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CONSTRUCTION OF HADAMARD STATES BY CHARACTERISTIC CAUCHY PROBLEM

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We construct Hadamard states for Klein–Gordon fields in a spacetime M_0 equal to the interior of the future lightcone C from a base point p in a globally hyperbolic spacetime (M, g).

Under some regularity conditions at the future infinity of C, we identify a boundary symplectic space of functions on C, which allows us to construct states for Klein–Gordon quantum fields in M_0 from states on the CCR algebra associated to the boundary symplectic space. We formulate the natural microlocal condition on the boundary state on C, ensuring that the bulk state it induces in M_0 satisfies the Hadamard condition.

Using pseudodifferential calculus on the cone C, we construct a large class of Hadamard states on the boundary with pseudodifferential covariances and characterize the pure states among them. We then show that these pure boundary states induce pure Hadamard states in M_0.

1. Introduction

Hadamard states are widely accepted as physically admissible states for noninteracting quantum fields on a curved spacetime, one of the main reasons being their link with the renormalization of the stress–energy tensor, a basic step in the formulation of semiclassical Einstein equations. Furthermore, they are nowadays considered a necessary ingredient in the perturbative formulation of interacting (nonlinear) theories (see the recent review articles [Khavkine and Moretti 2015; Hollands and Wald 2015]).

For Klein–Gordon fields, the construction of Hadamard states amounts to finding bisolutions of the Klein–Gordon equation (called in this context *two-point functions* and denoted here by $\lambda^{\pm}$) with a specified wavefront set (that is, verifying the *microlocal spectrum condition*) and satisfying additionally a positivity property [Radzikowski 1996].

There exist several ways to construct Hadamard states for Klein–Gordon fields: the first method relies on the Fulling–Narcowich–Wald deformation argument [Fulling et al. 1981], which reduces the construction of Hadamard states on an arbitrary spacetime to the case of ultrastatic spacetimes, where vacuum or thermal states are easily shown to be Hadamard states.

The second approach, worked out in [Junker 1995; Junker and Schrohe 2002; Gérard and Wrochna 2014], uses pseudodifferential calculus on a fixed Cauchy surface Σ in (M, g) and relies on the construction of a parametrix for the Cauchy problem on Σ. To use pseudodifferential calculus, some restrictions on Σ and on the behavior of the metric g at spatial infinity are necessary. On the other hand, the method

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produces a large classes of rather explicit Hadamard states, whose covariances, expressed in terms of Cauchy data, are pseudodifferential operators.

Another method, initiated by Moretti [2006; 2008] applies to *conformal field equations*, like the conformal wave equation, on an *asymptotically flat vacuum spacetime* (M_0, g_0). By asymptotic flatness, there exists a metric $\tilde{g}_0$, conformal to g_0, and a spacetime $(M, \tilde{g})$ such that $(M_0, \tilde{g}_0)$ can be causally embedded as an open set in $(M, \tilde{g})$ with the boundary $C = \partial M_0$ of M_0 being *null* in $(M, \tilde{g})$. States on the *boundary symplectic space*, containing the traces on C of solutions of the wave equation in M_0, naturally induce states inside M_0.

This method has been successfully applied in [Moretti 2006; 2008] to construct a distinguished Hadamard state for asymptotically flat vacuum spacetimes with past time infinity and then extended to several other geometrical situations in [Dappiaggi et al. 2009; 2011; Brum and Jorás 2015]. Further results also include generalizations to Maxwell fields [Dappiaggi and Siemssen 2013] and linearized gravity [Benini et al. 2014].

In the present paper we rework the above strategy systematically in terms of the associated *characteristic Cauchy problem* in order to construct a large class of Hadamard states (instead of a preferred single one) and to characterize the pure ones. For the sake of clarity, we do not impose geometrical assumptions on M_0 that allow one to correctly embed it in a larger spacetime M.

Instead we go the other way around and work in an a priori arbitrary globally hyperbolic spacetime (M, g), fix a base point p and consider the interior of the future lightcone

$$C := \partial J^+(p)\backslash\{p\}$$

as the spacetime M_0 of main interest, that is, $M_0 := I^+(p)$, where $I^+(p)$ (resp. $J^+(p)$) is the timelike (resp. causal) shadow of p; see [Wald 1984, Section 8.1].

We make the following assumption on the geometry of C.

Hypothesis 1.1. We assume that there exists $f \in C^\infty(M)$ such that:

(1) $C \subset f^{-1}(\{0\})$, $\nabla_a f \neq 0$ on C, $\nabla_a f(p) = 0$ and $\nabla_a \nabla_b f(p) = -2g_{ab}(p)$.

(2) The vector field $\nabla^a f$ is complete on C.

Using Hypothesis 1.1 one can construct coordinates (f, s, θ) near C such that $C \subset \{f = 0\}$ and

$$g\restriction_C = -2\, df\, ds + h(s, \theta)\, d\theta^2,$$

where $h(s, \theta)\, d\theta^2$ is a Riemannian metric on $\mathbb{S}^{d-1}$.

This choice of coordinates allows one to identify C with $\widetilde{C} := \mathbb{R} \times \mathbb{S}^{d-1}$. A natural space of smooth functions on $\widetilde{C}$ is then provided by $\mathcal{H}(\widetilde{C})$ — the intersection of Sobolev spaces of all orders, defined using the standard metric $m(\theta)\, d\theta^2$ on $\mathbb{S}^{d-1}$.

We consider the Klein–Gordon operator $P = -\Box_g + r(x)$ (with $r(x) \in C^\infty(M)$ real-valued) and its restriction on M_0, denoted by $P_0 := P\restriction_{M_0}$. The *bulk-to-boundary correspondence* can be expressed in this setup as follows. For an appropriate choice of $\beta(s, \theta) \in C^\infty(M_0)$, the restriction map

$$\rho\phi := (\beta^{-1}\phi)\restriction_C, \quad \phi \in C^\infty_{\rm sc}(M_0),$$

is a monomorphism[1] between the symplectic space of smooth, space-compact solutions of P_0 (endowed with the usual symplectic form induced by the causal propagator) and $\mathscr{H}(\widetilde{C})$, equipped with the symplectic form

$$\bar{g}_1 \sigma_C g_2 := \int_{\mathbb{R}\times\mathbb{S}^{d-1}} (\partial_s \bar{g}_1 g_2 - \bar{g}_1\, \partial_s g_2)|m|^{1/2}(\theta)\, ds\, d\theta, \quad g_1, g_2 \in \mathscr{H}(\widetilde{C}). \tag{1-1}$$

Thus, a quasifree state on $(\mathscr{H}(\widetilde{C}), \sigma_C)$ with two-point functions $\lambda^\pm$ induces a unique quasifree state on the usual symplectic space associated to P_0.

Product-type pseudodifferential operators. In [Gérard and Wrochna 2014] we constructed Hadamard states whose two-point functions on a Cauchy surface Σ are pseudodifferential operators. In the present case, the obvious difference is that on the cone C the coordinate s is distinguished both from the point of view of the microlocal spectrum condition (from now on abbreviated (μsc)) and in the expression (1-1) for the symplectic form. This suggests that one should rather consider *product-type* pseudodifferential operators $\Psi^{p_1,p_2}(\widetilde{C})$ with symbols satisfying estimates

$$|\partial_s^{\alpha_1} \partial_\sigma^{\beta_1} \partial_\theta^{\alpha_2} \partial_\eta^{\beta_2} a(s, \theta, \sigma, \eta)| \in O(\langle\sigma\rangle^{p_1-|\beta_1|}\langle\eta\rangle^{p_2-|\beta_2|})$$

in the covariables $\xi = (\sigma, \eta)$ relative to the decomposition $\widetilde{C} = \mathbb{R}\times\mathbb{S}^{d-1}$. Actually, to cope with the issue that σ_C is defined using an operator $D_s := \mathrm{i}^{-1}\partial_s$ whose spectrum is not separated from $\{0\}$ (analogously to the infrared problem in massless theories), we need to introduce a larger class $\widetilde{\Psi}^{p_1,p_2}(\widetilde{C})$ that includes some operators whose symbol is discontinuous at $\eta = 0$. Namely, we set

$$\widetilde{\Psi}^{p_1,p_2}(\widetilde{C}) := \Psi^{p_1,p_2}(\widetilde{C}) + B^{-\infty}\Psi^{p_2}(\widetilde{C}),$$

where $B^{-\infty}\Psi^{p_2}(\widetilde{C})$ is the class of pseudodifferential operators of order p_2 (in the θ variables) with values in operators on $\mathbb{R}$ that infinitely increase Sobolev regularity. Then, for instance, $|D_s| \otimes \mathbb{1}_\theta \in \widetilde{\Psi}^{1,0}(\widetilde{C})$ although it is not in the pseudodifferential class $\Psi^{1,0}(\widetilde{C})$.

Summary of results. Our main results can be summarized as follows. We always assume Hypothesis 1.1. If E and F are topological vector spaces, we write $T : E \to F$ to mean $T : E \to F$ is linear and continuous.

(1) For pairs[2] of two-point functions $\lambda^\pm$ on C satisfying $\lambda^\pm : \mathscr{H}(C) \to \mathscr{H}(C)$, we give in Theorem 5.3 conditions on $\mathrm{WF}(\lambda^\pm)$ that guarantee that the corresponding two-point functions on M_0 satisfy (μsc). This is essentially an adaptation of the results of [Moretti 2008] to our framework.

(2) In Theorem 7.4 we construct a large class of Hadamard states by specifying their two-point functions $\lambda^\pm \in \widetilde{\Psi}^{0,0}(\widetilde{C})$ on the cone.

(3) In Theorem 8.2 we characterize the subclass of Hadamard states constructed in (2), which additionally are pure on the symplectic space $(\mathscr{H}(\widetilde{C}), \sigma_C)$ on the cone. It turns out that they can be parametrized by a single operator in $\widetilde{\Psi}^{-\infty,0}(\widetilde{C})$.

[1] By monomorphism of symplectic spaces we mean an injective linear map that intertwines the symplectic forms.

[2] We work with charged fields, in which case it is natural to associate a *pair* of two-point functions to a quasifree state; see Section 3B1. The charged and neutral approaches are equivalent.

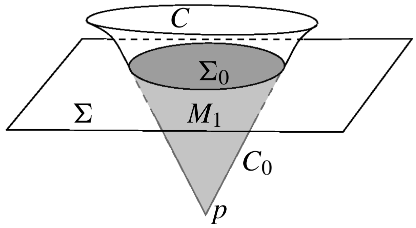

Figure 1. The Cauchy surface Σ in the future of p.

(4) In Theorem 8.4 we prove that if $\dim M \geq 4$ then the pure states considered in (3) induce pure states in the interior M_0 of the cone.

In Section 2C we argue that Hypothesis 1.1 covers the case when M_0 is an asymptotically flat vacuum spacetime with future time infinity, after a conformal transformation. Thus, our result (4) solves an open question of Moretti [2008] for $\dim M \geq 4$.

Characteristic Cauchy problem. The proof of our main result (4) relies on rather standard results on the *characteristic Cauchy problem* (also called the *Goursat problem* in the literature) in appropriate Sobolev spaces.

Let Σ be a Cauchy surface for (M, g) in the future of $\{p\}$ and $\Sigma_0 := \Sigma \cap M_0$. We set

$$M_1 := I^-(\Sigma_0; M) \cap M_0 \quad \text{and} \quad C_0 := (J^-(\Sigma_0; M) \cap C) \cup \{p\};$$

see Figure 1. M_1 is relatively compact in M with $\partial M_1 = \Sigma_0 \cup C_0$, Σ_0 and C_0 are compact in M with smooth boundary $\partial\Sigma_0 = \partial C_0$. We denote by $H_0^1(\Sigma_0)$ and $H_0^1(C_0)$ the respective restricted Sobolev spaces of order 1, i.e., the spaces of distributions in $H^1(\Sigma_0)$ and $H^1(C_0)$ that vanish on the boundary.

If $f \in H_0^1(\Sigma_0) \oplus L^2(\Sigma_0)$ is a pair of Cauchy data, we denote by $e_{\Sigma_0} f$ its extension by 0 to Σ and by $u = U_{\Sigma_0} f$ the restriction to M_1 of the solution of the Cauchy problem

$$\begin{cases} Pu = 0 & \text{in } M, \\ \rho u = e_{\Sigma_0} f & \text{on } \Sigma, \end{cases}$$

where $\rho u = (u\restriction_\Sigma, \mathrm{i}^{-1}\partial_\nu u\restriction_\Sigma)$. By standard energy estimates one obtains that

$$U_{\Sigma_0} : H_0^1(\Sigma_0) \oplus L^2(\Sigma_0) \to H^1(M_1)$$

is continuous.

In Section 8C we prove the following result.

Theorem 1.2. *The map*

$$T : H_0^1(\Sigma_0) \oplus L^2(\Sigma_0) \to H_0^1(C_0), \qquad f \mapsto (U_{\Sigma_0} f)\restriction_{C_0},$$

is a homeomorphism. Moreover, if $\dim M \geq 4$ *then* $T(C_0^\infty(\Sigma_0) \oplus C_0^\infty(\Sigma_0))$ *is dense in* $|D_s|^{-1/2} L^2(\widetilde{C})$.

The first part of Theorem 1.2 is equivalent to the existence and uniqueness of solutions in M_1 of the characteristic Cauchy problem

$$\begin{cases} Pu = 0 & \text{in } M_1, \\ u\restriction_{C_0} = \varphi, & \varphi \in H^1_0(C_0). \end{cases}$$

The proof proceeds by reduction to a case already considered by Hörmander [1990b], namely when the characteristic surface is the graph of a Lipschitz function defined on a *compact* domain. Beside [Hörmander 1990b] there is a considerable literature on the characteristic Cauchy problem for the Klein–Gordon equation, for example [Bär and Wafo 2015; Cagnac 1981; Dossa 2002; Nicolas 2006]; let us also mention related works on the Dirac equation [Nicolas 2002; Häfner and Nicolas 2011; Joudioux 2011]. The first part of Theorem 1.2 could actually also be deduced from [Bär and Wafo 2015, Theorem 23].

The second part of Theorem 1.2 asserts that there is no loss of information on the level of purity of states when going from the cone C to its interior M_0. The precise form of the statement comes from the fact that the one-particle Hilbert space associated to our Hadamard states, namely, the completion of $\mathcal{H}(\widetilde{C})$ for the inner product $(\cdot \mid (\lambda^+ + \lambda^-)\,\cdot)$, equals $|D_s|^{-1/2}L^2(\widetilde{C})$. The validity of this result appears to be very delicate; it would be for instance problematic for $|D_s|^{-\alpha}L^2(\widetilde{C})$ with $\alpha < \frac{1}{2}$ instead of $\alpha = \frac{1}{2}$ and we do not know whether it holds for $d < 3$. The generalization of Theorem 1.2 to other geometrical situations is thus an interesting open problem, particularly relevant for the quantum field theoretical bulk-to-boundary correspondence.

Plan of the paper. In Section 2 we fix the geometric setup and outline the construction of null coordinates near the cone C. In Section 3 we briefly review the Klein–Gordon field in M_0 and the definition of Hadamard states. Section 4 is devoted to the so-called bulk-to-boundary correspondence, i.e., to the definition of a convenient symplectic space $(\mathcal{H}(\widetilde{C}), \sigma_C)$ of functions on C, containing the traces on C of space-compact solutions in M_0.

In Section 5, we formulate the Hadamard condition on C, that is, the natural microlocal condition on the two-point functions of a quasifree state on $(\mathcal{H}(\widetilde{C}), \sigma_C)$ that ensures that the induced state in M_0 is a Hadamard state.

Section 6 is devoted to the pseudodifferential calculus on $\mathbb{R} \times \mathbb{S}^{d-1}$, more precisely to the "product-type" classes associated to bihomogeneous symbols. We also describe more general operator classes, which are pseudodifferential only in the variables in $\mathbb{S}^{d-1}$.

In Section 7 we construct large classes of Hadamard states on the cone, whose covariances belong to the operator classes introduced in Section 6. In Section 8 we characterize *pure* Hadamard states and show that they induce pure states in M_0. Finally in Section 9 we discuss the invariance of our classes of Hadamard states under change of null coordinates on C. Various technical results are collected in the Appendix.

2. Geometric setup

In this section we describe our geometrical setup and construct null coordinates near the cone C.

2A. ***Future lightcone.*** We consider a globally hyperbolic spacetime (M, g) of dimension $\dim M = d+1$. If $K \subset M$, then $I^{\pm}(K; M)$ and $J^{\pm}(K; M)$ denote the future/past timelike and causal, respectively, shadow of K in M; see, e.g., [Wald 1984, Chapter 8] or [Bär et al. 2007, Section 1.3] for more details. If the spacetime M is clear from the context these sets will simply be denoted by $I^{\pm}(K)$ and $J^{\pm}(K)$.

As outlined in the introduction, we fix a base point $p \in M$ and consider

$$C = \partial J^{+}(p) \backslash \{p\} \quad \text{and} \quad M_0 = I^{+}(p),$$

so that C is the future lightcone from p, with tip removed, and M_0 is the interior of C. From [Wald 1984, Section 8.1] we know that M_0 is open, with

$$\overline{M}_0 = J^{+}(p), \quad \partial M_0 = \partial J^{+}(p) = C \cup \{p\}.$$

We assume Hypothesis 1.1, i.e., that there exists $f \in C^{\infty}(M)$ such that:

(1) $C \subset f^{-1}(\{0\})$, $\nabla_a f \neq 0$ on C, $\nabla_a f(p) = 0$ and $\nabla_a \nabla_b f(p) = -2 g_{ab}(p)$.

(2) The vector field $\nabla^a f$ is complete on C.

It follows that C is a smooth hypersurface, although $\overline{C}$ is not smooth. Moreover, since C is a null hypersurface, $\nabla^a f$ is tangent to C.

2B. ***Causal structure.*** We now collect some useful results on the causal structure of M_0 and M.

Lemma 2.1. *Let $K \subset M_0$ be compact. Then*:

$$J^{-}(K) \cap J^{+}(p) \quad \textit{is compact}, \tag{2-1}$$

$$J^{+}(K) \cap C = \varnothing. \tag{2-2}$$

Proof. Equation (2-1) follows from [Bär et al. 2007, Lemma A.5.7]. Moreover, if $V \subset M_0$ is open with $K \subset V$, we have $J^{+}(K) \subset I^{+}(V) \subset M_0$. Since $\partial J^{-}(p) = \partial M_0$ and M_0 is open, this implies (2-2). □

The following lemma is due to Moretti [2006, Theorem 4.1(a)]. If $K \subset M_0$, the notation $J^{\pm}(K; M_0)$ or $J^{\pm}(K; M)$ is used in place of $J^{\pm}(K)$ to specify which causal structure one refers to.

Lemma 2.2. *The Lorentzian manifold (M_0, g) is globally hyperbolic. Moreover,*

$$J^{+}(K; M_0) = J^{+}(K; M) \quad \textit{and} \quad J^{-}(K; M_0) = J^{-}(K; M) \cap M_0 \quad \textit{for all } K \subset M_0. \tag{2-3}$$

The next proposition is also due to Moretti [2008, Lemma 4.3].

Proposition 2.3. *Let $K \subset M_0$ be compact. Then there exists a neighborhood U_1 of p in M such that no null geodesic starting from K intersects $\overline{C} \cap U_1$.*

2C. *Asymptotically flat spacetimes.* In what follows we explain the relation between Hypothesis 1.1 and the geometrical assumptions met in the literature on Hadamard states [Moretti 2006; 2008; Dappiaggi and Siemssen 2013; Benini et al. 2014].

