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(ELLIPTIC CASE)**





## A COUNTEREXAMPLE TO THE HOPF–OLEINIK LEMMA (ELLIPTIC CASE)

DARYA E. APUSHKINSKAYA AND ALEXANDER I. NAZAROV

*Dedicated to Professor M.V. Safonov*

We construct a new counterexample to the Hopf–Oleinik boundary point lemma. It shows that for convex domains, the  $C^{1,\text{Dini}}$  assumption on  $\partial\Omega$  is the necessary and sufficient condition providing the estimates of Hopf–Oleinik type.

### 1. Introduction

The influence of the properties of a domain on the behavior of a solution is one of the most important topics in the qualitative analysis of partial differential equations.

The significant result in this field is the Hopf–Oleinik lemma, known also as the “boundary point principle”. This celebrated lemma states:

**Lemma.** *Let  $u$  be a nonconstant solution to a second-order homogeneous uniformly elliptic nondivergence equation with bounded measurable coefficients, and let  $u$  attain its extremum at a point  $x^0$  located on the boundary of a domain  $\Omega \subset \mathbb{R}^n$ . Then  $(\partial u / \partial \mathbf{n})(x^0)$  is necessarily nonzero provided that  $\partial\Omega$  satisfies the proper assumptions at  $x^0$ .*

This result was established in a pioneering paper of S. Zaremba [1910] for the Laplace equation in a 3-dimensional domain  $\Omega$  having an interior touching ball at  $x^0$  and generalized by G. Giraud [1932; 1933] to equations with Hölder-continuous leading coefficients and continuous lower-order coefficients in domains  $\Omega$  belonging to the class  $C^{1,\alpha}$  with  $\alpha \in (0, 1)$ .

Notice that a related assertion about the negativity on  $\partial\Omega$  of the normal derivative of the Green’s function corresponding to the Dirichlet problem for the Laplace operator was proved much earlier for 2-dimensional smooth domains by C. Neumann [1888] (see also [Korn 1901]). The result of [Neumann 1888] was extended for operators with lower-order coefficients by L. Lichtenstein [1924]. The same version of the boundary point principle for the Laplacian and 3-dimensional domains satisfying a more flexible interior paraboloid condition was obtained by M. V. Keldysch and M. A. Lavrent’ev [1937].

A crucial step in studying the boundary point principle was made by E. Hopf [1952] and O. A. Oleĭnik [1952], who simultaneously and independently proved the statement for the general elliptic equations with bounded coefficients and domains satisfying an interior ball condition at  $x^0$ .

Later the efforts of many mathematicians were focused on the generalization of the boundary point principle in several directions (for the details, we refer the reader to [Alvarado et al. 2011; Alvarado

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2011] and references therein). Among these directions are the extension of the class of operators and the class of solutions, as well as the weakening of assumptions on the boundary.

The widening of the class of operators to singular/degenerate ones was made in the papers [Kamynin and Himčenko 1975; 1977; Alvarado et al. 2011], while the uniform elliptic operators with unbounded lower-order coefficients were studied in [Safonov 2010; Nazarov 2012] (see also [Nazarov and Uraltseva 2009]). We mention also the publications [Tolksdorf 1983; Mikayelyan and Shahgholian 2015], where the boundary point principle was established for a class of degenerate quasilinear operators including the  $p$ -Laplacian.

We note that before 2010, all the results were formulated for classical solutions, i.e.,  $u \in C^2(\Omega)$ . The class of solutions was expanded in [Safonov 2010] to strong generalized solutions with Sobolev's second-order derivatives. The latter requirement seems to be natural in the study of nondivergent elliptic equations.

The reduction of the assumptions on the boundary of  $\Omega$  up to  $C^{1, \text{Dini}}$ -regularity was realized for various elliptic operators in the papers [Widman 1967; Himčenko 1970; Lieberman 1985] (see also [Safonov 2008]). A weakened form of the Hopf–Oleinik lemma (the existence of a boundary point  $x^1$  in any neighborhood of  $x^0$  and a direction  $\ell$  such that  $(\partial u / \partial \ell)(x^1) \neq 0$ ) was proved in [Nadirashvili 1983] for a much wider class of domains including all Lipschitz ones. We mention also the paper [Sweers 1997], where the behavior of superharmonic functions near the boundary of a 2-dimensional domain with corners is described in terms of the main eigenfunction of the Dirichlet Laplacian.

The sharpness of some requirements was confirmed by corresponding counterexamples constructed in [Widman 1967; Himčenko 1970; Kamynin and Himčenko 1975; Safonov 2008; Alvarado et al. 2011; Nazarov 2012]. In particular, the counterexamples from [Widman 1967; Himčenko 1970; Safonov 2008] show that the Hopf–Oleinik result fails for domains lying entirely in non-Dini paraboloids.

The main result of our paper is a new counterexample (see Theorem 4.2) showing the sharpness of the Dini condition for the boundary of  $\Omega$ . The simplest version of this counterexample can be formulated as follows:

**Counterexample.** *Let  $\Omega$  be a convex domain in  $\mathbb{R}^n$ , let  $\partial\Omega$  in a neighborhood of the origin be described by the equation  $x_n = F(x')$  with  $F \geq 0$  and  $F(0) = 0$ , and let  $u \in W_{n, \text{loc}}^2(\Omega) \cap C(\bar{\Omega})$  be a solution of the uniformly elliptic equation*

$$-a^{ij}(x)D_i D_j u = 0 \quad \text{in } \Omega.$$

*Suppose also that  $u|_{\partial\Omega}$  vanishes at a neighborhood of the origin. If, in addition, the function*

$$\delta(r) = \sup_{|x'| \leq r} \frac{F(x')}{|x'|}$$

*is not Dini-continuous at zero, then  $(\partial u / \partial \mathbf{n})(0) = 0$ .*

Thus, it turns out that for convex domains, the Dini-continuity assumption on  $\delta(r)$  is necessary and sufficient for the validity of the boundary point principle. We emphasize that in our counterexample the Dini condition fails for the supremum of  $F(x')/|x'|$ , while in all the previous results of this kind, it fails for the infimum of  $F(x')/|x'|$ . In other words, we show that violating the Dini condition just in one direction causes the failure of the Hopf–Oleinik lemma.

**Notation and conventions.** Throughout the paper we use the following notation:

- $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$  is a point in  $\mathbb{R}^n$ .
- $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$ .
- $|x|, |x'|$  are the Euclidean norms in the corresponding spaces.
- $\chi_{\mathcal{E}}$  denotes the characteristic function of the set  $\mathcal{E} \subset \mathbb{R}^n$ .
- $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with boundary  $\partial\Omega$ .
- $\mathcal{P}_{r,h}(\bar{x}') = \{x \in \mathbb{R}^n : |x' - \bar{x}'| < r, 0 < x_n < h\}$  and  $\mathcal{P}_r(\bar{x}') = \mathcal{P}_{r,r}(\bar{x}')$ .
- $\mathcal{P}_{r,h} = \mathcal{P}_{r,h}(0)$  and  $\mathcal{P}_r = \mathcal{P}_r(0)$ .
- $B_r(x^0)$  is the open ball in  $\mathbb{R}^n$  with center  $x^0$  and radius  $r$ ;  $B_r = B_r(0)$ .
- For  $r_1 < r_2$ , we define the annulus  $\mathcal{B}(x^0, r_1, r_2) = B_{r_2}(x^0) \setminus \overline{B_{r_1}(x^0)}$ .
- $v_+ = \max\{v, 0\}$  and  $v_- = \max\{-v, 0\}$ .
- $\|\cdot\|_{\infty, \Omega}$  denotes the norm in  $L_{\infty}(\Omega)$ .
- We adopt the convention that the indices  $i$  and  $j$  run from 1 to  $n$ . We also adopt the convention regarding summation with respect to repeated indices.
- $D_i$  denotes the operator of (weak) differentiation with respect to  $x_i$ .
- $D = (D', D_n) = (D_1, \dots, D_{n-1}, D_n)$ .
- $\mathcal{L}$  is a linear uniformly elliptic operator with measurable coefficients

$$\mathcal{L}u \equiv -a^{ij}(x)D_i D_j u + b^i(x)D_i u, \quad v\mathcal{I}_n \leq (a^{ij}(x)) \leq v^{-1}\mathcal{I}_n, \tag{1}$$

where  $\mathcal{I}_n$  is the  $n \times n$  identity matrix. We define  $\mathbf{b}(x) = (b^1(x), \dots, b^n(x))$ .

- We use the letters  $C$  and  $N$  (with or without indices) to denote various constants. To indicate that, say,  $C$  depends on some parameters, we list them in parentheses:  $C(\dots)$ .

**Definition 1.1.** We say that a function  $\sigma : [0, 1] \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{D}_1$  if

- $\sigma$  is increasing,  $\sigma(0) = 0$ , and  $\sigma(1) = 1$ ;
- $\sigma(t)/t$  is summable and decreasing.

