

# ANALYSIS & PDE

Volume 9

No. 4

2016

SILVIA FREUND AND STEFAN TEUFEL

**PEIERLS SUBSTITUTION FOR MAGNETIC BLOCH BANDS**

# PEIERLS SUBSTITUTION FOR MAGNETIC BLOCH BANDS

SILVIA FREUND AND STEFAN TEUFEL

We consider the one-particle Schrödinger operator in two dimensions with a periodic potential and a strong constant magnetic field perturbed by slowly varying, nonperiodic scalar and vector potentials  $\phi(\varepsilon x)$  and  $A(\varepsilon x)$  for  $\varepsilon \ll 1$ . For each isolated family of magnetic Bloch bands we derive an effective Hamiltonian that is unitarily equivalent to the restriction of the Schrödinger operator to a corresponding almost invariant subspace. At leading order, our effective Hamiltonian can be interpreted as the Peierls substitution Hamiltonian widely used in physics for nonmagnetic Bloch bands. However, while for nonmagnetic Bloch bands the corresponding result is well understood, both on a heuristic and on a rigorous level, for magnetic Bloch bands it is not clear how to even define a Peierls substitution Hamiltonian beyond a formal expression. The source of the difficulty is a topological obstruction: in contrast to the nonmagnetic case, magnetic Bloch bundles are generically not trivializable. As a consequence, Peierls substitution Hamiltonians for magnetic Bloch bands turn out to be pseudodifferential operators acting on sections of nontrivial vector bundles over a two-torus, the reduced Brillouin zone. Part of our contribution is the construction of a suitable Weyl calculus for such pseudodifferential operators.

As an application of our results we construct a new family of canonical one-band Hamiltonians  $H_{\theta,q}^B$  for magnetic Bloch bands with Chern number  $\theta \in \mathbb{Z}$  that generalizes the Hofstadter model  $H_{\text{Hof}}^B = H_{0,1}^B$  for a single nonmagnetic Bloch band. It turns out that  $H_{\theta,q}^B$  is isospectral to  $H_{\text{Hof}}^{q^2 B}$  for any  $\theta$  and all spectra agree with the Hofstadter spectrum depicted in his famous (black and white) butterfly. However, the resulting Chern numbers of subbands, corresponding to Hall conductivities, depend on  $\theta$  and  $q$ , and thus the models lead to different colored butterflies.

## 1. Introduction

We consider perturbations of the self-adjoint Schrödinger operator

$$H_{B_0,\Gamma} = \frac{1}{2}(-i\nabla_x - A_0)^2 + V_\Gamma,$$

densely defined on  $L^2(\mathbb{R}^2)$ , where  $A_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $V_\Gamma : \mathbb{R}^2 \rightarrow \mathbb{R}$  act as multiplication operators. Here  $A_0(x) = (-B_0 x_2, 0)$  is the vector potential of a constant magnetic field  $B_0 \in \mathbb{R}$  and the scalar potential  $V_\Gamma$  is assumed to be periodic with respect to a Bravais lattice  $\Gamma \subset \mathbb{R}^2$ . The spectral properties of the operator  $H_{B_0,\Gamma}$  are extremely sensitive to the relation between the numerical value of  $B_0 \in \mathbb{R}$  and the area  $|\Gamma|$  of one lattice cell. When  $B_0$  and  $\Gamma$  are commensurable, in the sense that  $B_0|\Gamma|/2\pi = p/q \in \mathbb{Q}$ , the operator  $H_{B_0,\Gamma}$  is unitarily equivalent by an explicit unitary transformation  $\mathcal{F}_q$  to a countable direct sum of multiplication operators by real-valued continuous functions  $E_n : \mathbb{T}_q^* \rightarrow \mathbb{R}$  with  $E_n(k) \leq E_{n+1}(k)$

---

This work was supported by the German Science Foundation within the SFB TR 71.  
*MSC2010:* 81Q05.

*Keywords:* Schrödinger equation, magnetic field, periodic potential, Bloch bundle.

for all  $k \in \mathbb{T}_q^*$  and  $n \in \mathbb{N} = \{1, 2, \dots\}$ . Here the two-dimensional torus  $\mathbb{T}_q^*$  is the Pontryagin dual of a subgroup  $\Gamma_q$  of  $\Gamma$ . In summary, it holds that

$$\widehat{H}_{B_0, \Gamma} := \mathcal{F}_q H_{B_0, \Gamma} \mathcal{F}_q^* = \sum_{n=1}^{\infty} E_n P_n \quad \text{on } \mathcal{H} := \mathcal{F}_q L^2(\mathbb{R}^3) = L^2(\mathbb{T}_q^*; \mathcal{H}_\Gamma) \cong \bigoplus_{n=1}^{\infty} L^2(\mathbb{T}_q^*), \quad (1)$$

where  $P_n$  is the orthogonal projection onto the  $n$ -th summand in the direct sum. As a consequence, the spectrum  $\sigma(H_{B_0, \Gamma}) = \bigcup_{n=1}^{\infty} E_n(\mathbb{T}_q^*)$  is a union of intervals and purely absolutely continuous. If, on the other hand,  $B_0|\Gamma|/2\pi \notin \mathbb{Q}$ , then it is expected that  $\sigma(H_{B_0, \Gamma})$  is a set of Cantor type, i.e., a closed nowhere-dense set of zero Lebesgue measure. The proof of this so-called ten martini problem was given only recently [Avila and Jitomirskaya 2009] and it only applies to simple tight-binding models on  $\ell^2(\mathbb{Z}^2)$ . The most prominent picture of this commensurability problem is the fractal Hofstadter butterfly, a plot of the spectrum of such a simple tight binding model as a function of the magnetic field  $B_0$ ; see Figure 2 in Section 7.

The physical meaning of the operator  $H_{B_0, \Gamma}$  is that of a Hamiltonian for a single particle constrained to move in a planar two-dimensional crystalline lattice under the influence of a constant magnetic field of strength  $B_0$  perpendicular to the plane. However, from the point of view of physical applications and experiments, a constant magnetic field  $B_0$  is a highly idealized situation that can be realized only approximately. The distinction between rational and irrational magnetic fields  $B_0$  is a purely mathematical one. Thus it is of genuine interest to understand perturbations of  $H_{B_0, \Gamma}$  by potentials  $A^\varepsilon(x) := A(\varepsilon x)$  and  $\Phi^\varepsilon(x) := \Phi(\varepsilon x)$  corresponding to magnetic and electric fields  $B^\varepsilon(x) := \varepsilon(\text{curl } A)(\varepsilon x)$  and  $\mathcal{E}^\varepsilon(x) := \varepsilon(\nabla\Phi)(\varepsilon x)$  that are small and slowly varying in the asymptotic limit  $\varepsilon \ll 1$ . Here  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$  are smooth functions. We therefore consider the self-adjoint Schrödinger operator

$$H_{B_0, \Gamma}^\varepsilon = \frac{1}{2}(-i\nabla_x - A_0 - A^\varepsilon)^2 + V_\Gamma + \Phi^\varepsilon$$

for a fixed rational value of  $B_0|\Gamma|/2\pi = p/q$  in the asymptotic limit  $\varepsilon \ll 1$  as a perturbation of the simple block structure (1). It follows by well-known techniques of adiabatic perturbation theory that parts of the block decomposition (1) are stable under such perturbations: Assuming, for example, for a single function  $E_n$  the gap condition  $E_{n-1}(k) < E_n(k) < E_{n+1}(k)$  for all  $k \in \mathbb{T}_q^*$ , one can construct from  $P_n$  an orthogonal projection  $\Pi_n^\varepsilon$  such that  $\|[\Pi_n^\varepsilon, \widehat{H}_{B_0, \Gamma}^\varepsilon]\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(\varepsilon^\infty)$ . While the restriction  $P_n \widehat{H}_{B_0, \Gamma} P_n$  of the unperturbed operator to one of its invariant subspaces  $\text{ran } P_n$  acts as multiplication by the function  $E_n$ , the restriction  $\Pi_n^\varepsilon \widehat{H}_{B_0, \Gamma}^\varepsilon \Pi_n^\varepsilon$  of the perturbed operator  $\widehat{H}_{B_0, \Gamma}^\varepsilon$  to one of its almost invariant subspaces  $\text{ran } \Pi_n^\varepsilon$  a priori has no simple form. The ‘‘Peierls substitution rule’’, widely used in physics, suggests that  $\Pi_n^\varepsilon \widehat{H}_{B_0, \Gamma}^\varepsilon \Pi_n^\varepsilon$  is unitarily equivalent to a pseudodifferential operator with principal part

$$E_n(k - A(i\varepsilon\nabla_k)) + \Phi(i\varepsilon\nabla_k)$$

acting on some space of functions on the torus  $\mathbb{T}_q^*$ . The main result of our paper is to turn this claim into a precise statement and to prove it: we show that the blocks  $\Pi_n^\varepsilon \widehat{H}_{B_0, \Gamma}^\varepsilon \Pi_n^\varepsilon$  of the perturbed operator are unitarily equivalent to pseudodifferential operators acting on spaces of sections of possibly nontrivial

vector bundles over the torus with principal part given by the Peierls substitution rule. A special case of our main result, [Theorem 5.1](#), is the following statement:

**Theorem 1.1.** *Let  $A, \Phi$  be smooth bounded functions with bounded derivatives of any order and  $B_0|\Gamma|/2\pi = p/q \in \mathbb{Q}$ . For any simple Bloch function  $E_n$  of the unperturbed Hamiltonian  $H_{B_0,\Gamma}$  satisfying the gap condition, there exist for  $\varepsilon > 0$  small enough*

- an orthogonal projection  $\Pi_n^\varepsilon$ ,
- a line bundle  $\Xi_\theta$  over the torus  $\mathbb{T}_q^*$  with connection  $\nabla^\theta$  and Chern number  $\theta \in \mathbb{Z}$ ,
- a unitary map  $U^\varepsilon : \text{ran } \Pi_n^\varepsilon \rightarrow L^2(\Xi_\theta)$ , and
- a pseudodifferential operator  $E_n^\varepsilon \in \mathcal{L}(L^2(\Xi_\theta))$  with

$$\|E_n^\varepsilon - (E_n(k - A(i\varepsilon\nabla_k^\theta)) + \Phi(i\varepsilon\nabla_k^\theta))\|_{\mathcal{L}(L^2(\Xi_\theta))} = \mathcal{O}(\varepsilon)$$

such that  $\|[\Pi_n^\varepsilon, \widehat{H}_{B_0,\Gamma}^\varepsilon]\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}(\varepsilon^\infty)$  and

$$\|U^\varepsilon \Pi_n^\varepsilon \widehat{H}_{B_0,\Gamma}^\varepsilon \Pi_n^\varepsilon U^{\varepsilon*} - E_n^\varepsilon\|_{\mathcal{L}(L^2(\Xi_\theta))} = \mathcal{O}(\varepsilon^\infty). \tag{2}$$

In [Theorem 5.1](#) we actually consider a more general situation, where a single band  $E_n$  is replaced by a finite family of bands. Then  $\Xi_\theta$  becomes a vector bundle of finite rank and the Peierls substitution Hamiltonian is a pseudodifferential operator with matrix-valued symbol. We also compute the subprincipal symbol of  $E_n^\varepsilon$  explicitly, which contains important information for transport and magnetic properties of electron gases in periodic media.

[Theorem 5.1](#), and its special case [Theorem 1.1](#), were shown before for the case  $B_0 = 0$  [[Panati et al. 2003a](#)]. There one has  $\theta = 0$  and  $\Xi_0$  is a trivial vector bundle over the torus  $\mathbb{T}_q^*$ . For the case  $B_0 \neq 0$ , the validity and the meaning of Peierls substitution, even on a purely heuristic level, were a matter of debate (see, e.g., [[Zak 1986; 1991](#)]) and, to our knowledge, not even a precise conjecture was stated in the literature.

Before giving more details, let us mention that the systematic or even rigorous analysis of two-dimensional systems with periodic potential and magnetic field is a continuing theme in theoretical physics, for example [[Peierls 1933; Blount 1962; Zak 1968; Hofstadter 1976; Thouless et al. 1982; Sundaram and Niu 1999; Gat and Avron 2003b](#)], and also in mathematical physics and mathematics, for example [[Dubrovin and Novikov 1980a; 1980b; Novikov 1981; Buslaev 1987; Bellissard 1988; Guillot et al. 1988; Helffer and Sjöstrand 1989; Rammal and Bellissard 1990; Helffer et al. 1990; Helffer and Sjöstrand 1990a; 1990b; Nenciu 1991; Gérard et al. 1991; Hövermann et al. 2001; Panati et al. 2003a; Dimassi et al. 2004; Panati 2007; Avila and Jitomirskaya 2009; De Nittis and Panati 2010; De Nittis and Lein 2011; Stiepan and Teufel 2013](#)]. We can mention here only a small part of the enormous literature and we refer to [[Nenciu 1991](#)] for a review of the mathematical and physical literature to that point.

Most of the mathematical literature is concerned with the problem of recovering the spectrum and sometimes the density of states of the perturbed Hamiltonian  $H_{B_0,\Gamma}^\varepsilon$ . In some cases this is done by constructing isospectral effective Hamiltonians in the spirit of the Peierls substitution rule; see, e.g., [[Rammal and Bellissard 1990; Helffer et al. 1990; Helffer and Sjöstrand 1989; 1990a; 1990b; Gérard](#)

et al. 1991]. With a few exceptions, most notably [Rammal and Bellissard 1990], the limiting cases  $B_0 = 0$  and  $B_0 \rightarrow \infty$  were considered. More recently, the question of constructing unitarily equivalent effective Hamiltonians was taken up in [Panati et al. 2003a; De Nittis and Panati 2010; De Nittis and Lein 2011] and the limiting regimes  $B_0 = 0$  and  $B_0 \rightarrow \infty$  are fully understood by now even on a mathematical level. For a thorough discussion of the question of why unitary equivalence is important also from a physics point of view, we refer to [De Nittis and Panati 2010]. Let us mention here only one example: The two canonical models for effective Hamiltonians for the asymptotic regimes  $B_0 = 0$  and  $B_0 \rightarrow \infty$  are exactly isospectral. This is known as the duality of the Hofstadter model; see, e.g., [Gat and Avron 2003a]. However, they are not unitarily equivalent and describe different physics.

The problem of constructing unitarily equivalent effective Hamiltonians in the intermediate regime of finite  $B_0 \neq 0$  was, to our knowledge, completely open up to now<sup>1</sup> and its solution is the main content of our paper. While we use the same basic approach that was applied in [Panati et al. 2003a; De Nittis and Panati 2010] for the cases  $B_0 = 0$  and  $B_0 \rightarrow \infty$ , namely adiabatic perturbation theory [Panati et al. 2003b], there is a major geometric obstruction in extending these methods to perturbations around finite values of  $B_0$  such that  $B_0|\Gamma|/2\pi = p/q \in \mathbb{Q}$ , which we briefly explain. In all cases the projections  $P_n$  in (1) act on  $L^2(\mathbb{T}_q^*, \mathcal{H}_\Gamma)$  fiberwise, that is, they are given by projection-valued functions  $P_n : \mathbb{T}_q^* \rightarrow \mathcal{L}(\mathcal{H}_\Gamma)$ ,  $k \mapsto P_n(k)$ . For an isolated simple band  $E_n$  the corresponding projection-valued function  $P_n(\cdot)$  is smooth and defines a complex line bundle over  $\mathbb{T}_q^*$ , the so-called Bloch bundle associated with the Bloch band  $E_n$ . For  $B_0 = 0$  the Bloch bundles are trivial and the effective operator  $E_n^\varepsilon$  is a pseudodifferential operator acting on  $L^2(\mathbb{T}_1^*)$ , the space of  $L^2$ -sections of the trivial line bundle over the torus  $\mathbb{T}_1^*$ . The Bloch bundles for  $B_0 \neq 0$  are not trivial in general and  $E_n^\varepsilon$  has to be understood as a pseudodifferential operator acting on the sections of a nontrivial line bundle  $\Xi_\theta$  over the torus  $\mathbb{T}_q^*$ .

An important shortcoming of our result is, however, that we cannot allow for the case of a perturbation by a constant magnetic field  $B$ , corresponding to a linear vector potential  $A$ , in all steps of the derivation. While an (almost) invariant subspace and the corresponding (almost) block structure of the perturbed Hamiltonian can still be established in this case, and also the effective Hamiltonian  $\text{Op}^\theta(E_n(k - A(r)) + \Phi(r))$  remains well defined for linear  $A$ , the unitary map intertwining the (almost) invariant subspace and the reference space, as we construct it, no longer exists. For  $\theta = 0$  this problem actually disappears, and we recover the results for nonmagnetic Bloch bands with constant small magnetic fields  $B$  obtained in [De Nittis and Panati 2010; De Nittis and Lein 2011]. Note, however, that the physically relevant situation where  $B$  and also  $E = -\nabla\Phi$  are constant over a macroscopic volume containing  $\varepsilon^{-2}$  lattice sites is included in all of our results.

Let us mention that some of the physically relevant questions can be answered without establishing Peierls substitution in our sense of unitary equivalence. There are, in particular, semiclassical and algebraic approaches that allow for direct computation of many relevant quantities without the detour via Peierls substitution. The modified semiclassical equations of motion for magnetic Bloch bands [Sundaram and Niu 1999] became the starting point for a large number of quantitative results; see, e.g., [Xiao et al.

<sup>1</sup> It was observed in [Dimassi et al. 2004] that the method of [Panati et al. 2003a] can be directly applied also to magnetic Bloch bands if one assumes that the magnetic Bloch bundles are trivial. But this assumption is generically not satisfied.

2010] and references therein. This approach was rigorously derived and extended in [Stiepan and Teufel 2013; Teufel 2012]. In [Gat and Avron 2003b] the authors apply Bohr–Sommerfeld quantization with phases modified by the Berry curvature and the Rammal–Wilkinson term in order to compute the orbital magnetization in the Hofstadter model. For the case where  $B$  is constant or periodic and  $\Phi = 0$ , the algebraic approach of Bellissard and coworkers [Bellissard 1988; Rammal and Bellissard 1990; Bellissard et al. 1991] provides a powerful tool for expansions to all orders for eigenvalues, free energies and quantities derived from there. This approach can also cope with random perturbations and has developed into a very general machinery; see, e.g., [Bellissard et al. 1994; Schulz-Baldes and Teufel 2013] and references therein.

We end the introduction with a short outline of the paper. In Section 2 we give a precise formulation of the setup and introduce all relevant quantities and assumptions. In Section 3 we briefly formulate the result on the existence and the construction of almost invariant subspaces. We do not give a proof here, since nothing interesting changes with respect to the nonmagnetic case at this point. In Section 4 we analyze in detail the structure of magnetic Bloch bundles. As a result we can construct the reference space for the effective Hamiltonian and the unitary map from the almost invariant subspace to this reference space. This analysis is one key ingredient of our main result, which we formulate and prove in Section 5. The result and its proof are based on geometric Weyl calculi for operators acting on sections of nontrivial vector bundles, the other key ingredients, which are developed in Section 6. In the final Section 7, we explicitly compute Peierls substitution Hamiltonians for magnetic subbands of the Hofstadter Hamiltonian. The Hofstadter model is the canonical model for a single nonmagnetic Bloch band perturbed by a constant magnetic field  $B_0$ . As a result we find a new two-parameter family  $H_{\theta,q}^B$  (see (32)) of Hofstadter-like Hamiltonians indexed by integers  $\theta \in \mathbb{Z}$  and  $q \in \mathbb{N}$ . The operator  $H_{\theta,q}^B$  can be viewed as the canonical model for a magnetic Bloch band with Chern number  $\theta$  and originating from a Bloch band split into  $q$  magnetic subbands. Like the Hofstadter model itself, all  $H_{\theta,q}^B$  are representations of an element of the noncommutative torus algebra, the abstract Hofstadter operator. As a consequence they are all isospectral and lead to the same black and white butterfly, Figure 2. But the transport properties encoded in the Chern numbers of spectral bands depend on  $\theta$  and  $q$  and they give rise to different colored butterflies; see Figure 4. The results of Section 7 and a more detailed analysis presented in [Amr et al. 2015] suggest that our main theorem, Theorem 5.1, also holds for perturbations by magnetic fields with potentials  $A$  of linear growth.

## 2. Perturbed periodic and magnetic Schrödinger operators

We consider perturbations of a one-particle Schrödinger operator with a periodic potential and a constant magnetic field in two dimensions. The unperturbed operator is given by

$$H_{\text{MB}} = \frac{1}{2}(-i\nabla_x - A^{(0)}(x))^2 + V_{\tilde{\Gamma}}(x)$$

with domain  $H_{A^{(0)}}^2(\mathbb{R}^2)$ , a magnetic Sobolev space. Here

$$A^{(0)}(x) := \mathcal{B}_0 x \quad \text{with} \quad \mathcal{B}_0 := \begin{pmatrix} 0 & -B_0 \\ 0 & 0 \end{pmatrix}$$

and  $V_{\tilde{\Gamma}}$  is periodic with respect to a Bravais lattice

$$\tilde{\Gamma} := \{a\tilde{\gamma}_1 + b\tilde{\gamma}_2 \in \mathbb{R}^2 \mid a, b \in \mathbb{Z}\}$$

spanned by a basis  $(\tilde{\gamma}_1, \tilde{\gamma}_2)$  of  $\mathbb{R}^2$ , i.e.,  $V_{\tilde{\Gamma}}(x + \tilde{\gamma}) = V_{\tilde{\Gamma}}(x)$  for all  $\tilde{\gamma} \in \tilde{\Gamma}$ . We will later assume that  $B_0 \in \mathbb{R}$  satisfies a commensurability condition, so that  $H_{\text{MB}}$  obtains a magnetic Bloch band structure.

The full Hamiltonian is a perturbation of  $H_{\text{MB}}$  by “small” magnetic and electric fields of order  $\varepsilon$ . More precisely, let  $A^{(1)}$  be a linear vector potential of an additional constant magnetic field  $B_1$  and let  $A^{(2)}$  and  $\Phi$  be bounded vector and scalar potentials; then the full Hamiltonian  $H^\varepsilon$  reads

$$H^\varepsilon = \frac{1}{2}(-i\nabla_x - A^{(0)}(x) - \varepsilon A^{(1)}(x) - A^{(2)}(\varepsilon x))^2 + V_{\tilde{\Gamma}}(x) + \Phi(\varepsilon x) \tag{3}$$

with domain  $H_{A^{(0)} + \varepsilon A^{(1)}}^2(\mathbb{R}^2)$ , where

$$H_A^m := \{f \in L^2(\mathbb{R}^2) \mid (i\nabla_x + A(x))^\alpha f \in L^2(\mathbb{R}^2) \text{ for all } \alpha \in \mathbb{N}_0^2 \text{ with } |\alpha| \leq m\}$$

and  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ .

**Assumption 1.** Assume that  $A^{(2)} \in C_b^\infty(\mathbb{R}^2, \mathbb{R}^2)$  satisfies the gauge condition  $A^{(2)}(x) \cdot \tilde{\gamma}_2 = 0$  for all  $x \in \mathbb{R}^2$  and that  $\Phi \in C_b^\infty(\mathbb{R}^2, \mathbb{R})$ . Let  $V_{\tilde{\Gamma}} : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a measurable function such that  $V_{\tilde{\Gamma}}(x + \tilde{\gamma}) = V_{\tilde{\Gamma}}(x)$  for all  $\tilde{\gamma} \in \tilde{\Gamma}$  and that the operator of multiplication by  $V_{\tilde{\Gamma}}$  is relatively  $(-i\nabla - A^{(0)} - \varepsilon A^{(1)})^2$ -bounded with relative bound smaller than 1 for all  $\varepsilon > 0$  small enough.

