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SLIM IBRAHIM, NADER MASMOUDI AND KENJI NAKANISHI

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# CORRECTION TO THE ARTICLE SCATTERING THRESHOLD FOR THE FOCUSING NONLINEAR KLEIN–GORDON EQUATION

SLIM IBRAHIM, NADER MASMOUDI AND KENJI NAKANISHI

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This article resolves some errors in the paper "Scattering threshold for the focusing nonlinear Klein–Gordon equation", *Anal. PDE* **4**:3 (2011), 405–460. The errors are in the energy-critical cases in two and higher dimensions.

## 1. The errors and the missing ingredient

This article resolves some errors in [Ibrahim et al. 2011]. One correction affects also [Ibrahim et al. 2014; 2015]; henceforth, we refer to these papers by their years only. The major errors are the following three, one in [2011, Section 2] for the existence of mass-shifted ground state in the two-dimensional energy-critical case, and two in [2011, Section 5] for the nonlinear profile decomposition in the higher-dimensional energy-critical case:

(1) In the proof of [2011, Lemma 2.6], it is not precluded that the weak limit $Q$ in [2011, (2-67)] is zero. Hence the existence of $Q$ in the case $c \le 1$ is not proved.

(2) In [2011, (5-56)], we do not have $\|\vec{V}_n(\tau_n) - \vec{V}_\infty(\tau_n)\|_{L^2_x} \to 0$ when $h_\infty = 0$, $\tau_\infty = \pm\infty$ and $\liminf_{n\to\infty} |\tau_n h_n^2| > 0$. Indeed, assuming that $\tau_n h_n^2 \to m \in [-\infty, \infty]$ after extraction of a subsequence, we have
$$\|\vec{V}_n(\tau_n) - \vec{V}_\infty(\tau_n)\|_{L^2_x} \to \begin{cases} \|(e^{im/(2|\nabla|)} - 1)\psi\|_{L^2_x} & (|m| < \infty), \\ \sqrt{2}\|\psi\|_{L^2_x} & (m = \pm\infty). \end{cases} \tag{1-1}$$

(3) In the proof of [2011, Lemma 5.6], the global bound [2011, (5-96)] does not follow from the uniform bound on finite time intervals, since the required largeness of $n$ depends on the size of the interval $I$.

(1) is concerned only with a very critical case of exponential nonlinearity in two dimensions ($d = 2$). More precisely, it is problematic only if
$$0 < \limsup_{|u|\to\infty} e^{-\kappa_0|u|^2}|u|^2 f(u) < \infty, \tag{1-2}$$

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where $\kappa_0$ is the exponent in [2011, (1-29)]. Errors (2)–(3) are crucial only in the $H^1$ critical case of higher dimensions $d \geq 3$, with $h_\infty = 0$: the concentration by scaling in the nonlinear profile, where we need to modify the definition of the nonlinear concentrating waves and then solve the massless limit problem for the nonlinear Klein–Gordon equation (NLKG) (see Theorem 3.1 below). In the other case, i.e., with the subcritical or exponential nonlinearity or with $h_\infty = 1$, we still need to take care of (3), but it is a rather superficial change.

## 2. Correction for (1)

We do not know if [2011, Lemma 2.6] holds true in the very critical case (1-2). So we add the assumption

$$\limsup_{|u|\to\infty} e^{-\kappa_0|u|^2}|u|^2 f(u) \in \{0, \infty\} \tag{2-1}$$

in [2011, Proposition 1.2(3)] and [2011, Lemma 2.6]. The existence of $Q$ was used in [2011] only to characterize the threshold energy $m$, so the rest of the paper is not affected by it.

In [2014, (1.24)], the existence of $Q$ is mentioned to characterize the threshold $m^{(c)}$. It should also be restricted by (2-1), but the rest of [2014] does not really need $Q$. Removing $Q$, [2014, (2.3)] should be replaced with

$$m \leq H_p^{(c)}(\varphi), \tag{2-2}$$

[2014, (2.6)] should be replaced with

$$m \leq J^{(c)}(\lambda\varphi) = H_p^{(c)}(\lambda\varphi) \leq H_p^{(c)}(\varphi), \tag{2-3}$$

and [2014, (2.7)] with

$$\begin{aligned} \ddot{y} = (2+p)\|\dot{u}\|_{L^2}^2 + 2p(H_p^{(1)}(u) - m) &= (4+\varepsilon)\|\dot{u}\|_{L^2}^2 + (1-c)\varepsilon\|u\|_{L^2}^2 + 2p(H_p^{(c)}(u) - m) \\ &\geq \left(1 + \tfrac{1}{4}\varepsilon\right)\dot{y}^2/y + (1-c)\varepsilon y. \end{aligned} \tag{2-4}$$

The existence of $Q$ is also mentioned in [2015, Theorem 5.1]. It should be also restricted by (2-1). The rest of [2015] remains unaffected.

We still need to prove [2011, Lemma 2.6] under the new restriction (2-1). If the limit (2-1) is infinite, then [2015, Theorem 1.5(B)] implies $C^\star_{\mathrm{TM}}(F) = \infty > 1$. In this case, the proof of [2011, Lemma 2.6] remains valid. If the limit (2-1) is zero, then [2015, Theorem 1.5(B)] implies $C^\star_{\mathrm{TM}}(F) < \infty$. In this case, we do not argue as in [2011], but rely on the compactness [2015, Theorem 1.5(C)]. Let $\varphi_n \in H^1(\mathbb{R}^2)$ be a normalized maximizing sequence for $C^\star_{\mathrm{TM}}(F)$, i.e.,

$$\|\varphi_n\|_{L^2} = 1, \quad \kappa_0\|\nabla\varphi_n\|_{L^2}^2 \leq 4\pi, \quad 2F(\varphi_n) \to C := C^\star_{\mathrm{TM}}(F) \in (0, \infty). \tag{2-5}$$

