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THE INTERIOR C^2 ESTIMATE FOR THE MONGE–AMPÈRE EQUATION IN DIMENSION $n = 2$

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We introduce a new auxiliary function, and establish the interior C^2 estimate for the Monge–Ampère equation in dimension $n = 2$, which was first proved by Heinz by a geometric method.

1. Introduction

We consider the convex solution of the Monge–Ampère equation

$$\det D^2 u = f(x) \quad \text{in } B_R(0) \subset \mathbb{R}^2. \tag{1-1}$$

When the solution u is convex, (1-1) is elliptic. It is well known that the interior C^2 estimate is an important problem for elliptic equations. For the Monge–Ampère equation in dimension $n = 2$, the corresponding interior C^2 estimate was established by Heinz [1959], and for higher dimensions $n \geq 3$ Pogorelov [1978] constructed his famous counterexample, namely irregular solutions to Monge–Ampère equations.

Later, Urbas [1990] generalized the counterexample for σ_k Hessian equations with $k \geq 3$. So the interior C^2 estimate of the σ_2 Hessian equation

$$\sigma_2(D^2 u) = f \quad \text{in } B_R(0) \subset \mathbb{R}^n \tag{1-2}$$

is an interesting problem, where $\sigma_2(D^2u) = \sigma_2(\lambda(D^2u)) = \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1}\lambda_{i_2}$, the eigenvalues of D^2u are $\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)$, and $f > 0$. For $n = 2$, (1-2) is the Monge–Ampère equation (1-1). For $n = 3$ and $f \equiv 1$, (1-2) can be viewed as a special Lagrangian equation, and Warren and Yuan [2009] obtained the corresponding interior C^2 estimate in their celebrated paper. Moreover, the problem is still open for general f with $n \geq 4$ and nonconstant f with $n = 3$.

Moreover, Pogorelov-type estimates for the Monge–Ampère equations and the σ_k Hessian equation ($k \geq 2$) were derived by Pogorelov [1978] and Chou and Wang [2001], respectively, and see [Guan et al. 2015; Li et al. 2016] for some generalizations.

In this paper, we introduce a new auxiliary function, and establish the interior C^2 estimate as follows:

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Theorem 1.1. *Let $u \in C^4(B_R(0))$ be a convex solution of the Monge–Ampère equation* (1-1) *in dimension $n = 2$, where $0 < m \le f \le M$ in $B_R(0)$. Then*

$$|D^2u(0)| \le C_1 e^{C_2 \sup |Du|^2/R^2}, \tag{1-3}$$

where C_1 is a positive constant depending only on m, M, $R \sup |\nabla f|$ and $R^2 \sup |\nabla^2 f|$, and C_2 is a positive constant depending only on m and M.

Remark 1.2. By Trudinger's gradient estimates [1997], we can bound $|D^2u(0)|$ in terms of u. In fact, we get from the convexity of u that

$$\sup_{B_{R/2}(0)} |Du| \le \frac{\operatorname{osc}_{B_R(0)} u}{R/2} \le \frac{4 \sup_{B_R(0)} |u|}{R}$$

and

$$|D^2u(0)| \le C_1 e^{C_2 \sup_{B_{R/2}(0)} |Du|^2/(R/2)^2} \le C_1 e^{16 C_2 \sup_{B_R(0)} |u|^2/R^4}. \tag{1-4}$$

Remark 1.3. The result was first proved by Heinz [1959]. In fact, Heinz's proof depends on the strict convexity of solutions and the geometry of convex hypersurfaces in dimension two. Our proof, which is based on a suitable choice of auxiliary functions, is elementary and avoids geometric computations on the graph of solutions.

Remark 1.4. The interior C^2 estimate of the σ_2 Hessian equation (1-2) in higher dimensions is a longstanding problem. As we all know, it is hard to find a corresponding geometry in higher dimensions, so we cannot generalize Heinz's proof or Warren and Yuan's proof to higher dimensions. But the method in this paper and the optimal concavity in [Chen 2013] is helpful for this problem.

The rest of the paper is organized as follows. In Section 2, we give the calculations of the derivatives of eigenvalues and eigenvectors with respect to the matrix. In Section 3, we introduce a new auxiliary function, and prove Theorem 1.1.

2. Derivatives of eigenvalues and eigenvectors

In this section, we give the calculations of the derivatives of eigenvalues and eigenvectors with respect to the matrix. We expect the following result is known to many people; for example see [Andrews 2007] for a similar result. For completeness, we give the result and a detailed proof.

Proposition 2.1. *Let $W = \{W_{ij}\}$ be an $n \times n$ symmetric matrix with eigenvalues $\lambda(W) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and with corresponding continuous eigenvector field $\tau^i = (\tau^i_1, \dots, \tau^i_n) \in \mathbb{S}^{n-1}$. Suppose that $W = \{W_{ij}\}$ is diagonal with $\lambda_i = W_{ii}$ and corresponding eigenvector $\tau^i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{S}^{n-1}$ with the 1 in the i-th slot. If λ_k is distinct from the other eigenvalues, then we have, at the diagonal matrix W,*

$$\frac{\partial \tau^k_i}{\partial W_{pq}} = \begin{cases} 0 & \text{if } i = k \text{ for all } p, q, \\ \dfrac{1}{\lambda_k - \lambda_i} & \text{if } i \ne k,\ p = i,\ q = k, \\ 0 & \text{otherwise}; \end{cases} \tag{2-1}$$

$$\frac{\partial^2\tau_k^k}{\partial W_{pk}\partial W_{pk}} = -\frac{1}{(\lambda_k-\lambda_p)^2} \quad \textit{if } p\neq k; \tag{2-2}$$

$$\frac{\partial^2\tau_i^k}{\partial W_{ik}\partial W_{ii}} = \frac{1}{(\lambda_k-\lambda_i)^2} \quad \textit{if } i\neq k; \qquad \frac{\partial^2\tau_i^k}{\partial W_{ik}\partial W_{kk}} = -\frac{1}{(\lambda_k-\lambda_i)^2} \quad \textit{if } i\neq k; \tag{2-3}$$

$$\frac{\partial^2\tau_i^k}{\partial W_{iq}\partial W_{qk}} = \frac{1}{\lambda_k-\lambda_i}\frac{1}{\lambda_k-\lambda_q} \quad \textit{if } i\neq k,\ i\neq q,\ q\neq k; \tag{2-4}$$

$$\frac{\partial^2\tau_i^k}{\partial W_{pq}\partial W_{rs}} = 0 \quad \textit{otherwise}. \tag{2-5}$$

Proof. From the definitions of eigenvalue and eigenvector of a matrix W, we have

$$(W-\lambda_k I)\tau^k \equiv 0,$$

where τ^k is the eigenvector of W corresponding to the eigenvalue λ_k. That is, for $i=1,\dots,n$,

$$(W_{ii}-\lambda_k)\tau_i^k + \sum_{j\neq i} W_{ij}\tau_j^k = 0. \tag{2-6}$$

When $W=\{W_{ij}\}$ is diagonal and λ_k is distinct from the other eigenvalues, λ_k and τ^k are C^2 at the matrix W. In fact,

$$\tau_k^k = 1 \quad \text{and} \quad \tau_i^k = 0 \ \text{ for } i\neq k \quad \text{at } W. \tag{2-7}$$