Under these conditions,  $H_{\text{MB}}$  and  $H^\varepsilon$  are essentially self-adjoint on  $C_0^\infty(\mathbb{R}^2)$ , and self-adjoint on  $H_{A^{(0)}}^2(\mathbb{R}^2)$  and  $H_{A^{(0)} + \varepsilon A^{(1)}}^2(\mathbb{R}^2)$ , respectively. Note that any  $V_{\tilde{\Gamma}} \in L_{\text{loc}}^2(\mathbb{R}^2)$  satisfies [Assumption 1](#).

**The band structure of  $H_{\text{MB}}$ .** The magnetic translation of functions on  $\mathbb{R}^2$  by  $\tilde{\gamma}_j$  is defined by

$$(\tilde{T}_j \psi)(x) := e^{i\langle x, \mathcal{B}_0 \tilde{\gamma}_j \rangle} \psi(x - \tilde{\gamma}_j). \tag{4}$$

On  $L^2(\mathbb{R}^2)$  the magnetic translations are unitary and leave invariant the magnetic momentum operator and the periodic potential:

$$\tilde{T}_j^{-1}(-i\nabla - A^{(0)})\tilde{T}_j = (-i\nabla - A^{(0)}) \quad \text{and} \quad \tilde{T}_j^{-1}V_{\tilde{\Gamma}}\tilde{T}_j = V_{\tilde{\Gamma}}, \quad \text{and thus} \quad \tilde{T}_j^{-1}H_{\text{MB}}\tilde{T}_j = H_{\text{MB}}.$$

Because

$$\tilde{T}_1\tilde{T}_2 = e^{i\langle \tilde{\gamma}_2, \mathcal{B}_0 \tilde{\gamma}_1 \rangle} \tilde{T}_2\tilde{T}_1,$$

we only obtain a unitary representation of  $\tilde{\Gamma}$  if  $\langle \tilde{\gamma}_2, \mathcal{B}_0 \tilde{\gamma}_1 \rangle \in 2\pi\mathbb{Z}$ . Here  $\langle \tilde{\gamma}_2, \mathcal{B}_0 \tilde{\gamma}_1 \rangle = B_0 \tilde{\gamma}_1 \wedge \tilde{\gamma}_2$  is the magnetic flux through the unit cell  $M$  of the lattice  $\Gamma$  with oriented volume  $\tilde{\gamma}_1 \wedge \tilde{\gamma}_2$ .

**Assumption 2.** The flux of  $B_0$  per unit cell satisfies  $\langle \tilde{\gamma}_2, \mathcal{B}_0 \tilde{\gamma}_1 \rangle = 2\pi p/q \in 2\pi\mathbb{Q}$ .

By passing to the sublattice  $\Gamma \subset \tilde{\Gamma}$  spanned by the basis  $(\gamma_1, \gamma_2) := (q\tilde{\gamma}_1, \tilde{\gamma}_2)$  and defining the magnetic translations  $T_1$  and  $T_2$  analogously, we achieve  $\langle \gamma_2, \mathcal{B}_0\gamma_1 \rangle = 2\pi p \in 2\pi\mathbb{Z}$ . Hence

$$T : \Gamma \rightarrow \mathcal{L}(L^2(\mathbb{R}^2)), \quad \gamma = n_1\gamma_1 + n_2\gamma_2 \mapsto T_\gamma := T_1^{n_1} T_2^{n_2}, \tag{5}$$

is a unitary representation of  $\Gamma$  on  $L^2(\mathbb{R}^2)$  satisfying

$$T_\gamma^{-1} H_{\text{MB}} T_\gamma = H_{\text{MB}} \tag{6}$$

for all  $\gamma \in \Gamma$ . Before we introduce the Bloch–Floquet transformation in order to exploit the translation invariance of  $H_{\text{MB}}$ , we first define a number of useful function spaces. Let

$$\mathcal{H}_f := \{f \in L^2_{\text{loc}}(\mathbb{R}^2) \mid T_\gamma f = f \text{ for all } \gamma \in \Gamma\},$$

which, equipped with the inner product  $\langle f, g \rangle_{\mathcal{H}_f} := \int_M \overline{f(y)}g(y) \, dy$ , is a Hilbert space. Analogously, for  $m \in \mathbb{N}$ ,

$$\mathcal{H}^m_{A(0)}(\mathbb{R}^2) := \{f \in \mathcal{H}_f \mid (-i\nabla - A^{(0)})^\alpha f \in \mathcal{H}_f \text{ for all } \alpha \in \mathbb{N}^2_0 \text{ with } |\alpha| \leq m\}$$

is a Hilbert space with inner product

$$\langle f, g \rangle_{\mathcal{H}^m_{A(0)}(\mathbb{R}^2)} := \sum_{|\alpha| \leq m} \langle (-i\nabla - A^{(0)})^\alpha f, (-i\nabla - A^{(0)})^\alpha g \rangle_{\mathcal{H}_f}.$$

Let  $\Gamma^*$  be the dual lattice of  $\Gamma$ , i.e., the  $\mathbb{Z}$ -span of the unique basis  $(\gamma_1^*, \gamma_2^*)$  such that  $\gamma_i^* \cdot \gamma_j = 2\pi \delta_{ij}$ . By  $M$  and  $M^*$  we denote the centered fundamental cells of  $\Gamma$  and  $\Gamma^*$ , respectively. On  $\mathcal{H}_f$  a unitary representation of the dual lattice  $\Gamma^*$  is given by

$$\tau : \Gamma^* \rightarrow \mathcal{L}(\mathcal{H}_f), \quad \gamma^* \mapsto \tau(\gamma^*) \quad \text{with} \quad (\tau(\gamma^*)f)(y) := e^{iy \cdot \gamma^*} f(y).$$

Finally, let the space of  $\tau$ -equivariant functions be

$$\mathcal{H}_\tau := \{f \in L^2_{\text{loc}}(\mathbb{R}^2, \mathcal{H}_f) \mid f(k - \gamma^*) = \tau(\gamma^*)f(k) \text{ for all } \gamma^* \in \Gamma^*\}$$

equipped with the inner product  $\langle f, g \rangle_{\mathcal{H}_\tau} = \int_{M^*} \langle f(k), g(k) \rangle_{\mathcal{H}_f} dk$ , where  $dk$  is the normalized Lebesgue measure on  $M^*$  and  $\mathcal{H}_\tau$  is a Hilbert space.

For  $\psi \in C^\infty_0(\mathbb{R}^2)$ , the magnetic Bloch–Floquet transformation is defined by

$$(\mathcal{U}_{\text{BF}}\psi)(k, y) := \sum_{\gamma \in \Gamma} e^{-i(y-\gamma) \cdot k} (T_\gamma \psi)(y). \tag{7}$$

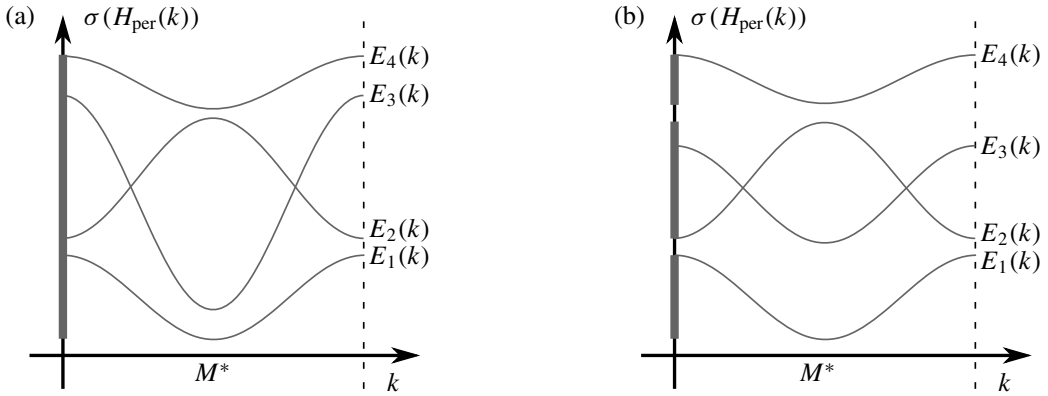
It extends uniquely to a unitary mapping  $\mathcal{U}_{\text{BF}} : L^2(\mathbb{R}^2) \rightarrow \mathcal{H}_\tau$  and its inverse is given by

$$(\mathcal{U}_{\text{BF}}^{-1}\phi)(x) = \int_{M^*} e^{ik \cdot x} \phi(k, x) \, dk.$$

Because of (6) the operator  $H_{\text{MB}}$  fibers in the magnetic Bloch–Floquet representation as

$$H_{\text{BF}}^0 := \mathcal{U}_{\text{BF}} H_{\text{MB}} \mathcal{U}_{\text{BF}}^* = \int_{M^*}^\oplus H_{\text{per}}(k) \, dk,$$





**Figure 1.** Two sketches of Bloch bands. Note that  $k \in \mathbb{R}^2$ , so the graphs of the Bloch bands are really surfaces. In (a) the families  $\{E_1(k)\}$ ,  $\{E_2(k), E_3(k)\}$  and  $\{E_4(k)\}$  are all isolated, but none of them is strictly isolated. In (b) they are all strictly isolated.

where

$$H_{\text{per}}(k) := \frac{1}{2}(-i\nabla_y - A^{(0)}(y) + k)^2 + V_\Gamma(y)$$

acts for any fixed  $k \in M^*$  on the  $k$ -independent domain  $\mathcal{H}_{A^{(0)}}^2(\mathbb{R}^2) \subset \mathcal{H}_f$ . The domain  $H_{A^{(0)}}^2(\mathbb{R}^2)$  of  $H_{\text{MB}}$  is mapped to

$$\mathcal{U}_{\mathcal{F}} H_{A^{(0)}}^2(\mathbb{R}^2) =: L_\tau^2(\mathbb{R}^2, \mathcal{H}_{A^{(0)}}^2(\mathbb{R}^2)) = L_{\text{loc}}^2(\mathbb{R}^2, \mathcal{H}_{A^{(0)}}^2(\mathbb{R}^2)) \cap \mathcal{H}_\tau.$$

As  $H_{\text{per}}(k)$  basically describes a Schrödinger particle in a box, it is bounded from below and has a compact resolvent for every  $k \in M^*$ . Hence  $H_{\text{per}}(k)$  has discrete spectrum with eigenvalues  $E_n(k)$  of finite multiplicity that accumulate at infinity. So let

$$E_1(k) \leq E_2(k) \leq \dots$$

be the eigenvalues, repeated according to their multiplicity. In the following,  $k \mapsto E_n(k)$  will be called the  $n$ -th band function or just the  $n$ -th Bloch band; see Figure 1. Since  $H_{\text{per}}(k)$  is  $\tau$ -equivariant, i.e.,

$$H_{\text{per}}(k - \gamma^*) = \tau(\gamma^*) H_{\text{per}}(k) \tau(\gamma^*)^{-1},$$

and  $\tau(\gamma^*)$  is unitary, the Bloch bands  $E_n(k)$  are  $\Gamma^*$ -periodic functions.

The effective Hamiltonians that we construct will be associated with isolated families of Bloch bands of the unperturbed operator  $H_{\text{per}}(k)$ .

**Definition 2.1.** A family of bands  $\{E_n(k)\}_{n \in I}$  with  $I = [I_-, I_+] \cap \mathbb{N}$  is called isolated, or synonymously is said to satisfy the gap condition, if

$$\inf_{k \in M^*} \text{dist}(\bigcup_{n \in I} \{E_n(k)\}, \bigcup_{m \notin I} \{E_m(k)\}) =: c_g > 0.$$

We say that  $\{E_n(k)\}_{n \in I}$  is strictly isolated with strict gap  $d_g$  if, for

$$\sigma_I := \overline{\bigcup_{n \in I} \bigcup_{k \in M^*} \{E_n(k)\}},$$

we have that

$$\inf_{m \notin I, k \in M^*} \text{dist}(E_m(k), \sigma_I) := d_g > 0.$$

By  $P_I(k)$  we denote the spectral projection of  $H_{\text{per}}(k)$  corresponding to the isolated family of eigenvalues  $\{E_n(k)\}_{n \in I}$ . Because of the gap condition, the map

$$\mathbb{R}^2 \rightarrow \mathcal{L}(\mathcal{H}_f), \quad k \mapsto P_I(k),$$

is real analytic and with  $H_{\text{per}}(k)$  also  $\tau$ -equivariant. This family of projections defines a vector bundle over the torus  $\mathbb{T}^* := \mathbb{R}^2 / \Gamma^*$ .

**Definition 2.2.** Let the bundle  $\pi : \Xi_\tau \rightarrow \mathbb{T}^*$  with typical fiber  $\mathcal{H}_f$  be given by

$$\Xi_\tau := (\mathbb{R}^2 \times \mathcal{H}_f) / \sim_\tau,$$

where

$$(k, \varphi) \sim_\tau (k', \varphi') \iff k' = k - \gamma^* \quad \text{and} \quad \varphi' = \tau(\gamma^*)\varphi \quad \text{for some } \gamma^* \in \Gamma^*.$$

The Bloch bundle  $\Xi_{\text{Bl}}$  associated to the isolated family  $\{E_n(k)\}_{n \in I}$  of Bloch bands is the subbundle given by

$$\Xi_{\text{Bl}} := \{(k, \varphi) \in \mathbb{R}^2 \times \mathcal{H}_f \mid \varphi \in P(k)\mathcal{H}_f\} / \sim_\tau. \tag{8}$$

Hence, the  $L^2$ -sections of  $\Xi_\tau$  are in one-to-one correspondence with elements of  $\mathcal{H}_\tau$  and the  $L^2$ -sections of the Bloch bundle are in one-to-one correspondence with functions  $f \in \mathcal{H}_\tau$  that satisfy  $P_I(k)f(k) = f(k)$  for all  $k \in \mathbb{R}^2$ .

**$H^\varepsilon$  as a pseudodifferential operator on  $\mathcal{H}_\tau$ .** The operator of multiplication by  $x$  on  $L^2(\mathbb{R}^2)$  is mapped under the Bloch–Floquet transformation to the operator  $i\nabla_k^\tau := \mathcal{U}_{\mathcal{F}} x \mathcal{U}_{\mathcal{F}}^*$ . A simple computation shows that  $i\nabla_k^\tau$  acts as the gradient with domain  $H_{\text{loc}}^1(\mathbb{R}^2, \mathcal{H}_f) \cap \mathcal{H}_\tau \subset \mathcal{H}_\tau$ . Hence, by the functional calculus for self-adjoint operators, the full Hamiltonian  $H^\varepsilon$  takes the form

$$H_{\text{BF}}^\varepsilon := \mathcal{U}_{\text{BF}} H^\varepsilon \mathcal{U}_{\text{BF}}^* = \frac{1}{2}(-i\nabla_y - A^{(0)}(y) + k - A(i\varepsilon\nabla_k^\tau))^2 + V_\Gamma(y) + \Phi(i\varepsilon\nabla_k^\tau),$$

where we put  $A := A^{(1)} + A^{(2)}$  and use that  $\varepsilon A^{(1)}(x) = A^{(1)}(\varepsilon x)$  due to linearity. One key step for the following analysis is to interpret  $H_{\text{BF}}^\varepsilon$  as a pseudodifferential operator with operator-valued symbol

$$H(k, r) := \frac{1}{2}(-i\nabla_y - A^{(0)}(y) + k - A(r))^2 + V_\Gamma(y) + \Phi(r) \tag{9}$$

under the quantization map  $k \mapsto k$  and  $r \mapsto i\varepsilon\nabla_k^\tau$ . To make this precise, note that  $H(k, r)$  is a  $\tau$ -equivariant symbol taking values in the self-adjoint operators on  $\mathcal{H}_f$  with domain  $\mathcal{H}_{A^{(0)}}^2$  independent of  $(k, r)$ . For the convenience of the reader we briefly give the definitions of the relevant symbol classes and refer to [Teufel 2003, Appendix B] for details on the  $\tau$ -quantization.

**Definition 2.3.** A function  $w : \mathbb{R}^4 \rightarrow [0, \infty)$  satisfying, for some  $C, N > 0$ ,

$$w(x) \leq C \langle x - y \rangle^N w(y) \quad \text{for all } x, y \in \mathbb{R}^4,$$

is called an order function. Here  $\langle x \rangle := (1 + |x|^2)^{1/2}$ .

Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be Hilbert spaces and  $w$  an order function. Then by  $S^w(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$  we denote the space functions  $f \in C^\infty(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ , which satisfy

$$\|f\|_{w,\alpha,\beta} := \sup_{(k,r) \in \mathbb{R}^4} w(k,r)^{-1} \|(\partial_k^\alpha \partial_r^\beta f)(k,r)\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} < \infty \quad \text{for all } \alpha, \beta \in \mathbb{N}_0^2.$$

Functions in  $S^w(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$  are called operator-valued symbols with order function  $w$ . For the constant order function  $w(k,r) \equiv 1$  we write  $S^1(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)) := S^{w \equiv 1}(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ .

Let  $\tau_j : \Gamma^* \rightarrow \mathcal{L}(\mathcal{H}_j)$ ,  $j = 1, 2$ , be unitary representations. A symbol  $f \in S^w(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$  is called  $(\tau_1, \tau_2)$ -equivariant if

$$f(k - \gamma^*, r) = \tau_2(\gamma^*) f(k, r) \tau_1(\gamma^*)^{-1} \quad \text{for all } \gamma^* \in \Gamma^* \text{ and } (k, r) \in \mathbb{R}^4.$$

The corresponding space is denoted by  $S^w_{(\tau_1, \tau_2)}(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$  and equipped with the Fréchet metric induced by the family of seminorms  $\|\cdot\|_{w,\alpha,\beta}$ .

We denote by  $S^w_{(\tau_1, \tau_2)}(\varepsilon, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$  the space of uniformly bounded functions

$$f : [0, \varepsilon_0) \rightarrow S^w_{(\tau_1, \tau_2)}(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)).$$

If  $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$  and  $\tau_1 = \tau_2$ , we write  $S^w_\tau(\varepsilon, \mathcal{L}(\mathcal{H}))$  instead.

**Proposition 2.4.** *Let  $w_A(k, r) := 1 + |k - A(r)|^2$ . Then the operator-valued function  $(k, r) \mapsto H(k, r)$  defined in (9) is a symbol  $H \in S^w_{(\tau_1, \tau_2)}(\mathcal{L}(\mathcal{H}^2_{A(0)}, \mathcal{H}_f))$  with  $\tau_1 = \tau|_{\mathcal{H}^2_{A(0)}}$  and  $\tau_2 = \tau$ .*

*Proof.* Since  $H(k, r) = H_{\text{per}}(k - A(r)) + \Phi(r)$ , all claims can be checked explicitly on  $H_{\text{per}}$  using **Assumption 1**: the  $(\tau_1, \tau_2)$ -equivariance of  $H$  follows from the  $(\tau_1, \tau_2)$ -equivariance of  $H_{\text{per}}$ , and  $H_{\text{per}} \in S^w_{(\tau_1, \tau_2)}(\mathcal{L}(\mathcal{H}^2_{A(0)}, \mathcal{H}_f))$  with  $w_0(k, r) := 1 + |k|^2$  implies  $H \in S^w_{(\tau_1, \tau_2)}(\mathbb{R}^4, \mathcal{L}(\mathcal{H}^2_{A(0)}, \mathcal{H}_f))$ . See [De Nittis and Panati 2010, Lemma 3.8] for details on the last argument.  $\square$

Note that the Weyl quantization of a symbol  $f$  in  $S^w_{(\tau_1, \tau_2)}(\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$  defines an operator  $\text{Op}^{(\tau_1, \tau_2)}(f)$  that maps  $\mathcal{H}_1$ -valued,  $\tau_1$ -equivariant functions to  $\mathcal{H}_2$ -valued,  $\tau_2$ -equivariant functions. For details on this  $\tau$ -quantization, see [Teufel 2003, Appendix B]. For a general introduction to pseudodifferential operators with operator-valued symbols in the same context, we refer to [Gérard et al. 1991].

Since the  $(\tau_1, \tau_2)$ -quantization  $\text{Op}^{(\tau_1, \tau_2)}(H)$  of  $H$  restricted to the space of smooth  $\tau$ -equivariant functions with values in  $\mathcal{H}^2_{A(0)}(\mathbb{R}^2)$  agrees with the restriction of  $H^\varepsilon_{\mathcal{F}}$ , and since both operators are essentially self-adjoint on this subspace, their closures agree and we will identify them in the following.

### 3. Almost invariant subspaces

The first step of space-adiabatic perturbation theory is the construction of the almost invariant subspace  $\Pi_I^\varepsilon \mathcal{H}_\tau$  associated with an isolated family of Bloch bands  $\{E_n(k)\}_{n \in I}$ . Here  $\Pi_I^\varepsilon$  is an orthogonal projection almost commuting with  $H^\varepsilon_{\mathcal{F}}$ . This concept goes back to [Nenciu 2002] and the general construction was introduced in [Nenciu and Sordoni 2004; Martinez and Sordoni 2002] based on techniques developed already in [Helffer and Sjöstrand 1990a]. The application to the case of nonmagnetic Bloch bands including the  $\tau$ -equivariant Weyl calculus was worked out in [Panati et al. 2003a; Teufel 2003]. Since

these methods carry over to the case of magnetic Bloch bands without difficulties — see also [Dimassi et al. 2004; Stiepan 2011] — we skip the details of the proof. Note, however, that we add a new observation to the statement: under the assumption of a strict gap and for sufficiently small perturbations, the resulting projection  $\Pi_I^\varepsilon$  actually commutes with  $H_{\text{BF}}^\varepsilon$ , since it turns out to be a spectral projection.

**Theorem 3.1.** *Let Assumptions 1 and 2 hold and let  $\{E_n(k)\}_{n \in I}$  be an isolated family of Bloch bands. Then there exists an orthogonal projection  $\Pi_I^\varepsilon \in \mathcal{L}(\mathcal{H}_\tau)$  such that  $H_{\text{BF}}^\varepsilon \Pi_I^\varepsilon$  is a bounded operator and*

$$\|[H_{\text{BF}}^\varepsilon, \Pi_I^\varepsilon]\| = \mathcal{O}(\varepsilon^\infty).$$

Moreover,  $\Pi_I^\varepsilon$  is close to a pseudodifferential operator  $\text{Op}^\tau(\pi)$ :

$$\|\Pi_I^\varepsilon - \text{Op}^\tau(\pi)\| = \mathcal{O}(\varepsilon^\infty), \tag{10}$$

where  $\pi \in S_\tau^1(\varepsilon, \mathcal{L}(\mathcal{H}_f)) := S_\tau^{w=1}(\varepsilon, \mathcal{L}(\mathcal{H}_f))$  with principal symbol  $\pi_0(k, r) = P_I(k - A(r))$ .

If  $\{E_n(k)\}_{n \in I}$  is strictly isolated with gap  $d_g$  and  $\|\Phi\|_\infty < \frac{1}{2}d_g$ , then (10) holds for  $\Pi_I^\varepsilon$  being the spectral projection of  $H_{\text{BF}}^\varepsilon$  associated to the interval  $[\inf E_I - \frac{1}{2}d_g, \sup E_I + \frac{1}{2}d_g]$ . In particular,  $[H_{\text{BF}}^\varepsilon, \Pi_I^\varepsilon] = 0$  in this case.

