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COMMUTATORS WITH FRACTIONAL DIFFERENTIATION AND NEW CHARACTERIZATIONS OF BMO-SOBOLEV SPACES

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For $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\alpha \in (0, 1)$, let D^α be the fractional differential operator and T be the singular integral operator. We obtain a necessary and sufficient condition on the function b to guarantee that $[b, D^\alpha T]$ is a bounded operator on a function space such as $L^p(\mathbb{R}^n)$ and $L^{p,\lambda}(\mathbb{R}^n)$ for any $1 < p < \infty$. Furthermore, we establish a necessary and sufficient condition on the function b to guarantee that $[b, D^\alpha T]$ is a bounded operator from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. This is a new theory. Finally, we apply our general theory to the Hilbert and Riesz transforms.

1. Introduction

For $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, denote by B the multiplication operator defined by $Bf(x) = b(x) f(x)$ for any measurable function f. If T is a linear operator on some measurable function space, then the commutator formed by B and T is defined by $[b, T] f(x) := (BT - TB) f(x)$. Let $0 \le \alpha \le 1$. The commutators we are interested in here are of the form

$$[b, T_\alpha] f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+\alpha}} (b(x) - b(y)) f(y)\, dy,$$

where Ω is homogeneous of degree zero, integrable on S^{n-1}.

The case $\alpha = 1$ was first investigated by Calderón [1965] and now is well known as Calderón's first-order commutator. Calderón proved that $b \in \mathrm{Lip}(\mathbb{R}^n)$ (Lipschitz space) is a sufficient condition for the L^p-boundedness of $[b, T_1]$ when Ω satisfies some assumptions but may fail to have any regularity. However, this result has inspired many mathematicians to find new proofs, to make generalizations and to find further applications. We refer the reader to [Calderón 1980; Coifman and Meyer 1975; 1978; Cohen 1981; Hofmann 1994; 1998], among numerous references, for its development and applications. We would like to single out the work by Coifman and Meyer [1975], who found a new proof of Calderón's first-order commutator by reducing the commutator estimates to continuity of multilinear operators, which was used to deal with higher-order commutators in the same paper and has since been widely developed.

Let us comment on the main idea of Calderón's proof for future convenience. Firstly, the special properties such as locality of Lipschitz functions enable Calderón to use a rotation method to reduce

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commutator estimates in the higher-dimensional cases to the one-dimensional case. Secondly, the one-dimension commutator is just the commutator formed by b and dH/dx, the derivative of the Hilbert transform. Then Calderón exploited the special properties of the Hilbert transform as being closely related to analytic functions and used a characterization of the Hardy space $H^1(\mathbb{R})$ in terms of the Lusin square function to prove his theorem. It is the second part that has been reproved by Coifman and Meyer using techniques from multilinear analysis.

The case $\alpha = 0$ was first studied by Coifman, Rochberg and Weiss [Coifman et al. 1976], who showed that $b \in \mathrm{BMO}(\mathbb{R}^n)$, the bounded mean oscillation space, is a sufficient and necessary condition for the L^p-boundedness of $[b, T_0]$ when $\Omega \in \mathrm{Lip}(S^{n-1})$ (see also [Janson 1978; Uchiyama 1978]). For rough Ω, similar results have also been obtained in [Álvarez et al. 1993; Hu 2003; Chen and Ding 2015]. It is worth mentioning that the operator $[b, T_0]$ has a different character from $[b, T_1]$, whose research actually was inspired by the factorization of Hardy spaces.

The case $0 < \alpha < 1$ was first investigated by Segovia and Wheeden [1971], who obtained an analogue for differentiation of a fractional order of the one-dimensional version of Calderón's result [1965]. Murray [1985] improved the results of [Stein and Weiss 1971], more or less along the research line initiated by Calderón, by extending the commutator with derivatives of the Hilbert transform to those with fractional derivatives of the Hilbert transform. It turns out that these commutators with fractional differentiation are closely related to BMO-Sobolev spaces. Let $0 < \alpha \leq 1$, and consider the fractional differentiation operators defined for f by

$$\widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

The fractional Laplacian can be defined in a distributional sense for functions that are not differentiable as long as $\hat{f}$ is not too singular at the origin or, in terms of the variable x, as long as

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(|1+|x|)^\alpha}\, dx < \infty.$$

For a function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the extension $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that satisfies the equation

$$u(x, 0) = f(x), \quad \triangle_x u + \frac{1-\alpha}{y} u_y + u_{yy} = 0.$$

Caffarelli and Silvestre [2007] showed that

$$C D^\alpha f = \lim_{y \to 0^+} -y^{1-\alpha} u_y = \frac{1}{\alpha} \lim_{y \to 0} \frac{u(x, y) - u(x, 0)}{y^\alpha}$$

for some C depending on n and α.

Let I_α be the Riesz potential operator of order α. The Sobolev space $I_\alpha(\mathrm{BMO})$ is the image of BMO under I_α. Equivalently, a locally integrable function b is in $I_\alpha(\mathrm{BMO})$ if and only if $D^\alpha b \in \mathrm{BMO}$. Strichartz [1980] showed that, for $\alpha \in (0, 1)$, $I_\alpha(\mathrm{BMO})$ is a space of functions modulo constants that is properly contained in Lip_α, while Lip_1 is properly contained in $I_1(\mathrm{BMO})$.

Murray studied it only in the one-dimensional case, the commutators $[b, T_\alpha]$ formed by b and $D^\alpha H$ or D^α, and showed that $b \in I_\alpha(\mathrm{BMO})(\mathbb{R})$ is equivalent to the L^p-boundedness of $[b, T_\alpha]$. Calderón's original proof did not work well in this new situation. Instead, Murray used special properties of one-dimensional commutators to represent them in a way that techniques of multilinear analysis developed in [Coifman and Meyer 1975] could come into play. In the meantime, she showed that $b \in \mathrm{Lip}(\mathbb{R})$ is also a necessary condition for L^p-boundedness of $[b, T_1]$, thus providing a converse of Calderón's results on $\mathbb{R}$. In the review of [Murray 1985] in Math Reviews, Y. Meyer indicates that the results there apply to functions on $\mathbb{R}^n$. However, a direct perusal of [Murray 1985] reveals that the paper only tackles the case $n = 1$. (Meyer might have known how to treat $n > 1$.) Maybe, it can in particular be applied to $[b, D^\alpha]$ on $\mathbb{R}^n$ for $n > 1$. But we think the techniques may fail for $[b, T_\alpha]$ on $\mathbb{R}^n$ for $n > 1$. The reason is that the higher-dimensional commutators are much more complicated due to the presence of Ω, which cannot be represented easily.

In the case of $0 < \alpha < 1$, by applying an off-diagonal $T1$ theorem (see [Hofmann 1998]), Q. Chen and Z. Zhang [2004] obtained the (L^p, L^q) bounds for the operator $[b, T_\alpha]$ with $\Omega \in \mathrm{Lip}(S^{n-1})$ and $D^\alpha b \in L^r(\mathbb{R}^n)$, where $1 < r < \infty$ and $1/p + 1/r = 1/q$. However, they pointed out that they do not know whether the off-diagonal $T1$ theorem is true for $r = \infty$, so the (L^p, L^p)-boundedness of the operator $[b, T_\alpha]$ cannot be obtained in [Chen and Zhang 2004]. We think there are two reasons that the (L^p, L^p)-boundedness of the operator $[b, T_\alpha]$ cannot be obtained in [Chen and Zhang 2004]. Firstly, Calderón's rotation method is of no use, since the elements in $I_\alpha(\mathrm{BMO})(\mathbb{R}^n)$ are not local and do not enjoy the properties of Lipschitz functions. Secondly, the $T1$ theorem developed by David and Journé [1984], which is a powerful tool for the commutators $[b, dH/dx]$ and $[b, D^\alpha]$ to give an alternate proof, does not work well in general situations, such as the cases where the operators are rough or the cases where the weak-boundedness property (WBP) is not easy to verify.

Here we use Fourier transform estimates and Littlewood–Paley theory developed in [Duoandikoetxea and Rubio de Francia 1986] to get the L^p-boundedness of $[b, T_\alpha]$ with rough kernel for all $1 < p < \infty$, which can be stated as follows.

Theorem 1.1. *Suppose $\alpha \in (0, 1)$ and $b \in I_\alpha(\mathrm{BMO})$. If $\Omega \in L\log^+ L(S^{n-1})$ having the mean value zero property, that is,*

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0, \tag{1-1}$$

then there is a constant C such that, for $1 < p < \infty$,

$$\|[b, T_\alpha]f\|_{L^p} \le C\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}.$$

We will prove this result in Section 2.

Remark 1.2. Our arguments depend heavily on the Fourier transform estimates, which is not a surprise from the historical point of view of techniques in handling rough operators [Duoandikoetxea and Rubio de Francia 1986]. But, as Murray has pointed out, the cases $0 < \alpha < 1$ are fundamentally different: the underlying details turn out to be very subtle and quite different from the cases of $\alpha = 0$ and $\alpha = 1$. Furthermore, we believe some modifications of the method in the present paper should provide an alternate proof of Calderón's first-order commutator estimate.

As applications to partial differential equations have been found in the cases $\alpha = 0$, 1 and Murray's one-dimensional result in the case $0 < \alpha < 1$ (see [Calderón 1980; Chiarenza et al. 1991; Di Fazio and Ragusa 1991; 1993; Murray 1987; Lewis and Silver 1988; Lewis and Murray 1991; 1995; Taylor 1991; 1997; 2015]), we also expect applications of our results to fractional-order partial differential equations (see for instance [Silvestre 2007; Caffarelli and Silvestre 2007; Caffarelli and Stinga 2016] on fractional elliptic equations).

Definition 1.3. A measurable function $f \in L^p(\mathbb{R}^n)$, $p \in (1, \infty)$, belongs to the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ with $\lambda \in [0, n)$ if the following norm is finite:

$$\|f\|_{L^{p,\lambda}} = \left(\sup_{x \in \mathbb{R}^n, r>0} \frac{1}{r^\lambda} \int_{Q(x,r)} |f(y)|^p \, dy \right)^{1/p},$$

where $Q(x, r)$stands for any cube of radius r and centered at x_0. When $\lambda = 0$, $L^{p,\lambda}(\mathbb{R}^n)$ coincides with the Lebesgue space $L^p(\mathbb{R}^n)$.

It is well known that the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ [1938] is connected to certain problems in elliptic PDEs. Later, the Morrey spaces were found to have many important applications to the Navier–Stokes equations, the Schrödinger equations, elliptic equations and potential analysis (see [Chiarenza and Frasca 1987; Kato 1992; Taylor 1992; Ruiz and Vega 1991; Shen 2003; Di Fazio et al. 1999; Palagachev and Softova 2004; Deng et al. 2005; Adams and Xiao 2004; 2011; 2012]).

