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ADVECTION-DIFFUSION EQUATIONS WITH DENSITY CONSTRAINTS

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In the spirit of the macroscopic crowd motion models with hard congestion (i.e., a strong density constraint $\rho \le 1$) introduced by Maury et al. some years ago, we analyze a variant of the same models where diffusion of the agents is also taken into account. From the modeling point of view, this means that individuals try to follow a given spontaneous velocity, but are subject to a Brownian diffusion, and have to adapt to a density constraint which introduces a pressure term affecting the movement. From the point of view of PDEs, this corresponds to a modified Fokker–Planck equation, with an additional gradient of a pressure (only living in the saturated zone $\{\rho = 1\}$) in the drift. We prove existence and some estimates, based on optimal transport techniques.

1. Introduction

In the past few years modeling crowd behavior has become a very active field of applied mathematics. Beyond their importance in real life applications, these modeling problems serve as basic ideas to understand many other phenomena coming for example from biology (cell migration, tumor growth, pattern formations in animal populations, etc.), particle physics and economics. A first nonexhaustive list of references for these problems is [Chalons 2007; Colombo and Rosini 2005; Coscia and Canavesio 2008; Cristiani et al. 2014; Dogbé 2008; Helbing 1992; Helbing and Molnár 1995; Hughes 2002; 2003; Maury and Venel 2009]. A very natural question in all these models is the study of congestion: in many practical situations, a high number of individuals could try to occupy the same spot, which could be impossible, or lead to strong negative effects on the motion, because of natural limitations on the crowd density.

These phenomena have been studied by using different models, which could be either “microscopic” (based on ODEs on the motion of a high number of agents) or “macroscopic” (describing the agents via their density and velocity, typically with Eulerian formalism). Let us concentrate on the macroscopic models, where the density ρ plays a crucial role. These very same models can be characterized either by “soft congestion” effects (i.e., the higher the density, the slower the motion), or by “hard congestion” (i.e., an abrupt threshold effect: if the density touches a certain maximal value, the motion is strongly affected, while nothing happens for smaller values of the density). See [Maury et al. 2011] for comparison between the different classes of models. This last class of models, due to the discontinuity in the congestion effects, presents new mathematical difficulties, which cannot be analyzed with the usual techniques from conservation laws (or, more generally, evolution PDEs) used for soft congestion.

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A very powerful tool to attack macroscopic hard congestion problems is the theory of optimal transportation (see [Villani 2003; Santambrogio 2015]), as we can see in [Maury et al. 2010; 2011; Roudneff-Chupin 2011; Santambrogio 2012a]. In this framework, the density of the agents solves a continuity equation (with velocity field taking into account the congestion effects), and can be seen as a curve in the Wasserstein space.

Our aim in this paper is to endow the macroscopic hard congestion models of [Maury et al. 2010; 2011; Roudneff-Chupin 2011; Santambrogio 2012a] with diffusion effects. In other words, we will study an evolution equation where particles

- have a spontaneous velocity field $u_t(x)$ which depends on time and on their position, and is the velocity they would follow in the absence of the other particles;
- must adapt their velocity to the existence of an incompressibility constraint which prevents the density to go beyond a given threshold;
- are subject to some diffusion effect.

This can be considered as a model for a crowd where a part of the motion of each agent is driven by a Brownian motion. Implementing this new element into the existing models could give a better approximation of reality; as usual when one adds a stochastic component, this can be a (very) rough approximation of unpredictable effects which are not already handled by the model, and this could work well when dealing with large populations.

Anyway, we do not want to discuss here the validity of this hard congestion model and we are mainly concerned with its mathematical analysis. In particular, we will consider existence and regularity estimates, while we do not treat the uniqueness issue. Uniqueness is considered in [Di Marino and Mészáros 2016], and one can observe that the insertion of diffusion dramatically simplifies the picture as far as uniqueness is concerned.

We also underline that one of the goals of the current paper (and of the work just cited) is to better "prepare" these hard congestion crowd motion models for a possible analysis in the framework of mean field games (see [Lasry and Lions 2006a; 2006b; 2007], and also [Santambrogio 2012b]). These MFG models usually involve a stochastic term, also implying regularizing effects, which are useful in the mathematical analysis of the corresponding PDEs.

The existing first-order models in light of the work of Maury, Roudneff-Chupin and Santambrogio. Some macroscopic models for crowd motion with density constraints and "hard congestion" effects were studied in [Maury et al. 2010; 2011]. We briefly present them as follows:

- The density of the population in a bounded (convex) domain $\Omega \subset \mathbb{R}^d$ is described by a probability measure $\rho \in \mathcal{P}(\Omega)$. The initial density $\rho_0 \in \mathcal{P}(\Omega)$ evolves in time, and ρ_t denotes its value at each time $t \in [0, T]$.
- The spontaneous velocity field of the population is a given time-dependent field, denoted by u_t. It represents the velocity that each individual would like to follow in the absence of the others. Ignoring the density constraint, this would give rise to the continuity equation $\partial_t \rho_t + \nabla \cdot (\rho_t u_t) = 0$. We observe that in the original work [Maury et al. 2010] the vector field $u_t(x)$ was taken of the form $-\nabla D(x)$ (independent

of time and of gradient form), but we try here to be more general (see [Roudneff-Chupin 2011], where the nongradient case is studied under some stronger regularity assumptions).

- The set of admissible densities will be denoted by $\mathcal{K} := \{\rho \in \mathcal{P}(\Omega) : \rho \le 1\}$. In order to guarantee that $\mathcal{K}$ is neither empty nor trivial, we suppose $|\Omega| > 1$.

- The set of admissible velocity fields with respect to the density ρ is characterized by the sign of the divergence of the velocity field on the saturated zone. We need to suppose also that all admissible velocity fields are such that no mass exists from the domain. So formally we set

$$\operatorname{adm}(\rho) := \bigl\{ v : \Omega \to \mathbb{R}^d : \nabla \cdot v \ge 0 \text{ on } \{\rho = 1\} \text{ and } v \cdot n \le 0 \text{ on } \partial\Omega \bigr\}.$$

- We consider the projection operator P in $L^2(\mathcal{L}^d)$:

$$P_{\operatorname{adm}(\rho)}[u] \in \operatorname{argmin}_{v \in \operatorname{adm}(\rho)} \int_\Omega |u - v|^2 \,\mathrm{d}x.$$

Note that we could have used the Hilbert space $L^2(\rho)$ instead of $L^2(\mathcal{L}^d)$; this would be more natural in this kind of evolution equation, as $L^2(\rho)$ is interpreted in a standard way as the tangent space to the Wasserstein space $\mathcal{W}_2(\Omega)$. Yet, these two projections turn out to be the same in this case, as the only relevant zone is $\{\rho = 1\}$. This is just formal, and would require more rigorous definitions (in particular of the divergence constraint in $\operatorname{adm}(\rho)$; see below). Anyway, to clarify, we choose to use the $L^2(\mathcal{L}^d)$-projection; in this way the vector fields are considered to be defined Lebesgue-a.e. on the whole Ω (and not only on $\{\rho > 0\}$) and the dependence of the projected vector field on ρ only passes through the set $\operatorname{adm}(\rho)$.

- Finally we solve the modified continuity equation

$$\partial_t \rho_t + \nabla \cdot (\rho_t P_{\operatorname{adm}(\rho_t)}[u_t]) = 0 \tag{1-1}$$

for ρ, where the main point is that ρ is advected by a vector field, compatible with the constraints, which is the closest to the spontaneous one.

The problem in solving (1-1) is that the projected field has very low regularity; it is a priori only L^2 in x, and it does not depend smoothly on ρ either (since a density 1 and a density $1-\varepsilon$ give very different projection operators). By the way, its divergence is not well defined either. To handle this issue we need to redefine the set of admissible velocities by duality. Taking a test function $p \in H^1(\Omega)$, $p \ge 0$ a.e., we obtain by integration by parts the equality

$$\int_\Omega v \cdot \nabla p \,\mathrm{d}x = -\int_\Omega (\nabla \cdot v) p \,\mathrm{d}x + \int_{\partial\Omega} p v \cdot n \,\mathrm{d}\mathcal{H}^{d-1}(x).$$

For vector fields v which do not let mass go through the boundary $\partial\Omega$, we have (in an a.e. sense) $v \cdot n = 0$. This leads to the definition

$$\operatorname{adm}(\rho) = \left\{ v \in L^2(\Omega; \mathbb{R}^d) : \int_\Omega v \cdot \nabla p \,\mathrm{d}x \le 0 \text{ for all } p \in H^1(\Omega) \text{ with } p \ge 0,\ p(1-\rho) = 0 \text{ a.e.} \right\}.$$

(Indeed, for a smooth vector field with vanishing normal component on the boundary, this is equivalent to imposing $\nabla \cdot v \ge 0$ on the set $\{\rho = 1\}$.)

Now, if we set

$$\operatorname{press}(\rho) := \{p \in H^1(\Omega) : p \geq 0,\ p(1-\rho) = 0 \text{ a.e.}\},$$

we observe that, by definition, $\operatorname{adm}(\rho)$ and $\nabla \operatorname{press}(\rho)$ are two convex cones which are dual to each other in $L^2(\Omega; \mathbb{R}^d)$. Hence we always have a unique orthogonal decomposition

$$u = v + \nabla p, \quad v \in \operatorname{adm}(\rho), \quad p \in \operatorname{press}(\rho), \quad \int_\Omega v \cdot \nabla p \, \mathrm{d}x = 0. \tag{1-2}$$

In this decomposition (as is the case every time we decompose on two dual convex cones), $v = P_{\operatorname{adm}(\rho)}[u]$. These will be our mathematical definitions for $\operatorname{adm}(\rho)$ and for the projection onto this cone.

Via this approach (introducing the new variable p and using its characterization from the previous line), for a given desired velocity field $u : [0, T] \times \Omega \to \mathbb{R}^d$, the continuity equation (1-1) can be rewritten as a system for the pair of variables (ρ, p), namely

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t (u_t - \nabla p_t)) = 0 & \text{in } [0, T] \times \Omega, \\ p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0 & \text{in } [0, T] \times \Omega, \\ \rho_t (u_t - \nabla p_t) \cdot n = 0 & \text{on } [0, T] \times \partial\Omega. \end{cases} \tag{1-3}$$

This system is endowed with the initial condition $\rho(0, x) = \rho_0(x)$ (for $\rho_0 \in \mathcal{K}$). As far as the spatial boundary $\partial\Omega$ is concerned, we put no-flux boundary conditions to preserve the mass in Ω.

Note that in the above system we withdrew the condition $\int (u_t - \nabla p_t) \cdot \nabla p_t = 0$, as it is a consequence of the system (1-3) itself. Informally, this can be seen as follows. For an arbitrary $p_0 \in \operatorname{press}(\rho_{t_0})$, we have that $t \mapsto \int_\Omega p_0 \rho_t$ is maximal at $t = t_0$ (where it is equal to $\int_\Omega p_0$). Differentiating this quantity with respect to t at $t = t_0$, using (1-3), we get the desired orthogonality condition at $t = t_0$. For a rigorous proof of this fact (which holds for a.e. t_0), we refer to Proposition 4.7 in [Di Marino et al. 2016].

A diffusive counterpart. The goal of our work is to study a second-order model of crowd movements with hard congestion effects, where beside the transport factor a nondegenerate diffusion is present as well. The diffusion is the consequence of a randomness (a Brownian motion) in the movement of the crowd.

With the ingredients introduced so far, we modify the Fokker–Planck equation $\partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t u_t) = 0$ in order to take into account the density constraint $\rho_t \leq 1$. Assuming enough regularity for the velocity field u, we observe that the Fokker–Planck equation is derived from a motion given by the stochastic ODE $\mathrm{d}X_t = u_t(X_t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t$ (where B_t is the standard d-dimensional Brownian motion), but is macroscopically represented by the advection of the density ρ_t by the vector field $-\nabla \rho_t / \rho_t + u_t$. Projecting onto the set of admissible velocities raises a natural question: should we project only u_t, and *then* apply the diffusion, or project the whole vector field, including $-\nabla \rho_t / \rho_t$? But this is not a real issue, since, at least formally, $\nabla \rho_t / \rho_t = 0$ on the saturated set $\{\rho_t = 1\}$ and

$$P_{\operatorname{adm}(\rho_t)}[-\nabla \rho_t / \rho_t + u_t] = P_{\operatorname{adm}(\rho_t)}[-\nabla \rho_t / \rho_t] + P_{\operatorname{adm}(\rho_t)}[u_t] = 0 + P_{\operatorname{adm}(\rho_t)}[u_t].$$

Rigorously, this corresponds to the fact that the heat kernel preserves the constraint $\rho \leq 1$. As a consequence, we consider the modified Fokker–Planck-type equation

$$\partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t P_{\operatorname{adm}(\rho_t)}[u_t]) = 0, \tag{1-4}$$

which can also be written equivalently for the variables (ρ, p) as

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t(u_t - \nabla p_t)) = 0 & \text{in } [0, T] \times \Omega, \\ p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0 & \text{in } [0, T] \times \Omega. \end{cases} \tag{1-5}$$

As usual, these equations are complemented by no-flux boundary conditions and by an initial datum $\rho(0, x) = \rho_0(x)$.

