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THE BORDERLINES OF INVISIBILITY AND VISIBILITY IN CALDERÓN'S INVERSE PROBLEM

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We consider the determination of a conductivity function in a two-dimensional domain from the Cauchy data of the solutions of the conductivity equation on the boundary. We prove uniqueness results for this inverse problem, posed by Calderón, for conductivities that are degenerate, that is, they may not be bounded from above or below. Elliptic equations with such coefficient functions are essential for physical models used in transformation optics and the study of metamaterials, e.g., the zero permittivity materials. In particular, we show that the elliptic inverse problems can be solved in a class of conductivities which is larger than L^∞. Also, we give new counterexamples for the uniqueness of the inverse conductivity problem.

We say that a conductivity is visible if the inverse problem is solvable so that the conductivity inside of the domain can be uniquely determined, up to a change of coordinates, using the boundary measurements. The original counterexamples for the inverse problem are related to the invisibility cloaking. This means that there are conductivities for which a part of the domain is shielded from detection via boundary measurements and even the existence of the shielded domain is hidden. Such conductivities are called invisibility cloaks.

In the present paper, we identify the borderline of the visible conductivities and the borderline of invisibility cloaking conductivities. Surprisingly, these borderlines are not the same. We show that between the visible and the cloaking conductivities, there are the electric holograms. These are conductivities which create an illusion of a nonexisting body. Such conductivities give counterexamples for the uniqueness of the inverse problem which are less degenerate than the previously known ones. These examples are constructed using transformation optics and the inverse maps of the optimal blow-up maps. The proofs of the uniqueness results for inverse problems are based on the complex geometrical optics and the Orlicz space techniques.

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Keywords: inverse conductivity problem, invisibility, quasiconformal mappings.

1. Introduction and main results

Invisibility cloaking has been a very topical subject in recent studies in mathematics, physics, and material science [Ammari et al. 2013; Alu and Engheta 2005; Greenleaf et al. 2007c; 2003a; 2003c; Milton and Nicorovici 2006; Leonhardt 2006; Liu 2013; Liu and Sun 2013; Pendry et al. 2006]. By invisibility cloaking we mean the possibility, both theoretical and practical, of shielding a region or object from detection via electromagnetic fields.

Counterexamples for inverse problems and the proposals for invisibility cloaking are closely related. In 2003, before the appearance of practical possibilities for cloaking, it was shown in [Greenleaf et al. 2003c] that passive objects can be coated with a layer of material with a degenerate conductivity which makes the object undetectable by electrostatic boundary measurements in such a way that the coated object appears in all measurements the same as a body made of homogeneous material. These constructions were based on blow-up maps and gave counterexamples for uniqueness of the inverse conductivity problem in the three- and higher-dimensional cases. In the two-dimensional case, the mathematical theory of the cloaking examples for the conductivity equation have been studied in [Kohn et al. 2008; 2010]. Besides for the conductivity equation, these results can be applied for other physical models based on elliptic equations.

The interest in cloaking was raised in particular in 2006 when it was realized that practical cloaking constructions are possible using so-called metamaterials which allow fairly arbitrary specification of electromagnetic material parameters. The construction of Leonhardt [2006] was based on conformal mapping on a nontrivial Riemannian surface. At the same time, Pendry et al. [2006] proposed a cloaking construction for Maxwell's equations using a blow-up map and the idea was demonstrated in laboratory experiments [Schurig et al. 2006]. Cloaking for the conductivity equation has been demonstrated in laboratory experiments by Yang et al. [2012]. In the now very large literature, there are also other suggestions for cloaking based on negative material parameters [Alu and Engheta 2005; Milton and Nicorovici 2006].

In this paper, we consider both new counterexamples and uniqueness results for inverse problems. We study Calderón's inverse problem in the two-dimensional case, that is, the question of whether an unknown conductivity distribution inside a domain can be determined from the voltage and current measurements made on the boundary of a simply connected domain $\Omega \subset \mathbb{R}^2$; see [Borcea 2002]. Mathematically the measurements correspond to the knowledge of the *Dirichlet-to-Neumann map* Λ_σ (or the quadratic form) associated to σ, i.e., the map taking the Dirichlet boundary values of the solution of the conductivity equation

$$\nabla \cdot \sigma(x) \nabla u(x) = 0 \quad \text{in } \Omega \tag{1-1}$$

to the corresponding Neumann boundary values,

$$\Lambda_\sigma : u|_{\partial\Omega} \mapsto \nu \cdot \sigma \nabla u|_{\partial\Omega}. \tag{1-2}$$

In the classical theory of the problem, the conductivity σ is bounded uniformly from above and below. The problem was originally proposed by Calderón [1980]. Sylvester and Uhlmann [1987] proved the unique identifiability of the conductivity in dimensions three and higher for isotropic conductivities which are C^∞-smooth, and Nachman [1988] gave a reconstruction method. In three dimensions or higher, unique identifiability of the conductivity is proven in [Haberman and Tataru 2013] for C^1-conductivities;

for earlier studies on the topic, see [Greenleaf et al. 2003b; Päivärinta et al. 2003]. The problem has also been solved with measurements only on a part of the boundary [Kenig et al. 2007].

In two dimensions, the first global solution of the inverse conductivity problem is due to Nachman [1996] for conductivities with two derivatives. In this seminal paper, the $\bar{\partial}$ technique was used for the first time in the study of Calderón's inverse problem. The smoothness requirements were reduced in [Brown and Uhlmann 1997] to Lipschitz conductivities. Finally, in [Astala and Päivärinta 2006] the uniqueness of the inverse problem was proven in the form that the problem was originally formulated in [Calderón 1980], i.e., for general isotropic conductivities in L^∞ which are bounded from below and above by positive constants. Stability of this reconstruction is studied in [Alessandrini 1988; Barceló et al. 2001; 2007] and the numerical solutions in [Astala et al. 2011; Isaacson et al. 2004; Knudsen et al. 2007; 2009; Mueller and Siltanen 2012; Siltanen et al. 2000].

The Calderón problem with anisotropic, i.e., matrix-valued, conductivities that are uniformly bounded from above and below, has been studied in two dimensions [Sylvester 1990; Nachman 1996; Lassas et al. 2003; Lassas and Uhlmann 2001; Astala et al. 2005; Imanuvilov et al. 2010] and in dimensions $n \geq 3$ [Lee and Uhlmann 1989; Lassas and Uhlmann 2001; Dos Santos Ferreira et al. 2009]. For example, for the anisotropic inverse conductivity problem in the two-dimensional case, it is known that the Dirichlet-to-Neumann map determines a regular conductivity tensor only up to a diffeomorphism $F : \overline{\Omega} \to \overline{\Omega}$; i.e., one can obtain an image of the interior of Ω in deformed coordinates. This implies that the inverse problem is not uniquely solvable, but the nonuniqueness of the problem can be characterized. This makes it possible, e.g., to find the unique conductivity that is closest to isotropic ones [Kolehmainen et al. 2005; 2010; 2013]. We note that this problem in higher dimensions is presently solved only in special cases, when the conductivity is real analytic; see [Lassas et al. 2003; Lassas and Uhlmann 2001].

In this work, we will study the inverse conductivity problem in the two-dimensional case with degenerate conductivities. Such conductivities appear in physical models where the medium varies continuously from a perfect conductor to a perfect insulator or in high-contrast problems [Borcea et al. 1996; Borcea 1999]. As an example, we may consider a case where the conductivity goes to zero or to infinity near ∂D, where $D \subset \Omega$ is a smooth open set. We ask what kind of degeneracy prevents solving the inverse problem; that is, we study what is the border of visibility. We also ask what kind of degeneracy makes it possible to coat an arbitrary object so that it appears the same as a homogeneous body in all static measurements; that is, we study what is the border of the invisibility cloaking. *Surprisingly, these borders are not the same.* We identify these borderlines and show that between them there are the *electric holograms*, that is, the conductivities creating an illusion of a *nonexistent* body (see Figure 1). These conductivities are counterexamples for the unique solvability of inverse problems for which even the topology of the domain cannot be determined using boundary measurements. Our main results for the uniqueness of the inverse problem are given in Theorems 1.8, 1.9, and 1.11 and the counterexamples are formulated in Theorems 1.6 and 1.7.

The cloaking constructions have given rise to the design technique called *transformation optics*. The metamaterials built to operate at microwave frequencies [Schurig et al. 2006] and near the optical frequencies [Ergin et al. 2010] are inherently prone to dispersion, so that realistic cloaking must currently be considered as occurring at a very narrow range of wavelengths. Fortunately, in many physical applications

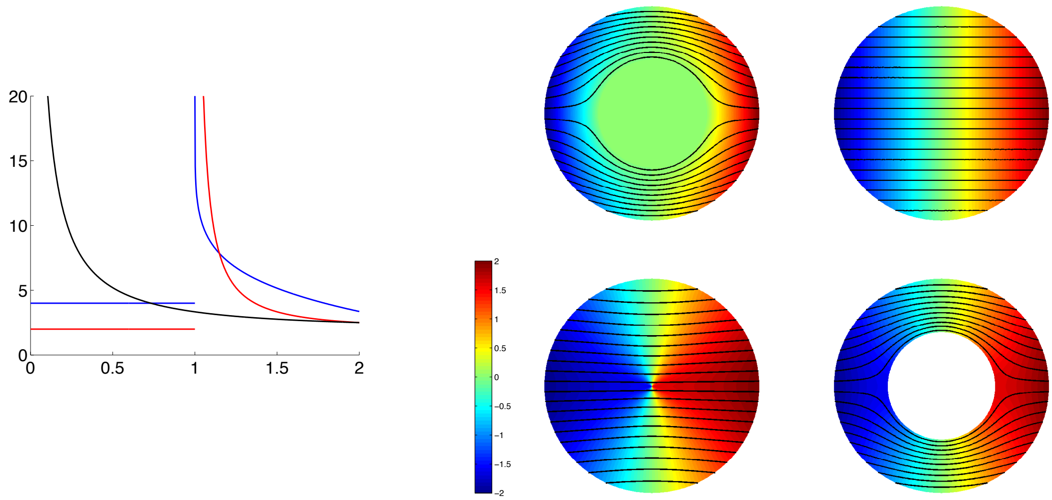

Figure 1. Left: tr(σ) of three radial and singular conductivities on the positive x axis. The curves correspond to the invisibility cloaking conductivity (red), with the singularity $\sigma^{22}(x, 0) \sim (|x| - 1)^{-1}$ for $|x| > 1$, a visible conductivity (blue) with a log log-type singularity at $|x| = 1$, and an electric hologram (black) with the conductivity having the singularity $\sigma^{11}(x, 0) \sim |x|^{-1}$. **Right, top:** All measurements on the boundary of the invisibility cloak (left) coincide with the measurements for the homogeneous disc (right). The color shows the value of the solution u with the boundary value $u(x, y)|_{\partial B(2)} = x$ and the black curves are the integral curves of the current $-\sigma \nabla u$. **Right, bottom:** All measurements on the boundary of the electric hologram (left) coincide with the measurements for an isolating disc covered with the homogeneous medium (right). The solutions and the current lines corresponding to the boundary value $u|_{\partial B(2)} = x$ are shown.

the materials need to operate only near a single frequency. The cloaking-type constructions have also inspired suggestions for possible devices producing extreme effects on wave propagation, including invisibility cloaks for magnetostatics [Gömöry et al. 2012], acoustics [Chen and Chan 2007a], quantum mechanics [Greenleaf et al. 2007a, 2008; 2011a], field rotators [Chen and Chan 2007b], electromagnetic wormholes [Greenleaf et al. 2007b], invisible sensors [Alu and Engheta 2009; Greenleaf et al. 2011b], perfect absorbers [Landy et al. 2008], and cloaked wave amplifiers [Greenleaf et al. 2012]. We also note that the differential equations with degenerate coefficients modeling cloaking devices have turned out to have interesting properties, such as nonexistence results for solutions with nonzero sources and local [Greenleaf et al. 2007c; Liu and Zhou 2011] and nonlocal [Lassas and Zhou 2011; Nguyen 2012] hidden boundary conditions. For reviews on the topic, see [Greenleaf et al. 2009a; 2009b].

Finally, the classical assumption that the electromagnetic material parameters (i.e., the coefficient functions in the elliptic equations) are uniformly bounded from below by positive constants is not valid in the modern study of materials, e.g., on the optical frequencies [Capolino 2009]. Thus one of the aims of this paper is to bring the mathematical study of elliptic equations and inverse problems closer to current topics in optics and imaging methods in physics.

The structure of the paper is the following. The main results and the formulation of the boundary measurements are presented in the first section. The proofs for the existence of the solutions of the direct problem as well as for the new counterexamples and the invisibility cloaking examples with a nonsmooth background are given in Section 2. The uniqueness result for isotropic conductivities is proven in Sections 3–4 and the reduction of the general problem to the isotropic case is shown in Section 5. In Sections 3–5, the degeneracy of the conductivity yields that the exponentially growing solutions, the standard tools used to study Calderón's inverse problem, cannot be constructed using purely microlocal or functional analytic methods. Because of this, we will extensively need the topological properties of the solutions: By Stoilow's theorem, the solutions are compositions of analytic functions and homeomorphisms. Using this, the continuity properties of the weakly monotone maps, and Orlicz estimates holding for homeomorphisms, we prove the existence of the solutions in the Sobolev–Orlicz spaces. These spaces are chosen so that we can obtain the subexponential asymptotics for the families of exponentially growing solutions needed in the $\bar\partial$ technique used to solve the inverse problem.

1A. ***Definition of measurements and solvability.*** Let $\Omega\subset\mathbb{R}^2$ be a bounded simply connected domain with a C^∞-smooth boundary. Let $\Sigma=\Sigma(\Omega)$ be the class of measurable matrix-valued functions $\sigma:\Omega\to M$, where M is the set of generalized matrices $\boldsymbol{m}$ of the form

$$\boldsymbol{m}=U\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}U^t,$$

where $U\in\mathbb{R}^{2\times 2}$ is an orthogonal matrix, $U^{-1}=U^t$ and $\lambda_1,\lambda_2\in[0,\infty)$. We denote by $W^{s,p}(\Omega)$ and $H^s(\Omega)=W^{s,2}(\Omega)$ the standard Sobolev spaces.

In the following, let $dm(z)$ denote the Lebesgue measure in $\mathbb{C}$ and $|E|$ be the Lebesgue measure of the set $E\subset\mathbb{C}$. Instead of defining the Dirichlet-to-Neumann operator for the above conductivities, we consider the corresponding quadratic forms.

Definition 1.1. Let $h\in H^{1/2}(\partial\Omega)$. The *Dirichlet-to-Neumann quadratic form* corresponding to the conductivity $\sigma\in\Sigma(\Omega)$ is given by

$$L_\sigma[h]=\inf A_\sigma[u],\quad \text{where } A_\sigma[u]=\int_\Omega \sigma(z)\nabla u(z)\cdot\nabla u(z)\,dm(z), \tag{1-3}$$

and the infimum is taken over real-valued $u\in L^1(\Omega)$ such that $\nabla u\in L^1(\Omega;\mathbb{R}^2)$ and $u|_{\partial\Omega}=h$. In the case where $L_\sigma[h]<\infty$ and $A_\sigma[u]$ reaches its minimum at some u, we say that u is a $W^{1,1}(\Omega)$ solution of the conductivity problem.

When σ is smooth and bounded from below and above by positive constants, $L_\sigma[h]$ is the quadratic form corresponding the Dirichlet-to-Neumann map (1-2),

$$L_\sigma[h]=\int_{\partial\Omega} h\Lambda_\sigma h\,dS, \tag{1-4}$$

where dS is the length measure on $\partial\Omega$. Physically, $L_\sigma[h]$ corresponds to the power needed to keep voltage h at the boundary. For smooth conductivities bounded from below, for every $h\in H^{1/2}(\partial\Omega)$, the integral $A_\sigma[u]$ always has a unique minimizer $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = h$, which is also a distributional solution to (1-1). Conversely, for functions $u \in H^1(\Omega)$, their traces lie in $H^{1/2}(\partial\Omega)$. It is for this reason that we chose to consider the $H^{1/2}$-boundary functions also in the most general case. We interpret that the Dirichlet-to-Neumann form corresponds to an idealization of the boundary measurements for $\sigma \in \Sigma(\Omega)$.

We note that the conductivities studied in the context of cloaking are not even in L^1_{loc}. As σ is unbounded, it is possible that $L_\sigma[h] = \infty$. Even if $L_\sigma[h]$ is finite, the minimization problem in (1-3) may generally have no minimizer and even if they exist, the minimizers need not be distributional solutions to (1-1). However, if the singularities of σ are not too strong, minimizers satisfying (1-1) do always exist. To show this, we need to define suitable subclasses of degenerate conductivities.

Let $\sigma \in \Sigma(\Omega)$. We start with precise quantities describing the possible degeneracy or loss of uniform ellipticity. First, a natural measure of the anisotropy of the conductivity σ at $z \in \Omega$ is

$$k_\sigma(z) = \sqrt{\frac{\lambda_1(z)}{\lambda_2(z)}},$$

where $\lambda_1(z)$ and $\lambda_2(z)$ are the eigenvalues of the matrix $\sigma(z)$ with $\lambda_1(z) \ge \lambda_2(z)$. If we want to simultaneously control both the size and the anisotropy, this is measured by the *ellipticity* $K(z) := K_\sigma(z)$ of $\sigma(z)$, i.e., the smallest number $1 \le K(z) \le \infty$ such that

$$\frac{1}{K(z)}|\xi|^2 \le \xi \cdot \sigma(z)\xi \le K(z)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2. \tag{1-5}$$

For a general, positive matrix-valued function $\sigma(z)$, we have at $z \in \Omega$ that

$$K(z) = k_\sigma(z) \max\big\{(\det\sigma(z))^{1/2}, (\det\sigma(z))^{-1/2}\big\}. \tag{1-6}$$

Consequently, we always have the following simple estimates.

Lemma 1.2. *For any measurable matrix function $\sigma(z)$, we have*

$$\tfrac{1}{4}\big(\operatorname{tr}\sigma(z) + \operatorname{tr}(\sigma(z)^{-1})\big) \le K(z) \le \operatorname{tr}\sigma(z) + \operatorname{tr}(\sigma(z)^{-1}).$$

Proof. Let $\lambda_{\max}$ and $\lambda_{\min}$ be the eigenvalues of $\sigma = \sigma(z)$. Then $K(z) = \max(\lambda_{\max}, \lambda_{\min}^{-1})$. Since $\operatorname{tr}\sigma(z) = \lambda_{\max} + \lambda_{\min}$ and $\operatorname{tr}(\sigma(z)^{-1}) = \lambda_{\max}^{-1} + \lambda_{\min}^{-1}$, the claim follows. □

Due to Lemma 1.2, we use the quantity $\operatorname{tr}\sigma(z) + \operatorname{tr}(\sigma(z)^{-1})$ as a measure of size and anisotropy of $\sigma(z)$.

For the degenerate elliptic equations, it may be that the optimization problem (1-3) has a minimizer which satisfies the conductivity equation but this solution may not have the standard $W^{1,2}_{\mathrm{loc}}$ regularity. Therefore more subtle smoothness estimates are required. We start with the exponentially integrable conductivities, and the natural energy estimates they require. As an important consequence, we will see the correct Sobolev–Orlicz regularity to work with. These observations are based on the following elementary inequality.

Lemma 1.3. *Let $K \ge 1$ and $A \in \mathbb{R}^{2\times 2}$ be a symmetric matrix satisfying*

$$\frac{1}{K}|\xi|^2 \le \xi \cdot A\xi \le K|\xi|^2, \quad \xi \in \mathbb{R}^2.$$

Then for every $p > 0$,

$$\frac{|\xi|^2}{\log(e+|\xi|^2)} + \frac{|A\xi|^2}{\log(e+|A\xi|^2)} \le \frac{2}{p}(\xi \cdot A\xi + e^{pK}).$$

Proof. Since $K \ge 1$ and $t \mapsto t/\log(e+t)$ is an increasing function, we have

$$\begin{aligned}\frac{|\xi|^2}{\log(e+|\xi|^2)} &\le \frac{K\xi \cdot A\xi}{\log(e+K\xi \cdot A\xi)} \\ &\le \frac{1}{p}\left(\frac{\xi \cdot A\xi}{\log(e+\xi \cdot A\xi)}\right) pK \\ &\le \frac{1}{p}(\xi \cdot A\xi + e^{pK}),\end{aligned}$$

where the last estimate follows from the inequality

$$ab \le a\log(e+a) + e^b, \quad a, b \ge 0.$$

Moreover, as K is at least as large as the maximal eigenvalue of A, we have $|A\xi|^2 \le K\xi \cdot A\xi$. Thus we see as above that

$$\frac{|A\xi|^2}{\log(e+|A\xi|^2)} \le \frac{K\xi \cdot A\xi}{\log(e+K\xi \cdot A\xi)} \le \frac{1}{p}(\xi \cdot A\xi + e^{pK}).$$

Adding the above estimates together proves the claim. □

Lemma 1.3 implies in particular that if $\sigma(z)$ is symmetric-matrix-valued function satisfying (1-5) for a.e. $z \in \Omega$ and $u \in W^{1,1}(\Omega)$, then we always have

$$\begin{aligned} p\int_\Omega \frac{|\nabla u(z)|^2}{\log(e+|\nabla u(z)|^2)}\,dm(z) &\le \int_\Omega \nabla u(z)\cdot\sigma(z)\nabla u(z)\,dm(z) + \int_\Omega e^{pK(z)}\,dm(z), \\ p\int_\Omega \frac{|\sigma(z)\nabla u(z)|^2}{\log(e+|\sigma(z)\nabla u(z)|^2)}\,dm(z) &\le \int_\Omega \nabla u(z)\cdot\sigma(z)\nabla u(z)\,dm(z) + \int_\Omega e^{pK(z)}\,dm(z).\end{aligned} \tag{1-7}$$

Note that these inequalities are valid whether u is a solution of the conductivity equation or not!

Due to (1-7), we see that to analyze finite energy solutions corresponding to a singular conductivity of exponentially integrable ellipticity, we are naturally led to consider the regularity gauge

$$Q(t) = \frac{t^2}{\log(e+t)}, \quad t \ge 0. \tag{1-8}$$

We say accordingly that f belongs to the Orlicz space $W^{1,Q}(\Omega)$ (see the Appendix) if f and its first distributional derivatives are in $L^1(\Omega)$ and

$$\int_\Omega \frac{|\nabla f(z)|^2}{\log(e+|\nabla f(z)|)}\,dm(z) < \infty.$$

The first existence result for solutions corresponding to degenerate conductivities is given as follows.

Theorem 1.4. *Let $\sigma(z)$ be a measurable symmetric-matrix-valued function. Suppose further that for some $p > 0$,*

$$\int_\Omega \exp\big(p\big(\operatorname{tr}\sigma(z) + \operatorname{tr}(\sigma(z)^{-1})\big)\big)\, dm(z) = C_1 < \infty. \tag{1-9}$$

Then, if $h \in H^{1/2}(\partial\Omega)$ is such that $L_\sigma[h] < \infty$ and there is a unique $w \in W^{1,1}(\Omega)$, $w|_{\partial\Omega} = h$ such that

$$A_\sigma[w] = \inf\big\{A_\sigma[v] : v \in W^{1,1}(\Omega),\, v|_{\partial\Omega} = h\big\}. \tag{1-10}$$

Moreover, w satisfies the conductivity equation

$$\nabla \cdot \sigma \nabla w = 0 \quad \text{in } \Omega \tag{1-11}$$

in the sense of distributions, and it has the regularity $w \in W^{1,Q}(\Omega) \cap C(\Omega)$.

Note that if σ is bounded near $\partial\Omega$ then $L_\sigma[h] < \infty$ for all $h \in H^{1/2}(\partial\Omega)$. Theorem 1.4 is proven in Theorem 2.1 and Corollary 2.3 in a more general setting.

Theorem 1.4 yields that for conductivities satisfying (1-9) and equal to 1 near $\partial\Omega$, we can define the Dirichlet-to-Neumann map

$$\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad \Lambda_\sigma(u|_{\partial\Omega}) = \nu \cdot \sigma\nabla u|_{\partial\Omega}, \tag{1-12}$$

where u satisfies (1-1). Many inverse scattering problems (see [Colton and Kress 2013]) can also be formulated in terms of Λ_σ.

The reader should consider the exponential condition (1-9) as being close to the optimal one, still allowing uniqueness in the inverse problem. Indeed, in view of Theorem 1.7 and Section 1E below, the most general situation where the Calderón inverse problem can be solved involves conductivities whose singularities satisfy a physically interesting small relaxation of the condition (1-9). Before solving inverse problems for conductivities satisfying (1-9), we discuss some counterexamples.

1B. *Counterexamples for the unique solvability of the inverse problem.* Let $F : \Omega_1 \to \Omega_2$, $y = F(x)$, be an orientation-preserving homeomorphism between domains $\Omega_1, \Omega_2 \subset \mathbb{C}$ for which F and its inverse F^{-1} are at least $W^{1,1}$-smooth and let $\sigma(x) = [\sigma^{jk}(x)]_{j,k=1}^2 \in \Sigma(\Omega_1)$ be a conductivity on Ω_1. Then the map F pushes σ forward to a conductivity $(F_*\sigma)(y)$, defined on Ω_2 and given by

$$(F_*\sigma)(y) = \frac{1}{[\det DF(x)]} DF(x)\sigma(x)\, DF(x)^t, \quad x = F^{-1}(y). \tag{1-13}$$

The main methods for constructing counterexamples to Calderón's problem are based on the following principle.

Proposition 1.5. *Assume that $\sigma, \tilde\sigma \in \Sigma(\Omega)$ satisfy* (1-9), *and let $F : \Omega \to \widetilde\Omega$ be a homeomorphism so that F and F^{-1} are $W^{1,Q}$-smooth and C^1-smooth near the boundary, and $F|_{\partial\Omega} = \mathrm{id}$. Suppose that $\tilde\sigma = F_*\sigma$. Then $L_\sigma = L_{\tilde\sigma}$.*

This proposition generalizes the observation of L. Tartar expanded upon in [Kohn and Vogelius 1984] to less smooth diffeomorphisms and conductivities and it follows from Lemma 2.4 proven later.