Let us consider two globally hyperbolic spacetimes (M_0, g_0) and (M, g), where M_0 is an embedded submanifold of M. One introduces the following set of assumptions:

Hypothesis 2.4. Suppose the spacetime (M, g) is such that

(1) there exists $\Omega \in C^\infty(M)$ with $\Omega > 0$ on M_0 and $g\restriction_{M_0} = \Omega^2\restriction_{M_0} g_0$,

(2) there exists $i^- \in M$ such that $J^+(i^-; M)$ is closed and

$$M_0 = J^+(i^-; M)\backslash \partial J^+(i^-; M),$$

(3) g_0 solves the vacuum Einstein equations at least in a neighborhood of

$$\mathscr{I}^- := \partial J^+(i^-; M)\backslash\{i^-\},$$

(4) $\Omega = 0$ and $d\Omega \neq 0$ on $\mathscr{I}^-$, $d\Omega(i^-) = 0$ and $\nabla_a\nabla_b\Omega(i^-) = -2g_{ab}(i^-)$,

(5) if $n^a := g^{ab}\nabla_b\Omega$, then there exists $\omega \in C^\infty(M)$ with $\omega > 0$ on $M_0 \cup \mathscr{I}^-$ and

  (a) $\nabla_a(\omega^4 n^a) = 0$ on $\mathscr{I}^-$,

  (b) the vector field $\omega^{-1}n$ is complete on $\mathscr{I}^-$.

Above, the symbols ∇_a refer to the metric g.

One says that (M_0, g_0) is an *asymptotically flat vacuum spacetime with past time infinity* i^- if there exists a spacetime (M, g) such that M_0 is an embedded submanifold of M and Hypothesis 2.4 is satisfied.[3]

Lemma 2.5. *Suppose (M_0, g_0) is an asymptotically flat vacuum spacetime with past time infinity i^- and let (M, g) satisfy Hypothesis 2.4. Then Hypothesis 1.1 is satisfied for $p := i^-$ and $f = \omega\Omega$.*

Note that actually only conditions (1), (2), (4) and (5b) in Hypothesis 2.4 are needed in Lemma 2.5.

In the present paper we construct Hadamard states for the Klein–Gordon operator $P = -\Box_g + r(x)$ in $(M_0, g\restriction_{M_0})$ for any smooth, real-valued r. In the special case of the conformal wave operator $P = -\Box_g + (n-2)/(4(n-1))\,R$ (with R the scalar curvature) this yields, however, also Hadamard states on (M_0, g_0), since the two metrics are conformally related; see Appendix A2.

2D. *Null coordinates near C.* For later use it is convenient to introduce null coordinates near C. The construction seems to be well known; we sketch it for the reader's convenience. Note however the estimates in Lemma 2.6, which will be useful later on.

We first choose normal coordinates $(y^0, \bar{y})$ at p such that $C = \{(y^0, \bar{y}) \mid (y^0)^2 - |\bar{y}|^2 = 0,\ y^0 > 0\}$ on a neighborhood of p.

Set

$$v := y^0 + |\bar{y}|, \quad w := y^0 - |\bar{y}|, \quad \psi := \frac{\bar{y}}{|\bar{y}|} \in \mathbb{S}^{d-1}, \tag{2-4}$$

[3] Note that we consider here only globally hyperbolic spacetimes; see [Moretti 2008, Appendix A] for a more general definition.

so that on a neighborhood of p one has $C = \{w = 0,\ v > 0\}$. Abusing notation slightly, we denote by $\psi^1, \dots, \psi^{d-1}$ coordinates on $\mathbb{S}^{d-1}$ and use the same letter for their pullback to local coordinates on M near p. We set

$$S := \{w = 0,\ v = \epsilon_0\}, \tag{2-5}$$

where $\epsilon_0 > 0$ will be chosen to be small enough. Note that $S \subset C$ is diffeomorphic to $\mathbb{S}^{d-1}$.

Lemma 2.6. (1) *There exists a unique solution $s \in C^\infty(C)$ of*

$$\begin{cases} (\nabla^a f \nabla_a s)\restriction_C = -1, \\ s\restriction_S = 0. \end{cases}$$

(2) *There exists unique solutions $\theta^j \in C^\infty(C)$, $1 \le j \le d-1$, of*

$$\begin{cases} (\nabla^a f \nabla_a \theta^j)\restriction_C = 0, \\ \theta^j\restriction_S = \psi^j. \end{cases}$$

(3) *Moreover, there exists $0 < \epsilon_0 < \epsilon_1$ and $k, \tilde{\theta}^j \in C^\infty(]-\epsilon_1, \epsilon_1[\times \mathbb{S}^{d-1})$ such that*

$$s(v, \psi) = \tfrac{1}{2}\ln(v) + k(v, \psi) \quad \textit{and} \quad \theta^j(v, \psi) = \tilde{\theta}^j(v, \psi) \quad \textit{on }]0, \epsilon_0[\times \mathbb{S}^{d-1}.$$

Proof. The proof is given in Appendix A4. □

It remains to extend s and θ^j to smooth functions on a neighborhood of C.

We argue as in [Wald 1984, Section 11.1]: for $s_0 \in \mathbb{R}$, the submanifold $S_{s_0} = \{s = s_0\} \subset C$ is spacelike, of codimension 2 in M. At a given point of S_{s_0} the orthogonal to its tangent space is two-dimensional and timelike, and hence contains two null lines. One of them is generated by $\nabla^a f$; the other is transverse to C. We extend (s, θ) to a neighborhood of C by imposing that (s, θ) are constant along the above family of null geodesics, transverse to C.

Lemma 2.7. *The functions (f, s, θ) constructed above are a system of local coordinates near C with $C \subset \{f = 0\}$ and*

$$g\restriction_C = -2\,df\,ds + h_{ij}(s, \theta)\,d\theta^i\,d\theta^j, \tag{2-6}$$

where $h_{ij}(s, \theta)\,d\theta^i\,d\theta^j$ is a smooth, s-dependent Riemannian metric on $\mathbb{S}^{d-1}$.

Proof. The proof will be given in Appendix A3. □

2E. *Estimates on traces.* In this subsection we derive estimates, in the coordinates (s, θ) on C constructed above, for the restriction to C of a smooth, space-compact function in M. These estimates will be applied later to traces on C of solutions of the Klein–Gordon equation in M_0.

We recall that $C^\infty_{\mathrm{sc}}(M)$ denotes the space of smooth *space compact* functions, i.e., the space of $\phi \in C^\infty(M)$ such that $\operatorname{supp}\phi \subset J^+(K) \cup J^-(K)$ for some compact $K \subset M$.

We will slightly abuse notation by writing $\phi(x^0, \dots, x^d)$ for the function ϕ expressed in some coordinate system $(x^0, \dots, x^d)$ near p. We will similarly write, for example, $\phi(v, \psi)$ or $\phi(s, \theta)$ for $\phi \in C^\infty(C)$.

By Lemma 2.1 we see that $\operatorname{supp}\phi \cap \overline{C}$ is compact in $\overline{C}$ if $\phi \in C^\infty_{\mathrm{sc}}(M)$. This means that it suffices to control the derivatives in (s, θ) of $\phi\restriction_C\ (s, \theta)$ near $s = -\infty$, that is, of $\phi\restriction_C\ (v, \psi)$ near $v = 0$. Clearly the only task is to control what happens near p, that is, when $s \to -\infty$. We first derive estimates in the coordinates (v, ψ) introduced in (2-4) in a neighborhood of $v = 0$. If $\phi \in C^\infty_{\rm sc}(M)$ we denote by $\phi(y^0, \bar{y})$ the function ϕ expressed in normal coordinates at p, which is defined on a neighborhood of 0. We then set

$$\hat{\phi}(v, \psi) = \phi\big(\tfrac{1}{2}v, \tfrac{1}{2}v\psi\big) \in C^\infty(]-\epsilon_1, \epsilon_1[\times \mathbb{S}^{d-1}) \quad \text{for some } \epsilon_1 > 0,$$

so that

$$\phi\restriction_C = \hat{\phi}\restriction_{\{v>0\}}.$$

We denote by S^0 the space of functions $u(v, \psi) \in C^\infty(]-\epsilon_1, \epsilon_1[\times \mathbb{S}^{d-1})$ which are bounded with all derivatives.

Lemma 2.8. (1) *If $\phi \in C^\infty_{\rm sc}(M)$ then $\hat{\phi}(v, \psi)$ belongs to S^0.*

(2) *Let $|h| = \det[h_{ij}]$. Then $|h|(v, \psi) = v^{2(d-1)} r_0(v, \psi)$ for $r_0, r_0^{-1} \in S^0$.*

Proof. Considering the map $\chi : \mathbb{S}^{d-1} \to \mathbb{R}^d$, $\psi \mapsto \psi$, and still denoting by ψ some coordinates, on $\mathbb{S}^{d-1}$ we have

$$\partial_v \tilde{\phi} = \tfrac{1}{2}(\partial_{y^0}\phi - \psi \cdot \partial_{\bar{y}}\phi) \quad \text{and} \quad \partial_{\psi^i}\tilde{\phi} = \tfrac{1}{2} v \partial_{\psi^i}\chi^j \, \partial_{\bar{y}^j}\phi.$$

From this we obtain (1). To prove (2) we need to express $h_{ij} = \langle \partial_{\theta^i} \mid g\, \partial_{\theta^j} \rangle$ on C. An easy computation using the estimates in Lemma 2.6 shows that on C we have

$$\partial_{\theta^i} = a_i^j(v, \psi)\, \partial_{\psi^j} + v r_0(v, \psi)\, \partial_v,$$

where $a_i^j, r_0 \in S^0$ and $[a^{ij}](v, \psi)$ is invertible. Plugging this into (A-9), we obtain

$$[h_{ij}](v, \psi) = v^2\big({}^t[a_i^j](v, \psi)[m_{ij}](\psi)[a_i^j](v, \psi) + v[b_{ij}](v, \psi)\big),$$

where $b_{ij} \in S^0$. This implies (2). □

Later we will also need the following lemma. We denote by $m_{ij}(\theta)\, d\theta^i\, d\theta^j$ the standard Riemannian metric on $\mathbb{S}^{d-1}$ and set

$$\beta(s, \theta) := |m|^{1/4}(\theta)|h|^{-1/4}(s, \theta). \tag{2-7}$$

Lemma 2.9. *Let*

$$\tilde{\phi}(s, \theta) := \beta^{-1}(s, \theta)\phi\restriction_C(s, \theta), \quad \phi \in C^\infty_{\rm sc}(M).$$

Then for all $s_1 \in \mathbb{R}$ one has

$$\partial_s^\alpha \partial_\theta^\beta \tilde{\phi} \in O(\mathrm{e}^{s(d-1)}), \quad s \in]-\infty, s_1], \quad \textit{for all } \alpha, \beta.$$

Proof. We note that $\beta^{-1} = v^{(d-1)/2} r_0(v, \psi)$, for $r_0, r_0^{-1} \in S^0$. From this and Lemma 2.8, it follows that if $\phi \in C_0^\infty(M)$ then $\tilde{\phi}(v, \psi) \in v^{(d-1)/2} S^0$. It remains to estimate the derivatives of $\tilde{\phi}$ with respect to s and θ. By a standard computation we obtain, for $u \in C^\infty(]-\epsilon_1, \epsilon_1[\times \mathbb{S}^{d-1})$,

$$\partial_{\theta^i} u = a_i^j(v, \psi)\, \partial_{\psi^j} u + v r_i(v, \psi)\, \partial_v u, \quad \text{and} \quad \partial_s u = v(1 + v r_0(v, \psi))\, \partial_v u + v b^j(v, \psi)\, \partial_{\psi^j} u$$

for $r_0, r_i, b^j, a_i^j \in S^0$ and $[a_i^j]$ invertible. From this point on the lemma is a routine computation. □

3. Klein–Gordon fields inside the future lightcone

3A. ***Klein–Gordon equation in M_0.*** We fix a smooth real function $r \in C^\infty(M)$ and consider the *Klein–Gordon operator* on (M, g)

$$P(x, D_x) = -\nabla^a \nabla_a + r(x) \quad \text{acting on } C^\infty(M).$$

We denote by $E_\pm$ in $\mathscr{D}'(M \times M)$ the retarded and advanced Green's functions for P, by $E = E_+ - E_-$ in $\mathscr{D}'(M \times M)$ the Pauli–Jordan commutator function and by $\mathrm{Sol}_{\mathrm{sc}}(P)$ the space of smooth, complex-valued, space-compact solutions of

$$P(x, D_x)\phi = 0 \quad \text{in } M.$$

Recall that we have set in Section 2A

$$M_0 := I^+(p)$$

and, by Lemma 2.2, we know that (M_0, g) is globally hyperbolic.

We denote by $P_0 = -\nabla^a \nabla_a + r(x)$ the restriction of P to M_0, by $E_0 \in \mathscr{D}'(M_0 \times M_0)$ the Pauli–Jordan function for P_0 and by $\mathrm{Sol}_{\mathrm{sc}}(P_0)$ the space of smooth, complex-valued, space-compact solutions of

$$P_0(x, D_x)\phi_0 = 0 \quad \text{in } M_0.$$

By the global hyperbolicity of (M_0, g) we know that $\mathrm{Sol}_{\mathrm{sc}}(P_0) = E_0 \mathscr{D}(M_0)$. From (2-3) and the uniqueness of $E_{0\pm}$ we obtain that $E_{0\pm} = E_\pm\restriction_{M_0 \times M_0}$; hence,

$$E_0 = E\restriction_{M_0 \times M_0} .$$

It follows that any $\phi_0 \in \mathrm{Sol}_{\mathrm{sc}}(P_0)$ uniquely extends to $\phi \in \mathrm{Sol}_{\mathrm{sc}}(P)$; in fact,

$$\phi_0 = E_0 f_0, \;\; f_0 \in \mathscr{D}(M_0) \quad \Longrightarrow \quad \phi_0 = E f_0\restriction_{M_0} . \tag{3-1}$$

As usual we equip $\mathrm{Sol}_{\mathrm{sc}}(P_0)$ with the symplectic form

$$\bar{\phi}_1 \sigma_0 \phi_2 := \int_{\Sigma_0} \overline{\nabla_a \phi_1} \phi_2 - \bar{\phi}_1 \nabla_a \phi_2 n^a \, d\sigma_h, \tag{3-2}$$

where $\Sigma_0 \subset M_0$ is a Cauchy hypersurface for (M_0, g) (see Appendix A1 for notation). It is well known that

$$E_0 : (C_0^\infty(M_0)/P_0 C_0^\infty(M_0), E_0) \to (\mathrm{Sol}_{\mathrm{sc}}(P_0), \sigma_0)$$

is a symplectomorphism.

3B. ***Hadamard states in M_0.*** We first briefly recall some standard facts and refer, for example, to [Gérard and Wrochna 2014, Section 2] for details and notation.

3B1. *Covariances of a quasifree state.* If $(\mathcal{Y}, \sigma)$ is a complex symplectic space, the *complex covariances* $\Lambda^{\pm} \in L_{\mathrm{h}}(\mathcal{Y}, \mathcal{Y}^*)$ of a (gauge-invariant) quasifree state ω on $\mathrm{CCR}(\mathcal{Y}, \sigma)$ (the polynomial CCR *-algebra of $(\mathcal{Y}, \sigma)$) are defined by

$$\omega(\psi(y_1)\psi^*(y_2)) =: (y_1 \mid \Lambda^+ y_2), \quad \omega(\psi^*(y_2)\psi(y_1)) =: (y_1 \mid \Lambda^- y_2), \quad y_1, y_2 \in \mathcal{Y}.$$

From the CCR we obtain that $\Lambda^+ - \Lambda^- = \mathrm{i}\sigma =: q$, and the necessary and sufficient condition for $\Lambda^{\pm}$ to be the complex covariances of a (gauge-invariant) quasifree state is that $\Lambda^{\pm} \geq 0$.

If $(\mathcal{Y}, \sigma) = (C_0^\infty(M_0)/PC_0^\infty(M_0), E_0)$, the complex covariances of a state ω are induced from *two-point functions*, still denoted by $\Lambda^{\pm}$, such that

$$\Lambda^{\pm} \in \mathcal{D}'(M_0 \times M_0), \quad P\Lambda^{\pm} = \Lambda^{\pm} P = 0,$$

where we identify operators on $C_0^\infty(M_0)$ with sesquilinear forms using the scalar product

$$(u \mid v) := \int_{M_0} \bar{u} v \, d\mu_g, \quad u, v \in C_0^\infty(M_0).$$

3B2. *Hadamard condition.* We now recall the *Hadamard condition* for quasifree states. We denote by T^*M the cotangent bundle of M and by $Z = \{(x, 0)\} \subset T^*M$ the zero section. The *principal symbol* of P is $p(x, \xi) = \xi_a g^{ab}(x) \xi_b$; the set

$$\mathcal{N} := \{(x, \xi) \in T^*M \backslash Z : p(x, \xi) = 0\}$$

is called the *characteristic manifold* of p.

The Hamilton vector field of p will be denoted by H_p, whose integral curves inside $\mathcal{N}$ are called *bicharacteristics*.

We will use the notation $X = (x, \xi)$ for points in $T^*M \backslash Z$ and write $X_1 \sim X_2$ if $X_1 = (x_1, \xi_1)$ and $X_2 = (x_2, \xi_2)$ are in $\mathcal{N}$ and X_1 and X_2 lie on the same bicharacteristic of p.

Let us fix a time orientation and denote by $V_{x\pm} \subset T_x M$ for $x \in M$ the open future/past lightcones and $V_{x\pm}^*$ the dual cones

$$V_x^{*\pm} := \{\xi \in T_x^* M : \xi \cdot v > 0 \text{ for all } v \in V_{x\pm} \text{ with } v \neq 0\}.$$

The set $\mathcal{N}$ has two connected components invariant under the Hamiltonian flow of p, namely

$$\mathcal{N}^{\pm} := \{X \in \mathcal{N} : \xi \in V_x^{*\pm}\}.$$

Definition 3.1. A quasifree state ω on $\mathrm{CCR}(C_0^\infty(M_0)/PC_0^\infty(M_0), E_0)$ with two-point functions $\Lambda^{\pm}$ satisfies the *microlocal spectrum condition* if

$$\mathrm{WF}(\Lambda^{\pm})' \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm}. \tag{μsc}$$

Quasifree states satisfying (μsc) are called *Hadamard states*.

This form of the Hadamard condition was shown in [Sahlmann and Verch 2001] to be equivalent to older definitions [Radzikowski 1996]; we refer the reader to [Sanders 2010; Wrochna 2013] for a discussion on equivalent formulations of the microlocal spectrum condition.

4. Bulk-to-boundary correspondence

4A. ***Boundary symplectic space.*** We equip C with the coordinates (s,θ) constructed in Section 2D and hence identify C with

$$\widetilde{C} := \mathbb{R}\times\mathbb{S}^{d-1}. \tag{4-1}$$

We denote by $H^k(\widetilde{C})$, $k\in\mathbb{N}$, the Sobolev space

$$H^k(\widetilde{C}) := \left\{ g\in\mathscr{D}'(\mathbb{R}\times\mathbb{S}^{d-1}) : \int |\partial_s^\alpha\partial_\theta^\beta g|^2 |m|^{1/2}\, ds\, d\theta < \infty,\ \alpha+|\beta|\le k \right\},$$

and extend the definition of $H^k(\widetilde{C})$ to $k\in\mathbb{R}$ in the usual way. The space $H^0(\widetilde{C})$ will be denoted simply by $L^2(\widetilde{C})$. We set also

$$\mathscr{H}(\widetilde{C}) := \bigcap_{k\in\mathbb{R}} H^k(\widetilde{C}) \quad\text{and}\quad \mathscr{H}'(\widetilde{C}) := \bigcup_{k\in\mathbb{R}} H^k(\widetilde{C}),$$

equipped with their canonical topologies.