**Remark 1.2.** Our assumption about the decay of  $\sigma(t)/t$  is not restrictive. Indeed, for any increasing function  $\sigma : [0, 1] \rightarrow \mathbb{R}_+$  satisfying  $\sigma(0) = 0$  and  $\sigma(1) = 1$  and having summable  $\sigma(t)/t$ , we can define

$$\tilde{\sigma}(t) = t \sup_{\tau \in [t, 1]} \frac{\sigma(\tau)}{\tau}, \quad t \in (0, 1).$$

It is easy to see that  $\tilde{\sigma} \in \mathcal{D}_1$ ,  $\tilde{\sigma}(t)/t$  decreases and  $\sigma(t) \leq \tilde{\sigma}(t)$  for all  $t \in (0, 1]$ .

**Definition 1.3.** Let a function  $\sigma$  belong to the class  $\mathcal{D}_1$ . We define the function  $\mathcal{J}_{\sigma}$  as

$$\mathcal{J}_{\sigma}(s) := \int_0^s \frac{\sigma(\tau)}{\tau} d\tau. \tag{2}$$

**Remark 1.4.** The decreasing of  $\sigma(t)/t$  implies

$$\sigma(t) \leq \mathcal{J}_\sigma(t) \quad \forall t \in [0, 1]. \quad (3)$$

In addition, for  $t \leq t_0 \leq 1$ , we have

$$\sigma(t/t_0) = \frac{\sigma(t/t_0)}{t/t_0} \cdot t/t_0 \leq \frac{\sigma(t)}{t} \cdot t/t_0 = \frac{\sigma(t)}{t_0}, \quad (4)$$

and, similarly,

$$\mathcal{J}_\sigma(t/t_0) \leq \frac{\mathcal{J}_\sigma(t)}{t_0}. \quad (5)$$

**Definition 1.5.** We say that a function  $\zeta$  satisfies the Dini condition at zero if

$$|\zeta(r)| \leq C\sigma(r),$$

and  $\sigma$  belongs to the class  $\mathcal{D}_1$ .

## 2. Preliminaries

**Properties of  $\Omega$ .** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ . Without loss of generality, we may assume  $0 \in \partial\Omega$ .

Suppose that  $\Omega$  is locally convex in a neighborhood of the origin. Without restriction, the latter means that for some  $0 < \mathcal{R}_0 \leq 1$ , we have

$$\mathcal{P}_{\mathcal{R}_0} \cap \Omega = \{(x', x_n) \in \mathbb{R}^n : |x'| \leq \mathcal{R}_0, F(x') < x_n < \mathcal{R}_0\},$$

where  $F$  is a convex nonnegative function satisfying  $F(0) = 0$ .

For  $r \in (0, \mathcal{R}_0)$ , we define the functions  $\delta = \delta(r)$  and  $\delta_1 = \delta_1(r)$  by the formulas

$$\delta(r) := \max_{|x'| \leq r} \frac{F(x')}{|x'|}, \quad \delta_1(r) := \max_{|x'| \leq r} |\nabla F(x')|. \quad (6)$$

**Lemma 2.1.** *The following statements hold:*

- (a)  $\delta_1(r) \rightarrow 0$  as  $r \rightarrow 0$  if and only if  $\delta(r) \rightarrow 0$  as  $r \rightarrow 0$ .
- (b)  $\delta_1(r)$  satisfies the Dini condition at zero if and only if  $\delta(r)$  satisfies the Dini condition at zero.

*Proof.* By the convexity of  $F$ , we have for any  $x'$  and  $z'$ , the estimate

$$F(z') \geq F(x') + \nabla F(x') \cdot (z' - x'). \quad (7)$$

Therefore,

$$|\nabla F(x')| \geq \nabla F(x') \cdot \frac{x'}{|x'|} \geq \frac{F(x')}{|x'|},$$

and, consequently,

$$\delta_1(r) \geq \delta(r). \quad (8)$$

On the other hand, for any  $r < \frac{1}{2}\mathcal{R}_0$ , we can find a point  $x'_*$  such that

$$|\nabla F(x'_*)| = \delta_1(r).$$

Choosing

$$z' = x'_* + r \frac{\nabla F(x'_*)}{|\nabla F(x'_*)|},$$

we easily deduce from (7) the inequalities

$$|z'| \leq 2r \quad \text{and} \quad F(z') \geq r\delta_1(r),$$

which provide

$$\delta(2r) \geq \delta(|z'|) \geq \frac{1}{2}\delta_1(r). \tag{9}$$

Combining (8) and (9), we conclude that statement (a) is obvious and the integrals

$$\int_0^{\mathcal{R}_0} \frac{\delta(r)}{r} dr \quad \text{and} \quad \int_0^{\mathcal{R}_0} \frac{\delta_1(r)}{r} dr$$

converge simultaneously. □

If  $\delta(r)$  does not converge to zero as  $r \rightarrow 0$ , we can easily see that the domain  $\Omega$  is contained in a dihedral wedge with the angle less than  $\pi$  and the edge going through the origin. For this case, the statement of Theorem 4.2 is proved already in [Apushkinskaya and Nazarov 2000, Theorem 4.3]. For this reason, we will assume throughout this paper that

$$\delta(r) \rightarrow 0 \quad \text{as} \quad r \rightarrow 0. \tag{10}$$

In view of (10), it is evident that  $\delta$  and  $\delta_1$  are moduli of continuity at the origin of the functions  $F(x')/|x'|$  and  $|\nabla F(x')|$ , respectively.

**Properties of  $\mathcal{X}(\Omega)$ .** Let  $\mathcal{X}(\Omega)$  be a function space with the norm  $\|\cdot\|_{\mathcal{X},\Omega}$ . For  $\Omega_1 \subset \Omega$ , we will assume

$$\|f\|_{\mathcal{X},\Omega_1} = \|f \cdot \chi_{\Omega_1}\|_{\mathcal{X},\Omega}.$$

We suppose that  $\mathcal{X}(\Omega)$  has the following properties:

- (i) For an arbitrary measurable function  $g$  defined in  $\Omega$  and any function  $f \in \mathcal{X}(\Omega)$ , the inequality  $|g(x)| \leq |f(x)|$  implies  $g \in \mathcal{X}(\Omega)$  and  $\|g\|_{\mathcal{X},\Omega} \leq \|f\|_{\mathcal{X},\Omega}$ .
- (ii) For  $f_k \in \mathcal{X}(\Omega)$ , the convergence  $f_k \searrow 0$  a.e. in  $\Omega$  implies  $\|f_k\|_{\mathcal{X},\Omega} \rightarrow 0$ .

Using the terminology of the classic monograph of Kantorovich and Akilov [1982], we may say that  $\mathcal{X}(\Omega)$  is the ideal functional space with order continuous monotone norm (see [Kantorovich and Akilov 1982, §3, Chapter IV, Part I] for more details).

We will also assume that

- (iii)  $\mathcal{X}_{\text{loc}}(\Omega)$  contains the Orlicz space  $L_{\Phi,\text{loc}}(\Omega)$  with  $\Phi(\xi) = e^\xi - \xi - 1$ .

Finally, the basic assumption about  $\mathcal{X}(\Omega)$  is the Aleksandrov-type maximum principle. Namely, we denote by  $\mathcal{W}_{\mathcal{X},\text{loc}}^2(\Omega)$  the set of the functions  $u$  satisfying  $D(Du) \in \mathcal{X}_{\text{loc}}(\Omega)$ , and suppose that if  $u \in \mathcal{W}_{\mathcal{X},\text{loc}}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ ,  $u|_{\partial\Omega} \leq 0$ , and  $\mathbf{b} \in \mathcal{X}(\Omega)$  then

$$u \leq N_0(n, \nu, \|\mathbf{b}\|_{\mathcal{X},\Omega}) \cdot \text{diam}(\Omega) \cdot \|(\mathcal{L}u)_+\|_{\mathcal{X},\{u>0\}}. \tag{11}$$

**Remark 2.2.** It is well known from [Aleksandrov 1960; 1963; Bakel'man 1961] (see also the survey [Nazarov 2005] for further references) that  $L_n(\Omega)$  has property (11). It is also evident that properties (i)–(iii) are satisfied in  $L_n(\Omega)$ . Therefore,  $L_n(\Omega)$  can be treated as a “basic” example of  $\mathcal{X}(\Omega)$ . As other examples of the space  $\mathcal{X}(\Omega)$ , we mention some Lebesgue weighted spaces with power weights (see [Nazarov 2001]).

