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We investigate the interior nodal sets \mathcal{N}_λ of Steklov eigenfunctions on connected and compact surfaces with boundary. The optimal vanishing order of Steklov eigenfunctions is shown to be $C\lambda$. The singular sets \mathcal{S}_λ consist of finitely many points on the nodal sets. We are able to prove that the Hausdorff measure $H^0(\mathcal{S}_\lambda)$ is at most $C\lambda^2$. Furthermore, we obtain an upper bound for the measure of interior nodal sets, $H^1(\mathcal{N}_\lambda) \leq C\lambda^{3/2}$. Here the positive constants C depend only on the surfaces.

1. Introduction

Let (\mathcal{M}, g) be a smooth, connected and compact surface with smooth boundary $\partial\mathcal{M}$. The main goal of this paper is to obtain an upper bound of interior nodal sets

$$\mathcal{N}_\lambda = \{z \in \mathcal{M} \mid e_\lambda = 0\}$$

for Steklov eigenfunctions, which satisfy

$$\begin{cases} \Delta_g e_\lambda = 0, & z \in \mathcal{M}, \\ \partial e_\lambda(z)/\partial \nu = \lambda e_\lambda(z), & z \in \partial\mathcal{M}, \end{cases} \quad (1-1)$$

where ν is a unit outward normal on $\partial\mathcal{M}$. The Steklov eigenfunctions were introduced by Steklov in 1902 for bounded domains in the plane. They interpret the steady state temperature distribution in domains where the heat flux on the boundary is proportional to the temperature. They also have applications in quite a few physical fields, such as fluid mechanics, electromagnetism and elasticity. In particular, the model (1-1) was studied by Calderón [1980] as solutions can be regarded as eigenfunctions of the Dirichlet-to-Neumann map. The interior nodal sets of Steklov eigenfunctions represent the stationary points in \mathcal{M} . In the context of quantum mechanics, nodal sets are the sets where a free particle is least likely to be found.

It is well known that the spectrum λ_j of the Steklov eigenvalue problem is discrete with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \quad \text{and} \quad \lim_{j \rightarrow \infty} \lambda_j = \infty.$$

There exists an orthonormal basis $\{e_{\lambda_j}\}$ of eigenfunctions such that

$$e_{\lambda_j} \in C^\infty(\mathcal{M}) \quad \text{and} \quad \int_{\partial\mathcal{M}} e_{\lambda_j} e_{\lambda_k} dV_g = \delta_j^k.$$

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Estimating the Hausdorff measure of nodal sets has always been an important subject concerning the study of eigenfunctions. This subject centers around the famous Yau conjecture. Recently, much work has been devoted to the bounds of nodal sets

$$Z_\lambda = \{z \in \partial\mathcal{M} \mid e_\lambda(z) = 0\}$$

of Steklov eigenfunctions on the boundary. Bellová and Lin [2015] proved $H^{m-1}(Z_\lambda) \leq C\lambda^6$ with C depending only on \mathcal{M} if \mathcal{M} is an $m+1$ -dimensional analytic manifold. Zelditch [2014] improved their results and gave the optimal upper bound $H^{m-1}(Z_\lambda) \leq C\lambda$ for analytic manifolds using microlocal analysis. For the smooth manifold \mathcal{M} , Wang and Zhu [2015] recently established a lower bound

$$H^{m-1}(Z_\lambda) \geq C\lambda^{(3-m)/2}.$$

Before presenting our results for interior nodal sets, let’s briefly review the literature about the nodal sets of classical eigenfunctions. The interested reader may refer to the book [Han and Lin 2008] and survey [Zelditch 2008] for detailed accounts about this subject. Let e_λ be L^2 normalized eigenfunctions of the Laplace–Beltrami operator on compact manifolds (\mathcal{M}, g) without boundary,

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda. \tag{1-2}$$

Yau’s conjecture states that, for any smooth manifolds, one should control the upper and lower bounds of nodal sets of classical eigenfunctions as

$$c\lambda \leq H^{n-1}(\mathcal{N}_\lambda) \leq C\lambda, \tag{1-3}$$

where C and c depend only on the manifold \mathcal{M} . The conjecture is only verified for real analytic manifolds, by Donnelly and Fefferman [1988]. Lin [1991] also showed the upper bound for analytic manifolds by a different approach. For smooth manifolds, the conjecture is still not settled. For the lower bound of nodal sets with $n \geq 3$, Colding and Minicozzi [2011] and Sogge and Zelditch [2011; 2012] independently obtained that

$$H^{n-1}(\mathcal{N}_\lambda) \geq C\lambda^{(3-n)/2}$$

for smooth manifolds. See also [Hezari and Sogge 2012] for deriving the same bound by adapting the idea in [Sogge and Zelditch 2011]. For the upper bound, Hardt and Simon [1989] gave an exponential upper bound

$$H^{n-1}(\mathcal{N}_\lambda) \leq Ce^{\lambda \ln \lambda}.$$

In surfaces, better results have been obtained. Brüning [1978] and Yau (unpublished) derived the same lower bound as (1-3). The best estimate to date for the upper bound is

$$H^1(\mathcal{N}_\lambda) \leq C\lambda^{3/2}$$

by Donnelly and Fefferman [1990a] and Dong [1992] using different methods.

Let us return to the Steklov eigenvalue problem (1-1). By the maximum principle, there exist nodal sets in the manifold \mathcal{M} and those sets must intersect the boundary $\partial\mathcal{M}$. Thus it is natural to study the size

of interior nodal sets in \mathcal{M} . We can also ask Yau-type questions about the Hausdorff measure of nodal sets. The natural and corresponding conjecture for Steklov eigenfunctions should be exactly the same as (1-3). See also the open questions in the survey by Girouard and Polterovich [2014]. Recently, Sogge, Wang and the author [Sogge et al. 2015] obtained a lower bound for interior nodal sets

$$H^{n-1}(\mathcal{N}_\lambda) \geq C\lambda^{(2-n)/2}$$

for n -dimensional manifolds \mathcal{M} . Very recently, Polterovich, Sher and Toth [Polterovich et al. 2015] verified the Yau-type conjecture for (1-1) on real analytic Riemannian surfaces.

An interesting topic related to the measure of nodal sets is about doubling inequalities. Based on doubling inequalities, one can obtain the vanishing order of eigenfunctions, which characterizes how fast the eigenfunctions vanish. For the classical eigenfunctions of (1-2), Donnelly and Fefferman [1988; 1990b] obtained that the maximal vanishing order of e_λ is of order at most $C\lambda$ everywhere. To achieve it, a doubling inequality

$$\int_{\mathbb{B}(z_0, 2r)} e_\lambda^2 \leq C e^\lambda \int_{\mathbb{B}(z_0, r)} e_\lambda^2 \tag{1-4}$$

is derived using Carleman estimates, where $\mathbb{B}(p, c)$ denotes a ball centered at p with radius c . The doubling estimate (1-4) plays an important role in obtaining the bounds of nodal sets for analytic manifolds in [Donnelly and Fefferman 1988] and the upper bound of nodal sets for smooth surfaces in [Donnelly and Fefferman 1990a]. For the Steklov eigenfunctions, we obtain a doubling inequality on the boundary $\partial\mathcal{M}$ and derive that the sharp vanishing order is less than $C\lambda$ on the boundary $\partial\mathcal{M}$. For Steklov eigenfunctions in \mathcal{M} , we are also able to get the doubling inequality; see Proposition 5. With the aid of doubling estimates and Carleman inequalities, the following optimal vanishing order for Steklov eigenfunctions can be obtained:

Theorem 1. *The vanishing order of the Steklov eigenfunction e_λ of (1-1) in \mathcal{M} is everywhere less than $C\lambda$.*

Its sharpness can be seen in the case that the manifold \mathcal{M} is a ball. Notice that the doubling estimates in Proposition 5 and the vanishing order in Theorem 1 hold for any n -dimensional compact manifolds.

Singular sets

$$\mathcal{S}_\lambda = \{z \in \mathcal{M} \mid e_\lambda = 0, \nabla e_\lambda = 0\}$$

are contained in nodal sets. In Riemannian surfaces, those singular sets consist of finitely many points in the 1-dimensional nodal sets. It is interesting to count the number of those singular sets. Based on a Carleman inequality with singularities, we are able to show an upper bound of singular sets.

Theorem 2. *Let (\mathcal{M}, g) be a smooth, connected and compact surface with smooth boundary $\partial\mathcal{M}$. Then*

$$H^0(\mathcal{S}_\lambda) \leq C\lambda^2 \tag{1-5}$$

holds for Steklov eigenfunctions in (1-1).

For the nodal sets of Steklov eigenfunctions, we are able to build a similar type of Carleman inequality as [Donnelly and Fefferman 1990a], and show the following result:

Theorem 3. *Let (\mathcal{M}, g) be a smooth, connected and compact surface with smooth boundary $\partial\mathcal{M}$. Then*

$$H^1(\mathcal{N}_\lambda) \leq C\lambda^{3/2} \tag{1-6}$$

holds for Steklov eigenfunctions in (1-1).

