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SHARP WEIGHTED NORM ESTIMATES BEYOND CALDERÓN–ZYGMUND THEORY

FRÉDÉRIC BERNICOT, DOROTHEE FREY AND STEFANIE PETERMICHL

We dominate nonintegral singular operators by adapted sparse operators and derive optimal norm estimates in weighted spaces. Our assumptions on the operators are minimal and our result applies to an array of situations, whose prototypes are Riesz transforms or multipliers, or paraproducts associated with a second-order elliptic operator. It also applies to such operators whose unweighted continuity is restricted to Lebesgue spaces with certain ranges of exponents (p_0, q_0) with $1 \leq p_0 < 2 < q_0 \leq \infty$. The norm estimates obtained are powers α of the characteristic used by Auscher and Martell. The critical exponent in this case is $\mathfrak{p} = 1 + p_0/q_0'$. We prove $\alpha = 1/(p - p_0)$ when $p_0 < p \leq \mathfrak{p}$ and $\alpha = (q_0 - 1)/(q_0 - p)$ when $\mathfrak{p} \leq p < q_0$. In particular, we are able to obtain the sharp A_2 estimates for nonintegral singular operators which do not fit into the class of Calderón–Zygmund operators. These results are new even in Euclidean space and are the first ones for operators whose kernel does not satisfy any regularity estimate.

1. Introduction

In the last ten years, it has been of great interest to obtain optimal operator norm estimates in Lebesgue spaces endowed with Muckenhoupt weights. One asks for the growth of the norm of certain operators, such as the Hilbert transform or the Hardy–Littlewood maximal function, with respect to a characteristic assigned to the weight. Originally, the main motivation for sharp estimates of this type came from certain important applications to partial differential equations. See for example Fefferman, Kenig and Pipher [Fefferman et al. 1991] and Astala, Iwaniec and Saksman [Astala et al. 2001]. Indeed, a long-standing regularity problem has been solved through the optimal weighted norm estimate of the Beurling–Ahlfors operator, a classical Calderón–Zygmund operator; see [Petermichl and Volberg 2002]. Since then, the area has been developing rapidly. Advances have greatly improved conceptual understanding of classical objects such as Calderón–Zygmund operators. The latter are now understood in several different ways, one of them being through pointwise control by so-called sparse operators; see, most recently, [Lacey 2015; Lerner and Nazarov 2015]. We bring this circle of ideas to the wide range of nonintegral singular operators, such as those considered in [Auscher and Martell 2007a]. Under minimal assumptions, we now demonstrate control by well-chosen sparse operators and derive optimal norm estimates in weighted spaces.

From a historic standpoint, Hunt, Muckenhoupt and Wheeden [Hunt et al. 1973] proved that the Hilbert transform is bounded on L^2_ω if and only if the weight ω satisfies the so-called A_2 condition. Then the

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extension for $p \in (1, \infty)$ of the class A_p for weights was made legitimate by the characterization of the Hardy–Littlewood maximal operator on L^p_ω. These classes, as well as the "dual classes" RH_q (describing a reverse Hölder property), originally in Euclidean space, are only defined in terms of volume of balls, so this entire theory has been extended to the doubling framework. Calderón–Zygmund operators have been proved to be bounded on L^p_ω if $\omega \in A_p$. More recently, the so-called A_p conjecture (which is now solved) was about the sharp dependence of this operator norm with respect to the A_p characteristic of the weight. This conjecture was solved by Petermichl and Volberg [2002] for the Beurling–Ahlfors transform, by Petermichl [2007; 2008] for the Hilbert and Riesz transforms (see also the alternative proof by Lacey, Petermichl and Reguera [Lacey et al. 2010]) and by Hytönen [2012] for arbitrary Calderón–Zygmund operators. The idea of dyadic shift [Petermichl 2000] and the seminal articles on two-weight questions of dyadic operators by Nazarov, Treil and Volberg [Nazarov et al. 1999; 2008] were very influential in this area at that point. While [Nazarov et al. 1999] influenced earlier proofs, [Nazarov et al. 2008] was important for later proofs. Recently Lerner [2010; 2013a; 2013b] has obtained an alternate proof of this result, by exploiting the notion of *local mean oscillation* rather than dyadic shift in order to control the norm of a Calderón–Zygmund operator by the norm of some specific operators, called *sparse operators*. Most recently, Lacey [2015] and Lerner and Nazarov [2015] gave another proof, which gets around the use of local oscillation through pointwise control.

Simultaneously, in recent years people were also interested in weighted estimates for nonintegral singular operators in a space of homogeneous type. Even on Euclidean space, Riesz transforms $\nabla L^{-1/2}$ may be considered in several situations where we do not have pointwise regularity estimates of an integral kernel, for example $L = -\operatorname{div}(A\nabla)$ with bounded coefficients A, or $L = -\Delta + V$ with some potential V. The situation is even more difficult if we are looking at Riesz transforms on bounded subsets (with Neumann–Dirichlet conditions), second-order elliptic operators on Lipschitz domains, and Riesz transforms on Riemannian manifolds, for example. For all such operators, there is only a range of exponents (p_0, q_0) where we have L^p estimates for the semigroup $(e^{-tL})_t$ and its gradient for $p \in (p_0, q_0)$. Weighted estimates for such operators are more delicate, naturally restricted to these same ranges of p. We refer the reader to [Auscher and Martell 2007a] for a recent survey about weighted estimates.

In this current work, we aim to combine these two fashionable problems and give a modern approach to singular nonkernel operators. This setting had been resistant to many of the ideas developed in recent years. Indeed, we are going to adapt the approach of Lacey [2015] in order to be able to deal with nonintegral singular operators. The main idea relies on defining a suitable maximal operator and then controlling the operator by *sparse operators* (whose definition is modified from the previous works). We describe our method in a very general setting given by a space of homogeneous type, equipped with a semigroup. However, we point out that even in the Euclidean case our results are new, since they do not rely on any pointwise regularity estimates of the kernel of the considered operators. Moreover, we modify the maximal operator that we are going to use: instead of the maximal truncated operator used by Lacey [2015], we use truncation in the "frequency" point of view (where the notion of "frequencies" has to be understood in terms of the semigroup). Simultaneously, we will use a slightly weaker notion of sparse operators; both of these facts will allow us to give a proof which is simpler than Lacey's proof. However, we are not able to recover the full A_2 result in its generality: indeed, the assumptions we need require that the considered operator satisfies a suitable decomposition in the frequency point of view (see Remark 1.5). As shown in Section 3, that covers the main prototypes of operators. It is interesting to observe that the proof of these sharp weighted estimates can be substantially simplified in our situation and extended to operators whose kernel does not satisfy any regularity estimate. Recently, Bui, Conde-Alonso, Duong and Hormozi [Bui et al. 2015] have extended Lerner's approach for operators with kernels having L^{p_0}-L^∞ regularity estimates (which corresponds to the case $q_0 = \infty$, as we will see in Section 3D). We emphasize that this work is the first one where we are able to consider the case $q_0 < \infty$ and where no regularity is required on the eventual "kernel", which (as shown in the examples in Section 3) will allow us to deal with various situations in terms of operators and ambient spaces.

1A. ***The setting.*** Let M be a locally compact, separable metric space equipped with a Borel measure μ that is finite on compact sets and strictly positive on any nonempty open set. For a measurable subset Ω of M, we shall often denote $\mu(\Omega)$ by $|\Omega|$.

For all $x \in M$ and $r > 0$, let $B(x,r)$ be the open ball for the metric d with centre x and radius r. For a ball B of radius r and $\lambda > 0$, denote by λB the ball concentric with B and with radius λr. We sometimes denote by $r(B)$ the radius of the ball B. Finally, we will use $u \lesssim v$ to say that there exists a constant C (independent of the important parameters) such that $u \le Cv$, and $u \simeq v$ to say that $u \lesssim v$ and $v \lesssim u$. Moreover, for a subset $\Omega \subset M$ of finite and nonvanishing measure and $f \in L^1_{\mathrm{loc}}(M,\mu)$, we denote by $\fint_\Omega f\,d\mu = (1/|\Omega|)\int_\Omega f\,d\mu$ the average of f on Ω. We let $\mathcal{M}$ be the uncentred Hardy–Littlewood maximal operator. For $p \in [1,\infty)$, we abbreviate by $\mathcal{M}_p$ the operator defined by $\mathcal{M}_p(f) := \big(\mathcal{M}(|f|^p)\big)^{1/p}$ for $f \in L^p_{\mathrm{loc}}(M,\mu)$.

We shall assume that (M,d,μ) satisfies the volume doubling property, that is,

$$|B(x,2r)| \lesssim |B(x,r)| \quad \text{for all } x \in M,\ r > 0. \tag{VD}$$

It follows that there exists $\nu > 0$ such that

$$|B(x,r)| \lesssim \left(\frac{r}{s}\right)^{\nu} |B(x,s)| \quad \text{for all } x \in M,\ r \ge s > 0, \tag{VD$_\nu$}$$

which implies

$$|B(x,r)| \lesssim \left(\frac{d(x,y)+r}{s}\right)^{\nu} |B(y,s)| \quad \text{for all } x,y \in M,\ r \ge s > 0.$$

An easy consequence of (VD) is that balls with a nonempty intersection and comparable radii have comparable measures.

Let us recall that, for $0 \le \theta < \frac{\pi}{2}$, a linear operator L with dense domain $\mathcal{D}_2(L)$ in $L^2(M,\mu)$ is called *θ-accretive* if the spectrum $\sigma(L)$ of L is contained in the closed sector $S_{\theta+} := \{\zeta \in \mathbb{C} : |\arg\zeta| \le \theta\} \cup \{0\}$, and $\langle Lg, g\rangle \in S_{\omega+}$ for all $g \in \mathcal{D}_2(L)$.

We suppose that there exists an unbounded operator L on $L^2(M,\mu)$ satisfying these assumptions:

Assumptions on L. *Let L be an injective, θ-accretive operator with dense domain $\mathcal{D}_2(L)$ in $L^2(M,\mu)$, where $0 \le \theta < \frac{\pi}{2}$. We assume that there exist two exponents $1 \le p_0 < 2 < q_0 \le \infty$ such that, for all balls B_1, B_2 of radius $\sqrt{t}$,*

$$\|e^{-tL}\|_{L^{p_0}(B_1)\to L^{q_0}(B_2)} \lesssim |B_1|^{-1/p_0}|B_2|^{1/q_0}e^{-cd(B_1,B_2)^2/t}. \tag{1-1}$$

As a consequence, L is a maximal accretive operator on $L^2(M,\mu)$, and therefore has a bounded H^∞ functional calculus on $L^2(M,\mu)$. The assumption $\theta < \frac{\pi}{2}$ implies that $-L$ is the generator of an analytic semigroup $(e^{-tL})_{t>0}$ in $L^2(M,\mu)$ (see [Albrecht et al. 1996; Kato 1966] for definitions and further considerations). The last part in the assumption means that the considered semigroup satisfies L^{p_0}-L^{q_0} off-diagonal estimates, an extension of L^2-L^2 Davies–Gaffney estimates. In situations where pointwise heat kernel bounds fail (see below for examples), this has turned out to be an appropriate replacement.

In this work, we study weighted estimates for nonintegral singular operators satisfying some cancellation with respect to this operator. We consider a linear (or sublinear) operator T satisfying the following properties:

Assumptions. (a) *T is well-defined as a bounded operator in L^2.*

(b) (L^{p_0}-L^{q_0} off-diagonal estimates) *There exists $N_0 \in \mathbb{N}$ such that, for all integers $N \ge N_0$ and all balls B_1, B_2 of radius $\sqrt{t}$,*

$$\|T(tL)^N e^{-tL}\|_{L^{p_0}(B_1)\to L^{q_0}(B_2)} \lesssim |B_1|^{-1/p_0}|B_2|^{1/q_0}\left(1+\frac{d(B_1,B_2)^2}{t}\right)^{-\frac{\nu+1}{2}}.$$

(c) *There exists an exponent $p_1 \in [p_0, 2)$ such that, for all $x \in M$ and $r > 0$,*

$$\left(\fint_{B(x,r)} |Te^{-r^2L}f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \inf_{y\in B(x,r)} \mathcal{M}_{p_1}(Tf)(y) + \inf_{y\in B(x,r)} \mathcal{M}_{p_1}(f)(y).$$

Item (b) encodes the fact that the operator T has some cancellation property which interacts well with the cancellation of the considered semigroup. Item (c) is a property which allows us to get off-diagonal estimates for the low-frequency part of the operator T. We point out that (b) and (c) are the main assumptions and were already used in numerous works to replace the notion of Calderón–Zygmund operators (see, e.g., [Auscher 2007; Auscher et al. 2004] and references therein).

We will assume the above throughout the paper. We abbreviate the setting with (M,μ,L,T).

1B. *Results.* Consider the setting (M,μ,L,T) satisfying the above assumptions. Then we claim that such an operator satisfies weighted boundedness. Indeed, such an operator satisfies the three following properties:

- T is bounded on L^2.
- For every $r > 0$ and some integer N large enough, $T(I-e^{-r^2L})^N$ satisfies L^{p_0}-L^{q_0} off-diagonal estimates (outside the diagonal); see Corollary 4.2 for a precise statement.
- T satisfies the Cotlar-type inequality of Assumption (c) for some $p_1 < 2$.

We then know from [Auscher 2007, Theorems 1.1 and 1.2] — see also the earlier results in [Auscher and Martell 2007a; Blunck and Kunstmann 2003; Auscher et al. 2004] — that T is bounded in L^p for every $p \in (p_0, q_0)$. By [Auscher and Martell 2007a, Theorem 3.13], T also satisfies some weighted estimates: for every $p \in (p_0, q_0)$ and every weight $\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$ (see Section 6 for a precise definition of this class of weights), the operator T is bounded in L^p_ω. However, it is not clear from these previous results how the quantity $\|T\|_{L^p_\omega \to L^p_\omega}$ depends on the weight ω. The methods used do not tend to give optimal estimates.

Our main result is the following:

Theorem 1.1. *Consider the setting (M, μ, L, T) as above. For $p \in (p_0, q_0)$, there exists a constant c_p such that, for every weight $\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$,*

$$\|T\|_{L^p_\omega \to L^p_\omega} \le c_p \big([\omega]_{A_{p/p_0}} [\omega]_{\mathrm{RH}_{(q_0/p)'}}\big)^{\alpha},$$

with

$$\alpha := \max\left\{\frac{1}{p - p_0}, \frac{q_0 - 1}{q_0 - p}\right\}. \tag{1-2}$$

In particular, by defining the specific exponent

$$\mathfrak{p} := 1 + \frac{p_0}{q_0'} \in (p_0, q_0),$$

we have $\alpha = 1/(p - p_0)$ if $p \in (p_0, \mathfrak{p}]$, and $\alpha = (q_0 - 1)/(q_0 - p)$ if $p \in [\mathfrak{p}, q_0)$.

