ANALYSIS & PDE

Volume 9 No. 3 2016

SATOSHI MASAKI AND JUN-ICHI SEGATA

ON THE WELL-POSEDNESS OF THE GENERALIZED KORTEWEG–DE VRIES EQUATION IN SCALE-CRITICAL $\hat{L}^r$-SPACE

msp

ANALYSIS AND PDE
Vol. 9, No. 3, 2016
dx.doi.org/10.2140/apde.2016.9.699

ON THE WELL-POSEDNESS OF THE GENERALIZED KORTEWEG–DE VRIES EQUATION IN SCALE-CRITICAL $\hat{L}^r$-SPACE

SATOSHI MASAKI AND JUN-ICHI SEGATA

The purpose of this paper is to study local and global well-posedness of the initial value problem for the generalized Korteweg–de Vries (gKdV) equation in $\hat{L}^r = \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{\hat{L}^r} = \|\hat{f}\|_{L^{r'}} < \infty\}$. We show (large-data) local well-posedness, small-data global well-posedness, and small-data scattering for the gKdV equation in the scale-critical $\hat{L}^r$-space. A key ingredient is a Stein–Tomas-type inequality for the Airy equation, which generalizes the usual Strichartz estimates for $\hat{L}^r$-framework.

1. Introduction

We consider the initial value problem for the generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u = \mu \partial_x(|u|^{\alpha-1}u), & t, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1-1}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $u_0 : \mathbb{R} \to \mathbb{R}$ is a given function, and $\mu \in \mathbb{R}\backslash\{0\}$ and $\alpha > 1$ are constants. We say that (1-1) is defocusing if $\mu > 0$ and focusing if $\mu < 0$.

The class of equations (1-1) arises in several fields of physics. Equation (1-1) with $\alpha = 2$ is the notable Korteweg–de Vries equation [1895], which models long waves propagating in a channel. Equation (1-1) with $\alpha = 3$ is also well-known as the modified Korteweg–de Vries equation, which describes a time evolution for the curvature of certain types of helical space curves [Lamb 1977].

Equation (1-1) has the following scale invariance: if $u(t,x)$ is a solution to (1-1), then

$$u_\lambda(t,x) := \lambda^{\frac{2}{\alpha-1}} u(\lambda^3 t, \lambda x)$$

is also a solution to (1-1) with initial data $u_\lambda(0,x) = \lambda^{\frac{2}{\alpha-1}} u_0(\lambda x)$ for any $\lambda > 0$. In what follows, a Banach space for initial data is referred to as a *scale-critical space* if its norm is invariant under $u_0(x) \mapsto \lambda^{\frac{2}{\alpha-1}} u_0(\lambda x)$.

The purpose of this paper is to study (large-data) local well-posedness, small-data global well-posedness and scattering for (1-1) in a scale-critical space $\hat{L}^{\frac{\alpha-1}{2}}$. For $r \in [1,\infty]$, the function space $\hat{L}^r$ is defined by

$$\hat{L}^r = \hat{L}^r(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{\hat{L}^r} = \|\hat{f}\|_{L^{r'}} < \infty\},$$

MSC2010: primary 35Q53, 35B40; secondary 35B30.
Keywords: generalized Korteweg–de Vries equation, scattering problem.

where $\hat{f}$ stands for the Fourier transform of f with respect to the space variable and r' denotes the Hölder conjugate of r. We use the conventions $1' = \infty$ and $\infty' = 1$. Our notion of well-posedness consists of existence, uniqueness, and continuity of the data-to-solution map. We also consider the persistence property of the solution; that is, the solution describes a continuous curve in the function space X whenever $u_0 \in X$.

Local well-posedness of the initial value problem (1-1) in a scale-subcritical Sobolev space $H^s(\mathbb{R})$, $s > s_\alpha := \frac{1}{2} - \frac{2}{\alpha-1}$, has been studied by many authors [Bourgain 1993; Grünrock 2005b; Guo 2009; Kato 1983; Kenig et al. 1993; 1996; Kishimoto 2009; Molinet and Ribaud 2003], where s_α, a scale-critical exponent, is the unique number such that $\dot{H}^{s_\alpha}$ becomes scale critical. A fundamental work on local well-posedness is due to Kenig, Ponce, and Vega [Kenig et al. 1993]. They proved that (1-1) is locally well-posed in $H^s(\mathbb{R})$ with $s > \frac{3}{4}$ $\left(\alpha = 2,\ s_2 = -\frac{3}{2}\right)$, $s \geqslant \frac{1}{4}$ $\left(\alpha = 3,\ s_3 = -\frac{1}{2}\right)$, $s \geqslant \frac{1}{12}$ $\left(\alpha = 4,\ s_4 = -\frac{1}{6}\right)$ and $s \geqslant s_\alpha$ $(\alpha \geqslant 5)$. Introducing Fourier restriction norms, Bourgain [1993] obtained local (and global[1]) well-posedness of the KdV equation (i.e., (1-1) with $\alpha = 2$) in $L^2(\mathbb{R})$. In [Kenig et al. 1996], Kenig, Ponce, and Vega improved the previous results for the KdV equation to $H^s(\mathbb{R})$ with $s > -\frac{3}{4}$. Further, Guo [2009] and Kishimoto [2009] extended the result of Kenig et al. in $H^{-\frac{3}{4}}(\mathbb{R})$. (See also [Buckmaster and Koch 2015] on the existence of a weak solution to the KdV equation at H^{-1}.) Grünrock [2005b] has shown local well-posedness of the quartic KdV equation ((1-1) with $\alpha = 4$) in H^s with $s > s_4$. Notice that all of the above results are based on the contraction mapping principle for the corresponding integral equation. Hence, a data-solution map associated with (1-1) is Lipschitz continuous.[2]

Concerning the well-posedness of (1-1) in the scale-critical $\dot{H}^{s_\alpha}$-space, Kenig et al. [1993] proved local well-posedness and global well-posedness for small data in the scale-critical space $\dot{H}^{s_\alpha}$ when $\alpha \geqslant 5$. Since the scale-critical exponent s_α is negative in the mass-subcritical case $\alpha < 5$, the well-posedness of (1-1) in $\dot{H}^{s_\alpha}$ becomes rather a difficult problem. Tao [2007] proved local well-posedness and global well-posedness for small data for (1-1) with the quartic nonlinearity[3] $\alpha = 4$ in $\dot{H}^{s_4}$. Later on, the above results were extended to a homogeneous Besov space $\dot{B}^{s_\alpha}_{2,\infty}$ by Koch and Marzuola [2012] $(\alpha = 4)$ and Strunk [2014] $(\alpha \geqslant 5)$. As far as we know, local well-posedness and small-data global well-posedness of (1-1) in $\dot{H}^{s_\alpha}$ for the mass-subcritical case $\alpha < 5$ were open except for the case $\alpha = 4$.

Local and global well-posedness for a class of nonlinear dispersive equations is currently being intensively investigated also in the framework of $\hat{L}^r$-space. For the one-dimensional nonlinear Schrödinger equation,

$$\begin{cases} i\partial_t v - \partial_x^2 v = \mu |v|^{\alpha-1} v, & t, x \in \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1-2}$$

where $\mu \in \mathbb{R}\backslash\{0\}$, Grünrock [2005a] has shown local and global well-posedness for (1-2) with $\alpha = 3$ in $\hat{L}^r$. Hyakuna and Tsutsumi [2012] extended Grünrock's result in $\hat{L}^r$ to all mass-subcritical cases $1 < \alpha < 5$. Grünrock and Vega [Grünrock 2004; Grünrock and Vega 2009] proved local well-posedness

[1] Since (1-1) preserves the L^2-norm of a solution in t, local well-posedness in L^2 yields global well-posedness in L^2 if $\alpha < 5$.

[2] In fact, if the nonlinear term is analytic, then the data-solution map associated with (1-1) is analytic.

[3] Strictly speaking, the local well-posedness is shown not for $\mu\partial_x(|u|^3 u)$ but for $\mu\partial_x(u^4)$. These two are not necessarily equivalent.

for the modified KdV equation (i.e., (1-1) with $\alpha = 3$) in $\hat{H}^r_s$, where

$$\hat{H}^r_s = \{ f \in \mathcal{S}' : \|f\|_{\hat{H}^r_s} = \|(1+\xi^2)^{\frac{s}{2}} \hat{f}(\xi)\|_{L^{r'}_\xi} < \infty \}.$$

However, the above results are not in scale-critical settings.

It would be interesting to compare the scale-critical space $\hat{L}^{\frac{\alpha-1}{2}}$ with some other scale-critical spaces in view of symmetries.[4] Other than the scaling, the $\hat{L}^{\frac{\alpha-1}{2}}$-norm is invariant under the three group operations

(i) translation in physical space, $(T_a f)(x) = f(x-a)$, where $a \in \mathbb{R}$,

(ii) translation in Fourier space, $(P_\xi f)(x) = e^{-ix\xi} f(x)$, where $\xi \in \mathbb{R}$,

(iii) Airy flow, $(\mathrm{Ai}(t) f)(x) = e^{-t\partial_x^3} f(x)$, where $t \in \mathbb{R}$.

The critical Lebesgue space $L^{\frac{\alpha-1}{2}}$ is invariant under the former two symmetries but not under the Airy flow. The critical Sobolev space $\dot{H}^{s_\alpha}$ (or homogeneous Triebel–Lizorkin and homogeneous Besov spaces $\dot{A}^{s_\alpha}_{2,q}$, with $1 \leqslant q \leqslant \infty$, more generally) is not invariant with respect to P_ξ if $s_\alpha \neq 0$. The critical weighted Lebesgue space $\dot{H}^{0,-s_\alpha} := L^2(\mathbb{R}, |x|^{-2s_\alpha}\,dx)$ is not invariant with respect to T_a and $\mathrm{Ai}(t)$. Further, when $\alpha = 5$, these four spaces coincide with L^2, which is invariant under the above three symmetries. Thus, among the above four critical spaces, $\hat{L}^{\frac{\alpha-1}{2}}$ possesses the richest symmetries, and, in some sense, $\hat{L}^{\frac{\alpha-1}{2}}$ is close to L^2-space. Inclusion relations between these spaces are summarized in Appendix B.

Local well-posedness. Before we state our main results, we introduce several notations.

Definition 1.1. Let $(s,r) \in \mathbb{R} \times [1,\infty]$. A pair (s,r) is said to be *acceptable* if $\frac{1}{r} \in \left[0, \frac{3}{4}\right)$ and

$$s \in \begin{cases} \left[-\frac{1}{2r}, \frac{2}{r}\right] & \text{if } 0 \leqslant \frac{1}{r} \leqslant \frac{1}{2}, \\ \left(\frac{2}{r} - \frac{5}{4}, \frac{5}{2} - \frac{3}{r}\right) & \text{if } \frac{1}{2} < \frac{1}{r} < \frac{3}{4}. \end{cases}$$

For an interval $I \subset \mathbb{R}$ and an acceptable pair (s,r), we define a function space $X(I; s, r)$ of space-time functions with the norm

$$\|f\|_{X(I;s,r)} = \left\| |D_x|^s f \right\|_{L_x^{p(s,r)}(\mathbb{R}; L_t^{q(s,r)}(I))},$$

where the exponents in the above norm are given by

$$\frac{2}{p(s,r)} + \frac{1}{q(s,r)} = \frac{1}{r}, \quad -\frac{1}{p(s,r)} + \frac{2}{q(s,r)} = s, \tag{1-3}$$

or equivalently,

$$\begin{pmatrix} \frac{1}{p(s,r)} \\ \frac{1}{q(s,r)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} s \\ \frac{1}{r} \end{pmatrix}.$$

We refer to $X(I; s, r)$ as an *$\hat{L}^r$-admissible space*.

Our main theorems are as follows.

[4]Here, a symmetry is an isometric bijection which possesses a group structure. Some of them are also "symmetries of (1-1)" in such a sense that an image of a solution of (1-1) again solves the equation.

Theorem 1.2 (local well-posedness in $\hat{L}^{\frac{\alpha-1}{2}}$). *For* $\frac{21}{5} < \alpha < \frac{23}{3}$, *the problem* (1-1) *is locally well-posed in* $\hat{L}^{\frac{\alpha-1}{2}}$. *Namely, for any* $u_0 \in \hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})$, *there exists an interval* $I = I(u_0)$ *such that a unique solution*

$$u \in C(I; \hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})) \cap \bigcap_{(s,\frac{\alpha-1}{2})\ \text{acceptable}} X\Big(I; s, \frac{\alpha-1}{2}\Big) \tag{1-4}$$

to (1-1) *exists. Furthermore, for any given subinterval* $I' \subset I$, *there exists a neighborhood* V *of* u_0 *in* $\hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})$ *such that the map* $u_0 \mapsto u$ *from* V *into the class defined by* (1-4) *with* I' *instead of* I *is Lipschitz continuous.*

Remark 1.3. Theorem 1.2 (and all results below) holds for more general nonlinearity of the form $\partial_x G(u)$ with $G \in \operatorname{Lip}\alpha$. For the precise condition on G, see Remark 3.5.

The proof of Theorem 1.2 is based on a contraction argument, with the help of a space-time estimate for the Airy equation in $\hat{L}^r$. A key ingredient is a Stein–Tomas-type inequality for the Airy equation, a special case of [Grünrock 2004, Corollary 3.6]:

$$\big\| |D_x|^{\frac{1}{r}} e^{-t\partial_x^3} f \big\|_{L^r_{t,x}(I\times\mathbb{R})} \leqslant C \|f\|_{\hat{L}^{\frac{r}{3}}}, \tag{1-5}$$

where $r \in (4, \infty]$. This inequality is a generalization of a well-known Strichartz estimate,

$$\big\| |D_x|^{\frac{1}{6}} e^{-t\partial_x^3} f \big\|_{L^6_{t,x}(I\times\mathbb{R})} \leqslant C \|f\|_{L^2}.$$

Moreover, interpolations between the above Stein–Tomas-type inequality (1-5) and the Kenig–Ruiz estimate or Kato's local smoothing effect give us the following generalized Strichartz estimate for the Airy equation in $\hat{L}^r$-framework (Proposition 2.1): if (s, r) is an acceptable pair then there exists C such that

$$\|e^{-t\partial_x^3} f\|_{X(\mathbb{R};s,r)} \leqslant C \|f\|_{\hat{L}^r} \tag{1-6}$$

for $f \in \hat{L}^r$. Furthermore, combining the homogeneous estimate and the Christ–Kiselev lemma (Lemma 2.6), we also obtain a generalized version of inhomogeneous Strichartz estimates. The estimate (1-5) can be regarded as a kind of restriction estimate of the Fourier transform, which goes back to Stein [Fefferman 1970] and Tomas [1975] (for more information on the restriction theorem, see, e.g., [Tao et al. 1998]). It is worth mentioning that the $\hat{L}^r$-spaces have naturally come out in this context.

