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SOME COUNTEREXAMPLES TO SOBOLEV REGULARITY FOR DEGENERATE MONGE–AMPÈRE EQUATIONS

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We construct a counterexample to $W^{2,1}$ regularity for convex solutions to

$$\det D^2u \le 1, \quad u|_{\partial\Omega} = \text{const.}$$

in two dimensions. We also prove a result on the propagation of singularities of the form $|x_2|/|\log x_2|$ in two dimensions. This generalizes a classical result of Alexandrov and is optimal by example.

1. Introduction

In this paper we investigate the $W^{2,1}$ regularity of convex Alexandrov solutions to degenerate Monge–Ampère equations of the form

$$\det D^2u(x) = \rho(x) \le 1 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \text{const.}, \tag{1}$$

where Ω is a bounded convex domain in $\mathbb{R}^n$.

In the case that ρ also has a strictly positive lower bound, $W^{2,1}$ estimates were first obtained by De Philippis and Figalli [2013]. They showed that $\Delta u \log^k(2 + \Delta u)$ is integrable for any k. It was subsequently shown in [De Philippis et al. 2013; Schmidt 2013] that D^2u is in fact $L^{1+\epsilon}$ for some ϵ depending on dimension and $\|1/\rho\|_{L^\infty(\Omega)}$. These estimates are optimal in light of two-dimensional examples due to Wang [1995] with the homogeneity

$$u(\lambda x_1, \lambda^\alpha x_2) = \lambda^{1+\alpha}u(x_1, x_2).$$

These estimates fail when ρ degenerates. In three and higher dimensions, it is not hard to construct solutions to (1) that have a Lipschitz singularity on part of a hyperplane, so the second derivatives concentrate (see Section 2). However, in two dimensions, a classical result of Alexandrov [1942] shows that Lipschitz singularities of convex solutions to $\det D^2u \le 1$ propagate to the boundary. Thus, in two dimensions, solutions to (1) are C^1 and D^2u has no jump part. However, this leaves open the possibility that D^2u has nonzero Cantor part.

The main result of this paper is the construction of a solution to (1) in two dimensions that is not $W^{2,1}$. This negatively answers an open problem stated in both [De Philippis and Figalli 2014] and [Figalli 2015], which was motivated by potential applications to the semigeostrophic equation. We also prove that, in two

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dimensions, singularities that are logarithmically slower than Lipschitz propagate. This result generalizes the theorem of Alexandrov and is optimal by example.

The $W^{2,1}$ estimates mentioned above have applications to the global existence of weak solutions to the semigeostrophic equation [Ambrosio et al. 2012; 2014]. In this context, the density ρ solves a continuity equation that preserves L^∞ bounds. This is the only regularity property of ρ that is globally preserved, due to nonlinear coupling between ρ and the velocity field. It is therefore useful to obtain estimates that depend on L^∞ bounds for ρ but not on its regularity.

To apply the results in [De Philippis and Figalli 2013; De Philippis et al. 2013] one must assume that ρ is supported in the whole space. However, in physically interesting cases, the initial density is compactly supported. It is thus natural to ask what one can show about solutions to (1). Our construction shows that, even in two dimensions, one must rely more on the specific structure of the semigeostrophic equation to obtain existence results for compactly supported initial data.

The idea of our construction is to start with a one-dimensional convex function of x_2 in the half-space $\{x_1 < 0\}$ whose second derivative has nontrivial Cantor part, and extend to a convex function on $\mathbb{R}^2$ which lifts from these values without generating too much Monge–Ampère measure. To accomplish this we start with a "building block" v_1 that agrees with $|x_2|$ in $\{|x_2| \geq (x_1)_+^\alpha\}$ for some $\alpha > 1$, and in the cusp $\{|x_2| < (x_1)_+^\alpha\}$ grows with the homogeneity

$$v_1(\lambda x_1, \lambda^\alpha x_2) = \lambda^\alpha v_1(x_1, x_2).$$

By superposing vertically translated rescalings of (a smoothed version of) v_1 in a self-similar way, we obtain our example.

Our main theorem is:

Theorem 1.1. *For all $n \geq 2$, there exist solutions to* (1) *that are not $W^{2,1}$.*

Remark 1.2. It is obvious that solutions to (1) in one dimension are $C^{1,1}$.