**Remark 2.3.** Unlike the natural properties (i)–(ii), assumption (iii) is a rather “technical” one. Without (iii), our arguments from the proof of Step 3 in Theorem 4.1 are not applicable to the approximating operator  $\mathcal{L}_\varepsilon$ . So, we cannot withdraw (iii) in abstract setting. However, in all known examples of  $\mathcal{X}(\Omega)$ , property (iii) is satisfied.

**Remark 2.4.** Some of the statements that will be referred to in the sequel were proved earlier just for the case  $\mathcal{X}(\Omega) = L_n(\Omega)$ . However, if all the arguments are based only on the Aleksandrov-type maximum principle, these statements remain valid for an arbitrary considered space  $\mathcal{X}(\Omega)$ . In such cases, we will refer to this remark without any further explanation.

We also need the following convergence lemmas.

**Lemma 2.5.** *Let  $\{f_j\}$  be a sequence of measurable functions on  $\Omega$ , and let  $f \in \mathcal{X}(\Omega)$ . Suppose also that  $f_j \rightarrow 0$  in measure on  $\Omega$ , and  $|f_j(x)| \leq |f(x)|$ .*

*Then*

$$\|f_j\|_{\mathcal{X},\Omega} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (12)$$

*Proof.* We argue by a contradiction. Suppose (12) fails. Then there exists a subsequence  $\{f_{j_k}\}$  satisfying

$$\|f_{j_k}\|_{\mathcal{X},\Omega} \geq \varepsilon > 0 \quad \forall k \in \mathbb{N}. \quad (13)$$

Due to the Riesz theorem, there exists also a subsubsequence  $\{f_{j_{k_l}}\}$  such that

$$f_{j_{k_l}} \rightarrow 0 \quad \text{a.e. in } \Omega.$$

For simplicity of notation, we renumber the latter subsequence  $\{f_{j_{k_l}}\}$  and denote its elements again by  $f_j$ .

Setting  $\tilde{f}_k := \sup_{j \geq k} |f_j|$ , we can easily see that  $\tilde{f}_k \searrow 0$  a.e. in  $\Omega$ . Now, taking into account properties (i) and (ii) of the space  $\mathcal{X}(\Omega)$ , we immediately get a contradiction with inequalities (13).  $\square$

**Lemma 2.6.** *Let  $f \in \mathcal{X}(\Omega)$ , and let  $\mu(\rho) := \sup_{x \in \Omega} \|f\|_{\mathcal{X}, B_\rho(x) \cap \Omega}$ .*

*Then*

$$\mu(\rho) \rightarrow 0 \quad \text{as } \rho \rightarrow 0.$$

*Proof.* For every  $\rho > 0$ , there exists a point  $x^* = x^*(\rho) \in \Omega$  such that

$$\|f\|_{\mathcal{X}, B_\rho(x^*) \cap \Omega} \geq \frac{1}{2} \mu(\rho).$$

Next, for the sequence  $f_\rho := f \cdot \chi_{B_\rho(x^*)}$ , it is evident that  $|f_\rho| \rightarrow 0$  in measure on  $\Omega$ . An application of Lemma 2.5 finishes the proof.  $\square$

**Remark 2.7.** We call  $\mu(\rho) := \sup_{x \in \Omega} \|f\|_{\mathcal{X}, B_\rho(x) \cap \Omega}$  the modulus of continuity of the function  $f$  in  $\mathcal{X}(\Omega)$ .



**Lemma 2.8.** *Let  $D(Du) \in \mathcal{X}(\Omega)$ , let  $\mathcal{L}$  be defined by (1), and let  $\mathcal{L}u \in \mathcal{X}(\Omega)$ . There exists the family of operators*

$$\mathcal{L}_\varepsilon = -a_\varepsilon^{ij}(x)D_iD_j + b_\varepsilon^i(x)D_i$$

*with smooth coefficients  $a_\varepsilon^{ij}$  and bounded coefficients  $b_\varepsilon^i$  satisfying*

$$v\mathcal{I}_n \leq (a_\varepsilon^{ij}(x)) \leq v^{-1}\mathcal{I}_n, \quad x \in \Omega, \tag{14}$$

$$|b_\varepsilon^i(x)| \leq |b^i(x)|, \quad x \in \Omega, \tag{15}$$

$$\|(\mathcal{L} - \mathcal{L}_\varepsilon)u\|_{\mathcal{X},\Omega} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{16}$$

*Proof.* We start with the extension of  $a^{ij}$  on the whole  $\mathbb{R}^n$  by the identity matrix and denote by  $a_\varepsilon^{ij}$  the standard mollification of extended functions  $a^{ij}$ . By construction, the coefficients  $a_\varepsilon^{ij}$  are smooth functions converging as  $\varepsilon \rightarrow 0$  to  $a^{ij}$  a.e. in  $\Omega$ . Moreover, it is clear that inequalities (14) are true.

Further, we set

$$\tilde{b}_\varepsilon^i(x) := \min\{|b^i(x)|, \varepsilon^{-1}\} \cdot \text{sign } b^i(x). \tag{17}$$

In view of (17), it is evident that  $\tilde{b}_\varepsilon^i D_i u$  converges as  $\varepsilon \rightarrow 0$  to  $b^i D_i u$  a.e. in  $\Omega$ . We claim that it is possible to change  $\tilde{b}_\varepsilon^i$  such that the ‘‘corrected coefficients’’  $b_\varepsilon^i$  satisfy

$$|b_\varepsilon^i D_i u| \leq |b^i D_i u| \quad \text{in } \Omega. \tag{18}$$

Indeed, if  $|\tilde{b}_\varepsilon^i D_i u| \leq |b^i D_i u|$  in  $\Omega$  then (18) holds with  $b_\varepsilon^i \equiv \tilde{b}_\varepsilon^i$ . Otherwise, consider a point  $x^0 \in \Omega$ , where  $|\tilde{b}_\varepsilon^i(x^0)D_i u(x^0)| > |b^i(x^0)D_i u(x^0)|$ .

- (a) Let  $\tilde{b}_\varepsilon^i(x^0)D_i u(x^0) > b^i(x^0)D_i u(x^0) \geq 0$ . In this case, we decrease all the coefficients  $\tilde{b}_\varepsilon^i(x^0)$  corresponding to the positive summands such that the sums  $b_\varepsilon^i D_i u$  and  $b^i D_i u$  become equal.
- (b) Let  $\tilde{b}_\varepsilon^i(x^0)D_i u(x^0) < b^i(x^0)D_i u(x^0) \leq 0$ . In this case, we decrease all the coefficients  $\tilde{b}_\varepsilon^i(x^0)$  corresponding to the negative summands such that the sums  $b_\varepsilon^i D_i u$  and  $b^i D_i u$  become equal.
- (c) Finally, let  $\tilde{b}_\varepsilon^i(x^0)D_i u(x^0)$  and  $b^i(x^0)D_i u(x^0)$  have different signs. In this case, we apply to  $-b_\varepsilon^i(x^0)$  the arguments from case (a) or from case (b), respectively.

Due to construction, the ‘‘corrected sum’’  $b_\varepsilon^i D_i u$  also converges as  $\varepsilon \rightarrow 0$  to  $b^i D_i u$  a.e. in  $\Omega$ , and the pointwise inequalities (15) hold true.

Finally, taking into account (18) and applying Lemma 2.5, we get (16). □

### 3. Gradient estimates near the boundary

**Lemma 3.1.** *Let  $\mathcal{N} \subset \mathbb{R}_+^n$  be an open set, let  $\gamma = v/\sqrt{n-1}$ , let  $\rho > 0$ , and let*

$$\Pi_\rho = \{y \in \mathbb{R}^n : |y_i| < \rho \text{ for } i = 1, \dots, n-1; 0 < y_n < \gamma\rho\}.$$

*We assume that  $|b| \in \mathcal{X}(\mathcal{N})$  and a function  $v$  satisfies the conditions*

$$v \in \mathcal{W}_{\mathcal{X},\text{loc}}^2(\mathcal{N}), \quad v \geq 0 \quad \text{in } \Pi_\rho, \quad v \geq k = \text{constant} > 0 \quad \text{on } \partial\mathcal{N} \cap \bar{\Pi}_\rho.$$

Then

$$v \geq C_1 k - C_2 k \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} - C_3 \rho \|(\mathcal{L}v)_-\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} \quad \text{in } \mathcal{N} \cap B_{\frac{\gamma\rho}{4}}(z),$$

where  $z = (0, \dots, 0, \frac{1}{2}\gamma\rho)$ , while  $C_1 = \frac{1}{16}(1-\gamma^2)$ ,  $C_2 = C_2(n, \nu, \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N}})$ , and  $C_3 = C_3(n, \nu, \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N}})$ .

*Proof.* The proof is similar in spirit to [Apushkinskaya and Ural'tseva 1995, Lemma 1].