Taking the first derivative of (2-6), we have

$$\left(\frac{\partial W_{ii}}{\partial W_{pq}} - \frac{\partial\lambda_k}{\partial W_{pq}}\right)\tau_i^k + (W_{ii}-\lambda_k)\frac{\partial\tau_i^k}{\partial W_{pq}} + \sum_{j\neq i}\left(\frac{\partial W_{ij}}{\partial W_{pq}}\tau_j^k + W_{ij}\frac{\partial\tau_j^k}{\partial W_{pq}}\right) = 0.$$

Hence, for $i=k$, we get, from (2-7),

$$\frac{\partial\lambda_k}{\partial W_{pq}} = \frac{\partial W_{kk}}{\partial W_{pq}} = \begin{cases} 1 & \text{if } p=k,\ q=k, \\ 0 & \text{otherwise,} \end{cases} \tag{2-8}$$

and, for $i\neq k$,

$$(W_{ii}-\lambda_k)\frac{\partial\tau_i^k}{\partial W_{pq}} + \sum_{j\neq i}\frac{\partial W_{ij}}{\partial W_{pq}}\tau_j^k = 0,$$

then

$$\frac{\partial\tau_i^k}{\partial W_{pq}} = \frac{1}{\lambda_k-\lambda_i}\frac{\partial W_{ik}}{\partial W_{pq}} = \begin{cases} \frac{1}{\lambda_k-\lambda_i} & \text{if } p=i,\ q=k, \\ 0 & \text{otherwise.} \end{cases} \tag{2-9}$$

Since $\tau^k\in\mathbb{S}^{n-1}$, we have

$$1 = |\tau^k|^2 = (\tau_1^k)^2 + \cdots + (\tau_k^k)^2 + \cdots + (\tau_n^k)^2. \tag{2-10}$$

Taking the first derivative of (2-10), and using (2-7),

$$\frac{\partial \tau_k^k}{\partial W_{pq}} = 0 \quad \text{for all } (p,q). \tag{2-11}$$

For $i = k$, taking the second derivative of (2-6), and using (2-7),

$$\left(\frac{\partial^2 W_{kk}}{\partial W_{pq}\partial W_{rs}} - \frac{\partial^2 \lambda_k}{\partial W_{pq}\partial W_{rs}}\right)\tau_k^k + \sum_{j\neq k}\left(\frac{\partial W_{kj}}{\partial W_{pq}}\frac{\partial \tau_j^k}{\partial W_{rs}} + \frac{\partial W_{kj}}{\partial W_{rs}}\frac{\partial \tau_j^k}{\partial W_{pq}}\right) = 0,$$

hence

$$\frac{\partial^2 \lambda_k}{\partial W_{pq}\partial W_{rs}} = \sum_{j\neq k}\left(\frac{\partial W_{kj}}{\partial W_{pq}}\frac{\partial \tau_j^k}{\partial W_{rs}} + \frac{\partial W_{kj}}{\partial W_{rs}}\frac{\partial \tau_j^k}{\partial W_{pq}}\right) = \begin{cases} \dfrac{1}{\lambda_k - \lambda_q} & \text{if } p = k,\ q \neq k,\ r = q,\ s = k, \\ \dfrac{1}{\lambda_k - \lambda_s} & \text{if } r = k,\ s \neq k,\ p = s,\ q = k, \\ 0 & \text{if otherwise.} \end{cases} \tag{2-12}$$

For $i \neq k$,

$$\left(\frac{\partial W_{ii}}{\partial W_{pq}} - \frac{\partial \lambda_k}{\partial W_{pq}}\right)\frac{\partial \tau_i^k}{\partial W_{rs}} + \left(\frac{\partial W_{ii}}{\partial W_{rs}} - \frac{\partial \lambda_k}{\partial W_{rs}}\right)\frac{\partial \tau_i^k}{\partial W_{pq}} + (W_{ii} - \lambda_k)\frac{\partial^2 \tau_i^k}{\partial W_{pq}\partial W_{rs}} + \sum_{j\neq i}\left(\frac{\partial W_{ij}}{\partial W_{pq}}\frac{\partial \tau_j^k}{\partial W_{rs}} + \frac{\partial W_{ij}}{\partial W_{rs}}\frac{\partial \tau_j^k}{\partial W_{pq}}\right) = 0,$$

then

$$\frac{\partial^2 \tau_i^k}{\partial W_{ik}\partial W_{ii}} = \frac{1}{\lambda_k - \lambda_i}\frac{\partial \tau_i^k}{\partial W_{ik}} = \frac{1}{\lambda_k - \lambda_i}\frac{1}{\lambda_k - \lambda_i} \qquad \text{if } i \neq k; \tag{2-13}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{iq}\partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i}\frac{\partial W_{iq}}{\partial W_{iq}}\frac{\partial \tau_q^k}{\partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i}\frac{1}{\lambda_k - \lambda_q} \qquad \text{if } i \neq k,\ i \neq q,\ q \neq k; \tag{2-14}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{ik}\partial W_{kk}} = \frac{1}{\lambda_k - \lambda_i}\left(-\frac{\partial \lambda_k}{\partial W_{kk}}\frac{\partial \tau_i^k}{\partial W_{ik}}\right) = -\frac{1}{\lambda_k - \lambda_i}\frac{1}{\lambda_k - \lambda_i} \qquad \text{if } i \neq k; \tag{2-15}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{pq}\partial W_{rs}} = 0 \qquad \text{otherwise.} \tag{2-16}$$

From (2-10), we have

$$2\tau_k^k \frac{\partial^2 \tau_k^k}{\partial W_{pq}\partial W_{rs}} + 2\sum_{i\neq k}\frac{\partial \tau_i^k}{\partial W_{pq}}\frac{\partial \tau_i^k}{\partial W_{rs}} = 0,$$

then

$$\frac{\partial^2 \tau_k^k}{\partial W_{pq}\partial W_{rs}} = -\sum_{i\neq k}\frac{\partial \tau_i^k}{\partial W_{pq}}\frac{\partial \tau_i^k}{\partial W_{rs}} = \begin{cases} -\dfrac{1}{\lambda_k - \lambda_p}\dfrac{1}{\lambda_k - \lambda_p} & \text{if } p \neq k,\ q = k,\ r = p,\ s = q, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Proposition 2.1 is finished. □

Example 2.2. When $n=2$, the matrix $\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ has two eigenvalues

$$\lambda_1 = \frac{(u_{11}+u_{22})+\sqrt{(u_{11}-u_{22})^2+4u_{12}u_{21}}}{2} \quad \text{and} \quad \lambda_2 = \frac{(u_{11}+u_{22})-\sqrt{(u_{11}-u_{22})^2+4u_{12}u_{21}}}{2}$$

with $\lambda_1 \geq \lambda_2$. If $\lambda_1 > \lambda_2$,

$$\left(\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} - \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0,$$

we get

$$\xi_1 = \frac{(u_{22}-u_{11})-\sqrt{(u_{11}-u_{22})^2+4u_{12}u_{21}}}{2} \quad \text{and} \quad \xi_2 = -u_{21}.$$

Then the eigenvector τ corresponding to λ_1 is

$$\tau = -\frac{(\xi_1, \xi_2)}{\sqrt{{\xi_1}^2+{\xi_2}^2}}.$$

By direct calcutions, we can get the derivatives of eigenvalues and the eigenvector τ with respect to the matrix, and this verifies Proposition 2.1.