Remark 1.3. In our examples, the support of ρ is irregular. In particular, in the higher-dimensional examples, the support of ρ is a cusp revolved around an axis, and in the two-dimensional example, the support of ρ has a very irregular "fractal" geometry.

In, e.g., [Daskalopoulos and Savin 2009; Guan 1997] the authors obtain interesting regularity results when ρ degenerates in a specific way, motivated by applications to prescribed Gauss curvature.

Our second result concerns the behavior of solutions to (1) near a single line segment in $\mathbb{R}^2$. Since Lipschitz singularities propagate, D^2u cannot concentrate on a line segment. (In our two-dimensional counterexample to $W^{2,1}$ regularity, D^2u concentrates on a family of horizontal rays.) On the other hand, by modifying an example in [Wang 1995] one can construct, for any $\epsilon > 0$, a solution to (1) that grows like $|x_2|/|\log x_2|^{1+\epsilon}$, with second derivatives not in $L\log^{1+\epsilon} L$ (see Section 4).

It is natural to ask whether one can take $\epsilon \leq 0$. We show that this is not possible. Indeed, we construct a family of barriers that agree with $|x_2|/|\log x_2|$ away from arbitrarily thin cusps around the x_1-axis, where we can make the Monge–Ampère measure as large as we like. By sliding these barriers we prove that singularities of the form $|x_2|/|\log x_2|$ propagate. Our second theorem is:

Theorem 1.4. *Assume that u is convex on $\mathbb{R}^2$ and that $\det D^2u \le 1$. Then if $u(0)=0$ and $u \ge c|x_2|/|\log x_2|$ in a neighborhood of the origin for some $c>0$, then u vanishes on the x_1-axis.*

Remark 1.5. Note that we assume the growth in a neighborhood of 0. For a Lipschitz singularity it is enough to assume the growth at a point, which automatically extends to a neighborhood by convexity. (See, e.g., [Figalli and Loeper 2009] for a short proof that Lipschitz singularities propagate.)

Remark 1.6. Theorem 1.4 shows that a solution to $\det D^2u \ge 1$ in two dimensions cannot separate from a tangent plane more slowly than $r^2e^{-1/r}$ in any fixed direction. This quantifies the classical result that such functions are strictly convex. The idea is that if not, then after subtracting a tangent plane we have $0 \le u \le C|x_1| + x_2^2 e^{-1/|x_2|}$ near the origin. Taking the Legendre transform one obtains $u^* \ge c|x_2|/|\log x_2|$ near the origin. Applying Theorem 1.4 to u^* gives a contradiction of the strict convexity of u.

The paper is organized as follows. In Section 2 we construct simple examples of solutions to (1) in the case $n \ge 3$ which have a Lipschitz singularity on a hyperplane. In Section 3 we construct a solution to (1) in two dimensions whose second derivatives have nontrivial Cantor part. This proves Theorem 1.1. In Section 4 we first construct examples showing that Theorem 1.4 is optimal. We then construct barriers related to these examples. Finally, we use the barriers to prove Theorem 1.4.

2. The case $n \ge 3$

In this section we construct simple examples of solutions to (1) in three and higher dimensions that have a Lipschitz singularity on a hyperplane. Denote $x \in \mathbb{R}^n$ by (x', x_n) and let $r = |x'|$. More precisely:

Proposition 2.1. *In dimension $n \ge 3$, for any $\alpha \ge \frac{n}{n-2}$ there exists a solution to* (1) *that is a positive multiple of $|x_n|$ in $\{|x_n| \ge (r-1)_+^{\alpha}\}$.*

Proof. Let $h(r) = (r-1)_+$. We search for a convex function $u = u(r, x_n)$ in $\{|x_n| < h(r)^\alpha\}$, with $\alpha > 1$, that glues "nicely" across the boundary to $|x_n|$. To that end we look for a function with the homogeneity

$$u(1+\lambda t, \lambda^\alpha x_n) = \lambda^\alpha u(1+t, x_n),$$

so that $\partial_n u$ is invariant under the rescaling. Let

$$u(r, x_n) = \begin{cases} h(r)^\alpha + h(r)^{-\alpha} x_n^2, & |x_n| < h(r)^\alpha, \\ 2|x_n|, & |x_n| \ge h(r)^\alpha. \end{cases}$$