*Proof.* The construction of  $\Pi_I^\varepsilon$  is given in [Teufel 2003, Proposition 5.16] for general Hamiltonians with symbol  $\tilde{H} \in S_{(\tau_1, \tau_2)}^w(\mathbb{R}^4, \mathcal{L}(\mathcal{D}, \mathcal{H}_f))$  for  $w(k, r) = 1 + |k|^2$ , where  $\tilde{H}(k, r)$  is pointwise a self-adjoint operator on  $\mathcal{H}_f$  with domain  $\mathcal{D}$ . In the case  $A^{(1)} = 0$  it applies verbatim also to our Hamiltonian, since then  $H \in S_{(\tau_1, \tau_2)}^w(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_{A^{(0)}}, \mathcal{H}_f))$ . The slight modification that allows one to include also a linear term  $A^{(1)} \neq 0$  is worked out in [De Nittis and Panati 2010, Theorem 3.12(1)], where the order function  $w$  is replaced by  $w_A$ . Note that their assumption (D) on the triviality of the Bloch bundle is not used in the proof of [De Nittis and Panati 2010, Theorem 3.12(1)]. We remark that the construction of  $\Pi_I^\varepsilon$  for nonzero  $A^{(0)}$  and  $A^{(1)} = 0$  was also done in [Stiepan 2011].

The statement for strictly isolated bands follows from inspecting, for example, the proof of [Teufel 2003, Proposition 5.16], from where the following notation is also borrowed. Under the assumption of a strict gap, the Moyal resolvent  $R(\zeta)$  can be constructed globally on  $\mathcal{U}_{z_0} = \mathbb{R}^4$  and for  $\zeta$  in a fixed positively oriented circle  $\Lambda \subset \mathbb{C}$  encircling  $[\inf E_I - \frac{1}{2}d_g, \sup E_I + \frac{1}{2}d_g]$ . But then [Teufel 2003, (5.28)] implies  $\text{Op}^\tau(R(\zeta)) = (\text{Op}^{(\tau_1, \tau_2)}(H) - \zeta)^{-1} + \mathcal{O}(\varepsilon^\infty)$  and thus, by [ibid., (5.38)],

$$\text{Op}^\tau(\pi) = \frac{i}{2\pi} \oint_\Lambda \text{Op}^\tau(R(\zeta)) d\zeta = \frac{i}{2\pi} \oint_\Lambda (H_{\text{BF}}^\varepsilon - \zeta)^{-1} d\zeta + \mathcal{O}(\varepsilon^\infty). \quad \square$$

### 4. Magnetic Bloch bundles

With respect to the (almost) invariant subspace  $\Pi_I^\varepsilon \mathcal{H}_\tau$  associated to an isolated family of Bloch bands, the Hamiltonian thus takes the (almost) block diagonal form

$$H_{\text{BF}}^\varepsilon = \Pi_I^\varepsilon H_{\text{BF}}^\varepsilon \Pi_I^\varepsilon + (1 - \Pi_I^\varepsilon) H_{\text{BF}}^\varepsilon (1 - \Pi_I^\varepsilon) + \mathcal{O}(\varepsilon^\infty),$$

where  $\mathcal{O}(\varepsilon^\infty)$  holds in the operator norm. For strictly isolated bands,  $\mathcal{O}(\varepsilon^\infty)$  can be replaced by zero and the prefix ‘‘almost’’ can be dropped. The remaining task is to show that the block  $\Pi_I^\varepsilon H_{\text{BF}}^\varepsilon \Pi_I^\varepsilon$  is

unitarily equivalent to an effective Hamiltonian  $H_{\text{eff}}$  given by Peierls substitution on some simple reference space  $\mathcal{H}_{\text{ref}}$ .

Let us quickly summarize how this is achieved in the case  $B_0 \equiv 0$  in [Panati et al. 2003a; Teufel 2003]. The smoothness of  $H(k, r)$  and the gap condition imply the smoothness of the spectral projection  $P_I(k - A(r))$ . In particular,  $P_I(k - A(r))$  has constant rank  $m \in \mathbb{N}$ . It is thus natural to choose  $\mathcal{H}_{\text{ref}}$  as the  $\mathbb{C}^m$ -valued functions over the torus  $\mathbb{T}^* = \mathbb{R}^2 / \Gamma^*$ , i.e.,  $\mathcal{H}_{\text{ref}} = L^2(\mathbb{T}^*, \mathbb{C}^m)$ . As in the case of  $\Pi_I^\varepsilon$ , the unitary map  $U^\varepsilon : \Pi_I^\varepsilon \mathcal{H}_\tau \rightarrow \mathcal{H}_{\text{ref}}$  is constructed perturbatively order by order as the quantization of a semiclassical symbol  $u(k, r) \asymp \sum_{j=0}^\infty \varepsilon^j u_j(k, r)$ . The starting point of the construction is a unitary map  $u_0(k, r) : P_I(k - A(r))\mathcal{H}_f \rightarrow \mathbb{C}^m$  that is smooth and right- $\tau$ -equivariant:

$$u_0(k - \gamma^*, r) = u_0(k, r)\tau(\gamma^*)^{-1} \quad \text{for all } k \in \mathbb{R}^2 \text{ and } \gamma^* \in \Gamma^*.$$

In geometric terms this means that we seek a  $U(m)$ -bundle isomorphism between the Bloch bundle  $\Xi_{\text{Bl}}$  and the trivial bundle over the torus  $\mathbb{T}^*$  with fiber  $\mathbb{C}^m$ . But such an isomorphism exists if and only if the Bloch bundle is trivial. It was shown in [Helffer and Sjöstrand 1989] for the case  $m = 1$  and in [Panati 2007] also for  $m \geq 1$  that, in the case  $B_0 = 0$ , time-reversal-invariance implies that the Bloch bundle associated to any isolated family of Bloch bands is indeed trivial and hence an appropriate  $u_0$  always exists.

However,  $H_{\text{MB}}$  is no longer time-reversal-invariant when  $B_0 \neq 0$  and the Bloch bundle is in general a nontrivial vector bundle over the torus. Indeed, its nonvanishing Chern numbers are closely related to the quantum Hall effect, as was first discovered in the seminal paper [Thouless et al. 1982]. The nontriviality of magnetic Bloch bundles is the main obstruction for defining Peierls substitution for magnetic Bloch bands in any straightforward way.

Let us start with a rough sketch of our strategy for overcoming this obstruction. Our reference space  $\mathcal{H}_{\text{ref}} = \mathcal{H}_\alpha$  now contains sections of a nontrivial vector bundle  $\Xi_\alpha$  over  $\mathbb{T}^*$  with typical fiber  $\mathbb{C}^m$  that is isomorphic to the Bloch bundle  $\Xi_{\text{Bl}}$ . According to a result of Panati [2007],  $\Xi_\alpha$  is uniquely characterized, up to isomorphisms, by its rank  $m \in \mathbb{N}$  and its Chern number  $\theta \in \mathbb{Z}$ . Of course we could just glue together local trivializations of  $\Xi_{\text{Bl}}$  by suitable transition functions in order to construct such a bundle  $\Xi_\alpha$ . However, for the definition of the map  $U^\varepsilon : \Pi_I^\varepsilon \mathcal{H}_\tau \rightarrow \mathcal{H}_\alpha$  and for the construction of an appropriate pseudodifferential calculus on  $\mathcal{H}_\alpha$ , it will be essential to have an explicit characterization of  $\Xi_\alpha$  with certain additional properties. To this end, we first explicitly define a global trivialization of the extended Bloch bundle given by

$$\Xi'_{\text{Bl}} := \{(k, \varphi) \in \mathbb{R}^2 \times \mathcal{H}_f \mid \varphi \in P_I(k)\mathcal{H}_f\} \tag{11}$$

over the contractible base space  $\mathbb{R}^2$ , i.e., an orthonormal basis  $(\varphi_1(k), \dots, \varphi_m(k))$  of  $P(k)\mathcal{H}_f$  depending smoothly on  $k \in \mathbb{R}^2$ . For this we use the parallel transport with respect to the Berry connection  $\nabla_k^{\text{B}} = P_I(k)\nabla_k P_I(k) + P_I^\perp(k)\nabla_k P_I^\perp(k)$ . Then  $\Xi_\alpha := (\mathbb{R}^2 \times \mathbb{C}^m) / \sim_\alpha$  is defined in terms of the “transition function”  $\alpha : \mathbb{R}^2 / \Gamma^* \times \Gamma^* \rightarrow \mathcal{L}(\mathbb{C}^m)$  defined by  $\varphi(k - \gamma^*) =: \alpha(k, \gamma^*)\tau(\gamma^*)\varphi(k)$ . But the functions  $\varphi_j(k)$  are not  $\tau$ -equivariant and their derivatives of order  $n$  grow like  $|k|^n$ . Thus they cannot be used directly to define a symbol of the form  $u_0(k, r)_{ij} = |e_i\rangle\langle\varphi_j(k - A(r))|$ . However, they do give the starting point for the perturbative construction of a unitary  $U_1^\varepsilon : \Pi_I^\varepsilon \mathcal{H}_\tau \rightarrow P_I \mathcal{H}_\tau$  by setting  $u_0(k, r)_{ij} := |\varphi_i(k)\rangle\langle\varphi_j(k - A(r))|$ ,

which is a good  $\tau$ -equivariant symbol. From the frame  $(\varphi_1(k), \dots, \varphi_m(k))$  we also get a bundle isomorphism between  $\Xi_{\text{BI}}$  and  $\Xi_\alpha$ , that is, a unitary map

$$U_\alpha : P_I \mathcal{H}_\tau \rightarrow \mathcal{H}_\alpha, \quad \varphi(k) \mapsto (U_\alpha \varphi)_j(k) := \langle \varphi_j(k), \varphi(k) \rangle_{\mathcal{H}_f},$$

where  $P_I \mathcal{H}_\tau = \{f \in \mathcal{H}_\tau \mid f(k) = P_I(k) f(k)\}$  contains the  $L^2$ -sections of the Bloch bundle. But  $U_\alpha$  is not a pseudodifferential operator and thus it is not clear a priori if

$$H_{\text{eff}} := U_\alpha U_1^\varepsilon \Pi_I^\varepsilon \text{Op}^\tau(H) \Pi_I^\varepsilon U_1^{\varepsilon*} U_\alpha^*$$

is a pseudodifferential operator and how its principal symbol looks. This problem will be solved by introducing a Weyl quantization adapted to the geometry of the Bloch bundle, for which the action of  $U_\alpha$  is explicit.

After this rough sketch of the general strategy, let us start with the construction of the frame  $(\varphi_1(k), \dots, \varphi_m(k))$ . For this we need a lemma on the properties of the Berry connection.

**Lemma 4.1.** *On the trivial bundle  $\mathbb{R}^2 \times \mathcal{H}_f$  the Berry connection*

$$\nabla_k^{\text{B}} := P_I(k) \nabla_k P_I(k) + P_I^\perp(k) \nabla_k P_I^\perp(k)$$

is a metric connection.

For arbitrary  $x, y \in \mathbb{R}^2$  let  $t^{\text{B}}(x, y)$  be the parallel transport with respect to the Berry connection along the straight line from  $y$  to  $x$ . Then  $t^{\text{B}}(x, y) \in \mathcal{L}(\mathcal{H}_f)$  is unitary, satisfies

$$t^{\text{B}}(x, y) = P_I(x) t^{\text{B}}(x, y) P_I(y) + P_I^\perp(x) t^{\text{B}}(x, y) P_I^\perp(y) \tag{12}$$

and is  $\tau$ -equivariant:

$$t^{\text{B}}(x - \gamma^*, y - \gamma^*) = \tau(\gamma^*) t^{\text{B}}(x, y) \tau(\gamma^*)^{-1}. \tag{13}$$

*Proof.* Let  $\psi, \phi : \mathbb{R}^2 \rightarrow \mathcal{H}_f$  be smooth functions; then a simple computation yields

$$\nabla \langle \psi(k), \phi(k) \rangle_{\mathcal{H}_f} = \langle \nabla^{\text{B}} \psi(k), \phi(k) \rangle_{\mathcal{H}_f} + \langle \psi(k), \nabla^{\text{B}} \phi(k) \rangle_{\mathcal{H}_f},$$

showing that  $\nabla^{\text{B}}$  is metric. As a consequence,  $t^{\text{B}}(x, y) \in \mathcal{L}(\mathcal{H}_f)$  is unitary. Let  $x(s) := y + s(x - y)$ ,  $s \in [0, 1]$ , be the straight line from  $y$  to  $x$ . Then  $t^{\text{B}}(x(s), y) =: t^{\text{B}}(s)$  is the unique solution of

$$\frac{d}{ds} t^{\text{B}}(s) = [(x - y) \cdot \nabla P_I(x(s)), P_I(x(s))] t^{\text{B}}(s) \quad \text{with} \quad t^{\text{B}}(0) = \mathbf{1}_{\mathcal{H}_f}. \tag{14}$$

From this and  $\nabla P_I = P_I(\nabla P_I) P_I^\perp + P_I^\perp(\nabla P_I) P_I$ , one easily computes that

$$\frac{d}{ds} (t^{\text{B}}(s)^* P_I(x(s)) t^{\text{B}}(s)) = 0,$$

which implies  $t^{\text{B}}(s)^* P_I(x(s)) t^{\text{B}}(s) = P_I(y)$  for all  $s \in [0, 1]$ , and thus (12). Now  $t^{\text{B}}(x(s) - \gamma^*, y - \gamma^*) =: \tilde{t}^{\text{B}}(s)$  is the unique solution of

$$\frac{d}{ds} \tilde{t}^{\text{B}}(s) = [(x - y) \cdot \nabla P_I(x(s) - \gamma^*), P_I(x(s) - \gamma^*)] \tilde{t}^{\text{B}}(s) \quad \text{with} \quad \tilde{t}^{\text{B}}(0) = \mathbf{1}_{\mathcal{H}_f}. \tag{15}$$

Thus, the  $\tau$ -equivariance of  $t^{\text{B}}(x, y)$  follows from comparing (14) and (15) and using the  $\tau$ -equivariance of the projection  $P_I(k)$ . □

**Proposition 4.2.** *Let  $\{E_n(k)\}_{n \in I}$  be an isolated family of Bloch bands with  $|I| = m$ . There are functions  $\varphi_j \in C^\infty(\mathbb{R}^2, \mathcal{H}_f)$ ,  $j = 1, \dots, m$ , such that  $(\varphi_1(k), \dots, \varphi_m(k))$  is an orthonormal basis of  $P_I(k)\mathcal{H}_f$  for all  $k \in \mathbb{R}^2$  and having the following property: there is a function  $\alpha : \mathbb{R}/\mathbb{Z} \rightarrow \mathcal{L}(\mathbb{C}^m)$  taking values in the unitary matrices such that*

$$\varphi(k - \gamma^*) = \alpha(\kappa_2)^{n_1} \tau(\gamma^*)\varphi(k)$$

for all  $\gamma^* =: n_1\gamma_1^* + n_2\gamma_2^* \in \Gamma^*$ ,  $k \in \mathbb{R}^2$  and  $\kappa_2 := \langle k, \gamma_2 \rangle / (2\pi)$ . If the rank  $m$  of the Bloch bundle is 1, then  $\varphi = \varphi_1$  can be chosen so that

$$\alpha(\kappa_2) = e^{-i2\pi\theta\kappa_2} = e^{-i\theta\langle k, \gamma_2 \rangle}, \tag{16}$$

where  $\theta \in \mathbb{Z}$  is the Chern number of the Bloch bundle.

*Proof.* Note that if the Bloch bundle is trivial then any trivializing frame  $(\varphi_j(k))_{j=1, \dots, m}$  would do the job and  $\alpha \equiv \mathbf{1}_{m \times m}$ . In general, we construct a trivializing frame of the extended Bloch bundle  $\mathcal{E}'_{\text{Bl}}$  (see (11)) by using the parallel transport with respect to the Berry connection.

Throughout this proof, we use instead of cartesian coordinates the coordinates  $\kappa_j := \langle k, \gamma_j \rangle / (2\pi)$ , namely  $k = \kappa_1\gamma_1^* + \kappa_2\gamma_2^*$ . In particular, we also identify  $\gamma^* = (n_1, n_2) \in \Gamma^*$  with  $(n_1, n_2) \in \mathbb{Z}^2$ .

Let  $\kappa_2 \mapsto (h_1(\kappa_2), \dots, h_m(\kappa_2))$  be a smooth,  $\tau_2$ -equivariant, orthonormal frame of  $\mathcal{E}'_{\text{Bl}}|_{\kappa_1=0}$ , i.e.,  $h_j(\kappa_2 - n_2) = \tau((0, n_2))h_j(\kappa_2)$  and  $(h_1(\kappa_2), \dots, h_m(\kappa_2))$  is an orthonormal basis of  $P_I((0, \kappa_2))\mathcal{H}_f$ . Since every complex vector bundle over the circle is trivial, such a frame always exists. Now we define a global frame of  $E'_{\text{Bl}}$  by parallel transport of  $h$  along the  $\gamma_1^*$ -direction,

$$\tilde{\varphi}_j(\kappa_1, \kappa_2) := t^{\text{B}}((\kappa_1, \kappa_2), (0, \kappa_2))h_j(\kappa_2).$$

By Lemma 4.1, the functions  $\tilde{\varphi}_j : \mathbb{R}^2 \rightarrow \mathcal{H}_f$  are smooth and  $(\tilde{\varphi}_1(k), \dots, \tilde{\varphi}_m(k))$  is an orthonormal basis of  $P_I(k)\mathcal{H}_f$  for all  $k \in \mathbb{R}^2$ . Since  $\tau(\gamma^*) : \text{ran } P_I(k) \rightarrow \text{ran } P_I(k + \gamma^*)$  is unitary for all  $k \in \mathbb{R}^2$ , we have that

$$\tilde{\varphi}_j(k - \gamma^*) =: \sum_{i=1}^m \tilde{\alpha}_{ji}(k, \gamma^*)\tau(\gamma^*)\tilde{\varphi}_i(k) \tag{17}$$

with a unitary  $m \times m$  matrix  $\tilde{\alpha}(k, \gamma^*) = (\tilde{\alpha}_{ji}(k, \gamma^*))_{j,i=1, \dots, m}$ . The  $\tau$ -equivariance of  $h$  implies

$$\tilde{\alpha}((0, \kappa_2), (0, n_2)) = \mathbf{1}_{m \times m} \quad \text{for all } \kappa_2 \in \mathbb{R} \text{ and } n_2 \in \mathbb{Z}.$$

From the  $\tau$ -equivariance (13) of the parallel transport, this also implies

$$\tilde{\alpha}(k, (0, n_2)) = \mathbf{1}_{m \times m} \quad \text{for all } k \in \mathbb{R}^2 \text{ and } n_2 \in \mathbb{Z}, \tag{18}$$

since

$$\begin{aligned} & t^{\text{B}}((\kappa_1, \kappa_2 - n_2), (0, \kappa_2 - n_2))\tau((0, n_2))t^{\text{B}}((0, \kappa_2), (\kappa_1, \kappa_2)) \\ &= \tau((0, n_2))t^{\text{B}}((\kappa_1, \kappa_2), (0, \kappa_2))\tau((0, n_2))^{-1}\tau((0, n_2))t^{\text{B}}((0, \kappa_2), (\kappa_1, \kappa_2)) = \tau((0, n_2)). \end{aligned}$$

From the definition (17) it follows that  $\tilde{\alpha}$  satisfies the cocycle condition

$$\tilde{\alpha}(k - \tilde{\gamma}^*, \gamma^*)\tilde{\alpha}(k, \tilde{\gamma}^*) = \tilde{\alpha}(k, \gamma^* + \tilde{\gamma}^*) \quad \text{for all } k \in \mathbb{R}^2 \text{ and } \gamma^*, \tilde{\gamma}^* \in \Gamma^*, \tag{19}$$

which, for  $\gamma^* = (0, n_2)$  and  $\tilde{\gamma}^* = (n_1, 0)$ , together with (18) implies

$$\tilde{\alpha}(k, (n_1, 0)) = \tilde{\alpha}(k, (n_1, n_2)) \quad \text{for all } k \in \mathbb{R}^2 \text{ and } n_1, n_2 \in \mathbb{Z}.$$

Hence,  $\tilde{\alpha}$  does not depend on  $n_2$  and we write  $\tilde{\alpha}(k, n_1)$  in the following. But then the cocycle condition (19) with  $\gamma^* = (n_1, 0)$  and  $\tilde{\gamma}^* = (0, n_2)$  implies

$$\tilde{\alpha}((\kappa_1, \kappa_2 - n_2), n_1)\tilde{\alpha}((\kappa_1, \kappa_2), 0) = \tilde{\alpha}((\kappa_1, \kappa_2), n_1),$$

and thus the periodicity of  $\tilde{\alpha}$  as a function of  $\kappa_2$ .

Next we introduce the  $m \times m$ -matrix-valued connection coefficients of the Berry connection as

$$\begin{pmatrix} \tilde{\mathcal{A}}_{ji}^1(k) \\ \tilde{\mathcal{A}}_{ji}^2(k) \end{pmatrix} := -\frac{i}{2\pi} \begin{pmatrix} \langle \tilde{\varphi}_i(k), \partial_{\kappa_1} \tilde{\varphi}_j(k) \rangle_{\mathcal{H}_f} \\ \langle \tilde{\varphi}_i(k), \partial_{\kappa_2} \tilde{\varphi}_j(k) \rangle_{\mathcal{H}_f} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{\mathcal{A}}_{ji}^2(k) \end{pmatrix},$$

where  $\tilde{\mathcal{A}}_{ji}^1(k) = 0$  because the  $\tilde{\varphi}_i$  are parallel along the  $\gamma_1^*$ -direction. From (18) we infer that  $\tilde{\mathcal{A}}^2$  is periodic in the  $\gamma_2^*$ -direction, that is, that  $\tilde{\mathcal{A}}^2(\kappa_1, \kappa_2 + n_2) = \tilde{\mathcal{A}}^2(\kappa_1, \kappa_2)$  for all  $k \in \mathbb{R}^2$  and  $n_2 \in \mathbb{Z}$ .

If we differentiate both sides of (17) with respect to  $\kappa_\ell$  and then project on  $\tilde{\varphi}_s(k - \gamma^*)$ , we obtain

$$\begin{aligned} 2\pi i \tilde{\mathcal{A}}_{js}^\ell(k - \gamma^*) &= \sum_{i=1}^m (\langle \tilde{\varphi}_s(k - \gamma^*), \partial_{\kappa_\ell} \tilde{\alpha}_{ji}(k, n_1) \tau(\gamma^*) \tilde{\varphi}_i(k) + \tilde{\alpha}_{ji}(k, n_1) \tau(\gamma^*) \partial_{\kappa_\ell} \tilde{\varphi}_i(k) \rangle) \\ &= \sum_{i=1}^m \partial_{\kappa_\ell} \tilde{\alpha}_{ji}(k, n_1) \overline{\tilde{\alpha}_{si}(k, n_1)} + 2\pi i \sum_{i,n=1}^m \tilde{\alpha}_{ji}(k, n_1) \tilde{\mathcal{A}}_{in}^\ell(k) \overline{\tilde{\alpha}_{sn}(k, n_1)}. \end{aligned}$$

Since  $\tilde{\mathcal{A}}_{ji}^1(k) = 0$ , the matrix  $\tilde{\alpha}(k, n_1)$  is independent of  $\kappa_1$  and satisfies the linear, first-order ODE

$$\partial_{\kappa_2} \tilde{\alpha}(\kappa_2, n_1) = 2\pi i (\tilde{\mathcal{A}}^2(0, \kappa_2) \tilde{\alpha}(\kappa_2, n_1) - \tilde{\alpha}(\kappa_2, n_1) \tilde{\mathcal{A}}^2(n_1, \kappa_2)). \tag{20}$$

Since  $\tilde{\alpha}(\kappa_2, \cdot) : \mathbb{Z} \rightarrow \mathcal{L}(\mathbb{C}^m)$  is a group homomorphism for every  $\kappa_2 \in \mathbb{R}/\mathbb{Z}$ , we can put  $\tilde{\alpha}(\kappa_2, n_1) = \alpha(\kappa_2)^{n_1}$  with  $\alpha(\kappa_2) := \tilde{\alpha}(\kappa_2, 1)$ . This proves the statement of the lemma for the case  $m > 1$  by setting  $\varphi := \tilde{\varphi}$ .