3. Proof of Theorem 1.1

Now we start to prove Theorem 1.1.

Let $\tau(x) = \tau(D^2u(x)) = (\tau_1, \tau_2) \in \mathbb{S}^1$ be the continuous eigenvector field of $D^2u(x)$ corresponding to the largest eigenvalue. Let

$$\Sigma := \big\{x \in B_R(0) : r^2 - |x|^2 + \langle x, \tau(x)\rangle^2 > 0,\ r^2 - \langle x, \tau(x)\rangle^2 > 0\big\}, \tag{3-1}$$

where $r = R/\sqrt{2}$. It is easy to show that Σ is an open set and $B_r(0) \subset \Sigma \subset B_R(0)$. We introduce a new auxiliary function in Σ as follows:

$$\phi(x) = \eta(x)^\beta g\big(\tfrac{1}{2}|Du|^2\big)u_{\tau\tau}, \tag{3-2}$$

where $\eta(x) = \big(r^2 - |x|^2 + \langle x, \tau(x)\rangle^2\big)\big(r^2 - \langle x, \tau(x)\rangle^2\big)$ with $\beta = 4$ and $g(t) = e^{c_0 t/r^2}$ with $c_0 = 32/m$. In fact, $\langle x, \tau(x)\rangle$ is invariant under rotations of the coordinates, and so is $\eta(x)$.

From the definition of Σ, we know $\eta(x) > 0$ in Σ, and $\eta = 0$ on $\partial\Sigma$. Assume the maximum of $\phi(x)$ in Σ is attained at $x_0 \in \Sigma$. By rotating the coordinates, we can assume $D^2u(x_0)$ is diagonal. In the following, we let $\lambda_i = u_{ii}(x_0)$, $\lambda = (\lambda_1, \lambda_2)$. Without loss of generality, we can assume $\lambda_1 \geq \lambda_2$ and $\tau(x_0) = (1, 0)$.

If

$$\eta\lambda_1 \leq 10^3\left(1 + M + r\sup|\nabla f| + \frac{M}{m}\frac{\sup|Du|}{r}\right)r^4 =: \Theta,$$

then we easily get

$$\begin{aligned} u_{\tau(0)\tau(0)}(0) &\leq \frac{1}{r^{4\beta}}\phi(0) \leq \frac{1}{r^{4\beta}}\phi(x_0) \\ &\leq \Theta e^{c_0 \sup|Du|^2/r^2} \\ &\leq 10^3(1 + M + r\sup|\nabla f|)e^{(c_0+2M/m)\sup|Du|^2/r^2}. \end{aligned}$$

Hence we get

$$|u_{\xi\xi}(0)| \le u_{\tau(0)\tau(0)}(0) \le 10^3(1+M+r\sup|\nabla f|)e^{(c_0+2M/m)\sup|Du|^2/r^2} \quad \text{for all } \xi \in \mathbb{S}^1.$$

This completes the proof of Theorem 1.1 under the condition $\eta\lambda_1 \le \Theta$.

Now, we assume $\eta\lambda_1 \ge \Theta$. Then we have

$$\lambda_1 = \frac{\eta\lambda_1}{\eta} \ge \frac{\Theta}{r^4} = 10^3\biggl(1+M+r\sup|\nabla f| + \frac{M}{m}\frac{\sup|Du|}{r}\biggr). \tag{3-3}$$

From (1-1), we have

$$\lambda_2 = \frac{f}{\lambda_1} \le \frac{M}{\lambda_1} < \lambda_1.$$

Hence λ_1 is distinct from the other eigenvalue, and $\tau(x)$ is C^2 at x_0. Moreover, the test function

$$\varphi = \beta\log\eta + \log g\bigl(\tfrac{1}{2}|Du|^2\bigr) + \log u_{11} \tag{3-4}$$

attains the local maximum at x_0. In the following, all the calculations are at x_0.

Then, we get

$$0 = \varphi_i = \beta\frac{\eta_i}{\eta} + \frac{g'}{g}\sum_k u_k u_{ki} + \frac{u_{11i}}{u_{11}},$$

so we have

$$\frac{u_{11i}}{u_{11}} = -\beta\frac{\eta_i}{\eta} - \frac{g'}{g}u_i u_{ii} \quad \text{if } i = 1, 2. \tag{3-5}$$

At x_0, we also have

$$\begin{aligned} 0 \ge \varphi_{ii} &= \beta\biggl(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\biggr) + \frac{g''g - g'^2}{g^2}\sum_k u_k u_{ki}\sum_l u_l u_{li} + \frac{g'}{g}\sum_k (u_{ki}u_{ki} + u_k u_{kii}) + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} \\ &= \beta\biggl(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\biggr) + \frac{g'}{g}\biggl(u_{ii}^2 + \sum_k u_k u_{kii}\biggr) + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2}, \end{aligned}$$

since $g''g - g'^2 = 0$. Let

$$F^{11} = \frac{\partial \det D^2u}{\partial u_{11}} = \lambda_2, \quad F^{22} = \frac{\partial \det D^2u}{\partial u_{22}} = \lambda_1,$$
$$F^{12} = \frac{\partial \det D^2u}{\partial u_{12}} = 0, \quad F^{21} = \frac{\partial \det D^2u}{\partial u_{21}} = 0.$$

Then from (1-1) we get

$$\lambda_2 = \frac{f}{\lambda_1}. \tag{3-6}$$

Differentiating (1-1) once, we get

$$F^{11}u_{11i} + F^{22}u_{22i} = f_i,$$

then

$$u_{22i} = \frac{1}{F^{22}}(f_i - F^{11}u_{11i}) = \frac{f_i}{\lambda_1} - \frac{f}{\lambda_1}\frac{u_{11i}}{u_{11}}. \tag{3-7}$$

