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BOUNDED SOLUTIONS TO THE ALLEN–CAHN EQUATION
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ALBERTO ENCISO AND DANIEL PERALTA-SALAS

We make use of the flexibility of infinite-index solutions to the Allen–Cahn equation to show that, given any compact hypersurface Σ of $\mathbb{R}^d$ with $d \geqslant 3$, there is a bounded entire solution of the Allen–Cahn equation on $\mathbb{R}^d$ whose zero level set has a connected component diffeomorphic (and arbitrarily close) to a rescaling of Σ. More generally, we prove the existence of solutions with a finite number of compact connected components of prescribed topology in their zero level sets.

1. Introduction

The study of the analogies between the level sets of the solutions to the Allen–Cahn equation

$$\Delta u + u - u^3 = 0$$

in $\mathbb{R}^d$ and minimal hypersurfaces in $\mathbb{R}^d$ was greatly fostered by De Giorgi's 1978 conjecture that all the level sets of any entire solution to the Allen–Cahn equation that is monotone in one direction have to be hyperplanes for $d \leqslant 8$. This is a natural counterpart of the Bernstein problem for minimal hypersurfaces, which asserts that any minimal graph in $\mathbb{R}^d$ must be a hyperplane provided that $d \leqslant 8$. Ghoussoub and Gui [1998] and Ambrosio and Cabré [2000] proved De Giorgi's conjecture for $d = 2, 3$, and the work of Savin [2009] showed that it is also true for $4 \leqslant d \leqslant 8$ under a weak additional technical assumption. Del Pino, Kowalczyk and Wei [del Pino et al. 2011] employed the Bombieri–De Giorgi–Giusti hypersurface to show that the statement of De Giorgi's conjecture does not hold for $d \geqslant 9$.

In dimension 2, it is well known [Dancer 2005] that the monotonicity hypothesis can be relaxed to the assumption that the solution u is *stable*, i.e., that its Morse index is 0. Let us recall that the *Morse index* of u is the maximal dimension of a vector space $V \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (|\nabla v|^2 - v^2 + 3u^2 v^2)\, dx < 0$$

for all nonzero $v \in V$. Remarkably, it has been shown recently [Pacard and Wei 2013] that in dimension 8 (actually, in any even dimension $d \geqslant 8$) there are bounded stable solutions to the Allen–Cahn equation whose level sets are not hyperplanes, but rather they are asymptotic to a minimal cone. For the role of minimal cones in the Allen–Cahn equation, see also [Cabré and Terra 2009] and references therein. In dimensions $d \leqslant 7$, the level sets of stable solutions to the Allen–Cahn equation are conjectured to all be hyperplanes [Pacard and Wei 2013].

The analysis and possible classification of bounded entire solutions to the Allen–Cahn equation is an important open problem, where the Morse index of the solutions plays a key role. Unlike the stable case [Dancer 2005], the structure of solutions with finite Morse index can be very complex; in fact, in dimension 3 a result of del Pino, Kowalczyk and Wei [del Pino et al. 2013] ensures that, under mild technical assumptions, given any embedded complete minimal surface in $\mathbb{R}^3$ with finite total curvature there is a bounded entire solution to the Allen–Cahn equation with a level set that is close to a large rescaling of this minimal surface, and that the Morse index of this solution coincides with the genus of the surface. Also in this direction, the existence of solutions to the Allen–Cahn equation with a level set close to a nondegenerate minimal hypersurface was proved by Pacard and Ritoré [2003] provided that the ambient space is a compact Riemannian manifold (instead of $\mathbb{R}^d$). Furthermore, Agudelo, del Pino and Wei [Agudelo et al. 2015] have recently constructed bounded entire axisymmetric solutions on $\mathbb{R}^3$ of arbitrarily large index that have multiple catenoidal ends.

Generally speaking, it is expected [del Pino et al. 2012; 2013] that the condition that the Morse index of the solution be finite should play a similar role to the finite total curvature assumption in the study of minimal hypersurfaces in Euclidean spaces. In particular, it is well known that there are many infinite-index solutions to the Allen–Cahn equation [Dancer 2001; Cabré and Terra 2009; Kowalczyk et al. 2015], and this abundance of solutions should translate into a wealth of possible level sets.

Our objective in this paper is to explore the flexibility of bounded entire solutions to the Allen–Cahn equation of infinite index by showing that there are bounded solutions to the Allen–Cahn equation on $\mathbb{R}^d$ with level sets of any compact topology. Specifically, given a compact hypersurface Σ without boundary of $\mathbb{R}^d$, we will show that there is a rescaling of Σ that is arbitrarily close to a connected component of the nodal set of a bounded entire solution of the Allen–Cahn equation. Furthermore, this level set is *structurally stable*, in the sense that any function on $\mathbb{R}^d$ which is sufficiently close to u in the C^1 norm in a neighborhood of this set will also have a zero level set of the same topology. In view of the existing literature, we are particularly interested in the case of high-dimension d.