We set $S(I; r) := X(I; 0, r)$. The $S(I; r)$-norm is the so-called *scattering norm*. It is understood that a key for obtaining a closed estimate for the corresponding integral equation, from which local well-posedness immediately follows, is to bound the scattering norm $S\big(I; \frac{\alpha-1}{2}\big)$. In the proof of Theorem 1.2, the scattering norm is handled by means of the above generalized Strichartz estimate (1-6). Notice that the pair $\big(0, \frac{\alpha-1}{2}\big)$ is acceptable only if $\alpha > \frac{21}{5}$, which leads to our restriction. For the upper bound on α, see Remark 4.1 below. Alternatively, Sobolev's embedding also yields a bound on the scattering norm, provided $\alpha \geqslant 5$. In such case, we obtain local well-posedness in $\dot{H}^{s_\alpha}$ as in [Kenig et al. 1993] (see Remark 4.4).

Persistence of regularity. We establish two persistence-of-regularity-type results for $\hat{L}^{\frac{\alpha-1}{2}}$-solutions given in Theorem 1.2. More specifically, we consider persistence of $\hat{L}^r$-regularity for $r \neq \frac{\alpha-1}{2}$ and $\dot{H}^s$-regularity for $-1 < s < \alpha$. These results yield local well-posedness in other $\hat{L}^r$-like spaces such as $\hat{L}^{r_1} \cap \hat{L}^{r_2}$, where $r_1 \leqslant \frac{\alpha-1}{2} \leqslant r_2$, and $\dot{H}^s \cap \hat{L}^{\frac{\alpha-1}{2}}$.

Theorem 1.4 (persistence of $\hat{L}^r$-regularity). *Assume $\frac{21}{5} < \alpha < \frac{23}{3}$. Let $u_0 \in \hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})$ and let $u \in C(I; \hat{L}^{\frac{\alpha-1}{2}}(\mathbb{R}))$ be a corresponding solution given in Theorem 1.2. If $u_0 \in \hat{L}_x^{\frac{\alpha_0-1}{2}}$ for some $\frac{21}{5} < \alpha_0 < \frac{23}{3}$, where $\alpha_0 \neq \alpha$, then*

$$u \in C(I; \hat{L}_x^{\frac{\alpha_0-1}{2}}(\mathbb{R})) \cap \bigcap_{(s,\frac{\alpha_0-1}{2})\text{ acceptable}} X\Big(I; s, \frac{\alpha_0-1}{2}\Big).$$

Theorem 1.5 (persistence of $\dot{H}^s$-regularity). *Assume $\frac{21}{5} < \alpha < \frac{23}{3}$. Let $u_0 \in \hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})$ and let $u \in C(I, \hat{L}^{\frac{\alpha-1}{2}}(\mathbb{R}))$ be a corresponding solution given in Theorem 1.2. If $u_0 \in \dot{H}_x^\sigma(\mathbb{R})$ for some $-1 < \sigma < \alpha$,*

$$|D_x|^\sigma u \in C(I; L^2(\mathbb{R})) \cap \bigcap_{(s,2)\text{ acceptable}} X(I; s, 2).$$

As a corollary, we obtain the following well-posedness results.

Corollary 1.6. *We have the following.*

(i) *If $\frac{21}{5} < \alpha < \frac{23}{3}$ then* (1-1) *is locally well-posed in $\hat{L}^{r_1} \cap \hat{L}^{r_2}$ as long as $\frac{8}{5} < r_1 \leqslant \frac{\alpha-1}{2} \leqslant r_2 < \frac{10}{3}$.*

(ii) *If $\frac{21}{5} < \alpha < 5$ then* (1-1) *is locally well-posed in $\dot{H}^{s_\alpha} \cap \hat{L}^{\frac{\alpha-1}{2}}$, where $s_\alpha = \frac{1}{2} - \frac{2}{\alpha-1}$.*

Since $\hat{L}^{\frac{\alpha-1}{2}} \subset \dot{H}^{s_\alpha}$ does not hold (see Lemma B.2), the second is weaker than well-posedness in $\dot{H}^{s_\alpha}$.

Here we remark that an $\hat{L}^{\frac{\alpha-1}{2}}$-solution has conserved quantities, provided the solution has appropriate regularity. More precisely, when $u_0 \in \hat{L}^{\frac{\alpha-1}{2}} \cap L^2$, a solution $u(t)$ has a conserved mass

$$M[u(t)] := \|u(t)\|_{L^2}^2.$$

Similarly, if $u_0 \in \hat{L}^{\frac{\alpha-1}{2}} \cap \dot{H}^1$ then the energy

$$E[u(t)] := \tfrac{1}{2}\|\partial_x u(t)\|_{L^2}^2 + \frac{\mu}{\alpha+1}\|u(t)\|_{L^{\alpha+1}}^{\alpha+1}$$

is invariant.

Blowup and scattering. We next consider long time behavior of solutions given in Theorem 1.2. To this end, we give the definitions of blowup and scattering of (1-1) for the initial data $u_0 \in \hat{L}_x^r$. Set

$$T_{\max} := \sup\big\{T > 0 : \text{the solution } u \text{ to (1-1) can be extended to } [0, T)\big\},$$
$$T_{\min} := \sup\big\{T > 0 : \text{the solution } u \text{ to (1-1) can be extended to } (-T, 0]\big\}.$$

Denote the lifespan of $u(t)$ as $(-T_{\min}, T_{\max})$. We say a solution $u(t)$ blows up in finite time for positive (resp. negative) time direction if $T_{\max} < +\infty$ (resp. $T_{\min} < +\infty$). We say a solution $u(t)$ scatters for positive time direction if $T_{\max} = +\infty$ and there exists a unique function $u_+ \in \hat{L}_x^r$ such that

$$\lim_{t\to+\infty} \|u(t) - e^{-t\partial_x^3} u_+\|_{\hat{L}_x^r} = 0,$$

where $e^{-t\partial_x^3} u_+$ is a solution to the Airy equation $\partial_t v + \partial_x^3 v = 0$ with initial condition $v(0, x) = u_+$. The scattering of u for negative time direction is defined in a similar fashion.

Roughly speaking, a solution scatters if a linear dispersion effect dominates the nonlinear interaction. A typical case is when the data (and the corresponding solution) is small. Here, we state this small-data scattering for (1-1).

Theorem 1.7 (small-data scattering). *Let $\frac{21}{5}<\alpha<\frac{23}{3}$. There exists $\varepsilon_0>0$ such that if $u_0\in\hat{L}_x^{\frac{\alpha-1}{2}}(\mathbb{R})$ satisfies*

$$\|u_0\|_{\hat{L}_x^{(\alpha-1)/2}}\leqslant\varepsilon_0,$$

then the solution $u(t)$ to (1-1) *given in Theorem 1.2 is global in time and scatters for both time directions. Moreover,*

$$\|u\|_{L_t^\infty(\mathbb{R};\hat{L}_x^{(\alpha-1)/2})}+\|u\|_{S(\mathbb{R};\frac{\alpha-1}{2})}\leqslant 2\|u_0\|_{\hat{L}_x^{(\alpha-1)/2}}.$$

We now give criterion for blowup and scattering.

Theorem 1.8 (blowup criterion). *Assume $\frac{21}{5}<\alpha<\frac{23}{3}$. Let $u_0\in\hat{L}^{\frac{\alpha-1}{2}}$ and let $u(t)$ be a corresponding unique solution of* (1-1) *given in Theorem 1.2. If $T_{\max}<\infty$ then*

$$\|u\|_{S([0,T);\frac{\alpha-1}{2})}\to\infty$$

as $T\uparrow T_{\max}$. A similar statement is true for negative time direction.

Theorem 1.9 (scattering criterion). *Assume $\frac{21}{5}<\alpha<\frac{23}{3}$. Let $u_0\in\hat{L}^{\frac{\alpha-1}{2}}$ and let $u(t)$ be a corresponding unique solution of* (1-1) *given in Theorem 1.2. The solution $u(t)$ scatters forward in time if and only if $T_{\max}=+\infty$ and $\|u\|_{S([0,\infty);\frac{\alpha-1}{2})}<\infty$. A similar statement is true for negative time direction.*

Finally, we give a criterion for scattering in terms of the energy. We note that if an $\hat{L}^{\frac{\alpha-1}{2}}$-solution $u(t)$ scatters (in the $\hat{L}^{\frac{\alpha-1}{2}}$-sense) as $t\to\pm\infty$ and if $u_0\in\hat{L}^{\frac{\alpha_0-1}{2}}$ (resp. if $u_0\in\dot{H}^\sigma$) then $u(t)$ scatters as $t\to\pm\infty$ also in the $\hat{L}^{\frac{\alpha_0-1}{2}}$-sense (resp. $\dot{H}^\sigma$-sense).

Theorem 1.10. *Let $\frac{21}{5}<\alpha<\frac{23}{3}$. If $u_0\in\hat{L}^{\frac{\alpha-1}{2}}\cap H^1$ satisfies $u_0\neq 0$ and $E[u_0]\leqslant 0$ then $u(t)$ does not scatter as $t\to\pm\infty$.*

The rest of the paper is organized as follows. In Section 2, we prove some linear space-time estimates for solutions to the Airy equation, in $\hat{L}^r$-framework. The generalized Strichartz estimates are established in Propositions 2.1 and 2.5. Section 3 is devoted to several nonlinear estimates. We also introduce several function spaces to work with in this section. Then, in Section 4, we prove our theorems. In Appendix A, we prove a fractional chain rule in space-time function space (Lemma 3.7). Finally in Appendix B, we briefly collect some inclusion relations for $\hat{L}^r$.

The following notation will be used throughout this paper: $|D_x|^s=(-\partial_x^2)^{\frac{s}{2}}$ and $\langle D_x\rangle^s=(I-\partial_x^2)^{\frac{s}{2}}$ denote the Riesz and Bessel potentials of order $-s$, respectively. For $1\leqslant p,q\leqslant\infty$ and $I\subset\mathbb{R}$, let us define a space-time norm

$$\|f\|_{L_t^qL_x^p(I)}=\big\|\|f(t,\cdot)\|_{L_x^p(\mathbb{R})}\big\|_{L_t^q(I)},$$
$$\|f\|_{L_x^pL_t^q(I)}=\big\|\|f(\cdot,x)\|_{L_t^q(I)}\big\|_{L_x^p(\mathbb{R})}.$$

2. Linear estimates for the Airy equation

In this section, we consider the space-time estimates of solutions to the Airy equation

$$\begin{cases}\partial_t u+\partial_x^3 u=F(t,x), & t\in I, x\in\mathbb{R},\\ u(0,x)=f(x), & x\in\mathbb{R},\end{cases}\tag{2-1}$$

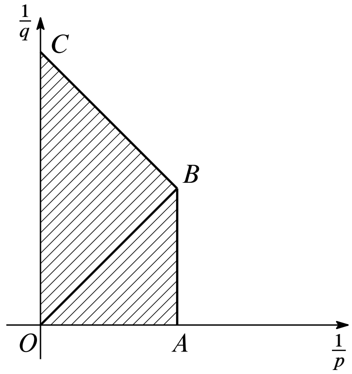

Figure 1. The range of (p, q) satisfying the assumptions of Proposition 2.1.

where $I \subset \mathbb{R}$ is an interval and $F : I \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are given functions.

Let $\{e^{-t\partial_x^3}\}_{t\in\mathbb{R}}$ be an isometric isomorphism group in $\widehat{L}^r$ defined by $e^{-t\partial_x^3} = \mathcal{F}^{-1}e^{it\xi^3}\mathcal{F}$, or more precisely by

$$(e^{-t\partial_x^3} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi+it\xi^3} \hat{f}(\xi)\, d\xi.$$

Using the group, the solution to (2-1) can be written as

$$u(t) = e^{-t\partial_x^3} f + \int_0^t e^{-(t-t')\partial_x^3} F(t')\, dt'.$$

We first show a homogeneous estimate associated with (2-1).

Proposition 2.1. *Let I be an interval. Let (p, q) satisfy*

$$0 \leqslant \frac{1}{p} < \frac{1}{4}, \quad 0 \leqslant \frac{1}{q} < \frac{1}{2} - \frac{1}{p}.$$

Then, for any $f \in \widehat{L}^r$,

$$\big\| |D_x|^s e^{-t\partial_x^3} f \big\|_{L_x^p L_t^q(I)} \leqslant C \|f\|_{\widehat{L}^r}, \tag{2-2}$$

where

$$\frac{1}{r} = \frac{2}{p} + \frac{1}{q}, \quad s = -\frac{1}{p} + \frac{2}{q},$$

and the positive constant C depends only on r and s.

Figure 1 shows the range of (p, q) satisfying the assumptions of Proposition 2.1, where $A = \left(\frac{1}{4}, 0\right)$, $B = \left(\frac{1}{4}, \frac{1}{4}\right)$, and $C = \left(0, \frac{1}{2}\right)$. The line segments OA and OC are included, but the other parts of the border are excluded.

To prove Proposition 2.1, we show three lemmas. The first one is a Stein–Tomas-type estimate.

Lemma 2.2 (Stein–Tomas-type estimate). *For any $r \in (4, \infty]$, there exists a positive constant C depending only on r such that for any $f \in \widehat{L}^{\frac{r}{3}}$,*

$$\big\| |D_x|^{\frac{1}{r}} e^{-t\partial_x^3} f \big\|_{L_{t,x}^r(I)} \leqslant C \|f\|_{\widehat{L}^{r/3}}. \tag{2-3}$$

Proof of Lemma 2.2. Although a more general version is proved in [Grünrock 2004, Corollary 3.6], here we give a direct proof, based on the fact that the exponents for the space variable and time variable on the left-hand side coincide.