For  $m = 1$  we evaluate the solution of (20) in order to obtain an explicit expression for  $\alpha$ ,

$$\tilde{\alpha}(\kappa_2, 1) = \exp\left(2\pi i \int_0^{\kappa_2} ds (\tilde{\mathcal{A}}^2(0, s) - \tilde{\mathcal{A}}^2(1, s))\right).$$

Introducing the curvature of the Berry connection,

$$\Omega(k) = \frac{|M^*|}{2\pi} \partial_{\kappa_1} \tilde{\mathcal{A}}^2(k),$$

by Stokes' theorem we have

$$2\pi \int_0^{\kappa_2} (\tilde{\mathcal{A}}^2(1, s) - \tilde{\mathcal{A}}^2(0, s)) ds = \frac{4\pi^2}{|M^*|} \int_0^{\kappa_2} \int_0^1 \Omega(p, s) dp ds =: \bar{\Omega}(\kappa_2)$$

and thus

$$\tilde{\alpha}(\kappa_2, 1) = e^{-i\bar{\Omega}(\kappa_2)}.$$



To obtain the simpler form claimed in the lemma, we put

$$\varphi(k) := e^{ik_1(2\pi\kappa_2\theta - \bar{\Omega}(\kappa_2))} \tilde{\varphi}(k),$$

where  $\theta := \bar{\Omega}(1)/(2\pi)$  is the Chern number of the Bloch bundle. Hence,

$$\varphi(k - \gamma^*) = e^{-i2\pi\theta\kappa_2n_1} \tau(\gamma^*)\varphi(k). \quad \square$$

**Proposition 4.3.** *Let Assumptions 1 and 2 hold with  $A^{(1)} = 0$  and let  $\{E_n(k)\}_{n \in I}$  be an isolated family of Bloch bands. Then there exists a unitary operator  $U_1^\varepsilon \in \mathcal{L}(\mathcal{H}_\tau)$  such that*

$$U_1^\varepsilon \Pi_I^\varepsilon U_1^{\varepsilon*} = P_I$$

and  $U_1^\varepsilon = \text{Op}^\tau(u) + \mathcal{O}_0(\varepsilon^\infty)$ , where  $u \asymp \sum_{j \geq 0} \varepsilon^j u_j$  belongs to  $S_\tau^1(\varepsilon, \mathcal{L}(\mathcal{H}_f))$  and has the  $\tau$ -equivariant principal symbol  $u_0(k, r) = \sum_{i=1}^m |\varphi_i(k)\langle \varphi_i(k - A(r)) | + u_0^1(k, r)$ .

*Proof.* We only need to show that a  $\tau$ -equivariant principal symbol  $u_0(k, r)$  of the form claimed above exists. Then the proof works line by line as the proof of [Teufel 2003, Proposition 5.18]; see also [Panati et al. 2003a]. However, according to Lemma 4.1,

$$u_0(k, r) := t^B(k, k - A(r)) = t^B((\kappa_1, \kappa_2), (\kappa_1 - A_1(r), \kappa_2))$$

is  $\tau$ -equivariant and has the desired form. Here we use the choice of gauge  $\gamma_2 \cdot A(r) = 0$  and write as before  $A(r) = A_1(r)\gamma_1^*$ . Note that at this point we have to assume  $A^{(1)} \equiv 0$ , because otherwise the  $\kappa_2$ -derivatives of  $u_0$  would become unbounded functions of  $r$  and  $u_0 \notin S_\tau^w$  for all order functions  $w$ .  $\square$

**Definition 4.4.** Using the matrix-valued function  $\alpha$  constructed in Proposition 4.2, we define

$$\mathcal{H}_\alpha := \{f \in L_{\text{loc}}^2(\mathbb{R}^2, \mathbb{C}^m) \mid f(k - \gamma^*) = \alpha(\kappa_2)^{-n_1} f(k) \text{ for all } k \in \mathbb{R}^2, \gamma^* \in \Gamma^*\}$$

with inner product  $\langle f, g \rangle_{\mathcal{H}_\alpha} = \int_{M^*} dk \langle f(k), g(k) \rangle_{\mathbb{C}^m}$ .

Using the orthonormal frame  $(\varphi_1(k), \dots, \varphi_m(k))$  constructed in Proposition 4.2, we define the unitary maps

$$\begin{aligned} U_\alpha(k) : P_I(k)\mathcal{H}_f &\rightarrow \mathbb{C}^m, & f &\mapsto (U_\alpha(k)f)_i := \langle \varphi_i(k), f \rangle_{\mathcal{H}_f}, \\ U_\alpha : P_I\mathcal{H}_\tau &\rightarrow \mathcal{H}_\alpha, & f &\mapsto (U_\alpha f)(k)_i := \langle \varphi_i(k), f(k) \rangle_{\mathcal{H}_f}. \end{aligned}$$

In the same way that  $P_I\mathcal{H}_\tau$  is the space of  $L^2$ -sections of the Bloch bundle  $\Xi_{\text{Bl}}$ , the space  $\mathcal{H}_\alpha$  is the space of  $L^2$ -sections of a bundle  $\Xi_\alpha$ .

**Definition 4.5.** Let

$$\Xi_\alpha := (\mathbb{R}^2 \times \mathbb{C}^m) / \sim_\alpha, \tag{21}$$

where

$$(k, \lambda) \sim_\alpha (k', \lambda') \iff k' = k - \gamma^* \quad \text{and} \quad \lambda' = \alpha(\kappa_2)^{-n_1} \lambda \quad \text{for some } \gamma^* = (n_1, n_2) \in \Gamma^*.$$

On sections of  $\Xi_\alpha$  we define the connection  $\nabla^\alpha := U_\alpha \nabla^B U_\alpha^*$ .

It was shown by Panati [2007] that even for  $m > 1$  the bundle  $\Xi_\alpha$  is, up to isomorphisms, uniquely determined by its Chern number

$$\theta := \frac{1}{2\pi} \int_{M^*} \text{tr}(\Omega(k)) dk.$$

However, we use a canonical form for  $\alpha$  only in the case  $m = 1$ , where a canonical choice is (16).

### 5. The effective Hamiltonian as a pseudodifferential operator

Combining the unitary maps  $U_1^\varepsilon : \Pi_I^\varepsilon \mathcal{H}_\tau \rightarrow P_I \mathcal{H}_\tau$  and  $U_\alpha : P_I \mathcal{H}_\tau \rightarrow \mathcal{H}_\alpha$  into

$$U^\varepsilon : \Pi_I^\varepsilon \mathcal{H}_\tau \rightarrow \mathcal{H}_\alpha, \quad U^\varepsilon := U_\alpha U_1^\varepsilon,$$

we find that the block  $\Pi_I^\varepsilon H_{\text{BF}}^\varepsilon \Pi_I^\varepsilon$  of  $H_{\text{BF}}^\varepsilon$  is unitarily equivalent to the effective Hamiltonian

$$H_I^{\text{eff}} := U^\varepsilon \Pi_I^\varepsilon H_{\text{BF}}^\varepsilon \Pi_I^\varepsilon U^{\varepsilon*}$$

acting on the space  $\mathcal{H}_\alpha$  of  $L^2$ -sections of  $\Xi_\alpha$ . The remaining problem is to compute explicitly an asymptotic expansion of  $H_I^{\text{eff}}$  in powers of  $\varepsilon$ , where the leading-order term should be given by Peierls substitution,

$$H_I^{\text{eff}} = E_I(k - A(i\varepsilon \nabla_k^\alpha)) + \Phi(i\varepsilon \nabla_k^\alpha) + \mathcal{O}(\varepsilon)$$

with

$$E_I(k)_{ij} = \langle \varphi_i(k), H_{\text{per}}(k) \varphi_j(k) \rangle.$$

Note that  $\nabla^\alpha$  is the only natural connection on sections of  $\Xi_\alpha$ , as the flat connection, used implicitly for Peierls substitution in the nonmagnetic case, is not at our disposal. It will be a considerable effort in itself to properly define the pseudodifferential operator  $E_I(k - A(i\varepsilon \nabla_k^\alpha)) + \Phi(i\varepsilon \nabla_k^\alpha)$  as an operator on  $\mathcal{H}_\alpha$ .

In the nonmagnetic case the problem of expanding  $H_{\text{eff}}$  is much simpler. Then not only the Hamiltonian  $H_{\text{BF}}^\varepsilon = \text{Op}^\tau(H)$  and the projection  $\Pi_I^\varepsilon = \text{Op}^\tau(\pi) + \mathcal{O}(\varepsilon^\infty)$  are  $\mathcal{O}(\varepsilon^\infty)$ -close to pseudodifferential operators, but also the intertwining unitary  $U^\varepsilon = \text{Op}^\tau(u) + \mathcal{O}(\varepsilon^\infty)$ . Moreover,  $\mathcal{H}_\alpha$  contains periodic functions and  $H_I^{\text{eff}}$  is close to a semiclassical pseudodifferential operator  $h_I^{\text{eff}}(k, i\varepsilon \nabla_k)$  with an asymptotic expansion of its symbol computable using the Moyal product:

$$\begin{aligned} H_I^{\text{eff}} &= U^\varepsilon \Pi_I^\varepsilon H_{\text{BF}}^\varepsilon \Pi_I^\varepsilon U^{\varepsilon*} = \text{Op}^\tau(u) \text{Op}^\tau(\pi) \text{Op}^\tau(H) \text{Op}^\tau(\pi) \text{Op}^\tau(u^*) + \mathcal{O}(\varepsilon^\infty) \\ &= \text{Op}^\tau(\underbrace{u \sharp \pi \sharp H \sharp \pi \sharp u^*}_{=: h_I^{\text{eff}}}) + \mathcal{O}(\varepsilon^\infty). \end{aligned}$$

In our magnetic case, however, we cannot proceed like this. Although the operators  $\Pi_I^\varepsilon$  and  $U_1^\varepsilon$  are again nearly pseudodifferential operators, this is no longer true for  $U_\alpha$ . The symbol for this operator would have to be  $u_\alpha(k, r) = \sum_{i=1}^m |\langle \varphi_i(k), r \rangle|$ , which is in no suitable symbol class because its derivatives of order  $n$  grow like  $|k|^n$ . So we have to deal with the fact that our effective Hamiltonian is of the form

$$H_I^{\text{eff}} = U^\varepsilon \Pi_I^\varepsilon \text{Op}^\tau(H) \Pi_I^\varepsilon U^{\varepsilon*} = U_\alpha P_I \text{Op}^\tau(h) P_I U_\alpha^* + \mathcal{O}(\varepsilon^\infty).$$

Our solution is to replace the  $\tau$ -quantized operator  $\text{Op}^\tau(\mathfrak{h}) = \mathfrak{h}(k, i\varepsilon\nabla_k^\tau)$  by a ‘‘Berry quantized’’ operator  $\text{Op}^B(h) = h(k, i\varepsilon\nabla_k^B)$  (see (26)) with a modified symbol  $h$ . Because of the unitary equivalence  $\nabla^\alpha = U_\alpha \nabla^B U_\alpha^*$ , one expects and we will show that  $U_\alpha h(k, i\varepsilon\nabla_k^B) U_\alpha^* = h_I^{\text{eff}}(k, i\varepsilon\nabla_k^\alpha)$  with  $h_I^{\text{eff}}(k, r)_{ij} := \langle \varphi_i(k), h(k, r) \varphi_j(k) \rangle$ . We postpone the detailed definitions of the new quantizations and the proofs of their relevant properties to Section 6. In a nutshell the quantization maps are defined as follows:

- For  $h \in S_\tau^1(\varepsilon, \mathcal{L}(\mathcal{H}_\tau))$  we put  $\text{Op}^B(h) = h(k, i\varepsilon\nabla_k^B)$  acting on  $\mathcal{H}_\tau$ .
- For  $h \in S_\alpha(\varepsilon, \mathcal{L}(\mathbb{C}^m))$  (see Definition 6.11) we put  $\text{Op}^\alpha(h) = h(k, i\varepsilon\nabla_k^\alpha)$  acting on  $\mathcal{H}_\alpha$ .
- For  $m = 1$  and  $\Gamma^*$ -periodic  $h \in S^1(\varepsilon, \mathcal{L}(\mathbb{C}))$  we put  $\text{Op}^\theta(h) = h(k, i\varepsilon\nabla_k^\theta)$  acting on  $\mathcal{H}_\alpha$ , where  $\nabla_k^\theta := \nabla_k + i\theta/(2\pi) \langle k, \gamma_1 \rangle \gamma_2$ .

The last quantization will only be used for the case  $m = 1$  in order to obtain an explicit expression for  $H_I^{\text{eff}}$ . Note that changing the connection from  $\nabla^\alpha$  to  $\nabla^\theta$  makes the quantization rule independent of  $\varphi_1$ . Moreover,  $\nabla^\theta$  is canonical in the sense that its curvature tensor  $R^\theta(X, Y) = i\theta|M|/(2\pi) (X_1 Y_2 - X_2 Y_1)$  is constant.

All in all, the steps leading to a representation of the effective Hamiltonian  $H_I^{\text{eff}}$  as a pseudodifferential operator are

$$\begin{aligned} H_I^{\text{eff}} &:= U^\varepsilon \Pi_I^\varepsilon H_{\text{BF}}^\varepsilon \Pi_I^\varepsilon U^{\varepsilon*} = U^\varepsilon \Pi_I^\varepsilon \text{Op}^\tau(H) \Pi_I^\varepsilon U^{\varepsilon*} = U_\alpha P_I \text{Op}^\tau(\mathfrak{h}) P_I U_\alpha^* + \mathcal{O}(\varepsilon^\infty) \\ &= U_\alpha P_I \text{Op}^B(h) P_I U_\alpha^* + \mathcal{O}(\varepsilon^\infty) = \text{Op}^\alpha(h_I^{\text{eff}}) + \mathcal{O}(\varepsilon^\infty). \end{aligned}$$

In the following theorem we collect our main results:

**Theorem 5.1.** *Let Assumptions 1 and 2 hold with  $A^{(1)} = 0$  and let  $\{E_n(k)\}_{n \in I}$  be an isolated family of Bloch bands. Then there exist an orthogonal projection  $\Pi_I^\varepsilon \in \mathcal{L}(\mathcal{H}_\tau)$  and a unitary map  $U^\varepsilon \in \mathcal{L}(\Pi_I^\varepsilon \mathcal{H}_\tau, \mathcal{H}_\alpha)$  such that*

$$\| [H_{\text{BF}}^\varepsilon, \Pi_I^\varepsilon] \|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty) \tag{22}$$

and, with  $H_I^{\text{eff}} := U^\varepsilon \Pi_I^\varepsilon H_{\text{BF}}^\varepsilon \Pi_I^\varepsilon U^{\varepsilon*}$ ,

$$\| (e^{-iH_{\text{BF}}^\varepsilon t} - U^{\varepsilon*} e^{-iH_I^{\text{eff}} t} U^\varepsilon) \Pi_I^\varepsilon \|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^\infty |t|). \tag{23}$$

If  $\{E_n(k)\}_{n \in I}$  is strictly isolated with gap  $d_g$  and  $\|\Phi\|_\infty < \frac{1}{2}d_g$ , then the expressions in (22) and (23) vanish exactly.

There is an  $\alpha$ -equivariant symbol  $h_I^{\text{eff}} \in S_\alpha(\varepsilon, \mathcal{L}(\mathbb{C}^m))$  such that

$$\| H_I^{\text{eff}} - \text{Op}^\alpha(h_I^{\text{eff}}) \|_{\mathcal{L}(\mathcal{H}_\alpha)} = \mathcal{O}(\varepsilon^\infty). \tag{24}$$

The asymptotic expansion of the symbol  $h_I^{\text{eff}}$  can be computed, in principle, to any order in  $\varepsilon$ . Its principal symbol is given by

$$h_0(k, r) = E_I(k - A(r)) + \Phi(r) \mathbf{1}_{m \times m},$$

where

$$E_I(k)_{ij} := \langle \varphi_i(k), H_{\text{per}}(k) \varphi_j(k) \rangle_{\mathcal{H}_\ell}$$

and  $(\varphi_1(k), \dots, \varphi_m(k))$  is the orthonormal frame of the extended Bloch bundle constructed in [Proposition 4.2](#). Thus, Peierls substitution is the leading-order approximation to the restriction of the Hamiltonian to an isolated family of bands:

$$\|H_I^{\text{eff}} - \text{Op}^\alpha(h_0)\|_{\mathcal{L}(\mathcal{H}_\alpha)} = \mathcal{O}(\varepsilon).$$

*Proof.* The projection  $\Pi_I^\varepsilon$  was constructed in [Theorem 3.1](#). The unitary  $U^\varepsilon := U_\alpha U_1^\varepsilon$  is obtained from  $U_1^\varepsilon$ , constructed in [Proposition 4.3](#), and  $U_\alpha$ , given in [Definition 4.4](#). Statement (23) follows from (22) by standard time-dependent perturbation theory.

Now the operator  $H_0 := U_1^\varepsilon \Pi_I^\varepsilon H_{\text{BF}}^\varepsilon \Pi_I^\varepsilon U_1^{\varepsilon*}$  is, by construction, asymptotic to the  $\tau$ -quantization of the semiclassical symbol  $\mathfrak{h} := u \sharp \pi \sharp H \sharp \pi \sharp u^* \in S_\tau^1(\varepsilon)$  with principal symbol

$$\mathfrak{h}_0(k, r) = \langle \varphi_i(k - A(r)), (H_{\text{per}}(k - A(r)) + \Phi(r))\varphi_j(k - A(r)) \rangle_{\mathcal{H}_r} |\varphi_i(k)\rangle \langle \varphi_j(k)|.$$

As sketched before and as to be shown in [Corollary 6.9](#), one can approximate  $\text{Op}^\tau(\mathfrak{h})$  by the Berry quantization  $\text{Op}^B(h)$  of a modified symbol  $h$  up to an error of order  $\varepsilon^\infty$ . More precisely, in [Corollary 6.9](#) we show that there is a sequence of symbols  $h_n \in S_\tau^1$  with  $h_0 = \mathfrak{h}_0$  such that, for any  $N \in \mathbb{N}$ ,

$$\left\| \sum_{n=0}^N \varepsilon^n \text{Op}^B(h_n) - \text{Op}^\tau(\mathfrak{h}) \right\| = \mathcal{O}(\varepsilon^{N+1}).$$

As we will show in [Proposition 6.13](#), the Berry quantization transforms in an explicit way under the unitary mapping  $U_\alpha$  to the reference space  $\mathcal{H}_\alpha$ . Namely, it holds that  $U_\alpha \text{Op}^B(h_n) U_\alpha^* = \text{Op}^\alpha(h_n^{\text{eff}})$  with

$$(h_n^{\text{eff}})_{ij}(k, r) = \langle \varphi_i(k), h_n(k, r)\varphi_j(k) \rangle.$$

Then (24) holds for any resummation  $h_I^{\text{eff}}$  of the asymptotic series  $\sum \varepsilon^n h_n^{\text{eff}}$ . □

As stated in the theorem, one can compute order by order the asymptotic expansion of  $h_I^{\text{eff}}$  using the explicit expansions of the symbols  $\pi$  and  $u$  and expanding Moyal products. We now show how to compute the subprincipal symbol  $h_1$  in a special case, and for this we adopt the notation introduced in the proof of [Theorem 5.1](#). According to [Corollary 6.9](#) there are two contributions to  $h_1$ , namely

$$h_1(k, r) = h_{1,c} + \mathfrak{h}_1 := -\frac{1}{2}i(\nabla_r \mathfrak{h}_0(k, r) \cdot M(k) + M(k) \cdot \nabla_r \mathfrak{h}_0(k, r)) + \mathfrak{h}_1,$$

where

$$M(k) := [\nabla P_I(k), P_I(k)].$$

While one could compute  $h_1$  also for general isolated families of bands, this is more cumbersome and the result is rather complicated. We therefore specialize to the case  $m = 1$ , i.e., to a single nondegenerate isolated band  $E_n$ . Then

$$\mathfrak{h}_0(k, r) = (E_n(k - A(r)) + \Phi(r))P_I(k)$$

and, using the  $\varphi$  corresponding to (16), we obtain that the Berry connection coefficient  $\mathcal{A}_1(k) = -i/(2\pi) \langle \varphi_n(k), \partial_{k_1} \varphi_n(k) \rangle$  is a periodic function of  $k_2$  and independent of  $k_1$ . Hence, introducing the

kinetic momentum  $\tilde{k} := k - A(r)$  and, recalling that  $A(r) = A_1(r)\gamma_1^*$ , we have  $\mathcal{A}_1(\tilde{k}) = \mathcal{A}_1(k)$ . Using this and specializing to the case  $\Gamma = \mathbb{Z}^2$  for the moment, one finds for the subprincipal symbol of

$$\mathfrak{h} = u \sharp \pi \sharp H \sharp \pi \sharp u^* = P_I \sharp u \sharp H \sharp u^* \sharp P_I,$$

by the same reasoning as in the proof of [Teufel 2003, Corollary 5.12], the expression

$$\begin{aligned} h_1(k, r) = & \left( -\mathcal{A}_1(\tilde{k})(\partial_2 E_n(\tilde{k})B(r) - \partial_{r_1} E_n(\tilde{k})) + (\mathcal{A}_2(k) - \mathcal{A}_2(\tilde{k}))(\partial_2 \Phi(r) \right. \\ & \left. - \partial_1 E_n(\tilde{k})B(r)) + B(r) \operatorname{Re}(i\langle \partial_1 \varphi_n(\tilde{k}), (H_{\text{per}} - E_n)(\tilde{k}) \partial_2 \varphi_n(\tilde{k}) \rangle_{\mathfrak{H}_f}) \right) P_I(k) \\ & - \frac{1}{2} i \nabla_r (E_n(\tilde{k}) + \Phi(r)) M(k), \end{aligned}$$

where  $\tilde{k} := k - A(r)$  and  $B = \operatorname{curl} A = \partial_2 A_1$ . Using  $P_I(k) \nabla P_I(k) P_I(k) = 0$ , the last term in  $h_1$  cancels exactly  $h_{1,c}$  in  $h_1$  and we find

$$\begin{aligned} h_1^{\text{eff}}(k, r) &= \langle \varphi_n(k), h_1(k, r) \varphi_n(k) \rangle \\ &= -\mathcal{A}_1(\tilde{k})(\partial_2 E_n(\tilde{k})B(r) - \partial_{r_1} E_n(\tilde{k})) + (\mathcal{A}_2(k) - \mathcal{A}_2(\tilde{k}))(\partial_2 \Phi(r) - \partial_1 E_n(\tilde{k})B(r)) + B(r) \mathcal{M}(\tilde{k}), \end{aligned}$$

with

$$\mathcal{M}(\tilde{k}) := \operatorname{Re}(i\langle \partial_1 \varphi_n(\tilde{k}), (H_{\text{per}} - E_n)(\tilde{k}) \partial_2 \varphi_n(\tilde{k}) \rangle_{\mathfrak{H}_f})$$

the Rammal–Wilkinson term. To get a nicer expression we compute the symbol with respect to the  $\theta$ -quantization. According to Proposition 6.14 we have to add

$$\begin{aligned} -\left( \mathcal{A}_1(k) \partial_{r_1} h_0^{\text{eff}}(k, r) + \left( \mathcal{A}_2(k) - \frac{\theta k_1}{2\pi} \right) \partial_{r_2} h_0^{\text{eff}}(k, r) \right) \\ = -\mathcal{A}_1(k)(\partial_{r_1} E_n(\tilde{k}) + \partial_1 \Phi(r)) - \left( \mathcal{A}_2(k) - \frac{\theta k_1}{2\pi} \right) (\partial_2 \Phi(r) - \partial_1 E_n(\tilde{k})B(r)). \end{aligned}$$

In summary we have

$$h_1^{\text{eff},\theta}(k, r) = -\mathcal{A}_1(\tilde{k})(\partial_1 \Phi(r) + \partial_2 E_n(\tilde{k})B(r)) - \left( \mathcal{A}_2(\tilde{k}) - \frac{\theta k_1}{2\pi} \right) (\partial_2 \Phi(r) - \partial_1 E_n(\tilde{k})B(r)) + B(r) \mathcal{M}(\tilde{k}),$$

where we note that the combination  $\mathcal{A}_2(\tilde{k}) - \theta k_1/(2\pi)$  is a  $\Gamma^*$ -periodic function.

