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RESONANCES FOR LARGE ONE-DIMENSIONAL "ERGODIC" SYSTEMS

FRÉDÉRIC KLOPP

Dedicated to Johannes Sjöstrand on the occasion of his seventieth birthday.

The present paper is devoted to the study of resonances for one-dimensional quantum systems with a potential that is the restriction to some large box of an ergodic potential. For discrete models, both on a half-line and on the whole line, we study the distributions of the resonances in the limit when the size of the box goes to infinity. For periodic and random potentials, we analyze how the spectral theory of the limit operator influences the distribution of the resonances.

Dans cet article, nous étudions les résonances d'un système unidimensionnel plongé dans un potentiel qui est la restriction à un grand intervalle d'un potentiel ergodique. Pour des modèles discrets sur la droite et la demie droite, nous étudions la distribution des résonances dans la limite de la taille de boîte infinie. Pour des potentiels périodiques et aléatoires, nous analysons l'influence de la théorie spectrale de l'opérateur limite sur la distribution des résonances.

0. Introduction

Consider $V : \mathbb{Z} \to \mathbb{R}$ a bounded potential and, on $\ell^2(\mathbb{Z})$, the Schrödinger operator $H = -\Delta + V$ defined by

$$(Hu)(n) = u(n+1) + u(n-1) + V(n)u(n) \quad \text{for all } n \in \mathbb{Z}$$

for $u \in \ell^2(\mathbb{Z})$.

The potentials V we will deal with are of two types:

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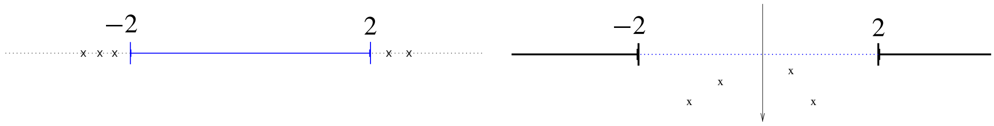

Figure 1. The meromorphic continuation.

- V periodic;
- $V = V_\omega$, the random Anderson model, i.e., the entries of the diagonal matrix V are independent, identically distributed, nonconstant random variables.

The spectral theory of such models has been studied extensively (see, e.g., [Kirsch 2008]) and it is well known that

- when V is periodic, the spectrum of H is purely absolutely continuous;
- when $V = V_\omega$ is random, the spectrum of H is almost surely pure point, i.e., the operator only has eigenvalues; moreover, the eigenfunctions decay exponentially at infinity.

Pick $L \in \mathbb{N}^*$. The main object of our study is the operator

$$H_L = -\Delta + V\mathbf{1}_{[\![-L+1,L]\!]} \tag{0-1}$$

when L is large. Here, $[\![-L+1, L]\!]$ is the integer interval $\{-L+1, \dots, L\}$, and $\mathbf{1}_{[\![a,b]\!]}(n) = 1$ if $a \le n \le b$ and $\mathbf{1}_{[\![a,b]\!]}(n) = 0$ if not.

For L large, the operator H_L is a simple Hamiltonian modeling a large sample of periodic or random material in the void. It is well known in this case (see, e.g., [Zworski 2002]) that not only is the spectrum of H_L of importance but also its (quantum) resonances, which we will now define.

As $V\mathbf{1}_{[\![-L+1,L]\!]}$ has finite rank, the essential spectrum of H_L is the same as that of the discrete Laplace operator, that is, $[-2, 2]$, and it is purely absolutely continuous. Outside this absolutely continuous spectrum, H_L has only discrete eigenvalues associated to exponentially decaying eigenfunctions.

We are interested in the resonances of the operator H_L in the limit when $L \to +\infty$. They are defined to be the poles of the meromorphic continuation of the resolvent of H_L through $(-2, 2)$, the continuous spectrum of H_L (see Figure 1, Theorem 1.3 and, e.g., [loc. cit.]). The resonances widths, that is, their imaginary part, play an important role in the large time behavior of e^{-itH_L}, especially the resonances of smallest width that give the leading order contribution (see [loc. cit.]).

Quantum resonances are basic objects in quantum theory. They have been the focus of a vast number of studies, both mathematical and physical (see, e.g., [loc. cit.] and references therein). Our purpose here is to study the resonances of H_L in the asymptotic regime $L \to +\infty$. As $L \to +\infty$, H_L converges to H in the strong resolvent sense. Thus, it is natural to expect that the differences in the spectral nature between the cases V periodic and V random should reflect into differences in the behavior of the resonances in both cases. We shall see below that this is the case. To illustrate this as simply as possible, we begin by stating three theorems, one for periodic potentials and two for random potentials, that underline these different behaviors. These results can be considered as paradigmatic for our main results, presented in Section 1.

The scattering theory or the closely related questions of resonances for the operator (0-1) or for closely related one-dimensional models have already been discussed in various works, both in the mathematical and physical literature (see, e.g., [Faris and Tsay 1989; 1994; Lifshits et al. 1988; Kunz and Shapiro 2006; Texier and Comtet 1999; Comtet and Texier 1997; Kunz and Shapiro 2008; Barra and Gaspard 1999; Kottos 2005; Titov and Fyodorov 2000]). We will make more comments on the literature as we develop our results in Section 1.

0A. *When V is periodic.* Assume that V is p-periodic ($p \in \mathbb{N}^*$) and does not vanish identically. Consider $H = -\Delta + V$ and let $\Sigma_{\mathbb{Z}}$ be its spectrum, $\Sigma_{\mathbb{Z}}^{\circ}$ be its interior and $E \mapsto N(E)$ be its integrated density of states, i.e., the number of states of the system per unit of volume below energy E (see Section 1B and, e.g., [Teschl 2000] for precise definitions and details).

Theorem 0.1. *There exist*

- $\mathfrak{D}$, *a discrete (possibly empty) set of energies in* $(-2, 2) \cap \Sigma_{\mathbb{Z}}^{\circ}$,
- *a function* h *that is real analytic in a complex neighborhood of* $(-2, 2)$ *and that does vanish on* $(-2, 2) \setminus \mathfrak{D}$

such that, for $I \subset (-2, 2) \setminus \mathfrak{D}$ *a compact interval such that either* $I \cap \Sigma_{\mathbb{Z}} = \varnothing$ *or* $I \subset \Sigma_{\mathbb{Z}}^{\circ}$, *there exists* $c_0 > 0$ *such that, for* L *sufficiently large with* $L \in p\mathbb{N}$, *one has*:

- *If* $I \cap \Sigma_{\mathbb{Z}} = \varnothing$, *then* H_L *has no resonance in* $I + i[-c_0, 0]$.
- *If* $I \subset \Sigma_{\mathbb{Z}}^{\circ}$, *one has:*
  - *There are plenty of resonances in* $I + i[-c_0, 0]$; *more precisely,*

$$\frac{1}{2L}\#\{z \in I + i[-c_0, 0] \mid z \text{ a resonance of } H_L\} = \int_I dN(E) + o(1), \tag{0-2}$$

    where $o(1) \to 0$ *as* $L \to +\infty$.
  - *Let* $(z_j)_j$ *be the resonances of* H_L *in* $I + i[-c_0, 0]$ *ordered by increasing real part; then*

$$L \cdot \operatorname{Re}(z_{j+1} - z_j) \asymp 1 \quad \text{and} \quad L \cdot \operatorname{Im} z_j = h(\operatorname{Re} z_j) + o(1), \tag{0-3}$$

    the estimates in (0-3) being uniform for all the resonances in $I + i[-c_0, 0]$ *when* $L \to +\infty$.

After rescaling their width by L, resonances are nicely interspaced points lying on an analytic curve (see Figure 2). We give a more precise description of the resonances in Theorem 1.7 and Propositions 1.8 and 1.9. In particular, we describe the set of energies $\mathfrak{D}$ and the resonances near these energies: they lie further away from the real axis, the maximal distance being of order $L^{-1} \log L$ (see Figure 3). Theorem 0.1 only describes the resonances closest to the real axis. In Section 1B, we also give results on the resonances located deeper in the lower half of the complex plane.

0B. *When V is random.* Assume now that $V = V_\omega$ is the Anderson potential, i.e., its entries are i.i.d. and distributed uniformly on [0, 1] for concreteness. Consider $H = -\Delta + V_\omega$. Let Σ be its almost sure spectrum (see, e.g., [Pastur and Figotin 1992] for this and the following notions), $E \mapsto n(E)$ its density of states (i.e., the derivative of the integrated density of states; see also Section 1B) and $E \mapsto \rho(E)$

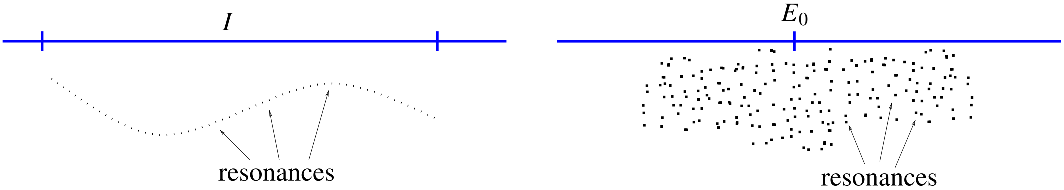

Figure 2. The rescaled resonances for the periodic (left) and the random (right) potential.

its Lyapunov exponent (see also Section 1C). The Lyapunov exponent is known to be continuous and positive; the density of states satisfies $n(E) > 0$ for a.e. $E \in \Sigma$ (see, e.g., [Bougerol and Lacroix 1985]).

Define $H_{\omega,L} := -\Delta + V_\omega \mathbf{1}_{[\![-L+1,L]\!]}$. We prove:

Theorem 0.2. *Pick $I \subset (-2,2)$ a compact interval.*

- *If $I \cap \Sigma = \varnothing$ then there exists $c_I > 0$ such that ω-a.s., for L sufficiently large,*

$$\{z \text{ a resonance of } H_{\omega,L} \text{ in } I + i(-c_I, 0]\} = \varnothing.$$

- *If $I \subset \Sigma^\circ$ then, for any $c > 0$, ω-a.s. one has*

$$\lim_{L\to+\infty} \frac{1}{L} \#\{z \text{ a resonance of } H_{\omega,L} \text{ in } I + i(-\infty, -e^{-2cL}]\} = \int_I \min\left(\frac{c}{\rho(E)}, 1\right) n(E)\, dE.$$

As the first statement of Theorem 0.2 is clear, let us discuss the second. Define $c_+ := \max_{E\in I} \rho(E)$. For $c \geq c_+$, ω-a.s. for L large the number of resonances in the strip $\{\operatorname{Re} z \in I,\ \operatorname{Im} z \leq -e^{-2cL}\}$ is approximately $2L \int_I n(E)\, dE$; thus, in $\{\operatorname{Re} z \in I,\ -e^{2c_+L} \leq \operatorname{Im} z < 0\}$, one finds at most $o(L)$ resonances. We shall see that, for $\delta > 0$, ω-a.s. for L large the strip $\{\operatorname{Re} z \in I,\ -e^{(2c_+ +\delta)L} \leq \operatorname{Im} z < 0\}$ actually contains no resonances (see Theorem 1.13).

Define $c_- := \min_{E\in I} \rho(E)$. For $c \leq c_-$, ω-a.s. for L large the strip $\{\operatorname{Re} z \in I,\ \operatorname{Im} z \leq -e^{-2cL}\}$ contains approximately $2cL \int_I n(E)/\rho(E)\, dE$ resonances. We shall see that, for $\kappa \in [0,1)$, the number of resonances in the strip $\{\operatorname{Re} z \in I,\ \operatorname{Im} z \leq -e^{-L^\kappa}\}$ is $O(L^\kappa)$, thus $o(L)$ (see Theorem 1.17).

One can also describe the resonances locally. Fix $E_0 \in (-2,2) \cap \Sigma^\circ$ such that $n(E_0) > 0$. Let $(z_l^L(\omega))_l$ be the resonances of $H_{\omega,L}$. We first rescale them. Define

$$x_l^L(\omega) = 2Ln(E_0)(\operatorname{Re} z_l^L(\omega) - E_0) \quad \text{and} \quad y_l^L(\omega) = -\frac{1}{2L\rho(E_0)} \log|\operatorname{Im} z_l^L(\omega)|. \tag{0-4}$$

Consider now the two-dimensional point process

$$\xi_L(E_0,\omega) = \sum_{z_l^L \text{ resonances of } H_{\omega,L}} \delta_{(x_l^L(\omega),\, y_l^L(\omega))}.$$

We prove:

Theorem 0.3. *The point process ξ_L converges weakly to a Poisson process of intensity 1 in $\mathbb{R} \times [0,1]$.*

In the random case, the structure of the (properly rescaled) resonances is quite different from that in the periodic case (see Figure 2). The real parts of the resonances are scaled in such a way that their average spacing becomes of order one. By Theorem 0.2, the imaginary parts are typically exponentially small (in L); when the resonances are rescaled as in (0-4), their imaginary parts are rewritten on a logarithmic scale so as to become of order 1 too. Once rescaled in this way, the local picture of the resonances of $H_{\omega,L}$ is that of a two-dimensional cloud of Poisson points (see the right-hand side of Figure 2).

Theorem 0.3 is the analogue for resonances of the well-known result on the distribution of eigenvalues and localization centers for the Anderson model in the localized phase (see, e.g., [Minami 1996; Killip and Nakano 2007; Germinet and Klopp 2014]).

As in the case of the periodic potential, Theorem 0.3 only describes the resonances closest to the real axis. In Section 1C, we also give results on resonances located deeper in the lower half of the complex plane. Up to distances of order $L^{-\infty}$ to the real axis, the cloud of resonances (once properly rescaled) will have the same Poissonian behavior as described above (see Theorem 1.10).

Besides proving Theorems 0.1 and 0.3, the goal of the paper is to describe the statistical properties of the resonances and relate them (the distribution of the resonances and of the widths) to the spectral characteristics of $H = -\Delta + V$, and possibly to the distribution of its eigenvalues (see, e.g., [Germinet and Klopp 2011]).

As they can be analyzed in a very similar way, we will discuss three models:

- The model H_L defined above.
- Its analogue on the half-line $\mathbb{N}$, i.e., on H_L, we impose an additional Dirichlet boundary condition at 0.
- The “half-infinite” model on $\ell^2(\mathbb{Z})$, that is,
$$H^\infty = -\Delta + W, \quad \text{where} \quad \begin{cases} W(n) = 0 & \text{for } n \geq 0, \\ W(n) = V(n) & \text{for } n \leq -1, \end{cases} \tag{0-5}$$
where V is chosen as above, periodic or random.

Though in the present paper we restrict ourselves to discrete models, it is clear that continuous one-dimensional models can be dealt with essentially using the methods developed here.

1. The main results

We now turn to our main results, a number of which were announced in [Klopp 2012]. Pick $V : \mathbb{Z} \to \mathbb{R}$ a bounded potential and, for $L \in \mathbb{N}$, consider the operators

- $H_L^{\mathbb{Z}} = -\Delta + V\mathbf{1}_{[\![0,L]\!]}$ on $\ell^2(\mathbb{Z})$;
- $H_L^{\mathbb{N}} = -\Delta + V\mathbf{1}_{[\![0,L]\!]}$ on $\ell^2(\mathbb{N})$ with Dirichlet boundary conditions at 0;
- H^∞, defined in (0-5).

Remark 1.1. Here, by “Dirichlet boundary condition at 0”, we mean that $H_L^{\mathbb{N}}$ is the operator $H_L^{\mathbb{Z}}$ restricted to the subspace $\ell^2(\mathbb{N})$, i.e., if $\Pi : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ is the orthogonal projector on $\ell^2(\mathbb{N})$, one has $H_L^{\mathbb{N}} = \Pi H_L^{\mathbb{Z}} \Pi$. In the literature, this is sometime called “Dirichlet boundary condition at -1” (see, e.g., [Teschl 2000]).

For the sake of simplicity, in the half-line case we only consider Dirichlet boundary conditions at 0. But the proofs show that these are not crucial; any selfadjoint boundary condition at 0 would do and, mutatis mutandis, the results would be the same.

Note also that by a shift of the potential V, replacing L by $L+L'$, studying $H_L^{\mathbb{Z}}$ is equivalent to studying $H_{L,L'} = -\Delta + V\mathbf{1}_{[\![-L',L]\!]}$ on $\ell^2(\mathbb{Z})$. Thus, to derive the results of Section 0 from those in the present section, it suffices to consider the models above, in particular $H_L^{\mathbb{Z}}$.

For the models $H_L^{\mathbb{N}}$ and $H_L^{\mathbb{Z}}$, we start with a discussion of the existence of a meromorphic continuation of the resolvent, then study the resonances when V is periodic and finally turn to the case when V is random.

As H^∞ is not a relatively compact perturbation of the Laplacian, the existence of a meromorphic continuation of its resolvent depends on the nature of V; so, it will be discussed when specializing to V periodic or random.

Remark 1.2 (notations). In the sequel, we write $a \lesssim b$ if for some $C > 0$ (independent of the parameters coming into a or b) one has $a \le Cb$. We write $a \asymp b$ if $a \lesssim b$ and $b \lesssim a$.

1A. ***The meromorphic continuation of the resolvent.*** One proves the well-known and simple:

Theorem 1.3. *The operator-valued functions $z \mapsto (z - H_L^{\mathbb{N}})^{-1}$ and $z \mapsto (z - H_L^{\mathbb{Z}})^{-1}$ for $z \in \mathbb{C}^+$ admit a meromorphic continuation from $\mathbb{C}^+$ to $\mathbb{C} \setminus ((-\infty, -2] \cup [2, +\infty))$ through* $(-2, 2)$ (see Figure 1) *with values in the operators from l^2_{comp} to l^2_{loc}.*

Moreover, the number of poles of each of these meromorphic continuations in the lower half-plane is at most equal to L.

The resonances are defined to be the poles of this meromorphic continuation (see Figure 1).

1B. ***The periodic case.*** We assume that, for some $p > 0$, one has

$$V_{n+p} = V_n \quad \text{for all } n \ge 0. \tag{1-1}$$

Let $\Sigma_{\mathbb{N}}$ be the spectrum of $H^{\mathbb{N}} = -\Delta + V$ acting on $\ell^2(\mathbb{N})$ with Dirichlet boundary condition at 0 and $\Sigma_{\mathbb{Z}}$ be the spectrum of $H^{\mathbb{Z}} = -\Delta + V$ acting on $\ell^2(\mathbb{Z})$. One has the following description for these spectra:

- $\Sigma_{\mathbb{Z}}$ is a union of intervals, i.e., $\Sigma_{\mathbb{Z}} := \sigma(H) = \bigcup_{j=1}^{p} [E_j^-, E_j^+]$, where $E_j^- < E_j^+$ $(1 \le j \le p)$ and $a_{j-1}^+ \le E_j^-$ $(2 \le j \le p)$ (see, e.g., [van Moerbeke 1976]); the spectrum of $H^{\mathbb{Z}}$ is purely absolutely continuous and the spectral resolution can be obtained via a Bloch–Floquet decomposition (see, e.g., [loc. cit.]).
- On $\ell^2(\mathbb{N})$ (see, e.g., [Pavlov 1994]), one has
  - $\Sigma_{\mathbb{N}} = \Sigma_{\mathbb{Z}} \cup \{v_j \mid 1 \le j \le n\}$ and $\Sigma_{\mathbb{Z}}$ is the absolutely continuous spectrum of H;
  - the $(v_j)_{0 \le j \le n}$ are isolated simple eigenvalues associated to exponentially decaying eigenfunctions.

It may happen that some of the gaps are closed, i.e., that the number of connected components of $\Sigma_{\mathbb{Z}}$ be strictly less than p. There still is a natural way to write $\Sigma_{\mathbb{Z}} := \sigma(H) = \bigcup_{j=1}^{p}[E_j^-, E_j^+]$ (see Section 4A1), but in this case, for some of the j, one has $E_{j-1}^+ = E_j^-$; we shall call the energies $E_{j-1}^+ = E_j^-$ *closed gaps* (see Definition 4.5). The existence of closed gaps is nongeneric (see [van Moerbeke 1976]).

The operators $H^\bullet$ (for $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$) admit an integrated density of states defined by

$$N(E) = \lim_{L\to+\infty} \frac{\#\{\text{eigenvalues of } (-\Delta+V)|_{[\![-L,L]\!]\cap\bullet} \text{ in } (-\infty, E]\}}{\#([\![-L, L]\!]\cap\bullet)}. \tag{1-2}$$

Here, the restriction of $-\Delta + V$ to $[\![-L, L]\!]\cap\bullet$ is taken with Dirichlet boundary conditions; this is for concreteness as it is known that, in the limit $L \to +\infty$, other selfadjoint boundary conditions would yield the same result for the limit (1-2).

The integrated density of states is the same for $H^{\mathbb{N}}$ and $H^{\mathbb{Z}}$ (see, e.g., [Pastur and Figotin 1992]). It defines the distribution function of some probability measure on $\Sigma_{\mathbb{Z}}$ that is real analytic on $\Sigma_{\mathbb{Z}}^\circ$. Let n denote the density of states of $H^{\mathbb{N}}$ and $H^{\mathbb{Z}}$, that is, $n(E) = dN(E)/dE$.

Remark 1.4. When L gets large, as $H_L^{\mathbb{N}}$ tends to $H^{\mathbb{N}}$ in the strong resolvent sense, interesting phenomena for the resonances of $H_L^{\mathbb{N}}$ should take place near energies in $\Sigma_{\mathbb{N}}$.

Define τ_k to be the shift by k steps to the left, that is, $\tau_k V(\cdot) = V(\cdot + k)$. Then, for $(\ell_L)_L$ such that $l_L \to +\infty$ and $L - \ell_L \to +\infty$ when $L \to +\infty$, $\tau_{l_L}^* H_L^{\mathbb{Z}} \tau_{l_L}$ tend to $H^{\mathbb{Z}}$ in the strong resolvent sense. Thus, interesting phenomena for the resonances of $H_L^{\mathbb{Z}}$ should take place near energies in $\Sigma_{\mathbb{Z}}$.

1B1. *Resonance-free regions.* We start with a description of resonance-free regions near the real axis. To this end, we introduce some operators on the positive and the negative half-lattice.

Above we have defined $H_{\mathbb{N}}$; we shall need another auxiliary operator. On $\ell^2(\mathbb{Z}_-)$ (where $\mathbb{Z}_- = \{n \le 0\}$), consider the operator $H_k^- = -\Delta + \tau_k V$ with Dirichlet boundary condition at 0 (where τ_k is defined to be the shift by k steps to the left, that is, $\tau_k V(\cdot) = V(\cdot + k)$). Let $\Sigma_k^- = \sigma(H_k^-)$.

As is the case for $H^{\mathbb{N}}$, one knows that $\sigma_{\text{ess}}(H_k^-) = \Sigma_{\mathbb{Z}}$ and that $\sigma_{\text{ess}}(H_k^-)$ is purely absolutely continuous (see, e.g., [Teschl 2000, Chapter 7]). H_k^- may also have discrete eigenvalues in $\mathbb{R} \setminus \Sigma_{\mathbb{Z}}$.

We prove:

Theorem 1.5. *Let I be a compact interval in* $(-2, 2)$.

(1) *If $I \subset \mathbb{R} \setminus \Sigma_{\mathbb{N}}$ (resp. $I \subset \mathbb{R} \setminus \Sigma_{\mathbb{Z}}$), then there exists $c > 0$ such that, for L sufficiently large, $H_L^{\mathbb{N}}$ (resp. $H_L^{\mathbb{Z}}$) has no resonances in the rectangle* $\{\operatorname{Re} z \in I,\ \operatorname{Im} z \in [-c, 0]\}$.

(2) *If $I \subset \Sigma_{\mathbb{Z}}$, then there exists $c > 0$ such that, for L sufficiently large, $H_L^{\mathbb{N}}$ and $H_L^{\mathbb{Z}}$ have no resonances in the rectangle* $\{\operatorname{Re} z \in I,\ \operatorname{Im} z \in [-c/L, 0]\}$.

(3) *Fix $0 \le k \le p-1$ and assume the compact interval I is such that $\{v_j\} = I^\circ \cap \Sigma_{\mathbb{N}} = I \cap \Sigma_{\mathbb{N}}$ and $I \cap \Sigma_{\mathbb{Z}} = \varnothing$* (the $(v_j)_j$ are as defined in the beginning of Section 1B).

(a) *If $I \cap \Sigma_k^- = \varnothing$ then there exists $c > 0$ such that, for L sufficiently large with $L \equiv k \bmod p$, $H_L^{\mathbb{N}}$ has a unique resonance in the rectangle $\{\operatorname{Re} z \in I,\ -c \le \operatorname{Im} z \le 0\}$; moreover, this resonance, say z_j, is simple and satisfies $\operatorname{Im} z_j \asymp -e^{-\rho_j L}$ and $|z_j - \lambda_j| \asymp e^{-\rho_j L}$ for some $\rho_j > 0$ independent of L.*

(b) *If* $I \cap \Sigma_k^- \neq \varnothing$ *then there exists* $c > 0$ *such that, for* L *sufficiently large with* $L \equiv k \bmod p$, $H_L^{\mathbb{N}}$ *has no resonance in the rectangle* $\{\operatorname{Re} z \in I,\ -c \le \operatorname{Im} z \le 0\}$.

So, below the spectral interval $(-2, 2)$, there exists a resonance-free region of width at least of order L^{-1}. For $H_L^{\mathbb{N}}$, if $L \equiv k \bmod p$ each discrete eigenvalue of $H^{\mathbb{N}}$ that is not an eigenvalue of H_k^- generates a resonance for $H_L^{\mathbb{N}}$ exponentially close to the real axis (when L is large). When the eigenvalue of H_k^- is also an eigenvalue of $H^{\mathbb{N}} = H_0^+$; it may also generate a resonance but only much further away in the complex plane, at least at a distance of order 1 to the real axis.

In case (3a) of Theorem 1.5, one can give an asymptotic expansion for the resonances (see Section 5B1).

We now turn to the description of the resonances of $H_L^\bullet$ near $[-2, 2]$. To this end, it will be useful to introduce a number of auxiliary functions and operators.

1B2. *Some auxiliary functions.* To H_k^- defined above, we associate N_k^-, the distribution function of its spectral measure (which is a probability measure), i.e., for $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$, we define $\int_{\mathbb{R}} \varphi(\lambda)\, dN_k^-(\lambda) := \varphi(H_k^-)(0,0)$, where $(\varphi(H_k^-)(x,y))_{(x,y)\in(\mathbb{Z}_-)^2}$ denotes the kernel of the operator $\varphi(H_k^-)$.

On $\Sigma_{\mathbb{Z}}^\circ$, the spectral measure dN_k^- admits a density with respect to the Lebesgue measure, say n_k^-, and this density is real analytic (see Proposition 5.6).

For $E \in \Sigma_{\mathbb{Z}}^\circ$, define

$$S_k^-(E) := \text{p.v.}\left(\int_{\mathbb{R}} \frac{dN_k^-(\lambda)}{\lambda - E}\right) = \lim_{\varepsilon \to 0+}\left(\int_{-\infty}^{E-\varepsilon} \frac{dN_k^-(\lambda)}{\lambda - E} - \int_{E+\varepsilon}^{+\infty} \frac{dN_k^-(\lambda)}{\lambda - E}\right). \tag{1-3}$$

The existence and analyticity of the Cauchy principal value S_k^- on $\Sigma_{\mathbb{Z}}^\circ$ is guaranteed by the analyticity of n_k^- (see, e.g., [King 2009]). Moreover, for $E \in \Sigma_{\mathbb{Z}}^\circ$, one has

$$S_k^-(E) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{dN_k^-(\lambda)}{\lambda - E - i\varepsilon} - i\pi n_k^-(E). \tag{1-4}$$

In the lower half-plane $\{\operatorname{Im} E < 0\}$, define the function

$$\Xi_k^-(E) := \int_{\mathbb{R}} \frac{dN_k^-(\lambda)}{\lambda - E} + e^{-i \arccos(E/2)} = \int_{\mathbb{R}} \frac{dN_k^-(\lambda)}{\lambda - E} + \frac{E}{2} + \sqrt{\left(\frac{E}{2}\right)^2 - 1}, \tag{1-5}$$

where

- in the first formula, the function $z \mapsto \arccos z$ is the analytic continuation to the lower half-plane of the branch of $\arccos z$ taking values in $[-\pi, 0]$ on the interval $[-1, 1]$;
- in the second formula, the branch of the square root $z \mapsto \sqrt{z^2 - 1}$ has positive imaginary part for $z \in (-1, 1)$.

The function Ξ_k^- is analytic in $\{\operatorname{Im} E < 0\}$ and in a neighborhood of $(-2, 2) \cap \Sigma_{\mathbb{Z}}^\circ$. Moreover, Ξ_k^- vanishes identically if and only if $V \equiv 0$ (see Proposition 5.7).

From now on we assume that $V \not\equiv 0$. In this case, in $\{\operatorname{Im} E < 0\}$ and on $(-2, 2) \cap \Sigma_{\mathbb{Z}}^\circ$, the analytic function Ξ_k^- has only finitely many zeros, each of finite multiplicity (see Proposition 5.7).

We shall need the analogues of the above-defined functions for the already-introduced operator $H_0^+ := H^{\mathbb{N}} = -\Delta + V$ considered on $\ell^2(\mathbb{N})$ with Dirichlet boundary conditions at 0. We define the function N_0^+ as the distribution function of the spectral measure of H_0^+, i.e., for $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$, we define $\int_{\mathbb{R}} \varphi(\lambda)\, dN_0^+(\lambda) := \varphi(H_0^+)(0,0)$. In the same way as we have defined n_k^-, S_k^- and Ξ_k^- from H_k^-, one can define n_0^+, S_0^+ and Ξ_0^+ from H_0^+. They also satisfy Proposition 5.6, relation (1-4) and Proposition 5.7.

For the description of the resonances, it will be convenient to define the following functions on $\Sigma_{\mathbb{Z}}^\circ$:

$$c^{\mathbb{N}}(E) := i + \frac{\Xi_k^-(E)}{\pi n_k^-(E)} = \frac{1}{\pi n_k^-(E)}(S_k^-(E) + e^{-i \arccos(E/2)}) \tag{1-6}$$

and

$$c^{\mathbb{Z}}(E) := \frac{\dfrac{(S_0^+(E) + e^{-i\arccos(E/2)})(S_k^-(E) + e^{-i\arccos(E/2)})}{n_0^+(E) n_k^-(E)} - \pi^2}{\dfrac{\pi(S_0^+(E) + e^{-i\arccos(E/2)})}{n_0^+(E)} + \dfrac{\pi(S_k^-(E) + e^{-i\arccos(E/2)})}{n_k^-(E)}}. \tag{1-7}$$

We shall see that the zeros of $c^\bullet - i$ play a special role for the resonances of $H_L^\bullet$; therefore, we define

$$\mathfrak{D}^\bullet = \{z \in \Sigma_{\mathbb{Z}}^\circ \mid c^\bullet(z) = i\}. \tag{1-8}$$

The set $\mathfrak{D}$ introduced in Theorem 0.1 is the set $\mathfrak{D}^{\mathbb{Z}} \cap (-2, 2)$.

Remark 1.6. Before describing the resonances, let us explain why the operators H_0^+ and H_k^- naturally occur in this study. They respectively are the strong resolvent limits (when $L \to +\infty$ with $L \in p\mathbb{N} + k$) of the operator $H_L^{\mathbb{Z}}$ restricted to $[\![0, L]\!]$ with Dirichlet boundary conditions at 0 and L “seen” from the left- and the right-hand side, respectively.

Indeed, define H_L to be the operator $H_L^{\mathbb{N}}$ restricted to $[\![0, L]\!]$ with Dirichlet boundary conditions at L (see Remark 1.1). Note that H_L is also the operator $H_L^{\mathbb{Z}}$ restricted to $[\![0, L]\!]$ with Dirichlet boundary conditions at 0 and L.

Clearly, the operator H_0^+ is the strong resolvent limit of H_L when $L \to +\infty$.

If $\tilde{\tau}_L$ denotes the translation by $-L$ that unitarily maps $\ell^2([\![0, L]\!])$ into $\ell^2([\![-L, 0]\!])$, then $\widetilde{H}_L = \tilde{\tau}_L H_L \tilde{\tau}_L^*$ converges in the strong resolvent sense to H_k^- when $L \to +\infty$ and $L \equiv k \bmod p$. Indeed, $\tau_L V = \tau_k V$ as V is p-periodic.

1B3. *Description of the resonances closest to the real axis.* Let $(\lambda_l)_{0 \le l \le L} = (\lambda_l^L)_{0 \le l \le L}$ be the eigenvalues of H_L (that is, the eigenvalues of $H_L^{\mathbb{N}}$ or $H_L^{\mathbb{Z}}$ restricted to $[\![0, L]\!]$ with Dirichlet boundary conditions; see Remark 1.1) listed in increasing order. They are described in Theorem 4.2; those away from the edges of $\Sigma_{\mathbb{Z}}$ are shown to be nicely interspaced points at a distance roughly L^{-1} from one another.

We first state our most general result describing the resonances in a uniform way. We then derive two corollaries describing the behavior of the resonance, first far from the set of exceptional energies $\mathfrak{D}^\bullet$ and second close to an exceptional energy.

Pick a compact interval $I \subset (-2, 2) \cap \Sigma_{\mathbb{Z}}^\circ$. For $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$ and $\lambda_l \in I$, for L large, define the complex number

$$\tilde{z}_l^\bullet = \lambda_l + \frac{1}{\pi n(\lambda_l) L} \cot^{-1} \circ c^\bullet \left[\lambda_l + \frac{1}{\pi n(\lambda_l) L} \cot^{-1} \circ c^\bullet \left(\lambda_l - i \frac{\log L}{L} \right) \right], \tag{1-9}$$

where the branch of $\cot^{-1}$ is the inverse of the branch of $z \mapsto \cot z$ that maps $[0, \pi) \times (0, -\infty)$ onto $\mathbb{C}^+ \setminus \{i\}$.

Note that, by Proposition 5.8, for L sufficiently large we know that, for any l such that $\lambda_l \in I$, one has

$$\operatorname{Im} c^\bullet\left(\lambda_l - i\frac{\log L}{L}\right) \in (0, +\infty) \setminus \{1\}$$

and

$$\operatorname{Im} c^\bullet\left[\lambda_l + \frac{1}{\pi n(\lambda_l) L}\cot^{-1} \circ c^\bullet\left(\lambda_l - i\frac{\log L}{L}\right)\right] \in (0, +\infty) \setminus \{1\}.$$

Thus, the formula (1-9) defines $\tilde{z}_l^\bullet$ properly and in a unique way. Moreover, as the zeros of $E \mapsto c^\bullet(E) - i$ are of finite order, one checks that

$$-\log L \lesssim L \cdot \operatorname{Im} \tilde{z}_l^\bullet \lesssim -1 \quad \text{and} \quad 1 \lesssim L \cdot \operatorname{Re}(\tilde{z}_{l+1}^\bullet - \tilde{z}_l^\bullet), \tag{1-10}$$

where the implicit constants are uniform for l such that $\lambda_l \in I$.

We prove:

Theorem 1.7. *Pick $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$ and $k \in \{0, \dots, p-1\}$. Let $E_0 \in (-2, 2) \cap \Sigma_{\mathbb{Z}}^\circ$.*

Then there exists $\eta_0 > 0$ and $L_0 > 0$ such that, for $L > L_0$ satisfying $L = k \bmod p$, for each $\lambda_l \in I := [E_0 - \eta_0, E_0 + \eta_0]$, there exists a unique resonance of $H_L^\bullet$, say $z_l^\bullet$, in the rectangle

$$\left[\tfrac{1}{2}\operatorname{Re}(\tilde{z}_l^\bullet + \tilde{z}_{l-1}^\bullet), \tfrac{1}{2}\operatorname{Re}(\tilde{z}_l^\bullet + \tilde{z}_{l+1}^\bullet)\right] + i[-\eta_0, 0];$$

this resonance is simple and it satisfies $|z_l^\bullet - \tilde{z}_l^\bullet| \lesssim 1/(L \log L)$.

This result calls for a few comments. First, the picture one gets for the resonances can be described as follows (see also Figure 3). As long as λ_l stays away from any zero of $E \mapsto c^\bullet(E) - i$, the resonances are nicely spaced points, as the following proposition proves.

Proposition 1.8. *Pick $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$ and $k \in \{0, \dots, p-1\}$. Let $I \subset (-2, 2) \cap \Sigma_{\mathbb{Z}}^\circ$ be a compact interval such that $I \cap \mathfrak{D}^\bullet = \varnothing$.*

Then, for L sufficiently large and each $\lambda_l \in I$, the resonance $z_l^\bullet$ admits the expansion

$$z_l^\bullet = \lambda_l + \frac{1}{\pi n(\lambda_l) L}\cot^{-1} \circ c^\bullet(\lambda_l) + O\left(\frac{1}{L^2}\right), \tag{1-11}$$

where the remainder term is uniform in l.

The proof of Proposition 1.8 actually yields a complete asymptotic expansion in powers of L^{-1} for the resonances in this zone (see Section 5B5).

Proposition 1.8 implies Theorem 0.1: we choose $\bullet = \mathbb{Z}$ and $k = 0$, then the set $\mathfrak{D}$ of exceptional points in Theorem 0.1 is exactly $\mathfrak{D}^{\mathbb{Z}} \cap (-2, 2)$; to obtain (0-3), it suffices to use the asymptotic form of the Dirichlet eigenvalues given by Theorem 4.2.

Near the zeros of $E \mapsto c^\bullet(E) - i$, the resonances take a "plunge" into the lower half of the complex plane (see Figure 3) and their imaginary part becomes of order $L^{-1}\log L$. Indeed, Theorem 1.7 and (1-9) imply:

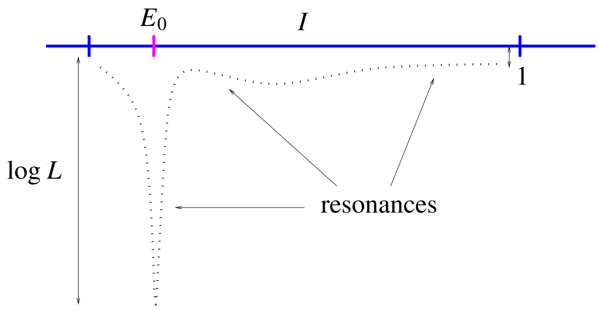

Figure 3. The resonances close to the real axis in the periodic case (after rescaling their imaginary parts by L).

Proposition 1.9. *Pick $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$ and $k \in \{0, \dots, p-1\}$. Let $E_0 \in \mathfrak{D}^\bullet$ be a zero of $E \mapsto c^\bullet(E) - i$ of order q in $(-2, 2) \cap \Sigma_{\mathbb{Z}}^\circ$.*

Then, for $\alpha > 0$ and L sufficiently large, if l is such that $|\lambda_l - E_0| \le L^{-\alpha}$, the resonance $z_l^\bullet$ satisfies

$$\operatorname{Im} z_l^\bullet = \frac{q}{2\pi n(\lambda_l)} \frac{\log\big(|\lambda_l - E_0|^2 + \big(q \log L/(2\pi n(\lambda_l)L)\big)^2\big)}{2L}(1 + o(1)), \tag{1-12}$$

where the remainder term is uniform in l such that $|\lambda_l - E_0| \le L^{-\alpha}$.

When $\bullet = \mathbb{Z}$, the asymptotic (1-12) shows that there can be a “resonance” phenomenon for resonances: when the two functions Ξ_k^- and Ξ_0^+ share a zero at the same real energy, the maximal width of the resonances increases; indeed, the factor in front of $L^{-1} \log L$ is proportional to the multiplicity of the zero of $\Xi_k^- \Xi_0^+$.

1B4. *Description of the low-lying resonances.* The resonances found in Theorem 1.7 are not necessarily the only ones: deeper in the lower complex plane, one may find more resonances. They are related to the zeros of Ξ_k^- when $\bullet = \mathbb{N}$ and of $\Xi_k^- \Xi_0^+$ when $\bullet = \mathbb{Z}$ (see Proposition 5.8).

We now study what happens below the line $\{\operatorname{Im} z = -\eta_0\}$ (see Theorem 1.7) for the resonances of $H_L^{\mathbb{N}}$ and $H_L^{\mathbb{Z}}$.

The functions Ξ_k^- and Ξ_0^+ are analytic in the lower half-plane and, by Proposition 5.7, they don’t vanish in an neighborhood of $-i\infty$. Hence, the functions Ξ_k^- and Ξ_0^+ have only finitely many zeros in the lower half-plane.

We prove:

Theorem 1.10. *Pick $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$ and $k \in \{0, \dots, p-1\}$. Let $(E_j^\bullet)_{1 \le j \le J}$ be the zeros of $E \mapsto c^\bullet(E) - i$ in $I + i(-\infty, 0)$. Pick $E_0 \in (-2, 2) \cap \Sigma_{\mathbb{Z}}^\circ$.*

There exists $\eta_0 > 0$ such that, for $I = E_0 + [-\eta_0, \eta_0]$ and L sufficiently large with $L \equiv k \bmod p$, one has:

- *If $E_0 \notin \{\operatorname{Re} E_j^\bullet \mid 1 \le j \le J\}$, then in the rectangle $I + i(-\infty, 0]$ the only resonances of $H_L^{\mathbb{N}}$ and $H_L^{\mathbb{Z}}$ are those given by Theorem 1.7.*
- *If $E_0 \in \{\operatorname{Re} E_j^\bullet \mid 1 \le j \le J\}$, then*

- *in the rectangle* $I+i[-\eta_0,0]$, *the only resonances of* $H_L^{\mathbb{N}}$ *and* $H_L^{\mathbb{Z}}$ *are those given by Theorem 1.7*;
- *in the strip* $I+i[-\infty,-\eta_0]$, *the resonances of* $H_L^{\bullet}$ *are contained in* $\bigcup_{j=1}^{J} D(E_j^{\bullet}, e^{-\eta_0 L})$;
- *in* $D(E_j^{\bullet}, e^{-\eta_0 L})$, *the number of resonances* (*counted with multiplicity*) *is equal to the order of* $E_j^{\bullet}$ *as a zero of* $E \mapsto c^{\bullet}(E)-i$.

We see that the total number of resonances below a compact subset of $(-2,2)\cap\overset{\circ}{\Sigma}_{\mathbb{Z}}$ that do not tend to the real axis when $L\to+\infty$ is finite. These resonances are related to the resonances of H^∞, to which we turn now.

1B5. *The half-line periodic perturbation.* Fix $p\in\mathbb{N}^*$. On $\ell^2(\mathbb{Z})$, we now consider the operator $H^\infty=\Delta+V$, where $V(n)=0$ for $n\geq 0$ and $V(n+p)=V(n)$ for $n\leq -1$. We prove:

Theorem 1.11. *The resolvent of* H^∞ *can be analytically continued from the upper half-plane through* $(-2,2)\cap\overset{\circ}{\Sigma}_{\mathbb{Z}}$ *to the lower half-plane. The resulting operator does not have any poles in the lower half-plane or on* $(-2,2)\cap\overset{\circ}{\Sigma}_{\mathbb{Z}}$.

The resolvent of H^∞ *can be analytically continued from the upper half-plane through* $(-2,2)\setminus\Sigma_{\mathbb{Z}}$ (*resp.* $\overset{\circ}{\Sigma}_{\mathbb{Z}}\setminus[-2,2]$) *to the lower half-plane*; *the poles of the continuation through* $(-2,2)\setminus\Sigma_{\mathbb{Z}}$ (*resp.* $\overset{\circ}{\Sigma}_{\mathbb{Z}}\setminus[-2,2]$) *are exactly the zeros of the function* $E\mapsto 1-e^{i\theta(E)}\int_{\mathbb{R}} 1/(\lambda-E)\,dN_{p-1}^-(\lambda)$ *when continued from the upper half-plane through* $(-2,2)\setminus\Sigma_{\mathbb{Z}}$ (*resp.* $\overset{\circ}{\Sigma}_{\mathbb{Z}}\setminus[-2,2]$) *to the lower half-plane.*

Remark 1.12. In Theorem 1.11 and below, every time we consider the analytic continuation of a resolvent through some open subset of the real line we implicitly assume the open subset to be nonempty.

In Figure 4, to illustrate Theorem 1.11, assuming that $\Sigma_{\mathbb{Z}}$ (in blue) has a single gap that is contained in $(-2,2)$, we have drawn the various analytic continuations of the resolvent of H^∞ and the presence or absence of resonances for the different continuations.

Using the same arguments as in the proof of Proposition 5.7, one easily sees that the continuations of the function $E\mapsto 1-e^{i\theta(E)}\int_{\mathbb{R}} 1/(\lambda-E)\,dN_{p-1}^-(\lambda)$ to the lower half-plane through $(-2,2)\setminus\Sigma_{\mathbb{Z}}$ and $\overset{\circ}{\Sigma}_{\mathbb{Z}}\setminus[-2,2]$ have at most finitely many zeros and that these zeros are away from the real axis.

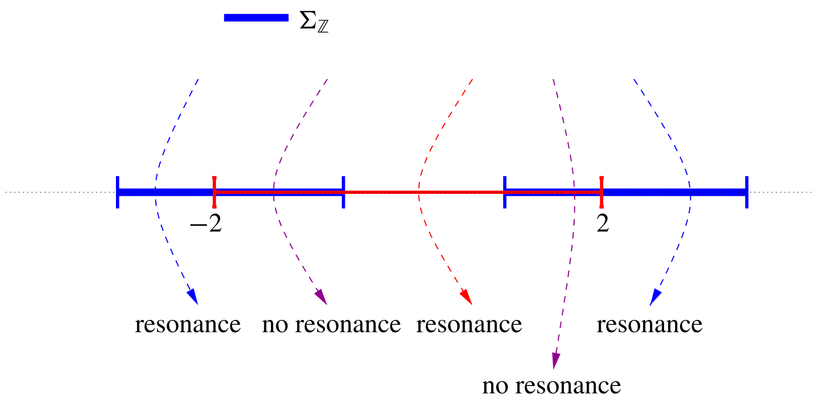

Figure 4. The analytic continuation of the resolvent and resonances for H^∞.

This also implies that the spectrum on H^∞ in $[-2,2]\cup\Sigma_{\mathbb{Z}}$ is purely absolutely continuous except possibly at the points of $\partial\Sigma_{\mathbb{Z}}\cup\{-2,2\}$, where $\partial\Sigma_{\mathbb{Z}}$ is the set of edges of $\Sigma_{\mathbb{Z}}$.

1C. *The random case.* We now turn to the random case. Let $V=V_\omega$, where $(V_\omega(n))_{n\in\mathbb{Z}}$ are bounded independent and identically distributed random variables. Assume that the common law of the random variables admits a bounded compactly supported density, say g.

