# Analysis & PDE

Volume 9 No. 2 2016



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ANALYSIS AND PDE
Vol. 9, No. 2, 2016
dx.doi.org/10.2140/apde.2016.9.353

# ON CHARACTERIZATION OF THE SHARP STRICHARTZ INEQUALITY FOR THE SCHRÖDINGER EQUATION

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We study the extremal problem for the Strichartz inequality for the Schrödinger equation on $\mathbb{R} \times \mathbb{R}^2$. We show that the solutions to the associated Euler–Lagrange equation are exponentially decaying in the Fourier space and thus can be extended to be complex analytic. Consequently, we provide a new proof of the characterization of the extremal functions: the only extremals are Gaussian functions, as investigated previously by Foschi, Hundertmark and Zharnitsky.

## 1. Introduction

We begin with some notation. For a Schwarz function $f$ on $\mathbb{R}^d$, $d \geq 1$, define the Fourier transform

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx, \quad \xi \in \mathbb{R}^d.$$

The inverse of the Fourier transform,

$$\mathcal{F}^{-1}(f)(x) = f^{\vee}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\, d\xi, \quad x \in \mathbb{R}^d.$$

The linear Strichartz inequality for the Schrödinger equation [Keel and Tao 1998; Tao 2006] asserts that

$$\|e^{it\Delta} f\|_{L^{2+4/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \leq C_d \|f\|_{L^2(\mathbb{R}^d)}, \tag{1}$$

where $e^{it\Delta} f(x) = (1/(2\pi)^d) \int_{\mathbb{R}^d} e^{ix\cdot\xi + it|\xi|^2} \hat{f}(\xi)\, d\xi$. We specify $d = 2$ and consider

$$\|e^{it\Delta} f\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)} \leq \boldsymbol{R} \|f\|_{L^2(\mathbb{R}^2)}, \tag{2}$$

where

$$\boldsymbol{R} := \sup\left\{ \frac{\|e^{it\Delta} f\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{R}^2)}} : f \in L^2,\ f \neq 0 \right\}. \tag{3}$$

We define an extremal function or extremal to (2) to be a nonzero function $f \in L^2$ such that the inequality is optimized, in the sense that

$$\|e^{it\Delta} f\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{R}^2)} = \boldsymbol{R} \|f\|_{L^2(\mathbb{R}^2)}. \tag{4}$$

The extremal problem of (2) concerns:

(i) Whether there exists an extremal function?

*MSC2010:* 35J10.
*Keywords:* Schrödinger equation, Strichartz inequality and extremals.

(ii) How to characterize the extremal functions? What are the explicit forms of extremal functions? Are they unique up to the symmetry of the inequality?

From Foschi [2007] and Hundertmark and Zharnitsky [2006], it is known that the Gaussian functions are the only extremal functions of the linear Strichartz inequality (2) for the dimensions $d = 1, 2$. Here Gaussian functions $\mathbb{R}^d \to \mathbb{C}$, $d = 1, 2$, are of the form

$$e^{A|x|^2+B\cdot x+C}$$

with $A, C \in \mathbb{C}$, $B \in \mathbb{C}^d$ and the real part of $A$ negative. The existence of extremizers was established previously by Kunze [2003] for the Strichartz inequality (1) when $d = 1$. When $d \geq 3$, existence of extremizers is proved by the second author in [Shao 2009] .

In this note, we are interested in the problem of how to characterize extremals for (2) via the study of the associated Euler–Lagrange equation. We show that the solutions of this generalized Euler–Lagrange equation enjoy fast decay in the Fourier space and thus can be extended to be complex analytic; see Theorem 1.1. Then, as an easy consequence, we give an alternative proof that all extremal functions to (2) are Gaussians, based on solving a functional equation of extremizers derived in [Foschi 2007]; see (7) and Theorem 1.2. Indeed, in the proof given below we use the information that $f$ is twice continuously differentiable, i.e., $f \in C^2$, which can be lowered to continuity by a more refined argument. The functional inequality (7) is a key ingredient in Foschi's proof. To prove $f$ in (7) to be a Gaussian function, local integrability of $f$ is assumed in [Foschi 2007], which is further reduced to measurable functions in [Charalambides 2013].