So, in summary, we obtain the following corollary:

**Corollary 5.2.** *Let Assumptions 1 and 2 hold with  $A^{(1)} = 0$  and let  $E(k) \equiv E_n(k)$  be an isolated nondegenerate Bloch band. Then there is a  $\Gamma^*$ -periodic symbol  $h^{\text{eff},\theta} \in S^1(\varepsilon, \mathcal{L}(\mathbb{C}))$  such that, for the effective Hamiltonian  $H_n^{\text{eff}} := H_{I=\{n\}}^{\text{eff}}$  from Theorem 5.1, it holds that*

$$\|H_n^{\text{eff}} - \operatorname{Op}^\theta(h^{\text{eff},\theta})\|_{\mathcal{L}(\mathfrak{H}_\alpha)} = \mathcal{O}(\varepsilon^\infty). \tag{25}$$

The asymptotic expansion of the symbol  $h^{\text{eff},\theta}$  can be computed, in principle, to any order in  $\varepsilon$ . Its principal symbol is given by

$$h_0(k, r) = E(\tilde{k}) + \Phi(r),$$

and its subprincipal symbol by

$$h_1(k, r) = \mathcal{A}(k, r) \cdot (B(r)\nabla E(\tilde{k})^\perp - \nabla\Phi(r)) + B(r)\mathcal{M}(\tilde{k}),$$

where  $\tilde{k} := k - A(r)$ ,  $\nabla E(\tilde{k})^\perp = (-\partial_2 E(\tilde{k}), \partial_1 E(\tilde{k}))$  and

$$\mathcal{M}(k) = -\operatorname{Im}(\langle \partial_1 \varphi(k), (H_{\text{per}} - E)(k) \partial_2 \varphi(k) \rangle_{\mathfrak{H}_\ell}).$$

The Berry connection coefficient  $\mathcal{A}$  is given by

$$\mathcal{A}(k, r) = \mathcal{A}_1(\tilde{k})\gamma_1 + \left( \mathcal{A}_2(\tilde{k}) - \frac{\theta}{2\pi} \langle k, \gamma_1 \rangle \right) \gamma_2,$$

where the components  $\mathcal{A}_j$  are computed from the function  $\varphi$  constructed in Proposition 4.2 as

$$\mathcal{A}_j(k) = -\frac{i}{2\pi} \langle \varphi(k), \partial_{\kappa_j} \varphi(k) \rangle := -\frac{i}{2\pi} \langle \varphi(k), \gamma_j^* \cdot \nabla \varphi(k) \rangle.$$

The two terms in the subprincipal symbol have the following physical meaning: Since  $\nabla E_n(k)$  is the velocity of a particle with quasimomentum  $k$  in the  $n$ -th band, the term in brackets is the Lorentz force on the particle. Since the  $\theta$ -quantization takes into account the integrated curvature of the Berry connection of  $2\pi\theta$  per lattice cell of  $\Gamma^*$ , the curvature form of the effective Berry connection coefficient  $\mathcal{A}$  integrates to zero. The second term in  $h_1$  is a correction to the energy, known as the Rammal–Wilkinson term. For the case  $\theta = 0$  we recover the first-order correction to Peierls substitution established in [Panati et al. 2003a].

## 6. Weyl quantization on the Bloch bundle

In this section we construct quantization schemes that map suitable symbols to pseudodifferential operators that act on sections of possibly nontrivial bundles. Our construction is related to and motivated by similar constructions in the literature [Pflaum 1998a; 1998b; Safarov 1997; Sharafutdinov 2004; 2005; Hansen 2011]. As opposed to the case of functions on  $\mathbb{R}^n$ , the relation between a pseudodifferential operator acting on sections of a vector bundle and its symbol becomes more subtle. If one defines a corresponding pseudodifferential calculus in local coordinates, as is done in [Hörmander 1985], for example, one can associate a symbol to an operator which is unique only up to an error of order  $\varepsilon$ . To define a full symbol, one has to take into account the geometry of the vector bundle. This means that instead of local coordinates, one must use a connection on the vector bundle and a connection on the base space. This idea goes back to Widom [1978; 1980], who was the first to develop a complete isomorphism between such pseudodifferential operators and their symbols. However, while he showed how to recover the full symbol from a pseudodifferential operator and proved that this map is bijective, he did not provide an explicit integral formula for the quantization map. His work was developed further by Pflaum [1998b] and Safarov [1997]. Pflaum [1998b] constructs a quantization map which maps symbols that are sections of endomorphism bundles to operators between the sections of the corresponding bundles. In his quantization formulas he uses a cutoff function so that he can use the exponential map corresponding to a given connection on the manifold that may not be defined globally. A geometric symbol calculus

for pseudodifferential operators between sections of vector bundles can also be found in [Sharafutdinov 2004; 2005], where the author moreover introduces the notion of a geometric symbol in comparison to a coordinatewise symbol. A semiclassical variant of this calculus can be found in [Hansen 2011]. When we compute the symbol  $f$  such that  $\text{Op}^\tau(f) = \text{Op}^B(f) + \mathcal{O}(\varepsilon^\infty)$ , one could say, using the language of [Sharafutdinov 2004; 2005], that  $f$  is the geometric symbol with respect to the Berry connection of the operator  $\text{Op}^\tau(f)$ .

While Safarov [1997] and Pflaum [1998a] provide formulas for the Weyl quantization, this is done only for pseudodifferential operators on manifolds and not for operators between sections of vector bundles. Moreover, Safarov and Pflaum consider only Hörmander symbol classes [1985]. In the following we define semiclassical Weyl calculi for more general symbol classes and include the case of bundles with an infinite-dimensional Hilbert space as the typical fiber. In addition we prove a Calderón–Vaillancourt-type theorem establishing  $L^2$ -boundedness and provide explicit formulas relating the different symbols of an operator corresponding to different quantization maps. However, our constructions are specific to bundles over the torus. Requiring periodicity conditions for symbols and functions allows us to project the calculus from the cover  $\mathbb{R}^2$  to the quotient  $\mathbb{R}^2/\mathbb{Z}^2$ , an approach already used in [Gérard and Nier 1998; Panati et al. 2003a; Teufel 2003]. A similar approach was also applied in [Asch et al. 1994], where the authors consider the Bochner Laplacian acting on sections of a line bundle with connection over the torus. In our calculus, the Bochner Laplacian  $-\Delta_k$  corresponding to a connection is obtained by quantization of the symbol  $f(k, r) = r^2$  for  $\varepsilon = 1$  using the same connection.

**The Berry quantization.** The basic idea of the “Berry quantization” is to map multiplication by  $r$  to the covariant derivative  $i\varepsilon \nabla_k^B$ . In contrast to the  $\tau$ -quantization, where  $r$  is mapped to  $i\varepsilon \nabla_k$ , this has two advantages. Since  $i\varepsilon \nabla_k^B$  is a connection on the Bloch bundle, it leaves invariant its space of sections. As a consequence,  $f(k, i\varepsilon \nabla_k^B)$  commutes with  $P_I$  if and only if  $f(k, r)$  commutes with  $P_I(k)$  for all  $(k, r) \in M^* \times \mathbb{R}^2$ . Moreover, the connection  $\nabla_k^B$  restricted to sections of the Bloch bundle is unitarily equivalent to the connection  $\nabla_k^\alpha$  on the bundle  $\Xi_\alpha$  via the unitary map  $U_\alpha$ .

As in [Panati et al. 2003a; Teufel 2003], a symbol  $f_\varepsilon \in S^w(\varepsilon, \mathcal{L}(\mathcal{H}_f))$  is called  $\tau$ -equivariant (more precisely,  $(\tau_1, \tau_2)$ -equivariant) if

$$f_\varepsilon(q - \gamma, p) = \tau_2(\gamma) f_\varepsilon(q, p) \tau_1(\gamma)^{-1} \quad \text{for all } \gamma \in \Gamma.$$

The spaces of  $\tau$ -equivariant symbols are denoted by  $S_\tau^w(\varepsilon, \mathcal{L}(\mathcal{H}_f))$ .

Using the parallel transport  $t^B(x, y)$  with respect to the Berry connection introduced in Lemma 4.1, we define the Berry quantization  $\text{Op}_\chi^B(f) \in \mathcal{L}(\mathcal{H}_\tau)$  for  $\tau$ -equivariant symbols  $f \in S_\tau^1(\mathcal{L}(\mathcal{H}_f))$  as

$$\begin{aligned} & (\text{Op}_\chi^B(f)\psi)(k) \\ &= \frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} e^{i(k-y)r/\varepsilon} \chi(k-y) t^B(k, \tfrac{1}{2}(k+y)) f(\tfrac{1}{2}(k+y), r) t^B(\tfrac{1}{2}(k+y), y) \psi(y) dy \right) dr. \end{aligned} \quad (26)$$

Here, in contrast to the usual Weyl quantization rule, we take into account that  $\psi$  is a section of a vector bundle with connection  $\nabla^B$  and that the symbol  $f(\cdot, r)$  is really a section of its endomorphism bundle. So, for  $f(\frac{1}{2}(k+y), r)$  to act on  $\psi(y)$  we first need to map  $\psi(y)$  into the correct fiber of the bundle,

which is done by the parallel transport  $t^B(\frac{1}{2}(k+y), y)$ . However, since the derivatives of  $t^B(x, y)$  are not uniformly bounded, we introduce a cutoff function  $\chi$  in the definition. The choice of this cutoff function has only an effect of order  $\mathcal{O}(\varepsilon^\infty)$  on the operator, but it simplifies the following analysis considerably.

**Definition 6.1.** A function  $\chi \in C^\infty(\mathbb{R}^2)$  is called a smooth cutoff function if  $\text{supp } \chi$  is compact,  $\chi \equiv 1$  in a neighborhood of 0, and  $0 \leq \chi \leq 1$ .

Since we need  $\text{Op}_\chi^B(f)$  only for  $f \in S_\tau^1(\mathcal{L}(\mathcal{H}_f))$  as an operator on  $\mathcal{H}_\tau$ , we do not follow the usual routine and show that it is well defined on distributions for general symbol classes. We also do not develop a full Moyal calculus for products of such pseudodifferential operators, although this could be done easily with the tools we provide.

For all steps the following simple lemma will be crucial. It states that the cutoff function in the definition of  $\text{Op}_\chi^B(f)$  ensures that all derivatives of the parallel transport in the integral remain bounded uniformly.

**Lemma 6.2.** *There are constants  $c_\alpha$  such that*

$$\|\partial_x^\alpha t^B(x, y)\| \leq c_\alpha \quad \text{for all } x, y \in \mathbb{R}^2 \text{ with } |x - y| < 1.$$

*Proof.* This follows from the smoothness of  $t^B$  and its  $\tau$ -equivariance (13). □

Before we prove  $\mathcal{H}_\tau$ -boundedness we first show that  $\text{Op}_\chi^B(f)$  is well defined on smooth functions.

**Proposition 6.3.** *Let  $f \in S_\tau^1(\mathcal{L}(\mathcal{H}_f))$  and  $\psi \in C^\infty(\mathbb{R}^2, \mathcal{H}_f) \cap \mathcal{H}_\tau$ . Then  $\text{Op}_\chi^B(f)\psi \in C^\infty(\mathbb{R}^2, \mathcal{H}_f) \cap \mathcal{H}_\tau$ .*

*Proof.* First note that, because of the cutoff function, the  $y$ -integral in (26) extends only over a bounded region. Thus one can use

$$e^{-iy \cdot r/\varepsilon} = \left( \frac{1 - \varepsilon^2 \Delta_y}{1 + r^2} \right)^N e^{-iy \cdot r/\varepsilon}$$

and integration by parts in order to show  $r$ -integrability of the inner integral. Therefore  $(\text{Op}_\chi^B(f)\psi)(k)$  is well defined and its smoothness follows immediately, since, by dominated convergence, we can differentiate under the integral and still get enough decay in  $r$  by the above trick. The  $\tau$ -equivariance of  $(\text{Op}_\chi^B(f)\psi)(k)$  can be checked directly using the  $\tau$ -equivariance of  $\psi$ ,  $t^B$  and  $f$ . □

**Proposition 6.4.** *Let  $f \in S_\tau^1(\mathcal{L}(\mathcal{H}_f))$ . Then  $\text{Op}_\chi^B(f) \in \mathcal{L}(\mathcal{H}_\tau)$  with*

$$\|\text{Op}_\chi^B(f)\|_{\mathcal{L}(\mathcal{H}_\tau)} \leq c_\chi \|f\|_{\infty, (4,1)},$$

where the constant  $c_\chi$  depends only on  $\chi$  and

$$\|f\|_{\infty, (4,1)} := \sum_{|\beta| \leq 4, |\beta'| \leq 1} \sup_{k \in M^*, r \in \mathbb{R}^2} \|\partial_k^\beta \partial_r^{\beta'} f(k, r)\|_\infty.$$

*Proof.* Let  $\tilde{\chi} : \mathbb{R}^2 \rightarrow [0, 1]$  be a cutoff function such that  $\text{supp } \tilde{\chi} \subset \{|r| < 1\}$  and  $\sum_{j \in \mathbb{Z}^2} \tilde{\chi}_j(r) \equiv 1$ , where  $\tilde{\chi}_j(r) := \tilde{\chi}(r - j)$ , and let  $f_j := \tilde{\chi}_j f$ . If we can show that  $\text{Op}_\chi^B(f_j) \in \mathcal{L}(\mathcal{H}_\tau)$  and

$$\sup_{j \in \mathbb{Z}^2} \sum_{i \in \mathbb{Z}^2} \|\text{Op}_\chi^B(f_j)^* \text{Op}_\chi^B(f_i)\|_{\mathcal{L}(\mathcal{H}_\tau)}^{1/2} \leq M \quad \text{and} \quad \sup_{j \in \mathbb{Z}^2} \sum_{i \in \mathbb{Z}^2} \|\text{Op}_\chi^B(f_j) \text{Op}_\chi^B(f_i)^*\|_{\mathcal{L}(\mathcal{H}_\tau)}^{1/2} \leq M, \quad (27)$$



then, according to the Cotlar–Stein lemma — see [Dimassi and Sjöstrand 1999, Lemma 7.10] — it follows that  $\sum_{j \in \mathbb{Z}^2} \text{Op}_\chi^B(f_j)$  converges strongly to a bounded operator  $F \in \mathcal{L}(\mathcal{H}_\tau)$  with  $\|F\|_{\mathcal{L}(\mathcal{H}_\tau)} \leq M$ . However, the following lemma shows that  $F = \text{Op}_\chi^B(f)$ :

**Lemma 6.5.** *If  $\psi \in C^\infty(\mathbb{R}^2, C^\infty(\mathbb{R}^2, \mathcal{H}_f) \cap \mathcal{H}_\tau)$  then there is a constant  $C$  such that, for all  $f \in S_\tau^1(\mathcal{L}(\mathcal{H}_f))$  with  $\text{supp } f \subset \mathbb{R}^2 \times \{|r| > R\}$ ,*

$$\|\text{Op}_\chi^B(f)\psi(k)\|_{\mathcal{H}_f} \leq \frac{C}{R^2} \|f\|_{\infty, (4,0)}.$$

*Proof.* Proceed as in the proof of Proposition 6.3, using

$$e^{-iy \cdot r/\varepsilon} = \frac{\varepsilon^4}{r^4} \Delta_y^2 e^{-iy \cdot r/\varepsilon}$$

instead. □

Hence, on the dense set  $\psi \in C^\infty(\mathbb{R}^2, \mathcal{H}_f) \cap \mathcal{H}_\tau$  the sequence  $\sum_j \text{Op}_\chi^B(f_j)\psi$  converges uniformly and thus also in the norm of  $\mathcal{H}_\tau$  to  $\text{Op}_\chi^B(f)\psi$ .

So we are left to show (27), which follows immediately once we can show

$$\|\text{Op}_\chi^B(f_j)^* \text{Op}_\chi^B(f_i)\|_{\mathcal{L}(\mathcal{H}_\tau)} \leq C(|i - j| + 1)^{-4} \|f_i\|_{\infty, (4,1)} \|f_j\|_{\infty, (4,1)} \tag{28}$$

and the analogous second bound for all  $i, j \in \mathbb{Z}^2$ . Let  $\phi, \psi \in \mathcal{H}_\tau$ ; then

$$\begin{aligned} & \langle \phi, \text{Op}_\chi^B(f_j)^* \text{Op}_\chi^B(f_i)\psi \rangle_{\mathcal{H}_\tau} \\ &= \frac{1}{(2\pi\varepsilon)^4} \int_{M^*} dq \int_{\mathbb{R}^8} dy dk dr dr' e^{ik(r-r')/\varepsilon} e^{i(qr' - yr)/\varepsilon} \chi(q-k)\chi(k-y)\phi^*(q)t^B(k, \frac{1}{2}(q+k)) \\ & \quad \times f_j^*(\frac{1}{2}(q+k), r')t^B(\frac{1}{2}(q+k), k)t^B(k, \frac{1}{2}(k+y))f_i(\frac{1}{2}(k+y), r)t^B(\frac{1}{2}(k+y), y)\psi(y). \end{aligned}$$

Because of the cutoff functions, the domains of integration for  $k$  and  $y$  are also restricted to compact convex sets  $M^* \subset M_k \subset M_y$ , respectively.

For  $|i - j| > 2$ ,  $f_i$  and  $f_j$  have disjoint  $r$ -support and

$$e^{ik \cdot (r-r')/\varepsilon} = \left( \frac{-\varepsilon^2 \Delta_k}{|r - r'|^2} \right)^2 e^{ik \cdot (r-r')/\varepsilon} \quad \text{for } r - r' \neq 0.$$

Now we insert this into the above integral, integrate by parts, take the norm into the integral and obtain, for  $|i - j| > 2$ ,

$$\begin{aligned} & |\langle \phi, \text{Op}_\chi^B(f_j)^* \text{Op}_\chi^B(f_i)\psi \rangle| \\ & \leq \frac{\varepsilon^4}{(2\pi\varepsilon)^4} \int_{M^*} dq \int_{M_k} dk \int_{M_y} dy \int_{\mathbb{R}^4} dr dr' \frac{1}{|r - r'|^4} \sum_{\beta_1, \dots, \beta_8} |\partial_k^{\beta_1} \chi(q-k)| |\partial_k^{\beta_2} \chi(k-y)| \\ & \quad \times \|\phi^*(q)\| \|\partial_k^{\beta_3} t^B(k, \frac{1}{2}(q+k))\| \|\partial_k^{\beta_4} f_j^*(\frac{1}{2}(q+k), r')\| \|\partial_k^{\beta_5} t^B(\frac{1}{2}(q+k), k)\| \|\partial_k^{\beta_6} t^B(k, \frac{1}{2}(k+y))\| \\ & \quad \times \|\partial_k^{\beta_7} f_i(\frac{1}{2}(k+y), r)\| \|\partial_k^{\beta_8} t^B(\frac{1}{2}(k+y), y)\| \|\psi(y)\| \end{aligned}$$

$$\leq c \|f_j\|_{\infty,4} \|f_i\|_{\infty,4} \sum_{\beta_1, \beta_2} \int_{M^*} dq \int_{M_k} dk \int_{M_y} dy \int_{\text{supp } \tilde{\chi}_i} dr \int_{\text{supp } \tilde{\chi}_j} dr' \times \frac{\|\phi(q)\| \|\psi(y)\|}{|r - r'|^4} |\partial_k^{\beta_1} \chi(q - k)| |\partial_k^{\beta_2} \chi(k - y)|.$$

Here the sum  $\sum_{\beta_1, \dots, \beta_8}$  runs over a finite number of multi-indices and we used [Lemma 6.2](#). Moreover, we have that, because of the  $\tau$ -equivariance,

$$\|f_j\|_{\infty,4} := \sum_{|\beta| \leq 4} \sup_{\substack{k \in M_y \\ r \in \mathbb{R}^2}} \|\partial_k^\beta f_j(k, r)\| = \sum_{|\beta| \leq 4} \sup_{\substack{k \in M^* \\ r \in \mathbb{R}^2}} \|\partial_k^\beta f(k, r)\|.$$

For the remaining integral we get

$$\begin{aligned} & \int_{M^*} dq \int_{M_k} dk \int_{M_y} dy \int_{\text{supp } \tilde{\chi}_i} dr \int_{\text{supp } \tilde{\chi}_j} dr' \frac{\|\phi(q)\| \|\psi(y)\|}{|r - r'|^4} |\partial_k^{\beta_1} \chi(q - k)| |\partial_k^{\beta_2} \chi(k - y)| \\ & \leq \frac{c_2}{(|i - j| - 2)^4} \int_{M_k} dk (\|\phi_{M^*}\| * \partial_k^{\beta_1} \chi)(k) (\|\psi_{M_y}\| * \partial_k^{\beta_2} \chi)(k) \\ & \leq \frac{c_2}{(|i - j| - 2)^4} \|\phi_{M^*}\|_2 \|\psi_{M_y}\|_2 \|\partial_k^{\beta_1} \chi\|_1 \|\partial_k^{\beta_2} \chi\|_1 \\ & \leq \frac{c_3}{(|i - j| - 2)^4} \|\phi\|_{\mathcal{H}_\tau} \|\psi\|_{\mathcal{H}_\tau}, \end{aligned}$$

where we used the Cauchy–Schwarz and Young inequalities in the next-to-last step. Here  $\phi_{M^*}(q) := \phi(q) \mathbf{1}_{M^*}(q)$  and  $\psi_{M_y}(q) := \psi(q) \mathbf{1}_{M_y}(q)$ .

In order to obtain a bound uniform in  $\varepsilon$  on  $\|\text{Op}_\chi^B(f_j)^* \text{Op}_\chi^B(f_i)\|_{\mathcal{H}_\tau}$  for all  $i$  and  $j$  directly, observe that one can get the factor  $\varepsilon^4 / (|r - r'|^2 |k - y| |q - k|)$  from appropriate integrations by parts also in  $r$  and  $r'$ , using

$$e^{i(k-y)r/\varepsilon} = \frac{-i\varepsilon(k-y) \cdot \nabla_r}{|k-y|^2} e^{i(k-y)r/\varepsilon}.$$

The remaining expression can be bounded as before, noting that  $1/|r - r'|^2$  is integrable on  $\mathbb{R}^4$  and that  $\partial_k^\beta \chi(k)/|k|$  is integrable on  $\mathbb{R}^2$ . In summary, we can conclude [\(28\)](#), which finishes the proof.  $\square$

Next we check that the choice of the cutoff function only has an effect of order  $\mathcal{O}(\varepsilon^\infty)$ .