Differentiating (1-1) twice, we get

$$\begin{aligned}
F^{11}u_{1111}+F^{22}u_{2211} &= f_{11}-2\frac{\partial^2 \det D^2u}{\partial u_{11}\partial u_{22}}u_{111}u_{221}-2\frac{\partial^2 \det D^2u}{\partial u_{12}\partial u_{21}}u_{112}^2\\
&= f_{11}-2u_{111}u_{221}+2u_{112}^2\\
&= f_{11}+2u_{112}^2-2u_{111}\left(\frac{f_1}{\lambda_1}-\frac{f}{\lambda_1}\frac{u_{111}}{u_{11}}\right)\\
&= f_{11}+2u_{112}^2-2f_1\frac{u_{111}}{u_{11}}+2f\left(\frac{u_{111}}{u_{11}}\right)^2
\end{aligned}\tag{3-8}$$

and

$$\begin{aligned}
F^{11}u_{1112}+F^{22}u_{2212} &= f_{12}-\frac{\partial^2 \det D^2u}{\partial u_{11}\partial u_{22}}u_{111}u_{222}-\frac{\partial^2 \det D^2u}{\partial u_{22}\partial u_{11}}u_{221}u_{112}-2\frac{\partial^2 \det D^2u}{\partial u_{12}\partial u_{21}}u_{121}u_{212}\\
&= f_{12}-u_{111}u_{222}-u_{112}u_{221}+2u_{112}u_{221}\\
&= f_{12}-u_{111}\left(\frac{f_2}{\lambda_1}-\frac{f}{\lambda_1}\frac{u_{112}}{u_{11}}\right)+u_{112}\left(\frac{f_1}{\lambda_1}-\frac{f}{\lambda_1}\frac{u_{111}}{u_{11}}\right)\\
&= f_{12}+f_1\frac{u_{112}}{u_{11}}-f_2\frac{u_{111}}{u_{11}}.
\end{aligned}\tag{3-9}$$

Hence

$$\begin{aligned}
0 &\ge \sum_{i=1}^{2}F^{ii}\varphi_{ii}\\
&= \beta\sum_i F^{ii}\left(\frac{\eta_{ii}}{\eta}-\frac{\eta_i^2}{\eta^2}\right)+\frac{g'}{g}\sum_i F^{ii}u_{ii}^2+\frac{g'}{g}\sum_k u_kf_k+\frac{1}{u_{11}}\sum_i F^{ii}u_{11ii}-\sum_i F^{ii}\left(\frac{u_{11i}}{u_{11}}\right)^2\\
&= \beta\lambda_2\left(\frac{\eta_{11}}{\eta}-\frac{\eta_1^2}{\eta^2}\right)+\beta\lambda_1\left(\frac{\eta_{22}}{\eta}-\frac{\eta_2^2}{\eta^2}\right)+\frac{g'}{g}(\lambda_1+\lambda_2)f+\frac{g'}{g}(u_1f_1+u_2f_2)\\
&\qquad +\frac{1}{u_{11}}\left(f_{11}+2u_{112}^2-2f_1\frac{u_{111}}{u_{11}}+2f\left(\frac{u_{111}}{u_{11}}\right)^2\right)-\lambda_2\left(\frac{u_{111}}{u_{11}}\right)^2-\lambda_1\left(\frac{u_{112}}{u_{11}}\right)^2\\
&\ge \beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta}+\lambda_1\frac{\eta_{22}}{\eta}\right)-\beta\frac{f}{\lambda_1}\frac{\eta_1^2}{\eta^2}-\beta\lambda_1\frac{\eta_2^2}{\eta^2}+\frac{f}{\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2-2\frac{f_1}{u_{11}}\frac{u_{111}}{u_{11}}\\
&\qquad +\frac{\lambda_1}{2}\left(\frac{u_{112}}{u_{11}}\right)^2+\frac{\lambda_1}{2}\left(\beta\frac{\eta_2}{\eta}+\frac{g'}{g}u_2u_{22}\right)^2+\frac{g'}{g}f\lambda_1-\frac{g'}{g}|\nabla u||\nabla f|-\frac{|f_{11}|}{\lambda_1}\\
&\ge \beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta}+\lambda_1\frac{\eta_{22}}{\eta}\right)-\beta\frac{f}{\lambda_1}\frac{\eta_1^2}{\eta^2}-\beta\lambda_1\frac{\eta_2^2}{\eta^2}+\frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2\\
&\qquad +\frac{\lambda_1}{2}\left(\frac{u_{112}}{u_{11}}\right)^2+\frac{\beta^2}{2}\lambda_1\frac{\eta_2^2}{\eta^2}+\beta f\frac{g'}{g}\frac{\eta_2}{\eta}u_2+\frac{g'}{g}f\lambda_1-\frac{g'}{g}|\nabla u||\nabla f|-\frac{|f_{11}|}{\lambda_1}-2\frac{f_1^2}{f\lambda_1}.
\end{aligned}\tag{3-10}$$

Lemma 3.1. *Under the condition $\eta\lambda_1\ge\Theta$ we have, at x_0,*

$$\beta\frac{f}{\lambda_1}\frac{\eta_1^2}{\eta^2}\le\frac{8\beta f r^6}{\eta^2\lambda_1}+\frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2 \quad \textit{and} \quad \beta f\frac{g'}{g}\frac{\eta_2}{\eta}u_2\ge -4\beta f\frac{g'}{g}\frac{r^4}{\eta}\frac{|u_2|}{r},\tag{3-11}$$

and

$$\beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta}+\lambda_1\frac{\eta_{22}}{\eta}\right)\geq -\frac{1}{2}\frac{g'}{g}f\lambda_1-\beta\lambda_1\left(\frac{\eta_2}{\eta}\right)^2-\frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2-\frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2-2\beta f\frac{r^2}{\eta\lambda_1}$$
$$-4\beta|f_{12}|\frac{r^4}{\eta\lambda_1}-32\beta\frac{f_1^2r^4}{\eta\lambda_1^3}-8\beta\frac{|f_1f_2|r^4}{\eta\lambda_1^3}-24\beta\frac{|f_1|r^3}{\eta\lambda_1}-\left(\frac{6\beta|f_1|^2r^4}{\eta\lambda_1}+\frac{12\beta fr^2}{\eta\lambda_1}\right)$$
$$-\left(\frac{8\beta fr^4|f_2|^2}{\eta\lambda_1^3}+\frac{(48\beta)^2fr^6}{\eta^2\lambda_1}+\frac{32\beta^2|f_2|^2r^8}{\eta^2\lambda_1 f}\right). \tag{3-12}$$

Proof. At x_0, $\tau=(\tau_1,\tau_2)=(1,0)$. Then from Proposition 2.1 we get

$$\langle x,\partial_i\tau\rangle=\sum_{m=1}^{2}x_m\frac{\partial\tau_m}{\partial x_i}=\sum_{m=1}^{2}x_m\frac{\partial\tau_m}{\partial u_{pq}}u_{pqi}=x_2\frac{\partial\tau_2}{\partial u_{pq}}u_{pqi}=x_2\frac{u_{12i}}{\lambda_1-\lambda_2}\quad\text{if } i=1,2. \tag{3-13}$$

From the definition of η, then we have, at x_0,

$$\eta=(r^2-|x|^2+\langle x,\tau\rangle^2)(r^2-\langle x,\tau\rangle^2)=(r^2-x_2^2)(r^2-x_1^2). \tag{3-14}$$