Roughly speaking, we can consider this equation to describe the law of a motion where each agent solves the stochastic differential equation

$$\mathrm{d}X_t = (u_t(X_t) - \nabla p_t(X_t))\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t.$$

This last statement is just formal and there are several issues defining a stochastic ODE like this. Indeed, the pressure variable is also an unknown, and globally depends on the law ρ_t of X_t. Hence, if we wanted to see this evolution as a superposition of individual motions, each agent should somehow predict the evolution of the pressure in order to solve his own equation. This calls to mind some notions from the stochastic control formulation of mean field games, as introduced by J.-M. Lasry and P.-L. Lions, even if here there are no strategic issues for the players. For mean field games with density constraints, we refer to [Cardaliaguet et al. 2015; Mészáros and Silva 2015; Santambrogio 2012b].

However, in this paper we will not consider any microscopic or individual problems, but only study the parabolic PDE (1-5).

Structure of the paper and main results. The main goal of the paper is to provide an existence result, with some extra estimates, for the Fokker–Planck equation (1-5) via time discretization, using the so-called splitting method (the two main ingredients of the equation, i.e., the advection with diffusion on one hand, and the density constraint on the other hand, are treated one after the other). In Section 2 we will collect some preliminary results, including what we need from optimal transport and from the previous works about density-constrained crowd motion, in particular on the projection operator onto the set $\mathcal{K}$. In Section 3 we will provide the aforementioned existence result, by a splitting scheme and some entropy bounds; the solution will be a curve of measures in $\mathrm{AC}^2([0, T]; \mathcal{W}_2(\Omega))$ (absolutely continuous curves with square-integrable speed). In Section 4 we will make use of BV estimates to justify that the solution just built is also $\mathrm{Lip}([0, T]; \mathcal{W}_1(\Omega))$ and satisfies a global BV bound $\|\rho_t\|_{\mathrm{BV}} \leq C$ (provided that $\rho_0 \in \mathrm{BV}$); this requires us to combine BV estimates on the Fokker–Planck equation (which are available depending on the regularity of the vector field u) with BV estimates on the projection operator on $\mathcal{K}$ (which have been recently proven in [De Philippis et al. 2016]). Section 5 presents a short review of alternative approaches, all discretized in time, but based either on gradient-flow techniques (the JKO scheme, see [Jordan et al. 1998]) or on different splitting methods. Finally, in the Appendix we detail the BV estimates on the Fokker–Planck equation (without any density constraint) that we could find; this seems to be a delicate matter, interesting in itself, and we are not aware of the sharp assumptions on the vector field u to guarantee the BV estimate that we need.

2. Preliminaries

Basic definitions and general facts on optimal transport. Here we collect some tools from the theory of optimal transportation, Wasserstein spaces, its dynamical formulation and more, which will be used later on. We formulate our problem either in a compact convex domain $\Omega \subset \mathbb{R}^d$ with smooth boundary or in the d-dimensional flat torus $\Omega := \mathbb{T}^d$ (although we will not adapt all our notation to the torus case). We refer to [Villani 2003; Santambrogio 2015] for more details. Given two probability measures $\mu, \nu \in \mathcal{P}(\Omega)$ and $p \geq 1$ we define the usual Wasserstein metric by means of the Monge–Kantorovich optimal transportation problem

$$W_p(\mu, \nu) := \inf\left\{ \int_{\Omega\times\Omega} |x-y|^p \, \mathrm{d}\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\}^{1/p},$$

where $\Pi(\mu, \nu) := \{\gamma \in \mathcal{P}(\Omega \times \Omega) : (\pi^x)_\#\gamma = \mu, (\pi^y)_\#\gamma = \nu\}$ and π^x and π^y denote the canonical projections from $\Omega \times \Omega$ onto Ω. This quantity happens to be a distance on $\mathcal{P}(\Omega)$ which metrizes the weak-$*$ convergence of probability measures; we denote by $\mathcal{W}_p(\Omega) := (\mathcal{P}(\Omega), W_p)$ the space of probabilities on Ω endowed with this distance.

Moreover, in the quadratic case $p = 2$ and under the assumption $\mu \ll \mathcal{L}^d$ (the d-dimensional Lebesgue measure on Ω), Y. Brenier [1987; 1991] showed that actually the optimal $\bar{\gamma}$ in the above problem (the existence of which is obtained simply by the direct method of calculus of variations) is induced by a map which is the gradient of a convex function, i.e., there exists $S : \Omega \to \Omega$ and $\psi : \Omega \to \mathbb{R}$ convex such that $S = \nabla\psi$ and $\bar{\gamma} := (\mathrm{id}, S)_\#\mu$. The function ψ is obtained as $\psi(x) = \frac{1}{2}|x|^2 - \varphi(x)$, where φ is the so-called Kantorovich potential for the transport from μ to ν, and is characterized as the solution of a dual problem that we will not develop here. In this way, the optimal transport map S can also be written as $S(x) = x - \nabla\varphi(x)$. Later, in the 1990s, R. McCann [1997] introduced a notion of interpolation between probability measures: the curve $\mu_t := ((T-t)x + ty)_\#\bar{\gamma}$, for $t \in [0, T]$ ($T > 0$ is given), gives a constant speed geodesic in the Wasserstein space connecting $\mu_0 := \mu$ and $\mu_T := \nu$.

Based on this notion of interpolation, J.-D. Benamou and Y. Brenier [2000] used some ideas from fluid mechanics to give a dynamical formulation to the Monge–Kantorovich problem. They showed that

$$\frac{1}{pT^{p-1}} W_p^p(\mu, \nu) = \inf\{\mathcal{B}_p(E, \mu) : \partial_t\mu + \nabla \cdot E = 0,\ \mu_0 = \mu,\ \mu_T = \nu\}.$$

Here $\mathcal{B}_p$ is a functional defined on pairs (E, μ), where E is a d-dimensional vector measure on $[0, T] \times \Omega$ and $\mu = (\mu_t)_t$ is a Borel-measurable family of probability measures on Ω. This functional is defined to be finite only if $E = E_t \otimes \mathrm{d}t$ (i.e., it is induced by a measurable family of vector measures on Ω: we have $\int_{[0,T]\times\Omega} \xi(t, x) \cdot \mathrm{d}E(t, x) = \int_0^T \mathrm{d}t \int_\Omega \xi(t, x) \cdot \mathrm{d}E_t(x)$ for all test functions $\xi \in C^0([0, T] \times \Omega; \mathbb{R}^d)$) and in this case it is defined through

$$\mathcal{B}_p(E, \mu) := \begin{cases} \displaystyle\int_0^T \int_\Omega \frac{1}{p}|v_t|^p \, \mathrm{d}\mu_t(x) \, \mathrm{d}t & \text{if } E_t = v_t \cdot \mu_t, \\ +\infty & \text{otherwise.} \end{cases}$$

It is well known that $\mathcal{B}_p$ is jointly convex and lower semicontinuous with respect to the weak-$*$ convergence of measures (see Section 5.3.1 in [Santambrogio 2015]) and that, if $\partial_t \mu + \nabla \cdot E = 0$, then $\mathcal{B}_p(E, \mu) < +\infty$ implies that $t \mapsto \mu_t$ is a curve in $\mathrm{AC}^p([0, T]; \mathcal{W}_p(\Omega))$.[1] In particular it is a continuous curve and the initial and final conditions on μ_0 and μ_T are well defined.

Coming back to curves in Wasserstein spaces, it is well known (see [Ambrosio et al. 2008] or Section 5.3 in [Santambrogio 2015]) that for any distributional solution μ_t (being a continuous curve in $\mathcal{W}_p(\Omega)$) of the continuity equation $\partial_t \mu + \nabla \cdot E = 0$ with $E_t = v_t \cdot \mu_t$, we have the relations

$$|\mu'|_{W_p}(t) \le \|v_t\|_{L^p_{\mu_t}} \quad \text{and} \quad W_p(\mu_t, \mu_s) \le \int_s^t |\mu'|_{W_p}(\tau)\,\mathrm{d}\tau,$$

where we denote by $|\mu'|_{W_p}(t)$ the metric derivative with respect to W_p of the curve μ_t (see for instance [Ambrosio and Tilli 2004] for general notions about curves in metric spaces and their metric derivative). For curves μ_t that are geodesics in $\mathcal{W}_p(\Omega)$ we have the equality

$$W_p(\mu_0, \mu_1) = \int_0^1 |\mu'|_{W_p}(t)\,\mathrm{d}t = \int_0^1 \|v_t\|_{L^p_{\mu_t}}\,\mathrm{d}t.$$

The last equality is in fact the Benamou–Brenier formula with the optimal velocity field v_t being the density of the optimal E_t with respect to the optimal μ_t. This optimal velocity field v_t can be computed as $v_t := (S - \mathrm{id}) \circ (S_t)^{-1}$, where $S_t := (1-t)\,\mathrm{id} + tS$ is the transport in McCann's interpolation (we assume here that the initial measure μ_0 is absolutely continuous, so that we can use transport maps instead of plans). This expression can be obtained if we consider that in this interpolation particles move with constant speed $S(x) - x$, but x represents here a Lagrangian coordinate, and not an Eulerian one: if we want to know the velocity at time t at a given point, we have to find out first the original position of the particle passing through that point at that time.

In the sequel we will also need the notion of entropy of a probability density, and for any probability measure $\varrho \in \mathcal{P}(\Omega)$ we define it as

$$\mathcal{E}(\varrho) := \begin{cases} \displaystyle\int_\Omega \varrho(x) \log \varrho(x)\,\mathrm{d}x & \text{if } \varrho \ll \mathcal{L}^d, \\ +\infty & \text{otherwise.} \end{cases}$$

We recall that this functional is lower semicontinuous and geodesically convex in $\mathcal{W}_2(\Omega)$.

As we will mainly be working with absolutely continuous probability measures (with respect to Lebesgue), we often identify measures with their densities.

Projection problems in Wasserstein spaces. Our analysis strongly relies on the projection operator $P_{\mathcal{K}}$ in the sense of W_2. Here $\mathcal{K} := \{\rho \in \mathcal{P}(\Omega) : \rho \le 1\}$ and

$$P_{\mathcal{K}}[\mu] := \mathrm{argmin}_{\rho \in \mathcal{K}}\, \tfrac{1}{2} W_2^2(\mu, \rho).$$

We recall the main properties of the projection operator $P_{\mathcal{K}}$ (see [Maury et al. 2010; Santambrogio 2012a; De Philippis et al. 2016]).

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[1] Here $\mathrm{AC}^p([0, T]; \mathcal{W}_p(\Omega))$ denotes the class of absolutely continuous curves in $\mathcal{W}_p(\Omega)$ with metric derivative in L^p. See the connection with the functional $\mathcal{B}_p$.

- As long as Ω is compact, for any probability measure μ, the minimizer in $\min_{\rho\in\mathcal{K}} \frac{1}{2}W_2^2(\mu, \rho)$ exists and is unique, and the operator $P_{\mathcal{K}}$ is continuous (it is even $C^{0,1/2}$ for the W_2 distance).
- The projection $P_{\mathcal{K}}[\mu]$ saturates the constraint $\rho \le 1$, in the sense that for any $\mu \in \mathcal{P}(\Omega)$ there exists a measurable set $B \subseteq \Omega$ such that $P_{\mathcal{K}}[\mu] = \mathbb{1}_B + \mu^{\mathrm{ac}}\mathbb{1}_{B^c}$, where μ^{ac} is the absolutely continuous part of μ.
- The projection is characterized in terms of a pressure field, in the sense that $\rho = P_{\mathcal{K}}[\mu]$ if and only if there exists a Lipschitz function $p \ge 0$, with $p(1-\rho) = 0$, and such that the optimal transport map S from ρ to μ is given by $S := \mathrm{id} - \nabla\varphi = \mathrm{id} + \nabla p$.
- There is (as proven in [De Philippis et al. 2016]) a quantified BV estimate: if $\mu \in \mathrm{BV}$ (in the sense that it is absolutely continuous and that its density belongs to $\mathrm{BV}(\Omega)$), then $P_{\mathcal{K}}[\mu]$ is also BV and

$$\mathrm{TV}(P_{\mathcal{K}}[\mu], \Omega) \le \mathrm{TV}(\mu, \Omega).$$

This last BV estimate will be crucial in Section 4, and it is important to have it in this very form (other estimates of the form $\mathrm{TV}(P_{\mathcal{K}}[\mu], \Omega) \le a\,\mathrm{TV}(\mu, \Omega) + b$ would not be as useful as this one, as they cannot be easily iterated).

3. Existence via a splitting-type algorithm (Main Scheme)

Similarly to the approach in [Maury et al. 2011] (see the algorithm (13) and Theorem 3.5) for a general nongradient vector field, we will build a theoretical algorithm, after time-discretization, to produce a solution of (1-5). Let us remark that splitting-type methods have been widely used in other contexts as well; see for instance [Clément and Maas 2011], which deals with splitting methods for Fokker–Planck equations and for more general gradient flows in metric and Wasserstein spaces, or [Laborde 2015], where a splitting-like approach is used to attack PDEs which are not gradient flows but “perturbations” of gradient flows.