Set $H_\omega^{\mathbb{N}}=-\Delta+V_\omega$ on $\ell^2(\mathbb{N})$ (with Dirichlet boundary condition at 0 for concreteness). Let $\sigma(H_\omega^{\mathbb{N}})$ be the spectrum of $H_\omega^{\mathbb{N}}$. Consider also $H_\omega^{\mathbb{Z}}=-\Delta+V_\omega$ acting on $\ell^2(\mathbb{Z})$. Then one knows (see, e.g., [Kirsch 2008]) that, ω-almost surely,

$$\sigma(H_\omega^{\mathbb{Z}})=\Sigma:=[-2,2]+\operatorname{supp} g. \tag{1-13}$$

One has the following description for the spectra $\sigma(H_\omega^{\mathbb{N}})$ and $\sigma(H_\omega^{\mathbb{Z}})$:

- ω-almost surely, $\sigma(H_\omega^{\mathbb{Z}})=\Sigma$; the spectrum is purely punctual; it consists of simple eigenvalues associated to exponentially decaying eigenfunctions (Anderson localization; see, e.g., [Pastur and Figotin 1992; Kirsch 2008]); one can prove that, under the assumptions made above, the whole spectrum is dynamically localized (see, e.g., [Cycon et al. 1987] and references therein).
- For $H_\omega^{\mathbb{N}}$ (see, e.g., [Pastur and Figotin 1992; Carmona and Lacroix 1990]), one has, ω-almost surely, $\sigma(H_\omega^{\mathbb{N}})=\Sigma\cup K_\omega$, where
  - Σ is the essential spectrum of $H_\omega^{\mathbb{N}}$ and it consists of simple eigenvalues associated to exponentially decaying eigenfunctions;
  - the set K_ω is the discrete spectrum of $H_\omega^{\mathbb{N}}$, which may be empty and depends on ω.

1C1. *The integrated density of states and the Lyapunov exponent.* It is well known (see, e.g., [Pastur and Figotin 1992]) that the integrated density of states of H, say $N(E)$, is defined as the limit

$$N(E)=\lim_{L\to+\infty}\frac{\#\{\text{eigenvalues of } H_\omega^{\mathbb{Z}}|_{[\![-L,L]\!]} \text{ in } (-\infty,E]\}}{2L+1}. \tag{1-14}$$

The above limit does not depend on the boundary conditions used to define the restriction $H_\omega^{\mathbb{Z}}|_{[\![-L,L]\!]}$. It defines the distribution function of a probability measure supported on Σ. Under our assumptions on the random potential, N is known to be Lipschitz continuous ([Pastur and Figotin 1992; Kirsch 2008]). Let $n(E)=dN(E)/dE$ be its derivative; it exists for almost all energies. If one assumes more regularity on g, the density of the random variables $(\omega_n)_n$, then the density of states n can be shown to exist everywhere and to be regular (see, e.g., [Cycon et al. 1987]).

One also defines the Lyapunov exponent, say $\rho(E)$, as

$$\rho(E):=\lim_{L\to+\infty}\frac{\log\|T_L(E,\omega)\|}{L+1},$$

where

$$T_L(E;\omega):=\begin{pmatrix}E-V_\omega(L) & -1\\ 1 & 0\end{pmatrix}\times\cdots\times\begin{pmatrix}E-V_\omega(0) & -1\\ 1 & 0\end{pmatrix} \tag{1-15}$$

For any E, ω-almost surely, the Lyapunov exponent is known to exist and to be independent of ω (see, e.g., [Cycon et al. 1987; Pastur and Figotin 1992; Carmona and Lacroix 1990]). It is positive at all energies. Moreover, by the Thouless formula [Cycon et al. 1987], it is continuous for all E and is the harmonic conjugate of $n(E)$.

For $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$, we now define $H^\bullet_{\omega,L}$ to be the operator $-\Delta^\bullet + V_\omega \mathbf{1}_{[\![0,L]\!]}$. The goal of the next sections is to describe the resonances of these operators in the limit $L \to +\infty$.

As in the case of a periodic potential V, the resonances are defined as the poles of the analytic continuation of $z \mapsto (H^\bullet_{\omega,L} - z)^{-1}$ from $\mathbb{C}^+$ through $(-2, 2)$ (see Theorem 1.3).

1C2. *Resonance-free regions.* We again start with a description of the resonance-free region near a compact interval in $(-2, 2)$. As in the periodic case, the size of the $H^\bullet_{\omega,L}$-resonance-free region below a given energy will depend on whether this energy belongs to $\sigma(H^\bullet_\omega)$ or not. We prove:

Theorem 1.13. *Fix $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$. Let I be a compact interval in $(-2, 2)$. Then, ω-a.s., one has:*

(1) *For $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$, if $I \subset \mathbb{R} \setminus \sigma(H^\bullet_\omega)$ then there exists $C > 0$ such that, for L sufficiently large, there are no resonances of $H^\bullet_{\omega,L}$ in the rectangle $\{\operatorname{Re} z \in I,\ 0 \geq \operatorname{Im} z \geq -1/C\}$.*

(2) *If $I \subset \Sigma^\circ$, then for $\varepsilon \in (0, 1)$ there exists $L_0 > 0$ such that, for $L \geq L_0$, there are no resonances of $H^\bullet_{\omega,L}$ in the rectangle $\{\operatorname{Re} z \in I,\ 0 \geq \operatorname{Im} z \geq -e^{-2\eta_\bullet \rho L(1+\varepsilon)})\}$, where*
   - *ρ is the maximum of the Lyapunov exponent $\rho(E)$ on I,*
   - $\eta_\bullet = \begin{cases} 1 & \text{if } \bullet = \mathbb{N}, \\ \frac{1}{2} & \text{if } \bullet = \mathbb{Z}. \end{cases}$

(3) *Pick $v_j = v_j(\omega) \in K_\omega$* (see the description of the spectrum of $H^{\mathbb{N}}_\omega$ just above Section 1C1) *and assume that $\{v_j\} = I^\circ \cap \sigma(H^{\mathbb{N}}_\omega) = I \cap \sigma(H^{\mathbb{N}}_\omega)$ and $I \cap \Sigma = \varnothing$; then there exists $c > 0$ such that, for L sufficiently large, $H^{\mathbb{N}}_{\omega,L}$ has a unique resonance in $\{\operatorname{Re} z \in I,\ -c \leq \operatorname{Im} z \leq 0\}$; moreover, this resonance, say z_j, is simple and satisfies $\operatorname{Im} z_j \asymp -e^{-\rho_j(\omega)L}$ and $|z_j - \lambda_j| \asymp e^{-\rho_j(\omega)L}$ for some $\rho_j(\omega) > 0$ independent of L.*

When comparing point (2) of this result with Theorem 1.5(2), it is striking that the width of the resonance-free region below Σ is much smaller in the random case (it is exponentially small in L) than in the periodic case (it is polynomially small in L). This a consequence of the localized nature of the spectrum, i.e., of the exponential decay of the eigenfunctions of $H^\bullet_\omega$.

1C3. *Description of the resonances closest to the real axis.* We will now see that below the resonance-free strip exhibited in Theorem 1.13 one does find resonances — actually, many of them. We prove:

Theorem 1.14. *Fix $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$. Let I be a compact interval in $(-2, 2) \cap \overset{\circ}{\Sigma}$.*

(1) *For any $\kappa \in (0, 1)$, ω-a.s. one has*
$$\frac{\#\{z \text{ resonance of } H^\bullet_{\omega,L} \mid \operatorname{Re} z \in I,\ 0 > \operatorname{Im} z \geq -e^{-L^\kappa}\}}{L} \to \int_I n(E)\, dE.$$

(2) *For $E \in I$ such that $n(E) > 0$ and $\lambda \in (0, 1)$, define the rectangle*
$$R^\bullet(E, \lambda, L, \varepsilon, \delta) := \left\{ z \in \mathbb{C} \,\middle|\, n(E)|\operatorname{Re} z - E| \leq \tfrac{1}{2}\varepsilon,\ -e^{\eta_\bullet \rho(E)\delta L} \leq e^{2\eta_\bullet \rho(E)\lambda L} \operatorname{Im} z \leq -e^{-\eta_\bullet \rho(E)\delta L} \right\},$$

where $\eta^\bullet$ is as defined in Theorem 1.13; then ω-a.s. one has

$$\lim_{\delta\to0^+}\lim_{\varepsilon\to0^+}\lim_{L\to+\infty}\frac{\#\{z \text{ resonances of } H^\bullet_{\omega,L} \text{ in } R^\bullet(E,\lambda,L,\varepsilon,\delta)\}}{L\varepsilon\delta}=1. \tag{1-16}$$

(3) *For $E\in I$ such that $n(E)>0$, define*

$$R^\bullet_\pm(E,1,L,\varepsilon,\delta)=\left\{z\in\mathbb{C}\;\middle|\; n(E)|\operatorname{Re} z-E|\le\tfrac12\varepsilon,\ -e^{-2\eta_\bullet\rho(E)(1\pm\delta)L}\le\operatorname{Im} z<0\right\};$$

then ω-a.s. one has

$$\lim_{\delta\to0^+}\lim_{\varepsilon\to0^+}\lim_{L\to+\infty}\frac{\#\{\text{resonances in } R^\bullet_\pm(E,1,L,\varepsilon,\delta)\}}{L\varepsilon\delta}=\begin{cases}1 & \text{if } \pm=-,\\ 0 & \text{if } \pm=+.\end{cases} \tag{1-17}$$

(4) *For $c>0$, ω-a.s. one has*

$$\lim_{L\to+\infty}\frac{\#\{z \text{ resonances of } H^\bullet_{\omega,L} \text{ in } I+i(-\infty,-e^{-2cL}]\}}{L}=\int_I\min\left(\frac{c}{\rho(E)},1\right)n(E)\,dE. \tag{1-18}$$

The striking fact is that the resonances are much closer to the real axis than in the periodic case; the lifetime of these resonances is much larger. The resonant states are quite stable, with lifetimes that are exponentially large in the width of the random perturbation. Point (4) is an integral version of point (2). Let us also note here that when $\bullet=\mathbb{Z}$, Theorem 1.14(4) is the statement of Theorem 0.2.

Note that the rectangles $R^\bullet(E,\lambda,L,\varepsilon,\delta)$ are very stretched along the real axis; their side-length in the imaginary part is exponentially small in L whereas their side-length in the real part is of order 1.

To understand Theorem 1.14(2), rescale the resonances of $H^\bullet_{\omega,L}$, say $(z^\bullet_{l,L}(\omega))_l$, as

$$x^\bullet_l=x^\bullet_{l,L}(E,\omega)=n(E)L(\operatorname{Re} z^\bullet_{l,L}(\omega)-E)\quad\text{and}\quad y^\bullet_l=y^\bullet_{l,L}(E,\omega)=-\frac{1}{2\eta_\bullet\rho(E)L}\log|\operatorname{Im} z^\bullet_{l,L}(\omega)|. \tag{1-19}$$

For $\lambda\in(0,1)$, this rescaling maps the rectangle $R^\bullet(E,\lambda,L,\varepsilon,\delta)$ into $\left\{|x|\le\tfrac12 L\varepsilon,\ |y-\lambda|\le\tfrac12\delta\right\}$ and the rectangles $R^\bullet_\pm(E,1,L,\varepsilon,\delta)$ are mapped into $\{|x|\le L\varepsilon/2,\ 1\mp\delta\le y\}$, respectively. The denominator of the quotient in (1-16) is just the area of the rescaled $R^\bullet(E,\lambda,L,\varepsilon,\delta)$ for $\lambda\in(0,1)$ or the rescaled $R^\bullet_+(E,1,L,\varepsilon,\delta)\setminus R^\bullet_-(E,1,L,\varepsilon,0)$. So, (2) states that, in the limit with ε and δ small and L large, the rescaled resonances become uniformly distributed in the rescaled rectangles.

We see that the structure of the set of resonances is very different from the one observed in the periodic case (see Figure 2). We will now zoom in on the resonance even more so as to make this structure clearer. We consider the two-dimensional point process $\xi^\bullet_L(E,\omega)$ defined by

$$\xi^\bullet_L(E,\omega)=\sum_{z^\bullet_{l,L}\text{ resonance of } H^\bullet_{\omega,L}}\delta_{(x^\bullet_l,y^\bullet_l)}, \tag{1-20}$$

where $x^\bullet_l$ and $y^\bullet_l$ are defined by (1-19).

We prove:

Theorem 1.15. *Fix $E\in(-2,2)\cap\Sigma^\circ$ such that $n(E)>0$. Then the point process $\xi^\bullet_L(E,\omega)$ converges weakly to a Poisson process in $\mathbb{R}\times(0,1]$ with intensity 1. That is, for any $p\ge0$, if $(I_n)_{1\le n\le p}$ (resp.*

$(C_n)_{1\le n\le p}$) *are disjoint intervals of the real line* $\mathbb{R}$ (*resp.* $[0,1]$), *then*

$$\lim_{L\to+\infty}\mathbb{P}\big(\big\{\omega\;\big|\;\#\{j\mid x^{\bullet}_{l,L}(E,\omega)\in I_n,\ y^{\bullet}_{l,L}(E,\omega)\in C_n\}=k_n\ \text{for}\ n=1,\dots,p\big\}\big)=\prod_{n=1}^{p}e^{-\mu_n}\frac{(\mu_n)^{k_n}}{k_n!},$$

where $\mu_n:=|I_n||C_n|$ *for* $1\le n\le p$.

This is the analogue of the celebrated result on the Poisson structure of the eigenvalues and localization centers of a random system (see, e.g., [Molchanov 1982; Minami 1996; Germinet and Klopp 2014]).

When considering the model for $\bullet=\mathbb{Z}$, Theorem 1.15 is Theorem 0.3.

In [Klopp 2011], we proved decorrelation estimates that can be used in the present setting to prove:

Theorem 1.16. *Fix* $E\in(-2,2)\cap\Sigma^\circ$ *and* $E'\in(-2,2)\cap\Sigma^\circ$ *such that* $E\neq E'$, $n(E)>0$ *and* $n(E')>0$. *Then the limits of the processes* $\xi^{\bullet}_L(E,\omega)$ *and* $\xi^{\bullet}_L(E',\omega)$ *are stochastically independent.*

Due to the rescaling, the above results only give a picture of the resonances in a zone of the type

$$E+L^{-1}[-\varepsilon^{-1},\varepsilon^{-1}]-i[e^{-2\eta_\bullet(1+\varepsilon)\rho(E)L},e^{-2\varepsilon\eta_\bullet\rho(E)L}] \tag{1-21}$$

for $\varepsilon>0$ arbitrarily small.

When L gets large, this rectangle is of a very small width and located very close to the real axis. Theorems 1.14, 1.15 and 1.16 describe the resonances lying closest to the real axis. As a comparison between points (1) and (2) in Theorem 1.14 shows, these resonances are the most numerous.

One can get a number of other statistics (e.g., the distribution of the spacings between the resonances) using the techniques developed for the study of the spectral statistics of a random system in the localized phase (see [Germinet and Klopp 2011; 2014; Klopp 2013]) combined with the analysis developed in Section 6.

1C4. *The description of the low-lying resonances.* It is natural to question what happens deeper in the complex plane. To answer this question, fix an increasing sequence of scales $(\ell_L)_L$ such that

$$\frac{\ell_L}{\log L}\to+\infty\quad\text{as } L\to+\infty\qquad\text{and}\qquad\frac{\ell_L}{L}\to 0\quad\text{as } L\to+\infty. \tag{1-22}$$

We first show that there are only a few resonances below the line $\{\operatorname{Im} z=e^{-\ell_L}\}$, namely:

Theorem 1.17. *Pick* $(\ell_L)_L$ *a sequence of scales satisfying* (1-22) *and* I *as above.*

Then, ω *almost surely, for* L *large one has*

$$\big\{z\ \text{resonances of}\ H^{\bullet}_{\omega,L}\ \text{in}\ \{\operatorname{Re} z\in I,\ \operatorname{Im} z\le -e^{-\ell_L}\}\big\}=O(\ell_L). \tag{1-23}$$

As we shall show now, after proper rescaling the structure of these resonances is the same as that of the resonances closer to the real axis.

Fix $E\in I$ such that $n(E)>0$. Recall that $(z^{\bullet}_{l,L}(\omega))_l$ are the resonances of $H_{\omega,L}$. We now rescale the resonances using the sequence $(\ell_L)_L$; this rescaling will select resonances that are further away from the real axis. Define

$$x_l^\bullet = x_{l,\ell_L}^\bullet(\omega) = n(E)\ell_L(\operatorname{Re} z_{l,L}^\bullet(\omega) - E) \quad \text{and} \quad y_j^\bullet = y_{l,\ell_L}^\bullet(\omega) = \frac{1}{2\eta_\bullet \ell_L \rho(E)} \log|\operatorname{Im} z_{l,L}^\bullet(\omega)|. \tag{1-24}$$

Consider now the two-dimensional point process

$$\xi_{L,\ell}^\bullet(E,\omega) = \sum_{z_{l,L}^\bullet \text{ resonance of } H_{\omega,L}^\bullet} \delta_{(x_{l,\ell_L}^\bullet, y_{l,\ell_L}^\bullet)}. \tag{1-25}$$

We prove the following analogue of the results of Theorems 1.14, 1.15 and 1.16 for resonances lying further away from the real axis.

Theorem 1.18. *Fix $E \in (-2,2) \cap \Sigma^\circ$ and $E' \in (-2,2) \cap \Sigma^\circ$ such that $E \neq E'$, $n(E) > 0$ and $n(E') > 0$. Fix a sequence of scales $(\ell_L)_L$ satisfying (1-22). Then one has*:

(1) *For $\lambda \in (0,1]$, ω-almost surely,*

$$\lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \lim_{L \to +\infty} \frac{\#\{z \text{ resonances of } H_{\omega,L}^\bullet \text{ in } R^\bullet(E,\lambda,\ell_L,\varepsilon,\delta)\}}{\ell_L \varepsilon \delta} = 1,$$

where $R^\bullet(E,\lambda,L,\varepsilon,\delta)$ is as defined in Theorem 1.14.

(2) *The point processes $\xi_{L,\ell}^\bullet(E,\omega)$ and $\xi_{L,\ell}^\bullet(E',\omega)$ converge weakly to Poisson processes in $\mathbb{R} \times (0,+\infty)$ of intensity* 1.

(3) *The limits of the processes $\xi_{L,\ell}^\bullet(E,\omega)$ and $\xi_{L,\ell}^\bullet(E',\omega)$ are stochastically independent.*

Point (1) shows that, in (1-23), one actually has

$$\{z \text{ resonances of } H_{\omega,L}^\bullet \text{ in } \{\operatorname{Re} z \in I,\ \operatorname{Im} z \le -e^{-\ell_L}\}\} \asymp \ell_L.$$

Notice also that the effect of the scaling (1-24) is to select resonances that live in the rectangle

$$E + \ell_L^{-1}[-\varepsilon^{-1}, \varepsilon^{-1}] - i[e^{-2\eta_\bullet(1+\varepsilon)\rho(E)\ell_L}, e^{-2\varepsilon\eta_\bullet\rho(E)\ell_L}]$$

This rectangle is now much further away from the real axis than the one considered in Section 1C3.

Modulo rescaling, the picture one gets for resonances in such rectangles is the same we got above in the rectangles (1-21). This description is valid almost all the way from distances to the real axis that are exponentially small in L up to distances that are of order $e^{-(\log L)^\alpha}$, $\alpha > 1$ (see (1-22)).

1C5. *Deep resonances.* One can also study the resonances that are even further away from the real axis in a way similar to what was done in the periodic case in Section 1B4. Define the random potentials on $\mathbb{N}$ and $\mathbb{Z}$

$$\widetilde{V}_{\omega,L}^{\mathbb{N}}(n) = \begin{cases} \omega_{L-n} & \text{for } 0 \le n \le L, \\ 0 & \text{for } L+1 \le n, \end{cases}$$
$$\widetilde{V}_{\omega,\tilde\omega,L}^{\mathbb{Z}}(n) = \begin{cases} 0 & \text{for } n \le -1, \\ \tilde\omega_n & \text{for } 0 \le n \le \left[\tfrac12 L\right], \\ \omega_{L-n} & \text{for } \left[\tfrac12 L\right]+1 \le n \le L, \\ 0 & \text{for } L+1 \le n, \end{cases} \tag{1-26}$$

where $\omega = (\omega_n)_{n\in\mathbb{N}}$ and $\tilde\omega = (\tilde\omega_n)_{n\in\mathbb{N}}$ are i.i.d. and satisfy the assumptions of the beginning of Section 1C.

Consider the operators

- $\widetilde{H}^{\mathbb{N}}_{\omega,L} = -\Delta + \widetilde{V}^{\mathbb{N}}_{\omega,L}$ on $\ell^2(\mathbb{N})$ with Dirichlet boundary condition at 0,
- $\widetilde{H}^{\mathbb{Z}}_{\omega,\tilde\omega,L} = -\Delta + \widetilde{V}^{\mathbb{Z}}_{\omega,\tilde\omega,L}$ on $\ell^2(\mathbb{Z})$.

Clearly, the random operator $\widetilde{H}^{\mathbb{N}}_{\omega,L}$ (resp. $\widetilde{H}^{\mathbb{Z}}_{\omega,L}$) has the same distribution as $H^{\mathbb{N}}_{\omega,L}$ (resp. $H^{\mathbb{Z}}_{\omega,L}$). Thus, for the low lying resonances, we are now going to describe those of $\widetilde{H}^{\mathbb{N}}_{\omega,L}$ (resp. $\widetilde{H}^{\mathbb{Z}}_{\omega,L}$) instead of those of $H^{\mathbb{N}}_{\omega,L}$ (resp. $H^{\mathbb{Z}}_{\omega,L}$).

Remark 1.19. The reason for this change of operators is the same as the one why, in the case of the periodic potential, we had to distinguish various auxiliary operators depending on the congruence of L modulo the period p: this gives a meaning to the limiting operators when $L \to +\infty$.

Define the probability measure $dN_\omega(\lambda)$ using its Borel transform by, for $\operatorname{Im} z \neq 0$,

$$\int_{\mathbb{R}} \frac{dN_\omega(\lambda)}{\lambda - z} := \langle \delta_0, (H^{\mathbb{N}}_\omega - E)^{-1}\delta_0\rangle. \tag{1-27}$$

Consider the function

$$\Xi_\omega(E) = \int_{\mathbb{R}} \frac{dN_\omega(\lambda)}{\lambda - E} + e^{-i\arccos(E/2)} = \int_{\mathbb{R}} \frac{dN_\omega(\lambda)}{\lambda - E} + \tfrac{1}{2}E + \sqrt{\left(\tfrac{1}{2}E\right)^2 - 1}, \tag{1-28}$$

where the choice of $z \mapsto \arccos z$ and $z \mapsto \sqrt{z^2-1}$ are those described after (1-5).

This random function Ξ_ω is the analogue of Ξ^-_k in the periodic case. One has the analogue of Proposition 5.7:

Proposition 1.20. *If $\omega_0 \neq 0$, one has $\Xi_\omega(E) \sim -\omega_0 E^{-2}$ as $|E| \to \infty$, $\operatorname{Im} E < 0$. Thus, ω-almost surely, Ξ_ω does not vanish identically in $\{\operatorname{Im} E < 0\}$.*

Pick $I \subset \Sigma^\circ \cap (-2, 2)$ compact. Then, ω-almost surely, the number of zeros of Ξ_ω (counted with multiplicity) in $I + i(-\infty, \varepsilon]$ is asymptotic to $\int_I n(E)/\rho(E)\, dE\, |\log\varepsilon|$ as $\varepsilon \to 0^+$; moreover, ω-almost surely, there exists $\varepsilon_\omega > 0$ such that all the zeros of Ξ_ω in $I + i[-\varepsilon_\omega, 0)$ are simple.

It seems reasonable to believe that, except for the zero at $-i\infty$, ω-almost surely all the zeros of Ξ_ω are simple; we do not prove it.

For the "deep" resonances, we then prove:

Theorem 1.21. *Fix $I \subset \Sigma^\circ \cap (-2, 2)$ a compact interval. There exists $c > 0$ such that, with probability 1, there exists $c_\omega > 0$ such that, for L sufficiently large, one has:*

1. *For each resonance of $\widetilde{H}^{\mathbb{N}}_{\omega,L}$ (resp. $\widetilde{H}^{\mathbb{Z}}_{\omega,\tilde\omega,L}$) in $I + i(-\infty, -e^{-cL}]$, say E, there exists a unique zero of Ξ_ω (resp. $\Xi_\omega \Xi_{\tilde\omega}$), say $\widetilde{E}$, such that $|E - \widetilde{E}| \le e^{-c_\omega L}$.*
2. *Reciprocally, to each zero (counted with multiplicity) of Ξ_ω (resp. $\Xi_\omega\Xi_{\tilde\omega}$) in the rectangle $I + i(-\infty, -e^{-cL}]$, say $\widetilde{E}$, one can associate a unique resonance of $\widetilde{H}^{\mathbb{N}}_{\omega,L}$ (resp. $\widetilde{H}^{\mathbb{Z}}_{\omega,\tilde\omega,L}$), say E, such that $|E - \widetilde{E}| \le e^{-c_\omega L}$.*

One can combine this result with the description of the asymptotic distribution of the resonances given by Theorem 1.18 to obtain the asymptotic distributions of the zeros of the function Ξ_ω near a point $E - i\varepsilon$ when $\varepsilon \to 0^+$. Indeed, let $(z_l(\omega))_l$ be the zeros of Ξ_ω in $\{\operatorname{Im} E < 0\}$. Rescale the zeros:

$$x_{l,\varepsilon}(\omega) = n(E)|\log\varepsilon|(\operatorname{Re} z_l(\omega) - E) \quad \text{and} \quad y_{l,\varepsilon}(\omega) = -\frac{1}{2\rho(E)|\log\varepsilon|}\log|\operatorname{Im} z_l(\omega)|; \tag{1-29}$$

and consider the two-dimensional point process $\xi_\varepsilon(E,\omega)$ defined by

$$\xi_\varepsilon(E,\omega) = \sum_{z_l(\omega)\text{ zeros of }\Xi_\omega} \delta_{(x_{l,\varepsilon},y_{l,\varepsilon})}. \tag{1-30}$$

Then one has:

Corollary 1.22. *Fix $E \in I$ such that $n(E) > 0$. Then the point process $\xi_\varepsilon(E,\omega)$ converges weakly to a Poisson process in $\mathbb{R}\times\mathbb{R}$ with intensity* 1.

The function Ξ_ω has been studied in [Kunz and Shapiro 2006; 2008], where the average density of its zeros was computed. Here we obtain a more precise result.

1C6. *The half-line random perturbation.* On $\ell^2(\mathbb{Z})$, we now consider the operator $H_\omega^\infty = -\Delta + V_\omega$, where $V_\omega(n) = 0$ for $n \ge 0$, $V_\omega(n) = \omega_n$ for $n \le -1$ and $(\omega_n)_{n\ge 0}$ are i.i.d. and have the same distribution as above. The spectral theory of the continuous analogue of H_ω^∞, i.e., the Schrödinger operator on the real line with a random potential on the half-line, was studied in [Carmona 1983].

Recall that Σ is the almost sure spectrum of $H_\omega^{\mathbb{Z}}$ (on $\ell^2(\mathbb{Z})$). We prove:

Theorem 1.23. *First, ω-almost surely, the resolvent of H_ω^∞ does not admit an analytic continuation from the upper half-plane through $(-2,2)\cap\Sigma^\circ$ to any subset of the lower half plane. Nevertheless, ω-almost surely, the spectrum of H_ω^∞ in $(-2,2)\cap\Sigma^\circ$ is purely absolutely continuous.*

Second, ω-almost surely, the resolvent of H_ω^∞ does admit a meromorphic continuation from the upper half-plane through $(-2,2)\setminus\Sigma$ to the lower half-plane; the poles of this continuation are exactly the zeros of the function $E \mapsto 1 - e^{i\theta(E)}\int_{\mathbb{R}} 1/(\lambda - E)\,dN_\omega(\lambda)$ when continued from the upper half-plane through $(-2,2)\setminus\Sigma$ to the lower half-plane.

Third, ω-almost surely, the spectrum of H_ω^∞ in $\Sigma^\circ\setminus[-2,2]$ is pure point associated to exponentially decaying eigenfunctions; hence, the resolvent of H_ω^∞ cannot be continued through $\Sigma^\circ\setminus[-2,2]$.

In Figure 5, to illustrate Theorem 1.23, assuming that $\Sigma_{\mathbb{Z}}$ (in blue) has a single gap that is contained in $(-2,2)$, we have drawn the analytic continuation of the resolvent of H_ω^∞ and the associated resonances; we also indicate the real intervals of the spectrum through which the resolvent of H_ω^∞ does not admit an analytic continuation and the spectral type of H_ω^∞ in the intervals.

Let us also note here that if $0 \in \operatorname{supp} g$ (where g is the density of the random variables defining the random potential) then, by (1-13), one has $[-2,2]\subset\Sigma$. In this case, there is no possibility to continue the resolvent of H_ω^∞ to the lower half-plane passing through $[-2,2]$.

Comparing Theorem 1.23 to Theorem 1.11, we see that, as for the operator H^∞, when continued through $(-2,2)\cap\Sigma^\circ$ the operator H_ω^∞ does not have any resonances, but for very different reasons.

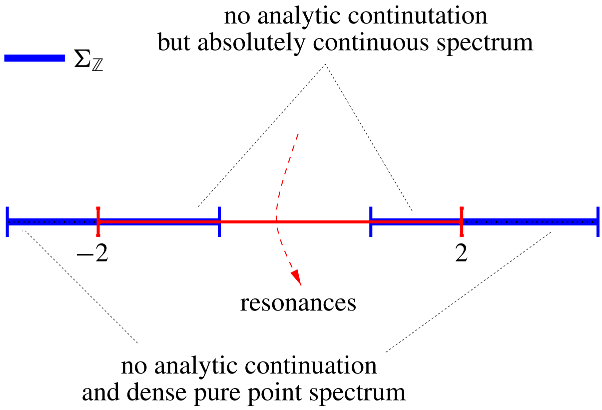

Figure 5. The analytic continuation of the resolvent and resonances for H^∞_ω.

When one does the continuation through $(-2, 2) \setminus \Sigma$, one sees that the number of resonances is finite; "near" the real axis, the continuation of the function $E \mapsto 1 - e^{i\theta(E)} \int_{\mathbb{R}} 1/(\lambda - E)\, dN_\omega(\lambda)$ has nontrivial imaginary part and near ∞ it does not vanish.

Theorem 1.23 also shows that the equation studied in [Kunz and Shapiro 2006; 2008], i.e., $\Xi_\omega(E) = 0$, does not describe the resonances of H^∞_ω as is claimed in these papers: these resonances do not exist as there is no analytic continuation of the resolvent of H^∞_ω through $(-2, 2) \cap \Sigma$! As is shown in Theorem 1.21, the solutions to the equation $\Xi_\omega(E) = 0$ give an approximation to the resonances of $H^{\mathbb{N}}_{\omega,L}$ (see Theorem 1.21).

1D. ***Outline of and reading guide to the paper.*** In the present section, we shall explain the main ideas leading to the proofs of the results presented above.

In Section 2, we prove Theorem 1.3; this proof is classical. As a consequence of the proof, one sees that, in the case of the half-lattice $\mathbb{N}$ (resp. lattice $\mathbb{Z}$), the resonances are the eigenvalues of a rank-one (resp. rank-two) perturbation of $(-\Delta + V)|_{[\![0,L]\!]}$ with Dirichlet boundary condition. The perturbation depends in an explicit way on the resonance. This yields a closed equation for the resonances in terms of the eigenvalues and normalized eigenfunctions of the Dirichlet restriction $(-\Delta + V)|_{[\![0,L]\!]}$. To obtain a description of the resonances we then are in need of a "precise" description of the eigenvalues and normalized eigenfunctions. Actually, the only information needed on the normalized eigenfunctions is their weight at the point L (and the point 0 in the full lattice case), 0 and L being the endpoints of $[\![0, L]\!]$.

In Section 3, we solve the two equations obtained previously under the condition that the weight of the normalized eigenfunctions at L (and 0) be much smaller than the spacing between the Dirichlet eigenvalues. This condition entails that the resonance equation we want to solve essentially factorizes and become very easy to solve (see Theorems 3.1, 3.2 and 3.3), i.e., it suffices to solve it near any given Dirichlet eigenvalue.

For periodic potentials, the condition that the eigenvalue spacing is much larger than the weight of the normalized eigenfunctions at L (and 0) is not satisfied: both quantities are of the same order of magnitude (see Theorem 4.2) for the Dirichlet eigenvalues in the bulk of the spectrum, i.e., the vast majority of them. This is a consequence of the extended nature of the eigenfunctions in this case. Therefore, we find another way to solve the resonance equation. This way goes through a more precise description of the Dirichlet eigenvalues and normalized eigenfunctions which is the purpose of Theorem 4.2. We use this description to reduce the resonance equation to an effective equation (see Theorem 5.1) up to errors of order $O(L^{-\infty})$. It is important to obtain errors of at most that size. Indeed, the effective equation may have solutions to any order (the order is finite and only depends on V but it is unknown); thus, to obtain solutions to the true equation from solutions to the effective equation with a good precision, one needs the two equations to differ by at most $O(L^{-\infty})$. We then solve the effective equation and, in Section 5B, prove the results of Section 1B.

On the other hand, for random potentials, it is well known that the eigenfunctions of the Dirichlet restriction $(-\Delta + V)|_{[\![0,L]\!]}$ are exponentially localized and, for most of them localized, far from the edge of $[\![0, L]\!]$. Thus, their weight at L (and 0 in the full lattice case) is typically exponentially small in L; the eigenvalue spacing however is typically of order L^{-1}. We can then use the results of Section 3 to solve the resonance equation. The real part of a given resonance is directly related to a Dirichlet eigenvalue and its imaginary part to the weight of the corresponding eigenfunction at L (and 0 in the full lattice case). The main difficulty is to find the asymptotic behavior of this weight. Indeed, while it is known that, in the random case, eigenfunctions decay exponentially away from a localization center and that, for the full random Hamiltonian (i.e., the Hamiltonian on the line or half-line with a random potential), at infinity this decay rate is given by the Lyapunov exponent, to the best of our knowledge, before the present work, it was not known at which length scale this Lyapunov behavior sets in (with a good probability). Answering this question is the purpose of Theorems 6.4 and 6.5 proved in Section 6C: we show that, for the one-dimensional Anderson model, for $\delta > 0$ arbitrary, on a box of size L sufficiently large, all the eigenfunctions exhibit an exponential decay (we obtain both an upper and a lower bound on the eigenfunctions) at a rate equal to the Lyapunov exponent at the corresponding energy (up to an error of size δ) as soon as one is at a distance δL from the corresponding localization center.

These bounds give estimates on the weight of most eigenfunctions at the point L (and 0 in the full lattice case); this is directly related to the distance of the corresponding localization center to the points L (and 0). One can then transform the known results on the statistics of the (rescaled) eigenvalues and (rescaled) localization centers into statistics of the (rescaled) resonances. This is done in Section 6B and proves most of the results in Section 1C.

Finally, Section 6D is devoted to the study of the full line Hamiltonian obtained from the free Hamiltonian on one half-line and a random Hamiltonian on the other half-line; it contains in particular the proof of Theorem 1.23.

2. The analytic continuation of the resolvent

Resonances for Jacobi matrices were considered in various works (see, e.g., [Brown et al. 2005; Iantchenko and Korotyaev 2012] and references therein). For the sake of completeness, we provide an independent proof of Theorem 1.3. It follows standard ideas that were first applied in the continuous setting, i.e., for

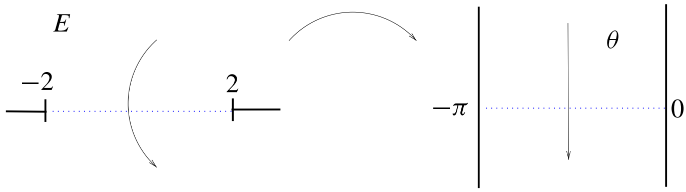

Figure 6. The mapping $E \mapsto \theta(E)$.

partial differential operators instead of finite difference operators (see, e.g., [Sjöstrand and Zworski 1991] and references therein).

The proof relies on the fact that the resolvent of the free Laplace operator can be continued holomorphically from $\mathbb{C}^+$ to $\mathbb{C} \setminus ((-\infty, -2] \cup [2, +\infty))$ as an operator valued function from l^2_{comp} to l^2_{loc}. This is an immediate consequence of the fact that, by discrete Fourier transformation, $-\Delta$ is the Fourier multiplier by the function $\theta \mapsto 2\cos\theta$.

Indeed, for $-\Delta$ on $\ell^2(\mathbb{Z})$ and $\operatorname{Im} E > 0$, one has, for $(n, m) \in \mathbb{Z}$ (assume $n - m \geq 0$),

$$\begin{aligned}\langle \delta_n, (-\Delta - E)^{-1}\delta_m \rangle &= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-i(n-m)\theta}}{2\cos\theta - E}\,d\theta = \frac{1}{2i\pi}\int_{|z|=1} \frac{z^{n-m}}{z^2 - Ez + 1}\,dz \\ &= \frac{1}{2\sqrt{\left(\frac{1}{2}E\right)^2 - 1}}\left(\tfrac{1}{2}E - \sqrt{\left(\tfrac{1}{2}E\right)^2 - 1}\right)^{n-m} = \frac{e^{i(n-m)\theta(E)}}{\sin\theta(E)},\end{aligned} \tag{2-1}$$

where $E = 2\cos\theta(E)$ and $\theta = \theta(E)$ is chosen so that $\operatorname{Im}\theta > 0$ and $\operatorname{Re}\theta \in (-\pi, 0)$ for $\operatorname{Im} E > 0$. The choice satisfies $\theta(\bar{E}) = \overline{\theta(E)}$.

The map $E \mapsto \theta(E)$ can be continued analytically from $\mathbb{C}^+$ to the cut plane $\mathbb{C} \setminus ((-\infty, -2] \cup [2, +\infty))$ as shown in Figure 6.

The continuation is one-to-one and onto from $\mathbb{C} \setminus ((-\infty, -2] \cup [2, +\infty))$ to $(-\pi, 0) + i\mathbb{R}$. It defines a choice of $E \mapsto \arccos\left(\frac{1}{2}E\right) = \theta(E)$.

Clearly, using (2-1), this continuation yields an analytic continuation of $R_0^{\mathbb{Z}} := (-\Delta - E)^{-1}$ from $\{\operatorname{Im} E > 0\}$ to $\mathbb{C} \setminus ((-\infty, -2] \cup [2, +\infty))$ as an operator from l^2_{comp} to l^2_{loc}.

Let us now turn to the half-line operator, i.e., $-\Delta$ on $\mathbb{N}$ with Dirichlet condition at 0. Pick E such that $\operatorname{Im} E > 0$ and set $E = 2\cos\theta$, where $\theta = \theta(E)$ is chosen as above. If, for $v \in \mathbb{C}^{\mathbb{N}}$ bounded and $n \geq -1$, one sets $v_{-1} = 0$ and

$$[R_0^{\mathbb{N}}(E)(v)]_n = \frac{1}{2i\sin\theta(E)}\sum_{j=-1}^{n} v_j \sin((n-j)\theta(E)) - e^{i\theta(E)}\frac{\sin((n+1)\theta(E))}{2i\sin\theta(E)}\sum_{j\geq 0} e^{ij\theta(E)} v_j, \tag{2-2}$$

then, for $\operatorname{Im} E > 0$, a direct computations shows that:

(1) For $v \in \ell^2(\mathbb{N})$, the vector $R_0^{\mathbb{N}}(E)(v)$ is in the domain of the Dirichlet Laplacian on $\ell^2(\mathbb{N})$, i.e., $[R_0^{\mathbb{N}}(E)(v)]_{-1} = 0$.

(2) For $n \geq 0$, one checks that

$$[R_0^{\mathbb{N}}(E)(v)]_{n+1} + [R_0^{\mathbb{N}}(E)(v)]_{n-1} - E[R_0^{\mathbb{N}}(E)(v)]_n = v_n. \tag{2-3}$$

(3) $R_0^{\mathbb{N}}(E)$ defines a bounded map from $\ell^2(\mathbb{N})$ to itself.

Thus, $R_0^{\mathbb{N}}(E)$ is the resolvent of the Dirichlet Laplacian on $\mathbb{N}$ at energy E for $\operatorname{Im} E > 0$.

Using the continuation of $E \mapsto \theta(E)$, (2-2) yields an analytic continuation of the resolvent $R_0^{\mathbb{N}}(E)$ as an operator from l^2_{comp} to l^2_{loc}.

Remark 2.1. Note that the resolvent $R_0^{\mathbb{N}}(E)$ at an energy E with $\operatorname{Im} E < 0$ is given by (2-2) with $\theta(E)$ replaced by $-\theta(E)$. For (2-2), one has to assume that $(v_j)_{j\in\mathbb{N}}$ decays fast enough at ∞.

To deal with the perturbation V, we proceed in the same way on $\mathbb{Z}$ and on $\mathbb{N}$. Set $V^L = V\mathbf{1}_{[\![0,L]\!]}$ (viewed as a function on $\mathbb{N}$ or $\mathbb{Z}$ depending on the case). Letting $R_0(E)$ be either $R_0^{\mathbb{Z}}(E)$ or $R_0^{\mathbb{N}}(E)$, we compute

$$-\Delta + V^L - E = (-\Delta - E)(1 + R_0(E)V^L) = (1 + V^L R_0(E))(-\Delta_L - E).$$

Thus it suffices to check that the operator $R_0(E)V^L$ (resp. $V^L R_0(E)$) can be analytically continued as an operator from l^2_{loc} to l^2_{loc} (resp. l^2_{comp} to l^2_{comp}). This follows directly from (2-2) and the fact V^L has finite rank.

To complete the proof of Theorem 1.3, we just note that, since

- $E \mapsto R_0(E)V^L$ (resp. $E \mapsto V^L R_0(E)$) is a finite-rank, operator-valued function, analytic on the connected set $\mathbb{C} \setminus ((-\infty, -2] \cup [2, +\infty))$,
- -1 is not an eigenvalue of $R_0(E)V^L$ (resp. $V^L R_0(E)$) for $\operatorname{Im} E > 0$,

by the Fredholm principle, the set of energies E for which -1 is an eigenvalue of $R_0(E)V^L$ (resp. $V^L R_0(E)$) is discrete. Hence, the set of resonances is discrete.

This completes the proof of the first part of Theorem 1.3. To prove the second part, we will first write a characteristic equation for resonances. The bound on the number of resonances will then be obtained through a bound on the number of solutions to this equation.

2A. ***A characteristic equation for resonances.*** In the literature, we did not find a characteristic equation for the resonances in a form suitable for our needs. The characteristic equation we derive will take different forms depending on whether we deal with the half-line or the full line operator. But in both cases, the coefficients of the characteristic equation will be constructed from the spectral data (i.e., the eigenvalues and eigenfunctions) of the operator H_L (see Remark 1.6).

2B. ***In the half-line case.*** We first consider $H_L^{\mathbb{N}}$ on $\ell^2(\mathbb{N})$ and prove:

Theorem 2.2. *Consider the operator H_L defined as $H_L^{\mathbb{N}}$ restricted to $[\![0, L]\!]$ with Dirichlet boundary conditions at L and define*:

- $(\lambda_j)_{0\leq j\leq L} = (\lambda_j(L))_{0\leq j\leq L}$ *are the Dirichlet eigenvalues of $H_L^{\mathbb{N}}$ ordered so that $\lambda_j < \lambda_{j+1}$.*
- $a_j^{\mathbb{N}} = a_j^{\mathbb{N}}(L) = |\varphi_j(L)|^2$, *where $\varphi_j = (\varphi_j(n))_{0\leq n\leq L}$ is a normalized eigenvector associated to λ_j.*

Then an energy E is a resonance of $H_L^{\mathbb{N}}$ if and only if

$$S_L(E) := \sum_{j=0}^{L} \frac{a_j^{\mathbb{N}}}{\lambda_j - E} = -e^{-i\theta(E)}, \quad E = 2\cos\theta(E), \tag{2-4}$$

$\theta(E)$ being chosen so that $\operatorname{Im}\theta(E) > 0$ *and* $\operatorname{Re}\theta(E) \in (-\pi, 0)$ *when* $\operatorname{Im} E > 0$.

Let us note that

$$a_j^{\mathbb{N}}(L) > 0 \quad \text{for all } 0 \le j \le L \qquad \text{and} \qquad \sum_{j=0}^{L} a_j^{\mathbb{N}}(L) = \sum_{j=0}^{L} |\varphi_j(L)|^2 = 1. \tag{2-5}$$

Proof of Theorem 2.2. By the proof of the first statement of Theorem 1.3 (see the beginning of Section 2), we know that an energy E is a resonance if and only if -1 if an eigenvalue of $R_0(E)V^L$, where $R_0(E)$ is defined by (2-2). Pick E an resonance and let $u = (u_n)_{n\ge 0}$ be a resonant state that is an eigenvector of $R_0(E)V^L$ associated to the eigenvalue -1. As $V_n^L = 0$ for $n \ge L+1$, (2-2) yields that, for $n \ge L+1$, $u_n = \beta e^{in\theta(E)}$ for some fixed $\beta \in \mathbb{C}^*$. As $u = -R_0(E)V^L u$, for $n \ge L+1$ it satisfies $u_{n+1} + u_{n-1} = E u_n$. Thus, $u_{L+1} = e^{i\theta(E)} u_L$ and, by (2-3), u is a solution to the eigenvalues problem

$$\begin{cases} u_{n+1} + u_{n-1} + V_n u_n = E u_n & \text{for all } n \in [\![0, L]\!], \\ u_{-1} = 0, \\ u_{L+1} = e^{i\theta(E)} u_L. \end{cases}$$

This can be equivalently be rewritten as

$$\begin{pmatrix} V_0 & 1 & 0 & \cdots & & 0 \\ 1 & V_1 & 1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & & \\ & 0 & 1 & V_{L-1} & & 1 \\ 0 & \cdots & 0 & 1 & & V_L + e^{i\theta(E)} \end{pmatrix} \begin{pmatrix} u_0 \\ \\ \vdots \\ \\ u_L \end{pmatrix} = E \begin{pmatrix} u_0 \\ \\ \vdots \\ \\ u_L \end{pmatrix}. \tag{2-6}$$

The matrix in (2-6) is the Dirichlet restriction of $H_L^{\mathbb{N}}$ to $[\![0, L]\!]$ perturbed by the rank-one operator $e^{i\theta(E)}\delta_L \otimes \delta_L$. Thus, by rank-one perturbation theory (see, e.g., [Simon 1995]), an energy E is a resonance if and only if satisfies (2-4).

This completes the proof of Theorem 2.2. □

Proof of Theorem 1.3. Let us now complete the proof of Theorem 1.3 for the operator on the half-line. Let us first note that, for $\operatorname{Im} E > 0$, the imaginary part of the left-hand side of (2-4) is positive by (2-7). On the other hand, the imaginary part of the right-hand side of (2-4) is equal to $-e^{\operatorname{Im}\theta(E)}\sin(\operatorname{Re}\theta(E))$ and, thus, is negative (recall that $\operatorname{Re}\theta(E) \in (-\pi, 0)$ (see Figure 1). Thus, as already emphasized, (2-4) has no solution in the upper half-plane or on the interval $(-2, 2)$.

Clearly, (2-4) is equivalent to the polynomial equation of degree $2L+2$ in the variable $z = e^{-i\theta(E)}$

$$\prod_{k=0}^{L} (z^2 - 2\lambda_k z + 1) - \sum_{j=0}^{L} a_j^{\mathbb{N}} \prod_{\substack{0 \le k \le L \\ k \ne j}} (z^2 - 2\lambda_k z + 1) = 0. \tag{2-7}$$

We are looking for the solutions to (2-7) in the upper half-plane. As the polynomial in the right-hand side of (2-7) has real coefficients, its zeros are symmetric with respect to the real axis. Moreover, one notices that, by (2-5), 0 is a solution to (2-7). Hence, the number of solutions to (2-7) in the upper half-plane is bounded by L. This completes the proof of Theorem 1.3. □

2C. *On the whole line.* Now consider $H_L^{\mathbb{Z}}$ on $\ell^2(\mathbb{Z})$. We prove:

Theorem 2.3. *Using the notations of Theorem 2.2, an energy E is a resonance of $H_L^{\mathbb{Z}}$ if and only if*

$$\det\left(\sum_{j=0}^{L}\frac{1}{\lambda_j-E}\begin{pmatrix}|\varphi_j(L)|^2 & \overline{\varphi_j(0)}\varphi_j(L)\\ \varphi_j(0)\overline{\varphi_j(L)} & |\varphi_j(0)|^2\end{pmatrix}+e^{-i\theta(E)}\right)=0, \tag{2-8}$$

where $\det(\cdot)$ denotes the determinant of a square matrix, $E=2\cos\theta(E)$ and $\theta(E)$ is chosen as in Theorem 2.2.

