ANALYSIS & PDE

Volume 9 No. 5 2016

VINCENT BRUNEAU AND NICOLAS POPOFF

ON THE NEGATIVE SPECTRUM OF THE ROBIN LAPLACIAN IN CORNER DOMAINS

msp

ANALYSIS AND PDE
Vol. 9, No. 5, 2016
dx.doi.org/10.2140/apde.2016.9.1259

ON THE NEGATIVE SPECTRUM OF THE ROBIN LAPLACIAN IN CORNER DOMAINS

VINCENT BRUNEAU AND NICOLAS POPOFF

Dedicated to Monique Dauge on the occasion of her 60th birthday

For a bounded corner domain Ω, we consider the attractive Robin Laplacian in Ω with large Robin parameter. Exploiting multiscale analysis and a recursive procedure, we have a precise description of the mechanism giving the bottom of the spectrum. It allows also the study of the bottom of the essential spectrum on the associated tangent structures given by cones. Then we obtain the asymptotic behavior of the principal eigenvalue for this singular limit in any dimension, with remainder estimates. The same method works for the Schrödinger operator in $\mathbb{R}^n$ with a strong attractive δ-interaction supported on $\partial\Omega$. Applications to some Ehrling-type estimates and the analysis of the critical temperature of some superconductors are also provided.

1. Introduction

1A. ***Context: Robin Laplacian with large parameter.*** Let M be a Riemannian manifold of dimension n without boundary and Ω an open domain of M (in practice one may think $M = \mathbb{R}^n$ or $M = \mathbb{S}^n$). We are interested in the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{on } \Omega, \\ \partial_\nu u - \alpha u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here $\alpha \in \mathbb{R}$ is the Robin parameter and ∂_ν denotes the outward normal to the boundary of Ω. We assume that Ω belongs to a general class of corner domains defined recursively, such as in [Dauge 1988]. This class of corner domains of M, precisely defined in Section 2, consists of open bounded sets $\Omega \subset M$ such that each point in $\partial\Omega$ can be associated with a *tangent cone*. We ask the sections of these tangent cones

MSC2010: 35J10, 35P15, 47F05, 81Q10.
Keywords: Robin Laplacian, eigenvalues estimates, corner domains.

to satisfy the same property, that is, as open sets of $\mathbb{S}^{n-1}$ to themselves be corner domains. The corner domains of $\mathbb{S}^0$ being its nonempty subsets, this leads to a natural recursive definition of corner domains; see [Dauge 1988; Bonnaillie-Noël et al. 2016a, Section 3] for a more complete description and examples. Note that these domains include various possible geometries, like regular domains, polyhedra and circular cones.

We denote by $\mathcal{Q}_\alpha[\Omega]$ the quadratic form of the Robin Laplacian on Ω with parameter α:

$$\mathcal{Q}_\alpha[\Omega](u) := \|\nabla u\|^2_{L^2(\Omega)} - \alpha\|u\|^2_{L^2(\partial\Omega)}, \quad u \in H^1(\Omega). \tag{2}$$

Since Ω is bounded and is the finite union of Lipschitz domains (see [Dauge 1988, Lemma AA.9]), the trace injection from $H^1(\Omega)$ into $L^2(\partial\Omega)$ is compact and the quadratic form $\mathcal{Q}_\alpha[\Omega]$ is lower semibounded. We define $L_\alpha[\Omega]$, its self-adjoint extension, whose spectrum is a sequence of eigenvalues, and we denote by $\lambda(\Omega, \alpha)$ the first one. It is the principal eigenvalue of the system (1).

The study of the spectrum of $L_\alpha[\Omega]$ has received some attention in the past years, in particular for the singular limit $\alpha \to +\infty$. This problem appeared first in a model of reaction diffusion for which the absorption mechanism competes with a boundary term [Lacey et al. 1998], and more recently it was established that the understanding of $\lambda(\Omega, \alpha)$ provides information on the critical temperature of surface superconductivity under zero magnetic field [Giorgi and Smits 2007]. Let us mention that such models are also used in the quantum Hall effect and topological insulators to justify the appearance of edge states (see [Asorey et al. 2015]).

In view of the quadratic form, it is not difficult to see that $\lambda(\Omega, \alpha) \to -\infty$ as $\alpha \to +\infty$ (while in the limit $\alpha \to -\infty$ they converge to those of the Dirichlet Laplacian). When $\Omega \subset \mathbb{R}^n$ is smooth, $\lambda(\Omega, \alpha) \le -\alpha^2$ for all $\alpha \ge 0$; see [Giorgi and Smits 2007, Theorem 2.1]. More precisely, it is known that $\lambda(\Omega, \alpha) \sim C_\Omega \alpha^2$ as $\alpha \to +\infty$ for some particular domains: for smooth domains, $C_\Omega = -1$ (see [Lacey et al. 1998; Lou and Zhu 2004] and [Daners and Kennedy 2010] for higher eigenvalues), and, for planar polygonal domains with corners of opening $(\theta_k)_{k=1,\dots,N}$,

$$C_\Omega = -\max_{0<\theta_k<\pi}\bigl(1, \sin^{-2}\tfrac{1}{2}\theta_k\bigr).$$

This last formula, conjectured in [Lacey et al. 1998], is proved in [Levitin and Parnovski 2008]. For general domains Ω having a piecewise smooth boundary it is natural to study the operator on tangent spaces and, from homogeneity reasons (see Lemma 3.2), one expects that $\lambda(\Omega, \alpha) \sim C_\Omega \alpha^2$ when $\alpha \to +\infty$, with some negative constant C_Ω. Levitin and Parnovski [2008] consider domains with corners satisfying the uniform interior cone condition. For each $x \in \partial\Omega$, they introduce $E(\Pi_x)$, the bottom of the spectrum of the Robin Laplacian on an infinite model cone Π_x (if x is a regular point, it is a half-space) and show

$$\lim_{\alpha\to+\infty} \frac{\lambda(\Omega, \alpha)}{\alpha^2} = \inf_{x\in\partial\Omega} E(\Pi_x). \tag{3}$$

But we have no guarantee concerning the finiteness of $E(\Pi_x)$ and, moreover, even if it is finite, we don't know if their infimum over $\partial\Omega$ is reached. Then an important question is to understand more precisely the influence of the geometry (of the boundary) of Ω in the asymptotic behavior of $\lambda(\Omega, \alpha)$ in order to give meaning to (3) (in particular proving that $\inf_{x\in\partial\Omega} E(\Pi_x)$ is finite) and, if possible, to obtain some remainder estimates.

1B. ***Local energies on admissible corner domains.*** In this article, our purpose is to develop a framework in the study of such asymptotics by introducing the *local energy* function $x \mapsto E(\Pi_x)$ on the recursive class of corner domains (see [Dauge 1988]). The natural tangent structures are given by dilation-invariant domains, more succinctly referred as *cones*. When the domain is a convenient cone Π, the quadratic form in (2) may still be defined on $H^1(\Pi)$. By immediate scaling, $\mathcal{Q}_\alpha[\Pi]$ is unitarily equivalent to $\alpha^2\mathcal{Q}_1[\Pi]$. Therefore the case where the parameter is equal to 1 plays an important role and we write $\mathcal{Q}[\Pi] = \mathcal{Q}_1[\Pi]$. For a general cone, we don't know whether $\mathcal{Q}[\Pi]$ is lower semibounded, and we define

$$E(\Pi) = \inf_{\substack{u \in H^1(\Pi) \\ u \neq 0}} \frac{\mathcal{Q}[\Pi](u)}{\|u\|^2},$$

the ground state energy of the Robin Laplacian on Π. For $x \in \overline{\Omega}$, denote by Π_x the tangent cone at x. When Π_x is the full space (corresponding to interior points), there is no boundary and $E(\Pi_x) = 0$, whereas, when Π_x is a half-space (corresponding to regular points of the boundary), it is easy to see that $E(\Pi_x) = E(\mathbb{R}_+) = -1$ (see [Daners and Kennedy 2010]). Moreover, when Π_x is an infinite planar sector $\mathcal{S}_\theta$ of opening θ, $E(\Pi_x)$ is given by

$$E(\mathcal{S}_\theta) = \begin{cases} -\sin^{-2}\frac{1}{2}\theta & \text{if } \theta \in (0, \pi), \\ -1 & \text{if } \theta \in (\pi, 2\pi); \end{cases} \tag{4}$$

see [Lacey et al. 1998; Levitin and Parnovski 2008]. No such explicit expressions are available for general cones in higher dimensions. In view of (3), we introduce the infimum of local energy $E(\Pi_x)$ for $x \in \overline{\Omega}$, which, from the above remarks, is also the infimum on the boundary:

$$\mathscr{E}(\Omega) := \inf_{x \in \partial\Omega} E(\Pi_x). \tag{5}$$

Our goal is to prove the finiteness of $\mathscr{E}(\Omega)$ (and firstly of $E(\Pi_x)$ for $x \in \overline{\Omega}$) for admissible corner domains and to give an estimate of $\lambda(\Omega, \alpha) - \alpha^2\mathscr{E}(\Omega)$ for α large. In view of the above particular cases, the local energy is clearly discontinuous (even for smooth domains it is piecewise constant with values in $\{0, -1\}$). We will use a recursive procedure in order to prove the finiteness and the lower semicontinuity of the local energy in the general case. It relies also on a multiscale analysis to get an estimate of the first eigenvalue, as developed in [Bonnaillie-Noël et al. 2016a] for the semiclassical magnetic Laplacian. Unlike [Bonnaillie-Noël et al. 2016a], where the complexity of model problems limits the study to dimension 3, for the Robin Laplacian we have a good understanding of the ground state energy on corner domains in any dimension. Moreover these techniques allow an analog spectral study of the Schrödinger operator with δ-interaction supported on closed corner hypersurfaces and on conical surfaces.

1C. ***Results for the Robin Laplacian.*** We define below generic notions associated with cones.

Definition 1.1. A cone Π is a domain of $\mathbb{R}^n$ which is dilation invariant:

$$\rho x \in \Pi \quad \text{for all } x \in \Pi,\ \rho > 0.$$

The section of a cone Π is $\Pi \cap \mathbb{S}^{n-1}$, generically denoted by ω. We say that two cones Π_1 and Π_2 are equivalent, and we write $\Pi_1 \equiv \Pi_2$, if they can be deduced one from another by a rotation. Given a cone Π, there exists $d \in \mathbb{N}$ with $0 \le d \le n$ such that

$$\Pi \equiv \mathbb{R}^{n-d} \times \Gamma, \quad \text{with } \Gamma \text{ a cone in } \mathbb{R}^d.$$

When d is minimal for such an equivalence, we say that Γ is the reduced cone of Π. When $d = n$, so that $\Pi = \Gamma$, we say that Π is irreducible.

In the following, $\mathfrak{P}_n$ denotes the class of admissible cones of $\mathbb{R}^n$ and $\mathfrak{D}(M)$ denotes the class of admissible corner domains on a given Riemannian manifold M without boundary. We refer to Section 2 for precise definitions of these classes of domains.

Theorem 1.2. *Let $\Pi \in \mathfrak{P}_n$ be an admissible cone.*

(1) *$E(\Pi) > -\infty$ and the Robin Laplacian $L[\Pi]$ is well defined as the Friedrichs extension of $Q[\Pi]$ with form domain $D(Q[\Pi]) = H^1(\Pi)$.*

(2) *Let Γ be the reduced cone of Π. Then the bottom of the essential spectrum of $L[\Gamma]$ is $\mathscr{E}(\omega)$, where ω is the section of Γ.*

This theorem generalizes to cones having no regular section the result of [Pankrashkin 2016], where the bottom of the essential spectrum is proved to be -1 for cones with regular section (as discussed at the end of Section 1A, in this case $\mathscr{E}(\omega) = -1$).

The crucial point of this theorem is to show that the Robin Laplacian on a cone, far from the origin, can be linked to the Robin Laplacian on the section of the cone, with a parameter related to the distance to the origin.

Notice that this theorem provides an effective procedure to compute the bottom of the essential spectrum for Laplacians on cones. In particular, as shown by Remark 6.4, we obtain that [Levitin and Parnovski 2008, Theorem 3.5] is incorrect in dimension $n \ge 3$; indeed, we construct a cone which contains an hyperplane passing through the origin for which the bottom of the essential spectrum (then of the spectrum) of the Robin Laplacian is below -1.