The outline of the paper is as follows. Section 2 is devoted to reducing the Steklov eigenvalue problem into an equivalent elliptic equation without boundary. Then we obtain the optimal doubling inequality and show Theorem 1. In Section 3, we establish the Carleman inequality with singularities at finitely many points. Under additional assumptions on those singular points, a stronger Carleman inequality is derived. We measure the singular sets in Section 4. Sections 5, 6 and 7 are devoted to obtaining the nodal length of Steklov eigenfunctions. Under the condition of slow growth of L^2 norm, we find out the nodal length in Section 6. Based on a similar type of Calderón and Zygmund decomposition procedure, we show the slow growth at almost every point. Then the measure of nodal sets is derived by summing up the nodal length in each small square. The letters c, C, C_i, d_i denote generic positive constants and do not depend on λ . They may vary in different lines and sections.

2. Vanishing order of Steklov eigenfunctions

In this section, we will reduce the Steklov eigenvalue problem to an equivalent model on a boundaryless manifold. The presence of eigenvalues on the boundary $\partial\mathcal{M}$ will be reflected in the coefficient functions of a second-order elliptic equation. Let $d(z) = \text{dist}\{z, \partial\mathcal{M}\}$ denote the geodesic distance function from $x \in \mathcal{M}$ to the boundary $\partial\mathcal{M}$. Since \mathcal{M} is smooth, there exists a ρ -neighborhood of $\partial\mathcal{M}$ in \mathcal{M} such that $d(x)$ is smooth in the neighborhood. Let's denote it as \mathcal{M}_ρ . We extend $d(z)$ smoothly in \mathcal{M} by

$$\delta(z) = \begin{cases} d(z), & z \in \mathcal{M}_\rho, \\ l(z), & z \in \mathcal{M} \setminus \mathcal{M}_\rho, \end{cases} \tag{2-1}$$

where $l(z)$ is a smooth function in $\mathcal{M} \setminus \mathcal{M}_\rho$. Note that the extended function $\delta(z)$ is a smooth function in \mathcal{M} . We first reduce the Steklov eigenvalue problem into an elliptic equation with Neumann boundary condition. Let

$$v(z) = e_\lambda \exp\{\lambda\delta(z)\}.$$

It is known that $v(z) = e_\lambda(z)$ on $\partial\mathcal{M}$. For $z \in \partial\mathcal{M}$, we have $\nabla_g \delta(z) = -\nu(z)$. Recall that $\nu(z)$ is the unit outer normal on $z \in \partial\mathcal{M}$. We can check that the new function $v(z)$ satisfies

$$\begin{cases} \Delta_g v + b(z) \cdot \nabla_g v + q(z)v = 0 & \text{in } \mathcal{M}, \\ \partial v / \partial \nu = 0 & \text{on } \partial\mathcal{M}, \end{cases} \tag{2-2}$$

with

$$\begin{cases} b(z) = -2\lambda \nabla_g \delta(z), \\ q(z) = \lambda^2 |\nabla_g \delta(z)|^2 - \lambda \Delta_g \delta(z). \end{cases} \tag{2-3}$$

In order to get rid of the boundary condition, we attach two copies of \mathcal{M} along the boundary and consider the double manifold $\bar{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}$. The metric g extends to $\bar{\mathcal{M}}$ with Lipschitz-type singularity along $\partial\mathcal{M}$, since the lift metric g' of g on \mathcal{M} to the double manifold $\bar{\mathcal{M}}$ is Lipschitz. There also exists a canonical

involutive isometry $\mathcal{F} : \bar{\mathcal{M}} \rightarrow \bar{\mathcal{M}}$ that interchanges the two copies of \mathcal{M} . Then the function $v(x)$ can be extended to $\bar{\mathcal{M}}$ by $v \circ \mathcal{F} = v$. Therefore, $v(z)$ satisfies

$$\Delta_{g'} v + \bar{b}(z) \cdot \nabla_{g'} v + \bar{q}(z)v = 0 \quad \text{in } \bar{\mathcal{M}}. \tag{2-4}$$

From (2-3), one can see that

$$\begin{cases} \|\bar{b}\|_{W^{1,\infty}(\bar{\mathcal{M}})} \leq C\lambda, \\ \|\bar{q}\|_{W^{1,\infty}(\bar{\mathcal{M}})} \leq C\lambda^2. \end{cases} \tag{2-5}$$

After this procedure, we can instead study the nodal sets for the second-order elliptic equation (2-4) with assumption (2-5). Note that $\bar{\mathcal{M}}$ is a manifold without boundary.

We present a brief proof of [Theorem 1](#). It is a small modification of the argument in [\[Zhu 2015\]](#), where the sharp vanishing order of Steklov eigenfunctions on the boundary $\partial\mathcal{M}$ is shown to be less than $C\lambda$. To achieve it, we derive the double inequality in a neighborhood of the boundary by quantitative Carleman estimates.

Proof of Theorem 1. Recall the strategy in [\[Zhu 2015\]](#); we do an even reflection in a small neighborhood of the boundary. Then we deal with a second-order elliptic equation with a Lipschitz-continuous leading coefficient function and satisfying the same conditions as (2-5). By the regularity argument for dealing with a Lipschitz metric in [\[Donnelly and Fefferman 1990b\]](#), the same Carleman estimates in [\[Zhu 2015\]](#) hold for (2-4). Let $r(z)$ be the distance function from z to the fixed point z_0 . If $u \in C_0^\infty(\mathbb{B}_{r_0}(z_0) \setminus \{z_0\})$ and $\tau > C_1(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})$, following the arguments in [\[Zhu 2015\]](#) and choosing the test function $\tilde{\phi}(z) = \ln r(z) - r^\epsilon(z)$ there instead, we have the Carleman inequality

$$C\|r^2 e^{\tau\phi(r)} (\Delta_{g'} u + \bar{b} \cdot \nabla_{g'} u + \bar{q}u)\|_{L^2} \geq \tau^{3/2} \|r^{\epsilon/2} e^{\tau\phi(r)} u\|_{L^2} + \tau^{1/2} \|r^{1+\epsilon/2} e^{\tau\phi(r)} \nabla u\|_{L^2},$$

where $\phi(r) = -\ln r(z) + r^\epsilon(z)$. See also, e.g., [\[Bakri and Casteras 2014\]](#) for similar estimates on manifolds with smooth metric. In particular, we have the following lemma:

Lemma 4. *Let $u \in C_0^\infty(\frac{1}{2}\epsilon_1 < r < \epsilon_0)$. If $\tau > C_1(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})$. Then*

$$\int r^4 e^{2\tau\phi(r)} |\Delta_{g'} u + \bar{b} \cdot \nabla_{g'} u + \bar{q}u|^2 dr d\omega \geq C_2 \tau^3 \int r^\epsilon e^{2\tau\phi(r)} u^2 dr d\omega, \tag{2-6}$$

where $\phi(r) = -\ln r(z) + r^\epsilon(z)$ and $0 < \epsilon_0, \epsilon_1, \epsilon < 1$ are some fixed constants. Moreover, (r, ω) are the standard polar coordinates.

Using this Carleman estimate and choosing suitable test functions, a Hadamard three-ball result can be obtained in $\bar{\mathcal{M}}$ following the arguments in [\[Zhu 2015\]](#). There exist constants r_0, C and $0 < \gamma < 1$ depending only on $\bar{\mathcal{M}}$ such that, for any solutions of (2-4), $0 < r < r_0$ and $z_0 \in \bar{\mathcal{M}}$, one has

$$\int_{\mathbb{B}(z_0,r)} v^2 \leq e^{C(1+\|\bar{b}\|_{W^{1,\infty}}+\|\bar{q}\|_{W^{1,\infty}}^{1/2})} \left(\int_{\mathbb{B}(z_0,2r)} v^2 \right)^{1-\gamma} \left(\int_{\mathbb{B}(z_0,r/2)} v^2 \right)^\gamma. \tag{2-7}$$

Based on a propagation of smallness argument using the three-ball result and Carleman estimates (2-6), as that in [\[Zhu 2015\]](#), taking the assumptions (2-5) into account, we are able to obtain the doubling inequality in $\bar{\mathcal{M}}$.

Proposition 5. *There exist constants r_0 and C depending only on $\bar{\mathcal{M}}$ such that, for any $0 < r < r_0$ and $z_0 \in \bar{\mathcal{M}}$,*

$$\|v\|_{L^2(\mathbb{B}(z_0, 2r))} \leq e^{C\lambda} \|v\|_{L^2(\mathbb{B}(z_0, r))} \tag{2-8}$$

for any solutions of (2-4).