So, an energy E is a resonance of $H_L^{\mathbb{Z}}$ if and only if $-e^{-i\theta(E)}$ belongs to the spectrum of the 2×2 matrix

$$\Gamma_L(E):=\sum_{j=0}^{L}\frac{1}{\lambda_j-E}\begin{pmatrix}|\varphi_j(L)|^2 & \overline{\varphi_j(0)}\varphi_j(L)\\ \varphi_j(0)\overline{\varphi_j(L)} & |\varphi_j(0)|^2\end{pmatrix}. \tag{2-9}$$

Proof of Theorem 2.3. The proof is the same as that of Theorem 2.2 except that now E is a resonance if there exists a nontrivial solution u to the eigenvalues problem

$$\begin{cases} u_{n+1}+u_{n-1}+V_nu_n=Eu_n & \text{for all } n\in[\![0,L]\!],\\ u_{-1}=e^{i\theta(E)}u_0\\ u_{L+1}=e^{i\theta(E)}u_L.\end{cases}$$

This can equivalently be rewritten as

$$\begin{pmatrix} V_0+e^{i\theta(E)} & 1 & 0 & \cdots & 0\\ 1 & V_1 & 1 & 0 & \\ \vdots & \ddots & \ddots & \ddots & \\ & 0 & 1 & V_{L-1} & 1\\ 0 & \cdots & 0 & 1 & V_L+e^{i\theta(E)}\end{pmatrix}\begin{pmatrix}u_0\\ \\ \vdots\\ \\ u_L\end{pmatrix}=E\begin{pmatrix}u_0\\ \\ \vdots\\ \\ u_L\end{pmatrix}.$$

Thus, using rank-one perturbations twice, we find that an energy E is a resonance if and only if

$$\left(1+e^{i\theta(E)}\sum_{j=0}^{L}\frac{|\varphi_j(0)|^2}{\lambda_j-E}\right)\left(1+e^{i\theta(E)}\sum_{j=0}^{L}\frac{|\varphi_j(L)|^2}{\lambda_j-E}\right)=e^{2i\theta(Ek)}\sum_{0\le j,j'\le L}\frac{\varphi_j(L)\varphi_{j'}(0)\overline{\varphi_{j'}(L)\varphi_j(0)}}{(\lambda_j-E)(\lambda_{j'}-E)},$$

that is, if and only if (2-8) holds. This completes the proof of Theorem 2.3. □

Let us now complete the proof of Theorem 1.3 for the operator on the full line. Let us first show that (2-8) has no solution in the upper half-plane. If $-e^{-i\theta(E)}$ belongs to the spectrum of the matrix defined by (2-8) and $u \in \mathbb{C}^2$ is a normalized eigenvector associated to $-e^{-i\theta(E)}$, one has

$$\sum_{j=0}^{L} \frac{1}{\lambda_j - E} \left| \left\langle \begin{pmatrix} \varphi_j(L) \\ \varphi_j(0) \end{pmatrix}, u \right\rangle \right|^2 = -e^{-i\theta(E)}.$$

This is impossible in the upper half-plane and on $(-2, 2)$ as the two sides of the equation have imaginary parts of opposite signs.

Note that

$$\sum_{j=0}^{L} \begin{pmatrix} \varphi_j(L) \\ \varphi_j(0) \end{pmatrix} \overline{\begin{pmatrix} \varphi_j(L) & \varphi_j(0) \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note also that $-e^{-i\theta(E)}$ is an eigenvalue of (2-8) if and only if it satisfies

$$1 + e^{i\theta(E)} \sum_{j=0}^{L} \frac{|\varphi_j(L)|^2 + |\varphi_j(0)|^2}{\lambda_j - E} = -\tfrac{1}{2} e^{2i\theta(E)} \sum_{0 \le j, j' \le L} \frac{1}{(\lambda_j - E)(\lambda_{j'} - E)} \begin{vmatrix} \varphi_j(0) & \varphi_{j'}(0) \\ \varphi_j(L) & \varphi_{j'}(L) \end{vmatrix}^2. \tag{2-10}$$

As the eigenvalues of H_L are simple, one computes

$$\sum_{0 \le j, j' \le L} \frac{1}{(\lambda_j - E)(\lambda_{j'} - E)} \begin{vmatrix} \varphi_j(0) & \varphi_{j'}(0) \\ \varphi_j(L) & \varphi_{j'}(L) \end{vmatrix}^2 = 2 \sum_{0 \le j \le L} \frac{1}{\lambda_j - E} \sum_{j' \ne j} \frac{1}{\lambda_{j'} - \lambda_j} \begin{vmatrix} \varphi_j(0) & \varphi_{j'}(0) \\ \varphi_j(L) & \varphi_{j'}(L) \end{vmatrix}^2. \tag{2-11}$$

Thus, (2-10) is equivalent to the polynomial equation of degree $2(L+1)$ in the variable $z = e^{-i\theta(E)}$

$$z \prod_{k=0}^{L} (z^2 - \lambda_k z + 1) - \sum_{j=0}^{L} (2a_j^{\mathbb{Z}} z + b_j^{\mathbb{Z}}) \prod_{\substack{0 \le k \le L \\ k \ne j}} (z^2 - \lambda_k z + 1) = 0, \tag{2-12}$$

where we have defined

$$a_j^{\mathbb{Z}} := \frac{1}{2}(|\varphi_j(L)|^2 + |\varphi_j(0)|^2) = \frac{1}{2} \left\| \begin{pmatrix} \varphi_j(L) \\ \varphi_j(0) \end{pmatrix} \right\|^2 = \frac{1}{2} \left\| \begin{pmatrix} |\varphi_j(L)|^2 & \overline{\varphi_j(0)} \varphi_j(L) \\ \varphi_j(0) \overline{\varphi_j(L)} & |\varphi_j(0)|^2 \end{pmatrix} \right\| \tag{2-13}$$

and

$$b_j^{\mathbb{Z}} := \sum_{j' \ne j} \frac{1}{\lambda_{j'} - \lambda_j} \begin{vmatrix} \varphi_j(0) & \varphi_{j'}(0) \\ \varphi_j(L) & \varphi_{j'}(L) \end{vmatrix}^2.$$

The sequence $(a_j^{\mathbb{Z}})_j$ also satisfies (2-5). Taking $|E|$ to $+\infty$ in (2-11), one notes that

$$\sum_{j=0}^{L} b_j^{\mathbb{Z}} = 0 \quad \text{and} \quad \sum_{j=0}^{L} \lambda_j b_j^{\mathbb{Z}} = -\frac{1}{2} \sum_{0 \le j, j' \le L} \begin{vmatrix} \varphi_j(0) & \varphi_{j'}(0) \\ \varphi_j(L) & \varphi_{j'}(L) \end{vmatrix}^2 = -1. \tag{2-14}$$

We are looking for the solutions to (2-12) in the upper half-plane. As the polynomial in the right-hand side of (2-12) has real coefficients, its zeros are symmetric with respect to the real axis. Moreover, one notices that, by (2-14), 0 is a root of order two of the polynomial in (2-12). Hence, as the polynomial has degree $2L + 3$, the number of solutions to (2-12) in the upper half-plane is bounded by L. This completes the proof of Theorem 1.3.

3. General estimates on resonances

By Theorems 2.2 and 2.3, we want to solve equations (2-4) and (2-8) in the lower half-plane. We first derive some general estimates for zones in the lower half-plane free of solutions to equations (2-4) and (2-8) (i.e., resonant-free zones for the operators $H_L^{\mathbb{N}}$ and $H_L^{\mathbb{Z}}$) and then a result on the existence of solutions to equations (2-4) and (2-8) (i.e., resonances for the operators $H_L^{\mathbb{N}}$ and $H_L^{\mathbb{Z}}$).

3A. ***General estimates for resonant-free regions.*** We keep the notations of Theorems 2.2 and 2.3. To simplify the notations in the theorems of this section, we will write a_j for either $a_j^{\mathbb{N}}$ when solving (2-4) or $a_j^{\mathbb{Z}}$ when solving (2-8). We will specify the superscript only when there is risk of confusion.

We first prove:

Theorem 3.1. *Fix $\delta > 0$. Then there exists $C > 0$ (independent of V and L) such that, for any L and $j \in \{0, \dots, L\}$ with $-4+\delta \le \lambda_{j-1}+\lambda_j < \lambda_{j+1}+\lambda_j \le 4-\delta$, equations (2-4) and (2-8) have no solution in the set* (see Figure 7)

$$U_j := \left\{E \in \mathbb{C} \,\middle|\, \operatorname{Re} E \in \left[\tfrac{1}{2}(\lambda_j+\lambda_{j-1}), \tfrac{1}{2}(\lambda_j+\lambda_{j+1})\right],\ 0 \ge C\cdot\theta'_\delta \operatorname{Im} E > -a_j d_j^2 |\sin \operatorname{Re}\theta(E)|\right\}, \quad (3\text{-}1)$$

where the map $E \mapsto \theta(E)$ is as defined in Section 2 and we have set

$$d_j := \min(\lambda_{j+1}-\lambda_j,\ \lambda_j-\lambda_{j-1},\ 1) \quad \textit{and} \quad \theta'_\delta := \max_{|E|\le 2-\delta/2} |\theta'(E)|. \quad (3\text{-}2)$$

In Theorem 3.1 there are no conditions on the numbers $(a_j)_j$ or $(d_j)_j$ except their being positive. In our application to resonances, this holds. Theorem 3.1 becomes optimal when $a_j \ll d_j^2$. In our application to resonances, for periodic operators one has $a_j \asymp L^{-1}$ and $d_j \asymp L^{-1}$ (see Theorem 5.2), and for random operators one has $a_j \asymp e^{-cL}$ and $d_j \gtrsim L^{-4}$ (see Theorem 6.4 and (6-10)). Thus, in the random case Theorem 3.1 will provide an optimal strip free of resonances, whereas in the periodic case we will use a much more precise computation (see Theorem 5.1) to obtain sharp results.

When $a_j \ll d_j^2$, one proves the existence of another resonant-free region near a energy λ_j, namely:

Theorem 3.2. *Fix $\delta > 0$. Pick $j \in \{0, \dots, L\}$ such that $-4+\delta < \lambda_{j-1}+\lambda_j < \lambda_{j+1}+\lambda_j < 4-\delta$. There exists $C > 0$ (depending only on δ) such that, for any L, if $a_j \le d_j^2/C^2$ then equations (2-4) and (2-8) have no solution in the set* (see Figure 7)

$$\widetilde{U}_j := \left\{E\in\mathbb{C} \,\middle|\, \operatorname{Re} E \in \left[\tfrac{1}{2}(\lambda_j+\lambda_{j-1}), \lambda_j - Ca_j\right] \cup \left[\lambda_j + Ca_j, \tfrac{1}{2}(\lambda_j+\lambda_{j+1})\right],\ -Ca_j \le \operatorname{Im} E \le -\frac{a_j d_j^2}{C}\right\}$$
$$\cup \left\{E \in \mathbb{C} \,\middle|\, \operatorname{Re} E \in \left[\tfrac{1}{2}(\lambda_j+\lambda_{j-1}), \tfrac{1}{2}(\lambda_j+\lambda_{j+1})\right],\ -\frac{d_j^2}{C} \le \operatorname{Im} E \le -Ca_j\right\}. \quad (3\text{-}3)$$

Theorem 3.2 becomes optimal when a_j is small and d_j is of order one. This will be sufficient to deal with the isolated eigenvalues for both the periodic and the random potential. It will also be sufficient to give a sharp description of the resonant-free region for random potentials. For the periodic potential, we will rely on much more precise computations (see Theorem 5.1).

Note that Theorem 3.2 guarantees that, if d_j is not too small, outside R_j (see Theorem 3.3) resonances are quite far below the real axis.

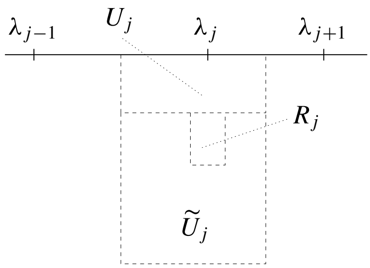

Figure 7. The resonance-free zones U_j and $\widetilde{U}_j$.

Proof of Theorem 3.1. The basic idea of the proof is that, for E close to λ_j, $S_L(E)$ and the matrix $\Gamma_L(E)$ are either large or have a very small imaginary part while, as $-4 < \lambda_{j-1}+\lambda_j < \lambda_{j+1}+\lambda_j < 4$, $e^{-i\theta(E)}$ has a large imaginary part. Thus, (2-4) and (2-8) have no solution in this region.

We start with (2-4). Pick $E \in U_j$ for some C large to be chosen later on. Assume first that $|E-\lambda_j| \le a_j d_j (2+C_0 d_j)^{-1}$ for $C_0 := 2e^{1/C}$. Recall that $0 < a_j, d_j \le 1$. Note that, for C sufficiently large, for $E \in U_j$, one has

$$|\operatorname{Im} e^{-i\theta(E)}| = e^{\operatorname{Im}\theta(E)}|\sin\operatorname{Re}\theta(E)| = e^{\operatorname{Im}[\theta(E)-\theta(\operatorname{Re} E)]}|\sin\operatorname{Re}\theta(E)| \\ \ge e^{\theta'_\delta \operatorname{Im} E}|\sin\operatorname{Re}\theta(E)| \ge e^{-1/C}|\sin\operatorname{Re}\theta(E)| \tag{3-4}$$

and

$$|e^{-i\theta(E)}| \le 1 \le e^{1/C}. \tag{3-5}$$

One estimates

$$|S_L(E)| \ge \frac{a_j}{|\lambda_j - E|} - \sum_{k\ne j}\frac{a_k}{|\lambda_k - E|} \ge \frac{2}{d_j} + C_0 - \sum_{k\ne j}\frac{2a_k}{\min_{k\ne j}|\lambda_k-\lambda_j|} \ge C_0 = 2e^{1/C}. \tag{3-6}$$

Thus, comparing (3-6) and (3-5), we see that (2-4) has no solution in $U_j \cap \{|E-\lambda_j| \le a_j d_j (2+Cd_j)^{-1}\}$.

Assume now that $|E-\lambda_j| > a_j d_j (2+C_0 d_j)^{-1}$. Then, for $E \in U_j$, one has

$$|\operatorname{Im} E| \le \frac{1}{\theta'_\delta C} a_j d_j^2 |\sin\operatorname{Re}\theta(E)|. \tag{3-7}$$

Thus, for $E \in U_j \cap \{|E-\lambda_j| > a_j d_j (2+C_0 d_j)^{-1}\}$, one computes

$$\begin{aligned} |\operatorname{Im} S_L(E)| &\le |\operatorname{Im} E|\left(\frac{a_j}{|\lambda_j - \operatorname{Re} E|^2 + |\operatorname{Im} E|^2} + \frac{4}{d_j^2 + |\operatorname{Im} E|^2}\right) \\ &\le \frac{1}{\theta'_\delta C} a_j d_j^2 |\sin(\operatorname{Re}\theta(E))| \left(\frac{(2+C_0 d_j)^2 a_j}{a_j^2 d_j^2} + \frac{4}{d_j^2}\right) \\ &\le \frac{4}{\theta'_\delta C}(1+e^{1/C})^2 |\sin(\operatorname{Re}\theta(E))| \le \frac{1}{2} e^{-1/C} |\sin(\operatorname{Re}\theta(E))| \end{aligned} \tag{3-8}$$

provided C satisfies $8e^{1/C}(1+e^{1/C})^2 < \theta'_\delta C$.

Hence, the comparison of (3-4) with (3-8) shows that (2-4) has no solution in

$$U_j \cap \{|E-\lambda_j| > a_j d_j (2+C_0 d_j)^{-1}\}$$

if we choose C large enough (independent of $(a_j)_j$ and $(\lambda_j)_j$). Thus, we have proved that, for some $C>0$ large enough (independent of $(a_j)_j$ and $(\lambda_j)_j$), (2-4) has no solution in U_j.

Let us now turn to the case of (2-8). The basic ideas are the same as for (2-4). Consider the matrix $\Gamma_L(E)$ defined by (2-9). The summands in (2-9) are hermitian, of rank 1, and their norm is given by (2-13).

Assume that $E \in U_j$ is a solution to (2-8). Define the vectors

$$v_j := a_j^{-1/2} \begin{pmatrix} \varphi_j(L) \\ \varphi_j(0) \end{pmatrix} \quad \text{for } j \in \{0, \dots, L\}.$$

Here, $a_j = a_j^{\mathbb{Z}}$.

Note that, by definition of a_j, one has $\|v_j\|^2 = 2$. Pick u in C^2 a normalized eigenvector of $\Gamma_L(E)$ associated to the eigenvalue $-e^{-i\theta(E)}$. Thus, u satisfies

$$\sum_{j=0}^{L} \frac{a_j \langle v_j, u\rangle v_j}{\lambda_j - E} = -e^{-i\theta(E)} u. \tag{3-9}$$

Note that, by assumption, one has

$$\sup_{E\in U_j} \left\| \sum_{k\neq j} \frac{a_k \langle v_k, u\rangle v_k}{\lambda_k - E} \right\| \lesssim \frac{1}{d_j} \quad \text{and} \quad \left| \operatorname{Im}\left(\sum_{k\neq j} \frac{a_k |\langle v_k, u\rangle|^2}{\lambda_k - E} \right) \right| \lesssim \frac{|\operatorname{Im} E|}{d_j^2}, \tag{3-10}$$

where the constants are independent of C, the one defining U_j.

Taking the (real) scalar product of (3-9) with $\bar{u}$, and then the imaginary part, we obtain

$$-\frac{a_j |\langle v_j, u\rangle|^2 \operatorname{Im} E}{|\lambda_j - E|^2} + \operatorname{Im}(e^{-i\theta(E)}) = O\left(\frac{|\operatorname{Im} E|}{d_j^2} \right).$$

Thus, for $E \in U_j$, as $a_j \le 1$, for C in (3-1) sufficiently large (depending only on δ),

$$\frac{a_j |\langle v_j, u\rangle|^2 |\operatorname{Im} E|}{|\lambda_j - E|^2} \ge \frac{1}{2} |\sin \operatorname{Re} \theta(E)|.$$

Hence, for a solution to (2-8) in U_j and u as above, one has

$$|\lambda_j - E| \le |\langle v_j, u\rangle| \sqrt{\frac{2a_j |\operatorname{Im} E|}{|\sin \operatorname{Re} \theta(E)|}} \le 2 \sqrt{\frac{a_j |\operatorname{Im} E|}{|\sin \operatorname{Re} \theta(E)|}}.$$

Hence, by the definition of U_j, for C large we get

$$\left| \frac{a_j}{\lambda_j - E} \right| \ge \frac{C\theta'_\delta}{d_j} \gg \frac{1}{d_j}. \tag{3-11}$$

By (3-10), the operator $\Gamma_L(E)$ can be written as

$$\Gamma_L(E) = \frac{a_j}{\lambda_j - E} v_j \otimes v_j + R_j(E) + i I_j(E), \tag{3-12}$$

where $R_j(E)$ and $I_j(E)$ are selfadjoint (I_j is nonnegative) and satisfy

$$\|R_j(E)\| \lesssim \frac{1}{d_j} \quad \text{and} \quad \|I_j(E)\| \lesssim \frac{|\text{Im}\, E|}{d_j^2}. \tag{3-13}$$

An explicit computation shows that the eigenvalues of the two-by-two matrix $\frac{a_j}{\lambda_j - E} v_j \otimes v_j + R_j(E)$ satisfy

$$\lambda = \frac{a_j}{\lambda_j - E}\left(1 + O\left(\frac{d_j}{C\theta'_\delta}\right)\right) \quad \text{or} \quad |\text{Im}\, \lambda| \lesssim \frac{|\text{Im}\, E|}{a_j},$$

where the implicit constants are independent of the one defining U_j.

Thus, by (3-12), using (3-11) and the second estimate in (3-13), we see that the eigenvalues of the matrix $\Gamma_L(E)$ satisfy

$$\lambda = \frac{a_j}{\lambda_j - E}\left(1 + O\left(\frac{d_j}{C\theta'_\delta}\right)\right) \quad \text{or} \quad |\text{Im}\, \lambda| \le \frac{2}{C\theta'_\delta}.$$

Clearly, for C large, no such value can be equal to $-e^{-i\theta(E)}$, being too large — by (3-11) — in the first case or having too small imaginary part in the second. The proof of Theorem 3.1 is complete. □

Proof of Theorem 3.2. Again, we start with the solutions to (2-4). For $z \in \widetilde{U}_j$, we compute

$$\text{Im}\, S_L(E) = \sum_{k=0}^{L} \frac{a_k \,\text{Im}\, E}{(\lambda_k - \text{Re}\, E)^2 + \text{Im}^2 E} = \frac{a_j \,\text{Im}\, E}{(\lambda_j - \text{Re}\, E)^2 + \text{Im}^2 E} + \sum_{\substack{0 \le k \le L \\ k \ne j}} \frac{-a_k \,\text{Im}\, E}{(\lambda_k - \text{Re}\, E)^2 + \text{Im}^2 E}. \tag{3-14}$$

When $-d_j^2/C \le \text{Im}\, E \le -Ca_j$, the second equality above and (2-5) yield, for C sufficiently large,

$$0 \le \text{Im}\, S_L(E) \lesssim \frac{a_j}{|\text{Im}\, E|} + \frac{|\text{Im}\, E|}{d_j^2 + \text{Im}^2 E} \le \frac{2}{C}. \tag{3-15}$$

On the other hand, for some $K > 0$, one has

$$|\text{Im}\, e^{-i\theta(E)}| \ge |\text{Im}\, e^{-i\theta(\text{Re}\, E)}| - \frac{K d_j^2}{C}.$$

Now, since under the assumptions of Theorem 3.2 one has

$$\min_{E \in [(\lambda_j + \lambda_{j-1})/2, (\lambda_j + \lambda_{j+1})/2]} |\text{Im}\, e^{-i\theta(E)}| \ge \tfrac{1}{4} \min\left(\sqrt{16 - (\lambda_j + \lambda_{j-1})^2}, \sqrt{16 - (\lambda_j + \lambda_{j+1})^2}\right), \tag{3-16}$$

we obtain that (2-4) has no solution in $\widetilde{U}_j \cap \{-d_j/C \le \text{Im}\, E \le -Ca_j\}$.

Now pick $E \in \widetilde{U}_j$ such that $-Ca_j \le \text{Im}\, E \le -a_j d_j^2/C$. Then (3-5) and (2-5) yield, for C sufficiently large,

$$\text{Im}\, S_L(E) \lesssim \frac{a_j \,\text{Im}\, E}{C^2 a_j^2 + \text{Im}^2 E} + \frac{Ca_j}{d_j^2} \le \frac{1}{C} + \frac{1}{2C}.$$

The imaginary part of $e^{-i\theta(E)}$ is estimated as above. Thus, for C sufficiently large, (2-4) has no solution in $\widetilde{U}_j \cap \{-Ca_j \le \operatorname{Im} E \le -a_j d_j^2/C\}$.

The case of (2-8) is studied in exactly the same way except that, as in the proof of Theorem 3.1, one has to replace the study of $S_L(E)$ by that of $\langle \Gamma_L(E)u, u\rangle$ for u a normalized eigenvector of $\Gamma_L(E)$ associated to $-e^{-i\theta(E)}$ and, thus, the coefficient a_k in (3-14) gets multiplied by a factor $|\langle v_k, u\rangle|^2$ that is bounded by 2.

This completes the proof of Theorem 3.2. □

3B. *The resonances near an "isolated" eigenvalue.* We will now solve (2-4) near a given λ_j under the additional assumptions that $a_j \ll d_j^2$. By Theorems 3.1 and 3.2, we will do so in the rectangle R_j (see Theorem 3.3 and Figure 7). Actually, we prove that in R_j there is exactly one resonance and give an asymptotic for this resonance in terms of a_j, d_j and λ_j. This result is going to be applied to the case of random V and to that of isolated eigenvalues (for any V).

Using the notations of Section 3, for $j \in \{0, \dots, L\}$ we define

$$S_{L,j}(E) := \sum_{k \ne j} \frac{a_k^{\mathbb{N}}}{\lambda_k - E} \quad \text{and} \quad \Gamma_{L,j}(E) := \sum_{k \ne j} \frac{1}{\lambda_k - E} \begin{pmatrix} |\varphi_k(L)|^2 & \overline{\varphi_k(0)}\varphi_k(L) \\ \varphi_k(0)\overline{\varphi_k(L)} & |\varphi_k(0)|^2 \end{pmatrix}. \tag{3-17}$$

We prove:

Theorem 3.3. *Pick $j \in \{0, \dots, L\}$ such that $-4 < \lambda_{j-1} + \lambda_j < \lambda_{j+1} + \lambda_j < 4$. There exists $C > 1$ (depending only on $(\lambda_{j-1} + \lambda_j) + 4$ and $4 - (\lambda_{j+1} + \lambda_j)$) such that, for any L, if $a_j \le d_j^2/C$, (2-4) and (2-8) have exactly one solution in the set*

$$R_j := \left\{ E \in \mathbb{C} \,\middle|\, \operatorname{Re} E \in \lambda_j + Ca_j[-1, 1],\ -Ca_j \le \operatorname{Im} E \le -\frac{a_j d_j^2}{C} \right\}. \tag{3-18}$$

Moreover, the solution to (2-4), say $z_j^{\mathbb{N}}$, satisfies

$$z_j^{\mathbb{N}} = \lambda_j + \frac{a_j^{\mathbb{N}}}{S_{L,j}(\lambda_j) + e^{-i\theta(\lambda_j)}} + O((a_j^{\mathbb{N}} d_j^{-1})^2) \tag{3-19}$$

and the solution to (2-8), say $z_j^{\mathbb{Z}}$, satisfies

$$z_j^{\mathbb{Z}} = \lambda_j + \left\langle \begin{pmatrix} \overline{\varphi_j(L)} \\ \varphi_j(0) \end{pmatrix}, (\Gamma_{L,j}(\lambda_j) + e^{-i\theta(\lambda_j)})^{-1} \begin{pmatrix} \overline{\varphi_j(L)} \\ \varphi_j(0) \end{pmatrix} \right\rangle + O((a_j^{\mathbb{Z}} d_j^{-1})^2). \tag{3-20}$$

Note that, if $a_j^{\mathbb{N}} d_j^{-2}$ is small, (3-19) gives the asymptotic of the width of the solution $z_j^{\mathbb{N}}$, namely,

$$\operatorname{Im} z_j^{\mathbb{N}} = \frac{a_j^{\mathbb{N}} \sin\theta(\lambda_j)}{[S_{L,j}(\lambda_j) + \cos\theta(\lambda_j)]^2 + \sin^2\theta(\lambda_j)} (1 + o(1)). \tag{3-21}$$

Recall that $\sin\theta(\lambda_j) < 0$ (see Theorem 2.2). For $H_L^{\mathbb{Z}}$, using the bounds (3-28) and (3-29), we see that the asymptotic of the imaginary part of the solution $z_j^{\mathbb{Z}}$ satisfies

$$-\frac{1}{C}a_j^{\mathbb{Z}} \le \operatorname{Im} z_j^{\mathbb{Z}} \le -Ca_j^{\mathbb{Z}}d_j^2. \tag{3-22}$$

This and (3-21) will be useful when $a_j^\bullet \ll d_j^2$, as will be the case for random potentials. The case when $a_j^\bullet$ and d_j are of the same order of magnitude requires more information. This is the case that we meet in the next section when dealing with periodic potentials.

The proof of Theorem 3.3 also yields the behavior of the functions $E \mapsto S_L(E) + e^{-i\theta(E)}$ and $E \mapsto \det(\Gamma_L(E) + e^{-i\theta(E)})$ near their zeros in R_j and, in particular, shows the following:

Proposition 3.4. *Fix $\delta > 0$. Under the assumptions of Theorem 3.3, there exists $c > 0$ such that, for $-4+\delta < \lambda_{j-1} + \lambda_j < \lambda_{j+1} + \lambda_j < 4 - \delta$, one has*

$$\inf_{0<r<ca_j^{\mathbb{N}}d_j^{-1}} \min_{|E-z_j^{\mathbb{N}}|=r} \frac{|S_L(E)+e^{-i\theta(E)}|}{r} \ge c \quad \textit{and} \quad \inf_{0<r<ca_j^{\mathbb{Z}}d_j^{-1}} \min_{|E-z_j^{\mathbb{Z}}|=r} \frac{|\det(\Gamma_L(E)+e^{-i\theta(E)})|}{r} \ge c.$$

Proposition 3.4 is a consequence of the analogues of (3-24) and (3-30) on the rectangles

$$\widetilde{R}_j = \tilde{z}_j + ca_j^\bullet d_j^{-1}[-1,1]\times[-1,1]$$

for $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$ and c sufficiently small.

Proof of Theorem 3.3. Let us start with (2-4). To prove the statement in (2-4), in R_j we compare the function $E \mapsto S_L(E) + e^{-i\theta(E)}$ to the function

$$E \mapsto \tilde{S}_{L,j}(E) = \frac{a_j^{\mathbb{N}}}{\lambda_j - E} + S_{L,j}(\lambda_j) + e^{-i\theta(\lambda_j)}.$$

Clearly, in $\mathbb{C}$, the equation $\tilde{S}_{L,j}(E) = 0$ admits a unique solution, given by

$$\tilde{z}_j = \lambda_j + \frac{a_j^{\mathbb{N}}}{S_{L,j}(\lambda_j) + e^{-i\theta(\lambda_j)}}.$$

For $E \in \partial R_j$, the boundary of R_j, one has

$$\begin{aligned} |\tilde{S}_{L,j}(E)| \ge \frac{1}{2C} \quad &\text{and} \quad \left|\frac{a_j^{\mathbb{N}}}{\lambda_j - E}\right| \ge \frac{1}{2C}, \\ |e^{-i\theta(E)} - e^{-i\theta(\lambda_j)}| \le Ca_j^{\mathbb{N}} \quad &\text{and} \quad |S_{L,j}(E) - S_{L,j}(\lambda_j)| \le Ca_j^{\mathbb{N}}d_j^{-2}. \end{aligned} \tag{3-23}$$

Hence, as $d_j \le 1$, one gets

$$\max_{E\in\partial R_j} \frac{|\tilde{S}_{L,j}(E) - S_L(E) - e^{-i\theta(E)}|}{|\tilde{S}_{L,j}(E)|} \le 4Ca_j^{\mathbb{N}}d_j^{-2}.$$

Thus, by Rouché's theorem, (2-4) has a unique solution in R_j.

To obtain the asymptotics of the solution, it suffices to use Rouché’s theorem again with the functions $\tilde{S}_{L,j}$ and $S_L(E)+e^{-i\theta(E)}$ on the smaller rectangle $\widetilde{R}_j = \tilde{z}_j + K(a_j^{\mathbb{N}} d_j^{-1})^2[-1,1]\times[-1,1]$. One then estimates

$$\max_{E\in\partial\widetilde{R}_j} \frac{|\tilde{S}_{L,j}(E) - S_L(E) - e^{-i\theta(E)}|}{|\tilde{S}_{L,j}(E)|} \le 4CK^{-1}. \tag{3-24}$$

Thus, for K sufficiently large, this completes the proof of the statements on the solutions to (2-4) contained in Theorem 3.3.

Let us turn to (2-8). On R_j, we now compare $\Gamma_L(E)+e^{-i\theta(E)}$ to the matrix-valued function

$$E \mapsto \widetilde{\Gamma}_{L,j}(E) := \frac{1}{\lambda_j - E}\begin{pmatrix} |\varphi_j(L)|^2 & \overline{\varphi_j(0)}\varphi_j(L) \\ \varphi_j(0)\overline{\varphi_j(L)} & |\varphi_j(0)|^2 \end{pmatrix} + \Gamma_{L,j}(\lambda_j) + e^{-i\theta(\lambda_j)}.$$

The matrix

$$\begin{pmatrix} |\varphi_j(L)|^2 & \overline{\varphi_j(0)}\varphi_j(L) \\ \varphi_j(0)\overline{\varphi_j(L)} & |\varphi_j(0)|^2 \end{pmatrix}$$

has rank 1 and can be diagonalized as

$$\begin{pmatrix} |\varphi_j(L)|^2 & \overline{\varphi_j(0)}\varphi_j(L) \\ \varphi_j(0)\overline{\varphi_j(L)} & |\varphi_j(0)|^2 \end{pmatrix} = P_j \begin{pmatrix} a_j^{\mathbb{Z}} & 0 \\ 0 & 0 \end{pmatrix} P_j^*,$$

where $a_j^{\mathbb{Z}}$ is given by (2-13) and

$$P_j = \frac{1}{\sqrt{a_j^{\mathbb{Z}}}}\begin{pmatrix} \varphi_j(L) & -\overline{\varphi_j(0)} \\ \varphi_j(0) & \overline{\varphi_j(L)} \end{pmatrix}.$$

Thus, $\widetilde{\Gamma}_{L,j}(E)$ is unitarily equivalent to

$$M := \frac{1}{\lambda_j - E}\begin{pmatrix} a_j^{\mathbb{Z}} & 0 \\ 0 & 0 \end{pmatrix} + P_j^*\Gamma_{L,j}(\lambda_j)P_j + e^{-i\theta(\lambda_j)}. \tag{3-25}$$

As $P_j^*\Gamma_{L,j}(\lambda_j)P_j$ is real and the imaginary part of $e^{-i\theta(\lambda_j)}$ does not vanish, $M_0 := P_j^*\Gamma_{L,j}(\lambda_j)P_j + e^{-i\theta(\lambda_j)}$ is invertible. By rank-1 perturbation theory (see, e.g., [Simon 2005]), we know that M is invertible if and only if $a_j^{\mathbb{Z}}[M_0^{-1}]_{11} + \lambda_j \ne E$ (where $[M]_{11}$ is the upper right coefficient of the 2×2 matrix M). In this case, one has

$$M^{-1} = M_0^{-1} - \frac{a_j^{\mathbb{Z}}}{a_j^{\mathbb{Z}}[M_0^{-1}]_{11} + \lambda_j - E} M_0^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_0^{-1}. \tag{3-26}$$

Hence, 0 is an eigenvalue of M if and only if

$$\begin{aligned} E &= \lambda_j + a_j^{\mathbb{Z}}[(P_j^*\Gamma_{L,j}(\lambda_j)P_j + e^{-i\theta(\lambda_j)})^{-1}]_{11} \\ &= \lambda_j + \left\langle \begin{pmatrix} \overline{\varphi_j(L)} \\ \varphi_j(0) \end{pmatrix}, (\Gamma_{L,j}(\lambda_j) + e^{-i\theta(\lambda_j)})^{-1}\begin{pmatrix} \overline{\varphi_j(L)} \\ \varphi_j(0) \end{pmatrix} \right\rangle. \end{aligned} \tag{3-27}$$

Note that, as $\Gamma_{L,j}(\lambda_j)$ is real symmetric and $\|\Gamma_{L,j}(\lambda_j)\| \le C d_j^{-1}$, one has

$$\left|\left\langle \begin{pmatrix} \overline{\varphi_j(L)} \\ \varphi_j(0) \end{pmatrix}, (\Gamma_{L,j}(\lambda_j)+e^{-i\theta(\lambda_j)})^{-1} \begin{pmatrix} \overline{\varphi_j(L)} \\ \varphi_j(0) \end{pmatrix} \right\rangle\right| \le \frac{a_j^{\mathbb{Z}}}{|\sin\theta(\lambda_j)|} \tag{3-28}$$

and

$$\operatorname{Im}\left(\left\langle \begin{pmatrix} \overline{\varphi_j(L)} \\ \varphi_j(0) \end{pmatrix}, (\Gamma_{L,j}(\lambda_j)+e^{-i\theta(\lambda_j)})^{-1} \begin{pmatrix} \overline{\varphi_j(L)} \\ \varphi_j(0) \end{pmatrix} \right\rangle\right) \le \frac{a_j^{\mathbb{Z}} d_j^2 \sin\theta(\lambda_j)}{1+d_j^2}. \tag{3-29}$$

Using (3-25), (3-26), (3-28) and (3-29),we see that, for $E \in \partial R_j$, the boundary of R_j, $\widetilde{\Gamma}_{L,j}(E)$ is invertible and that one has

$$\|[\widetilde{\Gamma}_{L,j}(E)]^{-1}\| \le 2C \quad \text{and} \quad \|\Gamma_{L,j}(E)-\Gamma_{L,j}(\lambda_j)\| \le C a_j^{\mathbb{Z}} d_j^{-2}.$$

Hence, as $d_j \le 1$, taking (3-23) into account, one gets

$$\max_{E\in\partial R_j} \|1-[\widetilde{\Gamma}_{L,j}(E)]^{-1}(\Gamma_L(E)+e^{-i\theta(E)})\| \le 4C^2 a_j^{\mathbb{Z}} d_j^{-2}.$$

In the same way, one proves

$$\max_{E\in\partial \widetilde{R}_j} \|1-[\widetilde{\Gamma}_{L,j}(E)]^{-1}(\Gamma_L(E)+e^{-i\theta(E)})\| \lesssim K^{-1}, \tag{3-30}$$

where we recall that $\widetilde{R}_j = \tilde{z}_j + K(a_j^{\mathbb{N}} d_j^{-1})^2[-1,1]\times[-1,1]$.

Thus, we can apply Rouché's theorem to compare the following two functions on ∂R_j and $\partial\widetilde{R}_j$ (for K sufficiently large):

$$\det(\widetilde{\Gamma}_{L,j}(E)) \quad \text{and} \quad \det(\Gamma_L(E)+e^{-i\theta(E)}),$$

as

$$\frac{|\det(\widetilde{\Gamma}_{L,j}(E))-\det(\Gamma_L(E)+e^{-i\theta(E)})|}{|\det(\widetilde{\Gamma}_{L,j}(E))|} = \left|1-\det\left(1-\left[1-[\widetilde{\Gamma}_{L,j}(E)]^{-1}(\Gamma_L(E)+e^{-i\theta(E)})\right]\right)\right|.$$

We then conclude as in the case of (2-4). This completes the proof of Theorem 3.3. □

Combining Theorems 3.3, 3.1 and 3.2, we get a pretty clear picture of the resonances near the Dirichlet eigenvalues in $(-2,2)$ as long as the associated $a_j^\bullet$ and d_j behave correctly. As said, this and the knowledge of the spectral statistics for random operators will enable us to prove the results described in Section 1C. For the periodic case, Theorems 3.1, 3.2 and 3.3 will prove not to be sufficient. As we shall see, in this case, $a_j^\bullet$ and d_j are of the same order of magnitude. Thus, neighboring Dirichlet eigenvalues have a sizable effect on the location of resonances. Therefore, in the next section, we compute the Dirichlet spectral data for the truncated periodic potential.

4. The Dirichlet spectral data for periodic potentials

As we did not find any suitable reference for this material, we first derive a suitable description of the spectral data (i.e., the $(a_j^\bullet)_j$ and $(\lambda_j)_j$) for the Dirichlet restriction of a periodic operator to the interval $[\![0,L]\!]$ when L becomes large.

Consider a potential $V:\mathbb{N}\to\mathbb{R}$ such that, for some $p\geq 1$, one has $V_k=V_{k+p}$ for all $k\geq 0$. We assume p to be minimal, i.e., to be the period of V. In our first result, we describe the spectrum of $H^{\mathbb{Z}}=-\Delta+V$ on $\ell^2(\mathbb{Z})$ and $H^{\mathbb{N}}=-\Delta+V$ on $\ell^2(\mathbb{N})$ (with Dirichlet boundary conditions at 0). In the second result we turn to H_L, the Dirichlet restriction $H^{\mathbb{N}}$ to $[\![0, L]\!]$ and describe its spectral data, i.e., its eigenvalues and eigenfunctions.

We recall:

Theorem 4.1. *The spectrum of $H^{\mathbb{Z}}$, say $\Sigma_{\mathbb{Z}}$, is a union of at most p disjoint intervals that all consist in purely absolutely continuous spectrum.*

The spectrum of $H^{\mathbb{N}}$ is the union of $\Sigma_{\mathbb{Z}}$ and at most finitely many simple eigenvalues outside $\Sigma_{\mathbb{Z}}$, say $(v_j)_{0\leq j\leq n}$. $\Sigma_{\mathbb{Z}}$ consists of purely absolutely continuous spectrum and the eigenfunctions associated to $(v_j)_{0\leq j\leq n}$, say $(\psi_j)_{0\leq j\leq n}$, are exponentially decaying at infinity.

Except for the exponential decay of the eigenfunctions, the proof of the statement for the periodic operator on $\mathbb{Z}$ and $\mathbb{N}$ is classical and can, e.g., be found in a more general setting in [Teschl 2000, Chapters 2, 3 and 7] (see also [van Moerbeke 1976; Reed and Simon 1980]). The exponential decay is an immediate consequence of Floquet theory for the periodic Hamiltonian on $\mathbb{Z}$ and the fact that the eigenvalues lie in gaps of $\Sigma_{\mathbb{Z}}$.

For $H^{\mathbb{Z}}$, one can define its Bloch quasimomentum (see the beginning of Section 4A for details), which we denote by θ_p; it is continuous and strictly increasing on $\Sigma_{\mathbb{Z}}$ and real analytic on $\Sigma_{\mathbb{Z}}^{\circ}$, the interior of $\Sigma_{\mathbb{Z}}$. Decompose $\Sigma_{\mathbb{Z}}$ into its connected components, i.e., $\Sigma_{\mathbb{Z}}=\bigcup_{r=1}^{q} B_r$, where $q\leq p$. Let c_q be the number of closed gaps contained in q. Then θ_p is continuous and strictly increasing on B_r and real analytic on B_r°, the interior of the r-th band. Moreover, on this set, its derivative can be expressed in terms of the density of states, defined in (1-2) as

$$n(\lambda)=\frac{1}{\pi}\theta_p'(\lambda). \tag{4-1}$$

We first describe the eigenvalues of H_L.

Theorem 4.2. *One has:*

(1) *For any $k\in\{0,\dots,p-1\}$, there exists $h_k:\Sigma_{\mathbb{Z}}\to\mathbb{R}$, a continuous function that is real analytic in a neighborhood of $\Sigma_{\mathbb{Z}}^{\circ}$ such that, for L sufficiently large with $L\equiv k \bmod p$,*

   (a) *for $1\leq r\leq q$, the function h_k maps B_r into $(-(c_r+1)\pi,(c_r+1)\pi)$;*

   (b) *the function*

$$\theta_{p,L}:=\theta_p-\frac{h_k}{L-k} \tag{4-2}$$

   is continuous and strictly monotonous on each B_r $(1\leq r\leq q)$;

   (c) *for $1\leq r\leq q$, the eigenvalues of H_L in B_r, the r-th band of $\Sigma_{\mathbb{Z}}$, say $(\lambda_j^r)_j$, are the solutions (in $\Sigma_{\mathbb{Z}}$) to the quantization conditions*

$$\theta_{p,L}(\lambda_j^r)=\frac{j\pi}{L-k},\quad j\in\mathbb{Z}. \tag{4-3}$$

(2) *There exists $c > 0$ such that, if λ is an eigenvalue of H_L outside $\Sigma_{\mathbb{Z}}$, then for $L = Np + k$ sufficiently large there exists $\lambda_\infty \in \Sigma_0^+ \cup \Sigma_k^- \setminus \Sigma_{\mathbb{Z}}$ such that one has $|\lambda - \lambda_\infty| \le e^{-cL}$.*

Recall that Σ_0^+ and Σ_k^- are the spectra of H_0^+ and H_k^-, respectively, defined in Section 1B2.

In Theorem 4.2, when solving (4-3), one has to do it for each band B_r and, for each band and each j such that $j\pi/(L-k) \in \theta_{p,L}(B_r)$, (4-3) admits a unique solution. But, it may happen that one has two solutions to (4-3) for a given j belonging to neighboring bands. In the sequel, to simplify the notations, we will not distinguish between the different bands, i.e., we will write eigenvalues $(\lambda_j)_j$ not referring to the band they belong to.

Let us now describe the associated eigenfunctions.

Theorem 4.3. *Recall that $(\lambda_j)_j$ are the eigenvalues of H_L in $\Sigma_{\mathbb{Z}}$ (enumerated as in Theorem 4.2).*

(1) *There exist $p+2$ positive functions, say f_0^+, $(f_k^-)_{0\le k\le p-1}$ and $\tilde f$, that are real analytic in a neighborhood of $\Sigma_{\mathbb{Z}}^\circ$ such that there exists $\sigma_r \in \{+1, -1\}$ such that, for $L = Np + k$ sufficiently large and λ_j in B_r°, the interior of r-th band of $\Sigma_{\mathbb{Z}}$, one has*

$$|\varphi_l(L)|^2 = \frac{f_k^-(\lambda_j)}{L-k}\left(1 + \frac{\tilde f(\lambda_j)}{L-k}\right)^{-1}, \quad |\varphi_l(0)|^2 = \frac{f_0^+(\lambda_j)}{f_k^-(\lambda_j)}|\varphi_l(L)|^2,$$
$$\text{and}\quad \varphi_l(L)\overline{\varphi_l(0)} = \sigma_r e^{i\pi l}|\varphi_l(L)||\varphi_l(0)| = \sigma_r e^{i(L-k)\theta_p(\lambda_j) - h_k(\lambda_j)}|\varphi_l(L)||\varphi_l(0)|. \tag{4-4}$$

(2) *Let λ be an eigenvalue of H_L outside $\Sigma_{\mathbb{Z}}$ (see Theorem 4.2(2)). If φ is a normalized eigenfunction associated to λ and H_L, one has one of the following alternatives for L large:*

(a) *If $\lambda_\infty \in \Sigma_0^+ \setminus \Sigma_k^-$, one has*

$$|\varphi(L)| \asymp e^{-cL} \quad \text{and} \quad |\varphi(0)| \asymp 1. \tag{4-5}$$

(b) *If $\lambda_\infty \in \Sigma_k^- \setminus \Sigma_0^+$, one has*

$$|\varphi(L)| \asymp 1 \quad \text{and} \quad |\varphi(0)| \asymp e^{-cL}. \tag{4-6}$$

(c) *If $\lambda_\infty \in \Sigma_k^- \cap \Sigma_0^+$, one has*

$$|\varphi(L)| \asymp 1 \quad \text{and} \quad |\varphi(0)| \asymp 1. \tag{4-7}$$

For later use, let us define $\theta_{p,L}$, $f_{0,L}$ and $f_{k,L}$ by

$$f_{k,L}(\lambda) = f_k^-(\lambda)\left(1 + \frac{\tilde f(\lambda)}{L-k}\right)^{-1} \quad \text{and} \quad f_{0,L}(\lambda) = f_0^+(\lambda)\left(1 + \frac{\tilde f(\lambda)}{L-k}\right)^{-1}, \tag{4-8}$$

where θ_p, h_k, f_0, f_k and $\tilde f$ are as defined in Theorem 4.2.

