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LOCAL ANALYTIC REGULARITY IN THE LINEARIZED CALDERÓN PROBLEM

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We show that the linearized local Dirichlet-to-Neumann map at a real-analytic potential for measurements made at an analytic open subset of the boundary is injective.

1. Introduction

In this paper, we consider the *linearized* Calderón problem with local partial data and related problems. We first briefly review Calderón's problem including the case of partial data. For a more complete review, see [Uhlmann 2009].

Calderón's problem is, roughly speaking, the question of whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is also called electrical impedance tomography. We describe the problem more precisely below.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current, the equation for the potential is given by

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega \tag{1-1}$$

since, by Ohm's law, $\gamma \nabla u$ represents the current flux. Given a potential $f \in H^{1/2}(\partial\Omega)$ on the boundary, the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= f. \end{aligned} \tag{1-2}$$

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The Dirichlet-to-Neumann (DN) map, or voltage-to-current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\bigg|_{\partial\Omega}, \tag{1-3}$$

where ν denotes the unit outer normal to $\partial\Omega$. The inverse problem is to determine γ knowing Λ_γ.

The local Calderón problem, or the Calderón problem with partial data, is the question of whether one can determine the conductivity by measuring the DN map on subsets of the boundary for voltages supported in subsets of the boundary. In this paper, we consider the case when the support of the voltages and the induced current fluxes are measured in the same open subset Γ. More conditions on this open set will be stated later. If $\gamma \in C^\infty(\overline{\Omega})$, the DN map is a classical pseudodifferential operator of order 1. It was shown in [Sylvester and Uhlmann 1986] that its full symbol computed in boundary normal coordinates near a point of Γ determines the Taylor series of γ at the point giving another proof of the result of Kohn and Vogelius [1984]. In particular, this shows that real-analytic conductivities can be determined by the local DN map. This result was generalized in [Lee and Uhlmann 1989] to the case of anisotropic conductivities using a factorization method related to the methods of this paper. Interior determination was shown in dimension $n \geq 3$ for C^2 conductivities [Sylvester and Uhlmann 1987]. This was extended to C^1 conductivities in [Haberman and Tataru 2013]. In two dimensions, uniqueness was proven for C^2 conductivities in [Nachman 1996] and for merely L^∞ conductivities in [Astala and Päivärinta 2006]. The case of partial data in dimension $n \geq 3$ was considered in [Bukhgeim and Uhlmann 2002; Kenig et al. 2007; Isakov 2007; Kenig and Salo 2013; Imanuvilov and Yamamoto 2013]. The two-dimensional case was solved in [Imanuvilov et al. 2010]. See [Kenig and Salo 2014] for a review. However, it is not known at the present whether one can uniquely determine the conductivity if one measures the DN map on an arbitrarily open subset of the boundary applied to functions supported in the same set. We refer to these types of measurements as the local DN map.

The map $\gamma \to \Lambda_\gamma$ is not linear. In this paper, we consider the linearization of the partial-data problem at a real-analytic conductivity for real-analytic Γ. We prove that the linearized map is injective. In fact, we prove a more general statement (see Theorem 1.6)

As in many works on Calderón's problem, one can reduce the problem to a similar one for the Schrödinger equation (see for instance [Uhlmann 2009]). This result uses that one can determine from the DN map the conductivity and the normal derivative of the conductivity. This result is only valid for the local DN map. One can then consider the more general problem of determining a potential from the corresponding DN map. The same is valid for the case of partial data and the linearization. It was shown in [Dos Santos Ferreira et al. 2009] that the linearization of the local DN map at the 0 potential is injective. We consider the linearization of the local DN map at any real-analytic potential assuming that the local DN map is measured on an open real-analytic set. We now describe more precisely our results in this setting.

Consider the Schrödinger operator $P = \Delta - V$ on the open set $\Omega \Subset \mathbb{R}^n$, where the boundary $\partial\Omega$ is smooth (and later assumed to be analytic in the most interesting region). Assume that 0 is not in the spectrum of the Dirichlet realization of P. Let G and K denote the corresponding Green and Poisson operators. Let $\gamma : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$ be the restriction operator and ν the exterior normal. If $x_0 \in \partial\Omega$, we can choose local coordinates $y = (y_1, \dots, y_n)$, centered at x_0 so that Ω is given by $y_n > 0$ and $\nu = -\partial_{y_n}$. If $\partial\Omega$ is analytic near x_0, we can choose the coordinates to be analytic.

The Dirichlet-to-Neumann (DN) operator is

$$\Lambda = \gamma \partial_\nu(x, \partial_x) K. \tag{1-4}$$

Consider a smooth deformation of smooth real-valued potentials

$$\begin{aligned} \operatorname{neigh}(0, \mathbb{R}) \ni t \mapsto P_t = \Delta - V_t, \\ V_t(x) = V(t, x) \in C^\infty(\operatorname{neigh}(0, \mathbb{R}) \times \overline{\Omega}; \mathbb{R}). \end{aligned} \tag{1-5}$$

Let G_t and K_t be the Green and Poisson kernels for P_t so that

$$\begin{pmatrix} P_t \\ \gamma \end{pmatrix} : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)$$

has the inverse

$$\begin{pmatrix} G_t & K_t \end{pmatrix}.$$

Then, denoting t-derivatives by dots,

$$\begin{pmatrix} \dot{G}_t & \dot{K}_t \end{pmatrix} = -\begin{pmatrix} G_t & K_t \end{pmatrix} \begin{pmatrix} \dot{P}_t \\ 0 \end{pmatrix} \begin{pmatrix} G_t & K_t \end{pmatrix} = -\begin{pmatrix} G_t \dot{P}_t G_t & G_t \dot{P}_t K_t \end{pmatrix};$$

that is,

$$\dot{G} = -G\dot{P}G, \qquad \dot{K} = -G\dot{P}K, \tag{1-6}$$

and consequently,

$$\dot{\Lambda} = -\gamma \partial_\nu G \dot{P} K. \tag{1-7}$$

Using the Green formula, we see that

$$\gamma \partial_\nu G = K^{\mathrm{t}}, \tag{1-8}$$

where K^{t} denotes the transposed operator.

In fact, write the Green formula,

$$\int_\Omega ((Pu_1)u_2 - u_1 Pu_2)\, dx = \int_{\partial\Omega} (\partial_\nu u_1 u_2 - u_1 \partial_\nu u_2) S(dx),$$

put $u_1 = Gv$ and $u_2 = Kw$ for $v \in C^\infty(\overline{\Omega})$ and $w \in C^\infty(\partial\Omega)$,

$$\int_\Omega vKw = \int_{\partial\Omega} (\gamma \partial_\nu G v) w S(dx),$$

and (1-8) follows.

Equation (1-7) becomes

$$\dot{\Lambda} = -K^{\mathrm{t}} \dot{P} K = K^{\mathrm{t}} \dot{V} K. \tag{1-9}$$

The linearized Calderón problem is: if $V_t = V + tq$, determine q from $\dot{\Lambda}_{t=0}$. The corresponding partial-data problem is to recover q or some information about q from local information about $\dot{\Lambda}_{t=0}$. From now on, we restrict the attention to $t = 0$. In this paper, we shall study the following linearized baby problem. Assume that V and $\partial\Omega$ are analytic near some point $x_0 \in \partial\Omega$. We also assume that V is smooth. If $\dot{\Lambda}$ (for $t = 0$) is an analytic pseudodifferential operator near x_0, can we conclude that q is analytic near x_0? Here,

$$\dot{\Lambda} = K^{\mathrm{t}} q K, \tag{1-10}$$

and we shall view the right-hand side as a Fourier integral operator acting on q.

Actually this problem is overdetermined in the sense that the symbol of a pseudodifferential operator on the boundary is a function of $2(n-1)$ variables while q is a function on n variables and $2(n-1) \geq n$ for $n \geq 2$ with equality precisely for $n = 2$. In order to have a nonoverdetermined problem, we shall only consider the symbol $\sigma_{\dot{\Lambda}}(y', \eta')$ of $\dot{\Lambda}$ along a half-ray in η'; i.e., we look at $\sigma_{\dot{\Lambda}}(y', t\eta'_0)$ for some fixed $\eta'_0 \neq 0$ and for some local coordinates as above. Assuming this restricted symbol to be a classical analytic symbol near $y' = 0$ and the potential $V = V_0$ to be analytic near $y = 0$ (i.e., near x_0), we shall show that q is real-analytic up to the boundary near x_0 (corresponding to $y = 0$).

In order to formulate the result more precisely, we first make some remarks about the analytic singular support of the Schwartz kernels of K and $K^{\mathrm{t}} q K$ and then we recall the notion of classical analytic pseudodifferential operators. Assume that $W \subset \mathbb{R}^n$ is an open neighborhood of $x_0 \in \partial\mathcal{O}$ and that

$$\partial\Omega \text{ and } V \text{ are analytic in } W. \tag{1-11}$$

For simplicity, we shall use the same symbol to denote operators and their Schwartz kernels. Then:

Lemma 1.1. *The Schwartz kernel $K(x, y')$ is analytic with respect to y', locally uniformly on the set*

$$\{(x, y') \in \overline{\Omega} \times (\partial\Omega \cap W) : x \neq y'\}.$$

Proof. Using (1-8), we can write $K(x, y') = \gamma\partial_\nu u(y')$, where $u = G(x, \cdot\,)$ solves the Dirichlet problem

$$(\Delta - V)u = \delta(\,\cdot - x), \quad \gamma u = 0,$$

and from analytic regularity for elliptic boundary-value problems, we get the lemma. (When $x \in \partial\Omega$, we view $G(x, y)$ away from $y = x$ as the limit of $G(x_j, y)$ when $\Omega \ni x_j \to x$.) $\square$

Lemma 1.2. *The Schwartz kernel $(K^{\mathrm{t}} q K)(x', y')$ is analytic on the set*

$$\{(x', y') \in (\partial\Omega \cap W)^2 : x' \neq y'\}. \tag{1-12}$$

Proof. Let (x'_0, y'_0) belong to the set (1-12). After decomposing q into a sum of two terms, we may assume that $x'_0 \notin \operatorname{supp}(q)$ or that $y'_0 \notin \operatorname{supp}(q)$. In the first case, it follows from Lemma 1.1 that $(K^{\mathrm{t}} q K)(x', y')$ is analytic in x' uniformly for (x', y') in a neighborhood of (x'_0, y'_0), and since the kernel is symmetric, we can exchange the roles of x' and y' and conclude that $(K^{\mathrm{t}} q K)(x', y')$ is analytic in y' uniformly for (x', y') in a neighborhood of (x'_0, y'_0). In the second case, we have the same conclusion about analyticity in x' and in y' separately. It then follows that $(KqK)(x', y')$ is analytic near (x'_0, y'_0) (by using the Fourier–Bros–Iagolnitzer (FBI) definition of the analytic wave-front set and which can also (most likely) be deduced from a classical result on logarithmic convexity of Reinhardt domains [Hörmander 1990, Theorem 2.4.6]). $\square$

Remark 1.3. By the same proof, $K^{\mathrm{t}}qK(x', y')$ is analytic near

$$\{(x', x') \in (\partial\Omega \cap W)^2 : (x', 0) \notin \operatorname{supp} q\}.$$

We next define the notion of a symbol up to exponentially small contributions. For that purpose, we assume that X is an analytic manifold and consider an operator

$$A : C_0^\infty(X) \to C^\infty(X) \tag{1-13}$$

that is also continuous

$$\mathcal{E}'(X) \to \mathcal{D}'(X). \tag{1-14}$$

Assume (as we have verified for $K^{\mathrm{t}}qK$ with n replaced by $n-1$ and with $X = \partial\Omega \cap W$) that the distribution kernel $A(x, y)$ is analytic away from the diagonal. After restricting to a local analytic coordinate chart, we may assume that $X \subset \mathbb{R}^n$ is an open set. The symbol of A is formally given on T^*X by

$$\sigma_A(x, \xi) = e^{-ix\cdot\xi} A(e^{i(\,\cdot\,)\cdot\xi}) = \int e^{-i(x-y)\cdot\xi} A(x, y)\, dy.$$

In the usual case of C^∞-theory, we give a meaning to this symbol up to $\mathcal{O}(\langle\xi\rangle^{-\infty})$ by introducing a cutoff $\chi(x, y) \in C^\infty(X \times X)$ that is properly supported and equal to 1 near the diagonal. In the analytic category, we would like to have an exponentially small indeterminacy, and the use of special cutoffs becoming more complicated, we prefer to make a contour deformation.

For x in a compact subset of X, let $r > 0$ be small enough and define for $\xi \neq 0$

$$\sigma_A(x, \xi) = \int_{x+\Gamma_{r,\xi}} e^{i(y-x)\cdot\xi} A(x, y)\, dy, \tag{1-15}$$

where

$$\Gamma_{r,\xi} : B(0, r) \ni t \mapsto t + i\chi\Big(\frac{t}{r}\Big) r \frac{\xi}{|\xi|} \in \mathbb{C}^n$$

and $\chi \in C^\infty(B(0, 1); [0, 1])$ is a radial function that vanishes on $B(0, \frac{1}{2})$ and is equal to 1 near $\partial B(0, 1)$. Thus, the contour $x + \Gamma_{r,\xi}$ coincides with $\mathbb{R}^n$ near $y = x$ and becomes complex for t close to the boundary of $B(0, r)$. Notice that along this contour

$$|e^{i(y-x)\cdot\xi}| = e^{-\chi(t/r)r|\xi|}$$

is bounded by 1 and for t close to $\partial B(0, r)$ it is exponentially decaying in $|\xi|$. Thus, from Stokes' formula, it is clear that $\sigma_A(x, \xi)$ will change only by an exponentially small term if we modify r. More generally, for (x, ξ) in a conic neighborhood of a fixed point $(x_0, \xi_0) \in X \times S^{n-1}$, we change $\sigma_A(x, \xi)$ only by an exponentially small term if we replace the contour in (1-15) by $x_0 + \Gamma_{r,\xi_0}$, and we then get a function that has a holomorphic extension to a conic neighborhood of (x_0, ξ_0) in $\mathbb{C}^n \times (\mathbb{C}^n \setminus \{0\})$.

Remark 1.4. Instead of using contour deformation to define σ_A, we can use an almost-analytic cutoff in the following way. Choose $C > 0$ so that

$$1 = \int Ch^{-n/2} e^{-(y-t)^2/2h}\, dt,$$

and put

$$e_t(y) = \tilde{\chi}(y-t)Ch^{-n/2}e^{-(y-t)^2/2h},$$

where $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 near 0 and has its support in a small neighborhood of that point. Then if $\hat{\chi}$ is another cutoff of the same type, we see by contour deformation that

$$\sigma_A(x,\xi) = e^{-ix\cdot\xi} A\left(\int \hat{\chi}(t) e_t e^{(\cdot)\cdot\xi}\right)$$

up to an exponentially decreasing term.

Definition 1.5. We say that A is a classical analytic pseudodifferential operator of order $m \in \mathbb{R}$ if σ_A is a classical analytic symbol (cl.a.s.) of order m on $X \times \mathbb{R}^n$ in the following sense.

There exist holomorphic functions $p_{m-j}(x,\xi)$ on a fixed complex conic neighborhood V of $X \times \dot{\mathbb{R}}^n$ such that

$$p_k(x,\xi) \text{ is positively homogeneous of degree } k \text{ in } \xi, \tag{1-16}$$

$$\text{for all } K \Subset V \cap \{(x,\xi) : |\xi| = 1\}, \text{ there exists } C = C_K \text{ such that } |p_{m-j}(x,\xi)| \le C^{j+1} j^j \text{ on } K, \tag{1-17}$$

$$\begin{gathered}\text{for all } K \Subset X \text{ and every } C_1 > 0 \text{ large enough, there exists } C_2 > 0 \\ \text{such that } \Bigl|\sigma_A(x,\xi) - \sum_{0 \le j \le |\xi|/C_1} p_{m-j}(x,\xi)\Bigr| \le C_2 e^{-|\xi|/C_2} \text{ with } (x,\xi) \in K \times \mathbb{R}^n \text{ and } |\xi| \ge 1.\end{gathered} \tag{1-18}$$

The formal sum $\sum_0^\infty p_{m-j}(x,\xi)$ is called a formal cl.a.s. when (1-16) and (1-17) hold. We define cl.a.s. and formal cl.a.s. on open conic subsets of $X \times \dot{\mathbb{R}}^n$ and on other similar sets by the obvious modifications of the above definitions. If $p(x,\xi)$ is a cl.a.s. on $X \times \mathbb{R}^n$ and if $\xi_0 \in \dot{\mathbb{R}}^n$, then

$$q(x,\tau) := p(x,\tau\xi_0)$$

is a cl.a.s. on $X \times \mathbb{R}_+$.

The main result of this work is:

Theorem 1.6. *Let $x_0 \in \partial\Omega$, and assume that $\partial\Omega$ and V are analytic near that point. Let $q \in L^\infty(\Omega)$. Choose local analytic coordinates $y' = (y_1, \dots, y_{n-1})$ on* neigh$(x_0, \partial\Omega)$, *centered at x_0, so that the symbol $\sigma_{\dot\Lambda}(y', \eta')$ becomes well defined up to an exponentially small term on* neigh$(0) \times \dot{\mathbb{R}}^{n-1}$. *Let $\eta'_0 \in \dot{\mathbb{R}}^{n-1}$.*

If $\sigma_{\dot\Lambda}(y', \tau\eta'_0)$ is a cl.a.s. on neigh$(0, \mathbb{R}^{n-1}) \times \mathbb{R}_+$, *then q is analytic up to the boundary in a neighborhood of x_0.*

We also have the converse statement.

We have a simpler direct result.

Proposition 1.7. *Let x_0, $\partial\Omega$, and V be as in Theorem 1.6, and choose analytic coordinates as done there. If $q \in L^\infty(\Omega)$ is analytic up to the boundary near x_0, then $\dot\Lambda$ is an analytic pseudodifferential operator near $y' = 0$.*

We get the following immediate consequence.

Corollary 1.8. *Under the conditions of the previous theorem, the map*

$$q \to \dot{\Lambda}$$

is injective.

This follows from the previous result since q must be analytic on W and, if the Taylor series of q vanishes on W, then $q = 0$ on the set where q is analytic.

Most of the paper will be devoted to the proof of Theorem 1.6, and in Section 7, we will prove Proposition 1.7.

2. Heuristics and some remarks about the Laplace transform

Let us first explain heuristically why some kind of Laplace transform will appear. Assume that $x_0 \in \partial\Omega$ and that V and $\partial\Omega$ are analytic near that point. Choose local analytic coordinates

$$y = (y_1, \dots, y_{n-1}, y_n) = (y', y_n)$$

centered at x_0 such that the set Ω coincides near x_0 (i.e., $y = 0$) with the half-space $\mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y_n > 0\}$. Assume also (for this heuristic discussion) that we know that $q(y) = q(y', y_n)$ is analytic in y' and that the original Laplace operator remains the standard Laplace operator also in the y coordinates. Then up to a smoothing operator, the Poisson operator is of the form

$$Ku(y) = \frac{1}{(2\pi)^{n-1}} \int e^{i(y'-w')\cdot\eta' - y_n|\eta'|} a(y, \eta') u(w')\, dw'\, d\eta',$$

where the symbol a is equal to 1 to leading order. We can view K, q, and K^{t} as pseudodifferential operators in y' with operator-valued symbols. K has the operator-valued symbol

$$K(y', \eta') : \mathbb{C} \ni z \mapsto z e^{-y_n|\eta'|} a(y, \eta') \in L^2([0, +\infty[_{y_n}). \tag{2-1}$$

The symbol of multiplication with q is independent of η' and equals multiplication with $q(y', \cdot)$. The symbol of K^{t} is

$$K^{\mathrm{t}}(y', \eta') : L^2([0, +\infty[_{y_n}) \ni f(y_n) \mapsto \int_0^\infty e^{-y_n|\eta'|} a(y, -\eta') f(y_n)\, dy_n \in \mathbb{C}. \tag{2-2}$$

For simplicity, we set $a = 1$ in the following discussion. To leading order, the symbol of $\dot{\Lambda}$ is

$$\sigma_{\dot{\Lambda}}(y', \eta') = \int_0^\infty e^{-2y_n|\eta'|} q(y', y_n)\, dy_n = (\mathcal{L}q(y', \cdot))(2|\eta'|), \tag{2-3}$$

where

$$\mathcal{L}f(\tau) = \int_0^\infty e^{-t\tau} f(t)\, dt$$

is the Laplace transform.

Now we fix $\eta_0' \in \dot{\mathbb{R}}^{n-1}$ and assume that $\sigma_{\dot{\Lambda}}(y', \tau\eta_0')$ is a cl.a.s. on $\mathrm{neigh}(0, \mathbb{R}^{n-1}) \times \mathbb{R}_+$:

$$\sigma_{\dot{\Lambda}}(y', \tau\eta_0') \sim \sum_1^\infty n_k(y', \tau), \tag{2-4}$$

where n_k is analytic in y' in a fixed complex neighborhood of 0, (positively) homogeneous of degree $-k$ in τ, and satisfies

$$|n_k(y', \tau)| \le C^{k+1} k^k |\tau|^{-k}. \tag{2-5}$$

More precisely for $C > 0$ large enough, there exists $\widetilde{C} > 0$ such that

$$\left| \sigma_{\dot{\Lambda}}(y', \tau\eta_0') - \sum_1^{[|\eta'|/C]} n_k(y', \tau) \right| \le \widetilde{C} \exp(-\tau/\widetilde{C}) \tag{2-6}$$

on the real domain.

From (2-3), we also have

$$\left| (\mathcal{L}q(y', \cdot))(2|\eta_0'|\tau) - \sum_1^{[|\eta'|/C]} n_k(y', \tau) \right| \le \exp(-\tau/\widetilde{C}) \tag{2-7}$$

for $y' \in \mathrm{neigh}(0, \mathbb{R}^{n-1})$ and $\tau \ge 1$. In this heuristic discussion, we assume that (2-7) extends to $y' \in \mathrm{neigh}(0, \mathbb{C}^{n-1})$. It then follows that $q(y', y_n)$ is analytic for y_n in a neighborhood of 0, from the following certainly classic result about Borel transforms.

Proposition 2.1. *Let $q \in L^\infty([0, 1])$, and assume that for some $C, \widetilde{C} > 0$*

$$\left| \mathcal{L}q(\tau) - \sum_0^{[\tau/C]} q_k \tau^{-(k+1)} \right| \le e^{-\tau/\widetilde{C}}, \quad \tau > 0, \tag{2-8}$$

$$|q_k| \le \widetilde{C}^{k+1} k^k. \tag{2-9}$$

Then q is analytic in a neighborhood of $t = 0$. The converse also holds.

Proof. We shall first show the converse statement, namely that, if q is analytic near $t = 0$, then (2-8) and (2-9) hold. We start by computing the Laplace transform of powers of t.

For $\tau > 0$, $a > 0$, and $k \in \mathbb{N}$,

$$\int_0^\infty e^{-t\tau} t^k \, dt = \frac{k!}{\tau^{k+1}}. \tag{2-10}$$

In fact, the integral to the left is equal to

$$(-\partial_\tau)^k \left(\int_0^\infty e^{-t\tau} dt \right) = (-\partial_\tau)^k \left(\frac{1}{\tau} \right).$$

Next, for $a > 0$, we look at

$$\frac{1}{k!} \int_0^a e^{-t\tau} t^k \, dt = \frac{1}{\tau^{k+1}} \left(1 - \frac{\tau^{k+1}}{k!} \int_a^\infty e^{-t\tau} t^k \, dt \right) = \frac{1}{\tau^{k+1}} \left(1 - \int_{a\tau}^\infty e^{-s} \frac{s^k}{k!} \, ds \right). \tag{2-11}$$

First let $\tau \in]0, \infty[$ be large. For $0 < \theta < 1$ to be optimally chosen, we write for $s \geq 0$

$$\frac{s^k}{k!}e^{-s} = \theta^{-k} \underbrace{\frac{(\theta s)^k}{k!}e^{-\theta s}}_{\leq 1} e^{-(1-\theta)s} \leq \theta^{-k}e^{-(1-\theta)s}.$$

Thus,

$$\int_{a\tau}^{\infty} e^{-s}\frac{s^k}{k!}\,ds = \theta^{-k}\int_{a\tau}^{\infty} e^{-(1-\theta)s}\,ds = \frac{\theta^{-k}e^{-(1-\theta)a\tau}}{1-\theta}. \tag{2-12}$$

We will estimate this for $k \leq a\tau/\mathcal{O}(1)$. Under the a priori assumption that $\theta \leq 1 - 1/\mathcal{O}(1)$, we look for θ that minimizes the numerator

$$\theta^{-k}e^{-(1-\theta)a\tau} = e^{-[(1-\theta)a\tau + k\ln\theta]}.$$

Setting the derivative of the exponent equal to 0, we are led to the choice $\theta = k/(a\tau)$. Assume that

$$\frac{k}{a\tau} \leq \theta_0 < 1. \tag{2-13}$$

Then,

$$(1-\theta)a\tau + k\ln\theta = a\tau\Big(1 - \frac{k}{a\tau} + \frac{k}{a\tau}\ln\frac{k}{a\tau}\Big) = a\tau\Big(1 - f\Big(\frac{k}{a\tau}\Big)\Big),$$

where

$$f(x) = x + x\ln\frac{1}{x}, \quad 0 \leq x \leq 1.$$

Clearly $f(0) = 0$ and $f(1) = 1$, and for $0 < x < 1$, we have $f'(x) = \ln(1/x) > 0$, so f is strictly increasing on [0, 1]. In view of (2-13),

$$(1-\theta)a\tau + k\ln\theta \geq a\tau(1 - f(\theta_0)),$$

and (2-12) gives

$$\int_{a\tau}^{\infty} e^{-s}\frac{s^k}{k!}\,ds \leq \frac{e^{-a\tau(1-f(\theta_0))}}{1-\theta_0}. \tag{2-14}$$

Using this in (2-11), we get

$$\frac{1}{k!}\int_0^a e^{-t\tau}t^k\,dt = \frac{1}{\tau^{k+1}}(1 + \mathcal{O}(1)e^{-a\tau/C(\theta_0)}) \quad \text{for } \frac{k}{a\tau} \leq \theta_0 < 1, \text{ where } C(\theta_0) > 0. \tag{2-15}$$

Now, assume that $q \in C([0, 1])$ is analytic near $t = 0$. Then for $t \in [0, 2a]$, $0 < a \ll 1$, we have

$$q(t) = \sum_0^{\infty} \frac{q^{(k)}(0)}{k!}t^k,$$

where

$$\frac{|q^{(k)}(0)|}{k!} \leq \widetilde{C}\frac{1}{(2a)^k}, \tag{2-16}$$

so

$$\Big|q(t) - \sum_0^{[\tau/C]} \frac{q^{(k)}(0)}{k!}t^k\Big| \leq \widetilde{C}e^{-\tau/\widetilde{C}}, \quad 0 \leq t \leq a.$$

Hence,

$$\mathcal{L}q=\sum_{0}^{[\tau/C]}\frac{q^{(k)}(0)}{\tau^{k+1}}+\mathcal{O}(e^{-\tau/\widetilde{C}})+\underbrace{\mathcal{L}(1_{[a,1]}q)(\tau)}_{=\mathcal{O}(e^{-\tau/\widetilde{C}})}$$

and we obtain (2-8) with $q_k=q^{(k)}(0)$ while (2-9) follows from (2-16).

We now prove the direct statement in the proposition, so we take $q\in L^\infty([0,1])$ satisfying (2-8) and (2-9). For $a>0$ small, put

$$\tilde{q}(t)=q(t)-1_{[0,a]}(t)\sum_{0}^{\infty}\frac{q_k}{k!}t^k.$$

The proof of the converse part shows that

$$|\mathcal{L}\tilde{q}(\tau)|\le e^{-\tau/\widetilde{C}}, \tag{2-17}$$

where $\widetilde{C}$ is a new positive constant, and it suffices to show that

$$\tilde{q} \text{ vanishes in a neighborhood of } 0. \tag{2-18}$$

We notice that $\mathcal{L}\tilde{q}$ is a bounded holomorphic function in the right half-plane. We can therefore apply the Phragmén–Lindelöf theorem in each sector $\arg\tau\in[0,\frac{\pi}{2}]$ and $\arg\tau\in[-\frac{\pi}{2},0]$ to the holomorphic function

$$e^{\tau/\widetilde{C}}\mathcal{L}\tilde{q}(\tau)$$

and conclude that this function is bounded in the right half-plane:

$$|\mathcal{L}\tilde{q}(\tau)|\le\mathcal{O}(1)e^{-\Re\tau/\widetilde{C}},\quad \Re\tau\ge 0. \tag{2-19}$$

Now, $\mathcal{L}\tilde{q}(i\sigma)=\mathcal{F}\tilde{q}(\sigma)$, where $\mathcal{F}$ denotes the Fourier transform, and the Paley–Wiener theorem allows us to conclude that $\operatorname{supp}\tilde{q}\subset[1/\widetilde{C},1]$. □

3. The Fourier integral operator $q\mapsto\sigma_{\dot\Lambda}$

Assume that $\partial\Omega$ and V are analytic near the boundary point x_0. Let $y'=(y_1,\dots,y_{n-1})$ be local analytic coordinates on $\partial\Omega$, centered at x_0. Then we can extend y' to analytic coordinates $y=(y_1,\dots,y_{n-1},y_n)=(y',y_n)$ in a full neighborhood of x_0, where y' is an extension of the given coordinates on the boundary and such that Ω is given (near x_0) by $y_n>0$ and

$$-P=D_{y_n}^2+R(y,D_{y'}), \tag{3-1}$$

where R is a second-order elliptic differential operator in y' with positive principal symbol $r(y,\eta')$. (Here we neglect a contribution $f(y)\partial_{y_n}$, which can be eliminated by conjugation.) Then there is a neighborhood $W\subset\mathbb{R}^n$ of $y=0$ and a cl.a.s. $a(y,\xi')$ on $W\times\mathbb{R}^{n-1}$ of order 0 such that

$$Ku(y)=\frac{1}{(2\pi)^{n-1}}\iint e^{i(\phi(y,\xi')-\tilde{y}'\cdot\xi')}a(y,\xi')u(\tilde{y}')\,d\tilde{y}'\,d\xi'+K_au(y) \tag{3-2}$$

for $y \in W$ and $u \in C_0^\infty(W \cap \partial\Omega)$. The distribution kernel of K_a is analytic on $W \times (W \cap \partial\Omega)$, and we choose a realization of a that is analytic in y. Here ϕ is the solution of the Hamilton–Jacobi problem

$$\begin{gathered}(\partial_{y_n}\phi)^2 + r(y, \phi'_{y'}) = 0, \quad \Im \partial_{y_n}\phi > 0,\\ \phi(y', 0, \xi') = y' \cdot \xi'.\end{gathered} \tag{3-3}$$

This means that we choose ϕ to be the solution of

$$\partial_{y_n}\phi - ir(y, \phi'_{y'})^{1/2} = 0 \tag{3-4}$$

with the natural branch of $r^{1/2}$ with a cut along the real negative axis.

To see this, recall (by the analytic Wentzel–Kramers–Brillouin (WKB) method [Sjöstrand 1982, Chapter 9]) that we can construct the first term $K_{\mathrm{fop}}u$ in the right-hand side of (3-2) such that PK_{fop} has analytic distribution kernel and $\gamma K_{\mathrm{fop}} = 1$. It then follows from local analytic regularity in elliptic boundary-value problems that the remainder operator K_a has analytic distribution kernel.

We notice that

$$K(e^{ix'\cdot\xi'}) = e^{i\phi(y,\xi')}a(y, \xi') + \mathcal{O}(e^{-|\xi'|/C}) \tag{3-5}$$

since the first term to the right solves the problem

$$Pu = 0, \quad u|_{y_n=0} = e^{iy'\cdot\xi'},$$

with an exponentially small error in the first equation. K is a real operator, so $K(e^{ix'\cdot(-\xi')}) = \overline{K(e^{ix'\cdot\xi'})}$. It follows that

$$\phi(y, -\xi') = -\overline{\phi(y, \xi')}, \qquad a(y, -\xi') = \overline{a(y, \xi')} \tag{3-6}$$

without any error in the last equation when viewing a as a formal cl.a.s. Notice also that, since K is real, $K^{\mathrm{t}} = K^*$.

We shall now view $\dot{\Lambda} = K^{\mathrm{t}}qK = K^*qK$ as a pseudodifferential operator in the classical quantization. In this section, we proceed formally in order to study the associated geometry. A more efficient analytic description will be given later for the left composition with an FBI transform in x'. The symbol becomes

$$\sigma_{\dot{\Lambda}}(x', \xi') = e^{-ix'\cdot\xi'}\dot{\mathcal{N}}(e^{i(\cdot)\cdot\xi'}) = (2\pi)^{1-n}\iint e^{i(x'\cdot(\eta'-\xi')-\phi^*(y,\eta')+\phi(y,\xi'))}a^*(y, \eta')a(y, \xi')q(y)\,dy\,d\eta',$$

where in general we write $f^*(z) = \overline{f(\bar{z})}$ for the holomorphic extension of the complex conjugate of a function f.

Actually, rather than letting ξ' tend to ∞, we replace ξ' with ξ'/h, where the new ξ' is of length $\asymp 1$ and $h \to 0$. This amounts to viewing $\dot{\mathcal{N}}$ as a semiclassical pseudodifferential operator with semiclassical symbol $\sigma_{\dot{\Lambda}}(x', \xi'; h) = \sigma_{\dot{\Lambda}}(x', \xi'/h)$. Thus,

$$\begin{aligned}\sigma_{\dot{\Lambda}}(x', \xi'; h) &= e^{-ix'\cdot\xi'/h}\dot{\mathcal{N}}(e^{i(\cdot)\cdot\xi'/h})\\ &= (2\pi h)^{1-n}\iint e^{(i/h)(x'\cdot(\eta'-\xi')-\phi^*(y,\eta')+\phi(y,\xi'))}a^*(y, \eta'; h)a(y, \xi'; h)q(y)\,dy\,d\eta',\end{aligned}$$

where $a(y, \xi'; h) = a(y, \xi'/h)$ and similarly for a^*.

We have

$$\phi(y,\xi')=y'\cdot\xi'+\psi(y,\xi'),\qquad \phi^*(y,\eta')=y'\cdot\eta'+\psi^*(y,\eta'),\tag{3-7}$$

where

$$\Im\psi,\Im\psi^*\asymp y_n,\qquad \Re\psi,\Re\psi^*=\mathcal{O}(y_n^2)\tag{3-8}$$

uniformly on every compact set that does not intersect the zero section. Equation (3-6) tells us that $\Re\psi$ is odd and $\Im\psi$ is even with respect to the fiber variables ξ' (and also positively homogeneous of degree 1 of course). Using (3-7) in the formula for the symbol of $\dot\Lambda$, we get

$$\begin{aligned}\sigma_{\dot\Lambda}(x',\xi';h)&=(2\pi h)^{1-n}\iint e^{(i/h)\Phi_M(x',\xi',y,\eta')}a^*(y,\eta';h)a(y,\xi';h)q(y)\,dy\,d\eta'\\&=:Mq(x',\xi';h),\end{aligned}\tag{3-9}$$

where

$$\Phi_M(x',\xi',y,\eta')=(x'-y')\cdot(\eta'-\xi')+\psi(y,\xi')-\psi^*(y,\eta')\tag{3-10}$$

and η' are the fiber variables. We shall see that this is a nondegenerate phase function in the sense of Hörmander [1971] except for the fact that Φ_M is not homogeneous in η' alone, so $q\mapsto M_hq(x',\xi'):=Mq(x',\xi';h)$ is a semiclassical Fourier integral operator, at least formally.

We fix a vector $\xi_0'\in\dot{\mathbb{R}}^{n-1}$ and consider Φ_M in a neighborhood of $(x',y,\xi',\eta')=(0,0,\xi_0',\xi_0')\in\mathbb{C}^{4(n-1)+1}=\mathbb{C}^{4n-3}$. The critical set C_{Φ_M} of the phase Φ_M is given by $\partial_{\eta'}\Phi_M=0$, which means that $x'-y'-\partial_{\eta'}\psi^*(y,\eta')=0$ or equivalently

$$x'=y'+\partial_{\eta'}\psi^*(y,\eta').\tag{3-11}$$

This is a smooth submanifold of codimension $n-1$ in $\mathbb{C}^{4n-3}$ that is parametrized by $(y,\eta',\xi')\in\mathrm{neigh}((0,\xi_0',\xi_0'),\mathbb{C}^{3n-2})$. We also see that Φ_M is a nondegenerate phase function in the sense that $d\partial_{\eta_1'}\Phi_M,\dots,d\partial_{\eta_{n-1}'}\Phi_M$ are linearly independent on C_{Φ_M}. Using the above parametrization, we express the graph of the corresponding canonical relation $\kappa:\mathbb{C}^{2n}_{y,y^*}\to\mathbb{C}^{4(n-1)}_{x',\xi',x'^*,\xi'^*}$ (where we notice that $4(n-1)\ge 2n$ with equality for $n=2$ and strict inequality for $n\ge 3$):

$$\begin{aligned}\mathrm{graph}(\kappa)&=\{(x',\xi',\partial_{x'}\Phi_M,\partial_{\xi'}\Phi_M;\,y,-\partial_y\Phi_M):(x',\xi',y,\eta')\in C_{\Phi_M}\}\\&=\big\{\big(y'+\partial_{\eta'}\psi^*(y,\eta'),\xi',\eta'-\xi',\partial_{\xi'}\psi(y,\xi')-\partial_{\eta'}\psi^*(y,\eta');\\&\qquad y,-\partial_{y'}\psi(y,\xi')+\partial_{y'}\psi^*(y,\eta')+\eta'-\xi',-\partial_{y_n}\psi(y,\xi')+\partial_{y_n}\psi^*(y,\eta')\big)\big\}.\end{aligned}\tag{3-12}$$

The restriction to $y_n=0$ of this graph is the set of points

$$(y',\xi',\eta'-\xi',0;\,y',0,\eta'-\xi',-\partial_{y_n}\psi(y',0,\xi')+\partial_{y_n}\psi^*(y',0,\eta')).\tag{3-13}$$

It contains the point

$$(0,\xi_0',0,0;\,0,0,-2\partial_{y_n}\psi(0,\xi_0'))=(0,\xi_0',0,0;\,0,0,-2ir(0,\xi_0')^{1/2}).\tag{3-14}$$

The tangent space at a point where $y_n = 0$ is given by

$$\begin{aligned}\bigl\{\bigl(\delta_{y'} + \psi^{*\prime\prime}_{\eta',y_n}\delta_{y_n}, \delta_{\xi'}, \delta_{\eta'} - \delta_{\xi'}, (\psi''_{\xi',y_n}(y,\xi') - \psi^{*\prime\prime}_{\eta',y_n}(y,\eta'))\delta_{y_n};\\ \delta_y, (-\psi''_{y',y_n}(y,\xi') + \psi^{*\prime\prime}_{y',y_n}(y,\eta'))\delta_{y_n} + \delta_{\eta'} - \delta_{\xi'},\\ (-\psi''_{y_n,y}(y,\xi') + \psi^{*\prime\prime}_{y_n,y}(y,\eta'))\delta_y + (-\psi''_{y_n,\xi'}\delta_{\xi'} + \psi^{*\prime\prime}_{y_n,\eta'}\delta_{\eta'})\bigr)\bigr\}.\end{aligned} \tag{3-15}$$

From (3-15), we see that, at every point of graph(κ) with $y_n = 0$ and with $\eta' \approx \xi'$,

(1) the projection graph$(\kappa) \to \mathbb{C}^{2n}_{y,y^*}$ has surjective differential and

(2) the projection graph$(\kappa) \to \mathbb{C}^{4(n-1)}_{x',\xi',x'^*,\xi'^*}$ has injective differential.

In fact, since κ is a canonical relation, (1) and (2) are pointwise equivalent, so it suffices to verify (2). In other words, we have to show that, if

$$\begin{aligned}0 &= \delta_{y'} + \psi^{*\prime\prime}_{\eta',y_n}\delta_{y_n},\\ 0 &= \delta_{\xi'},\\ 0 &= \delta_{\eta'} - \delta_{\xi'},\\ 0 &= (\psi''_{\xi',y_n}(y,\xi') - \psi^{*\prime\prime}_{\eta',y_n}(y,\eta'))\delta_{y_n},\end{aligned} \tag{3-16}$$

then $\delta_{y'} = 0$, $\delta_{y_n} = 0$, $\delta_{\xi'} = 0$, and $\delta_{\eta'} = 0$.

When $y_n = 0$, we have $\psi^* = -\psi$, and when in addition $\eta' \approx \xi'$, we see that the $(n-1) \times 1$ matrix in the fourth equation is nonvanishing, so this equation implies that $\delta_{y_n} = 0$. Then the first equation gives $\delta_{y'} = 0$, and from the second and third equations, we get $\delta_{\xi'} = 0$ and $\delta_{\eta'} = 0$ and we have verified (2).

As an exercise, let us determine the image under κ of the complexified conormal bundle of the boundary, given by $y_n = 0$ and $y^{*\prime} = 0$. From (3-13), we see that this image is the set of all points

$$(x', \xi', 0, 0). \tag{3-17}$$

The subset of real points in (3-17) is the image of the set of points $(y', 0, 0, y_n^*)$ such that y' is real and $y_n^* \in -i\mathbb{R}_+$.

Now restrict (x', ξ') to the set of $(x', t\eta_0')$ with $x' \in \mathbb{C}^{n-1}$ and $t \in \mathbb{C}$, where $0 \neq \eta_0' \in \mathbb{R}^{n-1}$. This means that we restrict the symbol of $\dot{\mathcal{N}}$ to the radial direction $\xi' \in \mathbb{C}\eta_0'$ and consider

$$\begin{aligned}\sigma_{\dot\Lambda}(x', t\eta_0'; h) &= Mq(x', t\eta_0'; h) =: M_{\mathrm{new}}q(x', t; h)\\ &= (2\pi h)^{1-n} \iint e^{i\Phi_{M_{\mathrm{new}}}(x',t,y,\eta')/h} a^*(y,\eta';h) a(y,\xi';h) q(y)\, dy\, d\eta',\end{aligned} \tag{3-18}$$

where

$$\Phi_{M_{\mathrm{new}}}(x', t, y; \eta') = \Phi_M(x', t\eta_0', y; \eta') = \psi(y, t\eta_0') - \psi^*(y, \eta') + (x' - y') \cdot (\eta' - t\eta_0'). \tag{3-19}$$

We will soon drop the subscripts "new" when no confusion is possible. This is again a nondegenerate phase function. The new canonical relation $\kappa_{\mathrm{new}} : \mathbb{C}^{2n}_{y,y^*} \to \mathbb{C}^{2n}_{x',t,x'^*,t^*}$ has the graph

$$\begin{aligned}\operatorname{graph}(\kappa_{\mathrm{new}}) = \big\{ \big(& y' + \partial_{\eta'}\psi^*(y,\eta'), t, \eta' - t\eta_0', \eta_0' \cdot \partial_{\xi'}\psi(y, t\eta_0') - \eta_0' \cdot \partial_{\xi'}\psi^*(y,\eta'); \\ & y, -\partial_{y'}\psi(y,t\eta_0') + \partial_{y'}\psi^*(y,\eta') + \eta' - t\eta_0', -\partial_{y_n}\psi(y,t\eta_0') + \partial_{y_n}\psi^*(y,\eta')\big)\big\}. \end{aligned} \tag{3-20}$$

This graph is conic with respect to the dilations

$$\mathbb{R}_+ \ni \lambda \mapsto (x', \lambda t, \lambda x'^*, t^*; y, \lambda y^*).$$

The restriction of the graph to $y_n = 0$ is

$$\{(y', t, \eta' - t\eta_0', 0; y', 0, \eta' - t\eta_0', -\partial_{y_n}\psi(y', 0, t\eta_0') + \partial_{y_n}\psi^*(y', 0, \eta'))\},$$

where

$$\partial_{y_n}\psi(y', 0, \xi') = ir(y', 0, \xi')^{1/2}, \qquad \partial_{y_n}\psi^*(y', 0, \xi') = -ir(y', 0, \xi')^{1/2},$$

so the restriction is

$$\{(y', t, \eta' - t\eta_0', 0; y', 0, \eta' - t\eta_0', -i(r^{1/2}(y', 0, t\eta_0') + r^{1/2}(y', 0, \eta')))\}. \tag{3-21}$$

If we take $\eta = t\eta_0'$ and use that $r^{1/2}$ is homogeneous of degree 1 in the fiber variables, we get

$$\{(y', t, 0, 0; y', 0, 0, -2itr^{1/2}(y', 0, \eta_0'))\}. \tag{3-22}$$

This is the graph of a diffeomorphism

$$\operatorname{neigh}(0, \partial\Omega) \times (-i\mathbb{R}^+_{y_n^*}) \to \operatorname{neigh}(0; \partial\Omega) \times \mathbb{R}^+_t.$$

The tangent space at a point where $y_n = 0$ is given by

$$\begin{aligned}\big\{\big(& \delta_{y'} + (\psi^*)''_{\eta',y_n}\delta_{y_n}, \delta_t, \delta_{\eta'} - \delta_t\eta_0', \eta_0' \cdot (\psi''_{\xi',y_n} - (\psi^*)''_{\eta',y_n})\delta_{y_n}; \\ & \delta_y, (-\psi''_{y',y_n} + (\psi^*)''_{y',y_n})\delta_{y_n} + \delta_{\eta'} - \delta_t\eta_0', (-\psi''_{y_n,y} + (\psi^*)''_{y_n,y})\delta_y - \psi''_{y_n,\xi'}\delta_t\eta_0' + (\psi^*)''_{y_n,\eta'}\delta_{\eta'}\big)\big\}. \end{aligned} \tag{3-23}$$

The projection onto the first component is injective as can be seen exactly as in the proof of the property (2) stated after (3-15). Now κ_{new} is a canonical relation between spaces of the same dimension, so we conclude that κ_{new} is a canonical transformation or more precisely near each point of its graph. Combining this with the observation right after (3-22), we get:

Proposition 3.1. *Equation* (3-20) *is the graph of a bijective canonical transformation*

$$\kappa_{\mathrm{new}} : \operatorname{neigh}((0; 0, -i), \mathbb{C}^n_y \times \mathbb{C}^n_{y^*}) \to \operatorname{neigh}((0, 1; 0), \mathbb{C}^n_{x',t} \times \mathbb{C}^n_{x'^*,t^*}).$$

The neighborhoods can be taken to be conic with respect to the actions $\mathbb{R}_+ \ni \lambda \mapsto (y, \lambda y^*)$ and $\mathbb{R}_+ \ni \lambda \mapsto (x, \lambda t, \lambda x'^*, t^*)$, and κ_{new} intertwines the two actions (so κ_{new} is positively homogeneous of degree 1 *with y^* as the fiber variables on the departure side and with t and x'^* as the fiber variables on the arrival side).*

Basically, the same exercise as the one leading to (3-17) shows that the image under κ_{new} of the complexified conormal bundle, given by $y_n = 0$ and $(y^*)' = 0$, is the zero section

$$\{(x', t : (x'^*, t^*) = 0)\}. \tag{3-24}$$

Consider the image of $T^*\partial\Omega \times i\mathbb{R}^-_{y_n^*} = \{(y, y^*) : y', (y^*)' \in \mathbb{R}^{n-1},\ y_n = 0,\ y_n^* \in i\mathbb{R}^-\}$ under κ_{new}. On that image,

$$x' = y' \in \mathbb{R}^{n-1},$$
$$\eta' - t\eta_0' \in \mathbb{R}^{n-1},$$
$$t^* = \eta_0' \cdot \partial_{\xi'}\psi(y, t\eta_0') - \eta_0' \cdot \partial_{\xi'}\psi^*(y, \eta') = 0.$$

If we restrict the attention to $t \in \mathbb{R}_+$ so that $\eta' = (y^*)' + t\eta_0' \in \mathbb{R}^{n-1}$, we see that

$$y_n^* = -\partial_{y_n}\psi(y', 0, t\eta_0') + \partial_{y_n}\psi(y', 0, \eta') \in i\mathbb{R}^-.$$

Thus, the image contains locally

$$\{(x', t, (x^*)', 0) : x', (x^*)' \in \mathbb{R}^{n-1},\ t \in \mathbb{R}^+\},$$

which has the right dimension $2(n-1)+1$

Similarly, the image of $T^*\partial\Omega \times \text{neigh}(i\mathbb{R}^-_{y_n^*}, \mathbb{C}_{y_n^*})$ is obtained by dropping the reality condition on t but keeping that on $\eta' - t\eta_0'$, and we get

$$\kappa_{\text{new}}(T^*\partial\Omega \times \text{neigh}(i\mathbb{R}^-_{y_n^*}, \mathbb{C}_{y_n^*})) = \{(x', t, x'^*, 0) : x', (x^*)' \in \mathbb{R}^{n-1},\ t \in \text{neigh}(\mathbb{R}^+, \mathbb{C})\}. \tag{3-25}$$

4. Some function spaces and their FBI transforms

We continue to work locally near a point x_0 where the boundary is analytic, and we use analytic coordinates y centered at x_0 as specified in the beginning of Section 3.

We start by defining some piecewise-smooth I-Lagrangian manifolds, some of which will be associated with function spaces below.

- The cotangent space $T^*\Omega$ that we identify with $(\text{neigh}(0) \cap \mathbb{R}^n_+) \times \mathbb{R}^n$.
- The real conormal bundle $N^*\partial\Omega \subset T^*\mathbb{R}^n$. In the local coordinates y,

$$N^*\partial\Omega = \{(y, \eta) \in \mathbb{R}^{2n} : y_n = 0,\ \eta' = 0\}.$$

  It will sometimes be convenient to write $N^*\partial\Omega = \partial\Omega \times \mathbb{R}^*$, where of course the second expression appeals to the use of special coordinates as above. More invariantly, $N^*\partial\Omega$ is the inverse image of the zero-section in $T^*\partial\Omega$ for the natural projection map $\pi_{T^*\partial\Omega} : T^*_{\partial\Omega}\mathbb{R}^n \to T^*\partial\Omega$.

We will also need some complex sets.

- The complexified zero-section in the complexification $\widetilde{T^*\mathbb{R}^n} = \mathbb{C}^n_y \times \mathbb{C}^n_\eta$ defined to be

$$\text{neigh}(\bar{0}, \mathbb{C}^n) \times \{\eta = 0\} \subset \mathbb{C}^n_y \times \mathbb{C}^n_\eta.$$

  We denote it by $\mathbb{C}^n_y \times 0_\eta$ for short.

- The complexification $\widetilde{N^*\partial\Omega}$ of $N^*\partial\Omega$ defined to be

$$\{(y,\eta)\in\mathbb{C}^n_y\times\mathbb{C}^n_\eta : y\in \mathrm{neigh}(0,\mathbb{C}^n),\ \ y_n=0,\ \ \eta'=0\}.$$

- The space $\pi^{-1}(T^*\partial\Omega)$, where $\pi : T^*_{\partial\Omega}\mathbb{R}^n\otimes\mathbb{C}\to T^*\partial\Omega\otimes\mathbb{C}$ is the natural projection and $\otimes\mathbb{C}$ indicates fiberwise complexification. In special coordinates, it is $\{(y,\eta) : (y',\eta')\in\mathbb{R}^{2(n-1)},\ \ y_n=0,\ \ \eta_n\in\mathbb{C}\}$. We will denote it by $T^*\partial\Omega\times\mathbb{C}$ or $T^*\partial\Omega\times\mathbb{C}_{\eta_n}$ for simplicity. It contains the subset $T^*\partial\Omega\times\mathbb{C}^-_{\eta_n}$ (easy to define invariantly), where $\mathbb{C}^-$ is the open lower half-plane. Notice that

$$T^*\partial\Omega\times\partial\mathbb{C}_-=T^*\partial\Omega\times\mathbb{R}=T^*_{\partial\Omega}\mathbb{R}^n.$$

- The piecewise-smooth (Lipschitz) manifold

$$F=\overline{T^*\Omega}\cup(T^*\partial\Omega\times\mathbb{C}^-_{\eta_n}).$$

  Notice that the two components to the right have $T^*_{\partial\Omega}\mathbb{R}^n$ as their common boundary.

- The piecewise-smooth (Lipschitz) manifold $(\mathbb{C}^n_y\times 0_\eta)\cup\widetilde{N^*\partial\Omega}$, where the two constituents contain $\widetilde{\partial\Omega}\times 0_\eta$. Here $\widetilde{\partial\Omega}$ denotes a complexification of the boundary (near x_0).

Let

$$Tu(z;h)=Ch^{-3n/4}\int_{\mathbb{R}^n}e^{(i/h)\phi(z,y)}u(y)\,dy,\quad z\in\mathbb{C}^n, \tag{4-1}$$

be a standard FBI transform [Sjöstrand 1982], sending distributions with compact support on $\mathbb{R}^n$ to holomorphic functions on (in general some subdomains of) $\mathbb{C}^n$. For simplicity, we let ϕ be a holomorphic quadratic form so that T can also be viewed as a generalized Bargmann transform and a metaplectic Fourier integral operator (see for instance [Sjöstrand 1990]). We work under the standard assumptions

$$\Im\phi''_{y,y}>0,\qquad \det\phi''_{z,y}\neq 0. \tag{4-2}$$

We let $C>0$ be the unique positive constant for which $T : L^2(\mathbb{R}^2)\to H_{\Phi_0}(\mathbb{C}^n)$ is unitary, where

$$\Phi_0(z)=\sup_{y\in\mathbb{R}^n}-\Im\phi(z,y)=-\Im\phi(z,y(z)) \tag{4-3}$$

is a strictly plurisubharmonic (real) quadratic form on $\mathbb{C}^n$ and H_{Φ_0} is the complex Hilbert space $\mathrm{Hol}(\mathbb{C}^n)\cap L^2(e^{-2\Phi_0/h}L(dz))$ with $L(dz)$ denoting the Lebesgue measure on $\mathbb{C}^n\simeq\mathbb{R}^{2n}$. Let

$$\kappa_T : \mathbb{C}^{2n}\ni(y,-\phi'_y(z,y))\mapsto(z,\phi'_z(z,y))\in\mathbb{C}^{2n} \tag{4-4}$$

be the complex (linear) canonical transformation associated to T, and let

$$\Lambda_{\Phi_0}=\left\{\left(z,\frac{2}{i}\frac{\partial\Phi_0}{\partial z}(z)\right) : z\in\mathbb{C}^n\right\}$$

be the R-symplectic[1] and I-Lagrangian[2] manifold of $\mathbb{C}^{2n}$, actually a real-linear subspace since ϕ is quadratic. Then we know that

$$\Lambda_{\Phi_0} = \kappa_T(\mathbb{R}^{2n}). \tag{4-5}$$

More explicitly,

$$\kappa_T^{-1}\left(z, \frac{2}{i}\frac{\partial \Phi_0}{\partial z}\right) = (y(z), \eta(z)) \in \mathbb{R}^{2n}, \tag{4-6}$$

where $y(z)$ appeared in (4-3).

Let

$$\Phi_1^{\mathrm{ext}}(z) = \sup_{y \in \partial\mathbb{R}^n_+} -\Im\phi(z, y) = -\Im\phi(z, \tilde{y}(z)), \tag{4-7}$$

where $\tilde{y}(z) = (\tilde{y}'(z), 0)$ and $\tilde{y}'(z)$ is the unique point of maximum in $\mathbb{R}^{n-1}$ of $y' \mapsto -\Im\phi(z, y', 0)$. If $\operatorname{supp} u \subset \{y \in \mathbb{R}^n : y_n \geq 0\}$, then $Tu \in H^{\mathrm{loc}}_{\Phi_1}$, where

$$\Phi_1(z) = \sup_{y \in \mathbb{R}^n_+} -\Im\phi(z, y) = \begin{cases} \Phi_0(z) & \text{if } y_n(z) \geq 0, \\ \Phi_1^{\mathrm{ext}}(z) & \text{if } y_n(z) \leq 0. \end{cases} \tag{4-8}$$

Notice that

- $-\Im\partial_{y_n}\phi(z, \tilde{y}(z)) \geq 0$ in the first case and
- $-\Im\partial_{y_n}\phi(z, \tilde{y}(z)) \leq 0$ in the second case.

Notice that

$$\frac{2}{i}\frac{\partial \Phi_1}{\partial z}(z) = \frac{2}{i}\left(\frac{\partial}{\partial z}(-\Im\phi)\right)(z, \tilde{y}(z)) = \phi'_z(z, \tilde{y}(z))$$

and $\tilde{\eta}(z) = -\phi'_y(z, \tilde{y}(z))$ satisfies $\tilde{\eta}'(z) \in \mathbb{R}^{n-1}$. When $\Phi_1(z) = \Phi_1^{\mathrm{ext}}(z)$,

$$\tilde{\eta}'(z) \in \mathbb{R}^{n-1}, \qquad \Im\tilde{\eta}_n(z) \leq 0. \tag{4-9}$$

This means that

$$\Lambda_{\Phi_1^{\mathrm{ext}}} = \kappa_T(T^*\partial\Omega \times \mathbb{C}^*_{\eta_n})$$

and that

$$\Lambda_{\Phi_1} = \kappa_T(F), \tag{4-10}$$

where F was defined above:

$$F = \overline{T^*(\Omega)} \cup \{(y', 0; \eta', \eta_n) : (y', \eta') \in T^*\partial\Omega,\ \Im\eta_n \leq 0\}. \tag{4-11}$$

It is a Lipschitz manifold. The second component is a union of complex half-lines; consequently in the region where $\Phi_1 < \Phi_0$, Λ_{Φ_1} is a union of complex half-lines. If we project these lines to the complex z-space, we get a foliation of $\mathbb{C}^n_z$ into complex half-lines and the restriction of Φ_1 to each of these is harmonic.

[1] i.e., symplectic with respect to $\Re\sigma$, where $\sigma = d\zeta \wedge dz$ is the complex symplectic form

[2] i.e., Lagrangian with respect to $\Im\sigma$

We introduce the real hyperplane

$$H = \pi_z \kappa_T (T^*_{\partial\Omega}\mathbb{R}^n),$$

which is the common boundary of the two half-spaces

$$\begin{aligned} H_+ &= \pi_z \kappa_T (T^*\Omega), \\ H_- &= \pi_z \kappa_T (\{(y', 0; \eta) : (y', \eta') \in T^*\partial\Omega,\ \Im \eta_n < 0\}). \end{aligned}$$

Here, $\pi_z : \mathbb{C}^n_z \times \mathbb{C}^n_\zeta \to \mathbb{C}^n$ is the natural projection. We have

$$\Phi_0 - \Phi_1 \begin{cases} = 0 & \text{in } H_+, \\ \asymp \operatorname{dist}(z, H)^2 & \text{in } H_-. \end{cases} \tag{4-12}$$

Similarly, recall the definition of the complexified normal bundle $\widetilde{N^*\partial\Omega}$ at the beginning of this section. It is a $\mathbb{C}$-Lagrangian manifold.[3] We have $\kappa_T(\widetilde{N^*\partial\Omega}) = \Lambda_{\Phi_3}$, where Φ_3 is pluriharmonic:

$$\Phi_3(z) = \operatorname{vc}_{y' \in \mathbb{C}^{n-1}} (-\Im \phi(z, y', 0)).$$

Similarly $\kappa_T(\mathbb{C}^n_y \times 0_\eta)$ (with the notation from the beginning of this section) is of the form Λ_{Φ_4}, where

$$\Phi_4(z) = \operatorname{vc}_{y \in \mathbb{C}^n} (-\Im \phi(z, y)).$$

The complex zero-section $\mathbb{C}_y \times 0_\eta$ and $T^*\mathbb{R}^n$ intersect transversally along the real zero-section $\mathbb{R}^n_y \times 0_\eta$. Correspondingly, we check that

$$\Phi_0(z) - \Phi_4(z) \asymp \operatorname{dist}(z, \pi_z \circ \kappa_T (\mathbb{R}^n \times 0_\eta))^2. \tag{4-13}$$

Similarly,

$$\Phi_1^{\text{ext}}(z) - \Phi_3(z) \asymp \operatorname{dist}(z, \pi_z \circ \kappa_T ((\partial\Omega \times 0) \times \mathbb{C}^*_{\eta_n}))^2, \tag{4-14}$$

where $\partial\Omega \times 0$ denotes the zero-section in $T^*\partial\Omega$, so that

$$(\partial\Omega \times 0) \times \mathbb{C}^*_{\eta_n} = N^*\partial\Omega \otimes \mathbb{C}$$

is the fiberwise complexification of $N^*\partial\Omega$. (Here we work locally near $y = 0$.)

Let u be real-analytic in a neighborhood of $\overline{\Omega}$, and consider

$$v(z) = T(1_\Omega u)(z), \tag{4-15}$$

where we restrict our attention to $z \in \mathbb{C}^n$ such that the critical point $y_{\Phi_4}(z)$ in the definition of $\Phi_4(z)$ belongs to a small complex neighborhood of $\overline{\Omega}$ or equivalently to $z \in \mathbb{C}^n$ in a small neighborhood of $\kappa_T(\overline{\Omega} \times 0_\eta)$. By the method of steepest descent, we see that $v \in H^{\text{loc}}_{\Phi_5}$, where first of all $\Phi_5 \le \Phi_1$ and further

$$\Phi_5(z) = \Phi_4(z) \quad \text{when both} \begin{cases} \Re y_{\Phi_4}(z) \in \Omega, \\ |\Im y_{\Phi_4}(z)| \ll \operatorname{dist}(\Re y_{\Phi_4}(z), \partial\Omega), \end{cases} \tag{4-16}$$

$$\Phi_5(z) = \Phi_3(z) \quad \text{when both} \begin{cases} \Re y_{\Phi_4}(z) \notin \Omega, \\ |\Im y_{\Phi_4}(z)| \ll \operatorname{dist}(\Re y_{\Phi_4}(z), \partial\Omega). \end{cases} \tag{4-17}$$

[3] i.e., a holomorphic manifold that is Lagrangian for the complex symplectic form σ

Actually, in the last case, we can relax the condition that $y_{\Phi_4}(z)$ belongs to a small (u-dependent) neighborhood of $\overline{\Omega}$. The appropriate restriction is then that the critical point $y_{\Phi_3}(z) \in \widetilde{\partial\Omega}$ in the definition of Φ_3 belongs to a small (u-dependent) neighborhood of $\partial\Omega$.

5. Expressing M with the help of FBI transforms

From now on, we work with M_{new}, $\Phi_{M_{\text{new}}}$, and κ_{new} and we drop the corresponding subscript "new". Then from (3-18),

$$Mq(x', t) = \frac{1}{(2\pi h)^{n-1}} \iint e^{(i/h)\Phi_M(x',t,y,\eta')} a^*(y, \eta'; h) a(y, t\eta'_0; h) q(y)\, dy\, d\eta' \tag{5-1}$$

with Φ_M given in (3-19).

We want to express Mq with the help of Tq, where T is as in (4-1), and we start by recalling some general facts about metaplectic Fourier integral operators of this form, following [Sjöstrand 1982] for the local theory and [Sjöstrand 1990] for the simplified global theory in the metaplectic framework (i.e., all phases are quadratic and all amplitudes are constant). To start with, we weaken the assumptions on the quadratic phase in T and assume only that $\phi(x, y)$ is a holomorphic quadratic form on $\mathbb{C}^n \times \mathbb{C}^n$ satisfying the second part of (4-2):

$$\det \phi''_{x,y}(x, y) \neq 0. \tag{5-2}$$

To T we can still associate a linear canonical transformation κ_T as in (4-4). Let Φ_1 and Φ_2 be plurisubharmonic quadratic forms on $\mathbb{C}^n$ related by

$$\Lambda_{\Phi_2} = \kappa_T(\Lambda_{\Phi_1}). \tag{5-3}$$

Then we can define $T : H_{\Phi_1} \to H_{\Phi_2}$ as a bounded operator as in (4-1) with the modification that $\mathbb{R}^n$ should be replaced by a so-called good contour, which is an affine subspace of $\mathbb{C}^n$ of real dimension n, passing through the nondegenerate critical point $y_c(x)$ the function

$$y \mapsto -\Im\phi(x, y) + \Phi_1(y) \tag{5-4}$$

and along which this function is $\Phi_2(x) - (\asymp |y - y_c(x)|^2)$. (Actually in this situation, it would have been better to replace the power $h^{-3n/4}$ by $h^{-n/2}$ since we would then get a uniform bound on the norm.)

Remark 5.1. Recall also that, if only Φ_1 is given as above, the existence of a quadratic form Φ_2 as in (5-3) is equivalent to the fact that (5-4) has a nondegenerate critical point and the plurisubharmonicity of Φ_2 is equivalent to the fact that the signature of the critical point is $(n, -n)$ (which represents the maximal number of negative eigenvalues of the Hessian of a plurisubharmonic quadratic form). This in turn is equivalent to the existence of an affine good contour as above.

In this situation, $T : H_{\Phi_1} \to H_{\Phi_2}$ is bijective with the inverse

$$Sv(y) = T^{-1}v(y) = \widetilde{C} h^{-n/4} \int e^{-(i/h)\phi(z,y)} v(z)\, dz, \tag{5-5}$$

which can be realized the same way with a good contour, and here the constant $\widetilde{C}$ does not depend on the choice of Φ_j, $j = 1, 2$.

Remark 5.2. Let us introduce the formal adjoints of T and S,

$$T^{\mathrm{t}}v(y) = Ch^{-3n/4} \int_{\mathbb{R}^n} e^{(i/h)\phi(z,y)} v(x)\, dx, \quad y \in \mathbb{C}^n,$$
$$S^{\mathrm{t}}u(x) = \widetilde{C}h^{-n/4} \int e^{-(i/h)\phi(x,y)} u(y)\, dy.$$

Let Ψ_1 and Ψ_2 be plurisubharmonic quadratic forms such that $\kappa_{S^{\mathrm{t}}}(\Lambda_{\Psi_1}) = \Lambda_{\Psi_2}$. Then as above, $T^{\mathrm{t}} : H_{\Psi_2} \to H_{\Psi_1}$ and $S^{\mathrm{t}} : H_{\Psi_1} \to H_{\Psi_2}$ are bijective and $S^{\mathrm{t}} = \mathrm{const}(T^{\mathrm{t}})^{-1}$. We claim that S^{t} is the inverse of T^{t}. In fact, this statement is independent of the choice of Φ_j and Ψ_j as above, and we can choose them to be pluriharmonic in such a way that Λ_{Φ_j} intersects $\Lambda_{-\Psi_j}$ transversally for one value of j and then automatically for the other value. Then for $j = 1, 2$, we can define

$$\langle u \mid v \rangle = \int_{\gamma_j} u(x)v(x)\, dx$$

for $u \in H_{\Phi_j}$ and $v \in H_{\Psi_j}$ (or rather for functions that are $\mathcal{O}(e^{\Phi_j/h})$ and $e^{\Psi_j/h}$, respectively — the space of such functions is of dimension 1, which suffices for our purposes) if we let γ_j be a good contour for $\Phi_j + \Psi_j$. For $u = \mathcal{O}(e^{\Phi_2/h})$ and $v = \mathcal{O}(e^{\Psi_2/h})$ nonzero,

$$0 \neq \langle u \mid v \rangle = \langle TSu \mid v \rangle = \langle Su \mid T^{\mathrm{t}}v \rangle = \langle u \mid S^{\mathrm{t}}T^{t}v \rangle,$$

and knowing already that $S^{\mathrm{t}}T^{\mathrm{t}}$ is a multiple of the identity, we see that it has to be equal to the identity.

Now return to the discussion of an FBI transform T whose phase satisfies (4-2). When letting T act on suitable H_Φ-spaces, it has the inverse S in (5-5). However, if we let T act on $L^2(\mathbb{R}^n)$ so that $Tu \in H_{\Phi_0}$ (with $\Lambda_{\Phi_0} = \kappa_T(\mathbb{R}^{2n})$), the best possible contour in (5-5) is

$$\Gamma(y) = \{z \in \mathbb{C}^n : y(z) = y\}.$$

This follows from the property

$$\Phi_0(z) + \Im\phi(z, y) \asymp \mathrm{dist}(z, \Gamma(y))^2 \asymp |y(z) - y|^2, \tag{5-6}$$

so $\Phi_0(z) + \Im\phi(z, y) = 0$ on $\Gamma(y)$ and $e^{-(i/h)\phi(z,y)+(1/h)\Phi_0(z)}$ is bounded there. This is not sufficient for a straightforward definition of $Sv(y)$, $v \in H_{\Phi_0}$, since we would need some extra exponential decay along the contour near infinity, but it does suffice to give a precise meaning up to exponentially small errors of the formula

$$\widetilde{T}u = (\widetilde{T}S)Tu \tag{5-7}$$

in a local situation, where $\widetilde{T} : L^2 \to H_{\widetilde{\Phi}_0}$ is a second FBI transform and where $\widetilde{T}S : H_{\Phi_0} \to H_{\widetilde{\Phi}_0}$ is defined by means of a good contour.

Proposition 5.3. *Let* $(y_0, \eta_0) \in \mathbb{R}^{2n}$, $(z_0, \zeta_0) = \kappa_T(y_0, \eta_0)$, *and* $(w_0, \omega_0) = \kappa_{\widetilde{T}}(y_0, \eta_0)$. *We realize* Tu *and* $\widetilde{T}u$ ($\widetilde{T}Su$ *modulo exponentially small terms*) *in* H_{Φ_0, z_0} *and* $H_{\widetilde{\Phi}_0, w_0}$ (*and* $H_{\widetilde{\Phi}_0, w_0}$) *by choosing good contours restricted to neighborhoods of* y_0 *and* y_0 (*and* z_0), *respectively. Then* (5-7) *holds* (*modulo an exponentially small error*) *in* $H_{\widetilde{\Phi}_0, w_0}$. *Here* $u \in \mathcal{D}'(\mathbb{R}^n)$ *is either independent of* h *or of temperate growth in* $\mathcal{D}'(\mathbb{R}^n)$ *as a function of* h.

Proof. The left-hand side of (5-7) is

$$\operatorname{const} h^{-3n/4-n} \iiint e^{(i/h)(\tilde{\phi}(w,x)-\phi(z,x)+\phi(z,y))} u(y)\, dy\, dz\, dx,$$

and all good contours being homotopic, we can write it as

$$\widetilde{C} h^{-3n/4} \int \left(\operatorname{const} h^{-n} \iint e^{(i/h)(-\phi(z,x)+\phi(z,y))} e^{(i/h)\tilde{\phi}(w,x)}\, dx\, dz \right) u(y)\, dy.$$

The expression in the big parentheses is nothing but $T^{\mathrm{t}} S^{\mathrm{t}}(e^{(i/h)\tilde{\phi}(w,\cdot)})(y)$, which by Remark 5.2 is equal to $e^{(i/h)\tilde{\phi}(w,y)}$, and (5-7) follows. (In the proof, we have chosen not to spell out the various exponentially small errors due to the fact that the integration contours are confined to various small neighborhoods of certain points.) □

We now return to the operator M in (5-1). Choose adapted analytic coordinates centered at x_0 as in the beginning of Section 3. In that section (see (3-25)), we have seen that there is a well defined canonical transformation κ_M from a neighborhood of $(0, 0, -i) \in \mathbb{C}^{2n}_{y,\eta}$ to a neighborhood of $(0, 1, 0, 0)$ in $\mathbb{C}^{n-1}_{x'} \times \mathbb{C}_t \times \mathbb{C}^{n-1}_{x'^*} \times \mathbb{C}_{t^*}$ mapping $T^*\partial\Omega \times i\mathbb{R}_-$ to $\mathbb{R}^{n-1}_{x'} \times \mathbb{R}_t \times \mathbb{R}^{n-1}_{x'^*} \times \{t^* = 0\}$. This means that we have a microlocal description of Mq near $(0, 1, 0, 0)$ and not a local one near $x' = 0$ and $t = 0$. We shall therefore microlocalize in (x', x'^*) by means of an FBI transform in the x' variables.

Let

$$\widehat{T}u(w') = \widehat{C} h^{(1-n)/2} \int_{\mathbb{R}^{n-1}} e^{(i/h)\hat{\phi}(w',x')} u(x')\, dx', \quad w' \in \mathbb{C}^{n-1}, \tag{5-8}$$

be a second FBI transform as in (4-1) though acting on $n-1$ variables and with a different normalization. Assume (for concreteness) that

$$\kappa_{\widehat{T}}(\mathbb{C}^{n-1}_{x'} \times \{0\}) = \mathbb{C}^{n-1}_{w'} \times \{0\}. \tag{5-9}$$

Then

$$\kappa_{\widehat{T}}(T^*\mathbb{R}^{n-1}) = \Lambda_{\widehat{\Phi}_0}, \tag{5-10}$$

where $\widehat{\Phi}_0$ is a strictly plurisubharmonic quadratic form. In view of (5-9) and the fact that the zero-section $\mathbb{C}^{n-1} \times \{0\}$ is strictly positive with respect to the real phase space, we also know that

$$\widehat{\Phi}_0(w') \asymp |w'|^2 \tag{5-11}$$

or equivalently that the quadratic form $\widehat{\Phi}_0$ is strictly convex.

By slight abuse of notation, we also let $\widehat{T}$ act (as $\widehat{T} \oplus 0$) on functions of n variables by

$$\widehat{T}(u)(w', t) = (\widehat{T}u(\,\cdot\,, t))(w').$$

The presence of $\widehat{T}$ leads to a formula for $\widehat{T}M$ that is simpler than the one for M in (5-1):

$$\begin{aligned}\widehat{T}Mq(w',t) &= \widehat{T}\big(e^{-(i/h)(\,\cdot\,)\cdot t\eta_0'}K^{\mathrm{t}}qK\big(e^{(i/h)(\,\cdot\,)\cdot t\eta_0'}\big)\big)(w')\\ &= \widehat{C}h^{(1-n)/2}\iiint e^{(i/h)(\hat{\phi}(w',\tilde{x}')-\tilde{x}'\cdot t\eta_0')}K(y,\tilde{x}')q(y)K(y,x')e^{(i/h)x'\cdot t\eta_0'}\,dx'\,dy\,d\tilde{x}'\\ &= \int K\big(e^{(i/h)(\hat{\phi}(w',\cdot)-(\,\cdot\,)\cdot t\eta_0')}\big)(y)q(y)K\big(e^{(i/h)(\,\cdot\,)\cdot t\eta_0'}\big)(y)\,dy.\end{aligned}$$

Up to exponentially small errors, we have (see (3-5))

$$K\big(e^{(i/h)(\,\cdot\,)\cdot t\eta_0'}\big)(y) = e^{(i/h)\phi(y,t\eta_0')}a(y,t\eta_0';h)$$

and

$$K\big(e^{(i/h)\hat{\phi}(w',\cdot)}\big)(y) = e^{(i/h)\tilde{\psi}(w',t\eta_0',y)}b(w',y,t\eta_0';h),$$

where b is an elliptic analytic symbol of order 0 and ψ is the solution of the eikonal equation in y

$$\partial_{y_n}\tilde{\psi} = ir(y,\partial_{y'}\psi)^{1/2},\quad \tilde{\psi}|_{y_n=0} = \hat{\phi}(w',y') - y'\cdot t\eta_0'.$$

Thus, up to exponentially small errors, we get for $q\in L^\infty(\Omega)$

$$\widehat{T}Mq(w',t) = \int e^{(i/h)\psi(w',t,y)}c(w',t,y;h)q(y)\,dy,\quad (w',t)\in \operatorname{neigh}((0,1),\mathbb{C}^{n-1}\times\mathbb{C}),\tag{5-12}$$

where c is an elliptic analytic symbol of order 0 and

$$\psi(w',t,y) = \tilde{\psi}(w',t,y) + \phi(y,t\eta_0')$$

satisfies

$$\psi|_{y_n=0} = \hat{\phi}(w',y'),\tag{5-13}$$

$$\partial_{y_n}\psi|_{y_n=0} = i\big(r(y',0,\partial_{y'}\hat{\phi}(w',y') - t\eta_0')^{1/2} + r(y',0,t\eta_0')^{1/2}\big).\tag{5-14}$$

Assume for simplicity that $r(0,0,\eta_0') = \frac{1}{4}$. Then, at the point $(w'=0, t=1, y=0)$,

$$(\partial_{w'}\psi, \partial_t\psi, -\partial_{y'}\psi, -\partial_{y_n}\psi) = (0,0,0,-i),$$

so $\kappa_{\widehat{T}M}(0,0,-i) = (0,1,0,0)$.[4] Also, $\kappa_{\widehat{T}M} = \kappa_{\widehat{T}}\circ\kappa_M$ and

$$\begin{aligned}\kappa_M(0,0,-i) &= (0,1,0,0),\\ \kappa_{\widehat{T}}(0,1,0,0) &= (0,1,0,0).\end{aligned}$$

Recall from (3-25) that

$$\kappa_M : \operatorname{neigh}((0;0,-i), T^*\partial\Omega\times\mathbb{C}^-_{y_n^*}) \to \operatorname{neigh}\big((0,1;0,0), \mathbb{R}^{n-1}_{x'}\times\mathbb{C}_t\times\mathbb{R}^{n-1}_{x'^*}\times\{t^*=0\}\big),$$

so

$$\kappa_{\widehat{T}M} : \operatorname{neigh}((0,0,-i), T^*\partial\Omega\times\mathbb{C}^-_{y_n^*}) \to \operatorname{neigh}((0,1,0,0),\Lambda_{\widehat{\Phi}_0\oplus 0}).$$

[4] We can verify directly that $\det\partial_{w',t}\partial_y\psi \neq 0$.

On the other hand, we have seen in Section 4 that $\kappa_T(F)=\Lambda_{\Phi_1}$ and that the part $T^*\partial\Omega\times\mathbb{C}^-_{y_n^*}$ of F is mapped to $\Lambda_{\Phi_1^{\mathrm{ext}}}$. More locally,

$$\begin{aligned}\kappa_T &: \mathrm{neigh}((0,0,-i), T^*\partial\Omega\times\mathbb{C}^-_{y_n^*}) \to \mathrm{neigh}(\kappa_T(0,0,-i), \Lambda_{\Phi_1^{\mathrm{ext}}})\\ \kappa_S &: \mathrm{neigh}(\kappa_T(0,0,-i), \Lambda_{\Phi_1^{\mathrm{ext}}}) \to \mathrm{neigh}((0,0,-i), T^*\partial\Omega\times\mathbb{C}^-_{y_n^*}).\end{aligned}$$

Using also (3-25), we get

$$\kappa_{\widehat{T}MS}: \mathrm{neigh}(\pi_z\kappa_T(0,0,-i), \Lambda_{\Phi_1^{\mathrm{ext}}}) \to \mathrm{neigh}((0,1,0,0), \Lambda_{\widehat{\Phi}_0\oplus 0}). \tag{5-15}$$

We then also know that

$$\widehat{\Phi}_0(w') = \mathrm{vc}_{y,z}(-\Im\psi(w',t,y)+\Im\phi_T(z,y)).$$

This means that the formal composition

$$\widehat{T}MSv(w',t) = \widetilde{C}h^{-n/4}\iint e^{(i/h)(\psi(w',t,y)-\phi_T(z,y))}c(w',t,y;h)v(z)\,dz\,dy \tag{5-16}$$

gives a well defined operator

$$\widehat{T}MS: H_{\Phi_1^{\mathrm{ext}},\pi_z\kappa_T(0,0,-i)} \to H_{\widehat{\Phi}_0\oplus 0,(0,1)} \tag{5-17}$$

that can be realized with the help of a good contour.

We shall next show that

$$\widehat{T}Mu = (\widehat{T}MS)Tu \quad \text{in } H_{\widehat{\Phi}_0\oplus 0,(0,1)} \tag{5-18}$$

when u is supported in $\{y_n\ge 0\}$. The proof is the same as the one for (5-7). The right-hand side in (5-18) is equal to

$$\mathrm{const}\, h^{-n}\iiint e^{(i/h)(\psi(w',t,x)-\phi_T(z,x)+\phi_T(z,y))}c(w',t,x;h)u(y)\,dy\,dz\,dx,$$

where the y-integration is over $\mathbb{R}^n_+$, and we may assume without loss of generality that u has its support in a small neighborhood of $y=0$. The $dz\,dx$ integration is, to start with, over the good contour in (5-16). This last integration can be viewed as $T^{\mathrm{t}}S^{\mathrm{t}}$ acting on $e^{(i/h)\psi(w',t,\cdot)}c(w',t,\cdot\,;h)$, and here $T^{\mathrm{t}}S^{\mathrm{t}}$ is the identity operator that can be realized with a good contour, so we get

$$(\widehat{T}MS)Tu(w',t) = \int e^{(i/h)\psi(w',t,x)}c(w',t,x;h)u(x)\,dx = \widehat{T}Mu(w',t),$$

and we have verified (5-18).

Above, we have established (5-17) as the quantum version of (5-15). It follows by an easy adaptation of the exercise leading to (3-17) that

$$\kappa_M(\mathrm{neigh}((0,0,-i), \mathbb{C}^{n-1}\times\{0\}\times\mathbb{C}^-_{\eta_n})) = \mathrm{neigh}((0,0,1,0), \mathbb{C}^{n-1}_{x'}\times\{x'^*=0\}\times\mathbb{C}_t\times\{t^*=0\}), \tag{5-19}$$

and hence,

$$\kappa_{\widehat{T}MS}(\mathrm{neigh}(\kappa_T(0,0,-i), \Lambda_{\Phi_3})) = \mathrm{neigh}((0,0,1,0), \Lambda_{0\oplus 0}). \tag{5-20}$$

The quantum version of (5-20) is

$$\widehat{T}MS : H^{\mathrm{loc}}_{\Phi_3,\pi_z(\mathrm{neigh}(\kappa_T(0,0,-i)))} \to H^{\mathrm{loc}}_{0\oplus 0,(0,1)}. \tag{5-21}$$

We also know that $\widehat{T}MS$ is an elliptic Fourier integral operator. Consequently, (5-17) and (5-21) have continuous inverses. We also have the following result.

Proposition 5.4. *If $u \in H^{\mathrm{loc}}_{\Phi_1,\pi_z(\mathrm{neigh}(\kappa_T(0,0,-i)))}$ and $\widehat{T}MSu \in H^{\mathrm{loc}}_{0\oplus 0,(0,1)}$, then $u \in H^{\mathrm{loc}}_{\Phi_3,\pi_z(\mathrm{neigh}(\kappa_T(0,0,-i)))}$.*

6. End of the proof of the main result

We will work with FBI and Laplace transforms of functions that are independent of h or that have some special h-dependence. Consider a formal Fourier integral operator $u \mapsto Tu$, given by

$$Tu(x;h) = Ch^{\alpha} \int e^{(i/h)\phi(x,y)} u(y)\, dy, \tag{6-1}$$

where $\phi = \phi_T$ is a quadratic form on $\mathbb{C}^{2n}_{x,y}$ satisfying

$$\det \phi''_{xy} \neq 0 \tag{6-2}$$

and hence generating a canonical transformation that will be used below.

Proposition 6.1. *If u is independent of h,*

$$\left(hD_h + \frac{1}{h} P_\alpha(x, hD; h) \right) Tu = 0, \tag{6-3}$$

where

$$P_\alpha = p(x, hD) + ih\big(\alpha + \tfrac{1}{2}\operatorname{tr}(\phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1})\big), \tag{6-4}$$

$$\begin{aligned} p(x,\xi) &= \tfrac{1}{2}\phi''_{xx}x\cdot x + x\cdot(\xi - \phi''_{xx}x) + \tfrac{1}{2}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}(\xi - \phi''_{xx}x)\cdot(\xi - \phi''_{xx}x) \\ &= -\tfrac{1}{2}\phi''_{xx}x\cdot x + \tfrac{1}{2}\phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}\phi''_{xx}x\cdot x \\ &\qquad + x\cdot\xi - {\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}\phi''_{xx}x\cdot\xi + \tfrac{1}{2}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}\xi\cdot\xi. \end{aligned} \tag{6-5}$$

Proof. We have

$$\begin{aligned} hD_h\big(e^{(i/h)\phi(x,y)}\big) &= -\frac{1}{h} e^{(i/h)\phi(x,y)}, \\ hD_h(h^\alpha) &= \frac{\alpha}{i} h^\alpha, \\ hD_h Tu(x;h) &= -\frac{1}{h} h^\alpha \int e^{(i/h)\phi(x,y)}(ih\alpha + \phi(x,y))u(y)\, dy. \end{aligned}$$

Try to write $\phi(x,y) = p(x, \phi'_x(x,y))$ for a suitable quadratic form $p(x,\xi)$ (that will turn out to be the one given in (6-5)). We have

$$\phi(x,y) = \tfrac{1}{2}\phi''_{xx}x\cdot x + \phi''_{xy}y\cdot x + \tfrac{1}{2}\phi''_{yy}y\cdot y, \tag{6-6}$$

$$\phi'_x = \phi''_{xx}x + \phi''_{xy}y, \quad \text{i.e., } y = {\phi''_{xy}}^{-1}(\phi'_x - \phi''_{xx}x), \tag{6-7}$$

and using the last relation from (6-7) in (6-6), we get

$$\phi(x,y)=\tfrac{1}{2}\phi''_{xx}x\cdot x+\phi''_{yx}x\cdot {\phi''_{xy}}^{-1}(\phi'_x-\phi''_{xx}x)+\tfrac{1}{2}\phi''_{yy}{\phi''_{xy}}^{-1}(\phi'_x-\phi''_{xx}x)\cdot {\phi''_{xy}}^{-1}(\phi'_x-\phi''_{xx}x), \tag{6-8}$$

where the ϕ''_{yx} and ${\phi''_{xy}}^{-1}$ in the second term cancel and we get $p(x,\phi'_x)$ with p as in (6-5).

To verify (6-4), it suffices to notice that

$$\begin{aligned}
e^{-(i/h)\phi(x,y)}p(x,hD_x)\big(e^{(i/h)\phi(x,y)}\big)-p(x,\phi'_x)&=\tfrac{1}{2}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}hD_x\cdot(\phi'_x)\\
&=\tfrac{1}{2}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}hD_x\cdot(\phi''_{xx}x)\\
&=\frac{h}{2i}\phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}\partial_x\cdot x\\
&=\frac{h}{2i}\operatorname{tr}\big(\phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}\big). \qquad\square
\end{aligned}$$

Remark 6.2. Let $\kappa_T:(y,-\phi'_y(x,y))\mapsto(x,\phi'_x(x,y))$ be the canonical transformation associated to T, which can also be written

$$\kappa_T:(y,-(\phi''_{yx}x+\phi''_{yy}y))\mapsto(x,\phi''_{xx}x+\phi''_{xy}y)$$

or still $\kappa_T:(y,\eta)\mapsto(x,\xi)$, where

$$\begin{aligned}
x&=-{\phi''_{yx}}^{-1}(\eta+\phi''_{yy}y),\\
\xi&=(\phi''_{xy}-\phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy})y-\phi''_{xx}{\phi''_{yx}}^{-1}\eta.
\end{aligned}$$

We see that the following three statements are equivalent.

- κ_T maps the Lagrangian space $\eta=0$ to $\xi=0$.
- $\phi''_{xy}-\phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy}=0$.
- $p(x,0)=0$ and $p'_\xi(x,0)$ for all x.

Example 6.3. Consider

$$\widehat{T}\mathcal{L}u(x;h)=Ch^{(1-n)/2}\int e^{(i/h)(\phi(x',y')+ix_ny_n)}u(y)\,dy,\quad \phi=\phi_{\widehat{T}}.$$

If $P'(x',hD_{x'};h)$ is the operator associated to $\widehat{T}$ in $n-1$ variables, we get when u is independent of h

$$\left(hD_h+\frac{1}{h}(P'(x',hD_{x'};h)+x_nhD_{x_n})\right)\widehat{T}\mathcal{L}u=0. \tag{6-9}$$

Similarly (though not a direct consequence of Proposition 6.1 but rather of its method of proof), we have for $\mathcal{L}$ alone that

$$\left(hD_h+\frac{1}{h}x_nhD_{x_n}\right)\mathcal{L}u=0. \tag{6-10}$$

Example 6.4. Let T be as above, and assume that we are in the situation of Remark 6.2 so that $p(x,0)=0$ and $p'_\xi(x,0)=0$. Then

$$p(x,hD)=bhD\cdot hD,$$

where b is a constant symmetric matrix. Then

$$P_\alpha = p(x, hD) + ih(\alpha + f_0), \quad f_0 = \frac{n}{2},$$

and (6-3) reads

$$(hD_h + (hbD \cdot D + i(\alpha + f_0)))Tu = 0. \tag{6-11}$$

If $Tu = \sum_m^\infty h^k v_k \in H_0$ and u is independent of h, we can plug this expression into (6-11) and get the sequence of equations

$$\begin{aligned}
\left(\frac{m}{i} + i(\alpha + f_0)\right) v_m &= 0, \\
\left(\frac{m+1}{i} + i(\alpha + f_0)\right) v_{m+1} + bD \cdot D v_m &= 0, \\
\left(\frac{m+2}{i} + i(\alpha + f_0)\right) v_{m+2} + bD \cdot D v_{m+1} &= 0, \\
&\vdots
\end{aligned}$$

so unless $v \equiv 0$, we get $m = \alpha + f_0$. We can choose $v_m \in H_0$ arbitrarily, and $v_{m+1}, v_{m+2}, \dots$ are then uniquely determined.

Now, consider the situation in Theorem 1.6 and let $q \in L^\infty(\Omega)$ be independent of h and such that $\sigma_{\dot\Lambda}(y', t\eta_0')$ is a cl.a.s. on neigh$(\{0\} \times \mathbb{R}_+, \mathbb{R}^{n-1} \times \mathbb{R}_+)$ of order -1 (see (2-4)):

$$\sigma_{\dot\Lambda}(y', t\eta_0') \sim \sum_1^\infty n_k(y', t), \tag{6-12}$$

where $n_k(y', t)$ is homogeneous of degree $-k$ in t.

$$|n_k(y', t)| \le C^{k+1} k^k |t|^{-k}, \quad y' \in \text{neigh}(0, \mathbb{C}^{n-1}). \tag{6-13}$$

For the moment, we shall only work with formal cl.a.s. and neglect remainders in the asymptotic expansions. The semiclassical symbol of $\dot\Lambda$ is then

$$\sigma_{\dot\Lambda}(y', t\eta_0'/h) \sim \sum_1^\infty n_k(y', t/h) = \sum_1^\infty h^h n_k(y', t), \quad (y', t) \in \text{neigh}((0, 1), \mathbb{R}^{n-1} \times \mathbb{R}_+). \tag{6-14}$$

Recall that $\sigma_{\dot\Lambda}(y', t\eta_0'/h) = Mq(y', t; h)$. From (6-14), we infer that $\widehat{T}Mq$ is a cl.a.s. near $w' = 0$ and $t = 1$:

$$\widehat{T}Mq \sim \sum_1^\infty h^k m_k(w', t). \tag{6-15}$$

Formally,

$$\widehat{T}M = (\widehat{T}M\mathcal{L}^{-1})\mathcal{L}. \tag{6-16}$$

The canonical transformation $\kappa_{\mathcal{L}}$ is given by

$$(y, \eta) \mapsto (y', i\eta_n, \eta', iy_n).$$

It maps the complex manifold $\eta'=0$ and $y_n=0$ to the manifold $\{(z,0)\}$ and the point $(0;0,-i)$ to $(0,1;0)$, so $\kappa_{\mathcal{L}^{-1}}=\kappa_{\mathcal{L}}^{-1}$ maps $\zeta=0$ to $\eta'=0$ and $y_n=0$. We noticed in (3-24) (see (3-22)) that κ_M takes the complexified conormal bundle to the zero-section, and it maps the point $(0;0,-i)$ to $(0,1;0)$. Thus, $\kappa_{M\mathcal{L}^{-1}}$ maps the zero-section $\zeta=0$ to the zero-section and in particular $(0,1;0)$ to $(0,1;0)$. (We may notice that this is global in the sense that we can extend z_n to an annulus, and we then get t in an annulus.) Since $\kappa_{\widehat{T}}$ maps the zero-section to the zero-section, we have the same facts for $\kappa_{\widehat{T}M}$.

From the above, it is clear that $\widehat{T}M\mathcal{L}^{-1}$ maps formal cl.a.s. to formal cl.a.s.

Recalling (6-14) for $\sigma_{\hat{\Lambda}}(y',t\eta_0'/h)=Mq(y',t;h)$ and using that $\widehat{T}M\mathcal{L}^{-1}$ is an elliptic Fourier integral operator whose canonical transformation maps the zero-section to the zero-section, we see that there exists a unique formal cl.a.s.

$$v\sim\sum_1^\infty v_k(z',z_n)h^k,\qquad z\in\operatorname{neigh}((0,1),\mathbb{C}^n),\tag{6-17}$$

such that in the sense of formal stationary phase

$$\widehat{T}Mq=\widehat{T}M\mathcal{L}^{-1}v.\tag{6-18}$$

Now q is independent of h, so Mq satisfies a compatibility equation of the form

$$\Big(hD_h+\frac{1}{h}P_{\widehat{T}M}\Big)Mq=0.\tag{6-19}$$

This gives rise to a similar compatibility condition for v

$$\Big(hD_h+\frac{1}{h}P_{\mathcal{L}M^{-1}\widehat{T}^{-1}\widehat{T}M}\Big)v=0$$

or simply

$$\Big(hD_h+\frac{1}{h}P_{\mathcal{L}}\Big)v=0,$$

which is the same as (6-10):

$$(h\partial_h+z_n\partial_{z_n})v=0.\tag{6-20}$$

Application of this to (6-17) gives

$$(k+z_n\partial_{z_n})v_k=0,\tag{6-21}$$

i.e.,

$$v_k(z)=q_k(z')z_n^{-k},\qquad |q_k(z')|\le C^{k+1}k^k.\tag{6-22}$$

Thus,

$$v\sim\sum_1^\infty q_k(z')\Big(\frac{h}{z_n}\Big)^k=\sum_0^\infty q_{k+1}(z')\Big(\frac{h}{z_n}\Big)^{k+1},$$

and we see as in Section 2 that

$$v\sim\mathcal{L}\tilde{q}(z;h),\qquad \tilde{q}(y)=1_{[0,a]}(y_n)\sum_0^\infty\frac{q_{k+1}(y')}{k!}y_n^k,\tag{6-23}$$

with $a>0$ small enough to assure the convergence of the power series.

More precisely (and now we end the limitation to formal symbols), as in (5-18) and (5-7), we check that

$$\widehat{T}M\tilde{q} \equiv (\widehat{T}M\mathcal{L}^{-1})\mathcal{L}\tilde{q} \quad \text{in } H_{0,(0,1)} \tag{6-24}$$

(up to an exponentially small error). By the construction of $\tilde{q}$, the right-hand side is $\equiv \widehat{T}Mq$ in the same space.

Put $r = q - \tilde{q}$. Then

$$\widehat{T}Mr \equiv 0 \quad \text{in } H_{0,(0,1)}. \tag{6-25}$$

Now, we replace $\mathcal{L}$ with T and consider in light of (5-18)

$$(\widehat{T}MS)Tr \equiv 0 \quad \text{in } H_{0,(0,1)}, \tag{6-26}$$

which implies that $Tr \in H_{\Phi_1}$ satisfies

$$Tr \equiv 0 \quad \text{in } H_{\Phi_1^{\text{ext}},\pi_z\kappa_T(0;0,-i)}. \tag{6-27}$$

As we saw in Section 4, Λ_{Φ_1} contains the closure $\overline{\Gamma}$ of the complex curve

$$\Gamma = \kappa_T(\{(0;0,\eta_n) : \Im\eta_n < 0\}),$$

and $\kappa_T((0;0,-i)) \in \Gamma$. Consequently, $\Phi_1|_{\pi_z\Gamma}$ is harmonic and (6-27) and the maximum principle imply that

$$Tr \equiv 0 \quad \text{in } H_{\Phi_1} \text{ on } \pi_z(\overline{\Gamma}). \tag{6-28}$$

In particular,

$$Tr \equiv 0 \quad \text{in } H_{\Phi_1,0} \tag{6-29}$$

and a fortiori

$$Tr \equiv 0 \quad \text{in } H_{\Phi_0,0}. \tag{6-30}$$

This implies that $r = 0$ near $y = 0$. Hence, $q = \tilde{q}$ near $y = 0$, which gives the theorem.

7. Proof of Proposition 1.7

We choose local coordinates $y = (y', y_n)$ as in the beginning of Section 2. As in Proposition 1.7, we assume that q is analytic in a neighborhood of 0. We adopt the alternative definition of symbols in Remark 1.4. It will also be convenient to consider the semiclassical symbol of $\dot{\Lambda}$, $\sigma_{\dot{\Lambda}}(y', \eta'; h) = \sigma_{\dot{\Lambda}}(y', \eta'/h)$. For $y' \in \operatorname{neigh}(0, \mathbb{R}^{n-1})$,

$$\sigma_{\dot{\Lambda}}(y', \eta'; h) = -\partial_{y_n} GqK\left(\int \chi(t')e_{t'}(\,\cdot\,; h)e^{i(\,\cdot\,)\cdot\eta'/h}\,dt'\right)(y', 0)e^{-iy'\cdot\eta'/h}, \tag{7-1}$$

where χ and e_t were defined in Remark 1.4 with n there replaced by $n-1$. By analytic WKB (as we already used), we have up to an exponentially small error

$$K(e_{t'}(\,\cdot\,; h)e^{i(\,\cdot\,)\cdot\eta'/h}) = Ch^{(1-n)/2}a(y, \eta'; h)e^{i\phi(y,t,\eta')/h}, \tag{7-2}$$

where ϕ is the solution of the eikonal problem

$$\partial_{y_n}\phi = ir(y, \partial_{y'}\phi)^{1/2}, \quad \phi|_{y_n=0} = y'\cdot\eta' + \frac{i}{2}(y'-t)^2 \tag{7-3}$$

and a is an cl.a.s. of order 0 obtained from solving a sequence of transport equations with the "initial" condition $a(y', 0, \eta'; h) = 1$.

Using again the analytic WKB method, we can find a cl.a.s. b of order 0 in h that solves the following inhomogeneous problem up to exponentially small errors:

$$\begin{cases} (h^2\Delta - h^2V)\big(h^{(3-n)/2}b(y,t,\eta';h)e^{(i/h)\phi(y,t,\eta')}\big) = Ch^{(5-n)/2}ae^{(i/h)\phi}, \\ \qquad\qquad\qquad\qquad\qquad\qquad b(y',0,t,\eta';h) = 0. \end{cases}$$

Then up to exponentially small errors,

$$GqK(e_t(\,\cdot\,;h)e^{i(\,\cdot\,)\cdot\eta'/h}) \equiv h^{(3-n)/2}b(y,t,\eta';h)e^{(i/h)\phi(y,t,\eta')}$$

and similarly for the gradients, so

$$-(\partial_{y_n})_{y_n=0}GqK(e_t(\,\cdot\,;h)e^{i(\,\cdot\,)\cdot\eta'/h}) \equiv -h^{(3-n)/2}(\partial_{y_n}b)(y',0,t,\eta';h)e^{(i/h)(y'\cdot\eta'+(i/2)(y'-t')^2)}.$$

Multiplying with $\chi(t')$ and integrating in t', we see that $\sigma_{\hat\Lambda}(y',\eta';h)$ is a cl.a.s. in the semiclassical sense, and this implies that $\sigma_{\hat\Lambda}(y',\eta)$ is a cl.a.s.

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DISPERSIVE ESTIMATES FOR THE SCHRÖDINGER OPERATOR ON STEP-2 STRATIFIED LIE GROUPS

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The present paper is dedicated to the proof of dispersive estimates on stratified Lie groups of step 2 for the linear Schrödinger equation involving a sublaplacian. It turns out that the propagator behaves like a wave operator on a space of the same dimension p as the center of the group, and like a Schrödinger operator on a space of the same dimension k as the radical of the canonical skew-symmetric form, which suggests a decay rate $|t|^{-(k+p-1)/2}$. We identify a property of the canonical skew-symmetric form under which we establish optimal dispersive estimates with this rate. The relevance of this property is discussed through several examples.

1. Introduction

1A. *Dispersive inequalities.* Dispersive inequalities for evolution equations (such as the Schrödinger and wave equations) play a decisive role in the study of semilinear and quasilinear problems which appear in numerous physical applications. Proving dispersion amounts to establishing a decay estimate for the L^∞ norm of the solutions of these equations at time t in terms of some negative power of t and the L^1 norm of the data. In many cases, the main step in the proof of this decay in time relies on the application of a stationary phase theorem on an (approximate) representation of the solution. Combined with an abstract functional analysis argument known as the TT^*-argument, dispersion phenomena yield a range of estimates involving spacetime Lebesgue norms. Those inequalities, called Strichartz estimates, have proved to be powerful in the study of nonlinear equations (for instance, one can consult [Bahouri et al. 2011] and the references therein).

In the $\mathbb{R}^d$ framework, dispersive inequalities have a long history, beginning with [Brenner 1975; Pecher 1976; Segal 1976; Strichartz 1977]. They were subsequently developed by various authors, starting with [Ginibre and Velo 1995] (for a detailed bibliography, we refer to [Keel and Tao 1998; Tao 2006] and the references therein). Bahouri et al. [2000] generalize the dispersive estimates for the wave equation to the Heisenberg group $\mathbb{H}^d$ with an optimal rate of decay of order $|t|^{-1/2}$ (regardless of the dimension d) and prove that no dispersion occurs for the Schrödinger equation. Del Hierro [2005] proved optimal results for the time behavior of the Schrödinger and wave equations on H-type groups: if p is the dimension of the center of the H-type group, Del Hierro establishes sharp dispersive inequalities for the wave equation solution (with a decay rate of $|t|^{-p/2}$) as well as for the Schrödinger equation solution (with a $|t|^{-(p-1)/2}$

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decay). Compared with the $\mathbb{R}^d$ framework, there is an exchange in the rates of decay between the wave and the Schrödinger equations.

Strichartz estimates in other settings have been obtained in a number of works. One can first cite various results dealing with variable coefficient operators (see for instance [Kapitanski 1989; Smith 1998]) or studies concerning domains, such as [Burq et al. 2008; Ivanovici et al. 2014; Smith and Sogge 1995]. One can also refer to the result concerning the full laplacian on the Heisenberg group [Furioli et al. 2007], works in the framework of the real hyperbolic spaces [Anker and Pierfelice 2009; Banica 2007; Tataru 2001], or in the framework of compact and noncompact manifolds [Anton 2008; Banica and Duyckaerts 2007; Burq et al. 2004]; finally, one can mention the quasilinear framework studied in [Bahouri and Chemin 1999; 2003; Klainerman and Rodnianski 2005; Smith and Tataru 2005] and the references therein.

In this paper our goal is to establish optimal dispersive estimates for the solutions of the Schrödinger equation on step-2 stratified Lie groups. We shall emphasize in particular the key role played by the canonical skew-symmetric form in determining the rate of decay of the solutions. It turns out that the Schrödinger propagator on G behaves like a wave operator on a space of the same dimension as the center of G, and like a Schrödinger operator on a space of the same dimension as the radical of the canonical skew-symmetric form associated with the dual of the center. This unusual behavior of the Schrödinger propagator in the case of Lie algebras whose canonical skew-symmetric form is degenerate (known as Lie algebras which are not MW; see [Moore and Wolf 1973; Müller and Ricci 1996], for example) makes the analysis of the explicit representations of the solutions tricky and gives rise to uncommon dispersive estimates. It will also appear from our analysis that the optimal rate of decay is not always in accordance with the dimension of the center: we shall exhibit examples of step-2 stratified Lie groups with center of any dimension for which no dispersion occurs for the Schrödinger equation. We shall actually highlight that the optimal rate of decay in the dispersive estimates for solutions to the Schrödinger equation is, rather, related to the properties of the canonical skew-symmetric form.

1B. ***Stratified Lie groups.*** Let us recall here some basic facts about stratified Lie groups (see [Corwin and Greenleaf 1990; Folland 1989; Folland and Stein 1982; Stein and Weiss 1971] and the references therein for further details). A connected, simply connected, nilpotent Lie group G is called stratified if its left-invariant Lie algebra $\mathfrak{g}$ (assumed to be real-valued and of finite dimension n) is endowed with a vector space decomposition

$$\mathfrak{g} = \bigoplus_{1\le k\le\infty} \mathfrak{g}_k,$$

where all but finitely many of the $\mathfrak{g}_k$ are $\{0\}$, such that $[\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1}$. If there are p nonzero $\mathfrak{g}_k$ then the group is said to be of step p. Via the exponential map

$$\exp : \mathfrak{g} \to G,$$

which is in that case a diffeomorphism from $\mathfrak{g}$ to G, one identifies G and $\mathfrak{g}$. It turns out that, under this identification, the group law on G (which is generally not commutative) provided by the Campbell–Baker–Hausdorff formula, $(x, y) \mapsto x \cdot y$, is a polynomial map. In the following we shall denote by $\mathfrak{z}$ the center of G, which is simply the last nonzero $\mathfrak{g}_k$, and write

$$G = \mathfrak{v} \oplus \mathfrak{z}, \tag{1-1}$$

where $\mathfrak{v}$ is any subspace of G complementary to $\mathfrak{z}$.

The group G is endowed with a smooth left-invariant measure $\mu(x)$, the Haar measure, induced by the Lebesgue measure on $\mathfrak{g}$, which satisfies the fundamental translation invariance property

$$\int_G f(y)\, d\mu(y) = \int_G f(x \cdot y)\, d\mu(y) \quad \text{for all } f \in L^1(G, d\mu),\ x \in G.$$

Note that the convolution of two functions f and g on G is given by

$$f * g(x) := \int_G f(x \cdot y^{-1}) g(y)\, d\mu(y) = \int_G f(y) g(y^{-1} \cdot x)\, d\mu(y) \tag{1-2}$$

and as in the euclidean case we define Lebesgue spaces by

$$\|f\|_{L^p(G)} := \left(\int_G |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}$$

for $p \in [1, \infty[$ with the standard modification when $p = \infty$.

Since G is stratified, there is a natural family of dilations on $\mathfrak{g}$ defined for $t > 0$ as follows: if X belongs to $\mathfrak{g}$, we can decompose X as $X = \sum X_k$ with $X_k \in \mathfrak{g}_k$, and then

$$\delta_t X := \sum t^k X_k.$$

This allows us to define the dilation δ_t on the Lie group G via the identification by the exponential map:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\delta_t"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "\exp \circ \delta_t \circ \exp^{-1}"'] & G \end{tikzcd}$$

To avoid heaviness, we shall still denote by δ_t the map $\exp \circ \delta_t \circ \exp^{-1}$.

Observe that the action of the left-invariant vector fields X_k for X_k belonging to $\mathfrak{g}_k$ changes the homogeneity in the following way:

$$X_k(f \circ \delta_t) = t^k X_k(f) \circ \delta_t,$$

where by definition $X_k(f)(y) := df(y \cdot \exp(sX_k))/ds|_{s=0}$ and the Jacobian of the dilation δ_t is t^Q, where $Q := \sum_{1 \le k \le \infty} k \dim \mathfrak{g}_k$ is called the homogeneous dimension of G:

$$\int_G f(\delta_t y)\, d\mu(y) = t^{-Q} \int_G f(y)\, d\mu(y). \tag{1-3}$$

Let us also point out that there is a natural norm ρ on G, which is homogeneous in the sense that it respects dilations: $x \mapsto \rho(x)$ for $x \in G$ satisfies

$$\rho(x^{-1}) = \rho(x), \quad \rho(\delta_t x) = t\rho(x) \quad \text{for all } x \in G; \quad \rho(x) = 0 \iff x = 0.$$

We can define the Schwartz space $\mathcal{S}(G)$ as the set of smooth functions on G such that $x \mapsto \rho^p(x)\mathcal{X}^\alpha f(x)$ belongs to $L^\infty(G)$ for all α in $\mathbb{N}^d$ and p in $\mathbb{N}$, where $\mathcal{X}^\alpha$ denotes a product of $|\alpha|$ left-invariant vector fields. The Schwartz space $\mathcal{S}(G)$ has properties very similar to those of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, particularly density in Lebesgue spaces.

1C. ***The Fourier transform.*** The group G being noncommutative, its Fourier transform is defined by means of irreducible unitary representations. We devote this section to the introduction of the basic concepts that will be needed in the sequel. From now on, we assume that G is a step-2 stratified Lie group, meaning $\mathfrak{z} = \mathfrak{g}_2$, and we let $\mathfrak{v} = \mathfrak{g}_1$ in (1-1). We choose a scalar product on $\mathfrak{g}$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal.