Consider the barrier function

$$\psi(y) = k \left( \left(1 - \frac{y_n}{\gamma\rho}\right)^2 - \frac{|y'|^2}{\rho^2} \right).$$

An elementary computation gives

$$\mathcal{L}\psi \leq k \left( \frac{2(n-1)}{\rho^2} v^{-1} - \frac{2}{\gamma^2 \rho^2} v \right) + |\mathbf{b}| |D\psi| \leq N_1(n, \nu) |\mathbf{b}| \frac{k}{\rho} \quad \text{in } \Pi_\rho.$$

Moreover, setting

$$\mathcal{S}_1 = \{y \in \partial(\mathcal{N} \cap \Pi_\rho) : |y_i| = \rho \text{ for some } i = 1, \dots, n-1\},$$

$$\mathcal{S}_2 = \{y \in \partial(\mathcal{N} \cap \Pi_\rho) : y_n = \gamma\rho\},$$

we have

$$\psi|_{\mathcal{S}_1 \cup \mathcal{S}_2} \leq 0 \leq v,$$

$$\psi|_{\partial\mathcal{N} \cap \bar{\Pi}_\rho} \leq k \leq v|_{\partial\mathcal{N} \cap \bar{\Pi}_\rho}.$$

Applying inequality (11) in  $\mathcal{N} \cap \Pi_\rho$  to the difference  $\psi - v$ , we obtain

$$\psi - v \leq N_0 \cdot \text{diam}(\Pi_\rho) \cdot \|(\mathcal{L}\psi - \mathcal{L}v)_+\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} \quad \text{in } \mathcal{N} \cap \Pi_\rho,$$

and, consequently,

$$\begin{aligned} v &\geq k \left( \left(1 - \frac{\frac{3}{4}\gamma\rho}{\gamma\rho}\right)^2 - \frac{\gamma^2 \rho^2}{16\rho^2} \right) - C_2 k \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} - C_3 \rho \|(\mathcal{L}v)_-\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} \\ &= \frac{1}{16}(1-\gamma^2)k - C_2 k \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} - C_3 \rho \|(\mathcal{L}v)_-\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} \quad \text{in } \mathcal{N} \cap B_{\frac{\gamma\rho}{4}}(z). \quad \square \end{aligned}$$

Our next statement is a version of [Nazarov 2012, Theorem 2.3].

**Lemma 3.2.** *Let  $v \in \mathcal{W}_{\mathcal{X}, \text{loc}}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ , let  $v|_{\partial\Omega} = 0$ , and let  $|\mathbf{b}| \in \mathcal{X}(\Omega)$ . Suppose also that for all  $\rho \leq \rho_* \leq 1$ , the inequalities*

$$\|b^n\|_{\mathcal{X}, \mathcal{P}_\rho \cap \Omega} \leq \mathfrak{B}\sigma(\rho/\rho_*), \quad \|(\mathcal{L}v)_+\|_{\mathcal{X}, \mathcal{P}_\rho \cap \Omega} \leq \mathfrak{F}\sigma(\rho/\rho_*)$$

hold true. Here  $\mathfrak{B}$  and  $\mathfrak{F}$  are some positive constants, while the function  $\sigma$  belongs to  $\mathcal{D}_1$ .

Then

$$\sup_{0 < x_n < \rho} \frac{v(0, x_n)}{x_n} \leq C_4 (\rho^{-1} \sup_{\mathcal{P}_\rho \cap \Omega} v + \mathfrak{F}\mathcal{J}_\sigma(\rho/\rho_*)) \quad \forall \rho \leq \rho_*. \quad (19)$$

Here the constant  $C_4$  depends on  $n$ ,  $\nu$ ,  $\mathfrak{B}$ ,  $\sigma$ , and on the moduli of continuity of  $|\mathbf{b}'|$  in  $\mathcal{X}(\mathcal{P}_{\rho_*} \cap \Omega)$ , whereas  $\mathcal{J}_\sigma$  is a function defined by formula (2).

**Remark 3.3.** We recall that  $0 \in \partial\Omega$ .

*Proof.* First, we assume that  $\rho \leq \bar{\rho}$ , where  $\bar{\rho} \leq \rho_*$  will be fixed later. Following [Nazarov 2012], we introduce the sequence of cylinders  $\mathcal{P}_{\rho_k, h_k}$ , with  $k \geq 0$ , where  $\rho_k = 2^{-k} \rho$  and  $h_k = \zeta_k \rho_k$ , while the sequence  $\zeta_k \downarrow 0$  will be chosen later.

We set  $w_k = v - M_k x_n$ , where the quantities  $M_k$ , with  $k \geq 1$ , are defined as

$$M_k = \sup_{\mathcal{P}_{\rho_k, h_{k-1}} \cap \Omega} \frac{v(x)}{\max\{x_n, h_k\}} \geq \sup_{\{\mathcal{P}_{\rho_k, h_{k-1}} \setminus \mathcal{P}_{\rho_k, h_k}\} \cap \Omega} \frac{v(x)}{x_n}.$$

It is easy to see that  $w_k \leq 0$  on  $\partial\Omega \cap \bar{\mathcal{P}}_{\rho_k, h_k}$ , while the definition of  $M_k$  gives  $w_k \leq 0$  on the top of the cylinder  $\mathcal{P}_{\rho_k, h_k}$ .

Let  $x^0 \in \mathcal{P}_{\rho_k - h_k, h_k} \cap \Omega$ . Taking into account Remark 2.4, we apply the so-called “boundary growth lemma” (see, for instance, [Ladyzhenskaya and Ural'tseva 1985, Lemma 2.5'], [Safonov 2010, Lemma 2.6] or [Nazarov 2012, Lemma 2.2]) to the (positive) function  $M_k h_k - w_k$  in  $\mathcal{P}_{h_k}(x^0) \cap \Omega$ . It gives for  $x \in \mathcal{P}_{h_k/2, h_k}(x^0) \cap \Omega$ ,

$$M_k h_k - w_k(x) \geq M_k h_k [\vartheta - N_2 \|b\|_{\mathcal{X}, \mathcal{P}_{\rho_k} \cap \Omega}] - N_3 h_k \|(\mathcal{L}w_k)_+\|_{\mathcal{X}, \mathcal{P}_{h_k}(x^0) \cap \Omega}, \tag{20}$$

where  $\vartheta = \vartheta(n, \nu, \sigma, \mathfrak{B}) \in (0, 1)$  and the positive constant  $N_2$  depends on the same parameters as  $\vartheta$ , whereas the positive constant  $N_3$  is completely defined by the values of  $n, \nu$  and  $\mathfrak{B}$ . We suppose that  $\bar{\rho}$  is so small that the quantity in the square brackets is greater than  $\vartheta/2$ . Further, direct calculation shows that the assumptions of our lemma imply

$$\begin{aligned} \|(\mathcal{L}w_k)_+\|_{\mathcal{X}, \mathcal{P}_{h_k}(x^0) \cap \Omega} &\leq \|(\mathcal{L}v)_+\|_{\mathcal{X}, \mathcal{P}_{h_k}(x^0) \cap \Omega} + M_k \|b^n\|_{\mathcal{X}, \mathcal{P}_{h_k}(x^0) \cap \Omega} \\ &\leq (\mathfrak{F} + M_k \mathfrak{B}) \sigma(\rho_k / \rho_*). \end{aligned}$$

Substituting the last inequality into (20) and taking the supremum with respect to  $x^0$ , we obtain

$$\sup_{\mathcal{P}_{\rho_k - h_k, h_k} \cap \Omega} w_k \leq M_k h_k (1 - \vartheta/2 + N_2 \mathfrak{B} \sigma(\rho_k / \rho_*)) + N_3 h_k \mathfrak{F} \sigma(\rho_k / \rho_*).$$

Repeating previous arguments provides for all integers  $m \leq \rho_k / h_k$  the inequalities

$$\sup_{\mathcal{P}_{\rho_k - mh_k, h_k} \cap \Omega} w_k \leq M_k h_k \left( (1 - \vartheta/2)^m + N_2 \mathfrak{B} \frac{\sigma(\rho_k / \rho_*)}{\vartheta/2} \right) + N_3 h_k \mathfrak{F} \frac{\sigma(\rho_k / \rho_*)}{\vartheta/2}.$$

Setting  $m = \lfloor \rho_{k+1} / h_k \rfloor$ , we arrive at

$$\sup_{\mathcal{P}_{\rho_{k+1}, h_k} \cap \Omega} w_k \leq \frac{M_k h_k}{1 - \vartheta/2} \left( \exp\left(-\lambda \frac{\rho_{k+1}}{h_k}\right) + N_2 \mathfrak{B} \frac{\sigma(\rho_k / \rho_*)}{\vartheta/2} \right) + N_3 h_k \mathfrak{F} \frac{\sigma(\rho_k / \rho_*)}{(1 - \vartheta/2)\vartheta/2},$$

where  $\lambda = -\ln(1 - \vartheta/2) > 0$ .