To present a precise statement, let us agree to say that an ϵ-*rescaling* is a diffeomorphism of $\mathbb{R}^d$ that can be written as $\Phi = \Phi_1 \circ \Phi_2$, where Φ_2 is a rescaling and $\|\Phi_1 - \mathrm{id}\|_{C^1(\mathbb{R}^d)} < \epsilon$ (here we could have taken any other fixed C^k norm, though). By a *hypersurface* we will refer to a smoothly embedded codimension 1 submanifold of $\mathbb{R}^d$, so self-intersections will not be allowed. Furthermore, in what follows we will use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$ for the Japanese bracket.

Theorem 1.1. *Let Σ be any compact orientable hypersurface without boundary of $\mathbb{R}^d$ with $d \geqslant 3$, and take any $\epsilon > 0$. Then there is a bounded entire solution u of the Allen–Cahn equation in $\mathbb{R}^d$ such that its zero level set $u^{-1}(0)$ has a connected component given by $\Phi(\Sigma)$, where Φ is an ϵ-rescaling. This set is structurally stable. Furthermore, u falls off at infinity as $|u(x)| < C \langle x \rangle^{(1-d)/2}$ if $d \geqslant 4$ and is in $L^4(\mathbb{R}^3)$ if $d = 3$.*

It is worth mentioning that the result that we will actually prove (Theorem 4.1) is in fact stronger, in the sense that given any finite number of hypersurfaces $\Sigma_1, \ldots, \Sigma_N$ that are not linked (see Definition 2.1) we will show that there is a diffeomorphism Φ such that $\Phi(\Sigma_1) \cup \cdots \cup \Phi(\Sigma_N)$ is a union of connected

components of the nodal set of a bounded entire solution to the Allen–Cahn equation. The diffeomorphism Φ is not an ϵ-rescaling, although it does act on each hypersurface Σ_j as an ϵ-rescaling composed with a rigid motion.

The idea of the proof of the theorem is that, when u is small in a suitable sense, solutions to the Allen–Cahn equation behave as solutions to the Helmholtz equation

$$\Delta w + w = 0.$$

Hence, a key step of the proof is to establish an analog of Theorem 1.1 for solutions to the Helmholtz equations with a sharp fall-off rate at infinity, which is as $\langle x \rangle^{(1-d)/2}$ (Theorem 2.2). For this we combine a construction using the first eigenfunction of the domain bounded by Σ with a Runge-type theorem with decay conditions at infinity that generalizes the results that we proved in [Enciso and Peralta-Salas 2012; 2015] for Beltrami fields on $\mathbb{R}^3$. Using suitable weighted estimates for a convolution operator associated with the Helmholtz equation (Theorem 3.1), we then promote these solutions of the Helmholtz equation to solutions of the Allen–Cahn equation and show that the latter still possess a nodal set of the desired topology. From the method of proof it stems that the statement of Theorem 1.1 remains valid for much more general nonlinearities. (More precisely, one can replace u^3 by a smooth enough function $F(u)$ that behaves as $u^{1+\alpha}$ as $u \to 0$ with $\alpha > (d+1)/(d-1)$.) Observe that solutions of the Helmholtz equation were also used in [Gutiérrez 2004] to construct nontrivial solutions in some L^q space to the closely related Ginzburg–Landau equation on $\mathbb{R}^3$ and $\mathbb{R}^4$, and that in fact for $d = 3$ our proof will borrow estimates from this work.

2. Bounded solutions to the Helmholtz equation

In this section we will prove an analog of Theorem 1.1 for solutions to the Helmholtz equation on $\mathbb{R}^d$. We shall begin by introducing some notation.

Let us consider the function

$$G(x) := \beta |x|^{1-d/2} Y_{d/2-1}(|x|), \tag{2-1}$$

where $Y_{d/2-1}$ denotes the Bessel function of the second kind and we have set

$$\beta := \frac{2^{1-d/2}\pi}{|\mathbb{S}^{d-1}|\Gamma(d/2-1)},$$

with $|\mathbb{S}^{d-1}|$ the area of the unit $(d-1)$-sphere and Γ the Gamma function. A simple computation in spherical coordinates shows that $\Delta G + G = 0$ everywhere but at the origin and the asymptotics for Bessel functions show that

$$G(x) = -\frac{1}{|\mathbb{S}^{d-1}||x|^{d-2}} + O(|x|^{3-d})$$

as $x \to 0$. It then follows that G is a fundamental solution for the Helmholtz equation, so, if v is, say, a Schwartz function on $\mathbb{R}^d$, the convolution $G * v$ satisfies

$$\Delta(G * v) + G * v = v. \tag{2-2}$$

As we discussed in the introduction, we will prove a result that is considerably more general than Theorem 1.1, as it applies to an arbitrary number of hypersurfaces. There is, however, a topological condition that we must impose on these hypersurfaces, which is described in the following:

Definition 2.1. Let $\Sigma_1, \ldots, \Sigma_N$ be compact orientable hypersurfaces without boundary of $\mathbb{R}^d$. We will say that they are *not linked* if there are N pairwise disjoint contractible sets $S_1, \ldots, S_N$ such that each hypersurface Σ_j is contained in S_j.

