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FREE PLURIHARMONIC FUNCTIONS ON NONCOMMUTATIVE POLYBALLS

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We study free k-pluriharmonic functions on the noncommutative regular polyball $\boldsymbol{B_n}$, $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$, which is an analogue of the scalar polyball $(\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$. The regular polyball has a universal model $\boldsymbol{S} := \{\boldsymbol{S}_{i,j}\}$ consisting of left creation operators acting on the tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ of full Fock spaces. We introduce the class $\boldsymbol{\mathcal{T}_n}$ of k-multi-Toeplitz operators on this tensor product and prove that $\boldsymbol{\mathcal{T}_n} = \operatorname{span}\{\boldsymbol{\mathcal{A}_n^*}\boldsymbol{\mathcal{A}_n}\}^{\text{-SOT}}$, where $\boldsymbol{\mathcal{A}_n}$ is the noncommutative polyball algebra generated by $\boldsymbol{S}$ and the identity. We show that the bounded free k-pluriharmonic functions on $\boldsymbol{B_n}$ are precisely the noncommutative Berezin transforms of k-multi-Toeplitz operators. The Dirichlet extension problem on regular polyballs is also solved. It is proved that a free k-pluriharmonic function has continuous extension to the closed polyball $\boldsymbol{B_n^-}$ if and only if it is the noncommutative Berezin transform of a k-multi-Toeplitz operator in $\operatorname{span}\{\boldsymbol{\mathcal{A}_n^*}\boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|}$.

We provide a Naimark-type dilation theorem for direct products $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ of unital free semigroups, and use it to obtain a structure theorem which characterizes the positive free k-pluriharmonic functions on the regular polyball with operator-valued coefficients. We define the noncommutative Berezin (resp. Poisson) transform of a completely bounded linear map on $C^*(\boldsymbol{S})$, the C^*-algebra generated by $\boldsymbol{S}_{i,j}$, and give necessary and sufficient conditions for a function to be the Poisson transform of a completely bounded (resp. completely positive) map. In the last section of the paper, we obtain Herglotz–Riesz representation theorems for free holomorphic functions on regular polyballs with positive real parts, extending the classical result as well as the Korányi–Pukánszky version in scalar polydisks.

Introduction

A multivariable operator model theory and a theory of free holomorphic functions on polydomains which admit universal operator models have been recently developed in [Popescu 2013; 2016]. An important feature of these theories is that they are related, via noncommutative Berezin transforms, to the study of

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the operator algebras generated by the universal models as well as to the theory of functions in several complex variables. These results played a crucial role in our work on the curvature invariant [Popescu 2015a], the Euler characteristic [Popescu 2014], and the group of free holomorphic automorphisms on noncommutative regular polyballs [Popescu 2015b].

The main goal of the present paper is to continue our investigation along these lines and to study the class of free k-pluriharmonic functions of the form

$$F(\boldsymbol{X})=\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} a_{(\boldsymbol{\alpha};\boldsymbol{\beta})}\boldsymbol{X}_{1,\alpha_1}\cdots\boldsymbol{X}_{k,\alpha_k}\boldsymbol{X}_{1,\beta_1}^*\cdots\boldsymbol{X}_{k,\beta_k}^*,\quad a_{(\boldsymbol{\alpha};\boldsymbol{\beta})}\in\mathbb{C},$$

where the series converge in the operator norm topology for any $\boldsymbol{X}=\{X_{i,j}\}$ in the regular polyball $\boldsymbol{B_n}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. The results of this paper will play an important role in the hyperbolic geometry of noncommutative polyballs [Popescu $\geq$ 2016]. To present our results we need some notation and preliminaries on regular polyballs and their universal models.

Throughout this paper, $B(\mathcal{H})$ stands for the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. We let $B(\mathcal{H})^{n_1}\times_c\cdots\times_c B(\mathcal{H})^{n_k}$, where $n_i\in\mathbb{N}:=\{1,2,\dots\}$, be the set of all tuples $\boldsymbol{X}:=(X_1,\dots,X_k)$ in $B(\mathcal{H})^{n_1}\times\cdots\times B(\mathcal{H})^{n_k}$ with the property that the entries of $X_s:=(X_{s,1},\dots,X_{s,n_s})$ commute with the entries of $X_t:=(X_{t,1},\dots,X_{t,n_t})$ for any $s,t\in\{1,\dots,k\}$, $s\neq t$. Note that the operators $X_{s,1},\dots,X_{s,n_s}$ do not necessarily commute. Let $\boldsymbol{n}:=(n_1,\dots,n_k)$ and define the polyball

$$\boldsymbol{P_n}(\mathcal{H}):=[B(\mathcal{H})^{n_1}]_1\times_c\cdots\times_c[B(\mathcal{H})^{n_k}]_1,$$

where

$$[B(\mathcal{H})^n]_1:=\{(X_1,\dots,X_n)\in B(\mathcal{H})^n:\|X_1X_1^*+\cdots+X_nX_n^*\|<1\},\quad n\in\mathbb{N}.$$

If A is a positive invertible operator, we write $A>0$. The *regular polyball* on the Hilbert space $\mathcal{H}$ is defined by

$$\boldsymbol{B_n}(\mathcal{H}):=\{\boldsymbol{X}\in\boldsymbol{P_n}(\mathcal{H}):\boldsymbol{\Delta_X}(I)>0\},$$

where the *defect mapping* $\boldsymbol{\Delta_X}:B(\mathcal{H})\to B(\mathcal{H})$ is given by

$$\boldsymbol{\Delta_X}:=(\mathrm{id}-\Phi_{X_1})\circ\cdots\circ(\mathrm{id}-\Phi_{X_k})$$

and $\Phi_{X_i}:B(\mathcal{H})\to B(\mathcal{H})$ is the completely positive linear map defined by

$$\Phi_{X_i}(Y):=\sum_{j=1}^{n_i}X_{i,j}YX_{i,j}^*,\quad Y\in B(\mathcal{H}).$$

Note that if $k=1$ then $\boldsymbol{B_n}(\mathcal{H})$ coincides with the noncommutative unit ball $[B(\mathcal{H})^{n_1}]_1$. We remark that the scalar representation of the (*abstract*) *regular polyball* $\boldsymbol{B_n}:=\{\boldsymbol{B_n}(\mathcal{H}):\mathcal{H}\text{ is a Hilbert space}\}$ is $\boldsymbol{B_n}(\mathbb{C})=\boldsymbol{P_n}(\mathbb{C})=(\mathbb{C}^{n_1})_1\times\cdots\times(\mathbb{C}^{n_k})_1$.

Let H_{n_i} be an n_i-dimensional complex Hilbert space with orthonormal basis $e_1^i,\dots,e_{n_i}^i$. We consider the *full Fock space* of H_{n_i}, defined by $F^2(H_{n_i}):=\mathbb{C}1\oplus\bigoplus_{p\geq 1}H_{n_i}^{\otimes p}$, where $H_{n_i}^{\otimes p}$ is the (Hilbert) tensor product of p copies of H_{n_i}. Let $\mathbb{F}_{n_i}^+$ be the unital free semigroup on n_i generators $g_1^i,\dots,g_{n_i}^i$ and the identity g_0^i. Set $e_\alpha^i := e_{j_1}^i \otimes \cdots \otimes e_{j_p}^i$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $e_{g_0^i}^i := 1 \in \mathbb{C}$. The length of $\alpha \in \mathbb{F}_{n_i}^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0^i$ and $|\alpha| := p$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i$ with $j_1, \ldots, j_p \in \{1, \ldots, n_i\}$. We define the *left creation operator* $S_{i,j}$ acting on the Fock space $F^2(H_{n_i})$ by setting $S_{i,j} e_\alpha^i := e_{g_j^i \alpha}^i$, $\alpha \in \mathbb{F}_{n_i}^+$, and the operator $\boldsymbol{S}_{i,j}$ acting on the Hilbert tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting

$$\boldsymbol{S}_{i,j} := \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes S_{i,j} \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}},$$

where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. We define $\boldsymbol{S} := (\boldsymbol{S}_1, \ldots, \boldsymbol{S}_k)$, where $\boldsymbol{S}_i := (\boldsymbol{S}_{i,1}, \ldots, \boldsymbol{S}_{i,n_i})$, or write $\boldsymbol{S} := \{\boldsymbol{S}_{i,j}\}$. The noncommutative Hardy algebra $\boldsymbol{F}_{\boldsymbol{n}}^\infty$ (resp. the polyball algebra $\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}$) is the weakly closed (resp. norm closed) nonselfadjoint algebra generated by $\{\boldsymbol{S}_{i,j}\}$ and the identity. Similarly, we define the *right creation operator* $R_{i,j} : F^2(H_{n_i}) \to F^2(H_{n_i})$ by setting $R_{i,j} e_\alpha^i := e_{\alpha g_j^i}^i$ for $\alpha \in \mathbb{F}_{n_i}^+$, and the corresponding operator $\boldsymbol{R}_{i,j}$ acting on $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$. The polyball algebra $\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}$ is the norm closed nonselfadjoint algebra generated by $\{\boldsymbol{R}_{i,j}\}$ and the identity.

We proved in [Popescu 2016] (in a more general setting) that $\boldsymbol{X} \in B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ is a *pure* element in the regular polyball $\boldsymbol{B}_{\boldsymbol{n}}(\mathcal{H})^-$, i.e., $\lim_{q_i \to \infty} \Phi_{X_i}^{q_i}(I) = 0$ in the weak operator topology, if and only if there is a Hilbert space $\mathcal{D}$ and a subspace $\mathcal{M} \subset F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{D}$ invariant under each operator $\boldsymbol{S}_{i,j} \otimes I$ such that $X_{i,j}^* = (\boldsymbol{S}_{i,j}^* \otimes I)|_{\mathcal{M}^\perp}$, under an appropriate identification of $\mathcal{H}$ with $\mathcal{M}^\perp$. The k-tuple $\boldsymbol{S} := (\boldsymbol{S}_1, \ldots, \boldsymbol{S}_k)$, where $\boldsymbol{S}_i := (\boldsymbol{S}_{i,1}, \ldots, \boldsymbol{S}_{i,n_i})$, is an element in the regular polyball $\boldsymbol{B}_{\boldsymbol{n}}\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)^-$ and plays the role of *left universal model* for the abstract polyball $\boldsymbol{B}_{\boldsymbol{n}}^- := \{\boldsymbol{B}_{\boldsymbol{n}}(\mathcal{H})^- : \mathcal{H} \text{ is a Hilbert space}\}$. The existence of the universal model will play an important role in this paper, since it will make the connection between noncommutative function theory, operator algebras, and complex function theory in several variables.

Brown and Halmos [1963] showed that a bounded linear operator T on the Hardy space $H^2(\mathbb{D})$ is a Toeplitz operator if and only if $S^* T S = T$, where S is the unilateral shift. Expanding on this idea, a study of noncommutative multi-Toeplitz operators on the full Fock space with n generators $F^2(H_n)$ was initiated in [Popescu 1989; 1995] and has had an important impact in multivariable operator theory and the structure of free semigroup algebras (see [Davidson and Pitts 1998; Davidson et al. 2001; 2005; Popescu 2006; 2009; Kennedy 2011; 2013]).

In Section 1, we introduce and study the class $\boldsymbol{\mathcal{T}}_{\boldsymbol{n}}$, $\boldsymbol{n} := (n_1, \ldots, n_k) \in \mathbb{N}^k$, of k-multi-Toeplitz operators. A bounded linear operator T on the tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ of full Fock spaces is called a k-multi-Toeplitz operator with respect to the *right universal model* $\boldsymbol{R} = \{\boldsymbol{R}_{i,j}\}$ if

$$\boldsymbol{R}_{i,s}^* T \boldsymbol{R}_{i,t} = \delta_{st} T, \quad s, t \in \{1, \ldots, n_i\},$$

for every $i \in \{1, \ldots, k\}$. We associate with each k-multi-Toeplitz operator T a formal power series in several variables and show that we can recapture T from its noncommutative "Fourier series". Moreover, we characterize the noncommutative formal power series which are Fourier series of k-multi-Toeplitz operators (see Theorems 1.5 and 1.6). Using these results, we prove that the set of all k-multi-Toeplitz operators on $\bigotimes_{i=1}^k F^2(H_{n_i})$ coincides with

$$\operatorname{span}\{\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}^{\text{-SOT}} = \operatorname{span}\{\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}^{\text{-WOT}},$$

where $\mathcal{A}_n$ is the noncommutative polyball algebra.

In Section 2, we characterize the bounded free k-pluriharmonic functions on regular polyballs. We prove that a function $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{H})$ is a bounded free k-pluriharmonic function if and only if there is a k-multi-Toeplitz operator $A \in \mathcal{T}_n$ such that $F(\boldsymbol{X}) = \mathcal{B}_{\boldsymbol{X}}[A]$ for $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$, where $\mathcal{B}_{\boldsymbol{X}}$ is the noncommutative Berezin transform at $\boldsymbol{X}$ (see Section 1 for the definition). In this case, $A = \text{SOT-}\lim_{r\to 1} F(r\boldsymbol{S})$ and there is a completely isometric isomorphism of operator spaces

$$\Phi : \mathbf{PH}^\infty(\boldsymbol{B_n}) \to \mathcal{T}_n, \quad \Phi(F) := A,$$

where $\mathbf{PH}^\infty(\boldsymbol{B_n})$ is the operator space of all bounded free k-pluriharmonic functions on the polyball.

The Dirichlet extension problem [Hoffman 1962] on noncommutative regular polyballs is solved. We show that a mapping $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{H})$ is a free k-pluriharmonic function which has continuous extension (in the operator norm topology) to the closed polyball $\boldsymbol{B_n}(\mathcal{H})^-$, and write $F \in \mathbf{PH}^c(\boldsymbol{B_n})$, if and only if there exists a k-multi-Toeplitz operator $A \in \overline{\operatorname{span}}\{\mathcal{A}_n^* \mathcal{A}_n\}^{\|\cdot\|}$ such that $F(\boldsymbol{X}) = \mathcal{B}_{\boldsymbol{X}}[A]$ for $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. In this case, $A = \lim_{r\to 1} F(r\boldsymbol{S})$, where the convergence is in the operator norm, and the map

$$\Phi : \mathbf{PH}^c(\boldsymbol{B_n}) \to \overline{\operatorname{span}}\{\mathcal{A}_n^* \mathcal{A}_n\}^{\|\cdot\|}, \quad \Phi(F) := A,$$

is a completely isometric isomorphism of operator spaces.

In Section 3, we provide a Naimark-type dilation theorem [1943] for direct products $\boldsymbol{F}_n^+ := \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ of free semigroups. We show that a map $K : \boldsymbol{F}_n^+ \times \boldsymbol{F}_n^+ \to B(\mathcal{E})$ is a positive semidefinite left k-multi-Toeplitz kernel on $\boldsymbol{F}_n^+$ if and only if there exists a k-tuple of *commuting row isometries* $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ — i.e., the nonselfadjoint algebra $\operatorname{Alg}(V_i)$ commutes with $\operatorname{Alg}(V_s)$ for any $i, s \in \{1, \dots, k\}$ with $i \neq s$ — such that

$$K(\boldsymbol{\sigma}, \boldsymbol{\omega}) = P_{\mathcal{E}} \boldsymbol{V}_{\boldsymbol{\sigma}}^* \boldsymbol{V}_{\boldsymbol{\omega}}\big|_{\mathcal{E}}, \quad \boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}_n^+,$$

and $\mathcal{K} = \bigvee_{\boldsymbol{\omega} \in \boldsymbol{F}_n^+} \boldsymbol{V}_{\boldsymbol{\omega}} \mathcal{E}$. In this case, the minimal dilation is unique up to isomorphism. Here, we use the notation $\boldsymbol{V}_{\boldsymbol{\sigma}} := V_{1,\sigma_1} \cdots V_{k,\sigma_k}$ if $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_k) \in \boldsymbol{F}_n^+$, and $V_{i,\sigma_i} := V_{i,j_1} \cdots V_{i,j_p}$ if $\sigma_i = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $V_{i,g_0^i} := I$. For more information on kernels in various noncommutative settings we refer the reader to the work of Ball and Vinnikov [2003] (see also [Ball et al. 2016] and the references therein).

We prove a Schur-type result [1918], which states that a free k-pluriharmonic function F on the polyball $\boldsymbol{B_n}$ is positive if and only if a certain right k-multi-Toeplitz kernel Γ_{F_r} associated with the mapping $\boldsymbol{S} \mapsto F(r\boldsymbol{S})$ is positive semidefinite for any $r \in [0, 1)$. Our Naimark-type result for positive semidefinite right k-multi-Toeplitz kernels on $\boldsymbol{F}_n^+$ is used to provide a structure theorem for positive free k-pluriharmonic functions. We show that a free k-pluriharmonic function $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ with $F(0) = I$ is positive if and only if it has the form

$$F(\boldsymbol{X}) = \sum_{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}_{\boldsymbol{\beta}}^*,$$

where $\boldsymbol{V} = (V_1, \dots, V_k)$ is a k-tuple of commuting row isometries on a space $\mathcal{K} \supset \mathcal{E}$ and $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_k)$ is the reverse of $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$, i.e., $\tilde{\alpha}_i = g_{i_k}^i \cdots g_{i_1}^i$ if $\alpha_i = g_{i_1}^i \cdots g_{i_k}^i \in \mathbb{F}_{n_i}^+$. The general case, when $F(0) \geq 0$, is also considered. As a consequence of these results, we obtain a structure theorem for positive k-harmonic functions on the regular polydisk included in $[B(\mathcal{H})]_1 \times_c \cdots \times_c [B(\mathcal{H})]_1$, which extends the corresponding classical result in scalar polydisks [Rudin 1969].

In Section 4, we define the *free pluriharmonic Poisson kernel* on the regular polyball $\boldsymbol{B_n}$ by setting

$$\boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X}) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+, \, i \in \{1,\dots,k\} \\ |\alpha_i| = m_i^-, \, |\beta_i| = m_i^+}} \boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k} \otimes X_{1,\alpha_1} \cdots X_{k,\alpha_k} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^*$$

for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$, where the convergence is in the operator norm topology. Given a completely bounded linear map $\mu : \operatorname{span}\{\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}\} \to B(\mathcal{E})$, we introduce the *noncommutative Poisson transform* of μ to be the map $\boldsymbol{\mathcal{P}}\mu : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ defined by

$$(\boldsymbol{\mathcal{P}}\mu)(\boldsymbol{X}) := \hat{\mu}[\boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X})], \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}),$$

where the completely bounded linear map

$$\hat{\mu} := \mu \otimes \mathrm{id} : \operatorname{span}\{\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}\}^{-\|\cdot\|} \otimes_{\min} B(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$$

is uniquely defined by $\hat{\mu}(A \otimes Y) := \mu(A) \otimes Y$ for any $A \in \operatorname{span}\{\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}\}$ and $Y \in B(\mathcal{H})$. We remark that, in the particular case when $n_1 = \cdots = n_k = 1$, $\mathcal{H} = \mathcal{K} = \mathbb{C}$, $\boldsymbol{X} = (X_1, \dots, X_k) \in \mathbb{D}^k$, and μ is a complex Borel measure on $\mathbb{T}^k$ (which can be seen as a bounded linear functional on $C(\mathbb{T}^k)$), we have that the noncommutative Poisson transform of μ coincides with the classical Poisson transform of μ [Rudin 1969].

In Section 4, we give necessary and sufficient conditions for a function $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ to be the noncommutative Poisson transform of a completely bounded linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$, where $C^*(\boldsymbol{R})$ is the C^*-algebra generated by the operators $\boldsymbol{R}_{i,j}$. In this case, we show that there exist a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, of *doubly commuting row isometries* acting on a Hilbert space $\mathcal{K}$, i.e., $C^*(V_i)$ commutes with $C^*(V_j)$ if $i \neq j$, and bounded linear operators $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ such that

$$F(\boldsymbol{X}) = (W_1^* \otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W_2 \otimes I), \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}),$$

where

$$C_{\boldsymbol{X}}(\boldsymbol{V}) := (I \otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}) \prod_{i=1}^{k} (I - V_{i,1} \otimes X_{i,1}^* - \cdots - V_{i,n_i} \otimes X_{i,n_i}^*)^{-1}.$$

In particular, we obtain necessary and sufficient conditions for a function $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ to be the noncommutative Poisson transform of a completely positive linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$. In this case, we have the representation

$$F(\boldsymbol{X}) = (W^* \otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W \otimes I), \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

In Section 5, we introduce the *noncommutative Herglotz–Riesz transform* of a completely positive linear map $\mu : \operatorname{span}\{\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}\} \to B(\mathcal{E})$ as the map $\boldsymbol{H}\mu : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ defined by

$$(\boldsymbol{H}\mu)(\boldsymbol{X}) := \hat{\mu}\left(2 \prod_{i=1}^{k} (I - \boldsymbol{R}_{i,1}^* \otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I\right)$$

for $\boldsymbol{X} := (X_1, \dots, X_n) \in \boldsymbol{B_n}(\mathcal{H})$. The main result of this section provides necessary and sufficient conditions for a function f from the polyball $\boldsymbol{B_n}(\mathcal{H})$ to $B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ to admit a Herglotz–Riesz-type representation [Herglotz 1911; Riesz 1911], i.e.,

$$f(\boldsymbol{X}) = (\boldsymbol{H}\mu)(\boldsymbol{X}) + i\Im f(0), \quad \boldsymbol{X} \in \boldsymbol{B}_n(\mathcal{H}),$$

where $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ is a completely positive linear map with the property that $\mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_{\boldsymbol{n}}^+$. In this case, we show that there exist a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, of doubly commuting row isometries on a Hilbert space $\mathcal{K}$ and a bounded linear operator $W : \mathcal{E} \to \mathcal{K}$ such that

$$f(\boldsymbol{X}) = (W^* \otimes I)\left(2\prod_{i=1}^{k}(I - V_{i,1}^* \otimes X_{i,1} - \cdots - V_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I\right)(W \otimes I) + i\Im f(0)$$

and $W^* \boldsymbol{V}_{\boldsymbol{\alpha}}^* \boldsymbol{V}_{\boldsymbol{\beta}} W = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_{\boldsymbol{n}}^+$.

We remark that, in the particular case when $n_1 = \cdots = n_k = 1$, we obtain an operator-valued extension of the integral representation for holomorphic functions with positive real parts in polydisks [Korányi and Pukánszky 1963].

1. k-multi-Toeplitz operators on tensor products of full Fock spaces

In this section, we introduce the class $\boldsymbol{\mathcal{T}_n}$ of k-multi-Toeplitz operators on tensor products of full Fock spaces. We associate with each k-multi-Toeplitz operator T a formal power series in several variables and show that we can recapture T from its noncommutative Fourier series. Moreover, we characterize the noncommutative formal power series which are Fourier series of k-multi-Toeplitz operators and prove that $\boldsymbol{\mathcal{T}_n} = \text{span}\{\boldsymbol{\mathcal{A}_n^*}\boldsymbol{\mathcal{A}_n}\}^{\text{-SOT}}$, where $\boldsymbol{\mathcal{A}_n}$ is the noncommutative polyball algebra.

First, we recall (see [Popescu 1999; 2016]) some basic properties for a class of noncommutative Berezin-type transforms [1972] associated with regular polyballs. Let $\boldsymbol{X} = (X_1, \dots, X_k) \in \boldsymbol{B_n}(\mathcal{H})^-$ with $X_i := (X_{i,1}, \dots, X_{i,n_i})$. We use the notation $X_{i,\alpha_i} := X_{i,j_1} \cdots X_{i,j_p}$ if $\alpha_i = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $X_{i,g_0^i} := I$. The *noncommutative Berezin kernel* associated with any element $\boldsymbol{X}$ in the noncommutative polyball $\boldsymbol{B_n}(\mathcal{H})^-$ is the operator

$$\boldsymbol{K_X} : \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \overline{\boldsymbol{\Delta_X}(I)(\mathcal{H})}$$

defined by

$$\boldsymbol{K_X} h := \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\dots,k} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes \boldsymbol{\Delta}_X(I)^{1/2} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^* h, \quad h \in \mathcal{H},$$

where the defect operator $\boldsymbol{\Delta_X}(I)$ was defined in the introduction. A very important property of the Berezin kernel is that $\boldsymbol{K_X} X_{i,j}^* = (\boldsymbol{S}_{i,j}^* \otimes I)\boldsymbol{K_X}$ for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. The *Berezin transform at $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$* is the map $\boldsymbol{\mathcal{B}_X} : B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big) \to B(\mathcal{H})$ defined by

$$\boldsymbol{\mathcal{B}_X}[g] := \boldsymbol{K_X^*}(g \otimes I_{\mathcal{H}}) f \boldsymbol{K_X}, \quad g \in B\left(\bigotimes_{i=1}^k F^2(H_{n_i})\right).$$

If g is in the C^*-algebra $C^*(\boldsymbol{S})$ generated by $\boldsymbol{S}_{i,1}, \dots, \boldsymbol{S}_{i,n_i}$, where $i \in \{1, \dots, k\}$, we define the Berezin transform at $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})^-$ by

$$\mathcal{B}_{\boldsymbol{X}}[g] := \lim_{r\to 1} \boldsymbol{K}_{r\boldsymbol{X}}^*(g \otimes I_{\mathcal{H}})\boldsymbol{K}_{r\boldsymbol{X}}, \quad g \in C^*(\boldsymbol{S}),$$

where the limit is in the operator norm topology. In this case, the Berezin transform at $\boldsymbol{X}$ is a unital completely positive linear map such that

$$\mathcal{B}_{\boldsymbol{X}}(\boldsymbol{S}_{\boldsymbol{\alpha}}\boldsymbol{S}_{\boldsymbol{\beta}}^*) = \boldsymbol{X}_{\boldsymbol{\alpha}}\boldsymbol{X}_{\boldsymbol{\beta}}^*, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+,$$

where $\boldsymbol{S}_{\boldsymbol{\alpha}} := \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k}$ if $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$.

The Berezin transform will play an important role in this paper. More properties concerning noncommutative Berezin transforms and multivariable operator theory on noncommutative balls and polydomains can be found in [Popescu 1999; 2013; 2016]. For basic results on completely positive and completely bounded maps we refer the reader to [Paulsen 1986; Pisier 2001; Effros and Ruan 2000].

Definition 1.1. Let $\mathcal{E}$ be a Hilbert space. A bounded linear operator $A \in B\big(\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})\big)$ is called k-multi-Toeplitz with respect to the universal model $\boldsymbol{R} := (\boldsymbol{R}_1, \dots, \boldsymbol{R}_k)$, where $\boldsymbol{R}_i := (\boldsymbol{R}_{i,1}, \dots, \boldsymbol{R}_{i,n_i})$, if

$$(I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,s}^*)A(I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,t}) = \delta_{st}A, \quad s, t \in \{1, \dots, n_i\},$$

for every $i \in \{1, \dots, k\}$.