Let $f$ be an extremal function to (2) with the constant $\boldsymbol{R}$. Then $f$ satisfies the generalized Euler–Lagrange equation

$$\omega\langle g, f\rangle = \mathfrak{Q}(g, f, f, f) \quad \text{for all } g \in L^2, \tag{5}$$

where $\omega = \mathfrak{Q}(f, f, f, f)/\|f\|_{L^2}^2 > 0$ and $\mathfrak{Q}(f_1, f_2, f_3, f_4)$ is the integral

$$\int_{(\mathbb{R}^2)^4} \bar{\hat{f}}_1(\xi_1)\bar{\hat{f}}_2(\xi_2)\hat{f}_3(\xi_3)\hat{f}_4(\xi_4)\delta(\xi_1+\xi_2-\xi_3-\xi_4)\delta(|\xi_1|^2+|\xi_2|^2-|\xi_3|^2-|\xi_4|^2)\, d\xi_1\, d\xi_2\, d\xi_3\, d\xi_4 \tag{6}$$

for $f_i \in L^2(\mathbb{R}^2)$, $1 \leq i \leq 4$, and $\delta(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi\cdot x}\, dx$ in the distribution sense for $d = 1, 2$. The proof of (5) is standard; see, e.g., [Evans 2010, p. 489] or [Hundertmark and Lee 2012, Section 2] for similar derivations of Euler–Lagrange equations.

**Theorem 1.1.** *If $f$ solves the generalized Euler–Lagrange equation (5) for some $\omega > 0$, then there exists $\mu > 0$ such that*

$$e^{\mu|\xi|^2}\hat{f} \in L^2(\mathbb{R}^2).$$

*Furthermore, $f$ can be extended to be complex analytic on $\mathbb{C}^2$.*

To prove this theorem, we follow the argument in [Hundertmark and Shao 2012]. Similar reasoning has appeared previously in [Erdoğan et al. 2011; Hundertmark and Lee 2009]. It relies on a multilinear weighted Strichartz estimate and a continuity argument. See Lemmas 2.1 and 2.2.

Next we prove that the extremals to (2) are Gaussian functions. We start with the study of the functional equation derived in [Foschi 2007], which reads

$$f(x)f(y) = f(w)f(z) \tag{7}$$

for any $x, y, w, z \in \mathbb{R}^2$ such that

$$x + y = w + z \quad \text{and} \quad |x|^2 + |y|^2 = |w|^2 + |z|^2. \tag{8}$$

Note that $x, y, w, z \in \mathbb{R}^2$ satisfy the relation (8) if and only if these four points form a rectangle in $\mathbb{R}^2$ with vertices $x$, $y$, $w$ and $z$. Indeed, by (8), these four points $x$, $y$, $w$ and $z$ form a parallelogram on $\mathbb{R}^2$ and $x \cdot y = w \cdot z$. Secondly, $w - x$ is perpendicular to $z - x$, since $(w-x)\cdot(z-x) = w\cdot z - w\cdot x - x\cdot z + |x|^2 = w \cdot z - (x + y) \cdot x + |x|^2 = w \cdot z - y \cdot x = 0$. This proves that $x$, $y$, $w$ and $z$ form a rectangle on $\mathbb{R}^2$. In [Foschi 2007], it is proven that $f \in L^2$ satisfies (7) if and only if $f$ is an extremal function to (2). Basically, this comes from two aspects. One is that, in the Foschi's proof of the sharp Strichartz inequality, only the Cauchy–Schwarz inequality is used at one place besides equality. So the equality in the Strichartz inequality (2), or equivalently the equality in Cauchy-Schwarz, yields the same functional equation as (7), where $f$ is replaced by $\hat{f}$. The other one is that the Strichartz norm for the Schrödinger equation satisfies the identity

$$\|e^{it\Delta} f\|_{L^4(\mathbb{R}\times\mathbb{R}^2)} = C\|e^{it\Delta} f^\vee\|_{L^4(\mathbb{R}\times\mathbb{R}^2)} \tag{9}$$

for some $C > 0$.