Remark 1.2. In the case $q_0 = p_0'$, we have $\mathfrak{p} = 2$ and obtain a sharp L^2_ω inequality with an exponent

$$\alpha = \frac{1}{2 - p_0}.$$

Remark 1.3. If $p_0 = 1$ and $q_0 = \infty$, we obtain $\alpha = \max\{1, 1/(p-1)\}$ and so we reprove the A_2 conjecture for such operators. Note that we are then able to prove these sharp estimates in the case of the Riesz transform $T = \nabla L^{-1/2}$ in the situation where this operator does not fit the Calderón–Zygmund framework (there is no pointwise regularity estimate of the full kernel); see Section 3.

Remark 1.4. We also prove the optimality of such estimates (in terms of the growth with respect to the characteristic of the weight) for sparse operators, which are shown to control our operators. The optimality also still holds for the operator itself if we know some "lower off-diagonal" estimates.

Remark 1.5. On the Euclidean space $\mathbb{R}^\nu$, consider the canonical heat semigroup and an "arbitrary" Calderón–Zygmund operator T: a linear L^2-bounded operator with a kernel K satisfying the regularity estimates

$$|K(x, y) - K(z, y)| + |K(y, x) - K(z, x)| \lesssim \left(\frac{d(y, z)}{d(y, z) + d(x, y)}\right)^{\varepsilon} d(x, y)^{-\nu}$$

for some $\varepsilon > 0$ and all points x, y, z with $2d(y, z) \le d(x, z)$. Then it is well known that T is L^p-bounded for every $p \in (1, \infty)$. Consequently, we can check that our Assumptions are satisfied for $p_0 = 1$ and any $q_0 < \infty$ as large as we want. Unfortunately, it is unclear to us if our approach could recover the optimal A_p estimates for arbitrary Calderón–Zygmund operators (which would correspond to $q_0 = \infty$). It appears that Assumption (b) describes an extra property on the operator T, a kind of suitable frequency decomposition or representation (as Fourier multipliers or paraproducts, for example). It is interesting to observe that, under this extra property, we are going to detail an "elementary" proof of the sharp weighted estimates (simpler than all the existing proofs, such as [Lerner 2013b; Lacey 2015]), which has also the very important property that is extends to nonintegral operators with no regularity property on the kernel.

We remark that this extra property already appeared in [Duong and McIntosoh 1999, Theorem 3], where boundedness of the maximal operator $T^\#$ (see Section 4 for the definition) in the case $q_0 = \infty$ was shown, and that this is also the only place where we are using it. See also [Duong and McIntosoh 1999, Remark, p. 251]. Moreover, as illustrated in Section 3, this extra property is satisfied for the main prototype of Calderón–Zygmund operators.

2. Notation and preliminaries on approximation operators

2A. ***Notation.*** For $p \in [1, \infty)$, a subset $E \subset M$ and a measure λ on E, we write $L^p(E, d\lambda)$ for the Lebesgue space, equipped with the norm

$$\|f\|_{L^p(E,d\lambda)} = \left(\int_E |f|^p \, d\lambda\right)^{\frac{1}{p}}.$$

For convenience, we forget E if $E = M$ is the whole space and λ if $\lambda = \mu$ is the underlying measure. Thus, L^p stands for $L^p(M, \mu)$. For a positive function ω, we write L^p_ω for the weighted Lebesgue space, equipped with the norm

$$\|f\|_{L^p_\omega} = \left(\int_M |f|^p \omega \, d\mu\right)^{\frac{1}{p}}.$$

For a positive function $\rho : M \to (0, \infty)$, we identify the function ρ with the measure $\rho \, d\mu$ in the sense that, for every measurable subset $E \subset M$, we use

$$\rho(E) = \int_E \rho \, d\mu.$$

For a ball B, we let $S_0(B) = 2B$ and $S_j(B) = 2^{j+1}B \setminus 2^j B$ for $j \geq 1$. By extending the average notion to coronas, we let

$$\fint_{S_j(B)} f \, d\mu = |2^j B|^{-1} \int_{S_j(B)} f \, d\mu.$$

2B. ***Operator estimates.*** The building blocks of our analysis will be the following operators derived from the semigroup $(e^{-tL})_{t>0}$. They serve as a replacement for Littlewood–Paley operators.

Two different classes of elementary operators will be needed: $(P_t)_{t>0}$, corresponding to an approximation of the identity at scale $\sqrt{t}$ commuting with the heat semigroup, and $(Q_t)_{t>0}$, which satisfies some extra cancellation with respect to L.

Definition 2.1. Let $N > 0$ and set $c_N = \int_0^{+\infty} s^N e^{-s}\, ds/s$. For $t > 0$, define

$$Q_t^{(N)} := c_N^{-1} (tL)^N e^{-tL} \tag{2-1}$$

and

$$P_t^{(N)} := \int_1^{\infty} Q_{st}^{(N)} \frac{ds}{s} = \phi_N(tL), \tag{2-2}$$

with $\phi_N(x) := c_N^{-1} \int_x^{+\infty} s^N e^{-s}\, ds/s$ for $x \geq 0$.

Remarks 2.2. Let $p \in [p_0, q_0]$ with $p < \infty$ and $N > 0$.

(i) Note that $P_t^{(1)} = e^{-tL}$ and $Q_t^{(1)} = tLe^{-tL}$. The two families of operators $(P_t^{(N)})_{t>0}$ and $(Q_t^{(N)})_{t>0}$ are related by

$$t\partial_t P_t^{(N)} = tL\phi_N'(tL) = -Q_t^{(N)}.$$

(ii) If N is an integer, then $Q_t^{(N)} = (-1)^N c_N^{-1} t^N \partial_t^N e^{-tL}$ and $P_t^{(N)} = p(tL)e^{-tL}$, where p is a polynomial of degree $N-1$ with $p(0) = 1$.

(iii) By L^p analyticity of the semigroup and (1-1), we know that, for every integer $N > 0$ and every $t > 0$, $P_t^{(N)}$ and $Q_t^{(N)}$ satisfy off-diagonal estimates at the scale $\sqrt{t}$. See the arguments in [Hofmann et al. 2011, Proposition 3.1], for example.

(iv) The operators $P_t^{(N)}$ and $Q_t^{(N)}$ are bounded in L^p, uniformly in $t > 0$. See [Auscher and Martell 2007b, Theorem 2.3], taking into account (iii).

Proposition 2.3 (Calderón reproducing formula). *Let $N > 0$ and $p \in (p_0, q_0)$. For every $f \in L^p$,*

$$\lim_{t \to 0^+} P_t^{(N)} f = f \quad \text{in } L^p, \tag{2-3}$$

$$\lim_{t \to +\infty} P_t^{(N)} f = 0 \quad \text{in } L^p, \tag{2-4}$$

and

$$f = \int_0^{+\infty} Q_t^{(N)} f \frac{dt}{t} \quad \text{in } L^p. \tag{2-5}$$

In particular, it follows that as L^p-bounded operators we have the decomposition

$$P_t^{(N)} = \mathrm{Id} - \int_0^t Q_s^{(N)} \frac{ds}{s}. \tag{2-6}$$

3. Examples and applications

Our assumptions on L hold for a large variety of second-order operators, for example uniformly elliptic operators in divergence form and Schrödinger operators with singular potentials on $\mathbb{R}^n$, or the Laplace–Beltrami operator on a Riemannian manifold. For more precise examples of L and references, see Section 3B, where we give some examples of singular integral operators T that fit into our setting. See also [Auscher and Martell 2006].

3A. *Holomorphic functional calculus of L.* Let $0 \le \theta < \sigma < \pi$, where θ denotes the angle of accretivity of L. Define the open sector in the complex plane of angle σ by

$$S_\sigma^o := \{z \in \mathbb{C} : z \neq 0,\ |\arg z| < \sigma\}.$$

Denote by $H(S_\sigma^o)$ the space of all holomorphic functions on S_σ^o, and let

$$H^\infty(S_\sigma^o) := \{\varphi \in H(S_\sigma^o) : \|\varphi\|_\infty < \infty\}.$$

By our assumptions, L has a bounded H^∞ functional calculus on L^2. Blunck and Kunstmann [2003] showed that, under the assumption (1-1), the functional calculus can be extended to L^p for $p \in (p_0, q_0)$.

We now obtain the following weighted version: Let $\sigma > \theta$ and let $\varphi \in H^\infty(S_\sigma^o)$. Set $T = \varphi(L)$. We check our Assumptions. Item (a) is a restatement of the fact that L has a bounded H^∞ functional calculus on L^2. Since T commutes with $e^{-r^2 L}$, we can obtain (c) as a consequence of (1-1) (we do not detail this here; similar estimates are done in the sequel). Finally, for large enough N, by adapting [Auscher et al. 2008, Lemma 3.6] one can show that $\varphi(L)(tL)^N e^{-tL}$ satisfies L^{q_0}-L^{q_0} off-diagonal estimates. Combining this with L^{p_0}-L^{q_0} off-diagonal estimates for e^{-tL} gives (b). We therefore have:

Theorem 3.1. *Let $p \in (p_0, q_0)$ and $\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$. The operator L has a bounded holomorphic functional calculus in L^p_ω with, for every $\sigma > \theta$,*

$$\|\varphi(L)\|_{L^p_\omega \to L^p_\omega} \le c_{p,\sigma}\big([\omega]_{A_{p/p_0}}[\omega]_{\mathrm{RH}_{(q_0/p)'}}\big)^\alpha \|\varphi\|_\infty$$

for all $\varphi \in H^\infty(S_\sigma^o)$ and α as defined in (1-2).

3B. *Riesz transforms.* The L^p-boundedness of Riesz transforms on manifolds has been widely studied in recent years. We refer the reader to [Bernicot and Frey 2015] for recent work and references for more details about such operators.

Several situations fit into our setting; we can consider specific operators, or specific ambient spaces, or both. Let us give some examples; more can be studied, like Riesz transforms on bounded domains or those associated with Schrödinger operators.

Dirichlet forms. Let (M, d, μ) be a complete space of homogeneous type, as above. Consider a self-adjoint operator L on L^2 and the quadratic form $\mathscr{E}$ associated with L, that is,

$$\mathscr{E}(f, g) = \int_M f L g \, d\mu.$$

If $\mathscr{E}$ is a strongly local and regular Dirichlet form (see [Fukushima et al. 1994; Gyrya and Saloff-Coste 2011] for precise definitions) with a carré du champ structure, then, with Γ being equal to this carré du champ operator, assume that the Poincaré inequality (P_2) holds, that is,

$$\left(\fint_B \left| f - \fint_B f \, d\mu \right|^2 d\mu\right)^{\frac{1}{2}} \lesssim r \left(\fint_B d\Gamma(f, f)\right)^{\frac{1}{2}} \tag{P_2}$$

for every $f \in \mathscr{D}(\mathscr{E})$ and every ball $B \subset M$ with radius r.

If the heat semigroup $(e^{-tL})_{t>0}$ and its carré du champ $(\sqrt{t}\Gamma e^{-tL})_{t>0}$ satisfy L^{p_0}-L^{q_0} off-diagonal estimates, then it can be checked that our Assumptions are satisfied for the Riesz transform (see [Auscher et al. 2004])

$$\mathfrak{R} := \Gamma L^{-1/2} = c_k \Gamma\left(\int_0^\infty (tL)^k e^{-tL} \frac{dt}{\sqrt{t}}\right),$$

for some numerical constant c_k and every integer $k \geq 1$.[1]

In particular, for $p_0 = 1$ and $q_0 = \infty$, we get the following result:

Theorem 3.2. *Consider the Riesz transform $\mathfrak{R}$ in one of the following situations*:

- *Euclidean space or any doubling Riemannian manifold with bounded geometry and nonnegative Ricci curvature* (see [Li and Yau 1986]).
- *In a convex doubling subset of $\mathbb{R}^\nu$ with the Laplace operator associated with Neumann boundary conditions* (see [Wang and Yan 2013]).

Then, for every $p \in (1,\infty)$ and every weight $\omega \in A_p$, we have

$$\|\mathfrak{R}\|_{L^p_\omega \to L^p_\omega} \lesssim [\omega]_{A_p}^\alpha \quad \textit{with } \alpha = \max\left\{1, \frac{1}{p-1}\right\}.$$

Note that in these situations we only have Lipschitz regularity of the heat kernel; the full kernel of the Riesz transform does not satisfy any pointwise regularity estimate and so does not fit into the class of Calderón–Zygmund operators (as previously studied in [Lerner 2013b; Lacey 2015]).

Second-order divergence form operators. Consider a doubling Riemannian manifold (M, d, μ), equipped with the Riemannian gradient ∇ and its divergence operator $\mathrm{div} = \nabla^*$. To a complex, bounded, measurable, matrix-valued function $A = A(x)$, defined on M and satisfying the ellipticity (or accretivity) condition $\mathrm{Re}(A(x)) \geq \kappa I > 0$ a.e., we may define a second-order divergence form operator

$$L = L_A f := -\mathrm{div}(A\nabla f).$$

Then L is sectorial and satisfies the conservation property but may not be self-adjoint.

Assume that the Poincaré inequality (P_2) holds on (M, d, μ). If the semigroup $(e^{-tL})_{t>0}$ and its gradient $(\sqrt{t}\nabla e^{-tL})_{t>0}$ satisfy L^{p_0}-L^{q_0} off-diagonal estimates, then it can be checked that our Assumptions are satisfied for the Riesz transform

$$\mathfrak{R} := \nabla L^{-1/2} = c_k \int_0^\infty \nabla (tL)^k e^{-tL} \frac{dt}{\sqrt{t}}.$$

We refer the reader to [Auscher 2007] for a precise study in the Euclidean setting of the exponents p_0 and q_0 depending on the matrix-valued map A. For example, we have $p_0 = 1$ and $q_0 = \infty$ in dimension $\nu = 1$.

[1] It is known that the assumed Poincaré inequality (P_2) self-improves into a Poincaré inequality (P_{p_1}) for some $p_1 < 2$ (see [Keith and Zhong 2008]), which allows us to check Assumption (c).

3C. ***Paraproducts associated with L.*** Throughout this subsection we assume that the semigroup satisfies the conservation property, which means that $e^{-tL}1 = 1$ for every $t > 0$, as well as the fact that the semigroup is supposed to have a heat kernel with pointwise Gaussian bounds (which correspond to L^1-L^∞ estimates).