By the standard rearrangement and the $H^1$ boundedness, we may assume that the $\varphi_n$ are radially decreasing and $\varphi_n \to \varphi$ weakly in $H^1(\mathbb{R}^2)$ for some $\varphi$. By [2015, Theorem 1.5(C)], we have $2F(\varphi_n) \to 2F(\varphi) = C > 0$. In particular, $\varphi \neq 0$. Since $\kappa_0\|\nabla\varphi\|_{L^2}^2 \leq 4\pi$ and $\|\varphi\|_{L^2} \leq 1$ by the weak convergence, we deduce from the definition of $C^\star_{\mathrm{TM}}(F)$ that $\|\varphi\|_{L^2} = 1$ and $\varphi$ is a maximizer. Hence, for a Lagrange multiplier $\mu \geq 0$,

$$f'(\varphi) - C\varphi = -\mu\Delta\varphi. \tag{2-6}$$

That $\mu \neq 0$ is obvious by the decay order of $f'$ as $\varphi \to 0$. Hence $\mu > 0$ and so $\kappa_0 \|\nabla\varphi\|_{L^2}^2 = 4\pi$, since otherwise we could increase both $F(\varphi)$ and $\|\nabla\varphi\|_{L^2}^2$ by the $L^2$ scaling $\varphi_{1,-1}^{\lambda}$ with $\lambda > 0$, using the $L^2$ supercritical condition [2011, (1-21)]. Then $Q(x) := \varphi(\mu^{-1/2}x) \in H^2(\mathbb{R}^2)$ satisfies

$$-\Delta Q + CQ = f'(Q), \quad \kappa_0\|\nabla Q\|_{L^2}^2 = 4\pi, \quad 2F(Q) = C\|Q\|_{L^2}^2. \tag{2-7}$$

Hence $J^{(C)}(Q) = \frac{1}{2}\|\nabla Q\|_{L^2}^2 = 2\pi/\kappa_0$. The rest of the proof of [2011, Lemma 2.6], namely the proof of $m_{\alpha,\beta} = m_{0,1} = 2\pi/\kappa_0$, remains valid.

## 3. Correction for (2)–(3)

For (2)–(3), we do not have to modify the main results, but need to correct the proof, including the definition of the nonlinear profile decomposition. Henceforth, we always assume that $0 < h_n \to h_\infty$, $(t_n, x_n) \in \mathbb{R}^{1+d}$ and $\tau_n = -t_n/h_n \to \tau_\infty \in [-\infty, \infty]$ are sequences. The main problematic case is when the energy concentrates, namely $h_\infty = 0$, which can happen only in the energy critical case [2011, (1-28)]

$$d \geq 3, \quad f(u) = \frac{|u|^{2^\star}}{2^\star}, \quad 2^\star = \frac{2d}{d-2}. \tag{3-1}$$

First we modify the vector notation in [2011, (4-1)]. For any real-valued function $a(t,x)$, the complex-valued functions $\vec{a}$, $\vec{a}$ and $\vec{a}$ are defined by

$$\vec{a} := (\langle\nabla\rangle - i\partial_t)a, \quad \vec{a} := (\langle\nabla\rangle_n - i\partial_t)a, \quad \vec{a} := (\langle\nabla\rangle_\infty - i\partial_t)a, \tag{3-2}$$

where $\langle\nabla\rangle_* = \sqrt{h_*^2 - \Delta}$ as in [2011, (5-1)]. Hence $a$ is recovered from either of them by

$$a = \operatorname{Re}\langle\nabla\rangle^{-1}\vec{a} = \operatorname{Re}\langle\nabla\rangle_n^{-1}\vec{a} = \operatorname{Re}\langle\nabla\rangle_\infty^{-1}\vec{a}. \tag{3-3}$$

Note that $(\vec{a}, a)$ was denoted by $(\vec{a}, \hat{a})$ in [2011], but it was confusing. Indeed, $u_{(n)}$ in [2011, (5-55)] did not make sense if $h_\infty = 0$, since $\vec{u}_{(n)}$ in [2011, (5-54)] was not in the form [2011, (4-1)]. So we replace [2011, (5-54)] with

$$\vec{u}_{(n)} = T_n\vec{U}_{(n)}((t - t_n)/h_n), \tag{3-4}$$

where $\vec{U}_{(n)}$ is defined by

$$\vec{V}_n := e^{it\langle\nabla\rangle_n}\psi, \quad \vec{U}_{(n)} = \vec{V}_n - i\int_{\tau_\infty}^{t} e^{i(t-s)\langle\nabla\rangle_n} f'(U_{(n)})\,ds. \tag{3-5}$$

Then $u_{(n)} = h_n T_n U_{(n)}((t - t_n)/h_n)$ is a solution of NLKG satisfying

$$\lim_{t\to\tau_\infty} \|(\vec{u}_{(n)} - \vec{v}_n)(th_n + t_n)\|_{L_x^2} = 0. \tag{3-6}$$

In other words, we keep NLKG in defining the profiles, even if $h_\infty = 0$. Note that if $h_\infty = 1$ then $\vec{U}_{(n)} = \vec{U}_\infty$ and so $u_{(n)}$ is unchanged.

By the change of [2011, (5-54)] to (3-4), the problematic [2011, (5-56)] is replaced with

$$\|\vec{u}_n(0) - \vec{u}_{(n)}(0)\|_{L_x^2} = \left\| \int_{\tau_\infty h_n + t_n}^{0\ (=\tau_n h_n + t_n)} e^{-is\langle\nabla\rangle} f'(u_{(n)})\,ds \right\|_{L_x^2} \to 0. \tag{3-7}$$

In order to prove the last limit as well as the global Strichartz approximation for (3), we need the convergence in the massless limit of the $H^1$ critical NLKG:

**Theorem 3.1.** *Assume* [2011, (1-28)] *and* $h_\infty = 0$. *Let* $\vec{U}_\infty$ *be the solution of*

$$\vec{V}_\infty := e^{it|\nabla|}\psi, \quad \vec{U}_\infty = \vec{V}_\infty - i\int_{\tau_\infty}^t e^{i(t-s)|\nabla|} f'(U_\infty)\,ds. \tag{3-8}$$

*Let* $\vec{U}_{(n)}$ *be the solution of* (3-5) *and* $\vec{u}_{(n)}(t) := T_n\vec{U}_{(n)}((t-t_n)/h_n)$. *Suppose that* $U_\infty \in [W]_2^\bullet(J)$ *for some interval* $J$ *whose closure in* $[-\infty,\infty]$ *contains* $\tau_\infty$. *Then for any bounded subinterval* $I \subset J$ *we have, as* $n \to \infty$,

$$\begin{gathered} \|\vec{U}_{(n)} - \vec{U}_\infty\|_{L^\infty_{t\in I}L^2_x} + \|U_{(n)} - U_\infty\|_{([W]_2^\bullet\cap[M]_0)(J)} + \|u_{(n)}\|_{[W]_0(J)} \to 0, \\ \|u_{(n)}\|_{([W]_2\cap[M]_0)(h_nJ+t_n)} \sim \|U_\infty\|_{([W]_2^\bullet\cap[M]_0)(J)} + o(1). \end{gathered} \tag{3-9}$$

