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A COMPLETE STUDY OF THE LACK OF COMPACTNESS AND EXISTENCE RESULTS OF A FRACTIONAL NIRENBERG EQUATION VIA A FLATNESS HYPOTHESIS, I

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Dedicated to the memory of Professor Abbas Bahri who left us on January 10, 2016.

We consider a nonlinear critical problem involving the fractional Laplacian operator arising in conformal geometry, namely the prescribed σ-curvature problem on the standard n-sphere, $n \geq 2$. Under the assumption that the prescribed function is flat near its critical points, we give precise estimates on the losses of the compactness and we provide existence results. In this first part, we will focus on the case $1 < \beta \leq n - 2\sigma$, which is not covered by the method of Jin, Li, and Xiong (2014, 2015).

1. Introduction and main results

Fractional calculus has attracted the interest of a lot of scientists during the last decades. This is essentially due to its numerous applications in various domains: medicine, population modeling, biology, earthquakes, optics, signal processing, astrophysics, water waves, porous media, nonlocal diffusion, image reconstruction problems; see [Hajaiej et al. 2011] and the references [1, 2, 6, 7, 13, 14, 19, 22, 25, 36, 38, 41, 43, 45, 46, 58] therein.

Many important properties of the Laplacian are not inherited, or are only partially satisfied, by its fractional powers. This gave birth to many challenging and rich mathematical problems. However, the literature remained quite silent until the publication of the breakthrough paper of Caffarelli and Silvester [2007]. This seminal work has hugely contributed to unblocking a lot of difficult problems and opening the way for the resolution of many other ones. In this paper, we study another important fractional PDE whose resolution also requires some novelties because of the nonlocal properties of the operator present in it. More precisely, we investigate the existence of solutions for the Nirenberg fractional nonlinear equation

$$P_\sigma u = c(n,\sigma) K u^{(n+2\sigma)/(n-2\sigma)} \quad \text{for } u > 0 \text{ on } \mathbb{S}^n, \tag{1-1}$$

where $\sigma \in (0,1)$, K is a positive function defined on $(\mathbb{S}^n, g_{\mathbb{S}^n})$,

$$P_\sigma = \frac{\Gamma\left(B+\frac{1}{2}+\sigma\right)}{\Gamma\left(B+\frac{1}{2}-\sigma\right)}, \quad B = \sqrt{-\Delta_{g_{\mathbb{S}^n}} + \left(\frac{n-1}{2}\right)^2},$$

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Γ is the gamma function, $c(n,\sigma)=\Gamma\left(\frac{n}{2}+\sigma\right)/\Gamma\left(\frac{n}{2}-\sigma\right)$, and $\Delta_{g_{\mathbb{S}^n}}$ is the Laplace–Beltrami operator on $(\mathbb{S}^n, g_{\mathbb{S}^n})$. The operator P_σ can be seen more concretely on $\mathbb{R}^n$ using stereographic projection. The stereographic projection from $\mathbb{S}^n\setminus\{N\}$ to $\mathbb{R}^n$ is the inverse of $F:\mathbb{R}^n\to\mathbb{S}^n\setminus\{N\}$ defined by

$$F(x)=\left(\frac{2x}{1+|x|^2},\frac{|x|^2-1}{|x|^2+1}\right),$$

where N is the north pole of $\mathbb{S}^n$. For all $f\in C^\infty(\mathbb{S}^n)$, we have

$$(P_\sigma(f))\circ F=\left(\frac{2}{1+|x|^2}\right)^{-(n+2\sigma)/2}(-\Delta)^\sigma\left(\left(\frac{2}{1+|x|^2}\right)^{(n-2\sigma)/2}(f\circ F)\right), \tag{1-2}$$

where $(-\Delta)^\sigma$ is the fractional Laplacian operator (see page 117 of [Stein 1970], for example).

For $\sigma=1$, the classical Nirenberg problem consists of the following question: which function K on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ is the scalar curvature of a metric g that is conformal to $g_{\mathbb{S}^n}$? This is equivalent to solving

$$P_1 v+1=-\Delta_{g_{\mathbb{S}^n}}v+1=Ke^{2v}\quad\text{on } \mathbb{S}^2 \tag{1-3}$$

and

$$P_1 w+1=-\Delta_{g_{\mathbb{S}^n}}w+b(n)R_0 w=b(n)Kw^{(n+2)/(n-2)}\quad\text{on } \mathbb{S}^n,\ n\ge 3, \tag{1-4}$$

where $g=e^{2v}g_{\mathbb{S}^n}$, $b(n)=(n-2)/(4(n-1))$, and $w=e^{(n-2)v/4}$, and where $R_0=n(n-1)$ is the scalar curvature of $(\mathbb{S}^n, g_{\mathbb{S}^n})$.

To our knowledge, the very first contribution to this topic is due to D. Koutroufiotis [1972]. He has been able to solve the above Nirenberg problem (1-3) when K is an antipodally symmetric function which is close to 1. However, his approach only applies to $\mathbb{S}^2$. Following a self-contained method, Moser [1973] has solved the Nirenberg problem on $\mathbb{S}^2$ for all antipodally symmetric functions K which are just positive somewhere. Later on, Chang and Yang [1988] have succeeded in removing the symmetry assumption on K in dimension 2 and Bahri and Coron [1991] have extended these results to dimension 3.

Another important issue related to the study of the classical Nirenberg problem is the compactness of the solutions. This has first been addressed by Chang, Gursky and Yang [Chang et al. 1993], Han [1990] and Schoen and Zhang [1996], for $n=2$ or $n=3$.

Compactness and existence of solutions in higher dimensions have been established in the breakthrough papers of Li [1995; 1996]. Let us point out that the situation is completely different in higher dimensions ($n>3$). More precisely, when $n=2$ or $n=3$, a sequence of solutions of the Nirenberg problem cannot blow up at more than one point. If $n>3$, there could be blow ups at many points, which considerably complicates the study of the problem. Many aspects of this very interesting situation have been addressed in [Ambrosetti et al. 1999; Ben Ayed et al. 1996; Ben Mahmoud and Chtioui 2012; Chen and Lin 2001; Li 1995; 1996].

Another stimulating situation is the study of higher orders and fractional order conformally invariant pseudodifferential operators P_k^g on $(\mathbb{S}^n, g_{\mathbb{S}^n})$, which exist for all positive integers k if n is odd and for $k=\left\{1,\dots,\frac{n}{2}\right\}$ if n is even. These operators were first introduced by Graham, Jenne, Mason and Sparling [Graham et al. 1992]. Beyond the case P_1^g which corresponds to the operator associated to the classical Nirenberg problem discussed above, the operator P_2^g is the well known Paneitz operator; see [Abdelhedi and Chtioui 2006; Djadli et al. 2000; Paneitz 2008; Wei and Xu 2009] and references therein. Up to positive constants, $P_1^g(1)$ is the scalar curvature associated to g and $P_2^g(1)$ is the so-called Q-curvature.

In the last two decades, it has been realized that the conformal Laplacian P_1^g, and more generally P_k^g, play a central role in conformal geometry. As mentioned previously, the classical Nirenberg problem is naturally associated to the conformal Laplacian. Consequently, the higher order Nirenberg problems are associated to Graham, Jenne, Mason and Sparling operators (known as the GJMS operators). Recently, a recursive formula for GJMS operators and Q-curvature has been found by Juhl [2014; 2013] (see also [Fefferman and Graham 2013]). Moreover, Graham and Zworski [2003] have introduced a family of fractional order conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds thanks to a scattering theory.

After this seminal paper, new interpretations of the fractional operators and their associated Q-curvatures have been the subject of many studies; see for example [Chang and González 2011]. For the Q-curvature of order σ on general manifolds, we refer to [Chang and González 2011; González et al. 2012; González and Qing 2013; Graham and Zworski 2003; Qing and Raske 2006] and references therein. Prescribing Q-curvature of order σ on $\mathbb{S}^n$ can be interpreted as a generalization of the Nirenberg problem, called in this context the fractional Nirenberg problem.

For $0 < \sigma < 1$, this challenging problem was first addressed in [Jin et al. 2014; 2015]. In these two groundbreaking papers, the authors were able to show the existence of solutions of (1-1) and to derive some compactness properties. More precisely, thanks to a very subtle approach based on approximation of the solutions of (1-1) by a blow-up subcritical method, they proved the existence of solutions for the critical fractional Nirenberg problem (1-1) (see Theorems 1.1 and 1.2 of [Jin et al. 2014]). Their method is based on tricky variational tools; in particular, they have established many interesting fractional functional inequalities. Their main hypothesis is the so-called flatness condition. Namely, let $K : \mathbb{S}^n \to \mathbb{R}$ be a C^2 positive function. We say that K satisfies the flatness condition $(f)_\beta$ if for each critical point y of K there exist $b_i = b_i(y) \in \mathbb{R}^*$ for $i \le n$, with $\sum_{i=1}^n b_i \ne 0$, such that in some geodesic normal coordinate centered at y we have

$$K(x) = K(y) + \sum_{i=1}^{n} b_i |(x-y)_i|^\beta + R(x-y), \tag{1-5}$$

where $\sum_{s=0}^{[\beta]} |\nabla^s R(y)||y|^{-\beta-s} = o(1)$ as y tends to zero. Here ∇^s denotes all possible derivatives of order s and $[\beta]$ is the integer part of β. However, they were only able to handle the case $n-2\sigma < \beta < n$ in the flatness hypothesis. This excludes some very interesting functions K. In fact, note that an important class of functions, which is worth including in any results of existence for (1-1), are the Morse functions (C^2 having only nondegenerate critical points). Such functions can be written in the form $(f)_\beta$ with $\beta = 2$. Since Jin, Li and Xiong require $n-2\sigma < \beta < n$ $(0 < \sigma < 1)$, their theorems do not apply to this relevant class of functions. Moreover, they require some additional technical assumptions (K antipodally symmetric in Theorem 1.1 and $K \in C^{1,1}$ positive in Theorem 1.2 of [Jin et al. 2014]).

Motivated by [Jin et al. 2014; 2015] and aiming to include a larger class of functions K in the existence results for (1-1), we develop in this paper a self-contained approach which enables us to include all the plausible cases ($1 < \beta < n$). Our method hinges on a readapted characterization of critical points at infinity. The approach is different for $1 < \beta \le n - 2\sigma$ and $n - 2\sigma \le \beta < n$. In this work, we handle the first case.

The spirit of our method goes back to the work of Bahri [1989] and Bahri and Coron [1991]. Nevertheless, the nonlocal properties of the fractional Laplacian involve many additional obstacles and require some novelties in the proofs. Note that in [Abdelhedi and Chtioui 2013], the first two authors have given an existence result for $n = 2$, $0 < \sigma < 1$, through an Euler–Hopf-type formula. In their paper, they assumed that K is a Morse function satisfying the nondegeneracy condition

$$\Delta K(y) \neq 0 \quad \text{whenever } \nabla K(y) = 0. \tag{nd}$$

We point out that the criterion of [Abdelhedi and Chtioui 2013] has an equivalent in dimension 3 (see [Abdelhedi and Chtioui ≥ 2016]). However, the same method cannot be generalized to higher dimensions $n \ge 4$ under the condition (nd), since the corresponding index counting criteria, when taking into account all the critical points at infinity, are always equal to 1. Recently, Y. Chen, C. Liu and Y. Zheng [Chen et al. 2016] proved an existence result for $n \ge 4$, under the (nd) condition and another topological condition, in the case where the index counting criteria, when taking into account all the critical points at infinity, are equal to 1, but a partial one is not equal to 1.

Convinced that the nondegeneracy assumption would exclude some interesting class of functions K, we opted for the flatness hypothesis used in [Jin et al. 2014; 2015]. But again, in order to include all plausible cases (both $1 < \beta \le n - 2\sigma$ and $n - 2\sigma \le \beta < n$), we need to develop a new line of attack with new ideas. This is essentially due to the structure of the multiple blow-up points, which is much more complicated than in the classical setting. Many new phenomena emerge. More precisely, it turns out that the strong interaction between the bubbles, in the case where $n - 2\sigma < \beta < n$, forces all blow-up points to be single, while in the case where $1 < \beta < n - 2\sigma$ such an interaction of two bubbles is negligible with respect to the self interaction, and if $\beta = n - 2\sigma$ there is a phenomenon of balance that is the interaction of two bubbles of the same order with respect to the self interaction. In order to state our results, we need the following notations and assumptions. Let

$$\mathcal{K} = \{y \in \mathbb{S}^n \mid \nabla K(y) = 0\}, \qquad \mathcal{K}_{n-2\sigma} = \{y \in \mathcal{K} \mid \beta = \beta(y) = n - 2\sigma\},$$

$$\mathcal{K}^+ = \left\{ y \in \mathcal{K} \,\Big|\, -\sum_{k=1}^{n} b_k(y) > 0 \right\}, \qquad \tilde{i}(y) = \sharp\{b_k = b_k(y) \mid 1 \le k \le n \text{ and } b_k < 0\}.$$

For each p-tuple, $1 \le p \le \sharp\mathcal{K}$, of distinct points $\tau_p := (y_{l_1}, \dots, y_{l_p}) \in (\mathcal{K}_{n-2\sigma})^p$, we define a $p \times p$ symmetric matrix $M(\tau_p) = (m_{ij})$ by

$$\begin{aligned} m_{ii} &= \frac{n-2\sigma}{n} \, \tilde{c}_1 \, \frac{-\sum_{k=1}^{n} b_k(y_{l_i})}{K(y_{l_i})^{n/(2\sigma)}}, \\ m_{ij} &= 2^{(n-2\sigma)/2} \, c_1 \, \frac{-G(y_{l_i}, y_{l_j})}{(K(y_{l_i}) K(y_{l_j}))^{(n-2\sigma)/(4\sigma)}}, \end{aligned} \tag{1-6}$$

where

$$G(y_{l_i}, y_{l_j}) = \frac{1}{(1-\cos d(y_{l_i}, y_{l_j}))^{(n-2\sigma)/2}},$$
$$c_1 = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{(n+2\sigma)/2}} \quad \text{and} \quad \tilde{c}_1 = \int_{\mathbb{R}^n} \frac{|x_1|^{n-2}}{(1+|x|^2)^n}\, dx. \tag{1-7}$$

Here x_1 is the first component of x in some geodesic normal coordinate system. Let $\rho(\tau_p)$ be the least eigenvalue of $M(\tau_p)$.

$$\text{Assume that } \rho(\tau_p) \neq 0 \text{ for each } \tau_p \in (\mathcal{K}_{n-2\sigma})^p,\ 1 \leq p \leq \sharp\mathcal{K}. \tag{A$_1$}$$

Now, we introduce the following sets:

$$\mathcal{C}^\infty_{n-2\sigma} := \{\tau_p = (y_{l_1}, \dots, y_{l_p}) \in (\mathcal{K}_{n-2\sigma})^p \mid 1 \leq p \leq \sharp\mathcal{K},\ y_i \neq y_j \text{ for all } i \neq j \text{ and } \rho(\tau_p) > 0\},$$
$$\mathcal{C}^\infty_{<n-2\sigma} := \{\tau_p = (y_{l_1}, \dots, y_{l_p}) \in (\mathcal{K}^+ \setminus \mathcal{K}_{n-2\sigma})^p \mid 1 \leq p \leq \sharp\mathcal{K} \text{ and } y_i \neq y_j \text{ for all } i \neq j\}.$$

For any $\tau_p = (y_{l_1}, \dots, y_{l_p}) \in (\mathcal{K})^p$, we write

$$i(\tau_p)_\infty = p - 1 + \sum_{j=1}^{p} (n - \tilde{i}(y_{l_j})).$$

Theorem 1.1. *Assume that K satisfies (A$_1$) and $(f)_\beta$ with $1 < \beta \leq n - 2\sigma$. If*

$$\sum_{\tau_p \in \mathcal{C}^\infty_{n-2\sigma}} (-1)^{i(\tau_p)_\infty} + \sum_{\tau'_p \in \mathcal{C}^\infty_{<n-2\sigma}} (-1)^{i(\tau'_p)_\infty} - \sum_{(\tau_p, \tau'_p) \in \mathcal{C}^\infty_{n-2\sigma} \times \mathcal{C}^\infty_{<n-2\sigma}} (-1)^{i(\tau_p)_\infty + i(\tau'_p)_\infty} \neq 1,$$

then (1-1) has at least one solution.

In Part II, we will address the case $n - 2\sigma \leq \beta < n$, following another approach and recovering the main existence results of [Jin et al. 2014; 2015]. More precisely, we will prove:

Theorem 1.2. *Assume that K satisfies (A$_1$) for each $p \geq 1$ and $(f)_\beta$ with $n - 2\sigma \leq \beta < n$. If*

$$\sum_{y \in \mathcal{K}^+ \setminus \mathcal{K}_{n-2\sigma}} (-1)^{i(y)_\infty} + \sum_{\tau_p \in \mathcal{C}^\infty_{n-2\sigma}} (-1)^{i(\tau_p)_\infty} \neq 1,$$

then (1-1) has at least one solution.

We organize the remainder of our paper as follows. Section 2 is devoted to recalling some preliminary results related to the variational structure associated to problem (1-1). In Section 3, we characterize the critical points at infinity of the associated variational problem. In Section 4, we give the proofs of the main results. The characterization of critical points at infinity requires some technical results, which, for the convenience of the reader, are given in the Appendix.

2. Preliminary results

Problem (1-1) has a variational structure; see Section 3 of [Jin et al. 2015], as well as [Chen and Zheng 2014; 2015; Chen et al. 2016; Jin et al. 2014]. The Euler–Lagrange functional associated to (1-1) is

$$J(u) = \frac{\|u\|^2}{\left(\int_{\mathbb{S}^n} K u^{2n/(n-2\sigma)}\right)^{(n-2\sigma)/n}} \quad \text{for } u \in H^\sigma(\mathbb{S}^n), \tag{2-1}$$

where $H^\sigma(\mathbb{S}^n)$ is the completion of $C^\infty(\mathbb{S}^n)$ by means of the norm

$$\|u\| = \left(\int_{\mathbb{S}^n} P_\sigma u u\right)^{1/2}. \tag{2-2}$$

Problem (1-1) is equivalent to finding the critical points of J subjected to the constraint $u \in \Sigma^+$, where

$$\Sigma^+ = \{u \in \Sigma \mid u \geq 0\} \quad \text{and} \quad \Sigma = \{u \in H^\sigma(\mathbb{S}^n) \mid \|u\| = 1\}.$$

The exponent $2n/(n-2\sigma)$ is critical for the Sobolev embedding $H^\sigma(\mathbb{S}^n) \to L^q(\mathbb{S}^n)$. This embedding is continuous and not compact. The functional J does not satisfy the Palais–Smale condition on Σ^+, but the sequences which violate the Palais–Smale condition are known. In order to describe them, let us introduce some notation. For $a \in \mathbb{S}^n$ and $\lambda > 0$, let

$$\delta_{a,\lambda}(x) = \bar{c}\,\frac{\lambda^{(n-2\sigma)/2}}{\left(1 + \frac{1}{2}(\lambda^2-1)\left(1-\cos(d(x,a))\right)\right)^{(n-2\sigma)/2}}, \tag{2-3}$$

where $d(\cdot\,,\cdot)$ is the distance induced by the standard metric of $\mathbb{S}^n$ and $\bar{c}$ is chosen so that $\delta_{a,\lambda}$ is the family of solutions for

$$P_\sigma u = u^{(n+2\sigma)/(n-2\sigma)} \quad \text{for } u > 0 \text{ on } \mathbb{S}^n; \tag{2-4}$$

see page 1113 of [Jin et al. 2014]. For $\varepsilon > 0$ and $p \in \mathbb{N}^*$, we define the set $V(p, \varepsilon)$ of potential critical points at infinity to be the set of $u \in \Sigma$ for which there exist $a_1, \dots, a_p \in \mathbb{S}^n$, $\alpha_1, \dots, \alpha_p > 0$, and $\lambda_1, \dots, \lambda_p > \varepsilon^{-1}$ satisfying

$$\begin{aligned} \left\| u - \sum_{i=1}^p \alpha_i \delta_{a_i,\lambda_i} \right\| &< \varepsilon, \\ \left| J(u)^{n/(n-2\sigma)} \alpha_i^{2/(n-2\sigma)} K(a_i) - 1 \right| &< \varepsilon \quad \text{for all } i, j = 1, \dots, p, \\ \varepsilon_{ij} &< \varepsilon \quad \text{for all } i \neq j, \end{aligned}$$

where

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{(2\sigma-n)/2}.$$

Following [Li and Zhu 1995; Brezis and Coron 1985], the failure of the Palais–Smale condition can be described as follows.

Proposition 2.1. *Assume that J has no critical points Σ^+. Let (u_k) be a sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ goes to zero. Then there exist an integer $p \in \mathbb{N}^*$, a sequence $(\varepsilon_k) > 0$ which tends to zero, and an extracted subsequence of the u_k, again denoted (u_k), such that $u_k \in V(p, \varepsilon_k)$.*

If u is a function in $V(p, \varepsilon)$, one can find an optimal representation, following the ideas introduced in [Bahri 1996]. Namely, we have:

Proposition 2.2. *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon \le \varepsilon_p$ and $u \in V(p, \varepsilon)$, then the minimization problem*

$$\min_{\alpha_i > 0, \lambda_i > 0, a_i \in \mathbb{S}^n} \left\| u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} \right\|$$

has a unique solution (α, λ, a) up to a permutation.

If we denote

$$v := u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)},$$

then v belongs to $H^\sigma(\mathbb{S}^n)$ and, arguing as in page 175 of [Bahri 1989], satisfies the condition

$$\langle v, \varphi_i \rangle = 0 \quad \text{for } \varphi_i = \delta_i, \frac{\partial \delta_i}{\partial \lambda_i}, \frac{\partial \delta_i}{\partial a_i} \text{ and } i = 1, \dots, p, \tag{V_0}$$

where $\delta_i = \delta_{a_i, \lambda_i}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $H^\sigma(\mathbb{S}^n)$ defined by

$$\langle u, v \rangle = \int_{\mathbb{S}^n} v P_\sigma u.$$

We say $v \in (V_0)$ if v satisfies (V_0). The following Morse lemma completely gets rid of the v-contributions.

Proposition 2.3. *There is a C^1 map which, to each $(\alpha_i, a_i, \lambda_i)$ such that $\sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)}$ belongs to $V(p, \varepsilon)$, associates $\bar{v} = \bar{v}(\alpha, a, \lambda)$ such that $\bar{v}$ is unique and satisfies*

$$J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + \bar{v}\right) = \min_{v \in (V_0)} \left\{ J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + v\right) \right\}.$$

Moreover, there exists a change of variables $v - \bar{v} \to V$ such that

$$J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + v\right) = J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} + \bar{v}\right) + \|V\|^2.$$

Furthermore, under the assumption $(f)_\beta$, $1 < \beta \le n$, there exists $c > 0$ such that the following holds:

$$\|\bar{v}\| \le c \sum_{i=1}^{p} \left(\frac{1}{\lambda_i^{n/2}} + \frac{1}{\lambda_i^{\beta}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{(n+2\sigma)/(2n)}}{\lambda_i^{(n+2\sigma)/2}} \right) \\ + c \begin{cases} \sum_{k \ne r} \varepsilon_{kr}^{(n+2\sigma)/(2(n-2\sigma))} (\log \varepsilon_{kr}^{-1})^{(n+2\sigma)/(2n)} & \text{if } n \ge 3, \\ \sum_{k \ne r} \varepsilon_{kr} (\log \varepsilon_{kr}^{-1})^{(n-2\sigma)/n} & \text{if } n < 3. \end{cases}$$

To conclude this section, we state the definition of critical point at infinity.

Definition 2.4. A critical point at infinity of J on Σ^+ is a limit of a flow-line $u(s)$ of the equation

$$\frac{\partial u}{\partial s} = -\partial J(u(s)), \quad u(0) = u_0,$$

such that $u(s)$ remains in $V(p, \varepsilon(s))$ for $s \geq s_0$. Here $\varepsilon(s) > 0$ and $\to 0$ when $s \to +\infty$. Using Proposition 2.2, $u(s)$ can be written as

$$u(s) = \sum_{i=1}^{p} \alpha_i(s)\delta_{(a_i(s),\lambda_i(s))} + v(s).$$

Defining $\tilde{\alpha}_i := \lim_{s\to+\infty} \alpha_i(s)$ and $\tilde{y}_i := \lim_{s\to+\infty} a_i(s)$, we denote a critical point at infinity by

$$\sum_{i=1}^{p} \tilde{\alpha}_i \delta_{(\tilde{y}_i,\infty)} \quad \text{or} \quad (\tilde{y}_1, \dots, \tilde{y}_p)_\infty.$$

3. Characterization of the critical points at infinity for $1 < \beta \leq n - 2\sigma$

This section is devoted to the characterization of the critical points at infinity in $V(p, \varepsilon)$, $p \geq 1$, under the β-flatness condition with $1 < \beta \leq n - 2\sigma$. This characterization is obtained through the construction of a suitable pseudogradient at infinity for which the Palais–Smale condition is satisfied along the decreasing flow-lines, as long as these flow-lines do not enter the neighborhood of a finite number of critical points y_i, $i = 1, \dots, p$, of K such that

$$(y_1, \dots, y_p) \in \mathcal{P}^\infty := \mathcal{C}^\infty_{<n-2\sigma} \cup \mathcal{C}^\infty_{n-2\sigma} \cup (\mathcal{C}^\infty_{<n-2\sigma} \times \mathcal{C}^\infty_{n-2\sigma}).$$

Note that we say $(y_1, \dots, y_p) \in \mathcal{C}^\infty_{<n-2\sigma} \times \mathcal{C}^\infty_{n-2\sigma}$ if there exists $1 \leq s \leq p-1$ such that $(y_1, \dots, y_s) \in \mathcal{C}^\infty_{<n-2\sigma}$ and $(y_{s+1}, \dots, y_p) \in \mathcal{C}^\infty_{n-2\sigma}$. More precisely:

Theorem 3.1. *Assume that K satisfies (A_1) for each $p \geq 1$ and $(f)_\beta$, $1 < \beta \leq n - 2\sigma$. Let*

$$\beta := \max\{\beta(y) \mid y \in \mathcal{K}\}.$$

For each $p \geq 1$, there exists a pseudogradient W in $V(p, \varepsilon)$ and a constant $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} \in V(p, \varepsilon)$ such that

(i) $$\langle \partial J(u), W(u)\rangle \leq -c\biggl(\sum_{i=1}^{p} \frac{1}{\lambda_i^\beta} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j\neq i} \varepsilon_{ij}\biggr),$$

(ii) $$\Bigl\langle \partial J(u+\bar{v}), W(u) + \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W(u))\Bigr\rangle \leq -c\biggl(\sum_{i=1}^{p} \frac{1}{\lambda_i^\beta} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j\neq i} \varepsilon_{ij}\biggr).$$

Furthermore, $|W|$ is bounded in $V(p, \varepsilon)$ and the only case where the maximum of the λ_i is not bounded is when $a_i \in B(y_{l_i}, \rho)$ with $y_{l_i} \in \mathcal{K}$ for all $i = 1, \dots, p$, $(y_{l_1}, \dots, y_{l_p}) \in \mathcal{P}^\infty$ and ρ is a positive constant small enough such that for any $y \in \mathcal{K}$, the expansion $(f)_\beta$ holds in $B(y, \rho)$.

In order to prove Theorem 3.1, we state the following two results, which deal with two specific cases of Theorem 3.1. Let $\delta_i = \delta_{(a_i,\lambda_i)}$ and

$$V_1(p,\varepsilon) = \left\{ u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p,\varepsilon) \;\middle|\; a_i \in B(y_{l_i},\rho),\ y_{l_i} \in \mathcal{K} \setminus \mathcal{K}_{n-2\sigma} \text{ for all } i = 1,\dots,p \right\},$$
$$V_2(p,\varepsilon) = \left\{ u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p,\varepsilon) \;\middle|\; a_i \in B(y_{l_i},\rho),\ y_{l_i} \in \mathcal{K}_{n-2\sigma} \text{ for all } i = 1,\dots,p \right\}.$$

Proposition 3.2. *For $p \geq 1$, there exists a pseudogradient W_1 in $V_1(p,\varepsilon)$ and $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V_1(p,\varepsilon)$ such that*

$$\langle \partial J(u), W_1(u) \rangle \leq -c \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \right).$$

Furthermore, $|W_1|$ is bounded in $V_1(p,\varepsilon)$ and the only case where the maximum of the λ_i is not bounded is when $a_i \in B(y_{l_i},\rho)$ with $y_{l_i} \in \mathcal{K}^+$ for all $i = 1,\dots,p$, with $(y_{l_1},\dots,y_{l_p}) \in \mathcal{C}^{\infty}_{<n-2\sigma}$.

Proposition 3.3. *For $p \geq 1$ there exists a pseudogradient W_2 in $V_2(p,\varepsilon)$ and $c > 0$ independent of $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V_2(p,\varepsilon)$ such that*

$$\langle \partial J(u), W_2(u) \rangle \leq -c \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \right).$$

Furthermore, $|W_2|$ is bounded in $V_2(p,\varepsilon)$ and the only case where the maximum of the λ_i is not bounded is when $a_i \in B(y_{l_i},\rho)$ with $y_{l_i} \in \mathcal{K}^+$ for all $i = 1,\dots,p$, with $(y_{l_1},\dots,y_{l_p}) \in \mathcal{C}^{\infty}_{n-2\sigma}$.

In constructing the pseudogradient W, we will use the following notation. Let $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p,\varepsilon)$, such that $a_i \in B(y_{l_i},\rho)$ and $y_{l_i} \in \mathcal{K}$ for all $i = 1,\dots,p$. For simplicity, if a_i is close to a critical point y_{l_i}, we will assume that the critical point is at the origin, so we will confuse a_i with $(a_i - y_{l_i})$. Now, let $i \in \{1,\dots,p\}$ and let M_1 be a positive large constant. We say that

$$\begin{aligned} i \in L_1 &\quad \text{if } \lambda_i |a_i| \leq M_1, \\ i \in L_2 &\quad \text{if } \lambda_i |a_i| > M_1. \end{aligned}$$

For each $i \in \{1,\dots,p\}$, we define the vector fields

$$Z_i(u) = \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}, \tag{3-1}$$

$$X_i = \alpha_i \sum_{k=1}^{n} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_k} \int_{\mathbb{R}^n} b_k \frac{|x_k + \lambda_i (a_i)_k|^{\beta}}{(1 + \lambda_i |(a_i)_k|)^{\beta-1}} \frac{x_k}{(1+|x|^2)^{n+1}}\, dx, \tag{3-2}$$

where $(a_i)_k$ is the k-th component of a_i in some geodesic normal coordinate system. We claim that X_i is bounded. Indeed, the claim is trivial if $i \in L_1$. If $i \in L_2$, by elementary computation we have the estimate

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|x_k + \lambda_i (a_i)_k|^\beta x_k}{(1+|x|^2)^{n+1}}\,dx &= (\lambda_i|(a_i)_k|)^\beta \int_{\mathbb{R}^n} \left|1 + \frac{x_k}{\lambda_i((a_i)_k)}\right|^\beta \frac{x_k}{(1+|x|^2)^{n+1}}\,dx \\ &= c(\operatorname{sign} \lambda_i (a_i)_k)(\lambda_i|(a_i)_k|)^{\beta-1}(1+o(1)) \end{aligned}\tag{3-3}$$

for any k, $1 \le k \le n$, such that $\lambda_i|(a_i)_k| > M_1/\sqrt{n}$. Hence our claim is valid.

Proof of Theorem 3.1. In order to complete the construction of the pseudogradient W suggested in Theorem 3.1, it only remains (using Propositions 3.2 and 3.3) to focus attention on the two following subsets of $V(p, \varepsilon)$.

Subset 1. We consider here the case of $u = \sum_{i=1}^p \alpha_i\delta_i = \sum_{i\in I_1} \alpha_i\delta_i + \sum_{i\in I_2} \alpha_i\delta_i$ such that

$$I_1 \neq \varnothing, \quad I_2 \neq \varnothing, \quad \sum_{i\in I_1} \alpha_i\delta_i \in V_1(\sharp I_1, \varepsilon), \quad \text{and} \quad \sum_{i\in I_2} \alpha_i\delta_i \in V_2(\sharp I_2, \varepsilon).$$

Without loss of generality, we can assume here and in the sequel that

$$\lambda_1 \le \cdots \le \lambda_p.$$

We distinguish three cases.

Case 1: $u_1 := \sum_{i\in I_1} \alpha_i\delta_i \notin V_1^1(\sharp I_1, \varepsilon)$

$$= \left\{ u = \sum_{j=1}^{\sharp I_1} \alpha_j\delta_j \;\middle|\; a_j \in B(y_{l_j}, \rho),\ y_{l_j} \in \mathcal{K}^+ \text{ for } j = 1, \dots, \sharp I_1 \text{ and } y_{l_j} \neq y_{l_k} \text{ for all } j \neq k \right\}.$$

In this case, the pseudogradient $\widetilde{W}_1(u) := W_1(u_1)$, where W_1 is as defined in Proposition 3.2, does not increase the maximum of the λ_i, $i \in I_1$. Using Proposition 3.2, we have

$$\langle \partial J(u), \widetilde{W}_1(u)\rangle \le -c\biggl(\sum_{i\in I_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{\substack{j\neq i\\ i,j\in I_1}} \varepsilon_{ij} + \sum_{i\in I_1} \frac{|\nabla K(a_i)|}{\lambda_i} \biggr) + O\biggl(\sum_{i\in I_1,\, j\in I_2} \varepsilon_{ij} \biggr).\tag{3-4}$$

An easy calculation implies that

$$\varepsilon_{ij} = o\biggl(\frac{1}{\lambda_i^{\beta_i}}\biggr) + o\biggl(\frac{1}{\lambda_j^{\beta_j}}\biggr) \quad \text{for all } i \in I_1 \text{ and all } j \in I_2.\tag{3-5}$$

Fixing $i_0 \in I_1$, we define

$$J_1 := \bigl\{ i \in I_2 \mid \lambda_i^{n-2} \ge \tfrac{1}{2}\lambda_{i_0}^{\beta_{i_0}} \bigr\} \quad \text{and} \quad J_2 := I_2 \setminus J_1.$$

Using (3-4) and (3-5), we find that

$$\langle \partial J(u), \widetilde{W}_1(u)\rangle \le -c\biggl(\sum_{i\in I_1\cup J_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in I_1} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j\neq i\in I_1} \varepsilon_{ij} \biggr) + o\biggl(\sum_{i=1}^p \frac{1}{\lambda_i^{\beta_i}} \biggr).\tag{3-6}$$

Let k_i be an index such that

$$|(a_i)_{k_i}| = \max_{1\le j\le n} |(a_i)_j|. \tag{3-7}$$

From Lemma 3.4 we have

$$\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u)\Big\rangle \le c \sum_{j\ne i\in J_1} 2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\Big(\sum_{i\in J_1} \frac{1}{\lambda_i^{\beta_i}}\Big) + O\Big(\sum_{i\in J_1\cap L_2} \frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 2}}{\lambda_i^2}\Big). \tag{3-8}$$

Observe that for $i < j$, we have

$$2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + 2^j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \le -c\varepsilon_{ij}. \tag{3-9}$$

In addition, for $i \in J_1$ and $j \in J_2$ we have $\lambda_j \le \lambda_i$, so by (3-18) we obtain $\lambda_i \dfrac{\partial \varepsilon_{ij}}{\partial \lambda_i} \le -c\varepsilon_{ij}$. These estimates yield

$$\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u)\Big\rangle \le -c \sum_{\substack{j\ne i\\ i\in J_1,\, j\in J_1\cup J_2}} \varepsilon_{ij} + O\Big(\sum_{i\in J_1} \frac{1}{\lambda_i^{\beta_i}}\Big) + O\Big(\sum_{i\in J_1\cap L_2} \frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i-2}}{\lambda_i^2}\Big) + O\Big(\sum_{i\in J_1,\, j\in I_1} \varepsilon_{ij}\Big).$$

Taking $m_1 > 0$ small enough, using Lemma 3.5, (3-21), and (3-16) we get

$$\Big\langle \partial J(u), \sum_{i\in J_1} -2^i Z_i(u) + m_1 \sum_{i\in J_1\cap L_2} X_i(u)\Big\rangle \le -c\Big(\sum_{\substack{j\ne i\\ i\in J_1,\, j\in J_1\cup J_2}} \varepsilon_{ij} + \sum_{i\in J_1} \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + O\Big(\sum_{i\in J_1} \frac{1}{\lambda_i^{\beta_i}}\Big) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}}\Big),$$

and by (3-6) we obtain

$$\Big\langle \partial J(u), \widetilde{W}_1(u) + m_1\Big(\sum_{i\in J_1} -2^i Z_i(u) + m_1 \sum_{i\in J_1\cap L_2} X_i(u)\Big)\Big\rangle \le -c\Big(\sum_{i\in I_1\cup J_1} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i\ne j\in I_1} \varepsilon_{ij} + \sum_{\substack{j\ne i\\ i\in J_1,\, j\in J_1\cup J_2}} \varepsilon_{ij} \sum_{i\in I_1\cup J_1} \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}}\Big). \tag{3-10}$$

We need to add the remaining indices $i \in J_2$. Note that $\tilde{u} := \sum_{j\in J_2} \alpha_j \delta_j \in V_2(\sharp J_2, \varepsilon)$. Thus, the pseudogradient $\widetilde{W}_2(u) = W_2(\tilde{u})$, where W_2 is as defined in Proposition 3.3, satisfies

$$\langle \partial J(u), \widetilde{W}_2(u)\rangle \le -c\Big(\sum_{j\in J_2} \frac{1}{\lambda_j^{\beta_j}} + \sum_{\substack{i\ne j\\ i,j\in J_2}} \varepsilon_{ij} + \sum_{j\in J_2} \frac{|\nabla K(a_j)|}{\lambda_j}\Big) + O\Big(\sum_{i\in J_1,\, j\in J_2} \varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}}\Big), \tag{3-11}$$

since $|a_i - a_j| \ge \rho$ for $i \in I_1$ and $j \in J_2$.

From (3-10) and (3-11), for $W=\widetilde{W}_1+m_1\big(\widetilde{W}_2+\sum_{i\in J_1}-2^i Z_i+m_1\sum_{i\in J_1\cap L_2}X_i\big)$ we obtain

$$\langle \partial J(u),\, W(u)\rangle \le -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\ne j}\varepsilon_{ij}\Big).$$

Case 2:
$$u_1:=\sum_{i\in I_1}\alpha_i\delta_i\in V_1^1(\sharp I_1,\varepsilon)\quad\text{and}\quad u_2:=\sum_{i\notin I_2}\alpha_i\delta_i\notin V_2^1(\sharp I_2,\varepsilon),$$

where

$$V_2^1(\sharp I_2,\varepsilon):=\Big\{u=\sum_{j=1}^{\sharp I_2}\alpha_j\delta_j\;\Big|\; a_j\in B(y_{l_j},\rho),\, y_{l_j}\in\mathcal{K}^+ \text{ for all } j=1,\dots,\sharp I_2 \text{ and } \rho(y_{l_1},\dots,y_{\sharp I_2})>0\Big\}.$$

Let $V_1(u):=W_2(u_2)$. By Proposition 3.3, we get

$$\langle \partial J(u),\, V_1(u)\rangle \le -c\Big(\sum_{i\in I_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{\substack{i\ne j\\ i,j\in I_2}}\varepsilon_{ij}\Big)+O\Big(\sum_{i\in I_2,\, j\in I_1}\varepsilon_{ij}\Big). \tag{3-12}$$

Observe that $V_1(u)$ does not increase the maximum of the λ_i, $i\in I_2$, since $u_2\notin V_2^1(\sharp I_2,\varepsilon)$. Fix $i_0\in I_2$ and let

$$\widetilde{J}_1=\big\{i\in I_1\mid \lambda_i^{\beta_i}\ge \tfrac{1}{2}\lambda_{i_0}^{n-2}\big\}\quad\text{and}\quad \widetilde{J}_2=I_1\setminus\widetilde{J}_1.$$

Using (3-12) and (3-5), we get

$$\langle \partial J(u),\, V_1(u)\rangle \le -c\Big(\sum_{i\in I_2\cup\widetilde{J}_1}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{\substack{i\ne j\\ i,j\in I_2}}\varepsilon_{ij}\Big)+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big). \tag{3-13}$$

We need to add the indices i for $i\in\widetilde{J}_2$. Let $\tilde u:=\sum_{j\in\widetilde{J}_2}\alpha_j\delta_j$ and let $V_2(u):=W_1(\tilde u)$. By Proposition 3.2, we have

$$\langle \partial J(u),\, V_2(u)\rangle \le -c\Big(\sum_{j\in\widetilde{J}_2}\frac{1}{\lambda_j^{\beta_j}}+\sum_{j\in\widetilde{J}_2}\frac{|\nabla K(a_j)|}{\lambda_j}+\sum_{\substack{i\ne j\\ i,j\in\widetilde{J}_2}}\varepsilon_{ij}\Big)+O\Big(\sum_{j\in\widetilde{J}_2,\, i\notin\widetilde{J}_2}\varepsilon_{ij}\Big).$$

Observe that $I_1=\widetilde{J}_1\cup\widetilde{J}_2$ and we are in the case where for all $i\ne j\in I_1$, we have $|a_i-a_j|\ge\rho$. Thus by (3-16) and (3-5), we get

$$O\Big(\sum_{j\in\widetilde{J}_2,\, i\notin\widetilde{J}_2}\varepsilon_{ij}\Big)=o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big),$$

and hence

$$\langle \partial J(u),\, V_1(u)+V_2(u)\rangle \le -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2\cup\widetilde{J}_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\ne j}\varepsilon_{ij}\Big).$$

Let in this case $W = V_1 + V_2 + m_1 \sum_{i\in\widetilde{J}_1} X_i(u)$, m_1 small enough. Using the above estimate and Lemma 3.5, we find that

$$\langle \partial J(u), W(u)\rangle \le -c\biggl(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\ne j}\varepsilon_{ij}\biggr).$$

Case 3: $$u_1 \in V_1^1(\sharp I_1, \varepsilon) \quad\text{and}\quad u_2 \in V_2^1(\sharp I_2, \varepsilon).$$

For $i = 1, 2$, let $\widetilde{V}_i$ be the pseudogradient in $V(p,\varepsilon)$ defined by $\widetilde{V}_i(u) = W_i(u_i)$ where W_i is the vector field defined by Proposition 3.2 (for $i = 1$) or 3.3 (for $i = 2$) in $V_i^1(\sharp I_i, \varepsilon)$, and let in this case $W = \widetilde{V}_1 + \widetilde{V}_2$. Using Proposition 3.3, Proposition 3.2, and (3-5) we get

$$\langle \partial J(u), W(u)\rangle \le -c\biggl(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\ne j}\varepsilon_{ij}\biggr).$$

Notice that in the first and second cases, the maximum of the λ_i, $1 \le i \le p$, is a bounded function and hence the Palais–Smale condition is satisfied along the flow-lines of W. However in the third case all the λ_i, $1 \le i \le p$, will increase and go to $+\infty$ along the flow-lines generated by W.

<u>Subset 2</u>. We consider the case of $u = \sum_{i=1}^{p}\alpha_i\delta_i \in V(p,\varepsilon)$, such that there exist a_i not contained in $\bigcup_{y\in\mathcal{K}} B(y,\rho)$. Let i_1 be such that for any $i < i_1$, we have $a_i \in B(y_{\ell_i},\rho)$, $y_{\ell_i} \in \mathcal{K}$ and $a_{i_1} \notin \bigcup_{y\in\mathcal{K}} B(y,\rho)$. Let us define

$$u_1 = \sum_{i<i_1}\alpha_i\delta_i.$$

Observe that u_1 must be contained in $V_1(i_1 - 1, \varepsilon)$ or $V_2(i_1 - 1, \varepsilon)$, or else u_1 satisfies the condition of Subset 1. Thus we can apply the associated vector field, which we will denote by Y, and we then have the estimate

$$\langle \partial J(u), Y(u)\rangle \le -c\biggl(\sum_{i<i_1}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i<i_1}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{\substack{i\ne j\\ i,j<i_1}}\varepsilon_{ij}\biggr) + O\biggl(\sum_{i<i_1,\, j\ge i_1}\varepsilon_{ij}\biggr).$$

Now we define the vector field

$$Y' = \frac{1}{\lambda_{i_1}}\frac{\partial\delta_{i_1}}{\partial a_{i_1}}\frac{\nabla K(a_{i_1})}{|\nabla K(a_{i_1})|} - c'\sum_{i\ge i_1} 2^i Z_i.$$

Using Propositions 3.3, 3.2, and the fact that $|\nabla K(a_{i_1})| \ge c > 0$, we derive

$$\langle \partial J(u), Y'(u)\rangle \le -c\frac{1}{\lambda_{i_1}} + O\biggl(\sum_{i\ne i_1}\varepsilon_{ij}\biggr) - c'\sum_{j\ne i,\, i\ge i_1}\varepsilon_{ij} + o\biggl(\sum_{i\ge i_1}\frac{1}{\lambda_i}\biggr).$$

Taking $c' > 0$ large enough, we find

$$\langle \partial J(u), Y'(u)\rangle \le -c\biggl(\sum_{i=i_1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=i_1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\ne j,\, i\ge i_1}\varepsilon_{ij}\biggr).$$

Now let $W := Y' + m_1 Y$, where m_1 is a small positive constant; then we have

$$\langle \partial J(u), W(u)\rangle \leq -c\biggl(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\biggr).$$

Finally, observe that our pseudogradient W in $V(p, \varepsilon)$ satisfies Theorem 3.1(i), and it is bounded since $\|\lambda_i\, \partial\delta_i/\partial\lambda_i\|$ and $\|(1/\lambda_i)\,\partial\delta_i/\partial a_i\|$ are bounded. From the definition of W, the λ_i, $1 \leq i \leq p$, decrease along the flow-lines of W as long as these flow-lines do not enter the neighborhood of a finite number of critical points y_{l_i}, $i = 1, \dots, p$, of $\mathcal{K}$ such that $(y_{l_1}, \dots, y_{l_p}) \in \mathcal{P}^{\infty}$. Now, arguing as in Appendix 2 of [Bahri 1996], Theorem 3.1(ii) follows from (i) and Proposition 2.3. This complete the proof of Theorem 3.1. □

Proof of Proposition 3.2. In our construction of the pseudogradient W_1, we need the following lemmas. Write 1_A for the characteristic function of a set A.