We set

$$\bar{g}_1\sigma_C g_2 := \int_{\mathbb{R}\times\mathbb{S}^{d-1}} (\partial_s\bar{g}_1 g_2 - \bar{g}_1\,\partial_s g_2)|m|^{1/2}(\theta)\, ds\, d\theta, \quad g_1, g_2\in\mathscr{H}(\widetilde{C}). \tag{4-2}$$

Introducing the *charge* $q := \mathrm{i}\sigma_C$ we have

$$\bar{g}_1 q g_2 = 2(g_1 \mid D_s g_2)_{L^2(\widetilde{C})}, \quad g_1, g_2\in\mathscr{H}(\widetilde{C}),$$

where $D_s = \mathrm{i}^{-1}\partial_s$ is selfadjoint on $L^2(\widetilde{C})$ on its natural domain. Clearly $(\mathscr{H}(\widetilde{C}),\sigma_C)$ is a complex symplectic space.

4B. ***Bulk-to-boundary correspondence.***

Definition 4.1. Let $\beta\in C^\infty(\widetilde{C})$ be as defined in (2-7). We set

$$\rho : \mathrm{Sol}_{\mathrm{sc}}(P_0)\to C^\infty(\mathbb{R}\times\mathbb{S}^{d-1}), \quad \phi\mapsto \beta^{-1}(s,\theta)\phi\restriction_C(s,\theta).$$

Proposition 4.2. (1) *ρ maps $\mathrm{Sol}_{\mathrm{sc}}(P_0)$ into $\mathscr{H}(\widetilde{C})$.*

(2) *$\rho : (\mathrm{Sol}_{\mathrm{sc}}(P_0),\sigma)\to(\mathscr{H}(\widetilde{C}),\sigma_C)$ is a monomorphism, i.e.,*

$$\overline{\rho\phi}_1\sigma_C\rho\phi_2 = \bar{\phi}_1\sigma\phi_2 \quad \textit{for all } \phi_1,\phi_2\in\mathrm{Sol}_{\mathrm{sc}}(P_0).$$

Proof. Let ϕ_0 and ϕ be as in (3-1). By Lemma 2.1 and the support properties of E, we see that $\mathrm{supp}\,\phi\cap\overline{C}$ is compact in M. Therefore the restriction of ϕ to C equals the restriction of a smooth, compactly supported function to C. By Lemma 2.9 and the fact that $\rho\phi_0$ is supported in $]-\infty, s_1[\,\times\mathbb{S}^{d-1}$ for some s_1, we obtain that $\rho\phi_0\in\mathscr{H}(\widetilde{C})$, which proves (1).

We now prove (2). Let $\phi_{i,0}\in\mathrm{Sol}_{\mathrm{sc}}(P_0)$, $i=1,2$, be restrictions to M_0 of $\phi_i\in\mathrm{Sol}_{\mathrm{sc}}(P)$. We fix a Cauchy surface Σ_0 for (M_0,g) such that $\mathrm{supp}\,\phi_{i,0}\cap\Sigma_0\subset K\Subset M_0$. We can find a Cauchy surface Σ for (M,g) such that $\Sigma\cap K = \Sigma_0\cap K$. Denoting by

$$J_a(\phi_1,\phi_2) := \bar{\phi}_1\nabla_a\phi_2 - \overline{\nabla_a\phi_1}\phi_2,$$

the conserved current, we have

$$\bar{\phi}_{1,0}\sigma_0\phi_{2,0} = \bar{\phi}_1\sigma\phi_2,$$

where

$$\bar{\phi}_1\sigma\phi_2 = -\int_\Sigma J_a(\phi_1,\phi_2)n^a\,d\sigma_h$$

is the symplectic form on $\mathrm{Sol}_{\mathrm{sc}}(P)$. We now apply Stokes formula in the form (A-6) to the domain $U\subset M$ bounded by $\Sigma\cap K$, $\overline{C}$ and $\partial J^+(\Sigma\cap K)$, using that $\nabla_a J^a(\phi_1,\phi_2)=0$. The boundary term on $\Sigma\cap K$ yields $-\bar{\phi}_1\sigma\phi_2$; the boundary term on $\partial J^+(\Sigma\cap K)$ vanishes. To express the boundary term on $\overline{C}$, we use the coordinates (f,s,θ) constructed in Section 2D. We formally obtain the quantity

$$\bar{g}_1\hat{\sigma}g_2 = \int_{\mathbb{R}\times\mathbb{S}^{d-1}}(\partial_s\bar{g}_1 g_2 - \bar{g}_1\,\partial_s g_2)|h|^{1/2}(s,\theta)\,ds\,d\theta$$

for $g_i = (\phi_i)\restriction_C$. This equals $\overline{\rho\phi}_1\sigma_C\rho\phi_2$ by an easy computation.

To justify the use of Stokes formula, we need to take care of the fact that $\overline{C}$ is not smooth at p. This can be done as follows: for $0<\epsilon\ll 1$, we denote by U_ϵ some ϵ-neighborhood of p. We replace $\overline{C}$ by a smooth hypersurface C_ϵ, obtained by smoothly gluing $C\backslash U_\epsilon$ to a piece of a Cauchy surface Σ'_ϵ passing through U_ϵ. The contribution of the integral on Σ_ϵ is written using (A-4) and converges to 0 when $\epsilon\to 0$, using that ϕ_i are smooth functions. The contribution of the integral on $C\backslash U_\epsilon$ converges to $\overline{\rho\phi}_1\sigma_C\rho\phi_2$, using that $\rho\phi_i\in\mathscr{H}(\widetilde{C})$. This completes the proof of the proposition. ☐

4C. *Pullback of states from the boundary.* Since

$$\rho : (\mathrm{Sol}_{\mathrm{sc}}(P_0),\sigma_0)\to(\mathscr{H}(\widetilde{C}),\sigma_C)$$

is a monomorphism, we can pull back a quasifree state ω_C on $\mathrm{CCR}(\mathscr{H}(\widetilde{C}),\sigma_C)$ to a quasifree state ω_0 on $\mathrm{CCR}(C_0^\infty(M_0)/P_0C_0^\infty(M_0),E_0)$ by setting

$$\omega_0(\psi(u_1)\psi^*(u_2)) := \omega_C\,(\psi(\rho\circ E_0u_1)\psi^*(\rho\circ E_0u_2)),\quad u_1,u_2\in C_0^\infty(M_0). \tag{4-3}$$

If $\lambda^\pm\in L_{\mathrm{h}}(\mathscr{H}(\widetilde{C}),\mathscr{H}(\widetilde{C})^*)$ are the complex covariances of ω_C, then the complex covariances of ω_0 are (formally) given by

$$\Lambda^\pm := (\rho\circ E_0)^*\circ\lambda^\pm\circ(\rho\circ E_0). \tag{4-4}$$

5. Hadamard condition on the cone

In this section we formulate the natural boundary version of the bulk Hadamard condition (μsc).

5A. *Preparations.* We recall that $p(x,\xi)$ denotes the principal symbol of the Klein–Gordon operator P (or P_0).

Let $C\subset M$ be the forward lightcone introduced in Section 2A. We denote by $N^*C\subset T^*M\backslash Z$ the *conormal bundle* to C, namely,

$$N^*C := \{(x,\xi)\in T^*M\backslash Z : x\in C \text{ and } \xi=0 \text{ on } T_xC\}.$$

The fact that C is characteristic is equivalent to

$$N^*C \subset \mathcal{N}, \tag{5-1}$$

where $\mathcal{N}$ is the characteristic manifold of p. Since N^*C is Lagrangian, it is well known that (5-1) implies that N^*C is invariant under the flow of H_p. The projections on M of bicharacteristics starting from N^*C are (modulo reparametrization) *characteristic curves*, i.e., integral curves of the vector field $v^a = \nabla^a f$ if $f \in C^\infty(M)$ is some defining function of C, that is, $f = 0$ and $df \neq 0$ on C.

We will use the coordinates (f, s, θ) introduced in Section 2D, which, for ease of notation, will be denoted by $x = (r, s, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{d-1}$. The dual coordinates are denoted by $\xi = (\varrho, \sigma, \eta)$, elements of T^*M will sometimes be denoted by $X = (x, \xi)$ and elements of T^*C will be denoted by $Y = ((s, y), (\sigma, \eta))$.

In the above coordinates, we have

$$C = \{r = 0\} \quad \text{and} \quad N^*C = \{r = 0,\ \sigma = \eta = 0\}$$

and, from (2-6), we obtain that

$$p(x, \xi)\restriction_C = -2\varrho\sigma + h(s, y, \eta), \tag{5-2}$$

where we set $h(s, y, \eta) = h^{ij}(0, s, y)\eta_i\eta_j$. Note that $h(s, y, \eta)$ is *elliptic*, that is, $h(s, y, \eta) \geq c_0|\eta|^2$ for $c_0 > 0$, locally in (s, y), since $h_{ij}\, dy^i\, dy^j$ is Riemannian.

For later use let us extend the notation $X_1 \sim X_2$ introduced in Section 3B2. For $Y = (s, y, \sigma, \eta) \in T^*C$ and $X = (x, \xi) \in T^*M$, we will write $Y \sim X$ if

$$\sigma \neq 0 \quad \text{and} \quad ((0, s, y), ((2\sigma)^{-1}h(s, y, \eta), \sigma, \eta)) \sim X. \tag{5-3}$$

Recall also that the positive/negative energy components $\mathcal{N}^\pm$ of $\mathcal{N}$ were defined in Section 3B2.

Lemma 5.1. *Let $Y_1 = (s_1, y_1, \sigma_1, \eta_1) \in T^*C$ and $X_2 = (x_2, \xi_2) \in T^*M$ with $x_2 \notin C$. Then*:

(1) *There exists $\varrho_1 \in \mathbb{R}$ such that*

$$X_1 := ((0, s_1, y_1), (\varrho_1, \sigma_1, \eta_1)) \sim (x_2, \xi_2) =: X_2$$

if and only if $\sigma_1 \neq 0$, in which case $\varrho_1 = (2\sigma_1)^{-1}h(s_1, y_1, \eta_1)$ and $Y_1 \sim X_2$.

(2) *If $Y_1 \sim X_2$, then $X_2 \in \mathcal{N}^\pm$ if and only if $\pm\sigma_1 > 0$.*

Proof. Let $X_1 = ((0, s_1, y_1), (\varrho_1, \sigma_1, \eta_1)) \in \mathcal{N}$. By (5-2) we have

$$-2\varrho_1\sigma_1 + h(s_1, y_1, \eta_1) = 0.$$

If $\sigma_1 = 0$ then $h(s_1, y_1, \eta_1) = 0$, hence $\eta_1 = 0$ by ellipticity of h. Therefore $\sigma_1 = 0$ implies $X_1 \in N^*C$. Since $X_2 \sim X_1$ and N^*C is invariant under the flow of H_p, we also have $X_2 \in N^*C$, which contradicts the hypothesis that $x_2 \notin C$. Therefore, necessarily $\sigma_1 \neq 0$, and hence $\varrho_1 = (2\sigma_1)^{-1}h(s_1, y_1, \eta_1)$ and $Y_1 \sim X_2$. This proves (1).

To prove (2) we have to show that

$$\pm\sigma_1 > 0 \iff ((0, s_1, y_1), ((2\sigma_1)^{-1}h(s_1, y_1, \eta_1), \sigma_1, \eta_1)) \in \mathcal{N}^{\pm}. \tag{5-4}$$

Let us fix $(y_1, \eta_1) \in T^*\mathbb{S}^{d-1}$ and $\sigma_1 \in \mathbb{R}$. Since $\mathcal{N}^{\pm}$ are the two connected components of $\mathcal{N}$, it suffices, by connectivity, to prove (5-4) for s_1 in a neighborhood of $-\infty$, i.e., in a neighborhood of p in M. Recall that we introduced Gaussian normal coordinates $(y^0, \bar{y})$ near p with ∂_{y^0} future oriented. Let α be the one-form $(2\sigma_1)^{-1}h(s_1, y_1, \eta_1)\,dr + \sigma_1\,ds + \eta_1\,dy$. Then

$$((0, s_1, y_1), ((2\sigma_1)^{-1}h(s_1, y_1, \eta_1), \sigma_1, \eta_1)) \in \mathcal{N}^{\pm} \iff \mp\langle\alpha \mid g^{-1}\,dy^0\rangle > 0.$$

Since it suffices to check the sign of $\langle\alpha \mid g^{-1}\,dy^0\rangle$ near p, we can, by a simple approximation argument (see, e.g., (A-9)) replace g by the flat metric at p. We then have — see Lemma 2.6 and recall that $s = u$ and $r = f$ —

$$y^0 = v + w, \quad v = \mathrm{e}^s, \quad w = \mathrm{e}^{-s}r,$$

hence

$$\mp\langle\alpha \mid g^{-1}\,dy^0\rangle = \pm 2(\mathrm{e}^{-s_1}\sigma_1 + \mathrm{e}^{s_1}(2\sigma_1)^{-1}h(s_1, y_1, \eta_1))$$

has the same sign as $\pm\sigma_1$, which proves (5-4). □

Recall that $E \in \mathcal{D}'(M \times M)$ is the Pauli–Jordan commutator function for P and $\rho : \mathcal{D}(M) \to C^\infty(\widetilde{C})$, $u \mapsto u\restriction_C$, is (modulo a smooth, nonzero multiplicative factor) the operator of restriction to C, defined in Definition 4.1.

Let us recall some notation: identifying $T^*(M_1 \times M_2)$ with $T^*M_1 \times T^*M_2$, we write $(T^*M_1 \times T^*M_2)\backslash Z$ for the image of $T^*(M_1 \times M_2)\backslash Z$ under this identification. If $\Gamma \subset (T^*M_1 \times T^*M_2)\backslash Z$, one sets

$$\begin{aligned} {}_{M_1}\Gamma &:= \{(x_1, \xi_1) : (x_1, \xi_1, x_2, 0) \in \Gamma \text{ for some } x_2\} \subset T^*M_1\backslash Z_1, \\ \Gamma_{M_2} &:= \{(x_2, \xi_2) : (x_1, 0, x_2, \xi_2) \in \Gamma \text{ for some } x_1\} \subset T^*M_2\backslash Z_2, \end{aligned} \tag{5-5}$$

where Z_i is the zero section of T^*M_i.

Proposition 5.2. *Let $\chi \in C_0^\infty(M)$ with* $\operatorname{supp}\chi \subset M\backslash C$ *and $\psi \in C_0^\infty(\widetilde{C})$. Then*:

(1) $\mathrm{WF}(\psi\rho \circ E\chi)' \subset \{(Y_1, X_2) : y_1 \in \operatorname{supp}\psi,\ x_2 \in \operatorname{supp}\chi,\ Y_1 \sim X_2\}$, *where the notation $Y \sim X$ is as defined in* (5-3).

(2) $\psi\rho \circ E\chi : \mathcal{D}(M) \to \mathcal{D}(\widetilde{C})$ *extends continuously as* $\psi\rho \circ E\chi : \mathcal{D}'(M) \to \mathcal{D}'(\widetilde{C})$.

Proof. It is well known that

$$\begin{aligned} \operatorname{supp} E &\subset \{(x_1, x_2) : x_1 \in J(x_2)\}, \\ \mathrm{WF}(E)' &= \{(X_1, X_2) \in \mathcal{N} \times \mathcal{N} :\ X_1 \sim X_2\}. \end{aligned} \tag{5-6}$$

On the other hand, the distributional kernel of ρ equals

$$\delta(r_2) \otimes \delta(s_1, y_1, s_2, y_2)\beta^{-1}(s_1, y_1) \in \mathcal{D}'(\widetilde{C} \times M).$$

It follows that

$$\mathrm{WF}(\rho)' = \{(Y_1, X_2) : r_2 = 0,\ (s_1, y_1) = (s_2, y_2),\ (\sigma_1, \eta_1) = (\sigma_2, \eta_2),\ (\sigma_2, \eta_2) \neq (0, 0)\}. \tag{5-7}$$

Since $E : \mathscr{D}(M) \to \mathscr{E}(M)$, we see that $\psi\rho \circ E\chi : \mathscr{D}(M) \to \mathscr{D}(\widetilde{C})$. Moreover, there exists $\chi_1 \in C_0^\infty(M)$ such that $\psi\rho \circ E\chi = \psi\rho \circ \chi_1 E\chi$. We then have

$$_{\widetilde{C}}\mathrm{WF}(\rho)' = \mathrm{WF}(E)'_M = \varnothing$$

and it follows from [Hörmander 1990a, Chapter 8] and (5-6)–(5-7) that

$$\begin{aligned}\mathrm{WF}(\psi\rho \circ E\chi)' &\subset \mathrm{WF}(\psi\rho)' \circ \mathrm{WF}(E\chi)' \\ &\subset \{(Y_1, X_2) : ((0, s_1, y_1), (\varrho_1, \sigma_1, \eta_1)) \sim X_2 \text{ for some } \varrho_1,\ x_2 \in \operatorname{supp}\chi\}.\end{aligned}$$

Using that $\operatorname{supp}\chi \cap C = \varnothing$ and Lemma 5.1(1), this implies (1). Moreover, (1) implies that

$$\mathrm{WF}(\psi\rho \circ E\chi)'_M = \varnothing. \tag{5-8}$$

Again by [Hörmander 1990a], this implies that $\psi\rho \circ E\chi = \mathscr{D}(M) \to \mathscr{D}(\widetilde{C})$ extends continuously as $\psi\rho \circ E\chi : \mathscr{D}'(M) \to \mathscr{D}'(\widetilde{C})$. □

5B. *Hadamard condition on the cone.* Recall from Section 4B that we can associate to a quasifree state ω_C on $\mathrm{CCR}(\mathscr{H}(\widetilde{C}), \sigma_C)$ a quasifree state ω_0 on $\mathrm{CCR}(C_0^\infty(M_0)/PC_0^\infty(M_0), E_0)$. In this subsection we give natural conditions on the covariances $\lambda^\pm$ of ω_C which ensure that the induced state ω_0 satisfies the microlocal spectrum condition (μsc).

Recall that we denote by $Y = ((s, y), (\sigma, \eta))$ the points in $T^*\widetilde{C}$. We also denote by Δ the diagonal in $T^*\widetilde{C} \times T^*\widetilde{C}$ and we will use the notation $_{\widetilde{C}}\Gamma$ and $\Gamma_{\widetilde{C}}$ introduced in (5-5).

Theorem 5.3. *Let* $\lambda^\pm : \mathscr{H}(\widetilde{C}) \to \mathscr{H}(\widetilde{C})$ *and*

$$\Lambda^\pm := (\rho \circ E_0)^* \circ \lambda^\pm \circ (\rho \circ E_0).$$

Then:

(1) $\Lambda^\pm \in \mathscr{D}'(M_0 \times M_0)$.

(2) *If*

(i) $\mathrm{WF}(\lambda^\pm)' \cap \{(Y_1, Y_2) : \pm\sigma_1 < 0 \textit{ or } \pm\sigma_2 < 0\} = \varnothing$,

(ii) $\mathrm{WF}(\lambda^+ - \lambda^-)' \cap \{(Y_1, Y_2) : \sigma_1 \textit{ and } \sigma_2 \neq 0\} \subset \Delta$,

then

(iii) $\mathrm{WF}(\lambda^\pm)' \cap \{(Y_1, Y_2) : \pm\sigma_1 > 0 \textit{ and } \pm\sigma_2 > 0\} \subset \Delta$.

(3) *Assume moreover that* $\lambda^\pm : \mathscr{H}(\widetilde{C}) \to \mathscr{H}(\widetilde{C})$ *and* $_{\widetilde{C}}\mathrm{WF}(\lambda^\pm)' = \mathrm{WF}(\lambda^\pm)'_{\widetilde{C}} = \varnothing$. *Then, if* (i) *and* (iii) *in* (2) *hold*, $\Lambda^\pm$ *satisfy* (μsc).