In this section the spontaneous velocity field is a general vector field $u : [0, T] \times \Omega \to \mathbb{R}^d$ (not necessarily a gradient), which depends also on time. The only assumption we require on u is that

$$u \in L^\infty([0, T] \times \Omega; \mathbb{R}^d). \tag{U}$$

We work on a time interval $[0, T]$ and in a bounded convex domain $\Omega \subset \mathbb{R}^d$ (the case of the flat torus is even simpler and we will not discuss it in detail). We consider $\rho_0 \in \mathcal{P}^{\mathrm{ac}}(\Omega)$ to be given, which represents the initial density of the population, and we suppose $\rho_0 \in \mathcal{K}$.

Splitting using the Fokker–Planck equation. Let us consider the following scheme.

Main Scheme. Let $\tau > 0$ be a small time step with $N := \lfloor T/\tau \rfloor$. Let us set $\rho_0^\tau := \rho_0$. For every $k \in \{1, \dots, N\}$, define ρ_{k+1}^τ from ρ_k^τ by solving

$$\begin{cases} \partial_t \varrho_t - \Delta \varrho_t + \nabla \cdot (\varrho_t u_{t+k\tau}) = 0, & t \in]0, \tau], \\ \varrho_0 = \rho_k^\tau, \end{cases} \tag{3-1}$$

equipped with the no-flux boundary condition ($\varrho_t(\nabla\varrho_t - u_t)\cdot n = 0$ a.e. on $\partial\Omega$), and setting $\rho_{k+1}^\tau = P_{\mathcal{K}}[\tilde\rho_{k+1}^\tau]$, where $\tilde\rho_{k+1}^\tau = \varrho_\tau$. See Figure 1 below.

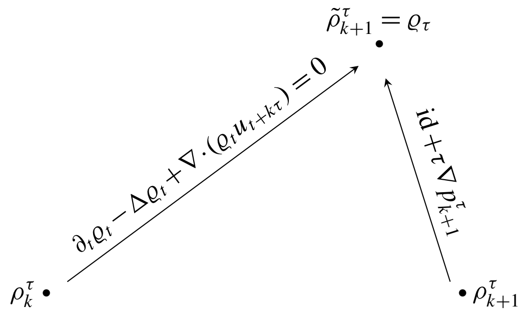

Figure 1. One time step.

Let us remark first that by classical results on parabolic equations (see for instance [Ladyzhenskaya et al. 1967]), since u satisfies the assumption (U), the equation (3-1) admits a unique distributional solution.

The above algorithm means to first follow the Fokker–Planck equation, ignoring the density constraint, for a time τ, then project. In order to state and prove the convergence of the scheme, we need to define some suitable interpolations of the discrete sequence of densities that we have just introduced.

First interpolation. We define the following curves of densities, velocities and momenta constructed with the help of the ρ^τ_k. First set

$$\rho^\tau_t := \begin{cases} \varrho_{2(t-k\tau)} & \text{if } t \in \big[k\tau, \big(k+\tfrac{1}{2}\big)\tau\big[, \\ \big(\mathrm{id} + 2((k+1)\tau - t)\nabla p^\tau_{k+1}\big)_{\#}\rho^\tau_{k+1} & \text{if } t \in \big[\big(k+\tfrac{1}{2}\big)\tau, (k+1)\tau\big[, \end{cases}$$

where ϱ_t is the solution of the Fokker–Planck equation (3-1) with initial datum ρ^τ_k and ∇p^τ_{k+1} arises from the projection of $\tilde{\rho}^\tau_{k+1}$, or more precisely, $(\mathrm{id} + \tau\nabla p^\tau_{k+1})$ is the optimal transport from ρ^τ_{k+1} to $\tilde{\rho}^\tau_{k+1}$. What are we doing? We are fitting into a time interval of length τ the two steps of our algorithm. First we follow the Fokker–Planck equation (3-1) at double speed, then we interpolate between the measure we reached and its projection following the geodesic between them. This geodesic is easily described as an image measure of ρ^τ_{k+1} through McCann's interpolation. By the construction it is clear that ρ^τ_t is a continuous curve in $\mathcal{P}(\Omega)$ for $t \in [0, T]$. We now define a family of time-dependent vector fields through

$$v^\tau_t := \begin{cases} -2\dfrac{\nabla\varrho_{2(t-k\tau)}}{\varrho_{2(t-k\tau)}} + 2u_t & \text{if } t \in \big[k\tau, \big(k+\tfrac{1}{2}\big)\tau\big[, \\ -2\nabla p^\tau_{k+1} \circ \big(\mathrm{id} + 2((k+1)\tau - t)\nabla p^\tau_{k+1}\big)^{-1} & \text{if } t \in \big[\big(k+\tfrac{1}{2}\big)\tau, (k+1)\tau\big[, \end{cases}$$

and, finally, we simply define the curve of momenta as $E^\tau_t := \rho^\tau_t v^\tau_t$.

Second interpolation. We define another interpolation as follows. Set

$$\tilde{\rho}^\tau_t := \varrho_{t-k\tau} \quad \text{if } t \in [k\tau, (k+1)\tau[,$$

where ϱ_t is (again) the solution of the Fokker–Planck equation (3-1) on the time interval $[0, \tau]$ with initial datum ρ^τ_k. Here we do not double its speed. We define the curve of velocities

$$\tilde{v}^\tau_t := -\frac{\nabla\varrho_{t-k\tau}}{\varrho_{t-k\tau}} + u_t \quad \text{if } t \in [k\tau, (k+1)\tau[,$$

and we build the curve of momenta by $\widetilde{E}^\tau_t := \tilde{\rho}^\tau_t \tilde{v}^\tau_t$.

Third interpolation. For each τ, we also define piecewise constant curves,

$$\hat{\rho}^\tau_t := \rho^\tau_{k+1} \quad \text{if } t \in [k\tau, (k+1)\tau[,$$
$$\hat{v}^\tau_t := \nabla p^\tau_{k+1} \quad \text{if } t \in [k\tau, (k+1)\tau[,$$

and $\widehat{E}^\tau_t := \hat{\rho}^\tau_t \hat{v}^\tau_t$. We remark that $p^\tau_{k+1}(1 - \rho^\tau_{k+1}) = 0$, hence the curve of momenta is just

$$\widehat{E}^\tau_t := \nabla p^\tau_{k+1} \quad \text{if } t \in [k\tau, (k+1)\tau[.$$

Mind the differences in the construction of ρ^τ_t, $\tilde{\rho}^\tau_t$ and $\hat{\rho}^\tau_t$ (and hence in the construction of v^τ_t, $\tilde{v}^\tau_t$ and $\hat{v}^\tau_t$, and E^τ_t, $\widetilde{E}^\tau_t$ and $\widehat{E}^\tau_t$):

(1) The first one is continuous in time for the weak-$*$ convergence, while the second and third ones are not.

(2) In the first construction we have taken into account the projection operator explicitly, while in the second one we see it just in an indirect manner (via the "jumps" occurring at every time of the form $t = k\tau$). The third interpolation is piecewise constant, and at every time it satisfies the density constraint.

(3) In the first interpolation the pair (ρ^τ, E^τ) solves the continuity equation, while in the other two it does not. This is not astonishing, as the continuity equation characterizes continuous curves in $\mathcal{W}_2(\Omega)$.

In order to prove the convergence of the scheme above, we will obtain uniform $\mathrm{AC}^2([0, T]; \mathcal{W}_2(\Omega))$ bounds for the curves ρ^τ. A key observation here is that the metric derivative (with respect to W_2) of the solution of the Fokker–Planck equation is comparable with the time differential of the entropy functional along the same solution (see Lemma 3.2). Now we state the main theorem of this section.

Theorem 3.1. *Let $\rho_0 \in \mathcal{K}$ and u be a given desired velocity field satisfying* (U). *Let us consider the interpolations introduced above. Then there exists a continuous curve $t \mapsto \rho_t \in \mathcal{W}_2(\Omega)$ for $t \in [0, T]$, and some vector measures $E, \widetilde{E}, \widehat{E} \in \mathfrak{M}([0, T] \times \Omega)$ such that the curves $\rho^\tau, \tilde{\rho}^\tau, \hat{\rho}^\tau$ converge uniformly in $\mathcal{W}_2(\Omega)$ to ρ and*

$$E^\tau \overset{*}{\rightharpoonup} E, \quad \widetilde{E}^\tau \overset{*}{\rightharpoonup} \widetilde{E}, \quad \widehat{E}^\tau \overset{*}{\rightharpoonup} \widehat{E} \quad \text{in } \mathfrak{M}([0, T] \times \Omega)^d \text{ as } \tau \to 0.$$

Moreover $E = \widetilde{E} - \widehat{E}$ and for a.e. $t \in [0, T]$ there exist time-dependent measurable vector fields $v_t, \tilde{v}_t, \hat{v}_t$ such that

(1) $E = \rho v, \quad \widetilde{E} = \rho \tilde{v}, \quad \widehat{E} = \rho \hat{v}$,

(2) $\displaystyle\int_0^T \left(\|v_t\|^2_{L^2_{\rho_t}} + \|\tilde{v}_t\|^2_{L^2_{\rho_t}} + \|\hat{v}_t\|^2_{L^2_{\rho_t}} \right) \mathrm{d}t < +\infty$,

(3) $v_t = \tilde{v}_t - \hat{v}_t$, *$\rho_t$-a.e.*, $\quad \widetilde{E}_t = \rho_t u_t - \nabla \rho_t \quad$ *and* $\quad \hat{v}_t = \nabla p_t$, *$\rho_t$-a.e.*,

where $p \in L^2([0,T]; H^1(\Omega))$, $p \geq 0$ *and* $p(1-\rho) = 0$ *a.e. in* $[0,T] \times \Omega$. *As a consequence, the pair* (ρ, p) *is a weak solution of the problem*

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t(u_t - \nabla p_t)) = 0 & \text{in } [0,T] \times \Omega, \\ p_t \geq 0,\ \rho_t \leq 1,\ p_t(1-\rho_t) = 0 & \text{in } [0,T] \times \Omega, \\ \rho_t(\nabla \rho_t - u_t + \nabla p_t) \cdot n = 0 & \text{on } [0,T] \times \partial\Omega, \\ \rho(0,\cdot) = \rho_0. \end{cases} \tag{3-2}$$

To prove this theorem we need the following tools.

Lemma 3.2. *Let us consider a solution* ϱ_t *of the Fokker–Planck equation on* $[0,T] \times \Omega$ *with the velocity field* u *satisfying* (U) *and with no-flux boundary conditions on* $[0,T] \times \partial\Omega$. *Then for any time interval* $]a,b[$ *we have the estimate*

$$\frac{1}{2}\int_a^b \int_\Omega \left| -\frac{\nabla \varrho_t}{\varrho_t} + u_t \right|^2 \varrho_t \, \mathrm{d}x \, \mathrm{d}t \leq \mathcal{E}(\varrho_a) - \mathcal{E}(\varrho_b) + \frac{1}{2}\int_a^b \int_\Omega |u_t|^2 \varrho_t \, \mathrm{d}x \, \mathrm{d}t. \tag{3-3}$$

In particular this implies

$$\frac{1}{2}\int_a^b |\varrho_t'|_{W_2}^2 \, \mathrm{d}t \leq \mathcal{E}(\varrho_a) - \mathcal{E}(\varrho_b) + \frac{1}{2}\int_a^b \int_\Omega |u_t|^2 \varrho_t \, \mathrm{d}x \, \mathrm{d}t, \tag{3-4}$$

where $|\varrho_t'|_{W_2}$ *denotes the metric derivative of the curve* $t \mapsto \varrho_t \in \mathcal{W}_2(\Omega)$.

Proof. To prove this inequality, we first make computations in the case where both u and ϱ are smooth, and ϱ is bounded from below by a positive constant. In this case we can write

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varrho_t) &= \int_\Omega (\log \varrho_t + 1)\partial_t \varrho_t \, \mathrm{d}x = \int_\Omega \log \varrho_t (\Delta \varrho_t - \nabla \cdot (\varrho_t u_t)) \, \mathrm{d}x \\ &= \int_\Omega \Big(-\frac{|\nabla \varrho_t|^2}{\varrho_t} + u_t \cdot \nabla \varrho_t \Big) \, \mathrm{d}x, \end{aligned}$$

where we use the conservation of mass (i.e., $\int_\Omega \partial_t \varrho_t \, \mathrm{d}x = 0$) and the boundary conditions in the integration by parts. We now compare this with

$$\begin{aligned} \frac{1}{2}\int_\Omega \left| -\frac{\nabla \varrho_t}{\varrho_t} + u_t \right|^2 \varrho_t \, \mathrm{d}x - \frac{1}{2}\int_\Omega |u_t|^2 \varrho_t \, \mathrm{d}x &= \int_\Omega \left(\frac{1}{2}\frac{|\nabla \varrho_t|^2}{\varrho_t} - \nabla \varrho_t \cdot u_t \right) \mathrm{d}x \\ &\leq \int_\Omega \left(\frac{|\nabla \varrho_t|^2}{\varrho_t} - \nabla \varrho_t \cdot u_t \right) \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varrho_t). \end{aligned}$$

This provides the first part of the statement, i.e., (3-3). If we combine this with the fact that the metric derivative of the curve $t \mapsto \varrho_t$ is always less than or equal to the $L^2_{\varrho_t}$ norm of the velocity field in the continuity equation, we also get

$$\frac{1}{2}|\varrho_t'|_{W_2}^2 - \frac{1}{2}\int_\Omega |u_t|^2 \varrho_t \leq -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varrho_t),$$

and hence (3-4).