It suffices to prove (2-3) for the case $I=\mathbb{R}$. For notational simplicity, we omit $\mathbb{R}$. The case $r=\infty$ follows from the Hausdorff–Young inequality. Let $r<\infty$. Squaring both sides, we may show that

$$\big\| \big| |D_x|^{\frac{1}{r}} e^{-t\partial_x^3} f \big|^2 \big\|_{L^{r/2}_{t,x}} \leqslant C \|f\|^2_{\hat{L}^{r/3}}. \tag{2-4}$$

The left-hand side of (2-4) is equal to

$$\left\| \iint_{\mathbb{R}^2} e^{ix(\xi-\eta)+it(\xi^3-\eta^3)} |\xi\eta|^{\frac{1}{r}} \hat{f}(\xi)\overline{\hat{f}(\eta)}\, d\xi\, d\eta \right\|_{L^{r/2}_{t,x}}.$$

Changing variables by $a=\xi-\eta$ and $b=\xi^3-\eta^3$, we have

$$\big\| \big| |D_x|^{\frac{1}{r}} e^{-t\partial_x^3} f \big|^2 \big\|_{L^{r/2}_{t,x}} = \left\| \iint_{\mathbb{R}^2} e^{ixa+itb} |\xi\eta|^{\frac{1}{r}} \hat{f}(\xi)\overline{\hat{f}(\eta)} \frac{1}{3|\xi^2-\eta^2|}\, da\, db \right\|_{L^{r/2}_{t,x}}.$$

We now use the Hausdorff–Young inequality to deduce that

$$\begin{aligned} \big\| \big| |D_x|^{\frac{1}{r}} e^{-t\partial_x^3} f |^2 \big\|_{L^{r/2}_{t,x}} &\leqslant C \big\| |\xi\eta|^{\frac{1}{r}} \hat{f}(\xi)\overline{\hat{f}(\eta)} |\xi^2-\eta^2|^{-1} \big\|_{L^{(r/2)'}_{a,b}} \\ &= C\left(\iint_{\mathbb{R}^2} \frac{|\xi\eta|^{\frac{1}{r-2}} |\hat{f}(\xi)|^{\frac{r}{r-2}} |\hat{f}(\eta)|^{\frac{r}{r-2}}}{|\xi-\eta|^{\frac{2}{r-2}} |\xi+\eta|^{\frac{2}{r-2}}}\, d\xi\, d\eta \right)^{1-\frac{2}{r}}. \end{aligned} \tag{2-5}$$

Notice that $\frac{r}{2}\geqslant 2$. We now split the integral region $\mathbb{R}^2$ into $\{\xi\eta\geqslant 0\}$ and $\{\xi\eta<0\}$. We only consider the first case, since the other can be treated essentially in the same way. For (ξ,η) with $\xi\eta\geqslant 0$, we have

$$\xi\eta \leqslant \frac{(\xi+\eta)^2}{4},$$

and so

$$\iint_{\xi\eta\geqslant 0} \frac{|\xi\eta|^{\frac{1}{r-2}} |\hat{f}(\xi)|^{\frac{r}{r-2}} |\hat{f}(\eta)|^{\frac{r}{r-2}}}{|\xi-\eta|^{\frac{2}{r-2}} |\xi+\eta|^{\frac{2}{r-2}}}\, d\xi\, d\eta \leqslant C \iint_{\xi\eta\geqslant 0} \frac{|\hat{f}(\xi)|^{\frac{r}{r-2}} |\hat{f}(\eta)|^{\frac{r}{r-2}}}{|\xi-\eta|^{\frac{2}{r-2}}}\, d\xi\, d\eta. \tag{2-6}$$

By the Hölder inequality and the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned} \iint_{\xi\eta\geqslant 0} \frac{|\hat{f}(\xi)|^{\frac{r}{r-2}} |\hat{f}(\eta)|^{\frac{r}{r-2}}}{|\xi-\eta|^{\frac{2}{r-2}}}\, d\xi\, d\eta &\leqslant \big\| |\hat{f}|^{\frac{r}{r-2}} \big\|_{L^{(r-2)/(r-3)}} \big\| (|\xi|^{-\frac{2}{r-2}} * |\hat{f}|^{\frac{r}{r-2}}) \big\|_{L^{r-2}} \\ &\leqslant C \|\hat{f}\|^{\frac{2r}{r-2}}_{L^{r/(r-3)}} = C\|f\|^{\frac{2r}{r-2}}_{\hat{L}^{r/3}_x} \end{aligned} \tag{2-7}$$

as long as $\frac{2}{r-2}<1$, that is, $r>4$. Combining (2-5), (2-6) and (2-7), we obtain the result. □

The second is a Kenig–Ruiz-type estimate [1983].

Lemma 2.3 (Kenig–Ruiz-type estimate). *There exists a universal constant C such that for any interval I and any $f\in L^2$,*

$$\big\| |D_x|^{-\frac{1}{4}} e^{-t\partial_x^3} f \big\|_{L^4_x L^\infty_t(I)} \leqslant C\|f\|_{L^2}. \tag{2-8}$$

Proof of Lemma 2.3. See [Kenig et al. 1991, Theorem 2.5]. □

The last estimate is an $\widehat{L}^q$-version of Kato's local smoothing effect [1983].

Lemma 2.4 (Kato's smoothing effect). *For any $q \in [2, \infty]$, there exists a positive constant C depending only on q such that for any interval I and for any $f \in \widehat{L}^q$,*

$$\big\| |D_x|^{\frac{2}{q}} e^{-t\partial_x^3} f \big\|_{L_x^\infty L_t^q(I)} \leqslant C \|f\|_{\widehat{L}^q}. \tag{2-9}$$

Proof of Lemma 2.4. We show (2-9) by slightly modifying the argument due to Kenig, Ponce, and Vega [Kenig et al. 1991, Theorem 2.5]. We prove (2-9) for the case $I = \mathbb{R}$ only.

The case $q = \infty$ is treated in Lemma 2.2. Hence, we may suppose $q < \infty$. A direct computation shows

$$\begin{aligned} |D_x|^{\frac{2}{q}} e^{-t\partial_x^3} f &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi + it\xi^3} |\xi|^{\frac{2}{q}} \hat{f}(\xi)\, d\xi \\ &= \frac{1}{3\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\eta^{\frac{1}{3}} + it\eta} |\eta|^{\frac{2}{3q}} \eta^{-\frac{2}{3}} \hat{f}(\eta^{\frac{1}{3}})\, d\eta, \end{aligned}$$

where we have used a change of variable $\eta = \xi^3$ to yield the last line. Take the L_t^q-norm and apply the Hausdorff–Young inequality to obtain

$$\big\| |D_x|^{\frac{2}{q}} e^{-t\partial_x^3} f \big\|_{L_t^q} \leqslant C \big\| e^{ix\eta^{\frac{1}{3}}} |\eta|^{\frac{2-q}{3q}} \hat{f}(\eta^{\frac{1}{3}}) \big\|_{L_\eta^{q'}} \leqslant C \|\hat{f}\|_{L^{q'}} = C \|f\|_{\widehat{L}^q}.$$

Since the right-hand side is independent of x, we obtain (2-9). □

Proof of Proposition 2.1. Interpolating (2-3), (2-8), and (2-9), we obtain (2-2). □

Next we show an inhomogeneous estimate associated with (2-1).

Proposition 2.5. *Let $\frac{4}{3} < r < 4$ and let (p_j, q_j) $(j = 1, 2)$ satisfy*

$$0 \leqslant \frac{1}{p_j} < \frac{1}{4}, \quad 0 \leqslant \frac{1}{q_j} < \frac{1}{2} - \frac{1}{p_j}.$$

Then, the inequalities

$$\Big\| \int_0^t e^{-(t-t')\partial_x^3} F(t')\, dt' \Big\|_{L_t^\infty(I;\widehat{L}_x^r)} \leqslant C_1 \big\| |D_x|^{-s_2} F \big\|_{L_x^{p_2'} L_t^{q_2'}(I)} \tag{2-10}$$

and

$$\Big\| |D_x|^{s_1} \int_0^t e^{-(t-t')\partial_x^3} F(t')\, dt' \Big\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C_2 \big\| |D_x|^{-s_2} F \big\|_{L_x^{p_2'} L_t^{q_2'}(I)} \tag{2-11}$$

hold for any F satisfying $|D_x|^{-s_2} F \in L_x^{p_2'} L_t^{q_2'}$, where

$$\frac{1}{r} = \frac{2}{p_1} + \frac{1}{q_1}, \quad s_1 = -\frac{1}{p_1} + \frac{2}{q_1} \qquad \frac{1}{r'} = \frac{2}{p_2} + \frac{1}{q_2}, \quad s_2 = -\frac{1}{p_2} + \frac{2}{q_2},$$

and where the constant C_1 depends on r, s_1 and I, and the constant C_2 depends on r, s_2 and I.

To prove Proposition 2.5, we employ the following lemma, which is essentially due to Christ and Kiselev [2001]. The version of this lemma that we use is the one presented in [Molinet and Ribaud 2004].

Lemma 2.6. *Let $I \subset \mathbb{R}$ be an interval and let $K : \mathcal{S}(I \times \mathbb{R}) \to C(\mathbb{R}^3)$. Assume that*

$$\left\| \int_I K(t,t')F(t')\,dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C \|F\|_{L_x^{p_2} L_t^{q_2}(I)}$$

for some $1 \leqslant p_1, p_2, q_1, q_2 \leqslant \infty$ *with* $\min(p_1, q_1) > \max(p_2, q_2)$. *Then*

$$\left\| \int_0^t K(t,t')F(t')\,dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C \|F\|_{L_x^{p_2} L_t^{q_2}(I)}.$$

Moreover, the case $q_1 = \infty$ and $p_2, q_2 < \infty$ is allowed.

Proof of Lemma 2.6. See [Molinet and Ribaud 2004, Lemma 2]. □

Proof of Proposition 2.5. We first prove the inequality (2-10). Since the group $\{e^{-t\partial_x^3}\}_{t\in\mathbb{R}}$ is isometric in $\widehat{L}^r$, the duality argument and Proposition 2.1 yield

$$\begin{aligned}
\left\| \int_0^t e^{-(t-t')\partial_x^3} F(t')\,dt' \right\|_{\widehat{L}_x^r} &= \left\| \int_0^t e^{t'\partial_x^3} F(t')\,dt' \right\|_{\widehat{L}_x^r} \\
&= \sup_{\|g\|_{\widehat{L}_x^{r'}}=1} \left(\int_{-\infty}^{\infty} \left(\int_0^t e^{t'\partial_x^3} F(t',x)\,dt' \right) g(x)\,dx \right) \\
&= \sup_{\|g\|_{\widehat{L}_x^{r'}}=1} \left(\int_0^t \int_{-\infty}^{\infty} |D_x|^{-s_2} F(t',x) |D_x|^{s_2} e^{-t'\partial_x^3} g(x)\,dt'\,dx \right) \\
&\leqslant \sup_{\|g\|_{\widehat{L}_x^{r'}}=1} \left\| |D_x|^{-s_2} F \right\|_{L_x^{p_2'} L_t^{q_2'}(I)} \left\| |D_x|^{s_2} e^{-t'\partial_x^3} g \right\|_{L_x^{p_2} L_t^{q_2}(I)} \\
&\leqslant C \sup_{\|g\|_{\widehat{L}_x^{r'}}=1} \left\| |D_x|^{-s_2} F \right\|_{L_x^{p_2'} L_t^{q_2'}(I)} \|g\|_{\widehat{L}_x^{r'}} \\
&= C \left\| |D_x|^{-s_2} F \right\|_{L_x^{p_2'} L_t^{q_2'}(I)},
\end{aligned} \tag{2-12}$$

where the constant C is independent of t. Hence we have (2-10).

Next we prove the inequality (2-11). Since the case $r = 2$ was already proved in [Kenig et al. 1993], we consider the case where $r \neq 2$. To prove (2-11), it suffices to show

$$\left\| |D_x|^{s_1} \int_I e^{-(t-t')\partial_x^3} F(t')\,dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C \left\| |D_x|^{-s_2} F \right\|_{L_x^{p_2'} L_t^{q_2'}(I)}. \tag{2-13}$$

Indeed, since $\min(p_1, q_1) > \max(p_2', q_2')$ follows from

$$\min(p_1, q_1) = \begin{cases} \frac{r}{r-1} & \text{if } \frac{4}{3} < r < 2, \\ r & \text{if } 2 < r < 4, \end{cases} \quad \max(p_2', q_2') = \begin{cases} r & \text{if } \frac{4}{3} < r < 2, \\ \frac{r}{r-1} & \text{if } 2 < r < 4, \end{cases}$$

we see that the combination of the Christ–Kiselev lemma (Lemma 2.6) with (2-13) implies (2-11). Therefore we concentrate our attention on proving (2-13). By Proposition 2.1,

$$\Big\| |D_x|^{s_1} \int_I e^{-(t-t')\partial_x^3} F(t')\,dt' \Big\|_{L_x^{p_1} L_t^{q_1}(I)} = \Big\| |D_x|^{s_1} e^{-t\partial_x^3} \int_I e^{t'\partial_x^3} F(t')\,dt' \Big\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C \Big\| \int_I e^{t'\partial_x^3} F(t')\,dt' \Big\|_{\widehat{L}_x^r}. \tag{2-14}$$

By the duality argument similar to (2-12), we obtain

$$\Big\| \int_I e^{t'\partial_x^3} F(t')\,dt' \Big\|_{\widehat{L}_x^r} \leqslant C \big\| |D_x|^{-s_2} F \big\|_{L_x^{p_2'} L_t^{q_2'}(I)}. \tag{2-15}$$

Combining (2-14) and (2-15), we obtain (2-13). □

3. Nonlinear estimates

In this section, we prove several nonlinear estimates which are used to prove main theorems. We introduce several function spaces. Let us recall that a pair $(s,r) \in \mathbb{R} \times [1,\infty]$ is said to be *acceptable* if $\frac{1}{r} \in \big[0, \frac{3}{4}\big)$ and

$$s \in \begin{cases} \big[-\frac{1}{2r}, \frac{2}{r}\big] & \text{if } 0 \leqslant \frac{1}{r} \leqslant \frac{1}{2}, \\ \big(\frac{2}{r} - \frac{5}{4}, \frac{5}{2} - \frac{3}{r}\big) & \text{if } \frac{1}{2} < \frac{1}{r} < \frac{3}{4}. \end{cases}$$

Definition 3.1. Let $(s,r) \in \mathbb{R} \times [1,\infty]$. A pair (s,r) is said to be *conjugate-acceptable* if $(1-s, r')$ is acceptable, where $\frac{1}{r'} = 1 - \frac{1}{r} \in [0,1]$.

Figure 2 shows the ranges of acceptable pairs (quadrangle $OABC$) and conjugate-acceptable pairs (quadrangle $DEFG$). Here, $O = (0,0)$, $A = \big(\frac{1}{2}, -\frac{1}{4}\big)$, $B = \big(\frac{3}{4}, \frac{1}{4}\big)$, $C = \big(\frac{1}{2}, 1\big)$, $D = (1,1)$, $E = \big(\frac{1}{2}, \frac{5}{4}\big)$, $F = \big(\frac{1}{4}, \frac{3}{4}\big)$, and $G = \big(\frac{1}{2}, 0\big)$.

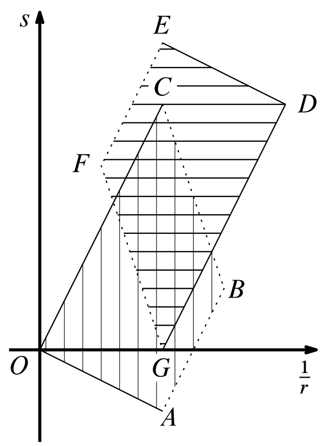

Figure 2. The ranges of acceptable pairs (quadrangle $OABC$) and conjugate-acceptable pairs (quadrangle $DEFG$).

