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BLOW-UP RESULTS FOR A STRONGLY PERTURBED SEMILINEAR HEAT EQUATION: THEORETICAL ANALYSIS AND NUMERICAL METHOD

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We consider a blow-up solution for a strongly perturbed semilinear heat equation with Sobolev subcritical power nonlinearity. Working in the framework of similarity variables, we find a Lyapunov functional for the problem. Using this Lyapunov functional, we derive the blow-up rate and the blow-up limit of the solution. We also classify all asymptotic behaviors of the solution at the singularity and give precise blow-up profiles corresponding to these behaviors. Finally, we attain the blow-up profile numerically, thanks to a new mesh-refinement algorithm inspired by the rescaling method of Berger and Kohn. Note that our method is applicable to more general equations, in particular those with no scaling invariance.

1. Introduction

We are concerned in this paper with blow-up phenomena arising in the nonlinear heat problem

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u + h(u), \\ u(\cdot, 0) = u_0 \in L^\infty(\mathbb{R}^n), \end{cases} \tag{1-1}$$

where $u(t) : x \mapsto u(x,t) \in \mathbb{R}$ for $x \in \mathbb{R}^n$ and Δ stands for the Laplacian in $\mathbb{R}^n$. The exponent $p > 1$ is subcritical (which means that $p < (n+2)/(n-2)$ if $n \geq 3$) and h is given by

$$h(z) = \mu \frac{|z|^{p-1}z}{\log^a(2+z^2)} \quad \text{with} \quad a > 0,\ \mu \in \mathbb{R}. \tag{1-2}$$

By standard results, the problem (1-1) has a unique classical solution $u(x,t)$ in $L^\infty(\mathbb{R}^n)$, which exists at least for small times. The solution $u(x,t)$ may develop singularities in some finite time. We say that $u(x,t)$ blows up in a finite time T if $u(x,t)$ satisfies (1-1) in $\mathbb{R}^n \times [0,T)$ and

$$\lim_{t\to T} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = +\infty.$$

T is called the blow-up time of $u(x,t)$. In such a blow-up case, a point $b \in \mathbb{R}^n$ is called a blow-up point of $u(x,t)$ if and only if there exist $(x_n, t_n) \to (b, T)$ such that $|u(x_n,t_n)| \to +\infty$ as $n \to +\infty$.

In the case $\mu = 0$, the equation (1-1) is the semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1}u. \tag{1-3}$$

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Problem (1-3) has been addressed in different ways in the literature. The existence of blow-up solutions has been proved by several authors (see [Fujita 1966; Levine 1973; Ball 1977]). Consider a solution $u(x,t)$ of (1-3) which blows up at a time T. The very first question to be answered is the blow-up rate, i.e., there are positive constants C_1 and C_2 such that

$$C_1(T-t)^{-\frac{1}{p-1}} \le \|u(t)\|_{L^\infty(\mathbb{R}^n)} \le C_2(T-t)^{-\frac{1}{p-1}} \quad \text{for all } t \in (0,T). \tag{1-4}$$

The lower bound in (1-4) follows by a simple argument based on Duhamel's formula (see [Weissler 1981]). For the upper bound, Giga and Kohn [1987] proved (1-4) for $1 < p < (3n+8)/(3n-4)$ or for nonnegative initial data with subcritical p.

Later, the estimate (1-4) was extended to all subcritical p without assuming nonnegativity for initial data u_0 by Giga, Matsui and Sasayama [Giga et al. 2004a]. The estimate (1-4) is a fundamental step to obtain more information about the asymptotic blow-up behavior, locally near a given blow-up point $\hat{b}$. Giga and Kohn [1989] showed that, for a given blow-up point $\hat{b} \in \mathbb{R}^n$,

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}} u(\hat{b} + y\sqrt{T-t}, t) = \pm\kappa,$$

where $\kappa = (p-1)^{-1/(p-1)}$, uniformly on compact sets of $\mathbb{R}^n$.

This result was specified by Filippas and Liu [1993] (see also [Filippas and Kohn 1992]) and Velázquez [1992; 1993] (see also [Herrero and Velázquez 1992a; 1992c; 1993]). Using the renormalization theory, Bricmont and Kupiainen [1994] showed the existence of a solution of (1-3) such that

$$\|(T-t)^{\frac{1}{p-1}} u(\hat{b} + z\sqrt{(T-t)|\log(T-t)|}, t) - f_0(z)\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as } t \to T, \tag{1-5}$$

where

$$f_0(z) = \kappa\left(1 + \frac{p-1}{4p}|z|^2\right)^{-\frac{1}{p-1}}. \tag{1-6}$$

Merle and Zaag [1997] obtained the same result through a reduction to a finite-dimensional problem. Moreover, they showed that the profile (1-6) is stable under perturbations of initial data (see also [Fermanian Kammerer et al. 2000; Fermanian Kammerer and Zaag 2000; Masmoudi and Zaag 2008] for related results).

In the program developed by those authors in the case $\mu = 0$, the invariance of (1-1) under the scaling transformation

$$\lambda \mapsto u_\lambda(\xi,\tau) = \lambda^{\frac{2}{p-1}} u(\lambda\xi, \lambda^2\tau)$$

played a crucial role. Indeed, this property is responsible for having an autonomous equation in similarity variables defined in (1-10) below (see (1-11) below when $\mu = 0$), which helps a lot.

A similar situation is available for the equation

$$\partial_t u = \Delta u + e^u$$

(see [Herrero and Velázquez 1993; Bebernes and Bricher 1992; Bressan 1990; 1992]).

With more general nonlinearities, namely with

$$\partial_t u = \Delta u + f(u) \tag{1-7}$$

with $f(u) \not\equiv |u|^{p-1}u$ and $f(u) \not\equiv e^u$, no result is available on the blow-up behavior. The first example available in the literature goes back to Giga and Kohn [1987], who considered (1-1) with a "weak" perturbation, namely

$$|h(z)| \le M(|z|^q + 1), \quad q \in [1, p). \tag{1-8}$$

They could extend various results from the case $h \equiv 0$.

In our paper, we aim at doing better, by considering "strong" (in comparison with (1-8)) perturbations, namely the case mentioned in (1-2). The resulting nonlinearity is so close to the power law $|u|^{p-1}u$ that it is not a priori clear if the perturbation is able to modify the blow-up behavior of the solution. A subtle point is the following:

When $\mu = 0$, the similarity variables' version of the PDE is autonomous (see (1-11) below with $\mu = 0$), and classical energy methods à la [Levine 1973] give a Lyapunov functional (see (1-16) below) whose role was crucial in the blow-up analysis performed by Giga and Kohn [1987; 1989] and later authors.

When $\mu \neq 0$, it is still possible to use the similarity variables, however, the resulting equation is not autonomous (see (1-11) below). Moreover, the size of the perturbations introduced by the h term is larger than in the "weak" case (1-8) and, more importantly, it is a priori larger than the correction computed for the solution when $\mu = \mathcal{O}(1/s^a)$ with $0 < a < 1$ as shown in Lemma 2.1, versus $1/s$ in the generic case when $\mu = 0$. New ideas are crucially needed, in particular to find a perturbed Lyapunov functional (see Theorem 1.1 below), and to go beyond the too-large perturbation term $1/s^a$ (we linearize around ϕ defined in (1-21)–(1-22) instead of κ).

Because of those difficulties and thanks to our new ideas, we believe that our paper gives a new framework to the study of blow-up for semilinear heat equations of the type (1-7) when the nonlinearity $f(u)$ could lack any scale invariance (exact, or approximate as in this case) at all.

In the case when the function h satisfies

$$|h(z)| \le M\left(\frac{|z|^p}{\log^a(2+z^2)} + 1\right) \quad \text{with} \quad a > 1 \tag{1-9}$$

and $M > 0$, the first author derived the existence of a Lyapunov functional in the similarity variables (1-10) for the problem (1-1), which is a crucial step in deriving the estimate (1-4). He also classified all possible blow-up behaviors of the solution when it approaches to singularity. Here, we aim at extending the results of [Nguyen 2015] to the case $a \in (0, 1]$. As we mentioned above, the first step is to derive the blow-up rate of the blow-up solution. As in [Giga et al. 2004a; Nguyen 2015], the key step is to find a Lyapunov functional in *similarity variables* for (1-1). More precisely, we introduce for all $b \in \mathbb{R}^n$ (b may be a blow-up point of u or not) the following *similarity variables*:

$$y = \frac{x-b}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad w_{b,T} = (T-t)^{\frac{1}{p-1}} u(x,t). \tag{1-10}$$

Hence $w_{b,T}$ satisfies, for all $s \geq -\log T$ and all $y \in \mathbb{R}^n$,

$$\partial_s w_{b,T} = \frac{1}{\rho}\operatorname{div}(\rho \nabla w_{b,T}) - \frac{w_{b,T}}{p-1} + |w_{b,T}|^{p-1} w_{b,T} + e^{-\frac{ps}{p-1}} h(e^{\frac{s}{p-1}} w_{b,T}), \tag{1-11}$$

where

$$\rho(y) = \left(\frac{1}{4\pi}\right)^{\frac{n}{2}} e^{-\frac{|y|^2}{4}} \tag{1-12}$$

and

$$|e^{-\frac{ps}{p-1}} h(e^{\frac{s}{p-1}} z)| \leq \frac{C_0}{s^a}(|z|^p + 1) \quad \text{for all } z \in \mathbb{R} \tag{1-13}$$

for some $C_0 > 0$.

Following the method introduced by Hamza and Zaag [2012a; 2012b] for perturbations of the semilinear wave equation, we introduce

$$\mathcal{J}_a[w](s) = \mathscr{E}[w](s) e^{\frac{\gamma}{a} s^{-a}} + \theta s^{-a}, \tag{1-14}$$

where γ and θ are positive constants, depending only on p, a, μ and n, which will be determined later, and

$$\mathscr{E}[w] = \mathscr{E}_0[w] + \mathscr{I}[w], \tag{1-15}$$

where

$$\mathscr{E}_0[w](s) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2 - \frac{1}{p+1}|w|^{p+1}\right)\rho\, dy \tag{1-16}$$

and

$$\mathscr{I}[w](s) = -e^{-\frac{(p+1)}{p-1}s} \int_{\mathbb{R}^n} H(e^{\frac{s}{p-1}} w)\rho\, dy, \quad H(z) = \int_0^z h(\xi)\, d\xi. \tag{1-17}$$

The main novelty of this paper is to allow values of a in $(0, 1]$, which is possible at the expense of taking the particular form (1-2) for the perturbation h. We aim at the following:

Theorem 1.1 (existence of a Lyapunov functional for (1-11)). *Let a, p, n and μ be fixed; consider w a solution of* (1-11). *Then there exist $\hat{s}_0 = \hat{s}_0(a, p, n, \mu) \geq s_0$, $\hat{\theta}_0 = \hat{\theta}_0(a, p, n, \mu)$ and $\gamma = \gamma(a, p, n, \mu)$ such that, if $\theta \geq \hat{\theta}_0$, then $\mathcal{J}_a$ satisfies the following inequality for all $s_2 > s_1 \geq \max\{\hat{s}_0, -\log T\}$*:

$$\mathcal{J}_a[w](s_2) - \mathcal{J}_a[w](s_1) \leq -\frac{1}{2}\int_{s_1}^{s_2}\int_{\mathbb{R}^n} (\partial_s w)^2 \rho\, dy\, ds. \tag{1-18}$$

As in [Giga et al. 2004a; Nguyen 2015], the existence of the Lyapunov functional is a crucial step for deriving the blow-up rate (1-4) and then the blow-up limit. In particular, we have the following:

Theorem 1.2. *Let a, p, n and μ be fixed and let u be a blow-up solution of* (1-1) *with a blow-up time T.*

(i) *<u>Blow-up rate</u>: There exists $\hat{s}_1 = \hat{s}_1(a, p, n, \mu) \geq \hat{s}_0$ such that, for all $s \geq s' = \max\{\hat{s}_1, -\log T\}$,*

$$\|w_{b,T}(y, s)\|_{L^\infty(\mathbb{R}^n)} \leq C, \tag{1-19}$$

where $w_{b,T}$ is as defined in (1-10) *and C is a positive constant depending only on n, p, μ and a bound of $\|w_{b,T}(\hat{s}_0)\|_{L^\infty}$.*

(ii) Blow-up limit: *If $\hat{a}$ is a blow-up point, then*

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u(\hat{a}+y\sqrt{T-t},t)=\lim_{s\to+\infty}w_{\hat{a},T}(y,s)=\pm\kappa \tag{1-20}$$

holds in L^2_ρ (L^2_ρ is the weighted L^2 space associated with the weight ρ of (1-12)) *and also uniformly on each compact subset of $\mathbb{R}^n$.*

Remark 1.3. We will not give the proof of Theorem 1.2 because its proof follows from Theorem 1.1 as in [Nguyen 2015]. Hence, we only give the proof of Theorem 1.1 and refer the reader to [Nguyen 2015, Section 2] for the proofs of (1-19) and (1-20).