Then ∇u is continuous on $\partial\{|x_n| < h(r)^\alpha\} \setminus \{r=1, x_n=0\}$. Furthermore, $\partial u|_{\{r=1, x_n=0\}}$ is the line segment between $\pm 2e_n$, which has measure zero. Thus, in the Alexandrov sense, $\det D^2u$ can be computed piecewise. In the cylindrical coordinates (r, x_n) one easily computes

$$\det D^2u = \begin{cases} \dfrac{1}{r^{n-2}}\left(2\alpha^{n-1}(\alpha-1)h(r)^{\alpha(n-2)-n}\left(1-\left(\dfrac{x_n}{h(r)^\alpha}\right)^2\right)^{n-1}\right), & |x_n| < h(r)^\alpha, \\ 0, & |x_n| \ge h(r)^\alpha. \end{cases}$$

For $\alpha \ge \frac{n}{n-2}$ the right-hand side is locally bounded. □

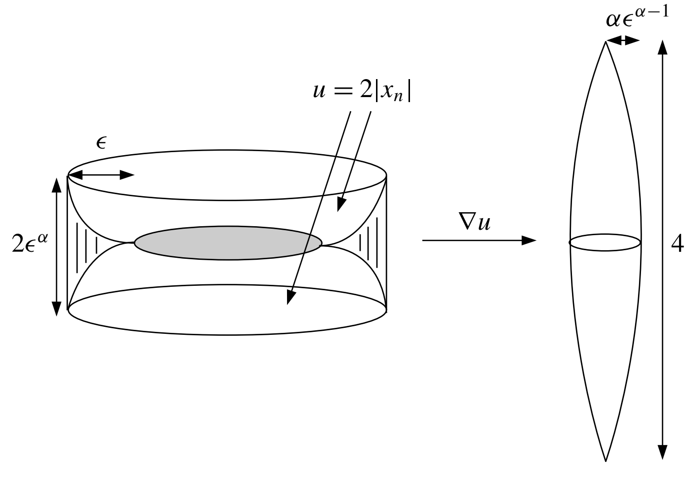

Figure 1. The gradient map of u decreases volume if $\alpha \geq \frac{n}{n-2}$.

Remark 2.2. The bound on α can be understood by looking at the gradient map of u, which takes a "ring" of volume like $h(r)^{1+\alpha}$ to a "football" with length of order 1 and radius of order $h(r)^{\alpha-1}$ (see Figure 1). Then impose that it decreases volume.

Remark 2.3. Observe that $\det D^2 u$ grows like dist.$^{n-2-n/\alpha}$ from its zero set. This is in a sense optimal; if $\det D^2 u < C|x_n|^{n-2}$ then one can modify Alexandrov's two-dimensional argument to show that the singularity has no extremal points.

3. The case $n = 2$

In this section we prove Theorem 1.1. We construct our example in several steps.

First, let $g(t)$ be a smooth, convex function such that $g(t) = \frac{1}{2}$ for $t \leq 0$ and $g(t) = t^{\alpha}$ for $t \geq 1$, where $\alpha > 1$. Then define

$$v_1(x_1, x_2) = \begin{cases} g(x_1) + \dfrac{1}{g(x_1)} x_2^2, & |x_2| < g(x_1), \\ 2|x_2|, & |x_2| \geq g(x_1). \end{cases}$$

It is easy to check that v_1 is a $C^{1,1}$ convex function, and in the Alexandrov sense,

$$\det D^2 v_1(x_1, x_2) = \begin{cases} 2\dfrac{g''(x_1)}{g(x_1)}\left(1 - \dfrac{x_2^2}{g(x_1)^2}\right), & |x_2| < g(x_1), \\ 0, & |x_2| \geq g(x_1). \end{cases}$$

In particular, $\det D^2 v_1$ is bounded, and decays like x_1^{-2} for x_1 large. Let v_λ be the rescalings defined by

$$v_\lambda(x_1, x_2) = \frac{1}{\lambda^{1+\alpha}} v_1(\lambda x_1, \lambda^{\alpha} x_2).$$

Observe that

$$\det D^2 v_\lambda(x_1, x_2) = \det D^2 v_1(\lambda x_1, \lambda^{\alpha} x_2),$$

and we have

$$v_\lambda = \frac{1}{\lambda}(x_1^\alpha + x_1^{-\alpha}x_2^2) \quad \text{in } \{x_1 \ge \lambda^{-1}\} \cap \{|x_2| \le x_1^\alpha\}. \tag{2}$$

In the following key lemma we show that any superposition of λ vertical translated copies of v_λ has bounded Monge–Ampère measure in $\{x_1 > 1/2\}$, and separates from its tangent planes when we step away from the x_2-axis.