Postponing the proof of the above theorem to the next section, we continue to correct [2011, Section 5]. Equation (3-7) in the case of $h_\infty = 0$ follows from the above estimate and $\tau_n \to \tau_\infty$ via Strichartz:

$$\begin{aligned} \left\| \int_{\tau_\infty h_n+t_n}^0 e^{-is\langle\nabla\rangle} f'(u_{(n)})\,ds \right\|_{L^2_x} &\lesssim \|f'(u_{(n)})\|_{[W^{*(1)}]_2(I_n)} \lesssim \|u_{(n)}\|^{2^\star-1}_{([W]_2\cap[M]_0)(I_n)} \\ &\lesssim \|U_\infty\|^{2^\star-1}_{[W]_2^\bullet\cap[M]_0(J_n)} + o(1) = o(1), \end{aligned} \tag{3-10}$$

where $I_n := (0, \tau_\infty h_n + t_n) \cup (\tau_\infty h_n + t_n, 0)$ and $J_n := (\tau_n, \tau_\infty) \cup (\tau_\infty, \tau_n)$.

We modify the definition of $ST$ in [2011, (5-59)–(5-60)] in the $\dot{H}^1$ critical case [2011, (1-28)] to

$$ST = [W]_2, \quad ST^* = [W^{*(1)}]_2 + L^1_t L^2_x, \quad ST^\diamond_\infty := \begin{cases} [W]_2 & (h^\diamond_\infty = 1), \\ [W]_2^\bullet & (h^\diamond_\infty = 0). \end{cases} \tag{3-11}$$

Indeed, $[K]_2$ and $[K^{*(1)}]_2$ norms are not needed in the $\dot{H}^1$ critical case. Then we simply discard the estimates [2011, (5-61)–(5-62)].

Next we reprove [2011, Lemma 5.5], extending it to unbounded intervals $I$. The above theorem implies that we can replace [2011, (5-64)] with the stronger[1]

$$\limsup_{n\to\infty} \|u^j_{(n)}\|_{ST(\mathbb{R})} \lesssim \|U^j_\infty\|_{ST^j_\infty(\mathbb{R})} \tag{3-12}$$

if $h^j_\infty = 0$, while it is trivial if $h^j_\infty = 1$. The proof of [2011, (5-65)] for $h^j_\infty = 1$ did not use the boundedness of $I$, so we may assume that all $h^j_\infty$ are 0. Then the above theorem implies that $\|u^{<k}_{(n)}\|_{[W]_0(\mathbb{R})} \to 0$ as $n \to \infty$, so it suffices to estimate the homogeneous norm $[W]_2^\bullet(\mathbb{R})$. We have

$$\|u^{<k}_{(n)}\|_{[W]_2^\bullet(\mathbb{R})} \sim \sum_{l=1}^d \left\| \sum_{j<k} \check{u}^{j,l}_{n,m} \right\|_{L^p_t \ell^2_{m\in\mathbb{Z}} L^q_x} \tag{3-13}$$

with $(1/p, 1/q, s) = W$ and

$$\check{u}^{j,l}_{n,m} := 2^{sm}\delta^l_m h^j_n T^j_n U^j_{(n)}((t-t^j_n)/h^j_n). \tag{3-14}$$

[1]Recall that $\widehat{U}^j_\infty$ in [2011] is denoted by $U^j_\infty$ in this correction according to (3-2).

Defining $\check{u}^{j,l}_{n,m,R}$ by [2011, (5-77)], we have

$$\|\check{u}^{j,l}_{n,m} - \check{u}^{j,l}_{n,m,R}\|_{L^p_t \ell^2_m L^q_x} \lesssim \|2^{sm}\delta^l_m U^j_{(n)}\|_{L^p_t \ell^2_m L^q_x(|t|+|m|+|x|>R)} \to 0 \quad \text{as} \ R \to \infty, \tag{3-15}$$

which is still uniform in $n$ since, by the above theorem, $U^j_{(n)}$ is approximated by $U^j_\infty$ in $[W]^\bullet_2(\mathbb{R})$, which is equivalent to the last norm without the restriction by $R$. Thus we obtain [2011, (5-65)] by the disjoint support property for large $n$.

According to the change of $u^j_{(n)}$, we replace the nonlinear decomposition [2011, (5-66)] with the simpler form

$$\lim_{n\to\infty}\Big\| f'(u^{<k}_{(n)}) - \sum_{j<k} f'(u^j_{(n)})\Big\|_{ST^*(I)} = 0, \tag{3-16}$$

which is the same as [2011, (5-66)] if $h^j_\infty = 1$. In that case, however, we used that $I$ was bounded in [2011, (5-82)]. We replace it with an interpolation between [2011, (4-84)] and

$$\|f'_S(u)\|_{[((1-\theta_0)K+\theta_0 W)^{*(1)}]_2(I)} \lesssim \|u\|_{[K]_2(I)}\|u\|^{p_1}_{[K]_0(I)} \lesssim \|u\|^{p_1+1}_{[K]_2(I)}, \tag{3-17}$$

where we can choose some $\theta_0 \in (0, 1)$ since $p_1 > 4/d$ (choosing $p_1$ close enough to $4/d$ if necessary). Since $Z := ((1-\theta_0)K + \theta_0 W)^{*(1)}$ is an interior dual-admissible exponent, we can find some $\theta_1 \in (0, 1)$ such that $\theta_1 Y + (1-\theta_1)Z$ is also a dual-admissible exponent. Interpolating (3-17) with [2011, (4-84)], we have

$$\|f'_S(u) - f'_S(v)\|_{[\theta_1 Y+(1-\theta_1)Z]_2(I)} \lesssim \|(u, v)\|^{p_1+1-\theta_1}_{[K]_2(I)\cap[Q]_{2p_1}(I)}\|u - v\|^{\theta_1}_{[P]_2(I)}. \tag{3-18}$$

Thus we obtain [2011, (5-66)] on any subset $I$ in the subcritical and exponential cases. In the $\dot{H}^1$ critical case [2011, (1-28)], we discard $u^j_{\langle n\rangle}$ in [2011, (5-85)] and prove (3-16) directly, putting

$$U^j_{n,R}(t, x) := \chi_R(t, x)U^j_{(n)}(t, x) \times \prod\big\{(1 - \chi_{h^{j,l}_n R})(t - t^{j,l}_n, x - x^{j,l}_n) \mid 1 \le l < k,\ h^l_n R < h^j_n\big\}. \tag{3-19}$$

It is still uniformly bounded in $([H]^\bullet_2 \cap [W]^\bullet_2)(\mathbb{R})$, and $U^j_{n,R} - \chi_R U^j_{(n)} \to 0$ in $[M]_0(\mathbb{R})$ as $n \to \infty$ thanks to the above theorem, as well as in $[L]_0$, and also $\chi_R U^j_{(n)} \to U^j_{(n)}$ as $R \to \infty$. Hence we may replace $u^j_{(n)}$ in (3-16) by $u^j_{(n),R} := h^j_n T^j_n U^j_{n,R}((t - t^j_n)/h^j_n)$, using [2011, (4-62)] for $d \le 5$, and a similar interpolation argument as above for $d \ge 6$; see (4-16)–(4-19) below. Then we obtain (3-16) by the disjoint support property, in the same way as [2011, (5-94)].