The next step is to minimize the local energy on a corner domain Ω and to prove that $\mathscr{E}(\Omega)$ is finite. Thanks to Theorem 1.2, we will be able to prove some monotonicity properties (on singular chains; see Section 2B for the definition), which, combined with continuity of the local energy (for the topology of singular chains), allow us to apply [Bonnaillie-Noël et al. 2016a, Section 3] and to obtain:

Theorem 1.3. *For any corner domain $\Omega \in \mathfrak{D}(M)$, the energy function $x \mapsto E(\Pi_x)$ is lower semicontinuous on $\overline{\Omega}$ and we have $\mathscr{E}(\Omega) > -\infty$.*

To get asymptotics of $\lambda(\Omega, \alpha)$ with control of the remainders, we need to control error terms when using change of variables and cut-off functions. However, the principal curvatures of the regular part of a corner domain may be unbounded in dimension $n \ge 3$ (think of a circular cone), so the standard estimates when using approximation of metrics may blow up. We use a multiscale analysis to overcome this difficulty and we get the following result:

Theorem 1.4. *Let $\Omega \in \mathfrak{D}(M)$ with $n \geq 2$ the dimension of M. Then there exists $\alpha_0 \in \mathbb{R}$, two constants $C^{\pm} > 0$ and two integers $0 \leq \overline{\nu} \leq \overline{\nu}_+ \leq n-2$ such that*

$$-C^- \alpha^{2-2/(2\overline{\nu}_+ +3)} \leq \lambda(\Omega, \alpha) - \alpha^2 \mathscr{E}(\Omega) \leq C^+ \alpha^{2-2/(2\overline{\nu}+3)} \quad \textit{for all } \alpha \geq \alpha_0.$$

The constant $\overline{\nu}$ corresponds to the degree of degeneracy of the curvatures near the minimizers of the local energy; its precise definition can be found in (29). The constant $\overline{\nu}_+$ describes the degeneracy of the curvatures globally in $\overline{\Omega}$; see Lemma 4.1. In particular, when Ω is polyhedral (that is, a domain with bounded curvatures on the regular part), $\overline{\nu} = \overline{\nu}_+ = 0$.

The proof of the lower bound relies on a multiscale partition of the unity where the size of the balls optimizes the error terms. The upper bound is less classical: using the concept of *singular chain*, we isolate a tangent "subreduced cone" for which the bottom of the spectrum corresponds to an isolated eigenvalue (below the essential spectrum). Then we construct recursive quasimodes, coming from this tangent "subreduced cone".

Note finally that for regular domains more precise asymptotics involving the mean curvature can be found ([Pankrashkin 2013; Helffer and Kachmar 2014] in dimension 2 and [Pankrashkin and Popoff 2015; 2016] for higher dimensions). A precise analysis is also done for particular polygonal geometries: the tunneling effect in some symmetry cases [Helffer and Pankrashkin 2015], and reduction to the boundary when the domain is the exterior of a convex polygon [Pankrashkin 2015]. In all these cases, the local energy is piecewise constant, and new geometric criteria appear near the set of minimizers. In fact, the local energy can be seen as a potential in the standard theory of the harmonic approximation [Dimassi and Sjöstrand 1999] and, under additional hypotheses on the local energy, it is reasonable to expect more precise asymptotics in higher dimensions. For polygons (dimension 2), another approach would consist in comparing the limit problem to a problem on a graph, in the spirit of [Grieser 2008], the nodes (resp. edges) corresponding to the vertices (resp. sides) of the polygons. But it is not clear how such an approach could be generalized to any dimension.

1D. ***Applications of the method for the Schrödinger operator with δ-interaction.*** Let $\Omega \in \mathfrak{D}(M)$ be a corner domain and let $S = \partial\Omega$ be its boundary. We consider $L^{\delta}_{\alpha}[M, S]$, the self-adjoint extension associated with the quadratic form

$$\mathcal{Q}^{\delta}_{\alpha}[M, S](u) := \|\nabla u\|^2_{L^2(M)} - \alpha \|u\|^2_{L^2(S)}, \quad u \in H^1(M).$$

The associated boundary problem is the Laplacian with the derivative jump condition across the closed hypersurface S: $[\partial_\nu u]_{\partial\Omega} = \alpha u$. It is well known (see, e.g., [Brasche et al. 1994]) that, since S is bounded, $L^{\delta}_{\alpha}[\mathbb{R}^n, S]$ is a relatively compact perturbation of $L_0 = -\Delta$ on $L^2(\mathbb{R}^n)$, and then

$$\sigma_{\text{ess}}(L^{\delta}_{\alpha}[\mathbb{R}^n, S]) = \sigma_{\text{ess}}(L_0) = [0, +\infty).$$

Moreover, $L^{\delta}_{\alpha}[\mathbb{R}^n, S]$ has a finite number of negative eigenvalues. If we denote by $\lambda^{\delta}(S, \alpha)$ the lowest one, by applying our techniques developed for the Robin Laplacian all the above results are still valid, replacing $\lambda(\Omega, \alpha)$ by $\lambda^{\delta}(S, \alpha)$. In particular, for $x \in S$, the tangent cone to Ω at x is Π_x and its boundary is denoted by S_x. We still define the tangent operator as $L_1^\delta[\mathbb{R}^n, S_x]$, and the associated local energy $E^\delta(S_x)$ at x, and their infimum $\mathscr{E}^\delta(S)$. Then:

Theorem 1.5. *Theorems 1.2–1.4 remain valid when replacing the Robin Laplacian $L_\alpha[\Omega]$ by the δ-interaction Laplacians $L_\alpha^\delta[M, S]$, $\lambda(\Omega, \alpha)$ by $\lambda^\delta(S, \alpha)$, $E(\Pi_x)$ by $E^\delta(S_x)$ and $\mathscr{E}(\Omega)$ by $\mathscr{E}^\delta(S)$.*

When x belongs to the regular part of S, S_x is an hyperplane and

$$E^\delta(\mathbb{R}^n, S_x) = E^\delta(\mathbb{R}, \{0\}) = -\tfrac{1}{4}; \tag{6}$$

see [Exner and Yoshitomi 2002]. Therefore $\mathscr{E}^\delta(S) = -\frac{1}{4}$ when S is regular, and we obtain the known main term of the asymptotic expansion of $\lambda^\delta(S, \alpha)$ proved in dimension 2 or 3 (see [Exner and Yoshitomi 2002; Exner and Pankrashkin 2014; Dittrich et al. 2016]).

To our best knowledge the only studies for δ-interactions supported on nonsmooth hypersurfaces are for broken lines and conical domains with circular section (see [Behrndt et al. 2014; Duchêne and Raymond 2014; Exner and Kondej 2015; Lotoreichik and Ourmières-Bonafos 2015]). In that case, we clearly have $\sigma(L_\alpha^\delta[\mathbb{R}^n, S]) = \alpha^2\sigma(L_1^\delta[\mathbb{R}^n, S])$ (see Lemma 3.2), and it is proved in the above references that the bottom of the essential spectrum of $L^\delta[\mathbb{R}^n, S]$ is $-\frac{1}{4}$. In view of our result, it remains true when the section of the conical surface is smooth. Moreover, our work seems to be the first result giving the main asymptotic behavior of $\lambda^\delta(S, \alpha)$ for interactions supported by general closed hypersurfaces with corners.

Remark 1.6. For the Robin Laplacian and the δ-interaction Laplacian, we can add a smooth positive weight function G in the boundary conditions. These conditions become, for the Robin condition, $\partial_\nu u = \alpha G(x)u$, and, for the δ-interaction case, $[\partial_\nu u] = \alpha G(x)u$. In our analysis, for $x \in \partial\Omega$ fixed, we have only to change α into $\alpha G(x)$ and, clearly, the results are still true by replacing $\mathscr{E}(\Omega)$ and $\mathscr{E}^\delta(S)$ by

$$\mathscr{E}_G(\Omega) := \inf_{x\in\partial\Omega} G(x)^2 E(\Pi_x), \quad \mathscr{E}_G^\delta(S) := \inf_{x\in S} G(x)^2 E^\delta(S_x).$$

For the Robin Laplacian, these cases were already considered in [Levitin and Parnovski 2008; Colorado and García-Melián 2011].

1E. ***Organization of the article.*** In Section 2, we recall the definitions of corner domains, in the spirit of [Dauge 1988; Maz′ja and Plamenevskiĭ 1977], and we give some properties proved in [Bonnaillie-Noël et al. 2016a]. Section 3 is devoted to the effects of perturbations on the quadratic form of the Robin Laplacian. It contains several technical lemmas used in the following sections.

Section 4 contains the proof of the lower bound of Theorem 1.4. It is based on a multiscale analysis in order to counterbalance the possible blow-up of curvatures in corner domains. In particular it involves the lower bound $\liminf_{\alpha\to+\infty} \lambda(\Omega, \alpha)/\alpha^2 \geq \mathscr{E}(\Omega)$ in any dimension, which is also used in Sections 5 and 6. Notice that in Section 4, at this stage of the analysis, the quantity $\mathscr{E}(\Omega)$ is still not known to be finite; its finiteness will be the recursive hypothesis of the next two sections.

Section 5 is a step in a recursive proof of Theorem 1.3 developed in Section 6. Then, when the finiteness of $\mathscr{E}(\Omega)$ is stated, Theorem 1.2 is a direct consequence of Lemmas 5.2 and 5.3 (see the end of Section 6A).

In Section 7, we prove the upper bound of Theorem 1.4. This is done by exploiting the results of Section 6 in order to find a tangent problem that admits an eigenfunction associated with $\mathscr{E}(\Omega)$. Then we construct recursive quasimodes, qualified either as *sitting* or *sliding*, from the language of [Bonnaillie-Noël et al. 2016a].

In Section 8 we give two possible applications of our results. A purely mathematical one concerns optimal estimates in compact injections of Sobolev spaces. In the second one we recall how, from the study of $\lambda(\Omega, \alpha)$, we derive properties on the critical temperature for zero fields for systems with enhanced surface superconductivity (where α^{-1} is related to the penetration depth).

2. Corner domains

Here we give some background of so-called admissible corner domains; see [Dauge 1988; Bonnaillie-Noël et al. 2016a].

2A. ***Tangent cones and recursive class of corner domains.*** Let M be a Riemannian manifold without boundary. We define recursively the class of admissible corner domains $\mathfrak{D}(M)$ and admissible cones $\mathfrak{P}_n$, in the spirit of [Dauge 1988]:

Initialization: $\mathfrak{P}_0$ has one element, $\{0\}$. $\mathfrak{D}(\mathbb{S}^0)$ is formed by all nonempty subsets of $\mathbb{S}^0$.

Recurrence: For $n \geq 1$,

(1) a cone Π (see Definition 1.1) belongs to $\mathfrak{P}_n$ if and only if the section of Π belongs to $\mathfrak{D}(\mathbb{S}^{n-1})$,

(2) $\Omega \in \mathfrak{D}(M)$ if and only if Ω is bounded and, for any $x \in \overline{\Omega}$, there exists a tangent cone $\Pi_x \in \mathfrak{P}_n$ to Ω at x.

By definition, Π_x is the tangent cone to Ω at $x \in \overline{\Omega}$ if there exists a local map $\psi_x : \mathcal{U}_x \mapsto \mathcal{V}_x$, where $\mathcal{U}_x$ and $\mathcal{V}_x$ are neighborhoods (called *map-neighborhoods*) of x in M and of 0 in $\mathbb{R}^n$, respectively, and ψ_x is a diffeomorphism such that

$$\psi_x(x) = 0, \quad (\mathrm{d}\psi_x)(x) = \mathbb{I}, \quad \psi_x(\mathcal{U}_x \cap \Omega) = \mathcal{V}_x \cap \Pi_x \quad \text{and} \quad \psi_x(\mathcal{U}_x \cap \partial\Omega) = \mathcal{V}_x \cap \partial\Pi_x. \tag{7}$$

In dimension 2, cones are half-planes, sectors and the full plane. The corner domains are (curvilinear) polygons on M with a finite number of vertices, each one of opening in $(0, \pi) \cup (\pi, 2\pi)$. This includes, of course, regular domains.

The key quantity in order to estimate errors when making a change of variables is

$$\kappa(x) = \|\mathrm{d}\psi\|_{W^{1,\infty}(\mathcal{U}_x)}. \tag{8}$$

It depends not only on x, but also on the choice of the local map. Note that, unlike for a regular domain, the curvature of the regular part of a corner domain may be unbounded (think of a circular cone). Therefore, $\kappa(x)$ is not bounded in general when picking an atlas of $\overline{\Omega}$. An important subclass of corner domains are those who are *polyhedral*: a cone is said to be polyhedral if its boundary is contained in a finite union of hyperplanes, and a domain is called polyhedral if all its tangent cones are polyhedral.

As proven in [Bonnaillie-Noël et al. 2016a], for a polyhedral domain it is possible to find an atlas such that κ is bounded. In the general case, we will have to control the possible blow-up of κ.