Lemma 3.1. *Let $\{x_{2,i}\}_{i=1}^N$ be fixed numbers with $|x_{2,i}| \le 1$ for all i, where N is any positive integer. Let*

$$w(x_1, x_2) = \sum_{i=1}^{N} v_N(x_1, x_2 - x_{2,i}).$$

Then

$$\det D^2 w < C(\alpha) \quad \text{in } \left\{x_1 > \tfrac{1}{2}\right\} \tag{3}$$

for some $C(\alpha)$ independent of N and the choice of $\{x_{2,i}\}$, and

$$w(2, x_2) > w(0, x_2) + \mu(\alpha) \quad \text{for all } |x_2| < 1, \tag{4}$$

for some $\mu(\alpha) > 0$ independent of N and the choice of $\{x_{2,i}\}$.

Proof. We first prove (3). Since $\det D^2 v_1$ is bounded we may assume that $N \ge 2$. Consider a point $p = (p_1, p_2) \in \left\{x_1 > \frac{1}{2}\right\}$. Since w is $C^{1,1}$, the curves $p_2 = x_{2,i} \pm p_1^\alpha$ don't contribute anything to $\det D^2 w$, so we may assume that $p_2 \ne x_{2,i} \pm p_1^\alpha$ for any i. Then in a neighborhood of p, a subset of $M \le N$ of the translates are not linear, and all are linear if in addition $|p_2| > 1 + p_1^\alpha$. Up to relabeling the indices and subtracting a linear function of x_2, by (2) we can write

$$w = \frac{M}{N}\left(x_1^\alpha + x_1^{-\alpha}\left(x_2^2 - 2x_2 \frac{1}{M}\sum_{i=1}^{M} x_{2,i} + \frac{1}{M}\sum_{i=1}^{M} x_{2,i}^2\right)\right)$$

in a neighborhood of p. Since $|x_{2,i}| \le 1$, one easily computes that

$$\det D^2 w(p) \le 2\alpha \frac{M^2}{N^2} p_1^{-2}\big(\alpha - 1 + (\alpha+1) p_1^{-2\alpha}(p_2^2 + 2|p_2| + 1)\big),$$

and $\det D^2 w(p) = 0$ if $|p_2| > 1 + p_1^\alpha$. We conclude that

$$\det D^2 w(p) < C(\alpha),$$

where $C(\alpha)$ does not depend on N.

To prove (4), since v_1 is monotone increasing in the e_1 direction, we have for $|x_2| \le 2$ that

$$v_1(2, x_2) - v_1(0, x_2) \ge v_1(2, x_2) - v_1(2^{1/\alpha}, x_2) \ge 2^{-\alpha}(2^\alpha - 2)^2.$$

Since $\alpha > 1$, the lower bound $\mu := 2^{-\alpha}(2^\alpha - 2)^2$ is strictly positive.

By (2) the same argument gives

$$v_N(2, x_2) - v_N(0, x_2) > \frac{\mu}{N}$$

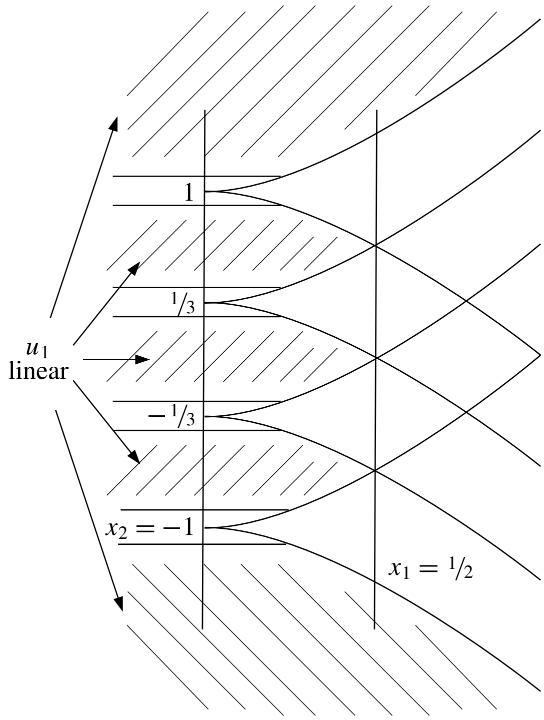

Figure 2. The function u_1 is a piecewise linear function of x_2 outside of the four equally spaced cusps between $x_2 = -1$ and $x_2 = 1$.