1C1. *Irreducible unitary representations.* Let us fix some notation, borrowed from [Ciatti et al. 2005] (see also [Corwin and Greenleaf 1990] or [Müller and Ricci 1996]). For any $\lambda \in \mathfrak{z}^\star$ (the dual of the center $\mathfrak{z}$) we define a skew-symmetric bilinear form on $\mathfrak{v}$ by

$$B(\lambda)(U, V) := \lambda([U, V]) \quad \text{for all } U, V \in \mathfrak{v}. \tag{1-4}$$

One can find a Zariski-open subset Λ of $\mathfrak{z}^\star$ such that the number of distinct eigenvalues of $B(\lambda)$ is maximum. We denote by k the dimension of the radical $\mathfrak{r}_\lambda$ of $B(\lambda)$. Since $B(\lambda)$ is skew-symmetric, the dimension of the orthogonal complement of $\mathfrak{r}_\lambda$ in $\mathfrak{v}$ is an even number, which we shall denote by $2d$. Therefore, there exists an orthonormal basis

$$(P_1(\lambda), \dots, P_d(\lambda), Q_1(\lambda), \dots, Q_d(\lambda), R_1(\lambda), \dots, R_k(\lambda))$$

such that the matrix of $B(\lambda)$ takes the form

$$\begin{pmatrix} 0 & \cdots & 0 & \eta_1(\lambda) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \eta_d(\lambda) & 0 & \cdots & 0 \\ -\eta_1(\lambda) & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\eta_d(\lambda) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where each $\eta_j(\lambda) > 0$ is smooth and homogeneous of degree 1 in $\lambda = (\lambda_1, \dots, \lambda_p)$ and the basis vectors are chosen to depend smoothly on λ in Λ. Decomposing $\mathfrak{v}$ as

$$\mathfrak{v} = \mathfrak{p}_\lambda + \mathfrak{q}_\lambda + \mathfrak{r}_\lambda$$

with

$$\mathfrak{p}_\lambda := \operatorname{Span}(P_1(\lambda), \dots, P_d(\lambda)), \quad \mathfrak{q}_\lambda := \operatorname{Span}(Q_1(\lambda), \dots, Q_d(\lambda)), \quad \mathfrak{r}_\lambda := \operatorname{Span}(R_1(\lambda), \dots, R_k(\lambda)),$$

any element $V \in \mathfrak{v}$ will be written in the following as $P + Q + R$ with $P \in \mathfrak{p}_\lambda$, $Q \in \mathfrak{q}_\lambda$ and $R \in \mathfrak{r}_\lambda$. We then introduce irreducible unitary representations of G on $L^2(\mathfrak{p}_\lambda)$

$$u_X^{\lambda,\nu}\phi(\xi) := \mathrm{e}^{-i\nu(R)-i\lambda(Z+[\xi+P/2,Q])}\phi(P+\xi), \quad \lambda \in \mathfrak{z}^*, \ \nu \in \mathfrak{r}_\lambda^*, \tag{1-5}$$

for any $x = \exp(X) \in G$ with $X = X(\lambda, x) := (P(\lambda, x), Q(\lambda, x), R(\lambda, x), Z(x))$ and $\phi \in L^2(\mathfrak{p}_\lambda)$. In order to shorten notation, we shall omit the dependence on (λ, x) whenever there is no risk of confusion.

1C2. *The Fourier transform.* In contrast with the euclidean case, the Fourier transform is defined on the bundle $\mathfrak{r}(\Lambda)$ above Λ whose fiber above $\lambda \in \Lambda$ is $\mathfrak{r}_\lambda^* \sim \mathbb{R}^k$. It is valued in the space of bounded operators on $L^2(\mathfrak{p}_\lambda)$. More precisely, the Fourier transform of a function f in $L^1(G)$ is defined as follows: for any $(\lambda, \nu) \in \mathfrak{r}(\Lambda)$,

$$\mathcal{F}(f)(\lambda, \nu) := \int_G f(x) u_{X(\lambda,x)}^{\lambda,\nu}\, d\mu(x).$$

Note that, for any (λ, ν), the map $u_{X(\lambda,x)}^{\lambda,\nu}$ is a group homomorphism from G into the group $U(L^2(\mathfrak{p}_\lambda))$ of unitary operators of $L^2(\mathfrak{p}_\lambda)$, so functions f of $L^1(G)$ have a Fourier transform $(\mathcal{F}(f)(\lambda, \nu))_{\lambda,\nu}$ that is a bounded family of bounded operators on $L^2(\mathfrak{p}_\lambda)$. One may check that the Fourier transform exchanges convolution, whose definition is recalled in (1-2), and composition:

$$\mathcal{F}(f \star g)(\lambda, \nu) = \mathcal{F}(f)(\lambda, \nu) \circ \mathcal{F}(g)(\lambda, \nu). \tag{1-6}$$

Further, the Fourier transform can be extended to an isometry from $L^2(G)$ onto the Hilbert space of two-parameter families $A = \{A(\lambda, \nu)\}_{(\lambda,\nu)\in\mathfrak{r}(\Lambda)}$ of operators on $L^2(\mathfrak{p}_\lambda)$ which are Hilbert–Schmidt for almost every $(\lambda, \nu) \in \mathfrak{r}(\Lambda)$, with $\|A(\lambda, \nu)\|_{\mathrm{HS}(L^2(\mathfrak{p}_\lambda))}$ measurable and with norm

$$\|A\| := \left(\int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*} \|A(\lambda, \nu)\|^2_{\mathrm{HS}(L^2(\mathfrak{p}_\lambda))} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda\right)^{\frac{1}{2}} < \infty,$$

where $|\mathrm{Pf}(\lambda)| := \prod_{j=1}^d \eta_j(\lambda)$ is the Pfaffian of $B(\lambda)$. We have the following Fourier–Plancherel formula: there exists a constant $\kappa > 0$ such that

$$\int_G |f(x)|^2\, dx = \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*} \|\mathcal{F}(f)(\lambda, \nu)\|^2_{\mathrm{HS}(L^2(\mathfrak{p}_\lambda))} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda. \tag{1-7}$$

Finally, we have an inversion formula as stated in the following proposition, proved in the Appendix.

Proposition 1.1. *There exists $\kappa > 0$ such that, for $f \in \mathcal{S}(G)$ and almost all $x \in G$, the following inversion formula holds*:

$$f(x) = \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*} \mathrm{tr}\Big((u_{X(\lambda,x)}^{\lambda,\nu})^\star \mathcal{F} f(\lambda, \nu)\Big) |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda. \tag{1-8}$$

1C3. *The sublaplacian.* Let $(V_1, \dots, V_m)$ be an orthonormal basis of $\mathfrak{g}_1$. The sublaplacian on G is defined by

$$\Delta_G := \sum_{j=1}^m V_j^2. \tag{1-9}$$

It is a self-adjoint operator which is independent of the orthonormal basis $(V_1, \dots, V_m)$, and homogeneous of degree 2 with respect to the dilations in the sense that

$$\delta_t^{-1} \Delta_G \delta_t = t^2 \Delta_G.$$

To write its expression in Fourier space, we consider the basis of Hermite functions $(h_n)_{n\in\mathbb{N}}$, normalized in $L^2(\mathbb{R})$ and satisfying, for all real numbers ξ,

$$h_n''(\xi) - \xi^2 h_n(\xi) = -(2n+1) h_n(\xi).$$

Then, for any multi-index $\alpha \in \mathbb{N}^d$, we define the functions $h_{\alpha,\eta(\lambda)}$ by

$$\begin{aligned} h_{\alpha,\eta(\lambda)}(\Xi) &:= \prod_{j=1}^{d} h_{\alpha_j,\eta_j(\lambda)}(\xi_j) \quad \text{for all } \Xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, \\ h_{n,\beta}(\xi) &:= \beta^{1/4} h_n(\beta^{1/2}\xi) \qquad \text{for all } (n, \beta) \in \mathbb{N} \times \mathbb{R}^+,\ \xi \in \mathbb{R}. \end{aligned} \tag{1-10}$$

The sublaplacian Δ_G defined in (1-9) satisfies

$$\mathcal{F}(-\Delta_G f)(\lambda, \nu) = \mathcal{F}(f)(\lambda, \nu)(H(\lambda) + |\nu|^2), \tag{1-11}$$

where $|\nu|$ denotes the euclidean norm of the vector ν in $\mathbb{R}^k$ and $H(\lambda)$ is the diagonal operator defined on $L^2(\mathbb{R}^d)$ by

$$H(\lambda) h_{\alpha,\eta(\lambda)} = \sum_{j=1}^{d} (2\alpha_j + 1)\eta_j(\lambda) h_{\alpha,\eta(\lambda)}.$$

In the following we shall denote the "frequencies" associated with $P_j^2(\lambda) + Q_j^2(\lambda)$ by

$$\zeta_j(\alpha, \lambda) := (2\alpha_j + 1)\eta_j(\lambda), \quad (\alpha, \lambda) \in \mathbb{N}^d \times \Lambda, \tag{1-12}$$

and those associated with $H(\lambda)$ by

$$\zeta(\alpha, \lambda) := \sum_{j=1}^{d} \zeta_j(\alpha, \lambda), \quad (\alpha, \lambda) \in \mathbb{N}^d \times \Lambda. \tag{1-13}$$

Note that Δ_G is directly related to the harmonic oscillator via $H(\lambda)$ since eigenfunctions associated with the eigenvalues $\zeta(\alpha, \lambda)$ are the products of 1-dimensional Hermite functions. Also observe that $\zeta(\alpha, \lambda)$ is smooth and homogeneous of degree 1 in $\lambda = (\lambda_1, \dots, \lambda_p)$. Moreover, $\zeta(\alpha, \lambda) = 0$ if and only if $B(\lambda) = 0$, or equivalently, by (1-4), $\lambda = 0$.

Notice also that there is a difference in homogeneity in the variables λ and ν. Namely, in the variable ν, the sublaplacian acts as in the euclidean case (homogeneity 2) while in λ, it has the homogeneity 1 of a wave operator.

Finally, for any smooth function Φ, we define the operator $\Phi(-\Delta_G)$ by the formula

$$\mathcal{F}(\Phi(-\Delta_G) f)(\lambda, \nu) := \Phi(H(\lambda) + |\nu|^2)\mathcal{F}(f)(\lambda, \nu), \tag{1-14}$$

which also reads

$$\mathcal{F}(\Phi(-\Delta_G)f)(\lambda,\nu)h_{\alpha,\eta(\lambda)} := \Phi(|\nu|^2+\zeta(\alpha,\lambda))\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}$$

for all $(\lambda,\nu)\in\mathfrak{r}(\Lambda)$ and $\alpha\in\mathbb{N}^d$.

1C4. *Strict spectral localization.* Let us introduce the following notion of spectral localization, which we shall call strict spectral localization and which will be very useful in the following.

Definition 1.2. A function f belonging to $L^1(G)$ is said to be strictly spectrally localized in a set $\mathcal{C}\subset\mathbb{R}$ if there exists a smooth function θ, compactly supported in $\mathcal{C}$, such that, for all $1\le j\le d$,

$$\mathcal{F}(f)(\lambda,\nu)=\mathcal{F}(f)(\lambda,\nu)\theta((P_j^2+Q_j^2)(\lambda)) \quad \text{for all } (\lambda,\nu)\in\mathfrak{r}(\Lambda). \tag{1-15}$$

Remark 1.3. One could expect the notion of spectral localization to relate to the laplacian instead of each individual vector field $P_j^2+Q_j^2$, assuming rather the less restrictive condition

$$\mathcal{F}(f)(\lambda,\nu)=\mathcal{F}(f)(\lambda,\nu)\theta(H(\lambda)) \quad \text{for all } (\lambda,\nu)\in\mathfrak{r}(\Lambda).$$

The choice we make here is more restrictive due to the anisotropic context (namely the fact that $\eta_j(\lambda)$ depends on j); in the case of the Heisenberg group or, more generally, H-type groups, the notion of "strict spectral localization" in a ring $\mathcal{C}$ of $\mathbb{R}^p$ actually coincides with the more usual definition of "spectral localization" since, as recalled in the next subsection, $\eta_j(\lambda)=4|\lambda|$ (for a complete presentation and more details on spectrally localized functions, we refer the reader to [Bahouri and Gallagher 2001; Bahouri et al. 2012a; 2012b]). Assumption (1-15) guarantees a lower bound, which roughly states that for $\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\lambda}$ to be nonzero we must have

$$(2\alpha_j+1)\eta_j(\lambda)\ge c>0 \quad \text{for all } j\in\{1,\dots,d\}, \tag{1-16}$$

hence each η_j must be bounded away from zero, rather than the sum over j. These lower bounds are important ingredients of the proof (see Section 3C).

1D. *Examples.*

Let us give a few examples of well-known stratified Lie groups with a step-2 stratification. Note that nilpotent Lie groups which are connected, simply connected and whose Lie algebra admits a step-2 stratification are called Carnot groups.

1D1. *The Heisenberg group.* The Heisenberg group $\mathbb{H}^d$ is defined as the space $\mathbb{R}^{2d+1}$ whose elements can be written $w=(x,y,s)$ with $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$, endowed with the product law

$$(x,y,s)\cdot(x',y',s')=(x+x',y+y',s+s'-2(x\mid y')+2(y\mid x')),$$

where $(\cdot\mid\cdot)$ denotes the euclidean scalar product on $\mathbb{R}^d$. In that case the center consists of the points of the form $(0,0,s)$ and is of dimension 1. The Lie algebra of left-invariant vector fields is generated by

$$X_j:=\partial_{x_j}+2y_j\partial_s,\quad Y_j:=\partial_{y_j}-2x_j\partial_s \quad \text{for } 1\le j\le d;\quad S:=\partial_s=\tfrac{1}{4}[Y_j,X_j].$$

The canonical skew-symmetric form $B(\lambda)(U,V)$ defined in (1-4) associated with the frequencies $\lambda\in\mathbb{R}^*$ is proportional to λ, since $[U,V]$ is proportional to ∂_s. Its radical reduces to $\{0\}$ with $\Lambda=\mathbb{R}^*$, and $|\eta_j(\lambda)| = 4|\lambda|$ for all $j \in \{1, \dots, d\}$. Note in particular that strict spectral localization and spectral localization are equivalent.

1D2. *H-type groups.* These groups are canonically isomorphic to $\mathbb{R}^{m+p}$ and are a multidimensional version of the Heisenberg group. The group law is of the form

$$(x^{(1)}, x^{(2)})\cdot(y^{(1)}, y^{(2)}) := \begin{pmatrix} x_j^{(1)}+y_j^{(1)}, & j = 1, \dots, m \\ x_k^{(2)}+y_k^{(2)}+\frac{1}{2}\langle x^{(1)}, U^{(k)}y^{(1)}\rangle, & k = 1, \dots, p \end{pmatrix},$$

where $U^{(j)}$ are $m \times m$ linearly independent, orthogonal, skew-symmetric matrices satisfying the property

$$U^{(r)}U^{(s)} + U^{(s)}U^{(r)} = 0$$

for every $r, s \in \{1, \dots, p\}$ with $r \neq s$. In that case the center is of dimension p and may be identified with $\mathbb{R}^p$, and the radical of the canonical skew-symmetric form associated with the frequencies λ is again $\{0\}$. For example, the Iwasawa subgroup of semisimple Lie groups of split rank 1 (see [Korányi 1985]) is of this type. On H-type groups, m is an even number, which we denote by $2l$, and the Lie algebra of left-invariant vector fields is spanned by the following vector fields, where we have written $z = (x, y)$ in $\mathbb{R}^l \times \mathbb{R}^l$: for $j = 1, \dots, l$ and $k = 1, \dots, p$,

$$X_j := \partial_{x_j} + \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{2l} z_l U_{l,j}^{(k)}\partial_{s_k}, \quad Y_j := \partial_{y_j} + \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{2l} z_l U_{l,j+l}^{(k)}\partial_{s_k} \quad \text{and} \quad \partial_{s_k}.$$

In that case, we have $\Lambda = \mathbb{R}^p \setminus \{0\}$ with $\eta_j(\lambda) = \sqrt{\lambda_1^2 + \cdots + \lambda_p^2}$ for all $j \in \{1, \dots, l\}$ (here, again, strict spectral localization and spectral localization are equivalent).

1D3. *Diamond groups.* These groups, which occur in crystal theory (for more details, consult [Ludwig 1995; Poguntke 1999]), are of the type $\Sigma \ltimes \mathbb{H}^d$, where Σ is a connected Lie group acting smoothly on $\mathbb{H}^d$. One can find examples for which the radical of the canonical skew-symmetric is of any dimension k, $0 \leq k \leq d$. For example, one can take for Σ the k-dimensional torus, acting on $\mathbb{H}^d$ by

$$\theta(w) := (\theta \cdot z, s) := (\mathrm{e}^{i\theta_1}z_1, \dots, \mathrm{e}^{i\theta_k}z_k, z_{k+1}, \dots, z_d, s), \quad w = (z, s),$$

where the element $\theta = (\theta_1, \dots, \theta_k)$ corresponds to the element $(\mathrm{e}^{i\theta_1}, \dots, \mathrm{e}^{i\theta_k})$ of $\mathbb{T}^k$. Then the product law on $G = \mathbb{T}^k \ltimes \mathbb{H}^d$ is given by

$$(\theta, w) \cdot (\theta', w') = \big(\theta + \theta', w.(\theta(w'))\big),$$

where $w.(\theta(w'))$ denotes the Heisenberg product of w by $\theta(w')$. As a consequence, the center of G is of dimension 1, since it consists of the points of the form $(0, 0, s)$ for $s \in \mathbb{R}$. Let us choose for simplicity $k = d = 1$; the algebra of left-invariant vector fields is generated by the vector fields ∂_θ, ∂_s, $\Gamma_{\theta,x}$ and $\Gamma_{\theta,y}$, where

$$\Gamma_{\theta,x} = \cos\theta\,\partial_x + \sin\theta\,\partial_y + 2(y\cos\theta - x\sin\theta)\partial_s,$$
$$\Gamma_{\theta,y} = -\sin\theta\,\partial_x + \cos\theta\,\partial_y - 2(y\sin\theta + x\cos\theta)\partial_s.$$

It is not difficult to check that the radical of $B(\lambda)$ is of dimension 1. In the general case, where $k \leq d$, the algebra of left-invariant vector fields is generated by the vector fields ∂_s, the $2(d-k)$ vectors

$$X_l = \partial_{x_l} + 2y_l\partial_s \quad \text{and} \quad Y_l = \partial_{y_l} - 2x_l\partial_s,$$

and the $3k$ vectors defined for $1 \leq j \leq k$ by ∂_{θ_j}, Γ_{θ_j,x_j} and Γ_{θ_j,y_j}, where

$$\begin{aligned}\Gamma_{\theta_j,x_j} &= \cos\theta_j\partial_{x_j} + \sin\theta_j\partial_{y_j} + 2(y_j\cos\theta_j - x_j\sin\theta_j)\partial_s,\\ \Gamma_{\theta_j,y_j} &= -\sin\theta_j\partial_{x_j} + \cos\theta_j\partial_{y_j} - 2(y_j\sin\theta_j + x_j\cos\theta_j)\partial_s,\end{aligned}$$

and this provides an example with a radical of dimension k.

1D4. *The tensor product of Heisenberg groups.* Consider $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$, the set of elements (w_1, w_2) in $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ that can be written as $(w_1, w_2) = (x_1, y_1, s_1, x_2, y_2, s_2)$ in $\mathbb{R}^{2d_1+1} \times \mathbb{R}^{2d_2+1}$, equipped with the product law

$$(w_1, w_2) \cdot (w_1', w_2') = (w_1 \cdot w_1', w_2 \cdot w_2'),$$

where $w_1 \cdot w_1'$ and $w_2 \cdot w_2'$ denote the product in $\mathbb{H}^{d_1}$ and $\mathbb{H}^{d_2}$, respectively. Clearly $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ is a step-2 stratified Lie group with center of dimension 2 and radical index null. Moreover, for $\lambda = (\lambda_1, \lambda_2)$ in the dual of the center, the canonical skew bilinear form $B(\lambda)$ has radical $\{0\}$ with $\Lambda = \mathbb{R}^* \times \mathbb{R}^*$, and one has $\eta_1(\lambda) = 4|\lambda_1|$ and $\eta_2(\lambda) = 4|\lambda_2|$. In that case, strict spectral localization is a more restrictive condition than spectral localization. Indeed, if f is spectrally localized, one has $\lambda_1 \neq 0$ *or* $\lambda_2 \neq 0$ on the support of $\mathcal{F}(f)(\lambda)$, while one has $\lambda_1 \neq 0$ *and* $\lambda_2 \neq 0$ on the support of $\mathcal{F}(f)(\lambda)$ if f is strictly spectrally localized.

1D5. *Tensor product of H-type groups.* The group $\mathbb{R}^{m_1+p_1} \otimes \mathbb{R}^{m_2+p_2}$ is easily verified to be a step-2 stratified Lie group with center of dimension $p_1 + p_2$, radical index null and a skew bilinear form $B(\lambda)$ defined on $\mathbb{R}^{m_1+m_2}$ with $m_1 = 2l_1$ and $m_2 = 2l_2$. The Zariski-open set associated with B is given by $\Lambda = (\mathbb{R}^{p_1} \setminus \{0\}) \times (\mathbb{R}^{p_2} \setminus \{0\})$ and, for $\lambda = (\lambda_1, \dots, \lambda_{p_1+p_2})$, we have

$$\begin{aligned}\eta_j(\lambda) &= \sqrt{\lambda_1^2 + \cdots + \lambda_{p_1}^2} && \text{for all } j \in \{1, \dots, l_1\},\\ \eta_j(\lambda) &= \sqrt{\lambda_{p_1+1}^2 + \cdots + \lambda_{p_1+p_2}^2} && \text{for all } j \in \{l_1+1, \dots, l_1+l_2\}.\end{aligned} \tag{1-17}$$

1E. *Main results.* The purpose of this paper is to establish optimal dispersive inequalities for the linear Schrödinger equation on step-2 stratified Lie groups associated with the sublaplacian. In view of (1-11) and the fact that the "frequencies" $\zeta(\alpha, \lambda)$ associated with $H(\lambda)$ given by (1-13) are homogeneous of degree 1 in λ, the Schrödinger operator on G behaves like a wave operator on a space of the same dimension p as the center of G, and like a Schrödinger operator on a space of the same dimension k as the radical of the canonical skew-symmetric form. By comparison with the classical dispersive estimates, the expected result would be a dispersion phenomenon with an optimal rate of decay of order $|t|^{-(k+p-1)/2}$. However, as will be seen through various examples, this anticipated rate is not always achieved. To reach this maximum rate of dispersion, we require a condition on $\zeta(\alpha, \lambda)$.

Assumption 1.4. *For each multi-index α in $\mathbb{N}^d$, the Hessian matrix of the map $\lambda \mapsto \zeta(\alpha, \lambda)$ satisfies*

$$\operatorname{rank} D_\lambda^2 \zeta(\alpha, \lambda) = p - 1,$$

where p is the dimension of the center of G.

Remark 1.5. As was observed in Section 1C3, $\zeta(\alpha, \lambda)$ is a smooth function, homogeneous of degree 1 on Λ. By homogeneity arguments, one therefore has $D_\lambda^2 \zeta(\alpha, \lambda)\lambda = 0$. It follows that

$$\operatorname{rank} D_\lambda^2 \zeta(\alpha, \lambda) \leq p - 1$$

always; hence, Assumption 1.4 may be understood as a maximal rank property.

Let us now present the dispersive inequality for the Schrödinger equation. Recall that the linear Schrödinger equation is as follows on G:

$$\begin{cases} (i\partial_t - \Delta_G) f = 0, \\ f|_{t=0} = f_0, \end{cases} \tag{1-18}$$

where the function f with complex values depends on $(t, x) \in \mathbb{R} \times G$.

Theorem 1. *Let G be a step-2 stratified Lie group with center of dimension p with $1 \leq p < n$ and radical index k. Assume that Assumption 1.4 holds. A constant C exists such that, if f_0 belongs to $L^1(G)$ and is strictly spectrally localized in a ring of $\mathbb{R}$ in the sense of Definition 1.2, then the associate solution f to the Schrödinger equation* (1-18) *satisfies*

$$\|f(t, \cdot)\|_{L^\infty(G)} \leq \frac{C}{|t|^{k/2}(1 + |t|^{(p-1)/2})} \|f_0\|_{L^1(G)} \tag{1-19}$$

for all $t \neq 0$ and the result is sharp in time.

The fact that a spectral localization is required in order to obtain the dispersive estimates is not surprising. Indeed, recall that in the $\mathbb{R}^d$ case, for instance, the dispersive estimate for the Schrödinger equation derives immediately (without any spectral localization assumption) from the fact that the solution $u(t)$ to the free Schrödinger equation on $\mathbb{R}^d$ with Cauchy data u_0 is, for $t \neq 0$,

$$u(t, \cdot) = u_0 * \frac{1}{(-2i\pi t)^{d/2}} \mathrm{e}^{-i|\cdot|^2/(4t)},$$

where $*$ denotes the convolution product in $\mathbb{R}^d$ (for a detailed proof of this fact, see for instance [Bahouri et al. 2011, Proposition 8.3]). However, proving dispersive estimates for the wave equation in $\mathbb{R}^d$ requires more elaborate techniques (including oscillating integrals), which involve an assumption of spectral localization in a ring. In the case of a step-2 stratified Lie group G, the main difficulty arises from the complexity of the expression of a Schrödinger propagator that mixes a wave operator behavior with that of a Schrödinger operator. This explains, on the one hand, the decay rate in the estimate (1-19) and on the other hand the hypothesis of strict spectral localization.

Let us now discuss Assumption 1.4. As mentioned above, there is no dispersion phenomenon for the Schrödinger equation on the Heisenberg group $\mathbb{H}^d$ (see [Bahouri et al. 2000]). Actually the same holds for the tensor product of Heisenberg groups $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ whose center is of dimension $p = 2$ and radical index null, and more generally for the case of step-2 stratified Lie groups, decomposable on nontrivial step-2 stratified Lie groups; indeed, we derive from Theorem 1 the following corollary:

Corollary 1.6. *Let $G = \bigotimes_{1\le m\le r} G_m$ be a decomposable, step-2 stratified Lie group where the groups G_m are nontrivial step-2 stratified Lie groups satisfying Assumption 1.4, of radical index k_m and with centers of dimension p_m. Then the dispersive estimate holds with rate $|t|^{-q}$:*

$$q := \frac{1}{2}\sum_{1\le m\le r}(k_m + p_m - 1) = \frac{1}{2}(k+p-r),$$

where p is the dimension of the center of G and k its radical index. Further, this rate is optimal.

Corollary 1.6 is a direct consequence of Theorem 1 and the simple observation that $\Delta_G = \bigotimes_{1\le m\le r} \Delta_{G_m}$. This result applies, for example, to the tensor product of Heisenberg groups, for which there is no dispersion, and to the tensor product of H-type groups $\mathbb{R}^{m_1+p_1} \otimes \mathbb{R}^{m_2+p_2}$, for which the dispersion rate is $t^{-(p_1+p_2-2)/2}$ (see [Del Hierro 2005]). Corollary 1.6 therefore shows that it can happen that the "best" rate of decay $|t|^{-(k+p-1)/2}$ is not reached, in particular for decomposable Lie groups. This suggests that Assumption 1.4 could be related with decomposability.

More generally, a large class of groups which do not satisfy the Assumption 1.4 is given by step-2 stratified Lie groups G for which $\zeta(0, \lambda)$ is a linear form on each connected component of the Zariski-open subset Λ. Of course, the Heisenberg group and any tensor product of Heisenberg groups is of that type. We then have the following result, which illustrates that there exist examples of groups without any dispersion and which do not satisfy Assumption 1.4.

Proposition 1.7. *Consider a step-2 stratified Lie group G whose radical index is null and for which $\zeta(0, \lambda)$ is a linear form on each connected component of the Zariski-open subset Λ. Then there exists $f_0 \in \mathcal{S}(G)$, $x \in G$ and $c_0 > 0$ such that*

$$|\mathrm{e}^{-it\Delta_G} f_0(x)| \ge c_0 \quad \textit{for all } t \in \mathbb{R}^+.$$

Finally, we point out that the dispersive estimate given in Theorem 1 can be regarded as a first step towards spacetime estimates of Strichartz type. However, due to the strict spectral localization assumption, the Besov spaces that should appear in the study (after summation over frequency bands) are naturally anisotropic; thus, proving such estimates is likely to be very technical, and is postponed to future works.

1F. ***Strategy of the proof of Theorem 1.*** In the statement of Theorem 1, there are two different results: the dispersive estimate in itself on the one hand, and its optimality on the other. Our strategy of proof is closely related to the method developed in [Bahouri et al. 2000; Del Hierro 2005], with additional, nonnegligible technicalities.

In the situation of [Bahouri et al. 2000], where the Heisenberg group $\mathbb{H}^d$ is considered, the authors prove that there is no dispersion by exhibiting explicitly Cauchy data f_0 for which the solution $f(t,\cdot)$ to the Schrödinger equation (1-18) satisfies

$$\|f(t,\cdot)\|_{L^q(\mathbb{H}^d)} = \|f_0\|_{L^q(\mathbb{H}^d)} \quad \text{for all } q \in [1,\infty]. \tag{1-20}$$

More precisely, they take advantage of the fact that the Kohn laplacian $\Delta_{\mathbb{H}^d}$ can be recast in the form

$$\Delta_{\mathbb{H}^d} = 4\sum_{j=1}^{d}(Z_j\overline{Z}_j + i\partial_s), \tag{1-21}$$

where $\{Z_1, \overline{Z}_1, \dots, Z_d, \overline{Z}_d, \partial_s\}$ is the canonical basis of the Lie algebra of left-invariant vector fields on $\mathbb{H}^d$ (see [Bahouri et al. 2012a] and the references therein for more details). This implies that, for a nonzero function f_0 belonging to $\mathrm{Ker}\big(\sum_{j=1}^d Z_j\overline{Z}_j\big)$, the solution of the Schrödinger equation on the Heisenberg group $f(t) = \mathrm{e}^{-it\Delta_{\mathbb{H}^d}} f_0$ actually solves the transport equation

$$f(z, s, t) = \mathrm{e}^{4dt\partial_s} f_0(z, s) = f_0(z, s + 4dt)$$

and hence satisfies (1-20). The arguments used in [Del Hierro 2005] for general H-type groups are similar to the ones developed in [Bahouri et al. 2000]: the dispersive estimate is obtained using an explicit formula for the solution, coming from Fourier analysis, combined with a stationary phase theorem. The Cauchy data used to prove the optimality is again in the kernel of an adequate operator, by a decomposition similar to (1-21).

As in [Bahouri et al. 2000; Del Hierro 2005], the first step of the proof of Theorem 1 consists in writing an explicit formula for the solution of the equation by use of the Fourier transform. Let us point out that, in the setting of [Bahouri et al. 2000; Del Hierro 2005], irreducible representations are isotropic with respect to the dual of the center of the group; this isotropy allows us to reduce to a one-dimensional framework and deduce the dispersive effect from a careful use of a stationary phase argument of [Stein 1986]. As we have already seen in Section 1C1, the irreducible representations are no longer isotropic in the general case of stratified Lie groups, and thus we adopt a more technical approach, making use of Schrödinger representation and taking advantage of some properties of Hermite functions appearing in the explicit representation of the solutions derived by Fourier analysis (see Section 3C). The optimality of the inequality is obtained as in [Bahouri et al. 2000; Del Hierro 2005], by an adequate choice of the initial data.

1G. ***Organization of the paper.*** The article is organized as follows. In Section 2, we write an explicit formulation of the solutions of the Schrödinger equation. Then, Section 3 is devoted to the proof of Theorem 1, and in Section 4 we discuss the optimality of the result and prove Proposition 1.7.

Finally, we mention that the letter C will be used to denote a universal constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \le CB$ for some constant C.

2. Explicit representation of the solutions

2A. ***The convolution kernel.*** Let f_0 belong to $\mathcal{S}(G)$ and let us consider $f(t, \cdot)$, the solution to the free Schrödinger equation (1-18). In view of (1-11), we have

$$\mathcal{F}(f(t, \cdot))(\lambda, \nu) = \mathcal{F}(f_0)(\lambda, \nu)\mathrm{e}^{it|\nu|^2 + itH(\lambda)},$$

which implies easily (arguing as in the Appendix) that $f(t, \cdot)$ belongs to $\mathcal{S}(G)$. Assuming that f_0 is strictly spectrally localized in the sense of Definition 1.2, there exists a smooth function θ compactly supported in a ring $\mathscr{C}$ of $\mathbb{R}$ such that if we define

$$\Theta(\lambda) := \prod_{j=1}^{d} \theta((P_j^2 + Q_j^2)(\lambda))$$

then

$$\mathscr{F}(f(t,\cdot))(\lambda,\nu) = \mathscr{F}(f_0)(\lambda,\nu)\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}.$$

Therefore, by the inverse Fourier transform (1-8), we deduce that the function $f(t,\cdot)$ may be decomposed in the following way:

$$f(t,x) = \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*} \operatorname{tr}\big((u^{\lambda,\nu}_{X(\lambda,x)})^\star \mathscr{F}(f_0)(\lambda,\nu)\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}\big)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda. \tag{2-1}$$

We set, for $X \in \mathbb{R}^n$,

$$k_t(X) := \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*} \operatorname{tr}\big(u^{\lambda,\nu}_X \Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}\big)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda. \tag{2-2}$$

The function k_t plays the role of a convolution kernel in the variables of the Lie algebra and we have the following result:

Proposition 2.1. *If the function k_t defined in* (2-2) *satisfies*

$$\|k_t\|_{L^\infty(\mathbb{R}^n)} \le \frac{C}{|t|^{k/2}(1+|t|^{(p-1)/2})} \quad \textit{for all } t \in \mathbb{R}, \tag{2-3}$$

then Theorem 1 holds.

Proof. We write, according to (2-1),

$$\begin{aligned} f(t,x) &= \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{y\in G} \operatorname{tr}\big((u^{\lambda,\nu}_{X(\lambda,x)})^* u^{\lambda,\nu}_{X(\lambda,y)}\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}\big) f_0(y)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,d\mu(y) \\ &= \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{y\in G} \operatorname{tr}\big(u^{\lambda,\nu}_{X(\lambda,y)}\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}\big) f_0(x\cdot y)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,d\mu(y). \end{aligned}$$

Note that we have used the property that the map $X \mapsto u^{\lambda,\nu}_X$ is a unitary representation, and the invariance of the Haar measure by translations.

Now we use the exponential law $y \mapsto Y = (P(\lambda,y), Q(\lambda,y), Z, R(\lambda,y))$ and the fact that $d\mu(y) = dY$, the Lebesgue measure; then we perform a linear orthonormal change of variables

$$(P(\lambda,y), Q(\lambda,y), R(\lambda,y)) \mapsto (\widetilde{P}, \widetilde{Q}, \widetilde{R}),$$

so that $d\mu(y) = dY = d\widetilde{P}\,d\widetilde{Q}\,dZ\,d\widetilde{R}$ and we write

$$\begin{aligned} f(t,x) = \kappa \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}_\lambda^*}\int_{(\widetilde{P},\widetilde{Q},Z,\widetilde{R})\in\mathbb{R}^n} &\operatorname{tr}\big(u^{\lambda,\nu}_{(\widetilde{P},\widetilde{Q},Z,\widetilde{R})}\Theta(\lambda)\mathrm{e}^{it|\nu|^2+itH(\lambda)}\big) \\ &\times f_0(x\cdot\exp(\widetilde{P},\widetilde{Q},Z,\widetilde{R}))|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda\,d\widetilde{P}\,d\widetilde{Q}\,dZ\,d\widetilde{R}. \end{aligned}$$

Thanks to the Fubini theorem and Young inequalities, we can write (dropping the tilde on the variables)

$$\begin{aligned}|f(t,x)| &= \left|\int_{(P,Q,Z,R)\in\mathbb{R}^n} k_t(P,Q,Z,R) f_0(x\cdot\exp(P,Q,Z,R))\,dP\,dQ\,dR\,dZ\right|\\ &\le \|k_t\|_{L^\infty(G)}\left|\int_{(P,Q,Z,R)\in\mathbb{R}^n} f_0(x\cdot\exp(P,Q,Z,R))\,dP\,dQ\,dR\,dZ\right|\\ &\le \|k_t\|_{L^\infty(G)}\|f_0\|_{L^1(G)}.\end{aligned}$$

Proposition 2.1 is proved. □

In the next subsections, we make preliminary work by transforming the expression of k_t and reducing the proof to statements equivalent to (2-3).

2B. ***Transformation of k_t: expression in terms of Hermite functions.*** Decomposing the operator $H(\lambda)$ in the basis of Hermite functions, and recalling notation (1-12) replaces (2-2) with

$$k_t(X)=\kappa\sum_{\alpha\in\mathbb{N}^d}\int_\Lambda\int_{\nu\in\mathfrak{r}^*_\lambda}\mathrm{e}^{it|\nu|^2+it\zeta(\alpha,\lambda)}\prod_{j=1}^d\theta(\zeta_j(\alpha,\lambda))\big(u_X^{\lambda,\nu}h_{\alpha,\eta(\lambda)}\mid h_{\alpha,\eta(\lambda)}\big)|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda,\qquad X\in\mathbb{R}^n.$$

Using the explicit form of $u_X^{\lambda,\nu}$ recalled in (1-5), we find the following result:

Lemma 2.2. *There is a constant $\tilde\kappa$ and a smooth function F such that, with the above notation, we have, for $t\neq 0$,*

$$k_t(P,Q,tZ,R)=\frac{\tilde\kappa\mathrm{e}^{-i|R|^2/(4t)}}{t^{k/2}}\sum_{\alpha\in\mathbb{N}^d}\int_\Lambda \mathrm{e}^{it\Phi_\alpha(Z,\lambda)}G_\alpha(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|F(\lambda)\,d\lambda,$$

where the phase Φ_α is given by

$$\Phi_\alpha(Z,\lambda):=\zeta(\alpha,\lambda)-\lambda(Z)$$

with notation (1-13) *and the function G_α is given by the following formula, for all $(P,Q,\eta)\in\mathbb{R}^{3d}$:*

$$G_\alpha(P,Q,\eta):=\prod_{j=1}^d\theta((2\alpha_j+1)\eta_j)g_{\alpha_j}(\sqrt{\eta_j}P_j,\sqrt{\eta_j}Q_j),\tag{2-4}$$

while, for each (ξ_1,ξ_2,n) in $\mathbb{R}^2\times\mathbb{N}$, using the notation (1-10),

$$g_n(\xi_1,\xi_2):=\mathrm{e}^{-i\xi_1\xi_2/2}\int_{\mathbb{R}}\mathrm{e}^{-i\xi_2\xi}h_n(\xi_1+\xi)h_n(\xi)\,d\xi.\tag{2-5}$$

Notice that $(g_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathbb{R}^2$ thanks to the Cauchy–Schwarz inequality and the fact that $\|h_n\|_{L^2(\mathbb{R})}=1$, and hence the same holds for $(G_\alpha)_{\alpha\in\mathbb{N}^d}$ (in $\mathbb{R}^{3d}$).

Proof. We begin by observing that, for $X=(P,Q,R,Z)$,

$$I:=(u_X^{\lambda,\nu}h_{\alpha,\eta(\lambda)}\mid h_{\alpha,\eta(\lambda)})=\mathrm{e}^{-i\nu(R)-i\lambda(Z)}\int_{\mathbb{R}^d}\mathrm{e}^{-i\lambda([\xi+P/2,Q])}h_{\alpha,\eta(\lambda)}(P+\xi)h_{\alpha,\eta(\lambda)}(\xi)d\xi,$$

with, in view of (1-4),

$$\lambda\big([\xi+\tfrac{1}{2}P, Q]\big)=B(\lambda)\big(\xi+\tfrac{1}{2}P, Q\big)=\sum_{1\le j\le d}\eta_j(\lambda)Q_j\big(\xi_j+\tfrac{1}{2}P_j\big).$$

As a consequence,

$$I=\mathrm{e}^{-i\nu(R)-i\lambda(Z)}\prod_{1\le j\le d}\int_{\mathbb{R}}\mathrm{e}^{-i\eta_j(\lambda)(\xi_j+P_j/2)Q_j}h_{\alpha_j,\eta_j(\lambda)}(P_j+\xi_j)h_{\alpha_j,\eta_j(\lambda)}(\xi_j)d\xi_j.$$

The change of variables $\tilde{\xi}_j=\sqrt{\eta_j(\lambda)}\xi_j$ gives, dropping the tilde for simplicity,

$$I=\mathrm{e}^{-i\nu(R)-i\lambda(Z)}\prod_{1\le j\le d}\int_{\mathbb{R}}\mathrm{e}^{-i\sqrt{\eta_j(\lambda)}Q_j(\xi_j+\sqrt{\eta_j(\lambda)}P_j/2)}h_{\alpha_j}(\xi_j+\sqrt{\eta_j(\lambda)}P_j)h_{\alpha_j}(\xi_j)\,d\xi_j,$$

which implies that

$$k_t(P,Q,tZ,R)=\kappa\sum_{\alpha\in\mathbb{N}^d}\int_{\mathfrak{r}(\Lambda)}\mathrm{e}^{-it\lambda(Z)-i\nu(R)}\mathrm{e}^{it\zeta(\alpha,\lambda)+it|\nu|^2}G_\alpha(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|\,d\nu\,d\lambda.$$

It is well known (see for instance Proposition 1.28 in [Bahouri et al. 2011]) that, for $t\neq 0$,

$$\int_{\mathbb{R}^k}\mathrm{e}^{-i(\nu|R)+it|\nu|^2}\,d\nu=\Big(\frac{i\pi}{t}\Big)^{\frac{k}{2}}\mathrm{e}^{-i|R|^2/(4t)},\tag{2-6}$$

where $(\cdot\,|\,\cdot)$ denotes the euclidean scalar product on $\mathbb{R}^k$. This implies that, for $t\neq 0$,

$$|k_t(P,Q,tZ,R)|\lesssim\frac{1}{|t|^{k/2}}\bigg|\sum_{\alpha\in\mathbb{N}^d}\int_\Lambda\mathrm{e}^{it\Phi_\alpha(Z,\lambda)}G_\alpha(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|F(\lambda)\,d\lambda\bigg|,$$

with F the Jacobian of the change of variables $f:\mathfrak{r}_\lambda^*\to\mathbb{R}^k$, which is a smooth function. Lemma 2.2 is proved. □

2C. ***Transformation of the kernel k_t: change of variable.*** We are then reduced to establishing that the kernel $\tilde{k}_t(P,Q,tZ)$, defined by

$$\tilde{k}_t(P,Q,tZ):=\sum_{\alpha\in\mathbb{N}^d}\int_\Lambda\mathrm{e}^{it\Phi_\alpha(Z,\lambda)}G_\alpha(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|F(\lambda)\,d\lambda,$$

satisfies

$$\|\tilde{k}_t\|_{L^\infty(G)}\le\frac{C}{1+|t|^{(p-1)/2}}\quad\text{for all } t\in\mathbb{R}.\tag{2-7}$$

To this end, let us define $m:=|\alpha|=\sum_{j=1}^d\alpha_j$ and, when $m\neq 0$, let us set $\gamma:=m\lambda\in\mathbb{R}^p$. By construction of $\eta(\lambda)$ (which is homogeneous of degree 1), we have

$$\eta(\lambda)=\tilde{\eta}_m(\gamma):=\frac{1}{m}\eta(\gamma)\quad\text{for all } m\neq 0.\tag{2-8}$$

Let us check that if λ lies in the support of $\theta(\zeta_j(\alpha,\cdot))$, then γ lies in a fixed ring $\mathscr{C}$ of $\mathbb{R}^p$, independent of α. On the one hand we note that there is a constant $C>0$ such that, on the support of $\theta(\zeta_j(\alpha,\lambda))$, the variable γ must satisfy

$$(2\alpha_j+1)\eta_j(\gamma)\le Cm \quad \text{for all } m\neq 0 \tag{2-9}$$

for all $\alpha\in\mathbb{N}^d$ such that $|\alpha|=m$. Since, for each j, we know that $\eta_j(\gamma)$ is positive and homogeneous of degree 1, we infer that the function $\eta_j(\gamma)$ goes to infinity with $|\gamma|$, so (2-9) implies that γ must remain bounded on the support of $\theta(\zeta_j(\alpha,\lambda))$. Moreover, thanks to (2-9) again, it is clear that the bound may be made uniform in m.

Now let us prove that γ may be bounded from below uniformly. We know that there is a positive constant c such that, for λ in the support of $\theta(\zeta_j(\alpha,\lambda))$, we have

$$\zeta_j(\alpha,\gamma)\ge cm \quad \text{for all } m\neq 0. \tag{2-10}$$

Writing $\gamma=|\gamma|\hat{\gamma}$ with $\hat{\gamma}$ on the unit sphere of $\mathbb{R}^p$, we find

$$|\gamma|\ge\frac{cm}{\zeta_j(\alpha,\hat{\gamma})}.$$

Defining

$$C_j:=\max_{|\hat{\gamma}|=1}\eta_j(\hat{\gamma})<\infty,$$

it is easy to deduce that if (2-10) is satisfied then

$$|\gamma|\ge\frac{cm}{(2m+d)\max_{1\le j\le d}C_j},$$

hence γ lies in a fixed ring of $\mathbb{R}^p$, independent of $\alpha\neq 0$. This fact will turn out to be important to perform the stationary phase argument.

Then we can rewrite the expression of $\tilde{k}_t(P,Q,tZ)$ in terms of the variable γ, which, in view of the homogeneity of the Pfaffian, produces the formula

$$\begin{aligned}\tilde{k}_t(P,Q,tZ)=\int_\Lambda &\mathrm{e}^{it\Phi_0(Z,\lambda)}G_0(P,Q,\eta(\lambda))|\mathrm{Pf}(\lambda)|F(\lambda)\,d\lambda\\ &+\sum_{m\in\mathbb{N}^*}\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=m}}m^{-p-d}\int\mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)}G_\alpha(P,Q,\tilde{\eta}_m(\gamma))|\mathrm{Pf}(\gamma)|F(\gamma/m)\,d\gamma.\end{aligned}$$

Note that the series in m is convergent, since the sum over $|\alpha|=m$ contributes a power m^{d-1}, whence a series of m^{-p-1}, which is convergent since $p\ge 1$. Since the functions G_α are uniformly bounded with respect to $\alpha\in\mathbb{N}^d$ and F is smooth, there is a positive constant C such that

$$\|\tilde{k}_t\|_{L^\infty(G)}\le C \quad \text{for all } t\in\mathbb{R}.$$

In order to establish the dispersive estimate, it suffices then to prove that

$$\|\tilde{k}_t\|_{L^\infty(G)}\le\frac{C}{|t|^{(p-1)/2}} \quad \text{for all } t\neq 0. \tag{2-11}$$

3. End of the proof of the dispersive estimate

In order to prove (2-11), we decompose $\tilde{k}_t$ into two parts, writing

$$\tilde{k}_t(P, Q, tZ) = k_t^1(P, Q, tZ) + k_t^2(P, Q, tZ),$$

with, for a constant c_0 to be fixed later on independently of m,

$$\begin{aligned} k_t^1(P, Q, tZ) := \int_{|\nabla_\lambda \Phi_0(Z,\lambda)| \le c_0} & e^{it\Phi_0(Z,\lambda)} G_0(P, Q, \eta(\lambda)) |\mathrm{Pf}(\lambda)| F(\lambda)\, d\lambda \\ & + \sum_{m\in\mathbb{N}^*} \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=m}} m^{-p-d} \int_{|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))| \le c_0} e^{it\Phi_\alpha(Z,\gamma/m)} G_\alpha(P, Q, \tilde{\eta}_m(\gamma)) \\ & \qquad \times F(\gamma/m) |\mathrm{Pf}(\gamma)|\, d\gamma. \end{aligned} \tag{3-1}$$

In the following subsections, we successively show (2-11) for k_t^1 and k_t^2.

3A. ***Stationary phase argument for k_t^1.*** To establish the estimate (2-11), let us first concentrate on k_t^1. As usual in this type of problem, we define, for each integral of the series defining k_t^1, a vector field that commutes with the phase, prove an estimate for each term and, finally, check the convergence of the series. More precisely, in the case when $\alpha \neq 0$ and $t > 0$ (the case $t < 0$ is dealt with exactly in the same manner), we consider the first-order operator

$$\mathcal{L}_\alpha^1 := \frac{\mathrm{Id} - i\nabla_\gamma(\Phi_\alpha(Z, \gamma/m)) \cdot \nabla_\gamma}{1 + t|\nabla_\gamma(\Phi_\alpha(Z, \gamma/m))|^2}.$$

Clearly we have

$$\mathcal{L}_\alpha^1 e^{it\Phi_\alpha(Z,\gamma/m)} = e^{it\Phi_\alpha(Z,\gamma/m)}.$$

Let us accept the next lemma for the time being.

Lemma 3.1. *For any integer N, there is a smooth function θ_N, compactly supported on a ring of $\mathbb{R}^p$, and a positive constant C_N such that, defining*

$$\psi_\alpha(\gamma) := G_\alpha(P, Q, \tilde{\eta}_m(\gamma)) F(\gamma/m) |\mathrm{Pf}(\gamma)| \tag{3-2}$$

and recalling (2-8), *we have*

$$|({}^t\mathcal{L}_\alpha^1)^N \psi_\alpha(\gamma)| \le C_N m^N \theta_N(\gamma) \big(1 + \big|t^{1/2} \nabla_\gamma(\Phi_\alpha(Z, \gamma/m))\big|^2\big)^{-N}.$$

Returning to k_t^1, let us define (recalling that γ belongs to a fixed ring $\mathscr{C}$)

$$\mathscr{C}_\alpha(Z) := \big\{\gamma \in \mathscr{C} \;\big|\; |\nabla_\gamma(\Phi_\alpha(Z, \gamma/m))| \le c_0\big\}$$

and let us write, for any integer N and $\alpha \neq 0$ (which we assume to be the case for the rest of the computations),

$$I_\alpha(Z) := \int_{\mathscr{C}_\alpha(Z)} e^{it\Phi_\alpha(Z,\gamma/m)} \psi_\alpha(\gamma)\, d\gamma = \int_{\mathscr{C}_\alpha(Z)} e^{it\Phi_\alpha(Z,\gamma/m)} ({}^t\mathcal{L}_\alpha^1)^N \psi_\alpha(\gamma)\, d\gamma, \tag{3-3}$$

where $\psi_\alpha(\gamma)$ has been defined in (3-2). Then, by Lemma 3.1, we find that for each integer N there is a constant C_N such that

$$|I_\alpha(Z)| \le C_N m^N \int_{\mathscr{C}_\alpha(Z)} \theta_N(\gamma)\big(1+t\big|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\big|^2\big)^{-N}\,d\gamma. \tag{3-4}$$

Then the end of the proof relies on three steps:

(1) a careful analysis of the properties of the support of the integral,

(2) a change of variables which leads to the estimate in $t^{-(p-1)/2}$,

(3) a control on m in order to prove the convergence of the sum over m.

Before entering into details for each step, let us observe that, by definition, we have

$$\Phi_\alpha\Big(Z,\frac{\gamma}{m}\Big) = \frac{1}{m}(\zeta(\alpha,\gamma)-\gamma(Z)),$$

with $\gamma(Z) = (A\gamma \mid Z) = (\gamma \mid {}^tAZ)$ for some invertible matrix A. Performing a change of variables in γ if necessary, we can assume without loss of generality that $A = \mathrm{Id}$. Thus we write

$$\nabla_\gamma\Big(\Phi_\alpha\Big(Z,\frac{\gamma}{m}\Big)\Big) = \frac{1}{m}(\nabla_\gamma\zeta(\alpha,\gamma)-Z). \tag{3-5}$$

3A1. *Analysis of the support of the integral defining $I_\alpha(Z)$.* Let us prove the following result:

Proposition 3.2. *One can choose the constant c_0 in* (3-1) *small enough such that, if γ belongs to $\mathscr{C}_\alpha(Z)$, then $\gamma\cdot Z \neq 0$.*

Proof. We write

$$\gamma\cdot Z = \gamma\cdot\nabla_\gamma\zeta(\alpha,\gamma) + \gamma\cdot(Z-\nabla_\gamma\zeta(\alpha,\gamma))$$

and, observing that, thanks to homogeneity arguments, $\gamma\cdot\nabla_\gamma\zeta(\alpha,\gamma) = \zeta(\alpha,\gamma)$, we deduce that, for any $\gamma\in\mathscr{C}_\alpha(Z)$,

$$|\gamma\cdot Z| \ge |\zeta(\alpha,\gamma)| - |\gamma|\,|Z-\nabla_\gamma\zeta(\alpha,\gamma)|.$$

Since, as argued above, γ belongs to a fixed ring and $\zeta(\alpha,\lambda)=0$ if and only if $\lambda=0$ (as noticed in Section 1C3), there is a positive constant c such that, for any $\gamma\in\mathscr{C}_\alpha(Z)$,

$$|\zeta(\alpha,\gamma)| \ge mc,$$

which implies, in view of the definition of $\mathscr{C}_\alpha(Z)$, that there is a positive constant $\tilde c$ depending only on the ring $\mathscr{C}$ such that

$$|\gamma\cdot Z| \ge mc - mc_0\tilde c.$$

This ensures the desired result, by choosing the constant c_0 in the definition of k_t^1 smaller than $c/\tilde c$. Proposition 3.2 is proved. $\square$

3A2. *A change of variables: the diffeomorphism $\mathscr{H}$.* We can assume without loss of generality (if not then the integral is zero) that $\mathscr{C}_\alpha(Z)$ is not empty and, in view of Proposition 3.2, we can write for any $\gamma \in \mathscr{C}_\alpha(Z)$ the orthogonal decomposition (since $Z \neq 0$)

$$\frac{1}{m}\nabla_\gamma \zeta(\alpha,\gamma) = \widetilde{\Gamma}_1 \widehat{Z}_1 + \widetilde{\Gamma}', \quad \text{with} \quad \widetilde{\Gamma}_1 := \Big(\frac{1}{m}\nabla_\gamma \zeta(\alpha,\gamma) \,\Big|\, \widehat{Z}\Big) \quad \text{and} \quad \widehat{Z}_1 := \frac{Z}{|Z|}. \tag{3-6}$$

Since $\widetilde{\Gamma}'$ is orthogonal to the vector Z, we infer that

$$|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))| = \frac{1}{m}|Z - \nabla_\gamma \zeta(\alpha,\gamma)| \geq |\widetilde{\Gamma}'|. \tag{3-7}$$

Let us consider an orthonormal basis $(\widehat{Z}_1, \dots, \widehat{Z}_p)$ in $\mathbb{R}^p$. Thanks to Proposition 3.2, we have $\gamma \cdot \widehat{Z}_1 \neq 0$ on the support of the integral defining $I_\alpha(Z)$. Obviously, the vector $\widetilde{\Gamma}'$ defined by (3-6) belongs to the vector space generated by $(\widehat{Z}_2, \dots, \widehat{Z}_p)$. To investigate the integral $I_\alpha(Z)$ defined in (3-3), let us consider the map $\mathscr{H} : \gamma \mapsto \tilde{\gamma}'$ defined by

$$\gamma \mapsto \mathscr{H}(\gamma) := (\gamma \cdot \widehat{Z}_1)\widehat{Z}_1 + \sum_{k=2}^{p} (\widetilde{\Gamma}' \cdot \widehat{Z}_k)\widehat{Z}_k =: \sum_{k=1}^{p} \tilde{\gamma}'_k \widehat{Z}_k \quad \text{for } \gamma \in \mathscr{C}_\alpha(Z). \tag{3-8}$$

Proposition 3.3. *The map $\mathscr{H}$ realizes a diffeomorphism from $\mathscr{C}_\alpha(Z)$ into a fixed compact set of $\mathbb{R}^p$.*

Proof. It is clear that the smooth function $\mathscr{H}$ maps $\mathscr{C}_\alpha(Z)$ into a fixed compact set $\mathscr{K}$ of $\mathbb{R}^p$ and that

$$\tilde{\gamma}'_1 = \gamma \cdot \widehat{Z}_1 \qquad \text{and} \qquad \tilde{\gamma}'_k = \frac{1}{m}\nabla_\gamma \zeta(\alpha,\gamma) \cdot \widehat{Z}_k \quad \text{for } 2 \leq k \leq p.$$

Now let us prove that, thanks to Assumption 1.4, the map $\mathscr{H}$ constitutes a diffeomorphism. Indeed, by straightforward computations we find that $D\mathscr{H}$, the differential of $\mathscr{H}$, satisfies

$$\begin{aligned}
&\langle D\mathscr{H}(\gamma)\widehat{Z}_1, \widehat{Z}_1\rangle = 1, \\
&\langle D\mathscr{H}(\gamma)\widehat{Z}_1, \widehat{Z}_k\rangle = \Big\langle \frac{1}{m} D^2_\gamma \zeta(\alpha,\gamma)\widehat{Z}_1, \widehat{Z}_k \Big\rangle \quad \text{for } 2 \leq k \leq p, \\
&\langle D\mathscr{H}(\gamma)\widehat{Z}_j, \widehat{Z}_k\rangle = \Big\langle \frac{1}{m} D^2_\gamma \zeta(\alpha,\gamma)\widehat{Z}_j, \widehat{Z}_k \Big\rangle \quad \text{for } 2 \leq j,k \leq p, \\
&\langle D\mathscr{H}(\gamma)\widehat{Z}_j, \widehat{Z}_1\rangle = 0 \qquad\qquad\qquad\qquad\quad \text{for } 2 \leq j \leq p.
\end{aligned}$$

Proving that $\mathscr{H}$ is a diffeomorphism amounts to showing that, for any $\gamma \in \mathscr{C}_\alpha(Z)$, the kernel of $D\mathscr{H}(\gamma)$ reduces to $\{0\}$. In view of the above formulas, if $V = \sum_{j=1}^{p} V_j \widehat{Z}_j$ belongs to the kernel of $D\mathscr{H}(\gamma)$ then $V_1 = V \cdot \widehat{Z}_1 = 0$ and $D^2_\gamma \zeta(\alpha,\gamma)V \cdot \widehat{Z}_k = 0$ for $2 \leq k \leq p$. Thus we can write $D^2_\gamma \zeta(\alpha,\gamma)V = \tau \widehat{Z}_1$ for some $\tau \in \mathbb{R}$. Since the function $\zeta(\alpha,\cdot)$ is homogeneous of degree 1, we have $D^2_\gamma \zeta(\alpha,\gamma)\gamma = 0$. We deduce that

$$0 = D^2_\gamma \zeta(\alpha,\gamma)\gamma \cdot V = \gamma \cdot D^2_\gamma \zeta(\alpha,\gamma)V = \tau \gamma \cdot \widehat{Z}_1.$$

Since $\gamma \cdot \widehat{Z}_1 \neq 0$ for all $\gamma \in \mathscr{C}_\alpha(Z)$, we find that $\tau = 0$ and therefore $D^2_\gamma \zeta(\alpha,\gamma)V = 0$. But Assumption 1.4 states that the Hessian $D^2_\gamma \zeta(\alpha,\gamma)$ is of rank $p-1$, so we conclude that V is collinear to γ. But we have seen that $V \cdot \widehat{Z}_1 = 0$, which contradicts the fact that $\gamma \cdot \widehat{Z}_1 \neq 0$. This entails that V is null and ends the proof of the proposition. $\square$

We can therefore perform the change of variables defined by (3-8) in the right-hand side of (3-4), to obtain

$$|I_\alpha(Z)| \le C_N m^N \int_{\mathcal{K}} \frac{1}{(1+t|\tilde{\gamma}'|^2)^N}\, d\tilde{\gamma}'\, d\tilde{\gamma}_1.$$

3A3. *End of the proof: convergence of the series.* Choosing $N = p-1$ implies, by the change of variables $\gamma^\sharp = t^{1/2}\tilde{\gamma}'$, that there is a constant C such that

$$|I_\alpha(Z)| \le C|t|^{-(p-1)/2} m^{p-1},$$

which gives rise to

$$\left| \int_{\mathscr{C}_\alpha(Z)} \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} \psi_\alpha(\gamma)\, d\gamma \right| \le C|t|^{-(p-1)/2} m^{p-1}.$$

We get in exactly the same way that

$$\left| \int_{|\nabla_\lambda \Phi_0(Z,\lambda)| \le c_0} \mathrm{e}^{it\Phi_0(Z,\lambda)} G_0(P,Q,\eta(\lambda)) |\mathrm{Pf}(\lambda)| F(\lambda)\, d\lambda \right| \le C|t|^{-(p-1)/2}.$$

Finally, returning to the kernel k_t^1 defined in (3-1), we get

$$|k_t^1(P,Q,tZ)| \le C|t|^{-(p-1)/2} + C|t|^{-(p-1)/2} \sum_{m\in\mathbb{N}^*} m^{d-1} m^{-d-p} m^{p-1} \le C|t|^{-(p-1)/2},$$

since the series over m is convergent. The dispersive estimate is thus proved for k_t^1.

3B. ***Stationary phase argument for k_t^2.*** We now prove (2-11) for k_t^2, which is easier since the gradient of the phase is bounded from below. We claim that there is a constant C such that

$$|k_t^2(P,Q,tZ)| \le \frac{C}{t^{(p-1)/2}}. \tag{3-9}$$

This can be achieved as above by means of adequate integrations by parts. Indeed, in the case when $\alpha \ne 0$, consider the first-order operator

$$\mathscr{L}_\alpha^2 := -i \frac{\nabla_\gamma(\Phi_\alpha(Z,\gamma/m)) \cdot \nabla_\gamma}{|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2}.$$

Note that, when $\alpha = 0$, the arguments are the same without performing the change of variable $\lambda = \gamma/m$. The operator $\mathscr{L}_\alpha^2$ obviously satisfies

$$\mathscr{L}_\alpha^2 \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} = t\mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)};$$

hence, by repeated integrations by parts, we get

$$\begin{aligned} J_\alpha(P,Q,tZ) &:= \int_{|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))| \ge c_0} \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} \psi_\alpha(\gamma)\, d\gamma \\ &= \frac{1}{t^N} \int_{|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))| \ge c_0} \mathrm{e}^{it\Phi_\alpha(Z,\gamma/m)} ({}^t\mathscr{L}_\alpha^2)^N \psi_\alpha(\gamma)\, d\lambda. \end{aligned}$$

Let us accept the following lemma for a while:

Lemma 3.4. *For any integer N, there is a smooth function θ_N compactly supported on a compact set of $\mathbb{R}^p$ such that*

$$|({}^t\mathcal{L}_\alpha^2)^N \psi_\alpha(\gamma)| \leq \frac{\theta_N(\gamma) m^N}{|\nabla_\gamma(\Phi_\alpha(Z, \gamma/m))|^N}.$$

One then observes that, if γ is in the support of the integral defining k_t^2, the lemma implies

$$|({}^t\mathcal{L}_\alpha^2)^N \psi_\alpha(\gamma)| \leq \frac{\theta_N(\gamma)}{c_0^N} m^N.$$

This estimate ensures the result as in Section 3A by taking $N = p - 1$.

3C. *Proofs of Lemmas 3.1 and 3.4.* Lemma 3.1 is an obvious consequence of the following Lemma 3.5, taking $(a, b) \equiv (0, 0)$. We omit the proof of Lemma 3.4, which consists in a straightforward modification of the arguments developed below.

Lemma 3.5. *For any integer N, one can write*

$$({}^t\mathcal{L}_\alpha^1)^N \psi_\alpha(\gamma) = f_{N,m}\big(\gamma, t^{1/2}\nabla_\gamma(\Phi_\alpha(Z, \gamma/m))\big),$$

with $|\alpha| = m$, and where $f_{N,m}$ is a smooth function supported on $\mathscr{C} \times \mathbb{R}^p$ with $\mathscr{C}$ a fixed ring of $\mathbb{R}^p$, such that for any pair $(a, b) \in \mathbb{N}^p \times \mathbb{N}^p$, there is a constant C (independent of m) such that

$$|\nabla_\gamma^a \nabla_\Theta^b f_{N,m}(\gamma, \Theta)| \leq C m^{N+|a|}(1 + |\Theta|^2)^{-N-|b|/2}.$$

Proof of Lemma 3.5. Let us prove the result by induction over N. We start with the case when N is equal to zero. Notice that in that case the function $f_{0,m}(\gamma, \Theta) = \psi_\alpha(\gamma)$ does not depend on the quantity $\Theta = t^{1/2}\nabla_\gamma(\Phi_\alpha(Z, \gamma/m))$, so we need to check that, for any $a \in \mathbb{N}^p$, there is a constant C such that

$$|\nabla_\gamma^a \psi_\alpha(\gamma)| \leq C m^{|a|} \tag{3-10}$$

when $|\alpha| = m$. The case when $a = 0$ is obvious thanks to the uniform bound on G_α. To deal with the case $|a| \geq 1$, we state the following technical result, which will be proved at the end of this section.

Lemma 3.6. *For any integer k, there is a constant C such that the following bound holds for the functions g_n, $n \in \mathbb{N}$, defined in* (2-5):

$$|(\xi_1 \partial_{\xi_1} + \xi_2 \partial_{\xi_2})^k g_n(\xi_1, \xi_2)| \leq C n^k \quad \textit{for all } (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Let us now compute $\nabla_\gamma^a \psi_\alpha(\gamma)$. Recall that, according to (3-2),

$$\psi_\alpha(\gamma) = G_\alpha(P, Q, \tilde{\eta}_m(\gamma)) F\Big(\frac{\gamma}{m}\Big) |\mathrm{Pf}(\gamma)| = F\Big(\frac{\gamma}{m}\Big) \prod_{j=1}^d \psi_{\alpha,j}(\gamma),$$

where

$$\psi_{\alpha,j}(\gamma) := \eta_j(\gamma) \tilde{\theta}((2\alpha_j + 1)\tilde{\eta}_{j,m}(\gamma)) g_{\alpha_j}\big(\sqrt{\tilde{\eta}_{j,m}(\gamma)}\, P_j, \sqrt{\tilde{\eta}_{j,m}(\gamma)}\, Q_j\big), \quad \tilde{\eta}_{j,m}(\gamma) := \frac{1}{m}\eta_j(\gamma).$$

We compute

$$\nabla_\gamma^a \psi_{\alpha,j}(\gamma) = \sum_{\substack{b\in\mathbb{N}^p\\ 0\le|b|\le|a|}} \binom{b}{a} \nabla_\gamma^b\big(\theta((2\alpha_j+1)\tilde\eta_{j,m}(\gamma))\big)\nabla_\gamma^{a-b}\big(\eta_j(\gamma) g_{\alpha_j}(\sqrt{\tilde\eta_{j,m}(\gamma)}P_j, \sqrt{\tilde\eta_{j,m}(\gamma)}Q_j)\big).$$

Let us assume first that $|a-b|=1$. Then we write, for some $1\le l\le p$,

$$\begin{aligned}\partial_{\gamma_l}&\big(\eta_j(\gamma) g_{\alpha_j}(\sqrt{\tilde\eta_{j,m}(\gamma)}P_j, \sqrt{\tilde\eta_{j,m}(\gamma)}Q_j)\big)\\ &= \partial_{\gamma_l}\eta_j(\gamma) g_{\alpha_j}(\sqrt{\tilde\eta_{j,m}(\gamma)}P_j, \sqrt{\tilde\eta_{j,m}(\gamma)}Q_j)\\ &\qquad + \eta_j(\gamma)\frac{\partial_{\gamma_l}\tilde\eta_{j,m}(\gamma)}{2\tilde\eta_{j,m}(\gamma)} \times ((\xi_1\partial_{\xi_1}+\xi_2\partial_{\xi_2})g_{\alpha_j})(\sqrt{\tilde\eta_{j,m}(\gamma)}P_j, \sqrt{\tilde\eta_{j,m}(\gamma)}Q_j).\end{aligned}$$

Next we use the fact that there is a constant C such that, on the support of $\theta((2\alpha_j+1)\tilde\eta_{j,m}(\gamma))$,

$$\tilde\eta_{j,m}(\gamma) \ge \frac{1}{Cm} \quad\text{and}\quad |\partial_{\gamma_l}\tilde\eta_{j,m}(\gamma)| \le \frac{C}{m},$$

so applying Lemma 3.6 gives

$$|\nabla_\gamma \psi_{\alpha,j}(\gamma)| \lesssim \alpha_j.$$

Recalling that $\alpha_j \le m$ and that $\psi_{\alpha,j}$ is uniformly bounded for all $j\in\{1,\dots,d\}$, this easily achieves the proof of the estimate (3-10) in the case $|a|=1$ by taking the product over j. Once we have noticed that

$$\alpha_1^{a_1}\cdots\alpha_d^{a_d} \lesssim (\alpha_1+\cdots+\alpha_d)^{a_1+\cdots+a_d},$$

the general case (when $|a|>1$) is dealt with identically; we omit the details.

Finally let us proceed with the induction: assume that for some integer N one can write

$$({}^t\mathcal{L}_\alpha^1)^{N-1}\psi_\alpha(\gamma) = f_{N-1,m}\big(\gamma, t^{1/2}\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\big),$$

where $f_{N-1,m}$ is a smooth function supported on $\mathscr{C}\times\mathbb{R}^p$, such that for any pair $(a,b)\in\mathbb{N}^p\times\mathbb{N}^p$ there is a constant C (independent of m) such that

$$|\nabla_\gamma^a\nabla_\Theta^b f_{N-1,m}(\gamma,\Theta)| \le Cm^{N-1+|a|}(1+|\Theta|^2)^{-(N-1)-|b|/2}. \tag{3-11}$$

We compute, for any function $\Psi(\gamma)$,

$$\begin{aligned}{}^t\mathcal{L}_\alpha^1\Psi(\gamma) &= i\frac{\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\cdot\nabla_\gamma\Psi(\gamma)}{1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2} + \frac{1+i\Delta(\Phi_\alpha(Z,\gamma/m))}{1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2}\Psi(\gamma)\\ &\qquad - 2it\sum_{1\le j,k\le p}\frac{\partial_{\gamma_j}\partial_{\gamma_k}(\Phi_\alpha(Z,\gamma/m))\partial_{\gamma_j}(\Phi_\alpha(Z,\gamma/m))\partial_{\gamma_k}(\Phi_\alpha(Z,\gamma/m))}{\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^2}\Psi(\gamma).\end{aligned}$$

We apply that formula to $\Psi := f_{N-1}\big(\gamma, t^{1/2}\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\big)$ and, estimating each of the three terms separately, we find (using the fact that $m \geq 1$)

$$\begin{aligned}
&\big|{}^t\mathscr{L}^1_\alpha\big(f_{N-1}\big(\gamma, t^{1/2}\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))\big)\big)\big| \\
&\quad \leq C\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-1} m^{N-1+1}\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-(N-1)} \\
&\qquad + C\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-1} m^{N-1}\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-(N-1)} \\
&\qquad\quad + Ct|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-2} m^{N-1}\big(1+t|\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))|^2\big)^{-(N-1)}
\end{aligned}$$

thanks to the induction assumption (3-11) along with (3-10) and the fact that, on $\mathscr{C}_\alpha(Z)$, all the derivatives of the function $\nabla_\gamma(\Phi_\alpha(Z,\gamma/m))$ are uniformly bounded with respect to α and Z. A similar argument allows us to control derivatives in γ and Θ, so Lemma 3.5 is proved. □

Proof of Lemma 3.6. By definition of g_n and using the change of variable

$$\xi \mapsto \xi - \tfrac{1}{2}\xi_1$$

we recover the Wigner-type formula

$$g_n(\xi_1,\xi_2) = \int_{\mathbb{R}} \mathrm{e}^{-i\xi_2\xi} h_n\big(\xi + \tfrac{1}{2}\xi_1\big) h_n\big(\xi - \tfrac{1}{2}\xi_1\big)\, d\xi.$$

Then an easy computation shows that, for all k,

$$|(\xi_1\partial_{\xi_1} + \xi_2\partial_{\xi_2})^k g_n(\xi_1,\xi_2)| \leq \int_{\mathbb{R}} \big|(\xi_1\partial_{\xi_1} + \xi\,\partial_\xi + 1)^k \big(h_n\big(\xi + \tfrac{1}{2}\xi_1\big) h_n\big(\xi - \tfrac{1}{2}\xi_1\big)\big)\big|\, d\xi.$$

By the Cauchy–Schwarz inequality (and a change of variables to transform $\xi + \frac{1}{2}\xi_1$ and $\xi - \frac{1}{2}\xi_1$ into (ξ,ξ')), it remains therefore to check that, for all k,

$$\|(\xi\,\partial_\xi)^k h_n\|_{L^2(\mathbb{R})} \leq C_k n^k.$$

This again reduces to checking that

$$\|\xi^{2k} h_n\|_{L^2(\mathbb{R})} + \|h_n^{(2k)}\|_{L^2(\mathbb{R})} \leq C_k n^k. \tag{3-12}$$

This estimate is a consequence of the identification of the domain of $\sqrt{H}$,

$$D(\sqrt{H}) = \{u \in L^2(\mathbb{R}) \mid \xi u, u' \in L^2(\mathbb{R})\},$$

which classically extends to powers of $\sqrt{H}$ as

$$D(H^{p/2}) = \{u \in L^2(\mathbb{R}) \mid \xi^{p-l} u^{(l)} \in L^2(\mathbb{R}),\ 0 \leq l \leq p\}.$$

Then (3-12) is finally obtained by applying this to $p = 2k$, recalling that $H^k h_n = (2n+1)^k h_n$. The lemma is proved. □

4. Optimality of the dispersive estimates

In this section, we first end the proof of Theorem 1 by proving the optimality of the dispersive estimates for groups satisfying Assumption 1.4. Then we prove Proposition 1.7.

4A. ***Optimality for groups satisfying Assumption 1.4.*** Let us now end the proof of Theorem 1 by establishing the optimality of the dispersive estimate (1-19). We use the fact that there always exists $\lambda^* \in \Lambda$ such that

$$\nabla_\lambda \zeta(0, \lambda^*) \neq 0, \tag{4-1}$$

where the function ζ is as defined in (1-12). Indeed, if not, the map $\lambda \mapsto \zeta(0, \lambda)$ would be constant, which is in contradiction with the fact that ζ is homogeneous of degree 1. We prove the following proposition, which yields the optimality of the dispersive estimate of Theorem 1.

Proposition 4.1. *Let* $\lambda^* \in \Lambda$ *satisfying* (4-1). *There is a function* $g \in \mathcal{D}(\mathbb{R}^p)$ *compactly supported in a connected open neighborhood of* λ^* *in* Λ *such that, for the initial data* f_0 *defined by*

$$\mathcal{F}(f_0)(\lambda, \nu) h_{\alpha,\eta(\lambda)} = \begin{cases} 0 & \text{if } \alpha \neq 0, \\ g(\lambda) h_{0,\eta(\lambda)} & \text{if } \alpha = 0, \end{cases} \qquad \text{for all } (\lambda, \nu) \in \mathfrak{r}(\Lambda), \tag{4-2}$$

there exists $c_0 > 0$ *and* $x \in G$ *such that*

$$|e^{-it\Delta_G} f_0(x)| \geq c_0 |t|^{-(k+p-1)/2}.$$

Proof. Let g be any smooth, compactly supported function over $\mathbb{R}^p$, and define f_0 by (4-2). For any point $x = e^X \in G$ in the form $X = (P = 0, Q = 0, Z, R)$, the inversion formula gives

$$e^{-it\Delta_G} f_0(x) = \kappa \int_{\lambda \in \Lambda} \int_{\nu \in \mathfrak{r}^*_\lambda} e^{it|\nu|^2 + it\zeta(0,\lambda) - i\lambda(Z) - i\nu(R)} g(\lambda) |\mathrm{Pf}(\lambda)| \, d\nu \, d\lambda.$$

To simplify notations, we set $\zeta_0(\lambda) := \zeta(0, \lambda)$. Setting $Z = tZ^*$ with $Z^* := \nabla_\lambda \zeta(0, \lambda^*) \neq 0$, we get, as in (2-6),

$$|e^{-it\Delta_G} f_0(x)| = c_1 |t|^{-k/2} \left| \int_{\lambda \in \mathbb{R}^p} e^{it(\lambda \cdot Z^* - \zeta_0(\lambda))} g(\lambda) |\mathrm{Pf}(\lambda)| \, d\lambda \right|$$

for some constant $c_1 > 0$. Without loss of generality, we can assume

$$\lambda^* = (1, 0, \dots, 0)$$

(if not, we perform a change of variables $\lambda \mapsto \Omega\lambda$, where Ω is a fixed orthogonal matrix), and we now shall perform a stationary phase in the variable λ', where we have written $\lambda = (\lambda_1, \lambda')$. For any fixed λ_1, the phase

$$\Phi_{\lambda_1}(\lambda', Z) := Z \cdot \lambda - \zeta_0(\lambda)$$

has a stationary point λ' if and only if $Z' = \nabla_{\lambda'} \zeta_0(\lambda)$ (with the same notation $Z = (Z_1, Z')$). We observe that the homogeneity of the function ζ_0 and the definition of Z^* imply that

$$Z^* = \nabla_\lambda \zeta_0(1, 0, \dots, 0) = \nabla_\lambda \zeta_0(\lambda_1, 0, \dots, 0) \quad \text{for all } \lambda_1 \in \mathbb{R};$$

hence, if $\lambda' = 0$, then the phase $\Phi_{\lambda_1}(0, Z^*)$ has a stationary point.

From now on we choose g supported near those stationary points $(\lambda_1, 0)$ and vanishing in the neighborhood of any other stationary point.

Let us now study the Hessian of Φ_{λ_1} in $\lambda' = 0$. Again because of the homogeneity of the function ζ_0, we have

$$[\operatorname{Hess} \zeta_0(\lambda)]\lambda = 0 \quad \text{for all } \lambda \in \mathbb{R}^p.$$

In particular, for all $\lambda_1 \neq 0$, $\operatorname{Hess} \zeta_0(\lambda_1, 0, \dots, 0)(\lambda_1, 0, \dots, 0) = 0$ and the matrix $\operatorname{Hess} \zeta_0(\lambda_1, 0, \dots, 0)$ in the canonical basis is of the form

$$\operatorname{Hess} \zeta_0(\lambda_1, 0, \dots, 0) = \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{Hess}_{\lambda',\lambda'} \zeta_0(\lambda_1, 0, \dots, 0) \end{pmatrix}.$$

Using that $\operatorname{Hess} \zeta_0(\lambda_1, 0, \dots, 0)$ is of rank $p-1$, we deduce that $\operatorname{Hess}_{\lambda',\lambda'} \zeta_0(\lambda_1, 0, \dots, 0)$ is also of rank $p-1$ and we conclude by the stationary phase theorem [Stein 1993, Chapter VIII.2], choosing g so that the remaining integral in λ_1 does not vanish. □

4B. ***Proof of Proposition 1.7.*** Assume that G is a step-2 stratified Lie group whose radical index is null and for which $\zeta(0, \lambda)$ is a linear form on each connected component of the Zariski-open subset Λ. Let g be a smooth nonnegative function supported in one of the connected components of Λ and define f_0 by

$$\mathcal{F}(f_0)(\lambda) h_{\alpha,\eta(\lambda)} = 0 \quad \text{for} \quad \alpha \neq 0 \qquad \text{and} \qquad \mathcal{F}(f_0)(\lambda) h_{0,\eta(\lambda)} = g(\lambda) h_{0,\eta(\lambda)}.$$

By the inverse Fourier formula, if $x = \mathrm{e}^X \in G$ is such that $X = (P = 0, Q = 0, tZ)$, then we have

$$\mathrm{e}^{-it\Delta_G}(x) = \kappa \int \mathrm{e}^{-it\lambda(Z)} \mathrm{e}^{it\zeta(0,\lambda)} g(\lambda) |\operatorname{Pf}(\lambda)| \, d\lambda.$$

Since $\zeta(0, \lambda)$ is a linear form on each connected component of Λ, there exists Z_0 in $\mathfrak{z}$ such that

$$-\lambda(Z_0) + \zeta(0, \lambda) = 0 \quad \text{for all } \lambda \in \mathfrak{z}^* \cap \operatorname{supp} g.$$

As a consequence, choosing $Z = Z_0$, we obtain

$$\mathrm{e}^{-it\Delta_G}(x) = \kappa \int g(\lambda) |\operatorname{Pf}(\lambda)| \, d\lambda \neq 0,$$

which ends the proof of the result.

Appendix: On the inversion formula in Schwartz space

This section is dedicated to the proof of the inversion formula in the Schwartz space $\mathcal{S}(G)$ (Proposition 1.1).

Proof. We first observe that, to establish (1-8), it suffices to prove that

$$f(0) = \kappa \int_{\lambda \in \Lambda} \int_{\nu \in \mathfrak{r}_\lambda^*} \operatorname{tr}(\mathcal{F}(f)(\lambda, \nu)) |\operatorname{Pf}(\lambda)| \, d\nu \, d\lambda. \tag{A-1}$$

Indeed, introducing the auxiliary function g defined by $g(x') := f(x \cdot x')$, which obviously belongs to $\mathcal{S}(G)$ and satisfies $\mathcal{F}(g)(\lambda, \nu) = u^{\lambda,\nu}_{X(\lambda,x^{-1})} \circ \mathcal{F}(f)(\lambda, \nu)$, and assuming (A-1) holds, we get

$$f(x) = g(0) = \kappa \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \operatorname{tr}(\mathcal{F}(g)(\lambda, \nu))|\operatorname{Pf}(\lambda)|\, d\nu\, d\lambda$$
$$= \kappa \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \operatorname{tr}(u^{\lambda,\nu}_{X(\lambda,x^{-1})} \mathcal{F}(f)(\lambda, \nu))|\operatorname{Pf}(\lambda)|\, d\nu\, d\lambda,$$

which is the desired result.

Let us now focus on (A-1). In order to compute the right-hand side of (A-1), we introduce

$$A := \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \operatorname{tr}(\mathcal{F}(f)(\lambda, \nu))|\operatorname{Pf}(\lambda)|\, d\nu\, d\lambda$$
$$= \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \int_{x\in G} \sum_{\alpha\in\mathbb{N}^d} \big(u^{\lambda,\nu}_{X(\lambda,x)} h_{\alpha,\eta(\lambda)} \;\big|\; h_{\alpha,\eta(\lambda)}\big)|\operatorname{Pf}(\lambda)| f(x) d\mu(x)\, d\nu\, d\lambda,$$

with the notation of Section 1C. In order to carry on the calculations, we need to resort to a Fubini argument, which comes from the identity

$$\sum_{\alpha\in\mathbb{N}^d} \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \|\mathcal{F}(f)(\lambda, \nu) h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)} |\operatorname{Pf}(\lambda)|\, d\nu\, d\lambda < \infty. \tag{A-2}$$

We postpone the proof of (A-2) to the end of this section. Thanks to (A-2), the order of integration does not matter and we can transform the expression of A: we use the fact that, for any $\alpha \in \mathbb{N}^d$,

$$(u^{\lambda,\nu}_{X(\lambda,x)} h_{\alpha,\eta(\lambda)} \mid h_{\alpha,\eta(\lambda)}) = \mathrm{e}^{-i\nu(R)-i\lambda(Z)} \int_{\mathbb{R}^d} \mathrm{e}^{-i\sum_j \eta_j(\lambda)(\xi_j+P_j/2)Q_j} h_{\alpha,\eta(\lambda)}(P+\xi) h_{\alpha,\eta(\lambda)}(\xi) d\xi,$$

where we have identified $\mathfrak{p}_\lambda$ with $\mathbb{R}^d$, and this gives rise to

$$A = \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \int_{x\in G} \int_{\xi\in\mathbb{R}^d} \sum_{\alpha\in\mathbb{N}^d} \mathrm{e}^{-i\nu(R)-i\lambda(Z)} \mathrm{e}^{-i\sum_j \eta_j(\lambda)(\xi_j+P_j/2)Q_j}$$
$$\times h_{\alpha,\eta(\lambda)}(P+\xi) h_{\alpha,\eta(\lambda)}(\xi) |\operatorname{Pf}(\lambda)| f(x)\, d\mu(x)\, d\xi\, d\nu\, d\lambda,$$

where we recall that

$$h_{\alpha,\eta(\lambda)}(\xi) = \prod_{j=1}^d h_{\alpha_j,\eta_j(\lambda)}(\xi_j) \quad \text{with} \quad h_{\alpha_j,\eta_j(\lambda)}(\xi_j) = \eta_j(\lambda)^{1/4} h_{\alpha_j}(\sqrt{\eta_j(\lambda)}\xi_j).$$

Performing the change of variables

$$\tilde\xi_j = \sqrt{\eta_j(\lambda)}\xi_j, \quad \widetilde{P}_j = \sqrt{\eta_j(\lambda)} P_j, \quad \widetilde{Q}_j = \sqrt{\eta_j(\lambda)} Q_j$$

for $j \in \{1, \dots, d\}$, we obtain, dropping the tilde on the variables,

$$A = \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}^*_\lambda} \int_{(P,Q,R,Z)\in\mathbb{R}^n} \int_{\xi\in\mathbb{R}^d} \sum_{\alpha\in\mathbb{N}^d} \mathrm{e}^{-i\nu(R)-i\lambda(Z)} \mathrm{e}^{-i\sum_l(\xi_l+P_l/2)\cdot Q_l} \prod_{j=1}^d h_{\alpha_j}(P_j+\xi_j) h_{\alpha_j}(\xi_j)$$
$$\times f(\eta^{-1/2}(\lambda)P, \eta^{-1/2}(\lambda)Q, R, Z)\, dP\, dQ\, dR\, dZ\, d\xi\, d\nu\, d\lambda,$$

with $\eta^{-1/2}(\lambda)P := (\eta_1^{-1/2}(\lambda)P_1, \dots, \eta_d^{-1/2}(\lambda)P_d)$ and similarly for Q.

Then using the change of variables $\xi'_j = \xi_j + P_j$ for $j \in \{1, \dots, d\}$ gives

$$A = \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}^*_\lambda}\int_{(\xi',Q,R,Z)\in\mathbb{R}^n}\int_{\xi\in\mathbb{R}^d}\sum_{\alpha\in\mathbb{N}^d} \mathrm{e}^{-i\nu(R)-i\lambda(Z)}\mathrm{e}^{-(i/2)\sum_l(\xi_l+\xi'_l)\cdot Q_l}\prod_{j=1}^d h_{\alpha_j}(\xi'_j)h_{\alpha_j}(\xi_j)$$
$$\times f(\eta^{-1/2}(\lambda)(\xi'-\xi), \eta^{-1/2}(\lambda)Q, R, Z)\,d\xi'\,dQ\,dR\,dZ\,d\xi\,d\nu\,d\lambda\,.$$

Because $(h_\alpha)_{\alpha\in\mathbb{N}^d}$ is a Hilbert basis of $L^2(\mathbb{R}^d)$, we have, for all $\phi \in L^2(\mathbb{R}^d)$,

$$\phi(\xi) = \sum_{\alpha\in\mathbb{N}^d}\int_{\xi'\in\mathbb{R}^d}\phi(\xi')h_\alpha(\xi')\,d\xi'\,h_\alpha(\xi),$$

which leads to

$$A = \int_{\lambda\in\Lambda}\int_{\nu\in\mathfrak{r}^*_\lambda}\int_{(Q,R,Z)\in\mathbb{R}^{d+k+p}}\int_{\xi\in\mathbb{R}^d}\mathrm{e}^{-i\nu(R)-i\lambda(Z)}\mathrm{e}^{-i\xi\cdot Q}f(0, \eta^{-1/2}(\lambda)Q, R, Z)\,dQ\,dR\,dZ\,d\xi\,d\nu\,d\lambda.$$

Applying the Fourier inversion formula successively on $\mathbb{R}^d$, $\mathbb{R}^k$ and $\mathbb{R}^p$ (and identifying $\mathfrak{r}(\Lambda)$ with $\mathbb{R}^p\times\mathbb{R}^k$), we conclude that there exists a constant $\kappa > 0$ such that

$$A = \kappa f(0),$$

which ends the proof of (A-1).

Let us conclude the proof by showing (A-2). We choose a nonnegative integer M. From the obvious fact that the function $(\mathrm{Id}-\Delta_G)^M f$ also belongs to $\mathcal{S}(G)$ (hence to $L^1(G)$), we get, in view of (1-11),

$$\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)} = (1+|\nu|^2+\zeta(\alpha,\lambda))^{-M}\mathcal{F}((\mathrm{Id}-\Delta_G)^M f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}.$$

In view of the definition of the Fourier transform on the group G, we thus have

$$\|\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}\|^2_{L^2(\mathfrak{p}_\lambda)}$$
$$= (1+|\nu|^2+\zeta(\alpha,\lambda))^{-2M}$$
$$\times\int_{\mathfrak{p}_\lambda}\biggl(\int_G(\mathrm{Id}-\Delta_G)^M f(x)u^{\lambda,\nu}_{X(\lambda,x)}h_{\alpha,\eta(\lambda)}(\xi)\,d\mu(x)\biggr)\biggl(\overline{\int_G(\mathrm{Id}-\Delta_G)^M f(x')u^{\lambda,\nu}_{X(\lambda,x')}h_{\alpha,\eta(\lambda)}(\xi)\,d\mu(x')}\biggr)\,d\xi.$$

Now, by Fubini's theorem, we get

$$\|\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}\|^2_{L^2(\mathfrak{p}_\lambda)}$$
$$= (1+|\nu|^2+\zeta(\alpha,\lambda))^{-2M}$$
$$\times\int_G\int_G(\mathrm{Id}-\Delta_G)^M f(x)\overline{(\mathrm{Id}-\Delta_G)^M f(x')}\bigl(u^{\lambda,\nu}_{X(\lambda,x)}h_{\alpha,\eta(\lambda)}\mid u^{\lambda,\nu}_{X(\lambda,x')}h_{\alpha,\eta(\lambda)}\bigr)_{L^2(\mathfrak{p}_\lambda)}\,d\mu(x)\,d\mu(x').$$

Since the operators $u^{\lambda,\nu}_{X(\lambda,x)}$ and $u^{\lambda,\nu}_{X(\lambda,x')}$ are unitary on $\mathfrak{p}_\lambda$ and the family $(h_{\alpha,\eta(\lambda)})_{\alpha\in\mathbb{N}^d}$ is a Hilbert basis of $\mathfrak{p}_\lambda$, we deduce that

$$\|\mathcal{F}(f)(\lambda,\nu)h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)} \le (1+|\nu|^2+\zeta(\alpha,\lambda))^{-M}\|(\mathrm{Id}-\Delta_G)^M f\|_{L^1(G)}.$$

Because

$$\mathrm{Card}\big(\{\alpha \in \mathbb{N}^d \mid |\alpha| = m\}\big) = \binom{m+d-1}{m} \leq C(m+1)^{d-1},$$

this ensures that

$$\sum_{\alpha\in\mathbb{N}^d} \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}_\lambda^*} \|\mathcal{F}(f)(\lambda,\nu) h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda$$
$$\lesssim \|(\mathrm{Id}-\Delta_G)^M f\|_{L^1(G)} \sum_m (m+1)^{d-1} \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}_\lambda^*} (1+|\nu|^2+\zeta(\alpha,\lambda))^{-M} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda.$$

Hence, taking $M = M_1 + M_2$ with $M_2 > \frac{1}{2}k$ implies that

$$\sum_{\alpha\in\mathbb{N}^d} \int_{\lambda\in\Lambda} \int_{\nu\in\mathfrak{r}_\lambda^*} \|\mathcal{F}(f)(\lambda,\nu) h_{\alpha,\eta(\lambda)}\|_{L^2(\mathfrak{p}_\lambda)} |\mathrm{Pf}(\lambda)|\, d\nu\, d\lambda$$
$$\lesssim \|(\mathrm{Id}-\Delta_G)^M f\|_{L^1(G)} \sum_m (m+1)^{d-1} \int_{\lambda\in\Lambda} (1+\zeta(\alpha,\lambda))^{-M_1} |\mathrm{Pf}(\lambda)|\, d\lambda.$$

Noticing that $\zeta(\alpha,\lambda) = 0$ if and only if $\lambda = 0$ and using the homogeneity of degree 1 of ζ yields that there exists $c > 0$ such that $\zeta(\alpha,\lambda) \geq cm|\lambda|$. Therefore, we can end the proof of (A-2) by choosing M_1 large enough and performing the change of variable $\mu = m\lambda$ in each term of the above series.

Proposition 1.1 is proved. □

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OBSTRUCTIONS TO THE EXISTENCE OF LIMITING CARLEMAN WEIGHTS

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We give a necessary condition for a Riemannian manifold to admit limiting Carleman weights in terms of its Weyl tensor (in dimensions 4 and higher), or its Cotton–York tensor in dimension 3. As an application, we provide explicit examples of manifolds without limiting Carleman weights and show that the set of such metrics on a given manifold contains an open and dense set.

1. Introduction

The inverse problem posed by Calderón asks for the determination of the conductivity of a medium by making voltage-to-current measurements in the boundary. The problem in the current form started with the seminal work of Calderón [1980] and research on it has been very intense. An outstanding problem is the case of anisotropic conductivities. At least in dimension $n \geqslant 3$, the right formalism seems to be the language of differential geometry. Namely for (M, g), a Riemannian manifold with boundary, and $\triangle_g$, the corresponding Laplace–Beltrami operator, does the Dirichlet-to-Neumann map determine the metric g up to a gauge transformation? The problem seemed out of reach, apart from the real analytic class (see [Kohn and Vogelius 1984; 1985]). However, a recent breakthrough in [Dos Santos Ferreira et al. 2009] allows one to solve several inverse problems in the Riemannian setting for a larger class of Riemannian manifolds. We refer to [Dos Santos Ferreira et al. 2009; 2013b; Salo 2013] for detailed accounts of these results, and recall the following theorem as an illustration. For reconstruction, see [Kenig et al. 2011] and for stability, see [Caro and Salo 2014].

Theorem 1.1 [Dos Santos Ferreira et al. 2009, Theorem 1.7; 2013a, Theorem 1.1]. *Let (M, g) be an admissible Riemannian manifold of dimension $n \geqslant 3$ with boundary and q_1, q_2 be two potentials in $L^{n/2}(M)$. Assume that 0 is not a Dirichlet eigenvalue for the corresponding Schrödinger operator $\mathcal{L}_{q_i} = -\triangle_g + q_i$. If $\Lambda_{q_1} = \Lambda_{q_2}$, then $q_1 = q_2$.*

A precise definition of admissibility is given in [Dos Santos Ferreira et al. 2009, Definition 1.5], but a necessary condition in that paper for a manifold (M, g) to be so was the existence of a so-called *limiting Carleman weight* (LCW for short). It turns out that this is a conformally invariant notion, as the following theorem shows:

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Keywords: Calderón problem, limiting Carleman weights, conformal geometry, Cotton–York tensor, Weyl tensor, conformal to a product.

Theorem 1.2 [Dos Santos Ferreira et al. 2009, Theorem 1.2]. *If (M, g) is an open manifold having a limiting Carleman weight, then some conformal multiple of the metric g, called $\tilde{g} \in [g]$, admits a parallel unit vector field. For simply connected manifolds, the converse is true.*

Recall that a vector field X is parallel if $\nabla X = 0$ and that in a simply connected manifold, X is parallel if and only if it is a Killing field (e.g., $\mathcal{L}_X g = 0$) and also a gradient field. It was proven in [Dos Santos Ferreira et al. 2009] that if $\tilde{g}$ admits a parallel vector field X, there exist local coordinates such that $X = \partial_1$ and

$$\tilde{g}(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix} \quad \text{and hence} \quad g(x) = e^{2f(x)} \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}.$$

In other words, around each point, $\tilde{g} = e \oplus g_0$, where g_0 is the metric of an $(n-1)$-manifold and e is the euclidean metric in $\mathbb{R}$.

Here we concentrate on the *local existence* of limiting Carleman weights for a given metric g. Thus we can consider the manifolds as being simply connected, and the existence of limiting Carleman weights is therefore equivalent to having parallel vector fields after a conformal change of the metric. This characterization is very elegant but it has the drawback that it requires information about the whole conformal class of g. It would be desirable to have a criterion that depends on the metric g itself in an invariant manner. It seems natural to look at this question in terms of the Weyl curvature tensor, which as a $(1, 3)$-tensor is a conformal invariant. In dimension $n \geqslant 4$, being conformally flat is equivalent to the Weyl tensor being zero.

For the case of parallel vector fields, we prove:

Theorem 1.3. *Let (M, g) be a Riemannian manifold of dimension $n \geqslant 4$. Assume that a metric $\tilde{g} \in [g]$ admits a parallel vector field. Then for any $p \in M$, there is a tangent vector $v \in T_pM$ such that the Weyl tensor of any metric in $[g]$ satisfies $W_p(v \wedge v^{\perp}) \subset v \wedge v^{\perp}$. In particular, for any $p \in M$, we have $W_p \in S^2(\Lambda^2(T_p^*M))$ has at least $n-1$ linearly independent eigenvectors that are simple.*

Recall that an element of $\Lambda_p^2(M)$ is simple if it is equal to $v \wedge w$ for $v, w \in T_pM$. In the above theorem, we are considering W_p as a curvature operator as defined, for instance, in [Besse 1987] and given a vector $v \in T_pM$, we define $v^{\perp} \in T_pM$ to be its orthogonal complement, that is, $v \oplus v^{\perp} = T_pM$. An *algebraic Weyl operator* (Weyl tensor) in a euclidean vector space V is a symmetric operator on the space $\Lambda^2 V$ that satisfies the Bianchi and the Ricci conditions (see Section 2, equations (3) and (4) for the definitions). To facilitate the reading, we include a brief overview of curvature operators in Section 2. We also give a special name to algebraic Weyl operators satisfying the condition in the above theorem.

Definition 1.4. Let W be a *Weyl tensor*. We say that W satisfies the *eigenflag condition* if and only if there is a vector $v \in V$ such that $W(v \wedge v^{\perp}) \subset v \wedge v^{\perp}$.

The following is an easy corollary of Theorem 1.3.

Corollary 1.5. *Let (M, g) be a 4-dimensional Riemannian manifold such that some $\tilde{g} \in [g]$ admits a parallel vector field. Then all the eigenvectors of the Weyl operator of g are simple.*

The theorem gives a simple algebraic condition to decide whether a given Riemannian manifold can admit a parallel vector field after a conformal change. Hence our theorem yields a quick way to decide that a given metric does not admit limiting Carleman weights; we illustrate this in Section 4 by showing that any manifold locally isometric to $\mathbb{CP}^2$ with its Fubini–Study metric does not fall into this class. However, the metric is analytic so Calderón's problem can be solved by unique continuation from the boundary, at least for analytic potentials.

Notice that conformal geometry in dimensions $n = 2$ and $n = 3$ is characterized differently. In dimension $n = 2$, every manifold is conformally flat due to the existence of isothermal coordinates. Dimension $n = 3$ is also special as conformal flatness is characterized by the vanishing of the Cotton tensor. Notice that in the presence of conformal flatness, direct proofs are available as long as the conformal parametrization is invertible. In analogy with higher dimensions, the existence of conformally parallel vector fields (and thus the existence of limiting Carleman weights) can be read algebraically from the Cotton–York tensor.

Theorem 1.6. *Let $n = 3$. If a metric $\tilde{g} \in [g]$ admits a parallel vector field then, for any $p \in M$, there is a tangent vector $v \in T_pM$ such that*

$$CY_p(v, v) = CY_p(w_1, w_2) = 0$$

for any pair of vectors $w_1, w_2 \in v^{\perp}$.

In the above theorem, the Cotton–York tensor CY is understood as a (0, 2)-tensor. The characterization can be read easily from the matrix representation of the Cotton–York tensor in any basis.

Corollary 1.7. *The above condition is equivalent to* $\det(CY_p) = 0$.

Finally, we end our study of the 3-dimensional case using Theorem 1.6 and Corollary 1.7 to determine which of the eight Thurston geometries admit limiting Carleman weights. The motivation for such a question spurs from the geometrization theorem, since any closed oriented 3-dimensional manifold arises as union of pieces admitting one of these eight geometries.

Theorem 1.8. *Among the eight Thurston geometries, only the* Nil *and* $\widetilde{\mathrm{SL}_2(\mathbb{R})}$*-geometries do not admit limiting Carleman weights. The others are admissible in the sense of* [Dos Santos Ferreira et al. 2009].

In the last section, we show that the set of metrics not admitting LCWs contains an open and dense subset of the space of all the metrics. A precise statement is contained in the next result:

Theorem 1.9. *Let U be an open submanifold of some compact manifold M without boundary having dimension $n \geqslant 3$. The set of Riemannian metrics on M for which no limiting Carleman weight exists on U contains an open and dense subset of the set of all metrics, endowed with the C^3-topology for $n = 3$, and the C^2-topology for $n \geqslant 4$.*

Remark 1.10. If a Riemannian metric on U admits an LCW, then Theorem 1.3 shows that its Weyl tensor satisfies the eigenflag condition at every point of U. We make use of that fact in our proof of Theorem 1.9, fixing a point p_0, and proving that the set of metrics whose Weyl tensor at p_0 does not satisfy the eigenflag condition is open and dense.

The proof of Theorem 1.9 gives indeed a constructive method for building explicit metrics that do not admit an LCW near any given Riemannian metric by adding a "bump" at a certain point. In Section 4 and the subsection beginning on page 584, we show explicit examples of classical homogeneous manifolds that do not admit local LCWs at any point of U.

In the companion paper [Angulo-Ardoy 2015], it is shown that the set of Riemannian metrics on U that do not admit a locally defined LCW at any point is also open and dense. This generalizes [Liimatainen and Salo 2012, Corollary 1.3], where it is proven that this set is residual.

2. Tensors in conformal geometry

The proof relies on the decomposition of the curvature tensor and its behaviour under conformal transformations. We denote by R, S and Ric the $(0,4)$-curvature, Schouten and Ricci tensors respectively, and by s the scalar curvature. Recall

$$S = \frac{1}{n-2}\Big(\mathrm{Ric} - \frac{1}{2(n-1)} s g\Big), \tag{1}$$

$$R = W + S \owedge g, \tag{2}$$

where $\owedge$ is the Kulkarni–Nomizu product of two symmetric 2-tensors, which is defined by

$$(\alpha \owedge \beta)_{ijkl} = \alpha_{ik}\beta_{jl} + \beta_{ik}\alpha_{jl} - \alpha_{il}\beta_{jk} - \alpha_{jk}\beta_{il},$$

and R and W are understood as $(0,4)$-tensors.

In the proof of Theorem 1.3, we consider W as an algebraic curvature operator; for a fuller treatment of such objects, we refer the reader to [Besse 1987], but for completeness we include here a short description. Consider the curvature at a point p as a $(0,4)$-tensor; its symmetries allow us to consider it as a symmetric linear endomorphism ρ_p of the space of bivectors $\Lambda^2(T_p^*M)$, that is, $\rho_p \in S^2(\Lambda^2(T_p^*M))$. Now the first Bianchi identity induces a projector onto the 4-forms, considered as symmetric endomorphisms of the space of bivectors:

$$b(R)(x,y,z,t) = \tfrac{1}{3}\big(R(x,y,z,t) + R(y,z,x,t) + R(z,x,y,t)\big), \tag{3}$$

so that $S^2(\Lambda^2(T_p^*M)) = \ker(b) \oplus \mathrm{Im}(b)$, where the elements of $\ker(b)$ are called the *algebraic curvature operators*. It turns out the Weyl tensors are curvature operators in the kernel of the Ricci contraction. That is, if we define $r : S^2(\Lambda^2(T_p^*M)) \to S^2(T_p^*(M))$ by

$$r(R)(x,y) = \mathrm{Tr}\big[R(x,\cdot\,,y,\cdot)\big] \tag{4}$$

then

$$\mathcal{W}(T_pM) = \ker(b) \cap \ker(r).$$

We would like to remark on one property of the space of Weyl tensors. Any rotation $\rho \in \mathrm{SO}(V)$ induces a rotation $B(\rho)$ on the space of bivectors, where $B(\rho)(v \wedge w) = \rho(v) \wedge \rho(w)$. The space of Weyl tensors is invariant under all such rotations (see [Besse 1987, 1.114]):

$$W_p \in \mathcal{W}(T_pM) \iff B(\rho) \circ W_p \circ B(\rho)^t \in \mathcal{W}(T_pM). \tag{5}$$

In our formulation of Theorem 1.3, we used the isomorphism induced by g between $\Lambda^2(T_p^*M)$ and $\Lambda^2(T_pM)$ to consider W_p as a symmetric endomorphism of the latter space. Thus, given a simple bivector $x \wedge y \in \Lambda^2(T_pM)$, we have that $W_p(x \wedge y)$ is the only bivector (not necessarily simple) such that

$$\big\langle W_p(x \wedge y), z \wedge t\big\rangle = \big\langle W_p(x,y)z, t\big\rangle$$

for any $z, t \in T_pM$, where the W_p in the right-hand side is considered as a $(1,3)$-tensor.

When dealing with a 4-dimensional manifold M, we make use of the Hodge operator (or, more precisely, of its equivalent in bivectors). This is a linear map $*:\Lambda^2_p M \to \Lambda^2_p M$ defined as

$$\langle *\omega, \tau\rangle = \langle \omega\wedge\tau, e_1\wedge e_2\wedge e_3\wedge e_4\rangle$$

for an oriented orthonormal basis $\{e_i\}$ of T_pM. Since $*$ is self-adjoint and $(*)^2\omega=\omega$ for any bivector, there is a splitting

$$\Lambda^2_p = \Lambda^+\oplus\Lambda^-$$

into eigenspaces with eigenvalues 1 and -1 respectively. Each eigenspace has dimension 3: Λ^+ is spanned by the bivectors $e_1\wedge e_2+e_3\wedge e_4$, $e_1\wedge e_3+e_4\wedge e_2$ and $e_1\wedge e_4+e_2\wedge e_3$ and Λ^- by the bivectors $e_1\wedge e_2-e_3\wedge e_4$, $e_1\wedge e_3-e_4\wedge e_2$ and $e_1\wedge e_4-e_2\wedge e_3$.

This gives a corresponding splitting for algebraic curvature operators R:

$$R=\begin{pmatrix} \frac{s}{12}\,\mathrm{Id}+W^+ & Z \\ Z^t & \frac{s}{12}\,\mathrm{Id}+W^- \end{pmatrix}, \tag{6}$$

where $W=W^+\oplus W^-$ and $Z=\big(\mathrm{Ric}-\frac{s}{4}g\big)\owedge g$ (see [Besse 1987, 1.126–1.128]).

Another important tensor in conformal geometry is the Cotton tensor. It is a (0, 3)-tensor defined as

$$C_{ijk}=(\nabla_i S)_{jk}-(\nabla_j S)_{ik}, \tag{7}$$

where the notation $(\nabla_a S)_{bc}$ stands for $(\nabla_{\partial_a} S)(\partial_b,\partial_c)$, so that

$$(\nabla_a S)_{bc}=\partial_a(S(\partial_b,\partial_c))-S(\nabla_a\partial_b,\partial_c)-S(\partial_b,\nabla_a\partial_c).$$

The Cotton tensor has the symmetries

$$\begin{aligned} C_{ijk}&=-C_{jik},\\ C_{ijk}+C_{jki}+C_{kij}&=0,\\ g^{ij}C_{ijk}&=0,\\ g^{ik}C_{ijk}&=0. \end{aligned} \tag{8}$$

The first three are straightforward, and the last follows from the second Bianchi identity (see [York 1971]).

If the metric is changed within its conformal class to $\tilde g=e^{2f}g$, the (1, 3)-Weyl tensor is unchanged, the (0, 4)-Weyl tensor changes as $\widetilde W=e^{2f}W$, and the Cotton tensor changes as

$$\widetilde C(x,y,z)=C(x,y,z)-W(x,y,z,\nabla f).$$

Indeed, conformal flatness is characterized, at any dimension $n\geqslant 3$, by the vanishing of both the Cotton and Weyl tensors at all points (see, for example, [Hertrich-Jeromin 2003, p. 5] for the classical proof and [Liimatainen and Salo 2015] for less regular metrics).

For $n\geqslant 4$, the Cotton tensor is the divergence of the Weyl tensor:

Proposition 2.1. *If $n\geqslant 3$, then $(\nabla_l W)^l_{ijk}=(n-3)C_{ijk}$.*

Thus the Cotton tensor vanishes if the Weyl tensor vanishes.

In dimension $n=3$, the Weyl tensor always vanishes, and conformal flatness has to be read directly from the Cotton tensor. This is conformally invariant, and it is equivalent to the so-called *Cotton–York* *tensor*. This new tensor is defined by considering the Cotton tensor as a map $C_p : T_pM \to \Lambda^2(T_p^*M)$ (thanks to the antisymmetry of C with respect to its first two entries) and composing with the Hodge star operator $* : \Lambda^2(T_p^*M) \to T_p^*M$. This gives a $(0, 2)$-tensor that turns out to be symmetric and trace-free, but not conformally invariant. The Cotton–York tensor also appears in the literature as a $(1, 1)$-tensor after raising one index.

In a patch with coordinates x^1, x^2, x^3, the Hodge star has the expression

$$*(dx^i \wedge dx^j) = \sum g_{lk} \frac{\epsilon^{ijl}}{\sqrt{\det(g)}}\, dx^k,$$

where ϵ^{ijl} is the signature of the permutation (i, j, l) (it takes the values 0, 1 and -1). So from

$$C = \sum C_{ijk} dx^i \otimes dx^j \otimes dx^k = \frac{1}{2} \sum C_{ijk} (dx^i \wedge dx^j) \otimes dx^k,$$

the following expression for the $(0, 2)$-version of the Cotton–York tensor follows:

$$CY_{ij} = \tfrac{1}{2} C_{kli} g_{jm} \frac{\epsilon^{klm}}{\sqrt{\det g}} = g_{jm} (\nabla_k S)_{li} \frac{\epsilon^{klm}}{\sqrt{\det g}}. \tag{9}$$

It follows from (8) that this tensor is symmetric and its trace is zero:

$$\begin{aligned} CY_{ij} &= CY_{ji}, \\ g^{ij} CY_{ij} &= CY_i^i = 0. \end{aligned}$$

Remark 2.2. The reader may notice, looking at (9), that the Cotton–York tensor is not conformally invariant. However, if the metric g is replaced by λg, the Cotton–York tensor is scaled by $\lambda^{-1/2}$ so, in particular, the determinant of the tensor is zero if and only if it is zero for any conformal metric. The $(1, 1)$-version of the Cotton–York tensor is not conformally invariant either. We remark that our computation of the scaling factor differs from the one found in the literature [York 1971].