We are now ready to state and prove the main result of this section. Notice that the proof of the theorem provides a satisfactory description of the structure of the diffeomorphism Ψ, as noted in Remark 2.3. The proof makes use of some techniques we introduced in [Enciso and Peralta-Salas 2013] to study the level sets of harmonic functions and in [Enciso and Peralta-Salas 2015] to construct Beltrami fields with prescribed vortex tubes. Throughout, diffeomorphisms are assumed to be of class C^∞ and connected with the identity, and B_R denotes the ball centered at the origin of radius R. Observe that, of course, for $N = 1$ the condition that the hypersurface be not linked is satisfied trivially.

Theorem 2.2. *Let $\Sigma_1, \ldots, \Sigma_N$ be compact orientable hypersurfaces without boundary of $\mathbb{R}^d$ that are not linked with $d \geqslant 3$. Then there is a function w satisfying the Helmholtz equation*

$$\Delta w + w = 0$$

in $\mathbb{R}^d$ and a diffeomorphism Ψ of $\mathbb{R}^d$ such that $\Psi(\Sigma_1), \ldots, \Psi(\Sigma_N)$ are structurally stable connected components of the zero set $w^{-1}(0)$. Furthermore, w falls off at infinity as $|\partial^\alpha w(x)| < C_\alpha \langle x \rangle^{(1-d)/2}$ for any multiindex α.

Proof. An easy application of Whitney's approximation theorem ensures that, by perturbing the hypersurfaces a little if necessary, we can assume that Σ_j is a real analytic hypersurface of $\mathbb{R}^d$. The fact that the hypersurfaces are not linked allow us now to rescale and translate them so that the (unique) precompact domains Ω_j that are bounded by each rescaled and translated real-analytic hypersurface, which we will call $\Sigma_j' := \partial\Omega_j$, are pairwise disjoint and their first Dirichlet eigenvalue $\lambda_1(\Omega_j)$ is 1. The first eigenvalue is always simple, so there is a unique eigenfunction ψ_j, modulo a multiplicative constant, that satisfies the eigenvalue equation

$$\Delta\psi_j + \psi_j = 0 \quad \text{in } \Omega_j, \qquad \psi_j|_{\Sigma_j'} = 0.$$

We can choose ψ_j so that it is positive in Ω_j.

Hopf's boundary point lemma shows that the gradient of ψ_j does not vanish on Σ_j:

$$\min_{x \in \Sigma_j} |\nabla\psi_j(x)| > 0. \tag{2-3}$$

Furthermore, as the hypersurface Σ_j is analytic, it is standard that ψ_j is analytic in an open neighborhood $\widetilde{\Omega}_j$ of the closure of Ω_j.

Our goal is to construct a solution w of the Helmholtz equation in $\mathbb{R}^d$ that approximates each function ψ_j in the set Ω_j. To this end, let us take a smooth function $\chi : \mathbb{R}^d \to \mathbb{R}$ that is equal to 1 in a narrow

neighborhood of the closure $\overline{\Omega}$ and is identically zero outside $\widetilde{\Omega}$, with

$$\widetilde{\Omega} := \bigcup_{j=1}^{N} \widetilde{\Omega}_j, \quad \Omega := \bigcup_{j=1}^{N} \Omega_j.$$

We can now define a smooth function w_1 on $\mathbb{R}^d$ by setting

$$w_1 := \sum_{j=1}^{N} \chi \psi_j.$$

Here we are assuming that $w_1 := 0$ outside $\widetilde{\Omega}$.

Let us now write

$$w_1 = w_1' + h$$

with

$$w_1'(x) = \int_{\mathbb{R}^d} G(x - y) f(y)\, dy, \tag{2-4}$$

where f is the compactly supported function $f := \Delta w_1 + w_1$ and G is the fundamental solution (2-1). By construction, h satisfies the homogeneous Helmholtz equation

$$\Delta h + h = 0.$$

The support of the function f is obviously contained in the open set $\widetilde{\Omega} \setminus \overline{\Omega}$. Therefore, an easy continuity argument ensures that one can approximate the integral (2-4) uniformly in the compact set $\overline{\Omega}$ by a finite Riemann sum of the form

$$w_2(x) := \sum_{n=1}^{M} c_n G(x - x_n). \tag{2-5}$$

Specifically, it is standard that for any $\delta > 0$ there is a large integer M, real numbers c_n and points $x_n \in \widetilde{\Omega} \setminus \overline{\Omega}$ such that the finite sum (2-5) satisfies