for $|x_2| \le 2$. Finally, since $|x_{2,i}| \le 1$, we have for $|x_2| < 1$ that

$$\sum_{i=1}^{N} \big(v_N(2, x_2 - x_{2,i}) - v_N(0, x_2 - x_{2,i}) \big) \ge \sum_{i=1}^{N} \frac{\mu}{N} = \mu > 0,$$

completing the proof $\square$

We can now complete the construction. Roughly, at stage k we superpose 2^{k+1} vertical translations of $v_{2^{k+1}}$, starting at the endpoints of the intervals removed up to the k-th stage in the construction of the Cantor set.

Proof of Theorem 1.1. Fix

$$\alpha := \frac{\log 3}{\log 2},$$

and define

$$u_1(x_1, x_2) = \sum_{i=0}^{3} v_4(x_1, x_2 - 1 + 2i/3).$$

Then u_1 is a piecewise linear function of x_2 outside of four equally spaced cusps in $\{x_1 > 0\}$ connected to thin strips in $\{x_1 < 0\}$ (see Figure 2).

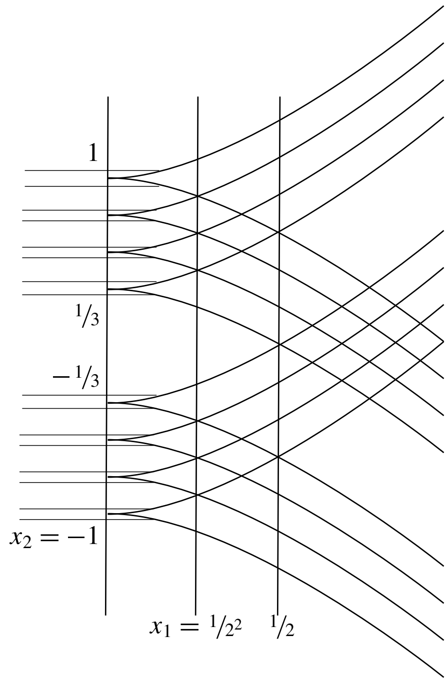

Figure 3. The function u_2 is obtained by superposing two rescaled copies of u_1, whose Hessians don't affect each other in $\{x_1 \le \frac{1}{2}\}$.

Define u_k inductively by

$$u_{k+1}(x_1, x_2) = \frac{1}{2^{1+\alpha}}\Big(u_k\big(2x_1, 3\big(x_2 + \tfrac{2}{3}\big)\big) + u_k\big(2x_1, 3\big(x_2 - \tfrac{2}{3}\big)\big)\Big).$$

We first claim that the det $D^2 u_k$ are uniformly bounded (in k) in $\{x_1 > \frac{1}{2}\}$. Indeed, each u_k is a sum of 2^{k+1} vertical translates of $v_{2^{k+1}}$ by values in $[-1, 1]$, so this follows from (3).

Next we show that the det $D^2 u_k$ are uniformly bounded in $\mathbb{R}^2$. Note that the u_k are linear functions of x_2 in $\{x_1 \le 1\} \times \{|x_2| > 2\}$, so in $\{x_1 \le \frac{1}{2}\}$, the rescaled copies of u_k in the definition of u_{k+1} are linear where the other is nontrivial (the determinants "don't interact"; see Figure 3). Since the rescaling $2^{-(1+\alpha)} u_k(2x_1, 3x_2)$ preserves Hessian determinants, we conclude that

$$\det D^2 u_{k+1}|_{\{x_1 \le 1/2\}} \le \sup_{x_1 \ge 0} \det D^2 u_k.$$

One easily checks that det $D^2 u_1$ is bounded, so the claim follows by induction.

Since $|v_\lambda|, |\nabla v_\lambda| < C R^\alpha/\lambda$ in B_R, the functions u_k are locally uniformly Lipschitz and bounded and thus converge locally uniformly to some u_∞. The right-hand sides det $D^2 u_k$ converge weakly to det $D^2 u_\infty$ (see [Gutiérrez 2001]), so

$$\det D^2 u_\infty < \Lambda < \infty$$

in all of $\mathbb{R}^2$.

Finally, let

$$u(x_1, x_2) = u_\infty((|x_1| - 1)_+, x_2)$$

be the function obtained by translating u_∞ to the right and reflecting over the x_2-axis.