3. Proof of Theorem 1.3

The $(1, 3)$-Weyl tensor is invariant under conformal changes of the metric. Thus, thanks to Theorem 1.2, we can assume that g admits a parallel vector field X. As in [Dos Santos Ferreira et al. 2009], we notice that in the appropriate semigeodesic coordinates, $X = e_1$ and the metric is written as

$$\tilde{g}(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}.$$

For any set of coordinates, e_1 is parallel if and only if $R_{1ijk} = 0$ (the sufficiency follows from Frobenius' theorem). Moreover, notice that $g_{1j} = 0$ for all $j \geqslant 2$. Thus, by the formula of the Schouten tensor, it holds that in these coordinates, $S_{1j} = 0$ for all $j \geqslant 2$. Now for $j, k, l \geqslant 2$,

$$(S \owedge g)_{1jkl} = S_{1k} g_{jl} + S_{jl} g_{1k} - S_{1l} g_{jk} - S_{jk} g_{1l} = 0,$$

and by the decomposition of the curvature tensor,

$$W_{1jkl} = R_{1jkl} - (S \owedge g)_{1jkl} = 0.$$

Recall that W acts on bivectors by

$$W(e_i \wedge e_j) = \sum_{k,l} W_{ijkl} e_k \wedge e_l.$$

Given $p \in M$, let $v = X_p = e_1$; thus $g_{1j} = \delta_{1j}$; in these coordinates, $e_1 \wedge e_1^\perp$ is invariant. In other words, for every $j, k, l \neq 1$,

$$\langle W(e_1 \wedge e_j), e_k \wedge e_l \rangle = 0 = W_{1jkl}.$$

Therefore $W(v \wedge v^\perp) \subset v \wedge v^\perp$, and the first part of Theorem 1.3 is proved. Finally, $v \wedge v^\perp$ is an $(n-1)$-dimensional subspace of simple bivectors; thus it contains $n-1$ linearly independent simple eigenbivectors of W.

Proof of Corollary 1.5. Let $v \in T_pM$ be the vector given by Theorem 1.3. Since $\Lambda^2(v^\perp)$ is orthogonal to $v \wedge v^\perp$, and $v \wedge v^\perp$ is invariant by W, we know that W also leaves $\Lambda^2(v^\perp)$ invariant. But $v^\perp$ being 3-dimensional implies that every element of $\Lambda^2(v^\perp)$ is simple, finishing the proof. □

4. Examples of manifolds without LCWs

This section provides explicit examples of Riemannian manifolds without any LCWs. Namely, this:

Theorem 4.1. *Let $\mathbb{CP}^2$ be the complex projective space with its Fubini–Study metric g_{can}. Then any subdomain $\Omega \subset \mathbb{CP}^2$ with boundary does not admit an LCW.*

Proof. Since $\mathbb{CP}^2$ is 4-dimensional, we will make use of the decomposition

$$\Lambda_p^2 \mathbb{CP}^2 = \Lambda^+ \oplus \Lambda^-$$

induced by the Hodge operator $* : \Lambda_p^2 \mathbb{CP}^2 \to \Lambda_p^2 \mathbb{CP}^2$ as was explained in Section 2.

Use $J : T_p\mathbb{CP}^2 \to T_p\mathbb{CP}^2$ to denote the canonical complex structure of $\mathbb{CP}^2$ and let $\{e_i\}$ be an orthonormal basis of $T_p\mathbb{CP}^2$, with $e_2 = Je_1,\ e_4 = Je_3$. A basis of $\Lambda_p^2\mathbb{CP}^2$ is given by

$$\phi_1 = e_1 \wedge e_2 + e_3 \wedge e_4, \quad \phi_2 = e_1 \wedge e_3 - e_2 \wedge e_4, \quad \phi_3 = e_1 \wedge e_4 + e_2 \wedge e_3 \tag{10}$$

for its self-dual component, and

$$\psi_1 = e_1 \wedge e_2 - e_3 \wedge e_4, \quad \psi_2 = e_1 \wedge e_3 + e_2 \wedge e_4, \quad \psi_3 = e_1 \wedge e_4 - e_2 \wedge e_3 \tag{11}$$

for its anti-self-dual part.

The curvature of $\mathbb{CP}^2$ is computed in several texts in Riemannian geometry; we give a quick overview here, but see [do Carmo 1992, p. 189] for more details. Viewing S^5 as the unit sphere in $\mathbb{C}^3$, and $\mathbb{CP}^2$ as the basis of a Riemannian submersion under the action of S^1 on S^5 given by $z \cdot (z_1, z_2, z_3) = (zz_1, zz_2, zz_3)$, the sectional curvature of a 2-plane in $\mathbb{CP}^2$ is

$$K(X, Y) = 1 + 3\cos^2\phi,$$

where X, Y is an orthonormal basis of the plane in $\mathbb{CP}^2$, and $\cos\phi$ is the hermitian product $\langle \overline{X}, i\overline{Y} \rangle$ of the horizontal lifts $\overline{X}, \overline{Y}$ of X, Y respectively to S^5. From here it is easy to see that the sectional curvatures of $\mathbb{CP}^2$ take values between 1 and 4. Since norms of horizontal lifts agree with those of the vectors in the base, $0 \leqslant \langle \overline{X}, i\overline{Y} \rangle \leqslant 1$. Therefore $K(X, Y) = 1$ only when $\langle \overline{X}, i\overline{Y} \rangle = 0$; since the complex structure of $\mathbb{CP}^2$ is induced by that of $\mathbb{C}^3$, this happens only when the plane $\sigma = \{X, Y\}$ satisfies $J\sigma \perp \sigma$. On the other hand, a 2-plane σ will have $K(\sigma) = 4$ if and only if σ is complex, i.e, $J\sigma = \sigma$.

To recover the full curvature operator from the sectional curvature, either use an explicit formula for the terms of the curvature in terms of the sectional curvatures, as the one in [Cheeger and Ebin 1975, p. 16], or continue using O'Neill's formula for the curvature terms $\langle R(x, y)z, w \rangle$ in $\mathbb{CP}^2$ in terms of the corresponding curvature terms in S^5 and O'Neill's A-tensor, as in [do Carmo 1992, p. 187, Exercise 10(a)]. The reader will also find [Sakai 1996, pp. 76–77] useful, which, in spite of defining the curvature tensor differently, makes explicit the relation between the complex structure of $\mathbb{CP}^2$ and the submersion $S^5 \to \mathbb{CP}^2$.

The only nonvanishing components of the curvature tensor are then

$$\langle R(e_1, e_2)e_1, e_2 \rangle = \langle R(e_3, e_4)e_3, e_4 \rangle = 4,$$
$$\langle R(e_1, e_3)e_1, e_3 \rangle = \langle R(e_1, e_4)e_1, e_4 \rangle = \langle R(e_2, e_3)e_2, e_3 \rangle = \langle R(e_2, e_4)e_2, e_4 \rangle = 1$$

for the sectional curvatures and

$$\langle R(e_1, e_2)e_3, e_4 \rangle = 2, \quad \langle R(e_1, e_3)e_2, e_4 \rangle = 1, \quad \langle R(e_1, e_4)e_2, e_3 \rangle = -1$$

for the mixed terms.

In the space of bivectors and with the ϕ_i, ψ_i as above, the curvature operator R_p satisfies

$$\begin{aligned} R_p(\phi_1) &= 6\phi_1, & R_p(\phi_2) &= 0, & R_p(\phi_3) &= 0, \\ R_p(\psi_1) &= 2\psi_1, & R_p(\psi_2) &= 2\psi_2, & R_p(\psi_3) &= 2\psi_3. \end{aligned}$$

Thus the curvature operator R_p of g_{can} is written as

$$R_p = \begin{pmatrix} 6E & 0 \\ 0 & 2I \end{pmatrix},$$

where I is the 3×3 identity matrix, and E is the matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A simple computation, using (6), yields

$$W_p^+ = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad W_p^- = 0.$$

Observe that every eigenvector of W_p belongs to either Λ^+ or Λ^-, which contain no simple eigenvectors. Hence no eigenvector of W_p is simple, which, by Corollary 1.5, implies that no subdomain of $(\mathbb{CP}^2, g_{fs})$ admits an LCW. □

Similar arguments can be used in higher dimensions to rule out domains in $\mathbb{CP}^n$ or other suitable symmetric spaces.

5. The 3-dimensional case

Restrictions on the Cotton–York tensor.

Proof of Theorem 1.6. Since Theorem 1.6 is formulated at some fixed point $p \in M$, we can assume that everything is local. Recall that in semigeodesic coordinates, the metric is independent of x_1, and that

$$g_{1j} = 0 = S_{1j} = S_{j1} = 0$$

if $j \neq 1$. It follows also that

$$0 = \Gamma^k_{1j} = \Gamma^k_{j1} = \Gamma^1_{jk}.$$

These identities simplify the expression of the Cotton–York tensor: if either i, l or k is equal to 1, then

$$(\nabla_k S)_{li} = \partial_k(S_{li}).$$

Now for $i \neq 1 \neq j$, we notice that $m \neq 1$ for each nonzero term in the sum:

$$CY_{ij} = g_{jm}(\nabla_k S)_{li}\frac{\epsilon^{klm}}{\sqrt{\det g}}.$$

Thus for $\epsilon^{klm} \neq 0$, necessarily k or l are equal to 1, and hence

$$CY_{ij} = g_{jm}\partial_k S_{li}\frac{\epsilon^{klm}}{\sqrt{\det g}} = 0$$

using that $\partial_1 S_{li} = 0 = S_{1i}$ for $i \neq 1$.

Similarly,

$$\sqrt{\det g}C_{11} = g_{1m}\partial_k S_{l1}\epsilon^{klm} = \partial_k S_{l1}\epsilon^{kl1} = 0.$$

These equations yield that $v = \partial/\partial_{x_1}$ is the vector required in Theorem 1.6. □

In fact, since the Cotton tensor is invariant after conformal changes of the metric, we can assume that M is isometric to $\mathbb{R} \times \Sigma$, where Σ is a surface. Taking coordinates (x_1, x_2, x_3), with $t = x_1$ and (x_2, x_3) isothermal coordinates of Σ, the metric reads as $g = dx_1^2 + e^f(dx_2^2 + dx_3^2)$ for some function $f(x_2, x_3)$ on Σ. In these coordinates, a simple expression of the full Cotton–York tensor is available. Namely, the Ricci tensor takes the values

$$\mathrm{Ric}_{1i} = 0, \quad \mathrm{Ric}_{22} = \mathrm{Ric}_{33} = -\tfrac{1}{2}(\Delta f), \quad \mathrm{Ric}_{23} = 0,$$

the scalar curvature is

$$s = -(\Delta f)e^{-f},$$

the Schouten tensor equals

$$\mathrm{Ric}_{11} = \tfrac{1}{4}(\Delta f)e^{-f}, \quad \mathrm{Ric}_{22} = \mathrm{Ric}_{33} = -\tfrac{1}{4}(\Delta f), \quad \mathrm{Ric}_{12} = \mathrm{Ric}_{13} = \mathrm{Ric}_{23} = 0,$$

and a further calculation using formula (9) yields the following explicit formula for the Cotton–York tensor:

$$CY_{12} = CY_{21} = -\tfrac{1}{4}\big(\Delta f\,\partial_3 f - \partial_3(\Delta f)\big)e^{-f},$$
$$CY_{13} = CY_{31} = \tfrac{1}{4}\big(\Delta f\,\partial_2 f - \partial_2(\Delta f)\big)e^{-f}.$$

The Cotton–York tensor of the product of $\mathbb{R}$ with a surface Σ in isothermal coordinates can also be expressed as

$$CY = \tfrac{1}{2} dx_1 \cdot (*ds),$$

where $\cdot$ is the symmetric product of forms, s is the scalar curvature of the surface, and $*$ is the Hodge star operator of the surface, which sends the 1-form ds to an orthogonal 1-form on Σ.

Proof of Corollary 1.7. Corollary 1.7 follows from this lemma:

Lemma 5.1. *Let V be a 3-dimensional euclidean space, and $A : V \to V$ be a symmetric endomorphism. Then there exists a 2-dimensional subspace P such that for any $v_1, v_2 \in P$, $w \in P^\perp$, we have*

$$\langle A v_1, v_2 \rangle = \langle A w, w \rangle = 0 \tag{12}$$

if and only if $\det(A) = \mathrm{Tr}(A) = 0$.

Proof. The "only if" part is clear: Let $e_1, e_2 \in P$ and $e_3 \in P^\perp$ form an orthonormal basis. The expression of A in these coordinates is

$$A = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{pmatrix}.$$

Thus the conditions on the determinant and the trace of A are obvious.

For the converse, first notice that since it is symmetric, we can diagonalize A. Our conditions imply the existence of $\lambda_1 \in \mathbb{R}$ and an orthonormal basis v_1, v_2, v_3 such that

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The desired plane P is the span of $\{v_1 + v_2, v_3\}$. Namely for $t_1, t_2 \in \mathbb{R}$,

$$\big\langle A\big(t_1(v_1+v_2)+t_2v_3\big), t_1(v_1+v_2)+t_2v_3\big\rangle = \lambda_1 t_1 \big\langle v_1 - v_2, t_1(v_1+v_2)+t_2v_3\big\rangle = 0,$$

and similarly

$$\big\langle A(v_1 - v_2), v_1 - v_2\big\rangle = \lambda_1 \big\langle v_1 + v_2, v_1 - v_2\big\rangle = 0. \qquad \square$$

Remark 5.2. The matrix expressions of the (1, 1)- and the (0, 2)-versions of the Cotton–York tensor are different at any point where the matrix for the metric is not the identity. However, the determinant will vanish for one of them if and only if it does for the other.

LCWs in the Thurston geometries. The rest of this section deals with the existence of LCWs among the eight Thurston geometries. A good reference for their definition and properties is the classical paper [Scott 1983]. We begin with the following six geometries:

- $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$: These three geometries are conformally flat, and consequently admit multiple LCWs.
- $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$: This case is obvious, with the LCW lying along the $\mathbb{R}$-direction.

- Sol: Recall that Sol can be seen as $\mathbb{R}^3$ with a metric given in the standard coordinates (x, y, z) by

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

The metric $\bar{g} = e^{-2z} \cdot g$ splits along ∂_x, and therefore g has an LCW.

The last two geometries have a different behaviour.

Theorem 5.3. $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ *and* Nil *do not admit LCWs.*

Proof. We start by recalling the properties we will need.

- $\widetilde{\mathrm{SL}_2(\mathbb{R})}$: Since our study is local, we will work directly in $\mathrm{SL}_2(\mathbb{R})$. Being a Lie group, $\mathrm{SL}_2(\mathbb{R})$ has a left-invariant metric defined by declaring the three matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

as an orthonormal basis of $T_I\,\mathrm{SL}_2(\mathbb{R})$. We will use E_1, E_2, E_3 to denote the left-invariant vector fields in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ agreeing with e_1, e_2, e_3 at the identity.

To write the metric in coordinates, we will use the Iwasawa decomposition that writes any element in $\mathrm{SL}_2(\mathbb{R})$ as an ordered product of three matrices of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that we can take θ, t and s as coordinates in a suitable neighbourhood of the identity matrix I, with ∂_θ, ∂_t and ∂_s agreeing with E_1, E_2 and E_3 at I, but not away from it. In fact, in these coordinates, a tedious calculation shows that the coefficients for the above-mentioned left-invariant metric are

$$\begin{gathered} g_{\theta\theta} = (4s^2+1)e^{2t} + \big((s^2-1)e^t + e^{-t}\big)^2, \quad g_{\theta s} = (s^2-1)e^t + e^{-t}, \\ g_{\theta t} = \big((s^2-1)e^t + e^{-t}\big)s + 2se^t, \quad g_{tt} = s^2+1, \quad g_{ts} = s, \quad g_{ss} = 1. \end{gathered} \tag{13}$$

To see this, write the orthonormal basis $\{E_i\}$ in terms of $\partial_\theta, \partial_t, \partial_s$.

Once we have an expression for the metric tensor in coordinates, computing the determinant of the Cotton–York tensor is a matter of following the definitions with a lot of care. The Ricci tensor is

$$\begin{aligned} \mathrm{Ric}_{\theta\theta} &= -8s^2e^{2t}, \\ \mathrm{Ric}_{\theta t} = \mathrm{Ric}_{t\theta} &= -4se^t, \\ \mathrm{Ric}_{tt} &= -2, \end{aligned}$$

the scalar curvature is $s = -2$, the Schouten tensor is

$$\begin{gathered} S_{\theta\theta} = -8s^2e^{2t} + \tfrac{1}{2}(4s^2+1)e^{2t} + \tfrac{1}{2}\big((s^2-1)e^t + e^{-t}\big)^2, \\ S_{\theta t} = \big(\tfrac{1}{2}s^3 - \tfrac{7}{2}s\big)e^t + \tfrac{1}{2}e^{-t}s, \quad S_{\theta s} = \tfrac{1}{2}(s^2-1)e^t + \tfrac{1}{2}e^{-t}, \\ S_{tt} = -\tfrac{3}{2} + \tfrac{1}{2}s^2, \quad S_{ts} = \tfrac{1}{2}s, \quad S_{ss} = \tfrac{1}{2}. \end{gathered} \tag{14}$$

The Cotton–York tensor of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ can be computed from these equations and formula (9), yielding

$$\begin{gathered} CY_{\theta\theta} = 4s^4e^{2t} - 28s^2e^{(2t)} + 8s^2 + 8e^{2t} + 4e^{(-2t)} - 12, \\ CY_{\theta t} = 4s^3e^t + 4se^{-t} - 14se^t, \quad CY_{\theta s} = 4s^2e^t + 4e^{-t} - 6e^t, \\ CY_{tt} = 4s^2 - 4, \quad CY_{ts} = 4s, \quad CY_{ss} = 4. \end{gathered} \tag{15}$$

When $s = t = 0$, this yields

$$CY_{(\theta,0,0)} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -4 & 0 \\ -2 & 0 & 4 \end{pmatrix},$$

with nonzero determinant. Since the metric is left-invariant, the same happens at any other point.

• Nil: This is the space of triangular matrices of the form

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\},$$

with the natural left-invariant metric. This turns out to be just $\mathbb{R}^3$ with the metric

$$g = dx^2 + dy^2 + (dz - x\,dy)^2.$$

Once again, we apply the standard formulas, and find the Ricci tensor

$$\mathrm{Ric} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2}x^2 - \frac{1}{2} & -\frac{1}{2}x \\ 0 & -\frac{1}{2}x & \frac{1}{2} \end{pmatrix},$$

the scalar curvature $s = -\frac{1}{2}$, the Schouten tensor,

$$S = \begin{pmatrix} -\frac{3}{8} & 0 & 0 \\ 0 & \frac{5}{8}x^2 - \frac{3}{8} & -\frac{5}{8}x \\ 0 & -\frac{5}{8}x & \frac{5}{8} \end{pmatrix},$$

and the Cotton–York tensor

$$CY = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -x^2 + \frac{1}{2} & x \\ 0 & x & -1 \end{pmatrix}.$$

The determinant of CY is $-\frac{1}{4}$, and there are no local LCWs in this space. □

6. Proof of Theorem 1.9 in dimensions $n \geqslant 4$

We divide the proof into two parts. First, we examine the set of algebraic Weyl operators satisfying the eigenflag condition. We prove that this set is semialgebraic (and, in fact, algebraic in dimension 4), and compute its codimension explicitly. Then, we see how to use this to approximate any metric by metrics whose Weyl tensor at a given point p_0 does not satisfy the eigenflag condition.

The algebraic part is contained in the following theorem.

Theorem 6.1. *The set $\mathcal{EW}$ of Weyl tensors that satisfy the eigenflag condition is a semialgebraic subset of the space of Weyl tensors with codimension*

$$\tfrac{1}{3}n^3 - n^2 - \tfrac{4}{3}n + 2.$$

In particular, the codimension is 2 for $n = 4$ and 12 for $n = 5$.

Remark 6.2. A semialgebraic subset of $\mathbb{R}^n$ is defined by equations and inequalities involving polynomials. We will need the Tarski–Seidenberg theorem, which states that the image of a semialgebraic set by a map given by polynomials is a semialgebraic set (see [Bochnak et al. 1998, Proposition 2.2.7]). At present, we do not know whether the set of Weyl tensors satisfying the eigenflag condition is an algebraic set; nonetheless, this will not be necessary for the purposes of this paper.

Dimension 4. Before proving Theorem 6.1, we recall the special structure of the Weyl operator in dimension 4. The curvature tensor in dimension 4 has the following decomposition induced by the Hodge operator $\star$ (see Section 2):

$$R = \begin{pmatrix} \frac{s}{12}\,\mathrm{Id} + W^+ & Z \\ Z^t & \frac{s}{12}\,\mathrm{Id} + W^- \end{pmatrix},$$

where W^+ (resp. W^-) is any symmetric traceless operator on the 3-dimensional space Λ^+ (resp. Λ^-). Reciprocally, any such operators appear as W^+ and W^- for some curvature operator.

Clearly there are no simple bivectors in Λ^+ or Λ^-. The Weyl operator can have simple eigenvectors only when W^+ and W^- share some eigenvalue since in that case W could have some eigenspace that would not be contained in Λ^+ or Λ^-.

In particular, if all the eigenvalues of W are different, all eigenvectors of W will be nonsimple. This gives the following argument for the density of Weyl operators in dimension 4 that do not satisfy the eigenflag condition.

Let $W_0 = W_0^+ \oplus W_0^-$ be a Weyl operator in $\mathcal{EW}$. We define a sequence of Weyl operators W_j having the same eigenvectors of W_0 and such that the corresponding eigenvalues of W_j converge to those of W_0. It is clear that we can choose the six eigenvalues of W_j to be different (thus assuring that $W_j \notin \mathcal{EW}$) and also such that the three eigenvalues of either W_j^+ or W_j^- add up to zero; this assures us that W_j is a Weyl operator, thus proving density of the complement of $\mathcal{EW}$.

Notice that this automatically implies the openness and denseness of the complement of $\mathcal{EW}$. Now we turn to the proof of Theorem 6.1.

Proof of Theorem 6.1 for $n = 4$. Let $W = W^+ \oplus W^-$ be a Weyl operator satisfying the eigenflag condition. Since $W \in \mathcal{EW}$, there is some $v \in V$ such that $W(v \wedge v^\perp) \subset v \wedge v^\perp$. This also implies that $\Lambda^2(v^\perp) = (v \wedge v^\perp)^\perp$ is an eigenspace of W.

We can perform a rotation in V so that $e_1 = v$ and $e_1 \wedge e_2$, $e_1 \wedge e_3$ and $e_1 \wedge e_4$ are eigenvectors of the Weyl operator with corresponding eigenvalues λ_{12}, λ_{13} and λ_{14}. Notice that the induced rotation in $\Lambda^2(V)$ leaves Λ^+ and Λ^- invariant.

We now compute $W(e_3 \wedge e_4)$: By the eigenflag condition,

$$W(e_3 \wedge e_4) \in \langle e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 \rangle.$$

By the choice of basis,

$$W(e_1 \wedge e_2 + e_3 \wedge e_4) = \lambda_{12}\, e_1 \wedge e_2 + W(e_3 \wedge e_4)$$

must lie in Λ^+. From $\lambda_{12}(e_1 \wedge e_2 + e_3 \wedge e_4) \in \Lambda^+$, it follows that

$$W(e_1 \wedge e_2 + e_3 \wedge e_4) - \lambda_{12}(e_1 \wedge e_2 + e_3 \wedge e_4) \in \langle e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 \rangle \cap \Lambda^+ = \{0\}.$$

Hence $W(e_3 \wedge e_4) = \lambda_{12} e_3 \wedge e_4$. Similarly, $W(e_2 \wedge e_4) = \lambda_{13} e_2 \wedge e_4$ and $W(e_2 \wedge e_3) = \lambda_{14} e_2 \wedge e_3$.

Thus in the basis of $\Lambda^2(V)$ as given in (10) and (11), W is written as

$$\begin{pmatrix} \lambda_{12} &&&&& \\ & \lambda_{13} &&&& \\ && \lambda_{14} &&& \\ &&& \lambda_{12} && \\ &&&& \lambda_{13} & \\ &&&&& \lambda_{14} \end{pmatrix},$$

and since both W^+ and W^- are traceless, $\lambda_{12} + \lambda_{13} + \lambda_{14} = 0$.

The dimension of the space of Weyl tensors in dimension 4 is 10. Let us now compute the dimension of $\mathcal{EW}$. By the above, the map

$$\Phi : \mathrm{SO}(V) \times \mathbb{R}^2 \to \mathcal{EW},$$

sending $(\rho, \lambda_{12}, \lambda_{13})$ to

$$B(\rho) \cdot \begin{pmatrix} \lambda_{12} &&&&& \\ & \lambda_{13} &&&& \\ && -\lambda_{12} - \lambda_{13} &&& \\ &&& \lambda_{12} && \\ &&&& \lambda_{13} & \\ &&&&& -\lambda_{12} - \lambda_{13} \end{pmatrix} \cdot B(\rho)^t,$$

is surjective, where $B(\rho)$ is the rotation on $\Lambda^2(V)$ induced by ρ.

This means that $\mathcal{EW}$ is the image of an algebraic set by an algebraic map, so it is a semialgebraic subset of $\mathcal{W}$ by the Tarski–Seidenberg theorem [Bochnak et al. 1998, Proposition 2.2.7]. The map is singular only if two of the three numbers λ_{12}, λ_{13} and $\lambda_{14} = -\lambda_{12} - \lambda_{13}$ coincide, or if all of them vanish. This implies that the map Φ is locally injective in an open set, and thus the dimension of $\mathcal{EW}$ is $\dim \mathrm{SO}(V) + 2 = 8$. $\square$

Remark 6.3. As mentioned before, we do not know whether $\mathcal{EW}$ is an algebraic set. However, in dimension 4, we have shown that operators in $\mathcal{EW}$ have at least one double eigenvalue. It follows that $\mathcal{EW}$ is contained in a proper algebraic set.

Theorem 6.4. *In dimension* 4, *the set of Weyl tensors having different eigenvalues and nonsimple eigenvectors is the complement of a proper algebraic set.*

Proof. The set of algebraic operators with at least one multiple eigenvalue is an algebraic set given by the equations

$$\Delta_t(\det(tW - I)) = 0,$$

where Δ_t is the discriminant of a polynomial in t. The discriminant of the characteristic polynomial of W vanishes exactly when the characteristic polynomial has nonsimple roots, which happens when the operator has eigenspaces of dimension greater than 1. □

Weyl tensors with the eigenflag condition in dimensions n ⩾ 5.

Proof of Theorem 6.1 for $n \geqslant 5$. As in dimension 4, we will find an algebraic map from a space of dimension smaller than $\dim \mathcal{W}$ whose image is exactly $\mathcal{EW}$ and use [Bochnak et al. 1998, Proposition 2.2.7] to show that $\mathcal{EW}$ is semialgebraic.

Let W be an algebraic Weyl operator with the eigenflag condition on the vector space V. We will build an orthonormal basis of V such that W is written conveniently.

By hypothesis, there is vector v such that $W(v \wedge v^{\perp}) \subset v \wedge v^{\perp}$. The operator $W|_{v\wedge v^{\perp}}$ is symmetric and diagonalizes in an orthonormal basis of bivectors contained in $v \wedge v^{\perp}$. All such eigenvectors are of the form $v \wedge w$, and two such bivectors $v \wedge w_1$ and $v \wedge w_2$ are orthogonal if and only if w_1 is orthogonal to w_2. We let $\{e_1 = v, e_2, \dots, e_n\}$ be an orthonormal basis of $v \wedge v^{\perp}$ such that $W|_{v\wedge v^{\perp}}$ is diagonal in the basis $e_1 \wedge e_k$, with eigenvalue λ_k.

Then, in this basis,

$$W = \begin{pmatrix} \lambda_2 & & & \\ & \ddots & & \\ & & \lambda_n & \\ & & & W_2 \end{pmatrix}.$$

In other words,

$$W = \sum \lambda_k e_{1k} \odot e_{1k} + W_2,$$

where W_2 is a symmetric operator on the vector space $\Lambda^2(v^{\perp})$ and $e_{ab} \odot e_{cd}$ denotes the symmetric endomorphism of $\Lambda^2 V$ sending $e_a \wedge e_b$ to $e_c \wedge e_d$ and vice versa; notice that we will use the same $\odot$ notation to indicate also the symmetric product in V; it will be clear from the context which situation applies.

Notice that

$$b(W) = 0, \quad b(e_{1k} \odot e_{1k}) = 0,$$

where b is the Bianchi projector defined as in (3); we obtain that W_2 is a curvature operator. It may not be a Weyl operator, because for the Ricci projector r introduced in (4),

$$r(e_{1k} \odot e_{1k}) = e_1 \odot e_1 + e_k \odot e_k. \tag{16}$$

Nonetheless, we can deduce that $\sum_{k=2}^{n}\lambda_k = 0$ because

$$0 = \langle r(W), e_1 \odot e_1\rangle = \sum_{k=2}^{n} \lambda_k \big\langle r(e_{1k} \odot e_{1k}), e_1 \odot e_1\big\rangle + \langle r(W_2), e_1 \odot e_1\rangle, \tag{17}$$

and $\langle r(W_2), e_1 \odot e_1\rangle = 0$ because W_2 is an operator on the orthogonal complement of e_1. Together with (16),

$$r(W_2) = -\sum_{k=2}^{n} \lambda_k\, r(e_{1k} \odot e_{1k}) = -\Bigg(\sum_{k=2}^{n} \lambda_k e_k \odot e_k\Bigg).$$

In other words, $W_2 \in \ker(b) \cap r^{-1}\big(-\sum_{k=2}^{n}\lambda_k e_k \odot e_k\big)$. We denote this (affine) space by $\mathcal{R}(\{\lambda_k\})$; its dimension will agree with the dimension of $\mathcal{W}(v^{\perp}) = \ker(b) \cap \ker(r)$.

Hence if $W \in \mathcal{EW}$, there exist an element $\rho \in \mathrm{SO}(V)$, numbers $\lambda_2, \dots, \lambda_n$ with $\sum_k \lambda_k = 0$, and a curvature operator $W_2 \in \mathcal{R}(\{\lambda_k\})$ such that

$$W = B(\rho) \cdot \left(\sum \lambda_k e_{1k} \odot e_{1k} + \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 & \\ & & & W_2 \end{pmatrix} \right) \cdot B(\rho)^t, \tag{18}$$

where remember that $B(\rho)$ is the map in bivectors induced by ρ. Let

$$\mathbb{S} = \Big\{ (\lambda_k)_{k=2,\dots,n} \;:\; \sum \lambda_k = 0 \Big\},$$

and define a map

$$\Phi : \mathrm{SO}(V) \times \mathbb{S} \times \mathcal{R}(\{\lambda_k\}) \to \mathcal{W}$$

by the above formula (18).

We know that

$$\sum \lambda_k e_{1k} \odot e_{1k} + \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 & \\ & & & W_2 \end{pmatrix}$$

is a Weyl tensor because it lies in the kernel of b and r, and conjugating by $B(\rho)$ produces another Weyl tensor by equation (5). It follows that $\Phi(\rho, \{\lambda_k\}, W_2)$ is always a Weyl tensor, and it is clear that it has the eigenflag property. Thus Φ is surjective onto $\mathcal{EW}$.

We will now compute the dimension of $\mathcal{EW}$. The dimension of the space of curvature operators is

$$\dim \mathcal{R}_n = \dim S^2(\Lambda^2 V) - \dim(\Lambda^4 V) = \tfrac{1}{12}n^4 - \tfrac{1}{12}n^2.$$

The dimension of the space of Weyl operators is

$$\dim \mathcal{W}_n = \dim \mathcal{R}_n - \dim S^2(V) = \tfrac{1}{12}n^4 - \tfrac{7}{12}n^2 - \tfrac{1}{2}.$$

The dimension of $\mathrm{SO}(V) \times \mathbb{S} \times \mathcal{R}(\{\lambda_k\})$ is thus the sum of

$$\dim \mathrm{SO}(V) = \binom{n}{2}, \quad \dim \mathbb{S} = n-2, \quad \dim \mathcal{R}(\{\lambda_k\}) = \tfrac{1}{12}(n-1)^4 - \tfrac{7}{12}(n-1)^2 - \tfrac{1}{2}.$$

However, the dimension of $\mathrm{SO}(V)\times\mathbb{S}\times\mathcal{R}(\{\lambda_k\})$ could be strictly greater than that of $\mathcal{EW}$. In order to prove that this is not the case, we show that Φ is finite-to-one when restricted to a nontrivial open subset $\mathcal{A}$ of $\mathrm{SO}(V)\times\mathbb{S}\times\mathcal{R}(\{\lambda_k\})$.

Let w be the projection from the curvature operators onto the Weyl tensors. Then $\mathcal{A}$ is the set of triples $(\rho,\{\lambda_k\},R)$ such that

- all λ_k for $k=2,\dots,n$ are different,
- the Weyl tensor $w(R)$ does not satisfy the eigenflag condition.

It is clear that $\mathcal{A}$ is open. In order to see that it is not empty, we use induction to find a Weyl tensor W_2 on the space $\partial_1^{\perp}$ that does not satisfy the eigenflag condition. The base case for the induction is dimension 4, which was done in the previous section. We fix arbitrary $\{\lambda_k\}$ whose sum is 0, and choose any rotation ρ. Let R_0 be any operator in $\mathcal{R}(\{\lambda_k\})$. Then $R_1=R_0+W_2-w(R_0)$ is a curvature operator in the affine space $\mathcal{R}(\{\lambda_k\})$ whose projection $w(R_1)$ to the space of Weyl tensors is W_2.

For $W\in\Phi(\mathcal{A})$, let us compute its preimages $(\rho,\{\lambda_k\},R_{n-1})$ in $\mathcal{A}$. The direction v_1 is a direction with the eigenflag property, and by the hypothesis, it is unique up to sign. The numbers λ_k for $k=2,\dots,n$ are the unique eigenvalues of $W|_{v_1\wedge v_1^{\perp}}$, up to change of order. The v_k are unit-vectors in $v_1^{\perp}$ such that $v_1\wedge v_k$ are eigenvectors of $W|_{v_1\wedge v_1^{\perp}}$ corresponding to the eigenvalues λ_k, and they are unique up to a change of sign. The basis v_k determines ρ uniquely and R_{n-1} is the unique remainder $B(\rho)^t\circ W\circ B(\rho)-\sum\lambda_k e_{1k}\odot e_{1k}$. It follows that $\Phi^{-1}(W)$ is finite for any W, and $\dim(\mathcal{EW})$ agrees with $\dim\big(\mathrm{SO}(V)\times\mathbb{S}\times\mathcal{R}(\{\lambda_k\})\big)$. Thus using the above formulae, we obtain that the codimension of $\mathcal{EW}$ inside $\mathcal{W}$ is

$$\tfrac{1}{3}n^3-n^2-\tfrac{4}{3}n+2. \qquad \square$$

Proof of Theorem 1.9 for $n=\dim M\geqslant 4$. We start with a precise statement of a folklore lemma in Riemannian geometry.

Lemma 6.5. *Let M be a Riemannian manifold with metric g and p any point in M, with $R(p)$ the curvature of the metric g at p.*

Then for any algebraic curvature operator R^0 close enough to $R(p)$, there exists a metric g' that agrees with g outside a neighbourhood of p and such that the curvature of g' at p is R^0.

Furthermore, we can choose g' such that

$$\|g'-g\|_{C^2}\leqslant C\|R^0-R(p)\|,$$

with a constant C independent of R^0.

Remark 6.6. The norm appearing in the left-hand side in the above inequality is computed in a fixed set of coordinates of p.

Proof. We use the following formula for the computation of the Riemannian curvature in terms of partial derivatives of g and the Christoffel symbols:

$$R_{ik\ell m}=\frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k\partial x^\ell}+\frac{\partial^2 g_{k\ell}}{\partial x^i\partial x^m}-\frac{\partial^2 g_{i\ell}}{\partial x^k\partial x^m}-\frac{\partial^2 g_{km}}{\partial x^i\partial x^\ell}\right)+g_{np}\big(\Gamma^n_{k\ell}\Gamma^p_{im}-\Gamma^n_{km}\Gamma^p_{i\ell}\big). \tag{19}$$

Take normal coordinates for the metric g at p. In these coordinates, the Christoffel symbols at p vanish.

In these coordinates, choose a smooth function φ with value 1 near p and value 0 in the complement of the domain of the coordinates. Define a new metric as

$$g'_{ij} = g_{ij} - \frac{1}{4} \sum_{k,h} R^*_{ihjk} x^h x^k \varphi(x)$$

in the coordinate patch, and by g outside of it, where $R^* = R^0 - R(p)$. If R^* is small enough, g' will still be positive definite. The Christoffel symbols are given by

$$\Gamma^m{}_{ij} = \frac{1}{2} g^{mk} \left(\frac{\partial}{\partial x^j} g_{ki} + \frac{\partial}{\partial x^i} g_{kj} - \frac{\partial}{\partial x^k} g_{ij} \right).$$

Thus, since the Christoffel symbols of g vanish, and we have added a quadratic perturbation to g, the Christoffel symbols of g' also vanish. We compute the curvature of g' at p using (19):

$$R'(p)_{iklm} = R(p)_{iklm} - \tfrac{1}{4}\big(R^*_{ikml} + R^*_{kilm} - R^*_{iklm} - R^*_{kiml}\big) = R(p)_{iklm} + R^*_{iklm} = R^0_{iklm}. \tag{20}$$

The C^2-norm of $g' - g$ is bounded by $C\|R^*\|$, with a constant C independent of R^*. □

Proof of Theorem 1.9 for $\dim M \geqslant 4$. Let $U \subset M$ for a compact manifold M. Denote by $\mathcal{O}$ the set of Riemannian metrics on M for which there is at least one point $p \in U$ such that the Weyl tensor W_p of g at p does not satisfy the eigenflag condition. By Theorem 1.3, $\mathcal{O}$ is contained in the set of metrics that do not admit an LCW on U.

Since the complement of $\mathcal{EW}$ is open, and the map that assigns its Weyl tensor to a Riemannian metric is continuous under C^2-deformations of the metric, $\mathcal{O}$ is open.

For density, fix an arbitrary point $p_0 \in U$ and consider a metric g such that $W(g)_{p_0} \in \mathcal{EW}$. By Theorem 6.1, we can find a Weyl tensor $\widetilde{W} \notin \mathcal{EW}$ such that $\|\widetilde{W} - W(g)_{p_0}\| < \epsilon$.

We choose $R_0 = R(g)_{p_0} - W(g)_{p_0} + \widetilde{W}$ and apply Lemma 6.5 to get a new metric g' that satisfies $\|g' - g\|_{C^2} \leqslant C\|\widetilde{W} - W(g)_{p_0}\| < C\epsilon$. The Weyl tensor of g' at p_0 is $\widetilde{W} \notin \mathcal{EW}$; thus g' is not in $\mathcal{O}$. Since ϵ is arbitrary, denseness of $\mathcal{O}$ follows. □

***Proof of Theorem 1.9 for n* = dim *M* = 3.** In this section, we use the Cotton tensor instead of the Weyl tensor.

The space of algebraic Cotton–York tensors at $p \in M$ consists of simply the symmetric, traceless operators on the euclidean space T_pM. It is obvious that the set of Cotton–York tensors with zero determinant is a proper algebraic subset of the set of all such tensors.

The following result is the equivalent of Lemma 6.5 for the Cotton tensor:

Lemma 6.7. *Let M be a Riemannian manifold with metric g and p any point in M.*

Then for any algebraic Cotton–York tensor CY^0 close enough to CY_p, we can find a metric g' that agrees with g outside a neighbourhood of p so that the Cotton–York tensor of g' at p is CY^0.

Furthermore, we can find the metric g' in such a way that the C^3-norm of $|g - g'|$ is bounded by a multiple of the norm of $CY^0 - CY_p$.

Proof. Our first goal is to find a formula that expresses the Cotton tensor at p in terms of the metric tensor and its derivatives. Take normal coordinates at p, so that g_p is the identity matrix, and the Christoffel symbols vanish at p. We start with the formula (19) for the curvature tensor and take derivatives.

We compute first the Schouten tensor in a neighbourhood of p:

$$S_{ab} = \tfrac{1}{2}\big(\delta_{ia}\delta_{lb} - \tfrac{1}{4}g_{ab}g^{il}\big)g^{km}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^\ell} + \frac{\partial^2 g_{k\ell}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{i\ell}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^\ell}\right) + Q(\Gamma), \tag{21}$$

where $Q(\Gamma)$ consists of terms like $\Gamma^n_{k\ell}\Gamma^p_{im}$.

The covariant derivative $\nabla_n S_{ab}(p) = (\partial/\partial x^n)S_{ab}(p)$ at p is

$$\nabla_n S_{ab}(p) = \frac{1}{2}\frac{\partial}{\partial x^n}\left(\frac{\partial^2 g_{ak}}{\partial x^k \partial x^b} + \frac{\partial^2 g_{kb}}{\partial x^a \partial x^k} - \frac{\partial^2 g_{ab}}{\partial x^k \partial x^k} - \frac{\partial^2 g_{kk}}{\partial x^a \partial x^b}\right) - \frac{1}{4}\frac{\partial}{\partial x^n}\left(\frac{\partial^2 g_{ik}}{\partial x^k \partial x^i} - \frac{\partial^2 g_{kk}}{\partial x^i \partial x^i}\right)\delta_{ab}. \tag{22}$$

The derivatives of $Q(\Gamma)$ vanish because one Christoffel symbol will remain in the final computation, and it evaluates to 0 at p.

The Cotton tensor at p is

$$\begin{aligned}
&C_{nab}(p)\\
&\quad= \frac{1}{2}\frac{\partial}{\partial x^n}\left(\frac{\partial^2 g_{ak}}{\partial x^k \partial x^b} + \frac{\partial^2 g_{kb}}{\partial x^a \partial x^k} - \frac{\partial^2 g_{ab}}{\partial x^k \partial x^k} - \frac{\partial^2 g_{kk}}{\partial x^a \partial x^b}\right) - \frac{1}{2}\frac{\partial}{\partial x^a}\left(\frac{\partial^2 g_{nk}}{\partial x^k \partial x^b} + \frac{\partial^2 g_{kb}}{\partial x^n \partial x^k} - \frac{\partial^2 g_{nb}}{\partial x^k \partial x^k} - \frac{\partial^2 g_{kk}}{\partial x^n \partial x^b}\right)\\
&\qquad - \frac{1}{4}\frac{\partial}{\partial x^n}\left(\frac{\partial^2 g_{ik}}{\partial x^k \partial x^i} - \frac{\partial^2 g_{kk}}{\partial x^i \partial x^i}\right) + \delta_{ab}\frac{1}{4}\frac{\partial}{\partial x^a}\left(\frac{\partial^2 g_{ik}}{\partial x^k \partial x^i} - \frac{\partial^2 g_{kk}}{\partial x^i \partial x^i}\right)\delta_{nb}\\
&\quad= \frac{1}{2}\left(\frac{\partial^3 g_{ak}}{\partial x^k \partial x^n \partial x^b} - \frac{\partial^3 g_{nk}}{\partial x^k \partial x^a \partial x^b} - \frac{\partial^3 g_{ab}}{\partial x^k \partial x^n \partial x^k} + \frac{\partial^3 g_{nb}}{\partial x^k \partial x^a \partial x^k}\right) - \frac{1}{4}\left(\frac{\partial^3 g_{ik}}{\partial x^k \partial x^i \partial x^n} - \frac{\partial^2 g_{kk}}{\partial x^i \partial x^i \partial x^n}\right)\delta_{ab}\\
&\qquad + \frac{1}{4}\left(\frac{\partial^3 g_{ik}}{\partial x^k \partial x^i \partial x^a} - \frac{\partial^2 g_{kk}}{\partial x^i \partial x^i \partial x^a}\right)\delta_{nb}.
\end{aligned} \tag{23}$$

If the A^{klm}_{ij} are small enough real numbers, symmetric under permutations of i, j and also under permutations of k, l, m (there are 60 different such terms), then

$$g'_{ij} = g_{ij} + \sum A^{klm}_{ij} x^k x^l x^m$$

defines a new metric g'.

The new Cotton tensor at 0 is

$$C'_{nab}(p) = C_{nab}(p) + \tfrac{1}{2}(A^{knb}_{ka} - A^{kab}_{kn} - A^{kkn}_{ab} + A^{kka}_{nb}) - \tfrac{1}{4}(A^{kin}_{ki} - A^{iin}_{kk})\delta_{ab} + \tfrac{1}{4}(A^{kia}_{ki} - A^{iia}_{kk})\delta_{nb}. \tag{24}$$

We define $\mathbb{A}$ to be the real vector space of dimension 60 whose coordinates are indexed by the tuples $(\{i, j\}, \{k, l, m\})$. The formula

$$A^{mlk}_{ij} \xrightarrow{L} \tfrac{1}{2}(A^{knb}_{ka} - A^{kab}_{kn} - A^{kkn}_{ab} + A^{kka}_{nb}) - \tfrac{1}{4}(A^{kin}_{ki} - A^{iin}_{kk})\delta_{ab} + \tfrac{1}{4}(A^{kia}_{ki} - A^{iia}_{kk})\delta_{nb}$$

defines a linear map $L : \mathbb{A} \to \mathcal{C}_p$ into the space of algebraic Cotton tensors (the (0, 3)-tensors with the symmetries (8)). It follows from (24) that the image of L consists of Cotton tensors, but it is a nice exercise to check it directly.

In order to show that we can prescribe the Cotton tensor at p, we just need to check that L is surjective. The map from the Cotton tensors to the Cotton–York tensors is a linear isomorphism, so we only need to check that the image of the above linear map has dimension 5. Let $L(e_{ij}^{klm})$ be the image by L of the basis vector $e_{ij}^{klm} \in \mathbb{A}$, with $A_{ij}^{kml} = 1$ and the other entries equal to 0. The reader may check, for instance, that $L(e_{11}^{122})$, $L(e_{11}^{123})$, $L(e_{11}^{222})$, $L(e_{11}^{223})$ and $L(e_{12}^{223})$ are linearly independent. $\square$

Proof of Theorem 1.9 for $\dim M = 3$. Let $U \subset M$ for a compact manifold M. This time, $\mathcal{O}$ is the set of Riemannian metrics on M for which there is at least one point $p \in U$ such that the Cotton–York tensor CY_p of g at p has nonzero determinant. By Theorem 1.6, $\mathcal{O}$ is contained in the set of metrics that do not admit an LCW on U.

Since the map that assigns its Cotton tensor to a Riemannian metric is continuous under C^3-deformations of the metric, $\mathcal{O}$ is open in the C^3-topology.

For density, let $\epsilon > 0$, fix an arbitrary point $p_0 \in U$ and consider a metric g such that its Cotton–York tensor $CY(g)_{p_0}$ at p_0 has zero determinant. Choose a symmetric traceless tensor with nonzero determinant CY^0 and such that $\|CY^0 - CY(g)_{p_0}\| < \epsilon$.

We apply Lemma 6.5 to get a new metric g' that satisfies $\|g' - g\|_{C^3} \leqslant C\|CY^0 - CY(g)_{p_0}\| < C\epsilon$ and whose Cotton–York tensor at p_0 is CY^0. It follows that g' is not in $\mathcal{O}$, and since ϵ is arbitrary, we deduce that $\mathcal{O}$ is dense. $\square$

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FINITE CHAINS INSIDE THIN SUBSETS OF $\mathbb{R}^d$

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In a recent paper, Chan, Łaba, and Pramanik investigated geometric configurations inside thin subsets of Euclidean space possessing measures with Fourier decay properties. In this paper we ask which configurations can be found inside thin sets of a given Hausdorff dimension without any additional assumptions on the structure. We prove that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{1}{2}(d+1)$ then, for each $k \in \mathbb{Z}^+$, there exists a nonempty interval I such that, given any sequence $\{t_1, t_2, \ldots, t_k : t_j \in I\}$, there exists a sequence of distinct points $\{x^j\}_{j=1}^{k+1}$ such that $x^j \in E$ and $|x^{i+1} - x^i| = t_j$ for $1 \leq i \leq k$. In other words, E contains vertices of a chain of arbitrary length with prescribed gaps.

1. Introduction

The problem of determining which geometric configurations one can find inside various subsets of Euclidean space is a classical subject. The basic problem is to understand how large a subset of Euclidean space must be to be sure that it contains the vertices of a congruent and possibly scaled copy of a given polyhedron or another geometric shape. In the case of a finite set, "large" refers to the number of points, while in infinite sets it refers to the Hausdorff dimension or Lebesgue density. The resulting class of problems has been attacked by a variety of authors using combinatorial, number theoretic, ergodic, and Fourier analytic techniques, creating a rich set of ideas and interactions.

We begin with a comprehensive result due to Tamar Ziegler [2006], which generalizes an earlier result due to Furstenberg, Katznelson and Weiss [Furstenberg et al. 1990]. See also [Bourgain 1986].

Theorem 1.1 [Ziegler 2006]. *Let $E \subset \mathbb{R}^d$ be of positive upper Lebesgue density, in the sense that*

$$\limsup_{R\to\infty} \frac{\mathcal{L}^d\{E \cap [-R, R]^d\}}{(2R)^d} > 0,$$

where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. Let E_δ denote the δ-neighborhood of E. Let $V = \{\mathbf{0}, v^1, v^2, \ldots, v^{k-1}\} \subset \mathbb{R}^d$, where $k \geq 2$ is a positive integer. Then there exists $l_0 > 0$ such that, for any $l > l_0$ and any $\delta > 0$, there exists $\{x^1, \ldots, x^k\} \subset E_\delta$ congruent to $lV = \{\mathbf{0}, lv^1, \ldots, lv^{k-1}\}$.

In particular, this result shows that we can recover every simplex similarity type and sufficiently large scaling inside a subset of $\mathbb{R}^d$ of positive upper Lebesgue density. It is reasonable to wonder whether the assumptions of Theorem 1.1 can be weakened, but the following result, due to Maga [2010], shows that

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the conclusion may fail even if we replace the upper Lebesgue density condition with the assumption that the set is of dimension d.

Theorem 1.2 [Maga 2010]. *For any $d \geq 2$ there exists a full-dimensional compact set $A \subset \mathbb{R}^d$ such that A does not contain the vertices of any parallelogram. If $d = 2$ then, given any triple of points x^1, x^2, x^3, $x^j \in A$, there exists a full-dimensional compact set $A \subset \mathbb{R}^2$ such that A does not contain the vertices of any triangle similar to $\triangle x^1 x^2 x^3$.*

In view of Maga's result, it is reasonable to ask whether interesting point configurations can be found inside thin sets under additional structural hypotheses. This question was recently addressed by Chan, Łaba, and Pramanik [Chan et al. 2013]. Before stating their result, we provide two relevant definitions.

Definition 1.3. Fix integers $n \geq 2$, $p \geq 3$ and $m = n\lceil \frac{1}{2}(p+1)\rceil$. Suppose $B_1, \dots, B_p$ are $n \times (m-n)$ matrices.

(a) We say that E contains a p-point $\mathscr{B}$-configuration if there exist vectors $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^{m-n} \setminus \vec{0}$ such that

$$\{z + B_j w\}_{j=1}^{p} \subset E.$$

(b) Moreover, given any finite collection of subspaces $V_1, \dots, V_q \subset \mathbb{R}^{m-n}$ with $\dim(V_i) < m-n$, we say that E contains a nontrivial p-point $\mathscr{B}$-configuration with respect to $(V_1, \dots, V_q)$ if there exist vectors $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^{m-n} \setminus \bigcup_{i=1}^{q} V_i$ such that

$$\{z + B_j w\}_{j=1}^{p} \subset E.$$

Definition 1.4. Fix integers $n \geq 2$, $p \geq 3$ and $m = n\lceil \frac{1}{2}(p+1)\rceil$. We say that a set of $n \times (m-n)$ matrices $\{B_1, \dots, B_p\}$ is nondegenerate if

$$\operatorname{rank}\begin{pmatrix} B_{i_2} - B_{i_1} \\ \vdots \\ B_{i_{m/n}} - B_{i_1} \end{pmatrix} = m - n$$

for any distinct indices $i_1, \dots, i_{m/n} \in \{1, \dots, p\}$.

Theorem 1.5 [Chan et al. 2013]. *Fix integers $n \geq 2$, $p \geq 3$ and $m = n\lceil \frac{1}{2}(p+1)\rceil$. Let $\{B_1, \dots, B_p\}$ be a collection of $n \times (m-n)$ nondegenerate matrices in the sense of Definition 1.4. Then, for any constant C, there exists a positive number $\epsilon_0 = \epsilon_0(C, n, p, B_1, \dots, B_p) \ll 1$ with the following property: Suppose the set $E \subset \mathbb{R}^n$ with $|E| = 0$ supports a positive, finite Radon measure μ with two conditions:*

(a) <u>Ball condition</u>: $\sup_{x \in E,\, 0<r<1} \mu(B(x,r))/r^{\alpha} \leq C$ *if* $n - \epsilon_0 < \alpha < n$.

(b) <u>Fourier decay</u>: $\sup_{\xi \in \mathbb{R}^n} |\hat{\mu}(\xi)|(1+|\xi|)^{\beta/2} \leq C$.

Then:

(i) *E contains a p-point $\mathscr{B}$-configuration in the sense of Definition 1.3(a).*

(ii) *Moreover, for any finite collection of subspaces $V_1, \dots, V_q \subset \mathbb{R}^{m-n}$ with $\dim(V_i) < m-n$, E contains a nontrivial p-point $\mathscr{B}$-configuration with respect to $(V_1, \dots, V_q)$ in the sense of Definition 1.3(b).*

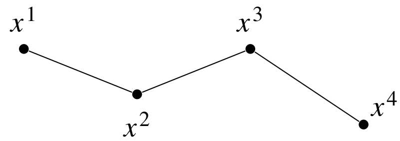

Figure 1. A 3-chain.

One can check that the Chan-Łaba–Pramanik result covers some geometric configurations but not others. For example, their nondegeneracy condition allows them to consider triangles in the plane, but not simplexes in $\mathbb{R}^3$ where three faces meet at one of the vertices at right angles, forming a three-dimensional corner. Most relevant to this paper is the fact that the conditions under which Theorem 1.5 holds are satisfied for chains (see Definition 1.6 below), but the conclusion requires decay properties for the Fourier transform of a measure supported on the underlying set. We shall see that, in the case of chains, such an assumption is not needed and the existence of a wide variety of chains can be established under an explicit dimensional condition alone.

Focus of this article. We establish that a set of sufficiently large Hausdorff dimension, *with no additional assumptions*, contains an arbitrarily long chain with vertices in the set and preassigned admissible gaps.

Definition 1.6 (see Figure 1). A k-chain in $E \subset \mathbb{R}^d$ with gaps $\{t_i\}_{i=1}^k$ is a sequence

$$\{x^1, x^2, \dots, x^{k+1} : x^j \in E,\ |x^{i+1} - x^i| = t_i,\ 1 \le i \le k\}.$$

We say that the chain is *nondegenerate* if all the x^j are distinct.

Our main result is the following:

Theorem 1.7. *Suppose that the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$, $d \ge 2$, is greater than $\frac{1}{2}(d+1)$. Then, for any $k \ge 1$, there exists an open interval $\tilde{I}$ such that for any $\{t_i\}_{i=1}^k \subset \tilde{I}$ there exists a nondegenerate k-chain in E with gaps $\{t_i\}_{i=1}^k$.*

In the course of establishing Theorem 1.7 we shall prove the following result, which is interesting in its own right and has a number of consequences for Falconer-type problems. See [Falconer 1985; Erdoğan 2005; Wolff 1999] for the background and the latest results pertaining to the Falconer distance problem.

Theorem 1.8. *Suppose that μ is a compactly supported, nonnegative Borel measure such that*

$$\mu(B(x,r)) \le Cr^{s_\mu} \tag{1-1}$$

for some $s_\mu \in \left(\frac{1}{2}(d+1), d\right]$, where $B(x,r)$ is the ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. Then, for any $t_1, \dots, t_k > 0$ and $\epsilon > 0$,

$$\mu \times \mu \times \cdots \times \mu\left\{(x^1, x^2, \dots, x^{k+1}) : t_i - \epsilon \le |x^{i+1} - x^i| \le t_i + \epsilon,\ i = 1, 2, \dots, k\right\} \le C\epsilon^k. \tag{1-2}$$

Corollary 1.9. *Given a compact set $E \subset \mathbb{R}^d$, $d \ge 2$, $k \ge 1$, define*

$$\Delta_k(E) = \{|x^1 - x^2|, |x^2 - x^3|, \dots, |x^k - x^{k+1}| : x^j \in E\}.$$

Suppose that the Hausdorff dimension of E is greater than $\frac{1}{2}(d+1)$. Then

$$\mathcal{L}^k(\Delta_k(E)) > 0.$$

Remark 1.10. Suppose that $E \subset \mathbb{R}^d$ has Hausdorff dimension $s > \frac{1}{2}(d+1)$ and is *Ahlfors–David regular*, i.e., there exists $C > 0$ such that, for every $x \in E$,

$$C^{-1} r^s \leq \mu(B(x, r)) \leq C r^s$$

(where μ is the restriction of the s-dimensional Hausdorff measure to E). Then, using the techniques in [Eswarathasan et al. 2011] along with Theorem 1.8, one can show that, for any sequence of positive real numbers $t_1, t_2, \dots, t_k$, the upper Minkowski dimension of

$$\{(x^1, x^2, \dots, x^{k+1}) \in E^{k+1} : |x^{j+1} - x^j| = t_j,\ 1 \leq j \leq k\}$$

does not exceed $(k+1)\dim_{\mathcal{H}}(E) - k$.

2. Proof of Theorem 1.7 and Theorem 1.8

The strategy for this section is as follows:

We begin by dividing both sides of (1-2) by ϵ^k. The left side becomes

$$\epsilon^{-k} \mu \times \cdots \times \mu\big\{(x^1, \dots, x^{k+1}) : t_i - \epsilon \leq |x^{i+1} - x^i| \leq t_i + \epsilon,\ i = 1, 2, \dots, k\big\}, \tag{2-1}$$

which can be interpreted as the density of ϵ-approximate chains in $E \times \cdots \times E$.

Theorem 1.8 gives an upper bound on this expression that is independent of ϵ. This is accomplished using an inductive argument on the chain length coupled with repeated application of an earlier result from [Iosevich et al. 2014], in which the authors establish $L^2(\mu)$ mapping properties of certain convolution operators. This upper bound is important in the final section, where we define a measure on the set of chains.

Next, we acquire a lower bound on (2-1). This result was already established in the case $k = 1$ in [Iosevich et al. 2012], where the authors show that the density of ϵ-approximate 1-chains with gap size t is bounded below, independent of ϵ, for all t in a nonempty open interval I. Using a pigeonholing argument, we extend the result in [Iosevich et al. 2012] to obtain a lower bound on (2-1) in the case that every gap is of equal size t for some $t \in I$. To obtain a lower bound on chains with variable gap size, we show that the density of ϵ-approximate k-chains is continuous as a function of gap sizes. Furthermore, we use the lower bound on chains with constant gaps to prove that this continuous function is not identically zero. We conclude that the density of ϵ-approximate k-chains is bounded below, independent of ϵ and independent of the gap sizes, as long as all gap sizes fall within some interval $\tilde{I}$ around t.

In the final section, we address the issue of nondegeneracy. To this end, we reinterpret the density of ϵ-approximate k-chains as a measure supported in E^{k+1} and show that it converges to a new measure, $\Lambda^k_{\vec{t}}$, as $\epsilon \downarrow 0$. This new measure is shown to be supported on "exact" k-chains ($\epsilon = 0$) with admissible gaps. We next show that the measure of the set of degenerate chains is 0, and we conclude that the mass of $\Lambda^k_{\vec{t}}$ is contained in nondegenerate k-chains.

We shall repeatedly use the following result, due to Iosevich, Sawyer, Taylor and Uriarte-Tuero:

Theorem 2.1 [Iosevich et al. 2014]. *Let $T_\lambda f(x) = \lambda * (f\mu)(x)$, where λ and μ are compactly supported, nonnegative Borel measures on $\mathbb{R}^d$. Suppose that μ satisfies* (1-1) *and, for some $\alpha > 0$,*

$$|\hat{\lambda}(\xi)| \le C|\xi|^{-\alpha}.$$

Suppose that ν is a compactly supported Borel measure supported on $\mathbb{R}^d$ satisfying (1-1) *with s_μ replaced by s_ν and suppose that $\alpha > d - s$, where $s = \frac{1}{2}(s_\mu + s_\nu)$. Then*

$$\|T_\lambda f\|_{L^2(\nu)} \le c\|f\|_{L^2(\mu)}.$$

In this article, we will use Theorem 2.1 with $\lambda = \sigma$, the surface measure on a $(d-1)$-dimensional sphere in $\mathbb{R}^d$. It is known — see [Stein 1993] — that

$$\hat{\sigma}(\xi) = O(|\xi|^{-(d-1)/2}).$$

Since the proof of Theorem 2.1 is short, we give the argument below for the sake of keeping the presentation as self-contained as possible. It is enough to show that

$$\langle T_{\lambda^\epsilon} f, g\nu\rangle \le C\|f\|_{L^2(\mu)} \cdot \|g\|_{L^2(\nu)}.$$

The left-hand side equals

$$\int \hat{\lambda}^\epsilon(\xi)\widehat{f\mu}(\xi)\widehat{g\nu}(\xi)\,d\xi.$$

By the assumptions of Theorem 2.1, the modulus of this quantity is bounded by

$$C\int |\xi|^{-\alpha}|\widehat{f\mu}(\xi)||\widehat{g\nu}(\xi)|\,d\xi$$

and applying Cauchy–Schwarz bounds this quantity by

$$C\biggl(\int |\widehat{f\mu}(\xi)|^2|\xi|^{-\alpha_\mu}\,d\xi\biggr)^{\frac{1}{2}} \cdot \biggl(\int |\widehat{g\nu}(\xi)|^2|\xi|^{-\alpha_\nu}\,d\xi\biggr)^{\frac{1}{2}} \tag{2-2}$$

for any $\alpha_\mu, \alpha_\nu > 0$ such that $\alpha = \frac{1}{2}(\alpha_\mu + \alpha_\nu)$.

By Lemma 2.5 below, the quantity (2-2) is bounded by $C\|f\|_{L^2(\mu)} \cdot \|g\|_{L^2(\nu)}$ after choosing, as we may, $\alpha_\mu > d - s_\mu$ and $\alpha_\nu > d - s_\nu$. This completes the proof of Theorem 2.1.

Proof of Theorem 1.8 and Corollary 1.9. Let $\epsilon > 0$. Divide both sides of (1-2) by ϵ^k and note that it suffices to establish the estimate

$$C_k^\epsilon(\mu) = \int\biggl(\prod_{i=1}^{k} \sigma_{t_i}^\epsilon(x^{i+1} - x^i)\,d\mu(x^i)\biggr)\,d\mu(x^{k+1}) \le c^k, \tag{2-3}$$

where c is independent of ϵ and $t_1, \dots, t_k > 0$. Here $\sigma_r^\epsilon(x) = \sigma_r * \rho_\epsilon(x)$, with σ_r the Lebesgue measure on the sphere of radius r, ρ a smooth cut-off function with $\int\rho = 1$ and $\rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$. Assume in addition that ρ is nonnegative and that $\rho(x) = \rho(-x)$.

Let σ denote the Lebesgue measure on the $(d-1)$-dimensional sphere in $\mathbb{R}^d$. Set $T_j^\epsilon = T_{\sigma_{t_j}}^\epsilon$, where $T_{\sigma_{t_j}}^\epsilon f(x) = \sigma_{t_j} * (f\mu)(x)$ was introduced in Theorem 2.1. Define

$$f_k^\epsilon(x) = T_k^\epsilon \circ \cdots \circ T_1^\epsilon(1)(x) \tag{2-4}$$

and

$$f_0^\epsilon(x) = 1.$$

It is important to note that $f_k(x)$ depends implicitly on the choices of $t_1, \ldots, t_k > 0$, and this choice will be made explicit throughout.

Observe that

$$f_{k+1}^\epsilon = T_{k+1}^\epsilon f_k^\epsilon. \tag{2-5}$$

Rewriting the left-hand side of (2-3), it suffices to show

$$C_k^\epsilon(\mu) = \int f_k^\epsilon(x)\, d\mu(x) \le c^k. \tag{2-6}$$

Using Cauchy–Schwarz (and keeping in mind that $\int d\mu(x) = 1$), we bound the left-hand side of (2-6) by

$$C_k^\epsilon(\mu) = \int f_k^\epsilon(x)\, d\mu(x) \le \| f_k^\epsilon \|_{L^2(\mu)}. \tag{2-7}$$

We now use induction on k to show that

$$\| f_k^\epsilon \|_{L^2(\mu)} \le c^k, \tag{2-8}$$

where c is the constant obtained in Theorem 2.1. For the base case, $k=0$, we have $\| f_0^\epsilon \|_{L^2(\mu)} = \int d\mu(x) = 1$. Next, we assume inductively that $\| f_k^\epsilon \|_{L^2(\mu)} \le c^k$.

We now show that, for any $t_{k+1} > 0$,

$$\| f_{k+1}^\epsilon \|_{L^2(\mu)} \le c^{k+1}.$$

First, use (2-5) to write

$$\| f_{k+1}^\epsilon \|_{L^2(\mu)} = \| T_{k+1}^\epsilon f_k^\epsilon \|_{L^2(\mu)}.$$

Next, use Theorem 2.1 with $\lambda = \sigma$, the Lebesgue measure on the sphere, and $\alpha = \frac{1}{2}(d-1)$ (see the comment immediately following Theorem 2.1 to justify this choice of α) to show that

$$\| T_{k+1}^\epsilon f_k^\epsilon \|_{L^2(\mu)} \le c \| f_k^\epsilon \|_{L^2(\mu)}$$

whenever $s_\mu > d - \alpha = \frac{1}{2}(d+1)$.

We complete the proof by applying the inductive hypothesis. This completes the verification of (2-8).

We now recover Corollary 1.9. Let $s_\mu \in \left(\frac{1}{2}(d+1), \dim(E)\right)$, and choose a probability measure μ with support contained in E which satisfies (1-1); the existence of such a measure is provided by Frostman's lemma (see [Falconer 1986], [Wolff 2003] or [Mattila 1995]).

Cover $\Delta_k(E)$ with cubes of the form

$$\bigcup_i \prod_{j=1}^{d} (t_{ij}, t_{ij} + \epsilon_i),$$

where $\prod$ denotes the Cartesian product. We have

$$1 = \mu \times \cdots \times \mu(E^{k+1}) \leq \sum_i \mu \times \cdots \times \mu\big\{(x^1, \ldots, x^{k+1}) : t_{ij} - \epsilon \leq |x^{j+1} - x^j| \leq t_{ij} + \epsilon_i,\ 1 \leq j \leq k\big\}.$$

By Theorem 1.8, the expression above is bounded by

$$C \sum_i \epsilon_i^k \tag{2-9}$$

and we conclude that (2-9) is bounded from below by $1/C > 0$. It follows that $\Delta_k(E)$ cannot have measure 0 and the proof of Corollary 1.9 is complete.

We now continue with the proof of Theorem 1.7.

Lower bound on $C_k^\epsilon(\mu)$. Let $s_\mu \in \big(\frac{1}{2}(d+1), \dim(E)\big)$, and choose a probability measure μ with support contained in E which satisfies (1-1).

We now establish the existence of a nonempty open interval $\tilde{I}$ such that

$$\liminf_{\epsilon \to 0} C_k^\epsilon(\mu) > 0, \tag{2-10}$$

where each t_i belongs to $\tilde{I}$ and $C_k^\epsilon(\mu)$ is as in (2-3).

Note that this positive lower bound alone establishes the existence of vertices $x^1, \ldots, x^{k+1} \in E$ such that $|x^{i+1} - x^i| = t_i$ for each $i \in \{1, \ldots, k\}$ (this follows, for instance, by Cantor's intersection theorem and the compactness of the set E). Extra effort is made in the next section in order to guarantee that we may take $x^1, \ldots, x^{k+1}$ distinct.

We first prove the estimate (2-10) in the case that all gaps are equal. This is accomplished using a pigeonholing argument on chains of length one. We then provide a continuity argument to show that the estimate holds for variable gap values t_i belonging to a nonempty open interval $\tilde{I}$. The second argument relies on the first precisely at the point when we show that the said continuous function is not identically equal to zero.

Lower bound for constant gaps. The proof of the estimate (2-10) in the case $k = 1$ was already established in [Iosevich et al. 2012] provided that μ satisfies the ball condition in (1-1) with $\frac{1}{2}(d+1) < s_\mu < \dim_{\mathcal{H}}(E)$. The existence of such measures is established by Frostman's lemma (see, e.g., [Falconer 1986], [Wolff 2003] or [Mattila 1995]).

More specifically, it is demonstrated in [Iosevich et al. 2012] that there exists $c(1) > 0$, $\epsilon_0 > 0$ and a nonempty open interval $I \subset (0, \text{diameter}(E))$ such that, if $t \in I$ and $0 < \epsilon < \epsilon_0$, then

$$C_1^\epsilon = \int \sigma_t^\epsilon * \mu(x)\, d\mu(x) > 2c(1).$$

To establish the estimate (2-10) for longer chains, we rely on the following lemmas:

Lemma 2.2. *Set*

$$G_{t,\epsilon}(1) = \{x \in E : \sigma_t^\epsilon * \mu(x) > c(1)\}.$$

There exists $m(1) \in \mathbb{Z}^+$ *such that, if* $t \in I$ *and* $0 < \epsilon < \epsilon_0$, *then*

$$\mu(G_{t,\epsilon}(1)) \geq 2^{-2m(1)}.$$

Lemma 2.3. *Set*

$$G_{t,\epsilon}(j+1) = \{x \in E : \sigma_t^\epsilon * \mu|_j(x) > c(j+1)\},$$

where $j \in \{1, \ldots, (k-1)\}$, $\mu|_j(x)$ *denotes restriction of the measure* μ *to the set* $G_{t,\epsilon}(j)$, *and*

$$c(j+1) = \tfrac{1}{2} c(j) \mu(G_{t,\epsilon}(j)).$$

Then there exists $m(j+1) \in \mathbb{Z}^+$ *such that if* $t \in I$ *and* $0 < \epsilon < \epsilon_0$, *then*

$$\mu(G_{t,\epsilon}(j+1)) > 2^{-2m(j+1)}.$$

We postpone the proof of Lemmas 2.2 and 2.3 momentarily, and we apply these lemmas to obtain a lower bound on $C_k^\epsilon(\mu)$.

We write

$$C_k^\epsilon(\mu) = \int f_k^\epsilon(x)\, d\mu(x),$$

where f_k^ϵ was introduced in (2-4) and here $t_1 = \cdots = t_k = t$.

Now

$$C_k^\epsilon(\mu) = \int f_k^\epsilon(x)\, d\mu(x) = \iint \sigma_t^\epsilon(x-y) f_{k-1}(y)\, d\mu(y)\, d\mu(x).$$

Integrating in x and restricting the variable y to the set $G_{t,\epsilon}(1)$, we write

$$C_k^\epsilon(\mu) \geq \int_{G_{t,\epsilon}(1)} \sigma_t^\epsilon * \mu(y) f_{k-1}(y)\, d\mu(y) \geq c(1) \int_{G_{t,\epsilon}(1)} f_{k-1}(y)\, d\mu(y) = c(1) \int f_{k-1}(y)\, d\mu_1(y).$$

To achieve a lower bound, we iterate this process. For each $j \in \{2, \ldots, k-1\}$ we have

$$\begin{aligned}\int f_{k-j}^\epsilon(x)\, d\mu_j(x) = \iint \sigma_t^\epsilon(x-y) f_{k-j-1}(y)\, d\mu(y)\, d\mu_j(x) &\geq \int_{G_{t,\epsilon}(j+1)} \sigma_t^\epsilon * \mu_j(y) f_{k-j-1}(y)\, d\mu(y) \\ &\geq c(j+1) \int_{G_{t,\epsilon}(j+1)} f_{k-j-1}(y)\, d\mu(y) \\ &= c(j+1) \int f_{k-j-1}(y)\, d\mu_{j+1}(y).\end{aligned}$$

It follows that

$$C_k^\epsilon(\mu) \geq \left(\prod_{j=1}^{k-1} c(i)\right) \iint \sigma_t^\epsilon(x-y)\, d\mu_{k-1}(y)\, d\mu(x) \geq \left(\prod_{j=1}^{k} c(i)\right) \mu(G_{t,\epsilon}(k)),$$

and we are done in light of Lemma 2.3.

Given Lemmas 2.2 and 2.3, we have shown that, for all $t \in I$ and all $0 < \epsilon < \epsilon_0$, we have

$$\liminf_{\epsilon \to 0} C_k^{\epsilon}(\mu) > 0, \tag{2-11}$$

where all gap lengths $t_1, \dots, t_k$ are constantly equal to t. This concludes the proof of the estimate (2-10) in the case of constant gaps.

We now proceed to the proofs of Lemmas 2.2 and 2.3.

Proof of Lemma 2.2. We write

$$2c(1) < \int \sigma_t^{\epsilon} * \mu(x)\, d\mu(x) \le \left(\int_{(G_{t,\epsilon}(1))^c} \sigma_t^{\epsilon} * \mu(x)\, d\mu(x) \right) + \left(\int_{G_{t,\epsilon}(1)} \sigma_t^{\epsilon} * \mu(x)\, d\mu(x) \right) = \mathcal{I} + \mathcal{II},$$

where A^c denotes the complement of a set $A \subset E$.

We first observe that

$$\mathcal{I} \le c(1).$$

Next we estimate $\mathcal{II}$. Let $m \in \mathbb{Z}^+$ and write

$$G_{t,\epsilon}(1) = \{x \in E : c(1) < \sigma_t^{\epsilon} * \mu(x) \le 2^m\} \cup \{x \in E : 2^m \le \sigma_t^{\epsilon} * \mu(x)\}.$$

Then

$$\begin{aligned} \mathcal{II} &= \int_{\{x \in E : c(1) < \sigma_t^{\epsilon} * \mu(x) \le 2^m\}} \sigma_t^{\epsilon} * \mu(x)\, d\mu(x) + \int_{\{x \in E : 2^m \le \sigma_t^{\epsilon} * \mu(x)\}} \sigma_t^{\epsilon} * \mu(x)\, d\mu(x) \\ &\le 2^m \mu(G_{t,\epsilon}(1)) + \sum_{l=m} 2^{l+1} \cdot \mu\big(\{x \in E : 2^l \le \sigma_t^{\epsilon} * \mu(x) \le 2^{l+1}\}\big). \end{aligned}$$

We use Theorem 2.1 to estimate

$$\mu\big(\{x \in E : 2^l \le \sigma_t^{\epsilon} * \mu(x) \le 2^{l+1}\}\big) \le c_d \cdot 2^{-2l},$$

where the constant c_d depends only on the ambient dimension d. Now,

$$\mathcal{II} \le 2^m \mu(G_{t,\epsilon}(1)) + 2c_d \cdot \sum_{l=m} 2^l \cdot 2^{-2l} \lesssim 2^m \mu(G_{t,\epsilon}(1)) + 2^{-m}.$$

It follows that

$$2c(1) \le \mathcal{I} + \mathcal{II} \lesssim c(1) + 2^m \mu(G_{t,\epsilon}(1)) + 2^{-m}.$$

Taking $m \in \mathbb{Z}^+$ large enough, we conclude that

$$\mu(G_{t,\epsilon}(1)) \ge 2^{-2m}. \qquad \square$$

Proof of Lemma 2.3. We prove the lemma by induction on j. The base case, $j = 1$, was established in Lemma 2.2. Next, assume that there exists $m(j) \in \mathbb{Z}^+$ such that

$$2^{-m(j)} < \mu(G_{t,\epsilon}(j))$$

for all $0 < \epsilon < \epsilon_0$ and $t \in I$.

By the definition of $G_{t,\epsilon}(j)$,

$$c(j)\mu(G_{t,\epsilon}(j)) < \int_{G_{t,\epsilon}(j)} \sigma_t^\epsilon * \mu|_{G_{t,\epsilon}(j-1)}(x)\, d\mu(x).$$

Set $c(j+1) = \frac{1}{2}c(j)\mu(G_{t,\epsilon}(j))$. By assumption, $2c(j+1) = c(j)\mu(G_{t,\epsilon}(j)) \geq c(j)2^{-m(j)}$, and in particular this quantity is positive. Next, we obtain a bound from above:

$$\begin{aligned}\int_{G_{t,\epsilon}(j)} \sigma_t^\epsilon * \mu|_{G_{t,\epsilon}(j-1)}(x)\, d\mu(x) &\leq \int_{G_{t,\epsilon}(j)} \sigma_t^\epsilon * \mu(x)\, d\mu(x)\\ &= \int \sigma_t^\epsilon * \mu|_j(x)\, d\mu(x)\\ &= \left(\int_{(G_{t,\epsilon}(j+1))^c} \sigma_t^\epsilon * \mu|_j(x)\, d\mu(x)\right) + \left(\int_{G_{t,\epsilon}(j+1)} \sigma_t^\epsilon * \mu|_j(x)\, d\mu(x)\right)\\ &= \mathcal{I} + \mathcal{II}.\end{aligned}$$

First we observe that

$$\mathcal{I} \leq c(j+1).$$

Next, we estimate $\mathcal{II}$. Let $m \in \mathbb{Z}^+$ and write

$$G_{t,\epsilon}(j+1) = \{x \in E : c(j+1) < \sigma_t^\epsilon * \mu|_j(x) \leq 2^m\} \cup \{x \in E : 2^m \leq \sigma_t^\epsilon * \mu|_j(x)\}.$$

Then

$$\begin{aligned}\mathcal{II} &= \int_{\{x\in E: c(j+1)<\sigma_t^\epsilon*\mu|_j(x)\leq 2^m\}} \sigma_t^\epsilon * \mu|_j(x)\, d\mu(x) + \int_{\{x\in E:2^m\leq\sigma_t^\epsilon*\mu(x)\}} \sigma_t^\epsilon * \mu|_j(x)\, d\mu(x)\\ &\leq 2^m \cdot \mu(G_{t,\epsilon}(j+1)) + \sum_{l=m} 2^{l+1} \cdot \mu\big(\{x \in E : 2^l \leq \sigma_t^\epsilon * \mu|_j(x) \leq 2^{l+1}\}\big).\end{aligned}$$

We use Theorem 2.1 to estimate

$$\mu\big(\{x \in E : 2^l \leq \sigma_t^\epsilon * \mu|_j(x) \leq 2^{l+1}\}\big) \leq c_d \cdot 2^{-2l},$$

where the constant c_d depends only on the ambient dimension d and the choice of the measure μ. Now,

$$\mathcal{II} \leq 2^m \mu(G_{t,\epsilon}(j+1)) + 2c_d \cdot \sum_{l=m} 2^l \cdot 2^{-2l} \lesssim 2^m \mu(G_{t,\epsilon}(j+1)) + 2^{-m}.$$

It follows that

$$2c(j+1) \leq \mathcal{I} + \mathcal{II} \lesssim c(j+1) + 2^m \mu(G_{t,\epsilon}(j+1)) + 2^{-m}.$$

Taking $m \in \mathbb{Z}^+$ large enough, we conclude that

$$\mu(G_{t,\epsilon}(j+1)) \geq 2^{-2m}. \qquad \square$$

Lower bound for variable gaps. We now verify (2-10) in the case of variable gap lengths. In more detail, we show that, for all $k \in \mathbb{Z}^+$ and for values of t_i in a nonempty open interval $\tilde{I}$, we have

$$\liminf_{\epsilon \to 0} \int f_k^\epsilon(x)\, d\mu(x) > 0, \tag{2-12}$$

where f_k^ϵ is as defined in (2-4) with $0 < t_1, \dots, t_k \in \tilde{I}$.

The following lemma captures the strategy of the proof and establishes (2-12).

Lemma 2.4. *We have*

$$C_k^\epsilon(\mu) = \int f_k^\epsilon(x)\, d\mu(x) = M_k(t_1, \dots, t_k) - \sum_{j=1}^{k} R_{k,j}^\epsilon(t_1, \dots, t_k), \tag{2-13}$$

where

$$M_k(t_1, t_2, \dots, t_k) = \int \hat{\sigma}_{t_k}(\xi) \widehat{f_{k-1}\mu}(-\xi) \hat{\mu}(\xi)\, d\xi \tag{2-14}$$

is continuous and bounded below by a positive constant (*independent of* ϵ) *on* $\tilde{I} \times \cdots \times \tilde{I}$ *for a nonempty open interval* $\tilde{I}$, *and*

$$R_{k,j}^\epsilon(t_1, t_2, \dots, t_k) = \int \hat{\sigma}(t_j\xi)(1 - \hat{\rho}(\epsilon\xi)) \widehat{f_{j-1}\mu}(\xi) \widehat{g_{j+1}^\epsilon\mu}(-\xi)\, d\xi = \mathbb{O}(\epsilon^{\alpha(s-(d+1)/2)}) \tag{2-15}$$

for some $\alpha > 0$.

In proving the lemma, we utilize the notation

$$g_j^\epsilon(x) = T_j^\epsilon \circ \cdots \circ T_k^\epsilon(1)(x) \tag{2-16}$$

and

$$g_{k+1}(x) = 1. \tag{2-17}$$

It is important to note that $g_j(x)$ depends implicitly on the choices of $t_1, \dots, t_k > 0$, and this choice will be made explicit throughout.

First, we demonstrate (2-13) with repeated use of Fourier inversion. We again employ a variant of the argument in [Iosevich et al. 2012]. Write

$$\int f_k^\epsilon(x)\, d\mu(x) = \iint \sigma_{t_1}^\epsilon(x-y) g_2^\epsilon(y)\, d\mu(x)\, d\mu(y) = \iint (\sigma_{t_1} * \rho_\epsilon)(x-y) g_2^\epsilon(y)\, d\mu(x)\, d\mu(y).$$

Using Fourier inversion and properties of the Fourier transform, this is equal to

$$\iiint e^{2\pi i(x-y)\cdot\xi} \hat{\sigma}_{t_1}(\xi) \hat{\rho}_\epsilon(\xi) g_2^\epsilon(y)\, d\mu(x)\, d\mu(y)\, d\xi.$$

Simplifying further, we write

$$\begin{aligned}\int f_k^\epsilon(x)\,d\mu(x) &= \int \hat{\sigma}_{t_1}(\xi)\hat{\rho}(\epsilon\xi)\hat{\mu}(\xi)\widehat{g_2^\epsilon\mu}(-\xi)\,d\xi \\ &= \int \hat{\sigma}_{t_1}(\xi)\hat{\mu}(\xi)\widehat{g_2^\epsilon\mu}(-\xi)\,d\xi + \int \hat{\sigma}_{t_1}(\xi)(1-\hat{\rho}(\epsilon\xi))\hat{\mu}(\xi)\widehat{g_2^\epsilon\mu}(-\xi)\,d\xi \\ &= \int \hat{\sigma}_{t_1}(\xi)\hat{\mu}(\xi)\widehat{g_2^\epsilon\mu}(-\xi)\,d\xi + R_{k,1}^\epsilon(t_1,t_2,\dots,t_k).\end{aligned}$$

With repeated use of Fourier inversion, we get

$$\begin{aligned}\int f_k^\epsilon(x)\,d\mu(x) &= \int \hat{\sigma}_{t_j}(\xi)\cdot\widehat{f_{j-1}\mu}(-\xi)\cdot\widehat{g_{j+1}^\epsilon\mu}(\xi)\,d\xi + \sum_{l=1}^{j} R_{k,l}^\epsilon(t_1,t_2,\dots,t_k) \\ &\ \ \vdots \\ &= \int \hat{\sigma}_{t_k}(\xi)\cdot\widehat{f_{k-1}\mu}(-\xi)\cdot\hat{\mu}(\xi)\,d\xi + \sum_{l=1}^{k} R_{k,l}^\epsilon(t_1,t_2,\dots,t_k) \\ &= M_k(t_1,t_2,\dots,t_k) + \sum_{l=1}^{k} R_{k,l}^\epsilon(t_1,t_2,\dots,t_k).\end{aligned}$$

We now prove that $M_k(t_1,t_2,\dots,t_k)$ is continuous on any compact set away from $(t_1,\dots,t_k)=\vec{0}$ and that

$$R_{k,j}^\epsilon(t_1,\dots,t_k) = \mathbb{O}(\epsilon^{\alpha(s-(d+1)/2)}). \tag{2-18}$$

Once these are established, we observe that the lower bound on constant chains established in (2-11) combined with (2-18) implies that $M_k(t_1,\dots,t_k)$ is positive when $t_1=\cdots=t_k=t$ for any given $t\in I$. Fixing any such $t\in I$, it will then follow by continuity that $M_k(t_1,\dots,t_k)$ is bounded from below on $\tilde{I}\times\cdots\times\tilde{I}$, where $\tilde{I}$ is a nonempty interval.

We now use the dominated convergence theorem to verify the continuity of $M_k(t_1,\dots,t_k)$ on any compact set away from $(t_1,\dots,t_k)=\vec{0}$. Let $t_1,\dots,t_k>0$. Using properties of the Fourier transform and recalling the definition of f_j from (2-4) and g_j from (2-16), we write

$$M_k(t_1,t_2,\dots,t_k) = \int \hat{\sigma}_{t_j}(\xi)\cdot\widehat{f_{j-1}\mu}(-\xi)\cdot\widehat{g_{j+1}\mu}(\xi)\,d\xi$$

for any $j\in\{1,\dots,k\}$.

Let $h_1,\dots,h_k\in\mathbb{R}$ be such that $(h_1,\dots,h_k)\downarrow 0$. Let

$$\tilde{f}_j = T_{t_j+h_j}\circ\cdots\circ T_{t_1+h_1}(1) \quad\text{and}\quad \tilde{g}_j = T_{t_j+h_j}\circ\cdots\circ T_{t_k+h_k}(1).$$

We have

$$M_k(t_1+h_1,t_2+h_2,\dots,t_k+h_k) = \int \hat{\sigma}_{t_j+h_j}(\xi)\cdot\widehat{\tilde{f}_{j-1}\mu}(-\xi)\cdot\widehat{\tilde{g}_{j+1}\mu}(\xi)\,d\xi.$$

The integrand goes to 0 as h_j goes to 0. Now, for t_j in a compact set, the expression above is bounded by

$$C(t_j)\int |\xi|^{-(d-1)/2}|\widehat{\tilde{f}_{j-1}\mu}(-\xi)||\widehat{\tilde{g}_{j+1}\mu}(\xi)|\,d\xi.$$

To proceed, we will utilize the following calculation:

Lemma 2.5. *Let μ be a compactly supported Borel measure such that $\mu(B(x,r)) \le Cr^s$ for some $s\in(0,d)$. Suppose that $\alpha > d-s$. Then, for $f\in L^2(\mu)$,*

$$\int |\widehat{f\mu}(\xi)|^2|\xi|^{-\alpha}\,d\xi \le C'\|f\|^2_{L^2(\mu)}. \tag{2-19}$$

To prove Lemma 2.5, observe that

$$\int |\widehat{f\mu}(\xi)|^2|\xi|^{-\alpha}\,d\xi = C\iint f(x)f(y)|x-y|^{-d+\alpha}\,d\mu(x)\,d\mu(y) = \langle Tf, f\rangle, \tag{2-20}$$

where

$$Tf(x) = \int |x-y|^{-d+\alpha} f(y)\,d\mu(y)$$

and the inner product above is with respect to $L^2(\mu)$. The positive constant C appearing in (2-20) depends only on the ambient dimension d. Observe that

$$\int |x-y|^{-d+\alpha}\,d\mu(y) \approx \sum_{j>0} 2^{j(d-\alpha)}\int_{|x-y|\approx 2^{-j}} d\mu(y) \le C\sum_{j>0} 2^{j(d-\alpha-s)} \le C'$$

since $\alpha > d-s$.

By symmetry, $\int |x-y|^{-d+\alpha}\,d\mu(x) \le C'$. It follows by using Schur's test [1911] — see also Lemma 7.5 in [Wolff 2003] — that

$$\|Tf\|_{L^2(\mu)} \le C'\|f\|_{L^2(\mu)}.$$

This implies the conclusion of Lemma 2.5 by applying the Cauchy–Schwarz inequality to (2-20). We note that Lemma 2.5 can also be recovered from the fractal Plancherel estimate due to R. Strichartz [1990]. See also Theorem 7.4 in [Wolff 2003], where a similar statement is proved by the same method as above.

We already established, using [Iosevich et al. 2014], that finite compositions of the operators T_l applied to $L^2(\mu)$ functions are in $L_2(\mu)$. Using the Cauchy–Schwarz inequality and in light of Lemma 2.5, $M_k(t_1+h_1, t_2+h_2, \dots, t_k+h_k)$ is bounded. We proceed by applying the dominated convergence theorem. We have

$$\begin{aligned}\lim_{h_j\downarrow 0} & M_k(t_1+h_1, t_2+h_2, \dots, t_k+h_k)\\ &= \int \hat{\sigma}_{t_j}(\xi)\cdot\widehat{\tilde{g}_{j-1}\mu}(-\xi)\cdot\widehat{\tilde{f}_{j+1}\mu}(\xi)\,d\xi\\ &= \int \hat{\sigma}_{t_j}(\xi)\cdot(T_{t_{j-1}+h_{j-1}}\circ\cdots\circ T_{t_1+h_1}(1)\cdot\mu)\widehat{\ }(-\xi)\cdot(T_{t_{j+1}+h_{j+1}}\circ\cdots\circ T_{t_k+h_k}(1)\cdot\mu)\widehat{\ }(\xi)\,d\xi.\end{aligned}$$

We then rewrite the procedure, isolating $\hat{\sigma}_{t_j}$ for each $j\in\{1,\dots,k\}$, and repeat the process above a total of k times.

Bounding the remainder. Next, we wish to show that $\lim_{\epsilon\downarrow 0} R_k^\epsilon(t_1,\dots,t_k)=0$. Fix $\epsilon>0$. Recall that $R_k^\epsilon(t_1,\dots,t_k)$ is equal to

$$\int (1-\hat\rho(\epsilon\xi))\hat\sigma(t\xi)\hat\mu(\xi)\widehat{f_k\mu}(-\xi)\,d\xi.$$

We consider the integral over $|\xi|<(1/\epsilon)^\alpha$ and the integral over $|\xi|>(1/\epsilon)^\alpha$ separately, where $\alpha\in(0,1)$ will be determined. Assume that $s>\frac{1}{2}(d+1)$.

Lemma 2.6. *Let $\rho:\mathbb{R}^d\to\mathbb{R}$ satisfy the following properties: $\rho\geq 0$, $\rho(x)=\rho(-x)$, the support of ρ is contained in $\{x:|x|<c\}$, and $\int\rho=1$. Then*

$$0\leq 1-\hat\rho(\xi)\leq 2\pi c|\xi|.$$

To prove Lemma 2.6, write

$$\hat\rho(\xi)=\int\cos(2\pi x\cdot\xi)\rho(x)\,dx.$$

We observe that $\cos x+|x|>1$, and conclude that the lemma follows when $|x|<c$. It follows that

$$\int_{|\xi|<(1/\epsilon)^\alpha}|\hat\rho(\epsilon\xi)-1||\hat\sigma(t\xi)||\hat\mu(\xi)||\widehat{f_k\mu}(-\xi)|\,d\xi\lesssim\epsilon^{1-\alpha}\int|\hat\sigma(t\xi)||\hat\mu(\xi)||\widehat{f_k\mu}(-\xi)|\,d\xi\lesssim\epsilon^{1-\alpha},$$

where the last line is justified in the estimation of $M_k(t)$ above.

It remains to estimate the quantity

$$\int_{|\xi|>(1/\epsilon)^\alpha}|\hat\sigma(t\xi)||\hat\mu(\xi)||\widehat{f_k\mu}(-\xi)|\,d\xi.$$

Proceeding as in the estimation of $M_k(t)$ above, we bound the integral above by

$$Ct^{-(d-1)/2}\int_{|\xi|>(1/\epsilon)^\alpha}|\xi|^{-(d-1)/2}|\hat\mu(\xi)||\widehat{f_k\mu}(-\xi)|\,d\xi$$

and then use Cauchy–Schwarz to bound it further by

$$Ct^{-(d-1)/2}\left(\int_{|\xi|>(1/\epsilon)^\alpha}|\xi|^{-(d-1)/2}|\hat\mu(\xi)|^2\,d\xi\right)^{\frac12}\left(\int_{|\xi|>(1/\epsilon)^\alpha}|\xi|^{-(d-1)/2}|\widehat{f_k\mu}(\xi)|^2\,d\xi\right)^{\frac12}.$$

We have already shown that the second integral is finite. The first integral is bounded by

$$\sum_{j>\alpha\log_2(1/\epsilon)}2^{-j(d-1)/2}\int_{2^j\leq|\xi|<2^{j+1}}|\hat\mu(\xi)|^2\,d\xi.$$

We may choose a smooth cut-off function ψ such that the inner integral is bounded by

$$\int|\hat\mu(\xi)|^2\hat\psi(2^{-j}\xi)\,d\xi.$$

By Fourier inversion, this integral is equal to

$$2^{dj}\iint\psi(2^j(x-y))\,d\mu(x)\,d\mu(y)\leq C2^{j(d-s)}.$$

Returning to the sum, we now have the estimate

$$C \sum_{j>\alpha \log_2(1/\epsilon)} 2^{-j(d-1)/2} \cdot 2^{j(d-s)} \le C \sum_{j>\alpha \log_2(1/\epsilon)} 2^{j(d+1)/2-s}.$$

As long as $s > \frac{1}{2}(d+1)$, this is $\ll \epsilon^{\alpha(s-(d+1)/2)}$. Thus $R_k^\epsilon(t_1, \dots, t_k)$ tends to 0 with ϵ as long as $\dim_{\mathcal{H}}(E) > \frac{1}{2}(d+1)$.

In conclusion, we have

$$\lim_{\epsilon \downarrow 0} \int \left(\prod_{j=1}^{k} \sigma_{t_j}^\epsilon (x^{i+1} - x^i)\, d\mu(x^i) \right) d\mu(x^{k+1}) > c_k > 0 \tag{2-21}$$

for all $t_j \in \tilde{I}$.

To complete the proof of Theorem 1.7, it remains to verify that E contains a nondegenerate k-chain with prescribed gaps. This is the topic of the next section.

3. Nondegeneracy

An important issue we have not yet addressed is that the chains we have found may be degenerate. As an extreme example, consider the case where $t_i = 1$ for all i. Then included in our chain count are chains which simply bounce back and forth between two different points. We now take steps to ensure that we can indeed find chains with distinct vertices.

We verified above that there exists a nonempty open interval $\tilde{I}$ such that

$$\lim_{\epsilon \downarrow 0} \int \left(\prod_{j=1}^{k} \sigma_{t_j}^\epsilon (x^{i+1} - x^i)\, d\mu(x^i) \right) d\mu(x^{k+1})$$

is bounded above and below for $t_1, \dots, t_k \in \tilde{I}$. The upper bound appears in (2-3) and the lower bound appears in (2-21).

From here onward, we fix $t_1, \dots, t_k \in \tilde{I}$ and set $\vec{t} = (t_1, \dots, t_k)$. We now define a nonnegative Borel measure on the set of k-chains with the gaps $\vec{t}$. Let $\Lambda_{\vec{t}}^k$ denote a nonnegative Borel measure, defined as

$$\Lambda_{\vec{t}}^k(A) = \lim_{\epsilon \downarrow 0} \int_A \left(\prod_{j=1}^{k} \sigma_{t_j}^\epsilon (x^{i+1} - x^i)\, d\mu(x^i) \right) d\mu(x^{k+1}),$$

where $A \subset E \times \cdots \times E$, the $(k+1)$-fold product of the set E.

It follows that $\Lambda_{\vec{t}}^k$ is a finite measure which is not identically zero:

$$0 < \Lambda_{\vec{t}}^k(E \times \cdots \times E). \tag{3-1}$$

The strategy we use to demonstrate the existence of nondegenerate k-chains in E is as follows: We first show that $\Lambda_{\vec{t}}^k$ has support contained in the set of k-chains. This is accomplished by showing that the measure has support contained in all "approximate" k-chains. We then show that the measure of the set of degenerate chains is zero. It follows, since the $\Lambda_{\vec{t}}^k$-measure of the set of k-chains is positive and the $\Lambda_{\vec{t}}^k$-measure of the set of degenerate k-chains is zero, that the set of nondegenerate k-chains in E is nonempty.

For each test entry $n \in \mathbb{Z}^+$, define the sets of $(1/n)$-approximate k-chains and the set of exact k-chains as

$$A_{n,k} = \left\{(x^1, \dots, x^{k+1}) \in E \times \cdots \times E : t_i - \frac{1}{n} \le |x^{i+1} - x^i| \le t_i + \frac{1}{n} \text{ for each } i = 1, \dots, k\right\}$$

and

$$A_k = \left\{(x^1, \dots, x^{k+1}) \in E \times \cdots \times E : |x^{i+1} - x^i| = t_i \text{ for each } i = 1, \dots, k\right\}.$$

Observe that

$$\bigcap_n A_{n,k} = A_k.$$

We now observe that the support of Λ_{t}^k is contained in the set of all approximate chains. This follows immediately from the observation that

$$\Lambda_{\vec{t}}^k (A_{n,k}^c) = 0$$

for each $n \in \mathbb{Z}^+$, where $A_{n,k}^c$ denotes the complement of the set $A_{n,k}$ in $E \times \cdots \times E$.

Next, we observe that the support of $\Lambda_{\vec{t}}^k$ is contained in the set of exact chains. Indeed, it follows from the previous equation that

$$\Lambda_{\vec{t}}^k \left(\bigcup_n A_{n,k}^c\right) \le \sum_n \Lambda_{\vec{t}}^k (A_{n,k}^c) = 0.$$

Recalling (3-1), we conclude that

$$0 < \Lambda_{\vec{t}}^k (E \times \cdots \times E) = \Lambda_{\vec{t}}^k \left(\bigcup_n A_{n,k}^c\right) + \Lambda_{\vec{t}}^k \left(\bigcap_n A_{n,k}\right), \tag{3-2}$$

and so

$$\Lambda_{\vec{t}}^k (A_k) = \Lambda_{\vec{t}}^k \left(\bigcap_n A_{n,k}\right) > 0.$$

Since $t_1, \dots, t_k \in \tilde{I}$ were chosen arbitrarily, we have shown that $\Lambda_{\vec{t}}^k (A_k) > 0$ whenever $\vec{t} = (t_1, \dots, t_k)$ and $t_i \in \tilde{I}$.

We now verify that the set of degenerate chains has $\Lambda_{\vec{t}}^k$-measure zero.

Lemma 3.1. *Let*

$$D_k = \{(x^1, ..., x^{k+1}) \in E \times \cdots \times E : x^i = x^j \textit{ for some } i \neq j\}.$$

Then

$$\Lambda_{\vec{t}}^k (D_k) = 0.$$

To prove the lemma, we first investigate the quantity

$$\int_{D_k} \left(\prod_{j=1}^k \sigma_{t_j}^{\epsilon} (x^{i+1} - x^i)\, d\mu(x^i)\right) d\mu(x^{k+1}).$$

By the definition of D_k, we can bound the quantity above by

$$\sum_{1\le m<n\le k+1}\int_{\{(x^1,\dots,x^{k+1}):x^m=x^n\}}\left(\prod_{j=1}^{k}\sigma^{\epsilon}_{t_j}(x^{i+1}-x^i)\,d\mu(x^i)\right)d\mu(x^{k+1}).$$

We can rewrite the integral as

$$\int_{(\mathbb{R}^d)^k}\int_{\{x:x=x^m\}}\left(\prod_{j=1}^{k}\sigma^{\epsilon}_{t_j}(x^{i+1}-x^i)\right)d\mu(x^n)\,d\mu(x^1)\cdots d\mu(x^{n-1})\,d\mu(x^{n+1})\cdots d\mu(x^{k+1}).$$

Since the inside integral is taken over a region of measure 0, this whole integral must be 0. This holds for every choice of m and n, and thus the entire sum must be 0. This completes the proof of the lemma.

In conclusion, we have shown that the set of exact k-chains has positive measure — $\Lambda^k_{\vec{t}}(A_k)>0$ — and that the set of degenerate chains has zero measure — $\Lambda^k_{\vec{t}}(D_k)=0$. It follows that $A_k\neq D_k$ and $A_k\neq\varnothing$. In other words, there exists a nonempty open interval $\tilde{I}$ and *distinct* elements $x^1,\dots,x^{k+1}\in E$ such that $|x^{i+1}-x^i|=t^i$ for each $i\in\{1,\dots,k\}$.

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ADVECTION-DIFFUSION EQUATIONS WITH DENSITY CONSTRAINTS

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In the spirit of the macroscopic crowd motion models with hard congestion (i.e., a strong density constraint $\rho \leq 1$) introduced by Maury et al. some years ago, we analyze a variant of the same models where diffusion of the agents is also taken into account. From the modeling point of view, this means that individuals try to follow a given spontaneous velocity, but are subject to a Brownian diffusion, and have to adapt to a density constraint which introduces a pressure term affecting the movement. From the point of view of PDEs, this corresponds to a modified Fokker–Planck equation, with an additional gradient of a pressure (only living in the saturated zone $\{\rho = 1\}$) in the drift. We prove existence and some estimates, based on optimal transport techniques.

1. Introduction

In the past few years modeling crowd behavior has become a very active field of applied mathematics. Beyond their importance in real life applications, these modeling problems serve as basic ideas to understand many other phenomena coming for example from biology (cell migration, tumor growth, pattern formations in animal populations, etc.), particle physics and economics. A first nonexhaustive list of references for these problems is [Chalons 2007; Colombo and Rosini 2005; Coscia and Canavesio 2008; Cristiani et al. 2014; Dogbé 2008; Helbing 1992; Helbing and Molnár 1995; Hughes 2002; 2003; Maury and Venel 2009]. A very natural question in all these models is the study of congestion: in many practical situations, a high number of individuals could try to occupy the same spot, which could be impossible, or lead to strong negative effects on the motion, because of natural limitations on the crowd density.

These phenomena have been studied by using different models, which could be either "microscopic" (based on ODEs on the motion of a high number of agents) or "macroscopic" (describing the agents via their density and velocity, typically with Eulerian formalism). Let us concentrate on the macroscopic models, where the density ρ plays a crucial role. These very same models can be characterized either by "soft congestion" effects (i.e., the higher the density, the slower the motion), or by "hard congestion" (i.e., an abrupt threshold effect: if the density touches a certain maximal value, the motion is strongly affected, while nothing happens for smaller values of the density). See [Maury et al. 2011] for comparison between the different classes of models. This last class of models, due to the discontinuity in the congestion effects, presents new mathematical difficulties, which cannot be analyzed with the usual techniques from conservation laws (or, more generally, evolution PDEs) used for soft congestion.

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A very powerful tool to attack macroscopic hard congestion problems is the theory of optimal transportation (see [Villani 2003; Santambrogio 2015]), as we can see in [Maury et al. 2010; 2011; Roudneff-Chupin 2011; Santambrogio 2012a]. In this framework, the density of the agents solves a continuity equation (with velocity field taking into account the congestion effects), and can be seen as a curve in the Wasserstein space.

Our aim in this paper is to endow the macroscopic hard congestion models of [Maury et al. 2010; 2011; Roudneff-Chupin 2011; Santambrogio 2012a] with diffusion effects. In other words, we will study an evolution equation where particles

- have a spontaneous velocity field $u_t(x)$ which depends on time and on their position, and is the velocity they would follow in the absence of the other particles;
- must adapt their velocity to the existence of an incompressibility constraint which prevents the density to go beyond a given threshold;
- are subject to some diffusion effect.

This can be considered as a model for a crowd where a part of the motion of each agent is driven by a Brownian motion. Implementing this new element into the existing models could give a better approximation of reality; as usual when one adds a stochastic component, this can be a (very) rough approximation of unpredictable effects which are not already handled by the model, and this could work well when dealing with large populations.

Anyway, we do not want to discuss here the validity of this hard congestion model and we are mainly concerned with its mathematical analysis. In particular, we will consider existence and regularity estimates, while we do not treat the uniqueness issue. Uniqueness is considered in [Di Marino and Mészáros 2016], and one can observe that the insertion of diffusion dramatically simplifies the picture as far as uniqueness is concerned.

We also underline that one of the goals of the current paper (and of the work just cited) is to better "prepare" these hard congestion crowd motion models for a possible analysis in the framework of mean field games (see [Lasry and Lions 2006a; 2006b; 2007], and also [Santambrogio 2012b]). These MFG models usually involve a stochastic term, also implying regularizing effects, which are useful in the mathematical analysis of the corresponding PDEs.

The existing first-order models in light of the work of Maury, Roudneff-Chupin and Santambrogio. Some macroscopic models for crowd motion with density constraints and "hard congestion" effects were studied in [Maury et al. 2010; 2011]. We briefly present them as follows:

- The density of the population in a bounded (convex) domain $\Omega \subset \mathbb{R}^d$ is described by a probability measure $\rho \in \mathcal{P}(\Omega)$. The initial density $\rho_0 \in \mathcal{P}(\Omega)$ evolves in time, and ρ_t denotes its value at each time $t \in [0, T]$.
- The spontaneous velocity field of the population is a given time-dependent field, denoted by u_t. It represents the velocity that each individual would like to follow in the absence of the others. Ignoring the density constraint, this would give rise to the continuity equation $\partial_t \rho_t + \nabla \cdot (\rho_t u_t) = 0$. We observe that in the original work [Maury et al. 2010] the vector field $u_t(x)$ was taken of the form $-\nabla D(x)$ (independent

of time and of gradient form), but we try here to be more general (see [Roudneff-Chupin 2011], where the nongradient case is studied under some stronger regularity assumptions).

- The set of admissible densities will be denoted by $\mathcal{K} := \{\rho \in \mathcal{P}(\Omega) : \rho \le 1\}$. In order to guarantee that $\mathcal{K}$ is neither empty nor trivial, we suppose $|\Omega| > 1$.

- The set of admissible velocity fields with respect to the density ρ is characterized by the sign of the divergence of the velocity field on the saturated zone. We need to suppose also that all admissible velocity fields are such that no mass exists from the domain. So formally we set

$$\mathrm{adm}(\rho) := \big\{ v : \Omega \to \mathbb{R}^d : \nabla \cdot v \ge 0 \text{ on } \{\rho = 1\} \text{ and } v \cdot n \le 0 \text{ on } \partial\Omega \big\}.$$

- We consider the projection operator P in $L^2(\mathcal{L}^d)$:

$$P_{\mathrm{adm}(\rho)}[u] \in \operatorname{argmin}_{v \in \mathrm{adm}(\rho)} \int_\Omega |u - v|^2 \,\mathrm{d}x.$$

Note that we could have used the Hilbert space $L^2(\rho)$ instead of $L^2(\mathcal{L}^d)$; this would be more natural in this kind of evolution equation, as $L^2(\rho)$ is interpreted in a standard way as the tangent space to the Wasserstein space $\mathcal{W}_2(\Omega)$. Yet, these two projections turn out to be the same in this case, as the only relevant zone is $\{\rho = 1\}$. This is just formal, and would require more rigorous definitions (in particular of the divergence constraint in $\mathrm{adm}(\rho)$; see below). Anyway, to clarify, we choose to use the $L^2(\mathcal{L}^d)$-projection; in this way the vector fields are considered to be defined Lebesgue-a.e. on the whole Ω (and not only on $\{\rho > 0\}$) and the dependence of the projected vector field on ρ only passes through the set $\mathrm{adm}(\rho)$.

- Finally we solve the modified continuity equation

$$\partial_t \rho_t + \nabla \cdot (\rho_t P_{\mathrm{adm}(\rho_t)}[u_t]) = 0 \tag{1-1}$$

for ρ, where the main point is that ρ is advected by a vector field, compatible with the constraints, which is the closest to the spontaneous one.

The problem in solving (1-1) is that the projected field has very low regularity; it is a priori only L^2 in x, and it does not depend smoothly on ρ either (since a density 1 and a density $1-\varepsilon$ give very different projection operators). By the way, its divergence is not well defined either. To handle this issue we need to redefine the set of admissible velocities by duality. Taking a test function $p \in H^1(\Omega)$, $p \ge 0$ a.e., we obtain by integration by parts the equality

$$\int_\Omega v \cdot \nabla p \,\mathrm{d}x = -\int_\Omega (\nabla \cdot v) p \,\mathrm{d}x + \int_{\partial\Omega} p v \cdot n \,\mathrm{d}\mathcal{H}^{d-1}(x).$$

For vector fields v which do not let mass go through the boundary $\partial\Omega$, we have (in an a.e. sense) $v \cdot n = 0$. This leads to the definition

$$\mathrm{adm}(\rho) = \left\{ v \in L^2(\Omega; \mathbb{R}^d) : \int_\Omega v \cdot \nabla p \,\mathrm{d}x \le 0 \text{ for all } p \in H^1(\Omega) \text{ with } p \ge 0,\ p(1-\rho) = 0 \text{ a.e.} \right\}.$$

(Indeed, for a smooth vector field with vanishing normal component on the boundary, this is equivalent to imposing $\nabla \cdot v \ge 0$ on the set $\{\rho = 1\}$.)

Now, if we set

$$\mathrm{press}(\rho) := \{p \in H^1(\Omega) : p \geq 0,\ p(1-\rho) = 0 \text{ a.e.}\},$$

we observe that, by definition, $\mathrm{adm}(\rho)$ and $\nabla\,\mathrm{press}(\rho)$ are two convex cones which are dual to each other in $L^2(\Omega; \mathbb{R}^d)$. Hence we always have a unique orthogonal decomposition

$$u = v + \nabla p, \quad v \in \mathrm{adm}(\rho), \quad p \in \mathrm{press}(\rho), \quad \int_\Omega v \cdot \nabla p \,\mathrm{d}x = 0. \tag{1-2}$$

In this decomposition (as is the case every time we decompose on two dual convex cones), $v = P_{\mathrm{adm}(\rho)}[u]$. These will be our mathematical definitions for $\mathrm{adm}(\rho)$ and for the projection onto this cone.