**Proposition 6.6.** *Let  $f \in S_\tau^1(\mathcal{L}(\mathcal{H}_f))$  and let  $\chi_1$  and  $\chi_2$  be two cutoff functions. Then*

$$\|\text{Op}_{\chi_1}^B(f) - \text{Op}_{\chi_2}^B(f)\| = \mathcal{O}(\varepsilon^\infty).$$

*Proof.* Let  $\bar{\chi} := \chi_1 - \chi_2$ ; then  $0 < c \leq |k| \leq C < \infty$  for all  $k \in \text{supp } \chi$ . We control the norm of  $\text{Op}_{\bar{\chi}}^B(f) = \text{Op}_{\chi_1}^B(f) - \text{Op}_{\chi_2}^B(f)$  as in the previous proof. So we have to estimate the integrals

$$\begin{aligned} & \langle \phi, \text{Op}_{\bar{\chi}}^B(f_j)^* \text{Op}_{\bar{\chi}}^B(f_i) \psi \rangle_{\mathcal{H}_\tau} \\ & = \frac{1}{(2\pi\varepsilon)^4} \int_{M^*} dq \int_{\mathbb{R}^8} dy dk dr dr' e^{i(k-y)r/\varepsilon} e^{i(q-k)r'/\varepsilon} \bar{\chi}(q-k) \bar{\chi}(k-y) \phi^*(q) t^B(k, \frac{1}{2}(q+k)) \\ & \quad \times f_j^*(\frac{1}{2}(q+k), r') t^B(\frac{1}{2}(q+k), k) t^B(k, \frac{1}{2}(k+y)) f_i(\frac{1}{2}(k+y), r) t^B(\frac{1}{2}(k+y), y) \psi(y). \end{aligned}$$

Using

$$e^{i(k-y)r/\varepsilon} = \left( \frac{-\varepsilon^2 \Delta_r}{|k-y|^2} \right)^N e^{i(k-y)r/\varepsilon} \quad \text{for } k-y \neq 0,$$

we can get any power of  $\varepsilon^2$  by integration by parts and estimating the remaining expression as in the previous proof. □

In the following we drop the subscript  $\chi$  in  $\text{Op}_\chi^B(f)$  in the notation whenever the statement is not affected by a change of order  $\varepsilon^\infty$ . Also note that  $\text{Op}^\tau(f) - \text{Op}_\chi^\tau(f) = \mathcal{O}(\varepsilon^\infty)$  for any cutoff function  $\chi$ .

Next we relate the  $\tau$ - and the Berry quantization by using a Taylor expansion of the parallel transport.

**Lemma 6.7.** *For  $\delta \in \mathbb{R}^2$  with  $|\delta| < \delta_0$  small enough, the parallel transport from  $z$  to  $z + \delta$  has a uniformly and absolutely convergent expansion*

$$t^B(z + \delta, z) = \sum_{n=0}^{\infty} t_n^{i_1, \dots, i_n}(z) \delta_{i_1} \cdots \delta_{i_n} := \sum_{n=0}^{\infty} \sum_{(i_1, \dots, i_n) \in \{1, 2\}^n} t_n^{i_1, \dots, i_n}(z) \delta_{i_1} \cdots \delta_{i_n},$$

where the coefficients  $t_n^{i_1, \dots, i_n} : \mathbb{R}^2 \rightarrow \mathcal{L}(\mathcal{H}_f)$  are real-analytic and  $\tau$ -equivariant. The first terms are, explicitly,

$$t_0 = \mathbf{1}_{\mathcal{H}_f} \quad \text{and} \quad t_1(z) = M(z) := [\nabla P_I(z), P_I(z)].$$

*Proof.* Note that  $t^B(z + \delta, z) = t(1)$ , where  $t(s)$  is the solution of

$$\frac{d}{ds} t(s) = [\delta \cdot \nabla P_I(z + s\delta), P_I(z + s\delta)] t(s) =: \delta \cdot M(z + s\delta) t(s) \quad \text{with} \quad t(0) = \mathbf{1}.$$

Since  $\delta \cdot M : \mathbb{R}^2 \rightarrow \mathcal{L}(\mathcal{H}_f)$  is smooth and uniformly bounded, the solution of this linear ODE is given by the uniformly convergent Dyson series

$$\begin{aligned} t^B(z + \delta, z) - \mathbf{1} &= \sum_{n=1}^{\infty} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} \delta \cdot M(z + t_1\delta) \cdots \delta \cdot M(z + t_n\delta) dt_n \cdots dt_1 \\ &= \sum_{n=1}^{\infty} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{t_1^{m_1} (\delta \cdot \nabla)^{m_1} \delta \cdot M(z)}{m_1!} \cdots \frac{t_n^{m_n} (\delta \cdot \nabla)^{m_n} \delta \cdot M(z)}{m_n!} dt_n \cdots dt_1, \end{aligned}$$

where in the second equality we inserted the uniformly convergent power series for the real-analytic function  $\delta \cdot M$ ,

$$\delta \cdot M(z + t\delta) = \sum_{m=0}^{\infty} \frac{t^m (\delta \cdot \nabla)^m \delta \cdot M(z)}{m!}. \quad \square$$

**Theorem 6.8.** *Let  $f \in S_t^1(\mathcal{L}(\mathcal{H}_f))$  and define, for  $n \in \mathbb{N}_0$ ,*

$$\mathfrak{f}_n(k, r) := \sum_{\substack{a, b \in \mathbb{N}_0 \\ a+b=n}} \frac{(-1)^a}{(2i)^n} t_a^{i_1, \dots, i_a}(k) (\partial_{r_{i_1}} \cdots \partial_{r_{i_a}} \partial_{r_{j_1}} \cdots \partial_{r_{j_b}} f)(k, r) (t_b^{j_1, \dots, j_b}(k))^*.$$

Then  $f_n \in \mathcal{S}_\tau^1(\mathcal{L}(\mathcal{H}_f))$  and

$$\left\| \sum_{n=0}^N \varepsilon^n \text{Op}^\tau(f_n) - \text{Op}^B(f) \right\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^{N+1}). \tag{29}$$

The first terms are, explicitly,  $f_0(k, r) = f(k, r)$  and

$$f_1(k, r) = \frac{1}{2}i(\nabla_r f(k, r) \cdot M(k) + M(k) \cdot \nabla_r f(k, r)),$$

where  $M(k) = [\nabla P_I(k), P_I(k)]$ . Moreover, if  $f$  has compact  $r$ -support, then

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N \varepsilon^n \text{Op}_\chi^\tau(f_n) = \text{Op}_\chi^B(f)$$

strongly in  $\mathcal{H}_\tau$ .

*Proof.* The idea is to insert the Taylor expansion of  $t^B$  from Lemma 6.7 into the integral in the definition (26). To this end first note that, with  $\delta := \frac{1}{2}(k - y)$ , we have that

$$t^B(k, \frac{1}{2}(k + y)) = t^B(\frac{1}{2}(k + y) + \delta, \frac{1}{2}(k + y)) \quad \text{and} \quad t^B(\frac{1}{2}(k + y), y) = t^B(\frac{1}{2}(k + y) - \delta, \frac{1}{2}(k + y))^*.$$

Assume that  $f$  has compact  $r$ -support for the moment. Then for  $\psi \in \mathcal{H}_\tau$  we get

$$\begin{aligned} (\text{Op}_\chi^B(f)\psi)(k) &= \frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^4} dy dr e^{i2\delta \cdot r/\varepsilon} \chi(k - y) \sum_{a=0}^\infty t_a^{i_1, \dots, i_a}(\frac{1}{2}(k + y)) \delta_{i_1} \dots \delta_{i_a} f(\frac{1}{2}(k + y), r) \\ &\quad \times \sum_{b=0}^\infty (-1)^b t_b^{j_1, \dots, j_b}(\frac{1}{2}(k + y))^* \delta_{j_1} \dots \delta_{j_b} \psi(y) \\ &= \frac{1}{(2\pi\varepsilon)^2} \sum_{a,b=0}^\infty (-1)^b \int_{\mathbb{R}^4} dy dr \left(\frac{\varepsilon}{2i}\right)^{a+b} (\partial_{r_{i_1}} \dots \partial_{r_{i_a}} \partial_{r_{j_1}} \dots \partial_{r_{j_b}} e^{i2\delta \cdot r/\varepsilon}) \\ &\quad \times \chi(k - y) t_a^{i_1, \dots, i_a}(\frac{1}{2}(k + y)) f(\frac{1}{2}(k + y), r) t_b^{j_1, \dots, j_b}(\frac{1}{2}(k + y))^* \psi(y) \\ &= \frac{1}{(2\pi\varepsilon)^2} \sum_{a,b=0}^\infty (-1)^a \left(\frac{\varepsilon}{2i}\right)^{a+b} \int_{\mathbb{R}^4} dy dr e^{i(k-y) \cdot r/\varepsilon} \chi(k - y) t_a^{i_1, \dots, i_a}(\frac{1}{2}(k + y)) \\ &\quad \times (\partial_{r_{i_1}} \dots \partial_{r_{i_a}} \partial_{r_{j_1}} \dots \partial_{r_{j_b}} f)(\frac{1}{2}(k + y), r) t_b^{j_1, \dots, j_b}(\frac{1}{2}(k + y))^* \psi(y) \\ &= \sum_{n=0}^\infty \varepsilon^n (\text{Op}_\chi^\tau(f_n)\psi)(k). \end{aligned}$$

Here we used that all sums and integrals converge absolutely and uniformly, so interchanging sums and integrals is no problem. Moreover, by the fact that  $\text{Op}_\chi^B(f)\psi$  is a uniformly bounded and  $\tau$ -equivariant function, the pointwise convergence implies also the strong convergence in  $\mathcal{H}_\tau$ .

In order to estimate  $\Delta_N \psi := (\sum_{n=0}^{N-1} \varepsilon^n \text{Op}_\chi^\tau(f_n) - \text{Op}_\chi^B(f))\psi$  in  $\mathcal{H}_\tau$ , we estimate, as in the previous proofs,  $|\langle \phi, \Delta_N \psi \rangle|$ . Write, for the remainder in the Taylor expansion,

$$\begin{aligned} t^B(z + \delta, z) &= \sum_{a=0}^{N-1} t_a^{i_1, \dots, i_a}(z) \delta_{i_1} \cdots \delta_{i_a} + \frac{(\partial_{i_1} \cdots \partial_{i_N} t^B)(z + \xi(\delta)\delta, z)}{N!} \delta_{i_1} \cdots \delta_{i_N} \\ &=: \sum_{a=0}^{N-1} t_a^{i_1, \dots, i_a}(z) \delta_{i_1} \cdots \delta_{i_a} + R_N^{i_1, \dots, i_N}(z, \delta) \delta_{i_1} \cdots \delta_{i_N}; \end{aligned}$$

then one term appearing in the estimate of  $|\langle \phi, \Delta_N \psi \rangle|$  is

$$\begin{aligned} &\frac{1}{(2\pi\varepsilon)^2} \int_{M^*} dk \int_{\mathbb{R}^4} dy dr e^{i2\delta \cdot r/\varepsilon} \chi(k-y) \phi^*(k) R_N^{i_1, \dots, i_N}(\tfrac{1}{2}(k+y), \delta) \delta_{i_1} \cdots \delta_{i_N} f(\tfrac{1}{2}(k+y), r) \psi(y) \\ &= \frac{1}{(2\pi\varepsilon)^2} \left(\frac{-\varepsilon}{2i}\right)^N \int_{M^*} dk \int_{\mathbb{R}^4} dy dr e^{i(k-y) \cdot r/\varepsilon} \chi(k-y) \phi^*(k) R_N^{i_1, \dots, i_N}(\tfrac{1}{2}(k+y), \delta) \\ &\qquad \qquad \qquad \times (\partial_{r_{i_1}} \cdots \partial_{r_{i_N}} f)(\tfrac{1}{2}(k+y), r) \psi(y). \end{aligned}$$

Such an expression can be bounded by a constant times  $\varepsilon^N \|\phi\| \|\psi\|$  by obtaining an integrable factor  $\varepsilon^2/(|r||k-y|)$  through additional integration by parts, as in the proof of Proposition 6.4. All other terms can be treated similarly, so we have shown (29) for  $f$  with compact  $r$ -support.

For the general statement we use again the Cotlar–Stein lemma on the family of almost orthogonal operators  $\Delta_{N,i} := \sum_{n=0}^{N-1} \varepsilon^n \text{Op}_\chi^\tau(f_{n,i}) - \text{Op}_\chi^B(f_i)$ . While this is very lengthy to write down, the estimates are completely analogous to those of Proposition 6.4, using integration by parts as before.  $\square$

Of course, we can reverse the roles of the two quantizations and obtain the reverse statement.

**Corollary 6.9.** *Let  $\mathfrak{f} \in S_\tau^1(\mathcal{L}(\mathcal{H}_\mathfrak{f}))$  and define*

$$f_n(k, r) := \sum_{a+b=n} \frac{(-1)^a}{(2i)^n} (t_a^{i_1, \dots, i_a}(k))^* (\partial_{r_{i_1}} \cdots \partial_{r_{i_a}} \partial_{r_{j_1}} \cdots \partial_{r_{j_b}} \mathfrak{f})(k, r) t_b^{j_1, \dots, j_b}(k) \quad \text{for } n \in \mathbb{N}_0.$$

Then  $f_n \in S_\tau^1(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_\mathfrak{f}))$  and

$$\left\| \sum_{n=0}^N \varepsilon^n \text{Op}^B(f_n) - \text{Op}^\tau(\mathfrak{f}) \right\|_{\mathcal{L}(\mathcal{H}_\tau)} = \mathcal{O}(\varepsilon^{N+1}).$$

The first terms are, explicitly,  $f_0(k, r) = \mathfrak{f}(k, r)$  and

$$f_1(k, r) = -\frac{1}{2}(i)(\nabla_r \mathfrak{f}(k, r) \cdot M(k) + M(k) \cdot \nabla_r \mathfrak{f}(k, r)).$$

While we do not use the following proposition explicitly, it sheds some light on the geometric significance of the Berry quantization. It states that  $\text{Op}^B(f)$  commutes with the projection  $P_I$  if and only if the symbol  $f(k, r)$  commutes pointwise with  $P_I(k)$ .

**Proposition 6.10.** *Let  $f \in S_\tau^1(\mathcal{L}(\mathcal{H}_\mathfrak{f}))$ . Then*

$$[f(k, r), P_I(k)] = 0 \quad \text{for all } (k, r) \in \mathbb{R}^4 \quad \iff \quad [\text{Op}^B(f), P_I] = 0.$$

*Proof.* It suffices to consider the commutator on the dense set  $C^\infty(\mathbb{R}^2, \mathcal{H}_f) \cap \mathcal{H}_\tau$ , so we can work with the integral definition (26) of  $\text{Op}^B(f)$ . For  $\psi \in C^\infty(\mathbb{R}^2, \mathcal{H}_f) \cap \mathcal{H}_\tau$  it follows from (12) that

$$\begin{aligned} & ([\text{Op}_\chi^B(f), P_I]\psi)(k) \\ &= \frac{1}{(2\pi)^2} \\ & \quad \times \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} e^{i(k-y)r} \chi(k-y) t^B(k, \frac{1}{2}(k+y)) [f(\frac{1}{2}(k+y), \varepsilon r), P_I(\frac{1}{2}(k+y))] t^B(\frac{1}{2}(k+y), y) \psi(y) dy \right) dr \end{aligned}$$

so the implication from left to right is obvious. To prove the reverse implication in detail is somewhat tedious. Since we don't use it, we only sketch the argument. Assume that  $[f(k, r), P_I(k)] = O(k, r) \neq 0$ . Then  $O \in S^1_\tau(\mathbb{R}^4, \mathcal{L}(\mathcal{H}_f))$  and one can show that  $\|\text{Op}_\chi^B(O)\| \geq C > 0$  for some  $C$  independent of  $\varepsilon$  by looking at the action of  $\text{Op}_\chi^B(O)$  on suitable coherent states. This even implies the stronger statement

$$[\text{Op}^B(f), P_I] = o(\varepsilon) \implies [f(k, r), P_I(k)] = 0 \quad \text{for all } (k, r) \in \mathbb{R}^4. \quad \square$$

**The  $\alpha$ -quantization and the  $\theta$ -quantization.** The other two quantizations we use are the  $\alpha$ -quantization and the effective quantization. The  $\alpha$ -quantization with respect to the connection  $\nabla^\alpha = U_\alpha \nabla^B U_\alpha^*$  is used to map  $\alpha$ -equivariant symbols in  $C^\infty(\mathbb{R}^4, \mathcal{L}(\mathbb{C}^m))$  to operators in  $\mathcal{L}(\mathcal{H}_\alpha)$ ; see Definition 4.4. For  $m = 1$  it can be replaced by the effective quantization with respect to the explicit connection  $\nabla_k^\theta := \nabla_k + i\theta/(2\pi) \langle k, \gamma_1 \rangle \gamma_2$ .

In both cases the construction is exactly the same as the one for the Berry quantization, which is to use the parallel transport of the desired connection in the definition of the quantization. Let

$$t^\alpha(x, y) : \mathbb{C}^m \rightarrow \mathbb{C}^m, \quad \lambda \mapsto t^\alpha(x, y)\lambda := U_\alpha(x) t^B(x, y) U_\alpha^*(y) \lambda,$$

be the parallel transport along the straight line from  $y$  to  $x$  with respect to the connection  $\nabla^\alpha = U_\alpha \nabla^B U_\alpha^*$ . Then  $\tau$ -equivariance of  $t^B$  implies  $\alpha$ -equivariance of  $t^\alpha$ , i.e.,

$$t^\alpha(x - \gamma^*, y - \gamma^*) = \alpha \left( \frac{\langle x, \gamma_2 \rangle}{2\pi} \right)^{-n_1} t^\alpha(x, y) \alpha \left( \frac{\langle y, \gamma_2 \rangle}{2\pi} \right)^{n_1}.$$

For  $m = 1$  we introduce the effective connection  $\nabla_k^\theta = \nabla_k + i\theta/(2\pi) \langle k, \gamma_1 \rangle \gamma_2$  and the corresponding  $\alpha$ -equivariant parallel transport

$$t^\theta(x, y) : \mathbb{C} \rightarrow \mathbb{C}, \quad \lambda \mapsto t^\theta(x, y)\lambda := e^{(i\theta/(4\pi))\langle x+y, \gamma_1 \rangle \langle y-x, \gamma_2 \rangle} \lambda.$$

We say that a symbol  $f \in C^\infty(\mathbb{R}^4, \mathcal{L}(\mathbb{C}^m))$  is  $\alpha$ -equivariant if

$$f(k - \gamma^*, r) = \alpha(\kappa_2)^{-n_1} f(k, r) \alpha(\kappa_2)^{n_1} \quad \text{for all } \gamma^* \in \Gamma^*, k, r \in \mathbb{R}^2,$$

where we again use the notation  $\kappa_j = \langle k, \gamma_j \rangle / (2\pi)$ . Note that for  $m = 1$  the  $\alpha$ -equivariant symbols are just the periodic symbols. However, for  $m > 1$  the  $\kappa_2$ -derivatives of an  $\alpha$ -equivariant symbol are in general unbounded as functions of  $\kappa_1$ . Thus we define the space of ‘‘bounded’’ symbols  $S_\alpha(\mathcal{L}(\mathbb{C}^m))$  as follows:

**Definition 6.11.** Let  $S_\alpha(\mathcal{L}(\mathbb{C}^m))$  be the space of  $\alpha$ -equivariant functions  $f \in C^\infty(\mathbb{R}^4, \mathcal{L}(\mathbb{C}^m))$  that satisfy

$$\sup_{k \in M^*, r \in \mathbb{R}^2} \|(\partial_k^\alpha \partial_r^\beta f)(k, r)\|_{\mathcal{L}(\mathbb{C}^m)} < \infty \quad \text{for all } \alpha, \beta \in \mathbb{N}_0^2.$$

As always,  $S_\alpha(\mathcal{L}(\mathbb{C}^m))$  is equipped with the corresponding Fréchet metric and  $S_\alpha(\varepsilon, \mathcal{L}(\mathbb{C}^m))$  denotes the space of uniformly bounded functions  $f : [0, \varepsilon_0) \rightarrow S_\alpha(\mathcal{L}(\mathbb{C}^m))$ .

In complete analogy to the Berry quantization, we define for  $\alpha$ -equivariant symbols  $f \in S_\alpha(\mathcal{L}(\mathbb{C}^m))$  and  $\psi \in \mathcal{H}_\alpha$  the  $\alpha$ -quantization by

$$\begin{aligned} &(\text{Op}_\chi^\alpha(f)\psi)(k) \\ &:= \frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} e^{i(k-y)r/\varepsilon} \chi(k-y) t^\alpha(k, \tfrac{1}{2}(k+y)) f(\tfrac{1}{2}(k+y), r) t^\alpha(\tfrac{1}{2}(k+y), y) \psi(y) dy \right) dr, \end{aligned}$$

and, for  $m = 1$ , the  $\theta$ -quantization by

$$\begin{aligned} &(\text{Op}_\chi^\theta(f)\psi)(k) \\ &:= \frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} e^{i(k-y)r/\varepsilon} \chi(k-y) t^\theta(k, \tfrac{1}{2}(k+y)) f(\tfrac{1}{2}(k+y), r) t^\theta(\tfrac{1}{2}(k+y), y) \psi(y) dy \right) dr \\ &= \frac{1}{(2\pi\varepsilon)^2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} e^{i(k-y)r/\varepsilon} \chi(k-y) e^{i\theta/(4\pi)\langle x+y, \gamma_1 \rangle \langle y-x, \gamma_2 \rangle} f(\tfrac{1}{2}(k+y), r) \psi(y) dy \right) dr. \end{aligned}$$

Now we can show all results of the previous section in a completely analogous way also for the  $\alpha$ - and the  $\theta$ -quantizations.

**Proposition 6.12.** Let  $f \in S_\alpha(\mathbb{R}^4, \mathcal{L}(\mathbb{C}^m))$ . Then  $\text{Op}_\chi^\alpha(f) \in \mathcal{L}(\mathcal{H}_\alpha)$ , with

$$\|\text{Op}_\chi^\alpha(f)\|_{\mathcal{L}(\mathcal{H}_\alpha)} \leq c_\chi \|f\|_{\infty, (4,1)} := c_\chi \sum_{\substack{|\beta| \leq 4 \\ |\beta'| \leq 1}} \sup_{\substack{k \in M^* \\ r \in \mathbb{R}^2}} \|\partial_k^\beta \partial_r^{\beta'} f(k, r)\|,$$

where the constant  $c_\chi$  depends only on  $\chi$ . For  $m = 1$  the same bound holds for  $\text{Op}_\chi^\theta(f)$ .

**Proposition 6.13.** Let  $f \in S_\tau^1(\mathcal{L}(\mathcal{H}_f))$  and

$$f_I(k, r)_{ij} := \langle \varphi_i(k), f(k, r) \varphi_j(k) \rangle.$$

Then  $f_I \in S_\alpha(\mathbb{C}^m)$  and

$$\text{Op}_\chi^\alpha(f_I) = U_\alpha \text{Op}_\chi^B(f) U_\alpha^*.$$

*Proof.* It follows directly from the definitions that  $f_I \in S_\alpha(\mathbb{C}^m)$ . The equality of the operators can be checked on the dense set  $C^\infty(\mathbb{R}^2) \cap \mathcal{H}_\alpha$  using their integral definitions and the fact that, again by definition,  $U_\alpha^*(x) t^\alpha(x, y) = t^B(x, y) U_\alpha^*(y)$ . □

For the case  $m = 1$  we can finally replace the  $\alpha$ - by the  $\theta$ -quantization if we suitably modify the symbol. To this end we introduce the Taylor series of the difference of the parallel transports as

$$t^{\theta*}(k, k + \delta) t^\alpha(k, k + \delta) =: \sum_{n=0}^\infty t_n^{i_1, \dots, i_n}(k) \delta_{i_1} \cdots \delta_{i_n},$$

where

$$t_0(k) \equiv 1 \quad \text{and} \quad t_1(k) = i \left( \mathcal{A}_1(k) \gamma_1 + \left( \mathcal{A}_2(k) - \frac{\theta}{2\pi} \langle k, \gamma_1 \rangle \right) \gamma_2 \right) =: i\mathcal{A}(k).$$

The proof of the following proposition is analogous to the proof of [Theorem 6.8](#). The expressions simplify a bit because for  $m = 1$  the symbol and the parallel transport commute.