Proof. To prove (1) it suffices to check that $\rho \circ E_0 : \mathscr{D}(M_0) \to \mathscr{H}(\widetilde{C})$. If $\chi \in C_0^\infty(M_0)$ then, by Lemma 2.1, $\rho \circ E_0\chi = \rho \circ \chi_1 E\chi$ for some $\chi_1 \in C_0^\infty(M)$. Since $E : \mathscr{D}(M) \to \mathscr{E}(M)$ and $\rho : \mathscr{D}(M) \to \mathscr{H}(\widetilde{C})$ are continuous, this proves (1).

To prove (2) we write

$$\begin{aligned}
\mathrm{WF}(\lambda^{\pm})' \cap \{\pm\sigma_1 > 0,\ \pm\sigma_2 > 0\} \\
&\subset (\mathrm{WF}(\lambda^{\mp})' \cap \{\pm\sigma_1 > 0,\ \pm\sigma_2 > 0\}) \cup (\mathrm{WF}(\lambda^{+} - \lambda^{-})' \cap \{\pm\sigma_1 > 0,\ \pm\sigma_2 > 0\}) \\
&\subset (\mathrm{WF}(\lambda^{\mp})' \cap \{\pm\sigma_1 > 0,\ \pm\sigma_2 > 0\}) \cup (\mathrm{WF}(\lambda^{+} - \lambda^{-})' \cap \{\sigma_1, \sigma_2 \neq 0\}).
\end{aligned}$$

The first set in the last line is empty by (i), and the second is contained in Δ by (ii).

To prove (3) we follow an argument due to Moretti [2008]. We treat only the case of λ^+, the case of λ^- being similar, and omit the $+$ superscript. Let $\chi_i \in C_0^\infty(M_0)$, $i = 1, 2$. By Proposition 2.3 there exists $\psi_i \in C_0^\infty(C)$ (and hence $\psi_i \equiv 0$ near p) such that any null geodesic starting from $\operatorname{supp} \chi_i$ intersects C in $\{\psi_i = 1\}$. We have:

$$\begin{aligned}
\chi_1 \Lambda \chi_2 = {} & \chi_1(\rho \circ E)^* \psi_1 \circ \lambda \circ \psi_2 (\rho \circ E)\chi_2 + \chi_1(\rho \circ E)^* \psi_1 \circ \lambda \circ (1 - \psi_2)(\rho \circ E)\chi_2 \\
&+ \chi_1(\rho \circ E)^*(1 - \psi_1) \circ \lambda \circ \psi_2(\rho \circ E)\chi_2 + \chi_1(\rho \circ E)^*(1 - \psi_1) \circ \lambda \circ (1 - \psi_2)(\rho \circ E)\chi_2 \\
=: {} & \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4.
\end{aligned}$$

By the properties of χ_i and ψ_i, we can find $\tilde{\chi}_i \in C_0^\infty(M)$ supported near p such that

(a) $(1 - \psi_i)(\rho \circ E)\chi_i = (1 - \psi_i)\rho \circ \tilde{\chi}_i E \chi_i$,

(b) no null geodesic from $\operatorname{supp} \chi_i$ intersects $\operatorname{supp} \tilde{\chi}_i$.

It follows from (b) and (5-6) that $\tilde{\chi}_i E \chi_i$ has a smooth, compactly supported kernel, hence

$$\tilde{\chi}_i E \chi_i : \mathcal{D}'(M) \to \mathcal{D}(M).$$

Since $(1 - \psi_i)\rho : \mathcal{D}(M) \to \mathcal{H}(\widetilde{C})$, we see that

$$(1 - \psi_i)\rho \circ E \chi_i : \mathcal{D}'(M) \to \mathcal{H}(\widetilde{C}), \tag{5-9}$$

hence

$$\chi_i(\rho \circ E)^*(1 - \psi_i) : \mathcal{H}'(\widetilde{C}) \to \mathcal{D}(M). \tag{5-10}$$

It remains to examine the properties of $\psi_i(\rho \circ E)\chi_i$. By Proposition 5.2, $\psi_i(\rho \circ E)\chi_i : \mathcal{D}'(M) \to \mathcal{E}'(\widetilde{C})$. Since $\mathcal{E}'(\widetilde{C}) \subset \mathcal{H}'(\widetilde{C})$ continuously, we have

$$\psi_i(\rho \circ E)\chi_i : \mathcal{D}'(M) \to \mathcal{H}'(\widetilde{C}), \tag{5-11}$$

hence

$$\chi_i(\rho \circ E)^* \psi_i : \mathcal{H}(\widetilde{C}) \to \mathcal{D}(M). \tag{5-12}$$

From (5-9)–(5-12) and the assumption that $\lambda : \mathcal{H}(\widetilde{C}) \to \mathcal{H}(\widetilde{C})$ it follows that $\Lambda_i : \mathcal{D}'(M_0) \to \mathcal{D}(M_0)$, which hence has a smooth kernel for $i = 2, 3, 4$, and $\mathrm{WF}(\chi_1 \Lambda \chi_2)' = \mathrm{WF}(\Lambda_1)'$.

To bound $\mathrm{WF}(\Lambda_1)'$ we choose $\tilde{\psi}_i \in C_0^\infty(\widetilde{C})$ such that $\tilde{\psi}_i \psi_i = \psi_i$ and write

$$\Lambda_1 = (\chi_1(\rho \circ E)\psi_1) \circ (\tilde{\psi}_1 \lambda \tilde{\psi}_2) \circ (\psi_2(\rho \circ E)_2) =: K_1^* \circ d \circ K_2,$$

where $K_i = \psi_i(\rho \circ E)\chi_i \in \mathscr{E}'(M \times \widetilde{C})$ and $d = \tilde{\psi}_1 c \tilde{\psi}_2 \in \mathscr{E}'(\widetilde{C} \times \widetilde{C})$. The distributions K_1, K_2 and d have compact support. Moreover, we have

$$\mathrm{WF}(d)'_{\widetilde{C}} = {}_{\widetilde{C}}\mathrm{WF}(d)' = \mathrm{WF}(K_1)'_M = {}_M\mathrm{WF}(K_2^*)' = \varnothing.$$

In fact, the first two equalities follow from the corresponding hypothesis on $\mathrm{WF}(c)'$ and the last two from (5-8). We can then apply the results in [Hörmander 1990a, Chapter 8] on the composition of kernels and obtain that $K_2^* \circ d \circ K_1$ is well defined and

$$\mathrm{WF}(K_2^* \circ d \circ K_1) \subset \mathrm{WF}(K_2^*)' \circ \mathrm{WF}(d)' \circ \mathrm{WF}(K_1)'.$$

Now we apply Proposition 5.2(1), the fact that $\mathrm{WF}(d)' \subset \mathrm{WF}(\lambda)'$ and Lemma 5.1(1). We obtain that, if $(X_1, X_2) \in \mathrm{WF}(\Lambda)'$, necessarily $X_1, X_2 \in \mathcal{N}_+$ and $X_1 \sim X_2$, which is exactly condition (μsc). $\square$

6. Pseudodifferential calculus

In this section we collect rather standard results on the pseudodifferential calculus on $\widetilde{C} = \mathbb{R} \times \mathbb{S}^{d-1}$. We will however need to consider *bihomogeneous* symbols on $\mathbb{R} \times \mathbb{S}^{d-1}$, i.e., symbols having different homogeneities in the covariables σ and η, dual to s and θ.

The reason for this is that the charge $q = -2D_s$ is not an elliptic differential operator in the usual sense (considered on $\widetilde{C}$), hence operators like $(q - z)^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{R}$ are not in the usual pseudodifferential classes.

For $k, k' \in \mathbb{R}$, we denote by $H^k(\mathbb{R})$ and $H^{k'}(\mathbb{S}^{d-1})$ the Sobolev spaces on $\mathbb{R}$ and $\mathbb{S}^{d-1}$ of orders k and k' and by $\|\cdot\|_k$ and $\|\cdot\|_{k'}$ their respective norms. Furthermore, we denote by $H^{k,k'}(\mathbb{R} \times \mathbb{S}^{d-1})$ the Sobolev space on $\mathbb{R} \times \mathbb{S}^{d-1}$ of biorder (k, k'), that is, the completion of $C_0^\infty(\mathbb{R} \times \mathbb{S}^{d-1})$ for the norm

$$\|\psi\|_{k,k'} := \|\langle D_s\rangle^k \langle D_\theta\rangle^{k'} \psi\|_2.$$

We set also, for $p \in \mathbb{R}$,

$$B^p(\mathbb{R}) = \bigcap_{k \in \mathbb{R}} B(H^k(\mathbb{R}), H^{k-p}(\mathbb{R})),$$

equipped with its natural topology.

6A. *Pseudodifferential operators on* $\mathbb{R} \times \mathbb{R}^{d-1}$.

Definition 6.1. Let $p_1, p_2 \in \mathbb{R}$.

(1) We denote by $S^{p_1,p_2}(\mathbb{R} \times \mathbb{R}^{d-1})$ the space of symbols $a \in C^\infty(T^*\mathbb{R} \times T^*\mathbb{R}^{d-1})$ such that

$$|\partial_s^{\alpha_1} \partial_\sigma^{\beta_1} \partial_y^{\alpha_2} \partial_\eta^{\beta_2} a| \in O(\langle\sigma\rangle^{p_1 - |\beta_1|} \langle\eta\rangle^{p_2 - |\beta_2|}), \quad \alpha_1, \beta_1 \in \mathbb{N},\ \alpha_2, \beta_2 \in \mathbb{N}^{d-1}.$$

(2) We denote by $B^{p_1}S^{p_2}(\mathbb{R} \times \mathbb{S}^{d-1})$ the space of $a \in C^\infty(T^*\mathbb{R}^{d-1}, B^{p_1}(\mathbb{R}))$ such that

$$\|\partial_y^{\alpha_2} \partial_\eta^{\beta_2} a\|_{p_1,k_1} \in O(\langle\eta\rangle^{p_2 - |\beta_2|}), \quad \alpha_2, \beta_2 \in \mathbb{N}^{d-1},$$

where $\|\cdot\|_{p_1,k_1}$ is any seminorm of a in $B^{p_1}(\mathbb{R})$.

Using the Weyl quantization on $\mathbb{R}\times\mathbb{R}^{d-1}$, we obtain a map

$$S^{p_1,p_2}(\mathbb{R}\times\mathbb{R}^{d-1}) \to B(C_0^\infty(\mathbb{R}\times\mathbb{R}^{d-1}), C^\infty(\mathbb{R}\times\mathbb{R}^{d-1})), \quad a\mapsto \mathrm{Op}(a),$$

whose range, denoted by $\Psi^{p_1,p_2}(\mathbb{R}\times\mathbb{R}^{d-1})$, is the space of pseudodifferential operators on $\mathbb{R}\times\mathbb{R}^{d-1}$ of biorder (p_1, p_2). Similarly, using the Weyl quantization on $\mathbb{R}^{d-1}$, we obtain a map

$$B^{p_1}S^{p_2}(\mathbb{R}\times\mathbb{R}^{d-1})B(C_0^\infty(\mathbb{R}\times\mathbb{R}^{d-1}), C^\infty(\mathbb{R}\times\mathbb{R}^{d-1})), \quad a\mapsto \mathrm{Op}(a),$$

whose range will be denoted by $B^{p_1}\Psi^{p_2}(\mathbb{R}\times\mathbb{R}^{d-1})$.

6B. ***Pseudodifferential operators on $\widetilde{C}$.*** Let $A : C_0^\infty(\widetilde{C}) \to C^\infty(\widetilde{C})$. If $\chi_i \in C^\infty(\mathbb{S}^{d-1})$, $i = 1, 2$, are cutoff functions supported in chart open sets $\Omega_i \subset \mathbb{S}^{d-1}$ and $\phi_i : \Omega_i \to \mathbb{R}^{d-1}$ are coordinate charts, then $\phi_1^* \circ \chi_1 A\chi_2 \circ \phi_2^{-1*} : C_0^\infty(\mathbb{R}\times\mathbb{R}^{d-1}) \to C^\infty(\mathbb{R}\times\mathbb{R}^{d-1})$.

Definition 6.2. (1) We denote by $\Psi^{p_1,p_2}(\widetilde{C})$ the space of operators $A : C_0^\infty(\widetilde{C}) \to C^\infty(\widetilde{C})$ such that, for any χ_i and ϕ_i as above, $\phi_1^* \circ \chi_1 A\chi_2 \circ \phi_2^{-1*} \in \Psi^{p_1,p_2}(\mathbb{R}\times\mathbb{R}^{d-1})$.

(2) We denote by $B^{p_1}\Psi^{p_2}(\widetilde{C})$ the space of operators $A : C_0^\infty(\widetilde{C}) \to C^\infty(\widetilde{C})$ such that, for any χ_i and ϕ_i as above, $\phi_1^* \circ \chi_1 A\chi_2 \circ \phi_2^{-1*} \in B^{p_1}\Psi^{p_2}(\mathbb{R}\times\mathbb{R}^{d-1})$.

(3) We set

$$\Psi^{-\infty,p_2}(\widetilde{C}) = \bigcap_{p_1\in\mathbb{R}} \Psi^{p_1,p_2}(\widetilde{C}) \quad\text{and}\quad B^{-\infty}\Psi^{p_2}(\widetilde{C}) = \bigcap_{p_1\in\mathbb{R}} B^{p_1}\Psi^{p_2}(\widetilde{C}).$$

(4) We set

$$\widetilde{\Psi}^{p_1,p_2}(\widetilde{C}) = \Psi^{p_1,p_2}(\widetilde{C}) + B^{-\infty}\Psi^{p_2}(\widetilde{C}).$$

Note that if one defines, analogously, $\widetilde{\Psi}^{-\infty,p_2}(\widetilde{C}) := \bigcap_{p_1\in\mathbb{R}} \widetilde{\Psi}^{p_1,p_2}(\widetilde{C})$, then actually $\widetilde{\Psi}^{-\infty,p_2}(\widetilde{C}) = B^{-\infty}\Psi^{p_2}(\widetilde{C})$. Moreover, it is easy to check that

$$\widetilde{\Psi}^{p_1,p_2}(\widetilde{C}) \circ \widetilde{\Psi}^{q_1,q_2}(\widetilde{C}) \subset \widetilde{\Psi}^{p_1+p_2,q_1+q_2}(\widetilde{C}).$$

We refer the reader to [Rodino 1975; Borsero and Schulz 2014; Ruzhansky and Turunen 2010] and references therein for more details on the pseudodifferential calculus on products of manifolds.[4]

6C. ***The Beals criterion.*** Let us denote by $\Psi^p(\mathbb{S}^{d-1})$ the classes of standard pseudodifferential operators on $\mathbb{S}^{d-1}$. It is well known that $\Psi^p(\mathbb{S}^{d-1})$ can be characterized by the Beals criterion, namely an operator $A : C^\infty(\mathbb{S}^{d-1}) \to C^\infty(\mathbb{S}^{d-1})$ belongs to $\Psi^p(\mathbb{S}^{d-1})$ if and only if

$$\mathrm{ad}_{f_1}\cdots\mathrm{ad}_{f_n}\,\mathrm{ad}_{X_1}\cdots\,\mathrm{ad}_{X_m} A : H^k(\mathbb{S}^{d-1}) \to H^{k-p+n}(\mathbb{S}^{d-1}), \quad n, m\in\mathbb{N},\ k\in\mathbb{Z}, \tag{6-1}$$

for any $f_i \in C^\infty(\mathbb{S}^{d-1})$ and smooth vector fields X_j on $\mathbb{S}^{d-1}$ [Ruzhansky and Turunen 2010]. Moreover, one can find a finite set of such f_i and X_j such that the topology on $\Psi^p(\mathbb{S}^{d-1})$ given by the collection of the norms of the multicommutators is equivalent to the standard topology on $\Psi^p(\mathbb{S}^{d-1})$, given by the

[4] Note however that the literature discusses mostly the case when both manifolds are compact.

symbol space topologies of the pullbacks $\phi_i^* \circ \chi_i A \chi_j \circ \phi_j$ in Definition 6.2, for a fixed covering of $\mathbb{S}^{d-1}$ by chart neighborhoods U_i.

These characterizations immediately carry over to the classes $B^{p_1}\Psi^{p_2}(\widetilde{C})$. In fact it is easy to see that $A \in B^{p_1}S^{p_2}(\widetilde{C})$ if and only if

$$\mathrm{ad}_{f_1} \cdots \mathrm{ad}_{f_n} \mathrm{ad}_{X_1} \cdots \mathrm{ad}_{X_m} A : H^{k,k'}(\widetilde{C}) \to H^{k-p_1,k'-p_2+n}(\mathbb{S}^{d-1}), \quad n, m \in \mathbb{N},\ k, k' \in \mathbb{Z}. \tag{6-2}$$

This result can be deduced from the previous one by considering the operators

$$((u_1| \otimes \mathbb{1}_{\mathbb{S}^{d-1}}) \circ A \circ (|u_2) \otimes \mathbb{1}_{\mathbb{S}^{d-1}}) : C^\infty(\mathbb{S}^{d-1}) \to C^\infty(\mathbb{S}^{d-1})$$

for $u_1 \in H^{-k+p_1}(\mathbb{R})$ and $u_2 \in H^k(\mathbb{R})$, which belong to $\Psi^{p_2}(\mathbb{S}^{d-1})$ if (6-2) holds. Applying the result recalled above about the equivalence of the standard topology and the topology given by the multicommutator norms, one obtains that $A \in B^{p_1}\Psi^{p_2}(\widetilde{C})$ if (6-2) holds.

In the usual case one can deduce from the Beals criterion standard results on the functional calculus for pseudodifferential operators, for example on complex powers of elliptic pseudodifferential operators [Bony 1997]. These results are easy to extend to the classes $B^{p_1}\Psi^{p_2}(\widetilde{C})$. We will need only a very simple one, which we now state. Recall that $\widetilde{\Psi}^{-\infty,0}(\widetilde{C}) = B^{-\infty}\Psi^0(\widetilde{C}) \subset B(L^2(\widetilde{C}))$. The spectrum of $b \in B(L^2(\widetilde{C}))$ is denoted by $\mathrm{spec}(b)$.

Proposition 6.3. *Let $b \in \widetilde{\Psi}^{-\infty,0}(\widetilde{C})$ and let be F holomorphic near* $\mathrm{spec}(b)$ *with $F(0) = 0$. Then $F(b) \in \widetilde{\Psi}^{-\infty,0}(\widetilde{C})$.*

Proof. The proof consists of expressing $F(b)$ as a contour integral and applying the Beals criterion to the resolvent $(b - z)^{-1}$. □

6D. *Essential support.* We denote by $\Psi^p_{\mathrm{ph}}(\mathbb{R})$, $p \in \mathbb{R}$, the class of global pseudodifferential operators on $\mathbb{R}$ with polyhomogeneous symbols.

Definition 6.4. The *essential support* of $a \in \Psi^{p_1,p_2}(\widetilde{C})$, denoted by $\mathrm{ess\,supp}(a) \subset T^*\mathbb{R}\backslash Z$, is defined by $(s_0, \sigma_0) \notin \mathrm{ess\,supp}(a)$ if there exists $b \in \Psi^0_{\mathrm{ph}}(\mathbb{R})$ that is elliptic at (s_0, σ_0) such that $b \circ a \in \Psi^{-\infty,p_2}(\widetilde{C})$.

Clearly $\mathrm{ess\,supp}(a)$ is a closed conic subset of $T^*\mathbb{R}\backslash Z$. Moreover, one can equivalently require that $a \circ b \in \Psi^{-\infty,p_2}(\widetilde{C})$ for some $b \in \Psi^0_{\mathrm{ph}}(\mathbb{R})$ that is elliptic at (s_0, σ_0).