The next step consists in obtaining an additional term in the asymptotic expansion given in Theorem 1.2(ii). Given b a blow-up point of $u(x,t)$, and up to changing u_0 by $-u_0$ and h by $-h$, we may assume that $w_{b,T}\to\kappa$ in L^2_ρ as $s\to+\infty$. As in [Nguyen 2015], we linearize $w_{b,T}$ around ϕ, where ϕ is the positive solution of the ordinary differential equation associated to (1-11),

$$\phi'=-\frac{\phi}{p-1}+\phi^p+e^{-\frac{ps}{p-1}}h(e^{\frac{s}{p-1}}\phi) \tag{1-21}$$

such that

$$\phi(s)\to\kappa \quad \text{as } s\to+\infty; \tag{1-22}$$

see [Nguyen 2015, Lemma A.3] for the existence of ϕ, and note that ϕ is unique. For the reader's convenience, we give in Lemma A.1 the expansion of ϕ as $s\to+\infty$.

Let us introduce $v_{b,T}=w_{b,T}-\phi(s)$; then $\|v_{b,T}(y,s)\|_{L^2_\rho}\to 0$ as $s\to+\infty$ and $v_{b,T}$ (or v for simplicity) satisfies the equation

$$\partial_s v=(\mathcal{L}+\omega(s))v+F(v)+H(v,s) \quad \text{for all } y\in\mathbb{R}^n,\ s\in[-\log T,+\infty),$$

where $\mathcal{L}=\Delta-\frac{1}{2}y\cdot\nabla+1$ and ω, F and H satisfy

$$\omega(s)=\mathcal{O}\left(\frac{1}{s^{a+1}}\right) \quad \text{and} \quad |F(v)|+|H(v,s)|=\mathcal{O}(|v|^2) \quad \text{as } s\to+\infty,$$

(see the beginning of Section 3 for the proper definitions of ω, F and G).

It is well known that the operator $\mathcal{L}$ is self-adjoint in $L^2_\rho(\mathbb{R}^n)$. Its spectrum is given by

$$\operatorname{spec}(\mathcal{L})=\left\{1-\tfrac{1}{2}m \;\middle|\; m\in\mathbb{N}\right\},$$

and it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are derived from Hermite polynomials:

For $n=1$, the eigenfunction corresponding to $1-\frac{1}{2}m$ is

$$h_m(y)=\sum_{k=0}^{[m/2]}\frac{m!}{k!(m-2k)!}(-1)^k y^{m-2k}, \tag{1-23}$$

For $n\ge 2$, we write the spectrum of $\mathcal{L}$ as

$$\operatorname{spec}(\mathcal{L})=\left\{1-\tfrac{1}{2}|m| \;\middle|\; |m|=m_1+\dots+m_n,\ (m_1,\dots,m_n)\in\mathbb{N}^n\right\}.$$

For $m = (m_1, \dots, m_n) \in \mathbb{N}^n$, the eigenfunction corresponding to $1 - \frac{1}{2}|m|$ is

$$H_m(y) = h_{m_1}(y_1) \cdots h_{m_n}(y_n), \tag{1-24}$$

where h_m is as defined in (1-23).

We also denote $c_m = c_{m_1} c_{m_2} \cdots c_{m_n}$ and $y^m = y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}$ for any $m = (m_1, \dots, m_n) \in \mathbb{N}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

In this way, we derive the following asymptotic behaviors of $w_{b,T}(y, s)$ as $s \to +\infty$:

Theorem 1.4 (classification of the behavior of $w_{b,T}$ as $s \to +\infty$). *Consider a solution $u(t)$ of* (1-1) *which blows-up at time T and b a blow-up point. Let $w_{b,T}(y, s)$ be a solution of* (1-11). *Then one of the following possibilities occurs*:

(i) $w_{b,T}(y, s) \equiv \phi(s)$.

(ii) *There exists $l \in \{1, \dots, n\}$ such that*, *up to an orthogonal transformation of coordinates*, *we have*

$$w_{b,T}(y, s) = \phi(s) - \frac{\kappa}{4ps}\left(\sum_{j=1}^{l} y_j^2 - 2l\right) + \mathcal{O}\left(\frac{1}{s^{a+1}}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right) \quad \textit{as } s \to +\infty.$$

(iii) *There exist an integer $m \geq 3$ and constants c_α not all zero such that*

$$w_{b,T}(y, s) = \phi(s) - e^{-(\frac{m}{2}-1)s} \sum_{|\alpha|=m} c_\alpha H_\alpha(y) + o(e^{-(\frac{m}{2}-1)s}) \quad \textit{as } s \to +\infty.$$

The convergence takes place in L^2_ρ as well as in $\mathscr{C}^{k,\gamma}_{\mathrm{loc}}$ for any $k \geq 1$ and some $\gamma \in (0, 1)$.

Remark 1.5. In [Nguyen 2015], we were unable to get this result in the case where h satisfies (1-9) with $a \in (0, 1]$. Here, by taking the particular form of the perturbation (see (1-2)), we are able to overcome technical difficulties in order to derive the result.

Remark 1.6. From Theorem 1.2(ii), we would naturally try to find an equivalent for $w - \kappa$ as $s \to +\infty$. A posteriori from our results in Theorem 1.4, we see that, in all cases, $\|w - \kappa\|_{L^2_\rho} \sim C/s^{a'}$ with $a' = \min\{a, 1\}$. This is indeed a new phenomenon observed in our (1-1) and which is different from the case of the unperturbed semilinear heat equation, where either $w - \kappa \equiv 0$ or $\|w - \kappa\|_{L^2_\rho} \sim C/s$ or $\|w - \kappa\|_{L^2_\rho} \sim Ce^{(1-m/2)s}$ for some even $m \geq 4$. This shows the originality of our paper. In our case, linearizing around κ would keep us trapped in the $1/s$ scale. In order to escape that scale, we forget the explicit function κ, which is not a solution of Equation (1-11) and linearize instead around the nonexplicit function ϕ, which happens to be an exact solution of (1-11). This way, we escape the $1/s$ scale and reach exponentially decreasing order.

Using the information obtained in Theorem 1.4, we can extend the asymptotic behavior of $w_{b,T}$ to larger regions. Particularly, we have the following:

Theorem 1.7 (convergence extension of $w_{b,T}$ to larger regions). *For all $K_0 > 0$:*

(i) *If Theorem 1.4(ii) occurs, then*

$$\sup_{|\xi|\le K_0} |w_{b,T}(\xi\sqrt{s}, s) - f_l(\xi)| = \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s}\right) \quad \textit{as } s \to +\infty, \tag{1-25}$$

where

$$f_l(\xi) = \kappa\left(1 + \frac{p-1}{4p}\sum_{j=1}^{l}\xi_j^2\right)^{-\frac{1}{p-1}} \quad \textit{for all } \xi \in \mathbb{R}^n \tag{1-26}$$

with l given in Theorem 1.4(ii).

(ii) *If Theorem 1.4(iii) occurs, then $m \ge 4$ is even and*

$$\sup_{|\xi|\le K_0} |w_{b,T}(\xi e^{(\frac{1}{2}-\frac{1}{m})s}, s) - \psi_m(\xi)| \to 0 \quad \textit{as } s \to +\infty, \tag{1-27}$$

where

$$\psi_m(\xi) = \kappa\left(1 + \kappa^{-p}\sum_{|\alpha|=m} c_\alpha \xi^\alpha\right)^{-\frac{1}{p-1}} \quad \textit{for all } \xi \in \mathbb{R}^n, \tag{1-28}$$

where c_α is the same as in Theorem 1.4 and the multilinear for $\sum_{|\alpha|=m} c_\alpha \xi^\alpha$ is nonnegative.

Remark 1.8. Note that Theorem 1.7 is analogous to the result obtained in [Velázquez 1992] for problem (1-1) without the perturbation. In particular, we follow the method of [loc. cit.] and care about the speed of the convergence, which was not given in that paper. Note also that the asymptotic profiles described in Theorem 1.7 are exactly the same as the ones derived in [loc. cit.] because we derived in this theorem the first-order approximation for the solution, unlike in Theorem 1.4, where we find the following terms in the expansion of the solution up to the second order. As in the unperturbed case ($h \equiv 0$), we expect that (1-25) is stable (see the previous remarks, particularly the paragraph after (1-5)) and (1-27) should correspond to unstable behaviors. The instability of (1-27) was proved only in one space dimension by Herrero and Velázquez [1992b; 1992d]. In particular, they proved the genericity of the asymptotic profile (1-25) in the one-dimensional case and announced the same for higher-dimensional cases, but they have never published it. While discussing numerical simulation for Equation (1-1) in one space dimension (see Section 4B below), we see that the numerical solutions exhibit only the behavior (1-25) and we could never obtain the behavior (1-27). This is probably due to the fact that the behavior (1-27) is unstable.

Remark 1.9. In [Nguyen and Zaag 2014], we constructed for the problem (1-1) with h given by (1-2) or (1-9) a solution which blows up in finite time at only one point and verifies the behavior (1-25) with $l = n$. The construction is inspired by the method of [Bricmont and Kupiainen 1994; Merle and Zaag 1997], relying on the reduction of the problem to a finite-dimensional one and a topological argument based on index theory.

At the end of this work, we give numerical confirmations for the asymptotic profile described in Theorem 1.7. For this purpose, we propose a new mesh-refinement method inspired by the rescaling algorithm of [Berger and Kohn 1988]. Note that their method was successful to solve blowing-up problems which are invariant under the transformation

$$\gamma \mapsto u_\gamma(\xi,\tau) = \gamma^{\frac{2}{p-1}} u(\gamma\xi, \gamma^2\tau) \quad \text{for all } \gamma > 0. \tag{1-29}$$

However, there are a lot of equations whose solutions blow up in finite time but which do not satisfy the property (1-29); one of them is (1-1) because of the presence of the perturbation term h. Although our method is very similar to Berger and Kohn's algorithm in spirit, it is better in the sense that it can be applied to a larger class of blowing-up problems which do not satisfy the rescaling property (1-29). To our knowledge, there are not many papers on the numerical blow-up profile, apart from [Berger and Kohn 1988] (see also [Nguyen 2014]), who already obtained numerical results for (1-1) without the perturbation term. For other numerical aspects, there are several studies for (1-1) in the unperturbed case; see, for example, [Abia, López-Marcos and Martínez 1998; 2001; Groisman and Rossi 2001; 2004; Groisman 2006; N'gohisse and Boni 2011; Kyza and Makridakis 2011; Cangiani et al. $\geq$ 2016] and the references therein. There is also the work of Baruch et al. [2010] studying standing-ring solutions.

This paper is organized as follows: Section 2 is devoted to the proof of Theorem 1.1. Theorem 1.2 follows from Theorem 1.1. Since all the arguments presented in [Nguyen 2015] remain valid for the case (1-9), except the existence of the Lyapunov functional for (1-11) (Theorem 1.1), we kindly refer the reader to [Nguyen 2015, Sections 2.3 and 2.4] for details of the proof. Section 3 deals with results on asymptotic behaviors (Theorems 1.4 and 1.7). In Section 4, we describe the new mesh-refinement method and give some numerical justifications for the theoretical results.

2. Existence of a Lyapunov functional for (1-11)

In this section, we mainly aim at proving that the functional $\mathscr{J}_a$ defined in (1-14) is a Lyapunov functional for (1-11) (Theorem 1.1). Note that this functional is far from being trivial and makes our main contribution.

In what follows, we denote by C a generic constant depending only on a, p, n and μ. We first give the following estimates on the perturbation term appearing in (1-11):

Lemma 2.1. *Let h be the function defined in* (1-2). *For all $\epsilon \in (0, p]$, there exists $C_0 = C_0(a,\mu,p,\epsilon) > 0$ and $\bar{s}_0 = \bar{s}_0(a,p,\epsilon) > 0$ large enough such that, for all $s \geq \bar{s}_0$,*

(i)
$$|e^{-\frac{ps}{p-1}} h(e^{\frac{s}{p-1}} z)| \leq \frac{C_0}{s^a}(|z|^p + |z|^{p-\epsilon}),$$
$$|e^{-\frac{(p+1)s}{p-1}} H(e^{\frac{s}{p-1}} z)| \leq \frac{C_0}{s^a}(|z|^{p+1} + 1),$$

where H is as defined in (1-17).