1C. ***Counterexample 1: invisibility cloaking.*** We consider here invisibility cloaking in a general background σ; that is, we aim to coat an arbitrary body with a layer of exotic material so that the coated body appears in measurements the same as the background conductivity σ. Usually one is interested in the case when the background conductivity σ is constant, isotropic, and $\sigma = 1 \cdot I$. However, we consider here a more general case and assume that σ is an L^∞-smooth conductivity in $\bar{B}(R) = \overline{B(R)}$, with $R = 2$, $\sigma(z) \geq c_0 I$, $c_0 > 0$. Here, $B(\rho)$ is an open two-dimensional disc of radius ρ and center zero and $\bar{B}(\rho)$ is its closure. Consider a homeomorphism

$$F : \bar{B}(2) \setminus \{0\} \to \bar{B}(2) \setminus \mathcal{K}, \tag{1-14}$$

where $\mathcal{K} \subset B(2)$ is a compact set which is the closure of a smooth open set and suppose F and its inverse F^{-1} are C^1-smooth in $\bar{B}(2) \setminus \{0\}$ and $\bar{B}(2) \setminus \mathcal{K}$, correspondingly. We also require that $F(z) = z$ for $z \in \partial B(2)$. The standard example of invisibility cloaking [Greenleaf et al. 2003c; Pendry et al. 2006] is the case when $\mathcal{K} = \bar{B}(1)$ and the map is given by

$$F_0(z) = \left(\frac{|z|}{2} + 1\right)\frac{z}{|z|}. \tag{1-15}$$

Using the map (1-14), we define a singular conductivity

$$\tilde{\sigma}(z) = \begin{cases} (F_*\sigma)(z) & \text{for } z \in B(2) \setminus \mathcal{K}, \\ \eta(z) & \text{for } z \in \mathcal{K}, \end{cases} \tag{1-16}$$

where $\eta(z) = [\eta^{jk}(x)]$ is any symmetric measurable matrix on $\mathcal{K}$ satisfying $c_1 I \leq \eta(z) \leq c_2 I$ with $c_1, c_2 > 0$. The conductivity $\tilde{\sigma}$ is called the cloaking conductivity obtained from the transformation map F and background conductivity σ and $\eta(z)$ is the conductivity of the cloaked (i.e., hidden) object.

In particular, choosing σ to be the constant conductivity $\sigma = 1$, $\mathcal{K} = \bar{B}(1)$, and F to be the map F_0 given in (1-15), we obtain the standard example of the invisibility cloaking. In dimensions $n \geq 3$, it was shown in [Greenleaf et al. 2003c] that the Dirichlet-to-Neumann map corresponding to $H^1(\Omega)$-solutions for the conductivity (1-16) coincide with the Dirichlet-to-Neumann map for $\sigma = 1$. In 2008, the analogous result was proven in the two-dimensional case in [Kohn et al. 2008]. For cloaking results for the Helmholtz equation with frequency $k \neq 0$ and for Maxwell's system in dimensions $n \geq 3$, see results in [Greenleaf et al. 2007c]. We note also that John Ball [1982] has used the push-forward by the analogous radial blow-up maps to study the discontinuity of solutions of partial differential equations, in particular the appearance of cavitation in nonlinear elasticity.

In the sequel, we consider cloaking results using measurements given in Definition 1.1. As we have formulated the boundary measurements in a new way, that is, in terms of the Dirichlet-to-Neumann forms L_σ associated to the class $W^{1,1}(\Omega)$, we present in Section 2D the complete proof of the following proposition, extending [Greenleaf et al. 2003c, Theorem 3]:

Theorem 1.6. (i) *Let $\sigma \in L^\infty(B(2))$ be a scalar conductivity, $\sigma(x) \geq c_0 > 0$, $\mathcal{K} \subset B(2)$ be a relatively compact open set with smooth boundary and*

$$F : \bar{B}(2) \setminus \{0\} \to \bar{B}(2) \setminus \mathcal{K}$$

be a homeomorphism. Assume that F and F^{-1} are C^1-smooth in $\overline{B}(2)\setminus\{0\}$ and $\overline{B}(2)\setminus\mathcal{K}$, correspondingly, and $F|_{\partial B(2)}=\mathrm{id}$. Moreover, assume there is $C_0>0$ such that

$$\|DF^{-1}(x)\|\le C_0 \quad \textit{for all } x\in\overline{B}(2)\setminus\mathcal{K}.$$

Let $\tilde\sigma$ be the conductivity defined in (1-16). *Then the boundary measurements for $\tilde\sigma$ and σ coincide in the sense that $L_{\tilde\sigma}=L_\sigma$.*

(ii) *Let $\tilde\sigma$ be a cloaking conductivity of the form* (1-16) *obtained from the transformation map F and the background conductivity σ, where F and σ satisfy the conditions in* (i). *Then*

$$\mathrm{tr}(\tilde\sigma)\notin L^1(B(2)\setminus\mathcal{K}). \tag{1-17}$$

The result (1-17) is optimal in the following sense. When F is the map F_0 in (1-15) and $\sigma=1$, the eigenvalues of the cloaking conductivity $\tilde\sigma$ in $B(2)\setminus\overline{B}(1)$ behave asymptotically as $|z|-1$ and $(|z|-1)^{-1}$ as $|z|\to 1$. This cloaking conductivity has so strong a degeneracy that (1-17) holds. On the other hand,

$$\mathrm{tr}(\tilde\sigma)\in L^1_{\mathrm{weak}}(B(2)), \tag{1-18}$$

where L^1_{weak} is the weak-L^1 space. We note that in the case when $\sigma=1$, $\det(\tilde\sigma)$ is identically 1 in $B(2)\setminus\overline{B}(1)$.

The formula (1-18) for the blow-up map F_0 in (1-15) and Theorem 1.6 identify the *borderline of the invisibility* for the trace of the conductivity: Any cloaking conductivity $\tilde\sigma$ satisfies $\mathrm{tr}(\tilde\sigma)\notin L^1(B(2))$ and there is an example of a cloaking conductivity for which $\mathrm{tr}(\tilde\sigma)\in L^1_{\mathrm{weak}}(B(2))$. Thus the borderline of invisibility is the same as the border between the space L^1 and the weak-L^1 space.

1D. ***Counterexample 2: illusion of a nonexistent obstacle.*** Next we consider new counterexamples for the inverse problem which could be considered as creating an illusion of a nonexisting obstacle. The example is based on a radial shrinking map, that is, a mapping $B(2)\setminus\overline{B}(1)\to B(2)\setminus\{0\}$. The suitable maps are the inverse maps of the blow-up maps $F_1:B(2)\setminus\{0\}\to B(2)\setminus\overline{B}(1)$, which are constructed by Iwaniec and Martin [2001] and have the optimal smoothness. Alternative constructions for such blow-up maps have also been proposed by Kauhanen et al. [2003]. Using the properties of these maps and defining a conductivity $\sigma_1=(F_1^{-1})_*1$ on $B(2)\setminus\{0\}$, we will later prove the following result.

Theorem 1.7. *Let γ_1 be a conductivity in $B(2)$ which is identically* 1 *in $B(2)\setminus\overline{B}(1)$ and zero in $B(1)$ and $\mathcal{A}:[1,\infty]\to[0,\infty]$ be any strictly increasing positive smooth function with $\mathcal{A}(1)=0$ which is sublinear in the sense that*

$$\int_1^\infty \frac{\mathcal{A}(t)}{t^2}\,dt<\infty. \tag{1-19}$$

Then there is a conductivity $\sigma_1\in\Sigma(B_2)$ satisfying $\det(\sigma_1)=1$ and

$$\int_{B(2)}\exp\big(\mathcal{A}\big(\mathrm{tr}(\sigma_1)+\mathrm{tr}(\sigma_1^{-1})\big)\big)\,dm(z)<\infty \tag{1-20}$$

such that $L_{\sigma_1}=L_{\gamma_1}$, i.e., the boundary measurements corresponding to σ_1 and γ_1 coincide.

We observe that, for instance, the function $\mathcal{A}_0(t) = t/(1+\log t)^{1+\varepsilon}$ satisfies (1-19) and for such a weight function, we have $\sigma_1 \in L^1(B_2)$. The proof of Theorem 1.7 is given in Section 2D.

Note that γ_1 corresponds to the case when $B(1)$ is a perfect insulator which is surrounded with constant conductivity 1. Thus Theorem 1.7 can be interpreted by saying that there is a relatively weakly degenerated conductivity satisfying integrability condition (1-20) that creates in the boundary observations an illusion of an obstacle that does not exist (see [Lai et al. 2009] for related results based on use of negative medium). Thus the conductivity can be considered as "electric hologram". As the obstacle can be considered as a "hole" in the domain, we can say also that even the topology of the domain cannot be detected. In other words, Calderón's program to image the conductivity inside a domain using the boundary measurements cannot work within the class of degenerate conductivities satisfying (1-19) and (1-20).

1E. ***Positive results for Calderón's inverse problem.*** Let us formulate our first main theorem which deals with inverse problems for anisotropic conductivities where both the trace and the determinant of the conductivity can be degenerate.

Theorem 1.8. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with smooth boundary. Let $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ be matrix-valued conductivities in Ω which satisfy the integrability condition*

$$\int_\Omega \exp\Big(p\big(\operatorname{tr}\sigma(z) + \operatorname{tr}(\sigma(z)^{-1})\big)\Big)\, dm(z) < \infty$$

for some $p > 1$. Moreover, assume that

$$\int_\Omega \mathcal{E}(q \det \sigma_j(z))\, dm(z) < \infty \quad \textit{for some } q > 0, \tag{1-21}$$

where $\mathcal{E}(t) = \exp(\exp(\exp(t^{1/2} + t^{-1/2})))$ and $L_{\sigma_1} = L_{\sigma_2}$. Then there is a $W^{1,1}_{\mathrm{loc}}$-homeomorphism $F : \Omega \to \Omega$ satisfying $F|_{\partial\Omega} = \mathrm{id}$ such that

$$\sigma_1 = F_* \sigma_2. \tag{1-22}$$

Equation (1-22) can be stated as saying that σ_1 and σ_2 are the same up to a change of coordinates; that is, the underlying manifold structures corresponding to these conductivities are the same; see [Lee and Uhlmann 1989; Lassas and Uhlmann 2001].

In the case when the conductivities are isotropic, we can improve the result of Theorem 1.8. The following theorem is our second main result for uniqueness of the inverse problem. For the earlier conjectures on the problem, see [Ingerman 2000].

Theorem 1.9. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with smooth boundary. If $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ are isotropic conductivities, i.e., $\sigma_j(z) = \gamma_j(z) I$, $\gamma_j(z) \in [0, \infty]$ satisfying*

$$\int_\Omega \exp\Big(\exp\Big(q\Big(\gamma_j(z) + \frac{1}{\gamma_j(z)}\Big)\Big)\Big)\, dm(z) < \infty \quad \textit{for some } q > 0, \tag{1-23}$$

and $L_{\sigma_1} = L_{\sigma_2}$, then $\sigma_1 = \sigma_2$.

Let us next consider anisotropic conductivities with bounded determinant but more degenerate ellipticity function $K_\sigma(z)$ defined in (1-5), and ask how far can we then generalize Theorem 1.8. Motivated by the counterexample given in Theorem 1.7, we consider the following class: we say that $\sigma \in \Sigma(\Omega)$ has an (at most) *exponentially degenerated anisotropy* with a weight $\mathcal{A}$, denoted $\sigma \in \Sigma_{\mathcal{A}} := \Sigma_{\mathcal{A}}(\Omega)$, if $\sigma(z) \in \mathbb{R}^{2\times 2}$ for a.e. $z \in \Omega$ and

$$\int_\Omega \exp\big(\mathcal{A}\big(\operatorname{tr}\sigma + \operatorname{tr}(\sigma^{-1})\big)\big)\, dm(z) < \infty. \tag{1-24}$$

In view of Theorem 1.7, for obtaining uniqueness for the inverse problem, we need to consider weights that are strictly increasing positive smooth functions $\mathcal{A} : [1, \infty] \to [0, \infty]$, $\mathcal{A}(1) = 0$, with

$$\int_1^\infty \frac{\mathcal{A}(t)}{t^2}\, dt = \infty \qquad \text{and} \qquad t\mathcal{A}'(t) \to \infty \quad \text{as } t \to \infty. \tag{1-25}$$

We say that $\mathcal{A}$ has *almost linear growth* if (1-25) holds. The point here is the first condition, that is, the divergence of the integral. The second condition is a technicality, which is satisfied by all weights one encounters in practice (which do not oscillate too much); the condition guarantees that the Sobolev-gauge function $P(t)$ defined below in (1-26) is equivalent to a convex function for large t; see [Astala et al. 2009, Lemma 20.5.4].

Note, in particular, that affine weights $\mathcal{A}(t) = pt - p$, $p > 0$, satisfy the condition (1-25). To develop uniqueness results for inverse problems within the class $\Sigma_{\mathcal{A}}$, the first problems we face are to establish the right Sobolev–Orlicz regularity for the solutions u of finite energy, $A_\sigma[u] < \infty$, and to solve the Dirichlet problem with given boundary values.

To start with this, we need the counterpart of the gauge $Q(t)$ defined in (1-8). In the case of a general weight $\mathcal{A}$, we define

$$P(t) = \begin{cases} t^2 & \text{for } 0 \leqslant t < 1, \\ \dfrac{t^2}{\mathcal{A}^{-1}(\log(t^2))} & \text{for } t \geqslant 1, \end{cases} \tag{1-26}$$

where $\mathcal{A}^{-1}$ is the inverse function of $\mathcal{A}$. As an example, note that if $\mathcal{A}$ is affine, $\mathcal{A}(t) = pt - p$ for some number $p > 0$, then the condition (1-24) takes us back to the exponentially integrable distortion of Theorem 1.8, while $P(t) = t^2\big(1 + (1/p)\log^+(t^2)\big)^{-1}$ is equivalent to the gauge function $Q(t)$ used in (1-8).

The inequalities (1-7) corresponding to the case when $\mathcal{A}$ is affine can be generalized for the following result holding for general gauge $\mathcal{A}$ satisfying (1-25).

Lemma 1.10. *Suppose $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $\mathcal{A}$ satisfies the almost linear growth condition* (1-25). *Then*

$$\int_\Omega \big(P(|\nabla u|) + P(|\sigma\nabla u|)\big)\, dm \le 2\int_\Omega e^{\mathcal{A}(\operatorname{tr}\sigma + \operatorname{tr}(\sigma^{-1}))}\, dm(z) + 2\int_\Omega \nabla u \cdot \sigma\nabla u\, dm$$

for every measurable function of symmetric matrices $\sigma(z) \in \mathbb{R}^{2\times 2}$.

Proof. We have, in fact, pointwise estimates. For these, note first that the conditions for $\mathcal{A}(t)$ imply that $P(t) \le t^2$ for every $t \ge 0$. Hence, if $|\nabla u(z)|^2 \le \exp\mathcal{A}\big(\operatorname{tr}\sigma(z) + \operatorname{tr}(\sigma^{-1}(z))\big)$ then

$$P(|\nabla u(z)|) \le \exp\mathcal{A}\big(\operatorname{tr}\sigma(z) + \operatorname{tr}(\sigma^{-1}(z))\big). \tag{1-27}$$

If, however, $|\nabla u(z)|^2 > \exp \mathcal{A}\big(\operatorname{tr}\sigma(z) + \operatorname{tr}(\sigma^{-1}(z))\big)$, then

$$P(|\nabla u(z)|) = \frac{|\nabla u(z)|^2}{\mathcal{A}^{-1}(\log|\nabla u(z)|^2)} \le \frac{|\nabla u(z)|^2}{\operatorname{tr}(\sigma^{-1}(z))} \le \nabla u(z)\cdot\sigma(z)\nabla u(z). \tag{1-28}$$

Thus at a.e. $z \in \Omega$, we have

$$P(|\nabla u(z)|) \le \exp \mathcal{A}\big(\operatorname{tr}\sigma(z) + \operatorname{tr}(\sigma^{-1}(z))\big) + \nabla u(z)\cdot\sigma(z)\nabla u(z). \tag{1-29}$$

Similar arguments give pointwise bounds for $P(|\sigma(z)\nabla u(z)|)$. Summing these estimates and integrating these pointwise estimates over Ω proves the claim. □

In the following, we say that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is in the Orlicz space $W^{1,P}(\Omega)$ if

$$\int_\Omega P(|\nabla u(z)|)\, dm(z) < \infty.$$

There are further important reasons that make the gauge $P(t)$ a natural and useful choice. For instance, in constructing a minimizer for the energy $A_\sigma[u]$, we are faced with the problem of possible equicontinuity of Sobolev functions with $A_\sigma[u]$ uniformly bounded. In view of Lemma 1.10, this is reduced to describing those weight functions $\mathcal{A}(t)$ for which the condition $P(|\nabla u(z)|) \in L^1(\Omega)$ implies that the continuity modulus of u can be estimated. As we will see later in (3-14), this follows for weakly monotone functions u (in particular, for homeomorphisms), as soon as the divergence condition

$$\int_1^\infty \frac{P(t)}{t^3}\, dt = \infty \tag{1-30}$$

is satisfied; that is, $P(t)$ has almost quadratic growth. In fact, note that the divergence of the integral $\int_1^\infty (\mathcal{A}(t)/t^2)\, dt$ is equivalent to

$$\int_1^\infty \frac{P(t)}{t^3}\, dt = \frac{1}{2}\int_1^\infty \frac{\mathcal{A}'(t)}{t}\, dt = \frac{1}{2}\int_1^\infty \frac{\mathcal{A}(t)}{t^2}\, dt = \infty, \tag{1-31}$$

where we have used the substitution $\mathcal{A}(s) = \log(t^2)$. Thus the condition (1-25) is directly connected to the smoothness properties of solutions of finite energy for conductivities satisfying (1-24).

We are now ready to formulate our third main theorem for uniqueness for the inverse problem, which gives a sharp result for singular anisotropic conductivities with a determinant bounded from above and below by positive constants.

Theorem 1.11. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with smooth boundary and $\mathcal{A} : [1, \infty) \to [0, \infty)$ be a strictly increasing smooth function satisfying the almost linear growth condition* (1-25)*. Let $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ be matrix-valued conductivities in Ω which satisfy the integrability condition*

$$\int_\Omega \exp\big(\mathcal{A}\big(\operatorname{tr}\sigma(z) + \operatorname{tr}(\sigma(z)^{-1})\big)\big)\, dm(z) < \infty. \tag{1-32}$$

Moreover, suppose that $c_1 \le \det(\sigma_j(z)) \le c_2$, with $z \in \Omega$, $j = 1, 2$, for some $c_1, c_2 > 0$, and $L_{\sigma_1} = L_{\sigma_2}$. Then there is a $W^{1,1}_{\mathrm{loc}}$-homeomorphism $F : \Omega \to \Omega$ satisfying $F|_{\partial\Omega} = \mathrm{id}$ such that

$$\sigma_1 = F_*\sigma_2.$$

We note that the determination of σ from L_σ in Theorems 1.8, 1.9, and 1.11 is constructive in the sense that one can write an algorithm which constructs σ from Λ_σ. For example, for nondegenerate scalar conductivities, such a construction has been numerically implemented in [Astala et al. 2011].

Let us next discuss the borderline of the visibility somewhat formally. Below we say that a conductivity is *visible* if there is an algorithm which reconstructs the conductivity σ from the boundary measurements L_σ, possibly up to a change of coordinates. In other words, for visible conductivities, one can use the boundary measurements to produce an image of the conductivity in the interior of Ω in some deformed coordinates. For simplicity, let us consider conductivities with $\det\sigma$ bounded from above and below. Then, Theorems 1.7 and 1.11 can be interpreted by saying that the almost linear growth condition (1-25) for the weight function $\mathcal{A}$ gives the *borderline of visibility* for the trace of the conductivity matrix: If $\mathcal{A}$ satisfies (1-25), the conductivities satisfying the integrability condition (1-32) are visible. However, if $\mathcal{A}$ does not satisfy (1-25), we can construct a conductivity in Ω satisfying the integrability condition (1-32) which appears as if an obstacle (which does not exist in reality) would have been included in the domain.

Thus the borderline of the visibility is between any spaces $\Sigma_{\mathcal{A}_1}$ and $\Sigma_{\mathcal{A}_2}$, where $\mathcal{A}_1$ satisfies condition (1-25) and $\mathcal{A}_2$ does not satisfy it. Examples of such gauge functions are $\mathcal{A}_1(t) = t(1+\log t)^{-1}$ and $\mathcal{A}_2(t) = t(1+\log t)^{-1-\varepsilon}$ with $\varepsilon > 0$.

Summarizing the results, in terms of the trace of the conductivity, we have identified the borderline of visible conductivities and the borderline of invisibility cloaking conductivities. Moreover, these borderlines are not the same and between the visible and the invisibility cloaking conductivities, there are conductivities creating electric holograms.

2. Proofs for the existence and uniqueness of the solution of the direct problem and for the counterexamples

First we show that under the conditions (1-24) and (1-25), the Dirichlet problem for the conductivity equation admits a unique solution u with finite energy $A_\sigma[u]$.

2A. ***The Dirichlet problem.*** In this section we prove Theorem 1.4. In fact, we prove it in a more general setting than it was stated.

Theorem 2.1. *Let $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$, where $\mathcal{A}$ satisfies the almost linear growth condition* (1-25). *Then, if $h \in H^{1/2}(\partial\Omega)$ is such that $L_\sigma[h] < \infty$ and $X = \{v \in W^{1,1}(\Omega) : v|_{\partial\Omega} = h\}$, there is a unique $w \in X$ satisfying* (1-10). *Moreover, w satisfies the conductivity equation*

$$\nabla \cdot \sigma \nabla w = 0 \quad \text{in } \Omega \tag{2-1}$$

in the sense of distributions, and has the regularity $w \in W^{1,P}(\Omega)$.

Proof. For $N > 0$, define $\Omega_N = \{x \in \Omega : \|\sigma(x)\| + \|\sigma(x)^{-1}\| \le N\}$. Let $w_n \in X$ be such that

$$\lim_{n\to\infty} A_\sigma[w_n] = C_0 = \inf\{A_\sigma[v] : v \in X\} = L_\sigma[h] < \infty$$

and $A_\sigma[w_n] < C_0 + 1$. Then by Lemma 1.10,

$$\int_\Omega P(|\nabla w_n(x)|)\, dm(x) + \int_\Omega P(|\sigma(x)\nabla w_n(x)|)\, dm(x) \le 2(C_1 + C_0 + 1) = C_2, \tag{2-2}$$

where

$$C_1 = \int_\Omega e^{\mathcal{A}(K(z))}\,dm(z).$$

By [Astala et al. 2009, Lemmas 20.5.3, 20.5.4], there is a convex and unbounded function $\Phi : [0,\infty) \to \mathbb{R}$ such that

$$\Phi(t) \le P(t) + c_0 \le 2\Phi(t)$$

with some $c_0 > 0$ and moreover, the function $t \mapsto \Phi(t^{5/8})$ is convex and increasing. This implies that $P(t) \ge c_1 t^{8/5} - c_2$ for some $c_1 > 0$, $c_2 \in \mathbb{R}$. Thus (2-2) yields that for all $1 < q \le 8/5$,

$$\|\nabla w_n\|_{L^q(\Omega)} \le C_3 = C_3(q, C_0, C_1) \quad \text{for } n \in \mathbb{Z}_+.$$

Using the Poincaré inequality in $L^q(\Omega)$ and that $(w_n - w_1)|_{\partial\Omega} = 0$, we see that

$$\|w_n - w_1\|_{L^q(\Omega)} \le C_4 C_3.$$

Thus, there is C_5 such that $\|w_n\|_{W^{1,q}(\Omega)} < C_5$ for all n. By restricting to a subsequence of $(w_n)_{n=1}^\infty$, which we denote in the sequel also by w_n, we see, using the Banach–Alaoglu theorem, that w_n converges as $n \to \infty$ to a limit in $W^{1,q}(\Omega)$. We denote this limit by w. As $W^{1,q}(\Omega)$ embeds compactly to $H^s(\Omega)$ for $s < 2(1 - q^{-1})$, we see that $\|w_n - w\|_{H^s(\Omega)} \to 0$ as $n \to \infty$ for all $s \in \left(\frac{1}{2}, \frac{3}{4}\right)$. Thus $w_n|_{\partial\Omega} \to w|_{\partial\Omega}$ in $H^{s-1/2}(\partial\Omega)$ as $n \to \infty$. This implies that $w|_{\partial\Omega} = h$ and $w \in X$. Moreover, for any $N > 0$,

$$\frac{1}{N}\int_{\Omega_N} |\nabla w_n(x)|^2\,dm(x) \le \int_{\Omega_N} \nabla w_n(x)\cdot\sigma(x)\nabla w_n(x)\,dm(x) \le C_0 + 1.$$

This implies that $\nabla w_n|_{\Omega_N}$ are uniformly bounded in $L^2(\Omega_N)$. Thus by restricting to a subsequence, we can assume that $\nabla w_n|_{\Omega_N}$ converges weakly in $L^2(\Omega_N)^2$ as $n \to \infty$. Clearly, the weak limit must be $\nabla w|_{\Omega_N}$. Since the norm

$$V \mapsto \left(\int_{\Omega_N} V\cdot\sigma V\,dm\right)^{1/2}$$

in $L^2(\Omega_N)^2$ is weakly lower semicontinuous, we see that

$$\int_{\Omega_N} \nabla w(x)\cdot\sigma(x)\nabla w(x)\,dm(x) \le \liminf_{n\to\infty} \int_{\Omega_N} \nabla w_n(x)\cdot\sigma(x)\nabla w_n(x)\,dm(x) \le C_0.$$

As this holds for all N, we see by applying the monotone convergence theorem as $N \to \infty$ that (1-10) holds. Thus w is a minimizer of A_σ in X.