A few more notations are necessary. If $\omega, \gamma \in \mathbb{F}_n^+$, we say that $\gamma <_r \omega$ if there is $\sigma \in \mathbb{F}_n^+ \setminus \{g_0\}$ such that $\omega = \sigma\gamma$. In this case, we set $\omega \setminus_r \gamma := \sigma$. Similarly, we say that $\gamma <_l \omega$ if there is $\sigma \in \mathbb{F}_n^+ \setminus \{g_0\}$ such that $\omega = \gamma\sigma$ and set $\omega \setminus_l \gamma := \sigma$. We denote by $\tilde{\alpha}$ the reverse of $\alpha \in \mathbb{F}_n^+$, i.e., $\tilde{\alpha} = g_{i_k} \cdots g_{i_1}$ if $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$. Notice that $\gamma <_r \omega$ if and only if $\tilde{\gamma} <_l \tilde{\omega}$. In this case we have $(\omega \setminus_r \gamma)^\sim = \tilde{\omega} \setminus_l \tilde{\gamma}$. We say that ω is *right comparable* with γ, and write $\omega \sim_{\mathrm{rc}} \gamma$, if any one of the conditions $\omega <_r \gamma$, $\gamma <_r \omega$ or $\omega = \gamma$ holds. In this case, we define

$$c_r^+(\omega, \gamma) := \begin{cases} \omega \setminus_r \gamma & \text{if } \gamma <_r \omega, \\ g_0 & \text{if } \omega <_r \gamma \text{ or } \omega = \gamma, \end{cases} \qquad \text{and} \qquad c_r^-(\omega, \gamma) := \begin{cases} \gamma \setminus_r \omega & \text{if } \omega <_r \gamma, \\ g_0 & \text{if } \gamma <_r \omega \text{ or } \omega = \gamma. \end{cases}$$

Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ be in $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$. We say that $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are right comparable, and write $\boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}$, if for each $i \in \{1, \dots, k\}$, any one of the conditions $\omega_i <_r \gamma_i$, $\gamma_i <_r \omega_i$ or $\omega_i = \gamma_i$ holds. In this case, we define

$$c_r^+(\boldsymbol{\omega}, \boldsymbol{\gamma}) := (c_r^+(\omega_1, \gamma_1), \dots, c_r^+(\omega_k, \gamma_k)) \quad \text{and} \quad c_r^-(\boldsymbol{\omega}, \boldsymbol{\gamma}) := (c_r^-(\omega_1, \gamma_1), \dots, c_r^-(\omega_k, \gamma_k)). \tag{1-1}$$

Similarly, we say that $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are *left comparable*, and write $\boldsymbol{\omega} \sim_{\mathrm{lc}} \boldsymbol{\gamma}$, if $\tilde{\boldsymbol{\omega}} \sim_{\mathrm{rc}} \tilde{\boldsymbol{\gamma}}$. The definitions of $c_l^+(\boldsymbol{\omega}, \boldsymbol{\gamma})$ and $c_l^-(\boldsymbol{\omega}, \boldsymbol{\gamma})$ are now clear. Note that

$$c_r^+(\boldsymbol{\omega}, \boldsymbol{\gamma})^\sim = c_l^+(\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\gamma}}) \quad \text{and} \quad c_r^-(\boldsymbol{\omega}, \boldsymbol{\gamma})^\sim = c_l^-(\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\gamma}}).$$

For each $m \in \mathbb{Z}$, we set $m^+ := \max\{m, 0\}$ and $m^- := \max\{-m, 0\}$.

Lemma 1.2. *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)$ be k-tuples in $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ such that $\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+$ for $i \in \{1, \dots, k\}$ with $|\alpha_i| = m_i^-$, $|\beta_i| = m_i^+$ and $m_i \in \mathbb{Z}$. If $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ are k-tuples in $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, then the inner product*

$$\left\langle \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^* (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k), e_{\omega_1}^1 \otimes \cdots \otimes e_{\omega_k}^k \right\rangle$$

is different from zero if and only if $\boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}$ and $(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_k) = (c_r^+(\boldsymbol{\omega}, \boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega}, \boldsymbol{\gamma}))$.

Proof. Under the conditions of the lemma, $\boldsymbol{S}_{i,\alpha_i} \boldsymbol{S}_{j,\beta_j}^* = \boldsymbol{S}_{j,\beta_j}^* \boldsymbol{S}_{i,\alpha_i}$ for any $i, j \in \{1, \dots, k\}$, $\alpha_i \in \mathbb{F}_{n_i}^+$ and $\beta_j \in \mathbb{F}_{n_j}^+$. Note that the inner product is different from zero if and only if $\beta_i \omega_i = \alpha_i \gamma_i$ for any $i \in \{1, \dots, k\}$. Let $m_i \in \mathbb{Z}$ and assume that $|\alpha_i| = m_i^- > 0$. Then $\beta_i = g_0^i$ and, consequently, $\omega_i = \alpha_i \gamma_i$. This shows that $\gamma_i <_r \omega_i$, $c_r^+(\omega_i, \gamma_i) = \alpha_i$ and $c_r^-(\omega_i, \gamma_i) = g_0^i$. In the case when $|\beta_i| = m_i^+ > 0$, we have $\alpha_i = g_0^i$ and $\beta_i \omega_i = \gamma_i$. Consequently, $\omega_i <_r \gamma_i$, $c_r^+(\omega_i, \gamma_i) = g_0^i$ and $c_r^-(\omega_i, \gamma_i) = \beta_i$. When $\alpha_i = \beta_i = g_0^i$, we have $\omega_i = \gamma_i$. Therefore, the scalar product above is different from zero if and only if $\boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}$ and $(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_k) = (c_r^+(\boldsymbol{\omega}, \boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega}, \boldsymbol{\gamma}))$. $\square$

If $\beta_i, \gamma_i \in \mathbb{F}_{n_i}^+$ and, for each $i \in \{1, \dots, k\}$, $\beta_i <_\ell \gamma_i$ or $\beta_i = \gamma_i$, then we write $\boldsymbol{\beta} \leq_\ell \boldsymbol{\gamma}$.

Lemma 1.3. *Given a k-tuple $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, the sequence*

$$\{\boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^* (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k)\}$$

consists of orthonormal vectors if $\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+$, $i \in \{1, \dots, k\}$ with $m_i \in \mathbb{Z}$, $|\alpha_i| = m_i^-$, $|\beta_i| = m_i^+$ and $\boldsymbol{\beta} \leq_\ell \boldsymbol{\gamma}$.

Proof. First, note that $\boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^* (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k) \neq 0$ if and only if $S_{i,\beta_i}^*(e_{\gamma_i}^i) \neq 0$ for each $i \in \{1, \dots, k\}$, which is equivalent to $\beta_i <_\ell \gamma_i$ or $\beta_i = \gamma_i$. Therefore, $\boldsymbol{\beta} \leq_\ell \boldsymbol{\gamma}$.

Fix $i \in \{1, \dots, k\}$ and $\gamma_i \in \mathbb{F}_{n_i}^+$. We prove that the sequence $\{S_{i,\alpha_i} S_{i,\beta_i}^* e_{\gamma_i}^i\}$ consists of orthonormal vectors if $\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+$ have the following properties:

(i) If $|\alpha_i| > 0$ then $\beta_i = g_0^i$, and if $|\beta_i| > 0$ then $\alpha_i = g_0^i$.

(ii) $\beta_i \leq_\ell \gamma_i$.

Indeed, let (α_i, β_i) and (α_i', β_i') be two distinct pairs with the above-mentioned properties. First, we consider the case when $g_0^i \neq \beta_i <_\ell \gamma_i$. Then $\alpha_i = g_0^i$ and, consequently, $S_{i,\alpha_i} S_{i,\beta_i}^* e_{\gamma_i}^i = e_{\gamma_i \backslash_\ell \beta_i}^i$. Similarly, if $g_0^i \neq \beta_i' <_\ell \gamma_i$ then $\alpha_i' = g_0^i$ and, consequently, $S_{i,\alpha_i'} S_{i,\beta_i'}^* e_{\gamma_i}^i = e_{\gamma_i \backslash_\ell \beta_i'}^i$. Since $(\alpha_i, \beta_i) \neq (\alpha_i', \beta_i')$, we must have $\beta_i \neq \beta_i'$, which implies $e_{\gamma_i \backslash_\ell \beta_i}^i \perp e_{\gamma_i \backslash_\ell \beta_i'}^i$. On the other hand, if $\beta_i' = g_0^i$ then $\alpha_i' \in \mathbb{F}_{n_i}^+$ and $S_{i,\alpha_i'} S_{i,\beta_i'}^* e_{\gamma_i}^i = e_{\alpha_i'} e_{\gamma_i} \perp e_{\gamma_i \backslash_\ell \beta_i'}^i$. It follows that $S_{i,\alpha_i'} S_{i,\beta_i'}^* e_{\gamma_i}^i \perp S_{i,\alpha_i'} S_{i,\beta_i'}^* e_{\gamma_i}^i$.

The second case is when $\beta_i = g_0^i$. Then $\alpha_i \in \mathbb{F}_{n_i}^+$ and $S_{i,\alpha_i} S_{i,\beta_i}^* e_{\gamma_i}^i = e_{\alpha_i} e_{\gamma_i}$. As we saw above, $S_{i,\alpha_i'} S_{i,\beta_i'}^* e_{\gamma_i}^i$ is equal to either $e_{\alpha_i'} e_{\gamma_i}$ (when $\beta_i' = g_0^i$) or $e_{\gamma_i \backslash_\ell \beta_i'}^i$ (when $g_0^i \neq \beta_i' <_\ell \gamma_i$). In each case, we have $S_{i,\alpha_i'} S_{i,\beta_i'}^* e_{\gamma_i}^i \perp S_{i,\alpha_i'} S_{i,\beta_i'}^* e_{\gamma_i}^i$, which completes the proof of our assertion. Using this result one can easily complete the proof of the lemma. $\square$

We associate with each k-multi-Toeplitz operator $A \in B\big(\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})\big)$ a formal power series

$$\varphi_A(\boldsymbol{S}) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^*,$$

where the coefficients are given by

$$\langle A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h, \ell \rangle := \langle A(h \otimes x), \ell \otimes y \rangle, \quad h, \ell \in \mathcal{E}, \tag{1-2}$$

and $x := x_1 \otimes \cdots \otimes x_k$, $y = y_1 \otimes \cdots \otimes y_k$ with

$$\begin{cases} x_i = e_{\beta_i}^i \ \text{ and } \ y_i = 1 & \text{if } \ m_i \geq 0, \\ x_i = 1 \ \text{ and } \ y_i = e_{\alpha_i}^i & \text{if } \ m_i < 0, \end{cases} \tag{1-3}$$

for every $i \in \{1, \dots, k\}$.

The next result shows that a k-multi-Toeplitz operator is uniquely determined by is Fourier series.

Theorem 1.4. *If A, B are k-multi-Toeplitz operators on $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$, then $A = B$ if and only if the corresponding formal Fourier series $\varphi_A(\boldsymbol{S})$ and $\varphi_B(\boldsymbol{S})$ are equal. Moreover, $Aq = \varphi_A(\boldsymbol{S})q$ for any vector-valued polynomial*

$$q = \sum_{\substack{\omega_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\omega_i| \leq p_i}} h_{(\omega_1,\dots,\omega_k)} \otimes e_{\omega_1}^1 \otimes \cdots \otimes e_{\omega_k}^k,$$

where $h_{(\omega_1,\dots,\omega_k)} \in \mathcal{E}$ and $(p_1, \dots, p_k) \in \mathbb{N}^k$.

Proof. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ be k-tuples in $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, and let $h, h' \in \mathcal{E}$. Since A is a k-multi-Toeplitz operator on $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$, we have

$$\begin{aligned} &\langle A(h \otimes e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k), h' \otimes e_{\omega_1}^1 \otimes \cdots \otimes e_{\omega_k}^k \rangle \\ &\qquad = \big\langle A(I_{\mathcal{E}} \otimes \boldsymbol{R}_{1,\tilde{\gamma}_1} \cdots \boldsymbol{R}_{k,\tilde{\gamma}_k})(h \otimes 1), (I_{\mathcal{E}} \otimes \boldsymbol{R}_{1,\tilde{\omega}_1} \cdots \boldsymbol{R}_{k,\tilde{\omega}_k})(h' \otimes 1) \big\rangle \\ &\qquad = \begin{cases} \langle A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} h, h' \rangle & \text{if } \boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $c_r^+(\boldsymbol{\omega}, \boldsymbol{\gamma})$ and $c_r^-(\boldsymbol{\omega}, \boldsymbol{\gamma})$ are defined by (1-1). Consequently,

$$A(h \otimes e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k) = \sum_{\substack{\boldsymbol{\omega} = (\omega_1,\dots,\omega_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \\ \boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}}} A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} h \otimes e_{\omega_1}^1 \otimes \cdots \otimes e_{\omega_k}^k$$

is a vector in $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$. Hence, we deduce that, for each $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, the series

$$\sum_{\substack{\boldsymbol{\omega} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \\ \boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}}} A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))}^* A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} \tag{1-4}$$

is WOT-convergent. Due to Lemma 1.3, given $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, the sequence $\{\boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^*(e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k)\}$, where $\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+$, $i \in \{1, \dots, k\}$ with $m_i \in \mathbb{Z}$, $|\alpha_i| = m_i^-$,

$|\beta_i| = m_i^+$ and $\boldsymbol{\beta} \leq_\ell \boldsymbol{\gamma}$, consists of orthonormal vectors. Note that, in this case, we also have $\boldsymbol{\alpha} \sim_{\rm rc} \boldsymbol{\beta}$ and, consequently, $A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} = A_{(c_r^+(\boldsymbol{\alpha},\boldsymbol{\beta});c_r^-(\boldsymbol{\alpha},\boldsymbol{\beta}))}$. Hence, and taking into account that the series (1-4) is WOT-convergent, we deduce that

$$
\begin{aligned}
&\varphi_A(\boldsymbol{S})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}) \\
&\qquad := \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h \otimes \boldsymbol{S}_{1,\alpha_1}\cdots \boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}^*_{1,\beta_1}\cdots \boldsymbol{S}^*_{k,\beta_k}(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k})
\end{aligned}
$$

is a convergent series in $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ be k-tuples in $\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}$. According to Lemma 1.2, the inner product

$$
\left\langle \boldsymbol{S}_{1,\alpha_1}\cdots \boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}^*_{1,\beta_1}\cdots \boldsymbol{S}^*_{k,\beta_k}(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \right\rangle
$$

is different from zero if and only if $\boldsymbol{\omega} \sim_{\rm rc} \boldsymbol{\gamma}$ and $(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k) = (c_r^+(\boldsymbol{\omega}, \boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega}, \boldsymbol{\gamma}))$. Now, using (1-4), one can see that

$$
\begin{aligned}
&\left\langle \varphi_A(\boldsymbol{S})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \right\rangle \\
&\quad = \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} \begin{aligned}[t] &\langle A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h, h' \rangle \\ &\times \left\langle \boldsymbol{S}_{1,\alpha_1}\cdots \boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}^*_{1,\beta_1}\cdots \boldsymbol{S}^*_{k,\beta_k}(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \right\rangle \end{aligned} \\
&\quad = \begin{cases} \langle A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} h, h' \rangle & \text{if } \boldsymbol{\omega} \sim_{\rm rc} \boldsymbol{\gamma}, \\ 0 & \text{otherwise,} \end{cases} \\
&\quad = \langle A(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \rangle
\end{aligned}
$$

for any $h, h' \in \mathcal{E}$, and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ in $\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}$, which shows that $Aq = \varphi_A(\boldsymbol{S})q$ for any vector-valued polynomial in $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$. Therefore, if the formal Fourier series $\varphi_A(\boldsymbol{S})$ and $\varphi_B(\boldsymbol{S})$ are equal, then $A = B$. $\square$

When $\mathcal{G}$ is a Hilbert space, $C_{(\alpha;\beta)} \in B(\mathcal{G})$, and the series

$$
\Sigma_1 := \sum_{m\in\mathbb{Z}, m<0} \sum_{\substack{\alpha,\beta\in\mathbb{F}^+_n\\ |\alpha|=m^-,\, |\beta|=m^+}} C_{(\alpha;\beta)} \quad \text{and} \quad \Sigma_2 := \sum_{m\in\mathbb{Z}, m\geq 0} \sum_{\substack{\alpha,\beta\in\mathbb{F}^+_n\\ |\alpha|=m^-,\, |\beta|=m^+}} C_{(\alpha;\beta)}
$$

are convergent in the operator topology, we say that the series

$$
\sum_{m\in\mathbb{Z}} \sum_{\substack{\alpha,\beta\in\mathbb{F}^+_n\\ |\alpha|=m^-,\, |\beta|=m^+}} C_{(\alpha;\beta)} := \Sigma_1 + \Sigma_2
$$

is convergent in the operator topology. In what follows, we show how we can recapture the k-multi-Toeplitz operators from their Fourier series. Moreover, we characterize the formal series which are Fourier series of k-multi-Toeplitz operators. Let $\mathcal{P}$ denote the set of all vector-valued polynomials in $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$, i.e., each $p \in \mathcal{P}$ has the form

$$q = \sum_{\substack{\omega_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\omega_i| \le p_i}} h_{(\omega_1,\dots,\omega_k)} \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k},$$

where $h_{(\omega_1,\dots,\omega_k)} \in \mathcal{E}$ and $(p_1, \dots, p_k) \in \mathbb{N}^k$.

Theorem 1.5. *Let* $\{A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\}$ *be a family of operators in* $B(\mathcal{E})$, *where* $\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+$, $|\alpha_i| = m_i^-$, $|\beta_i| = m_i^+$, $m_i \in \mathbb{Z}$ *and* $i \in \{1, \dots, k\}$. *Then*

$$\varphi(\boldsymbol{S}) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}^*_{1,\beta_1} \cdots \boldsymbol{S}^*_{k,\beta_k}$$

is the formal Fourier series of a k-multi-Toeplitz operator on $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$ *if and only if the following conditions are satisfied*:

(i) *For each* $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, *the series*

$$\sum_{\substack{\boldsymbol{\omega} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \\ \boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}}} A^*_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))}$$

is WOT-convergent.

(ii) *If* $\mathcal{P}$ *is the set of all vector-valued polynomials in* $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$, *then*

$$\sup_{r \in [0,1)} \sup_{p \in \mathcal{P}, \|p\| \le 1} \|\varphi(r\boldsymbol{S})p\| < \infty.$$

Moreover, if there is a k-multi-Toeplitz operator $A \in B\big(\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})\big)$ *such that* $\varphi(\boldsymbol{S}) = \varphi_A(\boldsymbol{S})$, *then the following statements hold*:

(a) $$\varphi(r\boldsymbol{S}) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes r^{\sum_{i=1}^k (|\alpha_i| + |\beta_i|)} \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}^*_{1,\beta_1} \cdots \boldsymbol{S}^*_{k,\beta_k}$$

is convergent in the operator norm topology, and its sum, which does not depend on the order of the series, is an operator in

$$\operatorname{span}\{f^* g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|},$$

where $\boldsymbol{\mathcal{A}_n}$ *is the polyball algebra.*

(b) $A = \text{SOT-}\lim_{r \to 1} \varphi(r\boldsymbol{S})$ *and*

$$\|A\| = \sup_{r \in [0,1)} \|\varphi(r\boldsymbol{S})\| = \lim_{r \to 1} \|\varphi(r\boldsymbol{S})\| = \sup_{q \in \mathcal{P}, \|q\| \le 1} \|\varphi(\boldsymbol{S})q\|.$$

Proof. First, we assume that $A \in B\big(\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})\big)$ is a k-multi-Toeplitz operator and $\varphi(\boldsymbol{S}) = \varphi_A(\boldsymbol{S})$, where the coefficients $A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}$ are given by (1-2) and (1-3). Note that (i) follows from the proof of Theorem 1.4. Moreover, from the same proof and Lemma 1.3 we have $\varphi_A(\boldsymbol{S})(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})$ and, consequently, $\varphi_A(r\boldsymbol{S})(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})$, $r\in[0,1)$, are vectors in $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$ and

$$\lim_{r\to 1}\varphi_A(r\boldsymbol{S})(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})=\varphi_A(\boldsymbol{S})(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})=A(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k}) \tag{1-5}$$

for any $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_k)$ in $\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}$ and $h\in\mathcal{E}$. Note also that, due to (i) and Lemma 1.3, we have $\varphi_A(r_1\boldsymbol{S}_1,\dots,r_k\boldsymbol{S}_k)(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})\in\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$ for any $r_i\in[0,1)$, $i\in\{1,\dots,k\}$. Now, we show that the series

$$\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\otimes\left(\prod_{i=1}^k r_i^{|\alpha_i|+|\beta_i|}\right)\boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}^*_{1,\beta_1}\cdots\boldsymbol{S}^*_{k,\beta_k}$$

is convergent in the operator norm topology and its sum is in $\operatorname{span}\{f^*g : f,g\in B(\mathcal{E})\otimes_{\min}\boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|}$, where $\boldsymbol{\mathcal{A}_n}$ is the polyball algebra. We denote the series above by $\varphi_A(r_1\boldsymbol{S}_1,\dots,r_k\boldsymbol{S}_k)$. Since A is a k-multi-Toeplitz operator, it is also a 1-multi-Toeplitz operator with respect to $R_k:=(R_{k,1},\dots,R_{k,n_k})$, the right creation operators on the Fock space $F^2(H_{n_k})$. Applying Theorem 1.4 to 1-multi-Toeplitz operators, we deduce that A has a unique Fourier representation

$$\psi_A(S_k):=\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_k,\beta_k\in\mathbb{F}^+_{n_k}\\ |\alpha_k|=m_k^-,\,|\beta_k|=m_k^+}} C_{(\alpha_k;\beta_k)}\otimes S_{k,\alpha_k}S^*_{k,\beta_k},$$

where $C_{(\alpha_k;\beta_k)}\in B\big(\mathcal{E}\otimes\bigotimes_{i=1}^{k-1}F^2(H_{n_i})\big)$. Moreover, we can prove that, for any $r_k\in[0,1)$,

$$\psi_A(r_kS_k):=\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_k,\beta_k\in\mathbb{F}^+_{n_k}\\ |\alpha_k|=m_k^-,\,|\beta_k|=m_k^+}} r_k^{|\alpha_k|+|\beta_k|}C_{(\alpha_k;\beta_k)}\otimes S_{k,\alpha_k}S^*_{k,\beta_k} \tag{1-6}$$

is convergent in the operator norm topology. Indeed, since $\psi_A(S_k)$ is the Fourier representation of the 1-multi-Toeplitz operator A with respect to $R_k:=(R_{k,1},\dots,R_{k,n_k})$, item (i) implies, in the particular case when $\gamma_k=g^k_0$, that $\sum_{\alpha_k\in\mathbb{F}^+_{n_k}}C^*_{(\alpha_k;g^k_0)}C_{(\alpha_k;g^k_0)}$ is WOT-convergent. Since A^* is also a 1-multi-Toeplitz operator, we can similarly deduce that the series $\sum_{\beta_k\in\mathbb{F}^+_{n_k}}C_{(g^k_0;\beta_k)}C^*_{(g^k_0;\beta_k)}$ is WOT-convergent. Since $S_{k,1},\dots,S_{k,n_k}$ are isometries with orthogonal ranges, we have

$$\Big\|\sum_{\alpha_k\in\mathbb{F}^+_{n_k},|\alpha_k|=m}C_{(\alpha_k;g^k_0)}\otimes r_k^{|\alpha_k|}S_{k,\alpha_k}\Big\|=r_k^m\Big\|\sum_{\alpha_k\in\mathbb{F}^+_{n_k}}C^*_{(\alpha_k;g^k_0)}C_{(\alpha_k;g^k_0)}\Big\|^{1/2},$$

$$\Big\|\sum_{\beta_k\in\mathbb{F}^+_{n_k},|\beta_k|=m}C_{(g^k_0;\beta_k)}\otimes r_k^{|\alpha_k|}S^*_{k,\beta_k}\Big\|=r_k^m\Big\|\sum_{\alpha_k\in\mathbb{F}^+_{n_k}}C_{(g^k_0;\beta_k)}C^*_{(g^k_0;\beta_k)}\Big\|^{1/2},$$

for any $m\in\mathbb{N}$. Now, it is clear that the series defining $\psi_A(r_kS_k)$ is convergent in the operator norm topology and, consequently, $\psi_A(r_kS_k)$ belongs to

$$\operatorname{span}\left\{f^*g : f,g\in B\left(\mathcal{E}\otimes\bigotimes_{i=1}^{k-1}F^2(H_{n_i})\right)\otimes_{\min}\mathcal{A}_{n_k}\right\}^{-\|\cdot\|},$$

where $\mathcal{A}_{n_k}$ is the noncommutative disc algebra generated by $S_{k,1}, \dots, S_{k,n_k}$ and the identity. For each $i \in \{1, \dots, k\}$, we set $\mathcal{E}_i := \mathcal{E} \otimes F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_i})$.

The next step in our proof is to show that

$$\psi_A(r_k S_k) = \mathcal{B}^{\text{ext}}_{r_k S_k}[A] := (I_{\mathcal{E}_{k-1}} \otimes K^*_{r_k S_k})(A \otimes I_{F^2(H_{n_k})})(I_{\mathcal{E}_{k-1}} \otimes K_{r_k S_k}), \tag{1-7}$$

where $K_{r_k S_k} : F^2(H_{n_k}) \to F^2(H_{n_k}) \otimes \mathcal{D}_{r_k S_k} \subset F^2(H_{n_k}) \otimes F^2(H_{n_k})$ is the noncommutative Berezin kernel defined by

$$K_{r_k S_k}\xi := \sum_{\beta_k \in \mathbb{F}^+_{n_k}} e^k_{\beta_k} \otimes \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}\xi, \quad \xi \in F^2(H_{n_k}),$$

and $\mathcal{D}_{r_k S_k} := \overline{\Delta_{r_k S_k}(I)(F^2(H_{n_k}))}$. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ be k-tuples in $\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}$, set $q := \max\{|\gamma_k|, |\omega_k|\}$, and define the operator

$$Q_q := \sum_{m_k \in \mathbb{Z}, |m_k| \le q} \sum_{\substack{\alpha_k, \beta_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m_k^-,\ |\beta_k| = m_k^+}} C_{(\alpha_k;\beta_k)} \otimes S_{k,\alpha_k} S^*_{k,\beta_k}.$$

Since $\psi_A(S_k)p = Ap$ for any polynomial $p \in \mathcal{P}$, a careful computation reveals that

$$\begin{aligned}
&\big\langle \mathcal{B}^{\text{ext}}_{r_k S_k}[A](h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \big\rangle \\
&\quad = \big\langle (A \otimes I_{F^2(H_{n_k})})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes K_{r_k S_k}(e^k_{\gamma_k})), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes K_{r_k S_k}(e^k_{\omega_k}) \big\rangle \\
&\quad = \Big\langle (A \otimes I_{F^2(H_{n_k})}) \Big(\sum_{\alpha_k \in \mathbb{F}^+_{n_k}} h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes e^k_{\alpha_k} \otimes \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\alpha_k}(e^k_{\gamma_k}) \Big), \\
&\qquad\qquad \sum_{\beta_k \in \mathbb{F}^+_{n_k}} h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes e^k_{\beta_k} \otimes \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}(e^k_{\omega_k}) \Big\rangle \\
&\quad = \Big\langle \sum_{\alpha_k \in \mathbb{F}^+_{n_k}} A(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes e^k_{\alpha_k}) \otimes \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\alpha_k}(e^k_{\gamma_k}), \\
&\qquad\qquad \sum_{\beta_k \in \mathbb{F}^+_{n_k}} h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes e^k_{\beta_k} \otimes \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}(e^k_{\omega_k}) \Big\rangle \\
&\quad = \sum_{m=0}^\infty \sum_{p=0}^\infty \sum_{\substack{\alpha_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m}} \sum_{\substack{\beta_k \in \mathbb{F}^+_{n_k} \\ |\beta_k| = p}} \big\langle A(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes e^k_{\alpha_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes e^k_{\beta_k} \big\rangle \\
&\qquad\qquad \times \big\langle \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\alpha_k}(e^k_{\gamma_k}), \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}(e^k_{\omega_k}) \big\rangle \\
&\quad = \sum_{m=0}^q \sum_{p=0}^q \sum_{\substack{\alpha_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m}} \sum_{\substack{\beta_k \in \mathbb{F}^+_{n_k} \\ |\beta_k| = p}} \big\langle Q_q(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes e^k_{\alpha_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes e^k_{\beta_k} \big\rangle \\
&\qquad\qquad \times \langle \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\alpha_k}(e^k_{\gamma_k}), \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}(e^k_{\omega_k}) \rangle \\
&\quad = \sum_{m=0}^\infty \sum_{p=0}^\infty \sum_{\substack{\alpha_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m}} \sum_{\substack{\beta_k \in \mathbb{F}^+_{n_k} \\ |\beta_k| = p}} \big\langle Q_q(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes e^k_{\alpha_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes e^k_{\beta_k} \big\rangle \\
&\qquad\qquad \times \big\langle \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\alpha_k}(e^k_{\gamma_k}), \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}(e^k_{\omega_k}) \big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \big\langle (Q_q \otimes I_{F^2(H_{n_k})})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes K_{r_k S_k}(e^k_{\gamma_k})), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes K_{r_k S_k}(e^k_{\omega_k}) \big\rangle \\
&= \big\langle \mathcal{B}^{\mathrm{ext}}_{r_k S_k}[Q_q](h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \big\rangle \\
&= \sum_{m_k \in \mathbb{Z}, |m_k| \le q} \ \sum_{\substack{\alpha_k, \beta_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m_k^-, \, |\beta_k| = m_k^+}} \big\langle (C_{(\alpha_k;\beta_k)} \otimes r_k^{|\alpha_k|+|\beta_k|} S_{k,\alpha_k} S^*_{k,\beta_k})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \big\rangle \\
&= \big\langle \psi_A(r_k S_k)(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \big\rangle
\end{aligned}
$$

for any $h, h' \in \mathcal{E}$. Consequently, (1-7) holds for any $r_k \in [0, 1)$. Hence, and using the fact the noncommutative Berezin kernel K_{rS_k} is an isometry, we deduce that

$$
\|\psi_A(r_k S_k)\| \le \|A\|, \quad r_k \in [0, 1).
$$

Moreover, one can show that

$$
A = \text{SOT-}\lim_{r_k \to 1} \psi_A(r_k S_k).
$$

Indeed, due to (i) (for 1-multi-Toeplitz operators), we have $\|\psi_A(r_k S_k)p - \psi_A(S_k)p\| \to 0$ as $r_k \to 1$ for any polynomial $p \in \mathcal{E}_{k-1} \otimes F^2(H_{n_k})$ with coefficients in $\mathcal{E}_{k-1}$. Since $\psi_A(S_k)p = Ap$ and $\|\psi_A(r_k S_k)\| \le \|A\|$ for any $r_k \in [0, 1)$, an approximation argument proves our assertion.