(ii)
$$|(p+1)e^{-\frac{(p+1)s}{p-1}} H(e^{\frac{s}{p-1}} z) - e^{-\frac{ps}{p-1}} h(e^{\frac{s}{p-1}} z) z| \leq \tfrac{C_0}{s^{a+1}}(|z|^{p+1} + 1).$$

Proof. Note that (i) obviously follows from the estimate

$$\forall q > 0,\ \ \forall b > 0, \quad \frac{|z|^q}{\log^b(2 + e^{\frac{2s}{p-1}} z^2)} \leq \frac{C}{s^b}(|z|^q + 1) \quad \text{for all } s \geq \bar{s}_0, \tag{2-1}$$

where $C = C(b, q) > 0$ and $\bar{s}_0 = \bar{s}_0(b, q) > 0$.

In order to derive the estimate (2-1), by considering the first case $z^2 e^{\frac{s}{p-1}} \geq 4$ then the case $z^2 e^{\frac{s}{p-1}} \leq 4$, we would obtain (2-1).

Part (ii) directly follows from an integration by parts and the estimate (2-1). Indeed, we have

$$H(\xi) = \int_0^\xi h(x)\,dx = \mu \int_0^\xi \frac{|x|^{p-1}x}{\log^a(2+x^2)}\,dx$$
$$= \frac{\mu|\xi|^{p+1}}{(p+1)\log^a(2+\xi^2)} + \frac{2a\mu}{p+1}\int_0^\xi \frac{|x|^{p+1}x}{(2+x^2)\log^{a+1}(2+x^2)}\,dx.$$

Replacing ξ by $e^s/(p-1)z$ and using (2-1), we then derive (ii). This ends the proof of Lemma 2.1. □

We assert that Theorem 1.1 is a direct consequence of the following lemma:

Lemma 2.2. *Let a, p, n and μ be fixed and w be a solution of* (1-11). *There exists $\tilde{s}_0 = \tilde{s}_0(a, p, n, \mu) \geq s_0$ such that the functional of $\mathscr{E}$ defined in* (1-15) *satisfies the following inequality, for all $s \geq \max\{\tilde{s}_0, -\log T\}$*:

$$\frac{d}{ds}\mathscr{E}[w](s) \leq -\frac{1}{2}\int_{\mathbb{R}^n} w_s^2 \rho\,dy + \gamma s^{-(a+1)}\mathscr{E}[w](s) + Cs^{-(a+1)}, \tag{2-2}$$

where $\gamma = 4C_0(p+1)/(p-1)^2$ and C_0 is given in Lemma 2.1.

Let us first derive Theorem 1.1 from Lemma 2.2, which we will prove later.

Proof of Theorem 1.1, given Lemma 2.2. Differentiating the functional $\mathscr{J}$ defined in (1-14), we obtain

$$\begin{aligned}\frac{d}{ds}\mathscr{J}_a[w](s) &= \frac{d}{ds}\{\mathscr{E}[w](s)e^{\frac{\gamma}{a}s^{-a}} + \theta s^{-a}\}\\ &= \frac{d}{ds}\mathscr{E}[w](s)e^{\frac{\gamma}{a}s^{-a}} - \gamma s^{-(a+1)}\mathscr{E}[w](s)e^{\frac{\gamma}{a}s^{-a}} - a\theta s^{-(a+1)}\\ &\leq -\frac{1}{2}e^{\frac{\gamma}{a}s^{-a}}\int_{\mathbb{R}^n} w_s^2\rho\,dy + [Ce^{\frac{\gamma}{a}s^{-a}} - a\theta]s^{-(a+1)} \quad \text{(using (2-2))}.\end{aligned}$$

Choosing θ large enough that $Ce^{\gamma\tilde{s}_0^{-a}/a} - a\theta \leq 0$ and noticing that $e^{\gamma s^{-a}/a} \geq 1$ for all $s > 0$, we derive

$$\frac{d}{ds}\mathscr{J}_a[w](s) \leq -\frac{1}{2}\int_{\mathbb{R}^n} w_s^2\rho\,dy \quad \text{for all } s \geq \tilde{s}_0.$$

This implies the inequality (1-18) and concludes the proof of Theorem 1.1, assuming that Lemma 2.2 holds. □

Proof of Lemma 2.2. Multiplying (1-11) by $w_s\rho$ and integrating by parts,

$$\int_{\mathbb{R}^n}|w_s|^2\rho = -\frac{d}{ds}\left\{\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2 - \frac{1}{p+1}|w|^{p+1}\right)\rho\,dy\right\} + e^{-\frac{ps}{p-1}}\int_{\mathbb{R}^n} h(e^{\frac{s}{p-1}}w)w_s\rho\,dy.$$

For the last term of the above expression, we obtain

$$
\begin{aligned}
e^{-\frac{ps}{p-1}} &\int_{\mathbb{R}^n} h(e^{\frac{s}{p-1}} w) w_s \rho\, dy \\
&= e^{-\frac{(p+1)s}{p-1}} \int_{\mathbb{R}^n} h(e^{\frac{s}{p-1}} w)\Big(e^{\frac{s}{p-1}} w_s + \frac{e^{\frac{s}{p-1}}}{p-1} w\Big)\rho\, dy - \frac{1}{p-1} e^{-\frac{ps}{p-1}} \int_{\mathbb{R}^n} h(e^{\frac{s}{p-1}} w) w\rho\, dy \\
&= e^{-\frac{p+1}{p-1}s} \frac{d}{ds} \int_{\mathbb{R}^n} H(e^{\frac{s}{p-1}} w)\rho\, dy - \frac{1}{p-1} e^{-\frac{ps}{p-1}} \int_{\mathbb{R}^n} h(e^{\frac{s}{p-1}} w) w\rho\, dy.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\int_{\mathbb{R}^n} |w_s|^2 \rho\, dy = &-\frac{d}{ds}\Big\{\int_{\mathbb{R}^n}\Big(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2 - \frac{1}{p+1}|w|^{p+1}\Big)\rho\, dy\Big\} \\
&+ \frac{d}{ds}\Big\{e^{-\frac{p+1}{p-1}s} \int_{\mathbb{R}^n} H(e^{\frac{s}{p-1}} w)\rho\, dy\Big\} \\
&+ \frac{p+1}{p-1} e^{-\frac{p+1}{p-1}s} \int_{\mathbb{R}^n} H(e^{\frac{s}{p-1}} w)\rho\, dy \\
&- \frac{1}{p-1} e^{-\frac{ps}{p-1}} \int_{\mathbb{R}^n} h(e^{\frac{s}{p-1}} w) w\rho\, dy.
\end{aligned}
$$

From the definition of the functional $\mathscr{E}$ given in (1-15), we derive a first identity in the following:

$$
\begin{aligned}
&\frac{d}{ds}\mathscr{E}[w](s) \\
&\quad = -\int_{\mathbb{R}^n} |w_s|^2 \rho\, dy + \frac{p+1}{p-1} e^{-\frac{p+1}{p-1}s} \int_{\mathbb{R}^n} H(e^{\frac{s}{p-1}} w)\rho\, dy - \frac{1}{p-1} e^{-\frac{ps}{p-1}} \int_{\mathbb{R}^n} h(e^{\frac{s}{p-1}} w) w\rho\, dy. \quad (2\text{-}3)
\end{aligned}
$$

A second identity is obtained by multiplying (1-11) by $w\rho$ and integrating by parts:

$$
\begin{aligned}
&\frac{d}{ds}\int_{\mathbb{R}^n} |w|^2 \rho\, dy \\
&\quad = -4\Big\{\int_{\mathbb{R}^n}\Big(\frac{1}{2}|\nabla w|^2 + \frac{1}{2(p-1)}|w|^2 - \frac{1}{p+1}|w|^{p+1}\Big)\rho\, dy - e^{-\frac{(p+1)s}{p-1}} \int_{\mathbb{R}^n} H(e^{\frac{s}{p-1}} w)\rho\, dy\Big\} \\
&\qquad + \Big(2 - \frac{4}{p+1}\Big) \int_{\mathbb{R}^n} |w|^{p+1}\rho\, dy - 4e^{-\frac{p+1}{p-1}s} \int_{\mathbb{R}^n} H(e^{\frac{s}{p-1}} w)\rho\, dy + 2e^{-\frac{ps}{p-1}} \int_{\mathbb{R}^n} h(e^{\frac{s}{p-1}} w) w\rho\, dy.
\end{aligned}
$$

Using the definition of $\mathscr{E}$ given in (1-15) again, we rewrite the second identity as follows:

$$
\begin{aligned}
\frac{d}{ds}\int_{\mathbb{R}^n} |w|^2 \rho\, dy = &-4\mathscr{E}[w](s) + 2\frac{p-1}{p+1}\int_{\mathbb{R}^n} |w|^{p+1}\rho\, dy \\
&- 4e^{-\frac{p+1}{p-1}s} \int_{\mathbb{R}^n} H(e^{\frac{s}{p-1}} w)\rho\, dy + 2e^{-\frac{ps}{p-1}} \int_{\mathbb{R}^n} h(e^{\frac{s}{p-1}} w) w\rho\, dy. \quad (2\text{-}4)
\end{aligned}
$$

From (2-3), we estimate

$$
\frac{d}{ds}\mathscr{E}[w](s) \le -\int_{\mathbb{R}^n} |w_s|^2 \rho\, dy + \frac{1}{p-1}\int_{\mathbb{R}^n}\big\{\big|(p+1)e^{-\frac{(p+1)s}{p-1}} H(e^{\frac{s}{p-1}} w) - e^{-\frac{ps}{p-1}} h(e^{\frac{s}{p-1}} w) w\big|\big\}\rho\, dy.
$$

Using Lemma 2.1(ii), we have, for all $s \geq \bar{s}_0$,

$$\frac{d}{ds}\mathscr{E}[w](s) \leq -\int_{\mathbb{R}^n} |w_s|^2 \rho\, dy + \frac{C_0 s^{-(a+1)}}{p-1}\int_{\mathbb{R}^n} |w|^{p+1}\rho\, dy + Cs^{-(a+1)}. \tag{2-5}$$

On the other hand, by (2-4) we have

$$\int_{\mathbb{R}^n} |w|^{p+1}\rho\, dy \leq \frac{2(p+1)}{p-1}\mathscr{E}[w](s) + \frac{p+1}{p-1}\int_{\mathbb{R}^n} |w_s w|\rho\, dy \\ + \frac{2(p+1)}{p-1}\int_{\mathbb{R}^n}\big(|e^{-\frac{p+1}{p-1}s}H(e^{\frac{s}{p-1}}w)| + |e^{-\frac{ps}{p-1}}h(e^{\frac{s}{p-1}}w)w|\big)\rho\, dy.$$

Using Lemma 2.1(i) and the fact that $|w_s w| \leq \epsilon(|w_s|^2 + |w|^{p+1}) + C(\epsilon)$ for all $\epsilon > 0$, we obtain

$$\int_{\mathbb{R}^n} |w|^{p+1}\rho\, dy \leq \frac{2(p+1)}{p-1}\mathscr{E}[w](s) + \epsilon\int_{\mathbb{R}^n} |w_s|^2\rho\, dy + (\epsilon + Cs^{-a})\int_{\mathbb{R}^n} |w|^{p+1}\rho\, dy + C.$$

Taking $\epsilon = \frac{1}{4}$ and s_1 large enough that $Cs^{-a} \leq \frac{1}{4}$ for all $s \geq s_1$, we have

$$\int_{\mathbb{R}^n} |w|^{p+1}\rho\, dy \leq \frac{4(p+1)}{p-1}\mathscr{E}[w](s) + \frac{1}{2}\int_{\mathbb{R}^n} |w_s|^2\rho\, dy + C \quad \text{for all } s > s_1. \tag{2-6}$$

Substituting (2-6) into (2-5) yields (2-2) with $\tilde{s}_0 = \max\{\bar{s}_0, s_1\}$. This concludes the proof of Lemma 2.2 and Theorem 1.1 also. □