In order to prove the same estimates without artificial smoothness and lower bound assumptions, we can act by approximation. We approximate the density ϱ_a by smooth and strictly positive densities ϱ_a^k (by convolution, so that we guarantee in particular $\mathcal{E}(\varrho_a^k) \to \mathcal{E}(\varrho_a)$), and the vector field u with smooth vector fields u^k (strongly in $L^4([a,b]\times\Omega)$, keeping the L^∞ bound). If we call ϱ^k the corresponding solution of the Fokker–Planck equation, it satisfies (3-3). This implies a uniform bound (with respect to k) for $\sqrt{\varrho^k}$ in $L^2([a,b]; H^1(\Omega))$, and hence a uniform bound on ϱ^k in $L^2([a,b]\times\Omega)$. From these bounds and the uniqueness of the solution of the Fokker–Planck equation with L^∞ drift, we deduce $\varrho^k \to \varrho$. The semicontinuity of the left-hand side in (3-3) and of the entropy term at $t = b$, together with the convergence of the entropy at $t = a$ and the convergence $\int_a^b \int_\Omega |u^k|^2 \varrho^k \,\mathrm{d}x\,\mathrm{d}t \to \int_a^b \int_\Omega |u|^2 \varrho \,\mathrm{d}x\,\mathrm{d}t$ (because we have a product of weak and strong convergence in L^2), allow us to pass (3-3) to the limit. ☐

Corollary 3.3. *From the inequality* (3-4) *we deduce that*

$$\mathcal{E}(\varrho_b) - \mathcal{E}(\varrho_a) \le \frac{1}{2}\int_a^b \int_\Omega |u_t|^2 \varrho_t \,\mathrm{d}x\,\mathrm{d}t,$$

and hence in particular for u satisfying (U), *we have*

$$\mathcal{E}(\varrho_b) - \mathcal{E}(\varrho_a) \le \frac{1}{2}\|u\|_{L^\infty}^2 (b-a).$$

As a consequence, *if $\varrho_a \le 1$*, *then we have*

$$\mathcal{E}(\varrho_b) \le \frac{1}{2}\|u\|_{L^\infty}^2 (b-a).$$

The same estimate can be applied to the curve $\tilde{\rho}^\tau$, with $a = k\tau$ and $b \in \,]k\tau, (k+1)\tau[$, thus obtaining $\mathcal{E}(\tilde{\rho}_t^\tau) \le C\tau$ for every t.

Lemma 3.4. *For any $\rho \in \mathcal{P}(\Omega)$ we have $\mathcal{E}(P_{\mathcal{K}}[\rho]) \le \mathcal{E}(\rho)$.*

Proof. We can assume $\rho \ll \mathcal{L}^d$, otherwise the claim is straightforward. As we pointed out in Section 2, we know that there exists a measurable set $B \subseteq \Omega$ such that

$$P_{\mathcal{K}}[\rho] = \mathbb{1}_B + \rho \mathbb{1}_{B^c}.$$

Hence it is enough to prove that

$$\int_B \rho \log \rho \,\mathrm{d}x \ge 0 = \int_B P_{\mathcal{K}}[\rho] \log P_{\mathcal{K}}[\rho] \,\mathrm{d}x,$$

as the entropies on B^c coincide. As the mass of ρ and $P_{\mathcal{K}}[\rho]$ are the same on all of Ω, and they coincide on B^c, we have

$$\int_B \rho(x)\,\mathrm{d}x = \int_B P_{\mathcal{K}}[\rho]\,\mathrm{d}x = |B|.$$

Then, by Jensen's inequality we have

$$\frac{1}{|B|}\int_B \rho \log \rho \,\mathrm{d}x \ge \left(\frac{1}{|B|}\int_B \rho \,\mathrm{d}x\right) \log\left(\frac{1}{|B|}\int_B \rho\,\mathrm{d}x\right) = 0.$$

The entropy decay follows. ☐

To analyze the pressure field we need the following result.

Lemma 3.5. *Let $\{p^\tau\}_{\tau>0}$ be a bounded sequence in $L^2([0,T];H^1(\Omega))$ and $\{\rho^\tau\}_{\tau>0}$ a sequence of piecewise constant curves valued in $\mathcal{W}_2(\Omega)$ which satisfy $W_2(\rho^\tau(a),\rho^\tau(b)) \le C\sqrt{b-a+\tau}$ for all $a<b\in[0,T]$ for a fixed constant C. Suppose that*

$$p^\tau \ge 0, \quad p^\tau(1-\rho^\tau)=0, \quad \rho^\tau \le 1,$$

and that

$$p^\tau \rightharpoonup p \quad \text{weakly in } L^2([0,T];H^1(\Omega)) \quad \text{and} \quad \rho^\tau \to \rho \quad \text{uniformly in } \mathcal{W}_2(\Omega).$$

Then $p(1-\rho)=0$ a.e. in $[0,T]\times\Omega$.

The proof of this result is the same as in Step 3 of Section 3.2 of [Maury et al. 2010] (see also [Roudneff-Chupin 2011] and Lemma 4.6 in [Di Marino et al. 2016]). We omit it in order not to overburden the paper.

The reader can note the strong connection with the classical Aubin–Lions lemma [Aubin 1963], applied to the compact injection of L^2 into H^{-1}. Indeed, from the weak convergence of p^τ in $L^2([0,T];H^1(\Omega))$, we just need to provide strong convergence of ρ^τ in $L^2([0,T];H^{-1}(\Omega))$. If instead of the quasi-Hölder assumption of the above lemma we suppose a uniform bound of $\{\rho^\tau\}_\tau$ in $\mathrm{AC}^2([0,T];\mathcal{W}_2(\Omega))$ (which is not so different), then the statement really can be deduced from the Aubin–Lions lemma. Indeed, the sequence $\{\rho^\tau\}$ is bounded in $L^\infty([0,T];L^2(\Omega))$ and its time derivative would be bounded in $L^2([0,T];H^{-1}(\Omega))$. This strongly depends on the fact that the H^{-1} distance can be controlled by the W_2 distance as soon as the measures have uniformly bounded densities (see [Loeper 2006; Maury et al. 2010]), a tool which is also crucial in the proofs in [Maury et al. 2010; Roudneff-Chupin 2011; Di Marino et al. 2016]. Then, the Aubin–Lions lemma guarantees compactness in $C^0([0,T];H^{-1}(\Omega))$, which is more than what we need.

Lemma 3.6. *Let us consider the previously defined interpolations. Then we have the following facts.*

(i) *For every $\tau>0$ and k we have*

$$\max\{W_2^2(\rho_k^\tau,\tilde\rho_{k+1}^\tau),\,W_2^2(\rho_k^\tau,\rho_{k+1}^\tau)\} \le \tau C(\mathcal{E}(\rho_k^\tau)-\mathcal{E}(\rho_{k+1}^\tau))+C\tau^2,$$

where $C>0$ only depends on $\|u\|_{L^\infty}$.

(ii) *There exists a constant C, only depending on ρ_0 and $\|u\|_{L^\infty}$, such that*

$$\mathcal{B}_2(E^\tau,\rho^\tau)\le C, \quad \mathcal{B}_2(\widetilde{E}^\tau,\tilde\rho^\tau)\le C \quad \text{and} \quad \mathcal{B}_2(\widehat{E}^\tau,\hat\rho^\tau)\le C.$$

(iii) *For the curve $[0,T]\ni t\mapsto\rho_t^\tau$ we have that*

$$\int_0^T |(\rho_t^\tau)'|_{W_2}^2\,\mathrm{d}t \le C,$$

for a $C>0$ independent of τ. Here we denote by $|(\rho_t^\tau)'|_{W_2}$ the metric derivative of the curve ρ^τ at t in $\mathcal{W}_2$. In particular, we have a uniform Hölder bound on ρ^τ, namely $W_2(\rho^\tau(a),\rho^\tau(b))\le C\sqrt{b-a}$ for every $b>a$.

(iv) *E^τ, $\widetilde{E}^\tau$, $\widehat{E}^\tau$ are uniformly bounded sequences in $\mathfrak{M}([0,T]\times\Omega)^d$.*

Proof. (i) First, by the triangle inequality and by the fact that $\rho^\tau_{k+1} = P_{\mathcal{K}}[\tilde\rho^\tau_{k+1}]$ we have that

$$W_2(\rho^\tau_k, \rho^\tau_{k+1}) \le W_2(\rho^\tau_k, \tilde\rho^\tau_{k+1}) + W_2(\tilde\rho^\tau_{k+1}, \rho^\tau_{k+1}) \le 2W_2(\rho^\tau_k, \tilde\rho^\tau_{k+1}). \tag{3-5}$$

We use (as before) the notation ϱ_t, $t \in [0, \tau]$ for the solution of the Fokker–Planck equation (3-1) with initial datum ρ^τ_k; in particular we have $\varrho_\tau = \tilde\rho^\tau_{k+1}$. Using Lemma 3.2 and since $\varrho_0 = \rho^\tau_k$ and $\varrho_\tau = \tilde\rho^\tau_{k+1}$ we have by (3-4) and $W_2(\rho^\tau_k, \tilde\rho^\tau_{k+1}) \le \int_0^\tau |\varrho'_t|_{W_2}\,\mathrm{d}t$ that

$$\begin{aligned} W_2^2(\rho^\tau_k, \tilde\rho^\tau_{k+1}) &\le \left(\tau^{1/2}\left(\int_0^\tau |\varrho'_t|^2_{W_2}\,\mathrm{d}t\right)^{1/2}\right)^2 \le 2\tau(\mathcal{E}(\varrho_0) - \mathcal{E}(\varrho_\tau)) + \tau\int_0^\tau\int_\Omega |u_{k\tau+t}|^2\varrho_t\,\mathrm{d}x\,\mathrm{d}t \\ &\le 2\tau(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\tilde\rho^\tau_{k+1})) + C\tau^2 \le 2\tau(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau^2, \end{aligned}$$

where $C > 0$ is a constant depending just on $\|u\|_{L^\infty}$. We have also used the fact that $\mathcal{E}(\rho^\tau_{k+1}) \le \mathcal{E}(\tilde\rho^\tau_{k+1})$, a consequence of Lemma 3.4.

Now by means of (3-5) we obtain

$$W_2^2(\rho^\tau_k, \rho^\tau_{k+1}) \le \tau C(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau^2.$$

(ii) We use Lemma 3.2 on the intervals of type $\left[k\tau, \left(k + \frac{1}{2}\right)\tau\right[$ and the fact that on each interval of type $\left[\left(k + \frac{1}{2}\right)\tau, (k+1)\tau\right[$ the curve ρ^τ_t is a constant speed geodesic. In particular, on these intervals we have

$$|(\rho^\tau)'|_{W_2} = \|v^\tau_t\|_{L^2_{\rho^\tau_t}} = 2\tau\|\nabla p^\tau_{k+1}\|_{L^2_{\rho^\tau_{k+1}}} = 2W_2(\rho^\tau_{k+1}, \tilde\rho^\tau_{k+1}).$$

On the other hand we also have

$$\tau^2\|\nabla p^\tau_{k+1}\|^2_{L^2_{\rho^\tau_{k+1}}} = W_2^2(\rho^\tau_{k+1}, \tilde\rho^\tau_{k+1}) \le W_2^2(\rho^\tau_k, \tilde\rho^\tau_{k+1}) \le \tau C(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau^2.$$

Hence we obtain

$$\begin{aligned} &\int_{k\tau}^{(k+1)\tau} \|v^\tau_t\|^2_{L^2(\rho^\tau_t)}\,\mathrm{d}t \\ &\quad= \int_{k\tau}^{(k+1/2)\tau}\int_\Omega 4\left|-\frac{\nabla\varrho_{2(t-k\tau)}}{\varrho_{2(t-k\tau)}} + u_{2t-k\tau}\right|^2 \varrho_{2(t-k\tau)}(x)\,\mathrm{d}x\,\mathrm{d}t + 4\int_{(k+1/2)\tau}^{(k+1)\tau}\int_\Omega |\nabla p^\tau_{k+1}|^2\rho^\tau_{k+1}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le C(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau + 2\tau\|\nabla p^\tau_{k+1}\|^2_{L^2_{\rho^\tau_{k+1}}} \\ &\quad\le C(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau. \end{aligned}$$

Hence by adding up we obtain

$$\mathcal{B}_2(E^\tau, \rho^\tau) \le \sum_k \bigl(C(\mathcal{E}(\rho^\tau_k) - \mathcal{E}(\rho^\tau_{k+1})) + C\tau\bigr) = C(\mathcal{E}(\rho^\tau_0) - \mathcal{E}(\rho^\tau_{N+1})) + CT \le C.$$

The estimates on $\mathcal{B}_2(\widetilde{E}^\tau, \tilde\rho^\tau)$ and $\mathcal{B}_2(\widehat{E}^\tau, \hat\rho^\tau)$ are completely analogous and arise from the previous computations.