Taking the first derivative of η, we get

$$\begin{aligned}\eta_i&=(-2x_i+2\langle x,\tau\rangle\langle x,\tau\rangle_i)(r^2-\langle x,\tau\rangle^2)+(r^2-|x|^2+\langle x,\tau\rangle^2)(-2\langle x,\tau\rangle\langle x,\tau\rangle_i)\\&=\big(-2x_i+2x_1(\delta_{i1}+\langle x,\partial_i\tau\rangle)\big)(r^2-x_1^2)+(r^2-x_2^2)\big(-2x_1(\delta_{i1}+\langle x,\partial_i\tau\rangle)\big)\\&=\begin{cases}-2x_1(r^2-x_2^2)+2x_1\langle x,\partial_1\tau\rangle(x_2^2-x_1^2) & \text{if } i=1,\\ -2x_2(r^2-x_1^2)+2x_1\langle x,\partial_2\tau\rangle(x_2^2-x_1^2) & \text{if } i=2.\end{cases}\end{aligned}$$

Hence

$$\begin{aligned}\beta\frac{f}{\lambda_1}\frac{\eta_1^2}{\eta^2}&=\beta\frac{f}{\lambda_1}\left(\frac{-2x_1(r^2-x_2^2)}{\eta}+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{u_{112}}{\lambda_1-\lambda_2}\right)^2\\&\leq\beta\frac{f}{\lambda_1}\left(\frac{8r^6}{\eta^2}+\frac{8r^8}{\eta^2}\left(\frac{u_{112}}{u_{11}}\right)^2\right)\leq\frac{8\beta fr^6}{\eta^2\lambda_1}+\frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2.\end{aligned} \tag{3-15}$$

Also we have

$$\begin{aligned}\frac{\eta_2}{\eta}&=\frac{-2x_2(r^2-x_1^2)}{\eta}+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{u_{221}}{\lambda_1-\lambda_2}\\&=\frac{-2x_2(r^2-x_1^2)}{\eta}+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{1}{\lambda_1-\lambda_2}\left(\frac{f_1}{\lambda_1}-\frac{f}{\lambda_1}\frac{u_{111}}{u_{11}}\right)\\&=\frac{-2x_2(r^2-x_1^2)}{\eta}\left(1-x_1\frac{(x_2^2-x_1^2)(r^2-x_2^2)}{\eta}\frac{1}{\lambda_1-\lambda_2}\frac{f_1}{\lambda_1}\right)+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{1}{\lambda_1-\lambda_2}\frac{f}{\lambda_1}\left(\beta\frac{\eta_1}{\eta}+\frac{g'}{g}u_1u_{11}\right)\\&=\frac{-2x_2(r^2-x_1^2)}{\eta}\left(1-x_1\frac{(x_2^2-x_1^2)(r^2-x_2^2)}{\eta}\frac{1}{\lambda_1-\lambda_2}\left(\frac{f_1}{\lambda_1}+\frac{g'}{g}u_1f\right)\right)\\&\qquad+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{1}{\lambda_1-\lambda_2}\frac{f}{\lambda_1}\beta\left(\frac{-2x_1(r^2-x_2^2)}{\eta}+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{u_{112}}{\lambda_1-\lambda_2}\right)\end{aligned}$$

$$= \frac{-2x_2(r^2-x_1^2)}{\eta}\left(1 - x_1\frac{(x_2^2-x_1^2)(r^2-x_2^2)}{\eta}\frac{1}{\lambda_1-\lambda_2}\left(\frac{f_1}{\lambda_1}+\frac{g'}{g}u_1 f - \frac{f}{\lambda_1}\beta\frac{2x_1(r^2-x_2^2)}{\eta}\right)\right) + \left(2x_1x_2\frac{x_2^2-x_1^2}{\eta}\right)^2\frac{f}{(\lambda_1-\lambda_2)^2}\beta\left(-\beta\frac{\eta_2}{\eta}-\frac{g'}{g}u_2u_{22}\right),$$

then we get

$$\left(1+\beta^2\left(2x_1x_2\frac{x_2^2-x_1^2}{\eta}\right)^2\frac{f}{(\lambda_1-\lambda_2)^2}\right)\frac{\eta_2}{\eta} = \frac{-2x_2(r^2-x_1^2)}{\eta}\left(1 - x_1\frac{(x_2^2-x_1^2)(r^2-x_2^2)}{\eta}\frac{1}{\lambda_1-\lambda_2}\left(\frac{f_1}{\lambda_1}+\frac{g'}{g}u_1 f - \frac{f}{\lambda_1}\beta\frac{2x_1(r^2-x_2^2)}{\eta}\right)\right) + \frac{-2x_2(r^2-x_1^2)}{\eta}\frac{2x_1^2x_2(x_2^2-x_1^2)^2(r^2-x_2^2)}{\eta^2}\frac{f}{(\lambda_1-\lambda_2)^2}\beta\frac{g'}{g}u_2\frac{f}{\lambda_1}. \tag{3-16}$$

Hence

$$\begin{aligned}\beta f\frac{g'}{g}\frac{\eta_2}{\eta}u_2 &\geq -\beta f\frac{g'}{g}\frac{2r^3}{\eta}\left(1+\frac{2r^5}{\eta\lambda_1}\left(\frac{|f_1|}{\lambda_1}+\frac{g'}{g}|u_1|f+\frac{2\beta f r^3}{\eta\lambda_1}\right)+\frac{16\beta r^9}{\eta^2}\frac{f^2}{\lambda_1^3}\frac{g'}{g}|u_2|\right)|u_2| \\ &\geq -\beta f\frac{g'}{g}\frac{2r^3}{\eta}\left(1+\frac14+\frac14+\frac14+\frac14\right)|u_2| \\ &= -4\beta f\frac{g'}{g}\frac{r^4}{\eta}\frac{|u_2|}{r}.\end{aligned}\tag{3-17}$$

In fact, $\eta_2/\eta \approx -2x_2(r^2-x_1^2)/\eta$ if $\eta\lambda_1$ is big enough, and we get from (3-16)

$$\begin{aligned}\frac{\eta_2^2}{\eta^2} &\geq \left(\left(1+\beta^2\left(2x_1x_2\frac{x_2^2-x_1^2}{\eta}\right)^2\frac{f}{(\lambda_1-\lambda_2)^2}\right)\frac{\eta_2}{\eta}\right)^2\left(1-\beta^2\left(2x_1x_2\frac{x_2^2-x_1^2}{\eta}\right)^2\frac{f}{(\lambda_1-\lambda_2)^2}\right)^2 \\ &\geq \left(\frac{-2x_2(r^2-x_1^2)}{\eta}\right)^2\left(1-\frac{2r^5}{\eta\lambda_1}\left(\frac{|f_1|}{\lambda_1}+\frac{g'}{g}|u_1|f+\frac{2\beta f r^3}{\eta\lambda_1}\right)-\frac{8\beta r^9}{\eta^2\lambda_1^2}\frac{g'}{g}|u_2|\frac{f^2}{\lambda_1}\right)^2\left(1-\beta^2\frac{16r^8}{\eta^2}\frac{f}{\lambda_1^2}\right)^2 \\ &\geq \left(\frac{-2x_2(r^2-x_1^2)}{\eta}\right)^2\left(1-\frac{1}{10^3}-\frac{64}{10^3}-\frac{1}{10}-\frac{1}{10^3}\right)^2\left(1-\frac{1}{10^3}\right)^2 \\ &\geq \frac12\left(\frac{-2x_2(r^2-x_1^2)}{\eta}\right)^2.\end{aligned}$$