Via this approach (introducing the new variable p and using its characterization from the previous line), for a given desired velocity field $u : [0, T] \times \Omega \to \mathbb{R}^d$, the continuity equation (1-1) can be rewritten as a system for the pair of variables (ρ, p), namely

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t(u_t - \nabla p_t)) = 0 & \text{in } [0, T] \times \Omega, \\ p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0 & \text{in } [0, T] \times \Omega, \\ \rho_t(u_t - \nabla p_t) \cdot n = 0 & \text{on } [0, T] \times \partial\Omega. \end{cases} \tag{1-3}$$

This system is endowed with the initial condition $\rho(0, x) = \rho_0(x)$ (for $\rho_0 \in \mathcal{K}$). As far as the spatial boundary $\partial\Omega$ is concerned, we put no-flux boundary conditions to preserve the mass in Ω.

Note that in the above system we withdrew the condition $\int (u_t - \nabla p_t) \cdot \nabla p_t = 0$, as it is a consequence of the system (1-3) itself. Informally, this can be seen as follows. For an arbitrary $p_0 \in \mathrm{press}(\rho_{t_0})$, we have that $t \mapsto \int_\Omega p_0 \rho_t$ is maximal at $t = t_0$ (where it is equal to $\int_\Omega p_0$). Differentiating this quantity with respect to t at $t = t_0$, using (1-3), we get the desired orthogonality condition at $t = t_0$. For a rigorous proof of this fact (which holds for a.e. t_0), we refer to Proposition 4.7 in [Di Marino et al. 2016].

A diffusive counterpart. The goal of our work is to study a second-order model of crowd movements with hard congestion effects, where beside the transport factor a nondegenerate diffusion is present as well. The diffusion is the consequence of a randomness (a Brownian motion) in the movement of the crowd.

With the ingredients introduced so far, we modify the Fokker–Planck equation $\partial_t \rho_t - \Delta\rho_t + \nabla \cdot (\rho_t u_t) = 0$ in order to take into account the density constraint $\rho_t \leq 1$. Assuming enough regularity for the velocity field u, we observe that the Fokker–Planck equation is derived from a motion given by the stochastic ODE $\mathrm{d}X_t = u_t(X_t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t$ (where B_t is the standard d-dimensional Brownian motion), but is macroscopically represented by the advection of the density ρ_t by the vector field $-\nabla\rho_t/\rho_t + u_t$. Projecting onto the set of admissible velocities raises a natural question: should we project only u_t, and *then* apply the diffusion, or project the whole vector field, including $-\nabla\rho_t/\rho_t$? But this is not a real issue, since, at least formally, $\nabla\rho_t/\rho_t = 0$ on the saturated set $\{\rho_t = 1\}$ and

$$P_{\mathrm{adm}(\rho_t)}[-\nabla\rho_t/\rho_t + u_t] = P_{\mathrm{adm}(\rho_t)}[-\nabla\rho_t/\rho_t] + P_{\mathrm{adm}(\rho_t)}[u_t] = 0 + P_{\mathrm{adm}(\rho_t)}[u_t].$$

Rigorously, this corresponds to the fact that the heat kernel preserves the constraint $\rho \leq 1$. As a consequence, we consider the modified Fokker–Planck-type equation

$$\partial_t \rho_t - \Delta\rho_t + \nabla \cdot (\rho_t P_{\mathrm{adm}(\rho_t)}[u_t]) = 0, \tag{1-4}$$

which can also be written equivalently for the variables (ρ, p) as

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t (u_t - \nabla p_t)) = 0 & \text{in } [0, T] \times \Omega, \\ p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0 & \text{in } [0, T] \times \Omega. \end{cases} \tag{1-5}$$

As usual, these equations are complemented by no-flux boundary conditions and by an initial datum $\rho(0, x) = \rho_0(x)$.

Roughly speaking, we can consider this equation to describe the law of a motion where each agent solves the stochastic differential equation

$$\mathrm{d}X_t = (u_t(X_t) - \nabla p_t(X_t))\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t.$$

This last statement is just formal and there are several issues defining a stochastic ODE like this. Indeed, the pressure variable is also an unknown, and globally depends on the law ρ_t of X_t. Hence, if we wanted to see this evolution as a superposition of individual motions, each agent should somehow predict the evolution of the pressure in order to solve his own equation. This calls to mind some notions from the stochastic control formulation of mean field games, as introduced by J.-M. Lasry and P.-L. Lions, even if here there are no strategic issues for the players. For mean field games with density constraints, we refer to [Cardaliaguet et al. 2015; Mészáros and Silva 2015; Santambrogio 2012b].

However, in this paper we will not consider any microscopic or individual problems, but only study the parabolic PDE (1-5).

Structure of the paper and main results. The main goal of the paper is to provide an existence result, with some extra estimates, for the Fokker–Planck equation (1-5) via time discretization, using the so-called splitting method (the two main ingredients of the equation, i.e., the advection with diffusion on one hand, and the density constraint on the other hand, are treated one after the other). In Section 2 we will collect some preliminary results, including what we need from optimal transport and from the previous works about density-constrained crowd motion, in particular on the projection operator onto the set $\mathcal{K}$. In Section 3 we will provide the aforementioned existence result, by a splitting scheme and some entropy bounds; the solution will be a curve of measures in $\mathrm{AC}^2([0, T]; \mathcal{W}_2(\Omega))$ (absolutely continuous curves with square-integrable speed). In Section 4 we will make use of BV estimates to justify that the solution just built is also $\mathrm{Lip}([0, T]; \mathcal{W}_1(\Omega))$ and satisfies a global BV bound $\|\rho_t\|_{\mathrm{BV}} \leq C$ (provided that $\rho_0 \in \mathrm{BV}$); this requires us to combine BV estimates on the Fokker–Planck equation (which are available depending on the regularity of the vector field u) with BV estimates on the projection operator on $\mathcal{K}$ (which have been recently proven in [De Philippis et al. 2016]). Section 5 presents a short review of alternative approaches, all discretized in time, but based either on gradient-flow techniques (the JKO scheme, see [Jordan et al. 1998]) or on different splitting methods. Finally, in the Appendix we detail the BV estimates on the Fokker–Planck equation (without any density constraint) that we could find; this seems to be a delicate matter, interesting in itself, and we are not aware of the sharp assumptions on the vector field u to guarantee the BV estimate that we need.

2. Preliminaries

Basic definitions and general facts on optimal transport. Here we collect some tools from the theory of optimal transportation, Wasserstein spaces, its dynamical formulation and more, which will be used later on. We formulate our problem either in a compact convex domain $\Omega \subset \mathbb{R}^d$ with smooth boundary or in the d-dimensional flat torus $\Omega := \mathbb{T}^d$ (although we will not adapt all our notation to the torus case). We refer to [Villani 2003; Santambrogio 2015] for more details. Given two probability measures $\mu, \nu \in \mathcal{P}(\Omega)$ and $p \geq 1$ we define the usual Wasserstein metric by means of the Monge–Kantorovich optimal transportation problem

$$W_p(\mu, \nu) := \inf\left\{ \int_{\Omega\times\Omega} |x-y|^p \,\mathrm{d}\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\}^{1/p},$$

where $\Pi(\mu, \nu) := \{\gamma \in \mathcal{P}(\Omega \times \Omega) : (\pi^x)_\#\gamma = \mu, (\pi^y)_\#\gamma = \nu\}$ and π^x and π^y denote the canonical projections from $\Omega \times \Omega$ onto Ω. This quantity happens to be a distance on $\mathcal{P}(\Omega)$ which metrizes the weak-$*$ convergence of probability measures; we denote by $\mathcal{W}_p(\Omega) := (\mathcal{P}(\Omega), W_p)$ the space of probabilities on Ω endowed with this distance.

Moreover, in the quadratic case $p = 2$ and under the assumption $\mu \ll \mathcal{L}^d$ (the d-dimensional Lebesgue measure on Ω), Y. Brenier [1987; 1991] showed that actually the optimal $\bar{\gamma}$ in the above problem (the existence of which is obtained simply by the direct method of calculus of variations) is induced by a map which is the gradient of a convex function, i.e., there exists $S : \Omega \to \Omega$ and $\psi : \Omega \to \mathbb{R}$ convex such that $S = \nabla\psi$ and $\bar{\gamma} := (\mathrm{id}, S)_\#\mu$. The function ψ is obtained as $\psi(x) = \frac{1}{2}|x|^2 - \varphi(x)$, where φ is the so-called Kantorovich potential for the transport from μ to ν, and is characterized as the solution of a dual problem that we will not develop here. In this way, the optimal transport map S can also be written as $S(x) = x - \nabla\varphi(x)$. Later, in the 1990s, R. McCann [1997] introduced a notion of interpolation between probability measures: the curve $\mu_t := ((T-t)x + ty)_\#\bar{\gamma}$, for $t \in [0, T]$ ($T > 0$ is given), gives a constant speed geodesic in the Wasserstein space connecting $\mu_0 := \mu$ and $\mu_T := \nu$.

Based on this notion of interpolation, J.-D. Benamou and Y. Brenier [2000] used some ideas from fluid mechanics to give a dynamical formulation to the Monge–Kantorovich problem. They showed that

$$\frac{1}{pT^{p-1}} W_p^p(\mu, \nu) = \inf\{\mathcal{B}_p(E, \mu) : \partial_t\mu + \nabla\cdot E = 0,\ \mu_0 = \mu,\ \mu_T = \nu\}.$$

Here $\mathcal{B}_p$ is a functional defined on pairs (E, μ), where E is a d-dimensional vector measure on $[0, T] \times \Omega$ and $\mu = (\mu_t)_t$ is a Borel-measurable family of probability measures on Ω. This functional is defined to be finite only if $E = E_t \otimes \mathrm{d}t$ (i.e., it is induced by a measurable family of vector measures on Ω: we have $\int_{[0,T]\times\Omega} \xi(t, x)\cdot \mathrm{d}E(t, x) = \int_0^T \mathrm{d}t \int_\Omega \xi(t, x)\cdot \mathrm{d}E_t(x)$ for all test functions $\xi \in C^0([0, T] \times \Omega; \mathbb{R}^d)$) and in this case it is defined through

$$\mathcal{B}_p(E, \mu) := \begin{cases} \displaystyle\int_0^T \int_\Omega \frac{1}{p}|v_t|^p \,\mathrm{d}\mu_t(x)\,\mathrm{d}t & \text{if } E_t = v_t \cdot \mu_t, \\ +\infty & \text{otherwise.} \end{cases}$$

It is well known that $\mathcal{B}_p$ is jointly convex and lower semicontinuous with respect to the weak-$*$ convergence of measures (see Section 5.3.1 in [Santambrogio 2015]) and that, if $\partial_t \mu + \nabla \cdot E = 0$, then $\mathcal{B}_p(E, \mu) < +\infty$ implies that $t \mapsto \mu_t$ is a curve in $\mathrm{AC}^p([0, T]; \mathcal{W}_p(\Omega))$.[1] In particular it is a continuous curve and the initial and final conditions on μ_0 and μ_T are well defined.

Coming back to curves in Wasserstein spaces, it is well known (see [Ambrosio et al. 2008] or Section 5.3 in [Santambrogio 2015]) that for any distributional solution μ_t (being a continuous curve in $\mathcal{W}_p(\Omega)$) of the continuity equation $\partial_t \mu + \nabla \cdot E = 0$ with $E_t = v_t \cdot \mu_t$, we have the relations

$$|\mu'|_{W_p}(t) \le \|v_t\|_{L^p_{\mu_t}} \quad \text{and} \quad W_p(\mu_t, \mu_s) \le \int_s^t |\mu'|_{W_p}(\tau)\,\mathrm{d}\tau,$$

where we denote by $|\mu'|_{W_p}(t)$ the metric derivative with respect to W_p of the curve μ_t (see for instance [Ambrosio and Tilli 2004] for general notions about curves in metric spaces and their metric derivative). For curves μ_t that are geodesics in $\mathcal{W}_p(\Omega)$ we have the equality

$$W_p(\mu_0, \mu_1) = \int_0^1 |\mu'|_{W_p}(t)\,\mathrm{d}t = \int_0^1 \|v_t\|_{L^p_{\mu_t}}\,\mathrm{d}t.$$

The last equality is in fact the Benamou–Brenier formula with the optimal velocity field v_t being the density of the optimal E_t with respect to the optimal μ_t. This optimal velocity field v_t can be computed as $v_t := (S - \mathrm{id}) \circ (S_t)^{-1}$, where $S_t := (1-t)\,\mathrm{id} + tS$ is the transport in McCann's interpolation (we assume here that the initial measure μ_0 is absolutely continuous, so that we can use transport maps instead of plans). This expression can be obtained if we consider that in this interpolation particles move with constant speed $S(x) - x$, but x represents here a Lagrangian coordinate, and not an Eulerian one: if we want to know the velocity at time t at a given point, we have to find out first the original position of the particle passing through that point at that time.

In the sequel we will also need the notion of entropy of a probability density, and for any probability measure $\varrho \in \mathcal{P}(\Omega)$ we define it as

$$\mathcal{E}(\varrho) := \begin{cases} \displaystyle\int_\Omega \varrho(x) \log \varrho(x)\,\mathrm{d}x & \text{if } \varrho \ll \mathcal{L}^d, \\ +\infty & \text{otherwise.} \end{cases}$$

We recall that this functional is lower semicontinuous and geodesically convex in $\mathcal{W}_2(\Omega)$.

As we will mainly be working with absolutely continuous probability measures (with respect to Lebesgue), we often identify measures with their densities.

Projection problems in Wasserstein spaces. Our analysis strongly relies on the projection operator $P_{\mathcal{K}}$ in the sense of W_2. Here $\mathcal{K} := \{\rho \in \mathcal{P}(\Omega) : \rho \le 1\}$ and

$$P_{\mathcal{K}}[\mu] := \operatorname{argmin}_{\rho \in \mathcal{K}} \tfrac{1}{2} W_2^2(\mu, \rho).$$

We recall the main properties of the projection operator $P_{\mathcal{K}}$ (see [Maury et al. 2010; Santambrogio 2012a; De Philippis et al. 2016]).

[1] Here $\mathrm{AC}^p([0, T]; \mathcal{W}_p(\Omega))$ denotes the class of absolutely continuous curves in $\mathcal{W}_p(\Omega)$ with metric derivative in L^p. See the connection with the functional $\mathcal{B}_p$.

• As long as Ω is compact, for any probability measure μ, the minimizer in $\min_{\rho\in\mathcal{K}} \frac{1}{2} W_2^2(\mu, \rho)$ exists and is unique, and the operator $P_{\mathcal{K}}$ is continuous (it is even $C^{0,1/2}$ for the W_2 distance).

• The projection $P_{\mathcal{K}}[\mu]$ saturates the constraint $\rho \le 1$, in the sense that for any $\mu \in \mathcal{P}(\Omega)$ there exists a measurable set $B \subseteq \Omega$ such that $P_{\mathcal{K}}[\mu] = \mathbb{1}_B + \mu^{\mathrm{ac}} \mathbb{1}_{B^c}$, where μ^{ac} is the absolutely continuous part of μ.

• The projection is characterized in terms of a pressure field, in the sense that $\rho = P_{\mathcal{K}}[\mu]$ if and only if there exists a Lipschitz function $p \ge 0$, with $p(1-\rho) = 0$, and such that the optimal transport map S from ρ to μ is given by $S := \mathrm{id} - \nabla\varphi = \mathrm{id} + \nabla p$.

• There is (as proven in [De Philippis et al. 2016]) a quantified BV estimate: if $\mu \in \mathrm{BV}$ (in the sense that it is absolutely continuous and that its density belongs to $\mathrm{BV}(\Omega)$), then $P_{\mathcal{K}}[\mu]$ is also BV and

$$\mathrm{TV}(P_{\mathcal{K}}[\mu], \Omega) \le \mathrm{TV}(\mu, \Omega).$$

This last BV estimate will be crucial in Section 4, and it is important to have it in this very form (other estimates of the form $\mathrm{TV}(P_{\mathcal{K}}[\mu], \Omega) \le a\, \mathrm{TV}(\mu, \Omega) + b$ would not be as useful as this one, as they cannot be easily iterated).

3. Existence via a splitting-type algorithm (Main Scheme)

Similarly to the approach in [Maury et al. 2011] (see the algorithm (13) and Theorem 3.5) for a general nongradient vector field, we will build a theoretical algorithm, after time-discretization, to produce a solution of (1-5). Let us remark that splitting-type methods have been widely used in other contexts as well; see for instance [Clément and Maas 2011], which deals with splitting methods for Fokker–Planck equations and for more general gradient flows in metric and Wasserstein spaces, or [Laborde 2015], where a splitting-like approach is used to attack PDEs which are not gradient flows but "perturbations" of gradient flows.

In this section the spontaneous velocity field is a general vector field $u : [0, T] \times \Omega \to \mathbb{R}^d$ (not necessarily a gradient), which depends also on time. The only assumption we require on u is that

$$u \in L^\infty([0, T] \times \Omega; \mathbb{R}^d). \tag{U}$$

We work on a time interval $[0, T]$ and in a bounded convex domain $\Omega \subset \mathbb{R}^d$ (the case of the flat torus is even simpler and we will not discuss it in detail). We consider $\rho_0 \in \mathcal{P}^{\mathrm{ac}}(\Omega)$ to be given, which represents the initial density of the population, and we suppose $\rho_0 \in \mathcal{K}$.

Splitting using the Fokker–Planck equation. Let us consider the following scheme.

Main Scheme. Let $\tau > 0$ be a small time step with $N := \lfloor T/\tau \rfloor$. Let us set $\rho_0^\tau := \rho_0$. For every $k \in \{1, \ldots, N\}$, define ρ_{k+1}^τ from ρ_k^τ by solving

$$\begin{cases} \partial_t \varrho_t - \Delta \varrho_t + \nabla \cdot (\varrho_t u_{t+k\tau}) = 0, & t \in]0, \tau], \\ \varrho_0 = \rho_k^\tau, \end{cases} \tag{3-1}$$

equipped with the no-flux boundary condition ($\varrho_t(\nabla \varrho_t - u_t) \cdot n = 0$ a.e. on $\partial\Omega$), and setting $\rho_{k+1}^\tau = P_{\mathcal{K}}[\tilde{\rho}_{k+1}^\tau]$, where $\tilde{\rho}_{k+1}^\tau = \varrho_\tau$. See Figure 1 below.

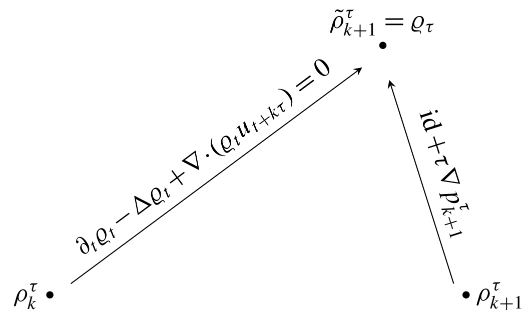

Figure 1. One time step.

Let us remark first that by classical results on parabolic equations (see for instance [Ladyzhenskaya et al. 1967]), since u satisfies the assumption (U), the equation (3-1) admits a unique distributional solution.

The above algorithm means to first follow the Fokker–Planck equation, ignoring the density constraint, for a time τ, then project. In order to state and prove the convergence of the scheme, we need to define some suitable interpolations of the discrete sequence of densities that we have just introduced.

First interpolation. We define the following curves of densities, velocities and momenta constructed with the help of the ρ^τ_k. First set

$$\rho^\tau_t := \begin{cases} \varrho_{2(t-k\tau)} & \text{if } t \in \big[k\tau,\, \big(k+\tfrac{1}{2}\big)\tau\big[, \\ \big(\mathrm{id} + 2((k+1)\tau - t)\nabla p^\tau_{k+1}\big)_\# \rho^\tau_{k+1} & \text{if } t \in \big[\big(k+\tfrac{1}{2}\big)\tau,\, (k+1)\tau\big[, \end{cases}$$

where ϱ_t is the solution of the Fokker–Planck equation (3-1) with initial datum ρ^τ_k and ∇p^τ_{k+1} arises from the projection of $\tilde{\rho}^\tau_{k+1}$, or more precisely, $(\mathrm{id} + \tau\nabla p^\tau_{k+1})$ is the optimal transport from ρ^τ_{k+1} to $\tilde{\rho}^\tau_{k+1}$. What are we doing? We are fitting into a time interval of length τ the two steps of our algorithm. First we follow the Fokker–Planck equation (3-1) at double speed, then we interpolate between the measure we reached and its projection following the geodesic between them. This geodesic is easily described as an image measure of ρ^τ_{k+1} through McCann's interpolation. By the construction it is clear that ρ^τ_t is a continuous curve in $\mathcal{P}(\Omega)$ for $t \in [0, T]$. We now define a family of time-dependent vector fields through

$$v^\tau_t := \begin{cases} -2\dfrac{\nabla\varrho_{2(t-k\tau)}}{\varrho_{2(t-k\tau)}} + 2u_t & \text{if } t \in \big[k\tau,\, \big(k+\tfrac{1}{2}\big)\tau\big[, \\ -2\nabla p^\tau_{k+1} \circ \big(\mathrm{id} + 2((k+1)\tau - t)\nabla p^\tau_{k+1}\big)^{-1} & \text{if } t \in \big[\big(k+\tfrac{1}{2}\big)\tau,\, (k+1)\tau\big[, \end{cases}$$

and, finally, we simply define the curve of momenta as $E^\tau_t := \rho^\tau_t v^\tau_t$.

Second interpolation. We define another interpolation as follows. Set

$$\tilde{\rho}^\tau_t := \varrho_{t-k\tau} \quad \text{if } t \in [k\tau, (k+1)\tau[,$$

where ϱ_t is (again) the solution of the Fokker–Planck equation (3-1) on the time interval $[0, \tau]$ with initial datum ρ^τ_k. Here we do not double its speed. We define the curve of velocities

$$\tilde{v}^\tau_t := -\frac{\nabla\varrho_{t-k\tau}}{\varrho_{t-k\tau}} + u_t \quad \text{if } t \in [k\tau, (k+1)\tau[,$$

and we build the curve of momenta by $\widetilde{E}^\tau_t := \tilde\rho^\tau_t \tilde v^\tau_t$.

Third interpolation. For each τ, we also define piecewise constant curves,

$$\hat\rho^\tau_t := \rho^\tau_{k+1} \quad \text{if } t \in [k\tau, (k+1)\tau[,$$
$$\hat v^\tau_t := \nabla p^\tau_{k+1} \quad \text{if } t \in [k\tau, (k+1)\tau[,$$

and $\widehat{E}^\tau_t := \hat\rho^\tau_t \hat v^\tau_t$. We remark that $p^\tau_{k+1}(1-\rho^\tau_{k+1}) = 0$, hence the curve of momenta is just

$$\widehat{E}^\tau_t := \nabla p^\tau_{k+1} \quad \text{if } t \in [k\tau, (k+1)\tau[.$$

Mind the differences in the construction of ρ^τ_t, $\tilde\rho^\tau_t$ and $\hat\rho^\tau_t$ (and hence in the construction of v^τ_t, $\tilde v^\tau_t$ and $\hat v^\tau_t$, and E^τ_t, $\widetilde{E}^\tau_t$ and $\widehat{E}^\tau_t$):

(1) The first one is continuous in time for the weak-$*$ convergence, while the second and third ones are not.

(2) In the first construction we have taken into account the projection operator explicitly, while in the second one we see it just in an indirect manner (via the "jumps" occurring at every time of the form $t = k\tau$). The third interpolation is piecewise constant, and at every time it satisfies the density constraint.

(3) In the first interpolation the pair (ρ^τ, E^τ) solves the continuity equation, while in the other two it does not. This is not astonishing, as the continuity equation characterizes continuous curves in $\mathcal{W}_2(\Omega)$.

In order to prove the convergence of the scheme above, we will obtain uniform $\mathrm{AC}^2([0,T];\mathcal{W}_2(\Omega))$ bounds for the curves ρ^τ. A key observation here is that the metric derivative (with respect to W_2) of the solution of the Fokker–Planck equation is comparable with the time differential of the entropy functional along the same solution (see Lemma 3.2). Now we state the main theorem of this section.

Theorem 3.1. *Let $\rho_0 \in \mathcal{K}$ and u be a given desired velocity field satisfying* (U). *Let us consider the interpolations introduced above. Then there exists a continuous curve $t \mapsto \rho_t \in \mathcal{W}_2(\Omega)$ for $t \in [0,T]$, and some vector measures $E, \widetilde{E}, \widehat{E} \in \mathfrak{M}([0,T]\times\Omega)$ such that the curves $\rho^\tau, \tilde\rho^\tau, \hat\rho^\tau$ converge uniformly in $\mathcal{W}_2(\Omega)$ to ρ and*

$$E^\tau \overset{*}{\rightharpoonup} E, \quad \widetilde{E}^\tau \overset{*}{\rightharpoonup} \widetilde{E}, \quad \widehat{E}^\tau \overset{*}{\rightharpoonup} \widehat{E} \quad \text{in } \mathfrak{M}([0,T]\times\Omega)^d \text{ as } \tau \to 0.$$

Moreover $E = \widetilde{E} - \widehat{E}$ and for a.e. $t \in [0,T]$ there exist time-dependent measurable vector fields $v_t, \tilde v_t, \hat v_t$ such that

(1) $E = \rho v, \quad \widetilde{E} = \rho\tilde v, \quad \widehat{E} = \rho \hat v,$

(2) $\displaystyle\int_0^T \left(\|v_t\|^2_{L^2_{\rho_t}} + \|\tilde v_t\|^2_{L^2_{\rho_t}} + \|\hat v_t\|^2_{L^2_{\rho_t}} \right) \mathrm{d}t < +\infty,$

(3) $v_t = \tilde v_t - \hat v_t$, *$\rho_t$-a.e.*, $\quad \widetilde{E}_t = \rho_t u_t - \nabla\rho_t \quad$ *and* $\quad \hat v_t = \nabla p_t$, *$\rho_t$-a.e.*,

where $p \in L^2([0,T]; H^1(\Omega))$, $p \geq 0$ *and* $p(1-\rho) = 0$ *a.e. in* $[0,T] \times \Omega$. *As a consequence, the pair* (ρ, p) *is a weak solution of the problem*

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t (u_t - \nabla p_t)) = 0 & \text{in } [0,T] \times \Omega, \\ p_t \geq 0,\ \rho_t \leq 1,\ p_t(1-\rho_t) = 0 & \text{in } [0,T] \times \Omega, \\ \rho_t (\nabla \rho_t - u_t + \nabla p_t) \cdot n = 0 & \text{on } [0,T] \times \partial\Omega, \\ \rho(0,\cdot) = \rho_0. \end{cases} \tag{3-2}$$

To prove this theorem we need the following tools.

Lemma 3.2. *Let us consider a solution* ϱ_t *of the Fokker–Planck equation on* $[0,T] \times \Omega$ *with the velocity field* u *satisfying* (U) *and with no-flux boundary conditions on* $[0,T] \times \partial\Omega$. *Then for any time interval* $]a,b[$ *we have the estimate*

$$\frac{1}{2} \int_a^b \int_\Omega \left| -\frac{\nabla \varrho_t}{\varrho_t} + u_t \right|^2 \varrho_t \,\mathrm{d}x \,\mathrm{d}t \leq \mathcal{E}(\varrho_a) - \mathcal{E}(\varrho_b) + \frac{1}{2} \int_a^b \int_\Omega |u_t|^2 \varrho_t \,\mathrm{d}x \,\mathrm{d}t. \tag{3-3}$$

In particular this implies

$$\frac{1}{2} \int_a^b |\varrho_t'|_{W_2}^2 \,\mathrm{d}t \leq \mathcal{E}(\varrho_a) - \mathcal{E}(\varrho_b) + \frac{1}{2} \int_a^b \int_\Omega |u_t|^2 \varrho_t \,\mathrm{d}x \,\mathrm{d}t, \tag{3-4}$$

where $|\varrho_t'|_{W_2}$ *denotes the metric derivative of the curve* $t \mapsto \varrho_t \in \mathcal{W}_2(\Omega)$.

Proof. To prove this inequality, we first make computations in the case where both u and ϱ are smooth, and ϱ is bounded from below by a positive constant. In this case we can write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\varrho_t) &= \int_\Omega (\log \varrho_t + 1) \partial_t \varrho_t \,\mathrm{d}x = \int_\Omega \log \varrho_t (\Delta \varrho_t - \nabla \cdot (\varrho_t u_t)) \,\mathrm{d}x \\ &= \int_\Omega \Big(-\frac{|\nabla \varrho_t|^2}{\varrho_t} + u_t \cdot \nabla \varrho_t \Big) \,\mathrm{d}x, \end{aligned}$$

where we use the conservation of mass (i.e., $\int_\Omega \partial_t \varrho_t \,\mathrm{d}x = 0$) and the boundary conditions in the integration by parts. We now compare this with

$$\begin{aligned} \frac{1}{2} \int_\Omega \left| -\frac{\nabla \varrho_t}{\varrho_t} + u_t \right|^2 \varrho_t \,\mathrm{d}x - \frac{1}{2} \int_\Omega |u_t|^2 \varrho_t \,\mathrm{d}x &= \int_\Omega \left(\frac{1}{2} \frac{|\nabla \varrho_t|^2}{\varrho_t} - \nabla \varrho_t \cdot u_t \right) \mathrm{d}x \\ &\leq \int_\Omega \left(\frac{|\nabla \varrho_t|^2}{\varrho_t} - \nabla \varrho_t \cdot u_t \right) \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\varrho_t). \end{aligned}$$

This provides the first part of the statement, i.e., (3-3). If we combine this with the fact that the metric derivative of the curve $t \mapsto \varrho_t$ is always less than or equal to the $L^2_{\varrho_t}$ norm of the velocity field in the continuity equation, we also get

$$\frac{1}{2} |\varrho_t'|_{W_2}^2 - \frac{1}{2} \int_\Omega |u_t|^2 \varrho_t \leq -\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\varrho_t),$$

and hence (3-4).

In order to prove the same estimates without artificial smoothness and lower bound assumptions, we can act by approximation. We approximate the density ϱ_a by smooth and strictly positive densities ϱ_a^k (by convolution, so that we guarantee in particular $\mathcal{E}(\varrho_a^k) \to \mathcal{E}(\varrho_a)$), and the vector field u with smooth vector fields u^k (strongly in $L^4([a,b]\times\Omega)$, keeping the L^∞ bound). If we call ϱ^k the corresponding solution of the Fokker–Planck equation, it satisfies (3-3). This implies a uniform bound (with respect to k) for $\sqrt{\varrho^k}$ in $L^2([a,b]; H^1(\Omega))$, and hence a uniform bound on ϱ^k in $L^2([a,b]\times\Omega)$. From these bounds and the uniqueness of the solution of the Fokker–Planck equation with L^∞ drift, we deduce $\varrho^k \to \varrho$. The semicontinuity of the left-hand side in (3-3) and of the entropy term at $t=b$, together with the convergence of the entropy at $t=a$ and the convergence $\int_a^b\int_\Omega |u^k|^2\varrho^k\,\mathrm{d}x\,\mathrm{d}t \to \int_a^b\int_\Omega |u|^2\varrho\,\mathrm{d}x\,\mathrm{d}t$ (because we have a product of weak and strong convergence in L^2), allow us to pass (3-3) to the limit. □

Corollary 3.3. *From the inequality* (3-4) *we deduce that*

$$\mathcal{E}(\varrho_b) - \mathcal{E}(\varrho_a) \le \frac{1}{2}\int_a^b\int_\Omega |u_t|^2\varrho_t\,\mathrm{d}x\,\mathrm{d}t,$$

and hence in particular for u satisfying (U), *we have*

$$\mathcal{E}(\varrho_b) - \mathcal{E}(\varrho_a) \le \frac{1}{2}\|u\|_{L^\infty}^2(b-a).$$

As a consequence, if $\varrho_a \le 1$, then we have

$$\mathcal{E}(\varrho_b) \le \frac{1}{2}\|u\|_{L^\infty}^2(b-a).$$

The same estimate can be applied to the curve $\tilde\rho^\tau$, with $a = k\tau$ and $b \in\,]k\tau, (k+1)\tau[$, thus obtaining $\mathcal{E}(\tilde\rho_t^\tau) \le C\tau$ for every t.

Lemma 3.4. *For any $\rho \in \mathcal{P}(\Omega)$ we have $\mathcal{E}(P_{\mathcal{K}}[\rho]) \le \mathcal{E}(\rho)$.*

Proof. We can assume $\rho \ll \mathcal{L}^d$, otherwise the claim is straightforward. As we pointed out in Section 2, we know that there exists a measurable set $B \subseteq \Omega$ such that

$$P_{\mathcal{K}}[\rho] = \mathbb{1}_B + \rho\mathbb{1}_{B^c}.$$

Hence it is enough to prove that

$$\int_B \rho\log\rho\,\mathrm{d}x \ge 0 = \int_B P_{\mathcal{K}}[\rho]\log P_{\mathcal{K}}[\rho]\,\mathrm{d}x,$$

as the entropies on B^c coincide. As the mass of ρ and $P_{\mathcal{K}}[\rho]$ are the same on all of Ω, and they coincide on B^c, we have

$$\int_B \rho(x)\,\mathrm{d}x = \int_B P_{\mathcal{K}}[\rho]\,\mathrm{d}x = |B|.$$

Then, by Jensen's inequality we have

$$\frac{1}{|B|}\int_B \rho\log\rho\,\mathrm{d}x \ge \left(\frac{1}{|B|}\int_B \rho\,\mathrm{d}x\right)\log\left(\frac{1}{|B|}\int_B \rho\,\mathrm{d}x\right) = 0.$$

The entropy decay follows. □

To analyze the pressure field we need the following result.

Lemma 3.5. *Let $\{p^\tau\}_{\tau>0}$ be a bounded sequence in $L^2([0,T]; H^1(\Omega))$ and $\{\rho^\tau\}_{\tau>0}$ a sequence of piecewise constant curves valued in $\mathcal{W}_2(\Omega)$ which satisfy $W_2(\rho^\tau(a), \rho^\tau(b)) \le C\sqrt{b-a+\tau}$ for all $a < b \in [0,T]$ for a fixed constant C. Suppose that*

$$p^\tau \ge 0, \quad p^\tau(1-\rho^\tau) = 0, \quad \rho^\tau \le 1,$$

and that

$$p^\tau \rightharpoonup p \quad \text{weakly in } L^2([0,T]; H^1(\Omega)) \quad \text{and} \quad \rho^\tau \to \rho \quad \text{uniformly in } \mathcal{W}_2(\Omega).$$

Then $p(1-\rho) = 0$ a.e. in $[0,T] \times \Omega$.

The proof of this result is the same as in Step 3 of Section 3.2 of [Maury et al. 2010] (see also [Roudneff-Chupin 2011] and Lemma 4.6 in [Di Marino et al. 2016]). We omit it in order not to overburden the paper.

The reader can note the strong connection with the classical Aubin–Lions lemma [Aubin 1963], applied to the compact injection of L^2 into H^{-1}. Indeed, from the weak convergence of p^τ in $L^2([0,T]; H^1(\Omega))$, we just need to provide strong convergence of ρ^τ in $L^2([0,T]; H^{-1}(\Omega))$. If instead of the quasi-Hölder assumption of the above lemma we suppose a uniform bound of $\{\rho^\tau\}_\tau$ in $\mathrm{AC}^2([0,T]; \mathcal{W}_2(\Omega))$ (which is not so different), then the statement really can be deduced from the Aubin–Lions lemma. Indeed, the sequence $\{\rho^\tau\}$ is bounded in $L^\infty([0,T]; L^2(\Omega))$ and its time derivative would be bounded in $L^2([0,T]; H^{-1}(\Omega))$. This strongly depends on the fact that the H^{-1} distance can be controlled by the W_2 distance as soon as the measures have uniformly bounded densities (see [Loeper 2006; Maury et al. 2010]), a tool which is also crucial in the proofs in [Maury et al. 2010; Roudneff-Chupin 2011; Di Marino et al. 2016]. Then, the Aubin–Lions lemma guarantees compactness in $C^0([0,T]; H^{-1}(\Omega))$, which is more than what we need.

Lemma 3.6. *Let us consider the previously defined interpolations. Then we have the following facts.*

(i) *For every $\tau > 0$ and k we have*

$$\max\bigl\{W_2^2(\rho_k^\tau, \tilde\rho_{k+1}^\tau),\, W_2^2(\rho_k^\tau, \rho_{k+1}^\tau)\bigr\} \le \tau C(\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\rho_{k+1}^\tau)) + C\tau^2,$$

where $C > 0$ only depends on $\|u\|_{L^\infty}$.

(ii) *There exists a constant C, only depending on ρ_0 and $\|u\|_{L^\infty}$, such that*

$$\mathcal{B}_2(E^\tau, \rho^\tau) \le C, \quad \mathcal{B}_2(\widetilde{E}^\tau, \tilde\rho^\tau) \le C \quad \text{and} \quad \mathcal{B}_2(\widehat{E}^\tau, \hat\rho^\tau) \le C.$$

(iii) *For the curve $[0,T] \ni t \mapsto \rho_t^\tau$ we have that*

$$\int_0^T |(\rho_t^\tau)'|_{W_2}^2 \,\mathrm{d}t \le C,$$

for a $C > 0$ independent of τ. Here we denote by $|(\rho_t^\tau)'|_{W_2}$ the metric derivative of the curve ρ^τ at t in $\mathcal{W}_2$. In particular, we have a uniform Hölder bound on ρ^τ, namely $W_2(\rho^\tau(a), \rho^\tau(b)) \le C\sqrt{b-a}$ for every $b > a$.

(iv) *$E^\tau, \widetilde{E}^\tau, \widehat{E}^\tau$ are uniformly bounded sequences in $\mathfrak{M}([0,T] \times \Omega)^d$.*

Proof. (i) First, by the triangle inequality and by the fact that $\rho^\tau_{k+1} = P_{\mathcal{K}}[\tilde\rho^\tau_{k+1}]$ we have that

$$W_2(\rho^\tau_k, \rho^\tau_{k+1}) \le W_2(\rho^\tau_k, \tilde\rho^\tau_{k+1}) + W_2(\tilde\rho^\tau_{k+1}, \rho^\tau_{k+1}) \le 2W_2(\rho^\tau_k, \tilde\rho^\tau_{k+1}). \tag{3-5}$$

We use (as before) the notation ϱ_t, $t \in [0, \tau]$ for the solution of the Fokker–Planck equation (3-1) with initial datum ρ^τ_k; in particular we have $\varrho_\tau = \tilde\rho^\tau_{k+1}$. Using Lemma 3.2 and since $\varrho_0 = \rho^\tau_k$ and $\varrho_\tau = \tilde\rho^\tau_{k+1}$ we have by (3-4) and $W_2(\rho^\tau_k, \tilde\rho^\tau_{k+1}) \le \int_0^\tau |\varrho'_t|_{W_2}\,\mathrm{d}t$ that

$$\begin{aligned} W_2^2(\rho^\tau_k, \tilde\rho^\tau_{k+1}) &\le \left(\tau^{1/2}\left(\int_0^\tau |\varrho'_t|^2_{W_2}\,\mathrm{d}t\right)^{1/2}\right)^2 \le 2\tau(\mathcal{E}(\varrho_0) - \mathcal{E}(\varrho_\tau)) + \tau\int_0^\tau\int_\Omega |u_{k\tau+t}|^2\varrho_t\,\mathrm{d}x\,\mathrm{d}t \\ &\le 2\tau(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\tilde\rho^\tau_{k+1})) + C\tau^2 \le 2\tau(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau^2, \end{aligned}$$

where $C > 0$ is a constant depending just on $\|u\|_{L^\infty}$. We have also used the fact that $\mathcal{E}(\rho^\tau_{k+1}) \le \mathcal{E}(\tilde\rho^\tau_{k+1})$, a consequence of Lemma 3.4.

Now by means of (3-5) we obtain

$$W_2^2(\rho^\tau_k, \rho^\tau_{k+1}) \le \tau C(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau^2.$$

(ii) We use Lemma 3.2 on the intervals of type $\big[k\tau, \big(k+\frac12\big)\tau\big[$ and the fact that on each interval of type $\big[\big(k+\frac12\big)\tau, (k+1)\tau\big[$ the curve ρ^τ_t is a constant speed geodesic. In particular, on these intervals we have

$$|(\rho^\tau)'|_{W_2} = \|v^\tau_t\|_{L^2_{\rho^\tau_t}} = 2\tau\|\nabla p^\tau_{k+1}\|_{L^2_{\rho^\tau_{k+1}}} = 2W_2(\rho^\tau_{k+1}, \tilde\rho^\tau_{k+1}).$$

On the other hand we also have

$$\tau^2\|\nabla p^\tau_{k+1}\|^2_{L^2_{\rho^\tau_{k+1}}} = W_2^2(\rho^\tau_{k+1}, \tilde\rho^\tau_{k+1}) \le W_2^2(\rho^\tau_k, \tilde\rho^\tau_{k+1}) \le \tau C(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau^2.$$

Hence we obtain

$$\begin{aligned} &\int_{k\tau}^{(k+1)\tau} \|v^\tau_t\|^2_{L^2(\rho^\tau_t)}\,\mathrm{d}t \\ &\quad= \int_{k\tau}^{(k+1/2)\tau}\int_\Omega 4\left|-\frac{\nabla\varrho_{2(t-k\tau)}}{\varrho_{2(t-k\tau)}} + u_{2t-k\tau}\right|^2 \varrho_{2(t-k\tau)}(x)\,\mathrm{d}x\,\mathrm{d}t + 4\int_{(k+1/2)\tau}^{(k+1)\tau}\int_\Omega |\nabla p^\tau_{k+1}|^2\rho^\tau_{k+1}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le C(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau + 2\tau\|\nabla p^\tau_{k+1}\|^2_{L^2_{\rho^\tau_{k+1}}} \\ &\quad\le C(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau. \end{aligned}$$

Hence by adding up we obtain

$$\mathcal{B}_2(E^\tau, \rho^\tau) \le \sum_k \big(C(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau\big) = C(\mathcal{E}(\rho^\tau_0) - \mathcal{E}(\rho^\tau_{N+1})) + CT \le C.$$

The estimates on $\mathcal{B}_2(\widetilde{E}^\tau, \tilde\rho^\tau)$ and $\mathcal{B}_2(\widehat{E}^\tau, \hat\rho^\tau)$ are completely analogous and arise from the previous computations.

(iii) The estimate on $\mathcal{B}_2(E^\tau, \rho^\tau)$ implies a bound on $\int_0^T |(\rho^\tau_t)'|^2_{W_2}\,\mathrm{d}t$ because v^τ is a velocity field for ρ^τ (i.e., the pair (E^τ, ρ^τ) solves the continuity equation).

(iv) In order to estimate the total mass of E we write

$$|E^\tau|([0,T]\times\Omega) = \int_0^T\int_\Omega |v_t^\tau|\rho_t^\tau\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\left(\int_\Omega |v_t^\tau|^2\rho_t^\tau\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega \rho_t^\tau\,\mathrm{d}x\right)^{1/2}\mathrm{d}t$$
$$\le \sqrt{T}\left(\int_0^T\int_\Omega |v_t^\tau|^2\rho_t^\tau\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \le C.$$

The bounds on $\widetilde{E}^\tau$ and $\widehat{E}^\tau$ rely on the same argument. □

Proof of Theorem 3.1. We use the tools from Lemma 3.6.

Step 1. By the bounds on the metric derivative of the curves ρ_t^τ we get compactness, i.e., there exists a curve $[0,T]\ni t\mapsto\rho_t\in\mathcal{P}(\Omega)$ such that ρ^τ (up to subsequences) converges uniformly in $[0,T]$ with respect to W_2, in particular weakly-$*$ in $\mathcal{P}(\Omega)$ for all $t\in[0,T]$. It is easy to see that $\tilde\rho^\tau$ and $\hat\rho^\tau$ are converging to the same curve. Indeed we have $\tilde\rho_t^\tau=\rho^\tau_{\tilde s(t)}$ and $\hat\rho_t^\tau=\rho^\tau_{\hat s(t)}$ for $|\tilde s(t)-t|\le\tau$ and $|\hat s(t)-t|\le\tau$, which implies $W_2(\rho_t^\tau,\tilde\rho_t^\tau), W_2(\rho_t^\tau,\hat\rho_t^\tau)\le C\tau^{1/2}$. This provides the convergence to the same limit.

Step 2. By the boundedness of E^τ, $\widetilde{E}^\tau$ and $\widehat{E}^\tau$ in $\mathfrak{M}([0,T]\times\Omega)^d$, we have the existence of E, $\widetilde{E}$, $\widehat{E}$ in $\mathfrak{M}([0,T]\times\Omega)^d$ such that (up to a subsequence) $E^\tau \overset{*}{\rightharpoonup} E$, $\widetilde{E}^\tau \overset{*}{\rightharpoonup} \widetilde{E}$, $\widehat{E}^\tau \overset{*}{\rightharpoonup} \widehat{E}$ as $\tau\to0$. Now we show that $E=\widetilde{E}-\widehat{E}$. Indeed, let us show that for any test function $f\in\mathrm{Lip}([0,T]\times\Omega)^d$ we have

$$\left|\int_0^T\int_\Omega f_t\cdot(E_t^\tau-(\widetilde{E}_t^\tau+\widehat{E}_t^\tau))(\mathrm{d}x,\mathrm{d}t)\right|\to0$$

as $\tau\to0$. First, for each $k\in\{0,\dots,N\}$ we have that

$$\int_{k\tau}^{(k+1/2)\tau}\int_\Omega f_t\cdot E_t^\tau(\mathrm{d}x,\mathrm{d}t) = \int_{k\tau}^{(k+1)\tau}\int_\Omega f_{(t+k\tau)/2}\cdot(-\nabla\varrho_{t-k\tau}+u_t\varrho_{t-k\tau})(\mathrm{d}x,\mathrm{d}t)$$
$$= \int_{k\tau}^{(k+1)\tau}\int_\Omega f_t\cdot\widetilde{E}_t^\tau(\mathrm{d}x,\mathrm{d}t)+\int_{k\tau}^{(k+1)\tau}\int_\Omega (f_{(t+k\tau)/2}-f_t)\cdot\widetilde{E}_t^\tau(\mathrm{d}x,\mathrm{d}t)$$

and

$$\int_{(k+1/2)\tau}^{(k+1)\tau}\int_\Omega f_t\cdot E_t^\tau(\mathrm{d}x,\mathrm{d}t)$$
$$= \int_{k\tau}^{(k+1)\tau}\int_\Omega - f_{(t+(k+1)\tau)/2}\circ\left(\mathrm{id}+((k+1)\tau-t)\nabla p_{k+1}^\tau\right)\cdot\nabla p_{k+1}^\tau\rho_{k+1}^\tau(\mathrm{d}x,\mathrm{d}t)$$
$$= -\int_{k\tau}^{(k+1)\tau}\int_\Omega f_t\cdot\widehat{E}_t^\tau(\mathrm{d}x,\mathrm{d}t)+\int_{k\tau}^{(k+1)\tau}\int_\Omega\left(f_t-f_{(t+(k+1)\tau)/2}\circ(\mathrm{id}+((k+1)\tau-t))\right)\cdot\hat v_t^\tau\hat\rho_t^\tau(\mathrm{d}x,\mathrm{d}t).$$

This implies that

$$\begin{aligned}\left|\int_0^T\int_\Omega f_t\cdot(E_t^\tau-\tilde{E}_t^\tau+\widehat{E}_t^\tau)(\mathrm{d}x,\mathrm{d}t)\right|\\ &\le\sum_k\int_{k\tau}^{(k+1)\tau}\mathrm{Lip}(f)\tau\int_\Omega|\widetilde{E}_t^\tau|(\mathrm{d}x,\mathrm{d}t)+\sum_k\int_{k\tau}^{(k+1)\tau}\mathrm{Lip}(f)\tau\int_\Omega(1+|\hat{v}_t^\tau|)|\widehat{E}_t^\tau|(\mathrm{d}x,\mathrm{d}t)\\ &\le\tau C\,\mathrm{Lip}(f)\big(|\widetilde{E}^\tau|([0,T]\times\Omega)+|\widehat{E}^\tau|([0,T]\times\Omega)+\mathcal{B}_2(\widehat{E},\hat{\rho})\big)\\ &\le\tau C\,\mathrm{Lip}(f),\end{aligned}$$

for a uniform constant $C>0$. Letting $\tau\to 0$, we prove the claim.

Step 3. The bounds on $\mathcal{B}_2(E^\tau,\rho^\tau)$, $\mathcal{B}_2(\widetilde{E}^\tau,\tilde{\rho}^\tau)$ and $\mathcal{B}_2(\widehat{E}^\tau,\hat{\rho}^\tau)$ pass to the limit by semicontinuity and allow us to conclude that E, $\widetilde{E}$ and $\widehat{E}$ are vector-valued Radon measures absolutely continuous with respect to ρ. Hence there exist v_t, $\tilde{v}_t$ and $\hat{v}_t$ such that $E=\rho v$, $\widetilde{E}=\rho\tilde{v}$ and $\widehat{E}=\rho\hat{v}$.

Step 4. We now look at the equations satisfied by E, $\widetilde{E}$ and $\widehat{E}$. First we use $\partial_t\rho^\tau+\nabla\cdot E^\tau=0$, pass to the limit as $\tau\to 0$ and get

$$\partial_t\rho+\nabla\cdot E=0.$$

Then, we use $\widetilde{E}^\tau=-\nabla\tilde{\rho}^\tau+u_t\tilde{\rho}^\tau$, pass to the limit again as $\tau\to 0$ and get

$$\widetilde{E}=-\nabla\rho+u_t\rho.$$

To justify this limit, the only delicate point is passing to the limit the term $u_t\tilde{\rho}^\tau$, since u is only L^∞, and $\tilde{\rho}^\tau$ converges weakly as measures, and we are a priori only allowed to multiply it by continuous functions. Yet, we remark that by Corollary 3.3 we have that $\mathcal{E}(\tilde{\rho}_t^\tau)\le C\tau$ for all $t\in[0,T]$. In particular, this provides, for each t, uniform integrability for $\tilde{\rho}_t^\tau$ and turns the weak convergence as measures into weak convergence in L^1. This allows multiplication by u_t in the weak limit.

Finally, we look at $\widehat{E}^\tau$. There exists a piecewise constant (in time) function p^τ (defined as p_{k+1}^τ on every interval $]k\tau,(k+1)\tau]$) such that $p^\tau\ge 0$, $p^\tau(1-\hat{\rho}^\tau)=0$,

$$\int_0^T\int_\Omega|\nabla p^\tau|^2(\mathrm{d}x,\mathrm{d}t)=\int_0^T\int_\Omega|\nabla p^\tau|^2\hat{\rho}^\tau(\mathrm{d}x,\mathrm{d}t)=\int_0^T\int_\Omega|\hat{v}^\tau|^2\hat{\rho}^\tau(\mathrm{d}x,\mathrm{d}t)\le C \tag{3-6}$$

and $\widehat{E}^\tau=\nabla p^\tau\hat{\rho}^\tau=\nabla p^\tau$. The bound (3-6) implies that p^τ is uniformly bounded in $L^2(0,T;H^1(\Omega))$. Since for every t we have $|\{p_t^\tau=0\}|\ge|\{\hat{\rho}_t^\tau<1\}|\ge|\Omega|-1$, we can use a suitable version of Poincaré's inequality, and get a uniform bound in $L^2([0,T];L^2(\Omega))=L^2([0,T]\times\Omega)$. Therefore, there exists $p\in L^2([0,T]\times\Omega)$ such that $p^\tau\rightharpoonup p$ weakly in L^2 as $\tau\to 0$. In particular we have $\widehat{E}=\nabla p$. Moreover it is clear that $p\ge 0$ and by Lemma 3.5 we obtain $p(1-\rho)=0$ a.e. as well. Indeed, the assumptions of the lemma are easily checked: we only need to estimate $W_2(\hat{\rho}^\tau(a),\hat{\rho}^\tau(b))$ for $b>a$, but we have

$$W_2(\hat{\rho}^\tau(a),\hat{\rho}^\tau(b))=W_2(\rho^\tau(k_a\tau),\rho^\tau(k_b\tau))\le C\sqrt{k_b-k_a}\quad\text{for } k_b\tau\le b+\tau \text{ and } k_a\ge a.$$

Once we have $\widehat{E} = \nabla p$ with $p(1-\rho) = 0$, $p \in L^2([0, T]; H^1(\Omega))$ and $\rho \in L^\infty$, we can also write

$$\widehat{E} = \nabla p = \rho \nabla p.$$

If we sum up our results, using $E = \widetilde{E} - \widehat{E}$, we have

$$\partial_t \rho - \Delta \rho + \nabla \cdot (\rho(u - \nabla p)) = 0 \quad \text{with } p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0 \text{ a.e. in } [0, T] \times \Omega.$$

As usual, this equation is satisfied in a weak sense, with no-flux boundary conditions. □

4. Uniform Lip([0, T]; $\mathcal{W}_1$) and BV estimates

In this section we provide uniform estimates for the curves ρ^τ, $\tilde{\rho}^\tau$ and $\hat{\rho}^\tau$ in the form of uniform BV (in space) bounds on $\tilde{\rho}^\tau$ (which implies the same bound for $\hat{\rho}^\tau$) and uniform Lipschitz bounds in time for the W_1 distance on ρ^τ. This means a small improvement compared to the previous section concerning time regularity, as we have Lipschitz instead of AC^2, even if we need to replace W_2 with W_1. It is also important for space regularity. Indeed, from Lemma 3.2 one could deduce that the solution ρ of the Fokker–Planck equation (1-5) satisfies $\sqrt{\rho} \in L^2([0, T]; H^1(\Omega))$ and, using $\rho \leq 1$, also $\rho \in L^2([0, T]; H^1(\Omega))$. Yet, this is just an integrable estimate in t, while the BV estimate of this section is uniform in the time variable.

Nevertheless there is a price to pay for this improvement: we have to assume higher regularity for the velocity field. These uniform-in-time W_1-Lipschitz bounds are based both on BV estimates for the Fokker–Planck equation (see Lemma A.1 in the Appendix) and for the projection operator $P_{\mathcal{K}}$ (see [De Philippis et al. 2016]). The assumption on u is, essentially, that we need to control the growth of the total variation of the solutions of the Fokker–Planck equation (3-1), and we need to iterate this bound along time steps.

We will discuss in the Appendix the different BV estimates on the Fokker–Planck equation that we were able to find. The desired estimate is true whenever $\|u_t\|_{C^{1,1}(\Omega)}$ is uniformly bounded and $u_t \cdot n = 0$ on $\partial\Omega$. It seems to be an open problem to obtain similar estimates under the sole assumption that u is Lipschitz continuous. Of course, we will also assume $\rho_0 \in \mathrm{BV}(\Omega)$. Despite these extra regularity assumptions, we think these estimates have their own interest, exploiting some finer properties of the solutions of the Fokker–Planck equation and of the Wasserstein projection operator.

Before entering into the details of the estimates, we want to discuss why we concentrate on BV estimates (instead of Sobolev ones) and on W_1 (instead of W_p, $p > 1$). The main reason is the role of the projection operator. Indeed, even if $\rho \in W^{1,p}(\Omega)$, we do not have in general $P_{\mathcal{K}}[\rho] \in W^{1,p}$ because the projection creates some jumps at the boundary of $\{P_{\mathcal{K}}[\rho] = 1\}$. This prevents us from obtaining any $W^{1,p}$ estimate for $p > 1$. On the other hand, [De Philippis et al. 2016] exactly proves a BV estimate on $P_{\mathcal{K}}[\rho]$ and paves the way to BV bounds for our equation. Concerning the regularity in time, we observe that the velocity field in the Fokker–Planck equation contains a term in $\nabla\rho/\rho$. Since the metric derivative in $\mathcal{W}_p$ is given by the L^p norm (with respect to ρ_t) of the velocity field, it is clear that estimates in $\mathcal{W}_p$ for $p > 1$ would require spatial $W^{1,p}$ estimates on the solution itself, which are impossible for $p > 1$ in this splitting scheme. We stress that this does not mean that uniform $W^{1,p}$ are impossible for the solution of (1-5); it only means that they are not uniform along the approximation used in our Main Scheme to build such a solution.

The precise result that we prove is the following.

Theorem 4.1. *Let us suppose that* $\|u_t\|_{C^{1,1}} \le C$ *and* $\rho_0 \in \mathrm{BV}(\Omega)$. *Then using the notations from the Main Scheme and Theorem 3.1 one has* $\|\tilde\rho^\tau_t\|_{\mathrm{BV}} \le C$ *and* $W_1(\rho^\tau_k, \rho^\tau_{k+1}) \le C\tau$. *As a consequence we also have* $\rho \in \mathrm{Lip}([0,T]; \mathcal{W}_1) \cap L^\infty([0,T]; \mathrm{BV}(\Omega))$.

To prove this theorem we need the following lemmas.

Lemma 4.2. *Suppose* $\|u_t\|_{\mathrm{Lip}} \le C$ *and* $u_t \cdot n = 0$ *on* $\partial\Omega$. *Then for the solution* ϱ *of* (A-1) *with velocity field* $v = u$ *we have the estimate*

$$\|\varrho_t\|_{L^\infty} \le \|\varrho_0\|_{L^\infty} e^{Ct},$$

where $C = \|\nabla \cdot u_t\|_{L^\infty}$.

Proof. Standard comparison theorems for parabolic equations allow us to prove the results once we notice that $f(t,x) := \|\varrho_0\|_{L^\infty} e^{Ct}$ is a supersolution of the Fokker–Planck equation, i.e.,

$$\partial_t f_t \ge \Delta f_t - \nabla \cdot (f_t u_t).$$

Indeed, in the above equation the Laplacian term vanishes as f is constant in x, $\partial_t f_t = C f_t$ and $\nabla \cdot (f_t u_t) = f_t \nabla \cdot u_t + \nabla f_t \cdot u_t = f_t \nabla \cdot u_t \le C f_t$, where $C = \|\nabla \cdot u_t\|_{L^\infty}$. From this inequality, and from $\rho_0 \le f_0$, we deduce $\rho_t \le f_t$ for all t. □

We remark that the above lemma implies in particular that after every step in the Main Scheme we have $\tilde\rho^\tau_{k+1} \le e^{\tau c} \le 1 + C\tau$, where $c := \|\nabla \cdot u\|_{L^\infty}$. We note the following corollary as well.

Corollary 4.3. *Along the iterations of our Main Scheme, for every k we have* $W_1(\tilde\rho^\tau_{k+1}, \rho^\tau_{k+1}) \le \tau C$ *for a constant* $C > 0$ *independent of* τ.

Proof. With the saturation property of the projection (see Section 2 or [De Philippis et al. 2016]), we know that there exists a measurable set $B \subseteq \Omega$ such that $\rho^\tau_{k+1} = \tilde\rho^\tau_{k+1} \mathbb{1}_B + \mathbb{1}_{\Omega\setminus B}$. On the other hand we know that

$$\begin{aligned} W_1(\tilde\rho^\tau_{k+1}, \rho^\tau_{k+1}) &= \sup_{\substack{f \in \mathrm{Lip}_1(\Omega) \\ 0 \le f \le \mathrm{diam}(\Omega)}} \int_\Omega f(\tilde\rho^\tau_{k+1} - \rho^\tau_{k+1})\,\mathrm{d}x \\ &= \sup_{\substack{f \in \mathrm{Lip}_1(\Omega) \\ 0 \le f \le \mathrm{diam}(\Omega)}} \int_{\Omega\setminus B} f(\tilde\rho^\tau_{k+1} - 1)\,\mathrm{d}x \le \tau C |\Omega| \,\mathrm{diam}(\Omega). \end{aligned}$$

We use the fact that the competitors f in the dual formula can be taken to be positive and bounded by the diameter of Ω, just by adding a suitable constant. This implies as well that C is dependent on c, $|\Omega|$ and $\mathrm{diam}(\Omega)$. □

Proof of Theorem 4.1. First we take care of the BV estimate. Lemma A.1 guarantees, for $t \in \,]k\tau, (k+1)\tau[$, that we have $\mathrm{TV}(\tilde\rho^\tau_t) \le C\tau + e^{C\tau}\,\mathrm{TV}(\rho^\tau_k)$. Together with the BV bound on the projection that we presented in Section 2 (taken from [De Philippis et al. 2016]), this can be iterated, providing a uniform bound (depending on $\mathrm{TV}(\rho_0)$, T and $\sup_t \|u_t\|_{C^{1,1}}$) on $\|\tilde\rho^\tau_t\|_{\mathrm{BV}}$. Passing this estimate to the limit as $\tau \to 0$ we get $\rho \in L^\infty([0,T];\mathrm{BV}(\Omega))$.

Then we estimate the behavior of the interpolation curve $\hat\rho^\tau$ in terms of W_1. We estimate

$$W_1(\rho^\tau_k, \tilde\rho^\tau_{k+1}) \le \int_{k\tau}^{(k+1)\tau} |(\tilde\rho^\tau_t)'|_{W_1}\,\mathrm{d}t \le \int_{k\tau}^{(k+1)\tau}\int_\Omega \left(\frac{|\nabla\tilde\rho^\tau_t|}{\tilde\rho^\tau_t} + |u_t|\right)\tilde\rho^\tau_t\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \int_{k\tau}^{(k+1)\tau} \|\tilde\rho^\tau_t\|_{\mathrm{BV}}\,\mathrm{d}t + C\tau \le C\tau.$$

Hence, we obtain

$$W_1(\rho^\tau_k, \rho^\tau_{k+1}) \le W_1(\rho^\tau_k, \tilde\rho^\tau_{k+1}) + W_1(\tilde\rho^\tau_{k+1}, \rho^\tau_{k+1}) \le \tau C.$$

This in particular means, for $b > a$,

$$W_1(\hat\rho^\tau(a), \hat\rho^\tau(b)) \le C(b - a + \tau).$$

We can pass this relation to the limit, using that, for every t, we have $\hat\rho^\tau_t \to \rho_t$ in $\mathcal{W}_2(\Omega)$ (and hence also in $\mathcal{W}_1(\Omega)$, since $W_1 \le W_2$), getting

$$W_1(\rho(a), \rho(b)) \le C(b - a),$$

which means that ρ is Lipschitz continuous in $\mathcal{W}_1(\Omega)$. □

5. Variations on a theme: some reformulations of the Main Scheme

In this section we propose some alternative approaches to study the problem (1-5). The general idea is to discretize in time, and give a way to produce a measure ρ^τ_{k+1} starting from ρ^τ_k. Observe that the interpolations ρ^τ, $\tilde\rho^\tau$ and $\hat\rho^\tau$ proposed in the previous sections are only technical tools to state and prove a convergence result, and the most important point is exactly the definition of ρ^τ_{k+1}.

The alternative approaches proposed here explore different ideas, more difficult to implement than the one that we presented in Section 3, and/or restricted to some particular cases (for instance when u is a gradient). They have their own modeling interest and this is the main reason justifying their sketchy presentation.

Variant 1: transport, diffusion then projection. We recall that the original splitting approach for the equation without diffusion [Maury et al. 2011; Roudneff-Chupin 2011] exhibited an important difference compared to what we did in Section 3. Indeed, in the first phase of each time step (i.e., before the projection) the particles follow the vector field u and $\tilde\rho^\tau_{k+1}$ was not defined as the solution of a continuity equation with advection velocity given by u_t, but as the image of ρ^τ_k via a straight-line transport: $\tilde\rho^\tau_{k+1} := (\mathrm{id} + \tau u_{k\tau})_\#\rho^\tau_k$. One can wonder whether it is possible to follow a similar approach here.

A possible way to proceed is as follows. Take a random variable X distributed according to ρ^τ_k, and define $\tilde\rho^\tau_{k+1}$ as the law of $X + \tau u_{k\tau}(X) + B_\tau$, where B is a Brownian motion, independent of X. This exactly means that every particle moves starting from its initial position X, following a displacement ruled by u, but adding a stochastic effect in the form of the value at time τ of a Brownian motion. We can check that this means

$$\tilde{\rho}_{k+1}^{\tau} := \eta_\tau * ((\mathrm{id} + \tau u_{k\tau})_{\#}\rho_k^\tau),$$

where η_τ is a Gaussian kernel with zero mean and variance τ, i.e., $\eta_\tau(x) := \frac{1}{(4\tau\pi)^{d/2}} e^{-|x|^2/(4\tau)}$. Then we define

$$\rho_{k+1}^{\tau} := P_{\mathcal{K}}[\tilde{\rho}_{k+1}].$$

Despite the fact that this scheme is very natural and essentially not that different from the Main Scheme, we have to be careful with the analysis. First we have to quantify somehow the distance $W_p(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau)$ for some $p \geq 1$ and show that this is of order τ in some sense. Second, we need to be careful when performing the convolution with the heat kernel (or adding the Brownian motion, which is the same). This requires working either in the whole space (which was not our framework) or in a periodic setting ($\Omega = \mathbb{T}^d$, the flat torus, which is quite restrictive). Otherwise, the "explicit" convolution step should be replaced with some other construction, such as following the heat equation (with Neumann boundary conditions) for a time τ. But this brings us back to a situation very similar to the Main Scheme, with the additional difficulty that we do not really have estimates on $(\mathrm{id} + \tau u_{k\tau})_{\#}\rho_k^\tau$.

Variant 2: gradient flow techniques for gradient velocity fields. In this section we assume that the velocity field of the population is given by the opposite of the gradient of a function, $u_t = -\nabla V_t$. A typical example is given when we take for V the distance function to the exit (see the discussions in [Maury et al. 2010] about this type of question). We start from the case where V does not depend on time, and we suppose $V \in W^{1,1}(\Omega)$. In this particular case — beside the splitting approach — the problem has a variational structure, hence it is possible to show the existence by the means of gradient flows in Wasserstein spaces.

Since the celebrated paper of Jordan, Kinderlehrer and Otto [Jordan et al. 1998], we know that the solutions of the Fokker–Planck equation (with a gradient vector field) can be obtained with the help of the gradient flow of a perturbed entropy functional with respect to the Wasserstein distance W_2. This formulation of the Jordan–Kinderlehrer–Otto (JKO) scheme was also used in [Maury et al. 2010] for the first-order model with density constraints. It is easy to combine the JKO scheme with density constraints to study the second-order/diffusive model. As a slight modification of the model from [Maury et al. 2010], we can consider the following discrete implicit Euler (or JKO) scheme. As usual, we fix a time step $\tau > 0$, $\rho_0^\tau = \rho_0$ and for all $k \in \{1, 2, \dots, \lfloor N/\tau \rfloor\}$ we just need to define ρ_{k+1}^τ. We take

$$\rho_{k+1}^{\tau} = \mathrm{argmin}_{\rho \in \mathcal{P}(\Omega)} \left\{ \int_\Omega V(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_k^\tau) \right\}, \tag{5-1}$$

where $I_{\mathcal{K}}$ is the indicator function of $\mathcal{K}$, which is

$$I_{\mathcal{K}}(x) := \begin{cases} 0 & \text{if } x \in \mathcal{K}, \\ +\infty & \text{otherwise.} \end{cases}$$

The usual techniques from [Jordan et al. 1998; Maury et al. 2010] can be used to identify that the system (1-5) is the gradient flow of the functional

$$\rho \mapsto J(\rho) := \int_\Omega V(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho)$$

and that the above discrete scheme converges (up to a subsequence) to a solution of (1-5), thus proving existence. The key estimate for compactness is

$$\frac{1}{2\tau} W_2^2(\rho_{k+1}^\tau, \rho_k^\tau) \le J(\rho_k^\tau) - J(\rho_{k+1}^\tau),$$

which can be summed up (as on the right-hand side we have a telescopic series), thus obtaining the same bounds on $\mathcal{B}_2$ that we used in Section 3.

Note that whenever $D^2 V \ge \lambda I$, the functional $\rho \mapsto \int_\Omega V(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho)$ is λ-geodesically convex. This allows us to use the theory in [Ambrosio et al. 2008] to prove not only existence, but also uniqueness for this equation, and even stability (contractivity or exponential growth on the distance between two solutions) in $\mathcal{W}_2$. Yet, we underline that the techniques of [Di Marino and Mészáros 2016] also give the same result. Indeed, that article contains two parts. In the first part, the equation with density constraints for a given velocity field u is studied, under the assumption that $-u$ has some monotonicity properties: $(-u_t(x) + u_t(y)) \cdot (x - y) \ge \lambda |x - y|^2$ (which is the case for the gradients of λ-convex functions). In this case standard Grönwall estimates on the W_2 distance between two solutions are proved, and it is not difficult to add diffusion to that result (as the heat kernel is already contractant in $\mathcal{W}_2$). In the second part, via different techniques (mainly using the adjoint equation, and proving somehow L^1 contractivity), the uniqueness result is provided for arbitrary L^∞ vector fields u, but with the crucial help of the diffusion term in the equation.

It is also possible to study a variant where V depends on time. We assume for simplicity that $V \in \mathrm{Lip}([0, T] \times \Omega)$ (this is a simplification; less regularity in space, such as $W^{1,1}$, could be sufficient). In this case we define

$$\begin{aligned} J_t(\rho) &:= \int_\Omega V_t(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho), \\ \rho_{k+1}^\tau &= \mathrm{argmin}_{\rho \in \mathcal{P}(\Omega)} \Big\{ J_{k\tau}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_k^\tau) \Big\}. \end{aligned} \tag{5-2}$$

The analysis proceeds similarly, with the only exception being that we get

$$\frac{1}{2\tau} W_2^2(\rho_{k+1}^\tau, \rho_k^\tau) \le J_{k\tau}(\rho_k^\tau) - J_{k\tau}(\rho_{k+1}^\tau),$$

which is no longer a telescopic series. Yet, we have $J_{k\tau}(\rho_{k+1}^\tau) \ge J_{(k+1)\tau}(\rho_{k+1}^\tau) + \mathrm{Lip}(V)\tau$, and we can go on with a telescopic sum plus a remainder of the order of τ. In the case where u_t is the opposite of the gradient of a λ-convex function V_t, one could consider approximation by functions which are piecewise constant in time and use the standard theory of gradient flows.

We remark here that [Alexander et al. 2014] gave another approach for dealing with first-order crowd motion models as limits of nonlinear-diffusion equations with gradient drift. This approach could plausibly be used also in the case where we add a simple diffusion term to the models studied in that paper.

Variant 3: transport then gradient flow-like step with the penalized entropy functional. We present now a different scheme, which combines some of the previous approaches. It could formally provide a solution of the same equation, but presents some extra difficulties.

We define now $\tilde{\rho}^{\tau}_{k+1} := (\mathrm{id} + \tau u_{k\tau})_{\#}\rho^{\tau}_{k}$ and with the help of this we define

$$\rho^{\tau}_{k+1} := \mathrm{argmin}_{\rho\in\mathcal{K}}\, \mathcal{E}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \tilde{\rho}^{\tau}_{k+1}).$$

In the last optimization problem we minimize strictly convex and lower semicontinuous functionals, and hence we have existence and uniqueness of the solution. The formal reason for this scheme being adapted to the equation is that we perform a step of a JKO scheme in the spirit of [Jordan et al. 1998] (without the density constraint) or of [Maury et al. 2010] (without the entropy term). This should let a term $-\Delta\rho - \nabla\cdot(\rho\nabla p)$ appear in the evolution equation. The term $\nabla\cdot(\rho u)$ is due to the first step (the definition of $\tilde{\rho}^{\tau}_{k+1}$). To explain a little bit more for the unexperienced reader, we consider the optimality conditions for the above minimization problem. Following [Maury et al. 2010], we can say that $\rho\in\mathcal{K}$ is optimal if and only if there exists a constant $\ell\in\mathbb{R}$ and a Kantorovich potential φ for the transport from ρ to ρ^{τ}_{k} such that

$$\rho = \begin{cases} 1 & \text{on } (\ln\rho + \varphi/\tau) < \ell, \\ 0 & \text{on } (\ln\rho + \varphi/\tau) > \ell, \\ \in [0,1] & \text{on } (\ln\rho + \varphi/\tau) = \ell. \end{cases}$$

We then define $p = (\ell - \ln\rho - \varphi/\tau)_+$ and we get $p\in\mathrm{press}(\rho)$. Moreover, ρ-a.e., $\nabla p = -\nabla\rho/\rho - \nabla\varphi/\tau$. We then use the fact that the optimal transport is of the form $T = \mathrm{id} - \nabla\varphi$ and obtain a situation as sketched in Figure 2.

Notice that

$$(\mathrm{id} + \tau u_{k\tau})^{-1} \circ (\mathrm{id} + \tau(\nabla p + \nabla\rho/\rho)) = \mathrm{id} - \tau(u_{(k+1)\tau} - \nabla p - \nabla\rho/\rho) + o(\tau)$$

provided u is regular enough. Formally we can pass to the limit $\tau\to 0$ and have

$$\partial_t\rho - \Delta\rho + \nabla\cdot(\rho(u - \nabla p)) = 0.$$

Yet, this turns out to be quite naïve, because we cannot get proper estimates on $W_2(\rho^{\tau}_{k}, \rho^{\tau}_{k+1})$. Indeed, this is mainly due to the hybrid nature of the scheme, i.e., a gradient flow for the diffusion and the projection part on the one hand and a free transport on the other hand. The typical estimate in the JKO scheme

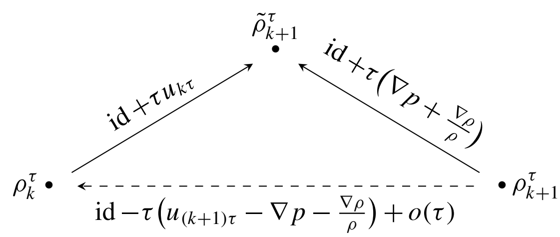

Figure 2. One time step.

comes from the fact that one can bound $W_2(\rho_k^\tau, \rho_{k+1}^\tau)^2/\tau$ with the opposite of the increment of the energy, and that this gives rise to a telescopic sum. Yet, this is not the case whenever the base point for a new time step is not equal to the previous minimizer. Moreover, the main difficulty here is the fact that the energy we consider implicitly takes the value $+\infty$, due to the constraint $\rho \in \mathcal{K}$, and hence no estimate is possible whenever $\tilde{\rho}_{k+1}^\tau \notin \mathcal{K}$. As a possible way to overcome this difficulty, one could approximate the discontinuous functional $I_{\mathcal{K}}$ with some finite energies of the same nature (for instance power-like entropies, even if the best choice would be an energy which is Lipschitz for the distance W_2). These kinds of difficulties are a matter of current study, in particular for mixed systems and/or multiple populations.

Appendix: BV-type estimates for the Fokker–Planck equation

Here we present some total variation (TV) decay results (in time) for the solutions of the Fokker–Planck equation. Some are very easy, some trickier. The goal is to look at those estimates which can be easily iterated in time and combined with the decay of the TV via the projection operator, as we did in Section 4.

Let us take a vector field $v : [0, +\infty[\times \Omega \to \mathbb{R}^d$ (we will choose later which regularity we need) and consider in Ω the problem

$$\begin{cases} \partial_t \rho_t - \Delta\rho_t + \nabla\cdot(\rho_t v_t) = 0 & \text{in }]0, +\infty[\times \Omega, \\ \rho_t(\nabla\rho_t - v_t)\cdot n = 0 & \text{on } [0, +\infty[\times \partial\Omega, \\ \rho(0, \cdot) = \rho_0 & \text{in } \Omega, \end{cases} \tag{A-1}$$

for $\rho_0 \in \mathrm{BV}(\Omega) \cap \mathcal{P}(\Omega)$.

Lemma A.1. *Suppose $\|v_t\|_{C^{1,1}} \le C$ for all $t \in [0, +\infty[$. Suppose that either $\Omega = \mathbb{T}^d$, or that Ω is convex and $v \cdot n = 0$ on $\partial\Omega$. Then we have the total variation decay estimate*

$$\int_\Omega |\nabla\rho_t|\,\mathrm{d}x \le C(t-s) + e^{C(t-s)} \int_\Omega |\nabla\rho_s|\,\mathrm{d}x \quad \textit{for all } 0 \le s \le t, \tag{A-2}$$

where $C > 0$ is a constant depending just on the $C^{1,1}$ norm of v.

Proof. First we remark that by the regularity of v the quantity

$$\|v\|_{L^\infty} + \|Dv\|_{L^\infty} + \|\nabla(\nabla\cdot v)\|_{L^\infty}$$

is uniformly bounded. Let us drop now the dependence on t in our notation and calculate in coordinates

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\rho|\,\mathrm{d}x &= \int_\Omega \frac{\nabla\rho}{|\nabla\rho|}\cdot\nabla(\partial_t\rho)\,\mathrm{d}x \\
&= \int_\Omega \frac{\nabla\rho}{|\nabla\rho|}\cdot\nabla(\Delta\rho - \nabla\cdot(v\rho))\,\mathrm{d}x = \int_\Omega \sum_j \frac{\rho_j}{|\nabla\rho|}\Big(\sum_i \rho_{iij} - (\nabla\cdot(v\rho))_j\Big)\,\mathrm{d}x \\
&= -\int_\Omega \sum_{i,j,k}\Big(\frac{\rho_{ij}^2}{|\nabla\rho|} - \frac{\rho_j\rho_k\rho_{ki}\rho_{ij}}{|\nabla\rho|^3}\Big)\,\mathrm{d}x + B_1 - \int_\Omega \sum_{j,i}\frac{\rho_j}{|\nabla\rho|}(v^i_{ij}\rho + v^i_i\rho_j + v^i_j\rho_i + v^i\rho_{ij})\,\mathrm{d}x \\
&\le B_1 + C + C\int_\Omega |\nabla\rho|\,\mathrm{d}x + \int_\Omega |\nabla\rho|\,|\nabla\cdot v|\,\mathrm{d}x + B_2 \le B_1 + B_2 + C + C\int_\Omega |\nabla\rho|\,\mathrm{d}x.
\end{aligned}$$

Here the B_i are the boundary terms, i.e.,

$$B_1 := \int_{\partial\Omega} \sum_{i,j} \frac{\rho_j n^i \rho_{ij}}{|\nabla\rho|} \, \mathrm{d}\mathcal{H}^{d-1} \quad \text{and} \quad B_2 := -\int_{\partial\Omega} (v \cdot n)|\nabla\rho| \, \mathrm{d}\mathcal{H}^{d-1}.$$

The constant $C > 0$ only depends on $\|v\|_{L^\infty} + \|\nabla \cdot v\|_{L^\infty} + \|\nabla(\nabla \cdot v)\|_{L^\infty}$. We used as well the fact that

$$-\int_{\Omega} \sum_{i,j,k} \left(\frac{\rho_{ij}^2}{|\nabla\rho|} - \frac{\rho_j \rho_k \rho_{ki} \rho_{ij}}{|\nabla\rho|^3} \right) \mathrm{d}x \le 0.$$

Now, it is clear that in the case of the torus the boundary terms B_1 and B_2 do not exist, hence we have the desired conclusion by Grönwall's lemma. In the case of the convex domain we have $B_2 = 0$ (because of the assumption $v \cdot n = 0$) and $B_1 \le 0$ because of the next lemma. □

Lemma A.2. *Suppose that $u : \Omega \to \mathbb{R}^d$ is a smooth vector field with $u \cdot n = 0$ on $\partial\Omega$, ρ is a smooth function with $\nabla\rho \cdot n = 0$ on $\partial\Omega$ and $\Omega \subset \mathbb{R}^d$ is a smooth convex set that we write as $\Omega = \{h < 0\}$ for a smooth convex function h with $|\nabla h| = 1$ on $\partial\Omega$ (so that $n = \nabla h$ on $\partial\Omega$). Then we have, on the whole boundary $\partial\Omega$, $\sum_{i,j} u^i_j \rho_j n^i = -\sum_{i,j} u^i h_{ij} \rho_j$.*

In particular, we have $\sum_{i,j} \rho_{ij} \rho_j n^i \le 0$.

Proof. The Neumann boundary assumption on u means $u(\gamma(t)) \cdot \nabla h(\gamma(t)) = 0$ for every curve γ valued in $\partial\Omega$ and for all t. Differentiating in t, we get

$$\sum_{i,j} u^i_j(\gamma(t))(\gamma'(t))^j h_i(\gamma(t)) + \sum_{i,j} u^i(\gamma(t)) h_{ij}(\gamma(t))(\gamma'(t))^j = 0.$$

Take a point $x_0 \in \partial\Omega$ and choose a curve γ with $\gamma(t_0) = x_0$ and $\gamma'(t_0) = \nabla\rho(x_0)$ (which is possible, since this vector is tangent to $\partial\Omega$ by assumption). This gives the first part of the statement. The second part, i.e., $\sum_{i,j} \rho_{ij} \rho_j n^i \le 0$, is obtained by taking $u = \nabla\rho$ and using that $D^2 h(x_0)$ is a positive definite matrix. □

Remark A.3. If we look attentively at the proof of Lemma A.1, we can see that we did not really exploit the regularizing effects of the diffusion term in the equation. This means that the given regularity estimate is the same that we would have without diffusion; in this case, the density ρ_t is obtained from the initial density as the image through the flow of v. Thus, the density depends on the determinant of the Jacobian of the flow, hence on the derivatives of v. It is normal that, if we want BV bounds on ρ_t, we need assumptions on two derivatives of v.

We would like to prove some form of BV estimates under weaker regularity assumptions on v, trying to exploit the diffusion effects. In particular, we would like to treat the case where v is only $C^{0,1}$. As we will see in the following lemma, this degenerates in some sense.

Lemma A.4. *Suppose that Ω is either the torus or a smooth convex set $\Omega = \{h < 0\}$ parametrized as a level set of a smooth convex function h. Let $v_t : \Omega \to \mathbb{R}^d$ be a vector field for $t \in [0, T]$, Lipschitz and bounded in space, uniformly in time. In the case of a convex domain, suppose $v \cdot n = 0$ on $\partial\Omega$. Let $H : \mathbb{R}^d \to \mathbb{R}$ be given by $H(z) := \sqrt{\varepsilon^2 + |z|^2}$. Now let ρ_t be the (sufficiently smooth) solution of the Fokker–Planck equation with homogeneous Neumann boundary condition.*

Then there exists a constant $C > 0$ (depending on v and Ω) such that

$$\int_\Omega H(\nabla\rho_t)\,\mathrm{d}x \le \int_\Omega H(\nabla\rho_0)\,\mathrm{d}x + C\varepsilon t + \frac{C}{\varepsilon}\int_0^t \|\rho_s\|^2_{L^\infty}\,\mathrm{d}s. \tag{A-3}$$

Proof. First let us discuss some properties of H. It is smooth, its gradient is $\nabla H(z) = z/H(z)$ and it satisfies $\nabla H(z)\cdot z \le H(z)$ for all $z \in \mathbb{R}^d$. Moreover its Hessian matrix is given by

$$[H_{ij}(z)]_{i,j\in\{1,\dots,d\}} = \left[\frac{\delta^{ij}H^2(z) - z^i z^j}{H^3(z)}\right]_{i,j\in\{1,\dots,d\}} = \frac{1}{H(z)}I_d - \frac{1}{H^3(z)}z\otimes z \quad \forall z \in \mathbb{R}^d,$$

where

$$\delta^{ij} = \begin{cases} 1 & \text{if } i = j,\\ 0 & \text{if } i \ne j,\end{cases}$$

is the Kronecker symbol. Note that, from this computation, the matrix $D^2H \ge 0$ is bounded from above by $1/H$, and hence by ε^{-1}. Moreover we introduce a uniform constant $C > 0$ such that

$$\|v\|^2_{L^\infty}|\Omega| + \|\nabla\cdot v\|_{L^\infty} + \|Dv\|_{L^\infty} \le C.$$

Now to show the estimate of this lemma we calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega H(\nabla\rho_t)\,\mathrm{d}x &= \int_\Omega \nabla H(\nabla\rho_t)\cdot\partial_t\nabla\rho_t\,\mathrm{d}x = \int_\Omega \nabla H(\nabla\rho_t)\cdot\nabla(\Delta\rho_t - \nabla\cdot(v_t\rho_t))\,\mathrm{d}x\\ &= \int_\Omega \nabla H(\nabla\rho_t)\cdot\nabla\Delta\rho_t\,\mathrm{d}x - \int_\Omega \nabla H(\nabla\rho_t)\cdot\nabla(\nabla\cdot(v_t\rho_t))\,\mathrm{d}x\\ &=: (I) + (II)\end{aligned}$$

Now we study each term separately and for simplicity we drop the t subscripts in the following. We start with the case of the torus, where there is no boundary term in the integration by parts:

$$\begin{aligned}(I) &= \int_\Omega \nabla H(\nabla\rho)\cdot\nabla\Delta\rho\,\mathrm{d}x = \int_\Omega\sum_{j,i} H_j(\nabla\rho)\rho_{jii}\,\mathrm{d}x = -\int_\Omega\sum_{j,i,k} H_{kj}(\nabla\rho)\rho_{ik}\rho_{ji}\,\mathrm{d}x,\\ (II) &= -\int_\Omega \nabla H(\nabla\rho)\cdot\nabla(\nabla\cdot(v\rho))\,\mathrm{d}x = -\int_\Omega\sum_{i,j} H_j(\nabla\rho)(v^i\rho)_{ij}\,\mathrm{d}x\\ &= \int_\Omega\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}v^i_j\rho\,\mathrm{d}x + \int_\Omega\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}v^i\rho_j\,\mathrm{d}x\\ &=: (II_a) + (II_b).\end{aligned}$$

First look at the term (II_a). Since the matrix H_{jk} is positive definite, we can apply a Young inequality for each index i and obtain

$$\begin{aligned}(II_a) = \int_\Omega\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}v^i_j\rho\,\mathrm{d}x &\le \frac12\int_\Omega\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}\rho_{ij}\,\mathrm{d}x + \frac12\int_\Omega\sum_{i,j,k} H_{jk}(\nabla\rho)v^i_j v^i_k\rho^2\,\mathrm{d}x\\ &\le \frac12|(I)| + C\|\rho\|^2_{L^2}\|D^2H\|_{L^\infty}.\end{aligned}$$

The L^2 norm in the second term will be estimated by the L^∞ norm for the sake of simplicity (see Remark A.5 below).

For the term (II_b) we first make a pointwise computation,

$$\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}v^i\rho_j = \frac{1}{H^3(\nabla\rho)}\sum_i \big(D_i^2\rho\cdot(\varepsilon^2 I_d + |\nabla\rho|^2 I_d - \nabla\rho\otimes\nabla\rho)\cdot\nabla\rho\big)v^i$$
$$= \frac{\varepsilon^2}{H^3(\nabla\rho)}\sum_i v^i D_i^2\rho\cdot\nabla\rho = -\varepsilon^2\sum_i v^i\partial_i\left(\frac{1}{H(\nabla\rho)}\right),$$

where $D_i^2\rho$ denotes the i-th row in the Hessian matrix of ρ, and we use $(|\nabla\rho|^2 I_d - \nabla\rho\otimes\nabla\rho)\cdot\nabla\rho = 0$.

Integrating by parts, we obtain

$$(II_b) = \varepsilon^2\int_\Omega(\nabla\cdot v)\frac{1}{H(\nabla\rho)}\,\mathrm{d}x \le C\varepsilon^2\|1/H\|_{L^\infty} \le C\varepsilon,$$

where we use $H(z)\ge\varepsilon$.

Summing up all the terms and using $\|D^2H\|\le\varepsilon^{-1}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega H(\nabla\rho_t)\,\mathrm{d}x \le -\frac{1}{2}|(I)| + C\|\rho_t\|_{L^\infty}^2\|D^2H\|_{L^\infty} + C\varepsilon \le C\varepsilon + C\|\rho_t\|_{L^\infty}^2\varepsilon^{-1},$$

which proves the claim.

If we switch to the case of a smooth bounded convex domain Ω, we have to handle boundary terms. These terms are

$$\int_{\partial\Omega}\sum_{i,j} H_j(\nabla\rho)\rho_{ij}n^i - \int_{\partial\Omega}\sum_{i,j} H_j(\nabla\rho)\rho v_j^i n^i,$$

where we ignore those terms which involve $n^i v^i$ (i.e., the integration by parts in (II_b), and the term $H_j(\nabla\rho)\rho_j n^i v^i$ in the integration by parts of (II_a)), since we have already supposed $v\cdot n = 0$. We use here Lemma A.2, which provides

$$\sum_{i,j} H_j(\nabla\rho)\rho_{ij}n^i - \rho H_j(\nabla\rho)v_j^i n^i = \frac{1}{H(\nabla\rho)}\sum_{i,j}(\rho_j\rho_{ij}n^i - \rho\rho_j v_j^i n^i)$$
$$= -\frac{1}{H(\nabla\rho)}\sum_{i,j}(\rho_j h_{ij}\rho_i - \rho\rho_j h_{ij}v^i).$$

If we use the fact that the matrix D^2h is positive definite and a Young inequality, we get $\sum_{i,j}\rho_j h_{ij}\rho_i \ge 0$ and

$$\rho\sum_{i,j}|\rho_j h_{ij}v^i| \le \frac{1}{2}\sum_{i,j}\rho_j h_{ij}\rho_i + \frac{1}{2}\sum_{i,j}\rho^2 v^j h_{ij}v^i,$$

which implies

$$\frac{1}{H(\nabla\rho)}\sum_{i,j}(\rho_j\rho_{ij}n^i - \rho\rho_j v_j^i n^i) \le \frac{\rho^2}{H(\nabla\rho)}\|D^2h\|_{L^\infty}|v|^2 \le \frac{C\|\rho\|_{L^\infty}^2}{\varepsilon}.$$

This provides the desired estimate on the boundary term. □

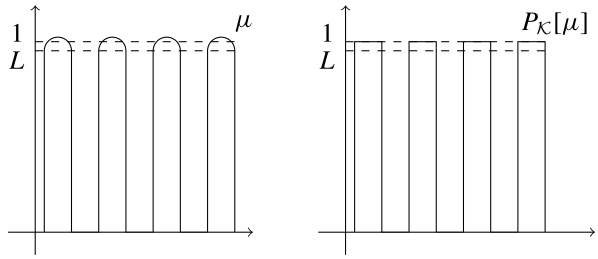

Figure 3. A counterexample to the decay of $\int_\Omega H(\nabla\rho)$, which corresponds to the total length of the graph.

Remark A.5. In the above proof, we needed to use the L^∞ norm of ρ only in the boundary term. When there is no boundary term, the L^2 norm is enough to handle the term (II_a). In both cases, the norm of ρ can be bounded in terms of the initial norm multiplied by e^{Ct}, where C bounds the divergence of v. On the other hand, in the torus case, one only needs to suppose $\rho_0 \in L^2$ and in the convex case $\rho_0 \in L^\infty$. Both assumptions are satisfied in the applications to crowd motion with density constraints.

We have seen that the constants in the above inequality depend on ε and explode as $\varepsilon \to 0$. This prevents us from obtaining a clean estimate on the BV norm in this context, but at least it proves that $\rho_0 \in \mathrm{BV} \Rightarrow \rho_t \in \mathrm{BV}$ for all $t > 0$ (to achieve this result, we just need to take $\varepsilon = 1$). Unfortunately, the quantity which is estimated is not the BV norm, but the integral $\int_\Omega H(\nabla\rho)$. This is not enough for the applications to Section 4, as it is unfortunately not true that the projection operator decreases the value of this other functional. (Here is a simple counterexample. Consider $\mu = g(x)\,\mathrm{d}x$ a BV density on $[0,2] \subset \mathbb{R}$, with g defined as follows. Divide the interval $[0,2]$ into $2K$ intervals J_i of length $2r$ (with $2rK = 1$); call t_i the center of each interval J_i (i.e., $t_i = i2r + r$, for $i = 0, \dots, 2K-1$) and set $g(x) = L + \sqrt{r^2 - (x - t_i)^2}$ on each J_i with i odd, and $g(x) = 0$ on J_i for i even, taking $L = 1 - \pi r/4$. It is not difficult to check that the projection of μ is equal to the indicator function of the union of all the intervals J_i with i odd, and that the value of $\int H(\nabla\rho)$ has increased by $K(2 - \pi/2)r = 1 - \pi/4$, i.e., by a positive constant. See Figure 3.)

If we pursue the value of the BV norm, we can provide the following estimate.

Lemma A.6. *Under the assumptions of Lemma A.4, if we suppose* $\rho_0 \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, *then, for* $t \le T$, *we have*

$$\int_\Omega |\nabla\rho_t|\,\mathrm{d}x \le \int_\Omega |\nabla\rho_0|\,\mathrm{d}x + C\sqrt{t}, \tag{A-4}$$

where the constant C *depends on* v, *on* T *and on* $\|\rho_0\|_{L^\infty}$.

Proof. Using the L^∞ estimate of Lemma 4.2, we will assume that $\|\rho_t\|_{L^\infty}$ is bounded by a constant (which depends on v, on T and on $\|\rho_0\|_{L^\infty}$). Then, we can write

$$\int_\Omega |\nabla\rho_t|\,\mathrm{d}x \le \int_\Omega H(\nabla\rho_t)\,\mathrm{d}x \le \int_\Omega H(\nabla\rho_0)\,\mathrm{d}x + C\varepsilon t + \frac{Ct}{\varepsilon} \le \int_\Omega (|\nabla\rho_0| + \varepsilon)\,\mathrm{d}x + C\varepsilon t + \frac{Ct}{\varepsilon}.$$

It is sufficient to choose, for fixed t, $\varepsilon = \sqrt{t}$, in order to prove the claim. □

Unfortunately, this $\sqrt{t}$ behavior is not suitable to be iterated, and the above estimate is useless for the sake of Section 4. The existence of an estimate (for v Lipschitz) of the form $\mathrm{TV}(\rho_t) \le \mathrm{TV}(\rho_0) + Ct$, or $\mathrm{TV}(\rho_t) \le \mathrm{TV}(\rho_0)e^{Ct}$, or even $f(\mathrm{TV}(\rho_t)) \le f(\mathrm{TV}(\rho_0))e^{Ct}$ for any increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$, seems to be an open question.

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ASYMPTOTIC STABILITY IN ENERGY SPACE FOR DARK SOLITONS OF THE LANDAU–LIFSHITZ EQUATION

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We prove the asymptotic stability in energy space of nonzero speed solitons for the one-dimensional Landau–Lifshitz equation with an easy-plane anisotropy

$$\partial_t m + m \times (\partial_{xx} m - m_3 e_3) = 0$$

for a map $m = (m_1, m_2, m_3) : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^2$, where $e_3 = (0, 0, 1)$. More precisely, we show that any solution corresponding to an initial datum close to a soliton with nonzero speed is weakly convergent in energy space as time goes to infinity to a soliton with a possible different nonzero speed, up to the invariances of the equation. Our analysis relies on the ideas developed by Martel and Merle for the generalized Korteweg–de Vries equations. We use the Madelung transform to study the problem in the hydrodynamical framework. In this framework, we rely on the orbital stability of the solitons and the weak continuity of the flow in order to construct a limit profile. We next derive a monotonicity formula for the momentum, which gives the localization of the limit profile. Its smoothness and exponential decay then follow from a smoothing result for the localized solutions of the Schrödinger equations. Finally, we prove a Liouville type theorem, which shows that only the solitons enjoy these properties in their neighbourhoods.

1. Introduction

We consider the one-dimensional Landau–Lifshitz equation

$$\partial_t m + m \times (\partial_{xx} m + \lambda m_3 e_3) = 0 \tag{LL}$$

for a map $m = (m_1, m_2, m_3) : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^2$, where $e_3 = (0, 0, 1)$ and $\lambda \in \mathbb{R}$. This equation was introduced by Landau and Lifshitz [1935]. It describes the dynamics of magnetization in a one-dimensional ferromagnetic material, for example in $CsNiF_3$ or TMNC (see, e.g., [Kosevich et al. 1990; Hubert and Schäfer 1998] and the references therein). The parameter λ accounts for the anisotropy of the material. The choices $\lambda > 0$ and $\lambda < 0$ correspond respectively to an easy-axis and an easy-plane anisotropy. In the isotropic case $\lambda = 0$, the equation is exactly the one-dimensional Schrödinger map equation, which has been intensively studied (see, e.g., [Guo and Ding 2008; Jerrard and Smets 2012]). In this paper, we study the Landau–Lifshitz equation with an easy-plane anisotropy ($\lambda < 0$). Performing, if necessary, a suitable scaling argument on the map m, we assume from now on that $\lambda = -1$. Our main goal is to prove the asymptotic stability for the solitons of this equation (see Theorem 1.1 below).

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The Landau–Lifshitz equation is Hamiltonian. Its Hamiltonian, the so-called Landau–Lifshitz energy, is given by

$$E(m) := \frac{1}{2}\int_{\mathbb{R}} \big(|\partial_x m|^2 + m_3^2\big).$$

In the sequel, we restrict our attention to the Hamiltonian framework in which the solutions m to (LL) have finite Landau–Lifshitz energy, i.e., belong to energy space

$$\mathcal{E}(\mathbb{R}) := \{\upsilon : \mathbb{R} \to \mathbb{S}^2 \mid \upsilon' \in L^2(\mathbb{R}) \text{ and } \upsilon_3 \in L^2(\mathbb{R})\}.$$

A soliton with speed c is a travelling-wave solution of (LL) having the form

$$m(x,t) := u(x-ct).$$

Its profile u is a solution to the ordinary differential equation

$$u'' + |u'|^2 u + u_3^2 u - u_3 e_3 + cu \times u' = 0. \tag{TWE}$$

The solutions of this equation are explicit. When $|c| \geq 1$, the only solutions with finite Landau–Lifshitz energy are the constant vectors in $\mathbb{S}^1 \times \{0\}$. In contrast, when $|c| < 1$, there exist nonconstant solutions u_c to (TWE), which are given by the formulae

$$[u_c]_1(x) = \frac{c}{\cosh((1-c^2)^{1/2}x)}, \quad [u_c]_2(x) = \tanh((1-c^2)^{1/2}x), \quad [u_c]_3(x) = \frac{(1-c^2)^{1/2}}{\cosh((1-c^2)^{1/2}x)},$$

up to the invariances of the problem, i.e., translations, rotations around the axis x_3 and orthogonal symmetries with respect to the plane $x_3 = 0$ (see [de Laire 2014] for more details).

Our goal is to study the asymptotic behaviour for solutions of (LL) which are initially close to a soliton in energy space. We endow $\mathcal{E}(\mathbb{R})$ with the metric structure corresponding to the distance introduced by de Laire and Gravejat [2015],

$$d_{\mathcal{E}}(f,g) := |\check{f}(0) - \check{g}(0)| + \|f' - g'\|_{L^2(\mathbb{R})} + \|f_3 - g_3\|_{L^2(\mathbb{R})},$$

where $f = (f_1, f_2, f_3)$ and $\check{f} = f_1 + if_2$ (and similarly for g). The Cauchy problem and the orbital stability of the travelling waves have been solved by de Laire and Gravejat [2015]. We are concerned with the asymptotic stability of travelling waves. The following theorem is our main result.

Theorem 1.1. *Let $\mathfrak{c} \in (-1,1) \setminus \{0\}$. There exists a positive number $\delta_{\mathfrak{c}}$, depending only on $\mathfrak{c}$, such that, if*

$$d_{\mathcal{E}}\big(m^0, u_{\mathfrak{c}}\big) \leq \delta_{\mathfrak{c}},$$

then there exist a number $\mathfrak{c}^* \in (-1,1) \setminus \{0\}$, and two functions $b \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ and $\theta \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ such that

$$b'(t) \to \mathfrak{c}^* \quad \textit{and} \quad \theta'(t) \to 0$$

as $t \to +\infty$, and for which the map

$$m_\theta := (\cos(\theta)m_1 - \sin(\theta)m_2, \sin(\theta)m_1 + \cos(\theta)m_2, m_3),$$

satisfies the convergences

$$\begin{aligned} \partial_x m_{\theta(t)}(\cdot + b(t), t) &\rightharpoonup \partial_x u_{\mathfrak{c}^*} && \textit{in } L^2(\mathbb{R}), \\ m_{\theta(t)}(\cdot + b(t), t) &\to u_{\mathfrak{c}^*} && \textit{in } L^\infty_{\mathrm{loc}}(\mathbb{R}), \\ m_3(\cdot + b(t), t) &\rightharpoonup [u_{\mathfrak{c}^*}]_3 && \textit{in } L^2(\mathbb{R}) \end{aligned}$$

as $t \to +\infty$.

Remarks. (i) Note that the case $\mathfrak{c} = 0$ — that is, black solitons — is excluded from the statement of Theorem 1.1. In this case, the map $\check{u}_0$ vanishes and we cannot apply the Madelung transform and the subsequent arguments. Orbital and asymptotic stability remain open problems for this case. Note that, to our knowledge, there is currently no available proof of the local well-posedness of (LL) in energy space, when u_0 vanishes and so the hydrodynamical framework can no longer be used.

(ii) Here, we state a weak convergence result and not a local strong convergence one, like the results given by Martel and Merle [2008a; 2008b] for the Korteweg–de Vries equation. In their situation, they can use two monotonicity formulae for the L^2 norm and the energy. This heuristically originates in the property that dispersion has negative speed in the context of the Korteweg–de Vries equation. In contrast, the possible group velocities for the dispersion of the Landau–Lifshitz equation are given by

$$v_g(k) = \pm \frac{1+2k^2}{\sqrt{1+k^2}},$$

where k is the wave number. Dispersion has both negative and positive speeds. A monotonicity formula remains for the momentum due to the existence of a gap in the possible group velocities, which satisfy the condition $|v_g(k)| \geq 1$. However, there is no evidence that one can establish a monotonicity formula for the energy.

Similar results were stated by Soffer and Weinstein [1989; 1990; 1992]. They provided the asymptotic stability of ground states for the nonlinear Schrödinger equation with a potential in a regime for which the nonlinear ground-state is a close continuation of the linear one. They rely on dispersive estimates for the linearized equation around the ground state in suitable weighted spaces, and they apply a fixed point argument. This strategy was successfully extended in particular by Buslaev, Perelman, C. Sulem and Cuccagna to the nonlinear Schrödinger equations without potential (see, e.g., [Buslaev and Perelman 1993; 1995; Buslaev and Sulem 2003; Cuccagna 2001]) and with a potential (see, e.g., [Gang and Sigal 2007]). We refer to the detailed historical survey by Cuccagna [2003] for more details. Later, Cuccagna [2011] proved a stronger result for the ground state satisfying the sufficient conditions for orbital stability of M. Weinstein, for seemingly generic nonlinear Schrödinger equation which has a smooth short range nonlinearity with the presence of a very short range and smooth linear potential. In addition, asymptotic stability in spaces of exponentially localized perturbations was studied by Pego and Weinstein [1994] (see also [Mizumachi 2001] for perturbations with algebraic decay).

Our strategy for establishing the asymptotic stability result in Theorem 1.1 is reminiscent of ideas developed by Martel and Merle [2006; 2008a; 2008b] for the Korteweg–de Vries equation, and successfully adapted by Béthuel, Gravejat and Smets in [Béthuel et al. 2014] for the Gross–Pitaevskii equation.

The main steps of the proof are similar to the ones for the Gross–Pitaevskii equation in [Béthuel et al. 2015]. Indeed, the solitons of the Landau–Lifshitz equation share many properties with the solitons of the Gross–Pitaevskii equation. In fact, the stereographic variable ψ defined by

$$\psi = \frac{u_1+iu_2}{1+u_3}$$

satisfies the equation

$$\partial_{xx}\psi + \frac{1-|\psi|^2}{1+|\psi|^2}\psi - ic\partial_x\psi = \frac{2\overline{\psi}}{1+|\psi|^2}(\partial_x\psi)^2,$$

which can be seen as a perturbation of the equation for the travelling waves of the Gross–Pitaevskii equation, namely

$$\partial_{xx}\Psi + (1-|\Psi|^2)\Psi - ic\partial_x\Psi = 0.$$

However, the analysis of the Landau–Lifshitz equation is much more difficult. Indeed, we rely on a Hasimoto like transform in order to relate the Landau–Lifshitz equation with a nonlinear Schrödinger equation. Doing so, we lose some regularity. We have to deal with a nonlinear equation at the L^2-level and not at the H^1-level as in the case of the Gross–Pitaevskii equation. This leads to important technical difficulties.

Returning to the proof of Theorem 1.1, we first translate the problem into the hydrodynamical formulation. Then, we prove the asymptotic stability in that framework. In fact, we begin by refining the orbital stability. Next, we construct a limit profile, which is smooth and localized. For the proof of the exponential decay of the limit profile, we cannot rely on the Sobolev embedding H^1 into L^∞ as was done in [Béthuel et al. 2015]. We use instead the results of Kenig, Ponce and Vega in [Kenig et al. 2003], and the Gagliardo–Nirenberg inequality (see the proof of Proposition 2.9 for more details). We also have to deal with the weak continuity of the flow in order to construct the limit profile. For the Gross–Pitaevskii equation, this property relies on the uniqueness in a weaker space (see [Béthuel et al. 2015]). There is no similar result at the L^2-level. Instead, we use the Kato smoothing effect. The asymptotic stability in the hydrodynamical variables then follows from a Liouville type theorem. It shows that the only smooth and localized solutions in the neighbourhood of the solitons are the solitons. Finally, we deduce the asymptotic stability in the original setting from the result in the hydrodynamical framework.

In Section 2 below, we explain the main tools and different steps for the proof. First, we introduce the hydrodynamical framework. Then, we state the orbital stability of the solitons under a new orthogonality condition. Next, we sketch the proof of the asymptotic stability for the hydrodynamical system and we state the main propositions. We finally complete the proof of Theorem 1.1.

In Sections 3 to 5, we give the proofs of the results stated in Section 2. In Section 3, we deal with the orbital stability in the hydrodynamical framework. In Section 4, we prove the localization and the smoothness of the limit profile. In the last section, we prove a Liouville type theorem. In a separate appendix, we show some facts used in the proofs, in particular the weak continuity of the (HLL) flow.

2. Main steps for the proof of Theorem 1.1

The hydrodynamical framework. We introduce the map $\check{m} := m_1 + i m_2$. Since m_3 belongs to $H^1(\mathbb{R})$, it follows from the Sobolev embedding theorem that

$$|\check{m}(x)| = (1 - m_3^2(x))^{1/2} \to 1$$

as $x \to \pm\infty$. As a consequence, the Landau–Lifshitz equation shares many properties with the Gross–Pitaevskii equation (see, e.g., [Béthuel et al. 2008]). One of these properties is the existence of a hydrodynamical framework for the Landau–Lifshitz equation. In terms of the maps $\check{m}$ and m_3, this equation may be written as

$$\begin{cases} i\partial_t \check{m} - m_3 \partial_{xx}\check{m} + \check{m}\partial_{xx} m_3 - \check{m} m_3 = 0, \\ \partial_t m_3 + \partial_x \langle i\check{m}, \partial_x \check{m}\rangle_{\mathbb{C}} = 0. \end{cases}$$

When the map $\check{m}$ does not vanish, one can write it as $\check{m} = (1 - m_3^2)^{1/2} \exp i\varphi$. The hydrodynamical variables $v := m_3$ and $w := \partial_x \varphi$ satisfy the system

$$\begin{cases} \partial_t v = \partial_x((v^2 - 1)w), \\ \partial_t w = \partial_x\Big(\dfrac{\partial_{xx} v}{1 - v^2} + v\dfrac{(\partial_x v)^2}{(1-v^2)^2} + v(w^2 - 1)\Big). \end{cases} \tag{HLL}$$

This system is similar to the hydrodynamical Gross–Pitaevskii equation (see, e.g., [Béthuel et al. 2015]).[1] We first study the asymptotic stability in the hydrodynamical framework.

In this framework, the Landau–Lifshitz energy is expressed as

$$E(\mathfrak{v}) := \int_{\mathbb{R}} e(\mathfrak{v}) := \frac{1}{2}\int_{\mathbb{R}} \Big(\frac{(v')^2}{1-v^2} + (1 - v^2)w^2 + v^2\Big), \tag{2-1}$$

where $\mathfrak{v} := (v, w)$ denotes the hydrodynamical pair. The momentum P, defined by

$$P(\mathfrak{v}) := \int_{\mathbb{R}} vw, \tag{2-2}$$

is also conserved by the Landau–Lifshitz flow. The momentum P and the Landau–Lifshitz energy E play an important role in the study of the asymptotic stability of the solitons. When $c \neq 0$, the function $\check{u}_c$ does not vanish. The hydrodynamical pair $Q_c := (v_c, w_c)$ is given by

$$v_c(x) = \frac{(1-c^2)^{1/2}}{\cosh((1-c^2)^{1/2}x)} \quad \text{and} \quad w_c(x) = \frac{c v_c(x)}{1 - v_c(x)^2} = \frac{c(1-c^2)^{1/2}\cosh((1-c^2)^{1/2}x)}{\sinh((1-c^2)^{1/2}x)^2 + c^2}. \tag{2-3}$$

The only invariances of (HLL) are translations and the opposite map $(v, w) \mapsto (-v, -w)$. We restrict our attention to the translation invariances. All the analysis developed below applies when the opposite map is also taken into account. For $a \in \mathbb{R}$, we define

$$Q_{c,a}(x) := Q_c(x - a) := (v_c(x - a), w_c(x - a)),$$

[1] The hydrodynamical terminology originates in the fact that the hydrodynamical Gross–Pitaevskii equation is similar to the Euler equation for an irrotational fluid (see, e.g., [Béthuel et al. 2014]).

a nonconstant soliton with speed c. We also set

$$\mathcal{NV}(\mathbb{R}) := \big\{ \mathfrak{v} = (v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \mid \max_{\mathbb{R}} |v| < 1 \big\}.$$

This nonvanishing space is endowed in the sequel with the metric structure provided by the norm

$$\|\mathfrak{v}\|_{H^1 \times L^2} := \big(\|v\|_{H^1}^2 + \|w\|_{L^2}^2 \big)^{1/2}.$$

Orbital stability. A perturbation of a soliton is provided by another soliton with a slightly different speed. This property follows from the existence of a continuum of solitons with different speeds. A solution corresponding to such a perturbation at initial time diverges from the soliton due to the different speeds of propagation, so that the standard notion of stability does not apply to solitons. The notion of orbital stability is tailored to deal with such situations. The orbital stability theorem below shows that a perturbation of a soliton at initial time remains a perturbation of the soliton, up to translations, for all time.