A list of examples can be found in [Bonnaillie-Noël et al. 2016a, Section 3.1]. Let us recall that, in dimension 2, all cones are polyhedral and therefore so are all corner domains, but this is not true anymore when $n \geq 3$: circular cones are typical examples of cones which are not polyhedral.

2B. ***Singular chains.*** For $x_0 \in \overline{\Omega}$, we denote by $\Gamma_{x_0} \in \mathfrak{P}_{d_0}$ the reduced cone of Π_{x_0} — see Definition 1.1 — and ω_{x_0} the section of Γ_{x_0}. A singular chain $\mathbb{X} = (x_0, \dots, x_p)$ is a sequence of points, with $x_0 \in \overline{\Omega}$, $x_1 \in \overline{\omega}_{x_0}$, and so on. We denote by $\mathfrak{C}(\Omega)$ the set of singular chains (in Ω), $\mathfrak{C}_{x_0}(\Omega)$ the set of chains initiated at x_0 and $\mathfrak{C}^*_{x_0}(\Omega)$ the set of $\mathbb{X} \in \mathfrak{C}_{x_0}(\Omega)$ such that $\mathbb{X} \neq (x_0)$. We denote by $l(\mathbb{X})$ the integer $p+1$ that is the length of the chain. Note that $1 \leq l(\mathbb{X}) \leq n+1$, and that $l(\mathbb{X}) \geq 2$ when $\mathbb{X} \in \mathfrak{C}^*_{x_0}(\Omega)$.

With a chain $\mathbb{X}$ is canonically associated a cone, denoted by $\Pi_{\mathbb{X}}$, called a tangent structure:

- If $\mathbb{X} = (x_0)$, then $\Pi_{\mathbb{X}} = \Pi_{x_0}$.
- If $\mathbb{X} = (x_0, x_1)$, write as above, in some adapted coordinates, $\Pi_{x_0} = \mathbb{R}^{n-d_0} \times \Gamma_{x_0}$. Let C_{x_1} be the tangent cone to ω_{x_0} at x_1. Then, in the adapted coordinates, $\Pi_{\mathbb{X}} = \mathbb{R}^{n-d_0} \times \langle x_1 \rangle \times C_{x_1}$, where $\langle x_1 \rangle$ is the vector space spanned by x_1 in Γ_{x_0}.
- And so on for longer chains.

We refer to [Bonnaillie-Noël et al. 2016a, Section 3.4] for complete definitions. Since singular chains are one of the tools of our analysis, we provide below some examples for a better understanding. In these examples, we assume for simplicity that Π_{x_0} is irreducible.

- If $x_1 \in \Pi_{x_0}$ (an interior point), then $\Pi_{(x_0,x_1)}$ is the full space.
- If x_1 is in the regular part of the boundary of ω_{x_0}, then C_{x_1} is a half-space of $\mathbb{R}^{n-1}$ and $\Pi_{(x_0,x_1)}$ is a half-space of $\mathbb{R}^n$. In particular, for a cone with regular section, all chains of length 2 are associated either with a half-space or the full space. The chains of length 3 are associated with the full space, and there are no longer chains.
- If $\Pi_{x_0} \subset \mathbb{R}^3$ is such that its section is a polygon and if x_1 is one of its vertices, then C_{x_1} is a two-dimensional sector, and $\Pi_{(0,x_1)}$ is a wedge. If x_2 is on the boundary of the sector C_{x_1}, then $\Pi_{(x_0,x_1,x_2)}$ is a half-space, but, if x_2 is on the interior of the sector, then $\Pi_{(0,x_1,x_2)} = \mathbb{R}^3$.

Given a cone $\Pi \in \mathfrak{P}_n$, we will also consider chains of Π, for example chains in $\mathfrak{C}_0(\Pi)$ are of the form $(0, x_1, \dots)$, where x_1 belongs to the closure of the section of the reduced cone of Π.

The main idea is to consider the local energy as a function not only defined on $\overline{\Omega}$, but also on singular chains: $\mathfrak{C}(\Omega) \ni \mathbb{X} \mapsto E(\Pi_{\mathbb{X}})$. In order to show regularity properties of this function, we define a partial order on singular chains: we say that $\mathbb{X} \leq \mathbb{X}'$ if $l(\mathbb{X}) \leq l(\mathbb{X}')$ and $x_k = x'_k$ for all $k \leq l(\mathbb{X})$. We also define a distance between cones through the action of isomorphisms:

$$\mathbb{D}(\Pi, \Pi') = \frac{1}{2}\Big\{ \min_{\substack{L \in \mathsf{BGL}_n \\ L\Pi = \Pi'}} \|L - \mathbb{I}_n\| + \min_{\substack{L \in \mathsf{BGL}_n \\ L\Pi' = \Pi}} \|L - \mathbb{I}_n\| \Big\}, \tag{9}$$

where BGL_n is the ring of linear isomorphisms L of $\mathbb{R}^n$ with norm $\|L\| \le 1$. Note that by definition the distance between two cones is $+\infty$ if they do not belong to the same orbit for the action of BGL_n on $\mathfrak{P}_n$.

We then define the natural distance, inherited on $\mathfrak{C}(\Omega)$, by $\mathbb{D}(\mathbb{X}, \mathbb{X}') = \|x_0 - x_0'\| + \mathbb{D}(\Pi_{\mathbb{X}}, \Pi_{\mathbb{X}'})$; see [Bonnaillie-Noël et al. 2016a, Definition 3.22]. Then [Bonnaillie-Noël et al. 2016a, Theorem 3.25] states that any function $F : \mathfrak{C}(\Omega) \to \mathbb{R}$, monotonous and continuous with respect to $\mathbb{D}$, is lower semicontinuous when restricted to $\overline{\Omega}$ (which corresponds to chains of length 1). We will show these two criteria; see Corollaries 6.2 and 6.3.

3. Change of variables and perturbation of the metric

This section contains mainly technical lemmas, which are useful in the following sections. We define the operator with metric and we show the influence of a change of variables from a corner domains toward tangent cones on the quadratic form.

3A. ***Change of variables and operator with metrics.*** We need to know how a change of variables transforms the quadratic form of the Robin Laplacian. Indeed, we will consider diffeomorphisms $\psi : \mathcal{O} \to \mathcal{O}'$, where $\mathcal{O}$ and $\mathcal{O}'$ are open sets, in these two situations:

- $\mathcal{O}$ and $\mathcal{O}'$ will be cones in $\mathfrak{P}_n$ and ψ will be a linear map on $\mathbb{R}^n$, or
- $\mathcal{O}$ and $\mathcal{O}'$ will be map-neighborhoods, respectively of a point in a closure of a corner domain and of 0 in the associated tangent cone.

This change of variables will induce a regular metric $\mathrm{G} : \mathcal{O}' \to \mathrm{GL}_n$. In the case where ψ is linear, G will be constant.

Let $L^2_{\mathrm{G}}(\mathcal{O}')$ be the space of the square-integrable functions for the weight $|\mathrm{G}|^{-1/2}$, endowed with its natural norm $\|v\|_{L^2_G} := \int_{\mathcal{O}'} |v|^2 |\mathrm{G}|^{-1/2}$. Due to the previous hypotheses, $L^2_{\mathrm{G}}(\mathcal{O}') = L^2(\mathcal{O}')$. Let $g = \mathrm{G}|_{\partial\mathcal{O}'}$ be the restriction of the metric to the boundary. We introduce the quadratic form

$$\mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}](v) = \int_{\mathcal{O}'} \langle \mathrm{G}\nabla v, \nabla v\rangle |\mathrm{G}|^{-1/2} - \alpha \int_{\partial\mathcal{O}'} |v|^2 |g|^{-1/2}.$$

Due to the above hypotheses on $\mathcal{O}'$ and G, we can define this quadratic form on $H^1(\mathcal{O}')$, endowed with the weighted norm $\|\cdot\|_{L^2_G}$.

Lemma 3.1. *Let $\mathcal{O}$ and $\mathcal{O}'$ be open sets and $\psi : \mathcal{O} \mapsto \mathcal{O}'$ a diffeomorphism as above. Let* $\mathrm{J} := \mathrm{d}(\psi^{-1})$ *be the Jacobian of ψ^{-1} and* $\mathrm{G} := \mathrm{J}^{-1}(\mathrm{J}^{-1})^\top$ *the associated metric. Then, for all $u \in H^1(\mathcal{O})$,*

$$\mathcal{Q}_\alpha[\mathcal{O}](u) = \mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}](u \circ \psi^{-1}) \quad \textit{and} \quad \|u\|_{L^2(\mathcal{O})} = \|u \circ \psi^{-1}\|_{L^2_G(\mathcal{O}')}.$$

Said differently, if we define $\mathrm{U} : u \mapsto u \circ \psi^{-1}$, then U is an isometry from $L^2(\mathcal{O})$ onto $L^2_G(\mathcal{O}')$, and $\mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}]\mathrm{U} = \mathcal{Q}_\alpha[\mathcal{O}]$. We will also use scaling on cones:

Lemma 3.2. *Let Π be a cone and $u \in H^1(\Pi)$. For $\alpha > 0$, we define $u_\alpha(x) := \alpha^{-n/2} u(x/\alpha)$. Then*

$$\|u_\alpha\|_{L^2} = \|u\|_{L^2} \quad \textit{and} \quad \mathcal{Q}_\alpha[\Pi](u) = \alpha^2 \mathcal{Q}[\Pi](u_\alpha).$$

In particular, $\mathcal{Q}_\alpha[\Pi]$ and $\alpha^2\mathcal{Q}[\Pi]$ are unitarily equivalent.

3B. *Approximation of metrics.* We will be led to consider situations where $\mathrm{J}-\mathbb{I}$ is small (and so is $\mathrm{G}-\mathbb{I}$). Therefore, for $v \in H^1(\mathcal{O}')$, we compute

$$\mathcal{Q}_\alpha[\mathcal{O}',\mathrm{G}](v)-\mathcal{Q}_\alpha[\mathcal{O}'](v)=\int_{\mathcal{O}'}\langle(\mathrm{G}-\mathbb{I})\nabla v,\nabla v\rangle|\mathrm{G}|^{-1/2}+\int_{\mathcal{O}'}|\nabla v|^2(|\mathrm{G}|^{-1/2}-1)+\alpha\int_{\partial\mathcal{O}'}|v|^2(|g|^{-1/2}-1)$$

and therefore

$$\begin{aligned}&\bigl|\mathcal{Q}_\alpha[\mathcal{O}',\mathrm{G}](v)-\mathcal{Q}_\alpha[\mathcal{O}'](v)\bigr|\\ &\qquad\le \bigl(\|\mathrm{G}-\mathbb{I}\|_{L^\infty_v}(\||\mathrm{G}|^{-1/2}-1\|_{L^\infty_v}+1)+\||\mathrm{G}|^{-1/2}-\mathbb{I}\|_{L^\infty_v}\bigr)\|\nabla v\|^2_{L^2}+\alpha\||g|^{-1/2}-1\|_{L^\infty_v}\|v\|_{L^2(\partial\mathcal{O}')},\end{aligned}$$

where $\|\cdot\|_{L^\infty_v}$ denotes the L^∞ norm on $\operatorname{supp} v$. Assume now that $\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}\le 1$; then there exists a universal constant $C>0$ such that

$$\bigl|\mathcal{Q}_\alpha[\mathcal{O}',\mathrm{G}](v)-\mathcal{Q}_\alpha[\mathcal{O}'](v)\bigr|\le C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}(\|\nabla v\|^2_{L^2}+\alpha\|v\|_{L^2(\partial\mathcal{O}')}). \tag{10}$$

This may be written as

$$\begin{aligned}&(1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|\nabla v\|^2_{L^2}-\alpha(1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|v\|_{L^2(\partial\mathcal{O}')}\\ &\qquad\le\mathcal{Q}_\alpha[\mathcal{O}',\mathrm{G}](v)\le(1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|\nabla v\|^2_{L^2}-\alpha(1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|v\|_{L^2(\partial\mathcal{O}')}\end{aligned}$$

That is, for $\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}$ small enough:

$$\begin{aligned}&(1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\biggl(\|\nabla v\|^2_{L^2}-\alpha\frac{1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}}{1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}}\|v\|_{L^2(\partial\mathcal{O}')}\biggr)\\ &\qquad\le\mathcal{Q}_\alpha[\mathcal{O}',\mathrm{G}](v)\le(1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\biggl(\|\nabla v\|^2_{L^2}-\alpha\frac{1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}}{1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}}\|v\|_{L^2(\partial\mathcal{O}')}\biggr).\end{aligned} \tag{11}$$

Similarly, we have a norm approximation: assuming that $\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}\le 1$,

$$(1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|v\|_{L^2}\le\|v\|_{L^2_G}\le(1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|v\|_{L^2}\quad\text{for all }\ v\in L^2(\mathcal{O}'). \tag{12}$$

By applying the previous inequality to tangent geometries with a constant metric, we will deduce the continuity of the local energy on strata in Section 6A.