Paraproducts with a BMO function. In recent works [Bernicot 2012; Frey 2013], several paraproducts have been studied in the context of a semigroup. They allow us to have (as is well known in Euclidean space) a decomposition of the pointwise product with two paraproducts and a resonant term (we also refer the reader to [Bailleul et al. 2015] for some applications of such paraproducts in the context of paracontrolled calculus for solving singular PDEs). Moreover, BMO spaces adapted to such a framework have been the focus of numerous works, so it is natural (as in the Euclidean setting) to study the linear operator given by the paraproduct of a BMO function.

Let us recall some definitions. A BMO_L function is a locally integrable function $f \in L^1_{\mathrm{loc}}$ such that

$$\|f\|_{\mathrm{BMO}_L} := \sup_B \left(\fint_B |f - e^{-r^2 L} f|^2 \, d\mu \right)^{\frac{1}{2}},$$

where we take the supremum over all balls B with radius $r > 0$. Such BMO spaces satisfy "standard" properties, such as the John–Nirenberg inequality and $T(1)$ theorem. In particular it is known (see [Bernicot and Zhao 2012; Bernicot and Martell 2015]) that, since the semigroup satisfies L^1-L^∞ off-diagonal estimates, the norm in BMO_L can be built through an L^p oscillation for any $p \in (1, \infty)$ and the corresponding norms are equivalent. For some integer k, the paraproduct under consideration is

$$\Pi_g(f) = \int_0^\infty Q_t^{(k)} (Q_t^{(k)} f \cdot P_t^{(k)} g) \, \frac{dt}{t}.$$

Using square function estimates, we then know that $Q_t^{(k)} g$ is uniformly bounded in L^∞ for $g \in \mathrm{BMO}_L$, so that Π_g is L^2-bounded. Assumptions (b) and (c) are also satisfied with $p_0 = 1$ and $q_0 = \infty$ (see details in the above references) and so we may apply Theorem 1.1 to the previous paraproduct for $g \in \mathrm{BMO}_L$.

Algebra property for fractional Sobolev spaces. Bernicot, Coulhon, and Frey [Bernicot et al. 2015] have used some paraproducts associated with such a framework involving a heat semigroup in order to study the algebra property for fractional Sobolev spaces. We refer to [Bernicot et al. 2015] for more details and references for other paraproducts associated with a semigroup. Then, up to some constant c_N, we have the product decomposition for two functions

$$fg = \Pi_g(f) + \Pi_f(g),$$

with the paraproduct defined by

$$\Pi_g(f) = \int_0^\infty Q_t^{(N)} f \cdot P_t^{(N)} g \, \frac{dt}{t}.$$

Fix a function $g \in L^\infty$; then, for $\alpha \in (0, 1)$, we are looking for the $\dot{L}^p_\alpha$-boundedness of Π_g, which corresponds the to L^p-boundedness of $T := L^{\alpha/2} \Pi_g L^{-\alpha/2}$. In [Bernicot et al. 2015], we gave different situations and criteria under which our Assumptions are satisfied. Mainly we considered the condition, introduced in [Auscher et al. 2004], that, for some $p \in (2, \infty]$,

$$\sup_{t>0} \big\| \sqrt{t} |\Gamma e^{-tL}| \big\|_{p \to p} < +\infty, \tag{G_p}$$

where Γ is the carré du champ associated with the operator L (and $|\Gamma \cdot|$ is its modulus). In this way, we may apply Theorem 1.1 to T and obtain a sharp algebra property for weighted fractional spaces, sharp with respect to the weight. We obtain the following estimates:

Theorem 3.3. *Let $(M, d, \mu, \mathscr{E})$ be a doubling metric measure Dirichlet space with a carré du champ* (see [Bernicot et al. 2015] for more details) *and assume that the heat semigroup e^{-tL} has a heat kernel with usual pointwise Gaussian estimates. For some $s \in (0, 1)$ and $p \in (1, \infty)$, consider the following weighted Leibniz rule: for every weight ω and all functions $f, g \in \{h \in L^\infty : L^{s/2}(h) \in L^p_\omega\}$,*

$$\|L^{s/2}(fg)\|_{L^p_\omega} \lesssim c(\omega)\big(\|L^{s/2} f\|_{L^p_\omega} \|g\|_\infty + \|f\|_\infty \|L^{s/2} g\|_{L^p_\omega}\big). \tag{3-1}$$

(a) (3-1) *is valid for $p \in (1, 2)$ and $s \in (0, 1)$ with every weight $\omega \in A_p \cap \mathrm{RH}_{(2/p)'}$ and a constant*

$$c(\omega) = ([\omega]_{A_p} [\omega]_{\mathrm{RH}_{(2/p)'}})^\alpha \quad \textit{with} \quad \alpha := \max\Big\{\frac{1}{p-1}, \frac{1}{2-p}\Big\}.$$

(b) *Under (G_q) for some $q \in (2, \infty)$,* (3-1) *is valid for $p \in (1, q)$ and $s \in (0, 1)$ with every weight $\omega \in A_p \cap \mathrm{RH}_{(q^-/p)'}$, where $q^- \in (p, q)$, and a constant*

$$c(\omega) = ([\omega]_{A_p} [\omega]_{\mathrm{RH}_{(q^-/p)'}})^\alpha \quad \textit{with} \quad \alpha := \max\Big\{\frac{1}{p-1}, \frac{q^- - 1}{q^- - p}\Big\}.$$

(c) *Under (G_∞),* (3-1) *is valid for $p \in (1, \infty)$ and $s \in (0, 1)$ with every weight $\omega \in A_p$ and a constant*

$$c(\omega) = [\omega]_{A_p}^\alpha \quad \textit{with} \quad \alpha := \max\Big\{\frac{1}{p-1}, 1\Big\}.$$

Other estimates can be obtained by combining the results of this paper with the other estimates of [Bernicot et al. 2015].

3D. ***Fourier multipliers.*** Let us also explain how we can recover the results of [Bui et al. 2015]. The main linear result [Bui et al. 2015, Theorem C] fits into our framework and corresponds to the particular case $q_0 = \infty$. Let us focus on the application to linear Fourier multipliers.

Consider a linear symbol m on $\mathbb{R}^\nu$ satisfying the Hörmander condition $M(s, l)$, which is

$$\sup_{R>0} \bigg(R^{s|\alpha|-\nu} \int_{R \le |\xi| \le 2R} |\partial_\xi^\alpha m(\xi)|^s \, d\xi\bigg)^{\frac{1}{s}} < \infty$$

for all $|\alpha| \le l$, some $s \in (1, 2]$ and $l \in (\nu/s, \nu)$. To this symbol we associate the linear Fourier multiplier

$$T(f) = T_m(f) : x \mapsto \int e^{ix.\xi} m(\xi) \widehat{f}(\xi) \, d\xi.$$

For every $r \in (\nu/l, \infty)$, [Bui et al. 2015, Lemma 5.2] shows that the kernel of T satisfies some L^r-L^∞ regularity off-diagonal estimates. So consider a smooth function ψ such that $\widehat{\psi}$ is supported on

$B(0,4)\setminus B(0,1)$ and well-normalized with $\int_0^\infty \hat{\psi}(t\xi)\,dt/t = 1$ for every ξ. Then, with the elementary operators

$$T_t(f) : x \mapsto \int e^{ix.\xi} m(\xi)\hat{\psi}(t\xi)\hat{f}(\xi)\,d\xi,$$

it can be proved that our Assumptions are satisfied for $p_0 = r$ and $q_0 = \infty$. Consequently, Theorem 1.1 allows us to regain [Bui et al. 2015, Theorem 5.3(a)]. Moreover, since T is self-adjoint, by duality we also deduce that the kernel of T satisfies some L^1-$L^{r'}$ off-diagonal estimates. Similarly, one can then show that our Assumptions are satisfied for $p_0 = 1$ and $q_0 = r'$. Consequently, Theorem 1.1 allows us to regain [Bui et al. 2015, Theorem 5.3(b)]. So we regain the same full result as [Bui et al. 2015, Theorem 5.3], with the exact same behaviour of the weighted estimates with respect to the weight.

The same comparison can be done for the linear part of their main result [Bui et al. 2015, Theorem C]. Under their assumptions (H1) and (H2), our Assumptions are satisfied with $q_0 = \infty$. We leave the details to the reader.

4. Unweighted boundedness of a certain maximal operator

Before introducing and studying a certain maximal operator related to T, we first explain some technical details of off-diagonal estimates.

4A. ***Off-diagonal estimates.*** We fix an integer $N > N_0$ (with N_0 as in our Assumptions) and write, for $t > 0$,

$$T_t := TQ_t^{(N)}.$$

Let $p \in (p_0, q_0)$. The Calderón reproducing formula (see Proposition 2.3) gives the identity

$$\mathrm{Id} = \int_0^\infty Q_t^{(N)}\,\frac{dt}{t}$$

in L^p. Since T is assumed to be sublinear, we can decompose the operator for $f \in L^p$ into

$$|T(f)| \leq \int_0^\infty |T_t(f)|\,\frac{dt}{t}. \tag{4-1}$$

Fix $t > 0$ and the elementary operator T_t. From Assumption (b) we know that T_t satisfies L^{p_0}-L^{q_0} off-diagonal estimates at the scale $\sqrt{t}$. Then consider a ball B of radius $r > 0$ with $r \leq \sqrt{t}$ and its dilated ball $\widetilde{B} := (\sqrt{t}/r)B$. We have $B \subset \widetilde{B}$ and $|\widetilde{B}| \lesssim (\sqrt{t}/r)^\nu |B|$, so

$$\left(\fint_B |T_t f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \left(\frac{\sqrt{t}}{r}\right)^{\frac{\nu}{q_0}} \sum_{j\geq 0} 2^{-j(\nu+1)} \left(\fint_{S_j(\widetilde{B})} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}. \tag{4-2}$$

Lemma 4.1. *Consider three parameters $r, \varepsilon, t > 0$. Let $N \in \mathbb{N}$ with $N > \max\{\frac{3}{2}\nu + 1,\ N_0\}$.*

(1) *If $r^2 < \varepsilon < t$, we have, for every ball B_r of radius r and the dilated ball $B_{\sqrt{\varepsilon}} = (\sqrt{\varepsilon}/r)B_r$,*

$$\left(\fint_{B_{\sqrt{\varepsilon}}} |T_t(I - e^{-r^2 L})^N f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \left(\frac{r^2}{t}\right)^{\frac{N}{2}} \sum_{l\geq 0} 2^{-l(\nu+1)} \left(\fint_{2^l B_r} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.$$

(2) *If* $t < \varepsilon < r^2$, *we have, for every ball* B_r *of radius* r, *every* $j \geq 3$, *every ball* $B_{\sqrt{\varepsilon}}$ *of radius* $\sqrt{\varepsilon}$ *included in* $S_j(B_r)$, *and every function* f *supported on* B_r,

$$\left(\fint_{B_{\sqrt{\varepsilon}}} |T_t(I-e^{-r^2L})^N f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim 2^{-j(\nu+1)}\left(\frac{t}{r^2}\right)^{\frac{1}{2}}\left(\fint_{B_r} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.$$

The same estimates are true for $T_t(I-P^{(N)}_{r^2})$ *in place of* $T_t(I-e^{-r^2L})^N$.

Proof. Consider the first case, $r^2 < \varepsilon < t$. We show the result for $T_t(I-P^{(N)}_{r^2})$, and then explain how to modify the proof in the case of $T_t(I-e^{-r^2L})^N$. By the definition of $P^{(N)}_{r^2}$, we have

$$I - P^{(N)}_{r^2} = \int_0^{r^2} Q^{(N)}_s\,\frac{ds}{s}.$$

Hence,

$$\left(\fint_{B_{\sqrt{\varepsilon}}} |T_t(I-P^{(N)}_{r^2})f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \int_0^{r^2}\left(\fint_{B_{\sqrt{\varepsilon}}} |T_t Q^{(N)}_s f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}\frac{ds}{s}.$$

Note that $T_t = TQ^{(N)}_t$ and $Q^{(N)}_t Q^{(N)}_s = (s/(s+t))^N Q^{(2N)}_{s+t}$ (up to a numerical constant) as well as $s+t \simeq t$. Using Assumption (b) and (4-2) for $s < r^2$ with $r^2 < \varepsilon < t$, we obtain that

$$\begin{aligned}\left(\fint_{B_{\sqrt{\varepsilon}}} |T_t Q^{(N)}_s f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} &\lesssim \left(\frac{s}{t}\right)^N\left(\fint_{B_{\sqrt{\varepsilon}}} |T_{s+t} f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}\\ &\lesssim \left(\frac{s}{t}\right)^N\left(\frac{t}{\varepsilon}\right)^{\frac{\nu}{2q_0}}\sum_{l\geq 0} 2^{-l(\nu+1)}\left(\fint_{2^l B_{\sqrt{t}}} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}},\end{aligned}$$

where we used that $s+t \lesssim t$. Since $s < r^2 < \varepsilon$, we can estimate $(s/t)^N(t/\varepsilon)^{\nu/2q_0}$ by $(s/t)^{N-\nu/2q_0}$, and then deduce that, for $k \geq 0$ such that $2^k r \simeq \sqrt{t}$,

$$\begin{aligned}\left(\frac{s}{t}\right)^{N-\frac{\nu}{2q_0}} 2^{-l(\nu+1)}\left(\fint_{2^l B_{\sqrt{t}}} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}} &\lesssim \left(\frac{r^2}{t}\right)^{\frac{N}{2}}\left(\frac{s}{t}\right)^{\frac{N}{2}-\frac{\nu}{2q_0}} 2^{-l(\nu+1)}\left(\fint_{2^l B_{\sqrt{t}}} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\\ &\lesssim \left(\frac{r^2}{t}\right)^{\frac{N}{2}}\left(\frac{s}{r^2}\right)^{\frac{\nu+1}{2}} 2^{-(l+k)(\nu+1)}\left(\fint_{2^{l+k} B_r} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}},\end{aligned}$$

where we used that N is sufficiently large that $\frac{1}{2}N - \nu/(2q_0) > \frac{1}{2}(\nu+1)$. We then conclude by summing over l and integrating over $s \in (0, r^2)$.