6E. *Wavefront set of kernels.* For $N = \mathbb{R}, \mathbb{S}^{d-1}, \mathbb{R}\times\mathbb{S}^{d-1}$, we denote by Δ_N the diagonal in $T^*N \times T^*N$ and by Z_N the zero section in T^*N.

For an operator $a \in \Psi^{p_1,p_2}(\mathbb{R} \times \mathbb{S}^{d-1})$, it is in general not true that $\mathrm{WF}(a)'$ is contained in the full diagonal $\Delta_{\mathbb{R}\times\mathbb{S}^{d-1}}$ (as would be the case for an operator in $\Psi^p(\mathbb{R} \times \mathbb{S}^{d-1})$). Instead one has the following estimate, which can be thought as a natural generalization of the usual estimate for the wavefront set of tensor products of distributions (in this case Schwartz kernels) [Borsero and Schulz 2014].

Lemma 6.5. *Let $a \in \Psi^{p_1,p_2}(\mathbb{R} \times \mathbb{S}^{d-1})$. Then*

$$\mathrm{WF}(a)' \subset \Delta_{\mathbb{R}} \times \Delta_{\mathbb{S}^{d-1}} \cup \Delta_{\mathbb{R}} \times (Z_{\mathbb{S}^{d-1}} \times Z_{\mathbb{S}^{d-1}}) \cup (Z_{\mathbb{R}} \times Z_{\mathbb{R}}) \times \Delta_{\mathbb{S}^{d-1}}.$$

Less precise estimates are valid for the $\widetilde{\Psi}^{p_1,p_2}(\mathbb{R} \times \mathbb{S}^{d-1})$ classes:

Lemma 6.6. (1) *Let $a \in B^{-\infty}\Psi^{p_2}(\widetilde{C})$. Then*

$$\mathrm{WF}(a)' \cap \{(Y_1, Y_2) : \sigma_1 \neq 0 \ \textit{or} \ \sigma_2 \neq 0\} = \varnothing.$$

(2) *Let $a \in \widetilde{\Psi}^{p_1,p_2}(\widetilde{C})$. Then*

$$_{\widetilde{C}}\mathrm{WF}(a)' = \mathrm{WF}(a)'_{\widetilde{C}} = \varnothing.$$

The proof is given in Appendix A5.

6F. ***Toeplitz pseudodifferential operators on $\widetilde{C}$.*** We recall that $\mathscr{H}(\widetilde{C}) = \bigcap_{m\in\mathbb{R}} H^m(\widetilde{C}) = \bigcap_{k\in\mathbb{R}} H^{k,k}(\widetilde{C})$. Let us set

$$L^2_{\pm}(\widetilde{C}) := \mathbb{1}_{\mathbb{R}^{\pm}}(D_s)L^2(\widetilde{C})$$

and denote by $i_{\pm} : L^2_{\pm}(\widetilde{C}) \to L^2(\widetilde{C})$ the corresponding isometric injection, so that $\pi_{\pm} := i_{\pm}i^*_{\pm} = \mathbb{1}_{\mathbb{R}^{\pm}}(D_s)$ is the orthogonal projection on $L^2_{\pm}(\widetilde{C})$ in $L^2(\widetilde{C})$. We also set

$$\mathscr{H}_{\pm}(\widetilde{C}) := i^*_{\pm}\mathscr{H}(\widetilde{C}) \subset \mathscr{H}(\widetilde{C}). \tag{6-3}$$

We will see in Section 7 that this provides a useful setup for the discussion of the positivity condition $\lambda^{\pm} \geq 0$ for the two-point functions of a Hadamard state.

Writing $\mathbb{1}_{\mathbb{R}^{\pm}} = \chi\mathbb{1}_{\mathbb{R}^{\pm}} + (1-\chi)\mathbb{1}_{\mathbb{R}^{\pm}}$ for a cutoff function $\chi \in C_0^{\infty}(\mathbb{R})$ equal to 1 near 0, we see that

$$\pi_{\pm} \in \widetilde{\Psi}^{0,0}(\widetilde{C}). \tag{6-4}$$

For $\alpha, \beta \in \{+, -\}$ and $p_1, p_2 \in \mathbb{R}$, we set

$$\widetilde{\Psi}^{p_1,p_2}_{\alpha\beta}(\widetilde{C}) := i_{\alpha} \circ \widetilde{\Psi}^{p_1,p_2}(\widetilde{C}) \circ i^*_{\beta}.$$

By (6-3) we see that $\widetilde{\Psi}^{p_1,p_2}_{\alpha\beta}(\widetilde{C}) : \mathscr{H}_{\beta}(\widetilde{C}) \to \mathscr{H}_{\alpha}(\widetilde{C})$. Moreover, if we set

$$R_{\alpha\beta} : \widetilde{\Psi}^{p_1,p_2}(\widetilde{C}) \to \widetilde{\Psi}^{p_1,p_2}_{\alpha\beta}(\widetilde{C}), \quad a \mapsto i^*_{\alpha} \circ a \circ i_{\beta}$$

then, using (6-4), we see that $R_{\alpha\beta}$ has right inverse

$$T_{\alpha\beta} : \widetilde{\Psi}^{p_1,p_2}_{\alpha\beta}(\widetilde{C}) \to \widetilde{\Psi}^{p_1,p_2}(\widetilde{C}), \quad a \mapsto i_{\alpha} \circ a \circ i^*_{\beta},$$

which allows us to identify $\widetilde{\Psi}^{p_1,p_2}_{\alpha\beta}(\widetilde{C})$ with $\operatorname{Ran} T_{\alpha\beta} \subset \widetilde{\Psi}^{p_1,p_2}(\widetilde{C})$. From (6-4) we also have

$$\widetilde{\Psi}^{p_1,p_2}_{\alpha\beta}(\widetilde{C}) \circ \widetilde{\Psi}^{q_1,q_2}_{\beta\gamma}(\widetilde{C}) \subset \widetilde{\Psi}^{p_1+q_1,p_2+q_2}_{\alpha\gamma}(\widetilde{C}). \tag{6-5}$$

7. Construction of Hadamard states on the cone

From the discussion in Section 5B, in particular Theorem 5.3, we are led to the following definition:

Definition 7.1. A pair of maps $\lambda^\pm : \mathscr{H}(\widetilde{C}) \to \mathscr{H}(\widetilde{C})$ is called a pair of *Hadamard two-point functions* on the cone C if

$$
{}_{\widetilde{C}}\mathrm{WF}(\lambda^\pm)' = \mathrm{WF}(\lambda^\pm)'_{\widetilde{C}} = \varnothing, \tag{Had-i}
$$

$$
\mathrm{WF}(\lambda^\pm)' \cap \{(Y_1, Y_2) : \pm\sigma_1 < 0 \;\text{ or }\; \pm\sigma_2 < 0\} = \varnothing, \tag{Had-ii}
$$

$$
\lambda^+ - \lambda^- = 2D_s, \tag{Had-iii}
$$

$$
\lambda^\pm \geq 0 \quad \text{on } \mathscr{H}(\widetilde{C}). \tag{Had-iv}
$$

As the name suggests, if $\lambda^\pm$ are Hadamard two-point functions on C in the sense of the above definition, then $\Lambda^\pm$ defined in (4-4) are Hadamard two-point functions on M_0 (as follows from Theorem 5.3).

We now discuss in more detail the various conditions in (Had-i)–(Had-iv). It is natural to consider pseudodifferential two-point functions, i.e., to assume that $\lambda^\pm \in \widetilde{\Psi}^{p_1,p_2}(\widetilde{C})$. Moreover to analyze conditions (Had-iii)–(Had-iv) it is convenient to reduce oneself to $\lambda^\pm$ of the form

$$
\lambda^\pm = (2|D_s|)^{1/2} c^\pm (2|D_s|)^{1/2}, \quad \text{where} \quad c^\pm \in \widetilde{\Psi}^{p_1,p_2}(\widetilde{C}), \tag{7-1}
$$

for $p_1, p_2 \in \mathbb{R}$. Note that, writing $(2|D_s|)^{1/2}$ as $\chi(D_s)(2|D_s|)^{1/2} + (1-\chi(D_s))(2|D_s|)^{1/2}$ for $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 near 0, we see that (7-1) implies that $\lambda^\pm \in \widetilde{\Psi}^{p_1+1,p_2}(\widetilde{C})$.

7A. *Wavefront set.*

We first analyze conditions (Had-i)–(Had-ii).

Proposition 7.2. *Assume that*

$$
\lambda^\pm = a^\pm + r^\pm, \quad a^\pm \in \Psi^{p_1,p_2}(\widetilde{C}), \quad r^\pm \in \widetilde{\Psi}^{-\infty,p_2}(\widetilde{C}), \quad (\mathbb{R}\times\mathbb{R}^\mp) \cap \operatorname{ess\,supp}(a^\pm) = \varnothing. \tag{7-2}
$$

Then $\lambda^\pm$ satisfies conditions (Had-i)–(Had-ii).

Proof. The fact that $\lambda^\pm$ satisfy (Had-i) follows from Lemma 6.6(2). Also, since, by Lemma 6.6(1), $r^\pm$ satisfy (Had-ii) we can assume that $\lambda^\pm = a^\pm$. We treat only the case of λ^+ and use the notation in the proof of Lemma 6.6. Let $\widetilde{Y}_1, \widetilde{Y}_2 \in T^*\widetilde{C}\backslash Z$ with $\tilde{\sigma}_1 \neq 0$ or $\tilde{\sigma}_2 \neq 0$. Let us assume that $\tilde{\sigma}_1 \neq 0$, the case $\tilde{\sigma}_2 \neq 0$ being similar, using the remark after Definition 6.4.

Since $(\mathbb{R}\times\mathbb{R}^+) \cap \operatorname{ess\,supp}(a^+) = \varnothing$, we can find a cutoff function χ_1 with $\chi_1(\tilde{s}_1) \neq 0$, a neighborhood V_1 of $\tilde{\sigma}_1$ and some $m_1 \in \Psi^0_{\mathrm{ph}}(\mathbb{R})$ elliptic at $(\tilde{s}_1, \tilde{\sigma}_1)$ such that $(1-m_1)(s, D_s)v_{\sigma,\lambda} \in O(\langle\lambda\rangle^{-\infty})$ in all $H^k(\mathbb{R})$ and $m_1(s, D_s) \circ a \in \widetilde{\Psi}^{-\infty,p_2}(\widetilde{C})$. The fact that $(\widetilde{Y}_1, \widetilde{Y}_2) \notin \mathrm{WF}(a)'$ then follows from the same arguments as in the proof of Lemma 6.6. □

In terms of $c^\pm$ appearing in (7-1), a natural condition implying (7-2) is

$$
\mathbb{1}_{\mathbb{R}^\mp}(D_s)c^\pm \in \widetilde{\Psi}^{-\infty,p_2}(\widetilde{C}), \tag{$\mu\mathrm{sc}_C$}
$$

which clearly implies that $\lambda^\pm$ satisfy (7-2).

Lemma 7.3. *Let $\lambda^\pm$ be given by* (7-1) *and such that* ($\mu\mathrm{sc}_C$) *holds. Then*

$$
c^\pm = \mathbb{1}_{\mathbb{R}^\pm}(D_s) + \widetilde{\Psi}^{-\infty,p_2}(\widetilde{C}).
$$

Proof. In terms of $c^{\pm}$, (Had-iii) becomes $c^{+}-c^{-}=\mathrm{sgn}(D_s)$. Let $\chi^{\pm}\in C^{\infty}(\mathbb{R})$ be cutoff functions equal to 1 near $\pm\infty$ and to 0 near $\mp\infty$. From ($\mu\mathrm{sc}_C$) and pseudodifferential calculus we obtain that

$$c^{\pm}=\chi^{\pm}(D_s)c^{\pm}\chi^{\mp}(D_s)+\widetilde{\Psi}^{-\infty,p_2}(\widetilde{C}). \tag{7-3}$$

Using successively (7-3) and $c^{+}-c^{-}=\mathrm{sgn}(D_s)$, we obtain

$$\begin{aligned}
c^{\pm}&=\chi^{\pm}c^{\pm}\chi^{\pm}+\widetilde{\Psi}^{-\infty,p_2}(\widetilde{C})\\
&=\chi^{\pm}(c^{\mp}\pm\mathrm{sgn}(D_s))\chi^{\pm}+\widetilde{\Psi}^{-\infty,p_2}(\widetilde{C})\\
&=\chi^{\pm}c^{\mp}\chi^{\pm}+\chi^{\pm}\chi^{\pm}+\widetilde{\Psi}^{-\infty,p_2}(\widetilde{C})\\
&=\chi^{\pm}\chi^{\mp}c^{\mp}\chi^{\mp}\chi^{\pm}+\chi^{\pm}+\widetilde{\Psi}^{-\infty,p_2}(\widetilde{C})\\
&=\chi^{\pm}+\widetilde{\Psi}^{-\infty,p_2}(\widetilde{C})\\
&=\mathbb{1}_{\mathbb{R}^{\pm}}(D_s)+\widetilde{\Psi}^{-\infty,p_2}(\widetilde{C}).
\end{aligned}$$

□

7B. *Positivity.* We now discuss conditions (Had-iii)–(Had-iv). In terms of $c^{\pm}$ they become

$$c^{+}-c^{-}=\mathrm{sgn}(D_s), \tag{7-4-iii}$$

$$c^{\pm}\geq 0 \quad \text{on } \mathscr{H}(\widetilde{C}). \tag{7-4-iv}$$

To analyze (7-4-iii)–(7-4-iv) we use the framework of Section 6F. We denote c^{+} simply by c and set

$$c_{\alpha\beta}=i_{\alpha}^{*}\circ c\circ i_{\beta},\quad \alpha,\beta\in\{+,-\},$$

so that

$$c=\sum_{\alpha,\beta\in\{+,-\}}i_{\alpha}c_{\alpha\beta}i_{\beta}^{*}. \tag{7-5}$$

Then (7-4-iii)–(7-4-iv) is equivalent to

$$\begin{pmatrix}c_{++}&c_{+-}\\c_{-+}&c_{--}\end{pmatrix}\geq 0\quad\text{and}\quad\begin{pmatrix}c_{++}-\mathbb{1}&c_{+-}\\c_{-+}&c_{--}+\mathbb{1}\end{pmatrix}\geq 0\quad\text{on } \mathscr{H}_{+}(\widetilde{C})\oplus\mathscr{H}_{-}(\widetilde{C}), \tag{7-6}$$

which is equivalent to

(i) $c_{++}\geq 0$, $c_{--}\geq\mathbb{1}$ and $c_{-+}=c_{+-}^{*}$.

(ii) $$\begin{aligned}&\left|(u_{+}\mid c_{+-}u_{-})\right|\leq(u_{+}\mid c_{++}u_{+})^{1/2}(u_{-}\mid c_{--}u_{-})^{1/2},\\&\left|(u_{+}\mid c_{+-}u_{-})\right|\leq(u_{+}\mid(c_{++}-\mathbb{1})u_{+})^{1/2}(u_{-}\mid(c_{--}+\mathbb{1})u_{-})^{1/2},\end{aligned}\qquad u_{\pm}\in\mathscr{H}_{\pm}(\widetilde{C}).$$

Condition (ii) above is implied by

$$\left|(u_{+}\mid c_{+-}u_{-})\right|\leq(u_{+}\mid(c_{++}-\mathbb{1})u_{+})^{1/2}(u_{-}\mid c_{--}u_{-})^{1/2},\quad u_{\pm}\in\mathscr{H}_{\pm}(\widetilde{C}).$$

We are now in position to prove the following theorem, which is the analog of [Gérard and Wrochna 2014, Theorem 7.5] in the present situation. It provides a rather large class of Hadamard two-point functions on C and hence, by Theorem 5.3, of Hadamard states on M_0.

Theorem 7.4. *Assume that*

$$c_{++} = \mathbb{1} + a_+^* a_+, \quad c_{--} = a_-^* a_-, \quad c_{+-} = c_{-+}^* = a_+^* d a_-$$

for $a_+ \in \widetilde{\Psi}_{++}^{-\infty,0}(\widetilde{C})$, $a_- \in \widetilde{\Psi}_{--}^{-\infty,0}(\widetilde{C})$, $d \in \widetilde{\Psi}_{+-}^{0,0}(\widetilde{C})$ *with* $\|d\|_{B(L^2_-(\widetilde{C}), L^2_+(\widetilde{C}))} \leq \mathbb{1}$.

Let c be given by (7-5), $\lambda^+ = (2|D_s|)^{1/2} c (2|D_s|)^{1/2}$ *and* $\lambda^- = \lambda^+ - 2D_s$. *Then* $\lambda^\pm$ *is a pair of Hadamard two-point functions on the cone.*

Proof. We set, as before, $\lambda^\pm = (2|D_s|)^{1/2} c^\pm (2|D_s|)^{1/2} \in \widetilde{\Psi}^{1,0}(\widetilde{C})$, so that $c^+ = c$ and $c^- = c - \mathrm{sgn}(D_s)$. Conditions (7-4-iii)–(7-4-iv) follow from the above discussion. It remains to check condition ($\mu\mathrm{sc}_C$). We embed the spaces $\widetilde{\Psi}_{\alpha\beta}^{p_1,p_2}(\widetilde{C})$ into $\widetilde{\Psi}^{p_1,p_2}(\widetilde{C})$ as explained at the end of Section 6F and we have

$$\begin{aligned} c^+ &= a_+^* a_+ + a_+^* d a_- + a_-^* d^* a_+ + a_-^* a_- + \mathbb{1}_{\mathbb{R}^+}(D_s), \\ c^- &= a_+^* a_+ + a_+^* d a_- + a_-^* d^* a_+ + a_-^* a_- + \mathbb{1}_{\mathbb{R}^-}(D_s), \end{aligned}$$

hence

$$\begin{aligned} \mathbb{1}_{\mathbb{R}^-}(D_s) c^+ &= a_+^* a_+ + a_+^* d a_- \in \widetilde{\Psi}^{-\infty,0}(\widetilde{C}), \\ \mathbb{1}_{\mathbb{R}^+}(D_s) c^- &= a_-^* d^* a_+ + a_-^* a_- \in \widetilde{\Psi}^{-\infty,0}(\widetilde{C}), \end{aligned}$$

and condition ($\mu\mathrm{sc}_C$) is satisfied. □

Remark 7.5. The special choice of vanishing a_+, a_- and d in Theorem 7.4 gives two-point functions

$$\lambda^\pm = \pm 2 \mathbb{1}_{\mathbb{R}^\pm}(D_s) D_s.$$

In the setting of asymptotically flat spacetimes with past time infinity i^-, these correspond to the Hadamard state found and further studied in [Moretti 2006; 2008].

8. Pure Hadamard states

In this section we first characterize *pure* Hadamard states on the cone C. We then prove that any pure Hadamard state ω_C on C induces a pure Hadamard state ω_0 in M_0.

8A. ***An abstract criterion for purity.*** Let $(\mathcal{Y}, \sigma)$ a complex symplectic space and ω a gauge invariant quasifree state on $\mathrm{CCR}(\mathcal{Y}, \sigma)$, with complex covariances $\lambda^\pm$.

Let $\mathcal{Y}^{\mathrm{cpl}}$ the completion of $\mathcal{Y}$ for the norm

$$\|y\|_\omega := (\bar{y} \cdot \lambda^+ y + \bar{y} \cdot \lambda^- y)^{1/2}. \tag{8-1}$$

Let us introduce the hermitian form $q = \mathrm{i}\sigma \in L_{\mathrm{h}}(\mathcal{Y}, \mathcal{Y}^*)$. Clearly q and $\lambda^\pm$ extend uniquely to $\mathcal{Y}^{\mathrm{cpl}}$. Then, by [Araki and Shiraishi 1971/72], ω is pure if and only if

(1) q is nondegenerate on $\mathcal{Y}^{\mathrm{cpl}}$,

(2) there exists an involution $\kappa : \mathcal{Y}^{\mathrm{cpl}} \to \mathcal{Y}^{\mathrm{cpl}}$ such that $\kappa^* q \kappa = q$, $q\kappa \geq 0$ and $\lambda^\pm = \frac{1}{2} q(\kappa \pm \mathbb{1})$.

