_\alpha\|_\infty$, $\max_{\alpha=1,\dots,N} \langle a_\alpha \rangle$ and the $\overline{H}_\alpha$).

Finally, property (15) is very natural from the point of view of traffic, since the limit corresponds to the case where all the traffic lights would be at the same position.

Related results. Achdou and Tchou [2015] studied a singular perturbation problem which has the same flavour as the one we are looking at in the present paper. More precisely, they consider the simplest network (a so-called junction) embedded in a star-shaped domain. They prove that the value function of an infinite horizon control problem converges, as the star-shaped domain "shrinks" to the junction, to the value function of a control problem posed on the junction. We borrow from them the idea of studying the cell problem on truncated domains with state constraints. We provide a different approach, which is in some sense more general because it can be applied to problems outside the framework of optimal control theory. Our approach relies in an essential way on the general theory developed in [IM].

The general theme of the lectures by P.-L. Lions [2013–2014] at the Collège de France was "Elliptic or parabolic equations and specified homogenization". As far as first-order Hamilton–Jacobi equations are concerned, the term "specified homogenization" refers to the problem of constructing correctors to cell problems associated with Hamiltonians that are typically the sum of a periodic one, H, and a compactly supported function f depending only on x, say. Lions exhibits sufficient conditions on f such that the effective Hamilton–Jacobi equation is not perturbed. In terms of flux limiters [IM], it corresponds to looking for sufficient conditions such that the effective flux limiter $\bar{A}$ given by Theorem 1.4 is (less than or) equal to $A_0 = \min H$.

Barles, Briani and Chasseigne [Barles et al. 2013, Theorem 6.1] considered the case

$$H(x, p) = \varphi\Big(\frac{x}{\varepsilon}\Big) H_R(p) + \Big(1 - \varphi\Big(\frac{x}{\varepsilon}\Big)\Big) H_L(p)$$

for some continuous increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{s\to-\infty} \varphi(s) = 0 \quad \text{and} \quad \lim_{s\to+\infty} \varphi(s) = 1.$$

They prove that u^ε converges towards a value function denoted by U^-, which they characterize as the solution to a particular optimal control problem. It is proved in [IM] that U^- is the solution of (6) with

$\overline{H}_\alpha = H_\alpha$ and $\bar{A}$ replaced by $A_I^+ = \max(A_0, A^*)$ with

$$A_0 = \max(\min H_R, \min H_L) \quad \text{and} \quad A^* = \max_{q\in[\min(p_R^0, p_L^0), \max(p_R^0, p_L^0)]} (\min(H_R(q), H_L(q))).$$

Giga and Hamamuki [2013] develop a theory which allows them in particular to prove existence and uniqueness for the following Hamilton–Jacobi equation (changing u to $-u$) in $\mathbb{R}^d$:

$$\begin{cases} \partial_t u + |\nabla u| = 0 & \text{for } x \neq 0, \\ \partial_t u + |\nabla u| + c = 0 & \text{at } x = 0. \end{cases}$$

The solutions of [Giga and Hamamuki 2013] are constructed as limits of the equation

$$\partial_t u^\varepsilon + |\nabla u^\varepsilon| + c\left(1 - \frac{|x|}{\varepsilon}\right)^+ = 0.$$

In the monodimensional case ($d = 1$), Theorem 1.7 implies that u^ε converges towards

$$\begin{cases} \partial_t u + |\nabla u| = 0 & \text{for } x \neq 0, \\ \partial_t u + \max(A, |\nabla u|) = 0 & \text{at } x = 0, \end{cases}$$

for some $A \in \mathbb{R}$. In view of Theorem 1.4, it is not difficult to prove that $A = \max(0, c)$. The Hamiltonian $\max(c, |\nabla u|)$ is identified in [Giga and Hamamuki 2013] and is referred to as the *relaxed* one.

It is known that homogenization of Hamilton–Jacobi equations is closely related to the study of the large time behaviour of solutions. Hamamuki [2013] discusses the large time behaviour of Hamilton–Jacobi equations with discontinuous source terms in two cases: for compactly supported ones and periodic ones. In our setting, we can address both, and even the sum of a periodic source term and a compactly supported one. It would be interesting to address such a problem in the case of traffic lights. Jin and Yu [2015] study the large time behaviour of the solutions of a Hamilton–Jacobi equations with an x-periodic Hamiltonian and what can be interpreted as a flux limiter depending periodically on time.

Further extensions. It is also possible to address the time homogenization problem of Theorem 1.12 with any finite number of junctions (with limiter functions $a_\alpha(t)$ that are piecewise constants — or continuous — and 1-periodic), either separated with distance of order $O(1)$ or with distance of order $O(\varepsilon)$, or mixing both, and even on a complicated network. See also [Jin and Yu 2015] for other connections between Hamilton–Jacobi equations and traffic light problems, and [Andreianov et al. 2010] for green waves modelling.

Note that the method presented in this paper can be readily applied (without modifying proofs) to the study of homogenization on a finite number of branches and not only two branches; the theory developed in [IM] should also be used for the limit problem.

Similar questions in higher dimensions with point defects of other codimensions will be addressed in future works.

Organization of the article. Section 2 is devoted to the proof of the convergence result (Theorem 1.7). Section 3 is devoted to the construction of correctors far from the junction point (Proposition 1.2), while the junction case, i.e., the proof of Theorem 4.6, is addressed in Section 4. We recall that Theorem 1.9 is a straightforward corollary of this stronger result. The proof of Theorem 4.6 makes use of a comparison principle which is expected but not completely standard. This is the reason why a proof is sketched in the Appendix, together with two others that are rather standard but included for the reader's convenience.

Notation. A ball centred at x of radius r is denoted by $B_r(x)$. If $\{u^\varepsilon\}_\varepsilon$ is locally bounded, the upper and lower relaxed limits are defined as

$$\begin{cases} \limsup_\varepsilon {}^* u^\varepsilon(X) = \limsup_{Y\to X,\varepsilon\to 0} u^\varepsilon(Y), \\ \liminf_{\varepsilon}{}_* u^\varepsilon(X) = \liminf_{Y\to X,\varepsilon\to 0} u^\varepsilon(Y). \end{cases}$$

In our proofs, constants may change from line to line.

2. Proof of convergence

This section is devoted to the proof of Theorem 1.7. We first construct barriers.

Lemma 2.1 (barriers). *There exists a nonnegative constant C such that, for any $\varepsilon > 0$,*

$$|u^\varepsilon(t,x) - u_0(x)| \le Ct \quad \textit{for } (t,x) \in (0,T)\times\mathbb{R}. \tag{16}$$

Proof. Let L_0 be the Lipschitz constant of the initial datum u_0. Taking

$$C = \sup_{\substack{(t,x)\in\mathbb{R}\times\mathbb{R} \\ |p|\le L_0}} |H(t,x,p)| < +\infty,$$

owing to (A0) and (A5), the functions $u^\pm(t,x) = u_0(x) \pm Ct$ are a super- and a sub-solution, respectively, of (1)–(2) and (16) follows via comparison principle. □

We can now prove the convergence theorem.

Proof of Theorem 1.7. We classically consider the upper and lower relaxed semilimits

$$\begin{cases} \bar{u} = \limsup_\varepsilon {}^* u^\varepsilon, \\ \underline{u} = \liminf_{\varepsilon}{}_* u^\varepsilon. \end{cases}$$

Notice that these functions are well defined because of Lemma 2.1. In order to prove convergence of u^ε towards u^0, it is sufficient to prove that $\bar{u}$ and $\underline{u}$ are a sub- and a super-solution, respectively, of (6) and (2). The initial condition follows immediately from (16). We focus our attention on the subsolution case, since the supersolution one can be handled similarly.

We first check that

$$\bar{u}(t,0) = \limsup_{(s,y)\to(t,0),\, y>0} \bar{u}(s,y) = \limsup_{(s,y)\to(t,0),\, y<0} \bar{u}(s,y). \tag{17}$$

This is a consequence of the stability of such a "weak continuity" condition; see [IM]. Indeed, it is shown in [IM] that classical viscosity solution can be viewed as a flux-limited one; in particular, u^ε solves

$$u_t^\varepsilon + H^-\Big(\frac{t}{\varepsilon}, \frac{0}{\varepsilon}, u_x^\varepsilon(t,0^+)\Big) \vee H^+\Big(\frac{t}{\varepsilon}, \frac{0}{\varepsilon}, u_x^\varepsilon(t,0^-))\Big) = 0 \quad \text{for } t > 0.$$

Since these ε-Hamiltonians are uniformly coercive and u^ε is continuous, we conclude that (17) holds true.

Let φ be a test function such that

$$(\bar{u}-\varphi)(t,x) < (\bar{u}-\varphi)(\bar{t},\bar{x}) = 0 \quad \text{for all } (t,x) \in B_{\bar{r}}(\bar{t},\bar{x}) \setminus \{(\bar{t},\bar{x})\}. \tag{18}$$

We argue by contradiction, by assuming that

$$\varphi_t(\bar{t},\bar{x}) + \overline{H}(\bar{x},\varphi_x(\bar{t},\bar{x})) = \theta > 0, \tag{19}$$

where

$$\overline{H}(\bar{x},\varphi_x(\bar{t},\bar{x})) := \begin{cases} \overline{H}_R(\varphi_x(\bar{t},\bar{x})) & \text{if } \bar{x} > 0, \\ \overline{H}_L(\varphi_x(\bar{t},\bar{x})) & \text{if } \bar{x} < 0, \\ F_{\bar{A}}(\varphi_x(\bar{t},0^-),\varphi_x(\bar{t},0^+)) & \text{if } \bar{x} = 0. \end{cases}$$

We only treat the case where $\bar{x}=0$, since the case $\bar{x}\neq 0$ is somewhat classical. This latter case is detailed in Section A in the Appendix for the reader's convenience. Using [IM, Proposition 2.5], we may suppose that

$$\varphi(t,x) = \phi(t) + \bar{p}_L x 1_{\{x<0\}} + \bar{p}_R x 1_{\{x>0\}}, \tag{20}$$

where ϕ is a C^1 function defined in $(0,+\infty)$. In this case, (19) becomes

$$\phi'(\bar{t}) + F_{\bar{A}}(\bar{p}_L,\bar{p}_R) = \phi'(\bar{t}) + \bar{A} = \theta > 0. \tag{21}$$

Let us consider a solution w of the equation

$$w_t + H(t,x,w_x) = \bar{A}, \tag{22}$$

provided by Theorem 1.9, which is in particular 1-periodic with respect to time. We recall that the function W is the limit of $w^\varepsilon = \varepsilon w(\,\cdot\,/\varepsilon)$ as $\varepsilon \to 0$. We claim that, if $\varepsilon > 0$ is small enough, the perturbed test function $\varphi^\varepsilon(t,x) = \phi(t) + w^\varepsilon(t,x)$ [Evans 1989] is a viscosity supersolution of

$$\varphi_t^\varepsilon + H\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\varphi_x^\varepsilon\Big) = \frac{\theta}{2} \quad \text{in } B_r(\bar{t},0)$$

for some sufficiently small $r > 0$. In order to justify this fact, let $\psi(t,x)$ be a test function touching φ^ε from below at $(t_1,x_1) \in B_r(\bar{t},0)$. In this way,

$$w\Big(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon}\Big) = \frac{1}{\varepsilon}(\psi(t_1,x_1) - \phi(t_1))$$

and

$$w(s,y) \ge \frac{1}{\varepsilon}(\psi(\varepsilon s,\varepsilon y) - \phi(\varepsilon s))$$

for (s,y) in a neighbourhood of $(t_1/\varepsilon, x_1/\varepsilon)$. Hence, from (21)–(22),

$$\psi_t(t_1,x_1) + H\Big(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon},\psi_x(t_1,x_1)\Big) \ge \bar{A} + \phi'(t_1) \ge \bar{A} + \phi'(\bar{t}) - \frac{\theta}{2} \ge \frac{\theta}{2}$$

provided r is small enough. Hence, the claim is proved.

Combining (7) from Theorem 1.9 with (18) and (20), we can fix $\kappa_r > 0$ and $\varepsilon > 0$ small enough so that

$$u^\varepsilon + \kappa_r \le \varphi^\varepsilon \quad \text{on } \partial B_r(\bar{t},0).$$

By the comparison principle the previous inequality holds in $B_r(\bar{t}, 0)$. Passing to the limit as $\varepsilon \to 0$ and $(t, x) \to (\bar{t}, \bar{x})$, we get the contradiction

$$\bar{u}(\bar{t}, 0) + \kappa_r \leq \varphi(\bar{t}, 0) = \bar{u}(\bar{t}, 0).$$

The proof of convergence is now complete. □

Remark 2.2. For the supersolution property, φ in (20) should be replaced with

$$\varphi(t, x) = \phi(t) + \hat{p}_L x 1_{\{x<0\}} + \hat{p}_R x 1_{\{x>0\}}.$$

3. Homogenized Hamiltonians

In order to prove Proposition 1.2, we first prove the following lemma. Even if the proof is standard, we give it in full detail since we will adapt it when constructing global correctors for the junction.

Lemma 3.1 (existence of a corrector). *There exists $\lambda \in \mathbb{R}$ for which there is a bounded* (*discontinuous*) *viscosity solution of* (3).

Remark 3.2. If H_α does not depend on t, then it is possible to construct a corrector which does not depend on time either. We leave the details to the reader.

Proof. For any $\delta > 0$, it is possible to construct a (possibly discontinuous) viscosity solution v^δ of

$$\begin{cases} \delta v^\delta + v^\delta_t + H_\alpha(t, x, p + v^\delta_x) = 0 & \text{in } \mathbb{R} \times \mathbb{R}, \\ v^\delta \ \text{ is } \mathbb{Z}^2\text{-periodic.} \end{cases}$$

First, the comparison principle implies

$$|\delta v^\delta| \leq C_\alpha, \tag{23}$$

where

$$C_\alpha = \sup_{(t,x) \in [0,1]^2} |H_\alpha(t, x, p)|.$$

Second, the function

$$m^\delta(x) = \sup_{t \in \mathbb{R}} (v^\delta)^*(t, x)$$

is a subsolution of

$$H_\alpha(t(x), x, p + m^\delta_x) \leq C_\alpha$$

(for some function $t(x)$). Assumptions (A3) and (A5) imply that there exists $C > 0$ independent of δ such that

$$|m^\delta_x| \leq C \quad \text{and} \quad v^\delta_t \leq C.$$

In particular, the comparison principle implies that, for all $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $h \geq 0$,

$$v^\delta(t + h, x) \leq v^\delta(t, x) + Ch.$$

Combining this inequality with the time-periodicity of v^δ yields

$$|v^\delta(t, x) - m^\delta(x)| \leq C;$$

in particular,

$$|v^\delta(t,x) - v^\delta(0,0)| \le C. \tag{24}$$

Hence, the half-relaxed limits

$$\bar{v} = \limsup_{\delta\to 0}{}^*(v^\delta - v^\delta(0,0)) \quad \text{and} \quad \underline{v} = \liminf_{\delta\to 0}{}_*(v^\delta - v^\delta(0,0))$$

are finite. Moreover, (23) implies that $\delta v^\delta(0,0) \to -\lambda$ (at least along a subsequence). Hence, the discontinuous stability of viscosity solutions implies that $\bar{v}$ is a $\mathbb{Z}^2$-periodic subsolution of (3) and $\underline{v}$ is a $\mathbb{Z}^2$-periodic supersolution of the same equation. Perron's method then allows us to construct a corrector between $\bar{v}$ and $\underline{v} + C$ with $C = \sup(\bar{v} - \underline{v})$. The proof of the lemma is now complete. □

The following lemma is completely standard; the proof is given in Section B in the Appendix for the reader's convenience.

Lemma 3.3 (uniqueness of λ). *The real number λ given by Lemma 3.1 is unique. If $\overline{H}_\alpha(p)$ denotes this real number, the function $\overline{H}_\alpha$ is continuous.*

Lemma 3.4 (coercivity of $\overline{H}_\alpha$). *The continuous function $\overline{H}_\alpha$ is coercive*:

$$\lim_{|p|\to+\infty} \overline{H}_\alpha(p) = +\infty.$$

Proof. In view of the uniform coercivity in p of H_α with respect to (t,x) (see (A3)), for any $R > 0$ there exists a positive constant C_R such that

$$|p| \ge C_R \quad \Longrightarrow \quad H_\alpha(t,x,p) \ge R \quad \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{25}$$

Let v^α be the discontinuous corrector given by Lemma 3.1 and $(\bar{t},\bar{x})$ the point of supremum of its upper semicontinuous envelope $(v^\alpha)^*$. Then we have

$$H_\alpha(\bar{t},\bar{x},p) \le \overline{H}_\alpha(p),$$

which implies

$$\overline{H}_\alpha(p) \ge R \quad \text{for } |p| \ge C_R. \tag{26}$$

The proof of the lemma is now complete. □

We first prove the quasiconvexity of $\overline{H}_\alpha$ under assumption (B-ii). We in fact prove more: the effective Hamiltonian is convex in this case.

Lemma 3.5 (convexity of $\overline{H}_\alpha$ under (B-ii)). *Assume* (A0)–(A5) *and* (B-ii). *Then the function $\overline{H}_\alpha$ is convex.*

Proof. For $p, q \in \mathbb{R}$, let v_p and v_q be solutions of (3) with $\lambda = \overline{H}_\alpha(p)$ and $\overline{H}_\alpha(q)$, respectively. We also set

$$u_p(t,x) = v_p(t,x) + px - t\overline{H}_\alpha(p)$$

and define u_q similarly.

Step 1: *u_p and u_q are locally Lipschitz continuous.* In this case, we have, almost everywhere,

$$\begin{cases}(u_p)_t + H_\alpha(t, x, (u_p)_x) = 0,\\ (u_q)_t + H_\alpha(t, x, (u_q)_x) = 0.\end{cases}$$

For $\mu \in [0, 1]$, let

$$\bar{u} = \mu u_p + (1 - \mu) u_q.$$

By convexity, we get, almost everywhere,

$$\bar{u}_t + H_\alpha(t, x, \bar{u}_x) \le 0. \tag{27}$$

We claim that the convexity of H_α (in the gradient variable) implies that $\bar{u}$ is a viscosity subsolution. To see this, we use an argument of [Bardi and Capuzzo-Dolcetta 1997, Proposition 5.1]. For $P = (t, x)$, we define a mollifier $\rho_\delta(P) = \delta^{-2}\rho(\delta^{-1}P)$ and set

$$\bar{u}_\delta = \bar{u} \star \rho_\delta$$

Then, by convexity, we get, with $Q = (s, y)$,

$$(\bar{u}_\delta)_t + H_\alpha(P, (\bar{u}_\delta)_x) \le \int dQ\, \{H_\alpha(P, \bar{u}_x(Q)) - H_\alpha(Q, \bar{u}_x(Q)\}\rho_\delta(P - Q).$$

The fact that $\bar{u}_x$ is locally bounded and the fact that H_α is continuous imply that the right-hand side goes to zero as $\delta \to 0$. We deduce (by stability of viscosity subsolutions) that (27) holds true in the viscosity sense. Then the comparison principle implies that

$$\mu \overline{H}_\alpha(p) + (1 - \mu)\overline{H}_\alpha(q) \ge \overline{H}_\alpha(\mu p + (1 - \mu) q). \tag{28}$$

Step 2: *u_p and u_q are continuous.* We proceed in two substeps:

Step 2.1: *the case of a single function u.* We first want to show that if $u = u_p$ is continuous and satisfies (27) almost everywhere, then u is a viscosity subsolution. To this end, we will use the structural assumptions satisfied by the Hamiltonian. The ones that were useful to prove the comparison principle will be also useful to prove the result we want, so we will revisit that proof. We also use the fact that

$$u(t, x) - px + t\overline{H}_\alpha(p) \quad \text{is bounded.} \tag{29}$$

For $\nu > 0$, we set

$$u^\nu(t, x) = \sup_{s \in \mathbb{R}} \left(u(s, x) - \frac{(t - s)^2}{2\nu} \right) = u(s_\nu, x) - \frac{(t - s_\nu)^2}{2\nu}.$$

As usual, we get from (29) that

$$|t - s_\nu| \le C\sqrt{\nu} \quad \text{with} \quad C = C(p, T) \tag{30}$$

for $t \in (-T, T)$. In particular $s_\nu \to t$ locally uniformly. If a test function φ touches u^ν from above at some point (t, x), then we have $\varphi_t(t, x) = -(t - s_\nu)/\nu$ and

$$\begin{aligned}\varphi_t(t,x) + H_\alpha(t,x,\varphi_x(t,x)) &\le H_\alpha(t,x,\varphi_x(t,x)) - H_\alpha(s_\nu,x,\varphi_x(t,x))\\ &\le \omega\big(|t-s_\nu|\big(1+\max(0, H_\alpha(s_\nu,x,\varphi_x(t,x)))\big)\big)\\ &\le \omega\bigg(\frac{(t-s_\nu)^2}{\nu} + |t-s_\nu|\bigg),\end{aligned} \tag{31}$$

where we have used (A2) in the third line. The right-hand side goes to zero as ν goes to zero since

$$\frac{(t-s_\nu)^2}{\nu} \to 0 \quad \text{locally uniformly with respect to } (t,x)$$

(recall u is continuous). Indeed, this can be checked for (t, x) replaced by (t_ν, x_ν) because, for any sequence $(t_\nu, s_\nu, x_\nu) \to (t, t, x)$, we have

$$u(t_\nu, x_\nu) \le u^\nu(t_\nu, x_\nu) = u(s_\nu, x_\nu) - \frac{(t_\nu - s_\nu)^2}{2\nu},$$

where the continuity of u implies the result. For a given $\nu > 0$, we see that (30) and (31) imply that

$$|\varphi_t|, |\varphi_x| \le C_{\nu,p}.$$

This implies in particular that u^ν is Lipschitz continuous, and then

$$u_t^\nu + H(t, x, u_x^\nu) \le o_\nu(1) \quad \text{a.e.},$$

where $o_\nu(1)$ is locally uniform with respect to (t, x).

Step 2.2: *application.* Applying Step 2.1, we get, for $z = p, q$,

$$(u_z^\nu)_t + H(t, x, (u_z^\nu)_x) \le o_\nu(1) \quad \text{a.e.},$$

where $o_\nu(1)$ is locally uniform with respect to (t, x). Step 1 implies that

$$\bar{u}^\nu := \mu u_p^\nu + (1-\mu) u_q^\nu$$

is a viscosity subsolution of

$$(\bar{u}^\nu)_t + H_\alpha(t, x, (\bar{u}^\nu)_x) \le o_\nu(1),$$

where $o_\nu(1)$ is locally uniform with respect to (t, x). In the limit $\nu \to 0$, we recover (by stability of subsolutions) that $\bar{u}$ is a viscosity subsolution, i.e., satisfies (27) in the viscosity sense. This then gives the same conclusion as in Step 1.

Step 3: *the general case.* To cover the general case, we simply replace u_p by $\tilde{u}_p$, the solution to the Cauchy problem

$$\begin{cases}(\tilde{u}_p)_t + H_\alpha(t, x, (\tilde{u}_p)_x) = 0 & \text{for } (t, x) \in (0, +\infty) \times \mathbb{R},\\ \tilde{u}_p(0, x) = px.\end{cases}$$

Then $\tilde{u}_p$ is continuous and satisfies $|\tilde{u}_p - u_p| \le C$. Proceeding similarly with $\tilde{u}_q$ and using Step 2, we deduce the desired inequality (28). The proof is now complete. □

We finally prove the quasiconvexity of $\overline{H}_\alpha$ under assumption (B-i).

Lemma 3.6 (quasiconvexity of $\overline{H}_\alpha$ under (B-i)). *Assume* (A0)–(A5) *and* (B-i). *Then the function* $\overline{H}_\alpha$ *is quasiconvex.*

Proof. We reduce quasiconvexity to convexity by composing with an increasing function γ; note that such a reduction was already used in optimization and in partial differential equations; see, for instance, [Lions 1981; Kawohl 1985].

We first assume that H_α satisfies

$$\begin{cases} H_\alpha \in C^2, \\ D^2_{pp} H_\alpha(x, p^0_\alpha) > 0, \\ D_p H_\alpha(x, p) < 0 & \text{for } p \in (-\infty, p^0_\alpha), \\ D_p H_\alpha(x, p) > 0 & \text{for } p \in (p^0_\alpha, +\infty), \\ H_\alpha(x, p) \to +\infty & \text{as } |p| \to +\infty \text{ uniformly with respect to } x \in \mathbb{R}. \end{cases} \tag{32}$$

For a function γ such that

$$\gamma \text{ is convex}, \quad \gamma \in C^2(\mathbb{R}) \quad \text{and} \quad \gamma' \ge \delta_0 > 0,$$

we have

$$D^2_{pp}(\gamma \circ H_\alpha) > 0$$

if and only if

$$(\ln \gamma')'(\lambda) > -\frac{D^2_{pp} H_\alpha(x, p)}{(D_p H_\alpha(x, p))^2} \quad \text{for } p = \pi^\pm_\alpha(x, \lambda) \text{ and } \lambda \ge H_\alpha(x, p), \tag{33}$$

where $\pi^\pm_\alpha(x, \lambda)$ is the only real number r such that $\pm r \ge 0$ and $H_\alpha(x, r) = \lambda$. Because $D^2_{pp} H_\alpha(x, p^0_\alpha) > 0$, we see that the right-hand side is negative for λ close enough to $H_\alpha(x, p^0_\alpha)$ and it is indeed possible to construct such a function γ.

In view of Remark 3.2, we can construct a solution of $\delta v^\delta + \gamma \circ H_\alpha(x, p + v^\delta_x) = 0$ with $-\delta v^\delta \to \overline{\gamma \circ H_\alpha}(p)$ as $\delta \to 0$, and a solution of

$$\gamma \circ H_\alpha(x, p + v_x) = \overline{\gamma \circ H_\alpha}(p).$$

This shows that

$$\overline{H}_\alpha = \gamma^{-1} \circ \overline{\gamma \circ H_\alpha}.$$

Thanks to Lemmas 3.4 and 3.5, we know that $\overline{\gamma \circ H_\alpha}$ is coercive and convex. Hence, $\overline{H}_\alpha$ is quasiconvex.

If now H_α does not satisfies (32) then, for all $\varepsilon > 0$, there exists $H^\varepsilon_\alpha \in C^2$ such that

$$\begin{cases} (D^2_{pp} H^\varepsilon_\alpha)(x, p^0_\alpha) > 0, \\ D_p H^\varepsilon_\alpha(x, p) < 0 & \text{for } p \in (-\infty, p^0_\alpha), \\ D_p H^\varepsilon_\alpha(x, p) > 0 & \text{for } p \in (p^0_\alpha, +\infty), \\ |H^\varepsilon_\alpha - H_\alpha| < \varepsilon. \end{cases}$$

Then we can argue as in the proof of continuity of $\overline{H}_\alpha$ and deduce that

$$\overline{H}_\alpha(p) = \lim_{\varepsilon \to 0} \overline{H}^\varepsilon_\alpha(p).$$

Moreover, the previous case implies that $\overline{H}^\varepsilon_\alpha$ is quasiconvex. Hence, so is $\overline{H}_\alpha$. The proof of the lemma is now complete. □

Proof of Proposition 1.2. Combine Lemmas 3.1, 3.3, 3.4, 3.5 and 3.6. □

4. Truncated cell problems

We consider the following problem: find $\lambda_\rho \in \mathbb{R}$ and w such that

$$\begin{cases} w_t + H(t, x, w_x) = \lambda_\rho & \text{for } (t, x) \in \mathbb{R} \times (-\rho, \rho), \\ w_t + H^-(t, x, w_x) = \lambda_\rho & \text{for } (t, x) \in \mathbb{R} \times \{-\rho\}, \\ w_t + H^+(t, x, w_x) = \lambda_\rho & \text{for } (t, x) \in \mathbb{R} \times \{\rho\}, \\ w \text{ is 1-periodic in } t. \end{cases} \tag{34}$$

Even if our approach is different, we borrow here an idea from [Achdou and Tchou 2015] by truncating the domain and considering correctors in $[-\rho, \rho]$ with $\rho \to +\infty$.

A comparison principle.

Proposition 4.1 (comparison principle for a mixed boundary value problem). *Let $\rho_2 > \rho_1 > \rho_0$ and $\lambda \in \mathbb{R}$ and v be a supersolution of the boundary value problem*

$$\begin{cases} v_t + H(t, x, v_x) \geq \lambda & \textit{for } (t, x) \in \mathbb{R} \times (\rho_1, \rho_2), \\ v_t + H^+(t, x, v_x) \geq \lambda & \textit{for } (t, x) \in \mathbb{R} \times \{\rho_2\}, \\ v(t, x) \geq U_0(t) & \textit{for } (t, x) \in \mathbb{R} \times \{\rho_1\}, \\ v \textit{ is 1-periodic in } t, \end{cases} \tag{35}$$

where U_0 is continuous and, for $\varepsilon_0 > 0$, let u be a subsolution of

$$\begin{cases} u_t + H(t, x, u_x) \leq \lambda - \varepsilon_0 & \textit{for } (t, x) \in \mathbb{R} \times (\rho_1, \rho_2), \\ u_t + H^+(t, x, u_x) \leq \lambda - \varepsilon_0 & \textit{for } (t, x) \in \mathbb{R} \times \{\rho_2\}, \\ u(t, x) \leq U_0(t) & \textit{for } (t, x) \in \mathbb{R} \times \{\rho_1\}, \\ u \textit{ is 1-periodic in } t. \end{cases} \tag{36}$$

Then $u \leq v$ in $\mathbb{R} \times [\rho_1, \rho_2]$.

Remark 4.2. A similar result holds true if the Dirichlet condition is imposed at $x = \rho_2$ and junction conditions

$$v_t + H^-(t, x, v_x) \geq \lambda \qquad \text{at } x = \rho_1,$$
$$u_t + H^-(t, x, u_x) \leq \lambda - \varepsilon_0 \quad \text{at } x = \rho_1,$$

are imposed at $x = \rho_1$.

The proof of Proposition 4.1 is very similar to (in fact simpler than) the proof of the comparison principle for Hamilton–Jacobi equations on networks contained in [IM]. The main difference lies in the fact that, in our case, u and v are global in time and the space domain is bounded. A sketch of the proof is provided in Section C in the Appendix, shedding some light on the main differences. Here, the parameter $\varepsilon_0 > 0$ in (36) is used in place of the standard correction term $-\eta/(T-t)$ for a Cauchy problem.

Correctors on truncated domains.

Proposition 4.3 (existence and properties of a corrector on a truncated domain). *There exists a unique $\lambda_\rho \in \mathbb{R}$ such that there exists a solution $w^\rho = w$ of* (34). *Moreover, there exists a constant $C > 0$ independent of $\rho \in (\rho_0, +\infty)$ and a function $m^\rho : [-\rho, \rho] \to \mathbb{R}$ such that*

$$\begin{cases} |\lambda_\rho| \le C, \\ |m^\rho(x) - m^\rho(y)| \le C|x-y| & \textit{for } x, y \in [-\rho, \rho], \\ |w^\rho(t,x) - m^\rho(x)| \le C & \textit{for } (t,x) \in \mathbb{R} \times [-\rho, \rho]. \end{cases} \tag{37}$$

Proof. In order to construct a corrector on the truncated domain, we proceed classically by considering

$$\begin{cases} \delta w^\delta + w^\delta_t + H(t,x,w^\delta_x) = 0 & \text{for } (t,x) \in \mathbb{R} \times (-\rho, \rho), \\ \delta w^\delta + w^\delta_t + H^-(t,x,w^\delta_x) = 0 & \text{for } (t,x) \in \mathbb{R} \times \{-\rho\}, \\ \delta w^\delta + w^\delta_t + H^+(t,x,w^\delta_x) = 0 & \text{for } (t,x) \in \mathbb{R} \times \{\rho\}, \\ w^\delta \text{ is 1-periodic in } t. \end{cases} \tag{38}$$

A discontinuous viscosity solution of (38) is constructed by Perron's method (in the class of 1-periodic functions in time) since $\pm\delta^{-1}C$ are trivial super- and sub-solutions if C is chosen to be

$$C = \sup_{\substack{t \in \mathbb{R} \\ x \in \mathbb{R}}} |H(t,x,0)|.$$

In particular, the solution w^δ satisfies, by construction,

$$|w^\delta| \le \frac{C}{\delta}. \tag{39}$$

We next consider

$$m^\delta(x) = \sup_{t \in \mathbb{R}} (w^\delta)^*(t,x).$$

We remark that the supremum is reached since w^δ is periodic in time; we also remark that m^δ is a viscosity subsolution of

$$H(t(x), x, m^\delta_x) \le C, \quad x \in (-\rho, \rho),$$

(for some function $t(x)$). In view of (A3), we conclude that m^δ is globally Lipschitz continuous and

$$|m^\delta_x| \le C \tag{40}$$

for some constant C which still only depends on H. Assumption (A3) also implies that

$$w^\delta_t \le C$$

(with C only depending on H). In particular, the comparison principle implies that, for all $t \in \mathbb{R}$, $x \in (-\rho, \rho)$ and $h \geq 0$,

$$w^\delta(t+h, x) \leq w^\delta(t, x) + Ch.$$

Combining this information with the periodicity of w^δ in t, we conclude that, for $t \in \mathbb{R}$ and $x \in (-\rho, \rho)$,

$$|w^\delta(t, x) - m^\delta(x)| \leq C.$$

In particular,

$$|w^\delta(t, x) - w^\delta(0, 0)| \leq C.$$

We then consider

$$\overline{w} = \limsup_\delta {}^*(w^\delta - w^\delta(0, 0)) \quad \text{and} \quad \underline{w} = \liminf_\delta {}_*(w^\delta - w^\delta(0, 0)).$$

We next remark that (39) and (40) imply that there exists $\delta_n \to 0$ such that

$$\begin{aligned} m^{\delta_n} - m^{\delta_n}(0) &\to m^\rho \quad \text{as } n \to +\infty, \\ \delta_n w^{\delta_n}(0, 0) &\to -\lambda_\rho \quad \text{as } n \to +\infty, \end{aligned}$$

(the first convergence being locally uniform). In particular, λ, $\overline{w}$, $\underline{w}$ and m^ρ satisfy

$$\begin{aligned} |\lambda_\rho| &\leq C, \\ |\overline{w} - m^\rho| &\leq C, \\ |\underline{w} - m^\rho| &\leq C, \\ |m^\rho_x| &\leq C. \end{aligned}$$

Discontinuous stability of viscosity solutions of Hamilton–Jacobi equations implies that $\overline{w} - 2C$ and $\underline{w}$ are a sub- and a super-solution, respectively, of (34), and

$$\overline{w} - 2C \leq \underline{w}.$$

Perron's method is used once again in order to construct a solution w^ρ of (34) which is 1-periodic in time. In view of the previous estimates, λ_ρ, m^ρ and w^ρ satisfy (37). Proving the uniqueness of λ_ρ is classical, so we skip it. The proof of the proposition is now complete. □

Proposition 4.4 (first definition of the effective flux limiter). *The map $\rho \mapsto \lambda_\rho$ is nondecreasing and bounded in* $(0, +\infty)$. *In particular*,

$$\bar{A} = \lim_{\rho \to +\infty} \lambda_\rho$$

exists and $\bar{A} \geq \lambda_\rho$ for all $\rho > 0$.

Proof. For $\rho' > \rho > 0$, we see that the restriction of $w^{\rho'}$ to $[-\rho, \rho]$ is a subsolution, as a consequence of [IM, Proposition 2.15]. The boundedness of the map follows from Proposition 4.3. □

We next prove that we can control w^ρ from below under appropriate assumptions on $\bar{A}$.

Proposition 4.5 (control of slopes on a truncated domain). *Assume first that $\bar{A} > \min \overline{H}_R$. Then, for all $\delta > 0$, there exists $\rho_\delta > 0$ and $C_\delta > 0$ (independent of ρ) such that, for $x \geq \rho_\delta$ and $h \geq 0$,*

$$w^\rho(t, x+h) - w^\rho(t, x) \geq (\bar{p}_R - \delta)h - C_\delta. \tag{41}$$

If we now assume that $\bar{A} > \min \overline{H}_L$ then, for $x \leq -\rho_\delta$ and $h \geq 0$,

$$w^\rho(t, x-h) - w^\rho(t, x) \geq (-\bar{p}_L - \delta)h - C_\delta \tag{42}$$

for some $\rho_\delta > 0$ and $C_\delta > 0$ as above.

Proof. We only prove (41), since the proof of (42) follows along the same lines. Let $\delta > 0$. In view of (A5), we know that there exists ρ_δ such that

$$|H(t, x, p) - H_R(t, x, p)| \leq \delta \quad \text{for } x \geq \rho_\delta. \tag{43}$$

Assume that $\bar{A} > \min \overline{H}_R$. Then Proposition 1.2 implies that we can pick p_R^δ such that

$$\overline{H}_R(p_R^\delta) = \overline{H}_R^+(p_R^\delta) = \lambda_\rho - 2\delta$$

for $\rho \geq \rho_0$ and $\delta \leq \delta_0$, by choosing ρ_0 large enough and δ_0 small enough.

We now fix $\rho \geq \rho_\delta$ and $x_0 \in [\rho_\delta, \rho]$. In view of Proposition 1.2 applied to $p = p_R^\delta$, we know that there exists a corrector v_R solving (3) with $\alpha = R$. Since it is $\mathbb{Z}^2$-periodic, it is bounded and $w_R = p_R^\delta x + v_R(t, x)$ solves

$$(w_R)_t + H_R(t, x, (w_R)_x) = \lambda_\rho - 2\delta \quad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

In particular, the restriction of w_R to $[\rho_\delta, \rho]$ satisfies (see [IM, Proposition 2.15])

$$\begin{cases} (w_R)_t + H_R(t, x, (w_R)_x) \leq \lambda_\rho - 2\delta & \text{for } (t, x) \in \mathbb{R} \times (\rho_\delta, \rho), \\ (w_R)_t + H_R^+(t, x, (w_R)_x) \leq \lambda_\rho - 2\delta & \text{for } (t, x) \in \mathbb{R} \times \{\rho\}. \end{cases}$$

In view of (43), this implies

$$\begin{cases} (w_R)_t + H(t, x, (w_R)_x) \leq \lambda_\rho - \delta & \text{for } (t, x) \in \mathbb{R} \times (\rho_\delta, \rho), \\ (w_R)_t + H^+(t, x, (w_R)_x) \leq \lambda_\rho - \delta & \text{for } (t, x) \in \mathbb{R} \times \{\rho\}. \end{cases}$$

Now we remark that $v = w^\rho - w^\rho(0, x_0)$ and $u = w_R - w_R(0, x_0) - 2C - 2\|v_R\|_\infty$ satisfy

$$v(t, x_0) \geq -2C \geq u(t, x_0),$$

where C is given by (37). Thanks to the comparison principle from Proposition 4.1, we thus get, for $x \in [x_0, \rho]$,

$$w^\rho(t, x) - w^\rho(t, x_0) \geq p_R^\delta(x - x_0) - C_\delta,$$

where C_δ is a large constant which does not depend on ρ. In particular, we get (41), reducing δ if necessary. $\square$

Construction of global correctors. We now state and prove a result which implies Theorem 1.9, stated in the introduction.

Theorem 4.6 (existence of a global corrector for the junction). *Assume* (A0)–(A5) *and either* (B-i) *or* (B-ii).

(i) General properties: *There exists a solution* w *of* (4) *with* $\lambda = \bar{A}$ *such that, for all* $(t,x) \in \mathbb{R}^2$,

$$|w(t,x) - m(x)| \le C \tag{44}$$

for some globally Lipschitz continuous function m, *and*

$$\bar{A} \ge A_0.$$

(ii) Bound from below at infinity: *If* $\bar{A} > \max_{\alpha=L,R}(\min \overline{H}_\alpha)$ *then there exists* $\delta_0 > 0$ *such that, for every* $\delta \in (0,\delta_0)$, *there exists* $\rho_\delta > \rho_0$ *such that* w *satisfies*

$$\begin{cases} w(t,x+h) - w(t,x) \ge (\bar{p}_R - \delta)h - C_\delta & \text{for } x \ge \rho_\delta \text{ and } h \ge 0, \\ w(t,x-h) - w(t,x) \ge (-\bar{p}_L - \delta)h - C_\delta & \text{for } x \le -\rho_\delta \text{ and } h \ge 0. \end{cases} \tag{45}$$

The first line of (45) *also holds if we have only* $\bar{A} > \min \overline{H}_R$, *while the second line of* (45) *also holds if we have only* $\bar{A} > \min \overline{H}_L$.

(iii) Rescaling w: *For* $\varepsilon > 0$, *we set*

$$w^\varepsilon(t,x) = \varepsilon w(\varepsilon^{-1}t, \varepsilon^{-1}x).$$

Then (*along a subsequence* $\varepsilon_n \to 0$) *we have that* w^ε *converges locally uniformly towards a function* $W = W(x)$ *which satisfies*

$$\begin{cases} |W(x) - W(y)| \le C|x-y| & \text{for all } x, y \in \mathbb{R}, \\ \overline{H}_R(W_x) = \bar{A} \quad \text{and} \quad \hat{p}_R \ge W_x \ge \bar{p}_R & \text{for } x \in (0,+\infty), \\ \overline{H}_L(W_x) = \bar{A} \quad \text{and} \quad \hat{p}_L \le W_x \le \bar{p}_L & \text{for } x \in (-\infty,0). \end{cases} \tag{46}$$

In particular, we have $W(0) = 0$ *and*

$$\hat{p}_R x 1_{\{x>0\}} + \hat{p}_L x 1_{\{x<0\}} \ge W(x) \ge \bar{p}_R x 1_{\{x>0\}} + \bar{p}_L x 1_{\{x<0\}}. \tag{47}$$

Proof. We consider (up to some subsequence)

$$\overline{w} = \limsup_{\rho\to+\infty}{}^*(w^\rho - w^\rho(0,0)), \quad \underline{w} = \liminf_{\rho\to+\infty}{}_*(w^\rho - w^\rho(0,0)) \quad \text{and} \quad m = \lim_{\rho\to+\infty}(m^\rho - m^\rho(0)).$$

We derive from (37) that $\underline{w}$ and $\overline{w}$ are finite and

$$m - C \le \underline{w} \le \overline{w} \le m + C.$$

Moreover, discontinuous stability of viscosity solutions implies that $\overline{w} - 2C$ and $\underline{w}$ are a sub- and a super-solution, respectively, of (4) with $\lambda = \bar{A}$ (recall Proposition 4.4). Hence, a discontinuous viscosity solution w of (4) can be constructed by Perron's method (in the class of functions that are 1-periodic in time).

Using (37) again, w and m satisfy (44). We also get (45) from Proposition 4.5 (use (37) and pass to the limit with m instead of w if necessary).

We now study $w^\varepsilon(t,x) = \varepsilon w(\varepsilon^{-1}t, \varepsilon^{-1}x)$. Note that (37) implies in particular that

$$w^\varepsilon(t,x) = \varepsilon m(\varepsilon^{-1}x) + O(\varepsilon).$$

In particular, we can find a sequence $\varepsilon_n \to 0$ such that

$$w^{\varepsilon_n}(t,x) \to W(x) \quad \text{locally uniformly as } n \to +\infty,$$

with $W(0) = 0$. Arguing as in the proof of convergence away from the junction point (see the case $\bar{x} \neq 0$ in Section A in the Appendix), we deduce that W satisfies

$$\begin{aligned} \overline{H}_R(W_x) &= \bar{A} \quad \text{for } x > 0, \\ \overline{H}_L(W_x) &= \bar{A} \quad \text{for } x < 0. \end{aligned}$$

We also deduce from (45) that, for all $\delta > 0$ and $x > 0$,

$$W_x \geq \bar{p}_R - \delta$$

in the case where $\bar{A} > \min \overline{H}_R$. Assume now that $\bar{A} = \min \overline{H}_R$. This implies that

$$\bar{p}_R \leq W_x \leq \hat{p}_R$$

and, in all cases, we thus get (47) for $x > 0$.

Similarly, we can prove for $x < 0$ that

$$\hat{p}_L \leq W_x \leq \bar{p}_L$$

and the proof of (46) of is achieved. This implies (47). The proof of Theorem 4.6 is now complete. □

Proof of Theorem 1.4. Let $\bar{A}$ denote the limit of A_ρ (see Proposition 4.4). We want to prove that $\bar{A} = \inf E$, where we recall that

$$E = \{\lambda \in \mathbb{R} : \text{there exists a subsolution } w \text{ of } (4)\}.$$

In view of (4), subsolutions are assumed to be periodic in time; we will see that they also automatically satisfy some growth conditions at infinity, see (48) below.

We argue by contradiction, by assuming that there exist $\lambda < \bar{A}$ and a subsolution w_λ of (4). The function

$$m_\lambda(x) = \sup_{t\in\mathbb{R}} (w_\lambda)^*(t,x)$$

satisfies

$$H(t(x), x, (m_\lambda)_x) \leq C$$

(for some function $t(x)$). Assumption (A3) implies that m_λ is globally Lipschitz continuous. Moreover, since w_λ is 1-periodic in time and $(w_\lambda)_t \leq C$,

$$|w_\lambda(t,x) - m_\lambda(x)| \leq C.$$

Hence,

$$w_\lambda^\varepsilon(t,x) = \varepsilon w_\lambda(\varepsilon^{-1}t, \varepsilon^{-1}x)$$

has a limit W^λ which satisfies

$$\overline{H}_R(W_x^\lambda) \le \lambda \quad \text{for } x > 0.$$

In particular, for $x > 0$,

$$W_x^\lambda \le \hat{p}_R^\lambda := \max\{p \in \mathbb{R} : \overline{H}_R(p) = \lambda\} < \bar{p}_R,$$

where $\bar{p}_R$ is as defined in (8). Similarly,

$$W_x^\lambda \ge \hat{p}_L^\lambda := \min\{p \in \mathbb{R} : \overline{H}_L(p) = \lambda\} > \bar{p}_L$$

with $\bar{p}_L$ as defined in (9). These two inequalities imply in particular that, for all $\delta > 0$, there exists $\tilde{C}_\delta$ such that

$$w_\lambda(t,x) \le \begin{cases} (\hat{p}_R^\lambda + \delta)x + \tilde{C}_\delta & \text{for } x > 0, \\ (\hat{p}_L^\lambda + \delta)x + \tilde{C}_\delta & \text{for } x < 0. \end{cases} \tag{48}$$

In particular,

$$w_\lambda < w \quad \text{for } |x| \ge R$$

if δ is small enough and R is large enough. Hence,

$$w_\lambda < w + C_R \quad \text{for } x \in \mathbb{R}.$$

Note finally that $u(t,x) = w(t,x) + C_R - \bar{A}t$ is a solution and $u_\lambda(t,x) = w_\lambda(t,x) - \lambda t$ is a subsolution of (1) with $\varepsilon = 1$ and $u_\lambda(0,x) \le u(0,x)$. Hence, the comparison principle implies that

$$w_\lambda(t,x) - \lambda t \le w(t,x) - \bar{A}t + C_R.$$

Dividing by t and letting $t \to +\infty$, we get the contradiction

$$\bar{A} \le \lambda.$$

The proof is now complete. □

5. Proof of Theorem 1.12

This section is devoted to the proof of Theorem 1.12. As pointed out in Remark 1.13 above, the notion of solutions for (1) has to first be made precise, because the Hamiltonian is discontinuous with respect to time.

Notion of solutions for (1). For $\varepsilon = 1$, a function u is a *solution* of (1) if it is globally Lipschitz continuous (in space and time) and it solves successively the Cauchy problems on time intervals $[\tau_i + k, \tau_{i+1} + k)$ for $i = 0, \ldots, K$ and $k \in \mathbb{N}$.

Because of this definition and approach, we have to show that, if the initial datum u_0 is globally Lipschitz continuous, then the solution to the successive Cauchy problems is also globally Lipschitz continuous (which of course ensures its uniqueness from the classical comparison principle). See Lemma 5.1 below.

Proof of Theorem 1.12(i). In view of the proof of Theorem 1.7, the reader can check that it is enough to get a global Lipschitz bound on the solution u^ε and to construct a global corrector in this new framework. The proof of these two facts is postponed; see Lemmas 5.1 and 5.2 following this proof. Notice that half-relaxed limits are not necessary anymore and that the reasoning can be completed by considering locally converging subsequences of $\{u^\varepsilon\}$. Notice also that the perturbed test function method of [Evans 1989] still works. As usual, if the viscosity subsolution inequality is not satisfied at the limit, this implies that the perturbed test function is a supersolution except at times $\varepsilon(\mathbb{Z}+\{\tau_0,\dots,\tau_K\})$. Still, a localized comparison principle in each slice of times for each Cauchy problem is sufficient to conclude. □

Lemma 5.1 (global Lipschitz bound). *The function u^ε is equi-Lipschitz continuous with respect to time and space.*

Proof. It is enough to get the result for $\varepsilon = 1$, since $u(t,x) = \varepsilon^{-1}u^\varepsilon(\varepsilon t, \varepsilon x)$ satisfies the equation with $\varepsilon = 1$ and the initial condition

$$u_0^\varepsilon(x) = \varepsilon^{-1}U_0^\varepsilon(\varepsilon x)$$

is equi-Lipschitz continuous. For the sake of clarity, we drop the ε superscript in u_0^ε and simply write u_0.

We first derive bounds on the time interval $[\tau_0,\tau_1) = [0,\tau_1)$. In order to do so, we assume that the initial data satisfies $|(u_0)_x| \le L$. Then, as usual, there is a constant $C > 0$ such that

$$u^\pm(t,x) = u_0(x) \pm Ct$$

are super- and sub-solutions of (1) and (10) with H given by (C1) with, for instance,

$$C := \max\Bigl(\max_{\alpha=1,\dots,N}\|a_\alpha\|_\infty,\ \max_{\alpha=0,\dots,N}\Bigl(\max_{|p|\le L}|\overline{H}_\alpha(p)|\Bigr)\Bigr). \tag{49}$$

Let u be the standard (continuous) viscosity solution of (1) on the time interval $(0,\tau_1)$ with initial data given by u_0 (recall that $\varepsilon = 1$). Then, for any $h > 0$ small enough, we have $-Ch \le u(h,x) - u(0,x) \le Ch$. The comparison principle implies, for $t \in (0,\tau_1 - h)$,

$$-Ch \le u(t+h,x) - u(t,x) \le Ch,$$

which shows the Lipschitz bound in time, on the time interval $[0,\tau_1)$,

$$|u_t| \le C. \tag{50}$$

From the Hamilton–Jacobi equation, we now deduce the following Lipschitz bound in space on the time interval $(0,\tau_1)$:

$$|\overline{H}_\alpha(u_x(t,\cdot))|_{L^\infty(b_\alpha,b_{\alpha+1})} \le C \quad \text{for } \alpha = 0,\dots,N. \tag{51}$$

We can now derive bounds on the time interval $[\tau_1,\tau_2)$ as follows. We deduce first that (51) still holds true at time $t = \tau_1$. Combined with our definition (49) of the constant C, we also deduce that

$$v^\pm(t,x) = u(\tau_1,x) \pm C(t-\tau_1)$$

are sub- and super-solutions of (6) for $t \in (\tau_1,\tau_2)$, where H is given by (C1). Reasoning as above, we get bounds (50) and (51) on the time interval $[\tau_1,\tau_2)$.

Such reasoning can be used iteratively to get the Lipschitz bounds (50) and (51) for $t \in [0, +\infty)$. The proof of the lemma is now complete. □

Lemma 5.2. *The conclusion of Theorem 4.6 still holds true in this new framework.*

Proof. The proof proceeds in several steps.

Step 1: *construction of a time-periodic corrector* w^ρ *on* $[-\rho, \rho]$. We first construct a Lipschitz corrector on a truncated domain. This, too, requires several steps.

Step 1.1: *first Cauchy problem on* $(0, +\infty)$. The method presented in the proof of Proposition 4.3, using a term δw^δ, has the inconvenience that it would not clearly provide a Lipschitz solution. In order to stick to our notion of globally Lipschitz solutions, we simply solve the Cauchy problem for $\rho > \rho_0 := \max_{\alpha=1,\dots,N} |b_\alpha|$,

$$\begin{cases} w_t^\rho + H(t, x, w_x^\rho) = 0 & \text{on } (0, +\infty) \times (-\rho, \rho)\,, \\ w_t^\rho + \overline{H}_N^-(w_x^\rho) = 0 & \text{on } (0, +\infty) \times \{-\rho\}, \\ w_t^\rho + \overline{H}_0^+(w_x^\rho) = 0 & \text{on } (0, +\infty) \times \{\rho\}, \\ w^\rho(0, x) = 0 & \text{for } x \in [-\rho, \rho]. \end{cases} \tag{52}$$

As in the proof of the previous lemma, we get global Lipschitz bounds with a constant C (independent of $\rho > 0$ and the distances $\ell_\alpha = b_{\alpha+1} - b_\alpha$):

$$|w_t^\rho|,\ |\overline{H}_\alpha(w_x^\rho(t, \cdot))|_{L^\infty((b_\alpha, b_{\alpha+1})\cap(-\rho,\rho))} \le C \quad \text{for } \alpha = 0, \dots, N. \tag{53}$$

Arguing as in [Forcadel et al. 2009a], for instance, we deduce that there exists a real number λ_ρ with

$$|\lambda_\rho| \le C$$

and a constant C_0 (that depends on ρ) such that

$$|w^\rho(t, x) + \lambda_\rho t| \le C_0. \tag{54}$$

Details are given in Section D in the Appendix for the reader's convenience.

Step 1.2: *getting global sub- and super-solutions.* Let us now define the following function (up to some subsequence $k_n \to +\infty$):

$$w_\infty^\rho(t, x) = \lim_{k_n \to +\infty} (w^\rho(t + k_n, x) + \lambda_\rho k_n),$$

which still satisfies (53) and (54). Then we also define the two functions

$$\overline{w}_\infty^\rho(t, x) = \inf_{k\in\mathbb{Z}} (w_\infty^\rho(t + k, x) + k\lambda_\rho) \quad \text{and} \quad \underline{w}_\infty^\rho(t, x) = \sup_{k\in\mathbb{Z}} (w_\infty^\rho(t + k, x) + k\lambda_\rho).$$

They still satisfy (53) and (54) and are a super- and a sub-solution, respectively, of the problem in $\mathbb{R} \times [-\rho, \rho]$. They moreover satisfy that $\overline{w}_\infty^\rho(t, x) + \lambda_\rho t$ and $\underline{w}_\infty^\rho(t, x) + \lambda_\rho t$ are 1-periodic in time, which implies the bounds

$$|\overline{w}_\infty^\rho(t, x) - \overline{w}_\infty^\rho(0, x) + \lambda_\rho t| \le C \quad \text{and} \quad |\underline{w}_\infty^\rho(t, x) - \underline{w}_\infty^\rho(0, x) + \lambda_\rho t| \le C.$$

Step 1.3: *a new Cauchy problem on* $(0,+\infty)$ *and construction of a time-periodic solution.* We note that $\overline{w}^\rho_\infty + 2C_0 \geq \underline{w}^\rho_\infty$ and we now solve the Cauchy problem with new initial data $\underline{w}^\rho_\infty(0,x)$ instead of the zero initial data and call $\tilde{w}^\rho$ the solution of this new Cauchy problem. From the comparison principle, we get

$$\underline{w}^\rho_\infty \leq \tilde{w}^\rho \leq \overline{w}^\rho_\infty + 2C_0.$$

In particular,

$$\tilde{w}^\rho(1,x) \geq \underline{w}^\rho_\infty(1,x) \geq \tilde{w}^\rho(0,x) - \lambda_\rho.$$

This implies, by comparison, that

$$\tilde{w}^\rho(k+1,x) \geq \tilde{w}^\rho(k,x) - \lambda_\rho. \tag{55}$$

Moreover $\tilde{w}^\rho$ still satisfies (53) (indeed with the same constant because, by construction, this is also the case for $\underline{w}^\rho_\infty$). We now define (up to some subsequence $k_n \to +\infty$)

$$\tilde{w}^\rho_\infty(t,x) = \lim_{k_n\to+\infty} (\tilde{w}^\rho(t+k_n,x) + \lambda_\rho k_n),$$

which, because of (55) and the fact that $\tilde{w}^\rho(t,x) + \lambda_\rho t$ is bounded, satisfies

$$\tilde{w}^\rho_\infty(k+1,x) + \lambda = \tilde{w}^\rho_\infty(k,x)$$

and then $\tilde{w}^\rho_\infty(t,x) + \lambda_\rho t$ is 1-periodic in time. Moreover $\tilde{w}^\rho_\infty$ is still a solution of the Cauchy problem and satisfies (53). We define

$$w^\rho := \tilde{w}^\rho_\infty + \lambda_\rho t,$$

which satisfies (37) and then provides the analogue of the function given in Proposition 4.3.

Step 2: *construction of* w *on* $\mathbb{R}$. The result of Theorem 4.6 still holds true for

$$w = \lim_{\rho\to+\infty} (w^\rho - w^\rho(0,0)),$$

which is globally Lipschitz continuous in space and time and satisfies (53) with $\rho = +\infty$, and

$$\bar{A} = \lim_{\rho\to+\infty} \lambda_\rho. \qquad \square$$

Proof of (12) *from Theorem 1.12.* We recall that $\overline{H}_L = \overline{H}_0$ and $\overline{H}_R = \overline{H}_1$ and set $a = a_1$ and (up to translation) $b_1 = 0$.

Step 1: *the convex case; identification of* $\bar{A}$.

Step 1.1: *a convex subcase.* We first work in the particular case where both $\overline{H}_\alpha$ for $\alpha = L, R$ are convex and given by the Legendre–Fenchel transform of convex Lagrangians L_α which satisfy, for some compact interval I_α,

$$L_\alpha(p) = \begin{cases} \text{finite} & \text{if } q \in I_\alpha, \\ +\infty & \text{if } q \notin I_\alpha. \end{cases} \tag{56}$$

Then it is known (see for instance the section on optimal control in [IM]) that the solution of (1) on the time interval $[0, \varepsilon\tau_1)$ is given by

$$u^\varepsilon(t,x) = \inf_{y\in\mathbb{R}}\left(\inf_{X\in S_{0,y;t,x}}\left\{u^\varepsilon(0,X(0)) + \int_0^t L_\varepsilon(s,X(s),\dot X(s))\,ds\right\}\right) \tag{57}$$

with

$$L_\varepsilon(s,x,p) = \begin{cases} \overline{H}_L^*(p) & \text{if } x<0,\\ \overline{H}_R^*(p) & \text{if } x>0,\\ \min\bigl(-a(\varepsilon^{-1}s), \min_{\alpha=L,R} L_\alpha(0)\bigr) & \text{if } x=0,\end{cases}$$

and, for $s<t$, the set of trajectories

$$S_{s,y;t,x} = \{X\in \mathrm{Lip}((s,t);\mathbb{R}) : X(s)=y,\ X(t)=x\}.$$

Combining this formula with the other one on the time interval $[\varepsilon\tau_1, \varepsilon\tau_2)$, and iterating on all necessary intervals, we get that (57) is a representation formula of the solution u^ε of (1) for all $t>0$. We also know (see the section on optimal control in [IM]), that the optimal trajectories from $(0,y)$ to (t_0,x_0) intersect the axis $x=0$ at most on a time interval $[t_1^\varepsilon, t_2^\varepsilon]$ with $0\le t_1^\varepsilon\le t_2^\varepsilon\le t_0$. If this interval is not empty, then we have $t_i^\varepsilon\to t_i^0$ for $i=1,2$ and we can easily pass to the limit in (57). In general, u^ε converges to u^0 given by the formula

$$u^0(t,x) = \inf_{y\in\mathbb{R}}\left(\inf_{X\in S_{0,y;t,x}}\left\{u^0(0,X(0)) + \int_x^t L_0(s,X(s),\dot X(s))\,ds\right\}\right)$$

with

$$L_0(s,x,p) = \begin{cases} \overline{H}_L^*(p) & \text{if } x<0,\\ \overline{H}_R^*(p) & \text{if } x>0,\\ \min\bigl(-\langle a\rangle, \min_{\alpha=L,R} L_\alpha(0)\bigr) & \text{if } x=0,\end{cases}$$

and, from [IM], we see that u^0 is the unique solution of (6) and (2) with $\bar A = \langle a\rangle$.

Step 1.2: *the general convex case.* The general case of convex Hamiltonians is recovered, because, for Lipschitz continuous initial data u_0, we know that the solution is globally Lipschitz continuous. Therefore, we can always modify the Hamiltonians $\overline{H}_\alpha$ outside some compact intervals so that the modified Hamiltonians satisfy (56).

Step 2: *general quasiconvex Hamiltonians; identification of* $\bar A$.

Step 2.1: *subsolution inequality.* From Theorem 2.10 in [IM], we know that $w(t,0)$, as a function of time only, satisfies, in the viscosity sense,

$$w_t(t,0) + a(t) \le \bar A \quad \text{for all } t\notin \bigcup_{i=1,\dots,K+1} \tau_i + \mathbb{Z}.$$

Using the 1-periodicity in time of w, we see that the integration in time on one period implies

$$\langle a\rangle \le \bar A. \tag{58}$$

Step 2.2: *supersolution inequality.* Recall that $\bar{A} \geq \langle a \rangle \geq A_0 := \max_{\alpha=L,R} \min(\overline{H}_\alpha)$. If $\bar{A} = A_0$, then obviously we get $\bar{A} = \langle a \rangle$. Hence, it remains to treat the case $\bar{A} > A_0$.

Step 2.3: *construction of a supersolution for* $x \neq 0$. Recall that $\bar{p}_R$ and $\bar{p}_L$ are defined in (8) and (9) and the minimum of $\overline{H}_\alpha$ is reached for $\bar{p}^0_\alpha$, $\alpha = R, L$. Since $\bar{A} > A_0$, there exists some $\delta > 0$ such that

$$\bar{p}_L + 2\delta < \bar{p}^0_L \quad \text{and} \quad \bar{p}^0_R < \bar{p}_R - 2\delta. \tag{59}$$

If w denotes a global corrector given by Lemma 5.2 (or Theorem 4.6), let us define

$$\underline{w}_R(t,x) = \inf_{h \geq 0}(w(t,x+h) - \bar{p}^0_R h) \quad \text{for } x \geq 0$$

and similarly

$$\underline{w}_L(t,x) = \inf_{h \geq 0}(w(t,x-h) + \bar{p}^0_L h) \quad \text{for } x \leq 0.$$

From (45) with $\rho_\delta = 0$, we deduce that we have, for some $\bar{h} \geq 0$,

$$w(t,x) \geq \underline{w}_R(t,x) = w(t,x+\bar{h}) - \bar{p}^0_R \bar{h} \geq w(t,x) + (\bar{p}_R - \delta - \bar{p}^0_R)\bar{h} - C_\delta.$$

From (59), this implies

$$0 \leq \bar{h} \leq \frac{C_\delta}{\delta} \tag{60}$$

and, using the fact that w is globally Lipschitz continuous, we deduce that, for $\alpha = R$,

$$w \geq \underline{w}_\alpha \geq w - C_1. \tag{61}$$

Moreover, by construction — as an infimum of (globally Lipschitz continuous) supersolutions — $\underline{w}_R$ is a (globally Lipschitz continuous) supersolution of the problem in $\mathbb{R} \times (0, +\infty)$. We also have, for $x = y + z$ with $z \geq 0$,

$$\begin{aligned}
\underline{w}_R(t,x) - \underline{w}_R(t,y) &= w(t,x+\bar{h}) - \bar{p}^0_R \bar{h} - \underline{w}_R(t,y) \\
&\geq w(t,x+\bar{h}) - \bar{p}^0_R \bar{h} - (w(t,y+\bar{h}+z) - \bar{p}^0_R(\bar{h}+z)) \\
&\geq \bar{p}^0_R z = \bar{p}^0_R (x-y),
\end{aligned}$$

which shows that

$$(\underline{w}_R)_x \geq \bar{p}^0_R. \tag{62}$$

Similarly (and we can also use a symmetry argument to see it), we get that $\underline{w}_L$ is a (globally Lipschitz continuous) supersolution in $\mathbb{R} \times (-\infty, 0)$, it satisfies (61) with $\alpha = L$, and

$$(\underline{w}_L)_x \leq \bar{p}^0_L. \tag{63}$$

We now define

$$\underline{w}(t,x) = \begin{cases} \underline{w}_R(t,x) & \text{if } x > 0, \\ \underline{w}_L(t,x) & \text{if } x < 0, \\ \min(\underline{w}_L(t,0), \underline{w}_R(t,0)) & \text{if } x = 0, \end{cases} \tag{64}$$

which, by construction is lower semicontinuous and satisfies (61), and is a supersolution for $x \neq 0$.

Step 2.4: *checking the supersolution property at* $x = 0$. Let φ be a test function touching $\underline{w}$ from below at $(t_0, 0)$ with $t_0 \notin \bigcup_{i=1,\dots,K+1} \tau_i + \mathbb{Z}$. We want to check that

$$\varphi_t(t_0, 0) + F_{a(t_0)}(\varphi_x(t_0, 0^-), \varphi_x(t_0, 0^+)) \geq \bar{A}. \tag{65}$$

We may assume that

$$\underline{w}(t_0, 0) = \underline{w}_R(t_0, 0),$$

since the case $\underline{w}(t_0, 0) = \underline{w}_L(t_0, 0)$ is completely similar. Let $\bar{h} \geq 0$ be such that

$$\underline{w}_R(t_0, 0) = w(t_0, 0 + \bar{h}) - \bar{p}_R^0 \bar{h}.$$

We distinguish two cases. Assume first that $\bar{h} > 0$. Then we have, for all $h \geq 0$,

$$\varphi(t, 0) \leq w(t, 0 + h) - \bar{p}_R^0 h,$$

with equality for $(t, h) = (t_0, \bar{h})$. This implies the viscosity inequality

$$\varphi_t(t_0, 0) + \overline{H}_R(\bar{p}_R^0) \geq \bar{A},$$

which implies (65), because $F_{a(t_0)}(\varphi_x(t_0, 0^-), \varphi_x(t_0, 0^+)) \geq a(t_0) \geq A_0 \geq \min \overline{H}_R = \overline{H}_R(\bar{p}_R^0)$.

Assume now that $\bar{h} = 0$. Then we have $\varphi \leq \underline{w} \leq w$, with equality at $(t_0, 0)$. This immediately implies (65).

Step 2.5: *conclusion.* We deduce that $\underline{w}$ is a supersolution on $\mathbb{R} \times \mathbb{R}$. Now let us consider a C^1 function $\psi(t)$ such that

$$\psi(t) \leq \underline{w}(t, 0),$$

with equality at $t = t_0$. Because of (62) and (63), we see that

$$\varphi(t, x) = \psi(t) + \bar{p}_L^0 x 1_{\{x<0\}} + \bar{p}_R^0 x 1_{\{x>0\}}$$

satisfies

$$\varphi \leq \underline{w},$$

with equality at $(t_0, 0)$. This implies (65) and, at almost every point t_0 where the Lipschitz continuous function $\underline{w}(t, 0)$ is differentiable, we have

$$\underline{w}_t(t_0, 0) + a(t_0) \geq \bar{A}.$$

Because w is 1-periodic in time, we get, after an integration on one period,

$$\langle a \rangle \geq \bar{A}. \tag{66}$$

Together with (58), we deduce that $\langle a \rangle = \bar{A}$, which is the desired result, for $N = 1$. ☐

Proof of (13) *in Theorem 1.12.* . We simply remark, using the subsolution viscosity inequality at each junction condition, that, for $\alpha = 1, \dots, N$,

$$\bar{A} \geq \langle a_\alpha \rangle,$$

which is the desired result. This achieves the proof of (12) and (13). $\square$

Proof of the monotonicity of $\bar{A}$ in Theorem 1.12. Let $N \geq 2$ and, for $i = c, d$, let us assume some given $b_1^i < \cdots < b_N^i$. and let us call w^i a global corrector given by Lemma 5.2 (or Theorem 4.6) with $\lambda = \bar{A}^i$ and $H = H^i$ for $i = c, d$, respectively.

We let $\ell_\alpha^i = b_{\alpha+1}^i - b_\alpha^i > 0$ and assume that

$$0 < \ell_{\alpha_0}^d - \ell_{\alpha_0}^c =: \delta_{\alpha_0} \quad \text{for some } \alpha_0 \in \{1, \ldots, N-1\}$$

and

$$\ell_\alpha^d = \ell_\alpha^c \quad \text{for all } \alpha \in \{1, \ldots, N-1\} \backslash \{\alpha_0\}.$$

Calling $\bar{p}_{\alpha_0}^0$ a point of global minimum of $\overline{H}_{\alpha_0}$, we define

$$\tilde{w}^d(t,x) = \begin{cases} w^c(t, x - b_{\alpha_0}^d + b_{\alpha_0}^c) & \text{if } x \leq b_{\alpha_0}^d + \ell_{\alpha_0}^c/2 =: x_-, \\ w^c(t, x_- - b_{\alpha_0}^d + b_{\alpha_0}^c) + \bar{p}_{\alpha_0}^0(x - x_-) & \text{if } x_- \leq x \leq x_+, \\ w^c(t, x - b_{\alpha_0+1}^d + b_{\alpha_0+1}^c) + \bar{p}_{\alpha_0}^0(x_+ - x_-) & \text{if } x \geq b_{\alpha_0+1}^d - \ell_{\alpha_0}^c/2 =: x_+. \end{cases}$$

Recall that w^i, $i = c, d$, are globally Lipschitz continuous in space and time. This shows that $\tilde{w}^d$ is also Lipschitz continuous in space and time by construction, because it is continuous at $x = x_-, x_+$. Moreover, $\tilde{w}^d$ is 1-periodic in time. We now want to check that $\tilde{w}^d$ is a subsolution of the equation satisfied by w^d with $\bar{A}^c$ on the right-hand side instead of $\bar{A}^d$. We only have to check it for all times $\bar{t} \notin \{\tau_0, \ldots, \tau_K\}$ and $\bar{x} \in [x_-, x_+]$, i.e., we have to show that

$$\tilde{w}_t^d(\bar{t}, \bar{x}) + \overline{H}_{\alpha_0}(\tilde{w}_x^d(\bar{t}, \bar{x})) \leq \bar{A}^c \quad \text{for all } \bar{x} \in [x_-, x_+]. \tag{67}$$

Assume that φ is a test function touching $\tilde{w}^d$ from above at such a point $(\bar{t}, \bar{x})$ with $\bar{x} \in [x_-, x_+]$. Then this implies in particular that $\psi(t,x) = \varphi(t,x) - \bar{p}_{\alpha_0}^0(x - x_-)$ touches $\tilde{w}^d(\,\cdot\,, x_-) = w^c(\,\cdot\,, x_0)$ from above at time $\bar{t}$ with $x_0 = b_{\alpha_0}^c + \ell_{\alpha_0}^c/2$. Recall that w^c is a solution of

$$w_t^c + \overline{H}_{\alpha_0}(w_x^c) = \bar{A}^c \quad \text{on } (b_{\alpha_0}^c, b_{\alpha_0+1}^c).$$

From the characterization of subsolutions (see Theorem 2.10 in [IM]), we then deduce that

$$\psi_t(\bar{t}) + \overline{H}_{\alpha_0}(\bar{p}_{\alpha_0}^0) \leq \bar{A}^c.$$

If $\bar{x} \in (x_-, x_+)$, then we have $\varphi_x(\bar{t}, \bar{x}) = \bar{p}_{\alpha^0}^0$. This means, in particular,

$$\varphi_t + \overline{H}_{\alpha_0}(\varphi_x) \leq \bar{A}^c \quad \text{at } (\bar{t}, \bar{x}) \text{ if } \bar{x} \in (x_-, x_+). \tag{68}$$

Now, using (68), and Theorem 2.10 in [IM] again, we deduce that we have, in the viscosity sense,

$$\tilde{w}_t^d(\bar{t}, \bar{x}) + \max\bigl(\overline{H}_{\alpha_0}^-(\tilde{w}_x^d(\bar{t}, \bar{x}^+)), \overline{H}_{\alpha_0}^+(\tilde{w}_x^d(\bar{t}, \bar{x}^-))\bigr) \leq \bar{A}^c \quad \text{for } \bar{x} = x_\pm. \tag{69}$$

Therefore, (68) and (69) imply (67).

Let us now call H^d the Hamiltonian in assumption (C1) constructed with the points $\{b_\alpha^d\}_{\alpha=1,\ldots,N}$. Then we have

$$\tilde{w}_t^d + H^d(t, x, \tilde{w}_x^d) \leq \bar{A}^c \quad \text{for all } t \notin \{\tau_0, \ldots \tau_K\}.$$

Note that the proof of Theorem 1.4 is unchanged for the present problem and then Theorem 1.4 still holds true. This shows that

$$\bar{A}^d \leq \bar{A}^c, \tag{70}$$

which is the expected monotonicity. The proof is now complete. □

Remark 5.3. In the previous proof, it would also be possible to compare the subsolution given by the restriction of $\tilde{w}^d$ on some interval $[-\rho, \rho]$ for $\rho > 0$ large enough (see Proposition 2.16 in [IM]) with the approximation $w^{d,\rho}$ of w^d on $[-\rho, \rho]$ with $\bar{A}^d \geq \bar{A}^d_\rho \to \bar{A}^d$ as $\rho \to +\infty$. The comparison for large times would imply $\bar{A}^d_\rho \leq \bar{A}^c$. As $\rho \to +\infty$, this would give the same conclusion (70).

Proof of (14) *in Theorem 1.12.* Let w be a global corrector associated to $\bar{A}$.