Foschi [2007] is able to show that all the solutions to (7) are Gaussians under the assumption that $f$ is a locally integrable function. This can be viewed as an investigation of the Cauchy functional equation (7) for functions supported on the paraboloids. To characterize the extremals for the Tomas–Stein inequality for the sphere in $\mathbb{R}^3$, [Christ and Shao 2012] studies the same functional equation (7) for functions supported on the sphere and prove that they are exponentially affine functions. Charalambides [2013] generalizes the analysis in [Christ and Shao 2012] to some general hypersurfaces in $\mathbb{R}^n$ that include the sphere, paraboloids and cones as special examples and proves that the solutions are exponentially affine functions. In [Charalambides 2013; Christ and Shao 2012], the functions are assumed to be measurable functions.

By the analyticity established in Theorem 1.1, equations (7) and (8) have the following easy consequence, which recovers the result in [Foschi 2007; Hundertmark and Zharnitsky 2006].

**Theorem 1.2.** *Suppose that $f$ is an extremal function to (2). Then*

$$f(x) = e^{A|x|^2 + B\cdot x + C}, \tag{10}$$

*where $A, C \in \mathbb{C}$, $B \in \mathbb{C}^2$ and $\Re(A) < 0$.*

Let $f$ be an extremal function to (2). Then, by Theorem 1.1, $f$ is continuous. This, together with (7) and (8), implies that any nontrivial $f$ is nowhere vanishing on $\mathbb{R}^2$; see, e.g., [Foschi 2007, Lemma 7.13]. For any $a \in \mathbb{R}^2$, there is a disk $D(a, r) \subset \mathbb{C}^2$, $r > 0$, such that $f$ is $C^2$ by Theorem 1.1 and $f$ is nowhere vanishing. Then $\log f$ is $C^2$ on $D(a, r)$; see, e.g., [Krantz 1992, Lemma 6.1.9]. Similar claims can be made for $\log f^2$. Then, up to a multiple of $2\pi$,

$$\log f^2(a) = \log f(a) + \log f(a).$$

After restriction to $\mathbb{R}^2$, $f$ satisfies (7) for $x$, $y$, $w$ and $z$ satisfying (8). So, by taking $r$ sufficiently small,

$$\log f(x) + \log f(y) = \log f(w) + \log f(z)$$

for $x, y, w, z \in B(a, r) \subset \mathbb{R}^2$ related as in (8). Since $\log f$ is twice differentiable, it is not hard to see that $\log f$ is a quadratic polynomial on $B(a, r)$. So $\log f$ is a quadratic polynomial on $\mathbb{R}^2$. Indeed, let $a = 0$ and $\phi(x_1) = \log f(x_1, 0)$, $\psi(0, x_2) = \log f(0, x_2)$. Then, since the four points $(x_1, x_2)$, $(x_2, -x_1)$, $(x_1 + x_2, x_2 - x_1)$ and $(0, 0)$ satisfy (8), we see that

$$[\phi(x_1) + \psi(x_2)] + [\phi(x_2) + \psi(-x_1)] = [\phi(x_1 + x_2) + \psi(x_2 - x_1)] + \log f(0, 0).$$

By differentiating firstly in $x_1$ and then in $x_2$, we see that $\phi'' = \psi''$ is a constant. Thus $f$ is a quadratic polynomial. It is easy to see that this argument generalizes to any $a \in \mathbb{R}^2$.

## 2. Complex analyticity

In this section, we show that the solutions to the generalized Euler–Lagrange equation (5) can be extended to be complex analytic.

We define

$$\begin{aligned} \eta &:= (\eta_1, \eta_2, \eta_3, \eta_4) \in (\mathbb{R}^2)^4, \\ a(\eta) &:= \eta_1 + \eta_2 - \eta_3 - \eta_4, \\ b(\eta) &:= |\eta_1|^2 + |\eta_2|^2 - |\eta_3|^2 - |\eta_4|^2. \end{aligned}$$

Let $\varepsilon \geq 0$ and $\mu \geq 0$. For $\xi \in \mathbb{R}^2$, define

$$F(\xi) := F_{\mu,\varepsilon}(\xi) = \frac{\mu|\xi|^2}{1 + \varepsilon|\xi|^2}. \tag{11}$$