Recently, Chen, Ding and Wang gave a criterion of the boundedness of a general linear or sublinear operator on Morrey spaces:

Theorem A [Chen et al. 2012]. *Let $0 < \lambda < n$. Suppose that $\Omega \in L^q(S^{n-1})$ for $q > n/(n-\lambda)$ and S is a sublinear operator satisfying $|Sf(x)| \le C \int_{\mathbb{R}^n} |\Omega(x-y)| \, |f(y)|/|x-y|^n \, dy$. Let $1 < p < \infty$. If the operator S is bounded on $L^p(\mathbb{R}^n)$, then S is bounded on $L^{p,\lambda}(\mathbb{R}^n)$.*

Clearly, $b \in I_\alpha(\mathrm{BMO}) \subset \mathrm{Lip}_\alpha$ for $0 < \alpha < 1$ implies $|[b, T_\alpha]f(x)| \le C \int_{\mathbb{R}^n} |\Omega(x-y)| \, |f(y)|/|x-y|^n \, dy$. Since $\Omega \in L^q(S^{n-1}) \subset L \log^+ L(S^{n-1})$ for $q > n/(n-\lambda)$, applying Theorem A and Theorem 1.1, we get:

Corollary 1.4. *Let $0 < \lambda < n$. Suppose $\alpha \in (0, 1)$ and $b \in I_\alpha(\mathrm{BMO})$. If $\Omega \in L^q(S^{n-1})$ for $q > n/(n-\lambda)$ and satisfies* (1-1), *then there is a constant C such that, for $1 < p < \infty$,*

$$\|[b, T_\alpha]f\|_{L^{p,\lambda}} \le C \|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^{p,\lambda}}.$$

Pérez [1995] gave a simple example to show that the commutator $[b, T_0]$ is not of weak type $(1, 1)$ when $b \in \mathrm{BMO}$. However, if $0 < \alpha < 1$, $b \in I_\alpha(\mathrm{BMO})$ and $\Omega \in \mathrm{Lip}(S^{n-1})$, it is easy to verify that $k(x, y) = (\Omega(x-y)/|x-y|^{n+\alpha})(b(x) - b(y))$ is a standard kernel. Moreover, $\Omega \in \mathrm{Lip}(S^{n-1}) \subset L \log^+ L(S^{n-1})$, we apply Theorem 1.1 (the L^2-boundedness of $[b, T_\alpha]$) to see $[b, T_\alpha]$ is a generalized Calderón–Zygmund operator. So the weak type $(1, 1)$-boundedness of $[b, T_\alpha]$ is a natural consequence. Therefore, it will be interesting to give a necessary condition for the $L^1 \to L^{1,\infty}$ bounds of $[b, T_\alpha]$, which is our main aim in this part. Moreover, we will also give the necessity of the $L^{p,\lambda}$-boundedness of the commutator $[b, T_\alpha]$.

The following useful characterization of $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ is due to Meyers [1964]:

Theorem B. *Let $\alpha \in (0,1]$. A locally integrable function b is in $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ if and only if there is a constant C such that, for any cube Q,*

$$\frac{1}{|Q|^{1+\alpha/n}}\int_Q |b(x)-b_Q|\,dx \le C.$$

We first give a necessary condition for the $L^{p,\lambda}$ bounds of $[b, T_\alpha]$.

Theorem 1.5. *Suppose $\alpha \in (0,1]$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ satisfies* (1-1) *or*

$$\int_{S^{n-1}} \Omega(x')x'_j\,d\sigma(x') = 0, \tag{1-2}$$

for $j = 1, 2, \dots, n$. Assume Ω is not identically zero. If $[b, T_\alpha]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$ for some $1 < p < \infty$ and $0 \le \lambda < n$, then $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

Remark 1.6. In particular, if $[b, T_\alpha]$ is a bounded on $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, then $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

Remark 1.7. Since the structure of Ω is complicated and cannot be represented easily, the idea of proving Theorem 1.5 is very different from Murray's method [1985], where the proof depends on a special property of the Hilbert transform H.

Theorem 1.8. *Suppose $\alpha \in (0,1]$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ satisfies* (1-1) *or* (1-2). *Assume Ω is not identically zero. If $[b, T_\alpha]$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, then $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.*

Remark 1.9. As far as we know, this is the first example of a necessary condition for the $L^1 \to L^{1,\infty}$-boundedness of an operator.

The proof of Theorems 1.5 and 1.8 will be given in Sections 3 and 4, respectively.

Moreover, in the course of showing the main result, in conjunction with Calderón's first-order estimates, we obtain the characterizations of $\mathrm{Lip}(\mathbb{R}^n)$ in terms of the L^p-, $(L^1, L^{1,\infty})$- and $L^{p,\lambda}$-boundedness of commutators. If $b \in \mathrm{Lip}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$, then by Theorem 2 in [Calderón 1965] it is easy to check that $[b, T_1]$ is a Calderón–Zygmund operator, so the weak type $(1,1)$-boundedness of $[b, T_1]$ is a natural consequence. Applying Calderón's conclusion [1965, Theorem 2] and Theorems A, 1.5 and 1.8 for the case of $\alpha = 1$, we give the characterizations for the Calderón commutator $[b, T_1]$ as follows.

Corollary 1.10. *Let $1 < p < \infty$ and $0 < \lambda < n$. Suppose that $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ satisfy* (1-2); *then the following four statements are equivalent*:

(i) $b \in \mathrm{Lip}(\mathbb{R}^n)$;

(ii) $[b, T_1]$ *is bounded on* $L^p(\mathbb{R}^n)$;

(iii) $[b, T_1]$ *is bounded from* $L^1(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$;

(iv) $[b, T_1]$ *is bounded on* $L^{p,\lambda}(\mathbb{R}^n)$.

For the case of $\alpha \in (0,1)$, in conjunction with Theorems 1.1, 1.5 and 1.8, we get:

Theorem 1.11. *Suppose $\alpha \in (0,1)$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ satisfy the mean value zero property. Let $1 < p < \infty$ and $0 < \lambda < n$. Then the implications* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) *hold for the following four statements*:

(i) $[b, T_\alpha]$ *is bounded on* $L^p(\mathbb{R}^n)$;

(ii) $[b, T_\alpha]$ *is bounded from* $L^1(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$;

(iii) $[b, T_\alpha]$ *is bounded on* $L^{p,\lambda}(\mathbb{R}^n)$;

(iv) $[b, T_\alpha]$ *is bounded from* $L^\infty(\mathbb{R}^n)$ *to* $\mathrm{BMO}(\mathbb{R}^n)$.

We will prove Theorem 1.11 in Section 5.

Let T_α and T be the operators which are defined respectively by

$$T_\alpha f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega((x-y)')}{|x-y|^{n+\alpha}} f(y)\, dy, \quad 0 < \alpha < 1, \tag{1-3}$$

and

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\widetilde{\Omega}((x-y)')}{|x-y|^{n}} f(y)\, dy. \tag{1-4}$$

We will give a relation between $[b, T_\alpha]$ and $[b, D^\alpha T]$ for the case of $0 < \alpha < 1$.

Proposition 1.12. *Let* $0 < \alpha < 1$. *For any fixed operator* T_α *defined by* (1-3) *with* $\Omega \in L^2(S^{n-1})$ *satisfying* (1-1), *there exists a singular integral operator* T *defined by* (1-4) *with* $\widetilde{\Omega} \in L^2_\alpha(S^{n-1})$ *satisfying* (1-1) *such that* $T_\alpha = D^\alpha T$. *Conversely, for any fixed singular integral operator* T *with* $\widetilde{\Omega} \in L^2_\alpha(S^{n-1})$ *satisfying* (1-1), *there exists an operator* T_α *with* $\Omega \in L^2(S^{n-1})$ *satisfying* (1-1) *such that* $T_\alpha = D^\alpha T$.

We will prove Proposition 1.12 in Section 6.

In particular, for any fixed singular integral operator T with $\widetilde{\Omega} \in C^2(S^{n-1})$ satisfying (1-1), there exists an operator T_α with $\Omega \in C^1(S^{n-1})$ satisfying (1-1) such that $T_\alpha = D^\alpha T$. Then, applying the result of Proposition 1.12, we get:

Corollary 1.13. *Suppose* $\alpha \in (0, 1)$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ *and* $\widetilde{\Omega} \in C^2(S^{n-1})$ *satisfy* (1-1). *Let* $1 < p < \infty$ *and* $0 < \lambda < n$. *Then the implications* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) *hold for the following four statements*:

(i) $[b, D^\alpha T]$ *is bounded on* $L^p(\mathbb{R}^n)$;

(ii) $[b, D^\alpha T]$ *is bounded from* $L^1(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$;

(iii) $[b, D^\alpha T]$ *is bounded on* $L^{p,\lambda}(\mathbb{R}^n)$;

(iv) $[b, D^\alpha T]$ *is bounded from* $L^\infty(\mathbb{R}^n)$ *to* $\mathrm{BMO}(\mathbb{R}^n)$.