By the above, $\sigma\nabla w_n \to \sigma\nabla w$ weakly in $L^2(\Omega_N)$ as $n \to \infty$ for all N. As noted above, there is a convex function $\Phi : [0,\infty) \to \mathbb{R}$ such that

$$\Phi(t) \le P(t) + c_0 \le 2\Phi(t), \quad c_0 > 0,$$

and $\Phi(t)$ is increasing for large values of t. Thus it follows from the semicontinuity results for integral operators, [Attouch et al. 2006, Theorem 13.1.2], Lebesgue's monotone convergence theorem, and (2-2) that

$$\begin{aligned}\int_\Omega \big(\Phi(|\nabla w|)+\Phi(|\sigma\nabla w|)\big)\,dm(x) &\le \lim_{N\to\infty}\liminf_{n\to\infty}\int_{\Omega_N}\big(\Phi(|\nabla w_n|)+\Phi(|\sigma\nabla w_n|)\big)\,dm\\ &\le \lim_{N\to\infty}\liminf_{n\to\infty}\int_{\Omega_N}\big(P(|\nabla w_n|)+P(|\sigma\nabla w_n|)\big)\,dm+2c_0|\Omega|\\ &\le C_2+2c_0|\Omega|.\end{aligned}$$

It follows from the above and the inequality $P(t)\ge c_1t^{8/5}-c_2$ that $\sigma(x)\nabla w(x)\in L^1(\Omega)$. Consider next $\phi\in C_0^\infty(\Omega)$. As $w+t\phi\in X$, $t\in\mathbb{R}$, and as w is a minimizer of A_σ in X, it follows that

$$\frac{d}{dt}A_\sigma[w+t\phi]\Big|_{t=0}=2\int_\Omega \nabla\phi(x)\cdot\sigma(x)\nabla w(x)\,dm(x)=0.$$

This shows that the conductivity equation (2-1) is valid in the sense of distributions.

Next, assume that w and $\tilde w$ are both minimizers of A_σ in X. Using the convexity of A_σ, we see that then the second derivative of $t\mapsto A_\sigma[tw+(1-t)\tilde w]$ vanishes at $t=0$. This implies that $\nabla(w-\tilde w)=0$ for a.e. $x\in\Omega$. As w and $\tilde w$ coincide at the boundary, this yields that $w=\tilde w$ and thus the minimizer is unique. □

The fact that the minimizer w is continuous will be proven in the next subsection.

2B. ***The Beltrami equation.*** It is natural to ask if the minimizer w in (1-10) is the only solution of finite σ-energy $A_\sigma[w]$ to the boundary value problem

$$\begin{aligned}\nabla\cdot\sigma\nabla w&=0 \quad \text{in } \Omega,\\ w|_{\partial\Omega}&=h.\end{aligned}\tag{2-3}$$

It turns out that this is the case and to prove this we introduce one of the basic tools in this work, the Beltrami differential equation.

To this end, recall the Hodge-star operator $*$, which in two dimensions is just the rotation

$$*=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

Note that $\nabla\cdot(*\nabla w)=0$ for all $w\in W^{1,1}(\Omega)$ and recall that $\Omega\subset\mathbb{C}$ is simply connected. If $\sigma(x)=[\sigma^{jk}(x)]_{j,k=1}^2\in\Sigma_{\mathcal{A}}(\Omega)$, where $\mathcal{A}$ satisfies (1-19), and if $u\in W^{1,1}(\Omega)$ is a distributional solution to the conductivity equation

$$\nabla\cdot\sigma(x)\nabla u(x)=0,\tag{2-4}$$

then by Lemma 1.10, we have $P(\nabla u)$, $P(\sigma\nabla u)\in L^1(\Omega)$ and thus in particular $\sigma\nabla u\in L^1(\Omega)$. By (2-4) and the Poincaré lemma, there is a function $v\in W^{1,1}(\Omega)$ such that

$$\nabla v=*\,\sigma(x)\nabla u(x).\tag{2-5}$$

Then

$$\nabla\cdot\sigma^*(x)\nabla v=0\quad \text{in } \Omega,\qquad \sigma^*(x)=*\,\sigma(x)^{-1}*^t.\tag{2-6}$$

In particular, the above shows that $u, v\in W^{1,P}(\Omega)$. Moreover, an explicit calculation (see, e.g., [Astala et al. 2009, Formula (16.20)]) reveals that the function $f=u+iv$ satisfies

$$\partial_{\bar z}f=\mu\,\partial_z f+\nu\,\overline{\partial_z f},\tag{2-7}$$

where

$$\mu=\frac{\sigma^{22}-\sigma^{11}-2i\sigma^{12}}{1+\operatorname{tr}(\sigma)+\det(\sigma)},\qquad \nu=\frac{1-\det(\sigma)}{1+\operatorname{tr}(\sigma)+\det(\sigma)},\tag{2-8}$$

and $\partial_{\bar z}=\frac{1}{2}(\partial_{x_1}+i\,\partial_{x_2})$ with $\partial_z=\frac{1}{2}(\partial_{x_1}-i\,\partial_{x_2})$. Note that $|\mu(z)|+|\nu(z)|<1$ for a.e. z. Summarizing, for $\sigma\in\Sigma_{\mathcal{A}}(\Omega)$, any distributional solution $u\in W^{1,1}(\Omega)$ of (2-4) is the real part of a solution f of (2-7). Conversely, the real part of any solution $f\in W^{1,1}(\Omega)$ of (2-7) satisfies (2-4), while the imaginary part is a solution to (2-6) and as $\sigma\in\Sigma_{\mathcal{A}}(\Omega)$, (2-4)–(2-6) and Lemma 1.10 yield that $u,v\in W^{1,P}(\Omega)$, and hence $f\in W^{1,P}(\Omega)$.

Furthermore, the ellipticity bound of $\sigma(z)$ is closely related to the distortion of the mapping f. Indeed, in the case when $\sigma(z_0)=\operatorname{diag}(\lambda_1,\lambda_2)$, a direct computation shows that

$$K_\sigma(z_0)=K_{\mu,\nu}(z_0),\qquad \text{where } K_{\mu,\nu}(z)=\frac{1+|\mu(z)|+|\nu(z)|}{1-(|\mu(z)|+|\nu(z)|)}\tag{2-9}$$

and $K_\sigma(z)$ is the ellipticity of $\sigma(z)$ defined in (1-5). Using the chain rule for the complex derivatives, which can be written as

$$\partial(v\circ F)=(\partial v)\circ F\cdot\partial F+(\bar\partial v)\circ F\cdot\overline{\bar\partial F},\tag{2-10}$$

$$\bar\partial(v\circ F)=(\partial v)\circ F\cdot\bar\partial F+(\bar\partial v)\circ F\cdot\overline{\partial F},\tag{2-11}$$

we see that $|\mu(z)|$ and $|\nu(z)|$ do not change in an orthogonal rotation of the coordinate axis, $z\mapsto\alpha z$, where $\alpha\in\mathbb{C}$, $|\alpha|=1$. Since, for any $z_0\in\Omega$ there exists an orthogonal rotation of the coordinate axis so that matrix $\sigma(z_0)$ is diagonal in the rotated coordinates, we see that the identity (2-9) holds for all $z_0\in\Omega$.

Equation (2-7) is also equivalent to the Beltrami equation

$$\bar\partial f(z)=\tilde\mu(z)\,\partial f(z)\quad\text{in }\Omega,\tag{2-12}$$

with the Beltrami coefficient

$$\tilde\mu(z)=\begin{cases}\mu(z)+\nu(z)\,\partial_z f(x)(\overline{\partial_z f(x)})^{-1} & \text{if } \partial_z f(x)\neq 0,\\ \mu(z) & \text{if } \partial_z f(x)=0\end{cases}\tag{2-13}$$

satisfying $|\tilde\mu(z)|\le|\mu(z)|+|\nu(z)|$ pointwise. We define the distortion of f at z to be

$$K(z,f):=K_{\tilde\mu}(z)=\frac{1+|\tilde\mu(z)|}{1-|\tilde\mu(z)|}\le K_\sigma(z),\qquad z\in\Omega.\tag{2-14}$$

Below we will also use the notation $K(z,f)=K_f(z)$.

In the sequel we will use frequently these different interpretations of the Beltrami equation. Note that

$$K(z,f)=\frac{1+|\tilde\mu(z)|}{1-|\tilde\mu(z)|}$$

so that

$$K(z,f)=\frac{|\partial f|+|\bar\partial f|}{|\partial f|-|\bar\partial f|}.$$

As $\|Df\|^2 = (|\partial f| + |\bar\partial f|)^2$ and $J(z, f) = |\partial f|^2 - |\bar\partial f|^2$, this yields the distortion equality (see, e.g., [Astala et al. 2009, Formula (20.3)])

$$\|Df(z)\|^2 = K(z, f)\, J(z, f) \quad \text{for a.e. } z \in \Omega. \tag{2-15}$$

We will use extensively the fact that if a homeomorphism $F : \Omega \to \Omega'$, $F \in W^{1,1}(\Omega)$, is a finite distortion mapping with the distortion $K_F \in L^1(\Omega)$ then by [Hencl et al. 2005] or [Astala et al. 2009, Theorem 21.1.4] the inverse function $H = F^{-1} : \Omega' \to \Omega$ is in $W^{1,2}(\Omega')$ and its derivative DH satisfies

$$\|DH\|_{L^2(\Omega')} \le 2\|K_F\|_{L^1(\Omega)}. \tag{2-16}$$

We will also need a few basic notions (see [Astala et al. 2009]) from the theory of Beltrami equations. As the coefficients μ, ν are defined only in the bounded domain Ω, outside Ω we set $\mu(z) = \nu(z) = 0$ and $\sigma(z) = 1$, and consider global solutions to (2-7) in $\mathbb{C}$. In particular, we consider the case when Ω is the unit disc $\mathbb{D} = B(1)$. We say that a solution $f \in W^{1,1}_{\text{loc}}(\mathbb{C})$ of (2-7) in $z \in \mathbb{C}$ is a *principal solution* if

(1) $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphism of $\mathbb{C}$ and

(2) $f(z) = z + \mathcal{O}(1/z)$ as $z \to \infty$.

The existence of principal solutions is a fundamental fact that holds true in quite wide generality. Further, with the principal solution one can classify all solutions, of sufficient regularity, to the Beltrami equation. These facts are summarized in the following version of Stoilow's factorization theorem (see [Astala et al. 2009, Theorem 20.5.2] for the proof).

Theorem 2.2. *Suppose $\mu(z)$ is supported in the unit disk $\mathbb{D}$, $|\mu(z)| < 1$ a.e. and*

$$\int_{\mathbb{D}} \exp(\mathcal{A}(K_\mu(z)))\, dm(z) < \infty, \qquad K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|},$$

where $\mathcal{A}$ satisfies the almost linear growth condition (1-25)*. Then the equation*

$$\bar\partial \Phi(z) = \mu(z)\, \partial \Phi(z), \quad z \in \mathbb{C}, \tag{2-17}$$

$$\Phi(z) = z + \mathcal{O}(1/z) \quad \text{as } z \to \infty, \tag{2-18}$$

has a unique solution in $\Phi \in W^{1,1}_{\text{loc}}(\mathbb{C})$. The solution $\Phi : \mathbb{C} \to \mathbb{C}$ is a homeomorphism and satisfies $\Phi \in W^{1,P}_{\text{loc}}(\mathbb{C})$. Moreover, when $\Omega_1 \subset \mathbb{C}$ is open, every solution of the equation

$$\bar\partial f(z) = \mu(z)\, \partial f(z), \quad z \in \Omega_1, \tag{2-19}$$

with the regularity $f \in W^{1,P}_{\text{loc}}(\Omega_1)$, can be written as $f = H \circ \Phi$, where Φ is the solution to (2-17)–(2-18) *and H is a holomorphic function in $\Omega_1' = \Phi(\Omega_1)$.*

Below we combine this result with the Poincaré lemma to analyze the solutions of the conductivity equation in the simply connected domain Ω.

Corollary 2.3. *Let $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$, where $\mathcal{A}$ satisfies* (1-25)*, and $u \in W^{1,1}_{\text{loc}}(\Omega)$ satisfy*

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega \quad \text{and} \quad \int_\Omega \nabla u(x) \cdot \sigma(x) \nabla u(x)\, dm(x) < \infty. \tag{2-20}$$

Then there exists a homeomorphism $\Phi : \mathbb{C} \to \mathbb{C}$, $\Phi \in W^{1,P}_{\mathrm{loc}}(\mathbb{C})$, *and a harmonic function* w, *defined in the domain* $\Omega' = \Phi(\Omega)$, *such that* $u = w \circ \Phi$. *In particular*, $u : \Omega \to \mathbb{R}$ *is continuous.*

Proof. Let $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be the conjugate function of u, described in (2-5), and set $f = u + iv$. Then by Lemma 1.10, we have $f \in W^{1,P}(\Omega)$, and Theorem 2.2 yields that $f = H \circ \Phi$, where $\Phi : \mathbb{C} \to \mathbb{C}$ is a homeomorphism with $\Phi \in W^{1,P}_{\mathrm{loc}}(\mathbb{C})$ and H is holomorphic in $\Phi(\Omega)$. Thus the real part $u = (\operatorname{Re} H) \circ \Phi$ has the required factorization with $w = \operatorname{Re} H$. □

Theorem 2.1 and Corollary 2.3 yield Theorem 1.4.

2C. ***Invariance of the Dirichlet-to-Neumann form under coordinate transformations.*** In this section, we assume that $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$, where $\mathcal{A}$ satisfies (1-25). We say that $F : \Omega \to \Omega'$ satisfies the condition $\mathcal{N}$ if for any measurable set $E \subset \Omega$, we have $|E| = 0 \Rightarrow |F(E)| = 0$. Also, we say that F satisfies the condition $\mathcal{N}^{-1}$ if for any measurable set $E \subset \Omega$, we have $|F(E)| = 0 \Rightarrow |E| = 0$.

Let $\sigma \in \Sigma_{\mathcal{A}}(\mathbb{C})$ be such that σ is constantly 1 in $\mathbb{C} \setminus \Omega$. Let

$$\hat{\mu}(z) = \frac{\sigma^{11}(z) - \sigma^{22}(z) + 2i\sigma^{12}(z)}{\sigma^{11}(z) + \sigma^{22}(z) + 2\sqrt{\det \sigma(z)}} \tag{2-21}$$

be the Beltrami coefficient associated to the isothermal coordinates corresponding to σ; see, e.g., [Sylvester 1990; Astala et al. 2009, Theorem 10.1.1]. A direct computation shows that $K_{\hat{\mu}}(z) = K_\sigma(z)$ and thus

$$\exp(\mathcal{A}(K_{\hat{\mu}})) \in L^1_{\mathrm{loc}}(\mathbb{C}),$$

and by Theorem 2.2, there exists a homeomorphism $F : \mathbb{C} \to \mathbb{C}$ satisfying (2-17)–(2-18) with the Beltrami coefficient $\hat{\mu}$ such that $F \in W^{1,P}_{\mathrm{loc}}(\mathbb{C})$. Due to the choice of $\hat{\mu}$, the conductivity $F_*\sigma$ is isotropic; see, e.g., [Sylvester 1990; Astala et al. 2009, Theorem 10.1.1]. Let us next consider the properties of the map F. First, as

$$\exp(\mathcal{A}(K_{\hat{\mu}})) \in L^1_{\mathrm{loc}}(\mathbb{C}),$$

it follows from [Kauhanen et al. 2003] that the function F satisfies the condition $\mathcal{N}$. Moreover, the fact that $K_F = K_{\hat{\mu}} \in L^1_{\mathrm{loc}}(\mathbb{C})$ implies by (2-16) that its inverse $H = F^{-1}$ is in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$. This yields by [Astala et al. 2009, Theorem 3.3.7] that F^{-1} satisfies the condition $\mathcal{N}$. In particular, the above yields that both F and F^{-1} are in $W^{1,P}_{\mathrm{loc}}(\mathbb{C})$.

The following lemma formulates the invariance of the Dirichlet-to-Neumann forms in the diffeomorphisms satisfying the above properties.

Lemma 2.4. *Assume that* $\Omega, \widetilde{\Omega} \subset \mathbb{C}$ *are bounded, simply connected domains with smooth boundaries and that* $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$ *and* $\tilde{\sigma} \in \Sigma_{\mathcal{A}}(\widetilde{\Omega})$, *where* $\mathcal{A}$ *satisfies* (1-25). *Let* $F : \Omega \to \widetilde{\Omega}$ *be a homeomorphism so that* F *and* F^{-1} *are* $W^{1,P}$*-smooth and* F *satisfies conditions* $\mathcal{N}$ *and* $\mathcal{N}^{-1}$. *Assume that* F *and* F^{-1} *are* C^1*-smooth near the boundary and assume that* $\rho = F|_{\partial\Omega}$ *is* C^2*-smooth. Also, suppose* $\tilde{\sigma} = F_*\sigma$. *Then*

$$L_{\tilde{\sigma}}[\tilde{h}] = L_\sigma[\tilde{h} \circ \rho]$$

for all $\tilde{h} \in H^{1/2}(\partial\widetilde{\Omega})$.

Proof. As F has the properties $\mathcal{N}$ and $\mathcal{N}^{-1}$, we have the area formula

$$\int_{\widetilde{\Omega}} H(y)\, dm(y) = \int_{\Omega} H(F(x))\, J(x, F)\, dm(x) \tag{2-22}$$

for all simple functions $H : \widetilde{\Omega} \to \mathbb{C}$, where $J(x, F)$ is the Jacobian determinant of F at x. Thus (2-22) holds for all $H \in L^1(\widetilde{\Omega})$.

Let $\tilde{h} \in H^{1/2}(\partial\widetilde{\Omega})$ and assume that $L_{\tilde{\sigma}}[\tilde{h}] < \infty$. Let $\tilde{u} : \widetilde{\Omega} \to \mathbb{R}$ be the unique minimizer of $A_{\tilde{\sigma}}[v]$ in $\widetilde{X} = \{\tilde{v} \in W^{1,1}(\widetilde{\Omega}) : \tilde{v}|_{\partial\widetilde{\Omega}} = \tilde{h}\}$. Then $\tilde{u}$ is the solution of the conductivity equation

$$\nabla \cdot \tilde{\sigma} \nabla \tilde{u} = 0, \quad \tilde{u}|_{\partial\widetilde{\Omega}} = \tilde{h}. \tag{2-23}$$

We define $h = \tilde{h} \circ F|_{\partial\Omega}$ and $u = \tilde{u} \circ F : \Omega \to \mathbb{C}$.

By Corollary 2.3, $\tilde{u}$ can be written in the form $\tilde{u} = \tilde{w} \circ \widetilde{G}$, where $\tilde{w}$ is harmonic and $\widetilde{G} \in W^{1,1}_{\text{loc}}(\mathbb{C})$ is a homeomorphism $\widetilde{G} : \mathbb{C} \to \mathbb{C}$.

By the Gehring–Lehto theorem (see [Astala et al. 2009, Corollary 3.3.3]), a homeomorphism $F \in W^{1,1}_{\text{loc}}(\Omega)$ is differentiable almost everywhere in Ω, say in the set $\Omega \setminus A$, where A has Lebesgue measure zero. Similar arguments for $\widetilde{G}$ show that $\widetilde{G}$ and the solution $\tilde{u}$ are differentiable almost everywhere, say in the set $\widetilde{\Omega} \setminus A'$, where A' has Lebesgue measure zero.

Since F has the property $\mathcal{N}^{-1}$, we see that $A'' = A' \cup F^{-1}(A') \subset \Omega$ has measure zero, and for $x \in \Omega \setminus A''$, the chain rule gives

$$Du(x) = (D\tilde{u})(F(x)) \cdot DF(x). \tag{2-24}$$

Note that the facts that F is a map with an exponentially integrable distortion and that $\tilde{u}$ is a real part of a map with an exponentially integrable distortion, do not generally imply, at least according to the knowledge of the authors, that their composition u is in $W^{1,1}_{\text{loc}}(\Omega)$. To overcome this problem, we define for $m > 1$,

$$\widetilde{\Omega}_m = \big\{ y \in \widetilde{\Omega} : \|DF^{-1}(y)\| + \|DF(F^{-1}(y))\| + \|\tilde{\sigma}(y)\| + |\nabla \tilde{u}(y)| < m \big\}$$

and $\Omega_m = F^{-1}(\widetilde{\Omega}_m)$. Then $\nabla u|_{\Omega_m} \in L^2(\Omega_m)$ and $\|\sigma\| < m^5$ in Ω_m; see (1-13).

Now for any $m > 0$,

$$\int_{\widetilde{\Omega}_m} \nabla \tilde{u}(y) \cdot \tilde{\sigma}(y) \nabla \tilde{u}(y)\, dm(y) \le A_{\tilde{\sigma}}[\tilde{u}] < \infty. \tag{2-25}$$

Due to the definition of $\tilde{\sigma} = F_* \sigma$, we see by using formulae (2-22) and (2-24) that

$$\int_{\Omega_m} \nabla u(x) \cdot \sigma(x) \nabla u(x)\, dm(x) = \int_{\widetilde{\Omega}_m} \nabla \tilde{u}(y) \cdot \tilde{\sigma}(y) \nabla \tilde{u}(y)\, dm(y). \tag{2-26}$$

Letting $m \to \infty$ and using the monotone convergence theorem, we see that

$$\int_{\Omega} \nabla u(x) \cdot \sigma(x) \nabla u(x)\, dm(x) = \int_{\widetilde{\Omega}} \nabla \tilde{u}(y) \cdot \tilde{\sigma}(y) \nabla \tilde{u}(y)\, dm(y) = A_{\tilde{\sigma}}[\tilde{u}] < \infty. \tag{2-27}$$

By Lemma 1.10, this implies that $u \in W^{1,P}(\Omega) \subset W^{1,1}(\Omega)$.

Clearly, as $\rho = F|_{\partial\Omega}$ is C^2-smooth, $h := \tilde{h} \circ F \in H^{1/2}(\partial\Omega)$ and $u|_{\partial\Omega} = h$. Thus

$$u \in X = \{ w \in W^{1,1}(\Omega) : w|_{\partial\Omega} = h \}.$$

Since $\tilde{u}$ is a minimizer of $A_{\tilde{\sigma}}$ in $\widetilde{X}$, and u satisfies

$$A_\sigma[u] \le A_{\tilde{\sigma}}[\tilde{u}] = L_{\tilde{\sigma}}(\tilde{h}),$$

we see that

$$L_\sigma[h] \le L_{\tilde{\sigma}}[\tilde{h}].$$

Changing the roles of $\tilde{\sigma}$ and σ, we obtain an opposite inequality, and prove the claim. □

In particular, if $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$, $\tilde{\sigma} \in \Sigma_{\mathcal{A}}(\widetilde{\Omega})$ and F are as in Lemma 2.4 and in addition to that, σ and $\tilde{\sigma}$ are bounded near $\partial\Omega$ and $\partial\widetilde{\Omega}$ respectively and $\rho = F|_{\partial\Omega} : \partial\Omega \to \partial\widetilde{\Omega}$ is C^2-smooth, then the quadratic forms L_σ and $L_{\tilde{\sigma}}$ can be written in terms of the Dirichlet-to-Neumann maps $\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ and $\Lambda_{\tilde{\sigma}} : H^{1/2}(\partial\widetilde{\Omega}) \to H^{-1/2}(\partial\tilde{\Omega})$ as in formula (1-4). Then, Lemma 2.4 implies that

$$\Lambda_{\tilde{\sigma}} = \rho_* \Lambda_\sigma, \tag{2-28}$$

where $\rho_*\Lambda_\sigma$ is the push-forward of Λ_σ in ρ defined by

$$(\rho_*\Lambda_\sigma)(\tilde{h}) = j \cdot \big((\Lambda_\sigma(\tilde{h} \circ \rho)) \circ \rho^{-1}\big)$$

for $\tilde{h} \in H^{1/2}(\partial\tilde{\Omega})$, where $j(z)$ is the Jacobian of the map $\rho^{-1} : \partial\widetilde{\Omega} \to \partial\Omega$.

2D. *Counterexamples revisited.* In this section we give the proofs of the claims stated in Section 1B. We start by proving Theorem 1.6. Since the change of variables used in the integration is singular, we present the arguments in detail.

Proof of Theorem 1.6. (i) Our aim is first to show that we have $L_\sigma[h] \le L_{\tilde{\sigma}}[h]$ and then to prove the opposite inequality. The proofs of these inequalities are based on different techniques due to the fact that $\tilde{\sigma}$ is not even in $L^1(B(2))$.