3C. *Functions with small support.* The following lemma compares the quadratic form with a metric to the one without metric for functions concentrated near the origin of a tangent cone:

Lemma 3.3. *Let $\Omega\in\mathfrak{D}(M)$, let $x_0\in\overline{\Omega}$, and let $\psi_{x_0}:\mathcal{U}_{x_0}\to\mathcal{V}_{x_0}$ be a map-neighborhood of x_0. Let* G *be the associated metric, defined in Lemma 3.1. Then there exist universal positive constants c and C such that, for all $r\in(0,c/\kappa(x_0))$ with $\mathcal{B}(0,r)\subset\mathcal{V}_{x_0}$, and all $v\in H^1(\Pi_{x_0})$ compactly supported in $\mathcal{B}(0,r)$,*

$$(1-Cr\kappa(x_0))\mathcal{Q}_{\alpha^-}[\Pi_{x_0}](v)\le\mathcal{Q}_\alpha[\Pi_{x_0},\mathrm{G}](v)\le(1+Cr\kappa(x_0))\mathcal{Q}_{\alpha^+}[\Pi_{x_0}](v), \tag{13}$$

where

$$\alpha^\pm(r,x_0)=\alpha\frac{1\mp Cr\kappa(x_0)}{1\pm Cr\kappa(x_0)} \tag{14}$$

and

$$\bigl|\|v\|_{L^2}-\|v\|_{L^2_G}\bigr|\le Cr\kappa(x_0)\|v\|_{L^2}.$$

Here $\kappa(x)$ is as defined in (8).

Proof. Let J be the Jacobian of $\psi_{x_0}^{-1}$. Since v is supported in a ball $\mathcal{B}(0,r)$ and $\mathrm{J}(0)=\mathbb{I}$, by the direct Taylor inequality we get $\|\mathrm{J}-\mathbb{I}\|_{L^\infty(\mathcal{B}(0,r))} \le r\|\mathrm{J}\|_{W^{1,\infty}(\mathcal{O})} = r\kappa(x_0)$. We use (10), and we follow the same steps leading to (11) and (12). □

Remark 3.4. When the quadratic forms are negative, the above inequality implies

$$\mathcal{Q}_{\alpha^-}[\Pi_{x_0}](v) \le \mathcal{Q}_\alpha[\Pi_{x_0},\mathrm{G}](v) \le \mathcal{Q}_{\alpha^+}[\Pi_{x_0}](v). \tag{15}$$

The following lemma will be useful when studying the essential spectrum of tangent operators:

Lemma 3.5. *Let $\Omega \in \mathfrak{D}(M)$ and choose $x_0 \in \overline{\Omega}$ such that $E(\Pi_{x_0})$ is finite. Let $\mathcal{U}_{x_0}$ be a map-neighborhood of x_0. Then*

$$\limsup_{\alpha\to+\infty} \inf_{\substack{u\in H^1(\Omega),\ \|u\|=1\\ \operatorname{supp} u\subset\mathcal{U}_{x_0}}} \alpha^{-2}\mathcal{Q}_\alpha[\Omega](u) \le E(\Pi_{x_0}).$$

This property is still true if $\Omega \in \mathfrak{P}_n$.

Proof. Obviously, $E(\Pi_{x_0})<0$. Let $\epsilon>0$ be such that $E(\Pi_{x_0})+\epsilon<0$. Note that

$$\frac{E(\Pi_{x_0})+\epsilon}{E(\Pi_{x_0})+\frac{1}{2}\epsilon} \in (0,1). \tag{16}$$

The functions in $H^1(\Pi_{x_0})$ with compact support are dense in $H^1(\Pi_{x_0})$, therefore there exists $v_\epsilon \in H^1(\Pi_{x_0})$ with compact support such that $\|v_\epsilon\|=1$ and $\mathcal{Q}[\Pi_{x_0}](v_\epsilon) < E(\Pi_{x_0})+\frac{1}{2}\epsilon$. Let $\mathcal{V}_{x_0}=\psi_{x_0}(\mathcal{U}_{x_0})$; we choose $r>0$ such that

$$\mathcal{B}(0,r)\subset\mathcal{V}_{x_0} \quad \text{and} \quad r\le\frac{c}{\kappa(x_0)}, \tag{17a}$$

$$\left(\frac{1-Cr\kappa(x_0)}{1+Cr\kappa(x_0)}\right)^2\left(E(\Pi_{x_0})+\frac{\epsilon}{2}\right) < E(\Pi_{x_0})+\epsilon. \tag{17b}$$

Conditions (17a) will allow us to apply Lemma 3.3. Note that (17b) is possible because of (16). The reason for this last condition will appear later. The value $\alpha^+=\alpha^+(x_0,r)$ is well defined in (14). The (normalized) test function

$$v_{\epsilon,\alpha^+}(x) := (\alpha^+)^{n/2} v_\epsilon(\alpha^+ x)$$

satisfies

$$\mathcal{Q}_{\alpha^+}[\Pi_{x_0}](v_{\epsilon,\alpha^+}) = (\alpha^+)^2\mathcal{Q}[\Pi_{x_0}](v_\epsilon) \tag{18}$$

(see Lemma 3.2) and its support is

$$\operatorname{supp} v_{\epsilon,\alpha^+} = (\alpha^+)^{-1}\operatorname{supp} v_\epsilon.$$

Therefore there exists α large enough such that

$$\operatorname{supp} v_{\epsilon,\alpha^+} \subset \mathcal{B}(0,r), \tag{19}$$

so we can apply Lemma 3.3. Therefore, by combining (18) with estimates (13), we get

$$\begin{aligned}\mathcal{Q}_\alpha[\Pi_{x_0}, \mathrm{G}](v_{\epsilon,\alpha^+}) &\leq (1+cr\kappa(x_0))\mathcal{Q}_{\alpha^+}[\Pi_{x_0}](v_{\epsilon,\alpha^+})\\ &= (1+cr\kappa(x_0))(\alpha^+)^2\mathcal{Q}[\Pi_{x_0}](v_\epsilon)\\ &\leq (1+cr\kappa(x_0))(\alpha^+)^2\big(E(\Pi_{x_0})+\tfrac{1}{2}\epsilon\big).\end{aligned}$$

Due to (17a) and (19), we can define

$$u_{\epsilon,\alpha} := v_{\epsilon,\alpha^+}\circ\psi_{x_0}^{-1},$$

with $\operatorname{supp} u_{\epsilon,\alpha}\subset\mathcal{U}_{x_0}$, and Lemma 3.1 gives $\mathcal{Q}_\alpha[\Omega](u_{\epsilon,\alpha}) = \mathcal{Q}_\alpha[\Pi_{x_0}, G](v_{\epsilon,\alpha})$. Moreover, $\|u_{\epsilon,\alpha}\|^2 = \|v\|^2_{L^2_G}\leq 1+Cr\kappa(x_0)$; therefore, keeping in mind that for ϵ small enough $E(\Pi_{x_0})+\frac{1}{2}\epsilon<0$, we get

$$\frac{\mathcal{Q}_\alpha[\Omega](u_{\epsilon,\alpha})}{\|u_{\epsilon,\alpha}\|^2}\leq(\alpha^+)^2\Big(E(\Pi_{x_0})+\frac{\epsilon}{2}\Big)=\Big(\frac{1-Cr\kappa(x_0)}{1+Cr\kappa(x_0)}\Big)^2\alpha^2\Big(E(\Pi_{x_0})+\frac{\epsilon}{2}\Big).$$

Setting $u = u_{\epsilon,\alpha}/\|u_{\epsilon,\alpha}\|$ and using (17b), we have proved

$$\mathcal{Q}_\alpha[\Omega](u)\leq E(\Pi_{x_0})+\epsilon$$

and we get the lemma. Since, locally, a cone of $\mathfrak{P}_n$ satisfies the same properties as a corner domain, the above proof works when Ω is a cone. ☐

Remark 3.6. As a direct consequence of the previous lemma, the min-max principle would provide a rough upper bound for $\limsup_{\alpha\to+\infty}\lambda(\alpha,\Omega)/\alpha^2$ by $\mathscr{E}(\Omega)$. But, at this stage, we still don't know whether $\mathscr{E}(\Omega)$ is finite or not when Ω is an n-dimensional corner domain.

4. Lower bound: multiscale partition of the unity

In this section, we prove the lower bound of Theorem 1.4 for any domain $\Omega\in\mathfrak{D}(M)$. We note at this point that this lower bound has interest only when $\mathscr{E}(\Omega)>-\infty$, which is not proved yet.

It relies on a multiscale partition of the unity of the domain by balls. Near each of these balls, we will perform a change of variables toward the tangent cone at the center of the ball, and we will estimate the remainder. However, the curvature of the boundary near each center of a ball may be large as this one is close to a conical point. We will counterbalance this effect by choosing balls of radius smaller with regard to the distances to conical points.

The following lemma is a consequence of [Bonnaillie-Noël et al. 2016a, Section 3.4.4 and Lemma B.1]:

Lemma 4.1. *Let $\Omega\in\mathfrak{D}(M)$ and let $\overline{\nu}_+$ be the smallest integer satisfying*

$$l(\mathbb{X})\geq\overline{\nu}_+\implies\Pi_{\mathbb{X}}\text{ is polyhedral}\qquad\text{for all }\mathbb{X}\in\mathfrak{C}(\Omega).$$

For each sequence of scales $(\delta_k)_{0\leq k\leq\overline{\nu}_+}$ in $(0,+\infty)$ there exists $h_0>0$, an integer $L>0$ and a constant $c(\Omega)>0$ such that, for all $h\in(0,h_0)$, there exists an h-dependent finite set of points $\mathcal{P}\subset\overline{\Omega}$ such that, for all $p\in\mathcal{P}$, there exists $0\leq k\leq\overline{\nu}_+$ such that:

- *The ball $\mathcal{B}(p, 2h^{\delta_0+\dots+\delta_k})$ is contained in a map-neighborhood of p.*
- *The curvature associated with this map-neighborhood* (*defined by* (8)) *satisfies*

$$\kappa(p) \le \frac{c(\Omega)}{h^{\delta_0+\dots+\delta_{k-1}}}.$$

- $\overline{\Omega} \subset \bigcup_{p\in\mathcal{P}} \mathcal{B}(p, h^{\delta_0+\dots+\delta_k})$, *and each point of* $\overline{\Omega}$ *belongs to at most* L *of these balls.*

We will need the standard IMS formula;[1] see for example [Simon 1983, Lemma 3.1]:

Lemma 4.2. *Let* $\chi_1, \dots, \chi_N \in \mathcal{C}^\infty(\overline{\Omega})$ *be such that* $\sum_{l=1}^N \chi_l^2 = 1$. *Then*

$$\|\nabla u\|^2 = \sum_{l=1}^N \|\nabla(\chi_l u)\|^2 - \sum_{l=1}^N \|u\nabla\chi_l\|^2 \quad \textit{for all } u \in H^1(\Omega).$$

We set $h = \alpha^{-1}$ and we now choose a partition of unity $(\chi_p)_{p\in\mathcal{P}}$ associated with the balls provided by the previous lemma; each χ_p is C^∞ and is supported in the ball $\mathcal{B}(p, 2\alpha^{-(\delta_0+\dots+\delta_k)})$, and

$$\begin{cases} \sum_{p\in\mathcal{P}} \chi_p^2 = 1 & \text{on } \overline{\Omega}, \\ \sum_{p\in\mathcal{P}} \|\nabla\chi_p\|_\infty^2 \le C(\Omega)\alpha^{2\delta} & \text{with } \delta = \delta_0 + \cdots + \delta_{\bar{\nu}_+}. \end{cases} \tag{20}$$

We apply Lemma 4.2 together with the uniform estimates of gradients (20):

$$\mathcal{Q}_\alpha[\Omega](u) = \sum_{p\in\mathcal{P}} \mathcal{Q}_\alpha[\Omega](\chi_p u) - \sum_{p\in\mathcal{P}} \|u\nabla\chi_p\|^2 \ge \sum_{p\in\mathcal{P}} \mathcal{Q}_\alpha[\Omega](\chi_p u) - C(\Omega)\alpha^{2\delta}\|u\|^2.$$

Therefore we are left with the task of estimating $\mathcal{Q}_\alpha[\Omega](\chi_p u)$ from below for each $p \in \mathcal{P}$. Let ψ_p be a local map on $\mathcal{B}(p, 2\alpha^{-(\delta_0+\dots+\delta_k)})$ and $v_p := (\chi_p u) \circ \psi_p^{-1}$. Let G_p be the associated metric. Then we deduce from Lemmas 3.1 and 3.3 that (recall that the quadratic forms are negative)

$$\begin{aligned}
\frac{\mathcal{Q}_\alpha[\Omega](\chi_p u)}{\|\chi_p u\|^2} &= \frac{\mathcal{Q}_\alpha[\Pi_p, \mathrm{G}_p](v_p)}{\|v_p\|_{\mathrm{G}_p}^2} \\
&\ge (1 + C\alpha^{-(\delta_0+\dots+\delta_k)}\kappa(p)) \frac{\mathcal{Q}_{\alpha^-}[\Pi_p](v_p)}{\|v_p\|^2} \\
&\ge (1 + C\alpha^{-(\delta_0+\dots+\delta_k)}\kappa(p))(\alpha^-)^2 E(\Pi_c) \ge (1 + C'\alpha^{-(\delta_0+\dots+\delta_k)}\kappa(p))\alpha^2\mathscr{E}(\Omega) \\
&= \alpha^2\mathscr{E}(\Omega) + O(\alpha^{2-\delta_k}),
\end{aligned}$$

where we have used Lemma 4.1 to control $\kappa(p)$.