Remark 1.14. We will give an application of Theorem 1.1 and Corollary 1.13 to Riesz transforms. In fact, for $0 < \alpha < 1$, since $\widehat{D^\alpha R_j f}(\xi) = -i\xi_j|\xi|^{\alpha-1}\hat{f}(\xi)$ a trivial computation gives

$$\eta(\alpha)\Big(\text{p.v.} \frac{x_j}{|x|^{n+1+\alpha}}\Big)^{\wedge}(\xi) = i\xi_j|\xi|^{\alpha-1}, \quad \text{where } \eta(\alpha) = \frac{1-n-\alpha}{2\pi} \frac{\Gamma\big(\frac{1}{2}(n+\alpha-1)\big)}{\pi^{n/2+\alpha-1}\Gamma\big(\frac{1}{2}(1-\alpha)\big)}.$$

From the above facts, we get

$$[b, D^\alpha R_j] f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega_j(x-y)}{|x-y|^{n+\alpha}} (b(x) - b(y)) f(y)\, dy,$$

where $\Omega_j(x') = \eta(\alpha)x_j/|x|$, $j = 1, 2, \ldots, n$. Since $\Omega_j(x')$ is in $L\log^+ L(S^{n-1})$ and satisfies the mean value zero property, by Theorem 1.1 we get, for $1 < p < \infty$,

$$\|[b, D^\alpha R_j]\|_{L^p} \le C\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}, \quad j = 1, 2, \ldots, n.$$

Now suppose that $[b, D^\alpha R_j]$ are bounded operators from L^∞ to BMO for $j = 1, 2, \ldots, n$. The vanishing moment of Ω_j gives $[b, D^\alpha R_j](1)(x) = -D^\alpha R_j b(x) = -R_j D^\alpha(b)(x) \in \mathrm{BMO}$, $j = 1, 2, \ldots, n$. Since $R_j : \mathrm{BMO} \to \mathrm{BMO}$ and $\sum_{j=1}^n R_j^2 f = -f$, we get

$$\|D^\alpha b\|_{\mathrm{BMO}} = \Big\|\sum_{j=1}^n R_j^2 D^\alpha b\Big\|_{\mathrm{BMO}} \le C\sum_{j=1}^n \Big\|(R_j D^\alpha b)\Big\|_{\mathrm{BMO}} \le C.$$

This gives that $D^\alpha b \in \mathrm{BMO}$. Then, applying Corollary 1.13, for $\alpha \in (0, 1)$, $1 < p < \infty$ and $0 < \lambda < n$ the following five statements are equivalent:

(i) $b \in I_\alpha(\mathrm{BMO})$;

(ii) $[b, D^\alpha R_j]$, $j = 1, \ldots, n$, are bounded on $L^p(\mathbb{R}^n)$;

(iii) $[b, D^\alpha R_j]$, $j = 1, \ldots, n$, are bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$;

(iv) $[b, D^\alpha R_j]$, $j = 1, \ldots, n$, are bounded on $L^{p,\lambda}(\mathbb{R}^n)$;

(v) $[b, D^\alpha R_j]$, $j = 1, \ldots, n$, are bounded from $L^\infty(\mathbb{R})$ to $\mathrm{BMO}(\mathbb{R}^n)$.

The following results show that if we assume some conditions on T, then we may characterize the commutator $[b, D^\alpha T]$ directly.

Corollary 1.15. *Suppose $\alpha \in (0, 1)$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If T is a bounded, invertible operator on* BMO, *then when $\widetilde{\Omega} \in C^2(S^{n-1})$ satisfies* (1-1), *for $1 < p < \infty$ and $0 < \lambda < n$ the following five statements are equivalent*:

(i) $b \in I_\alpha(\mathrm{BMO})$;

(ii) $[b, D^\alpha T]$ *is bounded on* $L^p(\mathbb{R}^n)$;

(iii) $[b, D^\alpha T]$ *is bounded from* $L^1(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$;

(iv) $[b, D^\alpha T]$ *is bounded on* $L^{p,\lambda}(\mathbb{R}^n)$;

(v) $[b, D^\alpha T]$ *is bounded from* $L^\infty(\mathbb{R}^n)$ *to* $\mathrm{BMO}(\mathbb{R}^n)$.

Proof. (i) $\Longrightarrow$ (ii) follows from Theorem 1.1 and (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) follows from Corollary 1.13, so it remains to prove (v) $\Longrightarrow$ (i). If $[b, T_\alpha]$ is bounded from L^∞ to BMO, the vanishing moment of Ω gives $[b, D^\alpha T](1)(x) = -T D^\alpha b(x) \in \mathrm{BMO}$. Since T is a bounded, invertible operator on BMO, we get $D^\alpha b \in \mathrm{BMO}$. □

Remark 1.16. Since H is a bounded, invertible operator on $\mathrm{BMO}(\mathbb{R})$, by Corollary 1.15 we have for $\alpha \in (0, 1)$, $1 < p < \infty$ and $0 < \lambda < n$ that the following five statements are equivalent:

(i) $b \in I_\alpha(\mathrm{BMO})$;

(ii) $[b, D^\alpha H]$ is bounded on $L^p(\mathbb{R})$;

(iii) $[b, D^\alpha H]$ is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$;

(iv) $[b, D^\alpha H]$ is bounded on $L^{p,\lambda}(\mathbb{R})$;

(v) $[b, D^\alpha H]$ is bounded from $L^\infty(\mathbb{R})$ to $\mathrm{BMO}(\mathbb{R})$.

2. Proof of Theorem 1.1

We first prove Theorem 1.1 by a key lemma, whose proof will be given below. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be a radial function such that $\operatorname{supp}\phi \subset \left\{\frac{1}{2} \le |\xi| \le 2\right\}$ and

$$\sum_{l\in\mathbb{Z}} \phi^3(2^{-l}\xi) = 1 \quad \text{for any } |\xi| > 0.$$

Define the multiplier S_l by $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\hat{f}(\xi)$ for all $l \in \mathbb{Z}$.

Lemma 2.1. *Suppose that $\Omega(x')$ satisfies* (1-1). *Let*

$$K_j(x) = \frac{\Omega(x')}{|x|^{n+\alpha}} \chi_{\{2^j \le |x| < 2^{j+1}\}}(x)$$

for $j \in \mathbb{Z}$. Define the multiplier T_j^l ($l \in \mathbb{Z}$) by $\widehat{T_j^l f}(\xi) = \phi(2^{j-l}\xi)\widehat{K_j}(\xi)\hat{f}(\xi)$. Set

$$V_l f(x) = \sum_{j\in\mathbb{Z}} [b, S_{l-j} T_j^l S_{l-j}](f)(x).$$

Let $0 < \alpha < 1$. For $b \in I_\alpha(\mathrm{BMO})(\mathbb{R}^n)$, the following conclusions hold:

(i) *If $\Omega \in L^\infty(S^{n-1})$, then there exists $0 < \tau < 1$ such that*

$$\|V_l f\|_{L^2} \le C\|\Omega\|_{L^\infty} 2^{-\tau|l|} \|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^2} \quad \text{for } l \in \mathbb{Z}. \tag{2-1}$$

(ii) *If $\Omega \in L^1(S^{n-1})$ then, for $1 < p < \infty$,*

$$\|V_l f\|_{L^p} \le C\|\Omega\|_{L^1} \|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^p} \quad \text{for } l \in \mathbb{Z}. \tag{2-2}$$

The constants C in (2-1) *and* (2-2) *are independent of l.*

Proof of Theorem 1.1. Let us now finish the proof of Theorem 1.1 by using Lemma 2.1.

Let $E_0 = \{x' \in S^{n-1} : |\Omega(x')| < 2\}$ and $E_d = \{x' \in S^{n-1} : 2^d \le |\Omega(x')| < 2^{d+1}\}$ for $d \in \mathbb{N}$. For $d \ge 0$, let

$$\Omega_d(y') = \Omega(y')\chi_{E_d}(y') - \frac{1}{|S^{n-1}|}\int_{E_d} \Omega(x')\, d\sigma(x'),$$

Then $\Omega(y') = \sum_{d\ge 0} \Omega_d(y')$. Since Ω satisfies (1-1),

$$\int_{S^{n-1}} \Omega_d(y')\, d\sigma(y') = 0 \quad \text{for all } d \ge 0.$$

Set

$$K_{j,d}(x) = \frac{\Omega_d(x)}{|x|^{n+\alpha}} \chi_{\{2^j \le |x| < 2^{j+1}\}}(x)$$

and define $T^l_{j,d}$ and $V_{l,d}$ in the same way as T^l_j and V_l are defined in Lemma 2.1, replacing K_j by $K_{j,d}$. With the notations above, it is easy to see that

$$[b, T_\alpha]f(x) = \sum_{d\ge 0}\sum_{l\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}[b, S_{l-j}T^l_{j,d}S_{l-j}]f(x) = \sum_{d\ge 0}\sum_{l\in\mathbb{Z}} V_{l,d}f(x).$$

By interpolating between (2-1) and (2-2), there exists $0<\theta<1$ such that

$$\|V_{l,d}f\|_{L^p} \le C\|\Omega_d\|_\infty 2^{-\theta|l|}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p} \quad \text{for } l\in\mathbb{Z}. \tag{2-3}$$

Taking a large positive integer N such that $N>2\theta^{-1}$,

$$\|[b, T_\alpha]f\|_{L^p} \le \sum_{d\ge 0}\sum_{Nd<|l|}\|V_{l,d}f\|_{L^p} + \sum_{d\ge 0}\sum_{0\le|l|\le Nd}\|V_{l,d}f\|_{L^p} =: J_1+J_2.$$

For J_1, using (2-3) we get

$$J_1 \le C\|D^\alpha b\|_{\mathrm{BMO}}\sum_{d\ge 0}2^d\sum_{|l|>Nd}2^{-\theta|l|}\|f\|_{L^p} \le C\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}.$$

Finally, by (2-2) we get

$$\begin{aligned} J_2 &\le C\|D^\alpha b\|_{\mathrm{BMO}}\sum_{d\ge 0}\sum_{0\le|l|<Nd}2^d\sigma(E_d)\|f\|_{L^p} \\ &\le C\|D^\alpha b\|_{\mathrm{BMO}}\sum_{d\ge 0}2^d\sigma(E_d)\|f\|_{L^p} \\ &\le C\|\Omega\|_{L\log^+L}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}. \end{aligned}$$

Combining the estimates of J_1 and J_2, we get

$$\|[b, T_\alpha]f\|_{L^p} \le C(1+\|\Omega\|_{L\log^+L})\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

which is exactly the required conclusion of Theorem 1.1. □

Proof of Lemma 2.1. Hence, to finish the proof of Theorem 1.1, it remains to prove Lemma 2.1. Let us begin by giving some lemmas and their proofs, which will play a key role in the proof.

Lemma 2.2 [Christ and Journé 1987]. *Let $\Theta_j f(x) := \int_{\mathbb{R}^n}\psi_j(x,y)f(y)\,dy$, where $\psi_j(x,y)$ satisfies the standard kernel conditions, i.e., for some $\gamma>0$ and $C>0$,*

$$|\psi_j(x,y)| \le C\frac{2^{j\gamma}}{(2^{-j}+|x-y|)^{n+\gamma}} \tag{2-4}$$

and

$$|\psi_j(x,y+h)-\psi_j(x,y)| \le C\frac{|h|^\gamma}{(2^{-j}+|x-y|)^{n+\gamma}}, \quad |h|\le 2^j, \tag{2-5}$$

for all $x, y\in\mathbb{R}^n$ and $j\in\mathbb{Z}$. Suppose that $du(x,t)=\sum_{j\in\mathbb{Z}}|\Theta_j 1(x)|^2\,dx\,\delta_{2^{-j}}(t)$ is a Carleson measure, where $\delta_{2^{-j}}(t)$ is Dirac mass at the point $t=2^{-j}$. Then $\sum_{j\in\mathbb{Z}}\|\Theta_j f\|^2_{L^2}\le C\|f\|^2_{L^2}$.