From this discussion we immediately obtain the following lemma:

Lemma 8.1. *Let $(\mathcal{Y}_i, \sigma_i)$, $i = 1, 2$, be two complex symplectic spaces and $\rho : \mathcal{Y}_1 \to \mathcal{Y}_2$ an injective map such that $\rho^*\sigma_2\rho = \sigma_1$. Let ω_2 be a pure, gauge-invariant quasifree state on* $\mathrm{CCR}(\mathcal{Y}_2, \sigma_2)$. *Let ω_1 be the gauge-invariant, quasifree state on* $\mathrm{CCR}(\mathcal{Y}_1, \sigma_1)$ *defined by the complex covariances*

$$\lambda_1^{\pm} = \rho^* \lambda_2^{\pm} \rho.$$

Then, if $\rho\mathcal{Y}_1$ is dense in $\mathcal{Y}_2$ for the norm $\|\cdot\|_{\omega_2}$ defined in (8-1), *the state ω_1 is pure on* $\mathrm{CCR}(\mathcal{Y}_1, \sigma_1)$.

8B. ***Pure Hadamard states on the cone.*** The following theorem is the exact analog of [Gérard and Wrochna 2014, Theorem 7.10]. In what follows we will use the notations introduced in Section 6F.

Theorem 8.2. *Let $\lambda^{\pm}$ be the two-point functions of a state ω_C on $(\mathcal{H}(\widetilde{C}), \sigma_C)$ of the form* (7-1) *and satisfying* ($\mu\mathrm{sc}_C$). *Then ω_C is pure if and only if there exists $a \in \widetilde{\Psi}^{-\infty,0}_{-+}(\widetilde{C})$ such that*

$$c^+ = \begin{pmatrix} \mathbb{1} + a^*a & a^*(\mathbb{1} + aa^*)^{1/2} \\ (\mathbb{1} + aa^*)^{1/2}a & aa^* \end{pmatrix}.$$

Proof. We consider the pair $c^{\pm}$ obtained from $\lambda^{\pm}$, write as before c^+ for c and identify c with the matrix

$$\begin{pmatrix} c_{++} & c_{+-} \\ c_{-+} & c_{--} \end{pmatrix}.$$

Arguing as in the proof of [Gérard and Wrochna 2014, Theorem 7.10], we obtain that the state ω_C on $(\mathcal{H}(\widetilde{C}), \sigma_C)$ with covariances $\lambda^{\pm}$ is pure if and only if

$$c = \begin{pmatrix} \mathbb{1} + a^*a & a^*(\mathbb{1} + aa^*)^{1/2} \\ (\mathbb{1} + aa^*)^{1/2}a & aa^* \end{pmatrix} \tag{8-2}$$

for some $a : L^2_+(\widetilde{C}) \to L^2_-(\widetilde{C})$. This proves the "if".

Let us now prove the "only if". Since we assumed that $c^{\pm} \in \widetilde{\Psi}^{0,0}(\widetilde{C})$ satisfy ($\mu\mathrm{sc}_C$), we obtain that

$$a^*a \in \widetilde{\Psi}^{-\infty,0}_{++}(\widetilde{C}), \quad (\mathbb{1} + aa^*)^{1/2}a \in \widetilde{\Psi}^{-\infty,0}_{-+}(\widetilde{C}). \tag{8-3}$$

We claim that

$$(\mathbb{1} + aa^*)^{-1/2} \in \mathbb{1} + \widetilde{\Psi}^{-\infty,0}_{--}(\widetilde{C}). \tag{8-4}$$

Let us prove (8-4). We use the operators $R_{\alpha\beta}$ and $T_{\alpha\beta}$ defined at the end of Section 6F. We first embed aa^* into $\widetilde{\Psi}^{-\infty,0}(\widetilde{C})$, i.e., consider $b = T_{--}(a^*a)$. Then $b \geq 0$ on $L^2(\widetilde{C})$ and, applying Proposition 6.3 to $F(z) = (1+z)^{1/2} - 1$, we obtain that $(\mathbb{1} + b)^{-1/2} - \mathbb{1} \in \widetilde{\Psi}^{-\infty,0}(\widetilde{C})$. Writing b as a 2×2 matrix acting on $L^2_+(\widetilde{C}) \oplus L^2_-(\widetilde{C})$ we see that $R_{++}((\mathbb{1} + b)^{1/2}) = (\mathbb{1} + aa^*)^{1/2}$, which proves (8-4). From (8-4) and (8-3) we obtain that $a \in \widetilde{\Psi}^{-\infty,0}_{-+}(\widetilde{C})$. □

In the next lemma we identify the completion of $\mathcal{H}(\widetilde{C})$ for the norm (8-1) associated to any Hadamard state considered in Theorem 8.2.

Let us first fix some notation. For $a : L^2_+(\widetilde{C}) \to L^2_-(\widetilde{C})$ we denote by $c^+(a)$ the operator defined in (8-2) and set $c^-(a) = c^+(a) - \mathrm{sgn}(D_s)$ and

$$\lambda^{\pm}(a) = (2|D_s|)^{1/2} c^{\pm}(a) (2|D_s|)^{1/2}. \tag{8-5}$$

If $\mathscr{H}$ is a Hilbert space and $h \geq 0$ is a selfadjoint operator on $\mathscr{H}$ with $\operatorname{Ker} h = \{0\}$, we denote by $h\mathscr{H}$ the completion of $\operatorname{Dom} h^{-1}$ (the range of h) for the norm $\|h^{-1}u\|_{\mathscr{H}}$.

Lemma 8.3. *Let $a : L^2_+(\widetilde{C}) \to L^2_-(\widetilde{C})$. Then the completion of $\mathscr{H}(\widetilde{C})$ for the norm $(\,\cdot\,|\,(\lambda^+(a)+\lambda^-(a))\,\cdot\,)^{1/2}$ equals $|D_s|^{-1/2}L^2(\widetilde{C})$.*

Proof. By (8-5) and the definition of $|D_s|^{-1/2}L^2(\widetilde{C})$, it suffices to prove that the completion of $\mathscr{H}(\widetilde{C})$ for the norm $\big(u \mid (c^+(a)+c^-(a))u\big)^{1/2}$ equals $L^2(\widetilde{C})$. Let

$$u(a) = \begin{pmatrix} (\mathbb{1}+aa^*)^{1/2} & a \\ a^* & (\mathbb{1}+a^*a)^{1/2} \end{pmatrix}$$

and note that

$$u(a)^* c^{\pm}(0) u(a) = c^{\pm}(a). \tag{8-6}$$

Moreover, using the identity $af(a^*a) = f(aa^*)a$, valid for any Borel function f, we obtain that $u(a)^{-1} = u(-a)$, hence $u(a) : L^2(\widetilde{C}) \to L^2(\widetilde{C})$ is boundedly invertible. By (8-6), it suffices to treat the case $a = 0$, which is obvious since $c^+(0)+c^-(0) = \mathbb{1}$. □

8C. *Pure Hadamard states in M_0.* Our main result concerns the purity of the states induced in the bulk. We postpone the introduction of the key technical ingredients of the proof to Section 8D for the sake of self-consistency of our results on the characteristic Cauchy problem.

Theorem 8.4. *Assume that* $\dim M \geq 4$. *Let ω_C be a pure Hadamard state on* $\mathrm{CCR}(\mathscr{H}(\widetilde{C}), \sigma_C)$ *as in Theorem 8.2. Then the state ω induced by ω_C on* $\mathrm{CCR}(C_0^\infty(M_0)\,/\,PC_0^\infty(M_0), E_0)$ *is a pure state.*

Proof. The proof relies on Lemma 8.1 and on some results on the characteristic Cauchy problem in M_0, proved below in Section 8D. Recall that the map $\rho : \mathrm{Sol}_{\mathrm{sc}}(P_0) \to \mathscr{H}(\widetilde{C})$ was introduced in Definition 4.1. By Lemmas 8.1 and 8.3 it suffices to check that $\rho(\mathrm{Sol}_{\mathrm{sc}}(P_0))$ is dense in $|D_s|^{-1/2}L^2(\widetilde{C})$. Since $C_0^\infty(\mathbb{R}\times\mathbb{S}^{d-1})$ is dense in $|D_s|^{-1/2}L^2(\widetilde{C})$, it suffices, for $w \in C_0^\infty(\mathbb{R}\times\mathbb{S}^{d-1})$, to find a sequence $\phi_n \in \mathrm{Sol}_{\mathrm{sc}}(P_0)$ such that $\rho\phi_n \to w$ in $|D_s|^{-1/2}L^2(\widetilde{C})$.

We will use freely the notation introduced in Section 8D. We first fix a Cauchy surface Σ in (M, g) as in Section 8D2 to the future of $\operatorname{supp} w$. Note that, since w vanishes near $s = -\infty$, we know that w belongs to the space $\widetilde{H}^1_0(\widetilde{C}_0)$ introduced in Proposition 8.8. By Theorem 8.7 and Proposition 8.8, there exists f in the energy space $\mathscr{E}_0(\Sigma_0)$ such that $w = R \circ Tf$. Since $C_0^\infty(\Sigma_0)\oplus C_0^\infty(\Sigma_0)$ is dense in $\mathscr{E}_0(\Sigma_0)$, there exists a sequence $f_n \in C_0^\infty(\Sigma_0)\oplus C_0^\infty(\Sigma_0)$ such that $f_n \to f$ in $\mathscr{E}_0(\Sigma_0)$. By Theorem 8.7 and Proposition 8.8 we have $R\circ Tf_n \to w$ in $\widetilde{H}^1_0(\widetilde{C}_0)$, hence also $R\circ Tf_n \to w$ in $|D_s|^{-1/2}L^2(\widetilde{C})$, by Remark 8.9.

Let $\phi_n \in \mathrm{Sol}_{\mathrm{sc}}(P_0)$ be the solution with Cauchy data f_n on Σ_0. Then $\rho\phi_n = R\circ Tf_n \to w$ in $|D_s|^{-1/2}L^2(\widetilde{C})$, which completes the proof of the theorem. □

8D. *A characteristic Cauchy problem in M_0.* From Lemma 8.1 we see that, to deduce purity of the bulk state from the purity of the boundary state, the range of ρ in $\mathscr{H}(\widetilde{C})$ should be sufficiently large. One way to ensure this is to solve a *characteristic Cauchy problem* in M_0, that is, to construct an inverse for ρ. If M has a *compact* Cauchy surface, the characteristic problem was shown to be well posed in *energy spaces* by Hörmander [1990b]. With some care those results can be used in our situation.

8D1. *Characteristic Cauchy problem for compact Cauchy surfaces.* We recall an important result of [Hörmander 1990b] on the characteristic Cauchy problem in energy spaces, whose framework is as follows:

One considers a spacetime $(\widetilde{M}, \tilde{g})$ for $\widetilde{M} = \mathbb{R} \times \widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is a smooth *compact* manifold and $\tilde{g} = -\tilde{\beta}(t, \mathrm{x})\, dt^2 + \tilde{h}_{ij}(t, \mathrm{x})\, d\mathrm{x}^i\, d\mathrm{x}^j$. One also fixes a real function $\tilde{r} \in C^\infty(\widetilde{M})$.

If $\widetilde{\Sigma}_1$ is a Cauchy hypersurface in $(\widetilde{M}, \tilde{g})$, we will denote by

$$\widetilde{U}_{\widetilde{\Sigma}_1} : C^\infty(\widetilde{\Sigma}_1) \oplus C^\infty(\widetilde{\Sigma}_1) \to C^\infty(\widetilde{M})$$

the Cauchy evolution operator for $-\Box_{\tilde{g}} + \tilde{r}$, so that $\phi = \widetilde{U}_{\widetilde{\Sigma}_1} f$ solves

$$\begin{cases} -\Box_{\tilde{g}}\phi + \tilde{r}\phi = 0, \\ \phi\restriction_{\widetilde{\Sigma}_1} = f^0, \\ n^\mu \nabla_\mu \phi\restriction_{\widetilde{\Sigma}_1} = f^1. \end{cases}$$

A hypersurface $\widetilde{C}$ of the form

$$\widetilde{C} = \{(F(\mathrm{x}), \mathrm{x}) : \mathrm{x} \in \widetilde{\Sigma}\}, \quad F \text{ Lipschitz}, \tag{8-7}$$

is called *spacelike* (resp. *weakly spacelike*) if

$$\sup_{\mathrm{x}\in\widetilde{\Sigma}} \bigl(-\beta^{-1}(F(\mathrm{x}), \mathrm{x}) + \partial_i F(\mathrm{x}) h^{ij}(F(\mathrm{x}), \mathrm{x}) \partial_j F(\mathrm{x})\bigr) < 0 \quad (\text{resp. } \le 0).$$

If F is smooth then of course $\widetilde{C}$ is spacelike (resp. weakly spacelike) if and only if all tangent vectors at each point of $\widetilde{C}$ are spacelike (resp. spacelike or null).

Since $\widetilde{\Sigma}$ is compact and F Lipschitz, the Sobolev space $H^1(\widetilde{C})$ and of course $L^2(\widetilde{C})$ are well defined, for example by identifying $\widetilde{C}$ with $\widetilde{\Sigma}$ and using the Riemannian metric $\tilde{h}_{ij}(0, \mathrm{x})\, d\mathrm{x}^i\, d\mathrm{x}^j$ on $\widetilde{\Sigma}$ to equip $\widetilde{C}$ with a density $d\nu_{\widetilde{C}}$.

One also needs the measure

$$d\nu^0_{\widetilde{C}} = (\beta^{-1} - h^{ij}\, \partial_i \widetilde{F}\, \partial_j \widetilde{F})\, d\nu_{\widetilde{C}},$$

which vanishes if $\widetilde{C}$ is a null hypersurface.

We now set

$$\mathscr{E}(\widetilde{C}) := H^1(\widetilde{C}) \oplus L^2(\widetilde{C}, d\nu^0_{\widetilde{C}}). \tag{8-8}$$

Note that if $\widetilde{C}$ is spacelike (i.e., a Cauchy hypersurface), then $\mathscr{E}(\widetilde{C}) = H^1(\widetilde{C}) \oplus L^2(\widetilde{C})$.

Theorem 8.5 [Hörmander 1990b]. *Let $\widetilde{\Sigma}_1$ be any Cauchy hypersurface in $\widetilde{M}$ and let $\widetilde{C}$ be weakly spacelike of the form* (8-7). *Then the map*

$$\widetilde{T} : \mathscr{E}(\widetilde{\Sigma}_1) \to \mathscr{E}(\widetilde{C}), \quad f \mapsto ((\widetilde{U}_{\widetilde{\Sigma}_1} f)\restriction_{\widetilde{C}}, (\beta^{-1}\partial_t \widetilde{U}_{\widetilde{\Sigma}_1} f)\restriction_{\widetilde{C}}),$$

is a homeomorphism.

Note that, if $\widetilde{C}$ is characteristic, then $L^2(\widetilde{C}, d\nu^0_{\widetilde{C}}) = \{0\}$ and $\mathscr{E}(\widetilde{C}) = H^1(\widetilde{C})$, so one obtains as a particular case the solvability of the characteristic Cauchy problem in energy spaces.

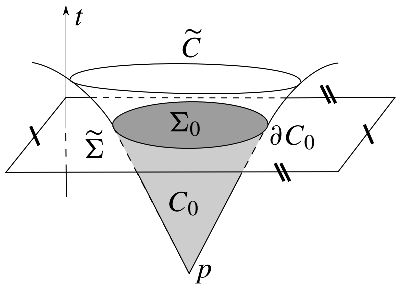

Figure 2. The modified cone $\widetilde{C}$.

8D2. *Embedding M_0 into $\widetilde{M}$.* We will use Hörmander's result, recalled above, to solve a characteristic Cauchy problem in M_0 in an arbitrary neighborhood of p. The first task is to locally embed M into a spacetime $\widetilde{M}$ as above.

We fix a Cauchy hypersurface Σ to the future of p and identify M with $\mathbb{R}\times\Sigma$ equipped with

$$g = -\beta(t,\mathrm{x})\,dt^2 + h_{ij}(t,\mathrm{x})\,d\mathrm{x}^i\,d\mathrm{x}^j.$$

We set $\Sigma_0 = \Sigma\cap M_0$ and fix an open, precompact set U such that $J^-(\Sigma_0)\cap J^+(p)\subset U$.

The following lemma shows that, over U, C can be parametrized by Σ.

Lemma 8.6. *There exists a bounded, Lipschitz function F defined on Σ such that*

$$\overline{C}\cap U = \{(t,\mathrm{x}) : t = F(\mathrm{x})\}\cap U.$$

Proof. The proof is given in Appendix A6. □

We next embed Σ_0 into a smooth compact manifold $\widetilde{\Sigma}$. We consider the spacetime $\widetilde{M} = \mathbb{R}\times\widetilde{\Sigma}$ and extend F to a Lipschitz function $\widetilde{F}$ on $\widetilde{\Sigma}$ and g to a metric $\tilde{g}$ as in Section 8D1. We set

$$\widetilde{C} = \{t = \widetilde{F}(\mathrm{x})\}\subset\widetilde{M}$$

and define

$$C_0 := (J^-(\Sigma_0; M)\cap C)\cup\{p\}. \tag{8-9}$$

C_0 is an open subset of $\overline{C}$, with $\overline{C}_0$ compact in M and

$$\partial\Sigma_0 = \partial C_0. \tag{8-10}$$

We claim that we can choose the embedding $\Sigma_0\subset\widetilde{\Sigma}$ and the extensions $\widetilde{F}$ and $\tilde{g}$ so that

$$J^-(\widetilde{\Sigma}\backslash\overline{\Sigma}_0; \widetilde{M})\cap\overline{C}_0 = \varnothing, \tag{8-11}$$

$$\widetilde{C}\ \text{ is weakly spacelike in }\ \widetilde{M}. \tag{8-12}$$

This is clearly possible by modifying Σ, F and g only outside a large open set U and using that the embedding of (M_0, g) into (M, g) is causally compatible; see (2-3).

The situation is summarized in Figure 2. Identification symbols (a single and double bar) are used to stress that $\widetilde{\Sigma}$ is compact.

8D3. *Sobolev spaces.* We now recall some well-known facts about Sobolev spaces. If Ω is a relatively compact open set in a compact manifold X with smooth boundary $\partial\Omega$, then $H^1_0(\Omega)$ — defined as the closure of $C^\infty_0(\Omega)$ in $H^1(\Omega)$ — can also be characterized as $H^1_0(\Omega) = \{u \in H^1(\Omega) : u\restriction_{\partial\Omega} = 0\}$. The restriction operator $r_\Omega : H^1(X) \to H^1(\Omega)$ is surjective from $H^1_{\partial\Omega}(X) = \{u \in H^1(X) : u\restriction_{\partial\Omega} = 0\}$ to $H^1_0(\Omega)$, with right inverse $e_\Omega : H^1_0(\Omega) \to H^1_{\partial\Omega}(X)$ equal to the extension by 0 in $X\backslash\Omega$.

We set $\mathscr{E}_0(\Omega) := H^1_0(\Omega) \oplus L^2(\Omega)$ and $\mathscr{E}_{\partial\Omega}(X) = H^1_{\partial\Omega}(X) \oplus L^2(X)$. We will still denote the operator $r_\Omega \oplus r_\Omega : \mathscr{E}_{\partial\Omega}(X) \to \mathscr{E}_0(\Omega)$ by r_Ω and $e_\Omega \oplus e_\Omega : \mathscr{E}_0(\Omega) \to \mathscr{E}_{\partial\Omega}(X)$ by e_Ω.