Let $0 < r < 2$ and

$$\mathcal{K}(r) = \mathcal{K} \cup F(\overline{B}(r)).$$

Moreover, let $\tilde{\sigma}_r$ be a conductivity that coincides with $\tilde{\sigma}$ in $B(2) \setminus \mathcal{K}(r)$ and is zero in $\mathcal{K}(r)$. Similarly, let σ_r be a conductivity that coincides with σ in $B(2) \setminus \overline{B}(r)$ and is zero in $\overline{B}(r)$. For these conductivities, we define the quadratic forms $A^r : W^{1,1}(B(2)) \to \mathbb{R}_+ \cup \{0, \infty\}$ and $\tilde{A}^r : W^{1,1}(B(2)) \to \mathbb{R}_+ \cup \{0, \infty\}$,

$$A^r[v] = \int_{B(2)\setminus\overline{B}(r)} \nabla v \cdot \sigma \nabla v \, dm(x), \quad \tilde{A}^r[v] = \int_{B(2)\setminus\mathcal{K}(r)} \nabla v \cdot \tilde{\sigma} \nabla v \, dm(x).$$

If we minimize $\tilde{A}^r[v]$ over $v \in W^{1,1}(B(2))$ with $v|_{\partial B(2)} = h$, we see that minimizers exist and that the restriction of any minimizer to $B(2) \setminus \overline{\mathcal{K}}(r)$ is the function $\tilde{u}_r \in W^{1,2}(B(2) \setminus \mathcal{K}(r))$ satisfying

$$\nabla \cdot \tilde{\sigma} \nabla \tilde{u}_r = 0 \quad \text{in } B(2) \setminus \mathcal{K}(r), \qquad \tilde{u}_r|_{\partial B(2)} = h, \qquad \nu \cdot \tilde{\sigma} \nabla \tilde{u}_r|_{\partial \mathcal{K}(r)} = 0.$$

Analogous equations hold for the minimizer u^r of A^r. As σ in $\overline{B}(2) \setminus B(r)$ and $\tilde{\sigma}$ in $\overline{B(2) \setminus \mathcal{K}(r)}$ are bounded from above and below by positive constants, we see using the change of variables and the chain rule that

$$L_{\sigma_r}[h] = L_{\tilde{\sigma}_r}[h] \quad \text{for } h \in H^{1/2}(\partial B(2)). \tag{2-29}$$

As $\sigma(x) \geq \sigma_r(x)$ and $\tilde{\sigma}(x) \geq \tilde{\sigma}_r(x)$ for all $x \in B(2)$,

$$L_\sigma[h] \geq L_{\sigma_r}[h], \quad L_{\tilde{\sigma}}[h] \geq L_{\tilde{\sigma}_r}[h]. \tag{2-30}$$

Let us consider the minimization problem (1-3) for σ. It is solved by the unique minimizer $u \in W^{1,1}(B(2))$ satisfying

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } B(2), \quad u|_{\partial B(2)} = h.$$

As $\sigma, \sigma^{-1} \in L^\infty(B(2))$, we have $u \in W^{1,2}(B(2))$ and Morrey's theorem [1938] yields that the solution u is $C^{0,\alpha}$-smooth in the open ball $B(2)$ for some $\alpha > 0$. Thus $u|_{B(R)}$ is in the *Royden algebra*

$$\mathcal{R}(B(R)) = C(B(R)) \cap L^\infty(B(R)) \cap W^{1,2}(B(R))$$

for all $R < 2$; see [Astala et al. 2009, p. 77].

By, e.g., [Iwaniec and Martin 2001, p. 443], for any $0 < R < 2$, the p-capacity of the disc $B(r)$ in $B(R)$ goes to zero as $r \to 0$ for all $p > 1$. Using this, and that $u \in W^{1,2}(B(2)) \subset L^q(B(2))$ for $q < \infty$, we see that (see [Kohn et al. 2008] for explicit estimates in the case when $\sigma = 1$)

$$\lim_{r \to 0} L_{\sigma_r}[h] = L_\sigma[h];$$

that is, the effect of an insulating disc of radius r in the boundary measurements vanishes as $r \to 0$. This and the inequalities (2-29) and (2-30) yield $L_{\tilde{\sigma}}[h] \geq L_\sigma[h]$. Next we consider the opposite inequality.

Let $\tilde{u} = u \circ F^{-1}$ in $B(2) \setminus \mathcal{K}$. As F is a homeomorphism, we see that if $x \to 0$ then $d(F(x), \mathcal{K}) \to 0$ and vice versa. Thus, as u is continuous at zero, we see that $\tilde{u} \in C(B(2) \setminus \mathcal{K}^{\text{int}})$ and $\tilde{u}$ has the constant value $u(0)$ on $\partial\mathcal{K}$. Moreover, as $F^{-1} \in C^1(B(2) \setminus \mathcal{K})$, we have $\|DF^{-1}\| \leq C_0$ in $B(2) \setminus \mathcal{K}$ and u is in the Royden algebra $\mathcal{R}(B(R))$ for all $R < 2$; we have by [Astala et al. 2009, Theorem 3.8.2] that the chain rule holds implying that $D\tilde{u} = ((Du) \circ F^{-1}) \cdot DF^{-1}$ a.e. in $B(2) \setminus \mathcal{K}$. Let $0 < R' < R'' < 2$. Then

$$|D\tilde{u}(z)| \leq C_0 \|Du\|_{C(\bar{B}(R''))} \quad \text{for } z \in F(B(R'')) \setminus \mathcal{K}.$$

As F and F^{-1} are C^1-smooth up to $\partial B(2)$, we have $\tilde{u} \in W^{1,1}(B(2) \setminus B(R'))$. These give $\tilde{u} \in W^{1,1}(B(2) \setminus \mathcal{K})$. Let $\tilde{v} \in W^{1,1}(B(2))$ be a function that coincides with $\tilde{u}$ in $B(2) \setminus \mathcal{K}$ and with $u(0)$ in $\mathcal{K}$.

Again, using the chain rule and the area formula as in the proof of Lemma 2.4, we see that $\tilde{A}^r[\tilde{v}] = A^r[u]$ for $r > 1$. Applying the monotone convergence theorem twice, we obtain

$$L_{\tilde{\sigma}}[h] \leq A_{\tilde{\sigma}}[\tilde{v}] = \lim_{r \to 0} \tilde{A}^r[\tilde{v}] = \lim_{r \to 0} A^r[u] = L_\sigma[h]. \tag{2-31}$$

As we have already proven the opposite inequality, this proves the claim (i).

(ii) Assume that $\tilde{\sigma}$ is a cloaking conductivity obtained by the transformation map F and the background conductivity $\sigma \in L^\infty(B(2))$, $\sigma \geq c_1 > 0$, but that opposite to the claim, we have $\operatorname{tr}(\tilde{\sigma}) \in L^1(B(2) \setminus \mathcal{K})$. Using formula (1-6) and the facts $\det(\tilde{\sigma}) = \det(\sigma \circ F^{-1})$ is bounded from above and below by strictly positive constants and $\operatorname{tr}(\tilde{\sigma}) \in L^1(B(2) \setminus \mathcal{K})$, we see that

$$\operatorname{tr}(\tilde{\sigma}^{-1}) = \operatorname{tr}(\tilde{\sigma}) / \det(\tilde{\sigma}) \in L^1(B(2) \setminus \mathcal{K}).$$

Hence by Lemma 1.2, $K_{\tilde\sigma} \in L^1(B(2)\setminus\mathcal{K})$. Let $G : B(2)\setminus\mathcal{K} \to B(2)\setminus\{0\}$ be the inverse map of F. Using the formulas (1-5), (1-13), and (2-15), we see that

$$\|\tilde\sigma(y)\| = \frac{\|DF(x)\cdot\sigma(x)\cdot DF(x)^t\|}{J(x,F)} \geq \frac{\|DF(x)\|^2}{J(x,F)K_\sigma(x)} = \frac{K_F(x)}{K_\sigma(x)}, \quad x = F^{-1}(y).$$

As $K_G = K_F \circ F^{-1}$ (see [Astala et al. 2009, Formula (2.15)] and $\|\tilde\sigma(y)\| \leq K_{\tilde\sigma}(y)$), the above yields $K_G \in L^1(B(2)\setminus\mathcal{K})$. Hence, we see using (2-16) that $F = G^{-1}$ is in $W^{1,2}(B(2)\setminus\{0\})$ and

$$\|DF\|_{L^2(B(2)\setminus\{0\})} \leq 2\|K_G\|_{L^1(B(2)\setminus\mathcal{K})}.$$

By the removability of singularities in Sobolev spaces (see [Kilpeläinen et al. 2000]), this implies that $F : B(2)\setminus\{0\} \to B(2)\setminus\mathcal{K}$ can be extended to a function $F^{\text{ext}} : B(2) \to \mathbb{C}$, $F^{\text{ext}} \in W^{1,2}(B(2))$. As the distortion K_F of the map F is finite a.e., the map F^{ext} is also a finite distortion map; see [Astala et al. 2009, Definition 20.0.3]. Thus, as $F^{\text{ext}} \in W^{1,2}_{\text{loc}}(B(2))$, it follows from the continuity theorem of finite distortion maps [Astala et al. 2009, Theorem 20.1.1] or [Manfredi 1994] that $F^{\text{ext}} : B(2) \to \mathbb{C}$ is continuous. Let $y_0 = F(0)$. Then the set $F^{\text{ext}}(\bar B(2)) = (\bar B(2)\setminus\mathcal{K}) \cup \{y_0\}$ is not closed as ∂K contains more that one point and thus it is not compact. This is a contradiction with the fact that F^{ext} is continuous. This proves the claim (ii). □

Next we prove the claim concerning the last counterexample.

Proof of Theorem 1.7. Let us start by reviewing the properties of the Iwaniec–Martin maps. Let $\mathcal{A}_1 : [1,\infty] \to [0,\infty]$ be a strictly increasing positive smooth function with $\mathcal{A}_1(1) = 0$ which satisfies the condition (1-19). Then by [Iwaniec and Martin 2001, Theorem 11.2.1], there exists a $W^{1,1}$-homeomorphism $F : B(2)\setminus\{0\} \to B(2)\setminus\bar B(1)$ with Beltrami coefficient μ satisfying

$$\int_{B(2)\setminus\{0\}} \exp\bigl(\mathcal{A}_1(K_\mu(z))\bigr)\,dm(z) < \infty, \quad \text{where } K_\mu(z) := \frac{1+|\mu(z)|}{1-|\mu(z)|}. \tag{2-32}$$

The function F can be obtained using the construction procedure of [Astala et al. 2009, Section 20.3] (see [Iwaniec and Martin 2001, Theorem 11.2.1] for the original construction) as follows: Let $S(t)$ be solution of the equation

$$\mathcal{A}_1(S(t)) = 1 + \log(t^{-1}), \quad 0 < t \leq 1. \tag{2-33}$$

Then $S : (0,1] \to [1,\infty)$ is a well-defined decreasing function, $S(1) = 1$ and with suitably chosen $c_1 > 0$, the function

$$F(z) = \frac{z}{|z|}\rho(|z|), \quad \rho(s) = 1 + c_1\left(\exp\left(\int_0^s \frac{dt}{tS(t)}\right) - 1\right), \tag{2-34}$$

is a homeomorphism $F : B(2)\setminus\{0\} \to B(2)\setminus\bar B(1)$. We say that F is the Iwaniec–Martin map corresponding to the weight function $\mathcal{A}_1(t)$.

Next let $\mathcal{A} : [1,\infty] \to [0,\infty]$ be a strictly increasing positive smooth function with $\mathcal{A}(1) = 0$ which satisfies the condition (1-19) and let F_1 be the Iwaniec–Martin map corresponding to the weight function $\mathcal{A}_1(t) = \mathcal{A}(4t)$.

Using the inverse of the map F_1, we define $\sigma_1 = (F_1^{-1})_* 1$ on $B(2) \setminus \{0\}$ and consider this function as an a.e. defined measurable function on $B(2)$. Using the definition of push-forward, (2-32), we see that $\det(\sigma_1) = 1$ and

$$K_{\sigma_1}(z) = K(F_1^{-1}(z), F_1^{-1}) = K_\mu(z).$$

Thus Lemma 1.2 and the fact that F_1 satisfies (2-32) with the weight function $\mathcal{A}_1(t) = \mathcal{A}(4t)$ yield that σ_1 satisfies (1-20) with the weight function $\mathcal{A}(t)$.

Recall that the conductivity γ_1 is identically 1 in $B(2) \setminus \bar{B}(1)$ and zero in $\bar{B}(1)$. Next, we consider the minimization problem (1-3) with the conductivities γ_1 and σ_1. To this end, we make analogous definitions to the proof of Theorem 1.6. For $1 < r < 2$, let γ_r be a conductivity that is 1 in $B(2) \setminus B(r)$ and is zero in $B(r)$. Similarly, let σ_r be a conductivity that coincides with σ_1 in $B(2) \setminus B(r-1)$ and is zero in $B(r-1)$.

As in (2-29) and (2-30), we see for $h \in H^{1/2}(\partial B(2))$ and $r > 1$, that

$$L_{\sigma_r}[h] = L_{\gamma_r}[h], \quad L_{\sigma_r}[h] \le L_{\sigma_1}[h], \quad L_{\gamma_r}[h] \le L_{\gamma_1}[h]. \tag{2-35}$$

Let $h \in H^{1/2}(\partial B(2))$. For $1 \le r < 2$, the solution of the boundary value problem

$$\Delta w_r = 0 \quad \text{in } B(2) \setminus \bar{B}(r), \qquad w_r|_{\partial B(2)} = h, \qquad \partial_\nu w_r|_{\partial B(r)} = 0$$

satisfies $L_{\gamma_r}[h] = \|\nabla w_r\|^2_{L^2(B(2)\setminus \bar{B}(r))}$ and it is easy to see that

$$\lim_{r\to 0} L_{\gamma_r}[h] = L_{\gamma_1}[h] \quad \text{for } h \in H^{1/2}(\partial B(2)). \tag{2-36}$$

Let $w = w_1$. Note that $w \in W^{1,2}(B(2) \setminus \bar{B}(1))$.

Let us consider the function $v = w \circ F_1$. As F_1 is C^1-smooth in $\bar{B}(2) \setminus \{0\}$ and the function w is C^1-smooth in $\bar{B}(R) \setminus \bar{B}(1)$ for all $1 < R < 2$, we have by the chain rule that

$$Dv(x) = (Dw)(F_1(x)) \cdot DF_1(x)$$

for all $x \in B(2) \setminus \{0\}$. As $Dw \in L^2(B(2) \setminus B(R))$ and $Dw \in L^\infty(B(R) \setminus \bar{B}(1))$ for all $1 < R < 2$, and

$$DF_1(x) = \frac{\rho(|x|)}{|x|}(I - P(x)) + \rho'(|x|)P(x),$$

where

$$P(x) : y \mapsto |x|^{-2}(x \cdot y)x$$

is the projector to the radial direction at the point x, using (2-34) we see that $\|DF_1(x)\| \le C|x|^{-1}$ with some $C > 0$ and

$$Dv \in L^p(B(2) \setminus \{0\}) \quad \text{for any } p \in (1, 2). \tag{2-37}$$

Thus $v \in W^{1,p}(B(2)\setminus\{0\})$ with any $p \in (1, 2)$ and by the removability of singularities in Sobolev spaces (see, e.g., [Kilpeläinen et al. 2000, Theorem 4.6 and p. 241]), the function v can be considered as a measurable function in $B(2)$ for which $v \in W^{1,p}(B(2))$. Thus v is in the domain of definition of the quadratic form A_{σ_1}.

As $w \in C^1(\bar B(R)\setminus \bar B(1))$ for all $1<R<2$ and F_1 is C^1-smooth in $\bar B(2)\setminus \bar B(1))$, we can again use the chain rule, the area formula, and the monotone convergence theorem to obtain

$$\begin{aligned} L_{\sigma_1}[h] \le A_{\sigma_1}[v] &= \lim_{R\to 2}\lim_{\rho\to 0}\int_{B(R)\setminus \bar B(\rho)} \nabla v\cdot \sigma_1 \nabla v\, dm(x) \\ &= \lim_{R\to 2}\lim_{\rho\to 0}\int_{F_1(B(R)\setminus \bar B(\rho))} \nabla w\cdot \gamma_1 \nabla w\, dm(x) = L_{\gamma_1}[h]. \end{aligned} \tag{2-38}$$

Next, consider the inequality opposite to (2-38). We have by (2-35) and (2-36) that

$$L_{\sigma_1}[h] \ge \lim_{r\to 1} L_{\sigma_r}[h] = \lim_{r\to 1} L_{\gamma_r}[h] = L_{\gamma_1}[h]. \tag{2-39}$$

The above inequalities prove the claim. □

3. Complex geometric optics solutions

In what follows, we assume that $\mathcal{A}$ satisfies the almost linear growth condition (1-25).

3A. ***Existence and properties of the complex geometric optics solutions.*** Let us start with the observation that if $\sigma_0 \in \Sigma(\Omega_0)$ is a conductivity in a smooth simply connected domain $\Omega_0 \subset \mathbb{C}$, and σ_1 is a conductivity in a larger smooth domain Ω_1 which coincides with σ_0 in Ω_0 and is 1 in $\Omega_1\setminus\Omega_0$, then L_{σ_0} determines L_{σ_1} by the formula

$$L_{\sigma_1}[h] = \inf\left\{ \int_{\Omega_1\setminus\Omega_0} |\nabla v|^2\, dm(z) + L_{\sigma_0}[v|_{\partial\Omega_0}] \;\middle|\; v\in W^{1,2}(\Omega_1\setminus\overline{\Omega_0}),\ v|_{\partial\Omega_1} = h \right\}.$$

This observation implies that we may consider inverse problems by assuming that the conductivity σ is the identity near $\partial\Omega$ without loss of generality. Also, we may assume that $\Omega=\mathbb{D}$, which we do below. We note that boundary values of the isotropic conductivity can also be directly determined from Λ_σ; see [Alessandrini 1990].

The main result of this section is the following uniqueness and existence theorem for the complex geometrical optics solutions.

Theorem 3.1. *Let $\sigma\in\Sigma_{\mathcal{A}}(\mathbb{C})$ be a conductivity such that $\sigma(x)=1$ for $x\in\mathbb{C}\setminus\Omega$. Then for every $k\in\mathbb{C}$, there is a unique solution $u(\,\cdot\,,k)\in W^{1,P}_{\mathrm{loc}}(\mathbb{C})$, where P is given in* (1-26), *for*

$$\nabla_z\cdot\sigma(z)\nabla_z u(z,k) = 0 \qquad \text{in } \mathbb{C}, \tag{3-1}$$

$$u(z,k) = e^{ikz}\Big(1+\mathcal{O}\Big(\frac{1}{z}\Big)\Big) \quad \text{as } |z|\to\infty. \tag{3-2}$$

We point out that the regularity $u\in W^{1,P}_{\mathrm{loc}}(\mathbb{C})$ is optimal in the sense that the standard slightly stronger assumption $u\in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ would not be valid for the solutions; see [Astala et al. 2009, Section 20.4.6].

We prove Theorem 3.1 in several steps. Recalling the reduction to the Beltrami equation (2-7), we start with the following lemma, where we define

$$B_{\mathcal{A}}(\mathbb{D}) = \left\{ \mu\in L^\infty(\mathbb{C}) \;\middle|\; \mathrm{supp}(\mu)\subset\bar{\mathbb{D}},\ 0\le\mu(x)<1 \text{ a.e., and } \int_{\mathbb{D}} \exp(\mathcal{A}(K_\mu(z)))\, dm(z) < \infty \right\}.$$

Lemma 3.2. *Assume that $\mu \in B_{\mathcal{A}}(\mathbb{D})$ and $f \in W^{1,P}_{\rm loc}(\mathbb{C})$ satisfies*

$$\bar\partial f(z) = \mu(z)\,\partial f(z) \qquad \text{for a.e. } z \in \mathbb{C}, \tag{3-3}$$

$$f(z) = \beta e^{ikz}\Big(1+\mathcal{O}\Big(\frac{1}{z}\Big)\Big) \quad \text{for } |z| \to \infty, \tag{3-4}$$

where $\beta \in \mathbb{C}\setminus\{0\}$ and $k \in \mathbb{C}$. Then

$$f(z) = \beta e^{ik\Phi(z)}, \tag{3-5}$$

where $\Phi \in W^{1,P}_{\rm loc}(\mathbb{C})$ is a homeomorphism $\Phi : \mathbb{C} \to \mathbb{C}$, $\bar\partial\Phi(z) = 0$ for $|z| > 1$, $K(z,\Phi) = K(z, f)$ for a.e. $z \in \mathbb{C}$, and

$$\Phi(z) = z + \mathcal{O}\Big(\frac{1}{z}\Big) \quad \text{for } |z| \to \infty. \tag{3-6}$$

Proof. By Theorem 2.2, we have for f the Stoilow factorization $f = h \circ \Phi$, where $h : \mathbb{C} \to \mathbb{C}$ is a holomorphic function and Φ is the principal solution of (3-3). This and the formulae (3-4) and (3-6) imply

$$\frac{h(\Phi(z))}{\beta e^{ik\Phi(z)}} = \frac{f(z)}{\beta e^{ik\Phi(z)}} \to 1 \quad \text{when } |z| \to \infty.$$

Thus, $h(\zeta) = \beta e^{ik\zeta}$ for all $\zeta \in \mathbb{C}$, and f has the representation (3-5). The claimed properties of Φ follow from the formula (3-5) and the similar properties of f. □

Next we consider case where $\beta = 1$. Below we will use the fact that if $\Phi : \mathbb{C} \to \mathbb{C}$ is a homeomorphism such that $\Phi \in W^{1,1}_{\rm loc}(\mathbb{C})$, we have $\Phi(z) - z = o(1)$ as $z \to \infty$ and that if Φ is analytic outside the disc $\bar B(r)$, $r > 0$, then by [Astala et al. 2009, Theorem 2.10.1 and (2.61)],

$$|\Phi(z)| \le |z| + 3r \quad \text{for } z \in \mathbb{C} \qquad \text{and} \qquad |\Phi(z) - z| \le r \quad \text{for } |z| > 2r. \tag{3-7}$$

In particular, the map Φ defined in Lemma 3.2 satisfies this with $r = 1$.

Lemma 3.3. *Assume that $\nu, \mu : \mathbb{C} \to \mathbb{C}$ are measurable functions satisfying*

$$\mu(z) = \nu(z) = 0 \quad \text{for } z \in \mathbb{C}\setminus\mathbb{D}, \tag{3-8}$$

$$|\mu(z)| + |\nu(z)| < 1 \quad \text{for a.e. } z \in \mathbb{D}, \tag{3-9}$$

and that $K_{\mu,\nu}(z)$ defined in (2-9) satisfies

$$\int_{\mathbb{D}} \exp(\mathcal{A}(K_{\mu,\nu}(z)))\,dm(z) < \infty. \tag{3-10}$$

Then for $k \in \mathbb{C}$, the equation

$$\partial_{\bar z} f = \mu\,\partial_z f + \nu\,\overline{\partial_z f}, \quad z \in \mathbb{C}, \tag{3-11}$$

has at most one solution $f \in W^{1,P}_{\rm loc}(\mathbb{C})$ satisfying

$$f(z) = e^{ikz}\Big(1+\mathcal{O}\Big(\frac{1}{z}\Big)\Big) \quad \text{for } |z| \to \infty. \tag{3-12}$$

Proof. Observe that we can write (3-11) in the form

$$\partial_{\bar z} f = \tilde\mu\, \partial_z f, \quad z \in \mathbb{C}, \tag{3-13}$$

where the coefficient $\tilde\mu$ is given by (2-13). Since $|\tilde\mu(z)| \le |\mu(z)| + |\nu(z)|$, we see that $\tilde\mu \in B_{\mathcal{A}}(\mathbb{D})$.

Next, assume (3-13) has two solutions f_1 and f_2 having the asymptotics (3-12). Let $\varepsilon > 0$ and consider the function

$$f_\varepsilon(z) = f_1(z) - (1+\varepsilon) f_2(z).$$

Then, $f_\varepsilon \in W^{1,P}_{\mathrm{loc}}(\mathbb{C})$, the function f_ε satisfies (3-11), and

$$f_\varepsilon(z) = -\varepsilon e^{ikz}\Big(1 + \mathcal{O}\Big(\frac{1}{z}\Big)\Big) \quad \text{for } |z| \to \infty.$$

By Lemma 3.2 and (3-7), there is $\Phi_\varepsilon(z)$ such that

$$f_\varepsilon(z) = f_1(z) - (1+\varepsilon) f_2(z) = -\varepsilon e^{ik\Phi_\varepsilon(z)}$$

and $|\Phi_\varepsilon(z)| \le |z| + 3$. Then for any $z \in \mathbb{C}$, we have that

$$f_1(z) - f_2(z) = \lim_{\varepsilon \to 0} f_\varepsilon(z) = 0.$$

Thus $f_1 = f_2$. ☐

3B. *Proof of Theorem 3.1.* In the following, we use general facts for *weakly monotone* mappings, and to this end, we recall some basic facts. Let $\Omega \subset \mathbb{C}$ be open and $u \in W^{1,1}(\Omega)$ be real-valued. We say that u is weakly monotone if both of the functions $u(x)$ and $-u(x)$ satisfy the maximum principle in the following weak sense: for any $a \in \mathbb{R}$ and relatively compact open sets $\Omega' \subset \Omega$,

$$\max(u(z) - a, 0) \in W^{1,1}_0(\Omega') \text{ implies that } u(z) \le a \text{ for a.e. } z \in \Omega';$$

see [Iwaniec and Martin 2001, Section 7.3]. We remark that if $f \in W^{1,1}_{\mathrm{loc}}(\Omega_1)$ and $f : \Omega_1 \to \Omega_2$ is a homeomorphism, where $\Omega_1, \Omega_2 \subset \mathbb{C}$ are open, the real part of f is weakly monotone. By [Astala et al. 2009, Lemma 20.5.8], if $f \in W^{1,1}(\Omega)$ is the solution of the Beltrami equation $\bar\partial f = \mu\, \partial f$ with a Beltrami coefficient μ satisfying $|\mu(z)| < 1$ for a.e. $z \in \mathbb{C}$, then the real and the imaginary parts of f are weakly monotone functions. An important property of weakly monotone functions is that their modulus of continuity can be estimated in an explicit way. Let $M(t) = M_P(t)$ be the P-modulus, that is, the function determined by the condition: for $M = M(t)$, we have

$$\int_1^{1/t} P(sM) \frac{ds}{s^3} = P(1) \quad \text{for all } t \in [0, \infty);$$

see (1-30) and [Iwaniec and Martin 2001, Section 7.5]. The function $M_P : [0,\infty) \to [0,\infty)$ is continuous at zero and $M_P(0) = 0$. Then by [Iwaniec and Martin 2001, Theorem 7.5.1], it holds that if $z', z \in \Omega$ satisfy $B(z, r) \subset \Omega$, $r < 1$, and $|z' - z| < r/2$, and $f \in W^{1,P}(\Omega)$ is a weakly monotone function, then for almost every $z, z' \in B(z,r)$, we have

$$|f(z') - f(z)| \leq 32\pi r \, \|Df\|_{(P,r)} M_P\left(\frac{|z-z'|}{2r}\right), \tag{3-14}$$

where

$$\|\nabla f\|_{(P,r)} = \inf\left\{\frac{1}{\lambda} \,\middle|\, \lambda > 0,\ \frac{1}{\pi r^2}\int_{B(z,r)} P(\lambda |Df(x)|)\, dm(z) \leq P(1)\right\}.$$

As we will see, this can be used to estimate the modulus of continuity of principal solutions of Beltrami equations corresponding to $\mu \in B_{\mathcal{A}}(\mathbb{D})$.