$$\|w_1' - w_2\|_{C^0(\Omega)} < \delta. \tag{2-6}$$

Let us now take a large ball B_R containing the closure of the set $\widetilde{\Omega}$. We shall next show that there is a finite number of points $\{x_n'\}_{n=1}^{M'}$ in $\mathbb{R}^d \setminus \overline{B}_R$ and constants c_n' such that the finite linear combination

$$w_3(x) := \sum_{n=1}^{M'} c_n' G(x - x_n') \tag{2-7}$$

approximates the function w_2 uniformly in Ω:

$$\|w_2 - w_3\|_{C^0(\Omega)} < \delta. \tag{2-8}$$

Here δ is the same arbitrarily small constant as above.

Consider the space $\mathcal{V}$ of all finite linear combinations of the form (2-7), where x_n' can be any point in $\mathbb{R}^d \setminus \bar{B}_R$ and the constants c_n' take arbitrary values. Restricting these functions to the set Ω, we can regard $\mathcal{V}$ as a subspace of the Banach space $C^0(\Omega)$ of continuous functions on Ω.

By the Riesz–Markov theorem, the dual of $C^0(\Omega)$ is the space $\mathcal{M}(\Omega)$ of the finite signed Borel measures on $\mathbb{R}^d$ whose support is contained in the set Ω. Let us take any measure $\mu \in \mathcal{M}(\Omega)$ such that $\int_{\mathbb{R}^d} f \, d\mu = 0$ for all $f \in \mathcal{V}$. We now define a function $F \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ as

$$F(x) := \int_{\mathbb{R}^d} G(x - x') \, d\mu(x'),$$

so that F satisfies the equation

$$\Delta F + F = \mu.$$

Notice that F is identically zero on $\mathbb{R}^d \setminus \bar{B}_R$ by the definition of the measure μ and that F satisfies the elliptic equation

$$\Delta F + F = 0$$

in $\mathbb{R}^d \setminus \bar{\Omega}$, so F is analytic in this set. Hence, since $\mathbb{R}^d \setminus \bar{\Omega}$ is connected and contains the set $\mathbb{R}^d \setminus B_R$, by analyticity the function F must vanish on the complement of Ω. It then follows that the measure μ also annihilates the function $G(\,\cdot - y)$ with $y \notin \bar{\Omega}$ because

$$0 = F(y) = \int_{\mathbb{R}^d} G(y - x') \, d\mu(x').$$

Therefore

$$\int_{\mathbb{R}^d} w_2 \, d\mu = 0,$$

which implies that w_2 can be uniformly approximated on Ω by elements of the subspace $\mathcal{V}$, due to the Hahn–Banach theorem. Accordingly, there is a finite set of points $\{x_n'\}_{n=1}^{M'}$ in $\mathbb{R}^d \setminus \bar{B}_R$ and reals c_n' such that the function w_3 defined by (2-7) satisfies the estimate (2-8).

To complete the proof of the theorem, notice that the function

$$w_4 := w_3 + h$$

satisfies the equation

$$\Delta w_4 + w_4 = 0 \tag{2-9}$$

in the ball B_R, whose interior contains Ω. Let us take hyperspherical coordinates $r := |x|$ and $\omega := x/|x| \in \mathbb{S}^{d-1}$ in B_R. Expanding the function w_4 (with respect to the angular variables) in a series of spherical harmonics and using (2-9), we immediately obtain that w_4 can be written in the ball as a Fourier–Bessel series of the form

$$w_4 = \sum_{l=0}^{\infty} \sum_{m \in I_l} c_{lm} j_l(r) Y_{lm}(\omega),$$

where j_l denotes a d-dimensional hyperspherical Bessel function, Y_{lm} are spherical harmonics on $\mathbb{S}^{d-1}$ and I_l is a finite set that depends on l and whose explicit expression will not be needed here.

Since the above series converges in $L^2(B_R)$, for any $\delta > 0$ there is an integer l_0 such that the finite sum

$$w := \sum_{l=0}^{l_0} \sum_{m \in I_l} c_{lm} j_l(r) Y_{lm}(\omega)$$

approximates the function w_4 in an L^2 sense:

$$\|w - w_4\|_{L^2(B_R)} < \delta. \tag{2-10}$$

By the properties of Bessel functions, w is smooth in $\mathbb{R}^d$, falls off as

$$|\partial^\alpha w(x)| \leqslant C_\alpha \langle x \rangle^{(1-d)/2}$$

at infinity for any multiindex α and satisfies the equation

$$\Delta w + w = 0 \tag{2-11}$$

in the whole space.

