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NICOLA SOAVE AND ALESSANDRO ZILIO

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MULTIDIMENSIONAL ENTIRE SOLUTIONS FOR AN ELLIPTIC SYSTEM MODELLING PHASE SEPARATION

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For the system of semilinear elliptic equations

$$\Delta V_i = V_i \sum_{j\neq i} V_j^2, \quad V_i > 0 \text{ in } \mathbb{R}^N,$$

we devise a new method to construct entire solutions. The method extends the existence results already available in the literature, which are concerned with the 2-dimensional case, also to higher dimensions $N \geq 3$. In particular, we provide an explicit relation between orthogonal symmetry subgroups, optimal partition problems of the sphere, the existence of solutions and their asymptotic growth. This is achieved by means of new asymptotic estimates for competing systems and new sharp versions for monotonicity formulae of Alt–Caffarelli–Friedman type.

1. Introduction

The elliptic systems

$$\begin{cases} \Delta V_i = V_i \sum_{j\neq i} V_j^2, \\ V_i \geq 0, \end{cases} \quad \text{in } \mathbb{R}^N,\ i = 1, \dots, k, \tag{1-1}$$

which arise in the blow-up analysis of phase-separation phenomena in coupled Schrödinger equations, has attracted increasing attention in recent years, and by now many results concerning existence and qualitative properties of the solutions are available. For a detailed explanation about how (1-1) appears, we refer to [Berestycki et al. 2013a; 2013b; Soave and Zilio 2016]. We prove the existence of *N-dimensional solutions* to (1-1) in $\mathbb{R}^N$ for any $N \geq 2$. By this, we mean that we construct solutions in $\mathbb{R}^N$ which cannot be obtained from solutions in lower dimensions by adding a dependence on some "mute" variable. Our results extend the construction developed in [Berestycki et al. 2013b], which concerns the planar case $N = 2$. In this perspective, we mention that previous results contained in [Berestycki et al. 2013a; 2013b] only regard the existence of solutions in dimension $N = 1$ or 2, and the question of the existence in higher dimensions was up to now open.

In order to state our main results, we introduce some notation. We denote by $\mathcal{O}(N)$ the orthogonal group of $\mathbb{R}^N$ and by $\mathfrak{S}_k$ the symmetric group of permutations of $\{1, \dots, k\}$. Let us assume that there

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exists a homomorphism $h:\mathcal{G}\to\mathfrak{S}_k$, where $\mathcal{G}<\mathcal{O}(N)$ is a nontrivial subgroup. We define the *equivariant right action* of $\mathcal{G}$ on $H^1(\mathbb{R}^N,\mathbb{R}^k)$ in the following way:

$$\begin{aligned}&\mathcal{G}\times H^1(\mathbb{R}^N,\mathbb{R}^k)\to H^1(\mathbb{R}^N,\mathbb{R}^k),\\ &\qquad (g,\boldsymbol{u})\mapsto g\cdot\boldsymbol{u}:=(u_{(h(g))^{-1}(1)}\circ g,\dots,u_{(h(g))^{-1}(k)}\circ g),\end{aligned}\tag{1-2}$$

where $\circ$ denotes the usual composition of functions, and we used the vector notation $\boldsymbol{u}:=(u_1,\dots,u_k)$. The set

$$H_{(\mathcal{G},h)}:=\{\boldsymbol{u}\in H^1(\mathbb{R}^N,\mathbb{R}^k):\boldsymbol{u}=g\cdot\boldsymbol{u}\ \text{ for all }\ g\in\mathcal{G}\}$$

is the subspace of the $(\mathcal{G},h)$-equivariant functions.

Definition 1.1. For $k\in\mathbb{N}$, a nontrivial subgroup $\mathcal{G}<\mathcal{O}(N)$, and a homomorphism $h:\mathcal{G}\to\mathfrak{S}_k$, we write that the triplet $(k,\mathcal{G},h)$ is admissible if there exists a $(\mathcal{G},h)$-equivariant function $\boldsymbol{u}$ with the following properties:

(i) $u_i\ge 0$ and $u_i\not\equiv 0$ for every i;

(ii) $u_iu_j\equiv 0$ for every $i\ne j$;

(iii) there exist $g_2,\dots,g_k\in\mathcal{G}$ such that

$$u_i=u_1\circ g_i\quad\text{for }\ i=2,\dots,k.$$

Remark 1.2. Notice that, if $(k,\mathcal{G},h)$ is admissible triplet, then all the $(\mathcal{G},h)$-equivariant functions satisfy (iii) in the previous definition with the same symmetries g_i; indeed, by (iii) and equivariance we deduce that $(h(g_i))^{-1}(i)=1$ for every i, so that any equivariant function satisfies

$$v_i=v_{(h(g_i))^{-1}(i)}\circ g_i=v_1\circ g_i\quad\text{for all }\ i=1,\dots,k.\tag{1-3}$$

This tells us that any equivariant function associated to an admissible triplet is completely determined by its first component: if we know that $\boldsymbol{v}$ is $(\mathcal{G},h)$-equivariant and that $(k,\mathcal{G},h)$ is an admissible triplet, then (1-3) holds true, and hence $v_2,\dots,v_k$ can be obtained by knowing v_1 and $g_2,\dots,g_k$.

We also underline the fact that there may exist symmetries in $\mathcal{G}$ whose corresponding permutation is the identity. In this case, these symmetries are imposed on the single components.

Finally, we observe that the definition of admissible triplet implicitly imposes several restrictions on $(k,\mathcal{G},h)$. For instance, by (iii) we immediately deduce that h can never be the trivial homomorphism $\mathcal{G}\to\mathfrak{S}_k$, $g\mapsto\text{id}$ for all g. Moreover, we also deduce that $\mathcal{G}$ has at least k different elements.

Let $(k,\mathcal{G},h)$ be an admissible triplet. We let $\Lambda_{(\mathcal{G},h)}$ be the set of those $\varphi\in H^1(\mathbb{S}^{N-1},\mathbb{R}^k)$ such that

$$\varphi\text{ is the restriction on }\mathbb{S}^{N-1}\text{ of a }(\mathcal{G},h)\text{-equivariant function fulfilling Definition 1.1(i)–(iii).}\tag{1-4}$$

We consider the minimization problem

$$\ell_{(k,\mathcal{G},h)}:=\inf_{\varphi\in\Lambda_{(\mathcal{G},h)}}\frac{1}{k}\sum_{i=1}^k\left(\sqrt{\left(\frac{N-2}{2}\right)^2+\frac{\int_{\mathbb{S}^{n-1}}|\nabla_\theta\varphi_i|^2}{\int_{\mathbb{S}^{n-1}}\varphi_i^2}}-\frac{N-2}{2}\right),\tag{1-5}$$

where ∇_θ denotes the tangential gradient on $\mathbb{S}^{N-1}$.

Theorem 1.3. *For any admissible pair* $(k, \mathcal{G}, h)$, *there exists a solution* $\boldsymbol{V}$ *of* (1-1) *with* k *components in* $\mathbb{R}^N$ *satisfying the following properties*:

- $\boldsymbol{V}$ *is* $(\mathcal{G}, h)$*-equivariant*;
- *we have*

$$\lim_{r\to+\infty} \frac{1}{r^{N-1+2\ell(k,\mathcal{G},h)}} \int_{\partial B_r} \sum_{i=1}^{k} V_i^2 \in (0, +\infty). \tag{1-6}$$

Here and in the rest of the paper $B_r(x_0)$ denotes the ball of centre x_0 and radius r; when $x_0 = 0$, we simply write B_r for the sake of simplicity.

Since the theorem is quite general, we think that it is worthwhile to spend some time making some explicit examples. This will be done in Section 2.1. For the moment, we anticipate that with our result we can recover Theorems 1.3 and 1.6 in [Berestycki et al. 2013b], and moreover we can produce a wealth of new solutions existing only in dimensions $N \geq 3$.

We also observe that condition (1-6) establishes that the solution $\boldsymbol{V}$ grows at infinity, in quadratic mean, like the power $|x|^{\ell(k,\mathcal{G},h)}$. It is worth remarking that for any solution $\boldsymbol{V}$ to (1-1) it is possible to define the *growth rate* as the uniquely determined value $d \in (0, +\infty]$ such that

$$\lim_{r\to+\infty} \frac{1}{r^{N-1+2m}} \int_{\partial B_r} \sum_{i=1}^{k} V_i^2 = \begin{cases} +\infty & \text{if } m < d, \\ 0 & \text{if } m > d; \end{cases}$$

see Proposition 1.5 in [Soave and Terracini 2015] and its proof. Therein, it is also shown that $\boldsymbol{V}$ *has algebraic growth*, i.e., it satisfies the pointwise upper bound

$$V_1(x) + \cdots + V_k(x) \leq C(1 + |x|^{\alpha}) \quad \text{for all } x \in \mathbb{R}^N \tag{1-7}$$

for some $C, \alpha \geq 1$, if and only if its growth rate d is finite; we point out moreover that, as shown in [Soave and Zilio 2014], the system does indeed admit solutions with exponential (i.e., nonalgebraic) growth.

Theorem 1.3 not only specifies the growth rate of the function ($d = \ell(k, \mathcal{G}, h)$), but also states that, for this precise growth rate, the limit

$$\lim_{r\to+\infty} \frac{1}{r^{N-1+2d}} \int_{\partial B_r} \sum_{i=1}^{k} V_i^2$$

is positive and finite. In this perspective we can prove that the solutions of Theorem 1.3 have minimal growth rate among all the possible $(\mathcal{G}, h)$-equivariant solutions.

Theorem 1.4. *Let* $(k, \mathcal{G}, h)$ *be an admissible pair and let* $\boldsymbol{V}$ *be a* $(\mathcal{G}, h)$*-equivariant solution of* (1-1). *Then the growth rate of* $\boldsymbol{V}$ *is at least* $\ell(k, \mathcal{G}, h)$.

Both the proofs of Theorems 1.3 and 1.4 exploit the hidden relationship between the elliptic system (1-1) and optimal partition problems of type (1-5). This relationship arises for instance by means of the validity of the following modification of the celebrated Alt–Caffarelli–Friedman monotonicity formula, tailor-made for the study of $(\mathcal{G}, h)$-equivariant solutions.

For $\boldsymbol{V} \in H^1(\mathbb{R}^N, \mathbb{R}^k)$ and $i = 1, \dots, k$ we define

$$J_i(r) := \int_{B_r} \frac{|\nabla V_i|^2 + V_i^2 \sum_{j \neq i} V_j^2}{|x|^{N-2}}.$$

Proposition 1.5. *Let $(k, \mathcal{G}, h)$ be an admissible triplet. There exists a constant $C > 0$ depending only on N and $(k, \mathcal{G}, h)$ such that, for any $(\mathcal{G}, h)$-equivariant solution $\boldsymbol{V}$ of* (1-1), *the function*

$$r \mapsto \frac{1}{r^{2k\ell(k,\mathcal{G},h)}} e^{-Cr^{-1/2}} J_1(r) \cdots J_k(r)$$

is monotone nondecreasing for $r > 1$ (we recall that $\ell(k, \mathcal{G}, h)$ has been defined in (1-5)).