Recall that

$$\bar{A} \geq \bar{A}_0 := \max_{\alpha=1,\dots,N} \langle a_\alpha \rangle \geq A_0 := \max_{\alpha=1,\dots,N} A_0^\alpha \quad \text{with} \quad A_0^\alpha = \max_{\beta=\alpha-1,\alpha} (\min \overline{H}_\beta). \tag{71}$$

Our goal is to prove (14), i.e., that $\bar{A} = \bar{A}_0$ when all the distances ℓ_α are large enough. Let us assume that

$$\bar{A} > \bar{A}_0.$$

Step 1: *considering another corrector with the same* $\langle \hat{a}_\alpha \rangle = \bar{A}_0$. Let $\mu_\alpha \geq 0$ be such that

$$\hat{a}_\alpha = \mu_\alpha + a_\alpha \quad \text{with} \quad \langle \hat{a}_\alpha \rangle = \bar{A}_0 \quad \text{for all } \alpha = 1, \dots, N.$$

Let us call $\hat{w}$ the corresponding corrector with associated constant $\hat{A}$. Then Theorem 1.4 (still valid here) implies that

$$\hat{A} \geq \bar{A} > \bar{A}_0.$$

We also split the set $\{1, \dots, N\}$ into two disjoint sets,

$$I_0 = \{\alpha \in \{1, \dots, N\} : \bar{A}_0 = A_0^\alpha\}$$

and

$$I_1 = \{\alpha \in \{1, \dots, N\} : \bar{A}_0 > A_0^\alpha\}.$$

Note that, by (71), if $\alpha \in I_0$ then $\langle a_\alpha \rangle = A_0^\alpha$ and then, by (C3), we have $a_\alpha(t) = \text{const} = A_0^\alpha$ for all $t \in \mathbb{R}$. For later use, we then claim that $\hat{w}$ satisfies

$$\hat{w}_t(t, x) + \max\big(\overline{H}_\alpha^-(\hat{w}_x(t, x^+)), \overline{H}_{\alpha-1}^+(\hat{w}_x(t, x^-))\big) = \hat{A} \quad \text{for all } (t, x) \in \mathbb{R} \times \{b_\alpha\} \tag{72}$$

and not only for $t \in \mathbb{R} \backslash (\mathbb{Z} + \{\tau_0, \dots, \tau_K\})$. Let us show this for subsolutions (the proof being similar for supersolutions). Let φ be a test function touching $\hat{w}$ from above at some point $(\bar{t}, \bar{x}) = (j + \tau_k, b_\alpha)$ for some $j \in \mathbb{Z}$, $k \in \{0, \dots, K\}$. Assume also that the contact between φ and $\hat{w}$ only holds at that point $(\bar{t}, \bar{x})$. The proof is a variant of a standard argument. For $\eta > 0$, let us consider the test function

$$\varphi_\eta(t, x) = \varphi(t, x) + \frac{\eta}{\bar{t} - t} \quad \text{for } t \in (-\infty, \bar{t}).$$

Then, for $r > 0$ fixed, we have

$$\inf_{\substack{(t,x)\in\overline{B_r(\bar t,\bar x)}\\ t<\bar t}} (\varphi_\eta - \hat w)(t,x) = (\varphi_\eta - \hat w)(t_\eta, x_\eta)$$

with

$$\begin{cases} P_\eta = (t_\eta, x_\eta) \to (\bar t, \bar x) = \overline{P} & \text{as } \eta \to 0,\\ \varphi_t(\overline{P}) \le \limsup_{\eta\to 0} (\varphi_\eta)_t(P_\eta). \end{cases}$$

This implies that $\hat w$ is a relaxed viscosity subsolution at $(\bar t, \bar x)$ in the sense of Definition 2.2 in [IM]. By Proposition 2.5 in [IM], we deduce that $\hat w$ is also a standard (i.e., not relaxed) viscosity subsolution at $(\bar t, \bar x)$. Finally, we get (72).

Step 2: *defining a space supersolution.* Let us define the function

$$M(x) = \inf_{t\in\mathbb{R}} \hat w(t,x).$$

Because $\hat w$ is globally Lipschitz continuous, we deduce that M is also globally Lipschitz continuous. Moreover, we have the viscosity supersolution inequality

$$\overline{H}_\alpha(M_x(x)) \ge \hat A > \bar A_0 \quad \text{for all } x \in (b_\alpha, b_{\alpha+1}),\ \alpha = 0, \dots, N.$$

Let us call, for $\alpha = 0, \dots, N$,

$$\begin{aligned} \bar p_{\alpha,R} &= \min E_{\alpha,R} \quad \text{with} \quad E_{\alpha,R} = \{p \in \mathbb{R} : \overline{H}^+_\alpha(p) = \overline{H}_\alpha(p) = \bar A_0\},\\ \bar p_{\alpha,L} &= \max E_{\alpha,L} \quad \text{with} \quad E_{\alpha,L} = \{p \in \mathbb{R} : \overline{H}^-_\alpha(p) = \overline{H}_\alpha(p) = \bar A_0\}. \end{aligned}$$

Let us now consider $\alpha = 0, \dots, N$ and two points $x_- < x_+$ with $x_\pm \in (b_\alpha, b_{\alpha+1})$. Let us assume that there is a test function $\varphi^\pm$ touching M from below at $x_\pm$. Then we have

$$\overline{H}_\alpha(\varphi^\pm_x(x_\pm)) \ge \hat A > \bar A_0$$

with

$$\varphi^\pm_x(x_\pm) \ge \bar p_{\alpha,R} \quad \text{or} \quad \varphi^\pm_x(x_\pm) \le \bar p_{\alpha,L}.$$

Moreover, if $\bar A_0 > \min \overline{H}_\alpha$, then we have

$$\bar p_{\alpha,L} < \bar p^0_\alpha < \bar p_{\alpha,R}$$

for any $\bar p^0_\alpha$ which is a point of global minimum of $\overline{H}_\alpha$.

Step 3: *a property of the space supersolution.* We now claim that the following case is impossible:

$$p^- := \varphi^-_x(x_-) < \varphi^+_x(x_+) =: p^+ \quad \text{and} \quad \inf_{[p^-,p^+]} \overline{H}_\alpha < \hat A.$$

Indeed, if $\bar p \in (p^-, p^+)$ is such that $\overline{H}_\alpha(\bar p) < \hat A$, then the geometry of the graph of the function M implies that

$$\inf_{x\in[x_-,x_+]} (M(x) - x\bar p) = M(\bar x) - \bar x \bar p \quad \text{for some } \bar x \in (x_-, x_+)$$

and then we have the viscosity supersolution inequality, at $\bar{x}$,

$$\overline{H}_\alpha(\bar{p}) \geq \hat{A},$$

which leads to a contradiction. Therefore (in either case, $\bar{A}_0 > \min \overline{H}_\alpha$ or $\bar{A}_0 = \min \overline{H}_\alpha$), it is possible to check that there is a point $\bar{x}_\alpha \in [b_\alpha, b_{\alpha+1}]$ such that the Lipschitz continuous function M satisfies, in the viscosity sense,

$$\begin{cases} M_x \geq \bar{p}_{\alpha,R} & \text{in } (b_\alpha, \bar{x}_\alpha), \\ -M_x \geq -\bar{p}_{\alpha,L} & \text{in } (\bar{x}_\alpha, b_{\alpha+1}). \end{cases}$$

Moreover, from Theorem 4.6(ii) (see Lemma 5.2), we deduce from $\hat{A} > \max(\min H_N, \min H_0)$ that

$$\bar{x}_N = +\infty \quad \text{and} \quad \bar{x}_0 = -\infty.$$

In particular, we deduce that there exists at least one $\alpha_0 \in \{1, \dots, N\}$ such that

$$\bar{x}_{\alpha_0} - b_{\alpha_0} \geq \tfrac{1}{2}\ell_{\alpha_0} \quad \text{and} \quad b_{\alpha_0} - \bar{x}_{\alpha_0-1} \geq \tfrac{1}{2}\ell_{\alpha_0-1}. \tag{73}$$

Step 4: *the case* $\alpha_0 \in I_0$. In this case, we see that there exists a time $\bar{t}$ such that the test function

$$\varphi(t,x) = \begin{cases} \bar{p}_{\alpha_0,R}(x - b_{\alpha_0}) & \text{for } x \geq b_{\alpha_0}, \\ \bar{p}_{\alpha_0-1,L}(x - b_{\alpha_0}) & \text{for } x \leq b_{\alpha_0}, \end{cases}$$

is a test function touching (up to some additive constant) $\hat{w}$ from below at $(\bar{t}, b_{\alpha_0})$. By (72), this implies

$$\bar{A}_0 = \max(\overline{H}_{\alpha_0}(\bar{p}_{\alpha_0,R}), \overline{H}_{\alpha_0-1}(\bar{p}_{\alpha_0-1,L})) \geq \hat{A} \geq \bar{A}.$$

This is a contradiction.

Step 5: *consequences on* $\hat{w}$. From the fact that $\hat{w}$ is 1-periodic in time and C-Lipschitz continuous in time (with a constant C depending only on $\max_{\alpha=1,\dots,N} \|\hat{a}_\alpha\|_\infty$ and the $\overline{H}_\alpha$; see (49)), we deduce that we have

$$\begin{cases} \hat{w}(t,x+h) - \hat{w}(t,x) \geq \bar{p}_{\alpha,R}h - 2C & \text{for } x,\ x+h \in (b_\alpha, \bar{x}_\alpha), \\ \hat{w}(t,x-h) - \hat{w}(t,x) \geq -\bar{p}_{\alpha,R}h - 2C & \text{for } x,\ x+h \in (\bar{x}_\alpha, b_{\alpha+1}). \end{cases} \tag{74}$$

Step 6: *the case* $\alpha_0 \in I_1$*; definition of a spacetime supersolution.* Proceeding similarly to Step 3 of the proof of (12), we define

$$\underline{\hat{w}}_{\alpha_0,R}(t,x) = \inf_{\ell_{\alpha_0}/4 \geq h \geq 0} (\hat{w}(t,x+h) - \bar{p}^0_{\alpha_0}h) \quad \text{for } b_{\alpha_0} \leq x \leq b_{\alpha_0} + \tfrac{1}{4}\ell_{\alpha_0}$$

and

$$\underline{\hat{w}}_{(\alpha_0-1),L}(t,x) = \inf_{\ell_{(\alpha_0-1)}/4 \geq h \geq 0} (\hat{w}(t,x-h) + \bar{p}^0_{\alpha_0-1}h) \quad \text{for } b_{\alpha_0} - \tfrac{1}{4}\ell_{\alpha_0-1} \leq x \leq b_{\alpha_0}.$$

From (74), we deduce that we have, for some $\bar{h} \in \left[0, \tfrac{1}{4}\ell_{\alpha_0}\right]$,

$$\hat{w}(t,x) \geq \underline{\hat{w}}_{\alpha_0,R}(t,x) = \hat{w}(t,x+\bar{h}) - \bar{p}^0_{\alpha_0}\bar{h} \geq \hat{w}(t,x) + (\bar{p}_{\alpha_0,R} - \bar{p}^0_{\alpha_0})\bar{h} - 2C,$$

which implies

$$0 \leq \bar{h} \leq \frac{2C}{\bar{p}_{\alpha_0,R} - \bar{p}^0_{\alpha_0}}.$$

As in Step 3 of the proof of (12), if

$$\frac{\ell_{\alpha_0}}{4} > \frac{2C}{\bar{p}_{\alpha_0,R} - \bar{p}^0_{\alpha_0}}, \tag{75}$$

this implies that $\underline{\hat{w}}_{\alpha_0,R}$ is a supersolution for $x \in \left(b_{\alpha_0}, b_{\alpha_0} + \frac{1}{4}\ell_{\alpha_0}\right)$. Similarly, if

$$\frac{\ell_{\alpha_0-1}}{4} > \frac{2C}{\bar{p}^0_{\alpha_0-1} - \bar{p}_{\alpha_0-1,L}} \tag{76}$$

then $\underline{\hat{w}}_{\alpha_0-1,L}$ is a supersolution for $x \in \left(b_{\alpha_0} - \frac{1}{4}\ell_{\alpha_0-1}, b_{\alpha_0}\right)$. We now define

$$\underline{\hat{w}}(t,x) = \begin{cases} \underline{\hat{w}}_{\alpha_0,R}(t,x) & \text{if } x \in \left(b_{\alpha_0}, b_{\alpha_0} + \frac{1}{4}\ell_{\alpha_0}\right), \\ \underline{\hat{w}}_{\alpha_0-1,L}(t,x) & \text{if } x \in \left(b_{\alpha_0} - \frac{1}{4}\ell_{\alpha_0-1}, b_{\alpha_0}\right), \\ \min(\underline{\hat{w}}_{\alpha_0-1,L}(t,b_{\alpha_0}),\ \underline{\hat{w}}_{\alpha_0,R}(t,b_{\alpha_0})) & \text{if } x = b_{\alpha_0}. \end{cases}$$

Then, as in Steps 4 and 5 of the proof of (12), we deduce that $\underline{\hat{w}}$ is a supersolution up to the junction point $x = b_{\alpha_0}$ and that

$$\bar{A}_0 = \langle \hat{a}_{\alpha_0} \rangle \geq \hat{A} \geq \bar{A}.$$

This is a contradiction.

Step 7: *conclusion.* If (75) and (76) hold true for any $\alpha_0 \in I_1$, then we deduce that $\bar{A} \leq \bar{A}_0$, which implies $\bar{A} = \bar{A}_0$. This ends the proof of (14) in Theorem 1.12. □

Proof of (15) *in Theorem 1.12.* Let us consider

$$\bar{a}(t) = \max_{\alpha=1,\dots,N} a_\alpha(t)$$

and $(w, \bar{\bar{A}})$ a solution (given by Theorem 4.6 (see also Lemma 5.2)) of

$$\begin{cases} w_t + \overline{H}_0(w_x) = \bar{\bar{A}} & \text{if } x < 0, \\ w_t + \overline{H}_N(w_x) = \bar{\bar{A}} & \text{if } x > 0, \\ w_t(t,0) + \max\left(\bar{a}(t), \overline{H}_N^-(w_x(t,0^+)), \overline{H}_0^+(w_x(t,0^-))\right) = \bar{\bar{A}} & \text{if } x = 0, \\ w \text{ is 1-periodic in } t. \end{cases}$$

From Theorem 1.12, we also know that

$$\bar{\bar{A}} = \langle \bar{a} \rangle.$$

For $N \geq 2$, we set $\ell = (\ell_1, \dots, \ell_{N-1}) \in (0, +\infty)^{N-1}$ and consider $b_0 = -\infty < b_1 < \cdots < b_N < b_{N+1} = +\infty$ with

$$\ell_\alpha = b_{\alpha+1} - b_\alpha \quad \text{for } \alpha = 1, \dots, N-1.$$

We now call $(w^\ell, \bar{A}^\ell)$ a global corrector, given by Theorem 4.6 (see also Lemma 5.2). The remainder of the proof is divided into several steps.

Step 1: *bound from above on* $\bar{A}^\ell$. We define

$$\tilde{w}(t,x)=\begin{cases} w(t,x-b_1) & \text{if } x\le b_1,\\ w(t,0)+\bar{p}^0_\alpha(x-b_\alpha)+\sum_{\beta=1,\dots,\alpha-1}\bar{p}^0_\beta(b_{\beta+1}-b_\beta) & \text{if } b_\alpha\le x\le b_{\alpha+1},\ \alpha\in\{1,\dots,N-1\},\\ w(t,x-b_N)+\sum_{\beta=1,\dots,N-1}\bar{p}^0_\beta(b_{\beta+1}-b_\beta) & \text{if } x\ge b_N.\end{cases}$$

Proceeding as in Step 1 of the proof of Theorem 1.12(ii), it is easy to check that $\tilde{w}$ is a subsolution of the equation satisfied by w^ℓ with $\bar{\bar{A}}$ on the right-hand side instead of $\bar{A}^\ell$. Then Theorem 1.4 implies that

$$\bar{A}^\ell\le\bar{\bar{A}}=\langle\bar{a}\rangle. \tag{77}$$

Step 2: *bound from below on* $\bar{A}^\ell$. From Theorem 2.10 in [IM], we deduce that we have, in the viscosity sense (in time only),

$$w^\ell_t(t,b_\alpha)+a_\alpha(t)\le\bar{A}^\ell\quad\text{for all } t\notin\bigcup_{k=0}^{K}\{\tau_k+\mathbb{Z}\}.$$

Let us call

$$\underline{A}=\liminf_{\ell\to 0}\bar{A}^\ell.$$

We also know that w^ℓ is 1-periodic and globally Lipschitz continuous with a constant which is independent of ℓ. Therefore, there exists a 1-periodic and Lipschitz continuous function $g=g(t)$ such that

$$w^\ell(t,b_\alpha)\to g(t)\quad\text{as } \ell\to 0\quad\text{for all } \alpha=1,\dots,N.$$

The stability of viscosity solutions implies, in the viscosity sense,

$$g'(t)+a_\alpha(t)\le\underline{A}\quad\text{for all } \alpha=1,\dots,N \text{ and } t\notin\bigcup_{k=0}^{K}\{\tau_k+\mathbb{Z}\}.$$

Because g is Lipschitz continuous, this inequality also holds for almost every $t\in\mathbb{R}$. This implies

$$g'(t)+\bar{a}(t)\le\underline{A}\quad\text{for a.e. } t\in\mathbb{R}.$$

An integration on one period gives

$$\langle\bar{a}\rangle\le\underline{A}. \tag{78}$$

Step 3: *conclusion.* Combining (77) with (78) finally yields that $\bar{A}^\ell\to\langle\bar{a}\rangle$ as $\ell\to 0$. The proof of (15) in Theorem 1.12 is now complete. □

Appendix: Proofs of some technical results

A. ***The case $\bar{x}\neq 0$ in the proof of convergence.*** We only deal with the subcase $\bar{x}>0$, since the subcase $\bar{x}<0$ is treated in the same way. Reducing $\bar{r}$ if necessary, we may assume that $B_{\bar{r}}(\bar{t},\bar{x})$ is compactly embedded in the set $\{(t,x)\in(0,+\infty)\times(0,+\infty):x>0\}$, because there exists a positive constant $c_{\bar{r}}$ such that

$$(t,x)\in B_{\bar{r}}(\bar{t},\bar{x})\implies x>c_{\bar{r}}. \tag{79}$$

Let $p=\varphi_x(\bar t,\bar x)$ and let $v^R=v^R(t,x)$ be a solution of the cell problem

$$v^R_t+H_R(t,x,p+v^R_x)=\overline{H}_R(p)\quad\text{in }\mathbb{R}\times\mathbb{R}. \tag{80}$$

We claim that, if $\varepsilon>0$ is small enough, the perturbed test function [Evans 1989]

$$\varphi^\varepsilon(t,x)=\varphi(t,x)+\varepsilon v^R\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\Big)$$

satisfies, in the viscosity sense, the inequality

$$\varphi^\varepsilon_t+H\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\varphi^\varepsilon_x\Big)\ge\frac{\theta}{2}\quad\text{in }B_r(\bar t,\bar x) \tag{81}$$

for sufficiently small $r>0$. To see this, let ψ be a test function touching φ^ε from below at (t_1,x_1) in $B_r(\bar t,\bar x)\subseteq B_{\bar r}(\bar t,\bar x)$. In this way, the function

$$\eta(s,y)=\frac{1}{\varepsilon}(\psi(\varepsilon s,\varepsilon y)-\varphi(\varepsilon s,\varepsilon y))$$

touches v^R from below at $(s_1,y_1)=(t_1/\varepsilon,x_1/\varepsilon)$ and (80) yields

$$\psi_t(t_1,x_1)-\varphi_t(t_1,x_1)+H_R\bigg(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon},p+\psi_x(t_1,x_1)-\varphi_x(t_1,x_1)\bigg)\ge\overline{H}_R(p). \tag{82}$$

Since (79) implies that $x/\varepsilon\to+\infty$, as $\varepsilon\to0$, uniformly with respect to $(t,x)\in B_{\bar r}(\bar t,\bar x)$, we can find, owing to (A5), an $\varepsilon_0>0$ independent of ψ and (t_1,x_1) such that the inequality

$$H\bigg(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon},\psi_x(t_1,x_1)\bigg)\ge H_R\bigg(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon},\psi_x(t_1,x_1)\bigg)-\frac{\theta}{4} \tag{83}$$

holds true for $\varepsilon<\varepsilon_0$. Combining (19), (82) and (83) and using the continuity of φ_x and φ_t, we have

$$\begin{aligned}\psi_t(t_1,x_1)&+H\bigg(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon},\psi_x(t_1,x_1)\bigg)\\&\ge\psi_t(t_1,x_1)+H_R\bigg(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon},p+\psi_x(t_1,x_1)-\varphi_x(t_1,x_1)\bigg)\\&\qquad+H_R\bigg(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon},\psi_x(t_1,x_1)\bigg)-H_R\bigg(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon},\varphi_x(\bar t,\bar x)+\psi_x(t_1,x_1)-\varphi_x(t_1,x_1)\bigg)-\frac{\theta}{4}\\&\ge\frac{\theta}{2}\end{aligned}$$

if r is sufficiently close to 0. The claim (81) is proved.

Since φ is strictly above $\bar u$, if ε and r are small enough then

$$u^\varepsilon+\kappa_r\le\varphi^\varepsilon\quad\text{on }\partial B_r(\bar t,\bar x)$$

for a suitable positive constant κ_r. By the comparison principle we deduce

$$u^\varepsilon+\kappa_r\le\varphi^\varepsilon\quad\text{in }B_r(\bar t,\bar x)$$

and, passing to the limit as $\varepsilon \to 0$ and $(t,x) \to (\bar{t},\bar{x})$ on both sides of the previous inequality, we produce the contradiction

$$\bar{u}(\bar{t},\bar{x}) < \bar{u}(\bar{t},\bar{x}) + \kappa_r \le \varphi(\bar{t},\bar{x}) = \bar{u}(\bar{t},\bar{x}).$$

B. ***Proof of Lemma 3.3.*** We first address uniqueness. Let us assume that we have two solutions of (3), (v^i,λ^i) for $i=1,2$. Let

$$u^i(t,x) = v^i(t,x) + px - \lambda^i t.$$

Then u^i solves

$$u^i_t + H_\alpha(t,x,u^i_x) = 0$$

with

$$u^1(0,x) \le u^2(0,x) + C.$$

The comparison principle implies

$$u^1 \le u^2 + C \quad \text{for all } t > 0$$

and then $\lambda^1 \ge \lambda^2$. Similarly, we get the reverse inequality and then $\lambda^1 = \lambda^2$.

We now turn to the continuity of the map $p \mapsto \overline{H}_\alpha(p)$. It follows from the stability of viscosity sub- and super-solutions, from the fact that the constant C in (24) is bounded for bounded p and from the comparison principle. This achieves the proof of the lemma.

C. ***Sketch of the proof of Proposition 4.1.*** Consider

$$M_\nu = \sup_{\substack{x\in[\rho_1,\rho_2]\\ s,t\in\mathbb{R}}} \left\{ u(t,x) - v(s,x) - \frac{(t-s)^2}{2\nu} \right\}.$$

We want to prove that

$$M = \lim_{\nu\to 0} M_\nu \le 0.$$

We argue by contradiction by assuming that $M > 0$. The supremum defining M_ν is reached; let s_ν, t_ν and x_ν denote a maximizer. Choose ν small enough so that $M_\nu \ge \frac{1}{2}M > 0$. We classically get

$$|t_\nu - s_\nu| \le C\sqrt{\nu}.$$

If there exists $\nu_n \to 0$ such that $x_{\nu_n} = \rho_1$ for all $n \in \mathbb{N}$, then

$$\tfrac{1}{2}M \le M_{\nu_n} \le U_0(t_{\nu_n}) - U_0(s_{\nu_n}) \le \omega_0(t_{\nu_n} - s_{\nu_n}) \le \omega_0(C\sqrt{\nu_n}),$$

where ω_0 denotes the modulus of continuity of U_0. The contradiction $M \le 0$ is obtained by letting n go to $+\infty$.

Hence, we can assume that, for ν small enough, $x_\nu > \rho_1$. Reasoning as in [IM, Theorem 7.8], we can easily reduce to the case where $H(t_\nu, x_\nu, \cdot)$ reaches its minimum for $p = p_0 = 0$. We can also consider the vertex test function G^γ associated with the single Hamiltonian H (using the notation of [IM], it corresponds to the case $N = 1$) and the free parameter γ. If $x_\nu < \rho_2$, then $G^\gamma(x,y)$ reduces to the standard test function $\frac{1}{2}(x-y)^2$.

We next consider

$$M_{\nu,\varepsilon} = \sup_{\substack{x,y\in[\rho_1,\rho_2]\cap\overline{B_r(x_\nu)}\\ s,t\in\mathbb{R}}} \left\{ u(t,x) - v(s,y) - \frac{(t-s)^2}{2\nu} - \varepsilon G^\gamma(\varepsilon^{-1}x, \varepsilon^{-1}y) - \varphi^\nu(t,s,x) \right\},$$

where $r = r_\nu$ is chosen so that $\rho_1 \notin \overline{B_r(x_\nu)}$ and φ^ν is the localization function

$$\varphi^\nu(t,s,x) = \tfrac{1}{2}((t-t_\nu)^2 + (s-s_\nu)^2 + (x-x_\nu)^2).$$

The supremum defining $M_{\nu,\varepsilon}$ is reached and, if (t,s,x,y) denotes a maximizer, then

$$(t,s,x,y) \to (t_\nu, s_\nu, x_\nu, x_\nu) \quad \text{as } (\varepsilon,\gamma) \to 0.$$

In particular, $x, y \in B_r(x_\nu)$ for ε and γ small enough. The remaining of the proof is completely analogous (in fact much simpler).

D. *Construction of λ_ρ in the proof of Lemma 5.2.* In order to get λ_ρ, it is enough to apply the following lemma:

Lemma D.1. *Let u be the solution of a Hamilton–Jacobi equation of evolution type subject to the initial condition $u(0,x) = 0$ and posed on a compact set K. Assume that*:

- *the comparison principle holds true*;
- *u is L-globally Lipschitz continuous in time and space*;
- *$u(k + \cdot, \cdot) + C$ is a solution for all $k \in \mathbb{N}$ and $C \in \mathbb{R}$.*

There then exists $\lambda \in \mathbb{R}$ such that

$$|u(t,x) - \lambda t| \le C_0$$

and

$$|\lambda| \le L,$$

where $C_0 = L(2 + 3\rho)$ if ρ denotes the diameter of K.

Proof. Define

$$\lambda^+(T) = \sup_{\tau\ge 0} \frac{u(\tau+T,0) - u(\tau,0)}{T} \quad \text{and} \quad \lambda^-(T) = \inf_{\tau\ge 0} \frac{u(\tau+T,0) - u(\tau,0)}{T}.$$

Note that $T \mapsto \pm T\lambda^\pm(T)$ is subadditive. The fact that u is L-Lipschitz continuous with respect to time implies that $\lambda^\pm(T)$ are both finite:

$$|\lambda^\pm(T)| \le L.$$

The ergodic theorem implies that $\lambda^\pm(T)$ converges towards $\lambda^\pm$ and

$$\lambda^+ = \inf_{T>0} \lambda^+(T) \quad \text{and} \quad \lambda^- = \sup_{T>0} \lambda^-(T).$$

If, moreover,

$$|\lambda^+(T) - \lambda^-(T)| \le \frac{C}{T}, \tag{84}$$

then the proof of the lemma is complete. Indeed, (84) implies in particular that $\lambda^+ = \lambda^-$ and

$$-\frac{C}{T} \le \lambda^-(T) - \lambda \le \lambda^+(T) - \lambda \le \frac{C}{T}.$$

This implies that $|u(t,0) - \lambda t| \le C$. Finally, we get

$$|u(t,x) - \lambda t| \le C + L\rho.$$

It remains to prove (84). There exists $k \in \mathbb{Z}$ and $\beta \in [0,1)$ such that $\tau^+ = k + \tau^- + \beta$. Moreover,

$$u(\tau^+, x) \le u(\tau^- + \beta, x) + u(\tau^+, 0) - u(\tau^- + \beta, 0) + 2L\rho,$$

where $\rho = \operatorname{diam} K$. Now note that $u(\tau^- + \beta + t, x) + D$ is a solution in $[\tau^+, +\infty)$ for all constant D. Hence, we get by comparison that, for all $t > 0$ and $x \in K$,

$$u(\tau^+ + t, x) \le u(\tau^- + \beta + t, x) + u(\tau^+, 0) - u(\tau^- + \beta, 0) + 2L\rho.$$

In particular,

$$u(\tau^+ + T, 0) - u(\tau^+, 0) \le u(\tau^- + \beta + T, 0) - u(\tau^- + \beta, 0) + 2L\rho \le u(\tau^- + T, 0) - u(\tau^-, 0) + 2L(1+\rho).$$

Finally, we get (after letting $\varepsilon \to 0$)

$$\lambda^+(T) \le \lambda^-(T) + \frac{2L(1+\rho)}{T}.$$

Similarly, we can get

$$\lambda^+(T) \ge \lambda^-(T) - \frac{2L(1+\rho)}{T}.$$

This implies (84) with $C = 2L(1+\rho)$. The proof of the lemma is now complete. □

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EXISTENCE AND CLASSIFICATION OF SINGULAR SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS WITH A GRADIENT TERM

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We completely classify the behaviour near 0, as well as at ∞ when $\Omega = \mathbb{R}^N$, of all positive solutions of $\Delta u = u^q|\nabla u|^m$ in $\Omega \setminus \{0\}$, where Ω is a domain in $\mathbb{R}^N$ ($N \geq 2$) and $0 \in \Omega$. Here, $q \geq 0$ and $m \in (0, 2)$ satisfy $m + q > 1$. Our classification depends on the position of q relative to the critical exponent $q_* := (N - m(N - 1))/(N - 2)$ (with $q_* = \infty$ if $N = 2$). We prove the following: if $q < q_*$, then any positive solution u has either (1) a removable singularity at 0, or (2) a *weak singularity* at 0 ($\lim_{|x|\to 0} u(x)/E(x) \in (0, \infty)$, where E denotes the fundamental solution of the Laplacian), or (3) $\lim_{|x|\to 0} |x|^{\vartheta} u(x) = \lambda$, where ϑ and λ are uniquely determined positive constants (a *strong singularity*). If $q \geq q_*$ (for $N > 2$), then 0 is a removable singularity for all positive solutions. Furthermore, for any positive solution in $\mathbb{R}^N \setminus \{0\}$, we show that it is either constant or has a nonremovable singularity at 0 (weak or strong). The latter case is possible only for $q < q_*$, where we use a new iteration technique to prove that all positive solutions are *radial*, nonincreasing and converging to *any nonnegative* number at ∞. This is in sharp contrast to the case of $m = 0$ and $q > 1$, when all solutions decay to 0. Our classification theorems are accompanied by corresponding existence results in which we emphasise the more difficult case of $m \in (0, 1)$, where new phenomena arise.

1. Introduction and main results

Let Ω be a domain in $\mathbb{R}^N$ with $N \geq 2$. We assume that $0 \in \Omega$ and set $\Omega^* := \Omega \setminus \{0\}$. We are concerned with the nonnegative solutions of nonlinear elliptic equations such as

$$-\Delta u + u^q|\nabla u|^m = 0 \quad \text{in } \Omega^*. \tag{1-1}$$

Unless otherwise stated, we always assume that $m, q \in \mathbb{R}$ satisfy

$$q \geq 0, \quad 0 < m < 2 \quad \text{and} \quad m + q > 1. \tag{1-2}$$

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Keywords: nonlinear elliptic equations, isolated singularities, Leray–Schauder fixed point theorem, Liouville-type result.

Our aim is to obtain a full classification of the behaviour near 0 (and also at ∞ if $\Omega = \mathbb{R}^N$) for all positive $C^1(\Omega^*)$-distributional solutions of (1-1), together with corresponding existence results. This study is motivated by a vast literature on the topic of isolated singularities. For instance, see [Brandolini et al. 2013; Brezis and Oswald 1987; Brezis and Véron 1980; Cîrstea 2014; Cîrstea and Du 2010; Friedman and Véron 1986; Nguyen Phuoc and Véron 2012; Serrin 1965; Vázquez and Véron 1980; 1985; Véron 1981; 1986; 1996] and their references. As a novelty of this article, we reveal new and distinct features of the profile of solutions of (1-1) near 0 (and at ∞ when $\Omega = \mathbb{R}^N$), arising from the introduction of the gradient factor in the nonlinear term. It can be seen from our proofs that more general problems could be considered. However, to avoid further technicalities, we restrict our attention to (1-1).

In a different but related direction, problems similar to (1-1) which include a gradient term have attracted considerable interest in a variety of contexts. Boundary value problems with measure data for (1-1) have recently been studied by [Marcus and Nguyen 2015]. With respect to boundary blow-up problems, equations like (1-1) arise in the study of stochastic control theory (see [Lasry and Lions 1989]). We refer to [Alarcón et al. 2012] for a large list of references when the domain is bounded and to [Felmer et al. 2013] when the domain is unbounded. In relation to viscous Hamilton–Jacobi equations, Bidaut-Véron and Dao [2012; 2013] have studied the parabolic version of (1-1) for $q = 0$. For the large-time behaviour of solutions of Dirichlet problems for subquadratic viscous Hamilton–Jacobi equations, see [Barles et al. 2010]. See [Brezis et al. 1986; Brezis and Friedman 1983; Oswald 1988] for the analysis of nonlinear parabolic versions of (1-1) with $m = 0$. If $\ell := m/(m+q)$ and $w := \ell^{m/(m-\ell)} u^{1/\ell}$, we rewrite (1-1) as

$$\Delta(w^\ell) = |\nabla w|^m \quad \text{in } \Omega^*, \tag{1-3}$$

where $\ell \in (0, 1]$ and $m \in (\ell, 2)$, from (1-2). The parabolic version of (1-3) has been studied in different exponent ranges in connection with various applications (most frequently describing thermal propagation phenomena in an absorptive medium); the case $\ell < 1$ is usually called *fast* diffusion, whereas $\ell > 1$ is *slow* diffusion. The fast diffusion case with singular absorption was analysed by Ferreira and Vazquez [2001] (see their references for the existence, uniqueness, regularity and asymptotic behaviour of solutions related problems). The parabolic form of equations like (1-3) also features in the study of the porous medium equation; see [Vázquez 1992; 2007] for a general introduction to this area.

We now return to problem (1-1). A solution of (1-1), which is a nonnegative $C^1(\Omega^*)$ function at the outset, is understood as in Definition 1.4. By the strong maximum principle (see Lemma 3.3), any solution of (1-1) is either identically zero or positive in Ω^*. The behaviour of solutions of (1-1) near zero is controlled by the fundamental solution of the Laplacian, denoted by E; see (1-11). For a positive solution u of (1-1), zero is a removable singularity if and only if $\lim_{|x|\to 0} u(x)/E(x) = 0$; see Lemma 3.11. If 0 is a nonremovable singularity, then $\lim_{|x|\to 0} u(x)/E(x) = \Lambda \in (0, \infty]$ and, as in [Véron 1986], we say that u has a *weak* (resp. *strong*) *singularity at* 0 if $\Lambda \in (0, \infty)$ (resp. $\Lambda = \infty$). The fundamental solution E, together with the nonlinear part of (1-1), plays a crucial role in the existence of solutions with nonremovable singularities at 0. We define

$$q_* := \frac{N - m(N-1)}{N-2} \quad \text{if } N \geq 3 \qquad \text{and} \qquad q_* := \infty \quad \text{if } N = 2. \tag{1-4}$$

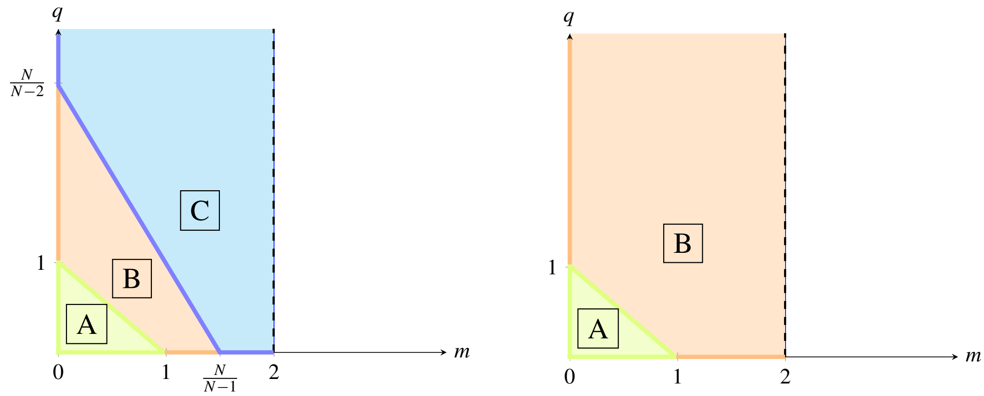

Figure 1. The left side (right side) picture pertaining to $N \geq 3$ ($N = 2$) illustrates different classification results established over various ranges of m and q. In region A, the dichotomy result of Serrin [1965, Theorem 1] is applicable. In this paper, we establish a trichotomy result (removable, weak or strong singularities) in Theorem 1.2(a) for region B, generalising the well-known result of [Véron 1981] for $m = 0$ and $q \in (1, N/(N-2))$ (the existence of weak singularities is also ascertained by [Nguyen Phuoc and Véron 2012] for $q = 0$ and $1 < m < N/(N-1)$). In region C, we obtain the removability result of Theorem 1.2(b), applicable for $N \geq 3$ (previously known in two cases: $m = 0$ and $q \geq N/(N-2)$, treated by [Brezis and Véron 1980]; and $q = 0$ and $N/(N-1) \leq m < 2$, due to [Nguyen Phuoc and Véron 2012]).

If (1-2) holds, we show that (1-1) admits solutions with weak (or strong) singularities at 0 if and only if $q < q_*$ (or, equivalently, $E^q|\nabla E|^m \in L^1(B_r(0))$ for some $r > 0$, where $B_r(0)$ denotes the ball centred at 0 of radius r). For $q < q_*$ and a smooth bounded domain Ω, we prove in Theorem 1.1 that (1-1) has solutions with any possible behaviour near 0 and a Dirichlet condition on $\partial\Omega$:

$$\lim_{|x|\to 0} \frac{u(x)}{E(x)} = \Lambda \qquad \text{and} \qquad u = h \quad \text{on } \partial\Omega. \tag{1-5}$$

Theorem 1.1 (existence I). *Let* (1-2) *hold,* $q < q_*$ *and* Ω *be a bounded domain with* C^1 *boundary. For any* $\Lambda \in [0, \infty]$ *and every nonnegative function* $h \in C(\partial\Omega)$*, there is a solution of* (1-1)+(1-5).

Theorem 1.1 is valid for $m = 0$ in (1-2) and $q \in (1, q_*)$, when the existence and uniqueness of the solution of (1-1) and (1-5) is known (see, for example, [Friedman and Véron 1986; Cîrstea and Du 2010, Theorem 1.2], where more general nonlinear elliptic equations are treated).

Since $m > 0$ in our framework, the presence of the gradient factor in the nonlinear term of (1-1) creates additional difficulties, especially for $0 < m < 1$, where new phenomena arise. By passing to the limit in approximating problems, we construct in Theorem 1.1 both the maximal and the minimal solution of (1-1)+(1-5) (see Remark 4.2).[1] If $m \geq 1$ in Theorem 1.1, then (1-1)+(1-5) has a *unique* solution (using Lemma 3.2 and Theorem 1.2(a)). In contrast, in Remark 4.3 we note that for $m \in (0, 1)$ the uniqueness

[1] The proof of Theorem 1.1 relies solely on (1-2) if $\Lambda = 0$ in (1-5).

of the solution of (1-1)+(1-5) may not necessarily hold.[2] In Section 2, using the Leray–Schauder fixed point theorem, we study separately the existence of radial solutions of (1-1) for $\Omega = B_R(0)$ with $R > 0$ and $m \in (0, 1)$. For such a domain Ω and h a nonnegative constant γ, the maximal and the minimal solution of (1-1)+(1-5) are both radial (see Remark 4.2). For $m \in (0, 1)$, we show that they do not coincide if $\Lambda = 0$ and $\gamma \in (0, \infty)$: the maximal solution is γ, whereas the minimal solution is provided by Theorem 2.2, which gives a radial solution u such that $u' > 0$ in $(0, R)$ and $u(R) = \gamma$. On the other hand, for any $\Lambda \in (0, \infty)$ and under the necessary assumption $q < q_*$, we construct a radial *nonincreasing* solution of (1-1) in $B_R(0) \setminus \{0\}$ satisfying $\lim_{r \to 0^+} u(r)/E(r) = \Lambda \in (0, \infty)$ and a Neumann boundary condition $u'(R) = 0$ (see Theorem 2.1).

Notice that, if (1-2) holds and $q < q_*$, then $u_0(x) = \lambda |x|^{-\vartheta}$ is a positive radial solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$ with a strong singularity at 0, where ϑ and λ are positive constants given by

$$\vartheta := \frac{2-m}{q+m-1} \quad \text{and} \quad \lambda := [\vartheta^{1-m}(\vartheta - N + 2)]^{1/(q+m-1)}. \tag{1-6}$$

In Theorem 1.2, we describe all the different behaviours near 0 of the positive solutions of (1-1).

Theorem 1.2 (classification I). *Let* (1-2) *hold.*

(a) *If $q < q_*$, then any positive solution u of* (1-1) *satisfies exactly one of the following*:

  (i) $\lim_{|x| \to 0} u(x) \in (0, \infty)$ *and u can be extended as a continuous solution of* (1-1) *in $\mathcal{D}'(\Omega)$, in the sense that $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ and*

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x + \int_\Omega |\nabla u|^m u^q \varphi \, \mathrm{d}x = 0 \quad \textit{for all } \varphi \in C^1_c(\Omega). \tag{1-7}$$

  (ii) $u(x)/E(x)$ *converges to a positive constant Λ as $|x| \to 0$ and, moreover,*

$$-\Delta u + u^q |\nabla u|^m = \Lambda \delta_0 \quad \textit{in } \mathcal{D}'(\Omega), \tag{1-8}$$

  where δ_0 denotes the Dirac mass at 0.

  (iii) $\lim_{|x| \to 0} |x|^{\vartheta} u(x) = \lambda$, *where ϑ and λ are as in* (1-6).

(b) *If $q \ge q_*$ for $N \ge 3$, then any positive solution of* (1-1) *satisfies only alternative* (i) *above.*

In Figure 1, we illustrate how our Theorem 1.2 fits into the literature by providing the classification results for the entire eligible range of $m \in [0, 2)$ and $q \in [0, \infty)$ satisfying (1-2) (that is, the regions B and C in Figure 1). We point out that (1-2) is essential for the conclusion of Theorem 1.2 to hold. Indeed, when (1-2) fails, such as in region A of Figure 1, Theorem 1 of [Serrin 1965] is applicable, so that any positive solution u of (1-1) satisfies exactly one of the following:

(1) The solution u can be defined at 0 and the resulting function is a continuous solution of (1-1) in the whole Ω.

(2) There exists a constant $C > 0$ such that $1/C \le u(x)/E(x) \le C$ near $x = 0$.

[2] If $0 < m < 1$, we cannot apply Lemma 3.2. The modified comparison principle in Lemma 3.1 requires the extra condition $|\nabla u_1| + |\nabla u_2| > 0$ in D, which restricts its applicability.

In Theorem 1.2 we reveal that the behaviour of solutions of (1-1) near 0 for (m, q) in region B is clearly distinct from that corresponding to region C (for $N \geq 3$). In the latter, (1-1) has no solutions with singularities at 0 (see Theorem 1.2(b)). Belonging to the region C, we distinguish the points on the critical line $q = q_* = (N - m(N-1))/(N-2)$, which joins the previously known critical values $N/(N-2)$ and $N/(N-1)$, corresponding to $m = 0$ and $q = 0$ in (1-1), respectively. When $N \geq 3$, Theorem 1.2(b) generalises the celebrated removability result of [Brezis and Véron 1980] for $m = 0$ and $q \geq N/(N-2)$, as well as the recent one of [Nguyen Phuoc and Véron 2012, Theorem A.2], where the special case $q = 0$ was treated: any positive $C^2(\overline{\Omega} \setminus \{0\})$ solution of $\Delta u = |\nabla u|^m$ in Ω^* remains bounded and it can be extended as a solution of the same equation in Ω when $N/(N-1) \leq m < 2$. If, in turn, $1 < m < N/(N-1)$ and $N \geq 2$, then Nguyen Phuoc and Véron [2012] ascertain the existence of positive solutions of $\Delta u = |\nabla u|^m$ in Ω^* with a weak singularity at zero. We note that our Theorem 1.2(a) provides a full classification of the behaviour near 0 for all positive solutions of (1-1), corresponding to the region B in Figure 1, extending the well-known trichotomy result of [Véron 1981] for $m = 0$ and $1 < q < N/(N-2)$ (see also [Brezis and Oswald 1987] for a different approach).

Our next goal is to fully understand the profile of all positive solutions of (1-1) in $\mathbb{R}^N \setminus \{0\}$, which we show to be *radial*. We stress that the introduction of the gradient factor in the nonlinear term of (1-1) gives rise to new difficulties. In particular, neither the Kelvin transform nor the moving plane method can be applied. To prove radial symmetry, we shall introduce a new iterative method. A key feature that distinguishes our problem from the case $m = 0$ is that any positive solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$ admits a limit at ∞, which may be *any* nonnegative number. This asymptotic pattern at ∞ is different compared to $m = 0$ in (1-1), when every positive solution of the equation

$$\Delta u = u^q \ \text{ in } \mathbb{R}^N \setminus \{0\} \quad \text{with} \quad q > 1 \tag{1-9}$$

must decay to 0 at ∞ (see Remark 3.5). Moreover, there are no positive solutions of (1-9) with a removable singularity at 0. For $q > 1$, Brezis [1984] showed that there exists a unique distributional solution ($u \in L^q_{\text{loc}}(\mathbb{R}^N)$) of $\Delta u = |u|^{q-1}u + f$ in $\mathbb{R}^N$ assuming only $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and, moreover, $u \geq 0$ a.e. provided that $f \geq 0$ a.e. in $\mathbb{R}^N$. The existence part of this result has been extended to the p-Laplace operator by [Boccardo et al. 1993] (for $q > p - 1 > 0$ and $p > 2 - 1/N$), whereas the question of uniqueness of solutions has been recently investigated by [D'Ambrosio et al. 2013].

We recall the profile of all positive solutions of (1-9) (see [Friedman and Véron 1986] for the results corresponding to the p-Laplace operator and $q > p - 1 > 0$):

- If $1 < q < N/(N-2)$, then *either* $u(x) = \lambda_0 |x|^{-\vartheta_0}$, where λ_0 and ϑ_0 correspond to λ and ϑ in (1-6) with $m = 0$ *or* u is a radial solution with a *weak singularity* at 0 and $\lim_{|x|\to\infty} u(x) = 0$. Moreover, for every $\Lambda \in (0, \infty)$, there exists a *unique* positive radial solution of (1-9) satisfying $\lim_{|x|\to 0} u(x)/E(x) = \Lambda$.
- If $q \geq N/(N-2)$ for $N \geq 3$, then there are no positive solutions of (1-9).

Compared to (1-9), our Theorem 1.3 reveals a much richer structure of solutions of (1-1) in $\mathbb{R}^N \setminus \{0\}$. There exist nonconstant positive solutions if and only if $q < q_*$ and, in this case, they must be radial, nonincreasing and satisfy

$$\lim_{|x|\to 0}\frac{u(x)}{E(x)}=\Lambda \quad \text{and} \quad \lim_{|x|\to\infty} u(x)=\gamma \tag{1-10}$$

with $\Lambda\in(0,\infty]$ and $\gamma\in[0,\infty)$. In addition, all solutions with a *strong singularity* at 0 are given in full by $u(x)=\lambda|x|^{-\vartheta}$ and $u_C(x)=Cu_1(C^{1/\vartheta}|x|)$ for $x\in\mathbb{R}^N\setminus\{0\}$. Here, $C>0$ is arbitrary and u_1 denotes the unique positive radial solution of (1-1) in $\mathbb{R}^N\setminus\{0\}$ with $\Lambda=\infty$ and $\gamma=1$ in (1-10). Theorem 1.3 gives a complete classification of all positive solutions of (1-1) in $\mathbb{R}^N\setminus\{0\}$.

Theorem 1.3 ($\Omega=\mathbb{R}^N$, existence and classification II). *Let* (1-2) *hold and u be any positive solution of* (1-1) *in $\mathbb{R}^N\setminus\{0\}$. The following assertions hold*:

(i) *If $q<q_*$ then, for any $\Lambda\in(0,\infty]$ and any $\gamma\in[0,\infty)$, there exists a unique positive radial solution of* (1-1) *in $\mathbb{R}^N\setminus\{0\}$, subject to* (1-10).

(ii) *If u is a nonconstant solution then $q<q_*$ and, moreover, u is radial, nonincreasing and satisfies* (1-10) *for some $\Lambda\in(0,\infty]$ and $\gamma\in[0,\infty)$. Furthermore, if $\Lambda=\infty$, then $\lim_{|x|\to 0}|x|^{\vartheta}u(x)=\lambda$, where ϑ and λ are given by* (1-6) (*with $u(x)=\lambda|x|^{-\vartheta}$ if $\gamma=0$*).

(iii) *If 0 is a removable singularity for u, then u must be constant. In particular, if $q\geq q_*$ and $N\geq 3$, then u is constant.*

Liouville-type theorems for nonlinear elliptic equations have received much attention (in relation to (1-1), we refer to [Farina and Serrin 2011; Filippucci 2009; Li and Li 2012; Mitidieri and Pokhozhaev 2001]). For a broad class of quasilinear elliptic equations with the nonhomogeneous term depending strongly on the gradient of the solution, Farina and Serrin [2011] establish that any $C^1(\mathbb{R}^N)$ solution must be constant. Their results apply for solutions unrestricted in sign and, in particular, for the p-Laplace model-type equation $\Delta_p u=|u|^{q-1}u|\nabla u|^m$ with $p>1$, $q>0$ and $m\geq 0$ under various restrictions on these parameters. With respect to (1-1), if $q>0$, $0\leq m<1$ and $q+m>1$, then the constant functions are the only nonnegative entire solutions of (1-1) (see [Filippucci 2009]). Furthermore, Farina and Serrin [2011] weakened the condition $m<1$ to $m<N/(N-1)$. In Theorem 1.3(iii), we further improve this Liouville-type result for (1-1) by changing the condition $m<N/(N-1)$ to $m<2$ as in (1-2). We give a short and elementary proof of Theorem 1.3(iii), which does not involve the test function method usually employed in the current literature (see Remark 3.14). Our technique relies on local estimates, the comparison principle, and the continuous extension at 0 of any solution of (1-1) with a removable singularity at 0 (see Lemma 3.13).

The proof of Theorem 1.3(i) relies on the (radial) maximal solution constructed in Theorem 1.1 for (1-1)+(1-5), where $\Omega=B_k(0)$ and $h\equiv\gamma$. For $\Lambda\in(0,\infty)$, we show that as $k\to\infty$ this solution converges to a positive radial solution $u_{\Lambda,\gamma}$ of (1-1) in $\mathbb{R}^N\setminus\{0\}$, subject to (1-10). The existence of the radial solution for $\Lambda=\infty$ is obtained as the limit of the $u_{j,\gamma}$ as $j\to\infty$. The uniqueness follows from the comparison principle (Lemma 3.1), based on $\lim_{r\to 0^+}u_1(r)/u_2(r)=1$ and $\lim_{r\to\infty}(u_1(r)-u_2(r))=0$ for any radial solutions u_1, u_2 satisfying (1-10).

The key ingredient in the proof of Theorem 1.3(ii) is Step 1 in Lemma 6.1: *any positive solution of* (1-1) *in $\mathbb{R}^N\setminus\{0\}$ admits a nonnegative limit at* ∞. We prove this fact using a new iterative technique, which we outline here. We take $(x_{n,1})$ with $|x_{n,1}| \nearrow \infty$ and $\lim_{n\to\infty} u(x_{n,1}) = a := \liminf_{|x|\to\infty} u(x)$. Given any sequence (x_n) in $\mathbb{R}^N$ with $|x_n| \nearrow \infty$, we show that, for any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that $u < \limsup_{j\to\infty} u(x_j) + \varepsilon$ in $\overline{B_{|x_n|/2}(x_n)}$ for every $n \geq N_\varepsilon$. Hence, for some $N_1 > 0$, we have $u < a + \varepsilon$ in $\overline{B_{|x_{n,1}|/2}(x_{n,1})}$ for all $n \geq N_1$. Moreover, by choosing $x_{n,2} \in \partial B_{|x_{n,1}|/2}(x_{n,1}) \cap \partial B_{|x_{n,1}|}(0)$, there exists $N_2 > N_1$ such that $u < a + 2\varepsilon$ on $\overline{B_{|x_{n,1}|/2}(x_{n,2})} \cup \overline{B_{|x_{n,1}|/2}(x_{n,1})}$ for all $n \geq N_2$. After a finite number of iterations K (independent of n and ε), we find $N_K > 0$ such that $u < a + K\varepsilon$ on $\partial B_{|x_{n,1}|}(0)$ for all $n \geq N_K$. Since $u(x) \leq \max_{|y|=\delta} u(y)$ for all $|x| \geq \delta > 0$ (see Lemma 3.6), we find that $\limsup_{|x|\to\infty} u(x) \leq a + K\varepsilon$. Letting $\varepsilon \to 0$, we find that there exists $\lim_{|x|\to\infty} u(x) = \gamma \in [0, \infty)$. If u is not a constant solution, then (1-10) holds for some $\Lambda \in (0, \infty]$. For $m \geq 1$, the radial symmetry of u is due to the uniqueness of the solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1-10), and the invariance of this problem under rotation. For $m \in (0, 1)$, we need to think differently (we cannot use Lemma 3.2). For any $\varepsilon > 0$ (and $\varepsilon < \gamma$ if $\gamma > 0$), we construct positive radial solutions u_ε and U_ε of (1-1) in $\mathbb{R}^N \setminus \{0\}$ with the properties

(P1) $u_\varepsilon \leq u \leq U_\varepsilon$ in $\mathbb{R}^N \setminus \{0\}$;

(P2) $u_\varepsilon(r)/E(r)$ and $U_\varepsilon(r)/E(r)$ converge to Λ as $r \to 0^+$;

(P3) $\lim_{r\to\infty} u_\varepsilon(r) = \max\{\gamma - \varepsilon, 0\}$ and $\lim_{r\to\infty} U_\varepsilon(r) = \gamma + \varepsilon$.

As $\varepsilon \to 0$, u_ε increases (U_ε decreases) to a positive radial solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1-10). The uniqueness of such a solution and (P1) prove that u is radial.

Notation. Let $B_R(x)$ denote the ball centred at x in $\mathbb{R}^N$ ($N \geq 2$) with radius $R > 0$. When $x = 0$, we simply write B_R instead of $B_R(0)$ and set $B_R^* := B_R \setminus \{0\}$. For abbreviation, we later use B^* in place of B_1^*. By ω_N, we denote the volume of the unit ball in $\mathbb{R}^N$. Let E denote the fundamental solution of the harmonic equation $-\Delta E = \delta_0$ in $\mathbb{R}^N$, namely

$$E(x) = \begin{cases} \dfrac{1}{N(N-2)\omega_N} |x|^{2-N} & \text{if } N \geq 3, \\ \dfrac{1}{2\pi} \log \dfrac{R}{|x|} & \text{if } N = 2. \end{cases} \tag{1-11}$$

For a bounded domain Ω of $\mathbb{R}^2$, we choose $R > 0$ large enough that Ω is included in B_R.

The concept of a solution for (1-1) in an open set D of $\mathbb{R}^N$ is made precise below, where we use $C_c^1(D)$ to denote the set of all functions in $C^1(D)$ with compact support in D.

Definition 1.4. By a *solution* (resp. *subsolution*, *supersolution*) of $\Delta u = u^q |\nabla u|^m$ in an open set $D \subseteq \mathbb{R}^N$, we mean a *nonnegative* function $u \in C^1(D)$ which satisfies

$$\int_D \nabla u \cdot \nabla \varphi \, dx + \int_D |\nabla u|^m u^q \varphi \, dx = 0 \quad (\text{resp. } \leq 0, \ \geq 0) \tag{1-12}$$

for every (nonnegative) function $\varphi \in C_c^1(D)$.

Outline. We divide the paper into six sections. In Section 2, we study the existence of radial solutions to (1-1) for $m \in (0, 1)$ and $\Omega = B_R$ with $R > 0$. Using the Leray–Schauder fixed point theorem, we prove that (a) there exist radial solutions with a weak singularity at 0 if and only if $q < q_*$ (see Theorem 2.1

and Lemma 2.5); and (b) for every $\gamma > 0$, there exists a nonconstant radial solution with a removable singularity at 0 satisfying $u(R) = \gamma$, assuming only (1-2); see Theorem 2.2. The case $m \in (0, 1)$ deserves special attention, since the failure of Lipschitz continuity in the gradient term yields a different version of the comparison principle (Lemma 3.1) compared to Lemma 3.2 for $m \geq 1$. Besides these comparison principles, Section 3 gives several auxiliary tools to be used later such as a priori estimates, a regularity result, and a spherical Harnack inequality. We prove Theorem 1.1 in Section 4 using a suitable perturbation technique. In Section 5 and Section 6, we establish the classification results of Theorem 1.2 and Theorem 1.3, respectively.

2. Existence of radial solutions when $m \in (0, 1)$

Here, we assume that $m \in (0, 1)$ and study the existence of positive radial solutions of (1-1) with $\Omega = B_R$ for $R > 0$. Without any loss of generality, we let $R = 1$ and consider the problem

$$u''(r) + (N-1)\frac{u'(r)}{r} = [u(r)]^q |u'(r)|^m \quad \text{for every } r \in (0, 1). \tag{2-1}$$

In Theorem 2.1, under sharp conditions, we prove that, for every $\Lambda \in (0, \infty)$, there exists a positive nonincreasing $C^2(0, 1]$ solution of (2-1), subject to

$$\lim_{r \to 0^+} \frac{u'(r)}{E'(r)} = \Lambda, \quad u'(1) = 0. \tag{2-2}$$

The first condition in (2-2) yields that $\lim_{r\to 0^+} u(r)/E(r) = \Lambda$, i.e., u has a weak singularity at 0.

Our central result is the following:

Theorem 2.1. *Assume that* $0 < m < 1$ *and* $1 - m < q < q_*$. *Then, for every* $\Lambda \in (0, \infty)$, *there exists a positive nonincreasing* $C^2(0, 1]$ *solution of* (2-1)+(2-2).

The proof of Theorem 2.1 is based on the transformation $w(s) = u(r)$ with $s = r^{2-N}$ if $N \geq 3$, and $w(s) = u(r)$ with $s = \ln(e/r)$ if $N = 2$. It is useful to introduce some notation:

$$C_N := \begin{cases} (N-2)^{m-2} & \text{if } N \geq 3, \\ e^{2-m} & \text{if } N = 2, \end{cases} \quad \text{and} \quad g_N(t) := \begin{cases} t^{-(q_*+1)} & \text{if } N \geq 3, \\ e^{(m-2)t} & \text{if } N = 2, \end{cases} \tag{2-3}$$

for all $t \in [1, \infty)$. For the definition of q_*, we refer to (1-4).

We see that u satisfies the differential equation in (2-1) if and only if

$$w''(s) = C_N g_N(s)[w(s)]^q |w'(s)|^m \quad \text{for all } s \in (1, \infty), \tag{2-4}$$

where the derivatives here are with respect to s. Moreover, (2-2) is equivalent to

$$\lim_{s\to\infty} w'(s) = \nu, \quad w'(1) = 0, \tag{2-5}$$

where $\Lambda = N(N-2)\omega_N \nu$ if $N \geq 3$, and $\Lambda = 2\pi\nu$ if $N = 2$.

In Lemma 2.4, we establish the assertion of Theorem 2.1 by proving that, for every $\nu \in (0, \infty)$, there exists a positive nondecreasing $C^2[1, \infty)$ solution of (2-4)+(2-5). Moreover, $w'(s) > 0$ for all $s \in (1, \infty)$

if $\nu \in (0, \nu_*]$, where we define

$$\nu_* := \left[(1-m)C_N \int_1^\infty t^q g_N(t)\,\mathrm{d}t\right]^{-\frac{1}{q+m-1}}. \tag{2-6}$$

We remark that $\nu_* < \infty$ since $t \mapsto t^q g_N(t) \in L^1[1, \infty)$.

The proof of Theorem 2.1 is given below, using the Leray–Schauder fixed point theorem. Adapting these ideas, we ascertain in Theorem 2.2 that, if $0 < m < 1$ and (1-2) holds, then, for every $\gamma > 0$, (2-1) admits a positive, increasing $C^2(0, 1]$ solution satisfying $u(1) = \gamma$. If, in turn, $m \geq 1$ in (1-2), then (2-1), subject to $u(1) = \gamma$, has a unique solution with a removable singularity at zero, namely $u \equiv \gamma$.

Theorem 2.2. *Let $0 < m < 1$ and $q > 1 - m$. Then, for every $\gamma > 0$, there exists a positive increasing $C^2(0, 1]$ solution of (2-1), subject to $u(1) = \gamma$.*

Theorem 2.2 is proved in Lemma 2.7.

Proof of Theorem 2.1. As mentioned above, Theorem 2.1 is equivalent to Lemma 2.4, whose proof relies essentially on the existence and uniqueness of a positive solution for a corresponding boundary value problem in Lemma 2.3.

Lemma 2.3. *Assume that $0 < m < 1$ and $1 - m < q < q_*$. Then, for any fixed integer $j \geq 2$ and every $\nu \in (0, \nu_*]$, there exists a unique positive $C^2[1, j]$ solution of the problem*

$$\begin{cases} w''(s) = C_N g_N(s)[w(s)]^q |w'(s)|^m & \text{for every } s \in (1, j), \\ w'(s) > 0 & \text{for every } s \in (1, j], \\ w'(1) = 0,\ w'(j) = \nu. \end{cases} \tag{2-7}$$

Proof. We first establish the uniqueness of a positive $C^2[1, j]$ solution of (2-7), followed by the proof of the existence of such a solution.

Uniqueness: Suppose that $w_{1,j}$ and $w_{2,j}$ are two positive $C^2[1, j]$ solutions of (2-7). For any $\varepsilon > 0$, we define $P_{j,\varepsilon}(s) = w_{1,j}(s) - (1+\varepsilon)w_{2,j}(s)$ for all $s \in [1, j]$. For abbreviation, we write P_ε instead of $P_{j,\varepsilon}$, since j is fixed. It suffices to show that, for every $\varepsilon > 0$, we have $P_\varepsilon \leq 0$ on $[1, j]$. Indeed, by letting $\varepsilon \to 0$ and interchanging $w_{1,j}$ and $w_{2,j}$, we find that $w_{1,j} = w_{2,j}$ in $[1, j]$. Suppose for contradiction that there exists $s_0 \in [1, j]$ such that $P_\varepsilon(s_0) = \max_{s\in[1,j]} P_\varepsilon(s) > 0$. We show that we arrive at a contradiction by analysing three cases:

Case 1: $s_0 = j$. That is, $P_\varepsilon(j) = \max_{s\in[1,j]} P_\varepsilon(s)$. From $P'_\varepsilon(j) = -\varepsilon\nu$, we have $P'_\varepsilon < 0$ on $(j-\delta, j)$ if $\delta > 0$ is small. This is a contradiction.

Case 2: $s_1 = 1$. It follows that $P_\varepsilon(s) > 0$ for every $s \in [1, 1+\delta]$ provided that $\delta > 0$ is small enough. Since $w_{1,j}$ and $w_{2,j}$ satisfy (2-7), for every $s \in (1, 1+\delta)$ we obtain that

$$\frac{|w'_{1,j}(s)|^{1-m}}{|w'_{2,j}(s)|^{1-m}} = \frac{\int_1^s g_N(t)[w_{1,j}(t)]^q\,\mathrm{d}t}{\int_1^s g_N(t)[w_{2,j}(t)]^q\,\mathrm{d}t} > (1+\varepsilon)^q. \tag{2-8}$$

Since $m + q > 1$, we get that $P'_\varepsilon > 0$ on $(1, 1+\delta)$, which contradicts $P_\varepsilon(1) = \max_{s\in[1,j]} P_\varepsilon(s)$.

Case 3: $s_0 \in (1, j)$. Using (2-7), $P_\varepsilon(s_0) > 0$, $P'_\varepsilon(s_0) = 0$ and $P''_\varepsilon(s_0) \le 0$, we arrive at a contradiction, since

$$0 \ge \frac{w''_{1,j}(s_0) - (1+\varepsilon)\, w''_{2,j}(s_0)}{C_N g_N(s_0)[w'_{2,j}(s_0)]^m} = (1+\varepsilon)^m [w_{1,j}(s_0)]^q - (1+\varepsilon)[w_{2,j}(s_0)]^q$$
$$> [w_{2,j}(s_0)]^q[(1+\varepsilon)^{m+q} - (1+\varepsilon)] > 0. \tag{2-9}$$

This completes the proof of uniqueness.

Existence: We apply the Leray–Schauder fixed point theorem (see [Gilbarg and Trudinger 1983, Theorem 11.6]) to a suitable homotopy that we construct below.

Step 1. *Construction of the homotopy.*

Let $\mathcal{B}$ denote the Banach space of $C^1[1, j]$ functions with the usual $C^1[1, j]$-norm. Let $\nu \in (0, \nu_*]$, where ν_* is given by (2-6). We define $f_\nu(x) := \frac{1}{2}(\nu + |x| - |x - \nu|)$ for all $x \in \mathbb{R}$, that is,

$$f_\nu(x) := \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } 0 \le x \le \nu, \\ \nu & \text{if } x \ge \nu. \end{cases} \tag{2-10}$$

Since ν is fixed, we will henceforth drop the index ν in f_ν. Let $w \in \mathcal{B}$ be arbitrary. We introduce the function $k = k_w : [0, \infty) \to \mathbb{R}$ given by

$$k_w(\mu) := \int_1^j g_N(t)\left(\mu + \int_1^t f(w'(\xi))\,\mathrm{d}\xi\right)^q \mathrm{d}t \quad \text{for every } \mu \in [0, \infty). \tag{2-11}$$

We see that, for any $w \in \mathcal{B}$, there exists a unique $\mu = \mu_w > 0$ such that

$$k_w(\mu_w) = \frac{\nu^{1-m}}{(1-m)C_N}. \tag{2-12}$$

Indeed, $\mu \mapsto k_w(\mu)$ is increasing and the right-hand side of (2-12) is larger than $k_w(0)$. Using that $\nu \in (0, \nu_*]$ and by a simple calculation, we obtain that $\nu < \mu_w \le \hat{\nu}$, where $\hat{\nu}$ is given by

$$\hat{\nu} := \left(\frac{\nu^{1-m}}{(1-m)C_N \int_1^2 g_N(t)\,\mathrm{d}t}\right)^{\frac{1}{q}}.$$

We now define $h_w : [1, j] \to \mathbb{R}$ by

$$h_w(t) := \int_1^t g_N(\tau)\left(\mu_w + \int_1^\tau f(w'(\xi))\,\mathrm{d}\xi\right)^q \mathrm{d}\tau \quad \text{for all } t \in [1, j]. \tag{2-13}$$

In particular, we have $h_w(j) = k_w(\mu_w)$. We prescribe our homotopy $H : \mathcal{B} \times [0, 1] \to \mathcal{B}$ to be

$$H[w, \sigma](s) = \sigma\left(\mu_w + \int_1^s [(1-m)C_N\, h_w(t)]^{1/(1-m)}\,\mathrm{d}t\right) \quad \text{for all } s \in [1, j], \tag{2-14}$$

where $w \in \mathcal{B}$ and $\sigma \in [0, 1]$ are arbitrary.

Step 2. *We claim that H is a compact operator from $\mathcal{B} \times [0, 1]$ to $\mathcal{B}$.*

We first show that $H : \mathcal{B} \times [0, 1] \to \mathcal{B}$ is continuous, i.e., if $(w_n, \sigma_n) \in \mathcal{B} \times [0, 1]$ such that $w_n \to w$ in $\mathcal{B}$ and $\sigma_n \to \sigma$ as $n \to \infty$, then $H[w_n, \sigma_n] \to H[w, \sigma]$ in $\mathcal{B}$. Since f in (2-10) is a continuous function, we have $f(w_n') \to f(w')$ as $n \to \infty$. From (2-13)–(2-14), it is enough to check that $\lim_{n\to\infty} \mu_{w_n} = \mu_w$. Suppose by contradiction that for a subsequence of w_n, relabelled w_n, we have $\lim_{n\to\infty} \mu_{w_n} = \tilde{\mu} \neq \mu_w$. Since $\mu_{w_n} \in (\nu, \hat{\nu}]$, we must have $\tilde{\mu} \in [\nu, \hat{\nu}]$. From (2-12) and the continuity of f, we have that

$$\frac{\nu^{1-m}}{(1-m)C_N} = k_{w_n}(\mu_{w_n}) \to k_w(\tilde{\mu}) \quad \text{as } n \to \infty.$$

But k_w is injective and thus $\tilde{\mu} = \mu_w$, which is a contradiction. This proves that $\lim_{n\to\infty} \mu_{w_n} = \mu_w$.

To see that H is compact, let $(w_n, \sigma_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B} \times [0, 1]$ and define $H_n(s) := H[w_n, \sigma_n](s)$ for all $s \in [1, j]$. We have $H_n \in C^2[1, j]$. We infer that $(H_n)_{n\in\mathbb{N}}$ is both uniformly bounded and equicontinuous in $\mathcal{B}$ since, from (2-12), we find that

$$\|H_n\|_{L^\infty(1,j)} \le j\hat{\nu}, \quad \|H_n'\|_{L^\infty(1,j)} \le \nu \quad \text{and} \quad \|H_n''\|_{L^\infty(1,j)} \le (j\hat{\nu})^q \nu^m \quad \text{for all } n \in \mathbb{N}. \tag{2-15}$$

Hence, the Arzelà–Ascoli theorem implies that $H : \mathcal{B} \times [0, 1] \to \mathcal{B}$ is compact.

Step 3. *The existence of a positive $C^2[1, j]$ solution of* (2-7), *completed.*

By the first two inequalities in (2-15), we have that $\|w\|_{C^1[1,j]}$ is bounded for all $(w, \sigma) \in \mathcal{B} \times [0, 1]$ satisfying $w = H[w, \sigma]$. From (2-14), we have $H[w, 0] = 0$ for all $w \in \mathcal{B}$. Therefore, the Leray–Schauder fixed point theorem implies the existence of $w_j \in \mathcal{B} = C^1[1, j]$ such that $H[w_j, 1] = w_j$. Thus, $\mu_{w_j} = w_j(1)$ and w_j satisfies

$$w_j(s) = w_j(1) + \int_1^s [(1-m)C_N h_{w_j}(t)]^{1/(1-m)}\, \mathrm{d}t \quad \text{for all } s \in [1, j]. \tag{2-16}$$

This gives that $w_j \in C^2[1, j]$. Using (2-12) and (2-13), we find that $w_j'(1) = 0$ and $w_j'(j) = \nu$. By twice differentiating (2-16), we get that

$$w_j'(s) = [(1-m)C_N\, h_{w_j}(s)]^{1/(1-m)}, \quad w_j''(s) = C_N |w_j'(s)|^m h_{w_j}'(s) > 0 \quad \text{for all } s \in (1, j). \tag{2-17}$$

It follows that $0 < w_j'(s) \le \nu$ for all $s \in (1, j]$, so that $f(w_j') = w_j'$ in $[1, j]$. Then we have

$$h_{w_j}(s) = \int_1^s g_N(\tau)[w_j(\tau)]^q\, \mathrm{d}\tau, \quad h_{w_j}'(s) = g_N(s)[w_j(s)]^q \quad \text{for all } s \in (1, j). \tag{2-18}$$

From (2-17)–(2-18), we conclude that w_j is a positive $C^2[1, j]$ solution of (2-7). □

Lemma 2.4. *If $0 < m < 1$ and $1 - m < q < q_*$, then for every positive constant ν there exists a positive $C^2[1, \infty)$ solution of the problem* (2-4)+(2-5).

Proof. We divide the proof into two cases.

Case 1: $\nu \in (0, \nu_*]$, where ν_* is given by (2-6). For each integer $j \ge 2$, let w_j denote the unique positive $C^2[1, j]$ solution of (2-7).

Fix $s \in [1, \infty)$ and write $j_s := \lceil s \rceil$, where $\lceil \cdot \rceil$ stands for the ceiling function.

Claim 1. *The function $j \mapsto w_j(s)$ is nonincreasing for $j \ge j_s$.*

Indeed, for every $\varepsilon > 0$ and $j \geq j_s$, we prove that $P_{j,\varepsilon} \leq 0$ on $[1, j]$, where we define $P_{j,\varepsilon}(t) := w_{j+1}(t) - (1+\varepsilon)w_j(t)$ for all $t \in [1, j]$. Fix $\varepsilon > 0$. Assume for contradiction that there exists $t_0 \in [1, j]$ such that $P_{j,\varepsilon}(t_0) = \max_{t\in[1,j]} P_{j,\varepsilon}(t) > 0$. By the same argument as in the uniqueness proof of Lemma 2.3, we derive a contradiction when $t_0 = 1$ or $t_0 \in (1, j)$. Suppose now that $t_0 = j$. Since $w''_{j+1}(t) > 0$ for all $t \in (1, j)$ and $w'_{j+1}(j+1) = \nu = w'_j(j)$, it follows that $P'_{j,\varepsilon}(j) < 0$. Thus, $P'_{j,\varepsilon}(t) < 0$ for all $t \in (j-\delta, j)$ if $\delta > 0$ is small enough. This contradicts $P_{j,\varepsilon}(j) = \max_{t\in[1,j]} P_{j,\varepsilon}(t)$, which proves that $P_{j,\varepsilon}(t) \leq 0$ for all $t \in [1, j]$. Letting $t = s$ and $\varepsilon \to 0$, we conclude Claim 1.

By Lemma 2.3, we have $w_j(s) \geq w_j(1) > \nu$ for all $s \in [1, j]$. Using Claim 1, for every $s \in [1, \infty)$, we can define $w_\infty(s) := \lim_{j\to\infty} w_j(s)$. We thus have $w_\infty \geq \nu$ on $[1, \infty)$.

Claim 2. *The function w_∞ is a positive $C^2[1, \infty)$ solution of* (2-4)+(2-5).