3. Blow-up behavior

This section is devoted to the proof of Theorems 1.4 and 1.7. Consider a blow-up point b and write w instead of $w_{b,T}$ for simplicity. From Theorem 1.2(ii) and up to changing the signs of w and h, we may assume that $\|w(y,s) - \kappa\|_{L^2_\rho} \to 0$ as $s \to +\infty$ uniformly on compact subsets of $\mathbb{R}^n$. As mentioned in the introduction, by setting $v(y,s) = w(y,s) - \phi(s)$ (ϕ is the positive solution of (1-21) such that $\phi(s) \to \kappa$ as $s \to +\infty$), we see that $\|v(y,s)\|_{L^2_\rho} \to 0$ as $s \to +\infty$ and v solves the equation

$$\partial_s v = (\mathscr{L} + \omega(s))v + F(v) + G(v,s) \quad \text{for all } y \in \mathbb{R}^n,\ s \in [-\log T, +\infty), \tag{3-1}$$

where $\mathscr{L} = \Delta - \frac{1}{2}y\cdot\nabla + 1$ and ω, F and G are given by

$$\begin{aligned} \omega(s) &= p(\phi^{p-1} - \kappa^{p-1}) + e^{-s}h'(e^{\frac{s}{p-1}}\phi),\\ F(v) &= |v+\phi|^{p-1}(v+\phi) - \phi^p - p\phi^{p-1}v,\\ G(v,s) &= e^{-\frac{ps}{p-1}}\big[h(e^{\frac{s}{p-1}}(v+\phi)) - h(e^{\frac{s}{p-1}}\phi) - e^{\frac{s}{p-1}}h'(e^{\frac{s}{p-1}}\phi)v\big]. \end{aligned}$$

By a direct calculation, we can show that

$$|\omega(s)| = \mathscr{O}\left(\frac{1}{s^{a+1}}\right) \quad \text{as } s \to +\infty \tag{3-2}$$

(see Lemma B.1 for the proof of this fact; note also that in the case where h is given by (1-9) and treated in [Nguyen 2015], we just obtain $|\omega(s)| = \mathscr{O}(s^{-a})$ as $s \to +\infty$, which was a major reason preventing us from deriving the result in the case $a \in (0,1]$) there.

Now, introducing

$$V(y,s)=\beta(s)v(y,s), \quad \text{where} \quad \beta(s)=\exp\left(-\int_s^{+\infty}\omega(\tau)\,d\tau\right), \tag{3-3}$$

V satisfies

$$\partial_s V=\mathscr{L}V+\bar{F}(V,s), \tag{3-4}$$

where $\bar{F}(V,s)=\beta(s)(F(V)+G(V,s))$ satisfies

$$\left|\bar{F}(V,s)-\frac{p}{2\kappa}V^2\right|=\mathbb{O}\left(\frac{V^2}{s^a}\right)+\mathbb{O}(|V|^3) \quad \text{as } s\to+\infty \tag{3-5}$$

(see [Nguyen 2015, Lemma C.1] for the proof of this fact; note that, in the case where h is given by (1-9), the first term in the right-hand side of (3-5) is $\mathbb{O}(V^2/s^{a-1})$).

Since $\beta(s)\to 1$ as $s\to+\infty$, each equivalent for V is also an equivalent for v. Therefore, it suffices to study the asymptotic behavior of V as $s\to+\infty$. More precisely, we claim the following:

Proposition 3.1 (classification of the behavior of V as $s\to+\infty$). *One of the following possibilities occurs*:

(i) $V(y,s)\equiv 0$.

(ii) *There exists* $l\in\{1,\dots,n\}$ *such that*, *up to an orthogonal transformation of coordinates*, *we have*

$$V(y,s)=-\frac{\kappa}{4ps}\left(\sum_{j=1}^{l}y_j^2-2l\right)+\mathbb{O}\left(\frac{1}{s^{a+1}}\right)+\mathbb{O}\left(\frac{\log s}{s^2}\right) \quad \textit{as } s\to+\infty.$$

(iii) *There exist an integer* $m\geq 3$ *and constants* c_α *not all zero such that*

$$V(y,s)=-e^{(1-\frac{m}{2})s}\sum_{|\alpha|=m}c_\alpha H_\alpha(y)+o(e^{(1-\frac{m}{2})s}) \quad \textit{as } s\to+\infty.$$

The convergence takes place in L^2_ρ *as well as in* $\mathscr{C}^{k,\gamma}_{\text{loc}}$ *for any* $k\geq 1$ *and* $\gamma\in(0,1)$.

Proof. Because we have the same equation (3-4) and a similar estimate (3-5) to the case treated in [Nguyen 2015], we do not give the proof and kindly refer the reader to Section 3 there. □

Let us derive Theorem 1.4 from Proposition 3.1.

Proof of Theorem 1.4. By the definition (3-3) of V, we see that given Proposition 3.1(i) it directly follows that $v(y,s)\equiv\phi(s)$, which is Theorem 1.4(i). Using Proposition 3.1(ii) and the fact that $\beta(s)=1+\mathbb{O}(1/s^a)$ as $s\to+\infty$, we see that, as $s\to+\infty$,

$$w(y,s)=\phi(s)+V(y,s)\left(1+\mathbb{O}\left(\frac{1}{s^a}\right)\right)=\phi(s)-\frac{\kappa}{4ps}\left(\sum_{j=1}^{l}y_j^2-2l\right)+\mathbb{O}\left(\frac{1}{s^{a+1}}\right)+\mathbb{O}\left(\frac{\log s}{s^2}\right),$$

which yields Theorem 1.4(ii).

Using Proposition 3.1(iii) and again the fact that $\beta(s) = 1 + \mathcal{O}(1/s^a)$ as $s \to +\infty$, we have

$$w(y,s) = \phi(s) - e^{(1-\frac{m}{2})s} \sum_{|\alpha|=m} c_\alpha H_\alpha(y) + o(e^{(1-\frac{m}{2})s}) \quad \text{as } s \to +\infty.$$

This concludes the proof of Theorem 1.4. □

We now give the proof of Theorem 1.7 from Theorem 1.4. Note that the derivation of Theorem 1.7 from Theorem 1.4 in the unperturbed case ($h \equiv 0$) was done by Velázquez [1992]. The idea to extend the convergence up to sets of the type $\{|y| \le K_0\sqrt{s}\}$ or $\{|y| \le K_0 e^{(1/2-1/m)s}\}$ is to estimate the effect of the convective term $-\frac{1}{2} y \cdot \nabla w$ in (1-11) in L^q_ρ spaces with $q > 1$. Since the proof of Theorem 1.7 is, in spirit, by the method given in [Velázquez 1992], all that we need to do is to control the strong perturbation term in (1-11). We therefore give the main steps of the proof and focus only on the new arguments. Note also that we only give the proof of Theorem 1.4(ii) because the proof of (iii) is exactly the same as in Proposition 34 in [Nguyen 2015].

Let us restate Theorem 1.7(i) in the following proposition:

Proposition 3.2 (asymptotic behavior in the $y/\sqrt{s}$ variable). *Assume that w is a solution of* (1-11) *which satisfies Theorem 1.4(ii). Then, for all $K > 0$,*

$$\sup_{|\xi|\le K} |w(\xi\sqrt{s}, s) - f_l(\xi)| = \mathcal{O}\left(\frac{1}{s^a}\right) + \mathcal{O}\left(\frac{\log s}{s}\right) \quad \text{as } s \to +\infty,$$

where $f_l(\xi) = \kappa\big(1 + ((p-1)/4p)\sum_{j=1}^{l} \xi_j^2\big)^{-1/(p-1)}$.

Proof. Define $q = w - \varphi$, where

$$\varphi(y,s) = \frac{\phi(s)}{\kappa}\left[\kappa\left(1 + \frac{p-1}{4ps}\sum_{j=1}^{l} y_j^2\right)^{-\frac{1}{p-1}} + \frac{\kappa l}{2ps}\right], \tag{3-6}$$

and ϕ is the unique positive solution of (1-21) satisfying (1-22).

Note that in [Velázquez 1992; Nguyen 2015], the authors took

$$\varphi(y,s) = \kappa\left(1 + \frac{p-1}{4ps}\sum_{j=1}^{l} y_j^2\right)^{-\frac{1}{p-1}} + \frac{\kappa l}{2ps}.$$

But this choice just works in the case where $a > 1$. In the particular case (1-2), we use in addition the factor $\phi(s)/\kappa$, which allows us to go beyond the order $1/s^a$ coming from the strong perturbation term in order to reach $1/s^{a+1}$ in many estimates in the proof.

Using Taylor's formula in (3-6) and Theorem 1.4(ii), we find that

$$\|q(y,s)\|_{L^2_\rho} = \mathcal{O}\left(\frac{1}{s^{a+1}}\right) + \mathcal{O}\left(\frac{\log s}{s^2}\right) \quad \text{as } s \to +\infty. \tag{3-7}$$

Straightforward calculations based on (1-11) yield

$$\partial_s q = (\mathcal{L} + \alpha)q + F(q) + G(q,s) + R(y,s) \quad \text{for all } (y,s) \in \mathbb{R}^n \times [-\log T, +\infty), \tag{3-8}$$

where

$$\begin{aligned}
\alpha(y,s) &= p(\varphi^{p-1}-\kappa^{p-1}) + e^{-s}h'(e^{\frac{s}{p-1}}\varphi),\\
F(q) &= |q+\varphi|^{p-1}(q+\varphi) - \varphi^p - p\varphi^{p-1}q,\\
G(q,s) &= e^{-\frac{ps}{p-1}}[h(e^{\frac{s}{p-1}}(q+\varphi)) - h(e^{\frac{s}{p-1}}\varphi) - e^{\frac{s}{p-1}}h'(e^{\frac{s}{p-1}}\varphi)q],\\
R(y,s) &= -\partial_s\varphi + \Delta\varphi - \frac{y}{2}\cdot\nabla\varphi - \frac{\varphi}{p-1} + \varphi^p + e^{-\frac{ps}{p-1}}h(e^{\frac{s}{p-1}}\varphi).
\end{aligned}$$

Let $K_0 > 0$ be fixed; we consider first the case $|y| \ge 2K_0\sqrt{s}$ and then $|y| \le 2K_0\sqrt{s}$ and make a Taylor expansion for $\xi = y/\sqrt{s}$ bounded. Simultaneously we obtain, for all $s \ge s_0$,

$$\begin{aligned}
\alpha(y,s) &\le \frac{C_1}{s^{a'}},\\
|F(q)| + |G(q,s)| &\le C_1(q^2 + \mathbf{1}_{\{|y|\ge 2K_0\sqrt{s}\}}),\\
|R(y,s)| &\le C_1\left(\frac{|y|^2+1}{s^{1+a'}} + \mathbf{1}_{\{|y|\ge 2K_0\sqrt{s}\}}\right),
\end{aligned}$$

where $a' = \min\{1,a\}$, $C_1 = C_1(M_0, K_0) > 0$ and M_0 is the bound of w in L^∞ norm. Note that we need to use in addition the fact that ϕ satisfies (1-21) to derive the bound for R (see Lemma B.2).