The expert reader will have already recognized the similarity with the original Alt–Caffarelli–Friedman monotonicity formula, proved in [Alt et al. 1984]; monotonicity formulae of Alt–Caffarelli–Friedman type for competing systems are key ingredients for the results in [Conti et al. 2005; Farina and Soave 2014; Noris et al. 2010; Soave and Terracini 2015; Soave and Zilio 2015; Wang 2014]. The previous result is, to our knowledge, the first example of a monotonicity formula under a symmetry constraint.

We review now the main known results regarding entire solutions of the system (1-1) which were already available, starting with the system with $k = 2$ components. The 1-dimensional problem was studied in [Berestycki et al. 2013a], where it is proved that there exists a solution satisfying the symmetry property $V_2(x) = V_1(-x)$, the monotonicity condition $V_1' > 0$ and $V_2' < 0$ in $\mathbb{R}$, and having at most linear growth, in the sense that there exists $C > 0$ such that

$$V_1(x) + V_2(x) \leq C(1 + |x|) \quad \text{for all } x \in \mathbb{R}^N.$$

Up to translations, scaling and exchange of the components, this is the unique solution in dimension $N = 1$; see [Berestycki et al. 2013b, Theorem 1.1]. The linear growth is the minimal admissible growth for nonconstant positive solutions of (1-1). Indeed, in any dimension $N \geq 1$, if (V_1, V_2) is a *nonnegative* solution of (1-1) (which means that the condition $V_i > 0$ is replaced by $V_i \geq 0$) and satisfies the sublinear growth condition

$$V_1(x) + V_2(x) \leq C(1 + |x|^\alpha) \quad \text{in } \mathbb{R}^N$$

for some $\alpha \in (0, 1)$ and $C > 0$, then one of V_1 and V_2 is 0 and the other has to be constant. This *Liouville-type theorem* has been proved by B. Noris et al. [2010, Proposition 2.6].

Differently from the problem in $\mathbb{R}$, in dimension $N = 2$, and hence in any dimension $N \geq 2$, the system (1-1) with $k = 2$ has infinitely many "geometrically distinct" solutions, i.e., solutions which cannot be obtained from each other by means of rigid motions, scalings or exchange of the components; see [Berestycki et al. 2013b, Theorem 1.3; Soave and Zilio 2014, Theorems 1.1 and 1.5]. These solutions can be distinguished according to their growth rates and symmetry properties. In particular, Berestycki et al. [2013b] proved the existence of solutions having algebraic growth, while the results in [Soave and Zilio 2014] concern solutions having exponential growth in x that are periodic in y.

Regarding systems with several components, the aforementioned existence results admit analogous counterparts for any $k \geq 3$; see [Berestycki et al. 2013b, Theorem 1.6; Soave and Zilio 2014, Theorem 1.8].

It is important to stress that the proofs in [Berestycki et al. 2013b; Soave and Zilio 2014] use the fact that the problem is posed in dimension $N = 2$, and apparently cannot be extended to higher dimensions (see Remark 4.4 for a more detailed discussion).

In parallel to the existence results, great efforts have been devoted to the analysis of the 1-dimensional symmetry of solutions under suitable assumptions; this, as explained in [Berestycki et al. 2013a], is inspired by some analogy with the derivation of (1-1) and of the Allen–Chan equation, for which symmetry results in the spirit of the celebrated De Giorgi's conjecture have been widely studied. In this context, we recall that, assuming $k = 2$ and $N = 2$, A. Farina [2014] proved that, if (V_1, V_2) has algebraic growth and $\partial_2 V_1 > 0$ in $\mathbb{R}^2$, then (V_1, V_2) is 1-dimensional. In the higher-dimensional case $N \geq 2$ with $k = 2$, Farina and the first author proved a Gibbons-type conjecture for (1-1); see [Farina and Soave 2014]. Furthermore, K. Wang [2014; 2015], as a product of his main results, showed that any solution of (1-1) with $k = 2$ having linear growth is 1-dimensional. We mention also [Berestycki et al. 2013a, Theorem 1.8; 2013b, Theorem 1.12], which are now included in Wang's result.

As far as the 1-dimensional symmetry for systems with $k > 2$ is concerned, we refer to [Soave and Terracini 2015, Theorem 1.3], where the main results in [Farina and Soave 2014; Wang 2014; 2015] are extended to systems with many components by means of improved Liouville-type theorems for multicomponent systems, which relate the number of nontrivial components of a nonnegative solution of the first equation in (1-1) and its growth rate. In this perspective, Theorem 1.4 is the counterpart of [Soave and Terracini 2015, Theorem 1.7] in a $(\mathcal{G}, h)$-equivariant setting. As a product of these two results, we can also derive the following:

Corollary 1.6. *For $k, N \in \mathbb{N}$, let*

$$\mathcal{L}_k(\mathbb{S}^{N-1}) := \inf_{(\omega_1,\ldots,\omega_k)\in\mathcal{P}_k} \sup_{i=1,\ldots,k} \lambda_1(\omega_i),$$

where $\mathcal{P}_k$ is the set of partitions of $\mathbb{S}^{N-1}$ in k open disjoint and connected sets, and λ_1 denotes the first eigenvalue of the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$. Also, let $(k, \mathcal{G}, h)$ be any admissible triplet with $\mathcal{G} < \mathcal{O}(N)$. Then

$$\mathcal{L}_k(\mathbb{S}^{N-1}) \leq \ell(k, \mathcal{G}, h).$$

It is tempting to conjecture that equality holds for an appropriate choice of $(\mathcal{G}, h)$, at least for some values of k, N. Indeed, in light of the known results in the literature, this is the case for $k = 2$ and $k = 3$, for every N. For $k = 2$, the only (up to isometries) optimal partition for $\mathcal{L}_2(\mathbb{S}^{N-1}) = 1$ is the partition of the sphere into two equal spherical cups [Alt et al. 1984]. This is clearly also an optimal partition for $\ell(2, \mathcal{G}, h)$ if $\mathcal{G}$ is equal to the group generated by the reflection T with respect to a hyperplane through the origin and $h(T)$ is defined as the permutation exchanging the indices 1 and 2. In the case $k = 3$, an optimal partition for $\mathcal{L}_3(\mathbb{S}^{N-1}) = \frac{3}{2}\left(N - \frac{1}{2}\right)$ is the so-called $\boldsymbol{Y}$-partition (see [Helffer et al. 2010; Soave and Terracini 2015]) which is then optimal also for $\ell(3, \mathcal{G}, h)$ if $\mathcal{G}$ is equal to the group generated by the rotation R of angle $\frac{2}{3}\pi$ around the x_N axis and $h(R)$ is the permutation mapping 1 into 2, 2 into 3 and 3 into 1.

To conclude, we mention also the contribution of Wang and Wei [2014], who considered the fractional analogue of (1-1). Such problems exhibit new interesting phenomena with respect to the local case. Moreover, we observe that our results, as those in [Berestycki et al. 2013b], seem to be somehow connected with those in [Wei and Weth 2007], which concern finite energy decaying solutions of a different problem.

Structure of the paper. in Section 2 we recall some known results needed for the rest of work, and which permit us to show, in Section 2.1, several concrete applications of Theorem 1.3. Section 3 is devoted to the proof of the equivariant Alt–Caffarelli–Friedman monotonicity formula, Proposition 1.5; finally, in Section 4, we give the proofs of the other main results, Theorems 1.3 and 1.4.

2. Preliminaries and application of Theorem 1.3

We introduce some notation and review some known results. Let $\beta > 0$, and let $\boldsymbol{U}$ be a solution to

$$\begin{cases} \Delta U_i = \beta U_i \sum_{j\neq i} U_j^2 & \text{in } B_R, \\ U_i > 0 & \text{in } B_R. \end{cases} \tag{2-1}$$

For $0 < r < R$, we set

$$\begin{aligned} H(\boldsymbol{U}, r) &:= \frac{1}{r^{N-1}} \int_{\partial B_r} \sum_{i=1}^k U_i^2, \\ E(\boldsymbol{U}, r) &:= \frac{1}{r^{N-2}} \int_{B_r} \sum_{i=1}^k |\nabla U_i|^2 + \beta \sum_{1\le i<j\le k} U_i^2 U_j^2, \\ N(\boldsymbol{U}, r) &:= \frac{E(\boldsymbol{U}, r)}{H(\boldsymbol{U}, r)} \quad \text{(the Almgren frequency function).} \end{aligned}$$

Under the previous notation, by Proposition 5.2 in [Berestycki et al. 2013b] it is known that $N(\boldsymbol{U}, \cdot)$ is monotone nondecreasing for $0 < r < R$,

$$\frac{d}{dr} H(\boldsymbol{U}, r) = \frac{2}{r} E(\boldsymbol{U}, r) + \frac{2\beta}{r^{N-1}} \int_{B_r} \sum_{i<j} U_i^2 U_j^2 > 0,$$

and, for any such r,

$$\int_1^r 2\beta \frac{\int_{B_s} \sum_{i<j} U_i^2 U_j^2}{s^{N-1} H(\boldsymbol{U}, s)} \, ds \le N(\boldsymbol{U}, r). \tag{2-2}$$

The frequency function, also called Almgren's quotient, gives information about the behaviour of the solutions with respect to radial dilations. Indeed, the possibility of defining a growth rate for any solution to (1-1) is a direct consequence of the monotonicity of $N(\boldsymbol{V}, \cdot)$. We recall that, as proved in [Soave and Terracini 2015, Proposition 1.5], for any solution $\boldsymbol{V}$ to (1-1) there exists a value $d \in (0, +\infty]$ such that

$$\lim_{r\to+\infty} \frac{(1/r^{N-1}) \int_{\partial B_r} \sum_{i=1}^k V_i^2}{r^{2d'}} = \begin{cases} +\infty & \text{if } d' < d, \\ 0 & \text{if } d' > d, \end{cases} \tag{2-3}$$

and $d < +\infty$ if and only if $\boldsymbol{V}$ has algebraic growth. We write that d is the *growth rate of* $\boldsymbol{V}$, and it is remarkable that

$$d = \lim_{r\to+\infty} N(\boldsymbol{V}, r); \tag{2-4}$$

again see [Soave and Terracini 2015, Proposition 1.5] (the result is stated in [Soave and Terracini 2015] for solutions with algebraic growth, but its proof works also without this assumption). Notice that on the left-hand side of (2-3) we have the quadratic average of $\boldsymbol{V}$ on spheres of increasing radius divided by a power of r^2; thus the name *growth rate*.