For an interval $I \subset \mathbb{R}$ and a conjugate-acceptable pair (s,r), we define a function space $Y(I;s,r)$ by

$$\|f\|_{Y(I;s,r)} = \big\| |D_x|^s f \big\|_{L_x^{\tilde{p}(s,r)}(\mathbb{R};L_t^{\tilde{q}(s,r)}(I))},$$

where the exponents are given by

$$\frac{2}{\tilde{p}(s,r)} + \frac{1}{\tilde{q}(s,r)} = 2 + \frac{1}{r}, \quad -\frac{1}{\tilde{p}(s,r)} + \frac{2}{\tilde{q}(s,r)} = s, \tag{3-1}$$

or equivalently,

$$\begin{pmatrix} \frac{1}{\tilde{p}(s,r)} \\ \frac{1}{\tilde{q}(s,r)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} s \\ 2+\frac{1}{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{p(s,r)} \\ \frac{1}{q(s,r)} \end{pmatrix} + \begin{pmatrix} \frac{4}{5} \\ \frac{2}{5} \end{pmatrix}.$$

With this terminology, Propositions 2.1 and 2.5 can be reformulated as follows:

Proposition 3.2. *Let I be an interval.*

(i) *Let (s,r) be an acceptable pair. Then, there exists a positive constant C depending only on s and r such that*

$$\|e^{-t\partial_x^3} f\|_{L^\infty(\mathbb{R};\hat{L}^r)} + \|e^{-t\partial_x^3} f\|_{X(\mathbb{R};s,r)} \leqslant C_{s,r}\|f\|_{\hat{L}^r}$$

for any $f \in \hat{L}^r$.

(ii) *Let (s_1,r) be an acceptable pair and let (s_2,r) be a conjugate-acceptable pair. Then, there exists a positive constant depending only on s_i and r such that for any $t_0 \in I \subset \mathbb{R}$ and any $F \in Y(I;s_2,r)$,*

$$\left\| \int_{t_0}^t e^{-(t-t')\partial_x^3} \partial_x F(t')\, dt' \right\|_{L_t^\infty(I;\hat{L}_x^r) \cap X(I;s_1,r)} \leqslant C\|F\|_{Y(I;s_2,r)}.$$

To handle $X(I;s,r)$- and $Y(I;s,r)$-spaces, the following lemma is useful.

Lemma 3.3. *Let $1 < p_i, q_i < \infty$ and $s_i \in \mathbb{R}$ for $i = 1,2$. Let p, q and s satisfy*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad s = \theta s_1 + (1-\theta)s_2$$

for some $\theta \in (0,1)$. Then, there exists a positive constant C, depending on $p_1, p_2, q_1, q_2, s_1, s_2$ and θ, such that

$$\big\| |D_x|^s f \big\|_{L_x^p L_t^q} \leqslant C \big\| |D_x|^{s_1} f \big\|_{L_x^{p_1} L_t^{q_1}}^{\theta} \big\| |D_x|^{s_2} f \big\|_{L_x^{p_2} L_t^{q_2}}^{1-\theta}$$

holds for any f such that $|D_x|^{s_1} f \in L_x^{p_1} L_t^{q_1}$ and $|D_x|^{s_2} f \in L_x^{p_2} L_t^{q_2}$.

Proof of Lemma 3.3. For $z \in \mathbb{C}$, define an operator $T_z = |D_x|^{zs_1+(1-z)s_2}$. Let $g(t)$ and $h(x)$ be $\mathbb{R}$-valued simple functions and $G_z(t)$ and $H_z(x)$ be extensions of these functions defined by

$$G_z(t) := |g(t)|^{\frac{1-(z/q_1+(1-z)/q_2)}{1-1/q}} \operatorname{sign} g(t)$$

and

$$H_z(x) := |h(x)|^{\frac{1-(z/p_1+(1-z)/p_2)}{1-1/p}} \operatorname{sign} h(x),$$

respectively, for $z \in \mathbb{C}$ with $0 \leqslant \operatorname{Re} z \leqslant 1$. Put

$$\Psi(z) := \iint_{\mathbb{R}^2} T_z f(t,x) G_z(t) H_z(x)\, dt\, dx.$$

By density and duality, it suffices to show

$$|\Psi(\theta)| \leqslant C \big\| |D_x|^{s_1} f \big\|_{L_x^{p_1} L_t^{q_1}}^{\theta} \big\| |D_x|^{s_2} f \big\|_{L_x^{p_2} L_t^{q_2}}^{1-\theta} \tag{3-2}$$

for any $f \in \mathcal{S}(\mathbb{R}^2)$ with compact Fourier support and any simple functions $g(t)$ and $h(x)$ such that $\|g\|_{L_t^{q'}} = \|h\|_{L_x^{p'}} = 1$.

Let us now prove (3-2). It is easy to see that $\Psi(z)$ is analytic in $0 < \operatorname{Re} z < 1$ and continuous in $0 \leqslant \operatorname{Re} z \leqslant 1$. By a variant of the multiplier theorem by Fernandez [1987, Theorem 6.4], we see that $|D_x|^{iy}$ is a bounded operator in $L_x^{p_1} L_t^{q_1}$ with norm $C(1+|y|)$. Therefore, for any $y \in \mathbb{R}$,

$$\begin{aligned} |\Psi(1+iy)| &\leqslant \big\| |D_x|^{iy(s_1-s_2)} (|D_x|^{s_1} f) \big\|_{L_x^{p_1} L_t^{q_1}} \|G_{1+iy} H_{1+iy}\|_{L_x^{p_1'} L_t^{q_1'}} \\ &\leqslant C(1+|y(s_1-s_2)|) \big\| |D_x|^{s_1} f \big\|_{L_x^{p_1} L_t^{q_1}} \|g\|_{L_t^{q'}} \|h\|_{L_x^{p'}} \\ &\leqslant C(1+|y(s_1-s_2)|) \big\| |D_x|^{s_1} f \big\|_{L_x^{p_1} L_t^{q_1}}. \end{aligned} \tag{3-3}$$

The same argument yields

$$|\Psi(iy)| \leqslant C(1+|y(s_1-s_2)|) \big\| |D_x|^{s_2} f \big\|_{L_x^{p_2} L_t^{q_2}}. \tag{3-4}$$

From (3-3), (3-4) and Hirschman's lemma [1952], we obtain (3-2) (see also [Stein 1956]). □

Estimates on nonlinearity. In this subsection, we establish an estimate on nonlinearity. For this, we introduce a Lipschitz μ-norm ($\mu > 0$) as follows. Write $\mu = N + \beta$ with $N \in \mathbb{Z}$ and $\beta \in (0,1]$. For a function $G : \mathbb{C} \to \mathbb{C}$, we define

$$\|G\|_{\operatorname{Lip}\mu} := \sum_{j=0}^{N} \sup_{z \in \mathbb{R}\setminus\{0\}} \frac{|G^{(j)}(z)|}{|z|^{\mu-j}} + \sup_{x \neq y} \frac{|G^{(N)}(x) - G^{(N)}(y)|}{|x-y|^{\beta}},$$

where $G^{(j)}$ is j-th derivative of G. We say $G \in \operatorname{Lip}\mu$ if $G \in C^N(\mathbb{R})$ and $\|G\|_{\operatorname{Lip}\mu} < \infty$.

The main estimate of this subsection is as follows:

Lemma 3.4. *Suppose that $G(z) \in \operatorname{Lip}\alpha$ for some $\frac{21}{5} < \alpha < \frac{23}{3}$. Let (s,r) be a pair which is acceptable and conjugate-acceptable. Then, the following two assertions hold*:

(i) *If $u \in S\big(I; \frac{\alpha-1}{2}\big) \cap X(I;s,r)$ then $G(u) \in Y(I;s,r)$. Moreover, there exists a constant C such that*

$$\|G(u)\|_{Y(I;s,r)} \leqslant C \|u\|_{S(I;\frac{\alpha-1}{2})}^{\alpha-1} \|u\|_{X(I;s,r)}$$

for any $u \in S\big(I; \frac{\alpha-1}{2}\big) \cap X(I;s,r)$.

(ii) *There exists a constant C such that*

$$\begin{aligned} \|G(u)-G(v)\|_{Y(I;s,r)} \leqslant{}& C\big(\|u\|_{X(I;s,r)} + \|v\|_{X(I;s,r)}\big)\big(\|u\|_{S(I;\frac{\alpha-1}{2})} + \|v\|_{S(I;\frac{\alpha-1}{2})}\big)^{\alpha-2} \|u-v\|_{S(I;\frac{\alpha-1}{2})} \\ &+ C\big(\|u\|_{S(I;\frac{\alpha-1}{2})} + \|v\|_{S(I;\frac{\alpha-1}{2})}\big)^{\alpha-1} \|u-v\|_{X(I;s,r)} \end{aligned}$$

for any $u, v \in S\big(I; \frac{\alpha-1}{2}\big) \cap X(I;s,r)$.

Remark 3.5. It is easy to see that $|z|^{\alpha-1}z \in \mathrm{Lip}\,\alpha$. The validity of the above lemma is the only assumption on the nonlinearity that we need. Hence, the all results of this article hold for an equation with generalized nonlinearity $\partial_t u + \partial_x^3 u = \partial_x(G(u))$, provided $G(z) \in \mathrm{Lip}\,\alpha$.

To prove the above lemma, we recall the following two lemmas.

Lemma 3.6. *Let I be an interval. Assume that $s \geqslant 0$. Let $p, q, p_i, q_i, \in (1,\infty)$ $(i = 1,2,3,4)$. Then,*

$$\big\| |D_x|^s (fg) \big\|_{L_x^p L_t^q(I)} \leqslant C\big(\big\| |D_x|^s f \big\|_{L_x^{p_1} L_t^{q_1}(I)} \|g\|_{L_x^{p_2} L_t^{q_2}(I)} + \|f\|_{L_x^{p_3} L_t^{q_3}(I)} \big\| |D_x|^s g \big\|_{L_x^{p_4} L_t^{q_4}(I)} \big),$$

provided that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4},$$

where the constant C is independent of I and f.

Proof of Lemma 3.6. If $s \in \mathbb{Z}$ then (the classical) Leibniz rule, Hölder's inequality, and Lemma 3.3 give us the result. By a similar argument, it suffices to consider the case $0 < s < 1$ to handle the general case. However, that case follows from [Kenig et al. 1993, Theorem A.8] and Lemma 3.3. □

Lemma 3.7. *Suppose that $\mu > 1$ and $s \in (0,\mu)$. Let $G \in \mathrm{Lip}\,\mu$. If $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$ satisfy*

$$\frac{1}{p} = \frac{\mu-1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{\mu-1}{q_1} + \frac{1}{q_2},$$

then there exists a positive constant C depending on $\mu, s, p_1, p_2, q_1, q_2$ and I such that

$$\big\| |D_x|^s G(f) \big\|_{L_x^p L_t^q(I)} \leqslant C \|G\|_{\mathrm{Lip}\,\mu} \|f\|_{L_x^{p_1} L_t^{q_1}(I)}^{\mu-1} \big\| |D_x|^s f \big\|_{L_x^{p_2} L_t^{q_2}(I)}$$

holds for any f satisfying $f \in L_x^{p_1} L_t^{q_1}(I)$ and $|D_x|^s f \in L_x^{p_2} L_t^{q_2}(I)$.

Although Lemma 3.7 is essentially the same as [Kenig et al. 1993, Theorem A.6; Christ and Weinstein 1991, Proposition 3.1], we give the proof of this lemma in Appendix A for self-containedness and in order to clarify the necessity of the assumption $G \in \mathrm{Lip}\,\mu$.

Proof of Lemma 3.4. We prove the second assertion since the first immediately follows from the second by letting $v = 0$. For simplicity, we name $S = S\big(I; \frac{\alpha-1}{2}\big)$, $L = X(I; s, r)$, and $N = Y(I; s, r)$.

Let us write

$$G(u) - G(v) = (u - v)\int_0^1 G'(\theta u + (1-\theta)v)\, d\theta.$$

Lemma 3.6 implies that

$$\begin{aligned} \|G(u) - G(v)\|_N &\leqslant C\|u - v\|_S \int_0^1 \big\| |D_x|^s \big(G'(\theta u + (1-\theta)v)\big) \big\|_{L_x^{p_1} L_t^{q_1}} d\theta \\ &\qquad + C\|u - v\|_L \int_0^1 \big\| \big(G'(\theta u + (1-\theta)v)\big) \big\|_{L_x^{p_2} L_t^{q_2}} d\theta \\ &=: I_1 + I_2, \end{aligned}$$

where

$$\begin{pmatrix} 1/p_1 \\ 1/q_1 \end{pmatrix} = \begin{pmatrix} 1/\tilde{p}(s,r) \\ 1/\tilde{q}(s,r) \end{pmatrix} - \begin{pmatrix} 1/p\big(0, \frac{\alpha-1}{2}\big) \\ 1/q\big(0, \frac{\alpha-1}{2}\big) \end{pmatrix} = (\alpha - 2)\begin{pmatrix} 1/p\big(0, \frac{\alpha-1}{2}\big) \\ 1/q\big(0, \frac{\alpha-1}{2}\big) \end{pmatrix} + \begin{pmatrix} 1/p(s,r) \\ 1/q(s,r) \end{pmatrix},$$

and

$$\begin{pmatrix} 1/p_2 \\ 1/q_2 \end{pmatrix} = \begin{pmatrix} 1/\tilde{p}(s,r) \\ 1/\tilde{q}(s,r) \end{pmatrix} - \begin{pmatrix} 1/p(s,r) \\ 1/q(s,r) \end{pmatrix} = (\alpha-1)\begin{pmatrix} 1/p\big(0,\frac{\alpha-1}{2}\big) \\ 1/q\big(0,\frac{\alpha-1}{2}\big) \end{pmatrix}.$$

It is easy to see that $\|G'\|_{\mathrm{Lip}(\alpha-1)} \leqslant \|G\|_{\mathrm{Lip}\,\alpha} < +\infty$. By the definition of $\|\cdot\|_{\mathrm{Lip}(\alpha-1)}$, we estimate I_2 as

$$\begin{aligned} I_2 &\leqslant C\|u-v\|_L \,\|G'\|_{\mathrm{Lip}(\alpha-1)} \int_0^1 \big\| |\theta u + (1-\theta)v|^{\alpha-1} \big\|_{L_x^{p_2} L_t^{q_2}} \, d\theta \\ &\leqslant C\|u-v\|_L \int_0^1 (\|u\|_S + \|v\|_S)^{\alpha-1} \, d\theta \\ &\leqslant C(\|u\|_S + \|v\|_S)^{\alpha-1} \|u-v\|_L. \end{aligned}$$

On the other hand, we see from Lemma 3.7 that

$$\big\| |D_x|^s \big(G'(\theta u + (1-\theta)v)\big) \big\|_{L_x^{p_1} L_t^{q_1}} \leqslant C\|G'\|_{\mathrm{Lip}(\alpha-1)} \|\theta u + (1-\theta)v\|_S^{\alpha-2} \,\|\theta u + (1-\theta)v\|_L$$

for any $\theta \in (0,1)$. Hence, we find the following estimate on I_1:

$$I_1 \leqslant C\|u-v\|_S \,\|G'\|_{\mathrm{Lip}(\alpha-1)} (\|u\|_S + \|v\|_S)^{\alpha-2} (\|u\|_L + \|v\|_L).$$

Collecting the above inequalities, we obtain the result. □

4. Proofs of the main theorems

In this section, we prove the main theorems. Recall the notation $S(I;r) = X(I;0,r)$. Now, take a number $s_L(\alpha)$ so that a pair $\big(s_L(\alpha), \frac{\alpha-1}{2}\big)$ is acceptable and conjugate-acceptable. We define $L\big(I;\frac{\alpha-1}{2}\big) = X\big(I; s_L(\alpha), \frac{\alpha-1}{2}\big)$ and $N\big(I;\frac{\alpha-1}{2}\big) = Y\big(I; s_L(\alpha), \frac{\alpha-1}{2}\big)$.