Lemma 2.3. *Let $\alpha \in (0,1)$ and $b \in I_\alpha(\mathrm{BMO})(\mathbb{R}^n)$. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be a radial function such that* $\operatorname{supp}\phi \subset \left\{\frac{1}{2} \le |\xi| \le 2\right\}$. *Define the multiplier operator S_l by $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\hat{f}(\xi)$ for $l \in \mathbb{Z}$. Then for $1 < p < \infty$ we have*

$$\left\|\left(\sum_{j\in\mathbb{Z}} 2^{2j\alpha}|[b, S_j]f|^2\right)^{1/2}\right\|_{L^p} \le C\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}.$$

Proof. Let $\widehat{\Phi} = \phi$ and $\Phi_{2^{-j}}(x) = 2^{jn}\Phi(2^j x)$; then $S_j f = \Phi_{2^{-j}} * f$. Let

$$k_j(x,y) = 2^{j\alpha}(b(x) - b(y))\Phi_{2^{-j}}(x-y);$$

then

$$2^{j\alpha}[b, S_j]f(x) = \int_{\mathbb{R}^n} k_j(x,y) f(y)\,dy.$$

It is easy to verify that $k_j(x,y)$ satisfies (2-4) and (2-5). Since

$$2^{j\alpha}[b, S_j]1 = 2^{j\alpha}S_j b = 2^{j\alpha}(|\xi|^\alpha|\xi|^{-\alpha}\phi(2^{-j}\xi)\hat{b})^\vee = (\hat{\sigma}(2^{-j}\xi)\widehat{D^\alpha b})^\vee =: S_j^\alpha(D^\alpha b),$$

where $\hat{\sigma}(\xi) = \phi(\xi)|\xi|^{-\alpha}$ and S_j^α is a multiplier defined by $S_j^\alpha f(x) = \sigma_{2^{-j}} * f(x)$, by $D^\alpha b \in \mathrm{BMO}$ we know

$$du(x,t) = \sum_{j\in\mathbb{Z}} |2^{j\alpha}[b, S_j]1(x)|^2\,dx\,\delta_{2^{-j}}(t)$$

is a Carleson measure. Thus, by Lemma 2.2 we get

$$\sum_{j\in\mathbb{Z}} \|2^{j\alpha}[b, S_j]f\|_{L^2}^2 \le C\|f\|_{L^2}^2. \tag{2-6}$$

Define the operator $\mathbb{T}$ by

$$\mathbb{T}f(x) = \int_{\mathbb{R}^n} \mathbb{K}(x,y) f(y)\,dy,$$

where $\mathbb{K} : (x,y) \mapsto \{k_j(x,y)\}_{j\in\mathbb{Z}}$ with $\|\mathbb{K}(x,y)\|_{\mathbf{C}\mapsto\ell^2} := \left(\sum_{j\in\mathbb{Z}} |k_j(x,y)|^2\right)^{1/2}$. Thus, (2-6) says that

$$\|\mathbb{T}f\|_{L^2(\ell^2)} \le C\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^2}.$$

On the other hand, for $b \in I_\alpha(\mathrm{BMO})$, it is easy to verify that, for $2|x - x_0| \le |x - y|$,

$$\left(\sum_{j\in\mathbb{Z}} |k_j(x,y) - k_j(x_0,y)|^2\right)^{1/2} \le C\|D^\alpha b\|_{\mathrm{BMO}} \frac{|x-x_0|^\alpha}{|x-y|^{n+\alpha}},$$

since $I_\alpha(\mathrm{BMO}) \subset \mathrm{Lip}_\alpha$ for $0 < \alpha < 1$. Then, by the result in [Grafakos 2004], we prove Lemma 2.3. □

Lemma 2.4. *Let $m_{\delta,j} \in C_0^\infty(\mathbb{R}^n)$, $0 < \delta < \infty$, for any fixed $j \in \mathbb{Z}$ and let $T_{\delta,j}$ be the multiplier operator defined by $\widehat{T_{\delta,j}f}(\xi) = m_{\delta,j}(\xi)\hat{f}(\xi)$. For $0 < \alpha < 1$, let $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ and let $[b, T_{\delta,j}]$ be the commutator of $T_{\delta,j}$. If, for some constants $A > 0$ and $0 < \beta < 1$,*

$$\|m_{\delta,j}\|_{L^\infty} \le CA2^{-j\alpha}\min\{\delta, \delta^{-\beta}\} \quad \textit{and} \quad \|\nabla m_{\delta,j}\|_{L^\infty} \le CA2^j 2^{-j\alpha},$$

then there exists a constant $0 < \lambda < 1$ *such that*

$$\|[b, T_{\delta,j}]f\|_{L^2} \leq CA \min\{\delta^{\lambda}, \delta^{-\beta\lambda}\}\|b\|_{\mathrm{Lip}_\alpha}\|f\|_{L^2},$$

where C *is independent of* δ *and* j.

Proof. Without loss of generality, we may assume that $\|b\|_{\mathrm{Lip}_\alpha} = 1$. Taking a $C_0^\infty(\mathbb{R}^n)$ radial function ϕ with $\operatorname{supp}\phi \subset \left\{\frac{1}{2} \leq |x| \leq 2\right\}$ and $\sum_{l\in\mathbb{Z}} \phi(2^{-l}x) = 1$ for any $|x| > 0$. Let $\phi_0(x) = \sum_{l=-\infty}^{0} \phi(2^{-l}x)$ and $\phi_l(x) = \phi(2^{-l}x)$ for positive integers l. Let $K_{\delta,j}(x) = m_{\delta,j}^{\vee}(x)$, the inverse Fourier transform of $m_{\delta,j}$. Split $K_{\delta,j}$ into

$$K_{\delta,j}(x) = K_{\delta,j}(x)\phi_0(x) + \sum_{l=1}^{\infty} K_{\delta,j}(x)\phi_l(x) =: \sum_{l=0}^{\infty} K_{\delta,j}^l(x).$$

Note that $\int_{\mathbb{R}^n} \widehat{\phi}(\eta)\, d\eta = 0$ and

$$\widehat{K_{\delta,j}^l}(x) = 2^{ln}\int_{\mathbb{R}^n} m_{\delta,j}(x-y)\widehat{\phi}(2^l y)\, dy = \int_{\mathbb{R}^n} m_{\delta,j}(x - 2^{-l}y)\widehat{\phi}(y)\, dy.$$

Thus,

$$\begin{aligned}\|\widehat{K_{\delta,j}^l}\|_{L^\infty} &\leq \left\|\int_{\mathbb{R}^n} (m_{\delta,j}(x-2^{-l}y) - m_{\delta,j}(x))\widehat{\phi}(y)\, dy\right\|_{L^\infty} \\ &\leq CA2^{-l}\|\nabla m_{\delta,j}\|_{L^\infty}\int_{\mathbb{R}^n} |y|\,|\widehat{\phi}(y)|\, dy \\ &\leq CA2^{-l}2^j2^{-j\alpha}.\end{aligned} \tag{2-7}$$

On the other hand, by the Young inequality,

$$\|\widehat{K_{\delta,j}^l}\|_{L^\infty} = \|\widehat{K_{\delta,j}} * \widehat{\phi_l}\|_{L^\infty} \leq \|\widehat{K_{\delta,j}}\|_{L^\infty}\|\widehat{\phi_l}\|_{L^1} \leq CA2^{-j\alpha}\min\{\delta, \delta^{-\beta}\}. \tag{2-8}$$

Therefore, by (2-7) and (2-8), for each $0 < \theta < 1$,

$$\|\widehat{K_{\delta,j}^l}\|_{L^\infty} \leq CA2^{-\theta l}2^{(\theta-\alpha)j}\min\{\delta^{1-\theta}, \delta^{-(1-\theta)\beta}\}. \tag{2-9}$$

Then, from (2-8), (2-9) and the Plancherel theorem, we get

$$\|T_{\delta,j}^l f\|_{L^2} \leq CA2^{-j\alpha}\min\{\delta, \delta^{-\beta}\}\|f\|_{L^2} \tag{2-10}$$

and

$$\|T_{\delta,j}^l f\|_{L^2} \leq CA2^{-\theta l}2^{(\theta-\alpha)j}\min\{\delta^{1-\theta}, \delta^{-(1-\theta)\beta}\}\|f\|_{L^2}. \tag{2-11}$$

Now we turn our attention to $[b, T_{\delta,j}^l]$, the commutator of the operator $T_{\delta,j}^l$. Decompose $\mathbb{R}^n$ into a grid of nonoverlapping cubes with side length 2^l. That is, $\mathbb{R}^n = \bigcup_{d=-\infty}^{\infty} Q_d$. Set $f_d = f\chi_{Q_d}$; then

$$f(x) = \sum_{d=-\infty}^{\infty} f_d(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

It is obvious that $\operatorname{supp}([b, T^l_{\delta,j}]f_d) \subset 2nQ_d$ and that the supports of $\{[b, T^l_{\delta,j}]f_d\}_{d=-\infty}^{+\infty}$ have bounded overlaps. So we have the almost orthogonality property

$$\|[b, T^l_{\delta,j}]f\|_{L^2}^2 \le C \sum_{d=-\infty}^{\infty} \|[b, T^l_{\delta,j}]f_d\|_{L^2}^2.$$

Thus, we may assume that $\operatorname{supp} f \subset Q$ for some cube with side length 2^l. Choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $0 \le \varphi \le 1$, $\operatorname{supp}\varphi \subset 100nQ$ and $\varphi = 1$ when $x \in 30nQ$. Set $\widetilde{Q} = 200nQ$ and $\tilde{b} = (b(x) - b_{\widetilde{Q}})\varphi(x)$; then

$$\|[b, T_{\delta,j}]f\|_{L^2} \le \sum_{l\ge 0} \|[b, T^l_{\delta,j}]f\|_{L^2} \le \sum_{l\ge 0} \|\tilde{b}T^l_{\delta,j}f\|_{L^2} + \sum_{l\ge 0} \|T^l_{\delta,j}(\tilde{b}f)\|_{L^2} =: I_1 + I_2.$$

For I_1, we have

$$I_1 \le \sum_{l\ge 0} \|\tilde{b}\|_{L^\infty} \|T^l_{\delta,j}f\|_{L^2} \le C \sum_{l\ge 0} 2^{l\alpha} \|b\|_{\mathrm{Lip}_\alpha} \|T^l_{\delta,j}f\|_{L^2}.$$