Let K be an arbitrary compact subset of $[1, \infty)$. We show that

$$w_j \to w_\infty \quad \text{uniformly in } K. \tag{2-19}$$

Let $j_K = j(K)$ be a large positive integer such that $K \subseteq [1, j]$ for all $j \geq j_K$. By Claim 1, we have $w_j \geq w_{j+1}$ in K for every $j \geq j_K$. Moreover, since $w_j \in C(K)$ and $0 \leq w'_j \leq \nu$ in K for all $j \geq j_K$, we obtain (2-19). In particular, $w_\infty \in C[1, \infty)$. From Lemma 2.3, w_j satisfies (2-16) with h_{w_j} given by (2-18). Using (2-19), we can let $j \to \infty$ in (2-16) to obtain that

$$w_\infty(s) = w_\infty(1) + \int_1^s \left[(1-m)C_N \int_1^t g_N(\tau)[w_\infty(\tau)]^q \, \mathrm{d}\tau\right]^{\frac{1}{1-m}} \mathrm{d}t \quad \text{for all } s \in (1, \infty). \tag{2-20}$$

Thus, $w_\infty \in C^2[1, \infty)$ satisfies (2-4) and $w'_\infty(1) = 0$.

It remains to prove that $\lim_{s\to\infty} w'_\infty(s) = \nu$. By using (2-20), we find that

$$w'_\infty(s) = \left[(1-m)C_N \int_1^s g_N(t)[w_\infty(t)]^q \, \mathrm{d}t\right]^{\frac{1}{1-m}} \quad \text{for every } s \in (1, \infty). \tag{2-21}$$

On the other hand, from (2-12) and (2-18), we have

$$\int_1^j g_N(t)[w_j(t)]^q \, \mathrm{d}t = h_{w_j}(j) = k_{w_j}(\mu_{w_j}) = \frac{\nu^{1-m}}{(1-m)C_N} \quad \text{for every } j \geq 2. \tag{2-22}$$

Since $w'_j(t) \leq \nu$ for all $t \in [1, j]$, we find that

$$w_j(t) \leq \nu t + w_j(1) - \nu \quad \text{for all } t \in [1, j].$$

Recall that $\nu < w_j(1) \leq w_2(1)$ for all $j \geq 2$. Consequently, we obtain that

$$g_N(t)[w_j(t)]^q \leq g_N(t)[\nu t + w_j(1) - \nu]^q \leq [w_2(1)]^q t^q g_N(t) \quad \text{for all } t \in [1, j] \text{ and } j \geq 2.$$

For every $t \in [1, \infty)$, it holds that $g_N(t)[w_j(t)]^q \to g_N(t)[w_\infty(t)]^q$ as $j \to \infty$. Thus, we can let $j \to \infty$ in (2-22) and use Lebesgue's dominated convergence theorem to find that

$$\int_1^\infty g_N(t)[w_\infty(t)]^q \, \mathrm{d}t = \frac{\nu^{1-m}}{(1-m)C_N}. \tag{2-23}$$

From (2-21) and (2-23), we conclude that $\lim_{s\to\infty} w'_\infty(s) = \nu$, proving Lemma 2.4 in Case 1.

Case 2: Let $\nu > \nu_*$, where ν_* is defined by (2-6). From Case 1, there exists a positive $C^2[1,\infty)$ solution w_* of (2-4)+(2-5) corresponding to $\nu = \nu_*$. If $N \ge 3$, then we define $r_* := (\nu/\nu_*)^{(m+q-1)/(q_*-q)} \in (1,\infty)$ and define $w : [1,\infty) \to (0,\infty)$ by

$$w(s) = \begin{cases} r_*^{(m+q_*-1)/(m+q-1)} w_*(s/r_*) & \text{for } r_* \le s < \infty, \\ r_*^{(m+q_*-1)/(m+q-1)} w_*(1) & \text{for } 1 \le s \le r_*. \end{cases} \tag{2-24}$$

If $N = 2$, we let $r_* := 1 + ((q+m-1)/(2-m)) \ln(\nu/\nu_*) \in (1,\infty)$ and define $w : [1,\infty) \to (0,\infty)$ by

$$w(s) = \begin{cases} \dfrac{\nu}{\nu_*} w_*(s+1-r_*) & \text{for } r_* \le s < \infty, \\ \dfrac{\nu}{\nu_*} w_*(1) & \text{for } 1 \le s \le r_*. \end{cases} \tag{2-25}$$

It is a simple exercise to check that w is a positive $C^2[1,\infty)$ solution of (2-4)+(2-5). □

Lemma 2.5. *Let* (1-2) *hold. If* (2-1) *has a solution with a weak singularity at* 0, *then* $q < q_*$.

Remark 2.6. Theorem 1.2(b) shows that $q < q_*$ is a necessary condition for the existence of solutions of (1-1) with a nonremovable singularity at 0 (see Section 5 for its proof).

Proof. We need only consider the nontrivial case $N \ge 3$. Suppose that $u \in C^2(0,1)$ is a positive solution of (2-1) such that $\lim_{r\to 0^+} u(r)/r^{2-N} =: \eta$ for some $\eta \in (0,\infty)$. Then u satisfies

$$\frac{d}{dr}(r^{N-1}u'(r)) = r^{N-1}[u(r)]^q |u'(r)|^m \ge 0 \quad \text{for all } r \in (0,1). \tag{2-26}$$

Hence, $r \mapsto r^{N-1}u'(r)$ is nondecreasing on (0, 1), so that it admits a limit as $r \to 0^+$. By l'Hôpital's rule, we obtain that

$$(0,\infty) \ni \eta = \lim_{r\to 0^+} r^{N-2}u(r) = -(N-2)^{-1} \lim_{r\to 0^+} r^{N-1}u'(r). \tag{2-27}$$

By integrating (2-26) over $\left(\varepsilon, \frac{1}{2}\right)$ for arbitrarily small $\varepsilon > 0$ and letting $\varepsilon \to 0^+$, we find that

$$2^{1-N}u'\left(\tfrac{1}{2}\right) + (N-2)\,\eta = \int_0^{1/2} r^{N-1}[u(r)]^q |u'(r)|^m \, \mathrm{d}r < \infty. \tag{2-28}$$

We use $A(r) \sim B(r)$ as $r \to 0^+$ to mean that $\lim_{r\to 0^+} A(r)/B(r) = 1$. By using (2-27), we have that

$$r^{N-1}[u(r)]^q |u'(r)|^m \sim (N-2)^m \eta^{q+m} r^{(N-1)(1-m)-q\,(N-2)} \quad \text{as } r \to 0^+.$$

This, jointly with (2-28), leads to $N - m\,(N-1) > q\,(N-2)$, which proves that $q < q_*$. □

Proof of Theorem 2.2. In view of the preliminary discussion in Section 2, Theorem 2.2 is equivalent to the following:

Lemma 2.7. *Let* $0 < m < 1$ *and* $m + q > 1$. *For any* $\gamma \in (0,\infty)$, *there exists a positive decreasing* $C^2[1,\infty)$ *solution of* (2-4), *subject to* $w(1) = \gamma$ *and* $\lim_{s\to\infty} w(s) > 0$.

Proof. We divide the proof into three steps and proceed similarly to Lemmas 2.3 and 2.4.

Step 1. *For every integer $j \geq 2$, there exists a unique positive $C^2[1, j]$ solution w_j of*

$$\begin{cases} w''(s) = C_N g_N(s)[w(s)]^q |w'(s)|^m & \text{for every } s \in (1, j), \\ w'(s) < 0 & \text{for every } s \in (1, j), \\ w(1) = \gamma, \\ w'(j) = 0. \end{cases} \tag{2-29}$$

To show uniqueness, we follow an argument similar to the uniqueness proof of Lemma 2.3 in Case 3. Keeping the same notation, we see that Case 2 there (that is, $\max_{s\in[1,j]} P_\varepsilon(s) = P_\varepsilon(1) > 0$) cannot happen due to $w(1) = \gamma$ in (2-29). Finally, in Case 1 (i.e., $s_0 = j$), we have $P_\varepsilon > 0$ on $[j - \delta, j]$ for $\delta > 0$ small enough, which implies (2-8) for all $s \in (j - \delta, j)$. Since $w'(s) < 0$ on $(1, j)$, it follows that $P'_\varepsilon < 0$ on $(j - \delta, j)$, which is a contradiction with $\max_{s\in[1,j]} P_\varepsilon(s) = P_\varepsilon(j)$.

Next, we show existence via the Leray–Schauder fixed point theorem. Let $\mathcal{B}$ denote the Banach space of $C^1[1, j]$ functions with the usual $C^1[1, j]$ norm. Let $\hat{f}(x) := \frac{1}{2}(\gamma + |x| - |x - \gamma|)$ for all $x \in \mathbb{R}$. We prescribe the homotopy $\hat{H} : \mathcal{B} \times [0, 1] \to \mathcal{B}$ as follows

$$\hat{H}[w, \sigma](s) = \sigma\left(\gamma - \int_1^s \left[C_N(1-m)\int_\tau^j g_N(t)(\hat{f}(w(t)))^q \,dt\right]^{\frac{1}{1-m}} d\tau\right) \quad \text{for all } s \in [1, j], \tag{2-30}$$

where $w \in \mathcal{B}$ and $\sigma \in [0, 1]$ are arbitrary. We show that $\hat{H}$ is a compact operator from $\mathcal{B} \times [0, 1]$ to $\mathcal{B}$ as in Step 2 in the existence proof of Lemma 2.3. We use that

$$\begin{aligned} &\|\hat{H}\|_{L^\infty(1,j)} \leq \gamma, \\ &\|\hat{H}'\|_{L^\infty(1,j)} \leq \left[C_N(1-m)\gamma^q \int_1^\infty g_N(t)\,dt\right]^{\frac{1}{1-m}}, \\ &\|\hat{H}''\|_{L^\infty(1,j)} \leq g_N(1)\left[C_N(1-m)^m\gamma^q\left(\int_1^\infty g_N(t)\,dt\right)^m\right]^{\frac{1}{1-m}}. \end{aligned} \tag{2-31}$$

Hence, $\|w\|_{C^1[1,j]}$ is bounded for all $(w, \sigma) \in \mathcal{B} \times [0, 1]$ satisfying $w = \hat{H}[w, \sigma]$. From (2-30), we have $\hat{H}[w, 0] = 0$ for all $w \in \mathcal{B}$. Therefore, by the Leray–Schauder fixed point theorem, there exists $w_j \in \mathcal{B} = C^1[1, j]$ such that $\hat{H}[w_j, 1] = w_j$. Thus, $w_j(1) = \gamma$, $w'_j(j) = 0$ and w_j satisfies

$$w_j(s) = \gamma - \int_1^s \left[C_N(1-m)\int_\tau^j g_N(t)(\hat{f}(w_j(t)))^q \,dt\right]^{\frac{1}{1-m}} d\tau \quad \text{for all } s \in [1, j]. \tag{2-32}$$

Clearly, $w'_j \leq 0$ in $[1, j]$ so that $w(s) \leq w(1) = \gamma$ in $[1, j]$.

To conclude Step 1, it remains to show that $w_j(s) > 0$ for all $s \in [1, j]$.

Claim 1. *If there exists $\hat{s} \in (1, j]$ such that $w_j(\hat{s}) = 0$, then $w_j = 0$ on $[\hat{s}, j]$.*

Indeed, since $w'_j \leq 0$ in $[1, j]$, it follows that $w_j(s) \leq 0$ in $[\hat{s}, j]$ and thus $\hat{f}(w_j(t)) = 0$ for all $t \in [\hat{s}, j]$. In particular, using (2-32), we find that

$$w_j(\hat{s}) - w_j(\xi) = \int_{\hat{s}}^\xi \left[C_N(1-m)\int_\tau^j g_N(t)(\hat{f}(w_j(t)))^q \,dt\right]^{\frac{1}{1-m}} d\tau = 0 \quad \text{for all } \xi \in [\hat{s}, j].$$

Claim 2. *We have* $w_j > 0$ *in* $[1, j]$.

If we suppose the contrary, then $\hat{s} \in (1, j]$, where we define $\hat{s} = \inf\{\xi \in (1, j] : w_j(\xi) = 0\}$. Then $w_j > 0$ on $[1, \hat{s})$ and $w_j = 0$ on $[\hat{s}, j]$. For any $\varepsilon \in (0, \gamma)$ small, there exists $\tilde{s} \in (1, \hat{s})$ such that $w_j(\tilde{s}) = \varepsilon$. Thus, by the mean value theorem, we have $-w_j'(\bar{s}) = \varepsilon/(\hat{s} - \tilde{s})$ for some $\bar{s} \in (\tilde{s}, \hat{s})$. Since $w_j = 0$ in $[\hat{s}, j]$ and $w_j \le \varepsilon$ on $[\tilde{s}, \hat{s}]$, by differentiating (2-32) we find that

$$\frac{\varepsilon}{\hat{s} - \tilde{s}} = -w_j'(\bar{s}) = \left[C_N(1-m) \int_{\bar{s}}^{\hat{s}} g_N(t)(w_j(t))^q \,\mathrm{d}t \right]^{\frac{1}{1-m}} \le [C_N(1-m) g_N(1)(\hat{s} - \tilde{s})\varepsilon^q]^{1/(1-m)}.$$

This yields that $\varepsilon \ge [(j-1)^{2-m} C_N(1-m) g_N(1)]^{-1/(q+m-1)}$. This is a contradiction, since $\varepsilon > 0$ can be made arbitrarily small. This proves Claim 2, completing the proof of Step 1.

To complete the proof of Lemma 2.7, we proceed as in Case 1 of Lemma 2.4.

Step 2. *For each fixed* $s \in [1, \infty)$, *the function* $j \mapsto w_j(s)$ *is nonincreasing whenever* $j \ge \lceil s \rceil$.

It suffices to prove that $P_{j,\varepsilon} \le 0$ in $[1, j]$ for every $\varepsilon > 0$, where $P_{j,\varepsilon}(t) := w_{j+1}(t) - (1+\varepsilon) w_j(t)$ for all $t \in [1, j]$. Assuming the contrary, we have $\max_{t \in [1,j]} P_{j,\varepsilon}(t) = P_{j,\varepsilon}(s_0) > 0$ for some $s_0 \in [1, j]$. We get a contradiction similarly to the proof of uniqueness of solutions to (2-29).

This shows that, for each $s \in [1, \infty)$, we may define $w_\infty(s) := \lim_{j \to \infty} w_j(s)$.

Step 3. *The function* w_∞ *is a positive decreasing* $C^2[1, \infty)$ *solution of* (2-4), *satisfying* $w_\infty(1) = \gamma$ *and* $\lim_{s \to \infty} w_\infty(s) > 0$.

The proof can be completed in the same way as Claim 2 in the proof of Lemma 2.4. We deduce that $w_j \to w_\infty$ uniformly in arbitrary compact sets of $[1, \infty)$. Hence w_∞ satisfies

$$w_\infty(s) = \gamma - \int_1^s \left[C_N(1-m) \int_\tau^\infty g_N(t)(w_\infty(t))^q \,\mathrm{d}t \right]^{\frac{1}{1-m}} \mathrm{d}\tau \quad \text{for all } s \in [1, \infty). \tag{2-33}$$

It follows that $w_\infty(1) = \gamma$ and $\lim_{s \to \infty} w_\infty'(s) = 0$. The fact that w_∞ is positive in $[1, \infty)$ follows as in Claim 2 of Step 1 above. We thus skip the details.

Finally, we show that $\lim_{s \to \infty} w_\infty(s) > 0$ by adjusting the proof of the positivity of w_∞. Suppose for contradiction that $\lim_{s \to \infty} w_\infty(s) = 0$. For any small $\varepsilon_1 > 0$, there exists $s_1 > 1$ large such that $w_\infty(s_1) = \varepsilon_1$. For any small $\varepsilon_2 \in (0, \gamma - \varepsilon_1)$, chosen independently of ε_1, there exists $\delta \in (0, 1)$ such that $w_\infty(s_1 - \delta) = \varepsilon_1 + \varepsilon_2$. By the mean value theorem, we have $-w_\infty'(s_2) = \varepsilon_2/\delta$ for some $s_2 \in (s_1 - \delta, s_1)$. Since $w_\infty \le \varepsilon_1 + \varepsilon_2$ in $[s_2, \infty)$, by differentiating (2-33) we find that

$$\varepsilon_2 \le -w_\infty'(s_2) \le \hat{C}^{1/(1-m)} (\varepsilon_1 + \varepsilon_2)^q/(1-m), \quad \text{where} \quad \hat{C} := C_N(1-m) \int_1^\infty g_N(t) \,\mathrm{d}t. \tag{2-34}$$

By taking $\varepsilon_1 \to 0$, we would get $\varepsilon_2 \ge \hat{C}^{-1/(q+m-1)}$. This is a contradiction, since ε_1 and ε_2 can be chosen arbitrarily small. This finishes the proof of Lemma 2.7. □

3. Auxiliary tools

We start with two comparison principles, to be used often in the paper.

Lemma 3.1 (comparison principle; see Theorem 10.1 in [Pucci and Serrin 2004]). *Let D be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. Let $\hat{B}(x, z, \boldsymbol{\xi}) : D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be continuous in $D \times \mathbb{R} \times \mathbb{R}^N$ and continuously differentiable with respect to $\boldsymbol{\xi}$ for $|\boldsymbol{\xi}| > 0$ in $\mathbb{R}^N$. Assume that $\hat{B}(x, z, \boldsymbol{\xi})$ is nondecreasing in z for fixed $(x, \boldsymbol{\xi}) \in D \times \mathbb{R}^N$. Let u_1 and u_2 be nonnegative $C^1(\bar{D})$ (distributional) solutions of*

$$\begin{cases} \Delta u_1 - \hat{B}(x, u_1, \nabla u_1) \geq 0 & \text{in } D, \\ \Delta u_2 - \hat{B}(x, u_2, \nabla u_2) \leq 0 & \text{in } D. \end{cases} \tag{3-1}$$

Suppose that $|\nabla u_1| + |\nabla u_2| > 0$ in D. If $u_1 \leq u_2$ on ∂D, then $u_1 \leq u_2$ in D.

The following result, given in [Pucci and Serrin 2007], is a version of Theorem 10.7(i) in [Gilbarg and Trudinger 1983] with the significant difference that $\hat{B}(x, z, \boldsymbol{\xi})$ is allowed to be singular at $\boldsymbol{\xi} = 0$ and that the class $C^1(D)$ is weakened to $W^{1,\infty}_{\text{loc}}(D)$.

Lemma 3.2 (comparison principle; see Corollary 3.5.2 in [Pucci and Serrin 2007]). *Let D be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. Assume that $\hat{B}(x, z, \boldsymbol{\xi}) : D \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz continuous with respect to $\boldsymbol{\xi}$ in $D \times \mathbb{R} \times \mathbb{R}^N$ and is nondecreasing in z for fixed $(x, \boldsymbol{\xi}) \in D \times \mathbb{R}^N$. Let u_1 and u_2 be (distribution) solutions in $W^{1,\infty}_{\text{loc}}(D)$ of* (3-1). *If $u_1 \leq u_2 + M$ on ∂D, where M is a positive constant, then $u_1 \leq u_2 + M$ in D.*

Throughout this section, we understand that (1-2) holds. In Lemma 3.3, we show that the strong maximum principle applies to (1-1) (as a simple consequence of Theorem 2.5.1 in [Pucci and Serrin 2007]). Subsequently, we present several ingredients to be invoked later, such as:

(i) A priori estimates (Lemma 3.4).

(ii) A regularity result (Lemma 3.8).

(iii) A spherical Harnack-type inequality (Lemma 3.9).

Lemma 3.3 (strong maximum principle). *If u is a solution of* (1-1) *such that $u(x_0) = 0$ for some $x_0 \in \Omega^*$, then $u \equiv 0$ in Ω^*.*

Proof. Using (1-2), we can easily find p such that $p > \max\{1/q, 1\}$ and $mp' > 1$, where p' denotes the Hölder conjugate of p, that is, $p' := p/(p-1)$. By Young's inequality, we have

$$z^q |\boldsymbol{\xi}|^m \leq \frac{z^{qp}}{p} + \frac{|\boldsymbol{\xi}|^{mp'}}{p'} \leq \frac{z^{qp}}{p} + \frac{|\boldsymbol{\xi}|}{p'}$$

for all $z \in \mathbb{R}^+$ and $\boldsymbol{\xi} \in \mathbb{R}^N$ satisfying $|\boldsymbol{\xi}| \leq 1$. Hence, by applying Theorem 2.5.1 in [Pucci and Serrin 2007], we conclude our claim. □

Lemma 3.4 (a priori estimates). *Fix $r_0 > 0$ such that $\bar{B}_{2r_0} \subset \Omega$. Let u be a positive (sub)solution of* (1-1). *Then there exist positive constants $C_1 = C_1(m, q)$ and $C_2 = C_2(r_0, u)$ such that*

$$u(x) \leq C_1 |x|^{-\vartheta} + C_2 \quad \text{for every } 0 < |x| \leq 2r_0, \tag{3-2}$$

where ϑ is given by (1-6). *In particular*, *we can take* $C_1 = [\vartheta^{1-m}(\vartheta+1)]^{1/(m+q-1)}$ *and* $C_2 = \max_{\partial B_{2r_0}} u$.

Proof. For any $\delta \in (0, 2r_0)$, we define the annulus $A_\delta := \{x \in \mathbb{R}^N : \delta < |x| < 2r_0\}$. We consider the radial function $F_\delta(x) = C_1(|x|-\delta)^{-\vartheta} + C_2$ on A_δ, where $C_1 := [\vartheta^{1-m}(\vartheta+1)]^{1/(m+q-1)}$ and $C_2 := \max_{\partial B_{2r_0}} u$. Our choice of C_1 ensures that F_δ is a (radial) supersolution to (1-1) in A_δ, that is,

$$F_\delta''(r) + (N-1)\frac{F_\delta'(r)}{r} \le [F_\delta(r)]^q |F_\delta'(r)|^m \quad \text{for all } \delta < r < r_0. \tag{3-3}$$

Indeed, to prove (3-3) it suffices to show that F_δ satisfies

$$F_\delta''(r) + (N-1)\frac{F_\delta'(r)}{r} \le C_1^{q+m}\vartheta^m (r-\delta)^{-[\vartheta(q+m)+m]} \quad \text{for all } \delta < r < 2r_0. \tag{3-4}$$

By a simple calculation, we see that (3-4) is equivalent to the inequality

$$\vartheta^{1-m}\Big[\vartheta - N + 2 + (N-1)\frac{\delta}{r}\Big] \le C_1^{m+q-1} \quad \text{for all } \delta < r < 2r_0. \tag{3-5}$$

Since (3-5) holds for our C_1, we obtain that F_δ is a supersolution to (1-1) in A_δ. We show that

$$u(x) \le F_\delta(|x|) \quad \text{for all } x \in A_\delta. \tag{3-6}$$

Clearly, (3-6) holds for every $x \in \partial A_\delta$. Using that $\nabla F_\delta \neq 0$ in A_δ, we can apply Lemma 3.1 to conclude that (3-6) holds. For any fixed $x \in B_{2r_0}^*$, we have $x \in A_\delta$ for all $\delta \in (0, |x|)$. Hence, by letting $\delta \to 0$ in (3-6), we obtain (3-2). This completes the proof. □

Remark 3.5. The presence of the gradient factor in (1-1) implies that every nonnegative constant is a solution of (1-1). Hence, the constant C_2 in (3-2) cannot be discarded nor made independent of u. This is in sharp contrast with the case $m = 0$ in (1-2), when it is known (see [Véron 1981, p. 227] or [Friedman and Véron 1986, Lemma 2.1]) that there exists a positive constant C_1, depending only on N and q, such that every positive solution of $\Delta u = u^q$ in Ω^* with $q > 1$ satisfies

$$u(x) \le C_1 |x|^{-2/(q-1)} \quad \text{for all } 0 < |x| \le r_0, \quad \text{where} \quad \bar{B}_{2r_0} \subset \Omega. \tag{3-7}$$

Since C_1 is independent of Ω, from (3-7) *any* positive solution of (1-9) decays to 0 at ∞.

Lemma 3.6. *If u is a positive solution of* (1-1) *in* $\mathbb{R}^N \setminus \{0\}$, *then for every* $\delta > 0$ *we have*

$$u(x) \le \max_{\partial B_\delta} u \quad \textit{for all } |x| \ge \delta. \tag{3-8}$$

Proof. We prove (3-8) for any fixed $\delta \in (0, 1)$ with $(N-1)/\delta > N-2$. For any fixed integer $k \ge 1$, we set $C_{k,\delta} = \big(\vartheta^{1-m}[\vartheta + 2 - N + k(N-1)/\delta]\big)^{1/(m+q-1)}$. Then, $C_{k,\delta}(k-|x|)^{-\vartheta}$ is a supersolution of (1-1) in $\delta < |x| < k$. If $m \in (0, 1)$, we define $f_{k,\delta}(x) := C_{k,\delta}(k-|x|)^{-\vartheta}$ for all $|x| \in (\delta, k)$. Since $\lim_{|x| \nearrow k} f_{k,\delta}(x) = \infty$, if $\varepsilon > 0$ is small then $u(x) \le f_{k,\delta}(x)$ for all $|x| \in [k-\varepsilon, k)$. Hence, by Lemma 3.1, we find that $u(x) \le f_{k,\delta}(x) + \max_{\partial B_\delta} u$ for all $|x| \in (\delta, k)$. For x fixed with $|x| \in (\delta, \infty)$, we have $\lim_{k\to\infty} f_{k,\delta}(x) = 0$ (since $m \in (0, 1)$) and (3-8) follows by letting $k \to \infty$.

If $m \in [1, 2)$, we denote by $f_{k,\delta}$ a positive radial solution of (1-1) in $\delta < |x| < k$ satisfying $f_{k,\delta}|_{\partial B_\delta} = 0$ and $\lim_{|x|\nearrow k} f_{k,\delta}(x) = \infty$. The existence of $f_{k,\delta}$ is obtained easily for $m \in [1, 2)$: for each integer $n \ge 1$, there is a unique positive radial solution $F_{n,k}$ of (1-1) for $|x| \in (\delta, k)$, subject to $F_{n,k}|_{\partial B_\delta} = 0$ and $F_{n,k}|_{\partial B_k} = n$ by using [Gilbarg and Trudinger 1983, Theorem 15.18], Lemma 3.2 and Lemma 3.3. Since δ is fixed, in the notation of $F_{n,k}$ we dropped the dependence on δ. We have $0 < F_{n,k}(r) \le F_{n+1,k}(r) \le C_{k,\delta}(k-r)^{-\vartheta}$ for all $r \in (\delta, k)$ and $F_{n,k}$ converges in $C^1_{\text{loc}}(\delta, k)$ as $n \to \infty$ to a positive radial solution $f_{k,\delta}$ of (1-1) in $\delta < |x| < k$ satisfying $f_{k,\delta}|_{\partial B_\delta} = 0$ and $\lim_{|x|\nearrow k} f_{k,\delta}(x) = \infty$. Moreover, $f_{k+1,\delta} \le f_{k,\delta}$ in (δ, k) and $f_{k,\delta}$ converges in $C^1_{\text{loc}}(\delta, \infty)$ as $k \to \infty$ to a nonnegative radial solution f_δ of (1-1) in $\delta < |x| < \infty$ with $f_\delta|_{\partial B_\delta} = 0$. Proceeding by contradiction, it can be shown that $f_\delta \equiv 0$ on (δ, ∞). We obtain (3-8) as for $m \in (0, 1)$, using Lemma 3.2 instead of Lemma 3.1. $\square$

Corollary 3.7. *Any positive $C^1(\mathbb{R}^N)$ solution of* (1-1) *in $\mathbb{R}^N$ must be constant.*

Proof. Let u be a positive solution of (1-1) in $\mathbb{R}^N$, that is $u \in C^1(\mathbb{R}^N)$ is a positive function satisfying (1-1) in $\mathscr{D}'(\mathbb{R}^N)$ (see Definition 1.4). Let $y \in \mathbb{R}^N$ be fixed. For any integer $k \ge 1$ and $\delta \in (0, 1)$ small, we define $f_{k,\delta}(z)$ for $|z| \in (\delta, k)$ as in Lemma 3.6. Similarly, we find that

$$u(x) \le f_{k,\delta}(x-y) + \max_{\partial B_\delta(y)} u \quad \text{for all } \delta < |x-y| < k. \tag{3-9}$$

Fix $x \in \mathbb{R}^N \setminus \{y\}$. For any small $\delta \in (0, |x-y|)$, by letting $k \to \infty$ in (3-9), we have $u(x) \le \max_{\partial B_\delta(y)} u$. Hence, $u(x) \le u(y)$ for all $x \in \mathbb{R}^N$. Since $y \in \mathbb{R}^N$ is arbitrary, we conclude that u is a constant. $\square$

Lemma 3.8 (a regularity result). *Fix $r_0 > 0$ such that $\bar{B}_{2r_0} \subset \Omega$. Let ζ and θ be nonnegative constants such that $\theta \le \vartheta$ and $\zeta = 0$ if $\theta = \vartheta$. Let u be a positive solution of* (1-1) *satisfying*

$$u(x) \le g(x) := d_1|x|^{-\theta}\left[\ln\left(\frac{1}{|x|}\right)\right]^\zeta + d_2 \quad \textit{for every } 0 < |x| \le 2r_0, \tag{3-10}$$

where d_1 and d_2 are positive constants. Then there exist constants $C > 0$ and $\alpha \in (0, 1)$ such that, for any x, x' in $\mathbb{R}^N$ with $0 < |x| \le |x'| < r_0$,

$$|\nabla u(x)| \le C\,\frac{g(x)}{|x|} \quad \textit{and} \quad |\nabla u(x) - \nabla u(x')| \le C\,\frac{g(x)}{|x|^{1+\alpha}}|x-x'|^\alpha. \tag{3-11}$$

Proof. We only show the first inequality in (3-11), which can then be used to obtain the second inequality as in [Cîrstea and Du 2010, Lemma 4.1]. Fix $x_0 \in B^*_{r_0}$ and define $v_{x_0} : B_1 \to (0, \infty)$ by

$$v_{x_0}(y) := \frac{u\big(x_0 + \frac{1}{2}|x_0|y\big)}{g(x_0)} \quad \text{for every } y \in B_1. \tag{3-12}$$

By a simple calculation, we obtain that v_{x_0} satisfies the equation

$$-\Delta v + \tilde{B}(y, v, \nabla v) = 0 \quad \text{in } B_1, \tag{3-13}$$

where $\tilde{B}(y, v, \nabla v)$ is defined by

$$\tilde{B}(y, v, \nabla v) = 2^{m-2}[|x_0|^\vartheta g(x_0)]^{m+q-1}[v(y)]^q|\nabla v(y)|^m \quad \text{for all } y \in B_1. \tag{3-14}$$

From (3-10) and (3-12), there exists a positive constant A_0, which depends on r_0, such that $v_{x_0}(y) \le A_0$ for all $y \in B_1$. Moreover, using the assumptions on θ and ζ, we infer that there exists a positive constant A_1, depending on r_0, such that $|x_0|^{\vartheta} g(x_0) \le A_1$ for all $0 < |x_0| < r_0$. Hence, using that $m \in (0, 2)$, we find a positive constant A_2, depending on r_0 but independent of x_0, such that

$$|\tilde{B}(y, v, \boldsymbol{\xi})| \le A_2(1 + |\boldsymbol{\xi}|)^2 \quad \text{for all } y \in B_1 \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N. \tag{3-15}$$

Then, by applying Theorem 1 in [Tolksdorf 1984], we obtain a constant A_3, which depends on N and A_2 but is independent of x_0, such that $|\nabla v_{x_0}(0)| \le A_3$. Since this is true for every $x_0 \in B^*_{r_0}$, we readily deduce the first inequality of (3-11). □

Lemma 3.9 (a spherical Harnack-type inequality). *Let $r_0 > 0$ be such that $\bar{B}_{2r_0} \subset \Omega$ and u be a positive solution of* (1-1). *Then there exists a positive constant C_0, depending on r_0, such that*

$$\max_{\partial B_r} u \le C_0 \min_{\partial B_r} u \quad \textit{for all } r \in (0, r_0). \tag{3-16}$$

Proof. Fix $x_0 \in B^*_{r_0}$. We define $v_{x_0} : B_1 \to \mathbb{R}$ as in (3-12). By Lemma 3.4, we know that (3-10) holds with $\theta = \vartheta$ and $\zeta = 0$. The proof of Lemma 3.8 shows that v_{x_0} is a solution of (3-13), where $\tilde{B}$ satisfies (3-15). Hence, by the Harnack inequality in [Trudinger 1967, Theorem 1.1], we have

$$\sup_{B_{1/3}} v_{x_0} \le C \inf_{B_{1/3}} v_{x_0}, \quad \text{or, equivalently,} \quad \sup_{B_{|x_0|/6}(x_0)} u \le C \inf_{B_{|x_0|/6}(x_0)} u, \tag{3-17}$$

where C is a positive constant independent of x_0 (but depending on A_2 and thus on r_0). Using (3-17) and a standard covering argument (see, for example, [Friedman and Véron 1986]), we conclude the proof of (3-16) with $C_0 = C^{10}$. □

As a consequence of Lemmas 3.8 and 3.9, we obtain the following:

Corollary 3.10. *Fix $r_0 > 0$ such that $\bar{B}_{4r_0} \subset \Omega$. Let u be a positive solution of* (1-1).

(a) *For any $0 < a < b \le \frac{3}{2}$, there exists a constant $C_{a,b}$, depending on r_0, such that*

$$\max_{ar \le |x| \le br} u(x) \le C_{a,b} \min_{ar \le |x| \le br} u(x) \quad \textit{for every } r \in (0, r_0). \tag{3-18}$$

(b) *There exists a positive constant C, depending on r_0, such that*

$$|\nabla u(x)| \le C \frac{u(x)}{|x|} \quad \textit{for all } 0 < |x| < r_0. \tag{3-19}$$

Proof. (a) For any $0 < a < b \le \frac{3}{2}$, we define $\mathcal{D}_{a,b} := \{y \in \mathbb{R}^N : a \le |y| \le b\}$. Since $\mathcal{D}_{a,b}$ is a compact set in $\mathbb{R}^N$, there exists a positive integer $k_{a,b}$ and $y_i \in \mathcal{D}_{a,b}$ with $i = 1, 2, \dots, k_{a,b}$ such that $\mathcal{D}_{a,b} \subseteq \bigcup_{i=1}^{k_{a,b}} B_{|y_i|/6}(y_i)$. Fix $r \in (0, r_0)$. Letting $x_i = r y_i$ for $i = 1, 2, \dots, k_{a,b}$, we find that

$$\mathcal{D}_{ar,br} := \{x \in \mathbb{R}^N : ar \le |x| \le br\} \subseteq \bigcup_{i=1}^{k_{a,b}} B_{|x_i|/6}(x_i).$$

By (3-17), there exists a positive constant $C = C(r_0)$ such that

$$\sup_{B_{|x_i|/6}(x_i)} u(x) \le C \inf_{B_{|x_i|/6}(x_i)} u(x) \quad \text{for all } i = 1, 2, \dots, k_{a,b}. \tag{3-20}$$

Hence, we obtain (3-18) with $C_{a,b} := C^{k_{a,b}}$.

(b) Fix $x_0 \in B^*_{r_0}$. In the definition of v_{x_0} in (3-12) and also in (3-14), we replace $g(x_0)$ by $u(x_0)$. By (a), the function v_{x_0} is bounded by a positive constant A_0, independent of x_0, since

$$v_{x_0}(y) := \frac{u\big(x_0 + \frac{1}{2}|x_0|y\big)}{u(x_0)} \le \frac{\max_{|x_0|/2 \le |y| \le 3|x_0|/2} u(y)}{\min_{|x_0|/2 \le |y| \le 3|x_0|/2} u(y)} \le A_0 \quad \text{for all } y \in B_1.$$

The proof of (3-19) can now be completed as in Lemma 3.8. □

We give a removability result for (1-1), which will be useful in the proof of Lemma 3.13, as well as to deduce that alternative (i) in Theorem 1.2(a) occurs when $\lim_{|x|\to 0} u(x)/E(x) = 0$.

Lemma 3.11. *Let u be a positive solution of* (1-1) *with* $\lim_{|x|\to 0} u(x)/E(x) = 0$. *Then there exists* $\lim_{|x|\to 0} u(x) \in (0, \infty)$ *and, moreover, u can be extended as a continuous solution of* (1-1) *in the whole* Ω. *If, in addition,* $0 < m < 1$, *then* $u \in C^1(\Omega)$.

Proof. As in [Cîrstea and Du 2010, Lemma 3.2(ii)], we obtain that $\limsup_{|x|\to 0} u(x) < \infty$. We show that (1-7) holds. Indeed, for $\varphi \in C^1_c(\Omega)$ fixed, let $R > 0$ be such that $\operatorname{Supp} \varphi \subset B_R \Subset \Omega$. Using the gradient estimates in Lemma 3.8 and $\limsup_{|x|\to 0} u(x) < \infty$, we can find positive constants C_1 and C_2 (depending on R), such that

$$|\nabla u|^m u^q \le C_1 |x|^{-m}(u + C_2) \quad \text{for all } 0 < |x| \le R.$$

Since $m < 2$, by [Serrin 1965, Theorem 1] we find that $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ and (1-7) holds.

We next prove that $\lim_{|x|\to 0} u(x) > 0$. Fix $r_0 > 0$ small such that $\bar{B}_{4r_0} \subset \Omega$. By using (3-19) in Corollary 3.10, there exists a positive constant C, depending on r_0, such that

$$\Delta u = u^q |\nabla u|^m \le C^m |x|^{-m} u^{m+q} \quad \text{in } B^*_{r_0}. \tag{3-21}$$

For each integer $k > 1/r_0$, let w_k denote the unique positive classical solution of the problem

$$\begin{cases} \Delta w = C^m |x|^{-m} w^{m+q} & \text{in } B_{r_0} \setminus \bar{B}_{1/k}, \\ w|_{\partial B_{1/k}} = \min_{\partial B_{1/k}} u, \\ w|_{\partial B_{r_0}} = \min_{\partial B_{r_0}} u. \end{cases} \tag{3-22}$$

By uniqueness, w_k must be radially symmetric. Using (3-21) and Lemma 3.2, we infer that

$$w_{k+1}(x) \le w_k(x) \le u(x) \quad \text{for every } 1/k \le |x| \le r_0. \tag{3-23}$$

Then $w_k \to w$ in $C^1_{\mathrm{loc}}(B^*_{r_0})$ as $k \to \infty$, where w is a positive radial solution of

$$\begin{cases} \Delta w = C^m |x|^{-m} w^{m+q} & \text{in } B^*_{r_0}, \\ \lim_{|x|\to 0} w(x)/E(x) = 0, \\ w|_{\partial B_{r_0}} = \min_{\partial B_{r_0}} u. \end{cases} \tag{3-24}$$

We have $\lim_{|x|\to 0} w(x) > 0$ (see, e.g., [Cîrstea 2014, Proposition 3.1(b)] if $N \ge 3$ and [Cîrstea 2014, Proposition 3.4(b)] if $N = 2$). From (3-23), we infer that $w \le u$ in $B^*_{r_0}$ and, hence, $\lim_{|x|\to 0} u(x) > 0$.

Finally, we show that $u \in C^1(\Omega)$ when $m \in (0, 1)$. In this case, we can choose $p \in (N, N/m)$. We show that $u \in W^{2,p}_{\mathrm{loc}}(B_{r_0})$, where $r_0 > 0$ is small so that $\bar{B}_{4r_0} \subset \Omega$. Since $u \in C^1(\Omega^*)$, we conclude that $u \in C^1(\Omega)$

using the continuous embedding $W^{2,p}(B_r) \subset C^1(B_r)$ for $r > 0$ (see, for example, Corollaries 9.13 and 9.15 in [Brezis 2011] or [Evans 2010, p. 270]).

Observe that $u^q|\nabla u|^m \in L^p(B_{r_0})$. Indeed, using (3-19), there exist constants $c_1, c_2 > 0$ such that

$$\int_{B_{r_0}} |\nabla u|^{mp} \mathrm{d}x \le c_1 \int_{B_{r_0}} |x|^{-mp}\,\mathrm{d}x \le c_2 r_0^{N-mp} < \infty \quad \text{since} \quad p < \frac{N}{m}. \tag{3-25}$$

Since $p > N$ and $u \in C(\bar{B}_{r_0})$, by Corollary 9.18 in [Gilbarg and Trudinger 1983, p. 243] there exists a unique solution $v \in W^{2,p}_{\mathrm{loc}}(B_{r_0}) \cap C(\bar{B}_{r_0})$ of the problem

$$\begin{cases} \Delta v = u^q|\nabla u|^m & \text{in } B_{r_0}, \\ v = u & \text{on } \partial B_{r_0}. \end{cases} \tag{3-26}$$

(The uniqueness of the solution $v \in W^{2,p}_{\mathrm{loc}}(B_{r_0}) \cap C(\bar{B}_{r_0})$ is valid for any $p > 1$.) We have $v \in W^{2,2}(D)$ for any subdomain $D \Subset B_{r_0}$ and, by Theorem 8.8 in [Gilbarg and Trudinger 1983, p. 183], $u \in W^{2,2}(D)$. By the uniqueness of the solution $v \in W^{2,2}_{\mathrm{loc}}(B_{r_0}) \cap C(\bar{B}_{r_0})$ of (3-26), it follows that $u = v$ and thus $u \in W^{2,p}_{\mathrm{loc}}(B_{r_0})$. Hence, u is in $C^1(\Omega)$, completing the proof of Lemma 3.11. □

Remark 3.12. If $u \in C^1(\mathbb{R}^N)$ is a positive solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$ then, by Lemma 3.11, u becomes a positive $C^1(\mathbb{R}^N)$ solution of (1-1) in $\mathbb{R}^N$ (and, by elliptic regularity theory, $u \in C^2(\mathbb{R}^N)$).

We are now ready to prove the first part of the assertion of Theorem 1.3(iii).

Lemma 3.13. *Let $\Omega = \mathbb{R}^N$. If 0 is a removable singularity for a positive solution u of* (1-1), *then u must be constant.*

Proof. Let u be a positive solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$ with a removable singularity at 0. By Lemma 3.11, we can extend u as a positive continuous solution of (1-1) in $\mathcal{D}'(\mathbb{R}^N)$. Moreover, using also Lemma 3.6, we find that $\sup_{\mathbb{R}^N} u = u(0) > 0$. We show that

$$u(0) = \limsup_{|y|\to\infty} u(y). \tag{3-27}$$

For any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $u(x) \le \limsup_{|y|\to\infty} u(y) + \varepsilon$ for all $|x| \ge R_\varepsilon$. Set $f_\varepsilon(x) = \varepsilon|x|^{2-N}$ if $N \ge 3$ and $f_\varepsilon(x) = (1/R_\varepsilon)\log(R_\varepsilon/|x|)$ if $N = 2$. Clearly, there exists $r_\varepsilon > 0$ small such that $u(x) \le f_\varepsilon(x)$ in $B^*_{r_\varepsilon}$. Fix $z \in \mathbb{R}^N \setminus \{0\}$. Then $0 < |z| < R_\varepsilon$ for every $\varepsilon > 0$ small and

$$u(z) \le f_\varepsilon(z) + \limsup_{|y|\to\infty} u(y) + \varepsilon. \tag{3-28}$$

Letting $\varepsilon \to 0$, we find that $u(0) \le \limsup_{|y|\to\infty} u(y) \le \sup_{\mathbb{R}^N} u = u(0)$. This proves (3-27).

If $u < u(0)$ in $\mathbb{R}^N \setminus \{0\}$, then (3-8) would imply that $u(z) \le \max_{|x|=1} u(x) < u(0)$ for all $|z| \ge 1$, which would contradict (3-27). Thus, there exists $z \in \mathbb{R}^N \setminus \{0\}$ such that $u(z) = u(0)$. Since u is a subharmonic function, by the strong maximum principle we have $u = u(0)$ on $\mathbb{R}^N$. □

Remark 3.14. For $m < 1$, Lemma 3.13 follows from Lemma 3.11, combined with either Corollary 3.7 or [Filippucci 2009, Theorem 2.2], whose proof uses a test function technique in [Mitidieri and Pokhozhaev 2001]. Moreover, if $m < N/(N-1)$, we regain Lemma 3.13 for the positive $C^1(\mathbb{R}^N)$ solutions of (1-1) using the results in [Farina and Serrin 2011, p. 4422].

4. Proof of Theorem 1.1

Let (1-2) hold and $q < q_*$. Assume that Ω is a bounded domain with C^1 boundary and $h \in C(\partial\Omega)$ is a nonnegative function. For any $n \geq 1$, we consider the perturbed problem

$$\Delta u = \frac{u^{q+1}}{\sqrt{u^2+1/n}} \frac{|\nabla u|^{m+2}}{|\nabla u|^2+1/n} \quad \text{in } \Omega^*. \tag{4-1}$$

Let $\Lambda \in [0, \infty)$. We shall prove the existence of a solution of (1-1)+(1-5) based on the following:

Lemma 4.1. *If $\Lambda \in [0, \infty)$, then there is a unique nonnegative solution $u_{\Lambda,n}$ of* (4-1)+(1-5).

Proof. The uniqueness follows from Lemma 3.2. Indeed, let $\hat{B}$ denote

$$\hat{B}(x, z, \boldsymbol{\xi}) = \hat{B}(z, \boldsymbol{\xi}) := \frac{z|z|^q}{\sqrt{z^2+1/n}} \frac{|\boldsymbol{\xi}|^{m+2}}{|\boldsymbol{\xi}|^2+1/n} \quad \text{for every } x \in \Omega^*, \ z \in \mathbb{R} \text{ and } \boldsymbol{\xi} \in \mathbb{R}^N.$$

We see that $\hat{B}$ is C^1 with respect to $\boldsymbol{\xi}$ in $\Omega^* \times \mathbb{R} \times \mathbb{R}^N$. By a simple calculation, we obtain that

$$\frac{\partial}{\partial z}\hat{B} = \frac{|\boldsymbol{\xi}|^{m+2}}{|\boldsymbol{\xi}|^2+1/n} \frac{|z|^q}{(z^2+1/n)^{3/2}} \left[qz^2 + \frac{q+1}{n} \right] \geq 0,$$

so that $\hat{B}$ is nondecreasing in z for fixed $(x, \boldsymbol{\xi}) \in \Omega^* \times \mathbb{R}^N$. Let $u_{\Lambda,n}$ and $\hat{u}_{\Lambda,n}$ denote two nonnegative solutions of (4-1)+(1-5). Fix $\varepsilon > 0$ arbitrary. If $\Lambda = 0$, then $u_{\Lambda,n} \leq \varepsilon E + \hat{u}_{\Lambda,n}$ in Ω^*. If $\Lambda \in (0, \infty)$ then $u_{\Lambda,n} \leq (1+\varepsilon)\hat{u}_{\Lambda,n}$, in Ω^* using $\lim_{|x|\to 0} u_{\Lambda,n}(x)/\hat{u}_{\Lambda,n}(x) = 1$ and Lemma 3.2. Hence, in both cases, letting $\varepsilon \to 0$ then interchanging $u_{\Lambda,n}$ and $\hat{u}_{\Lambda,n}$, we find that $u_{\Lambda,n} \equiv \hat{u}_{\Lambda,n}$.

The existence of a nonnegative solution $u_{\Lambda,n}$ for (4-1)+(1-5) is established in two steps.

Step 1. *For any integer $k \geq 2$, let $\mathcal{D}_k := \Omega \setminus \bar{B}_{1/k}$. There exists a unique nonnegative solution $u_{n,k} \in C^2(\mathcal{D}_k) \cap C(\bar{\mathcal{D}}_k)$ of the problem*

$$\begin{cases} \Delta u = \dfrac{u|u|^q}{\sqrt{u^2+1/n}} \dfrac{|\nabla u|^{m+2}}{|\nabla u|^2+1/n} & \text{in } \mathcal{D}_k := \Omega \setminus \bar{B}_{1/k}, \\ u = \Lambda E + \max_{\partial\Omega} h & \text{on } \partial B_{1/k}, \\ u = h & \text{on } \partial\Omega. \end{cases} \tag{4-2}$$

Moreover $u_{n,k}$ is positive in $\mathcal{D}_k$.

The existence assertion is a consequence of Theorem 15.18 in [Gilbarg and Trudinger 1983]. The conditions of their Theorem 14.1 and equation (10.36) can be checked easily. To see that the assumptions of [ibid., Theorem 15.5] are satisfied, we take $\theta = 1$ in (15.53) and use that $m \in (0, 2)$. The uniqueness and nonnegativity of the solution of (4-2) follows from Lemma 3.2. By Lemma 3.3, we obtain that $u_{n,k} > 0$ in $\mathcal{D}_k$. Observe also that $u_{n,k} \geq \min_{\partial\Omega} h$ in $\mathcal{D}_k$.

Step 2. *The limit of $u_{n,k}$ in $C^1_{\text{loc}}(\Omega^*)$ as $k \to \infty$ yields a nonnegative solution of* (4-1)+(1-5).

Since $\Lambda E + \max_{\partial\Omega} h$ is a supersolution of (4-2), we obtain that

$$0 < u_{n,k+1} \leq u_{n,k} \leq \Lambda E + \max_{\partial\Omega} h \quad \text{in } \mathcal{D}_k. \tag{4-3}$$

Thus, there exists $u_{\Lambda,n}(x) := \lim_{k\to\infty} u_{n,k}(x)$ for all $x \in \Omega^*$ and $u_{n,k} \to u_{\Lambda,n}$ in $C^1_{\mathrm{loc}}(\Omega^*)$ as $k \to \infty$ (see Lemma 3.8), where $u_{\Lambda,n}$ is a nonnegative solution of (4-1). We prove that $u_{\Lambda,n}$ satisfies (1-5). From (4-3) and Dini's theorem, we find that $u_{\Lambda,n} \in C(\overline{\Omega} \setminus \{0\})$ and $u_{\Lambda,n} = h$ on $\partial\Omega$.

If $\Lambda = 0$ then clearly $\lim_{|x|\to 0} u_{\Lambda,n}(x)/E(x) = 0$. If $\Lambda \in (0, \infty)$ then, by (4-3), we have

$$\limsup_{|x|\to 0} \frac{u_{\Lambda,n}(x)}{E(x)} \le \Lambda.$$

To end the proof of Step 2, we show that

$$\liminf_{|x|\to 0} \frac{u_{\Lambda,n}(x)}{E(x)} \ge \Lambda. \tag{4-4}$$

Fix $r_0 > 0$ small such that $\overline{B}_{4r_0} \subset \Omega$ and let k be any large integer such that $k > 1/r_0$. By Corollary 3.10(b), there exists a positive constant $C = C(r_0)$ such that

$$\Delta u_{n,k} = \frac{u_{n,k}^{q+1}}{\sqrt{u_{n,k}^2 + 1/n}} \, \frac{|\nabla u_{n,k}|^{m+2}}{|\nabla u_{n,k}|^2 + 1/n} \le u_{n,k}^q |\nabla u_{n,k}|^m \le C^m |x|^{-m} u_{n,k}^{m+q} \quad \text{in } B^*_{r_0}$$

for all $n \ge 1$ and every $k > 1/r_0$. Thus, $u_{n,k}$ is a supersolution of the problem

$$\begin{cases} \Delta w = C^m |x|^{-m} w^{m+q} & \text{in } B_{r_0} \setminus \overline{B}_{1/k}, \\ w = \Lambda E + \max_{\partial\Omega} h & \text{on } \partial B_{1/k}, \\ w = 0 & \text{on } \partial B_{r_0}. \end{cases} \tag{4-5}$$

On the other hand, (4-5) has a unique positive classical solution w_k. Then Lemma 3.2 gives that

$$w_k(x) \le u_{n,k}(x) \quad \text{for every } 1/k \le |x| \le r_0. \tag{4-6}$$

By [Cîrstea and Du 2010, Theorem 1.2], $\lim_{k\to\infty} w_k = w$ in $C^1_{\mathrm{loc}}(B^*_{r_0})$, where $w > 0$ in $B^*_{r_0}$ satisfies

$$\begin{cases} \Delta w = C^m |x|^{-m} w^{m+q} & \text{in } B^*_{r_0}, \\ \lim_{|x|\to 0} w(x)/E(x) = \Lambda, & \\ w = 0 & \text{on } \partial B_{r_0}. \end{cases} \tag{4-7}$$

By letting $k \to \infty$ in (4-6), we obtain that $w \le u_{\Lambda,n}$ in $B^*_{r_0}$, which leads to (4-4). ☐

Proof of Theorem 1.1, completed. Let $\Lambda \in [0, \infty)$ be arbitrary and $u_{\Lambda,n}$ denote the unique nonnegative solution of (4-1)+(1-5). By Lemmas 3.2 and 3.3, we obtain that

$$0 < u_{\Lambda,n+1} \le u_{\Lambda,n} \le \Lambda E + \max_{\partial\Omega} h \quad \text{in } \Omega^*. \tag{4-8}$$

Thus, $u_\Lambda(x) := \lim_{n\to\infty} u_{\Lambda,n}(x)$ exists for all $x \in \Omega^*$. By Lemma 3.8, we find that $u_{\Lambda,n} \to u_\Lambda$ in $C^1_{\mathrm{loc}}(\Omega^*)$ as $n \to \infty$, where u_Λ is a nonnegative solution of (1-1). Moreover, $u_\Lambda > 0$ in Ω^*, from Lemma 3.3. As before, $u_\Lambda \in C(\overline{\Omega} \setminus \{0\})$ and $u_\Lambda = h$ on $\partial\Omega$. If $\Lambda = 0$, then $\lim_{|x|\to 0} u_\Lambda(x)/E(x) = 0$. If $\Lambda \in (0, \infty)$ then, from the proof of Step 2, $w \le u_\Lambda$ in $B^*_{r_0}$, where w is the (unique) positive solution of (4-7). This and (4-8) prove that $\lim_{|x|\to 0} u_\Lambda(x)/E(x) = \Lambda$. Hence, u_Λ is a nonnegative solution of (1-1)+(1-5) such that $u_\Lambda \ge \min_{\partial\Omega} h$ in Ω^* and $u_\Lambda \in C^{1,\alpha}_{\mathrm{loc}}(\Omega^*)$ for some $\alpha \in (0, 1)$ (by Lemma 3.8).

We now prove Theorem 1.1 for $\Lambda = \infty$. For any $j \geq 1$, let $u_{j,n}$ denote the unique positive solution of (4-1)+(1-5) with $\Lambda = j$. By Lemmas 3.2 and 3.4, we find $C_1 > 0$ such that

$$0 < u_{j,n}(x) \leq u_{j+1,n}(x) \leq C_1|x|^{-\vartheta} + \max_{\partial\Omega} h \quad \text{for all } x \in \Omega^* \text{ and every } n \geq 2. \tag{4-9}$$

By Lemma 3.8, we have $u_{j,n} \to u_{\infty,n}$ in $C^1_{\mathrm{loc}}(\Omega^*)$ as $j \to \infty$, where $u_{\infty,n}$ is a solution of (4-1)+(1-5) with $\Lambda = \infty$. If u is any solution of (1-1)+(1-5) with $\Lambda = \infty$, then $u \leq u_{\infty,n+1} \leq u_{\infty,n}$ in Ω^*. (We use Theorem 1.2(a)(iii) for $u_{\infty,n}$.) We set $u_\infty(x) := \lim_{n\to\infty} u_{\infty,n}(x)$ for all $x \in \Omega^*$. Hence, $u_{\infty,n} \to u_\infty$ in $C^1_{\mathrm{loc}}(\Omega^*)$ as $n \to \infty$ and u_∞ is the maximal solution of (1-1)+(1-5) with $\Lambda = \infty$. □

Remark 4.2. For any $\Lambda \in [0,\infty) \cup \{\infty\}$, the solution of (1-1)+(1-5) constructed in the proof of Theorem 1.1, say $u_{\Lambda,h}$, is the *maximal* one, in the sense that any other (sub)solution is dominated by it. If $m \geq 1$, then $u_{\Lambda,h}$ is the only solution of (1-1) and (1-5) (by Lemma 3.2). If $0 < m < 1$, then we can construct the *minimal* solution of (1-1)+(1-5) using a similar perturbation argument. More precisely, for any integer $\xi \geq 1$, we consider the perturbed problem

$$\Delta u = u^q \left(|\nabla u|^2 + \frac{1}{\xi}\right)^{\frac{m}{2}} \quad \text{in } \Omega^*. \tag{4-10}$$

Under the assumptions of Theorem 1.1, it can be shown that (4-10), subject to (1-5), has a unique nonnegative solution $u_{\xi,\Lambda,h}$, which is dominated by any solution of (1-1)+(1-5) (using Lemma 3.2 for (4-10)). The existence of $u_{\xi,\Lambda,h}$ is obtained by proving Lemma 4.1 with (4-1) replaced by

$$\Delta u = \frac{u^{q+1}}{\sqrt{u^2 + 1/n}} \left(|\nabla u|^2 + \frac{1}{\xi}\right)^{\frac{m}{2}} \quad \text{in } \Omega^*. \tag{4-11}$$

The proof can be given as before and thus we skip the details. Moreover, $u_{\xi,\Lambda,h} \leq u_{\xi+1,\Lambda,h}$ in Ω^* and $u_{\xi,\Lambda,h}$ converges in $C^1_{\mathrm{loc}}(\Omega^*)$ as $\xi \to \infty$ to the minimal solution of (1-1)+(1-5). Furthermore, if $\Omega = B_\ell$ for some $\ell > 0$ and h is a nonnegative constant then, by construction, both the maximal solution and the minimal solution of (1-1)+(1-5) are radial.

Remark 4.3. For $m \in (0,1)$, the uniqueness of the solution of (1-1)+(1-5) may not necessarily hold (depending on Ω, h and Λ). Indeed, let $\Lambda \in (0,\infty)$ be arbitrary. Then there exists a nonincreasing solution u_1 of (2-1), subject to (2-2), such that $u_1'(r) = 0$ for all $r \in [r_1, 1]$ and $u_1' < 0$ on $(0, r_1)$ for some $r_1 \in (0,1]$ (see Theorem 2.1). If $\Lambda > 0$ is small, then $r_1 = 1$ (see Lemma 2.3) and, moreover, u_1 is the unique positive solution of (1-1)+(1-5) with $\Omega = B_1$ and $h \equiv u_1(r_1)$ (by Lemma 3.1).

By Theorem 2.2, there exists a positive, radial and increasing solution u_2 of (1-1) in $B^*_{r_1}$, subject to $u|_{\partial B_{r_1}} = u_1(r_1)$. Let $C := u_2(0)/u_1(r_1) \in (0,1)$ and $r_2 := r_1 C^{-1/\vartheta}$. We define $u_3 : (0, r_1 + r_2] \to (0,\infty)$ by

$$u_3(r) := \begin{cases} C u_1(C^{1/\vartheta} r) & \text{for } r \in (0, r_2), \\ u_2(r - r_2) & \text{for } r \in [r_2, r_1 + r_2]. \end{cases}$$

We observe that (1-1) in $B^*_{r_1+r_2}$, subject to $u|_{\partial B_{r_1+r_2}} = u_1(r_1)$ and $\lim_{|x|\to 0} u(x)/E(x) = \Lambda C^{1+(2-N)/\vartheta}$ has at least two distinct positive solutions: u_3 and the maximal solution, say u_4, as constructed in the proof of Theorem 1.1. We have $u_3 \neq u_4$, since $u_3'(r_2) = 0$ and $u_3 < u_1(r_1) \leq u_4$ on $[r_2, r_1+r_2)$.

5. Proof of Theorem 1.2

Let (1-2) hold. We first assume that $q < q_*$ and prove the claim of Theorem 1.2(a). Let u be any positive solution of (1-1). We write $\Lambda := \limsup_{|x|\to 0} u(x)/E(x)$ and analyse three cases: (I) $\Lambda = 0$; (II) $\Lambda \in (0,\infty)$; and (III) $\Lambda = \infty$. In Case (I), the claim follows from Lemma 3.11.

Case (II): $\Lambda \in (0,\infty)$. One can show the assertion of (ii) in Theorem 1.2(a) using an argument similar to [Friedman and Véron 1986, Theorem 1.1; Cîrstea and Du 2010, Theorem 5.1(b)]. We sketch the main ideas. Let $r_0 > 0$ be such that $\bar{B}_{4r_0} \subset \Omega$. For any $r \in (0, r_0)$ fixed, we define the function

$$V_{(r)}(\xi) := \frac{u(r\xi)}{E(r)} \quad \text{for all } \xi \in \mathbb{R}^N \text{ with } 0 < |\xi| < \frac{r_0}{r}.$$

We see that $V_{(r)}(\xi)$ satisfies the equation

$$\Delta V_{(r)}(\xi) = r^{2-m}[E(r)]^{q+m-1}[V_{(r)}(\xi)]^q |\nabla V_{(r)}(\xi)|^m \quad \text{for } 0 < |\xi| < \frac{r_0}{r}. \tag{5-1}$$

We prove that $\lim_{|x|\to 0} u(x)/E(x) = \Lambda$ by showing that, for every $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$\lim_{r\to 0^+} V_{(r)}(\xi) = G(\xi), \quad \text{where} \quad G(\xi) := \begin{cases} \Lambda |\xi|^{2-N} & \text{if } N \geq 3, \\ \Lambda & \text{if } N = 2. \end{cases} \tag{5-2}$$

For any $\xi \in \mathbb{R}^N \setminus \{0\}$, we define $W(\xi)$ by

$$W(\xi) := \begin{cases} |\xi|^{2-N} & \text{if } N \geq 3, \\ 1 + \ln(1/\min\{|\xi|, 1\}) & \text{if } N = 2. \end{cases}$$

Then by Lemma 3.8, there exist positive constants C_1, C and $\alpha \in (0,1)$ such that

$$0 < V_{(r)}(\xi) \leq C_1 W(\xi), \quad |\nabla V_{(r)}(\xi)| \leq C\frac{W(\xi)}{|\xi|} \quad \text{and} \quad |\nabla V_{(r)}(\xi) - V_{(r)}(\xi')| \leq C\frac{|\xi-\xi'|^\alpha}{|\xi|^{1+\alpha}} W(\xi) \tag{5-3}$$

for every $\xi, \xi' \in \mathbb{R}^N$ satisfying $0 < |\xi| \leq |\xi'| < r_0/r$. From the assumptions of Theorem 1.2, we infer that $\lim_{r\to 0^+} r^{2-m}[E(r)]^{q+m-1} = 0$. Thus, from (5-1) and (5-3), we find that, for any sequence $\bar{r}_n$ decreasing to zero, there exists a subsequence r_n such that

$$V_{(r_n)} \to V \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}) \qquad \text{and} \qquad \Delta V = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N \setminus \{0\}). \tag{5-4}$$

We set $\tilde{\Lambda}(r) := \sup_{|x|=r} u(x)/E(x)$ for $0 < r < r_0$. Then $\lim_{r\to 0^+} \tilde{\Lambda}(r) = \Lambda$ and there exists ξ_{r_n} on the $N-1$-dimensional sphere S^{N-1} in $\mathbb{R}^N$ such that $\tilde{\Lambda}(r_n) = u(r_n\xi_{r_n})/E(r_n)$. Passing to a subsequence, relabelled r_n, we have $\xi_{r_n} \to \xi_0$ as $n \to \infty$. We observe that

$$\frac{V_{(r_n)}(\xi)}{\tilde{\Lambda}(r_n|\xi|)} \leq \frac{E(r_n\,|\xi|)}{E(r_n)} \quad \text{for any } 0 < |\xi| < \frac{r_0}{r_n}, \tag{5-5}$$

with equality for $\xi = \xi_{r_n}$. Therefore, by letting $n \to \infty$ in (5-5) and using (5-4), we obtain that $V \le G$ in $\mathbb{R}^N \setminus \{0\}$ with $V(\xi_0) = G(\xi_0)$. Hence, $V = G$ in $\mathbb{R}^N \setminus \{0\}$. For $N \ge 3$, we also find that

$$\lim_{n\to\infty} \frac{(\nabla u)(r_n \xi)}{r_n^{1-N}} = -\frac{\Lambda}{N\omega_N} |\xi|^{-N}\xi \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{5-6}$$

Since $\{\bar{r}_n\}$ is an arbitrary sequence decreasing to 0, we conclude (5-2). Moreover,

$$\lim_{|x|\to 0} \frac{x\cdot \nabla u(x)}{|x|^{2-N}} = -\frac{\Lambda}{N\omega_N} \quad \text{and} \quad \lim_{|x|\to 0} \frac{|\nabla u(x)|}{|x|^{1-N}} = \frac{\Lambda}{N\omega_N}. \tag{5-7}$$

For $N \ge 3$, the claim of (5-7) follows easily from (5-6). For $N = 2$, one can follow the proof of Theorem 1.1 in [Friedman and Véron 1986] corresponding there to $p = N$ to obtain that $\lim_{r\to 0^+} r(\nabla u)(r\xi) = \Lambda \nabla E(\xi)$ for $\xi \in \mathbb{R}^N \setminus \{0\}$, which, for $|\xi| = 1$, gives (5-7).

To obtain (1-8), we use (5-7) and similar ideas in the proof of (5.1) in [Cîrstea and Du 2010].

Case (III): $\Lambda = \infty$. Using a contradiction argument based on Lemma 3.9 and the same argument as in [Brandolini et al. 2013, Corollary 4] or [Cîrstea 2014, Corollary 4.5], we find that $\lim_{|x|\to 0} u(x)/E(x) = \infty$. We next conclude the proof of Theorem 1.2(a) by showing that $\lim_{|x|\to 0} |x|^{\vartheta} u(x) = \lambda$.

Lemma 5.1. *Assume that* (1-2) *holds and $q < q_*$. Then any positive solution of* (1-1) *with a strong singularity at* 0 *satisfies* $\lim_{|x|\to 0} |x|^{\vartheta} u(x) = \lambda$, *where ϑ and λ are given by* (1-6).

Proof. We divide the proof into two steps.

Step 1. *We show that* $\liminf_{|x|\to 0} |x|^{\vartheta} u(x) > 0$.

Fix $r_0 > 0$ such that $\bar{B}_{4r_0} \subset \Omega$ and let C be a positive constant as in Corollary 3.10(b). Let k be a large integer such that $k > 1/r_0$. Consider the problem

$$\begin{cases} \Delta z = C^m |x|^{-m} z^{m+q} & \text{in } B_{r_0}^*, \\ z|_{\partial B_{r_0}} = \min_{\partial B_{r_0}} u. \end{cases} \tag{5-8}$$

Using (1-2) and $q < q_*$, by [Cîrstea and Du 2010, Theorem 1.2] we obtain a unique positive solution $z_k \in C^1(B_{r_0}^*)$ of (5-8) satisfying $\lim_{|x|\to 0} z_k(x)/E(x) = k$. Since $\lim_{|x|\to 0} u(x)/E(x) = \infty$, by (3-21) and Lemma 3.2 we find that $0 < z_k \le z_{k+1} \le u$ in $B_{r_0}^*$. We have $\lim_{k\to\infty} z_k = z_\infty$ in $C^1_{\text{loc}}(B_{r_0}^*)$ and z_∞ is a positive solution of (5-8) with $\lim_{|x|\to 0} z_\infty(x)/E(x) = \infty$ (see [Cîrstea and Du 2010, p. 197]). From $z_\infty \le u$ in $B_{r_0}^*$ and $\lim_{|x|\to 0} |x|^{\vartheta} z_\infty(x) > 0$ (see Theorem 1.1 in [Cîrstea and Du 2010]), we conclude Step 1.

Step 2. *We have* $\lim_{|x|\to 0} |x|^{\vartheta} u(x) = \lambda$, *where λ and ϑ are given by* (1-6).

We use a perturbation technique, as introduced in [Cîrstea and Du 2010], to construct a one-parameter family of sub- and supersolutions for (1-1). Fix $\varepsilon \in (0, \vartheta - N + 2)$. Observe that, if $N \ge 3$, then $q < q_*$ gives that $\vartheta > N - 2$. We define $\lambda_{\pm\varepsilon} > 0$ and $U_{\pm\varepsilon} : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ as follows:

$$U_{\pm\varepsilon}(x) = \lambda_{\pm\varepsilon} |x|^{-(\vartheta\pm\varepsilon)} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}, \quad \text{where} \quad \lambda_{\pm} := [(\vartheta\pm\varepsilon)^{1-m}(\vartheta - N + 2 \pm \varepsilon)]^{1/(q+m-1)}. \tag{5-9}$$

Clearly, we see that $\lambda_{\pm\varepsilon} \to \lambda$ as $\varepsilon \to 0$. By a direct computation, we find that

$$\Delta U_\varepsilon - U_\varepsilon^q |\nabla U_\varepsilon|^m \le 0 \le \Delta U_{-\varepsilon} - U_{-\varepsilon}^q |\nabla U_{-\varepsilon}|^m \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{5-10}$$

From Step 1, we obtain that $\lim_{|x|\to 0} u(x)/U_{-\varepsilon}(x) = \infty$. On the other hand, by the a priori estimates in Lemma 3.4 we have that $\lim_{|x|\to 0} u(x)/U_{\varepsilon}(x) = 0$. Since $\nabla U_{\pm\varepsilon} \neq 0$ in $\mathbb{R}^N \setminus \{0\}$, by (5-10) and the comparison principle in Lemma 3.1 we deduce that

$$u(x) \le U_\varepsilon(x) + \max_{\partial B_{r_0}} u \quad \text{and} \quad u(x) + \lambda r_0^{-\vartheta} \ge U_{-\varepsilon}(x) \quad \text{for all } 0 < |x| \le r_0, \tag{5-11}$$

where $r_0 \in (0, 1)$ is chosen so that $\bar{B}_{r_0} \subset \Omega$. Letting $\varepsilon \to 0$ in (5-11), we find that

$$\lambda(|x|^{-\vartheta} - r_0^{-\vartheta}) \le u(x) \le \lambda |x|^{-\vartheta} + \max_{\partial B_{r_0}} u \quad \text{for all } x \in B^*_{r_0}.$$

This concludes the proof of Step 2. □

Proof of Theorem 1.2, completed. It remains to show Theorem 1.2(b), that is, if $q \ge q_*$ for $N \ge 3$ then (1-1) has no positive solutions with singularities at 0. Indeed, when $q > q_*$, the a priori estimates in Lemma 3.4 give that $\lim_{|x|\to 0} u(x)/E(x) = 0$ for any solution of (1-1), proving the claim. If $q = q_*$, then $\vartheta = N - 2$, where ϑ is given by (1-6). For every $\varepsilon > 0$, we define U_ε as in (5-9) and, from the proof of Lemma 5.1, we see that

$$u(x) \le U_\varepsilon(x) + \max_{\partial B_{r_0}} u = [(N-2+\varepsilon)^{1-m}\varepsilon]^{1/(q+m-1)} |x|^{-(\vartheta+\varepsilon)} + \max_{\partial B_{r_0}} u \quad \text{for all } 0 < |x| \le r_0.$$

By letting $\varepsilon \to 0$, we find that $u(x) \le \max_{\partial B_{r_0}} u$ for every $0 < |x| \le r_0$, that is, 0 is a removable singularity for every solution of (1-1). Using Lemma 3.11, we finish the proof. □

6. Proof of Theorem 1.3

In this section, unless otherwise mentioned, we let $\Omega = \mathbb{R}^N$ in (1-1). Let (1-2) hold. If $q \ge q_*$ for $N \ge 3$ then, by Theorem 1.2(b), 0 is a removable singularity for all positive solutions of (1-1), which must be constant by Lemma 3.13. The assertion of Theorem 1.3(iii) is thus proved by Lemma 3.13. It remains to prove (i) and (ii) of Theorem 1.3.

(i) Let $q < q_*$. We divide the proof of Theorem 1.3(i) into two steps:

Uniqueness: From (3-8), any positive radial solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$ is nonincreasing. Furthermore, since it satisfies (2-1) for all $r \in (0, \infty)$, we see that it is convex. Hence, any positive radial solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$ satisfies only one of the following cases:

Case 1: There exists $r_u > 0$ such that $u'(r) = 0$ for all $r \ge r_u$ and $u' < 0$ on $(0, r_u)$.

Case 2: $u'(r) < 0$ for all $r > 0$.

We remark that Case 1 does happen for $m \in (0, 1)$, as can be seen from Theorem 2.1 (defining $u(r) = u(1)$ for $1 < r < \infty$). Let u_1 and u_2 denote any positive radial solutions of (1-1)+(1-10) for some $\Lambda \in (0, \infty]$ and $\gamma \in [0, \infty)$. (If $\gamma = 0$, then u_1 and u_2 are in Case 2.) Notice that $\lim_{r\to\infty}(u_1(r) - u_2(r)) = 0$ and $\lim_{r\to 0^+} u_1(r)/u_2(r) = 1$ (using Theorem 1.2(a) if $\Lambda = \infty$). If either u_1 or u_2 is in Case 2, then the uniqueness follows from Lemma 3.1, which is allowed because $|u_1'| + |u_2'| \neq 0$ in $\mathbb{R}^+$. Indeed, for every $\varepsilon > 0$, we have $u_1(r) \le (1+\varepsilon)u_2(r) + \varepsilon$ for every $r \in (0, \infty)$. Letting $\varepsilon \to 0$ then interchanging u_1

and u_2, we conclude that $u_1 \equiv u_2$. If both u_1 and u_2 are in Case 1, then $u_1 = u_2 = \gamma$ in $(\max(r_{u_1}, r_{u_2}), \infty)$. Using Lemma 3.1 on $(0, \max(r_{u_1}, r_{u_2})]$ as above, we find that $u_1 = u_2$ on $(0, \infty)$. (When $1 \le m < 2$, the proof of uniqueness of solutions can be made simpler by using Lemma 3.2 instead of Lemma 3.1, since we do not require that $|u_1'| + |u_2'| > 0$.)