In the previous discussion $\beta > 0$ was fixed. Let us now consider a sequence of parameters $\beta \to +\infty$ and a corresponding sequence $\{\boldsymbol{U}_\beta\}$ of solutions to (2-1). The asymptotic behaviour of the family $\{\boldsymbol{U}_\beta\}$ has been studied in [Berestycki et al. 2013a; Dancer et al. 2012; Noris et al. 2010; Soave and Zilio 2015; 2016; Tavares and Terracini 2012; Wei and Weth 2008] and many results are available. We only recall that, if the sequence is bounded in $L^\infty(B_R)$, then it is in turn uniformly bounded in $\mathrm{Lip}(B_R)$, and hence up to a subsequence it converges to a limit $\boldsymbol{U}$ in $\mathcal{C}^{0,\alpha}(B_R)$ and in $H^1_{\mathrm{loc}}(B_R)$ (see [Soave and Zilio 2015; Noris et al. 2010]). If $\boldsymbol{U} \not\equiv \boldsymbol{0}$, then $\boldsymbol{U}$ is Lipschitz continuous and $\{\boldsymbol{U} = \boldsymbol{0}\}$ has Hausdorff dimension $N-1$. Moreover, $H(\boldsymbol{U}, r)$ is nondecreasing and is nonzero for every $r > 0$ (see [Tavares and Terracini 2012]).

An important application of this asymptotic theory lies in the possibility of defining blow-down limits of entire solutions to (1-1). We recall part of [Berestycki et al. 2013b, Theorem 1.4] ($k = 2$) and [Soave and Terracini 2015, Theorem 1.4] (k arbitrary). Let $\boldsymbol{V}$ be a solution to (1-1), and for any $R > 0$ let us define the *blow-down family*

$$\boldsymbol{V}_R(x) := \frac{1}{H(\boldsymbol{V}, R)^{1/2}} \boldsymbol{V}(Rx).$$

If $\boldsymbol{V}$ has algebraic growth, i.e., its growth rate $d = N(\boldsymbol{V}, +\infty)$ is finite, then $\{\boldsymbol{V}_R\}$ converges, in $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$, as $R \to +\infty$ and up to a subsequence, to a homogeneous vector-valued function $\boldsymbol{V}_\infty$ with homogeneity degree d and such that

- the components $V_{i,\infty}$ are nonnegative and with disjoint support: $V_{i,\infty} V_{j,\infty} \equiv 0$ for every $i \neq j$;
- $V_{i,\infty} - V_{j,\infty}$ for any $i \neq j$ is harmonic in the interior of its support.

When $k = 2$, it then results that $(V_{1,\infty}, V_{2,\infty}) = (\Psi^+, \Psi^-)$, where Ψ is a homogenous harmonic polynomial in $\mathbb{R}^N$, and hence necessarily d is an integer.

2.1. ***A wealth of new solutions: applications of Theorem 1.3.*** We recall that, for any $k \geq 2$, problem (1-1) has several solutions in $\mathbb{R}^2$. Clearly, these are also solutions in higher dimensions, and up to now it was an open question whether or not there exist N-dimensional solutions of (1-1) in $\mathbb{R}^N$ with $N \geq 3$, i.e., solutions in $\mathbb{R}^N$ which cannot be obtained as solutions in $\mathbb{R}^{N-1}$ by adding a dependence on a variable. Theorem 1.3 gives a positive answer to these questions. In what follows we show how to use Theorem 1.3 as a recipe to construct entire solutions of (1-1).

A concrete example in $\mathbb{R}^3$ for $k = 2$. To start with a very concrete example, we focus on problem (1-1) in $\mathbb{R}^3$ with $k = 2$, and we examine the case where $\mathcal{G}$ is equal to the group of symmetries generated by the reflections T_1, T_2 and T_3 with respect to the planes $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$, respectively, and $h : \mathcal{G} \to \mathfrak{S}_k$ is defined on the generators of $\mathcal{G}$ by $h(T_i) = (1\ 2)$ for every i. We used here the standard notation $(1\ 2)$ to denote the cycle mapping 1 to 2, and 2 to 1. In order to check that this is an admissible triplet, we verify that

$$(u_1, u_2) = ((xyz)^+, (xyz)^-)$$

is a $(\mathcal{G}, h)$-equivariant function satisfying (i)–(iii) in Definition 1.1. For the equivariance, we explicitly observe that

$$\begin{aligned} T_i \cdot (u_1, u_2) &= (u_2 \circ T_i, u_1 \circ T_i) \quad && \text{(see (1-2))} \\ &= (u_1, u_2) && \text{(by definition of } \boldsymbol{u}\text{)} \end{aligned}$$

for every i, and since $\mathcal{G}$ is generated by T_1, T_2, T_3, this is sufficient to conclude that $\boldsymbol{u}$ is $(\mathcal{G}, h)$-equivariant. Points (i) and (ii) in Definition 1.1 are straightforward, and (iii) is satisfied since $u_2 = u_1 \circ T_i$ for any i. As a consequence, by Theorem 1.3 there exists a $(\mathcal{G}, h)$-equivariant solution (V_1, V_2) of (1-1) in $\mathbb{R}^3$ with $k = 2$ having growth rate equal to $\ell(k, \mathcal{G}, h) = N(\boldsymbol{V}, +\infty)$ (we recall that the growth rate is always equal to the limit at infinity of the Almgren frequency function; see (2-4)). Since the symmetries of $\mathcal{G}$ involve the 3 variables, this solution cannot be obtained by a 2-dimensional solution adding the dependence of 1 variable: $V_1 - V_2$ is not constant since $\boldsymbol{V}$ has growth rate $\ell(2, \mathcal{G}, h) > 0$; moreover, thanks to the symmetries T_1, T_2, T_3, we have that the function $V_1 - V_2$ vanishes on the set $\{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$. Since the projection of this set on any 2-dimensional subspace is equal to the entire subspace but $\boldsymbol{V}$ is nontrivial, we immediately deduce that the solution cannot be 2-dimensional.

In this particular case we can also explicitly compute $\ell(2, \mathcal{G}, h)$, in the following way: by minimality,

$$\ell(2, \mathcal{G}, h) \le \frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{\int_{\mathbb{S}^2} |\nabla_\theta (xyz)^+|^2}{\int_{\mathbb{S}^2} |(xyz)^+|^2}} - \frac{1}{2}\right) + \frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{\int_{\mathbb{S}^2} |\nabla_\theta (xyz)^-|^2}{\int_{\mathbb{S}^2} |(xyz)^-|^2}} - \frac{1}{2}\right),$$

and the right-hand side is equal to 3; indeed, since $\Phi := xyz$ is a homogeneous harmonic polynomial of degree 3, its angular part $\Phi|_{\mathbb{S}^2}$ solves

$$-\Delta_\theta \Phi|_{\mathbb{S}^2} = 12 \Phi|_{\mathbb{S}^2} \quad \text{in } \mathbb{S}^2,$$

and this permits us to carry out explicit computations. This means that Ψ (the blow-down limit) is a homogeneous harmonic polynomial of degree $\ell(2, \mathcal{G}, h) \le 3$. It is then necessary that $\Psi = \Phi = xyz$; to check this, we can simply consider all the homogeneous harmonic polynomials in $\mathbb{R}^3$ with degree at most 3, which have been classified, and observe that the only one being $(\mathcal{G}, h)$ equivariant is Φ. As a consequence, the degree of homogeneity of Ψ is $3 = \ell(2, \mathcal{G}, h)$.

General case in $\mathbb{R}^N$ with $k = 2$. The very same argument as before can be considered by taking any homogeneous harmonic polynomial Φ in $\mathbb{R}^N$ of degree $d \in \mathbb{N}$ with a nontrivial finite group of symmetries $\mathcal{G}$; by this we mean that there exists a group of symmetries with generators $T_1, \dots, T_m$ such that $\Phi^\pm \circ T_i = \Phi^\mp$. To any T_i we associate the cycle $(1\ 2)$. This induces a homomorphism $h : \mathcal{G} \to \mathfrak{S}_2$, and it is not difficult to check that $(2, \mathcal{G}, h)$ is an admissible triplet. Indeed, by assumption the pair $(u_1, u_2) = (\Phi^+, \Phi^-)$ fulfills (i)–(iii) in Definition 1.1, and is $(\mathcal{G}, h)$-equivariant: the equivariance follows by

$$\begin{aligned} T_i \cdot (u_1, u_2) &= (u_2 \circ T_i, u_1 \circ T_i) \quad \text{(see (1-2))} \\ &= (u_1, u_2) \end{aligned}$$

for any i. Points (i) and (ii) in Definition 1.1 are trivial, and (iii) is satisfied since $u_2 = u_1 \circ T_i$ for any i by assumption. If, as in the example above, the group $\mathcal{G}$ is chosen from the beginning so that the symmetries of $\mathcal{G}$ involve all the N variables, we obtain an N-dimensional solution to (1-1). Explicit cases where the previous argument is applicable are the following:

- At first, we show how we can recover Theorem 1.3 in [Berestycki et al. 2013b]. In dimension $N = 2$, we take $\Phi_d(x, y) := \mathfrak{Re}((x + iy)^d)$, with $d \in \mathbb{N}$. Then Φ_d is symmetric, in the previous sense, with respect to the group of symmetries generated by the reflections $T_1, \dots, T_d$ with respect to its nodal lines: $\Phi_d^{\pm} \circ T_i = \Phi_d^{\mp}$. By the previous argument, we find $(\mathcal{G}, h)$-equivariant solutions of the problem with growth rate $\ell(2, \mathcal{G}, h)$, which clearly are 2-dimensional. Reasoning as in our first example, it is not difficult in this case to check that $\ell(2, \mathcal{G}, h) = d$.