As a consequence of Theorem 4.2, we obtain:

Corollary 4.4. *For $\lambda \in \Sigma_{\mathbb{Z}}^{\circ}$, for $L \equiv k \bmod p$ sufficiently large, one has*

$$\frac{dN_k^-}{d\lambda}(\lambda) = n_k^-(\lambda) = f_k^-(\lambda)n(\lambda) = \frac{1}{\pi} f_k^-(\lambda)\theta_p'(\lambda) = \frac{1}{\pi} f_{k,L}(\lambda)\theta_{p,L}'(\lambda), \tag{4-9}$$

$$\frac{dN_0^+}{d\lambda}(\lambda) = n_0^+(\lambda) = f_0^+(\lambda)n(\lambda) = \frac{1}{\pi} f_0^+(\lambda)\theta_p'(\lambda) = \frac{1}{\pi} f_{0,L}(\lambda)\theta_{p,L}'(\lambda). \tag{4-10}$$

Here, θ_P, f_0^+ and f_k^- are the functions defined in Theorem 4.2.

Proof of Corollary 4.4. To prove the first equalities in (4-9) and (4-10), it suffices to prove that, for any $\chi \in \mathscr{C}_0^\infty(\Sigma_{\mathbb{Z}}^{\circ})$,

$$\langle \delta_0, \chi(H_k^-)\delta_0\rangle = \int_{\mathbb{R}} \chi(\lambda)\, dN_k^-(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}} \chi(\theta_p^{-1}(k)) f_k^-(\theta_p^{-1}(k))\, dk = \frac{1}{\pi}\int_{\mathbb{R}} \chi(\lambda) f_k^-(\lambda)\theta_p'(\lambda)\, d\lambda, \tag{4-11}$$

$$\langle \delta_0, \chi(H_0^+)\delta_0\rangle = \int_{\mathbb{R}} \chi(\lambda)\, dN_0^+(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}} \chi(\theta_p^{-1}(k)) f_0^+(\theta_p^{-1}(k))\, dk = \frac{1}{\pi}\int_{\mathbb{R}} \chi(\lambda) f_0^+(\lambda)\theta_p'(\lambda)\, d\lambda, \tag{4-12}$$

the full statement then following by standard density argument. The operator H_L converges to H_0^+ in the norm resolvent sense. Thus, we know that $\langle \delta_0, \chi(H_0^+)\delta_0\rangle = \lim_{L\to+\infty}\langle \delta_0, \chi(H_L)\delta_0\rangle$. Now, by Theorem 4.2, as χ is supported in $\Sigma_{\mathbb{Z}}^{\circ}$, using the Poisson formula one computes

$$\begin{aligned}\langle \delta_0, \chi(H_L)\delta_0\rangle &= \sum_j \chi(\lambda_j)|\varphi_j(0)|^2 = \frac{1}{L-k}\sum_l \chi\left(\theta_{p,L}^{-1}\left(\frac{l\pi}{L-k}\right)\right) f_{0,L}\left(\theta_{p,L}^{-1}\left(\frac{l\pi}{L-k}\right)\right)\\ &= \frac{1}{L-k}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}} e^{-i2\pi j\lambda}\chi\left(\theta_{p,L}^{-1}\left(\frac{\pi\lambda}{L-k}\right)\right) f_{0,L}\left(\theta_{p,L}^{-1}\left(\frac{\pi\lambda}{L-k}\right)\right) d\lambda\\ &= \frac{1}{\pi}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}} e^{-i2(L-k)j\theta_{p,L}(\lambda)}\chi(\lambda) f_{0,L}(\lambda)\theta_{p,L}'(\lambda)\, d\lambda.\end{aligned}$$

Thus, using the nonstationary phase, i.e., integrating by parts, one gets, for any $N \geq 2$,

$$\begin{aligned}\left|\langle \delta_0, \chi(H_L)\delta_0\rangle - \frac{1}{\pi}\int_{\mathbb{R}} \chi(\lambda) f_{0,L}(\lambda)\theta_{p,L}'(\lambda)\, d\lambda\right| &\leq \sum_{j\geq 1} C_{N,K}\|\chi\|_{\mathscr{C}^N}(|j|(L-k))^{-N}\\ &\leq C_{N,K}\|\chi\|_{\mathscr{C}^N}((L-k))^{-N}.\end{aligned} \tag{4-13}$$

Here we have used the analyticity of the functions $\theta_{p,L}$ and $f_{0,L}$.

To deal with H_k^-, we recall the operator $\widetilde{H}_L$ (which is unitarily equivalent to H_L) defined in Remark 1.6. One has $\langle \delta_L, H_L\delta_L\rangle = \langle \delta_0, \chi(\widetilde{H}_L)\delta_0\rangle$; thus, as H_k^- is the strong resolvent sense limit of $\widetilde{H}_L$, one gets $\langle \delta_0, \chi(H_k^-)\delta_0\rangle = \lim_{L\to+\infty}\langle \delta_L, \chi(H_L)\delta_L\rangle$.

Then (4-11) and (4-12) — and, thus, the first equalities in (4-9) and (4-10) — follow, as $\theta_{p,L}'$, $f_{0,L}$ and $f_{k,L}$ converge (locally uniformly on $\Sigma_{\mathbb{Z}}^{\circ}$) to θ_p', f_0^+ and f_k^-, respectively (see (4-8) and Theorem 4.2).

Let us now prove the second equalities in (4-9) and (4-10). To this end, we use an *almost analytic extension* (see [Mather 1971]) of χ, say $\tilde{\chi}$, that is, a function $\tilde{\chi} : \mathbb{C} \to \mathbb{C}$ satisfying

(1) $\tilde{\chi}(z) = \chi(z)$ for $z \in \mathbb{R}$,

(2) $\operatorname{supp}(\tilde{\chi}) \subset \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < 1\}$,

(3) $\tilde{\chi} \in \mathcal{S}(\{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < 1\})$,

(4) the family of functions $x \mapsto \partial\tilde{\chi}(x+iy)/\partial\bar{z} \cdot |y|^{-n}$ (for $0 < |y| < 1$) is bounded in $\mathcal{S}(\mathbb{R})$ for any $n \in \mathbb{N}$.

Moreover, $\tilde{\chi}$ can be chosen so that one has the following estimates: for $n \geq 0$, $\alpha \geq 0$, $\beta \geq 0$, there exists $C_{n,\alpha,\beta} > 0$ such that

$$\sup_{0<|y|\leq 1}\sup_{x\in\mathbb{R}}\left|x^\alpha \frac{\partial^\beta}{\partial x^\beta}\left(|y|^{-n}\frac{\partial\tilde{\chi}}{\partial\bar{z}}(x+iy)\right)\right| \leq C_{n,\alpha,\beta}\sup_{\beta'\leq n+\beta+2\alpha'\leq\alpha}\sup_{x\in\mathbb{R}}\left|x^{\alpha'}\frac{\partial^{\beta'}\chi}{\partial x^{\beta'}}(x)\right|. \tag{4-14}$$

By the definition of χ, the right-hand side of (4-14) is bounded uniformly in E complex.

Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$ and $\tilde{\chi}$ be an almost analytic extension of $\chi(x)$. Then, by [Helffer and Sjöstrand 1990; Klopp 1995], we know that, for any n and $\omega \in \Omega$,

$$\chi(H_\bullet) = \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\chi}}{\partial\bar{z}}(z)(z-H_\bullet)^{-1}\,dz\wedge d\bar{z}, \tag{4-15}$$

where $H_\bullet$ equals H_L, $\widetilde{H}_L$, H_0^+ or H_k^-.

Using the geometric resolvent equation (see, e.g., [Kirsch 2008, Theorem 5.20]) and the Combes–Thomas estimate (see, e.g., [Kirsch 2008, Theorem 11.2]), we know that for some $C > 0$, for $\operatorname{Im} z \neq 0$,

$$\left|\langle\delta_0, [(\widetilde{H}_L - z)^{-1} - (H_k^- - z)^{-1}]\delta_0\rangle\right| + \left|\langle\delta_0, [(H_L - z)^{-1} - (H_0^+ - z)^{-1}]\delta_0\rangle\right| \leq \frac{C}{|\operatorname{Im} z|}e^{-L|\operatorname{Im} z|/C}. \tag{4-16}$$

Plugging (4-16) into (4-15) and using (4-14), we get

$$\left|\sum_{j=0}^{L}\chi(\lambda_j)|\varphi_j(0)|^2 - \int_{\mathbb{R}}\chi(\lambda)\,dN_0^+(\lambda)\right| \leq \widetilde{C}_N\int_{|y|\leq 1}|y|^{N-1}e^{-L|y|/C}\,dy \leq C_N L^{-N}.$$

Thus, by (4-12) and (4-13), we obtain that, for $\chi \in \mathscr{C}_0^\infty(\Sigma_{\mathbb{Z}}^\circ)$ and any $N \geq 0$, there exists $C_N > 0$ such that

$$\begin{aligned}\left|\int_{\mathbb{R}}\chi(\lambda)[f_{0,L}(\lambda)\theta'_{p,L}(\lambda) - f_0^+(\lambda)\theta'_p(\lambda)]\,d\lambda\right| \\ = \left|\int_{\mathbb{R}}\chi(\lambda)f_{0,L}(\lambda)\theta'_{p,L}(\lambda)\,d\lambda - \int_{\mathbb{R}}\chi(\lambda)\,dN_0^+(\lambda)\right| \leq C_N L^{-N}.\end{aligned} \tag{4-17}$$

Now, by (4-3) and (4-8), the function $f_{0,L}\theta'_{p,L} - f_0^+\theta'_p$ admits an expansion in inverse powers of L that converges uniformly on compact subsets of $\Sigma_{\mathbb{Z}}^\circ$, namely,

$$f_{0,L}\theta'_{p,L} - f_0^+\theta'_p = \sum_{k\geq 1}L^{-k}\alpha_k.$$

Thus, (4-17) immediately yields that, for any $k \geq 1$, one has $\alpha_k \equiv 0$ on $\Sigma_{\mathbb{Z}}^\circ$. Hence, $f_{0,L}\theta'_{p,L} \equiv f_0^+\theta'_p$ on $\Sigma_{\mathbb{Z}}^\circ$. This completes the proof of Corollary 4.4. □

4A. ***The proofs of Theorems 4.2 and 4.3.*** We will first describe some objects from the spectral theory of $H^{\mathbb{Z}}$, use them to describe the spectral theory of $H^{\mathbb{N}}$, prove Theorem 4.2 and finally prove Theorem 4.3.

4A1. *The spectral theory of $H^{\mathbb{Z}}$.* This material is classical (see, e.g., [van Moerbeke 1976; Teschl 2000]); we only recall it for the reader's convenience. For $0 \le j \le p-1$, define $\widetilde{T}_j = \widetilde{T}_j(E)$ to be a monodromy matrix for the periodic finite difference operator $H^{\mathbb{Z}}$, that is,

$$\widetilde{T}_j(E) = T_{j+p-1,j}(E) = T_{j+p-1}(E)\cdots T_j(E) =: \begin{pmatrix} a_p^j(E) & b_p^j(E) \\ a_{p-1}^j(E) & b_{p-1}^j(E) \end{pmatrix}, \tag{4-18}$$

where

$$T_j(E) = \begin{pmatrix} E - V_j & -1 \\ 1 & 0 \end{pmatrix}. \tag{4-19}$$

The coefficients of $\widetilde{T}_j(E)$ are monic polynomials in the energy E; $a_p^j(E)$ has degree p and $b_p^j(E)$ has degree $p-1$. Clearly, $\det \widetilde{T}_j(E) = 1$. As $j \mapsto V_j$ is p-periodic, so is $j \mapsto \widetilde{T}_j(E)$. Moreover, for $j' < j$, one has

$$\widetilde{T}_j(E) T_{j,j}(E) = T_{j+p-1,j'+p-1}(E)\widetilde{T}_{j'}(E) = T_{j,j'}(E)\widetilde{T}_{j'}(E). \tag{4-20}$$

Thus, the discriminant $\underline{\Delta}(E) := \operatorname{tr} \widetilde{T}_j(E) = a_p^j(E) + b_{p-1}^j(E)$ is a polynomial of degree p that is independent of j; so are $\rho(E)$ and $\rho^{-1}(E)$, the eigenvalues of $\widetilde{T}_j(E)$. One defines the Bloch quasimomentum $E \mapsto \theta_p(E)$ by

$$\underline{\Delta}(E) = \rho(E) + \rho^{-1}(E) = 2\cos(p\theta_p(E)). \tag{4-21}$$

Let us recall some basic properties of the discriminant Δ and the coefficients of $\widetilde{T}_j$, the proofs of which can be found in [van Moerbeke 1976]:

(1) If $\Delta'(E) = 0$ then $|\underline{\Delta}(E)| \ge 2$.

(2) The zeros of Δ' are simple.

(3) E is a zero of Δ' such that $|\underline{\Delta}(E)| = 2$ if and only if $\widetilde{T}_j(E) \in \{+\mathrm{Id}, -\mathrm{Id}\}$ (for any j).

(4) The polynomials b_p^j and a_{p-1}^j only vanish in the set $\{|\underline{\Delta}(E)| \ge 2\}$; they keep a fixed sign in each of the connected components of the set $\{|\underline{\Delta}(E)| < 2\}$.

Note that $\underline{\Delta}(E)$ is real when E is real. Thus, for E real, $|\underline{\Delta}(E)| \le 2$ implies that $\rho^{-1}(E) = \overline{\rho(E)}$ and $|\underline{\Delta}(E)| > 2$ implies that $\rho(E)$ is real. When $|\underline{\Delta}(E)| \le 2$ we will fix $\rho(E) := e^{ip\theta_p(E)}$ and when $|\underline{\Delta}(E)| > 2$ we will fix $\rho(E)$ so that $|\rho(E)| < 1$.

E belongs to the spectrum of $H^{\mathbb{Z}}$ (i.e., $-\Delta + V$ on $\ell^2(\mathbb{Z})$) if and only if $|\underline{\Delta}(E)| \le 2$ (see, e.g., [Teschl 2000]).

Properties (1)–(3) above imply that, for E_0 a zero of Δ' such that $\underline{\Delta}(E_0) = \pm 2$, θ_p is real analytic near E_0 and $\theta_p'(E_0) \ne 0$.

Definition 4.5. E_0 is said to be a closed gap if and only if $|\underline{\Delta}(E_0)| = 2$ and $\Delta'(E_0) = 0$ or, equivalently, if and only if $\widetilde{T}_0(E_0)$ is diagonal.

Consider $\partial\Sigma_{\mathbb{Z}}$. It is the set of energies that are solutions to $|\underline{\Delta}(E)|=2$ where $\widetilde{T}_0(E)$ is not diagonal; it is also the set of roots of $|\underline{\Delta}(E)|=2$ that are not closed gaps. From the upper half of the complex plane, one can continue $E\mapsto\theta_p(E)$ analytically to the universal cover of $\mathbb{C}\setminus\partial\Sigma_{\mathbb{Z}}$. Each of the points in $\partial\Sigma_{\mathbb{Z}}$ is a branch point of θ_p of square root type. Moreover, for $E\notin\partial\Sigma_{\mathbb{Z}}$, there exist two linearly independent solutions to the eigenvalue equation $(-\Delta+V-E)u=0$, say $\varphi_\pm(E)$, satisfying, for $n\in\mathbb{Z}$,

$$\varphi_\pm(n+p,E)=e^{\pm ip\theta_p(E)}\varphi_\pm(n,E). \tag{4-22}$$

4A2. *The spectral theory of $H^{\mathbb{N}}$.* Let us now turn to the spectrum of the operator on the half-lattice.

The operator H_0^+. For the operator $H_0^+=H^{\mathbb{N}}$ (that is, $-\Delta+V$ on $\ell^2(\mathbb{N})$ with Dirichlet boundary conditions at 0), E is in the spectrum if and only if

- either $|\underline{\Delta}(E)|\le 2$,
- or $|\underline{\Delta}(E)|>2$ and $[\widetilde{T}_0(E)]^n\binom{1}{0}$ stays bounded as $n\to+\infty$.

The second condition is equivalent to requiring that $[\widetilde{T}_j(E)]^nT_{j-1}(E)\cdots T_0(E)\binom{1}{0}$ stay bounded as $n\to+\infty$.

When $|\underline{\Delta}(E)|\ne 2$ and $a^0_{p-1}(E)\ne 0$, one can diagonalize $\widetilde{T}_0(E)$ in the following way

$$\begin{pmatrix} a^0_{p-1}(E) & \rho(E)-a^0_p(E)\\ -a^0_{p-1}(E) & a^0_p(E)-\rho^{-1}(E)\end{pmatrix}\widetilde{T}_0(E)=\begin{pmatrix}\rho(E) & 0\\ 0 & \rho^{-1}(E)\end{pmatrix}\begin{pmatrix} a^0_{p-1}(E) & \rho(E)-a^0_p(E)\\ -a^0_{p-1}(E) & a^0_p(E)-\rho^{-1}(E)\end{pmatrix}. \tag{4-23}$$

Thus, using

$$\begin{vmatrix}\rho(E)-a^0_p(E) & -b^0_p(E)\\ -a^0_{p-1}(E) & \rho(E)-b^0_{p-1}(E)\end{vmatrix}=\begin{vmatrix}\rho(E)-a^0_p(E) & -b^0_p(E)\\ -a^0_{p-1}(E) & a^0_p(E)-\rho^{-1}(E)\end{vmatrix}=0 \tag{4-24}$$

for $n\in\mathbb{Z}$, one computes

$$(\widetilde{T}_0(E))^n=\begin{pmatrix}\tilde t^{11}_{0,n}(E) & \tilde t^{12}_{0,n}(E)\\ \tilde t^{21}_{0,n}(E) & \tilde t^{22}_{0,n}(E)\end{pmatrix}, \tag{4-25}$$

where

$$\begin{aligned}
\tilde t^{11}_{0,n}(E) &:= \rho^n(E)\frac{a^0_p(E)-\rho^{-1}(E)}{\rho(E)-\rho^{-1}(E)}+\rho^{-n}(E)\frac{\rho(E)-a^0_p(E)}{\rho(E)-\rho^{-1}(E)},\\
\tilde t^{12}_{0,n}(E) &:= (\rho^{-n}(E)-\rho^n(E))\frac{b^0_p(E)}{\rho(E)-\rho^{-1}(E)},\\
\tilde t^{21}_{0,n}(E) &:= (\rho^{n}(E)-\rho^{-n}(E))\frac{a^0_{p-1}(E)}{\rho(E)-\rho^{-1}(E)},\\
\tilde t^{22}_{0,n}(E) &:= \rho^{-n}(E)\frac{a^0_p(E)-\rho^{-1}(E)}{\rho(E)-\rho^{-1}(E)}+\rho^{n}(E)\frac{\rho(E)-a^0_p(E)}{\rho(E)-\rho^{-1}(E)}.
\end{aligned} \tag{4-26}$$

Clearly, the formulas (4-23), (4-25) and (4-26) stay valid even if $a^0_{p-1}(E)=0$. They also stay valid if $|\underline{\Delta}(E)|=2$ and $\Delta'(E)=0$. Indeed, by points (1)–(3) in Section 4A1, the functions $\rho-\rho^{-1}$, $a^0_p-\rho^{-1}$, $-\rho-a^0_p$, b^0_p and a^0_{p-1} are analytic near and have simple zeros at such points.

We have thus proved:

Lemma 4.6. *For $E \notin \partial\Sigma_{\mathbb{Z}}$, $(\widetilde{T}_0(E))^n$ has the form* (4-25)–(4-26)

Simple computations then show that E is in the spectrum of H_0^+ if and only if one of the following conditions is satisfied:

(1) $|\Delta(E)| \le 2$: moreover, the set $\{E \in \mathbb{R} \mid |\Delta(E)| \le 2\}$ is contained in the absolutely continuous spectrum of H_0^+.

(2) $|\Delta(E)| > 2$ and

$$a_{p-1}^0(E) = 0 \quad \text{and} \quad |a_p^0(E)| < 1. \tag{4-27}$$

Thus, on $\Sigma_{\mathbb{Z}}$, the spectrum of H_0^+ is purely absolutely continuous; it does not contain any embedded eigenvalues.

Note that, in case (2), $[\widetilde{T}_0(E)]^n\binom{1}{0}$ actually decays exponentially fast. In this case, E is an eigenvalue associated to the (nonnormalized) eigenfunction $(u_l)_{l\in\mathbb{N}}$, where, for $n \ge 0$ and $j \in \{0, \dots, p-1\}$,

$$u_{np+j}(E) = \left\langle T_{j-1}(E)\cdots T_0(E)\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\rangle \cdot [a_p^0(E)]^n = a_j(E)[a_p^0(E)]^n, \tag{4-28}$$

writing

$$T_{j-1}(E)\cdots T_0(E) =: \begin{pmatrix} a_j(E) & b_j(E) \\ a_{j-1}(E) b_{j-1}(E) & \end{pmatrix}. \tag{4-29}$$

It is well known that, for any j, the zeros of a_j and b_j are simple (see, e.g., [Teschl 2000, Section 4]), and the roots of a_{j+1} (resp. b_{j+1}) interlace those of a_j (resp. b_j). Let E' be an eigenvalue of H_0^+. Differentiating (4-24) at the energy E', we compute

$$b_p^0(E')\frac{da_{p-1}^0}{dE}(E') + (\rho(E') - \rho^{-1}(E'))\frac{d(\rho - a_p^0)}{dE}(E') = 0. \tag{4-30}$$

The eigenvalues of the operator H_k^-. Let us now turn to H_k^-. Recalling (4-29) and using the representation (4-25), we obtain that the eigenvalues of H_k^- outside $\Sigma_{\mathbb{Z}}$ satisfy

$$\begin{pmatrix} \rho(E) - a_p^0(E) & -a_{p-1}^0(E) \\ -b_p^0(E) & a_p^0(E) - \rho^{-1}(E) \end{pmatrix}\begin{pmatrix} a_{k+1}(E) \\ b_{k+1}(E)\end{pmatrix} = 0. \tag{4-31}$$

As for H_0^+, the eigenfunction associated to E and H_k^- decays exponentially fast. Indeed, the eigenvalues of H_k^- in the region $|\Delta(E)| > 2$ can be analyzed in the same way as we analyzed those of H_0^+, i.e., they are the energies such that $[\widetilde{T}_k(E)]^{-n}\binom{0}{1}$ stays bounded; this yields the quantization conditions $b_p^k(E) = 0$ and $|b_{p-1}^k(E)| < 1$. In this case, E is an eigenvalue associated to the (nonnormalized) eigenfunction $(u_l)_{-l\in\mathbb{N}}$, where, for $n \ge 0$ and $k \in \{0, \dots, p-1\}$,

$$u_{-np-k}(E) = b_k(E)[b_{p-1}^k(E)]^{-n}. \tag{4-32}$$

Common eigenvalues to H_0^+ and H_k^-. Assume now that E' is simultaneously an eigenvalue of H_k^- and H_0^+. In this case, one has $a_{p-1}^0(E')=0$, $|a_p^0(E')|<1$ and $b_p^0(E')b_{k+1}(E')=a_{k+1}(E')(\rho^{-1}(E')-\rho(E'))$. So (4-31) (see also (4-30)) becomes

$$\begin{pmatrix} d(\rho-a_p^0)(E')/dE & -da_{p-1}^0(E')/dE \\ -b_p^0(E) & a_p^0(E')-\rho^{-1}(E') \end{pmatrix}\begin{pmatrix} a_{k+1}(E') \\ b_{k+1}(E') \end{pmatrix}=0. \tag{4-33}$$

Hence, the analytic function $E\mapsto a_{k+1}(E)(a_p^0(E)-\rho(E))-b_{k+1}(E)a_{p-1}^0(E)$ has a root of order at least 2 at E'. It also implies that $a_{k+1}(E')\neq 0$. Indeed, if $a_{k+1}(E')=0$, (4-33) implies $b_{k+1}(E')=0$ as $da_{p-1}^0(E')/dE\neq 0$.

Conversely, if $E'\in\sigma(H_0^+)$ is such that $|\underline{\Delta}(E')|>2$ and $E\mapsto a_{k+1}(E)(a_p^0(E)-\rho(E))-b_{k+1}(E)a_{p-1}^0(E)$ has a root of order at least 2 at E', then (4-33) holds and E' is an eigenvalue of H_k^-.

We have thus proved:

Lemma 4.7. *$E_0\in\sigma(H_0^+)\cap\sigma(H_k^-)\setminus\mathbb{Z}$ if and only if $|\underline{\Delta}(E_0)|>2$ and E_0 is a double root of $E\mapsto a_{k+1}(E)(a_p^0(E)-\rho(E))-b_{k+1}(E)a_{p-1}^0(E)$.*

4A3. *The Dirichlet eigenvalues for a periodic potential*: *the proof of Theorem 4.2.* Let us now turn to the study of the eigenvalues and eigenvectors of H_L, i.e., to the proof of Theorem 4.2. We first prove the statements for the eigenvalues and then, in the next section, turn to the eigenvectors.

Recall that $L\equiv k \bmod p$; we write $L=Np+k$. By definition, E is an eigenvalue of $-\Delta+V$ on $[\![0,L]\!]$ with Dirichlet boundary conditions if and only if

$$\begin{aligned} 0&=\det\left(T_{L+1}(E)T_L(E)T_{L-1}(E)\cdots T_0(E)\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right)\\ &=\det\left(T_k(E)\cdots T_0(E)\cdot[\widetilde{T}_0(E)]^N\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right), \end{aligned} \tag{4-34}$$

where $\widetilde{T}_k(E)$ is the monodromy matrix defined above.

We use the notations of sections 4A2 and 4A1. Let us first show Theorem 4.2(1), namely:

Lemma 4.8. *For L large, one has*

$$\partial\Sigma_{\mathbb{Z}}\cap\sigma(H_L)=\{E_0\mid a_{k+1}(E_0)=a_{p-1}^0(E_0)=0 \text{ and } b_p^0(E_0)\neq 0\}.$$

Proof. For $E_0\in\partial\Sigma_{\mathbb{Z}}$, we know that $|\underline{\Delta}(E_0)|=2$ and $\widetilde{T}_0(E_0)$ is not diagonal. Assume $\underline{\Delta}(E_0)=2$ (the case $\underline{\Delta}(E_0)=-2$ is dealt with in the same way); hence, $\widetilde{T}_0(E_0)$ has a Jordan normal form, i.e., there exists a 2×2 invertible matrix P and $\alpha\in\mathbb{R}^*$ such that

$$\widetilde{T}_0(E_0)=P^{-1}\begin{pmatrix}1&0\\ \alpha&1\end{pmatrix}P,\quad\text{where}\quad P=\begin{pmatrix}p_{11}&p_{12}\\p_{21}&p_{22}\end{pmatrix}. \tag{4-35}$$

Thus, by (4-34), $E_0 \in \sigma(H_L)$ if and only if

$$0 = \left| \left(\begin{pmatrix} a_{k+1}(E_0) & b_{k+1}(E_0) \\ a_k(E_0) & b_k(E_0) \end{pmatrix} (\widetilde{T}_0(E_0))^N \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right|$$
$$= \left| \left(\begin{pmatrix} a_{k+1}(E_0) & b_{k+1}(E_0) \\ a_k(E_0) & b_k(E_0) \end{pmatrix} P^{-1} \begin{pmatrix} 1 & 0 \\ N\alpha & 1 \end{pmatrix} P \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right|; \tag{4-36}$$

that is,

$$0 = \left| \left(\begin{pmatrix} 1 & 0 \\ N\alpha & 1 \end{pmatrix} P \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P \begin{pmatrix} -b_{k+1}(E_0) \\ a_{k+1}(E_0) \end{pmatrix} \right) \right| = (\det P) a_{k+1}(E_0) - N\alpha p_{11}(-p_{11} b_{k+1}(E_0) + p_{12} a_{k+1}(E_0)).$$

For N large, this expression vanishes if and only if

$$(\det P) a_{k+1}(E_0) = 0 \quad \text{and} \quad \alpha p_{11}(-p_{11} b_{k+1}(E_0) + p_{12} a_{k+1}(E_0)) = 0.$$

Since P is invertible, $|b_{k+1}(E_0)| + |a_{k+1}(E_0)| \neq 0$ and $\alpha \neq 0$, one has $a_{k+1}(E_0) = 0$ and $p_{11} = 0$.

In this case, using $b_{k+1}(E_0) \neq 0$, we can then rewrite the eigenvalue equation (4-36) as

$$0 = \left| \left((\widetilde{T}_0(E_0))^N \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right| = \tilde{t}^{21}_{0,N}(E_0). \tag{4-37}$$

For $E \in \Sigma^\circ_{\mathbb{Z}}$ close to E_0, by (4-26) we have

$$t^{21}_{0,N}(E) = \frac{(\rho^N(E) - \rho^{-N}(E)) a^0_{p-1}(E)}{\rho(E) - \rho^{-1}(E)} = \rho^{N-1} \left(\sum_{j=0}^{N-1} \rho^{-2j}(E) \right) a^0_{p-1}(E).$$

As ρ is continuous at E_0 and $\rho^2(E_0) = 1$, taking E to E_0 we get

$$a^0_{p-1}(E_0) = 0.$$

As $\widetilde{T}_0(E_0)$ is not diagonal, this implies $b^0_p(E_0) \neq 0$. This completes the proof of Lemma 4.8. □

Now, pick $E \notin \partial\Sigma_{\mathbb{Z}}$. Then, by Lemma 4.6, the quantization condition (4-34) becomes

$$\begin{vmatrix} \rho^N(E) \dfrac{a^0_p(E) - \rho^{-1}(E)}{\rho(E) - \rho^{-1}(E)} + \rho^{-N}(E) \dfrac{\rho(E) - a^0_p(E)}{\rho(E) - \rho^{-1}(E)} & -b_{k+1}(E) \\ (\rho^N(E) - \rho^{-N}(E)) \dfrac{a^0_{p-1}(E)}{\rho(E) - \rho^{-1}(E)} & a_{k+1}(E) \end{vmatrix} = 0. \tag{4-38}$$

The eigenvalues outside of $\Sigma_{\mathbb{Z}}$. Let us first study the eigenvalues outside $\Sigma_{\mathbb{Z}}$, i.e., in the region $|\Delta(E)| > 2$. If, for $j \in \mathbb{N}$, we define

$$\begin{aligned} \alpha_j(E) &:= a_j(E) \frac{a^0_p(E) - \rho^{-1}(E)}{\rho(E) - \rho^{-1}(E)} + b_j(E) \frac{a^0_{p-1}(E)}{\rho(E) - \rho^{-1}(E)} \\ \text{and} \quad \beta_j(E) &:= a_j(E) \frac{\rho(E) - a^0_p(E)}{\rho(E) - \rho^{-1}(E)} - b_j(E) \frac{a^0_{p-1}(E)}{\rho(E) - \rho^{-1}(E)}, \end{aligned} \tag{4-39}$$

(4-38) can be rewritten as $\beta_{k+1}(E)+\rho^{2N}(E)\alpha_{k+1}(E)=0$; using

$$\alpha_{k+1}(E)+\beta_{k+1}(E)=a_{k+1}(E), \tag{4-40}$$

(4-38) becomes

$$\beta_{k+1}(E)=-\frac{\rho^{2N}(E)}{1-\rho^{2N}(E)}a_{k+1}(E). \tag{4-41}$$

We first show:

Lemma 4.9. *There exists $\eta>0$ such that, for L sufficiently large, $\sigma(H_L)\cap[(\Sigma_{\mathbb{Z}}+[-\eta,\eta])\setminus\Sigma_{\mathbb{Z}}]=\varnothing$.*

Proof. Using (4-39), we rewrite (4-41) as

$$a_{k+1}(E)(\rho(E)-a_p^0(E))-b_{k+1}(E)a_{p-1}^0(E)=\rho^{2N+1}(E)\frac{1-\rho^2(E)}{1-\rho^{2N}(E)}a_{k+1}(E). \tag{4-42}$$

Pick $E_0\in\partial\Sigma_Z$. Then, by our choice for ρ, for $\eta>0$ small we know that, for $E\in[E_0-\eta,E_0+\eta]\setminus\Sigma_{\mathbb{Z}}$, $\rho^2(E)=e^{-c_0\sqrt{|E-E_0|}(1+O(\sqrt{|E-E_0|}))}$. Hence, for $E\in[E_0-\eta,E_0+\eta]\setminus\Sigma_{\mathbb{Z}}$, one has

$$\left|\rho^{2N+1}(E)\frac{1-\rho^2(E)}{1-\rho^{2N}(E)}\right|\lesssim\min\Big(\sqrt{|E-E_0|},\ \frac{1}{N}\Big). \tag{4-43}$$

Thus, if $a_{k+1}(E_0)(\rho(E_0)-a_p^0(E_0))-b_{k+1}(E_0)a_{p-1}^0(E_0)\neq0$, (4-42) has no solution in $[E_0-\eta,E_0+\eta]\setminus\Sigma_{\mathbb{Z}}$ for η small and L sufficiently large.

Let us now assume that $a_{k+1}(E_0)(\rho(E_0)-a_p^0(E_0))-b_{k+1}(E_0)a_{p-1}^0(E_0)=0$.

- If $a_{k+1}(E_0)\neq0$, one computes

$$a_{k+1}(E)(\rho(E)-a_p^0(E))-b_{k+1}(E)a_{p-1}^0(E)=a_{k+1}(E_0)(\rho(E)-\rho(E_0))(1+o(1))$$

and

$$\rho^{2N+1}(E)\frac{1-\rho^2(E)}{1-\rho^{2N}(E)}a_{k+1}(E)=-(\rho(E)-\rho(E_0))a_{k+1}(E_0)\frac{\rho^{2(N+1)}(E)}{1-\rho^{2N}(E)}(1+o(1)).$$

Hence, for $\eta>0$ small and $E\in[E_0-\eta,E_0+\eta]\setminus\Sigma_{\mathbb{Z}}$, the two sides of (4-42) have opposite signs; there is no solution to (4-42) in this interval.

- If $a_{k+1}(E_0)=0$, then $b_{k+1}(E_0)\neq0$, $a_{p-1}^0(E_0)=0$, $\rho(E_0)=a_p^0(E_0)$ and $(a_{p-1}^0)'(E_0)\neq0$; one computes

$$a_{k+1}(E)(\rho(E)-a_p^0(E))-b_{k+1}(E)a_{p-1}^0(E)=-b_{k+1}(E_0)(a_{p-1}^0)'(E_0)(E-E_0)(1+o(1))$$

and, by (4-43), for $\eta>0$ small and $E\in[E_0-\eta,E_0+\eta]\setminus\Sigma_{\mathbb{Z}}$,

$$\left|\rho^{2N+1}(E)\frac{1-\rho^2(E)}{1-\rho^{2N}(E)}a_{k+1}(E)\right|\lesssim|E-E_0|\min\Big(\sqrt{|E-E_0|},\ \frac{1}{N}\Big).$$

Hence, for $\eta>0$ small and $E\in[E_0-\eta,E_0+\eta]\setminus\Sigma_{\mathbb{Z}}$, there is no solution to (4-42) in this interval.

This completes the proof of Lemma 4.9. □

In Lemma 4.8, we saw that, if $E_0\in\partial\Sigma_{\mathbb{Z}}$ satisfies

$$a_{k+1}(E_0)=0\quad\text{and}\quad a_{k+1}(E_0)(\rho(E_0)-a_p^0(E_0))-b_{k+1}(E_0)a_{p-1}^0(E_0)=0,$$

then E_0 is an eigenvalue of H_L for L large.

By Lemma 4.9, it now suffices to consider energies such that $|\Delta(E)| > 2+\eta$ for some $\eta > 0$. In this case, we note that the left-hand side in (4-41) is the left-hand side of the first equation in (4-31) (up to the factor $\rho-\rho^{-1}$, which does not vanish outside $\Sigma_{\mathbb{Z}}$). On the other hand, the right-hand side in (4-41) is uniformly exponentially small for large N on $\{E \in \mathbb{R} \mid |\Delta(E)| > 2+\eta\}$. Thus, for L large, the solutions to (4-41) are exponentially close to E', which is either an eigenvalue of H_0^+ or one of H_k^-. One distinguishes between the following cases:

(1) If E' is an eigenvalue of H_0^+ but not of H_k^-, then E' is a simple root of the function $E \mapsto \beta_{k+1}(E)$ (see Section 4A2); one has to distinguish two cases depending on whether $a_{k+1}(E')$ vanishes or not. Assume first $a_{k+1}(E')=0$; then, by (4-28), we know that the eigenvector of H_0^+ actually satisfies the Dirichlet boundary conditions at L; thus, E' is a solution to (4-41), i.e., an eigenvalue of H_L, and (4-28) gives a (nonnormalized) eigenvector.

Assume now that $a_{k+1}(E') \neq 0$; then, by Rouché's theorem, the unique solution to (4-41) close to E' satisfies

$$E - E' = -\frac{\rho^{2N}(E')}{\beta'_{k+1}(E')} a_{k+1}(E')(1+o(\rho^{2N}(E'))). \tag{4-44}$$

(2) If E' is an eigenvalue of H_k^- but not of H_0^+, mutatis mutandis, the analysis is the same as in point (1).

(3) If E' is an eigenvalue of both H_0^+ and H_k^-, then we are in a resonant tunneling situation. The analysis done in the Appendix shows that, near E', H_L has two eigenvalues, say $E_\pm$, satisfying, for some constant $\alpha > 0$,

$$E_\pm - E' = \pm\alpha\rho^N(E'))\big(1+O(N\rho(E')^N)\big). \tag{4-45}$$

This completes the proof of the statements of Theorem 4.2 for the eigenvalues outside $\Sigma_{\mathbb{Z}}$.

The eigenvalues inside $\Sigma_{\mathbb{Z}}$. We now study the eigenvalues in the region $\overset{\circ}{\Sigma}_{\mathbb{Z}}$. One can express $\rho(E)$ in terms of the Bloch quasimomentum $\theta_p(E)$ and use $\rho^{-1}(E) = \overline{\rho(E)}$. Notice that, on $\overset{\circ}{\Sigma}_{\mathbb{Z}}$, one has:

- $\operatorname{Im}\rho(E)$ does not vanish.
- The function $E \mapsto \rho(E)$ is real analytic.
- The functions $E \mapsto a_p^0(E)$, $E \mapsto a_{p-1}^0(E)$, $E \mapsto a_{k+1}(E)$ and $E \mapsto b_{k+1}(E)$ are real-valued polynomials.

We prove:

Lemma 4.10. *The function α_{k+1} is analytic and does not vanish on $\overset{\circ}{\Sigma}_{\mathbb{Z}}$.*

Proof. Assume that the function α_{k+1} vanishes at a point E_0 in $\overset{\circ}{\Sigma}_{\mathbb{Z}}$.

- If $\rho(E_0) \neq \rho^{-1}(E_0)$, then one has $a_{k+1}(E_0)(a_p^0(E_0)-\rho^{-1}(E_0))+b_{k+1}(E_0)a_{p-1}^0(E_0)=0$; as $\rho(E_0) \neq \rho^{-1}(E_0)$ and $E_0 \in \overset{\circ}{\Sigma}_{\mathbb{Z}}$, one has $\rho^{-1}(E_0) = \overline{\rho(E_0)} \notin \mathbb{R}$; thus, for

$$a_{k+1}(E_0)(a_p^0(E_0)-\rho^{-1}(E_0))-b_{k+1}(E_0)a_{p-1}^0(E_0)$$

to vanish, one needs $a_{k+1}(E_0)=0$ and $a^0_{p-1}(E_0)=0$ (as b_{k+1} and a_{k+1} don't vanish together); this implies that $\rho(E_0)=\pm 1$ and contradicts $\rho(E_0)\neq\rho^{-1}(E_0)$.

- If $\rho(E_0)=\rho^{-1}(E_0)$, such a point E_0 is a simple root of the three functions a^0_{p-1}, $\rho-\rho^{-1}$ and $a^0_p-\rho$ that are analytic near E_0 (see points (1)–(4) in Section 4A1). Moreover, one checks that the derivatives of these functions at that point are respectively real, purely imaginary and neither real nor purely imaginary; for E close to E_0, one has

$$\begin{aligned} a^0_{p-1}(E)&=A(E-E_0)(1+O(E-E_0)),\\ \rho(E)-\rho^{-1}(E)&=2iC(E-E_0)(1+O(E-E_0)),\\ a^0_p(E)-\rho^{-1}(E)&=(B+iC)(E-E_0)(1+O(E-E_0)),\quad\text{where}\quad (A,B,C)\in(\mathbb{R}^*)^3.\end{aligned}\tag{4-46}$$

Now, as a_{k+1} and b_{k+1} are real-valued and can't vanish at the same point, we see that $\alpha_{k+1}(E_0)\neq 0$.

This complete the proof of Lemma 4.10 □

Now, as $L=Np+k$, the characteristic equation (4-38) (valid for $E\in\Sigma^\circ_{\mathbb{Z}}$) becomes

$$\begin{aligned}\rho^{2N}(E)=e^{2iNp\theta_p(E)}&=-\frac{\overline{\alpha_{k+1}(E)}}{\alpha_{k+1}(E)}=-\frac{\beta_{k+1}(E)}{\overline{\beta_{k+1}(E)}}\\ &=\frac{a_{k+1}(E)(\rho(E)-a^0_p(E))-b_{k+1}(E)a^0_{p-1}(E)}{\overline{a_{k+1}(E)(\rho(E)-a^0_p(E))-b_{k+1}(E)a^0_{p-1}(E)}}=:e^{2ih_k(E)}.\end{aligned}\tag{4-47}$$

By Lemma 4.10, the function $E\mapsto h_k(E)$ defined in (4-47) is real analytic on $\Sigma^\circ_{\mathbb{Z}}$. Clearly, as we are inside $\Sigma_{\mathbb{Z}}$, ρ is real only at bands' edges or closed gaps, h_k takes values in $\pi\mathbb{Z}$ only at bands' edges or closed gaps. This implies Theorem 4.2(a). We prove:

Lemma 4.11. *The function h_k can be extended continuously from $\Sigma^\circ_{\mathbb{Z}}$ to $\Sigma_{\mathbb{Z}}$; for $E_0\in\partial\Sigma_{\mathbb{Z}}$, one has*

$$h_k(E_0)\in\begin{cases}\frac{\pi}{2}+\pi\mathbb{Z} & \text{if } a_{k+1}(E_0)\neq 0 \text{ and } a_{k+1}(E_0)(\rho(E_0)-a^0_p(E_0))-b_{k+1}(E_0)a^0_{p-1}(E_0)=0,\\ \pi\mathbb{Z} & \text{if not.}\end{cases}$$

The function $\theta_{p,L}$ is strictly increasing on the bands of $\Sigma_{\mathbb{Z}}$.

Proof. Pick $E_0\in\partial\Sigma_{\mathbb{Z}}$. It suffices to study the behavior of, for $E\in\Sigma_{\mathbb{Z}}$,

$$E\mapsto s(E):=a_{k+1}(E)(\rho(E)-a^0_p(E))-b_{k+1}(E)a^0_{p-1}(E)$$

near E_0 inside $\Sigma_{\mathbb{Z}}$. Write $E=E_0\pm t^2$ for t real and positive; here, the sign $\pm$ depends on whether E_0 is a left or right edge of $\Sigma_{\mathbb{Z}}$ and is chosen so that $E=E_0\pm t^2\in\Sigma^\circ_{\mathbb{Z}}$ for t small.

First, $t\mapsto\rho(E_0\pm t^2)$ is analytic near 0; thus, so is $t\mapsto s(E_0\pm t^2)$. Solving the characteristic equation $\rho^2(E)-\underline{\Delta}(E)\rho(E)+1=0$, one finds

$$\rho(E_0\pm t^2)=\rho(E_0)+iat+bt^2+O(t^3),\quad a\in\mathbb{R}^*,\ b\in\mathbb{R}.$$

Thus,

$$s(E_0\pm t^2)=s(E_0)+ia_{k+1}(E_0)\cdot a\cdot t+c\cdot t^2+O(t^3),$$

where

$$c := a'_{k+1}(E_0)(\rho(E_0) - a^0_p(E_0)) + a_{k+1}(E_0)(b - (a^0_p)'(E_0)) - (b'_{k+1}(E_0)a^0_{p-1}(E_0) + b_{k+1}(E_0)(a^0_{p-1})'(E_0)).$$

Hence:

- If $s(E_0) \neq 0$, then $s(E_0 \pm t^2) = s(E_0) + O(t)$; hence, $h_k(E_0 \pm t^2) = \pi n + O(t)$ for some $n \in \mathbb{Z}$.
- If $s(E_0) = 0$ and $a_{k+1}(E_0) \neq 0$, one has $s(E_0 \pm t^2) = i a_{k+1}(E_0) \cdot a \cdot t + O(t^2)$; thus, $h_k(E_0 \pm t^2) = \frac{\pi}{2} + \pi n + O(t)$ for some $n \in \mathbb{Z}$.
- If $s(E_0) = a_{k+1}(E_0) = 0$, one has $b_{k+1}(E_0) \neq 0$, $a^0_{p-1}(E_0) = 0$, $\rho(E_0) = a^0_p(E_0)$ and $(a^0_{p-1})'(E_0) \neq 0$; thus $s(E_0 \pm t^2) = -b_{k+1}(E_0)(a^0_{p-1})'(E_0)t^2 + 0(t^2)$; hence, $h_k(E_0 \pm t^2) = \pi n + O(t)$ for some $n \in \mathbb{Z}$.

This completes the proof of the statement of Lemma 4.11 on the function h_k.

Let us now control the monotony of $\theta_{p,L}$ (see Theorem 4.2) on the bands of $\Sigma_{\mathbb{Z}}$. It is well known that, keeping the above notations, $\theta_p(E_0 \pm t^2) - \theta_p(E_0) = \pm \alpha t(1 + t g_0(t))$ with $\alpha > 0$. The computations done in the previous paragraph show that $h_k(E_0 \pm t^2) = h_k(E_0) + a t^k (1 + t g_1(t))$, $k \geq 1$. Hence:

- If $k > 1$, we have $\theta_{p,L}(E_0 \pm t^2) - \theta_{p,L}(E_0) = \pm \alpha t(1 + t g_2(t))$.
- If $k = 1$, we have $\theta_{p,L}(E_0 \pm t^2) - \theta_{p,L}(E_0) = (\pm \alpha + a/(L-k)) t (1 + t g_2(t))$.

Hence, $\theta_{p,L}$ is strictly increasing inside the band near E_0 for L sufficiently large. Outside a neighborhood of the edges of a band, by analyticity of h_k, as the bands are compact, we have $|\theta'_{p,L} - \theta'_p| \lesssim L^{-1}$. As θ_p is strictly increasing on each band, $\theta_{p,L}$ is also strictly increasing outside a neighborhood of the edges of a band. This completes the proof of Lemma 4.11. □

One proves:

Lemma 4.12. *Let E_0 be a closed gap for $H^{\mathbb{Z}}$* (see Definition 4.5). *Then, for any $L = Np + k$,*

$$E_0 \in \sigma(H_L) \iff h_k(E_0) \in \pi\mathbb{Z} \iff a_{k+1}(E_0) = 0 \iff \alpha_{k+1}(E_0) \in i\mathbb{R}^*. \tag{4-48}$$

Proof. The proof of the first equivalence follows immediately from Definition 4.5 and the quantization condition (4-47); the second follows from (4-39) and the expansions in (4-46); the third follows from Lemma 4.11, (4-39) and (4-47). □

Let us note that, in particular, closed gaps where a_{k+1} vanishes are eigenvalues of H_L for all $L = Np + k$.

Remark 4.13. The characteristic equation (4-47) and the computations done at the end of the proof of Lemma 4.10 show that, for $L = Np + k$ large, an energy E_0 such that $\rho(E_0) = \rho^{-1}(E_0)$ is an eigenvalue of H_L if and only if $a_{k+1}(E_0) = 0$. This is an extension of Lemma 4.8.