Lemma 3.4. *Let $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p, \varepsilon)$ be such that $a_i \in B(y_{l_i}, \rho)$, $y_{l_i} \in \mathcal{K}$ for all $i = 1, \dots, p$. We then have*

$$\langle \partial J(u), Z_i(u)\rangle = -2c_2 J(u) \sum_{j\neq i} \alpha_i\alpha_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\biggl(\frac{1}{\lambda_i^{\beta_i}}\biggr) + 1_{L_2}(i)\, O\biggl(\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 2}}{\lambda_i^2}\biggr) + o\biggl(\sum_{j\neq i} \varepsilon_{ij}\biggr) + o\biggl(\sum_{j=1}^{p} \frac{1}{\lambda_j^{\beta_j}}\biggr),$$

with k_i defined as in (3-7).

Proof. Observe that for $k \in \{1, \dots, n\}$, if $\lambda_i |(a_i - y_{l_i})_k| > M_1/\sqrt{n}$, we have

$$\int_{\mathbb{R}^n} \frac{|x_k + \lambda_i (a_i - y_{l_i})_k|^{\beta_i - 1} x_k}{(1 + |x|^2)^n}\, dx = O\bigl((\lambda_i |(a_i - y_{l_i})_k|)^{\beta_i - 2}\bigr) \tag{3-14}$$

if M_1 is sufficiently large. If not, we have

$$\int_{\mathbb{R}^n} \frac{|x_k + \lambda_i (a_i - y_{l_i})_k|^{\beta_i - 1} |x_k|}{(1 + |x|^2)^n}\, dx = O(1).$$

Using the fact that the k_i defined in (3-7) satisfies $\lambda_i |(a_i - y_{l_i})_{k_i}| > M_1/\sqrt{n}$ if $i \in L_2$, Lemma 3.4 follows from Proposition A.1. □

Lemma 3.5. *Let $u = \sum_{i=1}^{p} \alpha_i \delta_i \in V(p, \varepsilon)$ be such that $a_i \in B(y_{l_i}, \rho)$, $y_{l_i} \in \mathcal{K}$ for all $i = 1, \dots, p$. We then have*

$$\langle \partial J(u), X_i(u)\rangle \leq O\biggl(\sum_{j\neq i} \frac{1}{\lambda_i}\biggl|\frac{\partial \varepsilon_{ij}}{\partial a_i}\biggr|\biggr) + 1_{L_1}(i)\, O\biggl(\frac{1}{\lambda_i^{\beta_i}}\biggr) - 1_{L_2}(i)\, c\biggl(\frac{1}{\lambda_i^{\beta_i}} + \frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 1}}{\lambda_i}\biggr) + o\biggl(\sum_{j=1}^{p} \frac{1}{\lambda_j^{\beta_j}}\biggr),$$

with k_i defined as in (3-7).

Proof. Using Proposition A.2, we have

$$\langle \partial J(u), X_i(u)\rangle \le -c\frac{1}{{\lambda_i}^{\beta_i}}\left(\int_{\mathbb{R}^n} b_{k_i}\frac{|x_k+\lambda_i(a_i-y_{l_i})_{k_i}|^{\beta_i}}{(1+\lambda_i|(a_i-y_{l_i})_{k_i}|)^{(\beta_i-1)/2}}\frac{x_{k_i}}{(1+|x|^2)^{n+1}}\,dx\right)^2 + O\left(\sum_{j\neq i}\frac{1}{\lambda_i}\left|\frac{\partial\varepsilon_{ij}}{\partial a_i}\right|\right)+o\left(\sum_{j=1}^{p}\frac{1}{\lambda_j^{\beta_j}}\right). \tag{3-15}$$

Using (3-3) and the fact that

$$\lambda_i|(a_i-y_{l_i})_{k_i}| > \frac{M_1}{\sqrt{n}} \quad \text{if } i\in L_2,$$

Lemma 3.5 follows. □

In order to construct the required pseudogradient, we have to divide the set $V_1(p,\varepsilon)$ into four different regions, construct an appropriate pseudogradient in each region, and then glue up through convex combinations. Let Z_1 and Z_2 be two vector fields. A convex combination of Z_1 and Z_2 is given by $\theta Z_1+(1-\theta)Z_2$, where θ is a cutoff function. Let

$$V_1^1(p,\varepsilon):=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon)\;\Big|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j,\ -\sum_{k=1}^{n}b_k(y_{l_i})>0, \text{ and } \lambda_i|a_i-y_{l_i}|<\delta \text{ for all } i=1,\dots,p\Big\},$$

$$V_1^2(p,\varepsilon):=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon)\;\Big|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j,\ \lambda_i|a_i-y_{l_i}|<\delta \text{ for all } i=1,\dots,p \text{ and } -\sum_{k=1}^{n}b_k(y_{l_i})<0 \text{ for some } i\Big\},$$

$$V_1^3(p,\varepsilon):=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon)\;\Big|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j \text{ and } \lambda_j|a_j-y_{l_j}|\ge\frac{\delta}{2}\text{ for some } j\Big\},$$

$$V_1^4(p,\varepsilon):=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{(a_i\lambda_i)}\in V_1(p,\varepsilon)\;\Big|\; y_{l_i}= y_{l_j}\text{ for some } i\neq j\Big\}.$$

Pseudogradient in $V_1^1(p,\varepsilon)$. Let $u=\sum_{i=1}^{p}\alpha_i\delta_i\in V_1^1(p,\varepsilon)$. For any $i\neq j$, we have $|a_i-a_j|>\rho$; therefore

$$\varepsilon_{ij}=O\left(\frac{1}{(\lambda_i\lambda_j)^{(n-2\sigma)/2}}\right)=o\left(\frac{1}{\lambda_i^{\beta_i}}\right)+o\left(\frac{1}{\lambda_j^{\beta_j}}\right), \tag{3-16}$$

since $\beta_i,\beta_j<n-2\sigma$. Let $W_1^1(u)=\sum_{i=1}^{p}Z_i(u)$. Using the fact that $|\nabla K(a_i)|/\lambda_i$ is small with respect to $1/{\lambda_i}^{\beta}$, we obtain from Proposition A.1

$$\langle\partial J(u), W_1^1(u)\rangle \le -c\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\right).$$

Pseudogradient in $V_1^2(p,\varepsilon)$. Let $u=\sum_{i=1}^{p}\alpha_i\delta_i\in V_1^2(p,\varepsilon)$. Without loss of generality, we can assume that $i=1,\dots,q$ are the indices which satisfy $-\sum_{k=1}^{n}b_k(y_{l_i})<0$. Let

$$I=\Big\{i\in\{1,\dots,p\}\mid \lambda_i^{\beta_i}\le\tfrac{1}{10}\min_{1\le j\le q}\lambda_j^{\beta_j}\Big\}.$$

In this region we define $W_1^2(u) = \sum_{i=1}^{q}(-Z_i)(u) + \sum_{i\in I} Z_i(u)$. Using a calculation similar to [Ben Mahmoud and Chtioui 2012], we obtain

$$\langle \partial J(u), W_1^2(u)\rangle \leq -c\biggl(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\biggr).$$

Pseudogradient in $V_1^3(p, \varepsilon)$. Let $u = \sum_{i=1}^{p}\alpha_i\delta_i \in V_1^3(p, \varepsilon)$. Without loss of generality, we can assume that $\lambda_1^{\beta_1} = \min\{\lambda_j^{\beta_j} \mid \lambda_j|a_j - y_{l_j}| \geq \delta\}$. Let

$$J := \bigl\{i \mid 1 \leq i \leq p \text{ and } \lambda_i^{\beta_i} \geq \tfrac{1}{2}\lambda_1^{\beta_1}\bigr\}.$$

Observe that if $i \notin J$ we have $\lambda_i|a_i - y_{l_i}| \geq \delta$. We write $u = \sum_{i\in J^C}\alpha_i\delta_i + \sum_{i\in J}\alpha_i\delta_i = u_1 + u_2$. Observe that u_1 has to satisfy one of the two above cases, that is, $u_1 \in V_1^1(\sharp J^C, \varepsilon)$ or $u_1 \in V_1^2(\sharp J^C, \varepsilon)$. Let $\widetilde{W}$ be a pseudogradient on $V_1^3(p, \varepsilon)$ defined by $\widetilde{W}(u) = W_1^1(u_1)$ if $u_1 \in V_1^1(\sharp J^C, \varepsilon)$, or $\widetilde{W}(u) = W_1^2(u_1)$ if $u_1 \in V_1^2(\sharp J^C, \varepsilon)$. In this region let $W_1^3(u) = \widetilde{W}(u) + X_1(u) + \sum_{i\in J\cap L_2} X_i(u) - M_1 Z_1(u)$. By Propositions A.1 and A.2, we have

$$\langle \partial J(u), W_1^3(u)\rangle \leq -c\biggl(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\biggr).$$

Pseudogradient in $V_1^4(p, \varepsilon)$. Finally, let $u = \sum_{i=1}^{p}\alpha_i\delta_i \in V_1^4(p, \varepsilon)$. Consider

$$B_k = \{j \mid 1 \leq j \leq p \text{ and } a_j \in B(y_{l_k}, \rho)\}.$$

In this case, there is at least one B_k which contains at least two indices. Without loss of generality, we can assume that $1, \ldots, q$ are the indices such that the set B_k, $1 \leq k \leq q$, contains at least two indices. We will decrease the λ_i for $i \in B_k$ with different speed. For this purpose, let

$$\chi : \mathbb{R} \to \mathbb{R}^+, \quad t \mapsto \begin{cases} 0 & \text{if } |t| \leq \tilde{\gamma}, \\ 1 & \text{if } |t| \geq 1. \end{cases}$$

Here $\tilde{\gamma}$ is a small constant. For $j \in B_k$, set $\overline{\chi}(\lambda_j) = \sum_{i\neq j,\, i\in B_k}\chi(\lambda_j/\lambda_i)$. Let

$$I_1 = \{i \mid 1 \leq i \leq p \text{ and } \lambda_i|a_i - y_{l_i}| \geq \delta\}.$$

We distinguish two cases:

Case 1: $I_1 \neq \varnothing$. Let in this case

$$J = \bigl\{j \mid 1 \leq j \leq p \text{ and } \lambda_j^{\beta_j} \geq \tfrac{1}{2}\min_{i\in I_1}\lambda_i^{\beta_i}\bigr\}.$$

Observe that, if $a_i \in B(y_{l_i}, \rho)$, we have $|\nabla K(a_i)| \sim \sum_{k=1}^{n}|b_k||(a_i - y_{l_i})_k|^{\beta_i - 1}$. So, if $i \in L_1$ we have $|\nabla K(a_i)|/\lambda_i \leq c/\lambda_i^{\beta_i}$, and if $i \in L_2$ we have

$$\frac{|\nabla K(a_i)|}{\lambda_i} \leq c\,\frac{|(a_i - y_{l_i})_k|^{\beta_i - 1}}{\lambda_i}.$$

Thus by Lemma 3.5 we obtain

$$\Big\langle \partial J(u), \sum_{i\in I_1} X_i(u)\Big\rangle \le -c_\delta\Big(\sum_{i\in J}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in J}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\in I_1\cap L_2}\frac{|(a_i-y_{l_i})|^{\beta_i-1}}{\lambda_i}\Big) + O\Big(\sum_{i\neq j,\ i\in I_1}\Big|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big|\Big) + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big).$$

Let $\widetilde{C} = \{(i, j) \mid \gamma \le \lambda_i/\lambda_j \le 1/\gamma\}$, where γ is a small positive constant. Observe that

$$\Big|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}\Big| = o(\varepsilon_{ij}) \quad \text{for all } (i, j)\in\widetilde{C},\, i\neq j.$$

This with (3-3) yields

$$\Big\langle \partial J(u), \sum_{i\in I_1} X_i(u)\Big\rangle \le -c_\delta\Big(\sum_{i\in J}\frac{1}{\lambda_i^{\beta_i}} + \sum_{i\in J}\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\in I_1\cap L_2}\frac{|(a_i-y_{l_i})|^{\beta_i-1}}{\lambda_i}\Big) + o\Big(\sum_{k=1}^{q}\sum_{\substack{i\neq j\in B_k\\ (i,j)\in\widetilde{C},\, i\in I_1}}\varepsilon_{ij}\Big) + O\Big(\sum_{k=1}^{q}\sum_{\substack{i\neq j\in B_k\\ (i,j)\notin\widetilde{C},\, i\in I_1}}\varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\Big). \tag{3-17}$$

For any $k = 1,\dots,q$, let $\lambda_{i_k} = \min\{\lambda_i \mid i\in B_k\}$. Define

$$\overline{Z} = -\sum_{k=1}^{q}\sum_{\substack{j\in B_k\\ (i_k,j)\notin\widetilde{C}}}\overline{\chi}(\lambda_j)Z_j - \gamma_1\sum_{k=1}^{q}\sum_{\substack{j\in B_k\\ (i_k,j)\in\widetilde{C}}}\overline{\chi}(\lambda_j)Z_j,$$

where γ_1 is a small positive constant. Using Lemma 3.4, we find that

$$\begin{aligned}\langle\partial J(u), \overline{Z}(u)\rangle \le{}& c\sum_{k=1}^{q}\sum_{\substack{i\neq j\\ j\in B_k,\,(j,i_k)\notin\widetilde{C}}}\overline{\chi}(\lambda_j)\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j}\\ &+ c\gamma_1\sum_{k=1}^{q}\sum_{\substack{i\neq j\\ j\in B_k,\,(j,i_k)\in\widetilde{C}}}\overline{\chi}(\lambda_j)\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j} + O\Big(\sum_{k=1}^{q}\sum_{\substack{j\in B_k\cap L_2\\ (j,i_k)\notin\widetilde{C}}}\Big(\frac{1}{\lambda_j^{\beta_j}} + \frac{|(a_j-y_{l_j})|^{\beta_j-2}}{\lambda_j^2}\Big)\Big)\\ &+ \gamma_1 O\Big(\sum_{k=1}^{q}\sum_{\substack{j\in B_k\cap L_2\\ (j,i_k)\in\widetilde{C}}}\Big(\frac{1}{\lambda_j^{\beta_j}} + \frac{|(a_j-y_{l_j})|^{\beta_j-2}}{\lambda_j^2}\Big)\Big).\end{aligned}$$

Observe that by using a direct calculation, we have

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \le -c\varepsilon_{ij} \quad \text{if } \lambda_i\ge\lambda_j,\ \lambda_i\sim\lambda_j, \text{ or } |a_i-a_j|\ge\delta_0>0. \tag{3-18}$$

Let $j \in B_k$, $1 \le k \le q$, and let i, $1 \le i \le p$, be such that $i \ne j$. If $i \notin B_k$, or $i \in B_k$ with $(i, j) \in \widetilde{C}$, then we have by (3-18)

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \le -c\varepsilon_{ij} \quad \text{and} \quad \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \le -c\varepsilon_{ij}.$$

In the case where $i \in B_k$ with $(i, j) \notin \widetilde{C}$ (assuming $\lambda_i \ll \lambda_j$), we have $\overline{\chi}(\lambda_j) - \overline{\chi}(\lambda_i) \ge 1$. Thus,

$$\overline{\chi}(\lambda_j)\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \overline{\chi}(\lambda_i)\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \le \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \le -c\varepsilon_{ij}.$$

We therefore have

$$\begin{aligned}\langle \partial J(u), \overline{Z}(u)\rangle \le -c\Biggl(\sum_{k=1}^{q} \sum_{\substack{i \ne j \\ j \in B_k,\, (j, i_k) \notin \widetilde{C}}} \varepsilon_{ij} + \gamma_1 \sum_{k=1}^{q} \sum_{\substack{i \ne j \\ j \in B_k,\, (j, i_k) \in \widetilde{C}}} \varepsilon_{ij}\Biggr) \\ + O\Biggl(\sum_{k=1}^{q} \sum_{\substack{j \in B_k \cap L_2 \\ (j, i_k) \notin \widetilde{C}}} \Biggl(\frac{1}{\lambda_j^{\beta_j}} + \frac{|(a_j - y_{l_j})|^{\beta_j - 2}}{\lambda_j^2}\Biggr)\Biggr) \\ + \gamma_1 O\Biggl(\sum_{k=1}^{q} \sum_{\substack{j \in B_k \cap L_2 \\ (j, i_k) \in \widetilde{C}}} \Biggl(\frac{1}{\lambda_j^{\beta_j}} + \frac{|(a_j - y_{l_j})|^{\beta_j - 2}}{\lambda_j^2}\Biggr)\Biggr). \qquad (3\text{-}19)\end{aligned}$$

Observe that if $j \in B_k$ with $(j, i_k) \in \widetilde{C}$, we have j or $i_k \in I_1$. Thus for M_1 large enough and γ_1 very small, we obtain from (3-17) and (3-19)

$$\begin{aligned}&\Bigl\langle \partial J(u), \sum_{i \in I_1} X_i + M_1 \overline{Z}(u)\Bigr\rangle \\ &\qquad \le -c\Biggl(\sum_{i \in J} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i \in J} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k=1}^{q} \sum_{\substack{i \ne j \\ j \in B_k}} \varepsilon_{ij}\Biggr) + O\Biggl(\sum_{k=1}^{q} \sum_{\substack{j \in B_k \\ (i_k, j) \notin \widetilde{C}}} \frac{1}{\lambda_j^{\beta_j}}\Biggr), \qquad (3\text{-}20)\end{aligned}$$

since

$$\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 2}}{\lambda_i^2} = o\Biggl(\frac{|(a_i - y_{l_i})_{k_i}|^{\beta_i - 1}}{\lambda_i}\Biggr) \quad \text{for any } i \in L_2 \qquad (3\text{-}21)$$

(as M_1 is large enough). Now, let in this region

$$W_1^4 := M_1\Bigl(\sum_{i \in I_1} X_i + M_1 \overline{Z}\Bigr) + \sum_{i \notin J}\Bigl(-\sum_{k=1}^{n} b_k\Bigr) Z_i.$$

We obtain from the above estimates

$$\langle \partial J(u), W_1^4(u)\rangle \le -c\Biggl(\sum_{i=1}^{p} \frac{1}{\lambda_i^{\beta_i}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \ne j} \varepsilon_{ij}\Biggr).$$

Case 2: $I_1 = \varnothing$. Let

$$I_2 = \{1\} \cup \{i \mid 1 \le i \le p \text{ and } \lambda_i \sim \lambda_1\}.$$

We write

$$u=\sum_{i\in I_2}\alpha_i\delta_i+\sum_{i\notin I_2}\alpha_i\delta_i:=u_1+u_2.$$

Observe that, for all $i\neq j\in I_2$ such that $i\neq j$, we have $|a_i-a_j|\geq\delta$. Indeed, if $|a_i-a_j|<\delta$, so $i,j\in B_k$, we get $|a_i-a_j|\leq|a_i-y_{l_i}|+|a_j-y_{l_i}|\leq 2\delta/\lambda_i$, since $I_1=\varnothing$ and $\lambda_i\sim\lambda_j$ for all $i,j\in I_2$. This implies that

$$\left(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j|a_i-a_j|^2\right)^{(n-2\sigma)/2}\leq c_1,$$

and hence $\varepsilon_{ij}\geq c$, which is a contradiction. Thus $u_1\in V_1^j(\sharp I_2,\varepsilon)$, $j=1$ or 2 or 3. Applying the associated pseudogradient denoted by $\overline{W}$, we obtain

$$\langle\partial J(u),\overline{W}(u)\rangle\leq -c\left(\sum_{i\in I_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in I_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{\substack{i\neq j\\ i,j\in I_2}}\varepsilon_{ij}\right)+O\left(\sum_{i\in I_2,\ j\notin I_2}\varepsilon_{ij}\right).$$

Let

$$J_2=\left\{i\mid 1\leq i\leq p,\ \lambda_i^{\beta_i}\geq\min_{j\in I_2}\lambda_j^{\beta_j}\right\}.$$

We can add to the above estimates all indices i such that $i\in J_2$. So, using the estimate (3-16) we obtain

$$\langle\partial J(u),\overline{W}(u)\rangle\leq -c\left(\sum_{i\in J_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in J_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{\substack{i\neq j\\ i,j\in I_2}}\varepsilon_{ij}\right)+o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\right)+O\left(\sum_{\substack{i,j\in B_k\\ i\in I_2,\ j\notin I_2}}\varepsilon_{ij}\right).$$

Let $M_1>0$ be large enough, then the above estimate and (3-19) yields

$$\begin{aligned}\langle\partial J(u),\,&M_1\overline{Z}(u)+\overline{W}(u)\rangle\\ &\leq -c\left(\sum_{i\in J_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\in J_2}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{k=1}^{q}\sum_{i\neq j\in B_k}\varepsilon_{ij}+\sum_{\substack{i\neq j\\ i,j\in I_2}}\varepsilon_{ij}\right)+O\left(\sum_{k=1}^{q}\sum_{\substack{i\in B_k\\ (i_k,i)\notin\widetilde{C}}}\frac{1}{\lambda_i^{\beta_i}}\right).\end{aligned}\tag{3-22}$$

By Step 3 in the proof of Proposition 3.3 below and (3-16), we have

$$\begin{aligned}\Big\langle\partial J(u),&\sum_{i\notin J_2}\Big(-\sum_{k=1}^{n}b_k\Big)Z_i(u)\Big\rangle\\ &\leq -c\left(\sum_{i\notin J_2}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i\notin J_2}\frac{|\nabla K(a_i)|}{\lambda_i}\right)+O\left(\sum_{k=1}^{q}\sum_{\substack{i\neq j\in B_k\\ i\notin J_2}}\varepsilon_{ij}\right)+o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}\right).\end{aligned}\tag{3-23}$$

Define

$$W_1^4(u)=M_1(M_1\overline{Z}(u)+\overline{W}(u))+\sum_{i\notin J_2}\Big(-\sum_{k=1}^{n}b_k\Big)Z_i(u).$$

Using (3-23), we get

$$\langle \partial J(u), W_1^4(u)\rangle \le -c\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{\beta_i}}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\Big),$$

since $1/\lambda_i^{\beta_i}=o(1/\lambda_{i_k}^{\beta_{i_k}})$ for all $i\in B_k$ such that $(i,i_k)\notin\widetilde{C}$.

The vector field W_1 in $V_1(p,\varepsilon)$ will be a convex combination of W_1^j, $j=1,\dots,4$. From the definitions of W_1^j, $j=1,\dots,4$, the only case where the maximum of the λ_i increases is when $a_i\in B(y_{l_i},\rho)$, $y_{l_i}\in\mathcal{K}^+$ for all $i=1,\dots,p$, with $y_{l_i}\neq y_{l_j}$ for all $i\neq j$. This concludes the proof of Proposition 3.2. □

Proof of Proposition 3.3. We divide the set $V_2(p,\varepsilon)$ into five sets:

$$V_2^1(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon)\;\Big|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j,\ -\sum_{k=1}^{n}b_k(y_{l_i})>0,\ \lambda_i|a_i-y_{l_i}|<\delta \text{ for all } i=1,\dots,p \text{ and } \rho(y_{l_i},\dots,y_{l_p})>0\Big\},$$

$$V_2^2(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon)\;\Big|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j,\ -\sum_{k=1}^{n}b_k(y_{l_i})>0,\ \lambda_i|a_i-y_{l_i}|<\delta \text{ for all } i=1,\dots,p \text{ and } \rho(y_{l_i},\dots,y_{l_p})<0\Big\},$$

$$V_2^3(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon)\;\Big|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j,\ \lambda_i|a_i-y_{l_i}|<\delta \text{ for all } i=1,\dots,p, \text{ and there exist } j \text{ such that } -\sum_{k=1}^{n}b_k(y_{l_j})<0\Big\},$$

$$V_2^4(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon)\;\Big|\; y_{l_i}\neq y_{l_j}\text{ for all } i\neq j, \text{ and there exist } j \text{ (at least) such that } \lambda_j|a_j-y_{l_j}|\ge\frac{\delta}{2}\Big\},$$

$$V_2^5(p,\varepsilon)=\Big\{u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2(p,\varepsilon)\;\Big|\;\text{such that there exist } i\neq j \text{ satisfying } y_{l_i}=y_{l_j}\Big\}.$$

We break up the proof into five steps.

Step 1. First, we consider the case $u=\sum_{i=1}^{p}\alpha_i\delta_{a_i\lambda_i}\in V_2^1(p,\varepsilon)$. We have, for any $i\neq j$, $|a_i-a_j|>\rho$ and therefore,

$$\varepsilon_{ij}=\Big(\frac{2}{(1-\cos d(a_i,a_j))\lambda_i\lambda_j}\Big)^{(n-2\sigma)/2}(1+o(1))=2^{(n-2\sigma)/2}\frac{G(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-2\sigma)/2}}(1+o(1)).$$

Here $G(a_i,a_j)$ is defined in (1-7). Thus,

$$\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}=-\frac{n-2\sigma}{2}\cdot 2^{(n-2\sigma)/2}\cdot\frac{G(a_i,a_j)}{(\lambda_i\lambda_j)^{(n-2\sigma)/2}}(1+o(1)).$$

Using Proposition A.1 with $\beta = n-2\sigma$ and the fact that $\alpha_i^{4\sigma/(n-2\sigma)} K(a_i) J(u)^{n/(n-2\sigma)} = 1+o(1)$ for all $i = 1, \dots, p$, we derive that

$$\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle = \frac{n-2\sigma}{2} J(u)^{1-n/2} \left(\frac{n-2\sigma}{n} \cdot \tilde{c}_1 \cdot \frac{\sum_{i=1}^p b_k}{K(a_i)^{n/(2\sigma)}} \frac{1}{\lambda_i^{n-2\sigma}} + c_1 2^{(n-2\sigma)/2} \sum_{i \neq j} \frac{G(y_{l_i}, y_{l_j})}{(K(a_i)K(a_j))^{(n-2\sigma)/(4\sigma)}} \frac{1}{(\lambda_i \lambda_j)^{(n-2\sigma)/2}} \right) + o\left(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \right).$$

Here $\tilde{c}_1 = c_0^{2n/(n-2\sigma)} \int_{\mathbb{R}^n} \frac{|(x_1)|^{n-2\sigma}}{(1+|x|^2)^n} \, dx$. Hence, using the fact that $|a_i - y_{l_i}| < \delta$ for δ very small, we get

$$\begin{aligned} \left\langle \partial J(u), \sum_{i=1}^p \alpha_i Z_i \right\rangle &\le -c\, {}^t\Lambda M(y_{l_1}, \dots, y_{l_p}) \Lambda + o\left(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \right) \\ &\le -c\rho(y_{l_1}, \dots, y_{l_p}) |\Lambda|^2 + o\left(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \right), \end{aligned}$$

where $\Lambda = {}^t\big(1/\lambda_1^{(n-2\sigma)/2}, \dots, 1/\lambda_p^{(n-2\sigma)/2}\big)$. Here $M(y_{l_1}, \dots, y_{l_p})$ is as defined in (1-6) and $\rho(y_{l_1}, \dots, y_{l_p})$ is the least eigenvalue of $M(y_{l_1}, \dots, y_{l_p})$. Using the fact that for all $i \neq j$, we have $\varepsilon_{ij} \le c/(\lambda_i \lambda_j)^{(n-2\sigma)/2}$, since $|a_i - a_j| \ge \delta$, we then obtain

$$\left\langle \partial J(u), \sum_{i=1}^p \alpha_i Z_i \right\rangle \le -c \left(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} \right).$$

In addition, for all $i = 1, \dots, p$, if $\lambda_i |a_i| < \delta$ then we have $|\nabla K(a_i)|/\lambda_i \sim |(a_i)_k|^{\beta-1}/\lambda_i \le c/\lambda_i^{\beta}$. Thus, we derive, for $W_2^1 := \sum_{i=1}^p \alpha_i Z_i$,

$$\langle \partial J(u), W_2^1 \rangle \le -c \left(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \right).$$

Step 2. Secondly, we study the case $u = \sum_{i=1}^p \alpha_i \delta_{a_i \lambda_i} \in V_2^2(p, \varepsilon)$. Since $\rho := \rho(y_{l_1}, \dots, y_{l_p})$ is the least eigenvalue of $M(y_{l_1}, \dots, y_{l_p})$, it satisfies

$$\rho = \inf_{X \in \mathbb{R}^p \setminus \{0\}} \left\{ \frac{{}^tX M(y_{l_1}, \dots, y_{l_p}) X}{\|X\|^2} \right\}. \tag{3-24}$$

Therefore, there exists an eigenvector $e = (e_i)_{i=1,\dots,p}$ associated to ρ such that $|e| = 1$ with $e_i > 0$, for all $i = 1, \dots, p$. Indeed,

$$\rho = {}^t e M(y_{l_1}, \dots, y_{l_p}) e = \sum_{i=1}^p m_{ii} e_i^2 + \sum_{i \neq j} m_{ij} e_i e_j \ge \sum_{i=1}^p m_{ii} |e_i|^2 + \sum_{i \neq j} m_{ij} |e_i| |e_j|, \tag{3-25}$$

since $m_{ij} < 0$ for $i \neq j$. Observe that if there exists $i_0 \neq j_0$ such that $e_{i_0} e_{j_0} < 0$, then the inequality in (3-25) will be strict. This is a contradiction with (3-24). Therefore $e_i e_j \geq 0$ for all $i \neq j$. Hence, we can work with $e = (e_1, \dots, e_p)$ such that $e_i \geq 0$, for all $i = 1 \dots, p$. Now, if there exists i_0 such that $e_{i_0} = 0$, then $M(y_{l_1}, \dots, y_{l_p})e = \rho e$ would imply that $\sum_{j \neq i_0} m_{j i_0} e_j = 0$ and $e_j = 0$, a contradiction. Thus, $e_i > 0$ for all $i = 1, \dots, p$.

Let $\gamma > 0$ such that for any $x \in B(e, \gamma) = \{y \in S^{p-1} \mid |y - e| \leq \gamma\}$, we have

$$ {}^t x M(y_{l_1}, \dots, y_{l_p}) x \leq \tfrac{1}{2} \rho(y_{l_1}, \dots, y_{l_p}). $$

Two cases may occur.

Case 1: $\Lambda / |\Lambda| \in B(e, \gamma)$, where $\Lambda = {}^t\big(1/\lambda_1^{(n-2\sigma)/2}, \dots, 1/\lambda_p^{(n-2\sigma)/2}\big)$.

In this case, we define $W_2^2 = -\sum_{i=1}^p \alpha_i Z_i$. As in Step 1, we find that

$$ \langle \partial J(u), W_2^2(u) \rangle \leq -c \bigg(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \bigg). $$

Case 2: $\Lambda / |\Lambda| \notin B(e, \gamma)$.

In this case, we define

$$ W_2^2 = -\frac{2}{n-2\sigma} |\Lambda| \sum_{i=1}^p \alpha_i \lambda_i^{n/2} \bigg(\frac{|\Lambda| e_i - \Lambda_i}{|\Lambda|} - \frac{\Lambda_i \langle |\Lambda| e - \Lambda, \Lambda \rangle}{|\Lambda|^3} \bigg) \frac{\partial \delta_{a_i \lambda_i}}{\partial \lambda_i}. $$

Using Proposition A.1, we find that

$$ \langle \partial J(u), W_2^2(u) \rangle = -c |\Lambda|^2 \frac{\partial}{\partial t} ({}^t\Lambda(t) M \Lambda(t)) \Big|_{t=0} + o\bigg(\sum_{i=1}^p \frac{1}{\lambda_i^{n-4}} \bigg) + o\bigg(\sum_{i \neq j} \varepsilon_{ij} \bigg), $$

where $M = M(y_{l_1}, \dots, y_{l_p})$ and $\Lambda(t) = \dfrac{(1-t)\Lambda + t|\Lambda| e}{\big|(1-t)\Lambda + t|\Lambda| e\big|} \Lambda$. Observe that

$$ {}^t\Lambda(t) M \Lambda(t) = \rho + \frac{(1-t)^2}{\big|(1-t)\Lambda + t|\Lambda| e\big|} ({}^t\Lambda M \Lambda - \rho |\Lambda|^2). $$

Thus we obtain $\dfrac{\partial}{\partial t}({}^t\Lambda(t) M \Lambda(t)) < -c$ and therefore,

$$ \langle \partial J(u), W_2^2(u) \rangle \leq -c \bigg(\sum_{i=1}^p \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \bigg). $$

Step 3. Now, we deal with the case $u = \sum_{i=1}^p \alpha_i \delta_{a_i \lambda_i} \in V_2^3(p, \varepsilon)$. Without loss of generality, we can assume that $1, \dots, q$ are the indices which satisfy $-\sum_{k=1}^n b_k(y_{l_i}) < 0$ for all $i = 1, \dots, q$. Let

$$ \widetilde{W}_2^1 = \sum_{i=1}^q -\alpha_i Z_i. $$

By Proposition A.1 and (3-18), we obtain

$$\langle \partial J(u), \widetilde{W}_2^1(u)\rangle \le -c\biggl(\sum_{i=1}^{q} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\, 1\le i\le q} \varepsilon_{ij}\biggr).$$

Set

$$I = \bigl\{ i \mid 1 \le i \le p \text{ and } \lambda_i \le \tfrac{1}{10} \min_{1\le j\le q} \lambda_j \bigr\}.$$

It is easy to see that we can add to the above estimates all indices i such that $i \notin I$. Thus

$$\langle \partial J(u), \widetilde{W}_2^1(u)\rangle \le -c\biggl(\sum_{i\notin I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\, i\notin I} \varepsilon_{ij}\biggr).$$

If $I \neq \varnothing$, in this case, we write

$$u = u_1 + u_2, \qquad u_1 = \sum_{i\in I} \alpha_i \delta_{a_i\lambda_i}, \qquad u_2 = \sum_{i\notin I} \alpha_i \delta_{a_i\lambda_i}.$$

Observe that u_1 must be contained in either $V_2^1(\sharp I, \varepsilon)$ or $V_2^2(\sharp I, \varepsilon)$. Thus we can apply the associated vector field which we denote by $\widetilde{W}_2^2$. We then have

$$\langle \partial J(u), \widetilde{W}_2^2(u)\rangle \le -c\biggl(\sum_{i\in I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\, i\in I} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i}\biggr) + O\biggl(\sum_{i\neq j,\, i\notin I} \varepsilon_{ij}\biggr).$$

Let in this subset $W_2^3 = \widetilde{W}_2^1 + m_1 \widetilde{W}_2^2$ for m_1 a small positive constant. We get

$$\langle \partial J(u), W_2^3(u)\rangle \le -c\biggl(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\biggr).$$

Step 4. We consider next the case $u = \sum_{i=1}^{p} \alpha_i \delta_{a_i\lambda_i} \in V_2^4(p, \varepsilon)$. Let

$$\lambda_{i_1} = \inf\{\lambda_j \mid \lambda_j |a_j| \ge \delta\}.$$

For $m_1 > 0$ small enough, we claim that

$$\langle \partial J(u), (X_{i_1} - m_1 Z_{i_1})(u)\rangle \le -c\biggl(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\neq i_1} \varepsilon_{i_1 j} + \sum_{i=1}^{p} \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}\biggr).$$

Indeed, for $i \neq j$, we have $|a_i - a_j| > \rho$, thus in Proposition A.2 the term $\left|\dfrac{1}{\lambda_i}\dfrac{\partial \varepsilon_{ij}}{\partial (a_i)_k}\right|$ is very small with respect to ε_{ij}. Hence,

$$\langle \partial J(u), X_{i_1}(u)\rangle \le -\frac{c}{\lambda_{i_1}^{n-2\sigma}}\biggl(\int_{\mathbb{R}^n} b_{k_{i_1}} \frac{|x_{k_{i_1}} + \lambda_{i_1}(a_{i_1})_{k_{i_1}}|^{\beta}}{(1+\lambda_{i_1}|(a_{i_1})_{k_{i_1}}|)^{(\beta-1)/2}} \frac{x_{k_{i_1}}}{(1+|x|^2)^{n+1}}\, dx\biggr)^2 + o\biggl(\frac{1}{\lambda_{i_1}^{n-2\sigma}} + \sum_{j\neq i_1} \varepsilon_{i_1 j}\biggr).$$

If $i_1 \in L_1$, in which case $\delta \le \lambda_{i_1}|a_{i_1}| \le M_1$, then an elementary calculation gives

$$\left(\int_{\mathbb{R}^n} b_{k_i} \frac{|x_{k_i} + \lambda_i (a_1)_{k_i}|^\beta}{(1+\lambda_i|(a_1)_{k_i}|)^{(\beta-1)/2}} \frac{x_{k_i}}{(1+|x|^2)^n}\, dx\right)^2 \ge c > 0. \tag{3-26}$$

Using (3-26), we get

$$\langle \partial J(u), X_{i_1}(u)\rangle \le -\frac{c}{\lambda_{i_1}^{n-2\sigma}} + o\Big(\sum_{j\neq i_1} \varepsilon_{i_1 j}\Big) \le -c \sum_{i=i_1}^{p} \frac{1}{\lambda_i^\beta} + o\Big(\sum_{j\neq i_1} \varepsilon_{i_1 j}\Big). \tag{3-27}$$

On the other hand, we have, by Proposition A.1 and (3-18),

$$\langle \partial J(u), Z_{i_1}(u)\rangle \le -c \sum_{j\neq i_1} \varepsilon_{i_1 j} + O\Big(\frac{1}{\lambda_{i_1}^{n-2\sigma}}\Big). \tag{3-28}$$

Using (3-27) and (3-28) our claim follows in this case.

If $i_1 \in L_2$, using (3-3), we find

$$\begin{aligned}\langle \partial J(u), X_{i_1}(u)\rangle &\le -c\Big(\frac{1}{\lambda_{i_1}^{n-2\sigma}} + \frac{|(a_{i_1})_{k_{i_1}}|^{\beta-1}}{\lambda_{i_1}}\Big) + o\Big(\sum_{j\neq i_1} \varepsilon_{i_1 j}\Big)\\ &\le -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \frac{|(a_{i_1})_{k_{i_1}}|^{\beta-1}}{\lambda_{i_1}}\Big) + o\Big(\sum_{j\neq i_1} \varepsilon_{i_1 j}\Big),\end{aligned}$$

and by Proposition A.1 and (3-3), we have

$$\langle \partial J(u), -Z_{i_1}(u)\rangle \le -c \sum_{j\neq i_1} \varepsilon_{i_1 j} + O\Big(\frac{|(a_{i_1})_{k_{i_1}}|^{\beta-2}}{\lambda_{i_1}^2}\Big).$$

Now using (3-21), we obtain

$$\begin{aligned}\langle \partial J(u), (X_{i_1} - m_1 Z_{i_1})(u)\rangle &\le -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\neq i_1} \varepsilon_{i_1 j} + \frac{|(a_{i_1})_k|^{\beta-1}}{\lambda_{i_1}}\Big)\\ &\le -c\Big(\sum_{i=i_1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\neq i_1} \varepsilon_{i_1 j} + \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}\Big),\end{aligned}$$

since $|\nabla K(a_{i_1})| \sim |(a_{i_1})_{k_i}|^{\beta-1}$. Thus, our claim follows.

Now let

$$I = \big\{ i \mid 1 \le i \le p \text{ and } \lambda_i < \tfrac{1}{10}\lambda_{i_1} \big\}.$$

We have

$$\langle \partial J(u), (X_{i_1} - m_1 Z_{i_1})(u)\rangle \le -c\Big(\sum_{i\notin I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{j\neq i,\, i\notin I} \varepsilon_{ij} + \frac{|\nabla K(a_{i_1})|}{\lambda_{i_1}}\Big).$$

Furthermore, using (3-3), we have

$$\Big\langle \partial J(u), \Big(X_{i_1} - m_1 Z_{i_1} + \sum_{i\notin I,\, i\in L_2} X_i\Big)(u)\Big\rangle \le -c\Big(\sum_{i\notin I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\notin I} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j,\, i\notin I} \varepsilon_{ij}\Big),$$

since for $i \notin I$ and $i \in L_1$, we have $|\nabla K(a_i)|/\lambda_i \le c/\lambda_i^{\beta}$. We need to add the remainder terms (if $I \neq \varnothing$). Let $u_1 = \sum_{i\in I} \alpha_i \delta_{a_i\lambda_i}$. For all $i \in I$ we have $\lambda_i|a_i| < \delta$. Thus, $u_1 \in V_2^j(\sharp I, \varepsilon)$ for $j = 1$ or 2 or 3, so we can apply the associated vector field which we will denote $\widetilde{W}_2^4$. We then have

$$\langle \partial J(u), \widetilde{W}_2^4\rangle \le -c\Big(\sum_{i\in I} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j,\, i,j\in I} \varepsilon_{ij} + \sum_{i\in I} \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + O\Big(\sum_{i\in I,\, j\notin I} \varepsilon_{ij}\Big).$$

Let $W_2^4 = X_{i_1} - m_1 Z_{i_1} + \sum_{i\notin I,\, i\in L_2} X_i + m_2 \widetilde{W}_2^4$ for $m_2 > 0$ small enough. We get

$$\langle \partial J(u), W_2^4(u)\rangle \le -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\Big).$$

<u>Step 5</u>. We study now the case $u = \sum_{i=1}^{p} \alpha_i \delta_{a_i\lambda_i} \in V_2^5(p, \varepsilon)$. Let

$$B_k = \{ j \mid 1 \le j \le p \text{ and } a_j \in B(y_{l_k}, \rho)\}.$$

In this case, there is at least one B_k which contains at least two indices. Without loss of generality, we can assume that $1, \dots, q$ are the indices such that the set B_k, $1 \le k \le q$, contains at least two indices. We will decrease the λ_i for $i \in B_k$ with different speed. For this purpose, let

$$\chi : \mathbb{R} \to \mathbb{R}^+, \qquad t \mapsto \begin{cases} 0 & \text{if } |t| \le \gamma', \\ 1 & \text{if } |t| \ge 1. \end{cases}$$

Here γ' is a small constant.

For $j \in B_k$, set $\overline{\chi}(\lambda_j) = \sum_{i\neq j,\, i\in B_k} \chi(\lambda_j/\lambda_i)$. Define

$$\widetilde{W}_2^5 = -\sum_{k=1}^{q} \sum_{j\in B_k} \alpha_j \overline{\chi}(\lambda_j) Z_j.$$

Using Proposition A.1 and (3-3), we obtain

$$\langle \partial J(u), \widetilde{W}_2^5(u)\rangle \le c\sum_{k=1}^{q}\Big(\sum_{i\neq j,\, j\in B_k} \overline{\chi}(\lambda_j)\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \sum_{j\in B_k,\, j\in L_1} \overline{\chi}(\lambda_j) O\Big(\frac{1}{\lambda_j^{n-2\sigma}}\Big) + \sum_{j\in B_k,\, j\in L_2} \overline{\chi}(\lambda_j) O\Big(\frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2}\Big)\Big).$$

For $j \in B_k$, with $k \le q$, if $\overline{\chi}(\lambda_j) \neq 0$, then there exists $i \in B_k$ such that $1/\lambda_j^{n-2\sigma} = o(\varepsilon_{ij})$ (for ρ small enough). Furthermore, for $j \in B_k$, if $i \notin B_k$ (or $i \in B_k$ with $\lambda_i \sim \lambda_j$), then we have, by (3-18),

$$\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \le -c\varepsilon_{ij} \quad\text{and}\quad \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \le -c\varepsilon_{ij}.$$

In the case where $i \in B_k$ (assuming $\lambda_i \ll \lambda_j$), we have $\overline{\chi}(\lambda_j) - \overline{\chi}(\lambda_i) \geq 1$. Thus

$$\overline{\chi}(\lambda_j)\lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \overline{\chi}(\lambda_i)\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c\varepsilon_{ij}.$$

Thus we obtain

$$\langle \partial J(u), \widetilde{W}_2^5(u)\rangle \leq -c \sum_{k=1}^{q} \sum_{j \in B_k} \overline{\chi}(\lambda_j)\Big(\sum_{i \neq j} \varepsilon_{ij} + \frac{1}{\lambda_j^{n-2\sigma}}\Big) + \sum_{k=1}^{q} \sum_{j \in B_k,\, j \in L_2} \overline{\chi}(\lambda_j) O\Big(\frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2}\Big). \tag{3-29}$$

We need to add the indices $j \in \big(\bigcup_{K=1}^{q} B_k\big)^C \cup \{j \in B_k \mid \overline{\chi}(\lambda_j) = 0\}$. Let

$$\lambda_{i_0} = \inf\{\lambda_i \mid i = 1, \ldots, p\}.$$

We distinguish two cases.