Let $Q = |q|$; we then use the above estimates and Kato's inequality, i.e., $\Delta f \cdot \operatorname{sign}(f) \le \Delta(|f|)$, to derive from (3-8) the following: for all $K_0 > 0$ fixed, there are $C_* = C_*(K_0, M_0) > 0$ and a time $s' > 0$ large enough such that, for all $s \ge s_* = \max\{s', -\log T\}$,

$$\partial_s Q \le \left(\mathscr{L} + \frac{C_*}{s^{a'}}\right)Q + C_*\left(Q^2 + \frac{|y|^2+1}{s^{1+a'}} + \mathbf{1}_{\{|y|\ge 2K_0\sqrt{s}\}}\right) \quad \text{for all } y \in \mathbb{R}^n. \tag{3-9}$$

Since

$$\left|w(y,s) - f_l\left(\frac{y}{\sqrt{s}}\right)\right| \le Q + \frac{C}{s^{a'}},$$

the conclusion of Proposition 3.2 follows if we show that

$$\forall K_0 > 0 \qquad \sup_{|y|\le K_0\sqrt{s}} Q(y,s) \to 0 \quad \text{as } s \to +\infty. \tag{3-10}$$

Let us now focus on the proof of (3-10) in order to conclude Proposition 3.2. For this purpose, we introduce the following norm: for $r \ge 0$, $q > 1$ and $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,

$$L^{q,r}_\rho(f) \equiv \sup_{|\xi|\le r}\left(\int_{\mathbb{R}^n} |f(y)|^q \rho(y-\xi)\,dy\right)^{\frac{1}{q}}.$$

Following the idea in [Velázquez 1992], we shall make estimates on solutions of (3-9) in the $L^{2,r(\tau)}_\rho$ norm, where $r(\tau) = K_0 e^{(\tau-\bar{s})/2} \le K_0\sqrt{\tau}$. In particular, we have the following:

Lemma 3.3. *Let s be large enough and let $\bar{s}$ be defined by $e^{s-\bar{s}} = s$. Then, for all $\tau \in [\bar{s}, s]$ and $K_0 > 0$,*

$$g(\tau) \le C_0\left(e^{\tau-\bar{s}}\epsilon(\bar{s}) + \int_{\bar{s}}^{(\tau-2K_0)_+} \frac{e^{\tau-t-2K_0}g^2(t)}{(1-e^{-(\tau-t-2K_0)})^{1/20}}\,dt\right),$$

where $g(\tau) = L_\rho^{2,r(K_0,\tau,\bar{s})}(Q(\tau))$, $r(K_0,\tau,\bar{s}) = K_0 e^{(\tau-\bar{s})/2}$, $\epsilon(s) = \mathbb{O}(1/s^{a+1}) + \mathbb{O}(\log s/s^2)$, $C_0 = C_0(C_*, M_0, K_0)$ *and* $z_+ = \max\{z, 0\}$.

Proof. Multiplying (3-9) by $\beta(\tau) = e^{\int_{\bar{s}}^{\tau} C_*/t^{a'}\,dt}$, we write $Q(y,\tau)$, for all $(y,\tau) \in \mathbb{R}^n \times [\bar{s}, s]$, in the integral form

$$Q(y,\tau) = \beta(\tau) S_{\mathscr{L}}(\tau-\bar{s}) Q(y,\bar{s}) + C_* \int_{\bar{s}}^{\tau} \beta(\tau) S_{\mathscr{L}}(\tau - t)\biggl(Q^2 + \frac{|y|^2}{t^{1+a'}} + \frac{1}{t^{1+a'}} + \mathbf{1}_{\{|y| \ge 2K_0\sqrt{t}\}}\biggr)\,dt,$$

where $S_{\mathscr{L}}$ is the linear semigroup corresponding to the operator $\mathscr{L}$.

Next, we take the $L_\rho^{2,r(K_0,\tau,\bar{s})}$ norms on both sides in order to get

$$\begin{aligned} g(\tau) &\le C_0 L_\rho^{2,r}[S_{\mathscr{L}}(\tau-\bar{s})Q(\bar{s})] + C_0 \int_{\bar{s}}^{\tau} L_\rho^{2,r}[S_{\mathscr{L}}(\tau-t)Q^2(t)]\,dt \\ &\quad + C_0 \int_{\bar{s}}^{\tau} L_\rho^{2,r}\biggl[S_{\mathscr{L}}(\tau-t)\biggl(\frac{|y|^2}{t^{1+a'}} + \frac{1}{t^{1+a'}}\biggr)\biggr]\,dt \\ &\quad + C_0 \int_{\bar{s}}^{\tau} L_\rho^{2,r}[S_{\mathscr{L}}(\tau-t)\mathbf{1}_{\{|y|\ge 2K_0\sqrt{t}\}}]\,dt \\ &\equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Proposition 2.3 in [Velázquez 1992] yields

$$\begin{aligned} |J_1| &\le C_0 e^{\tau-\bar{s}} \|Q(\bar{s})\|_{L^2_\rho} = e^{\tau-\bar{s}}\mathbb{O}(\epsilon(\bar{s})) \quad \text{as } \bar{s} \to +\infty, \\ |J_2| &\le \frac{C_0}{\bar{s}^{1+a'}} e^{\tau-\bar{s}} + C_0 \int_{\bar{s}}^{(\tau-2K_0)_+} \frac{e^{(\tau-t-2K_0)}}{(1-e^{-(\tau-t-2K_0)})^{1/20}} [L_\rho^{2,r(K_0,t,\bar{s})} Q(t)]^2\,dt, \\ |J_3| &\le \frac{C_0 e^{\tau-\bar{s}}}{\bar{s}^{1+a'}}(1 + (\tau - \bar{s})), \\ |J_4| &\le C_0 e^{-\delta\bar{s}}, \quad \text{where} \quad \delta = \delta(K_0) > 0. \end{aligned}$$

Putting together the estimates on J_i, $i = 1, 2, 3, 4$, we conclude the proof of Lemma 3.3. □

We now use the following Gronwall lemma:

Lemma 3.4 [Velázquez 1992]. *Let* ϵ, C, R *and* δ *be positive constants with* $\delta \in (0,1)$. *Assume that* $\mathscr{H}(\tau)$ *is a family of continuous functions satisfying*

$$\mathscr{H}(\tau) \le \epsilon e^{\tau} + C \int_0^{(\tau-R)_+} \frac{e^{\tau-s}\mathscr{H}^2(s)}{(1-e^{-(\tau-s-R)})^{\delta}}\,ds \quad \textit{for } \tau > 0.$$

Then there exist $\theta = \theta(\delta, C, R)$ *and* $\epsilon_0 = \epsilon_0(\delta, C, R)$ *such that, for all* $\epsilon \in (0, \epsilon_0)$ *and any* τ *for which* $\epsilon e^{\tau} \le \theta$, *we have*

$$\mathscr{H}(\tau) \le 2\epsilon e^{\tau}.$$

Applying Lemma 3.4 with $\mathscr{H} \equiv g$, we see from Lemma 3.3 that, for s large enough,

$$g(\tau) \le 2C_0 e^{\tau-\bar{s}}\epsilon(\bar{s}) \quad \text{for all } \tau \in [\bar{s}, s].$$

If $\tau = s$, then $e^{s-\bar{s}} = s$, $r = K_0\sqrt{s}$ and

$$g(s) \equiv L_\rho^{2,K_0\sqrt{s}}(Q(s)) = \mathbb{O}\left(\frac{1}{s^a}\right) + \mathbb{O}\left(\frac{\log s}{s}\right) \quad \text{as } s \to +\infty.$$

By using the regularizing effects of the semigroup $S_{\mathscr{L}}$ (see [Velázquez 1992, Proposition 2.3]), we then obtain

$$\sup_{|y| \le K_0\sqrt{s}/2} Q(y,s) \le C'(C_*, K_0, M_0) L_\rho^{2,K_0\sqrt{s}}(Q(s)) = \mathbb{O}\left(\frac{1}{s^a}\right) + \mathbb{O}\left(\frac{\log s}{s}\right) \quad \text{as } s \to +\infty,$$

which concludes the proof of Proposition 3.2. □

4. Numerical method

We give in this section a numerical study of the blow-up profile of (1-1) in one dimension. Though our method is very similar to Berger and Kohn's algorithm [1988] in spirit, it is better in the sense that it can be applied to equations which are not invariant under the transformation (1-29). Our method differs from Berger and Kohn's in the following way: we step the solution forward until its maximum value multiplied by a power of its mesh size reaches a preset threshold, where the mesh size and the preset threshold are linked; for the rescaling algorithm, the solution is stepped forward until its maximum value reaches a preset threshold, and the mesh size and the preset threshold do not need to be linked. For more clarity, we present in the next subsection the mesh-refinement technique applied to (1-1), then give various numerical experiments to illustrate the effectiveness of our method for the problem of the numerical blow-up profile. Note that our method is more general than Berger and Kohn's, in the sense that it applies to non-scale-invariant equations. However, when applied to the unperturbed case $F(u) = |u|^{p-1}u$, our method gives exactly the same approximation as that of [Berger and Kohn 1988].

4A. ***Mesh-refinement algorithm.*** As usually with numerical simulations of blow-up (see [Berger and Kohn 1988]), we will simulate the equation on a bounded interval (say $(-A, A)$ with $A > 0$) with homogeneous Dirichlet boundary conditions, rather than the whole line $\mathbb{R}$. This choice is reasonable for two reasons:

- If initial data on the line are symmetric and decreasing to zero at infinity, then this property persists in time; hence, we are close to the situation of a bounded interval $(-A, A)$ with $A > 0$ large and homogeneous Dirichlet condition.
- We believe that the blow-up on a bounded interval is the same as on the whole line, given that blow-up does not occur on the boundary, as is already known for the pure power and $\mu = 0$. Moreover, as in [Giga and Kohn 1987; Giga et al. 2004b], the results stated in the introduction can be extended to the case when the problem (1-1) is considered in a convex domain of $\mathbb{R}^n$ with Dirichlet condition. Thus, they hold for the problem (4-1).

For that reason we focus on the bounded interval case $(-A, A)$ here. For simplicity we will take $A = 1$. In this section, we describe our refinement algorithm to solve numerically the problem (1-1) with initial data $\varphi(x) > 0$, $\varphi(x) = \varphi(-x)$, $x\, d\varphi(x)/dx < 0$ for $x \neq 0$, which gives a positive symmetric and radially decreasing solution. Let us rewrite the problem (1-1) (with $\mu = 1$) as

$$\begin{cases} \partial_t u = \partial_x^2 u + F(u), & (x,t) \in (-1,1) \times (0,T), \\ u(1,t) = u(-1,t) = 0, & t \in (0,T), \\ u(x,0) = \varphi(x), & x \in (-1,1), \end{cases} \tag{4-1}$$

where $p > 1$ and

$$F(u) = u^p + \frac{u^p}{\log^a(2+u^2)} \quad \text{with} \quad a > 0. \tag{4-2}$$

Let δ and τ be the initial space and time steps, we define $C_\Delta = \tau/\delta^2$, $x_i = i\delta$, $t_n = n\tau$, $I = 1/\delta$ and $u_{i,n}$ as the approximation of $u(x_i, t_n)$, where $u_{i,n}$ is defined for all $n \geq 0$ and $i \in \{-I, \dots, I\}$ by

$$\begin{aligned} u_{i,n+1} &= u_{i,n} + C_\Delta[u_{i-1,n} - 2u_{i,n} + u_{i+1,n}] + \tau F(u_{i,n}), \\ u_{I,n} &= u_{-I,n} = 0, \\ u_{i,0} &= \varphi_i. \end{aligned} \tag{4-3}$$

Note that this scheme is first-order accurate in time and second-order in space, and it requires the stability condition $C_\Delta = \tau/\delta^2 \leq \frac{1}{2}$.

Our algorithm needs to fix the following parameters:

- $\lambda < 1$, the refining factor with λ^{-1} being a small integer.
- M, the threshold to control the amplitude of the solution.
- α, the parameter controlling the width of interval to be refined.

The parameters λ and M must satisfy the relation

$$M = \lambda^{-\frac{2}{p-1}} M_0, \quad \text{where} \quad M_0 = \delta^{\frac{2}{p-1}} \|\varphi\|_\infty. \tag{4-4}$$

Note that the relation (4-4) is important to make our method work. In [Berger and Kohn 1988], the typical choice is $M_0 = \|\varphi\|_\infty$, hence $M = \lambda^{-2/(p-1)} \|\varphi\|_\infty$.

In the initial step of the algorithm, we simply apply the scheme (4-3) until $\delta^{2/(p-1)} \|u(\cdot, t_n)\|_\infty$ reaches M (note that in [Berger and Kohn 1988] the solution is stepped forward until $\|u(\cdot, t_n)\|_\infty$ reaches M; in this first step, the thresholds of the two methods are the same, however, they will split after the second step; roughly speaking, for the threshold we shall use the quantity $\delta^{2/(p-1)} \|u(\cdot, t_n)\|_\infty$ in our method instead of the $\|u(\cdot, t_n)\|_\infty$ in [Berger and Kohn 1988]). Then, we use a linear interpolation in time to find τ_0^* such that

$$t_n - \tau \leq \tau_0^* \leq t_n \quad \text{and} \quad \delta^{\frac{2}{p-1}} \|u(\cdot, \tau_0^*)\| = M.$$

Afterward, we determine two grid points y_0^- and y_0^+ such that

$$\begin{cases} \delta^{\frac{2}{p-1}} u(y_0^- - \delta, \tau_0^*) < \alpha M \leq \delta^{\frac{2}{p-1}} u(y_0^-, \tau_0^*), \\ \delta^{\frac{2}{p-1}} u(y_0^+ + \delta, \tau_0^*) < \alpha M \leq \delta^{\frac{2}{p-1}} u(y_0^+, \tau_0^*). \end{cases} \tag{4-5}$$

Note that $y_0^- = -y_0^+$ because of the symmetry of the solution. This finishes the initial step.