In view of the definition and monotony of $\theta_{p,L}$, the quantization condition (4-47) is clearly equivalent to (4-3). This completes the proof Theorem 4.1 on the eigenvalues of H_L. Let us now turn to the computation of the associated eigenfunctions.

4A4. *The Dirichlet eigenfunctions for a truncated periodic potential*: *the proof of Theorem 4.3*. Recall that we assume $L = Np+k$. First, if $(u_l^j)_{l=0}^L$ is an eigenfunction associated to the eigenvalue λ_j, the eigenvalue equation reads

$$\begin{pmatrix} u_{l+1}^j \\ u_l^j \end{pmatrix} = T_l(\lambda_j)\begin{pmatrix} u_l^j \\ u_{l-1}^j \end{pmatrix} \quad \text{for } 0 \le l \le L, \quad \text{where} \quad u_{L+1}^j = u_{-1}^j = 0.$$

To normalize the solution, we assume that $u_0^j = 1$. The coefficients we want to compute are

$$|\varphi_j(L)|^2 = |u_L^j|^2 \left(\sum_{l=0}^L |u_l^j|^2\right)^{-1} \quad \text{and} \quad |\varphi_j(0)|^2 = \left(\sum_{l=0}^L |u_l^j|^2\right)^{-1}. \tag{4-49}$$

Fix $l = np+m$. Thus, using the notations of Section 4A3 and the expressions (4-25), (4-26) and (4-23), one computes

$$\begin{pmatrix} u_l^j \\ u_{l-1}^j \end{pmatrix} = T_{m-1,0}(\lambda_j)(\widetilde{T}_0(\lambda_j))^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_m(\lambda_j)\rho^n(\lambda_j) + \beta_m(\lambda_j)\rho^{-n}(\lambda_j) \\ \alpha_{m-1}(\lambda_j)\rho^n(\lambda_j) + \beta_{m-1}(\lambda_j)\rho^{-n}(\lambda_j) \end{pmatrix}, \tag{4-50}$$

where α_m and β_m are as defined in (4-39).

The eigenvectors associated to eigenvalues inside $\Sigma_{\mathbb{Z}}$. As $\rho^{-1}(\lambda_j) = \overline{\rho(\lambda_j)}$, $\beta_m(\lambda_j) = \overline{\alpha_m(\lambda_j)}$ and as the functions $(\alpha_m)_{0\le m\le p-1}$ do not vanish on $\Sigma_{\mathbb{Z}}^\circ$, we compute

$$|u_{np+m}^j|^2 = 2|\alpha_m(\lambda_j)|^2\left(1 + \operatorname{Re}\left[\frac{\alpha_m(\lambda_j)}{\overline{\alpha_m(\lambda_j)}}\rho^{2n}(\lambda_j)\right]\right). \tag{4-51}$$

As $L = Np+k$, using the quantization condition (4-47), we obtain that

$$\begin{aligned}\sum_{l=0}^L |u_l^j|^2 &\\ = 2\sum_{m=0}^k |\alpha_m(\lambda_j)|^2 &\left(1 + \operatorname{Re}\left[\frac{\alpha_m(\lambda_j)}{\overline{\alpha_m(\lambda_j)}}\rho^{2N}(\lambda_j)\right]\right) + 2\sum_{m=0}^{p-1}|\alpha_m(\lambda_j)|^2 \sum_{n=0}^{N-1}\left(1 + \operatorname{Re}\left[\frac{\alpha_m(\lambda_j)}{\overline{\alpha_m(\lambda_j)}}\rho^{2n}(\lambda_j)\right]\right)\\ = Npf(\lambda_j)&\left(1 + \frac{1}{Np}\tilde{f}(\lambda_j)\right),\end{aligned} \tag{4-52}$$

where we have defined

$$f(E) := \frac{2}{p}\sum_{m=0}^{p-1}|\alpha_m(E)|^2 \tag{4-53}$$

and, using the quantization condition (4-47), computed

$$\begin{aligned}\tilde{f}(E) := \frac{2}{f(E)}\operatorname{Re}&\left[\left(\sum_{m=0}^{p-1}\alpha_m^2(E)\right)\frac{1}{1-\rho^2(E)}\left(1 + \frac{\overline{\alpha_{k+1}(E)}}{\alpha_{k+1}(E)}\right)\right]\\ &+ \frac{2}{f(E)}\sum_{m=0}^k |\alpha_m(E)|^2\left(1 - \operatorname{Re}\left[\frac{\alpha_m(E)\overline{\alpha_{k+1}(E)}}{\overline{\alpha_m(E)}\alpha_{k+1}(E)}\right]\right).\end{aligned} \tag{4-54}$$

The function $E \mapsto f(E)$ is real analytic and does not vanish on $\Sigma_{\mathbb{Z}}^{\circ}$.

We prove:

Proposition 4.14. *For E_0, a closed gap, one has $\sum_{m=0}^{p-1} \alpha_m^2(E_0) = 0$.*

Proof. By the definition of (a_j, b_j) — see (4-29) — and that of $\alpha_j(E)$ — see (4-39) — the sequence $(\alpha_j(E))_{j\in\mathbb{Z}}$ satisfies the equation $\alpha_{j+1}+\alpha_{j-1}+(V_j-E)\alpha_j=0$. As $\widetilde{T}_0(E)=T_{p-1}(E)\cdots T_0(E)$, by (4-23), for $j\in\mathbb{Z}$ one has $\alpha_{j+p}(E)=\rho(E)\alpha_j(E)$. Hence, the column vector $A(E)=(\alpha_1(E),\dots,\alpha_p(E))^t$ satisfies

$$(H_\rho - E)A(E) = 0, \quad \text{where} \quad H_\rho = \begin{pmatrix} V_1 & 1 & 0 & \cdots & 0 & \rho(E) \\ 1 & V_2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & V_3 & 1 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & V_{p-1} & 1 \\ \rho^{-1}(E) & 0 & \cdots & 0 & 1 & V_p \end{pmatrix}.$$

Thus, we have

$$\langle (H_\rho - E)A(E), A(E)\rangle_{\mathbb{R}} = 0, \tag{4-55}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ denotes the real scalar product over $\mathbb{C}^p$, i.e.,

$$\left\langle \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix}, \begin{pmatrix} z_1' \\ \vdots \\ z_p' \end{pmatrix} \right\rangle_{\mathbb{R}} = \sum_{j=1}^{p} z_j z_j'.$$

The functions $E \mapsto A(E)$ and $E \mapsto \rho(E)$ being analytic over $\Sigma_{\mathbb{Z}}^{\circ}$ (see Section 4A1 and Lemma 4.10), one can differentiate (4-55) with respect to E to obtain

$$0 = -\langle A(E), A(E)\rangle_{\mathbb{R}} + (\rho(E)-\rho^{-1}(E))\big(\rho^{-1}(E)\rho'(E)\alpha_1(E)\alpha_p(E) - \alpha_p(E)\alpha_1'(E) + \alpha_1(E)\alpha_p'(E)\big). \tag{4-56}$$

Here we have used the fact that, if H_ρ^t is the transpose of the matrix H_ρ, then

$$H_\rho^t - H_\rho = (\rho(E)-\rho^{-1}(E)) \begin{pmatrix} 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

At E_0, a closed gap, one has $\rho(E_0) = \rho^{-1}(E_0)$. Hence, (4-56) implies

$$0 = \langle A(E_0), A(E_0)\rangle_{\mathbb{R}} = \sum_{m=0}^{p-1} \alpha_m^2(E_0).$$

This completes the proof of Proposition 4.14. □

In view of (4-54), the function $\tilde{f}$ is real analytic on $\Sigma_{\mathbb{Z}}^{\circ}$; indeed, the only poles of the function $E \mapsto [\rho(E) - \rho^{-1}(E)]^{-1}$ in $\Sigma_{\mathbb{Z}}^{\circ}$ are the closed gaps; they are simple poles of this function and, by Proposition 4.14, the real analytic function $E \mapsto \sum_{m=0}^{p-1} \alpha_m^2(E)$ vanishes at these poles.

Now that we have computed the normalization constant, let us compute the coefficient u_L^j defined in (4-49). As $L = Np + k$, the characteristic equation for λ_j — that is, (4-47) — reads

$$\alpha_{k+1}(\lambda_j)\rho^N(\lambda_j) = -\beta_{k+1}(\lambda_j)\rho^{-N}(\lambda_j) = -\overline{\alpha_{k+1}(\lambda_j)\rho^N(\lambda_j)}. \tag{4-57}$$

Hence, one computes

$$\begin{aligned} u_L^j &= \alpha_k(\lambda_j)\rho^N(\lambda_j) + \overline{\alpha_k(\lambda_j)\rho^N(\lambda_j)} = \rho^N(\lambda_j)\frac{\alpha_k(\lambda_j)\overline{\alpha_{k+1}(\lambda_j)} - \overline{\alpha_k(\lambda_j)}\alpha_{k+1}(\lambda_j)}{\overline{\alpha_{k+1}(\lambda_j)}} \\ &= \frac{-\rho^N(\lambda_j)a_{p-1}^0(\lambda_j)}{(\rho(\lambda_j) - \rho^{-1}(\lambda_j))\overline{\alpha_{k+1}(\lambda_j)}} = \frac{-e^{i[Np\theta_p(\lambda_j) - h_k(\lambda_j)]}a_{p-1}^0(\lambda_j)}{\left|a_{k+1}(\lambda_j)(a_p^0(\lambda_j) - \rho^{-1}(\lambda_j)) + b_{k+1}(\lambda_j)a_{p-1}^0(\lambda_j)\right|} \\ &= \frac{-e^{i\pi j}a_{p-1}^0(\lambda_j)}{\left|a_{k+1}(\lambda_j)(a_p^0(\lambda_j) - \rho^{-1}(\lambda_j)) + b_{k+1}(\lambda_j)a_{p-1}^0(\lambda_j)\right|}, \end{aligned} \tag{4-58}$$

where we have used the quantization condition satisfied by λ_j, the last equality in (4-47), and that

$$\begin{vmatrix} \alpha_{k+1}(\lambda_j) & \alpha_k(\lambda_j) \\ \overline{\alpha_{k+1}(\lambda_j)} & \overline{\alpha_k(\lambda_j)} \end{vmatrix} = \begin{vmatrix} \dfrac{a_{p-1}^0(\lambda_j)}{\rho(\lambda_j) - \rho^{-1}(\lambda_j)} & \dfrac{a_p^0(\lambda_j) - \rho^{-1}(\lambda_j)}{\rho(\lambda_j) - \rho^{-1}(\lambda_j)} \\ -\dfrac{a_{p-1}^0(\lambda_j)}{\rho(\lambda_j) - \rho^{-1}(\lambda_j)} & \dfrac{\rho(\lambda_j) - a_p^0(\lambda_j)}{\rho(\lambda_j) - \rho^{-1}(\lambda_j)} \end{vmatrix} \begin{vmatrix} b_{k+1}(\lambda_j) & b_k(\lambda_j) \\ a_{k+1}(\lambda_j) & a_k(\lambda_j) \end{vmatrix}$$

and

$$\begin{vmatrix} 1 & \dfrac{a_p^0(\lambda_j) - \rho^{-1}(\lambda_j)}{\rho(\lambda_j) - \rho^{-1}(\lambda_j)} \\ -1 & \dfrac{\rho(\lambda_j) - a_p^0(\lambda_j)}{\rho(\lambda_j) - \rho^{-1}(\lambda_j)} \end{vmatrix} = \begin{vmatrix} b_k(\lambda_j) & b_{k+1}(\lambda_j) \\ a_k(\lambda_j) & a_{k+1}(\lambda_j) \end{vmatrix} = 1.$$

Lemma 4.15. *Define the function* $\tilde{f}_k^-(E)$ *by*

$$\tilde{f}_k^-(E) := \frac{|a_{p-1}^0(E)|^2}{|a_{k+1}(E)(a_p^0(E) - \rho^{-1}(E)) + b_{k+1}(E)a_{p-1}^0(E)|^2}.$$

Then the function $\tilde{f}_k^-$ *does not vanish on* $\Sigma_{\mathbb{Z}}^{\circ}$.

Proof. By the definition of α_{k+1}, one has

$$\tilde{f}_k^-(E) = \frac{|a_{p-1}^0(E)|^2}{|\rho(E) - \rho^{-1}(E))|^2|\alpha_{k+1}(E)|^2}.$$

That this expression is well defined and does not vanish on $\Sigma_{\mathbb{Z}}^{\circ}$ follows from Lemma 4.10 and the computations made in the proof thereof. ☐

Plugging (4-58) into this and (4-51) into (4-49), recalling that $u_0^j = 1$, outside the bad closed gaps we obtain (4-4) if

- in addition to (4-53) and (4-54), we set $f_0^+(E) := 1/f(E)$ and $f_k^-(E) = f_0^+(E)\cdot \tilde{f}_k^-(E)$,
- we remember that the function a_{p-1}^0 only changes sign in the gaps of the spectrum $\Sigma_{\mathbb{Z}}$ (see point (4) in Section 4A1) and set σ_r to be the sign of $-a_{p-1}^0$ on B_r, the r-th band.

By (4-49) and (4-51), we obtain (4-4) using Lemma 4.15. This completes the proof of the statements in Theorem 4.3 on the eigenfunctions of H_L associated to eigenvalues in $\Sigma_{\mathbb{Z}}^\circ$.

Remark 4.16. To complete our study let us also see what happens to the eigenfunctions near the edges of the spectrum. Pick $E_0 \in \partial\Sigma_{\mathbb{Z}}$. One then knows that, for $E \in \Sigma_{\mathbb{Z}}$ with E close to E_0, one has

$$\theta_p(E) - \theta_p(E_0) = a\sqrt{|E-E_0|}(1+o(1)) \tag{4-59}$$

(see the proof of Lemma 4.11).

Let us rewrite $\tilde{f}$ (see (4-54)) as

$$\tilde{f}(E) = \frac{2}{f(E)}\Bigg[\sum_{m=0}^{p-1}|\alpha_m(E)|^2\cos\big(h_k(E)-2h_{m-1}(E)-p\theta_p(E)\big)\Bigg]\frac{\sin(h_k(E))}{\sin(p\theta_p(E))} + \frac{2}{f(E)}\sum_{m=0}^{k}|\alpha_m(E)|^2\big(1-\cos\big(2(h_k(E)-h_{m-1}(E))\big)\big). \tag{4-60}$$

Let us first show:

Lemma 4.17. *For any $0 \le m \le p-1$, $E \mapsto 2|\alpha_m(E)|^2/(pf(E))$ can be extended continuously from $\Sigma_{\mathbb{Z}}^\circ$ to $\Sigma_{\mathbb{Z}}$.*

Proof. For $p = 1$ there is nothing to be done as $2|\alpha_m(E)|^2/(pf(E)) \equiv 1$.

For $p \ge 2$, we note that for $0 \le m \le m+1 \le p-1$, as

$$\begin{vmatrix} a_{m+1}(E) & b_{m+1}(E) \\ a_m(E) & b_m(E) \end{vmatrix} = 1$$

by (4-29),

$$\begin{aligned} 0 &= a_{m+1}(E_0)(a_p^0(E_0) - \rho^{-1}(E_0)) + b_{m+1}(E_0)a_{p-1}^0(E_0) \\ &= a_m(E_0)(a_p^0(E_0) - \rho^{-1}(E_0)) + b_m(E_0)a_{p-1}^0(E_0) \end{aligned}$$

if and only if $a_{p-1}^0(E_0) = 0$ (as this implies $a_p^0(E_0) - \rho^{-1}(E_0) = 0$).

Let us assume this is the case. As $p \ge 2$, we know that $\sum_{j=0}^{p-1}|a_j(E_0)|^2 \ne 0$. By (4-46), for at least one $m_0 \in \{0,\dots,p-1\}$ one has $a_{m_0}(E_0) \ne 0$ and $\alpha_{m_0}(E) = bc^{-1}a_{m_0}(E_0) + O(\sqrt{|E-E_0|})$. Hence, $E \mapsto 2|\alpha_m(E)|^2/(pf(E))$ can be continued to E_0, setting

$$\frac{2|\alpha_m(E_0)|^2}{pf(E_0)} = \frac{|a_m(E_0)|^2}{|a_0(E_0)|^2+\cdots+|a_{p-1}(E_0)|^2}.$$

Actually, $f(E)$ can be continued at E_0 by setting

$$f(E_0) = |a_0(E_0)|^2 + \cdots + |a_{p-1}(E_0)|^2. \tag{4-61}$$

Let us now assume that $a^0_{p-1}(E_0) \neq 0$. We study the behavior of α_m near E_0. Recall (4-39). Then one has

(1) either $d_m := a_m(E_0)(a^0_p(E_0) - \rho^{-1}(E_0)) + b_m(E_0)a^0_{p-1}(E_0) \neq 0$, in which case, by (4-46), one has $\alpha_m(E) = (d_m c^{-1}/\sqrt{|E-E_0|})(1+o(1))$;

(2) or $d_m = a_m(E_0)(a^0_p(E_0) - \rho^{-1}(E_0)) + b_m(E_0)a^0_{p-1}(E_0) = 0$, in which case, since for some $A_m \in \mathbb{R}^*$ and $k_m \geq 1$ one has

$$a_m(E)(a^0_p(E) - \rho^{-1}(E_0)) + b_m(E)a^0_{p-1}(E) = A_m(E-E_0)^{k_m}(1+o(1)),$$

by (4-46), one can continue α_m to E_0 by setting $\alpha_m(E_0) = \frac{1}{2}a_m(E_0)$.

As $a^0_{p-1}(E_0) \neq 0$, we know that for some $m_0 \in \{0, \dots, p-1\}$ we are in case (a). Hence, one has

$$f(E) = \frac{2}{p|E-E_0|} \sum_{m=0}^{p-1} \left| a_m(E_0)(a^0_p(E_0) - \rho^{-1}(E_0)) + b_m(E_0)a^0_{p-1}(E_0) \right|^2 (1+o(1)) \tag{4-62}$$

and $E \mapsto 2|\alpha_m(E)|^2/(pf(E))$ can be continued to E_0, setting

$$\frac{2|\alpha_m(E_0)|^2}{pf(E_0)} = \frac{|d_m|^2}{|d_0|^2 + \cdots + |d_{p-1}|^2}$$

(using the notation introduced in point (a)).

This completes the proof of Lemma 4.17. □

By Lemma 4.11, we know that, for $1 \leq k \leq p$ and $E_0 \in \partial\Sigma_{\mathbb{Z}}$, one has $2h_k(E_0) \in \pi\mathbb{Z}$. Thus, for $1 \leq k \leq p$, $1 \leq m \leq p$ and $E_0 \in \partial\Sigma_{\mathbb{Z}}$, one has $\cos(h_k(E_0) - 2h_{m-1}(E_0) - p\theta_p(E_0))\sin(h_k(E_0)) = 0$. Using the expansions leading to the proof of Lemma 4.11, one gets

$$\cos(h_k(E) - 2h_{m-1}(E) - p\theta_p(E))\sin(h_k(E)) = c\sqrt{|E-E_0|}(1+o(1)).$$

Recalling (4-59) and the fact that $p\theta_p(E_0) \in \pi\mathbb{Z}$, Lemma 4.17 implies that $\tilde{f}$ can be extended continuously up to E_0. Hence, the expansion (4-52) again yields

$$\sum_{l=0}^{L} |u^j_l|^2 \asymp Npf(\lambda_j). \tag{4-63}$$

Let us now review the computation (4-58) in this case. We distinguish two cases:

(1) If $a^0_{p-1}(E_0) = 0$, then (4-58) and the fact that $a_{k+1}(E_0) \neq 0$ (this case was dealt with in point (1)), yields that, for $|\lambda_j - E_0|$ sufficiently small,

$$|u^j_L| \asymp \sqrt{|\lambda_j - E_0|}.$$

By (4-61) and (4-63), we obtain

$$|\varphi_j(L)|^2 \asymp \frac{|\lambda_j - E_0|}{Np} \quad \text{and} \quad |\varphi_j(0)|^2 \asymp \frac{1}{Np}. \tag{4-64}$$

(2) If $a^0_{p-1}(E_0) \neq 0$, then

(a) if $d_{k+1} \neq 0$ (see case (a) in the proof of Lemma 4.17), by (4-62) and (4-63) one has

$$|\varphi_j(0)|^2 \asymp \frac{|\lambda_j - E_0|}{Np} \quad \text{and} \quad |\varphi_j(L)|^2 \asymp \frac{|\lambda_j - E_0|}{Np}. \tag{4-65}$$

(b) if $d_{k+1} = 0$, by (4-62) and (4-63) one has

$$|\varphi_j(0)|^2 \asymp \frac{|\lambda_j - E_0|}{Np} \quad \text{and} \quad |\varphi_j(L)|^2 \asymp \frac{1}{Np}. \tag{4-66}$$

The eigenvectors associated to eigenvalues outside $\Sigma_{\mathbb{Z}}$. Let us now turn to the eigenfunctions associated to eigenvalues H_L in the gaps of $\Sigma_{\mathbb{Z}}$, i.e., in the region $\{E \mid |\underline{\Delta}(E)| > 2\}$. On $\mathbb{R} \setminus \Sigma_{\mathbb{Z}}$, the eigenvalue $E \mapsto \rho(E)$ is real-valued (recall that we pick it so that $|\rho(E)| < 1$) and so are all the functions $(\alpha_m)_{0 \le m \le p-1}$ and $(\beta_m)_{0 \le m \le p-1}$ (see (4-39)). For $0 \le m \le p-1$, (4-50) yields

$$|u^j_{np+m}|^2 = \alpha^2_m(E)\rho^{2n}(E) + \beta^2_m(E)\rho^{-2n}(E) + 2\alpha_m(E)\beta_m(E). \tag{4-67}$$

As when we studied the eigenvalues of H_L, let us now distinguish the cases when E is close to an eigenvalue of H^+_0 or to an eigenvalue of H^-_k:

(1) Pick E' an eigenvalue of H^+_0 but not an eigenvalue of H^-_k; then recall that $a^0_{p-1}(E') = 0 = a^0_p(E') - \rho(E')$. Thus, for $0 \le m \le p-1$, one has $\beta_m(E') = 0$. Assume that E is close to E'. As E satisfies (4-44), using (4-41), (4-67) becomes

$$|u^j_{np+m}|^2 = \rho^{2n}(E') \left| \alpha_m(E') - \frac{\beta'_m(E')}{\beta'_{k+1}(E')} a_{k+1}(E')[\rho(E') - \rho^{-1}(E')]\rho^{2(N-n)}(E') + O(\rho^{2N}(E)) \right|^2$$

for $0 \le m \le p-1$ if $0 \le n \le N-1$ and $0 \le m \le k$ if $n = N$.

Using (4-40), one computes

$$|u^j_{np+m}|^2 = \rho^{2n}(E') \left| a_m(E') - \frac{\beta'_m(E')}{\beta'_{k+1}(E')} a_{k+1}(E')\rho^{2(N-n)}(E') + O(\rho^{2N}(E)) \right|^2. \tag{4-68}$$

This yields

$$\sum_{l=0}^{L} |u^j_l|^2 = \sum_{m=0}^{p-1} \sum_{n=0}^{N-1} \rho^{2n}(E') a^2_m(E') + O(N\rho^{2N}(E)) = \frac{1}{1-\rho^2(E')} \sum_{m=0}^{p-1} a^2_m(E') + O(N\rho^{2N}(E)).$$

Moreover, by (4-49), (4-67) and (4-39), as $a^0_{p-1}(E') = 0 = a^0_p(E') - \rho(E')$, we obtain

$$\begin{aligned} |\varphi_j(L)|^2 &= \rho^{2N}(E') \frac{(1-\rho^2(E'))a^2_{k+1}(E')}{[\beta'_{k+1}(E')]^2 \sum_{m=0}^{p-1} a^2_m(E')} \begin{vmatrix} \beta'_k(E') & a_k(E') \\ \beta'_{k+1}(E') & a_{k+1}(E') \end{vmatrix}^2 + O(N\rho^{4N}(E)) \\ &= \gamma\rho^{2N}(E') + O(N\rho^{4N}(E)), \end{aligned}$$

where

$$\gamma := \frac{(1-\rho^2(E'))a_{k+1}^2(E')}{[\beta_{k+1}'(E')]^2\sum_{m=0}^{p-1}a_m^2(E')}\left(\frac{da_{p-1}^0}{dE}(E')\right)^2 > 0.$$

Hence, $|\varphi_j(L)|$ is exponentially small in L (recall $|\rho(E)| < 1$).

(2) If E' is an eigenvalue of H_k^- but not of H_0^+, then, inverting the parts of H_k^- and H_0^+, we see that $|\varphi_j(L)|$ is of order 1. A precise asymptotic can be computed but it won't be needed.

(3) If E' is an eigenvalue of H_0^+ and of H_k^-, the double well analysis done in the Appendix shows that, for normalized eigenvectors, say φ_j, $j = 1, 2$, associated to the two eigenvalues of H_L close to E', the four coefficients $|\varphi_j(0)|$ and $|\varphi_j(L)|$, $j = 1, 2$, are of order 1. Again, precise asymptotics can be computed but won't be needed.

This completes the description of the eigenfunctions given by Theorem 4.3 and completes the proof of this result.

5. Resonances in the periodic case

We are now in the state to prove the results stated in Section 1B. We first study the function $E \mapsto S_L(E)$ and $E \mapsto \Gamma_L(E)$ in the complex strip $I + i(-\infty, 0)$ for $I \subset \Sigma_{\mathbb{Z}}^\circ$.

5A. *The matrix Γ_L in the periodic case.*

Using Theorem 4.2, we first prove:

Theorem 5.1. *Fix $I \subset \Sigma_{\mathbb{Z}}^\circ$ a compact interval. There exists $\varepsilon_I > 0$ and $\sigma_I \in \{+1, -1\}$ such that, for any $N \geq 0$, there exists $C_N > 0$ such that, for L sufficiently large with $L \equiv k \bmod p$, one has*

$$\sup_{\substack{\operatorname{Re} E \in I \\ -\varepsilon_I < \operatorname{Im} E < 0}} |\Gamma_L(E) - \Gamma_L^{\mathrm{eff}}(E)| \leq C_N L^{-N}, \tag{5-1}$$

where

$$\Gamma_L^{\mathrm{eff}}(E) = -\frac{\theta_p'(E)}{\sin u_L(E)}\begin{pmatrix} e^{-iu_L(E)} f_k^-(E) & \sigma_I\sqrt{f_k^-(E)f_0^+(E)} \\ \sigma_I\sqrt{f_k^-(E)f_0^+(E)} & e^{-iu_L(E)} f_0^+(E)\end{pmatrix} + \begin{pmatrix} \int_{\mathbb{R}} 1/(\lambda - E)\, dN_k^-(\lambda) & 0 \\ 0 & \int_{\mathbb{R}} 1/(\lambda - E)\, dN_0^+(\lambda)\end{pmatrix} \tag{5-2}$$

and $u_L(E) := (L-k)\theta_{p,L}(E)$ (see (4-2)),

The sign σ_I only depends on the spectral band containing I.

Deeper in the lower half-plane, we obtain the following simpler estimate:

Theorem 5.2. *There exists $C > 0$ such that, for any $\varepsilon > 0$ and $L \geq 1$ sufficiently large with $L = Np + k$, one has*

$$\sup_{\substack{\operatorname{Re} E \in I \\ \operatorname{Im} E < -\varepsilon}} \left|\Gamma_L(E) - \begin{pmatrix} \int_{\mathbb{R}} 1/(\lambda - E)\, dN_k^-(\lambda) & 0 \\ 0 & \int_{\mathbb{R}} 1/(\lambda - E)\, dN_0^+(\lambda)\end{pmatrix}\right| \leq C\varepsilon^{-2}e^{-\varepsilon L/C}. \tag{5-3}$$

In Section 5B, the approximations (5-1) and (5-3) will be used to prove Theorems 1.5, 1.7 and 1.10.

Let us note that, as $\cot z = i + O(e^{-2i \operatorname{Im} z})$, for $\varepsilon \in (0, \varepsilon_I)$ the asymptotics given by Theorems 5.1 and 5.2 coincide in the region $\{\operatorname{Re} E \in I,\ \operatorname{Im} E \in (-\varepsilon_I, -\varepsilon)\}$; indeed, one has

$$\sup_{\substack{\operatorname{Re} E \in I \\ -\varepsilon_I < \operatorname{Im} E < -\varepsilon}} \left\| \frac{\theta_p'(E)}{\sin u_L(E)} \begin{pmatrix} e^{-iu_L(E)} f_k^-(E) & \sigma_I \sqrt{f_k^-(E) f_0^+(E)} \\ \sigma_I \sqrt{f_k^-(E) f_0^+(E)} & e^{-iu_L(E)} f_0^+(E) \end{pmatrix} \right\| \le e^{-\varepsilon L/C}.$$

Let us now turn to the proofs of Theorems 5.1 and 5.2.

5A1. *The proof of Theorem 5.1.* To prove Theorem 5.1, we split the sum $S_L(E)$ into two parts, one containing the Dirichlet eigenvalues “close” to $\operatorname{Re} E$, the other containing those “far” from $\operatorname{Re} E$. By “far”, we mean that the distance to $\operatorname{Re} E$ is bounded from below by a small constant independent of L. The “close” eigenvalues are then described by Theorem 4.2. For the “far” eigenvalues, the strong resolvent convergence of H_L to H_0^+, that of $\widetilde{H}_L$ to H_k^- (see Remark 1.6), and Combes–Thomas estimates enable us to compute the limit and to show that the prelimit and the limit are $O(L^{-\infty})$ close to each other. For the “close” eigenvalues, the sum occurring in (2-9), the definition of Γ_L, is a Riemann sum. We use the Poisson summation formula to obtain a precise approximation.

As I is a compact interval in $\Sigma_{\mathbb{Z}}^\circ$, we pick $\varepsilon > 0$ such that, for $E \in I$, one has $[E-6\varepsilon, E+6\varepsilon] \subset \Sigma_{\mathbb{Z}}^\circ$. Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$ be a nonnegative cut-off function such that $\chi \equiv 1$ on $[-4\varepsilon, 4\varepsilon]$ and $\chi \equiv 0$ outside $[-5\varepsilon, 5\varepsilon]$. For $E \in I$, define $\chi_E(\cdot) = \chi(\cdot - E)$.

We first give the asymptotic for the sum over the Dirichlet eigenvalues far from $\operatorname{Re} E$. We prove:

Lemma 5.3. *For any $N > 1$, there exists $C_N > 0$ such that, for L sufficiently large with $L \equiv k \bmod p$, one has*

$$\sup_{E \in \mathbb{C}} \left| \sum_{j=1}^{L} \frac{1 - \chi_{\operatorname{Re} E}(\lambda_j)}{\lambda_j - E} \begin{pmatrix} |\varphi_j(L)|^2 & \overline{\varphi_j(0)}\varphi_j(L) \\ \varphi_j(0)\overline{\varphi_j(L)} & |\varphi_j(0)|^2 \end{pmatrix} - \widetilde{M}(E) \right| \le C_N L^{-N}, \tag{5-4}$$

where

$$\widetilde{M}(E) := \begin{pmatrix} \int_{\mathbb{R}} (1 - \chi_{\operatorname{Re} E})(\lambda)/(\lambda - E)\, dN_k^-(\lambda) & 0 \\ 0 & \int_{\mathbb{R}} (1 - \chi_{\operatorname{Re} E})(\lambda)/(\lambda - E)\, dN_0^+(\lambda) \end{pmatrix}. \tag{5-5}$$

Proof of Lemma 5.3. Recall (see Theorem 2.2) that H_L is the operator H_0^+ restricted to $[\![0, L]\!]$ with Dirichlet boundary condition at L; as $L \equiv k \bmod p$, it is unitarily equivalent to the operator H_k^- restricted to $[\![-L, 0]\!]$ with Dirichlet boundary condition at $-L$ (see Remark 1.6).

Pick $\tilde{\chi} \in \mathscr{C}_0^\infty$ such that $\tilde{\chi} \equiv 1$ on $\sigma(H_0^+) \cup \sigma(H_k^-)$. First, we compute

$$\sum_{j=0}^{L} (1 - \chi_{\operatorname{Re} E})(\lambda_j) \frac{|\varphi_j(0)|^2}{\lambda_j - E} - \int_{\mathbb{R}} (1 - \chi_{\operatorname{Re} E})(\lambda) \frac{dN_0^+(\lambda)}{\lambda - E}$$
$$= \langle \delta_0, [\tilde{\chi}(1 - \chi_{\operatorname{Re} E})](H_L)(H_L - E)^{-1}\delta_0 \rangle - \langle \delta_0, [\tilde{\chi}(1 - \chi_{\operatorname{Re} E})](H_0^+)(H_0^+ - E)^{-1}\delta_0 \rangle,$$

$$\sum_{j=0}^{L}(1-\chi_{\operatorname{Re}E})(\lambda_j)\frac{|\varphi_j(L)|^2}{\lambda_j-E}-\int_{\mathbb{R}}(1-\chi_{\operatorname{Re}E})(\lambda)\,\frac{dN_k^-(\lambda)}{\lambda-E}$$
$$=\big\langle\delta_L,[\tilde\chi(1-\chi_{\operatorname{Re}E})](H_L)(H_L-E)^{-1}\delta_L\big\rangle-\big\langle\delta_L,[\tilde\chi(1-\chi_{\operatorname{Re}E})](H_k^-)(H_k^--E)^{-1}\delta_L\big\rangle,$$

and

$$\sum_{j=0}^{L}(1-\chi_{\operatorname{Re}E})(\lambda_j)\frac{\varphi_j(L)\overline{\varphi_j(0)}}{\lambda_j-E}=\big\langle\delta_L,[\tilde\chi(1-\chi_{\operatorname{Re}E})](H_L)(H_L-E)^{-1}\delta_0\big\rangle.$$

By the definition of $\chi_{\operatorname{Re}E}$, the function $\lambda\mapsto(\lambda-E)^{-1}\tilde\chi(\lambda)(1-\chi_{\operatorname{Re}E})(\lambda)$ is $\mathscr{C}_0^\infty$ on $\mathbb{R}$; moreover, its seminorms (see (4-14)) are bounded uniformly in $E\in\mathbb{C}$. Thus there exists an almost analytic extension of $[\tilde\chi(1-\chi_{\operatorname{Re}E})](\cdot)(\cdot-E)^{-1}$ such that, uniformly in E, one has (4-14).

In the same way as we obtained (4-16), we obtain

$$\big|\big\langle\delta_L,[(\widetilde H_L-z)^{-1}-(H_k^--z)^{-1}]\delta_L\big\rangle\big|+\big|\big\langle\delta_0,[(H_L-z)^{-1}-(H_0^+-z)^{-1}]\delta_0\big\rangle\big|+|\langle\delta_0,(H_L-z)^{-1}\delta_L\rangle|$$
$$\le\frac{C}{|\operatorname{Im}z|^2}e^{-L|\operatorname{Im}z|/C}. \tag{5-6}$$

Plugging (5-6) into (4-15) and using (4-14) for $[\tilde\chi(1-\chi_{\operatorname{Re}E})](\cdot)(\cdot-E)^{-1}$, we get

$$\sup_{\substack{L\ge1\\ L\equiv k \bmod p}} L^K\left|\sum_{j=0}^{L}(1-\chi_{\operatorname{Re}E})(\lambda_j)\frac{|\varphi_j(0)|^2}{\lambda_j-E}-\int_{\mathbb{R}}(1-\chi_{\operatorname{Re}E})(\lambda)\,\frac{dN_0^+(\lambda)}{\lambda-E}\right|<+\infty\quad\text{for all }K\in\mathbb{N}.$$

This entails (5-4) and completes the proof of Lemma 5.3. □

Let us now estimate the part of $\Gamma_L(E)$ associated to the Dirichlet eigenvalues close to $\operatorname{Re}E$. Define

$$\Gamma_L^\chi(E)=\sum_{j=1}^{L}\frac{\chi_{\operatorname{Re}E}(\lambda_j)}{\lambda_j-E}\begin{pmatrix}|\varphi_j(L)|^2 & \overline{\varphi_j(0)}\varphi_j(L)\\ \varphi_j(0)\overline{\varphi_j(L)} & |\varphi_j(0)|^2\end{pmatrix}. \tag{5-7}$$

We prove:

Lemma 5.4. *There exists $\varepsilon>0$ such that, for $N\ge1$, there exists C_N such that, for L sufficiently large with $L\equiv k \bmod p$, one has*

$$\sup_{\substack{\operatorname{Re}E\in I\\ -\varepsilon<\operatorname{Im}E<0}}|\Gamma_L^\chi(E)-\Gamma_L^{\text{eff}}(E)+\widetilde M(E)|\le C_NL^{-N},$$

where $\widetilde M$ is as defined in (5-5).

Clearly Lemmas 5.3 and 5.4 immediately yield Theorem 5.1.

Proof of Lemma 5.4. Recall that the quasimomentum θ_p defines a real analytic one-to-one monotonic map from the interior of each band of spectrum onto the set $(0,\pi)$, $(-\pi,0)$ or $(-\pi,\pi)$ (depending on the spectral band containing $I+[-4\varepsilon,4\varepsilon]$, where $\varepsilon>0$ has been fixed above) (see, e.g., [Teschl 2000]). Moreover, the derivative θ_p' is positive in the interior of a spectral band. Thus, for L sufficiently large, the real part of the derivative $\theta'_{p,L}$ (see (4-2)) is positive $I+[-3\varepsilon,3\varepsilon]$ and $\theta_{p,L}$ is real analytic one-to-one on a complex neighborhood of $(I+[-3\varepsilon,3\varepsilon])+i[-3\varepsilon,3\varepsilon]$ (possibly at the expense of reducing ε somewhat).

By (2-9), (4-8) and Theorem 4.2, one may write

$$\Gamma_L^\chi(E)=\frac{1}{L-k}\sum_{j\in\mathbb{Z}}\frac{\chi_{\operatorname{Re}E}\big(\theta_{p,L}^{-1}(\pi j/(L-k))\big)}{\theta_{p,L}^{-1}(\pi j/(L-k))-E}M\left(\theta_{p,L}^{-1}\left(\frac{\pi j}{L-k}\right)\right), \tag{5-8}$$

where

$$M(\lambda):=\begin{pmatrix} f_{k,L}(\lambda) & \sigma_I e^{i(L-k)\theta_{p,L}(\lambda)}\sqrt{f_{k,L}(\lambda)f_{0,L}(\lambda)} \\ \sigma_I e^{i(L-k)\theta_{p,L}(\lambda)}\sqrt{f_{k,L}(\lambda)f_{0,L}(\lambda)} & f_{0,L}(\lambda)\end{pmatrix} \tag{5-9}$$

and the matrix M is analytic in the rectangle $(I+[-3\varepsilon,3\varepsilon])+i[-3\varepsilon,3\varepsilon]$. Thus, the Poisson formula tells us that

$$\begin{aligned}\Gamma_L^\chi(E)&=\frac{1}{L-k}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}e^{-2i\pi jx}\frac{\chi_{\operatorname{Re}E}\big(\theta_{p,L}^{-1}(\pi x/(L-k))\big)}{\theta_{p,L}^{-1}(\pi x/(L-k))-E}M\left(\theta_{p,L}^{-1}\left(\frac{\pi x}{L-k}\right)\right)dx\\ &=\sum_{j\in\mathbb{Z}}\frac{1}{\pi}\int_{\mathbb{R}}e^{-2ij(L-k)\theta_{p,L}(\lambda)}\frac{\chi_{\operatorname{Re}E}(\lambda)}{\lambda-E}\theta'_{p,L}(\lambda)M(\lambda)\,d\lambda\\ &=\sum_{j\in\mathbb{Z}}\frac{1}{\pi}\int_{\mathbb{R}}M_{j,\chi}(E,\lambda,\lambda)\,d\lambda\end{aligned} \tag{5-10}$$

by the definition of $\chi_{\operatorname{Re}E}$; here, we have set

$$M_{j,\chi}(E,\lambda,\beta):=e^{-2ij(L-k)\theta_{p,L}(\beta+\operatorname{Re}E)}\frac{\chi(\lambda)}{\beta-i\operatorname{Im}E}\theta'_{p,L}(\beta+\operatorname{Re}E)M(\beta+\operatorname{Re}E).$$

Let us now study the individual terms in the last sum in (5-10). Recall that, on $[-4\varepsilon,4\varepsilon]$, χ is identically 1 and that $\lambda\mapsto\theta_{p,L}(\lambda+\operatorname{Re}E)$ and $\lambda\mapsto M(\lambda)$ are analytic in $(I+[-3\varepsilon,3\varepsilon])+i[-3\varepsilon,3\varepsilon]$; moreover, by (4-3), for some $\delta>0$ one has

$$\liminf_{L\to+\infty}\inf_{\lambda\in[-4\varepsilon,4\varepsilon]}\theta'_{p,L}(\lambda+\operatorname{Re}E)\ge\liminf_{L\to+\infty}\inf_{E\in I}\theta'_{p,L}(E)\ge\delta. \tag{5-11}$$

Recall also that $\operatorname{Im}E<0$. Consider $\tilde\chi:\mathbb{R}\to[0,1]$ smooth such that $\tilde\chi=1$ on $[-2\varepsilon,2\varepsilon]$ and $\tilde\chi=0$ outside $[-3\varepsilon,3\varepsilon]$.

In the complex plane, consider the paths $\gamma_\pm:R\to\mathbb{C}$ defined by

$$\gamma_\pm(\lambda)=\lambda\pm 2i\varepsilon\tilde\chi(\lambda).$$

As $-\varepsilon\le\operatorname{Im}E<0$, by contour deformation we have

$$\begin{aligned}\int_{\mathbb{R}}M_{j,\chi}(E,\lambda,\lambda)\,d\lambda&=\int_{\mathbb{R}}M_{j,\chi}(E,\lambda,\gamma_+(\lambda))\,d\lambda\\ &=-2i\pi e^{-2ij(L-k)\theta_{p,L}(E)}\theta'_{p,L}(E)M(E)+\int_{\mathbb{R}}M_{j,\chi}(E,\lambda,\gamma_-(\lambda))\,d\lambda.\end{aligned}$$

We then estimate:

• For $j<0$, using a nonstationary phase argument since the integrand is the product of a smooth function with an rapidly oscillating function (using $|j|(L-k)$ as the large parameter), one then estimates

$$\int_{\mathbb{R}} M_{j,\chi}(E,\lambda,\gamma_+(\lambda))\,d\lambda = O\big((|j|L)^{-\infty}\big).$$

The phase function is complex but its real part is nonpositive as $\operatorname{Im}\theta_{p,L}(\gamma_+(\cdot)+\operatorname{Re}E)\geq 0$ on the support of χ (by (5-11)). Note that the off-diagonal terms of $M(\lambda)$ also carry a rapidly oscillating exponential (see (5-9)) but it clearly does not suffice to counter the main one.

• In the same way, for $j>0$, one has

$$\int_{\mathbb{R}} M_{j,\chi}(E,\lambda,\gamma_-(\lambda))\,d\lambda = O((|j|L)^{-\infty}).$$

Thus, we compute

$$\int_{\mathbb{R}} M_{j,\chi}(E,\lambda,\lambda)\,d\lambda = O\big((|j|L)^{-\infty}\big) \quad \text{for } j<0, \tag{5-12}$$

$$\int_{\mathbb{R}} M_{j,\chi}(E,\lambda,\lambda)\,d\lambda = -2i\pi e^{-2ij(L-k)\theta_{p,L}(E)}\theta'_{p,L}(E)M(E)+O((|j|L)^{-\infty}) \quad \text{for } j>0. \tag{5-13}$$

Finally, for $j=0$, the contour deformation along γ_+ yields

$$\begin{aligned}\int_{\mathbb{R}} \frac{\chi(\lambda)}{\lambda-i\operatorname{Im}E}M(\lambda+\operatorname{Re}E)\,d\lambda &= \int_{\mathbb{R}} \frac{\chi_{\operatorname{Re}E}(\lambda)}{\lambda-E}\theta'_{p,L}(\lambda)\begin{pmatrix} f_{k,L}(\lambda) & 0 \\ 0 & f_{0,L}(\lambda)\end{pmatrix}d\lambda + O(L^{-\infty}) \\ &= \int_{\mathbb{R}} \frac{\chi_{\operatorname{Re}E}(\lambda)}{\lambda-E}\begin{pmatrix} dN_k^-(\lambda) & 0 \\ 0 & dN_0^+(\lambda)\end{pmatrix} + O(L^{-\infty})\end{aligned}$$

by Corollary 4.4.

Plugging this, (5-12) and (5-13) into (5-10) and computing the geometric sum immediately yields the asymptotic expansion (where the remainder term is uniform on the rectangle $I+i[-\varepsilon,0)$)

$$\begin{aligned}\Gamma_L^\chi(E) &= -2i\sum_{j>0} e^{-2ij(L-k)\theta_{p,L}(E)}\theta'_{p,L}(E)M(E)+\int_{\mathbb{R}} \frac{\chi_{\operatorname{Re}E}(\lambda)}{\lambda-E}\begin{pmatrix} dN_k^-(\lambda) & 0 \\ 0 & dN_0^+(\lambda)\end{pmatrix}+O(L^{-\infty}) \\ &= \frac{-e^{-i(L-k)\theta_{p,L}(E)}}{\sin((L-k)\theta_{p,L}(E))}\theta'_{p,L}(E)M(E)+\int_{\mathbb{R}} \frac{\chi_{\operatorname{Re}E}(\lambda)}{\lambda-E}\begin{pmatrix} dN_k^-(\lambda) & 0 \\ 0 & dN_0^+(\lambda)\end{pmatrix}+O(L^{-\infty}).\end{aligned} \tag{5-14}$$

This completes the proof of Lemma 5.4. □

5A2. *The proof of Theorem 5.2.* To prove (5-1), for $\operatorname{Im}E<-\varepsilon$ it suffices to write

$$\begin{aligned}\sum_{j=0}^{L}\frac{|\varphi_j(0)|^2}{\lambda_j-E}-\int_{\mathbb{R}}\frac{dN_0^+(\lambda)}{\lambda-E} &= \langle\delta_0,(H_L-E)^{-1}\delta_0\rangle-\langle\delta_0,(H_0^+-E)^{-1}\delta_0\rangle \\ &= \langle\delta_0,(H_L-E)^{-1}\delta_L\rangle\langle\delta_{L+1},(H_0^+-E)^{-1}\delta_0\rangle\end{aligned}$$

and

$$\sum_{j=0}^{L}\frac{|\varphi_j(L)|^2}{\lambda_j-E}-\int_{\mathbb{R}}\frac{dN_k^-(\lambda)}{\lambda-E}=\langle\delta_0,(H_L-E)^{-1}\delta_L\rangle\langle\delta_{L+1},(H_k^- -E)^{-1}\delta_0\rangle,$$

$$\sum_{j=0}^{L}\frac{\varphi_j(L)\overline{\varphi_j(0)}}{\lambda_j-E}=\langle\delta_L,(H_L-E)^{-1}\delta_0\rangle,$$

and to use the Combes–Thomas estimate (5-6). This completes the proof of Theorem 5.2.

5B. ***The proofs of Theorems 1.5, 1.7 and 1.10.*** We will now use Theorems 5.1 and 5.2 to prove Theorems 1.5, 1.7 and 1.10.

5B1. *The proof of Theorem 1.5.* The first statement of Theorem 1.5 is an immediate consequence of the characteristic equations for the resonances (2-4) and (2-8) and the description of the eigenvalues of H_L given in Theorem 4.2.