Given any $R' < R$ large enough for the set Ω to be contained in the ball $B_{R'}$, standard elliptic estimates allow us to pass from the L^2 bound (2-10) to a uniform estimate

$$\|w - w_4\|_{C^0(B_{R'})} < C\delta.$$

From this inequality and the bounds (2-6) and (2-8) we infer

$$\|w - w_1\|_{C^0(\Omega)} < C\delta. \tag{2-12}$$

Moreover, since w_1 also satisfies the Helmholtz equation in a neighborhood of the compact set $\overline{\Omega}$, standard elliptic estimates again imply that the uniform estimate (2-12) can be promoted to the C^1 bound

$$\|w - w_1\|_{C^1(\Omega)} < C\delta. \tag{2-13}$$

Finally, since $\Sigma_1 \cup \cdots \cup \Sigma_N$ is a union of components of the nodal set of w_1 and, by (2-3), the gradient of w_1 does not vanish on these hypersurfaces, the estimate (2-13) and a direct application of Thom's isotopy theorem [Abraham and Robbin 1967, Theorem 20.2] imply that there is a diffeomorphism Ψ of $\mathbb{R}^d$ such that

$$\Psi(\Sigma_1 \cup \cdots \cup \Sigma_N) \tag{2-14}$$

is a union of components of the zero set $w^{-1}(0)$. Moreover, the diffeomorphism Ψ is C^1-close to the identity. The structural stability of the set (2-14) for the function w also follows from Thom's isotopy theorem and the lower bound

$$\min_{x \in \Psi(\Sigma_1 \cup \cdots \cup \Sigma_N)} |\nabla w(x)| > 0,$$

as a consequence of the C^1 estimate (2-13) and the fact that the function w_1 satisfies the gradient condition (2-3). $\qquad\square$

Remark 2.3. It follows from the proof that there are rescalings Ψ_j^2, translations Ψ_j^3 and diffeomorphisms Ψ_j^1 with $\|\Psi_j^1 - \mathrm{id}\|_{C^1(\mathbb{R}^d)}$ arbitrarily small such that

$$\Psi(\Sigma_j) = (\Psi_j^1 \circ \Psi_j^2 \circ \Psi_j^3)(\Sigma_j).$$

In particular, if $N = 1$ the diffeomorphism Ψ can be assumed to be an ϵ-rescaling. A minor modification of the argument would have allowed us to take $\|\Psi_j^1 - \mathrm{id}\|_{C^k(\mathbb{R}^d)}$ arbitrarily small, with k any fixed number.

3. A weighted estimate for a convolution operator

In promoting solutions to the Helmholtz equation with sharp decay at infinity to solutions to the Allen–Cahn equation, the estimates that we establish in this section will play a key role.

Specifically, we will be interested in the convolution of the fundamental solution G, introduced in (2-1), with functions with certain decay rate at infinity. To quantify this, for any nonnegative integer k and any positive real ν let us denote by $C_\nu^k(\mathbb{R}^d)$ the closure of the space of Schwartz functions on $\mathbb{R}^d$ with respect to the metric

$$\|v\|_{k,\nu} := \max_{|\alpha| \leqslant k} \sup_{x \in \mathbb{R}^d} |\langle x \rangle^\nu \partial^\alpha v(x)|.$$

Clearly

$$\|vw\|_{k,\nu+\nu'} \leqslant C \|v\|_{k,\nu} \|w\|_{k,\nu'}$$

whenever $v \in C_\nu^k(\mathbb{R}^d)$ and $w \in C_{\nu'}^k(\mathbb{R}^d)$, where C is a constant that only depends on k. In particular,

$$\|v^s\|_{k,\nu} \leqslant C \|v\|_{k,\nu/s}^s. \tag{3-1}$$

The following theorem, which asserts that the convolution with G defines a bounded map

$$C_\nu^k(\mathbb{R}^d) \to C_{(d-1)/2}^k(\mathbb{R}^d)$$

for any $\nu > d$, provides the estimates that we need:

Theorem 3.1. *Suppose that $d \geqslant 3$. Then, for any $v \in C_\nu^k(\mathbb{R}^d)$ with $k \geqslant 0$ and $\nu > d$, one has*

$$\|G * v\|_{k,(d-1)/2} \leqslant C \|v\|_{k,\nu},$$

with a constant that depends on d and ν but not on v or k.

Proof. In view of the well-known asymptotics for Bessel functions when $d \geqslant 3$, there is a positive constant C such that G is bounded by

$$|G(x)| \leqslant \begin{cases} C|x|^{2-d} & \text{if } |x| < 1, \\ C|x|^{(1-d)/2} & \text{if } |x| > 1. \end{cases}$$

It then follows that

$$\begin{aligned}
|G * v(x)| &\leqslant \int_{\mathbb{R}^d} |G(z)| |v(x - z)| \, dz \\
&\leqslant C \|v\|_{0,\nu} \left(\int_{B_1} |z|^{2-d} \langle x - z \rangle^{-\nu} \, dz + \int_{\mathbb{R}^d \setminus B_1} |z|^{(1-d)/2} \langle x - z \rangle^{-\nu} \, dz \right).
\end{aligned} \tag{3-2}$$

For any fixed x, the first integral is convergent for any value of ν, while the second converges provided that $\nu > \frac{1}{2}(d+1)$. Since $\nu > d > \frac{1}{2}(d+1)$, we infer that $G * v(x)$ is well defined as a convergent integral for any $v \in C_\nu^0(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$, and it only remains to analyze its behavior for large $|x|$.