Take θ such that $\alpha < \theta < 1$ in (2-11); then, by (2-10) and (2-11),

$$\begin{aligned} I_1 &\le C\Big(\sum_{l<j} 2^{l\alpha} \|T^l_{\delta,j}f\|_{L^2} + \sum_{l\ge j} 2^{l\alpha} \|T^l_{\delta,j}f\|_{L^2} \Big) \\ &\le CA\Big(\sum_{l<j} 2^{(l-j)\alpha} \min\{\delta,\ \delta^{-\beta}\} + \sum_{l\ge j} 2^{(l-j)(\alpha-\theta)} \min\{\delta^{1-\theta},\ \delta^{-\beta(1-\theta)}\} \Big) \|f\|_{L^2} \\ &\le CA \min\{\delta^{1-\theta},\ \delta^{-\beta(1-\theta)}\} \|f\|_{L^2}, \end{aligned}$$

where C is independent of δ. Similarly, we can get

$$I_2 \le CA \min\{\delta^{1-\theta},\ \delta^{-\beta(1-\theta)}\} \|f\|_{L^2}.$$

Thus

$$\|[b, T_{\delta,j}]f\|_{L^2} \le CA \min\{\delta^{\lambda},\ \delta^{-\beta\lambda}\} \|f\|_{L^2}$$

with $0 < \lambda = 1 - \theta < 1$ and C independent of δ. □

Proof of (2-1) *in Lemma 2.1.* For $j \in \mathbb{Z}$, define the operator T_j by $T_j f = K_j * f$, where $K_j(x) = (\Omega(x')/|x|^{n+\alpha})\chi_{\{2^j \le |x| < 2^{j+1}\}}(x)$. Since $\Omega \in L^\infty(S^{n-1})$, for some $0 < \beta < 1$ we have

$$|\widehat{K_j}(\xi)| \le C\|\Omega\|_{L^\infty} 2^{-j\alpha} \min\{|2^j\xi|^{-\beta},\ |2^j\xi|\}$$

(see [Duoandikoetxea and Rubio de Francia 1986, pp. 551–552]). A trivial computation shows that $|\nabla\widehat{K_j}(\xi)| \le C\|\Omega\|_{L^1} 2^{(1-\alpha)j}$. Set

$$m_j(\xi) = \widehat{K_j}(\xi), \quad m^l_j(\xi) = m_j(\xi)\phi(2^{j-l}\xi).$$

Define the operator T^l_j by $\widehat{T^l_j f}(\xi) = m^l_j(\xi)\hat{f}(\xi)$. Thus $m^l_j \in C_0^\infty(\mathbb{R}^n)$ with

$$\|m^l_j\|_{L^\infty} \le C\|\Omega\|_{L^\infty} 2^{-j\alpha} \min\{2^{-\beta l},\ 2^l\} \quad \text{and} \quad \|\nabla m^l_j\|_{L^\infty} \le C\|\Omega\|_{L^\infty} 2^{(1-\alpha)j}. \tag{2-12}$$

Thus Lemma 2.4 with $\delta = 2^l$ and $I_\alpha(\mathrm{BMO}) \subset \mathrm{Lip}_\alpha$ for $0 < \alpha < 1$ says that, for some constant $0 < \lambda < 1$,

$$\|[b, T_j^l] f\|_{L^2} \le C\|\Omega\|_{L^\infty} \|D^\alpha b\|_{\mathrm{BMO}} \min\{2^{-\beta\lambda l},\ 2^{\lambda l}\}\|f\|_{L^2}, \quad l \in \mathbb{Z}. \tag{2-13}$$

By the Plancherel theorem, we get

$$\|T_j^l f\|_{L^2} \le C\|\Omega\|_{L^\infty} 2^{-j\alpha} \min\{2^{-\beta l},\ 2^l\}\|f\|_{L^2}. \tag{2-14}$$

For any $j, l \in \mathbb{Z}$ we may write

$$[b, S_{l-j} T_j^l S_{l-j}] f = [b, S_{l-j}](T_j^l S_{l-j} f) + S_{l-j}([b, T_j^l] S_{l-j} f) + S_{l-j} T_j^l([b, S_{l-j}] f).$$

Then

$$\begin{aligned}\|V_l f\|_{L^2} &\le \Big\|\sum_{j\in\mathbb{Z}} S_{l-j}([b, T_j^l] S_{l-j} f)\Big\|_{L^2} + \Big\|\sum_{j\in\mathbb{Z}} S_{l-j} T_j^l([b, S_{l-j}] f)\Big\|_{L^2} + \Big\|\sum_{j\in\mathbb{Z}} [b, S_{l-j}](T_j^l S_{l-j}] f)\Big\|_{L^2} \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Below we shall estimate I_i for $i = 1, 2, 3$. By Littlewood–Paley theory and (2-13), we get

$$\begin{aligned} I_1 &\le \Big(\sum_{j\in\mathbb{Z}} \|[b, T_j^l](S_{l-j} f)\|_{L^2}^2\Big)^{1/2} \\ &\le C\|\Omega\|_{L^\infty} \min\{2^{-\beta\lambda l},\ 2^{\lambda l}\}\|D^\alpha b\|_{\mathrm{BMO}} \Big(\sum_{j\in\mathbb{Z}} \|S_{l-j} f\|_{L^2}^2\Big)^{1/2} \\ &\le C\|\Omega\|_{L^\infty} \min\{2^{-\beta\lambda l},\ 2^{\lambda l}\}\|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^2}. \end{aligned} \tag{2-15}$$

Now we estimate I_2. By (2-14) and Lemma 2.3, we get

$$\begin{aligned} I_2 &\le \Big(\sum_{j\in\mathbb{Z}} \|T_j^l([b, S_{l-j}] f)\|_{L^2}^2\Big)^{1/2} \\ &\le C\|\Omega\|_{L^\infty} \min\{2^{-(\beta+\alpha)l},\ 2^{(1-\alpha)l}\} \Big(\sum_{j\in\mathbb{Z}} 2^{2j\alpha} \|[b, S_j] f\|_{L^2}^2\Big)^{1/2} \\ &\le C\|\Omega\|_{L^\infty} \min\{2^{-(\beta+\alpha)l},\ 2^{(1-\alpha)l}\}\|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^2}. \end{aligned} \tag{2-16}$$

Finally, by duality and (2-16) we get

$$I_3 \le C\|\Omega\|_{L^\infty} \min\{2^{-(\beta+\alpha)l},\ 2^{(1-\alpha)l}\}\|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^2}. \tag{2-17}$$

It follows from (2-15)–(2-17) that, for some constant $0 < \tau < 1$,

$$\|V_l f\|_{L^2} \le C\|\Omega\|_{L^\infty} 2^{-\tau|l|}\|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^2} \quad \text{for } l \in \mathbb{Z}.$$

This completes the proof of (2-1). $\square$

Proof of (2-2) *in Lemma 2.1.* Since $T_j^l = T_j S_{l-j}$ for any $j, l \in \mathbb{Z}$, we may write

$$[b, S_{l-j} T_j^l S_{l-j}] f = S_{l-j}([b, T_j] S_{l-j}^2 f) + S_{l-j} T_j([b, S_{l-j}^2] f) + [b, S_{l-j}](T_j S_{l-j}^2 f).$$

Thus,

$$\begin{aligned} \|V_l f\|_{L^p} &\le \Big\| \sum_{j\in\mathbb{Z}} S_{l-j}([b, T_j] S_{l-j}^2 f) \Big\|_{L^p} + \Big\| \sum_{j\in\mathbb{Z}} S_{l-j} T_j([b, S_{l-j}^2] f) \Big\|_{L^p} + \Big\| \sum_{j\in\mathbb{Z}} [b, S_{l-j}](T_j S_{l-j}^2 f) \Big\|_{L^p} \\ &=: L_1 + L_2 + L_3. \end{aligned}$$

Below we shall estimate L_i, $i = 1, 2, 3$. It is well known that, for any $g \in L^p(\mathbb{R}^n)$,

$$|[b, T_j] g(x)| \le C \|b\|_{\mathrm{Lip}_\alpha} M_\Omega g(x),$$

where

$$M_\Omega g(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)| \, |g(y)| \, dy.$$

From this we get, for $1 < p < \infty$,

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |[b, T_j] g_j|^2 \Big)^{1/2} \Big\|_{L^p} \le C \|\Omega\|_{L^1} \|b\|_{\mathrm{Lip}_\alpha} \Big\| \Big(\sum_{j\in\mathbb{Z}} |g_j|^2 \Big)^{1/2} \Big\|_{L^p}.$$

Then, by Littlewood–Paley theory and since $I_\alpha(\mathrm{BMO}) \subset \mathrm{Lip}_\alpha$ for $0 < \alpha < 1$, we have

$$L_1 \le C \Big\| \Big(\sum_{j\in\mathbb{Z}} |[b, T_j](S_{l-j}^2 f)|^2 \Big)^{1/2} \Big\|_{L^p} \le C \|\Omega\|_{L^1} \|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^p}.$$

For L_2, by a similar proof to that of [Chen and Zhang 2004, (1.13)], we get

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |T_j f_j|^2 \Big)^{1/2} \Big\|_{L^p} \le C \|\Omega\|_{L^1} \Big\| \Big(\sum_{j\in\mathbb{Z}} |D^\alpha f_j|^2 \Big)^{1/2} \Big\|_{L^p}.$$

Then, by Littlewood–Paley theory and the above inequality, we get

$$\begin{aligned} L_2 &\le C \|\Omega\|_{L^1} \Big\| \Big(\sum_{j\in\mathbb{Z}} |D^\alpha [b, S_{l-j}^2] f|^2 \Big)^{1/2} \Big\|_{L^p} \\ &\le C \|\Omega\|_{L^1} \Big\| \Big(\sum_{j\in\mathbb{Z}} |[b, D^\alpha] S_{l-j}^2 f|^2 \Big)^{1/2} \Big\|_{L^p} + C \|\Omega\|_{L^1} \Big\| \Big(\sum_{j\in\mathbb{Z}} |[b, D^\alpha S_{l-j}^2] f|^2 \Big)^{1/2} \Big\|_{L^p}. \end{aligned}$$

Note that the kernel of $[b, D^\alpha]$ is

$$K(x, y) = \eta(\alpha) \frac{b(x) - b(y)}{|x-y|^{n+\alpha}},$$

where $\eta(\alpha)$ is some normalization constant (see [Stein 1970]). Since $K(x, y)$ is antisymmetric, WBP is satisfied automatically. Also $[b, D^\alpha] 1 = D^\alpha b \in \mathrm{BMO}$ so, by the $T1$ theorem (see [David and Journé 1984]), $[b, D^\alpha]$ is bounded on $L^2(\mathbb{R}^n)$. It is easy to verify that $K(x, y)$ is a standard kernel; then, by the Calderón–Zygmund theorem (see [Grafakos 2004]), we get that $[b, D^\alpha]$ is bounded on $L^p(\ell^2(\mathbb{R}^n))$. Combining this with Lemma 2.3, we get

$$\begin{aligned}L_2 &\le C\|\Omega\|_{L^1}\|D^\alpha b\|_{\mathrm{BMO}}\bigg\|\bigg(\sum_{j\in\mathbb{Z}}|S_j f|^2\bigg)^{1/2}\bigg\|_{L^p} + C\|\Omega\|_{L^1}\bigg\|\bigg(\sum_{j\in\mathbb{Z}}|2^{j\alpha}[b, \bar{S}_j]f|^2\bigg)^{1/2}\bigg\|_{L^p}\\ &\le C\|\Omega\|_{L^1}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p},\end{aligned}$$

where $\bar{S}_j$ is the Littlewood–Paley operator given in the transform by multiplication with the function $|2^{-j}\xi|^\alpha\phi^2(2^{-j}\xi)$. By duality and the estimate of L_2, we get

$$L_3 \le C2^{-l\alpha}\|\Omega\|_{L^1}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}.$$