Now, we prove that the coefficients $C_{(\alpha_k;\beta_k)} \in B\big(\mathcal{E} \otimes \bigotimes_{i=1}^{k-1} F^2(H_{n_i})\big)$ of the Fourier series $\psi_A(S_k)$ are 1-multi-Toeplitz operators with respect to $R_{k-1} := (R_{k-1,1}, \ldots, R_{k-1,n_{k-1}})$. For each $i \in \{1, \ldots, k-1\}$, $s, t \in \{1, \ldots, n_i\}$, and any vector-valued polynomial $p \in \mathcal{E} \otimes \bigotimes_{i=1}^{k} F^2(H_{n_i})$ with coefficients in $\mathcal{E}$, Theorem 1.4 implies

$$
\begin{aligned}
\sum_{m_k \in \mathbb{Z}} \ \sum_{\substack{\alpha_k, \beta_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m_k^-, \, |\beta_k| = m_k^+}} &\big[(I_{\mathcal{E}_{k-2}} \otimes R^*_{i,s}) C_{(\alpha_k;\beta_k)} (I_{\mathcal{E}_{k-2}} \otimes R_{i,t}) \otimes S_{k,\alpha_k} S^*_{k,\beta_k}\big](p) \\
&= (I_{\mathcal{E}} \otimes \boldsymbol{R}^*_{i,s}) \psi_A(S_k) (I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,t})(p) \\
&= (I_{\mathcal{E}} \otimes \boldsymbol{R}^*_{i,s}) A (I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,t})(p) \\
&= \delta_{st} A(p) = \delta_{st} \psi_A(S_k)(p) \\
&= \delta_{st} \sum_{m_k \in \mathbb{Z}} \ \sum_{\substack{\alpha_k, \beta_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m_k^-, \, |\beta_k| = m_k^+}} (C_{(\alpha_k;\beta_k)} \otimes S_{k,\alpha_k} S^*_{k,\beta_k})(p).
\end{aligned}
$$

Hence, we deduce that

$$
(I_{\mathcal{E}_{k-2}} \otimes R^*_{i,s}) C_{(\alpha_k;\beta_k)} (I_{\mathcal{E}_{k-2}} \otimes R_{i,t}) = \delta_{st} C_{(\alpha_k;\beta_k)}
$$

for any $i \in \{1, \ldots, k-1\}$ and $s, t \in \{1, \ldots, n_i\}$, which proves that $C_{(\alpha_k;\beta_k)}$ is a 1-multi-Toeplitz operator with respect to $R_{k-1} := (R_{k-1,1}, \ldots, R_{k-1,n_{k-1}})$. Consequently, similarly to the first part of the proof, $C_{(\alpha_k;\beta_k)}$ has a Fourier representation

$$
\psi_{(\alpha_k;\beta_k)}(S_{k-1}) := \sum_{m_{k-1} \in \mathbb{Z}} \ \sum_{\substack{\alpha_{k-1}, \beta_{k-1} \in \mathbb{F}^+_{n_{k-1}} \\ |\alpha_{k-1}| = m_{k-1}^-, \, |\beta_{k-1}| = m_{k-1}^+}} C_{(\alpha_{k-1},\alpha_k;\beta_{k-1},\beta_k)} \otimes S_{k-1,\alpha_{k-1}} S^*_{k-1,\beta_{k-1}}, \tag{1-8}
$$

where $C_{(\alpha_{k-1},\alpha_k;\beta_{k-1},\beta_k)} \in B(\mathcal{E}_{k-2})$. Moreover, as above, one can prove that, for any $r_{k-1} \in [0, 1)$, the series $\psi_{(\alpha_k;\beta_k)}(r_{k-1}S_{k-1})$ is convergent in the operator norm topology, and its limit is an element in

$$\overline{\operatorname{span}}\{f^*g : f, g \in B(\mathcal{E}_{k-2}) \otimes_{\min} \mathcal{A}_{n_{k-1}}\}^{\|\cdot\|},$$

where $\mathcal{A}_{n_{k-1}}$ is the noncommutative disc algebra generated by $S_{k-1,1}, \dots, S_{k-1,n_{k-1}}$ and the identity. We also have

$$\lim_{r_{k-1}\to 1} \psi_{(\alpha_k;\beta_k)}(r_{k-1}S_{k-1})p = C_{(\alpha_k;\beta_k)}p$$

for any vector-valued polynomial $p \in \mathcal{E}_{k-2} \otimes F^2(H_{n_{k-1}})$. As in the first part of the proof, setting

$$\mathcal{B}^{\text{ext}}_{r_{k-1}S_{k-1}}[u] := (I_{\mathcal{E}_{k-2}} \otimes K^*_{r_{k-1}S_{k-1}})(u \otimes I_{F^2(H_{n_{k-1}})})(I_{\mathcal{E}_{k-2}} \otimes K_{r_{k-1}S_{k-1}}), \quad u \in B(\mathcal{E}_{n-1}),$$

one can prove that

$$\psi_{(\alpha_k;\beta_k)}(r_{k-1}S_{k-1}) = \mathcal{B}^{\text{ext}}_{r_{k-1}S_{k-1}}[C_{(\alpha_k;\beta_k)}] \quad \text{and} \quad \|\psi_{(\alpha_k,\beta_k)}(r_{k-1}S_{k-1})\| \le \|C_{(\alpha_k;\beta_k)}\| \tag{1-9}$$

for any $r_{k-1} \in [0, 1)$. Moreover, we can also show that

$$C_{(\alpha_k;\beta_k)} = \text{SOT-}\lim_{r_{k-1}\to 1} \psi_{(\alpha_k;\beta_k)}(r_{k-1}S_{k-1}).$$

Now, due to (1-6), (1-7), (1-8) and (1-9), we obtain

$$\begin{aligned}
&([\mathcal{B}^{\text{ext}}_{r_{k-1}S_{k-1}} \otimes \mathrm{id}_{B(F^2(H_{n_k}))}] \circ \mathcal{B}^{\text{ext}}_{r_kS_k})[A] \\
&\quad = \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_k,\beta_k\in\mathbb{F}^+_{n_k}\\ |\alpha_k|=m_k^-,\ |\beta_k|=m_k^+}} \mathcal{B}^{\text{ext}}_{r_{k-1}S_{k-1}}[C_{(\alpha_k;\beta_k)}] \otimes r_k^{|\alpha_k|+|\beta_k} S_{k,\alpha_k}S^*_{k,\beta_k} \\
&\quad = \sum_{m_k\in\mathbb{Z}} \sum_{m_{k-1}\in\mathbb{Z}} \sum_{\substack{\alpha_k,\beta_k\in\mathbb{F}^+_{n_k}\\ |\alpha_k|=m_k^-,\ |\beta_k|=m_k^+}} \sum_{\substack{\alpha_{k-1},\beta_{k-1}\in\mathbb{F}^+_{n_{k-1}}\\ |\alpha_{k-1}|=m_{k-1}^-,\ |\beta_{k-1}|=m_{k-1}^+}} r_k^{|m_k|}r_{k-1}^{|m_{k-1}|}C_{(\alpha_{k-1},\alpha_k;\beta_{k-1},\beta_k)} \\
&\qquad\qquad\qquad\qquad\qquad \otimes S_{k-1,\alpha_{k-1}}S^*_{k-1,\beta_{k-1}} \otimes S_{k,\alpha_k}S^*_{k,\beta_k},
\end{aligned}$$

where the series are convergent in the operator norm topology. Continuing this process, one can prove that there are some operators $C_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \in B(\mathcal{E})$ such that the series $\varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)$ given by

$$\sum_{m_k\in\mathbb{Z}} \cdots \sum_{m_1\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\ i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\ |\beta_i|=m_i^+}} r_k^{|m_k|}\cdots r_1^{|m_1|}C_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{1,\alpha_1}\cdots \boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}^*_{1,\beta_1}\cdots \boldsymbol{S}^*_{k,\beta_k}$$

is convergent in the operator norm topology and

$$\varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k) = [\mathcal{B}^{\text{ext}}_{r_1S_1} \otimes \mathrm{id}_{B(\bigotimes_{i=2}^k F^2(H_{n_i}))}] \circ [\mathcal{B}^{\text{ext}}_{r_2S_2} \otimes \mathrm{id}_{B(\bigotimes_{i=3}^k F^2(H_{n_i}))}] \circ \cdots \circ \mathcal{B}^{\text{ext}}_{r_kS_k}[A]. \tag{1-10}$$

Since the noncommutative Berezin kernels $K_{r_iS_i}$, $i \in \{1, \dots, k\}$, are isometries, we deduce that

$$\|\varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)\| \le \|A\|, \quad r_i \in [0, 1).$$

Note that the coefficients of the k-multi-Toeplitz operator $\varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)$ satisfy the relation

$$\langle r_k^{|m_k|} \cdots r_1^{|m_1|} C_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h, \ell\rangle = \langle \varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)(h \otimes x), (\ell \otimes y)\rangle, \tag{1-11}$$

where x, y are defined as in (1-3). Since A is a k-multi-Toeplitz operator, so is $Y_{r_k} := \mathcal{B}^{\text{ext}}_{r_k S_k}[A] = \psi_A(r_k S_k)$ and, iterating the argument, we deduce that

$$Y_{r_2,\dots,r_k} := [\mathcal{B}^{\text{ext}}_{r_2 S_2} \otimes \mathrm{id}_{B(\bigotimes_{i=3}^k F^2(H_{n_i}))}] \circ \cdots \circ \mathcal{B}^{\text{ext}}_{r_k S_k}[A]$$

is a k-multi-Toeplitz operator. In particular, $Y_{r_2,\dots,r_k}$ is a 1-multi-Toeplitz operator with respect to $R_1 := (R_{1,1}, \dots, R_{1,n_1})$. Applying the first part of the proof to $Y_{r_2,\dots,r_k}$, we deduce that

$$\text{SOT-}\lim_{r_1\to 1} [\mathcal{B}^{\text{ext}}_{r_1 S_1} \otimes \mathrm{id}_{B(\bigotimes_{i=2}^k F^2(H_{n_i}))}][Y_{r_2,\dots,r_k}] = Y_{r_2,\dots,r_k}.$$

Continuing this process, we obtain

$$\text{SOT-}\lim_{r_k\to 1} \cdots \text{SOT-}\lim_{r_1\to 1} [\mathcal{B}^{\text{ext}}_{r_1 S_1} \otimes \mathrm{id}_{B(\bigotimes_{i=2}^k F^2(H_{n_i}))}][Y_{r_2,\dots,r_k}] = A.$$

Consequently, using (1-10), (1-11) and (1-2), we deduce that

$$\langle C_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h, \ell\rangle = \langle A(h \otimes x), \ell \otimes y\rangle = \langle A_{(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k)} h, \ell\rangle,$$

which shows that $\varphi_A(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k) = \varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)$ for any $r_i \in [0, 1)$. Hence, we obtain

$$\varphi_A(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k) = \sum_{m_k\in\mathbb{Z}} \cdots \sum_{m_1\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} r_k^{|m_k|} \cdots r_1^{|m_1|} \times A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}^*_{1,\beta_1} \cdots \boldsymbol{S}^*_{k,\beta_k},$$

where the series are convergent in the operator norm topology. Moreover, due to (1-10), we have

$$\|\varphi_A(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)\| \le \|A\|, \quad r_i \in [0, 1).$$

Due to (1-5), we have

$$\lim_{r\to 1} \varphi_A(r\boldsymbol{S})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}) = A(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}).$$

Since $\|\varphi_A(r\boldsymbol{S}_1, \dots, r\boldsymbol{S}_k)\| \le \|A\|$, an approximation argument shows that

$$\text{SOT-}\lim_{r\to 1} \varphi_A(r\boldsymbol{S}_1, \dots, r\boldsymbol{S}_k) = A. \tag{1-12}$$

Let $\epsilon > 0$ and choose a vector-valued polynomial $q \in \mathcal{P}$ with $\|q\| = 1$ and $\|Aq\| > \|A\| - \epsilon$. Due to (1-12), there is $r_0 \in (0, 1)$ such that $\|\varphi_A(r_0\boldsymbol{S}_1, \dots, r_0\boldsymbol{S}_k)q\| > \|A\| - \epsilon$. Hence, we deduce that $\sup_{r\in[0,1)} \|\varphi_A(r\boldsymbol{S}_1, \dots, r\boldsymbol{S}_k)\| = \|A\|$.

Now, let $r_1, r_2 \in [0, 1)$ with $r_1 < r_2$. We already proved that $g(\boldsymbol{S}) := \varphi_A(r_2\boldsymbol{S}_1, \dots, r_2\boldsymbol{S}_k)$ is in $\text{span}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|}$. Due to the von Neumann-type inequality [1951] from [Popescu 2016], we have $\|g(r\boldsymbol{S})\| \le \|g(\boldsymbol{S})\|$ for any $r \in [0, 1)$. In particular, setting $r = r_1/r_2$, we deduce that

$$\|\varphi_A(r_1\boldsymbol{S}_1, \dots, r_1\boldsymbol{S}_k)\| \le \|\varphi_A(r_2\boldsymbol{S}_1, \dots, r_2\boldsymbol{S}_k)\|.$$

It is clear that $\lim_{r\to 1}\|\varphi_A(r\boldsymbol{S}_1,\dots,r\boldsymbol{S}_k)\| = \|A\|$. On the other hand, since $Aq = \varphi_A(\boldsymbol{S})q$ for any vector-valued polynomial $q \in \mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$, we deduce that $\|A\| = \sup_{q\in\mathcal{P},\|q\|\le 1}\|\varphi(\boldsymbol{S})q\|$.

Now we prove the converse of the theorem. Let $\{A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\}$ be a family of operators in $B(\mathcal{E})$, where $\alpha_i,\ \beta_i \in \mathbb{F}_{n_i}^+,\ |\alpha_i| = m_i^-,\ |\beta_i| = m_i^+,\ m_i \in \mathbb{Z}$ and $i \in \{1,\dots,k\}$, and assume that conditions (i) and (ii) hold. Note that, due to (i), $\varphi(\boldsymbol{S})p$ and $\varphi(r\boldsymbol{S})p,\ r \in [0,1)$, are vectors in $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$ and

$$\lim_{r\to 1}\varphi(r\boldsymbol{S})p = \varphi(\boldsymbol{S})p$$

for any $p \in \mathcal{P}$. Since $\sup_{p\in\mathcal{P},\|p\|\le 1}\|\varphi(r\boldsymbol{S})p\| < \infty$, there exists a unique bounded linear operator $A_r \in B\big(\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ such that $A_r p = \varphi(r\boldsymbol{S})p$ for any $p\in\mathcal{P}$. If $f \in \mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$ and $\{p_m\}$ is a sequence of polynomials $p_m \in \mathcal{P}$ such that $p_m \to f$ as $m\to\infty$, we set $A_r(f) := \lim_{m\to\infty}\varphi(r\boldsymbol{S})p_m$. Note that the definition is valid. On the other hand, note that

$$\sup_{p\in\mathcal{P},\|p\|\le 1}\|\varphi(\boldsymbol{S})p\| < \infty.$$

Indeed, this follows from the facts that $\lim_{r\to 1}\varphi(r\boldsymbol{S})p = \varphi(\boldsymbol{S})p$ and $\sup_{p\in\mathcal{P},\|p\|\le 1}\|\varphi(r\boldsymbol{S})p\| < \infty$. Consequently, there is a unique operator $A \in B\big(\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ such that $Ap = \varphi(\boldsymbol{S})p$ for any $p\in\mathcal{P}$. Since $\lim_{r\to 1}A_r p = \lim_{r\to 1}\varphi(r\boldsymbol{S})p = \varphi(\boldsymbol{S})p = Ap$ and $\sup_{r\in[0,1)}\|A_r\| < \infty$, we deduce that $A =$ SOT-$\lim_{r\to 1}A_r$.

Now we show that A is a k-multi-Toeplitz operator. First, note that $\boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}_{1,\beta_1}^*\cdots\boldsymbol{S}_{k,\beta_k}^*$ is a k-multi-Toeplitz operator for any $\alpha_i,\ \beta_i \in \mathbb{F}_{n_i}^+,\ i\in\{1,\dots,k\}$ with $m_i\in\mathbb{Z},\ |\alpha_i| = m_i^-$ and $|\beta_i| = m_i^+$. It is enough to check this on monomials of the form $h\otimes e_{\gamma_1}^1\otimes\cdots\otimes e_{\gamma_k}^k$. Consequently,

$$(I_{\mathcal{E}}\otimes \boldsymbol{R}_{i,s}^*)\varphi(r\boldsymbol{S})(I_{\mathcal{E}}\otimes\boldsymbol{R}_{i,t})p = \delta_{st}\varphi(r\boldsymbol{S})p,\quad s,t\in\{1,\dots,n_i\},$$

for any $p\in\mathcal{P}$ and every $i\in\{1,\dots,k\}$. Hence, A_r has the same property. Taking $r\to 1$, we conclude that A is a k-multi-Toeplitz operator. On the other hand, if $x := x_1\otimes\cdots\otimes x_k$, $y = y_1\otimes\cdots\otimes y_k$ satisfy (1-3) and $h,\ \ell\in\mathcal{E}$, we have

$$\begin{aligned}\langle A(h\otimes x),\ell\otimes y\rangle &= \lim_{r\to 1}\langle A_r(h\otimes x),\ell\otimes y\rangle\\ &= \lim_{r\to 1}\langle\varphi(r\boldsymbol{S})(h\otimes x),\ell\otimes y\rangle\\ &= \lim_{r\to 1}\big\langle r^{\sum_{i=1}^k|\alpha_i|+|\beta_i|}A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}h,\ell\big\rangle\\ &= \langle A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}h,\ell\rangle.\end{aligned}$$

Therefore,

$$\varphi(\boldsymbol{S}) := \sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\ i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\ |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\otimes\boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}_{1,\beta_1}^*\cdots\boldsymbol{S}_{k,\beta_k}^*$$

is the formal Fourier series of the k-multi-Toeplitz operator A on $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$. □

Theorem 1.6. *Let* $\{A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\}$ *be a family of operators in* $B(\mathcal{E})$, *where* $\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+$, $|\alpha_i| = m_i^-$, $|\beta_i| = m_i^+$, $m_i \in \mathbb{Z}$ *and* $i \in \{1, \dots, k\}$, *and let*

$$\varphi(\boldsymbol{S}) := \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{1,\alpha_1}\cdots \boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}_{1,\beta_1}^*\cdots \boldsymbol{S}_{k,\beta_k}^*$$

be the associated formal Fourier series. Then $\varphi(\boldsymbol{S})$ *is the formal Fourier series of a* k*-multi-Toeplitz operator* A *on* $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$ *if and only if the series defining* $\varphi(r\boldsymbol{S})$ *is convergent in the operator norm topology for any* $r \in [0, 1)$ *and*

$$\sup_{r\in[0,1)} \|\varphi(r\boldsymbol{S})\| < \infty.$$

Moreover, if A *is a* k*-multi-Toeplitz operator on* $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$, *then* $\varphi(r\boldsymbol{S}) = \boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A]$ *and*

$$\text{SOT-}\lim_{r\to 1} \boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A] = A, \quad \textit{where } \boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[u] := (I_{\mathcal{E}}\otimes \boldsymbol{K}_{r\boldsymbol{S}}^*)(u\otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})(I_{\mathcal{E}}\otimes \boldsymbol{K}_{r\boldsymbol{S}}),\ u\in B(\mathcal{E}_k),$$

and $\boldsymbol{K}_{r\boldsymbol{S}}$ *is the noncommutative Berezin kernel associated with* $r\boldsymbol{S} \in \boldsymbol{B_n}\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$.

Proof. Assume that $\varphi(\boldsymbol{S})$ is the formal Fourier series of a k-multi-Toeplitz operator A on $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$. Then Theorem 1.5 implies that $\varphi(r\boldsymbol{S})$ is convergent in the operator norm topology and

$$\|A\| = \sup_{r\in[0,1)} \|\varphi(r\boldsymbol{S})\|.$$

We recall that the noncommutative Berezin kernel associated with $r\boldsymbol{S} \in \boldsymbol{B_n}\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ is defined on $\bigotimes_{i=1}^k F^2(H_{n_i})$ with values in $\bigotimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}_{r\boldsymbol{S}} \subset \big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)\otimes\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$, where

$$\mathcal{D}_{r\boldsymbol{S}} := \overline{\boldsymbol{\Delta}_{r\boldsymbol{S}}(I)\bigg(\bigotimes_{i=1}^k F^2(H_{n_i})\bigg)}.$$

Let $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_k)$, $\boldsymbol{\omega} = (\omega_1,\dots,\omega_k) \in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, set $q := \max\{|\gamma_1|,\dots|\gamma_k|, |\omega_1|,\dots,|\omega_k|\}$, and define the operator

$$\Gamma_q := \sum_{m_1\in\mathbb{Z}, |m_1|\le q} \cdots \sum_{m_k\in\mathbb{Z}, |m_k|\le q} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_k}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{\boldsymbol{\alpha}}\boldsymbol{S}_{\boldsymbol{\beta}}^*,$$

where we use the notation $\boldsymbol{S}_{\boldsymbol{\alpha}} := \boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}$ if $\boldsymbol{\alpha} := (\alpha_1,\dots,\alpha_k) \in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$. We also set $e_{\boldsymbol{\alpha}} := e_{\alpha_1}^1\otimes\cdots\otimes e_{\alpha_k}^k$. Note that

$$\begin{aligned}
&\langle \boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A](h\otimes e_{\boldsymbol{\gamma}}), h'\otimes e_{\boldsymbol{\omega}}\rangle\\
&= \big\langle (I_{\mathcal{E}}\otimes \boldsymbol{K}_{r\boldsymbol{S}}^*)(A\otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})(I_{\mathcal{E}}\otimes \boldsymbol{K}_{r\boldsymbol{S}})(h\otimes e_{\boldsymbol{\gamma}}), h'\otimes e_{\boldsymbol{\omega}}\big\rangle\\
&= \Bigg\langle (A\otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})}) \sum_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} h\otimes e_{\boldsymbol{\alpha}}\otimes\boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\alpha}}^*(e_{\boldsymbol{\gamma}}), \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} h'\otimes e_{\boldsymbol{\beta}}\otimes\boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\beta}}^*(e_{\boldsymbol{\omega}})\Bigg\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}\ \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} \left\langle A(h\otimes e_{\boldsymbol{\alpha}})\otimes \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\alpha}}^*(e_{\boldsymbol{\gamma}}),\, h'\otimes e_{\boldsymbol{\beta}}\otimes \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\beta}}^*(e_{\boldsymbol{\omega}})\right\rangle\\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}\ \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} \langle A(h\otimes e_{\boldsymbol{\alpha}}),\, h'\otimes e_{\boldsymbol{\beta}}\rangle \left\langle \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\alpha}}^*(e_{\boldsymbol{\gamma}}),\, \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\beta}}^*(e_{\boldsymbol{\omega}})\right\rangle\\
&= \sum_{m_1\in\mathbb{Z},|m_1|\le q}\cdots\sum_{m_k\in\mathbb{Z},|m_k|\le q}\ \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_k}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} \langle \Gamma_q(h\otimes e_{\boldsymbol{\alpha}}),\, h'\otimes e_{\boldsymbol{\beta}}\rangle \left\langle \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\alpha}}^*(e_{\boldsymbol{\gamma}}),\, \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\beta}}^*(e_{\boldsymbol{\omega}})\right\rangle\\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}\ \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} \langle \Gamma_q(h\otimes e_{\boldsymbol{\alpha}}),\, h'\otimes e_{\boldsymbol{\beta}}\rangle \left\langle \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\alpha}}^*(e_{\boldsymbol{\gamma}}),\, \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\beta}}^*(e_{\boldsymbol{\omega}})\right\rangle\\
&= \left\langle (I_{\mathcal{E}}\otimes \boldsymbol{K}_{r\boldsymbol{S}}^*)(\Gamma_q\otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})(I_{\mathcal{E}}\otimes \boldsymbol{K}_{r\boldsymbol{S}})(h\otimes e_{\boldsymbol{\gamma}}),\, h'\otimes e_{\boldsymbol{\omega}}\right\rangle\\
&= \langle \boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[\Gamma_q](h\otimes e_{\boldsymbol{\gamma}}),\, h'\otimes e_{\boldsymbol{\omega}}\rangle\\
&= \sum_{m_1\in\mathbb{Z},|m_1|\le q}\cdots\sum_{m_k\in\mathbb{Z},|m_k|\le q}\ \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_k}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} \left\langle \left(A_{(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k)}\otimes r^{\sum_{i=1}^k(|\alpha_i|+|\beta_i|)}\boldsymbol{S}_{\boldsymbol{\alpha}}\boldsymbol{S}_{\boldsymbol{\beta}}^*\right)(h\otimes e_{\gamma}),\, h'\otimes e_{\boldsymbol{\omega}}\right\rangle\\
&= \langle \varphi_A(rS_1,\dots,rS_k)(h\otimes e_{\boldsymbol{\gamma}}),\, h'\otimes e_{\boldsymbol{\omega}}\rangle.
\end{aligned}
$$

Consequently, we obtain

$$
\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A] = \varphi_A(r\boldsymbol{S}_1,\dots,r\boldsymbol{S}_k),\qquad r\in[0,1),
$$

which proves the second part of the theorem.

To prove the converse, assume that $\{A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\}$ is a family of operators in $B(\mathcal{E})$, where $\alpha_i,\ \beta_i\in\mathbb{F}_{n_i}^+,\ |\alpha_i|=m_i^-,\ |\beta_i|=m_i^+,\ m_i\in\mathbb{Z}$ and $i\in\{1,\dots,k\}$, and let $\varphi(\boldsymbol{S})$ be the associated formal Fourier series. We also assume that $\varphi(r\boldsymbol{S})$ is convergent in the operator norm topology for each $r\in[0,1)$ and that

$$
M := \sup_{r\in[0,1)} \|\varphi(r\boldsymbol{S})\| < \infty.
$$

Note that $\varphi(r\boldsymbol{S})$ is a k-multi-Toeplitz operator and

$$
\varphi(r\boldsymbol{S})(h\otimes e_{\gamma_1}^1\otimes\cdots\otimes e_{\gamma_k}^k) = \sum_{\substack{\boldsymbol{\omega}=(\omega_1,\dots,\omega_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ \boldsymbol{\omega}\sim_{\text{rc}}\boldsymbol{\gamma}}} r^{\sum_{i=1}^k(|c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma})|+|c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma})|)} A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} h\otimes e_{\omega_1}^1\otimes\cdots\otimes e_{\omega_k}^k
$$

is a vector in $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$. Hence, we deduce that, for each $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$,

$$
\left\langle r^{\sum_{i=1}^k(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma})+c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} \sum_{\substack{\boldsymbol{\omega}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ \boldsymbol{\omega}\sim_{\text{rc}}\boldsymbol{\gamma}}} A^*_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} h,\, h\right\rangle \le \|\varphi(r\boldsymbol{S})\|^2\|h\|^2 \le M\|h\|^2
$$

for any $r\in[0,1)$ and $h\in\mathcal{E}$. Taking $r\to 1$, we get condition (i) of Theorem 1.5. Applying Theorem 1.5, we deduce that $\varphi(\boldsymbol{S})$ is the Fourier series of a k-multi-Toeplitz operator. $\square$

We remark that, due to Theorem 1.6, the order of the series in the definition of $\varphi_A(r\boldsymbol{S}_1, \dots, r\boldsymbol{S}_k)$ (see Theorem 1.5(a)) is irrelevant.