We will use these facts for $\Omega = \Sigma_0$, C_0 and $X = \widetilde{\Sigma}$, $\widetilde{C}$. If $\Omega = C_0$, then we use the notation in (8-8), i.e., $\mathscr{E}_0(C_0) = H^1_0(C_0) \oplus \{0\} \sim H^1_0(C_0)$, since C_0 is characteristic.

8D4. *Characteristic Cauchy problem.* In the theorem below, we denote by U_{Σ_0} the operator $U_{\widetilde{\Sigma}} \circ e_{\Sigma_0}$, that is, the Cauchy evolution operator (in $\widetilde{M}$) for Cauchy data in $\mathscr{E}_0(\Sigma_0)$ (extended by 0 in $\widetilde{\Sigma}\backslash\Sigma_0$).

Theorem 8.7. *The map*

$$T : \mathscr{E}_0(\Sigma_0) \to \mathscr{E}_0(C_0), \quad f \mapsto (U_{\Sigma_0} f)\restriction_{C_0},$$

is a homeomorphism.

Proof. We will prove the theorem by reducing ourselves to Theorem 8.5. We first claim that

$$T = r_{C_0} \circ \widetilde{T} \circ e_{\Sigma_0}. \tag{8-13}$$

In fact this follows from the fact that $e_{\Sigma_0} : \mathscr{E}_0(\Sigma_0) \to \mathscr{E}(\widetilde{\Sigma})$ is the extension by 0.

By Theorem 8.5, this implies that $T : \mathscr{E}_0(\Sigma_0) \to \mathscr{E}(C_0)$. Moreover, by finite speed of propagation, if $f \in C^\infty_0(\Sigma_0) \oplus C^\infty_0(\Sigma_0)$ then Tf vanishes near ∂C_0, hence T maps continuously $\mathscr{E}_0(\Sigma_0)$ into $\mathscr{E}_0(C_0)$.

We next claim that $S = r_{\Sigma_0} \circ \widetilde{T}^{-1} \circ e_{C_0}$ is a right inverse to T. In fact, let $g \in \mathscr{E}_0(C_0)$ and $\tilde{f} = \widetilde{T}^{-1} \circ e_{C_0} g = (\tilde{f}^0, \tilde{f}^1) \in \mathscr{E}(\widetilde{\Sigma})$. Since $\partial\Sigma_0 = \partial C_0$, we have $\tilde{f}^0\restriction_{\partial\Sigma_0} = g\restriction_{\partial C_0} = 0$, hence $e_{\Sigma_0} \circ r_{\Sigma_0} \tilde{f} \in \mathscr{E}(\widetilde{\Sigma})$. Since $\tilde{f} - e_{\Sigma_0} \circ r_{\Sigma_0} \tilde{f}$ vanishes on $\overline{\Sigma}_0$, we obtain by (8-11) and finite speed of propagation that

$$r_{C_0} \circ \widetilde{T}(\tilde{f} - e_{\Sigma_0} \circ r_{\Sigma_0} \tilde{f}) = 0,$$

hence $T \circ Sg = r_{C_0} \circ \widetilde{T} \tilde{f} = r_{C_0} \circ e_{C_0} g = g$. This completes the proof of the theorem. $\square$

8E. ***Sobolev space on the cone in null coordinates.*** Let us set

$$R : C^\infty(C) \to C^\infty(\mathbb{R} \times \mathbb{S}^{d-1}), \quad g \mapsto \beta^{-1} g(s, \theta).$$

The goal in this subsection is to describe more precisely the image of $H^1_0(C_0)$ under R.

We will denote by $\widetilde{C}_0 \subset \mathbb{R} \times \mathbb{S}^{d-1}$ the image of C_0 under the map $q \mapsto (s(q), \theta(q))$ for $q \in C$, where the coordinates (s, θ) are as constructed in Lemma 2.6. Using that $\partial C_0 = \partial\Sigma_0$ is spacelike and included in C, we easily obtain from Lemma 2.7 that $\widetilde{C}_0$ is of the form

$$\widetilde{C}_0 = \{(s, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1} : s < s_0(\theta)\}$$

for some smooth function s_0. To simplify notation, the measure $|m|^{1/2}(\theta)\, d\theta$ on $\mathbb{S}^{d-1}$ will be simply denoted by $d\theta$. We also set $r = \mathrm{e}^s$.

Proposition 8.8. *Assume $d = \dim M - 1 \geq 3$. Then the image of $H_0^1(C_0)$ under R equals the completion of $C_0^\infty(\widetilde{C}_0)$ under the norm*

$$\|\psi\|_1 := \left(\int_{\widetilde{C}_0} (r^{-1}|\partial_s\psi|^2 + r^{-1}|\partial_\theta\psi|^2 + r^{-1}|\psi|^2)\, ds\, d\theta\right)^{\frac{1}{2}}.$$

We will denote this space by $\widetilde{H}_0^1(\widetilde{C}_0)$.

Remark 8.9. Since $r \leq r_0$ on C_0, we see that $\widetilde{H}_0^1(\widetilde{C}_0)$ injects continuously into $|D_s|^{-1/2}L^2(\mathbb{R}\times\mathbb{S}^{d-1})$.

Proof. We recall that (v, ψ) (see (2-4)) are coordinates on C such that the topology in $H_0^1(C_0)$ is given by the norm

$$\left(\int_{C_0} (|v|^{d-1}|\partial_v g|^2 + |v|^{d-3}|\partial_\psi g|^2 + |v|^{d-1}|g|^2)\, dv\, d\psi\right)^{\frac{1}{2}}.$$

Recall that we have set $r = \mathrm{e}^s$. A function $g \in H_0^1(C_0)$ expressed in the coordinates (s, θ) or (r, θ) will still be denoted by g. Similarly, the image of $\widetilde{C}_0$ under the map $(s, \theta) \mapsto (\mathrm{e}^s, \theta)$ will still be denoted by $\widetilde{C}_0$.

From Lemma 2.6(3) and a routine computation, we see that an equivalent norm on $H_0^1(C_0)$ is

$$\left(\int_{\widetilde{C}_0} (r^{d-1}|\partial_r g|^2 + r^{d-3}|\partial_\theta g|^2 + r^{d-1}|g|^2\, dr\, d\theta)\right)^{\frac{1}{2}}. \tag{8-14}$$

Since $d = \dim M - 1 \geq 3$, Hardy's inequality $-\Delta \geq C|x|^{-2}$ holds on $L^2(\mathbb{R}^d)$. Considering (r, θ) as polar coordinates on $\mathbb{R}^d$, we obtain that

$$\int_{\widetilde{C}_0} r^{d-1}|\partial_r g|^2 + r^{d-3}|\partial_\theta g|^2\, dr\, d\theta \geq C\int_{\widetilde{C}_0} r^{d-3}|g|^2\, dr\, d\theta, \quad g \in H_0^1(C_0).$$

Therefore, adding a term $r^{d-3}|g|^2$ under the integral in (8-14) yields an equivalent norm on $H_0^1(C_0)$. Since r is bounded on $\widetilde{C}_0$, this term dominates the term $r^{d-1}|g|^2$ and we finally obtain that the topology of $H^1(C_0)$ is given by the norm

$$\left(\int_{\widetilde{C}_0} (r^{d-1}|\partial_r g|^2 + r^{d-3}|\partial_\theta g|^2 + \alpha r^{d-3}|g|^2)\, dr\, d\theta\right)^{\frac{1}{2}},$$

where the constant $\alpha > 0$ can be chosen arbitrarily large. Going back to coordinates (s, θ), we obtain the norm

$$\left(\int_{\widetilde{C}_0} (r^{d-2}|\partial_s g|^2 + r^{d-2}|\partial_\theta g|^2 + \alpha r^{d-2}|g|^2)\, ds\, d\theta\right)^{\frac{1}{2}}. \tag{8-15}$$

For two functions $m, n \in C^\infty(\mathbb{C}_0)$ we write $m \sim n$ if $m = r_0 n$ for some $r_0, r_0^{-1} \in S^0$, where the class S^0 is as defined in Section 2E. We have $\beta \sim r^{-(d-1)/2}$, hence

$$\partial_s\beta,\, \partial_\theta\beta \sim r^{-(d-1)/2}. \tag{8-16}$$

Setting $\psi = Rg = \beta^{-1}g$, we have

$$\partial_s g = \beta\,\partial_s\psi + (\partial_s\beta)\psi \quad \text{and} \quad \partial_\theta g = \beta\,\partial_\theta\psi + (\partial_\theta\beta)\psi.$$

Then, using (8-16) and choosing $\alpha \gg 1$ in (8-15), we obtain that (8-15) is equivalent to

$$\left(\int_{\widetilde{C}_0} (r^{-1}|\partial_s\psi|^2 + r^{-1}|\partial_\theta\psi|^2 + r^{-1}|\psi|^2)\, ds\, d\theta\right)^{\frac{1}{2}}. \tag{8-17}$$

This completes the proof of the proposition. □

9. Change of null coordinates

The map $\rho : \mathrm{Sol}_{\mathrm{sc}}(P_0) \to \mathcal{H}(\widetilde{C})$ introduced in Definition 4.1 depends on the choice of the null coordinates (s, θ) on C, i.e., on the choice of the initial hypersurface S used in Lemma 2.6 to construct (s, θ). In this section we discuss how our class of Hadamard states depends on the above choice.

9A. ***New null coordinates.*** We fix a reference hypersurface S in C, yielding null coordinates (s, θ) near C such that $g\restriction_C$ is given by (2-6) and $S = \{f = s = 0\}$.

We choose another hypersurface $\tilde{S}$ transverse to $\nabla^a f$ in C, hence

$$\tilde{S} = \{f = 0,\ s = b(\theta)\} \quad \text{for some } b \in C^\infty(\mathbb{S}^{d-1}). \tag{9-1}$$

Since $\nabla^a f\restriction_C = \partial_s$, we obtain that the new coordinates $(\tilde{s}, \tilde{\theta})$ obtained from Lemma 2.6 with S replaced by $\tilde{S}$ are given by

$$\tilde{\theta} = \theta, \quad \tilde{s}(s, \theta) = s - b(\theta). \tag{9-2}$$

We then have

$$g\restriction_C = -2\, df\, d\tilde{s} + \tilde{h}_{ij}(\tilde{s}, \theta)\, d\theta^i\, d\theta^j$$

and a standard computation shows that $|h|(\tilde{s}, \theta) = |h|(s, \theta)$, hence $\tilde{\beta}(\tilde{s}, \theta) = \beta(s, \theta)$. Denoting by $\tilde{\rho}$ the analog of ρ in Definition 4.1 for the new coordinates $(\tilde{s}, \theta)$ we then have

$$\tilde{\rho}\phi = U\rho\phi, \quad \phi \in \mathrm{Sol}_{\mathrm{sc}}(P_0), \tag{9-3}$$

where

$$U : \mathcal{H}(\widetilde{C}) \to \mathcal{H}(\widetilde{C}), \quad g \mapsto Ug(s, \theta) = g(s + b(\theta), \theta).$$

The map U is symplectic on $(\mathcal{H}(\widetilde{C}), \sigma_C)$ and unitary on $L^2(\widetilde{C})$ with $U^* D_s U = D_s$.

Proposition 9.1. *If $A \in \widetilde{\Psi}^{-\infty,p}(\widetilde{C})$ then $UAU^{-1} \in \widetilde{\Psi}^{-\infty,p}(\widetilde{C})$.*

Remark 9.2. The above invariance property does not hold for the classes $\Psi^{m,p}(\widetilde{C})$ since, for example, the classes $\Psi^{m,p}(\mathbb{R} \times \mathbb{R}^{d-1})$ are not even preserved by linear changes of variables $(s, y) \mapsto (s + Ay, y)$.

Proof. We will use the Beals criterion explained in Section 6C, which implies that $B \in \widetilde{\Psi}^{-\infty,p}(\widetilde{C})$ if and only if, for any functions $g_1, \dots g_n \in C^\infty(\mathbb{S}^{d-1})$ and smooth vector fields $X_1, \dots, X_m$ on $\mathbb{S}^{d-1}$ and any $N \in \mathbb{N},\ k, k' \in \mathbb{R}$, one has

$$\mathrm{ad}_{X_1} \cdots \mathrm{ad}_{X_m}\, \mathrm{ad}_{g_1} \cdots \mathrm{ad}_{g_n}\, B : H^{k,k'}(\widetilde{C}) \to H^{k+N,k'-p+n}(\widetilde{C}). \tag{9-4}$$

To simplify notation, we rewrite (9-4) as

$$\mathrm{ad}_{\overline{X}}^{\alpha}\, \mathrm{ad}_{\overline{g}}^{\beta}\, B : H^{k,k'}(\widetilde{C}) \to H^{k+N,k'+p+|\beta|}(\widetilde{C}), \tag{9-5}$$

denoting by $\bar{X}$ and $\bar{g}$ an arbitrary n-tuple of vector fields and m-tuple of functions, respectively.

If g is a function on $\mathbb{S}^{d-1}$, considered as a multiplication operator, and X is a vector field on $\mathbb{S}^{d-1}$, we have

$$U^{-1}gU = g, \quad U^{-1}XU = X + (X \cdot db)\partial_s, \quad U^{-1}\partial_s U = \partial_s. \tag{9-6}$$

Now let $A \in \widetilde{\Psi}^{-\infty,p}(\widetilde{C})$. For $\psi \in C^\infty(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$, let us denote by A_ψ the operator with distributional kernel $A(s_1, s_2, \theta_1, \theta_2)\psi(\theta_1, \theta_2)$. By the well-known properties of the pseudodifferential calculus on $\mathbb{S}^{d-1}$, we know that if $\psi = 1$ in some neighborhood of the diagonal then $A - A_\psi \in \widetilde{\Psi}^{-\infty,-\infty}(\widetilde{C})$ or, equivalently, maps $H^{k,k'}(\widetilde{C})$ into $H^{k+N,k'+N}(\widetilde{C})$ for any k, k' and N. Using (9-6) this implies that $U(A - A_\psi)U^{-1}$ has the same property, hence belongs to $\widetilde{\Psi}^{-\infty,-\infty}(\widetilde{C})$.

Therefore we can replace A by A_ψ and assume that the kernel of A is supported in $\mathbb{R}\times\mathbb{R}\times\Omega$, where Ω is an arbitrarily small neighborhood of the diagonal in $\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}$. Introducing a smooth partition of unity $1 = \sum_1^M \chi_i$ on $\mathbb{S}^{d-1}$, we see that we can replace A by $\chi A \chi$, where $\chi \in C^\infty(\mathbb{S}^{d-1})$ is supported in a small neighborhood of a point $\theta_0 \in \mathbb{S}^{d-1}$. We pick local coordinates $\theta_1, \dots, \theta_{d-1}$ near θ_0 and rewrite (9-5) as

$$\langle\partial_s\rangle^{k+N}\langle\partial_\theta\rangle^{k'-p+|\beta|}\,\mathrm{ad}^\alpha_{\bar{X}}\,\mathrm{ad}^\beta_{\bar{g}}\,A\langle\partial_s\rangle^{-k}\langle\partial_\theta\rangle^{-k'} \in B(L^2(\widetilde{C})). \tag{9-7}$$

We now set $A' = UAU^{-1}$. Note first that if the kernel of A is supported in $\mathbb{R}\times\mathbb{R}\times\Omega$ then so is the kernel of A', hence by the above discussion it suffices to check that A' satisfies (9-7). Let us set $U^{-1}XU = X'$ if X is a vector field on $\mathbb{S}^{d-1}$ and, in particular, $\partial'_\theta = U^{-1}\partial_\theta U = \partial_\theta + \partial_\theta b\partial_s$. Then an easy computation yields

$$\begin{aligned}\langle\partial_s\rangle^{k+N}\langle\partial_\theta\rangle^{k'-p+|\beta|}\,\mathrm{ad}^\alpha_{\bar{X}}\,\mathrm{ad}^\beta_{\bar{g}}\,UAU^{-1}\langle\partial_s\rangle^{-k}\langle\partial_\theta\rangle^{-k'}&\\ = U\langle\partial_s\rangle^{k+N}\langle\partial'_\theta\rangle^{k'-p+|\beta|}\,\mathrm{ad}^\alpha_{\bar{X}'}\,\mathrm{ad}^\beta_{\bar{g}}\,A\langle\partial_s\rangle^{-k}\langle\partial'_\theta\rangle^{-k'}U^{-1}.&\end{aligned} \tag{9-8}$$

Using (9-6) and the fact that $A \in \widetilde{\Psi}^{-\infty,p}(\widetilde{C})$, we obtain that

$$\mathrm{ad}^\alpha_{\bar{X}'}\,\mathrm{ad}^\beta_{\bar{g}}\,A \in \widetilde{\Psi}^{-\infty,p-|\beta|}(\widetilde{C}) \quad \text{and} \quad \langle\partial_s\rangle^{N}\langle\partial_\theta\rangle^{k'-p+|\beta|}\,\mathrm{ad}^\alpha_{\bar{X}'}\,\mathrm{ad}^\beta_{\bar{g}}\,A\langle\partial_s\rangle^{N}\langle\partial_\theta\rangle^{-k'} \in B(L^2(\widetilde{C}))$$

for any $N \in \mathbb{N}$. It follows that the left-hand side of (9-8) belongs to $B(L^2(\widetilde{C}))$ if, for any $s \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that

$$\langle\partial_s\rangle^{-N}\langle\partial'_\theta\rangle^{s}\langle\partial_\theta\rangle^{-s},\ \langle\partial_s\rangle^{-N}\langle\partial_\theta\rangle^{s}\langle\partial'_\theta\rangle^{-s} \in B(L^2(\widetilde{C})). \tag{9-9}$$

Let us now prove (9-9). The first statement of (9-9) is easy to check for $s \in \mathbb{N}$, using that $\partial'_\theta = \partial_\theta + \partial_\theta b\,\partial_s$. Conjugation by U gives the second statement for $s \in \mathbb{N}$. By duality and interpolation, we then obtain (9-9) for arbitrary s, which completes the proof of the proposition. □

From Proposition 9.1 and the fact that $U^* D_s U = D_s$, we immediately obtain the following result:

Proposition 9.3. *The classes of Hadamard states obtained in Theorems 7.4 and 8.2 are independent of the choice of the null coordinates* (s, θ).

Appendix

A1. ***Stokes formula.*** Let (M, g) an orientable, oriented pseudo-Riemannian manifold of dimension n. We denote by $d\,\mathrm{Vol}_g \in \bigwedge^n(M)$ the associated volume form and by $d\mu_g = |d\,\mathrm{Vol}_g|$ the associated density.

Let $\Sigma \subset M$ a smooth submanifold of codimension 1 and $\iota : \Sigma \to M$ the natural injection, which induces $\iota^* : \bigwedge(M) \to \bigwedge(\Sigma)$. From the orientation of M and a continuous transverse vector field $v \in T_\Sigma M$, we obtain an induced orientation of Σ. If $\Sigma \subset \partial U$ for an open set $U \subset M$ with piecewise smooth boundary ∂U, we choose v pointing outwards.

If $\omega \in \bigwedge^n(M)$ and $X \in TM$, then $X \lrcorner\, \omega \in \bigwedge^{n-1}(M)$ and one sets

$$\iota_X^* \omega := \iota^*(X \lrcorner\, \omega) \in \textstyle\bigwedge^{n-1}(\Sigma).$$

Similarly, if $\mu = |\omega|$ is a density on M, we set $\iota_X^* \mu := |\iota_X \omega|$, which is a density on Σ.