Case 1: There exists j such that $\overline{\chi}(\lambda_j) \neq 0$, $\lambda_{i_0} \sim \lambda_j$, and $\gamma' \leq \lambda_{i_0}/\lambda_j \leq 1$; then we observe in the above estimate $-1/\lambda_{i_0}^{n-2\sigma}$ and therefore $-\sum_{i=1}^{p} 1/\lambda_i^{n-2\sigma}$ and $-\sum_{k \neq r} \varepsilon_{kr}$. Thus we obtain

$$\langle \partial J(u), \widetilde{W}_2^5(u)\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij}\Big) + O\Big(\sum_{k=1}^{q} \sum_{j \in B_k,\, j \in L_2} \frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2}\Big).$$

Now let

$$W_2^5 = \widetilde{W}_2^5 + m_1 \sum_{i=1}^{p} X_i.$$

Using the above estimates with Proposition A.2 and (3-21), we obtain

$$\langle \partial J(u), W_2^5(u)\rangle \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i}\Big).$$

Case 2: For each $j \in B_k$, $1 \leq k \leq q$, we have $\lambda_{i_0} \ll \lambda_j$ (i.e., $\lambda_{i_0}/\lambda_j < \gamma'$), or if $\lambda_{i_0} \sim \lambda_j$ we have $\overline{\chi}(\lambda_j) = 0$. In this case we define

$$D = \Big(\{i \mid \overline{\chi}(\lambda_i) = 0\} \cup \Big(\bigcup_{k=1}^{q} B_k\Big)^C\Big) \cap \Big\{i \,\Big|\, \frac{\lambda_i}{\lambda_{i_0}} < \frac{1}{\gamma'}\Big\}.$$

It is easy to see that $i_0 \in D$ and if $i \neq j \in \{i \mid \overline{\chi}(\lambda_i) = 0\} \cup \big(\bigcup_{k=1}^{q} B_k\big)^C$ we have $a_i \in B(y_{l_i}, \rho)$ and $a_j \in B(y_{l_j}, \rho)$ with $y_{l_i} \neq y_{l_j}$. Let

$$u_1 = \sum_{i \in D} \alpha_i \delta_{a_i \lambda_i}.$$

Then u_1 has to satisfy one of the four subsets above, that is, $u_1 \in V_2^j(\sharp I, \varepsilon)$ for $j = 1, 2, 3$, or 4. Thus we can apply the associated vector field, which we will denote Y, and we have

$$\langle \partial J(u), Y(u)\rangle \leq -c\Big(\sum_{i \in D} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i \in D} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{\substack{i \neq j \\ i,j \in D}} \varepsilon_{ij}\Big) + O\Big(\sum_{i \in D,\, j \notin D} \varepsilon_{ij}\Big).$$

Observe that in the above estimates, we have the term $-1/\lambda_{i_0}^{n-2\sigma}$, thus we have $-\sum_{i=1}^{p} 1/\lambda_i^{n-2\sigma}$. Concerning the term $-\sum_{i\neq j}\varepsilon_{ij}$ for $i \in D$ and $j \in D^C$, we have

$$D^C = \left\{ i \;\Big|\; \frac{\lambda_i}{\lambda_{i_0}} > \frac{1}{\gamma'} \right\} \cup \left(\{ i \mid \overline{\chi}(\lambda_i) \neq 0 \} \cap \left(\bigcup_{k=1}^{q} B_k \right) \right).$$

If $j \in \{ i \mid \overline{\chi}(\lambda_i) \neq 0\} \cap \bigcup_{k=1}^{q} B_k$, then we have $(-\varepsilon_{ij})$ in the estimates (3-29). If $j \in \left\{ i \;\Big|\; \frac{\lambda_i}{\lambda_{i_0}} > \frac{1}{\gamma'} \right\}$, we can prove in this case that $|a_i - a_j| \geq \rho$. Thus

$$\varepsilon_{ij} \leq \frac{c}{(\lambda_i \lambda_j)^{(n-2\sigma)/2}} < \frac{c\gamma'^{(n-2\sigma)/2}}{(\lambda_{i_0}\lambda_i)^{(n-2\sigma)/2}} = o(\varepsilon_{i_0 i})$$

for γ' small enough. We derive that

$$\begin{aligned}&\big\langle \partial J(u), (\widetilde{W}_2^5 + m_1 Y)(u) \big\rangle \\ &\qquad \leq -c\left(\sum_{i\in D} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j} \varepsilon_{ij} \right) + \sum_{K=1}^{q} \sum_{j\in B_k,\, j\in L_2} \overline{\chi}(\lambda_j) O\left(\frac{|(a_j)_{k_i}|^{\beta-2}}{\lambda_j^2} \right),\end{aligned}$$

and hence, by (3-21), we get

$$\left\langle \partial J(u), \left(\widetilde{W}_2^5 + m_1 Y + m_2 \sum_{i=1,\, i\in L_2} X_i \right)(u) \right\rangle \leq -c\left(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2\sigma}} + \sum_{i\neq j} \varepsilon_{ij} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|}{\lambda_i} \right),$$

for m_1 and m_2 two small positive constants. In this case we define

$$W_2^5 := \widetilde{W}_2^5 + m_1 Y + m_2 \sum_{i=1,\, i\in L_2} X_i .$$

The vector field W_2 in $V_2(p, \varepsilon)$ will be a convex combination of W_2^j, $j = 1, \ldots, 5$. This concludes the proof of Proposition 3.3. □

Corollary 3.6. *Let $p \geq 1$. The critical points at infinity of J in $V(p, \varepsilon)$ correspond to*

$$(y_{l_1}, \ldots, y_{l_p})_\infty := \sum_{i=1}^{p} \frac{1}{K(y_{l_i})^{(n-2\sigma)/2}} \delta_{(y_{l_i}, \infty)},$$

where $(y_{l_1}, \ldots, y_{l_p}) \in \mathcal{P}^\infty$. Moreover, such a critical point at infinity has an index equal to

$$i(y_{l_1}, \ldots, y_{l_p})_\infty = p - 1 + \sum_{i=1}^{p} n - \tilde{i}(y).$$

4. Proof of Theorem 1.1

Using Corollary 3.6, the only critical points at infinity associated to problem (1-1) correspond to $w_\infty = (y_{i_1}, \ldots, y_{i_p}) \in \mathcal{P}^\infty$. We prove Theorem 1.1 by contradiction. Therefore, we assume that (1-1) has no solution. For any $w_\infty \in \mathcal{P}^\infty$, let $c(w)_\infty$ denote the associated critical value at infinity. Here we choose to consider a simplified situation where for any $w_\infty \neq w'_\infty$, we have $c(w)_\infty \neq c(w')_\infty$ and thus order the $c(w)_\infty$ with $w_\infty \in \mathcal{P}^\infty$ as

$$c(w_1)_\infty < \cdots < c(w_{k_0})_\infty.$$

For any $\bar{c} \in \mathbb{R}$, let $J_{\bar{c}} = \{u \in \Sigma^+ \mid J(u) \leq \bar{c}\}$. By using a deformation lemma (see [Bahri and Rabinowitz 1991]), we know that if $c(w_{k-1})_\infty < a < c(w_k)_\infty < b < c(w_{k+1})_\infty$, then

$$J_b \simeq J_a \cup W_u^\infty(w_k)_\infty, \tag{4-1}$$

where $W_u^\infty(w_k)_\infty$ denotes the unstable manifolds at infinity of $(w_k)_\infty$ (see [Bahri 1996]) and $\simeq$ denotes retracts by deformation.

Taking the Euler–Poincaré characteristic of both sides of (4-1), we find that

$$\chi(J_b) = \chi(J_a) + (-1)^{i(w_k)_\infty}, \tag{4-2}$$

where $i(w_k)_\infty$ denotes the index of the critical point at infinity $(w_k)_\infty$. Let

$$b_1 < c(w_1)_\infty = \min_{u \in \Sigma^+} J(u) < b_2 < c(w_2)_\infty < \cdots < b_{k_0} < c(w_{k_0})_\infty < b_{k_0+1}.$$

Since we have assumed that (1-1) has no solution, $J_{b_{k_0+1}}$ is a retract by deformation of Σ^+. Therefore $\chi(J_{b_{k_0+1}}) = 1$, since Σ^+ is a contractible set. Now using (4-2), after recalling that $\chi(J_{b_1}) = \chi(\varnothing) = 0$, we derive

$$1 = \sum_{j=1}^{k_0} (-1)^{i(w_j)_\infty}. \tag{4-3}$$

So, if (4-3) is violated, then (1-1) has a solution.

If there exists $w_\infty \neq w'_\infty$ such that $a < c(w)_\infty = c(w')_\infty < b$, then

$$J_b \simeq J_a \cup W_u^\infty(w)_\infty \cup W_u^\infty(w')_\infty. \tag{4-4}$$

By taking the Euler–Poincaré characteristic of both sides, we find that

$$\chi(J_b) = \chi(J_a) + (-1)^{i(w)_\infty} + (-1)^{i(w')_\infty}. \tag{4-5}$$

Repeating the same argument used above, we get a contradiction, completing the proof of Theorem 1.1.

Appendix

This appendix is devoted to some useful expansions of the gradient of J near a potential critical point at infinity consisting of p masses. These propositions are proved under some technical estimates of the different integral quantities, extracted from [Bahri 1989] (with some changes).

Proposition A.1. *Assume that K satisfies $(f)_\beta$, $1 < \beta < n$. For any $u = \sum_{j=1}^p \alpha_j \delta_j$ in $V(p, \varepsilon)$, the following expansions hold*:

(i) $$\left\langle \partial J(u), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle = -2c_2 J(u) \sum_{i\neq j} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o\Big(\sum_{i\neq j} \varepsilon_{ij}\Big) + o\Big(\frac{1}{\lambda_i}\Big),$$

where $c_2 = c_0^{2n/(n-2\sigma)} \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^{(n+2\sigma)/2}}$.

(ii) *If* $a_i \in B(y_{j_i}, \rho)$, $y_{j_i} \in \mathcal{K}$ *and* ρ *is a positive constant small enough, we have*

$$\begin{aligned}\left\langle \partial J(u), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle &= 2J(u)\Bigg(-c_2 \sum_{j\neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2\sigma}{2n} c_0^{2n/(n-2\sigma)} \beta \frac{\alpha_i}{K(a_i)} \frac{1}{\lambda_i^\beta} \\ &\qquad \times \sum_{k=1}^n b_k \int_{\mathbb{R}^n} \operatorname{sign}(x_k + \lambda_i(a_i - y_{j_i})_k) |x_k + \lambda_i(a_i - y_{j_i})_k|^{\beta-1} \frac{x_k}{(1+|x|^2)^n}\, dx \\ &\qquad + o\Big(\sum_{j\neq i} \varepsilon_{ij} + \sum_{j=1}^p \frac{1}{\lambda_j^\beta}\Big)\Bigg). \end{aligned} \tag{A-1}$$

(iii) *Furthermore, if* $\lambda_i |a_i - y_{j_i}| < \delta$, *for* δ *very small, we then have*

$$\left\langle \partial J(u), \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right\rangle = 2J(u)\Bigg(\frac{n-2\sigma}{2n} \beta c_3 \frac{\alpha_i}{K(a_i)} \frac{\sum_{k=1}^n b_k}{\lambda_i^\beta} - c_2 \sum_{j\neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o\Big(\sum_{j\neq i} \varepsilon_{ij} + \sum_{j=1}^p \frac{1}{\lambda_j^\beta}\Big)\Bigg), \tag{A-2}$$

where $c_3 = c_0^{2n/(n-2\sigma)} \int_{S^n} \frac{|x_1|^\beta}{(1+|x|^2)^n}\, dx$.

Proposition A.2. *Under condition* $(f)_\beta$, $1 < \beta < n$, *for each* $u = \sum_{j=1}^p \alpha_j \delta_j \in V(p, \varepsilon)$, *we have:*

(i) $$\left\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial a_i} \right\rangle = -c_5 J(u)^2 \alpha_i^{(n+2\sigma)/(n-2\sigma)} \frac{\nabla K(a_i)}{\lambda_i} + O\Big(\sum_{i\neq j} \frac{1}{\lambda_i} \Big|\frac{\partial \varepsilon_{ij}}{\partial a_i}\Big|\Big) + o\Big(\sum_{i\neq j} \varepsilon_{ij} + \frac{1}{\lambda_i}\Big),$$

where $c_5 = \int_{\mathbb{R}^n} \frac{dy}{(1+|y|^2)^n}$.

(ii) *If* $a_i \in B(y_{j_i}, \rho)$, $y_{j_i} \in \mathcal{K}$, *we have*

$$\begin{aligned}\left\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_k} \right\rangle &= -2(n-2\sigma) c_0^{2n/(n-2\sigma)} \alpha_i^{(n+2\sigma)/(n-2\sigma)} J(u)^2 \frac{1}{\lambda_i^\beta} \int_{\mathbb{R}^n} b_k |x_k + \lambda_i(a_i - y_{j_i})_k|^\beta \frac{x_k}{(1+|x|^2)^{n+1}}\, dy \\ &\qquad + o\Big(\sum_{i\neq j} \varepsilon_{ij}\Big) + o\Big(\sum_{i=1}^p \frac{1}{\lambda_i^\beta}\Big) + O\Big(\sum_{i\neq j} \frac{1}{\lambda_i} \Big|\frac{\partial \varepsilon_{ij}}{\partial a_i}\Big|\Big), \end{aligned}$$

where $k = 1, \ldots, n$ *and* $(a_i)_k$ *is the* k*-th component of* a_i *in some geodesic normal coordinate system.*

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ON POSITIVE SOLUTIONS OF THE (p, A)-LAPLACIAN WITH POTENTIAL IN MORREY SPACE

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We study qualitative positivity properties of quasilinear equations of the form

$$Q'_{A,p,V}[v] := -\operatorname{div}(|\nabla v|_A^{p-2} A(x)\nabla v) + V(x)|v|^{p-2}v = 0, \quad x \in \Omega,$$

where Ω is a domain in $\mathbb{R}^n$, $1 < p < \infty$, $A = (a_{ij}) \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^{n\times n})$ is a symmetric and locally uniformly positive definite matrix, V is a real potential in a certain local Morrey space (depending on p), and

$$|\xi|_A^2 := A(x)\xi \cdot \xi = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j, \quad x \in \Omega,\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

Our assumptions on the coefficients of the operator for $p \geq 2$ are the minimal (in the Morrey scale) that ensure the validity of the local Harnack inequality and hence the Hölder continuity of the solutions. For some of the results of the paper we need slightly stronger assumptions when $p < 2$.

We prove an Allegretto–Piepenbrink-type theorem for the operator $Q'_{A,p,V}$, and extend criticality theory to our setting. Moreover, we establish a Liouville-type theorem and obtain some perturbation results. Also, in the case $1 < p \leq n$, we examine the behaviour of a positive solution near a nonremovable isolated singularity and characterize the existence of the positive minimal Green function for the operator $Q'_{A,p,V}[u]$ in Ω.

1. Introduction

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$. The Allegretto–Piepenbrink (AP) theorem asserts that under some regularity assumptions on a real symmetric matrix A and a real potential V, the nonnegativity of the Dirichlet energy,

$$\int_\Omega \left(|\nabla u|_A^2 + V(x)|u|^2\right) \mathrm{d}x \geq 0 \quad \text{for all } u \in C_c^\infty(\Omega),$$

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is *equivalent* to the existence of a positive weak solution of the Schrödinger equation

$$-\operatorname{div}(A(x)\nabla v) + V(x)v = 0 \quad \text{in } \Omega, \tag{1-1}$$

where

$$|\xi|_A^2 := A(x)\xi\cdot\xi = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0 \quad \text{for all } x\in\Omega,\ \xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n. \tag{1-2}$$

After the original results in [Allegretto 1974; Piepenbrink 1974], a sequence of papers gradually relaxed the assumptions on A and V (see [Piepenbrink 1977; Moss and Piepenbrink 1978; Allegretto 1979; 1981]). It was established by Agmon [1983] that if $A \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n})$ is symmetric and locally uniformly positive definite in Ω, and $V \in L^q_{\mathrm{loc}}(\Omega)$ with $q > \frac{1}{2}n$, then the AP theorem holds true. If A is the identity matrix, further relaxation on the regularity of V is established in [Simon 1982, §C8], albeit some global condition on V^- is required there. We refer to [Lenz et al. 2009] and references therein for an up-to-date account.

A generalization of the AP theorem to certain quasilinear equations with A being the identity matrix and $V \in L^\infty_{\mathrm{loc}}(\Omega)$ has been carried out in [Pinchover and Tintarev 2007]. This was recently extended in [Pinchover and Regev 2015] to include Agmon's assumptions on the matrix A. More precisely, for $1<p<\infty$, A as above, and $V\in L^\infty_{\mathrm{loc}}(\Omega)$, the nonnegativity of the energy functional,

$$Q_{A,p,V}[u] := \int_\Omega \big(|\nabla u|_A^p + V(x)|u|^p\big)\,\mathrm{d}x \geq 0 \quad \text{for all } u\in C_c^\infty(\Omega), \tag{1-3}$$

is proved to be equivalent to the existence of a positive weak solution to the corresponding Euler–Lagrange quasilinear equation

$$Q'_{A,p,V}[u] := -\operatorname{div}\big(|\nabla v|_A^{p-2}A(x)\nabla v\big) + V(x)|v|^{p-2}v = 0 \quad \text{in } \Omega. \tag{1-4}$$

Clearly, the quasilinear equation (1-4) satisfies the homogeneity property of (1-1) but not the additivity (such an equation is sometimes called *half-linear*). Consequently, one expects that positive solutions of (1-4) would share some properties of positive solutions of (1-1).

An essential common implication of the various assumptions on A and V in the aforementioned results is the validity of the local Harnack inequality for positive solutions of (1-1) and (1-4). For instance, Agmon's assumption on V is optimal in the Lebesgue class of potentials for the Harnack inequality to be true. We stress that, when the Harnack inequality fails, the AP theorem might not be valid. Indeed, denote by $p' := p/(p-1)$ the conjugate index of p and suppose that A is the identity matrix. Let $V\in\mathcal{D}^{-1,p'}_{\mathrm{loc}}(\Omega)$, where $\mathcal{D}^{-1,p'}(\Omega)$ is the dual of $\mathcal{D}^{1,p}_0(\Omega)$, which is in turn defined as the closure of $C_c^\infty(\Omega)$ under the seminorm $\|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}$. If in addition to the nonnegativity of the energy functional one has that

$$\kappa\langle V, |u|^p\rangle \leq \int_\Omega |\nabla u|^p\,\mathrm{d}x \quad \text{for all } u\in C_c^\infty(\Omega)$$

for some positive constant κ, then the equation

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v) + \alpha V|v|^{p-2}v = 0 \quad \text{in } \Omega \tag{1-5}$$

admits a positive solution (in a certain weak sense) for any $\alpha \in (0, p^\sharp)$, where $p^\sharp < 1$ is given explicitly and depends only on p (see [Jaye et al. 2013, Theorem 1.2(i)], or [Jaye et al. 2012, Theorem 1.1(i)] for $p = 2$). Moreover, this range for α is optimal, as examples involving the Hardy potential reveal (see [Jaye et al. 2013, Remark 1.3], or [Jaye et al. 2012, Example 7.3] for $p = 2$). We note that under the above assumptions the local Harnack inequality for positive solutions of (1-5) is in general not valid.

The first aim of the present paper is to extend the AP theorem for the operator $Q'_{A,p,V}$ by relaxing significantly the condition $V \in L^\infty_{\text{loc}}(\Omega)$. In particular, under Agmon's (minimal) assumptions on the matrix A, we require V to lie in a certain local Morrey space, the largest such that the Harnack inequality for positive solutions (and hence the local Hölder continuity of solutions) holds true. This means that we assume (see for instance [Trudinger 1967, §5; Rakotoson and Ziemer 1990; Malý and Ziemer 1997] and also [Di Fazio 1988] for (1-1))

$$\sup_{\substack{y\in\omega \\ 0<r<\operatorname{diam}(\omega)}} \varphi_q(r) \int_{\omega\cap B_r(y)} |V| \,\mathrm{d}x < \infty \quad \text{for all } \omega \Subset \Omega, \tag{1-6}$$

where $\varphi_q(r)$ has the following behaviour near 0:

$$\varphi_q(r) \sim_{r\to 0} \begin{cases} r^{-n(q-1)/q} & \text{with } q > n/p \text{ if } p < n, \\ \log^{q(n-1)/n}(1/r) & \text{with } q > n \text{ if } p = n, \\ 1 & \text{if } p > n. \end{cases} \tag{1-7}$$

We prove, in addition, that the assertions of the AP theorem are equivalent to the existence of a weak solution $T \in L^{p'}_{\text{loc}}(\Omega; \mathbb{R}^n)$ of the first-order (nonlinear) divergence-type equation

$$-\operatorname{div}(AT) + (p-1)|T|_A^{p'} = V.$$

We refer to [Jaye et al. 2012, Theorem 1.3] for a related result, where A is the identity matrix and $p = 2$.

Recall that in general, functions in Morrey spaces cannot be approximated by functions in $C^\infty(\Omega)$, nor even by continuous functions (see [Zorko 1986]). Therefore, we cannot use an approximation argument to extend the AP theorem to our setting. Consequently, we need to start our study from the beginning of the topic and present in detail proofs involving new ideas.

Another aim of the paper is to extend to the above class of operators several classical results and tools that hold true in *general* bounded domains (see [Allegretto and Huang 1998; García-Melián and Sabina de Lis 1998; Pinchover and Regev 2015], where stronger regularity assumptions on the coefficients and the boundary are assumed). In particular, we prove the existence of the principal eigenvalue, establish its main properties, and study the relationships between the positivity of principal eigenvalue, the weak and strong maximum principles, and the (unique) solvability of the Dirichlet problem.

We then proceed to our main goal: establishing *criticality theory* for (1-4) with A and V satisfying the above assumptions. To present the main results of the paper, let us recall that if the inequality (1-3) holds true but cannot be improved, in the sense that one cannot add to its right-hand side a term of the form $\int_\Omega W|u|^p \,\mathrm{d}x$ with a nonnegative function $W \not\equiv 0$, then the nonnegative functional $Q_{A,p,V}$ is called *critical* in Ω. Furthermore, a sequence $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ is called a *null sequence* with respect to the nonnegative functional $Q_{A,p,V}$ in Ω if

(a) $u_k \geq 0$ for all $k \in \mathbb{N}$;

(b) there exists a fixed open set $K \Subset \Omega$ such that $\|u_k\|_{L^p(K)} = 1$ for all $k \in \mathbb{N}$;

(c) $\lim_{k\to\infty} Q_{A,p,V}[u_k] = 0$.

A positive function $\phi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a *ground state* of $Q_{A,p,V}$ in Ω if ϕ is an $L^p_{\mathrm{loc}}(\Omega)$ limit of a null sequence. Finally, a positive solution u of the equation $Q'_{A,p,V}[u] = 0$ in Ω is a *global minimal solution* if for any smooth compact subset K of Ω, and any positive supersolution $v \in C(\Omega \setminus \mathrm{int} K)$ of the equation $Q'_{A,p,V}[u] = 0$ in $\Omega \setminus K$, we have the implication

$$u \leq v \text{ on } \partial K \quad \Longrightarrow \quad u \leq v \text{ in } \Omega \setminus K.$$

The central result of this paper is summarized in the following theorem.

Main Theorem. *Let Ω be a domain in $\mathbb{R}^n$, where $n \geq 2$, and suppose that the functional $Q_{A,p,V}$ is nonnegative on $C_c^\infty(\Omega)$, where A is a symmetric and locally uniformly positive definite matrix in Ω, and*

$$\begin{cases} A \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^{n\times n}) \text{ and } V \text{ satisfies (1-6) with } \varphi_q \text{ as in (1-7)} & \text{if } p \geq 2, \\ A \in C^{0,\gamma}_{\mathrm{loc}}(\Omega; \mathbb{R}^{n\times n}),\ \gamma \in (0,1) \text{ and } V \text{ satisfies (1-6) with } \varphi_q \sim_{r\to 0} r^q, q > n & \text{if } p < 2. \end{cases}$$

Then the following assertions are equivalent:

(1) *$Q_{A,p,V}$ is critical in Ω.*

(2) *$Q_{A,p,V}$ admits a null sequence in Ω.*

(3) *There exists a ground state ϕ which is a positive weak solution of (1-4).*

(4) *There exists a unique* (*up to a multiplicative constant*) *positive supersolution v of (1-4) in Ω.*

(5) *There exists a global minimal solution u of (1-4) in Ω.*

In particular, $\phi = c_1 v = c_2 u$ for some positive constants c_1, c_2.

Moreover, if $1 < p \leq n$, then the above assertions are equivalent to

(6) *Equation (1-4) does not admit a positive minimal Green function.*

Remark 1.1. The additional regularity assumptions on A and V for the case $1 < p < 2$ in the Main Theorem seems to be technical, and might be nonessential. However, these assumptions guarantee the Lipschitz continuity of solutions of (1-4) (in fact they guarantee that solutions are $C^{1,\alpha}$; see [Lieberman 1993, Theorem 5.3]), a property which (as in [Pinchover and Tintarev 2007; Pinchover and Regev 2015]) is essential for the proof of the Main Theorem in this range of p. On the other hand, throughout the paper we do not use the boundary point lemma, which was an essential tool in [García-Melián and Sabina de Lis 1998; Pinchover and Tintarev 2007; Pinchover and Regev 2015].

The structure of the article is presented next. In Section 2A we define the local Morrey space of potentials V we are going to work with, and also present an uncertainty-type inequality for such potentials due to C. B. Morrey for $p = 2$, and D. R. Adams (see [Malý and Ziemer 1997, §1.3]) for $1 < p < \infty$, that holds true in this space. This is the key property that is used in [Malý and Ziemer 1997; Trudinger 1967] in order to extend Serrin's elliptic regularity theory [1964] for such equations. In Section 2C we recall several well-known local regularity and compactness properties of (sub/super)solutions of equation (1-4) found in [Malý and Ziemer 1997; Pucci and Serrin 2007].

In Section 3 we deal with bounded domains. Firstly, in Section 3A we establish some helpful lemmas, including the estimate (3-6) that extends to our case, a well-known inequality of P. Lindqvist [1990] proved for the p-Laplace equation and concerns the positivity of the corresponding I functional of Anane [1987] (see also [Díaz and Saá 1987]). We note that (3-6) replaces throughout our paper Picone's identity of Allegretto and Huang [1998]; a key tool in [Pinchover and Tintarev 2007; Pinchover and Regev 2015]. In addition, we prove in Section 3A the weak lower semicontinuity and the coercivity for two functionals related to the solvability of the Dirichlet problem in bounded domains. In Section 3B we use the results from Section 3A to prove the existence, simplicity and isolation of the principal eigenvalue λ_1 in a *general* bounded domain. Then we extend the main result in [García-Melián and Sabina de Lis 1998] concerning the equivalence of λ_1 being positive, the validity of the weak/strong maximum principle, and the existence of a unique positive solution for the Dirichlet problem

$$Q'_{A,p,V}[v] = g \ \text{ in } \ \omega, \quad v \in W_0^{1,p}(\omega), \quad \text{where } \ g \in L^{p'}(p;\omega) \ \text{ is nonnegative.}$$

In passing from local to global, the results in bounded domains of Section 3 are exploited in the last two sections. More precisely, in Section 4A we establish the AP theorem while in Section 4B we prove among other results the equivalence of the first four statements of the Main Theorem. In addition, we prove a Poincaré-type inequality for critical operators, and a Liouville comparison principle, generalizing results in [Pinchover and Tintarev 2007] and [Pinchover 2007; Pinchover et al. 2008], respectively (see also [Pinchover and Regev 2015]).

The last two statements of the Main Theorem are treated in Section 5C after establishing a suitable weak comparison principle (WCP) in Section 5A, and the behaviour of positive solutions near an isolated singularity in Section 5B.

We emphasize here, that generally speaking, we omit straightforward proofs that follow exactly the same steps as in the aforementioned papers, provided the needed tools have been obtained.

2. Preliminaries

In this section we fix our setting and notation, introduce some definitions, and review basic local regularity results of solutions of the equation (1-4).

Throughout the paper we assume that

- $1 < p < \infty$.
- Ω is a domain (an open and connected set) in $\mathbb{R}^n$, where $n \geq 2$.
- $A = (a_{ij}) \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^{n\times n})$ is a *symmetric* and *locally uniformly positive definite* matrix.

The assumptions on A imply in particular that

$$a_{ij}(x) = a_{ji}(x) \quad \text{for a.e. } x \in \Omega \text{ and } i, j = 1, \dots, n, \tag{S}$$

$$\forall \omega \Subset \Omega \ \exists \theta_\omega > 0 \qquad \theta_\omega |\xi| \leq |\xi|_A \leq \theta_\omega^{-1} |\xi| \quad \text{for a.e. } x \in \omega \text{ and all } \xi \in \mathbb{R}^n, \tag{E}$$

where we have set

$$|\xi|_A := \sqrt{A(x)\xi\cdot\xi} = \sqrt{\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j} \quad \text{for a.e. } x\in\Omega \text{ and } \xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n.$$

Moreover, we adopt the following notation:

- q' is the conjugate index of $q\in(1,\infty)$, i.e., $q'=q/(q-1)$.
- $\omega \Subset \Omega$ means ω is a subdomain of Ω with compact closure in Ω.
- $B_r(y) := \{x\in\mathbb{R}^n : |x-y|<r\}$, where $r>0$ and $y\in\mathbb{R}^n$.
- $\mathcal{L}^n(E)$ is the Lebesgue measure of a measurable set $E\subset\mathbb{R}^n$.
- $\langle f\rangle_\omega$ is the mean value of a function f in ω.
- $\operatorname{supp}\{f\}$ is the support of f.
- $f^+ := \max\{f,0\}$ and $f^- := -\min\{f,0\}$ are the positive and negative parts of f, respectively.
- γ and γ' will always stand for numbers in $(0,1)$.
- I_n is the identity matrix of size $n\times n$.
- $C(a,b,\dots)$ is a positive constant depending only on $a,b,\dots$, and may be different from line to line.

2A. ***Local Morrey spaces.*** In the present subsection we introduce a certain class of Morrey spaces that depend on the index p, where $1<p<\infty$. It is the class of spaces where the potential V of the operator $Q'_{A,p,V}$ belongs to.

Definition 2.1. Let $q\in[1,\infty]$ and $\omega\Subset\mathbb{R}^n$. For a measurable, real valued function f defined in ω, we set

$$\|f\|_{M^q(\omega)} := \sup_{\substack{y\in\omega\\ r<\operatorname{diam}(\omega)}} \frac{1}{r^{n/q'}}\int_{\omega\cap B_r(y)} |f|\,\mathrm{d}x.$$

We write then $f\in M^q_{\mathrm{loc}}(\Omega)$ if for any $\omega\Subset\Omega$ we have $\|f\|_{M^q(\omega)}<\infty$.

Remark 2.2. Note that $M^1_{\mathrm{loc}}(\Omega)\equiv L^1_{\mathrm{loc}}(\Omega)$ and $M^\infty_{\mathrm{loc}}(\Omega)\equiv L^\infty_{\mathrm{loc}}(\Omega)$, but $L^q_{\mathrm{loc}}(\Omega)\subsetneq M^q_{\mathrm{loc}}(\Omega)\subsetneq L^1_{\mathrm{loc}}(\Omega)$ for any $q\in(1,\infty)$.

For the regularity theory of equations with coefficients in Morrey spaces we refer to the monographs [Malý and Ziemer 1997; Morrey 1966], and also to [Rakotoson 1991; Byun and Palagachev 2013] for further regularity issues. For generalizations of the Morrey spaces and other applications to analysis and systems of equations we refer to [Peetre 1969; Adams and Xiao 2012; 2013].

Next we define a special local Morrey space $M^q_{\mathrm{loc}}(p;\Omega)$ which depends on the values of the exponent p.

Definition 2.3. For $p\neq n$, we define

$$M^q_{\mathrm{loc}}(p;\Omega) := \begin{cases} M^q_{\mathrm{loc}}(\Omega) & \text{with } q>n/p \text{ if } p<n,\\ L^1_{\mathrm{loc}}(\Omega) & \text{if } p>n,\end{cases}$$

while for $p = n$, $f \in M^q_{\mathrm{loc}}(n; \Omega)$ means that for some $q > n$ and any $\omega \Subset \Omega$ we have

$$\|f\|_{M^q(n;\omega)} := \sup_{\substack{y\in\omega \\ 0<r<\operatorname{diam}(\omega)}} \varphi_q(r) \int_{\omega\cap B_r(y)} |f|\,\mathrm{d}x < \infty,$$

where $\varphi_q(r) := \log(\operatorname{diam}(\omega)/r)^{q/n'}$ and $0 < r < \operatorname{diam}(\omega)$.

In what follows we will frequently use the following key fact (sometimes called an uncertainty-type inequality) originally due to Morrey and further generalized by Adams (see [Morrey 1966, Lemmas 5.2.1 and 5.4.2] for $p = 2$, [Trudinger 1967, Lemma 5.1] for $1 < p < n$, and [Rakotoson and Ziemer 1990; Malý and Ziemer 1997, Corollary 1.95]).

Theorem 2.4 (Morrey–Adams theorem). *Let $\omega \Subset \mathbb{R}^n$, and suppose that $V \in M^q(p; \omega)$.*

(i) *There exists a constant $C(n, p, q) > 0$ such that, for any $\delta > 0$ and all $u \in W^{1,p}_0(\omega)$,*

$$\int_\omega |V|\,|u|^p\,\mathrm{d}x \le \delta \|\nabla u\|^p_{L^p(\omega;\mathbb{R}^n)} + \frac{C(n, p, q)}{\delta^{n/(pq-n)}} \|V\|^{pq/(pq-n)}_{M^q(p;\omega)} \|u\|^p_{L^p(\omega)}. \tag{2-1}$$

(ii) *For any $\omega' \Subset \omega$ with Lipschitz boundary there exist positive constant $C(n, p, q, \omega', \omega)$ and δ_0 such that, for any $0 < \delta \le \delta_0$ and all $u \in W^{1,p}(\omega')$,*

$$\int_{\omega'} |V|\,|u|^p\,\mathrm{d}x \le \delta \|\nabla u\|^p_{L^p(\omega';\mathbb{R}^n)} + C(n, p, q, \delta, \|V\|_{M^q(p;\omega)}) \|u\|^p_{L^p(\omega')}.$$

Proof. (i) The case where $p \le n$ is contained in [Malý and Ziemer 1997]. In particular, for $p < n$ this follows from [Malý and Ziemer 1997, Corollary 1.95] (see also inequality (3.11) therein), while for $p = n$ one repeats that proof using Theorem 1.94 instead of Theorem 1.93 of that work. Thus, we only need to argue for $p > n$. In this case our assumption reads $V \in L^1(\omega)$. Recall also that by the Sobolev embedding theorem we have $W^{1,p}_0(\omega) \subset C(\omega)$. It follows that

$$\int_\omega |V|\,|u|^p\,\mathrm{d}x \le \|V\|_{L^1(\omega)} \|u\|^p_{L^\infty(\omega)} \le C(n, p) \|V\|_{L^1(\omega)} \|\nabla u\|^n_{L^p(\omega;\mathbb{R}^n)} \|u\|^{p-n}_{L^p(\omega)},$$

where we have used the Gagliardo–Nirenberg inequality (see for example [DiBenedetto 2002, §IX, Theorem 1.1]). The result follows by applying Young's inequality:

$$ab \le \delta a^{p/n} + \frac{p-n}{p} \left(\frac{n}{p\delta}\right)^{n/(p-n)} b^{p/(p-n)},$$

with $a = \|\nabla u\|^n_{L^p(\omega)}$, $b = C(n, p) \|V\|_{L^1(\omega)} \|u\|^{p-n}_{L^p(\omega)}$.

(ii) Let $\omega' \Subset \omega$ with $\partial\omega'$ being Lipschitz. We may then consider the extension operator (see for example [Evans and Gariepy 1992, §4.4])

$$E : W^{1,p}(\omega') \to W^{1,p}_0(\omega)$$

such that for any $u \in W^{1,p}(\omega')$ to have

$$\begin{cases} Eu = u & \text{in } \omega', \\ \|Eu\|_{L^p(\omega)} \le C(n, p, \omega', \omega)\|u\|_{L^p(\omega')}, \\ \|\nabla(Eu)\|_{L^p(\omega;\mathbb{R}^n)} \le C(n, p, \omega', \omega)\|u\|_{W^{1,p}(\omega';\mathbb{R}^n)}. \end{cases} \tag{2-2}$$

Thus, if $\delta > 0$ and $u \in W^{1,p}(\omega')$, it follows from (2-1) that

$$\int_\omega |V||Eu|^p \,\mathrm{d}x \le \delta\|\nabla(Eu)\|^p_{L^p(\omega;\mathbb{R}^n)} + \frac{C(n, p, q)}{\delta^{n/(pq-n)}}\|V\|^{pq/(pq-n)}_{M^q(p;\omega)}\|Eu\|^p_{L^p(\omega)}.$$

Applying (2-2) to the latter inequality yields (ii). □

2B. *Regularity assumptions on A and V.* We are now ready to introduce our regularity hypotheses on the coefficients of the operator $Q'_{A,p,V}$. Throughout the paper we assume that

$$\text{the matrix } A \text{ satisfies (S), (E), and the potential } V \text{ is in } M^q_{\mathrm{loc}}(p;\Omega). \tag{H0}$$

In the sequel, in the case $1 < p < 2$, we sometimes make the following stronger hypothesis:

$$A \in C^{0,\gamma}_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n}) \text{ satisfies (S), (E), and } V \in M^q_{\mathrm{loc}}(\Omega), \text{ where } q > n. \tag{H1}$$

2C. *The (p, A)-Laplacian with a potential term in $M^q_{\mathrm{loc}}(p;\Omega)$.* For a vector field $T \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ we define

$$\operatorname{div}_A T := \operatorname{div}(AT),$$

where $\operatorname{div}(AT)$ is meant in the distributional sense.

In this paper we are interested in the (p, A)-Laplacian equation plus a potential term, that is

$$Q'_{A,p,V}[v] := -\operatorname{div}_A(|\nabla v|^{p-2}_A \nabla v) + V|v|^{p-2}v = 0 \quad \text{in } \Omega. \tag{2-3}$$

This is the Euler–Lagrange equation associated with the functional

$$Q_{A,p,V}[u] := \int_\Omega (|\nabla u|^p_A + V|u|^p)\,\mathrm{d}x, \qquad u \in C^\infty_c(\Omega). \tag{2-4}$$

Definition 2.5. Assume that A and V satisfy (H0). A function $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *solution* of (2-3) in Ω if

$$\int_\Omega |\nabla v|^{p-2}_A A\nabla v\cdot\nabla u\,\mathrm{d}x + \int_\Omega V|v|^{p-2}vu\,\mathrm{d}x = 0 \quad \text{for all } u \in C^\infty_c(\Omega), \tag{2-5}$$

a *supersolution* of (2-3) in Ω if

$$\int_\Omega |\nabla v|^{p-2}_A A\nabla v\cdot\nabla u\,\mathrm{d}x + \int_\Omega V|v|^{p-2}vu\,\mathrm{d}x \ge 0 \quad \text{for all nonnegative } u \in C^\infty_c(\Omega), \tag{2-6}$$

and a subsolution if the reverse inequality holds. A *strict supersolution* of (2-3) in Ω is a supersolution which is not a solution.

Remark 2.6. The above definition makes sense because of condition (E), the Morrey–Adams theorem (Theorem 2.4), and Hölder's inequality. In light of our assumptions on A and V, and by a density argument, one can replace $C_c^\infty(\Omega)$ in Definition 2.5 by $W_c^{1,p}(\Omega)$, the space of all $L^p(\Omega)$ functions having compact support in Ω and first-order weak partial derivatives in $L^p(\Omega)$.

The following theorem follows from [Malý and Ziemer 1997, Theorem 3.14] for the case $p \le n$, and from [Pucci and Serrin 2007, Theorem 7.4.1] for the case $p > n$.

Theorem 2.7 (Harnack inequality). *Under hypothesis (H0), any nonnegative solution v of (2-3) in Ω satisfies the local Harnack inequality. Namely, for any $\omega' \Subset \omega \Subset \Omega$ there holds*

$$\sup_{\omega'} v \le C \inf_{\omega'} v, \tag{2-7}$$

where C is a positive constant depending only on n, p, q, $\operatorname{dist}(\omega', \omega)$, θ_ω, and $\|V\|_{M^q(\omega)}$ (and not on v).

Remark 2.8 (local Hölder continuity). A standard consequence of Theorem 2.7 is the following regularity assertion, found in [Malý and Ziemer 1997, Theorem 4.11] for $p \le n$ and in [Pucci and Serrin 2007, Theorem 7.4.1] for $p > n$:

> *Under hypothesis (H0), any solution v of (2-3) in Ω is locally Hölder continuous of order γ (depending on n, p, q, and θ_ω), and for any $\omega' \Subset \omega \Subset \Omega$, we have*
>
> $$[v]_{\gamma,\omega'} \le C \sup_{\omega} |v|, \tag{2-8}$$
>
> *where C is a positive constant depending only on n, p, q, $\operatorname{dist}(\omega', \omega)$, θ_ω, and $\|V\|_{M^q(\omega)}$. Here $[v]_{\gamma,\omega'}$ is the Hölder seminorm of v in ω'.*

Remark 2.9 (local Lipschitz continuity). Later on, when proving Theorem 4.12 for $p < 2$, we will need conditions under which the local Lipschitz continuity of solutions is guaranteed. In other words, in the case $p < 2$ we will need conditions that ensure the local boundedness of the modulus of the gradient of a solution of (2-3). This and more are provided by [Lieberman 1993, Theorem 5.3]:

> *Under hypothesis (H1), any solution v of (2-3) in Ω is of class $C^{1,\gamma'}_{\mathrm{loc}}(\Omega)$ for some $\gamma' \in (0, 1)$ depending only on n, p, γ, q and θ_ω.*

In particular, we will use the fact that, whenever $\omega' \Subset \omega \Subset \Omega$,

$$\sup_{\omega'} |\nabla v| \le C \sup_{\omega} |v|$$

for some positive constant C, depending only on n, p, γ, q, $\operatorname{dist}(\omega', \omega)$, θ_ω, $\|A\|_{C^{0,\gamma}(\omega)}$, and $\|V\|_{M^q(\omega)}$.

Remark 2.10 (weak Harnack inequality). For $p > n$, Theorem 2.7 holds true verbatim if v is merely a nonnegative supersolution of (2-3) in Ω (see [Pucci and Serrin 2007, Theorem 7.4.1]). For $p \le n$ we only have [Malý and Ziemer 1997, Theorem 3.13]:

Let $p \le n$ and set $s = n(p-1)/(n-p)$. Under hypothesis (H0), any nonnegative supersolution v of (2-3) in Ω satisfies the weak Harnack inequality, namely, for any $\omega' \Subset \omega \Subset \Omega$ and $0 < t < s$,

$$\|v\|_{L^t(\omega')} \le C \inf_{\omega'} v, \tag{2-9}$$

where C is a positive constant depending only on n, p, t, $\operatorname{dist}(\omega', \omega)$, $\mathcal{L}^n(\omega')$ and $\|V\|_{M^q(\omega)}$.

We conclude the section with the following important result that will be used several times throughout the paper.

Proposition 2.11 (Harnack convergence principle). *Consider a matrix $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ which satisfies conditions (S) and (E). Let $\{\omega_i\}_{i\in\mathbb{N}}$ be a sequence of Lipschitz domains such that $\omega_i \Subset \Omega$, $\omega_i \Subset \omega_{i+1}$ for $i \in \mathbb{N}$, and $\bigcup_{i\in\mathbb{N}} \omega_i = \Omega$, and fix a reference point $x_0 \in \omega_1$. Assume also that $\{\mathcal{V}_i\}_{i\in\mathbb{N}} \subset M^q(p; \omega_i)$ converges in $M^q_{\mathrm{loc}}(p; \Omega)$ to $\mathcal{V} \in M^q_{\mathrm{loc}}(p; \Omega)$. For each $i \in \mathbb{N}$, let v_i be a positive solution of the equation $Q'_{A,p,\mathcal{V}_i}[v] = 0$ in ω_i such that $v_i(x_0) = 1$.*

Then there exists $0 < \beta < 1$ such that, up to a subsequence, $\{v_i\}$ converges in $C^{0,\beta}_{\mathrm{loc}}(\Omega)$ to a positive solution v of the equation $Q'_{A,p,\mathcal{V}}[v] = 0$ in Ω.

Proof. The convergence in $C^{0,\beta}_{\mathrm{loc}}(\Omega)$ follows by the Arzelà–Ascoli theorem from the local Harnack inequality (2-7) and the local Hölder estimate (2-8).

Now pick an arbitrary $\omega \Subset \Omega$. We will show that a subsequence of $\{v_i\}_{i\in\mathbb{N}}$ converges weakly in $W^{1,p}(\omega)$ to a positive solution of $Q'_{A,p,\mathcal{V}}[u] = 0$ in Ω. Recall first that the definition of v_i being a positive weak solution to $Q'_{A,p,\mathcal{V}_i}[v] = 0$ in ω_i reads as

$$\int_{\omega_i} |\nabla v_i|_A^{p-2} A\nabla v_i \cdot \nabla u \,\mathrm{d}x + \int_{\omega_i} \mathcal{V}_i v_i^{p-1} u \,\mathrm{d}x = 0 \quad \text{for all } u \in W^{1,p}_0(\omega_i). \tag{2-10}$$

By Remark 2.8, v_i is also continuous for all $i \in \mathbb{N}$. Fix $k \in \mathbb{N}$. For $u \in C^\infty_c(\omega_k)$ we may thus pick $v_i|u|^p \in W^{1,p}_c(\omega_k)$. $i \ge k$, as a test function in (2-10) to get

$$\big\| |\nabla v_i|_A u \big\|^p_{L^p(\omega_k)} \le p\int_{\omega_k} |\nabla v_i|_A^{p-1} |u|^{p-1} v_i |\nabla u|_A \,\mathrm{d}x + \int_{\omega_k} |\mathcal{V}_i| v_i^p |u|^p \,\mathrm{d}x.$$

On the first term of the right-hand side we apply Young's inequality: $pab \le \varepsilon a^{p'} + [(p-1)/\varepsilon]^{p-1} b^p$, $\varepsilon \in (0,1)$, with $a = |\nabla v_i|_A^{p-1} |u|^{p-1}$ and $b = v_i |\nabla u|_A$. On the second term we apply the Morrey–Adams theorem (Theorem 2.4). We arrive at

$$\begin{aligned}(1-\varepsilon)\big\| |\nabla v_i|_A u \big\|^p_{L^p(\omega_k)} \\ \le ((p-1)/\varepsilon)^{p-1} \big\| v_i |\nabla u|_A \big\|^p_{L^p(\omega_k)} + \delta \|\nabla(v_i u)\|^p_{L^p(\omega_k;\mathbb{R}^n)} + C(n,p,q,\delta,\|\mathcal{V}\|_{M^q(p;\omega_{k+1})}) \|v_i u\|^p_{L^p(\omega_k)}.\end{aligned}$$

By (E) and the simple fact that

$$\|\nabla(v_i u)\|^p_{L^p(\omega_k;\mathbb{R}^n)} \le 2^{p-1}\big(\|v_i \nabla u\|^p_{L^p(\omega_k;\mathbb{R}^n)} + \|u \nabla v_i\|^p_{L^p(\omega_k;\mathbb{R}^n)}\big),$$

we end up with the following Caccioppoli estimate valid for all $i \ge k$ and any $u \in C_c^\infty(\omega_k)$:

$$\begin{aligned}((1-\varepsilon)\theta_{\omega_k}^{p} - 2^{p-1}\delta\theta_{\omega_k}^{-p})\big\| |\nabla v_i| u\big\|_{L^p(\omega_k)}^p \\ \le \big(((p-1)/\varepsilon)^{p-1}\theta_{\omega_k}^{-p} + 2^{p-1}\delta\big)\big\| v_i |\nabla u|\big\|_{L^p(\omega_k)}^p + C(n, p, q, \delta, \|\mathcal{V}\|_{M^q(p;\omega_{k+1})})\|v_i u\|_{L^p(\omega_k)}^p. \end{aligned} \tag{2-11}$$

Without loss of generality we assume that ω contains x_0. Picking $\omega' \Subset \Omega$ such that $\omega \subset \omega'$, we find $k \ge 1$ such that $\omega' \subset \omega_k$. Next we choose $\delta < (1-\varepsilon)2^{1-p}\theta_{\omega_k}^{2p}$ and specialize $u \in C_c^\infty(\omega_k)$ such that

$$\operatorname{supp}\{u\} \subset \omega', \quad 0 \le u \le 1 \text{ in } \omega', \quad u = 1 \text{ in } \omega \quad \text{and} \quad |\nabla u| \le 1/\operatorname{dist}(\omega', \omega) \text{ in } \omega. \tag{2-12}$$

Applying this to the Caccioppoli inequality (2-11), and using the fact that $\{v_i\}_{i\in\mathbb{N}}$ is bounded in the $L^\infty(\omega)$-norm uniformly in i (due to the local Harnack's inequality (2-7)), we conclude

$$\|\nabla v_i\|_{L^p(\omega;\mathbb{R}^n)}^p + \|v_i\|_{L^p(\omega)}^p \le C(n, p, q, \varepsilon, \delta, \operatorname{dist}(\omega', \omega), \theta_{\omega_k}, \|\mathcal{V}\|_{M^q(p;\omega_{k+1})}) \quad \text{for all } i \ge k.$$

So $\{v_i\}_{i\in\mathbb{N}}$ is bounded in $W^{1,p}(\omega)$. By weak compactness of $W^{1,p}(\omega)$, there exists a subsequence, still denoted by $\{v_i\}_{i\in\mathbb{N}}$, that converges weakly in $W^{1,p}(\omega)$ to a nonnegative function v with $v(x_0) = 1$.