Combining the estimates of L_1, L_2 and L_3, we get

$$\|V_l f\|_{L^p} \le C\|\Omega\|_{L^1}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p} \quad \text{for } l \in \mathbb{Z}.$$

This completes the proof of (2-2). □

3. Proof of Theorem 1.5

In the proof of Theorem 1.5, for $j = 1, \dots, 15$, A_j is a positive constant depending only on Ω, n, p, α, λ and A_i, $1 \le i < j$. We may assume $\|[b, T_\alpha]\|_{L^{p,\lambda}\to L^{p,\lambda}} = 1$. We want to prove that, for any fixed $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$,

$$M := \frac{1}{|B(x_0, r)|^{1+\alpha/n}}\int_{B(x_0,r)}|b(y) - a_0|\,dy \le A(p, \Omega, \alpha, \lambda), \tag{3-1}$$

where $a_0 = |B(x_0, r)|^{-1}\int_{B(x_0,r)} b(y)\,dy$. Since $[b - a_0, T_\alpha] = [b, T_\alpha]$, we may assume $a_0 = 0$. Let

$$f(y) = (\operatorname{sgn} b(y) - c_0)\chi_{B(x_0,r)}(y), \tag{3-2}$$

where $c_0 = (1/|B(x_0, r)|)\int_{B(x_0,r)} \operatorname{sgn} b(y)\,dy$. Then f has the following properties:

$$\int_{\mathbb{R}^n} f(y)\,dy = 0, \tag{3-3}$$

$$f(y)b(y) > 0, \tag{3-4}$$

$$\frac{1}{|B(x_0, r)|^{1+\alpha/n}}\int_{\mathbb{R}^n} f(y)b(y)\,dy = M. \tag{3-5}$$

Without loss of generality, we may assume that $|\Omega(x') - \Omega(y')| \le |x' - y'|$ for all x', $y' \in S^{n-1}$. Since Ω satisfies (1-1) or (1-2), there exists a positive number $A_1 < 1$ such that

$$\sigma(\Lambda) := \sigma\big(\{x' \in S^{n-1} : \Omega(x') \ge 2A_1\}\big) > 0, \tag{3-6}$$

where σ is the measure on S^{n-1} which is induced from the Lebesgue measure on $\mathbb{R}^n$. Then, for $x \in G = \{x \in \mathbb{R}^n : |x - x_0| > A_2 r = (2A_1^{-1} + 1)r \text{ and } (x - x_0)' \in \Lambda\}$,

$$\begin{aligned}|[b, T_\alpha] f(x)| &\geq \left| \int_{\mathbb{R}^n} \Omega((x-y)')|x-y|^{-n-\alpha} b(y) f(y)\, dy \right| - |b(x)| \left| \int_{\mathbb{R}^n} \Omega((x-y)')|x-y|^{-n-\alpha} f(y)\, dy \right| \\ &=: I_1(x) - I_2(x).\end{aligned}$$

For $I_1(x)$, noting that if $|y - x_0| < r$, we get $|(x - x_0)' - (x - y)'| \leq 2|y - x_0|/|x - x_0| \leq A_1$, then, since $\Omega \in \mathrm{Lip}(S^{n-1})$, we get $\Omega((x - y)') \geq A_1$. Thus it follows from (3-4) and (3-5) that

$$I_1(x) \geq A_1 \int_{B(x_0,r)} b(y) f(y) |y - x|^{-n-\alpha}\, dy \geq A_3 r^{n+\alpha} M |x - x_0|^{-n-\alpha}.$$

Since $\Omega \in \mathrm{Lip}(S^{n-1})$ and by (3-3), we have

$$I_2(x) \leq |b(x)| \int_{B(x_0,r)} |f(y)| \left| \frac{\Omega((x-y)')}{|x-y|^{n+\alpha}} - \frac{\Omega((x-x_0)')}{|x-x_0|^{n+\alpha}} \right| dy \leq A_4 r^{n+1} |b(x)|\, |x - x_0|^{-n-\alpha-1}.$$

Let $\theta = p/(n(p-1) + p\alpha + \lambda)$ and

$$F = \left\{ x \in G : |b(x)| > \frac{A_3 M r^{\alpha-1}}{2A_4} |x - x_0| \text{ and } |x - x_0| < M^\theta r \right\}.$$

This gives that $I_1(x) \geq 2I_2(x)$ when $x \in (G \setminus F) \cap \{x : |x - x_0| < M^\theta r\}$. Then we have

$$|[b, T_\alpha] f(x)| \geq I_1(x) - I_2(x) \geq \tfrac{1}{2} I_1(x) \quad \text{for } x \in (G \setminus F) \cap \{x : |x - x_0| < M^\theta r\}.$$

Hence,

$$\begin{aligned}
\|f\|^p_{L^{p,\lambda}} &\geq \|[b, T_\alpha] f\|^p_{L^{p,\lambda}} \\
&\geq \frac{1}{M^{\theta\lambda} r^\lambda} \int_{|x-x_0|<M^\theta r} |[b, T_\alpha] f(x)|^p\, dx \\
&\geq \frac{1}{M^{\theta\lambda} r^\lambda} \int_{(G\setminus F)\cap\{|x-x_0|<M^\theta r\}} \left(\tfrac{1}{2} A_3 M r^{\alpha+n} |x - x_0|^{-n-\alpha}\right)^p dx \\
&\geq \frac{1}{M^{\theta\lambda} r^\lambda} \int_{\{A_5(|F|+(B_2 r)^n)^{1/n}<|x-x_0|<M^\theta r\}\cap G} \left(\tfrac{1}{2} A_3 M r^{\alpha+n} |x - x_0|^{-n-\alpha}\right)^p dx \\
&= \frac{\sigma(\Lambda)}{M^{\theta\lambda} r^\lambda} \left(\frac{A_3 M r^{\alpha+n}}{2} \right)^p \int_{A_5(|F|+(A_2 r)^n)^{1/n}}^{M^\theta r} t^{-n(p-1)-p\alpha-1}\, dt \\
&= \frac{\sigma(\Lambda)}{M^{\theta\lambda} r^\lambda} \frac{\left(\tfrac{1}{2} B_3 M r^{\alpha+n}\right)^p}{(-n(p-1) - p\alpha)} \left((M^\theta r)^{-n(p-1)-p\alpha} - A_6 (|F| + (A_2 r)^n)^{(-n(p-1)-p\alpha)/n} \right).
\end{aligned}$$

Then, by $\|f\|_{L^{p,\lambda}} \leq C r^{(n-\lambda)/p}$ and an elementary computation, we have

$$(|F| + (A_2 r)^n)^{-(p-1)-p\alpha/n} \leq A_7 (M^{\theta(-n(p-1)-p\alpha)} + M^{\theta\lambda - p}) r^{-n(p-1)-p\alpha}.$$

Since $\lambda = p/\theta - n(p-1) - p\alpha$, we get

$$(|F| + (A_2 r)^n)^{-(p-1)-p\alpha/n} \leq A_8 M^{\theta(-n(p-1)-p\alpha)} r^{-n(p-1)-p\alpha}.$$

Then we have

$$|F| \geq A_9 M^{\theta n} r^n - (A_2 r)^n.$$

If $M \leq (2A_9^{-1}A_2^n)^{1/(\theta n)}$, then Theorem 1.5 is proved. If $M > (2A_9^{-1}A_2^n)^{1/(\theta n)}$, then

$$|F| \geq \tfrac{1}{2} A_9 M^{\theta n} r^n. \tag{3-7}$$

Now let $g(y) = \chi_{B(x_0,r)}(y)$. For $x \in F$,

$$\begin{aligned}|[b, T_\alpha]g(x)| &\geq |b(x)| \left| \int_{B(x_0,r)} \frac{\Omega((x-y)')}{|x-y|^{n+\alpha}} g(y)\, dy \right| - \int_{B(x_0,r)} |\Omega((x-y)')|\, |x-y|^{-n-\alpha} |b(y)|\, dy \\ &=: K_1 - K_2. \end{aligned} \tag{3-8}$$

For $y \in B(x_0, r)$ and $x \in F$ we have that $|x - x_0| \simeq |x - y|$ and $\Omega((x-y)') \geq A_1$. Now, regarding K_1, it follows that

$$K_1 \geq C|b(x)| \int_{B(x_0,r)} |x-y|^{-n-\alpha}\, dy \geq A_{10} |b(x)|\, |x-x_0|^{-n-\alpha} r^n. \tag{3-9}$$

For K_2, since $\Omega \in L^\infty(S^{n-1})$, we have

$$K_2 \leq C|x-x_0|^{-n-\alpha} \int_{B(x_0,r)} |b(y)|\, dy \leq A_{11} |x-x_0|^{-n-\alpha} r^{n+\alpha} M. \tag{3-10}$$

So, by (3-8)–(3-10) and since $|b(x)| > (A_3 M r^\alpha/(2A_4))|x-x_0|/r$ when $x \in F$, we get, for $x \in F$,

$$|[b, T_\alpha]g(x)| \geq A_{12} |x-x_0|^{1-n-\alpha} r^{n+\alpha-1} M - A_{11} |x-x_0|^{-n-\alpha} r^{n+\alpha} M. \tag{3-11}$$