Remark 4.1. If $\frac{27}{7} < \alpha < \frac{23}{3}$ then $s_L(\alpha)$ with the above property exists. Indeed, $s_L(\alpha) = \frac{3}{4} - \frac{1}{\alpha-1}$ works. Our upper bound on α comes from this point.

Local well-posedness in a scale-critical space. Let us prove Theorem 1.2. To prove this theorem, we show the following lemma.

Lemma 4.2. *Assume $\frac{21}{5} < \alpha < \frac{23}{3}$ and $u_0 \in \widehat{L}_x^{\frac{\alpha-1}{2}}$. Let $t_0 \in \mathbb{R}$ and I be an interval with $t_0 \in I$. Then, there exists a universal constant $\delta > 0$ such that, if a tempered distribution u_0 and an interval $I \ni t_0$ satisfy*

$$\varepsilon = \varepsilon(I; u_0, t_0) := \|e^{-(t-t_0)\partial_x^3} u_0\|_{S(I;\frac{\alpha-1}{2})} + \|e^{-(t-t_0)\partial_x^3} u_0\|_{L(I,\frac{\alpha-1}{2})} \leqslant \delta,$$

then there exists a unique solution $u \in C(I; \widehat{L}_x^{\frac{\alpha-1}{2}})$ to the initial value problem

$$\begin{cases} \partial_t u + \partial_x^3 u = \mu \partial_x (|u|^{\alpha-1} u), & t, x \in \mathbb{R}, \\ u(t_0, x) = u_0(x), & x \in \mathbb{R} \end{cases}$$

(*in the sense of the corresponding integral equation*) *that satisfies*

$$\|u\|_{S(I;\frac{\alpha-1}{2})} + \|u\|_{L(I;\frac{\alpha-1}{2})} \leqslant 2\varepsilon.$$

If $u_0 \in \hat{L}^{\frac{\alpha-1}{2}}$, in addition, then

$$\|u\|_{L^\infty(I;\hat{L}^{(\alpha-1)/2})} \leqslant \|u_0\|_{\hat{L}^{(\alpha-1)/2}} + C\varepsilon^\alpha$$

holds for some constant $C > 0$ and u belongs to all $\hat{L}^{\frac{\alpha-1}{2}}$-admissible spaces $X\big(I; s, \frac{\alpha-1}{2}\big)$.

Proof of Lemma 4.2. For $R > 0$, define a complete metric space

$$Z_R = \big\{u \in L\big(I; \tfrac{\alpha-1}{2}\big) \cap S\big(I; \tfrac{\alpha-1}{2}\big) : \|u\|_Z \leqslant R\big\},$$
$$\|u\|_Z := \|u\|_{L(I;\frac{\alpha-1}{2})} + \|u\|_{S(I;\frac{\alpha-1}{2})}, \quad d_Z(u,v) := \|u-v\|_Z.$$

For given tempered distribution u_0 with $e^{-(t-t_0)\partial_x^3}u_0 \in Z_\delta$ and $v \in Z_R$, we define

$$\Phi(v)(t) := e^{-(t-t_0)\partial_x^3}u_0 + \mu \int_{t_0}^t e^{-(t-t')\partial_x^3}\partial_x(|v|^{\alpha-1}v)(t')\,dt'.$$

We show that there exists $\delta > 0$ such that $\Phi : Z_{2\varepsilon} \to Z_{2\varepsilon}$ is a contraction map for any $0 < \varepsilon \leqslant \delta$.

To this end, we prove that there exist constants $C_1, C_2 > 0$ such that for any $u, v \in Z_R$,

$$\|\Phi(u)\|_Z \leqslant \|e^{-(t-t_0)\partial_x^3}u_0\|_Z + C_1R^\alpha, \tag{4-1}$$
$$d_Z(\Phi(u), \Phi(v)) \leqslant C_2R^{\alpha-1}d_Z(u,v). \tag{4-2}$$

Let $u \in Z_R$. We infer from Proposition 3.2(ii) that

$$\|\Phi(u)\|_Z \leqslant \|e^{-t\partial_x^3}u_0\|_Z + C\big\||u|^{\alpha-1}u\big\|_{N(I;\frac{\alpha-1}{2})}.$$

We then apply Lemma 3.4(i) with $r = \frac{\alpha-1}{2}$ and $s = s_L(\alpha)$ to obtain (4-1). A similar argument, employing Lemma 3.4(ii), shows (4-2).

Now let us choose $\delta > 0$ so that

$$C_1(2\delta)^{\alpha-1} \leqslant \tfrac{1}{2}, \quad C_2(2\delta)^{\alpha-1} \leqslant \tfrac{1}{2}. \tag{4-3}$$

Then, we conclude from (4-1), (4-2), and the smallness assumption that Φ is a contraction map on $Z_{2\varepsilon}$. Therefore, the Banach fixed point theorem ensures that there exists a unique solution $u \in Z_{2\varepsilon}$ to (1-1).

We now suppose that $u_0 \in \hat{L}^{\frac{\alpha-1}{2}}$. By means of Proposition 3.2, we have

$$\|u\|_{L^\infty(I,\hat{L}^{(\alpha-1)/2})} \leqslant \|u_0\|_{\hat{L}^{(\alpha-1)/2}} + C\varepsilon^\alpha$$

as in (4-1). The same argument shows $u \in X\big(I; s, \frac{\alpha-1}{2}\big)$ for any s such that $\big(s, \frac{\alpha-1}{2}\big)$ is acceptable. □

Proof of Theorem 1.2. By Lemma 4.2, we obtain a unique solution

$$u \in L_t^\infty([-T,T]; \hat{L}_x^{\frac{\alpha-1}{2}}) \cap S\big([-T,T]; \tfrac{\alpha-1}{2}\big) \cap L\big([-T,T]; \tfrac{\alpha-1}{2}\big)$$

for small $T = T(u_0) > 0$. We repeat the above argument to extend the solution, and then obtain a solution which has a maximal lifespan. The regularity property (1-4) and the continuous dependence of solution on the initial data are shown by a usual way. This completes Theorem 1.2. □

Blowup criterion and scattering criterion. In this subsection we prove Theorems 1.7, 1.8, and 1.9.

Proof of Theorem 1.8. Assume for contradiction that $T_{\max} < \infty$ and $\|u\|_{S([0,T_{\max});\frac{\alpha-1}{2})} < \infty$.

Step 1. We first show that the above assumption yields

$$\|u\|_{L([0,T_{\max});\frac{\alpha-1}{2})} < \infty.$$

Fix T so that $0 < T < T_{\max}$. Let $s_L(\alpha)$ be as in the previous section (see Remark 4.1). If we take $\theta \in (0,1)$ so that $\left(\theta s_L(\alpha), \frac{\alpha-1}{2}\right)$ is conjugate-acceptable then it follows from Proposition 3.2 that

$$\|u\|_{L([0,T];\frac{\alpha-1}{2})} \leqslant C\|u_0\|_{\hat{L}^{\frac{\alpha-1}{2}}} + C\left\||u|^{\alpha-1}u\right\|_{Y([0,T];\theta s_L(\alpha),\frac{\alpha-1}{2})}.$$

Then, Lemma 3.4(i) with $r = \frac{\alpha-1}{2}$ and Lemma 3.3 give us

$$\|u\|_{L([0,T];\frac{\alpha-1}{2})} \leqslant C\|u_0\|_{\hat{L}^{\frac{\alpha-1}{2}}} + C\|u\|^{\alpha-\theta}_{S([0,T];\frac{\alpha-1}{2})}\|u\|^{\theta}_{L([0,T];\frac{\alpha-1}{2})}.$$

By assumption,

$$\|u\|_{S([0,T];\frac{\alpha-1}{2})} \leqslant \|u\|_{S([0,T_{\max});\frac{\alpha-1}{2})} < +\infty$$

for any $T \in (0, T_{\max})$. Plugging this to the previous estimate, we see that there exist constants $A, B > 0$ such that

$$\|u\|_{L([0,T];\frac{\alpha-1}{2})} \leqslant A + B\|u\|^{\theta}_{L([0,T];\frac{\alpha-1}{2})}$$

for any $T \in (0, T_{\max})$, which gives us the desired bound since $\theta < 1$.

Step 2. Let $t_0 \in (0, T_{\max})$. Since

$$u(t) = e^{-(t-t_0)\partial_x^3}u(t_0) + \mu\int_{t_0}^{t} e^{-(t-t')\partial_x^3}\partial_x(|u|^{\alpha-1}u)(t')\,dt'$$

for $t \in (0, T_{\max})$, the above estimate yields the following bound on $e^{-(t-t_0)\partial_x^3}u_0$:

$$\|e^{-(t-t_0)\partial_x^3}u(t_0)\|_{S([t_0,T_{\max});\frac{\alpha-1}{2})\cap L([t_0,T_{\max};\frac{\alpha-1}{2})}$$
$$\leqslant \|u\|_{S([t_0,T_{\max});\frac{\alpha-1}{2})\cap L([t_0,T_{\max};\frac{\alpha-1}{2})} + C\|u\|^{\alpha-1}_{S([t_0,T_{\max});\frac{\alpha-1}{2})}\|u\|_{L([t_0,T_{\max});\frac{\alpha-1}{2})} < \infty.$$

Step 3. Let us now prove that we can extend the solution beyond $T_{\max}$. Let δ be the constant given in Lemma 4.2. We see from the bound in the previous step that there exists $t_0 \in (0, T_{\max})$ such that

$$\|e^{-(t-t_0)\partial_x^3}u(t_0)\|_{S([t_0,T_{\max});\frac{\alpha-1}{2})} + \|e^{-(t-t_0)\partial_x^3}u(t_0)\|_{L([t_0,T_{\max});\frac{\alpha-1}{2})} \leqslant \tfrac{1}{2}\delta.$$

Hence, one can take $\tau > 0$ so that

$$\|e^{-(t-t_0)\partial_x^3}u(t_0)\|_{S([t_0,T_{\max}+\tau);\frac{\alpha-1}{2})} + \|e^{-(t-t_0)\partial_x^3}u(t_0)\|_{L([t_0,T_{\max}+\tau);\frac{\alpha-1}{2})} \leqslant \delta.$$

Then, just as in the proof of Theorem 1.2 (or Lemma 4.2), we can construct a solution $u(t)$ to (1-1) in the interval $(-T_{\min}, T_{\max}+\tau)$, which contradicts the definition of $T_{\max}$. □

Proof of Theorem 1.9. We first assume that $T_{\max} = +\infty$ and $\|u\|_{S([0,\infty);\frac{\alpha-1}{2})} < \infty$. Then, as in the first step of the proof of Theorem 1.8, one obtains $\|u\|_{L([0,\infty);\frac{\alpha-1}{2})} < \infty$. Since $\{e^{-t\partial_x^3}\}_{t\in\mathbb{R}}$ is an isometry in $\widehat{L}^{\frac{\alpha-1}{2}}$, it suffices to show that $\{e^{t\partial_x^3}u(t)\}_{t\in\mathbb{R}}$ is a Cauchy sequence in $\widehat{L}^{\frac{\alpha-1}{2}}$ as $t\to\infty$. Let $0<t_1<t_2$. By an argument similar to the proof of (4-2), we obtain

$$\begin{aligned}\big\|e^{t_2\partial_x^3}u(t_2)-e^{t_1\partial_x^3}u(t_1)\big\|_{\widehat{L}^{(\alpha-1)/2}} &\leqslant C\,\big\||u|^{\alpha-1}u\big\|_{N([t_1,\infty);\frac{\alpha-1}{2})}\\ &\leqslant C\,\|u\|^{\alpha-1}_{S([t_1,\infty);\frac{\alpha-1}{2})}\,\|u\|_{L([t_1,\infty);\frac{\alpha-1}{2})}\to 0 \quad \text{as } t_1\to\infty.\end{aligned}$$

Hence, we find that the solution to (1-1) scatters to a solution of the Airy equation as $t\to\infty$.

Conversely, if $u(t)$ scatters forward in time then we can choose $T>0$ so that

$$\|e^{-t\partial_x^3}u_+\|_{S([T,\infty);\frac{\alpha-1}{2})}+\|e^{-t\partial_x^3}u_+\|_{L([T,\infty);\frac{\alpha-1}{2})}\leqslant \tfrac{1}{2}\delta,$$

where $u_+=\lim_{t\to\infty}e^{t\partial_x^3}u(t)\in\widehat{L}^{\frac{\alpha-1}{2}}$ and δ is the constant given in Lemma 4.2. Moreover, it holds for sufficiently large $t_0\in[T,\infty)$ that

$$\begin{aligned}\big\|e^{-t\partial_x^3}(e^{t_0\partial_x^3}u(t_0)-u_+)\big\|_{S([T,\infty);\frac{\alpha-1}{2})}+\big\|e^{-t\partial_x^3}(e^{t_0\partial_x^3}u(t_0)-u_+)\big\|_{L([T,\infty);\frac{\alpha-1}{2})}&\\ \leqslant C\,\|e^{t_0\partial_x^3}u(t_0)-u_+\|_{\widehat{L}^{(\alpha-1)/2}}\leqslant \tfrac{1}{2}\delta&\end{aligned}$$

by means of (2-2). We then see that

$$\|e^{-(t-t_0)\partial_x}u(t_0)\|_{S([T,\infty);\frac{\alpha-1}{2})}+\|e^{-(t-t_0)\partial_x}u(t_0)\|_{L([T,\infty);\frac{\alpha-1}{2})}\leqslant\delta.$$

Then, Lemma 4.2 implies that $\|u\|_{S([T,\infty);\frac{\alpha-1}{2})}\leqslant 2\delta$. □

Proof of Theorem 1.7. By (2-2), we have

$$\|e^{-t\partial_x^3}u_0\|_{L(\mathbb{R};\frac{\alpha-1}{2})}+\|e^{-t\partial_x^3}u_0\|_{S(\mathbb{R};\frac{\alpha-1}{2})}\leqslant C\varepsilon.$$

Then, in light of Lemma 4.2, we see that u exists globally in time and satisfies $\|u\|_S\leqslant 2C\varepsilon$, provided ε is small compared with the constant δ given in Lemma 4.2. Theorem 1.9 ensures that u scatters for both time directions. □

Persistence of regularity. In this subsection, we prove Theorems 1.4 and 1.5, and then Theorem 1.10.