With the above corrections, we now reprove [2011, Lemma 5.6]. First, [2011, (5-100)] holds for any subset $I \subset \mathbb{R}$, by the above improvement of [2011, Lemma 5.5]. Now, thanks to the change of $u^j_{(n)}$, [2011, (5-101)] is simplified to

$$\mathrm{eq}(u^{<k}_{(n)}) = f'(u^{<k}_{(n)}) - \sum_{j<k} f'(u^j_{(n)}), \tag{3-20}$$

which is vanishing by (3-16). Hence we obtain [2011, (5-103)]. We also obtain [2011, (5-104)] on $\mathbb{R}$ by the same nonlinear estimates as we used above. Then, applying [2011, Lemma 4.5] on $\mathbb{R}$, we obtain the desired [2011, Lemma 5.6].

Section 6 of [2011] is almost unchanged, except for the obvious modification in [2011, (6-6)] due to the change of $u_{(n)}$, namely

$$\vec{u}^j_{(n)} = T^j_n \vec{U}^j_{(n)}((t-t^j_n)/h^j_n), \tag{3-21}$$

and the notational change in [2011, (6-7)–(6-9)] from $(\vec{U}^0_\infty, \widehat{U}^0_\infty)$ to $(\vec{U}^0_\infty, U^0_\infty)$ due to (3-2). Since the case $h_\infty = 0$ is eliminated in the proof of [2011, Lemma 6.1], the errors (2)–(3) do not affect the rest of the paper.

## 4. Massless limit of scattering for the critical NLKG

It remains to prove Theorem 3.1. Throughout this section, we assume [2011, (1-28)]. The main idea is to decompose the time interval into a bounded subinterval and neighborhoods of $\pm\infty$. On the bounded part, we have strong convergence in the massless limit. In the neighborhoods of $t = \pm\infty$, we do not have strong convergence, but the Strichartz norms are uniformly controlled via the asymptotic free profiles.

The first ingredient concerns the uniform Strichartz bound for free waves.

**Lemma 4.1.** *Let $\vec{v}_n = e^{it\langle\nabla\rangle} T_n \psi$, $h_\infty = 0$, $\vec{V}_\infty = e^{it|\nabla|}\psi$, and let $Z \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right) \times [0, 1)$ satisfy $\mathrm{reg}^0(Z) = 1$ and $\mathrm{str}^0(Z) \le 0$, namely a wave-admissible Strichartz exponent except for the energy norm. Then we have*

$$\limsup_{n\to\infty} \|v_n\|_{[Z]_2(0,\infty)} \lesssim \|V_\infty\|_{[Z]^\bullet_2(0,\infty)} \quad \textit{and} \quad \lim_{n\to\infty} \|P_{<1} v_n\|_{[Z]_2(0,\infty)} = 0, \tag{4-1}$$

*where $P_{<a}$ denotes the smooth cut-off for the Fourier region $|\xi| < 2a$ defined by $P_{<a}\varphi = a^d \Lambda_0(ax) * \varphi$, with $\Lambda_0 \in \mathcal{S}(\mathbb{R}^d)$ in the proof of* [2011, Lemma 5.1]. *If $Z_3 = 0$, then we have also $\|v_n\|_{[Z]_0(0,\infty)} \to \|V_\infty\|_{[Z]_0(0,\infty)}$.*

*Proof.* Let $\vec{v}_n(t) = T_n \vec{V}_n(t/h_n)$. The Strichartz estimate for the Klein–Gordon and the wave equations

$$\|v_n\|_{[Z]_2(0,\infty)} \lesssim \|T_n\psi\|_{L^2} = \|\psi\|_{L^2}, \quad \|V_\infty\|_{[Z]^\bullet_2(0,\infty)} \lesssim \|\psi\|_{L^2} \tag{4-2}$$

implies that it suffices to consider $\psi$ in a dense subset of $L^2(\mathbb{R}^d)$. Hence we may assume that $\mathcal{F}\psi$ is $C^\infty$ with a compact $\mathrm{supp}\,\mathcal{F}\psi \not\ni 0$. Since $0 < \langle\xi\rangle_n - \langle\xi\rangle_\infty \le h_n^2/|\xi|$,

$$|(e^{it\langle\xi\rangle_n}\langle\xi\rangle_n^{-1} - e^{it|\xi|}|\xi|^{-1})| \lesssim |t| h_n^2 |\xi|^{-2} + h_n^2 |\xi|^{-3}, \tag{4-3}$$

and so, under the above assumption on $\psi$, for any $s \in \mathbb{R}$ and any sequence $S_n > 0$,

$$\|V_n - V_\infty\|_{L^\infty(0,S_n;H^s)} \le \langle S_n\rangle h_n^2 C(s,\psi). \tag{4-4}$$

Hence, by Sobolev in $x$ and Hölder in $t$,

$$\|V_n - V_\infty\|_{([Z]^\bullet_2 \cap [Z]_0)(0,S_n)} \le \langle S_n\rangle^{1+Z_1} h_n^2 C(s,\psi). \tag{4-5}$$

We deduce that if $S_n \to \infty$ and $S_n^{1+Z_1} h_n^2 \to 0$ then, using the (approximate) scale-invariance of $[Z]^\bullet_2$,

$$\begin{aligned}
\|v_n\|_{[Z]_2(0,h_nS_n)} &\sim \|v_n\|_{[Z]^\bullet_2(0,h_nS_n)} + \|P_{<1}v_n\|_{[Z]_0(0,h_nS_n)},\\
\|v_n\|_{[Z]^\bullet_2(0,h_nS_n)} &\sim \|V_n\|_{[Z]^\bullet_2(0,S_n)} \to \|V_\infty\|_{[Z]^\bullet_2(0,\infty)},\\
\|P_{<1}v_n\|_{[Z]_0(0,h_nS_n)} &\sim \|h_n^{Z_3} P_{<h_n} V_n\|_{[Z]_0(0,S_n)} \to 0,
\end{aligned}$$

and similarly, if $Z_3 = 0$ then $\|v_n\|_{[Z]_0(0,h_nS_n)} = \|V_n\|_{[Z]_0(0,S_n)} \to \|V_\infty\|_{[Z]_0(0,\infty)}$.