One can see that the doubling estimate holds in \mathcal{M} if $\mathbb{B}(z_0, 2r) \subset \mathcal{M}$. By standard elliptic estimates, one can have the L^∞ norm doubling inequality

$$\|v\|_{L^\infty(\mathbb{B}(z_0, 2r))} \leq e^{C\lambda} \|v\|_{L^\infty(\mathbb{B}(z_0, r))}.$$

Since $\bar{\mathcal{M}}$ is compact, we can derive that

$$\|v\|_{L^\infty(\mathbb{B}(z_0, r))} \geq r^{C\lambda}$$

for any $z_0 \in \bar{\mathcal{M}}$, which implies the vanishing order for v is less than $C\lambda$. So is the vanishing order of u . This completes [Theorem 1](#). □

3. Carleman estimates

This section is devoted to establishing Carleman inequalities involving weighted functions at finitely many points. From this section on, $\bar{\mathcal{M}}$ is a compact Riemannian surface. We construct suitable conformal coordinate charts near $\partial\mathcal{M} \subset \bar{\mathcal{M}}$ following the arguments in [\[Donnelly and Fefferman 1990a, p. 342–343\]](#), where the same construction is established for a Lipschitz double manifold. By the Riemann mapping theory in [\[Jost 1984\]](#), we first construct charts around $\partial\mathcal{M} \subset \mathcal{M}$. We map a half disk centered on the x -axis in the (x, y) -plane into the manifold \mathcal{M} with the x -axis mapped to $\partial\mathcal{M}$. Thus, the metric is locally given as $\bar{g}(x, y)(dx^2 + dy^2)$ with $y > 0$. The differentiable structure and the definition of the metric on the double manifold $\bar{\mathcal{M}}$ correspond to reflection about the x -axis. Thus, we have the required the conformal charts with $\bar{g}(x, |y|)(dx^2 + dy^2)$ on the double manifold $\bar{\mathcal{M}}$. Then we will consider the behavior of v in a conformal coordinate patch. There exists a finite number N of conformal charts (\mathcal{U}_i, ϕ_i) with $\phi_i : \mathcal{U}_i \subset \bar{\mathcal{M}} \rightarrow \mathcal{V}_i \subset \mathbb{R}^2$ and $i \in \{1, 2, \dots, N\}$. On each of these charts, the metric is conformally flat and there exists a positive function g_i such that $g' = g_i(x, y)(dx^2 + dy^2)$. By the compactness of the surface, there are positive constants c and C such that $0 < c < g_i < C$ for each i . Under this equivalent metric, $\Delta_{g'} = g_i^{-1}\Delta$, where Δ is the Euclidean Laplacian. Hence, (2-4) can be written as

$$\Delta v + \bar{b}(z) \cdot \nabla v + \bar{q}(z)v = 0 \quad \text{in } \mathcal{V}_i, \tag{3-1}$$

where ∇ is the Euclidean gradient and $z = (x, y)$. We use the same notations $\bar{b}(z)$ and $\bar{q}(z)$ as in (3-1), since they satisfy the same conditions as (2-5). They only differ by some function about g_i .

By restricting to a small ball $\mathbb{B}(p, 3c)$ contained in the conformal chart, we consider v in the small ball. Let $\bar{v}(z) = v(cz)$. It follows from (3-1) that

$$\Delta \bar{v} + \tilde{b}(z) \cdot \nabla \bar{v} + \tilde{q}(z)\bar{v} = 0 \quad \text{in } \mathbb{B}_3, \tag{3-2}$$

with $\tilde{b} = c\bar{b}$ and $\tilde{q} = c^2\bar{q}$. If c is sufficiently small, \tilde{b} and \tilde{q} are arbitrarily small.

The crucial tool in [Donnelly and Fefferman 1990a] is a Carleman inequality for classical eigenfunctions involving weighted functions with singularities at finitely many points. We will obtain the corresponding Carleman inequality for the second-order elliptic equation (3-2). We adapt the approach in [Donnelly and Fefferman 1990a] to obtain the desirable Carleman estimate for (3-2).

Let $\mathcal{D} \subset \mathcal{C}$ be an open set and $\psi \in C^\infty(\mathcal{D})$ be a real-valued function. We introduce the differential operators

$$\partial = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Direct computation shows that $\bar{\partial} \partial \psi = \frac{1}{4} \Delta \psi$. By the Cauchy–Riemann equation, u is holomorphic if and only if $\bar{\partial} u = 0$. For completeness, we present the elementary inequality in [Donnelly and Fefferman 1990a].

Lemma 6. *Let Φ be a smooth positive function in \mathcal{D} . Then*

$$\int_{\mathcal{D}} |\bar{\partial} u|^2 \Phi \geq \frac{1}{4} \int_{\mathcal{D}} (\Delta \ln \Phi) |u|^2 \Phi. \tag{3-3}$$

Here the integral is taken with respect to the Lebesgue measure.

We want the weight function to involve those singular points. To specialize the choice of Φ , we construct the following function ψ_0 :

Lemma 7. *There exists a smooth function ψ_0 defined for $|z| > 1 - 2a$ satisfying the following properties:*

- (i) $a_1 \leq \psi_0(z) \leq a_2$ with constants $a_1, a_2 > 0$.
- (ii) $\psi_0 = 1$ on $\{|z| > 1\}$.
- (iii) $\Delta \ln \psi_0 \geq 0$ on $\{|z| > (1 - 2a)\}$.
- (iv) If $1 - 2a < |z| < 1 - a$, then $\Delta \ln \psi_0 \geq a_3 > 0$.

The existence of such a ψ_0 follows from existence and uniqueness theory of ordinary differential equations.

We assume that

$$D_l = \{z \mid |z - z_l| \leq \delta\}.$$

Let D_l be a finite collection of pairwise disjoint disks that are contained in a unit disk centered at the origin. Let

$$D_l(a) = \{z \mid |z - z_l| \leq (1 - 2a)\delta\}$$

be the smaller concentric disk. We define a smooth weight function $\Psi_0(z)$ as

$$\Psi_0(z) = \begin{cases} 1 & \text{if } z \notin \bigcup_l D_l, \\ \psi_0((z - z_l)/\delta) & \text{if } z \in D_l. \end{cases}$$

We also introduce the domain

$$A_l = \{(1 - 2a)\delta \leq |z - z_l| \leq (1 - a)\delta\}.$$

From the last lemma, $\Psi_0(z)$ satisfies these properties:

- (i) $a_1 \leq \Psi_0(z) \leq a_2$.
- (ii) $\Delta \ln \Psi_0 \geq 0$ for $z \in \mathbb{R}^2 \setminus \bigcup_l D_l(a)$.
- (iii) $\Delta \ln \Psi_0 \geq a_3 \delta^{-2}$ for $z \in A_l$.

Note that the a_i in the above are positive constants independent of λ . Let

$$A = \bigcup_l A_l.$$

Suppose that τ is a nonnegative constant. We introduce $\Phi(z) = \Psi_0(z)e^{\tau|z|^2}$. For $u \in C_0^\infty(\mathbb{R}^2 \setminus \bigcup_l D_l(a))$, we assume that \mathfrak{D} contains the support of u and $A \subset \mathfrak{D} \subset \mathbb{R}^2 \setminus \bigcup_l D_l(a)$. Obviously,

$$\ln \Phi(z) = \ln \Psi_0(z) + \tau|z|^2.$$

Substituting Φ in Lemma 6 gives that

$$\int_{\mathfrak{D}} |\bar{\partial}u|^2 \Psi_0(z) e^{\tau|z|^2} \geq C_1 \tau \int_{\mathfrak{D}} |u|^2 \Psi_0(z) e^{\tau|z|^2} + C_2 \delta^{-2} \int_A |u|^2 e^{\tau|z|^2}, \tag{3-4}$$

where we have used the properties (ii) and (iii) for Ψ_0 . The boundedness of $\Psi_0(z)$ yields that

$$\int_{\mathfrak{D}} |\bar{\partial}u|^2 e^{\tau|z|^2} \geq C_3 \tau \int_{\mathfrak{D}} |u|^2 e^{\tau|z|^2} + C_4 \delta^{-2} \int_A |u|^2 e^{\tau|z|^2}. \tag{3-5}$$

Define the holomorphic function

$$P(z) = \prod_l (z - z_l).$$

Then $\bar{\partial}(u/P) = \bar{\partial}u/P$. Replacing u by u/P in (3-5), it follows that

$$\int_{\mathfrak{D}} |\bar{\partial}u|^2 |P|^{-2} e^{\tau|z|^2} \geq C_3 \tau \int_{\mathfrak{D}} |u|^2 |P|^{-2} e^{\tau|z|^2} + C_4 \delta^{-2} \int_A |u|^2 |P|^{-2} e^{\tau|z|^2}. \tag{3-6}$$

We will establish a Carleman inequality for second-order elliptic equations like (3-2). Write $\tilde{b}(x) = (\tilde{b}_1(x), \tilde{b}_2(x))$. Let

$$u = \partial f + \frac{1}{2}(\tilde{b}_1 - i\tilde{b}_2)f,$$

where $f \in C_0^\infty(\mathbb{R}^2 \setminus \bigcup_l D_l(a))$ is a real-valued function. Then

$$\bar{\partial}u = \frac{1}{4} \left[\Delta f + \operatorname{div} \tilde{b} f + \tilde{b} \cdot \nabla f + i \left(\frac{\partial(\tilde{b}_1 f)}{\partial y} - \frac{\partial(\tilde{b}_2 f)}{\partial x} \right) \right].$$