When $\bullet=\mathbb{N}$, i.e., for the operator on the half-line, if $I\subset(-2,2)$ does not meet $\Sigma_{\mathbb{N}}$, there exists $C>0$ such that, for L sufficiently large, $\operatorname{dist}(I,\sigma(H_L))>1/C$. Thus, on the set $I-i[0,+\infty)$, one has $\operatorname{Im}S_L(E)\le\operatorname{Im}E/C$. Since on I one has $\operatorname{Im}\theta_p(E)>1/C$ (see Section 2), the characteristic equation (2-4) admits a solution E such that $\operatorname{Re}E\in I$ only if $\operatorname{Im}E<1/C^2$. This completes the proof of Theorem 1.5(1) for $\bullet=\mathbb{N}$.

For $\bullet=\mathbb{Z}$, i.e., to study (2-8), one reasons in the same way except that one replaces the study of $S_L(E)$ by that of $\langle\Gamma_L(E)u,u\rangle$ for u an arbitrary vector in $\mathbb{C}^2$ of unit length. This completes the proof of Theorem 1.5(1).

Point (3a) is an immediate consequence of Theorems 3.3 and 3.2 and the description of the eigenvalues of H_L outside $\Sigma_{\mathbb{Z}}$. Notice that, in the present case, d_j in Theorems 3.3 and 3.2 is bounded from below by a constant independent of L, and $a_j^\bullet$ is exponentially small and described by Theorem 4.2.

Point (3b) is an immediate consequence of the description of the eigenvalues of H_L outside $\Sigma_{\mathbb{Z}}$ in Theorems 5.2(2) and 3.1. Indeed, in the present case, d_j and $a_j^\bullet$ are both of order 1; thus, Theorem 3.1 guarantees, around the common eigenvalue for H_k^- and H_0^+, a rectangle of width of order 1 free of resonances.

Let us now turn to the proof of point (2). We first prove the following corollary of Theorem 5.1:

Corollary 5.5. *Fix $I\subset\Sigma_{\mathbb{Z}}^\circ$ compact. There exists $\eta_0>0$ such that, for L sufficiently large, one has*

$$\min_{\substack{\operatorname{Re}E\in I\\ \operatorname{Im}E\in[-\eta_0/L,0)}}|S_L(E)+e^{-i\theta(E)}|\ge\eta_0\quad\textit{and}\quad\min_{\substack{\operatorname{Re}E\in I\\ \operatorname{Im}E\in[-\eta_0/L,0)}}|\det(\Gamma_L(E)+e^{-i\theta(E)})|\ge\eta_0.\tag{5-15}$$

Clearly, Corollary 5.5 implies that neither (2-4) nor (2-8) can have a solution in $I+i]-\eta_0/L,0]$. This proves Theorem 1.5(2).

Before proving Corollary 5.5, we first prove Propositions 5.7 and 5.8, as these will be used in the proof of Corollary 5.5.

5B2. *Results on the auxiliary functions defined in Section 1B2.* Recall that N_k^- is defined in Section 1B2. We prove:

Proposition 5.6. *For $k \in \{0, \dots, p-1\}$, dN_k^- is a positive measure that is absolutely continuous on $\Sigma_{\mathbb{Z}}$. Moreover, its density, say $E \mapsto n_k^-(E)$, is real analytic on $\Sigma_{\mathbb{Z}}^\circ$ and there exists $f_k^- : \Sigma_{\mathbb{Z}}^\circ \to \mathbb{R}$ a positive real analytic function such that, on $\Sigma_{\mathbb{Z}}^\circ$, one has $n_k^-(E) = f_k^-(E) n(E)$.*

Proof. Proposition 5.6 is an immediate consequence of Theorems 5.1 and 5.2 and Corollary 4.4. □

For Ξ_k^- defined in (1-5), we prove:

Proposition 5.7. *Ξ_k^- vanishes identically if and only if $V \equiv 0$, i.e., V vanishes identically. Moreover, if $V \not\equiv 0$ then there exists $\xi_k^- \neq 0$ and $\alpha_k^- \in \{2, 3, \dots\}$ such that $\Xi_k^-(E) \sim \xi_k^- E^{-\alpha_k^-}$ as $|E| \to \infty$, $\operatorname{Im} E < 0$.*

Proof. We will do the proofs for the function Ξ_k^-. Proposition 5.7 is an immediate consequence of the fact that, in the lower half-plane, the function $E \mapsto -e^{-i \arccos(E/2)} = -\frac{1}{2}E - \sqrt{\frac{1}{4}E^2 - 1}$ (i.e., the choice of it defined above) is equal to the Stieltjes (or Borel) transform of the spectral measure associated to the Dirichlet Laplacian on $\mathbb{N}$ and the vector δ_0; this follows from a direct computation (see Remark 2.1 and (2-2) for $n = 0$). Now, if one lets W be the symmetric of $\tau_k V$ with respect to 0, the spectral measure dN_k^- is also the spectral measure of the Schrödinger operator $H_k = -\Delta + W$ on $\mathbb{N}$ associated to δ_0. The equality of the Borel transforms implies the equality of the measures but δ_0 is cyclic for both operators, so the operators have equal spectral measures. This implies that the two operators are equal and, thus, the symmetric of $\tau_k V$ has to vanish identically on $\mathbb{N}$. As V is periodic, V must vanish identically.

As for the second point, if the function Ξ_k^- were to vanish to infinite order at $E = -i\infty$, as each of the terms $\int_{\mathbb{R}} 1/(\lambda - E)\, dN_k^-(\lambda)$ and $-\frac{1}{2}E - \sqrt{\frac{1}{4}E^2 - 1}$ admits an infinite asymptotic expansion in powers of E^{-1}, these two expansions would be equal. The n-th coefficient of these expansions are the n-th moments of the spectral measures of H_k and $-\Delta_0^+$, respectively (associated to the cyclic vector δ_0). So these moments would coincide and, thus, the spectral measures would coincide. One concludes as above. □

For $c^\bullet$ defined in (1-6) and (1-7), we prove:

Proposition 5.8. *Pick $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$. Let $I \subset (-2, 2) \cap \Sigma_{\mathbb{Z}}^\circ$ be a compact interval.*

There exists a neighborhood of I such that, in this neighborhood, the function $E \mapsto c^\bullet(E)$ is analytic and has a positive imaginary part.

The function $c^{\mathbb{N}}$ (resp. $c^{\mathbb{Z}}$) takes the value i only at the zeros of Ξ_k^- (resp. $\Xi_k^- \Xi_0^+$).

Proof. On $\{\operatorname{Im} E < 0\}$, define the functions

$$g_k^-(E) := i + \frac{\Xi_k^-(E)}{\pi n_k^-(E)} = \frac{1}{\pi n_k^-(E)}(S_k^-(E) + e^{-i \arccos(E/2)}), \tag{5-16}$$

$$g_0^+(E) := i + \frac{\Xi_0^+(E)}{\pi n_0^+(E)} = \frac{1}{\pi n_0^+(E)}(S_0^+(E) + e^{-i \arccos(E/2)}). \tag{5-17}$$

First, the analyticity of g_k^- and g_0^+ is clear; indeed, all the functions involved are analytic and the functions n_0^+ and n_k^- stay positive on $\Sigma_{\mathbb{Z}}^\circ$. Moreover, these functions can be analytically continued through

$(-2,2)\cap\Sigma^\circ_{\mathbb{Z}}$. By (1-4), for E real one has $\operatorname{Im} g_k^-(E)=\operatorname{Im} g_0^+(E)=\operatorname{Im} e^{-i\theta(E)}$, which is positive (see Section 2). Thus the functions $E\mapsto g_k^-(E)$ and $E\mapsto g_0^+(E)$ do not vanish on I. Moreover, as

$$\frac{g_0^+(E)g_k^-(E)-1}{g_0^+(E)+g_k^-(E)}=-\frac{1}{g_0^+(E)+g_k^-(E)}+\frac{1}{1/g_0^+(E)+1/g_k^-(E)},\tag{5-18}$$

this function has a positive imaginary part on I.

This proves the first two properties of $c^\bullet$ stated in Proposition 5.8. By the very definition of $c^\bullet$ and g_k^-, the last property stated in Proposition 5.8 is obviously satisfied in the case of the half-line; for the full line, i.e., if $\bullet=\mathbb{Z}$, the last property is a consequence of the computation

$$\begin{aligned}c^{\mathbb{Z}}(E)-i&=\frac{g_0^+(E)g_k^-(E)-1}{g_0^+(E)+g_k^-(E)}-i=\frac{(g_0^+(E)-i)(g_k^-(E)-i)}{g_0^+(E)+g_k^-(E)}\\&=\frac{\Xi_0^+(E)\Xi_k^-(E)}{2i\pi^2n_0^+(E)n_k^-(E)+\pi n_k^-(E)\Xi_0^+(E)+\pi n_0^+(E)\Xi_k^-(E)}.\end{aligned}\tag{5-19}$$

This completes the proof of Proposition 5.8. □

5B3. *The proof of Corollary 5.5.* In view of Theorem 5.1, to obtain (5-15) it suffices to prove that there exists $\eta_0>0$ such that, for L sufficiently large, one has

$$\min_{\substack{\operatorname{Re}E\in I\\ \operatorname{Im}E\in[-\eta_0/L,0)}}\left|\frac{\theta'_{p,L}(E)f_k^-(E)e^{-iu_L(E)}}{\sin u_L(E)}-\int_{\mathbb{R}}\frac{dN_k^-(\lambda)}{\lambda-E}-e^{-i\theta(E)}\right|\ge\eta_0,$$

where $u_L(E):=(L-k)\theta_{p,L}(E)$.

We compute

$$\frac{\theta'_{p,L}(E)f_k^-(E)e^{-iu_L(E)}}{\sin u_L(E)}-\int_{\mathbb{R}}\frac{dN_k^-(\lambda)}{\lambda-E}-e^{-i\theta(E)}=\theta'_{p,L}(E)f_k^-(E)(\cot u_L(E)-g_k^-(E)),\tag{5-20}$$

where g_k^- is as defined in (5-16).

Thus,

$$\left|\frac{\theta'_{p,L}(E)f_k^-(E)e^{-iu_L(E)}}{\sin u_L(E)}-\int_{\mathbb{R}}\frac{dN_k^-(\lambda)}{\lambda-E}-e^{-i\theta(E)}\right|\gtrsim|\cot u_L(E)-g_k^-(E)|$$

as, for η sufficiently small and $L\ge1$, one has

$$0<\min_{\substack{\operatorname{Re}E\in I\\ \operatorname{Im}E\in[-\eta/L,0)}}|\theta'_{p,L}(E)f_k^-(E)|\le\max_{\substack{\operatorname{Re}E\in I\\ \operatorname{Im}E\in[-\eta/L,0)}}|\theta'_{p,L}(E)f_k^-(E)|<+\infty.$$

Now notice that by Corollary 4.4, for $E\in I$, one has

$$\operatorname{Im}\left(\int_{\mathbb{R}}\frac{dN_k^-(\lambda)}{\lambda-E}\right)=-\theta'_{p,L}(E)f_k^-(E)=-\frac{1}{\pi}n_k^-(E).\tag{5-21}$$

Thus, as $E\mapsto\operatorname{Im} e^{-i\theta(E)}$ is positive on I, the analytic function $E\mapsto g_k^-(E)$ has positive imaginary part larger than, say $2\tilde\eta$ on I; hence, it has imaginary part larger than, say, $\tilde\eta$ in some neighborhood of $I+\overline{D(0,\eta_0)}$ (for sufficiently small $\eta_0>0$). Let M be the maximum modulus of this function on $I+\overline{D(0,\eta_0)}$. Then, as $\max_{\operatorname{Re} E\in I,\,\operatorname{Im} E\in[-\eta_0/L,0)}|\theta'_{p,L}(E)|\lesssim 1$, one has

$$\max_{\substack{\operatorname{Re} E\in I\\ \operatorname{Im} E\in[-\eta_0/L,0)\\ |\cot(u_L(E))|<2M}} |\operatorname{Im}\cot u_L(E)|\lesssim (M^2+1)\eta_0.$$

Possibly reducing η_0, this guarantees that, for $\operatorname{Re} E\in I$ and $\operatorname{Im} E\in[-\eta_0/L,0)$, one has

$$|\cot u_L(E)-g_k^-(E)|\geq 2M-M\geq M \quad\text{or}\quad \operatorname{Im}(\cot u_L(E)-g_k^-(E))\leq -\tilde\eta+\tfrac12\tilde\eta=-\tfrac12\tilde\eta.$$

This completes the proof of the first lower bound in (5-15) in Corollary 5.5.

To prove the second bound in (5-15), using (5-2) we compute

$$\begin{aligned}\frac{\det(\Gamma_L^{\mathrm{eff}}(E)+e^{-i\theta(E)})}{n_k^-(E)n_0^+(E)}&=(\cot u_L(E)-g_k^-(E))(\cot u_L(E)-g_0^+(E))-\frac{1}{\sin^2 u_L(E)}\\ &=-(g_0^+(E)+g_k^-(E))\left(\cot u_L(E)-\frac{g_0^+(E)g_k^-(E)-1}{g_0^+(E)+g_k^-(E)}\right),\end{aligned}\tag{5-22}$$

where g_k^- and g_0^+ are defined by (5-16) and (5-17).

Using Proposition 5.8, one then concludes the nonvanishing of $E\mapsto\det(\Gamma_L^{\mathrm{eff}}(E)+e^{-i\theta(E)})$ in the complex rectangle $\{\operatorname{Re} E\in I,\ \operatorname{Im} E\in[-\eta_0/L,0)\}$ (for η_0 sufficiently small) in the same way as above. This completes the proof of Corollary 5.5.

5B4. *The proof of Theorem 1.7.* To solve (2-4) and (2-8), by Theorem 5.1, we first solve the equations

$$\frac{\theta'_{p,L}(E)f_k^-(E)e^{-iu_L(E)}}{\sin u_L(E)}=\int_{\mathbb{R}}\frac{dN_k^-(\lambda)}{\lambda-E}-e^{-i\theta(E)}\quad\text{and}\quad\det(\Gamma_L^{\mathrm{eff}}(E)+e^{-i\theta(E)})=0\tag{5-23}$$

in a rectangle $I+i[-\eta,-\tilde\eta/L]$. Indeed, in such a rectangle, by Theorem 5.1 equations (2-4) and (2-8) are equivalent to

$$\begin{aligned}&\frac{\theta'_{p,L}(E)f_k^-(E)e^{-iu_L(E)}}{\sin u_L(E)}=\int_{\mathbb{R}}\frac{dN_k^-(\lambda)}{\lambda-E}-e^{-i\theta(E)}+O(L^{-\infty})\\ \text{and}\quad&\det(\Gamma_L^{\mathrm{eff}}(E)+e^{-i\theta(E)})=O(L^{-\infty}),\end{aligned}\tag{5-24}$$

respectively, where the terms $O(L^{-\infty})$ are analytic in a rectangle $\tilde I+i[-2\eta,-0)$ (where $I\subset\tilde I$) and the bound $O(L^{-\infty})$ holds in the supremum norm.

Thanks to (5-20) for $\bullet=\mathbb{N}$ and to (5-22) for $\bullet=\mathbb{Z}$, to solve the equations (5-23) it suffices to solve

$$\cot u_L(E)=c^\bullet(E),\tag{5-25}$$

where we recall $u_L(E):=(L-k)\theta_{p,L}(E)$, g_0^+ and g_k^- are as defined in (5-17) and (5-16), respectively, and, as in Section 1B3, we have set

- $c^{\mathbb{N}}(E):=g_k^-(E)$ in the case of the half-line,

- $c^{\mathbb{Z}}(E) := \dfrac{g_0^+(E)g_k^-(E)-1}{g_0^+(E)+g_k^-(E)}$ in the case of the line.

We want to solve (5-25) is a rectangle $I+i[-\varepsilon,0)$ for some ε small but fixed. Using Proposition 5.8, we pick ε so small that, in the rectangle $I+i[-\varepsilon,0]$, the only zeros of $c^\bullet-i$ are those on the real line and $\operatorname{Im} c^\bullet$ is positive in $I+i[-\varepsilon,0)$.

To solve (5-25), we change variables $u=(L-k)\theta_{p,L}(E)$, that is, we write

$$E=\theta_{p,L}^{-1}\left(\frac{u}{L-k}\right).$$

As, for L_0 sufficiently large, $\inf_{L\geq L_0,\,E\in I+i[-\varepsilon,0)}\operatorname{Re}\theta'_{p,L}(E)>c>0$, at the cost of possibly reducing ε this real analytic change of variables maps $I+[-\varepsilon,\varepsilon]+i[-\varepsilon,0)$ into, say, D_L such that $I_L+i[-\eta(L-k),0]\subset D_L$ (for some $\eta>0$), where $I_L=(L-k)\theta_{p,L}\big(I+\big[-\frac12\varepsilon,\frac12\varepsilon\big]\big)$; the inverse change of variable maps $I_L+i[-\eta(L-k),0]$ into some domain, say $\widetilde{D}_L$, such that $I+[-\varepsilon',\varepsilon']+i[-\varepsilon',0]\subset\widetilde{D}_L$ (for some $0<\varepsilon'<\varepsilon$). Now, to find all the solutions to (5-25) in $I+i[-\varepsilon',0)$, we first solve the following equation in $I_L+i[-\eta(L-k),0]$:

$$\cot u=c^\bullet\circ\theta_{p,L}^{-1}\left(\frac{u}{L-k}\right) \tag{5-26}$$

As $u\mapsto\cot u$ is π-periodic, we split $I_L+i[-\eta(L-k),0]$ into vertical strips of the type

$$l\pi+[0,\pi]+i[-\eta(L-k),0],\qquad l_-\leq l\leq l_+,\ (l_-,l_+)\in\mathbb{Z}^2.$$

Without loss of generality, we may assume that $I_L=[l_-,l_+]\pi$. To solve (5-26) on the rectangle $l\pi+[0,\pi]+i[-\eta(L-k),0]$, we shift u by $l\pi$ and solve the following equation on $[0,\pi]+i[-\eta(L-k),0]$:

$$\cot u=c^\bullet_{l,L}(u),\qquad\text{where}\quad c^\bullet_{l,L}(\,\cdot\,):=c^\bullet\circ\theta_{p,L}^{-1}\left(\frac{\cdot+l\pi}{L-k}\right). \tag{5-27}$$

In proving Theorem 1.5, we have already shown that, for some $\tilde\eta>0$ (independent of L sufficiently large and $l_-\leq l\leq l_+$), (5-27) does not have a solution in $[0,\pi]+i[-\tilde\eta,0]$. The cotangent is an analytic one-to-one mapping from $[0,\pi)+i(-\infty,0]$ to $\mathbb{C}^+\setminus\{i\}$. Thus, for L sufficiently large and $\tilde\eta$ sufficiently small, the cotangent defines a one-to-one mapping from $[0,\pi)+i[-\eta(L-k),-\tilde\eta]$ onto $T_L=\overline{D(z_+,r_+)\setminus D(z_-,r_-)}$, analytic in the interior of $[0,\pi)+i[-\eta(L-k),-\tilde\eta]$ and continuous up to the boundary, where we have defined

$$z_+=i\frac{e^{4\eta(L-k)}+1}{e^{4\eta(L-k)}-1},\qquad z_-=i\frac{e^{4\tilde\eta}-1}{e^{4\tilde\eta}-1},\qquad r_+=\frac{2e^{2\tilde\eta}}{e^{4\tilde\eta}-1},\qquad r_-=\frac{2e^{2\eta(L-k)}}{e^{4\eta(L-k)}-1}.$$

Moreover, the boundaries $\{0\}+i[-\eta(L-k),-\tilde\eta]$ and $\{\pi\}+i[-\eta(L-k),-\tilde\eta]$ are mapped onto the interval $[z_-+ir_-,z_++ir_+]$.

Let $\widetilde{Z}^\bullet$ denote the finite set of zeros of $E\mapsto c^\bullet(E)-i$ in I. Then, by a Taylor expansion near the zeros of $c-i$, we know that, for η sufficiently small, there exist $\varepsilon_0>0$ and $\tilde k\geq1$ such that, for L sufficiently large:

- For $\varepsilon \in (0, \varepsilon_0)$, there exists $0 < \eta_-$ such that, for $l_- \le l \le l_+$, if one has
$$\left|\theta_{p,L}^{-1}\left(\frac{l\pi}{L-k}\right) - \widetilde{E}\right| \ge \varepsilon \quad \text{for all } \widetilde{E} \in \widetilde{Z}^\bullet,$$
then one has $\eta_- \le |\operatorname{Im} c^\bullet_{l,L}(u) - 1|$ for all $u \in [0,\pi] + i[-\eta(L-k), 0]$.
- For $u \in [0,\pi] + i[-\eta(L-k), 0]$ and $\widetilde{E}$ the point in $\widetilde{Z}^\bullet$ closest to $\theta_{p,L}^{-1}(l\pi/(L-k))$, one has
$$\varepsilon_0 \le (1 - \operatorname{Im} c^\bullet_{l,L}(u)) \cdot \left[\left|\theta_{p,L}^{-1}\left(\frac{\operatorname{Re} u + l\pi}{L-k}\right) - \widetilde{E}\right| + \frac{|\operatorname{Im} u|}{L-k}\right]^{-\tilde{k}} \le \frac{1}{\varepsilon_0}, \tag{5-28}$$
where $\tilde{k}$ is the order of $\widetilde{E}$ as a zero of $E \mapsto c^\bullet(E) - i$.

As a consequence of the above description of $c^\bullet_{l,L}$, we obtain:

Lemma 5.9. *There exists $\tilde{\eta}$ and η small such that, for L sufficiently large, for all $l_- \le l \le l_+$, $u \mapsto c^\bullet_{l,L}(u)$ maps the rectangle $[0,\pi] + i[-\eta(L-k), -\tilde{\eta}]$ into a compact subset of $D(z_+, r_+) \setminus D(z_-, r_-)$ in such a way that*
$$\sup_{u \in \partial([0,\pi]+i[-\eta(L-k),-\tilde{\eta}])} |\cot u - c^\bullet_{l,L}(u)| \gtrsim \left(\left|\widetilde{E} - \theta_{p,L}^{-1}\left(\frac{l\pi}{L-k}\right)\right| + \frac{\tilde{\eta}}{L-k}\right)^{\tilde{k}}, \tag{5-29}$$
where $\widetilde{E}$ is the root of $E \mapsto c^\bullet(E) - i$ closest to $\theta_{p,L}^{-1}(l\pi/(L-k))$ and $\tilde{k}$ is the order of this root.

Note that, under the assumptions of Lemma 5.9, (5-29) implies that
$$\sup_{u \in \partial([0,\pi]+i[-\eta(L-k),-\tilde{\eta}])} |\cot u - c^\bullet_{l,L}(u)| \gtrsim L^{-\tilde{k}}.$$

Thus we can define the analytic mapping $\cot^{-1} \circ c^\bullet_{l,L}$ on $[0,\pi] + i[-\eta(L-k), -\tilde{\eta}]$; it maps the rectangle $[0,\pi] + i[-\eta(L-k), -\tilde{\eta}]$ into a compact subset of $(0,\pi) + i(-\eta(L-k), -\tilde{\eta})$. Equation (5-27) on $[0,\pi] + i[-\eta(L-k), -\tilde{\eta}]$ is, thus, equivalent to the fixed point equation on the same rectangle,
$$u = \cot^{-1} \circ c^\bullet_{l,L}(u) \tag{5-30}$$

We note that, for $\alpha \in (0,1)$ and L sufficiently large, if for some $\widetilde{E} \in \widetilde{Z}^\bullet$ of multiplicity $\tilde{k}$ one has $|\theta_{p,L}^{-1}(l\pi/(L-k)) - \widetilde{E}| < L^{-\alpha}$, then (5-27) has no solution in $[0,\pi] + i[-\eta(L-k), -\tilde{\eta}]$ outside of the set
$$R_{l,L} := [0,\pi] + i\left[-\eta(L-k), \frac{\alpha\tilde{k}}{4} \log\left[\left|\theta_{p,L}^{-1}\left(\frac{l\pi}{L-k}\right) - \widetilde{E}\right| + \frac{1}{L}\right]\right].$$

Indeed, for $u \in ([0,\pi] + i[-\eta(L-k), -\tilde{\eta}]) \setminus R_{l,L}$, by (5-28), that is, for
$$0 \le \operatorname{Re} u \le \pi \quad \text{and} \quad -\frac{\alpha\tilde{k}}{4} \log L \le \frac{\alpha\tilde{k}}{4} \log\left[\left|\theta_{p,L}^{-1}\left(\frac{l\pi}{L-k}\right) - \widetilde{E}\right| + \frac{1}{L}\right] \le \operatorname{Im} u \le -\tilde{\eta},$$
one has $|c^\bullet_{l,L}(u) - i| \lesssim L^{-\alpha\tilde{k}}$ and $|\cot u - i| \gtrsim L^{-\alpha\tilde{k}/2}$.

So if for some $\widetilde{E} \in \widetilde{Z}^\bullet$ one has $|\theta_{p,L}^{-1}(l\pi/(L-k)) - \widetilde{E}| < L^{-\alpha}$, it suffices to solve (5-30) on $R_{l,L}$. We compute the derivative of $c_{l,L}^\bullet$ in the interior of $R_{l,L}$:

$$\frac{d}{du}(\cot^{-1} \circ c_{l,L}^\bullet)(u) = -\frac{1}{L-k} \frac{c' \circ \theta_{p,L}^{-1}((u+l\pi)/(L-k))}{1+(c_{l,L}^\bullet(u))^2} \cdot \frac{1}{\theta_{p,L}'\big(\theta_{p,L}^{-1}((u+l\pi)/(L-k))\big)}$$
$$= \frac{1}{L-k} \frac{c' \circ \theta_{p,L}^{-1}((u+l\pi)/(L-k))}{c_{l,L}^\bullet(u)-i} \cdot \frac{1}{c_{l,L}^\bullet(u)+i} \cdot \frac{1}{\theta_{p,L}'\big(\theta_{p,L}^{-1}((u+l\pi)/(L-k))\big)}.$$

Thus, fixing $\alpha \in (0, 1)$:

- If l is such that for some $\widetilde{E} \in \widetilde{Z}^\bullet$ one has $|\theta_{p,L}^{-1}(l\pi/(L-k)) - \widetilde{E}| < L^{-\alpha}$, then for $u \in R_{l,L}$ we estimate

$$\begin{aligned}\left|\frac{d}{du}(\cot^{-1} \circ c_{l,L}^\bullet)(u)\right| & \\ &\lesssim \frac{1}{L-k}\left[\left|\theta_{p,L}^{-1}\left(\frac{l\pi}{L-k}\right) - \widetilde{E}\right| + \frac{|\operatorname{Im} u|}{L-k}\right]^{-1} \\ &\lesssim \frac{1}{(L-k)|\theta_{p,L}^{-1}(l\pi/(L-k)) - \widetilde{E}| + \big|\log\big[|\theta_{p,L}^{-1}(l\pi/(L-k)) - \widetilde{E}| + \tilde{\eta}/(L-k)\big]\big|} \\ &\lesssim \frac{1}{\log L}. \end{aligned} \tag{5-31}$$

- If l is such that for all $\widetilde{E} \in \widetilde{Z}^\bullet$ one has $|\theta_{p,L}^{-1}(l\pi/(L-k)) - \widetilde{E}| \geq L^{-\alpha}$, for $u \in [0, \pi] + i[-\eta(L-k), -\tilde{\eta}]$ we estimate

$$\begin{aligned}\left|\frac{d}{du}(\cot^{-1} \circ c_{l,L}^\bullet)(u)\right| &\lesssim \frac{1}{L-k}\left[\left|\theta_{p,L}^{-1}\left(\frac{l\pi}{L-k}\right) - \widetilde{E}\right| + \frac{|\operatorname{Im} u|}{L-k}\right]^{-1} \\ &\lesssim \frac{1}{(L-k)|\theta_{p,L}^{-1}(l\pi/(L-k)) - \widetilde{E}|} \lesssim \frac{1}{L^{1-\alpha}}. \end{aligned} \tag{5-32}$$

Hence, for L sufficiently large, $\cot^{-1} \circ c_{l,L}^\bullet$ is a contraction on $R_{l,L}$. Equation (5-30) thus admits a unique solution, say $\tilde{u}_{l,L}^\bullet$, in the rectangle $[0, \pi] + i[-\eta(L-k), -\tilde{\eta}]$. This solution is a simple root of $u \mapsto u - \cot^{-1} \circ c_{l,L}^\bullet(u)$. Hence, $\tilde{u}_{l,L}^\bullet$ is the only solution to (5-27) in $[0, \pi] + i[-\eta(L-k), -\tilde{\eta}]$.

By (5-24), for L sufficiently large and $l_- \leq l \leq l_+$, both the equations

$$\begin{aligned} S_L \circ \theta_{p,L}^{-1}\left(\frac{u+l\pi}{L-k}\right) + e^{-i\theta(\theta_{p,L}^{-1}((u+l\pi)/(L-k)))} &= 0, \\ \det\left(\Gamma_L \circ \theta_{p,L}^{-1}\left(\frac{u+l\pi}{L-k}\right) + e^{-i\theta(\theta_{p,L}^{-1}((u+l\pi)/(L-k)))}\right) &= 0, \end{aligned} \tag{5-33}$$

can be rewritten as

$$u = \cot^{-1}(c_{l,L}^\bullet(u) + O(L^{-\infty})) = \cot^{-1} \circ c_{l,L}^\bullet(u) + O(L^{-\infty}) \tag{5-34}$$

in $[0, \pi] + i[-\eta(L-k), -\tilde{\eta}]$.

Thus each of the equations in (5-33) admits a single solution in $[0,\pi]+i[-\eta(L-k),-\tilde\eta]$ and this root is simple; moreover, this solution, say $u_{l,L}$, satisfies $|u^\bullet_{l,L}-\tilde u^\bullet_{l,L}|=O(L^{-\infty})$; indeed, the bounds (5-31) and (5-32) guarantee that one can apply Rouché's theorem on the disk $D(\tilde u^\bullet_{l,L},L^{-k})$ for any $k\geq 0$.

Thus, we have proved:

Lemma 5.10. *Pick I as above. Then there exists $\eta>0$ such that, for L sufficiently large with $L=Np+k$, the resonances in $I+i[-\eta,0]$ are the energies $(z^\bullet_l)_{l_-\leq l\leq l_+}$ defined by*

$$z^\bullet_l=\theta^{-1}_{p,L}\left(\frac{u^\bullet_{l,L}+l\pi}{L-k}\right), \tag{5-35}$$

belonging to $I+i[-\eta,0]$.

Let us complete the proof of Theorem 1.14, that is, prove that, for η sufficiently small and L sufficiently large such that $L\equiv k \bmod p$, $z^\bullet_l$ is the unique resonance in $\left[\frac12\operatorname{Re}(\tilde z^\bullet_l+\tilde z^\bullet_{l-1}),\frac12\operatorname{Re}(\tilde z^\bullet_l+\tilde z^\bullet_{l+1})\right]+i[-\eta,0]$; recall that $\tilde z^\bullet_l$ is defined in (1-9).

We first note that the Taylor expansion of $\theta^{-1}_{p,L}$, (4-1) and the quantization condition (4-3) imply that

$$z^\bullet_l=\lambda_l+\frac{1}{\pi n(\lambda_l)L}u^\bullet_{l,L}+O\left(\left(\frac{\log L}{L}\right)^2\right)$$

as $\operatorname{Re}u_{l,L}\in[0,\pi)$ and $-\log L\lesssim\operatorname{Im}u_{l,L}\lesssim -1$.

Moreover, as

$$c^\bullet_{l,L}(u)=c^\bullet\left[\lambda_l+\frac{u}{\pi n(\lambda_l)L}+O\left(\frac{u^2}{L^2}\right)\right],$$

using (1-9) and (5-35) we compute

$$z^\bullet_l-\tilde z^\bullet_l=\frac{1}{\pi n(\lambda_l)L}\left(u^\bullet_{l,L}-\cot^{-1}\circ c^\bullet\left[\lambda_l+\frac{1}{\pi n(\lambda_l)L}\cot^{-1}\circ c^\bullet\left(\lambda_l-i\frac{\log L}{L}\right)\right]\right)+O\left(\left(\frac{\log L}{L}\right)^2\right).$$

Thus, one has

$$z^\bullet_l-\tilde z^\bullet_l=\frac{1}{\pi n(\lambda_l)L}\left(u^\bullet_{l,L}-\cot^{-1}\circ c^\bullet_{l,L}\left[\cot^{-1}\circ c^\bullet_{l,L}(-i\pi n(\lambda_l)\log L)\right]\right)+O\left(\left(\frac{\log L}{L}\right)^2\right).$$

As $u_{l,L}$ solves (5-34), sing (5-31) and (5-32) we thus obtain that

$$\begin{aligned}|z^\bullet_l-\tilde z^\bullet_l|&\lesssim\frac{1}{L\log L}\left|u^\bullet_{l,L}-\cot^{-1}\circ c^\bullet_{l,L}(-i\pi n(\lambda_l)\log L)\right|+\left(\frac{\log L}{L}\right)^2\\&\lesssim\frac{|u^\bullet_{l,L}|+\log L}{L\log^2 L}+\left(\frac{\log L}{L}\right)^2\lesssim\frac{1}{L\log L},\end{aligned}$$

using again $\operatorname{Re}u_{l,L}\in[0,\pi)$ and $-\log L\lesssim\operatorname{Im}u_{l,L}\lesssim -1$.

Taking into account (1-10), this completes the proof of Theorem 1.7.

5B5. *The proofs of Propositions 1.8 and 1.9.* Proposition 1.9 is an immediate consequence of Theorem 1.7, the definition (1-9) of $\tilde{z}_l^\bullet$ and the standard asymptotics of cot near $-i\infty$, i.e., $\cot z = i + 2ie^{-2iz} + O(e^{-4iz})$.

To prove Proposition 1.8, it suffices to notice that, under the assumptions of Proposition 1.8, the bound (5-32) on the derivative of $\cot^{-1} \circ c_{l,L}^\bullet$ on the rectangle $R_{l,L}$ becomes

$$\left| \frac{d}{du} (\cot^{-1} \circ c_{l,L}^\bullet)(u) \right| \lesssim \frac{1}{L}.$$

Thus, as a solution to (5-30), $u_{l,L}^\bullet$ admits an asymptotic expansion in inverse powers of L. Plugging this into (5-35) yields the asymptotic expansion for the resonance. Then (1-11) follows from the computation of the first terms.

5B6. *The proof of Theorem 1.10.* Theorem 1.10 is an immediate consequence of Theorem 5.2, the fact that the functions are analytic in the lower complex half-plane and have only finitely many zeros there, and the argument principle.

5C. ***The half-line periodic perturbation: the proof of Theorem 1.11.*** Using the same notations as above, we can write

$$H^\infty = \begin{pmatrix} H_{-1}^- & |\delta_{-1}\rangle\langle\delta_0| \\ |\delta_0\rangle\langle\delta_{-1}| & -\Delta_0^+ \end{pmatrix},$$

where $-\Delta_0^+$ is the Dirichlet Laplacian on $\ell^2(\mathbb{N})$.

Define the operators

$$\Gamma(E) := H_{-1}^- - E - \langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle |\delta_{-1}\rangle\langle\delta_{-1}|,$$
$$\widetilde{\Gamma}(E) := -\Delta_0^+ - E - \langle\delta_{-1}|(H_{-1}^- - E)^{-1}|\delta_{-1}\rangle |\delta_0\rangle\langle\delta_0|.$$

For $\operatorname{Im} E \neq 0$, $\langle\delta_{-1}|(H_{-1}^- - E)^{-1}|\delta_{-1}\rangle$ and $\langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle$ have a nonvanishing imaginary part of the same sign; hence, the complex number

$$\left(\langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle\right)^{-1} - \langle\delta_{-1}|(H_{-1}^- - E)^{-1}|\delta_{-1}\rangle$$

does not vanish. Thus, by rank-one perturbation theory, (see, e.g., [Simon 2005]), we know that $\Gamma(E)$ and $\widetilde{\Gamma}(E)$ are invertible and their inverses are given by

$$\Gamma^{-1}(E) := (H_{-1}^- - E)^{-1} + \frac{\left|(H_{-1}^- - E)^{-1}|\delta_{-1}\rangle\langle\delta_{-1}|(H_{-1}^- - E)^{-1}\right|}{\left(\langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle\right)^{-1} - \langle\delta_{-1}|(H_{-1}^- - E)^{-1}|\delta_{-1}\rangle} \tag{5-36}$$

and

$$\widetilde{\Gamma}^{-1}(E) := (-\Delta_0^+ - E)^{-1} + \frac{\left|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle\langle\delta_0|(-\Delta_0^+ - E)^{-1}\right|}{\left(\langle\delta_{-1}|(H_{-1}^- - E)^{-1}|\delta_{-1}\rangle\right)^{-1} - \langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle}. \tag{5-37}$$

Thus, for $\operatorname{Im} E \neq 0$, using Schur's complement formula we compute

$$(H^\infty - E)^{-1} = \begin{pmatrix} \Gamma(E)^{-1} & \gamma(E) \\ \gamma^*(\bar{E}) & \widetilde{\Gamma}(E)^{-1} \end{pmatrix}, \tag{5-38}$$

where $\gamma^*(\bar{E})$ is the adjoint of $\gamma(\bar{E})$ and

$$\gamma(E) := -|\Gamma(E)^{-1}|\delta_{-1}\rangle\langle\delta_0|(-\Delta_0^+ - E)^{-1}|.$$

Now, when coming from $\operatorname{Im} E > 0$ and passing through $(-2, 2) \cap \Sigma_{\mathbb{Z}}^{\circ}$, the complex numbers

$$\langle\delta_{-1}|(H_{-1}^- - E)^{-1}|\delta_{-1}\rangle \quad \text{and} \quad \langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle$$

keep imaginary parts of the same positive sign; thus, the two operator-valued functions $E \mapsto \Gamma^{-1}(E)$ and $E \mapsto (H^\infty - E)^{-1}$ can be analytically continued through $(-2, 2) \cap \Sigma_{\mathbb{Z}}^{\circ}$ from the upper to the lower complex half-plane (as operators from $\ell^2_{\text{comp}}(\mathbb{N})$ to $\ell^2_{\text{loc}}(\mathbb{N})$ and from $\ell^2_{\text{comp}}(\mathbb{Z})$ to $\ell^2_{\text{loc}}(\mathbb{Z})$, respectively).

When coming from the upper half-plane and passing through $(-2, 2) \setminus \Sigma_{\mathbb{Z}}$ and $\Sigma_{\mathbb{Z}}^{\circ} \setminus [-2, 2]$, (5-38) also provides an analytic continuation of $(H^\infty - E)^{-1}$. Definition (5-36) and formula (5-38) immediately show that the poles of these continuations only occur at the zeros of the function

$$E \mapsto 1 - \langle\delta_{-1}|(H_{-1}^- - E)^{-1}|\delta_{-1}\rangle\langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle = 1 - e^{i\theta(E)} \int_{\mathbb{R}} \frac{dN_{p-1}^-(\lambda)}{\lambda - E}$$

when continued from the upper half-plane through the sets $(-2, 2) \setminus \Sigma_{\mathbb{Z}}$ and $\Sigma_{Z}^{\circ} \setminus [-2, 2]$ (these sets are finite unions of open intervals).

This completes the proof of Theorem 1.11.

6. Resonances in the random case

As for the periodic potential, for the random potential we start with a description of the function $E \mapsto \Gamma_L(E)$ (see (2-9)), that is, with a description of the spectral data for the Dirichlet operator $H_{\omega,L}$.

6A. *The matrix Γ_L in the random case.* We recall a number of results on the Dirichlet eigenvalues of $H_{\omega,L}$ that will be used in our analysis.

It is well known that, under our assumptions, in dimension one the whole spectrum of H_ω is in the localization region (see, e.g., [Kunz and Souillard 1980; Cycon et al. 1987; Carmona and Lacroix 1990]), that is:

Theorem 6.1. *There exists $\rho > 0$ and $\alpha \in (0, 1)$ such that one has*

$$\sup_{\substack{L\in\mathbb{N}\cup\{+\infty\}\\ y\in[\![0,L]\!]\\ \operatorname{Im} E\neq 0}} \mathbb{E}\left\{ \sum_{x\in[\![0,L]\!]} e^{\rho|x-y|} |\langle\delta_x, (H_{\omega,L} - E)^{-1}\delta_y\rangle|^\alpha \right\} < \infty \tag{6-1}$$

and

$$\sup_{\substack{L\in\mathbb{N}\cup\{+\infty\}\\ y\in[\![0,L]\!]}} \mathbb{E}\left\{ \sum_{x\in[\![0,L]\!]} e^{\rho|x-y|} \sup_{\substack{\operatorname{supp} f\subset\mathbb{R}\\ |f|\leq 1}} |\langle\delta_x, f(H_{\omega,L})\delta_y\rangle| \right\} < \infty, \tag{6-2}$$

where $H_{\omega,+\infty} := H_\omega^{\mathbb{N}}$ and $[\![0, +\infty]\!] = \mathbb{N}$. The supremum is taken over the functions f that are Borelian and compactly supported.

As a consequence, one can define localization centers, e.g., by means of the following results:

Lemma 6.2 [Germinet and Klopp 2014]. *Fix $(l_L)_L$ a sequence of scales, i.e., $l_L \to +\infty$ as $L \to +\infty$. There exists $\rho > 0$ such that, for L sufficiently large, with probability larger than $1 - e^{-\ell_L}$, if*

(1) *$\varphi_{j,\omega}$ is a normalized eigenvector of $H_{\omega,L}$ associated to $E_{j,\omega}$ in Σ,*

(2) *$x_j(\omega) \in [\![0, L]\!]$ is a maximum of $x \mapsto |\varphi_{j,\omega}(x)|$ in $[\![0, L]\!]$,*

then for $x \in [\![0, L]\!]$ one has

$$|\varphi_{j,\omega}(x)| \le \sqrt{L} e^{2\ell_L} e^{-\rho|x-x_j(\omega)|}. \tag{6-3}$$

Note that Lemma 6.2 is of interest only if $\ell_L \lesssim L$; otherwise (6-3) is obvious. This result can, for example, be applied for the scales $l_L = 2 \log L$. In this case, the probability estimate of the bad sets (i.e., when the conclusions of Lemma 6.3 does not hold) is summable. The point $x_j(\omega)$ is a localization center for $E_{j,\omega}$ or $\varphi_{j,\omega}$. It is not defined uniquely, but, one easily shows that there exists $C > 0$ such that for any two localization centers, say x and x', one has $|x - x'| \le C \log L$ (see [Germinet and Klopp 2014]). For concreteness, we set the localization center associated to the eigenvalue $E_{j,\omega}$ to be the leftmost maximum of $x \mapsto \|\varphi_{j,\omega}\|_x$.

We show:

Lemma 6.3. *For any $p > 0$, there exist $C > 0$ and $L_0 > 0$ (depending on α and p) such that, for $L \ge L_0$, for any sequence satisfying (1-22), with probability at least $1 - L^{-p}$ there exist at most $C\ell_L$ eigenvalues having a localization center in $[\![0, \ell_L]\!] \cup [\![L - \ell_L, L]\!]$.*

We will now use the fact that we are dealing with one-dimensional systems to improve upon the estimate (6-3). We prove:

Theorem 6.4. *For any $\delta > 0$ and $p \ge 0$, there exist $C > 0$ and $L_0 > 0$ (depending on p and δ) such that, for $L \ge L_0$, with probability at least $1 - L^{-p}$ if $E_{j,\omega}$ is an eigenvalue in Σ associated to the eigenfunction $\varphi_{j,\omega}$ and the localization center $x_{j,\omega}$ then:*

- *If $x_{j,\omega} \in [\![0, L - C \log L]\!]$, one has*

$$-\rho(E_{j,\omega}) - \delta \le \frac{\log |\varphi_{j,\omega}(L)|}{L - x_{j,\omega}} \le -\rho(E_{j,\omega}) + \delta. \tag{6-4}$$

- *If $x_{j,\omega} \in [\![C \log L, L]\!]$, one has*

$$-\rho(E_{j,\omega}) - \delta \le \frac{\log |\varphi_{j,\omega}(0)|}{x_{j,\omega}} \le -\rho(E_{j,\omega}) + \delta. \tag{6-5}$$

To analyze the resonances of $H^{\mathbb{N}}_{\omega,L}$ (resp. $H^{\mathbb{Z}}_{\omega,L}$), we shall use (6-4) (resp. (6-4) and (6-5)).

We now use these estimates as the starting point of a short digression from the main theme of this paper. Let us first state a corollary to Theorem 6.4; we prove:

Theorem 6.5. *For any $\delta > 0$ and $p \ge 0$, for L sufficiently large (depending on p and δ), with probability at least $1 - L^{-p}$, if $E_{j,\omega}$ is an eigenvalue in Σ associated to the eigenfunction $\varphi_{j,\omega}$ and the localization* *center* $x_{j,\omega}$ *then, for* $|x - x_{j,\omega}| \geq \delta L$ *and* $1 \leq x \leq L$, *one has*

$$-\rho(E_{j,\omega}) - \delta \leq \frac{\log\bigl(|\varphi_{j,\omega}(x)| + |\varphi_{j,\omega}(x-1)|\bigr)}{|x - x_{j,\omega}|} \leq -\rho(E_{j,\omega}) + \delta. \tag{6-6}$$

Compare (6-6) to (6-3). There are two improvements. First, the unknown rate of decay ρ is replaced by the Lyapunov exponent $\rho(E_{j,\omega})$, which was expected to be the correct decay rate. Indeed, for the one-dimensional discrete Anderson model on the half-axis, it is well known (see, e.g., [Bougerol and Lacroix 1985; Carmona and Lacroix 1990; Pastur and Figotin 1992]) that, ω-almost surely, the spectrum is localized and the eigenfunctions decay exponentially at infinity at a rate given by the Lyapunov exponent. In Theorem 6.5, we state that, with good probability, this is true for finite volume restrictions.

Second, in (6-6), we get both an upper and lower bound on the eigenfunction. This is more precise than (6-3).

To our knowledge, such a result was not known until the present paper. The strategy that we use to prove this result can be applied in a more general one-dimensional setting to obtain analogues of (6-6) (see [Klopp $\geq$ 2016]).

We complement this with the much simpler:

Lemma 6.6. *For any* $C > 0$ *and* $p \geq 0$, *there exists* $K > 0$ *and* $L_0 > 0$ (*depending on* p *and* C) *such that, for* $L \geq L_0$, *with probability at least* $1 - L^{-p}$ *if* $E_{j,\omega}$ *is an eigenvalue in* Σ *associated to the eigenfunction* $\varphi_{j,\omega}$ *and the localization center* $x_{j,\omega}$ *then*:

- *If* $x_{j,\omega} \in [\![L - C\log L, L]\!]$, *one has* $L^{-K} \leq |\varphi_{j,\omega}(L)|$.
- *If* $x_{j,\omega} \in [\![0, C\log L]\!]$, *one has* $L^{-K} \leq |\varphi_{j,\omega}(0)|$.

The proof of this result is obvious and only uses the fact that the matrices in the cocycle defining the operator (see Section 6C) are bounded, that is, equivalently, that the solutions to the Schrödinger equation grow at most exponentially at a rate controlled by the potential.