Next we show that v is a solution of $Q'_{A,p,\mathcal{V}}[u] = 0$ in $\widetilde{\omega} \Subset \omega$ such that $x_0 \in \widetilde{\omega}$. First note that for a subsequence (that once more we do not rename) we have $v_i \to v$ a.e. in ω and in $L^p(\omega)$. For the potential term of the equation we note first that (up to a subsequence) $\mathcal{V}_i \to \mathcal{V}$ a.e. in ω. Thus, $\mathcal{V}_i v_i^{p-1} \to \mathcal{V} v^{p-1}$ a.e. in ω, while $|\mathcal{V}_i v_i^{p-1}| \le c|\mathcal{V}|$ a.e. in ω, where c is independent of i. Since $|\mathcal{V}| \in M^q_{\mathrm{loc}}(p; \Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ we may apply the dominated convergence theorem to get

$$\int_\omega \mathcal{V}_i v_i^{p-1} u \,dx \to \int_\omega \mathcal{V} v^{p-1} u \,dx \quad \text{for all } u \in C_c^\infty(\omega). \tag{2-13}$$

It remains to prove that

$$\xi_i := |\nabla v_i|_A^{p-2} A\nabla v_i \rightharpoonup_{i\to\infty} |\nabla v|_A^{p-2} A\nabla v =: \xi \quad \text{in } L^{p'}(\widetilde{\omega}; \mathbb{R}^n). \tag{2-14}$$

To this end, letting u be as in (2-12) but with ω and ω' replaced by $\widetilde{\omega}$ and ω respectively, we take $u(v_i - v)$ as a test function in (2-10) to obtain

$$\int_\omega u\xi_i \cdot \nabla(v_i - v)\,dx = -\int_\omega (v_i - v)\xi_i \nabla u \,dx - \int_\omega \mathcal{V}_i v_i^{p-1} u(v_i - v)\,dx. \tag{2-15}$$

We claim that

$$\int_\omega u\xi_i \cdot \nabla(v_i - v)\,dx \to_{i\to\infty} 0. \tag{2-16}$$

Indeed, by an argument similar to the one leading to (2-13), the second integral on the right of (2-15) converges to 0 as $i \to \infty$. For the first one, apply Holder's inequality to get

$$\begin{aligned}\left| -\int_\omega (v_i - v)\xi_i \nabla u \,dx \right| &\le \theta_\omega^{p/p'} \|(v_i - v)\nabla u\|_{L^p(\omega;\mathbb{R}^n)} \|\nabla v_i\|_{L^p(\omega;\mathbb{R}^n)}^{p/p'} \\ &\le C(p, \theta_\omega, \operatorname{dist}(\widetilde{\omega}, \omega))\|v_i - v\|_{L^p(\omega)} \|\nabla v_i\|_{L^p(\omega;\mathbb{R}^n)}^{p/p'},\end{aligned}$$

which also converges to 0 as $i \to \infty$ since the $\|\nabla v_i\|_{L^p(\omega;\mathbb{R}^n)}$ are uniformly bounded and $v_i \to v$ in $L^p(\omega)$.

Notice that, as in the case where $A = I_n$, we have, for any $X, Y \in \mathbb{R}^n$, $n \geq 1$,

$$\begin{aligned}(|X|_A^{p-2}AX - |Y|_A^{p-2}AY)\cdot(X-Y) &= |X|_A^p - |X|_A^{p-2}AX\cdot Y + |Y|_A^p - |Y|_A^{p-2}AY\cdot X \\ &\geq |X|_A^p - |X|_A^{p-1}|Y|_A + |Y|_A^p - |Y|_A^{p-1}|X|_A \\ &= (|X|_A^{p-1} - |Y|_A^{p-1})(|X|_A - |Y|_A) \geq 0. \end{aligned} \tag{2-17}$$

The above considerations imply that

$$0 \leq \mathcal{I}_i := \int_{\widetilde{\omega}} (\xi_i - \xi)\cdot\nabla(v_i - v)\,\mathrm{d}x \leq \int_\omega u(\xi_i - \xi)\cdot\nabla(v_i - v)\,\mathrm{d}x \to_{i\to\infty} 0,$$

where we have used (2-16) and the weak convergence in $L^{p'}(\omega;\mathbb{R}^n)$ of ∇v_i to ∇v. Thus $\lim_{i\to\infty}\mathcal{I}_i = 0$ and invoking a celebrated lemma of Maz'ya [1970] (see also Lemma 3.73 of [Heinonen et al. 1993]), (2-14) follows.

Hence, using Harnack's inequality, we have that v is a positive weak solution of $Q'_{A,p,\mathcal{V}}[u] = 0$ in $\widetilde{\omega}$ with $v(x_0) = 1$. We now use a standard Harnack chain argument and a diagonalization procedure to obtain a new subsequence (once again not renamed) $\{v_i\}_{i\in\mathbb{N}}$ such that $v_i \rightharpoonup v$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$ (and locally uniformly in Ω), where v is a positive weak solution of $Q'_{A,p,\mathcal{V}}[u] = 0$ in Ω. □

3. Principal eigenvalue and the maximum principle

Throughout the present section we fix a bounded domain ω in $\mathbb{R}^n$ and suppose that A is a uniformly elliptic, bounded matrix in ω and $V \in M^q(p;\omega)$. We consider, in ω, the operator $Q'_{A,p,V}$ defined in (2-3) and, for $u \in C_c^\infty(\omega)$, we let

$$Q_{A,p,V}[u;\omega] := \int_\omega \big(|\nabla u|_A^p + V(x)|u|^p\big)\,\mathrm{d}x.$$

Definition 3.1. We say that $\lambda \in \mathbb{R}$ is an *eigenvalue with an eigenfunction* v of the Dirichlet eigenvalue problem

$$\begin{cases} Q'_{A,p,V}[w] = \lambda|w|^{p-2}w & \text{in } \omega, \\ w = 0 & \text{on } \partial\omega, \end{cases} \tag{3-1}$$

if $v \in W_0^{1,p}(\omega)\setminus\{0\}$ satisfies

$$\int_\omega |\nabla v|_A^{p-2}A\nabla v\cdot\nabla u\,\mathrm{d}x + \int_\omega V|v|^{p-2}vu\,\mathrm{d}x = \lambda\int_\omega |v|^{p-2}vu\,\mathrm{d}x \quad \text{for all } u \in C_c^\infty(\omega). \tag{3-2}$$

Definition 3.2. A *principal eigenvalue* is an eigenvalue of (3-1) with a nonnegative eigenfunction.

The existence of a principal eigenvalue for the problem (3-1) and its variational characterization by the Rayleigh–Ritz variational formula

$$\lambda_1 = \lambda_1(Q_{A,p,V};\omega) := \inf_{u\in W_0^{1,p}(\omega)\setminus\{0\}} \frac{Q_{A,p,V}[u;\omega]}{\|u\|^p_{L^p(\omega)}} \tag{3-3}$$

is established in Theorem 3.9 below.

Consider first the equation

$$Q'_{A,p,V}[v] = g \quad \text{in } \omega, \qquad \text{where } g \in M^q(p;\omega) \text{ is nonnegative.} \tag{3-4}$$

By a solution of (3-4) we mean a function $v \in W^{1,p}_{\mathrm{loc}}(\omega)$ such that

$$\int_\omega |\nabla v|_A^{p-2} A\nabla v \cdot \nabla u \,\mathrm{d}x + \int_\omega V|v|^{p-2} vu \,\mathrm{d}x = \int_\omega gu \,\mathrm{d}x \quad \text{for all } u \in C_c^\infty(\omega).$$

A function $v \in W^{1,p}_{\mathrm{loc}}(\omega)$ is a supersolution of (3-4) if

$$\int_\omega |\nabla v|_A^{p-2} A\nabla v \cdot \nabla u \,\mathrm{d}x + \int_\omega V|v|^{p-2} vu \,\mathrm{d}x \geq \int_\omega gu \,\mathrm{d}x \quad \text{for all nonnegative } u \in C_c^\infty(\omega),$$

and a subsolution if the reverse inequality holds. One of our targets in the next subsection is to characterize, in terms of the strict positivity of the principal eigenvalue of problem (3-1), the following properties:

(a) the solvability in $W^{1,p}_0(\omega)$ of (3-4);

(b) the (generalized) weak maximum principle for (3-4);

(c) the strong maximum principle for (3-4).

Recall at this point that the (*generalized*) *weak maximum principle* for the operator $Q'_{A,p,V}$ asserts that a solution of (3-4) which is nonnegative on $\partial\omega$ is nonnegative in ω, while the *strong maximum principle* asserts that, in addition to the weak maximum principle, a solution of (3-4) which is nonnegative on $\partial\omega$ is either identically zero or strictly positive in ω.

3A. *Preparatory material.* We start with the following technical lemma, which generalizes computations found in [Anane 1987; Díaz and Saá 1987; Lindqvist 1990], where the case $V_1 = V_2 \equiv 0$ and $A = I_n$ is considered. This useful lemma replaces Picone's identity, which is a key tool in [Pinchover and Tintarev 2007; Pinchover and Regev 2015]. We note that in the present paper the lemma is used only for the case $V_1 = V_2$, but this assumption does not affect at all the volume of computations of the general case.

Lemma 3.3. *Let $g_i, V_i \in M^q(p;\omega)$, where $i = 1, 2$. There exists a positive constant c_p, depending only on p, such that the following assertions hold true*:

(i) *Suppose that $w_1, w_2 \in W^{1,p}_0(\omega) \setminus \{0\}$ are nonnegative solutions of*

$$Q'_{A,p,V_1}[w;\omega] = g_1 \quad \textit{and} \quad Q'_{A,p,V_2}[w;\omega] = g_2, \tag{3-5}$$

respectively, and let $w_{i,h} := w_i + h$, where h is a positive constant, and $i = 1, 2$. Then

$$\begin{aligned} I_h &:= \int_\omega \left(\frac{g_1 - V_1 w_1^{p-1}}{w_{1,h}^{p-1}} - \frac{g_2 - V_2 w_2^{p-1}}{w_{2,h}^{p-1}} \right) (w_{1,h}^p - w_{2,h}^p) \,\mathrm{d}x \\ &\geq c_p \begin{cases} \displaystyle\int_\omega (w_{1,h}^p + w_{2,h}^p) \left| \nabla \log \frac{w_{1,h}}{w_{2,h}} \right|_A^p \mathrm{d}x & \textit{if } p \geq 2, \\ \displaystyle\int_\omega (w_{1,h}^p + w_{2,h}^p) \left| \nabla \log \frac{w_{1,h}}{w_{2,h}} \right|_A^2 \big(|\nabla \log w_{1,h}|_A + |\nabla \log w_{2,h}|_A \big)^{p-2} \mathrm{d}x & \textit{if } p < 2. \end{cases} \end{aligned} \tag{3-6}$$

(ii) *In the particular case of nonnegative eigenfunctions, i.e.*,

$$w_1 := w_\lambda, \quad w_2 := w_\mu, \quad g_1 := \lambda |w_\lambda|^{p-2} w_\lambda, \quad g_2 = \mu |w_\mu|^{p-2} w_\mu,$$

with $\lambda, \mu \in \mathbb{R}$, *we have*

$$\int_\omega ((\lambda-\mu)-(V_1-V_2))(w_\lambda^p - w_\mu^p)\,\mathrm{d}x \\ \geq c_p \begin{cases} \displaystyle\int_\omega (w_\lambda^p + w_\mu^p)\left|\nabla \log \frac{w_\lambda}{w_\mu}\right|_A^p \mathrm{d}x & \textit{if } p \geq 2,\\ \displaystyle\int_\omega (w_\lambda^p + w_\mu^p)\left|\nabla \log \frac{w_\lambda}{w_\mu}\right|_A^2 \big(|\nabla \log w_\lambda|_A + |\nabla \log w_\mu|_A\big)^{p-2} \mathrm{d}x & \textit{if } p < 2. \end{cases}$$

(iii) *Suppose further that* ω *is Lipschitz, and let* $w_1, w_2 \in W^{1,p}(\omega)$ *be positive solutions of* (3-5) *such that* $w_1 = w_2 > 0$ *on* $\partial\omega$, *in the trace sense. Then*

$$\int_\omega \left(\frac{g_1}{w_1^{p-1}} - \frac{g_2}{w_2^{p-1}} - (V_1-V_2)\right)(w_1^p - w_2^p)\,\mathrm{d}x \\ \geq c_p \begin{cases} \displaystyle\int_\omega (w_1^p + w_2^p)\left|\nabla \log \frac{w_1}{w_2}\right|_A^p \mathrm{d}x & \textit{if } p \geq 2,\\ \displaystyle\int_\omega (w_1^p + w_2^p)\left|\nabla \log \frac{w_1}{w_2}\right|_A^2 \big(|\nabla \log w_1|_A + |\nabla \log w_2|_A\big)^{p-2} \mathrm{d}x & \textit{if } p < 2. \end{cases}$$

Proof. Set $\psi_{1,h} := (w_{1,h}^p - w_{2,h}^p)w_{1,h}^{1-p}$. It is easily seen that $\psi_{1,h} \in W_0^{1,p}(\omega)$ and, using it as a test function in the definition of w_1 being a solution of the first equation of (3-5), we get

$$\int_\omega (w_{1,h}^p - w_{2,h}^p)|\nabla(\log w_{1,h})|_A^p\,\mathrm{d}x - p\int_\omega w_{2,h}^p |\nabla(\log w_{1,h})|_A^{p-2} A\nabla(\log w_{1,h})\cdot\nabla\left(\log\frac{w_{2,h}}{w_{1,h}}\right)\mathrm{d}x \\ = \int_\omega \frac{g_1 - V_1 w_1^{p-1}}{w_{1,h}^{p-1}}(w_{1,h}^p - w_{2,h}^p)\,\mathrm{d}x.$$

In the same fashion we set $\psi_{2,h} := (w_{2,h}^p - w_{1,h}^p)w_{2,h}^{1-p}$ and use it as a test function in the definition of w_2 being a solution of the second equation of (3-5) to obtain

$$\int_\omega (w_{2,h}^p - w_{1,h}^p)|\nabla(\log w_{2,h})|_A^p\,\mathrm{d}x - p\int_\omega w_{1,h}^p |\nabla(\log w_{2,h})|_A^{p-2} A\nabla(\log w_{2,h})\cdot\nabla\left(\log\frac{w_{1,h}}{w_{2,h}}\right)\mathrm{d}x \\ = \int_\omega \frac{g_2 - V_2 w_2^{p-1}}{w_{2,h}^{p-1}}(w_{2,h}^p - w_{1,h}^p)\,\mathrm{d}x.$$

Adding these we arrive at

$$\int_\omega w_{1,h}^p\left(|\nabla(\log w_{1,h})|_A^p - |\nabla(\log w_{2,h})|_A^p - p|\nabla(\log w_{2,h})|_A^{p-2}A\nabla(\log w_{2,h})\cdot\nabla\left(\log\frac{w_{1,h}}{w_{2,h}}\right)\right)\mathrm{d}x \\ + \int_\omega w_{2,h}^p\bigg(|\nabla(\log w_{2,h})|_A^p - |\nabla(\log w_{1,h})|_A^p \\ - p|\nabla(\log w_{1,h})|_A^{p-2}A\nabla(\log w_{1,h})\cdot\nabla\left(\log\frac{w_{2,h}}{w_{1,h}}\right)\bigg)\mathrm{d}x = I_h. \quad (3\text{-}7)$$

Now we use the following inequality, found in [Lindqvist 1990, Lemma 4.2] for A being the identity matrix I_n; cf. [Pinchover et al. 2008, (2.19)] (the proof is essentially the same and we omit it):

For all vectors $\alpha, \beta \in \mathbb{R}^n$ and a.e. $x \in \omega$, we have

$$|\alpha|_A^p - |\beta|_A^p - p|\beta|_A^{p-2} A(x)\beta \cdot (\alpha - \beta) \geq C(p) \begin{cases} |\alpha - \beta|_A^p & \text{if } p \geq 2, \\ |\alpha - \beta|_A^2 (|\alpha|_A + |\beta|_A)^{p-2} & \text{if } p < 2. \end{cases} \tag{3-8}$$

Applying this to both terms of the left-hand side of (3-7), we obtain the inequality of part (i).

To prove part (ii), take $g_1 = \lambda |w_1|^{p-2} w_1$ and $g_2 = \mu |w_2|^{p-2} w_2$ for some $\lambda, \mu \in \mathbb{R}$, and rename w_1 and w_2 to w_λ and w_μ, respectively. The integrand of I_h in this case satisfies, for all $0 < h < 1$,

$$\left| \left((\lambda - V_1) \left(\frac{w_\lambda}{w_{\lambda,h}} \right)^{p-1} - (\mu - V_2) \left(\frac{w_\mu}{w_{\mu,h}} \right)^{p-1} \right) (w_{\lambda,h}^p - w_{\mu,h}^p) \right| \\ \leq (|\lambda - V_1| + |\mu - V_2|)((w_\lambda + 1)^p + (w_\mu + 1)^p) \in L^1(\omega),$$

by Theorem 2.4(i). As $h \to 0$, we have

$$\left((\lambda - V_1) \left(\frac{w_\lambda}{w_{\lambda,h}} \right)^{p-1} - (\mu - V_2) \left(\frac{w_\mu}{w_{\mu,h}} \right)^{p-1} \right) (w_{\lambda,h}^p - w_{\mu,h}^p) \to (\lambda - \mu - V_1 + V_2)(w_\lambda^p - w_\mu^p)$$

a.e. in ω. By applying the dominated convergence theorem and the Fatou lemma to the inequality of part (i), we get the desired estimate. Part (iii) follows from part (i) by setting $h = 0$. □

We modify to our case a well-known lemma on the negative part of a supersolution (see [Agmon 1983, Lemma 2.7] or [Pinchover et al. 2008, Lemma 2.4], for example).

Lemma 3.4. *Let $\mathcal{V} \in M_{\mathrm{loc}}^q(p; \Omega)$. If $v \in W_{\mathrm{loc}}^{1,p}(\Omega)$ is a supersolution of $Q'_{A,p,\mathcal{V}}[u] = 0$ in Ω, then v^- is a $W_{\mathrm{loc}}^{1,p}(\Omega)$ subsolution of the same equation.*

Proof. Though this argument is quite standard, we add it for completeness and since it requires the use of the Morrey–Adams theorem in the final limit argument. Following the steps of the proof in [Agmon 1983], we define

$$\varphi_\varepsilon := \frac{v_\varepsilon - v}{2v_\varepsilon}\varphi \quad \text{and} \quad v_\varepsilon := (v^2 + \varepsilon^2)^{1/2},$$

with φ being an arbitrary nonnegative function in $C_c^\infty(\Omega)$. It is straightforward to see that

$$\nabla v_\varepsilon \cdot \nabla \varphi \leq \nabla v \cdot \nabla \left(\frac{v}{v_\varepsilon} \varphi \right) \quad \text{a.e. in } \Omega,$$

and then

$$\tfrac{1}{2} \nabla (v_\varepsilon - v) \cdot \nabla \varphi \leq -\nabla v_\varepsilon \cdot \nabla \varphi_\varepsilon \quad \text{a.e. in } \Omega. \tag{3-9}$$

Thus, taking $\varphi_\varepsilon \in W_c^{1,p}(\Omega)$ as a test function in the definition of $v \in W_{\mathrm{loc}}^{1,p}(\Omega)$ being a supersolution of $Q'_{A,p,\mathcal{V}}[u] = 0$ in Ω, and then applying (3-9), we conclude that we only need to show that we can take the limit $\varepsilon \to 0$ in the inequality

$$\frac{1}{2} \int_\Omega |\nabla v|_A^{p-2} A \nabla (v_\varepsilon - v) \cdot \nabla \varphi \, \mathrm{d}x - \int_\Omega \mathcal{V} |v|^{p-2} v \varphi_\varepsilon \, \mathrm{d}x \leq 0. \tag{3-10}$$

Note that, since $\nabla(v_\varepsilon - v)/2 \to \nabla v^-$ and $v\varphi_\varepsilon \to -v^-\varphi$ as $\varepsilon \to 0$, this would readily give

$$\int_\Omega |\nabla v^-|_A^{p-2} A\nabla v^- \cdot \nabla\varphi \,\mathrm{d}x + \int_\Omega \mathcal{V}|v^-|^{p-2}v^-\varphi \,\mathrm{d}x \le 0 \quad \text{for all nonnegative } \varphi \in C_c^\infty(\Omega).$$

However, the justification of taking the limit inside both integrals in (3-10) is verified by the dominated convergence theorem. For the first one we use Hölder's inequality, while for the second we apply first Hölder's inequality and then the Morrey–Adams theorem. □

Definition 3.5. Let $(X, \|\cdot\|_X)$ be a Banach space. A functional $J : X \to \mathbb{R} \cup \{\infty\}$ is said to be *coercive* if $J[u] \to \infty$ as $\|u\|_X \to \infty$. The functional J is said to be (*sequentially*) *weakly lower semicontinuous* if $J[u] \le \liminf_{k\to\infty} J[u_k]$ whenever $u_k \rightharpoonup u$.

We have:

Proposition 3.6. (a) *Let $\omega \Subset \mathbb{R}^n$, $\mathcal{V} \in M^q(p;\omega)$ and $\mathcal{G} \in L^{p'}(\omega)$. Define the functional*

$$J : W_0^{1,p}(\omega) \to \mathbb{R} \cup \{\infty\}, \qquad J[u] := Q_{A,p,\mathcal{V}}[u;\omega] - \int_\omega \mathcal{G}u \,\mathrm{d}x. \tag{3-11}$$

Then J is weakly lower semicontinuous in $W_0^{1,p}(\omega)$.

(b) *Let $\omega \Subset \omega' \Subset \mathbb{R}^n$ with ω Lipschitz, and let $\mathcal{G}, \mathcal{V} \in M^q(p;\omega')$. Define the functional*

$$\bar{J} : W^{1,p}(\omega) \to \mathbb{R} \cup \{\infty\}, \qquad \bar{J}[u] := Q_{A,p,\mathcal{V}}[u;\omega] - \int_\omega \mathcal{G}|u| \,\mathrm{d}x. \tag{3-12}$$

Then $\bar{J}$ is weakly lower semicontinuous in $W^{1,p}(\omega)$.

Proof. We first prove statement (b). Let $u, \{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\omega)$ be such that $u_k \rightharpoonup u$ in $W^{1,p}(\omega)$. By the uniform boundedness principle, we have

$$K := \sup_{k\in\mathbb{N}} \|u_k\|_{W^{1,p}(\omega)} < \infty,$$

and thus, by the compact embedding of $W^{1,p}(\omega)$ in $L^p(\omega)$, and by passing to a subsequence we may assume that $u_k \to u$ in $L^p(\omega)$ and a.e. in ω.

Let $\delta > 0$. By Minkowski's inequality and the Morrey–Adams theorem (Theorem 2.4(ii)), we have

$$\begin{aligned}
\left(\int_\omega \mathcal{V}^\pm |u_k|^p \,\mathrm{d}x\right)^{1/p} &- \left(\int_\omega \mathcal{V}^\pm |u|^p \,\mathrm{d}x\right)^{1/p} \\
&\le \left(\int_\omega \mathcal{V}^\pm |u_k - u|^p \,\mathrm{d}x\right)^{1/p} \\
&\le \left(\delta \|\nabla(u_k - u)\|^p_{L^p(\omega;\mathbb{R}^n)} + C(n,p,q,\delta,\|\mathcal{V}^\pm\|_{M^q(p;\omega')})\|u_k - u\|^p_{L^p(\omega)}\right)^{1/p} \\
&\le \delta^{1/p}(K + \|\nabla u\|_{L^p(\omega;\mathbb{R}^n)}) + C(n,p,q,\delta,\|\mathcal{V}^\pm\|_{M^q(p;\omega')})\|u_k - u\|_{L^p(\omega)}.
\end{aligned} \tag{3-13}$$

This shows that

$$\limsup_{k\to\infty} \int_\omega \mathcal{V}^\pm |u_k|^p \,\mathrm{d}x \le \int_\omega \mathcal{V}^\pm |u|^p \,\mathrm{d}x.$$

On the other hand, by Fatou's lemma, we have

$$\int_\omega \mathcal{V}^{\pm}|u|^p \,\mathrm{d}x \le \liminf_{k\to\infty} \int_\omega \mathcal{V}^{\pm}|u_k|^p \,\mathrm{d}x.$$

The last two inequalities imply

$$\lim_{k\to\infty} \int_\omega \mathcal{V}|u_k|^p \,\mathrm{d}x = \int_\omega \mathcal{V}|u|^p \,\mathrm{d}x,$$

The weak lower semicontinuity of the gradient term follows from the convexity of the Lagrangian $\zeta \mapsto |\zeta|^p_{A(x)}$. We deduce then

$$Q_{A,p,\mathcal{V}}[u] \le \liminf_{k\to\infty} Q_{A,p,\mathcal{V}}[u_k]. \tag{3-14}$$

For the last term of J, we work similarly:

$$\begin{aligned}\int_\omega \mathcal{G}^{\pm}|u_k| \,\mathrm{d}x - \int_\omega \mathcal{G}^{\pm}|u| \,\mathrm{d}x \\ &\le \|\mathcal{G}^{\pm}\|^{1/p'}_{L^1(\omega)} \biggl(\int_\omega \mathcal{G}^{\pm}|u_k - u|^p \,\mathrm{d}x\biggr)^{1/p} \\ &\le \delta^{1/p}\|\mathcal{G}^{\pm}\|^{1/p'}_{L^1(\omega)}(K + \|\nabla u\|_{L^p(\omega;\mathbb{R}^n)}) + C(n, p, q, \delta, \|\mathcal{G}^{\pm}\|_{M^q(p;\omega')})\|u_k - u\|_{L^p(\omega)},\end{aligned}$$

and thus

$$\limsup_{k\to\infty} \int_\omega \mathcal{G}^{\pm}|u_k| \,\mathrm{d}x \le \int_\omega \mathcal{G}^{\pm}|u| \,\mathrm{d}x.$$

On the other hand,

$$\int_\omega \mathcal{G}^{\pm}|u| \,\mathrm{d}x \le \liminf_{k\to\infty} \int_\omega \mathcal{G}^{\pm}|u_k| \,\mathrm{d}x.$$

The last two inequalities imply

$$\lim_{k\to\infty} \int_\omega \mathcal{G}|u_k| \,\mathrm{d}x = \int_\omega \mathcal{G}|u| \,\mathrm{d}x,$$

and thus $\bar{J}$ is weakly lower semicontinuous in $W^{1,p}(\omega)$.

For the proof of the weak lower semicontinuity of J in $W^{1,p}_0(\omega)$, one follows the same steps, but uses Theorem 2.4(i) in (3-13), in order to obtain (3-14). Note that, since we require in this case that $\mathcal{G} \in L^{p'}(\omega)$, the functional $I(u) := \int_\omega \mathcal{G}u \,\mathrm{d}x$ is weakly continuous since it is a bounded linear functional. □

Proposition 3.7. (a) *Let $\omega \Subset \omega' \Subset \mathbb{R}^n$, where ω is Lipschitz, and $\mathcal{G}, \mathcal{V} \in M^q(p; \omega')$. If $\mathcal{V}$ is nonnegative, then for any $f \in W^{1,p}(\omega)$ we have that $\bar{J}$ is coercive in*

$$\mathcal{A} := \{u \in W^{1,p}(\omega) \mid u = f \text{ on } \partial\omega\}.$$

(b) *Let $\omega \Subset \mathbb{R}^n$, $\mathcal{V} \in M^q(p; \omega)$ and $\mathcal{G} \in L^{p'}(\omega)$. Assume that for some $\varepsilon > 0$ we have*

$$Q_{A,p,\mathcal{V}}[u; \omega] \ge \varepsilon\|u\|^p_{L^p(\omega)} \quad \text{for all } u \in W^{1,p}_0(\omega). \tag{3-15}$$

Then J is coercive in $W^{1,p}_0(\omega)$.

Proof. (a) Fix $t \in \mathbb{R}$, and suppose that $u \in \mathcal{A}$ is such that $\bar{J}[u] \le t$. It is enough to prove that

$$\|u\|_{W^{1,p}(\omega)} := \|u\|_{L^p(\omega)} + \|\nabla u\|_{L^p(\omega;\mathbb{R}^n)} \le C, \tag{3-16}$$

with C independent of u. To this end, from $\bar{J}[u] \le t$ and since $\mathcal{V} \ge 0$ a.e. in ω, we readily deduce

$$\int_\omega |\nabla u|_A^p \,\mathrm{d}x \le t + \int_\omega \mathcal{G}|u| \,\mathrm{d}x \le t + \|\mathcal{G}\|_{L^1(\omega)}^{1/p'} \left(\int_\omega |\mathcal{G}|\,|u|^p \,\mathrm{d}x \right)^{1/p} \le t + C\|u\|_{W^{1,p}(\omega)} \tag{3-17}$$

for some positive constant C that depends only on n, p, q, ω, $\|\mathcal{G}\|_{M^q(p;\omega')}$ and $\|\mathcal{G}\|_{L^1(\omega)}$, where we have used Theorem 2.4(ii) in the last inequality. Thus, applying also assumption (E), we obtain

$$\|\nabla u\|^p_{L^p(\omega;\mathbb{R}^n)} \le c_1 + c_2\|u\|_{W^{1,p}(\omega)}, \tag{3-18}$$

where c_1 and c_2 are positive constants independent of u. Next observe that $u - f \in W_0^{1,p}(\omega)$, so that

$$\|u\|_{L^p(\omega)} \le \|u - f\|_{L^p(\omega)} + \|f\|_{L^p(\omega)} \le C_P\|\nabla(u - f)\|_{L^p(\omega;\mathbb{R}^n)} + \|f\|_{L^p(\omega)}$$

for a positive constant C_P depending only on n and ω, because of the Poincaré inequality in $W_0^{1,p}(\omega)$. Using (E) we have, successively,

$$\begin{aligned}
\|u\|_{L^p(\omega)} &\le C_P\big(\|\nabla u\|_{L^p(\omega;\mathbb{R}^n)} + \|\nabla f\|_{L^p(\omega;\mathbb{R}^n)}\big) + \|f\|_{L^p(\omega)} \\
&\le \frac{C_P}{\theta_\omega}\left(\left(\int_\omega |\nabla u|_A^p \,\mathrm{d}x\right)^{1/p} + \|\nabla f\|_{L^p(\omega;\mathbb{R}^n)}\right) + \|f\|_{L^p(\omega)} \\
&\le \frac{C_P}{\theta_\omega}\big((t + C\|u\|_{W^{1,p}(\omega)})^{1/p} + \|\nabla f\|_{L^p(\omega;\mathbb{R}^n)}\big) + \|f\|_{L^p(\omega)},
\end{aligned}$$

with C as in (3-17). This implies the estimate

$$\|u\|^p_{L^p(\omega)} \le c_3 + c_4\|u\|_{W^{1,p}(\omega)}, \tag{3-19}$$

where c_3 and c_4 are positive constants independent of u. Now, (3-18) and (3-19) give

$$\|u\|^p_{W^{1,p}(\omega)} \le c_5 + c_6\|u\|_{W^{1,p}(\omega)},$$

for some positive constants c_5 and c_6 that are independent of u. This implies, in turn, $\|u\|_{W^{1,p}(\omega)} \le \max\{1, (c_5 + c_6)^{1/(p-1)}\}$, and (3-16) is proved.

(b) Let us prove the coercivity of J in $W_0^{1,p}(\omega)$. Assume that $J[u] \le t$ in (3-15); then, by applying Hölder's inequality, we obtain

$$\varepsilon\|u\|^p_{L^p(\omega)} \le t + \int_\omega \mathcal{G}u \,\mathrm{d}x \le t + \|\mathcal{G}\|_{L^{p'}(\omega)}\|u\|_{L^p(\omega)}.$$

This implies the estimate

$$\|u\|_{L^p(\omega)} \le m := \max\left\{1, \left(\frac{t + \|\mathcal{G}\|_{L^{p'}(\omega)}}{\varepsilon}\right)^{1/(p-1)}\right\}. \tag{3-20}$$

From $J[u] \le t$, applying once more Hölder's inequality and the Morrey–Adams theorem (Theorem 2.4(i)) we get

$$\begin{aligned}\int_\omega |\nabla u|_A^p \, \mathrm{d}x &\le t + \int_\omega \mathcal{G} u \, \mathrm{d}x + \int_\omega |\mathcal{V}||u|^p \, \mathrm{d}x \\ &\le t + \|\mathcal{G}\|_{L^{p'}(\omega)} \|u\|_{L^p(\omega)} + \delta \|\nabla u\|^p_{L^p(\omega;\mathbb{R}^n)} + C' \|u\|^p_{L^p(\omega)},\end{aligned} \tag{3-21}$$

where $C' = C_{n,p,q} \delta^{-n/(pq-n)} \|\mathcal{V}\|^{pq/(pq-n)}_{M^q(p;\omega)}$. Thus, from (3-20), (3-21) and assumption (E) we have, for $\delta < \theta^p_\omega$,

$$(\theta^p_\omega - \delta) \|\nabla u\|^p_{L^p(\omega;\mathbb{R}^n)} \le t + \|\mathcal{G}\|_{L^{p'}(\omega)} m + C' m^p,$$

which, together with (3-20), implies $\|u\|_{W^{1,p}(\omega)} \le C$. □

Remark 3.8. Propositions 3.6 and 3.7 will be used to prove the existence of a minimizer for the Rayleigh–Ritz variational problem (3-3), and to establish the weak comparison principle using the sub/supersolution method (see Section 5A).

3B. *Existence, properties and characterization of the positivity of λ_1.* The following theorem generalizes several results in the literature concerning the principal eigenvalue λ_1 (see [Allegretto and Huang 1998, Theorem 2.1; Anane 1987, Proposition 2; García-Melián and Sabina de Lis 1998, Lemma 3; Pinchover and Regev 2015, Lemma 6.4], for example). Note that our results apply to a general bounded domain, and in particular, the boundary point lemmas are not used in the proof (cf. [García-Melián and Sabina de Lis 1998, Lemma 3] and [Pinchover and Regev 2015]). In addition, we do not need any further regularity assumption on the entries of the matrix A as in the aforementioned references, while the potential V is far from being bounded.

Theorem 3.9. *Let ω be a bounded domain in $\mathbb{R}^n$, and assume that A is a uniformly elliptic, bounded matrix in ω, and $V \in M^q(p;\omega)$. Then the operator $Q'_{A,p,V}$ in ω admits a principal eigenvalue λ_1 given by the Rayleigh–Ritz variational formula (3-3). Moreover, λ_1 is the only principal eigenvalue, it is simple and an isolated eigenvalue in $\mathbb{R}$.*

Proof. We define λ_1 by (3-3) and prove that it is a principal eigenvalue. Using the Morrey–Adams theorem (Theorem 2.4) with $\delta = \theta^p_\omega$ one sees that

$$\lambda_1 \ge -C(n, p, q) \theta_\omega^{-np/(pq-n)} \|V\|^{pq/(pq-n)}_{M^q(p;\omega)} > -\infty.$$

In particular, setting $\mathcal{V} := V - \lambda_1 + \varepsilon$, with $\varepsilon > 0$, we get that

$$Q_{A,p,\mathcal{V}}[u;\omega] \ge \varepsilon \|u\|^p_{L^p(\omega)} \quad \text{for all } u \in W^{1,p}_0(\omega).$$

Applying Propositions 3.6(a) and 3.7(b) with $\mathcal{G} \equiv 0$, we get that $Q_{A,p,V-\lambda_1+\varepsilon}[\,\cdot\,;\omega]$ is coercive and weakly lower semicontinuous in $W^{1,p}_0(\omega)$ and, consequently, also in $W^{1,p}_0(\omega) \cap \{\|u\|_{L^p(\omega)} = 1\}$. Hence, the infimum

$$\varepsilon = \inf_{u \in W^{1,p}_0(\omega)\setminus\{0\}} \frac{Q_{A,p,V-\lambda_1+\varepsilon}[u;\omega]}{\|u\|^p_{L^p(\omega)}}$$

is attained in $W_0^{1,p}(\omega)\setminus\{0\}$ (see [Struwe 2008, Theorem 1.2], for example), and thus λ_1 is attained in $W_0^{1,p}(\omega)\setminus\{0\}$.

Let v_1 be a minimizer of (3-3). It is quite standard to see that v_1 is a solution of (3-1) with $\lambda=\lambda_1$. Since $|v_1|\in W_0^{1,p}(\omega)\setminus\{0\}$, it follows that $\big|\nabla(|v_1|)\big|_A=|\nabla v_1|_A$ a.e. in ω. This implies that $|v_1|$ is also a minimizer of (3-3) and thus a nonnegative solution of (3-1) with $\lambda=\lambda_1$. By the Harnack inequality, and the Hölder continuity of $|v_1|$, we obtain that $|v_1|$ is strictly positive in ω. In light of the homogeneity of the eigenvalue problem (3-1), we may assume that v_1 is strictly positive in ω.

To prove the simplicity of λ_1, we assume that $v_2\in W_0^{1,p}(\omega)$ is another eigenfunction of (3-1) with $\lambda=\lambda_1$. Hence, v_2 is a minimizer of (3-3), and thus has a definite sign. Without loss of generality, we may assume that $v_2>0$ in ω. Applying Lemma 3.3(ii) with $V_1=V_2=V$, $\lambda=\mu=\lambda_1$ and $w_\lambda=v_1$, $w_\mu=v_2$ we obtain

$$0\ge c_p\begin{cases}\displaystyle\int_\omega (v_1^p+v_2^p)\left|\nabla\log\frac{v_1}{v_2}\right|_A^p\,\mathrm{d}x & \text{if } p\ge 2,\\[2ex] \displaystyle\int_\omega (v_1^p+v_2^p)\left|\nabla\log\frac{v_1}{v_2}\right|_A^2(|\nabla\log v_1|_A+|\nabla\log v_2|_A)^{p-2}\,\mathrm{d}x & \text{if } p<2,\end{cases}$$

from which, because of (E), we deduce $|v_2\nabla v_1-v_1\nabla v_2|=0$ a.e. in ω, which in turn implies the existence of a positive constant c such that $v_2=cv_1$ a.e. in ω.

Next we show that λ_1 is the only eigenvalue possessing a nonnegative eigenfunction associated to it. If $\lambda>\lambda_1$ is an eigenvalue with eigenfunction $\varepsilon v_\lambda\ge 0$, where $\varepsilon>0$ is small, then, by Lemma 3.3(ii) with $V_1=V_2=V$, $\mu=\lambda_1$ and $w_\mu=v_1$,

$$(\lambda-\lambda_1)\int_\omega(\varepsilon v_\lambda^p-v_1^p)\,\mathrm{d}x\ge 0,$$

which is a contradiction for ε small enough.

It remains thus to prove that λ_1 is an isolated eigenvalue in $\mathbb{R}$. Suppose that there exists a sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}\subset\mathbb{R}$ such that $\lambda_k\downarrow\lambda_1$, as $k\to\infty$. Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence of the associated normalized eigenfunctions. We claim that $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\omega)$. Indeed, by the Morrey–Adams theorem, we obtain for some $0<\delta<1$ that

$$\int_\omega|\nabla v_k|_A^p\,\mathrm{d}x\le|\lambda_k|+\int_\omega|V||v_k|^p\,\mathrm{d}x\le\delta\|\nabla v_k\|^p_{L^p(\omega;\mathbb{R}^n)}+C,\tag{3-22}$$

which implies our claim. Therefore, up to a subsequence, v_k converges weakly in $W_0^{1,p}(\omega)$ and also in $L^p(\omega)$.

Next we claim that $v_k\to w$ in $W_0^{1,p}(\omega)$. Since $v_k\rightharpoonup w$ in $W_0^{1,p}(\omega)$, it is enough to show that $\{\|\nabla v_k\|_{L^p(\omega;\mathbb{R}^n)}\}$ is a Cauchy sequence. Let $\varepsilon>0$ be arbitrary. The inequality

$$|a^p-b^p|\le p|a-b|(a^{p-1}+b^{p-1}),\qquad a,b\ge 0,$$

together with the Hölder inequality and the Morrey–Adams theorem imply, for all sufficiently large $k,l\in\mathbb{N}$,

$$
\begin{aligned}
\left|\int_\omega |\nabla v_k|_A^p \,\mathrm{d}x - \int_\omega |\nabla v_l|_A^p \,\mathrm{d}x\right| &\\
&\le |\lambda_k - \lambda_l| + \int_\omega |V|\,\big||v_k|^p - |v_l|^p\big| \,\mathrm{d}x \\
&\le \varepsilon + p\int_\omega |V|\,|v_k - v_l|\,\big||v_k|^{p-1} + |v_l|^{p-1}\big| \,\mathrm{d}x. \\
&\le \varepsilon + C(p)\bigg(\int_\omega |V|\,|v_k - v_l|^p \,\mathrm{d}x\bigg)^{1/p}\bigg(\int_\omega |V|\,|v_k|^p \,\mathrm{d}x + \int_\omega |V|\,|v_l|^p \,\mathrm{d}x\bigg)^{1/p'}. \qquad (3\text{-}23)
\end{aligned}
$$

Applying first the Morrey–Adams theorem and then (3-22), we see that both integrals on the second factor of (3-23) are uniformly bounded in k and l, respectively. For the first factor we use again the Morrey–Adams theorem to arrive at

$$
\left|\int_\omega |\nabla v_k|_A^p \,\mathrm{d}x - \int_\omega |\nabla v_l|_A^p \,\mathrm{d}x\right| \le \varepsilon + C_1\bigg(\varepsilon\int_\omega |\nabla(v_k - v_l)|^p \,\mathrm{d}x + C_2\varepsilon^{n/(n-pq)}\int_\omega |v_k - v_l|^p \,\mathrm{d}x\bigg)^{1/p}, \quad (3\text{-}24)
$$

where C_1 and C_2 are positive constants independent of k and l. The convergence in $L^p(\omega)$ of v_k to v implies that there exists $m_\varepsilon \in \mathbb{N}$ such that

$$
\int_\omega |v_k - v_l|^p \,\mathrm{d}x \le \varepsilon^{n/(pq-n)+1} \quad \text{for all } k, l \ge m_\varepsilon.
$$

Coupling this with (3-24) implies that $\{\|\nabla v_k\|_{L^p(\omega;\mathbb{R}^n)}\}$ is a Cauchy sequence.

By a similar argument, one shows that

$$
Q_{A,p,V}[w] = \lambda_1 \|w\|^p_{L^p(\omega)},
$$

hence, w is a minimizer of the Rayleigh–Ritz variational problem (3-3), and hence an eigenfunction of (3-1) with $\lambda = \lambda_1$. The simplicity of λ_1 implies that $w = \pm v$, where $v > 0$ is the normalized principal eigenfunction with an eigenvalue λ_1. Without loss of generality, we may assume that $v_k \to v$ in $W_0^{1,p}(\omega)$.

Set $\omega_k^- := \{x \in \omega \mid v_k < 0\}$. By Lemma 3.4 (with $\mathcal{V} = V - \lambda_k$) we have that v_k^- is a subsolution of $Q'_{A,p,V-\lambda_k}[u] = 0$ in ω, and thus, from (3-2),

$$
\begin{aligned}
\int_\omega |\nabla v_k^-|_A^p \,\mathrm{d}x &\le \int_\omega |V - \lambda_k|\,|v_k^-|^p \,\mathrm{d}x \\
&\le \delta\|\nabla v_k^-\|^p_{L^p(\omega;\mathbb{R}^n)} + C(n,p,q)\delta^{-n/(pq-n)}\|V - \lambda_k\|^{pq/(pq-n)}_{M^q(p;\omega)}\|v_k^-\|^p_{L^p(\omega)}
\end{aligned}
$$

for any $\delta > 0$, where we have used Theorem 2.4. For $\delta < \theta_\omega^p$ we deduce, because of assumption (E), that

$$
(\theta_\omega^p - \delta)\|\nabla v_k^-\|^p_{L^p(\omega;\mathbb{R}^n)} \le C(n,p,q)\delta^{-n/(pq-n)}\|V - \lambda_k\|^{pq/(pq-n)}_{M^q(p;\omega)}\|v_k^-\|^p_{L^p(\omega)}.
$$

Since $v_k^- \equiv 0$ in $\omega \setminus \omega_k^-$, we use Poincaré's inequality

$$
\|v_k^-\|_{L^p(\omega)} \le \bigg(\frac{\mathcal{L}^n(\omega_k^-)}{\mathcal{L}^n(B_1)}\bigg)^{1/n}\|\nabla v_k^-\|_{L^p(\omega;\mathbb{R}^n)}, \qquad (3\text{-}25)
$$

to get

$$(\theta_\omega^p - \delta)\|\nabla v_k^-\|^p_{L^p(\omega;\mathbb{R}^n)} \le C(n,p,q)\delta^{-n/(pq-n)}\|V-\lambda_k\|^{pq/(pq-n)}_{M^q(p;\omega)}\left(\frac{\mathcal{L}^n(\omega_k^-)}{\mathcal{L}^n(B_1)}\right)^{p/n}\|\nabla v_k^-\|^p_{L^p(\omega;\mathbb{R}^n)}.$$

Canceling $\|\nabla v_k^-\|^p_{L^p(\omega;\mathbb{R}^n)}$, rearranging and raising to the power n/p, we arrive at

$$\mathcal{L}^n(\omega_k^-) \ge C(n,p,q)\mathcal{L}^n(B_1)(\theta_\omega^p - \delta)^{n/p}\delta^{n^2/[p(pq-n)]}\|V-\lambda_k\|^{-nq/(pq-n)}_{M^q(p;\omega)}. \tag{3-26}$$

Notice that $\|V-\lambda_1\|_{M^q(p;\omega)}$ is a strictly positive number. Indeed, assume that $\|V-\lambda_1\|_{M^q(p;\omega)} = 0$. Then v_1 is a nontrivial solution of the Dirichlet problem for the (p, A)-Laplace operator which is false under our assumptions on A (see [Heinonen et al. 1993; Pucci and Serrin 2007], for example).