Lemma 4.2 provides

$$\mathcal{Q}_\alpha[\Omega](u) \ge \left(\alpha^2\mathscr{E}(\Omega) + \sum_{k=0}^{\bar{\nu}_+} O(\alpha^{2-\delta_k}) + O(\alpha^{2\delta})\right)\|u\|^2 \quad \text{for all } u \in H^1(\Omega).$$

Recall that $\delta = \sum_{k=0}^{\bar{\nu}_+} \delta_k$; these remainders are optimized by choosing $\delta_0 = \cdots = \delta_{\bar{\nu}_+}$ and $2 - \delta_0 = 2\delta = 2(\bar{\nu}_+ + 1)\delta_0$, that is, $\delta_0 = 2/(2\bar{\nu}_+ + 3)$. We deduce from the min-max principle that there exists $\alpha_0 \in \mathbb{R}$

[1] IMS stands for *Ismagilov, Morgan and Simon.*

and $C^{-} > 0$ such that

$$\lambda(\Omega, \alpha) \geq \alpha^2 \mathscr{E}(\Omega) - C^{-} \alpha^{2-2/(2\overline{\nu}_{+}+3)} \quad \text{for all } \alpha \geq \alpha_0, \tag{21}$$

which is the lower bound of Theorem 1.4.

5. Tangent operator

In this section we describe the Robin Laplacian on a cone Π, linking some parts of its spectrum with its section ω.

5A. *Semiboundedness of the operator on tangent cones.*

Lemma 5.1. *Let $\Pi \in \mathfrak{P}_n$ and let ω be its section. Let $R \geq 0$, and let $u \in H^1(\Pi)$ with support in $\mathcal{B}(0, R)^{\complement}$.*[2] *Then*

$$\mathcal{Q}[\Pi](u) \geq \left(\inf_{r>R} \frac{\lambda(\omega, r)}{r^2} \right) \|u\|^2_{L^2(\Pi)}.$$

Proof. Let $\varphi : (r, \theta) \mapsto r\theta$ be the change of variables from $\mathbb{R}_+ \times \omega$ into Π and denote by $v(r, \theta) := u \circ \varphi^{-1}$ the function associated with the change of variables. We have

$$\|\nabla u\|^2_{L^2(\Pi)} = \int_{r>R} \left(|\partial_r v|^2 + \frac{1}{r^2} \|\nabla_\theta v(r, \cdot)\|^2_{L^2(\omega)} \right) r^{n-1} \, \mathrm{d}r;$$

therefore,

$$\begin{aligned}
\mathcal{Q}[\Pi](u) &\geq \int_{r>R} \frac{1}{r^2} \|\nabla_\theta v(r, \cdot)\|^2_{L^2(\omega)} r^{n-1} \, \mathrm{d}r - \int_{r>R} \|v(r, \cdot)\|^2_{L^2(\partial\omega)} r^{n-2} \, \mathrm{d}r \\
&= \int_{r>R} \frac{1}{r^2} \mathcal{Q}_r[\omega](v(r, \cdot)) r^{n-1} \, \mathrm{d}r \geq \int_{r>R} \frac{1}{r^2} \lambda(\omega, r) \|v(r, \cdot)\|^2_{L^2(\omega)} r^{n-1} \, \mathrm{d}r \\
&\geq \inf_{r>R} \frac{\lambda(\omega, r)}{r^2} \int_{r>R} \|v(r, \cdot)\|^2_{L^2(\omega)} r^{n-1} \, \mathrm{d}r
\end{aligned}$$

and the lemma follows. □

We now prove the following:

Lemma 5.2. *Let $\Pi \in \mathfrak{P}_n$ be such that its section ω satisfies $\mathscr{E}(\omega) > -\infty$. Then $E(\Pi) > -\infty$ and the Robin Laplacian $L[\Pi]$ is well defined as the Friedrichs extension of $Q[\Pi]$ with form domain $D(Q[\Pi]) = H^1(\Pi)$.*

Proof. Since $\mathscr{E}(\omega)$ is supposed to be finite, (21) implies

$$\liminf_{r \to +\infty} \frac{\lambda(\omega, r)}{r^2} \geq \mathscr{E}(\omega). \tag{22}$$

Let χ_1 and χ_2 be two regular cut-off functions defined on $\mathbb{R}_+$ such that $\operatorname{supp} \chi_1 \subset [0, 2R)$, $\chi_1 = 1$ on $[0, R]$ and $\chi_1^2 + \chi_2^2 = 1$. Lemma 4.2 provides

$$\mathcal{Q}[\Pi](u) = \sum_{i=1,2} \mathcal{Q}[\Pi](\chi_i u) - \sum_{i=1,2} \|\nabla \chi_i u\|^2. \tag{23}$$

[2] $R = 0$ is included, with $\mathcal{B}(0, 0) = \varnothing$.

Denote by D_0^R the set of functions in $H^1(\Pi\cap\mathcal{B}(0,2R))$ supported in $\mathcal{B}(0,2R)$. Since $\Pi\cap\mathcal{B}(0,2R)$ is a corner domain, D_0^R has compact injection into $L^2(\partial\Pi\cap\mathcal{B}(0,2R))$; see [Dauge 1988, Corollary AA.15]. We deduce the existence of a constant $C_1(R)\in\mathbb{R}$ such that

$$\mathcal{Q}[\Pi](\chi_1 u)\geq C_1(R)\|\chi_1 u\|^2_{L^2(\Pi\cap\mathcal{B}(0,2R))}=C_1(R)\|\chi_1 u\|^2_{L^2(\Pi)}.$$

Let $\epsilon>0$; from (22) we deduce the existence of $R>0$ such that

$$\frac{\lambda(\omega,r)}{r^2}\geq\mathscr{E}(\omega)-\epsilon\quad\text{for all } r>R$$

and therefore Lemma 5.1 gives

$$\mathcal{Q}(\chi_2 u)\geq(\mathscr{E}(\omega)-\epsilon)\|\chi_2 u\|^2_{L^2(\Pi)}.$$

There exists $C_2>0$ such that $\sum_i\|\nabla\chi_i\|^2\leq C_2R^{-2}$ for all $R>0$. Therefore we deduce that there exists $C_3=C_3(R,\epsilon,\omega)\in\mathbb{R}$ such that

$$\mathcal{Q}[\Pi](u)\geq C_3\|u\|^2_{L^2(\Pi)}.$$

We deduce that the quadratic form is lower semibounded and the operator $L[\Pi]$ is well defined as the self-adjoint extension of $\mathcal{Q}[\Pi]$, and its form domain is $H^1[\Pi]$. □

5B. ***Bottom of the essential spectrum for irreducible cones.*** Let $\Pi\in\mathfrak{P}_m$ with $m\geq n$, and let Γ be its reduced cone. In some suitable coordinates, we may write

$$\Pi=\mathbb{R}^{m-n}\times\Gamma$$

with $\Gamma\in\mathfrak{P}_n$ an irreducible cone. The associated Robin Laplacian admits the decomposition

$$L[\Pi]=-\Delta_{\mathbb{R}^{m-n}}\otimes\mathbb{I}_n+\mathbb{I}_{m-n}\otimes L[\Gamma].\tag{24}$$

In particular,

$$\mathfrak{S}(L[\Pi])=[E(\Gamma),+\infty).$$

Moreover, if $E(\Gamma)$ is a discrete eigenvalue for $L[\Gamma]$ and u is an associated eigenfunction (with exponential decay), then $\mathbb{I}\otimes u$ is called an L^∞-generalized eigenfunction for $L[\Pi]$ (this is linked to the notion of L^∞-spectral pair). Therefore we are led to investigate the bottom of the essential spectrum of $L[\Gamma]$. We prove:

Lemma 5.3. *Let $\Gamma\in\Pi_n$ be an irreducible cone of section ω such that $\mathscr{E}(\omega)>-\infty$. Then the bottom of the essential spectrum of $L[\Pi]$ is $\mathscr{E}(\omega)$.*

Proof. From Persson's lemma [1960], the bottom of the essential spectrum of $L[\Gamma]$ is the limit, as $R\to+\infty$, of

$$\Sigma(R):=\inf_{\substack{\Psi\in H^1(\Gamma),\,\Psi\neq 0\\ \operatorname{supp}(\Psi)\cap\mathcal{B}(0,R)=\varnothing}}\frac{\mathcal{Q}[\Gamma](\Psi)}{\|\Psi\|^2}.$$

Lower bound. From Lemma 5.1, we get directly

$$\liminf_{R\to+\infty} \Sigma(R) \geq \liminf_{R\to+\infty} \frac{\lambda(\omega, R)}{R^2}$$

and we deduce from (22) that

$$\liminf_{R\to+\infty} \Sigma(R) \geq \mathscr{E}(\omega).$$

Upper bound. By scaling — see Lemma 3.2 — we immediately have

$$\Sigma(R) = R^{-2} \inf_{\substack{\Psi\in H^1(\Gamma),\, \Psi\neq 0\\ \operatorname{supp}(\Psi)\cap\mathcal{B}(0,1)=\varnothing}} \frac{\mathcal{Q}_R[\Gamma](\Psi)}{\|\Psi\|^2}.$$

Each point x in $\overline{\Gamma} \setminus \overline{\mathcal{B}(0,1)}$ has a tangent cone Π_x. If we let $x_1 := x/|x| \in \bar{\omega}$, and let C_{x_1} be the tangent cone to ω at x_1, then $\Pi_x \equiv \mathbb{R} \times C_{x_1}$. Therefore, by tensor decomposition of the Robin Laplacian (see (24)), $E(C_{x_1}) = E(\Pi_x)$. Thus the finiteness of $\mathscr{E}(\omega)$ implies the finiteness of $E(\Pi_x)$, and from Lemma 3.5 we have

$$\limsup_{R\to+\infty} \Sigma(R) \leq E(\Pi_x) \quad \text{for all } x \in \overline{\Gamma} \setminus \overline{\mathcal{B}(0,1)}. \tag{25}$$

Using moreover that

$$\inf_{x\in\overline{\Gamma}\setminus\overline{\mathcal{B}(0,1)}} E(\Pi_x) = \inf_{x_1\in\partial\omega} E(C_{x_1}) = \mathscr{E}(\omega), \tag{26}$$

and taking the infimum in (25) over $x \in \overline{\Gamma} \setminus \overline{\mathcal{B}(0,1)}$, we deduce

$$\limsup_{R\to+\infty} \Sigma(R) \leq \mathscr{E}(\omega),$$

and the lemma follows. □

6. Infimum of the local energies in corner domains

6A. ***Finiteness of the infimum of the local energies.*** In this section, we prove the finiteness of $\mathscr{E}(\Omega)$ for any $\Omega \in \mathfrak{D}(M)$ and for any n-dimensional manifold M without boundary, by induction on the dimension n.

In dimension 1, bounded domains are intervals and the associated tangent cones are either half-lines or the full line whose associated energies are respectively -1 and 0 (by explicit computations), therefore the infimum of the local energies is finite.

Let $n \geq 2$ be fixed and let us assume that, for any corner domain ω of an $n-1$-dimensional Riemannian manifold without boundary, we have

$$\mathscr{E}(\omega) > -\infty.$$

We want to prove that the same holds in dimension n.

As a consequence of the recursive hypothesis, $E(\Pi)$ is finite for all $\Pi \in \mathfrak{P}_n$ — see Lemma 5.2 — and we can study the regularity of the local energy with respect to the geometry of a cone:

Proposition 6.1. *Assume the recursive hypothesis in dimension $n-1$. Then the map $\Pi \mapsto E(\Pi)$ is continuous on $\mathfrak{P}_n$ for the distance $\mathbb{D}$ defined in* (9).