Plugging the above u into (3-6), we obtain

$$\begin{aligned} & \int_{\mathfrak{D}} \left[|\Delta f + \tilde{b} \cdot \nabla f|^2 + |\operatorname{div} \tilde{b} f|^2 + \left| \frac{\partial(\tilde{b}_1 f)}{\partial y} - \frac{\partial(\tilde{b}_2 f)}{\partial x} \right|^2 \right] |P|^{-2} e^{\tau|z|^2} \\ & \geq C_3 \tau \int_{\mathfrak{D}} |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} - C_3 \tau \int_{\mathfrak{D}} |\tilde{b}|^2 |f|^2 |P|^{-2} e^{\tau|z|^2} \\ & \quad + C_4 \delta^{-2} \int_A |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} - C_4 \delta^{-2} \int_A |\tilde{b}|^2 |f|^2 |P|^{-2} e^{\tau|z|^2}. \end{aligned} \tag{3-7}$$

If we choose $u = f$ in (3-6), we get

$$\int_{\mathfrak{D}} |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} \geq C_3 \tau \int_{\mathfrak{D}} |f|^2 |P|^{-2} e^{\tau|z|^2}. \tag{3-8}$$

Since the norm of \tilde{b} is chosen small enough, it is smaller than τ , which will be chosen large enough. With the aid of (3-8), we can incorporate the terms involving \tilde{b} in the left-hand side of (3-7) into the first term in the right-hand side of (3-7):

$$\begin{aligned} & \int_{\mathfrak{D}} |\Delta f + \tilde{b} \cdot \nabla f|^2 |P|^{-2} e^{\tau|z|^2} \\ & \geq C_5 \tau \int_{\mathfrak{D}} |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} + C_4 \delta^{-2} \int_A |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} - C_4 \delta^{-2} \int_A |\tilde{b}|^2 |f|^2 |P|^{-2} e^{\tau|z|^2}. \end{aligned} \tag{3-9}$$

Furthermore, if $u = f$, the inequality (3-6) implies that

$$\int_{\mathfrak{D}} |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} \geq C_4 \delta^{-2} \int_A |f|^2 |P|^{-2} e^{\tau|z|^2}. \tag{3-10}$$

Applying (3-10) to the last term in the right-hand side of (3-9) gives that

$$\int_{\mathfrak{D}} |\Delta f + \tilde{b} \cdot \nabla f|^2 |P|^{-2} e^{\tau|z|^2} \geq C_6 \tau^2 \int_{\mathfrak{D}} |f|^2 |P|^{-2} e^{\tau|z|^2} + C_7 \delta^{-2} \int_A |\nabla f|^2 |P|^{-2} e^{\tau|z|^2}. \tag{3-11}$$

We continue to get a refined estimate for the last term of (3-11). In order to achieve this goal, we need the following hypotheses for the geometry of the disk D_l and the parameter $\tau > 1$:

- (R1) The radius δ of each disk D_l is less than $a_4 \tau^{-1}$.
- (R2) The distance between any two distinct z_l is at least $2a_5 \tau^{1/2} \delta$.
- (R3) The total number of disks D_l is at most $a_6 \tau$.

Under the those assumptions, we have these comparison estimates from [Donnelly and Fefferman 1990a]:

Lemma 8. *If \bar{z}_1 and \bar{z}_2 are any points in the same component A_l of A , then:*

- (i) $a_7 < e^{\tau|\bar{z}_1|^2} / e^{\tau|\bar{z}_2|^2} < a_8$.
- (ii) $a_9 < |P(\bar{z}_1)| / |P(\bar{z}_2)| < a_{10}$.

We also need the following Poincaré-type inequality on each annulus: if $f \in C^\infty(A_l)$ and f vanishes on the inner boundary of A_l , then

$$\int_{A_l} |\nabla f|^2 \geq a_{11} \delta^{-2} \int_{A_l} |f|^2. \tag{3-12}$$

The proof of (3-12) can be found in [Donnelly and Fefferman 1990a]. Let $z_l \in A_l$ be chosen arbitrarily. By Lemma 8, it follows that

$$\int_{A_l} |\nabla f|^2 |P(z)|^{-2} e^{\tau|z|^2} \geq C_8 \sum_l e^{\tau|z_l|^2} |P(z_l)|^{-2} \int_{A_l} |\nabla f|^2.$$

Since $f \in C_0^\infty(\mathbb{R}^2 \setminus \bigcup_l D_l(a))$, the inequality (3-12) yields that

$$\int_{A_l} |\nabla f|^2 |P(z)|^{-2} e^{\tau|z|^2} \geq C_9 \sum_l e^{\tau|z_l|^2} |P(z_l)|^{-2} \delta^{-2} \int_{A_l} |f|^2.$$

Using Lemma 8 again, we obtain

$$\int_{A_l} |\nabla f|^2 |P(z)|^{-2} e^{\tau|z|^2} \geq C_{10} \delta^{-2} \int_A |f|^2 |P(z)|^{-2} e^{\tau|z|^2}.$$

Substituting the last inequality into the last term in (3-11) leads to

$$\int_{\mathfrak{D}} |\Delta f + \tilde{b} \cdot \nabla f|^2 |P|^{-2} e^{\tau|z|^2} \geq C_6 \tau^2 \int_{\mathfrak{D}} |f|^2 |P|^{-2} e^{\tau|z|^2} + C_{11} \delta^{-4} \int_A |f|^2 |P|^{-2} e^{\tau|z|^2}. \tag{3-13}$$

We summarize the above arguments in the following proposition:

Proposition 9. Assume $f \in C_0^\infty(\mathbb{R}^2 \setminus \bigcup_l D_l(a))$. Then:

(i) It holds that

$$\int_{\mathfrak{D}} |\Delta f + \tilde{b} \cdot \nabla f|^2 |P|^{-2} e^{\tau|z|^2} \geq C \tau^2 \int_{\mathfrak{D}} |f|^2 |P|^{-2} e^{\tau|z|^2}. \tag{3-14}$$

(ii) If the additional assumptions (R1)–(R3) for D_l hold, the stronger inequality (3-13) is satisfied.

4. Measure of singular sets

Let \mathcal{M} be a compact smooth surface. In Section 2, we have shown that the Steklov eigenfunction e_λ vanishes at all points to order at most $C\lambda$. By the implicit function theorem, outside the singular sets, the nodal set is locally a 1-dimensional C^1 manifold. Adapting the arguments in [Donnelly and Fefferman 1990a] for (3-2), we can estimate those singular points in a quantitative way. We are able to obtain an upper bound for the singular points in terms of the eigenvalue λ .

Lemma 10. Singular sets consist of at most finitely many points.

Proof. Without loss of generality, we assume that $0 \in \mathcal{S}_\lambda$ and choose normal coordinates (x, y) at the origin. Next we prove there are finitely many singular points in $\bar{\mathcal{M}}$. Using Taylor expansion, we expand v locally at the origin. Then $v(x, y) = F_j(x, y) + W_{j+1}(x, y)$, where $F_j(x, y)$ consists of the leading nonvanishing term with homogenous order $j \geq 2$ and $W_{j+1}(x, y)$ is a higher-order reminder term. Since

$\Delta v + \bar{b}(z) \cdot \nabla v + \bar{q}(z)v = 0$ and the coordinate is normal, we obtain that $\Delta F_j = 0$. Under polar coordinates, we find that $F_j = r^j(a_1 \cos(j\theta) + a_2 \sin(j\theta))$. Obviously, $r^{-1} \partial F_j / \partial \theta$ and $\partial F_j / \partial r$ have no common zero if $r \neq 0$. Since

$$|\nabla F_j|^2 = \left| \frac{\partial F_j}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial F_j}{\partial \theta} \right|^2,$$

there exists a small neighborhood \mathcal{U} of the origin such that $\mathcal{U} \cap \mathcal{S}_\lambda = \emptyset$. Since $\bar{\mathcal{M}}$ is compact, the lemma follows. □

We plan to count the number of singular points in a sufficiently small ball. Let $p \in \bar{\mathcal{M}}$. Consider a geodesic ball $\mathbb{B}(p, c\lambda^{-1/2})$. If c is small enough, then this geodesic ball is contained in a conformal chart. If we choose

$$w(z) = v(c\lambda^{-1/2}z)$$

with c sufficiently small then, from (3-1), w satisfies

$$\Delta w + \hat{b}(x) \cdot \nabla w + \hat{q}(x)w = 0 \quad \text{in } \mathbb{B}(0, 4), \tag{4-1}$$

with $\hat{b}(x) = c\lambda^{-1/2}\bar{b}(x)$ and $\hat{q}(x) = c^2\lambda^{-1}\bar{q}(x)$. From (2-5), we obtain

$$\begin{cases} \|\hat{b}\|_{W^{1,\infty}(\mathbb{B}(0,4))} \leq c\lambda^{1/2}, \\ \|\hat{q}\|_{W^{1,\infty}(\mathbb{B}(0,4))} \leq c^2\lambda, \end{cases} \tag{4-2}$$

with c sufficiently small.