Since $\|g\|_{L^{p,\lambda}} \leq C r^{(n-\lambda)/p}$, by (3-11) and Hölder's inequality we have

$$\begin{aligned} A_{13} r^{(n-\lambda)/p} &\geq \|[b, T_\alpha]g\|_{L^{p,\lambda}} \\ &\geq \left(\frac{1}{(M^\theta r)^\lambda} \int_{\{\frac{1}{4} A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r\}} |[b, T_\alpha]g(x)|^p\, dx \right)^{1/p} \\ &\geq \frac{1}{(M^\theta r)^{\lambda/p + n/p'}} \int_{F \cap \{\frac{1}{4} A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r\}} |[b, T_\alpha]g(x)|\, dx \\ &\geq A_{12} \frac{M r^{n+\alpha-1}}{(M^\theta r)^{\lambda/p+n/p'}} \int_{F \cap \{\frac{1}{4} A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r\}} |x-x_0|^{1-n-\alpha}\, dx \\ &\qquad - A_{11} \frac{r^{n+\alpha} M}{(M^\theta r)^{\lambda/p+n/p'}} \int_{F \cap \{\frac{1}{4} A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r\}} |x-x_0|^{-n-\alpha}\, dx \\ &=: L_1 - L_2. \end{aligned} \tag{3-12}$$

To estimate L_1 and L_2, we first prove that

$$\left| F \cap \left\{ \tfrac{1}{4} A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r \right\} \right| \geq \tfrac{1}{4} A_9 M^{\theta n} r^n. \tag{3-13}$$

Let

$$F = \left(F \cap \left\{ \tfrac{1}{4} A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r \right\} \right) \cup \left(F \cap \left\{ |x-x_0| < \tfrac{1}{4} A_9^{1/n} M^\theta r \right\} \right) =: E_1 \cup E_2.$$

Notice that

$$|E_2| \le \left|\left\{x : |x-x_0| < \tfrac{1}{4}A_9^{1/n}M^{\theta}r\right\}\right| \le \left(\tfrac{1}{4}\right)^n A_9 M^{\theta n} r^n.$$

If $|E_1| < \frac{1}{4}A_9 M^{\theta n} r^n$, then

$$|F| = |E_1| + |E_2| < \tfrac{1}{4}A_9 M^{\theta n} r^n + \left(\tfrac{1}{4}\right)^n A_9 M^{\theta n} r^n < \tfrac{1}{2}A_9 M^{\theta n} r^n.$$

This contradicts $|F| \ge \frac{1}{2}A_9 M^{\theta n} r^n$. This proves (3-13). Now we turn to give the estimates of L_1 and L_2. Since $|x-x_0| < M^{\theta}r$ and by (3-13),

$$\begin{aligned} L_1 &\ge A_{12}\left|F \cap \left\{\tfrac{1}{2}A_9^{1/n}M^{\theta}r < |x-x_0| < M^{\theta}r\right\}\right| \frac{Mr^{n+\alpha-1}}{(M^{\theta}r)^{\lambda/p+n/p'}}(M^{\theta}r)^{1-n-\alpha} \\ &\ge A_{14}\frac{M^{\theta(1-\alpha)+1}r^{(n-\lambda)/p}}{M^{\theta(\lambda/p+n/p')}}. \end{aligned} \tag{3-14}$$

For L_2, we have

$$\begin{aligned} L_2 &\le A_{11}\frac{r^{n+\alpha}M}{(M^{\theta}r)^{\lambda/p+n/p'}}\int_{F\cap\left\{\frac{1}{2}A_9^{1/n}M^{\theta}r<|x-x_0|<M^{\theta}r\right\}}|x-x_0|^{-n-\alpha}\,dx \\ &\le A_{11}\frac{r^{n+\alpha}M}{(M^{\theta}r)^{\lambda/p+n/p'}}\int_{\left\{\frac{1}{2}A_9^{1/n}M^{\theta}r<|x-x_0|<M^{\theta}r\right\}}|x-x_0|^{-n-\alpha}\,dx \\ &\le A_{15}\frac{r^{(n-\lambda)/p}M^{1-\alpha\theta}}{M^{\theta(\lambda/p+n/p')}}. \end{aligned} \tag{3-15}$$

Now (3-12) and (3-14)–(3-15) show that

$$A_{13} \ge (A_{14}M^{\theta(1-\alpha)} - A_{15}M^{-\alpha\theta})\frac{M}{M^{\theta(\lambda/p+n/p')}}.$$

Since $\theta = p/(n(p-1)+p\alpha+\lambda)$,

$$M^{\theta(\lambda/p+n/p')} = M^{1-p\alpha/(n(p-1)+p\alpha+\lambda)} = M^{1-\alpha\theta}.$$

Thus, we get

$$A_{13} \ge A_{14}M^{\theta} - A_{15}.$$

Therefore, $M \le A(p,\Omega,\alpha,\lambda)$ and we complete the proof of Theorem 1.5. □

4. Proof of Theorem 1.8

As in the proof of Theorem 1.8, let A_j, $j = 1, \dots, 14$, be positive constants depending only on Ω, n, α and A_i, $1 \le i < j$. Without loss of generality, we may assume that $\|[b, T_\alpha]\|_{L^1\to L^{1,\infty}} = 1$. For any fixed $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, we also set $a_0 := |B(x_0,r)|^{-1}\int_{B(x_0,r)} b(y)\,dy = 0$ since $[b-a_0, T_\alpha] = [b, T_\alpha]$. It is our aim to prove the inequality

$$M = \frac{1}{|B(x_0,r)|^{1+\alpha/n}}\int_{B(x_0,r)}|b(y)|\,dy \le A(n,\Omega,\alpha).$$

Let f be as defined in (3-2) and Λ be as defined in (3-6). Take

$$G = \{x \in \mathbb{R}^n : |x - x_0| > A_2 r = (2A_1^{-1} + 1)r \text{ and } (x - x_0)' \in \Lambda\}.$$

Then for $x \in G$ we have

$$\begin{aligned}
|[b, T_\alpha] f(x)| &\geq |T_\alpha(bf)(x)| - |b(x)|\,|T_\alpha f(x)| \\
&= \left| \int_{\mathbb{R}^n} \Omega((x-y)')|x-y|^{-n-\alpha} b(y) f(y)\, dy \right| - |b(x)| \left| \int_{\mathbb{R}^n} \Omega((x-y)')|x-y|^{-n-\alpha} f(y)\, dy \right| \\
&=: I_1(x) - I_2(x).
\end{aligned}$$

Similar to the proof of Theorem 1.8, we get

$$I_1(x) \geq A_3 r^{n+\alpha} M |x - x_0|^{-n-\alpha}$$

and

$$I_2(x) \leq A_4 r^{n+1} |b(x)|\,|x - x_0|^{-n-\alpha-1}.$$

Let

$$F = \left\{ x \in G : |b(x)| > \frac{A_3 M r^{\alpha-1}}{2A_4} |x - x_0| \text{ and } |x - x_0| < M^{1/(n+\alpha)} r \right\}.$$

Then we have $|[b, T_\alpha] f(x)| \geq \frac{1}{2} I_1(x)$ when $x \in (G \setminus F) \cap \{x : |x - x_0| < M^{1/(n+\alpha)} r\}$. Thus,

$$\begin{aligned}
\|f\|_{L^1} &\geq \int_{\{x \in \mathbb{R}^n : |[b, T_\alpha] f(x)| > 1\}} dx \\
&\geq \int_{(G \setminus F) \cap \{|x - x_0| < M^{1/(n+\alpha)} r\} \cap \{|[b, T_\alpha] f(x)| > 1\}} dx \\
&\geq \int_{(G \setminus F) \cap \{|x - x_0| < M^{1/(n+\alpha)} r\} \cap \{A_3 M r^{\alpha+n} |x - x_0|^{-n-\alpha} > 2\}} dx \\
&\geq \int_{\{A_6(|F| + (A_2 r)^n)^{1/n} < |x - x_0| < A_5 M^{1/(n+\alpha)} r\} \cap G} dx \\
&= \int_{A_6(|F| + (A_2 r)^n)^{1/n}}^{A_5 M^{1/(n+\alpha)} r} t^{n-1}\, dt \int_\Lambda d\sigma(x').
\end{aligned}$$

Since $\|f\|_{L^1} \leq r^n$, we then have

$$|F| \geq A_7 M^{n/(n+\alpha)} r^n - A_8 r^n.$$

If $M \leq (2A_8 A_7^{-1})^{(n+\alpha)/n}$, then Theorem 1.8 is proved. If $M > (2A_8 A_7^{-1})^{(n+\alpha)/n}$, then

$$|F| \geq \tfrac{1}{2} A_7 M^{n/(n+\alpha)} r^n. \tag{4-1}$$

Now, let $g(y) = \chi_{B(x_0, r)}(y)$. Similar to (3-11) in the proof of Theorem 1.5, for $x \in F$ we have

$$|[b, T_\alpha] g(x)| \geq A_9 |x - x_0|^{1-n-\alpha} r^{n+\alpha-1} M - A_{10} |x - x_0|^{-n-\alpha} r^{n+\alpha} M.$$

Since $\|g\|_{L^1} \le Cr^n$, we have

$$A_{11}r^n \ge \|g\|_{L^1} \ge \int_{\{x\in\mathbb{R}^n:|[b,T_\alpha]g(x)|>1\}} dx \ge \int_{F\cap\{x:|x-x_0|\ge(2A_{10}/A_9)r\}\cap\{x\in\mathbb{R}^n:|[b,T_\alpha]g(x)|>1\}} dx.$$

For $|x-x_0| \ge (2A_{10}/A_9)r$,

$$|[b,T_\alpha]g(x)| \ge \tfrac{1}{2}A_9|x-x_0|^{1-n-\alpha}r^{n+\alpha-1}M.$$

Thus,

$$\begin{aligned} A_{11}r^n &\ge \int_{F\cap\{x:|x-x_0|\ge(2A_{10}/A_9)r\}\cap\{x\in\mathbb{R}^n:(A_9/2)|x-x_0|^{1-n-\alpha}r^{n+\alpha-1}M>1\}} dx \\ &= \int_{F\cap\{x:|x-x_0|\ge(2A_{10}/A_9)r\}\cap\{x\in\mathbb{R}^n:|x-x_0|\le A_{12}M^{1/(n+\alpha-1)}r\}} dx \\ &= \int_{\{x\in F:A_{13}r\le|x-x_0|\le A_{12}M^{1/(n+\alpha-1)}r\}} dx. \end{aligned} \tag{4-2}$$

If $M \le (A_{13}/A_{12})^{n+\alpha-1}$, then we have proved Theorem 1.8. If $M > (A_{13}/A_{12})^{n+\alpha-1}$, then

$$\begin{aligned} \int_{\{x\in F:A_{13}r\le|x-x_0|\le A_{12}M^{1/(n+\alpha-1)}r\}} dx &= \int_{\{x\in F:|x-x_0|\le A_{12}M^{1/(n+\alpha-1)}r\}} dx - \int_{\{x\in F:|x-x_0|\le A_{13}r\}} dx \\ &=: K_1 - K_2. \end{aligned} \tag{4-3}$$