**Proposition 6.14.** *Let  $f \in S^1(\mathbb{R}^4, \mathbb{C})$  be a periodic symbol and define, for  $n \in \mathbb{N}_0$ ,*

$$f_n^\theta(k, r) := i^n t_n^{i_1, \dots, i_n}(k) (\partial_{r_{i_1}} \cdots \partial_{r_{i_n}} f)(k, r).$$

*Then  $f_n^\theta \in S^1(\mathbb{R}^4, \mathbb{C})$  is periodic and*

$$\left\| \sum_{n=0}^N \varepsilon^n \text{Op}^\theta(f_n^\theta) - \text{Op}^\alpha(f) \right\|_{\mathcal{L}(\mathcal{H}_\varepsilon)} = \mathcal{O}(\varepsilon^{N+1}). \tag{30}$$

*The first terms are, explicitly,  $f_0^\theta(k, r) = f(k, r)$  and*

$$f_1^\theta(k, r) = -\mathcal{A}(k) \cdot \nabla_r f(k, r).$$

### 7. Application to the Hofstadter model

In this section we apply the general theory developed in the previous sections to perturbations of magnetic subbands of the Hofstadter Hamiltonian [\[1976\]](#). The motivation for doing this is twofold. First it shows, in the simplest possible example, how magnetic Peierls substitution Hamiltonians can be explicitly computed and analyzed. Second, we will find strong support for the conjecture that [Theorem 5.1](#) is actually still valid for perturbations by small constant fields  $B$ . Note that the Hofstadter Hamiltonian and related tight-binding models served not only as model Hamiltonians for the illustration of general results on perturbed periodic Schrödinger operators but also gave rise to considerable mathematical work dedicated specifically to them, e.g., [\[Helffer and Sjöstrand 1989; 1990a; Helffer et al. 1990; Bellissard et al. 1991; Avila and Jitomirskaya 2009\]](#). For a recent overview of the mathematics and the physics literature on the Hofstadter Hamiltonian we refer to [\[De Nittis 2010\]](#).

The Hofstadter model is the canonical model for a single nonmagnetic Bloch band perturbed by a constant magnetic field  $B_0$ . It can be seen to arise from the tight-binding formalism in physics or, alternatively, from Peierls substitution for a nonmagnetic Bloch band. The Hofstadter Hamiltonian is the discrete magnetic Laplacian on the lattice  $\tilde{\Gamma} = \mathbb{Z}^2$ ,

$$H_{\text{Hof}}^{B_0} = D_1 + D_1^* + D_2 + D_2^* \quad \text{acting on } \ell^2(\mathbb{Z}^2).$$

Here  $D_1$  and  $D_2$  are the (dual) magnetic translations

$$(D_1\psi)(x) := \psi(x - e_1) \quad \text{and} \quad (D_2\psi)(x) := e^{iB_0\langle x, e_1 \rangle} \psi(x - e_2).$$



For  $B_0 = 2\pi p/q$  we define the corresponding magnetic Bloch–Floquet transformation on the lattice  $\Gamma = q\mathbb{Z} \times \mathbb{Z}$  as

$$\mathcal{U}_{\text{BF}} : \ell^2(\Gamma; \mathbb{C}^q) \rightarrow L^2(\mathbb{T}_q^*; \mathbb{C}^q), \quad (\mathcal{U}_{\text{BF}}\psi)(k)_j := \sum_{\gamma \in \Gamma} e^{i\gamma \cdot k} (T_\gamma \psi)((j, 0)) \quad \text{for } j = 0, \dots, q - 1,$$

where we recall that the magnetic translations  $T_\gamma$  were defined in (5). Note that the fiber space  $\mathcal{H}_f = \mathbb{C}^q$  is now finite-dimensional and thus we can drop the additional phase  $e^{-ik \cdot y}$  in the definition of  $\mathcal{U}_{\text{BF}}$ , which appeared in (7) to make the domain of  $H_{\text{per}}(k)$  independent of  $k$ . As a consequence, the range of  $\mathcal{U}_{\text{BF}}$  now contains periodic functions on  $\mathbb{T}_q^* = [0, 2\pi/q) \times [0, 2\pi)$  and  $\tau$ -equivariance becomes periodicity. A straightforward computation shows that the shift operators  $D_j$  become matrix-multiplication operators  $\widehat{D}_j := \mathcal{U}_{\text{BF}} D_j \mathcal{U}_{\text{BF}}^*$ ,

$$\widehat{D}_1(k) = \begin{pmatrix} 0 & 0 & 0 & \dots & e^{iqk_1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad \text{and} \quad \widehat{D}_2(k) = e^{ik_2} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & e^{iB_0} & 0 & \dots & 0 \\ 0 & 0 & e^{i2B_0} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & e^{i(q-1)B_0} \end{pmatrix}.$$

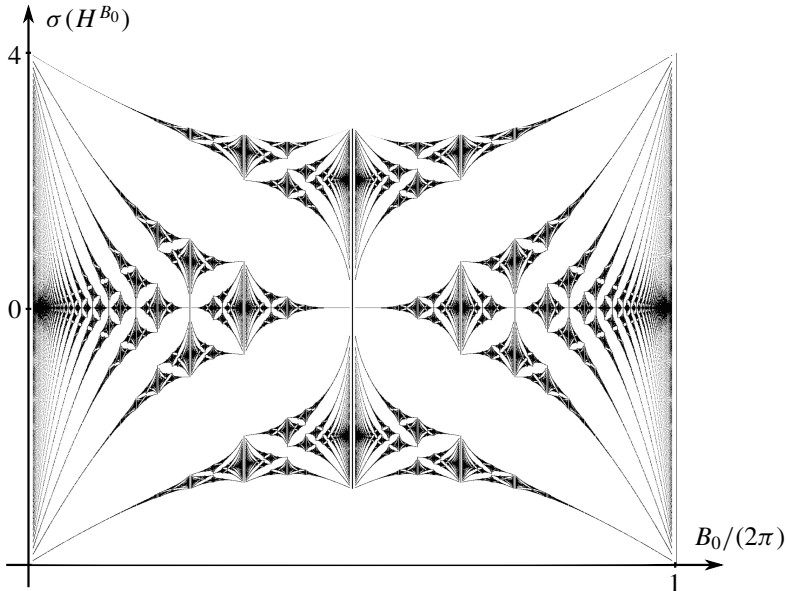
For the Hamiltonian one thus finds

$$\widehat{H}_{\text{Hof}}^{B_0}(k) = \begin{pmatrix} 2 \cos(k_2) & 1 & 0 & \dots & e^{iqk_1} \\ 1 & 2 \cos(k_2 - B_0) & 1 & \dots & 0 \\ 0 & 1 & 2 \cos(k_2 - 2B_0) & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & & & 1 \\ e^{-iqk_1} & 0 & \dots & 1 & 2 \cos(k_2 - (q-1)B_0) \end{pmatrix},$$

which is indeed  $2\pi/q$ -periodic in  $k_1$  and  $2\pi$ -periodic in  $k_2$ . The spectrum of  $\widehat{H}_{\text{Hof}}^{B_0}(k)$  consists of  $q$  distinct eigenvalue bands  $E_n(k)$ ,  $n = 1, \dots, q$ , with periodic spectral projections  $P_n(k)$ , defining the magnetic Bloch bands and Bloch bundles of the Hofstadter model. The spectrum of  $H_{\text{Hof}}^{B_0}$  is the union of the ranges of the functions  $E_n(k)$  and thus consists of  $q$  intervals. As a function of  $B_0$ , the spectrum is depicted in the famous Hofstadter butterfly (Figure 2). Note that for  $B_0 \notin 2\pi\mathbb{Q}$  the spectrum of  $H_{\text{Hof}}^{B_0}$  is a Cantor-type set, that is, a nowhere dense, closed set of Lebesgue measure zero; see [Avila and Jitomirskaya 2009].

Osadchy and Avron [2001] produced a colored version of the butterfly by coloring the gaps in the spectrum according to the sum of the Chern numbers of the overlying bands; see Figure 3. For example, for  $B_0 = 2\pi \frac{1}{3}$ , the top and the bottom bands have Chern number 1 each and the middle band has Chern number  $-2$ . Thus the gaps are labeled from top to bottom by 0 (white), 1 (red),  $-1$  (blue), and again 0 (white).

Now we apply the machinery developed in the previous sections to determine Peierls substitution Hamiltonians for magnetic subbands of  $H_{\text{Hof}}^{B_0}$ . Let  $B_0 = 2\pi p/q$ ; then  $\widehat{H}_{\text{Hof}}^{B_0}(k)$  is a matrix-valued function on the torus  $\mathbb{T}_q^* = [0, 2\pi/q) \times [0, 2\pi)$ , but its eigenvalue bands have period  $2\pi/q$  in both directions.



**Figure 2.** The black and white butterfly [Hofstadter 1976] showing the spectrum of  $H_{\text{Hof}}^{B_0}$  as a function of  $B_0$ . For rational values  $B_0 = 2\pi p/q$  the spectrum of  $H_{\text{Hof}}^{B_0}$  consists of  $q$  disjoint intervals if  $q$  is odd and of  $q - 1$  disjoint intervals if  $q$  is even.

Hence we can take as a model dispersion relation

$$E_q(k) := 2(\cos(qk_1) + \cos(qk_2)) = e^{iqk_1} + e^{-iqk_1} + e^{iqk_2} + e^{-iqk_2}.$$

This is, up to a constant factor, the leading-order part in the Fourier expansion of any Bloch band  $E_n(k)$  on  $\mathbb{T}_q^*$ . So we pick an isolated simple subband of  $\widehat{H}_{\text{Hof}}^{B_0}(k)$  with Chern number  $\theta \in \mathbb{Z}$  and approximate its dispersion by  $E_q(k)$ . If we now perturb  $B_0$  by an additional “small” constant magnetic field  $B = \text{curl } A(x)$  with  $A(x) = (0, Bx_1)$ , the Peierls substitution Hamiltonian for this subband is given as the  $\theta$ -quantization of  $E_q(k - A(r))$ ,

$$H_{\theta,q}^B := \text{Op}^\theta(E_q(k - A(r))) = e^{iq\mathfrak{K}_1} + e^{-iq\mathfrak{K}_1} + e^{iq\mathfrak{K}_2} + e^{-iq\mathfrak{K}_2},$$

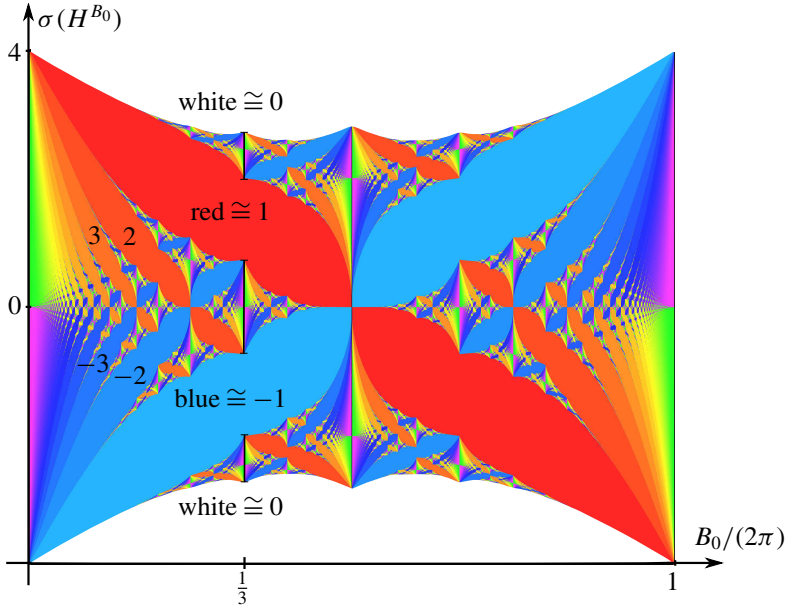
with

$$\mathfrak{K}_1 = k_1 \quad \text{and} \quad \mathfrak{K}_2 = k_2 - iB\nabla_1^\theta = k_2 - iB\partial_{k_1}$$

acting on

$$\mathcal{H}_\theta = \{f \in L_{\text{loc}}^2(\mathbb{R}^2) \mid f(k_1 - 2\pi/q, k_2) = e^{i\theta k_2} f(k_1, k_2) \text{ and } f(k_1, k_2 - 2\pi) = f(k_1, k_2)\}.$$

Here  $\nabla_k^\theta = (\partial_{k_1}, \partial_{k_2} + iq\theta k_1/(2\pi))$  and, due to our choice of gauge for the perturbing magnetic field, the operator  $H_{\theta,q}^B$  depends on  $\theta$  only through its domain. Note that this gauge is different from the one used in Theorem 5.1 and we use it to simplify the analysis of the resulting operator  $H_{\theta,q}^B$ . However, since Theorem 5.1 does not cover the case of a perturbation by a constant magnetic field anyway, our derivation of  $H_{\theta,q}^B$  is merely heuristic for any choice of gauge.



**Figure 3.** The colored butterfly for the Hofstadter Hamiltonian  $H_{\text{Hof}}^{B_0}$ , as first plotted in [Osadchy and Avron 2001]. The colored regions are open components of the resolvent set and the colors encode Chern numbers of overlying Bloch bundles. Physically, the Chern numbers represent the Hall conductivity of a corresponding noninteracting Fermi gas. For fixed  $B_0$ , i.e., in each vertical line, the Chern numbers of the single bands sum up to the total Chern number  $\theta = 0$ , as represented by the white region on bottom of the butterfly.

To determine the spectrum of  $H_{\theta,q}^B$ , it is sufficient to notice that it has the structure

$$U_1 + U_1^* + U_2 + U_2^*$$

with unitary operators  $U_1$  and  $U_2$  that satisfy

$$U_1 U_2 = e^{iq^2 B} U_2 U_1 =: e^{i\alpha} U_2 U_1. \tag{31}$$

The  $C^*$ -algebra  $\mathcal{N}_\alpha$  generated by two abstract elements  $\mathfrak{U}_1$  and  $\mathfrak{U}_2$  satisfying (31) is called the non-commutative torus. The mappings

$$\pi_{\theta,q}^B : \mathcal{N}_{q^2 B} \rightarrow \mathcal{L}(\mathcal{H}_\theta), \quad \mathfrak{U}_j \mapsto e^{iq^2 \mathcal{H}_j},$$

thus define a  $*$ -representation of  $\mathcal{N}_{q^2 B}$  into the bounded operators on  $\mathcal{H}_\theta$ . Accordingly, each operator  $H_{\theta,q}^B$  is a representation of the abstract element  $\mathfrak{H}^\alpha = \mathfrak{U}_1 + \mathfrak{U}_1^* + \mathfrak{U}_2 + \mathfrak{U}_2^*$  of  $\mathcal{N}_\alpha$  for  $\alpha = q^2 B$ . Since one can show that the representations  $\pi_{\theta,q}^B$  are  $*$ -isomorphisms onto their ranges (see [De Nittis 2010; Freund 2013; Amr et al. 2015]), this implies that the spectrum of  $H_{\theta,q}^B$  agrees with the spectrum of  $\mathfrak{H}^{q^2 B}$ . However, the latter is just the spectrum of  $H_{\text{Hof}}^{q^2 B}$ , i.e., it is again given by the black and white Hofstadter butterfly.

In order to associate Chern numbers with the spectral subbands of  $H_{\theta,q}^B$ , we now turn it by a suitable unitary transformation into matrix-multiplication form. Since  $H_{\theta,q}^B$  contains within  $e^{iq\mathfrak{k}_2}$  a shift by  $qB$  in the  $k_1$ -direction, this is possible if we assume this shift to be a rational fraction of the width  $2\pi/q$  of the Brillouin zone, that is,  $qB = (2\pi/q)\tilde{p}/\tilde{q}$  or  $B = (2\pi/q^2)\tilde{p}/\tilde{q}$  with  $\tilde{p}$  and  $\tilde{q}$  coprime. To this end we pass from  $\mathcal{H}_\theta$ , i.e., from complex-valued functions on the Brillouin zone  $M_q^* = [0, 2\pi/q) \times [0, 2\pi)$ , to  $\mathbb{C}^{\tilde{q}}$ -valued functions on the further reduced Brillouin zone  $M_{q,\tilde{q}}^* = [0, 2\pi/(q\tilde{q})) \times [0, 2\pi)$ . To define the corresponding unitary map  $U^B : \mathcal{H}_\theta \rightarrow L^2(M_{q,\tilde{q}}^*, \mathbb{C}^{\tilde{q}})$ , we let

$$M_j := \{(k_1, k_2) \in M_q^* \mid k_1 \in [(j-1)qB, (j-1)qB + 2\pi/(q\tilde{q}))\} \quad \text{for } j = 1, \dots, \tilde{q}$$

and define

$$(U^B \psi)_j(k) := e^{i\theta k_2(j-1)\tilde{p}/\tilde{q}} \psi(k_1 + (j-1)qB, k_2) \quad \text{for } k \in M_{q,\tilde{q}}^*.$$

Thus  $(U^B \psi)_j$  is obtained by restricting  $\psi \in \mathcal{H}_\theta$  to the region  $M_j$ , translating it to  $M_{q,\tilde{q}}^* = M_1$  and finally multiplying it by  $e^{i\theta k_2(j-1)\tilde{p}/\tilde{q}}$ . The last phase turns the translation by  $qB$  in the  $k_1$ -direction on  $\mathcal{H}_\theta$  into the cyclic permutation of components in  $L^2(M_{q,\tilde{q}}^*, \mathbb{C}^{\tilde{q}})$  multiplied by a phase. More precisely, we have

$$e^{iq\mathfrak{k}_1} \psi(k) = e^{iqk_1} \psi(k_1, k_2) \quad \text{and thus} \quad (U^B e^{iq\mathfrak{k}_1} \psi)_j(k) = e^{iq(k_1+(j-1)qB)} \psi_j(k),$$

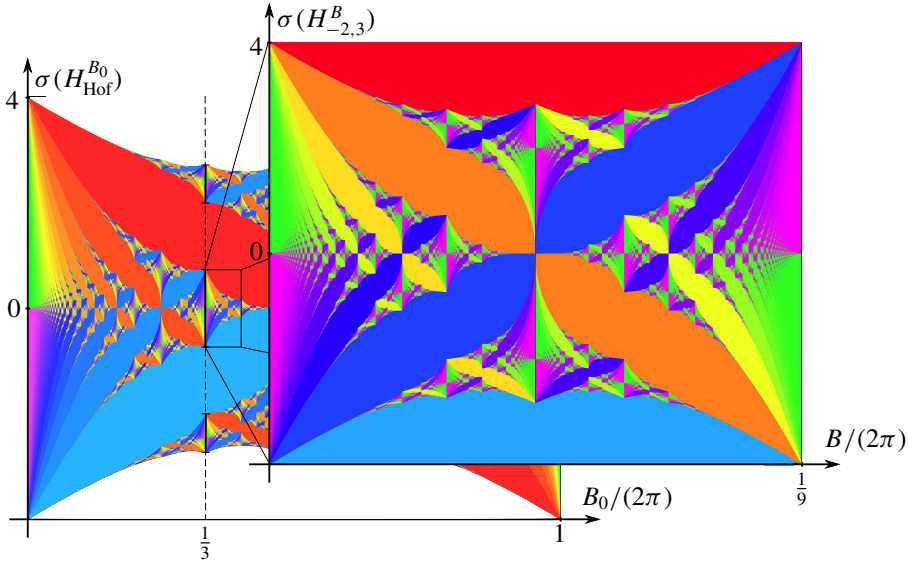
and

$$e^{iq\mathfrak{k}_2} \psi(k) = e^{iqk_2} \psi(k_1 + qB, k_2) \quad \text{and thus} \quad (U^B e^{iq\mathfrak{k}_2} \psi)_j(k) = e^{iqk_2} e^{-i\theta k_2 \tilde{p}/\tilde{q}} \psi_{j+1}(k).$$

Hence  $U^B H_{\theta,q}^B U^{B*}$  acts as the matrix-valued multiplication operator

$$H_{\theta,q}^B(k) = \begin{pmatrix} 2 \cos(qk_1) & e^{ik_2(q-\theta\tilde{p}/\tilde{q})} & 0 & \dots & e^{-ik_2(q-\theta\tilde{p}/\tilde{q})} \\ e^{-ik_2(q-\theta\tilde{p}/\tilde{q})} & 2 \cos(q(k_1+qB)) & e^{ik_2(q-\theta\tilde{p}/\tilde{q})} & \dots & 0 \\ 0 & e^{-ik_2(q-\theta\tilde{p}/\tilde{q})} & 2 \cos(q(k_1+2qB)) & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & & & e^{ik_2(q-\theta\tilde{p}/\tilde{q})} \\ e^{ik_2(q-\theta\tilde{p}/\tilde{q})} & 0 & \dots & e^{-ik_2(q-\theta\tilde{p}/\tilde{q})} & 2 \cos(q(k_1+(\tilde{q}-1)qB)) \end{pmatrix}. \quad (32)$$

Like the Hofstadter matrix  $\widehat{H}_{\text{Hof}}^{B_0}(k)$ , also  $H_{\theta,q}^B(k)$  has  $\tilde{q}$  distinct eigenvalue bands  $E_{\theta,q,n}^B(k)$ ,  $n = 1, \dots, \tilde{q}$ . By the isospectrality of  $H_{\theta,q}^B$  and  $H_{\text{Hof}}^{q^2 B}$ , the ranges of these band functions all agree. However, as functions they are, in general, distinct. The corresponding eigenprojections  $P_{\theta,q,n}^B(k)$  define line bundles over the torus  $M_{q,\tilde{q}}^*$  and one can compute their Chern numbers by integrating the curvature of the corresponding Berry connection  $P_{\theta,q,n}^B U^B \nabla_k^\theta U^{B*}$  over the reduced Brillouin zone  $M_{q,\tilde{q}}^*$ . Using a program from [Amr 2015], we did this numerically for a large number of values for  $\theta$ ,  $q$  and  $B$  and found that the Chern numbers of the subbands of  $H_{\theta,q}^B(k)$  always match the Chern numbers of the corresponding sub-subbands of the Hofstadter Hamiltonian. To make this more precise, recall that  $H_{\theta,q}^B(k)$  was derived as the Peierls substitution Hamiltonian for a magnetic subband of  $H_{\text{Hof}}^{B_0}$  for  $B_0 = 2\pi p/q$  with Chern number  $\theta$  perturbed by a small additional magnetic field  $B$ . The Chern numbers of the subbands of  $H_{\theta,q}^B(k)$  for



**Figure 4.** The operator  $H_{-2,3}^B$  is up to a constant factor and higher-order terms in the Fourier expansion of  $E_2(k)$  the leading order part of the Peierls substitution Hamiltonian for the middle band of  $H_{\text{Hof}}^{B_0}$  for  $B_0 = 2\pi \frac{1}{3}$ . This band has Chern number  $-2$ . As can be seen from the coloring, the Chern numbers of the subbands of  $H_{-2,3}^B$  for  $B/(2\pi) \in [0, \frac{1}{9}]$  exactly match the Chern numbers of the corresponding subbands of  $H_{\text{Hof}}^{B_0 + \tilde{B}}$ , where  $\tilde{B} = B(1 - 1/(1 + \pi/(3B))) = B(1 + \mathcal{O}(B))$ .