In the second case, when $t < \varepsilon < r^2$, we follow the same reasoning: with $\tau = \max\{s, t\}$,

$$\begin{aligned}\left(\fint_{B_{\sqrt{\varepsilon}}} |T_t Q^{(N)}_s f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} &\lesssim \left(\frac{\min\{s,t\}}{\tau}\right)^{N-\frac{\nu}{2q_0}}\left(\frac{\tau}{2^{2j}r^2}\right)^{\frac{\nu+1}{2}}\left(\frac{r^2}{\tau}\right)^{\frac{\nu}{2p_0}}\left(\fint_{B_r} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\\ &\lesssim \left(\frac{\min\{s,t\}}{\max\{s,t\}}\right)^{\frac{N}{2}-\frac{\nu}{2q_0}}\left(\frac{t}{r^2}\right)^{\frac{1}{2}} 2^{-j(\nu+1)}\left(\fint_{B_r} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}},\end{aligned}$$

where we used that $N > \nu + 1$. We may now integrate over s and obtain the desired result.

The modifications required for the case $T_t(I - e^{-r^2 L})^N$ are straightforward. We first observe that

$$(I - e^{-r^2 L})^N = \left(\int_0^{r^2} L e^{-sL} \, ds \right)^N = \int_0^{Nr^2} \alpha(s)(sL)^N e^{-sL} \, \frac{ds}{s}$$

with

$$\alpha(s) := s^{1-N} \left| \{ (s_1, \dots, s_N) \in (0, r^2)^N : s_1 + \cdots + s_N = s \} \right| \lesssim 1.$$

Define $\psi_s^{(N)}(L) := \alpha(s)(sL)^N e^{-sL}$. Then

$$(I - e^{-r^2 L})^N = \int_0^{Nr^2} \psi_s^{(N)}(L) \, \frac{ds}{s}.$$

Now we can also write $Q_t^{(N)} \psi_s^{(N)}(L) = (\min\{s,t\}/\max\{s,t\})^N \Theta_{s,t}$ with some operator $\Theta_{s,t}$ satisfying L^{p_0}-L^{q_0} off-diagonal estimates, and conclude as above. □

Considering the particular case $\varepsilon = r^2$, we may integrate over t the two inequalities of Lemma 4.1 and, from (4-1), deduce the following result:

Corollary 4.2. *For an integer $N > \max\{\frac{3}{2}\nu + 1, N_0\}$ and $r > 0$, $T(I - e^{-r^2 L})^N$ satisfies L^{p_0}-L^{q_0} (strictly) off-diagonal estimates at the scale $r > 0$: if B_1 and B_2 are two balls of radius $r > 0$ with $d(B_1, B_2) > 4r$, then for every function f supported on B_1 we have*

$$\left(\fint_{B_2} |T(I - e^{-r^2 L})^N f|^{q_0} \, d\mu \right)^{\frac{1}{q_0}} \lesssim \left(1 + \frac{d(B_1, B_2)}{r} \right)^{-(\nu+1)} \left(\fint_{B_1} |f|^{p_0} \, d\mu \right)^{\frac{1}{p_0}}.$$

4B. ***Maximal operator.*** We now fix an integer $N > \max\{\frac{3}{2}\nu + 1, \, N_0\}$ (and all the implicit constants may depend on it).

Definition 4.3. Define the maximal operator $T^\#$ of T by

$$T^\# f(x) = \sup_{\substack{B \text{ ball} \\ B \ni x}} \left(\fint_B \left| T \int_{r(B)^2}^\infty Q_t^{(N)} f \, \frac{dt}{t} \right|^{q_0} d\mu \right)^{\frac{1}{q_0}}, \quad x \in M,$$

for $f \in L^{q_0}_{\mathrm{loc}}$.

By definition of $P_t^{(N)} := \int_1^\infty Q_{st}^{(N)} \, ds/s$, we then have

$$T^\# f(x) = \sup_{\substack{B \text{ ball} \\ B \ni x}} \left(\fint_B |T P_{r(B)^2}^{(N)} f|^{q_0} \, d\mu \right)^{\frac{1}{q_0}}, \quad x \in M,$$

for $f \in L^{q_0}_{\mathrm{loc}}$.

Lemma 4.4. *Consider a sequence $(u_\varepsilon)_\varepsilon$ of L^2 functions which converges (in L^2) to some function $u \in L^2$ when ε tends to 0. Then, for almost every $x \in M$, we have*

$$|u(x)| \leq \liminf_{\varepsilon \to 0} \fint_{B(x,\varepsilon)} |u_\varepsilon| \, d\mu.$$

Proof. Due to the Lebesgue differentiation lemma, we know that

$$|u(x)| \le \liminf_{\varepsilon\to 0} \fint_{B(x,\varepsilon)} |u| \, d\mu.$$

Then we split, as follows:

$$\fint_{B(x,\varepsilon)} |u| \, d\mu \le \fint_{B(x,\varepsilon)} |u_\varepsilon| \, d\mu + \fint_{B(x,\varepsilon)} |u-u_\varepsilon| \, d\mu.$$

The second part is pointwise bounded by $\mathcal{M}[u_\varepsilon - u](x)$, which converges in L^2 to 0 (due to the L^2-boundedness of the maximal function), which allows us to conclude the proof. □

As a consequence of the previous lemma with the L^2-boundedness of T and Proposition 2.3, we deduce the following result:

Corollary 4.5. *For every function $f \in L^2$ we have, almost everywhere,*

$$|T(f)| \le T^{\#}(f).$$

Proposition 4.6. *The sublinear operator $T^{\#}$ is of weak type (p_0, p_0) and is bounded in L^p for every $p \in (p_0, 2]$.*

Remark 4.7. In the definition of the maximal operator, the previous boundedness still holds if we replace the average on the ball B by any average on λB for some constant $\lambda > 1$. In this case, the implicit constants will depend on λ.

Proof. We proceed in two steps:

Step 1 (L^2-boundedness of $T^{\#}$). We first claim that $T^{\#}$ satisfies the following Cotlar-type inequality ($p_1 \in [p_0, 2)$ is introduced in our Assumptions):

$$T^{\#} f(x) \lesssim \mathcal{M}_{p_1}(Tf)(x) + \mathcal{M}_{p_1} f(x), \quad x \in M. \tag{4-3}$$

Indeed,

$$T^{\#} f(x) = \sup_{\substack{B \text{ ball} \\ B \ni x}} \left(\fint_B |T P^{(N)}_{r(B)^2} f|^{q_0} \, d\mu \right)^{\frac{1}{q_0}}$$

and, since N is an integer, we have by Remark 2.2(ii), with $r = r(B)$, that

$$P^{(N)}_{r^2} = p(r^2 L) e^{-r^2 L}$$

with p a polynomial function. We then factor as

$$T P^{(N)}_{r^2} = (T e^{-r^2 L/2})(p(r^2 L) e^{-r^2 L/2}).$$

By Assumption (c), $Te^{-r^2L/2}$ satisfies some L^{p_1}-L^{q_0} estimates and, by Assumption (b) and Lemma 4.1, both $T(I - P^{(N)}_{r^2})$ and $p(r^2L)e^{-r^2L/2}$ satisfy L^{p_1}-L^{p_1} off-diagonal estimates at the scale r. We may compose these two estimates in order to obtain similar estimates as Assumption (c) for $TP^{(N)}_{r^2}$ and then directly obtain (4-3).

This in particular implies that $T^\#$ is bounded on L^2, since T is bounded on L^2 by assumption, and the Hardy–Littlewood maximal operator $\mathcal{M}_{p_1}$ is bounded on L^2 as $p_1 < 2$.

In the second step, we now use the extrapolation method of [Auscher 2007; Blunck and Kunstmann 2003] to show that $T^\#$ is of weak type (p_0, p_0), which by interpolation with the L^2-boundedness will conclude the proof of the proposition.

Step 2 (weak type (p_0, p_0) of $T^\#$). We apply [Auscher 2007, Theorem 1.1] (see also [Blunck and Kunstmann 2003]). As shown in Step 1, $T^\#$ is bounded on L^2. By assumption, we know that $(e^{-tL})_{t>0}$ satisfies L^{p_0}-L^2 off-diagonal estimates. It remains to show that $T^\#(I-e^{-tL})^N$ satisfies L^{p_0}-L^2 off-diagonal estimates (not including the diagonal), where we will use (for convenience, but it could be chosen differently) the same integer N as the one defining the maximal operator, which is chosen sufficiently large. More precisely, for a ball $B \subseteq M$ of radius r and a function $b \in L^{p_0}$ with $\operatorname{supp} b \subseteq B$, we will show that

$$|2^{j+1}B|^{-1/2}\|T^\#(I-e^{-r^2L})^N b\|_{L^2(S_j(B))} \lesssim c(j)|B|^{-1/p_0}\|b\|_{L^{p_0}(B)}, \quad j \geq 3, \tag{4-4}$$

with coefficients $c(j)$ satisfying $\sum_{j\geq 2} c(j)2^{\nu j} < \infty$.

For $x \in M$ and $\varepsilon > 0$, denote by $B_{x,\varepsilon}$ a ball of radius $\sqrt{\varepsilon}$ containing x. Then recall that $T_t = TQ_t^{(N)}$ and

$$T^\#(I-e^{-r^2L})^N b(x) \leq \sup_{\varepsilon>0}\left(\fint_{B_{x,\varepsilon}}\left|T\int_\varepsilon^\infty Q_t^{(N)}(I-e^{-r^2L})^N b\,\frac{dt}{t}\right|^{q_0} d\mu\right)^{\frac{1}{q_0}}.$$

Let $x \in S_j(B)$ for $j \geq 3$ and consider first the case $r^2 < \varepsilon$. Applying Lemma 4.1(1), we then deduce that

$$\left(\fint_{B_{x,\varepsilon}}|T_t(I-e^{-r^2L})^N b|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \left(\frac{r^2}{t}\right)^{\frac{N}{2}}\left(1+\frac{d(B,B_{x,\varepsilon})^2}{t}\right)^{-\frac{\nu+1}{2}}\left(\fint_B |b|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.$$

Since $\varepsilon < t$, it follows that either $2^j r \geq \sqrt{t}$, in which case $d(B, B_{x,\varepsilon}) \simeq 2^j r$, or $2^j r \leq \sqrt{t}$, in which case $d(B, B_{x,\varepsilon}) \leq 2\sqrt{t}$. So, in both situations, we have

$$1+\frac{d(B,B_{x,\varepsilon})^2}{t} \simeq 1+\frac{4^j r^2}{t}.$$

Consequently, we get

$$\left(\fint_{B_{x,\varepsilon}}|T_t(I-e^{-r^2L})^N b|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \left(\frac{r^2}{t}\right)^{\frac{N}{2}}\left(1+\frac{4^j r^2}{t}\right)^{-\frac{\nu+1}{2}}\left(\fint_B |b|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.$$

We then have to integrate along $t \in (\varepsilon, \infty)$ and we split the integral into two parts, depending on whether $t < 4^j r^2$ or $t > 4^j r^2$. We then obtain that

$$\left(\fint_{B_{x,\varepsilon}}\left|\int_\varepsilon^\infty T_t(I-e^{-r^2L})^N b\,\frac{dt}{t}\right|^{q_0}d\mu\right)^{\frac{1}{q_0}}$$
$$\lesssim\left(\int_\varepsilon^{4^jr^2}2^{-j(\nu+1)}\left(\frac{t}{r^2}\right)^{\frac14}\frac{dt}{t}+\int_{4^jr^2}^\infty\left(\frac{r^2}{t}\right)^{\frac N2}\frac{dt}{t}\right)\left(\fint_B|b|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}$$
$$\lesssim 2^{-j(\nu+1/2)}\left(\fint_B|b|^{p_0}\,d\mu\right)^{\frac{1}{p_0}},$$

which corresponds to the desired estimate (4-4) with $c(j)=2^{-j(\nu+1/2)}$.

Consider now the case $\varepsilon\le r^2$. Again, let $x\in S_j(B)$ for $j\ge 3$. We split the corresponding part of $T^{\#}(I-e^{-r^2L})^Nb(x)$ into

$$\sup_{\varepsilon<r^2}\left(\fint_{B_{x,\varepsilon}}|T(I-e^{-r^2L})^Nb|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}+\sup_{\varepsilon<r^2}\left(\fint_{B_{x,\varepsilon}}\left|\int_0^\varepsilon T_t(I-e^{-r^2L})^Nb\,\frac{dt}{\sqrt t}\right|^{q_0}d\mu\right)^{\frac{1}{q_0}}$$
$$=: I_1(x)+I_2(x). \quad (4\text{-}5)$$

Let $\widetilde S_j(B)$ be a slightly enlarged annulus such that $B_{x,\varepsilon}\subseteq\widetilde S_j(B)$ for $x\in S_j(B)$. We estimate the first term $I_1(x)$ in (4-5) against the maximal function, localized in $\widetilde S_j(B)$ due to the restriction of the supremum to small ε and the assumption $j\ge 3$. This gives, for $x\in S_j(B)$,

$$I_1(x)\lesssim\mathcal M_{q_0}\big(\mathbb 1_{\widetilde S_j(B)}T(I-e^{-r^2L})^Nb\big)(x).$$

By Hölder's inequality and Kolmogorov's lemma (see, e.g., [Duoandikoetxea 2001, Lemma 5.16]) for $\mathcal M_{q_0}$, we have

$$|2^{j+1}B|^{-1/2}\|I_1\|_{L^2(S_j(B))}\lesssim|2^{j+1}B|^{-1/2}\big\|\mathcal M_{q_0}\big(\mathbb 1_{\widetilde S_j(B)}T(I-e^{-r^2L})^Nb\big)\big\|_{L^2(2^jB)}$$
$$\lesssim|2^{j+1}B|^{-1/q_0}\|T(I-e^{-r^2L})^Nb\|_{L^{q_0}(\widetilde S_j(B))}.$$

By Corollary 4.2, we know that $T(I-e^{-r^2L})^N$ satisfies L^{p_0}-L^{q_0} (strictly) off-diagonal estimates at the scale r, thus giving (4-4) for this part with coefficients $c(j)=2^{-j(\nu+1)}$.

For I_2, on the other hand, we can directly estimate, using Lemma 4.1(2),

$$|B_{x,\varepsilon}|^{-1/q_0}\|T_t(I-e^{-r^2L})^Nb\|_{L^{q_0}(B_{x,\varepsilon})}\lesssim 2^{-j(\nu+1)}\left(\frac{t}{r^2}\right)^{\frac12}\left(\fint_B|b|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.$$

Therefore, we may then integrate over $t\in(0,\varepsilon)$. By taking the supremum over $\varepsilon\in(0,r^2)$ and over x, and using Minkowski's inequality, we obtain (4-4) also for I_2 with coefficients $c(j)=2^{-j(\nu+1)}$. □

5. Boundedness of the maximal operator by sparse operators

As done in previous works (see for example [Petermichl 2007; Hytönen 2012; Lerner 2010; 2013a; 2013b; Lacey 2015]), the analysis will involve a discrete stopping-time argument that relies on nice properties associated with a dyadic structure, which is by now well-known in the context of doubling space. We first recall the main results and then, by using this structure, we detail the stopping-time argument to bound the maximal operator $T^{\#}$ by some specific operators, called *sparse operators*.