Below, we use the unimodular function e_k given by

$$e_k(z) = e^{i(kz+\bar{k}\bar{z})}. \tag{3-15}$$

The following result shows the existence of the complex geometric solutions for degenerated conductivities.

Lemma 3.4. *Assume that μ and ν satisfy* (3-8)–(3-10) *and let $k \in \mathbb{C} \setminus \{0\}$. Then* (3-11) *has a solution $f \in W^{1,P}_{\mathrm{loc}}(\mathbb{C})$ satisfying the asymptotics* (3-12)*. Moreover, this solution can be written in the form*

$$f(z) = e^{ik\varphi(z)}, \tag{3-16}$$

where $\varphi : \mathbb{C} \to \mathbb{C}$ is a homeomorphism satisfying the asymptotics $\varphi(z) = z + \mathcal{O}(z^{-1})$. Moreover, for $R > 1$,

$$\int_{B(R)} P(|D\varphi(x)|)\, dm(x) \leq C_{\mathcal{A}}(R) \int_{B(R)} \exp(\mathcal{A}(K_{\mu,\nu}(z)))\, dm(z), \tag{3-17}$$

where $C_{\mathcal{A}}(R)$ depends on R and the weight function $\mathcal{A}$. In addition,

$$\bar{\partial}\varphi(z) = \mu(z)\, \partial\varphi(z) - \frac{\bar{k}}{k}\nu(z) e_{-k}(\varphi(z))\, \overline{\partial\varphi(z)} \quad \textit{for a.e. } z \in \mathbb{C}. \tag{3-18}$$

Proof. Let us approximate the functions μ and ν with functions

$$\mu_n(z) = \begin{cases} \mu(z) & \text{if } |\mu(z)| + |\nu(z)| \leq 1 - \frac{1}{n}, \\ \frac{\mu(z)}{|\mu(z)|}(1 - \frac{1}{n}) & \text{if } |\mu(z)| + |\nu(z)| > 1 - \frac{1}{n}, \end{cases} \tag{3-19}$$

$$\nu_n(z) = \begin{cases} \nu(z) & \text{if } |\mu(z)| + |\nu(z)| \leq 1 - \frac{1}{n}, \\ \frac{\nu(z)}{|\nu(z)|}(1 - \frac{1}{n}) & \text{if } |\mu(z)| + |\nu(z)| > 1 - \frac{1}{n}, \end{cases} \tag{3-20}$$

where $n \in \mathbb{Z}_+$. Consider the equations

$$\bar{\partial} f_n(z) = \mu_n(z) \partial f_n(z) + \nu_n(z) \overline{\partial f_n(z)} \quad \text{for a.e. } z \in \mathbb{C}, \tag{3-21}$$

$$f_n(z) = e^{ikz}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \quad \text{for } |z| \to \infty. \tag{3-22}$$

By Lemma 3.3, equations (3-21)–(3-22) have at most one solution $f_n \in W^{1,P}_{\mathrm{loc}}(\mathbb{C})$. The existence of the solutions can be seen as in the proof of [Astala et al. 2005, Lemma 3.5]; by [Astala et al. 2005, Lemma 3.2], solutions f_n for (3-21)–(3-22) can be constructed via the formula $f_n = h \circ g$, where g is the principal solution of $\bar\partial g = \hat\mu\,\partial g$, constructed in Theorem 2.2, and h is the solution of

$$\bar\partial h = (\hat\nu \circ g^{-1})\,\overline{\partial h}, \quad h(z) = e^{ikz}\Big(1+\mathcal{O}\Big(\frac{1}{z}\Big)\Big)$$

constructed in [Astala and Päivärinta 2006, Theorem 4.2], where $\hat\nu = (1+\nu_n)\bar\mu_n$ and $\hat\mu = \mu_n(1+\hat\nu)$, and moreover, it holds that $f_n \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$.

Let us now define the coefficient $\tilde\mu$ according to formula (2-13), and define an approximative coefficient $\tilde\mu_n$ using formula (2-13), where μ and ν are replaced by μ_n and ν_n and f by f_n. We can write (3-21) in the form

$$\bar\partial f_n(z) = \tilde\mu_n(z)\,\partial f_n(z) \quad \text{for a.e. } z \in \mathbb{C}, \tag{3-23}$$

where $|\tilde\mu_n| \le 1 - n^{-1}$.

By (3-22), (3-23), and Lemma 3.2, the function f_n can be written in the form

$$f_n(z) = e^{ik\varphi_n(z)}, \tag{3-24}$$

where φ_n is a homeomorphism, $\bar\partial\varphi_n(z) = 0$ for $|z| > 1$, $K(z,\varphi_n) = K(z, f_n)$ for a.e. $z \in \mathbb{C}$, and

$$\varphi_n(z) = z + \mathcal{O}\Big(\frac{1}{z}\Big) \quad \text{for } |z| \to \infty. \tag{3-25}$$

Then

$$|\bar\partial f_n(z)| = |\tilde\mu_n(z)|\,|\partial f_n(z)| \le |\tilde\mu(z)|\,|\partial f_n(z)|.$$

Let us consider next $a, b > 0$ and $0 \le t \le (ab)^{1/2}$. Using the definition (1-26) of $P(t)$, we see that

$$\begin{aligned} P(t) &\le \exp(\mathcal{A}(a)) && \text{for } t^2 \le e^{\mathcal{A}(a)}, \\ P(t) &\le \frac{ab}{\mathcal{A}^{-1}(\log\exp(\mathcal{A}(a)))} = b && \text{for } t^2 > e^{\mathcal{A}(a)}, \end{aligned}$$

which imply the inequality $P(t) \le b + \exp(\mathcal{A}(a))$. Due to the distortion equality (2-15), we can use this for $a = K(z,\varphi_n)$, $b = J(z,\varphi_n)$, and $t = |D\varphi_n(z)|$ and obtain

$$P(|D\varphi_n(z)|) \le J(z,\varphi_n) + \exp(\mathcal{A}(K(z,\varphi_n))). \tag{3-26}$$

Then, we see using (3-7) and the fact that φ_n is a homeomorphism that

$$\begin{aligned} \int_{B(R)} P(|D\varphi_n(z)|)\,dm(z) &\le \int_{B(R)} J(z,\varphi_n)\,dm(z) + \int_{B(R)} e^{\mathcal{A}(K(z,\varphi_n))}\,dm(z) \\ &\le m(\varphi_n(B(R))) + \int_{B(R)} \exp(\mathcal{A}(K_{\tilde\mu}(z)))\,dm(z) \\ &\le \pi(R+3)^2 + \int_{B(R)} \exp(\mathcal{A}(K_{\tilde\mu}(z)))\,dm(z) \end{aligned} \tag{3-27}$$

is finite by the assumption (3-10). We emphasize that the fact that φ_n is a homeomorphism is the essential fact which together with the inequality (3-26) yields the Orlicz estimate (3-27).

The estimate (3-27) together with the inequality (3-14) implies that the functions φ_n have uniformly bounded modulus of continuity in all compact sets of $\mathbb{C}$. Moreover, by (3-7), $|\varphi_n(z)| \le |z| + 3$.

Next we consider the Beltrami equation for φ. To this end, let $\psi \in C_0^\infty(\mathbb{C})$ and $R > 1$ be so large that $\operatorname{supp}(\psi) \subset B(R)$. Since the family $\{\varphi_n\}_{n=1}^\infty$ is uniformly bounded in the space $W^{1,P}(B(R))$ and $W^{1,P}(B(R)) \subset W^{1,q}(B(R))$ for some $q > 1$, we see that there is a subsequence φ_{n_j} that converges weakly in $W^{1,q}(B(R))$ to some limit φ when $j \to \infty$. Let us denote

$$\kappa_n(z) = -\frac{\bar{k}}{k}\nu_n(z)e_{-k}(\varphi_n(z)), \quad \kappa(z) = -\frac{\bar{k}}{k}\nu(z)e_{-k}(\varphi(z)).$$

Moreover, functions φ_n are uniformly bounded and have a uniformly bounded modulus of continuity in compact sets by (3-14) and thus by the Arzelà–Ascoli theorem, there is a subsequence, denoted also by φ_{n_j}, that converges uniformly to some function φ' in $B(R)$ for all $R > 1$. As φ_{n_j} converges in $C(\bar{B}(R))$ uniformly to φ' and weakly in $W^{1,q}(B(R))$ to φ, we see using convergence in distributions that $\varphi' = \varphi$. Thus, we see that

$$\lim_{j\to\infty} e_{-k}(\varphi_{n_j}(z)) = e_{-k}(\varphi(z)) \quad \text{uniformly for } z \in B(R),$$

and by the dominated convergence theorem $\kappa_n \to \kappa$ in $L^p(B(R))$, where $1/p + 1/q = 1$.

As $\varphi_n : \mathbb{C} \to \mathbb{C}$ is a homeomorphism and $\varphi_n \in W^{1,1}_{\text{loc}}(\mathbb{C})$, we can use chain rules (2-10) a.e. by the Gehring–Lehto theorem (see [Astala et al. 2009, Corollary 3.3.3]) and see using (3-21) and (3-24) that

$$\bar\partial\varphi_n(z) = \mu_n(z)\,\partial\varphi_n(z) - \frac{\bar{k}}{k}\nu_n(z)\,e_{-k}(\varphi_n(z))\,\overline{\partial\varphi_n(z)} \quad \text{for a.e. } z \in \mathbb{C}. \tag{3-28}$$

Recall that there is a convex function $\Phi : [0,\infty) \to [0,\infty)$ such that $\Phi(t) \le P(t) + c_0 \le 2\Phi(t)$. By [Attouch et al. 2006, Theorem 13.1.2], the map

$$\phi \mapsto \int_{B(R)} \Phi(|D\phi(x)|)\,dm(x)$$

is weakly lower semicontinuous in $W^{1,1}(B(R))$. By (3-27), the integral of $\Phi(|D\varphi_n|)$ is uniformly bounded in $n \in \mathbb{Z}_+$ over any disc $B(R)$. In particular, this yields that $\varphi \in W^{1,P}(B(R))$ for $R > 1$ and that (3-17) holds.

Furthermore, as $|\varphi(z)| \le |z| + 3$, this yields that

$$f(z) := e^{ik\varphi(z)} \in W^{1,P}_{\text{loc}}(\mathbb{C}). \tag{3-29}$$

Next define $\varphi_n(\infty) = \varphi(\infty) = \infty$. As φ_n and φ are conformal at infinity, we see using the Cauchy formula for $(\varphi_n(1/z) - \varphi(0))^{-1}$ that

$$\varphi(z) = z + \mathcal{O}\Big(\frac{1}{z}\Big) \quad \text{for } |z| \to \infty. \tag{3-30}$$

As $D\varphi_{n_j}$ converges weakly in $L^q(B(R))$ to $D\varphi$ and their norms are uniformly bounded, we have

$$\begin{aligned}\Big|\int_{\mathbb{C}}(\bar\partial\varphi - \mu\,\partial\varphi - \kappa\,\overline{\partial\varphi})\psi\,dm(z)\Big| &= \lim_{j\to\infty}\Big|\int_{\mathbb{C}}(\bar\partial\varphi_{n_j} - \mu\,\partial\varphi_{n_j} - \kappa\,\overline{\partial\varphi_{n_j}})\psi\,dm(z)\Big|\\ &\le \lim_{j\to\infty}\Big|\int_{\mathbb{C}} i\big((\mu_{n_j}-\mu)\,\partial\varphi_{n_j} + (\kappa_{n_j}-\kappa)\,\overline{\partial\varphi_{n_j}}\big)\psi\,dm(z)\Big|\\ &\le \lim_{j\to\infty}\big(\|\mu_{n_j}-\mu\|_{L^p(B(1))} + \|\kappa_{n_j}-\kappa\|_{L^p(B(1))}\big)\|\partial\varphi_{n_j}\|_{L^q(B(1))}\|\psi\|_{L^\infty(B(1))} = 0.\end{aligned}$$

This implies that $\varphi(z)$ satisfies (3-18).

Next we show that φ is a homeomorphism As $K(z) = K_{\nu,\mu} \in L^1_{\mathrm{loc}}(\mathbb{C})$, we have $K(z; \varphi_n) \in L^1_{\mathrm{loc}}(\mathbb{C})$; thus by (2-16), the inverse maps φ_n^{-1} satisfy $\varphi_n^{-1} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C}; \mathbb{C})$ and for all $R > 1$, the norms $\|\varphi_n^{-1}\|_{W^{1,2}(B(R))}$, $n \in \mathbb{Z}_+$, are uniformly bounded. Thus by the formula (3-14), the family $(\varphi_n^{-1})_{n=1}^{\infty}$ has a uniform modulus of continuity in compact sets. Hence, we see that there is a continuous function $\psi : \mathbb{C} \to \mathbb{C}$ such that $\varphi_n^{-1} \to \psi$ uniformly on compact sets when $n \to \infty$. As φ_n are conformal at infinity, we see, using again the Cauchy formula, that $\varphi_{n_j}^{-1} \to \psi$ uniformly on the Riemann sphere $\mathbb{S}^2$ as $j \to \infty$. Then,

$$\psi \circ \varphi(z) = \lim_{j\to\infty} \varphi_{n_j}^{-1}(\varphi(z)) = \lim_{j\to\infty} \varphi_{n_j}^{-1}(\varphi_{n_j}(z)) = z,$$

which implies that $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ is a continuous injective map and hence a homeomorphism.

As $\varphi : \mathbb{C} \to \mathbb{C}$ is a homeomorphism and $\varphi \in W^{1,1}_{\mathrm{loc}}(\mathbb{C})$, we can, by the Gehring–Lehto theorem, use chain rules (2-10) a.e. and see using (3-18) that $f(z) = e^{ik\varphi(z)}$ satisfies (3-11). By (3-30), $f(z)$ satisfies the asymptotics (3-12). This proves the claim. □

The above uniqueness and existence results have now proven Theorem 3.1.

4. Inverse conductivity problem with degenerate isotropic conductivity

In this section, we consider exponentially integrable scalar conductivities σ. In particular, we assume that σ is 1 in an open set containing $\mathbb{C} \setminus \mathbb{D}$ and its ellipticity function $K(z) = K_\sigma(z)$ of the conductivity σ satisfies an Orlicz space estimate

$$\int_{B(R_1)} \exp\bigl(\exp(qK(x))\bigr)\, dm(x) \le C_0 \quad \text{for some } C_0, q > 0, \tag{4-1}$$

with $R_1 = 1$. Note that by the John–Nirenberg lemma, (4-1) is satisfied if

$$\exp(qK(x)) \in \mathrm{BMO}(\mathbb{D}) \quad \text{for some } q > 0. \tag{4-2}$$

As noted before, we may assume without loss of generality that Ω is the unit disc $\mathbb{D}$.

4A. ***Estimates for principal solutions in Orlicz spaces.*** Let us consider next the principal solution of the Beltrami equation

$$\bar\partial \Phi(z) = \mu(z)\, \partial \Phi(z), \quad z \in \mathbb{C}, \tag{4-3}$$

$$\Phi(z) = z + O\Bigl(\frac{1}{z}\Bigr) \qquad \text{when } |z| \to \infty. \tag{4-4}$$

To this end, let $R_0 \ge 1$,

$$B^p_{\exp,N}(B(R_0)) = \Bigl\{\mu : \mathbb{C} \to \mathbb{C} \,\Big|\, |\mu(z)| < 1 \text{ for a.e. } z,\ \mathrm{supp}(\mu) \subset B(R_0) \text{ and } \int_{B(R_0)} \exp(pK_\mu(z))\, dm(z) \le N\Bigr\}$$

and

$$B^p_{\exp}(B(R_0)) = \bigcup_{N>0} B^p_{\exp,N}(B(R_0)).$$

The reason that we use the radius R_0 is to be able to apply the obtained results for the inverse function of the solution of the Beltrami equation satisfying another Beltrami equation with modified coefficients; see (4-45).

Assume that $p > 2$ and $\mu \in B^p_{\exp}(B(R_0))$. Then by [Astala et al. 2010, Theorem 1.1], we have the L^2-estimate

$$\|(\mu S)^m \mu\|_{L^2(\mathbb{C})} \le C(p,\beta) m^{-\beta/2} \int_{B(R_0)} \exp(pK_\mu(z))\, dm(z), \quad 2 < \beta < p, \tag{4-5}$$

where

$$S\phi(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial_w \phi(w)}{w - z}\, dm(w),$$

is the Beurling operator. Below, we use the operator $S_{B(R_0)}\phi = S\phi|_{B(R_0)}$. In particular, as Φ satisfies

$$\bar\partial \Phi = \bar\partial(\Phi - z) = \mu\, \partial(\Phi - z) + \mu = \mu S\, \bar\partial(\,\Phi - z) + \mu = \mu S\, \bar\partial\Phi + \mu,$$

(4-5) yields

$$\bar\partial \Phi = \sum_{m=0}^{\infty} (\mu S)^m \mu, \tag{4-6}$$

where the series converges in $L^2(\mathbb{C})$. To analyze the convergence more precisely, we need a refinement of the L^p-scale. In particular, we will use the Orlicz spaces $X^{j,q}(S)$, $j \in \mathbb{Z}_+$, $q \in \mathbb{R}$, $S \subset \mathbb{C}$ that are defined by

$$u \in X^{j,q}(S) \quad \text{if and only if} \quad \int_S M_{j,q}(u(x))\, dm(x) < \infty, \tag{4-7}$$

where

$$M_{j,q}(t) = |t|^j \log^q(e + |t|). \tag{4-8}$$

We use shorthand notations $X^q(S) = X^{2,q}(S)$ and $M_q(t) = M_{2,q}(t)$. Although (4-7)–(4-8) do not define a norm in $X^{j,q}(S)$, there is an equivalent norm

$$\|u\|_{X^{j,q}(S)} = \sup_v \left\{ \int_S |u(x)v(x)|\, dm(x) \;\middle|\; \int_D G_{j,q}(|v(x)|)\, dm(x) \le 1 \right\}, \tag{4-9}$$

where $G_{j,q}(t)$ is such a function that $(M_{j,q}, G_{j,q})$ are a Young complementary pair (see the Appendix) and, in particular, the following lemma holds.

Lemma 4.1. *Let $j = 1, 2, \dots,$ and $q \ge 0$.*

(i) *We have*

$$\int_{B(R_0)} M_{j,q}(u(x))\, dm(x) \le 2\|u\|^j_{X_{j,q}(B(R_0))} \log^q(e + \|u\|_{X^{j,q}(B(R_0))}).$$

(ii) *For*

$$\phi(t) = t^{1/j}(1 + 2\log^q(e + t^{-1/j})),$$

we have

$$\|u\|_{X^{j,q}(B(R_0))} \le \phi\left(\int_{B(R_0)} M_{j,q}(u(x))\, dm(x) \right).$$

Proof. (i) Let us denote $M(t) = M_{j,q}(t)$. For this function, we use the equivalent norms $\|u\|_M$ and $\|u\|_{(M)}$ defined in the Appendix. To show the claim, we use the inequality

$$\log(e + st) \le 2\log(e + s)\log(e + t), \quad t, s \ge 0. \tag{4-10}$$

Let us consider the function $w \in X^{j,q}(B(R_0))$. By (4-10) we have, for $k > 0$, that

$$\int_{B(R_0)} M_{j,q}(kw)\,dm = k^j \int_{B(R_0)} |w|^j \log^q(e+k|w|)\,dm \le 2k^j \log^q(e+k) \int_{B(R_0)} M_{j,q}(w)\,dm. \tag{4-11}$$

A function $u \in X^{j,q}(B(R_0))$ can be written as $u = kw$, where $k = \|u\|_{(M)}$ and $\|w\|_{(M)} = 1$. Then by (A-5)–(A-6), we have

$$\int_{B(R_0)} M_{j,q}(w)\,dm = 1,$$

and hence (4-11) and (A-4) yield the claim (i).

(ii) Using (4-11) and the definition (A-2) of the Orlicz norm, we see that for all $k > 0$,

$$\|u\|_{X^{j,q}(B(R_0))} \le \frac{1}{k}\Big(1 + \int_{B(R_0)} M_{j,q}(ku)\,dm\Big) \le \frac{1}{k}\Big(1 + 2k^j \log^q(e+k) \int_{B(R_0)} M_{j,q}(u)\,dm\Big).$$

Let $T = \int_{B(R_0)} M_{j,q}(u)\,dm$. Substituting $k = T^{-1/j}$ above, we obtain (ii). □

Theorem 4.2. *Assume that $\mu \in B^p_{\exp}(B(R_0))$, $2 < p < \infty$. Then the equations* (4-3)–(4-4) *have a unique solution $\Phi \in W^{1,1}_{\mathrm{loc}}(\mathbb{C})$ which, for $0 \le q \le p/4$, satisfies*

$$\bar\partial \Phi \in X^q(\mathbb{C}) \tag{4-12}$$

and the series (4-6) *converges in $X^q(\mathbb{C})$. The convergence of the series* (4-6) *in $X^q(\mathbb{C})$ is uniform for $\mu \in B^p_{\exp,N}(B(R_0))$ with any $N > 0$. Moreover, for $\mu \in B^p_{\exp,N}(B(R_0))$, the Jacobian $J_\Phi(z)$ of Φ satisfies*

$$\|J_\Phi\|_{X^{1,q}(B(R_0))} \le C, \tag{4-13}$$

where C depends only on p, q, N, and R_0. Moreover, let $s > 2$ and assume that $\mu_m, \tilde\mu_m \in B^p_{\exp,N}(B(R_0))$ and $0 \le q \le p/4$. Then we have the following implication:

$$\lim_{m\to\infty} \|\mu_m - \tilde\mu_m\|_{L^s(B(R_0))} = 0 \quad\Rightarrow\quad \lim_{m\to\infty} \|\bar\partial\Phi_m - \bar\partial\tilde\Phi_m\|_{X^q(\mathbb{C})} = 0, \tag{4-14}$$

where Φ_m and $\tilde\Phi_m$ are the solutions of (4-3)–(4-4) *corresponding to μ_m, $\tilde\mu_m$, respectively.*

Proof. Let $\Phi^\lambda(z)$, where $|\lambda| \le 1$, $z \in \mathbb{C}$, be the principal solution corresponding to the Beltrami coefficient $\lambda\mu$, that is, the solution with the Beltrami equation (2-17)–(2-18) with coefficient $\lambda\mu$. These solutions, in particular $\Phi^\lambda = \Phi^1$, exist and are unique by Theorem 2.2. It follows from [Astala et al. 2010, Theorems 1.1 and 5.1] that the Jacobian determinant $J\Phi^\lambda(z)$ of Φ^λ satisfies

$$\int_{B(R_0)} J\Phi^\lambda \log^{2q}(e + J\Phi^\lambda)\,dm(z) \le C < \infty, \tag{4-15}$$

where C is independent of λ and $\mu \in B^p_{\exp,N}(B(R_0))$ and depends only on N, p, and q. Thus (4-13) follows from Lemma 4.1(ii).

We showed already that when $p > 2$, we have $\bar\partial\Phi \in L^2(\mathbb{C})$ and that the series (4-6) converges in $L^2(\mathbb{C})$. To show the convergence of (4-6) in $X^q(\mathbb{C})$ and to prove (4-14), we present a few lemmas in terms of Orlicz spaces $X^q(B(R_0))$ and the function M_q defined in (4-8). Note that as μ vanishes in $\mathbb{C}\setminus B(R_0)$,

$$\|(\mu S)^n\mu\|_{X^q(\mathbb{C})} = \|(\mu S)^n\mu\|_{X^q(B(R_0))}.$$

Lemma 4.3. *Let $N \in \mathbb{Z}_+$, $2 < 2q < \beta < p$, and $\mu \in B^p_{\exp,N}(B(R_0))$. Then*

$$\int_{B(R_0)} M_q(\psi_n(x))\,dm(x) \le cn^{-(\beta-q)} < cn^{-q}, \tag{4-16}$$

where $\psi_n = (\mu S)^n\mu$ and $c > 0$ depends only on N, p, β, and q.

Proof. Let $E_n = \{z \in B(R_0) : |\psi_n(z)| \ge A^n\}$, where $A > 1$ is a constant to be chosen later. By (4-5),

$$\|\psi_n\|_{L^2(B(R_0))} \le C_{N,\beta,p}n^{-\beta/2}. \tag{4-17}$$

Thus

$$|E_n| \le C^2_{N,\beta,p}A^{-2n}n^{-\beta}. \tag{4-18}$$

Using (4-17), we obtain

$$\int_{B(R_0)\setminus E_n} |\psi_n|^2\log^q(e+|\psi_n|)\,dm \le \|\psi_n\|^2_{L^2(B(R_0))}\log^q(e+A^n) \le C_1n^{-\beta+q}, \tag{4-19}$$

where $C_1 = C^2_{N,\beta,p}\log^q(e+A)$.