It is clear that u is even in x_1 and x_2, and is a one-dimensional function $f(x_2)$ in the strip $\{|x_1| < 1\}$. It is easy to show that f' is the standard Cantor function (appropriately rescaled), so f'' has a nontrivial Cantor part. Indeed, $\partial_2 u_k(0, \cdot)$ jumps by 2^{1-k} over each of 2^{k+1} intervals of length $3^{-(k+1)}$ centered at the endpoints of the sets removed in the construction of the Cantor set. By (4) we also have

$$u(\pm 2, 0) > u(0, 0) + \mu.$$

Since u is even over both axes we conclude that

$$\{u < u(0, 0) + \mu\} \subset [-2, 2] \times [-C, C].$$

By convexity, u has bounded sublevel sets, completing the proof. □

4. A propagation result

In $\mathbb{R}^2$, the second derivatives of a solution to (1) cannot concentrate on a single line segment, since Lipschitz singularities propagate. (Compare to the example above, where the second derivatives concentrate on a family of horizontal rays.) In this section we investigate more closely how solutions to (1) can behave near a single line segment in $\mathbb{R}^2$.

We first construct, for any $\epsilon > 0$, examples that grow from the origin like $|x_2|/|\log x_2|^{1+\epsilon}$, with $D^2 u$ not in $L \log^{1+\epsilon} L$. We then construct a family of barriers related to these examples in the case $\epsilon = 0$. Finally, we use these barriers to prove that singularities of the form $|x_2|/|\log x_2|$ propagate.

Examples that grow logarithmically slower than Lipschitz.

Proposition 4.1. *For any $\alpha > 0$ there exists a solution to* (1) *in two dimensions that vanishes at* 0 *and lies above $c|x_2|/|\log x_2|^{1+1/\alpha}$, and whose Hessian is not in $L \log^{1+1/\alpha} L$.*

Proof. Let $\Omega_1 = \{|x_2| < h(x_1)e^{-1/x_1^\alpha}\}$ for some positive even function h to be determined. (By x^γ we mean $|x|^\gamma$). In Ω_1, define

$$u_0(x_1, x_2) = x_1^{\alpha+1} e^{-1/x_1^\alpha} + x_1^{\alpha+1} e^{1/x_1^\alpha} x_2^2.$$

We would like to glue this to a function of x_2 on $\Omega_2 = \mathbb{R}^2 \setminus \Omega_1$, which imposes the condition $\partial_1 u_0 = 0$ on the boundary. Computing, we find that

$$h^2(t) = \frac{1+(\alpha+1)t^\alpha/\alpha}{1-(\alpha+1)t^\alpha/\alpha} = 1 + 2\frac{\alpha+1}{\alpha} t^\alpha + O(t^{2\alpha}).$$

In this way we ensure that u_0 glues in a C^1 manner across $\partial\Omega_1$ to some function $g(|x_2|)$ in Ω_2 defined by

$$g(h(t)e^{-1/t^\alpha}) = t^{\alpha+1}(1 + h^2(t))e^{-1/t^\alpha}.$$

The agreement of derivatives on $\partial\Omega_1$ gives

$$g'(h(t)e^{-1/t^\alpha}) = 2t^{\alpha+1}h(t),$$

which upon differentiation and using the formula for h gives

$$g''(h(t)e^{-1/t^\alpha}) = 2(1+1/\alpha+o(1))e^{1/t^\alpha}t^{2\alpha+1}.$$

For $|z|$ small it follows that

$$g''(z) \geq \frac{1}{|z|\,|\log z|^{2+1/\alpha}},$$

giving the nonintegrability claimed (after, say, replacing x_1 by $(|x_1|-1)_+$).