(iii) The estimate on $\mathcal{B}_2(E^\tau, \rho^\tau)$ implies a bound on $\int_0^T |(\rho^\tau_t)'|^2_{W_2}\,\mathrm{d}t$ because v^τ is a velocity field for ρ^τ (i.e., the pair (E^τ, ρ^τ) solves the continuity equation).

(iv) In order to estimate the total mass of E we write

$$\begin{aligned}|E^\tau|([0,T]\times\Omega) &= \int_0^T\int_\Omega |v_t^\tau|\rho_t^\tau\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\left(\int_\Omega |v_t^\tau|^2\rho_t^\tau\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega \rho_t^\tau\,\mathrm{d}x\right)^{1/2}\mathrm{d}t\\ &\le \sqrt{T}\left(\int_0^T\int_\Omega |v_t^\tau|^2\rho_t^\tau\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \le C.\end{aligned}$$

The bounds on $\widetilde{E}^\tau$ and $\widehat{E}^\tau$ rely on the same argument. □

Proof of Theorem 3.1. We use the tools from Lemma 3.6.

Step 1. By the bounds on the metric derivative of the curves ρ_t^τ we get compactness, i.e., there exists a curve $[0,T]\ni t\mapsto \rho_t\in\mathcal{P}(\Omega)$ such that ρ^τ (up to subsequences) converges uniformly in $[0,T]$ with respect to W_2, in particular weakly-$*$ in $\mathcal{P}(\Omega)$ for all $t\in[0,T]$. It is easy to see that $\tilde\rho^\tau$ and $\hat\rho^\tau$ are converging to the same curve. Indeed we have $\tilde\rho_t^\tau=\rho^\tau_{\tilde s(t)}$ and $\hat\rho_t^\tau=\rho^\tau_{\hat s(t)}$ for $|\tilde s(t)-t|\le\tau$ and $|\hat s(t)-t|\le\tau$, which implies $W_2(\rho_t^\tau,\tilde\rho_t^\tau)$, $W_2(\rho_t^\tau,\hat\rho_t^\tau)\le C\tau^{1/2}$. This provides the convergence to the same limit.

Step 2. By the boundedness of E^τ, $\widetilde{E}^\tau$ and $\widehat{E}^\tau$ in $\mathfrak{M}([0,T]\times\Omega)^d$, we have the existence of E, $\widetilde{E}$, $\widehat{E}$ in $\mathfrak{M}([0,T]\times\Omega)^d$ such that (up to a subsequence) $E^\tau \overset{*}{\rightharpoonup} E$, $\widetilde{E}^\tau \overset{*}{\rightharpoonup} \widetilde{E}$, $\widehat{E}^\tau \overset{*}{\rightharpoonup} \widehat{E}$ as $\tau\to 0$. Now we show that $E=\widetilde{E}-\widehat{E}$. Indeed, let us show that for any test function $f\in\mathrm{Lip}([0,T]\times\Omega)^d$ we have

$$\left|\int_0^T\int_\Omega f_t\cdot(E_t^\tau-(\widetilde{E}_t^\tau+\widehat{E}_t^\tau))(\mathrm{d}x,\mathrm{d}t)\right|\to 0$$

as $\tau\to 0$. First, for each $k\in\{0,\dots,N\}$ we have that

$$\begin{aligned}\int_{k\tau}^{(k+1/2)\tau}\int_\Omega f_t\cdot E_t^\tau(\mathrm{d}x,\mathrm{d}t) &= \int_{k\tau}^{(k+1)\tau}\int_\Omega f_{(t+k\tau)/2}\cdot(-\nabla\varrho_{t-k\tau}+u_t\varrho_{t-k\tau})(\mathrm{d}x,\mathrm{d}t)\\ &= \int_{k\tau}^{(k+1)\tau}\int_\Omega f_t\cdot\tilde{E}_t^\tau(\mathrm{d}x,\mathrm{d}t)+\int_{k\tau}^{(k+1)\tau}\int_\Omega (f_{(t+k\tau)/2}-f_t)\cdot\widetilde{E}_t^\tau(\mathrm{d}x,\mathrm{d}t)\end{aligned}$$

and

$$\begin{aligned}&\int_{(k+1/2)\tau}^{(k+1)\tau}\int_\Omega f_t\cdot E_t^\tau(\mathrm{d}x,\mathrm{d}t)\\ &\quad=\int_{k\tau}^{(k+1)\tau}\int_\Omega -f_{(t+(k+1)\tau)/2}\circ\big(\mathrm{id}+((k+1)\tau-t)\nabla p_{k+1}^\tau\big)\cdot\nabla p_{k+1}^\tau\rho_{k+1}^\tau(\mathrm{d}x,\mathrm{d}t)\\ &\quad=-\int_{k\tau}^{(k+1)\tau}\int_\Omega f_t\cdot\widehat{E}_t^\tau(\mathrm{d}x,\mathrm{d}t)+\int_{k\tau}^{(k+1)\tau}\int_\Omega\big(f_t-f_{(t+(k+1)\tau)/2}\circ(\mathrm{id}+((k+1)\tau-t))\big)\cdot\hat{v}_t^\tau\hat\rho_t^\tau(\mathrm{d}x,\mathrm{d}t).\end{aligned}$$

This implies that

$$\begin{aligned}\left|\int_0^T\int_\Omega f_t\cdot(E_t^\tau-\tilde{E}_t^\tau+\widehat{E}_t^\tau)(\mathrm{d}x,\mathrm{d}t)\right|&\\ \le\sum_k\int_{k\tau}^{(k+1)\tau}\mathrm{Lip}(f)\tau\int_\Omega|\widetilde{E}_t^\tau|(\mathrm{d}x,\mathrm{d}t)&+\sum_k\int_{k\tau}^{(k+1)\tau}\mathrm{Lip}(f)\tau\int_\Omega(1+|\hat{v}_t^\tau|)|\widehat{E}_t^\tau|(\mathrm{d}x,\mathrm{d}t)\\ \le\tau C\,\mathrm{Lip}(f)\big(|\widetilde{E}^\tau|([0,T]\times\Omega)+|\widehat{E}^\tau|([0,T]\times\Omega)+\mathcal{B}_2(\widehat{E},\hat{\rho})\big)&\\ \le\tau C\,\mathrm{Lip}(f),&\end{aligned}$$

for a uniform constant $C>0$. Letting $\tau\to 0$, we prove the claim.

Step 3. The bounds on $\mathcal{B}_2(E^\tau,\rho^\tau)$, $\mathcal{B}_2(\widetilde{E}^\tau,\tilde{\rho}^\tau)$ and $\mathcal{B}_2(\widehat{E}^\tau,\hat{\rho}^\tau)$ pass to the limit by semicontinuity and allow us to conclude that E, $\widetilde{E}$ and $\widehat{E}$ are vector-valued Radon measures absolutely continuous with respect to ρ. Hence there exist v_t, $\tilde{v}_t$ and $\hat{v}_t$ such that $E=\rho v$, $\widetilde{E}=\rho\tilde{v}$ and $\widehat{E}=\rho\hat{v}$.

Step 4. We now look at the equations satisfied by E, $\widetilde{E}$ and $\widehat{E}$. First we use $\partial_t\rho^\tau+\nabla\cdot E^\tau=0$, pass to the limit as $\tau\to 0$ and get

$$\partial_t\rho+\nabla\cdot E=0.$$

Then, we use $\widetilde{E}^\tau=-\nabla\tilde{\rho}^\tau+u_t\tilde{\rho}^\tau$, pass to the limit again as $\tau\to 0$ and get

$$\widetilde{E}=-\nabla\rho+u_t\rho.$$

To justify this limit, the only delicate point is passing to the limit the term $u_t\tilde{\rho}^\tau$, since u is only L^∞, and $\tilde{\rho}^\tau$ converges weakly as measures, and we are a priori only allowed to multiply it by continuous functions. Yet, we remark that by Corollary 3.3 we have that $\mathcal{E}(\tilde{\rho}_t^\tau)\le C\tau$ for all $t\in[0,T]$. In particular, this provides, for each t, uniform integrability for $\tilde{\rho}_t^\tau$ and turns the weak convergence as measures into weak convergence in L^1. This allows multiplication by u_t in the weak limit.

Finally, we look at $\widehat{E}^\tau$. There exists a piecewise constant (in time) function p^τ (defined as p_{k+1}^τ on every interval $]k\tau,(k+1)\tau]$) such that $p^\tau\ge 0$, $p^\tau(1-\hat{\rho}^\tau)=0$,

$$\int_0^T\int_\Omega|\nabla p^\tau|^2(\mathrm{d}x,\mathrm{d}t)=\int_0^T\int_\Omega|\nabla p^\tau|^2\hat{\rho}^\tau(\mathrm{d}x,\mathrm{d}t)=\int_0^T\int_\Omega|\hat{v}^\tau|^2\hat{\rho}^\tau(\mathrm{d}x,\mathrm{d}t)\le C \tag{3-6}$$

and $\widehat{E}^\tau=\nabla p^\tau\hat{\rho}^\tau=\nabla p^\tau$. The bound (3-6) implies that p^τ is uniformly bounded in $L^2(0,T;H^1(\Omega))$. Since for every t we have $|\{p_t^\tau=0\}|\ge|\{\hat{\rho}_t^\tau<1\}|\ge|\Omega|-1$, we can use a suitable version of Poincaré's inequality, and get a uniform bound in $L^2([0,T];L^2(\Omega))=L^2([0,T]\times\Omega)$. Therefore, there exists $p\in L^2([0,T]\times\Omega)$ such that $p^\tau\rightharpoonup p$ weakly in L^2 as $\tau\to 0$. In particular we have $\widehat{E}=\nabla p$. Moreover it is clear that $p\ge 0$ and by Lemma 3.5 we obtain $p(1-\rho)=0$ a.e. as well. Indeed, the assumptions of the lemma are easily checked: we only need to estimate $W_2(\hat{\rho}^\tau(a),\hat{\rho}^\tau(b))$ for $b>a$, but we have

$$W_2(\hat{\rho}^\tau(a),\hat{\rho}^\tau(b))=W_2(\rho^\tau(k_a\tau),\rho^\tau(k_b\tau))\le C\sqrt{k_b-k_a}\quad\text{for } k_b\tau\le b+\tau \text{ and } k_a\ge a.$$

Once we have $\widehat{E} = \nabla p$ with $p(1-\rho) = 0$, $p \in L^2([0,T]; H^1(\Omega))$ and $\rho \in L^\infty$, we can also write

$$\widehat{E} = \nabla p = \rho \nabla p.$$

If we sum up our results, using $E = \widetilde{E} - \widehat{E}$, we have

$$\partial_t \rho - \Delta \rho + \nabla \cdot (\rho(u - \nabla p)) = 0 \quad \text{with } p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0 \text{ a.e. in } [0,T] \times \Omega.$$

As usual, this equation is satisfied in a weak sense, with no-flux boundary conditions. ☐

4. Uniform Lip$([0, T]; \mathcal{W}_1)$ and BV estimates

In this section we provide uniform estimates for the curves ρ^τ, $\tilde{\rho}^\tau$ and $\hat{\rho}^\tau$ in the form of uniform BV (in space) bounds on $\tilde{\rho}^\tau$ (which implies the same bound for $\hat{\rho}^\tau$) and uniform Lipschitz bounds in time for the W_1 distance on ρ^τ. This means a small improvement compared to the previous section concerning time regularity, as we have Lipschitz instead of AC^2, even if we need to replace W_2 with W_1. It is also important for space regularity. Indeed, from Lemma 3.2 one could deduce that the solution ρ of the Fokker–Planck equation (1-5) satisfies $\sqrt{\rho} \in L^2([0,T]; H^1(\Omega))$ and, using $\rho \leq 1$, also $\rho \in L^2([0,T]; H^1(\Omega))$. Yet, this is just an integrable estimate in t, while the BV estimate of this section is uniform in the time variable.

Nevertheless there is a price to pay for this improvement: we have to assume higher regularity for the velocity field. These uniform-in-time W_1-Lipschitz bounds are based both on BV estimates for the Fokker–Planck equation (see Lemma A.1 in the Appendix) and for the projection operator $P_{\mathcal{K}}$ (see [De Philippis et al. 2016]). The assumption on u is, essentially, that we need to control the growth of the total variation of the solutions of the Fokker–Planck equation (3-1), and we need to iterate this bound along time steps.

We will discuss in the Appendix the different BV estimates on the Fokker–Planck equation that we were able to find. The desired estimate is true whenever $\|u_t\|_{C^{1,1}(\Omega)}$ is uniformly bounded and $u_t \cdot n = 0$ on $\partial\Omega$. It seems to be an open problem to obtain similar estimates under the sole assumption that u is Lipschitz continuous. Of course, we will also assume $\rho_0 \in \mathrm{BV}(\Omega)$. Despite these extra regularity assumptions, we think these estimates have their own interest, exploiting some finer properties of the solutions of the Fokker–Planck equation and of the Wasserstein projection operator.