- Secondly, we construct infinitely many new solutions in $\mathbb{R}^3$. We take $\Phi_d(x, y) := \mathfrak{Re}((x + iy)^d)z$, with $d \in \mathbb{N}$. Let $T_1, \dots, T_d$ denote the reflections with respect to the nodal planes of $\mathfrak{Re}((x + iy)^d)$, and let T_z denote the reflection with respect to $\{z = 0\}$. Then $\Phi_d^{\pm} \circ T_i = \Phi_d^{\mp}$, so that the general argument above is applicable, and hence we find a $(\mathcal{G}, h)$-equivariant solution of (1-1) with growth rate $\ell(2, \mathcal{G}, h)$. As in the first example, since the nodal set of $V_1 - V_2$ has surjective projection on any 2-dimensional subspace, $\boldsymbol{V}$ is necessarily 3-dimensional. We can also check that $\ell(2, \mathcal{G}, h) = d + 1$. Since (Φ_d^+, Φ_d^-) is a $(\mathcal{G}, h)$-equivariant function, we have

$$\ell(2, \mathcal{G}, h) \le \frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{\int_{\mathbb{S}^2} |\nabla_\theta \Phi_d^+|^2}{\int_{\mathbb{S}^2} |\Phi_d^+|^2}} - \frac{1}{2}\right) + \frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{\int_{\mathbb{S}^2} |\nabla_\theta \Phi_d^-|^2}{\int_{\mathbb{S}^2} |\Phi_d^-|^2}} - \frac{1}{2}\right).$$

As in the previous example, we can prove that the right-hand side is equal to $d+1$. On the other hand, using the blow-down theorem and explicitly observing that the only $(\mathcal{G}, h)$-equivariant homogeneous harmonic polynomial in $\mathbb{R}^3$ with degree less than or equal to $d + 1$ is Φ_d, we conclude that $\ell(2, \mathcal{G}, h) = d + 1$.

- We conclude with the observation that the previous constructions can be extended in any dimensions. For instance we can consider the harmonic polynomial $\Phi = x_1 \cdots x_N$, together with the symmetry group generated by the reflections $T_1, \dots, T_N$ with respect to the coordinate planes $\{x_i = 0\}$, $i = 1, \dots, N$; notice that $\Phi^{\pm} \circ T_i = \Phi^{\mp}$ for any i. In the same way we could consider the harmonic polynomial $\Psi = \mathfrak{Re}((x_1 + ix_2)^d)x_3 \cdots x_N$, together with symmetry group generated by the reflections $T_1, \dots, T_d$ with respect to the nodal hyperplanes of $\mathfrak{Re}((x_1 + ix_2)^d)$, and by $R_3, \dots, R_N$, reflections with respect to the coordinate planes $\{x_i = 0\}$, $i = 3, \dots, N$.

The case $k \geq 3$ in $\mathbb{R}^2$. For $k \geq 3$ components, we first show how to recover Theorem 1.6 in [Berestycki et al. 2013b]. We thus focus for the moment on the dimension $N = 2$. Let $k \geq 3$ and, for any $m \in \mathbb{N}$, let $d = \frac{1}{2}mk$. We denote by R_d the rotation of angle π/d, by T_y the reflection with respect to $\{y = 0\}$

(this corresponds to complex conjugation in $\mathbb{C}$), and we consider the group $\mathcal{G} < \mathcal{O}(N)$ generated by R_d and T_y. We define a homomorphism $h : \mathcal{G} \to \mathfrak{S}_k$ (the group of permutations of $\{1, \dots, k\}$) letting

$$h(R_d) := (1\ 2\ \cdots\ d) \quad \text{and} \quad h(T_y) : i \mapsto k + 2 - i,$$

where the indexes are counted modulus k. We can explicitly check that $(k, \mathcal{G}, h)$ is an admissible triplet. Let us consider the function

$$\begin{aligned} u_1 &:= \begin{cases} r^d \cos(d\theta) & \text{in } \bigcup_{i=0}^{m-1} R_d^{ik}(\{-\pi/2d < \theta < \pi/2d\}), \\ 0 & \text{otherwise}, \end{cases} \\ u_2 &:= u_1 \circ R_d, \\ &\ \vdots \\ u_k &:= u_{k-1} \circ R_d = u_1 \circ R_d^{k-1}. \end{aligned}$$

It is $(\mathcal{G}, h)$-equivariant, as

$$\begin{aligned} R_d \cdot \boldsymbol{u} &= (u_k \circ R_d, u_1 \circ R_d, \dots, u_{k-1} \circ R_d) = \boldsymbol{u}, \\ T_y \cdot \boldsymbol{u} &= (u_1 \circ T_y, u_k \circ T_y, u_{k-1} \circ T_y, \dots, u_3 \circ T_y, u_2 \circ T_y) = \boldsymbol{u}. \end{aligned}$$

It clearly satisfies (i) and (ii) in Definition 1.1, and for (iii) it is sufficient to note that $u_j = u_1 \circ R_d^{j-1}$ for every $j = 2, \dots, k$. By Theorem 1.3, we obtain a $(\mathcal{G}, h)$-equivariant solution $\boldsymbol{V}$ of (1-1); the fact that $\boldsymbol{V}$ is 2-dimensional follows again from the symmetries: if $\boldsymbol{V}$ were 1-dimensional, then we could say that $\bigcup_{i \neq j}\{V_i - V_j = 0\}$ is the union of straight parallel lines. But, on the other hand, $\{V_2 - V_3 = 0\} = R_d(\{V_1 - V_2 = 0\})$, which cannot be parallel whenever $d > 1$, i.e., whenever $k \geq 3$.

To complete the analogy with the results in [Berestycki et al. 2013b], we still would have to prove that $N(\boldsymbol{V}, +\infty) = \ell(k, \mathcal{G}, h)$ is equal to d. Since we are in dimension $N = 2$, this can be done by means of explicit computations, following the line of reasoning already adopted in the previous examples. We decided to not stress this point for the sake of brevity.

The general case $k \geq 3$ in $\mathbb{R}^3$. The case $k \geq 3$ and $N \geq 3$ is intrinsically more involved, and hence we focus on some particular examples given by the groups of symmetries of the Platonic polyhedra. Let us consider for instance the group $\mathcal{G}_4 < \mathcal{O}(N)$ associated to the tetrahedron $\mathcal{T}$. It is known that this group is isomorphic to $\mathfrak{S}_4$. The isomorphism h_4 is obtained labelling all the vertices of $\mathcal{T}$, and associating to any $g \in \mathcal{G}_4$ the permutation induced on the vertices themselves. In order to define the function φ satisfying (i)–(iii) of Definition 1.1, we first take a tetrahedron with barycentre 0, and define on a face A a positive function $\tilde{\varphi}_1$ that is 0 on ∂A and symmetric with respect to all the transformations in $\mathcal{G}_4$ leaving A invariant. By rotation, we can define $\tilde{\varphi}_2$, $\tilde{\varphi}_3$ and $\tilde{\varphi}_4$ on the remaining faces. Now, considering the radial projection of the tetrahedron into the unit sphere $\mathbb{S}^2$, we obtain a function $(\varphi_1, \dots, \varphi_4)$ whose 1-homogeneous extension is by construction $(\mathcal{G}_4, h_4)$-equivariant, and satisfies (i)–(iii) of Definition 1.1. Thus $(4, \mathcal{G}_4, h_4)$ is an admissible triplet, and Theorem 1.3 yields the existence of a $(\mathcal{G}_4, h_4)$-equivariant solution for the system with 4 components in $\mathbb{R}^3$. Since the symmetries of the tetrahedron involve the dependence on 3 variables, this solution is not 2-dimensional.

In a similar way, one can construct $(\mathcal{G}_6, h_6)$-equivariant solutions with respect to the group of symmetries of the cube $\mathcal{G}_6$ (isomorphic to a subgroup of $\mathfrak{S}_8$ through an isomorphism h_6) for systems with $k = 3$ or $k = 6$ components. To this purpose, we consider a cube with barycentre 0 in $\mathbb{R}^3$, and we define on a face A a positive function $\tilde{\varphi}_1$ that is 0 on ∂A and symmetric with respect to all the transformations in $\mathcal{G}_6$ leaving A invariant. By rotation, we can define $\tilde{\varphi}_2, \dots, \tilde{\varphi}_6$ on the remaining faces. Considering the radial projection of the cube onto the unit sphere $\mathbb{S}^2$, we obtain a function $(\varphi_1, \dots, \varphi_6)$ whose 1-homogeneous extension is $(\mathcal{G}_6, h_6)$-equivariant and satisfies (i)–(iii) of Definition 1.1. Theorem 1.3 then gives a 3-dimensional $(\mathcal{G}_6, h_6)$-equivariant solution to (1-1) with 6 components in $\mathbb{R}^3$. In order to obtain a 3-component $(\mathcal{G}_6, h_6)$-equivariant solution, we proceed as in the previous discussion replacing $\tilde{\varphi}_1$ with $\tilde{\psi}_1 = \tilde{\varphi}_1 + \tilde{\varphi}_4$, where φ_4 has support on the face opposite to A in the cube. By rotation, we determine $\tilde{\psi}_2$ and $\tilde{\psi}_3$, each of them supported on the union of two opposite faces. As before, we can then consider the radial projection onto $\mathbb{S}^2$, and afterwards its 1-homogeneous extension (ψ_1, ψ_2, ψ_3), which is $(\mathcal{G}_6, h_6)$-equivariant and satisfies (i)–(iii) of Definition 1.1. For the equivariance, we recall that any isometry of the cube is identified by the faces that three given adjacent faces are mapped to (this is why we could construct solutions with cubical symmetry for systems with 3 components). In conclusion, by Theorem 1.3 we obtain a $(\mathcal{G}_6, h_6)$-equivariant solution of (1-1) with $k = 3$ components.

Arguing in a similar way, we may also obtain equivariant solutions with respect to the symmetries of the octahedron for systems with $k = 4$ and $k = 8$ components, and so on.

3. An Alt–Caffarelli–Friedman monotonicity formula for equivariant solutions

In this section we aim at proving Proposition 1.5. We always suppose that $(k, \mathcal{G}, h)$ is an admissible triplet, according to Definition 1.1. Moreover, we often omit the phrase "up to a subsequence" for simplicity. The proof is divided into several steps, and, as usual when dealing with Alt–Caffarelli–Friedman monotonicity formulae for competing systems, is based upon a control on an "approximated" optimal partition problem on $\mathbb{S}^{N-1}$. For any $\boldsymbol{u} \in H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$, we let

$$I_\beta(\boldsymbol{u}) := \frac{1}{k}\sum_{i=1}^{k} \gamma\left(\frac{\int_{\mathbb{S}^{n-1}} |\nabla_\theta u_i|^2 + \frac{1}{2}\beta u_i^2 \sum_{j\neq i} u_j^2}{\int_{\mathbb{S}^{n-1}} u_i^2}\right),$$

where

$$\gamma(t) := \sqrt{\left(\frac{N-2}{2}\right)^2 + t} - \left(\frac{N-2}{2}\right).$$

We denote by $\hat{H}_{(\mathcal{G},h)}$ the subspace of $(\mathcal{G}, h)$-equivariant functions in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$, and we introduce the optimal value

$$\ell_\beta(k, \mathcal{G}, h) := \inf_{\hat{H}_{(\mathcal{G},h)}} I_\beta.$$

In what follows, to keep the notation as simple as possible, we simply write ℓ and ℓ_β instead of $\ell(k, \mathcal{G}, h)$ and $\ell_\beta(k, \mathcal{G}, h)$, respectively.