Proof of Theorem 1.4. Let us prove that $u\in L\big(I;\frac{\alpha_0-1}{2}\big)$. As in the proof of Lemma 4.2, one deduces from Proposition 3.2 and Lemma 3.4(i) that

$$\begin{aligned}\|u\|_{L(I;\frac{\alpha_0-1}{2})}&\leqslant C\,\|u_0\|_{\widehat{L}^{(\alpha_0-1)/2}}+C\,\big\||u|^{\alpha-1}u\big\|_{N(I;\frac{\alpha_0-1}{2})}\\ &\leqslant C\,\|u_0\|_{\widehat{L}^{r_0}}+C\,\|u\|^{\alpha-1}_{S(I;\frac{\alpha-1}{2})}\,\|u\|_{L(I;\frac{\alpha_0-1}{2})}.\end{aligned}$$

Since we already know $\|u\|_{S(I;\frac{\alpha-1}{2})}<\infty$ by assumption, we have the desired bound

$$\|u\|_{L(I;\frac{\alpha_0-1}{2})}\leqslant 2C\,\|u_0\|_{\widehat{L}^{(\alpha_0-1)/2}}$$

for a sufficiently short interval I. Then, again by Proposition 3.2,

$$\|u\|_{L_t^\infty(I;\widehat{L}_x^{(\alpha_0-1)/2})\cap X(I;s,\frac{\alpha_0-1}{2})}\leqslant C_s\|u_0\|_{\widehat{L}^{(\alpha_0-1)/2}}+C_s\|u\|^{\alpha-1}_{S(I;\frac{\alpha-1}{2})}\,\|u\|_{L(I;\frac{\alpha_0-1}{2})}<+\infty$$

for any acceptable pair $\big(s,\frac{\alpha_0-1}{2}\big)$. Finite-time use of this argument yields the result. □

Proof of Theorem 1.5. Suppose that $0 < \sigma < \alpha$. Take a number ε so that $0 < \varepsilon < \min(1, \alpha - \sigma)$. Since $|D_x|^\sigma$ commutes with $e^{-t\partial_x^3}$ and since $(\varepsilon, 2)$ is acceptable and conjugate-acceptable, we see from Proposition 3.2 that

$$\big\| |D_x|^\sigma u(t) \big\|_{X(I;\varepsilon,2)} \leqslant C \big\| |D_x|^\sigma u_0 \big\|_{L^2} + C \big\| |D_x|^\sigma (|u|^{\alpha-1}u) \big\|_{Y(I;\varepsilon,2)}.$$

Since $\sigma + \varepsilon < \alpha$, arguing as in the proof of Lemma 3.6, one sees that

$$\begin{aligned} \big\| |D_x|^\sigma (|u|^{\alpha-1}u) \big\|_{Y(I;\varepsilon,2)} &= \big\| |D_x|^{\sigma+\varepsilon} (|u|^{\alpha-1}u) \big\|_{L_x^{\tilde{p}(\varepsilon,2)} L_t^{\tilde{q}(\varepsilon,2)}(I)} \\ &\leqslant C \|u\|^{\alpha-1}_{L_x^{p(0,(\alpha-1)/2)} L_t^{q(0,(\alpha-1)/2)}(I)} \big\| |D_x|^{\sigma+\varepsilon} u \big\|_{L_x^{p(\varepsilon,2)} L_t^{q(\varepsilon,2)}(I)} \\ &= C \|u\|^{\alpha-1}_{S(I;\frac{\alpha-1}{2})} \big\| |D_x|^\sigma u \big\|_{X(I;\varepsilon,2)}. \end{aligned}$$

Hence, we obtain an upper bound for $\big\| |D_x|^\sigma u \big\|_{X(I;\varepsilon,2)}$ for a small interval. Then, the result follows as in Theorem 1.4.

Next, let $-1 < \sigma < 0$. Set $\varepsilon = -\sigma \in (0, 1)$. As in the previous case, we have

$$\big\| |D_x|^\sigma u(t) \big\|_{X(I;\varepsilon,2)} \leqslant C \big\| |D_x|^\sigma u_0 \big\|_{L^2} + C \big\| |D_x|^\sigma (|u|^{\alpha-1}u) \big\|_{Y(I;\varepsilon,2)}$$

since $(\varepsilon, 2)$ is acceptable and conjugate-acceptable. Then,

$$\big\| |D_x|^\sigma (|u|^{\alpha-1}u) \big\|_{Y(I;\varepsilon,2)} = \big\| |u|^{\alpha-1}u \big\|_{L_x^{\tilde{p}(\varepsilon,2)} L_t^{\tilde{q}(\varepsilon,2)}(I)} \leqslant \|u\|^{\alpha-1}_{S(I;\frac{\alpha-1}{2})} \big\| |D_x|^\sigma u \big\|_{X(I;\varepsilon,2)}$$

by Hölder's inequality. The rest of the argument is the same. □

Remark 4.3. In the above proposition, the upper bound $s < \alpha$ is natural in view of the regularity that the nonlinearity $|u|^{\alpha-1}u$ possesses. When α is an odd integer, that is, if $\alpha = 5, 7$, then the nonlinearity u^5 or u^7 is analytic (in u) and so we can remove the upper bound and treat all $s > 0$. We omit the details.

Remark 4.4. By modifying the proof of Theorem 1.5, we easily reproduce the local well-posedness in $\dot{H}^{s_\alpha}$ for $\alpha \geqslant 5$. More precisely, by Lemma 3.3,

$$\|u\|_{S(I;\frac{\alpha-1}{2})} \leqslant \big\| |D_x|^{s_\alpha} u \big\|^{\frac{8}{5(\alpha-1)}}_{X(I;-\frac{1}{4},2)} \big\| |D_x|^{\frac{2(9-\alpha)}{(5\alpha-13)(\alpha-1)}} u \big\|^{\frac{5\alpha-13}{5(\alpha-1)}}_{L_{t,x}^{(5\alpha-13)/2}(I)}.$$

By Sobolev's embedding in space and Minkowski's inequality,

$$\begin{aligned} \big\| |D_x|^{\frac{2(9-\alpha)}{(5\alpha-13)(\alpha-1)}} u \big\|_{L_{t,x}^{(5\alpha-13)/2}(I)} &\leqslant C \big\| |D_x|^{s_\alpha - \frac{5\alpha-33}{4(5\alpha-13)}} u \big\|_{L_t^{(5\alpha-13)/2} L_x^{(4(5\alpha-13))/(5\alpha-17)}(I)} \\ &\leqslant C \big\| |D_x|^{s_\alpha} u \big\|_{X(I;-\frac{1}{4}+\frac{5}{5\alpha-13},2)}. \end{aligned}$$

Hence, estimating as in the proof of Theorem 1.5, we obtain a closed estimate in

$$|D_x|^{-s_\alpha} X(I;\varepsilon,2) \cap |D_x|^{-s_\alpha} X\big(I; -\tfrac{1}{4} + \tfrac{5}{5\alpha-13}, 2\big) \cap |D_x|^{-s_\alpha} X\big(I; -\tfrac{1}{4}, 2\big),$$

which yields local well-posedness in $\dot{H}^{s_\alpha}$.[5]

[5] Strictly speaking, we should work with pairs $\big(-\frac{1}{4} + \eta_1, 2\big)$ and $\big(-\frac{1}{4} + \frac{5}{5\alpha-13} - \eta_2, 2\big)$ for small $\eta_j = \eta_j(\alpha) > 0$ because the critical case $q\big(-\frac{1}{4}, 2\big) = \infty$ is excluded in Lemma 3.3. However, the modification is obvious.

Proof of Theorem 1.10. We suppose for contradiction that $u(t)$ scatters to $u_+ \in \widehat{L}^{\frac{\alpha-1}{2}}$ as $t \to \infty$. Since $u_0 \in H^1$, Theorems 1.4 and 1.5 imply that $u(t) \in C(\mathbb{R}; H^1)$. Further, $u(t)$ scatters also in H^1 and so we see that $\|\partial_x u(t)\|_{L^2} = \|\partial_x e^{t\partial_x^3} u(t)\|_{L^2} \to \|u_+\|_{\dot{H}^1}$ as $t \to \infty$.

On the other hand, by the Gagliardo–Nirenberg inequality and mass conservation,

$$\|u(t)\|_{L_x^{\alpha+1}} \leqslant C \|u_0\|_{L_x^2}^{\frac{2}{\alpha+1}} \left\| |D_x|^{\frac{2}{3(\alpha-1)}} u(t) \right\|_{L_x^{(3(\alpha-1))/2}}^{\frac{\alpha-1}{\alpha+1}}.$$

Since $u(t)$ scatters as $t \to \infty$, we see that $u \in X\big([0,\infty); \frac{2}{3(\alpha-1)}, \frac{\alpha-1}{2}\big)$ as in the proof of Theorem 1.9. Therefore, we can take a sequence $\{t_n\}_n$ with $t_n \to \infty$ as $n \to \infty$ so that $\|u(t_n)\|_{L^{\alpha+1}} \to 0$ as $n \to \infty$. Thus, by conservation of energy,

$$0 \geqslant E[u_0] = E[u(t_n)] = \tfrac{1}{2}\|\partial_x u(t_n)\|_{L^2}^2 - \frac{\mu}{\alpha+1}\|u(t_n)\|_{L^{\alpha+1}}^{\alpha+1} \to \tfrac{1}{2}\|u_+\|_{\dot{H}^1}^2$$

as $n \to \infty$. Hence, $E[u_0] < 0$ yields a contradiction. If $E[u_0] = 0$ then we see that $u_+ = 0$, and so $\|u_0\|_{L^2} = \|u_+\|_{L^2} = 0$. This contradicts $u_0 \neq 0$. □

Appendix A: Proof of Lemma 3.7

In this appendix, we prove Lemma 3.7. To prove this lemma, we need the following space-time bounds of the maximal function

$$(\mathcal{M}u)(x) = \sup_{R>0} \frac{1}{2R} \int_{x-R}^{x+R} |u(y)|\, dy.$$

Lemma A.1. *Let I be an interval. Assume $1 < p, q < \infty$.*

(i) *There exists a positive constant C depending on p, q and I such that*

$$\|\mathcal{M} f\|_{L_x^p L_t^q(I)} \leqslant C \|f\|_{L_x^p L_t^q(I)} \tag{A-1}$$

for any $f \in L_x^p L_t^q(I)$.

(ii) *There exists a positive constant C depending on p, q and I such that*

$$\|\mathcal{M} f_k\|_{L_x^p L_t^q \ell_k^2(I)} \leqslant C \|f_k\|_{L_x^p L_t^q \ell_k^2(I)} \tag{A-2}$$

for any $\{f_k\}_k \in L_x^p L_t^q \ell_k^2(I)$.

Proof of Lemma A.1. See [Fefferman and Stein 1971] for (A-1) and [Kenig et al. 1993, Lemma A.3(e)] for (A-2). □

Proof of Lemma 3.7. We follow [Sickel 1989] (see also [Runst and Sickel 1996]). Let $\{\varphi_k(D_x)\}_{k=-\infty}^{\infty}$ be a Littlewood–Paley decomposition with respect to the x-variable. From [Kenig et al. 1993, Lemma A.3], we see

$$\big\| |D_x|^s f \big\|_{L_x^p L_t^q} \sim \|2^{sk} \varphi_k(D_x) f\|_{L_x^p L_t^q \ell_k^2}. \tag{A-3}$$

Step 1. Write $\mu = N + \beta$ with $N \in \mathbb{Z}$ and $\beta \in (0, 1]$. We remark that $N \geqslant 1$ since $\mu > 1$. We first note that Taylor's expansion of G gives us

$$\begin{aligned} G(z) &= \sum_{l=0}^{N-1} \frac{G^{(l)}(a)}{\ell!}(z-a)^l + \int_a^z \frac{(z-v)^{N-1}}{(N-1)!} G^{(N)}(v)\,dv \\ &= \sum_{l=0}^{N} \frac{G^{(l)}(a)}{\ell!}(z-a)^l + \int_a^z \frac{(z-v)^{N-1}}{(N-1)!}\big(G^{(N)}(v) - G^{(N)}(a)\big)\,dv \\ &= \sum_{l=0}^{N}\sum_{j=0}^{l} \frac{(-1)^{l-j} G^{(l)}(a) a^{l-j}}{(\ell-j)!\,j!} z^j + \int_a^z \frac{(z-v)^{N-1}}{(N-1)!}\big(G^{(N)}(v) - G^{(N)}(a)\big)\,dv. \end{aligned}$$

Hence, applying the above expansion with $z = f(y)$ and $a = f(x)$,

$$\begin{aligned} \mathcal{F}^{-1}[\varphi_k \mathcal{F} G(f)](x) &= c\int_{\mathbb{R}^n} (\mathcal{F}^{-1}\varphi_k)(x-y) G(f(y))\,dy \\ &= c\sum_{l=0}^{N}\sum_{j=0}^{l} \frac{(-1)^{l-j} G^{(l)}(f(x))(f(x))^{l-j}}{(\ell-j)!\,j!} \int_{\mathbb{R}^n} (\mathcal{F}^{-1}\varphi_k)(x-y)(f(y))^j\,dy \\ &\quad + c\int_{\mathbb{R}^n} (\mathcal{F}^{-1}\varphi_k)(x-y) \int_{f(x)}^{f(y)} \frac{(f(y)-v)^{N-1}}{(N-1)!}\big(G^{(N)}(v) - G^{(N)}(f(x))\big)\,dv\,dy \\ &=: T_{1,k} + T_{2,k}. \end{aligned} \tag{A-4}$$

We first estimate $T_{1,k}$. Since $\int \mathcal{F}^{-1}\varphi_k(y)\,dy = \varphi_k(0) = 0$, the summand in $T_{1,k}$ vanishes if $j = 0$. By the estimate

$$|G^{(l)}(f(x))| \leqslant \|G\|_{\mathrm{Lip}\,\mu}\, |f(x)|^{\mu-l},$$

we have

$$\begin{aligned} \|2^{sk} T_{1,k}\|_{L_x^p L_t^q \ell_k^2} &\leqslant C\|G\|_{\mathrm{Lip}\,\mu} \sum_{j=1}^{N} \big\| |f|^{\mu-j}\, 2^{sk} \varphi_k(D_x)(f^j)| \big\|_{L_x^p L_t^q \ell_k^2} \\ &\leqslant C\|G\|_{\mathrm{Lip}\,\mu} \sum_{j=1}^{N} \|f\|_{L_x^{p_1} L_t^{q_1}}^{\mu-j} \big\| |D_x|^s (f^j)\big\|_{L_x^{p_{2,j}} L_t^{q_{2,j}}}, \end{aligned}$$

where

$$\frac{1}{p} = \frac{\mu-j}{p_1} + \frac{1}{p_{2,j}}, \qquad \frac{1}{q} = \frac{\mu-j}{q_1} + \frac{1}{q_{2,j}}.$$

Further, a recursive use of Lemma 3.6 yields

$$\big\| |D_x|^s (f^j)\big\|_{L_x^{p_{2,j}} L_t^{q_{2,j}}} \leqslant C_j \|f\|_{L_x^{p_1} L_t^{q_1}}^{j-1} \big\| |D_x|^s f\big\|_{L_x^{p_2} L_t^{q_2}}$$

for $j \geqslant 2$, which completes the estimate of $T_{1,k}$.