Let us begin the first refining step. Define

$$u^{(1)}(y^{(1)}, t^{(1)}) = u(y^{(1)}, \tau_0^* + t^{(1)}), \quad y^{(1)} \in (y_0^-, y_0^+),\ t^{(1)} \geq 0, \tag{4-6}$$

and set $\delta^{(1)} = \lambda\delta$, $\tau^{(1)} = \lambda^2\tau$ as the space and time step for the approximation of $u^{(1)}$ (note that $\tau^{(1)}/(\delta^{(1)})^2 = \tau/\delta^2 = C_\Delta$, which is a constant), $y_i^{(1)} = i\delta^{(1)}$, $t_n^{(1)} = n\tau^{(1)}$, $I_1 = y_0^+/\delta^{(1)}$ and $u_{i,n}^{(1)}$ as the approximation of $u^{(1)}(y_i^{(1)}, t_n^{(1)})$. Note that, in the unperturbed case, Berger and Kohn used the transformation (1-29) to define $u^{(1)}(y^{(1)}, t^{(1)}) = \lambda^{2/(p-1)}u(\lambda y^{(1)}, \tau_0^* + \lambda^2 t^{(1)})$ and then applied the same scheme for u to $u^{(1)}$. However, we can not do the same because (4-1) is not invariant under the transformation (1-29). Then applying the scheme (4-3) to $u^{(1)}$, we write

$$u_{i,n+1}^{(1)} = u_{i,n}^{(1)} + C_\Delta[u_{i-1,n}^{(1)} - 2u_{i,n}^{(1)} + u_{i+1,n}^{(1)}] + \tau^{(1)}F(u_{i,n}^{(1)}) \tag{4-7}$$

for all $n \geq 0$ and $i \in \{-I_1 + 1, \dots, I_1 - 1\}$.

Note that the computation of $u^{(1)}$ requires the initial data $u^{(1)}(y^{(1)}, 0)$ and the boundary condition $u^{(1)}(y_0^\pm, t^{(1)})$. For the initial condition, it is determined from $u(x, \tau_0^*)$ by using interpolation in space to get values at the new grid points. For the boundary condition, since $\tau^{(1)} = \lambda^2\tau$, we have from (4-6) that

$$u^{(1)}(y_0^\pm, n\tau^{(1)}) = u(y_0^\pm, \tau_0^* + n\lambda^2\tau). \tag{4-8}$$

Since u and $u^{(1)}$ will be stepped forward, each on its own grid ($u^{(1)}$ on (y_0^-, y_0^+) with the space and time steps $\delta^{(1)}$ and $\tau^{(1)}$, and u on $(-1, 1)$ with the space and time steps δ and τ), the relation (4-8) will provide us with the boundary values for $u^{(1)}$. In order to better understand how it works, let us consider an example with $\lambda = \frac{1}{2}$. After concluding the initial phase, the two solutions $u^{(1)}$ and u are stepped forward independently, each on its own grid; in other words, $u^{(1)}$ on (y_0^-, y_0^+) with the space and time steps $\delta^{(1)}$ and $\tau^{(1)}$, and u on $(-1, 1)$ with the space and time steps δ and τ. Then, using the linear interpolation in time for u, we get the boundary values for $u^{(1)}$ by (4-8), since $\tau^{(1)} = \lambda^2\tau = \frac{1}{4}\tau$. This means that u is stepped forward once every 4 time steps of $u^{(1)}$. After 4 steps forward of $u^{(1)}$, the values of u on the interval (y_0^-, y_0^+) must be updated to agree with the calculations of $u^{(1)}$. In other words, the approximation of u is used to assist in computing the boundary values for $u^{(1)}$. At each successive time step for u, the values of u on the interval (y_0^-, y_0^+) must be updated to make them agree with the more accurate fine grid solution $u^{(1)}$. When $(\delta^{(1)})^{2/(p-1)}\|u^{(1)}(\cdot, n\tau^{(1)})\|_\infty$ first exceeds M, we use a linear interpolation in time to find $\tau_1^* \in [\tau_{n-1}^{(1)}, \tau_n^{(1)}]$ such that $(\delta^{(1)})^{2/(p-1)}\|u^{(1)}(\cdot, \tau_1^*)\|_\infty = M$. On the interval where $(\delta^{(1)})^{2/(p-1)}\|u^{(1)}(\cdot, \tau_1^*)\|_\infty > \alpha M$, the grid is refined further and the entire procedure for $u^{(1)}$ is repeated to yield $u^{(2)}$, and so forth.

Before going to a general step, we would like to comment on relation (4-4). When $\delta^{2/(p-1)}\|u(\cdot, t)\|_\infty$ reaches the given threshold M in the initial phase, namely when $\delta^{2/(p-1)}\|u(\cdot, \tau_0^*)\|_\infty = M$, we want to refine the grid so that the maximum value of $(\delta^{(1)})^{2/(p-1)}u^{(1)}(y^{(1)}, 0)$ equals M_0. By (4-6), this request turns into $(\delta^{(1)})^{2/(p-1)}\|u(\cdot, \tau_0^*)\|_\infty = M_0$. Since $\delta^{(1)} = \lambda\delta$, it follows that $M = \lambda^{-2/(p-1)}M_0$, which yields (4-4).

Let $k \geq 0$; we set $\delta^{(k+1)} = \lambda^{-1}\delta^{(k)}$ and $\tau^{(k+1)} = \lambda^2\tau^{(k)}$ (note that $\tau^{(k+1)}/(\delta^{(k+1)})^2 = \tau^{(k)}/(\delta^{(k)})^2 = \cdots = \tau/\delta^2 = C_\Delta$), and let $y^{(k)}$ and $t^{(k)}$ be the variables of $u^{(k)}$, with $y_i^{(k)} = i\delta^{(k)}$ and $t_n^{(k)} = n\tau^{(k)}$. By convention, the index $k = 0$ means that $u^{(0)}(y^{(0)}, t^{(0)}) \equiv u(x,t)$, $\delta^{(0)} \equiv \delta$ and $\tau^{(0)} \equiv \tau$. The solution $u^{(k+1)}$ is related to $u^{(k)}$ by

$$u^{(k+1)}(y^{(k+1)}, t^{(k+1)}) = u^{(k)}(y^{(k+1)}, \tau_k^* + t^{(k+1)}), \tag{4-9}$$

where $y^{(k+1)} \in (y_k^-, y_k^+)$, $t^{(k+1)} \geq 0$, the time $\tau_k^* \in [t_{n-1}^{(k)}, t_n^{(k)}]$ satisfies

$$(\delta^{(k)})^{\frac{2}{p-1}} \|u^{(k)}(\cdot, \tau_k^*)\|_\infty = M,$$

and y_k^- and y_k^+ are two grid points determined by

$$\begin{cases} (\delta^{(k)})^{\frac{2}{p-1}} u^{(k)}(y_k^- - \delta^{(k)}, \tau_k^*) < \alpha M \leq (\delta^{(k)})^{\frac{2}{p-1}} u^{(k)}(y_k^-, \tau_k^*), \\ (\delta^{(k)})^{\frac{2}{p-1}} u^{(k)}(y_k^+ + \delta^{(k)}, \tau_k^*) < \alpha M \leq (\delta^{(k)})^{\frac{2}{p-1}} u^{(k)}(y_k^+, \tau_k^*). \end{cases} \tag{4-10}$$

The approximation of $u^{(k+1)}$ at the point $(y_i^{(k+1)}, t_n^{(k+1)})$, denoted by $u_{i,n}^{(k+1)}$, uses the scheme (4-3) with the space step $\delta^{(k+1)}$ and the time step $\tau^{(k+1)}$, which reads

$$u_{i,n+1}^{(k+1)} = u_{i,n}^{(k+1)} + C_\Delta[u_{i-1,n}^{(k+1)} - 2u_{i,n}^{(k+1)} + u_{i+1,n}^{(k+1)}] + \tau^{(k+1)} F(u_{i,n}^{(k+1)}) \tag{4-11}$$

for all $n \geq 1$ and $i \in \{-I_k + 1, \cdots, I_k - 1\}$, where $I_k = y_k^+/\delta^{(k+1)}$ (note that I_k is an integer since $\lambda^{-1} \in \mathbb{N}$).

As for the approximation of $u^{(k)}$, the computation of $u_{i,n}^{(k+1)}$ needs the initial data and the boundary condition. From (4-9) and the fact that $\tau^{(k+1)} = \lambda^2\tau^{(k)}$, we see that

$$u^{(k+1)}(y^{(k+1)}, 0) = u^{(k)}(y^{(k+1)}, \tau_k^*), \tag{4-12}$$

$$u^{(k+1)}(y_k^\pm, n\tau^{(k+1)}) = u^{(k)}(y_k^\pm, \tau_k^* + n\lambda^2\tau^{(k)}). \tag{4-13}$$

From (4-12), the initial data is simply calculated from $u^{(k)}(\cdot, \tau_k^*)$ by using a linear interpolation in space in order to assign values at new grid points. The essential step in this new mesh-refinement method is to determine the boundary condition through the identity (4-13), which means by a linear interpolation in time of $u^{(k)}$. Therefore, the previous solutions $u^{(k)}, u^{(k-1)}, \ldots$ are stepped forward independently, each on its own grid. More precisely, $\tau^{(k+1)} = \lambda^2\tau^{(k)} = \lambda^4\tau^{(k-1)} = \cdots$, so $u^{(k)}$ is stepped forward once every λ^{-2} time steps of $u^{(k+1)}$, $u^{(k-1)}$ once every λ^{-4} time steps of $u^{(k+1)}$, etc. On the other hand, the values of $u^{(k)}, u^{(k-1)}, \ldots$ must be updated to agree with the calculation of $u^{(k+1)}$. When $(\delta^{(k+1)})^{2/(p-1)}\|u^{(k+1)}(\cdot, \tau^{(k+1)})\|_\infty > M$, it is time for the next refining phase.

We would like to comment on the output of the refinement algorithm:

(i) Let τ_k^* be the time at which the refining takes place, then the ratio $\tau_k^*/\tau^{(k)}$, which indicates the number of time steps until $(\delta^{(k)})^{2/(p-1)}\|u^{(k)}\|_\infty$ reaches the given threshold M, tends to a constant as $k \to \infty$.

(ii) Let $u^{(k)}(\cdot, \tau_k^*)$ be the *refining solution*. If we plot $(\delta^{(k)})^{2/(p-1)}u^{(k)}(\cdot, \tau_k^*)$ on $(-1, 1)$, then their graphs eventually converge to a predicted one as $k \to \infty$.

(iii) Let (y_k^-, y_k^+) be the interval to be refined; then the quantity $(\delta^{(k)})^{-2}(y_k^+)^2$ behaves as a linear function of k.

These assertions can be well understood by the following theorem:

Theorem 4.1 (formal analysis). *Let u be a blowing-up solution to* (4-1); *then the output of the refinement algorithm satisfies*:

(i) *The ratio $\tau_k^*/\tau^{(k)}$ tends to a constant as $k \to \infty$, namely*

$$\frac{\tau_k^*}{\tau^{(k)}} \to \frac{(\lambda^{-2}-1)M^{1-p}}{C_\Delta(p-1)} \quad \text{as } k \to +\infty. \tag{4-14}$$

(ii) *Assume in addition that Theorem 1.7*(*i*) *holds. Defining $v^{(k)}(z) = (\delta^{(k)})^{2/(p-1)} u^{(k)}(zy_{k-1}^+, \tau_k^*)$ for all $k \geq 1$, we have*

$$\forall |z| < 1 \quad v^{(k)}(z) \sim M(1 + (\alpha^{1-p} - 1)\lambda^{-2}z^2)^{-\frac{1}{p-1}} \quad \text{as } k \to +\infty. \tag{4-15}$$

(iii) *The quantity $(\delta^{(k)})^{-2}(y_k^+)^2$ behaves as a linear function, namely*

$$(\delta^{(k)})^{-2}(y_k^+)^2 \sim \gamma k + B \quad \text{as } k \to +\infty, \tag{4-16}$$

where

$$\gamma = \frac{2M^{1-p}(\alpha^{1-p}-1)|\log\lambda|}{c_p(p-1)\lambda^2}, \qquad B = -\frac{M^{1-p}(\alpha^{1-p}-1)}{c_p(p-1)\lambda^2}\log\left(\frac{M^{1-p}\delta^2}{p-1}\right) \quad \text{and} \quad c_p = \frac{p-1}{4p}.$$

Remark 4.2. Note that there is no assumption on the value of a in the hypothesis in Theorem 4.1. It is understood in the sense that u blows up in finite time and its profile is described in Theorem 1.7.