Lemma 3.1. *Both ℓ and ℓ_β are positive and achieved (for all $\beta > 0$). It follows that $\ell_\beta \to \ell$ as $\beta \to +\infty$, and there exists a minimizer for ℓ_β, which solves*

$$\begin{cases} -\Delta_\theta u_{i,\beta} = \lambda_\beta u_{i,\beta} - \beta u_{i,\beta} \sum_{j\neq i} u_j^2 & \text{in } \mathbb{S}^{N-1}, \\ u_{i,\beta} > 0 & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} u_{i,\beta}^2 = 1 & \text{for all } i, \end{cases} \tag{3-1}$$

where $\lambda_\beta \geq 0$ and Δ_θ denotes the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$. Moreover, $\boldsymbol{u}_\beta \rightharpoonup \varphi$ weakly in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$ and φ is a nonnegative minimizer for ℓ.

Proof. Restricting ourselves to the subset of functions in $\hat{H}_{(\mathcal{G},h)}$ whose components have prescribed $L^2(\mathbb{S}^{N-1})$-norm equal to 1, it is easy to check that the functional I_β is weakly lower semicontinuous and coercive. Since $\hat{H}_{(\mathcal{G},h)}$ is also weakly closed, the direct method of the calculus of variations ensures the existence of a minimizer $\boldsymbol{u}_\beta$ for ℓ_β, which can be assumed to be nonnegative. By the Palais principle of symmetric criticality (notice that I_β is invariant under the action of any symmetry in $\mathcal{O}(N)$), the Lagrange multipliers rule, and the strong maximum principle, it follows that $\boldsymbol{u}_\beta$ satisfies

$$\begin{cases} -\Delta_\theta u_{i,\beta} + \sum_{j\neq i} \frac{1}{2}(1+\mu_{j,\beta}/\mu_{i,\beta})\beta u_{i,\beta} u_{j,\beta}^2 = \lambda_{i,\beta} u_{i,\beta} & \text{in } \mathbb{S}^{N-1}, \\ u_{i,\beta} > 0 & \text{in } \mathbb{S}^{N-1}, \end{cases}$$

where

$$\mu_{i,\beta} := \gamma'\bigg(\int_{\mathbb{S}^{n-1}} |\nabla_\theta u_{i,\beta}|^2 + \tfrac{1}{2}\beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2\bigg).$$

The equation for $u_{i,\beta}$ is nothing but (3-1): indeed, thanks to the symmetries in $\hat{H}(\mathcal{G}, h)$ (see Remark 1.2), we have $\mu_{i,\beta} = \mu_{j,\beta}$ and $\lambda_{i,\beta} = \lambda_{j,\beta} \geq 0$ for every $i \neq j$. Finally, $\ell_\beta > 0$ since otherwise $\boldsymbol{u}_\beta \equiv \boldsymbol{0}$, in contradiction with the normalization condition.

As far as ℓ is concerned, we introduce an auxiliary functional $I_\infty : \hat{H}_{(\mathcal{G},h)} \to (0, +\infty]$, defined by

$$I_\infty(\boldsymbol{u}) := \begin{cases} (1/k)\sum_{i=1}^k \gamma\big(\int_{\mathbb{S}^{n-1}} |\nabla u_i|^2 / \int_{\mathbb{S}^{n-1}} u_i^2\big) & \text{if } u_i u_j = 0 \text{ a.e. on } \mathbb{S}^{n-1} \text{ for any } i \neq j, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that I_β is increasing in β and converges pointwise to I_∞, implying that I_∞ is a weakly lower semicontinuous functional in the weakly closed set $\hat{H}_{(\mathcal{G},h)}$, and that I_β Γ-converges to I_∞ in the weak H^1-topology. Moreover, since the family $\{I_\beta\}$ is equicoercive, any sequence $\{\boldsymbol{u}_\beta\}$ of minimizers for I_β converges to a minimizer $\boldsymbol{u}$ of I_∞. Finally, by definition, $\ell > \ell_\beta$ for every $\beta > 0$, whence $\ell > 0$ follows. □

Further properties of the sequence $\{\boldsymbol{u}_\beta\}$ are collected in the next two lemmas.

Lemma 3.2. *The sequence $\{\boldsymbol{u}_\beta\}$ is uniformly bounded in* $\mathrm{Lip}(\mathbb{S}^{N-1})$. *Moreover, the sequence (λ_β) is bounded.*

Proof. Let $\{\boldsymbol{u}_\beta\}$ be a sequence of minimizers for ℓ_β satisfying (3-1), weakly converging to a minimizer $\boldsymbol{u}$ for ℓ. As $I_\beta(\boldsymbol{u}_\beta) = \ell_\beta \leq \ell$, there exists $C > 0$ such that

$$\int_{\mathbb{S}^{N-1}} \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 \leq C.$$

Moreover, by weak convergence, $\{\boldsymbol{u}_\beta\}$ is bounded in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$. Therefore, testing the first equation in (3-1) against $u_{i,\beta}$, we deduce that $\{\lambda_\beta\}$ is a bounded sequence of positive numbers, and this implies, through a Brézis–Kato argument (see for instance [Tavares 2010, page 124] for a detailed proof and [Brézis and Kato 1979] for the original argument), that $\{\boldsymbol{u}_\beta\}$ is uniformly bounded in $L^\infty(\mathbb{S}^{N-1}, \mathbb{R}^k)$. By the main results in [Soave and Zilio 2015], we infer that $\{\boldsymbol{u}_\beta\}$ is uniformly bounded[1] in $\mathrm{Lip}(\mathbb{S}^{N-1})$. □

Lemma 3.3. *We have $\boldsymbol{u}_\beta \to \varphi$ strongly in the $H^1(\mathbb{S}^{N-1})$ topology, in $C^{0,\alpha}(\mathbb{S}^{N-1})$ for every $0 < \alpha < 1$, and*

$$\lim_{\beta\to+\infty} \beta \int_{\mathbb{S}^{N-1}} u_{i,\beta}^2 u_{j,\beta}^2 = 0.$$

Moreover, $\lambda_\beta \to \ell(\ell+N-2)$ and

$$\begin{cases} -\Delta_\theta \varphi_i = \ell(\ell+N-2)\varphi_i & \text{in } \{\varphi_i > 0\}, \\ \int_{\mathbb{S}^{N-1}} \varphi_i^2 = 1. \end{cases}$$

Proof. Thanks to Lemma 3.2, we can simply apply Theorem 1.4 in [Noris et al. 2010]. To check that $\lambda_\beta \to \ell(\ell+N-2)$, we observe that, by boundedness, $\lambda_\beta \to \lambda_\infty \geq 0$ as $\beta \to +\infty$. Therefore, recalling that $\boldsymbol{u}_\beta \rightharpoonup \varphi$ in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$, for $i = 1, \dots, k$ we have

$$\begin{cases} -\Delta_\theta \varphi_i = \lambda_\infty \varphi_i & \text{in } \{\varphi_i > 0\}, \\ \int_{\mathbb{S}^{N-1}} \varphi_i^2 = 1. \end{cases}$$

This implies that

$$\ell = \frac{1}{k} \sum_i \sqrt{\left(\frac{N-2}{2}\right)^2 + \int_{\mathbb{S}^{N-1}} |\nabla_\theta \varphi_i|^2} - \frac{N-2}{2} = \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_\infty} - \frac{N-2}{2},$$

whence the thesis follows. □

The following result is the counterpart of Lemma 4.2 in [Wang 2014] in a $(\mathcal{G}, h)$-equivariant setting; see also Theorem 5.6 in [Berestycki et al. 2013b] for an analogous statement in dimension $N = 2$.

Lemma 3.4. *There exists a constant $C > 0$ such that*

$$\ell_\beta \geq \ell - C\beta^{-1/4}.$$

Before proving the lemma, we need a technical result. We recall that $\hat{H}_{(\mathcal{G},h)}$ denotes the set of $(\mathcal{G}, h)$-equivariant functions in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$.

[1] It is worth mentioning that the results in [Soave and Zilio 2015] are proved for the Laplace operator in the interior of subsets of $\mathbb{R}^N$, and their extension to a Riemannian setting presents some technical difficulties; the general extension of [Soave and Zilio 2015] to equations on manifolds will be the object a future contribution [Smit Vega Garcia et al. ≥ 2016]. We anticipate here the main argument: the key ingredients for the regularity results in [Soave and Zilio 2015] are elliptic estimates, an Almgren-type monotonicity formula and a sharp version of the Alt–Caffarelli–Friedman-type monotonicity formula. Thus, we need to extend these three tools for systems on $\mathbb{S}^{N-1}$. The elliptic theory is already available, as is the Almgren-type monotonicity formula (see for instance [Tavares and Terracini 2012, Section 7]). The Alt–Caffarelli–Friedman-type monotonicity formula represents the only obstruction, but it can be obtained by combining the results in [Teixeira and Zhang 2011] (an Alt–Caffarelli–Friedman-type monotonicity formula for scalar equations on Riemannian manifolds) and in [Soave and Zilio 2015] (the sharp version of the Alt–Caffarelli–Friedman-type monotonicity formula for systems in the euclidean space). Once these three tools are available, the proof proceeds as in [Soave and Zilio 2015].

Lemma 3.5. *Let $\boldsymbol{u} \in \hat{H}_{(\mathcal{G},h)}$. Then the function $\hat{\boldsymbol{u}}$, defined by*

$$\hat{u}_i = v_i^+ := \Big(u_i - \sum_{j \neq i} u_j \Big)^+,$$

also belongs to $\hat{H}_{(\mathcal{G},h)}$.