Next we will count the total order of the vanishing of singular points for w in the sufficiently small ball. We study w in (4-1).

Proposition 11. *Suppose $z_l \in \mathcal{S}_\lambda \cap \mathbb{B}(p, c\lambda^{-1/2})$, where v vanishes to order $n_l + 1$. Then $\sum_l n_l \leq C\lambda$.*

Proof. It suffices to count the number of singular points of w in a small Euclidean ball with radius $\frac{1}{10}$ centered at the origin. Suppose that w vanishes to order $n_l + 1$. Let $n_l = m_l + 1$. We first consider the case $n_l \geq 2$. Then $m_l \geq 1$. Define the polynomial

$$P(z) = \prod (z - z_l)^{m_l}$$

with $|z_l| < \frac{1}{10}$. Let $\mathcal{D} = \mathbb{B}(0, 2)$ and let D_l be small disjoint disks of radius δ centered at z_l . If $f \in C_0^\infty(\mathbb{R}^2 \setminus \bigcup_l D_l)$, the inequality (3-14) in Proposition 9 implies that

$$\int_{\mathcal{D}} (|\Delta f|^2 + |\hat{b} \cdot \nabla f|^2) |P|^{-2} e^{d_1\lambda|z|^2} \geq C_2\lambda^2 \int_{\mathcal{D}} |f|^2 |P|^{-2} e^{d_1\lambda|z|^2}, \tag{4-3}$$

where $\tau = d_1\lambda$. We choose a cut-off function $\theta(z)$ such that θw has compact support in \mathcal{D} . We select the cut-off function $\theta \in C_0^\infty(\mathcal{D} \setminus \bigcup_l D_l)$ with the following properties:

- (i) $\theta(z) = 1$ if $|z| < \frac{3}{2}$ and $|z - z_l| > 2\delta$.
- (ii) $|\nabla\theta| < C_3$ and $|\Delta\theta| < C_4$ if $|z| > \frac{3}{2}$.
- (iii) $|\nabla\theta| < C_5\delta^{-1}$ and $|\Delta\theta| < C_6\delta^{-2}$ if $|z - z_l| < 2\delta$.

Substituting $f = \theta w$ into (4-3) yields that

$$\int_{(|z| < 3/2) \cup (3/2 \leq |z| \leq 2)} |\Delta(\theta w) + \tilde{b} \cdot \nabla(\theta w)|^2 |P|^{-2} e^{d_1 \lambda |z|^2} \geq C_2 \lambda^2 \int_{|z| < 3/2} |w|^2 |P|^{-2} e^{d_1 \lambda |z|^2}.$$

From (4-1),

$$\Delta(\theta w) + \hat{b} \cdot \nabla(\theta w) = -\hat{q}\theta w + \Delta\theta w + 2\nabla\theta \cdot \nabla w + \hat{b} \cdot \nabla\theta w.$$

By the assumption on θ , we obtain

$$|\Delta\theta w| + |\nabla\theta \cdot \nabla w| + |\nabla\theta w| \leq C_7 \delta^{m_l} \quad \text{if } |z - z_l| \leq 2\delta.$$

Taking $\delta \rightarrow 0$, by the dominated convergence theorem, we have

$$c\lambda^2 \int_{|z| < 3/2} |w|^2 |P|^{-2} e^{d_1 \lambda |z|^2} + C_9(1 + \lambda)^2 \int_{3/2 \leq |z| \leq 2} (|w|^2 + |\nabla w|^2) |P|^{-2} e^{d_1 \lambda |z|^2} \geq C_{10} \lambda^2 \int_{|z| < 3/2} |w|^2 |P|^{-2} e^{d_1 \lambda |z|^2}. \quad (4-4)$$

Since c is sufficiently small, we can absorb the first term in the left-hand side of (4-4) into the right-hand side. Then

$$\int_{3/2 \leq |z| \leq 2} (|w|^2 + |\nabla w|^2) |P|^{-2} e^{d_1 \lambda |z|^2} \geq C_{11} \int_{|z| \leq 1/2} |w|^2 |P|^{-2} e^{d_1 \lambda |z|^2}. \quad (4-5)$$

Obviously, it follows that

$$\max_{|z| \geq 3/2} |P|^{-2} \int_{3/2 \leq |z| \leq 2} (|w|^2 + |\nabla w|^2) e^{d_1 \lambda |z|^2} \geq C_{11} \left(\min_{|z| \leq 1/2} |P|^{-2} \right) \int_{|z| \leq 1/2} |w|^2. \quad (4-6)$$

By standard elliptic theory, the last inequality implies

$$\max_{|z| \geq 3/2} |P|^{-2} e^{d_2 \lambda} \int_{|z| \leq 5/2} |w|^2 \geq C_{11} \left(\min_{|z| \leq 1/2} |P|^{-2} \right) \int_{|z| \leq 1/2} |w|^2. \quad (4-7)$$

We claim that

$$e^{d_3 \sum m_l} \leq \frac{\min_{|z| \leq 1/2} |P|^{-2}}{\max_{|z| \geq 3/2} |P|^{-2}}. \quad (4-8)$$

To prove (4-8), it suffices to verify

$$e^{-d_4 \sum m_l} \min_{|z| \geq 3/2} |P| \geq \max_{|z| \leq 1/2} |P| \quad (4-9)$$

away from the singular point z_l . Clearly,

$$\max_{|z| \leq 1/2} |P| \leq \left(\frac{1}{2}\right)^{\sum m_l}.$$

Since $z_l \in \mathbb{B}(0, \frac{1}{10})$, we have

$$\left(\frac{3}{4}\right)^{\sum m_l} \leq \min_{|z| \geq 3/2} |P|.$$

Combining the last two inequalities, we obtain (4-9). The claim is shown. Let's return to (4-7); we get

$$\frac{\min_{|z|\leq 1/2} |P|^{-2}}{\max_{|z|\geq 3/2} |P|^{-2}} \leq \frac{e^{d_5\lambda} \int_{|z|\leq 5/2} |w|^2}{C_{11} \int_{|z|\leq 1/2} |w|^2} \leq e^{d_6\lambda}, \tag{4-10}$$

where we applied doubling estimates in the last inequality. Thanks to (4-8), we obtain

$$\sum m_l \leq d_7\lambda.$$

Since $n_l = m_l + 1 \leq 2m_l$, we complete the lemma for $n_l \geq 2$.

If the vanishing order for the singular point is two, i.e., $n_l = 1$. We consider $Q(z) = \prod (z - z_l)^{n_l/2}$ instead of $P(z)$. In this case, $Q(z)$ may not be defined as a single-valued holomorphic function on \mathcal{C} . We pass to a finite-branched cover of the disk \mathcal{D} punctured at z_l . The Carleman estimates in the previous sections still work. The same conclusion will follow. \square

Based on the vanishing order estimate in Proposition 11, we are able to count the number of singular points.

Proof of Theorem 2. We cover the double manifold $\bar{\mathcal{M}}$ by geodesic balls with radius $C\lambda^{-1/2}$. Since $\bar{\mathcal{M}}$ is compact, the order of those balls is $C\lambda$. From Proposition 11, the conclusion in Theorem 2 follows. \square

Remark 12. Thanks to Proposition 11, we can actually show a stronger result. Let $z_l \in \mathcal{M}$ be a singular point with vanishing order $n_l + 1$. Then $\sum_l n_l \leq C\lambda^2$.

5. Growth of eigenfunctions

In this section, we will show that the eigenfunctions do not grow rapidly on too many small balls. We still restrict v to the small geodesic ball $\mathbb{B}(p, c\lambda^{-1/2})$ in the conformal chart. Let $w(z) = v(c\lambda^{-1/2}z)$. Then w satisfies the elliptic equation (4-1) with assumptions (4-2) in a Euclidean ball of radius 4 centered at the origin. If we suppose that w grows rapidly, that is,

$$C_1 \int_{(1-3a)\delta \leq |z-z_l| \leq (1-3a/4)\delta} w^2 \leq \int_{(1-3a/2)\delta \leq |z-z_l| \leq (1-a)\delta} w^2 \tag{5-1}$$

for all l and some large C_1 , then the following proposition is valid:

Proposition 13. *Suppose D_l are disks contained in a Euclidean ball of radius $\frac{1}{30}$ centered at the origin. Furthermore, assume that*

(R1) $\delta < d_1\lambda^{-1}$, and

(R2) $|z_l - z_k| > d_2\lambda^{1/2}\delta$ when $l \neq k$.

If (5-1) holds for all l , the number of disks D_l is less than $d_3\lambda$.