Define the weighted multilinear integral for $h_i \in L^2(\mathbb{R}^2)$, $1 \leq i \leq 4$, by

$$M_F(h_1, h_2, h_3, h_4) := \int_{(\mathbb{R}^2)^4} e^{F(\eta_1) - \sum_{j=2}^4 F(\eta_j)} \prod_{j=1}^4 |h(\eta_j)| \delta(a(\eta)) \delta(b(\eta))\, d\eta. \tag{12}$$

The multilinear estimate we need shows the weak interaction of Schrödinger waves between the high and low frequency. More precisely:

**Lemma 2.1.** *Let $h_i \in L^2(\mathbb{R}^2)$, $1 \leq i \leq 4$, and let $s > 1$ be a large number. If the Fourier transforms of $h_1$ and $h_2$ are supported in $\{\xi : |\xi| \leq s\}$ and $\{\xi : |\xi| \geq Ns\}$ with $N > 1$ a large number, respectively, then*

$$M_F(h_1, h_2, h_3, h_4) \leq C N^{-1/2} \prod_{j=1}^4 \|h_j\|_{L^2}. \tag{13}$$

*Proof.* The proof of this lemma needs the following two inequalities:

$$M_F(h_1, h_2, h_3, h_4) \le \int_{(\mathbb{R}^2)^4} \prod_{j=1}^{4} |h_j(\eta_j)| \delta(a(\eta))\delta(b(\eta))\, d\eta \tag{14}$$

and

$$\|e^{it\Delta} h_1 e^{it\Delta} h_2\|_{L^2_{t,x}} \le C N^{-1/2} \|h_1\|_{L^2} \|h_2\|_{L^2}. \tag{15}$$

Together with the Cauchy–Schwarz inequality and the $L^2 \to L^4$ Strichartz inequality, the inequality (13) follows from (14) and (15). Note that (15) is established in [Bourgain 1998]. Thus it remains to establish (14), where we follow [Erdoğan et al. 2011; Hundertmark and Shao 2012].

On the support of $\eta$ determined by $\delta(a(\eta))$ and $\delta(b(\eta))$, we have

$$\eta_1 + \eta_2 = \eta_3 + \eta_4 \quad \text{and} \quad |\eta_1|^2 + |\eta_2|^2 = |\eta_3|^2 + |\eta_4|^2.$$

Thus,

$$|\eta_1|^2 \le |\eta_2|^2 + |\eta_3|^2 + |\eta_4|^2.$$

Since the function $x \mapsto x/(1+\varepsilon x)$ is increasing on the interval $[0, \infty)$, we have

$$\frac{|\eta_1|^2}{1+\varepsilon|\eta_1|^2} \le \frac{\sum_{j=2}^4 |\eta_j|^2}{1+\sum_{j=2}^4 \varepsilon|\eta_j|^2} = \sum_{j=2}^{4} \frac{|\eta_j|^2}{1+\sum_{j=2}^4 \varepsilon|\eta_j|^2} \le \sum_{j=2}^{4} \frac{|\eta_j|^2}{1+\varepsilon|\eta_j|^2}.$$

This implies that $F(\eta_1) \le \sum_{j=2}^4 F(\eta_j)$, since $\mu \ge 0$. Hence,

$$e^{F(\eta_1) - \sum_{j=2}^4 F(\eta_j)} \le 1.$$

Therefore (14) follows by taking the absolute value in the integral. □

If $f \in L^2$ satisfies the generalized Euler–Lagrange equation (5), the following bootstrap lemma shows that $f$ gains certain regularity; namely, there is a constant $\mu > 0$ depending on the function $f$ such that $e^{\mu|\xi|^2} \hat{f} \in L^2$. This is enough to conclude that $f$ can be extended to be complex analytic.