Proof. As $u_i \in H^1(\mathbb{S}^{N-1})$, it follows straightforwardly that $\hat{\boldsymbol{u}} \in H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$. We have to show that it is also $(\mathcal{G}, h)$-equivariant, and to this end it is sufficient to show that $\boldsymbol{v}$ is $(\mathcal{G}, h)$-equivariant. This can be checked directly:

$$\begin{aligned} v_{(h(g))^{-1}(i)}(g(x)) &= u_{(h(g))^{-1}(i)}(g(x)) - \sum_{j \neq (h(g))^{-1}(i)} u_j(g(x)) = u_{(h(g))^{-1}(i)}(g(x)) - \sum_{j \neq i} u_{(h(g))^{-1}(j)}(g(x)) \\ &= v_i(x), \end{aligned}$$

where the last equality follows from the fact that $\boldsymbol{u}$ is $(\mathcal{G}, h)$-equivariant. □

Proof of Lemma 3.4. In order to simplify the notation, only in this proof we write ∇ and Δ instead of ∇_θ and Δ_θ, respectively. Let us consider the functions $\hat{\boldsymbol{u}}_\beta$, defined in Lemma 3.5. Since the components of $\hat{\boldsymbol{u}}_\beta$ have disjoint supports, we can use it as a competitor for ℓ. We aim at showing that $\hat{\boldsymbol{u}}_\beta$ is actually close enough to $\boldsymbol{u}_\beta$ in the energy sense, and in doing this we shall use many times the properties proved in Lemma 3.2. To be precise, we shall prove that there exists a constant $C > 0$ such that

$$1 - C\beta^{-1/2} \leq \int_{\mathbb{S}^{n-1}} \hat{u}_{i,\beta}^2 \leq 1 + C\beta^{-1/2}, \tag{3-2}$$

$$\int_{\mathbb{S}^{N-1}} |\nabla \hat{u}_{i,\beta}|^2 \leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4}. \tag{3-3}$$

Before we continue, let us point out that second estimate can be derived from an analogous one: there exists $C > 0$ independent of β and $\bar{\delta} > 0$ such that, for almost any $\delta \in (0, \bar{\delta})$, we have

$$\int_{\{\hat{u}_{i,\beta} > \delta\}} |\nabla \hat{u}_{i,\beta}|^2 \leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4} + C\delta.$$

Indeed, if the previous estimate is satisfied,

$$\int_{\mathbb{S}^{N-1}} |\nabla \hat{u}_{i,\beta}|^2 = \int_{\{\hat{u}_{i,\beta} > 0\}} |\nabla \hat{u}_{i,\beta}|^2 = \lim_{\delta \to 0^+} \int_{\{\hat{u}_{i,\beta} > \delta\}} |\nabla \hat{u}_{i,\beta}|^2 \leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4}.$$

Notice that in principle the value $\bar{\delta}$ could depend on β, but this is not a problem since C is, on the contrary, a universal constant.

Pointwise bounds. The boundedness of $\{\boldsymbol{u}_\beta\}$ in $\mathrm{Lip}(\mathbb{S}^{N-1})$ (Lemma 3.2) implies that there exists a constant $C_1 > 0$ such that

$$\beta^{1/2} u_{i,\beta} u_{j,\beta} \leq C_1 \quad \text{for all } i \neq j. \tag{3-4}$$

The proof is a straightforward adaptation of the one in [Soave and Zilio 2016, Theorem 1.1], which regards the same estimate in subsets of $\mathbb{R}^N$.

As a consequence we have that, for each $\theta \in \mathbb{S}^{N-1}$ and each $\beta > 0$,

$$u_{i,\beta}(\theta) \geq 2kC_1^{1/2}\beta^{-1/4} \quad \text{for at most one index } i, \tag{3-5}$$

where C_1 is the same constant as appears in (3-4). Indeed, assuming the contrary, there would exist two distinct indices $i \neq j$ satisfying the previous inequality, and hence

$$4k^2C_1\beta^{-1/2} \leq u_{i,\beta}(\theta)u_{j,\beta}(\theta) \leq C_1\beta^{-1/2},$$

a contradiction.

Finally, we observe that

$$\hat{u}_{i,\beta}(\theta) = 0 \implies u_{i,\beta}(\theta) \leq 2k(k-1)C_1^{1/2}\beta^{-1/4}. \tag{3-6}$$

If not, we have that (3-5) holds for i, and moreover

$$2k(k-1)C_1^{1/2}\beta^{-1/4} \leq u_{i,\beta}(\theta) \leq \sum_{j\neq i} u_{j,\beta}(\theta) \leq (k-1)\max_{j\neq i} u_{j,\beta}(\theta);$$

hence there exist two indexes for which (3-5) is satisfied in θ, a contradiction.

Integral bounds for the Laplacian. We prove that there exists a constant $C > 0$ (independent of β) such that

$$\int_{\mathbb{S}^{N-1}} |\Delta u_{i,\beta}| \leq C. \tag{3-7}$$

Indeed, directly from the equation and the divergence theorem,

$$0 = \int_{\mathbb{S}^{N-1}} (-\Delta u_{i,\beta}) = \int_{\mathbb{S}^{N-1}} \lambda_\beta u_{i,\beta} - \beta u_{i,\beta}\sum_{j\neq i} u_j^2;$$

that is,

$$0 \leq \int_{\mathbb{S}^{N-1}} \beta u_{i,\beta}\sum_{j\neq i} u_{j,\beta}^2 = \int_{\mathbb{S}^{N-1}} \lambda_\beta u_{i,\beta} \leq C,$$

as the functions $u_{i,\beta}$ are bounded in $L^\infty(\mathbb{S}^{N-1})$ and $\{\lambda_\beta\}$ is bounded. Consequently,

$$\int_{\mathbb{S}^{N-1}} |\Delta u_{i,\beta}| \leq \int_{\mathbb{S}^{N-1}} \lambda_\beta u_{i,\beta} + \beta u_{i,\beta}\sum_{j\neq i} u_{j,\beta}^2 \leq C.$$

Integral bounds for the competition term. Using (3-5) and the computations in the previous point, we deduce that

$$\begin{aligned}
&\int_{\mathbb{S}^{N-1}} \beta \sum_{i\neq j} u_{i,\beta}^2 u_{j,\beta}^2 \\
&\quad \leq \sum_{i\neq j}\biggl(\|u_{i,\beta}\|_{L^\infty(\{u_{i,\beta}\leq u_{j,\beta}\})} \int_{\{u_{i,\beta}\leq u_{j,\beta}\}} \beta u_{i,\beta}u_{j,\beta}^2 + \|u_{j,\beta}\|_{L^\infty(\{u_{j,\beta}<u_{i,\beta}\})} \int_{\{u_{j,\beta}<u_{i,\beta}\}} \beta u_{j,\beta}u_{i,\beta}^2 \biggr) \\
&\quad \leq C\beta^{-1/4}\sum_{i=1}^{k} \int_{\{u_{i,\beta}\leq u_{j,\beta}\}} \beta u_{i,\beta}\sum_{j\neq i} u_{j,\beta}^2 \leq C\beta^{-1/4}.
\end{aligned}$$

Integral bounds for the normal derivatives. For analogous reasons, we can show that there exists a constant $C > 0$ and $\bar{\delta} > 0$ small enough such that, for almost every $\delta \in (0, \bar{\delta})$,

$$\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} |\partial_\nu \hat{u}_{i,\beta}| \le C.$$

Firstly, since the function $\hat{u}_{i,\beta}$ is regular for β fixed, the set $\partial\{\hat{u}_{i,\beta} > \delta\}$ is regular for almost every $\delta > 0$, by Sard's lemma. Moreover, since $\hat{u}_{i,\beta}$ is nonnegative and regular, if $\delta < \bar{\delta}$ is small enough then

$$\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} |\partial_\nu \hat{u}_{i,\beta}| = -\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu \hat{u}_{i,\beta}. \tag{3-8}$$

Hence, for almost every $\delta \in (0, \bar{\delta})$ the set $\partial\{\hat{u}_{i,\beta} > \delta\}$ is regular, and (3-8) holds. With this choice we are in position to apply the divergence theorem, and consequently

$$\left|\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu \hat{u}_{i,\beta}\right| = \left|\int_{\{\hat{u}_{i,\beta}>\delta\}} \Delta \hat{u}_{i,\beta}\right| \le \int_{\{\hat{u}_{i,\beta}>\delta\}} \sum_{j=1}^{k} |\Delta u_{j,\beta}| \le C,$$

where C is independent of β by (3-7). With similar computations we also have the uniform estimate

$$\left|\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu u_{i,\beta}\right| \le C.$$

Estimates for the $L^2(\mathbb{S}^{N-1})$ norm. Thanks to (3-5) and (3-6), we have

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} (\hat{u}_{i,\beta} - u_{i,\beta})^2 &= \int_{\{\hat{u}_{i,\beta}>0\}} (\hat{u}_{i,\beta} - u_{i,\beta})^2 + \int_{\{\hat{u}_{i,\beta}=0\}} (\hat{u}_{i,\beta} - u_{i,\beta})^2 \\
&= \int_{\{u_{i,\beta}>\sum_{j\neq i} u_{j,\beta}\}} \Big(\sum_{j\neq i} u_{j,\beta}\Big)^2 + \int_{\{\hat{u}_{i,\beta}=0\}} u_{i,\beta}^2 \le C\beta^{-1/2},
\end{aligned}$$

whence (3-2) follows.

Estimates for the $H^1(\mathbb{S}^{N-1})$ seminorm. As a last step, we wish to estimate the L^2 norm of $\nabla \hat{u}_{i,\beta}$. Since $\partial\{\hat{u}_{i,\beta} > \delta\}$ is regular, we can apply the divergence theorem, deducing that

$$\begin{aligned}
\int_{\{\hat{u}_{i,\beta}>\delta\}} |\nabla \hat{u}_{i,\beta}|^2 &= \int_{\{\hat{u}_{i,\beta}>\delta\}} (-\Delta \hat{u}_{i,\beta})\hat{u}_{i,\beta} + \int_{\partial\{\hat{u}_{i,\beta}>\delta\}} (\partial_\nu \hat{u}_{i,\beta})\hat{u}_{i,\beta} \\
&= \underbrace{\int_{\{\hat{u}_{i,\beta}>\delta\}} (-\Delta u_{i,\beta})u_{i,\beta}}_{\text{(I)}} + \int_{\{\hat{u}_{i,\beta}>\delta\}} \Delta u_{i,\beta} \sum_{j\neq i} u_{j,\beta} \\
&\qquad + \underbrace{\int_{\{\hat{u}_{i,\beta}>\delta\}} \Delta\Big(\sum_{j\neq i} u_{j,\beta}\Big)\hat{u}_{i,\beta}}_{\text{(II)}} + \delta \int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu \hat{u}_{i,\beta}.
\end{aligned}$$

The first term (I) can be bounded, also recalling that $\lambda_\beta \ge 0$, using the equation

$$\begin{aligned}\int_{\{\hat u_{i,\beta}>\delta\}} (-\Delta u_{i,\beta}) u_{i,\beta} &= \int_{\{\hat u_{i,\beta}>\delta\}} \lambda_\beta u_{i,\beta}^2 - \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 \\ &\le \int_{\mathbb{S}^{N-1}} \lambda_\beta u_{i,\beta}^2 - \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 + \int_{\mathbb{S}^{N-1}\setminus\{\hat u_{i,\beta}>\delta\}} \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 \\ &= \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + \int_{\mathbb{S}^{N-1}\setminus\{\hat u_{i,\beta}>\delta\}} \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2.\end{aligned}$$