Proof. Let $\Pi \in \mathfrak{P}_n$ and let $(\Pi_k)_{k\in\mathbb{N}}$ be a sequence of cones with $\mathbb{D}(\Pi_k, \Pi) \to 0$ as $k \to +\infty$. This means that there exists a sequence $(\mathrm{J}_k)_{k\in\mathbb{N}}$ in GL_n with $\mathrm{J}_k(\Pi_k) = \Pi$, $\|\mathrm{J}_k\| \le 1$ and $\|\mathrm{J}_k - \mathbb{I}\| \to 0$ as $k \to +\infty$. Then, as a direct consequence of (11) and (12), we deduce that

$$\lim_{k\to+\infty} \frac{\mathcal{Q}[\Pi, \mathrm{G}_k](v)}{\|v\|^2_{L^2_{\mathrm{G}_k}}} = \frac{\mathcal{Q}[\Pi](v)}{\|v\|^2} \quad \text{for all } v \in H^1(\Pi).$$

Recall that the form domain of $\mathcal{Q}[\Pi, \mathrm{G}_k]$ is $H^1(\Pi)$; see Section 5A. Since $\mathcal{Q}[\Pi_k]$ and $\mathcal{Q}[\Pi, \mathrm{G}_k]$ are unitarily equivalent (see Lemma 3.1), we deduce that $E(\Pi_k) \to E(\Pi)$ as $k \to +\infty$. □

By definition of the distance on singular chains (see Section 2B), we get:

Corollary 6.2. *Assume the recursive hypothesis in dimension $n-1$. Let M be an n-dimensional manifold as above, and let $\Omega \in \mathfrak{D}(M)$ be a corner domain. Then the map $\mathbb{X} \mapsto E(\Pi_{\mathbb{X}})$ is continuous on $\mathfrak{C}(\Omega)$ for the distance $\mathbb{D}$. In particular, $x \mapsto E(\Pi_x)$ is continuous on each stratum of $\overline{\Omega}$.*

Let M be an n-dimensional manifold as above, let $\Omega \in \mathfrak{D}(M)$ and let $x_0 \in \partial\Omega$; in what follows, Γ_{x_0} is the reduced cone of Π_{x_0} and $\omega_{x_0} \in \mathfrak{D}(\mathbb{S}^{d-1})$ is its section, with $d \le n$. We note that (26) may be written as

$$\mathscr{E}(\omega_{x_0}) = \inf_{x_1\in\partial\omega_{x_0}} E(\Pi_{(x_0,x_1)}).$$

Therefore, Lemmas 5.2 and 5.3 show that

$$E(\Pi_{x_0}) \le E(\Pi_{(x_0,x_1)}) \quad \text{for all } x_1 \in \bar{\omega}_{x_0}.$$

We deduce by immediate recursion:

Corollary 6.3. *Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be two singular chains in $\mathfrak{C}(\Omega)$ satisfying $\mathbb{X}_1 \le \mathbb{X}_2$; we have*

$$E(\Pi_{\mathbb{X}_1}) \le E(\Pi_{\mathbb{X}_2}).$$

We combine this with Corollary 6.2 and we can apply [Bonnaillie-Noël et al. 2016a, Theorem 3.25] to get the lower semicontinuity of the local energy function $x \mapsto E(\Pi_x)$, and, from the compactness of $\overline{\Omega}$, we deduce that $\mathscr{E}(\Omega)$ is finite. This concludes the proof of Theorem 1.3 by induction.

As a consequence, Lemmas 5.2 and 5.3 imply Theorem 1.2.

6B. *Second energy level.* Note that for a cone which is not irreducible, the spectrum consists in essential spectrum, and Theorem 1.2 does not apply. However, there still exists a threshold in the spectrum: the second energy level of the tangent operator of a cone $\Pi \in \mathfrak{P}_n$ is defined as

$$\mathscr{E}^*(\Pi) := \inf_{\mathbb{X}\in\mathfrak{C}^*_0(\Pi)} E(\Pi_{\mathbb{X}}),$$

where we recall that $\mathfrak{C}^*_0(\Pi)$, defined in Section 2B, is the set of singular chains of Π of the form $\mathbb{X} = (0, \dots)$ and with $l(\mathbb{X}) \ge 2$, where $l(\mathbb{X})$ is the length of the chain.

Using Corollary 6.3 with $\mathbb{X}_1 = (0)$, then taking the infimum over the chain $\mathbb{X}_2 \geq \mathbb{X}_1$ with $l(\mathbb{X}_2) \geq 2$, we get $E(\Pi) \leq \mathscr{E}^*(\Pi)$. We also get $\mathscr{E}^*(\Pi) = \inf_{x_1 \in \partial\omega} E(\Pi_{(0,x_1)})$ and therefore, by (26),

$$\mathscr{E}(\omega) = \mathscr{E}^*(\Pi), \tag{27}$$

where ω is the section of the reduced cone of Π. The quantity $\mathscr{E}^*$ will be the discriminating value in the analysis carried out in Section 7.

6C. ***Examples.*** The inequality $E(\Pi) \leq \mathscr{E}^*(\Pi)$ is strict if and only if the operator on the reduced cone has eigenvalues below the essential spectrum. The presence (or absence) of a discrete spectrum is an interesting question in itself, and we describe here some examples for which this question has been studied. Due to the clear decomposition of the Robin Laplacian on a cone of the form $\mathbb{R}^{m-n} \times \Gamma$ — see (24) — we only treat the case of irreducible cones.

When Γ is the half-line, $E(\Gamma) = -1 < 0 = \mathscr{E}^*(\Gamma)$, and an associated eigenfunction is $x \mapsto e^{-x}$. The case of sectors is given by (4): the inequality is strict if and only if the sector is convex. In that case, an associated eigenfunction is $(x, y) \mapsto e^{-x/\sin\theta}$, where x denotes the variable associated with the axis of symmetry of the sector, and θ is the opening angle.

Pankrashkin [2016] provides geometrical conditions on three-dimensional cones with regular section. He shows that, when $\Gamma \in \mathfrak{P}_3$ is a cone such that $\mathbb{R}^3 \setminus \Gamma$ is convex, $E(\Gamma) = \mathscr{E}^*(\Gamma)$. On the other hand, if $\mathbb{R}^3 \setminus \Gamma$ is not convex, then $E(\Gamma)$ is a discrete eigenvalue below the essential spectrum.

Note finally that Levitin and Parnovski [2008] use a geometrical parameter to give a more explicit expression of $E(\Pi)$ when the section of Π is a polygonal domain that admits an inscribed circle.

Remark 6.4. In [Levitin and Parnovski 2008, Theorem 3.5], it is stated that the bottom of the spectrum of the Robin Laplacian on a cone which contains an hyperplane passing through the origin is -1. The following example shows that this statement is incorrect because the bottom of the essential spectrum could be below -1: Take a spherical polygon $\omega \subset \mathbb{S}^2$ such that

- ω is included in a hemisphere,
- ω has at least a vertex of opening $\theta \in (\pi, 2\pi)$.

Let $\Pi \subset \mathbb{R}^3$ be the cone of section ω, and let $\widetilde{\Pi}$ be its complement in $\mathbb{R}^3$. The cone $\widetilde{\Pi}$ contains a half-space, has an edge with opening angle $\tilde{\theta} = 2\pi - \theta \in (0, \pi)$. Then, from Theorem 1.2 and (4), we get that the bottom of the essential spectrum of $L[\Pi]$ is below $-\sin^{-2}\frac{1}{2}\tilde{\theta}$, and therefore $E(\widetilde{\Pi}) < -1$.

7. Upper bound: construction of quasimodes

In order to prove the upper bound of Theorem 1.4, we construct recursive quasimodes. The subsections below correspond to the following plan:

(A) Use the analysis of Section 6 to find a chain $\mathbb{X}_\nu = (x_0, \dots, x_\nu) \in \mathfrak{C}(\Omega)$ such that $L(\Pi_{\mathbb{X}_\nu})$ admits a generalized eigenfunction associated with the value $\mathscr{E}(\Omega)$, then construct a quasimode for $L_\alpha[\Pi_{\mathbb{X}_\nu}]$. We do this by using scaling and cut-off functions in a standard way.

(B) Use a recursive procedure (together with a multiscale analysis) to construct a quasimode on Π_{x_0}.

(C) Use this quasimode to construct a final quasimode on Ω, and choose the scales to optimize the remainders.

7A. ***A quasimode on a tangent structure.*** The next proposition uses the quantity $\mathscr{E}^*$ to state that there always exist a tangent structure that admits an L^∞-generalized eigenfunction associated with the ground state energy.

Proposition 7.1. *Let $\Pi \in \mathfrak{P}_n$. Then there exists $\mathbb{X} \in \mathfrak{C}_0(\Pi)$ satisfying*

$$E(\Pi_{\mathbb{X}}) = E(\Pi) \quad \text{and} \quad E(\Pi_{\mathbb{X}}) < \mathscr{E}^*(\Pi_{\mathbb{X}}). \tag{28}$$

Let $\Gamma_{\mathbb{X}} \in \mathfrak{P}_d$ be the irreducible cone of $\Pi_{\mathbb{X}}$. Then there exists an L^∞-generalized eigenfunction for $L[\Pi_{\mathbb{X}}]$ associated with $E(\Pi)$. Moreover it has the form $\mathbb{1} \otimes \Psi_{\mathbb{X}}$, in coordinates associated with the decomposition $\Pi_{\mathbb{X}} \equiv \mathbb{R}^{n-d} \times \Gamma_{\mathbb{X}}$, where $\Psi_{\mathbb{X}}$ has exponential decay.

Proof. The proof of the existence of $\mathbb{X}$ is recursive over the dimension d of the reduced cone of Π. The initialization is clear; indeed, when $d = 1$, we have that Π is a half-plane, $E(\Pi) = E(\mathbb{R}_+) = -1$ and $\mathscr{E}^*(\Pi) = E(\mathbb{R}) = 0$. Moreover, $\psi_{\mathbb{X}}(x) = e^{-x}$ provides an eigenfunction for $L[\mathbb{R}^+]$.

We now prove the heredity. First we find a chain $\mathbb{X}$ satisfying (28):

- If $E(\Pi) < \mathscr{E}^*(\Pi)$, then $\mathbb{X} = (0)$ and $\Pi_{\mathbb{X}} = \Pi$.

- If $E(\Pi) = \mathscr{E}^*(\Pi)$, we use Theorem 1.2: the function $x_1 \mapsto E(\Pi_{x_1})$ is lower semicontinuous on $\bar{\omega}$, where ω is the section of the reduced cone of Π. Therefore there exists $x_1 \in \partial\omega$ such that $\mathscr{E}^*(\Pi) = \mathscr{E}(\omega) = E(\Pi_{x_1}) = E(\Pi_{(0,x_1)})$. The dimension of the reduced cone of $\Pi_{(0,x_1)}$ is lower than that of Π; therefore, by the recursive hypothesis, there exists $\mathbb{X}' \in \mathfrak{C}_0(\Pi_{(0,x_1)})$ such that $E(\Pi_{\mathbb{X}'}) = E(\Pi_{(0,x_1)})$ and $E(\Pi_{\mathbb{X}'}) < \mathscr{E}^*(\Pi_{\mathbb{X}'})$. We write this chain in the form $\mathbb{X}' = (0, \mathbb{X}'')$, and the chain $\mathbb{X}'$ is pulled back into an element of $\mathfrak{C}_0(\Pi)$ by setting $\mathbb{X} = (0, x_1, \mathbb{X}'') \in \mathfrak{C}_0(\Pi)$. Note that $\Pi_{\mathbb{X}} = \Pi_{\mathbb{X}'}$, so that $E(\Pi_{(0,x_1)}) = E(\Pi_{\mathbb{X}}) = \mathscr{E}^*(\Pi) = E(\Pi)$ and $E(\Pi_{\mathbb{X}}) < \mathscr{E}(\Pi_{\mathbb{X}})$.

From Theorem 1.2 and (27), $E(\Pi_{\mathbb{X}}) < \mathscr{E}^*(\Pi_{\mathbb{X}})$ means that $E(\Pi_{\mathbb{X}})$ is an eigenvalue of $L(\Gamma_{\mathbb{X}})$ below the essential spectrum; therefore, there exists an associated eigenfunction $\Psi_{\mathbb{X}}$ with exponential decay, and $(y, z) \mapsto \Psi(z)$ for $(y, z) \in \mathbb{R}^{n-d} \times \Gamma_{\mathbb{X}}$ is clearly an L^∞-generalized eigenfunction for $L[\mathbb{R}^{n-d} \times \Gamma_{\mathbb{X}}]$. □

First, thanks to the lower semicontinuity of local energies, we choose $x_0 \in \partial\Omega$ such that $E(\Pi_{x_0}) = \mathscr{E}(\Omega)$. Then, using Proposition 7.1, we pick a singular chain $\mathbb{X}_\nu = (x_0, \dots, x_\nu)$ such that $L[\Pi_{\mathbb{X}_\nu}]$ has a generalized eigenfunction associated with $E(\Pi_{x_0})$. We let $\mathbb{X}_k = (x_0, \dots, x_k)$ for $0 \le k \le \nu$, and $\Pi_k := \Pi_{\mathbb{X}_k}$.