Theorem 1.7. *Let $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ and let $\boldsymbol{\mathcal{T}_n}$ be the set of all k-multi-Toeplitz operators on $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$. Then*

$$\boldsymbol{\mathcal{T}_n} = \overline{\operatorname{span}\{f^* g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}}^{\text{-SOT}} = \overline{\operatorname{span}\{f^* g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}}^{\text{-WOT}},$$

where $\boldsymbol{\mathcal{A}_n}$ is the polyball algebra.

Proof. Let

$$\mathcal{G} := \overline{\operatorname{span}\{f^* g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}}^{\|\cdot\|}.$$

According to Theorem 1.5, if $A \in \boldsymbol{\mathcal{T}_n}$ and $\varphi_A(\boldsymbol{S})$ is its Fourier series, then $\varphi_A(r\boldsymbol{S}) \in \mathcal{G}$ for any $r \in [0, 1)$ and $A = \text{SOT-lim}\, \varphi_A(r\boldsymbol{S})$. Consequently, $\boldsymbol{\mathcal{T}_n} \subseteq \bar{\mathcal{G}}^{\text{SOT}}$. Conversely, note that each monomial $\boldsymbol{S}_{\boldsymbol{\alpha}}^* \boldsymbol{S}_{\boldsymbol{\beta}}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, is a k-multi-Toeplitz operator. This shows that, for each $Y \in \mathcal{G}$,

$$(I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,s}^*) Y (I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,t}) = \delta_{st} Y, \quad s, t \in \{1, \dots, n_i\},$$

for every $i \in \{1, \dots, k\}$. Consequently, taking SOT-limits, we deduce that $\bar{\mathcal{G}}^{\text{SOT}} \subseteq \boldsymbol{\mathcal{T}_n}$, which proves that $\bar{\mathcal{G}}^{\text{SOT}} = \boldsymbol{\mathcal{T}_n}$.

Now, if $T \in \bar{\mathcal{G}}^{\text{WOT}}$, an argument as above shows that $T \in \boldsymbol{\mathcal{T}_n} = \bar{\mathcal{G}}^{\text{SOT}}$. Since $\bar{\mathcal{G}}^{\text{SOT}} \subseteq \bar{\mathcal{G}}^{\text{WOT}}$, we conclude that $\boldsymbol{\mathcal{T}_n} = \bar{\mathcal{G}}^{\text{SOT}} = \bar{\mathcal{G}}^{\text{WOT}}$. □

Corollary 1.8. *The set of all k-multi-Toeplitz operators on $\bigotimes_{i=1}^k F^2(H_{n_i})$ coincides with*

$$\overline{\operatorname{span}\{\boldsymbol{\mathcal{A}_n^*} \boldsymbol{\mathcal{A}_n}\}}^{\text{-SOT}} = \overline{\operatorname{span}\{\boldsymbol{\mathcal{A}_n^*} \boldsymbol{\mathcal{A}_n}\}}^{\text{-WOT}},$$

where $\boldsymbol{\mathcal{A}_n}$ is the polyball algebra.

2. Bounded free k-pluriharmonic functions and the Dirichlet extension problem

In this section, we show that the bounded free k-pluriharmonic functions on $\boldsymbol{B_n}$ are precisely the noncommutative Berezin transforms of k-multi-Toeplitz operators and solve the Dirichlet extension problem for the regular polyball $\boldsymbol{B_n}$.

Definition 2.1. A function F with operator-valued coefficients in $B(\mathcal{E})$ is called free k-pluriharmonic on the polyball $\boldsymbol{B_n}$ if it has the form

$$F(\boldsymbol{X}) = \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\, i \in \{1, \dots, k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} A_{(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_k)} \otimes \boldsymbol{X}_{1,\alpha_1} \cdots \boldsymbol{X}_{k,\alpha_k} \boldsymbol{X}_{1,\beta_1}^* \cdots \boldsymbol{X}_{k,\beta_k}^*,$$

where the series converge in the operator norm topology for any $\boldsymbol{X} = (X_1, \dots, X_k) \in \boldsymbol{B_n}(\mathcal{H})$, with $X_i := (X_{i,1}, \dots, X_{i,n_i})$, and any Hilbert space $\mathcal{H}$.

Due to the remark following Theorem 1.6, one can prove that the order of the series in the definition above is irrelevant. Note that any free holomorphic function on $\boldsymbol{B_n}$ is k-pluriharmonic. Indeed, according to [Popescu 2015b], any free holomorphic function on the polyball $\boldsymbol{B_n}$ has the form

$$f(\boldsymbol{X}) = \sum_{m_1\in\mathbb{N}} \cdots \sum_{m_k\in\mathbb{N}} \sum_{\substack{\alpha_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i}} A_{(\alpha_1,\dots,\alpha_k)} \otimes \boldsymbol{X}_{1,\alpha_1}\cdots \boldsymbol{X}_{k,\alpha_k}, \qquad \boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H}),$$

where the series converge in the operator norm topology. A function $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}\otimes\mathcal{H})$ is called bounded if

$$\|F\| := \sup_{\boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H})} \|F(\boldsymbol{X})\| < \infty.$$

A free k-pluriharmonic function is bounded if its representation on any Hilbert space is bounded. Denote by $\mathbf{PH}_{\mathcal{E}}^{\infty}(\boldsymbol{B_n})$ the set of all bounded free k-pluriharmonic functions on the polyball $\boldsymbol{B_n}$ with coefficients in $B(\mathcal{E})$. For each $m = 1, 2, \dots$, we define the norms $\|\cdot\|_m : M_m(\mathbf{PH}_{\mathcal{E}}^{\infty}(\boldsymbol{B_n})) \to [0,\infty)$ by setting

$$\|[F_{ij}]_m\|_m := \sup \|[F_{ij}(\boldsymbol{X})]_m\|,$$

where the supremum is taken over all n-tuples $\boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. It is easy to see that the norms $\|\cdot\|_m$, $m = 1, 2, \dots$, determine an operator space structure on $\mathbf{PH}_{\mathcal{E}}^{\infty}(\boldsymbol{B_n})$, in the sense of Ruan (see, e.g., [Effros and Ruan 2000]).

Let $\boldsymbol{\mathcal{T}_n}$ be the set of all k-multi-Toeplitz operators on $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$. According to Theorem 1.7, we have

$$\boldsymbol{\mathcal{T}_n} = \overline{\operatorname{span}}\{f^*g : f, g \in B(\mathcal{E})\otimes_{\min} \boldsymbol{\mathcal{A}_n}\}^{\text{-SOT}},$$

where $\boldsymbol{\mathcal{A}_n}$ is the polyball algebra. The main result of this section is the following characterization of bounded free k-pluriharmonic functions:

Theorem 2.2. *If $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E})\otimes_{\min} B(\mathcal{H})$, then the following statements are equivalent*:

(i) *F is a bounded free k-pluriharmonic function*;

(ii) *there exists $A\in\boldsymbol{\mathcal{T}_n}$ such that*

$$F(\boldsymbol{X}) = \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A] := (I_{\mathcal{E}}\otimes \boldsymbol{K}_{\boldsymbol{X}}^*)(A\otimes I_{\mathcal{H}})(I_{\mathcal{E}}\otimes \boldsymbol{K}_{\boldsymbol{X}}), \qquad \boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H}).$$

In this case, $A = \text{SOT-}\lim_{r\to 1} F(r\boldsymbol{S})$. *Moreover*, *the map*

$$\Phi : \mathbf{PH}_{\mathcal{E}}^{\infty}(\boldsymbol{B_n}) \to \boldsymbol{\mathcal{T}_n}, \qquad \Phi(F) := A,$$

is a completely isometric isomorphism of operator spaces.

Proof. Assume that F is a bounded free k-pluriharmonic function on $\boldsymbol{B_n}$ and has the representation from Definition 2.1. Then, for any $r\in[0,1)$,

$$F(r\boldsymbol{S}) \in \overline{\operatorname{span}}\{f^*g : f, g\in B(\mathcal{E})\otimes_{\min}\boldsymbol{\mathcal{A}_n}\}^{\text{-}\|\cdot\|}$$

and, due to the noncommutative von Neumann inequality [Popescu 1999], we have $\sup_{r\in[0,1)} \|F(r\boldsymbol{S})\| = \|F\|_\infty < \infty$. According to Theorem 1.6, $F(\boldsymbol{S})$ is the formal Fourier series of a k-multi-Toeplitz operator

$A\in B\big(\mathcal{E}\otimes\bigotimes_{i=1} F^2(H_{n_i})\big)$ and $A=\text{SOT-lim}_{r\to 1} F(r\boldsymbol{S})\in\boldsymbol{\mathcal{T}_n}$. Using the properties of the noncommutative Berezin kernel on polyballs, we have

$$F(r\boldsymbol{X})=(I_\mathcal{E}\otimes \boldsymbol{K}_{\boldsymbol{X}}^*)[F(r\boldsymbol{S})\otimes I_\mathcal{H}](I_\mathcal{E}\otimes \boldsymbol{K}_{\boldsymbol{X}}),\quad \boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H}).$$

Since the map $Y\mapsto Y\otimes I_\mathcal{H}$ is SOT-continuous on bounded subsets of $B\big(\mathcal{E}\otimes\bigotimes_{i=1} F^2(H_{n_i})\big)$, we deduce that

$$\text{SOT-}\lim_{r\to 1} F(r\boldsymbol{X})=(I_\mathcal{E}\otimes \boldsymbol{K}_{\boldsymbol{X}}^*)[A\otimes I_\mathcal{H}](I_\mathcal{E}\otimes \boldsymbol{K}_{\boldsymbol{X}})=\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A].$$

Since F is continuous in the norm topology on $\boldsymbol{B_n}(\mathcal{H})$, we have $F(r\boldsymbol{X})\to F(\boldsymbol{X})$ as $r\to 1$. Consequently, the relation above implies $F(\boldsymbol{X})=\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A]$, which completes the proof that (i) implies (ii).

To prove that (ii) implies (i), let $A\in\mathcal{T}_{\boldsymbol{n}}$ and $F(\boldsymbol{X}):=\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A]$ for $\boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H})$. Since A is a k-multi-Toeplitz operator, Theorem 1.5 shows that it has a formal Fourier series

$$\varphi(\boldsymbol{S}):=\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\otimes \boldsymbol{S}_{1,\alpha_1}\cdots \boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}_{1,\beta_1}^*\cdots \boldsymbol{S}_{k,\beta_k}^*$$

with the property that the series $\varphi(r\boldsymbol{S})$ is convergent in the operator norm topology to an operator in $\text{span}\{f^*g: f,g\in B(\mathcal{E})\otimes_{\min}\boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|}$. Moreover, we have $A=\text{SOT-lim}_{r\to 1}\varphi(r\boldsymbol{S})$ and

$$\|A\|=\sup_{r\in[0,1)}\|\varphi(r\boldsymbol{S})\|.$$

Hence, the map $\boldsymbol{X}\mapsto\varphi(\boldsymbol{X})$ is a k-pluriharmonic function on $\boldsymbol{B_n}(\mathcal{H})$. On the other hand, due to Theorem 1.6, we have $\varphi(r\boldsymbol{S})=\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A]$, where

$$\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[u]:=(I_\mathcal{E}\otimes \boldsymbol{K}_{r\boldsymbol{S}}^*)(u\otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})(I_\mathcal{E}\otimes \boldsymbol{K}_{r\boldsymbol{S}}),\quad u\in B(\mathcal{E}_k),$$

and $\boldsymbol{K}_{r\boldsymbol{S}}$ is the noncommutative Berezin kernel associated with $r\boldsymbol{S}\in \boldsymbol{B_n}\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$. Note that

$$\varphi(r\boldsymbol{X})=\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[\varphi(r\boldsymbol{S})]=(I_\mathcal{E}\otimes \boldsymbol{K}_{\boldsymbol{X}}^*)[\varphi(r\boldsymbol{S})\otimes I_\mathcal{H}](I_\mathcal{E}\otimes \boldsymbol{K}_{\boldsymbol{X}}).$$

Now, using continuity of φ on $\boldsymbol{B_n}(\mathcal{H})$ and the fact that $A=\text{SOT-lim}_{r\to 1}\varphi(r\boldsymbol{S})$, we deduce that

$$\varphi(\boldsymbol{X})=\text{SOT-}\lim_{r\to 1}\varphi(r\boldsymbol{X})=\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A]=F(\boldsymbol{X}),\quad \boldsymbol{X}\in \boldsymbol{B}_n(\mathcal{H}).$$

To prove the last part of the theorem, let $[F_{ij}]_m\in M_m(\mathbf{PH}_\mathcal{E}^\infty(\boldsymbol{B_n}))$ and use the noncommutative von Neumann inequality to obtain

$$\|[F_{ij}]_m\|=\sup_{\boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H})}\|[F_{ij}(\boldsymbol{X})]_m\|=\sup_{r\in[0,1)}\|[F_{ij}(r\boldsymbol{S})]_m\|.$$

On the other hand, $A_{ij}:=\text{SOT-lim}_{r\to 1} F_{ij}(rS)$ is a k-multi-Toeplitz operator and

$$F_{ij}(rS)=(I_\mathcal{E}\otimes \boldsymbol{K}_{r\boldsymbol{S}}^*)(A_{ij}\otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})(I_\mathcal{E}\otimes \boldsymbol{K}_{r\boldsymbol{S}}).$$

Hence, we obtain

$$\sup_{r\in[0,1)}\|[F_{ij}(r\boldsymbol{S})]_m\|\le\|[A_{ij}]_m\|.$$

Since $[A_{ij}]_m := \text{SOT-lim}_{r\to 1}[F_{ij}(rS)]_m$, we deduce that the inequality above is in fact an equality. This shows that Φ is a completely isometric isomorphisms of operator spaces. □

As a consequence, we can obtain the following Fatou-type result concerning the boundary behaviour of bounded k-pluriharmonic functions.

Corollary 2.3. *If $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ is a bounded free k-pluriharmonic function and $\boldsymbol{X}$ is a pure element in $\boldsymbol{B_n}(\mathcal{H})^-$, then the limit*

$$\text{SOT-}\lim_{r\to 1} F(r\boldsymbol{X})$$

exists.

Proof. If $\boldsymbol{X}$ is a pure element in $\boldsymbol{B_n}(\mathcal{H})^-$, then the noncommutative Berezin kernel $\boldsymbol{K_X}$ is an isometry (see [Popescu 2016]). Since F is free k-pluriharmonic function on $\boldsymbol{B_n}$, we have

$$F(r\boldsymbol{S}) \in \overline{\text{span}}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|}$$

and $F(r\boldsymbol{S})$ converges in the operator norm topology. Consequently,

$$F(r\boldsymbol{X}) = (I_\mathcal{E} \otimes \boldsymbol{K}_{\boldsymbol{X}}^*)[F(r\boldsymbol{S}) \otimes I_\mathcal{H}](I_\mathcal{E} \otimes \boldsymbol{K_X}).$$

Since F is bounded, Theorem 2.2 implies $\text{SOT-lim}_{r\to 1} F(r\boldsymbol{S}) = A \in \boldsymbol{\mathcal{T}_n}$ and $\sup_{0\le r<1} \|F(r\boldsymbol{S})\| < \infty$. Using these facts in the relation above, we conclude that $\text{SOT-lim}_{r\to 1} F(r\boldsymbol{X})$ exists. □

We denote by $\mathbf{PH}^c_\mathcal{E}(\boldsymbol{B_n})$ the set of all free k-pluriharmonic functions on $\boldsymbol{B_n}$ with operator-valued coefficients in $B(\mathcal{E})$, which have continuous extensions (in the operator norm topology) to the closed polyball $\boldsymbol{B_n}(\mathcal{H})^-$ for any Hilbert space $\mathcal{H}$. Throughout this section, we assume that $\mathcal{H}$ is an infinite-dimensional Hilbert space. In what follows we solve the Dirichlet extension problem for the regular polyballs.

Theorem 2.4. *If $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$, then the following statements are equivalent*:

(i) *F is a free k-pluriharmonic function on $\boldsymbol{B_n}(\mathcal{H})$ such that $F(r\boldsymbol{S})$ converges in the operator norm topology as $r \to 1$.*

(ii) *There exists $A \in \boldsymbol{\mathcal{P}} := \text{span}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|}$ such that*

$$F(\boldsymbol{X}) = \boldsymbol{\mathcal{B}}^{\text{ext}}_{\boldsymbol{X}}[A], \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

(iii) *F is a free k-pluriharmonic function on $\boldsymbol{B_n}(\mathcal{H})$ which has a continuous extension* (*in the operator norm topology*) *to the closed ball $\boldsymbol{B_n}(\mathcal{H})^-$.*

In this case, $A = \lim_{r\to 1} F(r\boldsymbol{S})$, where the convergence is in the operator norm. Moreover, the map

$$\Phi : \mathbf{PH}^c_\mathcal{E}(\boldsymbol{B_n}) \to \boldsymbol{\mathcal{P}}, \quad \Phi(F) := A,$$

is a completely isometric isomorphism of operator spaces.

Proof. Assume that (i) holds. Then F has a representation

$$F(\boldsymbol{X})=\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\ i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\ |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\otimes \boldsymbol{X}_{1,\alpha_1}\cdots \boldsymbol{X}_{k,\alpha_k}\boldsymbol{X}_{1,\beta_1}^*\cdots \boldsymbol{X}_{k,\beta_k}^*,$$

where the series converge in the operator norm topology for any $\boldsymbol{X}=(X_1,\dots,X_k)\in \boldsymbol{B_n}(\mathcal{H})$. Since the series defining $F(r\boldsymbol{S})$ converges in the operator topology, we deduce that

$$A:=\lim_{r\to 1}F(r\boldsymbol{S})\in\boldsymbol{\mathcal{P}}. \tag{2-1}$$

On the other hand, we have

$$\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[F(r\boldsymbol{S})]=(I_{\mathcal{E}}\otimes \boldsymbol{K}_{\boldsymbol{X}}^*)[F(r\boldsymbol{S})\otimes I_{\mathcal{H}}](I_{\mathcal{E}}\otimes \boldsymbol{K}_{\boldsymbol{X}})=F(r\boldsymbol{X})$$

for any $r\in[0,1)$ and $\boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H})$. Hence, and using (2-1), we deduce that

$$\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A]=\lim_{r\to 1}F(r\boldsymbol{X})=F(\boldsymbol{X}),$$

which proves (ii). Now we show that (ii) implies (i). Assuming (ii) and taking into account Theorem 1.7, one can see that A is a k-multi-Toeplitz operator. As in the proof of Theorem 2.2, the map defined by $F(\boldsymbol{X}):=\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A]$, $\boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H})$, is a bounded free k-pluriharmonic function. Moreover, we proved that

$$F(r\boldsymbol{S})=\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A],\qquad r\in[0,1), \tag{2-2}$$

$F(r\boldsymbol{S})\in\boldsymbol{\mathcal{P}}$ and also that $A=\text{SOT-}\lim_{r\to1}F(r\boldsymbol{S})$ and $\|A\|=\sup_{r\in[0,1)}\|F(r\boldsymbol{S})\|$. Since $A\in\boldsymbol{\mathcal{P}}$, there is a sequence of polynomials q_m in $\boldsymbol{S}_{\boldsymbol{\alpha}}^*\boldsymbol{S}_{\boldsymbol{\beta}}$ such that $q_m\to A$ in norm as $m\to\infty$. For any $\epsilon>0$, let $N\in\mathbb{N}$ be such that $\|A-q_m\|<\frac{1}{3}\epsilon$ for any $m\ge N$. Choose $\delta\in(0,1)$ such that $\|\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[q_N]-q_N\|<\frac{1}{3}\epsilon$ for any $r\in(\delta,1)$. Note that

$$\|\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A]-A\|\le\|\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A-q_N]\|+\|\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[q_N]-q_N\|+\|q_N-A\|\le\|A-q_N\|+\tfrac{2}{3}\epsilon<\epsilon$$

for any $r\in(\delta,1)$. Therefore, $\lim_{r\to1}\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A]=A$ in the norm topology. Hence, and due to (2-2), we deduce that $\lim_{r\to1}F(r\boldsymbol{S})=A$ in the norm topology, which shows that (i) holds. Since $\mathcal{H}$ is infinite-dimensional, that (iii) implies (i) is clear.

It remains to prove that (ii) implies (iii). We assume that (ii) holds. Then there exists $A\in\boldsymbol{\mathcal{P}}$ such that $F(\boldsymbol{X})=\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A]$ for all $\boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H})$. Due to Theorem 2.2, F is a bounded free k-pluriharmonic function on $\boldsymbol{B_n}(\mathcal{H})$. For any $\boldsymbol{Y}\in \boldsymbol{B_n}(\mathcal{H})^-$, one can show, as in the proof that (ii) implies (i), that $\widetilde{F}(\boldsymbol{Y}):=\lim_{r\to1}\boldsymbol{\mathcal{B}}_{r\boldsymbol{Y}}^{\text{ext}}[A]$ exists in the operator norm topology. Since $\|\boldsymbol{\mathcal{B}}_{r\boldsymbol{Y}}^{\text{ext}}[A]\|\le\|A\|$ for any $r\in[0,1)$, we deduce that $\|\widetilde{F}(\boldsymbol{Y})\|\le\|A\|$ for any $\boldsymbol{Y}\in \boldsymbol{B_n}(\mathcal{H})^-$. Note also that $\widetilde{F}$ is an extension of F. Lastly, we show that $\widetilde{F}$ is continuous on $\boldsymbol{B_n}(\mathcal{H})^-$. To this end, let $\epsilon>0$ and, due to the equivalence of (ii) and (i), we can choose $r_0\in[0,1)$ such that $\|A-F(r_0\boldsymbol{S})\|<\frac{1}{3}\epsilon$. Since $A-F(r_0\boldsymbol{S})\in\boldsymbol{\mathcal{P}}$, we deduce that

$$\|\widetilde{F}(\boldsymbol{Y})-F(r_0\boldsymbol{Y})\|=\|\lim_{r\to1}\boldsymbol{\mathcal{B}}_{r\boldsymbol{Y}}^{\text{ext}}[A]-F(r_0\boldsymbol{Y})\|\le\limsup_{r\to1}\|\boldsymbol{\mathcal{B}}_{r\boldsymbol{Y}}^{\text{ext}}[A]-F(r_0\boldsymbol{Y})\|\le\|A-F(r_0\boldsymbol{Y})\|<\tfrac{1}{3}\epsilon$$

for any $\boldsymbol{Y} \in \boldsymbol{B_n}(\mathcal{H})^-$. Since F is continuous on $\boldsymbol{B_n}(\mathcal{H})$, there is $\delta > 0$ such that $\|F(r_0\boldsymbol{Y}) - F(r_0\boldsymbol{W})\| < \frac{1}{3}\epsilon$ for any $\boldsymbol{W} \in \boldsymbol{B_n}(\mathcal{H})^-$ with $\|\boldsymbol{W} - \boldsymbol{Y}\| < \delta$. Now note that

$$\|\widetilde{F}(\boldsymbol{Y}) - \widetilde{F}(\boldsymbol{W})\| \leq \|\widetilde{F}(\boldsymbol{Y}) - F(r_0\boldsymbol{Y})\| + \|F(r_0\boldsymbol{Y}) - F(r_0\boldsymbol{W})\| + \|F(r_0\boldsymbol{W}) - \widetilde{F}(\boldsymbol{W})\| < \epsilon$$

for any $\boldsymbol{W} \in \boldsymbol{B_n}(\mathcal{H})^-$ with $\|\boldsymbol{W} - \boldsymbol{Y}\| < \delta$. □

3. Naimark-type dilation theorem for direct products of free semigroups

In this section, we provide a Naimark-type dilation theorem for direct products $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ of unital free semigroups, and use it to obtain a structure theorem which characterizes the positive free k-pluriharmonic functions on the regular polyball with operator-valued coefficients.

Consider the unital semigroup $\boldsymbol{F_n^+} := \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ with neutral element $\boldsymbol{g} := (g_0^1, \ldots, g_0^k)$. Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_k)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k)$ be in $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$. We say that $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are left comparable, and write $\boldsymbol{\omega} \sim_{\mathrm{lc}} \boldsymbol{\gamma}$, if for each $i \in \{1, \ldots, k\}$, one of the conditions $\omega_i <_l \gamma_i$, $\gamma_i <_l \omega_i$ or $\omega_i = \gamma_i$ holds (see the definitions preceding Lemma 1.2). In this case, we define

$$c_l^+(\boldsymbol{\omega}, \boldsymbol{\gamma}) := (c_l^+(\omega_1, \gamma_1), \ldots, c_l^+(\omega_k, \gamma_k)) \quad \text{and} \quad c_l^-(\boldsymbol{\omega}, \boldsymbol{\gamma}) := (c_l^-(\omega_1, \gamma_1), \ldots, c_l^-(\omega_k, \gamma_k)),$$

where

$$c_l^+(\omega, \gamma) := \begin{cases} \omega \backslash_l \gamma & \text{if } \gamma <_l \omega, \\ g_0 & \text{if } \omega <_l \gamma \text{ or } \omega = \gamma, \end{cases} \quad \text{and} \quad c_l^-(\omega, \gamma) := \begin{cases} \gamma \backslash_r \omega & \text{if } \omega <_l \gamma, \\ g_0 & \text{if } \gamma <_l \omega \text{ or } \omega = \gamma. \end{cases}$$

We say that $K : \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} \to B(\mathcal{E})$ is a *left k-multi-Toeplitz kernel* if $K(\boldsymbol{g}, \boldsymbol{g}) = I_{\mathcal{E}}$ and

$$K(\boldsymbol{\sigma}, \boldsymbol{\omega}) = \begin{cases} K(c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega}); c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega})) & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise.} \end{cases}$$

The kernel K is positive semidefinite if, for each $m \in \mathbb{N}$, any choice of $h_1, \ldots h_m \in \mathcal{E}$, and any $\boldsymbol{\sigma}^{(i)} := (\sigma_1^{(i)}, \ldots, \sigma_k^{(i)}) \in \boldsymbol{F_n^+}$, it satisfies the inequality

$$\sum_{i,j=1}^m \langle K(\boldsymbol{\sigma}^{(i)}, \boldsymbol{\sigma}^{(j)}) h_j, h_i \rangle \geq 0.$$

Definition 3.1. A map $K : \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} \to B(\mathcal{E})$ has a Naimark dilation if there exists a k-tuple of commuting row isometries $\boldsymbol{V} = (V_1, \ldots, V_k)$, $V_i = (V_{i,1}, \ldots, V_{i,n_i})$, on a Hilbert space $\mathcal{K} \supset \mathcal{E}$, i.e., the nonselfadjoint algebra $\mathrm{Alg}(V_i)$ commutes with $\mathrm{Alg}(V_s)$ for any $i, s \in \{1, \ldots, k\}$ with $i \neq s$, such that

$$K(\boldsymbol{\sigma}, \boldsymbol{\omega}) = P_{\mathcal{E}} \boldsymbol{V}_{\boldsymbol{\sigma}}^* \boldsymbol{V}_{\boldsymbol{\omega}}\big|_{\mathcal{E}}, \quad \boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F_n^+}.$$

The dilation is called minimal if $\mathcal{K} = \bigvee_{\boldsymbol{\omega} \in \boldsymbol{F_n^+}} \boldsymbol{V}_{\boldsymbol{\omega}} \mathcal{E}$.