If ∇_a is the Levi-Civita connection associated to g then

$$\nabla_a X^a \, d\,\mathrm{Vol}_g = d(X \lrcorner\, d\,\mathrm{Vol}_g),$$

which, applying Stokes formula

$$\int_U d\omega = \int_{\partial U} \iota^* \omega, \quad \omega \in \textstyle\bigwedge^{n-1}(M), \tag{A-1}$$

to $\omega = \iota_X^* \, d\,\mathrm{Vol}_g$ yields

$$\int_U \nabla_a X^a \, d\,\mathrm{Vol}_g = \int_{\partial U} \iota_X^* \, d\,\mathrm{Vol}_g \,. \tag{A-2}$$

Noncharacteristic boundaries. Assume first $\Sigma \subset \partial U$ is *noncharacteristic*, that is, the one-dimensional space

$$T_x(\Sigma)^{\mathrm{ann}} \subset T_x M^*$$

is not *null* (the superscript "ann" denotes the annihilator). It follows that the metric $h := \iota^* g$ on Σ is nondegenerate (in the Lorentzian case, one typically assume that Σ is spacelike; then $h = \iota^* g$ is Riemannian). Let $n \in T_\Sigma M$ be the unit, outward-pointing normal vector field to σ. Then

$$d\,\mathrm{Vol}_h = \iota_n^* \, d\,\mathrm{Vol}_g \quad \text{and} \quad \iota_X^* \, d\,\mathrm{Vol}_g = X^a n_a \, d\,\mathrm{Vol}_h, \tag{A-3}$$

hence

$$\int_\Sigma \iota_X^* \, d\,\mathrm{Vol}_g := \int_\Sigma X^a n_a \, d\sigma_h.$$

If all of ∂U is noncharacteristic, then from (A-2) we obtain Gauss's formula

$$\int_U \nabla_a X^a \, d\mu_g = \int_\Sigma X^a n_a \, d\sigma_h, \tag{A-4}$$

where $d\sigma_h = |d\,\mathrm{Vol}_h|$.

Characteristic boundaries. Assume now that Σ is characteristic. Then there is no normal vector field anymore. To express the right-hand side of (A-2), one chooses a defining function f for Σ, i.e., such that $f = 0$ and $df \neq 0$ on Σ, and completes f with coordinates $y^1, \dots, y^{n-1}$ such that $df \wedge dy^1 \wedge \cdots \wedge dy^{n-1}$ is positively oriented. Then, computing in the coordinates $f, y^1, \dots, y^{n-1}$, one sees that

$$\iota_X^* \, d\,\mathrm{Vol}_g = X^a \nabla_a f |g|^{1/2} \, dy^1 \wedge \cdots \wedge dy^{n-1},$$

hence

$$\int_\Sigma \iota_X^* \, d\,\mathrm{Vol}_g = \int_\Sigma X_a \nabla^a f |g|^{1/2} \, dy^1 \wedge \cdots \wedge dy^{n-1} \tag{A-5}$$

In the general case we can, for example, split ∂U as $\Sigma_1 \cup \Sigma_2$, where Σ_1 is noncharacteristic and Σ_2 is characteristic, and obtain

$$\int_U \nabla_a X^a \, d\mu_g = \int_{\Sigma_1} X^a n_a \, d\sigma_h + \int_{\Sigma_2} X_a \nabla^a f |g|^{1/2} \, dy^1 \wedge \cdots \wedge dy^{n-1}. \tag{A-6}$$

A2. *Conformal transformations.* In this section we briefly discuss conformal transformations of a globally hyperbolic spacetime (M, g). Let $\omega \in C^\infty(M)$ be strictly positive and consider the conformally related metric

$$g' = \omega^2 g.$$

Set

$$P = -\nabla^a \nabla_a + \frac{n-2}{4(n-1)} R,$$

where R is the scalar curvature. For this special choice of the lower-order terms, the conformal transformation $g \to g'$ amounts to

$$P' = \omega^{-n/2-1} P \omega^{n/2-1}.$$

This entails that the causal propagators are related by $E' = \omega^{-n/2+1} E \omega^{n/2+1}$. One concludes that multiplication by $\omega^{-n/2+1}$ induces a symplectic map

$$(\mathrm{Sol}_{\mathrm{sc}}(P), \sigma) \xrightarrow{\omega^{-n/2+1}} (\mathrm{Sol}_{\mathrm{sc}}(P'), \sigma'), \tag{A-7}$$

where σ and σ' are defined as in (3-2) using the respective volume densities.

We apply this discussion to (M_0, g) and the conformally related spacetime with metric $g' = \omega^2 g$. In the setting of Section 4A, there is a monomorphism of symplectic spaces

$$(\mathrm{Sol}_{\mathrm{sc}}(P_0), \sigma_0) \xrightarrow{\rho} (\mathscr{H}(\widetilde{C}), \sigma_C).$$

By (A-7) we also have a monomorphism

$$(\mathrm{Sol}_{\mathrm{sc}}(P_0'), \sigma_0') \xrightarrow{\rho \circ \omega^{n/2-1}} (\mathscr{H}(\widetilde{C}), \sigma_C).$$

Therefore, one can construct states for the conformally related spacetime using the bulk-to-boundary correspondence with a modified trace map $\rho' = \rho \circ \omega^{n/2-1}$.

A3. ***Proof of Lemma 2.7.*** We fix a point $q \in C$ and complete the coordinate $x^0 = f$ by local coordinates $\bar{x} = (x^1, \dots, x^d)$ near q. The functions s and θ_k defined on C are denoted by $s(\bar{x})$ and $\theta_k(\bar{x})$, since $\bar{x}$ are local coordinates on C. We denote by $h(\bar{x})$ the restriction of g^{-1} to T^*C. Note that the fact that C is null implies that $g^{00}(0, \bar{x}) \equiv 0$ and that from Lemma 2.6 we have

$$g^{i0}(\bar{x})\,\partial_i s(\bar{x}) = -1, \quad g^{i0}(\bar{x})\,\partial_i \theta_k(\bar{x}) = 0. \tag{A-8}$$

If X is a null vector, orthogonal to $C \cap \{s(\bar{x}) = s(q)\}$ and transverse to C, we obtain that

$$gX = \lambda\big(\tfrac{1}{2}\nabla_i s \nabla^i s, \nabla_i s\big), \quad \lambda \in \mathbb{R}.$$

Let us denote for the moment by $\tilde{s}$ and $\tilde{\theta}_k$ the extensions of s and θ_k outside C, which are constant along the flow of X. We obtain that, on C,

$$d\tilde{s} = \big(\tfrac{1}{2} ds \cdot h\, ds, ds\big), \quad d\tilde{\theta}_k = (ds \cdot h\, d\theta_k, d\theta_k).$$

Using also $df = (1, 0, \dots, 0)$ and (A-8), a routine computation leads to the following identities on C:

$$\begin{aligned}
&df \cdot g^{-1}\, df = d\tilde{s} \cdot g^{-1}\, d\tilde{s} = df \cdot g^{-1}\, d\tilde{\theta}_k = d\tilde{s} \cdot g^{-1}\, d\tilde{\theta}_k = 0,\\
&df \cdot g^{-1}\, d\tilde{s} = d\tilde{s} \cdot g^{-1}\, df = -1,\\
&d\tilde{\theta}_k \cdot g^{-1}\, d\tilde{\theta}_l = \partial_i \theta_k h^{ij} \partial_j \theta_l.
\end{aligned}$$

This implies that g is of the form (2-6) on C.

A4. ***Proof of Lemma 2.6.*** Since $(y^0, \bar{y})$ are normal coordinates, we have

$$g\restriction_C = -dv\, dw + \tfrac{1}{4} v^2 m_{ij}(\psi)\, d\psi^i\, d\psi^j + v^2 g_1, \tag{A-9}$$

where $m_{ij}(\psi)\, d\psi^i\, d\psi^j$ is the standard Riemannian metric on $\mathbb{S}^{d-1}$ and g_1 is a smooth pseudo-Riemannian metric in the arguments dv, dw and $v\, d\psi^i$.

We start by expressing f in the normal coordinates $(y^0, \bar{y})$. By Malgrange's preparation theorem [Hörmander 1990a, Theorem 7.5.6] one can write

$$f(y^0, \bar{y}) = m(y^0, \bar{y})((y^0)^2 - |\bar{y}|^2) + a(\bar{y})y^0 + b(\bar{y})$$

for m near $(0, 0)$ and $a, b \in C^\infty$ near 0. Since $C \subset f^{-1}(\{0\})$, we obtain that $b(\bar{y}) = a(\bar{y})|\bar{y}|$ and, since $b \in C^\infty(\mathbb{R}^d)$, necessarily $a \in O(|\bar{y}|^\infty)$. Moreover, from the Hessian of f at p we obtain that $m(0, 0) = 1$.

Going to coordinates (v, w, ψ), we obtain

$$f(v, w, \psi) = m(v, w, \psi)vw + wa(v, w, \psi)$$

for $a \in O(|w - v|^\infty)$. Using also that $m(0, 0, \psi) = 1$, it follows that

$$\partial_v f(v, 0, \psi) = \partial_{\psi^i} f(v, 0, \psi) = 0 \quad \text{and} \quad \partial_w f(v, 0, \psi) = v + r(v, \psi)$$

for $r \in O(|v|^2)$. Using (A-9) to express $(g^{-1})\restriction_C$, we obtain after an easy computation that

$$\nabla^a f = -2v\big((1 + va^0(v, \psi))\,\partial_v + va^i(v, \psi)\,\partial_{\psi^i}\big), \tag{A-10}$$

where a^0 and a^i are smooth, bounded functions near $v=0$.

Let us now prove (1). Using (A-10) we obtain the equation near p

$$(v+v^2a^0(v,\psi))\,\partial_v s+v^2a^i(v,\psi)\,\partial_{\psi^i}s=\tfrac{1}{2}$$

for smooth functions a^0 and a^i. We set $s=\frac{1}{2}\ln(vh(v,\psi))$ and obtain after an elementary computation

$$(1+va^0)\,\partial_v h+a^0h+va^i(v,\psi)\,\partial_{\psi^i}h=0,$$

which we can uniquely solve on $[-\epsilon_1,\epsilon_1]\times\mathbb{S}^{d-1}$ by fixing $h(0,\psi)$. We may fix $h(0,\psi)>0$ to ensure that $s(\epsilon_0,\psi)=0$. We obtain $s=\frac{1}{2}\ln v+\frac{1}{2}\ln h(v,\psi)$ for $h\in C^\infty([-\epsilon_1,\epsilon_1]\times\mathbb{S}^{d-1})$, $h>0$.

It remains to extend s globally to C. To do this it suffices to check that, for any $q\in C$, the integral curve of $\nabla^a f$ through q crosses S at one and only one point. By [Wald 1984, Corollary to Theorem 8.1.2] we know that q can be joined to p by a null geodesic γ. Locally, a null geodesic on C is, modulo reparametrization, an integral curve of $\nabla^a f$. Since $\nabla^a f$ is complete, the whole $\gamma\backslash\{p\}$ is an integral curve of $\nabla^a f$. Hence the integral curve of $\nabla^a f$ through q crosses S. Choosing ϵ_0 in (2-5) small enough, we can ensure that $\nabla^a f\nabla_a v>0$ on S, hence the integral curve through q crosses S at only one point. We can hence extend s globally to C as a C^∞ function.

The proof of (2) is similar. We obtain the equation near p

$$(v+v^2a^0(v,\psi))\,\partial_v\theta^j+v^2a^i(v,\psi)\,\partial_{\psi^i}\theta^j=0$$

or, equivalently,

$$(1+va^0(v,\psi))\,\partial_v\theta^j+va^i(v,\psi)\,\partial_{\psi^i}\theta^j=0,$$

which we can solve in $]-\epsilon_1,\epsilon_1[\times\mathbb{S}^{d-1}$ by imposing $\theta^j(\epsilon_0,\psi)=\psi^j$. The estimate (3) on θ^j is immediate. We extend θ^j to all of C by the same argument as before.

A5. *Proof of Lemma 6.6.* We use the characterization of the wavefront set of kernels using oscillatory test functions, which we now recall.

Let $(\tilde{s},\tilde{y})\in C$ and $\lambda\geq 1$. We set, for $(\sigma,\eta)\in\mathbb{R}\times\mathbb{R}^{d-1}$,

$$v_{\sigma,\lambda}(\cdot)=\chi(\cdot)\mathrm{e}^{\mathrm{i}\lambda\langle\cdot,\sigma\rangle}\in C_0^\infty(\mathbb{R})\quad\text{and}\quad w_{\eta,\lambda}(\cdot)=\psi(\cdot)\mathrm{e}^{\mathrm{i}\lambda\langle\cdot,\eta\rangle}\in C^\infty(\mathbb{S}^{d-1}),\tag{A-11}$$

where $\chi\in C_0^\infty(\mathbb{R})$ and $\psi\in C^\infty(\mathbb{S}^{d-1})$ are supported near $\tilde{s}$ and $\tilde{y}$, respectively. We set $u_{(\sigma,\eta),\lambda}=v_{\sigma,\lambda}\otimes w_{\eta,\lambda}$. Note that if V and W are small neighborhoods of $\tilde{\sigma}\in\mathbb{R}$ and $\tilde{\eta}\in\mathbb{R}^{d-1}$, respectively, then for $n_+=\max(n,0)$ we have, uniformly on $U=V\times W$,

$$\|u_{(\sigma,\eta),\lambda}\|_{k,k'}\in\begin{cases}O(\langle\lambda\rangle^{k_++k'_+}),\\ O(\langle\lambda\rangle^{k+k'_+}) & \text{if }\sigma_0\neq 0,\\ O(\langle\lambda\rangle^{k_++k'}) & \text{if }\eta_0\neq 0.\end{cases}\tag{A-12}$$

Let $\widetilde{Y}_1,\widetilde{Y}_2\in T^*C$. Then $(\widetilde{Y}_1,\widetilde{Y}_2)\notin\mathrm{WF}(a)'$ if there exist cutoff functions χ_i and ψ_i with $\chi_i(\tilde{s}_i),\psi_i(\tilde{y}_i)\neq 0$ and neighborhoods $U_i=V_i\times W_i$ of $(\tilde{\sigma}_i,\tilde{\eta}_i)$ such that

$$(u_{(\sigma_1,\eta_1),\lambda}\mid au_{(\sigma_2,\eta_2),\lambda})_{L^2(C)}\in O(\langle\lambda\rangle^{-\infty})\quad\text{uniformly for }(\sigma_i,\eta_i)\in U_i.\tag{A-13}$$

We first prove (1). Let $a \in B^{-\infty}\Psi^{p_2}(C)$ and $\widetilde{Y}_1, \widetilde{Y}_2 \in T^*C$ such that $\tilde{\sigma}_1 \neq 0$ or $\tilde{\sigma}_2 \neq 0$. Then (A-13) follows from (A-12) and the fact that $a : H^{k_1,k_2} \to H^{k_1+m,k_2+p_2}$ for any $m \geq 0$.

We now prove (2). If $a \in \Psi^{p_1,p_2}(C)$ the statement follows from Lemma 6.5. It remains to consider the case $a \in B^{-\infty}\Psi^{p_2}(C)$ and to prove that (A-13) holds if $(\tilde{\sigma}_1, \tilde{\eta}_1) = (0, 0)$ and $(\tilde{\sigma}_2, \tilde{\eta}_2) \neq 0$ or vice versa. If $\tilde{\sigma}_1 \neq 0$ or $\tilde{\sigma}_2 \neq 0$, we have already proved (A-13).

Assume now that $\tilde{\eta}_1 = 0$ and $\tilde{\eta}_2 \neq 0$, the other case being similar. Then we can find cutoff functions $g_i \in C_0^\infty(\mathbb{R}^{d-1})$ supported near $\tilde{\eta}_i$ with disjoint supports such that $(1 - g_i(\lambda^{-1}D_y))u_{(\sigma_i,\eta_i),\lambda} \in O(\lambda^{-\infty})$ in all $H^{k,k'}$ uniformly for $(\sigma_i, \eta_i) \in U$. It follows that

$$(u_{(\sigma_1,\eta_1),\lambda} \mid au_{(\sigma_2,\eta_2),\lambda})_{L^2(C)} = (u_{(\sigma_1,\eta_1),\lambda} \mid g_1(\lambda^{-1}D_y)ag_2(\lambda^{-1}D_y)u_{(\sigma_2,\eta_2),\lambda})_{L^2(C)} + O(\langle\lambda\rangle^{-\infty})$$

uniformly for $(\sigma_i, \eta_i) \in U_i$. By pseudodifferential calculus on $\mathbb{S}^{d-1}$, we know that $g_1(\lambda^{-1}D_y)ag_2(\lambda^{-1}D_y)$ is in $O(\langle\lambda\rangle^{-\infty})$ in $B(H^{k,k'})$ for any $k, k' \in \mathbb{R}$. Combined with (A-12), we obtain (A-13) also if $\tilde{\eta}_1 = 0$ and $\tilde{\eta}_2 \neq 0$. This completes the proof of the lemma.

A6. *Proof of Lemma 8.6.* Set $\gamma_{\mathrm{x}} = \{(s, \mathrm{x}) : s \leq 0\}$ for $\mathrm{x} \in \Sigma$. To prove that $\overline{C}$ is the graph of a function F over Σ we have to show that γ_{x} intersects $\overline{C}$ at one and only one point for each $\mathrm{x} \in \Sigma$. Then we have

$$F(\mathrm{x}) = \inf\{s \leq 0 : (s, \mathrm{x}) \in I^+(p)\}.$$

If $F(\mathrm{x}) = -\infty$ then $\gamma_{\mathrm{x}} \subset I^+(p) \cap J^-((0, \mathrm{x})) \subset J^+(p) \cap J^-((0, \mathrm{x}))$. This last set is compact by global hyperbolicity, which is a contradiction. Hence γ_{x} intersects $\overline{C}$. Moreover, if $(t_1, \mathrm{x}) \in \overline{C}$ then $(s, \mathrm{x}) \in J^-(p)$ for all $t_1 \leq s \leq 0$. This shows that γ_{x} intersects $\overline{C}$ at only one point, hence the function F is well defined, and bounded.

Let (T^0, x^0) be the coordinates of p. For $\mathrm{x} \neq \mathrm{x}^0$, C is smooth near $(F(\mathrm{x}), \mathrm{x})$ and ∂_t is transverse to C. By the implicit function theorem this implies that F is smooth near x. Moreover, if $K_1 \subset \Sigma$ is a compact set then dF is uniformly bounded on $K_1 \backslash \{x^0\}$. To prove this it suffices to introduce normal coordinates at p such that, near p, C becomes a neighborhood of the tip of the flat lightcone.

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ANALYSIS & PDE

Volume 9 No. 1 2016

Rademacher functions in Nakano spaces 1
SERGEY ASTASHKIN and MIECZYSŁAW MASTYŁO

Nonexistence of small doubly periodic solutions for dispersive equations 15
DAVID M. AMBROSE and J. DOUGLAS WRIGHT

The borderlines of invisibility and visibility in Calderón's inverse problem 43
KARI ASTALA, MATTI LASSAS and LASSI PÄIVÄRINTA

A characterization of 1-rectifiable doubling measures with connected supports 99
JONAS AZZAM and MIHALIS MOURGOGLOU

Construction of Hadamard states by characteristic Cauchy problem 111
CHRISTIAN GÉRARD and MICHAŁ WROCHNA

Global-in-time Strichartz estimates on nontrapping, asymptotically conic manifolds 151
ANDREW HASSELL and JUNYONG ZHANG

Limiting distribution of elliptic homogenization error with periodic diffusion and random potential 193
WENJIA JING

Blow-up results for a strongly perturbed semilinear heat equation: theoretical analysis and numerical method 229
VAN TIEN NGUYEN and HATEM ZAAG

2157-5045(2016)9:1;1-B