Before entering into the details of the estimates, we want to discuss why we concentrate on BV estimates (instead of Sobolev ones) and on W_1 (instead of W_p, $p > 1$). The main reason is the role of the projection operator. Indeed, even if $\rho \in W^{1,p}(\Omega)$, we do not have in general $P_{\mathcal{K}}[\rho] \in W^{1,p}$ because the projection creates some jumps at the boundary of $\{P_{\mathcal{K}}[\rho] = 1\}$. This prevents us from obtaining any $W^{1,p}$ estimate for $p > 1$. On the other hand, [De Philippis et al. 2016] exactly proves a BV estimate on $P_{\mathcal{K}}[\rho]$ and paves the way to BV bounds for our equation. Concerning the regularity in time, we observe that the velocity field in the Fokker–Planck equation contains a term in $\nabla\rho/\rho$. Since the metric derivative in $\mathcal{W}_p$ is given by the L^p norm (with respect to ρ_t) of the velocity field, it is clear that estimates in $\mathcal{W}_p$ for $p > 1$ would require spatial $W^{1,p}$ estimates on the solution itself, which are impossible for $p > 1$ in this splitting scheme. We stress that this does not mean that uniform $W^{1,p}$ are impossible for the solution of (1-5); it only means that they are not uniform along the approximation used in our Main Scheme to build such a solution.

The precise result that we prove is the following.

Theorem 4.1. *Let us suppose that* $\|u_t\|_{C^{1,1}} \le C$ *and* $\rho_0 \in \mathrm{BV}(\Omega)$. *Then using the notations from the Main Scheme and Theorem 3.1 one has* $\|\tilde\rho^\tau_t\|_{\mathrm{BV}} \le C$ *and* $W_1(\rho^\tau_k, \rho^\tau_{k+1}) \le C\tau$. *As a consequence we also have* $\rho \in \mathrm{Lip}([0,T]; \mathcal{W}_1) \cap L^\infty([0,T]; \mathrm{BV}(\Omega))$.

To prove this theorem we need the following lemmas.

Lemma 4.2. *Suppose* $\|u_t\|_{\mathrm{Lip}} \le C$ *and* $u_t \cdot n = 0$ *on* $\partial\Omega$. *Then for the solution* ϱ *of* (A-1) *with velocity field* $v = u$ *we have the estimate*

$$\|\varrho_t\|_{L^\infty} \le \|\varrho_0\|_{L^\infty} e^{Ct},$$

where $C = \|\nabla \cdot u_t\|_{L^\infty}$.

Proof. Standard comparison theorems for parabolic equations allow us to prove the results once we notice that $f(t,x) := \|\varrho_0\|_{L^\infty} e^{Ct}$ is a supersolution of the Fokker–Planck equation, i.e.,

$$\partial_t f_t \ge \Delta f_t - \nabla \cdot (f_t u_t).$$

Indeed, in the above equation the Laplacian term vanishes as f is constant in x, $\partial_t f_t = C f_t$ and $\nabla \cdot (f_t u_t) = f_t \nabla \cdot u_t + \nabla f_t \cdot u_t = f_t \nabla \cdot u_t \le C f_t$, where $C = \|\nabla \cdot u_t\|_{L^\infty}$. From this inequality, and from $\rho_0 \le f_0$, we deduce $\rho_t \le f_t$ for all t. □

We remark that the above lemma implies in particular that after every step in the Main Scheme we have $\tilde\rho^\tau_{k+1} \le e^{\tau c} \le 1 + C\tau$, where $c := \|\nabla \cdot u\|_{L^\infty}$. We note the following corollary as well.

Corollary 4.3. *Along the iterations of our Main Scheme, for every k we have* $W_1(\tilde\rho^\tau_{k+1}, \rho^\tau_{k+1}) \le \tau C$ *for a constant* $C > 0$ *independent of* τ.

Proof. With the saturation property of the projection (see Section 2 or [De Philippis et al. 2016]), we know that there exists a measurable set $B \subseteq \Omega$ such that $\rho^\tau_{k+1} = \tilde\rho^\tau_{k+1} \mathbb{1}_B + \mathbb{1}_{\Omega\setminus B}$. On the other hand we know that

$$\begin{aligned} W_1(\tilde\rho^\tau_{k+1}, \rho^\tau_{k+1}) &= \sup_{\substack{f \in \mathrm{Lip}_1(\Omega) \\ 0 \le f \le \mathrm{diam}(\Omega)}} \int_\Omega f(\tilde\rho^\tau_{k+1} - \rho^\tau_{k+1})\,\mathrm{d}x \\ &= \sup_{\substack{f \in \mathrm{Lip}_1(\Omega) \\ 0 \le f \le \mathrm{diam}(\Omega)}} \int_{\Omega\setminus B} f(\tilde\rho^\tau_{k+1} - 1)\,\mathrm{d}x \le \tau C |\Omega| \,\mathrm{diam}(\Omega). \end{aligned}$$

We use the fact that the competitors f in the dual formula can be taken to be positive and bounded by the diameter of Ω, just by adding a suitable constant. This implies as well that C is dependent on c, $|\Omega|$ and $\mathrm{diam}(\Omega)$. □

Proof of Theorem 4.1. First we take care of the BV estimate. Lemma A.1 guarantees, for $t \in\,]k\tau, (k+1)\tau[$, that we have $\mathrm{TV}(\tilde\rho^\tau_t) \le C\tau + e^{C\tau}\,\mathrm{TV}(\rho^\tau_k)$. Together with the BV bound on the projection that we presented in Section 2 (taken from [De Philippis et al. 2016]), this can be iterated, providing a uniform bound (depending on $\mathrm{TV}(\rho_0)$, T and $\sup_t \|u_t\|_{C^{1,1}}$) on $\|\tilde{\rho}_t^\tau\|_{\mathrm{BV}}$. Passing this estimate to the limit as $\tau \to 0$ we get $\rho \in L^\infty([0,T];\mathrm{BV}(\Omega))$.

Then we estimate the behavior of the interpolation curve $\hat{\rho}^\tau$ in terms of W_1. We estimate

$$\begin{aligned} W_1(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau) \le \int_{k\tau}^{(k+1)\tau} |(\tilde{\rho}_t^\tau)'|_{W_1}\,\mathrm{d}t &\le \int_{k\tau}^{(k+1)\tau}\int_\Omega \left(\frac{|\nabla \tilde{\rho}_t^\tau|}{\tilde{\rho}_t^\tau} + |u_t|\right)\tilde{\rho}_t^\tau\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_{k\tau}^{(k+1)\tau} \|\tilde{\rho}_t^\tau\|_{\mathrm{BV}}\,\mathrm{d}t + C\tau \le C\tau. \end{aligned}$$

Hence, we obtain

$$W_1(\rho_k^\tau, \rho_{k+1}^\tau) \le W_1(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau) + W_1(\tilde{\rho}_{k+1}^\tau, \rho_{k+1}^\tau) \le \tau C.$$

This in particular means, for $b > a$,

$$W_1(\hat{\rho}^\tau(a), \hat{\rho}^\tau(b)) \le C(b - a + \tau).$$

We can pass this relation to the limit, using that, for every t, we have $\hat{\rho}_t^\tau \to \rho_t$ in $\mathcal{W}_2(\Omega)$ (and hence also in $\mathcal{W}_1(\Omega)$, since $W_1 \le W_2$), getting

$$W_1(\rho(a), \rho(b)) \le C(b - a),$$

which means that ρ is Lipschitz continuous in $\mathcal{W}_1(\Omega)$. ☐

5. Variations on a theme: some reformulations of the Main Scheme

In this section we propose some alternative approaches to study the problem (1-5). The general idea is to discretize in time, and give a way to produce a measure ρ_{k+1}^τ starting from ρ_k^τ. Observe that the interpolations ρ^τ, $\tilde{\rho}^\tau$ and $\hat{\rho}^\tau$ proposed in the previous sections are only technical tools to state and prove a convergence result, and the most important point is exactly the definition of ρ_{k+1}^τ.

The alternative approaches proposed here explore different ideas, more difficult to implement than the one that we presented in Section 3, and/or restricted to some particular cases (for instance when u is a gradient). They have their own modeling interest and this is the main reason justifying their sketchy presentation.

Variant 1: transport, diffusion then projection. We recall that the original splitting approach for the equation without diffusion [Maury et al. 2011; Roudneff-Chupin 2011] exhibited an important difference compared to what we did in Section 3. Indeed, in the first phase of each time step (i.e., before the projection) the particles follow the vector field u and $\tilde{\rho}_{k+1}^\tau$ was not defined as the solution of a continuity equation with advection velocity given by u_t, but as the image of ρ_k^τ via a straight-line transport: $\tilde{\rho}_{k+1}^\tau := (\mathrm{id} + \tau u_{k\tau})_\# \rho_k^\tau$. One can wonder whether it is possible to follow a similar approach here.

A possible way to proceed is as follows. Take a random variable X distributed according to ρ_k^τ, and define $\tilde{\rho}_{k+1}^\tau$ as the law of $X + \tau u_{k\tau}(X) + B_\tau$, where B is a Brownian motion, independent of X. This exactly means that every particle moves starting from its initial position X, following a displacement ruled by u, but adding a stochastic effect in the form of the value at time τ of a Brownian motion. We can check that this means

$$\tilde{\rho}^{\tau}_{k+1} := \eta_\tau * ((\mathrm{id} + \tau u_{k\tau})_{\#}\rho^{\tau}_{k}),$$

where η_τ is a Gaussian kernel with zero mean and variance τ, i.e., $\eta_\tau(x) := \dfrac{1}{(4\tau\pi)^{d/2}} e^{-|x|^2/(4\tau)}$. Then we define

$$\rho^{\tau}_{k+1} := P_{\mathcal{K}}[\tilde{\rho}_{k+1}].$$

Despite the fact that this scheme is very natural and essentially not that different from the Main Scheme, we have to be careful with the analysis. First we have to quantify somehow the distance $W_p(\rho^{\tau}_{k}, \tilde{\rho}^{\tau}_{k+1})$ for some $p \geq 1$ and show that this is of order τ in some sense. Second, we need to be careful when performing the convolution with the heat kernel (or adding the Brownian motion, which is the same). This requires working either in the whole space (which was not our framework) or in a periodic setting ($\Omega = \mathbb{T}^d$, the flat torus, which is quite restrictive). Otherwise, the "explicit" convolution step should be replaced with some other construction, such as following the heat equation (with Neumann boundary conditions) for a time τ. But this brings us back to a situation very similar to the Main Scheme, with the additional difficulty that we do not really have estimates on $(\mathrm{id} + \tau u_{k\tau})_{\#}\rho^{\tau}_{k}$.

Variant 2: gradient flow techniques for gradient velocity fields. In this section we assume that the velocity field of the population is given by the opposite of the gradient of a function, $u_t = -\nabla V_t$. A typical example is given when we take for V the distance function to the exit (see the discussions in [Maury et al. 2010] about this type of question). We start from the case where V does not depend on time, and we suppose $V \in W^{1,1}(\Omega)$. In this particular case — beside the splitting approach — the problem has a variational structure, hence it is possible to show the existence by the means of gradient flows in Wasserstein spaces.

Since the celebrated paper of Jordan, Kinderlehrer and Otto [Jordan et al. 1998], we know that the solutions of the Fokker–Planck equation (with a gradient vector field) can be obtained with the help of the gradient flow of a perturbed entropy functional with respect to the Wasserstein distance W_2. This formulation of the Jordan–Kinderlehrer–Otto (JKO) scheme was also used in [Maury et al. 2010] for the first-order model with density constraints. It is easy to combine the JKO scheme with density constraints to study the second-order/diffusive model. As a slight modification of the model from [Maury et al. 2010], we can consider the following discrete implicit Euler (or JKO) scheme. As usual, we fix a time step $\tau > 0$, $\rho^{\tau}_{0} = \rho_0$ and for all $k \in \{1, 2, \ldots, \lfloor N/\tau \rfloor\}$ we just need to define ρ^{τ}_{k+1}. We take

$$\rho^{\tau}_{k+1} = \mathrm{argmin}_{\rho \in \mathcal{P}(\Omega)} \left\{ \int_\Omega V(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho^{\tau}_{k}) \right\}, \tag{5-1}$$

where $I_{\mathcal{K}}$ is the indicator function of $\mathcal{K}$, which is

$$I_{\mathcal{K}}(x) := \begin{cases} 0 & \text{if } x \in \mathcal{K}, \\ +\infty & \text{otherwise.} \end{cases}$$

The usual techniques from [Jordan et al. 1998; Maury et al. 2010] can be used to identify that the system (1-5) is the gradient flow of the functional

$$\rho \mapsto J(\rho) := \int_\Omega V(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho)$$

and that the above discrete scheme converges (up to a subsequence) to a solution of (1-5), thus proving existence. The key estimate for compactness is

$$\frac{1}{2\tau} W_2^2(\rho_{k+1}^\tau, \rho_k^\tau) \le J(\rho_k^\tau) - J(\rho_{k+1}^\tau),$$

which can be summed up (as on the right-hand side we have a telescopic series), thus obtaining the same bounds on $\mathcal{B}_2$ that we used in Section 3.