Theorem 3.2. *A map $K : \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} \to B(\mathcal{H})$ is a positive semidefinite left k-multi-Toeplitz kernel on the direct product $\boldsymbol{F_n^+}$ of free semigroups if and only if it admits a Naimark dilation.*

Proof. Let $\mathcal{K}_0$ be the vector space of all sums of tensor monomials $\sum_{\boldsymbol{\sigma}\in \boldsymbol{F}_n^+} e_{\boldsymbol{\sigma}}\otimes h_{\boldsymbol{\sigma}}$, where $\{h_{\boldsymbol{\sigma}}\}_{\boldsymbol{\sigma}\in \boldsymbol{F}_n^+}$ is a finitely supported sequence of vectors in $\mathcal{H}$. Define the sesquilinear form $\langle\cdot,\cdot\rangle_{\mathcal{K}_0}$ on $\mathcal{K}_0$ by setting

$$\left\langle \sum_{\boldsymbol{\omega}\in \boldsymbol{F}_n^+} e_{\boldsymbol{\omega}}\otimes h_{\boldsymbol{\omega}}, \sum_{\boldsymbol{\sigma}\in \boldsymbol{F}_n^+} e_{\boldsymbol{\sigma}}\otimes h'_{\boldsymbol{\sigma}}\right\rangle_{\mathcal{K}_0} := \sum_{\boldsymbol{\omega},\boldsymbol{\sigma}\in \boldsymbol{F}_n^+} \langle K(\boldsymbol{\sigma},\boldsymbol{\omega})h_{\boldsymbol{\omega}}, h'_{\boldsymbol{\sigma}}\rangle_{\mathcal{H}}, \quad h_{\boldsymbol{\omega}}, h'_{\boldsymbol{\sigma}}\in\mathcal{H}.$$

Since K is positive semidefinite, so is $\langle\cdot,\cdot\rangle_{\mathcal{K}_0}$. Set $\mathcal{N}:=\{f\in\mathcal{K}_0:\langle f,f\rangle=0\}$ and define the Hilbert space obtained by completing $\mathcal{K}_0/\mathcal{N}$ with the induced inner product. For each $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$, define the operator $V_{i,j}$ on $\mathcal{K}_0$ by setting

$$V_{i,j}\Bigl(\sum_{\boldsymbol{\sigma}\in \boldsymbol{F}_n^+} e_{\boldsymbol{\sigma}}\otimes h_{\boldsymbol{\sigma}}\Bigr) := \sum_{\boldsymbol{\sigma}=(\sigma_1,\dots,\sigma_k)\in \boldsymbol{F}_n^+} e_{\sigma_1}\otimes\cdots\otimes e_{\sigma_{i-1}}\otimes e_{g_j\sigma_i}\otimes e_{\sigma_{i+1}}\otimes\cdots\otimes e_{\sigma_k}\otimes h_{\boldsymbol{\sigma}}.$$

Note that if $p\in\{1,\dots,n_i\}$ then

$$\begin{aligned}
&\left\langle V_{i,j}\Bigl(\sum_{\boldsymbol{\omega}\in \boldsymbol{F}_n^+} e_{\boldsymbol{\omega}}\otimes h_{\boldsymbol{\omega}}\Bigr), V_{i,p}\Bigl(\sum_{\boldsymbol{\sigma}\in \boldsymbol{F}_n^+} e_{\boldsymbol{\sigma}}\otimes h'_{\boldsymbol{\sigma}}\Bigr)\right\rangle_{\mathcal{K}_0}\\
&\qquad = \sum_{\boldsymbol{\omega},\boldsymbol{\sigma}\in \boldsymbol{F}_n^+} \langle K(\sigma_1,\dots,\sigma_{i-1}, g_j\sigma_i,\sigma_{i+1},\dots,\sigma_k;\omega_1,\dots,\omega_{i-1},g_p\omega_i,\omega_{i+1},\dots,\omega_k)h_{\boldsymbol{\omega}}, h'_{\boldsymbol{\sigma}}\rangle_{\mathcal{H}}\\
&\qquad = \begin{cases} \sum_{\boldsymbol{\omega},\boldsymbol{\sigma}\in \boldsymbol{F}_n^+}\langle K(\boldsymbol{\sigma},\boldsymbol{\omega})h_{\boldsymbol{\omega}}, h'_{\boldsymbol{\sigma}}\rangle_{\mathcal{H}} & \text{if } j=p,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Hence and using the definition of $\langle\cdot,\cdot\rangle_{\mathcal{K}_0}$, we deduce that, for each $i\in\{1,\dots,k\}$, the operators $V_{i,1},\dots,V_{i,n_i}$ can be extended by continuity to isometries on $\mathcal{K}$ with orthogonal ranges. Note also that, if $i,s\in\{1,\dots,k\}$, $i\neq s$, $j\in\{1,\dots,n_i\}$ and $t\in\{1,\dots,n_s\}$, then

$$V_{i,j}V_{st}(e_{\sigma_1}\otimes\cdots\otimes e_{\sigma_k}\otimes h) = e_{\sigma_1}\otimes\cdots\otimes e_{\sigma_{i-1}}\otimes e_{g_j\sigma_i}\otimes e_{\sigma_{i+1}}\cdots\otimes e_{\sigma_{s-1}}\otimes e_{\sigma_{g_t\sigma_s}}\otimes e_{\sigma_{s+1}}\otimes\cdots\otimes e_{\sigma_k}\otimes h$$

when $i<s$. This shows that $V_{ij}V_{st}=V_{st}V_{ij}$. Since

$$\langle e_{\boldsymbol{g}}\otimes h, e_{\boldsymbol{g}}\otimes h'\rangle_{\mathcal{K}} = \langle K(\boldsymbol{g},\boldsymbol{g})h,h'\rangle_{\mathcal{H}} = \langle h,h'\rangle_{\mathcal{H}}, \quad h,h'\in\mathcal{H},$$

we can embed $\mathcal{H}$ into $\mathcal{K}$ by setting $h=e_{\boldsymbol{g}}\otimes h$. Note that, for any $\boldsymbol{\omega},\boldsymbol{\sigma}\in\boldsymbol{F}_n^+$ and $h,h'\in\mathcal{H}$, we have

$$\langle V_{\boldsymbol{\sigma}}^*V_{\boldsymbol{\omega}}h,h'\rangle_{\mathcal{K}} = \langle V_{\boldsymbol{\omega}}h, V_{\boldsymbol{\sigma}}h'\rangle_{\mathcal{K}} = \langle e_{\boldsymbol{\omega}}\otimes h, e_{\boldsymbol{\sigma}}\otimes h'\rangle_{\mathcal{K}} = \langle K(\boldsymbol{\sigma},\boldsymbol{\omega})h,h'\rangle_{\mathcal{H}}.$$

Therefore, $K(\boldsymbol{\sigma},\boldsymbol{\omega}) = P_{\mathcal{H}}\boldsymbol{V}_{\boldsymbol{\sigma}}^*\boldsymbol{V}_{\boldsymbol{\omega}}|_{\mathcal{H}}$ for any $\boldsymbol{\sigma},\boldsymbol{\omega}\in\boldsymbol{F}_n^+$. Since any element in $\mathcal{K}_0$ is a linear combination of vectors $\boldsymbol{V}_{\boldsymbol{\sigma}}h$, where $\boldsymbol{\sigma}\in\boldsymbol{F}_n^+$ and $h\in\mathcal{H}$, we deduce that $\mathcal{K}=\bigvee_{\boldsymbol{\omega}\in\boldsymbol{F}_n^+}\boldsymbol{V}_{\boldsymbol{\omega}}\mathcal{H}$, which proves the minimality of the Naimark dilation.

Now we prove the converse. Let $\boldsymbol{V}=(V_1,\dots,V_n)$ and $V_i=(V_{i,1},\dots,V_{i,n_i})$ be k-tuples of commuting row isometries on a Hilbert space $\mathcal{K}\supset\mathcal{H}$. Define $K:\boldsymbol{F}_n^+\times\boldsymbol{F}_n^+\to B(\mathcal{H})$ by setting $K(\boldsymbol{\sigma},\boldsymbol{\omega})=P_{\mathcal{H}}\boldsymbol{V}_{\boldsymbol{\sigma}}^*\boldsymbol{V}_{\boldsymbol{\omega}}|_{\mathcal{H}}$ for any $\boldsymbol{\sigma},\boldsymbol{\omega}\in\boldsymbol{F}_n^+$. Assume that $\boldsymbol{\sigma},\boldsymbol{\omega}\in\boldsymbol{F}_n^+$ and $\boldsymbol{\sigma}\sim_{\mathrm{lc}}\boldsymbol{\omega}$. Using the commutativity of the row isometries $V_1,\dots,V_k$, we can assume without loss of generality that there is $p\in\{1,\dots,k\}$ such that

$\omega_1 \leq_l \sigma_1, \ldots, \omega_p \leq_l \sigma_p, \sigma_{p+1} \leq_l \omega_{p+1}, \ldots, \sigma_k \leq_l \omega$. Since each $V_i = (V_{i,1}, \ldots, V_{i,n_i})$ is an isometry, we have $V_{i,t}^* V_{i,s} = \delta_{ts} I$. Consequently, and using the commutativity of the row isometries, we deduce that

$$\begin{aligned}
&\langle V_{1,\omega_1} \cdots V_{k,\omega_k} h,\, V_{1,\sigma_1} \cdots V_{k,\sigma_k} h' \rangle \\
&\qquad = \langle V_{2,\omega_2} \cdots V_{k,\omega_k} h,\, V_{1,\sigma_1 \backslash_l \omega_1} V_{2,\sigma_2} \cdots V_{k,\sigma_k} h' \rangle \\
&\qquad = \langle V_{2,\omega_2} \cdots V_{k,\omega_k} h,\, V_{2,\sigma_2} \cdots V_{k,\sigma_k} V_{1,\sigma_1 \backslash_l \omega_1} h' \rangle \\
&\qquad \;\vdots \\
&\qquad = \langle V_{p+1,\omega_{p+1}} \cdots V_{k,\omega_k} h,\, V_{p+1,\sigma_{p+1}} \cdots V_{k,\sigma_k} V_{1,\sigma_1 \backslash_l \omega_1} \cdots V_{p,\sigma_p \backslash_l \omega_p} h' \rangle \\
&\qquad = \langle V_{p+1,\omega_{p+1} \backslash_l \sigma_{p+1}} V_{p+2,\omega_{p+2}} \cdots V_{k,\omega_k} h,\, V_{p+2,\sigma_{p+2}} \cdots V_{k,\sigma_k} V_{1,\sigma_1 \backslash_l \omega_1} \cdots V_{p,\sigma_p \backslash_l \omega_p} h' \rangle \\
&\qquad = \langle V_{p+2,\omega_{p+2}} \cdots V_{k,\omega_k} V_{p+1,\omega_{p+1} \backslash_l \sigma_{p+1}} h,\, V_{p+2,\sigma_{p+2}} \cdots V_{k,\sigma_k} V_{1,\sigma_1 \backslash_l \omega_1} \cdots V_{p,\sigma_p \backslash_l \omega_p} h' \rangle \\
&\qquad \;\vdots \\
&\qquad = \langle V_{p+1,\omega_{p+1} \backslash_l \sigma_{p+1}} \cdots V_{k,\omega_k \backslash_l \sigma_k} h,\, V_{1,\sigma_1 \backslash_l \omega_1} \cdots V_{p,\sigma_p \backslash_l \omega_p} h' \rangle \\
&\qquad = \langle V_{1,\sigma_1 \backslash_l \omega_1}^* \cdots V_{p,\sigma_p \backslash_l \omega_p}^* V_{p+1,\omega_{p+1} \backslash_l \sigma_{p+1}} \cdots V_{k,\omega_k \backslash_l \sigma_k} h,\, h' \rangle
\end{aligned}$$

for any $h, h' \in \mathcal{H}$. Therefore, for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+$, we have

$$\begin{aligned}
K(\boldsymbol{\sigma}, \boldsymbol{\omega}) = P_{\mathcal{H}} \boldsymbol{V}_{\boldsymbol{\sigma}}^* \boldsymbol{V}_{\boldsymbol{\omega}}\big|_{\mathcal{H}} &= \begin{cases} P_{\mathcal{H}} \boldsymbol{V}_{c_l^+(\boldsymbol{\sigma},\boldsymbol{\omega})}^* \boldsymbol{V}_{c_l^-(\boldsymbol{\sigma},\boldsymbol{\omega})}\big|_{\mathcal{H}} & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise}, \end{cases} \\
&= \begin{cases} K(c_l^+(\boldsymbol{\sigma},\boldsymbol{\omega}); c_l^-(\boldsymbol{\sigma},\boldsymbol{\omega})) & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}$$

and $K(\boldsymbol{g}, \boldsymbol{g}) = I_{\mathcal{H}}$. This shows that K is a left k-multi-Toeplitz kernel on $\boldsymbol{F}_{\boldsymbol{n}}^+$. On the other hand, for any finitely supported sequence $\{h_{\boldsymbol{\omega}}\}_{\boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+}$ of elements in $\mathcal{H}$, we have

$$\sum_{\boldsymbol{\omega},\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \langle K(\boldsymbol{\sigma}, \boldsymbol{\omega}) h_{\boldsymbol{\omega}}, h_{\boldsymbol{\sigma}} \rangle = \sum_{\boldsymbol{\omega},\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \langle P_{\mathcal{H}} \boldsymbol{V}_{\boldsymbol{\sigma}}^* \boldsymbol{V}_{\boldsymbol{\omega}}|_{\mathcal{H}} h_{\boldsymbol{\omega}}, h_{\boldsymbol{\sigma}} \rangle = \Big\| \sum_{\boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \boldsymbol{V}_{\boldsymbol{\omega}} h_{\boldsymbol{\omega}} \Big\|^2 \geq 0.$$

Therefore, K is a positive semidefinite left k-multi-Toeplitz kernel on $\boldsymbol{F}_{\boldsymbol{n}}^+$. $\square$

We remark that the Naimark dilation provided in Theorem 3.2 is minimal. To prove the uniqueness of the minimal Naimark dilation, let $\boldsymbol{V}' = (V_1', \ldots, V_n')$, $V_i' = (V_{i,1}', \ldots, V_{i,n_i}')$, be a k-tuple of commuting row isometries on a Hilbert space $\mathcal{K}' \supset \mathcal{H}$ such that $K(\sigma, \omega) = P_{\mathcal{H}}^{\mathcal{K}'} (\boldsymbol{V}_{\boldsymbol{\sigma}}')^* \boldsymbol{V}_{\boldsymbol{\omega}}'\big|_{\mathcal{H}}$ for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+$ and with the property that $\mathcal{K} = \bigvee_{\boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \boldsymbol{V}_{\boldsymbol{\omega}}' \mathcal{H}$. For any $x, y \in \mathcal{H}$, we have

$$\langle \boldsymbol{V}_{\boldsymbol{\omega}} x, \boldsymbol{V}_{\boldsymbol{\sigma}} y \rangle_{\mathcal{K}} = \langle K(\boldsymbol{\sigma}, \boldsymbol{\omega}) x, y \rangle_{\mathcal{H}} = \langle P_{\mathcal{H}}^{\mathcal{K}'} (\boldsymbol{V}_{\boldsymbol{\sigma}}')^* \boldsymbol{V}_{\boldsymbol{\omega}}' x, y \rangle_{\mathcal{K}'} = \langle \boldsymbol{V}_{\boldsymbol{\omega}}' x, \boldsymbol{V}_{\boldsymbol{\sigma}}' y \rangle_{\mathcal{K}'}.$$

Consequently, the map

$$W\Big(\sum_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \boldsymbol{V}_{\boldsymbol{\sigma}} h_{\boldsymbol{\sigma}} \Big) := \sum_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \boldsymbol{V}_{\boldsymbol{\sigma}}' h_{\boldsymbol{\sigma}},$$

where $\{h_{\boldsymbol{\sigma}}\}_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+}$ is any finitely supported sequence of vectors in $\mathcal{H}$, is well-defined. Due to the minimality of the spaces $\mathcal{K}$ and $\mathcal{K}'$, the map extends to a unitary operator W from $\mathcal{K}$ onto $\mathcal{K}'$. Note also that

$WV_{i,j} = V'_{i,j}W$ for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, which completes the proof of the uniqueness of the minimal Naimark dilation.

We should mention that there is a dual Naimark-type dilation for positive semidefinite right k-multi-Toeplitz kernels. A kernel $\Gamma : \boldsymbol{F}_{\boldsymbol{n}}^+ \times \boldsymbol{F}_{\boldsymbol{n}}^+ \to B(\mathcal{E})$ is called *right k-multi-Toeplitz* if $\Gamma(\boldsymbol{g}, \boldsymbol{g}) = I_{\mathcal{E}}$ and

$$\Gamma(\boldsymbol{\sigma}, \boldsymbol{\omega}) = \begin{cases} \Gamma(c_r^+(\boldsymbol{\sigma}, \boldsymbol{\omega}); c_r^-(\boldsymbol{\sigma}, \boldsymbol{\omega})) & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{rc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise}, \end{cases}$$

where $c_r^+(\boldsymbol{\sigma}, \boldsymbol{\omega})$, $c_r^-(\boldsymbol{\sigma}, \boldsymbol{\omega})$ are defined by (1-1). We say that Γ has a Naimark dilation if there exists a k-tuple $\boldsymbol{W} = (W_1, \ldots, W_n)$, $W_i = (W_{i,1}, \ldots, W_{i,n_i})$, of commuting row isometries on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ such that $\Gamma(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) = P_{\mathcal{E}} \boldsymbol{W}_{\boldsymbol{\sigma}}^* \boldsymbol{W}_{\boldsymbol{\omega}}|_{\mathcal{E}}$ for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+$.

Theorem 3.3. *A map $\Gamma : \boldsymbol{F}_{\boldsymbol{n}}^+ \times \boldsymbol{F}_{\boldsymbol{n}}^+ \to B(\mathcal{H})$ is a positive semidefinite right k-multi-Toeplitz kernel on $\boldsymbol{F}_{\boldsymbol{n}}^+$ if and only if it admits a Naimark dilation. In this case, there is a minimal dilation which is uniquely determined up to isomorphism.*

Proof. We only sketch the proof, which is very similar to that of Theorem 3.2, pointing out the differences. First, $\mathcal{K}_0$ is the vector space of all sums of tensor monomials $\sum_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\tilde{\boldsymbol{\sigma}}} \otimes h_{\boldsymbol{\sigma}}$, where $\{h_{\boldsymbol{\sigma}}\}_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+}$ is a finitely supported sequence of vectors in $\mathcal{H}$, while the sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{K}_0}$ on $\mathcal{K}_0$ is defined by setting

$$\left\langle \sum_{\boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\tilde{\boldsymbol{\omega}}} \otimes h_{\boldsymbol{\omega}}, \sum_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\tilde{\boldsymbol{\sigma}}} \otimes h'_{\boldsymbol{\sigma}} \right\rangle_{\mathcal{K}_0} := \sum_{\boldsymbol{\omega}, \boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \langle \Gamma(\boldsymbol{\sigma}, \boldsymbol{\omega}) h_{\boldsymbol{\omega}}, h'_{\boldsymbol{\sigma}} \rangle_{\mathcal{H}}.$$

For each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, we define the operator $W_{i,j}$ on $\mathcal{K}_0$ by setting

$$W_{i,j}\left(\sum_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\tilde{\boldsymbol{\sigma}}} \otimes h_{\boldsymbol{\sigma}} \right) := \sum_{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_k) \in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\tilde{\sigma}_1} \otimes \cdots \otimes e_{\tilde{\sigma}_{i-1}} \otimes e_{g_j \tilde{\sigma}_i} \otimes e_{\tilde{\sigma}_{i+1}} \otimes \cdots \otimes e_{\tilde{\sigma}_k} \otimes h_{\boldsymbol{\sigma}}.$$

Taking into account the relations

$$c_r^+(\boldsymbol{\sigma}, \boldsymbol{\omega})^\sim = c_l^+(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) \quad \text{and} \quad c_r^-(\boldsymbol{\sigma}, \boldsymbol{\omega})^\sim = c_l^-(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}),$$

we deduce that

$$\begin{aligned} P_{\mathcal{H}} \boldsymbol{W}_{\boldsymbol{\sigma}}^* \boldsymbol{W}_{\boldsymbol{\omega}}\big|_{\mathcal{H}} &= \begin{cases} P_{\mathcal{H}} \boldsymbol{W}_{c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega})}^* \boldsymbol{W}_{c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega})}\big|_{\mathcal{H}} & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} \Gamma(c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega})^\sim; c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega})^\sim) & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} \Gamma(c_r^+(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}); c_r^-(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}})) & \text{if } \tilde{\boldsymbol{\sigma}} \sim_{\mathrm{rc}} \tilde{\boldsymbol{\omega}}, \\ 0 & \text{otherwise}, \end{cases} \\ &= \Gamma(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) \end{aligned}$$

for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+$. The rest of the proof is similar to that of Theorem 3.2. We leave it to the reader. □

Let F be a free k-pluriharmonic on the polyball $\boldsymbol{B_n}$ with operator-valued coefficients in $B(\mathcal{E})$ with representation

$$F(\boldsymbol{X}) = \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\ i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\ |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{X}_{1,\alpha_1}\cdots \boldsymbol{X}_{k,\alpha_k}\boldsymbol{X}_{1,\beta_1}^*\cdots \boldsymbol{X}_{k,\beta_k}^*, \tag{3-1}$$

where the series converge in the operator norm topology for any $\boldsymbol{X} = (X_1, \dots, X_k) \in \boldsymbol{B_n}(\mathcal{H})$, with $X_i := (X_{i,1}, \dots, X_{i,n_i})$, and any Hilbert space $\mathcal{H}$. We associate to F the kernel $\Gamma_F : \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} \to B(\mathcal{E})$ given by

$$\Gamma_F(\boldsymbol{\sigma}, \boldsymbol{\omega}) := \begin{cases} A_{(c_r^+(\boldsymbol{\sigma},\boldsymbol{\omega});c_r^-(\boldsymbol{\sigma},\boldsymbol{\omega}))} & \text{if } \boldsymbol{\sigma} \sim_{\text{rc}} \boldsymbol{\omega},\\ 0 & \text{otherwise.}\end{cases} \tag{3-2}$$

One can easily see that Γ_F is a right k-multi-Toeplitz kernel on $\boldsymbol{F_n^+}$. In what follows, we prove a Schur-type result for positive k-pluriharmonic functions in polyballs.

Theorem 3.4. *Let F be a k-pluriharmonic function on the regular polyball $\boldsymbol{B_n}$, with coefficients in $B(\mathcal{E})$. Then F is positive on $\boldsymbol{B_n}$ if and only if the kernel Γ_{F_r} is positive semidefinite for any $r \in [0, 1)$, where F_r stands for the mapping $\boldsymbol{X} \mapsto F(r\boldsymbol{X})$.*

Proof. For every $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, we set $e_{\boldsymbol{\gamma}} := e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}$ and $\boldsymbol{S_\gamma} := \boldsymbol{S}_{1,\gamma_1}\cdots \boldsymbol{S}_{k,\gamma_k}$. Let F be a k-pluriharmonic function with representation (3-1). Taking into account Lemma 1.2, for each $\boldsymbol{\gamma}, \boldsymbol{\omega} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, $r \in [0, 1)$, and $h, h' \in \mathcal{E}$, we have

$$\begin{aligned}\langle F(r\boldsymbol{S})(h\otimes e_{\boldsymbol{\gamma}}), h'\otimes e_{\boldsymbol{\omega}}\rangle &= \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\\ i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,|\beta_i|=m_i^+}} \langle A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}h, h'\rangle r^{\sum_{i=1}^k(|\alpha_i|+|\beta_i|}\langle \boldsymbol{S_\alpha S_\beta^*} e_{\boldsymbol{\gamma}}, e_{\boldsymbol{\omega}}\rangle\\ &= \begin{cases} r^{\sum_{i=1}^k(|\alpha_i|+|\beta_i|}\langle A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))}h, h'\rangle & \text{if } \boldsymbol{\omega} \sim_{\text{rc}} \boldsymbol{\gamma},\\ 0 & \text{otherwise,}\end{cases}\\ &= \langle \Gamma_{F_r}(\boldsymbol{\omega}, \boldsymbol{\gamma})h, h'\rangle.\end{aligned}$$

Hence, we deduce that the kernel Γ_{F_r} is positive semidefinite for any $r \in [0, 1)$ if and only if $F(r\boldsymbol{S}) \geq 0$ for any $r \in [0, 1)$. Now, let $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and let $r \in (0, 1)$ be such that $(1/r)\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. Since the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ is continuous in the operator norm and completely positive, so is $\mathrm{id}\otimes\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$. Consequently, we obtain

$$F(\boldsymbol{X}) = (\mathrm{id}\otimes\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}})[F(r\boldsymbol{S})] \geq 0, \qquad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

Note that if F is positive on $\boldsymbol{B_n}$ then $F(r\boldsymbol{S}) \geq 0$ for any $r \in [0, 1)$. □

Corollary 3.5. *Let $f : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ be a free holomorphic function. Then the following statements are equivalent*:

(i) $\Re f \geq 0$ *on the polyball* $\boldsymbol{B_n}$.

(ii) $\Re f(r\boldsymbol{S}) \geq 0$ *for any* $r \in [0, 1)$.

(iii) *The right k-multi Toeplitz kernel* $\Gamma_{\Re f_r}$ *is positive semidefinite for any* $r \in [0, 1)$.

Let us define the *free k-pluriharmonic Poisson kernel* by setting

$$\boldsymbol{\mathcal{P}}(\boldsymbol{Y}, \boldsymbol{X}) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} Y_{1,\tilde{\alpha}_1}^* \cdots Y_{k,\tilde{\alpha}_k}^* Y_{1,\tilde{\beta}_1} \cdots Y_{k,\tilde{\beta}_k} \otimes X_{1,\alpha_1} \cdots X_{k,\alpha_k} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^*$$

for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and any $\boldsymbol{Y} = (Y_1, \dots, Y_k)$ with $Y_i = (Y_{i,1}, \dots, Y_{i,n_i}) \in B(\mathcal{K})^{n_i}$ such that the series above is convergent in the operator norm topology. Let $\Omega \subset \boldsymbol{F_n^+} \times \boldsymbol{F_n^+}$ be the set of all $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k) \in \boldsymbol{F_n^+}$ are such that $\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+$, $|\alpha_i| = m_i^-$, $|\beta_i| = m_i^+$ for some $m_i \in \mathbb{Z}$.

Theorem 3.6. *A map* $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ *with* $F(0) = I$ *is a positive free k-pluriharmonic function on the regular polyball if and only if it has the form*

$$F(\boldsymbol{X}) = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}_{\boldsymbol{\beta}}^*,$$

where $\boldsymbol{V} = (V_1, \dots, V_k)$ *is a k-tuple of commuting row isometries on a space* $\mathcal{K} \supset \mathcal{E}$ *such that*

$$\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{S}_{\boldsymbol{\beta}}^* \geq 0, \quad r \in [0, 1),$$

and the series is convergent in the operator topology.

Proof. Assume that F is a positive free k-pluriharmonic function which has the representation (3-1) and $F(0) = I$. Due to Theorem 3.4, $F(r\boldsymbol{S}) \geq 0$ and the right k-multi-Toeplitz kernel Γ_{F_r} is positive semidefinite for any $r \in [0, 1)$. Taking limits as $r \to \infty$, we deduce that Γ_F is positive semidefinite as well. According to Theorem 3.3, Γ_F has a Naimark-type dilation. Therefore, there is a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$ of commuting row isometries on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ such that $\Gamma(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) = P_{\mathcal{E}} \boldsymbol{V}_{\boldsymbol{\sigma}}^* \boldsymbol{V}_{\boldsymbol{\omega}}|_{\mathcal{E}}$ for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F_n^+}$. Using (3-1) and (3-2), we deduce that

$$F(\boldsymbol{X}) = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}_{\boldsymbol{\beta}}^*,$$

where the convergence is in the norm topology. This shows, in particular, that $F(r\boldsymbol{S})$ is convergent.