Proof. We will use the stronger Carleman estimates in (3-13) in Proposition 9. We prove it by contradiction. Suppose that the collection $D_l = \{z \mid |z - z_l| \leq \delta\}$ are disjoint disks satisfying the hypotheses (R1)–(R3) in Section 3. Without loss of generality, we require that all the D_l are in a ball centered at the origin with radius $\frac{1}{30}$. As before, $D_l(a) = \{z \mid |z - z_l| \leq (1 - 2a)\delta\}$, where a is a suitably small positive constant. Let

\mathcal{D} be a ball centered at the origin with radius 2. We choose a cut-off function $\theta \in C_0^\infty(\mathcal{D} \setminus \bigcup_l D_l)$ and assume $\theta(z)$ satisfies the following properties:

- (i) $\theta(z) = 1$ if $|z| < 1$ and $|z - z_l| > (1 - \frac{3}{2}a)\delta$ for all l .
- (ii) $|\nabla\theta| + |\Delta\theta| < C_2$ if $|z| > 1$.
- (iii) $|\nabla\theta| < C_3\delta^{-1}$ and $|\Delta\theta| < C_4\delta^{-2}$ if $|z - z_l| < (1 - \frac{3}{2}a)\delta$.

Substituting $f = \theta w$ into (3-13) gives that

$$\int_{\mathcal{D}} |\Delta(\theta w) + \hat{b} \cdot \nabla(\theta w)|^2 |P|^{-2} e^{d_4\lambda|z|^2} \geq C_5\lambda^2 \int_{\mathcal{D}} |\theta w|^2 |P|^{-2} e^{d_4\lambda|z|^2} + C_6\delta^{-4} \int_A |\theta w|^2 |P|^{-2} e^{d_4\lambda|z|^2}. \tag{5-2}$$

We also assume $\tau = d_4\lambda$. Recall that $A = \bigcup_l A_l$ and $A_l = \{z \mid (1 - 2a)\delta \leq |z - z_l| \leq (1 - a)\delta\}$. We first consider the integral in the left-hand side of the last inequality. Again, by (4-1),

$$\Delta(\theta w) + \hat{b} \cdot \nabla(\theta w) = -\hat{q}\theta w + \Delta\theta w + 2\nabla\theta \cdot \nabla w + \hat{b} \cdot \nabla\theta w.$$

Thus,

$$|\Delta(\theta w) + \hat{b} \cdot \nabla(\theta w)|^2 \leq C(c\lambda^2\theta^2 w^2 + |\Delta\theta|^2 w^2 + |\nabla\theta|^2 |\nabla w|^2 + c\lambda |\nabla\theta|^2 w^2),$$

where c is sufficiently small. We will absorb the term involving $\theta^2 w^2$ into the right-hand side of (5-2). Since c is small enough, we get

$$\begin{aligned} \int_{\mathcal{D}} (|\Delta\theta|^2 w^2 + c|\nabla\theta|^2 w^2 + |\nabla\theta|^2 |\nabla w|^2) |P|^{-2} e^{d_4\lambda|z|^2} \\ \geq C_7\lambda^2 \int_{\mathcal{D}} |\theta w|^2 |P|^{-2} e^{d_4\lambda|z|^2} + C_8\delta^{-4} \int_A |\theta w|^2 |P|^{-2} e^{d_4\lambda|z|^2}. \end{aligned} \tag{5-3}$$

Using the properties of $\theta(z)$ and taking into account that each D_l lies in the ball centered at the origin with radius $\frac{1}{30}$, we obtain

$$\begin{aligned} \int_{\mathcal{D}} (|\Delta\theta|^2 w^2 + c|\nabla\theta|^2 w^2 + |\nabla\theta|^2 |\nabla w|^2) |P|^{-2} e^{d_4\lambda|z|^2} \\ \geq C_7\lambda^2 \int_{1/4 \leq |z| \leq 1/2} |w|^2 |P|^{-2} e^{d_4\lambda|z|^2} + C_9\delta^{-4} \sum_l \int_{(1-3a/2)\delta \leq |z-z_l| \leq (1-a)\delta} |w|^2 |P|^{-2} e^{d_4\lambda|z|^2}. \end{aligned} \tag{5-4}$$

Next we want to control the left-hand side of the last inequality. Write

$$\int_{\mathcal{D}} (|\Delta\theta|^2 w^2 + c|\nabla\theta|^2 w^2 + |\nabla\theta|^2 |\nabla w|^2) |P|^{-2} e^{d_4\lambda|z|^2} = I + \sum_l I_l, \tag{5-5}$$

where

$$\begin{aligned} I &= \int_{1 \leq |z| \leq 2} (|\Delta\theta|^2 w^2 + c|\nabla\theta|^2 w^2 + |\nabla\theta|^2 |\nabla w|^2) |P|^{-2} e^{d_4\lambda|z|^2}, \\ I_l &= \int_{(1-2a)\delta \leq |z-z_l| \leq (1-3a/2)\delta} (|\Delta\theta|^2 w^2 + c|\nabla\theta|^2 w^2 + |\nabla\theta|^2 |\nabla w|^2) |P|^{-2} e^{d_4\lambda|z|^2}. \end{aligned}$$

By standard elliptic estimates,

$$I \leq e^{d_5\lambda} \max_{|z| \geq 1} |P|^{-2} \int_{3/4 \leq |z| \leq 5/2} w^2. \tag{5-6}$$

Similarly, via elliptic estimates,

$$I_l \leq C_{10} \delta^{-4} \left(\max_{A_l} |P|^{-2} e^{d_4\lambda|z|} \right) \int_{(1-3a)\delta \leq |z-z_l| \leq (1-3a/4)\delta} w^2. \tag{5-7}$$

Thanks to [Lemma 8](#),

$$I_l \leq C_{11} \delta^{-4} \left(\min_{A_l} |P|^{-2} e^{d_4\lambda|z|} \right) \int_{(1-3a)\delta \leq |z-z_l| \leq (1-3a/4)\delta} w^2. \tag{5-8}$$

Combining these inequalities together in (5-4) leads to

$$\begin{aligned} & e^{d_5\lambda} \max_{|z| \geq 1} |P|^{-2} \int_{3/4 \leq |z| \leq 5/2} w^2 + C_{11} \delta^{-4} \sum_l \left(\min_{A_l} |P|^{-2} e^{d_4\lambda|z|} \right) \int_{(1-3a)\delta \leq |z-z_l| \leq (1-3a/4)\delta} w^2 \\ & \geq C_{12} \min_{|z| \leq 1/2} |P|^{-2} \int_{1/4 \leq |z| \leq 1/2} |w|^2 + C_{13} \delta^{-4} \sum_l \min_{A_l} \left(|P|^{-2} e^{d_4\lambda|z|^2} \right) \int_{(1-3a/2)\delta \leq |z-z_l| \leq (1-a)\delta} w^2. \end{aligned} \tag{5-9}$$

Performing similar arguments as for (4-8) shows that

$$\min_{|z| \leq 1/2} |P|^{-2} > \max_{|z| \geq 1} |P|^{-2} e^{d_5 \sum_l m_l}.$$

If the number of the D_l is $d_3\lambda$, then

$$\min_{|z| \leq 1/2} |P|^{-2} > \max_{|z| \geq 1} |P|^{-2} e^{d_6\lambda}. \tag{5-10}$$

We claim that

$$e^{C_{14}\lambda} \int_{1/4 \leq |z| \leq 1/2} w^2 \geq \int_{3/4 \leq |z| \leq 5/2} w^2. \tag{5-11}$$

We prove the claim by doubling estimates shown in [Proposition 5](#). We choose a ball $\mathbb{B}(x_0, \frac{1}{8}) \subset \{z \mid \frac{1}{4} \leq |z| \leq \frac{1}{2}\}$. It is clear that

$$\int_{1/4 \leq |z| \leq 1/2} w^2 \geq \int_{\mathbb{B}(x_0, 1/8)} w^2.$$

Using doubling estimates, we have

$$e^{C_{15}\lambda} \int_{\mathbb{B}(x_0, 1/8)} w^2 \geq \int_{\mathbb{B}(x_0, 2/8)} w^2.$$

By finite iterations, we can find a large ball $\mathbb{B}(x_0, 3)$ that contains $\{z \mid \frac{3}{4} \leq |z| \leq \frac{5}{2}\}$. This yields that

$$\int_{\mathbb{B}(x_0, 3)} w^2 \geq \int_{3/4 \leq |z| \leq 5/2} w^2.$$

Then the combination of these inequalities verifies the claim.

If we choose d_3 suitably large, since the number of disks D_l is $d_3\lambda$, also d_6 is suitably large. From the inequalities (5-10) and (5-11), it follows that

$$e^{d_5\lambda} \max_{|z|\geq 1} |P|^{-2} \int_{3/4\leq|z|\leq 5/2} w^2 < C_{12} \min_{|z|\leq 1/2} |P|^{-2} \int_{1/4\leq|z|\leq 1/2} w^2. \tag{5-12}$$

This contradicts the estimates (5-1) and (5-9). The proposition is proved. □

6. Growth estimates and nodal length

The purpose of this section is to find the connection between growth of eigenfunctions and nodal length. A suitable small growth in L^2 norm implies an upper bound on nodal length. We consider the second-order elliptic equations

$$\Delta \bar{w} + b^* \cdot \nabla \bar{w} + q^* \bar{w} = 0 \quad \text{in } \mathbb{B}(0, 4). \tag{6-1}$$

Assume that there exists a positive constant C such that $\|b^*\|_{W^{1,\infty}} \leq C$ and $\|q^*\|_{W^{1,\infty}} \leq C$. The following lemma relies on the Carleman estimates in Lemma 4. Suppose ϵ_1 is a sufficiently small positive constant.