Note that whenever $D^2V \ge \lambda I$, the functional $\rho \mapsto \int_\Omega V(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho)$ is λ-geodesically convex. This allows us to use the theory in [Ambrosio et al. 2008] to prove not only existence, but also uniqueness for this equation, and even stability (contractivity or exponential growth on the distance between two solutions) in $\mathcal{W}_2$. Yet, we underline that the techniques of [Di Marino and Mészáros 2016] also give the same result. Indeed, that article contains two parts. In the first part, the equation with density constraints for a given velocity field u is studied, under the assumption that $-u$ has some monotonicity properties: $(-u_t(x) + u_t(y)) \cdot (x - y) \ge \lambda |x - y|^2$ (which is the case for the gradients of λ-convex functions). In this case standard Grönwall estimates on the W_2 distance between two solutions are proved, and it is not difficult to add diffusion to that result (as the heat kernel is already contractant in $\mathcal{W}_2$). In the second part, via different techniques (mainly using the adjoint equation, and proving somehow L^1 contractivity), the uniqueness result is provided for arbitrary L^∞ vector fields u, but with the crucial help of the diffusion term in the equation.

It is also possible to study a variant where V depends on time. We assume for simplicity that $V \in \mathrm{Lip}([0, T] \times \Omega)$ (this is a simplification; less regularity in space, such as $W^{1,1}$, could be sufficient). In this case we define

$$J_t(\rho) := \int_\Omega V_t(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho),$$

$$\rho_{k+1}^\tau = \mathrm{argmin}_{\rho \in \mathcal{P}(\Omega)} \Big\{ J_{k\tau}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_k^\tau) \Big\}. \tag{5-2}$$

The analysis proceeds similarly, with the only exception being that we get

$$\frac{1}{2\tau} W_2^2(\rho_{k+1}^\tau, \rho_k^\tau) \le J_{k\tau}(\rho_k^\tau) - J_{k\tau}(\rho_{k+1}^\tau),$$

which is no longer a telescopic series. Yet, we have $J_{k\tau}(\rho_{k+1}^\tau) \ge J_{(k+1)\tau}(\rho_{k+1}^\tau) + \mathrm{Lip}(V)\tau$, and we can go on with a telescopic sum plus a remainder of the order of τ. In the case where u_t is the opposite of the gradient of a λ-convex function V_t, one could consider approximation by functions which are piecewise constant in time and use the standard theory of gradient flows.

We remark here that [Alexander et al. 2014] gave another approach for dealing with first-order crowd motion models as limits of nonlinear-diffusion equations with gradient drift. This approach could plausibly be used also in the case where we add a simple diffusion term to the models studied in that paper.

Variant 3: transport then gradient flow-like step with the penalized entropy functional. We present now a different scheme, which combines some of the previous approaches. It could formally provide a solution of the same equation, but presents some extra difficulties.

We define now $\tilde{\rho}^\tau_{k+1} := (\mathrm{id} + \tau u_{k\tau})_\# \rho^\tau_k$ and with the help of this we define

$$\rho^\tau_{k+1} := \operatorname{argmin}_{\rho\in\mathcal{K}} \mathcal{E}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \tilde{\rho}^\tau_{k+1}).$$

In the last optimization problem we minimize strictly convex and lower semicontinuous functionals, and hence we have existence and uniqueness of the solution. The formal reason for this scheme being adapted to the equation is that we perform a step of a JKO scheme in the spirit of [Jordan et al. 1998] (without the density constraint) or of [Maury et al. 2010] (without the entropy term). This should let a term $-\Delta\rho - \nabla\cdot(\rho\nabla p)$ appear in the evolution equation. The term $\nabla\cdot(\rho u)$ is due to the first step (the definition of $\tilde{\rho}^\tau_{k+1}$). To explain a little bit more for the unexperienced reader, we consider the optimality conditions for the above minimization problem. Following [Maury et al. 2010], we can say that $\rho \in \mathcal{K}$ is optimal if and only if there exists a constant $\ell \in \mathbb{R}$ and a Kantorovich potential φ for the transport from ρ to ρ^τ_k such that

$$\rho = \begin{cases} 1 & \text{on } (\ln\rho + \varphi/\tau) < \ell, \\ 0 & \text{on } (\ln\rho + \varphi/\tau) > \ell, \\ \in [0, 1] & \text{on } (\ln\rho + \varphi/\tau) = \ell. \end{cases}$$

We then define $p = (\ell - \ln\rho - \varphi/\tau)_+$ and we get $p \in \mathrm{press}(\rho)$. Moreover, ρ-a.e., $\nabla p = -\nabla\rho/\rho - \nabla\varphi/\tau$. We then use the fact that the optimal transport is of the form $T = \mathrm{id} - \nabla\varphi$ and obtain a situation as sketched in Figure 2.

Notice that

$$(\mathrm{id} + \tau u_{k\tau})^{-1} \circ (\mathrm{id} + \tau(\nabla p + \nabla\rho/\rho)) = \mathrm{id} - \tau(u_{(k+1)\tau} - \nabla p - \nabla\rho/\rho) + o(\tau)$$

provided u is regular enough. Formally we can pass to the limit $\tau \to 0$ and have

$$\partial_t\rho - \Delta\rho + \nabla\cdot(\rho(u - \nabla p)) = 0.$$

Yet, this turns out to be quite naïve, because we cannot get proper estimates on $W_2(\rho^\tau_k, \rho^\tau_{k+1})$. Indeed, this is mainly due to the hybrid nature of the scheme, i.e., a gradient flow for the diffusion and the projection part on the one hand and a free transport on the other hand. The typical estimate in the JKO scheme

$$\begin{tikzcd} & \tilde{\rho}^\tau_{k+1} & \\ \rho^\tau_k \arrow[ur, "\mathrm{id}+\tau u_{k\tau}"] & & \rho^\tau_{k+1} \arrow[ul, "\mathrm{id}+\tau(\nabla p+\frac{\nabla\rho}{\rho})"'] \arrow[ll, dashed, "\mathrm{id}-\tau\left(u_{(k+1)\tau}-\nabla p-\frac{\nabla\rho}{\rho}\right)+o(\tau)"] \end{tikzcd}$$

Figure 2. One time step.

comes from the fact that one can bound $W_2(\rho_k^\tau, \rho_{k+1}^\tau)^2/\tau$ with the opposite of the increment of the energy, and that this gives rise to a telescopic sum. Yet, this is not the case whenever the base point for a new time step is not equal to the previous minimizer. Moreover, the main difficulty here is the fact that the energy we consider implicitly takes the value $+\infty$, due to the constraint $\rho \in \mathcal{K}$, and hence no estimate is possible whenever $\tilde{\rho}_{k+1}^\tau \notin \mathcal{K}$. As a possible way to overcome this difficulty, one could approximate the discontinuous functional $I_{\mathcal{K}}$ with some finite energies of the same nature (for instance power-like entropies, even if the best choice would be an energy which is Lipschitz for the distance W_2). These kinds of difficulties are a matter of current study, in particular for mixed systems and/or multiple populations.

Appendix: BV-type estimates for the Fokker–Planck equation

Here we present some total variation (TV) decay results (in time) for the solutions of the Fokker–Planck equation. Some are very easy, some trickier. The goal is to look at those estimates which can be easily iterated in time and combined with the decay of the TV via the projection operator, as we did in Section 4.

Let us take a vector field $v : [0, +\infty[\times \Omega \to \mathbb{R}^d$ (we will choose later which regularity we need) and consider in Ω the problem

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t v_t) = 0 & \text{in }]0, +\infty[\times \Omega, \\ \rho_t (\nabla \rho_t - v_t) \cdot n = 0 & \text{on } [0, +\infty[\times \partial\Omega, \\ \rho(0, \cdot) = \rho_0 & \text{in } \Omega, \end{cases} \tag{A-1}$$

for $\rho_0 \in \mathrm{BV}(\Omega) \cap \mathcal{P}(\Omega)$.

Lemma A.1. *Suppose $\|v_t\|_{C^{1,1}} \le C$ for all $t \in [0, +\infty[$. Suppose that either $\Omega = \mathbb{T}^d$, or that Ω is convex and $v \cdot n = 0$ on $\partial\Omega$. Then we have the total variation decay estimate*

$$\int_\Omega |\nabla \rho_t| \,\mathrm{d}x \le C(t-s) + e^{C(t-s)} \int_\Omega |\nabla \rho_s| \,\mathrm{d}x \quad \text{for all } 0 \le s \le t, \tag{A-2}$$

where $C > 0$ is a constant depending just on the $C^{1,1}$ norm of v.

Proof. First we remark that by the regularity of v the quantity

$$\|v\|_{L^\infty} + \|Dv\|_{L^\infty} + \|\nabla(\nabla \cdot v)\|_{L^\infty}$$

is uniformly bounded. Let us drop now the dependence on t in our notation and calculate in coordinates

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \rho| \,\mathrm{d}x &= \int_\Omega \frac{\nabla \rho}{|\nabla \rho|} \cdot \nabla(\partial_t \rho) \,\mathrm{d}x \\ &= \int_\Omega \frac{\nabla \rho}{|\nabla \rho|} \cdot \nabla(\Delta \rho - \nabla \cdot (v\rho)) \,\mathrm{d}x = \int_\Omega \sum_j \frac{\rho_j}{|\nabla \rho|} \Big(\sum_i \rho_{iij} - (\nabla \cdot (v\rho))_j \Big) \,\mathrm{d}x \\ &= -\int_\Omega \sum_{i,j,k} \Big(\frac{\rho_{ij}^2}{|\nabla \rho|} - \frac{\rho_j \rho_k \rho_{ki} \rho_{ij}}{|\nabla \rho|^3} \Big) \,\mathrm{d}x + B_1 - \int_\Omega \sum_{j,i} \frac{\rho_j}{|\nabla \rho|} (v^i_{ij} \rho + v^i_i \rho_j + v^i_j \rho_i + v^i \rho_{ij}) \,\mathrm{d}x \\ &\le B_1 + C + C \int_\Omega |\nabla \rho| \,\mathrm{d}x + \int_\Omega |\nabla \rho| |\nabla \cdot v| \,\mathrm{d}x + B_2 \le B_1 + B_2 + C + C \int_\Omega |\nabla \rho| \,\mathrm{d}x. \end{aligned}$$

Here the B_i are the boundary terms, i.e.,

$$B_1 := \int_{\partial\Omega} \sum_{i,j} \frac{\rho_j n^i \rho_{ij}}{|\nabla\rho|} \,\mathrm{d}\mathcal{H}^{d-1} \quad \text{and} \quad B_2 := -\int_{\partial\Omega} (v\cdot n)|\nabla\rho| \,\mathrm{d}\mathcal{H}^{d-1}.$$

The constant $C > 0$ only depends on $\|v\|_{L^\infty} + \|\nabla\cdot v\|_{L^\infty} + \|\nabla(\nabla\cdot v)\|_{L^\infty}$. We used as well the fact that

$$-\int_\Omega \sum_{i,j,k} \left(\frac{\rho_{ij}^2}{|\nabla\rho|} - \frac{\rho_j\rho_k\rho_{ki}\rho_{ij}}{|\nabla\rho|^3} \right) \mathrm{d}x \le 0.$$

Now, it is clear that in the case of the torus the boundary terms B_1 and B_2 do not exist, hence we have the desired conclusion by Grönwall's lemma. In the case of the convex domain we have $B_2 = 0$ (because of the assumption $v\cdot n = 0$) and $B_1 \le 0$ because of the next lemma. □

Lemma A.2. *Suppose that* $u : \Omega \to \mathbb{R}^d$ *is a smooth vector field with* $u\cdot n = 0$ *on* $\partial\Omega$, ρ *is a smooth function with* $\nabla\rho\cdot n = 0$ *on* $\partial\Omega$ *and* $\Omega \subset \mathbb{R}^d$ *is a smooth convex set that we write as* $\Omega = \{h < 0\}$ *for a smooth convex function h with* $|\nabla h| = 1$ *on* $\partial\Omega$ (*so that* $n = \nabla h$ *on* $\partial\Omega$). *Then we have*, *on the whole boundary* $\partial\Omega$, $\sum_{i,j} u^i_j \rho_j n^i = -\sum_{i,j} u^i h_{ij}\rho_j$.

In particular, *we have* $\sum_{i,j} \rho_{ij}\rho_j n^i \le 0$.

Proof. The Neumann boundary assumption on u means $u(\gamma(t))\cdot\nabla h(\gamma(t)) = 0$ for every curve γ valued in $\partial\Omega$ and for all t. Differentiating in t, we get

$$\sum_{i,j} u^i_j(\gamma(t))(\gamma'(t))^j h_i(\gamma(t)) + \sum_{i,j} u^i(\gamma(t)) h_{ij}(\gamma(t))(\gamma'(t))^j = 0.$$

Take a point $x_0 \in \partial\Omega$ and choose a curve γ with $\gamma(t_0) = x_0$ and $\gamma'(t_0) = \nabla\rho(x_0)$ (which is possible, since this vector is tangent to $\partial\Omega$ by assumption). This gives the first part of the statement. The second part, i.e., $\sum_{i,j} \rho_{ij}\rho_j n^i \le 0$, is obtained by taking $u = \nabla\rho$ and using that $D^2 h(x_0)$ is a positive definite matrix. □

Remark A.3. If we look attentively at the proof of Lemma A.1, we can see that we did not really exploit the regularizing effects of the diffusion term in the equation. This means that the given regularity estimate is the same that we would have without diffusion; in this case, the density ρ_t is obtained from the initial density as the image through the flow of v. Thus, the density depends on the determinant of the Jacobian of the flow, hence on the derivatives of v. It is normal that, if we want BV bounds on ρ_t, we need assumptions on two derivatives of v.