To prove the converse, assume that $\boldsymbol{V} = (V_1, \dots, V_k)$ is a k-tuple of commuting row isometries on a space $\mathcal{K} \supset \mathcal{E}$ such that

$$\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{S}_{\boldsymbol{\beta}}^* \geq 0, \quad r \in [0, 1), \tag{3-3}$$

and the convergence is in the operator norm topology. Let $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and let $r \in (0, 1)$ be such that $(1/r)\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. Since the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ is continuous in the operator norm and completely positive, so is $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$. Consequently, we obtain

$$F(\boldsymbol{X}) := (\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}) \Bigg(\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{S}_{\boldsymbol{\beta}}^* \Bigg) \geq 0, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}). \qquad \square$$

We remark that the condition (3-3) is equivalent to the condition that the kernel defined by the relation $\Gamma_{rV}(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) := r^{|\boldsymbol{\sigma}|+|\boldsymbol{\omega}|} P_{\mathcal{E}} \boldsymbol{V}_{\boldsymbol{\sigma}}^* \boldsymbol{V}_{\boldsymbol{\omega}}|_{\mathcal{E}}$ for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+$ is positive semidefinite. We should also mention that one can find a version of the theorem above when the condition $F(0) = I$ is dropped. In this case, $F(0) = A \otimes I$ with $A \geq 0$ and we set

$$G_\epsilon := [(A + \epsilon I_{\mathcal{E}})^{-1/2} \otimes I](F + \epsilon I_{\mathcal{E}} \otimes I)[(A + \epsilon I_{\mathcal{E}})^{-1/2} \otimes I], \quad \epsilon > 0.$$

Since G_ϵ is a positive k-pluriharmonic function with $G_\epsilon(0) = I$, we can apply Theorem 3.6 to get a family $\boldsymbol{V}(\epsilon) = (V_1(\epsilon), \ldots, V_k(\epsilon))$ of k-tuples of commuting row isometries on a space $\mathcal{K}_\epsilon \supset \mathcal{E}$ such that

$$F(\boldsymbol{X}) = \lim_{\epsilon \to 0} \sum_{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega} (A + \epsilon I_{\mathcal{E}})^{1/2} \big[P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^*(\epsilon) \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}(\epsilon) \big|_{\mathcal{E}} \big] (A + \epsilon I_{\mathcal{E}})^{1/2} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}_{\boldsymbol{\beta}}^*,$$

where the convergence is in the operator norm topology.

Definition 3.7. A k-tuple $\boldsymbol{V} = (V_1, \ldots, V_k)$ of commuting row isometries $V_i = (V_{i,1}, \ldots, V_{i,n_i})$ is called *pluriharmonic* if the free k-pluriharmonic Poisson kernel $\boldsymbol{\mathcal{P}}(\boldsymbol{V}, r\boldsymbol{S})$ is a positive operator for any $r \in [0, 1)$.

Proposition 3.8. *Let $\boldsymbol{V} = (V_1, \ldots, V_k)$, $V_i = (V_{i,1}, \ldots, V_{i,n_i})$, be a k-tuple of commuting row isometries. Then $\boldsymbol{V}$ is pluriharmonic in each of the following cases*:

(i) $k = 1$ *and* $n_1 \in \mathbb{N}$.

(ii) $\boldsymbol{V}$ *is doubly commuting, i.e., the C^*-algebra $C^*(V_i)$ commutes with $C^*(V_s)$ if $i, s \in \{1, \ldots, k\}$ with $i \neq s$.*

(iii) $n_1 = \cdots = n_k = 1$.

Proof. It is easy to see that $\boldsymbol{V}$ is pluriharmonic if the condition in (i) is satisfied. Under the condition (ii), the proof that $\boldsymbol{V}$ is pluriharmonic is similar to the proof of Theorem 4.2(i), when we replace the universal operator $\boldsymbol{R}$ with $\boldsymbol{V}$. Now, we assume that $n_1 = \cdots = n_k = 1$. Then $\boldsymbol{V} = (V_1, \ldots, V_k)$, where $V_1, \ldots, V_k$ are commuting isometries on a Hilbert space $\mathcal{K}$. It is well known [Sz.-Nagy et al. 2010] that there is a k-tuple $\boldsymbol{U} = (U_1, \ldots, U_k)$ of commuting unitaries on a Hilbert space $\mathcal{G} \supset \mathcal{K}$ such that $U_i|_{\mathcal{K}} = V_i$ for $i \in \{1, \ldots, k\}$. Due to Fuglede's theorem (see [Douglas 1998]), the unitaries are doubly commuting. Due to (ii), $\boldsymbol{\mathcal{P}}(\boldsymbol{U}, r\boldsymbol{S})$ is a well-defined positive operator for any $r \in [0, 1)$, where the convergence defining the free k-pluriharmonic Poisson kernel $\boldsymbol{\mathcal{P}}(\boldsymbol{U}, r\boldsymbol{S})$ is in the operator norm topology. On the other hand, we have

$$\boldsymbol{\mathcal{P}}(\boldsymbol{V}, r\boldsymbol{S}) = (P_{\mathcal{K}} \otimes I) \boldsymbol{\mathcal{P}}(\boldsymbol{U}, r\boldsymbol{S})|_{\mathcal{K} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})} \geq 0,$$

which completes the proof. ∎

Proposition 3.9. *Let $\boldsymbol{V} = (V_1, \ldots, V_k)$ be a pluriharmonic tuple of commuting row isometries on a Hilbert space $\mathcal{K}$ and let $\mathcal{E} \subset \mathcal{K}$ be a subspace. Then the map*

$$F(\boldsymbol{X}) := (P_{\mathcal{E}} \otimes I) \boldsymbol{\mathcal{P}}(\boldsymbol{V}, \boldsymbol{X})|_{\mathcal{E} \otimes \mathcal{H}}, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}),$$

is a positive free k-pluriharmonic function on the polyball $\boldsymbol{B_n}$ with operator-valued coefficients in $B(\mathcal{E})$ and $F(0) = I$.

Moreover, *in the particular cases* (i) *and* (iii) *of Proposition 3.8*, *each positive free k-pluriharmonic function F with $F(0) = I$ has the form above.*

Proof. Since $\boldsymbol{V}$ is a tuple of commuting row isometries, the free k-pluriharmonic Poisson kernel $\boldsymbol{\mathcal{P}}(\boldsymbol{V}, r\boldsymbol{S})$ is a positive operator for any $r \in [0, 1)$ and so is the compression $(P_{\mathcal{E}} \otimes I)\boldsymbol{\mathcal{P}}(\boldsymbol{V}, r\boldsymbol{S})|_{\mathcal{E}\otimes\bigotimes_{i=1}^{k} F^2(H_{n_i})}$. Let $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and let $r \in (0, 1)$ be such that $(1/r)\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. Since the noncommutative Berezin transform $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ is continuous in the operator norm and completely positive, we deduce that

$$F(\boldsymbol{X}) := (P_{\mathcal{E}} \otimes I)\boldsymbol{\mathcal{P}}(\boldsymbol{V}, \boldsymbol{X})|_{\mathcal{E}\otimes\mathcal{H}} \geq 0, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}),$$

where the convergence of $\mathcal{P}(\boldsymbol{V}, \boldsymbol{X})$ is in the operator norm topology. Therefore, F is a positive free k-pluriharmonic function on the polyball $\boldsymbol{B_n}$ with operator-valued coefficients in $B(\mathcal{E})$ and $F(0) = I$. To prove the second part of this proposition, assume that F is a positive free k-pluriharmonic function with $F(0) = I$. According to Theorem 3.6, F has the form

$$F(\boldsymbol{X}) = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Omega} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}_{\boldsymbol{\beta}}^*,$$

where $\boldsymbol{V} = (V_1, \ldots, V_k)$ is a k-tuple of commuting row isometries on a space $\mathcal{K} \supset \mathcal{E}$ and the convergence of the series is in the operator norm topology. Since in the particular cases (i) and (ii) of Proposition 3.8 $\boldsymbol{V}$ is pluriharmonic, one can easily complete the proof. □

We remark that the theorem above contains, in particular, a structure theorem for positive k-harmonic functions on the regular polydisk included in $[B(\mathcal{H})]_1 \times_c \cdots \times_c [B(\mathcal{H})]_1$, which extends the corresponding classical result on scalar polydisks [Rudin 1969]. In the general case of the polyball it is unknown if all positive free k-pluriharmonic functions F with $F(0) = I$ have the form of Proposition 3.9.

4. Berezin transforms of completely bounded maps in regular polyballs

We define a class of noncommutative Berezin transforms of completely bounded linear maps and give necessary and sufficient conditions for a function to be the Poisson transform of a completely bounded or completely positive map.

Let $\mathcal{H}$ be a Hilbert space and identify the set $M_m(B(\mathcal{H}))$ of $m \times m$ matrices with entries from $B(\mathcal{H})$ with $B(\mathcal{H}^{(m)})$, where $\mathcal{H}^{(m)}$ is the direct sum of m copies of $\mathcal{H}$. Thus we have a natural C^*-norm on $M_m(B(\mathcal{H}))$. If $\mathcal{X}$ is an operator space, i.e., a closed subspace of $B(\mathcal{H})$, we consider $M_m(\mathcal{X})$ as a subspace of $M_m(B(\mathcal{H}))$ with the induced norm. Let $\mathcal{X}$, $\mathcal{Y}$ be operator spaces and let $u : \mathcal{X} \to \mathcal{Y}$ be a linear map. Define the map $u_m : M_m(\mathcal{X}) \to M_m(\mathcal{Y})$ by $u_m([x_{ij}]) := [u(x_{ij})]$. We say that u is completely bounded if $\|u\|_{\mathrm{cb}} := \sup_{m\geq 1} \|u_m\| < \infty$. If $\|u\|_{\mathrm{cb}} \leq 1$ then u is completely contractive; if u_m is an isometry for any $m \geq 1$ then u is completely isometric; and if u_m is positive for all m then u is called completely positive. For basic results concerning completely bounded maps and operator spaces we refer to [Paulsen 1986; Pisier 2001; Effros and Ruan 2000].

Let $\mathcal{K}$ be a Hilbert space and let $\mu : B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big) \to B(\mathcal{K})$ be a completely bounded map. It is well known (see, e.g., [Paulsen 1986]) that there exists a completely bounded linear map

$$\hat{\mu} := \mu \otimes \mathrm{id} : B\bigg(\bigotimes_{i=1}^k F^2(H_{n_i})\bigg) \otimes_{\min} B(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$$

such that $\hat{\mu}(f \otimes Y) := \mu(f) \otimes Y$ for $f \in B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ and $Y \in B(\mathcal{H})$. Moreover, $\|\hat{\mu}\|_{\mathrm{cb}} = \|\mu\|_{\mathrm{cb}}$ and, if μ is completely positive, then so is $\hat{\mu}$. We introduce the *noncommutative Berezin transform* associated with μ as the map

$$\boldsymbol{\mathcal{B}}_\mu : B\bigg(\bigotimes_{i=1}^k F^2(H_{n_i})\bigg) \times \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$$

defined by

$$\boldsymbol{\mathcal{B}}_\mu(A, \boldsymbol{X}) := \hat{\mu}[C_{\boldsymbol{X}}^*(A \otimes I_{\mathcal{H}})C_{\boldsymbol{X}}], \quad A \in B\bigg(\bigotimes_{i=1}^k F^2(H_{n_i})\bigg),\ \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}),$$

where the operator $C_{\boldsymbol{X}} \in B\big(\bigotimes_{i=1}^k F^2(H_{n_i}) \otimes \mathcal{H}\big)$ is defined by

$$C_{\boldsymbol{X}} := (I_{\bigotimes_{i=1}^k F^2(H_{n_i})} \otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}) \prod_{i=1}^k (I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*)^{-1}$$

and the *defect mapping* $\boldsymbol{\Delta}_{\boldsymbol{X}} : B(\mathcal{H}) \to B(\mathcal{H})$ is given by

$$\boldsymbol{\Delta}_{\boldsymbol{X}} := (\mathrm{id} - \Phi_{X_1}) \circ \cdots \circ (\mathrm{id} - \Phi_{X_k}),$$

where $\Phi_{X_i} : B(\mathcal{H}) \to B(\mathcal{H})$ is the completely positive linear map defined by

$$\Phi_{X_i}(Y) := \sum_{j=1}^{n_i} X_{i,j} Y X_{i,j}^*, \quad Y \in B(\mathcal{H}).$$

We need to show that the operator $I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*$ is invertible. Let $\boldsymbol{Y} = (Y_1, \ldots, Y_k)$ with $Y_i := (Y_{i,1}, \ldots, Y_{i,n_i}) \in B(\mathcal{H})^{n_i}$. We introduce the spectral radius of $\boldsymbol{Y}$ by setting

$$r(\boldsymbol{Y}) := \limsup_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \bigg\| \sum_{\substack{\alpha_i \in \mathbb{F}_{n_i}^+,\ |\alpha_i| = p_i \\ i \in \{1,\ldots,k\}}} Y_{\boldsymbol{\alpha}} Y_{\boldsymbol{\alpha}}^* \bigg\|^{\frac{1}{2(p_1+\cdots+p_k)}},$$

where $Y_{\boldsymbol{\alpha}} := Y_{1,\alpha_1} \cdots Y_{k,\alpha_k}$ for $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ and $Y_{i,\alpha_i} := Y_{i,j_1} \cdots Y_{i,j_p}$ for $\alpha_i = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$. We remark that, when $k = 1$, we recover the spectral radius of an n_i-tuple of operators, i.e., $r(Y_i) = \lim_{p\to\infty} \big\| \sum_{\beta_i \in \mathbb{F}_{n_i}^+, |\beta_i| = p} Y_{i,\beta_i} Y_{i,\beta_i}^* \big\|^{1/2p}$. Note also that

$$r(Y_i) = r(\boldsymbol{R}_{i,1} \otimes Y_{i,1}^* + \cdots + \boldsymbol{R}_{i,n_i} \otimes Y_{i,n_i}^*)$$

and $r(Y_i) \le r(\boldsymbol{Y})$ for any $i \in \{1, \ldots, k\}$. Consequently, if $r(\boldsymbol{Y}) < 1$ then $r(Y_i) < 1$ and the spectrum of $\boldsymbol{R}_{i,1} \otimes Y_{i,1}^* + \cdots + \boldsymbol{R}_{i,n_i} \otimes Y_{i,n_i}^*$ is included in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. In particular, when $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$, the noncommutative von Neumann inequality [Popescu 1999] implies $r(\boldsymbol{X}) \le r(t\boldsymbol{S}) = t$ for some $t \in (0, 1)$, which proves our assertion.

Proposition 4.1. *Let $\mathcal{B}_\mu$ be the noncommutative Berezin transform associated with a completely bounded linear map $\mu : B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big) \to B(\mathcal{K})$.*

(i) *If $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ is fixed, then*

$$\mathcal{B}_\mu(\,\cdot\,, \boldsymbol{X}) : B\bigg(\bigotimes_{i=1}^k F^2(H_{n_i})\bigg) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$$

is a completely bounded linear map with $\|\mathcal{B}_\mu(\,\cdot\,, \boldsymbol{X})\|_{\mathrm{cb}} \le \|\mu\|_{\mathrm{cb}} \|C_{\boldsymbol{X}}\|^2$.

(ii) *If μ is selfadjoint, then $\mathcal{B}_\mu(A^*, \boldsymbol{X}) = \mathcal{B}_\mu(A, \boldsymbol{X})^*$. Moreover, if μ is completely positive then so is the map $\mathcal{B}_\mu(\,\cdot\,, \boldsymbol{X})$.*

(iii) *If $A \in B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ is fixed, then the map*

$$\mathcal{B}_\mu(A, \cdot\,) : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$$

is continuous and $\|\mathcal{B}_\mu(A, \boldsymbol{X})\| \le \|\mu\|_{\mathrm{cb}} \|A\| \|C_{\boldsymbol{X}}\|^2$ for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$.

Proof. Parts (i) and (ii) follow easily from the definition of the noncommutative Berezin transform associated with μ. To prove (iii), let $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{B_n}(\mathcal{H})$ and note that

$$\begin{aligned}\|\mathcal{B}_\mu(A, \boldsymbol{X}) - \mathcal{B}_\mu(A, \boldsymbol{Y})\| &\le \|\mu\| \|C_{\boldsymbol{X}}^*(A \otimes I_{\mathcal{H}})(C_{\boldsymbol{X}} - C_{\boldsymbol{Y}})\| + \|\mu\| \|(C_{\boldsymbol{X}}^* - C_{\boldsymbol{Y}}^*)(A \otimes I_{\mathcal{H}})C_{\boldsymbol{Y}}\| \\ &\le \|\mu\| \|A\| \|C_{\boldsymbol{X}} - C_{\boldsymbol{Y}}\| (\|C_{\boldsymbol{X}}\| + \|C_{\boldsymbol{Y}}\|).\end{aligned}$$

The continuity of the map $\boldsymbol{X} \mapsto \mathcal{B}_\mu(A, \boldsymbol{X})$ will follow once we prove that $\boldsymbol{X} \mapsto C_{\boldsymbol{X}}$ is a continuous map on $\boldsymbol{B_n}(\mathcal{H})$. Note that

$$\|C_{\boldsymbol{X}} - C_{\boldsymbol{Y}}\| \le \|\boldsymbol{\Delta_X}(I)^{1/2}\| \bigg\| \prod_{i=1}^k (I - \boldsymbol{R}_{X_i^*})^{-1} - \prod_{i=1}^k (I - \boldsymbol{R}_{Y_i^*})^{-1} \bigg\| + \|\boldsymbol{\Delta_X}(I)^{1/2} - \boldsymbol{\Delta_Y}(I)^{1/2}\| \bigg\| \prod_{i=1}^k (I - \boldsymbol{R}_{X_i^*})^{-1} \bigg\|,$$

where $\boldsymbol{R}_{X_i^*} := I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*$. Since the maps $\boldsymbol{X} \mapsto \prod_{i=1}^k (I - \boldsymbol{R}_{X_i^*})^{-1}$ and $\boldsymbol{X} \mapsto \boldsymbol{\Delta_X}(I)^{1/2}$ are continuous on $\boldsymbol{B_n}(\mathcal{H})$ in the operator norm topology, our assertion follows. The inequality in (iii) is obvious. $\square$

We remark that the noncommutative Poisson transform introduced in [Popescu 1999] is in fact a particular case of the noncommutative Berezin transform associated with a linear functional. Indeed, let τ be the linear functional on $B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ defined by $\tau(A) := \langle A(1), 1\rangle$. If $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ is fixed, then $\mathcal{B}_\tau(\,\cdot\,, \boldsymbol{X}) : B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big) \to B(\mathcal{H})$ is a completely contractive linear map and

$$\langle \mathcal{B}_\tau(A, \boldsymbol{X})x, y\rangle = \langle C_{\boldsymbol{X}}^*(A \otimes I_{\mathcal{H}})C_{\boldsymbol{X}}(1 \otimes x), 1 \otimes y\rangle, \quad x, y \in \mathcal{H}.$$

Hence, we have

$$\mathcal{B}_\tau(A, \boldsymbol{X}) = \boldsymbol{K}_{\boldsymbol{X}}^*(A \otimes I)\boldsymbol{K}_{\boldsymbol{X}},$$

where $\boldsymbol{K_X}$ is the noncommutative Berezin kernel at $\boldsymbol{X}$. Note also that if $A \in B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ is fixed, then $\boldsymbol{\mathcal{B}}_\tau(A, \cdot) : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{H})$ is a bounded continuous map and $\|\boldsymbol{\mathcal{B}}_\tau(A, \boldsymbol{X})\| \le \|A\|$ for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$.

We mention that, if $n_1 = \cdots = n_k = 1$, $\mathcal{H} = \mathbb{C}$ and $\boldsymbol{X} = \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \mathbb{D}^k$, then we recover the Berezin transform of a bounded linear operator on the Hardy space $H^2(\mathbb{D}^k)$, i.e.,

$$\boldsymbol{\mathcal{B}}_\tau(A, \lambda) = \prod_{i=1}^k (1 - |\lambda_i|^2)\langle A k_{\boldsymbol{\lambda}}, k_{\boldsymbol{\lambda}}\rangle, \quad A \in B(H^2(\mathbb{D}^k)),$$

where $k_\lambda(\boldsymbol{z}) := \prod_{i=1}^k (1 - \bar{\lambda}_i z_i)^{-1}$ and $\boldsymbol{z} = (z_1, \ldots, z_k) \in \mathbb{D}^k$.

Define the set

$$\Lambda := \big\{(\boldsymbol{\sigma}, \boldsymbol{\omega}) \in \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} : \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega} \text{ and } (\boldsymbol{\sigma}, \boldsymbol{\omega}) = (c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega}), c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega}))\big\}. \tag{4-1}$$

Set $\tilde{\Lambda} := \{(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) : (\boldsymbol{\sigma}, \boldsymbol{\omega}) \in \Lambda\}$ and note that

$$\tilde{\Lambda} := \big\{(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) \in \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} : \tilde{\boldsymbol{\sigma}} \sim_{\mathrm{rc}} \tilde{\boldsymbol{\omega}} \text{ and } (\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) = (c_r^+(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}), c_r^-(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}))\big\}.$$

Moreover, we have $\Lambda = \tilde{\Lambda}$. In the case $(\boldsymbol{\sigma}, \boldsymbol{\omega}) \in \Lambda$, one can easily see that $c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega}) = c_r^+(\boldsymbol{\sigma}, \boldsymbol{\omega})$ and $c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega}) = c_r^-(\boldsymbol{\sigma}, \boldsymbol{\omega})$.

In what follows, we introduce the *noncommutative Poisson transform* of a completely positive linear map on the operator system

$$\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n} := \operatorname{span}\{\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}} : \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\},$$

where $\boldsymbol{R} := (\boldsymbol{R}_1, \ldots, \boldsymbol{R}_k)$ and $\boldsymbol{R}_i := (\boldsymbol{R}_{i,1}, \ldots, \boldsymbol{R}_{i,n_i})$ is the n_i-tuple of right creation operators (see Section 1). Regard $M_m(\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n})$ as a subspace of $M_m\big(B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)\big)$. Let $M_m(\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n})$ have the norm structure that it inherits from the (unique) norm structure on the C^*-algebra $M_m\big(B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)\big)$. We remark that

$$\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n} = \operatorname{span}\{\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}} : (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda\} = \operatorname{span}\{\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}} : (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda\},$$

where $\Lambda = \tilde{\Lambda}$ is given by (4-1). If $\mu : \boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n} \to B(\mathcal{E})$ is a completely bounded linear map, then there exists a unique completely bounded linear map

$$\hat{\mu} := \mu \otimes \mathrm{id} : \overline{\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n}}^{\|\cdot\|} \otimes_{\min} B(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$$

such that

$$\hat{\mu}(A \otimes Y) = \mu(A) \otimes Y, \quad A \in \boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n},\ Y \in B(\mathcal{H}).$$

Moreover, $\|\hat{\mu}\|_{\mathrm{cb}} = \|\mu\|_{\mathrm{cb}}$ and, if μ is completely positive, then so is $\hat{\mu}$.

We define the *free pluriharmonic Poisson kernel* by setting

$$\boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X}) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\ i \in \{1, \ldots, k\} \\ |\alpha_i| = m_i^-,\ |\beta_i| = m_i^+}} \boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k} \otimes X_{1,\alpha_1} \cdots X_{k,\alpha_k} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^*$$

for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$, where the convergence is in the operator norm topology. We need to show that the latter convergence holds. Indeed, note that, for each $i \in \{1, \dots, k\}$ and $r \in [0, 1)$, we have

$$\begin{aligned} W_i &:= \sum_{m_i \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+ \\ |\alpha_i| = m_i^-, \, |\beta_i| = m_i^+}} \boldsymbol{R}_{i,\tilde{\alpha}_i}^* \boldsymbol{R}_{i,\tilde{\beta}_i} \otimes r^{|\alpha_i| + |\beta_i|} \boldsymbol{S}_{i,\alpha_i} \boldsymbol{S}_{i,\beta_i}^* \\ &= \lim_{p_i \to \infty} \Bigg(\sum_{\substack{\alpha_i \in \mathbb{F}_{n_i} \\ 0 < |\alpha_i| \le p_i}} \boldsymbol{R}_{i,\tilde{\alpha}_i}^* \otimes r^{|\alpha_i|} \boldsymbol{S}_{i,\alpha_i} + \sum_{\substack{\beta_i \in \mathbb{F}_{n_i} \\ 0 \le |\beta_i| \le p_i}} \boldsymbol{R}_{i,\tilde{\beta}_i} \otimes r^{|\beta_i|} \boldsymbol{S}_{i,\beta_i}^* \Bigg), \end{aligned}$$

where the limit is in the operator norm topology. One can easily see that

$$\begin{aligned} W_1 \cdots W_k &= \boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S}) \\ &:= \lim_{p_1 \to \infty} \cdots \lim_{p_k \to \infty} \sum_{\substack{m_1 \in \mathbb{Z} \\ |m_1| \le p_1}} \cdots \sum_{\substack{m_k \in \mathbb{Z} \\ |m_k| \le p_k}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+, \, i \in \{1, \dots, k\} \\ |\alpha_i| = m_i^-, \, |\beta_i| = m_i^+}} \boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k} \\ &\qquad \otimes r^{\sum_{i=1}^k (|\alpha_i| + |\beta_i|)} \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \\ &\qquad \times \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^*. \end{aligned}$$

Therefore, the series defining $\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S})$, i.e.,

$$\sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+, \, i \in \{1, \dots, k\} \\ |\alpha_i| = m_i^-, \, |\beta_i| = m_i^+}} \boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k} \otimes r^{\sum_{i=1}^k (|\alpha_i| + |\beta_i|)} \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^*$$

are convergent in the operator norm topology. We remark that, due to the fact that the operators $W_1, \dots, W_k$ commute, the order of the limits above is irrelevant. Fix $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and let $r \in (0, 1)$ be such that $(1/r)\boldsymbol{X}$ is in $\boldsymbol{B_n}(\mathcal{H})$. Since the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ is continuous in the operator norm, so is $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$. Consequently, applying $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ to the relation above, we deduce that

$$\begin{aligned} &(\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}})[\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S})] \\ &\quad = \lim_{p_1 \to \infty} \cdots \lim_{p_k \to \infty} \sum_{\substack{m_1 \in \mathbb{Z} \\ |m_1| \le p_1}} \cdots \sum_{\substack{m_k \in \mathbb{Z} \\ |m_k| \le p_k}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+, \, i \in \{1, \dots, k\} \\ |\alpha_i| = m_i^-, \, |\beta_i| = m_i^+}} \boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k} \\ &\qquad \otimes X_{1,\alpha_1} \cdots X_{k,\alpha_k} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^*, \end{aligned}$$

where the limits are in the operator norm topology. This proves our assertion. Now, we introduce the *noncommutative Poisson transform* of a completely bounded linear map $\mu : \boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n} \to B(\mathcal{E})$ to be the map $\boldsymbol{\mathcal{P}}\mu : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ defined by

$$(\boldsymbol{\mathcal{P}}\mu)(\boldsymbol{X}) := \hat{\mu}[\boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X})], \qquad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

The next result contains some of the basic properties of the noncommutative Poisson kernel and the noncommutative Poisson transform.

Theorem 4.2. *Let $\mu : \boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n} \to B(\mathcal{E})$ be a completely bounded linear map.*

(i) *The map* $\boldsymbol{X} \mapsto \boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X})$ *is a positive* k*-pluriharmonic function on the polyball* $\boldsymbol{B_n}$, *with coefficients in* $B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$, *and has the factorization* $\mathcal{P}(\boldsymbol{R}, \boldsymbol{X}) = C_{\boldsymbol{X}}^* C_{\boldsymbol{X}}$, *where*

$$C_{\boldsymbol{X}} := (I_{\bigotimes_{i=1}^k F^2(H_{n_i})} \otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}) \prod_{i=1}^k (I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*)^{-1}.$$

(ii) *The noncommutative Poisson transform* $\boldsymbol{\mathcal{P}}\mu$ *is a free* k*-pluriharmonic function on the regular polyball* $\boldsymbol{B_n}$ *that coincides with the Berezin transform* $\boldsymbol{\mathcal{B}}_\mu(I, \cdot)$.

(iii) *If* μ *is a completely positive linear map*, *then* $\boldsymbol{\mathcal{P}}\mu$ *is a positive free* k*-pluriharmonic function on* $\boldsymbol{B_n}$.