The following theorem is a variant of the result by de Laire and Gravejat [2015] concerning sums of solitons. It is useful for the proof of the asymptotic stability.

Theorem 2.1. *Let $c \in (-1, 1) \setminus \{0\}$. There exists a positive number α_c, depending only on c, with the following properties. Given any $(v_0, w_0) \in X(\mathbb{R}) := H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that*

$$\alpha_0 := \|(v_0, w_0) - Q_{c,a}\|_{X(\mathbb{R})} \leq \alpha_c \tag{2-4}$$

for some $a \in \mathbb{R}$, there exist a unique global solution $(v, w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$ to (HLL) *with initial datum (v_0, w_0), and two maps $c \in \mathcal{C}^1(\mathbb{R}, (-1, 1) \setminus \{0\})$ and $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that the function ε defined by*

$$\varepsilon(\cdot\,, t) := \big(v(\cdot + a(t), t), w(\cdot + a(t), t) \big) - Q_{c(t)} \tag{2-5}$$

satisfies the orthogonality conditions

$$\langle \varepsilon(\cdot\,, t), \partial_x Q_{c(t)} \rangle_{L^2(\mathbb{R})^2} = \langle \varepsilon(\cdot\,, t), \chi_{c(t)} \rangle_{L^2(\mathbb{R})^2} = 0 \tag{2-6}$$

for any $t \in \mathbb{R}$. Moreover, there exist two positive numbers σ_c and A_c, depending only and continuously on c, such that

$$\max_{x \in \mathbb{R}} v(x, t) \leq 1 - \sigma_c, \tag{2-7}$$

$$\|\varepsilon(\cdot\,, t)\|_{X(\mathbb{R})} + |c(t) - c| \leq A_c \alpha^0, \tag{2-8}$$

$$|c'(t)| + |a'(t) - c(t)| \leq A_{\mathfrak{c}} \|\varepsilon(\cdot\,, t)\|_{X(\mathbb{R})}, \tag{2-9}$$

for any $t \in \mathbb{R}$.

Remark. In this statement, the function χ_c is a normalized eigenfunction associated to the unique negative eigenvalue of the linear operator

$$\mathcal{H}_c := E''(Q_c) + c P''(Q_c).$$

The operator $\mathcal{H}_c$ is self-adjoint on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, with domain $\mathrm{Dom}(\mathcal{H}_c) := H^2(\mathbb{R}) \times L^2(\mathbb{R})$ (see (A-42) for its explicit formula). It has a unique negative simple eigenvalue $-\tilde{\lambda}_c$, and its kernel is given by

$$\mathrm{Ker}(\mathcal{H}_c) = \mathrm{Span}(\partial_x Q_c). \tag{2-10}$$

Our statement of orbital stability relies on a different decomposition from that proposed by Grillakis, Shatah and Strauss in [Grillakis et al. 1987]. This modification is related to the proof of asymptotic stability. A key ingredient in the proof is the coercivity of the quadratic form G_c, which is defined in (2-46), under a suitable orthogonality condition. In case we use the orthogonality conditions in [Grillakis et al. 1987], the corresponding orthogonality condition for G_c is provided by the function $v_c^{-1} S \partial_c Q_c$ (see (2-40) for the definition of S), which does not belong to $L^2(\mathbb{R})$. In order to bypass this difficulty, we use the second orthogonality condition in (2-6) for which the corresponding orthogonality condition for G_c is given by the function $v_c^{-1} S \chi_c$, which does belong to $L^2(\mathbb{R})$ (see the appendix for more details). This alternative decomposition is inspired by the one used by Martel and Merle [2008a].

Concerning the proof of Theorem 2.1, we first establish an orbital stability theorem with the classical decomposition of Grillakis, Shatah and Strauss [Grillakis et al. 1987]. This appears as a particular case of the orbital stability theorem in [de Laire and Gravejat 2015] for sums of solitons. We next show that if we have orbital stability for some decomposition and orthogonality conditions, then we also have it for different decomposition and orthogonality conditions (see Section 2 for the detailed proof of Theorem 2.1).

Asymptotic stability for the hydrodynamical variables. The following theorem shows the asymptotic stability result in the hydrodynamical framework.

Theorem 2.2. *Let $\mathfrak{c} \in (-1, 1) \setminus \{0\}$. There exists a positive constant $\beta_{\mathfrak{c}} \leq \alpha_{\mathfrak{c}}$, depending only on $\mathfrak{c}$, with the following properties. Given any $(v_0, w_0) \in X(\mathbb{R})$ such that*

$$\|(v_0, w_0) - Q_{\mathfrak{c},\mathfrak{a}}\|_{X(\mathbb{R})} \leq \beta_{\mathfrak{c}},$$

for some $\mathfrak{a} \in \mathbb{R}$, there exist a number $\mathfrak{c}^* \in (-1, 1) \setminus \{0\}$ and a map $b \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that the unique global solution $(v, w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$ to (HLL) *with initial datum (v_0, w_0) satisfies*

$$\big(v(\cdot + b(t), t), w(\cdot + b(t), t)\big) \rightharpoonup Q_{\mathfrak{c}^*} \quad \textit{in } X(\mathbb{R}), \tag{2-11}$$

and

$$b'(t) \to \mathfrak{c}^*$$

as $t \to +\infty$.

Theorem 2.2 establishes a convergence to some orbit of the soliton. This result is stronger than the one given by Theorem 2.1 which only shows that the solution stays close to that orbit.

In the next subsections, we explain the main ideas of the proof, which follows the strategy developed by Martel and Merle [2008a; 2008b] for the Korteweg–de Vries equation.

Construction of a limit profile. Let $\mathfrak{c} \in (-1, 1) \setminus \{0\}$, and let $(v_0, w_0) \in X(\mathbb{R})$ be any pair satisfying the assumptions of Theorem 2.2. Since $\beta_\mathfrak{c} \leq \alpha_\mathfrak{c}$ in the assumptions of Theorem 2.2, we deduce from Theorem 2.1 that the unique solution (v, w) to (HLL) with initial datum (v_0, w_0) is global.

We take an arbitrary sequence of times $(t_n)_{n\in\mathbb{N}}$ tending to $+\infty$. In view of (2-8) and (2-9), we may assume, up to a subsequence, that there exist a limit perturbation $\varepsilon_0^* \in X(\mathbb{R})$ and a limit speed $c_0^* \in [-1, 1]$ such that

$$\varepsilon(\cdot, t_n) = \big(v(\cdot + a(t_n), t_n), w(\cdot + a(t_n), t_n)\big) - Q_{c(t_n)} \rightharpoonup \varepsilon_0^* \quad \text{in } X(\mathbb{R}), \tag{2-12}$$

and

$$c(t_n) \to c_0^* \tag{2-13}$$

as $n \to +\infty$. Our main goal is to show that

$$\varepsilon_0^* \equiv 0$$

(see Corollary 2.15). For that, we establish smoothness and rigidity properties for the solution of (HLL) with the initial datum $Q_{c_0^*} + \varepsilon_0^*$.

First, we require the constant $\beta_\mathfrak{c}$ to be sufficiently small so that, when the number α^0 which appears in Theorem 2.1 satisfies $\alpha^0 \leq \beta_\mathfrak{c}$, then we infer from (2-8) and (2-9) that

$$\min\{c(t)^2, a'(t)^2\} \geq \frac{\mathfrak{c}^2}{2}, \qquad \max\{c(t)^2, a'(t)^2\} \leq 1 + \frac{\mathfrak{c}^2}{2}, \tag{2-14}$$

and

$$\|v_\mathfrak{c}(\cdot) - v(\cdot + a(t), t)\|_{L^\infty(\mathbb{R})} \leq \min\left\{\frac{\mathfrak{c}^2}{4}, \frac{1-\mathfrak{c}^2}{16}\right\} \tag{2-15}$$

for any $t \in \mathbb{R}$. This yields, in particular, that $c_0^* \in (-1, 1) \setminus \{0\}$, and then, that $Q_{c_0^*}$ is well-defined and different from the black soliton.

By (2-8), we also have

$$|c_0^* - \mathfrak{c}| \leq A_\mathfrak{c} \beta_\mathfrak{c}, \tag{2-16}$$

and, applying again (2-8) well as (2-12) and the weak lower semicontinuity of the norm, we also know that the function

$$(v_0^*, w_0^*) := Q_{c_0^*} + \varepsilon_0^*$$

satisfies

$$\|(v_0^*, w_0^*) - Q_\mathfrak{c}\|_{X(\mathbb{R})} \leq A_\mathfrak{c} \beta_\mathfrak{c} + \|Q_\mathfrak{c} - Q_{c_0^*}\|_{X(\mathbb{R})}. \tag{2-17}$$

We next impose a supplementary smallness assumption on $\beta_\mathfrak{c}$ so that

$$\|(v_0^*, w_0^*) - Q_\mathfrak{c}\|_{X(\mathbb{R})} \leq \alpha_\mathfrak{c}. \tag{2-18}$$

By Theorem 2.1, there exists a unique global solution $(v^*, w^*) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$ to (HLL) with initial datum (v_0^*, w_0^*), and two maps $c^* \in \mathcal{C}^1(\mathbb{R}, (-1, 1) \setminus \{0\})$ and $a^* \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that the function ε^* defined by

$$\varepsilon^*(\cdot, t) := \big(v^*(\cdot + a^*(t), t), w(\cdot + a^*(t), t)\big) - Q_{c^*(t)} \tag{2-19}$$

satisfies the orthogonality conditions

$$\langle \varepsilon^*(\cdot\,,t), \partial_x Q_{c^*(t)}\rangle_{L^2(\mathbb{R})^2} = \langle \varepsilon^*(\cdot\,,t), \chi_{c^*(t)}\rangle_{L^2(\mathbb{R})^2} = 0, \tag{2-20}$$

as well as the estimates

$$\|\varepsilon^*(\cdot\,,t)\|_{X(\mathbb{R})} + |c^*(t)-\mathfrak{c}| + |a^{*\prime}(t) - c^*(t)| \le A_{\mathfrak{c}} \|(v_0^*, w_0^*) - Q_{\mathfrak{c}}\|_{X(\mathbb{R})}, \tag{2-21}$$

for any $t \in \mathbb{R}$.

We may take $\beta_{\mathfrak{c}}$ small enough such that, combining (2-16) with (2-17) and (2-21), we obtain

$$\min\{c^*(t)^2, (a^*)'(t)^2\} \ge \frac{\mathfrak{c}^2}{2}, \qquad \max\{c^*(t)^2, (a^*)'(t)^2\} \le 1 + \frac{\mathfrak{c}^2}{2}, \tag{2-22}$$

and

$$\big\| v_{\mathfrak{c}}(\cdot) - v^*(\cdot + a^*(t), t) \big\|_{L^\infty(\mathbb{R})} \le \min\Big\{\frac{\mathfrak{c}^2}{4}, \frac{1-\mathfrak{c}^2}{16}\Big\}, \tag{2-23}$$

for any $t \in \mathbb{R}$.

Finally, we use the weak continuity of the flow map for the Landau–Lifshitz equation. The proof relies on Proposition A.1 and follows the lines of the proof of Proposition 1 in [Béthuel et al. 2015].

Proposition 2.3. *Let $t \in \mathbb{R}$ be fixed. Then*

$$\big(v(\cdot + a(t_n), t_n + t), w(\cdot + a(t_n), t_n + t)\big) \rightharpoonup \big(v^*(\cdot\,,t), w^*(\cdot\,,t)\big) \quad \text{in } X(\mathbb{R}), \tag{2-24}$$

while

$$a(t_n + t) - a(t_n) \to a^*(t) \quad \text{and} \quad c(t_n + t) \to c^*(t) \tag{2-25}$$

as $n \to +\infty$. In particular, we have

$$\varepsilon(\cdot\,, t_n + t) \rightharpoonup \varepsilon^*(\cdot\,,t) \quad \text{in } X(\mathbb{R}) \tag{2-26}$$

as $n \to +\infty$.

Localization and smoothness of the limit profile. Our proof of the localization of the limit profile is based on a monotonicity formula.

Consider a pair (v, w) which satisfies the conclusions of Theorem 2.1 and suppose that (2-14) and (2-15) are true. Let R and t be two real numbers, and set

$$I_R(t) \equiv I_R^{(v,w)}(t) := \frac{1}{2}\int_{\mathbb{R}} [vw](x + a(t), t)\Phi(x - R)\,dx,$$

where Φ is the function defined on $\mathbb{R}$ by

$$\Phi(x) := \tfrac{1}{2}(1 + \mathrm{th}(\nu_{\mathfrak{c}} x)), \tag{2-27}$$

with $\nu_{\mathfrak{c}} := \sqrt{1-\mathfrak{c}^2}/8$.

Proposition 2.4. *Let $R \in \mathbb{R}$, $t \in \mathbb{R}$ and $\sigma \in [-\sigma_\mathfrak{c}, \sigma_\mathfrak{c}]$, with $\sigma_\mathfrak{c} := \sqrt{1-\mathfrak{c}^2}/4$. Under the above assumptions, there exists a positive number $B_\mathfrak{c}$, depending only on $\mathfrak{c}$, such that*

$$\frac{d}{dt}[I_{R+\sigma t}(t)] \geq \frac{1-\mathfrak{c}^2}{8} \int_\mathbb{R} \big[(\partial_x v)^2 + v^2 + w^2\big](x+a(t), t)\Phi'(x-R-\sigma t)\, dx - B_\mathfrak{c} e^{-2\nu_\mathfrak{c}|R+\sigma t|}. \tag{2-28}$$

In particular, we have

$$I_R(t_1) \geq I_R(t_0) - B_c e^{-2\nu_\mathfrak{c}|R|} \tag{2-29}$$

for any real numbers $t_0 \leq t_1$.

For the limit profile (v^*, w^*), we set $I_R^*(t) := I_R^{(v^*,w^*)}(t)$ for any $R \in \mathbb{R}$ and any $t \in \mathbb{R}$.

Proposition 2.5 [Béthuel et al. 2015]. *Given any positive number δ, there exists a positive number R_δ, depending only on δ, such that we have*

$$|I_R^*(t)| \leq \delta \quad \forall R \geq R_\delta,$$
$$|I_R^*(t) - P(v^*, w^*)| \leq \delta \quad \forall R \leq -R_\delta$$

for any $t \in \mathbb{R}$.

The proof of Proposition 2.5 is the same as that of Proposition 3 in [Béthuel et al. 2015].

From Propositions 2.4 and 2.5, as in [Béthuel et al. 2015] we derive:

Proposition 2.6 [Béthuel et al. 2015]. *Let $t \in \mathbb{R}$. There exists a positive constant $\mathcal{A}_\mathfrak{c}$ such that*

$$\int_t^{t+1} \int_\mathbb{R} \big[(\partial_x v^*)^2 + (v^*)^2 + (w^*)^2\big](x+a^*(s), s) e^{2\nu_\mathfrak{c}|x|}\, dx\, ds \leq \mathcal{A}_\mathfrak{c}.$$

We next consider the following map, which was introduced by de Laire and Gravejat [2015]:

$$\Psi := \frac{1}{2}\Big(\frac{\partial_x v}{(1-v^2)^{1/2}} + i(1-v^2)^{1/2} w\Big) \exp i\theta, \tag{2-30}$$

where

$$\theta(x,t) := -\int_{-\infty}^x v(y,t)w(y,t)\, dy. \tag{2-31}$$

The map Ψ solves the nonlinear Schrödinger equation

$$i\partial_t \Psi + \partial_{xx}\Psi + 2|\Psi|^2\Psi + \tfrac{1}{2}v^2\Psi - \mathrm{Re}\big(\Psi(1-2F(v,\overline{\Psi}))\big)(1-2F(v,\Psi)) = 0, \tag{2-32}$$

with

$$F(v,\Psi)(x,t) := \int_{-\infty}^x v(y,t)\Psi(y,t)\, dy, \tag{2-33}$$

while the function v satisfies the two equations

$$\begin{cases} \partial_t v = 2\partial_x \,\mathrm{Im}\big(\Psi(2F(v,\overline{\Psi}) - 1)\big), \\ \partial_x v = 2\,\mathrm{Re}\big(\Psi(1-2F(v,\overline{\Psi}))\big). \end{cases} \tag{2-34}$$

The local Cauchy problem for (2-32)–(2-34) was analyzed by de Laire and Gravejat [2015]. We recall the following proposition which shows the continuous dependence with respect to the initial datum of the solutions to the system of equations (2-32)–(2-34) (see [de Laire and Gravejat 2015] for the proof).

Proposition 2.7 [de Laire and Gravejat 2015]. *Suppose that the pairs* $(v^0, \Psi^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ *and* $(\tilde{v}^0, \widetilde{\Psi}^0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ *are such that*

$$\partial_x v^0 = 2\,\mathrm{Re}\big(\Psi^0\big(1-2F\big(v^0, \overline{\Psi^0}\big)\big)\big) \quad \textit{and} \quad \partial_x \tilde{v}^0 = 2\,\mathrm{Re}\big(\widetilde{\Psi}^0\big(1-2F\big(\tilde{v}^0, \overline{\widetilde{\Psi}^0}\big)\big)\big).$$

Given two solutions (v, Ψ) *and* $(\tilde{v}, \widetilde{\Psi})$ *in* $\mathcal{C}^0([0, T_*], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$, *with* $(\Psi, \widetilde{\Psi}) \in L^4([0, T_*], L^\infty(\mathbb{R}))^2$, *to* (2-32)–(2-34) *with initial data* (v^0, Ψ^0) *and* $(\tilde{v}^0, \widetilde{\Psi}^0)$ *respectively, for some positive time* T_*, *there exist a positive number* τ, *depending only on* $\|v^0\|_{L^2}$, $\|\tilde{v}^0\|_{L^2}$, $\|\Psi^0\|_{L^2}$ *and* $\|\widetilde{\Psi}^0\|_{L^2}$, *and a universal constant* A *such that we have*

$$\begin{aligned}\|v-\tilde{v}\|_{\mathcal{C}^0([0,T],L^2)} + \|\Psi-\widetilde{\Psi}\|_{\mathcal{C}^0([0,T],L^2)} + \|\Psi-\widetilde{\Psi}\|_{L^4([0,T],L^\infty)}& \\ \le A\big(\|v^0-\tilde{v}^0\|_{L^2} + \|\Psi^0-\widetilde{\Psi}^0\|_{L^2}\big)&\end{aligned} \tag{2-35}$$

for any $T \in [0, \min\{\tau, T_*\}]$. *In addition, there exists a positive number* B, *depending only on* $\|v^0\|_{L^2}$, $\|\tilde{v}^0\|_{L^2}$, $\|\Psi^0\|_{L^2}$ *and* $\|\widetilde{\Psi}^0\|_{L^2}$, *such that*

$$\|\partial_x v - \partial_x \tilde{v}\|_{\mathcal{C}^0([0,T],L^2)} \le B\big(\|v^0-\tilde{v}^0\|_{L^2} + \|\Psi^0 - \tilde{\Psi}^0\|_{L^2}\big) \tag{2-36}$$

for any $T \in [0, \min\{\tau, T_*\}]$.

This proposition plays an important role in the proof of not only the smoothing of the limit profile, but also the weak continuity of the hydrodynamical Landau–Lifshitz flow.

In order to prove the smoothness of the limit profile, we rely on the following smoothing type estimate for localized solutions of the linear Schrödinger equation (see [Béthuel et al. 2015; Escauriaza et al. 2008] for the proof of Proposition 2.8).

Proposition 2.8 [Béthuel et al. 2015; Escauriaza et al. 2008]. *Let* $\lambda \in \mathbb{R}$, *and consider a solution* $u \in \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}))$ *to the linear Schrödinger equation*

$$i\partial_t u + \partial_{xx} u = F, \tag{LS}$$

with $F \in L^2(\mathbb{R}, L^2(\mathbb{R}))$. *Then there exists a positive constant* K_λ, *depending only on* λ, *such that*

$$\lambda^2 \int_{-T}^{T} \int_{\mathbb{R}} |\partial_x u(x,t)|^2 e^{\lambda x}\,dx\,dt \le K_\lambda \int_{-T-1}^{T+1} \int_{\mathbb{R}} \big(|u(x,t)|^2 + |F(x,t)|^2\big) e^{\lambda x}\,dx\,dt \tag{2-37}$$

for any positive number T.

We apply Proposition 2.8 to Ψ^* as well as all its derivatives, where Ψ^* is the solution to (2-32) associated to the solution (v^*, w^*) of (HLL), and then express the result in terms of (v^*, w^*) to obtain:

Proposition 2.9. *The pair* (v^*, w^*) *is indefinitely smooth and exponentially decaying on* $\mathbb{R}\times\mathbb{R}$. *Moreover, given any* $k \in \mathbb{N}$, *there exists a positive constant* $A_{k,\mathfrak{c}}$, *depending only on* k *and* $\mathfrak{c}$, *such that*

$$\int_{\mathbb{R}} \big[(\partial_x^{k+1} v^*)^2 + (\partial_x^k v^*)^2 + (\partial_x^k w^*)^2\big](x + a^*(t), t) e^{\nu_{\mathfrak{c}}|x|}\,dx \le A_{k,\mathfrak{c}} \tag{2-38}$$

for any $t \in \mathbb{R}$.

The Liouville type theorem. We next establish a Liouville type theorem, which guarantees that the limit profile constructed above is exactly a soliton. In particular, we will show that $\varepsilon_0^* \equiv 0$.

The pair ε^* satisfies the equation

$$\partial_t \varepsilon^* = J\mathcal{H}_{c^*(t)}(\varepsilon^*) + J\mathcal{R}_{c^*(t)}\varepsilon^* + (a^{*\prime}(t) - c^*(t))(\partial_x Q_{c^*(t)} + \partial_x \varepsilon^*) - c^{*\prime}(t)\partial_c Q_{c^*(t)}, \tag{2-39}$$

where J is the symplectic operator

$$J = -2S\partial_x := \begin{pmatrix} 0 & -2\partial_x \\ -2\partial_x & 0 \end{pmatrix}, \tag{2-40}$$

and the remainder term $\mathcal{R}_{c^*(t)}\varepsilon^*$ is given by

$$\mathcal{R}_{c^*(t)}\varepsilon^* := E'(Q_{c^*(t)} + \varepsilon^*) - E'(Q_{c^*(t)}) - E''(Q_{c^*(t)})(\varepsilon^*).$$

We rely on the strategy developed by Martel and Merle [2008a] (see also [Martel 2006]), and then applied by Béthuel, Gravejat and Smets in [Béthuel et al. 2015] to the Gross–Pitaevskii equation. We define the pair

$$u^*(\,\cdot\,, t) := S\mathcal{H}_{c^*(t)}(\varepsilon^*(\,\cdot\,, t)). \tag{2-41}$$

Since $S\mathcal{H}_{c^*(t)}(\partial_x Q_{c^*(t)}) = 0$, we deduce from (2-39) that

$$\begin{aligned}\partial_t u^* = \mathcal{H}_{c^*(t)}(JSu^*) + S\mathcal{H}_{c^*(t)}(J\mathcal{R}_{c^*(t)}\varepsilon^*) - (c^*)'(t)S\mathcal{H}_{c^*(t)}(\partial_c Q_{c^*(t)}) \\ + (c^*)'(t)S\partial_c\mathcal{H}_{c^*(t)}(\varepsilon^*) + ((a^*)'(t) - c^*(t))S\mathcal{H}_{c^*(t)}(\partial_x \varepsilon^*).\end{aligned} \tag{2-42}$$

Decreasing further the value of $\beta_{\mathfrak{c}}$ if necessary, we have:

Proposition 2.10. *There exist two positive numbers A_* and R_*, depending only on $\mathfrak{c}$, such that we have*[2]

$$\frac{d}{dt}\left(\int_{\mathbb{R}} x u_1^*(x,t) u_2^*(x,t)\,dx\right) \geq \frac{1-\mathfrak{c}^2}{16}\|u^*(\,\cdot\,,t)\|^2_{X(\mathbb{R})} - A_*\|u^*(\,\cdot\,,t)\|^2_{X(B(0,R_*))} \tag{2-43}$$

for any $t \in \mathbb{R}$.

We give a second monotonicity type formula to dispose of the nonpositive local term $\|u^*(\cdot,t)\|^2_{X(B(0,R_*))}$ on the right-hand side of (2-43). If M is a smooth, bounded, two-by-two symmetric matrix-valued function, then

$$\frac{d}{dt}\langle Mu^*, u^*\rangle_{L^2(\mathbb{R})^2} = 2\langle SMu^*, \mathcal{H}_{c^*}(-2\partial_x u^*)\rangle_{L^2(\mathbb{R})^2} + \text{“superquadratic terms”}, \tag{2-44}$$

where S is the matrix

$$S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

[2] In (2-43), we use the notation

$$\|(f,g)\|^2_{X(\Omega)} := \int_\Omega ((\partial_x f)^2 + f^2 + g^2),$$

in which Ω denotes a measurable subset of $\mathbb{R}$.

For $c \in (-1, 1) \setminus \{0\}$, let M_c be given by

$$M_c := \begin{pmatrix} -\dfrac{2cv_c\partial_x v_c}{(1-v_c)^2} & -\dfrac{\partial_x v_c}{v_c} \\ -\dfrac{\partial_x v_c}{v_c} & 0 \end{pmatrix}. \tag{2-45}$$

We have the following lemma.

Lemma 2.11. *Let $c \in (-1, 1) \setminus \{0\}$ and $u \in X^3(\mathbb{R})$. Then*

$$\begin{aligned} G_c(u) &:= 2\langle SM_c u, \mathcal{H}_c(-2\partial_x u)\rangle_{L^2(\mathbb{R})^2} \\ &= 2\int_{\mathbb{R}} \mu_c\Big(u_2 - \frac{cv_c^2}{\mu_c}u_1 - \frac{2cv_c\partial_x v_c}{\mu_c(1-v_c^2)}\partial_x u_1\Big)^2 + 3\int_{\mathbb{R}} \frac{v_c^4}{\mu_c}\Big(\partial_x u_1 - \frac{\partial_x v_c}{v_c}u_1\Big)^2, \end{aligned} \tag{2-46}$$

where

$$\mu_c = 2(\partial_x v_c)^2 + v_c^2(1-v_c^2) > 0. \tag{2-47}$$

The functional G_c is a nonnegative quadratic form, and

$$\mathrm{Ker}(G_c) = \mathrm{Span}(Q_c). \tag{2-48}$$

We have indeed chosen the matrix M_c such that $M_c Q_c = \partial_x Q_c$ to obtain (2-48). Since Q_c does not vanish, we deduce from standard Sturm–Liouville theory that G_c is nonnegative, which is confirmed by the computation in Lemma 2.11.

By the second orthogonality condition in (2-20) and the fact that $\mathcal{H}_{c^*}(\chi_{c^*}) = -\tilde{\lambda}_{c^*}\chi_{c^*}$, we have

$$0 = \langle \mathcal{H}_{c^*}(\chi_{c^*}), \varepsilon^*\rangle_{L^2(\mathbb{R})^2} = \langle \mathcal{H}_{c^*}(\varepsilon^*), \chi_{c^*}\rangle_{L^2(\mathbb{R})^2} = \langle u^*, S\chi_{c^*}\rangle_{L^2(\mathbb{R})^2}. \tag{2-49}$$

On the other hand, we know that

$$\langle Q_{c^*}, S\chi_{c^*}\rangle = P'(Q_{c^*})(\chi_{c^*}) \neq 0, \tag{2-50}$$

so that the pair u^* is not proportional to Q_{c^*} under the orthogonality condition in (2-49). We claim the following coercivity property of G_c under this orthogonality condition.

Proposition 2.12. *Let $c \in (-1, 1) \setminus \{0\}$. There exists a positive number Λ_c, depending only and continuously on c, such that*

$$G_c(u) \geq \Lambda_c \int_{\mathbb{R}} [(\partial_x u_1)^2 + (u_1)^2 + (u_2)^2](x)e^{-2|x|}\,dx, \tag{2-51}$$

for any pair $u \in X(\mathbb{R})$ verifying

$$\langle u, S\chi_c\rangle_{L^2(\mathbb{R})^2} = 0. \tag{2-52}$$

Coming back to (2-44), we can prove the next proposition.

Proposition 2.13. *There exists a positive number B_*, depending only on $\mathfrak{c}$, such that*

$$\frac{d}{dt}\big(\langle M_{c^*(t)}u^*(\,\cdot\,,t),u^*(\,\cdot\,,t)\rangle_{L^2(\mathbb{R})^2}\big)\geq\frac{1}{B_*}\int_{\mathbb{R}}[(\partial_x u_1^*)^2+(u_1^*)^2+(u_2^*)^2](x,t)e^{-2|x|}\,dx\\ -B_*\|\varepsilon^*(\,\cdot\,,t)\|_{X(\mathbb{R})}^{1/2}\|u^*(\,\cdot\,,t)\|_{X(\mathbb{R})}^2 \quad (2\text{-}53)$$

for any $t\in\mathbb{R}$.

Propositions 2.10 and 2.13 have the following corollary.

Corollary 2.14. *Set*

$$N(t):=\frac{1}{2}\begin{pmatrix}0 & x\\ x & 0\end{pmatrix}+A_*B_*e^{2R_*}M_{c^*(t)}.$$

There exists a positive constant $\mathcal{A}_{\mathfrak{c}}$ such that we have

$$\frac{d}{dt}\big(\langle N(t)u^*(\,\cdot\,,t),u^*(\,\cdot\,,t)\rangle_{L^2(\mathbb{R})^2}\big)\geq\mathcal{A}_{\mathfrak{c}}\|u^*(\,\cdot\,,t)\|_{X(\mathbb{R})}^2 \quad (2\text{-}54)$$

for any $t\in\mathbb{R}$. Since

$$\int_{-\infty}^{+\infty}\|u^*(\,\cdot\,,t)\|_{X(\mathbb{R})}^2\,dt<+\infty, \quad (2\text{-}55)$$

there exists a sequence $(t_k^*)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty}\|u^*(\,\cdot\,,t_k^*)\|_{X(\mathbb{R})}^2=0. \quad (2\text{-}56)$$

In view of (2-20), (2-41) and the bound for $\mathcal{H}_{c^*}$ in (A-43), we have

$$\|\varepsilon^*(\,\cdot\,,t)\|_{X(\mathbb{R})}\leq A_{\mathfrak{c}}\|u^*(\,\cdot\,,t)\|_{X(\mathbb{R})}, \quad (2\text{-}57)$$

Hence, we can apply (2-56) and (2-57) in order to obtain

$$\lim_{k\to+\infty}\|\varepsilon^*(\,\cdot\,,t_k^*)\|_{X(\mathbb{R})}^2=0. \quad (2\text{-}58)$$

By (2-58) and the orbital stability in Theorem 2.1, this yields:

Corollary 2.15. $$\varepsilon_0^*\equiv 0.$$

At this stage we obtain (2-11) for some subsequence. We should extend this result for any sequence. The proof is exactly the same as the one done by Béthuel, Gravejat and Smets in [Béthuel et al. 2015] (see Subsection 1.3.4 in [Béthuel et al. 2015] for the details).

Proof of Theorem 1.1. We choose a positive number $\delta_{\mathfrak{c}}$ such that $\|(v_0,w_0)-Q_{\mathfrak{c}}\|_{X(\mathbb{R})}\leq\beta_{\mathfrak{c}}$, whenever $d_{\mathcal{E}}(m^0,u_{\mathfrak{c}})\leq\delta_{\mathfrak{c}}$. We next apply Theorem 2.2 to the solution $(v,w)\in\mathcal{C}^0(\mathbb{R},\mathcal{NV}(\mathbb{R}))$ to (HLL) corresponding to the solution m to (LL). This yields the existence of a speed $\mathfrak{c}^*$ and a position function b such that the convergences in Theorem 2.2 hold. In particular, since the weak convergence for m_3 is satisfied by Theorem 2.2, it is sufficient to show the existence of a phase function θ such that $\exp(i\theta(t))\partial_x\check{m}(\,\cdot+b(t),t)$ is weakly convergent to $\partial_x\check{u}_{\mathfrak{c}^*}$ in $L^2(\mathbb{R})$ as $t\to\infty$. The locally uniform convergence of $\exp(i\theta(t))\check{m}(\cdot+b(t),t)$ towards $\check{u}_{\mathfrak{c}^*}$ then follows from the Sobolev embedding theorem. We begin by constructing this phase function.

We fix a nonzero function $\chi\in\mathcal{C}_c^\infty(\mathbb{R},[0,1])$ such that χ is even. Using the explicit formula of $\check{u}_{\mathfrak{c}^*}$, we have

$$\int_{\mathbb{R}}\check{u}_{\mathfrak{c}^*}(x)\chi(x)\,dx=2\mathfrak{c}^*\int_{\mathbb{R}}\frac{\chi(x)}{\cosh\bigl(\sqrt{1-(\mathfrak{c}^*)^2}x\bigr)}\,dx\neq 0. \tag{2-59}$$

Decreasing the value of $\beta_{\mathfrak{c}}$ if needed, we deduce from the orbital stability in [de Laire and Gravejat 2015] that

$$\left|\int_{\mathbb{R}}\check{m}(x+b(t),t)\chi(x)\,dx\right|\geq|\mathfrak{c}^*|\int_{\mathbb{R}}\frac{\chi(x)}{\cosh\bigl(\sqrt{1-(\mathfrak{c}^*)^2}x\bigr)}\,dx\neq 0 \tag{2-60}$$

for any $t\in\mathbb{R}$.

Let $\Upsilon:\mathbb{R}^2\to\mathbb{R}$ be the $\mathcal{C}^1$ function defined by

$$\Upsilon(t,\theta):=\operatorname{Im}\biggl(e^{-i\theta}\int_{\mathbb{R}}\check{m}(x+b(t),t)\chi(x)\,dx\biggr).$$

From (2-60) we can find a number θ_0 such that $\Upsilon(0,\theta_0)=0$ and $\partial_\theta\Upsilon(0,\theta_0)>0$. Then, using the implicit function theorem, there exists a $\mathcal{C}^1$ function $\theta:\mathbb{R}\to\mathbb{R}$ such that $\Upsilon(t,\theta(t))=0$. In addition, using (2-60) another time, we can fix the choice of θ so that there exists a positive constant $A_{\mathfrak{c}^*}$ such that

$$\partial_\theta\Upsilon(t,\theta(t))=\operatorname{Re}\biggl(e^{-i\theta(t)}\int_{\mathbb{R}}\check{m}(x+b(t),t)\chi(x)\,dx\biggr)\geq A_{\mathfrak{c}^*}>0. \tag{2-61}$$

Differentiating the identity $\Upsilon(t,\theta(t))=0$ with respect to t, this implies that

$$|\theta'(t)|=\left|\frac{\partial_t\Upsilon(t,\theta(t))}{\partial_\theta\Upsilon(t,\theta(t))}\right|\leq\frac{1}{A_{\mathfrak{c}^*}}|\partial_t\Upsilon(t,\theta(t))| \tag{2-62}$$

for all $t\in\mathbb{R}$. Now, we differentiate the function Υ with respect to t, and we use the equation of $\check{m}$ to obtain

$$\begin{aligned}\partial_t\Upsilon(t,\theta(t))=\operatorname{Im}\biggl(e^{-i\theta}\int_{\mathbb{R}}\chi(x)\bigl(&\partial_x\check{m}(x+b(t),t)b'(t)-im_3(x+b(t),t)\partial_{xx}\check{m}(x+b(t),t)\\&+i\check{m}(x+b(t),t)\partial_{xx}m_3(x+b(t),t)-im_3(x+b(t),t)\check{m}(x+b(t),t)\bigr)\,dx\biggr).\end{aligned} \tag{2-63}$$

Since $b\in\mathcal{C}_b^1(\mathbb{R},\mathbb{R})$, and since both $\partial_x\check{m}$ and $\partial_t\check{m}$ belong to $\mathcal{C}_b^0(\mathbb{R},H^{-1}(\mathbb{R}))$, it follows that the derivative θ' is bounded on $\mathbb{R}$.

We denote by φ the phase function defined by

$$\varphi(x+b(t),t):=\varphi(b(t),t)+\int_0^x w(y+b(t),t)\,dy,$$

with $\varphi(b(t),t)\in[0,2\pi]$, which is associated to the function $\check{m}(x+b(t),t)$ for any $(x,t)\in\mathbb{R}^2$ in the way that

$$\check{m}(x+b(t),t)=(1-m_3^2(x+b(t),t))^{1/2}\exp(i\varphi(x+b(t),t)).$$

It is sufficient to prove that

$$\exp\bigl(i(\varphi(b(t),t)-\theta(t))\bigr)\to 1 \tag{2-64}$$

as $t\to\infty$ to obtain

$$\exp\bigl(i(\varphi(\cdot+b(t),t)-\theta(t))\bigr)\to\exp(i\varphi_{\mathfrak{c}^*}(\cdot)):=\exp\Bigl(i\int_0^{\cdot}w_{\mathfrak{c}^*}(y)\,dy\Bigr)\quad\text{in } L^\infty_{\mathrm{loc}}(\mathbb{R})$$

as $t\to\infty$. This implies, using Theorem 2.2 once again as well as the Sobolev embedding theorem, that

$$\begin{aligned} e^{-i\theta(t)}\partial_x\check{m}(\cdot+b(t),t)&\rightharpoonup\partial_x\check{u}_{\mathfrak{c}^*}\quad\text{in } L^2(\mathbb{R}),\\ e^{-i\theta(t)}\check{m}(\cdot+b(t),t)&\to\check{u}_{\mathfrak{c}^*}\qquad\text{in } L^\infty_{\mathrm{loc}}(\mathbb{R})\end{aligned} \tag{2-65}$$

as $t\to\infty$. Now let us prove (2-64). We have

$$\begin{aligned} &e^{-i\theta(t)}\int_{\mathbb{R}}\check{m}(x+b(t),t)\chi(x)\,dx\\ &\qquad=\exp\bigl(i[\varphi(b(t),t)-\theta(t)]\bigr)\int_{\mathbb{R}}(1-m_3^2(x+b(t),t))^{1/2}\exp\Bigl(i\int_0^x w(y+b(t),t)\,dy\Bigr)\chi(x)\,dx.\end{aligned}$$

We use the fact that $\Upsilon(t,\theta(t))=0$ to obtain

$$\begin{aligned} &\cos(\varphi(b(t),t)-\theta(t))\,\mathrm{Im}\Bigl(\int_{\mathbb{R}}(1-m_3^2(x+b(t),t))^{1/2}\exp\Bigl(i\int_0^x w(y+b(t),t)\,dy\Bigr)\chi(x)\,dx\Bigr)\\ &\quad+\sin(\varphi(b(t),t)-\theta(t))\,\mathrm{Re}\Bigl(\int_{\mathbb{R}}(1-m_3^2(x+b(t),t))^{1/2}\exp\Bigl(i\int_0^x w(y+b(t),t)\,dy\Bigr)\chi(x)\,dx\Bigr)=0.\end{aligned}$$

On the other hand, by (2-61), we have

$$\begin{aligned} &\cos(\varphi(b(t),t)-\theta(t))\,\mathrm{Re}\Bigl(\int_{\mathbb{R}}(1-m_3^2(x+b(t),t))^{1/2}\exp\Bigl(i\int_0^x w(y+b(t),t)\,dy\Bigr)\chi(x)\,dx\Bigr)\\ &\quad-\sin(\varphi(b(t),t)-\theta(t))\,\mathrm{Im}\Bigl(\int_{\mathbb{R}}(1-m_3^2(x+b(t),t))^{1/2}\exp\Bigl(i\int_0^x w(y+b(t),t)\,dy\Bigr)\chi(x)\,dx\Bigr)>0.\end{aligned}$$

We derive from Theorem 2.2 and (2-59) that

$$\mathrm{Im}\Bigl(\int_{\mathbb{R}}(1-m_3^2(x+b(t),t))^{1/2}\exp\Bigl(i\int_0^x w(y+b(t),t)\,dy\Bigr)\chi(x)\,dx\Bigr)\to\mathrm{Im}\Bigl(\int_{\mathbb{R}}\check{u}_{\mathfrak{c}^*}(x)\chi(x)\,dx\Bigr)=0,$$

and

$$\mathrm{Re}\Bigl(\int_{\mathbb{R}}(1-m_3^2(x+b(t),t))^{1/2}\exp\Bigl(i\int_0^x w(y+b(t),t)\,dy\Bigr)\chi(x)\,dx\Bigr)\to\mathrm{Re}\Bigl(\int_{\mathbb{R}}\check{u}_{\mathfrak{c}^*}(x)\chi(x)\,dx\Bigr)>0.$$

This is enough to derive (2-64).

Finally, we claim that $\theta'(t)\to 0$ as $t\to\infty$. Indeed, we can introduce (2-65) into (2-63), and we then obtain, using the equation satisfied by $\check{u}_{\mathfrak{c}^*}$, that

$$\partial_t\Upsilon(t,\theta(t))\to 0$$

as $t\to\infty$. By (2-62), this yields $\theta'(t)\to 0$ as $t\to\infty$, which finishes the proof of Theorem 1.1. □

3. Proof of the orbital stability

First, we recall the orbital stability theorem, which was established in [de Laire and Gravejat 2015] (see Corollary 2 and Propositions 2 and 4 in [de Laire and Gravejat 2015]).

Theorem 3.1. *Let $c \in (-1,1)\setminus\{0\}$ and $(v_0, w_0) \in X(\mathbb{R})$ satisfying* (2-4). *There exist a unique global solution $(v,w) \in \mathcal{C}^0(\mathbb{R}, \mathcal{NV}(\mathbb{R}))$ to* (HLL) *with initial datum (v_0, w_0), and two maps $c_1 \in \mathcal{C}^1(\mathbb{R}, (-1,1)\setminus\{0\})$ and $a_1 \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ such that the function ε_1, defined by* (2-5), *satisfies the orthogonality conditions*

$$\langle \varepsilon_1(\cdot\,,t), \partial_x Q_{c_1(t)}\rangle_{L^2(\mathbb{R})^2} = P'(Q_{c_1(t)})(\varepsilon_1(\cdot\,,t)) = 0 \tag{3-1}$$

for any $t \in \mathbb{R}$. Moreover, $\varepsilon_1(\cdot\,,t)$, $c_1(t)$ and $a_1(t)$ satisfy (2-7), (2-8) *and* (2-9) *for any $t \in \mathbb{R}$.*

With Theorem 3.1 at hand, we can provide the proof of Theorem 2.1.

Proof of Theorem 2.1. We consider the map

$$\Xi((v,w),\sigma,\mathfrak{b}) := \big(\langle \partial_x Q_{\sigma,\mathfrak{b}}, \varepsilon\rangle_{L^2\times L^2}, \langle \chi_{\sigma,\mathfrak{b}}, \varepsilon\rangle_{L^2\times L^2}\big),$$

where we have set $\varepsilon = (v,w) - Q_{\sigma,\mathfrak{b}}$, and $\chi_{\sigma,\mathfrak{b}} = \chi_\sigma(\cdot - \mathfrak{b})$ (we recall that χ_σ is the eigenfunction associated to the unique negative eigenvalue $-\tilde{\lambda}_\sigma$ of the operator $\mathcal{H}_\sigma$). The map Ξ is well-defined for, and depends smoothly on, $(v,w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, $\sigma \in (-1,1)\setminus\{0\}$ and $\mathfrak{b} \in \mathbb{R}$.

We fix $t \in \mathbb{R}$. In order to simplify the notation, we substitute $(c_1(t), a_1(t))$ by (c_1, a_1). We check that

$$\Xi(Q_{c_1,a_1}, c_1, a_1) = 0,$$

and we compute

$$\begin{cases} \partial_\sigma \Xi_1(Q_{c_1,a_1}, c_1, a_1) = 0, \\ \partial_\sigma \Xi_2(Q_{c_1,a_1}, c_1, a_1) = -\langle \chi_{c_1,a_1}, \partial_\sigma Q_{c_1,a_1}\rangle_{L^2\times L^2}. \end{cases}$$

Let $c \in (-1,1)\setminus\{0\}$ and suppose towards a contradiction that

$$\langle \chi_c, \partial_c Q_c\rangle_{L^2\times L^2} = 0.$$

Using the fact that $\mathcal{H}_c(\partial_c Q_c) = P'(Q_c)$, this gives

$$0 = \langle \chi_c, \partial_c Q_c\rangle_{L^2\times L^2} = -\frac{1}{\tilde{\lambda}_c}\langle \chi_c, \mathcal{H}_c(\partial_c Q_c)\rangle_{L^2\times L^2} = -\frac{1}{\tilde{\lambda}_c}\langle \chi_c, P'(Q_c)\rangle_{L^2\times L^2}.$$

Since $\mathcal{H}_c$ is self-adjoint, we also have

$$\langle \chi_c, \partial_x Q_c\rangle_{L^2\times L^2} = 0.$$

By Proposition 1 in [de Laire and Gravejat 2015], we infer that

$$0 > -\tilde{\lambda}_c \|\chi_c\|^2_{L^2\times L^2} = \langle \chi_c, \mathcal{H}_c(\chi_c)\rangle_{L^2\times L^2} \geq \Lambda_c \|\chi_c\|^2_{L^2\times L^2} > 0,$$

which provides the contradiction and shows that

$$\langle \chi_c, \partial_c Q_c\rangle_{L^2\times L^2} \neq 0 \tag{3-2}$$

for all $c \in (-1, 1) \setminus \{0\}$. In addition, we have

$$\begin{cases} \partial_b \Xi_1(Q_{c_1,a_1}, c_1, a_1) = \|\partial_x Q_{c_1}\|_{L^2}^2 = 2(1-c_1^2)^{1/2} > 0, \\ \partial_b \Xi_2(Q_{c_1,a_1}, c_1, a_1) = 0. \end{cases}$$

Therefore, the matrix

$$d_{\sigma,b}\Xi(Q_{c_1,a_1}, c_1, a_1) = \begin{pmatrix} 0 & \langle \chi_{c_1,a_1}, \partial_\sigma Q_{c_1,a_1}\rangle_{L^2\times L^2} \\ 2(1-c_1^2)^{1/2} & 0 \end{pmatrix}$$

is an isomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Then, we can apply the version of the implicit function theorem in [Béthuel et al. 2014] in order to find a neighbourhood $\mathcal{V}$ of Q_{c_1,a_1}, a neighbourhood $\mathcal{U}$ of (c_1, a_1), and a map $\gamma_{c_1,a_1} : \mathcal{U} \to \mathcal{V}$ such that

$$\Xi((v, w), \sigma, \mathfrak{b}) = 0 \Leftrightarrow (c(v, w), a(v, w)) := (\sigma, \mathfrak{b}) = \gamma_{c,a}(v, w) \quad \forall (v, w) \in \mathcal{V},\ \forall (\sigma, \mathfrak{b}) \in \mathcal{U}.$$

In addition, there exists a positive constant Λ, depending only on c_1, such that

$$\|\varepsilon(t)\|_X + |c(t) - c_1(t)| + |a(t) - a_1(t)| \le \Lambda \|\varepsilon_1(t)\|_X \le \Lambda_{c_1} A_{\mathfrak{c}} \alpha_0, \tag{3-3}$$

where $c(t) := c(v(t), w(t))$, $a(t) := a(v(t), w(t))$ and $\varepsilon(t) := (v(t), w(t)) - Q_{c(t),a(t)}$, for any fixed $t \in \mathbb{R}$. Using the fact that $(v(t), w(t))$ stays in a neighbourhood of $Q_{c_1(t),a_1(t)}$ for all $t \in \mathbb{R}$ by Theorem 3.1, and also the fact that c_1 satisfies (2-8), we are led to the following lemma.

Lemma 3.2. *Under the assumptions of Theorem 3.1, there exists a unique pair (a, c) of functions in $\mathcal{C}^0(\mathbb{R}, \mathbb{R}^2)$ such that*

$$\varepsilon(t) := (v(t), w(t)) - Q_{c(t),a(t)}$$

satisfies the orthogonality conditions

$$\langle \varepsilon(t), \partial_x Q_{c(t),a(t)}\rangle_{L^2\times L^2} = \langle \chi_{c(t),a(t)}, \varepsilon(t)\rangle_{L^2\times L^2} = 0. \tag{3-4}$$

Moreover, we have (2-8).

This completes the proof of orbital stability. Now, let us prove the continuous differentiability of the functions a and c, as well as the inequality

$$|c'(t)| + |a'(t) - c(t)| \le A_{\mathfrak{c}} \|\varepsilon(\cdot, t)\|_{X(\mathbb{R})}, \tag{3-5}$$

for all $t \in \mathbb{R}$. The $\mathcal{C}^1$ nature of a and c can be derived from a standard density argument as in [de Laire and Gravejat 2015]. Concerning (3-5), we can write the equations satisfied by ε, namely

$$\partial_t \varepsilon_v = \big((a'(t) - c(t))\partial_x v_{c,a} - c'(t)\partial_c v_{c,a}\big) + \partial_x\big(((v_{c,a} + \varepsilon_v)^2 - 1)(v_{c,a} + \varepsilon_w) - (v_{c,a}^2 - 1)w_{c,a}\big) \tag{3-6}$$

and

$$\partial_t \varepsilon_w = (a'(t) - c(t))\partial_x w_{c,a} - c'(t)\partial_c w_{c,a} \\ + \partial_x \left(\frac{\partial_{xx} v_{c,a} + \partial_{xx}\varepsilon_v}{1 - (v_{c,a} + \varepsilon_v)^2} + (v_{c,a} + \varepsilon_v)\frac{(\partial_x v_{c,a} + \partial_x \varepsilon_v)^2}{(1 - (v_{c,a} + \varepsilon_v)^2)^2} - \frac{\partial_{xx} v_{c,a}}{1 - v_{c,a}^2} - v_{c,a}\frac{(\partial_x v_{c,a})^2}{(1 - v_{c,a}^2)^2} \right) \\ + \partial_x \big((v_{c,a} + \varepsilon_v)((w_{c,a} + \varepsilon_w)^2 - 1) - v_{c,a}(w_{c,a}^2 - 1) \big). \quad (3\text{-}7)$$

We differentiate with respect to time the orthogonality conditions in (2-6) and we invoke equations (3-6) and (3-7) to write the identity

$$M \begin{pmatrix} c' \\ a' - c \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix}. \quad (3\text{-}8)$$

Here, M refers to the matrix of size 2 given by

$$M_{1,1} = \langle \partial_c Q_c, \chi_c \rangle_{L^2 \times L^2} + \langle \partial_c \chi_{c,a}, \varepsilon \rangle_{L^2 \times L^2},$$
$$M_{1,2} = \langle \chi_c, \partial_x Q_c \rangle_{L^2 \times L^2} - \langle \partial_x \chi_{c,a}, \varepsilon \rangle_{L^2 \times L^2},$$
$$M_{2,1} = -\langle \partial_x Q_c, \partial_c Q_c \rangle_{L^2 \times L^2} + \langle \partial_c \partial_x Q_{c,a}, \varepsilon \rangle_{L^2 \times L^2},$$
$$M_{2,2} = \|\partial_x Q_c\|_{L^2 \times L^2}^2 - \langle \partial_{xx} Q_{c,a}, \varepsilon \rangle_{L^2 \times L^2}.$$

The vectors Y and Z are defined by

$$Y = \big\langle \partial_x w_{c,a}, ((v_{c,a} + \varepsilon_v)^2 - 1)(w_{c,a} + \varepsilon_w) - (v_{c,a}^2 - 1)w_{c,a} \big\rangle_{L^2} \\ + \big\langle \partial_x v_{c,a}, ((w_{c,a} + \varepsilon_w)^2 - 1)(v_{c,a} + \varepsilon_v) - (w_{c,a}^2 - 1)v_{c,a} \big\rangle_{L^2} \\ - \left\langle \partial_{xx} v_{c,a}, \frac{\partial_{xx} v_{c,a} + \partial_{xx}\varepsilon_v}{1 - (v_{c,a} + \varepsilon_v)^2} - \frac{\partial_{xx} v_{c,a}}{1 - v_{c,a}^2} \right\rangle_{L^2} + c\langle \partial_x \chi_{c,a}, \varepsilon \rangle_{L^2 \times L^2}$$

and

$$Z = \big\langle \partial_{xx} v_{c,a}, ((v_{c,a} + \varepsilon_v)^2 - 1)(w_{c,a} + \varepsilon_w) - (v_{c,a}^2 - 1)w_{c,a} \big\rangle_{L^2} \\ + \big\langle \partial_{xx} w_{c,a}, ((w_{c,a} + \varepsilon_w)^2 - 1)(v_{c,a} + \varepsilon_v) - (w_{c,a}^2 - 1)v_{c,a} \big\rangle_{L^2} \\ - \left\langle \partial_{xxx} w_{c,a}, \frac{\partial_{xx} v_{c,a} + \partial_{xx}\varepsilon_v}{1 - (v_{c,a} + \varepsilon_v)^2} - \frac{\partial_{xx} v_{c,a}}{1 - v_{c,a}^2} \right\rangle_{L^2} + c\langle \partial_{xx} Q_{c,a}, \varepsilon \rangle_{L^2 \times L^2}.$$

We next decompose the matrix M as $M = D + H$, where D is the diagonal matrix of size 2 with diagonal coefficients

$$D_{1,1} = \langle \partial_c Q_c, \chi_c \rangle_{L^2 \times L^2} \neq 0,$$

by (3-2), and

$$D_{2,2} = \|\partial_x Q_{c(t)}\|_{L^2}^2 = 2(1 - c(t)^2)^{1/2},$$

so that D is invertible. Concerning the matrix H, we check that

$$\langle P'(Q_c), \partial_x Q_c \rangle_{L^2 \times L^2} = \langle \partial_x Q_c, \partial_c Q_c \rangle_{L^2 \times L^2} = 0.$$

Then,

$$H = \begin{pmatrix} \langle \partial_c \chi_{c,a}, \varepsilon \rangle_{L^2\times L^2} & -\langle \partial_x \chi_{c,a}, \varepsilon \rangle_{L^2\times L^2} \\ \langle \partial_c \partial_x Q_{c,a}, \varepsilon \rangle_{L^2\times L^2} & -\langle \partial_{xx} Q_{c,a}, \varepsilon \rangle_{L^2\times L^2} \end{pmatrix}.$$

It follows from the exponential decay of $Q_{c,a}$ and its derivatives that

$$|H| \leq A_{\mathfrak{c}} \|\varepsilon\|_{L^2\times L^2}.$$

We can make a further choice of the positive number α_c, such that the operator norm of the matrix $D^{-1}H$ is less than $1/2$. In this case, the matrix M is invertible and the operator norm of its inverse is uniformly bounded with respect to t. Coming back to (3-8), we are led to the estimate

$$|c'(t)| + |a'(t) - c(t)| \leq A_{\mathfrak{c}}(|Y(t)| + |Z(t)|). \tag{3-9}$$

It remains to estimate the quantities Y and Z. We write

$$\begin{aligned} &\left|\left\langle \partial_x w_{c,a}, ((v_{c,a}+\varepsilon_v)^2 - 1)(w_{c,a}+\varepsilon_w) - (v_{c,a}^2 - 1) w_{c,a} \right\rangle_{L^2}\right| \\ &\qquad\qquad = \left|\left\langle \partial_x w_{c,a}, (\varepsilon_v^2 + 2 v_{c,a}\varepsilon_v) w_{c,a} + \varepsilon_w((\varepsilon_v + v_{c,a})^2 - 1)\right\rangle_{L^2}\right| \\ &\qquad\qquad \leq A_{\mathfrak{c}} \|\varepsilon\|_{L^2\times L^2}. \end{aligned}$$

Arguing in the same way for the other terms in Y and Z, we obtain

$$|Y| + |Z| = \mathcal{O}(\|\varepsilon\|_{L^2\times L^2}),$$

which is enough to deduce (3-5) from (3-9).

To complete the proof, we show (2-7). Using the Sobolev embedding theorem of $H^1(\mathbb{R})$ into $\mathcal{C}^0(\mathbb{R})$, we can write

$$\max_{x\in\mathbb{R}} v(x,t) \leq \|v_{c(t)}\|_{L^\infty(\mathbb{R})} + \|v(\,\cdot\,,t) - v_{c(t),a(t)}\|_{L^\infty(\mathbb{R})} \leq \|v_{c(t)}\|_{L^\infty(\mathbb{R})} + \|\varepsilon(t)\|_{X(\mathbb{R})}.$$

By (2-3), $\|v_c\|_{L^\infty(\mathbb{R})} < 1$, so that (2-8) implies that there exists a small positive number γ_c such that $\|v_{c(t)}\|_{L^\infty(\mathbb{R})} \leq 1 - \gamma_c$. We obtain

$$\max_{x\in\mathbb{R}} v(x,t) \leq 1 - \gamma_c + \|\varepsilon(t)\|_{X(\mathbb{R})} \leq 1 - \gamma_c + \alpha_c.$$

For α_c small enough, the estimate (2-7) follows, with $\sigma_c := -\alpha_c + \gamma_c$. □

4. Proofs of localization and smoothness of the limit profile

Proof of Proposition 2.4. The proof relies on the conservation law for the density of momentum vw. Let R and t be two real numbers, and recall that

$$I_R(t) \equiv I_R^{(v,w)}(t) := \frac{1}{2}\int_{\mathbb{R}} [vw](x + a(t), t)\Phi(x - R)\,dx,$$

where Φ is the function defined on $\mathbb{R}$ by

$$\Phi(x) := \tfrac{1}{2}(1 + \operatorname{th}(\nu_{\mathfrak{c}} x)),$$

with $\nu_\mathfrak{c} := \sqrt{1-\mathfrak{c}^2}/8$. First, we deduce from the conservation law for vw (see Lemma 3.1 in [de Laire and Gravejat 2015] for more details) the identity

$$\frac{d}{dt}[I_{R+\sigma t}(t)] = -(a'(t)+\sigma)\int_{\mathbb{R}}[vw](x+a(t),t)\Phi'(x-R-\sigma t)\,dx + \int_{\mathbb{R}}\Big[v^2+w^2-3v^2w^2+\frac{3-v^2}{(1-v^2)^2}(\partial_x v)^2\Big](x+a(t),t)\Phi'(x-R-\sigma t)\,dx + \int_{\mathbb{R}}[\ln(1-v^2)](x+a(t),t)\Phi'''(x-R-\sigma t)\,dx. \tag{4-1}$$

Our goal is to provide a lower bound for the integrands on the right-hand side of (4-1).

Notice that the function Φ satisfies the inequality

$$|\Phi'''| \leq 4\nu_\mathfrak{c}^2\Phi'. \tag{4-2}$$

In view of the bound (2-14) on $a'(t)$ and the definition of $\sigma_\mathfrak{c}$, we obtain that

$$|a'(t)+\sigma|^2 \leq \frac{9+7\mathfrak{c}^2}{8}. \tag{4-3}$$

Hence, we deduce

$$\frac{d}{dt}[I_{R+\sigma t}(t)] \geq \int_{\mathbb{R}}\Big[4\nu_\mathfrak{c}^2\ln(1-v^2)+v^2+w^2-3v^2w^2 + (\partial_x v)^2 - \sqrt{\frac{9+7\mathfrak{c}^2}{8}}|vw|\Big](x+a(t),t)\Phi'(x-R-\sigma t)\,dx =: J_1+J_2. \tag{4-4}$$

At this step, we decompose the real line into two domains, $[-R_0, R_0]$ and its complement, where R_0 is to be defined below, and we denote by J_1 and J_2 the value of the integral on the right-hand side of (4-4) on each region. On $\mathbb{R}\setminus[-R_0, R_0]$, we bound the integrand pointwise from below by a positive quadratic form in (v, w). Exponentially small error terms arise from integration on $[-R_0, R_0]$.

For $|x| \geq R_0$, using Theorem 2.1 and the Sobolev embedding theorem, and choosing α_0 small enough and R_0 large enough, we obtain

$$|v(x+a(t),t)| \leq |\varepsilon_v(x,t)| + |v_{c(t)}(x)| \leq A_\mathfrak{c}\big(\alpha_0+\exp\big(-\sqrt{1-\mathfrak{c}^2}R_0\big)\big) \leq \tfrac{1}{12} \tag{4-5}$$

for any $t\in\mathbb{R}$. Using the fact that $\ln(1-s)\geq -2s$ for all $s\in\big[0,\frac{1}{2}\big]$ and introducing (4-5) in (4-4), we obtain

$$J_1 \geq \frac{1-\mathfrak{c}^2}{8}\int_{|x|\geq R_0}[v^2+w^2+(\partial_x v)^2](x+a(t),t)\Phi'(x-R-\sigma t)\,dx. \tag{4-6}$$

We next consider the case $x\in[-R_0, R_0]$. In that region, we have

$$|x-R-\sigma t| \geq -R_0+|R+\sigma t|.$$

Hence,

$$\Phi'(x-R-\sigma t) \leq 2\nu_\mathfrak{c}e^{2\nu_\mathfrak{c}R_0}e^{-2\nu_\mathfrak{c}|R+\sigma t|}. \tag{4-7}$$

Since the function $|\ln(\,\cdot\,)|$ is decreasing on $(0, 1]$, in view of (2-7) and (4-4),

$$|J_2| \le A_c \int_{|x|\le R_0} [v^2 + w^2 + (\partial_x v)^2](x + a(t), t)\Phi'(x - R - \sigma t)\, dx.$$

Then, by (4-7) and the control on the norm of (v, w) in $X(\mathbb{R})$ provided by the conservation of the energy, we obtain

$$|J_2| \le B_c e^{-2\nu_{\mathfrak{c}}|R+\sigma t|}.$$

This finishes the proof of (2-28). It remains to prove (2-29). For that, we distinguish two cases. If $R \ge 0$, we integrate (2-28) from $t = t_0$ to $t = (t_0 + t_1)/2$, choosing $\sigma = \sigma_{\mathfrak{c}}$ and $R = R - \sigma_{\mathfrak{c}} t_0$, and then from $t = (t_0 + t_1)/2$ to $t = t_1$ choosing $\sigma = -\sigma_{\mathfrak{c}}$ and $R = R + \sigma_{\mathfrak{c}} t_1$. If $R \le 0$, we use the same arguments for the reverse choices $\sigma = -\sigma_{\mathfrak{c}}$ and $\sigma = \sigma_{\mathfrak{c}}$. This implies (2-29), and finishes the proof of Proposition 2.4. □

Proof of Proposition 2.9. Let Ψ^* and v^* be the solutions of (2-32)–(2-34) expressed in terms of the hydrodynamical variables (v^*, w^*) as in (2-30). We split the proof into five steps.

Step 1. *There exists a positive number $A_{\mathfrak{c}}$, depending only on $\mathfrak{c}$, such that*

$$\int_t^{t+1} \int_{\mathbb{R}} |\partial_x \Psi^*(x + a^*(t), s)|^2 e^{\nu_{\mathfrak{c}}|x|}\, dx\, ds \le A_{\mathfrak{c}} \tag{4-8}$$

for any $t \in \mathbb{R}$.

By (2-23) and (2-30),

$$|\Psi^*| \le A_{\mathfrak{c}}(|\partial_x v^*| + |w^*|). \tag{4-9}$$

In view of Proposition 2.6 and the fact that $|a^*(t) - a^*(s)|$ is uniformly bounded for $s \in [t - 1, t + 2]$ by (2-22), this yields

$$\int_{t-1}^{t+2} \int_{\mathbb{R}} |\Psi^*(x + a^*(t), s)|^2 e^{2\nu_{\mathfrak{c}}|x|}\, dx\, ds \le A_{\mathfrak{c}}. \tag{4-10}$$

We define

$$F^* := -\tfrac{1}{2}(v^*)^2\Psi^* + \operatorname{Re}\big(\Psi^*(1 - 2F(v^*, \overline{\Psi}^*))\big)(1 - 2F(v^*, \Psi^*)).$$

We recall that $\|v^*\|_{L^\infty(\mathbb{R}\times\mathbb{R})} < 1 - \sigma_{\mathfrak{c}}$ by (2-23). Using the Cauchy–Schwarz inequality, the Sobolev embedding theorem and the control of the norm in $X(\mathbb{R})$ provided by the conservation of energy, we have $F(v^*, \Psi^*) \in L^\infty(\mathbb{R} \times \mathbb{R})$. Hence,

$$|F^*| \le A_{\mathfrak{c}}|\Psi^*|, \tag{4-11}$$

where $A_{\mathfrak{c}}$ is a positive number depending only on $\mathfrak{c}$. Then, by (4-10),

$$\int_{t-1}^{t+2} \int_{\mathbb{R}} |F^*(x + a^*(t), s)|^2 e^{2\nu_{\mathfrak{c}}|x|}\, dx\, ds \le A_{\mathfrak{c}} \tag{4-12}$$

for any $t \in \mathbb{R}$. Next, by Proposition 2.7, we have

$$\|\Psi^*\|_{L^4([t-1,t+2],L^\infty)} \le A_{\mathfrak{c}}. \tag{4-13}$$

Indeed, we fix $t \in \mathbb{R}$ and we denote by

$$(\Psi_1^0, v_1^0) := \big(\Psi^*(\cdot + a^*(t-1), t-1), v^*(\cdot + a^*(t-1), t-1)\big),$$
$$(\Psi_1(s), v_1(s)) := \big(\Psi^*(\cdot + a^*(t-1), t-1+s), v^*(\cdot + a^*(t-1), t-1+s)\big)$$

the corresponding solution to (2-32)–(2-34). Denote also by

$$(\Psi_2^0, v_2^0) := (\Psi_{c^*(t-1)}, v_{c^*(t-1)}),$$
$$(\Psi_2(s), v_2(s)) := \big(\Psi_{c^*(t-1)}(x - c^*(t-1)s), v_{c^*(t-1)}(x - c^*(t-1)s)\big)$$

the corresponding solution to (2-32)–(2-34), where $\Psi_{c^*(t)}$ is the solution to (2-32) associated to the soliton $Q_{c^*(t)}$. We have, by (2-35),

$$\|\Psi_1(s) - \Psi_2(s)\|_{L^4([0,\tau_\mathfrak{c}],L^\infty)} \le A\big(\|v_1^0 - v_2^0\|_{L^2} + \|\Psi_1^0 - \Psi_2^0\|_{L^2}\big).$$

Using (2-21), we obtain

$$\|\Psi_1(s) - \Psi_2(s)\|_{L^4([0,\tau_\mathfrak{c}],L^\infty)} \le A_\mathfrak{c},$$

where $\tau_\mathfrak{c} = \tau_\mathfrak{c}\big(\|v_1^0\|_{L^2}, \|v_2^0\|_{L^2}, \|\Psi_1^0\|_{L^2}, \|\Psi_2^0\|_{L^2}\big)$ depend only on $\mathfrak{c}$. Since $[0,3] \subseteq \bigcup_{0\le k\le 3/\tau_\mathfrak{c}} [k\tau_\mathfrak{c}, (k+1)\tau_\mathfrak{c}]$, we can infer (4-13) inductively.

In addition, by (4-9), we have

$$\|\Psi^*(\cdot + a^*(t), \cdot)\|_{L^\infty([t-1,t+2],L^2)} \le A_\mathfrak{c}. \tag{4-14}$$

Hence, applying the Cauchy–Schwarz inequality to the integral with respect to the time variable, (4-10), (4-13) and (4-14),

$$\begin{aligned}
&\int_{t-1}^{t+2}\int_\mathbb{R} |\Psi^*(x + a^*(t), s)|^4 e^{\nu_\mathfrak{c}|x|}\,dx\,ds \\
&\qquad \le \int_{t-1}^{t+2}\int_\mathbb{R} |\Psi^*(x + a^*(t), s)|^2 e^{\nu_\mathfrak{c}|x|}\,dx\,\|\Psi^*(s)\|^2_{L^\infty(\mathbb{R})}\,ds \\
&\qquad \le \|\Psi^*(\cdot + a^*(t), \cdot)e^{(\nu_\mathfrak{c}/2)|\cdot|}\|^2_{L^4([t-1,t+2],L^2(\mathbb{R}))}\|\Psi^*(\cdot + a^*(t), \cdot)\|^2_{L^4([t-1,t+2],L^\infty(\mathbb{R}))} \\
&\qquad \le \|\Psi^*(\cdot + a^*(t), \cdot)e^{\nu_\mathfrak{c}|\cdot|}\|_{L^2([t-1,t+2],L^2(\mathbb{R}))}\|\Psi^*(\cdot + a^*(t), \cdot)\|_{L^\infty([t-1,t+2],L^2(\mathbb{R}))} \\
&\qquad\qquad \|\Psi^*(\cdot + a^*(t), \cdot)\|^2_{L^4([t-1,t+2],L^\infty(\mathbb{R}))} \\
&\qquad \le A_\mathfrak{c}.
\end{aligned} \tag{4-15}$$

In order to use Proposition 2.8 on Ψ^*, it is sufficient to verify

$$\sup_{s\in[t-1,t+2]}\int_\mathbb{R} |\Psi^*(x + a^*(t), s)|^2 e^{2\nu_\mathfrak{c}|x|}\,dx\,ds \le A_\mathfrak{c}. \tag{4-16}$$

Indeed, using (4-16) and (4-13), we can write

$$\begin{aligned}\int_{t-1}^{t+2}\int_{\mathbb{R}} |\Psi^*(x+a^*(t),s)|^6 e^{2\nu_{\mathfrak{c}}|x|}\,dx\,ds \\ &\leq \big\|\Psi^*(\cdot+a^*(t),\cdot)e^{\nu_{\mathfrak{c}}|\cdot|}\big\|^2_{L^\infty([t-1,t+2],L^2(\mathbb{R}))}\|\Psi^*(\cdot+a^*(t),\cdot)\|^4_{L^4([t-1,t+2],L^\infty(\mathbb{R}))} \\ &\leq A_{\mathfrak{c}},\end{aligned}\tag{4-17}$$

which proves that Ψ^* satisfies the assumptions of Proposition 2.8. Then, we apply Proposition 2.8 with $u := \Psi^*(\cdot + a^*(t), \cdot + (t + 1/2))$, $T := 1/2$, $F := |u|^2u + F^*(\cdot, t + 1/2)$ and successively $\lambda := \pm\nu_{\mathfrak{c}}$, and we use (4-10) and (4-12) to obtain (4-8).

Now let us prove (4-16). First, we recall the next lemma stated by Kenig, Ponce and Vega [Kenig et al. 2003].

Lemma 4.1. *Let $a \in [-2,-1]$ and $b \in [2,3]$. Assume that $u \in C^0([a,b] : L^2(\mathbb{R}))$ is a solution of the inhomogeneous Schrödinger equation*

$$i\partial_t u + \partial_{xx} u = H,\tag{4-18}$$

with $H \in L^1([a,b] : L^2(e^{\beta x}\,dx))$, for some $\beta \in \mathbb{R}$, and

$$u_a \equiv u(\cdot, a),\ u_b \equiv u(\cdot, b) \in L^2(e^{\beta x}dx).\tag{4-19}$$

Then there exists a positive number K such that

$$\sup_{a\leq t\leq b}\|u(\cdot,t)\|_{L^2(e^{\beta x}dx)} \leq K\big(\|u_a\|_{L^2(e^{\beta x}dx)} + \|u_b\|_{L^2(e^{\beta x}dx)} + \|H\|_{L^1([a,b],L^2(e^{\beta x}dx))}\big).\tag{4-20}$$

In order to apply the lemma, we need to verify the existence of numbers a and b such that (4-19) holds for $u := \Psi^*(\cdot + a^*(t), \cdot + t)$ and such that $H := |u|^2u + F^*(\cdot, \cdot + t) \in L^1([a,b], L^2(e^{\beta x}\,dx))$ for $\beta = \pm\nu_{\mathfrak{c}}$ respectively and any $t \in \mathbb{R}$. Our first claim is a consequence of (4-10) and the Markov inequality. Indeed, there exist $s_0 \in [-2,-1]$ and $s_1 \in [2,3]$ such that

$$\int_{\mathbb{R}} |\Psi^*(x+a^*(t),s_j+t)|^2 e^{2\nu_{\mathfrak{c}}|x|}\,dx \leq A_{\mathfrak{c}} \quad \text{for } j = 0, 1.$$

For the second claim, due to (4-12) and the Cauchy–Schwarz estimate, it is sufficient to show that $|u|^2u \in L^1([-2,3], L^2(e^{\nu_{\mathfrak{c}}|x|}\,dx))$. To prove this we use the Cauchy–Schwarz inequality for the time variable, (4-10) and (4-13),

$$\begin{aligned}\int_{-2}^{3}\bigg(\int_{\mathbb{R}} |\Psi^*(x+a^*(t),s+t)|^6 e^{2\nu_{\mathfrak{c}}|x|}\,dx\bigg)^{1/2} ds \\ &\leq \big\|\Psi^*(\cdot+a^*(t),\cdot+t)e^{\nu_{\mathfrak{c}}|\cdot|}\big\|_{L^2([-2,3],L^2)}\|\Psi^*(\cdot+a^*(t),\cdot+t)\|^2_{L^4([-2,3],L^\infty)} \\ &\leq A_{\mathfrak{c}}.\end{aligned}$$

Now we may apply Lemma 4.1 with $a = s_0$ and $b = s_1$ to deduce (4-16). This finishes the proof of the first step.

In the next step, we prove that (4-8) remains true for all the derivatives of Ψ^* and v^*.

Step 2. *Let $k \geq 1$. There exists a positive number $A_{k,\mathfrak{c}}$, depending only on k and $\mathfrak{c}$, such that*

$$\int_t^{t+1} \int_{\mathbb{R}} |\partial_x^k \Psi^*(x + a^*(t), s)|^2 e^{\nu_{\mathfrak{c}}|x|} \, dx \, ds \leq A_{k,\mathfrak{c}} \tag{4-21}$$

and

$$\int_t^{t+1} \int_{\mathbb{R}} |\partial_x^k v^*(x + a^*(t), s)|^2 e^{\nu_{\mathfrak{c}}|x|} \, dx \leq A_{k,\mathfrak{c}} \tag{4-22}$$

for any $t \in \mathbb{R}$.

The proof of Step 2 is by induction on $k \geq 1$. We are going to differentiate (2-32) k times with respect to the space variable and write the resulting equation as

$$i\partial_t(\partial_x^k \Psi^*) + \partial_{xx}(\partial_x^k \Psi^*) = R_k(v^*, \Psi^*), \tag{4-23}$$

where $R_k(v^*, \Psi^*) = \partial_x^k(|\Psi^*|^2 \Psi^*) + \partial_x^k F^*$. We are going to prove by induction that (4-21), (4-22) and

$$\int_t^{t+1} \int_{\mathbb{R}} |R_k(v^*, \Psi^*)(x + a^*(t), s)|^2 e^{\nu_{\mathfrak{c}}|x|} \, dx \, ds \leq A_{k,\mathfrak{c}} \tag{4-24}$$

hold simultaneously for any $t \in \mathbb{R}$. Notice that (4-21) implies that $\partial_x^k \Psi^* \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}))$, while (4-24) implies that $R_k(v^*, \Psi^*) \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}))$. Therefore, if (4-21), (4-22) and (4-24) are established for some $k \geq 1$, then applying Proposition 2.8 to $\partial_x^k \Psi^*$ can be justified by a standard approximation procedure.

For $k = 1$, (4-21) is exactly (4-8). Equation (4-22) holds from Proposition 2.6 and the fact that $|a^*(t) - a^*(s)|$ is uniformly bounded for $s \in [t-1, t+2]$. Next, we write

$$\begin{aligned} R_1(v^*, \Psi^*) = {} & -v^* \partial_x v^* \Psi^* - \tfrac{1}{2}(v^*)^2 \partial_x \Psi^* + \mathrm{Re}\big(\partial_x \Psi^*(1 - 2F(v^*, \overline{\Psi}^*))\big)(1 - 2F(v^*, \Psi^*)) \\ & - 2v^*|\Psi^*|^2(1 - 2F(v^*, \Psi^*)) - 2v^*\Psi^* \, \mathrm{Re}\big(\Psi^*(1 - 2F(v^*, \overline{\Psi}^*)) - 2\partial_x(\Psi^*|\Psi^*|^2)\big). \end{aligned}$$

We will show that

$$\Psi^* \in L^\infty([t-1, t+2], L^\infty(\mathbb{R})) \tag{4-25}$$

in order to control the derivative of the cubic nonlinearity by $|\partial_x \Psi^*|$, and then we will use the fact that $F(v^*, \Psi^*) \in L^\infty(\mathbb{R} \times \mathbb{R})$, $\|v^*\|_{L^\infty(\mathbb{R}\times\mathbb{R})} < 1$ and the second equation in (2-34) to get

$$R_1(v^*, \Psi^*) \leq K\big(|\partial_x \Psi^*| + |\partial_x v^*||\Psi^*| + |\Psi^*|^2\big). \tag{4-26}$$

Let us prove (4-25). We define the function H on $\mathbb{R}$ by

$$H(s) := \frac{1}{2} \int_{\mathbb{R}} \big(|\partial_x \Psi^*(x, s)|^2 - |\Psi^*(x, s)|^4\big) \, dx.$$

We differentiate it with respect to s, integrate by parts and use (2-32) to obtain

$$\begin{aligned} H'(s) &= -\operatorname{Re}\biggl(\int_{\mathbb{R}} \partial_s \Psi^*(x,s)\bigl[\overline{\partial_{xx}\Psi^* + 2\Psi^*|\Psi^*|^2}\bigr](x,s)\,dx\biggr) \\ &= \operatorname{Re}\biggl(\int_{\mathbb{R}} \partial_s \Psi^*(x,s) F^*(x,s)\,dx\biggr) \\ &\leq \|\partial_s \Psi^*(s)\|_{H^{-1}(\mathbb{R})} \|F^*(s)\|_{H^1(\mathbb{R})}. \end{aligned} \tag{4-27}$$

We have

$$|\partial_x F^*| \leq K\bigl(|\partial_x \Psi^*| + |\partial_x v^*||\Psi^*| + |\Psi^*|^2\bigr),$$

using the fact that $F(v^*, \Psi^*) \in L^\infty(\mathbb{R}\times\mathbb{R})$, $\|v^*\|_{L^\infty(\mathbb{R}\times\mathbb{R})} < 1$ and the second equation in (2-34).

Hence, by (4-8), (4-10), (4-15) and the fact that $|\partial_x v^*| \leq |\Psi^*|$ on $\mathbb{R}\times\mathbb{R}$, we obtain

$$\|\partial_x F^*\|_{L^2([t-1,t+2],L^2(\mathbb{R}))} \leq A_{\mathfrak{c}}. \tag{4-28}$$

On the other hand, we infer

$$\|\partial_s \Psi^*\|_{L^2([t-1,t+2],H^{-1}(\mathbb{R}))} \leq A_{\mathfrak{c}} \tag{4-29}$$

from (2-32), (4-8), (4-12) and the fact that $\Psi^* \in L^4([t-1,t+2], L^\infty(\mathbb{R})) \cap L^8([t-1,t+2], L^4(\mathbb{R}))$.

Next, we integrate (4-27) between $t-1$ and $t+2$ and apply the Cauchy–Schwarz inequality to obtain $H \in W^{1,1}([t-1,t+2])$ for all $t \in \mathbb{R}$ using (4-28) and (4-29). Notice that all these computations can be justified by a standard approximation procedure. This yields, by the Sobolev embedding theorem, that $H \in L^\infty([t-1,t+2])$. We conclude that the derivative $\partial_x \Psi^* \in L^\infty([t-1,t+2], L^2(\mathbb{R}))$. Indeed, we can use the Gagliardo–Nirenberg inequality and the fact that Ψ^* is uniformly bounded in $L^2(\mathbb{R})$ by a positive number to write

$$\begin{aligned} H(s) &\geq \frac{1}{2}\int_{\mathbb{R}} |\partial_x \Psi^*(x,s)|^2\,dx - A\|\Psi^*(s)\|^3_{L^2(\mathbb{R})}\|\partial_x\Psi^*(\,\cdot\,)\|_{L^2(\mathbb{R})} \\ &\geq \frac{1}{2}\int_{\mathbb{R}} |\partial_x \Psi^*(x,s)|^2\,dx - AK^3\|\partial_x\Psi^*(\,\cdot\,)\|_{L^2(\mathbb{R})}. \end{aligned}$$

The function $x \mapsto \frac{1}{2}x^2 - AM^3x$ diverges to $+\infty$ when x goes to $+\infty$. Since H is bounded, we infer that $\|\partial_x\Psi^*(\,\cdot\,)\|_{L^2(\mathbb{R})}$ is uniformly bounded on $[t-1,t+2]$ for all $t \in \mathbb{R}$. This finishes the proof of (4-25) by the Sobolev embedding theorem. Then, by (4-26), (4-24) for $k=1$ is a consequence of (4-8), (4-15) and the fact that $|\partial_x v^*| \leq |\Psi^*|$ on $\mathbb{R}\times\mathbb{R}$.

Assume now that (4-21), (4-22) and (4-24) are satisfied for any integer $1 \leq k \leq k_0$ and any $t \in \mathbb{R}$. Let us prove these three estimates for $k = k_0+1$. We apply Proposition 2.8 with $u := \partial_x^{k_0}\Psi^*(\,\cdot + a^*(t), \cdot + (t+1/2))$, $T := 1/2$ and successively $\lambda := \pm\nu_{\mathfrak{c}}$. In view of (4-21), (4-23), (4-24) and the fact that $|a^*(t) - a^*(s)|$ is uniformly bounded for $s \in [t-1,t+2]$, this yields

$$\int_t^{t+1}\int_{\mathbb{R}} |\partial_x^{k_0+1}\Psi^*(x+a^*(t),s)|^2 e^{\nu_{\mathfrak{c}}|x|}\,dx\,ds \leq A_{\mathfrak{c}}, \tag{4-30}$$

so that (4-21) is satisfied for $k = k_0+1$.

Let $k \in \{1, \dots, k_0\}$. We use the induction hypothesis and (4-30) to infer that

$$\partial_x^{k-1} \Psi^* \in L^2([t, t+1], H^2(\mathbb{R})).$$

Also, we have

$$\partial_x^{k-1} \Psi^* \in H^1([t, t+1], L^2(\mathbb{R}))$$

using (4-23) and (4-24). This yields, by interpolation,

$$\partial_x^{k-1} \Psi^* \in H^{2/3}([t, t+1], H^{2/3}(\mathbb{R})).$$

Hence, using the Sobolev embedding theorem, we obtain

$$\partial_x^{k-1} \Psi^* \in L^\infty([t, t+1], L^\infty(\mathbb{R})) \quad \text{for all } t \in \mathbb{R}. \tag{4-31}$$

On the other hand, since $|\partial_x v^*| \le |\Psi^*|$, we have, by (4-25), that $\partial_x v^* \in L^\infty([t, t+1], L^\infty(\mathbb{R}))$. For $k \in \{2, \dots, k_0\}$, we differentiate the second equation in (2-34) k times and we use (4-31) to obtain

$$|\partial_x^k v^*| \le K \left(\sum_{j=1}^{k-1} |\partial_x^j \Psi^*| + \sum_{j=0}^{k-2} |\partial_x^j v^*| \right), \tag{4-32}$$

where K is a positive constant. By induction we infer from (4-31) that

$$\partial_x^k v^* \in L^\infty([t, t+1], L^\infty(\mathbb{R})) \quad \text{for all } t \in \mathbb{R}, \tag{4-33}$$

for all $k \in \{2, \dots, k_0\}$. Then, we just compute explicitly $R_{k_0+1}(v^*, \Psi^*)$ and we use (4-31) and (4-33) to obtain

$$|R_{k_0+1}(v^*, \Psi^*)| \le A_{k_0+1,\mathfrak{c},K} \left(\sum_{j=0}^{k_0+1} |\partial_x^j \Psi^*| + \sum_{j=1}^{k_0} |\partial_x^j v^*| \right).$$

Hence, by (4-21) for all $k \le k_0$, (4-22) and (4-30), we obtain (4-24) for $k = k_0 + 1$. Finally, we introduce (4-21) for all $k \le k_0 + 1$ and (4-22) for all $k \le k_0$ into (4-32) to deduce (4-22) for $k = k_0 + 1$. This finishes the proof of this step.

In order to finish the proof of Proposition 2.9, we now turn these L^2_{loc} in time estimates into L^∞ in time estimates, and then into uniform estimates.

Step 3. *Let $k \ge 0$. There exists a positive number $A_{k,\mathfrak{c}}$, depending only on k and $\mathfrak{c}$, such that*

$$\int_{\mathbb{R}} |\partial_x^k \Psi^*(x + a^*(t), t)|^2 e^{\nu_\mathfrak{c} |x|} \, dx \le A_{k,\mathfrak{c}} \tag{4-34}$$

for any $t \in \mathbb{R}$. In particular, we have

$$\left\| \partial_x^k \Psi^*(\cdot + a^*(t), t) e^{(\nu_\mathfrak{c}/2)|\cdot|} \right\|_{L^\infty(\mathbb{R})} \le A_{k,\mathfrak{c}} \tag{4-35}$$

for any $t \in \mathbb{R}$, and for a possibly different choice of the positive constant $A_{k,\mathfrak{c}}$.

Here, we use the Sobolev embedding theorem in time and (4-23) for the proof. By the Sobolev embedding theorem, we have

$$\big\|\partial_x^k\Psi^*(\cdot+a^*(t),t)e^{(\nu_\mathfrak{c}/2)|\cdot|}\big\|_{L^2(\mathbb{R})}^2 \leq K\Big(\big\|\partial_s\big(\partial_x^k\Psi^*(\cdot+a^*(t),s)e^{(\nu_\mathfrak{c}/2)|\cdot|}\big)\big\|_{L^2([t-1,t+1],L^2(\mathbb{R}))}^2 + \big\|\partial_x^k\Psi^*(\cdot+a^*(t),s)e^{(\nu_\mathfrak{c}/2)|\cdot|}\big\|_{L^2([t-1,t+1],L^2(\mathbb{R}))}^2\Big),$$

while, by (4-23),

$$\big\|\partial_s\big(\partial_x^k\Psi^*(\cdot+a^*(t),s)e^{(\nu_\mathfrak{c}/2)|\cdot|}\big)\big\|_{L^2([t-1,t+1],L^2(\mathbb{R}))}^2 \leq 2\Big(\big\|\partial_x^{k+2}\Psi^*(\cdot+a^*(t),s)e^{(\nu_\mathfrak{c}/2)|\cdot|}\big\|_{L^2([t-1,t+1],L^2(\mathbb{R}))}^2 + \big\|R_k(\Psi^*)(\cdot+a^*(t),s)e^{(\nu_\mathfrak{c}/2)|\cdot|}\big\|_{L^2([t-1,t+1],L^2(\mathbb{R}))}^2\Big),$$

so that we finally deduce (4-34) from (4-21) and (4-24). The estimate (4-35) follows from applying the Sobolev embedding theorem to (4-34).

The function v^* satisfies a similar inequality:

Step 4. *Let $k\in\mathbb{N}$. There exists a positive number $A_{k,\mathfrak{c}}$, depending only on k and $\mathfrak{c}$, such that*

$$\int_{\mathbb{R}}\big(\partial_x^k v^*(x+a^*(t),t)\big)^2 e^{\nu_\mathfrak{c}|x|}\,dx \leq A_{k,\mathfrak{c}} \tag{4-36}$$

and

$$\big\|\partial_x^k v^*(\cdot+a^*(t),t)e^{(\nu_\mathfrak{c}/2)|\cdot|}\big\|_{L^\infty(\mathbb{R})} \leq A_{k,\mathfrak{c}} \tag{4-37}$$

for any $t\in\mathbb{R}$.

The proof is similar to the proof of Step 3 using the first equation in (2-34) instead of (2-32). We use the Sobolev embedding theorem to write

$$\big\|\partial_x^k v^*(\cdot+a^*(t),t)e^{\nu_\mathfrak{c}|\cdot|}\big\|_{L^2(\mathbb{R})}^2 \leq K\Big(\big\|\partial_s\big(\partial_x^k v^*(\cdot+a^*(t),s)e^{\nu_\mathfrak{c}|\cdot|}\big)\big\|_{L^2([t-1,t+1],L^2(\mathbb{R}))}^2 + \big\|\partial_x^k v^*(\cdot+a^*(t),s)e^{\nu_\mathfrak{c}|\cdot|}\big\|_{L^2([t-1,t+1],L^2(\mathbb{R}))}^2\Big).$$

By the first equation in (2-34), (4-21), (4-23) and (4-33), we have

$$\big\|\partial_s\big(\partial_x^k v^*(\cdot+a^*(t),s)e^{\nu_\mathfrak{c}|\cdot|}\big)\big\|_{L^2([t-1,t+1],L^2(\mathbb{R}))}^2 \leq A_\mathfrak{c}.$$

This leads to (4-36). The uniform bound in (4-37) is then a consequence of the Sobolev embedding theorem.

Finally, we provide the estimates for the function w^*.

Step 5. *Let $k\in\mathbb{N}$. There exists a positive number $A_{k,\mathfrak{c}}$, depending only on k and $\mathfrak{c}$, such that*

$$\int_{\mathbb{R}}\big|\partial_x^k w^*(x+a^*(t),t)\big|^2 e^{\nu_\mathfrak{c}|x|}\,dx \leq A_{k,\mathfrak{c}} \tag{4-38}$$

and

$$\big\|\partial_x^k w^*(\cdot+a^*(t),t)e^{(\nu_\mathfrak{c}/2)|\cdot|}\big\|_{L^\infty(\mathbb{R})} \leq A_{k,\mathfrak{c}} \tag{4-39}$$

for any $t\in\mathbb{R}$.

The proof relies on the last two steps. First, we write

$$v^*\Psi^* = -\tfrac{1}{2}\partial_x((1-(v^*)^2)^{1/2}\exp i\theta^*).$$

Since $(1-v^*(x,t)^2)^{1/2}\exp i\theta^*(x,t) \to 1$ as $x \to -\infty$ for any $t \in \mathbb{R}$, we obtain the formula

$$2F(v^*,\Psi^*) = 1-(1-(v^*)^2)^{1/2}\exp i\theta^*. \tag{4-40}$$

Hence, using (2-30), we have

$$w^* = 2\,\mathrm{Im}\Big(\frac{\Psi^*(1-2F(v^*,\Psi^*))}{1-(v^*)^2}\Big). \tag{4-41}$$

Combining (2-7) and (4-40), we recall that

$$\frac{|1-2F(v^*,\Psi^*)|}{1-(v^*)^2} \le A_{\mathfrak{c}}. \tag{4-42}$$

Hence, we obtain

$$|w^*| \le A_{\mathfrak{c}}|\Psi^*|.$$

Then, (4-38) and (4-39) follow from (4-34) and (4-35) for $k=0$. For $k \ge 1$, we differentiate (4-41) k times with respect to the space variable, and using (4-35), (4-37) and (4-42), we are led to

$$|\partial_x^k w^*| \le A_{k,\mathfrak{c}}\Big(\sum_{j=0}^{k}|\partial_x^j\Psi^*| + \sum_{j=1}^{k-1}|\partial_x^j v^*|\Big).$$

We finish the proof of this step using Steps 3 and 4. This completes the proof of Proposition 2.9. □

5. Proof of the Liouville theorem

Proof of Proposition 2.10. First, by (2-38) and the explicit formula for v_c and w_c in (2-3), there exists a positive number $A_{k,\mathfrak{c}}$ such that

$$\int_{\mathbb{R}}\big((\partial_x^k\varepsilon_v^*(x,t))^2 + (\partial_x^k\varepsilon_w^*(x,t))^2\big)e^{\nu_{\mathfrak{c}}|x|}\,dx \le A_{k,\mathfrak{c}}, \tag{5-1}$$

for any $k \in \mathbb{N}$ and any $t \in \mathbb{R}$. In view of the formulae of $\mathcal{H}_c$ in (A-42) and for u^* in (2-41), a similar estimate holds for u^*, for a possibly different choice of the constant $A_{k,\mathfrak{c}}$. As a consequence, we are allowed to differentiate with respect to the time variable the quantity

$$\mathcal{I}^*(t) := \int_{\mathbb{R}} x u_1^*(x,t)u_2^*(x,t)\,dx$$

on the left-hand side of (2-43). Moreover, we can compute

$$\frac{d}{dt}(\mathcal{I}^*) = -2\int_{\mathbb{R}} \mu\langle \mathcal{H}_{c^*}(\partial_x u^*), u^*\rangle_{\mathbb{R}^2} + \int_{\mathbb{R}} \mu\langle \mathcal{H}_{c^*}(J\mathcal{R}_{c^*}\varepsilon^*), u^*\rangle_{\mathbb{R}^2} - (c^*)'\int_{\mathbb{R}} \mu\langle \mathcal{H}_{c^*}(\partial_c Q_{c^*}), u^*\rangle_{\mathbb{R}^2} + (c^*)'\int_{\mathbb{R}} \mu\langle \partial_c\mathcal{H}_{c^*}(\varepsilon^*), u^*\rangle_{\mathbb{R}^2} + ((a^*)' - c^*)\int_{\mathbb{R}} \mu\langle \mathcal{H}_{c^*}(\partial_x \varepsilon^*), u^*\rangle_{\mathbb{R}^2}, \quad (5\text{-}2)$$

where we have set $\mu(x) = x$ for any $x \in \mathbb{R}$.

At this stage, we split the proof into five steps. The proof of these steps is similar to the proof of Proposition 7 in [Béthuel et al. 2015].

Step 1. *There exist two positive numbers A_1 and R_1, depending only on $\mathfrak{c}$, such that*

$$\mathcal{I}_1^*(t) := -2\int_{\mathbb{R}} \mu\langle \mathcal{H}_{c^*}(\partial_x u^*), u^*\rangle_{\mathbb{R}^2} \ge \frac{1-\mathfrak{c}^2}{8}\|u^*(\,\cdot\,,t)\|^2_{X(\mathbb{R})} - A_1\|u^*(\,\cdot\,,t)\|^2_{X(B(0,R_1))} \quad (5\text{-}3)$$

for any $t \in \mathbb{R}$.

We introduce the explicit formula of the operator $\mathcal{H}_{c^*}$ in the definition of $\mathcal{I}_1^*(t)$ to obtain

$$\mathcal{I}_1^*(t) = 2\int_{\mathbb{R}} \mu\partial_x\Big(\frac{\partial_{xx}u_1^*}{1-v_{c^*}^2}\Big)u_1^* - 2\int_{\mathbb{R}} \mu\big(1-(c^*)^2-(5+(c^*)^2)v_{c^*}^2+2v_{c^*}^4\big)\frac{\partial_x u_1^*}{(1-v_{c^*}^2)^2}u_1^* + 2\int_{\mathbb{R}} \mu c^*\frac{1+v_{c^*}^2}{1-v_{c^*}^2}(\partial_x u_2^*)u_1^* - 2\int_{\mathbb{R}} \mu (c^*)^2\frac{(1+v_{c^*}^2)^2}{(1-v_{c^*}^2)^3}(\partial_x u_1^*)u_1^* + 2\int_{\mathbb{R}} \mu c^*\frac{1+v_{c^*}^2}{1-v_{c^*}^2}(\partial_x u_1^*)u_2^* - 2\int_{\mathbb{R}} \mu(1-v_{c^*}^2)(\partial_x u_2^*)u_2^*.$$

Integrating each term by parts, we obtain

$$\mathcal{I}_1^*(t) = \int_{\mathbb{R}} \iota_1^*(x,t)\,dx,$$

with

$$\iota_1^* = \bigg(\frac{2}{1-v_{c^*}^2} + 2x\frac{\partial_x v_{c^*} v_{c^*}}{1-v_{c^*}^2}\bigg)(\partial_x u_1^*)^2 - 2c^*\bigg(\frac{1+v_{c^*}^2}{1-v_{c^*}^2} + \frac{4x\partial_x v_{c^*} v_{c^*}}{\big(1-v_{c^*}^2\big)^2}\bigg)u_2^* u_1^* + (1-v_{c^*}^2 - 2x\partial_x v_{c^*} v_{c^*})(u_2^*)^2 + \frac{1+2((c^*)^2-3)v_{c^*}^2 + (2(c^*)^2-3)v_{c^*}^4 - 2v_{c^*}^6}{(1-v_{c^*}^2)^3}(u_1^*)^2 + 4x\partial_x v_{c^*} v_{c^*}\frac{((c^*)^2-3) + (2(c^*)^2-3)v_{c^*}^2 - 3v_{c^*}^4}{(1-v_{c^*}^2)^4}(u_1^*)^2.$$

Let δ be a small positive number. We next use the exponential decay of the function v_c and its derivatives to guarantee the existence of a radius R, depending only on $\mathfrak{c}$ and δ (in view of the bound on $c^* - \mathfrak{c}$ in

(2-21)), such that

$$\iota_1^*(x,t) \geq (2-\delta)(\partial_x u_1^*)^2(x,t) + \Big(\frac{1-\mathfrak{c}^2}{4} - \delta\Big)((u_1^*)^2(x,t) + (u_2^*)^2(x,t))$$

when $|x| \geq R$.

Then, we choose δ small enough and fix the number R_1 according to the value of the corresponding R, to obtain

$$\int_{|x|\geq R_1} \iota_1^*(x,t)\,dx \geq \frac{1-\mathfrak{c}^2}{8} \int_{|x|\geq R_1} \big((\partial_x u_1^*(x,t))^2 + u_1^*(x,t)^2 + u_2^*(x,t)^2\big)\,dx. \tag{5-4}$$

On the other hand, it follows from (2-3), and again (2-8), that

$$\int_{|x|\leq R_1} \iota_1^*(x,t)\,dx \geq \Big(\frac{1-\mathfrak{c}^2}{8} - A_1\Big) \int_{|x|\leq R_1} \big((\partial_x u_1^*(x,t))^2 + u_1^*(x,t)^2 + u_2^*(x,t)^2\big)\,dx$$

for a positive number A_1 depending only on $\mathfrak{c}$. Combining with (5-4), we obtain (5-3).

Step 2. *There exist two positive numbers A_2 and R_2, depending only on $\mathfrak{c}$, such that*

$$|\mathcal{I}_2^*(t)| := \Big|\int_{\mathbb{R}} \mu\langle \mathcal{H}_{c^*}(J\mathcal{R}_{c^*}\varepsilon^*), u^*\rangle_{\mathbb{R}^2}\Big| \leq \frac{1-\mathfrak{c}^2}{64}\|u^*(\cdot,t)\|_{X(\mathbb{R})}^2 + A_2\|u^*(\cdot,t)\|_{X(B(0,R_2))}^2 \tag{5-5}$$

for any $t \in \mathbb{R}$.

We refer to the proof of Step 2 in the proof of Proposition 7 in [Béthuel et al. 2015] for more details.

We infer the next step from (2-9), (2-57), the explicit formula of $\mathcal{H}_{c^*}$ in (A-42) and the exponential decay of the function $\partial_c Q_{c^*}$ and its derivatives.

Step 3. *There exist two positive numbers A_3 and R_3, depending only on $\mathfrak{c}$, such that*

$$|\mathcal{I}_4^*(t)| := \Big|(c^*)'\int_{\mathbb{R}} \mu\langle \mathcal{H}_{c^*}(\partial_c Q_{c^*}), u^*\rangle_{\mathbb{R}^2}\Big| \leq \frac{1-\mathfrak{c}^2}{64}\|u^*(\cdot,t)\|_{X(\mathbb{R})}^2 + A_3\|u^*(\cdot,t)\|_{X(B(0,R_3))}^2 \tag{5-6}$$

for any $t \in \mathbb{R}$.

We decompose the real line into two regions, $[-R, R]$ and its complement, for any $R > 0$. We use the fact that $|x| \leq e^{\nu_{\mathfrak{c}}|x|/4}$ for all $|x| \geq R$, to write

$$\begin{aligned}|\mathcal{I}_4^*(t)| \leq R|(c^*)'(t)| \int_{|x|\leq R} \big|\mathcal{H}_{c^*(t)}(\partial_c Q_{c^*(t)})(x)\big|\,|u^*(x,t)|\,dx \\ + \delta|(c^*)'(t)| \int_{|x|\geq R} \big|\mathcal{H}_{c^*(t)}(\partial_c Q_{c^*(t)})(x)\big|\,|u^*(x,t)|e^{\nu_{\mathfrak{c}}|x|/4}\,dx\end{aligned}$$

for any $t \in \mathbb{R}$. We deduce from (2-9), the explicit formula of $\mathcal{H}_{c^*}$ in (A-42) and the exponential decay of the function $\partial_c Q_{c^*}$ and its derivatives that

$$|\mathcal{I}_4^*(t)| \leq A_{\mathfrak{c}}\big(R\|u^*(\cdot,t)\|_{X(B(0,R))} + \delta\|u^*(\cdot,t)\|_{X(\mathbb{R})}\big)\|\varepsilon^*(\cdot,t)\|_{L^2(\mathbb{R})^2}$$

for any $t \in \mathbb{R}$. Hence, by (2-57),

$$|\mathcal{I}_4^*(t)| \leq A_{\mathfrak{c}}\Big(\frac{R^2}{\delta}\|u^*(\cdot,t)\|_{X(B(0,R))}^2 + 2\delta\|u^*(\cdot,t)\|_{X(\mathbb{R})}^2\Big).$$

We choose δ so that $2A_{\mathfrak{c}}\delta \leq (1-\mathfrak{c}^2)/64$, and we denote by R_4 the corresponding number R, to obtain (5-6), with $A_4 = A_{\mathfrak{c}} R_4^2/\delta$.

Similarly, we use (2-9), (2-21) and (2-57) to obtain:

Step 4. *There exists two positive numbers A_4 and R_4, depending only on $\mathfrak{c}$, such that*

$$|\mathcal{I}_3^*(t)| := \left|(c^*)' \int_{\mathbb{R}} \mu \langle \partial_c \mathcal{H}_{c^*}(\varepsilon^*), u^* \rangle_{\mathbb{R}^2}\right| \leq \frac{1-\mathfrak{c}^2}{64} \|u^*(\,\cdot\,,t)\|^2_{X(\mathbb{R})} + A_4 \|u^*(\,\cdot\,,t)\|^2_{X(B(0,R_4))} \tag{5-7}$$

for any $t \in \mathbb{R}$.

The last step follows from an argument as in Step 3.

Step 5. *There exist two positive numbers A_5 and R_5, depending only on $\mathfrak{c}$, such that*

$$\begin{aligned} |\mathcal{I}_5^*(t)| &:= \left|((a^*)' - c^*) \int_{\mathbb{R}} \mu \langle \mathcal{H}_{c^*}(\partial_x \varepsilon^*), u^* \rangle_{\mathbb{R}^2}\right| \\ &\leq \frac{1-\mathfrak{c}^2}{64} \|u^*(\,\cdot\,,t)\|^2_{X(\mathbb{R})} + A_5 \|u^*(\,\cdot\,,t)\|^2_{X(B(0,R_5))} \end{aligned} \tag{5-8}$$

for any $t \in \mathbb{R}$.

Finally, combining the estimates in Steps 1 to 5 with the identity (5-2), we obtain

$$\frac{d}{dt}(\mathcal{I}^*(t)) \geq \frac{1-\mathfrak{c}^2}{16} \|u^*(\,\cdot\,,t)\|^2_{X(\mathbb{R})} - (A_1 + A_2 + A_3 + A_4 + A_5) \|u^*(\,\cdot\,,t)\|^2_{X(B(0,R_*))},$$

allowing us to conclude the proof of (2-43) with

$$\begin{aligned} R_* &= \max\{R_1, R_2, R_3, R_4, R_5\}, \\ A_* &= A_1 + A_2 + A_3 + A_4 + A_5. \end{aligned}$$

□

Proof of Lemma 2.11. When $u \in H^3(\mathbb{R}) \times H^1(\mathbb{R})$, the function $\partial_x u$ is in the space $H^2(\mathbb{R}) \times L^2(\mathbb{R})$ which is the domain of $\mathcal{H}_c$. The scalar product on the right-hand side of (2-46) is well-defined in view of (2-45). Next, we use the formula for $\mathcal{H}_c$ in (A-42) to express $G_c(u)$ as

$$\begin{aligned} &\langle SM_c u, \mathcal{H}_c(-2\partial_x u) \rangle_{L^2(\mathbb{R})^2} \\ &= 2 \int_{\mathbb{R}} \frac{\partial_x v_c}{v_c} \left(\frac{1 - c^2 - (5 + c^2) v_c^2 + 2 v_c^4}{(1 - v_c^2)^2} + c^2 \frac{(1 + v_c^2)^2}{(1 - v_c^2)^3} - 2c^2 \frac{v_c^2 (1 + v_c^2)}{(1 - v_c^2)^3} \right) u_1 \partial_x u_1 \\ &\quad - 2 \int_{\mathbb{R}} \frac{\partial_x v_c}{v_c} \partial_x \left(\frac{\partial_{xx} u_1}{1 - v_c^2} \right) + 2 \int_{\mathbb{R}} \frac{\partial_x v_c (1 - v_c^2)}{v_c} u_2 \partial_x u_2 \\ &\quad + 2c \int_{\mathbb{R}} \left(2 \frac{v_c \partial_x v_c}{1 - v_c^2} u_1 \partial_x u_2 - \frac{\partial_x v_c (1 + v_c^2)}{v_c (1 - v_c^2)} \partial_x (u_1 u_2) \right). \end{aligned} \tag{5-9}$$

We recall that v_c solves the equation

$$\partial_{xx} v_c = (1 - c^2 - 2v_c^2) v_c, \tag{5-10}$$

which leads to

$$(\partial_x v_c)^2 = (1 - c^2 - v_c^2) v_c^2 \quad \text{and} \quad \partial_x \left(\frac{\partial_x v_c}{v_c} \right) = -v_c^2. \tag{5-11}$$

Then, the third integral on the right-hand side of (5-9) can be written as

$$2\int_{\mathbb{R}} \frac{\partial_x v_c(1-v_c^2)}{v_c} u_2 \partial_x u_2 = \int_{\mathbb{R}} \mu_c u_2^2, \tag{5-12}$$

with $\mu_c := 2(\partial_x v_c)^2 + (1-v_c^2)v_c^2$. Similarly, the last integral is given by

$$\int_{\mathbb{R}} \left(2\frac{v_c \partial_x v_c}{1-v_c^2} u_1 \partial_x u_2 - \frac{\partial_x v_c(1+v_c^2)}{v_c(1-v_c^2)} \partial_x(u_1 u_2) \right) = -\int_{\mathbb{R}} \left(v_c^2 u_1 u_2 + 2\frac{v_c \partial_x v_c}{1-v_c^2} u_2 \partial_x u_1 \right). \tag{5-13}$$

Combining (5-12) and (5-13) with (5-9), we obtain the identity

$$\langle SM_c u, \mathcal{H}_c(-2\partial_x u)\rangle_{L^2(\mathbb{R})^2} = I + \int_{\mathbb{R}} \mu_c \left(u_2 - \frac{c v_c^2}{\mu_c} u_1 - \frac{2c v_c \partial_x v_c}{\mu_c(1-v_c^2)} \partial_x u_1 \right)^2,$$

where

$$\begin{aligned} I = \int_{\mathbb{R}} 2\left(\frac{\partial_x v_c}{v_c} \left(\frac{1-c^2-(5+c^2)v_c^2+2v_c^4}{(1-v_c^2)^2} + c^2 \frac{1+v_c^2}{(1-v_c^2)^2} \right) - 2c^2 \frac{v_c^3 \partial_x v_c}{\mu_c(1-v_c^2)} \right) u_1 \partial_x u_1 \\ - \int_{\mathbb{R}} \frac{\partial_x v_c}{v_c} u_1 \partial_x \left(\frac{\partial_{xx} u_1}{1-v_c^2} \right) - c^2 \int_{\mathbb{R}} \frac{v_c^4}{\mu_c} u_1^2 - 4c^2 \int_{\mathbb{R}} \frac{(\partial_x v_c)^2 v_c^2}{\mu_c(1-v_c^2)^2} (\partial_x u_1)^2. \end{aligned}$$

Using (5-10) and (5-11), we finally deduce that

$$I = \frac{3}{2} \int_{\mathbb{R}} \frac{v_c^4}{\mu_c} \left(\partial_x u_1 - \frac{\partial_x v_c}{v_c} u_1 \right)^2,$$

which finishes the proof of (2-46). □

Proof of Proposition 2.12. We first rely on (2-3) and (2-46) to check that the quadratic form G_c is well-defined and continuous on $X(\mathbb{R})$. Next, setting

$$v = (v_c u_1, v_c u_2), \tag{5-14}$$

and using (5-10), we can express it as

$$G_c(u) = K_c(v) := \int_{\mathbb{R}} \frac{v_c^2}{\mu_c} \left(\partial_x v_1 - \frac{2\partial_x v_c}{v_c} v_1 \right)^2 + \int_{\mathbb{R}} \frac{\mu_c}{v_c^2} \left(v_2 + \frac{c\lambda_c}{\mu_c(1-v_c^2)} v_1 - 2\frac{c v_c \partial_x v_c}{\mu_c(1-v_c^2)} \partial_x v_1 \right)^2, \tag{5-15}$$

where we have set $\lambda_c := -\mu_c + 4(\partial_x v_c)^2$. From (2-48) and (5-14) we deduce that

$$\operatorname{Ker}(K_c) = \operatorname{Span}(v_c Q_c). \tag{5-16}$$

Let w be the pair defined in the following way

$$w = \left(v_1, v_2 - 2\frac{c v_c \partial_x v_c}{\mu_c(1-v_c^2)} \partial_x v_1 \right).$$

We compute

$$K_c(v) = \langle \mathcal{T}_c(w), w\rangle_{L^2(\mathbb{R})^2}, \tag{5-17}$$

with

$$\mathcal{T}_c(w) = \begin{pmatrix} -3\partial_x\Big(\dfrac{v_c^2}{\mu_c}\partial_x w_1\Big)+\Big(\dfrac{8v_c^4(\partial_x v_c)^2-2v_c^6(1-v_c^2)}{\mu_c^2}+\dfrac{4(\partial_x v_c)^2}{\mu_c}+\dfrac{c^2(2c^2-1+v_c^2)^2v_c^2}{\mu_c(1-v_c^2)^2}\Big)w_1-\dfrac{c(2c^2-1+v_c^2)}{(1-v_c^2)}w_2 \\ -\dfrac{c(2c^2-1+v_c^2)}{(1-v_c^2)}w_1+\dfrac{\mu_c}{v_c^2}w_2 \end{pmatrix}. \tag{5-18}$$

The operator $\mathcal{T}_c$ in (5-18) is self-adjoint on $L^2(\mathbb{R})^2$, with domain $\mathrm{Dom}(\mathcal{T}_c) = H^2(\mathbb{R}) \times L^2(\mathbb{R})$. In addition, combining (5-15) with (5-17) we deduce that $\mathcal{T}_c$ is nonnegative, with a kernel equal to

$$\mathrm{Ker}(\mathcal{T}_c) = \mathrm{Span}\bigg\{\bigg(v_c^2, \frac{2cv_c^2(\partial_x v_c)^2}{\mu_c(1-v_c^2)}\bigg)\bigg\}.$$

At this stage, we divide the proof into three steps.