Taking second derivatives of η, we get

$$\begin{aligned}\eta_{ii} &= (-2+2\langle x,\tau\rangle\langle x,\tau\rangle_{ii}+2\langle x,\tau\rangle_i\langle x,\tau\rangle_i)(r^2-\langle x,\tau\rangle^2) \\ &\quad + 2(-2x_i+2\langle x,\tau\rangle\langle x,\tau\rangle_i)(-2\langle x,\tau\rangle\langle x,\tau\rangle_i) \\ &\quad + (r^2-|x|^2+\langle x,\tau\rangle^2)(-2\langle x,\tau\rangle\langle x,\tau\rangle_{ii}-2\langle x,\tau\rangle_i\langle x,\tau\rangle_i) \\ &= \big(-2+2x_1\langle x,\tau\rangle_{ii}+2(\delta_{i1}+\langle x,\partial_i\tau\rangle)^2\big)(r^2-x_1^2) \\ &\quad + 2\big(-2x_i+2x_1(\delta_{i1}+\langle x,\partial_i\tau\rangle)\big)\big(-2x_1(\delta_{i1}+\langle x,\partial_i\tau\rangle)\big) \\ &\quad + (r^2-x_2^2)\big(-2x_1\langle x,\tau\rangle_{ii}-2(\delta_{i1}+\langle x,\partial_i\tau\rangle)^2\big),\end{aligned}$$

so

$$\eta_{11} = -2(r^2 - x_2^2) - 2x_1(x_1^2 - x_2^2)\langle x, \tau\rangle_{11} + (4x_2^2 - 12x_1^2)\langle x, \partial_1\tau\rangle + (2x_2^2 - 10x_1^2)\langle x, \partial_1\tau\rangle^2, \quad (3\text{-}18)$$

$$\eta_{22} = -2(r^2 - x_1^2) - 2x_1(x_1^2 - x_2^2)\langle x, \tau\rangle_{22} + 8x_1x_2\langle x, \partial_2\tau\rangle + (2x_2^2 - 10x_1^2)\langle x, \partial_2\tau\rangle^2. \quad (3\text{-}19)$$

Hence

$$\begin{aligned}\beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta} + \lambda_1\frac{\eta_{22}}{\eta}\right) = &-2\beta\left(\frac{f}{\lambda_1}\frac{r^2 - x_2^2}{\eta} + \lambda_1\frac{r^2 - x_1^2}{\eta}\right) - 2\beta\frac{x_1(x_1^2 - x_2^2)}{\eta}\left(\frac{f}{\lambda_1}\langle x, \tau\rangle_{11} + \lambda_1\langle x, \tau\rangle_{22}\right)\\ &+ \beta\frac{f}{\lambda_1}\left(\frac{x_2(4x_2^2 - 12x_1^2)}{\eta}\frac{u_{112}}{\lambda_1 - \lambda_2} + \frac{x_2^2(2x_2^2 - 10x_1^2)}{\eta}\left(\frac{u_{112}}{\lambda_1 - \lambda_2}\right)^2\right)\\ &+ \beta\lambda_1\left(\frac{8x_1x_2^2}{\eta}\frac{u_{221}}{\lambda_1 - \lambda_2} + \frac{x_2^2(2x_2^2 - 10x_1^2)}{\eta}\left(\frac{u_{221}}{\lambda_1 - \lambda_2}\right)^2\right). \quad (3\text{-}20)\end{aligned}$$

Direct calculations yield

$$\begin{aligned}\langle x, \tau\rangle_{11} &= \frac{\partial^2}{\partial x_1^2}\left(\sum_{m=1}^{2} x_m\tau_m\right) = 2\frac{\partial\tau_1}{\partial x_1} + \sum_{m=1}^{2} x_m\frac{\partial^2\tau_m}{\partial x_1^2}\\ &= 2\frac{\partial\tau_1}{\partial u_{pq}}u_{pq1} + \sum_{m=1}^{2} x_m\left(\frac{\partial\tau_m}{\partial u_{pq}}u_{pq11} + \frac{\partial^2\tau_m}{\partial u_{pq}\partial u_{rs}}u_{pq1}u_{rs1}\right)\\ &= 0 + x_1\frac{\partial^2\tau_1}{\partial u_{pq}\partial u_{rs}}u_{pq1}u_{rs1} + x_2\left(\frac{\partial\tau_2}{\partial u_{pq}}u_{pq11} + \frac{\partial^2\tau_2}{\partial u_{pq}\partial u_{rs}}u_{pq1}u_{rs1}\right)\\ &= -x_1\left(\frac{u_{112}}{\lambda_1 - \lambda_2}\right)^2 + x_2\left(\frac{1}{\lambda_1 - \lambda_2}\right)u_{1211} + 2x_2\left(-\frac{u_{112}u_{111}}{(\lambda_1 - \lambda_2)^2} + \frac{u_{112}u_{221}}{(\lambda_1 - \lambda_2)^2}\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\langle x, \tau\rangle_{22} &= \frac{\partial^2}{\partial x_2^2}\left(\sum_{m=1}^{2} x_m\tau_m\right) = 2\frac{\partial\tau_2}{\partial x_2} + \sum_{m=1}^{2} x_m\frac{\partial^2\tau_m}{\partial x_1^2}\\ &= 2\frac{\partial\tau_2}{\partial u_{pq}}u_{pq2} + \sum_{m=1}^{2} x_m\left(\frac{\partial\tau_m}{\partial u_{pq}}u_{pq22} + \frac{\partial^2\tau_m}{\partial u_{pq}\partial u_{rs}}u_{pq2}u_{rs2}\right)\\ &= 2\left(\frac{1}{\lambda_1 - \lambda_2}\right)u_{221} - x_1\left(\frac{u_{221}}{\lambda_1 - \lambda_2}\right)^2 + x_2\left(\frac{1}{\lambda_1 - \lambda_2}\right)u_{1222} + 2x_2\left(-\frac{u_{112}u_{221}}{(\lambda_1 - \lambda_2)^2} + \frac{u_{222}u_{221}}{(\lambda_1 - \lambda_2)^2}\right),\end{aligned}$$

then

$$\begin{aligned}\frac{f}{\lambda_1}\langle x, \tau\rangle_{11} + \lambda_1\langle x, \tau\rangle_{22} = &-x_1\frac{f}{\lambda_1}\left(\frac{u_{112}}{\lambda_1 - \lambda_2}\right)^2 + 2\lambda_1\left(\frac{u_{221}}{\lambda_1 - \lambda_2}\right) - x_1\lambda_1\left(\frac{u_{221}}{\lambda_1 - \lambda_2}\right)^2\\ &+ x_2\left(\frac{1}{\lambda_1 - \lambda_2}\right)\left(f_{12} + f_1\frac{u_{112}}{u_{11}} - f_2\left(\frac{u_{111}}{u_{11}}\right)\right)\\ &+ 2x_2\left(-\frac{u_{112}}{(\lambda_1 - \lambda_2)^2}f_1 + \frac{u_{221}}{(\lambda_1 - \lambda_2)^2}f_2\right). \quad (3\text{-}21)\end{aligned}$$