Proof. The fact that $\boldsymbol{X} \mapsto \boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X})$ is a free k-pluriharmonic function on the polyball $\boldsymbol{B_n}$ with coefficients in $\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n}$ was proved in the remarks preceding the theorem. Setting $\Lambda_i := \boldsymbol{R}_{i,1} \otimes r\boldsymbol{S}_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes r\boldsymbol{S}_{i,n_i}^*$ for each $i \in \{1, \dots, k\}$, we have

$$\begin{aligned} W_i &:= \sum_{m_i \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+ \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} \boldsymbol{R}_{i,\tilde{\alpha}_i}^* \boldsymbol{R}_{i,\tilde{\beta}_i} \otimes r^{|\alpha_i| + |\beta_i|} \boldsymbol{S}_{i,\alpha_i} \boldsymbol{S}_{i,\beta_i}^* \\ &= (I - \Lambda_i)^{-1} - I + (I - \Lambda_i^*)^{-1} \\ &= (I - \Lambda_i^*)^{-1} [(I - \Lambda_i) - (I - \Lambda_i^*)(I - \Lambda_i) + (I - \Lambda_i^*)](I - \Lambda_i)^{-1} \\ &= (I - \Lambda_i^*)^{-1} \left[I_{\bigotimes_{i=1}^k F^2(H_{n_i})} \otimes \left(I_{\bigotimes_{i=1}^k F^2(H_{n_i})} - \sum_{j=1}^{n_i} r^2 \boldsymbol{S}_{i,j} \boldsymbol{S}_{i,j}^* \right) \right] (I - \Lambda_i)^{-1}. \end{aligned}$$

Recall that $\boldsymbol{R}_{i,s}\boldsymbol{R}_{j,t} = \boldsymbol{R}_{j,t}\boldsymbol{R}_{i,s}$ and $\boldsymbol{R}_{i,s}\boldsymbol{R}_{j,t}^* = \boldsymbol{R}_{j,t}^*\boldsymbol{R}_{i,s}$ for any $i, j \in \{1, \dots, k\}$ with $i \neq j$ and for any $s \in \{1, \dots, n_i\}$ and $t \in \{1, \dots, n_j\}$. Similar commutation relations hold for the universal model $\boldsymbol{S}$. Since $\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S}) = W_1 \cdots W_k$ and $W_1, \dots, W_k$ are commuting positive operators, we deduce that

$$\begin{aligned} \boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S}) = \left(\prod_{i=1}^k (I - \boldsymbol{R}_{i,1}^* \otimes r\boldsymbol{S}_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes r\boldsymbol{S}_{i,n_i})^{-1} \right) \\ \times (I \otimes \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)) \prod_{i=1}^k (I - \boldsymbol{R}_{i,1} \otimes r\boldsymbol{S}_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes r\boldsymbol{S}_{i,n_i}^*)^{-1} \end{aligned}$$

for any $r \in [0, 1)$, and $\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S}) = C_{r\boldsymbol{S}}^* C_{r\boldsymbol{S}} \geq 0$. Now, let $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and let $r \in (0, 1)$ be such that $(1/r)\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. Since the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ is continuous in the operator norm and completely positive, so is $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$. Consequently, applying $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ to the relations above, we deduce that

$$\begin{aligned} \boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X}) &= (\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}})[\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S})] \\ &= \prod_{i=1}^k (I - \boldsymbol{R}_{i,1}^* \otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes X_{i,n_i})^{-1} (I \otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)) \prod_{i=1}^k (I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*)^{-1} \\ &= C_{\boldsymbol{X}}^* C_{\boldsymbol{X}}, \end{aligned}$$

which completes the proof of (i).

Using the results above and the continuity of $\hat{\mu}$ in the operator norm, we deduce that the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_\mu(I,\cdot)$ associated with μ coincides with the Poisson transform $\boldsymbol{\mathcal{P}}\mu$. Indeed, we have

$$\begin{aligned}\boldsymbol{\mathcal{B}}_\mu(I,\boldsymbol{X}) &= \hat{\mu}(C_{\boldsymbol{X}}^* C_{\boldsymbol{X}})\\ &= \sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} \mu(\boldsymbol{R}_{1,\tilde{\alpha}_1}^*\cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^*\boldsymbol{R}_{1,\tilde{\beta}_1}\cdots\boldsymbol{R}_{k,\tilde{\beta}_k})\otimes X_{1,\alpha_1}\cdots X_{k,\alpha_k}X_{1,\beta_1}^*\cdots X_{k,\beta_k}^*\\ &= \hat{\mu}(\boldsymbol{\mathcal{P}}(\boldsymbol{R},\boldsymbol{X}))\\ &= (\boldsymbol{\mathcal{P}}\mu)(\boldsymbol{X})\end{aligned}$$

for any $\boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H})$, where the convergence is in the operator norm topology of $B(\mathcal{K}\otimes\mathcal{H})$. This proves (ii). Note also that the Poisson transform $\boldsymbol{\mathcal{P}}\mu$ is a free k-pluriharmonic function on $\boldsymbol{B_n}$ with coefficients in $B(\mathcal{E})$. If μ is completely positive, then so is $\hat{\mu}$. Using the fact that $\hat{\mu}(C_{\boldsymbol{X}}^*C_{\boldsymbol{X}})=(\boldsymbol{\mathcal{P}}\mu)(\boldsymbol{X})$, we deduce (iii). $\square$

Consider the particular case when $n_1=\cdots=n_k=1$, $\mathcal{H}=\mathcal{K}=\mathbb{C}$, $\boldsymbol{X}=(X_1,\dots,X_k)$ with $X_j=r_je^{i\theta_j}\in\mathbb{D}$, and μ is a complex Borel measure on $\mathbb{T}^k$. Note that μ can be seen as a bounded linear functional on $C(\mathbb{T}^k)$. Consequently, there is a unique bounded linear functional $\hat{\mu}$ on the operator system generated by the monomials $S_1^{m_1^-}\cdots S_k^{m_k^-}S_1^{*\,m_1^+}\cdots S_k^{*\,m_k^+}$, where $m_1,\dots,m_k\in\mathbb{Z}$, and $S_1,\dots,S_k$ are the unilateral shifts acting on the Hardy space $H^2(\mathbb{T}^k)$, such that

$$\hat{\mu}(S_1^{m_1^-}\cdots S_k^{m_k^-}S_1^{*\,m_1^+}\cdots S_k^{*\,m_k^+})=\mu(e^{im_1^-\varphi_1}\cdots e^{im_k^-\varphi_k}e^{-im_1^+\varphi_1}\cdots e^{-im_k^+\varphi_k}),\quad m_1,\dots,m_k\in\mathbb{Z}.$$

Indeed, if p is any polynomial function of the form

$$p(z_1,\dots,z_k,\bar{z}_1,\dots,\bar{z}_k)=\sum a_{(m_1,\dots,m_k)}z_1^{m_1^-}\cdots z_k^{m_k^-}\bar{z}_1^{m_1^+}\cdots\bar{z}_k^{m_k^+},\quad (z_1,\dots,z_k)\in\mathbb{D}^k,$$

where $a_{(m_1,\dots,m_k)}\in\mathbb{C}$, then, due to the noncommutative von Neumann inequality [Popescu 1999], we have

$$\begin{aligned}|\hat{\mu}(p(S_1,\dots,S_k,S_1^*,\dots,S_k^*))| &= |\mu(p(e^{i\varphi_1},\dots,e^{i\varphi_k},e^{-i\varphi_1},\dots,e^{-i\varphi_k}))|\\ &\le \|\mu\|\,\|p(S_1,\dots,S_k,S_1^*,\dots,S_k^*)\|.\end{aligned}$$

Therefore, $\hat{\mu}$ is a bounded linear functional on the operator system $\operatorname{span}\{\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}^{-\|\cdot\|}$. Note that the noncommutative Poisson transform of $\hat{\mu}$, i.e., $B_{\hat{\mu}}(I,\cdot)$, coincides with the classical Poisson transform of μ. Indeed, for any $z=(r_1e^{i\theta_1},\dots,r_ke^{i\theta_k})\in\mathbb{D}^k$, we have

$$\begin{aligned}\boldsymbol{\mathcal{B}}_{\hat{\mu}}(I,z) &= \sum_{(p_1,\dots,p_k)\in\mathbb{Z}^k}\hat{\mu}(S_1^{p_1^-}\cdots p_k^{p_k^-}S_1^{*\,p_1^+}\cdots S_k^{*\,p_k^+})z_1^{p_1}\cdots z_k^{p_k}\\ &= \sum_{(p_1,\dots,p_k)\in\mathbb{Z}^k}\mu(\bar{\zeta}_1^{p_1}\cdots\bar{\zeta}^{p_k})z_1^{p_1}\cdots z_k^{p_k}\\ &= \sum_{(p_1,\dots,p_k)\in\mathbb{Z}^k}\left(\int_{\mathbb{T}^k}\bar{\zeta}_1^{p_1}\cdots\bar{\zeta}^{p_k}d\mu(\zeta)\right)z_1^{p_1}\cdots z_k^{p_k}\\ &= \int_{\mathbb{T}^k}\left(\sum_{(p_1,\dots,p_k)\in\mathbb{Z}^k}r_1^{|p_1|}\cdots r_k^{|p_k|}e^{ip_1(\theta_1-\varphi_1)}\dots e^{ip_k(\theta_k-\varphi_k)}\right)d\mu(\zeta)=\int_{\mathbb{T}^k}P(z,\zeta)\,d\mu(\zeta),\end{aligned}$$

where

$$P(z,\zeta)=P_{r_1}(\theta_1-\varphi_1)\cdots P_{r_k}(\theta_k-\varphi_k),\quad \zeta=(e^{i\varphi_1},\dots,e^{i\varphi_k})\in\mathbb{T}^k,$$

and $P_r(\theta-\varphi)=(1-r^2)/(1-2r\cos(\theta-\varphi)+r^2)$ is the Poisson kernel of the unit disc (see [Rudin 1969]).

We recall that Λ denotes the set of all pairs $(\boldsymbol{\alpha},\boldsymbol{\beta})\in\boldsymbol{F}_{\boldsymbol{n}}^+\times\boldsymbol{F}_{\boldsymbol{n}}^+$, where $\boldsymbol{F}_{\boldsymbol{n}}^+:=\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, with the property that $\boldsymbol{\alpha}\sim_{\mathrm{lc}}\boldsymbol{\beta}$ and $(\boldsymbol{\alpha},\boldsymbol{\beta})=(c_l^+(\boldsymbol{\alpha},\boldsymbol{\beta}),c_l^-(\boldsymbol{\alpha},\boldsymbol{\beta}))$. We remark that $(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda$ if and only if $(\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}})\in\Lambda$. As before, we use the notation $\tilde{\boldsymbol{\alpha}}=(\tilde{\boldsymbol{\alpha}}_1,\dots,\tilde{\boldsymbol{\alpha}}_k)$ if $\boldsymbol{\alpha}=(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_k)\in\mathbb{F}_{\boldsymbol{n}}^+$.

Throughout the rest of this section, we assume that $\mathcal{E}$ is a separable Hilbert space.

Lemma 4.3. *Let $\mu:\mathcal{R}_{\boldsymbol{n}}^*\mathcal{R}_{\boldsymbol{n}}\to B(\mathcal{E})$ be a completely bounded linear map. For each $r\in[0,1)$, define the linear map $\mu_r:\mathcal{R}_{\boldsymbol{n}}^*\mathcal{R}_{\boldsymbol{n}}\to B(\mathcal{E})$ by*

$$\mu_r(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}):=r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}\mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}),\quad(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda,$$

where $|\boldsymbol{\alpha}|:=|\alpha_1|+\cdots+|\alpha_k|$ if $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_{\boldsymbol{n}}^+$. Then

(i) *μ_r is a completely bounded linear map*;

(ii) $\|\mu\|_{\mathrm{cb}}=\sup_{0\le r<1}\|\mu_r\|_{\mathrm{cb}}=\lim_{r\to1}\|\mu_r\|_{\mathrm{cb}}$;

(iii) *$\mu_r(A)\to\mu(A)$ in the operator norm topology as $r\to1$ for any $A\in\mathcal{R}_{\boldsymbol{n}}^*\mathcal{R}_{\boldsymbol{n}}$*;

(iv) *if μ is completely positive, then so is μ_r for any $r\in[0,1)$.*

Proof. Let

$$p(\boldsymbol{R}^*,\boldsymbol{R}):=\sum_{\substack{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda'\subset\Lambda\\ \mathrm{card}(\Lambda')<\aleph_0}}a_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}},\quad a_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\in\mathbb{C},$$

and $0\le r_1<r_2\le1$. Using the noncommutative von Neumann inequality [Popescu 1999], we deduce that

$$\|\mu_{r_1}(p(\boldsymbol{R}^*,\boldsymbol{R}))\|=\|\mu(p(r_1\boldsymbol{R}^*,r_1\boldsymbol{R}))\|=\left\|\mu_{r_2}\left(p\left(\frac{r_1}{r_2}\boldsymbol{R}^*,\frac{r_1}{r_2}\boldsymbol{R}\right)\right)\right\|\le\|\mu_{r_2}\|\,\|p(\boldsymbol{R}^*,\boldsymbol{R})\|.$$

In particular, we have $\|\mu_r\|\le\|\mu\|$ for any $r\in[0,1)$. Similarly, passing to matrices over $\mathcal{R}_{\boldsymbol{n}}^*\mathcal{R}_{\boldsymbol{n}}$, one can show that $\|\mu_{r_1}\|_{\mathrm{cb}}\le\|\mu_{r_2}\|_{\mathrm{cb}}$ if $0\le r_1<r_2\le1$, and $\|\mu_r\|_{\mathrm{cb}}\le\|\mu\|_{\mathrm{cb}}$ for any $r\in[0,1)$. Now, one can easily see that $\mu_r(A)\to\mu(A)$ in the operator norm topology as $r\to1$ for any $A\in\mathcal{R}_{\boldsymbol{n}}^*\mathcal{R}_{\boldsymbol{n}}$, and $\|\mu\|_{\mathrm{cb}}=\sup_{0\le r<1}\|\mu_r\|_{\mathrm{cb}}$. Hence, and using the fact that the function $r\mapsto\|\mu_r\|_{\mathrm{cb}}$ is increasing for $r\in[0,1)$, we deduce that $\lim_{r\to1}\|\mu_r\|_{\mathrm{cb}}$ exists and it is equal to $\|\mu\|_{\mathrm{cb}}$.

To prove (iv), note that $\mu_r(p(\boldsymbol{R}^*,\boldsymbol{R}))=\mu\big(\boldsymbol{\mathcal{B}}_{r\boldsymbol{R}}[p(\boldsymbol{S}^*,\boldsymbol{S})]\big)$. Since the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_{r\boldsymbol{R}}$ and μ are completely positive linear maps and $p(\boldsymbol{R}^*,\boldsymbol{R})$ is unitarily equivalent to $p(\boldsymbol{S}^*,\boldsymbol{S})$, we deduce that μ_r is a completely positive linear map for each $r\in[0,1)$. □

Let F be a free k-pluriharmonic function on the polyball $\boldsymbol{B}_{\boldsymbol{n}}$, with operator-valued coefficients in $B(\mathcal{E})$, with representation

$$F(\boldsymbol{X})=\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\,i\in\{1,\dots,k\}\\|\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}}A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\otimes\boldsymbol{X}_{1,\alpha_1}\cdots\boldsymbol{X}_{k,\alpha_k}\boldsymbol{X}_{1,\beta_1}^*\cdots\boldsymbol{X}_{k,\beta_k}^*.$$

We associate to F and each $r \in [0, 1)$ the linear map $\nu_{F_r} : \mathcal{R}_{\boldsymbol{n}}^* \mathcal{R}_{\boldsymbol{n}} \to B(\mathcal{E})$ by setting

$$\nu_{F_r}(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) := r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}, \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda. \tag{4-2}$$

We remark that ν_{F_r} is uniquely determined by the radial function $r \mapsto F(r\boldsymbol{S})$. Indeed, note that, if $x := x_1 \otimes \cdots \otimes x_k$, $y = y_1 \otimes \cdots \otimes y_k$ satisfy (1-3) and $h, \ell \in \mathcal{E}$, we have

$$\langle F(r\boldsymbol{S})(h \otimes x), \ell \otimes y\rangle = \langle r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h, \ell\rangle = \langle \nu_{F_r}(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) h, \ell\rangle, \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda.$$

In what follows, we denote by $C^*(\boldsymbol{R})$ the C^*-algebra generated by the right creation operators $\boldsymbol{R}_{i,j}$, where $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$.

Theorem 4.4. *Let $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ be a free k-pluriharmonic function. Then the following statements are equivalent*:

(i) *There exists a completely bounded linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ such that $F = \boldsymbol{\mathcal{P}}\mu$.*

(ii) *The linear maps $\{\nu_{F_r}\}_{r\in[0,1)}$ associated with F are completely bounded and* $\sup_{0\le r<1} \|\nu_{F_r}\|_{\mathrm{cb}} < \infty$.

(iii) *There exist a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, of doubly commuting row isometries acting on a Hilbert space $\mathcal{K}$ and bounded linear operators $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ such that*

$$F(\boldsymbol{X}) = (W_1^* \otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W_2 \otimes I),$$

where

$$C_{\boldsymbol{X}}(\boldsymbol{V}) := (I \otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}) \prod_{i=1}^{k} (I - V_{i,1} \otimes X_{i,1}^* - \cdots - V_{i,n_i} \otimes X_{i,n_i}^*)^{-1}.$$

Moreover, *in this case we can choose μ such that $\|\mu\|_{\mathrm{cb}} = \sup_{0\le r<1} \|\nu_{F_r}\|_{\mathrm{cb}}$.*

Proof. Assume that (i) holds. Then

$$F(\boldsymbol{X}) = \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\} \\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} \mu(\boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k}) \otimes X_{1,\alpha_1} \cdots X_{k,\alpha_k} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^*$$

for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$, where the convergence is in the operator norm topology. Set $A_{(\boldsymbol{\alpha};\boldsymbol{\beta})} := \mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}})$ for any $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda$. Consequently, for each $r \in [0, 1)$, we have

$$\nu_{F_r}(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) := r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} \mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}), \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda.$$

We recall that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda$ if and only if $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \in \Lambda$. Applying Lemma 4.3, we deduce that $\{\nu_{F_r}\}$ is a completely bounded map and

$$\|\mu|_{\mathcal{R}_{\boldsymbol{n}}^* \mathcal{R}_{\boldsymbol{n}}}\|_{\mathrm{cb}} = \sup_{0\le r<1} \|\nu_{F_r}\|_{\mathrm{cb}} < \infty,$$

which proves that (i) implies (ii).

Now, we prove that (ii) implies (i). Assume that F is a free pluriharmonic function on $\boldsymbol{B_n}$ with coefficients in $B(\mathcal{E})$ and such that condition (ii) holds. Let $\{q_j\}$ be a countable dense subset of $\boldsymbol{\mathcal{R}_n^*\mathcal{R}_n}$. For instance, we can consider all the operators of the form

$$p(\boldsymbol{R}^*, \boldsymbol{R}) := \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda:\, |\boldsymbol{\alpha}|\le m, |\boldsymbol{\beta}|\le m} a_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}},$$

where $m \in \mathbb{N}$ and the coefficients $a_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ lie in some countable dense subset of the complex plane. For each j, we have $\|\nu_{F_r}(q_j)\| \le M\|q_j\|$ for any $r \in [0, 1)$, where $M := \sup_{0\le r<1} \|\nu_{F_r}\|_{\mathrm{cb}}$.

Due to the Banach–Alaoglu theorem [Douglas 1998], the ball $[B(\mathcal{E})]_M^-$ is compact in the w^*-topology. Since $\mathcal{E}$ is a separable Hilbert space, $[B(\mathcal{E})]_M^-$ is a metric space in the w^*-topology which coincides with the weak operator topology on $[B(\mathcal{E})]_M^-$. Consequently, the diagonal process guarantees the existence of a sequence $\{r_m\}_{m=1}^\infty$ such that $r_m \to 1$ and WOT-$\lim_{m\to 1} \nu_{F_{r_m}}(q_j)$ exists for each q_j. Fix $A \in \boldsymbol{\mathcal{R}_n^*\mathcal{R}_n}$ and $x, y \in \mathcal{E}$ and let us prove that $\{\langle \nu_{F_{r_m}}(A)x, y\rangle\}_{m=1}^\infty$ is a Cauchy sequence. Let $\epsilon > 0$ and choose q_j so that $\|q_j - A\| < \epsilon/(3M\|x\|\|y\|)$. Now we choose N so that $\left|\langle(\nu_{F_{r_m}}(q_j) - \nu_{F_{r_k}}(q_j))x, y\rangle\right| < \frac{1}{3}\epsilon$ for any $m, k > N$. Due to the fact that

$$\begin{aligned}\left|\langle(\nu_{F_{r_m}}(A)-\nu_{F_{r_k}}(A))x, y\rangle\right| &\le \left|\langle \nu_{F_{r_m}}(A-q_j)x, y\rangle\right| + \left|\langle(\nu_{F_{r_m}}(q_j)-\nu_{F_{r_k}}(q_j))x, y\rangle\right| + \left|\langle \nu_{F_{r_k}}(q_j-A)x, y\rangle\right| \\ &\le 2M\|x\|\|y\|\|A-q_j\| + \left|\langle(\nu_{F_{r_m}}(q_j)-\nu_{F_{r_k}}(q_j))x, y\rangle\right|,\end{aligned}$$

we deduce that $\left|\langle(\nu_{F_{r_m}}(A) - \nu_{F_{r_k}}(A))x, y\rangle\right| < \epsilon$ for $m, k > N$. Therefore,

$$\Phi(x, y) := \lim_{m\to\infty} \langle \nu_{F_{r_m}}(A)x, y\rangle$$

exists for any $x, y \in \mathcal{E}$ and defines a functional $\Phi : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ which is linear in the first variable and conjugate linear in the second. Moreover, we have $|\Phi(x, y)| \le M\|A\|\|x\|\|y\|$ for any $x, y \in \mathcal{E}$. Due to the Riesz representation theorem, there exists a unique bounded linear operator $B(\mathcal{E})$, which we denote by $\nu(A)$, such that $\Phi(x, y) = \langle \nu(A)x, y\rangle$ for $x, y \in \mathcal{E}$. Therefore,

$$\nu(A) = \text{WOT-}\lim_{r_m\to 1} \nu_{F_{r_m}}(A), \quad A \in \boldsymbol{\mathcal{R}_n^*\mathcal{R}_n},$$

and $\|\nu(A)\| \le M\|A\|$. Note that $\nu : \boldsymbol{\mathcal{R}_n^*\mathcal{R}_n} \to B(\mathcal{E})$ is a completely bounded map. Indeed, if $[A_{ij}]_m$ is an $m \times m$ matrix over $\boldsymbol{\mathcal{R}_n^*\mathcal{R}_n}$, then $[\nu(A_{ij})]_m =$ WOT-$\lim_{r_k\to 1}[\nu_{F_{r_k}}(A_{ij})]_m$. Hence, $\|[\nu(A_{ij})]_m\| \le M\|[A_{ij}]_m\|$ for all m, and so $\|\nu\|_{\mathrm{cb}} \le M$. Note also that $\nu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = A_{(\boldsymbol{\alpha};\boldsymbol{\beta})}$ for any $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda$, where $A_{(\boldsymbol{\alpha};\boldsymbol{\beta})}$ are the coefficients of F. By Wittstock's extension theorem [1981; 1984], there exists a completely bounded linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ which extends ν such that $\|\mu\|_{\mathrm{cb}} = \|\nu\|_{\mathrm{cb}}$. Since $F = \boldsymbol{\mathcal{P}}\mu$, the proof of (i) is complete.

Now, we prove the equivalence of (i) with (iii). If (i) holds, then according to Theorem 8.4 from [Paulsen 1986] there exist a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : C^*(\boldsymbol{R}) \to B(\mathcal{K})$, and bounded operators $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ with $\|\mu\| = \|W_1\|\|W_2\|$ such that

$$\mu(A) = W_1^*\pi(A)W_2, \quad A \in C^*(\boldsymbol{R}). \tag{4-3}$$

Set $V_{i,j} := \pi(\boldsymbol{R}_{i,j})$ for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$ and note that $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, is a k-tuple of doubly commuting row isometries. Using Theorem 4.2, one can easily see that the equality $F = \boldsymbol{\mathcal{P}}\mu$ implies the one from (iii). Now, we prove that (iii) implies (i). Since the k-tuple $\boldsymbol{V} = (V_1, \dots, V_n)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, consists of doubly commuting row isometries on a Hilbert space $\mathcal{K}$, the noncommutative von Neumann inequality [Popescu 1999] implies that the map $\pi : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ defined by

$$\pi(\boldsymbol{R}_{\boldsymbol{\alpha}} \boldsymbol{R}_{\boldsymbol{\beta}}^*) := \boldsymbol{V}_{\boldsymbol{\alpha}} \boldsymbol{V}_{\boldsymbol{\beta}}^*, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{F}_{\boldsymbol{n}}^+,$$

is a $*$-representation of $C^*(\boldsymbol{R})$. Define $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ by setting $\mu(A) := W_1^* \pi(A) W_2$, $A \in C^*(\boldsymbol{R})$, and note that μ is a completely bounded linear map. Using the relation

$$F(\boldsymbol{X}) = (W_1^* \otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W_2 \otimes I)$$

and the factorization $P(\boldsymbol{V}, \boldsymbol{X}) = C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})$ (see also Theorem 4.2), we deduce that $F(\boldsymbol{X}) = \boldsymbol{\mathcal{P}}\mu(\boldsymbol{X})$ for $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. □

We introduce the space $\mathbf{PH}^1(\boldsymbol{B_n})$ of all free k-pluriharmonic functions F on $\boldsymbol{B_n}$ such that the linear maps $\{\nu_{F_r}\}_{r \in [0,1)}$ associated with F are completely bounded and set $\|F\|_1 := \sup_{0 \le r < 1} \|\nu_{F_r}\|_{\text{cb}} < \infty$. As a consequence of Theorem 4.4, one can see that $\|\cdot\|_1$ is a norm on $\mathbf{PH}^1(\boldsymbol{B_n})$ and $(\mathbf{PH}^1(\boldsymbol{B_n}), \|\cdot\|_1)$ is a Banach space that can be identified with the Banach space $\text{CB}(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}, B(\mathcal{E}))$ of all completely bounded linear maps from $\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}$ to $B(\mathcal{E})$.

Corollary 4.5. *Let $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ be a free k-pluriharmonic function. Then the following statements are equivalent*:

(i) *There exists a completely positive linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ such that $F = \boldsymbol{\mathcal{P}}\mu$.*

(ii) *The linear maps $\{\nu_{F_r}\}_{r \in [0,1)}$ associated with F are completely positive.*

(iii) *There exist a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, of doubly commuting row isometries acting on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ and a bounded operator $W : \mathcal{E} \to \mathcal{K}$ such that*

$$F(\boldsymbol{X}) = (W^* \otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W \otimes I).$$

Proof. The proof is similar to that of Theorem 4.4. Note that for (i) implies (ii) we have to use Lemma 4.3(iv). For the converse, note that if ν_{F_r}, $r \in [0, 1)$, are completely positive linear maps then

$$\|\nu_{F_r}\|_{\text{cb}} = \|\nu_{F_r}(1)\| = \|\nu_{F_r}\| = \|A_{(\boldsymbol{g};\boldsymbol{g})}\|,$$

where $\boldsymbol{g} = (g_0^1, \dots, g_0^k)$ is the identity element in $\boldsymbol{F}_{\boldsymbol{n}}^+$. As in the proof of Theorem 4.4, we find a completely bounded map $\nu : \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}} \to B(\mathcal{E})$ such that

$$\nu(A) = \text{WOT-}\lim_{r_m \to 1} \nu_{F_{r_m}}(A), \quad A \in \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}.$$

Since ν_{F_r}, $r \in [0, 1)$, are completely positive linear maps, one can easily see that ν is completely positive. Using Arveson's extension theorem [1969], we find a completely positive map $\mu : C^*(\boldsymbol{R}) \to \mathbb{C}$ which extends ν and such that $\|\mu\|_{\text{cb}} = \|\nu\|_{\text{cb}}$. We also have that $F = \boldsymbol{\mathcal{P}}\mu$. Now, the proof that (iii) is equivalent to (i) uses Stinespring's representation theorem [1955] and is similar to the same equivalence from Theorem 4.4. We leave it to the reader. □

An open question remains. Is any positive free k-pluriharmonic function on the regular polyball $\boldsymbol{B_n}$ the Poisson transform of a completely positive linear map? The answer is positive if $k = 1$ (see [Popescu 2009]) and also when $n_1 = \cdots = n_k$ (see Section 3).