On the other hand, $\|V-\lambda_k\|_{M^q(p;\omega)} \to \|V-\lambda_1\|_{M^q(p;\omega)}$ as $k\to\infty$. Therefore, there exists $C>0$ such that

$$\|V-\lambda_k\|_{M^q(p;\omega)} \ge C\|V-\lambda_1\|_{M^q(p;\omega)} \quad \text{for all } k \ge k_0. \tag{3-27}$$

Consequently, (3-27) applied to (3-26) implies that

$$\mathcal{L}^n(\omega_k^-) \ge C > 0 \quad \text{for all } k \ge k_0$$

for a positive constant C independent on k.

With this at hand, the rest of the proof follows [Anane 1987, Théorème 2]. We include it for completeness: Let $\eta > 0$. Recalling that v is continuous in ω, we may pick a compact set $\omega_\eta \Subset \omega$ and $m_\eta > 0$, such that $\mathcal{L}^n(\omega \setminus \omega_\eta) < \eta$ and $v(x) \ge m_\eta$ for every $x \in \omega_\eta$. Up to subsequence that we don't rename, v_k converges to v a.e. in ω, and thus in ω_η. By the Egoroff theorem (see [Evans and Gariepy 1992, §1.2]) we have the existence of a measurable set $\omega' \subset \omega_\eta$ with $\mathcal{L}^n(\omega') < \eta$ such that v_k converges uniformly to v on $\omega_\eta \setminus \omega'$. Since $v \ge m_\eta > 0$ in ω_η we deduce that for any k large enough we have $v_k \ge 0$ on $\omega_\eta \setminus \omega'$. Thus, $\omega_k^- \subset \omega' \cup (\omega \setminus \omega_\eta)$, which implies that $\mathcal{L}^n(\omega_k^-) \le 2\eta$. Since $\eta > 0$ is arbitrary, for k large enough this contradicts our estimate $\mathcal{L}^n(\omega_k^-) \ge C_1$. □

We are now ready to prove the main result of this section. Extending the corresponding results in [García-Melián and Sabina de Lis 1998; Pinchover and Regev 2015], we have:

Theorem 3.10. *Let ω be a bounded domain, and assume that A is a uniformly elliptic, bounded matrix in ω, and $V \in M^q(p;\omega)$. Consider the following assertions*:

(α_1) *$Q'_{A,p.V}$ satisfies the weak maximum principle in ω.*

(α_2) *$Q'_{A,p.V}$ satisfies the strong maximum principle in ω.*

(α_3) $\lambda_1 > 0$.

(α_4) *The equation $Q'_{A,p,V}[v] = 0$ admits a positive strict supersolution in $W_0^{1,p}(\omega)$.*

(α_4') *The equation $Q'_{A,p,V}[v] = 0$ admits a positive strict supersolution in $W^{1,p}(\omega)$.*

(α_5) *For $0 \le g \in L^{p'}(\omega)$, there exists a unique nonnegative solution in $W_0^{1,p}(\omega)$ of $Q'_{A,p,V}[v] = g$.*

Then $\alpha_1 \Longleftrightarrow \alpha_2 \Longleftrightarrow \alpha_3 \Longrightarrow \alpha_4 \Longrightarrow \alpha_4'$, and $\alpha_3 \Longrightarrow \alpha_5 \Longrightarrow \alpha_4$.

Remark 3.11. In Corollary 4.14 we prove (imposing stronger regularity assumptions on A and V when $p < 2$) that in fact, $\alpha_4' \Longrightarrow \alpha_3$. Hence, under these additional assumptions for $p < 2$, all the above assertions are equivalent.

Proof. $\alpha_1 \Longrightarrow \alpha_2$: Let $v \in W^{1,p}(\omega)$ be a solution of (3-4) and suppose $v \geq 0$ on $\partial\omega$. The nonnegativity of g and the weak maximum principle implies that v is a nonnegative supersolution of (2-3) in ω. Suppose that for some $x_0, x_1 \in \omega$ we have $v(x_0) \neq 0$ and $v(x_1) = 0$ and let $\omega' \Subset \omega$ contain both x_0 and x_1. Recalling Remark 2.10, we apply the weak Harnack inequality if $p \leq n$, or the Harnack inequality if $p > n$, to get $v \equiv 0$ in ω'. This contradicts the assumption that $v(x_0) \neq 0$. Thus, if $v \neq 0$ we necessarily have $v > 0$ in ω.

$\alpha_2 \Longrightarrow \alpha_3$: Suppose that $\lambda_1 \leq 0$ and let $v \in W_0^{1,p}(\omega)$ be the corresponding principal eigenfunction. Then $u := -v$ is a supersolution of (2-3) in ω, satisfying $u = 0$ on $\partial\omega$, and $u \neq 0$. By the strong maximum principle, u is positive, which is absurd.

$\alpha_3 \Longrightarrow \alpha_1$: Let $v \in W^{1,p}(\omega)$ be a solution of (3-4) such that $v \geq 0$ on $\partial\omega$. Taking $v^- \in W_0^{1,p}(\omega)$ as a test function we see that

$$Q_{A,p,V}[v^-; \omega] = \int_{\omega^-} g v \, \mathrm{d}x,$$

where $\omega^- := \{x \in \omega \mid v < 0\}$. The nonnegativity of g gives $Q_{A,p,V}[v^-; \omega] \leq 0$, which implies that $\lambda_1 \leq 0$. Thus, we must have $v^- = 0$ a.e. in ω, or in other words $v \geq 0$ a.e. in ω.

$\alpha_3 \Longrightarrow \alpha_4$: Since $\lambda_1 > 0$, it follows that the principal eigenfunction is a positive strict supersolution of (2-3) in ω.

$\alpha_4 \Longrightarrow \alpha_4'$: This is trivial.

$\alpha_3 \Longrightarrow \alpha_5$: Consider the functional

$$J[u] := Q_{A,p,V}[u; \omega] - \int_\omega g u \, \mathrm{d}x, \qquad u \in W_0^{1,p}(\omega).$$

By Proposition 3.6(a), J is weakly lower semicontinuous in $W_0^{1,p}(\omega)$ and, by Proposition 3.7(b), J is coercive. Therefore, the corresponding Dirichlet problem admits a solution $v_1 \in W_0^{1,p}(\omega)$ (see [Struwe 2008, Theorem 1.2], for example). Since $\alpha_3 \Longrightarrow \alpha_2$, this solution is either zero or strictly positive.

If $v_1 = 0$, then $g = 0$, and by the uniqueness of the principal eigenvalue, equation (2-3) in ω does not admit a positive solution in $W_0^{1,p}(\omega)$. So, we may assume that $v_1 > 0$ and let $v_2 \in W_0^{1,p}(\omega)$ be another positive solution. Applying Lemma 3.3(i) with $g_1 = g_2 = g$ and $V_1 = V_2 = V$, we obtain

$$0 \geq \int_\omega g \left(\frac{1}{v_{1,h}^{p-1}} - \frac{1}{v_{2,h}^{p-1}} \right) (v_{1,h}^p - v_{2,h}^p) \, \mathrm{d}x \geq \int_\omega V \left(\left(\frac{v_1}{v_{1,h}} \right)^{p-1} - \left(\frac{v_2}{v_{2,h}} \right)^{p-1} \right) (v_{1,h}^p - v_{2,h}^p) \, \mathrm{d}x.$$

The integrand of the integral on the right converges to 0 a.e. in ω, and also it satisfies the following estimate for every $h < 1$:

$$\left| V \left(\left(\frac{v_1}{v_{1,h}} \right)^{p-1} - \left(\frac{v_2}{v_{2,h}} \right)^{p-1} \right) (v_{1,h}^p - v_{2,h}^p) \right| \leq 2|V|((v_1+1)^p + (v_2+1)^p) \in L^1(\omega).$$

Thus

$$\lim_{h\to 0}\int_\omega g\left(\frac{1}{v_{1,h}^{p-1}}-\frac{1}{v_{2,h}^{p-1}}\right)(v_{1,h}^p - v_{2,h}^p)\,\mathrm{d}x = 0,$$

which, together with Fatou's lemma, implies that the right-hand side of (3-6) equals zero. Thus, $v_2 = v_1$ a.e. in ω.

$\alpha_5 \Longrightarrow \alpha_4$: Let $v \in W_0^{1,p}(\omega)$ be a positive solution of (3-4) with $g \equiv 1$. Then v is readily a positive strict supersolution of (2-3) in ω. □

4. Positive global solutions

In the present section we pass from local to global properties of positive solutions of the equation (2-3) in Ω. In Section 4A we establish the AP theorem, while in Section 4B we prove among other results the equivalence of the first four statements of the Main Theorem.

4A. ***The AP theorem.*** In this subsection we prove the AP theorem for the operator $Q'_{A,p,V}$ under hypothesis (H0). We will add a couple of equivalent assertions to this theorem, regarding the first-order equation

$$-\mathrm{div}_A T + (p-1)|T|_A^{p'} = V \quad \text{in } \Omega, \tag{4-1}$$

where $\mathrm{div}_A T = \mathrm{div}(AT)$ and $T \in L_{\mathrm{loc}}^{p'}(\Omega;\mathbb{R}^n)$; see [Jaye et al. 2012, Theorem 1.3] for a similar study when $A = I_n$ and $p = 2$.

Definition 4.1. Suppose that the matrix A satisfies (S) and (E) and let $V \in L_{\mathrm{loc}}^1(\Omega)$. A vector field $T \in L_{\mathrm{loc}}^{p'}(\Omega;\mathbb{R}^n)$ is a *solution* of (4-1) in Ω if

$$\int_\Omega AT\cdot\nabla u\,\mathrm{d}x + (p-1)\int_\Omega |T|_A^{p'} u\,\mathrm{d}x = \int_\Omega Vu\,\mathrm{d}x \quad \text{for all } u \in C_c^\infty(\Omega), \tag{4-2}$$

a *subsolution* of (4-1) in Ω if

$$\int_\Omega AT\cdot\nabla u\,\mathrm{d}x + (p-1)\int_\Omega |T|_A^{p'} u\,\mathrm{d}x \le \int_\Omega Vu\,\mathrm{d}x \quad \text{for all nonnegative } u \in C_c^\infty(\Omega), \tag{4-3}$$

and a supersolution if the reverse inequality holds.

Remark 4.2. The additional assumption $V \in M_{\mathrm{loc}}^q(p;\Omega)$ allows the replacement of $C_c^\infty(\Omega)$ by $W_c^{1,p}(\Omega)$ in Definition 4.1.

Theorem 4.3 (the AP theorem). *Under hypothesis (H0), the following assertions are equivalent*:

($\mathcal{A}_1$) $Q_{A,p,V}[u] \ge 0$ *for all* $u \in C_c^\infty(\Omega)$.

($\mathcal{A}_2$) $Q'_{A,p,V}[w] = 0$ *admits a positive solution* $v \in W_{\mathrm{loc}}^{1,p}(\Omega)$.

($\mathcal{A}_3$) $Q'_{A,p,V}[w] = 0$ *admits a positive supersolution* $\tilde{v} \in W_{\mathrm{loc}}^{1,p}(\Omega)$.

($\mathcal{A}_4$) (4-1) *admits a solution* $T \in L_{\mathrm{loc}}^{p'}(\Omega;\mathbb{R}^n)$.

($\mathcal{A}_5$) (4-1) *admits a subsolution* $\widetilde{T} \in L_{\mathrm{loc}}^{p'}(\Omega;\mathbb{R}^n)$.

Proof. We prove $\mathcal{A}_1 \implies \mathcal{A}_2 \implies \mathcal{A}_j \implies \mathcal{A}_5 \implies \mathcal{A}_1$, where $j = 3, 4$.

$\mathcal{A}_1 \implies \mathcal{A}_2$: We fix a point $x_0 \in \Omega$ and let $\{\omega_i\}_{i\in\mathbb{N}}$ be a sequence of Lipschitz domains such that $x_0 \in \omega_1$, $\omega_i \Subset \omega_{i+1} \Subset \Omega$, $i \in \mathbb{N}$, and $\bigcup_{i\in\mathbb{N}} \omega_i = \Omega$. For $i \geq 2$, we consider the problem

$$\begin{cases} Q'_{A,p,V+1/i}[u] = f_i & \text{in } \omega_i, \\ u = 0 & \text{on } \partial\omega_i, \end{cases} \tag{4-4}$$

where $f_i \in C_c^\infty(\omega_i \setminus \bar{\omega}_{i-1}) \setminus \{0\}$ are nonnegative. Assertion $\mathcal{A}_1$ implies

$$\lambda_1(Q_{A,p,V+1/i}; \omega_i) \geq \frac{1}{i} \quad \text{for all } i \in \mathbb{N},$$

so that by Theorem 3.10 there exists a positive solution $v_i \in W_0^{1,p}(\omega_i)$ of (4-4). Since $\operatorname{supp}\{f_i\} \subset \omega_i \setminus \bar{\omega}_{i-1}$, setting $\omega_i' = \omega_{i-1}$ we have

$$\int_{\omega_i} |\nabla v_i|_A^{p-2} A\nabla v_i \cdot \nabla u \, dx + \int_{\omega_i} (V + 1/i) v_i^{p-1} u \, dx = 0 \quad \text{for all } u \in W_0^{1,p}(\omega_i'). \tag{4-5}$$

By Theorem 2.7, the solutions v_i we have obtained are continuous. We may thus normalize f_i so that $v_i(x_0) = 1$ for all $i \in \mathbb{N}$. To arrive to the desired conclusion we apply the Harnack convergence principle (Proposition 2.11) with $\mathcal{V}_i := V + 1/i$.

$\mathcal{A}_2 \implies \mathcal{A}_3$: This is immediate with $\tilde{v} = v$.

$\mathcal{A}_2 \implies \mathcal{A}_4$ and $\mathcal{A}_3 \implies \mathcal{A}_5$: Let v be a positive (super)solution of (2-3). By the weak Harnack inequality (Remark 2.10) if $p \leq n$, or by the Harnack inequality if $p > n$, we have $1/v \in L^\infty_{\text{loc}}(\Omega)$. Set

$$T := -|\nabla \log v|_A^{p-2} \nabla \log v,$$

and let $u \in C_c^\infty(\Omega)$. We may thus pick $|u|^p v^{1-p} \in W_c^{1,p}(\Omega)$ as a test function in (2-6) to get

$$(p-1)\int_\Omega |T|_A^{p'} |u|^p \, dx \leq p \int_\Omega |T|_A |u|^{p-1} |\nabla u|_A \, dx + \int_\Omega V|u|^p \, dx. \tag{4-6}$$

Note that from (4-6) we obtain $\mathcal{A}_1$ just by using Young's inequality $pab \leq (p-1)a^{p'} + b^p$ with $a = |T|_A |u|^{p-1}$ and $b = |\nabla u|_A$ in the first term of the right-hand side. Towards $\mathcal{A}_3$, we use instead Young's inequality

$$pab \leq \eta a^{p'} + \left(\frac{p-1}{\eta}\right)^{p-1} b^p, \tag{4-7}$$

with $\eta \in (0, p-1)$ and the above a, b. We arrive at

$$(p-1-\eta)\int_\Omega |T|_A^{p'} |u|^p \, dx \leq \left(\frac{p-1}{\eta}\right)^{p-1} \int_\Omega |\nabla u|_A^p \, dx + \int_\Omega |V||u|^p \, dx.$$

This, together with (E) and Theorem 2.4 imply, by specializing u, that $T \in L^{p'}_{\text{loc}}(\Omega; \mathbb{R}^n)$. Next we show that T is a (sub)solution of (4-1). To this end, for $u \in C_c^\infty(\Omega)$, or for nonnegative $u \in C_c^\infty(\Omega)$, we pick $uv^{1-p} \in W_c^{1,p}(\Omega)$ as a test function in (2-5) or (2-6), respectively, to obtain

$$-\int_\Omega AT \cdot \nabla u \, dx - (p-1)\int_\Omega |T|_A^{p'} u \, dx + \int_\Omega Vu \, dx = 0,$$

or $\geq$ in the supersolution case.

$\mathcal{A}_4 \Longrightarrow \mathcal{A}_5$: This is immediate with $\widetilde{T} = T$.

$\mathcal{A}_5 \Longrightarrow \mathcal{A}_1$: Suppose now that $T \in L^{p'}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and let $u \in C_c^\infty(\Omega)$. We compute

$$\begin{aligned} -\int_\Omega AT \cdot \nabla(|u|^p)\,\mathrm{d}x &= -p \int_\Omega |u|^{p-1} AT \cdot \nabla|u|\,\mathrm{d}x \\ &\leq p \int_\Omega |u|^{p-1} |T|_A |\nabla u|_A\,\mathrm{d}x \\ &\leq (p-1) \int_\Omega |u|^p |T|_A^{p'}\,\mathrm{d}x + \int_\Omega |\nabla u|_A^p\,\mathrm{d}x, \end{aligned}$$

where we have also used Young's inequality $pab \leq (p-1)a^{p'} + b^p$, with $a = |u|^{p-1}|T|_A$ and $b = |\nabla u|_A$. This readily implies

$$\int_\Omega |\nabla u|_A^p\,\mathrm{d}x \geq -\int_\Omega AT \cdot \nabla(|u|^p)\,\mathrm{d}x - (p-1)\int_\Omega |T|_A^{p'} |u|^p\,\mathrm{d}x \quad \text{for all } u \in C_c^\infty(\Omega). \tag{4-8}$$

If T is a subsolution of (4-1) then, testing (4-3) by $|u|^p$, one readily sees from (4-8) that $Q_{A,p,V}[u]$ is nonnegative for any $u \in C_c^\infty(\Omega)$. □

Remark 4.4. Inequality (4-8) with $A = I_n$ has been obtained in [Fleckinger et al. 1999].

4B. *Criticality theory.* In the present subsection we generalize several global positivity properties of the functional $Q_{A,p,V}$, where A and V satisfy (at least) our regularity assumption (H0). For the convenience of the reader, we recall the following terminology:

Definition 4.5. Assume that $Q_{A,p,V}$ is *nonnegative* in Ω (that is, $Q_{A,p,V}[u] \geq 0$ for all $u \in C_c^\infty(\Omega)$) with coefficients satisfying hypothesis (H0). Then $Q_{A,p,V}$ is called *subcritical* in Ω if there exists a nonnegative weight function $W \in M^q_{\mathrm{loc}}(p; \Omega) \setminus \{0\}$ such that

$$Q_{A,p,V}[u] \geq \int_\Omega W|u|^p\,\mathrm{d}x \quad \text{for all } u \in C_c^\infty(\Omega). \tag{4-9}$$

If this is not the case, then $Q_{A,p,V}$ is called *critical* in Ω.

The functional $Q_{A,p,V}$ is called *supercritical* in Ω if $Q_{A,p,V}$ is not nonnegative in Ω (that is, there exists $u \in C_c^\infty(\Omega)$ such that $Q_{A,p,V}[u] < 0$).

Definition 4.6. A sequence $\{u_k\} \subset W_0^{1,p}(\Omega)$ is called a *null sequence* with respect to the nonnegative functional $Q_{A,p,V}$ in Ω if

(a) $u_k \geq 0$ for all $k \in \mathbb{N}$,

(b) there exists a fixed open set $K \Subset \Omega$ such that $\|u_k\|_{L^p(K)} = 1$ for all $k \in \mathbb{N}$,

(c) $\lim_{k\to\infty} Q_{A,p,V}[u_k] = 0$.

We call a positive $\phi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ a *ground state* of $Q_{A,p,V}$ in Ω if ϕ is an $L^p_{\mathrm{loc}}(\Omega)$ limit of a null sequence.

Remark 4.7. Let $\omega \subset \mathbb{R}^n$ be a bounded domain, and suppose that A is a uniformly elliptic and bounded matrix in ω, and $V \in M^q(p;\omega)$. Let v_1 be the principal eigenfunction with eigenvalue λ_1. Set $C_K := \|v_1\|_{L^p(K)}$, where $K \Subset \omega$ is fixed. Then the constant sequence $\{C_K^{-1}v_1\}$ is a null sequence and $C_K^{-1}v_1$ is a ground state of $Q_{A,p,V-\lambda_1}$ in ω.

The following proposition states an elementary positivity property of the functional $Q_{A,p,V}$:

Proposition 4.8. *Suppose that $V_2 \geq V_1$ a.e. in Ω and $\mathcal{L}^n(\{V_2 > V_1\}) > 0$.*

(a) *If Q_{A,p,V_1} is nonnegative in Ω, then Q_{A,p,V_2} is subcritical in Ω.*

(b) *If Q_{A,p,V_2} is critical in Ω, then Q_{A,p,V_1} is supercritical in Ω.*

Proof. Part (b) follows from part (a) by contradiction, and, from the obvious relation

$$Q_{A,p,V_2}[u] = Q_{A,p,V_1}[u] + \int_\Omega (V_2 - V_1)|u|^p \,\mathrm{d}x \quad \text{for all } u \in C_c^\infty(\Omega),$$

part (a) is evident. □

Note here that Definitions 4.5 and 4.6, and also Proposition 4.8 make perfect sense if V is merely in $L^1_{\mathrm{loc}}(\Omega)$ for all values of p.

Now we connect the (sub)criticality of the functional $Q_{A,p,V}$ in Ω with the existence of positive weak (super)solutions for equation (2-3) in Ω, through the existence of ground states. Towards this we need to give sufficient conditions on A and V, under which a null sequence with respect to the nonnegative functional $Q_{A,p,V}$ will converge in L^p_{loc} to a function in $W^{1,p}_{\mathrm{loc}}$.

We need the following definition for the case $1 < p < 2$.

Definition 4.9. Suppose that $1 < p < 2$. A positive supersolution v of (2-3) will be called *regular* provided that v and $|\nabla v|$ are locally bounded a.e. in Ω.

Remark 4.10. Under hypothesis (H1) for $1 < p < 2$, any positive supersolution v of (2-3) satisfying $Q_{A,p,V}[v] = g \geq 0$ with $g \in L^{p'}_{\mathrm{loc}}(\Omega)$ is regular (see Remark 2.9).

We start with the following proposition, which gives us the intuition that any null sequence converges in some sense to *any* positive (regular if $p < 2$) (super)solution. Note that our proof for the case $p < 2$ is considerably shorter than the corresponding proof in [Pinchover and Tintarev 2007; Pinchover and Regev 2015].

Proposition 4.11. *Suppose that $\{u_k\} \subset W_0^{1,p}(\Omega)$ is a null sequence with respect to a nonnegative functional $Q_{A,p,V}$ in Ω with coefficients satisfying hypothesis (H0).*

Let v be a positive supersolution of the equation (2-3) in Ω. If $1 < p < 2$ we assume further that v is regular. Set $w_k := u_k/v$. Then $\{w_k\}$ is bounded in $W^{1,p}_{\mathrm{loc}}(\Omega)$, and $\nabla w_k \to 0$ in $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$.

Proof. Let $K \Subset \Omega$ be the set such that the null sequence $\{u_k\}$ satisfies $\|u_k\|_{L^p(K)} = 1$ for all $k \in \mathbb{N}$. Fix a Lipschitz domain ω such that $K \Subset \omega \Subset \Omega$.

By the Minkowski and Poincaré inequalities, and the weak Harnack inequality, we have

$$\begin{aligned}\|w_k\|_{L^p(\omega)} &\le \|w_k - \langle w_k\rangle_K\|_{L^p(\omega)} + \langle w_k\rangle_K(\mathcal{L}^n(\omega))^{1/p}\\ &\le C(n,p,\omega,K)\|\nabla w_k\|_{L^p(\omega;\mathbb{R}^n)} + \frac{1}{\inf_K v}\langle u_k\rangle_K(\mathcal{L}^n(\omega))^{1/p}.\end{aligned}$$

Since $\|u_k\|_{L^p(K)} = 1$, applying Holder's inequality we deduce

$$\|w_k\|_{L^p(\omega)} \le C(n,p,\omega,K)\|\nabla w_k\|_{L^p(\omega;\mathbb{R}^n)} + \frac{1}{\inf_K v}\left(\frac{\mathcal{L}^n(\omega)}{\mathcal{L}^n(K)}\right)^{1/p}. \tag{4-10}$$

Let

$$I(v,w_k) := C(p)\begin{cases}\displaystyle\int_\Omega v^p|\nabla w_k|_A^p\,\mathrm{d}x, & p\ge 2,\\ \displaystyle\int_\Omega |\nabla w_k|_A^2\big(|\nabla(vw_k)|_A + w_k|\nabla v|_A\big)^{p-2}\,\mathrm{d}x, & 1\le p<2,\end{cases}$$

where $C(p)$ is the constant in (3-8). We now use (3-8) with $\alpha = \nabla(w_k v) = \nabla u_k$ and $\beta = w_k\nabla v$ to obtain

$$\begin{aligned}I(v,w_k) &\le \int_\Omega |\nabla u_k|_A^p\,\mathrm{d}x - \int_\Omega w_k^p|\nabla v|_A^p\,\mathrm{d}x - \int_\Omega v|\nabla v|_A^{p-2}A\nabla v\cdot\nabla(w_k^p)\,\mathrm{d}x\\ &= \int_\Omega |\nabla u_k|_A^p\,\mathrm{d}x - \int_\Omega |\nabla v|_A^{p-2}A\nabla v\cdot\nabla(w_k^p v)\,\mathrm{d}x.\end{aligned} \tag{4-11}$$

Since v is a positive supersolution, we get

$$I(v,w_k) \le \int_\Omega |\nabla u_k|_A^p\,\mathrm{d}x + \int_\Omega V u_k^p\,\mathrm{d}x = Q_{A,p,V}[u_k]. \tag{4-12}$$

Suppose now that $p\ge 2$. Using the definition of I, and the weak Harnack inequality, we obtain from (4-12) that

$$c\int_\omega |\nabla w_k|^p\,\mathrm{d}x \le C(p)\int_\Omega v^p|\nabla w_k|_A^p\,\mathrm{d}x \le Q_{A,p,V}[u_k]\to 0 \quad\text{as } k\to\infty, \tag{4-13}$$

where $c>0$ is a positive constant. By the weak compactness of $W^{1,p}(\omega)$, we get for $p\ge 2$ that (up to a subsequence)

$$\nabla w_k \to 0 \quad \text{in } L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n). \tag{4-14}$$

By (4-10) and (4-13), we have that w_k is bounded in $W^{1,p}_{\mathrm{loc}}(\Omega)$ for any $p\ge 2$.

On the other hand, if $p<2$, then by the definition of I and (4-12), we get

$$C(p)\int_\Omega \frac{v^2|\nabla w_k|_A^2}{\big(|\nabla(vw_k)|_A + w_k|\nabla v|_A\big)^{2-p}}\,\mathrm{d}x \le Q_{A,p,V}[u_k]\to 0 \quad\text{as } k\to\infty.$$

For convenience we set $q_k = Q_{A,p,V}[u_k]$. By Hölder's inequality with conjugate exponents $2/p$ and $2/(2-p)$, we get

$$\begin{aligned}\int_\omega v^p|\nabla w_k|_A^p \,\mathrm{d}x &\le \left(\int_\Omega \frac{v^2|\nabla w_k|_A^2}{\big(|\nabla(vw_k)|_A + w_k|\nabla v|_A\big)^{2-p}}\,\mathrm{d}x\right)^{p/2}\left(\int_\omega \big(|\nabla(vw_k)|_A + w_k|\nabla v|_A\big)^p \,\mathrm{d}x\right)^{1-p/2}\\ &\le C(p)^{-1}q_k^{p/2}\left(\int_\omega v^p|\nabla w_k|_A^p\,\mathrm{d}x + \int_\omega w_k^p|\nabla v|_A^p\,\mathrm{d}x\right)^{1-p/2}\\ &\le C(p)^{-1}q_k^{p/2}\left(\int_\omega v^p|\nabla w_k|_A^p\,\mathrm{d}x + \int_\omega w_k^p|\nabla v|_A^p\,\mathrm{d}x + 1\right).\end{aligned}$$

Since v is locally bounded and locally bounded away from zero, $|\nabla v|$ is locally bounded, and A is uniformly elliptic and bounded in ω, we get using (4-10) that, for some positive constants c_j, $1 \le j \le 4$, that are independent of k,

$$c_1\int_\omega |\nabla w_k|^p\,\mathrm{d}x \le c_2 q_k^{p/2}\left(\int_\omega |\nabla w_k|^p\,\mathrm{d}x + \int_\omega w_k^p\,\mathrm{d}x + 1\right) \le c_2 q_k^{p/2}\left(c_3\int_\omega |\nabla w_k|^p\,\mathrm{d}x + c_4\right).$$

Since $q_k \to 0$ as $k \to \infty$, we conclude that also in the case $p < 2$ we have

$$\nabla w_k \to 0 \quad \text{in } L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n),$$

and thus by (4-10) we have that w_k is bounded in $W^{1,p}_{\mathrm{loc}}(\omega)$ for any $p < 2$. $\square$

Several consequences follow. In the following statement, uniqueness is meant up to a positive multiplicative constant.

Theorem 4.12. *Suppose that $Q_{A,p,V}$ is nonnegative in Ω with A and V satisfying hypothesis (H0) if $p \ge 2$, or (H1) if $1 < p < 2$. Then any null sequence with respect to $Q_{A,p,V}$ converges, in L^p_{loc} and a.e. in Ω, to a unique positive* (*regular if $p < 2$*) *supersolution of (2-3) in Ω. In particular, a ground state is the unique positive solution and the unique positive* (*regular if $p < 2$*) *supersolution of (2-3) in Ω, and so the ground state is C^γ if $p \ge 2$, or $C^{1,\gamma}$ if $1 < p < 2$.*

Remark 4.13. At this point we need to add the stronger assumption (H1) on A and V in the case $1 < p < 2$, since in this case we assume the existence of a positive regular (super)solution. In fact, the proof presented here for $p < 2$ applies under the least assumptions on A and V that ensure the Lipschitz continuity of positive solutions. This fails if we just keep the assumption (E) on the matrix A, even for $V \equiv 0$ (see [Jin et al. 2009]). To our knowledge, the least known assumptions on A and V ensuring the Lipschitz continuity of solutions are due to Lieberman [1993] (see our Remark 2.9).

Proof of Theorem 4.12. From the AP theorem we may fix a positive (regular if $p < 2$) supersolution $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and a positive (regular if $p < 2$) solution $\tilde v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of (2-3). Setting $w_k = u_k/v$, we have, by Proposition 4.11, that $\nabla w_k \to 0$ in $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. The Rellich–Kondrachov theorem implies (see the proof of [Lieb and Loss 2001, Theorem 8.11]) that, up to a subsequence, $w_k \to c$ for some $c \ge 0$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$. This implies in turn that, up to a further subsequence, $u_k \to cv$ a.e. in Ω and also in $L^p_{\mathrm{loc}}(\Omega)$. Consequently, $c = 1/\|v\|_{L^p(K)} > 0$. It follows that any null sequence $\{u_k\}$ converges (up to a positive multiplicative constant) to the same positive (regular if $p < 2$) supersolution v. Since the solution $\tilde{v}$ is a (regular if $p < 2$) supersolution, we see that $v = C\tilde{v}$ for some $C > 0$, and therefore it is also the unique positive solution of (2-3) in Ω. $\square$

We can now close the chain of implications between the assertions of Theorem 3.10 (see Remark 3.11).

Corollary 4.14. *Let $\omega \Subset \mathbb{R}^n$ and suppose that A is uniformly elliptic and bounded matrix in ω, and $V \in M^q(p; \omega)$. If $1 < p < 2$, we suppose in addition that A and V satisfy hypothesis (H1).*

If the equation $Q'_{A,p,V}[v] = 0$ admits a positive, regular, strict supersolution in $W^{1,p}(\omega)$, then the principal eigenvalue is strictly positive.

Hence, all assertions of Theorem 3.10 are equivalent (if by a supersolution we mean, when $p < 2$, a regular one).

Proof. $\alpha_4' \implies \alpha_3$: From the AP theorem we get $Q_{A,p,V}[u; \omega] \geq 0$ for all $u \in C_c^\infty(\omega)$, which implies that $\lambda_1 \geq 0$. Suppose that $\lambda_1 = 0$. Then, by Remark 4.7 and Theorem 4.12, the principal eigenfunction, which is a positive (regular if $p < 2$) solution of (2-3) in ω is the unique (regular if $p < 2$) positive supersolution of that equation. This shows that this equation cannot have a positive strict (regular if $p < 2$) supersolution. $\square$

In the next theorem we state characterizations of criticality, subcriticality and existence of a null sequence. We also state a useful Poincaré inequality in the case where $Q_{A,p,V}$ is critical. It generalizes the corresponding results in [Pinchover and Tintarev 2006; 2007; 2008; Pinchover and Regev 2015; Takáč and Tintarev 2008].

Theorem 4.15. *Suppose that $Q_{A,p,V}$ is nonnegative on $C_c^\infty(\Omega)$ with A and V satisfying hypothesis (H0) if $p \geq 2$, or (H1) if $1 < p < 2$. Then*

(i) *$Q_{A,p,V}$ is critical in Ω if and only if $Q_{A,p,V}$ admits a null sequence.*

(ii) *$Q_{A,p,V}$ admits a null sequence if and only if (2-3) admits a unique positive (regular if $p < 2$) supersolution.*

(iii) *$Q_{A,p,V}$ is subcritical in Ω if and only if there exists a strictly positive weight function $W \in C^0(\Omega)$ such that (4-9) holds true.*

(iv) *If $Q_{A,p,V}$ admits a ground state ϕ, then there exists a strictly positive weight function $W \in C^0(\Omega)$ such that, for every $\psi \in C_c^\infty(\Omega)$ with $\int_\Omega \phi\psi \, dx \neq 0$, the following Poincaré-type inequality holds:*

$$Q_{A,p,V}[u] + C\left|\int_\Omega u\psi \, dx\right|^p \geq \frac{1}{C}\int_\Omega W|u|^p \, dx \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

and some positive constant $C > 0$.

Remark 4.16. In the sequel (Lemma 4.22) we add the following accompaniment to (i): *if $Q_{A,p,V}$ is critical in Ω, then there exists a null sequence that converges locally uniformly in Ω to the ground state.*

Proof of Theorem 4.15. (i) If $Q_{A,p,V}$ is critical in Ω, we claim that, for any $\varnothing \neq K \Subset \Omega$,

$$c_K := \inf_{\substack{0 \le u \in C_c^\infty(\Omega) \\ \|u\|_{L^p(K)}=1}} Q_{A,p,V}[u] = 0. \tag{4-15}$$

To see this, pick $W \in C_c^\infty(K) \setminus \{0\}$ such that $0 \le W \le 1$. Then

$$c_K \int_\Omega W|u|^p \,\mathrm{d}x \le c_K \le Q_{A,p,V}[u] \quad \text{for all } u \in C_c^\infty(\Omega) \text{ with } \|u\|_{L^p(K)} = 1,$$

a contradiction to the criticality of $Q_{A,p,V}$ in case $c_K > 0$. Picking one such K, (4-15) implies the existence of a null sequence with respect to $Q_{A,p,V}$.

If $Q_{A,p,V}$ admits a null sequence then, by Theorem 4.12, (2-3) admits a unique positive solution v, which is also its unique (regular if $p < 2$) positive supersolution. Suppose now, to the contrary, that $Q_{A,p,V}$ is subcritical in Ω with a nonzero nonnegative weight W. By the AP theorem we obtain a positive solution w of the equation $Q'_{A,p,V-W}[u] = 0$, which is readily another positive supersolution of (2-3). This contradicts the uniqueness of v, and thus $Q_{A,p,V}$ has to be critical in Ω.

(ii) The sufficiency is captured by Theorem 4.12. To prove the necessity, let v be the unique positive (super)solution of $Q'_{A,p,V}$ in Ω. By part (i) we have that the nonexistence of null sequences with respect to $Q_{A,p,V}$ implies that $Q_{A,p,V}$ is subcritical in Ω. Now the same argument as in the proof of the necessity of the first statement of part (i) implies that v is not unique, a contradiction.

(iii) The necessity follows by the definition of subcriticality. On the other hand, the proof of the sufficiency of the first statement of (i) implies that $c_K > 0$ for any domain $K \Subset \Omega$. Using a standard partition of unity argument we arrive at a strictly positive W that satisfies (4-9) (see [Pinchover and Tintarev 2007, Lemma 3.1]).

(iv) The proof is identical to [Pinchover and Tintarev 2007, Theorem 1.6(4)] (and also [Pinchover and Regev 2015]). □

Corollary 4.17. *Suppose that for $i = 0, 1$ the functional Q_{A,p,V_i} is nonnegative in Ω with A and V_i satisfying hypothesis (H0) if $p \ge 2$, or (H1) if $1 < p < 2$. For $t \in (0, 1)$ set*

$$V_t := (1-t)V_0 + tV_1.$$

Then Q_{A,p,V_t} is nonnegative in Ω for all $t \in [0, 1]$. Moreover, if $\mathcal{L}^n(\{V_0 \neq V_1\}) > 0$, then Q_{A,p,V_t} is subcritical in Ω for any $t \in (0, 1)$.

Proof. The nonnegativity of Q_{A,p,V_t} for $t \in (0, 1)$ follows from the obvious relation

$$Q_{A,p,V_t}[u] = (1-t)Q_{A,p,V_0}[u] + tQ_{A,p,V_1}[u]. \tag{4-16}$$

Suppose now that $\{u_k\} \subset C_c^\infty(\Omega)$ is a null sequence with respect to Q_{A,p,V_t} in Ω for some $t \in (0, 1)$ such that $u_k \to \phi$ in $L^p_{\mathrm{loc}}(\Omega)$. It follows from (4-16) that $\{u_k\}$ is also a null sequence for Q_{A,p,V_0} and Q_{A,p,V_1} in Ω. By Theorem 4.12, ϕ is a solution of $Q'_{A,p,V_i}[u] = 0$ in Ω, for both values of i, which is impossible since $\mathcal{L}^n(\{V_0 \neq V_1\}) > 0$. □

Finally, we state generalizations of the corresponding results in [Pinchover and Tintarev 2007; Pinchover and Regev 2015]. We skip their proofs since they are essentially the same.

Proposition 4.18. *Suppose $\Omega' \subsetneq \Omega$ is a domain. Let A and V satisfy hypothesis (H0) if $p \geq 2$, or (H1) if $1 < p < 2$.*

(a) *If $Q_{A,p,V}$ is nonnegative in Ω, then $Q_{A,p,V}$ is subcritical in Ω'.*

(b) *If $Q_{A,p,V}$ is critical in Ω', then $Q_{A,p,V}$ is supercritical in Ω.*

Proposition 4.19. *Suppose that $Q_{A,p,V}$ is subcritical in Ω with A and V satisfying hypothesis (H0) if $p \geq 2$, or (H1) if $1 < p < 2$. Let $U \in L^\infty(\Omega) \setminus \{0\}$ be such that $U \ngeq 0$ and $\operatorname{supp}\{U\} \Subset \Omega$. Then there exist $\tau_+ > 0$ and $\tau_- \in [-\infty, 0)$ such that $Q_{A,p,V+tU}$ is subcritical in Ω if and only if $t \in (\tau_-, \tau_+)$ and $Q_{A,p,V+\tau_+ U}$ is critical in Ω.*

Proposition 4.20. *Suppose that $Q_{A,p,V}$ is critical in Ω with A and V satisfying hypothesis (H0) if $p \geq 2$, or (H1) if $1 < p < 2$. Denote by ϕ the corresponding ground state. Consider $U \in L^\infty(\Omega)$ such that $\operatorname{supp}\{U\} \Subset \Omega$. Then there exists $0 < \tau_+ \leq \infty$ such that $Q_{A,p,V+tU}$ is subcritical in Ω for $t \in (0, \tau_+)$ if and only if $\int_\Omega U|\phi|^p \,\mathrm{d}x > 0$.*

The following theorem extends the corresponding theorems in [Pinchover 2007; Pinchover and Regev 2015; Pinchover et al. 2008]; see some applications therein.

Theorem 4.21 (Liouville comparison theorem). *Suppose that for $i = 1, 2$ the functional Q_{A_i,p,V_i} is nonnegative in Ω with A_i and V_i satisfying hypothesis (H0) if $p \geq 2$, or (H1) if $1 < p < 2$. Suppose in addition that*:

(1) *Q_{A_2,p,V_2} admits a ground state ϕ in Ω.*

(2) *The equation $Q'_{A_1,p,V_1}[u] = 0$ in Ω admits a weak subsolution ψ with $\psi^+ \neq 0$.*

(3) *There exists $M > 0$ such that the matrix $(M\phi(x))^2 A_2(x) - (\psi^+(x))^2 A_1(x)$ is nonnegative-definite in $\mathbb{R}^n$ for almost every $x \in \Omega$.*

(4) *There exists $N > 0$ such that $|\nabla\psi|_{A_1(x)}^{p-2} \leq N^{p-2} |\nabla\phi|_{A_2(x)}^{p-2}$ for almost every x in $\Omega \cap \{\psi > 0\}$.*

Then the functional Q_{A_1,p,V_1} is critical in Ω, and ψ is the unique positive supersolution of $Q'_{A_1,p,V_1}[u] = 0$ in Ω.

We close this section by showing that the ground state is a locally uniform limit of a null sequence. This is a generalization of the second statement of [Pinchover and Regev 2015, Theorem 6.1(2)]. We give a detailed proof, as it utilizes many of the results presented above.

Lemma 4.22. *Suppose $Q_{A,p,V}$ is critical in Ω with A and V satisfying hypothesis (H0) if $p \geq 2$, or (H1) if $1 < p < 2$. Then $Q_{A,p,V}$ admits a null sequence that converges locally uniformly to the ground state.*

Proof. Let $\{\omega_i\}_{i\in\mathbb{N}}$ be a sequence of Lipschitz domains such that $\omega_i \Subset \Omega$, $\omega_i \Subset \omega_{i+1}$ for $i \in \mathbb{N}$, and $\bigcup_{i\in\mathbb{N}} \omega_i = \Omega$. We fix $x_0 \in \omega_1$ and a nonnegative $U \in C_c^\infty(\Omega) \setminus \{0\}$ with $\operatorname{supp}\{U\} \subset \omega_1$. By Proposition 4.19, for every $i \in \mathbb{N}$ there exists $t_i > 0$ such that the functional $Q_{A,p,V-t_i U}$ is critical in ω_i. For $i \in \mathbb{N}$ we denote by $\phi_i \in W^{1,p}(\omega_i)$ the corresponding ground states, normalized by $\phi_i(x_0) = 1$. The sequence of the t_i is strictly decreasing with i. Indeed, we have by Proposition 4.18 that $Q_{A,p,V-t_iU}$ has to be supercritical in ω_{i+1}. There thus exists $u \in C_c^\infty(\omega_{i+1})$ such that $Q_{A,p,V-t_iU}[u;\omega_{i+1}] < 0$. This in turn implies that

$$Q_{A,p,V-t_{i+1}U}[u;\omega_{i+1}] < (t_i - t_{i+1}) \int_{\omega_{i+1}} U|u|^p \,\mathrm{d}x.$$

The criticality of $Q_{A,p,V-t_{i+1}U}$ in ω_{i+1} implies by definition that $Q_{A,p,V-t_{i+1}U}$ is nonnegative in ω_{i+1} and thus $t_i > t_{i+1}$. Setting $t_\infty := \lim_{i\to\infty} t_i$, by Harnack's convergence principle (Proposition 2.11), up to a subsequence, $\{\phi_i\}_{i\in\mathbb{N}}$ converges locally uniformly to a positive solution v of the equation $Q'_{A,p,V-t_\infty U}[u] = 0$ in Ω. The AP theorem (Theorem 4.3) implies that $Q_{A,p,V-t_\infty U}$ is nonnegative in Ω. Clearly, $t_\infty \geq 0$. Let us show that in fact $t_\infty = 0$. If not then $V - t_\infty U \leq V$ a.e. in Ω and, since by our assumptions $Q_{A,p,V}$ is critical in Ω, Proposition 4.8(b) gives that $Q_{A,p,V-t_\infty U}$ is supercritical, contradicting its nonnegativity.

Summarizing, for each $i \in \mathbb{N}$ we have obtained a ground state $\phi_i \in W^{1,p}(\omega_i)$ of $Q_{A,p,V-t_iU}$ in ω_i, and the sequence $\{\phi_i\}_{i\in\mathbb{N}}$ converges locally uniformly to a positive solution v of the equation (2-3) in Ω. To conclude, we will show that $\{\phi_i\}_{i\in\mathbb{N}}$ is in fact a null sequence. Consider the principal eigenvalue $\lambda_1(Q_{A,p,V-t_iU_i};\omega_i)$, $i \in \mathbb{N}$, which is nonnegative. Suppose that for some $i \in \mathbb{N}$ we had $\lambda_1(Q_{A,p,V-t_iU_i};\omega_i) > 0$. Then the principal eigenfunction $v_1^{\omega_i} \in W_0^{1,p}(\omega_i)$ would be a positive, strict supersolution of the equation $Q'_{A,p,V-t_iU}[v;\omega_i] = 0$, which contradicts the fact that ϕ_i is the unique positive supersolution and also a solution of $Q'_{A,p,V-t_iU}[v;\omega_i] = 0$ (see Theorem 4.12). Thus, for each $i \in \mathbb{N}$, $\lambda_1(Q_{A,p,V-t_iU_i};\omega_i) = 0$ and, since ϕ_i is also the unique positive solution of $Q'_{A,p,V-t_iU}[v;\omega_i] = 0$ (see Theorem 4.12 again), we conclude $\phi_i = v_1^{\omega_i} \in W_0^{1,p}(\omega_i)$. Consequently,

$$\lim_{i\to\infty} Q_{A,p,V}[\phi_i] = \lim_{i\to\infty} t_i \int_{\Omega_1} U\phi_i^p \,\mathrm{d}x = 0.$$

After a further normalization, we may assume that for some $\varnothing \neq K \Subset \Omega$, there also holds $\|\phi_i\|_{L^p(K)} = 1$ for all $i \in \mathbb{N}$. $\square$

5. Positive solutions of minimal growth at infinity

The present section is devoted to the existence of positive solutions of the equation $Q'_{A,p,V}[v] = 0$ in $\Omega \setminus \{x_0\}$ that have minimal growth at infinity in Ω, and their role in criticality theory. For this purpose we extend in the following subsection the *weak comparison principle* (WCP) (cf. [García-Melián and Sabina de Lis 1998; Pinchover and Regev 2015]). Section 5B is devoted to the study of the behaviour of positive solutions near an isolated singularity. Finally, in Section 5C we study positive solutions of minimal growth at infinity in Ω, and prove the last two parts of the Main Theorem.