Existence: Let $\Lambda \in (0, \infty)$ and $\gamma \in [0, \infty)$ be fixed. For any integer $\ell \ge 2$, we denote by $u_{\Lambda,\gamma,\ell}$ the maximal nonnegative solution of (1-1)+(1-5) with $h \equiv \gamma$ and $\Omega = B_\ell$ constructed by Theorem 1.1. For brevity, we write u_ℓ instead of $u_{\Lambda,\gamma,\ell}$. Recall the notation $B_\ell^* := B_\ell(0) \setminus \{0\}$. From the proof of Theorem 1.1, $u_{n,\ell} \to u_\ell$ in $C^1_{\mathrm{loc}}(B_\ell^*)$ as $n \to \infty$, where $u_{n,\ell}$ stands here for the unique nonnegative solution of (4-1)+(1-5) with $h \equiv \gamma$ and $\Omega = B_\ell$. We observe that $u_{n,\ell}$ is radial by the rotation invariance of the operator and the symmetry of the domain and, hence, u_ℓ is radial, too. Since $u_{n,\ell}(r) \ge \gamma$ for all $r \in (0, \ell)$, by Lemma 3.2 we infer that $u_{n,\ell}(r) \le u_{n,\ell+1}(r)$ for every $r \in (0, \ell)$. Consequently, letting $n \to \infty$ and using also Lemma 3.1, we deduce that

$$\gamma \le u_\ell(r) \le u_{\ell+1}(r) \le \lambda r^{-\vartheta} + \gamma \quad \text{for all } 0 < r < \ell. \tag{6-1}$$

Thus, $u_\ell \to u_{\Lambda,\gamma}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $\ell \to \infty$, where $u_{\Lambda,\gamma}$ is a radial solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$. Letting $\ell \to \infty$ in (6-1), we find that $\lim_{r\to\infty} u_{\Lambda,\gamma}(r) = \gamma$. Since $u_\ell(1) \le \lambda + \gamma$, by Lemma 3.1 we get that $u_\ell(r) \le u_{\ell+1}(r) \le \Lambda E(r) + \lambda + \gamma$ for all $r \in (0, 1)$ and $\ell \ge 2$. Since $\lim_{r\to 0^+} u_\ell(r)/E(r) = \Lambda$, we obtain that $\lim_{r\to 0^+} u_{\Lambda,\gamma}(r)/E(r) = \Lambda$. Thus, $u_{\Lambda,\gamma}$ satisfies (1-10).

When $\Lambda = \infty$, we denote by $u_{j,\gamma}$ the radial solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1-10), where Λ is replaced by an integer $j \ge 2$. The above argument shows that $\gamma \le u_{j,\gamma}(r) \le u_{j+1,\gamma}(r) \le \lambda r^{-\vartheta} + \gamma$ in $(0, \infty)$, so that $u_{j,\gamma} \to u_{\infty,\gamma}$ in $C^1_{\mathrm{loc}}(0, \infty)$, where $u_{\infty,\gamma}$ is a radial solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$ satisfying (1-10) with $\Lambda = \infty$. This concludes the proof of Theorem 1.3(i).

(ii) In view of Theorem 1.2, we need to establish the following result:

Lemma 6.1. *Let* (1-2) *hold. If u is a positive nonconstant solution of* (1-1) *in* $\mathbb{R}^N \setminus \{0\}$, *then* $q < q_*$ *and there exists* $\lim_{|x|\to\infty} u(x) = \gamma$ *in* $[0, \infty)$. *Moreover, u is radially symmetric and nonincreasing in* $\mathbb{R}^N \setminus \{0\}$, *such that* $\lim_{r\to 0^+} u(r)/E(r) = \Lambda \in (0, \infty]$.

Proof. Let u be a positive nonconstant solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$. Then we have $q < q_*$ and $\lim_{|x|\to 0} u(x)/E(x) = \Lambda \in (0, \infty]$ by Theorem 1.2 and Lemma 3.13. We proceed in two steps:

Step 1. *There exists* $\lim_{|x|\to\infty} u(x)$ *in* $[0, \infty)$.

From (3-8), we have $\limsup_{|x|\to\infty} u(x) < \infty$.

Claim. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^N$ satisfying $|x_n| \nearrow \infty$ as $n \to \infty$ and* $L = \limsup_{n\to\infty} u(x_n)$. *Then, up to a subsequence, relabelled (x_n), we have for each $\varepsilon > 0$ that there exists $N_\varepsilon > 0$ such that $u(z) < L + \varepsilon$ for all $z \in \bar{B}_{|x_n|/2}(x_n)$ and every $n \ge N_\varepsilon$.*

Indeed, by defining $v_n(y) = u(x_n + y)$ for all $y \in B_{2|x_n|/3}$, we see that v_n satisfies (1-1) in $B_{2|x_n|/3}$. For any $R > 0$, there exists $n_R \ge 1$ such that $\frac{2}{3}|x_n| > R$ for all $n \ge n_R$. Since $(v_n)_{n\ge n_R}$ is uniformly bounded in B_R, as in Lemma 3.8, we find that $|\nabla v_n|$ is uniformly bounded in B_R. Then a subsequence of (v_n), relabelled (v_n), converges in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ to a nonnegative solution of $\Delta v = |\nabla v|^m v^q$ in $\mathbb{R}^N$, which is constant by Corollary 3.7. Thus, up to a subsequence, relabelled (x_n), we have that, for every $\varepsilon > 0$, there exists $N_\varepsilon \geq 1$ such that $|u(x_n+y)-u(x_n)| < \frac{1}{4}\varepsilon$ for all $n \geq N_\varepsilon$ and every $y \in B_1$. Let N_ε be large such that $u(x_n) \leq L+\frac{1}{4}\varepsilon$. Then $u(x_n+y) < L+\frac{1}{2}\varepsilon$ for all $n \geq N_\varepsilon$ and every $y \in B_1$. If $m \in (0, 1)$, we define $V_n(y)$ by $f_{k,\delta}(y)$ with $k = \frac{2}{3}|x_n|$ and $\delta = 1$ for every $1 < |y| < \frac{2}{3}|x_n|$, where $f_{k,\delta}$ is as in Lemma 3.6. If $m \geq 1$, then $V_n(y)$ denotes $C(2-|y|^{2-N})(2|x_n|/3+1-|y|)^{-\tau}$ for $N \geq 3$, and $C\ln(\tilde{C}|y|)\big(\frac{2}{3}|x_n|+1-|y|\big)^{-\tau}$ for $N = 2$, respectively, where C, $\tilde{C}$ and τ are positive constants independent of n. Taking C and $\tilde{C}$ large enough and τ sufficiently close to 0, we see that V_n is a supersolution of (1-1) in $1 < |y| < \frac{2}{3}|x_n|$, dominating $v_n(y)$ on $|y| = \frac{2}{3}|x_n|$. Hence, we find that

$$v_n(y) \leq V_n(y)+L+\tfrac{1}{2}\varepsilon \quad \text{for all } 1 \leq |y| < \tfrac{2}{3}|x_n| \text{ and every } n \geq N_\varepsilon. \tag{6-2}$$

Using that $\lim_{n\to\infty} V_n\big(\frac{1}{2}|x_n|\big) = 0$, we choose N_ε large so that $V_n\big(\frac{1}{2}|x_n|\big) < \frac{1}{2}\varepsilon$ for all $n \geq N_\varepsilon$. Since the maximum of v_n on $\bar{B}_{|x_n|/2}$ is achieved on $\partial B_{|x_n|/2}$, then, from (6-2), we conclude the claim.

We finish the proof of Step 1 by using the claim a finite number of times with the relabelling implicitly understood. Let $(x_{n,1})_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^N$ with $|x_{n,1}| \nearrow \infty$ and $\lim_{n\to\infty} u(x_{n,1}) = \liminf_{|x|\to\infty} u(x)$. The claim gives that, for any fixed $\varepsilon > 0$, there exists $N_1 = N_1(\varepsilon) > 0$ such that

$$u(z) < \liminf_{|x|\to\infty} u(x)+\varepsilon \quad \text{for all } z \in \bar{B}_{|x_{n,1}|/2}(x_{n,1}) \text{ whenever } n \geq N_1. \tag{6-3}$$

We choose $x_{n,2} \in \partial B_{|x_{n,1}|} \cap \partial B_{|x_{n,1}|/2}(x_{n,1})$. Thus, $|x_{n,2}| = |x_{n,1}| \nearrow \infty$ as $n \to \infty$. Since (6-3) holds for $z = x_{n,2}$ and all $n \geq N_1$, by applying the claim again there exists $N_2 > N_1$ such that

$$u(z) < \liminf_{|x|\to\infty} u(x)+2\varepsilon \quad \text{for all } z \in \bar{B}_{|x_{n,1}|/2}(x_{n,2}) \cup \bar{B}_{|x_{n,1}|/2}(x_{n,1}) \text{ and every } n \geq N_2.$$

We can repeat this process a finite number of times, say K, which is independent of n, such that for each $2 \leq i \leq K$ it generates a number N_i greater than N_{i-1} and a sequence $(x_{n,i})_{n\geq N_i}$ with $|x_{n,i}| = |x_{n,1}|$ with the property that $\partial B_{|x_{n,1}|} \subset \bigcup_{i=1}^K B_{|x_{n,1}|/2}(x_{n,i})$ and

$$u(z) < \liminf_{|x|\to\infty} u(x)+K\varepsilon \quad \text{for all } z \in \partial B_{|x_{n,1}|} \text{ and every } n \geq N_K. \tag{6-4}$$

In light of (3-8), we see that (6-4) implies that $u(z) \leq \liminf_{|x|\to\infty} u(x)+K\varepsilon$ for all $|z| \geq |x_{n,1}|$ and all $n \geq N_K$. Consequently, $\limsup_{|x|\to\infty} u(x) \leq \liminf_{|x|\to\infty} u(x)+K\varepsilon$. By taking $\varepsilon \to 0$, we obtain that $\limsup_{|x|\to\infty} u(x) = \liminf_{|x|\to\infty} u(x)$. This completes the proof of Step 1.

To conclude the proof of Lemma 6.1, we need only show:

Step 2. *The solution u is radial.*

Since $\lim_{|x|\to\infty} u(x) = \gamma \in [0, \infty)$, we have that u satisfies (1-10) for some $\Lambda \in (0, \infty]$. If $m \geq 1$, then (1-1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1-10), has a unique positive solution (by Lemma 3.2), which must be radial by the invariance of the problem under rotation.

Let us now assume that $m \in (0, 1)$. Let $\varepsilon \in (0, \gamma)$ be arbitrary. By Theorem 1.3(i), there exists a unique positive radial solution U_ε of (1-1) in $\mathbb{R}^N \setminus \{0\}$ such that $\lim_{r\to 0^+} U_\varepsilon(r)/E(r) = \Lambda$ and

$\lim_{r\to\infty} U_\varepsilon(r) = \gamma + \varepsilon$. From the proof of Theorem 1.3(i) (with γ there replaced by $\gamma + \varepsilon$ and $\ell > 1$ large such that $u(x) \le \gamma + \varepsilon$ for all $|x| \ge \ell$), we infer that $u \le U_\varepsilon$ in $\mathbb{R}^N \setminus \{0\}$.

Using Remark 4.2 and the same ideas as in the existence proof of Theorem 1.3(i), for any integer $\xi \ge 1$, we can construct the unique nonnegative radial solution $u_{\xi,\Lambda,\varepsilon}$ of

$$\begin{cases} \Delta u = u^q (|\nabla u|^2 + 1/\xi)^{m/2} & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim\limits_{|x|\to 0} u(x)/E(x) = \Lambda, \\ \lim_{|x|\to\infty} u(x) = \max\{\gamma - \varepsilon, 0\}. \end{cases} \tag{6-5}$$

By Lemma 3.2, we deduce that $u_{\xi,\Lambda,\varepsilon} \le u_{\xi+1,\Lambda,\varepsilon} \le u$ in $\mathbb{R}^N \setminus \{0\}$, since $\lim_{|x|\to 0} u_{\xi,\Lambda,\varepsilon}(x)/u(x) = 1$ and $\lim_{|x|\to\infty}(u_{\xi,\Lambda,\varepsilon}(x) - u(x))$ is either 0 if $\gamma = 0$ or $-\varepsilon$ if $\gamma > 0$. Thus, by defining $u_\varepsilon(r) := \lim_{\xi\to\infty} u_{\xi,\Lambda,\varepsilon}(r)$ for all $r \in (0, \infty)$, we obtain that u_ε is a positive radial solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$, satisfying $\lim_{r\to 0^+} u_\varepsilon(r)/E(r) = \Lambda$ and $\lim_{r\to\infty} u_\varepsilon(r) = \max\{\gamma - \varepsilon, 0\}$. Moreover, we have

$$u_{\varepsilon_2} \le u_{\varepsilon_1} \le u \le U_{\varepsilon_1} \le U_{\varepsilon_2} \quad \text{in } \mathbb{R}^N \setminus \{0\} \quad \text{for all } 0 < \varepsilon_1 < \varepsilon_2 < \gamma.$$

Letting ε tend to 0, we get that both u_ε and U_ε converge to a positive radial solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$, subject to (1-10). By the uniqueness of such a solution, we conclude that u is radial. □

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Since the submission of this paper, we have learned of a recent paper [Bidaut-Véron et al. 2014] on the local and global properties of solutions of quasilinear Hamilton–Jacobi equations, in which related questions are investigated.

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A TOPOLOGICAL JOIN CONSTRUCTION AND THE TODA SYSTEM ON COMPACT SURFACES OF ARBITRARY GENUS

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We consider the *Toda system* of Liouville equations on a compact surface Σ

$$\begin{cases} -\Delta u_1 = 2\rho_1\left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\right) - \rho_2\left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\right), \\ -\Delta u_2 = 2\rho_2\left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\right) - \rho_1\left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\right), \end{cases}$$

which arises as a model for nonabelian Chern–Simons vortices. Here h_1 and h_2 are smooth positive functions and ρ_1 and ρ_2 two positive parameters.

For the first time, the ranges $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $k \in \mathbb{N}$, and $\rho_2 \in (4\pi, 8\pi)$ are studied with a variational approach on surfaces with arbitrary genus. We provide a general existence result by using a new improved Moser–Trudinger-type inequality and introducing a topological join construction in order to describe the interaction of the two components u_1 and u_2.

1. Introduction

We are interested here in the *Toda system* on a compact surface Σ

$$\begin{cases} -\Delta u_1 = 2\rho_1\left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\right) - \rho_2\left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\right), \\ -\Delta u_2 = 2\rho_2\left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\right) - \rho_1\left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\right), \end{cases} \tag{1}$$

where Δ is the Laplace–Beltrami operator, ρ_1 and ρ_2 are two nonnegative parameters, $h_1, h_2 : \Sigma \to \mathbb{R}$ are smooth positive functions, and Σ is a compact orientable surface without boundary with a Riemannian metric g. For the sake of simplicity, we normalize the total volume of Σ so that $|\Sigma| = 1$.

The above system has been widely studied in the literature since it is motivated by problems in both differential geometry and mathematical physics. In geometry, it relates to the Frenet frame of holomorphic curves in $\mathbb{CP}^n$ [Bolton and Woodward 1997; Calabi 1953; Chern and Wolfson 1987]. In mathematical physics, it models nonabelian Chern–Simons theory in the self-dual case, when a scalar Higgs field is coupled to a gauge potential [Dunne 1995; Tarantello 2008; 2010; Yang 2001].

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Keywords: geometric PDEs, variational methods, min-max schemes.

Equation (1) is variational, and solutions correspond to critical points of the Euler–Lagrange functional $J_\rho : H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R}$ ($\rho = (\rho_1, \rho_2)$) given by

$$J_\rho(u_1, u_2) = \int_\Sigma Q(u_1, u_2)\, dV_g + \sum_{i=1}^{2} \rho_i \left(\int_\Sigma u_i \, dV_g - \log \int_\Sigma h_i e^{u_i} \, dV_g \right), \tag{2}$$

where $Q(u_1, u_2)$ is a quadratic form that has the expression

$$Q(u_1, u_2) = \tfrac{1}{3}(|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \cdot \nabla u_2). \tag{3}$$

The structure of J_ρ strongly depends on the range of the two parameters ρ_1 and ρ_2. An important tool in treating this kind of functional is the Moser–Trudinger inequality; see (7). For the Toda system, a similar sharp inequality was derived in [Jost and Wang 2001]:

$$4\pi \log \int_\Sigma e^{u_1 - \bar{u}_1} \, dV_g + 4\pi \log \int_\Sigma e^{u_2 - \bar{u}_2} \, dV_g \leq \int_\Sigma Q(u_1, u_2) \, dV_g + C_\Sigma, \quad (u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma); \tag{4}$$

here $\bar{u}_i$ stands for the average of u_i on Σ.

By means of the latter inequality, we immediately get existence of a critical point provided both ρ_1 and ρ_2 are less than 4π: indeed for these values, one can minimize J_ρ using standards methods of the calculus of variations. The case of larger ρ_i is subtler due to the fact that J_ρ is unbounded from below.

Before describing the main difficulties of (1), we consider its scalar counterpart: the Liouville equation

$$-\Delta u = 2\rho \left(\frac{h e^u}{\int_\Sigma h e^u \, dV_g} - 1 \right), \tag{5}$$

where h is a smooth positive function on Σ and ρ a positive real number.

Equation (5) appears in conformal geometry in the problem of prescribing the Gaussian curvature, whereas in mathematical physics it describes models in abelian Chern–Simons theory. The literature on (5) is broad with many results regarding existence, blow-up analysis, compactness, etc. [Malchiodi 2008b; Tarantello 2010].

As with many geometric problems, (5) presents blow-up phenomena. It was proved in [Brezis and Merle 1991; Li 1999; Li and Shafrir 1994] that, for a sequence of solutions $(u_n)_n$ that blow up around a point p, the following quantization property holds:

$$\lim_{r \to 0} \lim_{n \to +\infty} \int_{B_r(p)} h e^{u_n} \, dV_g = 4\pi.$$

Moreover, the limit function (after rescaling) can be viewed as the logarithm of the conformal factor of the stereographic projection from S^2 onto $\mathbb{R}^2$, composed with a dilation.

Concerning the Toda system (1), a sequence of solutions can blow up in three different ways: one component blows up and the other stays bounded, one component blows up faster than the other, or both components diverge at the same rate. Jost et al. [2006] proved that the volume quantizations in these scenarios are $(0, 4\pi)$ or $(4\pi, 0)$ in the first case, $(4\pi, 8\pi)$ or $(8\pi, 4\pi)$ for the second one, and $(8\pi, 8\pi)$

for the last situation. Moreover, each alternative may indeed occur [D'Aprile et al. 2015; 2014; del Pino et al. 2005; Esposito et al. 2005; Musso et al. 2013].

With this at hand, with some further analysis, it is possible to obtain a compactness property, namely that the set of solutions to (1) is bounded (in any smoothness norm) for (ρ_1, ρ_2) bounded away from multiples of 4π (see Theorem 2.1). This fact, combined with a monotonicity method from [Struwe 1985], allows one to attack problem (1) via min-max methods.

Let us now discuss the variational strategy for proving existence of solutions and how our result compares to the existing literature. The goal is to introduce min-max schemes based on the study of the sublevels of the Euler–Lagrange functional. Consider the scalar case (5), with Euler–Lagrange energy

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 \, dV_g + 2\rho\left(\int_\Sigma u \, dV_g - \log\int_\Sigma h e^u \, dV_g\right). \tag{6}$$

By the classical Moser–Trudinger inequality

$$8\pi \log\int_\Sigma e^{(u-\bar{u})} \, dV_g \leq \frac{1}{2}\int_\Sigma |\nabla u|^2 \, dV_g + C_{\Sigma,g}, \tag{7}$$

the latter energy is coercive if and only if $\rho < 4\pi$. A key result in treating this kind of problem without coercivity conditions (i.e., when $\rho > 4\pi$) is an improved version of (7), usually refereed to as the Chen–Li inequality and obtained in [Chen and Li 2001; Djadli 2008] (see also [Djadli and Malchiodi 2008]); roughly speaking, it states that, if the function e^u is *spread* (in a quantitative sense) among at least $(k+1)$ regions of Σ, $k \in \mathbb{N}$, then the constant in the left-hand side of (7) can be taken nearly $(k+1)$ times larger. This in turn implies that, for such functions u, $I_\rho(u)$ is bounded below even when $\rho < 4(k+1)\pi$. Therefore, if ρ satisfies the latter inequality and if $I_\rho(u)$ attains large negative values (i.e., when the lower bounds fail), e^u should be concentrated near at most k points of Σ; see [Djadli 2008] for a formal proof of this fact.

To describe such low sublevels, it is then natural to introduce the family of unit measures Σ_k that are supported at at most k points of Σ, known as *formal barycenters* of Σ of order k:

$$\Sigma_k = \left\{\sum_{i=1}^{k} t_i \delta_{x_i} : \sum_{i=1}^{k} t_i = 1, \ t_i \geq 0, \, x_i \in \Sigma \text{ for all } i = 1, \dots, k\right\}. \tag{8}$$

Endowed with the weak topology of distributions, Σ_1 is homeomorphic to Σ while, for $k \geq 2$, Σ_k is a stratified set (union of open manifolds of different dimensions). It is possible to show that the homology of Σ_k is always nontrivial and, using suitable test functions, that it injects into that of sufficiently low sublevels of I_ρ: this gives existence of solutions to (5) via suitable min-max schemes for every $\rho \notin 4\pi\mathbb{N}$.

Returning to the Toda system (1), a first existence result was presented in [Malchiodi and Ndiaye 2007] for $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $k \in \mathbb{N}$, and $\rho_2 < 4\pi$ (see also [Jost et al. 2006] for the case $k = 1$). When one of the two parameters is small, the system (1) resembles the scalar case (5) and one can adapt the above argument to this framework as well. When both parameters exceed the value 4π, the description of the low sublevels becomes more involved due to the interaction of the two components u_1 and u_2.

The first variational approach to understand this interaction was given by Malchiodi and Ruiz [2013], who obtained an existence result for $(\rho_1, \rho_2) \in (4\pi, 8\pi)^2$. This was done in particular by showing that, if both components of the system concentrate near the same point and with the same *rate*, then the constants in the left-hand side of (4) can be nearly doubled.

Later, the case of general parameters $(\rho_1, \rho_2) \notin \Lambda$ was considered in [Battaglia et al. 2015] but only for surfaces of positive genus. Using improved inequalities à la Chen and Li, it is possible to prove that, if $\rho_1 < 4(k+1)\pi$ and $\rho_2 < 4(l+1)\pi$, $k, l \in \mathbb{N}$, and if $J_\rho(u_1, u_2)$ is sufficiently low, then either e^{u_1} is close to Σ_k or e^{u_2} is close to Σ_l in the distributional sense. This (not mutually exclusive) alternative can be expressed in terms of the *topological join* of Σ_k and Σ_l. Recall that, given two topological spaces A and B, their join $A * B$ is defined as the family of elements of the form

$$A * B = \frac{\{(a, b, s) : a \in A,\ b \in B,\ s \in [0, 1]\}}{E}, \tag{9}$$

where E is an equivalence relation such that

$$(a_1, b, 1) \overset{E}{\sim} (a_2, b, 1) \quad \text{for all } a_1, a_2 \in A, b \in B, \qquad (a, b_1, 0) \overset{E}{\sim} (a, b_2, 0) \quad \text{for all } a \in A, b_1, b_2 \in B.$$

This construction allows one to map low sublevels of J_ρ into $\Sigma_k * \Sigma_l$, with the join parameter s expressing whether distributionally e^{u_1} is closer to Σ_k or e^{u_2} is closer to Σ_l.

The hypothesis on the genus of Σ in [Battaglia et al. 2015] was used in the following way: on such surfaces, one can construct two *disjoint* simple noncontractible curves γ_1 and γ_2 such that Σ retracts on each of them through continuous maps Π_1 and Π_2. By means of these retractions, low-energy sublevels may be described in terms of $(\gamma_1)_k$ or $(\gamma_2)_l$ only. On the other hand, one can build test functions modeled on $(\gamma_1)_k * (\gamma_2)_l$ for which each component u_i only concentrates near γ_i, to somehow *minimize* the interaction between the two components u_1 and u_2, due to the fact that γ_1 and γ_2 are disjoint.

We prove here the following result, which for the first time applies to surfaces of arbitrary genus when both parameters ρ_i are supercritical and one of them also arbitrarily large:

Theorem 1.1. *Let h_1 and h_2 be two positive smooth functions, and let Σ be any compact surface. Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $k \in \mathbb{N}$, and $\rho_2 \in (4\pi, 8\pi)$. Then problem* (1) *has a solution.*

Remark 1.2. Theorem 1.1 is new when Σ is a sphere and $k \geq 3$. As we already discussed, the case of surfaces with positive genus was covered in [Battaglia et al. 2015]. The case of $\Sigma \simeq S^2$ and $k = 1$ was covered in [Malchiodi and Ruiz 2013], while for $k = 2$ it was covered in [Lin et al. 2014]. In the latter paper, the authors indeed computed the Leray–Schauder degree of the equation for the range of ρ_i in Theorem 1.1. It turns out that the degree of (1) is 0 for the sphere when $k \geq 3$: since solutions do exist by Theorem 1.1, it means that either they are degenerate or that degrees of multiple ones cancel, so a global degree counting does not detect them. A similar phenomenon occurs for (5) on the sphere, when $\rho > 12\pi$ [Chen and Lin 2003]. Even for positive genus, we believe that our approach could be useful in computing the degree of the equation, as it happened in [Malchiodi 2008a] for the scalar equation (5). More precisely, we speculate that the degree should be computable as $1 - \chi(Y)$, where the set Y is given in (51). This is satisfied for example in the case of the sphere thanks to Lemma 5.4.

Other results on the degree of the system, but for different ranges of parameters, are available in [Malchiodi and Ruiz 2015].

As described above, in the situation of Theorem 1.1, it is natural to characterize low sublevels of the Euler–Lagrange energy J_ρ by means of the topological join $\Sigma_k * \Sigma_1$ (notice that $\Sigma_1 \simeq \Sigma$). However, differently from [Battaglia et al. 2015], we crucially take into account the interaction between the two components u_1 and u_2. As one can see from (3), the quadratic energy Q penalizes situations in which the gradients of the two components are aligned, and we would like to make a quantitative description of this effect. Our proof uses four new main ingredients.

• A refinement of the projection from low-energy sublevels onto the topological join $\Sigma_k * \Sigma_1$ from [Battaglia et al. 2015] (see Section 3), which uses the scales of concentration of the two components and which extends some construction in [Malchiodi and Ruiz 2013]. Having to deal with arbitrarily high values of ρ_1, differently from [Malchiodi and Ruiz 2013], we also need to take into account the stratified structure of Σ_k and the closeness in measure sense to its substrata.

• A new, scaling-invariant improved Moser–Trudinger inequality for system (1); see Proposition 3.5. This is inspired from another one in [Brezis and Merle 1991] for singular Liouville equations, i.e., of the form (5) but with Dirac masses on the right-hand side. The link between the two problems arises in the situation when one of the two components in (1) is much more concentrated than the other: in this case, the measure associated to its exponential function resembles a Dirac delta compared to the other one. The above improved inequality gives extra constraints to the projection on the topological join; see Proposition 3.7 and Corollary 3.8.

• A new set of test functions showing that the characterization of low-energy levels of J_ρ is sharp, as a subset Y of $\Sigma_k * \Sigma_1$. We need indeed to build test functions modeled on a set that contains $\Sigma_{k-1} * \Sigma_1$, and the stratified nature of Σ_{k-1} makes it hard to obtain uniform upper estimates on such functions.

• A new topological argument showing the noncontractibility of the above set Y, which we use then crucially to develop our min-max scheme. The fact that Y is simply connected and has Euler characteristic equal to 1 forces us to use rather sophisticated tools from algebraic topology.

We expect that our approach might extend to the case of general physical parameters ρ_1 and ρ_2, including the *singular Toda system*, in which Dirac masses (corresponding to ramification or vortex points) appear in the right-hand side of (1); see also [Battaglia 2015] for some results with this approach.

The paper is organized as follows. In Section 2, we recall some improved versions of the Moser–Trudinger inequality: first some that rely on the *macroscopic* spreading of the components u_1 and u_2 and then some refined ones, which are scaling-invariant. In Section 3, we derive a new — still scaling-invariant — improved version of the Moser–Trudinger inequality for systems, and we use it to find a characterization of low-energy levels of J_ρ by means of a subset Y of the topological join $\Sigma_k * \Sigma_1$. In Section 4, we construct then suitable test functions that show the optimality of the above characterization. In Section 5, we finally introduce the variational method to prove the existence of solutions.

Notation. The symbol $B_r(p)$ stands for the open metric ball of radius r and center p, while $A_p(r_1, r_2)$ is the open annulus of radii r_1 and r_2 and center p. For the complement of a set Ω in Σ, we will write Ω^c. Given a function $u \in L^1(\Sigma)$ and $\Omega \subset \Sigma$, the average of u on Ω is denoted by the symbol

$$\fint_{\Omega} u \, dV_g = \frac{1}{|\Omega|} \int_{\Omega} u \, dV_g,$$

while $\bar{u}$ stands for the average of u in Σ: since we are assuming $|\Sigma| = 1$, we have

$$\bar{u} = \int_{\Sigma} u \, dV_g = \fint_{\Sigma} u \, dV_g.$$

We also write

$$\mathcal{N}(f, D) = \frac{f}{\int_D f \, dV_g}.$$

The sublevels of the functional J_ρ will be denoted by

$$J_\rho^a = \{(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma) : J_\rho(u_1, u_2) \le a\}.$$

Throughout the paper, the letter C will stand for large constants that are allowed to vary among different formulas or even within the same line. To stress the dependence of the constants on some parameter, we add subscripts to C, as C_δ, etc. We will write $o_r(1)$ to denote quantities that tend to 0 as $r \to 0$ or $r \to +\infty$; we will similarly use the symbol $O_r(1)$ for bounded quantities.

2. Preliminaries

We begin by stating a compactness property that is needed in order to run the variational methods. Letting Λ be the set defined as

$$\Lambda = (4\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \cup 4\pi\mathbb{N}) \subseteq \mathbb{R}^2, \tag{10}$$

by the local blow-up in [Jost et al. 2006] and some analysis [Battaglia and Mancini 2015], one deduces:

Theorem 2.1 [Battaglia and Mancini 2015; Jost et al. 2006]. *For (ρ_1, ρ_2) in a fixed compact set of $\mathbb{R}^2 \setminus \Lambda$, the family of solutions to* (1) *is uniformly bounded in $C^{2,\beta}$ for some $\beta > 0$.*

In the next two subsections, we will discuss some improved versions of the Moser–Trudinger inequality (4) that hold under suitable assumptions on the components of the system. The first type of inequality relies on the spreading of the (exponentials of the) components over the surface [Battaglia et al. 2015]. The second one, from [Malchiodi and Ruiz 2013], relies instead on comparing the scales of concentration of the two components.

2.1. ***Macroscopic improved inequalities.*** Here comes the first kind of improved inequality: basically, if the *masses* of both e^{u_1} and e^{u_2} are *spread* on at least $k+1$ and $l+1$ different sets, then the logarithms in (4) can be multiplied by $k+1$ and $l+1$, respectively. Notice that this result was given in [Malchiodi and Ndiaye 2007] in the case $l = 0$ and in [Malchiodi and Ruiz 2013] in the case $k = l = 1$. The proof relies on localizing (4) by using cut-off functions near the regions of volume concentration. For (7), this was previously shown in [Chen and Li 1991].

Lemma 2.2 [Battaglia et al. 2015]. *Let $\delta > 0$, $\theta > 0$, $k, l \in \mathbb{N}$, and $\{\Omega_{1,i}, \Omega_{2,j}\}_{i\in\{0,\dots,k\}, j\in\{0,\dots,l\}} \subset \Sigma$ be such that*

$$d(\Omega_{1,i}, \Omega_{1,i'}) \geq \delta \quad \textit{for all } i, i' \in \{0, \dots, k\} \textit{ with } i \neq i',$$
$$d(\Omega_{2,j}, \Omega_{2,j'}) \geq \delta \quad \textit{for all } j, j' \in \{0, \dots, l\} \textit{ with } j \neq j'.$$

Then for any $\varepsilon > 0$, there exists $C = C(\varepsilon, \delta, \theta, k, l, \Sigma)$ such that any $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ satisfying

$$\int_{\Omega_{1,i}} e^{u_1}\, dV_g \geq \theta \int_\Sigma e^{u_1}\, dV_g \quad \textit{for all } i \in \{0, \dots, k\},$$
$$\int_{\Omega_{2,j}} e^{u_2}\, dV_g \geq \theta \int_\Sigma e^{u_2}\, dV_g \quad \textit{for all } j \in \{0, \dots, l\}$$

satisfies

$$4\pi(k+1) \log \int_\Sigma e^{u_1 - \bar{u}_1}\, dV_g + 4\pi(l+1) \log \int_\Sigma e^{u_2 - \bar{u}_2}\, dV_g \leq (1+\varepsilon) \int_\Sigma Q(u_1, u_2)\, dV_g + C.$$

As one can see, larger constants in the left-hand side of (4) can be helpful in obtaining lower bounds on the functional J_ρ even when the coefficients ρ_1 and ρ_2 exceed the threshold value $(4\pi, 4\pi)$. A consequence of this fact is that, when the energy $J_\rho(u_1, u_2)$ is large and negative, then e^{u_1} and e^{u_2} are forced to *concentrate* near certain points in Σ whose number depends on ρ_1 and ρ_2. To make this description rigorous, it is convenient to introduce some further notation.

We denote by $\mathcal{M}(\Sigma)$ the set of all Radon measures on Σ and introduce a distance on it by using duality versus Lipschitz functions; that is, we set

$$\boldsymbol{d}(\nu_1, \nu_2) = \sup_{\|f\|_{\mathrm{Lip}(\Sigma)} \leq 1} \left| \int_\Sigma f\, d\nu_1 - \int_\Sigma f\, d\nu_2 \right|, \qquad \nu_1, \nu_2 \in \mathcal{M}(\Sigma). \tag{11}$$

This is known as the *Kantorovich–Rubinstein distance*.

The following result was proven using the improved inequality from Lemma 2.2 (see previous page for $\mathcal{N}$):

Proposition 2.3 [Battaglia et al. 2015]. *Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$. Then, for any $\varepsilon > 0$, there exists $L > 0$ such that any $(u_1, u_2) \in J_\rho^{-L}$ satisfies either*

$$\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) < \varepsilon \quad \textit{or} \quad \boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_l) < \varepsilon.$$

When a measure is $\boldsymbol{d}$-close to an element in Σ_k (see (8)), it is then possible to map it continuously to a nearby element in this set. The next proposition collects some properties of this map from Proposition 2.2 in [Battaglia et al. 2015] and Lemma 2.3 in [Djadli and Malchiodi 2008] (together with the proof of Lemma 3.10).

Proposition 2.4. *Given $l \in \mathbb{N}$, for ε_l sufficiently small, there exists a continuous retraction*

$$\psi_l : \{\nu \in \mathcal{M}(\Sigma) : \boldsymbol{d}(\nu, \Sigma_l) < 2\varepsilon_l\} \to \Sigma_l.$$

Here continuity refers to the distance $\boldsymbol{d}$. In particular, if $\nu_n \rightharpoonup \nu$ in the sense of measures, with $\nu \in \Sigma_l$, then $\psi_l(\nu_n) \to \nu$.

Furthermore, the following property holds: given any $\varepsilon > 0$, there exists $\varepsilon' \ll \varepsilon$ with ε' depending on l and ε such that if $\boldsymbol{d}(\nu, \Sigma_{l-1}) > \varepsilon$ then there exist l points $x_1, \dots, x_l$ such that

$$d(x_i, x_j) > 2\varepsilon' \quad \textit{for } i \neq j, \qquad \int_{B_{\varepsilon'}(x_i)} \nu > \varepsilon' \quad \textit{for all } i = 1, \dots, l.$$

The alternative in Proposition 2.3 can be expressed naturally in terms of the topological join of $\Sigma_k * \Sigma_l$; see also the comments after (9). Indeed, we can define a map from the low sublevels J_ρ^{-L} onto this set.

Proposition 2.5 [Battaglia et al. 2015]. *Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$. Then for $L > 0$ sufficiently large there exists a continuous map*

$$\Psi : J_\rho^{-L} \to \Sigma_k * \Sigma_l.$$

Proof. The proof is carried out exactly as in Proposition 4.7 of [Battaglia et al. 2015]. We repeat here the argument for the reader's convenience as we will need to suitably modify it later on. By Proposition 2.3, we know that for any $\varepsilon > 0$, taking $L > 0$ sufficiently large, $(u_1, u_2) \in J_\rho^{-L}$ satisfies either $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) < \varepsilon$ or $\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_l) < \varepsilon$ (or both). Using then Proposition 2.4, it follows that either $\psi_k(\mathcal{N}(e^{u_1}, \Sigma))$ or $\psi_l(\mathcal{N}(e^{u_2}, \Sigma))$ is well-defined. We let $d_1 = \boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k)$ and $d_2 = \boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_l)$ and introduce a function $\tilde{s} = \tilde{s}(d_1, d_2)$ in the following way:

$$\tilde{s}(d_1, d_2) = f\left(\frac{d_1}{d_1 + d_2}\right),$$

where f is given by

$$f(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{4}], \\ 2z - \frac{1}{2} & \text{if } t \in (\frac{1}{4}, \frac{3}{4}), \\ 1 & \text{if } t \in [\frac{3}{4}, 1]. \end{cases}$$

We finally set

$$\Psi(u_1, u_2) = (1 - \tilde{s})\psi_k(\mathcal{N}(e^{u_1}, \Sigma)) + \tilde{s}\psi_l(\mathcal{N}(e^{u_2}, \Sigma)). \tag{12}$$

One just has to observe that, when one of the two ψ is not defined, the other necessarily is. Therefore, the map is well-defined by the equivalence relation of the topological join; see (9). □

2.2. *Scaling-invariant improved inequalities.* Malchiodi and Ruiz [2013] set up a tool to deal with situations to which Lemma 2.2 does not apply, for example in cases when both e^{u_1} and e^{u_2} are concentrated around only one point. They provided a definition of the *center* and the *scale of concentration* of such functions, to obtain new improved inequalities in terms of these. We are interested here in measures concentrated around possibly multiple points. We need therefore a localized version of the argument in [Malchiodi and Ruiz 2013], which applies to measures supported in a ball and sufficiently concentrated around its center.

Given $x_0 \in \Sigma$ and $r > 0$ small, consider the set

$$\mathcal{A}_{x_0,r} = \left\{ f \in L^1(B_r(x_0)) : f > 0 \text{ a.e. and } \int_{B_r(x_0)} f \, dV_g = 1 \right\},$$

endowed with the topology inherited from $L^1(\Sigma)$.

Fix a constant $R > 1$, and let $R_0 = 3R$. Define $\sigma : B_r(x_0) \times \mathscr{A}_{x_0,r} \to (0, +\infty)$ such that

$$\int_{B_{\sigma(x,f)}(x) \cap B_r(x_0)} f \, dV_g = \int_{(B_{R_0\sigma(x,f)}(x))^c \cap B_r(x_0)} f \, dV_g. \tag{13}$$

It is easy to check that $\sigma(x, f)$ is uniquely determined and continuous (both in $x \in B_r(x_0)$ and in $f \in L^1$). Moreover (see (3.2) in [Malchiodi and Ruiz 2013]), σ satisfies

$$d(x, y) \le R_0 \max\{\sigma(x, f), \sigma(y, f)\} + \min\{\sigma(x, f), \sigma(y, f)\}. \tag{14}$$

We now define $T : B_r(x_0) \times \mathscr{A}_{x_0,r} \to \mathbb{R}$ as

$$T(x, f) = \int_{B_{\sigma(x,f)}(x) \cap B_r(x_0)} f \, dV_g.$$

Lemma 2.6 ([Malchiodi and Ruiz 2013] with minor changes). *If $\bar{x} \in \overline{B_r(x_0)}$ is such that $T(\bar{x}, f) = \max_{y \in \overline{B_r(x_0)}} T(y, f)$, then $\sigma(\bar{x}, f) < 3\sigma(x, f)$ for any other $x \in \overline{B_r(x_0)}$.*

As a consequence of the previous lemma and of a covering argument, one can obtain the following:

Lemma 2.7 ([Malchiodi and Ruiz 2013] with minor changes). *There exists a fixed $\tau > 0$ such that*

$$\max_{x \in \overline{B_r(x_0)}} T(x, f) > \tau > 0 \quad \textit{for all } f \in \mathscr{A}_{x_0,r}.$$

Let us define $\sigma : \mathscr{A}_{x_0,r} \to \mathbb{R}$ by

$$\sigma(f) = 3 \min\{\sigma(x, f) : x \in \overline{B_r(x_0)}\},$$

which is obviously a continuous function.

Given τ as in Lemma 2.7, consider the set

$$S(f) = \{x \in \overline{B_r(x_0)} : T(x, f) > \tau, \ \sigma(x, f) < \sigma(f)\}. \tag{15}$$

If $\bar{x} \in \overline{B_r(x_0)}$ is such that $T(\bar{x}, f) = \max_{x \in \overline{B_r(x_0)}} T(x, f)$, then Lemmas 2.6 and 2.7 imply that $\bar{x} \in S(f)$. Therefore, $S(f)$ is a nonempty set for any $f \in \mathscr{A}_{x_0,r}$. Moreover, recalling (13) and the notation before it, from (14), we have that

$$\operatorname{diam}(S(f)) \le (R_0 + 1)\sigma(f). \tag{16}$$

We will now restrict ourselves to a class of functions in $L^1(B_r(x_0))$ that are almost entirely concentrated near the center x_0. In this case, one expects $\sigma(f)$ to be small and points in $S(f)$ to be close to x_0: see Remark 2.8 for precise estimates in this spirit. Given $\varepsilon > 0$ small, let us introduce the class of functions

$$\mathscr{C}_{\varepsilon,r}(x_0) = \left\{ f \in \mathscr{A}_{x_0,r} : \int_{B_\varepsilon(x_0)} f \, dV_g > 1 - \varepsilon \right\}. \tag{17}$$

Remark 2.8. For this class of functions, we claim that $T(x, f) \le \varepsilon$ when $d(x, x_0) > 2\varepsilon$. In fact, if $\sigma(x, f) \le d(x, x_0) - \varepsilon$, then we are done since

$$T(x, f) = \int_{B_{\sigma(x,f)}(x) \cap B_r(x_0)} f \, dV_g \le \int_{B_\varepsilon(x_0)^c \cap B_r(x_0)} f \, dV_g \le \varepsilon.$$

If this is not the case, i.e., $\sigma(x, f) > d(x, x_0) - \varepsilon$, then using $d(x, x_0) > 2\varepsilon$, we obtain

$$\begin{aligned} R_0\sigma(x, f) > R_0(d(x, x_0) - \varepsilon) &> \tfrac{1}{2} R_0\, d(x, x_0) \\ &> d(x, x_0) + \varepsilon. \end{aligned}$$

Similarly as before, we get

$$T(x, f) = \int_{(B_{R_0\sigma(x,f)}(x))^c \cap B_r(x_0)} f\, dV_g \le \int_{B_\varepsilon(x_0)^c \cap B_r(x_0)} f\, dV_g \le \varepsilon.$$

Being τ-universal, ε can be taken so small that $(T(x, f) - \tau)^+ = 0$ outside $B_{2\varepsilon}(x_0)$ for all $f \in \mathscr{C}_{\varepsilon,r}(x_0)$.

By the Nash embedding theorem, we can assume that $\Sigma \subset \mathbb{R}^N$ isometrically, $N \in \mathbb{N}$. Take an open tubular neighborhood $\Sigma \subset U \subset \mathbb{R}^N$ of Σ and $\delta > 0$ small enough so that

$$\mathrm{co}[B_x((R_0 + 1)\delta) \cap \Sigma] \subset U \quad \text{for all } x \in \Sigma, \tag{18}$$

where co denotes the convex hull in $\mathbb{R}^N$.

For $f \in \mathscr{C}_{\varepsilon,r}(x_0)$, we define now

$$\eta(f) = \frac{\int_\Sigma (T(x, f) - \tau)^+ (\sigma(f) - \sigma(x, f))^+ x\, dV_g}{\int_\Sigma (T(x, f) - \tau)^+ (\sigma(f) - \sigma(x, f))^+\, dV_g} \in \mathbb{R}^N,$$

which is well-defined; see Remark 2.8. The map η yields a sort of center of mass in $\mathbb{R}^N$ of the measure induced by f. Observe that the integrands become nonzero only on the set $S(f)$. However, whenever $\sigma(f) \le \delta$, (16) and (18) imply that $\eta(f) \in U$, and so we can define

$$\beta : \{f \in \mathscr{A}_{x_0,r} : \sigma(f) \le \delta\} \to \Sigma, \quad \beta(f) = P \circ \eta(f),$$

where $P : U \to \Sigma$ is the orthogonal projection.

We finally define the map $\psi : \mathscr{C}_{\varepsilon,r}(x_0) \to \Sigma \times (0, r)$, which will be the main tool of this subsection:

$$\psi(f) = (\beta, \sigma). \tag{19}$$

Roughly, this map expresses the center of mass of f and its scale of concentration around this point.

Malchiodi and Ruiz [2013] proved that, if both components (u_1, u_2) of the Toda system concentrate around the same point in Σ, with the same scale of concentration, then the constants in the left-hand side of (4) can be nearly doubled.

Remark 2.9. The core of the argument of the improved inequality in [Malchiodi and Ruiz 2013] consists of proving that

$$\psi(\mathscr{N}(e^{u_1}, B_r(x))) = \psi(\mathscr{N}(e^{u_2}, B_r(y)))$$

implies the existence of $\sigma > 0$ and of two balls $B_\sigma(z_1)$ and $B_\sigma(z_2)$ such that

$$\frac{\int_{B_\sigma(z_i)} e^{u_i}\, dV_g}{\int_\Sigma e^{u_i}\, dV_g} \ge \gamma_0, \quad \frac{\int_{(B_{R\sigma}(z_i))^c \cap B_r(z_i)} e^{u_i}\, dV_g}{\int_\Sigma e^{u_i}\, dV_g} \ge \gamma_0 \quad \text{for } i = 1, 2 \text{ with } d(z_1, z_2) \lesssim \sigma \tag{20}$$

for some fixed positive constant γ_0. Once this is achieved, the improved inequality is obtained by scaling arguments and Kelvin inversions (see Section 3 in [Malchiodi and Ruiz 2013] for full details).

Even when e^{u_1} and e^{u_2} are not necessarily concentrated near a single point, the assumptions of the next proposition still allow us to obtain (20) and hence again nearly double constants in the left-hand side of (4).

Proposition 2.10 ([Malchiodi and Ruiz 2013] with minor changes). *Let $\tilde{\varepsilon} > 0$ and $\delta' > 0$. Then there exist $R = R(\tilde{\varepsilon})$ and ψ as in definition* (19) *such that, for any $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ such that there exist $x, y \in \Sigma$ with*

$$\int_{B_r(x)} e^{u_1}\, dV_g \geq \delta' \int_\Sigma e^{u_1} dV_g, \qquad \int_{B_r(y)} e^{u_2}\, dV_g \geq \delta' \int_\Sigma e^{u_1}\, dV_g,$$
$$\mathcal{N}(e^{u_1}, B_r(x)) \in \mathscr{C}_{\varepsilon,r}(x), \qquad \mathcal{N}(e^{u_2}, B_r(y)) \in \mathscr{C}_{\varepsilon,r}(y)$$

and

$$\psi(\mathcal{N}(e^{u_1}, B_r(x))) = \psi(\mathcal{N}(e^{u_2}, B_r(y))), \tag{21}$$

the following inequality holds:

$$8\pi\left(\log \int_\Sigma e^{u_1-\bar{u}_1}\, dV_g + \log \int_\Sigma e^{u_2-\bar{u}_2}\, dV_g\right) \leq (1+\tilde{\varepsilon}) \int_\Sigma Q(u_1, u_2)\, dV_g + C \tag{22}$$

for some $C = C(\tilde{\varepsilon}, \delta', \Sigma)$.

Remark 2.11. (i) Condition (21) can be relaxed. In fact, let $C_1 > 1$ and $C_2 > 0$ be two positive constants and define

$$\psi(\mathcal{N}(e^{u_1}, B_r(x))) = (\beta_1, \sigma_1), \qquad \psi(\mathcal{N}(e^{u_2}, B_r(y))) = (\beta_2, \sigma_2).$$

Then, the result still holds true if

$$\frac{1}{C_1} \leq \frac{\sigma_1}{\sigma_2} \leq C_1, \qquad d(\beta_1, \beta_2) \leq C_2\sigma_1.$$

In such a case, the constant C would also depend on C_1 and C_2.

(ii) In the right-hand side of (22), one can actually integrate $Q(u_1, u_2)$ only in any set compactly containing $B_r(x) \cup B_r(y)$. This can be seen using suitable cut-off functions; see the comments before Lemma 2.2.

We can now improve this result for situations in which the first component of the system is concentrated around l points of Σ, $l \in \mathbb{N}$. The proof relies on combining the argument for Proposition 2.10 with the macroscopic improved inequality of Lemma 2.2 (see also Remark 2.11(ii)).

Proposition 2.12. *Let $\tilde{\varepsilon} > 0$, $\delta' > 0$, and $k \in \mathbb{N}$. Then there exist $R = R(\tilde{\varepsilon})$ and ψ as in definition* (19) *such that, for any $(u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma)$ with the property that there exist $\{x_i\}_{i\in\{1,\dots,k\}} \subset \Sigma$ and $y \in \Sigma$ with*

$$d(x_i, x_j) > 4\delta' \quad \text{for all } i, j \in \{1, \dots, k\} \text{ with } i \neq j,$$
$$\int_{B_{\delta'}(x_i)} e^{u_1}\, dV_g \geq \delta' \int_\Sigma e^{u_1}\, dV_g \quad \text{for } i = 1, \dots, k, \qquad \int_{B_{\delta'}(y)} e^{u_2}\, dV_g \geq \delta' \int_\Sigma e^{u_2}\, dV_g$$

such that

$$\mathcal{N}(e^{u_1}, B_{\delta'}(x_i)) \in \mathscr{C}_{\varepsilon,\delta'}(x_i) \quad \text{for } i = 1, \dots, k, \qquad \mathcal{N}(e^{u_2}, B_{\delta'}(y)) \in \mathscr{C}_{\varepsilon,\delta'}(y)$$

and

$$\psi(\mathcal{N}(e^{u_1}, B_{\delta'}(x_l))) = \psi(\mathcal{N}(e^{u_2}, B_{\delta'}(y))) \quad \textit{for some } l \in \{1, \dots, k\},$$

the following inequality holds:

$$4\pi(k+1)\log \int_\Sigma e^{u_1 - \bar{u}_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2 - \bar{u}_2}\, dV_g \le (1+\tilde{\varepsilon}) \int_\Sigma Q(u_1, u_2)\, dV_g + C$$

for some $C = C(\tilde{\varepsilon}, \delta', l, \Sigma)$.

In the next section, we will derive a new improved inequality for the Toda system with scaling-invariant features; see Proposition 3.5. The result is inspired by arguments developed in [Bartolucci and Malchiodi 2013] for the singular Liouville equation where a Dirac delta is involved (see Remark 3.6), and for the first time, this type of inequality is presented for a two-component problem.

3. A refined projection onto the topological join

Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$. By Proposition 2.5, we have the existence of a map Ψ from the low sublevels of J_ρ onto the topological join $\Sigma_k * \Sigma_1$; see (8) and (9). However, we will next need to also take into account the fine structure of the measures e^{u_1} and e^{u_2} as described in (19). For this reason, we will modify the map Ψ so that the join parameter s in (9) will depend on the local centers of mass and the local scales defined in (19) and (23). We will see in the sequel that this will provide extra information for describing functions in the low sublevels of J_ρ.

3.1. ***Construction.*** We start by defining the local centers of mass and the local scales of functions that are concentrated around l well-separated points of Σ.

Let $l \ge 2$, consider $0 < \varepsilon_l \ll \varepsilon_{l-1} \ll 1$ as given in Proposition 2.4, and suppose $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_l) < 2\varepsilon_l$ so that ψ_l is well-defined. Assume moreover $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{l-1}) > \varepsilon_{l-1}$. By the second part of Proposition 2.4, there exist $\varepsilon'_{l-1} \ll \varepsilon_{l-1}$ and l points $x_1^l, \dots, x_l^l$ such that

$$d(x_i^l, x_j^l) > 2\varepsilon'_{l-1} \quad \text{for } i \ne j, \qquad \int_{B_{\varepsilon'_{l-1}}(x_i^l)} e^{u_1}\, dV_g > \varepsilon'_{l-1} \int_\Sigma e^{u_1}\, dV_g \quad \text{for all } i = 1, \dots, l.$$

We then localize u_1 around the point x_i^l and define

$$f_{\mathrm{loc}}^{x_i^l}(u_1) = \frac{e^{u_1} \chi_{B_{\varepsilon'_{l-1}}(x_i^l)}}{\int_{B_{\varepsilon'_{l-1}}(x_i^l)} e^{u_1}\, dV_g}.$$

Given $\varepsilon > 0$, by the second assertion of Proposition 2.4, taking ε_l sufficiently small, one gets

$$\int_{B_\varepsilon(x_i^l)} f_{\mathrm{loc}}^{x_i^l}(u_1)\, dV_g > 1 - \varepsilon \quad \text{for } \boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_l) < 2\varepsilon_l.$$

It follows that $f_{\mathrm{loc}}^{x_i^l}(u_1) \in \mathscr{C}_{\varepsilon, \varepsilon'_{l-1}}(x_i^l)$ (see (17)), and hence, the map ψ in (19) is well-defined on $f_{\mathrm{loc}}^{x_i^l}(u_1)$. We then set

$$\big(\beta_{x_i^l}, \sigma_{x_i^l}\big) := \psi\big(f_{\mathrm{loc}}^{x_i^l}(u_1)\big). \tag{23}$$

In this way, starting from a function with $\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_l)<2\varepsilon_l$ and such that $\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_{l-1})>\varepsilon_{l-1}$, we obtain, around each point x_i^l, a notion of local center of mass and scale of concentration.

When $l=1$, we have to deal with just one point x_1^1 of Σ. We then apply the map ψ to the function $f_{\mathrm{loc}}^{x_1^1}$ directly.

As we discussed above, we would like to map low-energy sublevels of J_ρ into the topological join $\Sigma_k*\Sigma_1$ taking the above scales into account. More precisely, the parameter s in (9) will depend on the local scale $\sigma_{x_i^l}$ only of the points near the center of mass of e^{u_2} (in case of ambiguity, we will define a sort of averaged scale).

To proceed rigorously, let $0<\varepsilon_k\ll\varepsilon_{k-1}\ll\cdots\ll\varepsilon_1\ll 1$ be as before. We consider cut-off functions $\mathfrak{f}$, $\mathfrak{g}_l$, and $\mathfrak{h}$ for $l=1,\dots,k-1$ such that

$$\mathfrak{f}(t)=\begin{cases}0, & t\geq 2\varepsilon_k,\\ 1, & t\leq\varepsilon_k,\end{cases}$$

$$\mathfrak{g}_l(t)=\begin{cases}0, & t\geq 2\varepsilon_l,\\ 1, & t\leq\varepsilon_l,\end{cases}\qquad l=1,\dots,k-1, \tag{24}$$

$$\mathfrak{h}(t)=\begin{cases}0, & t\geq\frac{1}{8}\varepsilon'_{k-1},\\ 1, & t\leq\frac{1}{16}\varepsilon'_{k-1}.\end{cases} \tag{25}$$

We define now a *global scale* $\sigma_1(u_1)\in(0,1]$ for e^{u_1} in three steps. Suppose $\boldsymbol{d}(\mathcal{N}(e^{u_2},\Sigma),\Sigma_1)<2\varepsilon_1$ so that $\psi(f_{\mathrm{loc}}^z(u_2))=(\beta_z,\sigma_z)$ is well-defined.

First of all, we define an *averaged scale* for e^{u_1} by recurrence in the following way. If we have $\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_1)<2\varepsilon_1$, we set $C_1(u_1)=\sigma_{x_1^1}$. For $l\in\{2,\dots,k-1\}$, we define recursively

$$C_l(u_1)=\mathfrak{g}_{l-1}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_{l-1})\big)C_{l-1}(u_1)+\big(1-\mathfrak{g}_{l-1}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_{l-1})\big)\big)\frac{1}{l}\sum_{i=1}^{l}\sigma_{x_i}^{l}.$$

Secondly, we interpolate between $C_{k-1}(u_1)$ and the local scale of the closest point to β_z among the $\beta_{x_i^k}$ (provided they are well-defined), setting

$$B(u_1,u_2)=\mathfrak{h}\big(d\big(\beta_z,\{\beta_{x_1^k},\dots,\beta_{x_k^k}\}\big)\big)\sigma_x+\big(1-\mathfrak{h}\big(d\big(\beta_z,\{\beta_{x_1^k},\dots,\beta_{x_k^k}\}\big)\big)\big)\frac{1}{k}\sum_{i=1}^{k}\sigma_{x_i^k},$$

$$A(u_1,u_2)=\mathfrak{g}_{k-1}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_{k-1})\big)C_{k-1}(u_1)+\big(1-\mathfrak{g}_{k-1}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_{k-1})\big)\big)B(u_1,u_2),$$

where $x=x_j^k$ was chosen so that it realizes the minimum of $d(\beta_z,\{\beta_{x_1^k},\dots,\beta_{x_k^k}\})$: notice that, since $d(x_j^k,x_l^k)\geq 2\varepsilon'_{k-1}$ for $j\neq l$, by (25) the point realizing the latter minimum is unique if $\mathfrak{h}\neq 0$.

As a third and final step, to check whether e^{u_1} is $\boldsymbol{d}$-close to Σ_k, we set

$$\sigma_1(u_1)=\mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k)\big)A(u_1,u_2)+\big(1-\mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k)\big)\big).$$

We define next the *global scale* $\sigma_2(u_2)\in(0,1]$ of e^{u_2}. We will be interested here in functions concentrated near just one point of Σ. Therefore, we just need the single local scale $C_1(u_2)=\sigma_z$ if $\psi(f_{\mathrm{loc}}^z(u_2))=(\beta_z,\sigma_z)$ is well-defined. Moreover, we have to check the $\boldsymbol{d}$-closeness of e^{u_2} to Σ_1. Hence, the scale reads

$$\sigma_2(u_2) = \mathfrak{g}_1\big(\boldsymbol{d}\big(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1\big)\big)\sigma_z + \big(1 - \mathfrak{g}_1\big(\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1)\big)\big).$$

We can now specify the join parameter s in (9). Fix a constant $M \gg 1$, and consider the function

$$F_M(t) = \begin{cases} 0, & t \le 1/M, \\ \dfrac{t}{1+t}, & t \in [2/M, M], \\ 1, & t \ge 2M. \end{cases}$$

We then define

$$s(u_1, u_2) = F_M\left(\frac{\sigma_1(u_1)}{\sigma_2(u_2)}\right). \tag{26}$$

We now pass to considering the maps ψ_k and ψ_1 that are needed in the projection onto the join $\Sigma_k * \Sigma_1$; see (12). As mentioned in the introduction of this section, it is convenient to modify these maps in such a way that they take into account the local centers of mass defined in (19) and (23). More precisely, when e^{u_1} is concentrated in k well-separated points of Σ, we would rather consider the local centers of mass $\beta_{x_i^l}$ in (23) than the supports of the map ψ_k in Proposition 2.4.

Suppose $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) < 2\varepsilon_k$ so that ψ_k is well-defined, and suppose $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) > \varepsilon_{k-1}$ so that $\beta_{x_i^k}$ are defined for $i = 1, \dots, k$. Let

$$\psi_k(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) = \sum_{i=1}^{k} t_i \delta_{y_i}, \quad t_i \in [0, 1], \ y_i \in \Sigma.$$

Observe that, by construction and by the second statement in Proposition 2.4, $d(\beta_{x_i^k}, y_i) \to 0$ as $\varepsilon_k \to 0$. Hence, there exists a geodesic γ_i joining y_i and $\beta_{x_i^k}$ in unit time. We then perform an interpolation:

$$\widetilde{\psi}_k(\mathcal{N}(e^{u_1}, \Sigma)) = \begin{cases} \sum_{i=1}^k t_i \delta_{y_i} & \text{if } \boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) \le \varepsilon_{k-1}, \\ \sum_{i=1}^k t_i \delta_{\gamma_i(\varepsilon_{k-1}^{-1}\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1})-1)} & \text{if } \boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) \in (\varepsilon_{k-1}, 2\varepsilon_{k-1}), \\ \sum_{i=1}^k t_i \delta_{\beta_{x_i^k}} & \text{if } \boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_{k-1}) \ge 2\varepsilon_{k-1}. \end{cases} \tag{27}$$

For a function u_2 with $\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1) < 2\varepsilon_1$, letting $\psi_1(\mathcal{N}(e^{u_2}, \Sigma)) = \delta_z$, we let

$$\widetilde{\psi}_1(\mathcal{N}(e^{u_2}, \Sigma)) = \delta_{\beta_z}. \tag{28}$$

With these maps and this join parameter, we finally define the refined projection $\widetilde{\Psi} : J_\rho^{-L} \to \Sigma_k * \Sigma_1$ as

$$\widetilde{\Psi}(u_1, u_2) = (1 - s)\widetilde{\psi}_k(\mathcal{N}(e^{u_1}, \Sigma)) + s\widetilde{\psi}_1(\mathcal{N}(e^{u_2}, \Sigma)). \tag{29}$$

3.2. ***A new improved Moser–Trudinger inequality.*** Using the improved geometric inequality in [Bartolucci and Malchiodi 2013] for the singular Liouville equation, we can provide a dilation-invariant improved inequality for system (1). Before stating the main result, we prove some auxiliary lemmas; we first recall our notation on annuli at the end of Section 1.

Lemma 3.1. *Let $\gamma_0 > 0$, $\tau_0 > 0$, $z \in \Sigma$, and $r_2 > r_1 > 0$ (both small) be such that*

$$\frac{\int_{A_z(r_1,r_2)} e^{u_2}\,dV_g}{\int_\Sigma e^{u_2}\,dV_g} > \gamma_0 \quad \textit{and} \quad \sup_{y\in A_z(r_1,r_2)} \frac{\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\,dV_g}{\int_{A_z(r_1,r_2)} e^{u_2}\,dV_g} < 1-\tau_0. \tag{30}$$

Then for any $\varepsilon > 0$, there exist $C = C(\varepsilon, \tau_0, \gamma_0)$, $\tilde\tau_0 = \tilde\tau_0(\tau_0, \gamma_0)$, $\tilde r_1 \in [r_1/C, r_1/4]$, $\tilde r_2 \in [4r_2, Cr_2]$, and $\tilde u_2 \in H^1(\Sigma)$ such that

(a) *$\tilde u_2$ is constant in $B_{\tilde r_1}(z)$ and on $\partial B_{\tilde r_2}(z)$,*

(b) $$\int_{A_z(\tilde r_1,\tilde r_2)} |\nabla \tilde u_2|^2\,dV_g \le \int_{A_z(\tilde r_1,\tilde r_2)} |\nabla u_2|^2\,dV_g + \varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g,$$

(c) $$\sup_{y\in A_z(\tilde r_1,\tilde r_2)} \frac{\int_{B_{\tilde\tau_0 d(y,z)}(y)} e^{\tilde u_2}\,dV_g}{\int_{A_z(\tilde r_1,\tilde r_2)} e^{\tilde u_2}\,dV_g} < 1-\tilde\tau_0.$$

Proof. First of all, we modify u_2 so that it becomes constant in $B_{\tilde r_1}(z)$ and on $\partial B_{\tilde r_2}(z)$. Take $\varepsilon > 0$: we can find $C = C(\varepsilon)$ and properly chosen $\tilde r_1 \in [r_1/C, r_1/4]$ and $\tilde r_2 \in [4r_2, Cr_2]$ such that

$$\int_{A_z(\tilde r_1,2\tilde r_1)} |\nabla u_2|^2\,dV_g \le \varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g, \qquad \int_{A_z(\tilde r_2/2,\tilde r_2)} |\nabla u_2|^2\,dV_g \le \varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g.$$

We denote by $\bar u_2(\tilde r_1)$ and $\bar u_2(\tilde r_2)$ the averages

$$\bar u_2(\tilde r_1) = \fint_{A_z(\tilde r_1,2\tilde r_1)} u_2\,dV_g, \qquad \bar u_2(\tilde r_2) = \fint_{A_z(\tilde r_2/2,\tilde r_2)} u_2\,dV_g. \tag{31}$$

Now let χ be a cut-off function, with values in [0, 1], such that

$$\chi = \begin{cases} 0 & \text{in } B_{\tilde r_1}(z), \\ 1 & \text{in } A_z(2\tilde r_1, \tilde r_2/2), \\ 0 & \text{in } (B_{\tilde r_2}(z))^c, \end{cases}$$

and define

$$\tilde u_2 = \begin{cases} \chi(d(x,z))u_2 + (1-\chi(d(x,z))\bar u_2(\tilde r_1)) & \text{in } B_{2\tilde r_1}(z), \\ u_2 & \text{in } A_z(2\tilde r_1, \tilde r_2/2), \\ \chi(d(x,z))u_2 + (1-\chi(d(x,z))\bar u_2(\tilde r_2)) & \text{in } (B_{\tilde r_2/2}(z))^c. \end{cases} \tag{32}$$

By Poincaré's inequality, the Dirichlet energy of $\tilde u_2$ is bounded by

$$\int_{A_z(\tilde r_1,2\tilde r_1)} |\nabla \tilde u_2|^2\,dV_g \le \widetilde C\varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g, \qquad \int_{A_z(\tilde r_2/2,\tilde r_2)} |\nabla \tilde u_2|^2\,dV_g \le \widetilde C\varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g,$$

where $\widetilde C$ is a universal constant. Hence, one gets

$$\int_{A_z(\tilde r_1,\tilde r_2)} |\nabla \tilde u_2|^2\,dV_g \le \int_{A_z(\tilde r_1,\tilde r_2)} |\nabla u_2|^2\,dV_g + 2\widetilde C\varepsilon\int_\Sigma |\nabla u_2|^2\,dV_g.$$

We are left with proving that there exists $\tilde\tau_0 = \tilde\tau_0(\tau_0, \gamma_0)$ such that

$$\sup_{y\in A_z(\tilde r_1,\tilde r_2)} \frac{\int_{B_{\tilde\tau_0 d(y,z)}(y)} e^{\tilde u_2}\,dV_g}{\int_{A_z(\tilde r_1,\tilde r_2)} e^{\tilde u_2}\,dV_g} < 1-\tilde\tau_0. \tag{33}$$

If this isn't the case, there exist $(u_{2,n})_n \subset H^1(\Sigma)$ satisfying (30), $(\tilde r_{1,n})_n \subset [r_1/C, r_1/4]$, $(\tilde r_{2,n})_n \subset [4r_2, Cr_2]$, and cut-off functions $(\chi_n)_n$ and $(\tilde u_{2,n})_n \subset H^1(\Sigma)$ defined analogously to $\tilde u_2$ in (32) such that

$$\frac{e^{\tilde u_{2,n}}}{\int_{A_z(\tilde r_{1,n},\tilde r_{2,n})} e^{\tilde u_{2,n}}\, dV_g} \rightharpoonup \delta_{\bar x} \tag{34}$$

in the sense of measures for some $\bar x \in A_z(r_1/C, Cr_2)$. We distinguish between three situations.

Case 1. Suppose first that $\bar x \in A_z(r_1, 2r_2)$. By the choices of the cut-off functions and (32), as $\tilde u_{2,n}$ coincides with $u_{2,n}$ on $A_z(r_1/2, 2r_2)$, it follows that

$$\frac{e^{u_{2,n}}}{\int_{A_z(r_1,2r_2)} e^{u_{2,n}}\, dV_g} = \frac{e^{\tilde u_{2,n}}}{\int_{A_z(r_1,2r_2)} e^{\tilde u_{2,n}}\, dV_g} \rightharpoonup \delta_{\bar x}. \tag{35}$$

Case 1.1. Let $\bar x \in A_z(r_1, \frac{3}{2}r_2)$. To get a contradiction to (35), we prove that there exists $\bar\tau_0 = \bar\tau_0(\tau_0, \gamma_0)$ such that

$$\sup_{y\in A_z(r_1,(3/2)r_2)} \int_{B_{\bar\tau_0 d(y,z)}(y)} e^{u_{2,n}}\, dV_g \le (1-\bar\tau_0) \int_{A_z(r_1,2r_2)} e^{u_{2,n}}\, dV_g. \tag{36}$$

Let $\bar\tau_0 = \tau_0/2$. If $B_{\bar\tau_0 d(y,z)}(y) \subseteq A_z(r_1(1-\tau_0), r_2(1+\tau_0))$, we can use directly the second part of the assumption (30) on $u_{2,n}$ to get the bound on the left-hand side of (36) (taking $\bar\tau_0$ sufficiently small). Moreover, by the first part of (30) on $u_{2,n}$, we deduce

$$\int_{A_z(r_1,r_2)} e^{u_{2,n}}\, dV_g \ge \gamma_0 \int_\Sigma e^{u_{2,n}}\, dV_g \ge \gamma_0 \int_{A_z(r_1,2r_2)} e^{u_{2,n}}\, dV_g.$$

Given then $B_r(y) \subseteq A_z(r_2, 2r_2)$, since $B_r(y) \cap A_z(r_1, r_2) = \varnothing$, by the first inequality in (30),

$$\int_{B_r(y)} e^{u_{2,n}}\, dV_g \le (1-\gamma_0) \int_{A_z(r_1,2r_2)} e^{u_{2,n}}\, dV_g \quad \text{for any } B_r(y) \subseteq A_z(r_2, 2r_2). \tag{37}$$

Now if $B_{\bar\tau_0 d(y,z)}(y) \subseteq A_z(r_2, 2r_2)$, we exploit (37) to deduce the bound on the left-hand side of (36) taking a possibly smaller $\bar\tau_0$. This concludes the proof of the claim (36).

Case 1.2. Suppose $\bar x \in A_z(\frac{5}{4}r_2, 2r_2)$. Using again (37), we obtain a contradiction to (35).

Case 2. Consider now $\bar x \in A_z(r_1/2, r_2)$: reasoning exactly as in Case 1, we get a contradiction.

Case 3. We are left with the case $\bar x \in (A_z(r_1/2, 2r_2))^c$: notice that, differently from the previous two cases, the cut-off functions χ_n might not be identically equal to 1 near $\bar x_0$. For this choice of $\bar x$ and by (34),

$$\frac{\int_{A_z(r_1,r_2)} e^{\tilde u_{2,n}}\, dV_g}{\int_{A_z(\tilde r_{1,n},\tilde r_{2,n})} e^{\tilde u_{2,n}}\, dV_g} \to 0. \tag{38}$$

Using the definition of $\tilde u_{2,n}$ in $A_z(\tilde r_{2,n}/2, \tilde r_{2,n})$ given by (32) and applying Young's inequality with $1/p = \chi_n$ and $1/q = 1-\chi_n$, we have

$$e^{\tilde u_{2,n}} = e^{\chi_n u_{2,n}} e^{(1-\chi_n)\bar u_2(\tilde r_{2,n})} \le \chi_n e^{u_{2,n}} + (1-\chi_n) e^{\bar u_{2,n}(\tilde r_{2,n})} \quad \text{in } A_z(\tilde r_{2,n}/2, \tilde r_{2,n}). \tag{39}$$

Recall the notation in (31): by Jensen's inequality, it follows that

$$e^{\bar{u}_{2,n}(\tilde{r}_{2,n})} \le \fint_{A_z(\tilde{r}_{2,n}/2,\tilde{r}_{2,n})} e^{u_{2,n}}\,dV_g.$$

Therefore, integrating (39), one can show that

$$\int_{A_z(\tilde{r}_{2,n}/2,\tilde{r}_{2,n})} e^{\tilde{u}_{2,n}}\,dV_g \le 2\int_{A_z(\tilde{r}_{2,n}/2,\tilde{r}_{2,n})} e^{u_{2,n}}\,dV_g.$$

Similarly, we get

$$\int_{A_z(\tilde{r}_{1,n},2\tilde{r}_{1,n})} e^{\tilde{u}_{2,n}}\,dV_g \le 2\int_{A_z(\tilde{r}_{1,n},2\tilde{r}_{1,n})} e^{u_{2,n}}\,dV_g.$$

In conclusion, we have

$$\int_{A_z(\tilde{r}_{1,n},\tilde{r}_{2,n})} e^{\tilde{u}_{2,n}}\,dV_g \le 2\int_{\Sigma} e^{u_{2,n}}\,dV_g.$$

This, together with (38), implies that

$$\frac{\int_{A_z(r_1,r_2)} e^{u_{2,n}}\,dV_g}{\int_\Sigma e^{u_{2,n}}\,dV_g} \le 2\frac{\int_{A_z(r_1,r_2)} e^{\tilde{u}_{2,n}}\,dV_g}{\int_{A_z(\tilde{r}_{1,n},\tilde{r}_{2,n})} e^{\tilde{u}_{2,n}}\,dV_g} \to 0,$$

which is in contradiction with (30). Therefore we are done. □

Lemma 3.2. *Under the same assumptions of Lemma 3.1, let $\tilde{u}_2 \in H^1(\Sigma)$ be the function given there. Then property* (c) *can be extended to the following*: *there exists $\bar{\tau}_0 > 0$ such that*

$$\sup_{y\in B_{\tilde{r}_2}(z),\, y\neq z} \frac{\int_{B_{\tilde{\tau}_0 d(y,z)}(y)} e^{\tilde{u}_2}\,dV_g}{\int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\,dV_g} < 1-\bar{\tau}_0. \tag{40}$$

Proof. By property (c) of Lemma 3.1, we just have to show (40) for $y \in B_{\tilde{r}_1}(z)$. Observe that, by definition, $\tilde{u}_2$ is constant in $B_{\tilde{r}_1}(z)$. Therefore, for any $B_{\tilde{\tau}_0 d(y,z)}(y) \subseteq B_{\tilde{r}_1}(z)$, which implies $d(y,z) \le \tilde{r}_1$, we have

$$\int_{B_{\tilde{\tau}_0 d(y,z)}(y)} e^{\tilde{u}_2}\,dV_g = \frac{\tilde{\tau}_0^2 d(y,z)^2}{\tilde{r}_1^2}\int_{B_{\tilde{r}_1}(z)} e^{\tilde{u}_2}\,dV_g \le \tilde{\tau}_0^2\int_{B_{\tilde{r}_1}(z)} e^{\tilde{u}_2}\,dV_g \le \tilde{\tau}_0^2\int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\,dV_g,$$

and we conclude that (40) holds for $\tilde{\tau}_0$ small enough. For the same choice of $\tilde{\tau}_0$, we are left with the case $B := B_{\tilde{\tau}_0 d(y,z)}(y) \cap (B_{\tilde{r}_1}(z))^c \neq \varnothing$. The integral over B will be bounded by the integral over a larger ball with center shifted onto $\partial B_{\tilde{r}_1}(z)$. Using normal coordinates at z, consider the shift of center $y \mapsto \tilde{r}_1 y/d(y,z)$. Then we have, using the property (c),

$$\int_B e^{\tilde{u}_2}\,dV_g \le \int_{B_{\tilde{\tau}_0\tilde{r}_1}(\tilde{r}_1 y/d(y,z))} e^{\tilde{u}_2}\,dV_g \le (1-\tilde{\tau}_0)\int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\,dV_g.$$

Therefore, we get

$$\int_{B_{\tilde{\tau}_0 d(y,z)(y)}} e^{\tilde{u}_2}\,dV_g \le \tilde{\tau}_0^2\int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\,dV_g + \int_B e^{\tilde{u}_2} dV_g \le \tilde{\tau}_0^2\int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\,dV_g + (1-\tilde{\tau}_0)\int_{B_{\tilde{r}_2}(z)} e^{\tilde{u}_2}\,dV_g.$$

Taking $\bar{\tau}_0$ possibly smaller, we obtain the conclusion. □

We recall here an improved geometric inequality with $k = 1$ and $\alpha = 1$.