For concreteness, let us assume that $|x| > 2$. We shall next show that the integrals

$$I_1 := \int_{B_1} |z|^{2-d} \langle x - z \rangle^{-\nu} \, dz,$$

$$I_2 := \int_{B_{|x|/2}} |z|^{(1-d)/2} \langle x - z \rangle^{-\nu} \, dz,$$

$$I_3 := \int_{B_{2|x|} \setminus B_{|x|/2}} |z|^{(1-d)/2} \langle x - z \rangle^{-\nu} \, dz,$$

$$I_4 := \int_{\mathbb{R}^d \setminus B_{2|x|}} |z|^{(1-d)/2} \langle x - z \rangle^{-\nu} \, dz,$$

are then bounded as

$$I_j < C|x|^{(1-d)/2}, \tag{3-3}$$

where C does not depend on ν. In view of the inequality (3-2) and the fact that

$$\int_{\mathbb{R}^d \setminus B_1} |z|^{(1-d)/2} \langle x - z \rangle^{-\nu} \leqslant I_2 + I_3 + I_4,$$

this shows that the convolution with G is a bounded map $C_\nu^0(\mathbb{R}^d) \to C_{(d-1)/2}^0(\mathbb{R}^d)$. Since for any multiindex α we have

$$\partial^\alpha (G * v) = G * (\partial^\alpha v),$$

this immediately implies that the convolution with G is also a bounded map $C_\nu^k(\mathbb{R}^d) \to C_{(d-1)/2}^k(\mathbb{R}^d)$, thereby proving the theorem.

So it only remains to prove the estimate (3-3) for $1 \leqslant j \leqslant 4$. For this we start by using the elementary inequality

$$\langle x - z \rangle > \begin{cases} \frac{1}{2}|x| & \text{if } |z| < \frac{1}{2}|x|, \\ \frac{1}{2}|z| & \text{if } |z| > 2|x|, \end{cases}$$

to obtain, for $|x| > 2$,

$$I_1 < C|x|^{-\nu} \int_{B_1} |z|^{2-d} \, dz = C|x|^{-\nu} < C|x|^{-d} < C|x|^{(1-d)/2},$$

$$I_2 < C|x|^{-\nu} \int_{B_{|x|/2}} |z|^{(1-d)/2} \, dz = C|x|^{(d+1)/2 - \nu} < C|x|^{(1-d)/2},$$

$$I_4 < C \int_{\mathbb{R}^d \setminus B_{2|x|}} |z|^{(1-d)/2 - \nu} \, dz = C|x|^{(d+1)/2 - \nu} < C|x|^{(1-d)/2}.$$

To obtain these bounds we have used that $\nu > d$ by assumption.

To estimate I_3 we choose a Cartesian basis such that $x = |x|e_1$ and then use the rescaled variable $\bar{z} := z/|x|$ to write

$$I_3 = \int_{B_{2|x|} \setminus B_{|x|/2}} |z|^{(1-d)/2} \langle x - z \rangle^{-\nu} \, dz = |x|^{(d+1)/2-\nu} \int_{B_2 \setminus B_{1/2}} \frac{|\bar{z}|^{(1-d)/2} \, d\bar{z}}{(1/|x|^2 + |e_1 - \bar{z}|^2)^{\nu/2}}. \tag{3-4}$$

Denoting by B' the ball centered at e_1 of radius $\frac{1}{4}$, one can check that

$$\int_{B'} \frac{|\bar{z}|^{(1-d)/2} \, d\bar{z}}{(1/|x|^2 + |e_1 - \bar{z}|^2)^{\nu/2}} < C \int_0^{1/4} \frac{\rho^{d-1} \, d\rho}{(1/|x|^2 + \rho^2)^{\nu/2}}$$

$$= C|x|^{\nu-d} \int_0^{|x|/4} \frac{\bar{\rho}^{d-1} \, d\bar{\rho}}{(1 + \bar{\rho}^2)^{\nu/2}}$$

$$< C|x|^{\nu-d} \int_0^\infty \frac{\bar{\rho}^{d-1} \, d\bar{\rho}}{(1 + \bar{\rho}^2)^{\nu/2}}$$

$$< C|x|^{\nu-d},$$

where we have defined $\bar{\rho} := |x|\rho$, and we have used that the integral in $\bar{\rho}$ is convergent for any $\nu > d$. Plugging this into (3-4), one gets