Next, the dispersive decay of wave-type for the Klein–Gordon equation

$$\|e^{it\langle\nabla\rangle}\varphi\|_{B^0_{q,2}} \lesssim |t|^{-(d-1)\alpha}\|\varphi\|_{B^s_{q',2}}, \quad \alpha := \tfrac{1}{2} - \tfrac{1}{q} \in \left[0, \tfrac{1}{2}\right],\ s := (d+1)\alpha, \tag{4-6}$$

together with the embedding $L^{q'} \subset B^0_{q',2}$ implies that

$$\|v_n(t)\|_{B^\sigma_{q,2}} \lesssim |t|^{-(d-1)\alpha}\|\langle\nabla\rangle^{\sigma+s-1}T_n\psi\|_{L^{q'}} = |t|^{-(d-1)\alpha}h_n^{1-\alpha-\sigma}\|\langle\nabla\rangle_n^{\sigma+s-1}\psi\|_{L^{q'}}, \tag{4-7}$$

and so, putting $\alpha = \frac{1}{2} - Z_2$,

$$\begin{aligned}\|v_n\|_{[Z]_2(h_nS_n,\infty)} &\le C(\psi)h_n^{1-\alpha-Z_3}\|t^{-(d-1)\alpha}\|_{L_t^{1/Z_1}(h_nS_n,\infty)} \\ &\sim C(\psi)h_n^{1-\alpha-Z_3}(h_nS_n)^{Z_1-(d-1)\alpha} = C(\psi)S_n^{\alpha-1+Z_3} \to 0,\end{aligned} \tag{4-8}$$

where we used that $\mathrm{reg}^0(Z) = Z_3 - Z_1 + d\alpha = 1$ in the last identity and

$$\alpha - 1 + Z_3 = \mathrm{reg}^0(Z) + \mathrm{str}^0(Z) - 1 - Z_1 < 0 \tag{4-9}$$

in taking the limit. Note that the above exponent is zero at the energy space $Z = \left(0, \frac{1}{2}, 1\right)$, which is excluded by the assumption. The estimate in $[Z]_0(h_nS_n, \infty)$ for $Z_3 = 0$ is done in the same way. Combining them with the above estimates on $(0, h_nS_n)$ leads to the conclusion via a density argument. □

The second ingredient is convergence or propagation of small disturbance on finite intervals, which is uniformly controlled by the Strichartz norm of $U_\infty$.

**Lemma 4.2.** *For any $0 < M, \varepsilon < \infty$, there exists $\delta = \delta(\varepsilon, M) \in (0, 1)$ with the following property. Let $h_\infty = 0$ and let $U_\infty$ be a solution of NLW on some interval $J$ satisfying $\|U_\infty\|_{([H]^\bullet_2\cap[W]^\bullet_2)(J)} \le M$. Then, for any bounded subinterval $I \subset J$ with $0 \in I$ and any $\varphi_n \in L^2(\mathbb{R}^d)$ with $\|\varphi_n\|_{L^2} < \delta$, the unique solution $U_n$ of*

$$(\partial_t^2 - \Delta + h_n^2)U_n = f'(U_n), \quad \vec{U}_n(0) = \vec{U}_\infty(0) + \varphi_n, \tag{4-10}$$

*exists on $I$ for large $n$, satisfying*

$$\|\vec{U}_n - \vec{U}_\infty\|_{L_t^\infty L_x^2(I)} + \|U_n - U_\infty\|_{([W]^\bullet_2\cap[M]_0)(I)} < \varepsilon, \tag{4-11}$$

*and $\|h_nT_nU_n((t-t_n)/h_n)\|_{[W]_0(h_nI+t_n)} \lesssim \delta$ for large $n$.*

*Proof.* We give the detail only in the harder case $d \ge 6$, where we need the exotic Strichartz norms. Let $\gamma_n := U_n - U_\infty$ and $\vec{\gamma}_n := \vec{U}_n - \vec{U}_\infty$, then

$$(\partial_t^2 - \Delta)\gamma_n = f'(U_\infty + \gamma_n) - f'(U_\infty) - h_n^2U_n. \tag{4-12}$$

Note however that $\vec{\gamma}_n$ is not written only by $\gamma_n$. It suffices to prove the following:

**Claim.** There exist constants $\theta \in (0, 1)$ and $C > 1$ such that if

$$\|U_\infty\|_{([W]^\bullet_2\cap[\widetilde{M}]^\bullet_{2p})(0,S)} \le \eta, \quad \|\vec{\gamma}_n(0)\|_{L^2} \ll 1, \tag{4-13}$$

for some $0 < S < \infty$ and $0 < \eta \ll 1$, where $p = 2^\star - 2 = 4/(d-2)$, then

$$\|\vec{\gamma}_n\|_{L_t^\infty(0,S;L_x^2)} + \|\gamma_n\|_{[W]_2^\bullet(0,S)} \le C[\|\vec{\gamma}_n(0)\|_{L^2} + \|\vec{\gamma}_n(0)\|_{L^2}^\theta \eta^{(p+1)(1-\theta)}]. \tag{4-14}$$

*Proof of the claim.* The exotic Strichartz estimate for the wave equation yields, on the time interval $(0, S)$,

$$\|\gamma_n\|_{[\tilde{N}]_2^\bullet} \lesssim \|\bar{\gamma}_n(0)\|_{L^2} + \|f'(U_\infty+\gamma_n) - f'(U_\infty)\|_{[Y]_2} + \|h_n^2 U_n\|_{L_t^1 L_x^2}, \tag{4-15}$$

while the nonlinear estimate in the Besov space yields

$$\|f'(U_\infty+\gamma_n) - f'(U_\infty)\|_{[Y]_2} \lesssim \|(U_\infty,\gamma_n)\|_{[M]_0}^p \|\gamma_n\|_{[\tilde{N}]_2^\bullet} + \|(U_\infty,\gamma_n)\|_{[\tilde{M}]_{2p}^\bullet}^p \|\gamma_n\|_{[N]_0}, \tag{4-16}$$

and we have $\|\bar{\gamma}_n(0)\|_{L^2} \lesssim \|\vec{\gamma}_n(0)\|_{L^2} + o(1)$. The $L_t^1 L_x^2$ norm is estimated by

$$\|h_n^2 U_n\|_{L_t^1 L_x^2} \le \|h_n \vec{U}_n\|_{L_t^1 L_x^2} \le h_n S \|\vec{\gamma}_n + \vec{U}_\infty\|_{L_t^\infty L_x^2}. \tag{4-17}$$