We define

$$\bar{\nu} := \inf\{k \ge 0 : \Pi_k \text{ is polyhedral}\}. \tag{29}$$

The index $\bar{\nu}$ provides the shortest chain such that $\Pi_{\bar{\nu}}$ is polyhedral, with $\bar{\nu} = +\infty$ when Π_ν is not polyhedral. Moreover, when $\bar{\nu}$ is finite the tangent structure Π_k is polyhedral for all $\bar{\nu} \le k \le \nu$, and $\bar{\nu} \le n-2$, since any chain of length strictly larger than $n-2$ is associated either with a half-space or with the full space.

The tangent structure Π_ν is (in some suitable coordinates) $\mathbb{R}^p \times \Gamma_\nu$ with Γ_ν irreducible. We denote by π_{Γ_ν} the projection onto Γ_ν associated with this decomposition. Then, by Proposition 7.1, there exists an eigenfunction u with exponential decay for $L[\Gamma_\nu]$ associated with $E(\Pi_\nu)$.

Let $\chi \in \mathcal{C}^\infty(\mathbb{R}^+)$ be a cut-off function with compact support satisfying

$$\chi(r) = 1 \quad \text{if } r \le 1 \qquad \text{and} \qquad \chi(r) = 0 \quad \text{if } r \ge 2.$$

We define the scaled cut-off function

$$\chi_\alpha(r) = \chi(\alpha^\delta r),$$

where $\delta \in (0, 1)$ will be chosen later. The initial quasimode is

$$u_\nu(x) = \chi_\alpha(|x|) u(\pi_\Gamma(\alpha x)), \quad x \in \Pi_{\mathbb{X}_\nu}.$$

Standard computations show that

$$\frac{\mathcal{Q}_\alpha[\Pi_\nu](u_\nu)}{\|u_\nu\|^2} = \alpha^2 \mathscr{E}(\Omega) + \frac{\|\nabla(\chi_\alpha) u_\nu\|^2}{\|u_\nu\|^2};$$

in particular,

$$\frac{\mathcal{Q}_\alpha[\Pi_\nu](u_\nu)}{\|u_\nu\|^2} = \alpha^2 \mathscr{E}(\Omega) + O(\alpha^{2\delta}). \tag{30}$$

7B. ***Getting up along the chains.*** The previous section provides a quasimode u_ν for $L[\Pi_\nu]$. The aim of this section is a recursive decreasing procedure in order to get a quasimode for $L[\Pi_0]$. Therefore, this step is skipped if $\nu = 0$. This case happens when $E(\Pi_{x_0}) < \mathscr{E}^*(\Pi_{x_0})$, and the quasimode is called *sitting*, as was introduced in [Bonnaillie-Noël et al. 2016a]. Otherwise we suppose that $\nu \ge 1$, and we will construct quasimodes u_k defined on Π_k, for $0 \le k \le \nu$. These quasimodes are called *sliding*.

In what follows, $(d_k(\alpha))_{k=1,\dots,\nu}$ and $(r_k(\alpha))_{k=0,\dots,\nu}$ are positive sequences of shifts and radii (to be determined) going to 0 as $\alpha \to +\infty$.

Let $1 \le k \le \nu$ and assume that $u_k \in H^1(\Pi_k)$ is constructed and is supported in a ball $\mathcal{B}(0, r_k(\alpha))$. This is already done for $k = \nu$; see the last section. For $1 \le k \le \nu$, we define

$$v_k = d_k(\alpha)(0, x_k) \in \Pi_{k-1},$$

where $(0, x_k) \in \Pi_{k-1}$ are cylindrical coordinates associated with the decomposition $\Pi_{k-1} = \mathbb{R}^{p_k} \times \Gamma_{k-1}$. Intuitively, v_k is a point of Π_{k-1} satisfying $\|v_k\| = d_k(\alpha)$ and is collinear to $(0, x_k)$.

We construct u_{k-1} as follows:

- *Local map at v_k*: The tangent cone to Π_{k-1} at v_k is Π_k itself. Let $\psi_k : \mathcal{U}_{v_k} \mapsto \mathcal{V}_{v_k}$ be a local map. The map-neighborhoods $\mathcal{U}_{v_k}$ and $\mathcal{V}_{v_k}$ (of $v_k \in \Pi_{k-1}$ and $0 \in \Pi_k$, respectively) can be chosen of diameters smaller than $c_k d_k(\alpha)$, where c_k is the diameter of the map-neighborhood of x_k. Moreover, when $k \ge \overline{\nu}$, Π_k is polyhedral, so ψ_k is a translation. When this is not the case, by elementary scaling, $\kappa(v_k) \le \kappa(x_k)/d_k(\alpha)$; see [Bonnaillie-Noël et al. 2016a, Section 3] for more details on this process. Since the $(x_k)_{0 \le k \le \nu}$ are fixed, we can choose ν fixed map-neighborhoods associated with these points, and a constant $c(\Omega) > 0$ such that

$$\kappa(v_k) \le \begin{cases} c(\Omega)/d_k(\alpha) & \text{if } k \le \overline{\nu}, \\ c(\Omega) & \text{if } k \ge \overline{\nu} + 1. \end{cases} \tag{31}$$

We now add the constraint that

$$\frac{r_k(\alpha)}{d_k(\alpha)} \to 0 \quad \text{as } \alpha \to +\infty \qquad \text{if } k \le \overline{\nu}, \tag{32}$$

so that $r_k\kappa(v_k) \to 0$ for all $1 \le k \le \nu$, and we can define, for α large enough,

$$\tau_k := \frac{1 - Cr_k\kappa(v_k)}{1 + Cr_k\kappa(v_k)}, \tag{33}$$

where C is the constant appearing in Lemma 3.3.

• *Change of variables*: First we rescale u_k (the reason for this will appear later): let

$$\tilde{u}_k(x) = \tau_k(\alpha)^{n/2} u(\tau_k(\alpha)x). \tag{34}$$

This function satisfies

$$\|\tilde{u}_k\| = \|u_k\| \quad \text{and} \quad \mathcal{Q}_{\alpha_k^+}[\Pi_k](\tilde{u}_k) = \tau_k(\alpha)^2 \mathcal{Q}_\alpha[\Pi_k](u_k), \tag{35}$$

where $\alpha_k^+ = \tau_k(\alpha)\alpha$. Recall that $\operatorname{supp} u_k \subset \mathcal{B}(0, r_k(\alpha))$ by the recursive hypothesis on u_k. Then, due to (32), we have $c_k d_k(\alpha) > r_k(\alpha)/\tau_k(\alpha)$ for α large enough, and therefore

$$\operatorname{supp} \tilde{u}_k \subset \mathcal{B}(0, r_k(\alpha)/\tau_k(\alpha)) \subset \mathcal{V}_k.$$

As a consequence, we can define on $\mathcal{U}_k \cap \Pi_{k-1}$ the function

$$u_{k-1} = \tilde{u}_k \circ \psi_k. \tag{36}$$

We can extend this function by 0 outside its support so that $u_{k-1} \in H^1(\Pi_{k-1})$. Its support is inside a ball centered at 0 and of size $d_k + \operatorname{diam}(\mathcal{U}_k) = (1 + c_k)d_k$, so we set

$$r_{k-1} := (1 + c_k)d_k. \tag{37}$$

We derive from this recursive procedure a quasimode u_0 on Π_0, localized in a ball $\mathcal{B}(0, r_0(\alpha))$.

7C. *Quasimode on the initial domain Ω and choice of the scales.* Now we set $v_0 := x_0$, and we still define τ_0 by (33), then $\tilde{u}_0$ by (34) and u_{-1} by (36). Note that $\kappa(v_0)$ is constant since $v_0 = x_0$ is fixed. We compare $\mathcal{Q}_\alpha[\Pi_{k-1}](u_{k-1})$ with $\mathcal{Q}_\alpha[\Pi_k](u_k)$ for $0 \le k \le \nu$. We have, from Lemma 3.1,

$$\mathcal{Q}_\alpha[\Pi_k, \mathrm{G}_k](\tilde{u}_k) = \mathcal{Q}_\alpha[\Pi_{k-1}](u_{k-1}), \tag{38}$$

where $\mathrm{G}_k := \mathrm{J}_k^{-1}(\mathrm{J}_k^{-1})^\top$ is the associated metric with $\mathrm{J}_k := \mathrm{d}\psi_k^{-1}$.

Since, by construction, $r_k\kappa(v_k) \to 0$, we can apply Lemma 3.3, in particular the inequality (15):

$$\mathcal{Q}_\alpha[\Pi_k, G_k](\tilde{u}_k) \le \mathcal{Q}_{\alpha_k^+}[\Pi_k](\tilde{u}_k).$$

Combining this with (35) and (38) we get, for all $0 \le k \le \nu$,

$$\mathcal{Q}_\alpha[\Pi_{k-1}](u_{k-1}) \le \tau_k(\alpha)^2 \mathcal{Q}_\alpha[\Pi_k](u_k),$$

and therefore

$$\mathcal{Q}_\alpha[\Omega](u_{-1}) \le \prod_{k=0}^{\nu} \tau_k(\alpha)^2 \mathcal{Q}_\alpha[\Pi_\nu](u_\nu).$$

Recall that $\kappa(v_0)$ is fixed; we get, from (31),

$$\mathcal{Q}_\alpha[\Omega](u_{-1}) \le \left(1 + C\left(r_0 + \frac{r_1}{d_1} + \cdots + \frac{r_{\bar\nu}}{d_{\bar\nu}} + r_{\bar\nu+1} + \cdots + r_\nu\right)\right)\mathcal{Q}_\alpha[\Pi_\nu](u_\nu).$$

We now choose $r_k(\alpha) = \alpha^{-\sum_{p=0}^{k}\delta_k}$ when $k \le \bar\nu$ and $r_k = r_{\bar\nu}$ when $k > \bar\nu$, with $\delta_k > 0$. The shifts are set by (37), so that $r_k/d_k = O(\alpha^{-\delta_k})$ for all $1 \le k \le \bar\nu$. Moreover, the scale δ of Section 7A is related by $\delta = \sum_{k=0}^{\bar\nu}\delta_k$, and (30) provides

$$\mathcal{Q}_\alpha[\Omega](u_{-1}) \le \left(1 + \sum_{k=0}^{\bar\nu} O(\alpha^{-\delta_k})\right)(\alpha^2\mathscr{E}(\Omega) + O(\alpha^{2\delta})) = \alpha^2\mathscr{E}(\Omega) + \sum_{k=0}^{\bar\nu} O(\alpha^{2-\delta_k}) + O(\alpha^{2\delta}).$$

The error terms are the same as in Section 7C; therefore, we make the same choice of scales $\delta_k = 2/(2\bar\nu+3)$ for all $0 \le k \le \bar\nu$. By construction, u_{-1} is normalized, therefore the min-max theorem implies the upper bound of Theorem 1.4.

8. Applications

In the applications below, one must keep in mind that the finiteness of $\mathscr{E}(\Omega)$ is one of our results, and that this quantity can be made more explicit for particular geometries; see [Levitin and Parnovski 2008]. Moreover, this quantity goes to $-\infty$ as the corners of a domain Ω gets sharper: this is clear in dimension 2 since the local energy at a corner of opening θ goes to $-\infty$ as $\theta \to 0$; see (4). In higher dimension, it could be possible to use the approach from [Bonnaillie-Noël et al. 2016b] in order to show that the local energy goes to $-\infty$ for sharp cones (see the definition of a sharp cone therein).