From (3-20) and (3-21), we get

$$
\begin{aligned}
&\beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta}+\lambda_1\frac{\eta_{22}}{\eta}\right)\\
&\quad= -2\beta\left(\frac{f}{\lambda_1}\frac{r^2-x_2^2}{\eta}+\lambda_1\frac{r^2-x_1^2}{\eta}\right)-2\beta\frac{x_1x_2(x_1^2-x_2^2)}{\eta}\left(\frac{1}{\lambda_1-\lambda_2}\right)\left(f_{12}-f_2\left(\frac{u_{111}}{u_{11}}\right)\right)\\
&\qquad+\left(\frac{u_{112}}{\lambda_1-\lambda_2}\right)^2\left(2\beta\frac{f}{\lambda_1}\frac{x_1^2(x_1^2-x_2^2)}{\eta}+\beta\frac{f}{\lambda_1}\frac{x_2^2(2x_2^2-10x_1^2)}{\eta}\right)\\
&\qquad+\frac{u_{112}}{\lambda_1-\lambda_2}\left(-2\beta\frac{x_1(x_1^2-x_2^2)}{\eta}\left(x_2\frac{f_1}{\lambda_1}-2x_2\frac{f_1}{\lambda_1-\lambda_2}\right)+\beta\frac{f}{\lambda_1}\frac{x_2(4x_2^2-12x_1^2)}{\eta}\right)\\
&\qquad+\left(\frac{u_{221}}{\lambda_1-\lambda_2}\right)^2\left(2\beta\lambda_1\frac{x_1^2(x_1^2-x_2^2)}{\eta}+\beta\lambda_1\frac{x_2^2(2x_2^2-10x_1^2)}{\eta}\right)\\
&\qquad+\frac{u_{221}}{\lambda_1-\lambda_2}\left(-2\beta\frac{x_1(x_1^2-x_2^2)}{\eta}\left(2\lambda_1+2x_2\frac{f_2}{\lambda_1-\lambda_2}\right)+\beta\lambda_1\frac{8x_1x_2^2}{\eta}\right)\\
&\quad\geq -2\beta\lambda_1\frac{r^2-x_1^2}{\eta}-2\beta f\frac{r^2}{\eta\lambda_1}-2\beta|f_{12}|\frac{r^4}{\eta(\lambda_1-\lambda_2)}-2\beta|f_2|\frac{r^4}{\eta(\lambda_1-\lambda_2)}\left|\frac{u_{111}}{u_{11}}\right|\\
&\qquad-\left(\frac{u_{112}}{(\lambda_1-\lambda_2)}\right)^2\beta\frac{f}{\lambda_1}\frac{8r^4}{\eta}-\left|\frac{u_{112}}{\lambda_1-\lambda_2}\right|\left(6\beta\frac{|f_1|}{\lambda_1-\lambda_2}\frac{r^4}{\eta}+\beta\frac{f}{\lambda_1}\frac{12r^3}{\eta}\right)\\
&\qquad-\frac{1}{(\lambda_1-\lambda_2)^2}(u_{221})^2\beta\lambda_1\frac{8r^4}{\eta}-\frac{1}{\lambda_1-\lambda_2}|u_{221}|\left(4\beta\frac{r^4}{\eta}\frac{|f_2|}{\lambda_1}+\beta\lambda_1\frac{12r^3}{\eta}\right)\\
&\quad\geq -2\beta\lambda_1\frac{r^2-x_1^2}{\eta}-2\beta f\frac{r^2}{\eta\lambda_1}-4\beta|f_{12}|\frac{r^4}{\eta\lambda_1}-4\beta|f_2|\frac{r^4}{\eta\lambda_1}\left|\frac{u_{111}}{u_{11}}\right|\\
&\qquad-\left(\frac{u_{112}}{u_{11}}\right)^2\beta\frac{f}{\lambda_1}\frac{16r^4}{\eta}-\left|\frac{u_{112}}{u_{11}}\right|\left(12\beta\frac{|f_1|}{\lambda_1}\frac{r^4}{\eta}+\beta\frac{f}{\lambda_1}\frac{24r^3}{\eta}\right)\\
&\qquad-2\left(\frac{f_1^2}{\lambda_1^4}+\frac{f^2}{\lambda_1^4}\left(\frac{u_{111}}{u_{11}}\right)^2\right)\beta\lambda_1\frac{16r^4}{\eta}-\left(\frac{|f_1|}{\lambda_1^2}+\frac{f}{\lambda_1^2}\left|\frac{u_{111}}{u_{11}}\right|\right)\left(8\beta\frac{r^4}{\eta}\frac{|f_2|}{\lambda_1}+\beta\lambda_1\frac{24r^3}{\eta}\right)\\
&\quad\geq -2\beta\lambda_1\frac{r^2-x_1^2}{\eta}-2\beta f\frac{r^2}{\eta\lambda_1}-4\beta|f_{12}|\frac{r^4}{\eta\lambda_1}-32\beta\frac{f_1^2r^4}{\eta\lambda_1^3}-8\beta\frac{|f_1f_2|r^4}{\eta\lambda_1^3}-24\beta\frac{|f_1|r^3}{\eta\lambda_1}\\
&\qquad-\frac{\lambda_1}{2}\left(\frac{u_{112}}{u_{11}}\right)^2\left(\frac{32\beta fr^4}{\eta\lambda_1^2}+\frac{12\beta r^4}{\eta\lambda_1^2}+\frac{24\beta fr^4}{\eta\lambda_1^2}\right)-\frac{\lambda_1}{2}\left(\frac{12\beta|f_1|^2r^4}{\eta\lambda_1^2}+\frac{24\beta fr^2}{\eta\lambda_1^2}\right)\\
&\qquad-\frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2\left(\frac{64\beta fr^4}{\eta\lambda_1^2}+\frac{16\beta r^4}{\eta\lambda_1^2}+\frac{1}{4}+\frac{1}{4}\right)\\
&\qquad-\frac{f}{2\lambda_1}\left(\frac{16\beta r^4|f_2|^2}{\eta\lambda_1^2}+\left(\frac{48\beta r^3}{\eta}\right)^2+\left(\frac{8\beta r^4}{\eta}\frac{|f_2|}{f}\right)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\geq & -2\beta\lambda_1 \frac{r^2-x_1^2}{\eta} - 2\beta f \frac{r^2}{\eta\lambda_1} - 4\beta|f_{12}|\frac{r^4}{\eta\lambda_1} - 32\beta\frac{f_1^2r^4}{\eta\lambda_1^3} - 8\beta\frac{|f_1f_2|r^4}{\eta\lambda_1^3} - 24\beta\frac{|f_1|r^3}{\eta\lambda_1} \\
& - \frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2 - \left(\frac{6\beta|f_1|^2r^4}{\eta\lambda_1} + \frac{12\beta f r^2}{\eta\lambda_1}\right) \\
& - \frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2 - \left(\frac{8\beta f r^4|f_2|^2}{\eta\lambda_1^3} + \frac{(48\beta)^2 f r^6}{\eta^2\lambda_1} + \frac{32\beta^2|f_2|^2r^8}{\eta^2\lambda_1 f}\right).
\end{aligned}
$$