Proposition 3.3 [Bartolucci and Malchiodi 2013, Proposition 4.1]. *Let $p \in \Sigma$, and let $r > 0$ and $\tau_0 > 0$. Then for any $\varepsilon > 0$, there exists $C = C(\varepsilon, r)$ such that*

$$\log \int_{B_r(p)} d(x, p)^2 e^{2v} \, dV_g \le \frac{1+\varepsilon}{8\pi} \int_{B_r(p)} |\nabla v|^2 \, dV_g + C$$

for every function $v \in H_0^1(B_r(p))$ such that

$$\sup_{y \in B_r(p),\, y \neq p} \frac{\int_{B_{\tau_0 d(y,p)}(y)} d(x, p)^2 e^{2v} \, dV_g}{\int_{B_r(p)} d(x, p)^2 e^{2v} \, dV_g} < 1 - \tau_0.$$

We now state the new improved Moser–Trudinger inequality.

Remark 3.4. In what follows, the number r is supposed to be small but not tending to 0 while σ could be arbitrarily small.

Proposition 3.5. *Let $r > 0$, $\gamma_0 > 0$, and $\tau_0 > 0$. For any $\varepsilon > 0$, there exists $C = C(\varepsilon, r, \tau_0, \gamma_0)$ such that, if for some $\sigma \in (0, r/C^2)$ and $z \in \Sigma$*

$$\frac{\int_{B_{\sigma/2}(z)} e^{u_1} \, dV_g}{\int_\Sigma e^{u_1} \, dV_g} > \gamma_0, \qquad \frac{\int_{A_z(C\sigma, r/C)} e^{u_2} \, dV_g}{\int_\Sigma e^{u_2} \, dV_g} > \gamma_0 \tag{41}$$

and

$$\sup_{y \in A_z(C\sigma, r/C)} \frac{\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2} \, dV_g}{\int_{A_z(C\sigma, r/C)} e^{u_2} \, dV_g} < 1 - \tau_0, \tag{42}$$

then

$$4\pi \log \int_\Sigma e^{u_1 - \bar{u}_1} \, dV_g + 8\pi \log \int_\Sigma e^{u_2 - \bar{u}_2} \, dV_g \le \int_{B_r(z)} Q(u_1, u_2) \, dV_g + \varepsilon \int_\Sigma Q(u_1, u_2) \, dV_g + C.$$

Proof. Taking r sufficiently small, we may suppose that we have the Euclidean flat metric in the ball $B_{Cr}(z)$. Suppose for simplicity that $\bar{u}_1 = \bar{u}_2 = 0$ and that $z = 0$. Observe that we can write

$$\log \int_{B_r(0)} e^{u_2} \, dV_g = \log \int_{B_r(0)} |x|^2 e^{2(u_2/2 - \log|x|)} \, dV_g.$$

We wish to apply Proposition 3.3 to $u_2/2 - \log|x|$, so we need to modify this function in such a way that it becomes constant outside a given ball. Moreover, it will be useful to also replace it with a constant inside a smaller ball. In this process, we should not lose the volume-spreading property (42). By Lemma 3.1, this can be done, and we let $C = C(\varepsilon, \tau_0, \gamma_0)$, $\tilde{r}_1 \in [\sigma, C\sigma/4]$, $\tilde{r}_2 \in [4r/C, r]$, and $\tilde{u}_2 \in H^1(\Sigma)$ be as in the statement of the lemma. By property (a) in Lemma 3.1 and by Lemma 3.2, we are in position to apply

Proposition 3.3 to $(\tilde{u}_2 - \tilde{u}_2(\tilde{r}_2)) \in H_0^1(B_{\tilde{r}_2}(0))$ and get

$$\begin{aligned}
\log \int_\Sigma e^{u_2}\, dV_g &\le \log \int_{A_0(C\sigma, r/C)} e^{u_2}\, dV_g + C = \log \int_{A_0(C\sigma, r/C)} |x|^2 e^{2(u_2/2 - \log|x|)}\, dV_g + C \\
&\le \log \int_{B_{\tilde{r}_2}(0)} |x|^2 e^{2\tilde{u}_2}\, dV_g + C = \log \int_{B_{\tilde{r}_2}(0)} |x|^2 e^{2(\tilde{u}_2 - \tilde{u}_2(\tilde{r}_2))}\, dV_g + \tilde{u}_2(\tilde{r}_2) + C \\
&\le \frac{1+\varepsilon}{8\pi} \int_{A_0(\tilde{r}_1, \tilde{r}_2)} |\nabla \tilde{u}_2|^2\, dV_g + \tilde{u}_2(\tilde{r}_2) + C \\
&\le \frac{1+\varepsilon}{8\pi} \int_{A_0(\tilde{r}_1, \tilde{r}_2)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2\, dV_g + \varepsilon \int_\Sigma |\nabla u_2|^2\, dV_g + \tilde{u}_2(\tilde{r}_2) + C \\
&\le \frac{1}{8\pi} \int_{A_0(\sigma, r)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2\, dV_g + \varepsilon \int_\Sigma Q(u_1, u_2)\, dV_g + \tilde{u}_2(\tilde{r}_2) + C, \qquad (43)
\end{aligned}$$

where in the first row we exploited (41) while in the last one we used the definitions of $\tilde{r}_1$ and $\tilde{r}_2$. Observe that by the definition (32) of $\tilde{u}_2$ we have

$$\tilde{u}_2(\tilde{r}_2) = \fint_{A_z(\tilde{r}_2/2, \tilde{r}_2)} (\tfrac{1}{2}u_2 - \log|x|)\, dV_g.$$

Applying Hölder's and Poincaré's inequalities, one gets

$$\begin{aligned}
\fint_{A_z(\tilde{r}_2/2, \tilde{r}_2)} (\tfrac{1}{2}u_2 - \log|x|)\, dV_g &\le \fint_{A_z(\tilde{r}_2/2, \tilde{r}_2)} |u_2|\, dV_g + \widetilde{C}_r \le C_r \|u_2\|_{L^2(\Sigma)} + \widetilde{C}_r \\
&\le C_r \left(\int_\Sigma |\nabla u_2|^2\, dV_g \right)^{1/2} + \widetilde{C}_r \le \varepsilon \int_\Sigma |\nabla u_2|^2\, dV_g + \frac{\widetilde{C}_r C_r}{\varepsilon}. \qquad (44)
\end{aligned}$$

Inserting the latter estimate into (43), we deduce

$$\log \int_\Sigma e^{u_2}\, dV_g \le \frac{1}{8\pi} \int_{A_0(\sigma, r)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2\, dV_g + \varepsilon \int_\Sigma Q(u_1, u_2)\, dV_g + C. \qquad (45)$$

Using the fact that

$$\tfrac{1}{4}|\nabla u_2|^2 = Q(u_1, u_2) - \tfrac{1}{12}|\nabla(u_2 + 2u_1)|^2,$$

we obtain

$$\begin{aligned}
\int_{A_0(\sigma, r)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2\, dV_g &= \frac{1}{4} \int_{A_0(\sigma, r)} |\nabla u_2|^2\, dV_g - 2\pi \log\sigma + 2\pi \bar{u}_2(\sigma) + C \\
&= \int_{A_0(\sigma, r)} Q(u_1, u_2)\, dV_g - \frac{1}{12} \int_{A_0(\sigma, r)} |\nabla(u_2 + 2u_1)|^2\, dV_g \\
&\qquad - 2\pi \log\sigma + 2\pi \bar{u}_2(\sigma) + C, \qquad (46)
\end{aligned}$$

where $\bar{u}_2(\sigma) = \fint_{B_\sigma(0)} u_2\, dV_g$.

We claim now that for any $\tilde{\varepsilon} > 0$ one has

$$\int_{A_0(\sigma, r)} |\nabla(u_2 + 2u_1)|^2\, dV_g \ge 2\pi \left(\frac{2}{\tilde{\varepsilon}} (\bar{u}_2(\sigma) + 2\bar{u}_1(\sigma)) + \frac{1}{\tilde{\varepsilon}^2} \log\sigma \right) - \varepsilon \int_\Sigma Q(u_1, u_2)\, dV_g - C. \qquad (47)$$

Letting $v(x) = u_2(x) + 2u_1(x)$, we have to prove

$$\int_{A_0(\sigma,r)} |\nabla v|^2 \, dV_g \geq 2\pi \left(\frac{2}{\tilde{\varepsilon}} \bar{v}(\sigma) + \frac{1}{\tilde{\varepsilon}^2} \log \sigma \right),$$

where $\bar{v}(\sigma) = \bar{u}_2(\sigma) + 2\bar{u}_1(\sigma)$. Choose $k \in \mathbb{N}$ such that

$$\int_{A_0(2^k\sigma, 2^{k+1}\sigma)} |\nabla v|^2 \, dV_g \leq \varepsilon \int_\Sigma |\nabla v|^2 \, dV_g,$$

and define

$$\begin{cases} \tilde{u}(x) = \bar{v}(\sigma) & \text{if } x \in B_{2^k\sigma}(0), \\ \Delta \tilde{u}(x) = 0 & \text{if } x \in A_0(2^k\sigma, 2^{k+1}\sigma), \\ \tilde{u}(x) = v(x) & \text{if } x \notin B_{2^{k+1}\sigma}(0). \end{cases}$$

Then there exists a universal constant C_0 such that

$$\begin{aligned} \int_{A_0(2^k\sigma,r)} |\nabla \tilde{u}|^2 \, dV_g &\leq \int_{A_0(\sigma,r)} |\nabla v|^2 \, dV_g + C_0 \varepsilon \int_\Sigma |\nabla v|^2 \, dV_g \\ &\leq \int_{A_0(\sigma,r)} |\nabla v|^2 \, dV_g + C_0 \varepsilon \int_\Sigma Q(u_1, u_2) \, dV_g. \end{aligned}$$

Solving the Dirichlet problem in $A_0(2^k\sigma, r)$ with constant data $\bar{v}(\sigma)$ on $\partial B_{2^k\sigma}(0)$, one gets

$$\begin{cases} w(x) = A \log \sigma & \text{if } |x| > 2^k\sigma, \\ w(2^k\sigma) = A \log(2^k\sigma) = \bar{v}(\sigma) & \text{if } |x| = 2^k\sigma \end{cases}$$

for some constant A. We have that

$$\int_{A_0(2^k\sigma,r)} |\nabla w|^2 \, dV_g = 2\pi A^2 \log \frac{1}{2^k\sigma} - C = 2\pi \frac{\bar{v}(\sigma)^2}{\log(1/2^k\sigma)} - C.$$

Moreover,

$$\int_{A_0(2^k\sigma,r)} |\nabla w|^2 \, dV_g \leq \int_{A_0(2^k\sigma,r)} |\nabla \tilde{u}|^2 \, dV_g.$$

Finally, using Young's inequality

$$\bar{v}(\sigma) \log \frac{1}{\sigma} \leq \frac{1}{2} \left(\tilde{\varepsilon} \bar{v}(\sigma)^2 + \frac{1}{\tilde{\varepsilon}} \left(\log \frac{1}{\sigma} \right)^2 \right),$$

we end up with

$$\frac{\bar{v}(\sigma)^2}{\log(1/\sigma)} \geq \left(\frac{2}{\tilde{\varepsilon}} \bar{v}(\sigma) + \frac{1}{\tilde{\varepsilon}^2} \log \sigma \right).$$

Therefore, we conclude

$$\begin{aligned} 2\pi \left(\frac{2}{\tilde{\varepsilon}} \bar{v}(\sigma) + \frac{1}{\tilde{\varepsilon}^2} \log \sigma \right) - C &\leq 2\pi \frac{\bar{v}(\sigma)^2}{\log(1/\sigma)} - C = \int_{A_0(2^k\sigma,r)} |\nabla w|^2 \, dV_g \\ &\leq \int_{A_0(2^k\sigma,r)} |\nabla \tilde{u}|^2 \, dV_g \leq \int_{A_0(\sigma,r)} |\nabla v|^2 \, dV_g + C_0 \varepsilon \int_\Sigma Q(u_1, u_2) \, dV_g, \end{aligned}$$

which proves the claim (47).

Inserting (47) into (46), we have

$$\int_{A_0(\sigma,r)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2\,dV_g \le \int_{A_0(\sigma,r)} Q(u_1,u_2)\,dV_g - \tfrac{1}{12}2\pi\left(\frac{2}{\tilde{\varepsilon}}(\bar{u}_2(\sigma)+2\bar{u}_1(\sigma)) + \frac{1}{\tilde{\varepsilon}^2}\log\sigma\right) - 2\pi\log\sigma + 2\pi\bar{u}_2(\sigma) + \varepsilon\int_\Sigma Q(u_1,u_2)\,dV_g + C.$$

Choosing $\tilde{\varepsilon} = \frac{1}{6}$, we obtain

$$\int_{A_0(\sigma,r)} |\nabla(\tfrac{1}{2}u_2 - \log|x|)|^2\,dV_g \le \int_{A_0(\sigma,r)} Q(u_1,u_2)\,dV_g - 4\pi\bar{u}_1(\sigma) - 8\pi\log\sigma + \varepsilon\int_\Sigma Q(u_1,u_2)\,dV_g + C. \quad (48)$$

We use then (48) in (45) to get

$$8\pi\log\int_\Sigma e^{u_2}\,dV_g \le \int_{A_0(\sigma,r)} Q(u_1,u_2)\,dV_g - 4\pi\bar{u}_1(\sigma) - 8\pi\log\sigma + \varepsilon\int_\Sigma Q(u_1,u_2)\,dV_g + C. \quad (49)$$

For the first component, we consider the scalar local Moser–Trudinger inequality (see for example Proposition 2.3 of [Malchiodi and Ruiz 2013]), namely

$$\begin{aligned}\log\int_{B_{r/2}(0)} e^{u_1}\,dV_g &\le \frac{1}{16\pi}\int_{B_r(0)} |\nabla u_1|^2\,dV_g + \bar{u}_1(r) + \varepsilon\int_\Sigma |\nabla u_1|^2\,dV_g + C \\ &\le \frac{1}{4\pi}\int_{B_r(0)} Q(u_1,u_2)\,dV_g + \bar{u}_1(r) + \varepsilon\int_\Sigma Q(u_1,u_2)\,dV_g + C.\end{aligned}$$

Performing a dilation to $B_\sigma(0)$, one gets

$$4\pi\log\int_{B_{\sigma/2}(0)} e^{u_1}\,dV_g \le \int_{B_\sigma(0)} Q(u_1,u_2)\,dV_g + 4\pi\bar{u}_1(\sigma) + 8\pi\log\sigma + \varepsilon\int_\Sigma Q(u_1,u_2)\,dV_g + C.$$

We then use the assumption (41), and we obtain

$$4\pi\log\int_\Sigma e^{u_1}\,dV_g \le \int_{B_\sigma(0)} Q(u_1,u_2)\,dV_g + 4\pi\bar{u}_1(\sigma) + 8\pi\log\sigma + \varepsilon\int_\Sigma Q(u_1,u_2)\,dV_g + C. \quad (50)$$

Summing equations (49) and (50), we deduce

$$4\pi\log\int_\Sigma e^{u_1}\,dV_g + 8\pi\log\int_\Sigma e^{u_2}\,dV_g \le \int_{B_r(z)} Q(u_1,u_2)\,dV_g + \varepsilon\int_\Sigma Q(u_1,u_2)\,dV_g + C,$$

which concludes the proof. □

Remark 3.6. The above result is inspired by the work [Bartolucci and Malchiodi 2013] (see in particular Proposition 4.1 there), where the singular Liouville equation is considered. The authors derive a geometric inequality by means of the angular distribution of the conformal volume near the singularities. Somehow the singular equation can be seen as the limit case of the regular one. Roughly speaking, when one component is much more concentrated with respect to the other one, its effect resembles that of a Dirac delta.

3.3. ***Lower bounds on the functional J_ρ.*** We are going to exploit the improved inequality stated in Proposition 3.5 to derive new lower bounds of the energy functional J_ρ defined in (2); see Proposition 3.7. This will give us some extra constraints for the map from the low sublevels of J_ρ onto the topological join $\Sigma_k * \Sigma_1$; see (9).

Given a small $\delta > 0$, our aim is to describe the low sublevels of the functional J_ρ by means of the set

$$Y := (\Sigma_k * \Sigma_1) \setminus S \subseteq \Sigma_k * \Sigma_1, \tag{51}$$

where

$$S = \left\{ \left(\nu, \delta_z, \tfrac{1}{2}\right) \in \Sigma_k * \Sigma_1 : \nu = \sum_{i=1}^{k} t_i \delta_{x_i},\ d(x_i, x_j) \geq \delta \text{ for all } i \neq j,\ \delta \leq t_i \leq 1 - \delta \text{ for all } i,\ z \in \operatorname{supp}(\nu) \right\}. \tag{52}$$

We will show that there is a lower bound for J_ρ whenever $\widetilde{\Psi}$, which is defined in (29), has image inside S; see Proposition 3.7.

Consider $\mathscr{C}_{\varepsilon,r}(x_0)$ as given in (17), $f \in \mathscr{C}_{\varepsilon,r}(x_0)$, and ψ defined in (19). Before stating the next main result, we recall some properties of the map ψ; see Proposition 3.1 in [Malchiodi and Ruiz 2013] (with minor changes).

Fact. *Let $\psi(f) = (\beta, \sigma)$. Then given $R > 1$, there exists $p \in \Sigma$ with the properties*

$$d(p, \beta) \leq C'\sigma \quad \text{for some } C' = C'(R), \qquad \int_{B_\sigma(p) \cap B_r(x_0)} f \, dV_g > \tau, \qquad \int_{(B_{R\sigma}(p))^c \cap B_r(x_0)} f \, dV_g > \tau, \tag{53}$$

where τ depends only on R and Σ.

Recall also the distance $\boldsymbol{d}$ between measures in (11), the numbers $\varepsilon_i > 0$ in Proposition 2.4, the projections $\widetilde{\psi}_k$ and $\widetilde{\psi}_1$ in (27)–(28), and the definition of the parameter s in the topological join given by (26).

Proposition 3.7. *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4\pi, 8\pi)$ and that $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) < 2\varepsilon_k$ and $\boldsymbol{d}(\mathcal{N}(e^{u_2}, \Sigma), \Sigma_1) < \varepsilon_1$. Let*

$$\widetilde{\psi}_k(\mathcal{N}(e^{u_1}, \Sigma)) = \sum_{i=1}^{k} t_i \delta_{x_i}, \qquad \widetilde{\psi}_1(\mathcal{N}(e^{u_2}, \Sigma)) = \delta_{\beta_z}.$$

There exist $\delta > 0$ and $L > 0$ such that, if the properties

(1) *$d(x_i, x_j) \geq \delta$ for all $i \neq j$ and $t_i \in [\delta, 1-\delta]$ for all $i = 1, \dots, k$,*

(2) *$s(u_1, u_2) = \frac{1}{2}$, and*

(3) *$\beta_z = x_l$ for some $l \in \{1, \dots, k\}$*

hold true, then

$$J_\rho(u_1, u_2) \geq -L.$$

Proof. Suppose without loss of generality that $\bar{u}_1 = \bar{u}_2 = 0$. We first observe that exploiting the assumption $s(u_1, u_2) = \frac{1}{2}$ we deduce $\sigma_1(u_1) = \sigma_2(u_2)$. Secondly, it is not difficult to show that from property (1) it follows that $\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_{k-1})\geq 2\varepsilon_{k-1}$. Therefore, by the definition of $\widetilde{\psi}_k$, we deduce that $x_i=\beta_{x_i^k}$ for $i=1,\dots,k$, where the $\beta_{x_i^k}$ are the local centers of mass given by (23). Hence, we get

$$\widetilde{\psi}_k(\mathcal{N}(e^{u_1},\Sigma))=\sum_{i=1}^{k} t_i\delta_{\beta_{x_i^k}}.$$

Recalling that we have set (see Section 3.1)

$$\sigma_2(u_2)=\mathfrak{g}_1\big(\boldsymbol{d}(\mathcal{N}(e^{u_2},\Sigma),\Sigma_1)\big)\sigma_z+\big(1-\mathfrak{g}_1\big(\boldsymbol{d}(\mathcal{N}(e^{u_2},\Sigma),\Sigma_1)\big)\big),$$

using the fact that $\boldsymbol{d}(\mathcal{N}(e^{u_2},\Sigma),\Sigma_1)<\varepsilon_1$, by the definition of $\mathfrak{g}_1$ in (24), $\sigma_2(u_2)$ reduces to σ_z. We recall now also the definition of $\sigma_1(u_1)$, namely

$$\sigma_1(u_1)=\mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k)\big)A(u_1,u_2)+\big(1-\mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k)\big)\big)$$

with $A(u_1,u_2)$ defined in Section 3.1. Assuming $\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k)<2\varepsilon_k$ implies $\mathfrak{f}(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k))>0$. Again, using property (1), we obtain from $\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_{k-1})\geq 2\varepsilon_{k-1}$ that $\mathfrak{g}_{k-1}(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_{k-1}))=0$ and hence $A(u_1,u_2)=B(u_1,u_2)$ (see the notation before (26)). Moreover, the property (3) implies that $\mathfrak{h}(d(\beta_z,\{\beta_{x_1^k},\dots,\beta_{x_k^k}\}))=1$. Therefore, $B(u_1,u_2)=\sigma_{x_l^k}$. Hence, one finds

$$\sigma_{u_1}=\mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k)\big)\sigma_{x_l^k}+\big(1-\mathfrak{f}\big(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k)\big)\big).$$

We distinguish between two cases.

Case 1. Suppose first that $\mathfrak{f}(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k))=1$. In this case, we obtain $\sigma_{x_l^k}=\sigma_1(u_1)=\sigma_2(u_2)=\sigma_z$. By this fact and by property (3), we get $(\beta_{x_l^k},\sigma_{x_l^k})=(\beta_z,\sigma_z)$. Let $r=\delta/4$: from (53) and the definitions of β_z and $\beta_{x_i^k}$, there exists $\tilde{\gamma}_0>0$ such that

$$\int_{B_r(\beta_{x_i^k})} e^{u_1}\,dV_g\geq\tilde{\gamma}_0\int_\Sigma e^{u_1}\,dV_g\quad\text{for } i=1,\dots,k,\qquad \int_{B_r(\beta_z)} e^{u_2}\,dV_g\geq\tilde{\gamma}_0\int_\Sigma e^{u_2}\,dV_g. \tag{54}$$

Therefore, we are in position to apply Proposition 2.12 and get

$$4(k+1)\pi\log\int_\Sigma e^{u_1}\,dV_g+8\pi\log\int_\Sigma e^{u_2}\,dV_g\leq(1+\varepsilon)\int_\Sigma Q(u_1,u_2)\,dV_g+C_r.$$

The conclusion then follows from the expression of J_ρ and from the upper bounds on ρ_1 and ρ_2.

Case 2. Suppose now $\mathfrak{f}(\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k))<1$: we deduce immediately that $\boldsymbol{d}(\mathcal{N}(e^{u_1},\Sigma),\Sigma_k)\in(\varepsilon_k,2\varepsilon_k)$.

Given $\varepsilon>0$, let $R=R(\varepsilon)$ be such that Proposition 2.10 holds true. Let $C'=C'(R)$ and $\tau=\tau(R)$ be as in (53). Take $\tau_0=\tau/100$ and $\gamma_0=\tilde{\gamma}_0\tau$, where γ_0 is given as in (54), and let $C=C(\varepsilon,r,\tau_0,\gamma_0)$ be the constant obtained in Proposition 3.5. We then define $\widetilde{C}=\max\{C',C\}$. Moreover, observe that by construction $\sigma_{x_l^k}\leq\sigma_1(u_1)=\sigma_2(u_2)=\sigma_z$.

If $\sigma_{x_l^k}\leq\sigma_z\leq\widetilde{C}^8\sigma_{x_l^k}$, we still can apply Proposition 2.12 as before; see Remark 2.11. Consider now the case $\widetilde{C}^8\sigma_{x_l^k}\leq\sigma_z$. We distinguish between two situations.

Case 2.1. If r is as in Case 1, suppose that

$$\int_{B_{\widetilde{C}^4\sigma_{x_l^k}}(\beta_z)} e^{u_2}\,dV_g > \tau_0 \int_{B_r(\beta_z)} e^{u_2}\,dV_g \tag{55}$$

(the right side exceeds $\tilde{\gamma}_0\tau_0\int_\Sigma e^{u_2}\,dV_g$; see (54)). By the fact that $\widetilde{C}^4\sigma_{x_l^k} \ll \sigma_z$, from (53), we also get

$$\int_{(B_{R\widetilde{C}^4\sigma_{x_l^k}}(\beta_z))^c\cap B_r(\beta_z)} e^{u_2}\,dV_g > \tau_0 \int_{B_r(\beta_z)} e^{u_2}\,dV_g > \tilde{\gamma}_0\tau_0 \int_\Sigma e^{u_2}\,dV_g. \tag{56}$$

The conditions on the local scale of u_1, given by $(\beta_{x_l^k}, \sigma_{x_l^k}) = \psi(f_{\mathrm{loc}}^{x_l^k}(u_1))$, yield by (53) the existence of $p \in \Sigma$ such that

$$\int_{B_{\sigma_{x_l^k}}(p)} e^{u_1}\,dV_g > \tau \int_{B_r(\beta_{x_l^k})} e^{u_1}\,dV_g > \tilde{\gamma}_0\tau \int_\Sigma e^{u_1}\,dV_g,$$

$$\int_{(B_{R\sigma_{x_l^k}}(p))^c\cap B_r(\beta_{x_l^k})} e^{u_1}\,dV_g > \tau \int_{B_r(\beta_{x_l^k})} e^{u_1}\,dV_g > \tilde{\gamma}_0\tau \int_\Sigma e^{u_1}\,dV_g.$$

The latter formulas, together with (55) and (56), imply an improved Moser–Trudinger inequality (see Remarks 2.9 and 2.11):

$$8\pi\left(\log\int_\Sigma e^{u_1}\,dV_g + \log\int_\Sigma e^{u_2}\,dV_g\right) \le (1+\varepsilon)\int_{B_r(\beta_z)} Q(u_1,u_2)\,dV_g + C_0(\varepsilon, r, \tau, \tilde{\gamma}_0). \tag{57}$$

Case 2.2. Suppose now that the second situation occurs, namely

$$\int_{B_{\widetilde{C}^4\sigma_{x_l^k}}(z)} e^{u_2}\,dV_g \le \tau_0 \int_{B_r(\beta_z)} e^{u_2}\,dV_g. \tag{58}$$

The goal is to apply the improved inequality stated in Proposition 3.5. Take $\sigma = (C')^2\sigma_{x_l^k}$ and $A_{\beta_z}(C\sigma, r/C)$ as the annulus on which we will test the conditions (41)–(42). We start by considering (41). Observe that

$$\int_{B_{\sigma/2}(z)} e^{u_1}\,dV_g > \gamma_0 \int_\Sigma e^{u_1}\,dV_g$$

follows from (53) and (54) by the choice of σ and γ_0. Similarly, using the volume concentration of u_2 in $(B_{R\sigma_z}(p))^c \cap B_r(\beta_z)$ in (53) and (recalling the definition of $\widetilde{C}$) $C\sigma \ll R\sigma_z$, we get

$$\int_{A_{\beta_z}(C\sigma, r/C)} e^{u_2}\,dV_g > \gamma_0 \int_\Sigma e^{u_2}\,dV_g$$

by taking ε_1 sufficiently small in Proposition 3.7. We are left by proving condition (42), i.e.,

$$\sup_{y\in A_{\beta_z}(C\sigma, r/C)} \frac{\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\,dV_g}{\int_{A_{\beta_z}(C\sigma, r/C)} e^{u_2}\,dV_g} < 1-\tau_0.$$

If this is not the case, then there exists $y \in A_{\beta_z}(C\sigma, r/C)$ such that

$$\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\, dV_g \geq (1-\tau_0) \int_{A_{\beta_z}(C\sigma, r/C)} e^{u_2}\, dV_g.$$

Using the assumption (58) and $\sigma < \widetilde{C}^4 \sigma_{x_l^k}$, we get

$$\begin{aligned}\int_{B_{\tau_0 d(y,z)}(y)} e^{u_2}\, dV_g &\geq (1-\tau_0) \int_{A_{\beta_z}(C\sigma, r/C)} e^{u_2}\, dV_g \geq (1-\tau_0) \int_{A_{\beta_z}(C\sigma, r/C)} e^{u_2}\, dV_g \\ &= (1-\tau_0) \int_{B_r(\beta_z)} e^{u_2}\, dV_g - (1-\tau_0) \int_{B_{C\sigma}(\beta_z)} e^{u_2}\, dV_g \geq (1-2\tau_0) \int_{B_r(\beta_z)} e^{u_2}\, dV_g.\end{aligned}$$

Moreover, by the property of the local scale of u_2 given by $(\beta_z, \sigma_z) = \psi(f^z_{\mathrm{loc}}(u_2))$ (see (53)), we have

$$\int_{B_{\sigma_z(p)}} e^{u_2}\, dV_g > \tau \int_{B_r(\beta_z)} e^{u_2}\, dV_g, \qquad \int_{(B_{R\sigma_z}(p))^c \cap B_r(\beta_z)} e^{u_2}\, dV_g > \tau \int_{B_r(\beta_z)} e^{u_2}\, dV_g.$$

Notice that by the choice of τ_0 the three properties above cannot hold simultaneously. Hence, we have a contradiction. Finally, we are in position to apply Proposition 3.5 and deduce that

$$4\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \leq \int_{B_r(\beta_z)} Q(u_1, u_2)\, dV_g + \varepsilon \int_\Sigma Q(u_1, u_2)\, dV_g + C.$$

Observe that by the latter formula and by (57), in both Cases 2.1 and 2.2, we can assert that

$$4\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \leq \int_{B_r(\beta_z)} Q(u_1, u_2)\, dV_g + \varepsilon \int_\Sigma Q(u_1, u_2)\, dV_g + C. \tag{59}$$

Recall that under Case 2 we have $\boldsymbol{d}(\mathcal{N}(e^{u_1}, \Sigma), \Sigma_k) > \varepsilon_k$. By the second part of Proposition 2.4 (applied with $l = k+1$), there exist $\bar{\varepsilon}_k > 0$, depending only on ε_k, and $k+1$ points $\bar{x}_1, \ldots, \bar{x}_{k+1}$ such that

$$d(\bar{x}_i, \bar{x}_j) > 2\bar{\varepsilon}_k \quad \text{for } i \neq j, \qquad \int_{B_{\bar{\varepsilon}_k}(\bar{x}_i)} e^{u_1}\, dV_g > \bar{\varepsilon}_k \int_\Sigma e^{u_1}\, dV_g \quad \text{for all } i = 1, \ldots, k+1.$$

Without loss of generality, we can assume $\delta < \bar{\varepsilon}_k/8$. By this the choice of δ, there exist k points $\bar{y}_1, \ldots, \bar{y}_k$ such that

$$d(\bar{y}_i, \bar{y}_j) > \bar{\varepsilon}_k \quad \text{for } i \neq j, \qquad d(\bar{y}_i, \beta_{x_i^k}) > \delta \quad \text{for all } i = 1, \ldots, k,$$
$$\int_{B_{\bar{\varepsilon}_k}(\bar{y}_i)} e^{u_1}\, dV_g > \bar{\varepsilon}_k \int_\Sigma e^{u_1}\, dV_g \quad \text{for all } i = 1, \ldots, k.$$

We perform then a local Moser–Trudinger inequality for u_1 in each region (see (50)), and summing up, we have (recall that $r = \delta/4$)

$$4k\pi \log \int_\Sigma e^{u_1}\, dV_g \leq \int_{(B_r(\beta_{x_l^k}))^c} Q(u_1, u_2)\, dV_g + \varepsilon \int_\Sigma Q(u_1, u_2)\, dV_g + C_r, \tag{60}$$

where the average was estimated using Hölder's and Poincaré's inequalities as in (44). By summing (59) and (60), we deduce

$$4(k+1)\pi \log \int_\Sigma e^{u_1}\, dV_g + 8\pi \log \int_\Sigma e^{u_2}\, dV_g \le (1+\varepsilon) \int_\Sigma Q(u_1, u_2)\, dV_g + C,$$

so we conclude as in Case 1. □

By Proposition 3.7, we obtain:

Corollary 3.8. *Let S be as in* (52), *and let $Y = (\Sigma_k * \Sigma_1) \setminus S$. Then, for $\tilde{L} > 0$ large, $\widetilde{\Psi}$ (defined in* (29)) *maps the low sublevels $J_\rho^{-\tilde{L}}$ into the set Y.*

4. Test functions

We show that the lower bound in Proposition 3.7 is optimal; see also Corollary 3.8. In fact, we will construct suitable test functions modeled on Y on which J_ρ attains arbitrarily negative values.

To describe our construction, let us recall the test functions employed for the scalar case (5). When $\rho > 4\pi$, as mentioned in Section 1, the energy I_ρ in (6) is unbounded below. One can see that using test functions of the type

$$\varphi_{\lambda,z}(x) = \log\biggl(\frac{\lambda}{1+\lambda^2 d(x,z)^2}\biggr)^2, \tag{61}$$

for a given point $z \in \Sigma$ and for $\lambda > 0$, as $\lambda \to +\infty$, these satisfy the properties

$$e^{\varphi_{\lambda,z}} \rightharpoonup \delta_z \quad \text{and} \quad I_\rho(\varphi_{\lambda,z}) \to -\infty \quad (\rho > 4\pi), \tag{62}$$

holding uniformly in $z \in \Sigma$. More generally, if $\rho \in (4k\pi, 4(k+1)\pi)$, a natural family of test functions can be modeled on Σ_k [Djadli 2008; Djadli and Malchiodi 2008]. In fact, setting

$$\varphi_{\lambda,\nu}(x) = \log \sum_{i=1}^{k} t_i \biggl(\frac{\lambda}{1+\lambda^2 d(x,x_i)^2}\biggr)^2, \qquad \nu = \sum_{i=1}^{k} t_i \delta_{x_i}, \tag{63}$$

similarly to (62), for $\lambda \to +\infty$, one has uniformly in $\nu \in \Sigma_k$

$$\boldsymbol{d}(e^{\varphi_{\lambda,\nu}}, \nu) \to 0 \quad \text{and} \quad I_\rho(\varphi_{\lambda,\nu}) \to -\infty \quad (\rho \in (4k\pi, 4(k+1)\pi)).$$

When dealing with the energy functional J_ρ in (2), one can expect to interpolate between the $\varphi_{\lambda,\nu}$ for the component u_1 and the $\varphi_{\lambda,z}$ for u_2 when $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$. Therefore, the topological join $\Sigma_k * \Sigma_1$ represents a natural object to globally parametrize this family with the join parameter s playing the role of interpolation parameter. However, as mentioned in Section 1, the cross term in the quadratic energy penalizes gradients pointing in the same direction. By this reason, not all elements in $\Sigma_k * \Sigma_1$ will give rise to test functions with low energy. It will turn out that the subset Y of $\Sigma_k * \Sigma_1$ (see (51)) will be the right one at which to look.

4.1. ***A convenient deformation of $Y \cap \{s = \frac{1}{2}\}$.*** We construct here a continuous deformation of $Y \cap \{s = \frac{1}{2}\}$, which is relatively open in the join $\Sigma_k * \Sigma_1$, onto some closed subset: see Corollary 4.6. This will allow us to build test functions depending on a compact space of parameters, which is easier. Before doing this, we recall some facts from Section 3 of [Malchiodi 2008a].

There exists a deformation retract $H_0(\mathrm{t}, \cdot)$ of a neighborhood (with respect to the metric induced by $\boldsymbol{d}$ in (11)) of Σ_{k-1} in Σ_k onto Σ_{k-1}. To see this, one can take a positive δ_1 small enough and consider a nonincreasing continuous function $F_0 : (0, +\infty) \to (0, +\infty)$ such that

$$F_0(t) = \frac{1}{t} \quad \text{for } t \in (0, \delta_1], \qquad F_0(t) = \frac{1}{2\delta_1} \quad \text{for } t > 2\delta_1. \tag{64}$$

We then define $F : \Sigma_k \setminus \Sigma_{k-1} \to \mathbb{R}$ as

$$F\left(\sum_{i=1}^{k} t_i \delta_{x_i}\right) = \underbrace{\sum_{i \neq j} F_0(d(x_i, x_j))}_{F_1((x_i)_i)} + \underbrace{\sum_{i=1}^{k} \frac{1}{t_i(1-t_i)}}_{F_2((t_i)_i)}. \tag{65}$$

Notice that F is well-defined on $\Sigma_k \setminus \Sigma_{k-1}$ as it is invariant under permutation of the couples $(t_i, x_i)_{i=1,\dots,k}$. Observe also that it tends to $+\infty$ as its argument approaches Σ_{k-1}. Moreover, the gradient of F with respect to the metric of $\Sigma^k \times T_0$ (where T_0 is the simplex containing the k-tuple $T := (t_i)_i$) tends to $+\infty$ in norm as $\sum_{i=1}^{k} t_i \delta_{x_i}$ tends to Σ_{k-1}. It follows that, sending L to $+\infty$, we get a deformation retract of $F_L := \{F \geq L\} \cup \Sigma_{k-1}$ onto Σ_{k-1} for L sufficiently large. We then obtain H_0 by a reparametrization of the (positive) gradient flow of F.

We introduce now the set $\widetilde{Y}_{1/2} \subseteq Y \cap \{s = \frac{1}{2}\} \subseteq \Sigma_k * \Sigma_1$ defined as

$$\widetilde{Y}_{1/2} = \{(\nu, \delta_z, \tfrac{1}{2}) : \nu \in \Sigma_{k-1}\} \cup \{(\nu, \delta_z, \tfrac{1}{2}) : \nu \in \Sigma_k \setminus \Sigma_{k-1},\ z \notin \operatorname{supp}(\nu)\}.$$

Lemma 4.1. *There exists a continuous deformation $\widetilde{H}(\mathrm{t}, \cdot)$ of the set $Y \cap \{s = \frac{1}{2}\}$ onto $\widetilde{Y}_{1/2}$.*

Proof. Let $\delta > 0$ be as in (52). Consider $0 < \tilde{\delta} \ll \delta$, and let $\tilde{f} : (0, +\infty) \to (0, +\infty)$ be a nonincreasing continuous function given by

$$\tilde{f}(t) = \begin{cases} 1/t^2 & \text{in } t \leq \tilde{\delta}, \\ 0 & \text{in } t \geq 2\tilde{\delta}. \end{cases}$$

Moreover, recall the deformation retract $H_0(\mathrm{t}, \cdot)$ of a neighborhood of Σ_{k-1} in Σ_k onto Σ_{k-1} constructed above. To define $\widetilde{H}$, we distinguish among four situations, fixing $\hat{\delta} \ll \tilde{\delta}$ (in particular, we take $\hat{\delta}$ so small that H_0 is well-defined on the $3\hat{\delta}$-neighborhood of Σ_{k-1} in the metric $\boldsymbol{d}$).

(i) $\boldsymbol{d}(\nu, \Sigma_{k-1}) \leq \hat{\delta}$. Recall that elements in $Y \cap \{s = \frac{1}{2}\}$ are triples of the form $(\nu, \delta_z, \frac{1}{2})$ with $\nu \in \Sigma_k$. In this first case, we project ν onto Σ_{k-1} while δ_z remains fixed. If H_0 is the retraction described above, we simply define $\widetilde{H}$ to be

$$\widetilde{H}(\mathrm{t}, \nu, \delta_z, \tfrac{1}{2}) = (H_0(\mathrm{t}, \nu), \delta_z, \tfrac{1}{2}).$$

(ii) $\boldsymbol{d}(\nu, \Sigma_{k-1}) \in [\hat{\delta}, 2\hat{\delta}]$. Let

$$\nu_1(\mathrm{t}) = H_0(\mathrm{t}, \nu) = \sum_{i=1}^{k} t_i(\mathrm{t})\delta_{x_i(\mathrm{t})}.$$

If $\tilde{f}$ is as before, we introduce the following flow acting on the support of δ_z:

$$\frac{d}{d\mathrm{t}} z(\mathrm{t}) = \sum_{i=1}^{k} t_i(\mathrm{t}) f\big(d(z(\mathrm{t}), x_i(\mathrm{t}))\big)\nabla_z d(z(\mathrm{t}), x_i(\mathrm{t})). \tag{66}$$

To define $\widetilde{H}$ in this case, we interpolate from a constant motion in z and (66) depending on $\boldsymbol{d}(\nu, \Sigma_{k-1})$:

$$\widetilde{H}\big(\mathrm{t}, \nu, \delta_z, \tfrac{1}{2}\big) = \big(\nu_1(\mathrm{t}), \delta_{z(\mathrm{t}(\boldsymbol{d}(\nu, \Sigma_{k-1})-\hat{\delta})/\hat{\delta})}, \tfrac{1}{2}\big).$$

Notice that when $\boldsymbol{d}(\nu, \Sigma_{k-1}) = 2\hat{\delta}$ we get $z(\mathrm{t}(\boldsymbol{d}(\nu, \Sigma_{k-1}) - \hat{\delta})/\hat{\delta}) = z(\mathrm{t})$ and this point never intersects the support of $\nu_1(\mathrm{t})$ unless $\nu_1(\mathrm{t}) \in \Sigma_{k-1}$. Therefore, as for case (i), $\widetilde{H}(1, \nu, \delta_z, \frac{1}{2}) \in \widetilde{Y}_{1/2}$.

(iii) $\boldsymbol{d}(\nu, \Sigma_{k-1}) \in [2\hat{\delta}, 3\hat{\delta}]$. In this case, the evolution of ν interpolates between the projection onto Σ_{k-1} and staying fixed; i.e., we set

$$\nu_2(\mathrm{t}) = H_0\bigg(\mathrm{t}\frac{3\hat{\delta} - \boldsymbol{d}(\nu, \Sigma_{k-1})}{\hat{\delta}}, \nu\bigg)$$

and let $z(\mathrm{t})$ evolve according to (66) with $t_i(\mathrm{t})$ and $x_i(\mathrm{t})$ given by $\sum_{i=i}^{k} t_i(\mathrm{t})\delta_{x_i(\mathrm{t})} = \nu_2(\mathrm{t})$, so we define $\widetilde{H}$ as

$$\widetilde{H}\big(\mathrm{t}, \nu, \delta_z, \tfrac{1}{2}\big) = \big(\nu_2(\mathrm{t}), \delta_{z(\mathrm{t})}, \tfrac{1}{2}\big).$$

(iv) $\boldsymbol{d}(\nu, \Sigma_{k-1}) \geq 3\hat{\delta}$. The deformation $\widetilde{H}$ now leaves ν fixed while we let $z(\mathrm{t})$ evolve by (66) with $t_i(\mathrm{t}) \equiv t_i$ and $x_i(\mathrm{t}) \equiv x_i$:

$$\widetilde{H}\big(\mathrm{t}, \nu, \delta_z, \tfrac{1}{2}\big) = \big(\nu, \delta_{z(\mathrm{t})}, \tfrac{1}{2}\big).$$

Observe that in this case, by the definition of $\tilde{f}$ and by the choice of $\tilde{\delta}$, the latter flow of z does not intersect the support of ν and $d(z, z(1)) = O(\tilde{\delta})$. $\square$

We next *slice* the set $\widetilde{Y}_{1/2}$ in the second entry δ_z: for $p \in \Sigma$, we introduce $\widetilde{Y}_{(1/2,p)} \subseteq \Sigma_k$ given by

$$\widetilde{Y}_{(1/2,p)} = \big\{\nu \in \Sigma_k : \big(\nu, \delta_p, \tfrac{1}{2}\big) \in \widetilde{Y}_{1/2}\big\}, \tag{67}$$

so that

$$\widetilde{Y}_{1/2} = \bigcup_{p \in \Sigma} \big(\widetilde{Y}_{(1/2,p)}, \delta_p, \tfrac{1}{2}\big).$$

In Proposition 4.4, we will further deform $\widetilde{Y}_{(1/2,p)}$ to some compact subset of Σ_k (depending on p).

Let $\delta_2 > 0$ be a small number, $p \in \Sigma$, and χ_{δ_2} a cut-off function such that

$$\chi_{\delta_2} = \begin{cases} 0 & \text{in } B_{\delta_2}(p), \\ 1 & \text{in } (B_{2\delta_2}(p))^c. \end{cases} \tag{68}$$

We start by proving the following lemmas (we are extending the notation in (8) to any subset of Σ):

Lemma 4.2. *Let $p \in \Sigma$, and let $\delta_2 > 0$ be as before. There exists $\delta_3 > 0$ sufficiently small such that the above-defined map $H_0(\mathfrak{t}, \cdot)$ is a deformation retract of*

$$\left\{ \nu \in \widetilde{Y}_{(1/2,p)} : \int_\Sigma \chi_{\delta_2}\, d\nu \geq \delta_2,\ \boldsymbol{d}\left(\frac{\chi_{\delta_2}\nu}{\|\chi_{\delta_2}\nu\|}, \Sigma_{k-2} \right) \in (0, \delta_3) \right\} \cap \{ \boldsymbol{d}(\nu, \Sigma_{k-1}) < \delta_3 \}$$

onto $(\Sigma \setminus \{p\})_{k-1}$ with the property that for all $\mathfrak{t} \in [0, 1]$ we have $p \notin \operatorname{supp} H_0(\mathfrak{t}, \nu)$.

Proof. Let δ_1 be as in (64). We can assume that $\delta_1 \leq \delta_2/16$. We first prove that $H_0(\mathfrak{t}, \cdot)$ has the property that, as the $\boldsymbol{d}$-distance of ν from Σ_{k-1} tends to 0, the support of the measure $H_0(\mathfrak{t}, \nu)$ is contained in a shrinking neighborhood of the support of ν (uniformly in ν). We will then show that H_0 restricted to the particular set considered in the statement gives the desired deformation retract.

To prove the first assertion, we endow Σ^k, to which the k-tuple $X := (x_i)_i$ belongs, with the product metric and the simplex T_0, containing the k-tuple $T := (t_i)_i$, with its standard metric induced from $\mathbb{R}^k$. Then one can notice that, as the singularities of F_1 and F_2 behave like the inverse of the distance from the boundaries of their domains, there exists a constant C such that

$$\frac{1}{C} F_1(X)^2 - C \leq |\nabla_X F_1(X)| \leq C F_1(X)^2 + C, \qquad \frac{1}{C} F_2(T)^2 - C \leq |\nabla_T F_2(T)| \leq C F_2(T)^2 + C. \quad (69)$$

We now consider the evolution $\mathfrak{s} \mapsto \zeta(\nu, \mathfrak{s})$ with initial datum ν in a small neighborhood of Σ_{k-1}, where, we recall, F attains large values and its gradient does not vanish. If we evolve by the gradient of F, then X evolves by the gradient of F_1 and T by the gradient of F_2. By the last formula, we then have

$$\left| \frac{dX}{d\mathfrak{s}} \right| = |\nabla_X F_1| \leq C F_1(X)^2 + C.$$

On the other hand, still by (69), we have that

$$\frac{dF}{d\mathfrak{s}} = |\nabla_X F_1(X)|^2 + |\nabla_T F_2(T)|^2 \geq \frac{1}{C^2} F_1(X)^4 + \frac{1}{C^2} F_2(T)^4 - 2C.$$

Notice that this quantity is strictly positive if F is large enough (see (65)), which allows us to invert the function $\mathfrak{s} \mapsto F(\zeta(\nu, \mathfrak{s}))$. Therefore, if $\mathfrak{s}_\nu$ is the maximal time of existence for $\zeta(\nu, \mathfrak{s})$, we can write that

$$\int_0^{\mathfrak{s}_\nu} \left| \frac{dX}{d\mathfrak{s}} \right| d\mathfrak{s} = \int_{F(\nu)}^{\infty} \left| \frac{dX}{d\mathfrak{s}} \right| \frac{1}{dF/d\mathfrak{s}}\, dF.$$

By the above two inequalities, we deduce that

$$\int_0^{\mathfrak{s}_\nu} \left| \frac{dX}{d\mathfrak{s}} \right| d\mathfrak{s} \leq \int_{F(\nu)}^{\infty} \frac{C F_1(X)^2 + C}{F_1(X)^4/C^2 + F_2(T)^4/C^2 - 2C}\, dF.$$

By elementary inequalities, recalling that $F = F_1(X) + F_2(T)$, we also find

$$\int_0^{\mathfrak{s}_\nu} \left| \frac{dX}{d\mathfrak{s}} \right| d\mathfrak{s} \leq \widetilde{C} \int_{F(\nu)}^{\infty} \frac{1}{F^2 - \widetilde{C}}\, dF.$$

Therefore, as ν approaches Σ_{k-1}, namely for $F(\nu)$ large, we find that the displacement of X becomes smaller and smaller. This gives us the claim stated at the beginning of the proof.

Next, we observe that, by having $\nu \in \widetilde{Y}_{(1/2,p)}$ and $\boldsymbol{d}(\chi_{\delta_2}\nu/\|\chi_{\delta_2}\nu\|, \Sigma_{k-2}) > 0$ by assumption, it follows that there exists at most one point of the support of ν in the ball $B_{(3/4)\delta_2}(p)$ that does not coincide with p. Moreover, by the above claim, we have that the points outside $B_{\delta_2}(p)$ following the flow induced by F move by a distance of order $o_{\delta_3}(1)$ since $\boldsymbol{d}(\nu, \Sigma_{k-1}) < \delta_3$. Therefore, choosing δ_3 sufficiently small, we get the existence of at most one point in the ball $B_{(3/4)\delta_2}(p)$, different from p, even while the flow is acting.

By the choice of F_1 (see (64)–(65)) and by the choice $\delta_1 \le \delta_2/16$, we deduce that the point inside $B_{(3/4)\delta_2}(p)$ is not affected by the flow and in particular does not collapse onto p. □

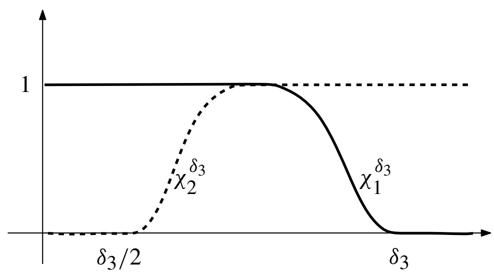
Lemma 4.3. *There exists a deformation retract $H(\mathrm{t}, \cdot)$ of $\{\nu \in \widetilde{Y}_{(1/2,p)} : \int_\Sigma \chi_{\delta_2}\, d\nu \ge \delta_2\}$ to the set*

$$\mathcal{B} := (\Sigma \setminus B_{\delta_2}(p))_k \cup \big\{\mathrm{card}\big((\mathrm{supp}(\nu)) \setminus B_{\delta_2}(p)\big) \le k-2\big\}.$$

Proof. Let us first consider a deformation retract that pushes points in $\Sigma \setminus \{p\}$ away from p. Define $H_1(\mathrm{t}, \cdot)$, $\mathrm{t} \in [0, 1]$, as follows: if $\nu = \sum_{i=1}^k t_i \delta_{x_i}$, $x_i \ne p$, then (using normal coordinates around p)

$$H_1(\mathrm{t}, \nu) = \sum_{i=1}^k t_i \delta_{x_{i,\mathrm{t}}}, \quad \text{where } x_{i,\mathrm{t}} = \begin{cases} \dfrac{x_i}{|x_i|}((1-\mathrm{t})|x_i| + \mathrm{t}\delta_2) & \text{if } d(p, x_i) < \delta_2, \\ x_i & \text{if } d(p, x_i) \ge \delta_2. \end{cases}$$

We next introduce two cut-off functions $\chi_1^{\delta_3}$ and $\chi_2^{\delta_3}$ ($\chi_2^{\delta_3}$ corresponds to the dashed graph):



For $\{\boldsymbol{d}(\nu, \Sigma_{k-1}) < \delta_3\}$, we define the deformation retract $H_2(\mathrm{t}, \cdot)$ as an *interpolation* between the homotopies H_0 and H_1, precisely

$$H_2(\mathrm{t}, \nu) = H_1\left(\mathrm{t}\chi_2^{\delta_3}\left(\boldsymbol{d}\left(\frac{\chi_{\delta_2}\nu}{\|\chi_{\delta_2}\nu\|}, \Sigma_{k-2}\right)\right), H_0\left(\mathrm{t}\chi_1^{\delta_3}\left(\boldsymbol{d}\left(\frac{\chi_{\delta_2}\nu}{\|\chi_{\delta_2}\nu\|}, \Sigma_{k-2}\right)\right), \nu\right)\right).$$

The introduction of the cut-off functions makes the deformation retract continuous with respect to the topology induced by the $\boldsymbol{d}$-distance.

For $\boldsymbol{d}(\nu, \Sigma_{k-1})$ arbitrary, we instead define H as

$$H(\mathrm{t}, \nu) = H_1\big(\mathrm{t}\chi_2^{\delta_3}(\boldsymbol{d}(\nu, \Sigma_{k-1})), H_2\big(\chi_1^{\delta_3}(\boldsymbol{d}(\nu, \Sigma_{k-1})), \nu\big)\big).$$

Again, notice that the cut-off functions in the first argument of H_1 give continuity in ν. □

The main result of this subsection is the following proposition: we retract $\widetilde{Y}_{(1/2,p)}$ to a set of measures $\Sigma_{k,p,\bar{\tau}}$ (see (70)) for which either the support is bounded away from p or for which there are at most $k-2$ points not closest to p. As we will see, these conditions will be helpful to find suitable test functions with low Euler–Lagrange energy; see the next subsections.

Proposition 4.4. *There exist $\bar{\tau} \gg 1$ and a retraction $\mathcal{R}_p$ of $\widetilde{Y}_{(1/2,p)}$ to the set*

$$\Sigma_{k,p,\bar{\tau}} = \left\{ \nu = \sum_{i=1}^{k} t_i \delta_{x_i} \in \Sigma_k : d(x_i, p) \geq \frac{1}{\bar{\tau}} \text{ for all } i \right\} \cup \left\{ \nu = \sum_{i=1}^{k} t_i \delta_{x_i} \in \Sigma_k : \operatorname{card}\{x_j : d(x_j, p) > \min_i d(x_i, p)\} \leq k-2 \right\}. \tag{70}$$

Proof. Recall first the definition (68) of χ_{δ_2}. We then extend the result in Lemma 4.3 to arbitrary values of $m_2(\nu) = \int_\Sigma \chi_{\delta_2}\, d\nu$, namely also for $m_2 < \delta_2$, finding a retraction onto $\mathcal{B}$. Consider normal coordinates around p. Define $m(\nu) = \big\| \nu\big(\chi_{\delta_2}(m_2(\nu)) + (1 - \chi_{\delta_2}(|x|))(1 - \chi_{\delta_2}(m_2(\nu)))\big) \big\|$, and let

$$T(\nu) = \begin{cases} \dfrac{\nu\big(\chi_{\delta_2}(m_2(\nu)) + (1 - \chi_{\delta_2}(|x|))(1 - \chi_{\delta_2}(m_2(\nu)))\big)}{m(\nu)} & \text{if } m_2(\nu) < 2\delta_2, \\ \nu & \text{if } m_2(\nu) \geq 2\delta_2. \end{cases}$$

We then define the retraction as

$$\widetilde{R}(\nu) = T\big(H(\chi_{\delta_2}(m_2(\nu)), \nu)\big).$$

Let $\nu_H = H(\chi_{\delta_2}(m_2(\nu)), \nu)$. To have $\widetilde{R}$ well-defined, we need to ensure that whenever T is acting, namely for $m_2(\nu_H) < 2\delta_2$, we have $m(\nu_H) > 0$. Clearly, it is enough to show that

$$\int_\Sigma (1 - \chi_{\delta_2})\, d\nu_H > 0. \tag{71}$$

We point out that

$$m_2(\nu_H) + \int_\Sigma (1 - \chi_{\delta_2})\, d\nu_H = 1.$$

Therefore, by $m_2 < 2\delta_2$, we obtain

$$\int_\Sigma (1 - \chi_{\delta_2})\, d\nu_H > 1 - 2\delta_2.$$

Finally, we construct a retraction of $\mathcal{B}$ onto $\Sigma_{k,p,\bar{\tau}}$. For $\nu \in \mathcal{B}$ with $\|(1 - \chi_{\delta_2})\nu\| > 0$, we define a parameter $\tau = \tau(\nu) \in (0, +\infty]$ in the following way:

$$\frac{1}{\tau} = \boldsymbol{d}\left(\frac{(1 - \chi_{\delta_2})\nu}{\|(1 - \chi_{\delta_2})\nu\|}, \delta_p \right). \tag{72}$$

Consider normal coordinates around p. Let $\bar{\tau} \gg 1$ be such that $1/\bar{\tau} \ll \delta_2 \ll 1$, and let $f : \mathcal{B} \times \Sigma \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be two smooth functions such that

$$f(\nu, x) = \begin{cases} 0 & \text{if } \tau = +\infty, \\ 1/\tau & \text{if } \tau < +\infty \text{ and } |x| \leq 1/\bar{\tau}, \\ |x| & \text{if } \tau < +\infty \text{ and } |x| \geq 2/\bar{\tau}, \end{cases} \qquad g(t) = \begin{cases} t & \text{if } t \leq 1/\bar{\tau}, \\ 1 & \text{if } t \geq 2/\bar{\tau}. \end{cases}$$

For $\nu = \sum_{i=1}^k s_i \delta_{y_i} \in \mathcal{B}$ with $\|(1 - \chi_{\delta_2})\nu\| > 0$, we consider $(1 - \chi_{\delta_2})\nu = \sum_{i=1}^k t_i \delta_{x_i}$ and then define

$$\tilde{\nu} = \frac{\sum_{i=1}^k t_i g(|x_i|) \delta_{(x_i/|x_i|) f(\nu, x_i)}}{\sum_{i=1}^k t_i g(|x_i|)}. \tag{73}$$

Observe that, for $d(x_i, p) \le 1/\bar\tau$ for all i, (73) reads as

$$\tilde\nu = \frac{\sum_{i=1}^k t_i |x_i| \delta_{(x_i/|x_i|)(1/\tau)}}{\sum_{i=1}^k t_i |x_i|}$$

while, for $d(x_i, p) \ge 2/\bar\tau$ for all i, we obtain $\tilde\nu = \sum_{i=1}^k t_i \delta_{x_i}$.

For a general $\nu \in \mathcal{B}$, the retraction is given by

$$\mathcal{R}_p(\nu) = (1 - m_2)\tilde\nu + \chi_{\delta_2}\nu. \tag{74}$$

Observe that, when $\|(1-\chi_{\delta_2})\nu\| = 0$, τ is not defined. However, the map $\mathcal{R}_p(\nu)$ is well-defined since in this case we have $m_2 = 1$. Notice furthermore that $\mathcal{R}_p(\nu) \in \Sigma_k$ since $\|\mathcal{R}_p(\nu)\| = 1$ and since we do not increase the number of points in the support of ν, due to the fact that the map $\nu \mapsto \tilde\nu$ does not affect the points x_i with $d(x_i, p) \ge 2/\bar\tau$, which was chosen such that $2/\bar\tau \ll \delta_2$. □

Remark 4.5. (i) With the above definitions, letting δ_2 tend to 0, one shows that the map $\mathcal{R}_p$ is homotopic to the identity on its domain.

(ii) The parameter δ_2 is chosen so that $\delta_2 \ll \delta$.

Combining Lemma 4.1 and Proposition 4.4 (applying its proof uniformly in $p \in \Sigma$), we obtain the following result; notice that by construction the retraction $\mathcal{R}_p$ from Proposition 4.4 depends continuously on p.

Corollary 4.6. *There exist $\bar\tau \gg 1$ and a continuous deformation $\mathcal{R}$ of $Y \cap \{s = \frac{1}{2}\}$ onto the set*

$$\bigcup_{p\in\Sigma} \{(\nu, \delta_p, \tfrac{1}{2}) : \nu \in \Sigma_{k,p,\bar\tau}\},$$

where $\Sigma_{k,p,\bar\tau}$ is as in (70).

In the next two subsections, we perform the construction of test functions using the above deformations.

4.2. *Test functions modeled on $\widetilde{Y}_{(1/2,p)} * \delta_p$.* In this subsection, we introduce a class of test functions parametrized on $\widetilde{Y}_{(1/2,p)} * \delta_p \subseteq Y$; see (67) and (51). The latter subset of Y is where the interaction between the two components of (1) is stronger and hence where more refined energy estimates will be needed. The remainder of Y will be taken care of in the next subsection.

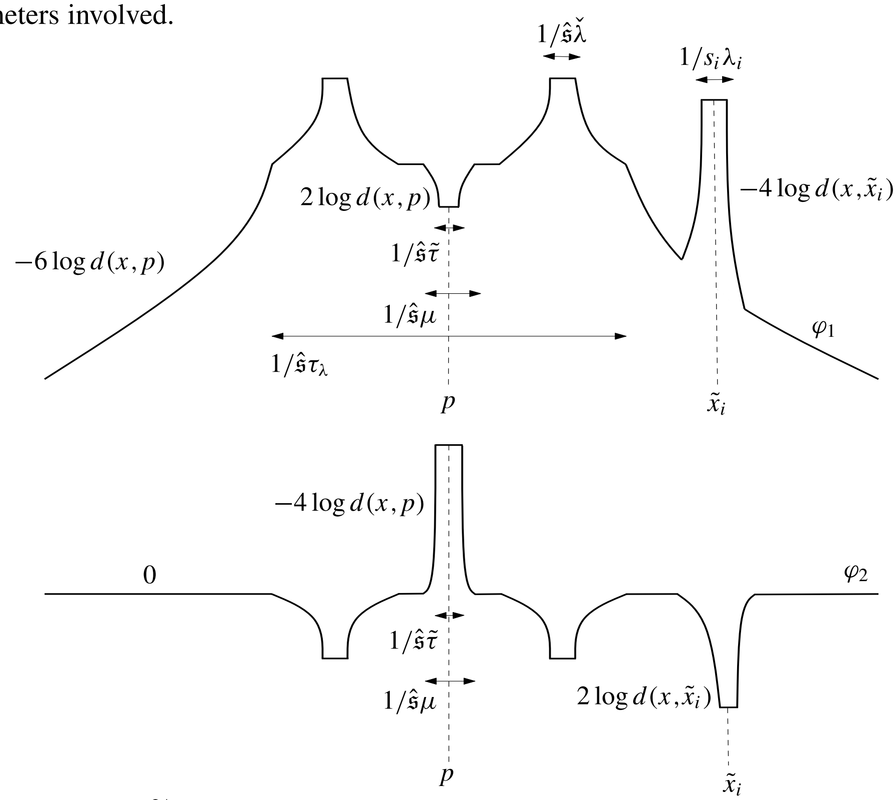
The retraction $\mathcal{R}_p$ defined in Proposition 4.4 will play a crucial role in the construction of the test functions. Indeed, starting from a measure in $\widetilde{Y}_{(1/2,p)}$ we will consider, through the map $\mathcal{R}_p$, a configuration belonging to $\Sigma_{k,p,\bar\tau}$; see (70). When considering $\widetilde{Y}_{(1/2,p)} * \delta_p$ and the corresponding join parameter s, our goal is to pass continuously from vector-valued functions (φ_1, φ_2) with $e^{\varphi_1} \simeq \hat\nu \in \Sigma_{k,p,\bar\tau}$ (in the distributional sense) to functions (φ_1, φ_2) with $e^{\varphi_2} \simeq \delta_p$. This needs to be done so that the energy $J_\rho(\varphi_1, \varphi_2)$ stays arbitrarily low.

As the formulas are rather involved, we first discuss the general ideas behind them. Our construction relies on superpositions of *regular bubbles* and *singular bubbles*. Regular bubbles are functions as in (61) that (roughly) optimize inequality (7) in the scalar case. Singular bubbles instead are profiles of solutions to (5) when a Dirac mass is present in the right-hand side: this singular version of (5) *shadows* system (1) when one component has a higher concentration than the other.

From the computational point of view, regular or singular bubbles behave like logarithmic functions of the distance from a point truncated at a proper scale, with coefficient -4 or -6, respectively. By this reason, we sometimes substitute an expression as in (61) (or in the subsequent formula) with truncated logarithms.

Another aspect of the construction is that, at a scale at which the function φ_i dominates, the gradient of the other component φ_j of (1) will behave like $-\frac{1}{2}\nabla\varphi_i$, the reason of which relies on the fact that this choice minimizes $Q(\varphi_1, \varphi_2)$ (see (3)) for φ_i fixed.

We introduce now the test functions (φ_1, φ_2) as in the figure below, starting by motivating the definitions of the parameters involved.



Consider $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(1/2,p)}$: recalling Proposition 4.4 and defining

$$\hat{\nu} := \mathscr{R}_p(\nu) = \sum_{i=i}^{k} t_i \delta_{x_i} \in \Sigma_{k,p,\bar{\tau}}, \tag{75}$$

let τ be as given in (72). Consider parameters $\tilde{\tau} \gg \mu \gg \lambda \gg 1$, and let $\mathfrak{s} \geq 1$ be a scaling parameter that will be used to deform one component into the other one: this will be chosen to depend on the join parameter s. Roughly speaking, φ_1 is made by a singular bubble at scale $1/\hat{\mathfrak{s}}\tau_\lambda$, where $\hat{\mathfrak{s}}$ is given by (78) (but one can think $\hat{\mathfrak{s}} = \mathfrak{s}$ for the moment) and

$$\tau_\lambda := \min\{\tau, \lambda\}, \tag{76}$$

on top of which we add regular bubbles at scales $1/s_i\lambda_i$ centered at points $\tilde{x}_i$ with $d(\tilde{x}_i, p) \geq 1/\hat{\mathfrak{s}}\tau$ for all i. The parameters s_i and λ_i are defined by (81) and (80) in order to get comparable integrals of e^{φ_1}

near all points $\tilde{x}_i$; we will discuss later why we sometimes take $\hat{\mathfrak{s}} \neq \mathfrak{s}$. The centers $\tilde{x}_i$ of the regular bubbles are defined as follows: letting $\bar{\delta}$ be small but fixed, we set in normal coordinates at p

$$\tilde{x}_i = \frac{1}{\tilde{s}_i} x_i, \qquad \tilde{s}_i = \begin{cases} \hat{\mathfrak{s}} & \text{if } d(x_i, p) \leq \bar{\delta}, \\ 1 & \text{if } d(x_i, p) \geq 2\bar{\delta}. \end{cases} \tag{77}$$

We point out that for $d(x_i, p) \leq \bar{\delta}$ we get $\tilde{x}_i = \frac{1}{\hat{\mathfrak{s}}} x_i$, which gives continuity when x_i approaches the *plateau* $\{d(\,\cdot\,, p) \leq 1/\tau_\lambda\}$. For $d(x_i, p) \geq \bar{\delta}$, instead the position of the points does not depend on $\mathfrak{s}$.

The effect of the increasing parameter $\mathfrak{s}$ depends on the starting configuration $\nu \in \widetilde{Y}_{(1/2,p)}$. In case we have points x_i on the plateau of the singular bubble, i.e., $d(x_i, p) \leq 1/\tau_\lambda$ for some i, the support of the singular and regular bubbles of φ_1 shrinks; moreover, the points $\tilde{x}_i$ approach p. On the other hand, φ_2 is (qualitatively) dilated by a factor of $1/\hat{\mathfrak{s}}$ so that e^{φ_2} loses concentration at the expense of e^{φ_1}.

In case we do not have points on the plateau, namely when $d(\tilde{x}_i, p) \geq 1/\tau_\lambda$ for all i, it is not convenient anymore to develop a singular bubble with center p as $\mathfrak{s}$ increases. To prevent this situation, we give an upper bound on $\hat{\mathfrak{s}}$ depending on τ. For $\tau_1 \geq 1$ large but fixed, we let $\widehat{P} : (0, +\infty) \to (0, +\infty)$ be a nondecreasing continuous function defined by

$$\begin{cases} \widehat{P}(t) = 1 & \text{for } t \leq \tau_1, \\ \widehat{P}(t) \to +\infty & \text{for } t \to 2\tau_1. \end{cases}$$

If τ is as in (72), we then define $\hat{\mathfrak{s}} = \hat{\mathfrak{s}}(\mathfrak{s}, \tau)$ as

$$\hat{\mathfrak{s}} = \begin{cases} \min\{\mathfrak{s}, \widehat{P}(\tau)\} & \text{if } \tau < 2\tau_1, \\ \mathfrak{s} & \text{if } \tau \geq 2\tau_1. \end{cases} \tag{78}$$

Notice that by construction of the retraction $\mathcal{R}_p$ (see Proposition 4.4) when there are no points on the plateau $\{d(\,\cdot\,, p) \leq 1/\tau_\lambda\}$, it follows that $\tau \leq C$ and therefore, taking $2\tau_1 > C$, we get $\hat{\mathfrak{s}} \leq \widehat{P}(C) < +\infty$.

In this situation, namely for $\hat{\mathfrak{s}}$ bounded from above, the second component φ_2 remains fixed when we start to concentrate the first component φ_1. To do this, we develop more and more concentrated bubbles around the points $\tilde{x}_i$; we introduce a parameter $\check{\lambda} = \check{\lambda}(\tau)$ so that $\check{\lambda} \to +\infty$ even for $\tau \leq 2\tau_1$ when $\mathfrak{s}$ increases. Let $\check{P} : (0, +\infty) \to (0, +\infty)$ be a nonincreasing continuous function such that

$$\begin{cases} \check{P}(t) \to +\infty & \text{for } t \to 2\tau_1, \\ \check{P}(t) = 1 & \text{for } t \geq 4\tau_1. \end{cases}$$

We then let

$$\check{\lambda} = \check{\mathfrak{s}}\lambda, \qquad \check{\mathfrak{s}} = \begin{cases} \mathfrak{s} & \text{if } \tau \leq 2\tau_1, \\ \min\{\mathfrak{s}, \check{P}(\tau)\} & \text{if } \tau > 2\tau_1. \end{cases} \tag{79}$$

To have a comparable integral of e^{φ_1} at each peak around $\tilde{x}_i$ for $i = 1, \ldots, k$, we impose the conditions

$$\begin{cases} \log \lambda_i - \log d(x_i, p) = \log \tau_\lambda + \log \check{\lambda} & \text{if } d(x_i, p) > 1/\tau_\lambda, \\ \lambda_i = \check{\lambda} & \text{if } d(x_i, p) \leq 1/\tau_\lambda \end{cases} \tag{80}$$

and

$$\log s_i + \log \tilde{s}_i = 2 \log \hat{\mathfrak{s}}, \tag{81}$$

which determine λ_i and s_i.

Recall the definitions of $\hat{\nu}$ in (75): motivated by the above discussion, we define the functions (φ_1, φ_2) as follows (see the figure on page 1995). The positive peaks of φ_1 are given by

$$v_1(x) = v_{1,1}(x) + v_{1,2}(x) = \log \sum_{i=1}^{k} t_i \frac{\max\left\{1, \min\left\{\left(\frac{4}{d(\tilde{x}_i, p)} d(x, \tilde{x}_i)\right)^{-4}, \left(\frac{4}{d(\tilde{x}_i, p)} \frac{1}{s_i \lambda_i}\right)^{-4}\right\}\right\}}{((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x, p)^2)^3},$$

where

$$v_{1,1}(x) = \log \sum_{i=1}^{k} t_i \max\left\{1, \min\left\{\left(\frac{4}{d(\tilde{x}_i, p)} d(x, \tilde{x}_i)\right)^{-4}, \left(\frac{4}{d(\tilde{x}_i, p)} \frac{1}{s_i \lambda_i}\right)^{-4}\right\}\right\},$$

$$v_{1,2}(x) = \log \frac{1}{((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x, p)^2)^3}.$$

The positive peak of φ_2 is instead defined by

$$v_2(x) = \log\left(\max\left\{1, \min\left\{(\hat{\mathfrak{s}}\mu\, d(x, p))^{-4}, \left(\frac{\mu}{\tilde{\tau}}\right)^{-4}\right\}\right\}\right).$$

We finally set

$$\varphi_{\lambda,\tilde{\tau},\mathfrak{s}}(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} := \begin{pmatrix} v_1(x) - \frac{1}{2} v_2(x) \\ -\frac{1}{2} v_{1,1}(x) + v_2(x) \end{pmatrix}. \tag{82}$$

The main result of this subsection is:

Proposition 4.7. *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$, let $\widetilde{\Psi}$ be defined in* (29), *and let $\varphi_{\lambda,\tilde{\tau},\mathfrak{s}}$ be defined in* (82), *with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(1/2,p)}$. Then for suitable values of $\tilde{\tau} \gg \mu \gg \lambda \gg 1$ and for $\mathfrak{s} = 1$, $\widetilde{\Psi}(\varphi_{\lambda,\tilde{\tau},1})$ is valued into the second component of the join $\Sigma_k * \Sigma_1$. Moreover, there is a value $\mathfrak{s}_{p,\nu} > 1$ of $\mathfrak{s}$, which depends continuously on p and ν such that $\widetilde{\Psi}(\varphi_{\lambda,\tilde{\tau},\mathfrak{s}_{p,\nu}})$ is valued into the first component of the join, and such that*

$$J_\rho(\varphi_{\lambda,\tilde{\tau},\mathfrak{s}}) \to -\infty \quad \text{as } \lambda \to +\infty \quad \text{uniformly in } \mathfrak{s} \in [1, \mathfrak{s}_{p,\nu}] \text{ and in } p \text{ and } \nu.$$

Proof. As some of the estimates are rather technical, most of the proof is postponed to the Appendix.