Lemma 14. *Suppose that w satisfies the growth estimate*

$$\int_{(1-3a/2)\epsilon_0 < r < (1-a)\epsilon_0} \bar{w}^2 \leq C_3 \int_{(1-3a)\epsilon_0 < r < (1-4a/3)\epsilon_0} \bar{w}^2, \tag{6-2}$$

where a and ϵ_0 are fixed small constants. Then, for $0 < \epsilon_1 < \frac{1}{100}\epsilon_0$, we have

$$\max_{r \leq \epsilon_1} |\bar{w}| \geq C_4 \left(\frac{\epsilon_1}{\epsilon_0} \right)^{C_5} \left(\int_{\mathbb{B}(0, (1-4/3a)\epsilon_0)} \bar{w}^2 \right)^{\frac{1}{2}}, \tag{6-3}$$

where f denotes the average of the integration.

Proof. We select a radial cut-off function $\theta \in C_0^\infty(\frac{1}{2}\epsilon_1 < r < (1 - \frac{11}{10}a)\epsilon_0)$ that satisfies the properties:

- (i) $\theta(r) = 1$ for $\frac{3}{4}\epsilon_1 < r < (1 - \frac{10}{9}a)\epsilon_0$.
- (ii) $|\nabla\theta| + |\Delta\theta| \leq C_6$ for $r > (1 - \frac{10}{9}a)\epsilon_0$.
- (iii) $|\nabla\theta| \leq C_7\epsilon_1^{-1}$ and $|\Delta\theta| < C_8\epsilon_1^{-2}$ for $r \leq \frac{3}{4}\epsilon_1$.

From (6-1), we get

$$\Delta(\theta\bar{w}) + b^* \cdot \nabla(\theta\bar{w}) + q^*\theta\bar{w} = \Delta\theta\bar{w} + 2\nabla\theta \cdot \nabla\bar{w} + b^* \cdot \nabla\theta\bar{w}.$$

Assume that $\tau > C$ is large enough. Substituting $u = \theta\bar{w}$ in Lemma 4 yields that

$$C_2\tau^3 \int r^\epsilon e^{2\tau\phi(r)} \theta^2 \bar{w}^2 dr d\omega \leq I, \tag{6-4}$$

where

$$I = \int r^4 e^{2\tau\phi(r)} |\Delta\theta\bar{w} + 2\nabla\theta \cdot \nabla\bar{w} + b^* \cdot \nabla\theta\bar{w}|^2 dr d\omega.$$

Note that $\phi(r)$ is a decreasing function. Furthermore, by the assumptions on $\theta(z)$, we obtain

$$I \leq e^{2\tau\phi(\epsilon_1/2)} \int_{\epsilon_1/2 < r < 3\epsilon_1/4} |\Delta\theta\bar{w} + 2\nabla\theta \cdot \nabla\bar{w} + b^* \cdot \nabla\theta\bar{w}|^2 r \, dr \, d\omega + e^{2\tau\phi((1-10a/9)\epsilon_0)} \int_{(1-10a/9)\epsilon_0 < r < (1-11a/10)\epsilon_0} |\Delta\theta\bar{w} + 2\nabla\theta \cdot \nabla\bar{w} + b^* \cdot \nabla\theta\bar{w}|^2 r \, dr \, d\omega.$$

By standard elliptic estimates, we derive that

$$I \leq C_9 e^{2\tau\phi(\epsilon_1/2)} \int_{\epsilon_1/4 < r < \epsilon_1} \bar{w}^2 r \, dr \, d\omega + C_{10} e^{2\tau\phi((1-10a/9)\epsilon_0)} \int_{(1-3a/2)\epsilon_0 < r < (1-a)\epsilon_0} \bar{w}^2 r \, dr \, d\omega. \tag{6-5}$$

Taking the inequality (6-4) and assumptions of θ into account, we have

$$\begin{aligned} C_{10} e^{2\tau\phi((1-10a/9)\epsilon_0)} \int_{(1-3a/2)\epsilon_0 < r < (1-a)\epsilon_0} \bar{w}^2 r \, dr \, d\omega + C_9 e^{2\tau\phi(\epsilon_1/2)} \int_{\epsilon_1/4 < r < \epsilon_1} \bar{w}^2 r \, dr \, d\omega \\ \geq C_2 \tau^3 \int_{3\epsilon_1/4 < r < (1-10a/9)\epsilon_0} r^\epsilon e^{2\tau\phi(r)} \bar{w}^2 \, dr \, d\omega \\ \geq C_2 \tau^3 \left(\left(1 - \frac{10}{9}a\right)\epsilon_0 \right)^{\epsilon-1} \int_{3\epsilon_1/4 < r < (1-10a/9)\epsilon_0} e^{2\tau\phi(r)} \bar{w}^2 r \, dr \, d\omega. \end{aligned} \tag{6-6}$$

Since ϵ and ϵ_0 are fixed positive constants, taking τ large enough we obtain

$$\frac{1}{2} C_2 \tau^3 \left(\left(1 - \frac{10}{9}a\right)\epsilon_0 \right)^{\epsilon-1} > C_{10}.$$

Taking the hypothesis (6-2) into consideration, we can incorporate the first term in the left-hand side of (6-6) into the right-hand side. It follows that

$$C_9 e^{2\tau\phi(\epsilon_1/2)} \int_{\epsilon_1/4 < r < \epsilon_1} \bar{w}^2 r \, dr \, d\omega \geq C_{10} e^{2\tau\phi((1-10a/9)\epsilon_0)} \int_{3\epsilon_1/4 < r < (1-10a/9)\epsilon_0} \bar{w}^2 r \, dr \, d\omega. \tag{6-7}$$

Fix such a τ ; adding the term

$$e^{2\tau\phi((1-10a/9)\epsilon_0)} \int_{r < 3\epsilon_1/4} \bar{w}^2 r \, dr \, d\omega$$

to both sides of the last inequality yields that

$$e^{2\tau\phi(\epsilon_1/2)} \int_{r < \epsilon_1} \bar{w}^2 r \, dr \, d\omega \geq C_{11} e^{2\tau\phi((1-10a/9)\epsilon_0)} \int_{r < (1-4a/3)\epsilon_0} \bar{w}^2 r \, dr \, d\omega, \tag{6-8}$$

where we have used the fact that ϕ is decreasing. Straightforward calculations show that

$$e^{2\tau(\phi((1-10a/9)\epsilon_0) - \phi(\epsilon_1/2))} \geq C_{13} \left(\frac{\epsilon_1}{\epsilon_0} \right)^{C_{12}}.$$

Thus,

$$\int_{r < \epsilon_1} \bar{w}^2 r \, dr \, d\omega \geq C_{13} \left(\frac{\epsilon_1}{\epsilon_0} \right)^{C_{12}} \int_{r < (1-4a/3)\epsilon_0} \bar{w}^2 r \, dr \, d\omega. \tag{6-9}$$

This completes the lemma. □

Our next goal is to find the relation between [Lemma 14](#) and nodal length. We assume that the estimate (6-2) exists. Then the conclusion (6-3) in [Lemma 14](#) holds. For $\epsilon_1 \leq \frac{1}{100}\epsilon$, if $|z| < \epsilon_1$ then using Taylor's expansion gives that

$$\left| \bar{w}(z) - \sum_{|\alpha| \leq C_5} \frac{1}{\alpha!} \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(0) z^\alpha \right| \leq \sup_{|z| \leq \epsilon_1} \sup_{|\alpha|=C_5+1} d_1 \left| \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(z) \right| \epsilon_1^{C_5+1},$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\partial/\partial z^\alpha = \partial/\partial z_1^{\alpha_1} \cdot \partial/\partial z_2^{\alpha_2}$. To control the right-hand side of the last inequality, by elliptic estimates and a rescaling argument we have

$$\left| \bar{w}(z) - \sum_{|\alpha| \leq C_5} \frac{1}{\alpha!} \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(0) z^\alpha \right| \leq d_2 \left(\int_{\mathbb{B}(0, (1-4a/3)\epsilon_0)} \bar{w}^2 \right)^{\frac{1}{2}} \left(\frac{\epsilon_1}{\epsilon_0} \right)^{C_5+1}.$$

Using the estimate (6-3) in [Lemma 14](#), we get

$$\left| \bar{w}(z) - \sum_{|\alpha| \leq C_5} \frac{1}{\alpha!} \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(0) z^\alpha \right| \leq d_3 \left(\frac{\epsilon_1}{\epsilon_0} \right) \max_{|z| \leq \epsilon_1} |\bar{w}|.$$

Choosing ϵ_1/ϵ_0 sufficiently small, by the triangle inequality we obtain

$$\sup_{|\alpha| \leq C_5} \left| \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(0) \right| \epsilon_1^{|\alpha|} \geq d_4 \max_{|z| \leq \epsilon_1} |\bar{w}|.$$

Applying again the estimate (6-3) to the right-hand side of the last inequality yields that

$$\sup_{|\alpha| \leq C_5} \left| \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(0) \right| \epsilon_0^{|\alpha|} \geq d_5 \left(\int_{\mathbb{B}(0, (1-4a/3)\epsilon_0)} \bar{w}^2 \right)^{\frac{1}{2}}. \tag{6-10}$$

By standard elliptic estimates, we also have

$$\sup_{|z| \leq \epsilon_0/2} \sup_{|\alpha| \leq C_5+1} \left| \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(z) \right| \epsilon_0^{|\alpha|} \leq d_6 \left(\int_{\mathbb{B}(0, (1-4a/3)\epsilon_0)} \bar{w}^2 \right)^{\frac{1}{2}}. \tag{6-11}$$

The basic relationship between derivatives and nodal length in two dimensions is shown in [\[Donnelly and Fefferman 1990a\]](#).