If $M \le A_{12}^{-(n+\alpha)(n+\alpha-1)}$, then Theorem 1.8 is proved. If $M > A_{12}^{-(n+\alpha)(n+\alpha-1)}$, we have

$$A_{12}M^{1/(n+\alpha-1)} \ge M^{1/(n+\alpha)}.$$

By (4-1), we get

$$K_1 \ge \int_{\{x\in F:|x-x_0|\le M^{1/(n+\alpha)}r\}} dx = \int_F dx = |F| \ge \tfrac{1}{2}A_7M^{n/(n+\alpha)}r^n$$

and

$$K_2 \le \int_{\{x\in F:|x-x_0|\le A_{13}r\}} dx \le A_{14}r^n.$$

Combining these estimates, from (4-2) and (4-3) we get

$$A_{11} \ge \tfrac{1}{2}A_7M^{n/(n+\alpha)} - A_{14}.$$

Then $M \le A(n,\Omega,\alpha)$. □

5. Proof of Theorem 1.11

Let

$$k(x,y) = \frac{\Omega(x-y)}{|x-y|^{n+\alpha}}(b(x)-b(y)).$$

Proof of (i) $\Longrightarrow$ (ii). Suppose that, for some $1 < p < \infty$,

$$\|[b,T_\alpha]f\|_{L^p} \le C\|f\|_{L^p}; \tag{5-1}$$

then, by Theorem 1.5 for $\lambda = 0$, we must have $b \in \mathrm{Lip}_\alpha$. If $\Omega \in \mathrm{Lip}(S^{n-1})$ and $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, there is a constant $C > 0$ such that, for all $x, x_0, y \in \mathbb{R}^n$ with $2|x - x_0| \le |y - x|$, the kernel $k(x, y)$ satisfies the inequality

$$|k(x, y) - k(x_0, y)| \le C|x - x_0|^\alpha |y - x|^{-n-\alpha}. \tag{5-2}$$

Applying (5-1) and (5-2), by using a Calderón–Zygmund decomposition and a trivial computation, we get

$$\|[b, T_\alpha]f\|_{L^{1,\infty}} \le C\|f\|_{L^1}. \qquad \square$$

Proof of (ii) $\Longrightarrow$ (iii). Suppose that $[b, T_\alpha]$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$; then by Theorem 1.8 we must have $b \in \mathrm{Lip}_\alpha$. So $k(x, y)$ satisfies (5-2). For fixed $x \in \mathbb{R}^n$, pick a cube $Q = Q(x_0, r)$ that contains x. Let $f = f_1 + f_2$, with $f_1 = f_{\chi_{2Q}}$ and $f_2 = f_{\chi_{(2Q)^c}}$. We select $a = [b, T_\alpha]f(x_0)$ and let $0 < \delta < 1$; then

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q \big||[b, T_\alpha]f(y)|^\delta - |a|^\delta\big|\,dy\right)^{1/\delta} &\le \left(\frac{1}{|Q|}\int_Q |[b, T_\alpha]f(y) - a|^\delta\,dy\right)^{1/\delta}\\ &\le \left(\frac{1}{|Q|}\int_Q |[b, T_\alpha]f_1(y)|^\delta\,dy\right)^{1/\delta} + \frac{1}{|Q|}\int_Q |[b, T_\alpha]f_2(y) - a|\,dy.\end{aligned}$$

Since $[b, T_\alpha] : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ and $0 < \delta < 1$, Kolmogorov's inequality [García-Cuerva and Rubio de Francia 1985, p. 485] yields

$$\left(\frac{1}{|Q|}\int_Q |[b, T_\alpha]f_1(y)|^\delta\,dy\right)^{1/\delta} \le \frac{1}{|Q|}\int_{\mathbb{R}^n} |f_1(y)|\,dy \le CMf(x).$$

By (5-2), it is easy to get

$$\frac{1}{|Q|}\int_Q |[b, T_\alpha]f_2(y) - a|\,dy \le CMf(x).$$

Combining these estimates, we get, for any fixed $x \in \mathbb{R}^n$,

$$\big(M^\sharp\big(|[b, T_\alpha]f|^\delta\big)\big)^{1/\delta}(x) \le CMf(x).$$

Applying this inequality we get, for $1 < p < \infty$ and $0 < \lambda < n$,

$$\big\|\big(M^\sharp\big(|[b, T_\alpha]f|^\delta\big)\big)\big\|^{1/\delta}_{L^{p/\delta,\lambda}} = \big\|\big(M^\sharp\big(|[b, T_\alpha]f|^\delta\big)\big)^{1/\delta}\big\|_{L^{p,\lambda}} \le C\|Mf\|_{L^{p,\lambda}} \le C\|f\|_{L^{p,\lambda}}.$$

(see [Chiarenza and Frasca 1987]). On the other hand,

$$\|[b, T_\alpha]f\|_{L^{p,\lambda}} = \||[b, T_\alpha]f|^\delta\|^{1/\delta}_{L^{p/\delta,\lambda}} \le \big\|M^\sharp\big(|[b, T_\alpha]f|^\delta\big)\big\|^{1/\delta}_{L^{p/\delta,\lambda}}.$$

Combining these estimates, we get

$$\|[b, T_\alpha]f\|_{L^{p,\lambda}} \le C\|f\|_{L^{p,\lambda}}. \qquad \square$$

Proof of (iii) $\Longrightarrow$ (iv). Suppose that $[b, T_\alpha]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$ for some $1 < p < \infty$ and $0 < \lambda < n$; then, by Theorem 1.5, we must have $b \in \mathrm{Lip}_\alpha$. So $k(x, y)$ satisfies (5-2). Let $f = f_1 + f_2$, with $f_1 = f_{\chi_{2Q}}$

and $f_2 = f\chi_{(2Q)^c}$. For any cube $Q = Q(x_0, r)$,

$$\frac{1}{|Q|}\int_Q |[b, T_\alpha]f(y) - [b, T_\alpha]f(x_0)|\,dy = \frac{1}{|Q|}\int_Q |[b, T_\alpha]f_1(y)|\,dy + \frac{1}{|Q|}\int_Q |[b, T_\alpha]f_2(y) - [b, T_\alpha]f(x_0)|\,dy.$$

By Hölder's inequality and since $[b, T_\alpha]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$, we get

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |[b, T_\alpha]f_1(y)|^p\,dy\right)^{1/p} &\le \frac{1}{r^{(n-\lambda)/p}} \sup_{t>0, x\in\mathbb{R}^n}\left(\frac{1}{t^\lambda}\int_{Q(x,t)\cap 2Q(x_0,r)} |f(y)|^p\,dy\right)^{1/p}\\ &\le \frac{C}{r^{(n-\lambda)/p}} r^{(n-\lambda)/p}\|f\|_{L^\infty} \le C\|f\|_{L^\infty}.\end{aligned}$$

By (5-2), it is easy to get

$$\frac{1}{|Q|}\int_Q |[b, T_\alpha]f_2(y) - [b, T_\alpha]f_2(x_0)|\,dy \le C\|f\|_{L^\infty}.$$

Combining these estimates, we get

$$\|[b, T_\alpha]f\|_{\mathrm{BMO}} \le C\|f\|_{L^\infty}. \qquad \square$$

6. Proof of Proposition 1.12

Denote by $\mathcal{H}_m$ the spaces of spherical harmonics of degree m and let $d_m = \dim \mathcal{H}_m$. If $\Omega \in L^2(S^{n-1})$ satisfies (1-1), then we can write

$$\Omega(x') = \sum_{m\ge 1}\sum_{j=1}^{d_m} a_{m,j} Y_{m,j}(x'),$$

where $\{Y_{m,j}\}_{j=1}^{d_m}$ denotes the normalized orthonormal basis of $\mathcal{H}_m$ (see [Calderón and Zygmund 1978] or [Stein and Weiss 1971]). Then

$$\sum_{m\ge 1}\sum_{j=1}^{d_m} a_{m,j}^2 < \infty.$$

By [Chen et al. 2003, p. 528], we have

$$(Y_{m,j}(x')|x|^{-n-\alpha})^\wedge(\xi) \backsimeq m^{-n/2-\alpha}|\xi|^\alpha Y_{m,j}(\xi').$$

Then we get

$$\widehat{T_\alpha f}(\xi) \backsimeq |\xi|^\alpha \sum_{m\ge 1}\sum_{j=1}^{d_m} m^{-n/2-\alpha} a_{m,j} Y_{m,j}(\xi')\hat{f}(\xi).$$

Using this, we get

$$\widehat{I_\alpha T_\alpha f}(\xi) \simeq \sum_{m\ge 1}\sum_{j=1}^{d_m} m^{-n/2-\alpha} a_{m,j} Y_{m,j}(\xi')\hat{f}(\xi).$$

Let

$$\Omega_0(\xi') = \sum_{m\geq 1}\sum_{j=1}^{d_m} m^{-n/2-\alpha} a_{m,j} Y_{m,j}(\xi').$$

It is easy to verify that Ω_0 satisfies (1-1) and

$$\sum_{m\geq 1}\sum_{j=1}^{d_m} m^n \|m^{-n/2-\alpha} a_{m,j} Y_{m,j}\|^2_{L^2(S^{n-1})} < \infty.$$

Then by [Stein and Weiss 1971, Theorem 4.7, p. 165] there exists a function $K(x) = \widetilde{\Omega}(x')/|x|^n$ such that $\widehat{K}(\xi) = \Omega_0(\xi')$ in the sense of principal value, where $\widetilde{\Omega}$ satisfies (1-1). Therefore, we get that

$$Tf(x) = I_\alpha T_\alpha f(x) = \text{p.v.}(K * f(x))$$

is a singular integral operator. In fact,

$$\widetilde{\Omega}(x') \simeq \sum_{m\geq 1}\sum_{j=1}^{d_m} m^{-\alpha} a_{m,j} Y_{m,j}(x'),$$

and

$$\|\widetilde{\Omega}\|^2_{L^2_\alpha(S^{n-1})} = \sum_{m\geq 1}\sum_{j=1}^{d_m} m^{2\alpha}(m^{-\alpha} a_{m,j})^2 < \infty.$$

This says that, for $0 < \alpha < 1$ and any operator T_α defined by (1-3) with $\Omega \in L^2(S^{n-1})$ satisfying (1-1), there exists a singular integral operator T defined by (1-4) with $\widetilde{\Omega} \in L^2_\alpha(S^{n-1})$ satisfying (1-1) such that $T_\alpha = D^\alpha T$. Conversely, for any fixed singular integral operator T with $\widetilde{\Omega} \in L^2_\alpha(S^{n-1})$ satisfying (1-1), there exists an operator T_α with $\Omega \in L^2(S^{n-1})$ satisfying (1-1) such that $T_\alpha = D^\alpha T$. $\square$

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