Concerning the first statement, when $\mathfrak{s} = 1$, by construction (see in particular Lemma A.2), one can see that most of the integral of e^{φ_2} is concentrated in a ball centered at p with radius of order $1/\tilde{\tau}$ while that of e^{φ_1} near at most k balls of larger scale. From the definitions of scales $\sigma_1(u_1)$ and $\sigma_2(u_2)$ in Section 3.1, it follows that for $\mathfrak{s} = 1$ the quantity $s(\varphi_1, \varphi_2)$ defined in (26) is equal to 1, provided we choose the parameters $\tilde{\tau} \gg \mu \gg \lambda \gg 1$ properly. By the way $\widetilde{\Psi}$ is defined, this implies our first statement.

As $\mathfrak{s}$ increases (see again Lemma A.2), the scale $\sigma_1(\varphi_1)$ (as defined in Section 3.1) decreases while, depending on τ, the scale of $\sigma_2(\varphi_2)$ reaches some positive value bounded away from 0. In particular for $\tau \geq 2\tau_1$ (recall (78)), by the estimates in Lemma A.2, for $\mathfrak{s} \simeq \log \tilde{\tau} - 2 \log \mu$, the scale $\sigma_2(\varphi_2)$ becomes of order 1. In any case, for $\mathfrak{s}$ sufficiently large, $s(\varphi_1, \varphi_2) = 0$, so $\widetilde{\Psi}$ maps the test function into the first component of the joint. As the scales $\sigma_1(\varphi_1)$ and $\sigma_2(\varphi_2)$ vary continuously in φ_1 and φ_2, $\mathfrak{s}_{p,\nu}$ can be chosen to depend continuously on p and ν.

Regarding the energy estimates, the most delicate situation is when τ is large, i.e., when $\hat{\mathfrak{s}} = \mathfrak{s}$; see (78). In this case, $\mathfrak{s}_{p,\nu} \simeq \log \tilde{\tau} - 2\log \mu$ and the computations are worked out in the Appendix. When τ instead is smaller than the fixed number $2\tau_1$ (see again (78)), the *singular* part of the first component of the test function (with slope $-6\log d(\,\cdot\,, p))$ has negligible contribution and the support of the measure $\hat{\nu}$ in (75) is bounded away from p by a fixed positive amount. In this case, the interaction between the two components is negligible, and similar estimates as those in Proposition 3.3 of [Battaglia et al. 2015] can be applied. □

We proceed now with parametrizing the above functions via the number s in the topological join. Ideally, one would like to have $\mathfrak{s}$ varying from 1 to $\mathfrak{s}_{p,\nu}$ as s decreases from 1 to 0. However, for this map to be well-defined on the topological join, we will need to eliminate the dependence of the test function on the first and second components of the join when $s = 1$ and $s = 0$, respectively. For this reason, we will need some extra deformations depending on s. The construction goes as follows, depending on three ranges of the join parameter s.

4.2.1. *The case* $s \in [\frac{1}{4}, \frac{3}{4}]$. Let $\varphi_{\lambda,\tilde{\tau},\mathfrak{s}}$ be defined in (82), with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(1/2,p)}$. We set

$$\Phi_\lambda(\nu, p, s) = \varphi_{\lambda,\tilde{\tau},2(1-\mathfrak{s}_{p,\nu})s+(3/2)\mathfrak{s}_{p,\nu}-1/2} \tag{83}$$

so that $\Phi_\lambda(\nu, p, \frac{1}{4}) = \varphi_{\lambda,\tilde{\tau},\mathfrak{s}_{p,\nu}}$ and $\Phi_\lambda(\nu, p, \frac{3}{4}) = \varphi_{\lambda,\tilde{\tau},1}$.

4.2.2. *The case* $s \in [0, \frac{1}{4}]$. Starting from test functions of the form $\varphi_{\lambda,\tilde{\tau},\mathfrak{s}_{p,\nu}}$, the goal will be to eliminate the dependence on the second component of the join, namely on the measure δ_p. To this end, we divide the interval $[0, \frac{1}{4}]$ in several subintervals in which we perform different operations on the test functions. Moreover, we want J_ρ to attend arbitrarily low values while doing these procedures. Notice that, in what follows, this range of the join parameter s will correspond to $\mathfrak{s} = \mathfrak{s}_{p,\nu}$, which is given in Proposition 4.7.

Step 1. Let $s \in [\frac{3}{16}, \frac{1}{4}]$. We flatten here the function v_2 in the second component of (82) by considering the deformation

$$\breve{\varphi}^t_{\lambda,\tilde{\tau}}(x) = \begin{pmatrix} \breve{\varphi}^t_1(x) \\ \breve{\varphi}^t_2(x) \end{pmatrix} := \begin{pmatrix} v_1(x) - \frac{1}{2}\, t\, v_2(x) \\ -\frac{1}{2} v_{1,1}(x) + t\, v_2(x) \end{pmatrix}, \quad t \in [0, 1].$$

We will then take

$$\Phi_\lambda(\nu, p, s) = \breve{\varphi}^t_{\lambda,\tilde{\tau}}(x), \quad t = 16(s - \tfrac{3}{16}). \tag{84}$$

It is easy to see that J_ρ attends arbitrarily low values on this deformation by minor modifications in the proof of Proposition 4.7.

Step 2. Let $s \in [\frac{1}{8}, \frac{3}{16}]$. Starting from $s = \frac{3}{16}$, we deform the test functions introduced in (82) to the standard test functions of the form given as in (63). Roughly speaking, the idea is to modify the profile of the first component φ_1 (see the figure on page 1995) by performing the following two continuous deformations. We first flatten the singular bubble $v_{1,2}$; see (82). On the other hand, we eliminate the dependence of the point p in the regular bubbles $v_{1,1}$. Therefore, we set

$$v^t_1(x) = v^t_{1,1}(x) + v^t_{1,2}(x),$$

where

$$v_{1,1}^t(x) = \log \sum_{i=1}^k t_i \max\left\{1, \min\left\{\left(\left(\frac{4}{d(\tilde{x}_i, p)}\right)^t d(x, \tilde{x}_i)\right)^{-4}, \left(\left(\frac{4}{d(\tilde{x}_i, p)}\right)^t \frac{1}{s_i\lambda_i}\right)^{-4}\right\}\right\}$$

and $v_{1,2}^t(x) = t\, v_{1,2}(x)$. Finally, recalling that we have flattened v_2 in Step 1, we consider

$$\widetilde{\varphi}_{\lambda,\tilde{\tau}}^t(x) = \begin{pmatrix} \widetilde{\varphi}_1^t(x) \\ \widetilde{\varphi}_2^t(x) \end{pmatrix} := \begin{pmatrix} v_1^t(x) \\ -\frac{1}{2} v_{1,1}^t(x) \end{pmatrix}, \quad t \in [0, 1]. \tag{85}$$

We will then take

$$\Phi_\lambda(\nu, p, s) = \widetilde{\varphi}_{\lambda,\tilde{\tau}}^t(x), \quad t = 16(s - \tfrac{1}{8}). \tag{86}$$

Concerning $\widetilde{\varphi}_1^t$, its peaks around $\tilde{x}_i$ for $i = 1, \dots, k$ are truncated at a scale $1/s_i\lambda_i$, with s_i given by (81) and λ_i to be chosen in the following way in order to have comparable volume at any $\tilde{x}_i$:

$$\begin{cases} \log\lambda_i + \log s_i - t\log d(\tilde{x}_i, p) = (t+1)\log\hat{\mathfrak{s}} + \log\check{\lambda} + t\log\tau_\lambda & \text{if } d(x_i, p) > 1/\tau_\lambda, \\ \lambda_i = \check{\lambda} & \text{if } d(x_i, p) \le 1/\tau_\lambda. \end{cases} \tag{87}$$

Observe that for $t = 0$ we again get (80). The following result holds true:

Proposition 4.8. *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$. Let $\widetilde{\varphi}_{\lambda,\tilde{\tau}}^t$ be defined as in* (85), *with $p \in \Sigma$ and $\nu \in \widetilde{Y}_{(1/2,p)}$. Then one has*

$$J_\rho(\widetilde{\varphi}_{\lambda,\tilde{\tau}}^t) \to -\infty \quad \textit{as } \lambda \to +\infty \quad \textit{uniformly in } t \in [0,1] \textit{ and in } p \textit{ and } \nu.$$

The most delicate case is when the set of the points on the plateau is not empty, i.e., for $I_1 \neq \varnothing$; see (121). We give the proof of the latter result just in this situation, skipping the case $I_1 = \varnothing$ where the singular bubble of the first component of the test function (with slope $-6\log d(\cdot, p)$) has negligible contribution and the estimates are rather easy. As observed in Case 1 of the proof of Proposition 4.7 (see (134)), for $I_1 \neq \varnothing$, we deduce $\hat{\mathfrak{s}} = \mathfrak{s}$ and $\check{\lambda} \le C\lambda$. Moreover, for this range of the join parameter s, we have $\mathfrak{s} = \mathfrak{s}_{p,\nu} \gg 1$. The proof will follow from the estimates below, which are obtained exactly as Lemmas A.1, A.2, and A.3 by using (81) and (87).

Lemma 4.9. *For $t \in [0, 1]$, we have that*

$$⨍_\Sigma \widetilde{\varphi}_1^t \, dV_g = O(1), \qquad ⨍_\Sigma \widetilde{\varphi}_2^t \, dV_g = O(1).$$

Lemma 4.10. *Recalling the notation in* (114), *for $t \in [0, 1]$, it holds that*

$$\int_\Sigma e^{\widetilde{\varphi}_1^t} \, dV_g \simeq_C \hat{\mathfrak{s}}^{2+2t} \tau_\lambda^{2t} \check{\lambda}^2, \qquad \int_\Sigma e^{\widetilde{\varphi}_2^t} \, dV_g \simeq_C 1.$$

Lemma 4.11. *Let $I_1, I_2 \subseteq I$ be as in* (121). *Then for $t \in [0, 1]$, we have*

$$\int_\Sigma Q(\widetilde{\varphi}_1^t, \widetilde{\varphi}_2^t) \, dV_g \le 8|I_1|\pi\big(\log\check{\lambda} - t\log\tau_\lambda + (1-t)\log\hat{\mathfrak{s}}\big) + \sum_{i\in I_2} 8\pi\big(\log s_i + \log\lambda_i - t\log d(\tilde{x}_i, p)\big)$$
$$+ 16t\pi \sum_{i\in I_2} \log d(\tilde{x}_i, p) + 24t^2\pi(\log\tau_\lambda + \log\hat{\mathfrak{s}}) + C,$$

for some $C = C(\Sigma)$.

Proof of Proposition 4.8. Using Lemmas 4.9, 4.10, and 4.11, the energy estimate we obtain is

$$J_\rho(\widetilde{\varphi}_1^t,\widetilde{\varphi}_2^t)\le 8|I_1|\pi\big(\log\check{\lambda}-t\log\tau_\lambda+(1-t)\log\hat{\mathfrak{s}}\big)+\sum_{i\in I_2}8\pi\big(\log s_i+\log\lambda_i-t\log d(\tilde{x}_i,p)\big)$$
$$+16t\pi\sum_{i\in I_2}\log d(\tilde{x}_i,p)+24t^2\pi(\log\tau_\lambda+\log\hat{\mathfrak{s}})-\rho_1\big((2+2t)\log\hat{\mathfrak{s}}+2t\log\tau_\lambda+2\log\check{\lambda}\big)+C$$

for some constant $C>0$. Inserting the condition (87), we obtain

$$J_\rho(\widetilde{\varphi}_1^t,\widetilde{\varphi}_2^t)\le 8|I_1|\pi\big(\log\check{\lambda}-t\log\tau_\lambda+(1-t)\log\hat{\mathfrak{s}}\big)+\sum_{i\in I_2}8\pi\big((t+1)\log\hat{\mathfrak{s}}+\log\check{\lambda}+t\log\tau_\lambda\big)$$
$$+16t\pi\sum_{i\in I_2}\log d(\tilde{x}_i,p)+24t^2\pi(\log\tau_\lambda+\log\hat{\mathfrak{s}})-\rho_1\big((2+2t)\log\hat{\mathfrak{s}}+2t\log\tau_\lambda+2\log\check{\lambda}\big)+C.$$

Notice that for $t=1$ we get exactly the estimate in (134) (recall that we have flattened v_2). The latter estimate can be rewritten as

$$J_\rho(\widetilde{\varphi}_1^t,\widetilde{\varphi}_2^t)\le\log\hat{\mathfrak{s}}\big(8(1-t)|I_1|\pi+8(t+1)|I_2|\pi+24t^2\pi-(2+2t)\rho_1\big)+\log\check{\lambda}\big(8(|I_1|+|I_2|)\pi-2\rho_1\big)$$
$$+\log\tau_\lambda\big(8t|I_2|\pi-8t|I_1|\pi+24t^2\pi-2t\rho_1\big)+16t\pi\sum_{i\in I_2}\log d(\tilde{x}_i,p)+C.$$

As observed in Case 1 of the proof of Proposition 4.7, by construction of $\Sigma_{k,p,\bar{\tau}}$ (see (70)), $|I_2|\le k-2$ whenever $|I_1|\ne\varnothing$. Therefore, we conclude that the latter estimate is uniformly large-negative in $t\in[0,1]$ since $\rho_1>4k\pi$ and by the fact that $\hat{\mathfrak{s}}=\hat{\mathfrak{s}}_{p,\nu}\gg\check{\lambda}\ge\tau_\lambda$. Observe that for $t=0$ we get

$$J_\rho(\widetilde{\varphi}_1^t,\widetilde{\varphi}_2^t)\le\log\hat{\mathfrak{s}}\big(8(|I_1|+|I_2|)\pi-2\rho_1\big)+\log\check{\lambda}\big(8(|I_1|+|I_2|)\pi-2\rho_1\big)+C,$$

which is the estimate one expects by considering standard bubbles as in (63); see for example part (i) of Proposition 4.2 in [Malchiodi and Ndiaye 2007]. □

Recall now the definition of $\hat{\nu}$ given in (75): $\hat{\nu}=\mathscr{R}_p(\nu)=\sum_{i=i}^k t_i\delta_{x_i}\in\Sigma_{k,p,\bar{\tau}}$. Notice that in the construction of the test functions (82), the points x_i are dilated according to (77) so deformed to the points $\tilde{x}_i$. Observe that for $t=0$ we obtain in (85) standard test functions as in (63). Roughly speaking, the first component resembles the form of $\varphi_{\lambda,\tilde{\nu}}$ (see (63)), where $\tilde{\nu}=\sum_{i=i}^k t_i\delta_{\tilde{x}_i}$.

In what follows, we will skip the energy estimates since they are quite standard for test functions as in (63); see for example part (i) of Proposition 4.2 in [Malchiodi and Ndiaye 2007].

Step 3. Consider $s\in[\frac{1}{16},\frac{1}{8}]$. We will deform here the points $\tilde{x}_i$ to the original points x_i. Observe that by construction (see (77)) we have $d(x_i,\tilde{x}_i)\le 2\bar{\delta}$ for all i. Hence, there exists a geodesic $\tilde{\gamma}_i$ joining $\tilde{x}_i$ and x_i in unit time, and we set $x_i^t=\tilde{\gamma}_i(t)$ with $t\in[0,1]$. Denoting by $\widehat{\varphi}_{\lambda,\bar{\tau}}^t=(\widehat{\varphi}_1^t,\widehat{\varphi}_2^t)$ the corresponding test functions, we will then take

$$\Phi_\lambda(\nu,p,s)=\widehat{\varphi}_{\lambda,\bar{\tau}}^t(x),\quad t=16(\tfrac{1}{8}-s).\tag{88}$$

Once we have deformed the points $\tilde{x}_i$ to the original ones x_i, i.e., for $t=1$, we get test functions for which the first component has the form of $\varphi_{\lambda,\mathscr{R}_p(\nu)}$.

Step 4. Consider $s \in [0, \frac{1}{16}]$. In this step, we eliminate the dependence on the map $\mathcal{R}_p$. Observe that $\mathcal{R}_p$ is homotopic to the identity map (see Remark 4.5), and let $\mathcal{H}_{\mathcal{R}_p} : \widetilde{Y}_{(1/2,p)} \times [0,1] \to \widetilde{Y}_{(1/2,p)}$ be a continuous map such that $\mathcal{H}_{\mathcal{R}_p}(\,\cdot\,, 0) = \mathcal{R}_p$ and $\mathcal{H}_{\mathcal{R}_p}(\,\cdot\,, 1) = \mathrm{Id}_{\widetilde{Y}_{(1/2,p)}}$. We consider then the deformation $\nu_t = \mathcal{H}_{\mathcal{R}_p}(\nu, t)$, and letting $\bar{\varphi}^t_{\lambda,\tilde{\tau}} = (\bar{\varphi}^t_1, \bar{\varphi}^t_2)$ be the corresponding test functions, we set

$$\Phi_\lambda(\nu, p, s) = \bar{\varphi}^t_{\lambda,\tilde{\tau}}(x), \quad t = 16(\tfrac{1}{16} - s). \tag{89}$$

Such a deformation will bring us to test functions that resemble the form of $\varphi_{\lambda,\nu}$.

4.2.3. *The case* $s \in [\frac{3}{4}, 1]$. The goal here will be to continuously deform the initial test functions in (82), with $\mathfrak{s} = 1$, to a configuration that does not depend on the measure ν; see (75). Furthermore, in this procedure, we want J_ρ to attend arbitrarily low values. For this purpose, we flatten v_1 (see (82)) by using the deformation

$$\varphi^t_{\lambda,\tilde{\tau}}(x) = \begin{pmatrix} \varphi^t_1(x) \\ \varphi^t_2(x) \end{pmatrix} := \begin{pmatrix} t v_1(x) - \frac{1}{2} v_2(x) \\ -\frac{1}{2} t v_{1,1}(x) + v_2(x) \end{pmatrix}, \quad t \in [0,1]. \tag{90}$$

We will then take

$$\Phi_\lambda(\nu, p, s) = \varphi^t_{\lambda,\tilde{\tau}}(x), \quad t = 4(1-s). \tag{91}$$

Proposition 4.12. *Suppose that* $\rho_1 \in (4k\pi, 4(k+1)\pi)$ *and* $\rho_2 \in (4\pi, 8\pi)$, *and let* $\varphi^t_{\lambda,\tilde{\tau}}$ *be defined as in* (90), *with* $p \in \Sigma$ *and* $\nu \in \widetilde{Y}_{(1/2,p)}$. *Then, one has*

$$J_\rho(\varphi^t_{\lambda,\tilde{\tau}}) \to -\infty \quad \textit{as } \lambda \to +\infty \quad \textit{uniformly in } t \in [0,1] \textit{ and in } p \textit{ and } \nu.$$

The latter result follows from the next estimates, which are obtained similarly as in Lemmas A.1, A.2, and A.3, using the fact that $\mathfrak{s} = 1$.

Lemma 4.13. *For* $t \in [0,1]$, *we have that*

$$\fint_\Sigma \varphi^t_1 \, dV_g = O(1), \qquad \fint_\Sigma \varphi^t_2 \, dV_g = O(1).$$

Lemma 4.14. *Recalling the notation in* (114), *there exists a constant* $C_1(\tau_\lambda, \lambda)$ *such that for* $t \in [0,1]$

$$\int_\Sigma e^{\varphi^t_1} \, dV_g \simeq_C \int_\Sigma e^{t v_1} \, dV_g = C_1(\tau_\lambda, \lambda), \qquad \int_\Sigma e^{\varphi^t_2} \, dV_g \simeq_C \int_\Sigma e^{v_2} \, dV_g \simeq_C \frac{\tilde{\tau}^2}{\mu^4}.$$

Lemma 4.15. *For* $t \in [0,1]$, *we have that*

$$\int_\Sigma Q(\varphi^t_1, \varphi^t_2) \, dV_g \le 8\pi(\log\tilde{\tau} - \log\mu) + C_2(\tau_\lambda, \lambda)$$

for some constant $C_2(\tau_\lambda, \lambda)$.

Proof of Proposition 4.12. Exploiting Lemmas 4.13, 4.14, and 4.15, we deduce

$$\begin{aligned} J_\rho(\varphi^t_1, \varphi^t_2) &\le 8\pi(\log\tilde{\tau} - \log\mu) - \rho_2(2\log\tilde{\tau} - 4\log\mu) + \widetilde{C}_1(\tau_\lambda, \lambda) + C_2(\tau_\lambda, \lambda) \\ &\le \log\tilde{\tau}(8\pi - 2\rho_2) + \log\mu(4\rho_2 - 8\pi) + \widetilde{C}_1(\tau_\lambda, \lambda) + C_2(\tau_\lambda, \lambda) \end{aligned}$$

for some constant $\widetilde{C}_1(\tau_\lambda, \lambda)$. The latter upper bound is large and negative since $\rho_2 > 4\pi$ and by the choice of the parameters $\tilde{\tau} \gg \mu \gg \lambda \ge \tau_\lambda$. $\square$

4.3. ***The global construction.*** In this subsection, we will perform a global construction of a family of test functions modeled on Y, relying on the estimates of the previous subsection. More precisely, as Y is not compact, we will consider a compact retraction of it.

Letting $(\mathfrak{D}, \frac{1}{2}) \subseteq (\Sigma_k \times \Sigma_1, \frac{1}{2})$ be the domain of the map $\mathcal{R}$ in Corollary 4.6, we extend it to $\{(\mathfrak{D}, s) : s \in (0, 1)\}$ fixing the second component and considering the same action of $\mathcal{R}$ on the first one.

Secondly, we retract the set Y to a subset where the (extended) map $\mathcal{R}$ is well-defined or where $s \in \{0, 1\}$. In order to do this, for $\nu = \sum_{i=1}^{k} t_i \delta_{x_i} \in \Sigma_k$, we let

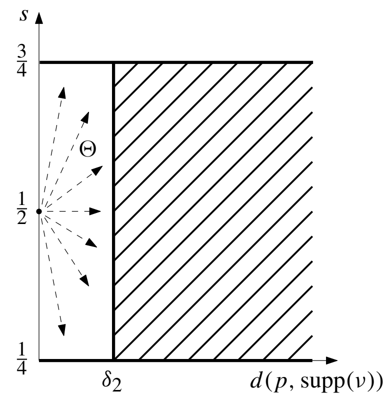
$$\mathcal{D}(\nu) = \min_{i=1,\dots,k,\, i \neq j} \{d(x_i, x_j), t_i, 1 - t_i\}.$$

Moreover, recall the choices of δ and δ_2 given in (52) and (68), respectively. Observe that for $\mathcal{D}(\nu) \leq \delta$ we are in the domain of $\mathcal{R}$. Moreover, for $\mathcal{D}(\nu) > \delta$ and $d(p, \operatorname{supp}(\nu)) \geq \delta_2$, the map $\mathcal{R}$ is still well-defined. The idea is then to retract the set Y to a subset where one of the above alternatives holds true or where $s \in \{0, 1\}$. We define now the retraction of Y in three steps.

Step 1. Let $\mathcal{D}(\nu) \geq 2\delta$. In this situation, we can deform a configuration (ν, δ_p, s) to a configuration $(\nu, \delta_{\tilde{p}}, \tilde{s}) \in Y$ (recall (51)) where either $d(\tilde{p}, \operatorname{supp}(\nu)) \geq \delta_2$ or $\tilde{s} \in \{0, 1\}$. Let

$$\Theta = (\Theta_1, \Theta_2) : [0, +\infty) \times [0, 1] \setminus \left\{\left(0, \tfrac{1}{2}\right)\right\} \to [0, +\infty) \times [0, 1] \setminus ((0, \delta_2) \times (0, 1))$$

be the radial projection as in



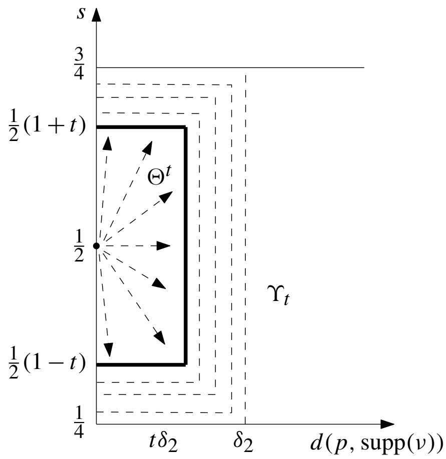
Observe now that by the fact that $\delta_2 \ll \delta$ (recall Remark 4.5), for $\mathcal{D}(\nu) \geq 2\delta$, we get the existence of a unique point $x_{j_p} \in \{x_1, \dots, x_k\}$ such that $d(p, x_{j_p}) \leq \delta_2$. To then get the above-described deformation, we define, in normal coordinates around x_{j_p}, the map

$$(\nu, \delta_p, s) \mapsto \left(\nu, \delta_{\Theta_1(d(p,\operatorname{supp}(\nu),s))(p/|p|)}, \Theta_2(d(p, \operatorname{supp}(\nu)), s)\right) \in \widetilde{\Upsilon}_\Theta,$$

where

$$\widetilde{\Upsilon}_\Theta = \left\{(\nu, \delta_p, s) : \mathcal{D}(\nu) \geq 2\delta,\ d(p, \operatorname{supp}(\nu)) \geq \delta_2\right\} \\ \cup \left\{(\nu, \delta_p, s) : \mathcal{D}(\nu) \geq 2\delta,\ d(p, \operatorname{supp}(\nu)) \leq \delta_2,\ s \in \{0, 1\}\right\}. \quad (92)$$

Step 2. Let $\mathfrak{D}(\nu) \in [\delta, 2\delta]$. In this range, we interpolate between the deformation Θ and the identity map. Consider the radial projection $\Theta^t = (\Theta_1^t, \Theta_2^t)$ given as in



with $t = (\mathfrak{D}(\nu) - \delta)/\delta$:

$$\Theta^t = (\Theta_1^t, \Theta_2^t) : [0, +\infty) \times [0, 1] \setminus \left\{\left(0, \tfrac{1}{2}\right)\right\} \to \Upsilon_t,$$

where

$$\Upsilon_t = [0, +\infty) \times [0, 1] \setminus \left((0, t\delta_2) \times \left(\tfrac{1}{2}(1-t), \tfrac{1}{2}(1+t)\right)\right).$$

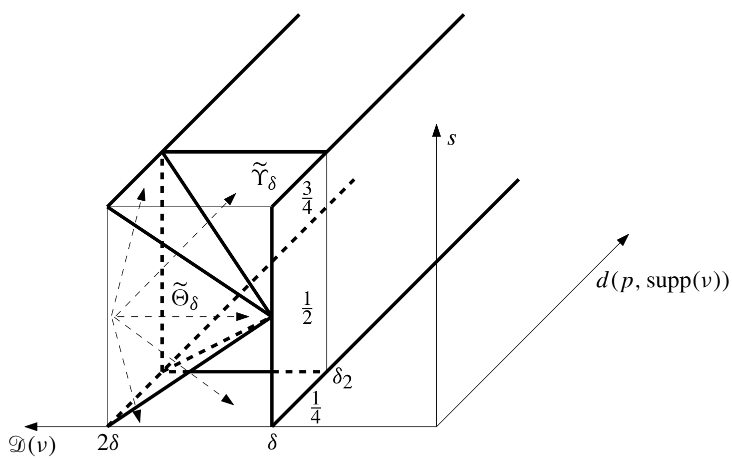
Observe that for $\mathfrak{D}(\nu) = 2\delta$ one gets $\Theta^t = \Theta^1 = \Theta$, while for $\mathfrak{D}(\nu) = \delta$ one deduces $\Theta^t = \Theta^0 = \mathrm{Id}$. We then set

$$(\nu, \delta_p, s) \mapsto \left(\nu, \delta_{\Theta_1^t(d(p, \mathrm{supp}(\nu), s))(p/|p|)}, \Theta_2^t(d(p, \mathrm{supp}(\nu)), s)\right).$$

Step 3. Let us now introduce the set we obtain after the deformation performed in Step 2:

$$\widetilde{\Upsilon}_\delta = \left\{(\nu, \delta_p, s) : \mathfrak{D}(\nu) = t \in [\delta, 2\delta],\ (p, s) \in \Upsilon_t\right\},$$

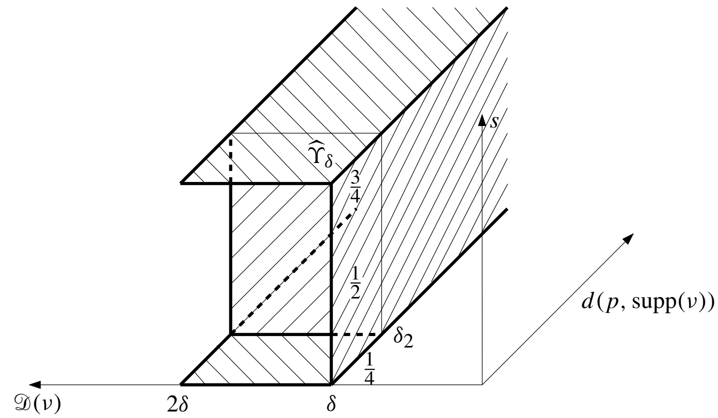
which we will deform using the radial projection $\widetilde{\Theta}_\delta : \widetilde{\Upsilon}_\delta \to \widehat{\Upsilon}_\delta$ given as in



where $\widehat{\Upsilon}_\delta$ is defined by

$$\widehat{\Upsilon}_\delta = \big\{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \in [\delta, 2\delta],\ d(p, \mathrm{supp}(\nu)) \le \delta_2,\ s \in \{0,1\}\big\} \cup \{(\nu, \delta_p, s) : \mathfrak{D}(\nu) = \delta\} \\ \cup \{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \in [\delta, 2\delta],\ d(p, \mathrm{supp}(\nu)) \ge \delta_2\}. \quad (93)$$

See the following figure, where $\partial\widehat{\Upsilon}_\delta$ is represented:



Construction of the test functions. Observing that for $\mathfrak{D}(\nu) \le \delta$ we are already in the domain of $\mathfrak{R}$ and recalling the sets (92) and (93), we have found a retraction $\mathcal{F} : Y \to Y_{\mathfrak{R}}$, where

$$\begin{aligned} Y_{\mathfrak{R}} &= \{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \le \delta\} \cup \widetilde{\Upsilon}_\delta \cup \widetilde{\Upsilon}_\Theta \\ &= \{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \le \delta\} \cup \{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \ge \delta,\ d(p, \mathrm{supp}(\nu)) \ge \delta_2\} \\ &\qquad \cup \big\{(\nu, \delta_p, s) : \mathfrak{D}(\nu) \ge \delta,\ d(p, \mathrm{supp}(\nu)) \le \delta_2,\ s \in \{0,1\}\big\}, \end{aligned} \quad (94)$$

on which the map $\mathfrak{R}$ is well-defined or where $s \in \{0, 1\}$.

Remark 4.16. By the way the retraction $\mathcal{F}$ is constructed, it is clear that we have indeed a deformation retract of the set Y onto $Y_{\mathfrak{R}}$, i.e., there exists a continuous map $\mathcal{F}_t : Y \times [0, 1] \to Y$ such that $\mathcal{F}_0 = \mathrm{Id}_Y$, $\mathcal{F}_1 = \mathcal{F} : Y \to Y_{\mathfrak{R}}$, and $\mathcal{F}_1(\xi) = \xi$ for all $\xi \in Y_{\mathfrak{R}}$.

We finally call $\Phi_\lambda = \Phi_\lambda(\nu, p, s)$ the test functions in Sections 4.2.1, 4.2.2, and 4.2.3 (see (83), (84), (86), (88), (89), and (91)) using as parameters $(\nu, p, s) \in Y_{\mathfrak{R}}$ (where we use the identification $p \simeq \delta_p$). By the estimates obtained in Section 4.2, the next result holds true.

Proposition 4.17. *Suppose that $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4\pi, 8\pi)$. Then we have*

$$J_\rho(\Phi_\lambda(\nu, p, s)) \to -\infty \quad \text{as } \lambda \to +\infty \quad \text{uniformly in } (\nu, p, s) \in Y_{\mathfrak{R}}.$$

The definition of Φ_λ reflects naturally the join element (ν, p, s) in the sense that, once composed with the map $\widetilde{\Psi}$ in (29), we obtain a map homotopic to the identity on $Y_{\mathfrak{R}}$; see the next section.

5. Proof of Theorem 1.1

In this section, we introduce the variational scheme that we will use to prove Theorem 1.1. As we already observed, the case of surfaces with positive genus was obtained in [Battaglia et al. 2015]. Therefore, from

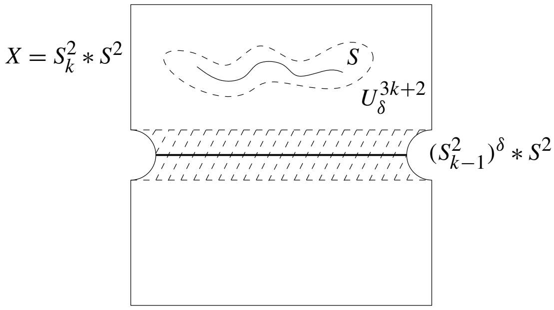


Figure 1. Here $X = S^2_k * S^2$ is the ambient space, $(S^2_{k-1})^\delta * S^2$ is a neighborhood of $S^2_{k-1} * S^2$ in X, S misses this neighborhood, and U_δ is a neighborhood of S in that complement.

now on, we will consider the case when Σ is homeomorphic to S^2. We will first analyze the topological structure of the set Y in (51) and then introduce a suitable min-max scheme.

5.1. ***On the topology of Y when Σ is a sphere.*** In this subsection, we will use the notation $\simeq$ for a homotopy equivalence and $\cong$ for an isomorphism. Consider the topological join $X = S^2_k * S^2$ (observe that $S^2_1 = S^2$), and recall the definition of its subset S given in (52), that is,

$$S = \left\{\left(\nu, \delta_y, \tfrac{1}{2}\right) \in S^2_k * S^2 : \nu \in S^2_k \setminus (S^2_{k-1})^\delta,\ y \in \operatorname{supp}(\nu)\right\},$$

where we have set

$$(S^2_{k-1})^\delta = \left\{\nu \in S^2_k : \nu = \sum_{i=1}^k t_i \delta_{x_i},\ d(x_i, x_j) < \delta \text{ for some } i \neq j\right\}$$
$$\cup \left\{\nu \in S^2_k : \nu = \sum_{i=1}^k t_i \delta_{x_i},\ t_i < \delta \text{ for some } i\right\} \cup \left\{\nu \in S^2_k : \nu = \sum_{i=1}^k t_i \delta_{x_i},\ t_i > 1 - \delta \text{ for some } i\right\}.$$

Notice that S is a smooth manifold of dimension $3k - 1$, with boundary of dimension $3k - 2$.

The key point of this subsection is to prove that the complementary subspace $Y = (S^2_k * S^2) \setminus S$ is not contractible; see Proposition 5.6. Before we do so, we establish some properties of Y and S. Below, U_δ will represent an open neighborhood of S not meeting $(S^2_{k-1})^\delta * S^2$ with the property that $\overline{U}_\delta$ is a manifold with boundary $\partial \overline{U}_\delta$, where both U_δ and $\overline{U}_\delta$ deformation-retract onto S and such that $\overline{U}_\delta \setminus S$ deformation-retracts onto $\partial \overline{U}_\delta$ (see Figure 1).

For a metric space $\mathscr{X}$, throughout this subsection, we use the notation for the k-tuples in $\mathscr{X}$

$$F(\mathscr{X}, k) := \{(x_1, \dots, x_k) \in \mathscr{X}^k : x_i \neq x_j,\ i \neq j\}$$

and $B(\mathscr{X}, n)$ to denote its quotient by the permutation action of the symmetric group. These are the ordered and unordered k-th *configuration spaces* of $\mathscr{X}$, respectively.

Lemma 5.1. *S is up to homotopy equivalence a degree-k covering of $B(S^2, k)$. Its homological dimension is at most k, and its mod-2 homology is completely described by*

$$H_*(S) \cong H_*(S^2) \otimes H_*(B(\mathbb{R}^2, k-1)).$$

Proof. The barycentric set S^2_k is a suitable quotient of

$$\Delta_{k-1} \times_{\mathfrak{S}_k} (S^2)^k,$$

with $\mathfrak{S}_k$ acting diagonally by permutations and $\Delta_{k-1} = \{(t_0, \dots, t_k) : t_i \in [0, 1], \ \sum t_i = 1\}$. The identification occurs when $x_i = x_j$ for some $i \neq j$ or when $t_i = 0$ for some i. When this happens, we are identifying with points in S^2_{k-1}. This means that, if $\dot{\Delta}_{k-1}$ is the open simplex, then

$$S^2_k \setminus S^2_{k-1} = \dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2, k), \tag{95}$$

where $F(S^2, k)$ is the configuration space of k distinct points on S^2. The action of $\mathfrak{S}_k$ on $F(S^2, k)$ is free, so we have a bundle projection

$$\dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2, k) \to B(S^2, k),$$

where $B(S^2, k) := F(S^2, k)/\mathfrak{S}_k$ is the configuration of k-unordered points on S^2. The preimages, being copies of the simplex, are contractible so that necessarily

$$S^2_k \setminus S^2_{k-1} \simeq B(S^2, k).$$

In fact, $\{1/k\}$ maps to $\dot{\Delta}_{k-1}$ with image $(1/k, \dots, 1/k)$ and the induced map

$$B(S^2, k) = \left\{\frac{1}{k}\right\} \times_{\mathfrak{S}_k} F(S^2, k) \to \dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2, k)$$

is an equivalence. To summarize, S can be deformed onto the subspace

$$W_k = \{([x_1, \dots, x_k], x) \in B(S^2, k) \times S^2 : x = x_i \text{ for some } i\}.$$

By projecting W_k onto $B(S^2, k)$, we get a covering. This implies that the homological dimension hd of W_k is that of $B(S^2, k)$, which is also the homological dimension of its covering space $F(S^2, k)$. We claim that this dimension is at most k. The projection onto the first coordinate $F(S^2, k) \to S^2$ is a bundle map with fiber $F(\mathbb{R}^2, k-1)$, so $\mathrm{hd}(F(S^2, k)) \leq 2 + \mathrm{hd}(F(\mathbb{R}^2, k-1))$. Since we also have a fibration $F(\mathbb{R}^2, k-1) \to F(\mathbb{R}^2, k-2)$ given by projecting onto the first $k-2$ entries, with fiber a copy of $\mathbb{R}^2 \setminus \{x_1, \dots, x_{k-2}\}$ that is a bouquet of circles, the claim follows immediately by induction, knowing that $F(\mathbb{R}^2, 2) \simeq S^1$.

Note that we can identify W_k with the quotient $F(S^2, k)/\mathfrak{S}_{k-1}$ where the symmetric group acts on the first $k-1$ coordinates. In particular in the case $k = 2$, $S \simeq W_2 = F(S^2, 2) \simeq S^2$.

By projecting W_k onto S^2 via the last coordinate, we get a bundle with fiber $B(\mathbb{R}^2, k-1)$. Let us look at the inclusion of the fiber over $\{\infty\} \in S^2 = \mathbb{R}^2 \cup \{\infty\}$ in this bundle

$$B(\mathbb{R}^2, k-1) \hookrightarrow W_k = F(S^2, k)/\mathfrak{S}_{k-1},$$
$$[x_1, \dots, x_{k-1}] \mapsto ([x_1, \dots, x_{k-1}], \infty).$$

Let S^∞ be the direct union of the S^n under inclusion: this is a contractible space. Now S^2 embeds in S^∞ and we have a map of quotients

$$F(S^2,k)/\mathfrak{S}_{k-1} \to F(S^\infty,k)/\mathfrak{S}_{k-1}.$$

The space on the right-hand side projects onto S^∞ with fiber $B(\mathbb{R}^\infty, k-1)$. Since the base space is contractible, there is a homotopy equivalence $F(S^\infty,k)/\mathfrak{S}_{k-1} \simeq B(\mathbb{R}^\infty, k-1)$. Let us consider the composition

$$B(\mathbb{R}^2,k-1) \xrightarrow{\iota} W_k = F(S^2,k)/\mathfrak{S}_{k-1} \to B(\mathbb{R}^\infty,k-1). \tag{96}$$

This composition is homotopic to the map induced on configuration spaces from the inclusion $\mathbb{R}^2 \subset \mathbb{R}^\infty$. It is a known useful fact that each embedding $B(\mathbb{R}^n,k) \hookrightarrow B(\mathbb{R}^{n+1},k)$ induces a monomorphism in mod-2 homology.[1] In the case $k=2$ for example, this is $B(\mathbb{R}^n,2) \simeq \mathbb{R}P^{n-1} \to B(\mathbb{R}^{n+1},2) \simeq \mathbb{R}P^n$. This then implies that $B(\mathbb{R}^2,k-1) \hookrightarrow B(\mathbb{R}^\infty,k-1)$ induces in homology mod-2 a monomorphism as well, which then means that the first portion of the composition in (96), which is inclusion of the fiber, injects in homology. Consider the Wang long exact sequence in homology associated to the bundle $W_k \to S^2$ [Mimura and Toda 1991, Theorem 2.5]:

$$H_{q+1}(W_k) \to H_{q-n+1}(B(\mathbb{R}^2,k-1)) \to H_q(B(\mathbb{R}^2,k-1)) \xrightarrow{\iota_*} H_q(W_k) \to H_{q-n}(B(\mathbb{R}^2,k-1))$$

with $n=2$ in our case. Since ι_* is a monomorphism, the long exact sequence splits into short exact sequences, and because we are working over a field, $H_q(W_k) \cong H_q(B(\mathbb{R}^2,k-1)) \oplus H_{q-2}(B(\mathbb{R}^2,k-1))$. Since $H_*(W_k) \cong H_*(S)$, the proof is complete. □

Remark 5.2. The top mod-2 homology group $H_k(S)$ is trivial if $k-1$ is not a binary power and is a copy of $\mathbb{Z}_2$ if $k-1$ is a binary power. This is because $H_{k-2}(B(\mathbb{R}^2,k-1))$ satisfies the same condition [Fuks 1970, p. 146], by Lemma 5.1.

Lemma 5.3. *Suppose $k \geq 3$. The manifold S defined in* (52) *is not orientable.*

Proof. We first observe that the manifold $S_k^2 \setminus S_{k-1}^2$ is not orientable for any $k \geq 2$. From the proof of Lemma 5.1,

$$S_k^2 \setminus S_{k-1}^2 = \dot{\Delta}_{k-1} \times_{\mathfrak{S}_k} F(S^2,k)$$

is a bundle over $B(S^2,k)$ with fiber the open simplex. Since $B(S^2,k)$ is orientable (because unordered configuration spaces of smooth manifolds are orientable if and only if the dimension of the manifold is even), the orientability of the total space is the same as the orientability of the bundle. But the braid generators of the fundamental group of $B(S^2,k)$ act (after restriction to the open simplex) by transpositions on the vertices of $\overline{\Delta}_{k-1}$ and this is orientation reversing, so the bundle is not orientable.

Now let V_k be the subset of $S_k^2 \setminus S_{k-1}^2$ of all sums $\sum t_i \delta_{x_i}$ with $x_i = \infty$ for some i. Again ∞ stands for the north pole of $S^2 = \mathbb{R}^2 \cup \{\infty\}$. Here $V_k \simeq B(\mathbb{R}^2,k-1)$. Note that $\pi_1(B(\mathbb{R}^2,k-1))$ embeds in $\pi_1(B(S^2,k))$ with similar braid generators. For the exact same reason as for $S_k^2 \setminus S_{k-1}^2$, V_k is not orientable.

[1] This follows from the work of Cohen [1976], who first calculated $H_*(B(\mathbb{R}^n,k);\mathbb{F})$ for all n and k and for $\mathbb{F} = \mathbb{Z}_2, \mathbb{Z}_p$, p odd.

Consider finally the manifold

$$\boldsymbol{S} = \{(\nu, \delta_y, \tfrac{1}{2}) \in S^2_k * S^2 : \nu \in S^2_k \setminus S^2_{k-1},\ \ y \in \operatorname{supp}(\nu)\}.$$

Then S is a codimension-0 submanifold of $\boldsymbol{S}$ (with boundary) that is also a deformation retract. Both S and $\boldsymbol{S}$ have the same orientation. But there is a bundle map $\boldsymbol{S} \to S^2$ with fiber V_k. It is easy to see now that the orientation of $\boldsymbol{S}$ is that of V_k. Indeed the bundle over the open upper hemisphere D of S^2 is trivial and thus homeomorphic to $V_k \times D$. This is an open subset of $\boldsymbol{S}$ that is nonorientable; thus, $\boldsymbol{S}$ must be nonorientable. □

Lemma 5.4. *Let $k \geq 3$. Then $Y = (S^2_k * S) \setminus S$ has the Euler characteristic of a contractible space*, *i.e.*, $\chi(Y) = 1$.

Proof. By the previous lemma, S is up to homotopy a degree-k covering of $B(S^2, k)$. This gives

$$\chi(S) = k\chi(B(S^2, k)) = k\frac{1}{k!}\chi(F(S^2, k)) = \frac{1}{(k-1)!}\chi(S^2)\chi(F(\mathbb{R}^2, k-1)) = 0.$$

Here what vanishes is $\chi(F(\mathbb{R}^2, k-1)) = 0$ since, letting $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, there are homeomorphisms

$$F(\mathbb{R}^2, k-1) = \mathbb{R}^2 \times F(\mathbb{R}^2 \setminus \{(0,0)\}, k-2) = \mathbb{R}^2 \times \mathbb{C}^* \times F(\mathbb{C}^* \setminus \{1\}, k-3)$$

and $\chi(\mathbb{C}^*) = \chi(S^1) = 0$.

On the other hand, S is a smooth $(3k-1)$-dimensional manifold with boundary. A neighborhood of S in $S^2_k * S^2$ is a $(3k+2)$-dimensional open manifold U_δ. This neighborhood is the union of two open subspaces A and B, where A is a fiberwise cone over the interior of S and B is a bundle over ∂S with fiber the cone over a hemisphere. The complement $\overline{U}_\delta \setminus S$ is the union of two subspaces $\widetilde{A}$ and $\widetilde{B}$, where $\widetilde{A}$ retracts onto an S^2-bundle over the interior of S while $\widetilde{B}$ is up to homotopy ∂S. Clearly $\widetilde{A} \cap \widetilde{B}$ retracts onto an S^2-bundle over ∂S. We can then write

$$\begin{aligned}\chi(U_\delta \setminus S) = \chi(\widetilde{A} \cup \widetilde{B}) &= \chi(\widetilde{A}) + \chi(\widetilde{B}) - \chi(\widetilde{A} \cap \widetilde{B}) = 2\chi(S) + \chi(\partial S) - 2\chi(\partial S) \\ &= 2\chi(S) - \chi(\partial S). \end{aligned} \tag{97}$$

We know that, for a manifold S of dimension m with boundary,

$$\chi(\partial S) = \chi(S) - (-1)^m \chi(S).$$

Since $\chi(S) = 0$, we get $\chi(\partial S) = 0$ and therefore $\chi(U_\delta \setminus S) = 0$ by (97).

Now cover $X = S^2_k * S^2$ by means of $U_\delta \simeq S$ and $Y = X \setminus S$. The inclusion-exclusion property of the Euler characteristic gives that

$$\chi(X) = \chi(U_\delta) + \chi(Y) - \chi(U_\delta \setminus S) = \chi(S) + \chi(Y) = \chi(Y)$$

so that $\chi(Y) = \chi(X)$. But $\chi(X) = 1$ since $\chi(X) = \chi(S^2_k * S^2) = \chi(S^2_k) + \chi(S^2) - \chi(S^2_k)\chi(S^2)$, and $\chi(S^2_k) = 1$ for $k \geq 3$ by the formula

$$\chi(Z_k) = 1 - \frac{1}{k!}(1-\chi)(2-\chi)\cdots(k-\chi)$$

for any surface Z [Malchiodi 2008a] and more generally for any simplicial complex Z [Kallel and Karoui 2011] with $\chi = \chi(Z)$. □

Lemma 5.5. *The set Y is simply connected.*

Proof. Using the same notation as in the proof of the previous lemma, we have the pushout

$$\begin{tikzcd} \widetilde{A} \cap \widetilde{B} \arrow[r] \arrow[d] & \widetilde{A} \arrow[d] \\ \widetilde{B} \arrow[r] & \overline{U}_\delta \setminus S \end{tikzcd}$$

Recall that $\widetilde{A}$ is up to homotopy an S^2-bundle over S, $\widetilde{B} \simeq \partial S$, and $\widetilde{A} \cap \widetilde{B}$ is an S^2-bundle over ∂S. This means that $\pi_1(\widetilde{A} \cap \widetilde{B}) = \pi_1(\partial S)$ and $\pi_1(\widetilde{A}) \cong \pi_1(S)$. We therefore have the pushout in the category of groups (by the van Kampen theorem)

$$\begin{tikzcd} \pi_1(\partial S) \arrow[r] \arrow[d, "\cong"] & \pi_1(S) \arrow[d] \\ \pi_1(\partial S) \arrow[r] & \pi_1(U_\delta \setminus S) \end{tikzcd}$$

which shows that $\pi_1(U_\delta \setminus S) \cong \pi_1(S) \cong \pi_1(U_\delta)$. Observe that we have used the fact that $\overline{U}_\delta \setminus S \simeq U_\delta \setminus S$ since we are removing the boundary from a manifold not intersecting S. On the other hand, we can use the same open covering of $X = S^2_k * S^2$ by U_δ and $Y = X \setminus S$. Since X is a join of connected spaces, it is 1-connected. The pushout of groups

$$\begin{tikzcd} \pi_1(U_\delta \setminus S) \arrow[r] \arrow[d, "\cong"] & \pi_1(X \setminus S) \arrow[d] \\ \pi_1(U_\delta) \arrow[r] & 0 \end{tikzcd}$$

implies that, because the left-hand vertical map is an isomorphism, the right-hand vertical map must be an isomorphism as well and $\pi_1(X \setminus S) = \pi_1(Y) = 0$. □

Despite the fact that Y is simply connected and has unit Euler characteristic, it is noncontractible.

Proposition 5.6. *Suppose $k \geq 2$ and $k \neq 4$. Then the subspace*

$$Y = (S^2_k * S^2) \setminus S$$

is not contractible.

Proof. We assume that Y is contractible and derive a contradiction. The main step is to prove that under this condition with mod-2 coefficients we must have

$$H_*(S) \cong H_{3k-1-*}(S^2_k), \quad 0 \leq * \leq k. \tag{98}$$

This will then be shown to be impossible.

The closed subset S has a neighborhood U_δ that is $(3k+2)$-dimensional with $(3k+1)$-dimensional boundary $\partial\overline{U}_\delta$. Using Poincaré's duality with mod-2 coefficients for the closed manifold $\partial\overline{U}_\delta$ gives us

$$H^*(\partial\overline{U}_\delta) \cong H_{3k+1-*}(\partial\overline{U}_\delta).$$

Since $\overline{U}_\delta \setminus S$ retracts onto $\partial\overline{U}_\delta$ and homology is dual to cohomology for finite-type spaces and field coefficients, we can conclude that

$$H_*(\overline{U}_\delta \setminus S) \cong H_{3k+1-*}(\overline{U}_\delta \setminus S), \quad * \geq 0. \tag{99}$$

Next we turn to the open covering of $X = S^2_k * S^2$ by U_δ and $Y = X \setminus S$. Using that $Y \cap U_\delta = U_\delta \setminus S$ and $U_\delta \simeq S$, the Mayer–Vietoris sequence for this union takes the form

$$H_*(U_\delta \setminus S) \to H_*(S) \oplus H_*(Y) \to H_*(X) \to H_{*-1}(U_\delta \setminus S) \to H_{*-1}(S) \oplus H_{*-1}(Y) \to H_{*-1}(X) \to \cdots.$$

Since Y has trivial reduced homology by assumption, the sequence becomes

$$H_*(U_\delta \setminus S) \to H_*(S) \to H_*(X) \to H_{*-1}(U_\delta \setminus S) \to H_{*-1}(S) \to H_{*-1}(X) \to \cdots. \tag{100}$$

But S has homological dimension k (see Lemma 5.1), so for $* > k+1$, we have the isomorphism $H_{*-1}(U_\delta \setminus S) \cong H_*(X)$. Since X is the third suspension of S^2_k, $H_*(X) \cong H_{*-3}(S^2_k)$ and thus

$$H_*(U_\delta \setminus S) \cong H_{*-2}(S^2_k), \quad * > k. \tag{101}$$

It is generally known [Kallel and Karoui 2011] that the barycentric set Z_k is $(2k+r-2)$-connected whenever Z is r-connected, $r \geq 1$. If $Z = S^2$, which is 1-connected, S^2_k is $(2k-1)$-connected and so X is $(2k+2)$-connected. In the range $* \leq 2k+2$, $\widetilde{H}_*(X) = 0$. The Mayer–Vietoris sequence (100) leads in this case to

$$H_*(U_\delta \setminus S) \cong H_*(S), \quad * < 2k+2.$$

Since S has no homology beyond degree k, we can focus on the range below so that

$$H_*(U_\delta \setminus S) \cong H_*(S), \quad 0 \leq * \leq k. \tag{102}$$

We can now combine all previous isomorphisms into one for $0 \leq * \leq k$:

$$H_*(S) \xrightarrow[(102)]{\cong} H_*(U_\delta \setminus S) \xrightarrow[(99)]{\cong} H_{3k+1-*}(U_\delta \setminus S) \xrightarrow[(101)]{\cong} H_{3k-1-*}(S^2_k).$$

This is the claim in (98). Note that S^2_k is $(3k-1)$-dimensional as a CW complex and is $(2k-1)$-connected, so its homology is nonzero only in the range $2k \leq * \leq 3k-1$.

The isomorphism $H_*(S) \cong H_{3k-1-*}(S^2_k)$ cannot hold. First let us check the case $k = 2$. In that case, we pointed out in the proof of Lemma 5.1 that $S \simeq F(S^2, 2) \simeq S^2$. Since $S^2_2 \simeq \Sigma^3\mathbb{R}P^2$ (the 3-fold suspension of $\mathbb{R}P^2$ [Kallel and Karoui 2011, Corollary 1.6]), the isomorphism obviously cannot hold: in fact, $H_1(S^2) = 0$ but $H_4(\Sigma^3\mathbb{R}P^2) = H_1(\mathbb{R}P^2) = \mathbb{Z}_2$.

Suppose that $k \geq 3$. According to Theorem 1.3 in [Kallel and Karoui 2011], S_k^2 has the same homology as (one desuspension) of the symmetric smash product $\overline{\mathrm{SP}}^k(S^3) = (S^3)^{\wedge k}/\mathfrak{S}_k$; i.e., $H_*(S_k^2) \cong H_{*+1}(\overline{\mathrm{SP}}^k(S^3))$. Combining this with (98), we get

$$H_*(S) \cong H_{3k-*}(\overline{\mathrm{SP}}^k(S^3)), \quad 0 \leq * \leq k. \tag{103}$$

We will show that this is impossible. To that end, we need to describe the groups on both sides of (103). We work again mod-2. From Lemma 5.1, we have that

$$H_*(S) \cong H_*(B(\mathbb{R}^2, k-1)) \oplus H_{*-2}(B(\mathbb{R}^2, k-1)), \quad * \geq 0.$$

(when $* - 2 < 0$ the corresponding group is zero). The mod-2 homology of $B(\mathbb{R}^2, k-1)$ has been computed by Fuks [1970], and it is best described as a subspace of the polynomial algebra (viewed as an infinite vector space generated by powers of the indicated generators)

$$\mathbb{Z}_2[a_{(1,2)}, a_{(3,4)}, \ldots, a_{(2^i-1,2^i)}, \ldots], \tag{104}$$

where the notation $a_{i,j}$ refers to a generator having homological degree i and a certain *filtration* degree j, both degrees being additive under multiplication of generators. Now the condition for an element $a^{k_1}_{(2^{i_1}-1,2^{i_1})} \cdots a^{k_r}_{(2^{i_r}-1,2^{i_r})} \in H_*(B(\mathbb{R}^2, k-1))$ is that its filtration degree is less than or equal to $k-1$, that is, if and only if $\sum_{i_s} k_{i_s} 2^{i_s} \leq k-1$.

For example, $\widetilde{H}_*(B(\mathbb{R}^2, 2)) = \mathbb{Z}_2\{a_{(1,2)}\}$ (one copy of $\mathbb{Z}_2$ generated by $a_{(1,2)}$ having homological degree 1 and filtration degree 2). Similarly $\widetilde{H}_*(B(\mathbb{R}^2, 4)) = \mathbb{Z}_2\{a_{(1,2)}, a^2_{(1,2)}, a_{(3,4)}\}$ so that

$$H_1(B(\mathbb{R}^2, 4)) = \mathbb{Z}_2\{a_{(1,2)}\}, \qquad H_2(B(\mathbb{R}^2, 4)) = \mathbb{Z}_2\{a^2_{(1,2)}\}, \qquad H_3(B(\mathbb{R}^2, 4)) = \mathbb{Z}_2\{a_{(3,4)}\}.$$

Now $H_*(B(\mathbb{R}^2, 5)) \cong H_*(B(\mathbb{R}^2, 4))$, and this turns out to be a general fact explained in Proposition 5.9 in more geometric terms.

On the other hand, the reduced groups $\widetilde{H}_*(\overline{\mathrm{SP}}^k(S^3))$ form a subvector space of the polynomial algebra

$$\mathbb{Z}_2[\iota_{(3,1)}, f_{(5,2)}, f_{(9,4)}, \ldots, f_{(2^{i+1}+1,2^i)}, \ldots] \tag{105}$$

consisting of those elements of second filtration degree precisely k (see the Appendix in [Kallel and Karoui 2011] and references therein). Here again $f_{(2^{i+1}+1,2^i)}$ denotes an element of homological degree $2^{i+1}+1$ and filtration degree 2^i. For example, (here $\iota = \iota_{(3,1)}$)

$$\widetilde{H}_*(\overline{\mathrm{SP}}^4\, S^3) = \mathbb{Z}_2\{\iota^4, \iota^2 f_{(5,2)}, f^2_{(5,2)}, f_{(9,4)}\},$$

which is better listed as

$$H_{12}(\overline{\mathrm{SP}}^4\, S^3) = \mathbb{Z}_2\{\iota^4\}, \qquad H_{11}(\overline{\mathrm{SP}}^4\, S^3) = \mathbb{Z}_2\{\iota^2 f_{(5,2)}\},$$
$$H_{10}(\overline{\mathrm{SP}}^4\, S^3) = \mathbb{Z}_2\{f^2_{(5,2)}\}, \qquad H_9(\overline{\mathrm{SP}}^4\, S^3) = \mathbb{Z}_2\{f_{(9,4)}\}.$$

This space $\overline{\mathrm{SP}}^4(S^3)$ is 8-connected, and more generally, $\overline{\mathrm{SP}}^k(S^3)$ is $2k$-connected [Kallel and Karoui 2011].

Let us now compare the groups in (103). When $* = 0$, $H_0(S) = \mathbb{Z}_2$ but so is $H_{3k}(\overline{\mathrm{SP}}^k(S^3))$ generated by the class $\iota^k_{(3,1)}$. Also when $* = 1$ and $k \geq 3$, $H_1(S) = H_1(B(\mathbb{R}^2, k-1)) = \mathbb{Z}_2$ but so is

$H_{3k-1}(\overline{\mathrm{SP}}^k(S^3))$ generated by $\{\iota^{k-2} f_{5,2}\}$. There is no contradiction yet. When $*=2$, we get the generator $a^2_{(1,2)} \in H_2(B(\mathbb{R}^2, k-1)) \cong \mathbb{Z}_2$ as soon as $k \geq 5$ ($a^2_{(1,2)}$ is in filtration 4). This gives that $H_2(S) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We claim however that $H_{3k-2}(\overline{\mathrm{SP}}^k(S^3)) = \mathbb{Z}_2$, which will give a contradiction in that case. Indeed a generator in filtration degree k in (105) is written as a finite product

$$\iota^{k_0} f^{k_1}_{5,2} \cdots f^{k_i}_{(2^{i+1}+1,2^i)} \cdots, \qquad \sum_{i \geq 0} k_i 2^i = k.$$

The homological degree of this class is $\sum_{i\geq 0} k_i(2^{i+1}+1) = 2\sum_{i\geq 0} k_i 2^i + \sum_{i \geq 0} k_i$. To obtain the rank of H_{3k-2}, we need to find all the possible sequences of integers $(k_0, k_1, k_2, \dots)$ such that $\sum_{i\geq 0} k_i 2^i = k$ and $2\sum_{i\geq 0} k_i 2^i + \sum_{i\geq 0} k_i = 3k-2$. We have to solve for

$$\sum_{i\geq 0} k_i 2^i = k = 2 + \sum_{i \geq 0} k_i.$$

This immediately gives that $k_i = 0$, $i \geq 2$. There is one and only one solution: $k_0 = k-4$ and $k_1 = 2$. And the group $H_{3k-2}(\overline{\mathrm{SP}}^k(S^3)) \cong \mathbb{Z}_2$ is generated by $\iota^{k-4} f^2_{5,2}$.

The isomorphism (103) cannot hold for $k \geq 5$. We are left to consider the case $k = 3$: here $H_3(S) = \mathbb{Z}_2$ but $H_6(\overline{\mathrm{SP}}^3(S^3)) = 0$, giving a contradiction.

In conclusion since the isomorphism (103) (equivalently (98)) cannot hold, Y must have nontrivial mod-2 homology and thus cannot be contractible as we had asserted. □

The next proposition treats the case $k = 4$: in preparation, we need the following lemma. Recall that S is a manifold with boundary embedded in $\overline{U}_\delta \subset S^2_k * S^2$. We can write $\overline{U}_\delta$ as the union of two sets $\bar{A}$ and $\bar{B}$, where $\bar{A}$ is a three-dimensional disk bundle over S and $\bar{A} \cap \bar{B}$ its restriction over ∂S. We refer to this bundle as the *normal disk bundle* and its boundary as the *sphere normal bundle*. Note that, in the proof of Lemma 5.4, we have used $\widetilde{A} = \bar{A} \setminus S$ and $\widetilde{B} = \bar{B} \setminus S$.

Lemma 5.7. *The sphere normal bundle over ∂S is orientable.*

Proof. We will view this bundle as an extension of a normal sphere bundle over the interior $\dot{S} := \mathrm{int}(S)$ that is orientable (in doing so, we give more details on the construction of $\bar{A}$ and $\bar{A} \cap \bar{B}$).

We recall that the join is given by the equivalence relation $X * Y = X \times Y \times I/\sim$, where $\sim$ are identifications at the endpoints of $I = [0, 1]$; see (9). The join contains the open dense subset $X \times Y \times (0, 1)$ (let us call it the *big cell*). This subset is a manifold of dimension $n + m + 1$ if X and Y are manifolds of dimensions n and m, respectively. In our case, S is a subset of the big cell

$$(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2 \times (0, 1) \subset (S^2_k \setminus (S^2_{k-1})^\delta) * S^2$$

and $\mathrm{int}(S)$ is regularly embedded as a differentiable submanifold. It therefore has a unit normal disk bundle (of dimension 3) in there. This is homeomorphic to a tubular neighborhood V^δ of $\mathrm{int}(S)$. Let us use the same name for the neighborhood and the normal bundle. The normal bundle of $\dot{S}$ in $(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2 \times (0, 1)$ is the normal bundle of $\dot{S}$ in $(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2 \times \{\frac{1}{2}\}$, to which we add a trivial line bundle. We can then consider directly $\dot{S}$ as a subset of $(S^2_k \setminus (S^2_{k-1})^\delta) \times S^2$ and show that it has an orientable rank-2 normal bundle there. Write $D_k := S^2_k \setminus (S^2_{k-1})^\delta$ and

$$S = \left\{ \left(\sum_{i=1}^{k} t_i \delta_{x_i}, x \right) \in D_k \times S^2 : x = x_i \text{ for some } i \right\}.$$

Define V^δ the neighborhood of S in $D_k \times S^2$ as

$$V^\delta = \left\{ \left(\sum_{i=1}^{k} t_i \delta_{x_i}, x \right) \in D_k \times S^2 : |x - x_i| < \frac{\delta}{2} \text{ for some and hence unique } x_i \right\}.$$

The choice of x_i is unique as x cannot be strictly within $\delta/2$ of two distinct x_i and x_j since $d(x_i, x_j) \geq \delta$ according to the definition of S. The neighborhood retracts back to S via the map

$$\left(\sum_{i=1}^{k} t_i \delta_{x_i}, x \right) \mapsto \left(\sum_{i=1}^{k} t_i \delta_{x_i}, x_i \right)$$

where $d(x, x_i) < \delta/2$. Consider the projection map $\pi : \dot{S} \to S^2$ sending $(\sum_{i=1}^{k} t_i \delta_{x_i}, x) \mapsto x_i$ if $d(x, x_i) < \delta/2$. We claim that the normal bundle of $\dot{S}$ in $D_k \times S^2$ is isomorphic to the pullback via π of the tangent bundle TS^2 over S^2. We assume δ to be less than the injectivity radius of S^2. Define a homeomorphism between the tubular neighborhood V^δ of $\dot{S}$ and a normal disk bundle of the pullback of TS^2 over $\dot{S}$ by sending $(\sum_{i=1}^{k} t_i \delta_{x_i}, x)$ with $d(x, x_i) < \delta/2$ for some i to the element in the pullback

$$\left(\left(\sum_{i=1}^{k} t_i \delta_{x_i}, x \right), \exp_{x_i}^{-1}(x) \right)$$

where $\exp_{x_i}$ is the exponential map at $x_i \in S^2$. This map is a homeomorphism onto its image, and the normal bundle to $\dot{S}$ in $D_k \times S^2$ is isomorphic to TS^2. Since TS^2 is orientable (although nontrivial), the normal bundle over $\dot{S}$ is orientable. This bundle can be extended to S by taking the closure of V^δ in $D_k \times S^2 := (S^2 \setminus (S^2_{k-1})^\delta) \times S^2 \times \{\frac{1}{2}\}$. This extension is orientable over all of S since it is orientable over the interior. By adding a line bundle, we get the normal bundle over S in the big cell. This bundle is orientable over all of S and in particular over ∂S. This is our claim. □

Proposition 5.8. *The subspace $Y = (S^2_k * S^2) \setminus S$ is not contractible for all $k \geq 2$.*

Proof. As before, we assume Y is contractible and derive a contradiction. We first show that for any field coefficients $\mathbb{F}$ and $* > k$

$$H_{*+3}(U_\delta \setminus S) \cong H_*(\partial S). \tag{106}$$

Write as before $\overline{U}_\delta \setminus S$ as the union $\widetilde{A} \cup \widetilde{B}$ with $\widetilde{A} \cap \widetilde{B}$ retracting onto the S^2-bundle over ∂S discussed earlier. The Mayer–Vietoris sequence for the union $\widetilde{A} \cup \widetilde{B}$ is given by

$$H_{n+1}(\widetilde{A} \cap \widetilde{B}) \to H_{n+1}(\widetilde{A}) \oplus H_{n+1}(\widetilde{B}) \to H_{n+1}(U_\delta \setminus S) \to H_n(\widetilde{A} \cap \widetilde{B}) \to H_n(\widetilde{A}) \oplus H_n(\widetilde{B}) \to H_n(U_\delta \setminus S).$$

As S has homological dimension at most k and $\widetilde{A}$ is an S^2-bundle over it, $H_n(\widetilde{A})$ vanishes for $n > k+2$. On the other hand, the S^2-bundle over ∂S is orientable (Lemma 5.7) and has a global section; this follows from the fact that the normal bundle over S has a trivial summand and hence there is a nonzero section over all S that we can restrict to ∂S. By the Gysin sequence [Hatcher 2002, §4.D], one has a splitting

$$H_n(\widetilde{A} \cap \widetilde{B}) \cong H_n(\partial S) \oplus H_{n-2}(\partial S), \quad n > 2.$$

Replacing in the Mayer–Vietoris sequence gives for $n > k+2$

$$\cdots \to \begin{matrix} H_{n+1}(\partial S) \\ \oplus \\ H_{n-1}(\partial S) \end{matrix} \xrightarrow{\phi_{n+1}} H_{n+1}(\partial S) \to H_{n+1}(U_\delta \setminus S) \to \begin{matrix} H_{n}(\partial S) \\ \oplus \\ H_{n-2}(\partial S) \end{matrix} \xrightarrow{\phi_n} H_n(\partial S) \to \cdots .$$

Now, in every inclusion of $\widetilde{A} \cap \widetilde{B}$ into $\widetilde{B}$, the fibers (i.e., S^2) contract to a point. Therefore, ϕ_n is trivial on the bottom group while restricted to the top group it is a bijection. This map is an epimorphism, and the long exact sequence for $n > k+2$ splits into short exact sequences

$$0 \to H_{n+1}(U_\delta \setminus S) \to H_n(\partial S) \oplus H_{n-2}(\partial S) \to H_n(\partial S) \to 0.$$

As vector spaces, we get $H_{n+1}(U_\delta \setminus S) \cong H_{n-2}(\partial S)$, which is our claim. Combined with (101), this yields

$$H_*(\partial S) \cong H_{*+1}(S_k^2), \quad * > k. \tag{107}$$

Next we look at the Mayer–Vietoris sequence for the union $S_k^2 = (S_k^2 \setminus S_{k-1}^2) \cup (S_{k-1}^2)^\delta$. It is shown in [Malchiodi 2008a] that $(S_{k-1}^2)^\delta \setminus S_{k-1}^2$ retracts onto $\partial (S_{k-1}^2)^\delta$ so that the long exact sequence becomes

$$\cdots \to H_{n+1}(\partial (S_{k-1}^2)^\delta) \to H_{n+1}(S_{k-1}^2) \oplus H_{n+1}(S_k^2 \setminus S_{k-1}^2) \to H_{n+1}(S_k^2) \to H_n(\partial (S_{k-1}^2)^\delta) \to \cdots .$$

Since the inclusion of S_{k-1}^2 in S_k^2 is contractible and since $S_k^2 \setminus S_{k-1}^2 \simeq B(S^2, k)$ has homological dimension k (see Lemma 5.1), for $n > k$, the short sequence

$$0 \to H_{n+1}(S_k^2) \to H_n(\partial (S_{k-1}^2)^\delta) \to H_n(S_{k-1}^2) \to 0$$

is exact and we have the splitting

$$H_*(\partial (S_{k-1}^2)^\delta) \cong H_*(S_{k-1}^2) \oplus H_{*+1}(S_k^2), \quad * > k. \tag{108}$$

Both isomorphisms (107) and (108) cannot hold simultaneously as we now explain.

A key point to observe is that ∂S is a degree-k regular covering of $\partial (S_{k-1}^2)^\delta$. A property of a covering $\pi : X \to Y$ is the existence of a transfer morphism $\mathrm{tr} : H_*(Y) \to H_*(X)$ so that $\pi_* \circ \mathrm{tr}$ is multiplication in $H_*(Y)$ by the degree of the covering, i.e., by k [Hatcher 2002, §3.G]. If the characteristic of the field of coefficients is prime to k, then this composite is nontrivial and $H_*(Y)$ injects into $H_*(X)$.

When $k = 4$, we have a degree-4 covering $\partial S \to \partial (S_3^2)^\delta$ so that with $\mathbb{F} = \mathbb{F}_3$-coefficients (the finite field with 3 elements) we must have a monomorphism $H_*(\partial (S_3^2)^\delta; \mathbb{F}_3) \hookrightarrow H_*(\partial S; \mathbb{F}_3)$. When $* > 4$, upon combining (107) and (108), we get a monomorphism

$$H_*(S_3^2; \mathbb{F}_3) \oplus H_{*+1}(S_4^2; \mathbb{F}_3) \to H_{*+1}(S_4^2; \mathbb{F}_3).$$

This leads immediately to a contradiction if $H_*(S_3^2; \mathbb{F}_3) \neq 0$ in that range of dimensions.

We know that $H_*(S_3^2) \cong H_{*+1}(\overline{\mathrm{SP}}^3(S^3))$. We therefore wish to show that $H_*(\overline{\mathrm{SP}}^3(S^3); \mathbb{F}_3) \neq 0$ for some $* \geq 6$. It turns out that old calculations of Nakaoka [1956] give us precisely the answer. Nakaoka's Theorem 15.5 states that

$$H^r(\mathrm{SP}^3(S^n); \mathbb{F}_3) \cong \mathbb{F}_3$$

for $r = 0, n, n+4k$ with $1 \leq k \leq [n/2]$ and $k \neq [n/4]$, $r = n+4k+1$ with $1 \leq k \leq [(2n-1)/4]$ and $k \neq [(n-1)/4]$, and $r = 2n$ with $n \equiv -2$ or $1 \pmod 4$. In our case $n = 3$, so $H^r(\mathrm{SP}^3(S^3); \mathbb{F}_3) \cong \mathbb{F}_3$ for $r = 0, 3, 7, 8$. Dually we obtain the same groups for $H_r(\mathrm{SP}^3(S^3); \mathbb{F}_3)$ (since working over a field). But $H_r(\mathrm{SP}^3(S^3); \mathbb{F}_3) \cong H_r(\overline{\mathrm{SP}}^3(S^3); \mathbb{F}_3)$ for $r > 3$ for the following three reasons:

- By construction, $H_r(\overline{\mathrm{SP}}^3(S^3); \mathbb{F}_3) = H_r(\mathrm{SP}^3(S^3), \mathrm{SP}^2(S^3); \mathbb{F}_3)$, $r \geq 1$.
- There is a splitting due originally to Steenrod (for any coefficients [Kallel and Karoui 2011]):
$$H_r(\mathrm{SP}^3(S^3)) \cong H_r(\mathrm{SP}^3(S^3), \mathrm{SP}^2(S^3)) \oplus H_r(\mathrm{SP}^2(S^3)).$$
- $H_r(\mathrm{SP}^2(S^3); \mathbb{F}_3) = 0$ if $r > 3$. In fact, from the covering $(S^3)^2 \to \mathrm{SP}^2(S^3)$, by a consequence of the transfer construction, $H_*(\mathrm{SP}^2(S^3); \mathbb{F}_3)$ is the subvector space of invariant cohomology classes in $H_*(S^3 \times S^3)$ under the induced permutation action interchanging the two spheres. Since S^3 is an odd sphere, the involution acts via $\tau_*([S^3] \otimes [S^3]) = -[S^3] \otimes [S^3]$ and the class $[S^3] \otimes [S^3]$ is not invariant so maps to 0 in $H_*(\mathrm{SP}^2(S^3); \mathbb{F}_3)$.