Lemma 15. *Suppose that \bar{w} satisfies (6-10) and (6-11). Then*

$$H^1(z \mid |z| \leq d_7 \bar{\epsilon} \text{ and } \bar{w}(z) = 0) \leq d_8 \bar{\epsilon}.$$

With the aid of the last lemma, we can readily obtain an upper nodal length estimate.

Proposition 16. *Let \bar{w} be the solution of (6-1). Suppose that $\bar{\epsilon} \leq \epsilon_0$ and w satisfies the growth condition*

$$\int_{(1-3a/2)\bar{\epsilon} < r < (1-a)\bar{\epsilon}} \bar{w}^2 \leq C_3 \int_{(1-3a)\bar{\epsilon} < r < (1-4a/3)\bar{\epsilon}} \bar{w}^2. \tag{6-12}$$

Then

$$H^1(z \mid |z| \leq d_9 \bar{\epsilon} \text{ and } \bar{w}(z) = 0) \leq d_{10} \bar{\epsilon}.$$

Proof. Since the inequalities (6-10) and (6-11) can be derived from (6-12) by Lemma 14, the proposition follows from the last lemma. □

7. Total nodal length

As Proposition 13 indicates, the eigenfunctions cannot grow rapidly on too many small balls. If they grow slowly, we have an upper bound on the local length of nodal sets by Proposition 16. In this section, we will link these two arguments together. To achieve it, we will employ a process of repeated subdivision and selection of squares. The idea is inspired by [Donnelly and Fefferman 1990a].

Assume that $\mathbb{B}(p, c\lambda^{-1/2})$ is a geodesic ball of the double manifold $\bar{\mathcal{M}}$. Choosing c to be small, it is contained in a conformal chart. Let $w(z) = v(c\lambda^{-1/2}z)$ with c sufficiently small. We know that w satisfies

$$\Delta w + \hat{b}(x) \cdot \nabla w + \hat{q}(x)w = 0 \quad \text{in } \mathbb{B}(0, 4). \tag{7-1}$$

We consider the square $P = \{(x, y) \mid \max(|x|, |y|) \leq \frac{1}{60}\}$ in $\mathbb{B}(0, 4)$ and divide it into a grid of closed squares P_l with side $\delta \leq a_1\lambda^{-1}$. If (5-1) holds for some point $z_l \in P_l$ and for some sufficiently large C_1 , we call P_l a square of rapid growth. With the aid of Proposition 13, we are able to obtain the following result:

Lemma 17. *There are at most $C\lambda^2$ squares with side δ where w is of rapid growth.*

Proof. Let I_1 be the collection of those indices l for which P_l is a square of rapid growth. For each $l \in I_1$, there exists some point $z_l \in P_l$ such that (5-1) holds. Let $|I_1|$ denote the cardinality of I_1 . Define

$$P_l^* = \{z \mid |z - z_l| < d_1\delta\lambda^{1/2}\}.$$

The collection of disks P_l^* covers the collection of squares P_l for $l \in I_1$. We choose a maximal collection of disjoint disks of P_l^* and denote it as I_2 . If $l \in I_2$, we define

$$P_l^{**} = \{z \mid |z - z_l| < 4d_1\delta\lambda^{1/2}\}.$$

Since the collection of disks in I_2 is maximal and they are disjoint, we obtain that

$$\bigcup_{l \in I_2} P_l^{**} \supseteq \bigcup_{l \in I_1} P_l^* \supseteq \bigcup_{l \in I_1} P_l.$$

Thus,

$$|I_2| \times 16d_1^2\delta^2\lambda \geq |I_1|\delta^2,$$

which implies

$$|I_2|\lambda \geq d_2|I_1|.$$

Recall from Proposition 13 that $|I_2| \leq d_3\lambda$. Therefore, we obtain the desirable estimate $|I_1| \leq d_4\lambda^2$. □

Now we introduce an iterative process of bisecting squares. We begin by dividing the square into a grid of squares $P_l(1)$ with side $\delta(1) = a_1\lambda^{-1}$, then separate them into two categories $R_l(1)$ and $S_l(1)$. $R_l(1)$ are those where w is of rapid growth and $S_l(1)$ are those where (5-1) fails for w . We continue to bisect each square $R_l(1)$ to obtain squares $P_l(2)$ with side $\delta(2) = \frac{1}{2}\delta(1)$. Again, we split $P_l(2)$ into the

subcollection $R_l(2)$ with rapid growth and $S_l(2)$ with slow growth. We repeat the process at each step k . Then there are squares $R_l(k)$ and $S_l(k)$ with $\delta(k) = \frac{1}{2^k}\delta(1)$. We count the number of $R_l(k)$ and $S_l(k)$ at step k :

Lemma 18. (i) *The number of squares $R_l(k)$ is at most $C_2\lambda^2$.*

(ii) *The number of squares $S_l(k)$ is at most $C_3\lambda^2$.*

Proof. The conclusion (i) follows directly from Lemma 17. We only need to show (ii). If $k = 1$, the conclusion (ii) follows because the total number of squares is at most of order λ^2 . If $k \geq 2$ then, by construction of those squares,

$$|S_l(k)| \leq 4|R_l(k - 1)| \leq C_4\lambda^2,$$

where we have used (i) in the last inequality. The lemma is done. □

The next lemma tells that almost every point lies in some $R_l(k)$ with slow growth. It is Lemma 6.3 in [Donnelly and Fefferman 1990a].

Lemma 19. $\bigcup_{k,l} S_l(k)$ covers the square P except for singular points $\mathcal{S} = \{z \in P \mid w(z) = 0, \nabla w = 0\}$.

We are ready to give the proof of Theorem 3.

Proof of Theorem 3. Consider $\bar{w}(z) = w(z_l + \epsilon_0^{-1}\delta(k)z)$. Then $\bar{w}(z)$ satisfies (6-1). Choosing a finite collection of $z_l \in S_l(k)$ and applying Proposition 16, we have

$$H^1(z \mid w(z) = 0 \text{ and } z \in S_l(k)) \leq C_5 2^{-k} \lambda^{-1}. \tag{7-2}$$

Furthermore, thanks to Lemma 19,

$$\begin{aligned} H^1(z \mid w(z) = 0 \text{ and } \max(|x|, |y|) \leq \frac{1}{60}) &\leq \sum_{l,k} H^1(z \mid w(z) = 0 \text{ and } z \in S_l(k)) \\ &\leq \lambda^2 \sum_k C_5 2^{-k} \lambda^{-1} \leq C_6 \lambda, \end{aligned} \tag{7-3}$$

where we have used (ii) in Lemma 18 and (7-2). Since $w(z) = v(c\lambda^{-1/2}z)$, by the rescaling argument, we obtain

$$H^1(\{v(z) = 0\} \cap \mathbb{B}(p, c\lambda^{-1/2})) \leq C_6 \lambda^{1/2}.$$

Finally, covering $\bar{\mathcal{M}}$ with order λ of geodesic balls with radius $c\lambda^{-1/2}$, we readily deduce that

$$H^1(z \in \bar{\mathcal{M}} \mid v(z) = 0) \leq C_7 \lambda^{3/2}.$$

Thus, so is $H^1(\mathcal{N}_\lambda)$. □

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ANALYSIS & PDE

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Peierls substitution for magnetic Bloch bands	773
SILVIA FREUND and STEFAN TEUFEL	
Dispersive estimates in \mathbb{R}^3 with threshold eigenstates and resonances	813
MARIUS BECEANU	
Interior nodal sets of Steklov eigenfunctions on surfaces	859
JIUYI ZHU	
Some counterexamples to Sobolev regularity for degenerate Monge–Ampère equations	881
CONNOR MOONEY	
Mean ergodic theorem for amenable discrete quantum groups and a Wiener-type theorem for compact metrizable groups	893
HUICHI HUANG	
Resonance free regions for nontrapping manifolds with cusps	907
KIRIL DATCHEV	
Characterizing regularity of domains via the Riesz transforms on their boundaries	955
DORINA MITREA, MARIUS MITREA and JOAN VERDERA	