Define $\underline{W}, O \in \left[0, \tfrac{1}{2}\right]^3$ by

$$\begin{aligned} \underline{W} &:= W - \frac{1}{2}\Big(0, \frac{1}{d}, 1\Big) = \Big(\frac{d-1}{2(d+1)}, \frac{d^2-2d-1}{2d(d+1)}, 0\Big), \\ O &:= W + p\underline{W} = \Big(\frac{(d+2)(d-1)}{2(d+1)(d-2)}, \frac{d^3+d^2-6d-4}{2(d-2)d(d+1)}, \frac{1}{2}\Big). \end{aligned} \tag{4-18}$$

Then $O$ is an interior dual exponent of the standard Strichartz, and so there is small $\theta \in (0, 1)$ such that $\theta Y + (1-\theta)O$ is also a dual exponent. Hence the standard Strichartz yields, for any wave-admissible exponent $Z$,

$$\|\gamma_n\|_{[Z]_2^\bullet} + \|\bar{\gamma}_n\|_{L_t^\infty L_x^2} \lesssim \|\bar{\gamma}_n(0)\|_{L^2} + \|f'(U_\infty+\gamma_n) - f'(U_\infty)\|_{[\theta Y+(1-\theta)O]_2^\bullet} + \|h_n^2 U_n\|_{L_t^1 L_x^2}, \tag{4-19}$$

where the nonlinear part is already estimated in $[Y]_2^\bullet$, while

$$\|f'(U_\infty+\gamma_n)\|_{[O]_2^\bullet} + \|f'(U_\infty)\|_{[O]_2^\bullet} \lesssim \eta^{p+1} + \|\gamma_n\|_{[W]_2^\bullet}^{p+1}. \tag{4-20}$$

Hence we have

$$\begin{gathered} \|\gamma_n\|_{[\tilde{N}]_2^\bullet} \lesssim \|\bar{\gamma}_n(0)\|_{L^2} + A + B, \\ \|\gamma_n\|_{[W]_2^\bullet \cap [\tilde{M}]_{2p}^\bullet} + \|\bar{\gamma}_n\|_{L_t^\infty L_x^2} \lesssim \|\bar{\gamma}_n(0)\|_{L^2} + A^\theta(\eta + \|\gamma_n\|_{[W]_2^\bullet})^{(1-\theta)(p+1)} + B, \\ A \lesssim (\eta + \|\gamma_n\|_{[\tilde{M}]_{2p}^\bullet})^p \|\gamma_n\|_{[\tilde{N}]_2^\bullet}, \\ B \lesssim S h_n \|\vec{\gamma}_n\|_{L_t^\infty L_x^2} + o(1). \end{gathered} \tag{4-21}$$

Assuming that $\|\gamma_n\|_{[\tilde{M}]_{2p}^\bullet} \ll 1$ and that $\|\vec{\gamma}_n\|_{L_t^\infty L_x^2}$ is bounded in $n$, we deduce from the above estimates that

$$\begin{gathered} A \ll \|\gamma_n\|_{[\tilde{N}]_2^\bullet} \lesssim \|\bar{\gamma}_n(0)\|_{L^2} + o(1), \quad B = o(1), \\ \|\gamma_n\|_{[W]_2^\bullet \cap [\tilde{M}]_{2p}^\bullet} + \|\bar{\gamma}_n\|_{L_t^\infty L_x^2} \lesssim \|\bar{\gamma}_n(0)\|_{L^2} + \|\bar{\gamma}_n(0)\|_{L^2}^\theta \eta^{(1-\theta)(p+1)} + o(1). \end{gathered} \tag{4-22}$$

It remains to prove the uniform bound on $\|\vec{\gamma}_n\|_{L_t^\infty L_x^2}$. Let $V_\infty$, $V_n$, $v_n$ be the free solutions defined by

$$\vec{V}_\infty := e^{it|\nabla|}\vec{U}_\infty(0), \quad \vec{V}_n := e^{it\langle\nabla\rangle_n}\vec{U}_n(0), \quad \vec{v}_n = T_n \vec{V}_n(t/h_n). \tag{4-23}$$

For any $0 < R_n \to 0$ such that $h_n/R_n \to 0$, we have

$$\|\mathscr{F}\vec{\gamma}_n\|_{L^\infty(0,S;L^2(|\xi|>R_n))} \lesssim \|\bar{\gamma}_n\|_{L^\infty(0,S;L^2_x)} + o(1). \tag{4-24}$$

For the lower frequency, we have, by the energy inequality, Hölder and Sobolev,

$$\begin{aligned}\|\vec{U}_n - \vec{V}_n\|_{L^\infty_t \dot{H}^{-1}_x(0,S)} &\lesssim \|f'(U_n)\|_{L^1_t \dot{H}^{-1}_x(0,S)} \\ &\lesssim S\|U_n\|^{p+1}_{L^\infty_t \dot{H}^1_x(0,S)} \lesssim S(\|\bar{U}_\infty\|_{L^\infty_t L^2_x(0,S)} + \|\bar{\gamma}_n\|_{L^\infty_t L^2_x(0,S)})^{p+1},\end{aligned} \tag{4-25}$$

and similarly $\|\vec{U}_\infty - \vec{V}_\infty\|_{L^\infty_t \dot{H}^{-1}_x(0,S)} \lesssim S\|\bar{U}_\infty\|^{p+1}_{L^\infty_t L^2_x}$. Since $|\langle\xi\rangle_n - \langle\xi\rangle_\infty| \le h_n$, we have also $\|\vec{V}_n(t) - \vec{V}_\infty(t)\|_{L^2_x} \lesssim |t| h_n \|\bar{U}_\infty(0)\|_{L^2} + \delta$. Hence

$$\begin{aligned}&\|\mathscr{F}\vec{\gamma}_n\|_{L^\infty(0,S;L^2(|\xi|<R_n))} \\ &\qquad \le R_n\|\vec{U}_n - \vec{V}_n\|_{L^\infty_t \dot{H}^{-1}_x(0,S)} + \|\vec{V}_n - \vec{V}_\infty\|_{L^\infty_t L^2_x(0,S)} + R_n\|\vec{V}_\infty - \vec{U}_\infty\|_{L^\infty_t \dot{H}^{-1}_x(0,S)} \\ &\qquad \lesssim o(1)S\|\bar{\gamma}_n\|^{p+1}_{L^\infty_t L^2_x(0,S)} + \delta + o(1)\end{aligned} \tag{4-26}$$