Therefore, for  $x \in \mathcal{P}_{\rho_{k+1}, h_k} \cap \Omega$ ,

$$\frac{w_k(x)}{\max\{x_n, h_{k+1}\}} \leq M_k \gamma_k + N_3 \mathfrak{F} \frac{\sigma(\rho_k / \rho_*)}{(1 - \vartheta/2)\vartheta/2} \cdot \frac{2\zeta_k}{\zeta_{k+1}}, \tag{21}$$

where

$$\gamma_k = \frac{1}{1 - \vartheta/2} \frac{2\zeta_k}{\zeta_{k+1}} \cdot \left( \exp\left(-\frac{\lambda}{2\zeta_k}\right) + N_2 \mathfrak{B} \frac{\sigma(\rho_k/\rho_*)}{\vartheta/2} \right).$$

Estimate (21) implies

$$\begin{aligned} M_{k+1} &\leq M_k(1 + \gamma_k) + N_3 \mathfrak{F} \frac{\sigma(\rho_k/\rho_*)}{(1 - \vartheta/2)\vartheta/2} \cdot \frac{2\zeta_k}{\zeta_{k+1}} \\ &\leq M_1 \cdot \prod_{j=1}^k (1 + \gamma_j) + 2N_3 \mathfrak{F} \cdot \sum_{j=1}^k \sigma(\rho_j/\rho_*) \frac{\zeta_j}{\zeta_{j+1}} \cdot \prod_{l=j}^{k-1} (1 + \gamma_l). \end{aligned}$$

We set  $\zeta_k = 1/(k + k_0)$  and choose  $k_0$  so large and  $\bar{\rho}/\rho_*$  so small that  $\gamma_1 \leq \frac{1}{2}$ . Note that  $k_0 = k_0(n, \nu, \sigma, \mathfrak{B})$ , whereas  $\bar{\rho}/\rho_*$  depends on the same parameters as  $k_0$  and, in addition, on the moduli of continuity of  $|\mathbf{b}'|$  in  $\mathcal{X}(\mathcal{P}_{\rho_*} \cap \Omega)$ .

Now we observe that the first term in  $\gamma_k$  forms a convergent series. The same is true for the second term, since

$$\sum_{k=1}^{\infty} \sigma(2^{-k} \rho/\rho_*) \asymp \int_0^{\infty} \sigma(2^{-s} \rho/\rho_*) ds \asymp \mathcal{J}_{\sigma}(\rho/\rho_*).$$

Therefore, the infinite product  $\Pi = \prod_k (1 + \gamma_k)$  also converges, and we obtain for  $k > 1$ , the inequality

$$\begin{aligned} M_k &\leq \Pi \cdot \left( M_1 + 2N_3 \mathfrak{F} \cdot \sum_{j=1}^k \sigma(\rho_j/\rho_*) \frac{\zeta_j}{\zeta_{j+1}} \right) \\ &\leq \Pi \cdot (M_1 + N_4(n, \nu, \sigma, \mathfrak{B}) \mathfrak{F} \mathcal{J}_{\sigma}(\rho/\rho_*)). \end{aligned} \quad (22)$$

Thus, all  $M_k$  are bounded. It remains only to note that

$$M_1 \leq \frac{1}{h_1} \sup_{\mathcal{P}_{\rho/2} \cap \Omega} v. \quad (23)$$

Combining (22) and (23), we arrive at

$$\sup_{0 < x_n < \rho/2} \frac{v(0, x_n)}{x_n} \leq N_5(n, \nu, \sigma, \mathfrak{B}) (\rho^{-1} \sup_{\mathcal{P}_{\rho/2} \cap \Omega} v + \mathfrak{F} \mathcal{J}_{\sigma}(\rho/\rho_*)). \quad (24)$$

Further, it is easy to find a majorant for  $v(0, x_n)/x_n$  for any  $x_n \in [\rho/2, \rho)$  since

$$\sup_{\rho/2 \leq x_n < \rho} \frac{v(0, x_n)}{x_n} \leq 2\rho^{-1} \sup_{\rho/2 \leq x_n < \rho} v(0, x_n) \leq 2\rho^{-1} \sup_{\mathcal{P}_{\rho} \cap \Omega} v. \quad (25)$$

Combining (24) and (25) implies (19) with  $C_4 = \max\{N_5, 2\}$  for  $\rho \leq \bar{\rho}$ .

Now, we consider  $\rho > \bar{\rho}$ . If  $x_n < \bar{\rho}$  then the estimate

$$\frac{v(0, x_n)}{x_n} \leq 2N_5 (\bar{\rho}^{-1} \sup_{\mathcal{P}_{\rho} \cap \Omega} v + \mathfrak{F} \mathcal{J}_{\sigma}(\rho/\rho_*)) \quad (26)$$

follows from the above arguments. Otherwise, i.e., for  $x_n \geq \bar{\rho}$ , inequality (26) is especially true. Thus, for  $\rho > \bar{\rho}$ , we again arrive at (19) with  $C_4 = \max\{N_5, 2\} \bar{\rho}^{-1}$ .  $\square$



### 4. Main results

Recall that  $\Omega$  satisfies the assumptions on page 442. Throughout this section, we shall suppose that  $\mathcal{L}$  is defined by (1),  $|\mathbf{b}| \in \mathcal{X}(\Omega)$ , and a function  $u$  satisfies the assumptions

$$u \in \mathcal{W}_{\mathcal{X},\text{loc}}^2(\Omega) \cap \mathcal{C}(\bar{\Omega}), \quad \mathcal{L}u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega \cap \bar{\mathcal{P}}_{\mathcal{R}_0}} = 0. \tag{27}$$

**Theorem 4.1.** *Let the inequality*

$$\sup_{x \in \mathcal{P}_{\mathcal{R}_0/2}} \|b^n\|_{\mathcal{X}, \mathcal{P}_\rho(x') \cap \Omega} \leq \mathfrak{B}\sigma(\rho/\mathcal{R}_0)$$

hold true for all  $\rho \leq \frac{1}{2}\mathcal{R}_0$ . Here  $\mathfrak{B}$  is a positive constant, and a function  $\sigma \in \mathcal{D}_1$  satisfies

$$\mathcal{J}_\sigma(t) = o(\delta(t)) \quad \text{as } t \rightarrow 0. \tag{28}$$

Then, there exists a sufficiently small positive number  $R_0$  completely defined by  $n, \nu, \mathcal{R}_0, \mathfrak{B}$ , by the functions  $\sigma, \delta$ , and by the moduli of continuity of  $|\mathbf{b}'|$  in  $\mathcal{X}(\Omega)$  such that for any  $r \in (0, \frac{1}{2}R_0)$ , we have

$$\text{osc}_{\Omega \cap \mathcal{P}_{r/4}} \frac{u(x)}{x_n} \leq (1 - \varkappa\delta(r)) \text{osc}_{\Omega \cap \mathcal{P}_{2r}} \frac{u(x)}{x_n}. \tag{29}$$

Here the constant  $\varkappa \in (0, 1)$  is completely determined by  $n, \nu$ .

*Proof.* The proof will be divided into 3 steps.

**Step 1:** Our arguments are adapted from [Apushkinskaya and Ural'tseva 1995, Lemma 2; Ural'tseva 1996, Lemma 3]. Let us denote

$$m^\pm = \sup_{\Omega \cap \mathcal{P}_{2r}} \pm \frac{u(x)}{x_n}, \quad \omega = m^+ + m^- = \text{osc}_{\Omega \cap \mathcal{P}_{2r}} \frac{u(x)}{x_n}.$$

Since  $u|_{\partial\Omega} = 0$ , we have  $m^\pm \geq 0$ . Therefore, at least one of the numbers  $m^\pm$  is not less than  $\frac{1}{2}\omega$ , and both of the numbers  $m^\pm$  are less than  $\omega$ .

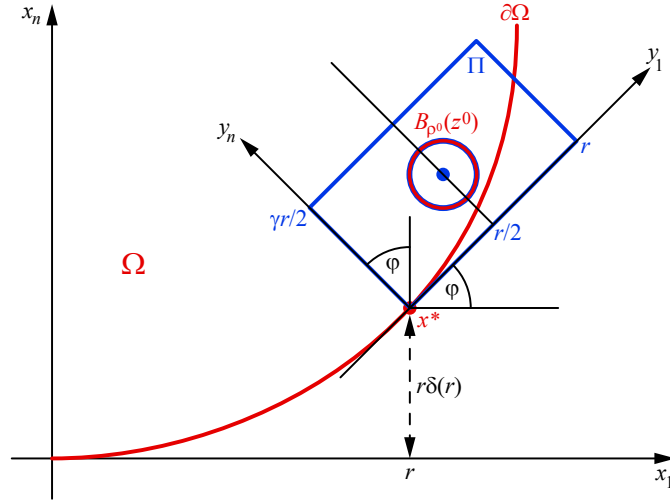
Let  $m^+ \geq \frac{1}{2}\omega$  for definiteness. Then we consider the nonnegative function  $v(x) = m^+x_n - u(x)$  in  $\Omega \cap \mathcal{P}_{2r}$ ; if  $m^- > \frac{1}{2}\omega$  then we consider the function  $v(x) = m^-x_n + u(x)$ .