The term (II) can be expanded further as

$$\begin{aligned}&\int_{\{\hat u_{i,\beta}>\delta\}} \Delta\Big(\sum_{j\neq i} u_{j,\beta}\Big)\hat u_{i,\beta} \\ &\qquad = -\int_{\{\hat u_{i,\beta}>\delta\}} \nabla\Big(\sum_{j\neq i} u_{j,\beta}\Big)\cdot\nabla \hat u_{i,\beta} + \delta\int_{\partial\{\hat u_{i,\beta}>\delta\}} \partial_\nu\Big(\sum_{j\neq i} u_{j,\beta}\Big) \\ &\qquad = \int_{\{\hat u_{i,\beta}>\delta\}} \Big(\sum_{j\neq i} u_{j,\beta}\Big)\Delta\hat u_{i,\beta} - \int_{\partial\{\hat u_{i,\beta}>\delta\}} \Big(\sum_{j\neq i} u_{j,\beta}\Big)\partial_\nu \hat u_{i,\beta} + \delta\int_{\partial\{\hat u_{i,\beta}>\delta\}} \partial_\nu\Big(\sum_{j\neq i} u_{j,\beta}\Big).\end{aligned}$$

Recalling the previous computations, and using again (3-5), we have

$$\begin{aligned}&\int_{\{\hat u_{i,\beta}>\delta\}} |\nabla \hat u_{i,\beta}|^2 \\ &\quad \le \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + \int_{\mathbb{S}^{N-1}\setminus\{\hat u_{i,\beta}>\delta\}} \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 + \int_{\{\hat u_{i,\beta}>\delta\}} \Delta u_{i,\beta} \sum_{j\neq i} u_{j,\beta} + \int_{\{\hat u_{i,\beta}>\delta\}} \Big(\sum_{j\neq i} u_{j,\beta}\Big)\Delta\hat u_{i,\beta} \\ &\qquad\qquad - \int_{\partial\{\hat u_{i,\beta}>\delta\}} \Big(\sum_{j\neq i} u_{j,\beta}\Big)\partial_\nu \hat u_{i,\beta} + \delta\int_{\partial\{\hat u_{i,\beta}>\delta\}} \partial_\nu u_{i,\beta} \\ &\quad \le \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4} + C\delta,\end{aligned}$$

which, as already observed, implies (3-3).

With (3-2) and (3-3) we are in position to complete the proof. By minimality, $\ell \le I_\infty(\hat{\boldsymbol u}_\beta)$ for every β, which gives

$$\begin{aligned}\ell &\le \frac1k \sum_{i=1}^k \gamma\left(\frac{\int_{\mathbb{S}^{N-1}} |\nabla \hat u_{i,\beta}|^2}{\int_{\mathbb{S}^{N-1}} \hat u_{i,\beta}^2}\right) \le \frac1k \sum_{i=1}^k \gamma\left(\frac{\int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4}}{1 - C\beta^{-1/2}}\right) \\ &\le \frac1k \sum_{i=1}^k \gamma\left(\int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + \frac12 \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2\right) + C\beta^{-1/4} = \ell_\beta + C\beta^{-1/4}. \qquad \square\end{aligned}$$

The proof of Proposition 1.5 can be obtained in a somewhat usual way.

Sketch of the proof of Proposition 1.5. Arguing as in [Conti et al. 2005, Section 7], or [Noris et al. 2010, Lemma 2.5], or else [Soave and Zilio 2015, Theorem 3.14], it is possible to check that

$$\frac{d}{dr}\log\left(\frac{J_1(r)\cdots J_k(r)}{r^{2k\ell}}\right)=-\frac{2k\ell}{r}+\frac{2}{r}\sum_i\gamma\left(\frac{r^2\int_{\partial B_r}|\nabla u_i|^2+\frac{1}{2}u_i^2\sum_{j\neq i}u_j^2}{\int_{\partial B_r}u_i^2}\right).$$

Changing variables in the integrals (see Theorem 3.14 in [Soave and Zilio 2015] for the details), we deduce that

$$\sum_i\gamma\left(\frac{r^2\int_{\partial B_r}|\nabla u_i|^2+\frac{1}{2}u_i^2\sum_{j\neq i}u_j^2}{\int_{\partial B_r}u_i^2}\right)\geq k\ell_{r^2},$$

where ℓ_{r^2} denotes the optimal value ℓ_β for $\beta=r^2$. Coming back to the previous equation and using Lemma 3.4, we conclude that

$$\frac{d}{dr}\log\left(\frac{J_1(r)\cdots J_k(r)}{r^{2k\ell}}\right)\geq\frac{2k}{r}(\ell_{r^2}-\ell)\geq-2kCr^{-3/2}$$

and, integrating, the thesis follows. □

4. Construction of equivariant solutions

For an admissible triplet $(k,\mathcal{G},h)$, we prove the existence of a $(\mathcal{G},h)$-equivariant solution to (1-1) with k components. We partially follow the method introduced in [Berestycki et al. 2013b], which consists in two steps:

- firstly, we prove the existence of a sequence of $(\mathcal{G},h)$-equivariant solutions $\boldsymbol{V}_R$, defined in balls of increasing radii $R\to+\infty$;
- secondly, we show that this sequence converges locally uniformly in $\mathbb{R}^N$ to a nontrivial solution.

With respect to [Berestycki et al. 2013b], we modify the construction conveniently choosing R from the beginning; this substantially simplifies the proof of the convergence of $\{\boldsymbol{V}_R\}$, and we refer to Remark 4.4 for more details. Finally, in the last part of the proof we characterize the growth of the solution using the Alt–Caffarelli–Friedman monotonicity formula for $(\mathcal{G},h)$-equivariant solutions.

By Lemma 3.1, we know that the optimal value ℓ (see (1-5)) is achieved by a nonnegative $(\mathcal{G},h)$-equivariant function $\varphi\in H^1(\mathbb{S}^{N-1},\mathbb{R}^k)$. Differently from the previous section, we take

$$\int_{\mathbb{S}^{N-1}}\varphi_i^2=\frac{1}{k}\iff\sum_{i=1}^k\int_{\mathbb{S}^{N-1}}\varphi_i^2=1. \tag{4-1}$$

This choice is possible, since the minimum problem for ℓ is invariant under scaling of type $t\mapsto t\varphi$ with $t\in\mathbb{R}$, and simplifies some computations.

Lemma 4.1. *For any $\beta>0$ there exists a $(\mathcal{G},h)$-equivariant solution $\{\boldsymbol{U}_\beta\}$ to the problem*

$$\begin{cases}\Delta U_{i,\beta}=\beta U_{i,\beta}\sum_{j\neq i}U_{j,\beta}^2 & \text{in } B_1,\\ U_{i,\beta}>0 & \text{in } B_1,\\ U_{i,\beta}=\varphi_i & \text{on } \partial B_1=\mathbb{S}^{N-1}.\end{cases}$$

Moreover,

(i) $U_{i,\beta}(0) = U_{j,\beta}(0)$ *for all* $i, j = 1, \dots, k$ *and* $\beta > 0$;

(ii) *letting*

$$\mathcal{E}_\beta(\boldsymbol{U}) = \int_{B_1} \sum_{i=1}^{k} |\nabla U_i|^2 + \beta \sum_{i<j} U_i^2 U_j^2,$$

the uniform estimate $\mathcal{E}_\beta(\boldsymbol{U}_\beta) \le \ell$ *holds*;

(iii) *there exists a Lipschitz continuous function* $\mathbf{0} \not\equiv \boldsymbol{U}_\infty$ *such that*, *up to a subsequence*, $\boldsymbol{U}_\beta \to \boldsymbol{U}_\infty$ *in* $\mathcal{C}^{0,\alpha}(B_1)$ *for every* $\alpha \in (0, 1)$ *and in* $H^1_{\mathrm{loc}}(B_1)$.

Proof. It is not difficult to check that the functional $\mathcal{E}_\beta$ admits a minimizer $\boldsymbol{U}_\beta$ in the H^1-weakly closed set of the $(\mathcal{G}, h)$-equivariant functions in $H^1(B_1, \mathbb{R}^k)$ with the prescribed boundary conditions. The fact that this minimizer solves the Euler–Lagrange equation is a consequence of Palais' principle of symmetric criticality. Property (i) follows straightforwardly by the equivariance (recall Remark 1.2). Concerning property (ii), we introduce the ℓ-homogeneous extension of φ, defined by

$$\phi(x) := |x|^\ell \varphi\Big(\frac{x}{|x|}\Big).$$

By minimality, $\mathcal{E}_\beta(\boldsymbol{U}_\beta) \le \mathcal{E}_\beta(\phi)$, so that it remains to check that $\mathcal{E}_\beta(\phi) \le \ell$. At first, since φ_i is an eigenfunction of $-\Delta_\theta$ on $\{\varphi_i > 0\}$ associated to the eigenvalue $\ell(\ell + N - 2)$, the function ϕ_i is harmonic in $\{\phi_i > 0\}$. Furthermore, by definition,

$$\sum_i \int_{\partial B_1} \phi_i^2 = 1$$

for every i. Therefore, using the Euler formula for homogeneous functions, we deduce that

$$\mathcal{E}_\beta(\phi) = \sum_i \int_{B_1} |\nabla\phi_i|^2 = \sum_i \int_{\{\phi_i>0\}\cap B_1} |\nabla\phi_i|^2 = \sum_i \int_{\partial B_1\cap\{\phi_i>0\}} \phi_i\,\partial_\nu\phi_i = \ell \sum_i \int_{\partial B_1\cap\{\phi_i>0\}} \phi_i^2 = \ell.$$

It remains to prove (iii). By (ii) and the boundary conditions, the sequence $\{\boldsymbol{U}_\beta\}$ is bounded in $H^1(B_1)$, and hence it converges weakly to some limit $\boldsymbol{U}_\infty$. By compactness of the trace operator, $\boldsymbol{U}_\infty \not\equiv \mathbf{0}$. All the functions $\boldsymbol{U}_\beta$ are nonnegative, subharmonic and have the same boundary conditions, and hence by the maximum principle they are uniformly bounded in $L^\infty(B_1)$. This, as recalled in Section 2, implies the thesis. □

We plan to use the solutions of Lemma 4.1 in order to construct entire solutions to (1-1). Our method is based on a simple blow-up argument. For a positive radius r_β to be determined, we introduce

$$V_{i,\beta}(x) := \beta^{1/2} r_\beta U_{i,\beta}(r_\beta x).$$

By definition, $\boldsymbol{V}_\beta$ solves

$$\Delta V_{i,\beta} = V_{i,\beta} \sum_{j\neq i} V_{j,\beta}^2 \quad \text{in } B_{1/r_\beta}.$$

A convenient choice of r_β is suggested by the following statement.

Lemma 4.2. *For any fixed $\beta > 1$ there exists a unique $r_\beta > 0$ such that*

$$\int_{\partial B_1} \sum_{i=1}^k V_{i,\beta}^2 = 1.$$

Moreover $r_\beta \to 0$, and consequently $B_{1/r_\beta} \to \mathbb{R}^N$, in the sense that for any compact $K \subset \mathbb{R}^N$ we have $K \Subset B_{1/r_\beta}$ provided β is sufficiently large.