**Lemma 2.2.** *If $f$ solves the generalized Euler–Lagrange equation (5) for some $\omega > 0$ and $\|f\|_{L^2} = 1$, then for $\hat{f}_> := \hat{f} 1_{|\xi| \ge s^2}$ with $s > 0$, there is a large constant $s \gg 1$ such that, for $\mu = s^{-4}$,*

$$\omega \|e^{F(\cdot)} \hat{f}_>\|_{L^2} \le o_1(1) \|e^{F(\cdot)} \hat{f}_>\|_{L^2} + C \|e^{F(\cdot)} \hat{f}_>\|_{L^2}^2 + C \|e^{F(\cdot)} \hat{f}_>\|_{L^2}^3 + o_2(1), \tag{16}$$

*where $\lim_{s\to\infty} o_i(1) = 0$ uniformly for all $\varepsilon > 0$, $i = 1, 2$, and the constant $C > 0$ is independent of $\varepsilon$ and $s$.*

*Proof.* Define $h(\xi) = e^{F(\xi)} \hat{f}(\xi)$ and $h_>(\xi) = e^{F(\xi)} \hat{f}_>$, where $\hat{f}_> = \hat{f} 1_{|\xi| \ge s^2}$. Let $P$ denote the symbol of differentiation $-i\partial_x$; under the Fourier transform, $\widehat{P} = |\xi|$. Correspondingly, we write $F(P)$ with the Fourier symbol $\mu|\xi|^2/(1+\varepsilon|\xi|^2)$.

We expand

$$\|e^{F(\cdot)} \hat{f}_>\|_{L^2}^2 = \langle e^{F(\cdot)} \hat{f}_>, e^{F(\cdot)} \hat{f}_> \rangle = \langle e^{2F(\cdot)} \hat{f}_>, \hat{f} \rangle = \langle e^{2F(P)} f_>, f \rangle.$$

Thus, in the generalized Euler–Lagrange equation (5), setting $g = e^{2F(P)} f_>$, we see that

$$\omega \| e^{F(P)} f_> \|_{L^2}^2 = Q(e^{2F(P)} f_>, f, f, f). \tag{17}$$

Since $\hat{f} = e^{-F(\xi)} h$ and $e^{2F(\xi)} \hat{f}_> = e^{F(\xi)} h_>$,

$$\begin{aligned} Q(e^{2F(P)} f_>, f, f, f) &= \int_{(\mathbb{R}^2)^4} e^{2F(\xi_1)} \overline{\hat{f}}_>(\xi_1) \overline{\hat{f}}_>(\xi_2) \hat{f}(\xi_3) \hat{f}_4(\xi_4) \delta(a(\xi)) \delta(b(\xi))\, d\xi \\ &= \int_{(\mathbb{R}^2)^4} \overline{e^{F(\xi_1)} h_>(\xi_1)}\, \overline{e^{-F(\xi_2)} h(\xi_2)} e^{-F(\xi_3)} h(\xi_3) e^{-F(\xi_4)} h(\xi_4) \delta(a(\xi)) \delta(b(\xi))\, d\xi \\ &= \int_{(\mathbb{R}^2)^4} e^{F(\xi_1) - \sum_{j=2}^4 F(\xi_j)} h_>(\xi_1) h(\xi_2) h(\xi_3) h(\xi_4) \delta(a(\xi)) \delta(b(\xi))\, d\xi, \end{aligned}$$

where $a(\xi) = \xi_1 + \xi_2 - \xi_3 - \xi_4$ and $b(\xi) = |\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 - |\xi_4|^2$ for $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in (\mathbb{R}^2)^4$. Thus,

$$\omega \| e^{F(P)} f_> \|_{L^2}^2 \le M_F(h_>, h, h, h). \tag{18}$$

Define

$$h_\sim = h 1_{s \le |\xi| \le s^2}, h_\ll = h 1_{|\xi| < s} \quad \text{and} \quad h_< = h_\ll + h_\sim .$$