The principal solution corresponding to the Beltrami coefficient $\lambda\mu$ can be written in the form

$$\Phi^\lambda(z) = z + \frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar\partial\Phi^\lambda(w)}{w-z}\,dm(w), \qquad \bar\partial\Phi^\lambda = (I-\lambda\mu S)^{-1}(\lambda\mu).$$

Expanding $\bar\partial_z\Phi^\lambda(z)$ as a power series in λ, we see that by (4-6) we can write, using any $0 < \rho < 1$,

$$\chi_{E_n}(z)\psi_n(z) = \frac{1}{2\pi i}\int_{|\lambda|=\rho} \lambda^{-n-2}\chi_{E_n}(z)\,\bar\partial_z\Phi^\lambda(z)\,d\lambda.$$

This gives

$$\|\chi_{E_n}\psi_n\|_{X^q(B(R_0))} \le \rho^{-(n+2)}\sup_{|\lambda|=\rho}\|\chi_{E_n}\bar\partial_z\Phi^\lambda\|_{X^q(B(R_0))}. \tag{4-20}$$

Using the facts that $|\lambda| = \rho$ and that the Beltrami coefficient Φ^λ is bounded by $|\lambda|$, we have, by the distortion equality (2-15), that

$$|\bar\partial_z\Phi^\lambda(z)|^2 \le \rho^2(1-\rho^2)^{-1}J\Phi^\lambda(z).$$

Hence,

$$I := \int_{E_n} M_q(\bar\partial_z\Phi^\lambda(z))\,dm(z) \le \frac{\rho^2}{1-\rho^2}\int_{E_n} J\Phi^\lambda\log^q\left(e+\left(\frac{\rho^2}{1-\rho^2}J\Phi^\lambda\right)^{1/2}\right)dm.$$

Next, let $\widehat{C}$ denote a generic constant which is a function of N, β, p, q and ρ but not of A. The above implies by (4-10), (4-15), and the inequality $\log(e+t^{1/2}) \le 1+\log(e+t)$, $t \ge 0$, that

$$\begin{aligned} I &\le \widehat{C}\int_{E_n} J\Phi^\lambda(z)(1+\log(e+J\Phi^\lambda))^q\,dm(z)\\ &\le \widehat{C}\Big(\int_{E_n} J\Phi^\lambda(z)\,dm(z)\Big)^{1/2}\Big(\int_{E_n} J\Phi^\lambda(z)(1+\log(e+J\Phi^\lambda(z)))^{2q}\,dm(z)\Big)^{1/2}\\ &\le \widehat{C}\Big(\int_{E_n} J\Phi^\lambda(z)\,dm(z)\Big)^{1/2}. \end{aligned}$$

By the area distortion theorem from [Astala 1994], as formulated in [Astala et al. 2009, Theorem 13.1.4], we have

$$\int_{E_n} J\Phi^\lambda(z)\,dm(z) \le |\Phi^\lambda(E_n)| \le \widehat{C}|E_n|^{1/M} \le \widehat{C}A^{-2n/M},$$

where $M=(1+\rho)/(1-\rho)>1$, and thus $I \le \widehat{C}A^{-n/M}$. By Lemma 4.1(ii), we also have the estimate

$$\|\chi_{E_n}\bar\partial\Phi^\lambda\|_{X^q} \le \widehat{C}A^{-n/M}.$$

Taking $\rho > e^{-1/2}$ and $A=e^M$, we see using (4-20) and Lemma 4.1 again that

$$\int_{E_n} M_q(\psi_n)\,dm(z) \le \widehat{C}e^{-n/2}$$

for sufficiently large $n\in\mathbb{Z}_+$. Thus the assertion follows from (4-19). □

Lemmas 4.1(ii) and 4.3 and the fact that μ vanishes outside $B(R_0)$ yield that for $q>1$ and $p>2q$, there is an $N>0$ such that the series (4-6) converges in $X^q(\mathbb{C})$, and moreover, convergence of the series (4-6) is uniform for $\mu\in B^p_{\exp,N}(B(R_0))$. Thus to prove Theorem 4.2 it remains to show (4-14).

Lemma 4.4. *Let* $2<2q<p$, $N>0$, $2<\beta<p$, $s>2$, $\mu,\nu\in B^p_{\exp,N}(B(R_0))$, *and* $B_n=(\mu S)^n\mu-(\nu S)^n\nu$. *Then*

$$\sup_{n\in\mathbb{Z}_+}\int_{\mathbb{C}} M_q(B_n(x))\,dm(x) \le C, \tag{4-21}$$

where $C>0$ *depends only on* N, p, *and* q. *Moreover, there is* $T>1$ *such that*

$$\|B_n\|_{L^2(\mathbb{C})} \le C_{N,\beta,p,s,T}\min\big(nT^n\|\mu-\nu\|_{L^s(B(R_0))},\, n^{-\beta/2}\big). \tag{4-22}$$

Proof. Lemmas 4.1 and 4.3 yield (4-21). Next, let us observe that for $z\in\mathbb{C}$,

$$B_n(z)=(\mu S)^n\mu-(\nu S)^n\nu=\sum_{j=0}^n A_j(z),\qquad A_j(z)=(\mu S)^j(\mu-\nu)(S\nu)^{n-j}\chi_{B(R_0)}.$$

As $\|\nu\|_{L^\infty}\le 1$ and $\|S\|_q:=\|S\|_{L^q(\mathbb{C})\to L^q(\mathbb{C})}<\infty$ for $1<q<\infty$, we have that

$$\begin{aligned}\int_{\mathbb{C}}|A_j(z)|^q\,dm(z) &\le (\|S\|_q^q)^j\int_{B(R_0)}|\mu(z)-\nu(z)|^q\,|((S\nu)^{n-j}\chi_{B(R_0)})(z)|^q\,dm(z)\\ &\le \|S\|_q^{jq}\Big(\int_{B(R_0)}|\mu(z)-\nu(z)|^{q\rho}dm(z)\Big)^{1/\rho}\Big(\int_{B(R_0)}|((S\nu)^{n-j}\chi_{B(R_0)})(z)|^{q\rho'}dm(z)\Big)^{1/\rho'},\end{aligned}$$

where $\rho^{-1} + (\rho')^{-1} = 1$ and $1 < \rho < \infty$. Thus

$$\|A_j(z)\|_{L^q(\mathbb{C})} \le (\|S\|_q)^j \|\mu - \nu\|_{L^{\rho q}(B(R_0))} (\|S\|_{q\rho'}^q)^{n-j} \|\nu\|_{L^{q\rho'}(B(R_0))}^q,$$

where $\|\nu\|_{L^{q\rho'}(B(R_0))} \le \pi R_0^2$. Thus by choosing $q = 2$ and ρ so that $s = q\rho > 2$ yielding $q\rho' = 2s/(s-2)$, we obtain

$$\|(\mu S)^n \mu - (\nu S)^n \nu\|_{L^2(\mathbb{C})} \le (n+1)\pi^2 R_0^4 (1 + \|S\|_{(2s/(s-2))}^2)^n \|\mu - \nu\|_{L^s(B(R_0))}.$$

This and (4-5) show that (4-22) is valid. □

Now we are ready to prove (4-14), which finishes the proof of Theorem 4.2. Let

$$B_{n,m} = (\mu_m S)^n \mu_m - (\tilde\mu_m S)^n \tilde\mu_m.$$

By the Schwarz inequality, we have that (4-21), (4-22) and Lemma 4.1 yield

$$\begin{aligned}\int_{B(R_0)} M_q(B_{n,m}(z))\, dm(z) &\le \int_{B(R_0)} |B_{n,m}|^2 \log^q(e + |B_{n,m}|)\, dm(z) \\ &\le \left(\int_{B(R_0)} M_{2q}(B_{n,m}(z))\, dm(z)\right)^{1/2} \|B_{n,m}\|_{L^2(B(R_0))} \\ &\le C \min\bigl(nT^n \|\mu_m - \tilde\mu_m\|_{L^s(B(R_0))}, n^{-\beta/2}\bigr), \end{aligned} \tag{4-23}$$

where C depends only on q, p, β, s, T, and N.

Let $\varepsilon > 0$. As μ_m and $\tilde\mu_m$ vanish outside $B(R_0)$,

$$\|\bar\partial \Phi_m - \bar\partial \tilde\Phi_m\|_{X^q(\mathbb{C})} = \|\bar\partial \Phi_m - \bar\partial \tilde\Phi_m\|_{X^q(B(R_0))} \le \sum_{n=0}^{\infty} \|B_{n,m}\|_{X^q(B(R_0))}.$$

Thus by (4-23) and Lemma 4.1(ii), we can take $n_0 \in \mathbb{N}$ so large that for all m,

$$\sum_{n=n_0}^{\infty} \|B_{n,m}\|_{X^q(B(R_0))} \le \frac{\varepsilon}{2}.$$

Applying again (4-23) and Lemma 4.1(ii), we can choose $\delta > 0$ so that

$$\sum_{n=0}^{n_0-1} \|B_{n,m}\|_{X^q(B(R_0))} \le \frac{\varepsilon}{2} \quad \text{when } \|\mu_m - \tilde\mu_m\|_{L^s(B(R_0))} \le \delta.$$

This proves Theorem 4.2. □

Lemma 4.5. *Assume that K_μ corresponding to μ supported in $\mathbb{D}$ satisfies* (4-1) *with q, $C_0 > 0$ and $R_1 = 1$. Let Φ be the principal solution of the Beltrami equation corresponding to μ. Then for all β, $R > 0$, the inverse function $\Psi = \Phi^{-1} : \mathbb{C} \to \mathbb{C}$ of Φ satisfies*

$$\int_{B(R)} \exp(\beta K_\mu(\Psi(z)))\, dm(z) < C,$$

where C depends only on q, C_0, β, and R.

Proof. Since Φ satisfies the condition $\mathcal{N}$ by [Astala et al. 2010, Corollary 4.3], we may change variable in integration to see that

$$\int_{B(R)} \exp(\beta K_\mu(\Psi(z)))\, dm(z) = \int_{\Psi(B(R))} \exp(\beta K_\mu(w)) J_\Phi(w)\, dm(w). \tag{4-24}$$

Using (3-7) for the function Φ and $R > 3$, we see that $\Psi(B(R)) \subset \tilde{B} = B(\tilde{R})$, $\tilde{R} = R + 1$. By (4-1), $\exp(K_\mu(z)) \in L^q(\tilde{B})$ for all $q > 1$ and thus by (4-13), $J_\Phi \in X^{1,q}(B(R))$ for $R > 0$.

Let us next use properties of Orlicz spaces and the notations discussed in the Appendix using a Young complementary pair (F, G), where

$$F(t) = \exp(t^{1/p}) - 1$$

and $G(t)$ satisfies $G(t) = C_p t(\log(1 + C_p t))^p$ for $t > T_p$ with suitable $C_p, T_p > 0$; see [Krasnosel'skiĭ and Rutickiĭ 1961, Theorem I.6.1].

By using $u(z) = \exp(\beta K_\mu(z))$ and $v = J_\Phi(z)$, we obtain from Young's inequality (A-7) the inequality

$$\begin{aligned}\int_{B(R)} &\exp(\beta K_\mu(\Psi(z)))\, dm(z) \\ &\le \int_{\tilde{B}} F(\exp(\beta K_\mu(w)))\, dm(w) + \int_{\tilde{B}} G(J_\Phi(w))\, dm(w) \\ &\le \int_{\tilde{B}} \exp((\exp(\beta K_\mu(w)))^{1/p})\, dm(w) + \int_{\tilde{B}} C_p J_\Phi(w)(\log(1 + C_p J_\Phi(w)))^p\, dm(w). \end{aligned} \tag{4-25}$$

We apply this by using $p > \beta/q$, so that

$$\big(\exp(\beta K_\mu(w))\big)^{1/p} \le \exp(q K_\mu(w)).$$

Thus

$$\int_{\tilde{B}} \exp\big(\big(\exp(\beta K_\mu(w))\big)^{1/p}\big)\, dm(w) \le \int_{\tilde{B}} \exp\big(\exp(q K_\mu(w))\big)\, dm(w) < \infty.$$

The last term in (4-25) is finite by (4-15), and thus the claim follows. □

4B. ***Asymptotics of the phase function of the exponentially growing solution.*** Let $\mu \in B^p_{\exp}(B(R_0))$, $k \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \mathbb{C}$ satisfy $|\lambda| \le 1$. Then using Lemmas 3.3 and 3.4, with the affine weight $\mathcal{A}(t) = pt - p$ corresponding to the gauge function Q, we see that the equation

$$\bar\partial_z f_k(z) = \lambda \mu(z)\, \overline{\partial_z f_k(z)} \qquad \text{for a.e. } z \in \mathbb{C}, \tag{4-26}$$

$$f_k(z) = e^{ikz}\Big(1 + O\Big(\frac{1}{z}\Big)\Big) \qquad \text{as } |z| \to \infty, \tag{4-27}$$

has the unique solution $f_k \in W^{1,Q}_{\mathrm{loc}}(\mathbb{C})$. Moreover, this solution can be written in the form

$$f_k(z) = e^{ik\varphi_k(z)}, \tag{4-28}$$

where $\varphi_k : \mathbb{C} \to \mathbb{C}$ is a homeomorphism satisfying

$$\bar\partial\varphi_k(z) = -\frac{\lambda\bar k}{k}\mu(z)e_{-k}(\varphi_k(z))\,\overline{\partial\varphi_k(z)} \quad \text{for a.e. } z \in \mathbb{C}, \tag{4-29}$$

$$\varphi_k(z) = z + \mathcal{O}\Big(\frac{1}{z}\Big) \quad \text{as } |z| \to \infty. \tag{4-30}$$

Below, we set $f_k(z) = f(z,k)$ and $\varphi_k(z) = \varphi(z,k)$ and estimate the next functions φ_k in the Orlicz space $X^q(\mathbb{C})$. The following lemma is a generalization of results of [Astala and Päivärinta 2006] to the Orlicz space setting.

Lemma 4.6. *Assume that $\nu \in B^p_{\exp}(B(R_0))$ for all $0 < p < \infty$. For $k \in \mathbb{C}\setminus\{0\}$, let $\Phi_k \in W^{1,1}(\mathbb{C})$ be the solution of*

$$\bar\partial\Phi_k(z) = -\frac{\bar k}{k}\nu(z)e_{-k}(z)\,\partial\Phi_k(z) \quad \textit{for a.e. } z \in \mathbb{C}, \tag{4-31}$$

$$\Phi_k(z) = z + \mathcal{O}\Big(\frac{1}{z}\Big). \tag{4-32}$$

Then for all $\varepsilon > 0$, there exists $C_0 > 0$ such that $\bar\partial_z\Phi_k(z) = g_k(z) + h_k(z)$, where $g_k, h_k \in X^q(\mathbb{C})$ are supported in $B(R_0)$ and

$$\sup_{k\in\mathbb{C}\setminus\{0\}} \|h_k\|_{X^q} < \varepsilon, \tag{4-33}$$

$$\sup_{k\in\mathbb{C}\setminus\{0\}} \|g_k\|_{X^q} < C_0, \tag{4-34}$$

$$\lim_{k\to\infty} \hat g_k(\xi) = 0, \tag{4-35}$$

where for all compact sets $S \subset \mathbb{C}$, the convergence in (4-35) *is uniform for $\xi \in S$.*

Proof. Let us define

$$\tilde\nu_k(z) = -\bar k k^{-1}\nu(z)$$

for $k \in \mathbb{C}\setminus\{0\}$. Note that then for any $p > 0$, there is $N > 0$ such that $\tilde\nu_k(\,\cdot\,,k)e_{-k}(\,\cdot\,) \in B^p_{\exp,N}(B(R_0))$ for all $k \in \mathbb{C}\setminus\{0\}$. By Theorem 4.2,

$$\lim_{n\to\infty}\Big\|\bar\partial\Phi_k - \sum_{n=0}^{\infty}(\tilde\nu_k e_{-k}S)^n(\tilde\nu_k e_{-k})\Big\|_{X^q(\mathbb{C})} = 0$$

uniformly in $k \in \mathbb{C}\setminus\{0\}$. For $m \in \mathbb{Z}_+$, we define

$$g_k(z) = g_k^{(m)}(z) = -\sum_{n=0}^{m}(\tilde\nu_k e_{-k}S)^n(\tilde\nu_k e_{-k}),$$

$$h_k(z) = h_k^{(m)}(z) = -\sum_{n=m+1}^{\infty}(\tilde\nu_k e_{-k}S)^n(\tilde\nu_k e_{-k}).$$

For given $\varepsilon > 0$, we can choose m so large that (4-33) holds for all $k \in \mathbb{C}\setminus\{0\}$, and then using Lemma 4.3, choose C_0 so that (4-34) holds for all $k \in \mathbb{C}\setminus\{0\}$.

Next, we show (4-35) when ε and m are fixed so that (4-33) and (4-34) hold. We can write

$$g_k(z) = -\sum_{n=0}^{m} e_{-nk} G_n, \quad G_n = \left(\frac{\bar{k}}{k}\right)^{n+1} \nu S_n(k)\nu \cdots \nu S_1(k)\nu,$$

where $S_j(k)$ is the Fourier multiplier

$$(S_j(k)\phi)^\wedge(\xi) = \boldsymbol{m}(\xi + jk)\hat{\phi}(\xi), \quad \boldsymbol{m}(\xi) = \frac{\bar{\xi}}{\xi}.$$

The proof of [Astala and Päivärinta 2006, Lemma 7.3] for $n \geq 1$ and the Riemann–Lebesgue lemma for $n = 0$ yield that for any $\tilde{\varepsilon} > 0$, there exists $R(n, \tilde{\varepsilon}) \geq 0$ such that, for $n \leq m$,

$$|\widehat{G}_n(\xi)| \leq (n+1)\kappa^n \tilde{\varepsilon} \quad \text{for } |\xi| > R(n, \tilde{\varepsilon}),$$

where $\kappa = \|\nu\|_{L^\infty} \leq 1$. Thus for $n \leq m$,

$$|\widehat{G}_n(\xi)| \leq (m+1)\tilde{\varepsilon} \quad \text{for } |\xi| > R_0 = \max_{n \leq m} R(n, \tilde{\varepsilon}),\ n = 0, 1, 2, \ldots, m. \tag{4-36}$$

As

$$(e_{-nk} G_n)^\wedge(\xi) = \widehat{G}_n(\xi - nk),$$

we see that for any $L > 0$, there is $k_0 > 0$ such that if $|k| > k_0$ then $j|k| - L > R_0$ for $1 \leq n \leq m$. Then it follows from (4-36) that if $|k| > k_0$, then

$$\sup_{|\xi|<L} |\hat{g}_k(\xi)| \leq (m+1)^2 \tilde{\varepsilon}.$$

This proves the limit (4-35), with the convergence being uniform for ξ belonging in a compact set. □

Proposition 4.7. *Assume that $\nu \in B^p_{\exp}(B(R_0))$ with $p > 4$ and $\Phi_k(z)$ is the solution of* (4-31)–(4-32). *Then*

$$\lim_{k\to\infty} \Phi_k(z) = z \quad \textit{uniformly for } z \in \mathbb{C}. \tag{4-37}$$

Proof. Step 1: We will first show that for all q with $4 < q < p$, we have $\bar{\partial}_z \Phi_k(z) \to 0$ weakly in $X^q(\mathbb{C})$ as $k \to \infty$. Let $\eta \in X^{-q}(\mathbb{C})$ and $\varepsilon_1 > 0$. By Theorem 4.2, there is $C_1 > 0$ such that

$$\sup_k \|\bar{\partial}\Phi_k\|_{X^q} \leq C_1.$$

Since $C_0^\infty(\mathbb{C})$ is dense in $X^{-q}(\mathbb{C})$ (see [Krasnosel’skiĭ and Rutickiĭ 1961, Section II.10]), we can find a function $\eta_0 \in C_0^\infty(\mathbb{C})$ such that

$$\|\eta - \eta_0\|_{X^{-q}} \leq \min(1, \varepsilon_1/C_1).$$

Then

$$|\langle \eta, \bar{\partial}\Phi_k\rangle| \leq |\langle \eta_0, \bar{\partial}\Phi_k\rangle| + \|\eta - \eta_0\|_{X^{-q}(\mathbb{C})} \|\bar{\partial}\Phi_k\|_{X^q(\mathbb{C})}, \tag{4-38}$$

where the second term on the right-hand side is smaller than ε_1. Moreover, by Lemma 4.6, we can write

$$\bar{\partial}\Phi_k = h_k + g_k$$

so that (4-33)–(4-35) are satisfied for $\varepsilon = \varepsilon_1(\|\eta\|_{X^{-q}} + 1)^{-1}$ and some $C_0 > 0$. Then $|\langle \eta_0, h_k\rangle| \leq \varepsilon_1$.

Since $\hat{\eta}_0$ is a rapidly decreasing function, $\hat{g}_k(\xi)$ is uniformly bounded for $\xi \in \mathbb{C}$ and $k \in \mathbb{C} \setminus \{0\}$ by Lemma 4.6, and $\hat{g}_k \to 0$ uniformly in all bounded domains as $k \to \infty$, we see that

$$\langle \eta_0, g_k \rangle = \langle \hat{\eta}_0, \hat{g}_k \rangle \to 0 \quad \text{as } k \to \infty. \tag{4-39}$$

Combining these, we see that $\langle \eta_0, \bar{\partial}\Phi_k \rangle \to 0$ as $k \to \infty$, and thus $\bar{\partial}_z \Phi_k(z) \to 0$ weakly in $X^q(\mathbb{C})$ as $k \to \infty$.

Step 2: Next we show the pointwise convergence

$$\lim_{k\to\infty} \bar{\partial}_z \Phi_k(z) = 0. \tag{4-40}$$

To this end, we observe that the function

$$\eta_z(w) = \frac{1}{\pi(w-z)} \chi_{B(R_0)}(w)$$

satisfies $\eta_z \in X^{-q}(\mathbb{C})$ for $q > 1$. Since $\Phi_k(z) - z = \mathcal{O}(1/z)$ and $\bar{\partial}\Phi_k$ is supported in $\overline{B(R_0)}$, we have

$$\Phi_k(z) = z - \frac{1}{\pi} \int_{B(R_0)} (w-z)^{-1}\, \bar{\partial}_w \Phi_k(w)\, dm(w) = z - \langle \eta_z, \bar{\partial}\Phi_k \rangle. \tag{4-41}$$

As $\bar{\partial}\Phi_k \to 0$ weakly in $X^q(\mathbb{C})$, we see (4-40) holds for all $z \in \mathbb{C}$.

Step 3: By (3-14) and (3-17), we see that the family $\{\Phi_k(z)\}_{k\in\mathbb{C}\setminus\{0\}}$ of homeomorphisms has a uniform modulus of continuity in compact sets. Moreover, since

$$\sup_k \|\bar{\partial}\Phi_k\|_{L^1(\mathbb{C})} \le \sup_k \|\bar{\partial}\Phi_k\|_{X^q(B(R_0))} = C_2 < \infty,$$

we obtain by (4-40), for $|z| > R_0 + 1$, that

$$|\Phi_k(z) - z| = |\langle \eta_z, \bar{\partial}\Phi_k \rangle| \le \frac{C}{|z|} \|\bar{\partial}\Phi_k\|_{L^1(\mathbb{C})} \le \frac{CC_2}{|z|}. \tag{4-42}$$

Thus, as the functions $\{\Phi_k(z)\}_{k\in\mathbb{C}\setminus\{0\}}$ are uniformly equicontinuous in compact sets, (4-42) and the pointwise convergence (4-40) yield the uniform convergence (4-37). □

4C. *Properties of the solutions of the nonlinear Beltrami equation.* Let $\lambda \in \mathbb{C}$, $|\lambda| \le 1$ and $\mu(z)$ be supported in $\overline{B(R_0)}$, $R_0 \ge 1$, and assume that $K = K_\mu$ satisfies (4-1) with $q, C_0 > 0$ and $R_1 = 1$. Motivated by Lemma 3.4, we consider next the solutions φ_k of the equation

$$\bar{\partial}_z \varphi_\lambda(z,k) = -\lambda \frac{\bar{k}}{k} \mu(z) e_{-k}(\varphi_\lambda(z,k))\, \overline{\partial_z \varphi_\lambda(z,k)}, \quad z \in \mathbb{C}, \tag{4-43}$$

$$\varphi_\lambda(z,k) = z + O\left(\frac{1}{z}\right) \quad \text{as } |z| \to \infty. \tag{4-44}$$

Let $\psi_\lambda(\,\cdot\,,k) = \varphi_\lambda(\,\cdot\,,k)^{-1}$ be the inverse function of $\varphi_\lambda(\,\cdot\,,k)$. A simple computation based on differentiation of the identity $\psi_\lambda(\varphi_\lambda(z,k),k) = z$ in the z-variable shows that

$$\bar{\partial}_z \psi_\lambda(z,k) = -\lambda \frac{\bar{k}}{k} \mu(\psi_\lambda(z,k)) e_{-k}(z)\, \partial_z \psi_\lambda(z,k), \quad z \in \mathbb{C}, \tag{4-45}$$

$$\psi_\lambda(z,k) = z + O\left(\frac{1}{z}\right) \quad \text{as } |z| \to \infty. \tag{4-46}$$

Define

$$\nu(z) = -\lambda \frac{\bar{k}}{k} \mu(z)$$

and consider the equations (4-43) and (4-45) simultaneously by defining the sets

$$B_\mu = \{(\varphi, \nu) : |\nu| \le |\mu| \text{ a.e. and } \varphi : \mathbb{C} \to \mathbb{C} \text{ is a homeomorphism with } \bar\partial\varphi = \nu\overline{\partial\varphi},\, \varphi(z) = z + O(z^{-1})\}$$

and, for Q defined by (1-8),

$$\mathcal{G}_\mu = \{g \in W^{1,Q}_{\mathrm{loc}}(\mathbb{C}) : \bar\partial g = (\nu \circ \varphi^{-1})\partial g,\ g(z) = z + O(z^{-1}),\ (\varphi, \nu) \in B_\mu\}.$$

Now $\exp(\exp(qK_\mu)) \in L^1(B(R_0))$ with some $0 < q < \infty$ and $|\nu| \le |\mu|$ almost everywhere. Then $K_\nu(z) \le K_\mu(z)$ almost everywhere. Let

$$\bar\partial\varphi = \nu\, \overline{\partial\varphi} \quad \text{in } \mathbb{C}, \qquad \varphi(z) = z + O(z^{-1}),$$

so that $\bar\partial\varphi = \tilde\nu\, \partial\varphi$ with $|\tilde\nu(z)| = |\nu(z)|$. Then for $\psi = \varphi^{-1}$, we have $K(z, \psi) = K_\nu(\psi(z))$; see (2-14). Thus by Lemma 4.5, we have

$$\sup_{g \in \mathcal{G}_\mu} \|\exp(\beta K(\,\cdot\,, g))\|_{L^1(B(R))} = \sup_{(\varphi,\nu) \in B_\mu} \|\exp(\beta K_\nu \circ \varphi^{-1})\|_{L^1(B(R))} < \infty \tag{4-47}$$

for all $\beta > 0$ and $R > 0$. Using this and Theorem 2.2, we see that the functions $g \in \mathcal{G}_\mu$ are homeomorphisms. Moreover, recall that $\mu \in B^p_{\exp}(B(R))$ for all $p \in (1, \infty)$. Thus for $g \in \mathcal{G}_\mu$, the condition $g \in W^{1,Q}_{\mathrm{loc}}(\mathbb{C})$ is equivalent to (see (4-7) and (4-8)) $Dg \in X^{-1}_{\mathrm{loc}}(\mathbb{C})$. Furthermore by (4-13), we have

$$\sup_{(\varphi,\nu) \in B_\mu} \|J_\varphi\|_{X^{1,q}(B(R))} < \infty \tag{4-48}$$

for all $q > 0$.