8A. ***On the optimal constant in relative bounds zero for the trace operator.*** The trace injection from $H^1(\Omega)$ into $L^2(\partial\Omega)$ being compact, the following relative 0-bound holds: for all $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\|u\|^2_{L^2(\partial\Omega)} \le \epsilon\|\nabla u\|^2_{L^2(\Omega)} + C(\epsilon)\|u\|^2_{L^2(\Omega)} \quad \text{for all } u \in H^1(\Omega). \tag{39}$$

This inequality is a particular case of Ehrling's lemma. It can be written as

$$\mathcal{Q}_{1/\epsilon}[\Omega](u) \ge -\frac{C(\epsilon)}{\epsilon}\|u\|^2_{L^2(\Omega)} \quad \text{for all } u \in H^1(\Omega).$$

Thus, by definition of $\lambda(\Omega, \alpha)$, for each $\epsilon > 0$ the best constant $C(\epsilon)$ in (39) is

$$C(\epsilon) = -\epsilon\lambda\left(\Omega, \frac{1}{\epsilon}\right).$$

From Theorem 1.4, we obtain that this constant is essentially $\epsilon^{-1}|\mathscr{E}(\Omega)|$. More precisely:

Proposition 8.1. *Let $\Omega \in \mathfrak{D}(M)$ be an admissible corner domain. Then there exist $\epsilon_0 > 0$ and $\gamma \in \left(0, \frac{2}{3}\right)$ such that, for all $\epsilon \in (0, \epsilon_0)$,*

$$\|u\|^2_{L^2(\partial\Omega)} \leq \epsilon \|\nabla u\|^2_{L^2(\Omega)} + \left(\frac{|\mathscr{E}(\Omega)|}{\epsilon} + O(\epsilon^{\gamma-1})\right)\|u\|^2_{L^2(\Omega)} \quad \textit{for all } u \in H^1(\Omega).$$

Let us recall that the finiteness of $\lambda(\Omega, \alpha)$ is closely related to the compactness of the injection of $H^1(\Omega)$ into $L^2(\partial\Omega)$ and, for some cusps, where $\lambda(\Omega, \alpha) = -\infty$, this injection is not compact (see [Nazarov and Taskinen 2011; Daners 2013]).

8B. ***Transition temperature of superconducting models.*** In the study of superconducting models, the physics literature has explored over the years the possibility of increasing the critical fields. Another more interesting and more recent idea is to increase the temperature below which the normal state (i.e., the critical point of the Ginzburg–Landau energy for which the material is nowhere in the superconducting state) is not stable. For zero fields associated to a superconducting body Ω, enhanced surface superconductivity is modeled via a negative penetration depth $b < 0$ and, following [Giorgi and Smits 2007], this critical temperature is given by

$$T_c^b(\Omega) = T_{c_0} - T_{c_0}\lambda\left(\Omega, \frac{\xi(0)}{|b|}\right), \tag{40}$$

where $\xi(0) > 0$ is the so-called coherence length at zero temperature, T_{c_0} is the vacuum zero field critical temperature for $b = \infty$ (corresponding to a superconductor surrounded by vacuum) and $\lambda(\Omega, \alpha)$ is the first eigenvalue of the Robin problem.

Thanks to Theorem 1.4, for $|b|$ small enough we have

$$T_c^b(\Omega) \geq T_{c_0} + T_{c_0}\frac{\xi(0)^2}{|b|^2}\left(|\mathscr{E}(\Omega)| + O(|b|^\gamma)\right)$$

for some $\gamma \in \left(0, \frac{2}{3}\right)$. Since $|\mathscr{E}(\Omega)| \geq 1$ and goes to $+\infty$ as the corners of $\partial\Omega$ become sharper, our results are consistent with the general physical principle of increase of $T_c^b(\Omega)$ due to confinement (see for instance [Montevecchi and Indekeu 2000, Section 4] and see [Yampolskii and Peeters 2000; Baelus et al. 2002] concerning superconducting properties of nanostructuring materials).

References

[Asorey et al. 2015] M. Asorey, A. P. Balachandran, and J. M. Perez-Pardo, "Edge states at phase boundaries and their stability", 2015. arXiv 1505.03461

[Baelus et al. 2002] B. J. Baelus, S. V. Yampolskii, F. M. Peeters, E. Montevecchi, and J. O. Indekeu, "Superconducting properties of mesoscopic cylinders with enhanced surface superconductivity", *Phys. Rev. B* **65**:2 (2002), art. ID #024510.

[Behrndt et al. 2014] J. Behrndt, P. Exner, and V. Lotoreichik, "Schrödinger operators with δ-interactions supported on conical surfaces", *J. Phys. A* **47**:35 (2014), art. ID #355202. MR 3254874 Zbl 1297.35154

[Bonnaillie-Noël et al. 2016a] V. Bonnaillie-Noël, M. Dauge, and N. Popoff, *Ground state energy of the magnetic Laplacian on general three-dimensional corner domains*, Mém. Soc. Math. Fr. (N.S.) **145**, Soc. Math. France, Paris, 2016.

[Bonnaillie-Noël et al. 2016b] V. Bonnaillie-Noël, M. Dauge, N. Popoff, and N. Raymond, "Magnetic Laplacian in sharp three dimensional cones", pp. 37–56 in *Spectral theory and mathematical physics* (Santiago, 2014), Operator Theory Advances and Application **254**, Birkhäuser, Boston, 2016.

[Brasche et al. 1994] J. F. Brasche, P. Exner, Y. A. Kuperin, and P. Šeba, "Schrödinger operators with singular interactions", *J. Math. Anal. Appl.* **184**:1 (1994), 112–139. MR 1275948 Zbl 0820.47005

[Colorado and García-Melián 2011] E. Colorado and J. García-Melián, "The behavior of the principal eigenvalue of a mixed elliptic problem with respect to a parameter", *J. Math. Anal. Appl.* **377**:1 (2011), 53–69. MR 2754808 Zbl 1209.35092

[Daners 2013] D. Daners, "Principal eigenvalues for generalised indefinite Robin problems", *Potential Anal.* **38**:4 (2013), 1047–1069. MR 3042694 Zbl 1264.35152

[Daners and Kennedy 2010] D. Daners and J. B. Kennedy, "On the asymptotic behaviour of the eigenvalues of a Robin problem", *Differential Integral Equations* **23**:7-8 (2010), 659–669. MR 2654263 Zbl 1240.35370

[Dauge 1988] M. Dauge, *Elliptic boundary value problems on corner domains: smoothness and asymptotics of solutions*, Lecture Notes in Mathematics **1341**, Springer, Berlin, 1988. MR 961439 Zbl 0668.35001

[Dimassi and Sjöstrand 1999] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series **268**, Cambridge University Press, 1999. MR 1735654 Zbl 0926.35002

[Dittrich et al. 2016] J. Dittrich, P. Exner, C. Kühn, and K. Pankrashkin, "On eigenvalue asymptotics for strong δ-interactions supported by surfaces with boundaries", *Asymptot. Anal.* **97**:1-2 (2016), 1–25. MR 3475116

[Duchêne and Raymond 2014] V. Duchêne and N. Raymond, "Spectral asymptotics of a broken δ-interaction", *J. Phys. A* **47**:15 (2014), art. ID #155203. MR 3191674 Zbl 1288.81044

[Exner and Kondej 2015] P. Exner and S. Kondej, "Gap asymptotics in a weakly bent leaky quantum wire", *J. Phys. A* **48**:49 (2015), art. ID # 495301. MR 3434828 Zbl 1330.81101

[Exner and Pankrashkin 2014] P. Exner and K. Pankrashkin, "Strong coupling asymptotics for a singular Schrödinger operator with an interaction supported by an open arc", *Comm. Partial Differential Equations* **39**:2 (2014), 193–212. MR 3169783 Zbl 1291.35247

[Exner and Yoshitomi 2002] P. Exner and K. Yoshitomi, "Asymptotics of eigenvalues of the Schrödinger operator with a strong δ-interaction on a loop", *J. Geom. Phys.* **41**:4 (2002), 344–358. MR 1888470 Zbl 1083.35034

[Giorgi and Smits 2007] T. Giorgi and R. Smits, "Eigenvalue estimates and critical temperature in zero fields for enhanced surface superconductivity", *Z. Angew. Math. Phys.* **58**:2 (2007), 224–245. MR 2305713 Zbl 1117.82052

[Grieser 2008] D. Grieser, "Spectra of graph neighborhoods and scattering", *Proc. Lond. Math. Soc.* (3) **97**:3 (2008), 718–752. MR 2448245 Zbl 1183.58027

[Helffer and Kachmar 2014] B. Helffer and A. Kachmar, "Eigenvalues for the Robin Laplacian in domains with variable curvature", 2014. To appear in *Trans. Amer. Math. Soc.* arXiv 1411.2700

[Helffer and Pankrashkin 2015] B. Helffer and K. Pankrashkin, "Tunneling between corners for Robin Laplacians", *J. Lond. Math. Soc.* (2) **91**:1 (2015), 225–248. MR 3338614 Zbl 1319.35129

[Lacey et al. 1998] A. A. Lacey, J. R. Ockendon, and J. Sabina, "Multidimensional reaction diffusion equations with nonlinear boundary conditions", *SIAM J. Appl. Math.* **58**:5 (1998), 1622–1647. MR 1637882 Zbl 0932.35120

[Levitin and Parnovski 2008] M. Levitin and L. Parnovski, "On the principal eigenvalue of a Robin problem with a large parameter", *Math. Nachr.* **281**:2 (2008), 272–281. MR 2387365 Zbl 1136.35060

[Lotoreichik and Ourmières-Bonafos 2015] V. Lotoreichik and T. Ourmières-Bonafos, "On the bound states of Schrödinger operators with δ-interactions on conical surfaces", 2015. arXiv 1510.05623

[Lou and Zhu 2004] Y. Lou and M. Zhu, "A singularly perturbed linear eigenvalue problem in C^1 domains", *Pacific J. Math.* **214**:2 (2004), 323–334. MR 2042936 Zbl 1061.35061

[Maz′ja and Plamenevskiĭ 1977] V. G. Maz′ja and B. A. Plamenevskiĭ, "Elliptic boundary value problems on manifolds with singularities", pp. 85–142 in *Problems in mathematical analysis, VI: Spectral theory, boundary value problems*, edited by N. N. Ural′ceva, Izdat. Leningrad. Univ., Leningrad, 1977. In Russian. MR 0509430

[Montevecchi and Indekeu 2000] E. Montevecchi and J. O. Indekeu, "Effects of confinement and surface enhancement on superconductivity", *Phys. Rev. B* **62** (2000), 14359–14372.

[Nazarov and Taskinen 2011] S. A. Nazarov and Y. Taskinen, "On the spectrum of the third boundary value problem in a domain with a peak", *Funktsional. Anal. i Prilozhen.* **45**:1 (2011), 93–96. In Russian; translated in *Funct. Anal. Appl.* **45**:77 (2011), 77–79. MR 2848745

[Pankrashkin 2013] K. Pankrashkin, "On the asymptotics of the principal eigenvalue for a Robin problem with a large parameter in planar domains", *Nanosystems: Phys. Chem. Math.* **4**:4 (2013), 474–483.

[Pankrashkin 2015] K. Pankrashkin, "On the Robin eigenvalues of the Laplacian in the exterior of a convex polygon", *Nanosystems: Phys. Chem. Math.* **6**:1 (2015), 46–56.

[Pankrashkin 2016] K. Pankrashkin, "On the discrete spectrum of robin laplacians in conical domains", *Math. Model. Nat. Phenom.* **11**:2 (2016), 100–110.

[Pankrashkin and Popoff 2015] K. Pankrashkin and N. Popoff, "Mean curvature bounds and eigenvalues of Robin Laplacians", *Calc. Var. Partial Differential Equations* **54**:2 (2015), 1947–1961. MR 3396438 Zbl 1327.35273

[Pankrashkin and Popoff 2016] K. Pankrashkin and N. Popoff, "An effective Hamiltonian for the eigenvalue asymptotics of the Robin Laplacian with a large parameter", *J. Math. Pures Appl.* (online publication March 2016).

[Persson 1960] A. Persson, "Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator", *Math. Scand.* **8** (1960), 143–153. MR 0133586 Zbl 0145.14901

[Simon 1983] B. Simon, "Semiclassical analysis of low lying eigenvalues, I: Nondegenerate minima: asymptotic expansions", *Ann. Inst. H. Poincaré Sect. A* (*N.S.*) **38**:3 (1983), 295–308. MR 708966

[Yampolskii and Peeters 2000] S. V. Yampolskii and F. M. Peeters, "Vortex structure of thin mesoscopic disks with enhanced surface superconductivity", *Phys. Rev. B* **62** (2000), 9663–9674.

Received 18 Jan 2016. Revised 30 Mar 2016. Accepted 29 Apr 2016.

VINCENT BRUNEAU: vbruneau@math.u-bordeaux1.fr
Institut de Mathematiques, Université de Bordeaux I, 351 Cours de La Libération, 33405 Talence, France

NICOLAS POPOFF: nicolas.popoff@math.u-bordeaux1.fr
Institut de Mathematiques, Université de Bordeaux I, 351 Cours de La Libération, 33405 Talence, France

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2016 is US $235/year for the electronic version, and $430/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 9 No. 5 2016

2157-5045(2016)9:5;1-7