5A. ***Preliminaries and reminder on dyadic analysis.*** We first recall several results about the construction of adjacent dyadic systems (see [Christ 1990; Sawyer and Wheeden 1992; Hytönen and Kairema 2012] for more details).

Definition 5.1. Let us fix some constants $0 < c_0 \leq C_0 < \infty$ and $\delta \in (0,1)$. A *dyadic system* (with parameters c_0, C_0, δ) is a family of open subsets $(Q^l_\alpha)_{\alpha \in \mathscr{A}_l, l \in \mathbb{Z}}$ satisfying the following properties:

- For every $l \in \mathbb{Z}$, the ambient space M is covered (up to a set with vanishing measure) by the disjoint union of the subsets at scale l, that is, there exists Z_l with $\mu(Z_l) = 0$ such that
$$M = \bigsqcup_{\alpha \in \mathscr{A}_l} Q^l_\alpha \sqcup Z_l.$$
- If $l \geq k$, $\alpha \in \mathscr{A}_k$ and $\beta \in \mathscr{A}_l$ then either $Q^l_\beta \subseteq Q^k_\alpha$ or $Q^k_\alpha \cap Q^l_\beta = \varnothing$.
- For every $l \in \mathbb{Z}$ and $\alpha \in \mathscr{A}_l$, there exists a point z^l_α with
$$B(z^l_\alpha, c_0\delta^l) \subseteq Q^l_\alpha \subseteq B(z^l_\alpha, C_0\delta^l) =: B(Q^l_\alpha). \tag{5-1}$$

For a cube Q^k_α, $k \in \mathbb{Z}$, $\alpha \in \mathscr{A}_k$, we call the unique cube Q^{k-1}_β, $\beta \in \mathscr{A}_{k-1}$, for which $Q^k_\alpha \subseteq Q^{k-1}_\beta$ the *parent* of Q^k_α. We denote the parent of $Q \in \mathscr{D}$ by Q^a and call Q a *child* of Q^a.

We refer the reader to [Hytönen and Kairema 2012] for a variant where the negligible Z_l does not appear if the subsets are not necessarily assumed to be open. We also refer to a very recent survey by Lerner and Nazarov [2015] about dyadic structures and how they are used for proving weighted estimates of singular operators.

Then we have the following result (see [Hytönen and Kairema 2012] and references therein):

Theorem 5.2. *There exist constants c_0, C_0, δ, finite constants $K = K(c_0, C_0, \delta)$ and $\rho = \rho(c_0, C_0, \delta)$, as well as a finite collection of families $\mathscr{D}^b$, $b = 1, 2, \ldots, K$, where each $\mathscr{D}^b$ is a dyadic system* (*with parameters c_0, C_0, δ*) *with the following extra property*: *for every ball $B = B(x,r) \subseteq M$, there exists $b \in \{1, \ldots, K\}$ and $Q \in \mathscr{D}^b$ with*
$$B \subseteq Q \quad \textit{and} \quad \operatorname{diam}(Q) \leq \rho r. \tag{5-2}$$

We define
$$\mathscr{D} := \bigcup_{b=1}^{K} \mathscr{D}^b,$$
and call a cube Q a *dyadic cube* whenever $Q \in \mathscr{D}$.

For every dyadic set $Q \in \mathscr{D}$, we let $\ell(Q) := \delta^k$, where the integer k is determined by
$$\delta^{k+1} \leq \operatorname{diam}(Q) < \delta^k.$$

This result means that in typical situations it is sufficient to consider a dyadic system instead of the whole collection of balls.

Definition 5.3. Given one of the previous dyadic systems $\mathscr{D}^k$ and a nonnegative weight $h \in L^1_{\mathrm{loc}}$, we define its corresponding maximal operator, weighted by h, by

$$\mathcal{M}_h^{\mathscr{D}^k}[f](x) := \sup_{x \in Q \in \mathscr{D}^k} \left(\frac{1}{h(Q)} \int_Q |f| \, h \, d\mu \right), \quad x \in M,$$

for every $f \in L^1_{\mathrm{loc}}(h \, d\mu)$.

Lemma 5.4. *Uniformly in $k \in \{1, \dots, K\}$ and in the weight h, the maximal operator $\mathcal{M}_h^{\mathscr{D}^k}$ is of weak type $(1, 1)$ and strong type (p, p) for the measure $h \, d\mu$ for every $p \in (1, \infty]$.*

We refer the reader to [Lerner and Nazarov 2015, Theorem 15.1] for a detailed proof of this result and more details. For completeness, we give a short proof here.

Proof. Since $\mathcal{M}_h^{\mathscr{D}_k}$ is L^∞-bounded (and so $L^\infty(h \, d\mu)$-bounded), it suffices by interpolation to check its weak $L^1(h \, d\mu)$-boundedness.

Fix a function $f \in L^1(h \, d\mu)$. For every $\lambda > 0$, we consider the set

$$\Omega_\lambda := \{x \in M : \mathcal{M}_h^{\mathscr{D}^k}[f](x) > \lambda\}.$$

Due to the properties of the dyadic system, there exists a collection $\mathscr{Q} := (P)_{P \in \mathscr{Q}} \subset \mathscr{D}^k$ of dyadic sets such that $\Omega_\lambda = \bigcup_{P \in \mathscr{Q}} P$ (up to a subset of measure zero) and such that each $P \in \mathscr{Q}$ is maximal in Ω_λ and, for every $P \in \mathscr{Q}$,

$$\frac{1}{h(P)} \int_P |f| h \, d\mu > \lambda.$$

Due to the maximality, the dyadic sets $P \in \mathscr{Q}$ are pairwise disjoint and so we conclude that

$$h(\Omega_\lambda) = \sum_{P \in \mathscr{Q}} h(P) \le \lambda^{-1} \sum_{P \in \mathscr{Q}} \int_P |f| h \, d\mu \le \lambda^{-1} \|f\|_{L^1(h \, d\mu)},$$

which leads to weak $L^1(h \, d\mu)$-boundedness, uniformly with respect to h. □

We will also need the weak type of a slight modification of the previous maximal function.

Lemma 5.5. *Fix $k \in \{1, \dots, K\}$ and consider the maximal function*

$$\mathcal{M}^*[f](x) := \sup_{x \in Q \in \mathscr{D}^k} \inf_{y \in Q} \mathcal{M}[f](y), \quad x \in M,$$

for every $f \in L^1_{\mathrm{loc}}(h \, d\mu)$. It follows that $\mathcal{M}^*[f] = \mathcal{M}[f]$ almost everywhere. Consequently, the maximal operator $\mathcal{M}^*$ is of weak type $(1, 1)$ and strong type (p, p) for every $p \in (1, \infty]$.

Proof. Indeed, since the quantity $\inf_{y \in Q} \mathcal{M} f(y)$ is decreasing with respect to Q, it follows that

$$\mathcal{M}^*[f](x) = \lim_{\substack{x \in Q \\ \operatorname{diam}(Q) \to 0}} \inf_{y \in Q} \mathcal{M}[f](y) = \mathcal{M}[f](x),$$

where we have used the Lebesgue differentiation lemma, which implies the last equality for almost every $x \in M$. □

5B. ***Upper estimates of the maximal operator with sparse operators.*** From the previous subsection we know that we have several dyadic grids $\mathscr{D}^b$ for $b \in \{1, \dots, K\}$. In the sequel, we define $\mathscr{D} := \bigcup_{b=1}^K \mathscr{D}^b$ and call any element of $\mathscr{D}$ a *dyadic set*.

Definition 5.6 (sparse collection). A collection of dyadic sets $\mathscr{S} := (P)_{P \in \mathscr{S}} \subset \mathscr{D}$ is said to be *sparse* if for each $P \in \mathscr{S}$ one has

$$\sum_{Q \in \mathrm{ch}_{\mathscr{S}}(P)} \mu(Q) \le \tfrac{1}{2}\mu(P), \tag{5-3}$$

where $\mathrm{ch}_{\mathscr{S}}(P)$ is the collection of $\mathscr{S}$-children of P, namely the maximal elements of $\mathscr{S}$ that are strictly contained in P.

For a dyadic cube $Q \in \mathscr{D}$, we denote by $5Q$ its neighbourhood

$$5Q := \{x \in M : d(x, Q) \le 4\ell(Q)\}.$$

Theorem 5.7. *Consider an exponent $p \in (p_0, q_0)$. There exists a constant $C > 0$ such that, for all $f \in L^p$ and $g \in L^{p'}$, both supported in $5Q_0$ for some $Q_0 \in \mathscr{D}$, there exists a sparse collection $\mathscr{S} \subset \mathscr{D}$ (depending on f and g) with*

$$\left|\int_{Q_0} Tf \cdot g \, d\mu\right| \le C \sum_{P \in \mathscr{S}} \mu(P) \left(\fint_{5P} |f|^{p_0} \, d\mu\right)^{\frac{1}{p_0}} \left(\fint_{5P} |g|^{q_0'} \, d\mu\right)^{\frac{1}{q_0'}}.$$

A careful examination of the proof shows that, indeed, if the initial ball Q_0 belongs to the dyadic grid $\mathscr{D}^b$ for some $b \in \{1, \dots, K\}$, then the whole sparse collection $\mathscr{S}$ belongs to the same dyadic grid $\mathscr{D}^b$. However, it will be important in Proposition 6.4 (to prove sharp weighted estimates for sparse operators) to play with the different dyadic grids.

Proof. Let $p \in (p_0, q_0)$. Suppose $f \in L^p$ and $g \in L^{p'}$, supported in $5Q_0$ for a dyadic set $Q_0 \in \mathscr{D}$. Fix the parameter $b \in \{1, \dots, K\}$ such that $Q_0 \in \mathscr{D}^b$. For some large enough constant η (which will be fixed later), define the subset

$$E = \left\{x \in Q_0 \;\middle|\; \max\{\mathcal{M}^*_{Q_0,p_0} f(x), T^\#_{Q_0} f(x)\} > \eta \left(\fint_{5Q_0} |f|^{p_0} \, d\mu\right)^{\frac{1}{p_0}}\right\},$$

where both $\mathcal{M}^*_{Q_0,p_0}$ and $T^\#_{Q_0}$ are defined relative to the initial subset $Q_0 \in \mathscr{D}^b$ as follows: for every $x \in Q_0$,

$$\mathcal{M}^*_{Q_0,p_0}[f](x) := \sup_{\substack{x \in Q \subset Q_0 \\ Q \in \mathscr{D}^b}} \inf_{y \in Q} \mathcal{M}_{p_0}[f](y)$$

and

$$T^\#_{Q_0} f(x) = \sup_{\substack{x \in Q \subset Q_0 \\ Q \in \mathscr{D}^b}} \left(\fint_Q \left| T \int_{\ell(Q)^2}^\infty Q_t^{(N)}(f) \, \frac{dt}{t}\right|^{q_0} d\mu\right)^{\frac{1}{q_0}}.$$

We extend both $\mathcal{M}^*_{Q_0,p_0}$ and $T^{\#}_{q_0}$ by 0 outside Q_0.

Due to the properties of dyadic subsets, we know that every $Q \in \mathcal{D}^b$ is contained in a ball with radius equivalent to $\ell(Q)$. Thus, up to some implicit constants, $\mathcal{M}^*_{Q_0,p_0}$ is bounded by the Hardy–Littlewood maximal function $\mathcal{M}_{p_0}$ (see Lemma 5.5) and $T^{\#}_{Q_0}$ is controlled by the maximal operator $T^{\#}$. So Proposition 4.6 yields that both $\mathcal{M}^*_{Q_0,p_0}$ and $T^{\#}_{Q_0}$ are of weak type (p_0, p_0).

Then it follows that $\mu(E) \lesssim (1/\eta)\mu(Q_0)$. So, if η is chosen large enough, then we know that E is an open proper subset of Q_0. In the sequel, all the implicit constants will only depend on the ambient space. For convenience, we only emphasize the dependence relative to η, which will be useful later to show how η can be fixed.

Consider a maximal dyadic covering of E, which is a collection of dyadic subsets $(B_j)_j \subset \mathcal{D}^b$ such that

- the collection covers E: $E = \bigsqcup_j B_j$, up to a set of null measure, with disjointness of the dyadic cubes;
- the dyadic cubes are maximal, in the sense that $B_j^a \cap E^c \neq \varnothing$ for every j, where we recall that B_j^a is the parent of B_j.