Proof. We have to find $r_\beta > 0$ such that $\beta r_\beta^2 H(\boldsymbol{U}_\beta, r_\beta) = 1$. The strict monotonicity of $H(\boldsymbol{U}_\beta, \cdot)$ (see Section 2) implies the strict monotonicity of the continuous function $r \mapsto \beta r^2 H(\boldsymbol{U}_\beta, r)$. By regularity, for any fixed β,

$$\lim_{r\to 0} \beta r^2 H(\boldsymbol{U}_\beta, r) = \lim_{r\to 0} \beta \frac{r^2}{r^{N-1}} \int_{\partial B_r} \sum_{i=1}^k U_{i,\beta}^2 = \beta \lim_{r\to 0} r^2 \cdot \sum_{i=1}^k U_{i,\beta}^2(0) = 0,$$

and, by the normalization (4-1), $\beta H(\boldsymbol{U}_\beta, 1) = \beta > 1$. This proves existence and uniqueness of r_β. If, by contradiction, $r_\beta \geq \bar r > 0$, then by Lemma 4.1(iii) and the monotonicity of $H(\boldsymbol{U}_\beta, \cdot)$ we would have

$$1 = \beta r_\beta^2 H(\boldsymbol{U}_\beta, r_\beta) \geq \beta \bar r^2 H(\boldsymbol{U}_\beta, \bar r) \geq \frac{\beta \bar r^2}{2} \frac{1}{\bar r^{N-1}} \int_{\partial B_{\bar r}} \sum_{i=1}^k U_{i,\infty} \geq \beta C,$$

which gives a contradiction for $\beta > 1/C$. In order to bound from below the second-to-last term, we recall that since $\mathbf{0} \not\equiv \boldsymbol{U}_\infty$ we have $H(\boldsymbol{U}_\infty, r) \neq 0$ for all $0 < r < 1$ (see Section 2). ☐

Lemma 4.3. *Up to a subsequence, $\boldsymbol{V}_\beta \to \boldsymbol{V}$ in $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^N)$, and $\boldsymbol{V}$ is an entire $(\mathcal{G}, h)$-equivariant solution of* (1-1) *with $N(\boldsymbol{V}, r) \leq \ell$ for every $r > 0$.*

Proof. Since $\mathcal{E}_\beta(\boldsymbol{U}_\beta) \leq \ell$ and $H(\boldsymbol{U}_\beta, 1) = 1$, by scaling and using the monotonicity of the Almgren quotient we have

$$N(\boldsymbol{V}_\beta, r) \leq N\Big(\boldsymbol{V}_\beta, \frac{1}{r_\beta}\Big) = N(\boldsymbol{U}_\beta, 1) \leq \frac{\mathcal{E}(\boldsymbol{U}_\beta)}{H(\boldsymbol{U}_\beta, 1)} \leq \ell \tag{4-2}$$

for every $0 < r < 1/r_\beta$, $\beta > 0$. Now let $r > 0$; then, for β sufficiently large,

$$\frac{d}{dr} \log H(\boldsymbol{V}_\beta, r) = \frac{2}{r} N_\beta(\boldsymbol{v}_\beta, r) + \frac{2}{r^{N-1} H(\boldsymbol{V}_\beta, r)} \int_{B_r} \sum_{i<j} V_i^2 V_j^2 \leq \frac{2\ell}{r} + \frac{2}{r^{N-1} H(\boldsymbol{V}_\beta, r)} \int_{B_r} \sum_{i<j} V_i^2 V_j^2.$$

Integrating the inequality for $r \in (1, R)$, and recalling (2-2), we infer that

$$\frac{H(\boldsymbol{V}_\beta, R)}{R^{2\ell}} \leq H(\boldsymbol{V}_\beta, 1) e^\ell = e^\ell \quad \text{for all } R \geq 1, \tag{4-3}$$

independently of β. By subharmonicity and standard elliptic estimates, we deduce that $\boldsymbol{V}_\beta$ converges in $\mathcal{C}^2(B_R)$ to some limit $\boldsymbol{V}^R$, and since R has been chosen arbitrarily, a diagonal selection gives convergence to an entire limit $\boldsymbol{V}$, which is clearly $(\mathcal{G}, h)$-equivariant. Since $\boldsymbol{V}$ solves (1-1) and

$$\int_{\partial B_1} \sum_{i=1}^k V_{i,\beta}^2 = 1 \qquad \text{and} \qquad V_{i,\beta}(0) = V_{j,\beta}(0) \quad \text{for all } i, j$$

(see Lemmas 4.1 and 4.2), all the components of $\boldsymbol{V}$ are nontrivial, and hence nonconstant. ☐

We now show that the growth rate of the solution is exactly equal to ℓ. In light of the upper bound on the Almgren quotient proved in the previous lemma, this is a consequence of Theorem 1.4.

Proof of Theorem 1.4. Let us assume by contradiction that there exists a $(\mathcal{G}, h)$-equivariant solution $\boldsymbol{V}$ with growth rate less than $\ell-\varepsilon$ for some $\varepsilon > 0$. By monotonicity it results $N(\boldsymbol{V}, r) \le N(\boldsymbol{V}, +\infty) \le \ell-\varepsilon$ for every $r > 0$. We consider the blow-down sequence

$$\boldsymbol{V}_R(x) = \frac{1}{\sqrt{H(\boldsymbol{V}, R)}}\boldsymbol{V}(Rx).$$

By Theorem 1.4 in [Soave and Terracini 2015], it converges in $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ to a limit $\boldsymbol{W}$, which is segregated, nonnegative, homogeneous with homogeneity degree $\delta := N(\boldsymbol{V}, +\infty) \le \ell-\varepsilon$, and such that $\Delta W_i = 0$ in $\{W_i > 0\}$. The uniform convergence entails the $(\mathcal{G}, h)$-equivariance, and hence the trace $\hat{\boldsymbol{w}}$ of $\boldsymbol{W}$ on the sphere $\mathbb{S}^{N-1}$ is an admissible competitor for ℓ, in the sense that $\ell \le I_\infty(\hat{\boldsymbol{w}})$ (I_∞ is defined in Lemma 3.1). The value $I_\infty(\hat{\boldsymbol{w}})$ can be computed explicitly; indeed, by harmonicity, homogeneity and symmetry, $\hat{w}_i$ is an eigenfunction of the Laplace–Beltrami operator $-\Delta_\theta$ on a subdomain of $\mathbb{S}^{N-1}$, associated to the eigenvalue $\delta(\delta+N-2)$. This, by definition, implies that $I_\infty(\hat{\boldsymbol{w}}) = \delta < \ell$, in contradiction with the minimality of ℓ. □

So far we proved the existence of a $(\mathcal{G}, h)$-equivariant solution having growth rate ℓ in the weak sense of (2-3). It remains to show that the stronger condition (1-6) holds. First we make the following remark.

Remark 4.4. Both Theorem 1.3 and [Berestycki et al. 2013b, Theorem 1.6] are based upon the same two-step procedure: construction of solutions in balls B_R of increasing radius, and passage to the limit as $R \to +\infty$. The main difference is in the fact that while in [Berestycki et al. 2013b] the authors prescribed the value of the functions on the boundary ∂B_R, we prescribed the value on ∂B_1, conveniently choosing r_β. This permits us to greatly simplify the proof of the convergence, since by the doubling property (4-3) the normalization on ∂B_1 is enough to have $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^N)$-convergence of our approximating sequence. In [Berestycki et al. 2013b, page 123], this compactness is proved in a different way, using fine tools such as Proposition 5.7 therein, which seems difficult to generalize to higher dimensions.

Lemma 4.5. *We have*

$$\lim_{r\to\infty} \frac{1}{r^{2\ell}} H(\boldsymbol{V}, r) \in (0, +\infty).$$

Proof. It is easy to prove that the limit exists and it is less than 1. Indeed

$$\frac{d}{dr}\log\frac{H(\boldsymbol{V}, r)}{r^{2\ell}} = \frac{H'(\boldsymbol{V}, r)}{H(\boldsymbol{V}, r)} - \frac{2\ell}{r} = \frac{2}{r}(N(\boldsymbol{V}, r)-\ell) \le 0,$$

and by construction $H(\boldsymbol{V}, 1) = 1$. Letting

$$L = \lim_{r\to\infty}\frac{H(\boldsymbol{V}, r)}{r^{2\ell}},$$

we are left to show that $L > 0$. Recalling that $N(\boldsymbol{V}, +\infty) = \ell$, we have

$$L = \lim_{r\to\infty}\left(\frac{E(\boldsymbol{V}, r)}{r^{2\ell}}\right)\cdot\lim_{r\to+\infty}\frac{H(\boldsymbol{V}, r)}{E(\boldsymbol{V}, r)} \ge \frac{1}{\ell}\liminf_{r\to\infty}\frac{E(\boldsymbol{V}, r)}{r^{2\ell}},$$

and the thesis follows if

$$\liminf_{r\to\infty} \frac{E(\boldsymbol{V}, r) + H(\boldsymbol{V}, r)}{r^{2\ell}} > 0.$$

To this aim, we note that with computations analogous to those in [Soave and Zilio 2016, conclusion of the proof of Theorem 1.5] we can prove that

$$\frac{E(\boldsymbol{V}, r) + H(\boldsymbol{V}, r)}{r^{2\ell}} \geq \frac{C}{r^{2\ell}} (J_1(r) \cdots J_k(r))^{1/k} = C \left(\frac{1}{r^{2\ell k}} J_1(r) \cdots J_k(r) \right)^{\frac{1}{k}},$$

where the integrals J_i are evaluated for the function $\boldsymbol{V}$. Since $\boldsymbol{V}$ is a $(\mathcal{G}, h)$-equivariant solution of (1-1), we are in position to apply the Alt–Caffarelli–Friedman monotonicity formula of Proposition 1.5, whence

$$\frac{E(\boldsymbol{V}, r) + H(\boldsymbol{V}, r)}{r^{2\ell}} \geq C(J_1(1) \cdots J_k(1))^{1/k} e^{Cr^{-1/2}} \geq Ce^{Cr^{-1/2}}$$

for every $r > 1$. $\square$

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NICOLA SOAVE: nicola.soave@gmail.com
Mathematisches Institut, Justus-Liebig-Universität Giessen, Arndtstrasse 2, 35392 Giessen, Germany

ALESSANDRO ZILIO: azilio@ehess.fr
Centre d'Analyse et de Mathématique Sociales, École des Hautes Études en Sciences Sociales, 190–198 Avenue de France, 75244 Paris CEDEX 13, France

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