As a consequence, $H_r(\overline{\mathrm{SP}}^3(S^3); \mathbb{F}_3) \cong \mathbb{F}_3$ for $r = 7, 8$, which gives a contradiction as we had asserted. □

Note that using the transfer property for the homology of a covering used in the proof of Proposition 5.8 we can give an alternative proof of Proposition 5.6 for k odd.

To conclude this topological discussion, it is worthwhile noting that Lemma 5.1 can be used to give a novel proof of the following result on the mod-2 homology of unordered configurations of points in $\mathbb{R}^n$:

Proposition 5.9. *For k odd and $n \geq 2$, one has*

$$H_*(B(\mathbb{R}^n, k); \mathbb{Z}_2) \cong H_*(B(\mathbb{R}^n, k-1); \mathbb{Z}_2).$$

Proof. All homology is with mod-2 coefficients. A starting point is the homology splitting

$$H_q(B(S^n, k)) \cong H_q(B(\mathbb{R}^n, k)) \oplus H_{q-n}(B(\mathbb{R}^n, k-1)). \tag{109}$$

One reference to this result is Theorem 18(1) of [Salvatore 2004]. It is also a special case of a similar result of Kallel, where one can replace the sphere by any closed manifold M and $\mathbb{R}^n$ by $M \setminus \{p\}$, its punctured version. Let $W_{n,k} := F(S^n, k)/\mathfrak{S}_{k-1}$ where $\mathfrak{S}_{k-1}$ acts by permutations on the first $k-1$ coordinates. By projecting onto the last coordinate, we obtain a bundle over S^n with fiber $B(\mathbb{R}^n, k-1)$. Precisely as in the proof of Lemma 5.1, we see that

$$H_*(W_{n,k}) \cong H_*(B(\mathbb{R}^n, k-1)) \oplus H_{*-n}(B(\mathbb{R}^n, k-1)). \tag{110}$$

Consider next the degree-k regular covering $\pi : W_{n,k} \to B(S^n, k) := F(S^n, k)/\mathfrak{S}_k$. There is a transfer morphism $\mathrm{tr} : H_*(B(S^n, k)) \to H_*(W_{n,k})$ so that the composite $\pi_* \circ \mathrm{tr}$ is multiplication by k. Since

k is odd and thus prime to the characteristic of the field $\mathbb{Z}_2$, multiplication by k is injective and $H_*(B(S^n, k))$ necessarily embeds in $H_*(W_{n,k})$; that is, (109) embeds into (110). But $H_*(B(\mathbb{R}^n, k-1))$ always embeds into $H_*(B(\mathbb{R}^n, k))$ (in fact for any coefficients as it is relatively easy to see). This means that $H_*(B(\mathbb{R}^n, k); \mathbb{Z}_2) \cong H_*(B(\mathbb{R}^n, k-1); \mathbb{Z}_2)$ if k is odd as claimed. It also means that $H_*(B(S^n, k)) \cong H_*(W_{n,k})$. □

5.2. ***Min-max scheme.*** To prove Theorem 1.1, we will run a min-max scheme based on (a retraction of) the set Y in (51). More precisely, we will consider the set $Y_{\mathfrak{R}}$ introduced in (94) on which the test functions Φ_λ are modeled. Some parts are quite standard and follow the ideas of [Ding et al. 1999] (see [Malchiodi 2008a] for a Morse-theoretical point of view). For the specific problem (1), the crucial step is Proposition 5.10, giving information on the topology of the low sublevels of J_ρ; see also the comments after the proof.

Given any $L > 0$, Proposition 4.17 guarantees us the existence of $\lambda > 1$ sufficiently large such that $J_\rho(\Phi_\lambda(\nu, p, s)) < -L$ for any $(\nu, p, s) \in Y_{\mathfrak{R}}$. Recalling $\widetilde{\Psi}$ in (29), we take L so large that Corollary 3.8 applies, i.e., such that $\widetilde{\Psi}(J_\rho^{-L}) \subseteq Y$. The crucial step in describing the topology of the low sublevels of J_ρ is:

Proposition 5.10. *Let L and λ be as above, and let $\mathcal{F}$ be the retraction given before* (94). *Then the composition*

$$Y_{\mathfrak{R}} \xrightarrow{\Phi_\lambda} J_\rho^{-L} \xrightarrow{\mathcal{F}\circ\widetilde{\Psi}} Y_{\mathfrak{R}}$$

is homotopically equivalent to the identity map on $Y_{\mathfrak{R}}$.

Proof. We divide the proof in three cases, depending on the values of the join parameter s.

Case 1. Let $s \in [\frac{3}{4}, 1]$. In this case, the test functions we are considering have the form $(\varphi_1^t, \varphi_2^t)$, $t = t(s)$, as defined in Section 4.2.3. Notice that, as discussed at the beginning of the proof of Proposition 4.7, most of the integral of $e^{\varphi_2^t}$ is localized near p and $\sigma_2(\varphi_2^t) \ll \sigma_1(\varphi_1^t)$ for these values of s, which again implies $s(\varphi_1^t, \varphi_2^t) = 1$; see (26). It turns out that, by the construction in Section 3.1, one has

$$\widetilde{\Psi}(\Phi_\lambda(\nu, p, s)) = \widetilde{\Psi}(\varphi_1^t, \varphi_2^t) = (*, \tilde{p}, 1),$$

where $*$ is an irrelevant element of Σ_k (recall that they are all identified when the join parameter equals 1; see (9)) and where $\tilde{p} \in \Sigma$ is a point close to p. If $p(\mathfrak{t}) : [0, 1] \to \Sigma$ is a geodesic joining p to $\tilde{p}$, one can realize the desired homotopy as

$$((\nu, p, s); \mathfrak{t}) \mapsto (\nu, p(\mathfrak{t}), (1-\mathfrak{t})s + \mathfrak{t}), \quad \mathfrak{t} \in [0, 1].$$

Case 2. Let $s \in [\frac{1}{4}, \frac{3}{4}]$. The test functions we are considering here are given in Section 4.2.1. For this range of s, the exponential of the first component φ_1 (see (82)) is well-concentrated around the points $\tilde{x}_i$; see (77). The exponential of the second component φ_2, depending on the value of s, will instead either be concentrated near p or will be spread over Σ in the sense that $\sigma_2(\varphi_2)$ might not be small. Recall the maps $\widetilde{\psi}_l$ given in Proposition 2.4 and the definition of $\hat{\nu}$ involved in the construction of the test functions given in (75): $\hat{\nu} = \mathcal{R}_p(\nu) = \sum_{i=i}^k t_i \delta_{x_i}$. We then have

$$\widetilde{\Psi}(\Phi_\lambda(\nu, p, s)) = \widetilde{\Psi}(\varphi_1, \varphi_2) = \begin{cases} (\widetilde{\psi}_k(\varphi_1), \widetilde{\psi}_1(\varphi_2), s(\varphi_1, \varphi_2)) & \text{if } \sigma_2(\varphi_2) \text{ small,} \\ (\widetilde{\psi}_k(\varphi_1), *, 0) & \text{otherwise,} \end{cases}$$

with $\widetilde{\psi}_1(\varphi_2)$ close to p (whenever defined, i.e., for $\sigma_2(\varphi_2)$ small) and $\widetilde{\psi}_k(\varphi_1)$ close to $\sum_{i=1}^k t_i \delta_{\tilde{x}_i}$ in the distributional sense. Furthermore, writing $\varphi_1 = \varphi_{1,\lambda}$ to emphasize the dependence on λ, it turns out that

$$\widetilde{\psi}_k(\varphi_{1,\lambda}) \to \sum_{i=1}^k t_i \delta_{\tilde{x}_i} \quad \text{as } \lambda \to +\infty,$$

which gives us the homotopy

$$(\nu; \mathfrak{t}) \mapsto \widetilde{\psi}_k(\varphi_{1,\lambda/\mathfrak{t}}), \quad \mathfrak{t} \in [0, 1].$$

Reasoning as in Step 3 of Section 4.2.2, we get a homotopy that deforms the points $\tilde{x}_i$ to the original ones x_i. Letting $\tilde{\gamma}_i$ be the geodesic joining $\tilde{x}_i$ and x_i in unit time, we consider

$$(\nu; \mathfrak{t}) \mapsto \sum_{i=1}^k t_i \delta_{\tilde{\gamma}_i(1-\mathfrak{t})}, \quad \mathfrak{t} \in [0, 1].$$

Notice that for $\mathfrak{t} = 0$ we get in the above homotopy $(\nu; 0) = \mathcal{R}_p(\nu)$. Observe now that $\mathcal{R}_p$ is homotopic to the identity map (see Remark 4.5), and let $\mathcal{H}_{\mathcal{R}_p}$ be the map introduced in Step 4 of Section 4.2.2, which realizes this homotopy. We then consider

$$(\nu; \mathfrak{t}) \mapsto \mathcal{H}_{\mathcal{R}_p}(\nu, 1 - \mathfrak{t}), \quad \mathfrak{t} \in [0, 1].$$

Finally, letting $\mathcal{H}$ be the concatenation of the above homotopies (rescaling the respective domains of definition) and letting $p(\mathfrak{t}) : [0, 1] \to \Sigma$ again be a geodesic joining p to $\widetilde{\psi}_1(\varphi_2)$ (whenever defined), we get the desired homotopy

$$((\nu, p, s); \mathfrak{t}) \mapsto \begin{cases} \big(\mathcal{H}(\nu; \mathfrak{t}), p(\mathfrak{t}), (1 - \mathfrak{t})s + \mathfrak{t}s(\varphi_1, \varphi_2)\big), \ \mathfrak{t} \in [0, 1] & \text{if } \sigma_2(\varphi_2) \text{ is small,} \\ \big(\mathcal{H}(\nu; \mathfrak{t}), p, (1 - \mathfrak{t})s\big), \ \mathfrak{t} \in [0, 1] & \text{otherwise.} \end{cases} \tag{111}$$

Case 3. Let $s \in [0, \frac{1}{4}]$. In this case, the test functions we are considering are as in Section 4.2.2. Notice that for this range of s we always get $\sigma_2(\widehat{\varphi}_2^t) \ll \sigma_1(\widehat{\varphi}_1^t)$ (see the beginning of the proof of Proposition 4.7) and therefore $s(\widehat{\varphi}_1^t, \widehat{\varphi}_2^t) = 0$. We have to further subdivide this case depending on the values of s due to the construction of the test functions in the Steps 1–4 of Section 4.2.2.

Emphasizing in the test functions the dependence on λ and recalling that $t = t(s)$, for $s \in [\frac{3}{16}, \frac{1}{4}]$, we get the property $\widetilde{\psi}_k(\breve{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \sum_{i=1}^k t_i \delta_{\tilde{x}_i}$ (see Step 1). When $s \in [\frac{1}{8}, \frac{3}{16}]$, one has by construction that $\widetilde{\psi}_k(\widetilde{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \sum_{i=1}^k t_i \delta_{\tilde{x}_i}$ (see Step 2). For $s \in [\frac{1}{8}, \frac{3}{16}]$, we instead get $\widehat{\psi}_k(\widetilde{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \sum_{i=1}^k t_i \delta_{\tilde{\gamma}_i}$ (see Step 3). Finally, when $s \in [\frac{1}{8}, \frac{3}{16}]$, we obtain $\overline{\psi}_k(\widetilde{\varphi}_{1,\lambda}^t) \xrightarrow{\lambda\to\infty} \mathcal{H}_{\mathcal{R}_p}(\nu, t)$ (see Step 4).

In any case, we then proceed analogously as in Step 2 and the desired homotopy is given as in the second part of (111). $\square$

In this situation, one says that the set J_ρ^{-L} *dominates* $Y_{\mathcal{R}}$ [Hatcher 2002, p. 528]. Recall now that Y is not contractible (see Proposition 5.6); $Y_{\mathcal{R}}$ being a deformation retract of Y (see Remark 4.16), we get that $Y_{\mathcal{R}}$ is not contractible too. Therefore, by the latter result, we deduce that

$$\Phi_\lambda(Y_{\mathcal{R}}) \text{ is not contractible in } J_\rho^{-L}.$$

Moreover, one can take λ large enough so that $\Phi_\lambda(Y_{\mathcal{R}}) \subset J_\rho^{-2L}$. We next define the topological cone over $Y_{\mathcal{R}}$ by the equivalence relation

$$\mathscr{C} = Y_{\mathcal{R}} \times [0, 1]/Y_{\mathcal{R}} \times \{0\},$$

where $Y_{\mathcal{R}} \times \{0\}$ is identified to a single point, and we consider the min-max value

$$m = \inf_{h\in\Gamma} \max_{\xi\in\mathscr{C}} J_\rho(h(\xi)),$$

where

$$\Gamma = \big\{h : \mathscr{C} \to H^1(\Sigma) \times H^1(\Sigma) : h(\nu, p, s) = \Phi_\lambda(\nu, p, s) \text{ for all } (\nu, p, s) \in \partial\mathscr{C} \simeq Y_{\mathcal{R}}\big\}. \tag{112}$$

First, we observe that the map from $\mathscr{C}$ to $H^1(\Sigma) \times H^1(\Sigma)$ defined by $(\,\cdot\,, t) \mapsto t\Phi_\lambda(\,\cdot\,)$ belongs to Γ; hence, this is a nonempty set. Moreover, by the choice of Φ_λ we have

$$\sup_{(\nu,p,s)\in\partial\mathscr{C}} J_\rho(h(\nu, p, s)) = \sup_{(\nu,p,s)\in Y_{\mathcal{R}}} J_\rho(\Phi_\lambda(\nu, p, s)) \le -2L.$$

The crucial point is to show that $m \ge -L$. Indeed, $\partial\mathscr{C}$ is contractible in $\mathscr{C}$ and hence in $h(\mathscr{C})$ for any $h \in \Gamma$. On the other hand by the fact that $Y_{\mathcal{R}}$ is not contractible and by Proposition 5.10, $\partial\mathscr{C}$ is not contractible in J_ρ^{-L}, so we deduce that $h(\mathscr{C})$ is not contained in J_ρ^{-L}. This being valid for any $h \in \Gamma$, we conclude that $m \ge -L$ necessarily.

It follows from standard variational arguments [Struwe 2000] that the functional J_ρ admits a Palais–Smale sequence at level m. However, this does not guarantee the existence of a critical point since it is not known whether the Palais–Smale condition holds. To bypass this problem, one needs a different argument, usually named the *monotonicity trick*. This technique was first introduced by Struwe [1985] (see also [Ding et al. 1999; Jeanjean 1999; Lucia 2007]) and has been used intensively, so we will be sketchy.

Let us take $\eta > 0$ such that $[\rho_1 - 2\eta, \rho_1 + 2\eta] \times [\rho_2 - 2\eta, \rho_2 + 2\eta] \subset \mathbb{R}^2 \setminus \Lambda$, where Λ is the set defined in (10). Consider then a parameter $\gamma \in [-\eta, \eta]$. It is easy to see that the above min-max geometry holds uniformly for any $\rho_\gamma = (\rho_1 + \gamma, \rho_2 + \gamma)$. In particular, for any $L > 0$, there exists λ large enough so that

$$\sup_{(\nu,p,s)\in\partial\mathscr{C}} J_{\rho_\gamma}(h(\nu, p, s)) < -2L, \qquad m_\gamma = \inf_{h\in\Gamma} \sup_{\xi\in\mathscr{C}} J_{\rho_\gamma}(h(\xi)) \ge -L. \tag{113}$$

In this setting, the following result is well-known:

Lemma 5.11. *The functional $J_{\rho_{\tilde\gamma}}$ possesses a bounded Palais–Smale sequence $(u_{1,n}, u_{2,n})_n$ at level $m_{\tilde\gamma}$ for almost every $\tilde\gamma \in \Upsilon = [-\eta, \eta]$.*

Standard arguments show that a bounded Palais–Smale sequence yields the existence of a critical point; see, e.g., Proposition 5.4 in [Malchiodi 2008b]. Consider now $\tilde{\gamma}_n \in \Upsilon$ such that $\tilde{\gamma}_n \to 0$, and let $(u_{1,n}, u_{2,n})_n$ denote the corresponding solutions. To conclude, it is then sufficient to apply the compactness result given in Theorem 2.1, which implies convergence of $(u_{1,n}, u_{2,n})$ to a solution of (1).

Appendix: Proof of Proposition 4.7

The energy estimates of Proposition 4.7 will follow from the next three lemmas.

Lemma A.1. *If φ_1 and φ_2 are defined as in* (82), *we have that*

$$\fint_\Sigma \varphi_1 \, dV_g = O(1), \qquad \fint_\Sigma \varphi_2 \, dV_g = O(1).$$

Proof. From elementary inequalities (see also the figure on page 1995), it is easy to show that there exists a constant C so that

$$|\varphi_1| + |\varphi_2| \le C\left(1 + \log \frac{1}{d(\cdot\,,p)} + \sum_i \frac{1}{d(\cdot\,,\tilde{x}_i)}\right).$$

As the logarithm of the distance from a fixed point is integrable, the conclusion easily follows. □

In the following, for positive numbers a and b, we will use the notation

$$a \simeq_C b \quad \Longleftrightarrow \quad \text{there exists } C > 1 \text{ such that } \frac{b}{C} \le a \le Cb. \tag{114}$$

Lemma A.2. *Under the above assumptions, one has*

$$\int_\Sigma e^{\varphi_1} \, dV_g \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2, \qquad \int_\Sigma e^{\varphi_2} \, dV_g \simeq_C \max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2 \mu^4}, 1\right\}.$$

Proof. Let $\tau \in (0, +\infty]$ be fixed, and let $\hat{\nu} \in \Sigma_{k,p,\bar{\tau}}$ be as in (75). For simplicity, we may assume that there is only one point in the support of $\hat{\nu}$, i.e., $\hat{\nu} = \delta_{x_j}$. The case of a general $\hat{\nu}$ is then treated in an analogous way. It is not difficult to show that the terms $-\frac{1}{2}v_2$ and $-\frac{1}{2}v_{1,1}$ do not affect the integrals of e^{φ_1} and e^{φ_2}, respectively, and that

$$\int_\Sigma e^{\varphi_1} \, dV_g \simeq_C \int_\Sigma e^{v_1} \, dV_g, \qquad \int_\Sigma e^{\varphi_2} \, dV_g \simeq_C \int_\Sigma e^{v_2} \, dV_g.$$

Therefore, it is enough to prove

$$\int_\Sigma e^{v_1} \, dV_g \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2, \qquad \int_\Sigma e^{v_2} \, dV_g \simeq_C \max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2 \mu^4}, 1\right\}. \tag{115}$$

We start by observing that, by definition, for $d(x_j, p) \le 4/\lambda_j$

$$v_1(x) = \log \frac{1}{((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x,p)^2)^3}.$$

By an elementary change of variables, we find

$$\int_\Sigma e^{v_1} \, dV_g = \int_\Sigma \frac{1}{((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x,p)^2)^3} \, dV_g \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^4. \tag{116}$$

By the definition of τ and $\hat{\nu} \in \Sigma_{k,p,\bar{\tau}}$ (see in particular (72) and (73)), recalling that $d(x_j, p) \le 4/\lambda_j$ and that $\lambda_j \ge \lambda$ by construction, we get

$$\frac{1}{\tau} \le d(x_j, p) \le \frac{4}{\lambda_j} \le \frac{C}{\lambda}. \tag{117}$$

By taking λ sufficiently large, we deduce $\tau \gg 1$. It follows that $\check{s} = 1$ and $\check{\lambda} = \lambda$; see (79). Moreover, by (117), we have

$$\frac{C}{\lambda} \le \tau_\lambda \le \lambda.$$

Therefore, we can rewrite (116) as

$$\int_\Sigma e^{v_1}\, dV_g = \int_\Sigma \frac{1}{((\hat{\mathfrak{s}}\tau)^{-2} + d(x, p)^2)^3}\, dV_g \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2$$

and the proof of the first part of (115) is concluded. Suppose now $d(x_j, p) > 4/\lambda_j$, and divide Σ into three subsets:

$$\mathscr{A} = A_{\tilde{x}_j}\left(\frac{1}{s_j\lambda_j}, \frac{d(\tilde{x}_j, p)}{4}\right), \qquad \mathscr{B} = B_{1/(s_j\lambda_j)}(\tilde{x}_j), \qquad \mathscr{C} = \Sigma \setminus (\mathscr{A} \cup \mathscr{B}).$$

We start by estimating

$$\int_{\mathscr{B}} e^{v_1}\, dV_g = \int_{B_{1/(s_j\lambda_j)}(\tilde{x}_j)} \frac{s_j^4 \lambda_j^4 d(\tilde{x}_j, p)^4}{((\hat{\mathfrak{s}}\tau_\lambda)^{-2} + d(x, p)^2)^3}\, dV_g.$$

Observe that if in the latter formula we substitute $d(x, p)$ with $d(\tilde{x}_j, p)$ we get negligible errors, which will be omitted. Therefore, we can rewrite it as

$$\begin{aligned}\int_{\mathscr{B}} e^{v_1}\, dV_g &= \int_{B_{1/(s_j\lambda_j)}(\tilde{x}_j)} \frac{s_j^4\lambda_j^4}{d(\tilde{x}_j, p)^2} \frac{1}{((\hat{\mathfrak{s}}\tau_\lambda d(\tilde{x}_j, p))^{-2} + 1)^3}\, dV_g \\ &= \frac{s_j^2\lambda_j^2}{d(\tilde{x}_j, p)^2} \frac{C}{((\hat{\mathfrak{s}}\tau_\lambda d(\tilde{x}_j, p))^{-2} + 1)^3} = s_j^2 \tilde{s}_j^2 \frac{\lambda_j^2}{d(x_j, p)^2} \frac{C}{((\hat{\mathfrak{s}}\tau_\lambda d(\tilde{x}_j, p))^{-2} + 1)^3},\end{aligned}$$

where in the last equality we have used (77). Exploiting now the conditions (80) and (81) and the assumption $d(x_j, p) > 4/\lambda_j$ and recalling that $d(x_j, p) \ge 1/\tau$ by definition (73), we conclude that

$$\int_{\mathscr{B}} e^{v_1}\, dV_g = \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2 \frac{C}{((\hat{\mathfrak{s}}\tau_\lambda d(\tilde{x}_j, p))^{-2} + 1)^3} \simeq_C \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2.$$

It is then not difficult to show that

$$\int_{\mathscr{A}} e^{v_1}\, dV_g \le \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2 C, \qquad \int_{\mathscr{C}} e^{v_1}\, dV_g \le \hat{\mathfrak{s}}^4 \tau_\lambda^2 \check{\lambda}^2 C$$

for some $C > 0$. This concludes the proof of the first part of (115).

For the second part of (115), similarly as before, we divide Σ into

$$\widetilde{\mathscr{A}} = A_p\left(\frac{1}{\hat{\mathfrak{s}}\tilde{\tau}}, \frac{1}{\hat{\mathfrak{s}}\mu}\right), \quad \widetilde{\mathscr{B}} = B_{1/(\hat{\mathfrak{s}}\tilde{\tau})}(p), \quad \widetilde{\mathscr{C}} = \Sigma \setminus (\widetilde{\mathscr{A}} \cup \widetilde{\mathscr{B}}).$$

For $x \in \widetilde{\mathscr{B}}$, we have $v_2(x) = \log(\mu/\tilde{\tau})^{-4}$; hence,

$$\int_{\widetilde{\mathscr{B}}} e^{v_2}\, dV_g = \int_{B_{1/(\hat{\mathfrak{s}}\tilde{\tau})}(p)} \left(\frac{\mu}{\tilde{\tau}}\right)^{-4} dV_g = \frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4} C. \tag{118}$$

Moreover, working in normal coordinates around p, one gets

$$\int_{\widetilde{\mathscr{A}}} e^{v_2}\, dV_g \le \frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4} C \tag{119}$$

for some $C > 0$. On the other hand, we have

$$\int_{\widetilde{\mathscr{C}}} e^{v_2}\, dV_g \simeq_C 1. \tag{120}$$

From (118), (119), and (120), it follows that

$$\int_\Sigma e^{v_2}\, dV_g \simeq_C \max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4}, 1\right\},$$

which concludes the proof of the second part of (115). $\square$

Recalling the definition of $\hat{\nu} \in \Sigma_{k,p,\bar{\tau}}$ in (75), we introduce now the following sets of indices. Let $I \subseteq \{1, \dots, k\}$ be given by

$$I = \left\{ i : d(x_i, p) > \frac{4}{\lambda_i} \right\}.$$

We then subdivide I into two subsets $I_1, I_2 \subseteq I$:

$$I_1 = \left\{ i : d(x_i, p) \le \frac{1}{\tau_\lambda} \right\}, \qquad I_2 = \left\{ i : d(x_i, p) > \frac{1}{\tau_\lambda} \right\}. \tag{121}$$

Lemma A.3. *Under the above assumptions, one has*

$$\int_\Sigma Q(\varphi_1, \varphi_2)\, dV_g \le 8\pi(\log\tilde{\tau} - \log\mu) + 8|I_1|\pi(\log\check{\lambda} - \log\tau_\lambda) + \sum_{i\in I_2} 8\pi(\log s_i + \log\lambda_i - \log d(\tilde{x}_i, p))$$
$$+ 16\pi \sum_{i\in I_2} \log d(\tilde{x}_i, p) + (24\pi\log\tau_\lambda + 24\pi\log\hat{\mathfrak{s}}) + C$$

for some $C = C(\Sigma)$.

Proof. We start by observing that, by definition, $\nabla v_{1,1} = 0$ in $\Sigma \setminus \bigcup_{i\in I} A_{\tilde{x}_i}(1/s_i\lambda_i, d(\tilde{x}_i, p)/4)$ while $\nabla v_2 = 0$ in $\Sigma \setminus A_p(1/\hat{\mathfrak{s}}\tilde{\tau}, 1/\hat{\mathfrak{s}}\mu)$. We next prove the following estimates on the gradients of $v_{1,1}$, $v_{1,2}$, and v_2:

$$|\nabla v_{1,1}(x)| \le \frac{4}{d_{\min}(x)} \quad \text{in} \bigcup_{i\in I} A_{\tilde{x}_i}\left(\frac{1}{s_i\lambda_i}, \frac{d(\tilde{x}_i, p)}{4}\right), \tag{122}$$

$$|\nabla v_2(x)| \le \frac{4}{d(x,p)} \quad \text{in } A_p\left(\frac{1}{\hat{\mathfrak{s}}\tilde{\tau}}, \frac{1}{\hat{\mathfrak{s}}\mu}\right), \tag{123}$$

$$|\nabla v_{1,2}(x)| \le \frac{6}{d(x,p)} \quad \text{for every } x \in \Sigma, \tag{124}$$

where $d_{\min}(x) = \min_{i \in I} d(x, \tilde{x}_i)$ and

$$|\nabla v_{1,2}(x)| \le C \hat{\mathfrak{s}} \tau_\lambda \quad \text{for every } x \in \Sigma, \tag{125}$$

where C is a constant independent of τ_λ and $\hat{\mathfrak{s}}$.

Concerning (122) and (123), we show the inequalities just for $v_{1,1}$ as for v_2 the proof is similar. We have that

$$\begin{aligned}
\nabla v_{1,1}(x) &= -4 \frac{\sum_{i=1}^k t_i \left(\frac{d(x,\tilde{x}_i)}{d(\tilde{x}_i,p)} \right)^{-5} \nabla_x \left(\frac{d(x,\tilde{x}_i)}{d(\tilde{x}_i,p)} \right)}{\sum_{j=1}^k t_j \left(\frac{d(x,\tilde{x}_j)}{d(\tilde{x}_j,p)} \right)^{-4}} = -4 \frac{\sum_{i=1}^k t_i \left(\frac{d(x,\tilde{x}_i)}{d(\tilde{x}_i,p)} \right)^{-4} \frac{\nabla_x d(x,\tilde{x}_i)}{d(x,\tilde{x}_i)}}{\sum_{j=1}^k t_j \left(\frac{d(x,\tilde{x}_j)}{d(\tilde{x}_j,p)} \right)^{-4}} \\
&= -4 \frac{\sum_{i=1}^k t_i \left(\frac{d(x,\tilde{x}_i)}{d(\tilde{x}_i,p)} \right)^{-4} \frac{\nabla_x d(x,\tilde{x}_i)}{d_{\min}(x)}}{\sum_{j=1}^k t_j \left(\frac{d(x,\tilde{x}_j)}{d(\tilde{x}_j,p)} \right)^{-4}}.
\end{aligned}$$

Exploiting the fact that $|\nabla_x d(x, \tilde{x}_i)| \le 1$, we obtain (122). Moreover, by direct computations, one gets (123). We consider now

$$\nabla v_{1,2}(x) = -3 \frac{\hat{\mathfrak{s}}^2 \tau_\lambda^2 \nabla_x (d^2(x, p))}{1 + \hat{\mathfrak{s}}^2 \tau_\lambda^2 d^2(x, p)}.$$

Using the estimate $|\nabla_x (d^2(x, p))| \le 2d(x, p)$, the properties (124) and (125) easily follow by the inequalities

$$\frac{\hat{\mathfrak{s}}^2 \tau_\lambda^2 d^2(x, p)}{1 + \hat{\mathfrak{s}}^2 \tau_\lambda^2 d^2(x, p)} \le 1, \qquad \frac{\hat{\mathfrak{s}} \tau_\lambda d(x, p)}{1 + \hat{\mathfrak{s}}^2 \tau_\lambda^2 d^2(x, p)} \le 1 \quad \text{for every } x \in \Sigma,$$

respectively. Recalling the definitions of φ_1 and φ_2 in (82) and that $v_1 = v_{1,1} + v_{1,2}$, we obtain

$$\begin{aligned}
\int_\Sigma Q(\varphi_1, \varphi_2)\, dV_g &= \frac{1}{3} \int_\Sigma \big(|\nabla \varphi_1|^2 + |\nabla \varphi_2|^2 + \nabla \varphi_1 \cdot \nabla \varphi_2 \big)\, dV_g \\
&= \frac{1}{3} \int_\Sigma \big(|\nabla v_1|^2 + \tfrac{1}{4} |\nabla v_2|^2 - \nabla v_1 \cdot \nabla v_2 \big)\, dV_g \\
&\qquad + \frac{1}{3} \int_\Sigma \big(|\nabla v_2|^2 + \tfrac{1}{4} |\nabla v_{1,1}|^2 - \nabla v_2 \cdot \nabla v_{1,1} \big)\, dV_g \\
&\qquad + \frac{1}{3} \int_\Sigma \big(\nabla v_1 - \tfrac{1}{2} \nabla v_2 \big) \cdot \big(\nabla v_2 - \tfrac{1}{2} \nabla v_{1,1} \big)\, dV_g \\
&= \frac{1}{4} \int_\Sigma |\nabla v_{1,1}|^2\, dV_g + \frac{1}{4} \int_\Sigma |\nabla v_2|^2\, dV_g + \frac{1}{3} \int_\Sigma |\nabla v_{1,2}|^2\, dV_g \\
&\qquad + \int_\Sigma \big(\tfrac{1}{6} \nabla v_{1,1} \cdot \nabla v_{1,2} - \tfrac{7}{12} \nabla v_{1,1} \cdot \nabla v_2 \big)\, dV_g.
\end{aligned} \tag{126}$$

We start by observing that the integral of the mixed terms is uniformly bounded. Indeed, we claim that

$$\nabla v_{1,1} \cdot \nabla v_2 = 0. \tag{127}$$

By the remark before (122), (127) will follow by proving $A_{\tilde{x}_i}(1/s_i \lambda_i, d(\tilde{x}_i, p)/4) \cap A_p(1/\hat{\mathfrak{s}} \tilde{\tau}, 1/\hat{\mathfrak{s}} \mu) = \varnothing$ for all $i \in I$. Recall the constant $\bar{\delta}$ in (77). Clearly, when all the points of the support of $\hat{\nu}$ are bounded away from p, i.e., $d(x_i, p) > \bar{\delta}$ for all i, we get the conclusion. Consider now the case $d(x_i, p) \le \bar{\delta}$ for some i, and observe that in this case $\tilde{s}_i = \hat{\mathfrak{s}}$; see (77). Moreover, by taking $\bar{\delta}$ sufficiently small, one has also $\check{s} \le C$ by the definition (79) (see also (117) and the motivation above it). To prove that the above two subsets are disjoint, one just has to ensure that $d(\tilde{x}_i, p) \gg 1/\hat{\mathfrak{s}}\mu$. We distinguish between two cases. Suppose first that $d(x_i, p) > 1/\tau_\lambda$. By the assumptions we have made and by (80), one gets

$$d(\tilde{x}_i, p) = \frac{1}{\tilde{s}_i} d(x_i, p) = \frac{1}{\hat{\mathfrak{s}}} d(x_i, p) \ge \frac{1}{\hat{\mathfrak{s}}\lambda_i} = \frac{1}{\hat{\mathfrak{s}} d(x_i, p)\tau_\lambda \check{\lambda}} \ge \frac{1}{C\hat{\mathfrak{s}}\tau_\lambda\check{\lambda}} = \frac{1}{C\hat{\mathfrak{s}}\tau_\lambda\check{s}\lambda} \ge \frac{1}{C\hat{\mathfrak{s}}\tau_\lambda\lambda} \gg \frac{1}{\hat{\mathfrak{s}}\mu}$$

by the choice of the parameters μ and λ. The case $d(x_i, p) \le 1/\tau_\lambda$ is treated in the same way with minor modifications. This conclude the proof of (127).

We claim now that

$$\int_\Sigma \nabla v_{1,1} \cdot \nabla v_{1,2}\, dV_g \le C. \tag{128}$$

We introduce the sets

$$A_i = \left\{ x \in \Sigma : d(x, \tilde{x}_i) = \min_{j \in I} d(x, x_j) \right\}. \tag{129}$$

By (122) and (125), we get

$$\begin{aligned}\int_\Sigma \nabla v_{1,1} \cdot \nabla v_{1,2}\, dV_g &\le \int_\Sigma \frac{C}{d_{\min}(x)\, d(x,p)}\, dV_g \le \sum_{i\in I} \int_{A_i} \frac{C}{d(x,\tilde{x}_i)\, d(x,p)}\, dV_g \\ &\le \sum_{i\in I} \int_{A_{\tilde{x}_i}(1/(s_i\lambda_i), d(\tilde{x}_i,p)/4)} \frac{C}{d(x,\tilde{x}_i)\, d(\tilde{x}_i,p)}\, dV_g \le C,\end{aligned}$$

which proves the claim (128).

Using the estimate (122), one has

$$\begin{aligned}\frac{1}{4}\int_\Sigma |\nabla v_{1,1}|^2\, dV_g &\le 4\int_\Sigma \frac{1}{d^2_{\min}(x)}\, dV_g \le 4\sum_{i\in I}\int_{A_i} \frac{1}{d^2(x,\tilde{x}_i)}\, dV_g \\ &\le 4\sum_{i\in I}\int_{A_{\tilde{x}_i}(1/(s_i\lambda_i), d(\tilde{x}_i,p)/4)} \frac{1}{d^2(x,\tilde{x}_i)}\, dV_g \\ &\le \sum_{i\in I} 8\pi(\log s_i + \log\lambda_i + \log d(\tilde{x}_i, p)) + C.\end{aligned} \tag{130}$$

Recalling the definitions of $I_1, I_2 \subseteq I$ given in (121), we observe that for $i \in I_1$ we get $\lambda_i = \check{\lambda}$ and $\tilde{s}_i = \hat{\mathfrak{s}}$; see (80) and (77), respectively. Moreover, taking into account (81), we deduce

$$\begin{aligned}\frac{1}{4}\int_\Sigma |\nabla v_{1,1}|^2\, dV_g &\le 8|I_1|\pi(\log\check{\lambda} - \log\tau_\lambda) + \sum_{i\in I_2} 8\pi(\log s_i + \log\lambda_i + \log d(\tilde{x}_i, p)) + C \\ &= 8|I_1|\pi(\log\check{\lambda} - \log\tau_\lambda) + \sum_{i\in I_2} 8\pi(\log s_i + \log\lambda_i - \log d(\tilde{x}_i, p)) \\ &\qquad + 16\pi\sum_{i\in I_2}\log d(\tilde{x}_i, p) + C.\end{aligned} \tag{131}$$

Similarly as for (130), by (123), we get

$$\frac{1}{4}\int_\Sigma |\nabla v_2|^2\, dV_g = 4\int_{A_p(1/(\hat{\mathfrak{s}}\tilde{\tau}),1/(\hat{\mathfrak{s}}\mu))} \frac{1}{d^2(x,p)}\, dV_g \le 8\pi(\log\tilde{\tau}-\log\mu)+C. \tag{132}$$

To estimate the term $|\nabla v_{1,2}|^2$, we consider $\Sigma = B_{1/\hat{\mathfrak{s}}\tau_\lambda}(p)\cup(\Sigma\setminus B_{1/\hat{\mathfrak{s}}\tau_\lambda}(p))$. From (124), we deduce that

$$\int_{B_{1/(\hat{\mathfrak{s}}\tau_\lambda)}(p)} |\nabla v_{1,2}|^2\, dV_g \le C.$$

Then using (124), one finds

$$\frac{1}{3}\int_{\Sigma\setminus B_{1/(\hat{\mathfrak{s}}\tau_\lambda)}(p)} |\nabla v_{1,2}|^2\, dV_g \le 12\int_{\Sigma\setminus B_{1/(\hat{\mathfrak{s}}\tau_\lambda)}(p)} \frac{1}{d^2(x,p)}\, dV_g \le 24\pi(\log\tau_\lambda+\log\hat{\mathfrak{s}})+C. \tag{133}$$

Finally, by (127) and (128) and by inserting (131), (132), and (133) into (126), we get the conclusion. □

Proof of Proposition 4.7. Using Lemmas A.1, A.2, and A.3, the energy estimate we get is

$$\begin{aligned} J_\rho(\varphi_1,\varphi_2) &\le 8\pi(\log\tilde{\tau}-\log\mu)+8|I_1|\pi(\log\check{\lambda}-\log\tau_\lambda)+\sum_{i\in I_2}8\pi(\log s_i+\log\lambda_i-\log d(\tilde{x}_i,p)) \\ &\qquad + 16\pi\sum_{i\in I_2}\log d(\tilde{x}_i,p)+(24\pi\log\tau_\lambda+24\pi\log\hat{\mathfrak{s}}) \\ &\qquad - \rho_1(4\log\hat{\mathfrak{s}}+2\log\tau_\lambda+2\log\check{\lambda})-\rho_2\log\max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\}+C \\ &\le 8\pi(\log\tilde{\tau}-\log\mu)+8|I_1|\pi(\log\check{\lambda}-\log\tau_\lambda)+\sum_{i\in I_2}8\pi(\log s_i+\log\tilde{s}_i+\log\lambda_i \\ &\qquad -\log d(x_i,p))+16\pi\sum_{i\in I_2}\log d(\tilde{x}_i,p)+(24\pi\log\tau_\lambda+24\pi\log\hat{\mathfrak{s}}) \\ &\qquad - \rho_1(4\log\hat{\mathfrak{s}}+2\log\tau_\lambda+2\log\check{\lambda})-\rho_2\log\max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\}+C \end{aligned}$$

for some constant $C>0$. Exploiting the conditions (80) and (81), we obtain

$$\begin{aligned} J_\rho(\varphi_1,\varphi_2) &\le 8\pi(\log\tilde{\tau}-\log\mu)+8|I_1|\pi(\log\check{\lambda}-\log\tau_\lambda)+\sum_{i\in I_2}8\pi(2\log\hat{\mathfrak{s}}+\log\check{\lambda}+\log\tau_\lambda) \\ &\qquad + 16\pi\sum_{i\in I_2}\log d(\tilde{x}_i,p)+(24\pi\log\tau_\lambda+24\pi\log\hat{\mathfrak{s}}) \\ &\qquad - \rho_1(4\log\hat{\mathfrak{s}}+2\log\tau_\lambda+2\log\check{\lambda})-\rho_2\log\max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\}+C. \end{aligned} \tag{134}$$

Recalling the definitions of I_1 and I_2 in (121), we distinguish between two cases.

Case 1. Suppose first that $I_1\neq\varnothing$. By construction, it follows that $\tau\gg 1$; see (72) and (73). Therefore, by (78), we get $\hat{\mathfrak{s}}=\mathfrak{s}$. On the other hand, using (79) and the definition of $\check{\lambda}$ under it, we deduce $\check{\lambda}\le C\lambda$.

For $\hat{\mathfrak{s}}\ll\tilde{\tau}/\mu^2$, we get in (134)

$$\max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\}=\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4}. \tag{135}$$

In this case, (134) can be rewritten as

$$J_\rho(\varphi_1,\varphi_2) \le \log\tilde{\tau}(8\pi-2\rho_2)+\log\lambda\big(8(|I_1|+|I_2|)\pi-2\rho_1\big)+\log\hat{\mathfrak{s}}\big(24\pi+16|I_2|\pi-4\rho_1+2\rho_2\big) \\ +\log\tau_\lambda\big(8|I_2|\pi-8|I_1|\pi+24\pi-2\rho_1\big)+\log\mu(4\rho_2-8\pi)+C. \quad (136)$$

Recalling that $\hat{\mathfrak{s}} \ll \tilde{\tau}/\mu^2$, the latter estimate is negative by the choice of the parameters $\tilde{\tau} \gg \mu \gg \lambda$ and $\rho_2 > 4\pi$.

When instead $\hat{\mathfrak{s}} = \tilde{\tau}/\mu^2 + O(1)$, we have

$$\max\left\{\frac{\tilde{\tau}^2}{\hat{\mathfrak{s}}^2\mu^4},1\right\}=1. \quad (137)$$

Considering now (134) and observing that $\log\hat{\mathfrak{s}} = \log\tilde{\tau} - 2\log\mu + C$, we end up with

$$J_\rho(\varphi_1,\varphi_2) \le \log\tilde{\tau}\big(32\pi+16|I_2|\pi-4\rho_1\big)+\log\lambda\big(8(|I_1|+|I_2|)\pi-2\rho_1\big) \\ +\log\tau_\lambda\big(8|I_2|\pi-8|I_1|\pi+24\pi-2\rho_1\big)+\log\mu\big(8\rho_1-56\pi-32|I_2|\pi\big)+C.$$

The crucial fact is that by construction of $\Sigma_{k,p,\tilde{\tau}}$ (see (70)) $|I_2| \le k-2$ whenever $|I_1| \ne \varnothing$. Hence, we conclude that

$$J_\rho(\varphi_1,\varphi_2) \le \log\tilde{\tau}(16k\pi-4\rho_1)+\log\lambda\big(8(|I_1|+|I_2|)\pi-2\rho_1\big)+\log\tau_\lambda\big(8|I_2|\pi-8|I_1|\pi+24\pi-2\rho_1\big) \\ +\log\mu\big(8\rho_1-56\pi-32|I_2|\pi\big)+C,$$

which is large-negative since $\rho_1 > 4k\pi$ and by the choice of the parameters.

Case 2. Suppose now $I_1 = \varnothing$. By construction, we deduce that $\tau \le C$; see (72) and (73). Therefore, using (78), we obtain $\hat{\mathfrak{s}} \le C$. In this case, the equality in (135) always holds true. Moreover, by (79), we have $\check{\lambda} = \mathfrak{s}\lambda$. Hence, (134) can be rewritten as

$$J_\rho(\varphi_1,\varphi_2) \le \log\mathfrak{s}(8|I_2|\pi-2\rho_1)+\log\tilde{\tau}(8\pi-2\rho_2)+\log\lambda(8|I_2|\pi-2\rho_1) \\ +\log\tau_\lambda\big(8|I_2|\pi+24\pi-2\rho_1\big)+\log\mu(4\rho_2-8\pi)+C.$$

Observing that $|I_2| \le k$, we conclude that the latter estimate is large-negative since $\rho_1 > 4k\pi$ and $\rho_2 > 4\pi$ and by the choice of the parameters. □

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WELL-POSEDNESS AND SCATTERING FOR THE ZAKHAROV SYSTEM IN FOUR DIMENSIONS

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The Cauchy problem for the Zakharov system in four dimensions is considered. Some new well-posedness results are obtained. For small initial data, global well-posedness and scattering results are proved, including the case of initial data in the energy space. None of these results are restricted to radially symmetric data.

1. Introduction and main results

Let $\alpha > 0$. The Zakharov system

$$\begin{cases} i\dot{u} - \Delta u = nu, \\ \ddot{n}/\alpha^2 - \Delta n = -\Delta |u|^2 \end{cases} \tag{1-1}$$

with initial data

$$u(0,x) = u_0, \quad n(0,x) = n_0, \quad \dot{n}(0,x) = n_1 \tag{1-2}$$

is considered as a simplified mathematical model for Langmuir waves in a plasma, which couples the envelope $u : \mathbb{R}^{1+d} \to \mathbb{C}$ of the electric field and the ion density $n : \mathbb{R}^{1+d} \to \mathbb{R}$, neglecting magnetic effects and the vector field character of the electric field; see [Sulem and Sulem 1999, Chapter V; Zakharov 1972].

The parameter $\alpha > 0$ is called the ion sound speed. Formally, as $\alpha \to \infty$, (1-1) reduces to the focusing cubic Schrödinger equation

$$i\dot{u} - \Delta u = |u|^2 u, \tag{1-3}$$

which is energy-critical in dimension $d = 4$; see for example [Kenig and Merle 2006; Killip and Visan 2010; Dodson 2014] and the references therein concerning recent developments on global-well-posedness, blow-up and scattering for (1-3). For rigorous results on the subsonic limit (as $\alpha \to \infty$) of (1-1) to (1-3) we refer the reader to [Schochet and Weinstein 1986; Ozawa and Tsutsumi 1992; Masmoudi and Nakanishi 2008].

Strong solutions (u, n) of the Zakharov system preserve the mass

$$\int_{\mathbb{R}^d} |u|^2 \, dx = \int_{\mathbb{R}^d} |u_0|^2 \, dx \tag{1-4}$$

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and the energy, with $D := \sqrt{-\Delta}$,

$$E(u,n,\dot n) = \int_{\mathbb{R}^d} |\nabla u|^2 + \frac{|D^{-1}\dot n|^2}{2\alpha^2} + \frac{|n|^2}{2} - n|u|^2\,dx = E(u_0,n_0,n_1). \tag{1-5}$$

In view of (1-5), a natural space for the initial data is the energy space

$$(u_0,n_0,n_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times \dot H^{-1}(\mathbb{R}^d). \tag{1-6}$$

For initial data in the energy space, the Zakharov system is known to be globally well-posed if $d = 1$ (see [Ginibre et al. 1997]) and locally well-posed if $d = 2, 3$ (see [Bourgain and Colliander 1996]). A low regularity local well-posedness theory has been developed in [Ginibre et al. 1997] in all dimensions, with further extensions in [Bejenaru et al. 2009] if $d = 2$, and in [Bejenaru and Herr 2011] if $d = 3$; see also the references therein for previous work. In the case of the torus $\mathbb{T}^d$, well-posedness results were proved in [Takaoka 1999; Kishimoto 2013].

In [Merle 1996] blow-up results in finite or infinite time for initial data of negative energy were proved if $d = 3$ and, if $d = 2$, blow-up in finite time was derived in [Glangetas and Merle 1994a; 1994b]. Concerning the final data problem in weighted Sobolev spaces, we refer to [Shimomura 2004; Ginibre and Velo 2006; Ozawa and Tsutsumi 1993/94].

Recently, the asymptotic behavior as $t \to \infty$ for the initial data problem was studied in dimension $d = 3$: In [Guo and Nakanishi 2014], small-data energy scattering in the radial case was obtained by using a normal form technique and the improved Strichartz estimates for radial functions from [Guo and Wang 2014]. In [Guo et al. 2013], a dichotomy between scattering and grow-up was obtained for radial solutions with energy below the ground state energy. In the nonradial case in dimension $d = 3$, scattering was obtained in [Hani et al. 2013] under the assumption that the initial data are small enough and have sufficient regularity and decay. This result was improved recently in [Guo et al. 2014a; Guo 2014], where scattering was shown for small initial data belonging to the energy space with some additional angular regularity.

In the present paper, we continue the analysis of the initial value problem (1-1) and focus on the energy-critical dimension $d = 4$. In particular, we will address the small-data global well-posedness and scattering problem in the energy space, i.e.,

$$(u_0,n_0,n_1) \in H^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4) \times \dot H^{-1}(\mathbb{R}^4), \tag{1-7}$$

with no additional symmetry or decay assumption.

We reduce the wave equation to a first-order equation as usual: let

$$N := n - \frac{iD^{-1}\dot n}{\alpha}; \tag{1-8}$$

then $n = \operatorname{Re} N = \frac{1}{2}(N + \overline{N})$ and the Zakharov system for (u, N) reads as follows:

$$\begin{cases} (i\partial_t - \Delta)u = \frac{1}{2}Nu + \frac{1}{2}\overline{N}u, \\ (i\partial_t + \alpha D)N = \alpha D|u|^2. \end{cases} \tag{1-9}$$

The Hamiltonian then becomes

$$E(u,n,\dot n)=E_Z(u,N):=\int_{\mathbb{R}^4}|\nabla u|^2+\tfrac{1}{2}|N|^2-\operatorname{Re}N|u|^2\,dx. \tag{1-10}$$

We will restrict ourselves to the system (1-9). Our first main result is a small-data global well-posedness and scattering result.

Theorem 1.1. *There exists $\varepsilon_0=\varepsilon_0(\alpha)>0$ such that, for any $(s,l)\in\mathbb{R}^2$ satisfying $(s,l)=(1,0)$ or*

$$l\ge 0,\quad s<4l+1,\quad (s,l)\neq(2,3),\quad \max\bigl(\tfrac{1}{2}(l+1),l-1\bigr)\le s\le\min\bigl(l+2,2l+\tfrac{11}{8}\bigr) \tag{1-11}$$

and any initial data $(u_0,N_0)\in H^s(\mathbb{R}^4)\times H^l(\mathbb{R}^4)$ satisfying

$$\|(u_0,N_0)\|_{H^{1/2}(\mathbb{R}^4)\times L^2(\mathbb{R}^4)}<\varepsilon_0, \tag{1-12}$$

there exists a unique global solution $(u,N)\in C(\mathbb{R};H^s(\mathbb{R}^4)\times H^l(\mathbb{R}^4))$ of (1-9) with some space-time integrability. The solution map is continuous in the norms

$$H^s\times H^l\to L^\infty(\mathbb{R};H^s\times H^l),\quad (u_0,N_0)\mapsto(u,N). \tag{1-13}$$

Moreover, there exist $(u^\pm,N^\pm)\in H^s(\mathbb{R}^4)\times H^l(\mathbb{R}^4)$ such that

$$\lim_{t\to\pm\infty}\bigl(\|u(t)-S(t)u^\pm\|_{H^s}+\|N(t)-W_\alpha(t)N^\pm\|_{H^l}\bigr)=0, \tag{1-14}$$

where $S(t)=e^{-it\Delta}$ and $W_\alpha(t)=e^{it\alpha D}$ are the free propagators.

In the above statement, we need the space-time integrability to ensure uniqueness. For example, for any $T>0$,

$$u\in L^2((0,T);B^{1/2}_{4,2}(\mathbb{R}^4)) \tag{1-15}$$

is sufficient for uniqueness on $[0,T]$, where $B^{1/2}_{4,2}$ is the inhomogeneous Besov space. See Propositions 3.1, 5.1 and 5.2 for more detail on the space-time integrability.

Very recently, we learned about independent work of Kato and Tsugawa [$\ge$ 2015]. By a different method, they prove the small data scattering for $l=s-\frac{1}{2}\ge 0$, using bilinear estimates in U^p-V^p spaces for the standard iteration. While their iteration scheme is more direct, our estimates are more elementary and we cover a wider range of (s,l).

Our second result is a large-data local well-posedness result for the same range of regularity (s,l) as above, except for the energy space $H^1(\mathbb{R}^4)\times L^2(\mathbb{R}^4)$.

Theorem 1.2. *Let $(s,l)\in\mathbb{R}^2$ satisfy (1-11). Then, for any $(u_0,N_0)\in H^s(\mathbb{R}^4)\times H^l(\mathbb{R}^4)$, there exists $T=T(u_0,N_0)>0$ and a unique local solution $(u,N)\in C([-T,T];H^s(\mathbb{R}^4)\times H^l(\mathbb{R}^4))$ to (1-9) satisfying some space-time integrability; (1-15) is enough for the uniqueness. Both $T>0$ and (u,N) depend continuously on (u_0,N_0).*

In dimension $d=4$, Ginibre, Tsutsumi and Velo [Ginibre et al. 1997] proved local well-posedness in the range $l\le s\le l+1,\ l>0,\ 2s>l+1$; see Figure 1 (right). Their method is the standard Picard iteration argument in the $X^{s,b}$ spaces. Theorem 1.2 gives further local well-posedness results in a new

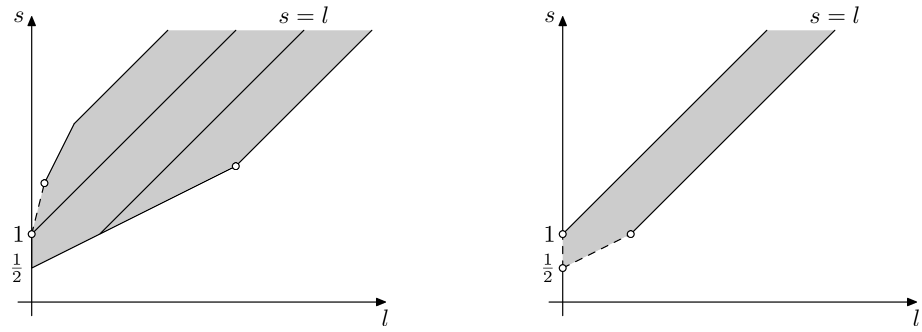

Figure 1. *Left*: the range of (s, l) obtained in Theorems 1.1 and 1.2. *Right*: the range of (s, l) obtained in [Ginibre et al. 1997].

region, indicated in Figure 1 (left), while Theorem 1.1 covers the same range of exponents as well as the energy space $(s, l) = (1, 0)$, which is missing from the large-data result, Theorem 1.2.

The proofs for Theorems 1.1 and 1.2 use the normal form technique and Strichartz estimates as in [Guo and Nakanishi 2014] and the follow-up papers [Guo et al. 2013; 2014a; Guo 2014] and related work on the Klein–Gordon–Zakharov system [Guo et al. 2014b; 2014c]. Our argument is somewhat simpler than [Ginibre et al. 1997] and it also implies some scattering results.

There is a qualitative difference in our proof between $s < l + 1$ and $s > l + 1$. Since the Strichartz norm of $W_\alpha(t)$ is worse than that of $S(t)$, for $s < l + 1$ we use only the H^l_x norm for N, while keeping the full Strichartz norm for u. For $s > l + 1$, however, this strategy is prevented by the normal form of u, so we need to modify the Strichartz norm for u, and to use that of N. Consequently, we cannot recover all the Strichartz norms of $S(t)$ for u, in spite of the scattering. See Proposition 5.2 for the precise statement. This is consistent with the fact that [Ginibre et al. 1997] is restricted to $s \le l + 1$ and $X^{s,b}$ implies the full range of Strichartz norm.

The energy space $(s, l) = (1, 0)$ is at the intersection of $s = l + 1$ and $l = 0$, where our multilinear estimates actually break down. More precisely, we cannot close any Strichartz bound for the normal form of u when $(s, l) = (1, 0)$. This is why $(1, 0)$ is excluded from Theorem 1.2. Fortunately enough, with the help of the conservation law (1-10) and using the well-posedness in nearby (s, l), we are still able to show global well-posedness and scattering in the energy space $(s, l) = (1, 0)$ for small data as in Theorem 1.1. Since the limit NLS (1-3) is critical in the energy space $H^1(\mathbb{R}^4)$, it may have blow-up with bounded $H^1 \times L^2$ norm for large data, which suggests that there may be essential difference between large and small data.

At the other excluded endpoint, $(s, l) = (2, 3)$, we can prove a strong ill-posedness result, both by instant exit and by nonexistence.

Theorem 1.3. *There exists a radial function $u_0 \in H^2(\mathbb{R}^4)$ such that, for any $\varepsilon > 0$, any $N_0 \in H^3(\mathbb{R}^4)$, and any $T_0 > 0$, the system* (1-9) *has no solution $(u, N) \in C([0, T_0]; \mathcal{S}'(\mathbb{R}^4)^2)$ satisfying $(u(0), N(0)) = (\varepsilon u_0, N_0)$, the equation* (1-9) *in the distribution sense, and*

$$(u, N) \in L^2((0, T_0); H^1(\mathbb{R}^4) \times H^3(\mathbb{R}^4)). \tag{1-16}$$

Moreover, the unique local solution $(u, N) \in C([-T, T]; H^2 \times H^2)$ *given by Theorem 1.2 satisfies* $N(t) \notin H^3(\mathbb{R}^4)$ *for all* $t \in [-T, T] \setminus \{0\}$.

Note that (1-16) is weaker than the usual weak solutions, as it does not require $(u(t), N(t)) \in H^s \times H^l$ for all t near 0. The above ill-posedness is due to the mismatch of regularity between u and N in the normal form for N.

The rest of paper is organized as follows. In Section 2, we recall the normal form reduction from [Guo and Nakanishi 2014], and then gather multilinear estimates used in the later sections. They easily follow from the Littlewood–Paley decomposition, Coifman–Meyer bilinear estimate, Strichartz and Sobolev inequalities. Using these estimates and the standard contraction argument, we first prove the small data scattering in $H^s \times H^l$ for $s \le l + 1$ in Section 3, and then the local well-posedness for large data in $H^{1/2} \times L^2$ in Section 4. In Section 5, we extend these results to higher regularity by persistence of regularity, except for the energy space $(s, l) = (1, 0)$. Theorem 1.1 for $(s, l) \neq (1, 0)$ follows from Propositions 3.1, 5.1 and 5.2. Similarly, Theorem 1.2 follows from Propositions 4.2, 5.1 and 5.2. In Section 6, we prove Theorem 1.1 in the energy space $(s, l) = (1, 0)$, using the results in $(s, 0)$ for $s < 1$ and in $(1, l)$ for $l > 0$. In Section 7, we prove the ill-posedness result, Theorem 1.3 at $(s, l) = (2, 3)$.

2. Normal form and multilinear estimates

In this section, we set up integral equations and basic estimates for solving the equation. Our analysis is based on the normal form reduction devised in [Guo and Nakanishi 2014].

2A. ***Review of the normal form reduction and notation from*** [Guo and Nakanishi 2014]. Let $\hat{\phi} = \mathscr{F}\phi$ denote the Fourier transform of ϕ. We use $S(t)$ and $W_\alpha(t)$ to denote the Schrödinger and wave semigroup, respectively:

$$S(t)\phi = \mathscr{F}^{-1}(e^{it|\xi|^2}\hat{\phi}) \quad \text{and} \quad W_\alpha(t)\phi = \mathscr{F}^{-1}(e^{i\alpha t|\xi|}\hat{\phi}).$$

Fix a radial, smooth, bump function $\eta_0 : \mathbb{R}^4 \to [0, 1]$ with support in the ball $B_{\frac{8}{5}}(0)$, which is equal to 1 in the smaller ball $B_{4/5}(0)$. For $k \in \mathbb{Z}$, let $\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$ and $\chi_{\le k}(\xi) = \eta_0(\xi/2^k)$, and let P_k and $P_{\le k}$ denote the corresponding Fourier multipliers.

For two functions u and v, and a fixed $K \in \mathbb{N}$, $K \ge 5$, we define the paraproduct-type operators

$$(uv)_{LH} := \sum_{k \in \mathbb{Z}} (P_{\le k-K} u)(P_k v), \quad (uv)_{HL} := (vu)_{LH}, \quad (uv)_{HH} := \sum_{\substack{|k_1 - k_2| \le K-1 \\ k_1, k_2 \in \mathbb{Z}}} (P_{k_1} u)(P_{k_2} v), \tag{2-1}$$

so that $uv = (uv)_{LH} + (uv)_{HL} + (uv)_{HH}$. We also define

$$\begin{aligned} (uv)_{\alpha L} &:= \sum_{\substack{|k - \log_2 \alpha| \le 1, \\ k \in \mathbb{Z}}} (P_k u)(P_{\le k-K} v), \quad (uv)_{L\alpha} := (vu)_{\alpha L}, \\ (uv)_{XL} &:= \sum_{\substack{|k - \log_2 \alpha| > 1, \\ k \in \mathbb{Z}}} (P_k u)(P_{\le k-K} v), \quad (uv)_{LX} := (vu)_{XL}, \end{aligned} \tag{2-2}$$

so that $(uv)_{HL} = (uv)_{\alpha L} + (uv)_{XL}$.

Moreover, for any of the bilinear operators $(uv)_*$ defined in (2-1)–(2-2), we denote its symbol (multiplier) by $\mathcal{P}_*$. We denote finite sums of these bilinear operators in the obvious way, e.g., $(uv)_{LH+HH} = (uv)_{LH} + (uv)_{HH}$. Henceforth, for simplicity, we replace the nonlinear term $\frac{1}{2}\operatorname{Re} Nu$ in (1-9) with Nu as in [Guo and Nakanishi 2014], because the complex conjugation here makes no essential difference for our arguments. With these notations, it was shown in [Guo and Nakanishi 2014] that (1-9) is equivalent — at least for smooth solutions — to the integral equation

$$\begin{aligned} u(t) = S(t)u_0 - S(t)\Omega(N,u)(0) + \Omega(N,u)(t) - i\int_0^t S(t-s)\Omega(\alpha D|u|^2, u)(s)\,ds \\ - i\int_0^t S(t-s)\Omega(N, Nu)(s)\,ds - i\int_0^t S(t-s)(Nu)_{LH+HH+\alpha L}(s)\,ds \end{aligned} \tag{2-3}$$

and

$$\begin{aligned} N(t) = W_\alpha(t)N_0 - W_\alpha(t)D\tilde{\Omega}(u,u)(0) + D\tilde{\Omega}(u,u)(t) - i\int_0^t W_\alpha(t-s)\alpha D(u\bar{u})_{HH+\alpha L+L\alpha}\,ds \\ - i\int_0^t W_\alpha(t-s)(D\tilde{\Omega}(Nu,u) + D\tilde{\Omega}(u,Nu))(s)\,ds, \end{aligned} \tag{2-4}$$

where Ω and $\tilde{\Omega}$ are the bilinear Fourier multiplication operators

$$\begin{aligned} \Omega(f,g) &= \mathcal{F}^{-1}\int \mathcal{P}_{XL}\frac{\hat{f}(\xi-\eta)\hat{g}(\eta)}{-|\xi|^2+\alpha|\xi-\eta|+|\eta|^2}\,d\eta, \\ \tilde{\Omega}(f,g) &= \mathcal{F}^{-1}\int \mathcal{P}_{XL+LX}\frac{\alpha\hat{f}(\xi-\eta)\hat{\bar{g}}(\eta)}{|\xi-\eta|^2-|\eta|^2-\alpha|\xi|}\,d\eta. \end{aligned}$$

The equations after normal form reduction can be written as

$$\begin{aligned} (i\partial_t + D^2)(u - \Omega(N,u)) &= (Nu)_{LH+HH+\alpha L} + \Omega(\alpha D|u|^2, u) + \Omega(N, Nu), \\ (i\partial_t + \alpha D)(N - D\tilde{\Omega}(u,u)) &= \alpha D|u|^2_{HH+\alpha L+L\alpha} + D\tilde{\Omega}(Nu,u) + D\tilde{\Omega}(u,Nu). \end{aligned} \tag{2-5}$$

2B. ***Function spaces and Strichartz estimates.*** Let $s, l \in \mathbb{R}$ and $1 \le p, q \le \infty$. We use $B^s_{p,q}$, $\dot{B}^s_{p,q}$ to denote the standard Besov space, with norms

$$\|f\|_{B^s_{p,q}} = \|P_{\le 0}f\|_p + \Big(\sum_{k=1}^{\infty} 2^{ksq}\|P_k f\|_p^q\Big)^{\frac{1}{q}}, \quad \|f\|_{\dot{B}^s_{p,q}} = \Big(\sum_{k=-\infty}^{\infty} 2^{ksq}\|P_k f\|_p^q\Big)^{\frac{1}{q}},$$

with obvious modifications if $q = \infty$, and we simply write $B^s_p = B^s_{p,2}$, $\dot{B}^s_p = \dot{B}^s_{p,2}$.

For exponents $s \le l+1$, we use the resolution spaces

$$\begin{aligned} u \in X^s &:= C(\mathbb{R}; H^s(\mathbb{R}^4)) \cap L^\infty(\mathbb{R}; H^s(\mathbb{R}^4)) \cap L^2(\mathbb{R}; B^s_4(\mathbb{R}^4)), \\ N \in Y^l &:= C(\mathbb{R}; H^l(\mathbb{R}^4)) \cap L^\infty(\mathbb{R}; H^l(\mathbb{R}^4)). \end{aligned} \tag{2-6}$$

For any Banach function space Z on $\mathbb{R}^{1+4}$ and any interval $I \subset \mathbb{R}$, the restriction of Z onto I is denoted by $Z(I)$. For example,

$$X^s([0,T]) = C([0,T]; H^s(\mathbb{R}^4)) \cap L^2((0,T); B^s_4(\mathbb{R}^4)). \tag{2-7}$$

We will use the following well-known Strichartz estimates for the wave and the Schrödinger equation in dimension $d = 4$:

Lemma 2.1 (Strichartz estimates; see [Keel and Tao 1998]). *For any $s \in \mathbb{R}$ and any functions $\phi(x)$ and $f(t,x)$, we have*

$$\begin{aligned}
\|S(t)\phi\|_{L^\infty_t H^s_x \cap L^2_t B^s_4} &\lesssim \|\phi\|_{H^s},\\
\left\| \int_0^t S(t-s) f(s)\,ds \right\|_{L^\infty_t L^2_x \cap L^2_t B^0_4} &\lesssim \|f\|_{L^1_t L^2_x + L^2_t B^0_{4/3}},\\
\|W_\alpha(t)\phi\|_{L^\infty_t L^2_x \cap L^2_t \dot{B}^{-5/6}_6} &\lesssim \|\phi\|_{L^2},\\
\left\| \int_0^t W_\alpha(t-s) f(s)\,ds \right\|_{L^\infty_t L^2_x \cap L^2_t \dot{B}^{-5/6}_6} &\lesssim \|f\|_{L^1_t L^2_x}.
\end{aligned}$$

2C. ***Multilinear estimates for quadratic and cubic terms.*** Next, we prove multilinear estimates for the nonlinear terms in (2-5) in the Besov spaces of $x \in \mathbb{R}^4$. For t, only Hölder's inequalities will be used, which need no explanation. In the following, we ignore the dependence of constants on (s,l), but distinguish by $C(K)$ when it is not uniform for K. The main tools are Littlewood–Paley theory and certain Coifman–Meyer-type bilinear Fourier multiplier estimates. Roughly speaking, the multipliers Ω and $\tilde{\Omega}$ act like

$$\Omega(f,g) \sim D^{-1}\langle D\rangle^{-1}(fg)_{XL} \quad \text{and} \quad \tilde{\Omega}(f,g) \sim D^{-1}\langle D\rangle^{-1}(f\bar{g})_{XL+LX}, \tag{2-8}$$

in product estimates in the Besov spaces. Hence the proof is reduced to usual computation of exponents as in the paraproduct. We only sketch the proof.

Lemma 2.2 (quadratic terms). *Let $K \geq 5$.*

(1) *Assume that $s, l \geq 0$. Then, for any $N(x)$ and $u(x)$,*

$$\begin{aligned}
\|(Nu)_{LH+\alpha L}\|_{B^s_{4/3}} &\lesssim \|N\|_{H^l}\|u\|_{B^s_4},\\
\|(Nu)_{HH}\|_{B^s_{4/3}} &\lesssim C(K)\|N\|_{H^l}\|u\|_{B^s_4}.
\end{aligned} \tag{2-9}$$

(2) *Assume $0 \leq l+1 \leq 2s$. Then, for any $u(x)$ and $v(x)$,*

$$\begin{aligned}
\|D(uv)_{HH}\|_{H^l} &\lesssim C(K)\|u\|_{B^s_4}\|v\|_{B^s_4},\\
\|D(uv)_{\alpha L+L\alpha}\|_{H^l} &\lesssim \|u\|_{B^s_4}\|v\|_{B^s_4}.
\end{aligned} \tag{2-10}$$

Proof. The estimates above follow directly from Bony's paraproduct and Hölder's inequality. For example,

$$\|P_k(Nu)_{LH}\|_{L^{4/3}} \lesssim \sum_{j=k-2}^{k+2} \|(P_{\leq j-K}N)(P_j u)\|_{L^{4/3}} \lesssim \sum_{j=k-2}^{k+2} \|N\|_{L^2}\|P_j u\|_{L^4}. \tag{2-11}$$

Then we sum up the squares with respect to k. The other estimates follow similarly. This argument loses the summability for HH at the 0 regularity ($s = l = 0$ for (1) and $s = l + 1 = 0$ for (2)), but then we can simply use Hölder in x together with the embeddings $B^0_p \subset L^p$ and $L^{p'} \subset B^0_{p'}$ for $2 \le p \le \infty$. □

Similarly to [Guo et al. 2013, Lemma 4.4; Guo et al. 2014b, Lemma 4.4], we will exploit in the proof of local well-posedness and persistence of regularity that the boundary contributions, as well as cubic terms, can be made small by choosing $K \ge 5$ large.

Lemma 2.3 (boundary terms). *There exist $\theta_j(s, l) \ge 0$ such that, for all $K \ge 5$ and for any $N(x)$, $u(x)$ and $v(x)$, we have the following*:

(1) *If $l \ge \max(0, s-2)$ and $(s, l) \ne (2, 0)$,*
$$\|\Omega(N, u)\|_{H^s} \lesssim 2^{-\theta_1 K}\|N\|_{H^l}\|u\|_{H^s}, \quad \theta_1 > 0 \textit{ for } s < l + 2. \tag{2-12}$$

(2) *If $l \le \min(2s-1, s+1)$ and $(s, l) \ne (2, 3)$,*
$$\|D\tilde{\Omega}(u, v)\|_{H^l} \lesssim 2^{-\theta_2 K}\|u\|_{H^s}\|v\|_{H^s}, \quad \theta_2 > 0 \textit{ for } l < s + 1. \tag{2-13}$$

(3) *If $l \ge \min(0, s-1)$ and $(s, l) \ne (1, 0)$,*
$$\|\Omega(N, u)\|_{B^s_4} \lesssim 2^{-\theta_3 K}\|N\|_{H^l}\|u\|_{B^s_4}, \quad \theta_3 > 0 \textit{ for } s < l + 1. \tag{2-14}$$

(4) *If $l \le \min\big(2s - \frac{1}{2}, s + \frac{3}{2}\big)$ and $(s, l) \ne \big(2, \frac{7}{2}\big)$,*
$$\|\langle D\rangle^l \tilde{\Omega}(u, v)\|_{\dot{B}^{1/6}_6} \lesssim 2^{-\theta_4 K}[\|u\|_{B^s_4}\|v\|_{H^s} + \|v\|_{B^s_4}\|u\|_{H^s}], \quad \theta_4 > 0 \textit{ for } l < s + \tfrac{3}{2}. \tag{2-15}$$

Proof. Since they are all straightforward, we prove only (2-14)–(2-15), leaving (2-12)–(2-13) to the reader. By [Guo and Nakanishi 2014, Lemma 3.5] and using (2-8) with Bernstein, we have
$$\begin{aligned}\|P_k\langle D\rangle D\Omega(P_{k_0}N, P_{k_1}u)\|_{L^p_x} &\lesssim \|P_{k_0}N\|_{L^{p_0}_x}\|P_{k_1}u\|_{L^{p_1}_x}\\ &\lesssim 2^{4k_0(1/p_0-1/q_0)+4k_1(1/p_1-1/q_1)}\|P_{k_0}N\|_{L^{q_0}_x}\|P_{k_1}u\|_{L^{q_1}_x}\end{aligned} \tag{2-16}$$
for any $k, k_0, k_1 \in \mathbb{Z}$ and any $p, p_0, p_1, q_0, q_1 \in [1, \infty]$ satisfying $1/p = 1/p_0 + 1/p_1$ and $q_j \le p_j$. The same estimate holds for the bilinear operator $\tilde{\Omega}$. For the low frequency part, say if $k_1 \le k_0 - K$, we can replace P_{k_1} with $P_{\le k_1}$. The above with $(p, p_0, p_1, q_0, q_1) = (4, 4, \infty, 2, 4)$ and the HL restriction $|k - k_0| \le 1$ in Ω yields
$$\|\Omega(N, u)\|_{B^s_4} \lesssim \Big\| 2^{k^+(s-1-l)}\|P_kN\|_{H^l}\sum_{k_1 \le k-K} 2^{k_1 - k_1^+ s}\|P_{k_1}u\|_{B^s_4}\Big\|_{l^2_k}, \tag{2-17}$$
where $k^+ := \max(k, 0)$, using $P_{\le 0}B^s_p \subset \dot{B}^0_{p,\infty}$ for the lower frequency component. The summation over $k_1 \le k - K$ is bounded by
$$\begin{cases} 2^{k-K} & \text{if } k \le K,\\ 2^{(1-s)^+(k-K)} & \text{if } k > K,\ s \ne 1,\\ k - K & \text{if } k > K,\ s = 1.\end{cases} \tag{2-18}$$

This and $\|P_k N\|_{H^l} \in \ell_k^2$ lead to (2-14), with the small factor $2^{-\theta_3 K}$ for $s < 1$ and for $1 \le s < l+1$. The conditions $l \ge 0$ and $l \ge s-1$ ensure uniform boundedness of the coefficient after the summation for $s < 1$ and for $s > 1$, respectively, while the endpoint $(s, l) = (1, 0)$ is excluded due to the logarithmic growth at $s = 1$. Similarly, with $(p, p_0, p_1, q_0, q_1) = (6, 6, \infty, 4, 2)$, we have

$$\|P_k \langle D\rangle^l \tilde{\Omega}(u, v)_{HL}\|_{\dot{B}_6^{1/6}} \lesssim 2^{k^+(l-1-s)-k/2} \sum_{k_1 \le k-K} 2^{2k_1 - k_1^+ s} \|P_k u\|_{B_4^s} \|P_{k_1} v\|_{H^s}. \tag{2-19}$$

Using this and $\|P_k u\|_{B_4^s} \in \ell_{k\ge 0}^2$ lead to (2-15), with the small factor for $s < 2$ and for $2 \le s < l - \frac{3}{2}$. □

Lemma 2.4 (cubic terms). *There exist $\theta_j(s, l) \ge 0$ such that, for all $K \ge 5$ and for any $M(x)$, $N(x)$, $u(x)$, $v(x)$ and $w(x)$, we have the following*:

(1) *If $s \ge \frac{1}{2}$, then $\theta_1 > 0$ and*

$$\|\Omega(D(uv), w)\|_{H^s} \lesssim 2^{-\theta_1 K}\big[\|u\|_{H^s}\|v\|_{B_4^{1/2}} + \|v\|_{H^s}\|u\|_{B_4^{1/2}}\big]\|w\|_{B_4^{1/2}}. \tag{2-20}$$

(2) *If $l \ge 0$, $-l < s \le l+2$, $s \le 2l+1$, and $(s, l) \ne (1, 0)$,*

$$\|\Omega(M, Nu)\|_{B_{4/3}^s} \lesssim 2^{-\theta_2 K}\|M\|_{H^l}\|N\|_{H^l}\|u\|_{B_4^s}, \quad \theta_2 > 0 \textit{ for } s < l+2. \tag{2-21}$$

(3) *If $s \ge \frac{1}{2}$, $-s < l \le s+1$, $l \le 2s$, and $(s, l) \ne (1, 2)$,*

$$\|D\tilde{\Omega}(Nu, v)\|_{H^l} + \|D\tilde{\Omega}(v, Nu)\|_{H^l} \lesssim 2^{-\theta_3 K}\|N\|_{H^l}\|u\|_{B_4^s}\|v\|_{B_4^s}, \quad \theta_3 > 0 \textit{ for } l < s+1. \tag{2-22}$$

Proof. For (2-20), we can use a standard product inequality for $s \ge \frac{1}{2}$:

$$\|uv\|_{B_{8/5}^s} \lesssim \|u\|_{H^s}\|v\|_{B_4^{1/2}} + \|v\|_{H^s}\|u\|_{B_4^{1/2}}, \tag{2-23}$$

which easily follows using $B_4^{1/2} \subset L^8$, e.g., by the paraproduct calculus. Putting $f := uv$ we obtain, from (2-16) with $(p, p_0, p_1, q_0, q_1) = \big(2, 2, \infty, \frac{8}{5}, 4\big)$,

$$\|P_k \Omega(Df, w)\|_{H^s} \lesssim 2^{k/2 - k^+} \sum_{k_1 \le k-K} 2^{k_1 - k_1^+/2}\|f\|_{B_{8/5}^s}\|w\|_{B_4^{1/2}}, \tag{2-24}$$

which leads to (2-20) with a small factor, in the same way as in the previous lemma.