Adding this to (4-24), we obtain

$$\|\vec{\gamma}_n\|_{L^\infty_t L^2_x(0,S)} \lesssim \|\bar{\gamma}_n\|_{L^\infty_t L^2_x(0,S)} + o(1)S\|\bar{\gamma}_n\|^{p+1}_{L^\infty_t L^2_x(0,S)} + \delta + o(1). \tag{4-27}$$

Combining this with the estimates (4-22), we deduce that both $\vec{\gamma}_n$ and $\bar{\gamma}_n$ are bounded in $L^\infty_t L^2_x(0, S)$. □

To prove (4-11) from this claim, we decompose $I$ into subintervals $I_j$ such that $\|U_\infty\|_{([W]^\bullet_2 \cap [\widetilde{M}]^\bullet_{2p})(I_j)} \le \eta$ for each $j$. Then applying the above claim iteratively to the subintervals for small $\delta > 0$ yields (4-11), where the bound on $[M]_0$ is derived by interpolation and Sobolev embedding of $[H]^\bullet_2$ and $[W]^\bullet_2$.

For the estimate in $[W]_0$, we have, by scaling,

$$\begin{aligned}&\|h_n T_n U_n((t-t_n)/h_n)\|_{[W]_0(h_n I + t_n)} \\ &\qquad \sim h_n^{1/2}\|U_n\|_{[W]_0(I)} \lesssim h_n^{1/2}\|U_n\|_{[W]^\bullet_2(I)} + \|P_{<1} v_n\|_{[W]_0(I)} + h_n^{1/2}\|P_{<h_n}(U_n - V_n)\|_{[W]_0(I)},\end{aligned} \tag{4-28}$$

where $\vec{V}_n := e^{it\langle\nabla\rangle_n}\vec{U}_n(0)$ and $\vec{v}_n = T_n\vec{V}_n(t/h_n)$. The first term on the right is vanishing since $\|U_n\|_{[W]^\bullet_2(I)}$ is bounded as shown above. The second term is $O(\delta)$ by Lemma 4.1. The third term is bounded — using Sobolev, Hölder and the same estimate as in (4-25) — by

$$|I|^{W_1} h_n^{1/2+d(1/2-W_2)}\|U_n - V_n\|_{L^\infty_t L^2_x(I)} \lesssim (|I|h_n)^{3/2-1/(d+1)}(\|\bar{U}_\infty\|_{L^\infty_t L^2_x(I)} + \varepsilon)^{p+1} = o(1), \tag{4-29}$$

hence (4-28) is $O(\delta)$ for large $n$. This concludes the proof of the lemma for $d \ge 6$.

The case $d \le 5$ is the same, but the nonlinear estimate is much simpler. In (4-13), $[\widetilde{M}]^\bullet_{2p}$ is replaced with $[M]_0$, and by the standard Strichartz we have

$$\|\gamma_n\|_{[W]^\bullet_2 \cap [M]_0} + \|\bar{\gamma}_n\|_{L^\infty_t L^2_x} \lesssim \|\bar{\gamma}_n(0)\|_{L^2} + \|f'(U_\infty + \gamma_n) - f'(U_\infty)\|_{[W^{*(1)}]^\bullet_2} + \|h_n^2 U_n\|_{L^1_t L^2_x} \tag{4-30}$$

and

$$\begin{aligned}\|f'(U_\infty + \gamma_n) - f'(U_\infty)\|_{[W^{*(1)}]^\bullet_2} &\lesssim \|(U_\infty, \gamma_n)\|^p_{[W]^\bullet_2 \cap [M]_0}\|\gamma_n\|_{[W]^\bullet_2 \cap [M]_0} \\ &\lesssim (\eta + \|\gamma_n\|_{[W]^\bullet_2 \cap [M]_0})^p \|\gamma_n\|_{[W]^\bullet_2 \cap [M]_0}.\end{aligned} \tag{4-31}$$

Then, estimating $\|h_n^2 U_n\|_{L_t^1 L_x^2(0,S)}$ in the same way as for $d \geq 6$, we obtain (4-14) without the last term. Equation (4-28) is the same as above. □

*Proof of Theorem 3.1.* Let $v_n$, $V_n$, $V_\infty$ be the free solutions defined by

$$\vec{V}_n = e^{it\langle\nabla\rangle_n}\psi, \quad \vec{V}_\infty = e^{it|\nabla|}\psi, \quad \vec{v}_n = T_n V_n((t-t_n)/h_n), \tag{4-32}$$

and

$$M := \|U_\infty\|_{[W]_2^\bullet(J)}. \tag{4-33}$$

First consider the case $\tau_\infty = \infty$. Let $0 < \varepsilon < 1$ and choose $S > 0$ so large that

$$\delta_0 := \|V_\infty\|_{([W]_2^\bullet \cap [M]_0)(S,\infty)} \leq \delta(\varepsilon, M), \tag{4-34}$$

where $\delta(\cdot\,,\cdot)$ is given by Lemma 4.2. Then Lemma 4.1 implies that

$$\|v_n\|_{([W]_2 \cap [M]_0)(h_n S + t_n,\infty)} \lesssim \delta_0 \tag{4-35}$$

for large $n$. If $\delta_0 \ll 1$, then the standard scattering argument for NLKG using the Strichartz norms implies that $u_{(n)}$ exists on $(h_n S + t_n, \infty)$, satisfying

$$\|\vec{u}_{(n)} - \vec{v}_n\|_{L_t^\infty L_x^2(h_n S+t_n,\infty)} + \|u_{(n)} - v_n\|_{([W]_2 \cap [M]_0)(h_n S+t_n,\infty)} \lesssim \delta_0^{2^\star-1} \ll \delta_0 \tag{4-36}$$

and also, for NLW,

$$\|\vec{U}_\infty - \vec{V}_\infty\|_{L_t^\infty L_x^2(S,\infty)} + \|U_\infty - V_\infty\|_{([W]_2^\bullet \cap [M]_0)(S,\infty)} \lesssim \delta_0^{2^\star-1} \ll \delta_0. \tag{4-37}$$