We split the integral $M_F(h_>, h, h, h)$ into the following pieces:

$$M_F(h_>, h_<, h_<, h_<) + \sum_{j_2, j_3, j_4} M_F(h_>, h_{j_2}, h_{j_3}, h_{j_4}) =: A + B,$$

where $h_{j_k}$ is either $h_>$ or $h_<$, but at least one is $h_>$. We further split $A$ into two terms,

$$M_F(h_>, h_\ll, h_<, h_<) + M_F(h_>, h_\sim, h_<, h_<);$$

we estimate this term by using Lemma 2.1:

$$A \lesssim s^{-1/2} \|h_>\|_{L^2} \|h_\ll\|_{L^2} \|h_<\|_{L^2}^2 + \|h_>\|_{L^2} \|h_\sim\|_{L^2} \|h_<\|_{L^2}^2 \lesssim \|h_>\|_{L^2} (s^{-1/2} \|h_\ll\|_{L^2} + \|h_\sim\|_{L^2}) \|h_<\|_{L^2}^2.$$

Since $\|f\|_{L^2} = 1$,

$$\begin{aligned} \|h_<\|_{L^2} &\le e^{\mu s^4} \|f\|_{L^2} = e^{\mu s^4}, \\ \|h_\ll\|_{L^2} &\le e^{\mu s^2}, \\ \|h_\sim\|_{L^2} &\le e^{\mu s^4} \|f_\sim\|_{L^2}, \end{aligned}$$

where $f_\sim$ is defined by $\hat{f}_\sim = \hat{f} 1_{s \le |\xi| \le s^2}$. Thus we have

$$A \lesssim e^{3\mu s^4} \|h_>\|_{L^2} (s^{-1/2} e^{\mu s^2 - \mu s^4} + \|f_\sim\|_{L^2}). \tag{19}$$

Similarly we estimate the term $B$. We split $B$ as $B_1 + B_2$, where $B_1 = \sum_{j_2, j_3, j_4} M_F(h_>, h_{j_2}, h_{j_3}, h_{j_4})$ contains exactly one $h_>$ in $\{h_{j_2}, h_{j_3}, h_{j_4}\}$, while $B_2 = \sum_{j_2, j_3, j_4} M_F(h_>, h_{j_2}, h_{j_3}, h_{j_4})$ contains two or more $h_>$.

To estimate $B_1$,

$$B_1 \lesssim e^{\mu s^4}\|h_>\|_{L^2}^2\|h_<\|_{L^2}(s^{-1/2}e^{\mu s^2-\mu s^4}+\|f_\sim\|_{L^2}) \lesssim e^{2\mu s^4}\|h_>\|_{L^2}^2(s^{-1/2}e^{\mu s^2-\mu s^4}+\|f_\sim\|_{L^2}). \quad (20)$$

To estimate $B_2$,

$$B_2 \lesssim \|h_>\|_{L^2}^3\|h_<\|_{L^2}+\|h_>\|_{L^2}^4 \lesssim e^{\mu s^4}\|h_>\|_{L^2}^3+\|h_>\|_{L^2}^4. \quad (21)$$

Thus, from (19), (20) and (21), we obtain

$$\|e^{F(\cdot)}\hat{f}_>\|_{L^2}^2$$
$$\lesssim e^{3\mu s^4}\|h_>\|_{L^2}(s^{-1/2}e^{\mu s^2-\mu s^4}+\|f_\sim\|_{L^2})+e^{2\mu s^4}\|h_>\|_{L^2}^2(s^{-1/2}e^{\mu s^2-\mu s^4}+\|f_\sim\|_{L^2})+e^{\mu s^4}\|h_>\|_{L^2}^3+\|h_>\|_{L^2}^4.$$

Since $\lim_{s\to\infty}\|f_\sim\|_{L^2}=0$, we take $s$ sufficiently large and set $\mu=s^{-4}$:

$$\omega\|e^{F(\cdot)}\hat{f}_>\|_{L^2} \le o_1(1)\|e^{F(\cdot)}\hat{f}_>\|_{L^2}+C\|e^{F(\cdot)}\hat{f}_>\|_{L^2}^2+C\|e^{F(\cdot)}\hat{f}_>\|_{L^2}^3+o_2(1), \quad (22)$$

which completes the proof of Lemma 2.2. □

**Remark 2.3.** Clearly the choice of $\mu$ in the preceding lemma depends on the function $f$ itself.

Now we conclude that $f$ in Lemma 2.2 gains certain regularity.