$$I_3 = |x|^{(d+1)/2-\nu} \left(\int_{B'} \frac{|\bar{z}|^{(1-d)/2} \, d\bar{z}}{(1/|x|^2 + |e_1 - \bar{z}|^2)^{\nu/2}} + \int_{B_2 \setminus (B_{1/2} \cup B')} \frac{|\bar{z}|^{(1-d)/2} \, d\bar{z}}{(1/|x|^2 + |e_1 - \bar{z}|^2)^{\nu/2}} \right)$$

$$< C|x|^{(d+1)/2-\nu}(|x|^{\nu-d} + C) < C|x|^{(1-d)/2}.$$

To obtain the first inequality we have used that $|e_1 - \bar{z}|^2 \geqslant \frac{1}{16}$ for all $\bar{z} \in B_2 \setminus (B_{1/2} \cup B')$, and the second inequality follows from the assumption $\nu > d$. This is the last estimate that we needed in (3-3) and thus the theorem follows. $\qquad \square$

In view of the structure of the nonlinearity of the Allen–Cahn equation, the following corollary will be useful:

Corollary 3.2. *For any $v \in C^k_{(d-1)/2}(\mathbb{R}^d)$ with $k \geqslant 0$ and $d \geqslant 4$, one has the estimate*

$$\|G * (v^3)\|_{k,(d-1)/2} \leqslant C\|v\|^3_{k,(d-1)/2}.$$

Proof. We can apply Theorem 3.1 with $\nu := \frac{1}{2}(3d - 3)$ because $\nu > d$ for all dimensions $d \geqslant 4$, thus implying that

$$\|G * (v^3)\|_{k,(d-1)/2} \leqslant C\|v^3\|_{k,(3d-3)/2} \leqslant C\|v\|^3_{k,(d-1)/2},$$

where we have used the relation (3-1). $\qquad \square$

4. Proof of Theorem 1.1

We are now ready to prove the main result of this paper, which reduces to Theorem 1.1 when $N = 1$.

Theorem 4.1. *Let $\Sigma_1, \ldots, \Sigma_N$ be compact orientable hypersurfaces without boundary of $\mathbb{R}^d$ that are not linked with $d \geqslant 3$, and let us take any positive integer k. Then there is a diffeomorphism Φ of $\mathbb{R}^d$ such*

that $\Phi(\Sigma_1), \ldots, \Phi(\Sigma_N)$ *are connected components of the level set* $u^{-1}(0)$ *of a smooth solution to the Allen–Cahn equation in* $\mathbb{R}^d$, *which is bounded as* $|\partial^\alpha u(x)| < C_\alpha \langle x \rangle^{(1-d)/2}$ *for any multiindex with* $|\alpha| < k$ *if* $d \geqslant 4$, *and if* $d = 3$ *then* u *is in* $L^4(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. *Furthermore, these level sets are structurally stable and the diffeomorphism* Φ *can be assumed to have the same structure as in Remark 2.3.*

Proof. Let us first assume that $d \geqslant 4$, since this will allow us to apply Corollary 3.2. By Theorem 2.2 there is a solution w to the Helmholtz equation

$$\Delta w + w = 0$$

on $\mathbb{R}^d$ such that $\Psi(\Sigma_1), \ldots, \Psi(\Sigma_N)$ are connected components of its zero set $w^{-1}(0)$, where Ψ is a diffeomorphism of $\mathbb{R}^d$. Moreover, $\|w\|_{k,(d-1)/2} < C$ and the above hypersurfaces are structurally stable, in the sense that there exist a large ball B_R and a positive constant η such that, if w' is any function with

$$\|w - w'\|_{C^1(B_R)} < \eta, \tag{4-1}$$

then there is a diffeomorphism Φ of $\mathbb{R}^d$ such that

$$\Phi(\Sigma_1) \cup \cdots \cup \Phi(\Sigma_N) \tag{4-2}$$

are structurally stable connected components of the level set $w'^{-1}(0)$. Furthermore, Φ is close to Ψ in the norm $C^1(\mathbb{R}^d)$.

Let us take a small positive constant ϵ that will be fixed later and consider the iterative scheme

$$u_0 := \delta w, \quad u_{n+1} := \delta w + G * (u_n^3), \tag{4-3}$$

where we have set

$$\delta := \frac{\epsilon}{2\|w\|_{k,(d-1)/2}}.$$

Our goal is to show that, if ϵ is small enough, u_n converges in $C^k_{(d-1)/2}(\mathbb{R}^d)$ to a function u that satisfies the Allen–Cahn equation

$$\Delta u + u - u^3 = 0$$

and is close to δw in a suitable norm.