Due to the definition of  $\delta$ , for any sufficiently small  $r > 0$ , we can find a point  $x^* \in \partial\mathcal{P}_r \cap \partial\Omega$  such that  $x_n^* = r\delta(r)$ . Without loss of generality, we may assume that  $x_1^* = r$  and  $x_\tau^* = 0$  for  $\tau = 2, \dots, n-1$ .

Next we assign to  $x^*$  a local orthogonal coordinate system  $y_1, \dots, y_n$  such that

- (a) the  $y_1$ -axis is directed along the projection of the vector  $(x_1^*, \dots, x_{n-1}^*)$  onto a tangential hyperplane to  $\partial\Omega$  at  $x^*$ ;
- (b) the  $y_2, \dots, y_{n-1}$ -axes are parallel to the  $x_2, \dots, x_{n-1}$ -axes, respectively;
- (c) the  $y_n$ -axis is directed inside  $\Omega$ .

Due to the extremal property of  $x^*$ , the axes  $y_1, \dots, y_{n-1}$  lie in the supporting hyperplane to  $\partial\Omega$  at  $x^*$ . Moreover, if  $x^*$  is a smooth point of  $\partial\Omega$  then  $y_n$  is directed along the inward normal to  $\partial\Omega$ .



**Figure 1.** Schematic view of  $\Pi$  and  $B_{\rho_0}(z^0)$ .

Setting  $\gamma = v/\sqrt{n-1}$ , we consider in  $y$ -coordinates the cylinder

$$\Pi := \{y \in \mathbb{R}^n : |y_1 - \frac{1}{2}r| < \frac{1}{2}r, |y_\tau| < \frac{1}{2}r, 0 < y_n < \frac{1}{2}\gamma r\},$$

and the ball  $B_{\rho_0}(z^0)$  with  $\rho_0 = \frac{1}{8}\gamma r$  and  $z^0 = (\frac{1}{2}r, 0, \dots, 0, \frac{1}{4}\gamma r)$ .

It should be emphasized that from now on, all considerations will be carried out in  $x$ -coordinates.

We claim that

$$B_{\rho_0}(z^0) \subset \Omega. \tag{30}$$

Indeed, assume that (30) fails. Then there is a point  $\hat{x} \in B_{\rho_0}(z^0)$  satisfying (in  $x$ -coordinates) the inequalities

$$F(\hat{x}') \geq \hat{x}_n \geq z_n^0 - \rho_0. \tag{31}$$

Since  $\hat{x} \in B_{\rho_0}(z^0)$ , it is clear that  $|\hat{x}'| \leq 2r$  and

$$F(\hat{x}') \leq 2r\delta(2r).$$

On the other hand, denoting by  $\varphi$  the angle between the  $x_n$ - and  $y_n$ -axes (see Figure 1), we conclude that

$$z_n^0 - \rho_0 = r\delta(r) + \frac{1}{2}r \sin \varphi + \frac{1}{4}\gamma r \cos \varphi - \frac{1}{8}\gamma r \geq \frac{1}{8}\gamma r(2 \cos \varphi - 1).$$

Thus (31) is transformed into

$$\gamma(2 \cos \varphi - 1) \leq 16\delta(2r). \tag{32}$$

In view of (10) and Lemma 2.1, one can choose  $R_0$  so small that  $\delta_1(R_0) \leq \frac{3}{4}$ . It guarantees for all  $r \leq \frac{1}{2}R_0$ , the inequalities

$$\cos \varphi = \frac{1}{\sqrt{1 + \tan^2 \varphi}} \geq \frac{1}{\sqrt{1 + \delta_1^2(r)}} \geq \frac{1}{\sqrt{1 + \delta_1^2(R_0)}} \geq \frac{4}{5}. \tag{33}$$

Now, combining (33) and (32), we get a contradiction with relation (10) provided  $\delta(R_0)$  is small enough. The proof of (30) is complete.

**Step 2:** With (30) at hand, we observe that

$$\inf\{x_n : x \in \Omega \cap \Pi\} \geq r\delta(r).$$

On the other hand, the condition  $u = 0$  for  $x \in \partial\Omega \cap \Pi$  gives the estimate

$$v = m^+ x_n \geq \frac{1}{2}\omega x_n \quad \text{on } \partial\Omega \cap \Pi.$$

Hence,

$$v \geq \frac{1}{2}\omega r\delta(r) =: k_0 \quad \text{on } \partial\Omega \cap \Pi. \tag{34}$$

So, we can apply Lemma 3.1 to the function  $v$  in cylinder  $\Pi$ . This gives the estimate

$$\inf_{B_{\rho_0}(z^0)} v \geq (k_0(C_1 - C_2\|\mathbf{b}\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}}) - C_3\omega r\|b^n\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}})_+,$$

where  $C_1, C_2$  and  $C_3$  are the constants from Lemma 3.1. Decreasing  $R_0$ , if necessary, we may assume that  $\|\mathbf{b}\|_{\mathcal{X},\Omega \cap \mathcal{P}_{R_0}} \leq C_1/(2C_2)$ . Thus, we arrive at

$$\inf_{B_{\rho_0}(z^0)} v \geq (k_0\frac{1}{2}C_1 - C_3\omega r\|b^n\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}})_+ =: k_1. \tag{35}$$

Consider now an arbitrary point  $\tilde{z} = (\tilde{z}', \frac{1}{4}r + \frac{1}{8}\rho_0)$  such that  $|\tilde{z}'| \leq \frac{1}{4}r$ . Observe also that  $B_{\rho_0}(\tilde{z}) \subset \Omega$ , otherwise we get a contradiction with the definition of  $\delta(r)$ .

We claim that

$$\inf_{B_{\rho_0/8}(\tilde{z})} v \geq (k_0\tilde{C}_1 - \tilde{C}_2\omega r\|b^n\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}})_+, \tag{36}$$

where  $\tilde{C}_1 = \tilde{C}_1(n, \nu)$ , whereas  $\tilde{C}_2$  is determined completely by  $n, \nu$ , and  $\|\mathbf{b}\|_{\mathcal{X},\Omega}$ . Indeed, due to the convexity of  $\Omega$ , for  $l$  running from 1 to a finite number  $\mathfrak{N} = \mathfrak{N}(n, \nu)$  chosen so that

$$\frac{4}{3\rho_0}|z^0 - \tilde{z}| \leq \mathfrak{N} \leq \frac{2}{\rho_0}|z^0 - \tilde{z}|, \tag{37}$$

and for points  $z^{[l]} := z^0 - (l/\mathfrak{N})(z^0 - \tilde{z})$ , we have  $B_{\rho_0}(z^{[l]}) \subset \Omega$ . It should be emphasized that the lower and the upper bounds in (37) do not depend on  $r$ .

In view of (35), we can compare in  $\mathcal{B}(z^{[1]}, \frac{1}{8}\rho_0, \rho_0)$  the function  $v$  with the standard barrier function

$$w(x) = k_1 \frac{|x - z^{[1]}|^{-s} - \rho_0^{-s}}{(\frac{1}{8}\rho_0)^{-s} - \rho_0^{-s}}.$$

If  $s = n\nu^{-2}$  then elementary calculation guarantees the estimates

$$\begin{aligned} \mathcal{L}w &\leq |\mathbf{b}||Dw| \leq c(n, \nu)k_1|\mathbf{b}|\rho_0^{-1} && \text{in } \mathcal{B}(z^{[1]}, \frac{1}{8}\rho_0, \rho_0), \\ w(x) &= k_1 \leq v(x) && \text{on the sphere } |x - z^{[1]}| = \frac{1}{8}\rho_0, \\ w(x) &= 0 \leq v(x) && \text{on the sphere } |x - z^{[1]}| = \rho_0. \end{aligned}$$

Applying the maximum principle (11) in  $\mathcal{B}(z^{[1]}, \frac{1}{8}\rho_0, \rho_0)$  to the difference  $w - v$  gives us the inequality

$$v(x) \geq (k_1(w(x) - 2cN_0\|\mathbf{b}\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}}) - N_0\frac{1}{4}\gamma r\omega\|b^n\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}})_+.$$

Since  $B_{\rho_0/8}(z^{[2]}) \subset \mathcal{B}(z^{[1]}, \frac{1}{8}\rho_0, \frac{7}{8}\rho_0)$ , the evident bound  $w \geq \theta(n, v)$  holds true in  $B_{\rho_0/8}(z^{[2]})$ .