Proof. As we will see in the proof, the statement (i) concerns the blow-up limit of the solution and (ii) is due to the blow-up profile stated in Theorem 1.7.

(i) If σ_k is the real time when the refinement from $u^{(k)}$ to $u^{(k+1)}$ takes place, we have, by (4-9),

$$\sigma_k = \tau_0^* + \tau_1^* + \cdots + \tau_k^*,$$

where τ_k^* is such that $(\delta^{(k)})^{2/(p-1)}\|u^{(k)}(\cdot, \tau_k^*)\|_\infty = M$. This means that

$$u^{(k)}(\cdot, \tau_k^*) = u(\cdot, \sigma_k). \tag{4-17}$$

On the other hand, from Theorem 1.7(i) and the definition (1-26) of f, we see that

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_{L^\infty} = \kappa. \tag{4-18}$$

Combining (4-18) and (4-17) yields

$$(T-\sigma_k)^{\frac{1}{p-1}}\|u^{(k)}(\cdot, \tau_k^*)\|_\infty = \kappa + o(1), \tag{4-19}$$

where $o(1)$ represents a term that tends to 0 as $k \to +\infty$.

Since $\|u^{(k)}(\cdot,\tau_k^*)\|_\infty = M(\delta^{(k)})^{-2/(p-1)}$, we then derive

$$T-\sigma_k = (M^{-1}\kappa)^{p-1}(\delta^{(k)})^2 + o(1). \tag{4-20}$$

By the definition of σ_k and (4-17), we infer that $\tau_k^* = \sigma_k - \sigma_{k-1}$ (we can think τ_k^* as the *live time* of $u^{(k)}$ in the k-th refining phase). Hence,

$$\begin{aligned}\frac{\tau_k^*}{\tau^{(k)}} = \frac{\sigma_k-\sigma_{k-1}}{\tau^{(k)}} &= \frac{1}{\tau^{(k)}}[(T-\sigma_{k-1})-(T-\sigma_k)]\\ &= \frac{1}{\tau^{(k)}}(M^{-1}\kappa)^{p-1}((\delta^{(k-1)})^2-(\delta^{(k)})^2)+o(1)\\ &= \frac{(\delta^{(k)})^2}{\tau^{(k)}}(M^{-1}\kappa)^{p-1}(\lambda^{-2}-1)+o(1).\end{aligned}$$

Since the ratio $\tau^{(k)}/(\delta^{(k)})^2$ is always fixed by the constant C_Δ, we finally obtain

$$\lim_{k\to+\infty}\frac{\tau_k^*}{\tau^{(k)}} = \frac{(\lambda^{-2}-1)M^{1-p}}{C_\Delta(p-1)},$$

which is the conclusion of Theorem 4.1(i).

(ii) By the symmetry of the solution, we have $y_{k-1}^- = y_{k-1}^+$. We then consider the following mapping: for all $k\ge 1$,

$$z\mapsto v^{(k)}(z) \quad \text{for all } |z|\le 1, \quad \text{where} \quad v^{(k)}(z) = \left(\delta^{(k)}\right)^{\frac{2}{p-1}}u^{(k)}(zy_{k-1}^+,\tau_k^*).$$

We will show that $v^{(k)}(z)$ converges to some fixed function as $k\to+\infty$. For this purpose, we first write $u^{(k)}(y^{(k)},\tau_k*)$ in terms of $w(\xi,s)$ thanks to (4-17) and (1-10):

$$u^{(k)}(y^{(k)},\tau_k^*) = u(y^{(k)},\sigma_k) = (T-\sigma_k)^{-\frac{1}{p-1}}w(\xi^{(k)},s_k), \tag{4-21}$$

where $\xi^{(k)} = y^{(k)}/\sqrt{T-\sigma_k}$ and $s_k = -\log(T-\sigma_k)$.

If we write Theorem 1.7(i) in the variable $y/\sqrt{s}$ through (1-10), we have the equivalence

$$\left\|w(y,s)-f\left(\frac{y}{\sqrt{s}}\right)\right\|_{L^\infty}\to 0 \quad \text{as } s\to+\infty, \tag{4-22}$$

where f is as given in (1-26).

From (4-22), (4-20) and (4-21), we derive

$$u^{(k)}(y^{(k)},\tau_k^*) = M\kappa^{-1}(\delta^{(k)})^{-\frac{2}{p-1}}f\left(\frac{y^{(k)}}{(M^{-1}\kappa)^{\frac{p-1}{2}}\delta^{(k)}\sqrt{s_k}}\right)+o(1).$$

Then, multiplying both of sides by $(\delta^{(k)})^{2/(p-1)}$ and replacing $y^{(k)}$ by zy_{k-1}^+, we obtain

$$(\delta^{(k)})^{\frac{2}{p-1}}u^{(k)}(zy_{k-1}^+,\tau_k^*) = M\kappa^{-1}f\left(\frac{zy_{k-1}^+}{(M^{-1}\kappa)^{\frac{p-1}{2}}\delta^{(k)}\sqrt{s_k}}\right)+o(1). \tag{4-23}$$

From the definition (4-10) of y^+_{k-1}, we may assume that

$$(\delta^{(k-1)})^{\frac{2}{p-1}} u^{(k-1)}(y^+_{k-1}, \tau^*_{k-1}) = \alpha M.$$

Combining this with (4-23), we have

$$\alpha = \kappa^{-1} f\left(\frac{y^+_{k-1}}{(M^{-1}\kappa)^{\frac{p-1}{2}} \delta^{(k-1)} \sqrt{s_{k-1}}}\right) + o(1).$$

Since $s_k = -\log(T - \sigma_k)$ and $\delta^{(k)} = \lambda^k \delta$, we have from (4-20) that

$$s_k = 2k|\log\lambda| - \log\left(\frac{M^{1-p}\delta^2}{p-1}\right) + o(1), \tag{4-24}$$

which implies $\lim_{k\to+\infty} s_{k-1}/s_k = 1$. Thus, it is reasonable to assume that $y^+_{k-1}/\sqrt{s_{k-1}}$ and $y^+_{k-1}/\sqrt{s_k}$ tend to the positive root ζ as $k \to +\infty$. Hence,

$$\alpha = \kappa^{-1} f\left(\frac{\zeta}{(M^{-1}\kappa)^{\frac{p-1}{2}} \delta^{(k)} \lambda^{-1}}\right) + o(1).$$

Using the definition (1-26) of f, we have

$$\alpha = \left(1 + c_p\left|\frac{\zeta}{(M^{-1}\kappa)^{\frac{p-1}{2}} \delta^{(k)}}\right|^2 \lambda^2\right)^{-\frac{1}{p-1}} + o(1),$$

which implies

$$\left|\frac{\zeta}{(M^{-1}\kappa)^{\frac{p-1}{2}} \delta^{(k)}}\right|^2 = \frac{1}{c_p}[(\alpha^{1-p} - 1)\lambda^{-2}] + o(1), \tag{4-25}$$

where c_p is the constant given in the definition (1-26) of f.

Substituting this into (4-23) and using the definition (1-26) of f again, we arrive at

$$v^{(k)}(z) = M\left(1 + c_p\left|\frac{\zeta}{(M^{-1}\kappa)^{\frac{p-1}{2}} \delta^{(k)}}\right|^2 z^2\right)^{-\frac{1}{p-1}} + o(1) = M(1 + (\alpha^{1-p} - 1)\lambda^{-2} z^2)^{-\frac{1}{p-1}} + o(1).$$

Let $k \to +\infty$; the conclusion of (ii) then follows.

(iii) From (4-25) and the fact that $y^+_k/\sqrt{s_k} \to \zeta$ as $k \to +\infty$, we have

$$(\delta^{(k)})^{-2}(y^+_k)^2 = \frac{(\alpha^{1-p} - 1)M^{1-p}}{c_p\lambda^2(p-1)} \log s_k + o(1).$$

Using (4-24), we then derive

$$(\delta^{(k)})^{-2}(y^+_k)^2 = \frac{2k|\log\lambda|(\alpha^{1-p} - 1)M^{1-p}}{c_p\lambda^2(p-1)} - \frac{(\alpha^{1-p} - 1)M^{1-p}}{c_p\lambda^2(p-1)} \log\left(\frac{M^{1-p}\delta^2}{p-1}\right) + o(1),$$

which yields the conclusion of (iii) and finishes the proof of Theorem 4.1. □

δ	0.040	0.020	0.010	0.005
M	0.320	0.160	0.080	0.040

Table 1. The value of M corresponds to the initial data and the initial space step.

4B. ***The numerical results.*** This subsection gives various numerical confirmations for the assertions stated in the previous subsection (Theorem 4.1). All the experiments reported here used $\varphi(x) = 2(1+\cos(\pi x))$ as the initial data, $\alpha = 0.6$ as the parameter for controlling the interval to be refined, $\lambda = \frac{1}{2}$ as the refining factor, $C_\Delta = \frac{1}{4}$ as the stability condition for the scheme (4-3), $p = 3$ and $a = 0.1$, 1 and 10 in the nonlinearity F given in (4-2). The threshold M is chosen to satisfy the condition (4-4). In Table 1, we give some values of M corresponding to the initial data and the initial space step δ. We generally stop the computation after 40 refining phases. Indeed, since $(\delta^{(k)})^{2/(p-1)}\|u^{(k)}(\cdot,\tau_k^*)\|_\infty = M$ and $\delta^{(k)} = \lambda\delta^{(k-1)}$, we have by induction that

$$\|u^{(k)}(\cdot,\tau_k^*)\|_\infty = (\delta^{(k)})^{-\frac{2}{p-1}}M = (\lambda\delta^{(k-1)})^{-\frac{2}{p-1}}M = \cdots = (\lambda^k\delta)^{-\frac{2}{p-1}}M.$$

With these parameters, we see that the corresponding amplitude of u approaches 10^{12} after 40 iterations.

4B(i). *The value $\tau_k^*/\tau^{(k)}$ tends to a constant as $k \to +\infty$.* It is convenient to denote the computed value of $\tau_k^*/\tau^{(k)}$ by $N^{(k)}$ and the predicted value given in the statement Theorem 4.1(i) by N^{pre}. Note that the value of N^{pre} does not depend on a, but depends on δ because of the relation (4-4). More precisely,

$$N^{\text{pre}}(\delta) = \frac{(1-\lambda^2)\|\varphi\|_\infty^{1-p}}{C_\Delta(p-1)\delta^2}.$$

Then, considering the quantity $N^{(k)}/N^{\text{pre}}$, theoretically it is expected to converge to 1 as k tends to infinity. Table 2 provides computed values of $N^{(k)}/N^{\text{pre}}$ at some selected indices of k, computing with $\delta = 0.005$ and three different values of a. According to the numerical results given in Table 2, the computed values in the cases $a = 10$ and $a = 1.0$ approach to 1 as expected, which gives us a numerical answer for the statement (4-18). However, the numerical results in the case $a = 0.1$ are not good due to the fact that the speed of convergence to the blow-up limit (4-18) is $1/|\log(T-t)|^{a'}$ with $a' = \min\{a, 1\}$ (see Theorem 1.4).

4B(ii). *The function $v^{(k)}(z)$ introduced in Theorem 4.1(ii) converges to a predicted profile as $k \to +\infty$.* As stated in Theorem 4.1(ii), if we plot $v^{(k)}(z)$ over the fixed interval $(-1, 1)$ then the graph of $v^{(k)}$

k	10	15	20	25	30	35	40
$a = 10$	1.0325	1.0203	1.0149	1.0117	1.0096	1.0080	1.0072
$a = 1.0$	0.9699	0.9771	0.9816	0.9845	0.9867	0.9885	0.9899
$a = 0.1$	0.5853	0.5885	0.5923	0.5957	0.5989	0.6016	0.6043

Table 2. The values of $N^{(k)}/N^{\text{pre}}$ at some selected indices of k, computing with $\delta = 0.005$ and three different values of a.