*Proof of Theorem 1.1.* Let $f \in L^2$ and $f \neq 0$. We normalize $f$ so that $\|f\|_{L^2}=1$. In Lemma 2.2, we choose $s$ sufficiently large such that $o_1(1) \le \frac{1}{2}\omega$ and $o_2(1) \le \frac{1}{2}M$, where $M=\sup\{G(x) : x \in [0,\infty)\}$, and

$$G(x) := \tfrac{1}{2}\omega x - Cx^2 - Cx^3, \quad x \in [0,\infty), \quad (23)$$

and $C$ is the same constant as in (16). It is easy to see that $0 \le M < \infty$. Then $G(x) \le M$ for all $x \in [0,\infty)$ by Lemma 2.2. Also the function $G$ is continuous on $[0,\infty)$. On the other hand, $G''(x) < 0$ for all $x \in (0,\infty)$; thus $G$ is concave. The line $G=\frac{1}{2}M$ intersects at two points of the positive $x$ axis, $x=x_0$ and $x=x_1>0$.

We define $H : (0,\infty) \to [0,\infty)$ via

$$H(\varepsilon)=\left(\int_{|\xi|\ge s^2}|e^{F_{s^{-4},\varepsilon}(\xi)}\hat{f}|^2\,d\xi\right)^{\frac{1}{2}}.$$

The function $H$ is continuous on $(0,\infty)$ by the dominated convergence theorem and $H(0,\infty)$ is connected. Hence $G^{-1}\big([0,\frac{1}{2}M]\big)$ is either contained in $[0,x_0]$ or $[x_1,\infty)$; only one alternative holds. For $\varepsilon=1$ and $s$ sufficiently large, $H(1) \ge x_1$ is impossible. Hence the first alternative holds.

Therefore $G^{-1}\big([0,\frac{1}{2}M]\big) \subset [0,x_0]$, which yields that

$$\|e^{F(\cdot)}\hat{f}_>\|_{L^2} \le C_0, \quad \text{that is,} \quad \|e^{s^{-4}|\xi|^2/(1+\varepsilon|\xi|^2)}\hat{f}_>\|_{L^2} \le C_0, \quad (24)$$

uniformly in all $\varepsilon > 0$. By the monotone convergence theorem,

$$\|e^{s^{-4}|\xi|^2}\hat{f}_>\|_{L^2} \le C_0 < \infty.$$

It is clear that $e^{s^{-4}|\xi|^2}\hat{f}1_{|\xi|\le s^2}\in L^2$. Therefore,

$$e^{s^{-4}|\xi|^2}\hat{f}\in L^2.$$

Let $\mu=s^{-4}$. This proves the first half of Theorem 1.1.

To prove that $f$ can be extended to be complex analytic on $\mathbb{C}^2$, we observe that, by the Cauchy–Schwarz inequality, for any $\lambda\in\mathbb{R}$,

$$e^{\lambda|\xi|}\hat{f}(\xi)=e^{\lambda|\xi|-\mu|\xi|^2}e^{\mu|\xi|^2}\hat{f}(\xi)\in L^2(\mathbb{R}^2). \tag{25}$$

So it is not hard to see that $f$ can be extended to be complex analytic on $\mathbb{C}^2$; see, e.g., [Reed and Simon 1975, Theorem IX.13]. Alternatively, analyticity can be obtained in the following way. Similarly to in (25) for $k\in\mathbb{N}\cup\{0\}$, $|\xi|^k e^{\lambda|\xi|}\hat{f}\in L^1(\mathbb{R}^2)$. For $z\in\mathbb{C}^2$, we choose $\lambda>|z|$, then

$$f(z)=(2\pi)^{-2}\int_{\mathbb{R}^2}e^{iz\cdot\xi-\lambda|\xi|}e^{\lambda|\xi|}\hat{f}(\xi)\,d\xi.$$

Then, by taking differentiation under the integral sign, complex analyticity follows. □

## Acknowledgements

The research of Jiang is supported by National Science Council Grant NSC 102-2115-M-007-0101-MY 2. Shao is supported in part by the NSF grant DMS-1160981. The authors would like to thank the anonymous referee for helpful comments and suggestions, which have been incorporated into this paper.

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Received 22 Apr 2014. Revised 28 Sep 2015. Accepted 16 Dec 2015.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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