$B = (2\pi/q^2)\tilde{p}/\tilde{q}$  agree with the Chern numbers of the subbands of  $H_{\text{Hof}}^{B_0 + \tilde{B}}$  into which the unperturbed subband of  $H_{\text{Hof}}^{B_0}$  splits. Here

$$\tilde{B} = B \left( 1 - \frac{1}{1 - 2\pi/(q\theta B)} \right) = B \left( 1 - \frac{1}{1 - q\tilde{q}/(\theta\tilde{p})} \right) = B + \mathcal{O}(B^2).$$

The situation is depicted in [Figure 4](#). Note, however, that for drawing the colored butterfly of  $H_{\theta,q}^B$  it is not feasible to compute all Chern numbers numerically by integrating the curvature of the Berry connection. This is because, for large denominators  $\tilde{q}$ , the matrix  $H_{\theta,q}^B(k)$  and the number of its subbands becomes large. Instead, in [\[Amr 2015\]](#) an algorithm was found that allows to compute the Chern numbers of  $H_{\theta,q}^B$  in a purely algebraic fashion, similar to the diophantine equations used for labeling the gaps of  $H_{\text{Hof}}^{B_0}$ . Also, the code to produce the colored butterfly of  $H_{-2,3}^B$  in [Figure 4](#) is taken from [\[Amr 2015\]](#) and based on a code originally developed by Daniel Osadchy. This algorithm, the details on the numerics, and a much more detailed study of the operator  $H_{\theta,q}^B$  will be presented elsewhere [\[Amr et al. 2015\]](#). There, we also show how to explicitly incorporate a better approximation to the true dispersion relation of a magnetic subband and the subprincipal symbol, as given in [Theorem 5.1](#), into the Peierls substitution Hamiltonian. Then the agreement in terms of Chern numbers depicted in [Figure 4](#) turns into a quantitative agreement also of the spectrum. We take these numerical results as an indication that [Theorem 5.1](#) also holds for perturbations by small constant magnetic fields.

## Acknowledgments

We are grateful to Abderramán Amr for his involvement in a related project [Amr 2015; Amr et al. 2015], which had important impact on Section 7. In particular, his code was used to produce Figure 4. We thank Giuseppe De Nittis, Jonas Lampart, Gianluca Panati and Jakob Wachsmuth for numerous very helpful discussions and for continued exchange about many questions closely related to the content of this work. We thank Max Lein for his careful reading of a preliminary version of the manuscript.

## References

- [Amr 2015] A. Amr, *Chern numbers in solid state physics*, Diploma thesis, Universität Tübingen, 2015.
- [Amr et al. 2015] A. Amr, G. De Nittis, and S. Teufel, “Twisted Hofstadter Hamiltonians”, in preparation, 2015.
- [Asch et al. 1994] J. Asch, H. Over, and R. Seiler, “Magnetic Bloch analysis and Bochner Laplacians”, *J. Geom. Phys.* **13**:3 (1994), 275–288. [MR 1269244](#) [Zbl 0804.58022](#)
- [Avila and Jitomirskaya 2009] A. Avila and S. Jitomirskaya, “The ten martini problem”, *Ann. of Math. (2)* **170**:1 (2009), 303–342. [MR 2521117](#) [Zbl 1166.47031](#)
- [Bellissard 1988] J. Bellissard, “ $C^*$  algebras in solid state physics: 2D electrons in a uniform magnetic field”, pp. 49–76 in *Operator algebras and applications, II*, edited by D. E. Evans and M. Takesaki, London Math. Soc. Lecture Note Ser. **136**, Cambridge Univ. Press, 1988. [MR 996451](#) [Zbl 0677.46055](#)
- [Bellissard et al. 1991] J. Bellissard, C. Kreft, and R. Seiler, “Analysis of the spectrum of a particle on a triangular lattice with two magnetic fluxes by algebraic and numerical methods”, *J. Phys. A* **24**:10 (1991), 2329–2353. [MR 1118535](#) [Zbl 0738.35047](#)
- [Bellissard et al. 1994] J. Bellissard, A. van Elst, and H. Schulz-Baldes, “The noncommutative geometry of the quantum Hall effect”, *J. Math. Phys.* **35**:10 (1994), 5373–5451. [MR 1295473](#) [Zbl 0824.46086](#)
- [Blount 1962] E. I. Blount, “Formalisms of band theory”, pp. 305–373 in *Solid state physics, XIII*, Academic Press, New York, 1962. [MR 0138464](#)
- [Buslaev 1987] V. S. Buslaev, “Semiclassical approximation for equations with periodic coefficients”, *Uspekhi Mat. Nauk* **42**:6 (1987), 77–98. In Russian; translated in *Russ. Math. Surveys* **42**:6 (1987), 97–125. [MR 933996](#) [Zbl 0698.35130](#)
- [De Nittis 2010] G. De Nittis, *Hunting colored (quantum) butterflies: a geometric derivation of the TKNN-equations*, Ph.D. thesis, Scuola Internazionale Superiore di Studi Avanzati, 2010, <https://tel.archives-ouvertes.fr/tel-00674271/document>.
- [De Nittis and Lein 2011] G. De Nittis and M. Lein, “Applications of magnetic  $\Psi$ DO techniques to SAPT”, *Rev. Math. Phys.* **23**:3 (2011), 233–260. [MR 2793476](#) [Zbl 1214.81093](#)
- [De Nittis and Panati 2010] G. De Nittis and G. Panati, “Effective models for conductance in magnetic fields: derivation of Harper and Hofstadter models”, preprint, 2010. [arXiv 1007.4786](#)
- [Dimassi and Sjöstrand 1999] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series **268**, Cambridge University Press, 1999. [MR 1735654](#) [Zbl 0926.35002](#)
- [Dimassi et al. 2004] M. Dimassi, J.-C. Guillot, and J. Ralston, “On effective Hamiltonians for adiabatic perturbations of magnetic Schrödinger operators”, *Asymptot. Anal.* **40**:2 (2004), 137–146. [MR 2104132](#) [Zbl 1130.81344](#)
- [Dubrovin and Novikov 1980a] B. A. Dubrovin and S. P. Novikov, “Ground states in a periodic field: magnetic Bloch functions and vector bundles”, *Dokl. Akad. Nauk SSSR* **253**:6 (1980), 1293–1297. In Russian; translated in *Soviet Math. Dokl.* **22** (1980), 240–244. [MR 583789](#) [Zbl 0489.46055](#)
- [Dubrovin and Novikov 1980b] B. A. Dubrovin and S. P. Novikov, “Ground states of a two-dimensional electron in a periodic magnetic field”, *Zh. Eksp. Teoret. Fiz.* **79**:3 (1980), 1006–1016. In Russian; translated in *Soviet Physics JETP* **52** (1980), 511–516. [MR 617904](#)
- [Freund 2013] S. Freund, *Effective Hamiltonians for magnetic Bloch bands*, Ph.D. thesis, Universität Tübingen, 2013, <https://bibliographie.uni-tuebingen.de/xmlui/handle/10900/37930>.
- [Gat and Avron 2003a] O. Gat and J. E. Avron, “Magnetic fingerprints of fractal spectra and duality of Hofstadter models”, *New J. Phys.* **5** (2003), 44.1–44.8.

- [Gat and Avron 2003b] O. Gat and J. E. Avron, “Semiclassical analysis and the magnetization of the Hofstadter model”, *Physical Review Letters* **91**:18 (2003), art. ID 186801.
- [Gérard and Nier 1998] C. Gérard and F. Nier, “Scattering theory for the perturbations of periodic Schrödinger operators”, *J. Math. Kyoto Univ.* **38**:4 (1998), 595–634. [MR 1669979](#) [Zbl 0934.35111](#)
- [Gérard et al. 1991] C. Gérard, A. Martinez, and J. Sjöstrand, “A mathematical approach to the effective Hamiltonian in perturbed periodic problems”, *Comm. Math. Phys.* **142**:2 (1991), 217–244. [MR 1137062](#) [Zbl 0753.35057](#)
- [Guillot et al. 1988] J.-C. Guillot, J. Ralston, and E. Trubowitz, “Semiclassical asymptotics in solid state physics”, *Comm. Math. Phys.* **116**:3 (1988), 401–415. [MR 937768](#) [Zbl 0672.35014](#)
- [Hansen 2011] S. Hansen, “Rayleigh-type surface quasimodes in general linear elasticity”, *Anal. PDE* **4**:3 (2011), 461–497. [MR 2872123](#) [Zbl 1264.35248](#)
- [Helffer and Sjöstrand 1989] B. Helffer and J. Sjöstrand, “Équation de Schrödinger avec champ magnétique et équation de Harper”, pp. 118–197 in *Schrödinger operators* (Sønderborg, 1988), edited by H. Holden and A. Jensen, Lecture Notes in Phys. **345**, Springer, Berlin, 1989. [MR 1037319](#) [Zbl 0699.35189](#)
- [Helffer and Sjöstrand 1990a] B. Helffer and J. Sjöstrand, *Analyse semi-classique pour l'équation de Harper, II: Comportement semi-classique près d'un rationnel*, Mém. Soc. Math. France (N.S.) **40**, Société Mathématique de France, Paris, 1990. [MR 1052373](#) [Zbl 0714.34131](#)
- [Helffer and Sjöstrand 1990b] B. Helffer and J. Sjöstrand, “On diamagnetism and de Haas-van Alphen effect”, *Ann. Inst. H. Poincaré Phys. Théor.* **52**:4 (1990), 303–375. [MR 1062904](#) [Zbl 0715.35070](#)
- [Helffer et al. 1990] B. Helffer, P. Kerdelhué, and J. Sjöstrand, *Le papillon de Hofstadter revisité*, Mém. Soc. Math. France (N.S.) **43**, Société Mathématique de France, Paris, 1990. [MR 1090467](#) [Zbl 0732.44004](#)
- [Hofstadter 1976] D. R. Hofstadter, “Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields”, *Phys. Rev. B* **14**:6 (1976), 2239–2249.
- [Hörmander 1985] L. Hörmander, *The analysis of linear partial differential operators, IV: Pseudodifferential operators*, Grundlehren der Math. Wissenschaften **274**, Springer, Berlin, 1985. [MR 781537](#) [Zbl 0601.35001](#)
- [Hövermann et al. 2001] F. Hövermann, H. Spohn, and S. Teufel, “Semiclassical limit for the Schrödinger equation with a short scale periodic potential”, *Comm. Math. Phys.* **215**:3 (2001), 609–629. [MR 1810947](#) [Zbl 1052.81039](#)
- [Martinez and Sordoni 2002] A. Martinez and V. Sordoni, “A general reduction scheme for the time-dependent Born–Oppenheimer approximation”, *C. R. Math. Acad. Sci. Paris* **334**:3 (2002), 185–188. [MR 1891055](#) [Zbl 1079.81524](#)
- [Nenciu 1991] G. Nenciu, “Dynamics of band electrons in electric and magnetic fields: rigorous justification of the effective Hamiltonians”, *Reviews of Modern Physics* **63** (1991), 91–128.
- [Nenciu 2002] G. Nenciu, “On asymptotic perturbation theory for quantum mechanics: almost invariant subspaces and gauge invariant magnetic perturbation theory”, *J. Math. Phys.* **43**:3 (2002), 1273–1298. [MR 1885006](#) [Zbl 1059.81064](#)
- [Nenciu and Sordoni 2004] G. Nenciu and V. Sordoni, “Semiclassical limit for multistate Klein–Gordon systems: almost invariant subspaces, and scattering theory”, *J. Math. Phys.* **45**:9 (2004), 3676–3696. [MR 2081813](#) [Zbl 1071.81043](#)
- [Novikov 1981] S. P. Novikov, “Magnetic Bloch functions and vector bundles: typical dispersion laws and their quantum numbers”, *Dokl. Akad. Nauk SSSR* **257**:3 (1981), 538–543. In Russian; translated in *Soviet Math. Dokl.* **23** (1981), 298–303. [Zbl 0483.46054](#)
- [Osadchy and Avron 2001] D. Osadchy and J. E. Avron, “Hofstadter butterfly as quantum phase diagram”, *J. Math. Phys.* **42**:12 (2001), 5665–5671. [MR 1866679](#) [Zbl 1019.81071](#)
- [Panati 2007] G. Panati, “Triviality of Bloch and Bloch–Dirac bundles”, *Ann. Henri Poincaré* **8**:5 (2007), 995–1011. [MR 2342883](#) [Zbl 05207467](#)
- [Panati et al. 2003a] G. Panati, H. Spohn, and S. Teufel, “Effective dynamics for Bloch electrons: Peierls substitution and beyond”, *Comm. Math. Phys.* **242**:3 (2003), 547–578. [MR 2020280](#) [Zbl 1058.81020](#)
- [Panati et al. 2003b] G. Panati, H. Spohn, and S. Teufel, “Space-adiabatic perturbation theory”, *Adv. Theor. Math. Phys.* **7**:1 (2003), 145–204. [MR 2014961](#)
- [Peierls 1933] R. Peierls, “Zur Theorie des Diamagnetismus von Leitungselektronen”, *Zeitschrift für Physik* **80**:11–12 (1933), 763–791. [Zbl 0006.19204](#)

- [Pflaum 1998a] M. J. Pflaum, “A deformation-theoretical approach to Weyl quantization on Riemannian manifolds”, *Lett. Math. Phys.* **45**:4 (1998), 277–294. [MR 1653420](#) [Zbl 0995.53057](#)
- [Pflaum 1998b] M. J. Pflaum, “The normal symbol on Riemannian manifolds”, *New York J. Math.* **4** (1998), 97–125. [MR 1640055](#) [Zbl 0903.35099](#)
- [Rammal and Bellissard 1990] R. Rammal and J. Bellissard, “An algebraic semi-classical approach to Bloch electrons in a magnetic field”, *Journal de Physique* **51** (1990), 1803–1830.
- [Safarov 1997] Y. Safarov, “Pseudodifferential operators and linear connections”, *Proc. London Math. Soc.* (3) **74**:2 (1997), 379–416. [MR 1425328](#) [Zbl 0872.35140](#)
- [Schulz-Baldes and Teufel 2013] H. Schulz-Baldes and S. Teufel, “Orbital polarization and magnetization for independent particles in disordered media”, *Comm. Math. Phys.* **319**:3 (2013), 649–681. [MR 3040371](#) [Zbl 1271.82022](#)
- [Sharafutdinov 2004] V. A. Sharafutdinov, “A geometric symbolic calculus for pseudodifferential operators, I”, *Mat. Tr.* **7**:2 (2004), 159–206. In Russian; translated in *Siberian Adv. Math.* **15** (2005), 81–125. [MR 2124544](#) [Zbl 1081.58016](#)
- [Sharafutdinov 2005] V. A. Sharafutdinov, “A geometric symbolic calculus for pseudodifferential operators, II”, *Mat. Tr.* **8**:1 (2005), 176–201. In Russian; translated in *Siberian Adv. Math.* **15** (2005), 71–95. [MR 1955026](#) [Zbl 1082.58025](#)
- [Stiepan 2011] H.-M. Stiepan, *Adiabatic perturbation theory for magnetic Bloch Bands*, Ph.D. thesis, Universität Tübingen, 2011.
- [Stiepan and Teufel 2013] H.-M. Stiepan and S. Teufel, “Semiclassical approximations for Hamiltonians with operator-valued symbols”, *Comm. Math. Phys.* **320**:3 (2013), 821–849. [MR 3057191](#) [Zbl 1268.81079](#)
- [Sundaram and Niu 1999] G. Sundaram and Q. Niu, “Wave-packet dynamics in slowly perturbed crystals: gradient corrections and Berry-phase effects”, *Phys. Rev. B* **59**:23 (1999), 14915–14925.
- [Teufel 2003] S. Teufel, *Adiabatic perturbation theory in quantum dynamics*, Lecture Notes in Mathematics **1821**, Springer, Berlin, 2003. [MR 2158392](#) [Zbl 1053.81003](#)
- [Teufel 2012] S. Teufel, “Semiclassical approximations for adiabatic slow-fast systems”, *Europhysics Letters* **98**:5 (2012), art. ID 50003.
- [Thouless et al. 1982] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, “Quantized Hall conductance in a two-dimensional periodic potential”, *Phys. Review Letters* **49**:6 (1982), 405–408.
- [Widom 1978] H. Widom, “Families of pseudodifferential operators”, pp. 345–395 in *Topics in functional analysis (essays dedicated to M. G. Kreĭn on the occasion of his 70-th birthday)*, edited by I. Gohberg and M. Kac, Adv. in Math. Suppl. Stud. **3**, Academic Press, New York, 1978. [MR 538027](#) [Zbl 0477.58035](#)
- [Widom 1980] H. Widom, “A complete symbolic calculus for pseudodifferential operators”, *Bull. Sci. Math.* (2) **104**:1 (1980), 19–63. [MR 560744](#) [Zbl 0434.35092](#)
- [Xiao et al. 2010] D. Xiao, M.-C. Chang, and Q. Niu, “Berry phase effects on electronic properties”, *Rev. Modern Phys.* **82**:3 (2010), 1959–2007. [MR 2734353](#) [Zbl 1243.82059](#)
- [Zak 1968] J. Zak, “Dynamics of Electrons in Solids in External Fields”, *Phys. Review* **168**:3 (1968), 686–695.
- [Zak 1986] J. Zak, “Effective hamiltonians and magnetic energy bands?”, *Phys. Letters A* **117**:7 (1986), 367–371.
- [Zak 1991] J. Zak, “Exact symmetry of approximate effective Hamiltonians”, *Phys. Rev. Letters* **67** (1991), 2565–2568.

Received 22 Jan 2014. Revised 22 Dec 2015. Accepted 26 Feb 2016.

SILVIA FREUND: [sima@fa.uni-tuebingen.de](mailto:sima@fa.uni-tuebingen.de)

Fachbereich Mathematik, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany

STEFAN TEUFEL: [stefan.teufel@uni-tuebingen.de](mailto:stefan.teufel@uni-tuebingen.de)

Fachbereich Mathematik, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany



# Analysis & PDE

[msp.org/apde](http://msp.org/apde)

## EDITORS

EDITOR-IN-CHIEF

Patrick Gérard

[patrick.gerard@math.u-psud.fr](mailto:patrick.gerard@math.u-psud.fr)

Université Paris Sud XI

Orsay, France

## BOARD OF EDITORS

Nicolas Burq	Université Paris-Sud 11, France <a href="mailto:nicolas.burq@math.u-psud.fr">nicolas.burq@math.u-psud.fr</a>	Werner Müller	Universität Bonn, Germany <a href="mailto:mueller@math.uni-bonn.de">mueller@math.uni-bonn.de</a>
Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy <a href="mailto:berti@sissa.it">berti@sissa.it</a>	Yuval Peres	University of California, Berkeley, USA <a href="mailto:peres@stat.berkeley.edu">peres@stat.berkeley.edu</a>
Sun-Yung Alice Chang	Princeton University, USA <a href="mailto:chang@math.princeton.edu">chang@math.princeton.edu</a>	Gilles Pisier	Texas A&M University, and Paris 6 <a href="mailto:pisier@math.tamu.edu">pisier@math.tamu.edu</a>
Michael Christ	University of California, Berkeley, USA <a href="mailto:mchrist@math.berkeley.edu">mchrist@math.berkeley.edu</a>	Tristan Rivière	ETH, Switzerland <a href="mailto:riviere@math.ethz.ch">riviere@math.ethz.ch</a>
Charles Fefferman	Princeton University, USA <a href="mailto:cf@math.princeton.edu">cf@math.princeton.edu</a>	Igor Rodnianski	Princeton University, USA <a href="mailto:irod@math.princeton.edu">irod@math.princeton.edu</a>
Ursula Hamenstaedt	Universität Bonn, Germany <a href="mailto:ursula@math.uni-bonn.de">ursula@math.uni-bonn.de</a>	Wilhelm Schlag	University of Chicago, USA <a href="mailto:schlag@math.uchicago.edu">schlag@math.uchicago.edu</a>
Vaughan Jones	U.C. Berkeley & Vanderbilt University <a href="mailto:vaughan.f.jones@vanderbilt.edu">vaughan.f.jones@vanderbilt.edu</a>	Sylvia Serfaty	New York University, USA <a href="mailto:serfaty@cims.nyu.edu">serfaty@cims.nyu.edu</a>
Vadim Kaloshin	University of Maryland, USA <a href="mailto:vadim.kaloshin@gmail.com">vadim.kaloshin@gmail.com</a>	Yum-Tong Siu	Harvard University, USA <a href="mailto:siu@math.harvard.edu">siu@math.harvard.edu</a>
Herbert Koch	Universität Bonn, Germany <a href="mailto:koch@math.uni-bonn.de">koch@math.uni-bonn.de</a>	Terence Tao	University of California, Los Angeles, USA <a href="mailto:tao@math.ucla.edu">tao@math.ucla.edu</a>
Izabella Laba	University of British Columbia, Canada <a href="mailto:ilaba@math.ubc.ca">ilaba@math.ubc.ca</a>	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA <a href="mailto:met@math.unc.edu">met@math.unc.edu</a>
Gilles Lebeau	Université de Nice Sophia Antipolis, France <a href="mailto:lebeau@unice.fr">lebeau@unice.fr</a>	Gunther Uhlmann	University of Washington, USA <a href="mailto:gunther@math.washington.edu">gunther@math.washington.edu</a>
László Lempert	Purdue University, USA <a href="mailto:lempert@math.purdue.edu">lempert@math.purdue.edu</a>	András Vasy	Stanford University, USA <a href="mailto:andras@math.stanford.edu">andras@math.stanford.edu</a>
Richard B. Melrose	Massachusetts Inst. of Tech., USA <a href="mailto:rbb@math.mit.edu">rbb@math.mit.edu</a>	Dan Virgil Voiculescu	University of California, Berkeley, USA <a href="mailto:dvv@math.berkeley.edu">dvv@math.berkeley.edu</a>
Frank Merle	Université de Cergy-Pontoise, France <a href="mailto:Frank.Merle@u-cergy.fr">Frank.Merle@u-cergy.fr</a>	Steven Zelditch	Northwestern University, USA <a href="mailto:zelditch@math.northwestern.edu">zelditch@math.northwestern.edu</a>
William Minicozzi II	Johns Hopkins University, USA <a href="mailto:minicozz@math.jhu.edu">minicozz@math.jhu.edu</a>	Maciej Zworski	University of California, Berkeley, USA <a href="mailto:zworski@math.berkeley.edu">zworski@math.berkeley.edu</a>
Clément Mouhot	Cambridge University, UK <a href="mailto:c.mouhot@dpms.cam.ac.uk">c.mouhot@dpms.cam.ac.uk</a>		

## PRODUCTION

[production@msp.org](mailto:production@msp.org)

Silvio Levy, Scientific Editor

---

See inside back cover or [msp.org/apde](http://msp.org/apde) for submission instructions.

---

The subscription price for 2016 is US \$235/year for the electronic version, and \$430/year (+\$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.


---

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

---

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2016 Mathematical Sciences Publishers

# ANALYSIS & PDE

Volume 9 No. 4 2016

---

Peierls substitution for magnetic Bloch bands	773
SILVIA FREUND and STEFAN TEUFEL	
Dispersive estimates in $\mathbb{R}^3$ with threshold eigenstates and resonances	813
MARIUS BECEANU	
Interior nodal sets of Steklov eigenfunctions on surfaces	859
JIUYI ZHU	
Some counterexamples to Sobolev regularity for degenerate Monge–Ampère equations	881
CONNOR MOONEY	
Mean ergodic theorem for amenable discrete quantum groups and a Wiener-type theorem for compact metrizable groups	893
HUICHI HUANG	
Resonance free regions for nontrapping manifolds with cusps	907
KIRIL DATCHEV	
Characterizing regularity of domains via the Riesz transforms on their boundaries	955
DORINA MITREA, MARIUS MITREA and JOAN VERDERA	