Step 1. *Let $c \in (-1, 1) \setminus \{0\}$. There exists a positive number Λ_1, depending continuously on c, such that*

$$\langle \mathcal{T}_c(w), w\rangle_{L^2(\mathbb{R})^2} \geq \Lambda_1 \int_{\mathbb{R}} (w_1^2 + w_2^2), \tag{5-19}$$

for any pair $w \in X^1(\mathbb{R})$ such that

$$\bigg\langle w, \bigg(v_c^2, \frac{2cv_c^2(\partial_x v_c)^2}{\mu_c(1-v_c^2)}\bigg)\bigg\rangle_{L^2(\mathbb{R})^2} = 0. \tag{5-20}$$

We claim that the essential spectrum of $\mathcal{T}_c$ is given by

$$\sigma_{\mathrm{ess}}(\mathcal{T}_c) = [\tau_c, +\infty), \tag{5-21}$$

with

$$\tau_c = \tau_{1,c} - \tfrac{1}{2}\tau_{2,c}^{1/2} > 0. \tag{5-22}$$

Here, we have set

$$\tau_{1,c} = \frac{4(1-c^2)+c^2(2c^2-1)^2}{2(3-2c^2)} + \frac{3-2c^2}{2}$$

and

$$\tau_{2,c} = \Big(\frac{4(1-c^2)+c^2(2c^2-1)^2}{3-2c^2} - (3-2c^2)\Big)^2 + 4c^2(2c^2-1)^2.$$

In particular, 0 is an isolated eigenvalue in the spectrum of $\mathcal{T}_c$. The inequality (5-19) follows with Λ_1 either equal to τ_c, or to the smallest positive eigenvalue of $\mathcal{T}_c$. In view of the analytic dependence on c of the operator $\mathcal{T}_c$, Λ_1 depends continuously on c.

Now, let us prove (5-21). We rely on the Weyl criterion. It follows from (2-47) and (5-10) that

$$\frac{\mu_c(x)}{v_c^2(x)} \to 3 - 2c^2 \quad \text{and} \quad \frac{(\partial_x v_c)^2(x)}{\mu_c(x)} \to \frac{1-c^2}{3-2c^2}$$

as $x \to \pm\infty$. Coming back to (5-18), we introduce the operator $\mathcal{T}_\infty$ given by

$$\mathcal{T}_\infty(w) = \begin{pmatrix} -\frac{3}{3-2c^2}\partial_{xx} w_1 + \frac{4(1-c^2)+c^2(2c^2-1)^2}{3-2c^2} w_1 - c(2c^2-1)w_2 \\ -c(2c^2-1)w_1 + (3-2c^2)w_2 \end{pmatrix}.$$

By the Weyl criterion, the essential spectrum of $\mathcal{T}_c$ is equal to the spectrum of $\mathcal{T}_\infty$.

We next apply again the Weyl criterion to establish that a real number λ belongs to the spectrum of $\mathcal{T}_\infty$ if and only if there exists a complex number ξ such that

$$\lambda^2 - \Big(\frac{3}{3-2c^2}|\xi|^2 + \frac{4(1-c^2)+c^2(2c^2-1)^2}{3-2c^2} + 3 - 2c^2\Big)\lambda + 3|\xi|^2 + 4(1-c^2) = 0.$$

This is the case if and only if

$$\lambda = \frac{4(1-c^2)+c^2(2c^2-1)^2+3|\xi|^2}{2(3-2c^2)} + \frac{3-2c^2}{2} \\ \pm \frac{1}{2}\Big(\Big(\frac{4(1-c^2)+c^2(2c^2-1)^2+3|\xi|^2}{3-2c^2} - (3-2c^2)\Big)^2 + 4c^2(2c^2-1)^2\Big)^{1/2}.$$

This leads to $\sigma_{\mathrm{ess}}(\mathcal{T}_c) = \sigma(\mathcal{T}_\infty) = [\tau_c, +\infty)$, with τ_c as in (5-22). This completes the proof of Step 1.

Step 2. *There exists a positive number Λ_2, depending continuously on c, such that*

$$K_c(v) \geq \Lambda_2 \int_{\mathbb{R}} ((\partial_x v_1)^2 + v_1^2 + v_2^2), \tag{5-23}$$

for any pair $v \in X^1(\mathbb{R})$ such that

$$\langle v, v_c^{-1} S\chi_c\rangle_{L^2(\mathbb{R})^2} = 0. \tag{5-24}$$

We start by improving the estimate in (5-19). Given a pair $w \in X^1(\mathbb{R})$, we observe that

$$\Big|\langle \mathcal{T}_c(w), w\rangle_{L^2(\mathbb{R})^2} - 3\int_{\mathbb{R}} \frac{v_c^2}{\mu_c}(\partial_x w_1)^2\Big| \leq A_c \int_{\mathbb{R}} (w_1^2 + w_2^2).$$

Here and in the sequel, A_c refers to a positive number depending continuously on c. For $0 < \tau < 1$, we have

$$\langle \mathcal{T}_c(w), w\rangle_{L^2(\mathbb{R})^2} \geq (1-\tau)\langle \mathcal{T}_c(w), w\rangle_{L^2(\mathbb{R})^2} + 3\tau\int_{\mathbb{R}} \frac{v_c^2}{\mu_c}(\partial_x w_1)^2 - A_c\tau\int_{\mathbb{R}} (w_1^2 + w_2^2).$$

Since $v_c^2/\mu_c \geq 1/(3-2c^2)$, this yields

$$\langle \mathcal{T}_c(w), w\rangle_{L^2(\mathbb{R})^2} \geq ((1-\tau)\Lambda_1 - A_c\tau)\int_{\mathbb{R}} (w_1^2 + w_2^2) + \frac{3\tau}{3-2c^2}\int_{\mathbb{R}} (\partial_x w_1)^2$$

under condition (5-20). For τ small enough, this leads to

$$\langle \mathcal{T}_c(w), w\rangle_{L^2(\mathbb{R})^2} \geq A_c \int_{\mathbb{R}} \big((\partial_x w_1)^2 + w_1^2 + w_2^2\big) \tag{5-25}$$

when w satisfies condition (5-20).

Since the pair w depends on the pair v, we can write (5-25) in terms of v. By (5-17), $K_c(v)$ is equal to the left-hand side of (5-25). We deduce that (5-25) may be expressed as

$$K_c(v) \geq A_c \int_{\mathbb{R}} \big((\partial_x v_1)^2 + v_1^2\big) + A_c \int_{\mathbb{R}} \Big(v_2 - \frac{2cv_c(\partial_x v_c)}{\mu_c(1-v_c^2)} \partial_x v_1\Big)^2.$$

We recall that, given two vectors a and b in a Hilbert space H, we have

$$\|a-b\|_H^2 \geq \tau \|a\|_H^2 - \frac{\tau}{1-\tau} \|b\|_H^2$$

for any $0 < \tau < 1$. Then, we deduce that

$$K_c(v) \geq A_c \int_{\mathbb{R}} \big((\partial_x v_1)^2 + v_1^2 + \tau v_2^2\big) - \frac{\tau A_c}{1-\tau} \int_{\mathbb{R}} \Big(\frac{v_c(\partial_x v_c)}{\mu_c(1-v_c^2)} \partial_x v_1\Big)^2.$$

We choose τ small enough so that we can infer from (2-3) that

$$K_c(v) \geq A_c \int_{\mathbb{R}} \big((\partial_x v_1)^2 + v_1^2 + v_2^2\big) \tag{5-26}$$

when w satisfies condition (5-20), i.e., when v is orthogonal to the pair

$$\mathfrak{v}_c = \bigg(v_c^2 - \partial_x\bigg(\frac{2cv_c^2(\partial_x v_c)^2}{\mu_c(1-v_c^2)}\bigg), \frac{2cv_c^2(\partial_x v_c)^2}{\mu_c(1-v_c^2)}\bigg). \tag{5-27}$$

Next, we verify that (5-26) remains true, decreasing possibly the value of A_c, when we replace this orthogonality condition by

$$\langle v, v_c Q_c \rangle_{L^2(\mathbb{R})^2} = 0. \tag{5-28}$$

We remark that

$$\langle \mathfrak{v}_c, v_c Q_c \rangle_{L^2(\mathbb{R})^2} \neq 0.$$

Indeed, we would deduce from (5-26) that

$$0 = K_c(v_c Q_c) \geq A_c \int_{\mathbb{R}} \big((\partial_x v_c^2)^2 + v_c^4 + (v_c w_c)^2\big) > 0,$$

which is impossible. In addition, the number $\langle \mathfrak{v}_c, v_c Q_c \rangle_{L^2(\mathbb{R})^2}$ depends continuously on c in view of (5-27). Given a pair $\tilde{v}$ satisfying (5-28), we denote by λ the real number such that $\mathfrak{v} = \lambda v_c Q_c + \tilde{v}$ is orthogonal to $\mathfrak{v}_c$. Since $v_c Q_c$ belongs to the kernel of K_c, using (5-26) we obtain

$$K_c(\tilde{v}) = K_c(\mathfrak{v}) \geq A_c \int_{\mathbb{R}} \big((\partial_x \mathfrak{v}_1)^2 + \mathfrak{v}_1^2 + \mathfrak{v}_2^2\big). \tag{5-29}$$

On the other hand, since $\tilde{v}$ satisfies (5-28), we have

$$\lambda = \frac{\langle \mathfrak{v}, v_c Q_c \rangle_{L^2(\mathbb{R})^2}}{\|v_c Q_c\|_{L^2(\mathbb{R})^2}^2}.$$

Using the Cauchy–Schwarz inequality, this yields

$$\lambda^2 \leq A_c \Big(\int_{\mathbb{R}} \big(v_c^4 + (v_c w_c)^2\big)\Big)\Big(\int_{\mathbb{R}} \big(\mathfrak{v}_1^2 + \mathfrak{v}_2^2\big)\Big).$$

Hence, by (2-3) and (5-29),

$$\lambda^2 \le A_c K_c(\mathfrak{v}) = A_c K_c(\tilde{v}).$$

Using (5-29), this leads to

$$\int_{\mathbb{R}} \big((\partial_x \tilde{v}_1)^2 + \tilde{v}_1^2 + \tilde{v}_2^2\big) \le 2\bigg(\lambda^2 \int_{\mathbb{R}} v_c^2\big((\partial_x v_c)^2 + v_c^2 + w_c^2\big) + \int_{\mathbb{R}} \big((\partial_x \mathfrak{v}_1)^2 + \mathfrak{v}_1^2 + \mathfrak{v}_2^2\big)\bigg) \le A_c K_c(\tilde{v}).$$

We finish the proof of this step applying again the same argument. We write $v = \lambda v_c S Q_c + \tilde{v}$, with $\langle \tilde{v}, v_c Q_c\rangle_{L^2(\mathbb{R})^2} = 0$. Since $v_c Q_c$ belongs to the kernel of K_c, we infer from the same argument that

$$K_c(v) = K_c(\tilde{v}) \ge \Lambda_2 \int_{\mathbb{R}} (\partial_x \tilde{v}_1)^2 + \tilde{v}_1^2 + \tilde{v}_2^2. \tag{5-30}$$

Using the orthogonality condition in (5-24), we obtain

$$\lambda = -\frac{\langle \tilde{v}, v_c^{-1} S \chi_c\rangle_{L^2(\mathbb{R})^2}}{\langle Q_c, S\chi_c\rangle_{L^2(\mathbb{R})^2}}.$$

By the Cauchy–Schwarz inequality, we are led to

$$\lambda^2 \le A_c \|v_c^{-1} S\chi_c\|_{L^2\times L^2}^2 \int_{\mathbb{R}} \big(\tilde{v}_1^2 + \tilde{v}_2^2\big).$$

Invoking the exponential decay of χ_c in (A-46), we deduce

$$\|v_c^{-1} S\chi_c\|_{L^2\times L^2}^2 \le A_c.$$

As a consequence, we can derive from (5-30) that

$$\lambda^2 \le A_c K_c(\tilde{v}) = A_c K_c(v).$$

Combining again with (5-30), we are led to

$$\int_{\mathbb{R}} \big((\partial_x v_1)^2 + v_1^2 + v_2^2\big) \le 2\bigg(\lambda^2 \int_{\mathbb{R}} v_c^2\big((\partial_x v_c)^2 + v_c^2 + w_c^2\big) + \int_{\mathbb{R}} \big((\partial_x \tilde{v}_1)^2 + \tilde{v}_1^2 + \tilde{v}_2^2\big)\bigg) \le A_c K_c(v).$$

which completes the proof of Step 2.

Step 3. *End of the proof.*

Since the pair v depends on the pair u as in (5-14), we can write (5-23) in terms of u. The left-hand side of (5-23) is equal to $G_c(u)$ by (5-15). Moreover, for the right-hand side, we have

$$\int_{\mathbb{R}} \big((\partial_x v_1)^2 + v_1^2 + v_2^2\big) = \int_{\mathbb{R}} v_c^2\big((\partial_x u_1)^2 + (2v_c^2 + c^2)u_1^2 + u_2^2\big).$$

We deduce that (5-23) may be written as

$$G_c(u) \ge A_c \int_{\mathbb{R}} v_c^2\big((\partial_x u_1)^2 + u_1^2 + u_2^2\big), \tag{5-31}$$

when $v_c u$ verifies the orthogonality condition (5-24), which means that u verifies the orthogonality condition (2-52). We recall that

$$v_c(x) \geq A_c e^{-|x|}$$

by (2-3), which is sufficient to obtain (2-51). This completes the proof of Proposition 2.12. □

Proof of Proposition 2.13. First we check that we are allowed to differentiate the quantity

$$\mathcal{J}^*(t) := \langle M_{c^*(t)} u^*(\cdot\,,t), u^*(\cdot\,,t)\rangle_{L^2(\mathbb{R})^2}.$$

Indeed, by (2-41), (5-1) and (A-42), there exists a positive number $A_{k,\mathfrak{c}}$ such that

$$\int_{\mathbb{R}} \big(\big(\partial_x^k u_1^*(x,t)\big)^2 + \big(\partial_x^k u_2^*(x,t)\big)^2\big) e^{\nu_{\mathfrak{c}}|x|}\,dx \leq A_{k,\mathfrak{c}}. \tag{5-32}$$

Next, using (2-42) and (2-45), we obtain

$$\begin{aligned} \frac{d}{dt}(\mathcal{J}^*) = {} & 2\langle SM_{c^*}u^*, \mathcal{H}_{c^*}(JSu^*)\rangle_{L^2(\mathbb{R})^2} + 2\langle SM_{c^*}u^*, \mathcal{H}_{c^*}(J\mathcal{R}_{c^*}\varepsilon^*)\rangle_{L^2(\mathbb{R})^2} \\ & + 2((a^*)' - c^*)\langle SM_{c^*}u^*, \mathcal{H}_{c^*}(\partial_x \varepsilon^*)\rangle_{L^2(\mathbb{R})^2} - 2(c^*)'\langle SM_{c^*}u^*, \mathcal{H}_{c^*}(\partial_c Q_{c^*})\rangle_{L^2(\mathbb{R})^2} \\ & + (c^*)'\langle \partial_c M_{c^*}u^*, u^*\rangle_{L^2(\mathbb{R})^2} + 2(c^*)'\langle M_{c^*}u^*, S\partial_c\mathcal{H}_{c^*}(\varepsilon^*)\rangle_{L^2(\mathbb{R})^2}. \end{aligned} \tag{5-33}$$

The proof of (2-53) is the same as in [Béthuel et al. 2015]. We will give only the main ideas of the proof. We will estimate all the terms on the right-hand side of (5-33) except the fourth term, which vanishes.

For the first one, we infer from Proposition 2.12 the following estimate.

Step 1. *There exists a positive number B_1, depending only on $\mathfrak{c}$, such that*

$$\mathcal{J}_1^*(t) := 2\langle SM_{c^*}u^*, \mathcal{H}_{c^*}(JSu^*)\rangle_{L^2(\mathbb{R})^2} \geq B_1 \int_{\mathbb{R}} \big[(\partial_x u_1^*)^2 + (u_1^*)^2 + (u_2^*)^2\big](x,t) e^{-2|x|}\,dx$$

for any $t \in \mathbb{R}$.

From (2-21), (2-57) and (5-1), we get an estimate for the second term.

Step 2. *There exists a positive number B_2, depending only on $\mathfrak{c}$, such that*

$$|\mathcal{J}_2^*(t)| := 2|\langle SM_{c^*}u^*, \mathcal{H}_{c^*}(J\mathcal{R}_{c^*}\varepsilon^*)\rangle_{L^2(\mathbb{R})^2}| \leq B_2 \|\varepsilon^*(\cdot\,,t)\|_{X(\mathbb{R})}^{1/2} \|u^*(\cdot\,,t)\|_{X(\mathbb{R})}^2$$

for any $t \in \mathbb{R}$.

For the third one, we use (2-21) to obtain:

Step 3. *There exists a positive number B_3, depending only on $\mathfrak{c}$, such that*

$$|\mathcal{J}_3^*(t)| := 2|(a^*)' - c^*|\big|\langle SM_{c^*}u^*, \mathcal{H}_{c^*}(\partial_x\varepsilon^*)\rangle_{L^2(\mathbb{R})^2}\big| \leq B_3 \|\varepsilon^*(\cdot\,,t)\|_{X(\mathbb{R})}^{1/2} \|u^*(\cdot\,,t)\|_{X(\mathbb{R})}^2$$

for any $t \in \mathbb{R}$.

We now prove the following statement for the fourth term.

Step 4. *We have*

$$\mathcal{J}_4^*(t) := 2(c^*)'\langle SM_{c^*}u^*, \mathcal{H}_{c^*}(\partial_c Q_{c^*})\rangle_{L^2(\mathbb{R})^2} = 0$$

for any $t \in \mathbb{R}$.

Since $\mathcal{H}_{c^*}(\partial_c Q_{c^*}) = P'(Q_{c^*}) = SQ_{c^*}$ and $M_{c^*}Q_{c^*} = S\partial_x Q_{c^*}$, we have

$$\begin{aligned}\langle SM_{c^*}u^*, \mathcal{H}_{c^*}(\partial_c Q_{c^*})\rangle_{L^2(\mathbb{R})^2} &= \langle M_{c^*}u^*, Q_{c^*}\rangle_{L^2(\mathbb{R})^2} = \langle u^*, S\partial_x Q_{c^*}\rangle_{L^2(\mathbb{R})^2}\\ &= \langle \varepsilon^*, \mathcal{H}_{c^*}(\partial_x Q_{c^*})\rangle_{L^2(\mathbb{R})^2} = 0.\end{aligned}$$

This is the reason why we do not need to establish a quadratic dependence of $(c^*)'(t)$ on ε^*.

Next, we use (2-3), (2-9), (2-21) and (2-45) to bound the fifth term.

Step 5. *There exists a positive number* B_5, *depending only on* $\mathfrak{c}$, *such that*

$$|\mathcal{J}_5^*(t)| := |(c^*)'||\langle \partial_c M_{c^*}u^*, u^*\rangle_{L^2(\mathbb{R})^2}| \leq B_5\|\varepsilon^*(\cdot\,,t)\|_{X(\mathbb{R})}^{1/2}\|u^*(\cdot\,,t)\|_{X(\mathbb{R})}^2$$

for any $t \in \mathbb{R}$.

Finally, we acquire a bound on the sixth term in the same way.

Step 6. *There exists a positive number* B_6, *depending only on* $\mathfrak{c}$, *such that*

$$|\mathcal{J}_6^*(t)| := |(c^*)'|\big|\langle M_{c^*}u^*, S\partial_c\mathcal{H}_{c^*}(\varepsilon^*)\rangle_{L^2(\mathbb{R})^2}\big| \leq B_6\|\varepsilon^*(\cdot\,,t)\|_{X(\mathbb{R})}^{1/2}\|u^*(\cdot\,,t)\|_{X(\mathbb{R})}^2$$

for any $t \in \mathbb{R}$.

We conclude the proof of Proposition 2.13 by combining the six previous steps to obtain (2-53), with $B_* := \max\{1/B_1, B_2 + B_3 + B_5 + B_6\}$. □

Proof of Corollary 2.14. Corollary 2.14 is a consequence of Propositions 2.10 and 2.13. We combine the two estimates (2-43) and (2-53) with the definition of $N(t)$ to obtain

$$\frac{d}{dt}\big(\langle N(t)u^*(\cdot\,,t), u^*(\cdot\,,t)\rangle_{L^2(\mathbb{R})^2}\big) \geq \Big(\frac{1-\mathfrak{c}^2}{16} - A_* B_*^2 e^{2R_*}\|\varepsilon^*(\cdot\,,t)\|_{X(\mathbb{R})}^{1/2}\Big)\|u^*(\cdot\,,t)\|_{X(\mathbb{R})}^2$$

for any $t \in \mathbb{R}$. In view of (2-21), we fix the parameter $\beta_{\mathfrak{c}}$ such that

$$\|\varepsilon^*(\cdot\,,t)\|_{X(\mathbb{R})}^{1/2} \leq \frac{1-\mathfrak{c}^2}{32A_* B_*^2 e^{2R_*}}$$

for any $t \in \mathbb{R}$, to obtain (2-54). In view of (2-3), (2-21) and (2-45), we notice that there exists a positive number $A_{\mathfrak{c}}$, depending only on $\mathfrak{c}$, such that

$$\|M_{c^*(t)}\|_{L^\infty(\mathbb{R})} \leq A_{\mathfrak{c}} \tag{5-34}$$

for any $t \in \mathbb{R}$. Moreover, since the map $t \mapsto \langle N(t)u^*(\cdot\,,t), u^*(\cdot\,,t)\rangle_{L^2(\mathbb{R})^2}$ is uniformly bounded by (5-32) and (5-34), the estimate (2-55) follows by integrating (2-54) from $t = -\infty$ to $t = +\infty$. Finally, statement (2-56) is a direct consequence of (2-55). □

Appendix A. Appendix

Weak continuity of the hydrodynamical flow. In this section, we prove the weak continuity of the hydrodynamical flow, which is stated in the following proposition.

Proposition A.1. *We consider a sequence* $(v_{n,0}, w_{n,0})_{n\in\mathbb{N}} \in \mathcal{NV}(\mathbb{R})^{\mathbb{N}}$, *and a pair* $(v_0, w_0) \in \mathcal{NV}(\mathbb{R})$ *such that*

$$v_{n,0} \rightharpoonup v_0 \quad \text{in } H^1(\mathbb{R}) \quad \text{and} \quad w_{n,0} \rightharpoonup w_0 \quad \text{in } L^2(\mathbb{R}) \tag{A-1}$$

as $n \to +\infty$. *We denote by* (v_n, w_n) *the unique solution to* (HLL) *with initial datum* $(v_{n,0}, w_{n,0})$ *and we assume that there exists a positive number* T_n *such that the solutions* (v_n, w_n) *are defined on* $(-T_n, T_n)$, *and satisfy the condition*

$$\sup_{n\in\mathbb{N}} \sup_{t\in(-T_n,T_n)} \max_{x\in\mathbb{R}} v_n(x,t) \le 1-\sigma \tag{A-2}$$

for a given positive number σ. *Then, the unique solution* (v, w) *to* (HLL) *with initial datum* (v_0, w_0) *is defined on* $(-T_{\max}, T_{\max})$, *with*[3]

$$T_{\max} = \liminf_{n\to+\infty} T_n,$$

and for any $t \in (-T_{\max}, T_{\max})$, *we have*

$$v_n(t) \rightharpoonup v(t) \quad \text{in } H^1(\mathbb{R}) \quad \text{and} \quad w_n(t) \rightharpoonup w(t) \quad \text{in } L^2(\mathbb{R}) \tag{A-3}$$

as $n \to +\infty$.

First we prove a weak continuity property of the flow of equations (2-32)–(2-34). Next, we deduce the weak convergence of w_n from (4-41).

More precisely, we consider now a sequence of initial conditions $(\Psi_{n,0}, v_{n,0}) \in L^2(\mathbb{R}) \times H^1(\mathbb{R})$, such that the norms $\|\Psi_{n,0}\|_{L^2}$ and $\|v_{n,0}\|_{L^2}$ are uniformly bounded with respect to n, and we assume that

$$\sup_{n\in\mathbb{N}} \|v_{n,0}\|_{L^\infty(\mathbb{R})} < 1. \tag{A-4}$$

Then, there exist two functions $\Psi_0 \in L^2(\mathbb{R})$ and $v_0 \in H^1(\mathbb{R})$ such that, going possibly to a subsequence,

$$\Psi_{n,0} \rightharpoonup \Psi_0 \quad \text{in } L^2(\mathbb{R}), \tag{A-5}$$

$$v_{n,0} \rightharpoonup v_0 \quad \text{in } H^1(\mathbb{R}), \tag{A-6}$$

and, for any compact subset K of $\mathbb{R}$,

$$v_{n,0} \to v_0 \quad \text{in } L^\infty(K) \tag{A-7}$$

as $n \to +\infty$. We claim that this convergence is conserved along the flow corresponding to equations (2-32)–(2-34).[4]

[3] See Theorem 1 in [de Laire and Gravejat 2015] for more details.

[4] We only consider here positive time but the proof remains valid for negative time.

Proposition A.2. *We consider two sequences* $(\Psi_{n,0})_{n\in\mathbb{N}} \in L^2(\mathbb{R})^{\mathbb{N}}$ *and* $(v_{n,0})_{n\in\mathbb{N}} \in H^1(\mathbb{R})^{\mathbb{N}}$, *and two functions* $\Psi_0 \in L^2(\mathbb{R})$ *and* $v_0 \in H^1(\mathbb{R})$, *such that assumptions* (A-4)–(A-7) *are satisfied*, *and we denote by* (Ψ_n, v_n) *and* (Ψ, v), *respectively*, *the unique global solutions to* (2-32)–(2-34) *with initial data* $(\Psi_{n,0}, v_{n,0})$ *and* (Ψ_0, v_0), *which we assume to be defined on* $[0, T]$ *for a positive number* T. *For any fixed* $t \in [0, T]$, *we have*

$$\Psi_n(\,\cdot\,, t) \rightharpoonup \Psi(\,\cdot\,, t) \quad \text{in } L^2(\mathbb{R}), \tag{A-8}$$

$$v_n(\,\cdot\,, t) \rightharpoonup v(\,\cdot\,, t) \quad \text{in } H^1(\mathbb{R}) \tag{A-9}$$

when $n \to +\infty$.

Proof. We split the proof into four steps.

Step 1. *There exist three functions* $\Phi \in L^2([0, T], L^2(\mathbb{R}))$ *and* $\mathfrak{v} \in L^2([0, T], H^1(\mathbb{R}))$ *such that*, *up to a further subsequence*,

$$\Psi_n(t) \rightharpoonup \Phi(t) \qquad \text{in } L^2(\mathbb{R}), \tag{A-10}$$

$$v_n(\,\cdot\,, t) \rightharpoonup \mathfrak{v}(\,\cdot\,, t) \quad \text{in } H^1(\mathbb{R}), \tag{A-11}$$

$$v_n(\,\cdot\,, t) \to \mathfrak{v}(\,\cdot\,, t) \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}) \tag{A-12}$$

for all $t \in [0, T]$, *and*

$$|\Psi_n|^2\Psi_n \rightharpoonup |\Phi|^2\Phi \quad \text{in } L^2([0, T], L^2(\mathbb{R})), \tag{A-13}$$

when $n \to +\infty$.

Proof. We recall that there exists a constant M such that

$$\|\Psi_{n,0}\|_{L^2} \le M \quad \text{and} \quad \|v_{n,0}\|_{H^1} \le M$$

uniformly on n. Applying Proposition 2.7 to the pairs (Ψ_n, v_n) and $(0, 0)$, we obtain

$$\|\Psi_n\|_{\mathcal{C}^0_T L^2_x} + \|v_n\|_{\mathcal{C}^0_T H^1_x} + \|\Psi_n\|_{L^4_T L^\infty_x} \le A\big(\|\Psi_{n,0}\|_{L^2} + \|v_{n,0}\|_{H^1}\big).$$

This leads to

$$\|\Psi_n\|_{L^4_T L^\infty_x} \le 2AM, \quad \|\Psi_n\|_{L^\infty_T L^2_x} \le 2AM \quad \text{and} \quad \|v_n\|_{L^\infty_T H^1_x} \le 2AM. \tag{A-14}$$

Hence, there exist two functions $\Phi \in L^\infty([0, T], L^2(\mathbb{R})) \cap L^4([0, T], L^\infty(\mathbb{R}))$ and $\mathfrak{v} \in L^\infty([0, T], H^1(\mathbb{R}))$ such that

$$\Psi_n \overset{*}{\rightharpoonup} \Phi \quad \text{in } L^\infty([0, T], L^2(\mathbb{R})),$$

$$v_n \overset{*}{\rightharpoonup} \mathfrak{v} \quad \text{in } L^\infty([0, T], H^1(\mathbb{R})).$$

Let us prove (A-10) and (A-11). We argue as in [Béthuel et al. 2015] and we introduce a cutoff function $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$ such that $\chi \equiv 1$ on $[-1, 1]$ and $\chi \equiv 0$ on $(-\infty, 2] \cup [2, +\infty)$. Set $\chi_p(\,\cdot\,) := \chi(\,\cdot\,/p)$ for any integer $p \in \mathbb{N}^*$. By (A-14), the sequences $(\chi_p\Psi_n)_{n\in\mathbb{N}}$ and $(\chi_p v_n)_{n\in\mathbb{N}}$ are bounded in $\mathcal{C}^0([0, T], L^2(\mathbb{R}))$ and

$\mathcal{C}^0([0,T], H^1(\mathbb{R}))$, respectively. In view of the Rellich–Kondrachov theorem, the sets $\{\chi_p \Psi_n(\cdot,t) \mid n \in \mathbb{N}\}$ and $\{\chi_p v_n(\cdot,t) \mid n \in \mathbb{N}\}$ are relatively compact in $H^{-2}(\mathbb{R})$ and $H^{-1}(\mathbb{R})$, respectively, for any fixed $t \in [0,T]$. In addition, since the pair (Ψ_n, v_n) is a solution to (2-32)–(2-34), we have that $(\partial_t \Psi_n, \partial_t v_n)$ belongs to $\mathcal{C}^0([0,T], H^{-2}(\mathbb{R}) \times H^{-1}(\mathbb{R}))$ and satisfies

$$\|\partial_t \Psi_n(\cdot,t)\|_{H^{-2}(\mathbb{R})} \le K_M \quad \text{and} \quad \|\partial_t v_n(\cdot,t)\|_{H^{-1}(\mathbb{R})} \le K_M.$$

This leads to the fact that the pair $(\chi_p \Psi_n, \chi_p v_n)$ is equicontinuous in $\mathcal{C}^0([0,T], H^{-2}(\mathbb{R}) \times H^{-1}(\mathbb{R}))$. Then, we apply the Arzelà–Ascoli theorem and the Cantor diagonal argument to find a further subsequence (independent of p), such that, for each $p \in \mathbb{N}^*$,

$$\chi_p \Psi_n \to \chi_p \Phi \quad \text{in } \mathcal{C}^0([0,T], H^{-2}(\mathbb{R})), \tag{A-15}$$

$$\chi_p v_n \to \chi_p \mathfrak{v} \quad \text{in } \mathcal{C}^0([0,T], H^{-1}(\mathbb{R})) \tag{A-16}$$

as $n \to +\infty$. Combining this with (A-14) we infer that (A-10) and (A-11) hold. By the Sobolev embedding theorem, (A-12) is a consequence of (A-11).

Now, let us prove (A-13). Using the Hölder inequality, we infer that

$$\int_0^T \int_{\mathbb{R}} |\Psi_n(x,t)|^6 \, dx \, dt \le \|\Psi_n\|^2_{L^\infty L^2_x} \|\Psi_n\|^4_{L^4_T L^\infty_x}.$$

By (A-14), we conclude that

$$\big\| |\Psi_n|^2 \Psi_n \big\|_{L^2_T L^2_x} \le M. \tag{A-17}$$

So, there exists a function $\Phi_1 \in L^2(\mathbb{R} \times [0,T])$ such that up to a further subsequence,

$$|\Psi_n|^2 \Psi_n \rightharpoonup \Phi_1 \quad \text{in } L^2(\mathbb{R} \times [0,T]).$$

Let us prove that $\Phi_1 \equiv |\Phi|^2 \Phi$. To obtain this it is sufficient to prove that, up to a subsequence,

$$\Psi_n \to \Phi \quad \text{in } L^2\big([0,T], L^2([-R,R])\big) \tag{A-18}$$

for any $R > 0$, i.e., the sequence (Ψ_n) is relatively compact in $L^2([-R,R] \times [0,T])$. Indeed, using the Hölder inequality, we obtain

$$\begin{aligned} \big\| |\Psi_n|^2 \Psi_n - |\Phi|^2 \Phi \big\|_{L^{6/5}_{T,R}} &= \big\| (\Psi_n - \Phi)(|\Psi_n|^2 + |\Phi|^2) + \Psi_n \Phi(\overline{\Psi}_n - \overline{\Phi}) \big\|_{L^{6/5}_{T,R}} \\ &\le 2 \big\| |\Psi_n - \Phi| (|\Psi_n|^2 + |\Phi|^2) \big\|_{L^{6/5}_{T,R}} \\ &\le 2 \|\Psi_n - \Phi\|_{L^2_{T,R}} \big(\|\Psi_n\|^2_{L^6_{T,R}} + \|\Phi\|^2_{L^6_{T,R}} \big) \end{aligned} \tag{A-19}$$

for any $R > 0$. By (A-17), (Ψ_n) is uniformly bounded in $L^6(\mathbb{R} \times [0,T])$ and $\Phi \in L^6(\mathbb{R} \times [0,T])$. Then

$$|\Psi_n|^2 \Psi_n \to |\Phi|^2 \Phi \quad \text{in } L^{6/5}([-R,R] \times [0,T]),$$

so that $\Phi_1 \equiv |\Phi|^2 \Phi$. Now, let us prove that the sequence (Ψ_n) is relatively compact in $L^2([-R,R] \times [0,T])$. The main point of the proof is the following claim.

Claim 1. *Let* Ψ *be a solution of* (2-32) *in*

$$\mathcal{C}^0([0,T],L^2(\mathbb{R}))\cap L^4([0,T],L^\infty(\mathbb{R})).$$

Then $\Psi\in L^2\big([0,T],H^{1/2}_{\mathrm{loc}}(\mathbb{R})\big)$.

Proof. The proof relies on the Kato smoothing effect for the linear Schrödinger group (see [Linares and Ponce 2009]). Let $S(t)=e^{it\partial_{xx}}$, and

$$\mathcal{F}(\Psi,v):=\tfrac{1}{2}v^2\Psi-\mathrm{Re}\big(\Psi(1-2F(v,\overline{\Psi}))\big)\big(1-2F(v,\Psi)\big). \tag{A-20}$$

We recall that there exists a positive constant M such that

$$\sup_{x\in\mathbb{R}}\int_{-\infty}^{+\infty}|D_x^{1/2}S(t)f(x)|^2\,dt\le M\|f\|_{L^2}^2 \tag{A-21}$$

and

$$\left\|\int_{\mathbb{R}}S(-t')D_x^{1/2}h(\,\cdot\,,t')\,dt'\right\|_{L^2}\le M\|h\|_{L^1_xL^2_t} \tag{A-22}$$

when $f\in L^2(\mathbb{R})$ and $h\in L^1(\mathbb{R},L^2(\mathbb{R}))$ (see [Linares and Ponce 2009] for more details). We prove that there exists a positive constant M such that

$$\|D_x^{1/2}\Psi\|_{L^\infty_xL^2_T}\le M\|\Psi_0\|_{L^2}+M\|\Psi\|_{L^2_{T,x}}\big(\|\Psi\|^2_{L^6_{T,x}}+T^{1/2}\big(\|v\|^2_{L^\infty_{T,x}}+\|1-2F(v,\Psi)\|^2_{L^\infty_{T,x}}\big)\big). \tag{A-23}$$

The claim is a consequence of this estimate, so that it is sufficient to prove (A-23).

We write

$$\Psi(x,t)=S(t)\Psi_0(x)+i\int_0^tS(t-t')\big(2(|\Psi|^2\Psi)(x,t')+\mathcal{F}(\Psi,v)(x,t')\big)\,dt'$$

for all $(x,t)\in\mathbb{R}$. First, using (A-21), we obtain

$$\sup_{x\in\mathbb{R}}\int_{-\infty}^{+\infty}\big|D_x^{1/2}S(t)\Psi_0(x)\big|^2\,dt\le M\|\Psi_0\|^2_{L^2}.$$

For the nonlinear term, we can argue as in [Goubet and Molinet 2009] to prove that

$$\left\|\int_0^tS(t-t')D_x^{1/2}g(\,\cdot\,,t')\,dt'\right\|_{L^\infty_xL^2_T}\le M\|g\|_{L^1_TL^2_x}. \tag{A-24}$$

Using a duality argument, it is equivalent to prove that for any smooth function h that satisfies $\|h\|_{L^1_xL^2_t}\le 1$, we have

$$\left|\int_{\mathbb{R}\times[0,T]^2}S(t-t')D_x^{1/2}g(x,t')\bar{h}(x,t)\,dt'\,dx\,dt\right|\le M\|g\|_{L^1_TL^2_x}. \tag{A-25}$$

Using the Cauchy–Schwarz and Strichartz estimates and (A-22), the left-hand side can be written as

$$\begin{aligned}\left|\int_{\mathbb{R}}\left(\int_0^T S(-t')D_x^{1/2}g(x,t')\,dt'\right)\left(\int_0^T \overline{S(-t)h(x,t)}\,dt\right)dx\right| \\ = \left|\int_{\mathbb{R}}\left(\int_0^T S(-t')g(x,t')\,dt'\right)\left(\int_0^T \overline{S(-t)D_x^{1/2}h(x,t)}\,dt\right)dx\right| \\ \le M\left\|\int_0^T S(-t')g(x,t')\,dt'\right\|_{L^2} \le M\|g\|_{L^1_TL^2_x}.\end{aligned}$$

This achieves the proof of (A-24). Similarly, we have

$$\left\|\int_0^t S(t-t')D_x^{1/2}g(\,\cdot\,,t')\,dt'\right\|_{L^\infty_xL^2_T} \le M\|g\|_{L^{5/6}_{T,x}}. \tag{A-26}$$

We next apply (A-24) and (A-26) on the nonlinear terms to obtain, using the Cauchy–Schwarz and Hölder estimates,

$$\left\|\int_0^t D_x^{1/2}S(t-t')(|\Psi|^2\Psi)(\,\cdot\,,t')\,dt'\right\|_{L^\infty_xL^2_T} \le M\|\Psi^3\|_{L^{6/5}_{T,x}} \le M\|\Psi\|_{L^2_{T,x}}\|\Psi\|^2_{L^6_{T,x}}$$

and

$$\begin{aligned}\left\|\int_0^t D_x^{1/2}S(t-t')\mathcal{F}(\Psi,v)(\,\cdot\,,t')\,dt'\right\|_{L^\infty_xL^2_T} &\le M\|\mathcal{F}(\Psi,v)\|_{L^1_TL^2_x} \\ &\le M\|\Psi\|_{L^1_TL^2_x}\big(\|v\|^2_{L^\infty_{T,x}}+\|1-2F(v,\Psi)\|^2_{L^\infty_{T,x}}\big) \\ &\le MT^{1/2}\|\Psi\|_{L^2_{T,x}}\big(\|v\|^2_{L^\infty_{T,x}}+\|1-2F(v,\Psi)\|^2_{L^\infty_{T,x}}\big).\end{aligned}$$

Since $v\in L^\infty([0,T],H^1(\mathbb{R}))$ and $\Psi\in L^\infty([0,T],L^2(\mathbb{R}))$, we know that $\Psi\in L^\infty([0,T],L^2(\mathbb{R}))$ and $F(\Psi,v)\in L^\infty(\mathbb{R}\times[0,T])$. Using the fact that $\Psi\in L^6(\mathbb{R}\times[0,T])$, we finish the proof of this claim. □

Applying this claim to the sequence (Ψ_n) yields that (Ψ_n) is uniformly bounded in $L^2([0,T],H^{1/2}_{\mathrm{loc}}(\mathbb{R}))$. On the other hand, we have $\mathcal{F}(\Psi_n,v_n)\in L^\infty([0,T],L^2(\mathbb{R}))$, since

$$v_n\in L^\infty([0,T],H^1(\mathbb{R})),\quad \Psi_n\in L^\infty([0,T],L^2(\mathbb{R}))\quad\text{and}\quad F(\Psi_n,v_n)\in L^\infty(\mathbb{R}\times[0,T]).$$

Then, using (2-32) and (A-17), we obtain that (Ψ_n) is uniformly bounded in $H^1([0,T],H^{-2}(\mathbb{R}))$. Hence, by interpolation, $(\Psi_n)\in H^{1/10}([0,T],H^{1/4}_{\mathrm{loc}}(\mathbb{R}))$, so that it converges in $L^2([-R,R]\times[0,T])$ for any $R>0$. This finishes the proofs of (A-18) and of Step 1. □

Step 2. *We have*

$$\mathcal{F}(\Psi_n,v_n)\rightharpoonup\mathcal{F}(\Phi,\mathfrak{v})\quad in\ L^2(\mathbb{R}), \tag{A-27}$$

for any $t\in[0,T]$, and

$$\mathcal{F}(\Psi_n,v_n)\to\mathcal{F}(\Phi,\mathfrak{v})\quad in\ L^1([0,T],L^2_{\mathrm{loc}}(\mathbb{R})). \tag{A-28}$$

Proof. Let $\phi \in L^2(\mathbb{R})$. We compute

$$\int_{\mathbb{R}} (v_n^2(x,t)\Psi_n(x,t) - \mathfrak{v}^2(x,t)\Phi(x,t))\phi(x)\,dx \\ = \int_{\mathbb{R}} (v_n^2(x,t) - \mathfrak{v}^2(x,t))\Psi_n(x,t)\phi(x)\,dx + \int_{\mathbb{R}} (\Psi_n(x,t) - \Phi(x,t))\mathfrak{v}^2(x,t)\phi(x)\,dx. \quad \text{(A-29)}$$

The second term on the right-hand side goes to 0 when n goes to $+\infty$, since $\mathfrak{v}^2(t)\phi \in L^2(\mathbb{R})$ for all t on the one hand and using (A-10) on the other hand. For the first term on the right-hand side, we consider a cutoff function χ with support in $[-1, 1]$ and let $\chi_R(x) = \chi(x/R)$ for all $(x, R) \in \mathbb{R} \times (0, +\infty)$. We set

$$I_n(t) := \int_{\mathbb{R}} (v_n^2(x,t) - \mathfrak{v}^2(x,t))\Psi_n(x,t)\phi(x)\,dx,$$
$$I_n^{(1)}(t) := \int_{\mathbb{R}} (v_n^2(x,t) - \mathfrak{v}^2(x,t))\Psi_n(x,t)\chi_R(x)\phi(x)\,dx,$$
$$I_n^{(2)}(t) := \int_{\mathbb{R}} (v_n^2(x,t) - \mathfrak{v}^2(x,t))\Psi_n(x,t)(1-\chi_R(x))\phi(x)\,dx,$$

so that $I_n(t) = I_n^{(1)}(t) + I_n^{(2)}(t)$. By the Cauchy–Schwarz inequality, we have

$$|I_n^{(1)}(t)| \le \|\Psi_n(t)\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|v_n^2(t) - \mathfrak{v}^2(t)\|_{L^\infty([-R,R])}. \quad \text{(A-30)}$$

Using (A-12) and (A-14), we infer that

$$I_n^{(1)}(t) \to 0 \quad \text{for any } t \in [0, T], \quad \text{(A-31)}$$

as $n \to +\infty$. Next, we write

$$|I_n^{(2)}(t)| \le \big(\|v_n(t)\|_{L^\infty(\mathbb{R})}^2 + \|\mathfrak{v}(t)\|_{L^\infty(\mathbb{R})}^2\big) \|\Psi_n(t)\|_{L^2(\mathbb{R})} \|(1-\chi_R)\phi\|_{L^2(\mathbb{R})}.$$

Since $\phi \in L^2(\mathbb{R})$, we have

$$\lim_{R\to\infty} \|(1-\chi_R)\phi\|_{L^2(\mathbb{R})} = 0.$$

In view of (A-14), this is sufficient to prove that

$$I_n(t) \to 0 \quad \text{(A-32)}$$

as $n \to +\infty$, for all $t \in [0, T]$. This yields

$$(v_n^2\Psi_n)(t) \rightharpoonup (\mathfrak{v}^2\Phi)(t) \quad \text{in } L^2(\mathbb{R}) \quad \text{(A-33)}$$

for any $t \in [0, T]$. Now, we prove

$$v_n^2\Psi_n \to \mathfrak{v}^2\Phi \quad \text{in } L^1([0,T], L^2_{\mathrm{loc}}(\mathbb{R})). \quad \text{(A-34)}$$

As in (A-29), we write

$$\|v_n^2\Psi_n - \mathfrak{v}^2\Phi\|_{L^1_T L^2_R} \le \|(v_n^2 - \mathfrak{v}^2)\Psi_n\|_{L^1_T L^2_R} + \|(\Psi_n - \Phi)\mathfrak{v}^2\|_{L^1_T L^2_R}.$$

For the first term on the right-hand side, we infer from the Cauchy–Schwarz inequality that

$$\begin{aligned}\|(v_n^2-\mathfrak{v}^2)\Psi_n\|_{L^1_T L^2_R} &\le \|v_n^2-\mathfrak{v}^2\|_{L^2_T L^2_R}\|\Psi_n\|_{L^2_T L^\infty_R}\\ &\le \|v_n-\mathfrak{v}\|_{L^4_T L^4_R}\big(\|v_n\|_{L^4_T L^4_R}+\|\mathfrak{v}\|_{L^4_T L^4_R}\big)T^{1/2}\|\Psi_n\|_{L^4_T L^\infty_R}.\end{aligned}$$

On the other hand, by (A-14), v_n is uniformly bounded on $L^2([0,T],H^1(\mathbb{R}))$. By the first equation of (2-34) and (A-14), v_n is uniformly bounded in $H^1([0,T],H^{-1}(\mathbb{R}))$. We deduce that v_n is uniformly bounded in $H^{1/3}([0,T],H^{1/3}(\mathbb{R}))$ and so that v_n converges to $\mathfrak{v}$ in $L^4([0,T],L^4([-R,R]))$ as $n\to+\infty$. Hence, using (A-14) once again, we obtain

$$\|(v_n^2-\mathfrak{v}^2)\Psi_n\|_{L^1_T L^2_R}\to 0$$

as $n\to+\infty$. For the second term we have, by the Cauchy–Schwarz inequality and the Sobolev embedding theorem,

$$\|(\Psi_n-\Phi)\mathfrak{v}^2\|_{L^1_T L^2_R}\le \|\Psi_n-\Phi\|_{L^2_T L^2_R}\|\mathfrak{v}^2\|_{L^2_T L^\infty_R}\le M^2T^{1/2}\|\Psi_n-\Phi\|_{L^2_T L^2_R}.$$

This yields, using (A-18),

$$\|(\Psi_n-\Phi)\mathfrak{v}^2\|_{L^1_T L^2_R}\to 0$$

as $n\to+\infty$, which proves (A-34). Next, we set

$$\mathcal{G}(v_n,\Psi_n)=\Psi_n(1-F(v_n,\overline{\Psi}_n))(1-F(v_n,\Psi_n)).$$

We have, by (2-33),

$$\partial_x F(v_n,\Psi_n)=v_n\Psi_n \quad\text{and}\quad \partial_x F(\mathfrak{v},\Phi)=\mathfrak{v}\Phi.$$

Using the same arguments as in the proof of (A-32), we obtain

$$\partial_x F(v_n,\Psi_n)\rightharpoonup \partial_x F(\mathfrak{v},\Phi)\quad\text{in } L^2(\mathbb{R}))$$

for any $t\in[0,T]$. Hence,

$$F(v_n,\Psi_n)\to F(\mathfrak{v},\Phi)\quad\text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}) \tag{A-35}$$

for any $t\in[0,T]$. Using (A-10), (A-35) and the same arguments as in the proof of (A-33), we conclude that

$$\mathcal{G}(v_n,\Psi_n)\rightharpoonup \mathcal{G}(\mathfrak{v},\Phi)\quad\text{in } L^2(\mathbb{R}) \tag{A-36}$$

for any $t\in[0,T]$. Next, we use (A-18) and (A-35) to prove that

$$\mathcal{G}(v_n,\Psi_n)\to \mathcal{G}(\mathfrak{v},\Phi)\quad\text{in } L^1([0,T],L^2_{\mathrm{loc}}(\mathbb{R})). \tag{A-37}$$

This finishes the proof of this step. □

Step 3. $(\Phi,\mathfrak{v})$ *is a weak solution of* (2-32)–(2-34).

Proof. By (A-18), we have

$$i\partial_t\Psi_n\to i\partial_t\Phi\quad\text{in } \mathcal{D}'(\mathbb{R}\times[0,T])\quad\text{and}\quad \partial^2_{xx}\Psi_n\to\partial^2_{xx}\Phi\quad\text{in } \mathcal{D}'(\mathbb{R}\times[0,T])$$

as $n \to +\infty$. It remains to invoke (A-13) and (A-35) and to take the limit $n \to +\infty$ in the expression

$$\int_0^T \int_{\mathbb{R}} \big(i\partial_t \Psi_n + \partial_{xx}^2 \Psi_n + 2|\Psi_n|^2\Psi_n + \tfrac{1}{2}v_n^2\Psi_n - \operatorname{Re}\big(\Psi(1-2F(v_n, \overline{\Psi_n}))\big)(1-2F(v_n, \Psi_n))\big)\bar{h} = 0,$$

where $h \in \mathcal{C}_c^\infty(\mathbb{R} \times [0, T])$, in order to establish that $(\Phi, \mathfrak{v})$ is a solution to (2-32) in the sense of distributions. In addition, using the same arguments as above and (A-35), we prove that $(\Phi, \mathfrak{v})$ is a solution to (2-34) in the sense of distributions. Moreover, we infer from (A-5) that $\Phi(\cdot, 0) = \Psi_0$ and from (A-6) that $\mathfrak{v}(\cdot, 0) = v_0$. □

In order to prove that the function $(\Phi, \mathfrak{v})$ coincides with the solution (Ψ, v) in Proposition A.2, it is sufficient, in view of the uniqueness result given by Proposition 2.7, to establish the following.

Step 4. $\Phi \in \mathcal{C}([0, T], L^2(\mathbb{R}))$ *and* $\mathfrak{v} \in \mathcal{C}([0, T], H^1(\mathbb{R}))$.

Proof. First, we prove that $\Phi \in \mathcal{C}([0, T], L^2(\mathbb{R}))$. This is a direct consequence of the identity

$$\Phi(x, t) = S(t)\Phi_0 + \int_0^t S(t-t')\big(2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathfrak{v})(\cdot, t')\big)\, dt'. \tag{A-38}$$

Indeed, let us define

$$G(\Phi, \mathfrak{v})(t) = \int_0^t S(t-t')\big(2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathfrak{v})(\cdot, t')\big)\, dt'.$$

Since $S(t)\Phi_0 \in \mathcal{C}([0, T], L^2(\mathbb{R}))$, it suffices to show $G(\Phi, \mathfrak{v}) \in \mathcal{C}([0, T], L^2(\mathbb{R}))$. We take $(t_1, t_2) \in [0, T]^2$ and write

$$\begin{aligned} G(\Phi, \mathfrak{v})(t_1) - G(\Phi, \mathfrak{v})(t_2) = \int_0^{t_1} (S(t_1-t') - S(t_2-t'))\big(2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathfrak{v})(\cdot, t')\big)\, dt' \\ - \int_{t_1}^{t_2} S(t-t')\big(2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathfrak{v})(\cdot, t')\big)\, dt'. \end{aligned}$$

For the second term on the right-hand side, we use the Strichartz and Cauchy–Schwarz inequalities to obtain

$$\begin{aligned} &\Big\| \int_{t_1}^{t_2} S(t-t')\big(2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathfrak{v})(\cdot, t')\big)\, dt' \Big\|_{L^2} \\ &\qquad \le M \big\| 2|\Phi|^2\Phi + \mathcal{F}(\Phi, \mathfrak{v}) \big\|_{L^1([t_1, t_2], L^2(\mathbb{R}))} \\ &\qquad \le M|t_1 - t_2|^{1/2} \| |\Phi|^2\Phi \|_{L^2_{T,x}} + M|t_1 - t_2| \|\mathcal{F}(\Phi, \mathfrak{v})\|_{L^\infty_T L^2_x}. \end{aligned} \tag{A-39}$$

For the first term, we write

$$S(t_1 - t') - S(t_2 - t') = S(t_1 - t')(1 - S(t_2 - t_1)).$$

Hence,

$$\begin{aligned} &\Big\| \int_0^{t_1} (S(t_1-t') - S(t_2-t'))\big(2(|\Phi|^2\Phi)(\cdot, t') + \mathcal{F}(\Phi, \mathfrak{v})(\cdot, t')\big)\, dt' \Big\|_{L^2} \\ &\qquad = \|(1 - S(t_2 - t_1))G(\Phi, \mathfrak{v})(t_1)\|_{L^2}. \end{aligned} \tag{A-40}$$

Taking the limit $t_2 \to t_1$ in (A-39) and (A-40), we obtain that $\Phi \in \mathcal{C}([0,T], L^2(\mathbb{R}))$.

Now, let us prove (A-38). Denote by $\widetilde{\Phi}$ the function given by the right-hand side of (A-38). We will prove that

$$\Psi_n(t) \rightharpoonup \widetilde{\Phi}(t) \quad \text{in } L^2(\mathbb{R}) \tag{A-41}$$

for all $t \in \mathbb{R}$. This yields $\Phi \equiv \widetilde{\Phi}$ by uniqueness of the weak limit. Let $R > 0$ and denote by χ_R the function defined in Step 2. Set

$$\begin{aligned}
G_n^{(1)}(\,\cdot\,,t) &= \int_0^t S(t-t')\chi_R\big(2(|\Psi_n|^2\Psi_n)(\,\cdot\,,t') + \mathcal{F}(\Psi_n, v_n)(\,\cdot\,,t')\big)\,dt',\\
G_n^{(2)}(\,\cdot\,,t) &= \int_0^t S(t-t')(1-\chi_R)\big(2(|\Psi_n|^2\Psi_n)(\,\cdot\,,t') + \mathcal{F}(\Psi_n, v_n)(\,\cdot\,,t')\big)\,dt',\\
G^{(1)}(\,\cdot\,,t) &= \int_0^t S(t-t')\chi_R\big(2(|\Phi|^2\Phi)(\,\cdot\,,t') + \mathcal{F}(\Phi, \mathfrak{v})(\,\cdot\,,t')\big)\,dt',\\
G^{(2)}(\,\cdot\,,t) &= \int_0^t S(t-t')(1-\chi_R)\big(2(|\Phi|^2\Phi)(\,\cdot\,,t') + \mathcal{F}(\Phi, \mathfrak{v})(\,\cdot\,,t')\big)\,dt',
\end{aligned}$$

for all $t \in \mathbb{R}$, so that $G(\Phi, \mathfrak{v}) = G^{(1)} + G^{(2)}$ and $G(\Psi_n, v_n) = G_n^{(1)} + G_n^{(2)}$. Since $S(t)\Psi_{n,0} \rightharpoonup S(t)\Phi_0$ in $L^2(\mathbb{R})$ as $n \to +\infty$ for all $t \in \mathbb{R}$, it is sufficient to show that

$$G(\Psi_n, v_n)(t) \rightharpoonup G(\Phi, \mathfrak{v})(t) \quad \text{in } L^2(\mathbb{R})$$

as $n \to +\infty$ for all $t \in \mathbb{R}$. Let $\varphi \in L^2(\mathbb{R})$. We write

$$\begin{aligned}
&\big(G(\Psi_n, v_n)(t) - G(\Phi, \mathfrak{v})(t), \varphi\big)_{L^2}\\
&\qquad = \int_{-\infty}^{+\infty} \big[G_n^{(1)}(x,t) - G^{(1)}(x,t)\big]\overline{\varphi(x)}\,dx + \int_{-\infty}^{+\infty} \big[G_n^{(2)}(x,t) - G^{(2)}(x,t)\big]\overline{\varphi(x)}\,dx\\
&\qquad = I_n^R(t) + J_n^R(t).
\end{aligned}$$

For the first integral, using the Cauchy–Schwartz inequality, the Strichartz estimates for the admissible pairs $(6,6)$ and $(\infty,2)$, the Hölder inequality and (A-19), there exists a positive constant M such that for all $t \in [0,T]$ we have

$$\begin{aligned}
|I_n^R(t)| &\le \|G_n^{(1)}(t) - G^{(1)}(t)\|_{L^2}\|\varphi\|_{L^2}\\
&\le M\|\varphi\|_{L^2}\Big(\big\||\Psi_n|^2\Psi_n - |\Phi|^2\Phi\big\|_{L^{6/5}_{T,R}} + \|\mathcal{F}(\Psi_n, v_n) - \mathcal{F}(\Phi, \mathfrak{v})\|_{L^1_T L^2_R}\Big)\\
&\le M\|\varphi\|_{L^2}\Big(\|\mathcal{F}(\Psi_n, v_n) - \mathcal{F}(\Phi, \mathfrak{v})\|_{L^1_T L^2_R} + \|\Psi_n - \Phi\|_{L^2_{T,R}}\big(\|\Psi_n\|^2_{L^6_{T,R}} + \|\Phi\|^2_{L^6_{T,R}}\big)\Big).
\end{aligned}$$

Then, using (A-18) and (A-28), we obtain for all $t \in \mathbb{R}$

$$|I_n^R(t)| \to 0 \quad \text{as } n \to \infty.$$

Next, using the Hölder inequality we have

$$|J_n^R(t)| \leq 2\biggl(\int_0^T\int_{-\infty}^{\infty}\bigl||\Psi_n|^2\Psi_n(x,t')-|\Phi|^2\Phi(x,t')\bigr|^{6/5}\,dx\,dt'\biggr)^{5/6}\biggl(\int_0^T\int_{|x|\geq R}|S(t-t')\varphi|^6\,dx\,dt'\biggr)^{1/6} + \int_0^T\biggl(\int_{-\infty}^{\infty}\bigl|\mathcal{F}(\Psi_n,v_n)(x,t')-\mathcal{F}(\Phi,\mathfrak{v})(x,t')\bigr|^2\,dx\biggr)^{1/2}dt' \sup_{t'\in[0,T]}\biggl(\int_{|x|\geq R}|S(t-t')\varphi(x)|^2\,dx\biggr)^{1/2}.$$

The terms on the right-hand side are bounded by a constant independent of n. Besides, since $(6, 6)$ and $(\infty, 2)$ are admissible pairs, we have

$$\|S(t)\varphi\|_{L^6_{T,x}} \leq M\|\varphi\|_{L^2(\mathbb{R})},$$
$$\|S(t)\varphi\|_{L^\infty_T L^2(\mathbb{R})} \leq M\|\varphi\|_{L^2(\mathbb{R})},$$

so that, by the dominated convergence theorem and the fact that $t\mapsto S(t)$ is uniformly continuous from $[0, T]$ to $L^2(\mathbb{R})$, we obtain

$$\lim_{R\to\infty}\int_0^T\int_{|x|\geq R}|S(t)\varphi|^6\,dx\,dt = \lim_{R\to\infty}\sup_{t\in[0,T]}\biggl(\int_{|x|\geq R}|S(t)\varphi(x)|^2\,dx\biggr)^{1/2} = 0.$$

Hence,

$$\lim_{R\to\infty}|J_n^R(t)| = 0 \quad \text{uniformly with respect to } n\in\mathbb{N}$$

for any $t\in[0, T]$. This completes the proof of (A-41) and then of (A-38). This leads to the fact that $\Phi\in\mathcal{C}^0([0, T], L^2(\mathbb{R}))$.

Now, let us prove that $\mathfrak{v}\in\mathcal{C}^0([0, T], H^1(\mathbb{R}))$. Since $(\Phi, \mathfrak{v})$ satisfies the first equation in (2-34), $\Phi\in L^\infty([0, T], L^2(\mathbb{R}))$ and $F(\Psi, \mathfrak{v})\in L^\infty([0, T], L^\infty(\mathbb{R}))$, we have $\mathfrak{v}\in H^1([0, T], H^{-1}(\mathbb{R}))$. This yields, using the Sobolev embedding theorem, $\mathfrak{v}\in\mathcal{C}^0([0, T], H^{-1}(\mathbb{R}))$. Let $(t_1, t_2)\in[0, T]^2$. We can write that

$$\begin{aligned}\int_{\mathbb{R}}|\mathfrak{v}(t_1,x)-\mathfrak{v}(t_2,x)|^2\,dx &= \bigl\langle \mathfrak{v}(t_1,x)-\mathfrak{v}(t_2,x),\, \mathfrak{v}(t_1,x)-\mathfrak{v}(t_2,x)\bigr\rangle_{H^{-1},H^1}\\ &\leq \|\mathfrak{v}(t_1,x)-\mathfrak{v}(t_2,x)\|_{H^{-1}}\|\mathfrak{v}(t_1,x)-\mathfrak{v}(t_2,x)\|_{H^1}.\end{aligned}$$

Since $\mathfrak{v}\in\mathcal{C}^0([0, T], H^{-1}(\mathbb{R}))\cap L^\infty([0, T], H^1(\mathbb{R}))$, we obtain $\mathfrak{v}\in\mathcal{C}^0([0, T], L^2(\mathbb{R}))$. Next, we write

$$\|F(\mathfrak{v},\Phi)(t_1)-F(\mathfrak{v},\Phi)(t_2)\|_{L^\infty(\mathbb{R})} \leq \|\mathfrak{v}(t_1)-\mathfrak{v}(t_2)\|_{L^2}\|\Phi(t_1)\|_{L^2} + \|\Phi(t_2)-\Phi(t_1)\|_{L^2}\|\mathfrak{v}(t_2)\|_{L^2}.$$

Using the fact that Φ, $\mathfrak{v}\in\mathcal{C}^0([0, T], L^2(\mathbb{R}))$, we infer that $F(\mathfrak{v}, \Phi)\in\mathcal{C}^0([0, T], L^\infty(\mathbb{R}))$. Then, by the second equation in (2-34), $\mathfrak{v}\in\mathcal{C}^0([0, T], H^1(\mathbb{R}))$. This finishes the proof of this step, and of Proposition A.2. □

Finally, we give the proof of Proposition A.1.

Proof of Proposition A.1. In view of Proposition A.2, it is sufficient to prove the convergence of w_n. The proof follows the arguments in the proof of (A-27). Let $\phi \in L^2(\mathbb{R})$. We rely on (4-41) to write

$$\begin{aligned}\int_{\mathbb{R}} &[w^*(t,x) - w_n(t,x)]\phi(x)\,dx \\ &= 2\int_{\mathbb{R}} \operatorname{Im}\Big(\frac{\Psi^*(t,x)(1-2F(v^*,\Psi^*)(t,x))}{1-(v^*)^2(t,x)} - \frac{\Psi_n(t,x)(1-2F(v_n,\Psi_n)(t,x))}{1-(v_n)^2(t,x)}\Big)\phi(x)\,dx \\ &= 2\int_{\mathbb{R}} \operatorname{Im}\Big(\frac{\Psi^*(t,x)}{1-(v^*)^2(t,x)} - \frac{\Psi_n(t,x)}{1-(v_n)^2(t,x)}\Big)\phi(x)\,dx \\ &\qquad - 4\int_{\mathbb{R}} \operatorname{Im}\Big(\frac{\Psi^*(t,x)F(v^*,\Psi^*)(t,x)}{1-(v^*)^2(t,x)} - \frac{\Psi_n(t,x)F(v_n,\Psi_n)(t,x)}{1-(v_n)^2(t,x)}\Big)\phi(x)\,dx\end{aligned}$$

for all $t \in [0, T]$. Then, we use the same arguments as in the proof of (A-27) to show that the two last terms on the right-hand side go to 0 when n goes to $+\infty$. This finishes the proof of the proposition. □

Exponential decay of χ_c. In this subsection, we recall the explicit formula and some useful properties of the operator $\mathcal{H}_c$, and then study its negative eigenfunction χ_c. For $c \in (-1, 1) \setminus \{0\}$, the operator $\mathcal{H}_c$ is given in explicit terms by

$$\mathcal{H}_c(\varepsilon) = \begin{pmatrix} \mathcal{L}_c(\varepsilon_v) + c^2\dfrac{(1+v_c^2)^2}{(1-v_c^2)^3}\varepsilon_v - c\dfrac{1+v_c^2}{1-v_c^2}\varepsilon_w \\ -c\dfrac{1+v_c^2}{1-v_c^2}\varepsilon_v + (1-v_c^2)\varepsilon_w \end{pmatrix}, \tag{A-42}$$

where $\varepsilon = (\varepsilon_v, \varepsilon_w)$ and

$$\mathcal{L}_c(\varepsilon_v) = -\partial_x\Big(\frac{\partial_x \varepsilon_v}{1-v_c^2}\Big) + \big(1 - c^2 - (5+c^2)v_c^2 + 2v_c^4\big)\frac{\varepsilon_v}{(1-v_c^2)^2}.$$

In view of (A-42), the operator $\mathcal{H}_c$ is an isomorphism from $H^2(\mathbb{R}) \times L^2(\mathbb{R}) \cap \operatorname{Span}(\partial_x Q_c)^\perp$ onto $\operatorname{Span}(\partial_x Q_c)^\perp$. In addition, there exists a positive number A_c, depending continuously on c, such that

$$\|\mathcal{H}_c^{-1}(f,g)\|_{H^{k+2}(\mathbb{R})\times H^k(\mathbb{R})} \le A_c\|(f,g)\|_{H^k(\mathbb{R})^2} \tag{A-43}$$

for any $(f, g) \in H^k(\mathbb{R})^2 \cap \operatorname{Span}(\partial_x Q_c)^\perp$ and any $k \in \mathbb{N}$.

The following proposition establishes the coercivity of the quadratic form H_c under suitable orthogonality conditions.

Proposition A.3. *Let $c \in (-1, 1) \setminus \{0\}$. There exists a positive number Λ_c, depending only on c, such that*

$$H_c(\varepsilon) \ge \Lambda_c\|\varepsilon\|^2_{H^1\times L^2} \tag{A-44}$$

for any pair $\varepsilon \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfying the two orthogonality conditions

$$\langle \partial_x Q_c, \varepsilon\rangle_{L^2\times L^2} = \langle \chi_c, \varepsilon\rangle_{L^2\times L^2} = 0. \tag{A-45}$$

Moreover, the map $c \mapsto \Lambda_c$ is uniformly bounded from below on any compact subset of $(-1, 1) \setminus \{0\}$.

The proof relies on standard Sturm–Liouville theory (see, e.g., the proof of Proposition 1 in [de Laire and Gravejat 2015] for more details).

Now, we turn to the analysis of the pair χ_c.

Lemma A.4. *The pair χ_c belongs to $\mathcal{C}^\infty(\mathbb{R}) \times \mathcal{C}^\infty(\mathbb{R})$. In addition, there exist two positive numbers A_c and a_c, depending continuously on c, such that $a_c > \sqrt{1-c^2}$ and*

$$|\partial_x^k \chi_c| \le A_c e^{-a_c|x|} \quad \textit{on } \mathbb{R} \textit{ for } k \in \{0, 1, 2\}. \tag{A-46}$$

Proof. We set $\chi_c := (\zeta_c, \xi_c)$. Since $\mathcal{H}_c(\chi_c) = -\tilde{\lambda}_c \chi_c$, we have the following system

$$-\partial_x\left(\frac{\partial_x \zeta_c}{1-v_c^2}\right) + \big(1 - c^2 - (5+c^2)v_c^2 + 2v_c^4\big)\frac{\zeta_c}{(1-v_c^2)^2} + c^2\frac{(1+v_c^2)^2}{(1-v_c^2)^3}\zeta_c - c\frac{1+v_c^2}{1-v_c^2}\xi_c = -\tilde{\lambda}_c\zeta_c, \tag{A-47}$$

$$c\frac{1+v_c^2}{1-v_c^2}\zeta_c = (1 - v_c^2 + \tilde{\lambda}_c)\xi_c. \tag{A-48}$$

It follows from standard elliptic theory that $\chi_c \in H^2(\mathbb{R}) \times L^2(\mathbb{R})$. Since the coefficients in (A-48) are smooth and bounded from above and below, we infer from a standard bootstrap argument that $\chi_c \in \mathcal{C}^\infty(\mathbb{R}) \times \mathcal{C}^\infty(\mathbb{R})$. Notice in particular that, by the Sobolev embedding theorem, χ_c and $\partial_x \chi_c$ are bounded on $\mathbb{R}$. Then, we deduce from the first statement in (5-11) that[5]

$$-\partial_{xx}\zeta_c + (1+\tilde{\lambda}_c)\zeta_c - c\xi_c = \mathcal{O}(v_c^2), \tag{A-49}$$

$$\zeta_c = \frac{1+\tilde{\lambda}_c}{c}\xi_c + \mathcal{O}(v_c^2). \tag{A-50}$$

Note that we have

$$B_c \exp\big(-\sqrt{1-c^2}|x|\big) \le v_c(x) \le A_c \exp\big(-\sqrt{1-c^2}|x|\big) \quad \text{for all } x \in \mathbb{R}, \tag{A-51}$$

where B_c and A_c are two positive numbers.

In order to prove (A-46), we now introduce (A-50) into (A-49) to obtain

$$-\partial_{xx}\zeta_c + b_c^2\zeta_c = \mathcal{O}\big(\exp\big(-2\sqrt{1-c^2}|x|\big)\big), \tag{A-52}$$

$$\xi_c = \frac{c}{1+\tilde{\lambda}_c}\zeta_c + \mathcal{O}\big(\exp\big(-2\sqrt{1-c^2}|x|\big)\big), \tag{A-53}$$

with $b_c^2 = \dfrac{1-c^2+2\tilde{\lambda}_c+(\tilde{\lambda}_c)^2}{1+\tilde{\lambda}_c} > 1-c^2$. Next, we set

$$g_c := -\partial_{xx}\zeta_c + b_c^2\zeta_c, \tag{A-54}$$

so that $g_c(x) = \mathcal{O}\big(\exp\big(-2\sqrt{1-c^2}|x|\big)\big)$ for all $x \in \mathbb{R}$. Using the variation of constants method, we obtain, for all $x \in \mathbb{R}$,

$$\zeta_c(x) = A(x)e^{b_c x} + A_c e^{b_c x} + B(x)e^{-b_c x} + B_c e^{-b_c x},$$

[5] The notation $\mathcal{O}(v_c^2)$ refers to a quantity bounded by $A_c v_c^2$ (pointwise), where the positive number A_c depends only on c.

with

$$A(x) = \frac{-1}{2b_c} \int_0^x e^{-b_c t} g_c(t)\, dt$$

and

$$B(x) = \frac{-1}{2b_c} \int_0^x e^{b_c t} g_c(t)\, dt.$$

Since $\zeta_c \in L^2(\mathbb{R})$, this leads to

$$\zeta_c(x) = \mathcal{O}\big(\exp\big(-2\sqrt{1-c^2}|x|\big) + \exp(-b_c|x|)\big).$$

Hence, we can take $a_c = \min\big\{2\sqrt{1-c^2}, b_c\big\}$ and invoke (A-50) to obtain (A-46) for $k = 0$. Using (5-10), (5-11), (A-47), (A-48) and (A-51), we extend (A-46) to $k \in 1, 2$. □

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ON THE WELL-POSEDNESS OF THE GENERALIZED KORTEWEG–DE VRIES EQUATION IN SCALE-CRITICAL $\hat{L}^r$-SPACE

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The purpose of this paper is to study local and global well-posedness of the initial value problem for the generalized Korteweg–de Vries (gKdV) equation in $\hat{L}^r = \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{\hat{L}^r} = \|\hat{f}\|_{L^{r'}} < \infty\}$. We show (large-data) local well-posedness, small-data global well-posedness, and small-data scattering for the gKdV equation in the scale-critical $\hat{L}^r$-space. A key ingredient is a Stein–Tomas-type inequality for the Airy equation, which generalizes the usual Strichartz estimates for $\hat{L}^r$-framework.

1. Introduction

We consider the initial value problem for the generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u = \mu \partial_x(|u|^{\alpha-1}u), & t, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1-1}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $u_0 : \mathbb{R} \to \mathbb{R}$ is a given function, and $\mu \in \mathbb{R}\backslash\{0\}$ and $\alpha > 1$ are constants. We say that (1-1) is defocusing if $\mu > 0$ and focusing if $\mu < 0$.

The class of equations (1-1) arises in several fields of physics. Equation (1-1) with $\alpha = 2$ is the notable Korteweg–de Vries equation [1895], which models long waves propagating in a channel. Equation (1-1) with $\alpha = 3$ is also well-known as the modified Korteweg–de Vries equation, which describes a time evolution for the curvature of certain types of helical space curves [Lamb 1977].

Equation (1-1) has the following scale invariance: if $u(t,x)$ is a solution to (1-1), then

$$u_\lambda(t,x) := \lambda^{\frac{2}{\alpha-1}} u(\lambda^3 t, \lambda x)$$

is also a solution to (1-1) with initial data $u_\lambda(0,x) = \lambda^{\frac{2}{\alpha-1}} u_0(\lambda x)$ for any $\lambda > 0$. In what follows, a Banach space for initial data is referred to as a *scale-critical space* if its norm is invariant under $u_0(x) \mapsto \lambda^{\frac{2}{\alpha-1}} u_0(\lambda x)$.

The purpose of this paper is to study (large-data) local well-posedness, small-data global well-posedness and scattering for (1-1) in a scale-critical space $\hat{L}^{\frac{\alpha-1}{2}}$. For $r \in [1,\infty]$, the function space $\hat{L}^r$ is defined by

$$\hat{L}^r = \hat{L}^r(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{\hat{L}^r} = \|\hat{f}\|_{L^{r'}} < \infty\},$$

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where $\hat{f}$ stands for the Fourier transform of f with respect to the space variable and r' denotes the Hölder conjugate of r. We use the conventions $1' = \infty$ and $\infty' = 1$. Our notion of well-posedness consists of existence, uniqueness, and continuity of the data-to-solution map. We also consider the persistence property of the solution; that is, the solution describes a continuous curve in the function space X whenever $u_0 \in X$.

Local well-posedness of the initial value problem (1-1) in a scale-subcritical Sobolev space $H^s(\mathbb{R})$, $s > s_\alpha := \frac{1}{2} - \frac{2}{\alpha-1}$, has been studied by many authors [Bourgain 1993; Grünrock 2005b; Guo 2009; Kato 1983; Kenig et al. 1993; 1996; Kishimoto 2009; Molinet and Ribaud 2003], where s_α, a scale-critical exponent, is the unique number such that $\dot{H}^{s_\alpha}$ becomes scale critical. A fundamental work on local well-posedness is due to Kenig, Ponce, and Vega [Kenig et al. 1993]. They proved that (1-1) is locally well-posed in $H^s(\mathbb{R})$ with $s > \frac{3}{4}$ $\left(\alpha = 2,\, s_2 = -\frac{3}{2}\right)$, $s \geqslant \frac{1}{4}$ $\left(\alpha = 3,\, s_3 = -\frac{1}{2}\right)$, $s \geqslant \frac{1}{12}$ $\left(\alpha = 4,\, s_4 = -\frac{1}{6}\right)$ and $s \geqslant s_\alpha$ $(\alpha \geqslant 5)$. Introducing Fourier restriction norms, Bourgain [1993] obtained local (and global[1]) well-posedness of the KdV equation (i.e., (1-1) with $\alpha = 2$) in $L^2(\mathbb{R})$. In [Kenig et al. 1996], Kenig, Ponce, and Vega improved the previous results for the KdV equation to $H^s(\mathbb{R})$ with $s > -\frac{3}{4}$. Further, Guo [2009] and Kishimoto [2009] extended the result of Kenig et al. in $H^{-\frac{3}{4}}(\mathbb{R})$. (See also [Buckmaster and Koch 2015] on the existence of a weak solution to the KdV equation at H^{-1}.) Grünrock [2005b] has shown local well-posedness of the quartic KdV equation ((1-1) with $\alpha = 4$) in H^s with $s > s_4$. Notice that all of the above results are based on the contraction mapping principle for the corresponding integral equation. Hence, a data-solution map associated with (1-1) is Lipschitz continuous.[2]

Concerning the well-posedness of (1-1) in the scale-critical $\dot{H}^{s_\alpha}$-space, Kenig et al. [1993] proved local well-posedness and global well-posedness for small data in the scale-critical space $\dot{H}^{s_\alpha}$ when $\alpha \geqslant 5$. Since the scale-critical exponent s_α is negative in the mass-subcritical case $\alpha < 5$, the well-posedness of (1-1) in $\dot{H}^{s_\alpha}$ becomes rather a difficult problem. Tao [2007] proved local well-posedness and global well-posedness for small data for (1-1) with the quartic nonlinearity[3] $\alpha = 4$ in $\dot{H}^{s_4}$. Later on, the above results were extended to a homogeneous Besov space $\dot{B}^{s_\alpha}_{2,\infty}$ by Koch and Marzuola [2012] $(\alpha = 4)$ and Strunk [2014] $(\alpha \geqslant 5)$. As far as we know, local well-posedness and small-data global well-posedness of (1-1) in $\dot{H}^{s_\alpha}$ for the mass-subcritical case $\alpha < 5$ were open except for the case $\alpha = 4$.

Local and global well-posedness for a class of nonlinear dispersive equations is currently being intensively investigated also in the framework of $\hat{L}^r$-space. For the one-dimensional nonlinear Schrödinger equation,

$$\begin{cases} i\partial_t v - \partial_x^2 v = \mu |v|^{\alpha-1} v, & t, x \in \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1-2}$$

where $\mu \in \mathbb{R}\backslash\{0\}$, Grünrock [2005a] has shown local and global well-posedness for (1-2) with $\alpha = 3$ in $\hat{L}^r$. Hyakuna and Tsutsumi [2012] extended Grünrock's result in $\hat{L}^r$ to all mass-subcritical cases $1 < \alpha < 5$. Grünrock and Vega [Grünrock 2004; Grünrock and Vega 2009] proved local well-posedness

[1] Since (1-1) preserves the L^2-norm of a solution in t, local well-posedness in L^2 yields global well-posedness in L^2 if $\alpha < 5$.

[2] In fact, if the nonlinear term is analytic, then the data-solution map associated with (1-1) is analytic.

[3] Strictly speaking, the local well-posedness is shown not for $\mu\partial_x(|u|^3 u)$ but for $\mu\partial_x(u^4)$. These two are not necessarily equivalent.

for the modified KdV equation (i.e., (1-1) with $\alpha = 3$) in $\hat{H}^r_s$, where

$$\hat{H}^r_s = \{ f \in \mathcal{S}' : \|f\|_{\hat{H}^r_s} = \|(1+\xi^2)^{\frac{s}{2}} \hat{f}(\xi)\|_{L^{r'}_\xi} < \infty \}.$$

However, the above results are not in scale-critical settings.

It would be interesting to compare the scale-critical space $\hat{L}^{\frac{\alpha-1}{2}}$ with some other scale-critical spaces in view of symmetries.[4] Other than the scaling, the $\hat{L}^{\frac{\alpha-1}{2}}$-norm is invariant under the three group operations

(i) translation in physical space, $(T_a f)(x) = f(x-a)$, where $a \in \mathbb{R}$,

(ii) translation in Fourier space, $(P_\xi f)(x) = e^{-ix\xi} f(x)$, where $\xi \in \mathbb{R}$,

(iii) Airy flow, $(\mathrm{Ai}(t) f)(x) = e^{-t\partial_x^3} f(x)$, where $t \in \mathbb{R}$.

The critical Lebesgue space $L^{\frac{\alpha-1}{2}}$ is invariant under the former two symmetries but not under the Airy flow. The critical Sobolev space $\dot{H}^{s_\alpha}$ (or homogeneous Triebel–Lizorkin and homogeneous Besov spaces $\dot{A}^{s_\alpha}_{2,q}$, with $1 \leqslant q \leqslant \infty$, more generally) is not invariant with respect to P_ξ if $s_\alpha \neq 0$. The critical weighted Lebesgue space $\dot{H}^{0,-s_\alpha} := L^2(\mathbb{R}, |x|^{-2s_\alpha}\, dx)$ is not invariant with respect to T_a and $\mathrm{Ai}(t)$. Further, when $\alpha = 5$, these four spaces coincide with L^2, which is invariant under the above three symmetries. Thus, among the above four critical spaces, $\hat{L}^{\frac{\alpha-1}{2}}$ possesses the richest symmetries, and, in some sense, $\hat{L}^{\frac{\alpha-1}{2}}$ is close to L^2-space. Inclusion relations between these spaces are summarized in Appendix B.

Local well-posedness. Before we state our main results, we introduce several notations.

Definition 1.1. Let $(s, r) \in \mathbb{R} \times [1, \infty]$. A pair (s, r) is said to be *acceptable* if $\frac{1}{r} \in [0, \frac{3}{4})$ and

$$s \in \begin{cases} \left[-\frac{1}{2r}, \frac{2}{r}\right] & \text{if } 0 \leqslant \frac{1}{r} \leqslant \frac{1}{2}, \\ \left(\frac{2}{r} - \frac{5}{4}, \frac{5}{2} - \frac{3}{r}\right) & \text{if } \frac{1}{2} < \frac{1}{r} < \frac{3}{4}. \end{cases}$$

For an interval $I \subset \mathbb{R}$ and an acceptable pair (s, r), we define a function space $X(I; s, r)$ of space-time functions with the norm

$$\|f\|_{X(I;s,r)} = \big\| |D_x|^s f \big\|_{L_x^{p(s,r)}(\mathbb{R}; L_t^{q(s,r)}(I))},$$

where the exponents in the above norm are given by

$$\frac{2}{p(s,r)} + \frac{1}{q(s,r)} = \frac{1}{r}, \quad -\frac{1}{p(s,r)} + \frac{2}{q(s,r)} = s, \tag{1-3}$$

or equivalently,

$$\begin{pmatrix} \frac{1}{p(s,r)} \\ \frac{1}{q(s,r)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} s \\ \frac{1}{r} \end{pmatrix}.$$

We refer to $X(I; s, r)$ as an *$\hat{L}^r$-admissible space*.

Our main theorems are as follows.

[4]Here, a symmetry is an isometric bijection which possesses a group structure. Some of them are also "symmetries of (1-1)" in such a sense that an image of a solution of (1-1) again solves the equation.

Theorem 1.2 (local well-posedness in $\hat{L}^{\frac{\alpha-1}{2}}$). *For* $\frac{21}{5} < \alpha < \frac{23}{3}$, *the problem* (1-1) *is locally well-posed in* $\hat{L}^{\frac{\alpha-1}{2}}$. *Namely, for any* $u_0 \in \hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})$, *there exists an interval* $I = I(u_0)$ *such that a unique solution*

$$u \in C(I; \hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})) \cap \bigcap_{(s,\frac{\alpha-1}{2}) \text{ acceptable}} X\Big(I; s, \frac{\alpha-1}{2}\Big) \tag{1-4}$$

to (1-1) *exists. Furthermore, for any given subinterval* $I' \subset I$, *there exists a neighborhood* V *of* u_0 *in* $\hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})$ *such that the map* $u_0 \mapsto u$ *from* V *into the class defined by* (1-4) *with* I' *instead of* I *is Lipschitz continuous.*

Remark 1.3. Theorem 1.2 (and all results below) holds for more general nonlinearity of the form $\partial_x G(u)$ with $G \in \operatorname{Lip} \alpha$. For the precise condition on G, see Remark 3.5.

The proof of Theorem 1.2 is based on a contraction argument, with the help of a space-time estimate for the Airy equation in $\hat{L}^r$. A key ingredient is a Stein–Tomas-type inequality for the Airy equation, a special case of [Grünrock 2004, Corollary 3.6]:

$$\big\| |D_x|^{\frac{1}{r}} e^{-t\partial_x^3} f \big\|_{L^r_{t,x}(I\times\mathbb{R})} \leqslant C \|f\|_{\hat{L}^{\frac{r}{3}}}, \tag{1-5}$$

where $r \in (4, \infty]$. This inequality is a generalization of a well-known Strichartz estimate,

$$\big\| |D_x|^{\frac{1}{6}} e^{-t\partial_x^3} f \big\|_{L^6_{t,x}(I\times\mathbb{R})} \leqslant C \|f\|_{L^2}.$$

Moreover, interpolations between the above Stein–Tomas-type inequality (1-5) and the Kenig–Ruiz estimate or Kato's local smoothing effect give us the following generalized Strichartz estimate for the Airy equation in $\hat{L}^r$-framework (Proposition 2.1): if (s, r) is an acceptable pair then there exists C such that

$$\|e^{-t\partial_x^3} f\|_{X(\mathbb{R};s,r)} \leqslant C \|f\|_{\hat{L}^r} \tag{1-6}$$

for $f \in \hat{L}^r$. Furthermore, combining the homogeneous estimate and the Christ–Kiselev lemma (Lemma 2.6), we also obtain a generalized version of inhomogeneous Strichartz estimates. The estimate (1-5) can be regarded as a kind of restriction estimate of the Fourier transform, which goes back to Stein [Fefferman 1970] and Tomas [1975] (for more information on the restriction theorem, see, e.g., [Tao et al. 1998]). It is worth mentioning that the $\hat{L}^r$-spaces have naturally come out in this context.

We set $S(I; r) := X(I; 0, r)$. The $S(I; r)$-norm is the so-called *scattering norm*. It is understood that a key for obtaining a closed estimate for the corresponding integral equation, from which local well-posedness immediately follows, is to bound the scattering norm $S\big(I; \frac{\alpha-1}{2}\big)$. In the proof of Theorem 1.2, the scattering norm is handled by means of the above generalized Strichartz estimate (1-6). Notice that the pair $\big(0, \frac{\alpha-1}{2}\big)$ is acceptable only if $\alpha > \frac{21}{5}$, which leads to our restriction. For the upper bound on α, see Remark 4.1 below. Alternatively, Sobolev's embedding also yields a bound on the scattering norm, provided $\alpha \geqslant 5$. In such case, we obtain local well-posedness in $\dot{H}^{s_\alpha}$ as in [Kenig et al. 1993] (see Remark 4.4).

Persistence of regularity. We establish two persistence-of-regularity-type results for $\hat{L}^{\frac{\alpha-1}{2}}$-solutions given in Theorem 1.2. More specifically, we consider persistence of $\hat{L}^r$-regularity for $r \neq \frac{\alpha-1}{2}$ and $\dot{H}^s$-regularity for $-1 < s < \alpha$. These results yield local well-posedness in other $\hat{L}^r$-like spaces such as $\hat{L}^{r_1} \cap \hat{L}^{r_2}$, where $r_1 \leqslant \frac{\alpha-1}{2} \leqslant r_2$, and $\dot{H}^s \cap \hat{L}^{\frac{\alpha-1}{2}}$.

Theorem 1.4 (persistence of $\hat{L}^r$-regularity). *Assume $\frac{21}{5} < \alpha < \frac{23}{3}$. Let $u_0 \in \hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})$ and let $u \in C(I; \hat{L}^{\frac{\alpha-1}{2}}(\mathbb{R}))$ be a corresponding solution given in Theorem 1.2. If $u_0 \in \hat{L}_x^{\frac{\alpha_0-1}{2}}$ for some $\frac{21}{5} < \alpha_0 < \frac{23}{3}$, where $\alpha_0 \neq \alpha$, then*

$$u \in C(I; \hat{L}_x^{\frac{\alpha_0-1}{2}}(\mathbb{R})) \cap \bigcap_{(s,\frac{\alpha_0-1}{2}) \text{ acceptable}} X\Big(I; s, \frac{\alpha_0-1}{2}\Big).$$

Theorem 1.5 (persistence of $\dot{H}^s$-regularity). *Assume $\frac{21}{5} < \alpha < \frac{23}{3}$. Let $u_0 \in \hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})$ and let $u \in C(I, \hat{L}^{\frac{\alpha-1}{2}}(\mathbb{R}))$ be a corresponding solution given in Theorem 1.2. If $u_0 \in \dot{H}_x^{\sigma}(\mathbb{R})$ for some $-1 < \sigma < \alpha$,*

$$|D_x|^{\sigma} u \in C(I; L^2(\mathbb{R})) \cap \bigcap_{(s,2) \text{ acceptable}} X(I; s, 2).$$

As a corollary, we obtain the following well-posedness results.

Corollary 1.6. *We have the following.*

(i) *If $\frac{21}{5} < \alpha < \frac{23}{3}$ then* (1-1) *is locally well-posed in $\hat{L}^{r_1} \cap \hat{L}^{r_2}$ as long as $\frac{8}{5} < r_1 \leqslant \frac{\alpha-1}{2} \leqslant r_2 < \frac{10}{3}$.*

(ii) *If $\frac{21}{5} < \alpha < 5$ then* (1-1) *is locally well-posed in $\dot{H}^{s_\alpha} \cap \hat{L}^{\frac{\alpha-1}{2}}$, where $s_\alpha = \frac{1}{2} - \frac{2}{\alpha-1}$.*

Since $\hat{L}^{\frac{\alpha-1}{2}} \subset \dot{H}^{s_\alpha}$ does not hold (see Lemma B.2), the second is weaker than well-posedness in $\dot{H}^{s_\alpha}$.

Here we remark that an $\hat{L}^{\frac{\alpha-1}{2}}$-solution has conserved quantities, provided the solution has appropriate regularity. More precisely, when $u_0 \in \hat{L}^{\frac{\alpha-1}{2}} \cap L^2$, a solution $u(t)$ has a conserved mass

$$M[u(t)] := \|u(t)\|_{L^2}^2.$$

Similarly, if $u_0 \in \hat{L}^{\frac{\alpha-1}{2}} \cap \dot{H}^1$ then the energy

$$E[u(t)] := \tfrac{1}{2}\|\partial_x u(t)\|_{L^2}^2 + \frac{\mu}{\alpha+1}\|u(t)\|_{L^{\alpha+1}}^{\alpha+1}$$

is invariant.

Blowup and scattering. We next consider long time behavior of solutions given in Theorem 1.2. To this end, we give the definitions of blowup and scattering of (1-1) for the initial data $u_0 \in \hat{L}_x^r$. Set

$$T_{\max} := \sup\big\{T > 0 : \text{the solution } u \text{ to (1-1) can be extended to } [0, T)\big\},$$
$$T_{\min} := \sup\big\{T > 0 : \text{the solution } u \text{ to (1-1) can be extended to } (-T, 0]\big\}.$$

Denote the lifespan of $u(t)$ as $(-T_{\min}, T_{\max})$. We say a solution $u(t)$ blows up in finite time for positive (resp. negative) time direction if $T_{\max} < +\infty$ (resp. $T_{\min} < +\infty$). We say a solution $u(t)$ scatters for positive time direction if $T_{\max} = +\infty$ and there exists a unique function $u_+ \in \hat{L}_x^r$ such that

$$\lim_{t \to +\infty} \|u(t) - e^{-t\partial_x^3} u_+\|_{\hat{L}_x^r} = 0,$$

where $e^{-t\partial_x^3} u_+$ is a solution to the Airy equation $\partial_t v + \partial_x^3 v = 0$ with initial condition $v(0, x) = u_+$. The scattering of u for negative time direction is defined in a similar fashion.

Roughly speaking, a solution scatters if a linear dispersion effect dominates the nonlinear interaction. A typical case is when the data (and the corresponding solution) is small. Here, we state this small-data scattering for (1-1).

Theorem 1.7 (small-data scattering). *Let $\frac{21}{5} < \alpha < \frac{23}{3}$. There exists $\varepsilon_0 > 0$ such that if $u_0 \in \hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})$ satisfies*

$$\|u_0\|_{\hat{L}_x^{(\alpha-1)/2}} \leqslant \varepsilon_0,$$

then the solution $u(t)$ to (1-1) *given in Theorem 1.2 is global in time and scatters for both time directions. Moreover,*

$$\|u\|_{L_t^\infty(\mathbb{R};\hat{L}_x^{(\alpha-1)/2})} + \|u\|_{S(\mathbb{R};\frac{\alpha-1}{2})} \leqslant 2\|u_0\|_{\hat{L}_x^{(\alpha-1)/2}}.$$

We now give criterion for blowup and scattering.

Theorem 1.8 (blowup criterion). *Assume $\frac{21}{5} < \alpha < \frac{23}{3}$. Let $u_0 \in \hat{L}^{\frac{\alpha-1}{2}}$ and let $u(t)$ be a corresponding unique solution of* (1-1) *given in Theorem 1.2. If $T_{\max} < \infty$ then*

$$\|u\|_{S([0,T);\frac{\alpha-1}{2})} \to \infty$$

as $T \uparrow T_{\max}$. A similar statement is true for negative time direction.

Theorem 1.9 (scattering criterion). *Assume $\frac{21}{5} < \alpha < \frac{23}{3}$. Let $u_0 \in \hat{L}^{\frac{\alpha-1}{2}}$ and let $u(t)$ be a corresponding unique solution of* (1-1) *given in Theorem 1.2. The solution $u(t)$ scatters forward in time if and only if $T_{\max} = +\infty$ and $\|u\|_{S([0,\infty);\frac{\alpha-1}{2})} < \infty$. A similar statement is true for negative time direction.*

Finally, we give a criterion for scattering in terms of the energy. We note that if an $\hat{L}^{\frac{\alpha-1}{2}}$-solution $u(t)$ scatters (in the $\hat{L}^{\frac{\alpha-1}{2}}$-sense) as $t \to \pm\infty$ and if $u_0 \in \hat{L}^{\frac{\alpha_0-1}{2}}$ (resp. if $u_0 \in \dot{H}^\sigma$) then $u(t)$ scatters as $t \to \pm\infty$ also in the $\hat{L}^{\frac{\alpha_0-1}{2}}$-sense (resp. $\dot{H}^\sigma$-sense).

Theorem 1.10. *Let $\frac{21}{5} < \alpha < \frac{23}{3}$. If $u_0 \in \hat{L}^{\frac{\alpha-1}{2}} \cap H^1$ satisfies $u_0 \neq 0$ and $E[u_0] \leqslant 0$ then $u(t)$ does not scatter as $t \to \pm\infty$.*

The rest of the paper is organized as follows. In Section 2, we prove some linear space-time estimates for solutions to the Airy equation, in $\hat{L}^r$-framework. The generalized Strichartz estimates are established in Propositions 2.1 and 2.5. Section 3 is devoted to several nonlinear estimates. We also introduce several function spaces to work with in this section. Then, in Section 4, we prove our theorems. In Appendix A, we prove a fractional chain rule in space-time function space (Lemma 3.7). Finally in Appendix B, we briefly collect some inclusion relations for $\hat{L}^r$.

The following notation will be used throughout this paper: $|D_x|^s = (-\partial_x^2)^{\frac{s}{2}}$ and $\langle D_x\rangle^s = (I - \partial_x^2)^{\frac{s}{2}}$ denote the Riesz and Bessel potentials of order $-s$, respectively. For $1 \leqslant p, q \leqslant \infty$ and $I \subset \mathbb{R}$, let us define a space-time norm

$$\|f\|_{L_t^q L_x^p(I)} = \big\| \|f(t,\cdot)\|_{L_x^p(\mathbb{R})} \big\|_{L_t^q(I)},$$
$$\|f\|_{L_x^p L_t^q(I)} = \big\| \|f(\cdot,x)\|_{L_t^q(I)} \big\|_{L_x^p(\mathbb{R})}.$$

2. Linear estimates for the Airy equation

In this section, we consider the space-time estimates of solutions to the Airy equation

$$\begin{cases} \partial_t u + \partial_x^3 u = F(t,x), & t \in I, x \in \mathbb{R}, \\ u(0,x) = f(x), & x \in \mathbb{R}, \end{cases} \tag{2-1}$$

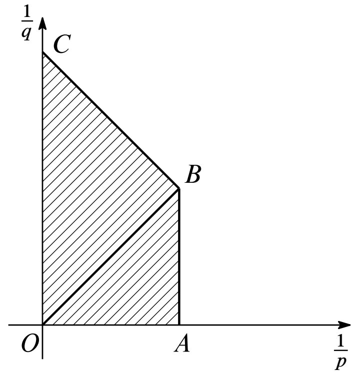

Figure 1. The range of (p, q) satisfying the assumptions of Proposition 2.1.

where $I \subset \mathbb{R}$ is an interval and $F : I \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are given functions.

Let $\{e^{-t\partial_x^3}\}_{t\in\mathbb{R}}$ be an isometric isomorphism group in $\hat{L}^r$ defined by $e^{-t\partial_x^3} = \mathcal{F}^{-1} e^{it\xi^3} \mathcal{F}$, or more precisely by

$$(e^{-t\partial_x^3} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi + it\xi^3} \hat{f}(\xi)\, d\xi.$$

Using the group, the solution to (2-1) can be written as

$$u(t) = e^{-t\partial_x^3} f + \int_0^t e^{-(t-t')\partial_x^3} F(t')\, dt'.$$

We first show a homogeneous estimate associated with (2-1).

Proposition 2.1. *Let I be an interval. Let (p, q) satisfy*

$$0 \leqslant \frac{1}{p} < \frac{1}{4}, \quad 0 \leqslant \frac{1}{q} < \frac{1}{2} - \frac{1}{p}.$$

Then, for any $f \in \hat{L}^r$,

$$\big\| |D_x|^s e^{-t\partial_x^3} f \big\|_{L_x^p L_t^q(I)} \leqslant C \|f\|_{\hat{L}^r}, \tag{2-2}$$

where

$$\frac{1}{r} = \frac{2}{p} + \frac{1}{q}, \quad s = -\frac{1}{p} + \frac{2}{q},$$

and the positive constant C depends only on r and s.

Figure 1 shows the range of (p, q) satisfying the assumptions of Proposition 2.1, where $A = \left(\frac{1}{4}, 0\right)$, $B = \left(\frac{1}{4}, \frac{1}{4}\right)$, and $C = \left(0, \frac{1}{2}\right)$. The line segments OA and OC are included, but the other parts of the border are excluded.

To prove Proposition 2.1, we show three lemmas. The first one is a Stein–Tomas-type estimate.

Lemma 2.2 (Stein–Tomas-type estimate). *For any $r \in (4, \infty]$, there exists a positive constant C depending only on r such that for any $f \in \hat{L}^{\frac{r}{3}}$,*

$$\big\| |D_x|^{\frac{1}{r}} e^{-t\partial_x^3} f \big\|_{L_{t,x}^r(I)} \leqslant C \|f\|_{\hat{L}^{r/3}}. \tag{2-3}$$

Proof of Lemma 2.2. Although a more general version is proved in [Grünrock 2004, Corollary 3.6], here we give a direct proof, based on the fact that the exponents for the space variable and time variable on the left-hand side coincide.

It suffices to prove (2-3) for the case $I=\mathbb{R}$. For notational simplicity, we omit $\mathbb{R}$. The case $r=\infty$ follows from the Hausdorff–Young inequality. Let $r<\infty$. Squaring both sides, we may show that

$$\big\| \big| |D_x|^{\frac{1}{r}} e^{-t\partial_x^3} f \big|^2 \big\|_{L^{r/2}_{t,x}} \leqslant C \|f\|^2_{\hat{L}^{r/3}}. \tag{2-4}$$

The left-hand side of (2-4) is equal to

$$\left\| \iint_{\mathbb{R}^2} e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^{\frac{1}{r}} \hat{f}(\xi)\overline{\hat{f}(\eta)}\, d\xi\, d\eta \right\|_{L^{r/2}_{t,x}}.$$

Changing variables by $a=\xi-\eta$ and $b=\xi^3-\eta^3$, we have

$$\big\| \big| |D_x|^{\frac{1}{r}} e^{-t\partial_x^3} f \big|^2 \big\|_{L^{r/2}_{t,x}} = \left\| \iint_{\mathbb{R}^2} e^{ixa+itb} |\xi\eta|^{\frac{1}{r}} \hat{f}(\xi)\overline{\hat{f}(\eta)} \frac{1}{3|\xi^2-\eta^2|}\, da\, db \right\|_{L^{r/2}_{t,x}}.$$

We now use the Hausdorff–Young inequality to deduce that

$$\begin{aligned} \big\| \big| |D_x|^{\frac{1}{r}} e^{-t\partial_x^3} f \big|^2 \big\|_{L^{r/2}_{t,x}} &\leqslant C \big\| |\xi\eta|^{\frac{1}{r}} \hat{f}(\xi)\overline{\hat{f}(\eta)} |\xi^2-\eta^2|^{-1} \big\|_{L^{(r/2)'}_{a,b}} \\ &= C\left(\iint_{\mathbb{R}^2} \frac{|\xi\eta|^{\frac{1}{r-2}} |\hat{f}(\xi)|^{\frac{r}{r-2}} |\hat{f}(\eta)|^{\frac{r}{r-2}}}{|\xi-\eta|^{\frac{2}{r-2}} |\xi+\eta|^{\frac{2}{r-2}}}\, d\xi\, d\eta \right)^{1-\frac{2}{r}}. \end{aligned} \tag{2-5}$$

Notice that $\frac{r}{2}\geqslant 2$. We now split the integral region $\mathbb{R}^2$ into $\{\xi\eta\geqslant 0\}$ and $\{\xi\eta<0\}$. We only consider the first case, since the other can be treated essentially in the same way. For (ξ,η) with $\xi\eta\geqslant 0$, we have

$$\xi\eta \leqslant \frac{(\xi+\eta)^2}{4},$$

and so

$$\iint_{\xi\eta\geqslant 0} \frac{|\xi\eta|^{\frac{1}{r-2}} |\hat{f}(\xi)|^{\frac{r}{r-2}} |\hat{f}(\eta)|^{\frac{r}{r-2}}}{|\xi-\eta|^{\frac{2}{r-2}} |\xi+\eta|^{\frac{2}{r-2}}}\, d\xi\, d\eta \leqslant C \iint_{\xi\eta\geqslant 0} \frac{|\hat{f}(\xi)|^{\frac{r}{r-2}} |\hat{f}(\eta)|^{\frac{r}{r-2}}}{|\xi-\eta|^{\frac{2}{r-2}}}\, d\xi\, d\eta. \tag{2-6}$$

By the Hölder inequality and the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned} \iint_{\xi\eta\geqslant 0} \frac{|\hat{f}(\xi)|^{\frac{r}{r-2}} |\hat{f}(\eta)|^{\frac{r}{r-2}}}{|\xi-\eta|^{\frac{2}{r-2}}}\, d\xi\, d\eta &\leqslant \big\| |\hat{f}|^{\frac{r}{r-2}} \big\|_{L^{(r-2)/(r-3)}} \big\| (|\xi|^{-\frac{2}{r-2}} * |\hat{f}|^{\frac{r}{r-2}}) \big\|_{L^{r-2}} \\ &\leqslant C \|\hat{f}\|^{\frac{2r}{r-2}}_{L^{r/(r-3)}} = C\|f\|^{\frac{2r}{r-2}}_{\hat{L}^{r/3}_x} \end{aligned} \tag{2-7}$$

as long as $\frac{2}{r-2}<1$, that is, $r>4$. Combining (2-5), (2-6) and (2-7), we obtain the result. □

The second is a Kenig–Ruiz-type estimate [1983].

Lemma 2.3 (Kenig–Ruiz-type estimate). *There exists a universal constant C such that for any interval I and any $f\in L^2$,*

$$\big\| |D_x|^{-\frac{1}{4}} e^{-t\partial_x^3} f \big\|_{L^4_x L^\infty_t(I)} \leqslant C\|f\|_{L^2}. \tag{2-8}$$

Proof of Lemma 2.3. See [Kenig et al. 1991, Theorem 2.5]. □

The last estimate is an $\widehat{L}^q$-version of Kato's local smoothing effect [1983].

Lemma 2.4 (Kato's smoothing effect). *For any $q \in [2,\infty]$, there exists a positive constant C depending only on q such that for any interval I and for any $f \in \widehat{L}^q$,*

$$\big\| |D_x|^{\frac{2}{q}} e^{-t\partial_x^3} f \big\|_{L_x^\infty L_t^q(I)} \leqslant C \|f\|_{\widehat{L}^q}. \tag{2-9}$$

Proof of Lemma 2.4. We show (2-9) by slightly modifying the argument due to Kenig, Ponce, and Vega [Kenig et al. 1991, Theorem 2.5]. We prove (2-9) for the case $I=\mathbb{R}$ only.

The case $q=\infty$ is treated in Lemma 2.2. Hence, we may suppose $q<\infty$. A direct computation shows

$$\begin{aligned} |D_x|^{\frac{2}{q}} e^{-t\partial_x^3} f &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi+it\xi^3} |\xi|^{\frac{2}{q}} \hat f(\xi)\, d\xi \\ &= \frac{1}{3\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\eta^{\frac13}+it\eta} |\eta|^{\frac{2}{3q}} \eta^{-\frac23} \hat f(\eta^{\frac13})\, d\eta, \end{aligned}$$

where we have used a change of variable $\eta=\xi^3$ to yield the last line. Take the L_t^q-norm and apply the Hausdorff–Young inequality to obtain

$$\big\| |D_x|^{\frac{2}{q}} e^{-t\partial_x^3} f \big\|_{L_t^q} \leqslant C \big\| e^{ix\eta^{\frac13}} |\eta|^{\frac{2-q}{3q}} \hat f(\eta^{\frac13}) \big\|_{L_\eta^{q'}} \leqslant C \|\hat f\|_{L^{q'}} = C\|f\|_{\widehat{L}^q}.$$

Since the right-hand side is independent of x, we obtain (2-9). □

Proof of Proposition 2.1. Interpolating (2-3), (2-8), and (2-9), we obtain (2-2). □

Next we show an inhomogeneous estimate associated with (2-1).

Proposition 2.5. *Let $\frac43 < r < 4$ and let (p_j, q_j) $(j=1,2)$ satisfy*

$$0 \leqslant \frac{1}{p_j} < \frac14, \quad 0 \leqslant \frac{1}{q_j} < \frac12 - \frac{1}{p_j}.$$

Then, the inequalities

$$\left\| \int_0^t e^{-(t-t')\partial_x^3} F(t')\, dt' \right\|_{L_t^\infty(I;\widehat{L}_x^r)} \leqslant C_1 \big\| |D_x|^{-s_2} F \big\|_{L_x^{p_2'} L_t^{q_2'}(I)} \tag{2-10}$$

and

$$\left\| |D_x|^{s_1} \int_0^t e^{-(t-t')\partial_x^3} F(t')\, dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C_2 \big\| |D_x|^{-s_2} F \big\|_{L_x^{p_2'} L_t^{q_2'}(I)} \tag{2-11}$$

hold for any F satisfying $|D_x|^{-s_2} F \in L_x^{p_2'} L_t^{q_2'}$, where

$$\frac1r = \frac{2}{p_1} + \frac{1}{q_1}, \quad s_1 = -\frac{1}{p_1} + \frac{2}{q_1} \qquad \frac{1}{r'} = \frac{2}{p_2} + \frac{1}{q_2}, \quad s_2 = -\frac{1}{p_2} + \frac{2}{q_2},$$

and where the constant C_1 depends on r, s_1 and I, and the constant C_2 depends on r, s_2 and I.

To prove Proposition 2.5, we employ the following lemma, which is essentially due to Christ and Kiselev [2001]. The version of this lemma that we use is the one presented in [Molinet and Ribaud 2004].

Lemma 2.6. *Let $I \subset \mathbb{R}$ be an interval and let $K : \mathcal{S}(I \times \mathbb{R}) \to C(\mathbb{R}^3)$. Assume that*

$$\left\| \int_I K(t,t')F(t')\,dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C \|F\|_{L_x^{p_2} L_t^{q_2}(I)}$$

for some $1 \leqslant p_1, p_2, q_1, q_2 \leqslant \infty$ *with* $\min(p_1,q_1) > \max(p_2,q_2)$. *Then*

$$\left\| \int_0^t K(t,t')F(t')\,dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C \|F\|_{L_x^{p_2} L_t^{q_2}(I)}.$$

Moreover, the case $q_1 = \infty$ and $p_2, q_2 < \infty$ is allowed.