Next, we estimate $T_{2,k}$. First note that

$$\left| \int_{f(x)}^{f(y)} \frac{(f(y)-v)^{N-1}}{(N-1)!}\big(G^{(N)}(v) - G^{(N)}(f(x))\big)\,dv \right| \leqslant C\|G\|_{\mathrm{Lip}\,\mu}\, |f(x)-f(y)|^{\mu}$$

by the definition of $\|G\|_{\mathrm{Lip}\,\mu}$. Further, for any $M > 0$, there exists C_M such that

$$|(\mathcal{F}^{-1}\varphi_k)(x-y)| = 2^k \big|(\mathcal{F}^{-1}\varphi_0)(2^k(x-y))\big| \leqslant C_M 2^k (1+2^k|x-y|)^{-M}.$$

Therefore,

$$|T_{2,k}| \leqslant C 2^k \|G\|_{\mathrm{Lip}\,\mu} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^\mu}{(1+2^k|x-y|)^M}\,dy \leqslant C \sum_{l=0}^{\infty} 2^{k-lM} (I_{k-l}^\mu f)(x),$$

where

$$I_k^\mu f(x) = \int_{|z|\leqslant 2^{-k}} |f(x+z)-f(x)|^\mu\,dz = 2^{-k}\int_{|z|\leqslant 1} |f(x+2^{-k}z)-f(x)|^\mu\,dz.$$

We now claim that

$$\|2^{k(s+1)}(I_k^\mu f)\|_{L_x^p L_t^q \ell_k^2} \leqslant C \big\| |D_x|^{\frac{s}{\mu}} f \big\|_{L_x^{\mu p} L_t^{\mu q}}^{\mu}. \tag{A-5}$$

This claim completes the proof. Indeed, combining the above estimates, we see that

$$\|2^{sk} T_{2,k}\|_{L_x^p L_t^q \ell_k^2} \leqslant C \sum_{l=0}^{\infty} 2^{l(s+1-M)} \big\|2^{(k-l)(s+1)}(I_{k-l}^\mu f)\big\|_{L_x^p L_t^q \ell_k^2} \leqslant C \big\| |D_x|^{\frac{s}{\mu}} f \big\|_{L_x^{\mu p} L_t^{\mu q}}^{\mu},$$

provided we choose $M > s+1$. By Lemma 3.3, we conclude that

$$\big\| |D_x|^{\frac{s}{\mu}} f \big\|_{L_x^{\mu p} L_t^{\mu q}} \leqslant \|f\|_{L_x^{p_1} L_t^{q_1}}^{1-\frac{1}{\mu}} \big\| |D_x|^s f \big\|_{L_x^{p_2} L_t^{q_2}}^{\frac{1}{\mu}}.$$

Step 2. We prove claim (A-5). Let Δ_h be the difference operator $\Delta_h f(x) = f(x+h) - f(x)$. Since $f = \sum_{m\in\mathbb{Z}} \varphi_{k+m}(D_x) f$ for any $k \in \mathbb{Z}$, one sees that

$$\begin{aligned}
\big\|2^{k(s+1)}(I_k^\mu f)(x)\big\|_{L_x^p L_t^q \ell_k^2} &= \bigg\| 2^{ks} \int_{|z|\leqslant 1} |\Delta_{2^{-k}z} f(x)|^\mu\,dz \bigg\|_{L_x^p L_t^q \ell_k^2} \\
&\leqslant \bigg\| 2^{ks} \int_{|z|\leqslant 1} \bigg|\Delta_{2^{-k}z} \sum_{m=-\infty}^{-1} \varphi_{k+m}(D) f(x)\bigg|^\mu dz \bigg\|_{L_x^p L_t^q \ell_k^2} \\
&\quad + \bigg\| 2^{ks} \int_{|z|\leqslant 1} \bigg|\Delta_{2^{-k}z} \sum_{m=0}^{\infty} \varphi_{k+m}(D) f(x)\bigg|^\mu dz \bigg\|_{L_x^p L_t^q \ell_k^2} \\
&=: A + B.
\end{aligned}$$

We estimate A. Take $a \in \big(\frac{1}{\mu}, 1\big)$. Let $k \in \mathbb{Z}$. If $m < 0$ and $|h| \leqslant 2^{-k}$ then we have

$$\begin{aligned}
\big|\Delta_h \mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x)\big| &\leqslant |h| \big|\nabla(\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f])(x+\theta h)\big| \\
&\leqslant 2^m \sup_{|y|\leqslant 2^{-k}} \bigg| \bigg(\nabla \mathcal{F}^{-1}\bigg[\varphi_0 \mathcal{F}\bigg[f\bigg(\frac{\cdot}{2^{k+m}}\bigg)\bigg]\bigg]\bigg)(2^{k+m}(x-y)) \bigg| \\
&\leqslant C_a 2^m \sup_{y\in\mathbb{R}} \frac{\big|\big(\nabla\mathcal{F}^{-1}\big[\varphi_0\mathcal{F}\big[f\big(\frac{\cdot}{2^{k+m}}\big)\big]\big]\big)(2^{k+m}(x-y))\big|}{1+|2^{k+m}y|^a} \\
&\leqslant C_a 2^m \sup_{y\in\mathbb{R}} \frac{|\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x-y)|}{1+|2^{k+m}y|^a}
\end{aligned}$$

for any $x \in \mathbb{R}$, where we have used the estimate

$$\sup_{y\in\mathbb{R}} \frac{|\nabla \mathcal{F}^{-1}[\varphi_0 \mathcal{F} f](x-y)|}{1+|y|^a} \leqslant C \sup_{y\in\mathbb{R}} \frac{|\mathcal{F}^{-1}[\varphi_0 \mathcal{F} f](x-y)|}{1+|y|^a}$$

(see [Runst and Sickel 1996, Section 2.1.6, Proposition 2(i)]) to obtain the last line. We define the Peetre–Fefferman–Stein maximal function by

$$\varphi_j^{*,a} f(x) := \sup_{y\in\mathbb{R}} \frac{|\mathcal{F}^{-1}[\varphi_j \mathcal{F} f](x-y)|}{1+|2^j y|^a}.$$

By the above estimates, we have

$$\begin{aligned} A &\leqslant C \Big\| 2^{ks} \sum_{m=-\infty}^{-1} \sup_{|z|\leqslant 1} \big|\Delta_{2^{-k}z} \varphi_{k+m}(D) f(x)\big|^{\mu} \Big\|_{L_x^p L_t^q \ell_k^2} \leqslant C \sum_{m=-\infty}^{-1} 2^{m\mu} \big\| 2^{k\frac{s}{\mu}} \varphi_{k+m}^{*,a} f \big\|_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2\mu}}^{\mu} \\ &\leqslant C \sum_{m=-\infty}^{-1} 2^{m(\mu-s)} \big\| 2^{(k+m)\frac{s}{\mu}} \varphi_{k+m}^{*,a} f \big\|_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2\mu}}^{\mu} \leqslant C \big\| 2^{k\frac{s}{\mu}} \varphi_k^{*,a} f \big\|_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2\mu}}^{\mu}, \end{aligned}$$

where we used the fact that $s<\mu$. Since $(\varphi_k^{*,a} f)(x) = \big(\varphi_0^{*,a}(\tilde{\varphi}_k(D_x) f)\big(\frac{\cdot}{2^k}\big)\big)(2^k x)$, [Triebel 1983, Lemma 2.3.6] yields

$$(\varphi_k^{*,a} f)(x) \leqslant C \big(\mathcal{M}[(\tilde{\varphi}_k(D_x) f)^{\frac{1}{a}}]\big)^a (x),$$

where $\tilde{\varphi}_k = \sum_{i=-1}^{1} \varphi_{k+i}$. Then, (A-2), the embedding $\ell^2 \hookrightarrow \ell^q$ $(2<q\leqslant\infty)$, and (A-3) lead us to

$$\begin{aligned} \big\| 2^{k\frac{s}{\mu}} \varphi_k^{*,a} f \big\|_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2\mu}} &\leqslant C \big\| 2^{k\frac{s}{a\mu}} \mathcal{M}[(\tilde{\varphi}_k(D_x) f)^{\frac{1}{a}}] \big\|_{L_x^{a\mu p} L_t^{a\mu q} \ell_k^{2a\mu}}^{a} \\ &\leqslant C \big\| 2^{k\frac{s}{a\mu}} (\tilde{\varphi}_k(D_x) f)^{\frac{1}{a}} \big\|_{L_x^{a\mu p} L_t^{a\mu q} \ell_k^{2}}^{a} \\ &\leqslant C \big\| 2^{k\frac{s}{\mu}} \tilde{\varphi}_k(D_x) f \big\|_{L_x^{\mu p} L_t^{\mu q} \ell_k^{2/a}} \leqslant C \big\| |D_x|^{\frac{s}{\mu}} f \big\|_{L_x^{\mu p} L_t^{\mu q}} \end{aligned}$$

since $\frac{1}{\mu} < a < 1$.

Let us proceed to the estimate of B. We first note that

$$\begin{aligned} &\int_{|z|\leqslant 1} \Big| \Delta_{2^{-k}z} \sum_{m=0}^{\infty} \varphi_{k+m}(D) f(x) \Big|^{\mu} dz \\ &\qquad = \int_{|z|\leqslant 1} \Big| \sum_{m=0}^{\infty} 2^{-\frac{\varepsilon}{\mu} m} 2^{\frac{\varepsilon}{\mu} m} \Delta_{2^{-k}z} \varphi_{k+m}(D) f(x) \Big|^{\mu} dz \\ &\qquad \leqslant C_\varepsilon \int_{|z|\leqslant 1} \sum_{m=0}^{\infty} 2^{\varepsilon m} \big| \Delta_{2^{-k}z} \varphi_{k+m}(D) f(x) \big|^{\mu} dz \\ &\qquad = C_\varepsilon \sum_{m=0}^{\infty} 2^{\varepsilon m} \int_{|z|\leqslant 1} \big| \Delta_{2^{-k}z} \varphi_{k+m}(D) f(x) \big|^{\mu} dz \\ &\qquad \leqslant C \sum_{m=0}^{\infty} 2^{\varepsilon m} \big(\sup_{|z|\leqslant 1} \big| \Delta_{2^{-k}z} \varphi_{k+m}(D) f(x) \big| \big)^{\mu(1-\lambda)} \int_{|z|\leqslant 1} \big| \Delta_{2^{-k}z} \varphi_{k+m}(D) f(x) \big|^{\mu\lambda} dz, \end{aligned}$$

where $\lambda \in (0,1)$. For $m \geqslant 0$ and $|h| \leqslant 2^{-k}$, the triangle inequality gives us

$$\left|\Delta_h \mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x)\right| \leqslant 2 \sup_{|y|\leqslant 2^{-k}} \left|\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x-y)\right| \leqslant C2^{ma}\varphi^{*,a}_{k+m}f(x),$$

where $a \in \left(\frac{1}{\mu}, 1\right)$. Further,

$$\int_{|z|\leqslant 1} \left|\Delta_{2^{-k}z}\varphi_{k+m}(D_x)f(x)\right|^{\mu\lambda} dz \leqslant C\mathcal{M}[|\varphi_{k+m}(D)_x f|^{\mu\lambda}](x).$$

Using these inequalities, one deduces from Hölder's inequality, the embedding $\ell^2 \hookrightarrow \ell^q$ $(2 < q \leqslant \infty)$, (A-2), and (A-3) that

$$\begin{aligned} B &\leqslant C\left\|2^{sk}\sum_{m=0}^{\infty} 2^{m\varepsilon}\mathcal{M}[|\varphi_{k+m}(D_x)f|^{\mu\lambda}]2^{ma\mu(1-\lambda)}(\varphi^{*,a}_{k+m}f)^{\mu(1-\lambda)}\right\|_{L^p_x L^q_t \ell^2_k} \\ &\leqslant C\sum_{m=0}^{\infty} 2^{m(\varepsilon+a\mu(1-\lambda))}\left\|2^{sk}\mathcal{M}[|\varphi_{k+m}(D_x)f|^{\mu\lambda}](\varphi^{*,a}_{k+m}f)^{\mu(1-\lambda)}\right\|_{L^p_x L^q_t \ell^2_k} \\ &\leqslant C\sum_{m=0}^{\infty} 2^{m(\varepsilon+a\mu(1-\lambda)-s)}\left\|\mathcal{M}[|2^{\frac{s}{\mu}k}\varphi_k(D_x)f|^{\mu\lambda}]\right\|_{L^{p/\lambda}_x L^{q/\lambda}_t \ell^{2/\lambda}_k}\left\|2^{\frac{s}{\mu}k}\varphi^{*,a}_k f\right\|^{\mu(1-\lambda)}_{L^{\mu p}_x L^{\mu q}_t \ell^{2\mu}_k} \\ &\leqslant C\sum_{m=0}^{\infty} 2^{m(\varepsilon+a\mu(1-\lambda)-s)}\left\||D_x|^{\frac{s}{\mu}}f\right\|^{\mu}_{L^{\mu p}_x L^{\mu q}_t} \\ &\leqslant \left\||D_x|^{\frac{s}{\mu}}f\right\|^{\mu}_{L^{\mu p}_x L^{\mu q}_t} \end{aligned}$$

as long as $\varepsilon + a\mu(1-\lambda) - s < 0$. Since $a \in \left(\frac{1}{\mu}, 1\right)$, we are able to choose $\lambda \in (0,1)$ and $\varepsilon > 0$ suitably. □

Appendix B: Inclusion relations of $\widehat{L}^r$

In this appendix, we briefly summarize some inclusion relations between $\widehat{L}^r$ and other frequently used spaces such as Lebesgue spaces or Sobolev spaces. Here, $\dot{H}^{0,s} = \dot{H}^{0,s}(\mathbb{R})$ stands for a weighted L^2-space with norm $\|f\|_{\dot{H}^{0,s}} = \left\||x|^s f\right\|_{L^2}$.