Thus we obtain

$$\|u_{(n)}\|_{([W]_2 \cap [M]_0)(h_n S+t_n,\infty)} \lesssim \|V_\infty\|_{([W]_2^\bullet \cap [M]_0)(S,\infty)} \sim \|U_\infty\|_{([W]_2^\bullet \cap [M]_0)(S,\infty)} \tag{4-38}$$

and, for large $n$,

$$\|\vec{U}_{(n)}(S) - \vec{V}_n(S)\|_{L_x^2} + \|\vec{V}_n(S) - \vec{V}_\infty(S)\|_{L_x^2} + \|\vec{V}_\infty(S) - \vec{U}_\infty(S)\|_{L_x^2} \ll \delta_0. \tag{4-39}$$

The next step is to go from $S$ to the negative time direction. If $J$ is bounded from below, then let $S' := \inf J$. Otherwise, choose $S' < S$ so that

$$\|U_\infty\|_{([W]_2^\bullet \cap [M]_0)(-\infty,S')} < \varepsilon. \tag{4-40}$$

Applying Lemma 4.2 to $U_\infty$ and $U_{(n)}$ backward in time from $t = S$, we obtain

$$\|\vec{U}_{(n)} - \vec{U}_\infty\|_{L_t^\infty L_x^2(S',S)} + \|U_{(n)} - U_\infty\|_{([W]_2^\bullet \cap [M]_0)(S',S)} < \varepsilon \tag{4-41}$$

and $\|u_{(n)}\|_{[W]_0(h_n S'+t_n, h_n S+t_n)} \lesssim \delta_0$ for large $n$.

If $J$ is unbounded from below, we have still to go from $S'$ to $-\infty$. The standard argument for small data scattering of NLW for $t \to -\infty$ implies that

$$\|\operatorname{Re}|\nabla|^{-1} e^{it|\nabla|}\vec{U}_\infty(S')\|_{([W]_2^\bullet \cap [M]_0)(-\infty,0)} \sim \|U_\infty\|_{([W]_2^\bullet \cap [M]_0)(-\infty,S')} < \varepsilon. \tag{4-42}$$

Then Lemma 4.1 applied backward in $t$ implies, for large $n$,

$$\|\operatorname{Re}\langle\nabla\rangle^{-1}e^{it\langle\nabla\rangle}T_n\vec{U}_\infty(S')\|_{([W]_2\cap[M]_0)(-\infty,0)} \lesssim \varepsilon. \tag{4-43}$$

Let $w_n$ be the solution of NLKG with $\vec{w}_n(0)=T_n\vec{U}_{(n)}(S')$. Then the above estimate together with $\|\vec{U}_{(n)}(S')-\vec{U}_\infty(S')\|_{L^2_x}<\varepsilon$ and the scattering for NLKG implies

$$\|w_n\|_{([W]_2\cap[M]_0)(-\infty,0)} \lesssim \varepsilon. \tag{4-44}$$

Since $w_n=h_nT_nU_{(n)}(t/h_n+S')=u_{(n)}(t+h_nS'+t_n)$, we deduce that

$$\begin{aligned}\|U_{(n)}\|_{([W]_2^\bullet\cap[M]_0)(-\infty,S')} &\sim \|u_{(n)}\|_{([W]_2^\bullet\cap[M]_0)(-\infty,h_nS'+t_n)}\\ &\lesssim \|u_{(n)}\|_{([W]_2\cap[M]_0)(-\infty,h_nS'+t_n)} = \|w_n\|_{([W]_2\cap[M]_0)(-\infty,0)} \lesssim \varepsilon.\end{aligned} \tag{4-45}$$

Thus we obtain, in the case $\tau_\infty=\infty$,

$$\|U_{(n)}-U_\infty\|_{([W]_2^\bullet\cap[M]_0)(J)}+\|u_n\|_{[W]_0(h_nJ+t_n)} \lesssim \varepsilon+\delta_0 \tag{4-46}$$

for large $n$. Since $\varepsilon$ and $\delta_0$ can be chosen as small as we wish, this implies

$$\lim_{n\to\infty}\|U_{(n)}-U_\infty\|_{([W]_2^\bullet\cap[M]_0)(J)}+\|u_n\|_{[W]_0(h_nJ+t_n)}=0 \tag{4-47}$$

and, by scaling,

$$\|u_{(n)}\|_{([W]_2\cap[M]_0)(h_nJ+t_n)} \sim \|U_\infty\|_{([W]_2^\bullet\cap[M]_0)(J)}+\|u_{(n)}\|_{[W]_0(h_nJ+t_n)} = \|U_\infty\|_{([W]_2^\bullet\cap[M]_0)(J)}+o(1). \tag{4-48}$$

Since $S\to\infty$ and $S'\to\inf J$ as $\varepsilon,\delta\to+0$, we also obtain

$$\lim_{n\to\infty}\|\vec{U}_{(n)}-\vec{U}_\infty\|_{L^\infty_tL^2_x(I)}=0 \tag{4-49}$$

for any finite subinterval $I$. The case $\tau_\infty=-\infty$ is the same by the time symmetry.

If $\tau_\infty\in\mathbb{R}$ then $\|\vec{U}_{(n)}(\tau_\infty)-\vec{U}_\infty(\tau_\infty)\|_{L^2_x}\to 0$. Hence the same argument as we used above to go from $S$ to $-\infty$ yields

$$0=\lim_{n\to\infty}\|\vec{U}_{(n)}-\vec{U}_\infty\|_{L^\infty_tL^2_x(S',\tau_\infty)}=\lim_{n\to\infty}\|U_{(n)}-U_\infty\|_{([W]_2^\bullet\cap[M]_0)(\inf J,\tau_\infty)} \tag{4-50}$$

for any $S'\in(\inf J,\tau_\infty)$, and also on $(\tau_\infty,\sup J)$ by the time symmetry. Thus we obtain (4-47) and (4-49) for any $\tau_\infty\in[-\infty,\infty]$. □

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SLIM IBRAHIM: ibrahim@math.uvic.ca
*Department of Mathematics and Statistics, University of Victoria, PO Box 3060 STN CSC, Victoria, BC V8P 5C3, Canada*

NADER MASMOUDI: masmoudi@courant.nyu.edu
*Courant Institute for Mathematical Sciences, New York University, New York, NY 10012-1185, United States*

KENJI NAKANISHI: nakanishi@ist.osaka-u.ac.jp
*Department of Pure and Applied Mathematics, Osaka University, Graduate School of Information Science and Technology, Osaka 560-0043, Japan*

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