Proof of Lemma 2.6. See [Molinet and Ribaud 2004, Lemma 2]. □

Proof of Proposition 2.5. We first prove the inequality (2-10). Since the group $\{e^{-t\partial_x^3}\}_{t\in\mathbb{R}}$ is isometric in $\widehat{L}^r$, the duality argument and Proposition 2.1 yield

$$\begin{aligned}
\left\| \int_0^t e^{-(t-t')\partial_x^3} F(t')\,dt' \right\|_{\widehat{L}_x^r} &= \left\| \int_0^t e^{t'\partial_x^3} F(t')\,dt' \right\|_{\widehat{L}_x^r} \\
&= \sup_{\|g\|_{\widehat{L}_x^{r'}}=1} \left(\int_{-\infty}^{\infty} \left(\int_0^t e^{t'\partial_x^3} F(t',x)\,dt' \right) g(x)\,dx \right) \\
&= \sup_{\|g\|_{\widehat{L}_x^{r'}}=1} \left(\int_0^t \int_{-\infty}^{\infty} |D_x|^{-s_2} F(t',x) |D_x|^{s_2} e^{-t'\partial_x^3} g(x)\,dt'\,dx \right) \\
&\leqslant \sup_{\|g\|_{\widehat{L}_x^{r'}}=1} \left\| |D_x|^{-s_2} F \right\|_{L_x^{p_2'} L_t^{q_2'}(I)} \left\| |D_x|^{s_2} e^{-t'\partial_x^3} g \right\|_{L_x^{p_2} L_t^{q_2}(I)} \\
&\leqslant C \sup_{\|g\|_{\widehat{L}_x^{r'}}=1} \left\| |D_x|^{-s_2} F \right\|_{L_x^{p_2'} L_t^{q_2'}(I)} \|g\|_{\widehat{L}_x^{r'}} \\
&= C \left\| |D_x|^{-s_2} F \right\|_{L_x^{p_2'} L_t^{q_2'}(I)},
\end{aligned} \tag{2-12}$$

where the constant C is independent of t. Hence we have (2-10).

Next we prove the inequality (2-11). Since the case $r = 2$ was already proved in [Kenig et al. 1993], we consider the case where $r \neq 2$. To prove (2-11), it suffices to show

$$\left\| |D_x|^{s_1} \int_I e^{-(t-t')\partial_x^3} F(t')\,dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C \left\| |D_x|^{-s_2} F \right\|_{L_x^{p_2'} L_t^{q_2'}(I)}. \tag{2-13}$$

Indeed, since $\min(p_1,q_1) > \max(p_2',q_2')$ follows from

$$\min(p_1,q_1) = \begin{cases} \frac{r}{r-1} & \text{if } \frac{4}{3} < r < 2, \\ r & \text{if } 2 < r < 4, \end{cases} \qquad \max(p_2',q_2') = \begin{cases} r & \text{if } \frac{4}{3} < r < 2, \\ \frac{r}{r-1} & \text{if } 2 < r < 4, \end{cases}$$

we see that the combination of the Christ–Kiselev lemma (Lemma 2.6) with (2-13) implies (2-11). Therefore we concentrate our attention on proving (2-13). By Proposition 2.1,

$$\left\| |D_x|^{s_1} \int_I e^{-(t-t')\partial_x^3} F(t')\,dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} = \left\| |D_x|^{s_1} e^{-t\partial_x^3} \int_I e^{t'\partial_x^3} F(t')\,dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C \left\| \int_I e^{t'\partial_x^3} F(t')\,dt' \right\|_{\hat{L}_x^r}. \tag{2-14}$$

By the duality argument similar to (2-12), we obtain

$$\left\| \int_I e^{t'\partial_x^3} F(t')\,dt' \right\|_{\hat{L}_x^r} \leqslant C \left\| |D_x|^{-s_2} F \right\|_{L_x^{p_2'} L_t^{q_2'}(I)}. \tag{2-15}$$

Combining (2-14) and (2-15), we obtain (2-13). □

3. Nonlinear estimates

In this section, we prove several nonlinear estimates which are used to prove main theorems. We introduce several function spaces. Let us recall that a pair $(s, r) \in \mathbb{R} \times [1, \infty]$ is said to be *acceptable* if $\frac{1}{r} \in \left[0, \frac{3}{4}\right)$ and

$$s \in \begin{cases} \left[-\frac{1}{2r}, \frac{2}{r}\right] & \text{if } 0 \leqslant \frac{1}{r} \leqslant \frac{1}{2}, \\ \left(\frac{2}{r} - \frac{5}{4}, \frac{5}{2} - \frac{3}{r}\right) & \text{if } \frac{1}{2} < \frac{1}{r} < \frac{3}{4}. \end{cases}$$

Definition 3.1. Let $(s, r) \in \mathbb{R} \times [1, \infty]$. A pair (s, r) is said to be *conjugate-acceptable* if $(1 - s, r')$ is acceptable, where $\frac{1}{r'} = 1 - \frac{1}{r} \in [0, 1]$.

Figure 2 shows the ranges of acceptable pairs (quadrangle $OABC$) and conjugate-acceptable pairs (quadrangle $DEFG$). Here, $O = (0, 0)$, $A = \left(\frac{1}{2}, -\frac{1}{4}\right)$, $B = \left(\frac{3}{4}, \frac{1}{4}\right)$, $C = \left(\frac{1}{2}, 1\right)$, $D = (1, 1)$, $E = \left(\frac{1}{2}, \frac{5}{4}\right)$, $F = \left(\frac{1}{4}, \frac{3}{4}\right)$, and $G = \left(\frac{1}{2}, 0\right)$.

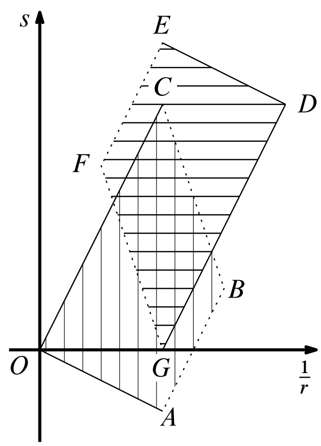

Figure 2. The ranges of acceptable pairs (quadrangle $OABC$) and conjugate-acceptable pairs (quadrangle $DEFG$).

For an interval $I \subset \mathbb{R}$ and a conjugate-acceptable pair (s, r), we define a function space $Y(I; s, r)$ by

$$\|f\|_{Y(I;s,r)} = \big\| |D_x|^s f \big\|_{L_x^{\tilde{p}(s,r)}(\mathbb{R}; L_t^{\tilde{q}(s,r)}(I))},$$

where the exponents are given by

$$\frac{2}{\tilde{p}(s,r)} + \frac{1}{\tilde{q}(s,r)} = 2 + \frac{1}{r}, \quad -\frac{1}{\tilde{p}(s,r)} + \frac{2}{\tilde{q}(s,r)} = s, \tag{3-1}$$

or equivalently,

$$\begin{pmatrix} \frac{1}{\tilde{p}(s,r)} \\ \frac{1}{\tilde{q}(s,r)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} s \\ 2 + \frac{1}{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{p(s,r)} \\ \frac{1}{q(s,r)} \end{pmatrix} + \begin{pmatrix} \frac{4}{5} \\ \frac{2}{5} \end{pmatrix}.$$

With this terminology, Propositions 2.1 and 2.5 can be reformulated as follows:

Proposition 3.2. *Let I be an interval.*

(i) *Let (s, r) be an acceptable pair. Then, there exists a positive constant C depending only on s and r such that*

$$\|e^{-t\partial_x^3} f\|_{L^\infty(\mathbb{R}; \hat{L}^r)} + \|e^{-t\partial_x^3} f\|_{X(\mathbb{R};s,r)} \leqslant C_{s,r} \|f\|_{\hat{L}^r}$$

for any $f \in \hat{L}^r$.

(ii) *Let (s_1, r) be an acceptable pair and let (s_2, r) be a conjugate-acceptable pair. Then, there exists a positive constant depending only on s_i and r such that for any $t_0 \in I \subset \mathbb{R}$ and any $F \in Y(I; s_2, r)$,*

$$\left\| \int_{t_0}^{t} e^{-(t-t')\partial_x^3} \partial_x F(t')\, dt' \right\|_{L_t^\infty(I; \hat{L}_x^r) \cap X(I; s_1, r)} \leqslant C \|F\|_{Y(I; s_2, r)}.$$

To handle $X(I; s, r)$- and $Y(I; s, r)$-spaces, the following lemma is useful.

Lemma 3.3. *Let $1 < p_i, q_i < \infty$ and $s_i \in \mathbb{R}$ for $i = 1, 2$. Let p, q and s satisfy*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad s = \theta s_1 + (1-\theta) s_2$$

for some $\theta \in (0, 1)$. Then, there exists a positive constant C, depending on $p_1, p_2, q_1, q_2, s_1, s_2$ and θ, such that

$$\big\| |D_x|^s f \big\|_{L_x^p L_t^q} \leqslant C \big\| |D_x|^{s_1} f \big\|_{L_x^{p_1} L_t^{q_1}}^{\theta} \big\| |D_x|^{s_2} f \big\|_{L_x^{p_2} L_t^{q_2}}^{1-\theta}$$

holds for any f such that $|D_x|^{s_1} f \in L_x^{p_1} L_t^{q_1}$ and $|D_x|^{s_2} f \in L_x^{p_2} L_t^{q_2}$.

Proof of Lemma 3.3. For $z \in \mathbb{C}$, define an operator $T_z = |D_x|^{z s_1 + (1-z) s_2}$. Let $g(t)$ and $h(x)$ be $\mathbb{R}$-valued simple functions and $G_z(t)$ and $H_z(x)$ be extensions of these functions defined by

$$G_z(t) := |g(t)|^{\frac{1-(z/q_1 + (1-z)/q_2)}{1-1/q}} \operatorname{sign} g(t)$$

and

$$H_z(x) := |h(x)|^{\frac{1-(z/p_1 + (1-z)/p_2)}{1-1/p}} \operatorname{sign} h(x),$$

respectively, for $z \in \mathbb{C}$ with $0 \leqslant \operatorname{Re} z \leqslant 1$. Put

$$\Psi(z) := \iint_{\mathbb{R}^2} T_z f(t,x) G_z(t) H_z(x)\, dt\, dx.$$

By density and duality, it suffices to show

$$|\Psi(\theta)| \leqslant C \big\| |D_x|^{s_1} f \big\|^{\theta}_{L_x^{p_1} L_t^{q_1}} \big\| |D_x|^{s_2} f \big\|^{1-\theta}_{L_x^{p_2} L_t^{q_2}} \tag{3-2}$$

for any $f \in \mathcal{S}(\mathbb{R}^2)$ with compact Fourier support and any simple functions $g(t)$ and $h(x)$ such that $\|g\|_{L_t^{q'}} = \|h\|_{L_x^{p'}} = 1$.

Let us now prove (3-2). It is easy to see that $\Psi(z)$ is analytic in $0 < \operatorname{Re} z < 1$ and continuous in $0 \leqslant \operatorname{Re} z \leqslant 1$. By a variant of the multiplier theorem by Fernandez [1987, Theorem 6.4], we see that $|D_x|^{iy}$ is a bounded operator in $L_x^{p_1} L_t^{q_1}$ with norm $C(1+|y|)$. Therefore, for any $y \in \mathbb{R}$,

$$\begin{aligned} |\Psi(1+iy)| &\leqslant \big\| |D_x|^{iy(s_1-s_2)} (|D_x|^{s_1} f) \big\|_{L_x^{p_1} L_t^{q_1}} \|G_{1+iy} H_{1+iy}\|_{L_x^{p_1'} L_t^{q_1'}} \\ &\leqslant C(1+|y(s_1-s_2)|) \big\| |D_x|^{s_1} f \big\|_{L_x^{p_1} L_t^{q_1}} \|g\|_{L_t^{q'}} \|h\|_{L_x^{p'}} \\ &\leqslant C(1+|y(s_1-s_2)|) \big\| |D_x|^{s_1} f \big\|_{L_x^{p_1} L_t^{q_1}}. \end{aligned} \tag{3-3}$$

The same argument yields

$$|\Psi(iy)| \leqslant C(1+|y(s_1-s_2)|) \big\| |D_x|^{s_2} f \big\|_{L_x^{p_2} L_t^{q_2}}. \tag{3-4}$$

From (3-3), (3-4) and Hirschman's lemma [1952], we obtain (3-2) (see also [Stein 1956]). □

Estimates on nonlinearity. In this subsection, we establish an estimate on nonlinearity. For this, we introduce a Lipschitz μ-norm ($\mu > 0$) as follows. Write $\mu = N + \beta$ with $N \in \mathbb{Z}$ and $\beta \in (0,1]$. For a function $G : \mathbb{C} \to \mathbb{C}$, we define

$$\|G\|_{\operatorname{Lip}\mu} := \sum_{j=0}^{N} \sup_{z \in \mathbb{R}\setminus\{0\}} \frac{|G^{(j)}(z)|}{|z|^{\mu-j}} + \sup_{x \neq y} \frac{|G^{(N)}(x) - G^{(N)}(y)|}{|x-y|^{\beta}},$$

where $G^{(j)}$ is j-th derivative of G. We say $G \in \operatorname{Lip}\mu$ if $G \in C^N(\mathbb{R})$ and $\|G\|_{\operatorname{Lip}\mu} < \infty$.

The main estimate of this subsection is as follows:

Lemma 3.4. *Suppose that $G(z) \in \operatorname{Lip}\alpha$ for some $\frac{21}{5} < \alpha < \frac{23}{3}$. Let (s,r) be a pair which is acceptable and conjugate-acceptable. Then, the following two assertions hold*:

(i) *If $u \in S\big(I; \frac{\alpha-1}{2}\big) \cap X(I;s,r)$ then $G(u) \in Y(I;s,r)$. Moreover, there exists a constant C such that*

$$\|G(u)\|_{Y(I;s,r)} \leqslant C \|u\|^{\alpha-1}_{S(I;\frac{\alpha-1}{2})} \|u\|_{X(I;s,r)}$$

for any $u \in S\big(I; \frac{\alpha-1}{2}\big) \cap X(I;s,r)$.

(ii) *There exists a constant C such that*

$$\begin{aligned} \|G(u)-G(v)\|_{Y(I;s,r)} \leqslant{}& C\big(\|u\|_{X(I;s,r)} + \|v\|_{X(I;s,r)}\big)\big(\|u\|_{S(I;\frac{\alpha-1}{2})} + \|v\|_{S(I;\frac{\alpha-1}{2})}\big)^{\alpha-2} \|u-v\|_{S(I;\frac{\alpha-1}{2})} \\ &+ C\big(\|u\|_{S(I;\frac{\alpha-1}{2})} + \|v\|_{S(I;\frac{\alpha-1}{2})}\big)^{\alpha-1} \|u-v\|_{X(I;s,r)} \end{aligned}$$

for any $u, v \in S\big(I; \frac{\alpha-1}{2}\big) \cap X(I;s,r)$.

Remark 3.5. It is easy to see that $|z|^{\alpha-1}z \in \mathrm{Lip}\,\alpha$. The validity of the above lemma is the only assumption on the nonlinearity that we need. Hence, the all results of this article hold for an equation with generalized nonlinearity $\partial_t u + \partial_x^3 u = \partial_x(G(u))$, provided $G(z) \in \mathrm{Lip}\,\alpha$.

To prove the above lemma, we recall the following two lemmas.

Lemma 3.6. *Let I be an interval. Assume that $s \geqslant 0$. Let $p, q, p_i, q_i, \in (1,\infty)$ $(i = 1,2,3,4)$. Then,*

$$\big\| |D_x|^s(fg)\big\|_{L_x^p L_t^q(I)} \leqslant C\Big(\big\| |D_x|^s f\big\|_{L_x^{p_1} L_t^{q_1}(I)} \|g\|_{L_x^{p_2} L_t^{q_2}(I)} + \|f\|_{L_x^{p_3} L_t^{q_3}(I)} \big\| |D_x|^s g\big\|_{L_x^{p_4} L_t^{q_4}(I)}\Big),$$

provided that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4},$$

where the constant C is independent of I and f.

Proof of Lemma 3.6. If $s \in \mathbb{Z}$ then (the classical) Leibniz rule, Hölder's inequality, and Lemma 3.3 give us the result. By a similar argument, it suffices to consider the case $0 < s < 1$ to handle the general case. However, that case follows from [Kenig et al. 1993, Theorem A.8] and Lemma 3.3. □

Lemma 3.7. *Suppose that $\mu > 1$ and $s \in (0,\mu)$. Let $G \in \mathrm{Lip}\,\mu$. If $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$ satisfy*

$$\frac{1}{p} = \frac{\mu-1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{\mu-1}{q_1} + \frac{1}{q_2},$$

then there exists a positive constant C depending on $\mu, s, p_1, p_2, q_1, q_2$ and I such that

$$\big\| |D_x|^s G(f)\big\|_{L_x^p L_t^q(I)} \leqslant C \|G\|_{\mathrm{Lip}\,\mu} \|f\|_{L_x^{p_1} L_t^{q_1}(I)}^{\mu-1} \big\| |D_x|^s f\big\|_{L_x^{p_2} L_t^{q_2}(I)}$$

holds for any f satisfying $f \in L_x^{p_1} L_t^{q_1}(I)$ and $|D_x|^s f \in L_x^{p_2} L_t^{q_2}(I)$.

Although Lemma 3.7 is essentially the same as [Kenig et al. 1993, Theorem A.6; Christ and Weinstein 1991, Proposition 3.1], we give the proof of this lemma in Appendix A for self-containedness and in order to clarify the necessity of the assumption $G \in \mathrm{Lip}\,\mu$.

Proof of Lemma 3.4. We prove the second assertion since the first immediately follows from the second by letting $v = 0$. For simplicity, we name $S = S\big(I; \frac{\alpha-1}{2}\big)$, $L = X(I; s, r)$, and $N = Y(I; s, r)$.

Let us write

$$G(u) - G(v) = (u - v)\int_0^1 G'(\theta u + (1-\theta)v)\, d\theta.$$

Lemma 3.6 implies that

$$\begin{aligned}\|G(u) - G(v)\|_N \leqslant{} & C\|u - v\|_S \int_0^1 \big\| |D_x|^s\big(G'(\theta u + (1-\theta)v)\big)\big\|_{L_x^{p_1} L_t^{q_1}}\, d\theta \\ & + C\|u - v\|_L \int_0^1 \big\|\big(G'(\theta u + (1-\theta)v)\big)\big\|_{L_x^{p_2} L_t^{q_2}}\, d\theta \\ =:{} & I_1 + I_2,\end{aligned}$$

where

$$\begin{pmatrix} 1/p_1 \\ 1/q_1 \end{pmatrix} = \begin{pmatrix} 1/\tilde{p}(s,r) \\ 1/\tilde{q}(s,r) \end{pmatrix} - \begin{pmatrix} 1/p\big(0, \frac{\alpha-1}{2}\big) \\ 1/q\big(0, \frac{\alpha-1}{2}\big) \end{pmatrix} = (\alpha-2)\begin{pmatrix} 1/p\big(0, \frac{\alpha-1}{2}\big) \\ 1/q\big(0, \frac{\alpha-1}{2}\big) \end{pmatrix} + \begin{pmatrix} 1/p(s,r) \\ 1/q(s,r) \end{pmatrix},$$

and

$$\begin{pmatrix} 1/p_2 \\ 1/q_2 \end{pmatrix} = \begin{pmatrix} 1/\tilde{p}(s,r) \\ 1/\tilde{q}(s,r) \end{pmatrix} - \begin{pmatrix} 1/p(s,r) \\ 1/q(s,r) \end{pmatrix} = (\alpha-1) \begin{pmatrix} 1/p\big(0, \frac{\alpha-1}{2}\big) \\ 1/q\big(0, \frac{\alpha-1}{2}\big) \end{pmatrix}.$$

It is easy to see that $\|G'\|_{\mathrm{Lip}(\alpha-1)} \leqslant \|G\|_{\mathrm{Lip}\,\alpha} < +\infty$. By the definition of $\|\cdot\|_{\mathrm{Lip}(\alpha-1)}$, we estimate I_2 as

$$\begin{aligned} I_2 &\leqslant C\|u-v\|_L \, \|G'\|_{\mathrm{Lip}(\alpha-1)} \int_0^1 \big\| |\theta u + (1-\theta)v|^{\alpha-1} \big\|_{L_x^{p_2} L_t^{q_2}} \, d\theta \\ &\leqslant C\|u-v\|_L \int_0^1 (\|u\|_S + \|v\|_S)^{\alpha-1} \, d\theta \\ &\leqslant C(\|u\|_S + \|v\|_S)^{\alpha-1} \|u-v\|_L. \end{aligned}$$

On the other hand, we see from Lemma 3.7 that

$$\big\| |D_x|^s \big(G'(\theta u + (1-\theta)v)\big) \big\|_{L_x^{p_1} L_t^{q_1}} \leqslant C\|G'\|_{\mathrm{Lip}(\alpha-1)} \|\theta u + (1-\theta)v\|_S^{\alpha-2} \, \|\theta u + (1-\theta)v\|_L$$

for any $\theta \in (0,1)$. Hence, we find the following estimate on I_1:

$$I_1 \leqslant C\|u-v\|_S \, \|G'\|_{\mathrm{Lip}(\alpha-1)} (\|u\|_S + \|v\|_S)^{\alpha-2} (\|u\|_L + \|v\|_L).$$

Collecting the above inequalities, we obtain the result. ☐

4. Proofs of the main theorems

In this section, we prove the main theorems. Recall the notation $S(I;r) = X(I;0,r)$. Now, take a number $s_L(\alpha)$ so that a pair $\big(s_L(\alpha), \frac{\alpha-1}{2}\big)$ is acceptable and conjugate-acceptable. We define $L\big(I; \frac{\alpha-1}{2}\big) = X\big(I; s_L(\alpha), \frac{\alpha-1}{2}\big)$ and $N\big(I; \frac{\alpha-1}{2}\big) = Y\big(I; s_L(\alpha), \frac{\alpha-1}{2}\big)$.

Remark 4.1. If $\frac{27}{7} < \alpha < \frac{23}{3}$ then $s_L(\alpha)$ with the above property exists. Indeed, $s_L(\alpha) = \frac{3}{4} - \frac{1}{\alpha-1}$ works. Our upper bound on α comes from this point.

Local well-posedness in a scale-critical space. Let us prove Theorem 1.2. To prove this theorem, we show the following lemma.

Lemma 4.2. *Assume* $\frac{21}{5} < \alpha < \frac{23}{3}$ *and* $u_0 \in \widehat{L}_x^{\frac{\alpha-1}{2}}$. *Let* $t_0 \in \mathbb{R}$ *and* I *be an interval with* $t_0 \in I$. *Then, there exists a universal constant* $\delta > 0$ *such that, if a tempered distribution* u_0 *and an interval* $I \ni t_0$ *satisfy*

$$\varepsilon = \varepsilon(I; u_0, t_0) := \|e^{-(t-t_0)\partial_x^3} u_0\|_{S(I;\frac{\alpha-1}{2})} + \|e^{-(t-t_0)\partial_x^3} u_0\|_{L(I,\frac{\alpha-1}{2})} \leqslant \delta,$$

then there exists a unique solution $u \in C(I; \widehat{L}_x^{\frac{\alpha-1}{2}})$ *to the initial value problem*

$$\begin{cases} \partial_t u + \partial_x^3 u = \mu \partial_x(|u|^{\alpha-1}u), & t, x \in \mathbb{R}, \\ u(t_0, x) = u_0(x), & x \in \mathbb{R} \end{cases}$$

(*in the sense of the corresponding integral equation*) *that satisfies*

$$\|u\|_{S(I;\frac{\alpha-1}{2})} + \|u\|_{L(I;\frac{\alpha-1}{2})} \leqslant 2\varepsilon.$$

If $u_0 \in \hat{L}^{\frac{\alpha-1}{2}}$, in addition, then

$$\|u\|_{L^\infty(I;\hat{L}^{(\alpha-1)/2})} \leqslant \|u_0\|_{\hat{L}^{(\alpha-1)/2}} + C\varepsilon^\alpha$$

holds for some constant $C > 0$ and u belongs to all $\hat{L}^{\frac{\alpha-1}{2}}$-admissible spaces $X\big(I; s, \frac{\alpha-1}{2}\big)$.

Proof of Lemma 4.2. For $R > 0$, define a complete metric space

$$Z_R = \big\{u \in L\big(I; \tfrac{\alpha-1}{2}\big) \cap S\big(I; \tfrac{\alpha-1}{2}\big) \,:\, \|u\|_Z \leqslant R\big\},$$
$$\|u\|_Z := \|u\|_{L(I;\frac{\alpha-1}{2})} + \|u\|_{S(I;\frac{\alpha-1}{2})}, \quad d_Z(u,v) := \|u-v\|_Z.$$

For given tempered distribution u_0 with $e^{-(t-t_0)\partial_x^3}u_0 \in Z_\delta$ and $v \in Z_R$, we define

$$\Phi(v)(t) := e^{-(t-t_0)\partial_x^3}u_0 + \mu\int_{t_0}^t e^{-(t-t')\partial_x^3}\partial_x(|v|^{\alpha-1}v)(t')\,dt'.$$

We show that there exists $\delta > 0$ such that $\Phi : Z_{2\varepsilon} \to Z_{2\varepsilon}$ is a contraction map for any $0 < \varepsilon \leqslant \delta$.

To this end, we prove that there exist constants $C_1, C_2 > 0$ such that for any $u, v \in Z_R$,

$$\|\Phi(u)\|_Z \leqslant \|e^{-(t-t_0)\partial_x^3}u_0\|_Z + C_1R^\alpha, \tag{4-1}$$
$$d_Z(\Phi(u),\Phi(v)) \leqslant C_2R^{\alpha-1}d_Z(u,v). \tag{4-2}$$

Let $u \in Z_R$. We infer from Proposition 3.2(ii) that

$$\|\Phi(u)\|_Z \leqslant \|e^{-t\partial_x^3}u_0\|_Z + C\big\||u|^{\alpha-1}u\big\|_{N(I;\frac{\alpha-1}{2})}.$$

We then apply Lemma 3.4(i) with $r = \frac{\alpha-1}{2}$ and $s = s_L(\alpha)$ to obtain (4-1). A similar argument, employing Lemma 3.4(ii), shows (4-2).

Now let us choose $\delta > 0$ so that

$$C_1(2\delta)^{\alpha-1} \leqslant \tfrac{1}{2}, \quad C_2(2\delta)^{\alpha-1} \leqslant \tfrac{1}{2}. \tag{4-3}$$

Then, we conclude from (4-1), (4-2), and the smallness assumption that Φ is a contraction map on $Z_{2\varepsilon}$. Therefore, the Banach fixed point theorem ensures that there exists a unique solution $u \in Z_{2\varepsilon}$ to (1-1).

We now suppose that $u_0 \in \hat{L}^{\frac{\alpha-1}{2}}$. By means of Proposition 3.2, we have

$$\|u\|_{L^\infty(I,\hat{L}^{(\alpha-1)/2})} \leqslant \|u_0\|_{\hat{L}^{(\alpha-1)/2}} + C\varepsilon^\alpha$$

as in (4-1). The same argument shows $u \in X\big(I; s, \frac{\alpha-1}{2}\big)$ for any s such that $\big(s, \frac{\alpha-1}{2}\big)$ is acceptable. □

Proof of Theorem 1.2. By Lemma 4.2, we obtain a unique solution

$$u \in L_t^\infty([-T,T]; \hat{L}_x^{\frac{\alpha-1}{2}}) \cap S\big([-T,T]; \tfrac{\alpha-1}{2}\big) \cap L\big([-T,T]; \tfrac{\alpha-1}{2}\big)$$

for small $T = T(u_0) > 0$. We repeat the above argument to extend the solution, and then obtain a solution which has a maximal lifespan. The regularity property (1-4) and the continuous dependence of solution on the initial data are shown by a usual way. This completes Theorem 1.2. □

Blowup criterion and scattering criterion. In this subsection we prove Theorems 1.7, 1.8, and 1.9.

Proof of Theorem 1.8. Assume for contradiction that $T_{\max} < \infty$ and $\|u\|_{S([0,T_{\max});\frac{\alpha-1}{2})} < \infty$.

Step 1. We first show that the above assumption yields

$$\|u\|_{L([0,T_{\max});\frac{\alpha-1}{2})} < \infty.$$

Fix T so that $0 < T < T_{\max}$. Let $s_L(\alpha)$ be as in the previous section (see Remark 4.1). If we take $\theta \in (0,1)$ so that $\left(\theta s_L(\alpha), \frac{\alpha-1}{2}\right)$ is conjugate-acceptable then it follows from Proposition 3.2 that

$$\|u\|_{L([0,T];\frac{\alpha-1}{2})} \leqslant C\|u_0\|_{\hat{L}^{\frac{\alpha-1}{2}}} + C\left\||u|^{\alpha-1}u\right\|_{Y([0,T];\theta s_L(\alpha),\frac{\alpha-1}{2})}.$$

Then, Lemma 3.4(i) with $r = \frac{\alpha-1}{2}$ and Lemma 3.3 give us

$$\|u\|_{L([0,T];\frac{\alpha-1}{2})} \leqslant C\|u_0\|_{\hat{L}^{\frac{\alpha-1}{2}}} + C\|u\|^{\alpha-\theta}_{S([0,T];\frac{\alpha-1}{2})}\|u\|^{\theta}_{L([0,T];\frac{\alpha-1}{2})}.$$

By assumption,

$$\|u\|_{S([0,T];\frac{\alpha-1}{2})} \leqslant \|u\|_{S([0,T_{\max});\frac{\alpha-1}{2})} < +\infty$$

for any $T \in (0, T_{\max})$. Plugging this to the previous estimate, we see that there exist constants $A, B > 0$ such that

$$\|u\|_{L([0,T];\frac{\alpha-1}{2})} \leqslant A + B\|u\|^{\theta}_{L([0,T];\frac{\alpha-1}{2})}$$

for any $T \in (0, T_{\max})$, which gives us the desired bound since $\theta < 1$.

Step 2. Let $t_0 \in (0, T_{\max})$. Since

$$u(t) = e^{-(t-t_0)\partial_x^3}u(t_0) + \mu\int_{t_0}^{t} e^{-(t-t')\partial_x^3}\partial_x(|u|^{\alpha-1}u)(t')\,dt'$$

for $t \in (0, T_{\max})$, the above estimate yields the following bound on $e^{-(t-t_0)\partial_x^3}u_0$:

$$\begin{aligned}\|e^{-(t-t_0)\partial_x^3}u(t_0)\|&_{S([t_0,T_{\max});\frac{\alpha-1}{2})\cap L([t_0,T_{\max};\frac{\alpha-1}{2})} \\ &\leqslant \|u\|_{S([t_0,T_{\max});\frac{\alpha-1}{2})\cap L([t_0,T_{\max};\frac{\alpha-1}{2})} + C\|u\|^{\alpha-1}_{S([t_0,T_{\max});\frac{\alpha-1}{2})}\|u\|_{L([t_0,T_{\max});\frac{\alpha-1}{2})} < \infty.\end{aligned}$$

Step 3. Let us now prove that we can extend the solution beyond $T_{\max}$. Let δ be the constant given in Lemma 4.2. We see from the bound in the previous step that there exists $t_0 \in (0, T_{\max})$ such that

$$\|e^{-(t-t_0)\partial_x^3}u(t_0)\|_{S([t_0,T_{\max});\frac{\alpha-1}{2})} + \|e^{-(t-t_0)\partial_x^3}u(t_0)\|_{L([t_0,T_{\max});\frac{\alpha-1}{2})} \leqslant \tfrac{1}{2}\delta.$$

Hence, one can take $\tau > 0$ so that

$$\|e^{-(t-t_0)\partial_x^3}u(t_0)\|_{S([t_0,T_{\max}+\tau);\frac{\alpha-1}{2})} + \|e^{-(t-t_0)\partial_x^3}u(t_0)\|_{L([t_0,T_{\max}+\tau);\frac{\alpha-1}{2})} \leqslant \delta.$$

Then, just as in the proof of Theorem 1.2 (or Lemma 4.2), we can construct a solution $u(t)$ to (1-1) in the interval $(-T_{\min}, T_{\max}+\tau)$, which contradicts the definition of $T_{\max}$. □

Proof of Theorem 1.9. We first assume that $T_{\max} = +\infty$ and $\|u\|_{S([0,\infty);\frac{\alpha-1}{2})} < \infty$. Then, as in the first step of the proof of Theorem 1.8, one obtains $\|u\|_{L([0,\infty);\frac{\alpha-1}{2})} < \infty$. Since $\{e^{-t\partial_x^3}\}_{t\in\mathbb{R}}$ is an isometry in $\hat{L}^{\frac{\alpha-1}{2}}$, it suffices to show that $\{e^{t\partial_x^3}u(t)\}_{t\in\mathbb{R}}$ is a Cauchy sequence in $\hat{L}^{\frac{\alpha-1}{2}}$ as $t\to\infty$. Let $0<t_1<t_2$. By an argument similar to the proof of (4-2), we obtain

$$\begin{aligned}\left\|e^{t_2\partial_x^3}u(t_2)-e^{t_1\partial_x^3}u(t_1)\right\|_{\hat{L}^{(\alpha-1)/2}} &\leqslant C\left\||u|^{\alpha-1}u\right\|_{N([t_1,\infty);\frac{\alpha-1}{2})}\\ &\leqslant C\|u\|^{\alpha-1}_{S([t_1,\infty);\frac{\alpha-1}{2})}\|u\|_{L([t_1,\infty);\frac{\alpha-1}{2})}\to 0 \quad \text{as } t_1\to\infty.\end{aligned}$$

Hence, we find that the solution to (1-1) scatters to a solution of the Airy equation as $t\to\infty$.

Conversely, if $u(t)$ scatters forward in time then we can choose $T>0$ so that

$$\|e^{-t\partial_x^3}u_+\|_{S([T,\infty);\frac{\alpha-1}{2})}+\|e^{-t\partial_x^3}u_+\|_{L([T,\infty);\frac{\alpha-1}{2})}\leqslant \tfrac{1}{2}\delta,$$

where $u_+=\lim_{t\to\infty}e^{t\partial_x^3}u(t)\in\hat{L}^{\frac{\alpha-1}{2}}$ and δ is the constant given in Lemma 4.2. Moreover, it holds for sufficiently large $t_0\in[T,\infty)$ that

$$\begin{aligned}\left\|e^{-t\partial_x^3}(e^{t_0\partial_x^3}u(t_0)-u_+)\right\|_{S([T,\infty);\frac{\alpha-1}{2})}+\left\|e^{-t\partial_x^3}(e^{t_0\partial_x^3}u(t_0)-u_+)\right\|_{L([T,\infty);\frac{\alpha-1}{2})}&\\ \leqslant C\|e^{t_0\partial_x^3}u(t_0)-u_+\|_{\hat{L}^{(\alpha-1)/2}}\leqslant \tfrac{1}{2}\delta&\end{aligned}$$

by means of (2-2). We then see that

$$\|e^{-(t-t_0)\partial_x}u(t_0)\|_{S([T,\infty);\frac{\alpha-1}{2})}+\|e^{-(t-t_0)\partial_x}u(t_0)\|_{L([T,\infty);\frac{\alpha-1}{2})}\leqslant\delta.$$

Then, Lemma 4.2 implies that $\|u\|_{S([T,\infty);\frac{\alpha-1}{2})}\leqslant 2\delta$. □

Proof of Theorem 1.7. By (2-2), we have

$$\|e^{-t\partial_x^3}u_0\|_{L(\mathbb{R};\frac{\alpha-1}{2})}+\|e^{-t\partial_x^3}u_0\|_{S(\mathbb{R};\frac{\alpha-1}{2})}\leqslant C\varepsilon.$$

Then, in light of Lemma 4.2, we see that u exists globally in time and satisfies $\|u\|_S\leqslant 2C\varepsilon$, provided ε is small compared with the constant δ given in Lemma 4.2. Theorem 1.9 ensures that u scatters for both time directions. □

Persistence of regularity. In this subsection, we prove Theorems 1.4 and 1.5, and then Theorem 1.10.

Proof of Theorem 1.4. Let us prove that $u\in L\big(I;\frac{\alpha_0-1}{2}\big)$. As in the proof of Lemma 4.2, one deduces from Proposition 3.2 and Lemma 3.4(i) that

$$\begin{aligned}\|u\|_{L(I;\frac{\alpha_0-1}{2})}&\leqslant C\|u_0\|_{\hat{L}^{(\alpha_0-1)/2}}+C\left\||u|^{\alpha-1}u\right\|_{N(I;\frac{\alpha_0-1}{2})}\\ &\leqslant C\|u_0\|_{\hat{L}^{r_0}}+C\|u\|^{\alpha-1}_{S(I;\frac{\alpha-1}{2})}\|u\|_{L(I;\frac{\alpha_0-1}{2})}.\end{aligned}$$

Since we already know $\|u\|_{S(I;\frac{\alpha-1}{2})}<\infty$ by assumption, we have the desired bound

$$\|u\|_{L(I;\frac{\alpha_0-1}{2})}\leqslant 2C\|u_0\|_{\hat{L}^{(\alpha_0-1)/2}}$$

for a sufficiently short interval I. Then, again by Proposition 3.2,

$$\|u\|_{L_t^\infty(I;\hat{L}_x^{(\alpha_0-1)/2})\cap X(I;s,\frac{\alpha_0-1}{2})}\leqslant C_s\|u_0\|_{\hat{L}^{(\alpha_0-1)/2}}+C_s\|u\|^{\alpha-1}_{S(I;\frac{\alpha-1}{2})}\|u\|_{L(I;\frac{\alpha_0-1}{2})}<+\infty$$

for any acceptable pair $\big(s,\frac{\alpha_0-1}{2}\big)$. Finite-time use of this argument yields the result. □

Proof of Theorem 1.5. Suppose that $0 < \sigma < \alpha$. Take a number ε so that $0 < \varepsilon < \min(1, \alpha - \sigma)$. Since $|D_x|^\sigma$ commutes with $e^{-t\partial_x^3}$ and since $(\varepsilon, 2)$ is acceptable and conjugate-acceptable, we see from Proposition 3.2 that

$$\big\||D_x|^\sigma u(t)\big\|_{X(I;\varepsilon,2)} \leqslant C\big\||D_x|^\sigma u_0\big\|_{L^2} + C\big\||D_x|^\sigma(|u|^{\alpha-1}u)\big\|_{Y(I;\varepsilon,2)}.$$

Since $\sigma + \varepsilon < \alpha$, arguing as in the proof of Lemma 3.6, one sees that

$$\begin{aligned}\big\||D_x|^\sigma(|u|^{\alpha-1}u)\big\|_{Y(I;\varepsilon,2)} &= \big\||D_x|^{\sigma+\varepsilon}(|u|^{\alpha-1}u)\big\|_{L_x^{\tilde{p}(\varepsilon,2)}L_t^{\tilde{q}(\varepsilon,2)}(I)} \\ &\leqslant C\|u\|^{\alpha-1}_{L_x^{p(0,(\alpha-1)/2)}L_t^{q(0,(\alpha-1)/2)}(I)}\big\||D_x|^{\sigma+\varepsilon}u\big\|_{L_x^{p(\varepsilon,2)}L_t^{q(\varepsilon,2)}(I)} \\ &= C\|u\|^{\alpha-1}_{S(I;\frac{\alpha-1}{2})}\big\||D_x|^\sigma u\big\|_{X(I;\varepsilon,2)}.\end{aligned}$$

Hence, we obtain an upper bound for $\big\||D_x|^\sigma u\big\|_{X(I;\varepsilon,2)}$ for a small interval. Then, the result follows as in Theorem 1.4.

Next, let $-1 < \sigma < 0$. Set $\varepsilon = -\sigma \in (0,1)$. As in the previous case, we have

$$\big\||D_x|^\sigma u(t)\big\|_{X(I;\varepsilon,2)} \leqslant C\big\||D_x|^\sigma u_0\big\|_{L^2} + C\big\||D_x|^\sigma(|u|^{\alpha-1}u)\big\|_{Y(I;\varepsilon,2)}$$

since $(\varepsilon, 2)$ is acceptable and conjugate-acceptable. Then,

$$\big\||D_x|^\sigma(|u|^{\alpha-1}u)\big\|_{Y(I;\varepsilon,2)} = \big\||u|^{\alpha-1}u\big\|_{L_x^{\tilde{p}(\varepsilon,2)}L_t^{\tilde{q}(\varepsilon,2)}(I)} \leqslant \|u\|^{\alpha-1}_{S(I;\frac{\alpha-1}{2})}\big\||D_x|^\sigma u\big\|_{X(I;\varepsilon,2)}$$

by Hölder's inequality. The rest of the argument is the same. □

Remark 4.3. In the above proposition, the upper bound $s < \alpha$ is natural in view of the regularity that the nonlinearity $|u|^{\alpha-1}u$ possesses. When α is an odd integer, that is, if $\alpha = 5, 7$, then the nonlinearity u^5 or u^7 is analytic (in u) and so we can remove the upper bound and treat all $s > 0$. We omit the details.

Remark 4.4. By modifying the proof of Theorem 1.5, we easily reproduce the local well-posedness in $\dot{H}^{s_\alpha}$ for $\alpha \geqslant 5$. More precisely, by Lemma 3.3,

$$\|u\|_{S(I;\frac{\alpha-1}{2})} \leqslant \big\||D_x|^{s_\alpha}u\big\|^{\frac{8}{5(\alpha-1)}}_{X(I;-\frac{1}{4},2)}\big\||D_x|^{\frac{2(9-\alpha)}{(5\alpha-13)(\alpha-1)}}u\big\|^{\frac{5\alpha-13}{5(\alpha-1)}}_{L_{t,x}^{(5\alpha-13)/2}(I)}.$$

By Sobolev's embedding in space and Minkowski's inequality,

$$\begin{aligned}\big\||D_x|^{\frac{2(9-\alpha)}{(5\alpha-13)(\alpha-1)}}u\big\|_{L_{t,x}^{(5\alpha-13)/2}(I)} &\leqslant C\big\||D_x|^{s_\alpha-\frac{5\alpha-33}{4(5\alpha-13)}}u\big\|_{L_t^{(5\alpha-13)/2}L_x^{(4(5\alpha-13))/(5\alpha-17)}(I)} \\ &\leqslant C\big\||D_x|^{s_\alpha}u\big\|_{X(I;-\frac{1}{4}+\frac{5}{5\alpha-13},2)}.\end{aligned}$$

Hence, estimating as in the proof of Theorem 1.5, we obtain a closed estimate in

$$|D_x|^{-s_\alpha}X(I;\varepsilon,2) \cap |D_x|^{-s_\alpha}X\big(I;-\tfrac{1}{4}+\tfrac{5}{5\alpha-13},2\big) \cap |D_x|^{-s_\alpha}X\big(I;-\tfrac{1}{4},2\big),$$

which yields local well-posedness in $\dot{H}^{s_\alpha}$.[5]

[5] Strictly speaking, we should work with pairs $\big(-\frac{1}{4}+\eta_1, 2\big)$ and $\big(-\frac{1}{4}+\frac{5}{5\alpha-13}-\eta_2, 2\big)$ for small $\eta_j = \eta_j(\alpha) > 0$ because the critical case $q\big(-\frac{1}{4},2\big) = \infty$ is excluded in Lemma 3.3. However, the modification is obvious.

Proof of Theorem 1.10. We suppose for contradiction that $u(t)$ scatters to $u_+ \in \hat{L}^{\frac{\alpha-1}{2}}$ as $t \to \infty$. Since $u_0 \in H^1$, Theorems 1.4 and 1.5 imply that $u(t) \in C(\mathbb{R}; H^1)$. Further, $u(t)$ scatters also in H^1 and so we see that $\|\partial_x u(t)\|_{L^2} = \|\partial_x e^{t\partial_x^3} u(t)\|_{L^2} \to \|u_+\|_{\dot{H}^1}$ as $t \to \infty$.

On the other hand, by the Gagliardo–Nirenberg inequality and mass conservation,

$$\|u(t)\|_{L_x^{\alpha+1}} \leqslant C \|u_0\|_{L_x^2}^{\frac{2}{\alpha+1}} \big\| |D_x|^{\frac{2}{3(\alpha-1)}} u(t) \big\|_{L_x^{(3(\alpha-1))/2}}^{\frac{\alpha-1}{\alpha+1}}.$$

Since $u(t)$ scatters as $t \to \infty$, we see that $u \in X\big([0,\infty); \frac{2}{3(\alpha-1)}, \frac{\alpha-1}{2}\big)$ as in the proof of Theorem 1.9. Therefore, we can take a sequence $\{t_n\}_n$ with $t_n \to \infty$ as $n \to \infty$ so that $\|u(t_n)\|_{L^{\alpha+1}} \to 0$ as $n \to \infty$. Thus, by conservation of energy,

$$0 \geqslant E[u_0] = E[u(t_n)] = \tfrac{1}{2}\|\partial_x u(t_n)\|_{L^2}^2 - \frac{\mu}{\alpha+1}\|u(t_n)\|_{L^{\alpha+1}}^{\alpha+1} \to \tfrac{1}{2}\|u_+\|_{\dot{H}^1}^2$$

as $n \to \infty$. Hence, $E[u_0] < 0$ yields a contradiction. If $E[u_0] = 0$ then we see that $u_+ = 0$, and so $\|u_0\|_{L^2} = \|u_+\|_{L^2} = 0$. This contradicts $u_0 \neq 0$. □

Appendix A: Proof of Lemma 3.7

In this appendix, we prove Lemma 3.7. To prove this lemma, we need the following space-time bounds of the maximal function

$$(\mathcal{M}u)(x) = \sup_{R>0} \frac{1}{2R} \int_{x-R}^{x+R} |u(y)|\, dy.$$

Lemma A.1. *Let I be an interval. Assume $1 < p, q < \infty$.*

(i) *There exists a positive constant C depending on p, q and I such that*

$$\|\mathcal{M} f\|_{L_x^p L_t^q(I)} \leqslant C \|f\|_{L_x^p L_t^q(I)} \tag{A-1}$$

for any $f \in L_x^p L_t^q(I)$.

(ii) *There exists a positive constant C depending on p, q and I such that*

$$\|\mathcal{M} f_k\|_{L_x^p L_t^q \ell_k^2(I)} \leqslant C \|f_k\|_{L_x^p L_t^q \ell_k^2(I)} \tag{A-2}$$

for any $\{f_k\}_k \in L_x^p L_t^q \ell_k^2(I)$.

Proof of Lemma A.1. See [Fefferman and Stein 1971] for (A-1) and [Kenig et al. 1993, Lemma A.3(e)] for (A-2). □

Proof of Lemma 3.7. We follow [Sickel 1989] (see also [Runst and Sickel 1996]). Let $\{\varphi_k(D_x)\}_{k=-\infty}^{\infty}$ be a Littlewood–Paley decomposition with respect to the x-variable. From [Kenig et al. 1993, Lemma A.3], we see

$$\big\| |D_x|^s f \big\|_{L_x^p L_t^q} \sim \|2^{sk} \varphi_k(D_x) f\|_{L_x^p L_t^q \ell_k^2}. \tag{A-3}$$

Step 1. Write $\mu = N + \beta$ with $N \in \mathbb{Z}$ and $\beta \in (0, 1]$. We remark that $N \geqslant 1$ since $\mu > 1$. We first note that Taylor's expansion of G gives us

$$
\begin{aligned}
G(z) &= \sum_{l=0}^{N-1} \frac{G^{(l)}(a)}{\ell!}(z-a)^l + \int_a^z \frac{(z-v)^{N-1}}{(N-1)!} G^{(N)}(v)\,dv \\
&= \sum_{l=0}^{N} \frac{G^{(l)}(a)}{\ell!}(z-a)^l + \int_a^z \frac{(z-v)^{N-1}}{(N-1)!} \bigl(G^{(N)}(v) - G^{(N)}(a)\bigr)\,dv \\
&= \sum_{l=0}^{N}\sum_{j=0}^{l} \frac{(-1)^{l-j} G^{(l)}(a) a^{l-j}}{(\ell-j)!\,j!} z^j + \int_a^z \frac{(z-v)^{N-1}}{(N-1)!} \bigl(G^{(N)}(v) - G^{(N)}(a)\bigr)\,dv.
\end{aligned}
$$

Hence, applying the above expansion with $z = f(y)$ and $a = f(x)$,

$$
\begin{aligned}
\mathcal{F}^{-1}[\varphi_k \mathcal{F} G(f)](x) &= c \int_{\mathbb{R}^n} (\mathcal{F}^{-1}\varphi_k)(x-y) G(f(y))\,dy \\
&= c \sum_{l=0}^{N}\sum_{j=0}^{l} \frac{(-1)^{l-j} G^{(l)}(f(x))(f(x))^{l-j}}{(\ell-j)!\,j!} \int_{\mathbb{R}^n} (\mathcal{F}^{-1}\varphi_k)(x-y)(f(y))^j\,dy \\
&\quad + c \int_{\mathbb{R}^n} (\mathcal{F}^{-1}\varphi_k)(x-y) \int_{f(x)}^{f(y)} \frac{(f(y)-v)^{N-1}}{(N-1)!} \bigl(G^{(N)}(v) - G^{(N)}(f(x))\bigr)\,dv\,dy \\
&=: T_{1,k} + T_{2,k}.
\end{aligned}
\tag{A-4}
$$

We first estimate $T_{1,k}$. Since $\int \mathcal{F}^{-1}\varphi_k(y)\,dy = \varphi_k(0) = 0$, the summand in $T_{1,k}$ vanishes if $j = 0$. By the estimate

$$
|G^{(l)}(f(x))| \leqslant \|G\|_{\mathrm{Lip}\,\mu} |f(x)|^{\mu-l},
$$

we have

$$
\begin{aligned}
\|2^{sk} T_{1,k}\|_{L_x^p L_t^q \ell_k^2} &\leqslant C \|G\|_{\mathrm{Lip}\,\mu} \sum_{j=1}^{N} \bigl\| |f|^{\mu-j}\, 2^{sk} \varphi_k(D_x)(f^j)| \bigr\|_{L_x^p L_t^q \ell_k^2} \\
&\leqslant C \|G\|_{\mathrm{Lip}\,\mu} \sum_{j=1}^{N} \|f\|_{L_x^{p_1} L_t^{q_1}}^{\mu-j} \bigl\| |D_x|^s (f^j) \bigr\|_{L_x^{p_{2,j}} L_t^{q_{2,j}}},
\end{aligned}
$$

where

$$
\frac{1}{p} = \frac{\mu-j}{p_1} + \frac{1}{p_{2,j}}, \qquad \frac{1}{q} = \frac{\mu-j}{q_1} + \frac{1}{q_{2,j}}.
$$

Further, a recursive use of Lemma 3.6 yields

$$
\bigl\| |D_x|^s (f^j) \bigr\|_{L_x^{p_{2,j}} L_t^{q_{2,j}}} \leqslant C_j \|f\|_{L_x^{p_1} L_t^{q_1}}^{j-1} \bigl\| |D_x|^s f \bigr\|_{L_x^{p_2} L_t^{q_2}}
$$

for $j \geqslant 2$, which completes the estimate of $T_{1,k}$.

Next, we estimate $T_{2,k}$. First note that

$$
\left| \int_{f(x)}^{f(y)} \frac{(f(y)-v)^{N-1}}{(N-1)!} \bigl(G^{(N)}(v) - G^{(N)}(f(x))\bigr)\,dv \right| \leqslant C \|G\|_{\mathrm{Lip}\,\mu} |f(x) - f(y)|^{\mu}
$$

by the definition of $\|G\|_{\mathrm{Lip}\,\mu}$. Further, for any $M>0$, there exists C_M such that

$$|(\mathcal{F}^{-1}\varphi_k)(x-y)| = 2^k\big|(\mathcal{F}^{-1}\varphi_0)(2^k(x-y))\big| \leqslant C_M 2^k(1+2^k|x-y|)^{-M}.$$

Therefore,

$$|T_{2,k}| \leqslant C2^k\|G\|_{\mathrm{Lip}\,\mu}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^\mu}{(1+2^k|x-y|)^M}\,dy \leqslant C\sum_{l=0}^{\infty}2^{k-lM}(I^\mu_{k-l}f)(x),$$

where

$$I^\mu_k f(x) = \int_{|z|\leqslant 2^{-k}}|f(x+z)-f(x)|^\mu\,dz = 2^{-k}\int_{|z|\leqslant 1}|f(x+2^{-k}z)-f(x)|^\mu\,dz.$$

We now claim that

$$\|2^{k(s+1)}(I^\mu_k f)\|_{L^p_xL^q_t\ell^2_k} \leqslant C\big\||D_x|^{\frac{s}{\mu}}f\big\|^\mu_{L^{\mu p}_xL^{\mu q}_t}. \tag{A-5}$$

This claim completes the proof. Indeed, combining the above estimates, we see that

$$\|2^{sk}T_{2,k}\|_{L^p_xL^q_t\ell^2_k} \leqslant C\sum_{l=0}^{\infty}2^{l(s+1-M)}\big\|2^{(k-l)(s+1)}(I^\mu_{k-l}f)\big\|_{L^p_xL^q_t\ell^2_k} \leqslant C\big\||D_x|^{\frac{s}{\mu}}f\big\|^\mu_{L^{\mu p}_xL^{\mu q}_t},$$

provided we choose $M>s+1$. By Lemma 3.3, we conclude that

$$\big\||D_x|^{\frac{s}{\mu}}f\big\|_{L^{\mu p}_xL^{\mu q}_t} \leqslant \|f\|^{1-\frac{1}{\mu}}_{L^{p_1}_xL^{q_1}_t}\,\||D_x|^s f\|^{\frac{1}{\mu}}_{L^{p_2}_xL^{q_2}_t}.$$

Step 2. We prove claim (A-5). Let Δ_h be the difference operator $\Delta_h f(x) = f(x+h)-f(x)$. Since $f=\sum_{m\in\mathbb{Z}}\varphi_{k+m}(D_x)f$ for any $k\in\mathbb{Z}$, one sees that

$$\begin{aligned}\|2^{k(s+1)}(I^\mu_k f)(x)\|_{L^p_xL^q_t\ell^2_k} &= \Big\|2^{ks}\int_{|z|\leqslant 1}|\Delta_{2^{-k}z}f(x)|^\mu\,dz\Big\|_{L^p_xL^q_t\ell^2_k}\\ &\leqslant \Big\|2^{ks}\int_{|z|\leqslant 1}\Big|\Delta_{2^{-k}z}\sum_{m=-\infty}^{-1}\varphi_{k+m}(D)f(x)\Big|^\mu\,dz\Big\|_{L^p_xL^q_t\ell^2_k}\\ &\quad+\Big\|2^{ks}\int_{|z|\leqslant 1}\Big|\Delta_{2^{-k}z}\sum_{m=0}^{\infty}\varphi_{k+m}(D)f(x)\Big|^\mu\,dz\Big\|_{L^p_xL^q_t\ell^2_k}\\ &=: A+B.\end{aligned}$$

We estimate A. Take $a\in\big(\frac{1}{\mu},1\big)$. Let $k\in\mathbb{Z}$. If $m<0$ and $|h|\leqslant 2^{-k}$ then we have

$$\begin{aligned}\big|\Delta_h\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x)\big| &\leqslant |h|\big|\nabla(\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f])(x+\theta h)\big|\\ &\leqslant 2^m\sup_{|y|\leqslant 2^{-k}}\Big|\Big(\nabla\mathcal{F}^{-1}\Big[\varphi_0\mathcal{F}\Big[f\Big(\frac{\cdot}{2^{k+m}}\Big)\Big]\Big]\Big)(2^{k+m}(x-y))\Big|\\ &\leqslant C_a2^m\sup_{y\in\mathbb{R}}\frac{\big|\big(\nabla\mathcal{F}^{-1}\big[\varphi_0\mathcal{F}\big[f\big(\frac{\cdot}{2^{k+m}}\big)\big]\big]\big)(2^{k+m}(x-y))\big|}{1+|2^{k+m}y|^a}\\ &\leqslant C_a2^m\sup_{y\in\mathbb{R}}\frac{|\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x-y)|}{1+|2^{k+m}y|^a}\end{aligned}$$

for any $x \in \mathbb{R}$, where we have used the estimate

$$\sup_{y\in\mathbb{R}} \frac{|\nabla \mathcal{F}^{-1}[\varphi_0 \mathcal{F} f](x-y)|}{1+|y|^a} \leqslant C \sup_{y\in\mathbb{R}} \frac{|\mathcal{F}^{-1}[\varphi_0 \mathcal{F} f](x-y)|}{1+|y|^a}$$

(see [Runst and Sickel 1996, Section 2.1.6, Proposition 2(i)]) to obtain the last line. We define the Peetre–Fefferman–Stein maximal function by

$$\varphi_j^{*,a} f(x) := \sup_{y\in\mathbb{R}} \frac{|\mathcal{F}^{-1}[\varphi_j \mathcal{F} f](x-y)|}{1+|2^j y|^a}.$$

By the above estimates, we have

$$\begin{aligned} A &\leqslant C \Big\| 2^{ks} \sum_{m=-\infty}^{-1} \sup_{|z|\leqslant 1} \big|\Delta_{2^{-k}z}\varphi_{k+m}(D) f(x)\big|^{\mu} \Big\|_{L_x^p L_t^q \ell_k^2} \leqslant C \sum_{m=-\infty}^{-1} 2^{m\mu} \big\| 2^{k\frac{s}{\mu}} \varphi_{k+m}^{*,a} f\big\|^{\mu}_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2\mu}} \\ &\leqslant C \sum_{m=-\infty}^{-1} 2^{m(\mu-s)} \big\| 2^{(k+m)\frac{s}{\mu}} \varphi_{k+m}^{*,a} f\big\|^{\mu}_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2\mu}} \leqslant C \big\| 2^{k\frac{s}{\mu}} \varphi_{k}^{*,a} f\big\|^{\mu}_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2\mu}}, \end{aligned}$$

where we used the fact that $s < \mu$. Since $(\varphi_k^{*,a} f)(x) = \big(\varphi_0^{*,a}(\tilde{\varphi}_k(D_x) f)\big(\frac{\cdot}{2^k}\big)\big)(2^k x)$, [Triebel 1983, Lemma 2.3.6] yields

$$(\varphi_k^{*,a} f)(x) \leqslant C \big(\mathcal{M}[(\tilde{\varphi}_k(D_x) f)^{\frac{1}{a}}]\big)^a (x),$$

where $\tilde{\varphi}_k = \sum_{i=-1}^{1} \varphi_{k+i}$. Then, (A-2), the embedding $\ell^2 \hookrightarrow \ell^q$ $(2 < q \leqslant \infty)$, and (A-3) lead us to

$$\begin{aligned} \big\| 2^{k\frac{s}{\mu}} \varphi_k^{*,a} f\big\|_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2\mu}} &\leqslant C \big\| 2^{k\frac{s}{a\mu}} \mathcal{M}[(\tilde{\varphi}_k(D_x) f)^{\frac{1}{a}}]\big\|^a_{L_x^{a\mu p} L_t^{a\mu q} \ell_k^{2a\mu}} \\ &\leqslant C \big\| 2^{k\frac{s}{a\mu}} (\tilde{\varphi}_k(D_x) f)^{\frac{1}{a}}\big\|^a_{L_x^{a\mu p} L_t^{a\mu q} \ell_k^{2}} \\ &\leqslant C \big\| 2^{k\frac{s}{\mu}} \tilde{\varphi}_k(D_x) f\big\|_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2/a}} \leqslant C \big\| |D_x|^{\frac{s}{\mu}} f\big\|_{L_x^{\mu p} L_t^{\mu q}} \end{aligned}$$

since $\frac{1}{\mu} < a < 1$.

Let us proceed to the estimate of B. We first note that

$$\begin{aligned} &\int_{|z|\leqslant 1} \Big|\Delta_{2^{-k}z} \sum_{m=0}^{\infty} \varphi_{k+m}(D) f(x)\Big|^{\mu} dz \\ &\qquad = \int_{|z|\leqslant 1} \Big| \sum_{m=0}^{\infty} 2^{-\frac{\varepsilon}{\mu}m} 2^{\frac{\varepsilon}{\mu}m} \Delta_{2^{-k}z}\varphi_{k+m}(D) f(x)\Big|^{\mu} dz \\ &\qquad \leqslant C_\varepsilon \int_{|z|\leqslant 1} \sum_{m=0}^{\infty} 2^{\varepsilon m} \big|\Delta_{2^{-k}z}\varphi_{k+m}(D) f(x)\big|^{\mu} dz \\ &\qquad = C_\varepsilon \sum_{m=0}^{\infty} 2^{\varepsilon m} \int_{|z|\leqslant 1} \big|\Delta_{2^{-k}z}\varphi_{k+m}(D) f(x)\big|^{\mu} dz \\ &\qquad \leqslant C \sum_{m=0}^{\infty} 2^{\varepsilon m} \big(\sup_{|z|\leqslant 1} \big|\Delta_{2^{-k}z}\varphi_{k+m}(D) f(x)\big|\big)^{\mu(1-\lambda)} \int_{|z|\leqslant 1} \big|\Delta_{2^{-k}z}\varphi_{k+m}(D) f(x)\big|^{\mu\lambda} dz, \end{aligned}$$

where $\lambda \in (0,1)$. For $m \geqslant 0$ and $|h| \leqslant 2^{-k}$, the triangle inequality gives us

$$\left|\Delta_h \mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x)\right| \leqslant 2 \sup_{|y|\leqslant 2^{-k}} \left|\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x-y)\right| \leqslant C2^{ma}\varphi_{k+m}^{*,a}f(x),$$

where $a \in \left(\frac{1}{\mu},1\right)$. Further,

$$\int_{|z|\leqslant 1} \left|\Delta_{2^{-k}z}\varphi_{k+m}(D_x)f(x)\right|^{\mu\lambda} dz \leqslant C\mathcal{M}[|\varphi_{k+m}(D)_x f|^{\mu\lambda}](x).$$

Using these inequalities, one deduces from Hölder's inequality, the embedding $\ell^2 \hookrightarrow \ell^q$ $(2 < q \leqslant \infty)$, (A-2), and (A-3) that

$$\begin{aligned}
B &\leqslant C\left\|2^{sk}\sum_{m=0}^{\infty} 2^{m\varepsilon}\mathcal{M}[|\varphi_{k+m}(D_x)f|^{\mu\lambda}]2^{ma\mu(1-\lambda)}(\varphi_{k+m}^{*,a}f)^{\mu(1-\lambda)}\right\|_{L_x^p L_t^q \ell_k^2} \\
&\leqslant C\sum_{m=0}^{\infty} 2^{m(\varepsilon+a\mu(1-\lambda))}\left\|2^{sk}\mathcal{M}[|\varphi_{k+m}(D_x)f|^{\mu\lambda}](\varphi_{k+m}^{*,a}f)^{\mu(1-\lambda)}\right\|_{L_x^p L_t^q \ell_k^2} \\
&\leqslant C\sum_{m=0}^{\infty} 2^{m(\varepsilon+a\mu(1-\lambda)-s)}\left\|\mathcal{M}[|2^{\frac{s}{\mu}k}\varphi_k(D_x)f|^{\mu\lambda}]\right\|_{L_x^{p/\lambda} L_t^{q/\lambda} \ell_k^{2/\lambda}} \left\|2^{\frac{s}{\mu}k}\varphi_k^{*,a}f\right\|_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2\mu}}^{\mu(1-\lambda)} \\
&\leqslant C\sum_{m=0}^{\infty} 2^{m(\varepsilon+a\mu(1-\lambda)-s)}\left\||D_x|^{\frac{s}{\mu}}f\right\|_{L_x^{\mu p} L_t^{\mu q}}^{\mu} \\
&\leqslant \left\||D_x|^{\frac{s}{\mu}}f\right\|_{L_x^{\mu p} L_t^{\mu q}}^{\mu}
\end{aligned}$$

as long as $\varepsilon + a\mu(1-\lambda) - s < 0$. Since $a \in \left(\frac{1}{\mu},1\right)$, we are able to choose $\lambda \in (0,1)$ and $\varepsilon > 0$ suitably. $\square$

Appendix B: Inclusion relations of $\widehat{L}^r$

In this appendix, we briefly summarize some inclusion relations between $\widehat{L}^r$ and other frequently used spaces such as Lebesgue spaces or Sobolev spaces. Here, $\dot{H}^{0,s} = \dot{H}^{0,s}(\mathbb{R})$ stands for a weighted L^2-space with norm $\|f\|_{\dot{H}^{0,s}} = \||x|^s f\|_{L^2}$.

Lemma B.1. *We have the following*:

(i) $L^r \hookrightarrow \widehat{L}^r$ *if* $1 \leqslant r \leqslant 2$ *and* $\widehat{L}^r \hookrightarrow L^r$ *if* $2 \leqslant r \leqslant \infty$.

(ii) $\dot{H}^{0,\frac{1}{r}-\frac{1}{2}} \hookrightarrow \widehat{L}^r$ *if* $1 < r \leqslant 2$ *and* $\widehat{L}^r \hookrightarrow \dot{H}^{0,\frac{1}{r}-\frac{1}{2}}$ *if* $2 \leqslant r < \infty$.

(iii) $\widehat{L}^r \hookrightarrow \dot{B}_{2,r'}^{\frac{1}{2}-\frac{1}{r}}$ *if* $1 \leqslant r \leqslant 2$ *and* $\dot{B}_{2,r'}^{\frac{1}{2}-\frac{1}{r}} \hookrightarrow \widehat{L}^r$ *if* $2 \leqslant r \leqslant \infty$.

Proof of Lemma B.1. The first assertion follows from the Hausdorff–Young inequality. The Sobolev embedding (in the Fourier side) yields the second. We omit the details.

The third is also immediate from the Hölder inequality. Indeed, if $2 \leqslant r \leqslant \infty$ then

$$\|\hat{f}\|_{L^{r'}(\{2^n\leqslant|\xi|\leqslant 2^{n+1}\})} \leqslant C2^{n(\frac{1}{2}-\frac{1}{r})}\|\hat{f}\|_{L^2(\{2^n\leqslant|\xi|\leqslant 2^{n+1}\})}$$

for any $n \in \mathbb{Z}$. Taking the $\ell_n^{r'}$-norm, we obtain the desired embedding. The case $1 \leqslant r \leqslant 2$ follows in the same way. $\square$

Let $\dot{H}^s = \dot{H}^s(\mathbb{R})$ be a homogeneous Sobolev space with norm

$$\|f\|_{\dot{H}^s} = \big\||\xi|^s \hat{f}\big\|_{L^2}.$$

Notice that the above inclusions are the same as for $\dot{H}^{\frac{1}{2}-\frac{1}{r}}$. Namely, we can replace $\widehat{L}^r$ with $\dot{H}^{\frac{1}{2}-\frac{1}{r}}$ in Lemma B.1 (except for the endpoint case $r = 1, \infty$ in (i)). Indeed, (i) is a Sobolev embedding, (ii) follows from Hardy's inequality, and a basic property of Besov spaces gives us (iii). However, there is no inclusion between $\widehat{L}^r$ and $\dot{H}^{\frac{1}{2}-\frac{1}{r}}$ for $r \neq 2$.

Lemma B.2. *For $1 \leqslant r \leqslant \infty$ ($r \neq 2$), $\widehat{L}^r \not\hookrightarrow \dot{H}^{\frac{1}{2}-\frac{1}{r}}$ and $\dot{H}^{\frac{1}{2}-\frac{1}{r}} \not\hookrightarrow \widehat{L}^r$.*

Proof of Lemma B.2. If $2 < r \leqslant \infty$, we have the following counterexamples: Let us define $f_n(x)$ by $\widehat{f_n}(\xi) = 1$ for $n \leqslant \xi \leqslant n+1$ and $\widehat{f_n}(\xi) = 0$ elsewhere. Then, $f_n(x)$ satisfies $\|f_n\|_{\dot{H}^{\frac{1}{2}-\frac{1}{r}}} \to \infty$ as $n \to \infty$, while $\|f_n\|_{\widehat{L}^r}$=1. Hence. $\widehat{L}^r \not\hookrightarrow \dot{H}^{\frac{1}{2}-\frac{1}{r}}$. On the other hand, for some $p \in \left(\frac{1}{2}, \frac{1}{r'}\right)$, take $g_n(x)$ ($n \geqslant 3$) so that $\widehat{g_n}(\xi) = \xi^{-\frac{1}{r'}} |\log \xi|^{-p}$ for $\frac{1}{n} \leqslant \xi \leqslant \frac{1}{2}$ and $\widehat{g_n}(\xi) = 0$ elsewhere. Then, $\|g_n\|_{\dot{H}^{\frac{1}{2}-\frac{1}{r}}}$ is bounded but $\|g_n\|_{\widehat{L}^r} \to \infty$ as $n \to \infty$. This shows $\dot{H}^{\frac{1}{2}-\frac{1}{r}} \not\hookrightarrow \widehat{L}^r$.

The case $1 < r < 2$ follows by duality.

Let us consider the case $r = 1$. We note that $\delta_0(x) \in \widehat{L}^1 \setminus \dot{H}^{-\frac{1}{2}}$, where $\delta_0(x)$ is the Dirac delta function. Therefore, $\widehat{L}^1 \not\hookrightarrow \dot{H}^{-\frac{1}{2}}$. On the other hand,

$$f_n(x) = \left(\log\left(1 + \tfrac{1}{n}\right)\right)^{-1} \mathcal{F}^{-1}[\mathbf{1}_{\{1 \leqslant \xi \leqslant 1+\frac{1}{n}\}}](x)$$

is a counterexample for $\dot{H}^{-\frac{1}{2}} \not\hookrightarrow \widehat{L}^1$. $\square$

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REGULARITY FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH VERY IRREGULAR KERNELS

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We prove Hölder regularity for a general class of parabolic integro-differential equations, which (strictly) includes many previous results. We present a proof that avoids the use of a convex envelope as well as give a new covering argument that is better suited to the fractional order setting. Our main result involves a class of kernels that may contain a singular measure, may vanish at some points, and are not required to be symmetric. This new generality of integro-differential operators opens the door to further applications of the theory, including some regularization estimates for the Boltzmann equation.

1. Introduction

We study the Hölder regularity for solutions of integro-differential equations of the form

$$u_t + b(x,t)\cdot\nabla u - \int_{\mathbb{R}^d} \big(u(x+h,t) - u(x,t)\big) K(x,h,t)\,\mathrm{d}h = f(x,t). \tag{1-1}$$

The integral may be singular at the origin and must be interpreted in the appropriate sense. These equations now appear in many contexts. Most notably, they appear naturally in the study of stochastic processes with jumps, which traditionally has been the main motivation for their interest. In the same way that pure jump processes contain the class of diffusions (processes with continuous paths) as particular limiting cases, (1-1) contains the usual second-order parabolic equations as particular limiting cases. This is due to the fact

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that the integral term becomes a second-order operator $a_{ij}(x,t)\,\partial_{ij}u$ as the order α (to be defined below) converges to 2. We note that the simplest choice of K is $K(h) = C_{d,\alpha}|h|^{-d-\alpha}$, which results in the equation

$$u_t + (-\Delta)^{\alpha/2}u = 0,$$

and converges to the usual heat equation $u_t - \Delta u = 0$ as $\alpha \to 2$ (recall that $(-\Delta)^{\alpha/2}$ is the operator whose Fourier symbol is $|\xi|^\alpha$).

The Hölder estimates that we obtain in this article are an integro-differential version of the celebrated result by Krylov and Safonov [1980] for parabolic equations with measurable coefficients. There are, in fact, several versions of these Hölder estimates for integro-differential equations, which were obtained in the last 10 years, and we briefly review them in Section 1A. Besides the elliptic/parabolic distinction, the difference between each version of the estimates is in the level of generality in the possible choices of the kernels $K(x,h,t)$. In this article, we obtain the estimates for a very generic class of kernels K, including nearly all previous results of this type.

The most common assumption in the literature is that for all x and t, the kernel K is comparable pointwise in terms of h to the kernel for the fractional Laplacian. More precisely,

$$(2-\alpha)\frac{\lambda}{|h|^{d+\alpha}} \le K(x,h,t) \le (2-\alpha)\frac{\Lambda}{|h|^{d+\alpha}}. \tag{1-2}$$

This is often accompanied by the symmetry assumption $K(x,h,t) = K(x,-h,t)$. It is important for the applications of these estimates that no regularity condition may be assumed for K with respect to x or t.

In this paper, we only assume a much weaker version of (1-2). The upper bound for K, in (1-2), is relaxed to hold only in average when we integrate all the values of h on an annulus, and it appears as assumption (A2). Also, for our work, the lower bound in (1-2) only needs to hold in a subset of values of h that has positive density, given as assumption (A3). We also make an assumption, (A4), which says that the odd part of K is under control if α is close to 1. The exact conditions are listed in Section 2. We prove that solutions of (1-1) are uniformly Hölder continuous, which we state in an informal way here and revisit more precisely in Section 7.

Theorem 1.1. *Let u solve* (1-1). *Assume that for every $x \in B_1$ and $t \in [-1,0]$, the kernel $K(x,\cdot,t)$ satisfies the assumptions* (A1), (A2), (A3) *and* (A4) *in Section 2. Assume also that f is bounded, b is bounded, and for $\alpha < 1$, we have $b \equiv 0$. Then for some $\gamma > 0$,*

$$[u]_{C^\gamma(Q_{1/2})} \le C(\|u\|_{L^\infty(\mathbb{R}^d\times[-1,0])} + \|f\|_{L^\infty(Q_1)}).$$

The constants C and γ depend on the constants μ, λ and Λ in (A1)–(A4), on the dimension d, on a lower bound for α (in particular, α can be arbitrarily close to 2), and on $\|b\|_{L^\infty}$.

Our purpose in developing Theorem 1.1 is not merely for the sake of generalization. An estimate with the level of generality given here can be used to obtain a priori estimates for the homogeneous Boltzmann equation. This is a novel application. None of the previous Hölder estimates for integral equations are appropriate to be applied to the Boltzmann equation.

As a byproduct of our proof of Theorem 1.1, we simplify and clarify some of the details regarding parabolic covering arguments (see the crawling ink spots of Section 6) as well as present a proof that does not invoke a convex envelope. Rather, we circumvent the oft-used gradient mapping of the convex envelope by using a mapping that associates points via their correspondence through parameters in an inf-convolution, modeled on the arguments of [Imbert and Silvestre 2013a], originating in [Cabré 1997; Savin 2007].

In Section 8, we apply this result to derive the $C^{1,\alpha}$ regularity for the parabolic Isaacs equation. This is a rather standard application of Hölder estimates for equations with rough coefficients, as in Theorem 1.1.

1A. ***Comparison with previous results and some discussion of* (1-1).** The Hölder estimates for integro-differential equations that take the form of (1-1) are a fractional-order version of the classical theorem by Krylov and Safonov [1980]. This is a fundamental result in the study of regularity properties of parabolic equations in nondivergence form, and has consequences for many aspects of the subsequent PDE theory. The classical theorem of De Giorgi, Nash and Moser concerns second-order parabolic equations in *divergence form*, in contrast with the theorem of Krylov and Safonov. The basic results for integro-differential equations in divergence form were developed earlier, and a small survey of this subject can be found in [Kassmann and Schwab 2014].

The simplest case of K would be $K(h) = (2-\alpha)|h|^{-d-\alpha}$, and this choice gives the operator $Lu(x) = -C_{d,\alpha}(-\Delta)^{\alpha/2}u(x)$, which is a multiple of the fractional Laplacian of order α (the operator whose Fourier symbol is $|\xi|^\alpha$). This operator (and its inverse, the Riesz potential of order α) have a long history, and have been fundamental to potential theory for about a century; see, for example, Landkof's book [1966]. In fact, the appearance of nonlocal operators similar to the one in (1-1) is in some sense generic among all linear operators that satisfy the positive global maximum principle (that is, the operator is nonpositive whenever it is evaluated at a positive maximum of a C^2 function). This has been known since the work of Courrège [1965]. He proved that any linear operator with the positive maximum principle must be of the form

$$Lu(x) = -c(x)u(x) + b(x)\cdot\nabla u(x) + \mathrm{Tr}\big(A(x)D^2u(x)\big) + \int_{\mathbb{R}^d}\big(u(x+h) - u(x) - \mathbb{1}_{B_1}(h)\nabla u(x)\cdot h\big)\mu(x,\mathrm{d}h),$$

where $c \geq 0$ is a function, $A \geq 0$ is a matrix, b is a vector, all of A, b, c are bounded, and $\mu(x,\cdot)$ is a Lévy measure that satisfies

$$\sup_x \int_{\mathbb{R}^d} \min(|h|^2, 1)\mu(x,\mathrm{d}h) < +\infty.$$

Heuristically from the point of view of jump-diffusion stochastic processes, b records the drift, A records the local covariance (or $\sqrt{A}$ is the diffusion matrix), and μ records the jumps.

The first Hölder regularity result for an equation of the form (1-1) was obtained in [Bass and Levin 2002a]. In that paper, the authors consider the elliptic equation (u constant in time), with symmetric kernels satisfying the pointwise bound (1-2) and without drift. Their proof uses probabilistic techniques involving a related Markov (pure jump) stochastic process. Other results using probabilistic techniques were [Bass and Kassmann 2005; Song and Vondraček 2004], where different assumptions on the kernels are considered. The first purely analytical proof was given in [Silvestre 2006]. This first generation of results consists only of elliptic problems. They are not *robust* in the sense that as order approaches 2, the constants in the estimates blow up (hence they do not recover the known second-order results). Furthermore, they all require a pointwise bound below for the kernels as in (1-2).

The first robust Hölder estimate for the elliptic problem was obtained in [Caffarelli and Silvestre 2009], which means that the estimate they proved has constants that do not blow up as the order α of the equation goes to 2. In that sense, it is the first true generalization of the theorem of Krylov and Safonov. It was the first of the series of papers [Caffarelli and Silvestre 2009; 2011a; 2011b] recreating the regularity theory for fully nonlinear elliptic equations in the nonlocal setting. As above, these results are only for the elliptic problem, and they require symmetric kernels that satisfy the pointwise assumption (1-2).

The first estimate for parabolic integro-differential equations, *in nondivergence form*, appeared, to the best of our knowledge, in [Silvestre 2011] (the divergence case had some earlier results such as [Bass and Levin 2002b; Chen and Kumagai 2003]). In this case, the kernels are symmetric and satisfy (1-2) with $\alpha = 1$. The focus of [Silvestre 2011] is on the interaction between the integro-differential part and the drift term. The proof can easily be extended to arbitrary values of α, but the estimate is not robust (it blows up as $\alpha \to 2$), and the details of this proof are explained in the lecture notes by one of the authors [Silvestre 2012b]. It is even possible to extend this proof to kernels that satisfy the upper bound in average like in our assumption (A2) below (see [Silvestre 2014b]). However, the estimates are not robust, and the lower bound in (1-2) is required.

The first robust estimate for parabolic equations appeared in [Chang Lara and Dávila 2014], which is a parabolic version of the result in [Caffarelli and Silvestre 2009]. The kernels are required to be symmetric and to satisfy the two pointwise inequalities (1-2) as an assumption.

Elliptic integro-differential equations with nonsymmetric kernels are studied in the articles [Chang Lara 2012; Chang Lara and Dávila 2012]. There, the kernels are decomposed into the sum of their even (symmetric) and odd parts. The symmetric part is assumed to satisfy (1-2), and there are appropriate assumptions on the odd part so that the symmetric part of the equations controls the odd part. This effectively makes the contribution to the equation from the odd part of the kernel a lower-order term.

The only articles where the lower bound in the kernels (1-2) is not required to hold at all points are [Bjorland et al. 2012; Guillen and Schwab 2012; Kassmann and Mimica 2013a; Kassmann et al. 2014]. These papers concern elliptic equations and the upper bound in (1-2) is still assumed to hold. It is important to point out that under the conditions in [Bjorland et al. 2012; Kassmann et al. 2014], the Harnack inequality is not true. There is, in fact, a counterexample in [Bogdan and Sztonyk 2005] (also discussed in [Kassmann et al. 2014]). The assumption in these works that was made to replace the pointwise lower bound on the kernels is more restrictive than our assumption (A3) below.

The main result in this article (see Theorems 7.1 and 7.2) generalizes nearly all previous Hölder estimates (for both elliptic and parabolic equations) for integro-differential equations with rough kernels in nondivergence form. It strictly contains the Hölder regularity results in [Bass and Levin 2002a; Bjorland et al. 2012; Caffarelli and Silvestre 2009; Chang Lara 2012; Chang Lara and Dávila 2012; 2014; Guillen and Schwab 2012; Kassmann et al. 2014]. There is an interesting new result given in [Kassmann and Mimica 2013b] that allows for kernels with a *logarithmic* growth at the origin (among other cases), corresponding in our context to the limit $\alpha \to 0$, and it is not contained in the result of this paper.

Our approach draws upon ideas from several previous papers. Moreover, we haven been able to simplify the ideas substantially, especially how to handle parabolic equations, and we do not follow the method in

[Chang Lara and Dávila 2014]. Our method allows us to make more general assumptions on the class of possible kernels. We would like to point out that we do not make any assumption for *simplicity* in this paper. Extending these results to a more singular family of kernels would require new ideas.

There are two possible directions that we did not pursue in this paper. We did not try to analyze singularities of the kernels of order more general than a power of $|h|$, as in [Kassmann and Mimica 2013b]. Also, it might be possible to extend our regularity results for equations with Hölder continuous drifts and $\alpha < 1$, as in [Silvestre 2012a]; although, we do point out that this technique does not work right away with the methods in this paper. We also point out that the results in this paper and all of the others mentioned (except for [Kassmann and Schwab 2014]), require that the Lévy measure — referred to above as $\mu(x, \mathrm{d}h)$ — has a nontrivial absolutely continuous part, $K\,\mathrm{d}h$, with respect to Lebesgue measure (our work allows for a measure with a density plus some singular part). Verifying the validity of, and finding a proof for, results similar to Theorem 1.1 in the case when μ may not have a density with respect to Lebesgue measure remains a significant open question in the integro-differential theory.

The importance of not assuming any regularity in x and t for the ingredients of (1-1) — the case of so-called bounded measurable coefficients — is for much more than simply mathematical generality. For example, because equations such as (1-1) often lack a "divergence structure" — i.e., admitting a representation as a weak formulation for functions in an energy space such as $H^{\alpha/2}$ — they can usually only be realized as classical solutions or as viscosity solutions (weak solutions). (We note that uniqueness for equations related to (1-1) is still an open question for the theory of viscosity solutions of integro-differential equations, and recent progress has been made in [Mou and Święch 2015].) That means that one of the few tools available for compactness arguments involving families of solutions are those provided in the space of continuous functions via Theorem 1.1. This is relevant for both the possibility of proving the existence of classical solutions as well as for analyzing fully nonlinear equations in a way that doesn't depend on the regularity of the coefficients. Indeed, both situations can be viewed as morally equivalent to studying linear equations with bounded measurable coefficients. For studying regularity of translation invariant equations, this arises by effectively differentiating the equation, which results in coefficients that depend upon the solution. In the fully nonlinear case, many situations involve operators that are a min-max of linear operators, and so the bounded measurable linear coefficients arise from choosing the operators that achieve the min-max for the given function at each given point — a situation in which you cannot assume any regular dependence in the x-variable. Such min-max representations turn out to be somewhat generic for fully nonlinear elliptic equations, as was noted in the recent work [Guillen and Schwab 2014, Section 4].

1B. ***Application: the homogeneous non-cut-off Boltzmann equation.*** In this section, we briefly explain an important application of our main result, which is not possible to obtain with any of the previously known estimates for integro-differential equations. This result is explained in detail in [Silvestre 2014a].

The Boltzmann equation is a well-known integral equation that models the evolution of the density of particles in dilute gases. In the space homogeneous case, the equation is

$$f_t = Q(f, f) := \int_{\mathbb{R}^n} \int_{\partial B_1} \big(f(v', t) f(v'_\star, t) - f(v_\star, t) f(v, t)\big) B(|v - v_\star|, \theta)\,\mathrm{d}\sigma\,\mathrm{d}v_\star. \tag{1-3}$$

Here v', $v'_\star$ and θ are defined by the relations

$$\begin{aligned} r &= |v_* - v| = |v'_* - v'|, & v' &= \tfrac{1}{2}(v + v_*) + \tfrac{1}{2} r\sigma, \\ \cos\theta &= \sigma \cdot \frac{v_* - v}{|v_* - v|}, & v'_* &= \tfrac{1}{2}(v + v_*) - \tfrac{1}{2} r\sigma. \end{aligned}$$

There are several modeling choices for the *cross-section* function B. From some physical considerations, it makes sense to consider $B(r,\theta) \approx r^\gamma |\theta|^{n-1+\alpha}$, with $\gamma > -n$ and $\alpha \in (0,2)$. Note that this cross section B is never integrable with respect to the variable $\sigma \in \partial B_1$. In order to avoid this difficulty, sometimes a (non-physical) cross section is used that is integrable. This assumption is known as *Grad's cut-off assumption*.

Until the middle of the 1990s, most works on the Boltzmann equation used Grad's cut-off assumption. The non-cut-off case, despite its relevance for physical applications, was not studied so much due to its analytical complexity. An important result that caused a better understanding of the non-cut-off case came with the paper of Alexandre, Desvillettes, Villani, and Wennberg [Alexandre et al. 2000], in which they obtained a lower bound on the entropy dissipation in terms of the Sobolev norm $\|f\|^{\alpha/2}_{\text{loc}}$. All regularity results for the non-cut-off case that came afterwards are based on a coercivity estimate that is a small variation of this entropy dissipation argument. So far, this was the only regularization mechanism that was known for the Boltzmann equation.

It turns our that we can split the right-hand side of the Boltzmann equation, (1-3), in two terms. The first one is an integro-differential operator, and the second is a lower-order term:

$$\begin{aligned} f_t &= Q_1(f,f) + Q_2(f,f) \\ &:= \int_{\mathbb{R}^n} \int_{\partial B_1} f(v'_\star, t)\big(f(v',t) - f(v,t)\big) B(|v - v_\star|, \theta)\, \mathrm{d}\sigma\, \mathrm{d}v_\star \\ &\qquad\qquad + f(v,t) \int_{\mathbb{R}^n} \int_{\partial B_1} \big(f(v'_\star, t) - f(v_\star, t)\big) B(|v - v_\star|, \theta)\, \mathrm{d}\sigma\, \mathrm{d}v_\star \\ &= \int_{\mathbb{R}^n} \big(f(v',t) - f(v,t)\big) K_f(v, v', t)\, \mathrm{d}v' + c f(v,t) [|v|^\gamma * f](v). \end{aligned}$$

The kernel K_f depends on f through a complicated change of variables given using the integral identity above. If one knew that f was a smooth positive function vanishing at infinity, then indeed it could be proved that $K_f(v, v', t) \approx |v - v'|^{-n-\alpha}$, and the first term would correspond to an integro-differential operator of order α in the usual sense satisfying (1-2). Unfortunately, this is not practical for obtaining basic a priori estimates for (1-3). In fact, there is very little we can assume a priori from the solution f to the Boltzmann equation, and it is not enough to conclude that K_f satisfies (1-2). Instead, all we know a priori about f is given by its *macroscopic quantities*: its mass (the integral of f), the energy (its second moment), and its entropy. The first two quantities are constant in time, whereas the third is monotone decreasing. It can be shown that K_f satisfies the hypotheses (A1), (A2), (A3) and (A4), depending on these macroscopic quantities only. Therefore, the results in this article can be used to obtain a priori estimates for solutions of the homogeneous, non-cut-off, Boltzmann equation, which is explained in [Silvestre 2014a]. It is a new regularization effect for the Boltzmann equation that is not based on coercivity estimates, as in [Alexandre et al. 2000].

Interestingly enough, the macroscopic quantities do not give much more information about K_f than what our assumptions (A1), (A2) and (A3) say. The kernels K_f will be symmetric, so, in fact, (A4) is redundant. In terms of this generalization, almost the full power of our main result is needed. The only nonessential points are that the kernels can be assumed to be symmetric, and the robustness of the estimates does not necessarily play a role.

1C. *Notation.*

- Our space variable x belongs to $\mathbb{R}^d$.
- The annulus is $R_r := B_{2r} \setminus B_r$.
- The parabolic cylinder Q_r is defined as

$$Q_r := B_r \times (-r^\alpha, 0], \text{ and with a different center, } Q_r(x,t) = Q_r + (x,t).$$

- The "α-growth" class is

$$\mathrm{Growth}(\alpha) = \big\{ v : \mathbb{R}^d \to \mathbb{R} \,\big|\, |v(x)| \le C(1+|x|)^{\alpha-\varepsilon} \text{ for some } C, \varepsilon > 0 \big\}.$$

- Pointwise $C^{1,1}$ is defined as

$$C^{1,1}(x) := \big\{ v : \mathbb{R}^d \to \mathbb{R} \,\big|\, \exists M(x) \text{ and } \varepsilon \text{ so that } |v(x+h) - v(x) - \nabla v(x) \cdot h| \le M(x)|h|^2 \text{ for } |h| < \varepsilon \big\},$$

  and over $\mathbb{R}^d$, we have

$$C^{1,1}(\mathbb{R}^d) := \big\{ v : \mathbb{R}^d \to \mathbb{R} \,\big|\, \|v\|_{L^\infty(\mathbb{R}^d)} < \infty, \|\nabla v\|_{L^\infty(\mathbb{R}^d)} < \infty, \text{ and } v \in C^{1,1}(x) \forall x \text{ with } M(x) \text{ independent of } x \big\}.$$

- The difference operator for the different possibilities of α is

$$\delta_y u(x) := \begin{cases} u(x+y) - u(x) & \text{if } \alpha < 1, \\ u(x+y) - u(x) - \mathbb{1}_{B_1}(y) \nabla u(x) \cdot y & \text{if } \alpha = 1, \\ u(x+y) - u(x) - \nabla u(x) \cdot y & \text{if } \alpha > 1. \end{cases}$$

- The class of kernels and corresponding linear operators are

$$\mathcal{K} := \{ K : \mathbb{R}^d \to \mathbb{R} \mid K \text{ satisfies assumptions (A1)–(A4)} \},$$

$$\mathcal{L} := \left\{ Lu(x) = \int_{\mathbb{R}^d} \delta_h u(x) K(h) \, \mathrm{d}h \,\middle|\, K \in \mathcal{K} \right\}.$$

We will try to stick to the following conventions for constants:

- Large constants will be upper case letters, e.g., C, and small constants will be lower case letters, e.g., c.
- If the value of a constant is not relevant for later arguments, then we will freely use the particular letter for the constant without regard to whether or not it was used previously or will be used subsequently.
- If the value of a constant is relevant to later arguments (e.g., in determining values of subsequent constants), then we will label the constant with a subscript, e.g., C_0, C_1, C_2, etc.

Note 1.2. The following observation is useful and applies for all values of α: if $u(x) = \varphi(x)$ and $u \ge \varphi$ everywhere, then $\delta_h u(x) \ge \delta_h \varphi(x)$ for all h. This implicitly assumes that for $\alpha \ge 1$, both u and φ are differentiable at x.

2. Classes of kernels and extremal operators

The kernel $K(x,h,t)$ in (1-1) is not assumed to have any regularity with respect to x or t. The best way to think about it is that for every value of x and t, we have a kernel $(K_{x,t}(h) = K(x,\cdot,t))$ that belongs to a certain class. This class of kernels is what we describe below.

2A. ***Assumptions on K.*** For each value of λ, Λ, μ and α, we consider the family of kernels $K : \mathbb{R}^d \to \mathbb{R}$ satisfying the following assumptions:

(A1) $K(h) \geq 0$ for all $h \in \mathbb{R}^d$.

(A2) For every $r > 0$,

$$\int_{B_{2r}\setminus B_r} K(h)\,\mathrm{d}h \leq (2-\alpha)\Lambda r^{-\alpha}. \tag{2-1}$$

(A3) For every $r > 0$, there exists a set A_r such that

- $A_r \subset B_{2r} \setminus B_r$,
- A_r is symmetric in the sense that $A_r = -A_r$,
- $|A_r| \geq \mu |B_{2r} \setminus B_r|$,
- $K(h) \geq (2-\alpha)\lambda r^{-d-\alpha}$ in A_r.

Equivalently,

$$\left|\left\{y \in B_{2r} \setminus B_r \mid K(h) \geq (2-\alpha)\lambda r^{-d-\alpha} \text{ and } K(-h) \geq (2-\alpha)\lambda r^{-d-\alpha}\right\}\right| \geq \mu |B_{2r} \setminus B_r|. \tag{2-2}$$

(A4) For all $r > 0$,

$$\left|\int_{B_{2r}\setminus B_r} hK(h)\,\mathrm{d}h\right| \leq \Lambda |1-\alpha| r^{1-\alpha}. \tag{2-3}$$

2B. ***Discussion of the assumptions.*** We stress that although our kernels can be zero for large sets of h, their corresponding integral operators are not rightfully described as "degenerate". One can draw an analogy with the second-order case in the context of diffusions. A diffusion process will satisfy uniform hitting-time estimates for measurable sets of positive measure whenever the diffusion matrix is comparable to the identity from below and above. In the context of our pure jump processes related to (1-1), these jump processes will still satisfy such uniform hitting-time estimates even though the kernels can be zero in many points (meaning that at the occurrence of any *one* jump, the process will have zero probability of jumping with certain values of h).

The first assumption, (A1), is unavoidable if one hopes to study examples of (1-1) that satisfy a comparison principle between sub- and supersolutions.

The second assumption, (A2), is mostly used to estimate an upper bound for the application of the operator, L, to a smooth test function. It is more general than assuming a pointwise upper bound, as was done in [Caffarelli and Silvestre 2009; Kassmann et al. 2014] and many others. It is also slightly more general than a corresponding bound obtained by integrating on spheres as

$$\int_{\partial B_r} K(h)\,\mathrm{d}S(h) \leq (2-\alpha)\Lambda r^{-1-\alpha}.$$

It is, however, a stronger hypothesis than

$$\int_{B_r} |h|^2 K(h)\,\mathrm{d}h \le \Lambda r^{2-\alpha}.$$

It is worth pointing out that (A2) implies

$$\int_{\mathbb{R}^d \setminus B_r} K(h)\,\mathrm{d}h \le \frac{2^\alpha}{2^\alpha - 1}(2-\alpha)\Lambda r^{-\alpha}.$$

The first factor blows up as $\alpha \to 0$ but not as $\alpha \to 2$. In fact, the proofs of all our regularity results fail for $\alpha \le 0$ exactly because the tails of the integrals become infinite. The question of what happens as $\alpha \to 0$ is interesting for the nonlocal theory, and some results are obtained in [Kassmann and Mimica 2013b] (note, there they do not use the typical normalization constant as in potential theory, where $C_{d,\alpha} \approx \alpha$ as $\alpha \to 0$, so the limit operator is *not* a multiple of the identity). We also have

$$\int_{\mathbb{R}^d} (1 \wedge |h|^2) K(h)\,\mathrm{d}h \le C(\alpha)\Lambda \tag{2-4}$$

for a constant $C(\alpha)$ that stays bounded as $\alpha \to 2$, and (2-1) can be thought of as a scale invariant, of order α, version of (2-4).

Note that the assumption (A2) does not preclude the kernel K from containing a singular measure. For example, the measure given by

$$\int_A K(h)\,\mathrm{d}h = \int_{A \cap \{h_1 = h_2 = \cdots = h_{d-1} = 0\}} (2-\alpha)\frac{\lambda}{|h_n|^{1+\alpha}}\,\mathrm{d}h_d$$

is a valid kernel K that satisfies (A2) (but not (A3)). In this case, K is a singular measure, but we abuse notation by writing it as if it was absolutely continuous with a density $K(h)$.

The example above corresponds to the operator

$$-\int_{\mathbb{R}^d} \delta_h u(x) K(h)\,\mathrm{d}h = (-\partial_{dd})^{\alpha/2} u.$$

As we mentioned before, this kernel satisfies the assumption (A2) but not (A3). However, the kernel of the operator

$$-\int_{\mathbb{R}^d} \delta_h u(x) K(h)\,\mathrm{d}h := (-\partial_{dd})^{\alpha/2} u(x) + (-\Delta)^{\alpha/2} u(x)$$

would satisfy both (A2) and (A3).

The third assumption, (A3), is stated in a form that does not require the kernel K to be positive along some prescribed rays or cone-like sets, as was done in [Kassmann et al. 2014]. The relaxation to (A3) from previous works is important to allow for situations where the positivity set of K may change from radius to radius. As mentioned above, it is equivalent to (2-2), which is the form we will actually invoke later on.

Finally, note that the assumption (A4) is automatic for symmetric kernels (i.e., when $K(h) = K(-h)$), since in that case the left-hand side is identically zero. This assumption is made in order to control the odd part of the kernels in a fashion that does not require us to split up L into two pieces involving the even and odd parts of K. It is also worth pointing out that even for $\alpha < 1$, the kernel K can have some asymmetry, but it must die out as $r \to \infty$.

There are two final facts that are important to point out. The first one is the observation that although each K may not be such that

$$Lu(x) = \int_{\mathbb{R}^d} \delta_h u(x) K(h)\,\mathrm{d}h$$

results in an operator that is scale invariant, i.e., $Lu(r\,\cdot)(x) = r^\alpha Lu(rx)$, the *family of* K that satisfy (A1)–(A4) is scale invariant. The second one is that some authors have worked with assumptions where the lower bound in (1-2) is only required for $|h| \le 1$. This does not effect our overall result because we can add and subtract the term

$$f(u;x) := (2-\alpha)\int_{\mathbb{R}^d} \delta_h u(x) \mathbb{1}_{\mathbb{R}^d\setminus B_1}(h)|h|^{-d-\alpha}\,\mathrm{d}h$$

from (1-1). Assuming K satisfies the lower bound of (1-2) only for $|h| \le 1$, this would result in an operator governed by $\widetilde{K}(h) = K(h) + \mathbb{1}_{\mathbb{R}^d\setminus B_1}(h)|h|^{-d-\alpha}$, and now $\widetilde{K}$ does satisfy the lower bound of (1-2) for all h. Furthermore, the term $f(u;\cdot)$ is controlled by $\|u\|_{L^\infty}$ and possibly $C|\nabla u|$ (depending on α) due to the fact that $\mathbb{1}_{\mathbb{R}^d\setminus B_1}(h)|h|^{-d-\alpha}$ is integrable, and hence these terms can be absorbed into the equation as a gradient term and bounded right-hand side. This pertains to, e.g., the results in [Chang Lara 2012].

2C. ***Extremal operators and useful observations.*** As mentioned above, $\mathcal{L}$ is the class of all integro-differential operators Lu of the form

$$Lu(x) = \int_{\mathbb{R}^d} \delta_h u(x) K(h)\,\mathrm{d}h,$$

where K is a kernel satisfying the assumptions (A1)–(A4) specified above. Sometimes we wish to refer to a kernel, K, instead of the operator, L, and so we also use $\mathcal{K}$ to denote the collection of all such kernels. Correspondingly, we define the extremal operators $M^+_{\mathcal{L}}$ and $M^-_{\mathcal{L}}$ as in [Caffarelli and Silvestre 2009]:

$$\begin{aligned} M^+_{\mathcal{L}} u(x) &= \sup_{L\in\mathcal{L}} Lu(x),\\ M^-_{\mathcal{L}} u(x) &= \inf_{L\in\mathcal{L}} Lu(x). \end{aligned}$$

In order to avoid notational clutter, we omit the subscript $\mathcal{L}$ in the rest of the paper. We note that when (1-1) holds for some kernel K satisfying the assumptions and with a bounded b and f, this also implies that the pair of inequalities

$$\begin{aligned} u_t + C_0|\nabla u| - M^- u &\ge -C_0,\\ u_t - C_0|\nabla u| - M^+ u &\le C_0 \end{aligned}$$

is simultaneously satisfied. The advantage of this new formulation is that it can be understood in the viscosity sense, whereas the original equation (1-1) only makes sense for classical solutions. Unless otherwise noted, we use the terms solution, subsolution, and supersolution to be interpreted in the viscosity sense (made precise below, in Definition 3.2). There may be instances when we need equations to hold in a classical sense, and in those cases, we will explicitly mention that need.

Remark 2.1. We emphasize that although (1-1) allows for K that are x-dependent, the class $\mathcal{L}$ — and hence the definition of $M^{\pm}$ — contains only those K that are independent of x. The desired inequalities are obtained because $\mathcal{L}$ contains *all possible* such K, and hence, for each x fixed, $K(x,\cdot) \in \mathcal{L}$.

It will be useful to know an important feature of $M^{\pm}$ regarding translations, rotations, and scaling. This is an important feature to keep in mind in the sense that for any *one* choice of a kernel to determine (1-1), K may not have any symmetry or scaling properties on its own. However, it is controlled by an extremal operator that does enjoy these properties. This is particularly relevant for intuition on what to expect from solutions of these equations.

Lemma 2.2. *M^+ (and hence M^-) obey the following*:

(i) *If $z \in \mathbb{R}^d$ is fixed, and $Tu := u(\cdot + z)$, then $M^+Tu(x) = M^+u(x+z)$ (translation invariance).*

(ii) *If R is a rotation or reflection on $\mathbb{R}^d$, then $M^+u(R\,\cdot)(x) = M^+u(Rx)$ (rotation invariance).*

(iii) *If $r > 0$, then $M^+u(r\,\cdot)(x) = r^{\alpha}M^+u(rx)$ (scaling).*

Proof of Lemma 2.2. Property (i) follows from a direct equality in $LTu(x) = Lu(x+z)$ whenever $K \in \mathcal{L}$ (importantly, note that $K \in \mathcal{L}$ requires $K(x,h) = K(h)$). Property (ii) follows because $\mathcal{L}$ is closed under composing K with a rotation or reflection. Property (iii) follows from the observation that if $K \in \mathcal{L}$, then

$$\widetilde{K}(h) := r^{-d-\alpha}K\left(\frac{h}{r}\right) \in \mathcal{L}$$

as well, combined with the fact that for L, $\widetilde{L}$ corresponding to K, $\widetilde{K}$, we have $Lu(r\,\cdot)(x) = r^{\alpha}\widetilde{L}u(rx)$. It is worth remarking that when $\alpha = 1$, one must be careful with rescaling the integral due to the presence of $\mathbb{1}_{B_1}(h)$. However, in this case the rescaling still holds because (A4) implies that

$$\int_{B_1\setminus B_r} hK(h)\,\mathrm{d}h = 0,$$

and this allows to keep the term $\mathbb{1}_{B_1}(h)$ fixed in $\widetilde{L}$ without effecting its value. $\square$

In the rest of this section, we make some elementary estimates that give us some bounds on $Lu(x)$ in terms of bounds for u and its derivatives. These estimates explain the need for the assumptions (2-1) and (2-3). We start with the following lemma.

Lemma 2.3. *Let K be a kernel satisfying assumptions* (A2) *and* (A4). *Then the following inequalities hold*:

$$\int_{B_r} |h|^2K(h)\,\mathrm{d}h \le C\Lambda r^{2-\alpha}, \tag{2-5}$$

$$\left|\int_{B_r} hK(h)\,\mathrm{d}h\right| \le C\Lambda r^{1-\alpha} \quad \text{if } \alpha < 1, \tag{2-6}$$

$$\left|\int_{\mathbb{R}^d\setminus B_r} hK(h)\,\mathrm{d}h\right| \le C\Lambda r^{1-\alpha} \quad \text{if } \alpha > 1, \tag{2-7}$$

$$\int_{\mathbb{R}^d\setminus B_r} K(h)\,\mathrm{d}h \le C\Lambda\frac{2-\alpha}{\alpha}r^{-\alpha}. \tag{2-8}$$

In this lemma, *the constant C is independent of all the other constants.*

Proof. The four assertions are all proved in a similar fashion, and they follow from a straightforward decomposition of the integrals in dyadic rings $B_{2^{k+1}r} \setminus B_{2^k r}$ followed by applications of (2-1) and (2-3). We will only write down explicitly the proof of (2-7) as an example.

Assume $\alpha > 1$. We use (2-3) and decompose the integral in dyadic rings $B_{2^{k+1}r} \setminus B_{2^k r}$:

$$\begin{aligned}\left|\int_{\mathbb{R}^d \setminus B_r} hK(h)\,\mathrm{d}h\right| &\le \sum_{k=0}^{\infty}\left|\int_{B_{2^{k+1}r}\setminus B_{2^k r}} hK(h)\,\mathrm{d}h\right| \\ &\le \sum_{k=0}^{\infty} \Lambda|1-\alpha|(2^k r)^{1-\alpha} \\ &\le \Lambda r^{1-\alpha}\frac{|1-\alpha|}{1-2^{1-\alpha}}.\end{aligned}$$

Since the last factor on the right is bounded uniformly for $\alpha \in (1,2)$, we have finished the proof. $\square$

Lemma 2.4. *Assume $\alpha \ge \alpha_0$. Let K be any kernel that satisfies* (2-1) *and* (2-3). *Let u be a function that is C^2 around the point x and $p = \nabla u(x)$. Moreover, assume that u satisfies the following bounds globally*:

$$D^2 u \le AI, \quad |u| \le B.$$

Then,

$$\int_{\mathbb{R}^d} \delta_h u(x)K(h)\,\mathrm{d}h \le C\left(\frac{B}{A}\right)^{-\alpha/2}\left(B + \left(\frac{B}{A}\right)^{1/2}|p|\right).$$

Here C is a constant that depends on Λ and α_0. Moreover, when $\alpha = 1$, we can drop the term depending on p and get

$$\int_{\mathbb{R}^d} \delta_h u(x)K(h)\,\mathrm{d}y \le C(AB)^{1/2}.$$

Proof. Since $\delta_h u(x)$ has a different form depending on $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$, we must divide the proof into these three cases.

We start with the case $\alpha < 1$. In this case $\delta_h u(x) = u(x+h) - u(x)$. Let $r > 0$ be arbitrary. Then

$$\begin{aligned}\int_{\mathbb{R}^d} \delta_h u(x)K(h)\,\mathrm{d}y &= \int_{B_r} \delta_h u(x)K(h)\,\mathrm{d}h + \int_{\mathbb{R}^d\setminus B_r} \delta_h u(x)K(h)\,\mathrm{d}h \\ &\le \int_{B_r} (p\cdot h + A|h|^2)K(h)\mathrm{d}h + \int_{\mathbb{R}^d\setminus B_r} 2B\,K(h)\,\mathrm{d}h. \qquad (2\text{-}9)\end{aligned}$$

Using (2-6), (2-5) and (2-8), we get

$$\le C\big(|p|r^{1-\alpha} + Ar^{2-\alpha} + Br^{-\alpha}\big). \qquad (2\text{-}10)$$

We finish the proof in the case $\alpha < 1$ by picking $r = (B/A)^{1/2}$.

The case $\alpha > 1$ is similar. In this case $\delta_h u(x) = u(x+h) - u(x) - p\cdot h$ and we get

$$\begin{aligned}\int_{\mathbb{R}^d} \delta_h u(x)K(h)\,\mathrm{d}h &= \int_{B_r} \delta_h u(x)K(h)\,\mathrm{d}h + \int_{\mathbb{R}^d\setminus B_r} \delta_h u(x)K(h)\,\mathrm{d}h \\ &\le \int_{B_r} A|h|^2 K(h)\,\mathrm{d}h + \int_{\mathbb{R}^d\setminus B_r} (p\cdot h + 2B)K(h)\,\mathrm{d}h.\end{aligned}$$

This time using (2-5), (2-7) and (2-8), we again arrive at (2-10), and conclude by picking the same $r = (B/A)^{1/2}$.

We are left with the case $\alpha = 1$. In this case,

$$\delta_h u(x) = u(x+h) - u(x) - p \cdot h \, \mathbb{1}_{B_1}(h).$$

For arbitrary $r > 0$, we have

$$\int_{\mathbb{R}^d} \delta_y u(x) K(h) \, \mathrm{d}h = \int_{B_r} \big(u(x+h) - u(x) - p \cdot h\big) K(y) \, \mathrm{d}h \\ + \int_{\mathbb{R}^d \setminus B_r} \big(u(x+h) - u(x)\big) K(h) \, \mathrm{d}h \pm \int_{B_1 \triangle B_r} h \cdot p \, K(h) \, \mathrm{d}h.$$

The last term on the right-hand side is equal to zero because of the assumption (2-3). Therefore, we can drop this term and use the other two to estimate the integral:

$$\begin{aligned} \int_{\mathbb{R}^d} \delta_h u(x) K(h) \, \mathrm{d}h &\le \int_{B_r} A|h|^2 K(h) \, \mathrm{d}h + \int_{\mathbb{R}^d \setminus B_r} 2B \, K(h) \, \mathrm{d}h. \\ &\le C(Ar + Br^{-1}), \end{aligned}$$

where the second inequality follows from (2-5) and (2-8). Picking $r = (B/A)^{1/2}$, we obtain

$$\int_{\mathbb{R}^d} \delta_h u(x) K(h) \, \mathrm{d}h \le C(AB)^{1/2}. \qquad \square$$

Remark 2.5. Lemma 2.4 requires an inequality to hold for $D^2 u$ in the full space $\mathbb{R}^d$. This does not require the function u to be C^2 globally. What it means is that $u(x) - \frac{1}{2} A|x|^2$ is concave.

Corollary 2.6. *Let $M_{\mathcal{L}}^+$ and $M_{\mathcal{L}}^-$ be the extremal operators defined above. Let $p = \nabla u(x)$ and assume that u satisfies the global bounds*

$$-A_- I \le D^2 u \le A_+ I, \qquad |u| \le B.$$

Then

$$\begin{aligned} M_{\mathcal{L}}^+ u(x) &\le C\Big(\frac{B}{A_+}\Big)^{-\alpha/2}\Big(B + \Big(\frac{B}{A_+}\Big)^{1/2}|p|\Big), \\ M_{\mathcal{L}}^- u(x) &\ge -C\Big(\frac{B}{A_-}\Big)^{-\alpha/2}\Big(B + \Big(\frac{B}{A_-}\Big)^{1/2}|p|\Big). \end{aligned}$$

Moreover, *if* $\alpha = 1$, *the estimate can be reduced to*

$$\begin{aligned} M_{\mathcal{L}}^+ u(x) &\le C(BA_+)^{1/2}, \\ M_{\mathcal{L}}^- u(x) &\ge -C(BA_-)^{1/2}. \end{aligned}$$

Proof. The estimate for $M_{\mathcal{L}}^+$ follows from taking the supremum in K in Lemma 2.4. The estimate for $M_{\mathcal{L}}^-$ follows then since

$$M_{\mathcal{L}}^- u(x) = -M_{\mathcal{L}}^+[-u](x). \qquad \square$$

3. Viscosity solutions

We use a standard definition of viscosity solutions for integral equations that is the parabolic version of the one in [Caffarelli and Silvestre 2009] and equivalent under most conditions to the parabolic version of [Barles and Imbert 2008].