Lemma B.1. *We have the following*:

(i) $L^r \hookrightarrow \widehat{L}^r$ *if* $1 \leqslant r \leqslant 2$ *and* $\widehat{L}^r \hookrightarrow L^r$ *if* $2 \leqslant r \leqslant \infty$.

(ii) $\dot{H}^{0,\frac{1}{r}-\frac{1}{2}} \hookrightarrow \widehat{L}^r$ *if* $1 < r \leqslant 2$ *and* $\widehat{L}^r \hookrightarrow \dot{H}^{0,\frac{1}{r}-\frac{1}{2}}$ *if* $2 \leqslant r < \infty$.

(iii) $\widehat{L}^r \hookrightarrow \dot{B}^{\frac{1}{2}-\frac{1}{r}}_{2,r'}$ *if* $1 \leqslant r \leqslant 2$ *and* $\dot{B}^{\frac{1}{2}-\frac{1}{r}}_{2,r'} \hookrightarrow \widehat{L}^r$ *if* $2 \leqslant r \leqslant \infty$.

Proof of Lemma B.1. The first assertion follows from the Hausdorff–Young inequality. The Sobolev embedding (in the Fourier side) yields the second. We omit the details.

The third is also immediate from the Hölder inequality. Indeed, if $2 \leqslant r \leqslant \infty$ then

$$\|\hat{f}\|_{L^{r'}(\{2^n\leqslant|\xi|\leqslant 2^{n+1}\})} \leqslant C2^{n(\frac{1}{2}-\frac{1}{r})}\|\hat{f}\|_{L^2(\{2^n\leqslant|\xi|\leqslant 2^{n+1}\})}$$

for any $n \in \mathbb{Z}$. Taking the $\ell_n^{r'}$-norm, we obtain the desired embedding. The case $1 \leqslant r \leqslant 2$ follows in the same way. □

Let $\dot{H}^s = \dot{H}^s(\mathbb{R})$ be a homogeneous Sobolev space with norm

$$\|f\|_{\dot{H}^s} = \big\| |\xi|^s \hat{f} \big\|_{L^2}.$$

Notice that the above inclusions are the same as for $\dot{H}^{\frac{1}{2}-\frac{1}{r}}$. Namely, we can replace $\widehat{L}^r$ with $\dot{H}^{\frac{1}{2}-\frac{1}{r}}$ in Lemma B.1 (except for the endpoint case $r = 1, \infty$ in (i)). Indeed, (i) is a Sobolev embedding, (ii) follows from Hardy's inequality, and a basic property of Besov spaces gives us (iii). However, there is no inclusion between $\widehat{L}^r$ and $\dot{H}^{\frac{1}{2}-\frac{1}{r}}$ for $r \neq 2$.

Lemma B.2. *For* $1 \leqslant r \leqslant \infty$ $(r \neq 2)$, $\widehat{L}^r \not\hookrightarrow \dot{H}^{\frac{1}{2}-\frac{1}{r}}$ *and* $\dot{H}^{\frac{1}{2}-\frac{1}{r}} \not\hookrightarrow \widehat{L}^r$.

Proof of Lemma B.2. If $2 < r \leqslant \infty$, we have the following counterexamples: Let us define $f_n(x)$ by $\widehat{f_n}(\xi) = 1$ for $n \leqslant \xi \leqslant n+1$ and $\widehat{f_n}(\xi) = 0$ elsewhere. Then, $f_n(x)$ satisfies $\|f_n\|_{\dot{H}^{\frac{1}{2}-\frac{1}{r}}} \to \infty$ as $n \to \infty$, while $\|f_n\|_{\widehat{L}^r}=1$. Hence. $\widehat{L}^r \not\hookrightarrow \dot{H}^{\frac{1}{2}-\frac{1}{r}}$. On the other hand, for some $p \in \big(\frac{1}{2}, \frac{1}{r'}\big)$, take $g_n(x)$ $(n \geqslant 3)$ so that $\widehat{g_n}(\xi) = \xi^{-\frac{1}{r'}} |\log \xi|^{-p}$ for $\frac{1}{n} \leqslant \xi \leqslant \frac{1}{2}$ and $\widehat{g_n}(\xi) = 0$ elsewhere. Then, $\|g_n\|_{\dot{H}^{\frac{1}{2}-\frac{1}{r}}}$ is bounded but $\|g_n\|_{\widehat{L}^r} \to \infty$ as $n \to \infty$. This shows $\dot{H}^{\frac{1}{2}-\frac{1}{r}} \not\hookrightarrow \widehat{L}^r$.

The case $1 < r < 2$ follows by duality.

Let us consider the case $r = 1$. We note that $\delta_0(x) \in \widehat{L}^1 \setminus \dot{H}^{-\frac{1}{2}}$, where $\delta_0(x)$ is the Dirac delta function. Therefore, $\widehat{L}^1 \not\hookrightarrow \dot{H}^{-\frac{1}{2}}$. On the other hand,

$$f_n(x) = \big(\log\big(1 + \tfrac{1}{n}\big)\big)^{-1} \mathcal{F}^{-1}[\mathbf{1}_{\{1 \leqslant \xi \leqslant 1+\frac{1}{n}\}}](x)$$

is a counterexample for $\dot{H}^{-\frac{1}{2}} \not\hookrightarrow \widehat{L}^1$. □

Acknowledgments

The authors express their deep gratitude to Professor Yoshio Tsutsumi for valuable comments on a preliminary version of the manuscript. They would particularly like to thank Professor Grünrock for drawing their attention to the article [Grünrock 2004] in which a more general version of Lemma 2.2 is proved. Part of this work was done while the authors were visiting the Department of Mathematics at the University of California, Santa Barbara, whose hospitality they gratefully acknowledge. Masaki is partially supported by JSPS, Grant-in-Aid for Young Scientists (B) 24740108. Segata is partially supported by JSPS, Strategic Young Researcher Overseas Visits Program for Accelerating Brain Circulation and by MEXT, Grant-in-Aid for Young Scientists (A) 25707004.

References

[Bourgain 1993] J. Bourgain, "Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II: the KdV-equation", *Geom. Funct. Anal.* **3**:3 (1993), 209–262. MR 1215780 Zbl 0787.35098

[Buckmaster and Koch 2015] T. Buckmaster and H. Koch, "The Korteweg–de Vries equation at H^{-1} regularity", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32**:5 (2015), 1071–1098. MR 3400442 Zbl 06507528

[Christ and Kiselev 2001] M. Christ and A. Kiselev, "Maximal functions associated to filtrations", *J. Funct. Anal.* **179**:2 (2001), 409–425. MR 1809116 Zbl 0974.47025

[Christ and Weinstein 1991] F. M. Christ and M. I. Weinstein, "Dispersion of small amplitude solutions of the generalized Korteweg–de Vries equation", *J. Funct. Anal.* **100**:1 (1991), 87–109. MR 1124294 Zbl 0743.35067

[Fefferman 1970] C. Fefferman, "Inequalities for strongly singular convolution operators", *Acta Math.* **124** (1970), 9–36. MR 0257819 Zbl 0188.42601

[Fefferman and Stein 1971] C. Fefferman and E. M. Stein, "Some maximal inequalities", *Amer. J. Math.* **93** (1971), 107–115. MR 0284802 Zbl 0222.26019

[Fernandez 1987] D. L. Fernandez, "Vector-valued singular integral operators on L^p-spaces with mixed norms and applications", *Pacific J. Math.* **129**:2 (1987), 257–275. MR 909030 Zbl 0634.42014

[Grünrock 2004] A. Grünrock, "An improved local well-posedness result for the modified KdV equation", *Int. Math. Res. Not.* **2004**:61 (2004), 3287–3308. MR 2096258 Zbl 1072.35161

[Grünrock 2005a] A. Grünrock, "Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS", *Int. Math. Res. Not.* **2005**:41 (2005), 2525–2558. MR 2181058 Zbl 1088.35063

[Grünrock 2005b] A. Grünrock, "A bilinear Airy-estimate with application to gKdV-3", *Differential Integral Equations* **18**:12 (2005), 1333–1339. MR 2174975 Zbl 1212.35412

[Grünrock and Vega 2009] A. Grünrock and L. Vega, "Local well-posedness for the modified KdV equation in almost critical $\widehat{H}^r_s$-spaces", *Trans. Amer. Math. Soc.* **361**:11 (2009), 5681–5694. MR 2529909 Zbl 1182.35197

[Guo 2009] Z. Guo, "Global well-posedness of Korteweg–de Vries equation in $H^{-3/4}(\mathbb{R})$", *J. Math. Pures Appl.* (9) **91**:6 (2009), 583–597. MR 2531556 Zbl 1173.35110

[Hirschman 1952] I. I. Hirschman, Jr., "A convexity theorem for certain groups of transformations", *J. Analyse Math.* **2**:2 (1952), 209–218. MR 0057936 Zbl 0052.06302

[Hyakuna and Tsutsumi 2012] R. Hyakuna and M. Tsutsumi, "On existence of global solutions of Schrödinger equations with subcritical nonlinearity for $\widehat{L}^p$-initial data", *Proc. Amer. Math. Soc.* **140**:11 (2012), 3905–3920. MR 2944731 Zbl 1283.35126

[Kato 1983] T. Kato, "On the Cauchy problem for the (generalized) Korteweg–de Vries equation", pp. 93–128 in *Studies in applied mathematics*, edited by V. Guillemin, Adv. Math. Suppl. Stud. **8**, Academic Press, New York, 1983. MR 759907 Zbl 0549.34001

[Kenig and Ruiz 1983] C. E. Kenig and A. Ruiz, "A strong type (2, 2) estimate for a maximal operator associated to the Schrödinger equation", *Trans. Amer. Math. Soc.* **280**:1 (1983), 239–246. MR 712258 Zbl 0525.42011

[Kenig et al. 1991] C. E. Kenig, G. Ponce, and L. Vega, "Oscillatory integrals and regularity of dispersive equations", *Indiana Univ. Math. J.* **40**:1 (1991), 33–69. MR 1101221 Zbl 0738.35022

[Kenig et al. 1993] C. E. Kenig, G. Ponce, and L. Vega, "Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle", *Comm. Pure Appl. Math.* **46**:4 (1993), 527–620. MR 1211741 Zbl 0808.35128

[Kenig et al. 1996] C. E. Kenig, G. Ponce, and L. Vega, "A bilinear estimate with applications to the KdV equation", *J. Amer. Math. Soc.* **9**:2 (1996), 573–603. MR 1329387 Zbl 0848.35114

[Kishimoto 2009] N. Kishimoto, "Well-posedness of the Cauchy problem for the Korteweg–de Vries equation at the critical regularity", *Differential Integral Equations* **22**:5-6 (2009), 447–464. MR 2501679 Zbl 1240.35461

[Koch and Marzuola 2012] H. Koch and J. L. Marzuola, "Small data scattering and soliton stability in $\dot{H}^{-1/6}$ for the quartic KdV equation", *Anal. PDE* **5**:1 (2012), 145–198. MR 2957553 Zbl 1267.35184

[Korteweg and de Vries 1895] D. J. Korteweg and G. de Vries, "On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves", *Phil. Mag.* (5) **39** (1895). JFM 26.0881.02

[Lamb 1977] G. L. Lamb, Jr., "Solitons on moving space curves", *J. Mathematical Phys.* **18**:8 (1977), 1654–1661. MR 0440173 Zbl 0351.35019

[Molinet and Ribaud 2003] L. Molinet and F. Ribaud, "On the Cauchy problem for the generalized Korteweg–de Vries equation", *Comm. Partial Differential Equations* **28**:11-12 (2003), 2065–2091. MR 2015413 Zbl 1059.35124

[Molinet and Ribaud 2004] L. Molinet and F. Ribaud, "Well-posedness results for the generalized Benjamin–Ono equation with small initial data", *J. Math. Pures Appl.* (9) **83**:2 (2004), 277–311. MR 2038121 Zbl 1084.35094

[Runst and Sickel 1996] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications **3**, Walter de Gruyter & Co., Berlin, 1996. MR 1419319 Zbl 0873.35001

[Sickel 1989] W. Sickel, "On boundedness of superposition operators in spaces of Triebel–Lizorkin type", *Czechoslovak Math. J.* **39(114)**:2 (1989), 323–347. MR 992137 Zbl 0693.46039

[Stein 1956] E. M. Stein, "Interpolation of linear operators", *Trans. Amer. Math. Soc.* **83** (1956), 482–492. MR 0082586 Zbl 0072.32402

[Strunk 2014] N. Strunk, "Well-posedness for the supercritical gKdV equation", *Commun. Pure Appl. Anal.* **13**:2 (2014), 527–542. MR 3117359 Zbl 1307.35266

[Tao 2007] T. Tao, "Scattering for the quartic generalised Korteweg–de Vries equation", *J. Differential Equations* **232**:2 (2007), 623–651. MR 2286393 Zbl 1171.35107

[Tao et al. 1998] T. Tao, A. Vargas, and L. Vega, "A bilinear approach to the restriction and Kakeya conjectures", *J. Amer. Math. Soc.* **11**:4 (1998), 967–1000. MR 1625056 Zbl 0924.42008

[Tomas 1975] P. A. Tomas, "A restriction theorem for the Fourier transform", *Bull. Amer. Math. Soc.* **81** (1975), 477–478. MR 0358216 Zbl 0298.42011

[Triebel 1983] H. Triebel, *Theory of function spaces*, Monographs in Mathematics **78**, Birkhäuser, Basel, 1983. MR 781540 Zbl 0546.46027

Received 28 Jul 2015. Revised 17 Nov 2015. Accepted 30 Jan 2016.

SATOSHI MASAKI: masaki@amath.hiroshima-u.ac.jp
Laboratory of Mathematics, Institute of Engineering, Hiroshima University, Higashihiroshima, Hiroshima 739-8527, Japan

JUN-ICHI SEGATA: segata@m.tohoku.ac.jp
Mathematical Institute, Tohoku University, 6-3, Aoba, Aramaki, Aoba-ku, Sendai 980-8578, Japan

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2016 is US $/year for the electronic version, and $/year (+$, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 9 No. 3 2016

2157-5045(2016)9:3;1-9