A first observation is that, if $\|u_n\|_{k,(d-1)/2} < \epsilon$ and ϵ is small enough, by the definition of δ we automatically have

$$\begin{aligned}
\|u_{n+1}\|_{k,(d-1)/2} &\leqslant \delta \|w\|_{k,(d-1)/2} + \|G * (u_n^3)\|_{k,(d-1)/2} \\
&\leqslant \delta \|w\|_{k,(d-1)/2} + C\|u_n\|_{k,(d-1)/2}^3 \leqslant \tfrac{1}{2}\epsilon + C\epsilon^3 < \epsilon.
\end{aligned} \tag{4-4}$$

Here we have used Corollary 3.2 to estimate $G * (u_n^3)$. Notice that the smallness that we have to impose on ϵ only depends on the constant that appears in Corollary 3.2. In particular, since the first function u_0 of the iteration satisfies

$$\|u_0\|_{k,(d-1)/2} = \tfrac{1}{2}\epsilon,$$

the induction property (4-4) then implies that

$$\|u_n\|_{k,(d-1)/2} < \epsilon \tag{4-5}$$

for all n.

To estimate the difference $u_{n+1} - u_n$, let us start by noticing that for any functions v, v' we have

$$\|v^3 - v'^3\|_{k,(3d-3)/2} = \max_{|\alpha|\leqslant k} \sup_{x\in\mathbb{R}^d} \langle x\rangle^{(3d-3)/2} \big|\partial^\alpha (v^2(v-v') + vv'(v-v') + v'^2(v-v'))\big|$$
$$\leqslant C\big(\|v\|^2_{k,(d-1)/2} + \|v'\|^2_{k,(d-1)/2}\big)\|v - v'\|_{k,(d-1)/2}.$$

It then follows from Theorem 3.1, the fact that $\frac{1}{2}(3d-3) > d$ when $d \geqslant 4$, and (4-5) that we can write

$$\|u_{n+1} - u_n\|_{k,(d-1)/2} = \|G * (u_n^3 - u_{n-1}^3)\|_{k,(d-1)/2}$$
$$\leqslant C\|u_n^3 - u_{n-1}^3\|_{k,(3d-3)/2} \leqslant C\epsilon^2\|u_n - u_{n-1}\|_{k,(d-1)/2}. \tag{4-6}$$

If ϵ is small enough that $C\epsilon^2 < \frac{1}{2}$, it is standard that (4-4) and (4-6) imply that u_n converges in $C^k_{(d-1)/2}(\mathbb{R}^d)$ to some function u with

$$\|u\|_{k,(d-1)/2} \leqslant \epsilon. \tag{4-7}$$

Since the map $v \mapsto G * (v^3)$ is continuous in $C^k_{(d-1)/2}(\mathbb{R}^d)$, from (4-3) we infer that u satisfies the integral equation

$$u = \delta w + G * (u^3). \tag{4-8}$$

As w is a solution of the Helmholtz equation and G is a fundamental solution satisfying (2-2), it then follows that

$$\Delta u + u = u^3,$$

so u is a solution of the Allen–Cahn equation, which is smooth by elliptic regularity.

One can now use the bound (4-7), the relation (4-8) and the definition of δ to write

$$\left\|w - \frac{u}{\delta}\right\|_{k,(d-1)/2} = \frac{1}{\delta}\|\delta w - u\|_{k,(d-1)/2} = \frac{1}{\delta}\|G * (u^3)\|_{k,(d-1)/2} \leqslant \frac{C}{\delta}\|u\|^3_{k,(d-1)/2} \leqslant C\epsilon^2. \tag{4-9}$$

In view of the stability estimate (4-1), if ϵ is small enough (namely, $C\epsilon^2 < \eta$), we infer that there is a diffeomorphism Φ, close to the diffeomorphism Ψ in the norm $C^1(\mathbb{R}^d)$, such that the hypersurfaces (4-2) are structurally stable connected components of the level set $u^{-1}(0)$. The theorem then follows for $d \geqslant 4$.

For $d = 3$ the proof is similar but employs the $L^4(\mathbb{R}^3)$ estimates for the inverse of the Helmholtz equation proved in [Gutiérrez 2004]. More precisely, as G is the real part of the fundamental solution of the Helmholtz equation that satisfies the Sommerfeld outgoing radiation condition (namely, $-e^{i|x|}/4\pi|x|$), the iteration (4-3) converges in $L^4(\mathbb{R}^3)$ by [Gutiérrez 2004]. The rest of the proof is just as in higher dimensions, replacing the estimate (4-9) in the ball B_R by the standard elliptic estimate

$$\left\|w - \frac{u}{\delta}\right\|_{C^k(B_R)} \leqslant C\left\|w - \frac{u}{\delta}\right\|_{L^4(\mathbb{R}^3)} \leqslant C\epsilon^2,$$

where the last bound is what one obtains directly from the iteration. Moreover by [Gutiérrez 2004, Proposition 1], it follows that any $L^4(\mathbb{R}^3)$ solution to the Allen–Cahn equation is bounded. $\qquad\square$

Acknowledgments

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