It remains to show that $\det D^2u_0$ is positive and bounded. One computes for

$$x_2^2 = s^2h(x_1)^2e^{-2/x_1^\alpha}, \quad s^2 < 1,$$

that

$$\det D^2u_0(x_1,x_2) = 2\alpha^2\big((1-s^2)+(\alpha+1)x_1^\alpha(1+s^2)/\alpha\big)+O(x_1^{2\alpha}),$$

completing the proof. □

Barriers. We now construct barriers that agree with $|x_2|/|\log x_2|$ except for in very thin cusps around the x_1-axis where the Monge–Ampère measure is as large as we like. Let

$$h_\alpha(t) = \begin{cases} 0, & t \leq 0, \\ \frac{1}{2}e^{-1/t^\alpha}, & t > 0, \end{cases}$$

where $\alpha > 0$ is large. Let $\Omega_{1,\alpha} = \{|x_2| < h_\alpha(x_1)\}$ be a thin cusp around the positive x_1-axis and let $\Omega_{2,\alpha}$ be its complement. Our barrier is

$$b_\alpha(x_1,x_2) = \begin{cases} x_1^\alpha e^{-1/x_1^\alpha} + x_1^\alpha e^{1/x_1^\alpha}x_2^2 & \text{in } \Omega_{1,\alpha}, \\ \frac{5}{2}|x_2|/|\log 2x_2| & \text{in } \Omega_{2,\alpha}. \end{cases}$$

Note that b_α is convex and bounded by 1 on $\Omega_{2,\alpha} \cap \{|x_2| < \frac{1}{4}\}$, and b_α is continuous across $\partial\Omega_{1,\alpha}$. Furthermore, on $\partial\Omega_{1,\alpha}$ one computes (from inside $\Omega_{1,\alpha}$) that

$$\partial_1 b_\alpha(x_1,x_2) = \alpha e^{-1/x_1^\alpha}\big(\tfrac{3}{4}x_1^{-1} + \tfrac{5}{4}x_1^{\alpha-1}\big) \geq 0,$$

so the derivatives have positive jumps across $\partial\Omega_{1,\alpha}$.

Set $x_2^2e^{2/x_1^\alpha} = a$. One computes in $\Omega_{1,\alpha}$ (where $a \leq \frac{1}{4}$) that

$$\begin{aligned}\det D^2b_\alpha &= 2\alpha^2x_1^{-2}\left((1-a) + \frac{\alpha-1+a(3\alpha+1)}{\alpha}x_1^\alpha + \frac{\alpha-1-a(\alpha+1)}{\alpha}x_1^{2\alpha}\right) \\ &\geq \tfrac{3}{2}\alpha^2x_1^{-2}.\end{aligned}$$

Finally, let $\Omega := \big(-\infty, \frac{1}{2}\big] \times \big[-\frac{1}{4}, \frac{1}{4}\big]$. We conclude that b_α are convex in Ω, with

$$\det D^2b_\alpha \geq 6\alpha^2 \quad \text{in } \Omega_{1,\alpha} \cap \Omega,$$

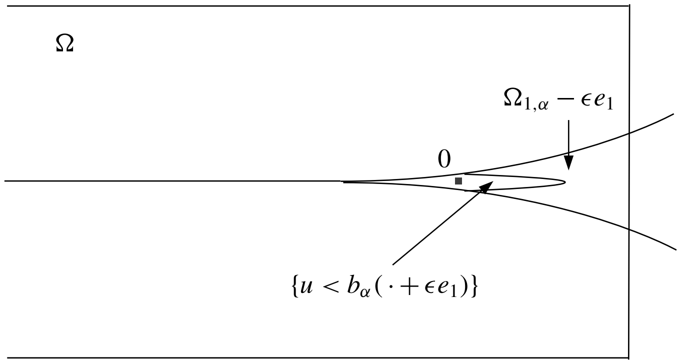

Figure 4. If $u > b_\alpha$ on the right edge of $\overline{\Omega_{1,\alpha}} \cap \Omega$, then we get a contradiction by sliding b_α to the left.

and furthermore

$$b_\alpha < \tfrac{5}{4} \cdot 2^{-\alpha} e^{-2^\alpha} \quad \text{for } \Omega_{1,\alpha} \cap \Omega.$$

Propagation. We prove Theorem 1.4 by sliding the barriers b_α from the right.

Proof of Theorem 1.4. By rescaling and multiplying by a constant, we may assume that

$$u \geq \tfrac{5}{2}|x_2|/|\log 2x_2| \quad \text{in } \left\{|x_2| < \tfrac{1}{4}\right\} \cap B_1,$$

with $u(0) = 0$ and $\det D^2 u < \Lambda$ for some large Λ. Choose α so large that $\alpha^2 > \Lambda$. Slide the barriers $b_\alpha(\cdot - te_1)$ from the right. Since $u \geq b_\alpha(\cdot - te_1)$ on $\partial(\Omega_{1,\alpha} + te_1) \cap \Omega$ for all $|t|$ small, it follows from the maximum principle that