5A. *Weak comparison principle* (*WCP*). We prove first a simple version of the WCP that holds true for the p-Laplacian operator with a *nonnegative* potential (see [Pucci and Serrin 2007, Theorem 2.4.1], for instance).

Lemma 5.1. *Let ω be a Lipschitz domain in $\mathbb{R}^n$. Suppose that A is a uniformly elliptic and bounded matrix in ω, and $\mathcal{G}, \mathcal{V} \in M^q(p;\omega)$ with $\mathcal{V} \geq 0$ a.e. in Ω. Suppose that v_1 (respectively v_2) is a subsolution* (*respectively supersolution*) *of the equation*

$$Q'_{A,p,\mathcal{V}}[v]=\mathcal{G} \quad \text{in } \omega. \tag{5-1}$$

If $v_1 \le v_2$ *a.e. on* $\partial\omega$ *in the trace sense*, *then* $v_1 \le v_2$ *a.e. in* ω.

Proof. Our assumption that $v_1 \le v_2$ a.e. on $\partial\omega$ implies $(v_2-v_1)^- \in W_0^{1,p}(\omega)$. Using this as a test function in the definitions of v_1 and v_2 being, respectively, sub- and supersolutions of (5-1), and subtracting the two resulting inequalities, we obtain

$$\int_\omega (|\nabla v_1|_A^{p-2}A\nabla v_1 - |\nabla v_2|_A^{p-2}A\nabla v_2)\cdot\nabla(v_2-v_1)^-\,\mathrm{d}x + \int_\omega \mathcal{V}(|v_1|^{p-2}v_1-|v_2|^{p-2}v_2)(v_2-v_1)^-\,\mathrm{d}x \le 0.$$

In other words,

$$\int_{\{v_2<v_1\}} \big((|\nabla v_1|_A^{p-2}A\nabla v_1 - |\nabla v_2|_A^{p-2}A\nabla v_2)\cdot(\nabla v_1-\nabla v_2)\,\mathrm{d}x + \mathcal{V}(|v_1|^{p-2}v_1-|v_2|^{p-2}v_2)(v_1-v_2)\big)\,\mathrm{d}x \le 0.$$

By (2-17) we have that each term of the sum of the integrand is nonnegative, with equality if and only if $\nabla v_1=\nabla v_2$ a.e. in the set $\{v_2<v_1\}$, or equivalently if $(v_2-v_1)^-=c\ge 0$ a.e. in ω. Since $(v_2-v_1)^-=0$ a.e. on $\partial\omega$ in the trace sense, we conclude $v_1\le v_2$ a.e. in ω. □

The following proposition deals with the sub/supersolution technique:

Proposition 5.2. *Let* ω *be a Lipschitz domain in* $\mathbb{R}^n$. *Assume that* A *is a uniformly elliptic and bounded matrix in* ω, *and* $g, V \in M^q(p;\omega)$, *where* $g\ge 0$ *a.e. in* ω. *Let* $f,\varphi,\psi \in W^{1,p}(\omega)\cap C(\bar\omega)$, *where* $f\ge 0$ *a.e. in* ω, *and*

$$\begin{cases} Q'_{A,p,V}[\psi] \le g \le Q'_{A,p,V}[\varphi] & \text{in } \omega, \text{ in the weak sense},\\ \psi\le f\le\varphi & \text{on } \partial\omega,\\ 0\le\psi\le\varphi & \text{in } \omega.\end{cases}$$

Then there exists a nonnegative solution $u\in W^{1,p}(\omega)\cap C(\bar\omega)$ *of*

$$\begin{cases} Q'_{A,p,V}[u]=g & \text{in } \omega,\\ u=f & \text{on } \partial\omega,\end{cases} \tag{5-2}$$

such that $\psi\le u\le\varphi$ *in* ω.

Moreover, *if* $f>0$ *a.e. in* $\partial\omega$, *then the solution* u *is the unique solution of* (5-2).

Proof. Consider the set

$$\mathcal{K} := \{v\in W^{1,p}(\omega)\cap C(\bar\omega) \mid 0\le\psi\le v\le\varphi \text{ in } \omega\}.$$

For any $x\in\omega$ and $v\in\mathcal{K}$ we define

$$G(x,v) := g(x)+2V^-(x)(v(x))^{p-1}.$$

Note that $G\in M^q(p;\omega)$ and $G\ge 0$ a.e. in ω. The map $T:\mathcal{K}\to W^{1,p}(\omega)$ defined by $T(v)=u$, where u is the solution of

$$\begin{cases} Q'_{A,p,|V|}[u]=G(x,v) & \text{in } \omega,\\ u=f & \text{in the trace sense on } \partial\omega,\end{cases} \tag{5-3}$$

is well-defined by Propositions 3.6 and 3.7. Indeed, consider the functionals

$$J, \bar{J} : W^{1,p}(\omega) \to \mathbb{R} \cup \{\infty\}$$

defined in (3-12) and (3-11), respectively, with $\mathcal{V} = |V|$ and $\mathcal{G} = G(x, v)$. Let

$$\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{A} := \{u \in W^{1,p}(\omega) \mid u = f \ \text{ on } \ \partial\omega\}$$

be such that

$$J[u_k] \downarrow m := \inf_{u\in\mathcal{A}} J[u].$$

Since $f \geq 0$, we have that $\{|u_k|\}_{k\in\mathbb{N}} \subset \mathcal{A}$ as well, which implies $m \leq J[|u_k|] = \bar{J}[u_k] \leq J[u_k]$, the latter inequality holds since $\mathcal{G} \geq 0$ a.e. in ω. In particular, it follows that $\inf_{u\in\mathcal{A}} \bar{J}[u] = m$. Letting $k \to \infty$, we deduce

$$\bar{J}[u_k] \to m.$$

But, by Proposition 3.6(b), we have that $\bar{J}$ is weakly lower semicontinuous and, by Proposition 3.7(a), it is also coercive. Since $\mathcal{A}$ is weakly closed, it follows (see [Struwe 2008, Theorem 1.2], for example) that m is achieved by a nonnegative function $u \in \mathcal{A}$ that satisfies $\bar{J}(u) = m$. Moreover, $J(u) = \bar{J}(u) = m$. So u is a minimizer of J on $\mathcal{A}$ and hence a solution of (5-3).

Observe that the map T is monotone. Indeed, let $v_1, v_2 \in \mathcal{K}$ be such that $v_1 \leq v_2$. Then, since $G(x, v)$ is increasing in v, we have

$$Q'_{A,p,|V|}[T(v_1); \omega] = g(x, v_1) \leq g(x, v_2) = Q'_{A,p,|V|}[T(v_2); \omega]$$

and, since $T(v_1) = f = T(v_2)$ on $\partial\omega$, we get from Lemma 5.1 with $\mathcal{V} = |V|$ and $\mathcal{G} = g(x, v_1)$ that $T(v_1) \leq T(v_2)$ in ω.

Let $v \in W^{1,p}(\omega) \cap C(\bar{\omega})$ be a subsolution of (5-2). Then $Q'_{A,p,|V|}[v] = Q'_{A,p,V}[v] + G(x, v) - g(x) \leq G(x, v)$ in ω, in the weak sense, and thus v is a subsolution of (5-3). On the other hand, $T(v)$ is a solution of (5-3). Lemma 5.1 with $\mathcal{V} = |V|$ and $\mathcal{G} = G(x, v)$ gives $v \leq T(v)$ a.e. in ω. This implies in turn that

$$Q'_{A,p,V}[T(v)] = g + 2V^- \big(|v|^{p-2}v - |T(v)|^{p-2}T(v)\big) \leq g \quad \text{in } \omega,$$

in the weak sense.

Summarizing, if v is a subsolution of (5-2), then $T(v)$ is a subsolution of (5-2) such that $v \leq T(v)$ a.e. in ω. In the same fashion, we can show that if $v \in W^{1,p}(\omega) \cap C(\bar{\omega})$ is a supersolution of (5-2) then $T(v)$ is a supersolution of (5-2) such that $v \geq T(v)$ a.e. in ω.

Defining the sequences

$$\underline{u}_0 := \psi, \quad \underline{u}_n := T(\underline{u}_{n-1}) = T^{(n)}(\psi) \qquad \text{and} \qquad \bar{u}_0 := \varphi, \quad \bar{u}_n := T(\bar{u}_{n-1}) = T^{(n)}(\varphi), \qquad n \in \mathbb{N},$$

we get from the above considerations that $\{\underline{u}_n\}$ and $\{\bar{u}_n\}$ increases and decreases, respectively, to functions $\underline{u}$ and $\bar{u}$ for every $x \in \omega$. Moreover, the convergence is clearly also in $L^p(\omega)$ (by Theorem 1.9 in [Lieb and Loss 2001]). Then, using an argument similar to the proof of Proposition 2.11, it follows that $\underline{u}$ and $\bar{u}$ are fixed points of T, and both solve (5-2) and satisfy $\psi \leq \underline{u} \leq \bar{u} \leq \varphi$ in ω.

The uniqueness claim follows from Lemma 3.3(iii). □

Finally, we extend the WCP (cf. [García-Melián and Sabina de Lis 1998; Pinchover and Regev 2015; Pucci and Serrin 2007]):

Theorem 5.3 (weak comparison principle). *Let $\omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose that A is a uniformly elliptic and bounded matrix in ω, and $g, V \in M^q(p;\omega)$ with $g \geq 0$ a.e. in ω. Assume that $\lambda_1 > 0$, where λ_1 is the principal eigenvalue of the operator $Q'_{A,p,V}$ defined by (3-3). Let $u_2 \in W^{1,p}(\omega) \cap C(\bar{\omega})$ be a solution of*

$$\begin{cases} Q'_{A,p,V}[u_2] = g & \text{in } \omega, \\ u_2 > 0 & \text{on } \partial\omega. \end{cases}$$

If $u_1 \in W^{1,p}(\omega) \cap C(\bar{\omega})$ satisfies

$$\begin{cases} Q'_{A,p,V}[u_1] \leq Q'_{A,p,V}[u_2] & \text{in } \omega, \\ u_1 \leq u_2 & \text{on } \partial\omega, \end{cases}$$

then $u_1 \leq u_2$ in ω.

Proof. Since u_2 is a supersolution of (2-3) in ω that is positive on $\partial\omega$, the strong maximum principle implies $u_2 > 0$ in $\bar{\omega}$. Let $c := \max\{1, \max_{\bar{\omega}} u_1 / \min_{\bar{\omega}} u_2\}$, then $u_1 \leq cu_2$ in $\bar{\omega}$. Consider now the problem

$$\begin{cases} Q'_{A,p,V}[v] = g & \text{in } \omega, \\ v = u_2 & \text{on } \partial\omega. \end{cases} \tag{5-4}$$

By the choice of c and our assumption we have that cu_2 is a supersolution of (5-4) such that $u_1 \leq u_2 \leq cu_2$ on $\partial\omega$, while u_1 is a subsolution of (5-4). Applying Proposition 5.2 with $\psi = u_1$ and $\varphi = cu_2$, we get a unique solution v of (5-4) such that $u_1 \leq v \leq cu_2$ in ω and $v = u_2$ on $\partial\omega$, in the trace sense. Clearly, v is a supersolution of (2-3) in ω that is positive on $\partial\omega$. Again, by the strong maximum principle, we get $v > 0$ in $\bar{\omega}$. By the uniqueness of the boundary problem (5-4) (Proposition 5.2), we have $v = u_2$. Hence, $u_1 \leq u_2$ in ω. □

5B. *Behaviour of positive solutions near an isolated singularity.* Using the weak comparison principle of Theorem 5.3 we study the behaviour of positive solutions near an isolated singular point. We have:

Theorem 5.4. *Let $p \leq n$ and $x_0 \in \Omega$. Suppose A and V satisfy hypothesis (H0) in Ω, and let u be a nonnegative solution of the equation $Q'_{A,p,V}[v] = 0$ in $\Omega \setminus \{x_0\}$.*

(1) *If u is bounded near x_0, then u can be extended to a nonnegative solution in Ω.*

(2) *If u is unbounded near x_0, then $\lim_{x \to x_0} u(x) = \infty$.*

Proof. (1) This is a special case of [Malý and Ziemer 1997, Theorem 3.16], which is in turn an extension to $V \in M^q_{\text{loc}}(p;\Omega)$ of [Serrin 1964, Theorem 10], where V is assumed to be in $L^q_{\text{loc}}(\Omega)$ for some $q > n/p$. In particular, this part of the theorem holds true for solutions of arbitrary sign in $\Omega \setminus o$, where o is a set having zero *p-capacity*.

(2) We follow the argument in [Fraas and Pinchover 2011] (for a slightly different argument see [Serrin 1964, p. 278]). Without loss of generality, we assume that $x_0 = 0$ and $B_1(0) \Subset \Omega$. For $r > 0$, we write the ball as $B_r := B_r(0)$, and the corresponding sphere as $S_r := \partial B_r$.

Since $\limsup_{x\to 0} u(x) = \infty$, there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \Omega$ converging to 0 such that $u(x_k) \to \infty$ as $k \to \infty$. Let $r_k = |x_k|$, where $k = 1, 2, \dots$, and consider the annular domains $\mathbb{A}_k := B_{3r_k/2} \setminus \bar{B}_{r_k/2}$. For each k we scale $\mathbb{A}_k$ to the fixed annulus $\mathbb{A}' := B_{3/2}(0) \setminus \bar{B}_{1/2}(0)$. Note next that if u is a solution of the equation $Q'_{A,p,V}[v] = 0$ in $\Omega \setminus \{0\}$ then, for any positive R, the function $u_R(x) := u(Rx)$ satisfies the equation

$$Q'_{A_R,p,V_R}[u_R] := -\operatorname{div}_{A_R}\big(|\nabla u_R|_{A_R}^{p-2} A_R(x)\nabla u_R\big) + V_R(x)|u_R|^{p-2}u_R = 0 \quad \text{in } \Omega_R, \tag{5-5}$$

where $A_R(x) := A(Rx)$, $V_R(x) := R^p V(Rx)$ and $\Omega_R := \{x/R \mid x \in \Omega \setminus \{0\}\}$. Applying thus the Harnack inequality in $\mathbb{A}'$, we have, for k sufficiently large,

$$\sup_{x\in\mathbb{A}_k} u(x) = \sup_{x\in\mathbb{A}'} u_{r_k}(x) \le C \inf_{x\in\mathbb{A}'} u_{r_k}(x) = C \inf_{x\in\mathbb{A}_k} u(x), \tag{5-6}$$

where the positive constant C is independent of r_k. To see this, for example in the case $p < n$, observe that $\|V_R\|_{M^q(\mathbb{A}')} = R^{p-n/q}\|V\|_{M^q(\mathbb{A}_R)}$ and, by our assumptions on q, we have that the exponent on R is nonnegative (it is in fact positive). Now from (5-6) we may readily deduce

$$\min_{S_{r_k}} u(x) \to \infty \quad \text{as } k \to \infty. \tag{5-7}$$

Let v be a fixed positive solution of the equation $Q'_{A,p,V}[w] = 0$ in B_1 and set, for $0 < r < 1$,

$$m_r := \min_{S_r} \frac{u(x)}{v(x)}.$$

Then, as in [Fraas and Pinchover 2011, Lemma 4.2], the WCP implies that the function m_r is monotone as $r \to 0$. This, together with (5-7), implies that m_r is monotone nonincreasing near 0. Therefore, $\lim_{r\to 0} m_r = \infty$ and, thus, $\lim_{x\to 0} u(x) = \infty$. □

Remark 5.5. The asymptotic behaviour of positive solutions of the equation $Q'_{A,p,V}[v] = 0$ near an isolated singular point remains open for further studies (see [Fraas and Pinchover 2011; 2013; Pinchover and Tintarev 2008] and the references therein for partial results).

5C. *Positive solutions of minimal growth and Green's function.* The following notion was introduced by Agmon [1983] in the linear case and was extended to p-Laplacian-type equations of the form (1-4) in [Pinchover and Tintarev 2007; Pinchover and Regev 2015].

Definition 5.6. Let K_0 be a compact subset of Ω. A positive solution u of (2-3) in $\Omega \setminus K_0$ is said to be a *positive solution of minimal growth in a neighbourhood of infinity in* Ω, and denoted by $u \in \mathcal{M}_{\Omega;K_0}$ if, for any smooth compact subset of Ω with $K_0 \Subset \operatorname{int} K$ and any positive supersolution $v \in C(\Omega \setminus \operatorname{int} K)$ of (2-3) in $\Omega \setminus K$, we have

$$u \le v \quad \text{on } \partial K \quad \Longrightarrow \quad u \le v \quad \text{in } \Omega \setminus K.$$

If $u \in \mathcal{M}_{\Omega;\varnothing}$, then u is called a *global minimal solution of* (2-3) *in* Ω.

We first prove that if $Q_{A,p,V}$ is nonnegative in Ω then $\mathcal{M}_{\Omega;\{x_0\}} \ne \varnothing$ for any $x_0 \in \Omega$. This result extends the corresponding results in [Pinchover and Tintarev 2007; 2008; Pinchover and Regev 2015].

Theorem 5.7. *Suppose that $Q_{A,p,V}$ is nonnegative in Ω, where A and V satisfy hypothesis (H0). Then, for any $x_0 \in \Omega$, the equation $Q'_{A,p,V}[v] = 0$ admits a solution $u \in \mathcal{M}_{\Omega;\{x_0\}}$.*

Proof. We fix a point $x_0 \in \Omega$ and let $\{\omega_i\}_{i\in\mathbb{N}}$ be a sequence of Lipschitz domains such that $x_0 \in \omega_1$, $\omega_i \Subset \omega_{i+1} \Subset \Omega$ for $i \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} \omega_i = \Omega$. Setting $r_1 := \sup_{x\in\omega_1} \operatorname{dist}(x; \partial\omega_1)$ (the inradius of ω_1), we define the open sets

$$U_i := \omega_i \setminus \bar{B}_{r_1/(i+1)}(x_0).$$

Pick a fixed reference point $x_1 \in U_1$ and note that $U_i \Subset U_{i+1}$, $i \in \mathbb{N}$, and also $\bigcup_{i\in\mathbb{N}} U_i = \Omega \setminus \{x_0\}$. Also let $f_i \in C_c^\infty\big(B_{r_1/i}(x_0) \setminus \bar{B}_{r_1/(i+1)}(x_0)\big) \setminus \{0\}$ be a sequence of nonnegative functions. The nonnegativity of $Q_{A,p,V}$ implies $\lambda_1(Q_{A,p,V+1/i}; U_i) > 0$, and thus, by Theorem 3.10, we obtain for each $i \in \mathbb{N}$ a positive solution v_i of

$$\begin{cases} Q'_{A,p,V+1/i}[v] = f_i & \text{in } U_i, \\ v = 0 & \text{on } \partial U_i. \end{cases}$$

Normalizing by $u_i(x) := v_i(x)/v_i(x_1)$, the Harnack convergence principle (Proposition 2.11) implies that $\{u_i\}_{i\in\mathbb{N}}$ admits a subsequence converging uniformly in compact subsets of $\Omega \setminus \{x_0\}$ to a positive solution u of the equation $Q'_{A,p,V}[w] = 0$ in $\Omega \setminus \{x_0\}$.

We claim that $u \in \mathcal{M}_{\Omega;\{x_0\}}$. To this end, let K be a compact smooth subset of Ω such that $x_0 \in \operatorname{int} K$, and let $v \in C(\Omega \setminus \operatorname{int} K)$ be a positive supersolution of (2-3) in $\Omega \setminus K$ with $u \le v$ on ∂K. Let $\delta > 0$. There then exists $i_K \in \mathbb{N}$ such that $\operatorname{supp}\{f_i\} \Subset K$ for all $i \ge i_K$ and, in addition, $u_i \le (1+\delta)v$ on $\partial(\omega_i \setminus K)$. The WCP (Theorem 5.3) implies $u_i \le (1+\delta)v$ in $\omega_i \setminus K$ and letting $i \to \infty$ we obtain $u \le (1+\delta)v$ in $\Omega \setminus K$. Since $\delta > 0$ is arbitrary, we conclude $u \le v$ in $\Omega \setminus K$. □

Definition 5.8. A function $u \in \mathcal{M}_{\Omega,\{x_0\}}$ having a nonremovable singularity at x_0 is called a *minimal positive Green function of $Q'_{A,V}$ in Ω with a pole at x_0*. We denote such a function by $G^{\Omega}_{A,V}(x, x_0)$.

The following theorem states that criticality is equivalent to the existence of a global minimal solution, that is, (1) $\Longleftrightarrow$ (5) in the Main Theorem presented in the introduction. It extends [Pinchover and Regev 2015, Theorem 9.6] and also [Pinchover and Tintarev 2007, Theorem 5.5; 2008, Theorem 5.8].

Theorem 5.9. *Suppose that $Q_{A,p,V}$ is nonnegative in Ω with A and V satisfying hypothesis (H0) if $p \ge 2$, or (H1) if $1 < p < 2$. Then $Q_{A,p,V}$ is subcritical in Ω if and only if (2-3) does not admit a global minimal solution in Ω. In particular, ϕ is a ground state of (2-3) in Ω if and only if ϕ is a global minimal solution of (2-3) in Ω.*

Proof. To prove necessity, let $Q_{A,p,V}$ be subcritical in Ω. Clearly (by the AP theorem) there exists a continuous positive strict supersolution v of (2-3) in Ω. We proceed by contradiction. Suppose there exists a global minimal solution u of (2-3) in Ω and fix K to be a compact smooth subset of Ω. Let $\varepsilon_{\partial K} := \min_{\partial K} v / \max_{\partial K} u$. Then $\varepsilon_{\partial K} u \le v$, and $\varepsilon_{\partial K}^{-1} v$ is also a positive continuous supersolution of (2-3) in Ω. Using it as a comparison function in the definition of $u \in \mathcal{M}_{\Omega;\varnothing}$, we get $\varepsilon_{\partial K} u \le v$ in $\Omega \setminus K$. Letting also $\varepsilon_K := \min_K v / \max_K u$, we readily have $\varepsilon_K u \le v$ in K. Consequently, by setting $\varepsilon := \min\{\varepsilon_{\partial K}, \varepsilon_K\}$ we have

$$\varepsilon u \le v \quad \text{in } \Omega.$$

Now we define

$$\varepsilon_0 := \max\{\varepsilon > 0 \mid \varepsilon u \le v \text{ in } \Omega\}$$

and note that, since $\varepsilon_0 u$ and v are, respectively, a continuous solution and a continuous strict supersolution of (2-3) in Ω, we have $\varepsilon_0 u \not\equiv v$. There thus exist $x_1 \in \Omega$ and $\delta, r > 0$ such that $B_r(x_1) \subset \Omega$ and

$$(1+\delta)\varepsilon_0 u(x) \le v(x) \quad \text{for all } x \in \bar{B}_r(x_1).$$

But, since $u \in \mathcal{M}_{\Omega;\varnothing}$, it follows that

$$(1+\delta)\varepsilon_0 u(x) \le v(x) \quad \text{for all } x \in \Omega \setminus \bar{B}_r(x_1).$$

Consequently, $(1+\delta)\varepsilon_0 u(x) \le v(x)$ in Ω, which contradicts the definition of ε_0. We note that in the proof of this part we did not use the further regularity assumption (H1).

To prove sufficiency, assume that $Q_{A,p,V}$ is critical in Ω with ground state ϕ satisfying $\phi(x_1) = 1$ for some $x_1 \in \Omega$. We will prove that $\phi \in \mathcal{M}_{\Omega;\varnothing}$. To this end, consider an exhaustion $\{\omega_i\}_{i\in\mathbb{N}}$ of Ω such that $x_0 \in \omega_1$ and $x_1 \in \Omega \setminus \omega_1$. Fix $j \in \mathbb{N}$ and let $f_j \in C_c^\infty(B_{r_1/j}(x_0)) \setminus \{0\}$ satisfy $0 \le f_j(x) \le 1$, where, as in the previous proof, we write r_1 for the inradius of ω_1. Let $v_{i,j}$ be a positive solution of

$$\begin{cases} Q'_{A,p,V}[v] = f_j & \text{in } \omega_i, \\ v = 0 & \text{on } \partial\omega_i. \end{cases}$$

The WCP (Theorem 5.3) ensures that the sequence $\{v_{i,j}\}_{i\in\mathbb{N}}$ is nondecreasing. If $\{v_{i,j}(x_1)\}$ is bounded, then the sequence converges to v_j, where v_j is such that $Q'_{A,p,V}[v_j] = f_j$ in Ω. Thus v_j would be a strict supersolution of (2-3), which contradicts Theorem 4.15, since the ground state ϕ is the only positive supersolution of $Q'_{A,p,V}[w] = 0$ in Ω. Therefore, $v_{i,j}(x_1) \to \infty$ as $i \to \infty$. Defining thus the normalized sequence $u_{i,j}(x) := v_{i,j}(x)/v_{i,j}(x_1)$, by the Harnack convergence principle (Proposition 2.11) we may extract a subsequence of $\{u_{i,j}\}$ that converges as $i \to \infty$ to a positive solution u_j of the equation (2-3) in Ω. Once again, by the uniqueness of the ground state, we have $u_j = \phi$.

Now let K be a smooth compact set of Ω and assume that $x_0 \in \operatorname{int}(K)$. Let $v \in C(\Omega \setminus \operatorname{int} K)$ be a positive supersolution of (2-3) in $\Omega \setminus K$ such that $\phi \le v$ on ∂K. Let $j \in \mathbb{N}$ be large enough that $\operatorname{supp}\{f_j\} \Subset K$. For any $\delta > 0$ there exists $i_\delta \in \mathbb{N}$ such that, for $i \ge i_\delta$,

$$\begin{cases} 0 = Q'_{A,p,V}[u_{i,j}] \le Q'_{A,p,V}[v] & \text{in } \omega_i \setminus K, \\ Q'_{A,p,V}[v] \ge 0 & \text{in } \omega_i \setminus K, \\ 0 \le u_{i,j} \le (1+\delta)v & \text{on } \partial(\omega_i \setminus K), \end{cases}$$

which implies that $\phi = u_j \le (1+\delta)v$ in $\Omega \setminus K$. Letting $\delta \to 0$ we obtain $\phi \le v$ in $\Omega \setminus K$. □

To conclude the paper, it remains to establish the equivalence between (1) and (6) of the Main Theorem.

Theorem 5.10. *Suppose that $Q_{A,p,V}$ is nonnegative in Ω with A and V satisfying hypothesis (H0) if $p \ge 2$, or (H1) if $1 < p < 2$. Let $u \in \mathcal{M}_{\Omega,\{x_0\}}$ for some $x_0 \in \Omega$.*

(a) *If u has a removable singularity at x_0, then $Q_{A,p,V}$ is critical in Ω.*

(b) *Let* $1 < p \leq n$ *and suppose that* u *has a nonremovable singularity at* x_0; *then* $Q_{A,p,V}$ *is subcritical in* Ω.

(c) *Let* $p > n$ *and suppose that* u *has a nonremovable singularity at* x_0. *Assume also that* $\lim_{x\to x_0} u(x) = c$, *where* c *is a positive constant. Then* $Q_{A,p,V}$ *is subcritical in* Ω.

Proof. (a) If u has a removable singularity at x_0, its continuous extension is a global minimal solution in Ω, and Theorem 5.9 assures that $Q_{A,p,V}$ is critical in Ω.

(b) Assume that u has a nonremovable singularity at x_0 and suppose for the sake of contradiction that $Q_{A,p,V}$ is critical in Ω. Theorem 5.9 implies the existence of a global minimal solution v of (2-3) in Ω. By Theorem 5.4 we have $\lim_{x\to x_0} u(x) = \infty$ and thus, by comparison, $v \leq \varepsilon u$ in Ω, where ε is an arbitrary positive constant. This implies that $v = 0$, a contradiction.

(c) Suppose that $Q_{A,p,V}$ is critical in Ω and let $v > 0$ be the corresponding global minimal solution. We may assume that $v(x_0) = c$. Since both u and v are continuous at x_0, it follows that for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, for all $0 < \delta < \delta_\varepsilon$,

$$(1-\varepsilon)u(x) \leq v(x) \leq (1+\varepsilon)u(x) \quad \text{for all } x \in \partial B_\delta(x_0).$$

Since u and v are positive solutions (in $\Omega \setminus \{x_0\}$ and Ω, respectively) of minimal growth at infinity in Ω, the above inequality implies that

$$(1-\varepsilon)u(x) \leq v(x) \leq (1+\varepsilon)u(x) \quad \text{for all } x \in \Omega \setminus \{x_0\}.$$

Letting $\varepsilon \to 0$, we get $u = v$ in Ω, which contradicts our assumption that u has a nonremovable singularity at x_0. □

Remark 5.11. For sufficient conditions ensuring that in the subcritical case with $p > n$ the limit of the Green function $G^{\Omega}_{A,V}(x, x_0)$ as $x \to x_0$ always exists and is positive, see [Fraas and Pinchover 2013].

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GEOMETRIC OPTICS EXPANSIONS FOR HYPERBOLIC CORNER PROBLEMS, I: SELF-INTERACTION PHENOMENON

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In this article we are interested in the rigorous construction of geometric optics expansions for hyperbolic corner problems. More precisely we focus on the case where self-interacting phases occur. Those phases are proper to the high frequency asymptotics for the corner problem and correspond to rays that can display a homothetic pattern after a suitable number of reflections on the boundary. To construct the geometric optics expansions in that framework, it is necessary to solve a new amplitude equation in view of initializing the resolution of the WKB cascade.

1. Introduction

The aim of this article is to give rigorous methods to construct geometric optics expansions for linear hyperbolic initial boundary value problems in the quarter-space. Such problems will be called corner problems and are of the form

$$\begin{cases} L(\partial)u^\varepsilon := \partial_t u^\varepsilon + A_1\partial_1 u^\varepsilon + A_2\partial_2 u^\varepsilon = 0, \\ B_1 u^\varepsilon{}_{|x_1=0} = g^\varepsilon, \quad B_2 u^\varepsilon{}_{|x_2=0} = 0, \quad u^\varepsilon{}_{|t\le 0} = 0, \end{cases} \qquad (x_1,x_2)\in\mathbb{R}_+\times\mathbb{R}_+,\ t\ge 0, \tag{1}$$

where the matrices A_1, A_2 are in $M_N(\mathbb{R})$ and where the boundary matrices B_1, B_2 are elements of $M_{p_1,N}(\mathbb{R})$ and $M_{p_2,N}(\mathbb{R})$ respectively (the values of the integers p_1 and p_2 will be made precise in Assumption 2.2).

We have, in this article, chosen to work with only two space dimensions in order to save some notations. However, all the following results can be generalized if one looks at problem (1) with extra space variables $x' \in \mathbb{R}^{d-2}$ (with, of course, the suitable modifications on the operator $L(\partial)$ to preserve hyperbolicity).

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This article can be seen, in some sense, as a complement to the paper by Sarason and Smoller [1974] in which the authors give intuitions and elements of proof about how to construct geometric optics expansions but where the construction is not performed rigorously. To our knowledge it is the only paper about this subject in the literature for general first-order systems and we shall rely on some of the deep ideas of this seminal work. In particular, the links between the phase generation by reflections and the geometry of the characteristic variety will be the foundation of the proofs in this article (see Section 3 and [Sarason and Smoller 1974, Section 6] for more details).

Indeed, in [Sarason and Smoller 1974] the authors give examples of corner problems whose characteristic variety is such that, according to their argumentation, the associated ansatz of the geometric optics expansion has to contain more phases than the analogous ansatz for each problem in the half-spaces $\{x_1 > 0\}$ and $\{x_2 > 0\}$. They also show that a new phenomenon, specific to corner problems, may happen for some characteristic variety configurations: the existence of "self-interacting phases". By self-interacting phases we mean that some phases can regenerate themselves after a suitable number of reflections on both sides of the corner. Such spectral configurations trap part of the solution in a periodically repeating pattern of reflections from one side to the other (see Definition 4.8 and Figure 6 for more details).

Our aim is to give a *rigorous* construction of the geometric optics expansion when self-interacting phases occur. This result is achieved in Theorem 4.27. The most interesting thing during this construction is the appearance of a new amplitude equation whose resolution is needed to initialize the resolution of the whole cascade of equations. More precisely, the resolution of the new amplitude equation requires the invertibility of an operator acting on the trace of one of the self-interacting amplitudes. This operator arises under the form $(I - \mathbb{T})$ and is reminiscent of Osher's invertibility assumption [1973] for proving an a priori estimate for (1). We show in Theorems 4.28 and 4.29 that a sufficient and necessary (in many meaningful cases) condition for the new amplitude equation to be solvable in $L^2(\mathbb{R}_+)$ is that the energy associated with the trapped information does not increase. Such a formulation matches with the naive (but intuitive) idea that if part of the information is trapped and increases after running through one cycle, then the associated geometric optics expansion will blow up after repeated cycles.

Inverting an operator of the form $(I - \mathbb{T})$ in view of constructing the geometric optics expansion is not surprising. Indeed, if we make the analogy with the analysis of the initial boundary value problem in the half-space, the necessary and sufficient condition to ensure strong well-posedness is the so-called uniform Kreiss–Lopatinskii condition (see [Kreiss 1970] and Assumption 2.11). When one wants to construct geometric optics expansions for such problems in a half-space, a "microlocalized" version of this condition arises [Williams 1996]. So one should expect that an analogous situation takes place for the corner problem and that the solvability condition we exhibit here is a microlocalized version of a stronger condition ensuring well-posedness of (1).

The full characterization of strong well-posedness for the corner problem has not been achieved yet. Some partial results are known, for example for symmetric corner problems with strictly dissipative boundary conditions (in that framework, the strong well-posedness can easily be obtained with few modifications of the proofs of [Lax and Phillips 1960; Benzoni-Gavage and Serre 2007] for half-space problems). However, there are, to our knowledge, few results concerning the general framework, that is to say, corner problems only satisfying the uniform Kreiss–Lopatinskii condition on each side. A fundamental contribution to this study is the article by Osher [1973]. In this paper, the author uses the invertibility of an operator of the form $(I - \mathbb{T}_\zeta)$ — here ζ denotes a time frequency — to establish a priori energy estimates. More precisely, he uses such an invertibility property to construct a "Kreiss-type symmetrizer" providing a priori energy estimates with a loss of regularity from the source terms to the solution. Unfortunately the number of losses in the estimates is not even explicit. However, some new results about the possibility to obtain energy estimates without loss can be found in [Benoit 2015].

We believe that, as for the half-space problems, the invertibility condition on $(I-\mathbb{T})$ is a microlocalized version of Osher's condition. It is also interesting to remark that the example given in Section 3E shows that the invertibility condition on $(I-\mathbb{T})$ may not be satisfied if we only impose the uniform Kreiss–Lopatinskii condition on either side of the corner. But, looking still at the example of Section 3E, we observe that the invertibility condition on $(I-\mathbb{T})$ is automatically satisfied if the boundary conditions are strictly dissipative.

The paper is organized as follows: In Section 2 we define some objects and introduce notations for dealing with geometric optics expansions for initial boundary value problems. We also give some known results about the well-posedness theory for the corner problem (1). In Section 3, we explain, and make complete, the phase generation process by reflection as studied in [Sarason and Smoller 1974]. We also briefly give an example of a 2×2 corner problem for which geometric optics expansions contain infinitely many phases.

Section 4 is devoted to the proof of our main result. Firstly, we give a rigorous framework for the description of the phases obtained by successive reflections. This framework has to be general enough to take into account self-interacting phases. Then we construct the geometric optics expansion. To do that, it is, in a first time, necessary to exhibit a global "tree" structure on the set of phases, then to find a way to initialize the resolution. As already mentioned, the initialization requires solving a new amplitude equation for the trace of a self-interacting amplitude. The derivation of this equation is performed in Section 4B2. Then we show that, once we have organized the set of phases and we have constructed one of the self-interacting amplitudes, we can construct all amplitudes associated with phases "close to" the self-interacting ones. A more precise study of the structure of the phase set then permits to determine all the phases in the geometric optics expansion.

The end of Section 4 aims at justifying the geometric optics expansion and then at giving a necessary and sufficient condition to ensure that the operator $(I - \mathbb{T})$ is invertible. We also give examples of corner problems with one loop and revisit some of the conclusions of [Sarason and Smoller 1974]. Finally, we make some comments on our results and give some prospects in Section 5.

2. An overview of well-posedness for half-space and corner problems

2A. ***Notations and definitions.*** Let

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}, \quad \partial\Omega_1 := \Omega \cap \{x_1 = 0\}, \quad \text{and} \quad \partial\Omega_2 := \Omega \cap \{x_2 = 0\}$$

be the quarter-space and both its edges. For $T > 0$, we will define

$$\Omega_T :=]-\infty, T] \times \Omega, \quad \partial\Omega_{1,T} :=]-\infty, T] \times \partial\Omega_1, \quad \text{and} \quad \partial\Omega_{2,T} :=]-\infty, T] \times \partial\Omega_2.$$

The used function spaces will be the usual Sobolev spaces $H^n(X)$, with the notations $L^2(X) = H^0(X)$ and $H^\infty(X) := \bigcap_n H^n(X)$, where X is some Banach space. But we will also need the weighted Sobolev spaces defined by $H^n_\gamma(X) := \{u \in \mathscr{D}'(X) \mid e^{-\gamma t} u \in H^n(X)\}$ for $\gamma > 0$.

At last, during the construction of the WKB expansion, to make sure that amplitudes are smooth enough, we shall need the source term in (1) to be flat at the corner. The associated space of profiles is thus defined as

$$H^n_f := \left\{g \in H^n(\mathbb{R} \times \mathbb{R}_+) \mid \forall k \leq n,\ \partial_x^k g(t,x)_{|x=0} = 0\right\} \quad \forall n \in \mathbb{N} \cup \{\infty\}. \tag{2}$$

The flat-at-the-corner weighted Sobolev spaces $H^n_{f,\gamma}$ are defined in a similar way.

Throughout $\mathscr{L}$ will be the symbol of the differential operator $L(\partial)$; i.e., for $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$,

$$\mathscr{L}(\tau, \xi) := \tau I + \sum_{j=1}^{2} \xi_j A_j.$$

The characteristic variety V of $L(\partial)$ is given by

$$V := \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 \mid \det \mathscr{L}(\tau, \xi) = 0\}.$$

In this article we choose to work with constantly hyperbolic operators. However, it has to be mentioned that the analysis of Section 4 is slightly easier in the particular framework of strictly hyperbolic operators. We thus assume the following property on $L(\partial)$:

Assumption 2.1. *There exists an integer $q \geq 1$, real-valued $\lambda_1, \dots, \lambda_q$ analytic on $\mathbb{R}^2 \setminus \{0\}$ and positive integers $\mu_1, \dots, \mu_q$ such that*

$$\det \mathscr{L}(\tau, \xi) = \prod_{j=1}^{q} (\tau + \lambda_j(\xi))^{\mu_j} \quad \forall \xi \in \mathbb{S}^1,$$

where the semisimple eigenvalues $\lambda_j(\xi)$ satisfy $\lambda_1(\xi) < \dots < \lambda_q(\xi)$.

Let us also assume that the boundary of Ω is noncharacteristic, and that the matrices B_1 and B_2 induce the good number of boundary conditions, that is to say:

Assumption 2.2. *We assume that the matrices A_1, A_2 are invertible. Then p_1 (resp. p_2), the number of lines of B_1 (resp. B_2), equals the number of positive eigenvalues of A_1 (resp. A_2).*

Moreover we also assume that B_1 and B_2 are of maximal rank.

Under Assumptions 2.1 and 2.2, we can define the resolvent matrices

$$\mathscr{A}_1(\zeta) := -A_1^{-1}(\sigma I + i\eta A_2) \quad \text{and} \quad \mathscr{A}_2(\zeta) := -A_2^{-1}(\sigma I + i\eta A_1),$$

where ζ denotes an element of the frequency space

$$\Xi := \left\{\zeta := (\sigma = \gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R},\ \gamma \geq 0\right\} \setminus \{(0,0)\}.$$

For convenience, we also introduce Ξ_0 the boundary of Ξ:

$$\Xi_0 := \Xi \cap \{\gamma = 0\}.$$

For $j=1,2$ and $\zeta \in (\Xi \setminus \Xi_0)$, we denote by $E_j^s(\zeta)$ the stable subspace of $\mathscr{A}_j(\zeta)$ and by $E_j^u(\zeta)$ its unstable subspace. These spaces are well-defined according to [Hersh 1963]. The stable subspace $E_j^s(\zeta)$ has dimension p_j, whereas $E_j^u(\zeta)$ has dimension $N-p_j$. Let us recall the following theorem due to Kreiss [1970] and generalized by Métivier [2000] for constantly hyperbolic operators:

Theorem 2.3 (block structure). *Under Assumptions 2.1–2.2, for all $\underline{\zeta} \in \Xi$, there exists a neighborhood $\mathscr{V}$ of $\underline{\zeta}$ in Ξ, integers $L_1, L_2 \geq 1$, two partitions $N = \nu_{1,1}+\cdots+\nu_{1,L_1} = \nu_{2,1}+\cdots+\nu_{2,L_2}$ with $\nu_{1,l}, \nu_{2,l} \geq 1$, and two invertible matrices T_1, T_2, regular on $\mathscr{V}$ such that for all $\zeta \in \mathscr{V}$,*

$$T_1(\zeta)^{-1}\mathscr{A}_1(\zeta)T_1(\zeta) = \operatorname{diag}\big(\mathscr{A}_{1,1}(\zeta), \dots, \mathscr{A}_{1,L_1}(\zeta)\big),$$
$$T_2(\zeta)^{-1}\mathscr{A}_2(\zeta)T_2(\zeta) = \operatorname{diag}\big(\mathscr{A}_{2,1}(\zeta), \dots, \mathscr{A}_{2,L_2}(\zeta)\big),$$

where the blocks $\mathscr{A}_{j,l}(\zeta)$ have size $\nu_{j,l}$ and satisfy one of the following alternatives:

(i) *All the elements in the spectrum of $\mathscr{A}_{j,l}(\underline{\zeta})$ have positive real part.*

(ii) *All the elements in the spectrum of $\mathscr{A}_{j,l}(\underline{\zeta})$ have negative real part.*

(iii) $\nu_{j,l}=1$, $\mathscr{A}_{j,l}(\underline{\zeta}) \in i\mathbb{R}$, $\partial_\gamma \mathscr{A}_{j,l}(\underline{\zeta}) \in \mathbb{R}\setminus\{0\}$, *and* $\mathscr{A}_{j,l}(\zeta) \in i\mathbb{R}$ *for all* $\zeta \in \mathscr{V} \cap \Xi_0$.

(iv) $\nu_{j,l} > 1$, *and there exists* $k_{j,l} \in i\mathbb{R}$ *such that*

$$\mathscr{A}_{j,l}(\underline{\zeta}) = \begin{bmatrix} k_{j,l} & i & & 0 \\ & \ddots & \ddots & \\ & & \ddots & i \\ 0 & & & k_{j,l} \end{bmatrix},$$

the coefficient in the lower left corner of $\partial_\gamma \mathscr{A}_{j,l}(\underline{\zeta})$ is real and nonzero, and moreover, $\mathscr{A}_{j,l}(\zeta) \in iM_{\nu_{j,l}}(\mathbb{R})$ for all $\zeta \in \mathscr{V} \cap \Xi_0$.

Thanks to this theorem it is possible to describe the four kinds of frequencies, one for each part of the boundary $\partial\Omega$:

Definition 2.4. For $j=1,2$, we denote by

(1) $\mathscr{E}_j$ the set of elliptic frequencies, that is to say, the set of $\underline{\zeta} \in \Xi_0$ such that Theorem 2.3 for the matrix $\mathscr{A}_j(\underline{\zeta})$ is satisfied with one block of type (i) and one block of type (ii) only;

(2) $\mathscr{H}_j$ the set of hyperbolic frequencies, that is to say, the set of $\underline{\zeta} \in \Xi_0$ such that Theorem 2.3 for the matrix $\mathscr{A}_j(\underline{\zeta})$ is satisfied with blocks of type (iii) only;

(3) $\mathscr{EH}_j$ the set of mixed frequencies, that is to say, the set of $\underline{\zeta} \in \Xi_0$ such that Theorem 2.3 for the matrix $\mathscr{A}_j(\underline{\zeta})$ is satisfied with one block of type (i), one of type (ii) and at least one of type (iii), but without a block of type (iv);

(4) $\mathscr{G}_j$ the set of glancing frequencies, that is to say, the set of $\underline{\zeta} \in \Xi_0$ such that Theorem 2.3 for the matrix $\mathscr{A}_j(\underline{\zeta})$ is satisfied with at least one block of type (iv).