Since $\mu(B_j) \le \mu(E) \lesssim \eta^{-1}\mu(Q_0)$, if η is chosen large enough then, using the doubling property of the measure μ, we deduce that we also have

$$\mu(B_j^a) \le \mu(Q_0).$$

Due to the properties of the dyadic system, we then deduce that B_j^a is included in Q_0, and so the maximality of B_j yields

$$\max\left\{\inf_{y\in B_j^a} \mathcal{M}_{p_0}[f](y),\ \left(\fint_{B_j^a}\left|T\int_{\ell(B_j^a)^2}^{\infty} Q_t^{(N)}(f)\,\frac{dt}{t}\right|^{q_0} d\mu\right)^{\frac{1}{q_0}}\right\} \le \eta\left(\fint_{5Q_0} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}. \tag{5-4}$$

We first initialize the collection $\mathcal{S} := \{Q_0\}$, which we are going to build in a recursive way. For $B \in \mathcal{D}$, define the operator T_B by

$$T_B f := T\int_0^{\ell(B)^2} Q_t^{(N)}(f\mathbb{1}_{5B})\,\frac{dt}{t}.$$

Step 1. In this step, we aim to show that, for some numerical constant C_0,

$$\left|\int_{Q_0} Tf\cdot g\,d\mu\right| \le C_0\eta|Q_0|\left(\fint_{Q_0} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\left(\fint_{Q_0} |g|^{q_0'}\,d\mu\right)^{\frac{1}{q_0'}} + \sum_j\left|\int_{B_j} T_{B_j} f\cdot g\,d\mu\right|. \tag{5-5}$$

Seeking that, write

$$\left|\int_{Q_0} Tf\cdot g\,d\mu\right| \le \left|\int_{Q_0\setminus E} Tf\cdot g\,d\mu\right| + \left|\int_E Tf\cdot g\,d\mu\right|.$$

For the first part, notice that $|Tf(x)| \le T^{\#}_{Q_0} f(x) \le \eta\big(⨍_{5Q_0} |f|^{p_0}\, d\mu\big)^{1/p_0}$ for a.e. $x \in Q_0 \setminus E$ by definition of E. Hence

$$\left| \int_{Q_0\setminus E} Tf \cdot g \, d\mu \right| \le \eta \mu(Q_0) \left(⨍_{5Q_0} |f|^{p_0}\, d\mu \right)^{\frac{1}{p_0}} \left(⨍_{Q_0} |g|^{q_0'}\, d\mu \right)^{\frac{1}{q_0'}}.$$

For the part on E, we use the covering to obtain

$$\begin{aligned} \left| \int_E Tf \cdot g \, d\mu \right| &\le \sum_j \left| \int_{B_j} T_{B_j} f \cdot g \, d\mu \right| + \left| \sum_j \int_{B_j} (T - T_{B_j}) f \cdot g \, d\mu \right| \\ &\le \sum_j \left| \int_{B_j} T_{B_j} f \cdot g \, d\mu \right| + \sum_j \mu(B_j) \left(⨍_{B_j} |(T - T_{B_j}) f|^{q_0}\, d\mu \right)^{\frac{1}{q_0}} \left(⨍_{B_j} |g|^{q_0'}\, d\mu \right)^{\frac{1}{q_0'}}. \end{aligned}$$

The first sum enters into the recursion and is acceptable in view of (5-5). For the second sum, we have

$$\left| (T - T_{B_j}) f \right| \le \left| T \int_{\ell(B_j)^2}^{\infty} Q_t^{(N)}(f) \, \frac{dt}{t} \right| + \left| T \int_0^{\ell(B_j)^2} Q_t^{(N)}(\mathbb{1}_{(5B_j)^c} f) \, \frac{dt}{t} \right|. \tag{5-6}$$

Using the doubling property, we can estimate the first term against the maximal operator and get

$$\begin{aligned} \left(⨍_{B_j} \left| T \int_{\ell(B_j)^2}^{\infty} Q_t^{(N)} f \, \frac{dt}{t} \right|^{q_0} d\mu \right)^{\frac{1}{q_0}} &\lesssim \left(⨍_{B_j^a} \left| T \int_{\ell(B_j)^2}^{\infty} Q_t^{(N)} f \, \frac{dt}{t} \right|^{q_0} \right)^{\frac{1}{q_0}} \\ &\lesssim \inf_{z \in B_j^a} T^{\#} f(z) + \left(⨍_{B_j^a} \left| T \int_{\ell(B_j)^2}^{\ell(B_j^a)^2} Q_t^{(N)} f \, \frac{dt}{t} \right|^{q_0} \right)^{\frac{1}{q_0}}. \end{aligned}$$

By the maximality of the dyadic cubes B_j, we know that B_j^a intersects E^c; hence, from (5-4), we have

$$\inf_{z \in B_j^a} T^{\#} f(z) \le \eta \left(⨍_{5Q_0} |f|^{p_0}\, d\mu \right)^{\frac{1}{p_0}}.$$

Moreover, we also know that for every dyadic set B_j we have

$$\inf_{y \in B_j^a} \mathcal{M}_{p_0}[f](y) \le \eta \left(⨍_{5Q_0} |f|^{p_0}\, d\mu \right)^{\frac{1}{p_0}},$$

which yields in particular that

$$\begin{aligned} \left(⨍_{B_j^a} \left| T \int_{\ell(B_j)^2}^{\ell(B_j^a)^2} Q_t^{(N)} f \, \frac{dt}{t} \right|^{q_0} \right)^{\frac{1}{q_0}} &\lesssim \int_{\ell(B_j)^2}^{\ell(B_j^a)^2} \left(⨍_{B_j^a} |T Q_t^{(N)} f|^{q_0}\, d\mu \right)^{\frac{1}{q_0}} \frac{dt}{t} \\ &\lesssim \eta \left(⨍_{5Q_0} |f|^{p_0}\, d\mu \right)^{\frac{1}{p_0}}. \end{aligned}$$

We do not detail this last inequality, since it is a simpler particular case of the next one.

For the second term in (5-6), we use the L^{p_0}-L^{q_0} off-diagonal estimates for $T_t = TQ_t^{(N)}$ from Assumption (b). We have that

$$(5B_j)^c \subset \bigcup_{k=2}^{\infty} S_k(B_j),$$

and can therefore decompose

$$\begin{aligned}\left(\fint_{B_j}\left|T\int_0^{\ell(B_j)^2} Q_t^{(N)}(f\mathbb{1}_{(5B_j)^c})\,\frac{dt}{t}\right|^{q_0} d\mu\right)^{\frac{1}{q_0}} &\le \int_0^{\ell(B_j)^2}\left(\fint_{B_j}|T_t(f\mathbb{1}_{(5B_j)^c})|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}\frac{dt}{t}\\ &\le \sum_{k\ge 2}\int_0^{\ell(B_j)^2}\left(\fint_{B_j}|T_t(f\mathbb{1}_{S_k(B_j)})|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}\frac{dt}{t}.\end{aligned}$$

For fixed $t\in(0,\ell(B_j)^2)$ we know that T_t satisfies L^{p_0}-L^{q_0} off-diagonal estimates at the scale $\sqrt{t}$. We then cover $S_k(B_j)$ by balls of radius $\sqrt{t}$, with a finite overlap property (by the doubling property of the measure). We then deduce that these balls R satisfy

$$d(R,B_j)\ge \ell(B_j)\quad\text{and}\quad d(R,B_j)\simeq d(S_k(B_j),B_j)\simeq 2^k\ell(B_j).$$

Moreover, the number of these balls needed to cover $S_k(B_j)$ is controlled by

$$\#\{R\}\lesssim\left(\frac{2^k\ell(B_j)}{\sqrt{t}}\right)^{\nu}. \tag{5-7}$$

By summing over such a covering, we get

$$\begin{aligned}\left(\fint_{B_j}|T_t(f\mathbb{1}_{S_k(B_j)})|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} &\lesssim \sum_R\left(1+\frac{d(R,B_j)^2}{t}\right)^{-\frac{\nu+1}{2}}\left(\fint_R|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\\ &\lesssim\left(1+\frac{4^k\ell(B_j)^2}{t}\right)^{-\frac{\nu+1}{2}}\left(\frac{2^k\ell(B_j)}{\sqrt{t}}\right)^{\frac{\nu}{p_0}}|2^kB_j|^{-\frac{1}{p_0}}\sum_R\left(\int_R|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.\end{aligned}$$

By Hölder's inequality with the bounded overlap property of the collection $\{R\}$ with (5-7), we then have

$$\sum_R\left(\int_R|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\lesssim\left(\int_{S_k(B_j)}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\left(\frac{2^k\ell(B_j)}{\sqrt{t}}\right)^{\frac{\nu}{p_0'}},$$

hence

$$\begin{aligned}\left(\fint_{B_j}|T_t(f\mathbb{1}_{S_k(B_j)})|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} &\lesssim\left(1+\frac{4^k\ell(B_j)^2}{t}\right)^{-\frac{\nu+1}{2}}\left(\frac{2^k\ell(B_j)}{\sqrt{t}}\right)^{\nu}\left(\fint_{S_k(B_j)}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\\ &\lesssim\left(\frac{\sqrt{t}}{2^k\ell(B_j)}\right)\left(\fint_{S_k(B_j)}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.\end{aligned}$$

We therefore get

$$\begin{aligned}\bigg(\fint_{B_j}\bigg|T\int_0^{\ell(B_j)^2} Q_t^{(N)}(f\mathbb{1}_{(5B_j)^c})\,\frac{dt}{t}\bigg|^{q_0}\,d\mu\bigg)^{\frac{1}{q_0}} &\lesssim \sum_{k=2}^{\infty}\bigg(\fint_{S_k(B_j)}|f|^{p_0}\,d\mu\bigg)^{\frac{1}{p_0}}\int_0^{\ell(B_j)^2}\bigg(\frac{\sqrt{t}}{2^k\ell(B_j)}\bigg)\,\frac{dt}{t}\\ &\lesssim \sum_{k=2}^{\infty}2^{-k}\bigg(\fint_{S_k(B_j)}|f|^{p_0}\,d\mu\bigg)^{\frac{1}{p_0}}\\ &\lesssim \sup_{k\ge 2}\bigg(\fint_{2^kB_j}|f|^{p_0}\,d\mu\bigg)^{\frac{1}{p_0}}\\ &\lesssim \inf_{z\in B_j^a}\mathcal{M}_{p_0}f(z)\lesssim \eta\bigg(\fint_{5Q_0}|f|^{p_0}\bigg)^{\frac{1}{p_0}},\end{aligned}\tag{5-8}$$

where we used (5-4).

On the other hand,

$$\bigg(\fint_{B_j}|g|^{q_0'}\,d\mu\bigg)^{\frac{1}{q_0'}}\le \inf_{z\in B_j}\mathcal{M}_{q_0'}g(z),$$

and, using $\bigcup_j B_j = E$, Kolmogorov's inequality, the fact that $\mu(E)\lesssim\mu(Q_0)$ (since η will be chosen larger than 1) and $\operatorname{supp} g\subseteq 5Q_0$,

$$\sum_j \mu(B_j)\inf_{z\in B_j}\mathcal{M}_{q_0'}[g](z)\le\int_E\mathcal{M}_{q_0'}[g](z)\,d\mu(z)\lesssim\mu(E)^{1-1/q_0'}\big\||g|^{q_0'}\big\|_1^{\frac{1}{q_0'}}\lesssim\mu(Q_0)\bigg(\fint_{5Q_0}|g|^{q_0'}\,d\mu\bigg)^{\frac{1}{q_0'}}.$$

Therefore, putting all the estimates together, we have shown that

$$\sum_j\mu(B_j)\bigg(\fint_{B_j}|(T-T_{B_j})f|^{q_0}\,d\mu\bigg)^{\frac{1}{q_0}}\bigg(\fint_{B_j}|g|^{q_0'}\,d\mu\bigg)^{\frac{1}{q_0'}}\lesssim\eta\mu(Q_0)\bigg(\fint_{5Q_0}|f|^{p_0}\bigg)^{\frac{1}{p_0}}\bigg(\fint_{5Q_0}|g|^{q_0'}\bigg)^{\frac{1}{q_0'}},$$

where the implicit constant only depends on the ambient space through previous numerical constants. This concludes the proof of (5-5).

Step 2 (recursion and conclusion). Starting from the initial dyadic cube Q_0, we have built a collection of dyadic cubes $(Q_1^j)_j$ such that

$$\bigg|\int_{Q_0}Tf\cdot g\,d\mu\bigg|\le C_0\eta\mu(Q_0)\bigg(\fint_{5Q_0}|f|^{p_0}\,d\mu\bigg)^{\frac{1}{p_0}}\bigg(\fint_{5Q_0}|g|^{q_0'}\,d\mu\bigg)^{\frac{1}{q_0'}}+\sum_j\bigg|\int_{Q_1^j}T_{Q_1^j}f^j\cdot g^j\,d\mu\bigg|,$$

where f^j and g^j are both supported in $5Q_1^j$ and are pointwise bounded by f and g, respectively. Moreover, the following properties hold:

(a) *Small measure*: for some numerical constant $\widetilde{K}$,

$$\sum_j\mu(Q_1^j)\le\frac{\widetilde{K}}{\eta}\mu(Q_0).$$

(b) *Disjointness and covering*: $(Q_1^j)_j$ are pairwise disjoint and included in Q_0.

We then add all these cubes to the collection $\mathcal{S}$, and rename $\mathcal{S} = \mathcal{S} \cup \bigcup_j \{Q_1^j\}$. And we iterate the procedure. For every cube Q_1^j, there exists a collection of dyadic cubes $(Q_2^{j,k})_k$ such that

$$\left|\int_{Q_1^j} Tf^j \cdot g^j \, d\mu\right| \\ \leq C_0 \eta \mu(Q_1^j) \left(\fint_{5Q_1^j} |f|^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \left(\fint_{5Q_1^j} |g|^{q_0'} \, d\mu \right)^{\frac{1}{q_0'}} + \sum_k \left| \int_{Q_2^{j,k}} T_{Q_2^{j,k}} f^{j,k} \cdot g^{j,k} \, d\mu \right|,$$

with the properties that $f^{j,k}$ and $g^{j,k}$ are pointwise bounded by f and g, and also:

(a) *Small measure*: $$\sum_k \mu(Q_2^{j,k}) \leq \frac{\widetilde{K}}{\eta} \mu(Q_1^j).$$

(b) *Disjointness and covering*: $(Q_2^{j,k})_k$ are pairwise disjoint and included in Q_1^j.

We then add all these cubes to the collection $\mathcal{S}$, to obtain $\mathcal{S} = \mathcal{S} \cup \bigcup_{j,k} \{Q_2^{j,k}\}$. We iterate this reasoning, which allows us to build the collection $\mathcal{S}$ with the property that

$$\left| \int_{Q_0} Tf \cdot g \, d\mu \right| \leq C_0 \eta \sum_{Q \in \mathcal{S}} \mu(Q) \left(\fint_{5Q} |f|^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \left(\fint_{5Q} |g|^{q_0'} \, d\mu \right)^{\frac{1}{q_0'}}.$$

Indeed, it is easy to check that the remainder term at the i-th step is an integral over a subset of measure which tends to 0 as i goes to ∞. So for fixed $f \in L^p$ and $g \in L^{p'}$ with $p' < \infty$, the remainder term also tends to 0.

It remains for us to check that this collection $\mathcal{S}$ is sparse.