Lemma 4.8. *The set $\mathcal{G}_\mu$ is relatively compact in the topology of uniform convergence.*

Proof. Let $(\varphi, \nu) \in B_\mu$ and $\psi = \varphi^{-1}$ and

$$\bar\partial g = (\nu \circ \varphi^{-1})\partial g, \quad g(z) = z + O(z^{-1}).$$

As μ is supported in $B(R_0)$, the function φ is analytic outside $\overline{B(R_0)}$; we see using (3-7) for the function φ that for $R > 0$, we have $\varphi(B(R)) \subset B(R + 3R_0)$, $\psi(B(R)) \subset B(R + 3R_0)$, and that ψ is analytic outside $\overline{B(4R_0)}$.

Thus (3-7) and the same arguments which we used to prove the estimate (3-27) yield that for $R > 0$,

$$\begin{aligned} \|Q(|Dg|)\|_{L^1(B(R))} &\le \pi(R + 3R_0)^2 + \int_{B(R)} \exp\bigl(qK_\nu(\psi(w)) - q\bigr)\, dm(w) \\ &\le \pi(R + 3R_0)^2 + \int_{B(R+3R_0)} \exp(qK_\nu(z) - q) J_\varphi(z)\, dm(z), \end{aligned} \tag{4-49}$$

where $Q(t) = |t|^2/\log(|t| + e)$. We will next use Young's inequality (A-7) with the admissible pair (F, G), where (see [Krasnosel'skiĭ and Rutickiĭ 1961, Chapter 1.3])

$$F(t) = e^t - t - 1, \quad G(t) = (1 + t)\log(1 + t) - t. \tag{4-50}$$

By Young's inequality, we have

$$\int_{B(R+3R_0)} \exp(qK_\nu(z)-q)J_\varphi(z)\,dm(z) \le \int_{B(R+3R_0)} \exp\big(\exp(qK_\nu(w)-q)\big)\,dm(w) + \int_{B(R+3R_0)} (1+J_\varphi(w))\log(1+J_\varphi(w))\,dm(w).$$

This, (4-1), (4-48), and (4-49) show that there is a constant $C(R,\mu)$ such that for $g\in\mathcal{G}_\mu$,

$$\|Q(|Dg|)\|_{L^1(B(R))} \le C(R,\mu). \tag{4-51}$$

As $g\in\mathcal{G}_\mu$ are homeomorphisms, this, (3-14) and (A-1)–(A-3) imply that functions $g\in\mathcal{G}_\mu$ are equicontinuous in compact sets of $\mathbb{C}$. As $\operatorname{supp}(\nu\circ\psi)\subset B(4R_0)$, functions $g\in\mathcal{G}_\mu$ are analytic outside the disc $B(4R_0)$ and the inequality (3-7) yields, for $R>0$ and $g\in\mathcal{G}_\mu$, that

$$g(B(R))\subset B(R+12R_0).$$

By the Arzelà–Ascoli theorem, these imply that the set $\{g|_{B(R)} : g\in\mathcal{G}_\mu\}$ is relatively compact in the topology of uniform convergence in $B(R)$ for any $R>0$. Thus by using a diagonalization argument, we see that for an arbitrary sequence $g_n\in\mathcal{G}_\mu$, $n=1,2,\dots$, there is a subsequence g_{n_j} which converges uniformly in all discs $B(R)$, $R>0$. Finally, by Young's inequality (see the Appendix), we get using the same notations as in (4-41) that for $|z|>4R_0+1$,

$$\begin{aligned} |g_k(z)-z| &= \left|\frac{1}{\pi}\int_{B(4R_0)} (w-z)^{-1}\,\bar\partial_w g_k(w)\,dm(w)\right| \\ &\le \frac{1}{\pi(|z|-4R_0)}\int_{B(4R_0)} \big(Q(|\bar\partial_w g_k(w)|)+G_0(1)\big)\,dm(w), \end{aligned} \tag{4-52}$$

where $Q(t)$ and $G_0(t)=|t|^2\log(|t|+1)$ form a Young complementary pair. Thus

$$|g_k(z)-z| \le \frac{C_\mu}{|z|-4R_0} \quad \text{for } |z|>4R_0+1.$$

Using this and the uniform convergence of g_{n_j} in all discs $B(R)$, $R>0$, we see that g_n has a subsequence converging uniformly in $\mathbb{C}$. □

Theorem 4.9. *Let $\lambda, k\in\mathbb{C}\setminus\{0\}$, $|\lambda|=1$. Assume that $\varphi_\lambda(z,k)$ satisfies* (4-43)–(4-44) *with μ supported in $\mathbb{D}$ which satisfies* (4-1) *with $q>0$ and $R_1=1$. Then*

$$\lim_{k\to\infty}\varphi_\lambda(z,k)=z$$

uniformly in $z\in\mathbb{C}$ and $|\lambda|=1$.

Proof. Let $\psi_\lambda(\,\cdot\,,k)$ be the inverse function of $\varphi_\lambda(\,\cdot\,,k)$. It is sufficient to show that

$$\lim_{k\to\infty}\psi_\lambda(z,k)=z$$

uniformly in $z\in\mathbb{C}$ and $|\lambda|=1$.

Then, $\psi_\lambda(\,\cdot\,,k)$ is the solution of (4-45)–(4-46). Define

$$\nu(z)=-\lambda\bar{k}k^{-1}\mu(z)$$

and note that $|\nu(z)|=|\mu(z)|$. Hence

$$\big(\varphi_\lambda(\,\cdot\,,k),\,\nu(\,\cdot\,)\,e_{-k}(\,\cdot\,)\big)\in B_\mu$$

and $\psi_\lambda(\,\cdot\,,k)\in\mathcal{G}_\mu$. Moreover, as $\varphi_\lambda(\,\cdot\,,k)$ is homeomorphism in $\mathbb{C}$ and analytic outside of $B(1)$, it follows from (3-7) with $r=1$ that $\varphi_\lambda(\,\cdot\,,k)$ maps the ball $B(1)$ into $B(4)$ and moreover, its inverse $\psi_\lambda(\,\cdot\,,k)$ maps the disc $B(4)$ into $B(5)$ and $\mathbb{C}\setminus B(4)$ into $\mathbb{C}\setminus B(1)$.

It follows from Lemma 4.8 that if the claim is not valid, there are sequences $(\lambda_n)_{n=1}^\infty$, $|\lambda_n|=1$, and $(k_n)_{n=1}^\infty$, $k_n\to\infty$, such that

$$\psi_\infty(z)=\lim_{n\to\infty}\psi_{\lambda_n}(z,k_n),\tag{4-53}$$

where the convergence is uniform, $z\in\mathbb{C}$, and $\psi_\infty(z)$ is not equal to z. Thus, to prove the claim, it is enough to show that any limit of form (4-53) satisfies $\psi_\infty(z)=z$. Note that by considering subsequences, we can assume that $\lambda_n\to\lambda$ and $\bar{k}_nk_n^{-1}\to\beta$ as $n\to\infty$, where $|\lambda|=|\beta|=1$. Next define $\nu_0(z)=-\lambda\beta\mu(z)$.

Let us consider the solution of

$$\bar\partial_z\Phi_\lambda(z,k)=\nu_0(\psi_\infty(z))\,e_{-k}(z)\,\partial_z\Phi_\lambda(z,k),\tag{4-54}$$

$$\Phi_\lambda(z,k)=z+O\Big(\frac{1}{z}\Big)\quad\text{as } |z|\to\infty.\tag{4-55}$$

We note that here $\nu_0(\psi_\infty(z))=0$ for $|z|>4$ as ν_0 is supported in $\bar{B}(1)$ and ψ_∞ maps $\mathbb{C}\setminus\bar{B}(4)$ into $\mathbb{C}\setminus\bar{B}(1)$. By Proposition 4.7, $\Phi_\lambda(z,k)\to z$ as $k\to\infty$ uniformly in $z\in\mathbb{C}$. Since for every $z\in\mathbb{C}$, the function $\eta_z:w\mapsto\chi_{B(4)}(w)(z-w)^{-1}$ is in $X^{-q}(\mathbb{C})$ for $q>1$, we obtain, using (4-41), that

$$\begin{aligned}|\psi_{\lambda_n}(z,k_n)-\Phi_\lambda(z,k_n)|&=\frac{1}{\pi}\bigg|\int_{B(4)}(w-z)^{-1}\,\bar\partial_w\big(\psi_{\lambda_n}(w,k_n)-\Phi_\lambda(w,k_n)\big)\,dm(w)\bigg|\\&\le\|\eta_z\|_{X^{-q}}\,\big\|\bar\partial(\psi_{\lambda_n}(\,\cdot\,,k_n)-\Phi_\lambda(\,\cdot\,,k_n))\big\|_{X^q(B(4))}.\end{aligned}\tag{4-56}$$

Let us next assume that we can prove that

$$\lim_{n\to\infty}\big\|\mu\circ\psi_{\lambda_n}(\,\cdot\,,k_n)-\mu\circ\psi_\infty(\,\cdot\,,k_n)\big\|_{L^s(\mathbb{C})}=0\quad\text{for some } s>2.\tag{4-57}$$

If this is the case, let $p\in(4q,\infty)$. By assumption (4-1) and Lemma 4.5, there is N such that the Beltrami coefficients of functions $\psi_{\lambda_n}(\,\cdot\,,k_n)$ are in $B^p_{\exp,N}(\mathbb{D})$ for all $n\in\mathbb{Z}_+$ and $p>4$. By Theorem 4.2 and (4-57),

$$\lim_{n\to\infty}\big\|\bar\partial\big(\psi_{\lambda_n}(\,\cdot\,,k_n)-\Phi_\lambda(\,\cdot\,,k_n)\big)\big\|_{X^q(\mathbb{C})}=0.$$

As $\lim_{n\to\infty}\Phi_\lambda(z,k_n)=z$ uniformly in $z\in\mathbb{C}$, this and (4-56) show that $\psi_\infty(z)=z$.

Thus, to prove the claim it is enough to show (4-57). First, as $\psi_{\lambda_n}(\,\cdot\,,k_n)\to\psi_\infty(\,\cdot\,)$ uniformly as $n\to\infty$ and as $\psi_{\lambda_n}(\,\cdot\,,k_n)$ maps $\mathbb{C}\setminus B(3)$ into $\mathbb{C}\setminus B(2)$, we see using the dominated convergence theorem that the formula (4-57) is valid when μ is replaced by a smooth compactly supported function. Next, let (F,G) be the complementary Young pair given by (4-50) and $E_F(B(R))$ be the closure of $L^\infty(B(R))$ in $X_F(B(R))$. By [Adams 1975, Theorem 8.21], the set $C_0^\infty(\mathbb{D})$ is dense in $E_F(\mathbb{D})$ with respect to the norm of X_F. Thus when μ is a nonsmooth Beltrami coefficient satisfying the assumption (4-1) and $\varepsilon > 0$, we can find a smooth function $\theta \in C_0^\infty(\mathbb{D})$, $\|\theta\|_\infty < 2$ such that $\|\mu - \theta\|_F < \varepsilon$. Then, since $|\mu - \theta|$ is supported in $\mathbb{D}$ and bounded by 3, we have

$$\begin{aligned}\big\|\mu\circ\psi_{\lambda_n}(\,\cdot\,,k_n)-\theta\circ\psi_{\lambda_n}(\,\cdot\,,k_n)\big\|^s_{L^s(\mathbb{C})} &= \int_{\mathbb{D}} |\mu(z)-\theta(z)|^s J_{g_n}(z)\,dm(z)\\ &\le 3^{s-1}\bigg(\int_{\mathbb{D}} F(|\mu(z)-\theta(z)|)\,dm(z)\bigg)\bigg(\int_{\mathbb{D}} G(J_{g_n}(z))\,dm(z)\bigg), \quad (4\text{-}58)\end{aligned}$$

where g_n is the inverse of the function $\psi_{\lambda_n}(\,\cdot\,,k_n)$. Then,

$$\int_{\mathbb{D}} G(J_{g_n})\,dm \le C\|J_{g_n}\|_{X^{1,1}(B(2))}$$

and by (4-48), $\|J_{g_n}\|_{X^{1,1}(\mathbb{D})}$ is uniformly bounded in n. Using (4-58) and (A-5), we see that (4-57) holds for all μ satisfying the assumption (4-1) and thus claim of the theorem follows. ☐

4D. $\bar{\partial}$*-equations in k-planes.* Let us consider a Beltrami coefficient $\mu \in B^p_{\exp}(\mathbb{D})$ and approximate μ with functions μ_n supported in $\mathbb{D}$ for which

$$\lim_{n\to\infty} \mu_n(z) = \mu(z) \quad \text{and} \quad \|\mu_n\|_\infty \le c_n < 1;$$

see, e.g., (3-19). Let $f_\mu(\,\cdot\,,k) \in W^{1,Q}_{\rm loc}(\mathbb{C})$ be the solution of the equations

$$\bar{\partial}_z f_\mu(z,k) = \mu(z)\,\overline{\partial_z f_\mu(z,k)} \qquad \text{for a.e. } z \in \mathbb{C}, \tag{4-59}$$

$$f_\mu(z,k) = e^{ikz}\Big(1 + \mathcal{O}_k\Big(\frac{1}{z}\Big)\Big) \quad \text{for } |z| \to \infty, \tag{4-60}$$

and $f_{\mu_n}(\,\cdot\,,k) \in W^{1,Q}_{\rm loc}(\mathbb{C})$ be the solution of the similar equations of Beltrami coefficients μ_n and μ; see Lemma 3.4. Here $\mathcal{O}_k(h(z))$ means a function of (z,k) that satisfies $|\mathcal{O}_k(h(z))| \le C(k)|h(z)|$ for all z with some constant $C(k)$ depending on $k \in \mathbb{C}$. Let

$$\varphi_\mu(z,k) = (ik)^{-1}\log(f_\mu(z,k)), \quad \varphi_{\mu_n}(z,k) = (ik)^{-1}\log(f_n(z,k));$$

see (3-5). Then by (3-7), we have

$$|\varphi_{\mu_n}(z,k)| \le |z| + 3, \quad |\varphi_\mu(z,k)| \le |z| + 3. \tag{4-61}$$

By the proof of Lemma 3.4, we see that by choosing a subsequence of μ_n, $n \in \mathbb{Z}_+$, which we continue to denote by μ_n, we can assume that

$$\lim_{n\to\infty} \varphi_{\mu_n}(z,k) = \varphi_\mu(z,k) \quad \text{uniformly in } (z,k) \in B(R)\times\{k_0\} \text{ for all } R > 0 \text{ and } k_0 \in \mathbb{C}. \tag{4-62}$$

Let us write the solutions f_{μ_n} and f_μ as

$$\begin{aligned} f_{\mu_n}(z,k) &= e^{ik\varphi_{\mu_n}(z,k)} = e^{ikz} M_{\mu_n}(z,k),\\ f_\mu(z,k) &= e^{ik\varphi_\mu(z,k)} \;= e^{ikz} M_\mu(z,k).\end{aligned}$$

Similar notations are introduced when μ is replaced by $-\mu$ etc. Let

$$h^{(+)}_{\mu_n}(z,k) = \tfrac{1}{2}\big(f_{\mu_n}(z,k) + f_{-\mu_n}(z,k)\big),$$
$$h^{(-)}_{\mu_n}(z,k) = \tfrac{1}{2}i\big(\overline{f_{\mu_n}(z,k)} - \overline{f_{-\mu_n}(z,k)}\big),$$

and

$$u^{(1)}_{\mu_n}(z,k) = h^{(+)}_{\mu_n}(z,k) - i h^{(-)}_{\mu_n}(z,k),$$
$$u^{(2)}_{\mu_n}(z,k) = -h^{(-)}_{\mu_n}(z,k) + i h^{(+)}_{\mu_n}(z,k).$$

Then by (4-61), $h^{(+)}_{\mu_n}(z,k)$ and $h^{(-)}_{\mu_n}(z,k)$ are uniformly bounded for $(z,k) \in B(R_1) \times B(R_2)$ for any $R_1, R_2 > 0$. By (4-62), we can define the pointwise limits

$$\lim_{n\to\infty} h^{(\pm)}_{\mu_n}(z,k) = h^{(\pm)}_{\mu}(z,k), \qquad \lim_{n\to\infty} u^{(j)}_{\mu_n}(z,k) = u^{(j)}_{\mu}(z,k), \qquad j = 1,2. \tag{4-63}$$

The above formulae imply

$$u^{(2)}_{\mu}(z,k) = i u^{(1)}_{-\mu}(z,k) \quad \text{and} \quad u^{(1)}_{\mu}(z,k) = -i u^{(2)}_{-\mu}(z,k). \tag{4-64}$$

Moreover, for

$$\tau_{\mu_n}(k) = \tfrac{1}{2}\big(\overline{t_{\mu_n}(k)} - \overline{t_{-\mu_n}(k)}\big), \quad \tau_{\mu_n}(k) = \tfrac{1}{2}\big(\overline{t_{\mu_n}(k)} - \overline{t_{-\mu_n}(k)}\big),$$

and

$$t_{\pm\mu_n}(k) = \frac{i}{2\pi}\int_{\partial\mathbb{D}} M_{\pm\mu_n}(z,k)\,dz, \quad t_{\pm\mu}(k) = \frac{i}{2\pi}\int_{\partial\mathbb{D}} M_{\pm\mu}(z,k)\,dz,$$

we see using the dominated convergence theorem that $\lim_{n\to\infty} t_{\mu_n}(k) = t_\mu(k)$ for all $k \in \mathbb{C}$, and hence

$$\lim_{n\to\infty} \tau_{\mu_n}(k) = \tau_\mu(k) \quad \text{for all } k \in \mathbb{C}. \tag{4-65}$$

Then, as $|\mu_n| \le c_n < 1$ correspond to conductivities σ_n satisfying $\sigma_n, \sigma_n^{-1} \in L^\infty(\mathbb{D})$, we have by [Astala and Päivärinta 2006, Formula (8.2)] the $\bar\partial$-equations with respect to the k-variables,

$$\bar\partial_k u^{(j)}_{\mu_n}(z,k) = -i\tau_{\mu_n}(k)\overline{u^{(j)}_{\mu_n}(z,k)}, \quad k \in \mathbb{C},\ j = 1,2; \tag{4-66}$$

see also [Nachman 1988; 1996] for a different formulation of such equations. For $z \in \mathbb{C}$, functions $u^{(j)}_{\mu_n}(z,\cdot)$, $n \in \mathbb{Z}_+$, are uniformly bounded in $B(R)$ for all $R > 0$; the limit (4-63) and the dominated convergence theorem imply that $u^{(j)}_{\mu_n}(z,\cdot) \to u^{(j)}_{\mu}(z,\cdot)$ as $n \to \infty$ in $L^p(B(R))$ for all $p < \infty$ and $R > 0$. Since the functions $|\tau_{\mu_n}(k)|$, $n \in \mathbb{Z}_+$, are uniformly bounded in compact sets, the pointwise limits (4-63), (4-65) and the equation (4-66) yield that

$$\bar\partial_k u^{(j)}_{\mu}(z,k) = -i\tau_\mu(k)\overline{u^{(j)}_{\mu}(z)}, \quad k \in \mathbb{C},\ j = 1,2, \tag{4-67}$$

holds for all $z \in \mathbb{C}$ in the sense of distributions and $u^{(j)}_{\mu}(z,\cdot) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for all $p < \infty$.

4E. ***Proof of uniqueness results for isotropic conductivities.***

Proof of Theorem 1.9. Let us consider isotropic conductivities σ_j, $j=1,2$. Due to the above proven results, the proof will go along the lines of Section 8 of [Astala and Päivärinta 2006], where L^∞-conductivities are considered, and its reformulation, presented in Section 18 of [Astala et al. 2009] in a quite straightforward way, when the changes explained below are made. The key point is the following proposition.

Proposition 4.10. *Assume that $\mu \in B^p_{\exp}(\mathbb{D})$ and let $f_{\pm\mu}(z,k)$ satisfy* (4-59)–(4-60) *with the Beltrami coefficients $\pm\mu$. Then $f_{\pm\mu}(z,k) = e^{izk} M_{\pm\mu}(z,k)$, where*

$$\operatorname{Re} \frac{M_{+\mu}(z,k)}{M_{-\mu}(z,k)} > 0 \tag{4-68}$$

for every $z, k \in \mathbb{C}$.

Proof. Let us consider the Beltrami coefficients $\mu_n(z)$, $n \in \mathbb{Z}_+$, defined in Section 4D that converge pointwise to $\mu(z)$ and satisfy $|\mu_n| \le c_n < 1$. By Lemma 3.2, the functions $M_{\pm\mu_n}(z,k)$ do not attain the value zero anywhere. By [Astala and Päivärinta 2006, Proposition 4.3], the inequality (4-68) holds for the functions $M_{\pm\mu_n}(z,k)$. Then, $f_{\pm\mu_n}(z,k) \to f_{\pm\mu}(z,k)$ as $n \to \infty$ for all $k, z \in \mathbb{C}$, and thus we see that

$$\operatorname{Re} \frac{M_{+\mu}(z,k)}{M_{-\mu}(z,k)} = \lim_{n\to\infty} \operatorname{Re} \frac{M_{+\mu_n}(z,k)}{M_{-\mu_n}(z,k)} \ge 0. \tag{4-69}$$

To show that the equality does not hold in (4-69), we assume the opposite. In this case, there are z_0 and k_0 such that

$$M_{+\mu}(z_0,k_0) = it M_{-\mu}(z_0,k_0) \tag{4-70}$$

for some $t \in \mathbb{R} \setminus \{0\}$. Then

$$f(z,k_0) = e^{ik_0 z}\big(M_{+\mu}(z,k_0) - it M_{-\mu}(z,k_0)\big)$$

is a solution of (4-59) and satisfies the asymptotics

$$f(z,k_0) = (1-it)e^{ik_0 z}\Big(1 + \mathcal{O}\Big(\frac{1}{z}\Big)\Big) \quad \text{for } |z| \to \infty.$$

By using (2-13) to write (4-59) in the form (3-13) and applying Lemma 3.2, we see that the solution $f(z,k_0)$ can be written in the form

$$f(z,k_0) = (1-it)e^{ik_0\varphi(z)}.$$

This is in contradiction with (4-70), which would be implied by $f(z_0,k_0)=0$, and thus proves (4-68). □

Let $f_{\pm\mu}(z,k)$ be as in Proposition 4.10 and use below for the functions defined in (4-63) the shorthand notation $u^{(1)}_\mu(z,k) = u_1(z,k)$ and $u^{(2)}_\mu(z,k) = u_2(z,k)$. Then $u_1(z,k)$ and $u_2(z,k)$ are solutions of (4-67). A direct computation shows also that

$$\nabla\cdot\sigma\nabla u_1(\,\cdot\,,k) = 0 \quad \text{and} \quad \nabla\cdot\frac{1}{\sigma}\nabla u_2(\,\cdot\,,k) = 0,$$

where

$$\sigma(z) = (1-\mu(z))/(1+\mu(z))$$

is the conductivity corresponding to μ. Note that the conductivity $1/\sigma(z) = (1+\mu(z))/(1-\mu(z))$ is the conductivity corresponding to $-\mu$.

Generally, the near field measurements, that is, the Dirichlet-to-Neumann map Λ_σ on $\partial\Omega$, determines the scattering measurements, in particular the scattered fields outside Ω; see [Nachman 1988]. In our setting, this means that we can use Lemma 5.1 and argue, e.g., as in the proof of Proposition 6.1 in [Astala and Päivärinta 2006], that Λ_σ determines uniquely the solutions $f_{\pm\mu}(z_0, k)$ and $\tau_{\pm\mu}(k)$ for $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and $k \in \mathbb{C}$. We note that a constructive method based on integral equations on $\partial\mathbb{D}$ to determine $f_{\pm\mu}(z_0, k)$ from Λ_σ is presented in [Astala et al. 2011].

As $u_j(z, \cdot)$, $j = 1, 2$, are bounded and nonvanishing functions which satisfy (4-67), we have $\bar\partial u_j(z, \cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{C})$. This implies that

$$\partial u_j(z, \cdot) \in \mathrm{BMO}_{\mathrm{loc}}(\mathbb{C}) \subset L^p_{\mathrm{loc}}(\mathbb{C}) \quad \text{for all } p < \infty$$

(see, e.g., [Astala et al. 2009, Theorem 4.6.5]), and hence $u_j(z, \cdot) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$.