Thus, by definition, Ξ_0 admits the decomposition

$$\Xi_0 = \mathscr{E}_j \cup \mathscr{EH}_j \cup \mathscr{H}_j \cup \mathscr{G}_j.$$

The study made in [Kreiss 1970; Métivier 2000] shows that the subspaces $E_1^s(\zeta)$ and $E_2^s(\zeta)$ admit a continuous extension up to Ξ_0. Moreover, if $\underline{\zeta} \in \Xi_0 \setminus (\mathscr{G}_1 \cup \mathscr{G}_2)$, one has the decomposition

$$\mathbb{C}^N = E_1^s(\underline{\zeta}) \oplus E_1^u(\underline{\zeta}) = E_2^s(\underline{\zeta}) \oplus E_2^u(\underline{\zeta}), \tag{3}$$

and for $j \in \{1, 2\}$,

$$E_j^s(\underline{\zeta}) = E_j^{s,e}(\underline{\zeta}) \oplus E_j^{s,h}(\underline{\zeta}),$$
$$E_j^u(\underline{\zeta}) = E_j^{u,e}(\underline{\zeta}) \oplus E_j^{u,h}(\underline{\zeta}).$$

where $E_j^{s,e}(\underline{\zeta})$ (resp. $E_j^{u,e}(\underline{\zeta})$) is the generalized eigenspace associated with eigenvalues of $\mathscr{A}_j(\underline{\zeta})$ with negative (resp. positive) real part, and where the spaces $E_j^{s,h}(\underline{\zeta})$ and $E_j^{u,h}(\underline{\zeta})$ are sums of eigenspaces of $\mathscr{A}_j(\underline{\zeta})$ associated with some purely imaginary eigenvalues of $\mathscr{A}_j(\underline{\zeta})$. From Assumption 2.2 we also have

$$\mathbb{C}^N = A_1 E_1^s(\underline{\zeta}) \oplus A_1 E_1^u(\underline{\zeta}) = A_2 E_2^s(\underline{\zeta}) \oplus A_2 E_2^u(\underline{\zeta}). \tag{4}$$

In fact, it is possible to give a more precise decomposition of the spaces $E_j^{s,h}(\underline{\zeta})$ and $E_j^{u,h}(\underline{\zeta})$. Indeed, let $\underline{\omega}_{m,j}$ be a purely imaginary eigenvalue of $\mathscr{A}_j(\underline{\zeta})$, that is,

$$\det(\underline{\tau} + \underline{\eta} A_1 + \underline{\omega}_{m,2} A_2) = \det(\underline{\tau} + \underline{\omega}_{m,1} A_1 + \underline{\eta} A_2) = 0.$$

Then, using Assumption 2.1, there exists an index $k_{m,j}$ such that

$$\underline{\tau} + \lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2}) = \underline{\tau} + \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta}) = 0,$$

where $\lambda_{k_{m,j}}$ is smooth in both variables. Let us then introduce the following classification:

Definition 2.5. The set of incoming (resp. outgoing) phases for the side $\partial\Omega_1$, denoted by $\mathfrak{I}_1$ (resp. $\mathfrak{O}_1$), is the set of indices m such that the group velocity $v_m := \nabla \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta})$ satisfies $\partial_1 \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta}) > 0$ (resp. $\partial_1 \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta}) < 0$).

Similarly, the set of incoming (resp. outgoing) phases for the side $\partial\Omega_2$, denoted by $\mathfrak{I}_2$ (resp. $\mathfrak{O}_2$), is the set of indices m such that the group velocity $v_m := \nabla \lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2})$ satisfies $\partial_2 \lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2}) > 0$ (resp. $\partial_2 \lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2}) < 0$).

Thanks to this definition, we can write the following decomposition of the stable and unstable components $E_j^{s,h}(\underline{\zeta})$ and $E_j^{u,h}(\underline{\zeta})$:

Lemma 2.6. *For all $\underline{\zeta} \in \mathscr{H}_j \cup \mathscr{EH}_j$, $j = 1, 2$, we have*

$$E_1^{s,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{I}_1} \ker \mathscr{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \quad E_1^{u,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{O}_1} \ker \mathscr{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \tag{5}$$

$$E_2^{s,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{I}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}), \quad E_2^{u,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{O}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}). \tag{6}$$

From Assumption 2.2 we can also write

$$A_1 E_1^{s,h}(\underline{\zeta}) = \bigoplus_{m\in\mathfrak{I}_1} A_1 \ker \mathscr{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \quad A_1 E_1^{u,h}(\underline{\zeta}) = \bigoplus_{m\in\mathfrak{O}_1} A_1 \ker \mathscr{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \tag{7}$$

$$A_2 E_2^{s,h}(\underline{\zeta}) = \bigoplus_{m\in\mathfrak{I}_2} A_2 \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}), \quad A_2 E_2^{u,h}(\underline{\zeta}) = \bigoplus_{m\in\mathfrak{O}_2} A_2 \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}). \tag{8}$$

We refer, for example, to [Coulombel and Guès 2010] for a proof of this lemma.

2B. ***Known results about strong well-posedness.*** We consider the corner problem with source terms in the interior of Ω_T and on either side of the boundary $\partial\Omega_T$ given by

$$\begin{cases} L(\partial)u = f & \text{on } \Omega_T, \\ B_1 u_{|x_1=0} = g_1 & \text{on } \partial\Omega_{1,T}, \\ B_2 u_{|x_2=0} = g_2 & \text{on } \partial\Omega_{2,T}, \\ u_{|t\le 0} = 0. \end{cases} \tag{9}$$

By strong well-posedness for the corner problem (9) we mean the following:

Definition 2.7. The corner problem (9) is said to be strongly well-posed if for $T > 0$ and for all $f \in L^2(\Omega_T)$ and $g_j \in L^2(\partial\Omega_{j,T})$, the corner problem (9) admits a unique solution $u \in L^2(\Omega_T)$ with traces in $L^2(\partial\Omega_{1,T})$ and $L^2(\Omega_{2,T})$ satisfying the energy estimate

$$\begin{aligned} \|u\|^2_{L^2(\Omega_T)} + \|u_{|x_1=0}\|^2_{L^2(\partial\Omega_{1,T})} + \|u_{|x_2=0}\|^2_{L^2(\partial\Omega_{2,T})} \\ \le C_T\big(\|f\|^2_{L^2(\Omega_T)} + \|g_1\|^2_{L^2(\partial\Omega_{1,T})} + \|g_2\|^2_{L^2(\partial\Omega_{2,T})}\big) \end{aligned} \tag{10}$$

for some constant C_T depending on T.

As we have already mentioned in the Introduction, the full characterization of strong well-posedness for the corner problem (9) has not been achieved yet. However, we have some partial results.

First of all, the strong well-posedness is proved in the particular framework of symmetric operators with strictly dissipative boundary conditions, that is, boundary conditions defined as follows:

Definition 2.8. For $j = 1, 2$, the boundary condition $B_j u_{|x_j=0} = g_j$ is said to be strictly dissipative if the inequality

$$\langle A_j v, v\rangle < 0 \quad \forall v \in \ker B_j \setminus \{0\}$$

holds and $\ker B_j$ is maximal (in the sense of inclusion) for this property.

We thus have the following result:

Theorem 2.9 [Benoit 2015, chapitre 4]. *Under Assumption 2.2, if the matrices A_1 and A_2 are symmetric and if the boundary conditions of the corner problem (9) are strictly dissipative, then under a certain algebraic condition on the matrix $A_1^{-1}A_2$, the corner problem (9) is strongly well-posed is the sense of Definition 2.7.*

We refer to [Benoit 2015, chapitre 4] for a proof of this result and for more details about the mentioned algebraic condition (see hypothèse 4.1.2 of [Benoit 2015]).[1] For more details about the algebraic condition we also refer to [Métivier and Rauch 2016] in which the necessity of imposing this condition is shown.

It is also easy to show (see [Benoit 2015, paragraphe 5.3.1]) that a necessary condition for (9) to be strongly well-posed is that each initial boundary value problem

$$\begin{cases} L(\partial)u = f, \\ B_j u_{|x_j=0} = g_j, \quad u_{|t\leq 0} = 0, \end{cases} \quad \text{on } \{x_j > 0, x_{3-j} \in \mathbb{R}\} \text{ for } j = 1, 2, \tag{11}$$

is strongly well-posed in the usual sense for initial boundary value problems in the half-space (see, for example, [Benzoni-Gavage and Serre 2007]).

This implies that Theorem 2.9 is not sharp (except for $N=2$, thanks to [Strang 1969]) because there exist nonstrictly dissipative boundary conditions leading to a strongly well-posed initial boundary value problem (11) (see, for example, [Benoit 2014, paragraphe 5.3]).

However, the set of the boundary conditions making (11) strongly well-posed has been characterized by [Kreiss 1970] and is composed of the boundary condition satisfying the so-called uniform Kreiss–Lopatinskii condition:

Definition 2.10. The initial boundary value problem (11) is said to satisfy the uniform Kreiss–Lopatinskii condition if for all $\zeta \in \Xi$, we have

$$\ker B_j \cap E_j^s(\zeta) = \{0\}.$$

So for the corner problem (9) to be strongly well-posed it is necessary that, for $j = 1, 2$, the initial boundary value problem (11) satisfies the "uniform" Kreiss–Lopatinskii condition. We thus make the assumption:

Assumption 2.11. *For all $\zeta \in \Xi$, we have*

$$\ker B_1 \cap E_1^s(\zeta) = \ker B_2 \cap E_2^s(\zeta) = \{0\}.$$

In particular, the restriction of B_1 (resp. B_2) to the stable subspace $E_1^s(\zeta)$ (resp. $E_2^s(\zeta)$) is invertible, and its inverse is denoted by $\phi_1(\zeta)$ (resp. $\phi_2(\zeta)$).

Unsurprisingly, the counterexample [Osher 1974a] shows that imposing the uniform Kreiss–Lopatinskii condition on each side of the boundary is not sufficient to ensure that the corner problem (9) is strongly well-posed.

3. The phase generation process and examples

Before constructing the geometric optics expansions, it is necessary to describe the expected phases in these expansions. Since the boundary of the domain Ω is not flat, we expect that it is possible to generate more phases than for half-space problems. Indeed, at the very first glance, we can think that a ray of

[1] We do not want to give more details about this condition because it is not used to construct the WKB expansion. Moreover, this condition will be satisfied by all our examples.

geometric optics can be reflected several times on the boundary of the domain, with different new phases generated at each reflection.

It is thus very important in order to postulate an ansatz to be able to describe all the phases that can be obtained by successive reflections on each side of the boundary.

Here, we shall go back to the discussion by Sarason and Smoller [1974] explaining this phenomenon and establishing a very strong link between the geometry of the characteristic variety of $L(\partial)$ and the phase generation process.

As already mentioned in the Introduction, we are interested here in corner problems which are homogeneous in the interior and on one side of the boundary. The only nonzero source term, which arises in the boundary condition on $\partial\Omega_1$, will be highly oscillating, and we want to understand which phases can be induced by this source term. We will here describe the phase generation process when the source term is taken on $\partial\Omega_1$; the arguments are the same for a source term on $\partial\Omega_2$.

3A. ***Source term induced phases.*** Our problem of study is

$$\begin{cases} L(\partial)u^\varepsilon = 0 & \text{on } \Omega_T, \\ B_1 u^\varepsilon{}_{|x_1=0} = g^\varepsilon & \text{on } \partial\Omega_{1,T}, \\ B_2 u^\varepsilon{}_{|x_2=0} = 0 & \text{on } \partial\Omega_{2,T}, \\ u^\varepsilon{}_{|t\leq 0} = 0, \end{cases} \tag{12}$$

where the source term on $\partial\Omega_{1,T}$ is given by

$$g^\varepsilon(t,x_2) := e^{\frac{i}{\varepsilon}\varphi(t,x_2)} g(t,x_2), \tag{13}$$

where the amplitude g belongs to H_f^∞ and is zero for negative times. The planar phase φ is defined by

$$\varphi(t,x_2) := \underline{\tau} t + \underline{\xi}_2 x_2$$

for two fixed real numbers $\underline{\tau} > 0$ and $\underline{\xi}_2$.

The fact that g belongs to H_f^∞ implies that g^ε is zero at the corner. Assume that g identically vanishes in a neighborhood of the corner. Then by finite speed of propagation for the half-space problem, we can, at least during a small time interval, see the corner problem (12) as a boundary value problem in the half-space $\{x_1 \geq 0\}$.

Geometric optics expansions for boundary value problems in the half-space have already been studied (see, for example, [Williams 1996]) and, going back over the existing analysis, we expect that the source term g^ε on the side $\partial\Omega_1$ induces in the interior of the domain several rays associated with the planar phases

$$\varphi^{0,k}(t,x) := \varphi(t,x_2) + \underline{\xi}_1^{0,k} x_1,$$

where the $(\underline{\xi}_1^{0,k})_k$ are the roots in the ξ_1-variable of the dispersion relation

$$\det \mathscr{L}(\underline{\tau}, \xi_1, \underline{\xi}_2) = 0. \tag{14}$$

An important remark to understand the phase generation process is that the $(\underline{\xi}_1^{0,k})_k$ are the intersection points (with the convention that complex roots are viewed as intersection points at infinity) between the line $\{(\underline{\tau}, \xi_1, \underline{\xi}_2), \xi_1 \in \mathbb{R}\}$ and the section of the characteristic variety V at $\tau = \underline{\tau}$.

Let us denote by p_r the number of real roots of (14) and by $2p_c$ the number of complex roots (which occur in conjugate pairs). We also assume that $(\underline{\tau}, \underline{\xi}_2)$ is not a glancing frequency for the matrix $\mathscr{A}_1$; hence p_r can be decomposed as $p_r^i + p_r^o$, where p_r^i (resp. p_r^o) is the number of real roots inducing an incoming (resp. outgoing) group velocity (see Definition 2.5). We thus have $p_1 = p_r^i + p_c$ and $N - p_1 = p_r^o + p_c$ by Lemma 2.6. Firstly we shall consider φ_i^0, one of the p_r^i phases with an incoming group velocity, and φ_o^0, one of the p_r^o phases with an outgoing group velocity. We also denote by v_i^0 and v_o^0 the associated group velocities. Phases associated with complex roots will be dealt with separately. The following discussion should be performed for each such real phase.

We shall study separately the influence of the phases φ_i^0 and φ_o^0 upon the generation of phases.

The phase φ_o^0: The phase φ_o^0, associated with an outgoing group velocity, describes the “past” of the information reflected on the side $\partial\Omega_1$ at the initial time. In other words, to know the origin of a point on the side $\partial\Omega_1$, it is sufficient to travel along the characteristic with group velocity v_o^0 by rewinding time back to $-\infty$.

This leads us to separate two cases, making, thus, more precise the Definition 2.5:

Definition 3.1. An outgoing group velocity $v = (v_1, v_2)$ for the side $\partial\Omega_1$ (i.e., $v_1 < 0$) is said to be

- outgoing-incoming if $v_2 > 0$,
- outgoing-outgoing if $v_2 < 0$.

First subcase: v_o^0 outgoing-outgoing. Let us fix a point on the side $\partial\Omega_1$ and we draw the characteristic line with group velocity v_o^0 passing through this point. Since v_o^0 is outgoing for each side of the boundary, the information at the considered point of $\partial\Omega_1$ can only come from information in the interior of the domain, which has been transported towards the side $\partial\Omega_1$; see Figure 1. But, without a source term in the interior of Ω, such information is zero. As a consequence, the amplitude u_o^0 associated with the phase φ_o^0 is zero, since according to Lax's lemma [1957] it satisfies the transport equation

$$\begin{cases} \partial_t u_o^0 + v_o^0 \cdot \nabla_x u_o^0 = 0, \\ u^0_{o_{|t\leq 0}} = 0. \end{cases}$$

Outgoing-outgoing phases do not have any influence on the WKB expansion or on the phase generation process and are therefore ignored from now on.

Second subcase: v_o^0 outgoing-incoming. Once again, we fix a point on the side $\partial\Omega_1$ and we draw the characteristic line with group velocity v_o^0 passing through this point. As in the subcase of an outgoing-outgoing, the information at the considered point of $\partial\Omega_1$ cannot come from the interior of the domain.

However, the characteristic associated with the group velocity v_o^0 hits the side $\partial\Omega_2$ when we rewind the time back to $-\infty$, so the information at the point of the side $\partial\Omega_1$ could a priori come from some

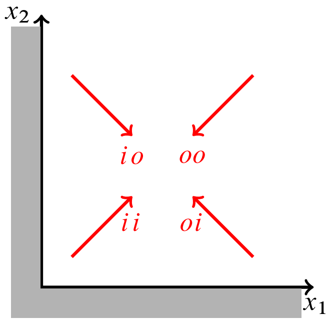

Figure 1. The four different kinds of phases.

information on the side $\partial\Omega_2$, which would have been transported towards the side $\partial\Omega_1$. But this is not possible at time $t = 0$ since the boundary condition on $\partial\Omega_2$ is homogeneous for negative times.

So, the amplitude associated with the outgoing-incoming phase φ_o^0 is zero at time $t = 0$ and even on a small time interval if g^ε identically vanishes near the corner. That is why we do not take into account the phase φ_o^0 *initially* in the phase generation process.

Let us stress here that the phase φ_o^0 is moved apart a priori only at time $t = 0$. Indeed, for some configuration of the characteristic variety, this phase can be generated at a future reflection on the side $\partial\Omega_2$, and will finally be included in the ansatz. We will make more comments on this point in Section 3C, after having precisely described which reflections are taken into account.

The phase φ_i^0: The phase φ_i^0 is associated with an incoming group velocity for the side $\partial\Omega_1$. Opposite to the phase φ_o^0, it describes the "future" of the source term g^ε. That is to say, when time goes to $+\infty$, part of the oscillations in g^ε is transported along the characteristic with group velocity v_i^0. So, the phase φ_i^0 carries a nonzero information and has to be taken into account in the phase generation process.

However, once again, we have to separate two subcases, according to the following refinement of the Definition 2.5:

Definition 3.2. An incoming group velocity $v = (v_1, v_2)$ for the side $\partial\Omega_1$ (i.e., $v_1 > 0$) is said to be

- incoming-incoming if $v_2 > 0$,
- incoming-outgoing if $v_2 < 0$.

The four kinds of (nonglancing) oscillating phases used in this analysis are drawn in Figure 1.

First subcase: v_i^0 incoming-incoming. We choose a point $(0, \underline{x}_2)$, on $\partial\Omega_1$ such that $g^\varepsilon(0, \underline{x}_2)$ is nonzero and we draw the characteristic with velocity v_i^0 passing through this point. When t goes to $+\infty$, the information transported along this ray will never hit the side $\partial\Omega_2$ and will be unable to generate new phases by reflection. So, when the group velocity v_i^0 is incoming-incoming, the phase generation process for the phase φ_i^0 stops.

Second subcase: v_i^0 incoming-outgoing. We fix a point $(0, \underline{x}_2) \in \partial\Omega_1$ with $g^\varepsilon(0, \underline{x}_2) \neq 0$, and we draw the characteristic with velocity v_i^0 passing through this point. As $v_{i,2}^0$ is negative, this ray will hit, after a while, the side $\partial\Omega_2$. We thus expect that this ray will give rise to reflected oscillations and that this reflection will create new phases. This reflection phenomenon and more specifically the new expected phases will be described in the next section. But before that, we will conclude the discussion on the phases induced by the source term g^ε by considering the possible complex-valued phases.

φ_k^0 **complex-valued:** We now deal with phases corresponding to roots in the ξ_1-variable to the dispersion relation (14) with nonzero imaginary part. Let us first introduce some vocabulary.

Definition 3.3. A phase φ_k^0 with $\underline{\xi}_1^{0,k} \in \mathbb{C} \setminus \mathbb{R}$ is said to be

- evanescent for the side $\partial\Omega_1$ if $\operatorname{Im} \underline{\xi}_1^{0,k} > 0$,
- explosive for the side $\partial\Omega_1$ if $\operatorname{Im} \underline{\xi}_1^{0,k} < 0$.

Thanks to the construction of geometric optics expansion for complex-valued phases made, for example, in [Marcou 2010; Lescarret 2007], the expected behavior of the amplitudes associated with these phases is a propagation of information in the normal direction to the side $\partial\Omega_1$. However, this propagation is exponentially decreasing (resp. increasing) according to the variable x_1 for the amplitudes linked with evanescent (resp. explosive) phases. In all that follows, as we are looking for amplitudes in $L^2(\Omega)$ so as in [Lescarret 2007; Marcou 2010; Williams 1996] we do not take into account explosive phases.

Thus, we only keep the evanescent phases. Since, for regularity considerations on the oscillating amplitudes, we are working with a source term in H_f^∞, this source term satisfies, in particular, $g(t,0) = 0$. Consequently, the information carried by evanescent phases will never hit the side $\partial\Omega_2$ and the evanescent phases for the side $\partial\Omega_1$ are, as well as the incoming-incoming ones, stopping conditions in the phase generation process.

To summarize, the phases induced directly by the source term g^ε are the incoming (for the side $\partial\Omega_1$) phases and the evanescent phases for the side $\partial\Omega_1$. Incoming-incoming and evanescent phases will not be reflected; thus we only have to study the reflections on $\partial\Omega_2$ associated with incoming-outgoing phases.

3B. ***The first reflection.*** We assume that the dispersion relation (14) has at least one solution in the ξ_1-variable generating an incoming-outgoing group velocity. We shall describe the reflection of one of these phases. Of course, to determine all the expected phases in the WKB expansion, the following discussion has to be repeated for each of these phases.

Let $\underline{\xi}_1^0$ be a fixed root in the ξ_1-variable to (14). We denote by v_i^0 the associated incoming-outgoing group velocity which corresponds to rays emanating from $\partial\Omega_1$ and hitting $\partial\Omega_2$ in finite time. Let us also assume that time is large enough so that the ray associated with v_i^0 and emanating from the support of g^ε has hit $\partial\Omega_2$. Once again, by (formal) finite speed of propagation arguments, the reflection of the ray can not hit immediately the side $\partial\Omega_1$. Thus during a small time, we can represent our situation as an initial boundary value problem in the half-space $\{x_2 \geq 0\}$ whose boundary source term has been turned on by the amplitude for the outgoing (for the side $\partial\Omega_2$) phase φ_i^0.

We thus have to determine the roots $(\underline{\xi}_2^{1,k})_k$ in the ξ_2-variable of the dispersion relation

$$\det \mathscr{L}(\underline{\tau}, \underline{\xi}_1^0, \xi_2) = 0. \tag{15}$$

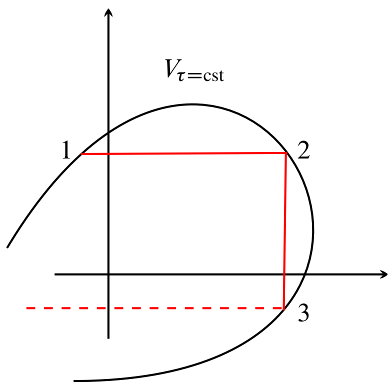

Figure 2. The geometry of the characteristic variety and the phase generation.

Let us stress that we already know one of them, that is, $\underline{\xi}_2$. For each of the new roots, we associate to it the phase

$$\varphi^{1,k}(t,x) := \underline{\tau}t + \underline{\xi}_1^0 x_1 + \underline{\xi}_2^{1,k} x_2.$$

It is interesting to note that the $(\underline{\xi}_2^{1,k})_k$ are the intersection points between $V \cap \{\tau = \underline{\tau}\}$ and the line $\{(\underline{\tau}, \underline{\xi}_1^0, \xi_2),\ \xi_2 \in \mathbb{R}\}$. Thus to determine the phases generated by the source term, it is necessary to consider the intersection of $V \cap \{\tau = \underline{\tau}\}$ with a horizontal line, and to determine the phases generated by the first reflection, we have to consider the intersection of $V \cap \{\tau = \underline{\tau}\}$ with a vertical line (see Figure 2). To determine the phases generated by the second reflection, we will have to consider the intersection with a horizontal line and so on. We see that this process strongly depends on the geometry of the characteristic variety V.

Repeating exactly the same arguments as those used for the phases induced by the source term, we claim that outgoing-outgoing and incoming-outgoing phases can be neglected (at least initially for incoming-outgoing phases). Consequently, for real roots of (15), we just have to consider those associated with an incoming-incoming or outgoing-incoming group velocity. Let φ_i^1 denote one of these phases and v_i^1 its group velocity.

v_i^1 incoming-incoming: In that case, as when the group velocity v_i^0 was incoming-incoming, the considered ray will never hit the side $\partial\Omega_1$, and it will never be reflected. The phase generation process for the phase φ_i^1 stops, and we are free to study the reflection(s) of another root of (15).

v_i^1 outgoing-incoming: The reflected ray travels towards $\partial\Omega_1$, it will hit $\partial\Omega_1$ after a while, and we will have to determine how it is reflected back. So the phase generation process for the phase φ_i^1 continues.

Concerning complex roots of (15) (if such roots exist), we only add in the WKB expansion those associated with evanescent phases for the side $\partial\Omega_2$ (that is to say, those satisfying $\operatorname{Im}\underline{\xi}_2^{1,k} > 0$). As for the complex-valued phases induced by the source term, they will never be reflected back, and the phase generation process for these phases stops.

3C. *Summary.* To summarize, the phase generation process is the following: We start from a source term on $\partial\Omega_1$ and we only study the reflections for the incoming phases that it induces. If all of the phases are incoming-incoming (or evanescent), then the process stops. Otherwise, we determine the reflections on $\partial\Omega_2$ of all the incoming-outgoing phases and we shall consider them into the ansatz. If one of these reflected phases is outgoing-incoming, we will determine its reflection on $\partial\Omega_1$, otherwise the phase generation process stops. This leads us to consider sequences of phases which are alternatively incoming-outgoing and outgoing-incoming until we find an incoming-incoming or evanescent phase during a reflection, which ends the sequence.

There are, of course, two possibilities: either each of these sequences of phases generated by successive reflections is finite, and then the number of phases in the ansatz will be finite (see the example of Section 3E), or at least one of these sequences is infinite, and then the number of phases in the ansatz is infinite (see Section 3D).

In all the preceding discussion, we used the tacit assumption that we never meet glancing phases. This assumption is satisfied in all our examples and it will be clearly stated in Theorem 4.27. Formally glancing phases should be stopping criterion as well as incoming-incoming and evanescent phases. However, how to include rigorously glancing modes in the WKB expansion is left for future studies.

Let us also stress that during a reflection on the side $\partial\Omega_1$ (resp. $\partial\Omega_2$), the fact that outgoing-incoming (resp. incoming-outgoing) phases are not considered does not prevent these phases from appearing in the WKB expansion.

Indeed, let $(\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$ be an incoming-outgoing phase generated by the source term g^ε and $(\underline{\tau}, \underline{\xi}_1, \tilde{\underline{\xi}}_2)$ be an outgoing-incoming phase also generated by the source term. This phase is a priori not taken into account in the WKB expansion at the first step of the phase generation process described above. Let us assume that the intersection between the characteristic variety $V \cap \{\tau = \underline{\tau}\}$ and the line $\{(\underline{\tau}, \underline{\xi}_1, \xi_2),\ \xi_2 \in \mathbb{R}\}$ contains a value of ξ_2, say $\tilde{\underline{\xi}}_2$, such that the associated oscillating phase is outgoing-incoming and that the intersection between $V \cap \{\tau = \underline{\tau}\}$ and the line $\{(\underline{\tau}, \xi_1, \tilde{\underline{\xi}}_2), \xi_1 \in \mathbb{R}\}$ contains the frequency $(\underline{\tau}, \tilde{\underline{\xi}}_1, \tilde{\underline{\xi}}_2)$ (in other words, it is equivalent to say that there exists a rectangle with sides parallel to the x- and y-axes whose corners are four points of $V \cap \{\tau = \underline{\tau}\}$). If the frequency $(\underline{\tau}, \tilde{\underline{\xi}}_1, \tilde{\underline{\xi}}_2)$ is associated with an incoming-outgoing group velocity, we remark that by applying the phase generation process (more precisely during the third reflection), we have to consider the frequency $(\underline{\tau}, \underline{\xi}_1, \tilde{\underline{\xi}}_2)$, which has been initially excluded.

Moreover, when we study the reflections of the phase associated with the frequency $(\underline{\tau}, \tilde{\underline{\xi}}_1, \tilde{\underline{\xi}}_2)$ on the side $\partial\Omega_1$, we are led to consider one more time the phase with frequency $(\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$. So, the phase associated with the frequency $(\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$ is "self-generating" or "self-interacting" because it is in the set of the phases that it generates. Such a configuration in the characteristic variety will be called a "loop". An explicit example of a corner problem with a loop will be given in Section 3E.

The fact that at each reflection there is more than one generated phase and this self-interaction phenomenon between the phases imply that there is no natural order on the set of phases as in the $N = 2$ framework. Indeed, when $N > 2$ we have to deal with a tree matching the phase generation at each reflection. Thus, constructing the WKB expansion when $N > 2$ will be less intuitive than when $N = 2$, a framework in which it is sufficient to use the order induced by the phase generation process. In Sections 4B1 and 4B4, we show how to overcome this lack of natural order in view of constructing the WKB expansion.

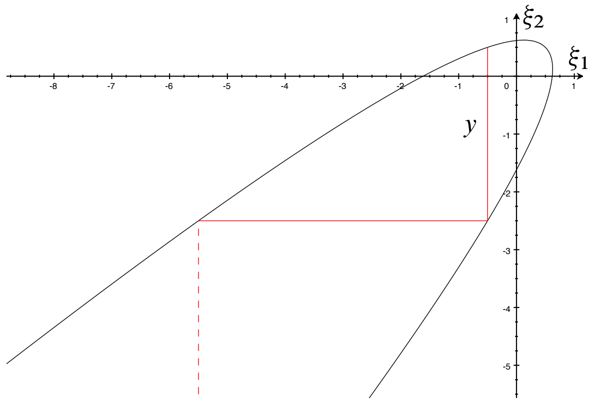

Figure 3. Phase generation for the corner problem of Section 3D.

3D. ***An example with infinitely many phases.*** The aim of this section is to illustrate the phase generation process and to give an explicit example of a corner problem whose geometric optics expansion contains an infinite number of phases. Moreover, this example will also stress the fact that the phase generation process is much simpler when $N = 2$, since it gives a natural order of construction of the WKB expansion.

Let us consider the corner problem (12) with

$$A_1 := \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

It is thus clear that $p_1 = p_2 = 1$; then we have to choose $B_1, B_2 \in M_{1,2}(\mathbb{R})$ in such a way that the boundary conditions on $\partial\Omega_1$ and $\partial\Omega_2$ are strictly dissipative [Strang 1969]. Moreover, one can easily check that this corner problem satisfies Assumptions 2.1 and 2.2.

We choose

$$g^{\varepsilon}(t, x_2) := e^{\frac{i}{\varepsilon}(t+\frac{1}{2}x_2)} g(t, x)$$

for the source term on $\partial\Omega_1$ in the corner problem (12). Then the phase generation process for this problem is precisely described in [Benoit 2015, paragraphe 6.6.1] and is illustrated in Figure 3. The phases that we have to consider form a "stairway" in a parabola (see Figure 3). The points of this stairway are labeled by two sequences $(\xi_{1,p})_{p\in\mathbb{N}}$ and $(\xi_{2,p})_{p\in\mathbb{N}}$ in such a way that points $(\xi_{1,p}, \xi_{2,p})_{p\in\mathbb{N}}$ match with points in the "top of the parabola", whereas points $(\xi_{1,p}, \xi_{2,p+1})_{p\in\mathbb{N}}$ match with points in the "bottom of the parabola". Finally we initialize at $\xi_{1,0} = -\frac{1}{2}$ and $\xi_{2,0} = \frac{1}{2}$. A simple computation shows that we have

$$\xi_{1,p} = -2p^2 - 3p - \tfrac{1}{2}, \quad \xi_{2,p} = -2p^2 - p + \tfrac{1}{2},$$

and

$$v_p = \frac{1}{4p^2+4p+2} \begin{bmatrix} 4p+1 \\ -(4p+3) \end{bmatrix}, \quad w_p = \frac{1}{4p^2+8p+5} \begin{bmatrix} -(4p+5) \\ 4p+3 \end{bmatrix}.$$

So all the points of the "top" are associated with incoming-outgoing group velocities, while points of the "bottom" are associated with outgoing-incoming group velocities. Thus according to the phase generation process described above, the number of phases in the expansion will be infinite.

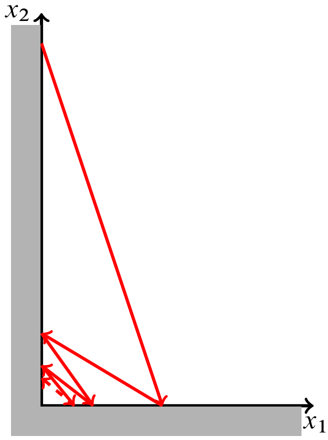

Figure 4. Appearance of the characteristics for problem of Section 3D.

We refer to [Benoit 2015, paragraphes 6.6.2 and 6.6.3] for a rigorous construction of the geometric optics expansion and a justification of its convergence towards the exact solution. The difficult part of this analysis does not come from the construction because we are in the simpler case $N = 2$ but it comes from the justification. Indeed, when infinitely many phases occur, to ensure that the WKB expansion (at a finite order in terms of powers of ε) makes sense, we have to ensure that a series converges.

Finally, we address the following phenomenon. If we fix a point $(0, y_p) \in \partial\Omega_1$ and follow the characteristic with group velocity v_p, then we will hit $\partial\Omega_2$ at time t_p^v in a point $(x_p, 0)$. Then if we start from $(x_p, 0) \in \partial\Omega_2$ and follow the characteristic line with group velocity w_p, we will hit $\partial\Omega_1$ at time t_p^w in a point $(y_{p+1}, 0)$. A simple computation shows that the considered sequences are given by

$$y_p = \frac{1}{4p+1} y_0, \quad x_p = \frac{1}{4p+3} y_0, \quad t_p^v = \frac{1}{v_{p,1}} y_p, \quad t_p^w = \frac{1}{v_{p,2}} x_p,$$

from which we deduce that from the starting point $(0, y_0)$, $y_0 > 0$, we will get closer and closer to the corner at each reflection and will reach the corner in an infinite time. A scheme illustrating the characteristic lines for this corner problem is given in Figure 4.

3E. ***An example with a loop.*** We consider the corner problem

$$\begin{cases} \partial_t u^\varepsilon + A_1 \partial_1 u^\varepsilon + A_2 \partial_2 u^\varepsilon = 0, \\ B_1 u^\varepsilon{}_{|x_1=0} = 0, \quad B_2 u^\varepsilon{}_{|x_2=0} = g^\varepsilon, \quad u^\varepsilon{}_{|t\le 0} = 0, \end{cases} \quad (x_1, x_2) \in \Omega, \tag{16}$$

with

$$A_1 := \begin{bmatrix} 3/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -3/\sqrt{2} & 0 \\ 0 & 0 & 5/7 \end{bmatrix}, \quad A_2 := \begin{bmatrix} -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

This system does not have any physical meaning and is composed of a "wave type" equation and a scalar transport equation. It is clear that the corner problem (16) satisfies Assumption 2.2 with $p_1 = 2$

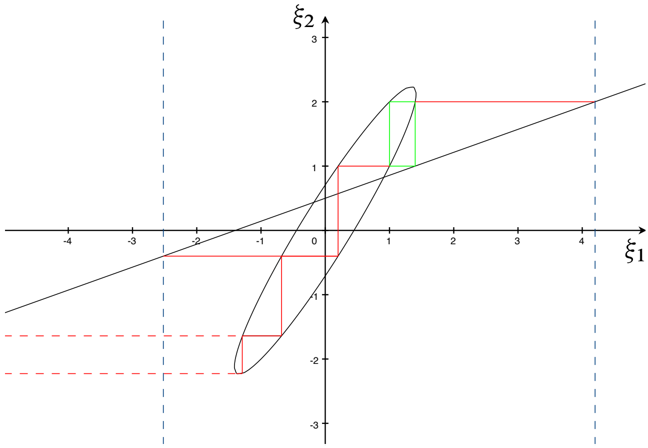

Figure 5. Section of the characteristic variety and the phase generation for corner problem (16).

and $p_2 = 1$. The corner problem (16) does not satisfy Assumption 2.1, but it is hyperbolic in the sense of geometrically regular hyperbolic systems (see [Métivier and Zumbrun 2005, Definition 2.2]). This hyperbolicity assumption is sufficient for our discussion as long as we do not have to consider the ansatz frequencies corresponding to intersection points[2] of the different sheets of the characteristic variety.

For the corner problem (16), the equation of the section of the characteristic variety with the plane $\{\tau = 1\}$ is given by

$$V_{\tau=1}:\ (5\xi_1^2 + 2\xi_2^2 - 6\xi_1\xi_2 + 1)\big(1 + \tfrac{5}{7}\xi_1 - 2\xi_2\big) = 0,$$

and is composed of an ellipse and a crossing line; see Figure 5. If we choose

$$g^\varepsilon(t, x_1) = e^{\frac{i}{\varepsilon}(t + \frac{7}{5}x_1)} g(t, x_1)$$

for the source term g^ε on $\partial\Omega_2$ in (16) then after the application of the phase generation process (see Figure 5), we obtain a loop as introduced in Section 3C. The four self-interacting phases and the associated group velocities are given by

$$\begin{aligned} &\varphi_1(t,x) := t + \tfrac{7}{5}x_1 + 2x_2, \quad v_1 = \begin{bmatrix} -1 \\ 1/5 \end{bmatrix}, \qquad \varphi_2(t,x) := t + \tfrac{7}{5}x_1 + x_2, \quad v_2 = \begin{bmatrix} 5/7 \\ -2 \end{bmatrix}, \\ &\varphi_3(t,x) := t + x_1 + x_2, \qquad v_3 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \varphi_4(t,x) := t + x_1 + 2x_2, \quad v_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \end{aligned} \tag{17}$$

[2]We have to stress that these intersection points, specific to geometrically regular hyperbolic systems, can, generically, induce an infinite number of phases in the WKB expansion. Indeed, let us assume that in a given intersection point, one of the sheets of the variety is associated with an incoming-outgoing group velocity, whereas the other sheet is associated with an outgoing-incoming group velocity. Then, using the fact that the group velocities are regular, one can find a neighborhood on each sheet such that the group velocity does not change type on this neighborhood. It immediately follows that if a ray of the geometric optics expansion contains a frequency in these neighborhoods, it is automatically attracted toward the intersection point by forming a "stairway", like in Section 3D. The fact that this phenomenon does not occur for the corner problem (16), and that the number of generated phases in finite, is somewhat very special.

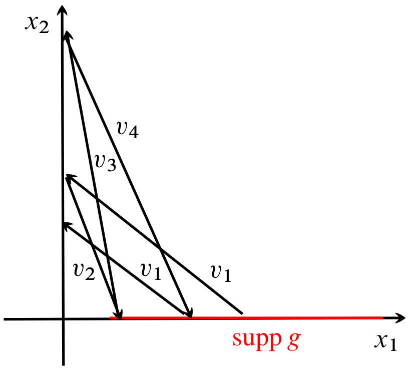

Figure 6. The loop.

where v_1 and v_3 are outgoing-incoming, whereas v_2 and v_4 are incoming-outgoing. The precise values of the *fifteen other expected phases* in the WKB expansion can be explicitly computable but they are not really important for our actual discussion. Let us just stress that, as it can be seen in Figure 5, there are four evanescent phases for the side $\partial\Omega_2$ but there are no oscillating phases of the form $(1, \xi_1, 0)$. Thus, in particular, the technical Assumption 4.9, used in the construction of the WKB expansion will be satisfied.

As in Section 3D we are not interested in the construction of the geometric optics expansion but we want to study the behavior of the rays associated with the phases $(\varphi_j)_{j=1,\dots,4}$ when T is large.

If we start from a point $(x_0, 0) \in \partial\Omega_2$ and make it travel along the characteristics with group velocity v_1, v_2, v_3 and v_4, then after one cycle the ray will hit $\partial\Omega_2$ after a time of travel t_0 in a point $(x_2, 0)$. Some computations, like those made in Section 3D, show that, for $x_0 > 0$, we have

$$x_{2p} = \beta^{-p} x_0, \quad t_p = \tilde{\alpha}\beta^{-p} x_0,$$

with

$$\beta^{-1} := \frac{1}{28} = \frac{v_{1,2}}{v_{1,1}} \frac{v_{2,1}}{v_{2,2}} \frac{v_{3,2}}{v_{3,1}} \frac{v_{4,1}}{v_{4,2}},$$

and $\tilde{\alpha}$ a nonrelevant parameter for our purpose. Since $\beta > 1$, the ray concentrates at the corner. Moreover the total time of travel towards the corner $\sum_{p\geq 0} t_p$ is the sum of a finite geometric sum so the ray reaches the corner in finite time.

We will come back in Section 4D to this example, and more precisely to the resolution of the new amplitude equation needed to construct the geometric optics expansion. Let us conclude this section by noting that Figure 6 depicts the characteristics associated with the group velocities $(v_j)_{j=1,\dots,4}$.

4. Geometric optics expansions for self-interacted trapped rays

Until the end of this paper, we will study the following hyperbolic corner problem with N equations:

$$\begin{cases} \partial_t u^\varepsilon + A_1 \partial_1 u^\varepsilon + A_2 \partial_2 u^\varepsilon = 0, \\ B_1 u^\varepsilon{}_{|x_1=0} = g^\varepsilon, \quad B_2 u^\varepsilon{}_{|x_2=0} = 0, \quad u^\varepsilon{}_{|t\leq 0} = 0, \end{cases} \qquad (t, x_1, x_2) \in \Omega_T, \tag{18}$$

where we recall that $A_1, A_2 \in M_N(\mathbb{R})$ with $N \geq 2$, $B_1 \in M_{p_1,N}(\mathbb{R})$ and $B_2 \in M_{p_2,N}(\mathbb{R})$. Our goal is to construct the WKB approximation to the solution u^ε to (18) when self-interacting phases occur. But before starting the construction of the geometric optics expansion, we shall give a precise and rigorous meaning of the phase generation process described in Section 3. This is the object of the following section.

4A. *General framework.* In this section we define a general framework wherein we can construct rigorously geometric optics expansions for corner problems. As already mentioned, the geometry of the characteristic variety influences the phase generation process and consequently it also influences the geometric optics expansion. Though not the most general, our framework will be general enough to take into account one-loop and self-interacting phases. Possible extensions are indicated at the end of this article.

4A1. *Definition of the frequency set and first properties.* Let us start with the definition of what we mean by a frequency set:

Definition 4.1. Let $\mathscr{I}$ be a subset of $\mathbb{N}$ and $\underline{\tau} \in \mathbb{R}$, $\underline{\tau} \neq 0$. A set indexed by $\mathscr{I}$,

$$\mathscr{F} := \left\{ f_i := (\underline{\tau}, \xi_1^i, \xi_2^i),\ i \in \mathscr{I} \right\},$$

will be a set of frequencies for the corner problem (18) if for all $i \in \mathscr{I}$, the frequency f_i satisfies

$$\det \mathscr{L}(f_i) = 0$$

and one of the following alternatives:

(i) $\xi_1^i, \xi_2^i \in \mathbb{R}$.

(ii) $\xi_1^i \in (\mathbb{C} \setminus \mathbb{R})$, $\xi_2^i \in \mathbb{R}$ and $\operatorname{Im} \xi_1^i > 0$.

(iii) $\xi_2^i \in (\mathbb{C} \setminus \mathbb{R})$, $\xi_1^i \in \mathbb{R}$ and $\operatorname{Im} \xi_2^i > 0$.

In all that follows, if $\mathscr{F}$ is a frequency set for the corner problem (18), we will define

$$\mathscr{F}_{\mathrm{os}} := \{ f_i \in \mathscr{F} \text{ satisfying (i)} \},$$
$$\mathscr{F}_{\mathrm{ev1}} := \{ f_i \in \mathscr{F} \text{ satisfying (ii)} \},$$
$$\mathscr{F}_{\mathrm{ev2}} := \{ f_i \in \mathscr{F} \text{ satisfying (iii)} \}.$$

It is clear that the sets $\mathscr{F}_{\mathrm{os}}$, $\mathscr{F}_{\mathrm{ev1}}$ and $\mathscr{F}_{\mathrm{ev2}}$ give a partition of $\mathscr{F}$. Moreover, to each $f_i \in \mathscr{F}_{\mathrm{os}}$, we can associate a group velocity $v_i := (v_{i,1}, v_{i,2})$. Let us recall that the group velocity v_i is defined in Definition 2.5. The set $\mathscr{F}_{\mathrm{os}}$ can be decomposed as

$$\mathscr{F}_{\mathrm{ii}} := \{ f_i \in \mathscr{F}_{\mathrm{os}} \mid v_{i,1},\ v_{i,2} > 0 \}, \qquad \mathscr{F}_{\mathrm{io}} := \{ f_i \in \mathscr{F}_{\mathrm{os}} \mid v_{i,1} > 0,\ v_{i,2} < 0 \},$$
$$\mathscr{F}_{\mathrm{oi}} := \{ f_i \in \mathscr{F}_{\mathrm{os}} \mid v_{i,1} < 0,\ v_{i,2} > 0 \}, \qquad \mathscr{F}_{\mathrm{oo}} := \{ f_i \in \mathscr{F}_{\mathrm{os}} \mid v_{i,1} < 0,\ v_{i,2} < 0 \}.$$
$$\mathscr{F}_{\mathrm{g}} := \{ f_i \in \mathscr{F}_{\mathrm{os}} \mid v_{i,1} = 0 \text{ or } v_{i,2} = 0 \},$$

The partition of $\mathscr{F}$ induces the following partition of $\mathscr{I}$:

$$\mathscr{I} = \mathscr{I}_{\mathrm{g}} \cup \mathscr{I}_{\mathrm{oo}} \cup \mathscr{I}_{\mathrm{io}} \cup \mathscr{I}_{\mathrm{oi}} \cup \mathscr{I}_{\mathrm{ii}} \cup \mathscr{I}_{\mathrm{ev1}} \cup \mathscr{I}_{\mathrm{ev2}},$$

where we have denoted by $\mathscr{I}_{\mathrm{io}}$ (resp. $_{\mathrm{g}}$, $_{\mathrm{oo}}$, $_{\mathrm{oi}}$, $_{\mathrm{ii}}$, $_{\mathrm{ev1}}$, $_{\mathrm{ev2}}$) the set of indices $i \in \mathscr{I}$ such that the corresponding frequency $f_i \in \mathscr{F}_{\mathrm{io}}$ (resp. $_{\mathrm{g}}$, $_{\mathrm{oo}}$, $_{\mathrm{oi}}$, $_{\mathrm{ii}}$, $_{\mathrm{ev1}}$, $_{\mathrm{ev2}}$).

From now on, the source term g^ε on the boundary in (18) is given by

$$g^\varepsilon(t,x_2) := e^{\frac{i}{\varepsilon}(\underline{\tau}t+\underline{\xi}_2x_2)}g(t,x_2), \tag{19}$$

where the amplitude g belongs to H_f^∞ and is zero for negative times.

The following definition gives a precise framework for the phase generation process described in Section 3. More precisely, this definition qualifies the frequency set that contains all (and only) the frequencies linked with the expected nonzero amplitudes in the WKB expansion of the solution to the corner problem (18).