$$u\left(\tfrac{1}{2}, x_2\right) \leq b_\alpha\left(\tfrac{1}{2}, x_2\right)$$

for some $\left(\tfrac{1}{2}, x_2\right) \in \overline{\Omega_{1,\alpha}} \cap \Omega$. (Indeed, if not, we can take $t = -\epsilon$ small and obtain

$$\{u < b_\alpha(\cdot + \epsilon e_1)\} \subset (\Omega_{1,\alpha} - \epsilon e_1) \cap \Omega,$$

which contradicts the Alexandrov maximum principle; see Figure 4). Taking $\alpha \to \infty$, we conclude that $u(e_1/2) = 0$.

By convexity, near each point on the x_1-axis where u is zero, there is a singularity of the same type as near the origin. We can apply the above argument at all such points to complete the proof. $\square$

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References

[Alexandrov 1942] A. Alexandroff, "Smoothness of the convex surface of bounded Gaussian curvature", *C. R.* (*Doklady*) *Acad. Sci. URSS* (*N.S.*) **36** (1942), 195–199. MR 0007626 Zbl 0061.37605

[Ambrosio et al. 2012] L. Ambrosio, M. Colombo, G. De Philippis, and A. Figalli, "Existence of Eulerian solutions to the semigeostrophic equations in physical space: the 2-dimensional periodic case", *Comm. Partial Differential Equations* **37**:12 (2012), 2209–2227. MR 3005541 Zbl 1258.35164

[Ambrosio et al. 2014] L. Ambrosio, M. Colombo, G. De Philippis, and A. Figalli, "A global existence result for the semi-geostrophic equations in three dimensional convex domains", *Discrete Contin. Dyn. Syst.* **34**:4 (2014), 1251–1268. MR 3117839 Zbl 1287.35064

[Daskalopoulos and Savin 2009] P. Daskalopoulos and O. Savin, "On Monge–Ampère equations with homogeneous right-hand sides", *Comm. Pure Appl. Math.* **62**:5 (2009), 639–676. MR 2494810 Zbl 1171.35341

[De Philippis and Figalli 2013] G. De Philippis and A. Figalli, "$W^{2,1}$ regularity for solutions of the Monge–Ampère equation", *Invent. Math.* **192**:1 (2013), 55–69. MR 3032325 Zbl 1286.35107

[De Philippis and Figalli 2014] G. De Philippis and A. Figalli, "The Monge–Ampère equation and its link to optimal transportation", *Bull. Amer. Math. Soc.* (*N.S.*) **51**:4 (2014), 527–580. MR 3237759

[De Philippis et al. 2013] G. De Philippis, A. Figalli, and O. Savin, "A note on interior $W^{2,1+\varepsilon}$ estimates for the Monge–Ampère equation", *Math. Ann.* **357**:1 (2013), 11–22. MR 3084340 Zbl 1280.35153

[Figalli 2015] A. Figalli, "Global existence for the semigeostrophic equations via Sobolev estimates for Monge–Ampère", lecture notes from CIME Summer Course on *Partial differential equations and geometric measure theory* (Cetraro, 2014), 2015, Available at http://cvgmt.sns.it/paper/2665/.

[Figalli and Loeper 2009] A. Figalli and G. Loeper, "C^1 regularity of solutions of the Monge–Ampère equation for optimal transport in dimension two", *Calc. Var. Partial Differential Equations* **35**:4 (2009), 537–550. MR 2496656 Zbl 1170.35400

[Guan 1997] P. Guan, "Regularity of a class of quasilinear degenerate elliptic equations", *Adv. Math.* **132**:1 (1997), 24–45. MR 1488238 Zbl 0892.35036

[Gutiérrez 2001] C. E. Gutiérrez, *The Monge–Ampère equation*, Progress in Nonlinear Differential Equations and their Applications **44**, Birkhäuser, Boston, 2001. MR 1829162 Zbl 0989.35052

[Schmidt 2013] T. Schmidt, "$\mathrm{W}^{2,1+\varepsilon}$ estimates for the Monge–Ampère equation", *Adv. Math.* **240** (2013), 672–689. MR 3046322 Zbl 1290.35136

[Wang 1995] X. J. Wang, "Some counterexamples to the regularity of Monge–Ampère equations", *Proc. Amer. Math. Soc.* **123**:3 (1995), 841–845. MR 1223269 Zbl 0822.35054

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