Definition 3.1 (cf. [Caffarelli and Silvestre 2011b, Definition 21 and (1.2)]). We say I is a nonlocal operator that is elliptic with respect to the class of operators in this article if $Iu(x)$ is well-defined for any function $u \in \text{Growth}(\alpha)$ such that $u \in C^2(x)$ and moreover,

$$M^-(u_1-u_2)(x) - C|\nabla(u_1-u_2)(x)| \le Iu_1(x) - Iu_2(x) \le M^+(u_1-u_2)(x) + C|\nabla(u_1-u_2)(x)|.$$

The constant C must be equal to zero if $\alpha \le 1$.

We say that I is translation invariant if $I[u(\cdot - x_0)] = Iu(\cdot - x_0)$.

Note that the operators M^+ and M^- in particular are nonlocal operators, uniformly elliptic with respect to this class. These are the only operators that are needed for the main result in this article (Theorem 1.1). The main result has implications to nonlinear equations in terms of operators, as in Definition 3.1, which are given in Section 8.

Definition 3.2 (cf. [Caffarelli and Silvestre 2009, Definition 2.2; Caffarelli and Silvestre 2011b, Definition 25]). Let I be a nonlocal operator as in Definition 3.1. Assume that $u \in \text{Growth}(\alpha)$. We say $u : \mathbb{R}^d \times [T_1, T_2]$ satisfies the following inequality in the viscosity sense, and also refer to it as a viscosity supersolution of

$$u_t - Iu \ge 0 \quad \text{in } \Omega \subset \mathbb{R}^d \times \mathbb{R}$$

if every time there exist a $C^{1,1}$ function $\varphi : D \subset \Omega \to \mathbb{R}$ so that $\varphi(x_0, t_0) = u(x_0, t_0)$ and also $u \ge \varphi$ in $D \cap \{t \le t_0\}$, then the auxiliary function

$$v(x) = \begin{cases} \varphi(x, t_0) & \text{if } (x, t_0) \in D, \\ u(x, t_0) & \text{if } (x, t_0) \notin D \end{cases}$$

satisfies

$$v_t(x_0, t_0) - Iv(x_0, t_0) \ge 0.$$

One of the most characteristic properties of viscosity solutions is that they obey the comparison principle. In the context of this article, we state it as follows.

Proposition 3.3. *Let I be a translation invariant nonlocal operator that is uniformly elliptic in the sense of Definition 3.1. Let $u, v \in \mathbb{R}^n \times [0, T]$ be two continuous functions such that*

- *for all $x \in \mathbb{R}^n$, we have $u(x, 0) \ge v(x, 0)$,*
- *for all $x \in \mathbb{R}^n \setminus B_1$ and $t \in [0, T]$, we have $u(x, t) \ge v(x, t)$,*
- *$u_t - Iu \ge 0$ and $v_t - Iv \le 0$ in $B_1 \times [0, T]$.*

The $u(x, t) \ge v(x, t)$ for all $x \in B_1$ and $t \in [0, T]$.

The proof of Proposition 3.3 is by now standard. We refer the reader to [Chang Lara and Dávila 2014, Corollary 3.1; Silvestre 2011, Lemmas 3.2, 3.3; Caffarelli and Silvestre 2009, Theorem 5.2; Barles and Imbert 2008] for the main ideas. For the purposes of this article, we do not use the full power of Proposition 3.3. We only use the comparison principle to compare a supersolution u with a special barrier function constructed in Section 5. This barrier function is explicit and is smooth, except on a sphere where it has an *angle* singularity. The comparison principle follows easily from Definition 3.2 when v is this special barrier function or any smooth subsolution of the equation.

In [Caffarelli and Silvestre 2009], and many subsequent works, it was frequently used that wherever a viscosity solution u can be touched with a C^2 test function from one side, the equation can be evaluated classically with the original u at that particular point (a notable departure from the second-order theory!). This fact plays a role in some measure estimates used to prove the regularity results in those works. With our current setting, it is not possible to evaluate the equation pointwise in u because of the gradient terms; however, many possible useful variations on that theme can be shown — similar to [Kassmann et al. 2014, Appendix 7.2]. In this case, the following lemma is what we will use to obtain pointwise evaluation of the regularized supersolution.

Lemma 3.4. *Assume u satisfies the following inequality in the viscosity sense*:

$$u_t + C_0|\nabla u| - M^- u \geq -C \quad \text{in } \Omega.$$

Assume also that there is a test function $\varphi : \mathbb{R}^d \times [t_1, t_2] \to \mathbb{R}$ so that $\varphi(x_0, t_0) = u(x_0, t_0)$ and $\varphi(x, t) \leq u(x, t)$ for all $t \in (t_0 - \varepsilon, t_0]$.

Then, the following inequality holds:

$$\varphi_t(x_0,t_0)+C_0|\nabla\varphi(x_0,t_0)|-M^-\varphi(x_0,t_0)-\inf\left\{\int_{\mathbb{R}^d}\big(u(x+y,t_0)-\varphi(x+y,t_0)\big)K(y)\,\mathrm{d}y \;:\; K\in\mathcal{K}\right\}\geq -C.$$

Proof. We can use φ as the test function for Definition 3.2 in any small domain $D = B_r(x_0) \times (t_0 - \varepsilon, t_0]$. Constructing the auxiliary function v, we observe that

$$v_t(x_0, t_0) = \varphi_t(x_0, t_0), \quad \nabla v(x_0, t_0) = \nabla\varphi(x_0, t_0),$$

$$\begin{aligned} M^- v(x_0, t_0) &= \inf\left\{\int_{\mathbb{R}^d} \delta_y \varphi(x) K(y)\,\mathrm{d}y + \int_{\mathbb{R}^d \setminus B_r} \big(u(x+y) - \varphi(x+y)\big) K(y)\,\mathrm{d}y \;:\; K \in \mathcal{K}\right\} \\ &\geq M^-\varphi(x_0, t_0) + \inf\left\{\int_{\mathbb{R}^d \setminus B_r} \big(u(x+y) - \varphi(x+y)\big) K(y)\,\mathrm{d}y \;:\; K \in \mathcal{K}\right\}. \end{aligned}$$

Observe that the last term is monotone increasing as $r \to 0$.

From Definition 3.2, for any $r > 0$, we have that $v_t(x_0, t_0) + C_0|\nabla v(x_0, t_0)| - M^- v(x_0, t_0) \geq -C_1$. The result of the lemma follows by taking $r \to 0$. ☐

4. Relating a pointwise value with an estimate in measure: the growth lemma

In order to obtain the Hölder continuity of u, we need to show the following point-to-measure lemma, which seems to originate in the work of Landis [1971] (in some circles, it is known as the *growth lemma*).

It is a cornerstone of regularity theory, it leads to the weak Harnack inequality, and it is one of the few places where the equation plays a fundamental role.

Lemma 4.1. *There exist positive constants A_0 and δ_0 depending on λ, Λ, d, α_0 and C_0 so that if $\alpha > \alpha_0$ and if $u : \mathbb{R}^d \times (-1, 0] \to \mathbb{R}$ is a function such that*

(1) *$u \geq 0$ in the whole space $\mathbb{R}^d \times (-1, 0]$,*

(2) *u is a supersolution in Q_1, i.e.,*

$$u_t + C_0|\nabla u| - M^- u(x) \geq 0 \quad \text{in } Q_1, \tag{4-1}$$

(3) $\min_{Q_{1/4}} u \leq 1$,

then

$$|\{u \leq A_0\} \cap Q_1| \geq \delta_0.$$

The following function, q, plays an important role in the proof of Lemma 4.1. It is actually an inf-convolution of u with a quadratic, and it is defined as

$$q(x, t) = \min_{y \in \overline{B}_1} u(y, t) + 64|x - y|^2. \tag{4-2}$$

Note that q is a nonnegative function. We will prove a collection of properties of the function q, which will lead us to the proof of Lemma 4.1.

The next barrier is used to find a bound for the rate at which q can decrease with respect to t.

Lemma 4.2. *For a universal constant C_1, the function*

$$\varphi(x, t) = \max(0, f(t) - 64|x|^2)$$

is a subsolution to

$$\varphi_t + C_0|\nabla\varphi| - M^-\varphi \leq 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0].$$

The inequality holds classically at all points where $\varphi > 0$.

Here $f(t)$ is the (unique) positive solution to the (backward) ODE

$$\begin{cases} f(0) = 0, \\ f'(t) = -C_1\big(f(t)^{1/2} + f(t)^{1-\alpha/2}\big), \end{cases} \tag{4-3}$$

where C_1 is a constant depending on Λ and α_0 (such that $\alpha \geq \alpha_0$).

Proof. Note that for every fixed value of $t \in (-\infty, 0]$, it holds that

$$\|\varphi\|_{L^\infty} = f(t), \quad \|\nabla\varphi\|_{L^\infty} \leq C\sqrt{f(t)}, \quad \text{and} \quad 0 \geq D^2\varphi \geq -128I.$$

Applying Corollary 2.6,

$$M^-\varphi \geq -Cf(t)^{1-\alpha/2}.$$

Then, at all points where $\varphi > 0$, we have

$$\varphi_t + C_0|\nabla\varphi| - M^-\varphi \leq f'(t) + C_0Cf(t)^{1/2} + Cf(t)^{1-\alpha/2}.$$

The lemma then follows by choosing C_1 so that f' dominates the right-hand side. □

It is worth commenting that the ODE for f in Lemma 4.2 has a unique solution that is strictly positive for $t < 0$. This function f is differentiable and locally Lipschitz. The universal constant C_2 of the following result is the Lipschitz constant of f in the interval $[-T, 0]$, where $f(T) = -4$.

Corollary 4.3. *Assume $x \in B_{1/8}$ and $q(x,t) < 3$. Then there are positive universal constants τ and C_2 such that for $s \in (t-\tau, t)$, we have $q(x,s) - q(x,t) < C_2(t-s)$.*

Proof. We let x, t, and s be fixed as stated. Let y be the point where the minimum for $q(x,t)$ is achieved in (4-2). Using the definition of q, we note that for all values of $z \in B_1$, we have $u(z,s) \geq q(x,s) - 64|x-z|^2$.

The point of the proof is to use the fact that u and φ are respectively super- and subsolutions of (4-1) on the time interval $(s, 0]$. In order to invoke a comparison result between them, we will make various choices involving τ and f to enforce φ to be below u at the initial time, s, and on the boundary, which is $\mathbb{R}^d \setminus B_1$.

We define the function

$$\bar{\varphi}(\bar{x}, \bar{t}) := \varphi(\bar{x} - x, \bar{t} - s + t_0),$$

where t_0 is a fixed time, yet to be chosen. We fix the constant τ so that

$$\tau < f^{-1}(3) - f^{-1}(4),$$

and we fix the time $t_0 < 0$ so that

$$f(t_0) = \min(q(x,s), 4).$$

Checking the boundary condition for $\bar{x} \notin B_1$ and $\bar{t} > s$, we see that $|x - \bar{x}| \geq \frac{7}{8}$ (as $x \in B_{1/8}$), and hence since $f(t_0) \leq 4 \leq 49$, we have (note f is decreasing)

$$\bar{\varphi}(\bar{x}, \bar{t}) = \varphi(\bar{x} - x, \bar{t} - s + t_0) = \max\bigl(0, f(\bar{t} - s + t_0) - 64|x - \bar{x}|^2\bigr) \leq \max(0, f(t_0) - 49) \leq 0.$$

Checking the initial condition at $\bar{t} = s$, we have (by the definition of t_0)

$$\bar{\varphi}(\bar{x}, s) = \varphi(\bar{x} - x, t_0) = \max(0, f(t_0) - 64|x - \bar{x}|^2) \leq \max(0, q(x,s) - 64|x - \bar{x}|^2) \leq u(\bar{x}, s),$$

from the definition of q.

Comparison therefore tells us that $u \geq \bar{\varphi}$ on $B_1 \times (s, 0)$, and, in particular, for $\bar{x} = y$ and $\bar{t} = t$,

$$u(y,t) \geq \varphi(x - y, t - s + t_0) \geq f(t - s + t_0) - 64|x - y|^2.$$

Hence

$$q(x,t) = u(y,t) + 64|x-y|^2 \geq f(t - s + t_0),$$

and we will use

$$q(x,t) \geq f(t - s + t_0) \geq f(t_0) - |f'(t_0)|(t-s).$$

In the case that $f(t_0) = q(x,s)$, we can conclude the corollary with $C_2 := \max\{f'(t) : t \in (-f^{-1}(4), 0)\}$. However, τ was chosen specifically so that it is impossible for $f(t_0) < q(x,s)$. Indeed we see that if it occurred that $f(t_0) = 4$ then because f is decreasing and $t - s \leq \tau$, it holds that

$$3 > q(x,t) \geq f(t - s + t_0) \geq f(t_0) + f(\tau + t_0) - f(t_0) \geq 4 + f(f^{-1}(3)) - 4 = 3,$$

which is a contradiction. Thus $f(t_0) = q(x,s)$ is the only possibility, and we conclude. $\square$

Corollary 4.3 should be interpreted as $q_t \geq -C_2$ everywhere. The next lemma gives us a bound above for q_t in a set of positive measure.

Lemma 4.4. *Under the assumptions of Lemma 4.1*, (*but assuming here* $u(0,0) = 1$) *the function* q *from* (4-2) *satisfies* $|\{q_t \leq A_1\} \cap Q_1| \geq \delta_1 > 0$, *where* A_1 *and* δ_1 *are universal constants.*

Proof. Since $u(0,0) = 1$, for any $x \in B_{1/4}$, we have $q(x,0) \leq 1 + 64|x|^2 < 5$. Moreover, the minimum is achieved at some $y \in B_{1/2}$ since $1 + 64|y-x|^2 > 5$ if $|y| > \frac{1}{2}$. By similar reasoning, we also have that for every $x \in B_{1/8}$, it holds that $q(x,0) < 2$. Corollary 4.3 implies that for $t \in (-\tau, 0]$,

$$q(x,t) \leq q(x,0) + C_2|t| < 2 + C_2|t|.$$

Thus if we restrict $t \in (-\tau', 0]$, where $\tau' = 1/C_2$, then we have that $q(x,t) < 3$ and a second application of Corollary 4.3 shows that $q(x,t) + C_2 t$ is monotone increasing. Thus $q_t(x,t)$ exists pointwise for a.e. $t \in (-\tau', 0]$ and q_t exists as a signed measure. Furthermore,

$$q_t(x,t) \geq -C_2 \quad \text{for a.e. } t \in (-\tau', 0].$$

Integrating the measure $q_t(x,t)$ and ignoring its singular part shows (note, $q \geq 0$ always)

$$\begin{aligned}
C = 2|B_{1/8}| &\geq \int_{B_{1/8}} q(x,0) - q(x,-\tau')\,\mathrm{d}x \\
&\geq \int_{-\tau'}^{0} \int_{B_{1/8}} q_t(x,s)\,\mathrm{d}x\,\mathrm{d}s \\
&\geq A_1 \big|\big((-\tau',0] \times B_{1/8}\big) \cap \{q_t > A_1\}\big| - C_2 \big|\big((-\tau',0] \times B_{1/8}\big) \setminus \{q_t > A_1\}\big| \\
&= -C_2 \tau' |B_{1/8}| + (A_1 + C_2)\big|\big((-\tau',0] \times B_{1/8}\big) \cap \{q_t > A_1\}\big|.
\end{aligned}$$

Therefore, rearranging shows that

$$\big|\big((-\tau',0] \times B_{1/8}\big) \cap \{q_t > A_1\}\big| \leq \frac{C + C_2\tau'}{A_1 + C_2}.$$

We can make the right-hand side arbitrarily small by choosing A_1 large. In particular, we choose A_1 sufficiently large (depending only on universal constants) so that we have

$$\big|\big((-\tau',0] \times B_{1/8}\big) \cap \{q_t \leq A_1\}\big| \geq \tfrac{1}{2}\tau'|B_{1/8}| =: \delta_1. \qquad \square$$

After Corollary 4.3 and Lemma 4.4, we obtain a set of positive measure where $|q_t|$ is bounded. At this point, we can use ideas from the stationary case to proceed with the rest of the proof.

The next lemma replaces Lemma 8.1 in [Caffarelli and Silvestre 2009]. We, in fact, prove a slightly modified version of the lemma, which enforces a quadratic growth of $\delta_h u$ simultaneously on *two* rings. In the proofs of Theorem 8.7 and Lemma 10.1 in [Caffarelli and Silvestre 2009], there is a cube decomposition plus a covering argument. It could be replaced by a double covering argument. In this paper, we will have a simpler covering argument using Vitali's lemma only once. This is possible thanks to the stronger measure estimate in the next lemma (in two simultaneous rings).

Lemma 4.5. *Let μ be the constant in* (2-2) *and $c_0 < 1$ be an arbitrary constant. Let y be the point in $B_{1/2}$ where the minimum of* (4-2) *is achieved and u satisfies* (4-1). *Assume that $x \in B_{1/4}$, $q(x,t) < 3$ and $q_t(x,t) < A_1$. Then, for A_2 sufficiently large* (*depending on C_1, μ_1, λ, Λ, c_0 and α_0 but not on α*), *we have that there exists some $r \le r_0$ so that both*

$$\left|\left\{h \in B_{2r} \setminus B_r \,:\, \delta_h u(y,t) \le A_2 r^2 \text{ and } \delta_{-h} u(y,t) \le A_2 r^2\right\}\right| \ge \tfrac{1}{2}\mu |B_{2r} \setminus B_r| \tag{4-4}$$

and

$$\left|\left\{h \in B_{2c_0 r} \setminus B_{c_0 r} \,:\, \delta_h u(y,t) \le A_2 (c_0 r)^2 \text{ and } \delta_{-h} u(y,t) \le A_2 (c_0 r)^2\right\}\right| \ge \tfrac{1}{2}\mu |B_{2c_0 r} \setminus B_{c_0 r}| \tag{4-5}$$

hold simultaneously for r and $c_0 r$. Here $r_0 = 4^{-1/(2-\alpha)}$, and we note that $r_0 \to 0$ as $\alpha \to 2$.

In Lemma 4.5, we abuse notation by writing

$$\delta_h u(y,t) = u(y+h,t) - u(y,t) - 128(x-y)\cdot h,$$

even though $\nabla u(y,t)$ may not exist. Note that if u happens to be differentiable at (y,t), then $\nabla u(y,t) = 128(x-y)$ because of (4-2). The value of c_0 will be selected as a universal constant in Lemma 4.7.

Proof. From the construction of x and y, we have that $u(y,t) = q(x,t) - 64|x-y|^2$. Moreover, $u(z,s) \ge q(x,s) - 64|x-z|^2$ for any $z \in \mathbb{R}^n$ and $s \le t$. Since we are assuming that $q_t(x,t) < A_1$ (in particular, that q_t exists at that point), there is an $\varepsilon > 0$ so that $q(x,s) > q(x,t) - A_1(t-s)$ for $s \in (t-\varepsilon, t]$. Consequently, $u(z,s) \ge q(x,t) - 64|x-z|^2 - A_1(t-s)$ for $s \in (t-\varepsilon, t]$.

Let

$$\varphi(z,s) := \max\bigl(q(x,t) - 64|x-z|^2 - A_1(t-s), -256\bigr).$$

The choice of the number -256 is made so that the maximum is always achieved by the paraboloid every time $z \in B_1$. From the analysis above, we have that $u \ge \varphi$ in $\mathbb{R}^n \times (t-\varepsilon, t]$ and $u(y,t) = \varphi(y,t)$. Note that since $q(x,t) < 3$, we have $|\nabla\varphi(y,t)| \le 16\sqrt{3}$. Also, from Lemma 2.4, since $D^2\varphi \ge -128I$, we have $M^-\varphi(y,t) \ge -C$ for some universal constant C. We apply Lemma 3.4 and we get

$$\begin{aligned}
0 &\le \varphi_t(y,t) + C_0|\nabla\varphi(y,t)| - M^-\varphi(y,t) - \inf\left\{\int_{\mathbb{R}^d} \bigl(u(y+h,t) - \varphi(y+h,t)\bigr) K(h)\,\mathrm{d}h : K \in \mathcal{K}\right\} \\
&\le A_1 + C_0|\nabla\varphi(y,t)| - M^-\varphi(y,t) - \inf\left\{\int_{\mathbb{R}^d} \bigl(u(y+h,t) - \varphi(y+h,t)\bigr) K(h)\,\mathrm{d}h : K \in \mathcal{K}\right\} \\
&\le C - \inf\left\{\int_{\mathbb{R}^d} \bigl(u(y+h,t) - \varphi(y+h,t)\bigr) K(h)\,\mathrm{d}h : K \in \mathcal{K}\right\}.
\end{aligned}$$

Note that $u(y+h,t) - \varphi(y+h,t) \ge 0$ for all values of $h \in \mathbb{R}^n$. We abuse notation by saying

$$\delta_h u(y,t) = u(y+h,t) - u(y,t) - h \cdot \nabla\varphi(y,t).$$

Note that

$$u(y+h,t) - \varphi(y+h,t) = \delta_h u(y,t) - \delta_h \varphi(y,t),$$

and $\delta_h \varphi(y,t) = -64|h|^2$ whenever $y + h \in B_1$.

Using that the integrand is positive, we can reduce its domain of integration to an arbitrary subset of $\mathbb{R}^n$:

$$\begin{aligned} C &\geq \inf\left\{ \int_{B_{r_0}} \big(u(y+h,t)-\varphi(y+h,t)\big)K(h)\,\mathrm{d}h : K \in \mathcal{K} \right\} \\ &= \inf\left\{ \int_{B_{r_0}} \big(\delta_h u(y,t)+64|h|^2\big)K(h)\,\mathrm{d}h : K \in \mathcal{K} \right\}. \end{aligned}$$

Let us define

$$w(h) := \delta_h u(x,t) + 64|h|^2 \geq 0$$

for $h \in B_{r_0}$. We have that there exists an admissible kernel K such that

$$C \geq \int_{B_{r_0}} w(h)K(h)\,\mathrm{d}h. \tag{4-6}$$

Let $r \leq r_0 = 4^{-1/(2-\alpha)}$. From (2-2), we know that

$$\big|\big\{h \in B_{2r} \setminus B_r \,:\, K(h) > (2-\alpha)\lambda r^{-d-\alpha} \text{ and } K(-h) > (2-\alpha)\lambda r^{-d-\alpha}\big\}\big| > \mu|B_{2r} \setminus B_r|. \tag{4-7}$$

To obtain a contradiction, let us assume that the result of the lemma is false. That is, for all $r \leq r_0$, either

$$\big|\big\{h \in B_{2r} \setminus B_r \,:\, w(h) > (A+64)r^2 \text{ or } w(-h) > (A+64)r^2\big\}\big| > \big(1-\tfrac{1}{2}\mu\big)|B_{2r} \setminus B_r| \tag{4-8}$$

or

$$\big|\big\{h \in B_{2c_0r} \setminus B_{c_0r} : w(h) > (A+64)(c_0r)^2 \text{ or } w(-h) > (A+64)(c_0r)^2\big\}\big| > \big(1-\tfrac{1}{2}\mu\big)|B_{2c_0r} \setminus B_{c_0r}|. \tag{4-9}$$

Therefore, the intersection of the set in (4-7) — with r appropriately chosen in each case — with either of that in (4-8) or (4-9) must have measure at least $\frac{1}{2}\mu|B_{2r} \setminus B_r|$ or $\frac{1}{2}\mu|B_{2c_0r} \setminus B_{c_0r}|$, depending on which of the two possibilities occurred. Let us set $\tilde{r}$ to be either r or c_0r, depending upon whether we will invoke (4-8) or (4-9). Let us call $G_{\tilde{r}}$ this intersection between the sets (4-7) and either (4-8) or (4-9). Note that $G_{\tilde{r}} \subset B_{2\tilde{r}} \setminus B_{\tilde{r}}$ and $G_{\tilde{r}}$ is symmetric (i.e., $G_{\tilde{r}} = -G_{\tilde{r}}$). Moreover, for all $h \in G_{\tilde{r}}$, either $w(h) > (A+64)\tilde{r}^2$ and $K(h) > (2-\alpha)\lambda\tilde{r}^{-d-\alpha}$ or $w(-h) > (A+64)\tilde{r}^2$ and $K(-h) > (2-\alpha)\lambda\tilde{r}^{-d-\alpha}$. Therefore

$$\begin{aligned} \int_{B_{2\tilde{r}} \setminus B_{\tilde{r}}} w(h)K(h)\,\mathrm{d}h &\geq \int_{G_{\tilde{r}}} w(h)K(h)\,\mathrm{d}h \\ &= \frac{1}{2}\int_{G_{\tilde{r}}} w(h)K(h) + w(-h)K(-h)\,\mathrm{d}h \\ &\geq \frac{1}{2}\int_{G_{\tilde{r}}} A\lambda(2-\alpha)\tilde{r}^{-d+2-\alpha}\,\mathrm{d}h \\ &\geq A\lambda(2-\alpha)\tilde{r}^{2-\alpha}\mu\omega_d, \end{aligned}$$

where ω_d is a constant depending on dimension only.

We invoke the contradiction assumption for each of the radii $r_j = 2^{-j-1}r_0$, with $j = 0, 1, 2, \ldots$. For each r_j, we get the estimates corresponding to $\tilde{r}_j$, which is either r_j or c_0r_j, depending on the case of the contradiction assumption. Partitioning B_{r_0}, we get

$$\begin{aligned}\int_{B_{r_0}} w(h)K(h)\,\mathrm{d}h &= \sum_{j=0}^{\infty} \int_{B_{2r_j}\setminus B_{r_j}} w(h)K(h)\,\mathrm{d}h \\ &\geq \frac{1}{2}\sum_{j=0}^{\infty} \int_{B_{2\tilde{r}_j}\setminus B_{\tilde{r}_j}} w(h)K(h)\,\mathrm{d}h \\ &\geq (A+64)\lambda(2-\alpha)\mu\omega_d \sum_{j=0}^{\infty} (\tilde{r}_j)^{2-\alpha} \\ &\geq (A+64)\lambda(2-\alpha)\mu\omega_d \sum_{j=0}^{\infty} (c_0 2^{-j-1} r_0)^{2-\alpha} \\ &= C(d)c_0^{2-\alpha}(A+64)\mu\lambda \frac{2-\alpha}{1-2^{\alpha-2}}.\end{aligned}$$

We get a contradiction with (4-6) if A is large enough. Note that the last factor is bounded away from zero, independently of α as long as $\alpha \in (0,2)$. Thus the value of $A = A_2$ is independent of α, and it is chosen to obtain this contradiction. ☐

The following geometric statement about functions will play a role in the proof of Lemma 4.1.

Lemma 4.6. *Let $u : \mathbb{R}^d \to \mathbb{R}$ be a continuous bounded function such that $\nabla u(0)$ exists. Let $q(x) = \min_{y\in\bar{B}_1} u(y) + 64|x-y|^2$. Assume the following conditions hold true*:

- *There is at least one point $x_0 \in \mathbb{R}^d$ for which $q(x_0) = u(0) + 64|x_0|^2 = \min_{y\in\bar{B}_1}\{u(y) + 64|x_0-y|^2\}$.*
- *If we consider the (symmetric) set*

$$G := \{h \in B_2 \setminus B_1 : \delta_h u(0) \leq A \text{ and } \delta_{-h} u(0) \leq A\},$$

then $|G| \geq \frac{1}{2}\mu|B_2 \setminus B_1|$. (Here, as in Lemma 4.5, $\delta_h u(y,t) = u(y+h,t) - u(y,t) - 128(x-y)\cdot h$.)

Then there are constants c_0 and C_4 depending on A and μ and d so that if for some pair of points x_1, y_1 we have

$$q(x_1) = u(y_1) + 64|x_1-y_1|^2,$$

then $|y_1| < c_0$ implies $|x_1 - x_0| < C_4$.

Proof. Assume $|y_1| < c_0$. Let p_0 and p_1 be the quadratic polynomials

$$\begin{aligned}p_0(z) &= q(x_0) - 64|x_0 - z|^2,\\ p_1(z) &= q(x_1) - 64|x_1 - z|^2.\end{aligned}$$

From the definition of q, we have that $p_0(z) \leq u(z)$ and $p_1(z) \leq u(z)$ for all $z \in \mathbb{R}^d$. Moreover, $p_0(y_0) = u(y_0)$ and $p_1(y_1) = u(y_1)$.

Observe that $p_1 - p_0$ is the affine function

$$p_1(z) - p_0(z) = q(x_1) - q(x_0) + 64(|x_0|^2 - |x_1|^2) + 128(x_1 - x_0)\cdot z.$$

Since $p_1(y_1) = u(y_1) \geq p_0(y_1)$, we have

$$p_1(y_1+z) - p_0(y_1+z) \geq 128(x_1-x_0)\cdot z.$$

Using that $u(y_1+z) \geq p_1(y_1+z) \geq p_0(y_1+z) + 128(x_1-x_0)\cdot z$, we get that

$$\begin{aligned}\delta_{(y_1+z)}u(0) &\geq \delta_{(y_1+z)}p_0(0) + 128(x_1-x_0)\cdot z\\ &\geq -64 + 128(x_1-x_0)\cdot z \quad \text{for } z\in B_1.\end{aligned}$$

Let us consider the following set, which is the intersection of a cone (whose vertex is at y_1, recall that $|y_1| < c_0$) and the ring $B_2\setminus B_1$:

$$H = \big\{h \in B_2\setminus B_1 \,:\, h = y_1 + z \text{ with } z\cdot(x_1-x_0) > c_0|z|\,|x_1-x_0|\big\}.$$

Observe that as $c_0 \to 0$, the set H approximates the intersection of the ring $B_2\setminus B_1$ with the half-space $\{z \,:\, z\cdot(x_1-x_0) > 0\}$. More precisely

$$\big|B_2\setminus B_1\setminus H\setminus -H\big| \leq Cc_0$$

for some constant C depending on dimension only.

Let us choose c_0 so that $Cc_0 < \frac{1}{2}\mu|B_2\setminus B_1|$. Then $H\cap G$ must have a positive measure (also $G\cap -H$, recall that G is symmetric), and so there exists some $h\in H\cap G$. Then

$$\begin{aligned}A \geq \delta_h u(0) &\geq -64 + 128(x_1-x_0)\cdot z\\ &> -64 + 128c_0|x_1-x_0|\,|z|\\ &\geq -64 + 64c_0|x_1-x_0|.\end{aligned}$$

Therefore $|x_1-x_0| < \big(\frac{1}{64}A+1\big)/c_0 =: C_4$. □

In the proof of Lemma 4.1, we will use the map $m : y\mapsto x$, which assigns the point x where the minimum is achieved in the definition of q. This maps plays the same role as the gradient map of the convex envelope of u in B_r does in an ABP-based proof of the growth lemma. This would be the purpose of [Caffarelli and Silvestre 2009, Lemma 8.4] or [Bjorland et al. 2012, Lemma 3.6]. In those cases, we would need to adjust u by a supporting hyperplane and argue using a convex envelope. In our approach, we work without invoking a convex envelope.

Note that after Corollary 4.3 and Lemma 4.4, where we obtain that $|q_t|$ is bounded in a set of positive measure, the rest of the proof of Lemma 4.1 should be interpreted as a nonlocal version of the method in [Savin 2007]. It is more flexible, and arguably more natural, than an ABP-based proof.

We are now in a position to prove Lemma 4.1.

Proof of Lemma 4.1. We assume $u(0,0) = 1$. The result follows for the assumption $\min_{Q_{1/2}} u = 1$ by a simple translation argument.

Let G be the set of points $(x,t)\in B_{1/8}\times(-\tau,0]$ so that $q_t \leq A_1$. From Lemma 4.4, we have a universal lower bound on its measure: $|G| > \delta_1$. For each point $(x,t)\in G$, there is at least one point $y\in B_1$ that realizes the minimum value for $q(x,t)$ in (4-2). For each fixed value of t, we define the map $m : y\mapsto x$.

This is a well-defined function if $u \in C^1$. In general, the function nature of m is not necessary, and we should think of m as a set mapping that sends values of y into a set of possible values of x (like the subdifferential of a convex function).

We note that if $y \in m^{-1}(G)$, we have $q_t(x,t) \le A_1$ for some $x \in G$, and we can apply Lemma 4.5, which was presented above. This gives a ball around y and a collection of points where u does not grow too much. For example, we can control the set

$$E_y := \{z \in B_{c_0 r}(y) : u(z,t) < A_2 + 43\}. \tag{4-10}$$

This is possible by starting with the ring from Lemma 4.5 and then noting that $r \le 1$, $u(y) < 3$ (since $q(x) \le 3$, see first line of the proof of Lemma 4.5), $\delta_{\pm h} u(y) \le A_2 r^2$, $|h| \le \frac{1}{2}$, $|x-y| \le \frac{5}{8}$, and $128|x-y|\,|h| \le 40$. Thus from Lemma 4.5, we see that

$$|E_y| = \left|\left\{z \in B_{c_0 r}(y) : u(z,t) < A_2 + 43\right\}\right| > \delta |B_{c_0 r}|. \tag{4-11}$$

Here δ is a constant that depends on dimension and the μ from Lemma 4.5. We note that we use the r^2-growth of $\delta_h u$ from Lemma 4.5 in a very rough fashion at this step. The importance of the r^2 comes later, in relationship to an upper bound on $|m(B_r)|$. We also note that we have used the ball $B_{c_0 r}$ instead of B_r. At this stage, both balls have the same estimate regarding the growth of u on a universal proportion of the set. However, only $B_{c_0 r}$ also has the necessary estimate for the size of $m(B_{c_0 r})$. This choice will be further illuminated below.

We need to estimate a set where u is not too large, and given the choice of E_y above, we see that a good candidate is

$$NL := \bigcup_{y \in m^{-1}(G)} E_y.$$

Thanks to (4-10) and (4-11), the measure of NL can be equivalently estimated via the size of

$$NLB := \bigcup_{y \in m^{-1}(G)} B_{c_0 r(y)}(y),$$

where $B_{c_0 r(y)}(y)$ is the good ball given in Lemma 4.5. Therefore, the only question is whether or not the set, NLB, has a measure that is comparable to B_1.

If $\{B_j\}$ is a Vitali subcovering of the collection $\{B_{r(y)}(y)\}_{y \in m^{-1}(G)}$, then we have

$$\bigcup_j 5B_j \supset m^{-1}(G),$$

and hence

$$m\left(\bigcup_j 5B_j\right) \supset m(m^{-1}(G)).$$

Also by subadditivity, we have that

$$\left| m\left(\bigcup_j B_j\right)\right| \le \sum_j |m(B_j)|.$$

In order to conclude, it would suffice to know that

$$|m(B_j)| \le C_3 |B_j|, \tag{4-12}$$

which allows us to compare $|NLB|$ back to $|G|$.

The inequality (4-12) follows from the following lemma.

Lemma 4.7. *Under the same conditions as in Lemma 4.5, $|m(B_{c_0 r}(y))| \le C_3 r^d$. Here r is the same value as in Lemma 4.5, c_0 is fixed from Lemma 4.6 and depends only on other universal constants, and C_3 depends on c_0, C_4 (of Lemma 4.6) and the constant A_1 of Lemma 4.5.*

In order to prove Lemma 4.7, we only use the equation through Lemma 4.5. Indeed, after fixing a time t and rescaling, it reduces to Lemma 4.6.

We simply sketch the main idea to show how Lemma 4.7 follows from Lemma 4.6.

Sketch of the proof of Lemma 4.7. Assume that u and q are as given in the statements of Lemmas 4.5 and 4.7. After a translation, we can assume that $y = 0$. We would then define the rescaled functions

$$\hat{u}(z) = r^{-2} u(rz) \quad \text{and} \quad \hat{q}(z) = r^{-2} q(rz) \quad \text{for } z \in B_2.$$

We note the definition of $\hat{q}$ will be through a minimum over $B_{1/r}$, but, in fact, restricting the minimum to B_1 changes nothing since $y = 0$ is such a point that gives the minimum for $\hat{x} = x/r$. Then Lemma 4.6 is applicable with the functions $\hat{u}$ and $\hat{q}$, with the point $x_0 = \hat{x} = x/r$, and the set $\hat{G} = r^{-1}G$, with G being the set arising from the outcome of Lemma 4.5. □

Lemma 4.7 gives (4-12) via the result of Lemma 4.5 and the choice of $c_0 r(y)$.

We will use the fact that m maps onto G as well as the fact that by construction of the subcover $\{B_j\}$, $m^{-1}(G)$ is contained in its union. Thus we see that

$$G = m(m^{-1}(G)) \subset m\Big(\bigcup_j B_j\Big) = \bigcup_j m(B_j),$$

and hence by the choice of $c_0 r(y)$ and the definition of E_y, with Lemmas 4.5 and 4.7, it holds that

$$|G| \le \Big|\bigcup_j m(B_j)\Big| \le \sum_j |m(B_j)| \le \sum_j C_3 |B_j| \le \sum_j \frac{C_3}{\delta} |E_{y_j}|.$$

Since the B_j were chosen to be disjoint, so are the corresponding E_{y_j}, and thus we can conclude

$$|NL| \ge \Big|\bigcup_j E_{y_j}\Big| = \sum_j |E_{y_j}| \ge \frac{\delta}{C_3} |G| \ge \frac{\delta \delta_1}{C_3}.$$

This finishes the proof of Lemma 4.1. □

5. A special barrier function

This section is concerned with the construction of a barrier function that is essential for all of the results regarding regularity of parabolic (and elliptic) equations in nondivergence form. In principle, one would expect our construction to be similar to the one presented in [Chang Lara and Dávila 2014, Lemma 4.2], but this is not actually the case. We deviate in some significant respects due to the additional generality allowed by assumptions (A2) and (A3). In this regard, our construction is more accurately described as a parabolic version of the barrier from [Kassmann et al. 2014, Section 5], where similar lower bounds on only small sets were allowed. Significant detail is required to carry over the ideas from [loc. cit.] to the parabolic setting. These additional difficulties involved in the construction of the barrier are, in fact, also related to the conditions under which the Harnack inequality fails for equations such as (1-1).

Because of the relative strength of the terms $|\nabla p|$ and $M^- p$ under rescaling, it is necessary to break the construction of the special barrier function into two cases: one with $\alpha \geq 1$ and the other with $\alpha < 1$. For the second case, we must remove the gradient term from the equation.

5A. *The main lemmas and the barrier.*

Lemma 5.1. *Let $\alpha \in [1, 2)$ and suppose $r \in (0, 1)$ is given. There exists $\varepsilon_0 > 0$, $q_0 > 0$ and a function $p : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ such that for all $\alpha \geq 1$,*

$$p_t + C_0|\nabla p| - M^- p \leq 0 \quad \text{in } \big(B_1 \times (0, \infty)\big) \setminus \big(B_r \times (0, r^\alpha]\big), \tag{5-1}$$

$$p \leq 1 \quad \text{in } B_r \times (0, r^\alpha], \tag{5-2}$$

$$p \leq 0 \quad \text{in } \big(\mathbb{R}^d \setminus B_1\big) \times (0, \infty) \text{ and } \big(\mathbb{R}^d \setminus B_r\big) \times \{0\}, \tag{5-3}$$

$$p \geq \varepsilon_0 r^{q_0} e^{-C_5(T - r^\alpha)} \quad \text{in } B_{3/4} \times [r^\alpha, T]. \tag{5-4}$$

The constants ε_0 and q_0 depend only on λ, Λ, μ, C_0, α_0 and dimension.

Lemma 5.2. *Let $\alpha \in [\alpha_0, 2)$ and suppose $r \in (0, 1)$ is given. Then the same statement of Lemma 5.1 remains true except* (5-1) *is replaced by*

$$p_t - M^- p \leq 0 \quad \text{in } \big(B_1 \times (0, \infty)\big) \setminus \big(B_r \times (0, r^\alpha]\big). \tag{5-5}$$

Remark 5.3. Note that the same constants ε_0 and q_0 can be chosen to work for both Lemmas 5.1 and 5.2.

Remark 5.4. The existence of the barrier is closely related to uniform estimates on hitting times of a Markov process, which are crucial to the proofs of the weak Harnack inequality and Hölder regularity in the probabilistic framework. These hitting-time estimates appear in the original work of Krylov and Safonov [1980; 1979], and they have become a standard technique in the probability literature (see the presentation in, e.g., the lecture notes [Bass 2004]). In other contexts, there exists an explicit barrier and this lemma looks deceivingly simple. For nonlocal equations whose kernels are allowed to vanish, this step is, in fact, highly nontrivial. Lemmas 5.1 and 5.2 have a probabilistic interpretation as the lower bound for the probability of the process to hit a ball between time 0 and r^α.

The strategy for this construction is to start with a yet-to-be-determined function, Φ, supported in B_1, and rescale Φ on the time interval $t \in (0, r^\alpha)$ as

$$p(x, t) = t^{-q_0} \Phi\left(\frac{rx}{t^{1/\alpha}}\right), \tag{5-6}$$

and then to use

$$p(x,t) = e^{-C_5(t-r^\alpha)} p(x, r^\alpha) = e^{-C_5(t-r^\alpha)} r^{-\alpha q_0} \Phi(x) \tag{5-7}$$

for $t \in (r^\alpha, \infty)$. The choice of $rx/t^{1/\alpha}$ is to make sure that p will be positive for all $|x| < 1$ when $t \geq r^\alpha$. The constants q_0 and C_5 are there to force the subsolution property in the regions where $M^- p$ cannot be made to be as large as we like. We now make some initial computations to illuminate our subsequent choices (note the use of Lemma 2.2):

$$p_t = -q_0 t^{-q_0-1} \Phi\left(\frac{rx}{t^{1/\alpha}}\right) - \tfrac{1}{\alpha} t^{-q_0-1/\alpha-1} \nabla\Phi\left(\frac{rx}{t^{1/\alpha}}\right) \cdot rx, \tag{5-8}$$

$$\nabla p = r t^{-q_0-1/\alpha} \nabla\Phi\left(\frac{rx}{t^{1/\alpha}}\right), \tag{5-9}$$

$$M^- p = t^{-q_0-1} r^\alpha M^- \Phi\left(\frac{rx}{t^{1/\alpha}}\right). \tag{5-10}$$

We want to satisfy (5-1), which then can be transformed to the new goal (at least for $t \in (0, r^\alpha)$):

$$t^{-q_0-1}\left(-q_0\Phi\left(\frac{rx}{t^{1/\alpha}}\right) - \tfrac{1}{\alpha} t^{-1/\alpha} \nabla\Phi\left(\frac{rx}{t^{1/\alpha}}\right) \cdot rx + r t^{1-1/\alpha} C_0 \left|\nabla\Phi\left(\frac{rx}{t^{1/\alpha}}\right)\right| - r^\alpha M^-\Phi\left(\frac{rx}{t^{1/\alpha}}\right)\right) \leq 0. \tag{5-11}$$

Switching out variables

$$z = \frac{rx}{t^{1/\alpha}},$$

for an appropriate set of z, we want

$$t^{-q_0-1}\big(-q_0\Phi(z) - \tfrac{1}{\alpha}\nabla\Phi(z)\cdot z + r t^{1-1/\alpha} C_0 |\nabla\Phi(z)| - r^\alpha M^-\Phi(z)\big) \leq 0. \tag{5-12}$$

We can now turn to the requirement for p to satisfy (5-1) when $t \geq r^\alpha$. The computations are similar to the case of $t \in [0, r^\alpha]$. Using (5-6),

$$\begin{aligned} p_t &= -C_5 e^{-C_5(t-r^\alpha)} r^{-\alpha q_0} \Phi(x), \\ \nabla p &= r^{-\alpha q_0} e^{-C_5(t-r^\alpha)} \nabla\Phi(x), \\ M^- p &= r^{-\alpha q_0} e^{-C_5(t-r^\alpha)} M^-\Phi(x). \end{aligned}$$

Then the goal (5-12) becomes

$$e^{-C_5(t-r^\alpha)} r^{-\alpha q_0}\big(-C_5\Phi(x) + C_0|\nabla\Phi(x)| - M^-\Phi(x)\big) \leq 0. \tag{5-13}$$

The function Φ and subsequently p will be built in a many-staged process. One of the key components is a special bump function, which acts as a barrier in the stationary setting. This construction proceeds similarly to that of [Kassmann et al. 2014], and we would like to point out that there, just as here, there are significant challenges for this construction due to the generality of the lower bound assumption in (2-2) (cf. the bump function in [Caffarelli and Silvestre 2009], where the lower bound on K holds globally).

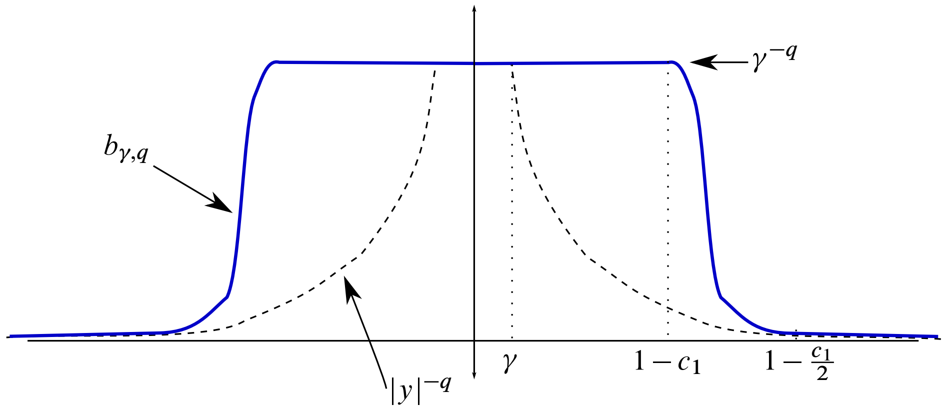

Figure 1. The function $b_{\gamma,q}$.

We start with a two-parameter family of auxiliary functions

$$b_{\gamma,q}(y) = \hat{b}(|y|)$$

and

$$\hat{b}(r) = \begin{cases} r^{-q} & \text{if } r \geq 1-\frac{c_1}{2}, \\ m_{\gamma,q}(r) & \text{if } 1-c_1 \leq r \leq 1-\frac{c_1}{2}, \\ \gamma^{-q} & \text{if } r \leq 1-c_1, \end{cases} \tag{5-14}$$

with $m_{\gamma,q}$ smooth and monotonically decreasing (so there will be a restriction between γ and c_1 both being small enough), and without loss of generality $m_{\gamma,q}$ will be such that

$$b_{\gamma,q}(y) \geq \min\{\gamma^{-q}, |y|^{-q}\} \quad \text{for all } \ y \in \mathbb{R}^d.$$

See Figure 1 for the graph of $b_{\gamma,q}$.

The key part of the construction is that there are choices of γ and q that make b a subsolution in a given small strip (and a subsequent truncation allows the equation to hold in a large set). We state this result for the choices of γ and q, and then we will prove it in Section 5B.

Lemma 5.5. *Let $C > 0$ be given. Then there exist a small constant c_1 and choices of γ_1 and q_1 (depending on C plus all other universal objects) such that*

$$M^- b_{\gamma_1,q_1}(x) \geq C q_1 |x|^{-q_1-\alpha} \quad \textit{for all } \ 1-\frac{c_1}{2} \leq |x| \leq 1, \tag{5-15}$$

for all $\alpha \in (\alpha_0, 2)$. The constant c_1 depends on the lower bound of K in (2-2).

Remark 5.6. Lemma 5.5 provides a subsolution to a stationary problem. It is a generalized version of [Caffarelli and Silvestre 2009, Corollary 9.2; Bjorland et al. 2012, Lemma 3.10; Kassmann et al. 2014, Lemmas 5.2 and 5.3] to the more general class of kernels in this article.

Now that we know the details of an equation for $b = b_{\gamma,q}$, we will continue the calculations, which will be useful to construct p. For the following, we assume that $1-\frac{c_1}{2} \leq |z| \leq 1$. We also note that

γ_1, q_1, and C will be determined subsequently:

$$b(z) = b_{\gamma_1,q_1}(z) = |z|^{-q_1} \quad \text{if } 1 - \tfrac{c_1}{2} < |z|, \tag{5-16}$$

$$\nabla b(z) = -q_1 z|z|^{-q_1-2}, \tag{5-17}$$

$$-\tfrac{1}{\alpha}\nabla b(z)\cdot z = \tfrac{1}{\alpha} q_1 |z|^{-q_1}, \tag{5-18}$$

$$C_0|\nabla b(z)| = C_0 q_1 |z|^{-q_1-1}, \tag{5-19}$$

$$-M^- b(z) \le -C q_1 |z|^{-q_1-\alpha}. \tag{5-20}$$

Now that we have sorted out the details regarding b_{γ_1,q_1}, we can proceed with the proof of Lemma 5.1. Some complications arise from the need to satisfy the boundary conditions in (5-3).

We will give the proof of Lemma 5.1 and then afterwards indicate the few steps that are modified to prove Lemma 5.2.

Proof of Lemma 5.1. We proceed with defining p in terms of Φ as described in (5-6) and (5-7). Note that this construction gives a function p that is unbounded around the origin $(0,0)$. To fix that, at the end of the proof, we have an extra truncation step.

In order to satisfy the boundary conditions (5-3), Φ will be the following truncated version of $b_{\gamma,q}$:

$$\Phi(z) = \max\{b_{\gamma,q}(z) - b_{\gamma,q}(e_1), 0\}.$$

This function Φ is zero outside of B_1 and strictly positive inside B_1. The properties of the function b will be used to make the value of $M^-\Phi$ large in $B_1 \setminus B_{1-c_2/2}$.

Recall the variable z,

$$z = \frac{rx}{t^{1/\alpha}}. \tag{5-21}$$

We need to verify (5-12) and (5-13) in order to account for the regions $t \in [0, r^\alpha]$ and $t \in (r^\alpha, \infty)$. We will need to select parameters and constants to work for both ranges of t. But we note that all of the parameters are such that they can be chosen to satisfy both conditions simultaneously.

Part 1: $t \in [0, r^\alpha]$.

Note the following relations for $z \in B_1$:

$$\nabla\Phi(z) = \nabla b(z),$$
$$M^-\Phi(z) \ge M^- b(z).$$

We need to find parameters so that (5-12) holds. The computation will be different in the three regions $|z| \le 1 - \frac{c_1}{2}$, $1 - \frac{c_1}{2} < |z| < 1$, and $|z| \ge 1$.

Replacing (5-17), (5-18), (5-19) and (5-20) in the left-hand side of (5-12), we get

$$-q_0\Phi(z) - \tfrac{1}{\alpha}\nabla\Phi(z)\cdot z + rt^{1-1/\alpha}C_0|\nabla\Phi(z)| - r^\alpha M^-\Phi(z)$$
$$\le -q_0\Phi(z) - \tfrac{1}{\alpha}\nabla b(z)\cdot z + rC_0|\nabla b(z)| - r^\alpha M^- b(z). \tag{5-22}$$

For the last inequality, we used that $t^{1-1/\alpha} \le 1$. This is because $t \le r^\alpha \le 1$ and $\alpha \ge 1$. When $\alpha < 1$, the negative power of t cannot be controlled and that is why we assume $C_0 = 0$ in those cases.

When $1-\frac{c_1}{2}<|z|<1$, we can ignore $-q_0 b(z)$, and instead focus on

$$-\tfrac{1}{\alpha}\nabla b(z)\cdot z+rC_0|\nabla b(z)|-r^\alpha M^- b(z)\le 0. \tag{5-23}$$

In light of (5-18), (5-19), (5-20), it will suffice to choose b so that

$$\tfrac{1}{\alpha}q_1|z|^{-q_1}+rC_0q_1|z|^{-q_1-1}-Cq_1r^\alpha|z|^{-q_1-\alpha}\le 0,$$

or more succinctly

$$q_1|z|^{-q_1}\bigl(\tfrac{1}{\alpha}+rC_0|z|^{-1}-Cr^\alpha|z|^{-\alpha}\bigr)\le 0. \tag{5-24}$$

After C is chosen to obtain (5-24) (recall $|z|\le 1$), then $b=b_{\gamma_1,q_1}$ can be fixed by Lemma 5.5. The resulting b will be smooth and bounded.

Switching now to the set $|z|\le 1-\frac{c_1}{2}$, inequality (5-22) then follows from

$$\bigl(-q_0\Phi(z)-\tfrac{1}{\alpha}\nabla b(z)\cdot z+rC_0|\nabla b(z)|-r^\alpha M^- b(z)\bigr)\le 0. \tag{5-25}$$

The function Φ is strictly positive in B_1 and, in particular, it is bounded below by a positive constant in $B_{1-c_1/2}$. Since C, γ_1, q_1 have all been fixed and all of the terms are bounded, we can then choose q_0 large enough so that (5-25) will also hold.

We are only left with the case $|z|\ge 1$. Note that because of the angle singularity of the function Φ on $|z|=1$, we cannot touch the function Φ from above with any smooth function at those points. Therefore, the points $|z|=1$ play no role in Φ satisfying (5-12) in the viscosity sense. If $|z|>1$, then $\Phi(z)=|\nabla\Phi(z)|=0$ and $M^-\Phi(z)\ge 0$ because z will be at a global minimum of Φ, and so (5-12) trivially holds.

Part 2: $t\in(r^\alpha,\infty)$.

We now need to make sure (5-13) holds. The procedure is similar to the first part.

In the region $1-\frac{c_1}{2}<|x|<1$, using (5-19) and (5-20), we get

$$-C_5\Phi(x)+C_0|\nabla\Phi(x)|-M^-\Phi(x)=-C_5\Phi(x)+C_0q_1|z|^{-q_1-1}-Cq_1|z|^{-q_1-\alpha}.$$

We ignore the term $-C_5\Phi(x)\le 0$ and use

$$-C_5\Phi(x)+C_0|\nabla\Phi(x)|-M^-\Phi(x)\le q_1|x|^{-q_1}(C_0|x|^{-1}-C|x|^{-\alpha})\le 0.$$

The last inequality holds provided that we choose C large enough (which can be done by choosing appropriate values of γ and q from Lemma 5.5).

In the region $|x|<1-\frac{c_1}{2}$, we use that b (note that γ and q are fixed in the previous step) is a given smooth function and $\Phi(x)\ge\bigl|1-\frac{c_1}{2}\bigr|^{-q}-1>0$. Therefore, picking a large enough C_5, we can make (5-13) hold.

If $|x|\ge 1$, then the equation holds just as in the first part of this proof, owing to the fact that z will be at a global minimum of Φ. Note that the constant C, which we use for picking γ_1 and q_1 in Lemma 5.5, needs to be large enough to satisfy the requirements of both Part 1 ($t\in[0,r^\alpha]$) and Part 2 ($t>r^\alpha$) of this proof.

Part 3: The truncation step.

Now there is one last step of truncation. At this stage, the function $t^{-q_0}\Phi(rx/t^{1/\alpha})$ has a singularity at $x=0$ and $t\to 0$, which of course violates requirement (5-2).

We define the function

$$\tilde{p}(x,t) := t^{-q_0}\Phi\left(\frac{rx}{t^{1/\alpha}}\right),$$

and p will be defined as a truncation of $\tilde{p}$ to be compatible with (5-2). Importantly, in this truncation we need to not destroy the equation satisfied by our choice of $\tilde{p}$ outside of $B_r \times [0, r^\alpha]$. That means that we should only truncate at a small enough t so that the support of $\tilde{p}(\cdot, t)$ is contained in B_r. This way, for such x outside of B_r, the desired equation is trivially satisfied because the equation will be evaluated where $\tilde{p}_t = 0$ and $\tilde{p}(x,t) = 0$, which is the global minimum for $\tilde{p}$, giving $\nabla\tilde{p} = 0$ and $M^-\tilde{p} \geq 0$. Given the scaling $z = rx/t^{1/\alpha}$ and that the support of Φ is in B_1, we see that a convenient choice for truncation will be when the graph of $t = (r|x|)^\alpha$ intersects the line $|x| = r$; hence at $t = r^{2\alpha}$.

Accordingly, we define (note for each t, we know that $\tilde{p}$ has its max at $x = 0$)

$$\begin{aligned} p(x,t) &= \frac{\min\{\tilde{p}(x,t), \tilde{p}(0, r^{2\alpha})\}}{\tilde{p}(0, r^{2\alpha})} \\ &= (r^{-2\alpha q_0}\Phi(0))^{-1}\min\{\tilde{p}(x,t), r^{-2\alpha q_0}\Phi(0)\}. \end{aligned}$$

This now gives a complete description of p for t in both $(0, r^\alpha)$ and $[r^\alpha, \infty)$ via (5-6) and (5-7) respectively.

The inequality (5-4) follows by a direct inspection using the expression (5-7) for $\tilde{p}$. We get that for $t > r^\alpha$ and $|x| \leq \frac{3}{4}$,

$$p(x,t) = \left(r^{-2\alpha q_0}\Phi(0)\right)^{-1} e^{-C_5(t-r^\alpha)} r^{-\alpha q_0}\Phi(x) \geq r^{\alpha q_0} e^{-C_5(t-r^\alpha)} \min_{B_{3/4}} \Phi.$$

We note that the truncation expression has shown that the choice of q for the lower bound requirement in (5-4) will be $q = \alpha q_0$. The choice of radius $\frac{3}{4}$ in (5-4) is irrelevant, since a similar lower bound would hold if $\frac{3}{4}$ is replaced by any other number smaller than 1.

This completes the proof of Lemma 5.1. □

We now mention where the proof of Lemma 5.2 deviates from the previous one.

Proof of Lemma 5.2. One needs to go back and remove the term $C_0|\nabla p|$ from all of the calculations. Note this was the only term affected by the factor $t^{1-\alpha/2}$, which would be unbounded if $\alpha < 1$. □

5B. ***The proof of Lemma 5.5.*** Lemma 5.5 will be attained in two stages, Lemmas 5.10 and 5.11. First we develop some auxiliary results related to b. We begin by making a useful observation about the behavior of $\delta_h b$.

Lemma 5.7. *Assume* $\alpha \in [1, 2)$. *If* $b = b_{\gamma,q}$ *is as in* (5-14), *then for some universal* r_0 *and* $C(q)$, *where* $|h| \leq r_0$ *and* $1 - \frac{c_1}{2} < |x| < 1$, *we have*

$$\delta_h b(x) \geq -q\frac{|h|^2}{|x|^{q+2}} + q(q+2)\frac{(h_1)^2}{|x|^{q+2}} - C(q)|h|^3$$

(*this is only relevant*, *and only invoked*, *for* $\alpha > 1$; *otherwise we would use a different expansion for* $\alpha < 1$).

Proof. This follows from Taylor's theorem. Note that h is restricted to be in a small set, B_{r_0}, and so actually $b(x) = |x|^{-q}$ and $b(x+h) \geq |x+h|^{-q}$. □

The next lemma says that our assumptions allow that for all $r \le r_1$, the set A_r intersects annuli centered at $-e_1$ in a uniformly nontrivial fashion. This feature is essential to be able to utilize the lower bounds on K in (2-2).

Lemma 5.8. *There exist constants c_1, c_2 and r_1 (all small), so that*

(i) *for any x so that $1 - c_1 < |x| < 1$,*

$$\left|A_{r_1} \cap B_{1-c_1}(-x)\right| \ge \tfrac{1}{4}\mu\left|B_{2r_1} \setminus B_{r_1}\right|,$$

(ii) *for all r,*

$$\left|A_r \cap \{h : (h_1)^2 \ge c_2|h|^2\}\right| \ge \tfrac{1}{2}\mu\left|B_{2r} \setminus B_r\right|.$$

Proof. We first note that by the symmetry of A_r,

$$\left|A_r \cap (B_{2r} \setminus B_r) \cap \{h : h \cdot x \le 0\}\right| \ge \tfrac{1}{2}\mu\left|B_{2r} \setminus B_r\right|. \tag{5-26}$$

Now we will establish (i). We first choose r_1 small enough so that

$$\left|\big((B_{2r_1} \setminus B_{r_1}) \cap \{h : h \cdot x \le 0\}\big) \setminus B_{|x|}(-x)\right| \le \tfrac{1}{8}\mu\left|(B_{2r_1} \setminus B_{r_1}) \cap \{h : h \cdot x \le 0\}\right|.$$

Note that this choice of r_1 can be done uniformly for all $1 - c_1 < |x| < 1$.

Let us define the failed set where A_r cannot reach $B_{1-c_1}(-x)$ as

$$F := \big((B_{2r_1} \setminus B_{r_1}) \cap \{h : h \cdot x \le 0\}\big) \setminus B_{1-c_1}(-x).$$

With r_1 fixed, we can choose c_1 small enough so that

$$|F| \le \tfrac{1}{4}\mu\left|(B_{2r_1} \setminus B_{r_1}) \cap \{h : h \cdot x \le 0\}\right|. \tag{5-27}$$

This is possible because

$$\begin{aligned} |F| &\le \left|\big((B_{2r_1} \setminus B_{r_1}) \cap \{h : h \cdot x \le 0\}\big) \setminus B_{|x|}(-x)\right| + \left|B_{|x|} \setminus B_{1-c_1}\right| \\ &\le \tfrac{1}{8}\mu\left|(B_{2r_1} \setminus B_{r_1}) \cap \{h : h \cdot x \le 0\}\right| + C(1 - (1 - c_1)^d). \end{aligned}$$

Finally, combining (5-26) with (5-27) we obtain (i).

To establish (ii), we note that

$$\begin{aligned} \left|A_r \cap \{h : (h_1)^2 \ge c_2|h|^2\}\right| &\ge |A_r| - \left|\{h \in B_{2r} \setminus B_r : h_1^2 < c_2|h|^2\}\right| \\ &\ge (\mu - Cc_2)|B_{2r} \setminus B_r| \end{aligned}$$

for a universal constant C. Thus, we simply take c_2 small enough so that $(\mu - Cc_2) \ge \frac{1}{2}\mu$. ☐

Note 5.9. If $\gamma_1 < \gamma_2$ and q is fixed, then for all y,

$$b_{\gamma_1,q}(y) \ge b_{\gamma_2,q}(y),$$

and the two functions are equal when $|y| \ge 1 - \frac{c_1}{2}$; hence

$$M^- b_{\gamma_1,q}(x) \ge M^- b_{\gamma_2,q}(x)$$

for all $|x| \ge 1 - \frac{c_1}{2}$.

Next we make the first choice of parameter for b. It is the selection of the exponent, q, and it only uses the information about the family $\mathcal{K}$ for α very close to 2.

Lemma 5.10. *Let $\gamma \le \gamma_0 = \frac{1}{4}$ be fixed. Let $C > 0$ be given. Then, there exist a $q_1 \ge 1$ and an α_1, depending only on C, γ_0, C_0, μ, d, λ, Λ, such that*

$$M^- b_{\gamma,q_1}(x) \ge C q_1 |x|^{-q_1-\alpha} \quad \text{for all } 1-\tfrac{c_1}{2} < |x| < 1,$$

for all orders, $\alpha \in (\alpha_1, 2)$ and for all $\gamma \le \gamma_0$.

Then once the q has been chosen, we can finish the definition of b by fixing the truncation height, γ^{-q}, to be large enough (so γ small enough). This allows us to fix one function that satisfies the special subsolution property for all $\alpha \in [\alpha_0, 2)$.

Lemma 5.11. *Let $C > 0$ and q_1 be as in Lemma 5.10. Then there exists a $\gamma_1 \le \gamma_0 = \frac{1}{4}$ such that*

$$M^- b_{\gamma_1,q_1}(x) \ge C q_1 |x|^{-q_1-\alpha} \quad \text{for all } 1-\tfrac{c_1}{2} < |x| < 1,$$

for all orders, $\alpha \in (\alpha_0, \alpha_1]$.

First we give the proof of Lemma 5.10.

Proof of Lemma 5.10 . Let x be any point such that $1-\frac{c_1}{2} < |x| < 1$. We begin with a few simplifying observations. First of all, there is no loss of generality in assuming $\alpha > 1$ for this lemma — indeed the end of the proof culminates with a choice of α_1 that is sufficiently close to 2 (hence $\delta_h b(x)$ uses only one case for $\alpha > 1$). Second, to simplify notation, we drop the γ, q dependence and denote $b_{\gamma,q}$ by b.

To obtain the bound we want, we only need the contribution of $\delta_h b(x)$ to $M^- b(x)$ in a small ball, $h \in B_{r_2}$, for some r_2 fixed with, say, $r_2 = \min\{r_0, \frac{c_1}{2}\}$, where r_0 originates in Lemma 5.7 and c_1 comes from Lemma 5.8. This is because the large curvature of the graph of b in the h_1-direction can be used to dominate the integral at the expense of all the other terms.

We also note that for $h \in \mathbb{R}^d \setminus B_{r_2}$, we have

$$\delta_h b(x) \ge \inf_{h \in \mathbb{R}^d \setminus B_{r_2}} \bigl(b(x+h) - b(x) - q|x|^{-q-2} x \cdot h\bigr) \ge -C_q\Bigl(1 + \frac{x}{|x|} \cdot h\Bigr).$$

Here $C_q = \max\bigl(q\bigl(1-\frac{c_1}{2}\bigr)^{-q-1}, \bigl(1-\frac{c_1}{2}\bigr)^{-q}\bigr)$.

Therefore, by Lemma 2.3, we see that

$$\int_{\mathbb{R}^d \setminus B_{r_2}} \delta_h b(x) K(h)\, \mathrm{d}h \ge -(2-\alpha) C_q \Lambda \Bigl(\frac{r_2^{-\alpha}}{\alpha} + r_2^{1-\alpha}\Bigr). \tag{5-28}$$

Furthermore, combining Lemmas 5.8 and 5.7, we see that on each ring, $B_{2^{-k}r} \setminus B_{2^{-k-1}r}$, we can enhance the positive contribution to $M^- f(x)$ by manipulating the term

$$\frac{q(q+2)}{|x|^{q+2}} \int_{B_{2^{-k}r} \setminus B_{2^{-k-1}r}} (h_1)^2 K(h)\, \mathrm{d}h.$$

By Lemma 5.8 and assumption (A3), we see that

$$\begin{aligned}
\int_{B_{2^{-k}r}\setminus B_{2^{-k-1}r}} (h_1)^2 K(h)\,\mathrm{d}h &\geq \int_{A_{2^{-k-1}r}} (h_1)^2 K(h)\,\mathrm{d}h \\
&\geq \int_{A_{2^{-k-1}r}\cap\{h:(h_1)^2\geq c_2|h|^2\}} c_2|h|^2 K(h)\,\mathrm{d}h \\
&\geq c_2(2^{-k-1}r)^2\lambda(2-\alpha)(2^{-k-1}r)^{-d-\alpha}\left|A_{2^{-k-1}r}\cap\{h\,:\,(h_1)^2\geq c_2|h|^2\}\right| \\
&\geq c_2(2^{-k-1}r)^2\lambda(2-\alpha)(2^{-k-1}r)^{-d-\alpha}\tfrac{1}{2}\mu\left|B_{2^{-k}r}\setminus B_{2^{-k-1}r}\right| \\
&= c_2\lambda(2-\alpha)\mu 2c(d)r^{2-\alpha}2^{-k(\alpha-2)},
\end{aligned}$$

where $c(d)$ is a purely dimensional constant that we use temporarily during this proof. Hence adding up the contribution along all of the rings, we see

$$\int_{B_{r_2}} (h_1)^2 K(h)\,\mathrm{d}h = \sum_{k=0}^{\infty}\int_{B_{2^{-k}r_2}\setminus B_{2^{-k-1}r_2}} (h_1)^2 K(h)\,\mathrm{d}h \geq \big(\lambda\mu c_2 c(d)\big) r_2^{2-\alpha}, \tag{5-29}$$

where we have collected various dimensional constants into $c(d)$ in such a way that is uniform for $\alpha\in(0,2)$. Note that

$$\sum_{k=0}^{\infty}(2-\alpha)2^{k(\alpha-2)} = \frac{2-\alpha}{1-2^{\alpha-2}} \leq 2$$

for all $\alpha\in(1,2)$.

We also estimate the following integral using assumption (A2):

$$\int_{B_{r_2}} |h|^3 K(h)\,\mathrm{d}h = \sum_k \int_{B_{2^{-k}r_2}\setminus B_{2^{-k-1}r_1}} |h|^3 K(h)\,\mathrm{d}h \leq \frac{(2-\alpha)2^\alpha}{1-2^{\alpha-3}} r_2^{3-\alpha}\Lambda. \tag{5-30}$$

Now we need to put all of the pieces together. We will use Lemma 5.7 to balance the terms of different orders in both $|h|$ and q. We will be invoking Lemma 2.3 as well as the bounds from (5-28)–(5-30):

$$\begin{aligned}
&\int_{\mathbb{R}^d} \delta_h b(x)K(h)\,\mathrm{d}h \\
&= \int_{B_{r_2}} \delta_h b(x)K(h)\,\mathrm{d}h + \int_{\mathbb{R}^d\setminus B_{r_2}} \delta_h b(x)K(h)\,\mathrm{d}h \\
&\geq \frac{q(q+2)}{|x|^{q+2}}\int_{B_{r_2}} (h_1)^2 K(h)\,\mathrm{d}h - \frac{q}{|x|^{q+2}}\int_{B_{r_2}} |h|^2 K(h)\,\mathrm{d}h - C(q)\int_{B_{r_2}} |h|^3 K(h)\,\mathrm{d}h + \int_{\mathbb{R}^d\setminus B_{r_2}} \delta_h f(x)K(h)\,\mathrm{d}h \\
&\geq \frac{q}{|x|^{q+2}}\big((q+2)(\lambda\mu c_2 c(d)) - C_d\Lambda\big)r_2^{2-\alpha} - (2-\alpha)\bigg(C_q\Lambda\bigg(\frac{r_2^{-\alpha}}{\alpha} + r_2^{1-\alpha}\bigg) - C(q)\frac{2^\alpha}{1-2^{\alpha-3}}r_2^{3-\alpha}\Lambda\bigg).
\end{aligned} \tag{5-31}$$

At this point, we note that the first term is the one that does not have the factor $(2-\alpha)$ in front. We will first choose q large to control the sign of this term. Hence we can choose $q = q_1$ large enough, depending only on the given constant C and the universal parameters, so that (recall C, with no subscript, was the parameter given in the statement of this lemma and $|x| < 1$)

$$\frac{q}{|x|^{q+2}}\big((q+2)(\lambda\mu c_2 c(d)) - C_d\Lambda\big)r_2^{2-\alpha} \geq 3Cq|x|^{-q-\alpha}r_2^{2-\alpha}.$$

Once q_1 has been fixed, we can now choose α_1 close enough to 2 so that the rest of the expression in (5-31) is small:

$$(2-\alpha)\left(C_q\Lambda\left(\frac{r_2^{-\alpha}}{\alpha} + r_2^{1-\alpha}\right) - C(q)\frac{2^\alpha}{1-2^{\alpha-3}}r_2^{3-\alpha}\Lambda\right) \leq Cq_1|x|^{-q_1-\alpha}r_2^{2-\alpha}.$$

(Recall that $r_2 = \min\{r_0, \frac{c_1}{2}\}$.) Thus we have achieved

$$\int_{\mathbb{R}^d} \delta_h b(x)K(h)\,\mathrm{d}h \geq 2Cq_1|x|^{-q_1-\alpha}r_2^{2-\alpha}.$$

The chosen value of α is sufficiently close to 2. We may choose α even closer to 2 so that $r_2^{2-\alpha} > \frac{1}{2}$ and

$$\int_{\mathbb{R}^d} \delta_h b(x)K(h)\,\mathrm{d}h \geq Cq_1|x|^{-q_1-\alpha}.$$

Taking an infimum over K yields the result. □

Remark 5.12. The underlying reason why the previous proof works is because if we fix the values of Λ, λ and μ, the following limit holds:

$$\lim_{\alpha\to 2} M^- b(x) = \mathcal{M}^-_{\tilde{\lambda},\tilde{\Lambda}}(D^2 b(x)),$$

where $\mathcal{M}^-$ is the classical minimal Pucci operator of order 2 and $\tilde{\lambda}$, $\tilde{\Lambda}$ are ellipticity constants that depend on λ, Λ, μ and dimension. The proof of this fact goes along the same lines as the proof of Lemma 5.10.

Remark 5.13. We note that the statement and proof of Lemma 5.10 here, combined with step 1 of the proof of Lemma 5.1, corrects an error in the construction of the similar barrier used in [Kassmann et al. 2014, Section 5], where the truncation step should have been done first, not at the end of the construction.

Now we can conclude this section with the proof of Lemma 5.11.

Proof of Lemma 5.11. Let x be any point such that $1-\frac{c_1}{2} < |x| < 1$. First of all, we note that q_1 has been fixed already, so we will drop it from the notation. Since we will be manipulating the choice of γ to obtain the desired bound on $M^- b_{\gamma,q_1}(x)$, it will be convenient to have bounds that transparently do not depend on γ. Therefore, as above, we keep $\gamma_0 = \frac{1}{4}$ fixed and we will use an auxiliary function to make some of the estimates. Let φ be any function in $C^2(\mathbb{R}^d)$ such that

$$0 \leq \varphi \leq b_{\gamma_0,q_1} \quad \text{in } \mathbb{R}^d,$$

and

$$\varphi(x) = |x|^{-q_1} \quad \forall\, |x| \geq 1-\tfrac{c_1}{2}.$$

We note that these definitions imply $\|\varphi\|_{C^2}$ can be chosen to be independent of γ (depending on universal parameters plus γ_0, q_1).

We now estimate the contributions from the positive and negative parts of $(\delta_h b(x))^{\pm}$ separately. The first estimate below is simply a use of the fact that by construction, φ touches b from below at x, and the second one uses (5-28):

$$\begin{aligned}\int_{\mathbb{R}^d}(\delta_h f(x))^- K(h)\,\mathrm{d}h &\le \int_{B_{r_1}} C(d)\big(\|\varphi\|_{C^{1,1}(B_{1/2}(x))}\big)|h|^2K(h)\,\mathrm{d}h+\int_{\mathbb{R}^d\setminus B_{r_1}}(\delta_h f(x))^- K(h)\,\mathrm{d}h\\ &\le C_dC(d)\|\varphi\|_{C^{1,1}(B_{1/2}(x))}\Lambda r_1^{2-\alpha}+C_d\frac{\Lambda}{\alpha}r_1^{-\alpha}+q_0C_d\Lambda r_1^{1-\alpha}. \qquad (5\text{-}32)\end{aligned}$$

Now we move to $(\delta_h b(x))^+$. Here we will use Lemma 5.8(i), the important feature being that there is *at least one* good ring where $(\delta_h b(x))^+$ will see the influence of the value of b on the set B_{1-c_2}. We alert the reader to a strange term in line (5-33) below, which arises simply as a worst case scenario of the three definitions of δ_h, and, for example, if $\alpha<1$, the term would not even be necessary. It does not harm the computation, and so we leave it there for any of the possible three cases of δ_h via α. Finally we note the important feature that we may only integrate on the set $h\in B_{1-c_1}(-x)$, which allows us to avoid the singularity of K at $h=0$. Also note if $h\in B_{1-c_1}(-x)$, then $|h|\le 2$:

$$\begin{aligned}&\int_{\mathbb{R}^d}(\delta_h f(x))^+K(h)\,\mathrm{d}h\\ &\ge\int_{A_{r_1}\cap B_{1-c_1}(-x)}(\delta_h f(x))^+K(h)\,\mathrm{d}h\\ &\ge\int_{A_{r_1}\cap B_{1-c_1}(-x)}(\gamma^{-q_1}-|x|^{-q_1})K(h)\,\mathrm{d}h-q_1|x|^{-q_1-1}\int_{B_{1-c_1}(-x)}|h|K(h)\,\mathrm{d}h \qquad (5\text{-}33)\\ &\ge\big(\gamma^{-q_1}-\big(1-\tfrac{c_1}{2}\big)^{-q_1}\big)(2-\alpha)\lambda\int_{A_{r_1}\cap B_{1-c_1}(-e_1)}|h|^{-d-\alpha}\,\mathrm{d}h-q_1\big(1-\tfrac{c_1}{2}\big)^{-q_1-1}\int_{B_{1-c_1}(-e_1)}2K(h)\,\mathrm{d}h \qquad (5\text{-}34)\\ &\ge\big(\gamma^{-q_1}-\big(1-\tfrac{c_1}{2}\big)^{-q_1}\big)(2-\alpha)\lambda r_1^{-d-\alpha}\tfrac14\mu\big|B_{2r_1}\setminus B_{r_1}\big|-q_1\big(1-\tfrac{c_1}{2}\big)^{-q_1-1}(2-\alpha)C(d,\alpha_0). \qquad (5\text{-}35)\end{aligned}$$

We note the use of (2-8) in the transition between the last two lines.

Recall that the values of c_1 and q_1 were fixed in Lemmas 5.8 and 5.10. In order to conclude the proof, we see that we can choose $\gamma=\gamma_1$ large enough so that when we add together the contributions from (5-32) and (5-35), the final estimate becomes greater than $C>q_1$ for all $\alpha\in(\alpha_0,\alpha_1)$. We note that it is crucial to have $\alpha\le\alpha_1<2$ in this case in order to keep α uniformly away from 2, which would cause problems. ☐

6. An estimate in L^ε — the weak Harnack inequality

The purpose of this section is to combine the point-to-measure estimate with the special barrier to prove the L^ε estimate, also called the *weak Harnack inequality*.

Theorem 6.1 (the L^ε estimate). *Assume $\alpha\ge\alpha_0>0$. Let u be a function such that*

$$\begin{gathered}u\ge 0\quad \textit{in } \mathbb{R}^d\times[-1,0],\\ u_t+C_0|\nabla u|-M^-u\le C\quad \textit{in } Q_1,\end{gathered}$$

and for the case $\alpha < 1$, further assume $C_0 = 0$. Then there are constants C_6 and ε such that

$$\left(\int_{B_{1/4}\times[-1,-2^{-\alpha}]} u^{\varepsilon}\mathrm{d}x\,\mathrm{d}t\right)^{1/\varepsilon} \le C_6(\inf_{Q_{1/4}} u + C).$$

The constants C_6 and ε depend on α_0, λ, Λ, C_0, d and μ.

Note that the L^{ε} norm of u is computed in the cylinder $B_{1/4}\times[-1,-2^{-\alpha}]$. This cylinder lies earlier in time than the cylinder $Q_{1/2}$, where the infimum is taken in the right-hand side of the inequality. This is natural due to the causality effect of parabolic equations. What should be noted in this case is that, due to the scaling of the equation, the size of these cylinders varies. Indeed, if $\alpha \in (1,2)$, then the time interval $[-1,-2^{-\alpha}]$ is longer than $\frac{1}{2}$ and certainly longer than $[-4^{-\alpha},0]$, which is the time span of $Q_{1/4}$. However, for small values of α, the length of $[-1,-2^{-\alpha}]$ becomes arbitrarily small and the time span of $Q_{1/4}$ is almost 1. We still have uniform choices of the constants C and ε because of the assumption $\alpha \ge \alpha_0 > 0$.

The basic building block of this proof is Lemma 4.1, which needs to be combined with Lemmas 5.1 and 5.2 as well as a covering argument. Since the work of Krylov and Safonov [1980], it is known that these ingredients lead to Theorem 6.1. However, there are several ways to organize the proof and there are some subtleties that we want to point out. Thus, we describe the full proof explicitly. We start with some preparatory lemmas.

The following lemma plays the role of Corollary 4.26 in [Imbert and Silvestre 2013b], which the reader can compare with [Chang Lara and Dávila 2014, Corollary 5.2]. Recall the notation $Q_r(x,t) = B_r(x)\times[t-r^{\alpha},t]$. We now define a time shift of the cylinder Q, which we call $\bar{Q}^m$. For any positive number m, we write $\bar{Q}^m$ to denote

$$\bar{Q}^m = B_r(x)\times(t,t+mr^{\alpha}).$$

The cylinder $\bar{Q}^m$ starts exactly where Q ends (see Figure 3). Moreover, its time span is enlarged by a factor m. Because of the order of causality, the information we have about the solution u in Q propagates to $\bar{Q}^m$. This is reflected in the following lemma.

Lemma 6.2 (stacked point estimate). *Let m be a positive integer. There exist $\delta_2 > 0$ and $N > 0$ depending only on λ, Λ, d, α_0 and m such that if for some cylinder $Q = Q_{\rho}(x_0,t_0) \subset Q_1$, we have*

$$u \ge 0 \text{ in } \mathbb{R}^d \times [-1,0], \tag{6-1}$$

$$u_t + C_0|\nabla u| - M^- u \ge 0 \quad \text{in } Q_1, \tag{6-2}$$

$$\big|\{u \ge N\}\cap Q_{\rho}(x_0,t_0)\big| \ge (1-\delta_2)|Q_{\rho}|, \tag{6-3}$$

$$B_{2\rho}(x_0)\times[t_0-\rho^{\alpha},t_0+m\rho^{\alpha}] \subset Q_1, \tag{6-4}$$

then $u \ge 1$ in $\bar{Q}^m = B_{\rho}(x_0)\times[t_0,t_0+m\rho^{\alpha}]$.

Proof. Let $\tilde{u}$ be the scaled function

$$\tilde{u}(x,t) = \frac{A_0}{N}u(\rho x + x_0, \rho^{\alpha}t + t_0),$$

where A_0 is the constant from Lemma 4.1.

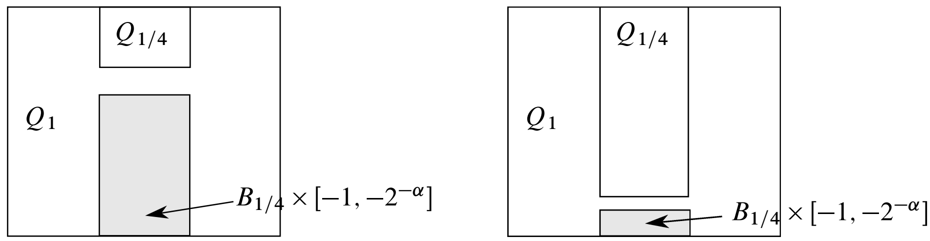

Figure 2. The cylinders $Q_{1/4}$ and $B_{1/4} \times [-1, -2^{-\alpha}]$ with large α (left) and small α (right).

Both u and $\tilde{u}$ satisfy (6-2). From our assumption (6-3), we have that

$$\big|\{\tilde{u} > A_0\} \cap Q_1\big| \ge (1-\delta_2)|Q_1|.$$

Applying the contrapositive of Lemma 4.1, we obtain that $\tilde{u} \ge 1$ in $Q_{1/4}$. Thus,

$$u \ge \frac{N}{A_0} \quad \text{in } Q_{\rho/4}(x_0, t_0).$$

Recall that u is a supersolution in Q_1 and $u \ge 0$ everywhere. We apply Lemmas 5.1 or 5.2 with $r = \frac{1}{2}$ to obtain the subsolution, p, and we can compare the functions $\tilde{u}$ and p. Writing this in terms of u gives

$$u(x,t) \ge \frac{N}{M} p\left(\frac{(x-x_0)}{\rho}, \frac{(t-t_0+(\rho/4)^\alpha)}{\rho^\alpha}\right).$$

The conclusion follows from taking N large enough, combined with the lower bound for p given in Lemma 5.1. □

The point of the previous lemma is that it can be combined with the *crawling ink spots* theorem. This is a covering argument that can be used as an alternative to the Calderón–Zygmund decomposition, and it is close to the original argument by Krylov and Safonov in [1980]. It has the cosmetic advantage that it does not use cubes but only balls. Moreover, the Calderón–Zygmund decomposition uses that we can tile the space with cubes, which is only true for $\alpha = 1$. In [Chang Lara and Dávila 2014], this difficulty is overcome by a special tiling with variable scaling, which is explained by the beginning of Section 4.2. It is a cumbersome construction to define rigorously. The use of the crawling ink spots theorem completely avoids this difficulty.

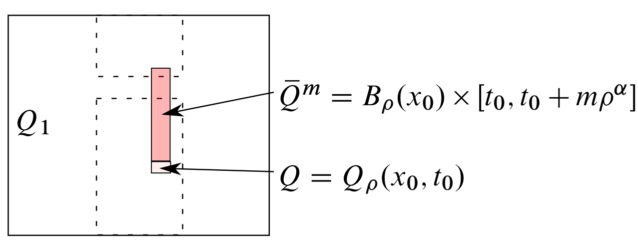

Figure 3. The cylinders involved in Lemma 6.2.

Theorem 6.3 (crawling ink spots). *Let $E \subset F \subset B_{1/2} \times \mathbb{R}$. We make the following two assumptions*:

- *For every point $(x,t) \in F$, there exists a cylinder $Q \subset B_1 \times \mathbb{R}$ so that $(x,t) \in Q$ and $|E \cap Q| \leq (1-\mu)|Q|$.*
- *For every cylinder $Q \subset B_1 \times \mathbb{R}$ such that $|E \cap Q| > (1-\mu)|Q|$, we have $\bar{Q}^m \subset F$.*

Then

$$|E| \leq \frac{m+1}{m}(1-c\mu)|F|.$$

Here c is an absolute constant depending on dimension only.

The proof of Theorem 6.3 will be presented in the Appendix. The crawling ink spots theorem is used with a value of m sufficiently large so that $\frac{m+1}{m}(1-c\delta) < 1$. In order to prove the L^ε estimate, we would want to apply Theorem 6.3 with

$$E = \{u \geq N^{k+1}\} \cap B_{1/2} \cap (-1, -2^{-\alpha}) \quad \text{and} \quad F = \{u \geq N^k\} \cap B_{1/2} \cap (-1, -2^{-\alpha}).$$

The problem is that the assumption of Theorem 6.3 is not implied by Lemma 6.2 because there is no way to ensure that $t + mr^\alpha \leq -2^{-\alpha}$. This is a difficulty that is nonexistent in the elliptic setting. Because of the time shift in all the point estimates, the conclusion of the crawling ink spots theorem may be spilling outside of the time interval $[-1, -2^{-\alpha}]$. There is no trivial workaround for this.

The purpose of the following lemma is to show that the cylinders $Q_\rho(x_0, \rho_0)$ that satisfy the condition of the crawling ink spots theorem are necessarily small, and consequently the amount of measure that *leaks* outside the cylinder $B_{1/4} \times [-1, -2^{-\alpha}]$ will decay exponentially.

Lemma 6.4. *Assume that*

$$\inf_{Q_{1/4}} u \leq 1,$$

$$u \geq 0 \quad \text{in } \mathbb{R}^d \times [-1, 0],$$

$$u_t + C_0|\nabla u| - M^- u \geq 0 \quad \text{in } Q_1,$$

and that there is a cylinder $Q_\rho(x_0, t_0)$ such that

$$Q_\rho(x_0, t_0) \subset B_{1/4} \times [-1, -2^{-\alpha}],$$

$$\big|\{u \geq N\} \cap Q_\rho(x_0, t_0)\big| \geq (1-\delta_2)|Q_\rho|.$$

Then $\rho < C\, N^{-\gamma}$ for some universal $\gamma > 0$ and $C > 0$.

Proof. Applying Lemma 4.1 rescaled to $Q_\rho(x_0, t_0)$, we obtain that $u \geq N/M$ in $Q_{\rho/4}(x_0, t_0)$. Just as in the proof of Lemma 6.2, we get

$$u(x,t) \geq \frac{N}{M} p\big(\tfrac{4}{3}(x - x_0), \big(\tfrac{4}{3}\big)^\alpha \big(t - t_0 + \big(\tfrac{1}{4}\rho\big)^\alpha\big)\big),$$

where p is the function from Lemmas 5.1 or 5.2 with $r = \frac{1}{3}\rho$. The reason for the factor $\frac{4}{3}$ is that since $x_0 \in B_{1/4}$, we know that $B_{3/4}(x_0) \subset B_1$.

We have that $x_0 \in B_{1/4}$, $t_0 \in [-1, -2^{-\alpha}]$ and $\rho \leq \min\big(\frac{1}{4}, (1-2^{-\alpha})^{1/\alpha}\big)$. Since $\inf_{Q_{1/4}} u \leq 1$, we have

$$\frac{M}{N} \geq \inf\big\{p(x,t) \,:\, x \in B_{2/3} \wedge t \in \big[(3^{-\alpha}(2^\alpha - 1) + \big(\tfrac{1}{3}\rho\big)^\alpha, \big(\tfrac{4}{3}\big)^\alpha + \big(\tfrac{1}{3}\rho\big)^\alpha\big]\big\} \geq c\rho^q,$$

which holds by (5-4) in Lemmas 5.1 and 5.2. Therefore $\rho < CN^{-\gamma}$, where $\gamma = \frac{1}{q}$ and q is the exponent from Lemma 5.1 or 5.2. □

Proof of Theorem 6.1. We start by noting that we can assume $C = 0$. Otherwise we consider $\tilde{u}(x,t) = u(x,t) - Ct$ instead. For every positive integer k, let

$$A_k := \{u > N^k\} \cap (B_{1/4} \times (-1, -2^{-\alpha})),$$

where N is the constant from Lemma 6.2. We apply Theorem 6.3 with

$$E = \{u \geq N^{k+1}\} \cap \big(B_{1/4} \times (-1, -2^{-\alpha})\big) \quad \text{and} \quad F = \{u \geq N^k\} \cap \big(B_{1/4} \times (-1, -2^{-\alpha} + CmN^{-\gamma\alpha k})\big),$$

where C and γ are the constants from Lemma 6.4.

Let us verify that both assumptions of Theorem 6.3 are satisfied. The first assumption in Theorem 6.3 is implied by Lemma 6.4 (at least when N and/or k are large). Indeed, any point

$$(x,t) \in B_{1/4} \times (-1, -2^{-\alpha} + mN^{-\gamma\alpha k})$$

is contained in some cylinder $Q_r(x_0, t_0)$ with large enough ρ so that $\rho > CN^{-k\gamma}$. Because of Lemma 6.2, whenever there is a cylinder Q such that $|A_{k+1} \cap Q| \geq (1-\delta)|Q|$, we know that $\bar{Q}^m \subset \{u > N^k\}$. Moreover, because of Lemma 6.4, the length in time of $\bar{Q}^m$ is less than $mCN^{-\gamma k}$. Therefore $\bar{Q}^m \subset F$. Thus, the second assumption of Theorem 6.3 holds as well.

Note that we allow the result of the crawling ink spots theorem to spill to the time interval

$$[-2^{-\alpha}, -2^{-\alpha} + CmN^{-\gamma\alpha k}].$$

Therefore,

$$|A_{k+1}| \leq \frac{m+1}{m}(1-c\delta)\big(|A_k| + CmN^{-\gamma\alpha k}\big).$$

We first pick m sufficiently large so that

$$\frac{m+1}{m}(1-c\delta) := 1 - \mu < 1.$$

Thus, we have

$$|A_{k+1}| \leq (1-\mu)\big(|A_k| + CmN^{-\gamma\alpha k}\big).$$

This already implies an exponential decay on $|A_k|$, which proves the theorem. □

7. Hölder continuity of solutions

We first state a Hölder continuity for parabolic integral equations without drift. In this case, $\alpha \in (0,2)$ can be arbitrarily small, although the estimates depend on its lower bound α_0.

Theorem 7.1 (Hölder estimates without drift). *Assume $\alpha \geq \alpha_0 > 0$. Let u be a bounded function in $\mathbb{R}^d \times [-1,0]$ such that*

$$\begin{aligned} u_t - M^+ u &\leq C \quad \text{in } Q_1, \\ u_t - M^- u &\geq -C \quad \text{in } Q_1. \end{aligned}$$

Then there are constants C_7 and γ, depending on n, λ, Λ and α_0, such that

$$\|u\|_{C^\gamma(Q_{1/2})} \le C_7\big(\|u\|_{L^\infty(\mathbb{R}^d\times[-1,0])} + C\big).$$

We can also include a drift term in the equation when $\alpha \ge 1$. This is stated in the next result.

Theorem 7.2 (Hölder estimates with drift). *Assume $\alpha \ge 1$. Let u be a bounded function in $\mathbb{R}^d \times [-1,0]$ such that*

$$\begin{aligned} u_t - C_0|\nabla u| - M^+u &\le C \quad \text{in } Q_1,\\ u_t + C_0|\nabla u| - M^-u &\ge -C \quad \text{in } Q_1. \end{aligned}$$

Then there are constants C_7 and γ, depending on n, λ, Λ, C_0, such that

$$\|u\|_{C^\gamma(Q_{1/2})} \le C_7\big(\|u\|_{L^\infty(\mathbb{R}^d\times[-1,0])} + C\big).$$

The proofs of these two theorems are essentially the same. The only difference is that when $\alpha \ge 1$, we can include a nonzero drift term in Theorem 6.1. Because of this, we write the proof only once, for Theorem 7.2, which applies to both theorems.

Proof of Theorem 7.2. We start by observing that we can reduce to the case $C \le \varepsilon_0$ and $\|u\|_{L^\infty} \le \frac{1}{2}$ by considering the function

$$\frac{1}{C/\varepsilon_0 + 2\|u\|_{L^\infty}} u(x,t).$$

We choose ε_0 sufficiently small, which will be specified below.

Our objective is to prove that for some $\gamma > 0$, which will also be specified below,

$$\operatorname*{osc}_{Q_r} u \le 2r^\gamma \tag{7-1}$$

for all $r \in (0,1)$. This proves the desired modulus of continuity at the point $(0,0)$. Since there is nothing special about the origin, we obtain the result of the theorem at every point in $Q_{1/2}$ using a standard scaling and translation argument. Note that since $\|u\|_{L^\infty} \le \frac{1}{2}$, we know a priori that (7-1) holds for all $r < 2^{-1/\gamma}$. We can make this threshold arbitrarily small by choosing a small value of γ.

In order to prove that (7-1) holds for all values of $r \in (0,1)$, we use induction. We assume that it holds for all $r \ge 8^{-k}$ and we show that it then holds for all $r \ge 8^{-(k+1)}$. Because of the observation in the previous paragraph, we can guarantee this inequality for the first few values of k by choosing a small value of γ. Thus, we are left to prove the inductive step.

Let

$$\tilde u(x,t) = \frac{1}{2}\,\frac{1}{8^{\gamma(k-1)}} u\bigg(\frac{8^{-(k-1)}}{2}x, \frac{8^{-\alpha(k-1)}}{2^\alpha}t\bigg).$$

This function $\tilde u$ is a scaled version of u so that the values of $\tilde u$ in Q_2 correspond to the values of u in $Q_{8^{-k+1}}$. Moreover, since (7-1) holds for $r \ge 8^{-k}$, we have that

$$\operatorname*{osc}_{Q_{2r}} \tilde u \le \min(r^\gamma, 1) \quad \text{for all } r \ge \tfrac{1}{8}. \tag{7-2}$$

Since $\operatorname{osc}_{Q_2} \tilde u \le 1$, for all $(x,t) \in Q_2$, we have that $\tilde u(x,t) \ge \max_{Q_2} \tilde u - \frac{1}{2}$ or $\tilde u(x,t) \le \min_{Q_2} \tilde u + \frac{1}{2}$. There may be points where both inequalities hold. The important thing is that at least one of the two inequalities holds at every point $(x,t) \in Q_2$. Therefore, one of the two inequalities will hold in at least half of the points (in measure) of the cylinder $B_{1/4}\times[-1,-2^{-\alpha}]$. Without loss of generality, let us assume it is the first of these inequalities that holds for most points (a similar argument works otherwise). That is, we have

$$\left|\{\tilde u \ge \max_{Q_2}\tilde u - \tfrac12\} \cap \left(B_{1/4}\times[-1,-2^{-\alpha}]\right)\right| \ge \tfrac12|B_{1/4}|\times(1-2^{-\alpha}).$$

Let v be the truncated function

$$v(x,t) := \left(\tilde u(x,t) - \max_{Q_2}\tilde u + 1\right)^+.$$

Note that $v \ge 0$ everywhere and $v = \tilde u(x,t) - \max_{Q_2}\tilde u + 1$ in Q_2. If $x \notin B_2$ and $t \in [-1,0]$, it can happen that $v(x,t) > \tilde u(x,t) - \max_{Q_2}\tilde u + 1$. We can estimate their difference using (7-2):

$$v(x,t) - \left(\tilde u(x,t) - \max_{Q_2}\tilde u + 1\right) \le \operatorname*{osc}_{B_{|x|}\times[-1,0]}\tilde u - 1 \le \left(\frac{|x|}{2}\right)^\gamma - 1 \quad \text{for any } x \notin B_2,\ t\in[-1,0]. \tag{7-3}$$

Note that for any fixed R, the right-hand side converges to zero uniformly for $2 \le |x| \le R$ as $\gamma \to 0$.

Inside Q_1, the function v satisfies the equation

$$\begin{aligned} v_t + C_0|\nabla v| - M^- v &\ge \tilde u_t + C_0|\nabla \tilde u| - M^-\tilde u + M^-(\tilde u - v) \\ &\ge -\varepsilon_0 + M^-(\tilde u - v) \\ &= -\varepsilon_0 + M^-\left((\tilde u - \max\tilde u + 1) - v\right) \\ &\ge -\varepsilon_0 - c(\gamma). \end{aligned}$$

Here $c(\gamma) = -\min_{Q_1} M^-\left((\tilde u - \max\tilde u + 1) - v\right) = \max_{Q_1} M^+\left(v - (\tilde u - \max\tilde u + 1)\right)$. We can estimate $c(\gamma)$ using (7-3) and assumption (A2), because

$$\begin{aligned} L\left(v - (\tilde u - \max\tilde u + 1)\right)(x) &= \int_{\mathbb{R}^d} \delta_h\left(v - (\tilde u - \max\tilde u + 1)\right)(x) K(h)\,\mathrm{d}h \\ &= \int_{|h|\ge 2}\left(v - (\tilde u - \max\tilde u + 1)\right)(h) K(h)\,\mathrm{d}h \\ &\le C\int_{2\le|h|\le R}(|h|^\gamma - 1)K(h)\,\mathrm{d}h + \int_{|h|\ge R} 2\|\tilde u\|_{L^\infty} K(h)\,\mathrm{d}h, \end{aligned} \tag{7-4}$$

where we note the use of the fact that $v - (\tilde u - \max\tilde u + 1) \equiv 0$ and also $\nabla\left(v - (\tilde u - \max\tilde u + 1)\right) \equiv 0$ in Q_2. Thus given any ρ, we can make $c(\gamma) < \rho$ by first choosing R large enough so that the tails of K are negligible outside of B_R — hence controlling the second term of (7-4) — and then choosing γ small enough so that second term of (7-4) is small enough. Since none of these choices depend upon the kernel, K, they hold for M^+, and hence $c(\gamma)$, as well.

Applying Theorem 6.1,

$$\begin{aligned} \min_{Q_{1/4}} v + \varepsilon_0 + c(\gamma) &\ge \frac{1}{C_6}\left(\int_{B_{1/4}\times[-1,-2^{-\alpha}]} v^\varepsilon \mathrm{d}x\,\mathrm{d}t\right)^{1/\varepsilon} \\ &\ge \frac{1}{C_6}\left(\frac12|B_{1/4}|(1-2^{-\alpha})\right)^{1/\varepsilon}\frac12. \end{aligned}$$

Let us choose $\varepsilon_0 > 0$ and $\gamma > 0$ sufficiently small so that

$$\delta := \frac{1}{C_6}\left(\frac12|B_{1/4}|(1-2^{-\alpha})\right)^{1/\varepsilon}\frac12 - \varepsilon_0 - c(\gamma) > 0.$$

Therefore, we obtained $\min_{Q_{1/4}} v \geq \delta$, which implies that $\operatorname{osc}_{Q_{1/4}} \tilde{u} \leq 1-\delta$. In terms of the original variables, this means that

$$\operatorname*{osc}_{Q_{8^{-k}}} u \leq 2 \times 8^{-\gamma(k-1)}(1-\delta).$$

Consequently, for any $r \in (8^{-k-1}, 8^{-k})$,

$$\operatorname*{osc}_{Q_r} u \leq 2 \times 8^{-\gamma(k-1)}(1-\delta).$$

Choosing γ sufficiently small so that

$$8^{-2\gamma} \geq (1-\delta)$$

implies that (7-1) holds for all $r > 2^{-k-1}$. This finishes the inductive step, and hence the proof.

Note that there is no circular dependence between the constants γ and ε_0. All conditions required in the proof are satisfied for any smaller value. We choose ε_0 and γ sufficiently small so that all these conditions are met. □

8. $C^{1,\gamma}$ regularity for nonlinear equations

It is by now standard that a Hölder regularity result as in Theorem 1.1 for kernels K that have rough dependence in x and t implies a $C^{1,\alpha}$ estimate for solutions to nonlinear equations. The following is a more precise statement.

Theorem 8.1. *Assume $\alpha_0 > 1$, $\alpha \in [\alpha_0, 2]$ and I is a translation-invariant nonlocal operator that is uniformly elliptic with respect to the class of kernels that satisfy* (A1), (A2), (A3) *and* (A4). *Let $u : \mathbb{R}^n \times [-T, 0] \to \mathbb{R}$ be a bounded viscosity solution of the equation*

$$u_t - Iu = f \quad \text{in } B_1 \times [-T, 0].$$

Then $u(\cdot, t) \in C^{1+\gamma}(B_{1/2})$ for all $t \in [-T/2, 0]$ and $u(x, \cdot) \in C^{(1+\gamma)/\alpha}([-T/2, 0])$ for all $x \in B_{1/2}$. Moreover, the following regularity estimate holds:

$$\sup_{t\in[-T/2,0]} \|u(\cdot,t)\|_{C^{1+\gamma}(B_{1/2})} + \sup_{x\in B_{1/2}} \|u(x,\cdot)\|_{C^{(1+\gamma)/\alpha}([-T/2,0])} \\ \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n\times[-T,0])} + \|f\|_{L^\infty(B_1\times[-T,0])} + I\big).$$

The constants C and γ depend only on λ, Λ, μ, n and α_0. Here $\gamma > 0$ is the minimum between $\alpha_0 - 1$ and the constant γ from Theorem 1.1 (*or Theorem 7.2*).

The proof of Theorem 8.1 is given in [Serra 2015] for the smaller class of symmetric kernels satisfying (1-2). His proof uses the main result in [Chang Lara and Dávila 2014], and the proof of Theorem 8.1 follows simply by replacing it with Theorem 7.2 in this paper. There is only one comment that needs to be made. In [Serra 2015], the following quantity is used a few times to control the tail of an integral operator

$$\|u\|_{L^1(\mathbb{R}^n,\omega_0)} := \int_{\mathbb{R}^n} u(x)(1+|x|)^{-n-\alpha_0}\,dx.$$

Because of our assumption (2-1), this quantity is not sufficient and needs to be replaced by

$$\max\{x \in \mathbb{R}^n \,:\, (1+|x|)^{\varepsilon-\alpha_0} u(x)\}$$

for some arbitrary small $\varepsilon > 0$. After this small modification, the proof in [Serra 2015] straightforwardly applies to prove Theorem 8.1 using Theorem 7.2.

The main example of a nonlinear integral operator I is the Isaacs operator from stochastic games:

$$I u(x) = \inf_i \sup_j \int_{\mathbb{R}^n} \delta_h u(x,t) K^{ij}(h)\, \mathrm{d}h.$$

Here, the kernels K^{ij} must satisfy the hypotheses (A1), (A2), (A3) and (A4) uniformly in i and j.

The result can also be extended to kernels $K^{ij}(x,h,t)$ that are not translation-invariant provided that they are continuous with respect to x and t. See [Serra 2015] for a discussion on this extension.

Appendix: The crawling ink spots theorem

We prove a version of the *crawling ink spots theorem* for fractional parabolic equations, which is a covering argument that first appeared in the original work of Krylov and Safonov [1979]. There it is indicated that the result was previously known by Landis, and it was Landis himself who came up with its suggestive name.

Let d_α be the parabolic distance of order α. By definition, it is

$$d_\alpha\big((x_0,t_0),(x_1,t_1)\big) = \max\big((2|t_1-t_2|)^{1/\alpha}, |x_1-x_2|\big).$$

The parabolic cylinders $Q_r(x,t)$ are balls of radius r centered at $\big(x, t-\frac{1}{2}r^\alpha\big)$ with respect to the distance d_α. The importance of this characterization is that it allows us to use the Vitali covering lemma, since this result is valid in arbitrary metric spaces.

Lemma A.1. *Let $\mu > 0$ and $E \subset F \subset B_1 \times \mathbb{R}$ be two open sets that satisfy the following two assumptions*:

- *For every point $(x,t) \in F$, there exists a cylinder $Q \subset B_1 \times \mathbb{R}$ so that $(x,t) \in Q$ and $|E \cap Q| \le (1-\mu)|Q|$.*
- *For every cylinder $Q \subset B_1 \times \mathbb{R}$ such that $|E \cap Q| > (1-\mu)|Q|$, we have $Q \subset F$.*

Then $|E| \le (1-c\mu)|F|$, where c is a constant depending on dimension only.

Proof. For every point $(x,t) \in F$, let Q^0 be the cylinder such that $(x,t) \in Q^0$ and $|E \cap Q^0| < (1-\mu)|Q^0|$.

Recall that F is an open set. Let us choose a maximal cylinder $Q^{(x,t)}$ such that $(x,t) \in Q^{(x,t)}$, $Q^{(x,t)} \subset Q^0$ and $Q^{(x,t)} \subset F$. Two things may happen; either $Q^{(x,t)} = Q^0$, in which case $|Q^{(x,t)} \cap E| < (1-\mu)|Q^{(x,t)}|$, or for any larger cylinder $Q^{(x,t)} \subset Q \subset Q^0$, we would have $Q \not\subset F$. In the latter case, we would have $|E \cap Q| \le (1-\mu)|Q|$ for *any* cylinder Q so that $Q^{(x,t)} \subset Q \subset Q^0$. In particular, the inequality holds for a decreasing sequence converging to $Q^{(x,t)}$ and therefore $|E \cap Q^{(x,t)}| \le (1-\mu)|Q^{(x,t)}|$.

In any case, we have constructed a cover $Q^{(x,t)}$ of the set F so that for all $(x,t) \in F$,

- $(x,t) \in Q^{(x,t)}$,
- $Q^{(x,t)} \subset F$,
- $|Q^{(x,t)} \cap E| \le (1-\mu)|Q^{(x,t)}|$.

Using the Vitali covering lemma, we can select a countable subcollection of cylinders Q_j such that $F \subset \bigcup_{j=1}^\infty 5Q_j$. Here each Q_j is one of the cylinders $Q^{(x,t)}$. We write $5Q_j$ to denote the cylinder expanded as a ball with respect to the metric d_α with the same center and five times the radius.

Since $Q_j \subset F$ and $|E \cap Q_j| \le (1-\mu)|Q_j|$, we have $|Q_j \cap (F \setminus E)| \ge \mu |Q_j|$. Therefore,

$$\begin{aligned}|F \setminus E| &\ge \sum_{j=1}^{\infty} |Q_j \cap (F \setminus E)| \\ &\ge \sum_{j=1}^{\infty} \mu |Q_j| \\ &= 5^{-d-\alpha} \mu \sum_{j=1}^{\infty} |5Q_j| \ge 5^{-d-\alpha} \mu |F|.\end{aligned}$$

The lemma follows with $c = 5^{-d-\alpha}$. $\square$

Lemma A.1 is not applicable directly to parabolic equations. What we need is a covering lemma so that if $|E \cap Q| \ge (1-\mu)|Q|$, then a time-shift of the cylinder Q is included in F instead of Q itself. This time-shift is given by the cylinders $\bar{Q}^m$, which we defined in Section 6.

We now give the proof of the *crawling ink spots* theorem.

Proof of Theorem 6.3. Let $\mathcal{Q}$ be the collection of cylinders $Q \subset B_1 \times \mathbb{R}$ such that $|E \cap Q| > (1-\mu)|Q|$. Let $G = \bigcup_{Q \in \mathcal{Q}} Q$. By construction, E and G satisfy the assumptions of Lemma A.1; thus $|E| \le (1-c\mu)|G|$. In order to prove this theorem, we are left to show that $|G| \le (m+1)/m|F|$. For that, we will see that

$$\Big|\bigcup_{Q \in \mathcal{Q}} \bar{Q}^m\Big| \ge \frac{m}{m+1}\Big|\bigcup_{Q \in \mathcal{Q}} Q \cup \bar{Q}^m\Big| \ge \frac{m}{m+1}|G|.$$

The second inequality above is trivial by the inclusion of the sets. The first inequality is not obvious since the cylinders may overlap. We justify this first inequality below.

From Fubini's theorem, the measure of any set $A \in B_1 \times \mathbb{R}$ is given by

$$|A| = \int_{B_1} \mathcal{L}_1(A \cap (\{x\} \times \mathbb{R}))\, \mathrm{d}x,$$

where $\mathcal{L}_1$ stands for the one-dimensional Lebesgue measure.

We finish the proof applying Fubini's theorem and noticing that for all $x \in B_1$,

$$\mathcal{L}_1\Big(\bigcup_{Q \in \mathcal{Q}} \bar{Q}^m \cap (\{x\} \times \mathbb{R})\Big) \ge \frac{m}{m+1}\mathcal{L}_1\Big(\bigcup_{Q \in \mathcal{Q}} (Q \cup \bar{Q}^m) \cap (\{x\} \times \mathbb{R})\Big).$$

This inequality follows from Lemma A.2, which is described below. $\square$

The following lemma is copied directly from [Imbert and Silvestre 2013b, Lemma 2.4.25]. An elementary proof is given there, which is independent of the rest of the text.

Lemma A.2. *Consider two (possibly infinite) sequences of real numbers $(a_k)_{k=1}^N$ and $(h_k)_{k=1}^N$ for $N \in \mathbb{N} \cup \{\infty\}$ with $h_k > 0$ for $k = 1, \dots, N$. Then*

$$\Big|\bigcup_{k=1}^{N} (a_k, a_k + (m+1)h_k)\Big| \le \frac{m}{m+1}\Big|\bigcup_{k=1}^{N} (a_k + h_k, a_k + (m+1)h_k)\Big|.$$

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