Decreasing  $R_0$ , if necessary, we ensure that  $\|b\|_{\mathcal{X}, \Omega \cap \mathcal{P}_{R_0}} \leq (4cN_0)^{-1}\theta$ . This implies

$$\inf_{B_{\rho_0/8}(z^{[2]})} v(x) \geq \left(\frac{1}{2}k_1\theta - N_0\frac{1}{4}\gamma r\omega\|b^n\|_{\mathcal{X}, \Omega \cap \mathcal{P}_{2r}}\right)_+ =: k_2.$$

Repeating this procedure for  $\mathcal{B}(z^{[l]}, \frac{1}{8}\rho_0, \rho_0)$  and  $l = 2, \dots, \mathfrak{N}$ , we arrive at (36) with  $\tilde{C}_1 = (\frac{1}{2}\theta)^{\mathfrak{N}}$  and

$$\tilde{C}_2 = N_0\frac{1}{4}\gamma \cdot \frac{1 - (\frac{1}{2}\theta)^{\mathfrak{N}}}{1 - \frac{1}{2}\theta}.$$

Furthermore, it is clear that

$$(k_0\tilde{C}_1 - \tilde{C}_2r\omega\|b^n\|_{\mathcal{X}, \Omega \cap \mathcal{P}_{2r}})_+ \geq \omega r \left(\frac{1}{2}\tilde{C}_1\delta(r) - \tilde{C}_2\mathfrak{B}\sigma(r/\mathcal{R}_0)\right)_+,$$

while inequalities (3) and (4) guarantee that

$$\sigma(r/\mathcal{R}_0) \leq \frac{\mathcal{J}_\sigma(r)}{\mathcal{R}_0}.$$

Decreasing  $R_0$  again and taking into account the assumption (28) and the above inequalities, we can transform (36) into the form

$$\inf_{B_{\rho_0/8}(\tilde{z})} v \geq \frac{1}{4}\tilde{C}_1\omega r\delta(r) =: \tilde{k}. \quad (38)$$

**Step 3:** Now, we take a small  $\eta > 0$ , define the set

$$\mathcal{A}_\eta := \mathcal{B}(\tilde{z}, \frac{1}{8}\rho_0, \tilde{z}_n) \cap \Omega \cap \{x \in \mathcal{P}_{\mathcal{R}_0} : F(x') + \eta < x_n < \mathcal{R}_0\}$$

and introduce in  $\mathcal{A}_\eta$  the barrier function

$$W(x) = \mu\tilde{k} \frac{|x - \tilde{z}|^{-s} - (\tilde{z}_n)^{-s}}{(\frac{1}{8}\rho_0)^{-s} - (\tilde{z}_n)^{-s}},$$

where  $s = nv^{-2}$  and  $0 < \mu \leq 1$ .

Notice that  $D(Du) \in \mathcal{X}(\mathcal{A}_\eta)$ . Using Lemma 2.8, we construct the family of operators  $\mathcal{L}_\varepsilon$  satisfying  $\|\mathcal{L}_\varepsilon u\|_{\mathcal{X}, \mathcal{A}_\eta} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Arguing in the spirit of the proof of Lemma 4.2 [Ladyzhenskaya and Ural'tseva 1988], we define  $v_1(x)$  and  $v_2(x)$  as solutions of the problems

$$\begin{cases} \mathcal{L}_\varepsilon v_1 = b_\varepsilon^i D_i W & \text{in } \mathcal{A}_\eta, \\ v_1 = v & \text{on } \partial\mathcal{A}_\eta, \end{cases} \quad \begin{cases} \mathcal{L}_\varepsilon v_2 = b_\varepsilon^i D_i W - b_\varepsilon^n m^+ & \text{in } \mathcal{A}_\eta, \\ v_2 = 0 & \text{on } \partial\mathcal{A}_\eta. \end{cases}$$

It is well known (see, for instance, [Krylov 2008, Chapter 6]) that  $D(Dv_1)$  and  $D(Dv_2)$  belong to the space  $\text{BMO}_{\text{loc}}(\mathcal{A}_\eta)$ . Moreover, the John–Nirenberg theorem [1961] (see also [Duoandikoetxea 2001, §4, Chapter 6]) implies that  $D(Dv_i)$ , with  $i = 1, 2$ , belong to the Orlicz space  $L_{\Phi, \text{loc}}(\mathcal{A}_\eta)$  with  $\Phi(\xi) = e^\xi - \xi - 1$ . So, taking into account the property (iii), we may conclude that  $v_i \in \mathcal{W}_{\mathcal{X}, \text{loc}}^2(\mathcal{A}_\eta)$ , with  $i = 1, 2$ .



Furthermore, in view of (38) and by direct calculation, we have the inequalities

$$\begin{aligned} \mathcal{L}_\varepsilon W &\leq b_\varepsilon^i D_i W && \text{in } \mathcal{A}_\eta, \\ W(x) = \mu \tilde{k} &\leq v(x) = v_1(x) && \text{on the sphere } |x - \tilde{z}| = \frac{1}{8} \rho_0, \\ W(x) = 0 &\leq v(x) = v_1(x) && \text{on } \partial \mathcal{A}_\eta \cap \{x \in \mathbb{R}^n : |x - \tilde{z}| = \tilde{z}_n\}. \end{aligned}$$

On the rest of  $\partial \mathcal{A}_\eta$ , we have  $x_n = F(x') + \eta$  and, consequently,  $\text{dist}\{x, \partial \Omega\} \leq \eta$ . Since  $u \in C(\bar{\Omega})$ , the latter inequality implies the estimate  $u \leq H(\eta)$  there, and therefore,

$$v_1(x) = v(x) = m^+ x_n - u \geq \frac{1}{2} \omega x_n - H(\eta),$$

where  $H$  is a nonnegative function tending to zero as  $\eta \rightarrow 0$ .

In addition, it is easy to verify that

$$W(x) \leq \mu N_6(n, \nu) \tilde{C}_1 \omega \delta(r) x_n \quad \text{in } \bar{\mathcal{B}}(\tilde{z}, \frac{1}{8} \rho_0, \tilde{z}_n).$$

Choosing  $\mu = \min\{1, (2N_6 \tilde{C}_1)^{-1}\}$ , we get

$$v_1(x) \geq W(x) - H(\eta) \quad \text{on } \partial \mathcal{A}_\eta.$$

The maximum principle (11) applied to the difference  $W - H(\eta) - v_1$  in  $\mathcal{A}_\eta$  provides the inequality

$$v_1(x) \geq W(x) - H(\eta) \geq \mu N_7(n, \nu) \tilde{C}_1 \omega \delta(r) (\tilde{z}_n - |x - \tilde{z}|) - H(\eta).$$

It follows from the last inequality with  $x = (\tilde{z}', x_n) \in \Omega$  and  $0 < x_n \leq \tilde{z}_n - \frac{1}{8} \rho_0 = \frac{1}{4} r$  that

$$v_1(\tilde{z}', x_n) \geq N_8(n, \nu) \omega \delta(r) x_n - H(\eta). \tag{39}$$

Next, we look for a majorant for  $v_2$ . With this aim in view, we extend the coefficients  $a_\varepsilon^{ij}$  continuously and the coefficients  $b_\varepsilon^i$  by zero to the whole annulus  $\mathcal{B}(\tilde{z}, \frac{1}{8} \rho_0, \tilde{z}_n)$ , and denote by  $\tilde{v}_2(x)$  the solution of the problem

$$\begin{aligned} \mathcal{L}_\varepsilon \tilde{v}_2 &= \begin{cases} (\mathcal{L}_\varepsilon v_2)_+ & \text{in } \mathcal{A}_\eta, \\ 0 & \text{in } \mathcal{B}(\tilde{z}, \frac{1}{8} \rho_0, \tilde{z}_n) \setminus \mathcal{A}_\eta, \end{cases} \\ \tilde{v}_2 &= 0 \quad \text{on } \partial \mathcal{B}(\tilde{z}, \frac{1}{8} \rho_0, \tilde{z}_n). \end{aligned}$$

The maximum principle guarantees

$$v_2 \leq \tilde{v}_2 \quad \text{in } \mathcal{A}_\eta. \tag{40}$$

Direct computations show that for  $\rho \leq \frac{1}{4} r$  the barrier function  $W$  satisfies in the set

$$\mathcal{E}_\rho := \mathcal{P}_\rho(\tilde{z}', 0) \cap \mathcal{B}(\tilde{z}, \frac{1}{8} \rho_0, \tilde{z}_n)$$

the following inequalities

$$\begin{aligned} |D_n W| &\leq |DW| \leq N_9(n, \nu) \mu \frac{\tilde{k}}{r} \leq N_9 \omega \delta(r), \\ |D' W| &\leq N_9 \mu \frac{\tilde{k} \rho}{r^2} \leq N_9 \omega \frac{\delta(r) \rho}{r}. \end{aligned}$$