Let us return to the resonances in the random case and the description of the function S_L. Recall that in (2-4) the values $(\lambda_j)_j$ are the eigenvalues $(E_{j,\omega})_{0\leq j\leq L}$ of $H_{\omega,L}$ and the coefficients $(a^\bullet_j)_j$ are defined in Theorem 2.2 and by (2-13). Thus, Theorem 6.4 describes the coefficients $(a^\bullet_j)_j$ coming into S_L and Γ_L (see (2-4) and (2-8)). Let us now state a few consequences of Theorem 6.4.

Fix a compact interval I in Σ, the almost sure spectrum of H_ω. For $\bullet \in \{\mathbb{N}, \mathbb{Z}\}$, define

$$d^\bullet_{j,\omega} = \begin{cases} L - x_{j,\omega} & \text{for } \bullet = \mathbb{N}, \\ \min(x_{j,\omega}, L - x_{j,\omega}) & \text{for } \bullet = \mathbb{Z}. \end{cases} \tag{6-7}$$

Taking $p > 2$ in Theorem 6.4 and using a Borel–Cantelli argument, we obtain:

ω-almost surely, for $\delta > 0$ and L sufficiently large,

$$\text{if } \lambda_j = E_{j,\omega} \in I \text{ and } d^\bullet_{j,\omega} \geq C\log L \text{ then } -2\rho(\lambda_j) - \delta \leq \frac{\log a^\bullet_j}{d^\bullet_{j,\omega}} \leq -2\rho(\lambda_j) + \delta. \tag{6-8}$$

This and the continuity of the Lyapunov exponent (see, e.g., [Bougerol and Lacroix 1985; Carmona and Lacroix 1990; Pastur and Figotin 1992]) guarantees that

$$\omega\text{-almost surely, for any } \delta>0 \text{ and } L \text{ large, one has } -2\eta_{\bullet}\sup_{E\in I}\rho(E)(1+\delta)L\le\inf_{\lambda_j\in I}\log a_j^{\bullet}, \tag{6-9}$$

where $\eta_{\bullet}$ is as defined in Theorem 1.13.

To use the analysis performed in Section 3, we also need a description for the $(\lambda_j)_j$, i.e., the Dirichlet eigenvalues of $H_{\omega,L}$. To this end, we will use the results of [Germinet and Klopp 2014; Klopp 2011; 2013] (see also [Germinet and Klopp 2011]).

We first recall the Minami estimate satisfied by $H_{\omega,L}$ (see, e.g., [Combes et al. 2009] and references therein): there exists $C>0$ such that, for $I\subset\mathbb{R}$, one has

$$\mathbb{P}\big(\operatorname{tr}(\mathbf{1}_I(H_{\omega,L}))\ge 2\big)\le\mathbb{E}\big(\operatorname{tr}(\mathbf{1}_I(H_{\omega,L}))\big[\operatorname{tr}(\mathbf{1}_I(H_{\omega,L}))-1\big]\big)\le C|I|^2(L+1)^2.$$

Here $\mathbf{1}_I(H)$ denotes the spectral projector for the selfadjoint operator H onto the energy interval I.

By a simple covering argument, this entails the estimate

$$\mathbb{P}(|\lambda_i-\lambda_j|\le L^{-q} \text{ for some } i\ne j)\le CL^{-q+2}.$$

Thus, for $q>3$, a Borel–Cantelli argument yields that

$$\omega\text{-almost surely, for } L \text{ sufficiently large, } \min_{i\ne j}|\lambda_i-\lambda_j|\ge L^{-q}. \tag{6-10}$$

6B. ***The proofs of the main results in the random case.*** We are now going to prove the results stated in Section 1C.

6B1. *The proof of Theorem 1.13.* As for Theorem 1.5, this result follows from Theorem 3.1. Point (1) is proved exactly as Theorem 1.5(1). Point (2) follows immediately from Theorem 3.1 and (6-9). This completes the proof of Theorem 1.13.

6B2. *The proof of Theorem 1.14.* Recall that $\kappa\in(0,1)$. To prove (1) we proceed as follows. The standard result guaranteeing the existence of the density of states N (see, e.g., [Bougerol and Lacroix 1985; Carmona and Lacroix 1990; Pastur and Figotin 1992]) implies that, ω-almost surely, one has

$$\frac{\#\{\lambda_j\in I\}}{L+1}\to\int_I dN(E). \tag{6-11}$$

This, in particular, shows that if $I\subset\Sigma^{\circ}$ is a compact interval then, ω-almost surely, for L sufficiently large I is covered by intervals of the form $[\lambda_j,\lambda_{j+1}]$ and their number is of size $\asymp L$ (actually this holds for $\lambda_j\in I+[-\varepsilon,\varepsilon]$ if $\varepsilon>0$ is chosen small enough). Moreover, the estimate (6-10) guarantees that $d_j\ge L^{-q}$ (for any $q>3$ fixed) for all $\lambda_j\in I$. Thus, Theorems 3.1, 3.2 and 3.3 and the estimate (6-8) guarantee that, ω-almost surely, all the resonances in the strip $I-i[e^{-L^{\kappa}},0)$ are described by Theorem 3.3. Indeed, for such a resonance the imaginary part must be larger than $-e^{-L^{\kappa}}$; thus, by Theorem 3.1, for every rectangle $\big[\frac12(\lambda_j+\lambda_{j-1}),\frac12(\lambda_j+\lambda_{j+1})\big]-i[e^{-L^{\kappa}},0)$ containing a resonance, one has $a_j^{\bullet}\lesssim e^{-L^{\kappa}}L^{2q}$ Thus, $a_j^{\bullet}\ll d_j^2$ and one can apply Theorem 3.3 to compute the resonance.

Let us count the number of those resonances. To this end, let $\ell_L = \tau L^\kappa$, where τ is to be chosen. By (6-8) and (6-10), ω-almost surely one has $a_j^\bullet \ll d_j^2$ for all j such that $\lambda_j \in I$ as long as the Dirichlet eigenvalue λ_j is associated to a localization center in $[\![0, L-\ell_L]\!]$ (actually this holds for $\lambda_j \in I + [-\varepsilon, \varepsilon]$ if $\varepsilon > 0$ is chosen small enough); thus, we can apply Theorems 3.3 and 3.2 to each of the $(\lambda_j)_j$ that are associated to a localization center in $[\![0, L-\ell_L]\!]$. By (3-19), each of these eigenvalues gives rise to a single simple resonance, the imaginary part of which is of size $\asymp a_j^\bullet$; they lie above the line $\{\operatorname{Im} z \geq e^{-\rho \ell_L} = e^{-L^\kappa}\}$ for $\tau\rho = 1$. Actually, the estimate (6-10) guarantees that $d_j \geq L^{-q}$ (for any $q > 3$ fixed) and Theorem 3.2 shows that these resonances are the only ones above the line $\operatorname{Im} z \geq -L^{-q}$. Moreover, by Lemma 6.3, we know there at most $C\ell_L$ eigenvalues λ_j that do not have their localization center in $[\![0, L-\ell_L]\!]$. Thus we obtain, ω-almost surely,

$$\lim_{L\to+\infty} \frac{1}{L} \#\{z \text{ resonance of } H_{\omega,L} \text{ with } \operatorname{Re} z \in I,\ \operatorname{Im} z \geq -e^{-L^\kappa}\} = \int_I dN(E).$$

Point (2) is proved in the same way. Pick $\lambda \in (0,1)$. In addition to what was used above, one uses the continuity of the density of states $E \mapsto n(E)$ and the Lyapunov exponent $E \mapsto \rho(E)$. Assume E is as in point (2). Then, ω-almost surely, the reasoning done above shows that, for any $\eta > 0$, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, for L sufficiently large one has

$$\begin{aligned}
&\#\left\{\lambda_l \text{ eigenvalue of } H^{\mathbb{N}}_{\omega,L} \text{ in } E + \frac{\varepsilon}{2n(E)}[-1+\eta, 1-\eta] \text{ with } -e^{\eta_\bullet \rho(E)\delta L} \lesssim e^{2\eta_\bullet \rho(E)\lambda L} a_l^\bullet \lesssim -e^{-\eta_\bullet \rho(E)\delta L}\right\} \\
&\qquad \leq \#\{z \text{ resonance of } H^\bullet_{\omega,L} \text{ in } R^\bullet(E, \lambda, L, \varepsilon, \delta)\} \\
&\qquad \leq \#\Big\{\lambda_l \text{ eigenvalue of } H^{\mathbb{N}}_{\omega,L} \text{ in } E + \frac{\varepsilon}{2n(E)}[-1-\eta, 1+\eta] \\
&\qquad\qquad\qquad \text{with } -e^{\eta_\bullet \rho(E)\delta L} \lesssim e^{2\eta_\bullet \rho(E)\lambda L} a_l^\bullet \lesssim -e^{-\eta_\bullet \rho(E)\delta L}\Big\}.
\end{aligned}$$

Using Theorem 6.4 and the continuity of the Lyapunov exponent in conjunction with the definition of a_j (see (2-4) and (2-13)), we obtain that, ω-almost surely, for any $\eta > 0$ there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, for L sufficiently large one has

$$\begin{aligned}
&\#\left\{\text{eigenvalue of } H^{\mathbb{N}}_{\omega,L} \text{ in } E + \frac{\varepsilon}{2n(E)}[-1+\eta, 1-\eta] \text{with localization center in } I^\bullet(L, \delta, -\eta)\right\} \\
&\quad \leq \#\{z \text{ resonance of } H^\bullet_{\omega,L} \text{ in } R^\bullet(E, \lambda, L, \varepsilon, \delta)\} \\
&\quad \leq \#\left\{\text{eigenvalue of } H^{\mathbb{N}}_{\omega,L} \text{ in } E + \frac{\varepsilon}{2n(E)}[-1-\eta, 1+\eta] \text{ with localization center in } I^\bullet(L, \delta, \eta)\right\},
\end{aligned}$$

where $I^{\mathbb{N}}(L, \lambda, \delta, \eta)$ is the interval — here $[r]$ denotes the integer part of $r \in \mathbb{R}$ —

$$I^{\mathbb{N}}(L, \lambda, \delta, \eta) = [L\lambda] + [\![-L\delta(1+\eta), L\delta(1+\eta)]\!]$$

and $I^{\mathbb{Z}}(L, \lambda, \delta, \eta)$ is the union of intervals

$$I^{\mathbb{Z}}(L, \lambda, \delta, \eta) = \left(\left[\tfrac{1}{2}L\lambda\right] + [\![-L\delta(1+\eta), L\delta(1+\eta)]\!]\right) \cup \left(\left[L\left(1 - \tfrac{1}{2}\lambda\right)\right] + [\![-L\delta(1+\eta), L\delta(1+\eta)]\!]\right).$$

Now, using the exponential localization of the eigenfunctions, one has that, ω-almost surely, for any $\eta > 0$ there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, for L sufficiently large one has

$$\begin{aligned}\#\left\{\text{eigenvalue of } H^{\mathbb{N}}_{\omega,L,\lambda,\delta,-2\eta,\bullet} \text{ in } E+\frac{\varepsilon}{2n(E)}[-1+2\eta,1-2\eta]\right\}\\ \leq \#\left\{z \text{ resonance of } H^{\bullet}_{\omega,L} \text{ in } R^{\bullet}(E,\lambda,L,\varepsilon,\delta)\right\}\\ \leq \#\left\{\text{eigenvalue of } H^{\mathbb{N}}_{\omega,L,\lambda,\delta,2\eta,\bullet} \text{ in } E+\frac{\varepsilon}{2n(E)}[-1-2\eta,1+2\eta]\right\},\end{aligned} \tag{6-12}$$

where $H^{\mathbb{N}}_{\omega,L,\lambda,\delta,\eta,\bullet} = (H^{\mathbb{N}}_{\omega,L})_{|I^{\bullet}(L,\lambda,\delta,\eta)}$ with Dirichlet boundary conditions at the edges of the interval $I^{\bullet}(L,\lambda,\delta,\eta)$.

This immediately yields point (2) for $\lambda \in (0,1)$, using (6-11) for the operators $H^{\mathbb{N}}_{\omega,L,\lambda,\delta,\eta,\bullet}$. The case $\lambda = 1$ is dealt with in the same way.

As already said, point (3) is an “integral” version of point (2). Using the same ideas as above, partitioning $I = \bigcup_{p=0}^{P} I_p$ so that $|I_p| \sim \varepsilon$ centered in E_p, one proves

$$\begin{aligned}\sum_{p=0}^{P}\#\left\{\text{eigenvalue of } H^{-}_{\omega,p,L,\bullet} \text{ in } E_p+\frac{\varepsilon}{2n(E_p)}[-1+2\eta,1-2\eta]\right\}\\ \leq \#\left\{z \text{ resonance of } H^{\bullet}_{\omega,L} \text{ in } I+[-e^{-L^{\kappa}},-e^{-cL}]\right\}\\ \leq \sum_{p=0}^{P}\#\left\{\text{eigenvalue of } H^{+}_{\omega,p,L,\bullet} \text{ in } E_p+\frac{\varepsilon}{2n(E_p)}[-1-2\eta,1+2\eta]\right\},\end{aligned}$$

where

- $H^{-}_{\omega,p,L,\bullet}$ is the operator $H^{\mathbb{N}}_{\omega}$ restricted to
  - $[\![2L^{\kappa}, (\inf(c\rho^{-1}(E_p),1)-\eta)L]\!]$ if $\bullet = \mathbb{N}$,
  - $[\![2L^{\kappa}, (\inf(c\rho^{-1}(E_p),1)/2-\eta)L]\!] \cup [\![(1-\inf(c\rho^{-1}(E_p),1)/2+\eta)L, L-2L^{\kappa}]\!]$ if $\bullet = \mathbb{Z}$;
- $H^{+}_{\omega,p,L,\bullet}$ is the operator $H^{\mathbb{N}}_{\omega}$ restricted to
  - $[\![L^{\kappa}/2, (\inf(c\rho^{-1}(E_p),1)+\eta)L]\!]$ if $\bullet = \mathbb{N}$,
  - $[\![L^{\kappa}/2, (\inf(c\rho^{-1}(E_p),1)/2+\eta)L]\!] \cup [\![(1-\inf(c\rho^{-1}(E_p),1)/2-\eta)L, L-L^{\kappa}/2]\!]$ if $\bullet = \mathbb{Z}$.

In the computation above, we used the continuity of both the density of states $E \mapsto n(E)$ and the Lyapunov exponent $E \mapsto \rho(E)$. Thus, we obtain

$$\begin{aligned}\#\left\{z \text{ resonance of } H^{\bullet}_{\omega,L} \text{ in } I+(-\infty,e^{-cL}]\right\}\\ = L\left(\sum_{p=0}^{P}\inf(c\rho^{-1}(E_p),1)n(E_p)|I_p|+o(1)\right)+\#\{z \text{ resonance of } H^{\bullet}_{\omega,L} \text{ in } I+(-\infty,e^{-L^{\kappa}}]\}.\end{aligned}$$

The last term being controlled by Theorem 1.17, one obtains point (3) as the Riemann sum in the right-hand side above converges to the integral in the right-hand side of (1-18) as $\varepsilon \to 0$. This completes the proof of Theorem 1.14.

6B3. *The proof of Theorem 1.15.* The proof of Theorem 1.15 relies on [Germinet and Klopp 2014, Theorem 1.13], which describes the local distribution of the eigenvalues and localization centers $(E_{j,\omega}, x_{j,\omega})$; namely, one has

$$\lim_{L\to+\infty} \mathbb{P}\Big(\big\{\omega \,\big|\, \#\{n \mid E_{j,\omega} \in E+L^{-1}I_n,\ x_{j,\omega} \in LC_n\} = k_n \ \text{ for } n = 1,\dots,p\big\}\Big) = \prod_{n=1}^{p} e^{-\tilde\mu_n}\frac{(\tilde\mu_n)^{k_n}}{k_n!}, \quad (6\text{-}13)$$

where $\tilde\mu_n := n(E)|I_n||C_n|$ for $1 \le n \le p$.

Recall that $(z_j^L(\omega))_j$ are the resonances of $H_{\omega,L}$. By the argument used in the proof of Theorem 1.14, we know that, ω-almost surely, all the resonances in $K_L := [E-\varepsilon, E+\varepsilon] + i[-e^{-L^\kappa}, 0]$ are constructed from the $(\lambda_j, a_j^\bullet)$ by formula (3-19). Thus, up to renumbering, the rescaled real and imaginary parts (see (1-19)) become

$$x_j = (\operatorname{Re} z_{l,L}^\bullet(\omega) - E)L = (\lambda_j - E)L + O(La_j) = (E_{j,\omega} - E)L + O(Le^{-L^\kappa}),$$
$$y_j = -\frac{1}{2L}\log|\operatorname{Im} z_{l,L}^\bullet(\omega)| = -\frac{\log a_j^\bullet}{2L} + O(1/L) = \rho(E)\frac{d_{j,\omega}^\bullet}{L} + o(1),$$

where $\lambda_j = E_{j,\omega}$ and $d_{j,\omega}^\bullet$ is defined as in (6-7); here we used the continuity of $E \mapsto \rho(E)$.

On the other hand, for the resonances below the line in $\{\operatorname{Im} z = -e^{-L^\kappa}\}$, one has $y_j \lesssim L^{\kappa-1}$. So all these resonances are "pushed upwards" towards the upper half-plane. Hence, the statement of Theorem 1.15 is an immediate consequence of (6-13).

6B4. *The proof of Theorem 1.16.* Using the computations of the previous section, as $E \neq E'$, Theorem 1.16 is a direct consequence of [Klopp 2011, Theorem 1.2] (see also [Germinet and Klopp 2014, Theorem 1.11]).

6B5. *The proof of Theorem 1.17.* Consider equations (2-4) and (2-8). By Theorem 6.4 and Lemma 6.3, ω-almost surely, for L large the number of $(a_j^\bullet)_j$ larger than $e^{-10\ell_L}$ is bounded by $C\ell_L$. Solving (2-4) and (2-8) in the strip $\{\operatorname{Re} E \in I,\ \operatorname{Im} E < -e^{-\ell_L}\}$, we can write $S_L(E) = S_L^-(E) + S_L^+(E)$, where

$$S_L^-(E) := \sum_{a_j^{\mathbb{N}} \le e^{-10\ell_L}} \frac{a_j^{\mathbb{N}}}{\lambda_j - E} \quad \text{and} \quad S_L^+(E) := \sum_{a_j^{\mathbb{N}} > e^{-10\ell_L}} \frac{a_j^{\mathbb{N}}}{\lambda_j - E},$$

and similarly decompose $\Gamma_L(E) = \Gamma_L^-(E) + \Gamma_L^+(E)$. For L large, one then has

$$\sup_{\operatorname{Im} E < -e^{-\ell_L}} \|S_L^-(E)\| + \|\Gamma_L^-(E)\| \le e^{-8\ell_L}. \quad (6\text{-}14)$$

The count of the number of resonances given by the proof of Theorems 2.2 and 2.3 then shows that the equations (2-4) and (2-8), where S_L and Γ_L are respectively replaced by S_L^+ and Γ_L^+, have at most $C\ell_L$ solutions in the lower half-plane. We will call the equations where S_L and Γ_L are replaced by S_L^+ and Γ_L^+ the $+$-equations. The analogues of Theorems 3.1, 3.2 and 3.3 for the $+$-equations and Theorem 6.4 show that the only solutions to the $+$-equations in the strip $\{\operatorname{Re} E \in I,\ -e^{-4\ell_L/5} < \operatorname{Im} E < -e^{-3\ell_L/4}\}$ are given by formulas (3-19) and (3-20) for the eigenvalues of the Dirichlet problem associated to a localization center in $[\![L-2\ell_L, L-\tfrac{1}{2}\ell_L]\!]$ if $\bullet = \mathbb{N}$ and in $[\![\tfrac{1}{2}\ell_L, 2\ell_L]\!] \cup [\![L-2\ell_L, L-\tfrac{1}{2}\ell_L]\!]$ if $\bullet = \mathbb{Z}$. Thus, these

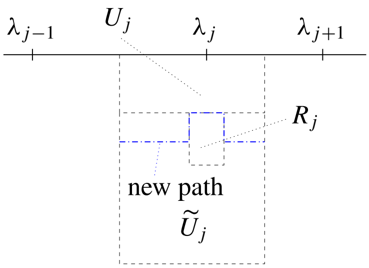

Figure 8. The new path (in blue).

zeros are simple and separated by a distance at least L^{-4} from each other (recall (6-10)). Moreover, we can cover the interval I by intervals of the type $\left[\frac{1}{2}(\lambda_j+\lambda_{j-1}), \frac{1}{2}(\lambda_j+\lambda_{j+1})\right]$, that is, one can write

$$I \subset \bigcup_{j=j^-}^{j^+} \left[\tfrac{1}{2}(\lambda_j+\lambda_{j-1}), \tfrac{1}{2}(\lambda_j+\lambda_{j+1})\right], \tag{6-15}$$

where $\lambda_{j^- -1} \notin I$, $\lambda_{1+j^+} \notin I$, $\lambda_{j^-} \in I$ and $\lambda_{j^+} \in I$.

Consider now the line $\{\operatorname{Im} E = -e^{-\ell_L}\}$ and its intersection with the vertical strip

$$\left[\tfrac{1}{2}(\lambda_j+\lambda_{j-1}), \tfrac{1}{2}(\lambda_j+\lambda_{j+1})\right] - i\mathbb{R}^+.$$

Three things may occur:

(1) $e^{-\ell_L} < a_j^\bullet d_j^2 |\sin\theta(\lambda_j)|/C$ (the constant C is defined in Theorem 3.1); then, on the interval

$$\left[\tfrac{1}{2}(\lambda_j+\lambda_{j-1}), \tfrac{1}{2}(\lambda_j+\lambda_{j+1})\right] - ie^{-\ell_L},$$

one has

$$|S_L^+(E) + e^{-i\theta(E)}| \gtrsim 1 \quad \text{and} \quad |\det(\Gamma_L^+(E) + e^{-i\theta(E)})| \gtrsim 1; \tag{6-16}$$

this follows from the proof of Theorem 3.1 (see in particular (3-5), (3-6), (3-7) and (3-8)) for some fixed $c > 0$; recall that, on the interval $I + ie^{-\ell_L}$, one has $|\sin\theta(E)| \gtrsim 1$.

(2) $e^{-\ell_L} > Ca_j^\bullet$ (the constant C is defined in Theorem 3.2); then, on the interval

$$\left[\tfrac{1}{2}(\lambda_j+\lambda_{j-1}), \tfrac{1}{2}(\lambda_j+\lambda_{j+1})\right] - ie^{-\ell_L},$$

one has again (6-16) for a possibly different constant; this follows from the proof of Theorem 3.2 (see in particular (3-15) and (3-16)).

(3) If we are neither in case (1) nor in case (2), then the line $\{\operatorname{Im} E = -e^{-\ell_L}\}$ may cross R_j (defined in Theorem 3.3; see also Figure 7); we change the contour $\{\operatorname{Im} E = -e^{-\ell_L}\}$ so as to enter $\widetilde{U}_j$ until we reach the boundary of R_j and then follow this boundary, getting closer to the real axis, turning around R_j and finally reaching the line $\{\operatorname{Im} E = -e^{-\ell_L}\}$ again on the other side of R_j and following it up to the boundary of $\widetilde{U}_j$ (see Figure 8); on this new line, the bound (6-16) again holds; moreover, this new line is closer to the real axis than the line $\{\operatorname{Im} E = -e^{-\ell_L}\}$.

Let us call $\mathscr{C}_\ell$ the path obtained by gluing together the paths constructed in points (1)–(3) for $j^- \le j \le j^+$ and the half-lines $\frac{1}{2}(\lambda_{j^-} + \lambda_{j^- -1}) - i[e^{-\ell_L}, +\infty)$ and $\frac{1}{2}(\lambda_{j^+} + \lambda_{j^+ +1}) - i[e^{-\ell_L}, +\infty)$ (see (6-15)). One can then apply Rouché's theorem to compare the +-equations to the equations (2-4) and (2-8): by (6-14) and (6-16), on the line $\mathscr{C}_\ell$ one has $|S_L^-| < |S_L^+ + e^{-i\theta}|$ and

$$|\det(\Gamma_L(E) + e^{-i\theta(E)}) \det(\Gamma_L^+(E) + e^{-i\theta(E)})| \le \tfrac{1}{2} |\det(\Gamma_L(E) + e^{-i\theta(E)})|.$$

Thus, the number of solutions to equations (2-4) and (2-8) below the line $\mathscr{C}_\ell$ is bounded by $C\ell_L$; as $\mathscr{C}_\ell$ lies above $\{\operatorname{Im} E = -e^{-\ell_L}\}$, in the half-plane $\{\operatorname{Im} E < -e^{-\ell_L}\}$ the equations (2-4) and (2-8) have at most $C\ell_L$ solutions. We have proved Theorem 1.17.

6B6. *The proof of Theorem 1.18.* The first point in Theorem 1.18 is proved in the same way as point (2) in Theorem 1.14 up to the change of scales, L being replaced by ℓ_L. Pick scales $(\ell'_L)_L$ satisfying (1-22) such that $\ell'_L \ll \ell_L$. One has:

Lemma 6.7. *Fix two sequences $(a_L)_L$ and $(b_L)_L$ such that $a_L < b_L$. With probability one, for L sufficiently large,*

$$\begin{aligned}\#\big\{\text{eigenvalue of } H_{\omega,\ell_L - 2\ell'_L/\rho} \text{ in } [a_L + e^{-\ell'_L}, b_L - e^{-\ell'_L}]\big\} \\ \le \#\big\{\text{eigenvalue of } H_{\omega,L} \text{ in } [a_L, b_L] \text{ with localization center in } [\![0, \ell_L]\!]\big\} \\ \le \#\big\{\text{eigenvalue of } H_{\omega,\ell_L + 2\ell'_L/\rho} \text{ in } [a_L - e^{-\ell'_L}, b_L + e^{-\ell'_L}]\big\},\end{aligned}$$

where ρ is given by Lemma 6.2.

Proof. To prove Lemma 6.7, we apply Lemma 6.2 to $L = \ell_L + \ell'_L$ (i.e., for the operator H_ω restricted to the interval $[\![0, \ell_L + \ell'_L]\!]$) and $l_L = \ell'_L$. The probability of the bad set is the $O(L^{-\infty})$, thus summable in L. Using the localization estimate (6-3), one proves that

- each eigenvalue of $H_{\omega,\ell_L - 2\ell'_L/\rho}$ is at a distance of at most $e^{-\ell'_L}$ of an eigenvalue of $H_{\omega,L}$ with localization center in $[\![0, \ell_L]\!]$;
- each eigenvalue of $H_{\omega,L}$ with localization center in $[\![0, \ell_L]\!]$ is at a distance of at most $e^{-\ell'_L}$ of an eigenvalue of $H_{\omega,\ell_L + 2\ell'_L/\rho}$.

Lemma 6.7 follows. □

The first point in Theorem 1.18 is then Theorem 1.14(2) for the operators $H_{\omega,\ell_L - 2\ell'_L/\rho}$ and $H_{\omega,\ell_L + 2\ell'_L/\rho}$ and the fact that $\ell'_L \ll \ell_L$.

The proof of the second statement in Theorem 1.18 is very similar to that of Theorem 1.15. Fix a compact interval I in Σ°. As ℓ_L satisfies (1-22), one can find $\ell'_L < \ell''_L$ also satisfying (1-22) such that $e^{-\ell''_L} \ll e^{-\ell_L} \ll e^{-\ell'_L}$. For the same reasons as in the proof of Theorem 1.15, after rescaling all the resonances in $I - i(-\infty, 0)$ outside the strip $I - i[e^{-\ell'_L}, e^{-\ell''_L})$ are then pushed to either 0 or $i\infty$ as $L \to +\infty$.

On the other hand, the resonances in the strip $I - i[e^{-\ell'_L}, e^{-\ell''_L})$ are described by (3-19). The rescaled real and imaginary parts of the resonances (see (1-24)) now become $x_j = (E_{j,\omega} - E)\ell_L + o(1)$ and $y_j = \rho(E) d_{j,\omega}/\ell_L + o(1)$.

Now, to compute the limit of $\mathbb{P}(\#\{j \mid x_j \in I,\ y_j \in J\} = k)$, using the exponential decay property (6-3) it suffices to use [Germinet and Klopp 2014, Theorem 1.14]. Let us note here that [Germinet and Klopp 2014, Condition (1.50)] on the scales $(\ell_L)_L$ is slightly stronger than (1-22). That condition (1-22) suffices is a consequence of the stronger localization property known in the present case (compare Theorem 6.4 to [Germinet and Klopp 2014, Assumption (Loc)]). This completes the proof of the second point in Theorem 1.18. The final statement in 1.18 is proved in exactly the same way as Theorem 1.16.

The proof of Theorem 1.18 is complete.

6B7. *The proofs of Proposition 1.20 and Theorem 1.21.* Localization for the operator $H^{\mathbb{N}}_\omega$ can be described by the following:

Lemma 6.8. *There exists $\rho > 0$ and $q > 0$ such that, ω-almost surely, there exists $C_\omega > 0$ such that, for L sufficiently large, if*

(1) *$\varphi_{j,\omega}$ is a normalized eigenvector of $H_{\omega,L}$ associated to $E_{j,\omega}$ in Σ,*

(2) *$x_j(\omega) \in \mathbb{N}$ is a maximum of $x \mapsto |\varphi_{j,\omega}(x)|$ in $\mathbb{N}$,*

then, for $x \in \mathbb{N}$, one has

$$|\varphi_{j,\omega}(x)| \le C_\omega(1 + |x_j(\omega)|^2)^{q/2} e^{-\rho|x - x_j(\omega)|}. \tag{6-17}$$

Moreover, the mapping $\omega \mapsto C_\omega$ is measurable and $\mathbb{E}(C_\omega) < +\infty$.

This result for our model is a consequence of Theorem 6.1 (see, e.g., [Kunz and Souillard 1980; Cycon et al. 1987; Carmona and Lacroix 1990]) and [Germinet and Klopp 2014, Theorem 6.1].

We thus obtain the representation for the function Ξ_ω

$$\Xi_\omega(E) = \sum_j \frac{|\varphi_{j,\omega}(0)|^2}{E_{j,\omega} - E} + e^{-i\arccos(E/2)}. \tag{6-18}$$

As $H^{\mathbb{N}}_\omega$ satisfies a Dirichlet boundary condition at -1, one has

$$|\varphi_{j,\omega}(0)| > 0 \quad \text{for all } j \qquad \text{and} \qquad \sum_j |\varphi_{j,\omega}(0)|^2 = 1. \tag{6-19}$$

As $E \to -i\infty$, the representation (6-18) yields

$$\Xi_\omega(E) = -E^{-2} \sum_j |\varphi_{j,\omega}(0)|^2 E_{j,\omega} + O(E^{-3}) = -E^{-2}\langle \delta_0, H^{\mathbb{N}}_\omega \delta_0\rangle + O(E^{-3}) = -\omega_0 E^{-2} + O(E^{-3}).$$

This proves the first point in Proposition 1.20.

As a direct consequence of Theorem 6.1 and the computation leading to Theorem 5.2 (see Section 5A2), we obtain that there exists $\tilde{c} > 0$ such that, for L sufficiently large, with probability at least $1 - e^{-\tilde{c}L}$ one has

$$\sup_{\operatorname{Im} E \le -e^{-\tilde{c}L}} \left| \int_{\mathbb{R}} \frac{dN_\omega(\lambda)}{\lambda - E} - \langle \delta_0, (H_{\omega,L} - E)^{-1}\delta_0\rangle \right| \le e^{-\tilde{c}L}. \tag{6-20}$$

Taking

$$L = L_\varepsilon \sim c^{-1}|\log \varepsilon| \tag{6-21}$$

for some sufficiently small $c>0$, this and Rouché's theorem implies that, with probability $1-\varepsilon^3$, the number of zeros of Ξ_ω (counted with multiplicity) in $I+i(-\infty,\varepsilon]$ is bounded

- from above by the number of resonances of H_{ω,L_ε} in $I^+_\varepsilon+i(-\infty,-\varepsilon-\varepsilon^2]$,
- from below by the number of resonances of H_{ω,L_ε} in $I^-_\varepsilon+i(-\infty,-\varepsilon+\varepsilon^2]$,

where $I^+_\varepsilon=[a-\varepsilon,b+\varepsilon]$ and $I^+_\varepsilon=[a+\varepsilon,b-\varepsilon]$ if $I=[a,b]$.

Here, to apply Rouché's theorem, we apply the same strategy as in the proof of Theorem 1.17 and construct a path bounding a region larger (resp. smaller) than $I^+_\varepsilon+i(-\infty,-\varepsilon-\varepsilon^2]$ (resp. $I^-_\varepsilon+i(-\infty,-\varepsilon+\varepsilon^2]$) on which one can guarantee $|S_L(E)+e^{-i\theta(E)}|\gtrsim 1$.

Now, we choose the constant c (see (6-21)) to be so small that $c<\min_{E\in I}\rho(E)$. Applying point (3) of Theorem 1.14 to H_{ω,L_ε} with this constant c, we obtain that the number of resonances of H_{ω,L_ε} in $I^+_\varepsilon+i(-\infty,\varepsilon-\varepsilon^2]$ (resp. $I^-_\varepsilon+i(-\infty,\varepsilon+\varepsilon^2]$) is bounded from above (resp. bounded from below) by

$$L_\varepsilon\int_I\min\left(\frac{c}{\rho(E)},1\right)n(E)\,dE\,(1+O(1))=\frac{|\log\varepsilon|}{c}\int_I\frac{c}{\rho(E)}n(E)\,dE\,(1+O(1))$$
$$=|\log\varepsilon|\int_I\frac{n(E)}{\rho(E)}\,dE\,(1+O(1)).$$

Hence, we obtain the second point of Proposition 1.20. The last point of this proposition is then an immediate consequence of the arguments developed to obtain the second point if one takes into account the following facts:

- The minimal distance between the Dirichlet eigenvalues of $H^{\mathbb{N}}_{\omega,L}$ is bounded from below by L^{-4} (see (6-10)).
- The growth of the function $E\mapsto S_L(E)+e^{-i\theta(E)}$ near the resonances (i.e., its zeros) is well controlled by Proposition 3.4.

Indeed, this implies that the resonances of $H^{\mathbb{N}}_{\omega,L}$ are simple in $I+i[-e^{-\sqrt{L}},0)$ (one can choose larger rectangles) and that near each resonance one can apply Rouché's theorem to control the zero of Ξ_ω. Note that this also yields, ω-almost surely, there exists c_ω such that

$$\min_{\substack{z \text{ zero of } \Xi_\omega\\ z\in I+i(-\varepsilon_\omega,0)}}\ \inf_{0<r<\varepsilon_\omega(\operatorname{Im}z)^{3/2}}\ \min_{|E-z|=r}\frac{|\Xi_\omega(E)|}{r}\gtrsim 1. \tag{6-22}$$

This completes the proof of Proposition 1.20.

Theorem 1.21 is a consequence of the following:

Theorem 6.9. *There exists $\tilde c>0$ such that, ω-almost surely, for $L\ge 1$ sufficiently large one has*

$$\sup_{\substack{\operatorname{Re}E\in I\\ \operatorname{Im}E<-e^{-\tilde cL}}}\left|\Gamma_{L,\omega,\tilde\omega}(E)-\begin{pmatrix}\int_{\mathbb{R}}1/(\lambda-E)\,dN_{\tilde\omega}(\lambda) & 0\\ 0 & \int_{\mathbb{R}}1/(\lambda-E)\,dN_\omega(\lambda)\end{pmatrix}\right|+\left|S_{L,\omega}(E)-\int_{\mathbb{R}}\frac{dN_\omega(\lambda)}{\lambda-E}\right|\le e^{-\tilde cL},$$

where $\Gamma_{L,\omega,\tilde\omega}(E)$ (resp. $S_{L,\omega}(E)$) is the matrix $\Gamma_L(E)$ (resp. the function $S_L(E)$)—see (2-9)—constructed from the Dirichlet data on $[\![0,L]\!]$ of $-\Delta+V^{\mathbb{Z}}_{\omega,\tilde\omega,L}$ (resp. $-\Delta+V^{\mathbb{N}}_{\omega,L}$) (see (1-26)) using formula (2-9) (resp. (2-4)).

Theorem 6.9 is proved exactly as Theorem 5.2 except that one uses the localization estimates (6-2) instead of the Combes–Thomas estimates.

Theorem 1.21 is then an immediate consequence of the estimate (6-20). Indeed, this implies that if z is a resonance for, e.g., $H^N_{\omega,L}$ in $I+i(-\infty, e^{\tilde{c}L}]$, then $|\Xi_\omega(z)| \le e^{-\tilde{c}L}$. By the last point of Proposition 1.20, ω-almost surely we know that the multiplicity of the zeros of Ξ_ω is bounded by N_ω. Moreover, for the zeros of Ξ_ω in $I+i(-\varepsilon_\omega, 0)$, we know the bound (6-22). This bound and (6-20) imply that

$$\max_{\substack{z \text{ zero of } \Xi_\omega \\ z\in I+i(-\varepsilon_\omega, e^{-\tilde{c}L})}} \max_{|E-z|=e^{-\tilde{c}L}} \frac{|\Xi_\omega(E)-(S_{\omega,L}(E)+e^{-i\theta(E)})|}{|\Xi_\omega(E)|} < e^{-\tilde{c}L}.$$

This yields Theorem 1.21(2) by an application of Rouché's theorem. Point (1) is obtained in the same way, using Proposition 3.4, which gives

$$\max_{\substack{z \text{ resonance of } H^{\mathbb{N}}_{\omega,L} \\ z\in I+i(-\varepsilon_\omega, e^{-\tilde{c}L})}} \max_{|E-z|=e^{-\tilde{c}L}} \frac{|\Xi_\omega(E)-(S_{\omega,L}(E)+e^{-i\theta(E)})|}{|S_{\omega,L}(E)+e^{-i\theta(E)}|} < e^{-\tilde{c}L}.$$

The case of $H^{\mathbb{Z}}_{\omega,\tilde{\omega},L}$ is dealt with in the same way.

This completes the proof of Theorem 1.21.

6C. *Estimates on the growth of eigenfunctions.* In the present section we are going to prove Theorems 6.4 and 6.5. At the end of the section, we also prove the simpler Lemma 6.3.

The proof of Theorem 6.4 relies on locally uniform estimates on the rate of growth of the cocycle (1-15) attached to the Schrödinger operator, which we present now. Define

$$T_L(E,\omega) = T(E,\omega_L)\cdots T(E,\omega_0), \tag{6-23}$$

where

$$T(E,\omega_j) = \begin{pmatrix} E-\omega_j & -1 \\ 1 & 0 \end{pmatrix}.$$

We start with an upper bound on the large deviations of the growth rate of the cocycle that is uniform in energy. Fix $\alpha > 1$ and $\delta \in (0,1)$. For one part, the proof of Theorem 6.4 relies on the following:

Lemma 6.10. *Let $I \subset \mathbb{R}$ be a compact interval. For any $\delta > 0$, there exists $L_\delta > 0$ and $\eta > 0$ such that, for $L \ge L_\delta$ and any $K > 0$, one has*

$$\mathbb{P}\left(\frac{\log \|T_L(E;\tau^k(\omega))u\|}{L+1} \le \rho(E)+\delta \text{ for all } 0\le k\le K,\ E\in I,\ \|u\|=1\right) \ge 1-Ke^{-\eta(L+1)}, \tag{6-24}$$

where we recall that $\tau : \Omega \to \Omega$ denotes the left shift (i.e., if $\omega = (\omega_n)_{n\ge 0}$ then $[\tau(\omega)]_n = \omega_{n+1}$ for $n \ge 0$) and $\tau^n = \tau\circ\cdots\circ\tau$ n times.

At the heart of this result is a large deviation principle for the growth rate of the cocycle (see [Bougerol and Lacroix 1985, Section I and Theorem 6.1]). As it also serves in the proof of Theorem 6.4, we recall it now. One has:

Lemma 6.11. *Let $I \subset \mathbb{R}$ be a compact interval. For any $\delta > 0$, there exists $L_\delta > 0$ and $\eta > 0$ such that, for $L \geq L_\delta$, one has*

$$\sup_{\substack{E \in I \\ \|u\|=1}} \mathbb{P}\left(\left|\frac{\log \|T_L(E;\omega)u\|}{L+1} - \rho(E)\right| \geq \delta\right) \leq e^{-\eta(L+1)}. \tag{6-25}$$

While this result is not stated as is in [Bougerol and Lacroix 1985], it can be obtained from their Lemma 6.2 and Theorem 6.1. Indeed, by inspecting the proof of these results, it is clear that the quantities involved (in particular, the principal eigenvalue of $T(z; E) = T(z)$ in [loc. cit., Theorem 4.3]) are continuous functions of the energy E. Thus, taking this into account, the proof of [loc. cit., Theorem 6.1] yields, for our cocycle, a convergence that is locally uniform in energy, that is, (6-25).

To prove Theorem 6.4, in addition to Lemma 6.10 we also need to guarantee a uniform lower bound on the growth rate of the cocycle. We need this bound at least on the spectrum of $H_{\omega,L}$ with a good probability. Actually, this is the best one can hope for: a uniform bound in the style of (6-24) will not hold.

We prove:

Lemma 6.12. *Fix I a compact interval and $\delta > 0$. Pick $u \in \mathbb{C}^2$ with $\|u\| = 1$. For $0 \leq j \leq L$, if $j \leq L-1$, define*

$$\mathcal{K}_j^+(\omega, L, \delta, u) := \left\{E \in I \;\middle|\; \left|\frac{\log \|T^{-1}_{L-(j+1)}(E, \tau^{j+1}(\omega))u\|}{L-j} - \rho(E)\right| > \delta\right\}$$

and, if $1 \leq j$, define

$$\mathcal{K}_j^-(\omega, L, \delta, u) := \left\{E \in I \;\middle|\; \left|\frac{\log \|T_{j-1}(E, \omega)u\|}{j} - \rho(E)\right| > \delta\right\};$$

finally, define $\mathcal{K}_L^+(\omega, L, \delta, u) = \varnothing = \mathcal{K}_0^-(\omega, L, \delta, u)$.

Recall that $(E_{j,\omega})_{0\leq j\leq L}$ are the eigenvalues of $H_{\omega,L}$ and let $x_{j,\omega}$ be the associated localization centers.

For $0 \leq \ell \leq L$, define the sets

$$\Omega_B^+(L, \ell, \delta, u) := \{\omega \mid L - x_{j,\omega} \geq \ell \text{ and } E_{j,\omega} \in \mathcal{K}^+_{x_{j,\omega}}(\omega, L, \delta, u) \text{ for some } j\}$$

and

$$\Omega_B^-(L, \ell, \delta, u) := \{\omega \mid x_{j,\omega} \geq \ell \text{ and } E_{j,\omega} \in \mathcal{K}^-_{x_{j,\omega}}(\omega, L, \delta, u) \text{ for some } j\}.$$

Then the sets $\Omega_B^\pm(L, \ell, \delta, u)$ are measurable and, for any $\delta > 0$, there exist $\eta > 0$ and $\ell_0 > 0$ such that, for $L \geq \ell \geq \ell_0$, one has

$$\max\bigl(\mathbb{P}(\Omega_B^+(L, \ell, \delta, u)),\ \mathbb{P}(\Omega_B^-(L, \ell, \delta, u))\bigr) \leq \frac{(L+1)|I|e^{-\eta(\ell-1)}}{1-e^{-\eta}}. \tag{6-26}$$

Here, the constant η is the one given by (6-25).

First, let us explain the meaning of Lemma 6.12. Since by Lemma 6.10 we already control the growth of the cocycle from above, we see that in the definitions of the sets $\mathcal{K}_j^-(\omega, L, \delta, u)$ and $\mathcal{K}_j^+(\omega, L, \delta, u)$ it would have sufficed to require

$$\frac{\log \|T_{j-1}(E,\omega)u\|}{j} - \rho(E) \le -\delta \quad \text{and} \quad \frac{\log \|T^{-1}_{L-(j+1)}(E,\tau^{j+1}(\omega))u\|}{L-(j+1)} - \rho(E) \le -\delta,$$

respectively.

Hence, what Lemma 6.12 measures is that the probability that the cocycle at energy $E_{n,\omega}$ leading from a localization center $x_{n,\omega}$ to either 0 or L decays at a rate smaller than the rate predicted by the Lyapunov exponent.

The sets $\Omega^{\pm}_B(L,\ell,\delta,u)$ are the sets of bad configurations, i.e., the events when the rate of decay of the solution is far from the Lyapunov exponent. Indeed, for ω outside $\Omega^{\pm}_B(L,\ell,\delta)$, i.e., if the reverse of the inequalities defining $\mathcal{K}^{\pm}_j(\omega,L,\delta,u)$ hold, when $j = x_{n,\omega}$ and $E = E_{n,\omega}$ we know that the eigenfunction $\varphi_{n,\omega}$ has to decay from the center of localization $x_{n,\omega}$ (which is a local maximum of its modulus) towards the edges of the intervals at a rate larger than $\gamma(E_{n,\omega}) - \delta$. The eigenfunction being normalized, at the localization center it is of size at least $L^{-1/2}$. This will entail the estimates (6-4) and (6-5) with a good probability.

There is a major difference in the uniformity in energy obtained in Lemmas 6.12 and 6.10. In Lemma 6.12, we do not get a lower bound on the decay rate that is uniform all over I: it is merely uniform over the spectrum inside I (which is sufficient for our purpose, as we shall see). The reason for this difference in the uniformity between Lemma 6.10 and 6.12 is the same that makes the Lyapunov exponent $E \mapsto \rho(E)$ in general only upper semicontinuous and not lower semicontinuous (in the present situation, it actually is continuous).

We postpone the proofs of Lemmas 6.10 and 6.12 to the end of this section and turn to the proofs of Theorems 6.4 and 6.5.