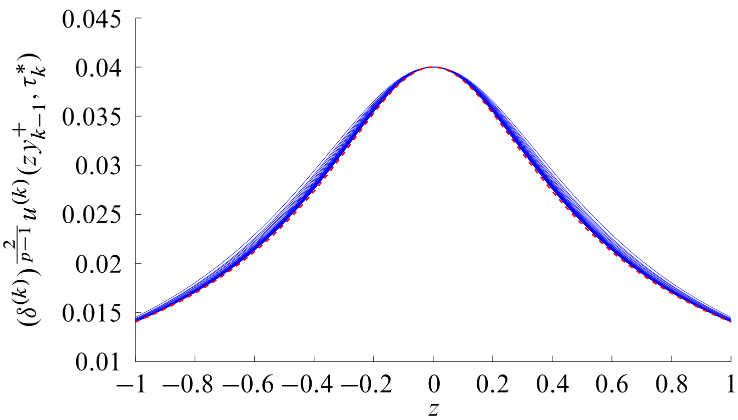

Figure 1. The graph of $v^{(k)}(z)$ at some selected indices of k, computing with $\delta = 0.005$ and $a = 10$. They converge to the predicted profile (the dash line) as stated in (4-15) as k increases.

would converge to the predicted one. Figure 1 gives us a numerical confirmation for this fact, computing with $\delta = 0.005$ and $a = 10$. Looking at Figure 1, we see that the graph of $v^{(k)}$ evidently converges to the predicted one given in the right-hand side of (4-15) as k increases. The last curve $v^{(40)}$ seemingly coincides with the prediction. Figure 2 shows the graph of $v^{(40)}$ and the predicted profile for another experiment with $\delta = 0.005$ and $a = 0.1$. They coincide to within plotting resolution.

In Table 3, we give the error in L^∞ between $v^{(k)}(z)$ at index $k = 40$ and the predicted profile given in the right-hand side of (4-15), namely

$$e_{\delta,a} = \sup_{z \in (-1,1)} \left| v^{(40)}(z) - M(1 + (\alpha^{1-p} - 1)\lambda^{-2} z^2)^{-\frac{1}{p-1}} \right|. \tag{4-26}$$

These numerical computations give us confirmation that the computed profiles v_k converges to the predicted one. Since the error $e_{\delta,a}$ tends to 0 as δ goes to 0, the numerical computations also answer to the stability of the blow-up profile stated in Theorem 1.7(i). In fact, the stability makes the solution visible in numerical simulations.

4B(iii). *The quantity* $(\delta^{(k)})^{-2}(y_k^+)^2$ *behaves like a linear function in* k. For making a quantitative comparison between our numerical results and the predicted behavior as stated in Theorem 4.1(iii), we plot the graph of $(\delta^{(k)})^{-2}(y_k^+)^2$ against k and denote by $\gamma_{\delta,a}$ the slope of this curve. Then, considering

δ	0.04	0.02	0.01	0.005
$a = 10$	0.002906	0.000789	0.000470	0.000238
$a = 1.0$	0.001769	0.000671	0.000359	0.000213
$a = 0.1$	0.002562	0.000687	0.000380	0.000235

Table 3. Error $e_{\delta,a}$ in L^∞ between the computed and predicted profiles, defined in (4-26).

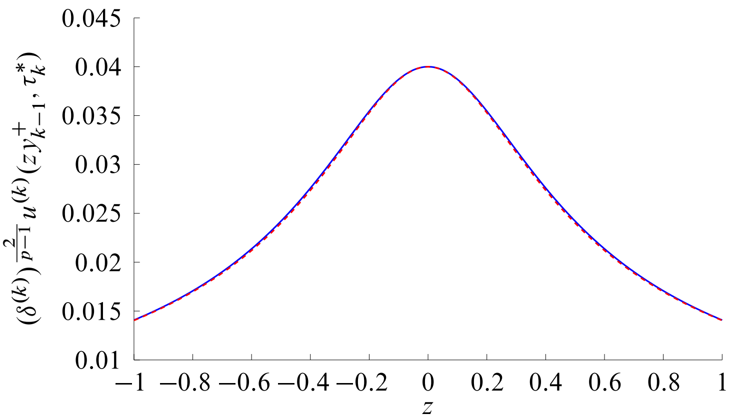

Figure 2. The graph of $v^{(k)}(z)$ at $k = 40$ and the predicted profile given in (4-15), computing with $\delta = 0.005$ and $a = 0.1$. They coincide to within plotting resolution.

the ratio $\gamma_{\delta,a}/\gamma$, where γ is as given in Theorem 4.1(iii). As expected, this ratio $\gamma_{\delta,a}/\gamma$ would approach 1. Figure 3 shows $(\delta^{(k)})^{-2}(y_k^+)^2$ as a function of k, computing with the initial space step $\delta = 0.005$ for different values of a. Looking at Figure 3, we see that the two middle curves, corresponding to the cases $a = 10$ and $a = 1$, behave like the predicted linear function (the top line), while this is not true in the case $a = 0.1$ (the bottom curve). In order to make this clearer, Table 4 lists the values of $\gamma_{\delta,a}/\gamma$, computing with various values of the initial space step δ for three different values of a. Here, the value of $\gamma_{\delta,a}$ is calculated for $20 \le k \le 40$. As Table 4 shows, the numerical values in the cases $a = 10$ and $a = 1$ agree with the prediction stated in Theorem 4.1(ii), while the numerical values in the case $a = 0.1$ are far from the predicted ones.

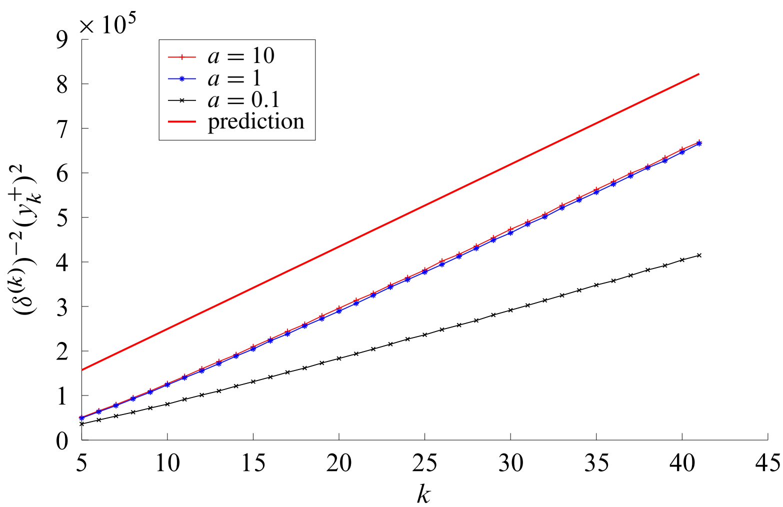

Figure 3. The graph of $(\delta^{(k)})^{-2}(y_k^+)^2$ against k, computing with $\delta = 0.005$ for three different values of a.

δ	0.04	0.02	0.01	0.005
$a = 10$	1.9514	1.1541	0.9991	0.9669
$a = 1.0$	1.9863	1.1436	1.0052	0.9682
$a = 0.1$	1.9538	0.8108	0.6417	0.5986

Table 4. The values of $\gamma_{\delta,a}/\gamma$, computing with various values of the initial space step δ for three different values of a.

Appendix A

The following lemma from [Nguyen 2015] gives the expansion of $\phi(s)$, the unique solution of (1-21) satisfying (1-22):

Lemma A.1. *Let ϕ be a positive solution of the ordinary differential equation*

$$\phi_s = -\frac{\phi}{p-1} + \phi^p + \frac{\mu \phi^p}{\log^a(2 + e^{\frac{2s}{p-1}} \phi^2)}.$$

If we assume in addition $\phi(s) \to \kappa$ as $s \to +\infty$, then $\phi(s)$ takes the form

$$\phi(s) = \kappa(1 + \eta_a(s))^{-\frac{1}{p-1}} \quad \text{as } s \to +\infty,$$

where

$$\eta_a(s) \sim C_* \int_s^{+\infty} \frac{e^{s-\tau}}{\tau^a}\, d\tau = \frac{C_*}{s^a}\left(1 + \sum_{j \geq 1} \frac{b_j}{s^j}\right)$$

with $C_* = \mu\big(\frac{1}{2}(p-1)\big)^a$ and $b_j = (-1)^j \prod_{i=0}^{j-1}(a+i)$.

Proof. See Lemma A.3 in [Nguyen 2015]. □

Appendix B

We aim at proving the following:

Lemma B.1 (estimate of $\omega(s)$). *We have*

$$|\omega(s)| = \mathcal{O}\left(\frac{1}{s^{a+1}}\right) \quad \text{as } s \to +\infty.$$

Proof. From Lemma A.1, we write

$$p(\phi(s)^{p-1} - \kappa^{p-1}) = -\frac{p\eta_a(s)}{p-1}(1 + \eta_a(s))^{-1} = -\frac{pC_*}{(p-1)s^a}(1 + \eta_a(s))^{-1} + \mathcal{O}\left(\frac{1}{s^{a+1}}\right).$$

A direct calculation yields

$$\begin{aligned} e^{-s} h'(e^{\frac{p}{p-1}} \phi(s)) &= \frac{\mu p \phi^{p-1}(s)}{\log^a(2 + e^{\frac{2s}{p-1}} \phi^2(s))} - \frac{2a\mu e^{\frac{2s}{p-1}} \phi^{p+1}(s)}{(2 + e^{\frac{2s}{p-1}} \phi^2(s)) \log^{a+1}(2 + e^{\frac{2s}{p-1}} \phi^2(s))} \\ &= \frac{pC_*}{(p-1)s^a}(1 + \eta_a(s))^{-1} + \mathcal{O}\left(\frac{1}{s^{a+1}}\right). \end{aligned}$$

Adding the two above estimates, we obtain the desired result. This ends the proof of Lemma B.1. ☐

Lemma B.2 (estimate of $R(y,s)$). *We have*

$$|R(y,s)| = \mathcal{O}\left(\frac{|y|^2+1}{s^{a'+1}}\right) \quad \textit{as } s \to +\infty$$

with $a' = \min\{1, a\}$.

Proof. Let us write $\varphi(y,s) = (\phi(s)/\kappa)\nu(y,s)$, where

$$\nu(y,s) = \kappa\left(1 + \frac{p-1}{4ps}\sum_{j=1}^{l} y_j^2\right)^{-\frac{1}{p-1}} + \frac{\kappa l}{2ps}.$$

Then, we write $R(y,s) = (\phi(s)/\kappa)R_1(y,s) + R_2(y,s)$, where

$$R_1(y,s) = \nu_s - \Delta\nu - \frac{y}{2}\cdot\nabla\nu - \frac{\nu}{p-1} + \nu^p,$$
$$R_2(y,s) = -\frac{\phi'}{\kappa}\nu - \frac{\phi}{\kappa}\nu^p + \phi^p\left(\frac{\nu}{\kappa}\right)^p + e^{-\frac{ps}{p-1}}h'\left(e^{\frac{s}{p-1}}\frac{\phi\nu}{\kappa}\right).$$

The term $R_1(y,s)$ is already treated in [Velázquez 1992] and it is bounded by

$$|R_1(y,s)| \le \frac{C(|y|^2+1)}{s^2} + C\mathbf{1}_{\{|y|\ge 2K_0\sqrt{s}\}}.$$

To bound R_2, we use the fact that ϕ satisfies (1-22) to write

$$R_2(y,s) = \frac{\nu\phi}{\kappa^p}(\kappa^{p-1} - \phi^{p-1})(\kappa^{p-1} - \nu^{p-1}) + e^{-\frac{ps}{p-1}}\left[h\left(e^{\frac{s}{p-1}}\frac{\phi\nu}{\kappa}\right) - h(e^{\frac{s}{p-1}}\phi)\right] + \left(1 - \frac{\nu}{\kappa}\right)e^{-\frac{ps}{p-1}}h(e^{\frac{s}{p-1}}\phi).$$

Noting that $\nu(y,s) = \kappa + \bar{\nu}(y,s)$ with $|\bar{\nu}(y,s)| \le (C/s)(|y|^2+1)$, uniformly for $y \in \mathbb{R}$ and $s \ge 1$, and recalling from Lemma A.1 that $\phi(s) = \kappa(1+\eta_a(s))^{-1/(p-1)}$, where $\eta_a(s) = \mathcal{O}(s^{-a})$, then using a Taylor expansion, we derive

$$|R_2(y,s)| \le C\left(\frac{|y|^2+1}{s^{a+1}} + \mathbf{1}_{\{|y|\ge 2K_0\sqrt{s}\}}\right).$$

This concludes the proof of Lemma B.2. ☐

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