So, in view of (15) and (10), we have for all  $\rho \leq \frac{1}{4}r$ , the bounds

$$\begin{aligned} \|(\mathcal{L}_\varepsilon \tilde{v}_2)_+\|_{\mathcal{X}, \varepsilon_\rho} &\leq \|b^n\|_{\mathcal{X}, \varepsilon_\rho} (m^+ + \|D_n W\|_{\infty, \varepsilon_\rho}) + \|b'\|_{\mathcal{X}, \varepsilon_\rho} \|D'W\|_{\infty, \varepsilon_\rho} \\ &\leq N_{10}(n, \nu) \omega \left( \mathfrak{B}\sigma \left( \frac{\rho}{\mathcal{R}_0} \right) + \frac{\delta(r)}{r} \rho \|b'\|_{\mathcal{X}, \mathcal{A}_\eta} \right). \end{aligned}$$

Since the function  $\rho \mapsto (\mathfrak{B}\sigma(\rho/\mathcal{R}_0) + (\delta(r)/r)\rho\|b'\|_{\mathcal{X}, \mathcal{A}_\eta})$  satisfies the Dini condition at zero, there exist the uniquely defined function  $\sigma_1 \in \mathcal{D}_1$  and a constant  $\mathfrak{B}_1$  such that

$$\mathfrak{B}\sigma \left( \frac{\rho}{\mathcal{R}_0} \right) + \frac{\delta(r)}{r} \rho \|b'\|_{\mathcal{X}, \mathcal{A}_\eta} = \mathfrak{B}_1 \sigma_1 \left( \frac{4\rho}{r} \right).$$

Thus, we may apply Lemma 3.2 to the function  $\tilde{v}_2$ . It gives for  $\rho = \frac{1}{4}r$  the estimate

$$\sup_{0 < x_n < r/4} \frac{\tilde{v}_2(\tilde{z}', x_n)}{x_n} \leq C_4 \left( \left( \frac{1}{4}r \right)^{-1} \sup_{\varepsilon_{r/4}} \tilde{v}_2 + N_{10} \omega \mathfrak{B}_1 \mathcal{J}_{\sigma_1}(1) \right). \quad (41)$$

It is easy to see that

$$\mathfrak{B}_1 \mathcal{J}_{\sigma_1}(1) = \mathfrak{B} \mathcal{J}_\sigma \left( \frac{r}{4\mathcal{R}_0} \right) + \frac{1}{4} \delta(r) \|b'\|_{\mathcal{X}, \mathcal{A}_\eta}.$$

Furthermore, applying (11) to  $\tilde{v}_2$  and to the operator  $\mathcal{L}_\varepsilon$  in  $\mathcal{B}(\tilde{z}, \frac{1}{8}\rho_0, \tilde{z}_n)$ , we obtain

$$\sup_{\varepsilon_{r/4}} \tilde{v}_2 \leq \sup_{\mathcal{B}(\tilde{z}, \rho_0/8, \tilde{z}_n)} \tilde{v}_2 \leq N_{11}(n, \nu, \|b\|_{\mathcal{X}, \Omega}) \omega r \left( \mathfrak{B}\sigma \left( \frac{r}{\mathcal{R}_0} \right) + \delta(r) \|b'\|_{\mathcal{X}, \mathcal{A}_\eta} \right).$$

Substitution of the above estimates in (41) and consideration of (3) provide

$$\sup_{0 < x_n < r/4} \frac{\tilde{v}_2(\tilde{z}', x_n)}{x_n} \leq N_{12} \omega \left( \mathfrak{B} \mathcal{J}_\sigma \left( \frac{r}{\mathcal{R}_0} \right) + \delta(r) \|b'\|_{\mathcal{X}, \mathcal{A}_\eta} \right), \quad (42)$$

where the constant  $N_{12}$  depends only on  $n, \nu$  and  $\|b\|_{\mathcal{X}, \Omega}$ .

Taking into account the inequality (5), the assumption (28), and the evident relation  $\|b'\|_{\mathcal{X}, \mathcal{A}} = o(1)$  as  $r \rightarrow 0$ , we decrease  $R_0$  such that the property

$$\mathfrak{B} \mathcal{J}_\sigma \left( \frac{r}{\mathcal{R}_0} \right) + \delta(r) \|b'\|_{\mathcal{X}, \mathcal{A}_\eta} \leq \frac{N_8}{2N_{12}} \delta(r) \quad (43)$$

holds true for all  $r \leq R_0$ .

Finally, combining (39)–(40) with (42)–(43), we arrive at the estimate

$$v_1(\tilde{z}', x_n) - v_2(\tilde{z}', x_n) \geq \frac{1}{2} N_8 \omega \delta(r) x_n - H(\eta) \quad (44)$$

for  $r \leq R_0$  and  $x = (\tilde{z}', x_n) \in \Omega$  with  $x_n \in [F(\tilde{z}') + \eta, \frac{1}{4}r]$ .

Considering in  $\mathcal{A}_\eta$  the function  $v_3(x) = v(x) - v_1(x) + v_2(x)$ , one can easily see that

$$\mathcal{L}_\varepsilon v_3 = -\mathcal{L}_\varepsilon u \rightarrow 0 \quad \text{in } \mathcal{X}(\mathcal{A}_\eta) \text{ as } \varepsilon \rightarrow 0.$$

In addition,  $v_3 = 0$  on  $\partial\mathcal{A}_\eta$ . Applying the maximum principle (11) to  $\pm v_3$  and to the operator  $\mathcal{L}_\varepsilon$ , we obtain that the difference  $v_1(x) - v_2(x)$  converges to  $v(x)$  uniformly in  $\mathcal{A}_\eta$ . Therefore, passing in (44) first to the limit as  $\varepsilon \rightarrow 0$  and then as  $\eta \rightarrow 0$ , we get

$$\frac{v(x)}{x_n} \geq \frac{1}{2} N_8 \omega \delta(r) \tag{45}$$

for  $r \leq R_0$  and  $x = (\tilde{z}', x_n) \in \Omega$  with  $x_n \in [F(\tilde{z}'), \frac{1}{4}r]$ .

Since  $\tilde{z}'$  can be chosen arbitrarily with only  $|\tilde{z}'| \leq \frac{1}{4}r$ , the estimate (45) gives (29) with  $\varkappa = \frac{1}{2}N_8$ .  $\square$

**Theorem 4.2** (main theorem). *Let the assumptions of Theorem 4.1 hold, and suppose*

$$\delta(r) = \max_{|x'| \leq r} \frac{F(x')}{|x'|}$$

*does not satisfy the Dini condition at zero.*

*Then for any function  $u$  satisfying (27), the equality*

$$\frac{\partial u}{\partial \mathbf{n}}(0) = 0$$

*holds true.*

*Proof.* Consider the sequence  $r_k = 8^{-k} R_0$ , with  $k \geq 0$ , where  $R_0$  is the constant from Theorem 4.1.

Applying Theorem 4.1 to  $u$  guarantees for  $k \geq 0$  the inequalities

$$\operatorname{osc}_{\Omega \cap \mathcal{P}_{r_{k+1}}} \frac{u(x)}{x_n} \leq (1 - \varkappa \delta(\frac{1}{2}r_k)) \operatorname{osc}_{\Omega \cap \mathcal{P}_{r_k}} \frac{u(x)}{x_n} \leq \operatorname{osc}_{\Omega \cap \mathcal{P}_{R_0}} \frac{u(x)}{x_n} \cdot \prod_{j=0}^k (1 - \varkappa \delta(\frac{1}{2}r_j)).$$

Since

$$\sum_{j=0}^{\infty} \ln(1 - \varkappa \delta(\frac{1}{2}r_j)) \asymp - \sum_{j=0}^{\infty} \delta(\frac{1}{2}r_j) \asymp - \int_0^{r_0} \frac{\delta(r)}{r} dr = -\infty,$$

we have

$$\prod_{j=0}^k (1 - \varkappa \delta(\frac{1}{2}r_j)) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We recall also that Lemma 3.2 implies the finiteness of the quantity  $\operatorname{osc}_{\Omega \cap \mathcal{P}_{R_0}} (u(x)/x_n)$ .

Thus, taking into account that  $u|_{\partial\Omega \cap \mathcal{P}_{R_0}} = 0$ , we get

$$\left| \frac{\partial u}{\partial \mathbf{n}}(0) \right| = \left| \lim_{x_n \rightarrow 0} \frac{u(0, x_n)}{x_n} \right| \leq \lim_{k \rightarrow \infty} \left| \operatorname{osc}_{\Omega \cap \mathcal{P}_{r_k}} \frac{u(x)}{x_n} \right| = 0. \quad \square$$

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