So consider $Q \in \mathcal{S}$. By the disjointness property of the selected dyadic cubes, it is clear that any child $\overline{Q} \in \mathrm{ch}_{\mathcal{S}}(Q)$ has been selected (strictly) after Q and in the collection $\mathcal{S}_Q$ *generated* by Q. Using the smallness property of the measure in the algorithm, we know that summing over all the cubes R selected strictly after Q in the collection generated by Q gives us

$$\sum_{R \in \mathcal{S}_Q} \mu(R) = \sum_{l \geq 1} \left(\frac{\widetilde{K}}{\eta} \right)^l \mu(Q) \leq \frac{\widetilde{K}}{\eta - \widetilde{K}} \mu(Q).$$

We then deduce that, by choosing η large enough, the selected collection is sparse. □

6. Boundedness of a sparse operator

Definition 6.1 (A_p weight). A measurable function $\omega : M \to (0, \infty)$ is an A_p weight for some $p \in (1, \infty)$ if

$$[\omega]_{A_p} := \sup_{\text{ball } B} \left(\fint_B \omega \, d\mu \right) \left(\fint_B \omega^{1-p'} \, d\mu \right)^{p-1} < \infty,$$

with p' the conjugate exponent $p' = p/(p-1)$. For $p = 1$, we extend this notion with the characteristic constant

$$[\omega]_{A_1} := \sup_{\text{ball } B} \left(\fint_B \omega \, d\mu \right) \bigl(\operatorname*{ess\,inf}_{x\in B} \omega(x) \bigr)^{-1}.$$

Definition 6.2 (RH_q weight). A measurable function $\omega : M \to (0,\infty)$ is an RH_q weight for some $q \in (1,\infty)$ if

$$[\omega]_{\mathrm{RH}_q} := \sup_{\text{ball } B} \left(\fint_B \omega^q \, d\mu \right)^{\frac{1}{q}} \left(\fint_B \omega \, d\mu \right)^{-1} < \infty.$$

For $q = \infty$, we extend this notion with the characteristic constant

$$[\omega]_{\mathrm{RH}_\infty} := \sup_{\text{ball } B} \bigl(\operatorname*{ess\,sup}_{x\in B} \omega(x) \bigr) \left(\fint_B \omega \, d\mu \right)^{-1}.$$

We recall some well-known properties of the weight.

Lemma 6.3. (a) *If $p \in (1,\infty)$ and ω is a weight, then $\omega \in A_p$ if and only if $\omega^{1-p'} \in A_{p'}$ with*

$$[\omega^{1-p'}]_{A_{p'}} = [\omega]_{A_p}^{p'-1}.$$

(b) (see [Johnson and Neugebauer 1991]) *If $q \in [1,\infty]$, $s \in [1,\infty)$ and ω is a weight, then $\omega \in A_q \cap \mathrm{RH}_s$ if and only if $\omega^s \in A_{s(q-1)+1}$ with*

$$[\omega^s]_{A_{s(q-1)+1}} \le [\omega]_{A_q}^s [\omega]_{\mathrm{RH}_s}^s.$$

We prove the following sharp weighted estimates for the "sparse" operators:

Proposition 6.4. *Let $p_0, q_0 \in [1,\infty]$ be two exponents with $p_0 < q_0$, and let $p \in (p_0, q_0)$. Suppose that S is a bounded operator on L^p and that there exists a constant $c > 0$ such that for all $f \in L^p$ and $g \in L^{p'}$ there exists a sparse collection $\mathcal{S}$ with*

$$|\langle S(f), g\rangle| \le c \sum_{P\in\mathcal{S}} \left(\fint_{5P} |f|^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \left(\fint_{5P} |g|^{q_0'} \, d\mu \right)^{\frac{1}{q_0'}} \mu(P).$$

Denote

$$r := \left(\frac{q_0}{p}\right)' \left(\frac{p}{p_0} - 1\right) + 1 \quad \textit{and} \quad \delta := \min\{q_0', \ p_0(r-1)\}.$$

Then there exists a constant $C = C(S, p, p_0, q_0)$ such that, for every weight $\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$, the operator S is bounded on L^p_ω with

$$\|S\|_{L^p_\omega \to L^p_\omega} \le C \bigl([\omega]_{A_{p/p_0}} [\omega]_{\mathrm{RH}_{(q_0/p)'}} \bigr)^{\alpha},$$

with

$$\alpha := \frac{1}{\delta} \left(\frac{q_0}{p}\right)' = \max\left\{ \frac{1}{p-p_0}, \ \frac{q_0-1}{q_0-p} \right\}.$$

In particular, by defining the specific exponent

$$\mathfrak{p} := 1 + \frac{p_0}{q_0'} \in (p_0, q_0),$$

we have $\alpha = 1/(p - p_0)$ *if* $p \in (p_0, \mathfrak{p}]$, *and* $\alpha = (q_0 - 1)/(q_0 - p)$ *if* $p \in [\mathfrak{p}, q_0)$.

Remark 6.5. The property $p_0 < \mathfrak{p}$ is equivalent to the condition $p_0 < q_0$, and the property $\mathfrak{p} < q_0$ is also equivalent to the condition $p_0 < q_0$. So the assumption guarantees us that

$$p_0 < \mathfrak{p} < q_0.$$

We note that, using extrapolation theory (as developed in [Auscher and Martell 2007a, Theorem 4.9]) and by tracking the behaviour of implicit constants with respect to the weights, a sharp weighted estimate for one particular exponent $p \in (p_0, q_0)$ allows us to get the sharp weighted estimates for all the exponents in the range $p \in (p_0, q_0)$. Here we are going to detail a proof which directly gives the weighted estimates for all such exponents.

Remarks 6.6. (1) In the case where $q_0 = p_0'$, it is $\mathfrak{p} = 2$ and we obtain sharp weighted estimates with the power

$$\alpha = \max\left\{\frac{1}{p - p_0}, \frac{1}{p + p_0 - pp_0}\right\}.$$

(2) In particular, in the situation where $p_0 = 1$ and $q_0 = \infty$, we recover the "usual" sharp behaviour, dictated by the A_2 conjecture, with the power

$$\alpha = \max\left\{1, \frac{1}{p-1}\right\}.$$

(3) In the case $q_0 = \infty$, we obtain

$$\alpha = \max\{1, (p - p_0)^{-1}\},$$

which is the same exponent as in [Bui et al. 2015] and allows us to regain their result (the linear part) as explained in Section 3D.

Remark 6.7. For a weight ω, we know (see Lemma 6.3 and [Auscher and Martell 2007a, Lemma 4.4]) that

$$\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'} \iff \sigma := \omega^{1-p'} \in A_{p'/q_0'} \cap \mathrm{RH}_{(p_0'/p')'}.$$

These are also equivalent to

$$\omega^{(q_0/p)'} \in A_r$$

with $r := (q_0/p)'(p/p_0 - 1) + 1$. We have the estimates on the characteristic constants

$$[\omega^{(q_0/p)'}]_{A_r} \lesssim \big([\omega]_{A_{p/p_0}}[\omega]_{\mathrm{RH}_{(q_0/p)'}}\big)^{(q_0/p)'} \quad \text{and} \quad [\sigma]_{A_{p'/q_0'}}[\sigma]_{\mathrm{RH}_{(p_0'/p')'}} \lesssim \big([\omega]_{A_{p/p_0}}[\omega]_{\mathrm{RH}_{(q_0/p)'}}\big)^{p'-1}.$$

Proof of Proposition 6.4. Let us define three weights,

$$\sigma := \omega^{1-p'}, \quad u := \sigma^{(p_0'/p')'} \quad \text{and} \quad v := \omega^{(q_0/p)'}.$$

Then $u = v^{1-r'}$ with

$$r := \left(\frac{q_0}{p}\right)'\left(\frac{p}{p_0} - 1\right) + 1.$$

Combining the previous remark with Lemma 6.3, the fact that $\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$ yields that $v \in A_r$ and so

$$\sup_{\text{ball } B} \left(\fint_B v\, d\mu\right)\left(\fint_B u\, d\mu\right)^{r-1} \leq [v]_{A_r} \lesssim [\omega]^{(q_0/p)'},$$

where we set

$$[\omega] := [\omega]_{A_{p/p_0}} [\omega]_{\mathrm{RH}_{(q_0/p)'}},$$

the characteristic constant of the weight ω in the class $A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$. Using the comparison between dyadic subsets with balls and the doubling property of the measure μ, we then deduce that

$$\sup_{Q \in \mathscr{D}} \left(\fint_Q v\, d\mu\right)\left(\fint_Q u\, d\mu\right)^{r-1} \lesssim [v]_{A_r} \lesssim [\omega]^{(q_0/p)'}. \tag{6-1}$$

We know that the dual space (with respect to the measure $d\mu$) of L^p_ω is $L^{p'}_\sigma$. So the desired L^p_ω-boundedness of S is equivalent to the inequality

$$|\langle S(f), g\rangle| \lesssim [\omega]^\alpha \|f\|_{L^p_\omega} \|g\|_{L^{p'}_\sigma}. \tag{6-2}$$

Let us fix two functions $f \in L^p_\omega$ and $g \in L^{p'}_\sigma$. Then, by assumption, there exists a sparse collection $\mathscr{S}$ such that

$$|\langle S(f), g\rangle| \leq c \sum_{P \in \mathscr{S}} \left(\fint_{5P} |f|^{p_0}\, d\mu\right)^{\frac{1}{p_0}} \left(\fint_{5P} |g|^{q_0'}\, d\mu\right)^{\frac{1}{q_0'}} \mu(P).$$

For every $P \in \mathscr{S}$, we know that there exists a dyadic cube $\overline{P}$ such that $5P \subset \overline{P}$ and $\mu(\overline{P}) \lesssim \mu(5P)$. We split $\mathscr{S}$ into K collections $(\mathscr{S}_k)_{k=1,\dots,K}$ for which $\overline{P} \in \mathscr{D}^k$. Each collection $\mathscr{S}_k$ is still sparse, since it is a subcollection of $\mathscr{S}$.

We now fix $k \in \{1, \dots, K\}$. For every $P \in \mathscr{S}_k$, we set $E_P \subset P$ to be the set of all $x \in P$ which are not contained in any $\mathscr{S}_k$-child of P. By the sparseness property of $\mathscr{S}_k$, we then have

$$\mu(P) \leq 2\mu(E_P)$$

and the sets $(E_P)_{P \in \mathscr{S}_k}$ are pairwise disjoint.

So we have

$$|\langle S(f), g\rangle| \lesssim \sum_{k=1}^{K} \sum_{P \in \mathscr{S}_k} \left(\fint_{\overline{P}} |f|^{p_0}\, d\mu\right)^{\frac{1}{p_0}} \left(\fint_{\overline{P}} |g|^{q_0'}\, d\mu\right)^{\frac{1}{q_0'}} \mu(E_P). \tag{6-3}$$

We then change the measure with the weight u as follows:

$$\left(\fint_{\overline{P}} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}} = \left(\fint_{\overline{P}} |u^{-1/p_0} f|^{p_0} u\,d\mu\right)^{\frac{1}{p_0}} = \left(\frac{1}{u(\overline{P})}\int_{\overline{P}} |u^{-1/p_0} f|^{p_0} u\,d\mu\right)^{\frac{1}{p_0}} \left(\fint_{\overline{P}} u\,d\mu\right)^{\frac{1}{p_0}}. \tag{6-4}$$

Similarly, we have

$$\left(\fint_{\overline{P}} |g|^{q_0'}\,d\mu\right)^{\frac{1}{q_0'}} = \left(\fint_{\overline{P}} |v^{-1/q_0'} g|^{q_0'} v\,d\mu\right)^{\frac{1}{q_0'}} = \left(\frac{1}{v(\overline{P})}\int_{\overline{P}} |v^{-1/q_0'} g|^{q_0'} v\,d\mu\right)^{\frac{1}{q_0'}} \left(\fint_{\overline{P}} v\,d\mu\right)^{\frac{1}{q_0'}}. \tag{6-5}$$

Set $\alpha := \delta^{-1}(q_0/p)'$, with $\delta := \min\{q_0', p_0(r-1)\}$ and $\beta := 1/p_0 - (r-1)/q_0'$. We note that $\beta \le 0$ is equivalent to $p \ge \mathfrak{p}$ and is also equivalent to $\delta = q_0'$; whereas $\beta \ge 0$ is equivalent to $p \le \mathfrak{p}$ and to $\delta = p_0(r-1)$. We are first going to detail the end of the proof in the case $\beta \le 0$ and then explain that the situation $\beta \ge 0$ is very similar.

Step 1 (the case $p \ge \mathfrak{p}$, i.e., $\beta \le 0$). Putting the two last estimates, (6-4) and (6-5), into (6-3) yields

$$|\langle S(f), g\rangle| \lesssim [\omega]^{\alpha} \sum_{k=1}^{K} \sum_{P\in\mathcal{S}_k} \left(\frac{1}{u(\overline{P})}\int_{\overline{P}} |u^{-1/p_0} f|^{p_0} u\,d\mu\right)^{\frac{1}{p_0}} \times \left(\frac{1}{v(\overline{P})}\int_{\overline{P}} |v^{-1/q_0'} g|^{q_0'} v\,d\mu\right)^{\frac{1}{q_0'}} \left(\fint_{\overline{P}} u\,d\mu\right)^{\beta} \mu(E_P), \tag{6-6}$$

where we used that

$$\left(\fint_{\overline{P}} u\,d\mu\right)^{\frac{r-1}{\delta}} \left(\fint_{\overline{P}} v\,d\mu\right)^{\frac{1}{\delta}} \lesssim [\omega]^{\delta^{-1}(q_0/p)'}, \tag{6-7}$$

which comes from (6-1).