Let us now consider the isotropic conductivities σ and $\tilde\sigma$ in $\Omega = \mathbb{D}$ which are equal to 1 near $\partial\mathbb{D}$ and satisfy (1-23). Assume that $\Lambda_\sigma = \Lambda_{\tilde\sigma}$. Then, by the above considerations, $\tau_{\pm\mu}(k) = \tau_{\pm\tilde\mu}(k)$ for $k \in \mathbb{C}$.

Let

$$\mu = (1-\sigma)/(1+\sigma) \quad \text{and} \quad \tilde\mu = (1-\tilde\sigma)/(1+\tilde\sigma)$$

be the Beltrami coefficients corresponding to σ and $\tilde\sigma$.

By applying Lemma 3.3 with $k = 0$, we see that $f_\mu(z, 0) = 1$ for all $z \in \mathbb{C}$ and hence $u_1(z, 0) = 1$. By Lemma 3.2, the map $z \mapsto f_\mu(z, k)$ is continuous. Thus

$$u_1 \in \mathcal{X}^p, \quad 1 < p < \infty,$$

where $\mathcal{X}^p$ is the space of functions $v(z, k)$, $(k, z) \in \mathbb{C}^2$ for which $v(z, \cdot) \in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ and $v(z, \cdot)$ are bounded for all $z \in \mathbb{C}$ and the function $v(\cdot, k)$ is continuous for all $k \in \mathbb{C}$. These properties are crucial in the following lemma, which is a reformulation of the properties of the functions $u_1(z, k)$, with $z, k \in \mathbb{C}$, proven in [Astala and Päivärinta 2006] for L^∞-conductivities.

Lemma 4.11. (i) *The functions $u_1(z, k)$ with $k \neq 0$ have the z-asymptotics*

$$u_1(z, k) = \exp(ikz + v(z; k)), \tag{4-71}$$

where $C(k) > 0$ is such that $|v(z, k)| \le C(k)$ for all $z \in \mathbb{C}$.

(ii) *The functions $u_1(z, k)$ have the k-asymptotics*

$$u_1(z, k) = \exp(ikz + k\varepsilon_\mu(k; z)), \quad k \neq 0, \tag{4-72}$$

where for each fixed z, we have $\varepsilon_\mu(k; z) \to 0$ as $k \to \infty$.

(iii) *Let $1 < p < \infty$. The $u_1(z, k)$ given in* (4-63) *is the unique function in $\mathcal{X}^p$ such that $u_1(z, k)$ is nonvanishing, $u_1(z, 0) = 1$ for all $z \in \mathbb{C}$, and $u_1(z, k)$ satisfies the $\bar\partial$-equation* (4-67) *with the asymptotics and* (4-71) *and* (4-72).

Proof. (i) Let us omit the (z,k)-variables in some expressions and define $u_1(z,k)=u_1$, $f_\mu(z,k)=f_\mu$, etc. By the definition of u_1,

$$u_1=\tfrac{1}{2}\big(f_\mu+f_{-\mu}+\overline{f_\mu}-\overline{f_{-\mu}}\big)=f_\mu\bigg(1+\frac{f_\mu-f_{-\mu}}{f_\mu+f_{-\mu}}\bigg)^{-1}\bigg(1+\frac{\overline{f_\mu}-\overline{f_{-\mu}}}{f_\mu+f_{-\mu}}\bigg), \tag{4-73}$$

where each factor is nonvanishing by Proposition 4.10. Thus (4-60) yields (4-71).

(ii) Let

$$F_t(z,k)=e^{-it/2}\Big(f_\mu(z,k)\cos\frac{t}{2}+if_{-\mu}(z,k)\sin\frac{t}{2}\Big),\quad t\in\mathbb{R}.$$

Then

$$\bar\partial_z F_t(z,k)=\mu(z)e^{-it}\,\overline{\partial_z F_t(z,k)}\quad\text{for } z\in\mathbb{C},$$
$$F_t(z,k)=e^{ikz}(1+O_k(z^{-1}))\qquad\text{as } z\to\infty.$$

Thus $F_t(z,k)=\exp(k\varphi_\lambda(z,k))$, where $\lambda=e^{-it}$ and $\varphi_\lambda(z,k)$ solves (4-43). Note that $f_\mu(z,k)=\exp(k\varphi_{\lambda_0}(z,k))$, where $\lambda_0=1$. Then

$$\frac{f_\mu-f_{-\mu}}{f_\mu+f_{-\mu}}+e^{it}=\frac{2e^{it}\,F_t}{f_\mu+f_{-\mu}}=\frac{\exp(k\varphi_\lambda(z,k))}{\exp(k\varphi_{\lambda_0}(z,k))}\,\frac{2e^{it}}{1+M_{-\mu}(z,k)/M_\mu(z,k)}. \tag{4-74}$$

By Theorem 4.9, we have, for $z\in\mathbb{C}$ and $k\in\mathbb{C}\setminus\{0\}$, that

$$e^{-|k|\varepsilon_1(k)}\le|M_{\pm\mu}(z,k)|\le e^{|k|\varepsilon_1(k)}, \tag{4-75}$$

and

$$e^{-|k|\varepsilon_2(k)}\le\inf_{|\lambda|=1}\left|\frac{\exp(k\varphi_\lambda(z,k))}{\exp(k\varphi_{\lambda_0}(z,k))}\right|\le\sup_{|\lambda|=1}\left|\frac{\exp(k\varphi_\lambda(z,k))}{\exp(k\varphi_{\lambda_0}(z,k))}\right|\le e^{|k|\varepsilon_2(k)}, \tag{4-76}$$

where $\varepsilon_j(k)\to 0$ as $k\to\infty$. Since $\mathrm{Re}(M_{-\mu}/M_\mu)>0$, estimates (4-74) and (4-75) yield for $z\in\mathbb{C}$, $k\ne 0$, that

$$\inf_{t\in\mathbb{R}}\left|\frac{f_\mu-f_{-\mu}}{f_\mu+f_{-\mu}}+e^{it}\right|\ge e^{-|k|\varepsilon(k)}\quad\text{and}\quad\frac{|f_\mu-f_{-\mu}|}{|f_\mu+f_{-\mu}|}\le 1-e^{-|k|\varepsilon(k)}.$$

This and (4-73) yield the k-asymptotics (4-72).

(iii) As observed above, the function $u_1(z,k)$ given in (4-63) satisfies the conditions stated in (iii).

Next, let $u_1(z,k)$ and $\tilde u_1(z,k)$ be two functions which satisfy the assumptions of the claim. Let us consider the logarithms

$$\delta_1(z,k)=\log u_1(z,k),\quad \tilde\delta_1(z,k)=\log\tilde u_1(z,k),\quad k,z\in\mathbb{C}.$$

As $u_1(z,\cdot)\in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ for some $p<\infty$ and $u_1(z,\cdot)$ is a bounded and nonvanishing function, we see that $\delta_1(z,\cdot)\in W^{1,p}_{\mathrm{loc}}(\mathbb{C})$. As $u_1(z,0)=1$, we have

$$\delta_1(z,0)=0\quad\text{for } z\in\mathbb{C}. \tag{4-77}$$

Moreover, $z\mapsto\delta_1(z,k)$ is continuous for any k. Let $k\ne 0$ be fixed. Then by (4-71),

$$\delta_1(z,k)=ikz+v(z,k),\quad z\in\mathbb{C}, \tag{4-78}$$

where $v(\cdot,k)$ is bounded and we see using elementary degree theory [O'Regan et al. 2006, Corollary 1.2.10] that the map $H_k:\mathbb{C}\to\mathbb{C}$, $H_k(z)=\delta_1(z,k)$, is surjective.

The function $\tilde{\delta}_1(z,k)$ has the same above properties as $\delta_1(z,k)$. Next we want to show that $\delta_1(z,k) = \tilde{\delta}_1(z,k)$ for all $z \in \mathbb{C}$ and $k \neq 0$. As the map $H_k : z \mapsto \delta_1(z,k)$ is surjective for all $k \neq 0$, this follows if we show that

$$w \neq z \text{ and } k \neq 0 \quad \Rightarrow \quad \delta_1(w,k) \neq \tilde{\delta}_1(z,k). \tag{4-79}$$

To this end, let $z, w \in \mathbb{C}$, $z \neq w$. Functions u_1 and $\tilde{u}_1$ satisfy the same equation (4-67) with the coefficient $\tau(k) = \tau_\mu(k)$. Subtracting these equations from each other, we see that the difference $g(k; w, z) = \delta_1(w,k) - \tilde{\delta}_1(z,k)$ satisfies

$$\begin{aligned} \bar{\partial}_k g(k; w, z) &= \gamma(k; w, z)\, g(k; w, z), \quad k \in \mathbb{C}, \\ \gamma(k; w, z) &= -i\tau(k) \exp\big(i \operatorname{Im} \delta_1(k; w, z)\big) E\big(i \operatorname{Im} g(k; w, z)\big), \end{aligned} \tag{4-80}$$

where

$$E(t) = (e^{-t} - 1)/t.$$

Here, $\gamma(\cdot; w, z)$ is a locally bounded function. As $w \neq z$, the principle of the argument for pseudoanalytic functions (see [Astala and Päivärinta 2006, Proposition 3.3]), (4-80), the boundedness of γ, and the asymptotics

$$g(k; w, z) = ik(w - z) + k\varepsilon(k, w, z),$$

where $\varepsilon(k, w, z) \to 0$ as $k \to \infty$, imply that $k \mapsto g(k; w, z)$ vanishes for one and only one value of $k \in \mathbb{C}$. Thus by (4-77), $g(k; w, z) = 0$ implies that $k = 0$, and hence (4-79) holds. Thus $\delta_1(z,k) = \tilde{\delta}_1(z,k)$ and $u_1(z,k) = \tilde{u}_1(z,k)$ for all $z \in \mathbb{C}$ and $k \neq 0$. □

Remark 4.12. Note that $\tau_{\pm\mu}(k)$ is determined by Λ_σ. Thus Lemma 4.11 means that $u_1(z,k)$ can be constructed as a unique complex curve $z \mapsto u_1(z, \cdot)$, $z \in \mathbb{C}$, in the space of the solutions of the $\bar{\partial}$-equation (4-67) which has the properties stated in (iii).

When $u_j(z,k)$ and $\tilde{u}_j(z,k)$, $j = 1, 2$, are functions corresponding to μ and $\tilde{\mu}$, the above shows that $u_1(z,k) = \tilde{u}_1(z,k)$. Using $\tau_{-\mu}$ instead of τ_μ and (4-64), we see by Lemma 4.11 that $u_2(z,k) = \tilde{u}_2(z,k)$ for all $z \in \mathbb{C}$ and $k \neq 0$.

Thus $f_{\pm\mu}(z,k) = f_{\pm\tilde{\mu}}(z,k)$ for all $z \in \mathbb{C}$ and $k \neq 0$. By [Astala et al. 2009, Theorem 20.4.12], the Jacobians of $f_{\pm\mu} \in W^{1,Q}_{\mathrm{loc}}(\mathbb{C})$ are nonvanishing almost everywhere. Thus we see using the Beltrami equation (4-59) and the fact that $f_{\pm\mu}(z,k) = f_{\pm\tilde{\mu}}(z,k)$ for all $z \in \mathbb{C}$ and $k \neq 0$ that $\mu = \tilde{\mu}$ almost everywhere. Hence $\sigma = \tilde{\sigma}$ a.e. This proves the claim of Theorem 1.9. □

5. Reduction of the inverse problem for an anisotropic conductivity to the isotropic case

In this section, we assume that the weight function $\mathcal{A}$ satisfies the almost linear growth condition (1-25). Let $\sigma = \sigma^{jk} \in \Sigma_{\mathcal{A}}(\mathbb{C})$ be a conductivity matrix such that $\sigma(z) = 1$ for z in $\mathbb{C} \setminus \Omega$ and in some neighborhood of $\partial\Omega$.

Let $z_0 \in \partial\Omega$, and define

$$\mathcal{H}_\sigma(z) = \int_{\eta_z} (\Lambda_\sigma(u|_{\partial\Omega}))(z')\, ds(z'), \tag{5-1}$$

where η_z is the path (oriented in the positive direction) from z_0 to z along $\partial\Omega$. This map is called the *σ-Hilbert transform*, and it can be considered a bounded map

$$\mathcal{H}_\sigma : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)/\mathbb{C}.$$

As shown in beginning of Section 2C, there exists a homeomorphism $F : \mathbb{C} \to \mathbb{C}$ such that $F(\Omega) = \tilde{\Omega}$, $\tilde{\sigma} = F_*\sigma$ is isotropic (i.e., a scalar function times the identity matrix), F and F^{-1} are $W^{1,P}$-smooth, and $F(z) = z + O(1/z)$. Moreover, F satisfies conditions $\mathcal{N}$ and $\mathcal{N}^{-1}$. Also, as $\sigma = 1$ near the boundary, we have that F and F^{-1} are C^∞-smooth near the boundary.

By the definition of $\tilde{\sigma} = F_*\sigma$, we see that

$$\det(\tilde{\sigma}(y)) = \det(\sigma(F^{-1}(y))) \tag{5-2}$$

for $y \in \widetilde{\Omega}$. Thus under the assumptions of Theorem 1.11, where $\det(\sigma), \det(\sigma)^{-1} \in L^\infty(\Omega)$, we see that the isotropic conductivity $\tilde{\sigma}$ satisfies $\tilde{\sigma}, \tilde{\sigma}^{-1} \in L^\infty(\Omega)$.

Let us next consider the case when the assumptions of Theorem 1.8 are valid and we have $\mathcal{A}(t) = pt - p$, with $p > 1$. Then, as F satisfies the condition $\mathcal{N}$, the area formula gives

$$\begin{aligned} I_1 &= \int_{\widetilde{\Omega}} \exp\biggl(\exp\biggl(q\biggl(\tilde{\sigma}(y) + \frac{1}{\tilde{\sigma}(y)}\biggr)\biggr)\biggr)\, dm(y) \\ &= \int_{\Omega} \exp\biggl(\exp\biggl(q\biggl(\det(\sigma(x))^{1/2} + \frac{1}{\det(\sigma(x))^{1/2}}\biggr)\biggr)\biggr) J_F(x)\, dm(x). \end{aligned} \tag{5-3}$$

In the case when $\mathcal{A}(t) = pt - p$, with $p > 1$, [Astala et al. 2010, Theorem 1.1] implies that

$$J_F \log^\beta(e + J_F) \in L^1(\Omega)$$

for $0 < \beta < p$. Then, Young's inequality (A-7) with the admissible pair (4-50) implies that

$$\begin{aligned} &\int_\Omega \exp\biggl(\exp\biggl(q\biggl(\det(\sigma(x))^{1/2} + \frac{1}{\det(\sigma(x))^{1/2}}\biggr)\biggr)\biggr) J_F(x)\, dm(x) \\ &\quad\le \biggl(\int_\Omega \exp\biggl(\exp\biggl(\exp\biggl(q\biggl(\det(\sigma)^{1/2} + \frac{1}{\det(\sigma)^{1/2}}\biggr)\biggr)\biggr)\biggr)\, dm\biggr)\biggl(\int_\Omega (1 + J_F)\log(1 + J_F)\, dm\biggr), \end{aligned} \tag{5-4}$$

and if conductivity σ satisfies (1-21), we see that I_1 is finite for some $q > 0$.

Thus under assumptions of Theorem 1.8, we see that I_1 is finite for the isotropic conductivity $\tilde{\sigma}$.

Let $\rho = F|_{\partial\Omega}$. It follows from Lemma 2.4 and (2-28) that $\rho_*\Lambda_\sigma = \Lambda_{\tilde{\sigma}}$. Then,

$$\mathcal{H}_{\tilde{\sigma}} h = \mathcal{H}_\sigma(h \circ \rho^{-1})$$

for all $h \in H^{1/2}(\partial\tilde{\Omega})$.

Next we seek a function $G_\Omega(z, k)$, with $z \in \mathbb{C} \setminus \Omega$, $k \in \mathbb{C}$, that satisfies

$$\bar{\partial}_z G_\Omega(z, k) = 0 \quad \text{for } z \in \mathbb{C} \setminus \overline{\Omega}, \tag{5-5}$$

$$G_\Omega(z, k) = e^{ikz}(1 + \mathcal{O}_k(z^{-1})) \quad \text{as } z \to \infty, \tag{5-6}$$

$$\operatorname{Im} G_\Omega(\,\cdot\,, k)|_{\partial\Omega} = \mathcal{H}_\sigma(\operatorname{Re} G_\Omega(\,\cdot\,, k)|_{\partial\Omega}). \tag{5-7}$$

To study it, we consider a similar function $G_{\widetilde\Omega}(\,\cdot\,,k):\mathbb{C}\setminus\widetilde\Omega\to\mathbb{C}$ corresponding to the scalar conductivity $\tilde\sigma$, which satisfies in the domain $\mathbb{C}\setminus\overline{\widetilde\Omega}$ the equations (5-5)–(5-6) and the boundary condition

$$\operatorname{Im} G_{\widetilde\Omega}(\,\cdot\,,k)=\mathcal{H}_{\tilde\sigma}(\operatorname{Re} G_{\widetilde\Omega}(\,\cdot\,,k))\qquad \text{on } z\in\partial\widetilde\Omega.$$

Below, let $\tilde\mu=(1-\tilde\sigma)/(1+\tilde\sigma)$ be the Beltrami coefficient corresponding to the conductivity $\tilde\sigma$.

Lemma 5.1. *Assume that $\sigma\in\Sigma_{\mathcal{A}}(\Omega)$ is 1 near $\partial\Omega$. Then for all $k\in\mathbb{C}$,*

(i) *For $k\in\mathbb{C}$ and $z\in\mathbb{C}\setminus\overline{\widetilde\Omega}$, we have $G_{\widetilde\Omega}(z,k)=W(z,k)$, where $W(\,\cdot\,,k)\in W^{1,P}_{\mathrm{loc}}(\mathbb{C})$ is the unique solution of*

$$\bar\partial_z W(z,k)=\tilde\mu(z)\overline{\partial_z W(z,k)}\qquad \text{for } z\in\mathbb{C},\tag{5-8}$$

$$W(z,k)=e^{ikz}(1+\mathcal{O}_k(z^{-1}))\quad \text{as } z\to\infty.\tag{5-9}$$

(ii) *The equations* (5-5)–(5-7) *have a unique solution $G_\Omega(\,\cdot\,,k)\in C^\infty(\mathbb{C}\setminus\Omega)$ and $G_\Omega(z,k)=G_{\widetilde\Omega}(F(z),k)$ for $z\in\mathbb{C}\setminus\Omega$.*

Proof. The definition of the Hilbert transform $\mathcal{H}_{\tilde\sigma}$ implies that any solution $G_{\widetilde\Omega}(z,k)$ of (5-5)–(5-7) can be extended to a solution $W(z,k)$ of (5-8). On other hand, the restriction of the solution $W(z,k)$ of (5-8)–(5-9) satisfies (5-5)–(5-7). The equations (5-8)–(5-9) have a unique solution by Theorem 3.1. As the solution $W(\,\cdot\,,k)$ is analytic in $\mathbb{C}\setminus\operatorname{supp}(\tilde\sigma)$, the claim (i) follows.

The claim (ii) follows immediately as $F:\mathbb{C}\setminus\overline\Omega\to\mathbb{C}\setminus\overline{\widetilde\Omega}$ is conformal, $F(z)=z+\mathcal{O}(1/z)$, and $\mathcal{H}_{\tilde\sigma}h=\mathcal{H}_\sigma(h\circ\rho)$ for all $h\in H^{1/2}(\partial\tilde\Omega)$. □

Lemma 5.2. *Assume that Ω is given and that $\sigma\in\Sigma_{\mathcal{A}}(\Omega)$ is 1 near $\partial\Omega$. Then the Dirichlet-to-Neumann form L_σ determines the values of the restriction $F|_{\mathbb{C}\setminus\Omega}$, the boundary $\partial\widetilde\Omega$, and the Dirichlet-to-Neumann map $\Lambda_{\tilde\sigma}$ of the isotropic conductivity $\tilde\sigma=F_*\sigma$ on $\widetilde\Omega$.*

Proof. When $\sigma=1$ near $\partial\Omega$, the Dirichlet-to-Neumann form L_σ determines the Dirichlet-to-Neumann map Λ_σ. By Lemma 3.4, we have $W(z,k)=\exp(ik\varphi(z,k))$, where by Theorem 4.9,

$$\lim_{k\to\infty}\sup_{z\in\mathbb{C}}|\varphi(z,k)-z|=0.\tag{5-10}$$

For $k\neq 0$, we choose the branch of the logarithm of $G(z,k)=W(F(z),k)$ so that it is a continuous function of $z\in\mathbb{C}\setminus\Omega$ and

$$\lim_{z\to\infty}(\log G(z,k)-ikz)=0.$$

Then,

$$\lim_{k\to\infty}(ik)^{-1}\log G(z,k)=\lim_{k\to\infty}\varphi(F(z),k)=F(z).\tag{5-11}$$

By Lemma 5.1, $G(z,k)$ can be constructed for any $z\in\mathbb{C}\setminus\Omega$ by solving the equations (5-5)–(5-9). Thus the restriction of F to $\mathbb{C}\setminus\Omega$ is determined by the values of the limit (5-11). As $\widetilde\Omega=\mathbb{C}\setminus F(\mathbb{C}\setminus\Omega)$ and $\Lambda_{\tilde\sigma}=(F|_{\partial\Omega})_*\Lambda_\sigma$, this proves the claim. □

Above we saw that if the assumptions of Theorem 1.8 for σ are satisfied then for the isotropic conductivity $\tilde\sigma=F_*\sigma$, we have $\tilde\sigma,\tilde\sigma^{-1}\in L^\infty(\widetilde\Omega)$. Also, under the assumptions of Theorem 1.8 for σ, the integral I_1 in (5-3) is finite for some $q>0$. Thus Theorems 1.8 and 1.11 follow by Theorem 1.9 and Lemma 5.2.

Appendix: Orlicz spaces

For the proofs of the facts discussed in this appendix, we refer to [Adams 1975; Krasnosel'skiĭ and Rutickiĭ 1961].

Let $F, G : [0, \infty) \to [0, \infty)$ be bijective convex functions. The pair (F, G) is called a Young complementary pair if

$$F'(t) = f(t), \quad G'(t) = g(t), \quad g = f^{-1}.$$

In the following, we will consider also extensions of these functions defined by $F, G : \mathbb{C} \to [0, \infty)$ by setting $F(t) = F(|t|)$ and $G(t) = G(|t|)$. By [Krasnosel'skiĭ and Rutickiĭ 1961, Section I.7.4], there are examples of such pairs for which

$$F(t) = \frac{1}{p} t^p \log^a t, \quad G(t) = \frac{1}{q} t^q \log^{-a} t,$$

where $p, q \in (1, \infty)$, $p^{-1} + q^{-1} = 1$ and $a \in \mathbb{R}$. We define that $u : D \to \mathbb{C}$, where $D \subset \mathbb{R}^2$, is in the *Orlicz class* $K_F(D)$ if

$$\int_{\mathbb{D}} F(|u(x)|)\, dm(x) < \infty. \tag{A-1}$$

The Orlicz space $X_F(D)$ is the smallest vector space containing the set $K_F(D)$. For a Young complementary pair (F, G), one can define for $u \in X_F(D)$ the norm

$$\|u\|_F = \sup\left\{ \int_D |u(x)v(x)|\, dm(x) \;\middle|\; \int_D G(u(x))\, dm(x) \le 1 \right\}. \tag{A-2}$$

There is also a *Luxenburg norm*

$$\|u\|_{(F)} = \inf\left\{ t > 0 \;\middle|\; \int_D F\left(\frac{u(x)}{t}\right) dm(x) \le 1 \right\}, \tag{A-3}$$

which is equivalent to the norm $\|u\|_F$, and one always has

$$\|u\|_{(F)} \le \|u\|_F \le 2\|u\|_{(F)}. \tag{A-4}$$

By [Adams 1975, Theorem 8.10], $L_X(D)$ is a Banach space with respect to the norm $\|u\|_{(F)}$. Moreover, it holds that (see [Krasnosel'skiĭ and Rutickiĭ 1961, Theorems II.9.5 and II.10.5])

$$\|u\|_{(F)} \le 1 \quad \Rightarrow \quad \int_D F(u(x))\, dm(x) \le \|u\|_F, \tag{A-5}$$

$$\|u\|_{(F)} \ge 1 \quad \Rightarrow \quad \int_D F(u(x))\, dm(x) \ge \|u\|_{(F)}. \tag{A-6}$$

We also recall Young's inequality [Krasnosel'skiĭ and Rutickiĭ 1961, Theorem II.9.3], $uv \le F(u) + G(v)$ for $u, v \ge 0$, which implies

$$\left| \int_D u(x)v(x)\, dm(x) \right| \le \|u\|_F \, \|u\|_G. \tag{A-7}$$

The set $K_F(D)$ is a vector space when F satisfies the Δ_2-condition, that is, there is $k > 1$ such that $F(2t) \le kF(t)$ for all $t \in \mathbb{R}_+$; see [Adams 1975, Lemma 8.8]. In this case, $X_F(D) = K_F(D)$.

We will use functions

$$M_{p,q}(t) = |t|^p (\log(1 + |t|))^q, \quad 1 \le p < \infty,\ q \in \mathbb{R},$$

and use for $F(t)=M_{p,q}(t)$ the notations $X_F(D)=X^{p,q}(D)$ and $\|u\|_F=\|u\|_{X^{p,q}(D)}$. For $p=2$, we define

$$M_{2,q}(t)=M_q(t), \quad X^{2,q}(D)=X^q(D).$$

Note that if D is bounded, $1<p<\infty$ and $0<\varepsilon<p-1$, then

$$L^{p+\varepsilon}(D)\subset X^{p,q}(D)\subset L^{p-\varepsilon}(D).$$

Finally, we note that the dual space of $X^q(D)$ is $X^{-q}(D)$ and

$$\left|\int_D u(x)v(x)\,dm(x)\right| \le \|u\|_{X^q(D)}\,\|v\|_{X^{-q}(D)}. \tag{A-8}$$

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