Now we just need to estimate $-2\beta\lambda_1(r^2-x_1^2)/\eta$. If $x_2^2 \leq \frac{1}{2}r^2$, we get

$$
-2\beta\lambda_1\frac{r^2-x_1^2}{\eta} = -\frac{8}{r^2-x_2^2}\lambda_1 \geq -\frac{16}{r^2}\lambda_1 \geq -\frac{1}{2}\frac{c_0}{r^2} f\lambda_1 = -\frac{1}{2}\frac{g'}{g} f\lambda_1.
$$

If $x_2^2 \geq \frac{1}{2}r^2$, we get

$$
-2\beta\lambda_1\frac{r^2-x_1^2}{\eta} = -\frac{8}{r^2-x_2^2}\lambda_1 \geq -\frac{x_2^2}{r^2-x_2^2}\frac{8}{r^2-x_2^2}\lambda_1 = -\beta\lambda_1\frac{1}{2}\left(\frac{2x_2}{r^2-x_2^2}\right)^2 \geq -\beta\lambda_1\left(\frac{\eta_2}{\eta}\right)^2.
$$

Hence

$$
-2\beta\lambda_1\frac{r^2-x_1^2}{\eta} \geq -\frac{1}{2}\frac{g'}{g} f\lambda_1 - \beta\lambda_1\left(\frac{\eta_2}{\eta}\right)^2 \tag{3-22}
$$

and

$$
\begin{aligned}
\beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta} + \lambda_1\frac{\eta_{22}}{\eta}\right) \geq & -\frac{1}{2}\frac{g'}{g} f\lambda_1 - \beta\lambda_1\left(\frac{\eta_2}{\eta}\right)^2 - \frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2 - \frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2 \\
& - 2\beta f\frac{r^2}{\eta\lambda_1} - 4\beta|f_{12}|\frac{r^4}{\eta\lambda_1} - 32\beta\frac{f_1^2r^4}{\eta\lambda_1^3} - 8\beta\frac{|f_1f_2|r^4}{\eta\lambda_1^3} - 24\beta\frac{|f_1|r^3}{\eta\lambda_1} \\
& - \left(\frac{6\beta|f_1|^2r^4}{\eta\lambda_1} + \frac{12\beta f r^2}{\eta\lambda_1}\right) \\
& - \left(\frac{8\beta f r^4|f_2|^2}{\eta\lambda_1^3} + \frac{(48\beta)^2 f r^6}{\eta^2\lambda_1} + \frac{32\beta^2|f_2|^2r^8}{\eta^2\lambda_1 f}\right).
\end{aligned} \tag{3-23}
$$

This concludes the proof of Lemma 3.1. □

Now we continue to prove Theorem 1.1. From (3-10) and Lemma 3.1, we get

$$
\begin{aligned}
0 \geq & \sum_{i=1}^{2} F^{ii}\varphi_{ii} \\
\geq & \frac{1}{2}\frac{g'}{g} f\lambda_1 - 2\beta f\frac{r^2}{\eta\lambda_1} - 4\beta|f_{12}|\frac{r^4}{\eta\lambda_1} - 32\beta\frac{f_1^2r^4}{\eta\lambda_1^3} - 8\beta\frac{|f_1f_2|r^4}{\eta\lambda_1^3} - 24\beta\frac{|f_1|r^3}{\eta\lambda_1} \\
& - \left(\frac{6\beta|f_1|^2r^4}{\eta\lambda_1} + \frac{12\beta f r^2}{\eta\lambda_1}\right) - \left(\frac{8\beta f r^4|f_2|^2}{\eta\lambda_1^3} + \frac{(48\beta)^2 f r^6}{\eta^2\lambda_1} + \frac{32\beta^2|f_2|^2r^8}{\eta^2\lambda_1 f}\right) \\
& - 4\beta f\frac{g'}{g}\frac{r^4}{\eta}\frac{|u_2|}{r} - \frac{g'}{g}|\nabla u||\nabla f| - \frac{|f_{11}|}{\lambda_1} - 2\frac{f_1^2}{f\lambda_1} - \frac{8\beta f r^6}{\eta^2\lambda_1}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{c_0 m}{2r^2}\lambda_1 - \frac{8r^2\cdot M}{\eta\lambda_1} - \frac{32r^2\cdot r^2|f_{12}|}{\eta\lambda_1} - \frac{128r^2\cdot r^2 f_1^2}{\eta\lambda_1^3} - \frac{32r^2\cdot r^2|f_1 f_2|}{\eta\lambda_1^3} - \frac{96r^2\cdot r|f_1|}{\eta\lambda_1}\\
&\quad - \frac{24r^2\cdot r^2|f_1|^2}{\eta\lambda_1} - \frac{48r^2\cdot M}{\eta\lambda_1} - \frac{32r^2\cdot M\cdot r^2|f_2|^2}{\eta\lambda_1^3} - \frac{192^2\cdot M\cdot r^6}{\eta^2\lambda_1} - \frac{512r^6\cdot r^2|f_2|^2\cdot 1/m}{\eta^2\lambda_1}\\
&\quad - \frac{16r^2\cdot c_0 m}{\eta}\frac{|u_2|}{r} - \frac{c_0}{r^2}\cdot r|\nabla f|\cdot\frac{|\nabla u|}{r} - \frac{|f_{11}|}{\lambda_1} - 2\frac{f_1^2}{m\lambda_1} - \frac{32Mr^6}{\eta^2\lambda_1}.
\end{aligned}
$$

So we get

$$
\eta\lambda_1 \leq C\left(1+\frac{|\nabla u|}{r}\right)r^4, \tag{3-24}
$$

where C is a positive constant depending only on c_0, m, M, $r|\nabla f|$ and $r^2|\nabla^2 f|$. So we easily get

$$
u_{\tau(0)\tau(0)}(0) \leq \frac{1}{r^{4\beta}}\phi(0) \leq \frac{1}{r^{4\beta}}\phi(x_0) \leq C\left(1+\frac{\sup|Du|}{r}\right)e^{c_0\sup|Du|^2/r^2} \leq Ce^{(c_0+2)\sup|Du|^2/r^2}
$$

and

$$
|u_{\xi\xi}(0)| \leq u_{\tau(0)\tau(0)}(0) \leq Ce^{(c_0+2)\sup|Du|^2/r^2} \quad \text{for all } \xi\in\mathbb{S}^1. \tag{3-25}
$$

This completes the proof of Theorem 1.1 under the condition $\eta\lambda_1 \geq \Theta$. Hence Theorem 1.1 holds.

Remark 3.2. The eigenvector field τ is important. In fact, it is well-defined when the largest eigenvalue is distinct from the others, and τ depends only on the adjoint matrix. For the Monge–Ampère equation in dimension $n \geq 3$, we do not know whether the largest eigenvalue is distinct from the others, so our method is not suitable for this case.

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