Since $\beta \le 0$ and $E_P \subset P \subset \overline{P}$ with $\mu(E_p) \ge \frac{1}{2}\mu(P) \ge c_\nu \mu(\overline{P})$, where c_ν is a constant only dependent on the doubling property of μ and constants of the dyadic system, we deduce that

$$\left(\fint_{\overline{P}} u\,d\mu\right)^{\beta} \le c_\nu^{-\beta}\left(\fint_{E_P} u\,d\mu\right)^{\beta}.$$

Then let us define two other weights,

$$\varpi := \sigma v^{p'/q_0'} \quad \text{and} \quad \rho := \omega u^{p/p_0}. \tag{6-8}$$

Since $u = v^{1-r'}$, an easy computation yields

$$u^{-\beta} \varpi^{1/p'} \rho^{1/p} = \sigma^{1/p'} \omega^{1/p} = 1. \tag{6-9}$$

By Hölder's inequality with $\gamma := 1/(1-\beta) \in [0,1]$ and the relation

$$1 = \frac{\gamma}{p} + \frac{\gamma}{p'} + (1-\gamma),$$

we have

$$\mu(E_P) = \int_{E_P} (u^{-\beta} \varpi^{1/p'} \rho^{1/p})^{\gamma} \, d\mu \le u(E_P)^{-\beta\gamma} \varpi(E_P)^{\gamma/p'} \rho(E_P)^{\gamma/p}. \tag{6-10}$$

Hence,

$$\left(\fint_{E_P} u \, d\mu \right)^{\beta} \mu(E_P) = u(E_P)^{\beta} \mu(E_P)^{1-\beta} \le \varpi(E_P)^{1/p'} \rho(E_P)^{1/p}.$$

So, coming back to (6-6) we then deduce that

$$|\langle S(f), g\rangle| \lesssim [\omega]^{\alpha} \sum_{k=1}^{K} \sum_{P \in \mathcal{S}_k} \left(\frac{1}{u(\overline{P})} \int_{\overline{P}} |u^{-1/p_0} f|^{p_0} u \, d\mu \right)^{\frac{1}{p_0}} \times \left(\frac{1}{v(\overline{P})} \int_{\overline{P}} |v^{-1/q_0'} g|^{q_0'} v \, d\mu \right)^{\frac{1}{q_0'}} \varpi(E_P)^{1/p'} \rho(E_P)^{1/p}.$$

With the dyadic weighted maximal function (see Lemma 5.4 for its definition) and since $E_P \subset P \subset \overline{P}$, we deduce that

$$\begin{aligned} |\langle S(f), g\rangle| &\lesssim [\omega]^{\alpha} \sum_{k=1}^{K} \sum_{P \in \mathcal{S}_k} \inf_{E_P} \mathcal{M}_u^{\mathcal{D}^k} \left(|u^{-1/p_0} f|^{p_0}\right)^{1/p_0} \inf_{E_P} \mathcal{M}_v^{\mathcal{D}^k} \left(|v^{-1/q_0'} g|^{q_0'}\right)^{1/q_0'} \varpi(E_P)^{1/p'} \rho(E_P)^{1/p} \\ &\lesssim [\omega]^{\alpha} \sum_{k=1}^{K} \sum_{P \in \mathcal{S}_k} \left(\int_{E_P} \mathcal{M}_u^{\mathcal{D}^k} \left(|u^{-1/p_0} f|^{p_0}\right)^{p/p_0} \rho \, d\mu \right)^{\frac{1}{p}} \left(\int_{E_P} \mathcal{M}_v^{\mathcal{D}^k} \left(|v^{-1/q_0'} g|^{q_0'}\right)^{p'/q_0'} \varpi \, d\mu \right)^{\frac{1}{p'}}. \end{aligned}$$

By Hölder's inequality and using the disjointness of the collection $(E_P)_{P \in \mathcal{S}_k}$, one gets

$$|\langle S(f), g\rangle| \lesssim [\omega]^{\alpha} \sum_{k=1}^{K} \left(\int \mathcal{M}_u^{\mathcal{D}^k} \left(|u^{-1/p_0} f|^{p_0}\right)^{p/p_0} \rho \, d\mu \right)^{\frac{1}{p}} \left(\int \mathcal{M}_v^{\mathcal{D}^k} \left(|v^{-1/q_0'} g|^{q_0'}\right)^{p'/q_0'} \varpi \, d\mu \right)^{\frac{1}{p'}}.$$

Since $p \in (p_0, q_0)$, the dyadic maximal function $\mathcal{M}_u^{\mathcal{D}^k}$ is $L^{p/p_0}(u \, d\mu)$-bounded (uniformly in the weight u; see Lemma 5.4) and similarly for the weight v, hence

$$|\langle S(f), g\rangle| \lesssim [\omega]^{\alpha} \left(\int |u^{-1/p_0} f|^{p} \rho \, d\mu \right)^{\frac{1}{p}} \left(\int |v^{-1/q_0'} g|^{p'} \varpi \, d\mu \right)^{\frac{1}{p'}}.$$

Due to the definition (6-8) of ρ and ϖ, we conclude

$$|\langle S(f), g\rangle| \lesssim [\omega]^{\alpha} \left(\int |f|^{p} \omega \, d\mu \right)^{\frac{1}{p}} \left(\int |g|^{p'} \sigma \, d\mu \right)^{\frac{1}{p'}},$$

which corresponds to (6-2).

Step 2 (the case $p \le \mathfrak{p}$, i.e., $\beta \ge 0$). In this situation, (6-7) still holds and, due to the choice of δ, it yields (instead of (6-6))

$$|\langle S(f), g\rangle| \lesssim [\omega]^{\alpha} \sum_{k=1}^{K} \sum_{P\in\mathcal{S}_k} \left(\frac{1}{u(\bar{P})}\int_{\bar{P}} |u^{-1/p_0} f|^{p_0} u\, d\mu\right)^{\frac{1}{p_0}} \times \left(\frac{1}{v(\bar{P})}\int_{\bar{P}} |v^{-1/q_0'} g|^{q_0'} v\, d\mu\right)^{\frac{1}{q_0'}} \left(\fint_{\bar{P}} v\, d\mu\right)^{\bar{\beta}} \mu(E_P), \quad (6\text{-}11)$$

with

$$\bar{\beta} := \frac{1}{q_0'} - \frac{1}{\delta} = \frac{1}{q_0'} - \frac{1}{p_0(r-1)} = -(r-1)\beta.$$

In particular, since we are in the situation $\beta \ge 0$, we know that $\bar{\beta} \le 0$. We can then reproduce a similar reasoning as in the first step, using the inequality

$$\left(\fint_{\bar{P}} v\, d\mu\right)^{\bar{\beta}} \lesssim \left(\fint_{E_P} v\, d\mu\right)^{\bar{\beta}}.$$

We use the same weights ϖ and ρ as defined in (6-8), and the exact same computations allow us to conclude since, by definition, $u = v^{1-r'}$, which implies

$$u^{-\beta} = v^{-\beta(1-r')} = v^{-\bar{\beta}}. \qquad \square$$

7. Sharpness of the weighted estimates for the "sparse operators"

We are going to show that the exponents we obtained previously are sharp for sparse operators. We do so only for dimension $n = 1$, since higher-dimensional cases follow through minor modifications.

So let us consider the Euclidean space $\mathbb{R}$, equipped with its natural metric and measure. We first state some easy estimates on specific weights. For $p > 1$, the weight $w_\alpha : x \mapsto |x|^\alpha$ belongs to A_p if and only if $-1 < \alpha < p-1$. One has

$$[w_{-1+\varepsilon}]_{A_p} \sim \varepsilon^{-1} \quad \text{and} \quad [w_{p-1-\varepsilon}]_{A_p} \sim \varepsilon^{-(p-1)}$$

as $\varepsilon \to 0$.

On the other hand, if $s > 1$ then $w_{-1/s+\varepsilon}$ is critical for RH_s. When $\varepsilon \to 0$,

$$[w_{-1/s+\varepsilon}]_{\mathrm{RH}_s} \sim \varepsilon^{-1/s}.$$

Having these sharp estimates, we are now going to prove the optimality of Proposition 6.4. Consider the particular sparse collection $\mathcal{S}$ of those dyadic intervals contained in $[0,1]$ that contain 0, namely $\mathcal{S} = \{I_n := [0, 2^{-n}] : n \in \mathbb{N}\}$. Then $\mathcal{S}$ is a sparse collection. We consider sharpness in the inequality

$$\sum_{I\in\mathcal{S}} |I| \langle |f|^{p_0}\rangle_I^{1/p_0} \langle |g|^{q_0'}\rangle_I^{1/q_0'} \lesssim \Phi([\omega]_{p_0,q_0,p}) \|f\|_{L^p_\omega} \|g\|_{L^{p'}_\sigma}, \qquad (7\text{-}1)$$

where $1 \le p_0 < 2 < q_0 \le \infty$ are fixed and, to simplify the notation, we denote by $\langle\cdot\rangle_I$ the average on the interval I.

Proposition 7.1. *For $p \in (p_0, q_0)$, there exist functions f and g such that, asymptotically as $r \to \infty$, the power function $\Phi(r) = r^\alpha$ is the best possible choice, where $\alpha = 1/(p - p_0)$ if $p \in (p_0, \mathfrak{p}]$ and $\alpha = (q_0 - 1)/(q_0 - p)$ if $p \in [\mathfrak{p}, q_0)$.*

Notice that for $q_0 = \infty$ the above sum corresponds to the pointwise-defined operator

$$Sf = \sum_{I \subseteq \mathscr{D}[0,1], 0 \in I} \langle |f^{p_0}| \rangle_I^{1/p_0} \chi_I$$

tested against g.

For convenience, we also will use the following notation (introduced in [Auscher and Martell 2007a]): for a weight ω,

$$[\omega]_{p_0,q_0,p} := [\omega]_{A_{p/p_0}} [\omega]_{\mathrm{RH}_{(q_0/p)'}}.$$

Proof. Let $p \in (p_0, \mathfrak{p}]$. Consider functions $f_\varepsilon := x \mapsto x^{-1/p_0+\varepsilon} \chi_{[0,1]}$ and $g_\varepsilon := x \mapsto x^{-1/p_0'+\varepsilon} \chi_{[0,1]}$. One calculates, for $I_n = [0, 2^{-n}]$ with $n \geqslant 0$, that

$$\langle |f_\varepsilon|^{p_0} \rangle_{I_n}^{1/p_0} = \frac{2^{n/p_0 - n\varepsilon}}{(p_0 \varepsilon)^{1/p_0}} \sim \varepsilon^{-1/p_0} 2^{-n\varepsilon} 2^{n/p_0}$$

and

$$\langle |g_\varepsilon|^{q_0'} \rangle_{I_n}^{1/q_0'} = \frac{2^{n/p_0' - n\varepsilon}}{(1 - q_0'/p_0' + q_0'\varepsilon)^{1/q_0'}} \sim 2^{-n\varepsilon} 2^{n/p_0'}$$

by noticing that $q_0'/p_0' < 1$.

Hence we obtain, for the left-hand side of (7-1),

$$\varepsilon^{-1/p_0} \sum_{n=0}^{\infty} 2^{-2n\varepsilon} = \varepsilon^{-1/p_0} \frac{1}{1 - \left(\frac{1}{4}\right)^\varepsilon} \sim \varepsilon^{-1/p_0} \varepsilon^{-1}.$$

Choose the weight $\omega_\varepsilon = w_{p/p_0 - 1 - \varepsilon} := x \mapsto x^{p/p_0 - 1 - \varepsilon}$, which is critical for A_{p/p_0}, with

$$[\omega_\varepsilon]_{A_{p/p_0}} \sim \varepsilon^{-(p/p_0 - 1)} \quad \text{as } \varepsilon \to 0.$$

We also notice that ω_ε is a power weight of positive exponent and therefore $[\omega_\varepsilon]_{\mathrm{RH}_{(q_0/p)'}} \sim 1$ as $\varepsilon \to 0$. Thus, $[\omega_\varepsilon]_{p_0,q_0,p} \sim \varepsilon^{-(p/p_0 - 1)}$ and $[\omega_\varepsilon]_{p_0,q_0,p}^{1/(p-p_0)} \sim \varepsilon^{-1/p_0}$. We calculate

$$\|f_\varepsilon\|_{L^p_{\omega_\varepsilon}} = \left(\int_0^1 x^{-1+(p-1)\varepsilon} \, dx \right)^{\frac{1}{p}} \sim \varepsilon^{-1/p}.$$

With $\sigma_\varepsilon = \omega_\varepsilon^{1-p'}$ we calculate

$$\|g_\varepsilon\|_{L^{p'}_{\sigma_\varepsilon}} = \left(\int_0^1 x^{(2p'-1)\varepsilon - 1} \, dx \right)^{\frac{1}{p'}} \sim \varepsilon^{-1/p'}.$$

Gathering the information gives $\varepsilon^{-1/p} \varepsilon^{-1/p'} \varepsilon^{-1/p_0}$ on the right-hand side and $\varepsilon^{-1} \varepsilon^{-1/p_0}$ on the left, showing that the choice of Φ cannot be improved for this range of p.

Now let $p \in [\mathfrak{p}, q_0)$. To treat this range, we apply what we have found before to the modified exponents $1 \le q_0' < 2 < p_0' \le \infty$. We have seen examples of sharpness for the sum

$$\sum_{I \in \mathcal{S}} |I| \langle |f|^{q_0'} \rangle_I^{1/q_0'} \langle |g|^{p_0} \rangle_I^{1/p_0} \sim [\omega]_{q_0', p_0', s}^{1/(s-q_0')} \|f\|_{L^s_\omega} \|g\|_{L^{s'}_\sigma}$$

when $q_0' \le s \le \mathfrak{p}(q_0', p_0')$. Indeed, with $f_\varepsilon := x \mapsto x^{-1/q_0'+\varepsilon} \chi_{[0,1]}$, $g_\varepsilon := x \mapsto x^{-1/q_0+\varepsilon} \chi_{[0,1]}$ and $\omega_\varepsilon := x \mapsto |x|^{s/q_0'-1-\varepsilon}$ we obtain that the left-hand side is of order $\varepsilon^{-1}\varepsilon^{-1/q_0'}$, and $\|f_\varepsilon\|_{L^s_{\omega_\varepsilon}} \sim \varepsilon^{-1/s}$ and $\|g_\varepsilon\|_{L^{s'}_\sigma} \sim \varepsilon^{-1/s'}$. Now observe that $[\mathfrak{p}(q_0', p_0')]' = \mathfrak{p}(p_0, q_0)$. Note also that, therefore, $\mathfrak{p}(p_0, q_0) \le s' \le q_0$. Using this for $s' = p$, it remains to calculate $[\sigma_\varepsilon]_{p_0,q_0,p}^{(q_0-1)/(q_0-p)}$, where $\sigma_\varepsilon = \omega_\varepsilon^{1-p}$:

$$\sigma_\varepsilon(x) = |x|^{(p'/q_0'-1-\varepsilon)(1-p)} = |x|^{-1/(q_0/p)'+(p-1)\varepsilon}.$$

This weight is of negative exponent and critical for $\mathrm{RH}_{(q_0/p)'}$ with $[\sigma_\varepsilon]_{p_0,q_0,p} \sim \varepsilon^{-1/(q_0/p)'}$. Therefore, $[\sigma_\varepsilon]_{p_0,q_0,p}^{(q_0-1)/(q_0-p)} \sim \varepsilon^{-1/q_0'}$. Gathering the information, we obtain that the left-hand side is of order $\varepsilon^{-1}\varepsilon^{-1/q_0'}$ and of order $\varepsilon^{-1/q_0'}\varepsilon^{-1/p}\varepsilon^{-1/p'}$ when using $\Phi(r) = r^\alpha$, showing that the estimate cannot be improved. □

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FRÉDÉRIC BERNICOT: frederic.bernicot@univ-nantes.fr
Laboratoire Jean Leray, CNRS - Universite de Nantes, 2, rue de la Houssinière, 44322 Nantes Cedex 3, France

DOROTHEE FREY: d.frey@tudelft.nl
Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, Netherlands

STEFANIE PETERMICHL: stefanie.petermichl@math.univ-toulouse.fr
Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse Cedex 9, France

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