We would like to prove some form of BV estimates under weaker regularity assumptions on v, trying to exploit the diffusion effects. In particular, we would like to treat the case where v is only $C^{0,1}$. As we will see in the following lemma, this degenerates in some sense.

Lemma A.4. *Suppose that* Ω *is either the torus or a smooth convex set* $\Omega = \{h < 0\}$ *parametrized as a level set of a smooth convex function h. Let* $v_t : \Omega \to \mathbb{R}^d$ *be a vector field for* $t \in [0, T]$, *Lipschitz and bounded in space*, *uniformly in time. In the case of a convex domain*, *suppose* $v\cdot n = 0$ *on* $\partial\Omega$. *Let* $H : \mathbb{R}^d \to \mathbb{R}$ *be given by* $H(z) := \sqrt{\varepsilon^2 + |z|^2}$. *Now let* ρ_t *be the* (*sufficiently smooth*) *solution of the Fokker–Planck equation with homogeneous Neumann boundary condition.*

Then there exists a constant $C > 0$ (depending on v and Ω) such that

$$\int_\Omega H(\nabla\rho_t)\,\mathrm{d}x \le \int_\Omega H(\nabla\rho_0)\,\mathrm{d}x + C\varepsilon t + \frac{C}{\varepsilon}\int_0^t \|\rho_s\|^2_{L^\infty}\,\mathrm{d}s. \tag{A-3}$$

Proof. First let us discuss some properties of H. It is smooth, its gradient is $\nabla H(z) = z/H(z)$ and it satisfies $\nabla H(z)\cdot z \le H(z)$ for all $z \in \mathbb{R}^d$. Moreover its Hessian matrix is given by

$$[H_{ij}(z)]_{i,j\in\{1,\dots,d\}} = \left[\frac{\delta^{ij}H^2(z) - z^i z^j}{H^3(z)}\right]_{i,j\in\{1,\dots,d\}} = \frac{1}{H(z)}I_d - \frac{1}{H^3(z)}z\otimes z \quad \forall z\in\mathbb{R}^d,$$

where

$$\delta^{ij} = \begin{cases} 1 & \text{if } i = j,\\ 0 & \text{if } i \ne j,\end{cases}$$

is the Kronecker symbol. Note that, from this computation, the matrix $D^2H \ge 0$ is bounded from above by $1/H$, and hence by ε^{-1}. Moreover we introduce a uniform constant $C > 0$ such that

$$\|v\|^2_{L^\infty}|\Omega| + \|\nabla\cdot v\|_{L^\infty} + \|Dv\|_{L^\infty} \le C.$$

Now to show the estimate of this lemma we calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega H(\nabla\rho_t)\,\mathrm{d}x &= \int_\Omega \nabla H(\nabla\rho_t)\cdot\partial_t\nabla\rho_t\,\mathrm{d}x = \int_\Omega \nabla H(\nabla\rho_t)\cdot\nabla(\Delta\rho_t - \nabla\cdot(v_t\rho_t))\,\mathrm{d}x\\ &= \int_\Omega \nabla H(\nabla\rho_t)\cdot\nabla\Delta\rho_t\,\mathrm{d}x - \int_\Omega \nabla H(\nabla\rho_t)\cdot\nabla(\nabla\cdot(v_t\rho_t))\,\mathrm{d}x\\ &=: (I) + (II)\end{aligned}$$

Now we study each term separately and for simplicity we drop the t subscripts in the following. We start with the case of the torus, where there is no boundary term in the integration by parts:

$$\begin{aligned}(I) &= \int_\Omega \nabla H(\nabla\rho)\cdot\nabla\Delta\rho\,\mathrm{d}x = \int_\Omega\sum_{j,i} H_j(\nabla\rho)\rho_{jii}\,\mathrm{d}x = -\int_\Omega\sum_{j,i,k} H_{kj}(\nabla\rho)\rho_{ik}\rho_{ji}\,\mathrm{d}x,\\ (II) &= -\int_\Omega \nabla H(\nabla\rho)\cdot\nabla(\nabla\cdot(v\rho))\,\mathrm{d}x = -\int_\Omega\sum_{i,j} H_j(\nabla\rho)(v^i\rho)_{ij}\,\mathrm{d}x\\ &= \int_\Omega\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}v^i_j\rho\,\mathrm{d}x + \int_\Omega\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}v^i\rho_j\,\mathrm{d}x\\ &=: (II_a) + (II_b).\end{aligned}$$

First look at the term (II_a). Since the matrix H_{jk} is positive definite, we can apply a Young inequality for each index i and obtain

$$\begin{aligned}(II_a) = \int_\Omega\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}v^i_j\rho\,\mathrm{d}x &\le \frac{1}{2}\int_\Omega\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}\rho_{ij}\,\mathrm{d}x + \frac{1}{2}\int_\Omega\sum_{i,j,k} H_{jk}(\nabla\rho)v^i_j v^i_k\rho^2\,\mathrm{d}x\\ &\le \frac{1}{2}|(I)| + C\|\rho\|^2_{L^2}\|D^2H\|_{L^\infty}.\end{aligned}$$

The L^2 norm in the second term will be estimated by the L^∞ norm for the sake of simplicity (see Remark A.5 below).

For the term (II_b) we first make a pointwise computation,

$$\begin{aligned}\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}v^i\rho_j &= \frac{1}{H^3(\nabla\rho)}\sum_i \big(D_i^2\rho\cdot(\varepsilon^2 I_d+|\nabla\rho|^2 I_d-\nabla\rho\otimes\nabla\rho)\cdot\nabla\rho\big)v^i\\ &= \frac{\varepsilon^2}{H^3(\nabla\rho)}\sum_i v^i D_i^2\rho\cdot\nabla\rho = -\varepsilon^2\sum_i v^i\partial_i\left(\frac{1}{H(\nabla\rho)}\right),\end{aligned}$$

where $D_i^2\rho$ denotes the i-th row in the Hessian matrix of ρ, and we use $(|\nabla\rho|^2 I_d-\nabla\rho\otimes\nabla\rho)\cdot\nabla\rho=0$.

Integrating by parts, we obtain

$$(II_b)=\varepsilon^2\int_\Omega(\nabla\cdot v)\frac{1}{H(\nabla\rho)}\,\mathrm{d}x\le C\varepsilon^2\|1/H\|_{L^\infty}\le C\varepsilon,$$

where we use $H(z)\ge\varepsilon$.

Summing up all the terms and using $\|D^2H\|\le\varepsilon^{-1}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega H(\nabla\rho_t)\,\mathrm{d}x\le-\frac{1}{2}|(I)|+C\|\rho_t\|_{L^\infty}^2\|D^2H\|_{L^\infty}+C\varepsilon\le C\varepsilon+C\|\rho_t\|_{L^\infty}^2\varepsilon^{-1},$$

which proves the claim.

If we switch to the case of a smooth bounded convex domain Ω, we have to handle boundary terms. These terms are

$$\int_{\partial\Omega}\sum_{i,j}H_j(\nabla\rho)\rho_{ij}n^i-\int_{\partial\Omega}\sum_{i,j}H_j(\nabla\rho)\rho v_j^i n^i,$$

where we ignore those terms which involve n^iv^i (i.e., the integration by parts in (II_b), and the term $H_j(\nabla\rho)\rho_j n^iv^i$ in the integration by parts of (II_a)), since we have already supposed $v\cdot n=0$. We use here Lemma A.2, which provides

$$\begin{aligned}\sum_{i,j}H_j(\nabla\rho)\rho_{ij}n^i-\rho H_j(\nabla\rho)v_j^in^i &= \frac{1}{H(\nabla\rho)}\sum_{i,j}(\rho_j\rho_{ij}n^i-\rho\rho_jv_j^in^i)\\ &= -\frac{1}{H(\nabla\rho)}\sum_{i,j}(\rho_jh_{ij}\rho_i-\rho\rho_jh_{ij}v^i).\end{aligned}$$

If we use the fact that the matrix D^2h is positive definite and a Young inequality, we get $\sum_{i,j}\rho_jh_{ij}\rho_i\ge0$ and

$$\rho\sum_{i,j}|\rho_jh_{ij}v^i|\le\frac{1}{2}\sum_{i,j}\rho_jh_{ij}\rho_i+\frac{1}{2}\sum_{i,j}\rho^2v^jh_{ij}v^i,$$

which implies

$$\frac{1}{H(\nabla\rho)}\sum_{i,j}(\rho_j\rho_{ij}n^i-\rho\rho_jv_j^in^i)\le\frac{\rho^2}{H(\nabla\rho)}\|D^2h\|_{L^\infty}|v|^2\le\frac{C\|\rho\|_{L^\infty}^2}{\varepsilon}.$$

This provides the desired estimate on the boundary term. □

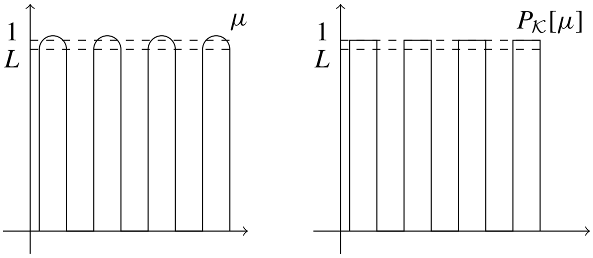

Figure 3. A counterexample to the decay of $\int_\Omega H(\nabla\rho)$, which corresponds to the total length of the graph.

Remark A.5. In the above proof, we needed to use the L^∞ norm of ρ only in the boundary term. When there is no boundary term, the L^2 norm is enough to handle the term (II_a). In both cases, the norm of ρ can be bounded in terms of the initial norm multiplied by e^{Ct}, where C bounds the divergence of v. On the other hand, in the torus case, one only needs to suppose $\rho_0 \in L^2$ and in the convex case $\rho_0 \in L^\infty$. Both assumptions are satisfied in the applications to crowd motion with density constraints.

We have seen that the constants in the above inequality depend on ε and explode as $\varepsilon \to 0$. This prevents us from obtaining a clean estimate on the BV norm in this context, but at least it proves that $\rho_0 \in \mathrm{BV} \Rightarrow \rho_t \in \mathrm{BV}$ for all $t > 0$ (to achieve this result, we just need to take $\varepsilon = 1$). Unfortunately, the quantity which is estimated is not the BV norm, but the integral $\int_\Omega H(\nabla\rho)$. This is not enough for the applications to Section 4, as it is unfortunately not true that the projection operator decreases the value of this other functional. (Here is a simple counterexample. Consider $\mu = g(x)\,\mathrm{d}x$ a BV density on $[0,2] \subset \mathbb{R}$, with g defined as follows. Divide the interval $[0,2]$ into $2K$ intervals J_i of length $2r$ (with $2rK = 1$); call t_i the center of each interval J_i (i.e., $t_i = i2r + r$, for $i = 0, \dots, 2K-1$) and set $g(x) = L + \sqrt{r^2 - (x-t_i)^2}$ on each J_i with i odd, and $g(x) = 0$ on J_i for i even, taking $L = 1 - \pi r/4$. It is not difficult to check that the projection of μ is equal to the indicator function of the union of all the intervals J_i with i odd, and that the value of $\int H(\nabla\rho)$ has increased by $K(2-\pi/2)r = 1 - \pi/4$, i.e., by a positive constant. See Figure 3.)

If we pursue the value of the BV norm, we can provide the following estimate.

Lemma A.6. *Under the assumptions of Lemma A.4, if we suppose* $\rho_0 \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, *then, for* $t \le T$, *we have*

$$\int_\Omega |\nabla\rho_t|\,\mathrm{d}x \le \int_\Omega |\nabla\rho_0|\,\mathrm{d}x + C\sqrt{t}, \tag{A-4}$$

where the constant C *depends on* v, *on* T *and on* $\|\rho_0\|_{L^\infty}$.

Proof. Using the L^∞ estimate of Lemma 4.2, we will assume that $\|\rho_t\|_{L^\infty}$ is bounded by a constant (which depends on v, on T and on $\|\rho_0\|_{L^\infty}$). Then, we can write

$$\int_\Omega |\nabla\rho_t|\,\mathrm{d}x \le \int_\Omega H(\nabla\rho_t)\,\mathrm{d}x \le \int_\Omega H(\nabla\rho_0)\,\mathrm{d}x + C\varepsilon t + \frac{Ct}{\varepsilon} \le \int_\Omega (|\nabla\rho_0| + \varepsilon)\,\mathrm{d}x + C\varepsilon t + \frac{Ct}{\varepsilon}.$$

It is sufficient to choose, for fixed t, $\varepsilon = \sqrt{t}$, in order to prove the claim. □

Unfortunately, this $\sqrt{t}$ behavior is not suitable to be iterated, and the above estimate is useless for the sake of Section 4. The existence of an estimate (for v Lipschitz) of the form $\mathrm{TV}(\rho_t) \le \mathrm{TV}(\rho_0) + Ct$, or $\mathrm{TV}(\rho_t) \le \mathrm{TV}(\rho_0)e^{Ct}$, or even $f(\mathrm{TV}(\rho_t)) \le f(\mathrm{TV}(\rho_0))e^{Ct}$ for any increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$, seems to be an open question.

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