For (2-21) and (2-22), we can use a standard product inequality:

$$\sigma \le \min(s, l, s+l-1) \implies \|Nu\|_{H^\sigma} \lesssim \|N\|_{H^l}\|u\|_{B_4^s}, \tag{2-25}$$

which holds for $s + l > 0$ unless $s = 1$ and $\sigma = l$. Putting $g := Nu$ we obtain, from (2-16) with $(p, p_0, p_1, q_0, q_1) = \big(\frac{4}{3}, 2, 4, 2, 2\big)$,

$$\|P_k \Omega(M, g)\|_{B_{4/3}^s} \lesssim \sum_{k_1 \le k-K} 2^{k^+(s-1-l)-k+k_1-k_1^+\sigma}\|P_k M\|_{H^l}\|P_{k_1} g\|_{H^\sigma}. \tag{2-26}$$

First, the low frequency part $k \le 0$ is bounded using Young on $\mathbb{Z}$ by

$$\begin{aligned}\|P_{\le 0}\Omega(M,g)\|_{B^s_{4/3}} \lesssim \|P_k\Omega(M,g)\|_{\ell^1_{k\le 0}L^{4/3}_x} \lesssim \|P_k M\|_{\ell^2_{k\le 0}H^l_x}\Big\|\sum_{k_1\le k-K} 2^{-k+k_1}\|P_{k_1}g\|_{H^\sigma_x}\Big\|_{\ell^2_{k\le 0}} \\ \lesssim 2^{-K}\|M\|_{H^l}\|g\|_{H^\sigma}. \qquad (2\text{-}27)\end{aligned}$$

For $0 < k \le K$, the summation over k_1 is bounded by $2^{k(s-l-1)-K}\|P_k M\|_{H^l} \in \ell^2_k$ with the small factor for $s < l+2$. For $K < k$, it is bounded by

$$\begin{cases} 2^{k(s-1-l-\sigma)}2^{-K(1-\sigma)} & \text{if } \sigma < 1, \\ 2^{k(s-2-l)} & \text{if } \sigma > 1. \end{cases} \qquad (2\text{-}28)$$

The case $\sigma < 1$ is fine if $\sigma = l$ by $s \le 2l+1$, if $\sigma \le s+1$ by $l \ge 0$, and if $\sigma = s+l-1$ by $l \ge 0$. In the critical case $s = 1$ for the product inequality, we have $s < 2l+1$ and $l > 0$ by the exclusion $(s,l) \ne (1,0)$, so that we can choose $\sigma = l - \varepsilon$. The case $\sigma > 1$ is fine by $s \le l+2$. Then the only remaining case is $(s,l) = (3,1)$, where we are forced to choose $\sigma = 1$; then we should replace (2-26) for $k > K$ with

$$\|P_k\Omega(M,g)\|_{B^s_{4/3}} \lesssim 2^{k(s-2-l)}\|P_k M\|_{H^l}\|P_{\le k-K}g\|_{H^1}, \qquad (2\text{-}29)$$

which is bounded using $\|P_k M\|_{H^l} \in \ell^2_k$. Thus we obtain (2-21).

Similarly, from (2-16) with $(p, p_0, p_1, q_0, q_1) = (2, 2, \infty, 2, 4)$, we have

$$\|P_k D\tilde{\Omega}(g,v)_{HL}\|_{H^l} + \|P_k D\tilde{\Omega}(v,g)_{LH}\|_{H^l} \lesssim \sum_{k_1\le k-K} 2^{k^+(l-1-\sigma)+k_1-k_1^+ s}\|P_k g\|_{H^\sigma}\|P_{k_1}v\|_{B^s_4}, \qquad (2\text{-}30)$$

for which the low frequencies $k \le K$ are easily bounded using the factor 2^{k_1}, while for $k > K$ the summation is bounded by

$$\begin{cases} 2^{k(l-s-\sigma)}2^{-K(1-s)} & \text{if } s < 1, \\ 2^{k(l-1-\sigma)}(k-K) & \text{if } s = 1, \\ 2^{k(l-1-\sigma)} & \text{if } s > 1. \end{cases} \qquad (2\text{-}31)$$

The case $s < 1$ is fine if $\sigma = s$ by $l \le 2s$, and if $\sigma = s+l-1$ by $s \ge \frac{1}{2}$. The case $s > 1$ is fine if $\sigma = s$ by $l \le s+1$, and obviously if $\sigma = l$. The critical case $s = 1$ is also fine, as none of the conditions is on the boundary, thanks to $(s,l) \ne (1,2)$.

For the other HL interaction, choosing $(p, p_0, p_1, q_0, q_1) = (2, 4, 4, 2, 4)$ we have

$$\|P_k D\tilde{\Omega}(g,v)_{LH}\|_{H^l} + \|P_k D\tilde{\Omega}(v,g)_{HL}\|_{H^l} \lesssim \sum_{k_1\le k-K} 2^{k^+(l-1-s)+k_1-k_1^+\sigma}\|P_k v\|_{B^s_4}\|P_{k_1}g\|_{H^\sigma}, \qquad (2\text{-}32)$$

which is also easy for $k \le K$. For $k > K$, the summation is bounded by

$$\begin{cases} 2^{k(l-\sigma-s)}2^{-K(1-\sigma)} & \text{if } \sigma < 1, \\ 2^{k(l-1-s)} & \text{if } \sigma > 1. \end{cases} \qquad (2\text{-}33)$$

The case $\sigma < 1$ is the same as the case $s < 1$ in (2-31). The case $\sigma > 1$ is OK by $l \le s+1$. When $l = s+1 \ge \frac{3}{2}$, we can choose $\sigma = \min(s, l, s+l-1) = s \ne 1$ thanks to $(s,l) \ne (1,2)$. In the critical case $s = 1$, we can choose $\sigma < \min(s, l, s+l-1) \le 1$ such that $l - s - \sigma < 0$, since $l < 2s = 2$. This concludes the proof of (2-22). □

3. Small data scattering for $s \le l + 1$

Using the multilinear estimates in the previous section, it is now easy to obtain global well-posedness and scattering for small initial data in $H^s \times H^l$ in the range (1-11) under $s \le l+1$. In Section 5 we will show that we only need smallness in $H^{1/2} \times L^2$ for all regularities by a persistence of regularity argument. Fix $K = 5$. As in [Guo and Nakanishi 2014, Section 4], for fixed initial data $(u_0, N_0) \in H^s \times H^l$ we define a mapping $(u, N) \mapsto (u', N') = \Phi_{u_0,N_0}(u, N)$ by the right-hand sides of the equations (2-3)–(2-4). Then, for small initial data (u_0, N_0), we see that Φ_{u_0,N_0} is a contraction in a small ball around 0 of $X^s \times Y^l$. Indeed, from the estimates in the previous section, we obtain

$$\begin{aligned} \|u'\|_{X^s} &\lesssim \|u_0\|_{H^s} + \|N\|_{Y^l}\|u\|_{X^s} + \|u\|_{X^s}^3 + \|N\|_{Y^l}^2\|u\|_{X^s}, \\ \|N'\|_{Y^l} &\lesssim \|N_0\|_{H^l} + \|u\|_{X^s}^2 + \|N\|_{Y^l}\|u\|_{X^s}^2, \end{aligned} \tag{3-1}$$

where we need $s \le l+1$ in using (2-14) for $\Omega(N, u)$. By the contraction mapping principle, we have a unique solution in a small ball in $X^s \times Y^l$, and the Lipschitz continuity of the solution map $H^s \times H^l \to X^s \times Y^l$ follows from the standard argument.

Now we derive scattering for (u, N) in $H^s \times H^l$, assuming we have (s, l) satisfying (1-11), that (u, N) is in $X^{1/2} \times Y^0$ with small norm, and the scattering of the transformed variables, namely, for

$$\Psi(u, N) := (u - \Omega(N, u), N - D\tilde{\Omega}(u, u)) \tag{3-2}$$

there exist $(u_\pm, N_\pm) \in H^s \times H^l$ with small norm in $H^{1/2} \times L^2$ such that

$$\Psi(u, N) - (S(t)u_\pm, W_\alpha(t)N_\pm) \to 0 \quad \text{in } H^s \times H^l \quad (t \to \pm\infty). \tag{3-3}$$

In the current case $s \le l+1$, the latter assumption, (3-3), obviously holds in view of the fact that $(u, N) \in X^s \times Y^l$ and the Strichartz estimate with the global bounds on the nonlinear terms.

The bilinear estimate for the normal form in Lemma 2.3 implies that the above transform Ψ is invertible for small data in $H^{1/2} \times L^2$ and bi-Lipschitz. More precisely, for any $(u', N') \in H^{1/2} \times L^2$, the inverse image $\Psi^{-1}(u', N')$ is the fixed points of the map

$$(u, N) \mapsto \Psi_{u',N'}(u, N) := (u' + \Omega(N, u), N' + D\tilde{\Omega}(u, u)). \tag{3-4}$$

Lemma 2.3 implies that $\Psi_{u',N'}$ is a contraction in a small ball of $H^{1/2} \times L^2$ if (u', N') is small, hence the unique small $(u, N) \in \Psi^{-1}(u', N')$ is given by the iteration

$$(u, N) = \lim_{k\to\infty} (\Psi_{u',N'})^k(0, 0). \tag{3-5}$$

By (3-3), we get

$$(u, N) - \Psi^{-1}(S(t)u_\pm, W_\alpha(t)N_\pm) \to 0 \quad \text{in } H^{1/2} \times L^2 \quad (t \to \pm\infty).$$

To show the scattering for (u, N), it suffices to show

$$\Psi^{-1}(S(t)u_\pm, W_\alpha(t)N_\pm) \to (S(t)u_\pm, W_\alpha(t)N_\pm) \quad \text{in } H^{1/2} \times L^2 \quad (t \to \pm\infty). \tag{3-6}$$

By the construction of the inverse, we get

$$(u^n_\pm(t), N^n_\pm(t)) \to \Psi^{-1}(S(t)u_\pm, W_\alpha(t)N_\pm) \quad \text{in } L^\infty_t(H^{1/2} \times L^2) \quad (n \to \infty), \tag{3-7}$$

where $(u^0_\pm, N^0_\pm) = (0, 0)$, and, for $n = 1, 2, \dots,$

$$\begin{aligned} u^{n+1}_\pm &= S(t)u_\pm + \Omega(N^n_\pm, u^n_\pm), \\ N^{n+1}_\pm &= W_\alpha(t)N_\pm + D\tilde{\Omega}(u^n_\pm, u^n_\pm). \end{aligned}$$

Thus, to show (3-6), it suffices to show for any n that

$$(u^n_\pm(t), N^n_\pm(t)) \to (S(t)u_\pm, W_\alpha(t)N_\pm) \quad \text{in } H^{1/2} \times L^2 \quad (t \to \pm\infty), \tag{3-8}$$

for which, by induction on n and bilinear estimates, it suffices to show

$$(\Omega(N_F, u_F), D\tilde{\Omega}(u_F, u_F)) \to 0 \quad \text{in } H^s \times H^l \quad (t \to \pm\infty) \tag{3-9}$$

for all free solutions (u_F, N_F) in $H^s \times H^l$. The density argument with the bilinear estimate allows us to restrict to the case $u_F(0), N_F(0) \in C^\infty_0(\mathbb{R}^4)$; then the above is almost obvious, by the dispersive decay of $S(t)$ and $W_\alpha(t)$ (we omit the details).

For higher regularity, $(s, l) \neq \left(\frac{1}{2}, 0\right)$, we do not have smallness in $H^s \times H^l$, so we should replace Lemma 2.3 with the estimates

$$\begin{aligned} \|\Omega(N, u)\|_{H^s} &\lesssim \|N\|_{H^l}\|u\|_{B_u}, \\ \|\Omega(N, u)\|_{B_u} &\lesssim \|N\|_{B_N}\|u\|_{B_u}, \\ \|D\tilde{\Omega}(u, u)_{HL}\|_{H^l} &\lesssim \|u\|_{H^s}\|u\|_{B_u}, \end{aligned} \tag{3-10}$$

where the Besov spaces B_u and B_N are defined by

$$B_u := B^{s-\varepsilon}_p, \quad B_N := B^{l-\varepsilon}_p, \quad \frac{1}{p} = \frac{1}{2} - \frac{\varepsilon}{4} \tag{3-11}$$

for some small $\varepsilon > 0$ such that $H^s \times H^l \subset B_u \times B_N$ by the sharp Sobolev embedding. The estimates (3-10) imply that $\Psi_{u',N'}$ is a contraction with respect to the equivalent norm

$$\|(u, N)\|_Z := \|u\|_{H^s} + \|N\|_{H^l} + \delta^{-2}\|u\|_{B_u} \tag{3-12}$$

for $0 < \delta \ll 1$ on the closed set

$$F := \left\{(u, N) \in H^s \times H^l \;\middle|\; \|(u, N)\|_Z \le 1/\delta,\ \|N\|_{B_N} \le \delta,\ \|u\|_{B_u} \le \delta^3\right\} \tag{3-13}$$

provided that $2(u', N') \in F$. Indeed, (3-10) yields, for any $(u, N) \in F$,

$$\|(\Omega(N, u), D\tilde{\Omega}(u, u))\|_{H^s \times H^l} \lesssim \delta^2, \quad \|\Omega(N, u)\|_{B_u} \lesssim \delta^4, \tag{3-14}$$

hence $\|(\Omega(N,u), D\tilde{\Omega}(u,u))\|_Z \lesssim \delta^2$ and $\Psi_{u',N'}(u,N) \in F$. For the difference, we have from (3-10), for any $(v,M) \in H^s \times H^l$,

$$\begin{aligned} \|(\Omega(N,v), D\tilde{\Omega}(u,v)_{HL})\|_{H^s\times H^l} &\lesssim \|(u,N)\|_{H^s\times H^l}\|v\|_{B_u} \lesssim \delta\|(v,M)\|_Z,\\ \|(\Omega(M,u), D\tilde{\Omega}(v,u)_{HL})\|_{H^s\times H^l} &\lesssim \|(v,M)\|_{H^s\times H^l}\|u\|_{B_u} \lesssim \delta^3\|(v,M)\|_Z,\\ \|\Omega(N,v)+\Omega(M,u)\|_{B_u} &\lesssim \|N\|_{B_N}\|v\|_{B_u} + \|M\|_{B_N}\|u\|_{B_u} \lesssim \delta^3\|(v,M)\|_Z. \end{aligned} \tag{3-15}$$

Since the scattering of $\Psi(u,N)$ implies $\|\Psi(u,N)\|_{B_u\times B_N} \to 0$ as $t\to\infty$, choosing $\delta > 0$ small enough ensures that $2\Psi(u,N) \in F$ for large t. Then (u,N) given by (3-5) is the same as the fixed point in F. Since we can take $\delta > 0$ arbitrarily small, (3-14) implies that $\|(u,N)-\Psi(u,N)\|_{H^s\times H^l} \to 0$ as $t\to\infty$, hence the scattering of (u,N) in $H^s \times H^l$.

Since all the estimates are uniform and global in time, the same argument works for the final state problem, namely to find the solution for a prescribed (small) scattering data at $t=\infty$. Thus we obtain:

Proposition 3.1. *Let $(s,l) \in \mathbb{R}^2$ satisfy (1-11), $s \le l+1$, and $(s,l) \ne (1,0)$. Then there exists $\varepsilon_1 = \varepsilon_1(s,l) > 0$ such that, for any $(u_0,N_0) \in H^s(\mathbb{R}^4)\times H^l(\mathbb{R}^4)$ satisfying $\|(u_0,N_0)\|_{H^s\times H^l} \le \varepsilon_1$, there exists a unique global solution $(u,N) \in X^s \times Y^l$ of (1-9). Moreover, there exists $(u^+,N^+) \in H^s\times H^l$ such that*

$$\lim_{t\to\infty} \|u(t)-S(t)u^+\|_{H^s_x} + \|N(t)-W_\alpha(t)N^+\|_{H^l_x} = 0. \tag{3-16}$$

Conversely, for any $(u^+,N^+) \in H^s\times H^l$ with $\|(u^+,N^+)\|_{H^s\times H^l} \le \varepsilon_1$, there exists a unique solution $(u,N) \in X^s\times Y^l$ satisfying (3-16). Both the maps $(u_0,N_0) \mapsto (u,N)$ and $(u^+,N^+) \mapsto (u,N)$ are Lipschitz continuous from the ε_1-ball into $X^s\times Y^l$.

The uniqueness without the smallness is proved in the next section. For the question of whether (u,N) obtained above really solves (1-9) before the normal form, see Remark 5.3.

4. Large data local well-posedness for $s < l+1$

For large data, the proof in the previous section does not immediately work, in particular at the endpoint $(s,l) = \left(\frac{1}{2},0\right)$. The main difficulty is the lack of flexibility in the choice of the Strichartz norm for the boundary term and the bilinear term $(Nu)_{LH}$. More precisely, $L^\infty_t H^l_x$ is the only choice among the Strichartz norms of $W_\alpha(t)$ for N to estimate $\Omega(N,u)$ in $L^\infty_t H^s_x$ and to avoid losing regularity in $(Nu)_{LH}$. For the former term, we can play with the frequency gap parameter K in the normal form to extract a small factor. For the latter term, we use the following:

Lemma 4.1. *Let $0 < T \le \infty$ and $N \in C([0,T); L^2(\mathbb{R}^4))$. Suppose that $W_\alpha(-t)N(t)$ is strongly convergent in L^2_x as $t \to T-0$. Then, for any $\varepsilon > 0$, there exists a finite increasing sequence $0 = T_0 < T_1 < \cdots < T_{n+1} = T$ such that*

$$\|N\|_{(L^\infty_t L^2_x + L^2_t L^4_x)(T_j,T_{j+1})} < \varepsilon \tag{4-1}$$

for each $j = 0,\dots,n$.

Note that the $L_t^2 L_x^4$ norm is not controlled by the Strichartz estimate for $W_\alpha(t)$, but it is bounded for nice initial data. The case $T = \infty$ will be used for large data scattering. For $T < \infty$, the assumption on N is equivalent to $N \in C([0, T]; L^2)$.

Proof. Put $N^+ := \lim_{t \to T-0} W_\alpha(-t)N(t) \in L_x^2$. By the strong convergence, there exists $T' \in (0, T)$ such that $\sup_{T' \le t < T} \|N(t) - W_\alpha(t)N^+\|_{L_x^2} < \frac{1}{4}\varepsilon$. Since $C_0^\infty \subset L_x^2$ is dense, there exists $N_0 \in C_0^\infty$ such that $\|N_0 - N^+\|_{L_x^2} < \frac{1}{4}\varepsilon$. The dispersive decay of $W_\alpha(t)$ implies that $W_\alpha(t)N_0 \in L_t^2 L_x^4(\mathbb{R})$. Define N' by

$$N'(t) := \begin{cases} P_{\le k} N(t) & \text{if } 0 \le t \le T', \\ W_\alpha(t)N_0 & \text{if } T' < t < T. \end{cases} \tag{4-2}$$

By the above choice of T' and N_0, we have $\|N - N'\|_{L^\infty((T',T);L_x^2)} < \frac{1}{2}\varepsilon$. Since $N \in C([0, T']; L_x^2)$ and $[0, T']$ is compact, $N'(t) \to N(t)$ in L_x^2 uniformly on $t \in [0, T']$ as $k \to \infty$. Hence, for large k, we have $\|N - N'\|_{L^\infty([0,T'];L_x^2)} < \frac{1}{2}\varepsilon$. Hence,

$$\|N - N'\|_{L^\infty([0,T);L_x^2)} < \tfrac{1}{2}\varepsilon, \quad N' \in L_t^2([0, T); L_x^4). \tag{4-3}$$

Choosing $T_1 < T_2 < \cdots < T_n$ appropriately ensures that $\|N'\|_{L_t^2 L_x^4(T_j, T_{j+1})} < \frac{1}{2}\varepsilon$ for each j, then we get the desired estimate. □

Now we are ready to prove the local well-posedness for large data in $H^{1/2} \times L^2$. For any initial data $(u_0, N_0) \in H^{1/2} \times L^2$, let

$$u_F := S(t)(u_0 - \Omega(N_0, u_0)), \quad N_F := W_\alpha(t)(N_0 - D\tilde{\Omega}(u_0, u_0)), \tag{4-4}$$

and apply Lemma 4.1 to N_F. Then, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\|u_F\|_{L_t^2 B_4^{1/2}(0,T)} + \|N_F\|_{L_t^\infty L_x^2 + L_t^2 L_x^4(0,T)} < \varepsilon. \tag{4-5}$$

Putting $\mathcal{H} := H^{1/2} \times L^2$ and $m := \|(u_F(0), N_F(0))\|_{\mathcal{H}}$, we look for a unique local solution on $(0, T)$ as a fixed point of the map Φ_{u_0,N_0} in the closed set

$$K_m^\varepsilon := \left\{(u, N) \in C([0, T]; \mathcal{H}) \;\middle|\; \|(u, N)\|_{L_t^\infty(0,T;\mathcal{H})} \le 2m,\ \|u\|_{L_t^2 B_4^{1/2}(0,T)} + \|N\|_{L_t^\infty L_x^2 + L_t^2 L_x^4(0,T)} \le 2\varepsilon\right\}.$$

From the multilinear estimates in Section 2, we have

$$\begin{aligned} \|\Omega(N, u)\|_{X^{1/2}} &\lesssim 2^{-\theta K} \|N\|_{L_t^\infty L_x^2} \|u\|_{X^{1/2}}, \\ \|\Omega(D|u|^2, u)\|_{L_t^1 H_x^{1/2}} &\lesssim 2^{-\theta K} \|u\|_{L_t^\infty H^{1/2}} \|u\|_{L_t^2 B_4^{1/2}}^2, \\ \|\Omega(N, Nu)\|_{L_t^2 B_{4/3}^{1/2}} &\lesssim 2^{-\theta K} \|N\|_{L_t^\infty L_x^2}^2 \|u\|_{L_t^2 B_4^{1/2}}, \\ \|D\tilde{\Omega}(u, u)\|_{L_t^\infty L_x^2} &\lesssim 2^{-\theta K} \|u\|_{L_t^\infty H_x^{1/2}}^2, \\ \|D\tilde{\Omega}(Nu, u)\|_{L_t^1 L_x^2} &\lesssim 2^{-\theta K} \|N\|_{L_t^\infty L_x^2} \|u\|_{L_t^2 B_4^{1/2}}^2, \end{aligned} \tag{4-6}$$

and the same estimate on $D\tilde{\Omega}(u, Nu)$, as well as for the difference. Taking K large makes these estimates contractive. For the remaining two terms,

$$\begin{aligned}\|D|u|^2_{HH+\alpha L+L\alpha}\|_{L^1_t L^2_x} &\lesssim C(K)\|u\|^2_{L^2_t B^{1/2}_4},\\ \|(Nu)_{LH+HH+\alpha L}\|_{L^2_t B^{1/2}_{4/3}+L^1_t H^{1/2}_x} &\lesssim C(K)\|N\|_{L^\infty_t L^2_x+L^2_t L^4_x}\|u\|_{L^2_t B^{1/2}_4},\end{aligned}\tag{4-7}$$

which is also made contractive on the interval $[0, T]$ by choosing $\varepsilon > 0$ small enough that $C(K)\varepsilon \ll 1$ after fixing K. Then Φ_{u_0,N_0} becomes a contraction on K^ε_m.

The uniqueness of the solution in the class $X^{1/2} \times Y^0$ is obtained in the same fashion: Let (u_j, N_j) for $j = 0, 1$ be two solutions in $X^{1/2} \times Y^0$. For any $\varepsilon > 0$, applying Lemma 4.1 we can find $T' \in (0, T)$ such that, for $j = 0, 1$,

$$\|u_j\|_{L^2_t B^{1/2}_4(0,T')} + \|N_j\|_{L^\infty_t L^2_x+L^2_t L^4_x(0,T')} < \varepsilon,\tag{4-8}$$

so that both the solutions belong to K^ε_m on $[0, T']$, hence $(u_0, N_0) = (u_1, N_1)$ as long as they are solutions in the above class.

The continuous dependence is also obtained in the same way, because

$$H^{1/2} \times L^2 \to X^{1/2} \times Y^0, \quad (u_0, N_0) \mapsto (u_F, N_F)\tag{4-9}$$

is continuous. Take a strongly convergent sequence of initial data. If the smallness condition (4-5) is satisfied by the limit, then so it is by those sufficiently close to the limit. Then we can estimate the difference from the limit in the same way as above, leading to the strong continuity.

We have worked at the lowest regularity $(s, l) = \left(\frac{1}{2}, 0\right)$, but the same argument works as long as we have the small factor $2^{-\theta K}$, namely for $|s - l| < 1$. Thus we obtain:

Proposition 4.2. *Let $(s, l) \in \mathbb{R}^2$ satisfy (1-11) and $|s - l| < 1$. For any $(u_0, N_0) \in H^s(\mathbb{R}^4) \times H^l(\mathbb{R}^4)$, there exists a unique local solution $(u, N) \in (X^s \times Y^l)([0, T])$ of (1-9) for some $T > 0$, where both T and (u, N) depend continuously on (u_0, N_0). More precisely, if $(u_{0,n}, N_{0,n}) \to (u_0, N_0)$ in $H^s \times H^l$, then $T_n \to T$ and, for any $0 < T' < T$, we have $\|u_n - u\|_{X^s([0,T'])} + \|N_n - N\|_{Y^l([0,T'])} \to 0$.*

5. Persistence of regularity except for $(s, l) = (1, 0)$

Once we have the unique solution at the lowest regularity $(s, l) = \left(\frac{1}{2}, 0\right)$, it gains as much regularity as the initial data. To prove this, we will focus on the derivation of a priori estimates, assuming that all relevant norms are finite, which is justified by the local well-posedness in higher regularity by Proposition 4.2.

For solutions $(u, N) \in (X^{1/2} \times Y^0)([0, T))$ with $(u(0), N(0)) \in H^s \times H^l$ and $0 < T \le \infty$, we will improve the regularity up to $H^s \times H^l$ by the following steps:

(1) Improve u to $s < l + 1$.

(2) Improve N to $l \le 2s - 1$, $l \le s + 1$ and $(s, l) \ne (2, 3)$ for $s < l + 1$.

(3) Improve u to $1 < s < 4l + 1$, $s \le 2l + \frac{11}{8}$ and $s \le l + 2$.

The persistence of regularity is a general phenomenon in nonlinear wave equations, but we encounter some difficulties. One is the same as in the previous section, which is solved by Lemma 4.1. Another difficulty for $s \geq l+1$ is that the normal form can not keep the full Strichartz norm of u, which is why we separate (3).

5A. ***Regularity upgrade for u in*** $\boldsymbol{s < l+1}$**.** Let $(s, l) \in \mathbb{R}^2$ satisfy (1-11) and $s < l+1$. Let (u_0, N_0) be in $H^s \times H^l$ and let $(u, N) \in (X^{1/2} \times Y^l)([0, T))$ be a solution for some $0 < T \leq 0$. If $T = \infty$, we also assume that N scatters in H^l_x. From the estimates in Section 2, we have, for $s < l+1$,

$$\begin{aligned} \|(Nu)_{LH+HH+\alpha L}\|_{L^2_t B^s_{4/3}+L^1_t H^s_x} &\leq C_1(K)\|N\|_{L^\infty_t L^2_x+L^2_t L^4_x}\|u\|_{L^2_t B^s_4}, \\ \|\Omega(N,u)\|_{L^\infty_t H^s_x} &\leq C_0 2^{-\theta K}\|N\|_{L^\infty_t L^2_x}\|u\|_{L^\infty_t H^s_x}, \\ \|\Omega(N,u)\|_{L^2_t B^s_4} &\leq C_0 2^{-\theta K}\|N\|_{L^\infty_t H^l_x}\|u\|_{L^2_t B^s_4}, \\ \|\Omega(D|u|^2,u)\|_{L^1_t H^s_x} &\leq C_0 2^{-\theta K}\|u\|_{L^\infty_t H^s_x}\|u\|^2_{L^2_t B^{1/2}_4}, \\ \|\Omega(N,Nu)\|_{L^2_t B^s_{4/3}} &\leq C_0 2^{-\theta K}\|N\|^2_{L^\infty_t H^l_x}\|u\|_{L^2_t B^s_4} \end{aligned} \tag{5-1}$$

for some constants $\theta(s,l) > 0$, $C_0(s,l) > 0$ and $C_1(K,s,l) > 0$. Note that $C_0 \to \infty$ as $(s,l) \to (1,0)$ in the third and the last estimates, and the small factor $2^{-\theta K}$ is lost for $s = l+1$ in the third estimate. Anyway, taking $K = K(s,l)$ large ensures smallness of the right side in the latter 4 estimates:

$$C_0 2^{-\theta K}\{\|N\|_{L^\infty_t H^l_x} + \|u\|^2_{L^2_t B^{1/2}_4} + \|N\|^2_{L^\infty_t H^l_x}\} \ll 1. \tag{5-2}$$

After fixing such K, choose $\varepsilon > 0$ such that $C_1(K)\varepsilon \ll 1$, and apply Lemma 4.1 to N, which yields a finite sequence $0 = T_0 < T_1 < \cdots < T_{n+1} = T$ such that

$$\|N\|_{(L^\infty_t L^2_x + L^2_t L^4_x)(T_j, T_{j+1})} < \varepsilon. \tag{5-3}$$

Then on each subinterval we obtain, from the above estimates,

$$\|u\|_{X^s(T_j,T_{j+1})} \leq C_2\|u(T_j)\|_{H^s} + \tfrac{1}{2}\|u\|_{X^s(T_j,T_{j+1})} \tag{5-4}$$

for some constant $C_2(s) > 0$. Hence, if $u(0) \in H^s$ then, by induction on j, we deduce that $u \in X^s([0,T))$. If $T = \infty$, this implies the scattering of u in H^s, via the argument in Section 3.

For continuous dependence on the initial data, consider a sequence of solutions (u_n, N_n) such that $(u_n(0), N_n(0)) \to (u(0), N(0))$ in $H^s \times H^l$, $u_n \to u$ in $X^{1/2}(I)$ and $N_n \to N$ in $Y^l(I)$ for some interval $I \subset [0,T)$. For large n, (u_n, N_n) satisfies similar bounds to (5-2) and (5-3) within I, with slightly bigger bounds. Then the same estimates as above for $(u_n - u, N_n - N)$ yield the convergence in $(X^s \times Y^l)(I)$.

5B. ***Regularity upgrade for N in*** $\boldsymbol{s < l+1}$**.** Let $(s, l) \in \mathbb{R}^2$ satisfy (1-11) and $s < l+1$. Let (u_0, N_0) be in $H^s \times H^l$ and let $(u, N) \in (X^s \times Y^{l'})([0, T))$ for some $0 < T \leq \infty$ and some $l' \in (s-1, l)$. From the estimates in Section 2, we have

$$\begin{aligned}\|D(|u|^2)_{HH+L\alpha+\alpha L}\|_{L_t^1 H_x^l} &\le C_1(K)\|u\|_{L_t^2 B_4^s}^2,\\ \|D\tilde{\Omega}(u,u)\|_{L_t^\infty H_x^l} &\le C_0\|u\|_{L_t^\infty H_x^s}^2,\\ \|D\tilde{\Omega}(Nu,u)\|_{L_t^1 H_x^l}+\|D\tilde{\Omega}(u,Nu)\|_{L_t^1 H_x^l} &\le C_0\|N\|_{L_t^\infty H_x^l}\|u\|_{L_t^2 B_4^s}^2\end{aligned} \tag{5-5}$$

for some constants $C_0(s,l)>0$ and $C_1(K,s,l)>0$, and the same for $D\tilde{\Omega}(u,Nu)$. Choose $\varepsilon>0$ so small that $C_0\varepsilon^2\ll 1$. Since $u\in L_t^2 B_4^s(0,T)$, there exists a finite sequence $0=T_0<T_1<\cdots<T_{n+1}=T$ such that

$$\|u\|_{L_t^2 B_4^s(T_j,T_{j+1})}<\varepsilon \tag{5-6}$$

for each j. Then on each subinterval we have, from the above estimates,

$$\|N\|_{L_t^\infty H_x^l(T_j,T_{j+1})}\le C_2\|N(T_j)\|_{H^l}+\tfrac{1}{2}\|N\|_{L_t^\infty H_x^l(T_j,T_{j+1})}+C_1(K)\varepsilon^2+C_0\|u\|_{L_t^\infty H_x^s(T_j,T_{j+1})}^2, \tag{5-7}$$

for some constant $C_2(l)>0$. Hence, if $N(0)\in H^l$ then, by induction on j, we deduce that $N\in L_t^\infty H^l(0,T)$. If $T=\infty$, then we have the scattering of N from the argument in Section 3. We also obtain the Strichartz norm of N using (2-15) for the normal form. We can also upgrade continuous dependence, using the same estimates for the difference from the limit; see the previous subsection for more detail. Combining the results in this and the previous subsections yields:

Proposition 5.1. *Let $(s,l)\in\mathbb{R}^2$ satisfy* (1-11) *and $s<l+1$. Let $(u,N)\in(X^{1/2}\times Y^0)(I)$ be a solution of* (1-9) *on an interval $I\subset\mathbb{R}$, and suppose that $(u(t_0),N(t_0))\in H^s(\mathbb{R}^4)\times H^l(\mathbb{R}^4)$ at some $t_0\in I$. Then $(u,N)\in(X^s\times Y^l)(I)$ and, moreover,*

$$N\in L_t^2(I;\dot{B}_6^{l-5/6}\cap\dot{B}_6^{-5/6}). \tag{5-8}$$

If $I\supset(t_0,\infty)$, then (u,N) scatters in $H^s\times H^l$ as $t\to\infty$. If $(u_n(t_0),N_n(t_0))\to(u(t_0),N(t_0))$ in $H^s\times H^l$ and the corresponding sequence of solutions $(u_n,N_n)\to(u,N)$ in $(X^{1/2}\times Y^0)(J)$ on some interval $t_0\in J\subset I$, then the convergence holds in $(X^s\times Y^l)(J)$. The same convergence result holds for the scattering data if $I\cap J\supset(t_1,\infty)$ for some $t_1<\infty$.

5C. ***Regularity upgrade for u in $s\ge l+1$.*** Let $(s,l)\in\mathbb{R}^2$ satisfy $s\ge l+1$. Then $l>0$ and $s>1$. Let $(u_0,N_0)\in H^s\times H^l$ and let $(u,N)\in(X^{s'}\times Y^l)([0,T))$ for some $0<T\le\infty$ and some $s'\in(1,s)$. In this case, the normal form estimate is not good enough to keep the full Strichartz bound of u. Hence we decompose

$$\begin{aligned}u&=u'+\Omega(N,u),\\ (i\partial_t-\Delta)u'&=(Nu)_{\widetilde{LH}}+\Omega(\alpha D|u|^2,u)+\Omega(N,Nu),\end{aligned} \tag{5-9}$$

where $\widetilde{LH}:=LH+HH+\alpha L$ for brevity, and look for closed estimates in

$$\begin{aligned}&u'\in X^s, && u\in X':=L_t^\infty H_x^s\cap L_t^{2/(1-\gamma)}L_x^\infty,\\ &N\in L_t^\infty H_x^l\cap L_t^{2/\gamma}\mathcal{B}, && \mathcal{B}:=\dot{B}_{q_1}^{l-5\gamma/6}\cap\dot{B}_{q_1}^{-5\gamma/6}\end{aligned} \tag{5-10}$$

with $1/q_1 := \frac{1}{2} - \frac{1}{3}\gamma$ for some $\gamma \in \left[0, \frac{3}{4}\right]$ satisfying

$$\gamma + 1 < s, \quad 2l + \tfrac{1}{2}\gamma + 1 \geq s. \tag{5-11}$$

Such γ exists if and only if $1 < s < 4l + 1$ and $s \leq 2l + \frac{11}{8}$. Also note that

$$X^s \subset L_t^{2/(1-\gamma)} B^s_{4/(1+\gamma)} \subset L_t^{2/(1-\gamma)} L_x^\infty \tag{5-12}$$

since $\gamma + 1 < s$. Similarly, $L_t^{2/\gamma}\mathcal{B}$ is a wave-Strichartz norm in H^l; see (5-8).

We write $(Nu)_{\widetilde{LH}} = (Nu')_{\widetilde{LH}} + (N\Omega(N, u))_{\widetilde{LH}}$. From the estimates in Section 2, we have, for $k > 2$,

$$\begin{aligned} \|(Nu')_{\widetilde{LH}}\|_{L_t^2 B^s_{4/3} + L_t^1 H_x^s} &\leq C_1(K)\|N\|_{L_t^\infty L_x^2 + L_t^2 L_x^4} \|u'\|_{L_t^2 B_4^s}, \\ \|\Omega(D|u|^2, u)\|_{L_t^1 H_x^s} &\leq C_0 2^{-\theta K} \|u\|^2_{L_t^2 B_4^{1/2}} \|u\|_{L_t^\infty H_x^s}, \\ \|P_{>k}\Omega(N, u)\|_{L_t^\infty H_x^s} &\leq C_0 \|N_{>k-1}\|_{L_t^\infty H_x^l} \|u\|_{L_t^\infty H_x^s} \end{aligned} \tag{5-13}$$

for some constants $C_0(s, l) > 0$, $\theta(s) > 0$ and $C_1(K, s, l) > 0$. We need some more estimates. Since $H^{l+2} \subset L_x^\infty$ we have, for $k > 2$,

$$\|P_{>k}\Omega(N, u)\|_{L_t^{2/(1-\gamma)} L_x^\infty} \leq C_0 \|N_{>k-1}\|_{L_t^\infty H_x^l} \|u\|_{L_t^{2/(1-\gamma)} L_x^\infty}. \tag{5-14}$$

It remains to estimate $\Omega(N, Nu)$ and $(N\Omega(N, u))_{\widetilde{LH}}$. If $\mathcal{B} \subset L_x^4$ then, for $k \gg \langle \log \alpha \rangle$,

$$\begin{aligned} \|P_{>k}\Omega(N, Nu)\|_{B^{l+2}_{4/3}} &\lesssim \|N_{>k-1}\|_{H_x^l} \|Nu\|_{L_x^4} \lesssim \|N_{>k-1}\|_{H_x^l} \|N\|_{\mathcal{B}} \|u\|_{L_x^\infty}, \\ \|P_{>k}(N\Omega(N, u))_{\widetilde{LH}}\|_{B^{l+2}_{4/3}} &\lesssim \|N\|_{\mathcal{B}} \|N_{>k-K-3}\|_{H_x^l} \|u\|_{L_x^\infty}. \end{aligned} \tag{5-15}$$

If $\mathcal{B} \not\subset L_x^4$ but $l \geq \frac{5}{6}\gamma$, then, putting

$$\frac{1}{q_2} := \frac{1}{q_1} - \frac{l - \frac{5}{6}\gamma}{4} = \frac{1}{2} - \frac{\gamma}{8} - \frac{l}{4}, \quad \frac{1}{q_3} := \frac{1}{2} + \frac{1}{q_2}, \tag{5-16}$$

we have $\mathcal{B} \subset L^{q_2}$ and $B^s_{4/3} \supset B^{2l+\gamma/2+1}_{4/3} \supset B^{l+2}_{q_3}$, and so

$$\begin{aligned} \|P_{>k}\Omega(N, Nu)\|_{B^s_{4/3}} &\lesssim \|N_{>k-1}\|_{H_x^l} \|Nu\|_{L^{q_2}} \lesssim \|N_{>k-1}\|_{H_x^l} \|N\|_{\mathcal{B}} \|u\|_{L_x^\infty}, \\ \|P_{>k}(N\Omega(N, u))_{\widetilde{LH}}\|_{B^s_{4/3}} &\lesssim \|N\|_{\mathcal{B}} \|N_{>k-K-3}\|_{H^l} \|u\|_{L_x^\infty}. \end{aligned} \tag{5-17}$$

If $l < \frac{5}{6}\gamma$ then, using

$$B^s_{4/3} \supset B^{2l+\gamma/2+1}_{4/3} \supset B^{2l-5\gamma/6+2}_{q_4}, \quad \frac{1}{q_4} := \frac{1}{2} + \frac{1}{q_1} = 1 - \frac{\gamma}{3}, \tag{5-18}$$

we have

$$\|P_{>k}(N\Omega(N, u))_{\widetilde{LH}}\|_{B^s_{4/3}} \lesssim \|N\|_{\mathcal{B}} \|\Omega(N, u)_{>k-K-2}\|_{H^{l+2}} \lesssim \|N\|_{\mathcal{B}} \|N_{>k-K-3}\|_{H^l} \|u\|_{L_x^\infty}. \tag{5-19}$$

For the other term, putting $\sigma := \frac{5}{6}\gamma - l > 0$ and $\beta := l/(l+\sigma) \in (0, 1)$, we have the complex interpolation

$$[H^l, B^{-\sigma}_{q_1}]_\beta = B^0_{q_5} \subset L^{q_5}, \quad [H^l, B^{-\sigma}_{q_1}]_{1-\beta} = B^{l-\sigma}_{q_6}, \tag{5-20}$$

where $1/q_5 := \frac{1}{2}(1-\beta)+\beta/q_1$ and $1/q_6 := 1/q_4 - 1/q_5$, whereas

$$\begin{aligned} \|P_{>k}\Omega(N, Nu)\|_{B^s_{4/3}} \lesssim \|P_{>k}\Omega(N, Nu)\|_{B^{l-\sigma+2}_{q_4}} &\lesssim \|N_{>k-1}\|_{B^{l-\sigma}_{q_6}} \|Nu\|_{L^{q_5}} \\ &\lesssim \|N_{>k-1}\|_{B^{l-\sigma}_{q_6}} \|N\|_{B^0_{q_5}} \|u\|_{L^\infty_x}. \end{aligned} \tag{5-21}$$

Hence, by the interpolation inequality,

$$\|\Omega(N, Nu)\|_{B^s_{4/3}} \lesssim \|N_{>k-1}\|^{\beta}_{H^l} \|N\|^{1-\beta}_{H^l} \|N\|_{\mathcal{B}} \|u\|_{L^\infty_x}. \tag{5-22}$$

Therefore, in any case we have some $\beta(l, \gamma) \in (0, 1]$ such that

$$\begin{aligned} \|P_{>k}\Omega(N, Nu)\|_{L^2_t B^s_{4/3}} &\le C_2 \|N_{>k-1}\|^{\beta}_{L^\infty_t H^l_x} \|N\|^{1-\beta}_{L^\infty_t H^l_x} \|N\|_{L^{2/\gamma}_t \mathcal{B}} \|u\|_{X'}, \\ \|P_{>k}(N\Omega(N, u))_{\widetilde{LH}}\|_{L^2_t B^s_{4/3}} &\le C_2 \|N_{>k-K-3}\|_{L^\infty_t H^l_x} \|N\|_{L^{2/\gamma}_t \mathcal{B}} \|u\|_{X'} \end{aligned} \tag{5-23}$$

for some constant $C_2(s, l) > 0$. Choose $K \gg 1$ so large that $C_0 2^{-\theta K} \|u\|^2_{L^2_t B^{1/2}_4} \ll 1$, and then choose $\varepsilon > 0$ so small and $k \gg K$ so large that

$$\begin{aligned} C_1(K)\varepsilon + C_0 \|N_{>k-1}\|_{L^\infty_t H^l_x} &\ll 1, \\ C_2 \|N_{>k-K-3}\|^{\beta}_{L^\infty_t H^l_x} \|N\|^{1-\beta}_{L^\infty_t H^l_x} \|N\|_{L^{2/\gamma}_t \mathcal{B}} &\ll 1. \end{aligned} \tag{5-24}$$

Applying Lemma 4.1, we obtain a finite sequence $0 = T_0 < T_1 < \cdots < T_{n+1} = T$ such that (5-3) holds. Then, from the above estimates, on each subinterval,

$$\begin{aligned} \|u'_{>k}\|_{X^s(T_j, T_{j+1})} &\le C_3 \|u'(T_j)\|_{H^s} + \delta \|u'\|_{X^s(T_j, T_{j+1})} + \delta \|u\|_{X'(T_j, T_{j+1})}, \\ \|\Omega(N, u)_{>k}\|_{X'(T_j, T_{j+1})} &\le \delta \|u\|_{X'(T_j, T_{j+1})} \end{aligned} \tag{5-25}$$

for some small constant $\delta > 0$, while the frequencies below k are bounded by $X^{1/2}$. Using $u = u' + \Omega(N, u)$ and $X^s \subset X'$, and adding the low frequencies, we obtain

$$\|u'\|_{X^s(T_j, T_{j+1})} + \|\Omega(N, u)\|_{X'(T_j, T_{j+1})} \le 2C_3 \|u'(T_j)\|_{H^s} + 2^{k(s-1/2)} \|u\|_{X^{1/2}(T_j, T_{j+1})}. \tag{5-26}$$

By induction on j starting from $\|u'(0)\|_{H^s} < \infty$, we thus obtain

$$\|u\|_{X'(0,T)} \lesssim \|u'\|_{X^s(0,T)} + \|\Omega(N, u)\|_{X'(0,T)} < \infty. \tag{5-27}$$

If $T = \infty$, then u is scattering, by the argument in Section 3. Thus we have obtained:

Proposition 5.2. *Let $(s, l) \in \mathbb{R}^2$ satisfy* (1-11) *and $s \ge l+1$. Let $(u, N) \in (X^{1/2} \times Y^0)(I)$ be a solution of* (1-9) *on an interval $I \subset \mathbb{R}$, and suppose that $(u(t_0), N(t_0)) \in H^s(\mathbb{R}^4) \times H^l(\mathbb{R}^4)$ at some $t_0 \in I$. Then we have $u - \Omega(N, u) \in X^s(I)$, as well as* (5-8), *and, for all $\gamma \in \left[0, \frac{3}{4}\right]$ satisfying* (5-11),

$$u \in C(I; H^s_x) \cap L^\infty_t(I; H^s_x) \cap L^{2/(1-\gamma)}(I; L^\infty_x). \tag{5-28}$$

We also have scattering and continuous dependence similar to Proposition 5.1, but in the space (5-28).

It is easy to replace L^∞_x with $B^s_{4/(1+\gamma)} + H^{l+2}$ using $u' \in X^s$ and (5-14).

5D. *Lipschitz continuity of the solution map.* Here we consider local Lipschitz continuity of the flow map. In the above arguments, the Lipschitz dependence is lost only when we seek time intervals with smallness, typically by Lemma 4.1. If $(u_0, N_0) \in H^s \times H^l$ with $s > \frac{1}{2}$ and $l > 0$, however, it is easy to see that (4-5) holds locally uniformly with respect to the initial data, because we can first dispose of the high frequencies using the higher regularity, and then the remaining low frequencies by Sobolev in x and Hölder in t.

Similarly, the regularity upgrading argument in Section 5A works uniformly if $l > 0$ and $T < \infty$, because of (5-3), and so does the argument in Section 5B for $s > \frac{1}{2}$, $l < \min(2s-1, s+1)$, and $T < \infty$, because of (5-6), as well as that in Section 5C for $l > 0$, $s < \min\big(2l + \frac{11}{8}, l+2\big)$, and $T < \infty$, because of (5-24) and (5-3).

Thus we obtain Lipschitz continuity of the flow map, locally both in time and in the initial data, for all the exponents (s,l) in the range and off the boundary. Since we need to decrease l for the uniform control in (5-24), γ in (5-10) cannot be on the boundary, namely $2l + \frac{1}{2}\gamma + 1 > s$, for the local Lipschitz estimate.

The Lipschitz continuity global in time and for the scattering is more tricky, because the $L_t^2 L_x^4$ norm in Lemma 4.1 is not bounded by the Strichartz estimate for the wave equation. For small data, we can obtain Lipschitz estimates directly from the contraction mapping argument, but then the smallness on $H^{1/2} \times L^2$ depends on (s,l), which tends to 0 as (s,l) approaches $s = 4l+1$, $(2,3)$ or (∞,∞). The regularity upgrading for N in Section 5B works well for $T = \infty$, because in (5-6) the number of subintervals can be uniformly bounded for each $\varepsilon > 0$, provided that $\|u\|_{L_t^2 B_4^s}$ is uniformly bounded. This yields a smallness condition in the form

$$\|(u_0, N_0)\|_{H^{1/2}\times L^2} \le \varepsilon_2(s,l), \tag{5-29}$$

where $\varepsilon_2(s,l) > 0$ is nondecreasing in l for global Lipschitz continuity in $H^s \times H^l$.

Remark 5.3. Strictly speaking, we need to prove that the solution to (2-5) obtained above is also a solution of (1-9) before the normal form. The easiest way is to use [Ginibre et al. 1997] for existence of solutions for smooth approximating initial data, taking the limit by the continuous dependence proved above. To be self-contained, however, we can directly show that smooth solutions of (2-5) solve (1-9). In fact, if $(u, N) \in (X^s \times Y^s)(I)$ with $s \gg 1$ is a solution of (2-5) on some interval I, then, by definition of Ω and $\tilde{\Omega}$, (2-5) reads

$$\begin{aligned} eq_u &:= (i\partial_t + D^2)u - Nu = -\Omega(eq_N, u) - \Omega(N, eq_u), \\ eq_N &:= (i\partial_t + \alpha D)N - \alpha D|u|^2 = -D\tilde{\Omega}(eq_u, u) - D\tilde{\Omega}(u, eq_u). \end{aligned}$$

Since $eq_u, eq_N \in C(I; H^{s-2})$ and Ω, $D\tilde{\Omega} : (H^{s-2})^2 \to H^{s-2}$ has a small factor due to K, we deduce that $eq_u = 0 = eq_N$ on I if K is large enough.

6. Small data scattering in the energy space

For $(s,l) = (1,0)$, the failure of Strichartz bound on the normal form $\Omega(N,u)$ cannot be compensated by regularity of N, and so there seems no way to close the estimates as above for $(s,l) = (1,0)$. Instead, we invoke the conservation laws with the weak compactness argument. This type of argument usually yields a weak result, typically without uniqueness. We can however obtain the strong well-posedness for small data as in Theorem 1.1, thanks to that both in the larger space $(s, 0)$ with $s < 1$, and in the smaller space $(1, l)$ with $l > 0$.

Assume that $(u_0, N_0) \in H^1 \times L^2$. By Proposition 3.1 there is $\varepsilon_0 := \varepsilon_1\big(\frac{1}{2}, 0\big) \ll 1$ such that, if $\|(u_0, N_0)\|_{H^{1/2}\times L^2} \le \varepsilon_0$, then there is a unique global solution (u, N) in $X^{1/2} \times Y^0$, satisfying

$$\|(u, N)\|_{X^{1/2}\times Y^0} \le C\varepsilon_0 \ll 1. \tag{6-1}$$

Proposition 5.1 implies that $(u, N) \in X^s \times Y^0$ for all $s \in \big[\frac{1}{2}, 1\big)$.

Fix a sequence $\{(u_{0,n}, N_{0,n})\} \subset \mathcal{S}(\mathbb{R}^4)$ such that

$$(u_{0,n}, N_{0,n}) \to (u_0, N_0) \quad \text{in } H^1 \times L^2 \qquad \text{and} \qquad \|(u_{0,n}, N_{0,n})\|_{H^{1/2}\times L^2} \le \varepsilon_0.$$

By Proposition 3.1, for each n, there is a unique global solution (u_n, N_n) satisfying (6-1) and, for all $\frac{1}{2} \le s < 1$,

$$\sup_n \|(u_n, N_n)\|_{X^s\times Y^0} < \infty. \tag{6-2}$$

Now we claim a uniform bound at the energy level:

$$\sup_{n,t} \|(u_n(t), N_n(t))\|_{H^1\times L^2} < \infty. \tag{6-3}$$

By Proposition 5.1, we have $(u_n, N_n) \in X^8 \times Y^9$ for all n, by which we can justify the conservation law $E_Z(u_n(t), N_n(t)) = E_Z(u_{0,n}, N_{0,n})$. Using (6-1) for N_n together with the Sobolev inequality $\|u\|_{L^4_x} \lesssim \|\nabla u\|_{L^2_x}$ yields

$$E_Z(u_n, N_n) = (1 - O(\varepsilon_0))\|\nabla u_n\|_2^2 + \tfrac{1}{2}\|N_n\|_2^2, \tag{6-4}$$

which, combined with the lower regularity bound (6-2), implies (6-3).

Next we prove convergence $u_n(t) \to u(t)$ in H^1_x as $n \to \infty$, locally uniformly in $\mathbb{R}$. Take any convergent sequence $t_n \to t_\infty$. From Propositions 3.1 and 5.1, we know that $u_n(t_n) \to u(t_\infty)$ in H^s_x for $s < 1$, and $N_n(t_n) \to N(t_\infty)$ in L^2_x. From (6-3), we have $\{u_n(t_n)\}_n$ is bounded in $H^1_x \subset L^4_x$, thus we get $u(t_\infty) \in H^1$, $u_n(t_n) \to u(t_\infty)$ weakly in H^1_x, and $|u_n(t_n)|^2 \to |u(t_\infty)|^2$ weakly in L^2_x. Since $N_n(t_n) \to N(t_\infty)$ strongly in L^2_x, we have $\int N_n(t_n)|u_n(t_n)|^2\,dx \to \int N(t_\infty)|u(t_\infty)|^2\,dx$, and so

$$E_Z(u(t_\infty), N(t_\infty)) \le \liminf_{n\to\infty} E_Z(u_n(t_n), N_n(t_n)) = \liminf_{n\to\infty} E_Z(u_{0,n}, N_{0,n}) = E_Z(u_0, N_0). \tag{6-5}$$

By the time reversibility we get $E_Z(u(t_\infty), N(t_\infty)) = E_Z(u_0, N_0)$. Indeed, if there is a $t_0 \in \mathbb{R}$ such that $E_Z(u(t_0), N(t_0)) < E_Z(u_0, N_0)$, then we solve the Zakharov system with initial data $(u(t_0), N(t_0))$ at $t = t_0$. By the uniqueness we get a contradiction. Then the equality in (6-5) implies $\|\nabla u_n(t_n)\|_{L^2} \to \|\nabla u(t_\infty)\|_{L^2}$, from which we conclude that $u_n(t_n) \to u(t_\infty)$ strongly in H^1_x, and so the locally uniform convergence $u_n \to u$ in $C(\mathbb{R}; H^1_x)$. Thus we obtain the unique global solution $(u, N) \in (C \cap L^\infty)(\mathbb{R}; H^1 \times L^2)$. Note that the smoothness of the approximate solutions (u_n, N_n) was used only to ensure the unique existence and the conservation law. Now that we have them for the solutions in the energy space, we can apply the above argument to a sequence of initial data in $H^1 \times L^2$, which implies continuous dependence of the initial data, locally uniformly in time.

By Propositions 3.1 and 5.1, (u, N) scatters to some (u^+, N^+) in $H^s \times L^2$ for all $s < 1$. Since $u(t) \in L^\infty(\mathbb{R}; H^1_x)$, we have $S(-t)u(t) \to u^+$ weakly in H^1 as $t \to +\infty$. Since $|u(t)|^2$ is bounded in $(H^1_x)^2 \subset B^1_{4/3}$, while $N(t)$ is vanishing in B^{-1}_4 as $t \to \infty$ due to the scattering in $L^2_x \subset B^{-1}_4$, we have

$$\int N(t)|u(t)|^2\,dx \to 0 \quad (t \to +\infty), \tag{6-6}$$

and so

$$\begin{aligned}\|\nabla u^+\|_2^2 + \tfrac{1}{2}\|N^+\|_2^2 &\le \liminf_{t\to+\infty} \|\nabla S(-t)u(t)\|_2^2 + \tfrac{1}{2}\|W_\alpha(-t)N(t)\|_2^2 \\ &= \liminf_{t\to+\infty} \|\nabla u(t)\|_2^2 + \tfrac{1}{2}\|N(t)\|_2^2 \\ &= \liminf_{t\to+\infty} E_Z(u(t), N(t)) = E_Z(u_0, N_0).\end{aligned}$$

To prove the equality above, we consider the final state problem. Following the argument in the first step, above, we fix a sequence $\{(u_n^+, N_n^+)\} \subset \mathcal{S}(\mathbb{R}^4)$ such that $(u_n^+, N_n^+) \to (u^+, N^+)$ in $H^1 \times L^2$. Then, by Proposition 3.1, we have a sequence of solutions $(\tilde{u}_n, \tilde{N}_n) \in X^{1/2} \times Y^0$ scattering to (u_n^+, N_n^+) as $t \to \infty$, which converges to (u, N) in $X^{1/2} \times Y^0$ as $n \to \infty$. The regularity is upgraded to $X^s \times Y^l$ for all (s, l) in Proposition 5.1. As in the first step, we have $\sup_{n,t} \|(\tilde{u}_n, \tilde{N}_n)\|_{H^1 \times L^2} < \infty$, hence $u_n(t) \to u(t)$ weakly in H^1_x. Thus, by (6-6),

$$E_Z(u(t), N(t)) \le \liminf_{n\to\infty} E_Z(\tilde{u}_n(t), \tilde{N}_n(t)) = \liminf_{n\to\infty} \|\nabla u_n^+\|_2^2 + \tfrac{1}{2}\|N_n^+\|_2^2 = \|\nabla u^+\|_2^2 + \tfrac{1}{2}\|N^+\|_2^2.$$

Hence, we get

$$\lim_{t\to+\infty} E_Z(u(t), N(t)) = \|\nabla u^+\|_2^2 + \tfrac{1}{2}\|N^+\|_2^2$$

and so $S(-t)u(t) \to u^+$ strongly in H^1_x, namely the scattering in H^1_x.

To show the continuity of the solution map in $L^\infty_t(\mathbb{R}; H^1_x)$, it remains to prove $u_n(t_n) - u(t_n) \to 0$ in H^1_x, in the case $t_n \to \infty$, for a sequence of solutions (u_n, N_n) in the energy space such that

$$(u_n(0), N_n(0)) \to (u(0), N(0)) \quad \text{in } H^1 \times L^2.$$

Since $S(-t)u(t) \to u^+$ in H^1_x, this is equivalent to showing $S(-t_n)u_n(t_n) \to u^+$ in H^1_x. We already know the H^s_x convergence for $s < 1$ as well as the weak convergence in H^1_x. Then the strong convergence is equivalent to $\|u_n(t_n)\|_{H^1_x} \to \|u^+\|_{H^1}$. Since

$$\|N_n(t_n)\|_{B^{-1}_4} \le \|N_n - N\|_{L^\infty_t L^2_x} + \|N(t_n)\|_{B^{-1}_4} \to 0, \tag{6-7}$$

we have $\int N_n(t_n)|u_n(t_n)|^2\,dx \to 0$, and so, as $n \to \infty$,

$$\begin{aligned}\|\nabla u_n(t_n)\|_2^2 + \tfrac{1}{2}\|N_n(t_n)\|_2^2 = E_Z(u_n(t_n), N_n(t_n)) + o(1) &= E_Z(u_n(0), N_n(0)) + o(1) \\ &= E_Z(u(0), N(0)) + o(1) \\ &= \|\nabla u^+\|_2^2 + \tfrac{1}{2}\|N^+\|_2^2 + o(1).\end{aligned} \tag{6-8}$$

Since $\|u_n(t_n)\|_2 \to \|u^+\|_2$ and $\|N_n(t_n)\|_2 \to \|N^+\|_2$, the above implies the strong convergence of $S(-t_n)u_n(t_n)$ in H_x^1, and thus $u_n \to u$ in $L_t^\infty(\mathbb{R}; H_x^1)$. This completes the proof of Theorem 1.1 in the case $(s,l)=(1,0)$.

7. Ill-posedness at $(s, l) = (2, 3)$

In this section, we prove Theorem 1.3. The main point is that the multilinear estimates fail only for the boundary quadratic term coming from the initial data. Exploiting the dispersive smoothing, we can prove that the other terms are more regular if the initial data is localized in space.

Proof of Theorem 1.3. First of all, for any initial data $(u_0, N_0) \in H^2 \times H^3$, we have a unique local solution for (s,l) in (1-11) satisfying $s \le 2$ and $l \le 3$, say $(u, N) \in (X^2 \times Y^2)([0,T])$, by Propositions 4.2 and 5.1. In the Duhamel formula (2-4), the first term on the right is obviously in $C(\mathbb{R}; H^3)$. The integral terms are regular thanks to the high regularity. Indeed,

$$\begin{aligned} \|D|u|^2_{HH+\alpha L+L\alpha}\|_{L_t^1 H_x^3} &\lesssim \|u\|^2_{L_t^2 B_4^2}, \\ \|D\tilde{\Omega}(Nu,u)\|_{L_t^1 H_x^3} &\lesssim \|N\|_{L_t^\infty H_x^2}\|u\|^2_{L_t^2 B_4^2}, \end{aligned} \tag{7-1}$$

and the same for $D\tilde{\Omega}(u, Nu)$. To bound $D\tilde{\Omega}(u,u)$ in H_x^3, we use local smoothing for u, assuming that

$$u_0 \in W^{2,1}(\mathbb{R}^4) = \{f \mid \partial^\alpha f \in L^1(\mathbb{R}^4) \text{ for } |\alpha| \le 2\}. \tag{7-2}$$

Then $S(t)u_0 \in C((0,\infty); B_p^2)$ for all $p > 2$ by the dispersive L_x^p decay estimate for $S(t)$. Moreover, in the Duhamel formula (2-3) of u, the terms except for $(Nu)_{\widetilde{LH}}$ easily gain better regularity by

$$\begin{aligned} \|\Omega(N,u)\|_{H_x^3} &\lesssim \|N\|_{H_x^2}\|u\|_{H_x^2}, \\ \|\Omega(D|u|^2,u)\|_{L_t^1 H_x^3} &\lesssim \|u\|_{L_t^\infty H_x^2}\|u\|^2_{L_t^2 B_4^2}, \\ \|\Omega(N,Nu)\|_{L_t^2 B^4_{4/3}} &\lesssim \|N\|^2_{L_t^\infty H_x^2}\|u\|_{L_t^2 B_4^2}. \end{aligned} \tag{7-3}$$

The remaining term is bounded in $C([0,T]; B_3^2)$ by

$$\left\| \int_0^t S(t-s)(Nu)_{\widetilde{LH}}\, ds \right\|_{B_3^2} \lesssim \int_0^t |t-s|^{2/3}\|(Nu)_{\widetilde{LH}}\|_{B^2_{3/2}}\, ds \lesssim \int_0^t |t-s|^{2/3}\|N(s)\|_{L_x^6}\|u(s)\|_{H_x^2}\, ds. \tag{7-4}$$

Gathering the above estimates, we obtain $u \in C((0,T]; H^2 \cap B_3^2)$. Since $B_3^2 \subset L^\infty$,

$$\|D\tilde{\Omega}(u,u)\|_{H^3} \lesssim \|u\|_{H^2}\|u\|_{B_3^2}, \tag{7-5}$$

and, plugging this into the above estimates for N, we deduce that

$$N - W_\alpha(t)D\tilde{\Omega}(u_0,u_0) \in C((0,T]; H_x^3) \tag{7-6}$$

if $u_0 \in H^2 \cap W^{2,1}(\mathbb{R}^4)$. Hence, it suffices to find such a u_0 such that $D\tilde{\Omega}(u_0,u_0) \notin H_x^3$. This is constructed in Lemma 7.1. Then $N(t) \notin H_x^3$ for all $0 < t < T$, namely the instant exit or the latter part of the theorem.

Thanks to the high regularity, it is easy to translate it to nonexistence. Indeed, if (u, N) is in $L^2((0,T); H^1 \times H^3)$ then, from the equation without the normal form,

$$Nu \in L^1_t H^1_x \implies u \in C_t H^1_x \cap L^2_t B^1_4 \implies D|u|^2 \in L^1_t L^2_x \implies N \in C_t L^2_x. \tag{7-7}$$

In particular, (u, N) belongs to the uniqueness class at $(s, l) = \left(\frac{1}{2}, 0\right)$. Hence it should be identical with the exiting solution obtained above, satisfying $N(t) \notin H^3$ for all $t \neq 0$, contradicting $N \in L^2_t((0,T); H^3_x)$. □

It remains to prove the failure of the bilinear estimate:

Lemma 7.1. *There is a radial* $u \in (H^2 \cap W^{2,1})(\mathbb{R}^4)$ *satisfying* $D\tilde{\Omega}(u,u) \notin H^3(\mathbb{R}^4)$.

This failure of the bilinear estimate comes from that $H^2(\mathbb{R}^4)$ is not an algebra, but we should be careful about cancellation in the nonlinearity. In fact, the proof below implies that $D\tilde{\Omega}(u,u)$ is bounded in H^3 for real-valued or purely imaginary $u \in H^2$.

Proof. Modulo a bounded operator, the symbol of $D\tilde{\Omega}$ can be approximated by

$$\frac{\alpha|\xi|}{|\xi-\eta|^2 - |\eta|^2 \mp \alpha|\xi|} = \frac{\alpha}{|\xi|} + \frac{\alpha(2\xi\cdot\eta \pm \alpha|\xi|)}{|\xi|(|\xi-\eta|^2 - |\eta|^2 \mp \alpha|\xi|)} \tag{7-8}$$

in the *XL* frequency, while, in the *LX* frequency,

$$\frac{\alpha|\xi|}{|\xi-\eta|^2 - |\eta|^2 \mp \alpha|\xi|} = \frac{-\alpha}{|\xi|} + \frac{\alpha(2\xi\cdot(\xi-\eta) \pm \alpha|\xi|)}{|\xi|(|\xi-\eta|^2 - |\eta|^2 \mp \alpha|\xi|)}, \tag{7-9}$$

where the second terms are $O(|\xi|^{-2}\langle \mathrm{Low}\rangle)$ for $|\xi| \gg 1$, and so bounded $H^2 \times H^2 \to H^3$ for high frequency. Hence, it suffices to construct $u \in H^2 \cap W^{1,2}$ such that $\operatorname{supp} \hat{u}(\xi) = 0$ for $|\xi| \lesssim 1$ and

$$(u\bar{u})_{HL} - (u\bar{u})_{LH} \notin H^2(\mathbb{R}^4). \tag{7-10}$$

Indeed, this is necessary and sufficient for $D\tilde{\Omega}(u,u) \notin H^3$ under the condition of $\operatorname{supp}\hat{u}$. Note that the left side is simply zero if $u(\mathbb{R}^4) \subset \mathbb{R}$ or $iu(\mathbb{R}^4) \subset \mathbb{R}$. The remaining is the antisymmetric part, which can be expanded by putting $u = v + iw$:

$$(u\bar{u})_{HL} - (u\bar{u})_{LH} = 2i[(wv)_{HL} - (wv)_{LH}]. \tag{7-11}$$

Now it is easy to avoid the cancellation considering the forms

$$v = \sum_{j>J} a_j \varphi_j, \quad w = \sum_{j>J} b_j \varphi_j, \quad \varphi_j(x) = \varphi(2^j x), \tag{7-12}$$

where $J \gg \log\alpha$, $\{a\}, \{b\} \subset [0,\infty)$, and $\varphi \in \mathcal{S}(\mathbb{R}^4; \mathbb{R})$ is a nonzero real-valued radial function satisfying

$$0 \le \hat{\varphi} \le 1, \quad \operatorname{supp}\hat{\varphi} \subset \left\{ \big||\xi| - 1\big| \ll 1 \right\}. \tag{7-13}$$

Put $c := \varphi(0) > 0$. Inserting the above ansatz expands the bilinear form

$$(vw)_{HL} - (vw)_{LH} = \sum_{j>J} \sum_{k>J}^{j-K} (a_j b_k - a_k b_j)\varphi_j \varphi_k. \tag{7-14}$$

Since $\mathscr{F}(\varphi_j\varphi_k)$ is supported around $|\xi| = 2^j$,

$$\|(7\text{-}14)\|_{H^2}^2 \sim \sum_{j>J}\Big\|2^{2j}\sum_{k\le j-K}(a_jb_k - a_kb_j)\varphi_j\varphi_k\Big\|_2^2. \tag{7-15}$$

Imposing a support condition on $\{a\}$ and $\{b\}$,

$$\operatorname{supp} a \cap \operatorname{supp} b = \varnothing, \tag{7-16}$$

we can decouple the above as

$$\|(7\text{-}14)\|_{H^2}^2 \sim \sum_{j>J}\Big\|2^{2j}\sum_{k\le j-K}a_jb_k\varphi_j\varphi_k\Big\|_2^2 + \sum_{j>J}\Big\|2^{2j}\sum_{k\le j-K}b_ja_k\varphi_j\varphi_k\Big\|_2^2. \tag{7-17}$$

By rescaling $x \mapsto 2^{-j}x$ and using $\varphi(2^{k-j}x) = c + O(|2^{k-j}x|)$, the L_x^2 norm is approximated by

$$\begin{aligned}\Big\|2^{2j}\sum_{k\le j-K}a_jb_k\varphi_j\varphi_k\Big\|_{L_x^2} &= \Big\|a_j\varphi(x)\sum_{k\le j-K}b_k\varphi(2^{k-j}x)\Big\|_{L_x^2}\\ &\ge c|a_j|\,\|\varphi\|_{L_x^2}\sum_{k\le j-K}b_k - C\|a_jx\varphi(x)\|_{L_x^2}\sum_{k\le j-K}b_k2^{k-j}.\end{aligned} \tag{7-18}$$

Fix $\theta \in \left(\frac{1}{2}, \frac{3}{4}\right)$ and let

$$a_j = \begin{cases} j^{-\theta} & \text{if } J < j \text{ is even,}\\ 0 & \text{otherwise,}\end{cases} \qquad b_j = \begin{cases} j^{-\theta} & \text{if } J < j \text{ is odd,}\\ 0 & \text{otherwise.}\end{cases} \tag{7-19}$$

Then, for $j > K + J$,

$$\sum_{k\le j-K} b_k \sim (j-K)^{1-\theta}, \qquad \sum_{k\le j-K} b_k2^{k-j} \lesssim 2^{-K}, \tag{7-20}$$

and so

$$\|(7\text{-}14)\|_{H^2} \gtrsim \|j^{-\theta}(j-K)^{1-\theta}\|_{\ell^2(j>J+K)} - C2^{-K}\|j^{-\theta}\|_{\ell^2(j>J)} = \infty, \tag{7-21}$$

since $-\theta < -\frac{1}{2} < 1-2\theta$. Also, we have

$$\|u\|_{H^2} \lesssim \|j^{-\theta}\|_{\ell^2(j>J)} < \infty, \quad \|u\|_{W^{2,1}} \lesssim \|2^{-2j}j^{-\theta}\|_{\ell^1(j>J)} < \infty. \tag{7-22}$$

Thus we have obtained an example $u \in H^2 \cap W^{2,1}$, as desired. □

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