6C1. *The proof of Theorem 6.4.* By Lemma 6.10, as $T_L(E,\omega) \in \mathrm{SL}(2,\mathbb{R})$, with probability at least $1 - KLe^{-\eta(L+1)}$, for $L \ge L_\delta$ and any $K > 0$, one also has

$$\forall 0 \le k \le K \ \ \forall E \in I \ \ \forall \|u\| = 1 \quad \frac{\log \|T^{-1}_L(E;\tau^k(\omega))u\|}{L+1} \le \rho(E) + \delta.$$

Now pick $\ell = C \log L$, where $C > 0$ is to be chosen later on. We know that, with probability $\mathbb{P}$ satisfying

$$\mathbb{P} \ge 1 - L^2 e^{-\eta\ell}, \tag{6-27}$$

for $L \ge L_\delta$, any $l \in [\ell, L]$ and any $k \in [0, L]$, one also has

$$\forall E \in I \ \ \forall \|u\| = 1 \ \ \frac{\log \|T^{-1}_l(E;\tau^k(\omega))u\|}{l+1} \le \rho(E) + \delta. \tag{6-28}$$

Let $\varphi_{j,\omega}$ be a normalized eigenfunction associated to the eigenvalue $E_{j,\omega} \in I$ with localization center $x_{j,\omega}$. By the definition of the localization center, one has

$$\frac{1}{L+1} \le \left\| \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}) \\ \varphi_{j,\omega}(x_{j,\omega}-1) \end{pmatrix} \right\|^2 \le 1 \quad \text{and} \quad \frac{1}{L+1} \le \left\| \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}+1) \\ \varphi_{j,\omega}(x_{j,\omega}) \end{pmatrix} \right\|^2 \le 1. \tag{6-29}$$

By the eigenvalue equation, for $x \in [\![0, L]\!]$ one has

$$\begin{pmatrix} \varphi_{j,\omega}(x) \\ \varphi_{j,\omega}(x-1) \end{pmatrix} = \begin{cases} T_{x-x_{j,\omega}}(E;\tau^{x_{j,\omega}}(\omega)) \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}) \\ \varphi_{j,\omega}(x_{j,\omega}-1) \end{pmatrix} & \text{if } x \geq x_{j,\omega}, \\ T^{-1}_{x_{j,\omega}-x}(E;\tau^{x}(\omega)) \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}) \\ \varphi_{j,\omega}(x_{j,\omega}-1) \end{pmatrix} & \text{if } x \leq x_{j,\omega}. \end{cases} \tag{6-30}$$

Hence, by (6-24) and (6-28), with probability at least $1 - 2L^2 e^{-\eta\ell} - L^{-p}$, if $|x_{j,\omega} - x| \geq \ell$ then for $x_{j,\omega} < x \leq L$ one has

$$\begin{aligned} \frac{e^{-(\rho(E_{j,\omega})+\delta)|x-x_{j,\omega}|}}{\sqrt{L+1}} &\leq e^{-(\rho(E_{j,\omega})+\delta)|x-x_{j,\omega}|} \left\| \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}) \\ \varphi_{j,\omega}(x_{j,\omega}-1) \end{pmatrix} \right\| \\ &\leq \left\| T_{x-x_{j,\omega}}(E;\tau^{x_{j,\omega}}(\omega)) \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}) \\ \varphi_{j,\omega}(x_{j,\omega}-1) \end{pmatrix} \right\| = \left\| \begin{pmatrix} \varphi_{j,\omega}(x) \\ \varphi_{j,\omega}(x-1) \end{pmatrix} \right\| \end{aligned} \tag{6-31}$$

and for $0 \leq x < x_{j,\omega}$ one has

$$\begin{aligned} \left\| \begin{pmatrix} \varphi_{j,\omega}(x) \\ \varphi_{j,\omega}(x-1) \end{pmatrix} \right\| &= \left\| T^{-1}_{x-x_{j,\omega}}(E;\tau^{x_{j,\omega}}(\omega)) \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}) \\ \varphi_{j,\omega}(x_{j,\omega}-1) \end{pmatrix} \right\| \\ &\geq e^{-(\rho(E_{j,\omega})+\delta)|x-x_{j,\omega}|} \left\| \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}) \\ \varphi_{j,\omega}(x_{j,\omega}-1) \end{pmatrix} \right\| \geq \frac{e^{-(\rho(E_{j,\omega})+\delta)|x-x_{j,\omega}|}}{\sqrt{L+1}} \end{aligned} \tag{6-32}$$

On the other hand, by the definition of the Dirichlet boundary conditions, we know that

$$\begin{pmatrix} \varphi_{j,\omega}(0) \\ \varphi_{j,\omega}(-1) \end{pmatrix} = \varphi_{j,\omega}(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varphi_{j,\omega}(L+1) \\ \varphi_{j,\omega}(L) \end{pmatrix} = \varphi_{j,\omega}(L) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus,

$$\varphi_{j,\omega}(0) T_{x_{j,\omega}-1}(E;\omega) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}) \\ \varphi_{j,\omega}(x_{j,\omega}-1) \end{pmatrix}$$

and

$$\varphi_{j,\omega}(L) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = T_{L-x_{j,\omega}-1}(E;\tau^{x_{j,\omega}+1}(\omega)) \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}+1) \\ \varphi_{j,\omega}(x_{j,\omega}) \end{pmatrix}.$$

Thus, for $\omega \notin \Omega^+_B(L,\ell,\delta,u_+) \cup \Omega^-_B(L,\ell,\delta,u_-)$, where we have set $u_- := \binom{0}{1}$ and $u_+ := \binom{1}{0}$, we know that

$$e^{-(\rho(E_{j,\omega})-\delta)(L-x_{j,\omega})} \leq \left\| T^{-1}_{L-x_{j,\omega}-1}(E;\tau^{x_{j,\omega}+1}(\omega)) u_+ \right\| \quad \text{and} \quad e^{-(\rho(E_{j,\omega})-\delta)x_{j,\omega}} \leq \| T_{x_{j,\omega}-1}(E;\omega) u_- \|.$$

Thus we obtain that, for $\omega \notin \Omega^+_B(L,\ell,\delta,u_+) \cup \Omega^-_B(L,\ell,\delta,u_-)$, one has

$$|\varphi_{j,\omega}(L)| = \left\| T^{-1}_{L-x_{j,\omega}}(E;\tau^{x_{j,\omega}+1}(\omega)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|^{-1} \left\| \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}+1) \\ \varphi_{j,\omega}(x_{j,\omega}) \end{pmatrix} \right\| \leq e^{-(\rho(E_{j,\omega})-\delta)(L-x_{j,\omega}-1)} \tag{6-33}$$

and

$$|\varphi_{j,\omega}(0)| = \left\| T_{x_{j,\omega}}(E;\tau^{x_{j,\omega}}(\omega)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|^{-1} \left\| \begin{pmatrix} \varphi_{j,\omega}(x_{j,\omega}) \\ \varphi_{j,\omega}(x_{j,\omega}-1) \end{pmatrix} \right\| \leq e^{-(\rho(E_{j,\omega})-\delta)(x_{j,\omega}-1)}. \tag{6-34}$$

The estimates given by Lemma 6.12 on the probability of $\Omega_B^+(L,\ell,\delta,u_+)$ and $\Omega_B^-(L,\ell,\delta,u_-)$ for $\ell = C\log L$ and the estimate (6-27) then imply that, with a probability at least $1-4L^2e^{-\eta(\ell-1)}-L^{-p}$, the bounds (6-31), (6-32), (6-33) and (6-34) hold. Thus, picking $\ell = C\log L$ for $C>0$ sufficiently large (depending only on η and, thus, on δ and p), these bounds hold with a probability at least $1-L^{-p}$. This completes the proof of Theorem 6.4.

Remark 6.13. One may wonder whether the uniform growth estimate given by Lemmas 6.10 and 6.12 is actually necessary in the proof of Theorem 6.4. That they are necessary is due to the fact that both the eigenvalue $E_{j,\omega}$ and the localization center $x_{j,\omega}$ (and, thus, the vector

$$\left\|\begin{pmatrix}\varphi_{j,\omega}(x_{j,\omega})\\ \varphi_{j,\omega}(x_{j,\omega}-1)\end{pmatrix}\right\|$$

also) depend on ω. Thus, (6-25) is not sufficient to estimate the second term in the left-hand sides of (6-31) and (6-32).

6C2. *The proof of Theorem 6.5.* To prove Theorem 6.5, we follow the strategy that led to the proof of Theorem 6.4. First, note that (6-31) and (6-32) provide the expected lower bounds on the eigenfunction with the right probability. As for the upper bound, by (6-30), using the conclusions of Theorem 6.4 and the bounds given by Lemma 6.10, we know that, e.g., for $0\le x<x_{j,\omega}$,

$$\left\|\begin{pmatrix}\varphi_{j,\omega}(x)\\ \varphi_{j,\omega}(x-1)\end{pmatrix}\right\| = \left\|T_x(E;\omega)\begin{pmatrix}1\\0\end{pmatrix}\right\||\varphi_{j,\omega}(0)| \le e^{(\rho(E_{j,\omega})+\delta)x}e^{-(\rho(E_{j,\omega})-\delta)x_{j,\omega}} \le e^{-(\rho(E_{j,\omega})-C\delta)|x-x_{j,\omega}|}$$

if $(1+C)x \le (C-1)x_{j,\omega}$, i.e., $2(1+C)^{-1}x_{j,\omega} \le x_{j,\omega}-x$.

For $x\ge x_{j,\omega}$ one reasons similarly and, thus, completes the proof of Theorem 6.5.

Remark 6.14. Actually, as the proof shows, the results one obtains are more precise than the claims made in Theorem 6.5 (see [Klopp $\ge$ 2016]).

6C3. *The proof of Lemma 6.12.* The proofs for the two sets $\Omega_B^\pm(L,\ell,\delta,u)$ are the same. We will only write out the one for $\Omega_B^+(L,\ell,\delta,u)$. Let us first address the measurability issue for $\Omega_B^+(L,\ell,\delta,u)$. The functions $\omega\mapsto E_{j,\omega}$ and $\omega\mapsto\varphi_{j,\omega}$ are continuous (as the eigenvalues and eigenvectors of finite-dimensional matrices depending continuously on the parameter $\omega=(\omega_j)_{0\le j\le L}$). Thus, for fixed j, the sets $\{\omega \mid E_{j,\omega}\in\mathcal{K}_j^-(\omega,L,\delta,u)\}$ and $\{\omega \mid x_{j,\omega}>j\}$ are open (we used the definition of $x_{j,\omega}$ as the leftmost localization center (see Theorem 6.4)). This yields the measurability of $\Omega_B^+(L,\ell,\delta,u)$.

We claim that

$$\frac{1}{L+1}\mathbf{1}_{\Omega_B^+(L,\ell,\delta,u)} \le \sum_{j=0}^{L+1-\ell}\langle\delta_j,\mathbf{1}_{\mathcal{K}_j^+(\omega,L,\delta,u)}(H_{\omega,L})\delta_j\rangle, \tag{6-35}$$

where $\mathbf{1}_{\mathcal{K}_j^+(\omega,L,\delta,u)}(H_{\omega,L})$ denotes the spectral projector associated to $H_{\omega,L}$ on the set $\mathcal{K}_j^+(\omega,L,\delta,u)$. Indeed, if one has $E_{j,\omega}\notin\mathcal{K}_{x_{j,\omega}}^+(\omega,L,\delta,u)$ for all j then the left-hand side of (6-35) vanishes and the right-hand side is nonnegative. On the other hand, if, for some j, one has $0\le x_{j,\omega}\le L-\ell$ and

$E_{j,\omega} \in \mathcal{K}^+_{x_{j,\omega}}(\omega, L, \delta, u)$, then we compute

$$\sum_{l=0}^{L-\ell} \langle \delta_l, \mathbf{1}_{\mathcal{K}_j^+(\omega,L,\delta,u)}(H_{\omega,L})\delta_l\rangle = \sum_{l=0}^{L-\ell} \sum_{\substack{k \\ E_{k,\omega}\in\mathcal{K}_j^+(\omega,L,\delta,u)}} |\varphi_{k,\omega}(l)|^2 \geq |\varphi_{j,\omega}(x_{j,\omega})|^2 \geq \frac{1}{L+1} \geq \frac{1}{L+1}\mathbf{1}_{\Omega_B^+(L,\ell,\delta,u)}$$

by the definition of $x_{j,\omega}$.

An important fact is that, by construction (see Lemma 6.12), the set of energies $\mathcal{K}_j^+(\omega, L, \delta, u)$ does not depend on ω_j. Hence, denoting by $\mathbb{E}_{\omega_j}(\,\cdot\,)$ the expectation with respect to ω_j and $\mathbb{E}_{\hat{\omega}_j}(\,\cdot\,)$ the expectation with respect to $\hat{\omega}_j = (\omega_k)_{k\neq j}$, we compute

$$\mathbb{E}\biggl(\sum_{j=0}^{L-\ell}\langle \delta_j, \mathbf{1}_{\mathcal{K}_j^+(\omega,L,\delta,u)}(H_{\omega,L})\delta_j\rangle\biggr) = \sum_{j=0}^{L-\ell}\mathbb{E}_{\hat{\omega}_j}\bigl(\mathbb{E}_{\omega_j}\bigl(\langle \delta_j, \mathbf{1}_{\mathcal{K}_j^+(\omega,L,\delta,u)}(H_{\omega,L})\delta_j\rangle\bigr)\bigr).$$

As ω_j is assumed to have a bounded, compactly supported distribution and as $\mathcal{K}_j^+(\omega, L, \delta, u)$ does not depend on ω_j, a standard spectral averaging lemma (see, e.g., [Simon 2005, Theorem 11.8]) yields

$$\mathbb{E}_{\omega_j}\bigl(\langle \delta_j, \mathbf{1}_{\mathcal{K}_j^+(\omega,L,\delta,u)}(H_{\omega,L})\delta_j\rangle\bigr) \leq |\mathcal{K}_j^+(\omega, L, \delta, u)|,$$

where $|\cdot|$ denotes the Lebesgue measure. Thus, we obtain

$$\mathbb{E}\biggl(\sum_{j=0}^{L-\ell}\langle \delta_j, \mathbf{1}_{\mathcal{K}_j^+(\omega,L,\delta,u)}(H_{\omega,L})\delta_j\rangle\biggr) \leq \sum_{j=0}^{L-\ell}\mathbb{E}_{\hat{\omega}_j}\bigl(|\mathcal{K}_j^+(\omega, L, \delta, u)|\bigr) = \sum_{j=0}^{L-\ell}\mathbb{E}\bigl(|\mathcal{K}_j^+(\omega, L, \delta, u)|\bigr). \tag{6-36}$$

By Lemma 6.11 and the Fubini–Tonelli theorem, we know that

$$\begin{aligned}\mathbb{E}\bigl(|\mathcal{K}_j^+(\omega, L, \delta, u)|\bigr) = \mathbb{E}\biggl(\int_I \mathbf{1}_{\mathcal{K}_j^+(\omega,L,\delta,u)}(E)\,dE\biggr) &= \int_I \mathbb{E}(\mathbf{1}_{\mathcal{K}_j^+(\omega,L,\delta,u)}(E))\,dE \\ &\leq |I|\sup_{E\in I}\mathbb{P}\biggl(\biggl|\frac{\log\|T_{L-(j+1)}^{-1}(E,\omega)u\|}{L-j} - \rho(E)\biggr| > \delta\biggr) \\ &\leq |I|r^{-\eta(L-j)}.\end{aligned}$$

Taking the expectation of both sides of (6-35) and plugging this into (6-36), we obtain

$$\mathbb{P}(\Omega_B^+(L, \ell, \delta, u)) \leq (L+1)|I|e^{-\eta(\ell-1)}\sum_{j=0}^{L-\ell}e^{-\eta j} \leq \frac{(L+1)|I|e^{-\eta(\ell-1)}}{1-e^{-\eta}}.$$

In the same way, one obtains

$$\mathbb{P}(\Omega_B^-(L, \ell, \delta, u)) \leq \frac{(L+1)|I|e^{-\eta(\ell-1)}}{1-e^{-\eta}}.$$

This completes the proof of Lemma 6.12.

Remark 6.15. This proof can be seen as the analogue for products of finitely many random matrices of the so-called Kotani trick (see, e.g., [Cycon et al. 1987]).

6C4. *The proof of Lemma 6.10.* The basic idea of this proof is to use the estimate (6-25), in particular, the exponentially small probability and some perturbation theory for the cocycles so as to obtain a uniform estimate.

Let η be given by (6-25). Fix $\eta' < \frac{1}{2}\eta$ and write

$$I = \bigcup_{j\in J}[E_j, E_{j+1}], \quad \text{where} \quad \tfrac{1}{2}e^{-\eta'(L+1)} \le E_{j+1} - E_j \le 2e^{-\eta'(L+1)}; \tag{6-37}$$

thus, $\#J \lesssim e^{\eta'(L+1)}$.

We now want to estimate what happens for $E \in [E_j, E_{j+1}]$. Using (1-15) and

$$\begin{pmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} E_j - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} = (E - E_j)\Delta T, \quad \text{where} \quad \Delta T := \left|\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|,$$

we compute

$$T_L(E,\omega) = T_L(E_j,\omega) + \sum_{l=1}^{L}(E - E_j)^l S_l, \tag{6-38}$$

where

$$\begin{aligned} S_l &:= \sum_{n_1<n_2<\cdots<n_l} T_{n_1}(E_j, \tau^{L-n_1}\omega) \times \Delta T \times T_{n_2-n_1-1}(E_j, \tau^{n_2}\omega) \times \Delta T \times \cdots \times \Delta T \times T_{L-n_l-1}(E_j, \tau^{n_l}\omega) \\ &= \sum_{n_1<n_2<\cdots<n_l} \prod_{m=2}^{l} \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, T_{n_m-n_{m-1}-1}(E_j, \tau^{n_m}\omega)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\rangle \left| T_{n_1}(E_j, \tau^{L-n_1}\omega)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right| T_{L-n_l-1}(E_j, \tau^{n_l}\omega). \end{aligned}$$

Clearly, as the random variables have compact support, one has the uniform bound

$$\sup_{\substack{E\in I \\ \omega\in\Omega}} \|T_L(E;\omega)\| \le e^{CL}. \tag{6-39}$$

Thus one has

$$\sup_{\omega\in\Omega} \|S_l\| \le L^l e^{CL}. \tag{6-40}$$

Hence, for l_0 fixed, one computes

$$\left\|\sum_{l=l_0}^{L}(E - E_j)^l S_l\right\| \le \sum_{l=l_0}^{L}(E - E_j)^l \|S_l\| \le \sum_{l=l_0}^{L} e^{-\eta'(L+1)l} L^l e^{CL} \le 1 \tag{6-41}$$

if $\eta' l_0 > 2C$ and L is sufficiently large (depending only on η' and C).

We now assume that l_0 satisfies $\eta' l_0 > 2C$ and pick $1 \le l \le l_0$. Pick $\delta_0 \in (0, 1)$ small, to be fixed later. Assume moreover that L is such that $\delta_0 L \ge L_\delta$, where L_δ is as defined in Lemma 6.11. Then, by Lemma 6.11, for $m \in \{2, \dots, l\}$ one has

(1) either $n_m - n_{m-1} \le L_\delta$, in which case one has

$$\|T_{n_m-n_{m-1}-1}(E_j, \tau^{n_m-1}\omega)\| \le e^{C(n_m-n_{m-1})};$$

(2) or $n_m - n_{m-1} \geq L_\delta$, in which case, by (6-25), with probability at least equal to $1 - e^{-\eta(n_m - n_{m-1})/2}$, one has

$$\|T_{n_m - n_{m-1}-1}(E_j, \tau^{n_m - 1}\omega)\| \leq e^{(n_m - n_{m-1})(\rho(E_j)+\delta)}.$$

Define

$$G_{n_1,\dots,n_l} = \{m \in \{2,\dots,l\} \mid n_m - n_{m-1} \geq L_\delta\} \quad \text{and} \quad B_{n_1,\dots,n_l} = \{2,\dots,l\} \setminus G_{n_1,\dots,n_l}.$$

By definition, one has

$$\sum_{m \in B_{n_1,\dots,n_l}} (n_m - n_{m-1}) \leq l L_\delta \quad \text{and} \quad \sum_{m \in G_{n_1,\dots,n_l}} (n_m - n_{m-1}) \geq L - l L_\delta. \tag{6-42}$$

For a fixed sequence $n_1 < n_2 < \cdots < n_m$, the random variables $(T_{n_{m'} - n_{m'-1}-1}(E_j, \tau^{n_{m'}}\omega))_{1 \leq m' \leq m}$ are independent. Hence, by (6-25), for a fixed $(m_1, \dots, m_K) \in G_{n_1,\dots,n_l}$, one has

$$\mathbb{P}\Big(\inf_{1 \leq k \leq K} \big\|T_{n_{m_k} - n_{m_k - 1}-1}(E_j, \tau^{n_{m_k}}\omega)\big\| \geq e^{(\rho(E_j)+\delta)(n_{m_k} - n_{m_k-1})}\Big) \leq e^{-\eta \sum_{k=1}^K n_{m_k} - n_{m_k-1}}.$$

Thus, for $\varepsilon \in (0,1)$, one has

$$\mathbb{P}\Bigg(\inf_{1 \leq k \leq K} \|T_{n_{m_k} - n_{m_k-1}-1}(E_j, \tau^{n_{m_k}-1}\omega)\| \geq e^{(\rho(E_j)+\delta)(n_{m_k} - n_{m_k-1})} \\ \text{for some } (m_1,\dots,m_K) \in G_{n_1,\dots,n_l} \text{ with } \sum_{k=1}^K n_{m_k} - n_{m_k-1} \geq \varepsilon L\Bigg) \leq L^l e^{-\eta\varepsilon L}.$$

Hence, with probability at least $1 - L^l e^{-\eta\varepsilon L}$, we know that there exists $(m_1,\dots,m_K) \in G_{n_1,\dots,n_l}$ such that

$$\sum_{k=1}^K n_{m_k} - n_{m_k-1} \geq L - l L_\delta - \varepsilon L \quad \text{and} \quad \|T_{n_{m_k} - n_{m_k-1}-1}(E_j, \tau^{n_{m_k}-1}\omega)\| \leq e^{(\rho(E_j)+\delta)(n_{m_k} - n_{m_k-1})}$$

for all $1 \leq k \leq K$. Using the estimates (6-42) and (6-39) for the remaining terms in the product below, for any given m-tuple $(n_1,\dots,n_m)$ one obtains

$$\mathbb{P}\Bigg(\prod_{m=1}^l \|T_{n_m - n_{m-1}-1}(E_j, \tau^{n_{m_k}-1}\omega)\| \leq e^{(\rho(E_j)+\delta)(1-\varepsilon)(L - lL_\delta) + C(\varepsilon L + lL_\delta)}\Bigg) \geq 1 - L^l e^{-\eta\varepsilon L}.$$

Hence, with probability at least $1 - l_0 L^{l_0} e^{-\eta\varepsilon L}$, for $1 \leq l \leq l_0$ we estimate

$$\begin{aligned}\|S_l\| &\leq \sum_{n_1 < n_2 < \cdots < n_l} \prod_{m=1}^l \|T_{n_m - n_{m-1}-1}(E_j, \tau^{n_{m_k}}\omega)\| \\ &\leq L^l e^{(\rho(E_j)+\delta)(1-\varepsilon)L + C\varepsilon L + (C - (\rho(E_j)+\delta)(1-\varepsilon)) l L_\delta} \\ &\leq L^l e^{[\rho(E_j)+\delta+(C-\rho(E_j)-\delta)\varepsilon]L + [C - (\rho(E_j)+\delta)(1-\varepsilon)]L_\delta l} \\ &\leq L^{l_0} e^{[\rho(E_j)+\delta+(C-\rho(E_j)-\delta)\varepsilon]L + [C - (\rho(E_j)+\delta)(1-\varepsilon)]L_\delta l_0}.\end{aligned} \tag{6-43}$$

It remains now to choose the quantities η', l_0 and ε so that the following requirements are satisfied:

$$\eta' l_0 > 2C, \quad (C-\rho(E_j)-\delta)\varepsilon \le \frac{\delta}{2}, \quad l_0 L^{l_0} e^{-\eta\varepsilon L} e^{\eta'(L+1)} \ll 1$$
$$\text{and} \quad \frac{[C-(\rho(E_j)+\delta)(1-\varepsilon)]L_\delta l_0}{L+1} \le \frac{\delta}{2(\rho(E_j)+\delta)}. \tag{6-44}$$

Fixing ε small, picking $0<\eta'<\frac{1}{3}\eta\varepsilon$ and setting $l_0 = L^\alpha$, where $\alpha\in(0,1)$, we see that all the conditions in (6-44) are satisfied for L sufficiently large. Moreover, one has

$$l_0 L^{l_0} e^{-\eta\varepsilon L} e^{\eta'(L+1)} \le e^{-\eta\varepsilon L/2}.$$

Plugging this and the last estimate in (6-43) into (6-38), we obtain that, with probability at least $1-e^{-\eta\varepsilon L/2}$, for any $j\in J$ (see (6-37)) and $E\in[E_j, E_{j+1}]$ one has

$$\|T_L(E,\omega)-T_L(E_j,\omega)\| \le 1+\sum_{l=1}^{l_0} e^{-\eta' l(L+1)} L^l e^{(\rho(E_j)+2\delta)L} \le 1+e^{(\rho(E_j)+2\delta)(L+1)}. \tag{6-45}$$

As ρ is continuous (see, e.g., [Bougerol and Lacroix 1985]), one gets that, for any $\delta>0$ and L sufficiently large, with probability at least $1-e^{-\eta\varepsilon L/2}$, one has, for any $E\in I$,

$$\|T_L(E,\omega)\| \lesssim e^{(\rho(E)+2\delta)(L+1)}.$$

Hence, as $T_L(E,\omega)\in \mathrm{SL}(2,\mathbb{R})$, one has $\|T_L^{-1}(E,\omega)\| \lesssim e^{(\rho(E)+2\delta)(L+1)}$.

Using the fact that the probability measure on Ω is invariant under the shift (it is a product measure), we obtain (6-24). This completes the proof of Lemma 6.10.

6C5. *The proof of Lemma 6.3.* Assume the realization ω is such that the conclusions of Lemma 6.2 hold in I for the scales $l_L = 2\log L$. Fix $\alpha>0$ and let $\mathscr{E}_{L,\omega}$ be the set of indices of the eigenvalues $(E_{j,\omega})_{0\le j\le L}$ of $H_{\omega,L}$ having a localization center in $[\![L-\ell_L, L]\!]$. Fix $C>\alpha>0$ and consider the projector $\Pi_C := \mathbf{1}_{[\![L-C\ell_L, L]\!]}$ in $\ell^2([\![0,L]\!])$.

Consider the Gram matrices

$$G(\mathscr{E}_{L,\omega}) = \big((\langle\varphi_{j,\omega},\varphi_{j,\omega}\rangle)\big)_{(n,m)\in\mathscr{E}_{L,\omega}\times\mathscr{E}_{L,\omega}} = \mathrm{Id}_N,$$

where $N=\#\mathscr{E}_{L,\omega}$, and

$$G_\pi(\mathscr{E}_{L,\omega}) = \big((\langle\Pi_C\varphi_{j,\omega},\Pi_C\varphi_{j,\omega}\rangle)\big)_{(n,m)\in\mathscr{E}_{L,\omega}\times\mathscr{E}_{L,\omega}}.$$

By definition, the rank of $G_\pi(\mathscr{E}_{L,\omega})$ is bounded by the rank of Π_C, i.e., by $C\ell_L$. Moreover, as by (6-3) one has $\|(1-\Pi_C)\varphi_{j,\omega}\| \le L^q e^{-\rho\eta C\ell_L}$, one has

$$\|\mathrm{Id}_N - G_\pi(\mathscr{E}_{L,\omega})\| \le L^{2+q} e^{-\rho\eta C\ell_L} \le L^{2+q-C\rho\eta}.$$

Thus, picking $C\eta\rho > q+2$ yields that, for L sufficiently large, $G_\pi(\mathscr{E}_{L,\omega})$ is invertible and its rank is N. This yields $\#\mathscr{E}_{L,\omega} = N \le C\ell_L$ and the proof of Lemma 6.3 is complete.

6D. ***The half-line random perturbation: the proof of Theorem 1.23.*** Using the same notations as in Section 5C, we can write

$$H^\infty = \begin{pmatrix} H^-_{\omega,-1} & |\delta_{-1}\rangle\langle\delta_0| \\ |\delta_0\rangle\langle\delta_{-1}| & -\Delta_0^+ \end{pmatrix},$$

where

- $-\Delta_0^+$ is the Dirichlet Laplacian on $\ell^2(\mathbb{N})$,
- $H^-_{\omega,-1} = -\Delta + V_\omega$ on $\ell^2(\{n \le -1\})$ with Dirichlet boundary conditions at 0.

Define the operators

$$\Gamma_\omega(E) := -\Delta_0^+ - E - \langle\delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle|\delta_0\rangle\langle\delta_0|,$$
$$\widetilde{\Gamma}_\omega(E) := H^-_{\omega,-1} - E - \langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle|\delta_{-1}\rangle\langle\delta_{-1}|.$$

For $\operatorname{Im} E \neq 0$, the numbers $\langle\delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle$ and $\langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle$ have nonvanishing imaginary parts of the same sign; hence, the complex number

$$(\langle\delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle)^{-1} - \langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle$$

does not vanish. Thus, by rank-one perturbation theory (see, e.g., [Simon 2005]), we thus know that $\Gamma_\omega(E)$ and $\widetilde{\Gamma}_\omega(E)$ are invertible for $\operatorname{Im} E \neq 0$ and that

$$\Gamma_\omega^{-1}(E) = (-\Delta_0^+ - E)^{-1} + \frac{|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle\langle\delta_0|(-\Delta_0^+ - E)^{-1}|}{\big(\langle\delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle\big)^{-1} - \langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle} \tag{6-46}$$

$$\widetilde{\Gamma}_\omega^{-1}(E) = (H^-_{\omega,-1} - E)^{-1} + \frac{|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle\langle\delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|}{\big(\langle\delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle\big)^{-1} - \langle\delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle}. \tag{6-47}$$

Thus, for $\operatorname{Im} E \neq 0$, using Schur's complement formula we compute

$$(H^\infty_\omega - E)^{-1} = \begin{pmatrix} \widetilde{\Gamma}_\omega^{-1}(E) & \gamma(E) \\ \gamma^*(\bar{E}) & \Gamma_\omega^{-1}(E) \end{pmatrix}, \tag{6-48}$$

where $\gamma^*(\bar{E})$ is the adjoint of $\gamma(\bar{E})$ and

$$\gamma(E) := -\big|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle\langle\delta_0|\Gamma_\omega^{-1}(E)\big|$$

6D1. *The continuation through* $(-2,2)\setminus\Sigma$. Let us start with the analytic continuation through $(-2,2)\setminus\Sigma$.

One easily checks that the function $E \mapsto \langle\delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle^{-1}$ is analytic outside Σ, the essential spectrum of $H^-_{\omega,-1}$, and has simple zeros at the isolated eigenvalues of $H^-_{\omega,-1}$. Hence, $E \mapsto \Gamma_\omega^{-1}(E)$ can be analytically continued near an isolated eigenvalue of $H^-_{\omega,-1}$ different from -2 and 2.

As for $\widetilde{\Gamma}_\omega^{-1}$, using the spectral decomposition of $(H^-_{\omega,-1} - E)^{-1}$, as for any eigenvector of $H^-_{\omega,-1}$, say φ, one has $\langle\delta_{-1}, \varphi\rangle \neq 0$; for E_0 an isolated eigenvalue of $H^-_{\omega,-1}$ different from -2 and 2, doing a polar decomposition of $\widetilde{\Gamma}_\omega^{-1}$ near E_0 one checks that $E \mapsto \widetilde{\Gamma}_\omega^{-1}(E)$ can be analytically continued to a neighborhood of E_0.

Finally let us check what happens with γ. We compute

$$\gamma(E) = -\langle\delta_{-1}|(H^-_{\omega,-1}-E)^{-1}|\delta_{-1}\rangle^{-1}\big|(H^-_{\omega,-1}-E)^{-1}|\delta_{-1}\rangle\langle\delta_0|(-\Delta_0^+-E)^{-1}\big|.$$

As $E\mapsto\langle\delta_{-1}|(H^-_{\omega,-1}-E)^{-1}|\delta_{-1}\rangle^{-1}(H^-_{\omega,-1}-E)^{-1}$ is analytic near any isolated eigenvalue of $H^-_{\omega,-1}$, we see that $E\mapsto\gamma(E)$ can be can be analytically continued to a neighborhood of an isolated eigenvalue of $H^-_{\omega,-1}$.

Hence, the representation (6-48) immediately shows that the resolvent $(H^\infty_\omega-E)^{-1}$ can be continued through $(-2,2)\setminus\Sigma$, the poles of the continuation being given by the zeros of the function

$$E\mapsto 1-\langle\delta_0|(-\Delta_0^+-E)^{-1}|\delta_0\rangle\langle\delta_{-1}|(H^-_{\omega,-1}-E)^{-1}|\delta_{-1}\rangle = 1-e^{i\theta(E)}\int_{\mathbb{R}}\frac{dN_\omega(\lambda)}{\lambda-E}.$$

6D2. *No continuation through* $(-2,2)\cap\Sigma^\circ$. Let us study the analytic continuation through $(-2,2)\cap\Sigma^\circ$. Considering the lower right coefficient of this matrix, we see that, when coming from upper half-plane through $(-2,2)\cap\Sigma^\circ$, $E\mapsto(H^\infty_\omega-E)^{-1}$ can be continued meromorphically to the lower half plane (as an operator from $\ell^2_{\text{comp}}(\mathbb{Z})$ to $\ell^2_{\text{loc}}(\mathbb{Z})$) only if $E\mapsto\Gamma^{-1}_\omega(E)$ can be continued meromorphically (as an operator from $\ell^2_{\text{comp}}(\mathbb{N})$ to $\ell^2_{\text{loc}}(\mathbb{N})$).

As $E\mapsto(-\Delta_0^+-E)^{-1}$ can be analytically continued (see Section 2), by (6-46) the meromorphic continuation of $E\mapsto\Gamma^{-1}_\omega(E)$ will exist if and only if the complex-valued map

$$E\mapsto g_\omega(E) := \frac{1}{\big(\langle\delta_{-1}|(H^-_{\omega,-1}-E)^{-1}|\delta_{-1}\rangle\big)^{-1}-\langle\delta_0|(-\Delta_0^+-E)^{-1}|\delta_0\rangle}$$

can be meromorphically continued from the upper half-plane through $(-2,2)\cap\Sigma^\circ$. Fix ω such that the spectrum of $H^-_{\omega,-1}$ is equal to Σ and pure point (this is almost sure; see, e.g., [Carmona and Lacroix 1990; Pastur and Figotin 1992]). As δ_{-1} is a cyclic vector for $H^-_{\omega,-1}$, for E an eigenvalue of $H^-_{\omega,-1}$ one then has

$$\lim_{\varepsilon\to0^+}\big(\langle\delta_{-1}|(H^-_{\omega,-1}-E-i\varepsilon)^{-1}|\delta_{-1}\rangle\big)^{-1}=0. \tag{6-49}$$

Hence, if the analytic continuation of g_ω would exist on $(-2,2)\cap\Sigma^\circ$ it would be equal to

$$g_\omega(E+i0) = -\frac{1}{\langle\delta_0|(-\Delta_0^+-E-i0)^{-1}|\delta_0\rangle}. \tag{6-50}$$

By analyticity of both sides, this in turn would imply that (6-50) holds on the whole upper half-plane; thus, in view of the definition of g_ω, that (6-49) holds on the whole upper half plane: this is absurd! Thus, we have proved that, ω-almost surely, $E\mapsto(H^\infty_\omega-E)^{-1}$ does not admit a meromorphic continuation through $(-2,2)\cap\Sigma^\circ$.

6D3. *Absolutely continuity of the spectrum of* H^∞_ω *in* $(-2,2)\cap\Sigma^\circ$. Let us now prove that the spectral measure of H^∞_ω in $(-2,2)\cap\Sigma^\circ$ is purely absolutely continuous. It suffices (see, e.g., [Teschl 2000, Section 2.5; Simon 2005, Theorem 11.6]) to prove that, for all $E\in(-2,2)\cap\Sigma^\circ$, one has

$$\limsup_{\varepsilon\to0^+}\big|\langle\delta_0,(H^\infty_\omega-E-i\varepsilon)^{-1}\delta_0\rangle\big|+\big|\langle\delta_{-1},(H^\infty_\omega-E-i\varepsilon)^{-1}\delta_{-1}\rangle\big|<+\infty.$$

Using (6-46), (6-47) and (6-48), for $\operatorname{Im} E \neq 0$ we compute

$$\langle \delta_{-1}, (H_\omega^\infty - E)^{-1}\delta_{-1}\rangle = \frac{\langle \delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle}{1 - \langle \delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle \cdot \langle \delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle}, \tag{6-51}$$

$$\langle \delta_{-n}, (H_\omega^\infty - E)^{-1}\delta_{m}\rangle = \frac{-\langle \delta_{-n}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle \langle \delta_0|(-\Delta_0^+ - E)^{-1}|\delta_m\rangle}{1 - \langle \delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle \cdot \langle \delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle} \tag{6-52}$$

for $n \geq 1$ and $m \leq 0$, and

$$\langle \delta_0, (H_\omega^\infty - E)^{-1}\delta_0\rangle = \frac{\langle \delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle}{1 - \langle \delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle \cdot \langle \delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle}. \tag{6-53}$$

Thus, to prove the absolute continuity of the spectral measure of H_ω^∞ in $(-2, 2) \cap \Sigma^\circ$, it suffices to prove that, for $E \in (-2, 2) \cap \Sigma^\circ$, one has

$$\limsup_{\varepsilon\to 0^+}\Bigg(\left|\frac{1}{\big(\langle \delta_{-1}|(H^-_{\omega,-1} - E - i\varepsilon)^{-1}|\delta_{-1}\rangle\big)^{-1} - \langle \delta_0|(-\Delta_0^+ - E - i\varepsilon)^{-1}|\delta_0\rangle}\right| \\ + \left|\frac{1}{\big(\langle \delta_0|(-\Delta_0^+ - E - i\varepsilon)^{-1}|\delta_0\rangle\big)^{-1} - \langle \delta_{-1}|(H^-_{\omega,-1} - E - i\varepsilon)^{-1}|\delta_{-1}\rangle}\right|\Bigg) < \infty.$$

This is the case, as

- the signs of the imaginary parts of $-\big(\langle \delta_{-1}|(H^-_{\omega,-1} - E - i\varepsilon)^{-1}|\delta_{-1}\rangle\big)^{-1}$ and $\langle \delta_0|(-\Delta_0^+ - E - i\varepsilon)^{-1}|\delta_0\rangle$ are the same (negative if $\operatorname{Im} E < 0$ and positive if $\operatorname{Im} E > 0$),
- for $E \in (-2, 2)$, $\langle \delta_0|(-\Delta_0^+ - E - i\varepsilon)^{-1}|\delta_0\rangle$ has a finite limit when $\varepsilon \to 0^+$,
- for $E \in (-2, 2)$, the imaginary part of $\langle \delta_0|(-\Delta_0^+ - E - i\varepsilon)^{-1}|\delta_0\rangle$ does not vanish in the limit $\varepsilon \to 0^+$.

So, we have proved the part of Theorem 1.23 concerning the absence of analytic continuation of the resolvent of H_ω^∞ through $(-2, 2) \cap \Sigma^\circ$ and the nature of its spectrum in this set.

6D4. *The spectrum of H_ω^∞ is pure point in $\Sigma^\circ \setminus [-2, 2]$.* Let us now prove the last part of Theorem 1.23. The proof relies again on (6-48). We pick $\beta \in \big(0, \tfrac{1}{2}\alpha\big)$, where α is determined by Theorem 6.1 for $H^-_{\omega,-1}$. Then, for $n \geq 1$ and $m \leq 0$, using the Cauchy–Schwartz inequality, for $\operatorname{Im} E \neq 0$ we compute

$$\begin{aligned} &\mathbb{E}\big(\big|\langle \delta_{-n}, (H_\omega^\infty - E)^{-1}\delta_m\rangle\big|^\beta\big)^2 \\ &\leq \big|\langle \delta_0|(-\Delta_0^+ - E)^{-1}|\delta_m\rangle\big|^2 \cdot \mathbb{E}\big(\big|\langle \delta_{-n}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle\big|^{2\beta}\big) \\ &\qquad \cdot \mathbb{E}\left(\left|\frac{1}{1 - \langle \delta_0|(-\Delta_0^+ - E)^{-1}|\delta_0\rangle \cdot \langle \delta_{-1}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle}\right|^{2\beta}\right). \end{aligned} \tag{6-54}$$

For a compact interval $J \subset (-2, 2) \setminus \Sigma$, we know that, for $n \geq 1$ and $m \leq 0$,

- $\sup_{\operatorname{Im} E\neq 0}\big|\langle \delta_0|(-\Delta_0^+ - E)^{-1}|\delta_m\rangle\big| \lesssim e^{-cm}$ by the Combes–Thomas estimates;
- $\sup_{\operatorname{Im} E\neq 0} \mathbb{E}\big(\big|\langle \delta_{-n}|(H^-_{\omega,-1} - E)^{-1}|\delta_{-1}\rangle\big|^{2\beta}\big) \lesssim e^{-2\beta\rho n}$ by the characterization (6-1) of localization in Σ for $H^-_{\omega,-1}$.

It suffices now to estimate the last term in (6-54) using a standard decomposition of rank-one perturbations (see, e.g., [Simon 2005; Aizenman and Molchanov 1993]); one writes

$$\frac{1}{1-\langle\delta_0|(-\Delta_0^+-E)^{-1}|\delta_0\rangle\cdot\langle\delta_{-1}|(H^-_{\omega,-1}-E)^{-1}|\delta_{-1}\rangle}=\frac{\omega_{-1}-b}{\omega_{-1}-a},$$

where a and b only depend on $(\omega_{-n})_{n\geq 2}$. Thus, as $(\omega_{-n})_{n\geq 1}$ have a bounded density, for $\operatorname{Im} E\neq 0$ one has

$$\mathbb{E}\left(\left|\frac{1}{1-\langle\delta_0|(-\Delta_0^+-E)^{-1}|\delta_0\rangle\cdot\langle\delta_{-1}|(H^-_{\omega,-1}-E)^{-1}|\delta_{-1}\rangle}\right|^{2\beta}\right)\leq\mathbb{E}_{(\omega_{-n})_{n\geq 2}}\mathbb{E}_{\omega_{-1}}\left(\left|\frac{\omega_{-1}-b}{\omega_{-1}-a}\right|^{2\beta}\right)\leq C_\beta<+\infty.$$

Thus, we have proved that, for a compact interval $J\subset\Sigma\setminus[-2,2]$, for $\beta\in\left(0,\frac{1}{2}\alpha\right)$ and some $\tilde\rho>0$, for $n\geq 1$ and $m\leq 0$ one has

$$\sup_{\substack{\operatorname{Im} E\neq 0\\ \operatorname{Re} E\in I}}\mathbb{E}\left(\left|\langle\delta_{-n},(H^\infty_\omega-E)^{-1}\delta_m\rangle\right|^\beta\right)<C_\beta e^{-\tilde\rho(m-n)}.$$

In the same way, using (6-51) and (6-53), one proves that

$$\sup_{\substack{\operatorname{Im} E\neq 0\\ \operatorname{Re} E\in I}}\mathbb{E}\left(\left|\langle\delta_0,(H^\infty_\omega-E)^{-1}\delta_0\rangle\right|^\beta+\left|\langle\delta_{-1},(H^\infty_\omega-E)^{-1}\delta_{-1}\rangle\right|^\beta\right)<+\infty.$$

Thus, we have proved that, for some $\tilde\rho>0$, one has

$$\sup_{\substack{\operatorname{Im} E\neq 0\\ \operatorname{Re} E\in I}}\sup_{m\in\mathbb{Z}}\mathbb{E}\left(\sum_{n\in\mathbb{Z}}e^{\tilde\rho(m-n)}\left|\langle\delta_{-n},(H^\infty_\omega-E)^{-1}\delta_m\rangle\right|^\beta\right)<+\infty.$$

Hence, we know that the spectrum of H^∞_ω in $\Sigma\setminus[-2,2]$ (as J can be taken arbitrarily, contained in this set) is pure point associated to exponentially decaying eigenfunctions (see, e.g., [Aizenman and Molchanov 1993; Aizenman 1994; Aizenman et al. 2001]). This completes the proof of Theorem 1.23.

Appendix

In this section we study the eigenvalues and eigenvectors of H_L (see Remark 1.6) near an energy E' that is an eigenvalue of both H_0^+ and H_k^- (see the ends of Sections 4A3 and 4A4). We keep the notations of Sections 4A3 and 4A4.

Let $\varphi^+\in\ell^2(\mathbb{N})$ (resp. $\varphi^-\in\ell^2(\mathbb{Z}_-)$) be normalized eigenvectors of H_0^+ (resp. H_k^-) associated to E_-. Thus, by (4-28) and (4-32), we can pick, for $n\geq 0$ and $l\in\{0,\dots,p-1\}$,

$$\varphi^+_{np+l}=ca_l(E')\rho^n(E')\quad\text{and}\quad\varphi^-_{-np-l}=c^-b_l(E')\rho^n(E').\tag{A-1}$$

Assume $L=Np+k$ and, for $l\in\{0,\dots,L\}$, define $\varphi^{\pm,L}\in\ell^2([\![0,L]\!])$ by

$$\varphi_l^{+,L}:=\varphi_l^+,\quad\varphi_{-1}^{+,L}=\varphi_{L+1}^{+,L}:=\varphi_{-1}^+=0,\quad\varphi_l^{-,L}:=\varphi_{l-L}^-\quad\text{and}\quad\varphi_{-1}^{-,L}=\varphi_{L+1}^{-,L}:=\varphi_0^-=0.\tag{A-2}$$

Thus, one has

$$H_L\varphi^{+,L} = E'\varphi^{+,L} + \varphi^{+}_{L+1}\delta_L, \quad H_L\varphi^{-,L} = E'\varphi^{-,L} + \varphi^{-}_{-L-1}\delta_0 \quad \text{and} \quad \langle\varphi^{+,L}, \varphi^{-,L}\rangle = O(N\rho^N(E)). \tag{A-3}$$

Recall that $a_k(E') \neq 0 \neq b_k(E')$ (see Sections 4A3 and 4A4); thus, by (A-1), one has

$$|\varphi^{-}_{-L-1}| \asymp |\rho(E')|^n \asymp |\varphi^{+}_{L+1}|. \tag{A-4}$$

Moreover, as H_L converges to H_0^+ in the strong resolvent sense, for $\varepsilon > 0$ sufficiently small and L sufficiently large, H_L has no spectrum in the compact $E' + \left[-2\varepsilon, -\frac{1}{2}\varepsilon\right] \cup \left[\frac{1}{2}\varepsilon, 2\varepsilon\right]$. Let Π_L be the spectral projector onto the interval $\left[-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\right]$, that is, $\Pi_L := 1/(2i\pi)\int_{|z-E'|=\varepsilon}(H_L - z)^{-1}\,dz$. By (A-3), one computes

$$(1 - \Pi_L)\varphi^{+,L} = \frac{\varphi^{+}_{L+1}}{2i\pi}\int_{|z-E'|=\varepsilon}(E' - z)^{-1}(H_L - z)^{-1}\delta_0\,dz.$$

Thus, one gets

$$\|(1 - \Pi_L)\varphi^{+,L}\| + \|(1 - \Pi_L)\varphi^{-,L}\| \lesssim |\rho(E')|^N. \tag{A-5}$$

Define

$$\tilde\chi^{+,L} = \frac{1}{\|\Pi_L\varphi^{+,L}\|}\Pi_L\varphi^{+,L} \quad \text{and} \quad \tilde\chi^{-,L} = \frac{1}{\|\Pi_L\varphi^{-,L}\|}\Pi_L\varphi^{-,L}.$$

The Gram matrix of $(\tilde\chi^{+,L}, \tilde\chi^{-,L})$ then reads $\mathrm{Id} + O(N\rho^N(E))$. Orthonormalizing $(\tilde\chi^{+,L}, \tilde\chi^{-,L})$ into $(\chi^{+,L}, \chi^{-,L})$ and computing the matrix elements of $\Pi_L(H_L - E')$ in this basis, we obtain

$$\begin{pmatrix} \varphi^{+}_{L+1}\langle\delta_L, \varphi^{+,L}\rangle & \varphi^{+}_{L+1}\langle\delta_0, \varphi^{+,L}\rangle \\ \varphi^{-}_{-L-1}\langle\delta_L, \varphi^{-,L}\rangle & \varphi^{-}_{-L-1}\langle\delta_0, \varphi^{-,L}\rangle \end{pmatrix} + O(N^2\rho^{2N}(E)) = \alpha\rho^N(E)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + O(N^2\rho^{2N}(E))$$

Thus, we obtain that the eigenvalues of H_L near E' are given by $E' \pm \alpha\rho^N(E) + O(N^2\rho^{2N}(E))$ and the eigenvectors by $\frac{1}{\sqrt{2}}(\varphi^{+,L} \pm \varphi^{-,L}) + O(\rho^N(E))$. In particular, their components at 0 and L are asymptotic to nonvanishing constants.

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