5. Herglotz–Riesz representations for free holomorphic functions with positive real parts

In this section, we introduce the noncommutative Herglotz–Riesz transform of a completely positive linear map $\mu : \boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n \to B(\mathcal{E})$ and obtain Herglotz–Riesz representation theorems for free holomorphic functions with positive real parts in regular polyballs.

Define the space

$$\mathbf{RH}(\boldsymbol{B_n}) := \operatorname{span}\{\Re f : f \in \operatorname{Hol}_{\mathcal{E}}(\boldsymbol{B_n})\},$$

where $\operatorname{Hol}_{\mathcal{E}}(\boldsymbol{B_n})$ is the set of all free holomorphic functions in the polyball $\boldsymbol{B_n}$ with coefficients in $B(\mathcal{E})$. Let $\tau : B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big) \to \mathbb{C}$ be the bounded linear functional defined by $\tau(A) = \langle A1, 1\rangle$. We remark that the radial function associated with $\varphi \in \mathbf{RH}(\boldsymbol{B_n})$, i.e., $r \mapsto \varphi(r\boldsymbol{R})$ for $r \in [0, 1)$, uniquely determines the family $\{\nu_{\varphi_r}\}_{r\in[0,1)}$ of linear maps $\nu_{\varphi_r} : \boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n \to B(\mathcal{E})$ defined by (4-2). Indeed, note that

$$\begin{aligned}\nu_{\varphi_r}(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) &:= (\mathrm{id} \otimes \tau)[(I \otimes \boldsymbol{R}_{\boldsymbol{\alpha}}^*)\varphi(r\boldsymbol{R})],\\ \nu_{\varphi_r}(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}) &:= (\mathrm{id} \otimes \tau)[\varphi(r\boldsymbol{R})(I \otimes \boldsymbol{R}_{\boldsymbol{\alpha}})],\end{aligned}$$

for any $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k) \in \boldsymbol{F}_n^+ := \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, where $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_k)$ and $\boldsymbol{R}_{\boldsymbol{\alpha}} := \boldsymbol{R}_{1,\alpha_1} \cdots \boldsymbol{R}_{k,\alpha_k}$, and $\nu_{\varphi_r}(\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}$ is different from $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_n^+$. Consider the space

$$\mathbf{RH}^1(\boldsymbol{B_n}) := \big\{\varphi \in \mathbf{RH}(\boldsymbol{B_n}) : \nu_{\varphi_r} \text{ is completely bounded and } \sup_{0\le r<1} \|\nu_{\varphi_r}\|_{\mathrm{cb}} < \infty\big\}.$$

If $\varphi \in \mathbf{RH}^1(\boldsymbol{B_n})$, we define $\|\varphi\|_1 := \sup_{0\le r<1} \|\nu_{\varphi_r}\|_{\mathrm{cb}}$. Denote by $\mathrm{CB}_0(\boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n, B(\mathcal{E}))$ the space of all completely bounded linear maps $\lambda : \boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n \to B(\mathcal{E})$ such that $\lambda(\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_n^+$.

Theorem 5.1. $(\mathbf{RH}^1(\boldsymbol{B_n}), \|\cdot\|_1)$ *is a Banach space which can be identified with the Banach space* $\mathrm{CB}_0(\boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n, B(\mathcal{E}))$. *Moreover, if* $\varphi : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ *is a function, then the following statements are equivalent*:

(i) φ *is in* $\mathbf{RH}^1(\boldsymbol{B_n})$.

(ii) *There is a unique completely bounded linear map* $\mu_\varphi \in \mathrm{CB}_0(\boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n, B(\mathcal{E}))$ *such that* $\varphi = \boldsymbol{\mathcal{P}}\mu_\varphi$.

(iii) *There exist a k-tuple* $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, *of doubly commuting row isometries on a Hilbert space* $\mathcal{K}$ *and bounded linear operators* $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ *such that*

$$\varphi(\boldsymbol{X}) = (W_1^* \otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W_2 \otimes I)$$

and $W_1^* \boldsymbol{V}_{\boldsymbol{\alpha}}^* \boldsymbol{V}_{\boldsymbol{\beta}} W_2 = 0$ *if* $\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}$ *is not equal to* $\boldsymbol{R}_{\boldsymbol{\gamma}}$ *or* $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ *for some* $\boldsymbol{\gamma} \in \boldsymbol{F}_n^+$.

Proof. Define the map $\Psi : \mathrm{CB}_0\,(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}},\, B(\mathcal{E})) \to \mathbf{RH}^1(\boldsymbol{B_n})$ by $\Psi(\mu) := \boldsymbol{\mathcal{P}}\mu$. To prove the injectivity of Ψ, let μ_1, μ_2 be in $\mathrm{CB}_0\,(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}},\, B(\mathcal{E}))$ such that $\Psi(\mu_1) = \Psi(\mu_2)$. Due to the uniqueness of the representation of a free k-pluriharmonic function and the definition of the noncommutative Poisson transform of a completely bounded map on $\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}$, we deduce that $\mu_1(\boldsymbol{R}_{\boldsymbol{\alpha}}) = \mu_2(\boldsymbol{R}_{\boldsymbol{\alpha}})$ and $\mu_1(\boldsymbol{R}_{\boldsymbol{\alpha}}^*) = \mu_2(\boldsymbol{R}_{\boldsymbol{\alpha}}^*)$ for $\boldsymbol{\alpha} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, and $\mu_1(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}) = \mu_2(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_{\boldsymbol{n}}^+$. Hence, we deduce that $\mu_1 = \mu_2$.

By Theorem 4.4, for any $\varphi \in \mathbf{RH}^1(\boldsymbol{B_n})$ there is a completely bounded linear map $\mu_\varphi \in \mathrm{CB}(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}},\, B(\mathcal{E}))$ such that $\varphi = \boldsymbol{\mathcal{P}}\mu_\varphi$ and $\|\varphi\|_1 = \|\mu_\varphi\|_{\mathrm{cb}}$. This proves that the map Ψ is surjective and $\|\boldsymbol{\mathcal{P}}\mu_\varphi\|_1 = \|\mu_\varphi\|_{\mathrm{cb}}$. Therefore, (i) is equivalent to (ii).

Now, the latter equivalence and Theorem 4.2 imply

$$\varphi(\boldsymbol{X}) = (\boldsymbol{\mathcal{P}}\mu_\varphi)(\boldsymbol{X}) = \hat{\mu}_\varphi(C_{\boldsymbol{X}}^* C_{\boldsymbol{X}}), \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}), \tag{5-1}$$

where

$$C_{\boldsymbol{X}} := (I_{\bigotimes_{i=1}^k F^2(H_{n_i})} \otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}) \prod_{i=1}^k (I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*)^{-1}.$$

Due to Wittstock's extension theorem [1984], there exists a completely bounded map $\Phi : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ that extends μ_φ with $\|\mu_\varphi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{cb}}$. According to Theorem 8.4 from [Paulsen 1986], there exist a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : C^*(\boldsymbol{R}) \to B(\mathcal{K})$, and bounded operators $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ with $\|\Phi\| = \|W_1\|\|W_2\|$ such that

$$\Phi(A) = W_1^* \pi(A) W_2, \quad A \in C^*(\boldsymbol{R}). \tag{5-2}$$

Set $V_{i,j} := \pi(\boldsymbol{R}_{i,j})$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$ and note that $\boldsymbol{V} = (V_1, \ldots, V_n)$ is a k-tuple of doubly commuting row isometries $V_i = (V_{i,1}, \ldots, V_{i,n_i})$. Using (5-1) and (5-2), one can deduce (iii). The proof that (iii) implies (i) is similar to the proof of the same implication from Theorem 4.4. ☐

Consider now the subspace of free holomorphic functions $\boldsymbol{H}^1(\boldsymbol{B_n}) := \mathrm{Hol}(\boldsymbol{B_n}) \bigcap \mathbf{PH}^1(\boldsymbol{B_n})$ together with the norm $\|\cdot\|_1$. Using Theorem 5.1, we can obtain the following weak analogue of the F. and M. Riesz theorem [Hoffman 1962] in our setting.

Corollary 5.2. $(\boldsymbol{H}^1(\boldsymbol{B_n}), \|\cdot\|_1)$ *is a Banach space which can be identified with the annihilator of* $\boldsymbol{\mathcal{R}}_n$ *in* $\mathrm{CB}_0\,(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}},\, B(\mathcal{E}))$, *i.e.*,

$$(\boldsymbol{\mathcal{R}}_n)^\perp := \{\mu \in \mathrm{CB}_0\,(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}},\, B(\mathcal{E})) : \mu(\boldsymbol{R}_{\boldsymbol{\alpha}}) = 0 \text{ for all } \boldsymbol{\alpha} \in \boldsymbol{F}_{\boldsymbol{n}}^+,\ |\boldsymbol{\alpha}| \geq 1\}.$$

Moreover, for each $f \in \boldsymbol{H}^1(\boldsymbol{B_n})$, *there is a unique completely bounded linear map* $\mu_f \in (\boldsymbol{\mathcal{R}}_n)^\perp$ *such that* $f = \boldsymbol{\mathcal{P}}\mu_f$.

Given a completely bounded linear map $\mu : \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}} \to B(\mathcal{E})$, we introduce the *noncommutative Fantappiè transform* of μ to be the map $\boldsymbol{\mathcal{F}}\mu : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ defined by

$$(\boldsymbol{\mathcal{F}}\mu)(\boldsymbol{X}) := \hat{\mu}\Bigg(\prod_{i=1}^k (I - \boldsymbol{R}_{i,1}^* \otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes X_{i,n_i})^{-1}\Bigg)$$

for $\boldsymbol{X} := (X_1, \dots, X_k) \in \boldsymbol{B_n}(\mathcal{H})$. We remark that the noncommutative Fantappiè transform is a linear map and $\boldsymbol{\mathcal{F}}\mu$ is a free holomorphic function in the open polyball $\boldsymbol{B_n}$ with coefficients in $B(\mathcal{E})$.

Let $\mu : \boldsymbol{\mathcal{R}_n^*\mathcal{R}_n} \to B(\mathcal{E})$ be a completely positive linear map. We introduce the *noncommutative Herglotz–Riesz transform* of μ on the regular polyball to be the map $\boldsymbol{H}\mu : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ defined by

$$(\boldsymbol{H}\mu)(\boldsymbol{X}) := \hat{\mu}\left(2\prod_{i=1}^k (I - \boldsymbol{R}_{i,1}^* \otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I\right)$$

for $\boldsymbol{X} := (X_1, \dots, X_k) \in \boldsymbol{B_n}(\mathcal{H})$. Note that $(\boldsymbol{H}\mu)(\boldsymbol{X}) = 2(\boldsymbol{\mathcal{F}}\mu)(\boldsymbol{X}) - \mu(I) \otimes I$.

Theorem 5.3. *Let f be a function from the polyball $\boldsymbol{B_n}(\mathcal{H})$ to $B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$. Then the following statements are equivalent*:

(i) *f is a free holomorphic function with $\Re f \geq 0$ and the linear maps $\{\nu_{\Re f_r}\}_{r\in[0,1)}$ associated with $\Re f$ are completely positive.*

(ii) *The function f admits a Herglotz–Riesz representation*

$$f(\boldsymbol{X}) = (\boldsymbol{H}\mu)(\boldsymbol{X}) + i\Im f(0),$$

where $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ is a completely positive linear map with the property that $\mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F_n^+}$.

(iii) *There exist a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, of doubly commuting row isometries on a Hilbert space $\mathcal{K}$ and a bounded linear operator $W : \mathcal{E} \to \mathcal{K}$ such that*

$$f(\boldsymbol{X}) = (W^* \otimes I)\left(2\prod_{i=1}^k (I - V_{i,1}^* \otimes X_{i,1} - \cdots - V_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I\right)(W \otimes I) + i\Im f(0)$$

and $W^*\boldsymbol{V}_{\boldsymbol{\alpha}}^*\boldsymbol{V}_{\boldsymbol{\beta}}W = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F_n^+}$.

Proof. We prove that (i) implies (ii). Let f have the representation $f(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha}\in\boldsymbol{F_n^+}} A_{(\boldsymbol{\alpha})} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}}$. Due to Corollary 4.5, there exists a completely positive linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ such that $\Re f = \boldsymbol{\mathcal{P}}\mu$. Consequently, we have

$$\mu(I) = \tfrac{1}{2}(A_{(\boldsymbol{g})} + A_{(\boldsymbol{g})}^*), \quad \mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}) = \tfrac{1}{2}A_{(\boldsymbol{\alpha})}^*, \quad \mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = \tfrac{1}{2}A_{(\boldsymbol{\alpha})} \quad \text{for all } \boldsymbol{\alpha} \in \boldsymbol{F_n^+},\ |\boldsymbol{\alpha}| \geq 1,$$

and $\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F_n^+}$. Using the definition of the Herglotz–Riesz transform, we obtain

$$\begin{aligned}
(\boldsymbol{H}\mu)(\boldsymbol{X}) &:= \hat{\mu}\left(2\prod_{i=1}^k (I - \boldsymbol{R}_{i,1}^* \otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I\right) \\
&= \sum_{\boldsymbol{\alpha}\in\boldsymbol{F_n^+}} A_{(\boldsymbol{\alpha})} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} + A_{(\boldsymbol{g})} \otimes I - \tfrac{1}{2}(A_{(\boldsymbol{g})} + A_{(\boldsymbol{g})}^*) \otimes I \\
&= f(\boldsymbol{X}) - \tfrac{1}{2}(A_{(\boldsymbol{g})}^* - A_{(\boldsymbol{g})}) \otimes I \\
&= f(\boldsymbol{X}) - i\Im f(0),
\end{aligned}$$

which proves (ii). Now we prove that (ii) implies (i). Assume that (ii) holds. Then

$$f(\boldsymbol{X}) = 2(\mathcal{F}\mu)(\boldsymbol{X}) - \mu(I)\otimes I - i\Im f(0)$$

is a free holomorphic function on the polyball $\boldsymbol{B_n}$. Taking into account that $\mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma}\in \boldsymbol{F}_{\boldsymbol{n}}^+$, and using Theorem 4.2, we deduce that

$$\begin{aligned}
&\tfrac{1}{2}(f(\boldsymbol{X}) + f(\boldsymbol{X})^*)\\
&\quad = \tfrac{1}{2}((\boldsymbol{H}\mu)(\boldsymbol{X}) + (\boldsymbol{H}\mu)(\boldsymbol{X})^*)\\
&\quad = \hat{\mu}\left(\prod_{i=1}^{k}(I - \boldsymbol{R}_{i,1}^*\otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^*\otimes X_{i,n_i})^{-1} - I + \prod_{i=1}^{k}(I - \boldsymbol{R}_{i,1}\otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i}\otimes X_{i,n_i}^*)^{-1}\right)\\
&\quad = \hat{\mu}(\mathcal{P}(\boldsymbol{R},\boldsymbol{X}))\ge 0.
\end{aligned}$$

Therefore, $\Re f$ is a free k-pluriharmonic function such that $\Re f = \mathcal{P}\mu$. Due to Corollary 4.5, we deduce that the linear maps $\{\nu_{\Re f_r}\}_{r\in[0,1)}$ associated with $\Re f$ are completely positive.

Now, we prove that (ii) implies (iii). Assume that (ii) holds. According to Stinespring's representation theorem [1955], there is a Hilbert space $\mathcal{K}$, a $*$-representation $\pi: C^*(\boldsymbol{R})\to B(\mathcal{K})$ and a bounded $W:\mathcal{E}\to\mathcal{K}$ with $\|\mu(I)\| = \|W\|^2$ such that $\mu(A) = W^*\pi(A)W$ for all $A\in C^*(\boldsymbol{R})$. Setting $V_{i,j} := \pi(\boldsymbol{R}_{ij})$ for all $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$, it is clear that the k-tuple $\boldsymbol{V} = (V_1,\dots,V_k)$, $V_i = (V_{i,1},\dots,V_{i,n_i})$, consists of doubly commuting row isometries. Note that, if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma}\in\boldsymbol{F}_{\boldsymbol{n}}^+$, then

$$W^*\boldsymbol{V}_{\boldsymbol{\alpha}}^*\boldsymbol{V}_{\boldsymbol{\beta}}W = W^*\pi(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}})W = \mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}) = 0.$$

Now, one can easily see that the relation $f(\boldsymbol{X}) = (\boldsymbol{H}\mu)(\boldsymbol{X}) + i\Im f(0)$ leads to the representation in (iii), which completes the proof.

It remains to prove that (iii) implies (ii). To this end, assume that (iii) holds. Since the k-tuple $\boldsymbol{V} = (V_1,\dots,V_k)$, $V_i = (V_{i,1},\dots,V_{i,n_i})$, consists of doubly commuting row isometries on a Hilbert space $\mathcal{K}$, the noncommutative von Neumann inequality [Popescu 1999] implies that the map $\pi: C^*(\boldsymbol{R})\to B(\mathcal{E})$ defined by

$$\pi(\boldsymbol{R}_{\boldsymbol{\alpha}}\boldsymbol{R}_{\boldsymbol{\beta}}^*) := \boldsymbol{V}_{\boldsymbol{\alpha}}\boldsymbol{V}_{\boldsymbol{\beta}}^*,\quad \boldsymbol{\alpha},\boldsymbol{\beta}\in\boldsymbol{F}_{\boldsymbol{n}}^+,$$

is a $*$-representation of $C^*(\boldsymbol{R})$. Define $\mu: C^*(\boldsymbol{R})\to B(\mathcal{E})$ by setting $\mu(A) := W^*\pi(A)W$. It is clear that μ is a completely positive linear map and (iii) implies

$$f(\boldsymbol{X}) = \hat{\mu}\left(2\prod_{i=1}^{k}(I - \boldsymbol{R}_{i,1}^*\otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^*\otimes X_{i,n_i})^{-1} - I\right) + i\Im f(0).$$

Note also that

$$\mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}) = W^*\pi(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}})W = W^*\boldsymbol{V}_{\boldsymbol{\alpha}}^*\boldsymbol{V}_{\boldsymbol{\beta}}W = 0$$

if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma}\in\boldsymbol{F}_{\boldsymbol{n}}^+$. This shows that (ii) holds. $\square$

In the particular case when $n_1 = \cdots = n_k = 1$, we obtain an operator-valued extension of Korányi–Pukánszky integral representation for holomorphic functions with positive real part on polydisks [Korányi and Pukánszky 1963].

In what follows, we say that f has a Herglotz–Riesz representation if Theorem 5.3(ii) is satisfied.

Theorem 5.4. *Let $f : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ be a function, where $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$. If f admits a Herglotz–Riesz representation, then f is a free holomorphic function with $\Re f \geq 0$.*

Conversely, if f is a free holomorphic function such that $\Re f \geq 0$, then there is a unique completely positive linear map $\mu : \boldsymbol{\mathcal{R}}_n^* + \boldsymbol{\mathcal{R}}_n \to B(\mathcal{E})$ such that

$$f(\boldsymbol{Y}) = (\boldsymbol{H}\mu)(k\boldsymbol{Y}) + i\Im f(0), \quad \boldsymbol{Y} \in \frac{1}{k}\boldsymbol{B_n}(\mathcal{H}).$$

Moreover,

$$f(\boldsymbol{X}) = 2\sum_{\boldsymbol{\alpha} \in \boldsymbol{F}_n^+} k^{|\boldsymbol{\alpha}|}\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} - \mu(I) \otimes I, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

Proof. The forward implication was proved in Theorem 5.3. We prove the converse. Assume that f has the representation

$$f(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha} \in \boldsymbol{F}_n^+} A_{(\boldsymbol{\alpha})} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}}, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}). \tag{5-3}$$

First we consider the case when $\frac{1}{2}(A_{(\boldsymbol{g})} + A_{(\boldsymbol{g})}^*) = I_{\mathcal{E}}$. Since $\Re f \geq 0$ and $\Re f(0) = I$, Theorem 3.6 shows that there is a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$ of commuting row isometries on a space $\mathcal{K} \supset \mathcal{E}$ such that

$$\Re f(\boldsymbol{X}) = \sum_{(\boldsymbol{\sigma}, \boldsymbol{\omega}) \in \boldsymbol{F}_n^+ \times \boldsymbol{F}_n^+} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}_{\boldsymbol{\beta}}^*.$$

The uniqueness of the representation of free k-pluriharmonic functions on $\boldsymbol{B_n}$ implies

$$P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} = \begin{cases} \frac{1}{2}A_{(\boldsymbol{\alpha})} & \text{if } \boldsymbol{\alpha} \in \boldsymbol{F}_n^+, \boldsymbol{\beta} = \boldsymbol{g}, \\ \frac{1}{2}A_{(\boldsymbol{\beta})}^* & \text{if } \boldsymbol{\beta} \in \boldsymbol{F}_n^+, \boldsymbol{\alpha} = \boldsymbol{g}, \\ 0 & \text{otherwise.} \end{cases} \tag{5-4}$$

Set $T_{i,j} := (1/k)V_{i,j}$ for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. According to Proposition 1.9 from [Popescu 2016], the k-tuple $\boldsymbol{T} := (T_1, \dots, T_k)$, $T_i := (T_{i,1}, \dots, T_{i,n_i})$, is in the closed polyball $\boldsymbol{B_n}(\mathcal{K})$. Using Theorem 7.2 from [Popescu 2016], we find a k-tuple $\boldsymbol{W} := (W_1, \dots, W_k)$ of doubly commuting row isometries on a Hilbert space $\mathcal{G} \supset \mathcal{K}$ such that $W_{i,j}^*|_{\mathcal{K}} = T_{i,j}^*$ for all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. Define the linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ by setting

$$\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}} \boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = P_{\mathcal{E}}\big[P_{\mathcal{K}} \boldsymbol{W}_{\tilde{\boldsymbol{\beta}}} \boldsymbol{W}_{\tilde{\boldsymbol{\alpha}}}^*|_{\mathcal{K}}\big]\big|_{\mathcal{E}}, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{F}_n^+.$$

Note that μ is a completely positive linear map with the property that $\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = (1/2k^{|\boldsymbol{\beta}|})A_{(\boldsymbol{\beta})}^*$ and $\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = (1/2k^{|\boldsymbol{\alpha}|})A_{(\boldsymbol{\alpha})}$ if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{F}_n^+$ with $\boldsymbol{\alpha} \neq \boldsymbol{g}$ and $\boldsymbol{\beta} \neq \boldsymbol{g}$, and $\mu(I) = I_{\mathcal{E}}$. Consequently, (5-3) and (5-4) imply

$$f(\boldsymbol{X}) = 2\sum_{\boldsymbol{\alpha} \in \boldsymbol{F}_n^+} k^{|\boldsymbol{\alpha}|}\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} - \mu(I) \otimes I, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

Setting $\boldsymbol{Y} := (1/k)\boldsymbol{X}$, we deduce that

$$\begin{aligned} f(\boldsymbol{Y}) &= 2\sum_{\boldsymbol{\alpha}\in \boldsymbol{F}_n^+} \mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*)\otimes k^{|\boldsymbol{\alpha}|}\boldsymbol{Y}_{\boldsymbol{\alpha}} - \mu(I)\otimes I \\ &= \hat{\mu}\left(2\prod_{i=1}^k (I-\boldsymbol{R}_{i,1}^*\otimes kY_{i,1}-\cdots-\boldsymbol{R}_{i,n_i}^*\otimes kY_{i,n_i})^{-1} - I\right) \\ &= (\boldsymbol{H}\mu)(k\boldsymbol{Y}) \end{aligned}$$

for any $\boldsymbol{Y}\in (1/k)\boldsymbol{B_n}(\mathcal{H})$, which completes the proof when $A_{(\boldsymbol{g})} = I_{\mathcal{E}}$. Now, we consider the case when $C_{(\boldsymbol{g})} := \frac{1}{2}(A_{(\boldsymbol{g})} + A_{(\boldsymbol{g})}^*) \geq 0$. For each $\epsilon > 0$, define the free holomorphic function

$$g_\epsilon := (C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}[f+\epsilon I_{\mathcal{E}}\otimes I_{\mathcal{H}}](C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}$$

and note that $\Re g_\epsilon(0) = I$. Applying the first part of the proof to g_ϵ, we find a completely positive linear map $\mu_\epsilon : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ with the property that

$$\mu_\epsilon(\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = \frac{1}{2k^{|\boldsymbol{\beta}|}}(C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}A_{(\boldsymbol{\beta})}^*(C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}$$

and

$$\mu_\epsilon(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = \frac{1}{2k^{|\boldsymbol{\alpha}|}}(C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}A_{(\boldsymbol{\alpha})}(C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}$$

if $\boldsymbol{\alpha},\boldsymbol{\beta}\in \boldsymbol{F}_n^+$ with $\boldsymbol{\alpha}\neq \boldsymbol{g}$ and $\boldsymbol{\beta}\neq\boldsymbol{g}$, and $\mu_\epsilon(I) = I_{\mathcal{E}}$. Setting $\nu_\epsilon(\xi) := (C_{(\boldsymbol{g})}+\epsilon I)^{1/2}\mu_\epsilon(\xi)(C_{(\boldsymbol{g})}+\epsilon I)^{1/2}$ for all $\xi\in C^*(\boldsymbol{R})$, one can easily see that ν_ϵ is a completely positive linear map with the property that $\nu_\epsilon(\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = (1/2k^{|\boldsymbol{\beta}|})A_{(\boldsymbol{\beta})}^*$ and $\nu_\epsilon(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = (1/2k^{|\boldsymbol{\alpha}|})A_{(\boldsymbol{\alpha})}$ if $\boldsymbol{\alpha},\boldsymbol{\beta}\in\boldsymbol{F}_n^+$ with $\boldsymbol{\alpha}\neq\boldsymbol{g}$ and $\boldsymbol{\beta}\neq\boldsymbol{g}$, and $\nu_\epsilon(I) = C_{(\boldsymbol{g})}+\epsilon I_{\mathcal{E}}$. Following the proofs of Theorem 4.4 and Corollary 4.5, we find a completely positive linear map $\nu : C^*(\boldsymbol{R})\to B(\mathcal{E})$ such that $\nu(\xi) = \text{WOT-}\lim_{\epsilon_k\to 0}\nu_{\epsilon_k}(\xi)$ for $\xi\in C^*(\boldsymbol{R})$, where $\{\epsilon_k\}$ is a sequence of positive numbers converging to zero. Consequently, we have $\nu(\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = (1/2k^{|\boldsymbol{\beta}|})A_{(\boldsymbol{\beta})}^*$ and $\nu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = (1/2k^{|\boldsymbol{\alpha}|})A_{(\boldsymbol{\alpha})}$ if $\boldsymbol{\alpha},\boldsymbol{\beta}\in\boldsymbol{F}_n^+$ with $\boldsymbol{\alpha}\neq\boldsymbol{g}$ and $\boldsymbol{\beta}\neq\boldsymbol{g}$, and $\nu(I) = C_{(\boldsymbol{g})}$. As in the first part of this proof, one can easily see that

$$f(\boldsymbol{Y}) = (\boldsymbol{H}\nu)(k\boldsymbol{Y}) + i\Im f(0), \quad \boldsymbol{Y}\in \frac{1}{k}\boldsymbol{B_n}(\mathcal{H}),$$

and

$$f(\boldsymbol{X}) = 2\sum_{\boldsymbol{\alpha}\in\boldsymbol{F}_n^+} k^{|\boldsymbol{\alpha}|}\nu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*)\otimes \boldsymbol{X}_{\boldsymbol{\alpha}} - \nu(I)\otimes I, \quad \boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H}). \qquad \square$$

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