Definition 4.2. The corner problem (18) is said to be complete for reflections if there exists a set of frequencies $\mathscr{F}$ satisfying the following properties:

(i) $\mathscr{F}$ contains the real roots (in the variable ξ_1) associated with incoming-outgoing or incoming-incoming group velocities and the complex roots with positive imaginary part to the dispersion equation

$$\det \mathscr{L}(\underline{\tau},\xi_1,\underline{\xi}_2)=0.$$

(ii) $\mathscr{F}_g=\varnothing$.[3]

(iii) If $(\underline{\tau},\xi_1^i,\xi_2^i)\in\mathscr{F}_{\mathrm{io}}$, then $\mathscr{F}$ contains all the roots (in the variable ξ_2), denoted by ξ_2^p, of the dispersion relation $\det\mathscr{L}(\underline{\tau},\xi_1^i,\xi_2)=0$ that satisfy one of the following two alternatives:

(iii′) $\xi_2^p\in\mathbb{R}$ and the frequency $(\underline{\tau},\xi_1^i,\xi_2^p)$ is associated with an outgoing-incoming group velocity or an incoming-incoming group velocity.

(iii″) $\operatorname{Im}\xi_2^p>0$.

(iv) If $(\underline{\tau},\xi_1^i,\xi_2^i)\in\mathscr{F}_{\mathrm{oi}}$, then $\mathscr{F}$ contains all the roots (in the variable ξ_1), denoted by ξ_1^p, of the dispersion relation $\det\mathscr{L}(\underline{\tau},\xi_1,\xi_2^i)=0$ that satisfy one of the following two alternatives:

(iv′) $\xi_1^p\in\mathbb{R}$ and the frequency $(\underline{\tau},\xi_1^p,\xi_2^i)$ is associated with an incoming-outgoing or an incoming-incoming group velocity.

(iv″) $\operatorname{Im}\xi_1^p>0$.

(v) $\mathscr{F}$ is minimal (for the inclusion) for the four preceding properties.

Remark. Property (i) establishes that the frequency set $\mathscr{F}$ contains all the incoming phases for $\partial\Omega_1$ that are induced by the source term g^ε.

Property (iii) (resp. (iv)) explains the generation by reflection on the side $\partial\Omega_2$ (resp. $\partial\Omega_1$) of a wave packet that emanates from the side $\partial\Omega_1$ (resp. $\partial\Omega_2$).

An immediate consequence of the minimality of $\mathscr{F}$ is that $\mathscr{F}_{\mathrm{oo}}$ is empty. In all that follows, we will assume that the dispersion relation $\det\mathscr{L}(\underline{\tau},\xi_1,\underline{\xi}_2)=0$ has at least one real solution $\underline{\xi}_1$ such that the group velocity for the frequency $\underline{f}:=(\underline{\tau},\underline{\xi}_1,\underline{\xi}_2)$ is incoming-outgoing. This assumption is, of course, not necessary. However, without this assumption, it is easy to see that the phase generation for the corner

[3] This restriction is probably not necessary. However, for a first work on this subject we did not want to add the technicality induced by the determination of amplitudes associated with glancing frequencies (see [Williams 2000] for such a construction). Incorporating glancing modes in the WKB expansion is left for future studies.

problem (18) is not richer than the phase generation for the standard boundary value problem in the half-space $\{x_1 \geq 0\}$. Indeed, the minimality of the frequency set $\mathscr{F}$ would imply, in this case,

$$\mathscr{F} = \mathscr{F}_{\mathrm{ii}} \cup \mathscr{F}_{\mathrm{ev1}} \qquad \text{and} \qquad \forall f_i \in \mathscr{F}, \quad \xi_2^i = \underline{\xi}_2.$$

For a corner problem that is complete for reflections, one can define the following functions. These functions are defined on the index set $\mathscr{I}$ and give, in the output, the indices "in the direct vicinity" of the input index:

$$\Phi,\ \Psi : \mathscr{I} \to \mathscr{P}_N(\mathscr{I}),$$

where $\mathscr{P}_N(\mathscr{I})$ denotes the power set of $\mathscr{I}$ with at most N elements. More precisely, for $i \in \mathscr{I}$ and $f_i = (\underline{\tau}, \xi_1^i, \xi_2^i)$,

$$\Phi(i) := \{ j \in \mathscr{I} \mid \xi_2^j = \xi_2^i \} \quad \text{and} \quad \Psi(i) := \{ j \in \mathscr{I} \mid \xi_1^j = \xi_1^i \}.$$

Thanks to these functions, the index set $\mathscr{I}$ can be seen as a graph. This graph structure will be more abstract than the description of $\mathscr{I}$ based on the wave packet reflections, but it will be easier to work with when we will construct the WKB expansion. This graph structure is defined by the following relation: two points $i, j \in \mathscr{I}$ are linked by an edge if and only if $i \in \Phi(j)$ or $i \in \Psi(j)$.

In terms of wave packet reflection, the set $\Phi(i)$ (resp. $\Psi(i)$) is the set of all indices of the phases that are considered in the reflection of the wave packet with phase associated to f_i on $\partial\Omega_2$. Let us stress that the index i is not necessarily the index of an incident ray but can be the index of one of the reflected rays.

It is easy to see that functions Φ and Ψ have the following properties. One can also check that these properties are independent of the concept of "loop" that will be introduced in the following section.

Proposition 4.3. *If the corner problem (18) is complete for reflections, then Φ and Ψ satisfy the following properties:*

(i) $\forall i \in \mathscr{I}$, *we have* $i \in \Psi(i)$ *and* $i \in \Phi(i)$.

(ii) $\forall i \in \mathscr{I}, \forall j \in \Psi(i), \forall k \in \Phi(i)$, *we have* $\Psi(i) = \Psi(j)$ *and* $\Phi(i) = \Phi(k)$.

(iii) $\forall i \in \mathscr{I}$, *we have* $\Phi(i) \cap \mathscr{I}_{\mathrm{ev2}} = \varnothing$ *and* $\Psi(i) \cap \mathscr{I}_{\mathrm{ev1}} = \varnothing$, *and*, $\forall i \in \mathscr{I}_{\mathrm{ev1}}, \forall j \in \mathscr{I}_{\mathrm{ev2}}$, *we have* $\Psi(i) \subset \mathscr{I}_{\mathrm{ev1}}$ *and* $\Phi(i) \subset \mathscr{I}_{\mathrm{ev2}}$.

(iv) $\forall i \in \mathscr{I}_{\mathrm{os}}$, *we have* $\#(\Phi(i) \cap \mathscr{I}_{\mathrm{ev1}} \cap \mathscr{I}_{\mathrm{io}} \cap \mathscr{I}_{\mathrm{ii}}) \leq p_1$ *and* $\#(\Psi(i) \cap \mathscr{I}_{\mathrm{ev2}} \cap \mathscr{I}_{\mathrm{oi}} \cap \mathscr{I}_{\mathrm{ii}}) \leq p_2$.

(v) $\forall i \in \mathscr{I}$, *we have, on one hand*, $\forall i_1, i_2 \in \Phi(i),\ i_1 \neq i_2$,

$$\Phi(i) \cap \Psi(i_1) = \{i_1\} \quad \textit{and} \quad \Psi(i_1) \cap \Psi(i_2) = \varnothing,$$

and on the other hand, $\forall j_1, j_2 \in \Psi(i),\ j_1 \neq j_2$,

$$\Psi(i) \cap \Phi(j_1) = \{j_1\} \quad \textit{and} \quad \Phi(j_1) \cap \Phi(j_2) = \varnothing.$$

Proof. Properties (i), (ii) and (v) are direct consequences of the definition of the functions Φ and Ψ. Property (iii) arises from the definition of the frequency set. Finally property (iv) is a consequence of the block structure theorem (see Theorem 2.3). $\square$

Thanks to functions Φ and Ψ it is easy to define the notion of two linked indices in the graph structure of $\mathscr{I}$. In terms of wave packet reflections, this notion means that the index i will be linked with the index j if and only if j is obtained from the wave packet associated with i after several reflections. In other terms, we can say that the index i generates the index j, or that i is the "father" of j. The following definition makes this notion more precise:

Definition 4.4. If $\underline{i} \in \mathscr{I}_{\text{io}}$, we say that the index $j \in \mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ev}1}$ (resp. $j \in \mathscr{I}_{\text{oi}} \cup \mathscr{I}_{\text{ev}2}$) is linked with the index $\underline{i}$ if there exists $p \in 2\mathbb{N}+1$ (resp. $p \in 2\mathbb{N}$) and a sequence of indices $\ell = (\ell_1, \ell_2, \dots, \ell_p) \in \mathscr{I}^p$ such that

$$\ell_1 \in \Psi(\underline{i}) \cap \mathscr{I}_{\text{oi}}, \quad \ell_2 \in \Phi(\ell_1) \in \mathscr{I}_{\text{io}}, \quad \dots, \quad j \in \Phi(\ell_p) \quad (\text{resp. } j \in \Psi(\ell_p)). \tag{α'}$$

We say that the index $j \in \mathscr{I}_{\text{ii}}$ is linked with the index $\underline{i}$, if there is a sequence of indices $\ell = (\ell_1, \ell_2, \dots, \ell_p) \in \mathscr{I}^p$ such that

$$\ell_1 \in \Psi(\underline{i}) \cap \mathscr{I}_{\text{oi}}, \quad \ell_2 \in \Phi(\ell_1) \cap \mathscr{I}_{\text{io}}, \quad \dots, \quad j \in \begin{cases} \Phi(\ell_p) & \text{if } p \text{ is odd,} \\ \Psi(\ell_p) & \text{if } p \text{ is even.} \end{cases} \tag{β'}$$

If $\underline{i} \in \mathscr{I}_{\text{oi}}$, we say that the index $j \in \mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ev}1}$ (resp. $j \in \mathscr{I}_{\text{oi}} \cup \mathscr{I}_{\text{ev}2}$) is linked with the index $\underline{i}$, if there exists $p \in 2\mathbb{N}$ (resp. $p \in 2\mathbb{N}+1$) and a sequence of indices $\ell = (\ell_1, \ell_2, \dots, \ell_p) \in \mathscr{I}^p$ such that

$$\ell_1 \in \Phi(\underline{i}) \cap \mathscr{I}_{\text{io}}, \quad \ell_2 \in \Psi(\ell_1) \in \mathscr{I}_{\text{oi}}, \quad \dots, \quad j \in \Phi(\ell_p) \quad (\text{resp. } j \in \Psi(\ell_p)). \tag{α''}$$

We say that the index $j \in \mathscr{I}_{\text{ii}}$ is linked with the index $\underline{i}$, if there exists a sequence of indices $\ell = (\ell_1, \ell_2, \dots, \ell_p) \in \mathscr{I}^p$ such that

$$\ell_1 \in \Phi(\underline{i}) \cap \mathscr{I}_{\text{io}}, \quad \ell_2 \in \Psi(\ell_1) \cap \mathscr{I}_{\text{oi}}, \quad \dots, \quad j \in \begin{cases} \Psi(\ell_p) & \text{if } p \text{ is odd,} \\ \Phi(\ell_p) & \text{if } p \text{ is even.} \end{cases} \tag{β''}$$

Finally, if $\underline{i} \in \mathscr{I}_{\text{ii}} \cup \mathscr{I}_{\text{ev}1} \cup \mathscr{I}_{\text{ev}2}$, there is no element of $\mathscr{I}$ linked with $\underline{i}$.

Moreover, we will say that an index $j \in \mathscr{I}$ is linked with the index $\underline{i}$ by a sequence of type H (for "horizontal") (resp. V (for "vertical")) and we will use the notation $i \underset{H}{\rightarrowtail} j$ (resp. $i \underset{V}{\rightarrowtail} j$) if the sequence $(\underline{i}, \ell_1, \ell_2, \dots, \ell_p, j)$ satisfies (α'') or (β'') (resp. (α') or (β')).

Let us comment a bit on this definition. In terms of wave packet reflections, if one fixes an index $\underline{i} \in \mathscr{I}_{\text{io}}$, an index j is linked with the index $\underline{i}$ if j comes from $\underline{i}$ after several reflections. More precisely, the incoming-outgoing ray associated with $\underline{i}$ hits the side $\partial\Omega_2$ and is reflected in the outgoing-incoming ray associated with the index ℓ_1. Then the ray of index ℓ_1 hits the side $\partial\Omega_1$, and generates the incoming-outgoing ray associated with the index ℓ_2. This ray hits the side $\partial\Omega_2$ and so on until the ray associated with the index ℓ_p generates by reflection the index j.

The distinction of cases based on the group velocity of the index j in the subcase (α') considers the fact that a ray associated with an index in $\mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ev}1}$ (resp. $\mathscr{I}_{\text{oi}} \cup \mathscr{I}_{\text{ev}2}$) can be generated by a ray associated with ℓ_p only during a reflection on the side $\partial\Omega_1$ (resp. $\partial\Omega_2$), or equivalently after an even (resp. odd) number of reflections, whereas a ray with an incoming-incoming group velocity can be generated by the ray ℓ_p during a reflection on the side $\partial\Omega_1$ or one the side $\partial\Omega_2$. That is the reason why the subcase (β') differs from the subcase (α').

If one rather sees the index set $\mathscr{I}$ with a graph structure, saying that j is linked with $\underline{i}$ is no more than saying that starting from $\underline{i}$ one can reach the index j by passing through the indices ℓ_i, with the following rule of travel: if one reaches ℓ_l by following a vertical (resp. horizontal) edge of the graph, then ℓ_{l+1} will be reached by following a horizontal (resp. vertical) edge. A sequence of type H (resp. V) just means that when we start from $\underline{i}$, the first edge is a horizontal (resp. vertical) one.

The following proposition is an immediate consequence of Definitions 4.2 and 4.4.

Proposition 4.5. *Let $\mathscr{F}$ be a complete-for-reflection frequency set indexed by $\mathscr{I}$. Let $\mathscr{I}_0$ be the set of indices in $\mathscr{I}$ generated by the source term g^ε; that is to say,*

$$\mathscr{I}_0 := \{i \in \mathscr{I}_{\mathrm{io}} \cup \mathscr{I}_{\mathrm{ii}} \cup \mathscr{I}_{\mathrm{ev1}} \mid \det \mathscr{L}(\underline{\tau}, \xi_1^i, \underline{\xi}_2) = 0\}.$$

Let $\mathscr{I}_{\mathscr{R}}$ be the set of indices in $\mathscr{I}$ linked with one of the elements of $\mathscr{I}_0$. Then

$$\mathscr{I}_{\mathscr{R}} = \mathscr{I}.$$

Proof. Let $\mathscr{F}_{\mathscr{R}}$ be the set of frequencies indexed by $\mathscr{I}_{\mathscr{R}}$. It is clear that the set $\mathscr{F}_{\mathscr{R}}$ satisfies properties (i)–(iv) of Definition 4.2. Let us describe the verification of property (iii) to be more convincing.

We fix $\underline{i} \in \mathscr{I}_{\mathscr{R}}$, an incoming-outgoing index. Let $\underline{\ell}$ be a sequence that links $\underline{i}$ to one of the indices of $\mathscr{I}_0$. Then indices in $\mathscr{I}_{\mathrm{oi}} \cap \Psi(\underline{i})$, $\mathscr{I}_{\mathrm{ii}} \cap \Psi(\underline{i})$ and $\mathscr{I}_{\mathrm{ev2}} \cap \Psi(\underline{i})$ are linked with an element in $\mathscr{I}_0$ by the sequence $(\underline{\ell}, \underline{i})$. As a consequence, these indices are in $\mathscr{I}_{\mathscr{R}}$. We just showed that $\mathscr{I}_{\mathscr{R}}$ satisfies property (iii) of Definition 4.2.

We now want to show that $\mathscr{I}_{\mathscr{R}} = \mathscr{I}$. By contradiction, we assume that there exists $j \in (\mathscr{I} \setminus \mathscr{I}_{\mathscr{R}})$. Firstly, if $j \in \mathscr{I}_{\mathrm{ev1}} \cup \mathscr{I}_{\mathrm{ev2}} \cup \mathscr{I}_{\mathrm{ii}}$, then the frequency set indexed by $\mathscr{I} \setminus \{j\}$ still satisfies properties (i)–(iv) in Definition 4.2. This fact contradicts the minimality of $\mathscr{F}$.

Then, if $j \in \mathscr{I}_{\mathrm{io}} \cup \mathscr{I}_{\mathrm{oi}}$, we construct the set of indices linked with j, and we denote this set by $\tilde{\mathscr{I}}$. Let $\tilde{\mathscr{F}}$ be the frequency set indexed by $\tilde{\mathscr{I}}$. The set $(\tilde{\mathscr{F}} \cup \mathscr{F}_{\mathscr{R}}) \setminus (\tilde{\mathscr{F}} \cap \mathscr{F}_{\mathscr{R}})$ satisfies properties (i)–(iv) in Definition 4.2 and is strictly included in $\mathscr{F}$ because $j \in (\tilde{\mathscr{F}} \cap \mathscr{F}_{\mathscr{R}})$. Once more, this fact is incompatible with the minimality of the frequency set $\mathscr{F}$. $\square$

Proposition 4.5 concludes the description of our formal framework for frequency sets. Let us stress that in this framework we do not assume that the number of phases in the WKB expansion is finite. The assumption "$\#\mathscr{F} < +\infty$" will only be used to make sure that the formal geometric optics expansion constructed in the following sections is relevant, in the sense that the expansion is well-defined and that it does indeed approximate the exact solution. But it will not be used to construct the WKB expansion, at least, at a formal level.

4A2. *Frequency sets with loops.* As mentioned in the beginning of this section, the aim of all that follows is to construct rigorous geometric optics expansions for corner problem where some amplitudes in the expansion display a self-interacting phenomenon. To do that, we will need to consider corner problems whose characteristic variety contains a "loop". By loop, we mean that it is possible to find at least four points on the section of the characteristic variety $V \cap \{\tau = \underline{\tau}\}$ such that if we draw the segments linking these points, we obtain a rectangle or a finite "stairway" (see Section 3C and [Sarason and Smoller 1974, Figure 8]).

Many kinds of loops are possible and few of them lead to a self-interaction phenomenon. That is why, in all that follows, we will assume that there is a unique loop and that this loop induces a self-interaction phenomenon. The uniqueness of the loop is probably not a necessary assumption, but it permits us to simplify many steps of the proof and to save a lot of combinatorial arguments. We refer to [Benoit 2015, paragraphe 6.10] for more details. The different kinds of loops are defined as follows:

Definition 4.6. Let $i \in \mathscr{I}$, $p \in 2\mathbb{N}+1$ and $\ell = (\ell_1, \dots, \ell_p) \in \mathscr{I}^p$ (we stress that elements of ℓ are not necessarily distinct).

- We say that the index $i \in \mathscr{I}$ admits a loop if there exists a sequence ℓ satisfying
$$\ell_1 \in \Phi(i), \quad \ell_2 \in \Psi(\ell_1), \quad \dots, \qquad i \in \Psi(\ell_p).$$
- A loop for an index i is said to be simple if the sequence ℓ does not contain a periodically repeated subsequence.
- An index $i \in \mathscr{I}_{\mathrm{io}}$ (resp. $i \in \mathscr{I}_{\mathrm{oi}}$) admits a self-interaction loop if i admits a simple loop and if the sequence (i, ℓ, i) is of type V (resp. H) according to Definition 4.4.

From now on, let us assume the following:

Assumption 4.7. *Let (18) be complete for the reflections. We assume that the frequency set $\mathscr{F}$ contains a unique loop of size* 3 *and that this loop is a self-interaction loop. More precisely, we want the following properties to be satisfied*:

(vi) $\exists (n_1, n_3) \in \mathscr{I}_{\mathrm{io}}^2$, $(n_2, n_4) \in \mathscr{I}_{\mathrm{oi}}^2$ *such that*
$$n_4 \in \Psi(n_1), \quad n_3 \in \Phi(n_4), \quad n_2 \in \Psi(n_3), \quad n_1 \in \Phi(n_2).$$

(vii) *Let $i \in \mathscr{I}$ be an index with a loop $\ell = (\ell_1, \dots, \ell_p)$. Then $p = 3$ and $\{i, \ell_1, \ell_2, \ell_3\} = \{n_1, n_2, n_3, n_4\}$.*

The fact that we restrict our attention to a loop of size 3 is just to simplify as much as possible the redaction of the proof. However, all of the following construction can be generalized to loops with more than three elements.

One of the main difficulties induced by the presence of a loop is that the definition of linked indices does not permit us anymore to define a partial order on the frequency set, as can be done in the case $N = 2$. Indeed, if one considers indices n_1 and n_3 defined in Assumption 4.7 then we have $n_1 \underset{V}{\succ\!\!\!\rightarrow} n_3$ and $n_3 \underset{V}{\succ\!\!\!\rightarrow} n_1$ but $n_1 \neq n_3$. We will see in Section 4B1 how this new difficulty can be overcome.

We conclude this section by defining what we mean by "trapped" and "self-interacting" rays.

Definition 4.8. A ray of the geometric optics expansion is said to be trapped if when we follow its characteristics, we never escape from a compact set.

A ray of the geometric optics expansion is said to be self-interacting if when we follow its characteristics, we can find a repeating sequence of group velocities.

So a trapped ray is a ray which will never escape to "infinity". The ray obtained by following the characteristic lines for the indices (n_1, n_2, n_3, n_4) is a self-interacting trapped ray, whereas, the ray described in Section 3D is non-self-interacting trapped ray.

To conclude this section let us give the following assumption about an extra special property which applies to the frequency set $\mathscr{F}$. This assumption will be useful in the beginning of Section 4B to derive the WKB cascade. Moreover, some extra comments about this assumption can be found in Section 5.

Assumption 4.9. *If $\mathscr{F}_{\mathrm{ev}1}$ (resp. $\mathscr{F}_{\mathrm{ev}1}$) is not empty then $\mathscr{F}_{\mathrm{os}}$ does not contain any element of the form $f_k = (\underline{\tau}, 0, \xi_2^k)$ (resp. $f_k = (\underline{\tau}, \xi_1^k, 0)$).*

4A3. *Some definitions and notation.* For $j \in \mathscr{I}_{\mathrm{ev}1} \cup \mathscr{I}_{\mathrm{ii}} \cup \mathscr{I}_{\mathrm{io}}$ (resp. $i \in \mathscr{I}_{\mathrm{ev}2} \cup \mathscr{I}_{\mathrm{ii}} \cup \mathscr{I}_{\mathrm{oi}}$), we denote by $f^j := (\underline{\tau}, \xi_1^j, \xi_j^2)$ the associated frequency. Let us recall that thanks to the uniform Kreiss–Lopatinskii condition, it is possible to define ϕ_1^j (resp. ϕ_2^j), the inverse of B_1 (resp. B_2) restricted to the stable subspace $E_1^s(i\underline{\tau}, \xi_2^j)$ (resp. $E_2^s(i\underline{\tau}, \xi_1^j)$).

To construct the amplitudes in the WKB expansion, we will need the following projectors and partial inverses:

Definition 4.10. For $j \in \{1,2\}$ and $f_k = (\underline{\tau}, \xi_1^k, \xi_2^k) \in \mathscr{F}$, let us denote by $P^k_{1,s,j}$ (resp. $P^k_{2,s,j}$), the projector on $E_j^{s,e}(i\underline{\tau}, \xi^k_{3-j})$ (resp. $E_i^{u,e}(i\underline{\tau}, \xi^k_{3-j})$) associated with the direct sum (3), and P_1^k (resp. P_2^k) the projector on $\ker \mathscr{L}(f_k)$ associated with the sums (5) (resp. (6)).

Let us denote by $Q^k_{1,s,j}$ (resp. $Q^k_{2,s,j}$), the projector on $E_j^{s,e}(i\underline{\tau}, \xi^k_{3-j})$ (resp. $E_j^{u,e}(i\underline{\tau}, \xi^k_{3-j})$) associated with the direct sum (4), and Q_j^k (resp. Q_2^k) the projector on $A_1 \ker \mathscr{L}(f_k)$ (resp. $A_2 \ker \mathscr{L}(f_k)$) associated with the sums (7) (resp. (8)).

Let R_j^k be the partial inverse of $\mathscr{L}(f_k)$, defined by the two relations

$$R_j^k \mathscr{L}(f_k) = I - P_j^k, \quad P_j^k R_j^k = R_j^k Q_j^k = 0. \tag{20}$$

Finally, to simplify the notation as much as possible, set

$$S_1^k := P_1^k \phi_1^k, \quad S_2^k := P_2^k \phi_2^k, \quad S_{s,1}^k := P_{s,i}^k \phi_1^k, \quad S_{s,2}^k := P_{s,2}^k \phi_2^k.$$

An important remark is that, for $k \in \mathscr{F}_{\mathrm{os}}$, if f_k is the associated frequency then $\operatorname{Ran} \mathscr{L}(f_k) = \ker Q_1^k = \ker Q_2^k$, and that, for $j \in \{1,2\}$, the projector Q_j^k induces an isomorphism from $\operatorname{Ran} P_j^k$ to $\operatorname{Ran} Q_j^k$.

We will have to solve transport equations, so the following variables will be convenient:

$$\forall j \in \mathscr{I}_{\mathrm{io}}, \quad t^j_{\mathrm{io}}(t, x_1) := t - \frac{1}{v_{j,1}} x_1, \quad x^j_{\mathrm{io}}(x_1, x_2) := x_2 - \frac{v_{j,2}}{v_{j,1}} x_1, \tag{21}$$

$$\forall j \in \mathscr{I}_{\mathrm{oi}}, \quad t^j_{\mathrm{oi}}(t, x_2) := t - \frac{1}{v_{j,2}} x_2, \quad x^j_{\mathrm{oi}}(x_1, x_2) := x_1 - \frac{v_{j,1}}{v_{j,2}} x_2. \tag{22}$$

4B. ***Construction of the WKB expansion.*** During all the construction, we will have to consider three kinds of phases, namely oscillating phases, evanescent phases for the side $\partial\Omega_1$ and evanescent phases for the side $\partial\Omega_2$. These will be denoted by

$$\begin{aligned}
\varphi_k(t,x) &:= \langle (t,x), f_k \rangle, && f_k \in \mathscr{F}_{\mathrm{os}},\\
\psi_{k,1}(t,x_2) &:= \langle (t,0,x_2), f_k \rangle, && f_k \in \mathscr{F}_{\mathrm{ev}1} \cup \mathscr{F}_{\mathrm{os}},\\
\psi_{k,2}(t,x_1) &:= \langle (t,x_1,0), f_k \rangle, && f_k \in \mathscr{F}_{\mathrm{ev}2} \cup \mathscr{F}_{\mathrm{os}}.
\end{aligned}$$

For a given amplitude $g \in H_f^\infty$, zero for negative times, we will work with a source term on the side $\partial\Omega_1$ of the form

$$g^\varepsilon(t,x_2) := e^{\frac{i}{\varepsilon}(\underline{\tau} t + x_2 \xi_2^{n_1})} g(t,x_2),$$

that is to say, a source term that "turns on" the index n_1 on the loop, and that has an incoming group velocity for the side $\partial\Omega_1$. So, we expect that the source term g^ε will generate a wave packet propagating towards the side $\partial\Omega_2$.

As in [Lescarret 2007], evanescent modes will be treated in a "monoblock" way, that is to say, that for an index $\underline{i} \in \mathscr{I}_{\text{ev}1}$ (resp. $\underline{i} \in \mathscr{I}_{\text{ev}2}$), all the indices $j \in \mathscr{I}_{\text{ev}1} \cap \Phi(\underline{i})$ (resp. $j \in \mathscr{I}_{\text{ev}2} \cap \Psi(\underline{i})$) will contribute to a single vector-valued amplitude. To write off the ansatz and to describe with enough precision the boundary conditions, it is useful to introduce the two equivalence relations $\underset{\Phi}{\sim}$ and $\underset{\Psi}{\sim}$ defined by

$$i \underset{\Phi}{\sim} j \iff j \in \Phi(i),$$
$$i \underset{\Psi}{\sim} j \iff j \in \Psi(i).$$

The fact that these relations are effectively equivalence relations is a direct consequence of Proposition 4.3.

Let $\mathfrak{C}_1$ (resp. $\mathfrak{C}_2$) be the set of equivalence classes for the relation $\underset{\Phi}{\sim}$ (resp. $\underset{\Psi}{\sim}$), and $\mathscr{R}_1$ (resp. $\mathscr{R}_2$) be a set of class representative for $\mathfrak{C}_1$ (resp. $\mathfrak{C}_2$). So $\mathscr{R}_1$ (resp. $\mathscr{R}_2$) is a set of indices which includes all the possible values for ξ_2 (resp. ξ_1) of the different frequencies. Let us define $\mathfrak{R}_1$ and $\mathfrak{R}_2$ by

$$\mathfrak{R}_1 := \{ i \in \mathscr{R}_1 \mid \Phi(i) \cap \mathscr{I}_{\text{ev}1} \neq \varnothing \}, \tag{23}$$
$$\mathfrak{R}_2 := \{ i \in \mathscr{R}_2 \mid \Psi(i) \cap \mathscr{I}_{\text{ev}2} \neq \varnothing \}. \tag{24}$$

$\mathfrak{R}_1$ (resp. $\mathfrak{R}_2$) is a set of class representative of the values in ξ_2 (resp. ξ_1) for which there is an evanescent mode for the side $\partial\Omega_1$ (resp. $\partial\Omega_2$). At last, without loss of generality, we can always assume that $n_1 \in \mathscr{R}_2$; in other words, we choose n_1 as a class representative of its equivalence class.

We take for the ansatz

$$u^\varepsilon(t,x) \sim \sum_{k \in \mathscr{I}_{\text{os}}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} \sum_{n \geq 0} \varepsilon^n u_{n,k}(t,x)$$
$$+ \sum_{k \in \mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}(t,x_2)} \sum_{n \geq 0} \varepsilon^n U_{n,k,1}\left(t,x,\frac{x_1}{\varepsilon}\right) + \sum_{k \in \mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}(t,x_1)} \sum_{n \geq 0} \varepsilon^n U_{n,k,2}\left(t,x,\frac{x_2}{\varepsilon}\right). \tag{25}$$

And we now want to determine the profiles $u_{n,k}$ and $U_{n,k,i}$. We are looking for oscillating profiles $u_{n,k}$ in the space $H^\infty(\Omega_T)$, whereas, the space for the evanescent profiles is (see [Lescarret 2007]):

Definition 4.11. For $i = 1, 2$, the set $P_{\text{ev},i}$ of evanescent profiles for the side $\partial\Omega_i$ is defined as functions $U(t,x,X_i) \in H^\infty(\Omega_T \times \mathbb{R}_+)$ for which there exists a positive δ such that $e^{\delta X_i} U(t,x,X_i) \in H^\infty(\Omega_T \times \mathbb{R}_+)$.

Plugging the ansatz (25) in the evolution equation of the corner problem (18) and identifying in terms of powers of ε leads us to solve the cascade of equations

$$\begin{cases} \mathscr{L}(d\varphi_k)u_{0,k}=0 & \forall k\in\mathscr{I}_{\mathrm{os}}, \\ i\mathscr{L}(d\varphi_k)u_{n+1,k}+L(\partial)u_{n,k}=0 & \forall n\in\mathbb{N},\ \forall k\in\mathscr{I}_{\mathrm{os}}, \\ L_k(\partial_{X_1})U_{0,k,1}=0 & \forall k\in\mathfrak{R}_1, \\ L_k(\partial_{X_1})U_{n+1,k,1}+L(\partial)U_{n,k,1}=0 & \forall n\in\mathbb{N},\ \forall k\in\mathfrak{R}_1, \\ L_k(\partial_{X_2})U_{0,k,2}=0 & \forall k\in\mathfrak{R}_2, \\ L_k(\partial_{X_2})U_{n+1,k,2}+L(\partial)U_{n,k,2}=0 & \forall n\in\mathbb{N},\ \forall k\in\mathfrak{R}_2, \end{cases} \tag{26}$$

where the "fast" differentiation operators $L_k(\partial_{X_1})$ and $L_k(\partial_{X_2})$ are given by

$$L_k(\partial_{X_1}):=A_1\big(\partial_{X_1}-\mathscr{A}_1(\underline{\tau},\xi_2^k)\big) \quad \text{for } k\in\mathfrak{R}_1,$$
$$L_k(\partial_{X_2}):=A_2\big(\partial_{X_2}-\mathscr{A}_2(\underline{\tau},\xi_1^k)\big) \quad \text{for } k\in\mathfrak{R}_2.$$

Then, plugging the ansatz (25) in the boundary conditions on the sides $\partial\Omega_1$ and $\partial\Omega_2$ gives

$$B_1\Bigg[\sum_{k\in\mathscr{I}_{\mathrm{os}}} e^{\frac{i}{\varepsilon}\psi_{k,1}}u_{n,k}(t,0,x_2)+\sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}}U_{n,k,1}(t,0,x_2,0)+\sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\underline{\tau}t}U_{n,k,2}\left(t,0,x_2,\frac{x_2}{\varepsilon}\right)\Bigg] = \delta_{n,0}e^{\frac{i}{\varepsilon}\psi_{n_1,1}}g, \tag{27}$$

and

$$B_2\Bigg[\sum_{k\in\mathscr{I}_{\mathrm{os}}} e^{\frac{i}{\varepsilon}\psi_{k,2}}u_{n,k}(t,x_1,0)+\sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}}U_{n,k,2}(t,x_1,0,0)+\sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\underline{\tau}t}U_{n,k,1}\left(t,x_1,0,\frac{x_1}{\varepsilon}\right)\Bigg]=0. \tag{28}$$

Let us study the first boundary condition. If there are no evanescent phases for the side $\partial\Omega_2$ then it simply reads

$$B_1\Bigg[\sum_{k\in\mathscr{I}_{\mathrm{os}}} e^{\frac{i}{\varepsilon}\psi_{k,1}}u_{n,k}(t,0,x_2)+\sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}}U_{n,k,1}(t,0,x_2,0)\Bigg]=\delta_{n,0}e^{\frac{i}{\varepsilon}\psi_{n_1,1}}g,$$

whereas if there are evanescent phases for the side $\partial\Omega_2$ we can use Assumption 4.9 to decouple (27) into

$$\begin{cases} B_1\Bigg[\sum_{k\in\mathscr{I}_{\mathrm{os}}} e^{\frac{i}{\varepsilon}\psi_{k,1}}u_{n,k}(t,0,x_2)+\sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}}U_{n,k,1}(t,0,x_2,0)\Bigg]=\delta_{n,0}e^{\frac{i}{\varepsilon}\psi_{n_1,1}}g, \\ B_1\sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\underline{\tau}t}U_{n,k,2}\left(t,0,x_2,\frac{x_2}{\varepsilon}\right)=0. \end{cases} \tag{29}$$

Indeed Assumption 4.9 implies the linear independence of the phases $\psi_{k,1}$ and $\underline{\tau}t$. The same reasoning for the boundary condition gives

$$\begin{cases} B_2\Bigg[\sum_{k\in\mathscr{I}_{\mathrm{os}}} e^{\frac{i}{\varepsilon}\psi_{k,2}}u_{n,k}(t,x_1,0)+\sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}}U_{n,k,2}(t,x_1,0,0)\Bigg]=0, \\ B_2\sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\underline{\tau}t}U_{n,k,1}\left(t,x_1,0,\frac{x_1}{\varepsilon}\right)=0. \end{cases} \tag{30}$$

Now, using again the linear independence of the phases, the boundary conditions (29) and (30) can be decomposed as the following cascades of equations:

$$\begin{cases} B_1\Big[\sum_{j\in\Phi(n_1)\cap\mathscr{I}_{\text{os}}} u_{n,j} + U_{n,n_1,1|X_1=0}\Big]_{|x_1=0} = \delta_{n,0}g & \forall n\in\mathbb{N},\ \text{if } n_1\in\mathfrak{R}_1,\\ B_1\Big[\sum_{j\in\Phi(n_1)} u_{n,j}\Big]_{|x_1=0} = \delta_{n,0}g & \forall n\in\mathbb{N},\ \text{if } n_1\notin\mathfrak{R}_1,\\ B_1\Big[\sum_{j\in\Phi(k)\cap\mathscr{I}_{\text{os}}} u_{n,j} + U_{n,k,1|X_1=0}\Big]_{|x_1=0} = 0 & \forall n\in\mathbb{N},\ \forall k\in\mathfrak{R}_1\setminus\{n_1\},\\ B_1\Big[\sum_{j\in\Phi(k)} u_{n,j}\Big]_{|x_1=0} = 0 & \forall n\in\mathbb{N},\ \forall k\notin\mathfrak{R}_1\setminus\{n_1\},\\ B_2\Big[\sum_{j\in\Psi(k)\cap\mathscr{I}_{\text{os}}} u_{n,j} + U_{n,k,2|X_2=0}\Big]_{|x_2=0} = 0 & \forall n\in\mathbb{N},\ \forall k\in\mathfrak{R}_2,\\ B_2\Big[\sum_{j\in\Psi(k)} u_{n,j}\Big]_{|x_2=0} = 0 & \forall n\in\mathbb{N},\ \forall k\notin\mathfrak{R}_2. \end{cases} \tag{31}$$

and

$$\begin{cases} B_1\sum_{k\in\mathfrak{R}_2} U_{n,k,2}\Big(t,0,x_2,\frac{x_2}{\varepsilon}\Big) = 0 & \forall n\in\mathbb{N},\\ B_2\sum_{k\in\mathfrak{R}_2} U_{n,k,1}\Big(t,x_1,0,\frac{x_1}{\varepsilon}\Big) = 0 & \forall n\in\mathbb{N}. \end{cases} \tag{32}$$

At last, plugging the ansatz (25) in the initial condition of the corner problem (18) leads us to solve

$$\forall n\in\mathbb{N},\qquad \begin{cases} u_{n,k|t=0} = 0 & \forall k\in\mathscr{I}_{\text{os}},\\ U_{n,k,1|t=0} = 0 & \forall k\in\mathfrak{R}_1,\\ U_{n,k,2|t=0} = 0 & \forall k\in\mathfrak{R}_2. \end{cases} \tag{33}$$

The main steps in the construction of the geometric optics expansion are the following. First, before solving the WKB cascade, we will describe a global structure on the set of indices $\mathscr{I}$. More precisely, this structure is based on a partition which takes into account the different relations that an index can have with the elements of the loop. We will thus be able to express $\mathscr{I}$ as a union of nonintersecting "trees" (or ordered sets by the relations $\underset{H}{\rightarrowtail}$ and $\underset{V}{\rightarrowtail}$, see Definition 4.4). Then, we will construct the amplitudes for the indices of the loop. To do this, we will need a new invertibility condition, which will be studied in Section 4D.

Thanks to the knowledge of the amplitudes associated with the loop, we will be able to construct the amplitudes in a direct neighborhood of the indices of the loop. In other terms, the new invertibility condition will be used to start the construction of the geometric optics expansion.

Then, to construct the remaining amplitudes, we will first make a more precise study of the structure of the trees that form $\mathscr{I}$. Using this more precise analysis, we will see that the construction of the amplitudes in these trees is rather easy because one can define a partial order on these trees.

The scheme of proof, and more precisely the order of construction of the amplitudes will be exactly the same for higher-order terms.

4B1. *Global structure of the set of indices $\mathscr{I}$.* In this section we will construct a partition of $\mathscr{I}$ based upon the position of the indices compared to the loop index n_1 and no more on the different kinds of elements in $\mathscr{I}$. More precisely, the partition will be based upon the different kinds of sequences that can link an index i to the loop index n_1.

The idea of the construction is the following; firstly, thanks to Proposition 4.5, we know that every index i in $\mathscr{I}$ is linked by a sequence of type V to one of the indices of $\mathscr{I}_0$ (cf. Definition 4.4). Without loss of generality, one can always assume that for all indices i the sequence linking i to the index in $\mathscr{I}_0$ does not start by the subsequence (n_4, n_2, n_3, n_1).

The following lemma is also immediate:

Lemma 4.12. *For all $i \in \mathscr{I}$, there exists at least one sequence of type V linking i to n_1. Equivalently, for all $i \in \mathscr{I}$,*

$$n_1 \underset{V}{\rightarrowtail} i,$$

where the notation $\underset{V}{\rightarrowtail}$ has been introduced in Definition 4.4.

Proof. It is sufficient to treat the case of indices i linked with i_0 for $i_0 \in \mathscr{I}_0 \setminus \{n_1\}$. For such indices, there exists a sequence, denoted by $\tilde{\ell}$, of type V linking i to i_0. By definition, $i_0 \in \Phi(n_1)$. So i is linked with n_1 by the type-V sequence defined by $\ell = (n_4, n_3, n_2, i_0, \tilde{\ell})$. $\square$

Now, let $i \in \mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}$, and let $\ell^i = (\ell_1, \ell_2, \dots, \ell_p)$ be a type-V sequence linking i to n_1. The way to construct the sets, denoted $A_{a_l}, B_{b_m}, C_{c_q}, D_{d_r}$, of the sought partition is based on the following algorithm:

Let $\mathtt{C}_1 := \#\Psi(n_1) - 2$, and

$$\Psi(n_1) \setminus \{n_1, n_4\} := \{a_1, a_2, \dots, a_{\mathtt{C}_1}\}.$$

Let $l \in \{1, \dots, \mathtt{C}_1\}$. We will say that $i \in A_{a_l}$ if and only if the sequence ℓ^i can be chosen such that $\ell_1 = a_l$.

At this stage, we have treated all the sequences that do not start with n_4. To treat the sequences that start with n_4, let $\mathtt{C}_4 := \#\Phi(n_4) - 2$, and

$$\Phi(n_4) \setminus \{n_3, n_4\} := \{b_1, b_2, \dots, b_{\mathtt{C}_4}\}.$$

Then for $m \in \{1, \dots, \mathtt{C}_2\}$, we will say that $i \in B_{b_m}$ if and only if the sequence ℓ^i can be chosen such that $\ell_1 = n_4$ and $\ell_2 = b_m$.

Consequently we have treated all the sequences ℓ^i except those starting with (n_4, n_3).

Finally let $\mathtt{C}_3 := \#\Psi(n_3) - 2$, $\mathtt{C}_2 := \#\Phi(n_2) - 2$ and

$$\Psi(n_3) \setminus \{n_2, n_3\} := \{c_1, c_2, \dots, c_{\mathtt{C}_3}\}, \quad \Phi(n_2) \setminus \{n_1, n_2\} := \{d_1, d_2, \dots, d_{\mathtt{C}_2}\}.$$

We define the sets C_{c_q} and D_{d_r} by the relations:

- For $q \in \{1, \dots, \mathtt{C}_3\}$, we have $i \in C_{c_q}$ if and only if the sequence ℓ^i can be chosen such that $\ell_1 = n_4$, $\ell_2 = n_3$ and $\ell_3 = c_q$.
- For $r \in \{1, \dots, \mathtt{C}_2\}$, we have $i \in D_{d_r}$ if and only if the sequence ℓ^i can be chosen such that $\ell_1 = n_4$, $\ell_2 = n_3$, $\ell_3 = n_2$ and $\ell_4 = d_r$.

This algorithm permits to consider all the possible sequences because no sequence starts with the subsequence (n_4, n_3, n_2, n_1). Then, we repeat this construction for all the potential sequences linking i to n_1.

It is thus clear that

$$(\mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}) = \Big(\bigcup_{l \le \mathtt{C}_1} A_{a_l}\Big) \cup \Big(\bigcup_{m \le \mathtt{C}_2} B_{b_m}\Big) \cup \Big(\bigcup_{q \le \mathtt{C}_3} C_{c_q}\Big) \cup \Big(\bigcup_{r \le \mathtt{C}_4} D_{d_r}\Big). \tag{34}$$

The sets A_{a_l} and B_{b_m} can be characterized as follows: A_{a_l} is the set of indices $i \in \mathscr{I}$ such that $a_l \underset{H}{\rightarrowtail} i$, whereas B_{b_m} is the set of indices $i \in \mathscr{I}$ such that $b_m \underset{V}{\rightarrowtail} i$. In terms of wave packet reflection, the set A_{a_l} gathers the indices obtained by reflection of the phase associated with the index a_l, this phase being obtained by reflection of the wave packet associated with n_1 on the side $\partial\Omega_2$. In a similar way, B_{b_m} gathers the indices obtained by reflection of the phase associated with the index b_j. The phase associated with b_j being obtained by reflection of the phase associated with n_4 on the side $\partial\Omega_1$. An analogous characterization stands for the sets C_{c_q} and D_{d_r}.

Lemma 4.13. *The decomposition*

$$(\mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}) = \Big(\bigcup_{l \le \mathtt{C}_1} A_{a_l}\Big) \cup \Big(\bigcup_{m \le \mathtt{C}_2} B_{b_m}\Big) \cup \Big(\bigcup_{n \le \mathtt{C}_3} C_{c_n}\Big) \cup \Big(\bigcup_{q \le \mathtt{C}_4} D_{d_q}\Big)$$

is a partition of $\mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}$.

Proof. Let us first define the "mirror" sequence of a sequence by the relation

$$\forall \ell = (\ell_1, \ell_2, \dots, \ell_p) \in \mathscr{I}^p, \quad \bar{\ell} := (\ell_p, \ell_{p-1}, \dots, \ell_1) \in \mathscr{I}^p.$$

Let $l, l' \in \{1, \dots, \mathtt{C}_1\}$, $l \neq l'$.

Proof of $A_{a_l} \cap A_{a_{l'}} = \varnothing$: We argue by contradiction. Let us assume that there exists $i \in A_{a_l} \cap A_{a_{l'}}$. Then by definition, there exists a type-H sequence $\ell = (\ell_1, \dots, \ell_p)$ linking i to a_l and a type-H sequence $\ell' = (\ell'_1, \dots, \ell'_{p'})$ linking i to $a_{l'}$. We now have to consider several cases depending on the oddness/evenness of p and p'.

<u>$p, p' \in 2\mathbb{N}$:</u> By the definition of type-H sequences, we have $i \in \Psi(\ell_p)$ and $i \in \Psi(\ell'_{p'})$. Thanks to property (ii) of Proposition 4.3, $\ell'_{p'} \in \Psi(\ell_p)$. The sequence $(\ell, \overline{\ell'})$ is consequently a type-H sequence linking a_l to $a_{l'}$. But $a_l \in \Phi(a_{l'})$, so the sequence $(\ell, \overline{\ell'}, a_{l'})$ is a loop for the index a_l with exactly $p + p' + 1$ elements. This contradicts Assumption 4.7.

$\underline{p \in 2\mathbb{N},\, p' \in 2\mathbb{N}+1}$: Now $i \in \Psi(\ell_p)$ and $i \in \Phi(\ell'_{p'})$ or equivalently $\ell'_{p'} \in \Phi(i)$. The sequence $(\ell, i, \overline{\ell'})$ is a type-H sequence linking a_l to $a_{l'}$. Then $(\ell, i, \overline{\ell'}, a_{l'})$ is a loop for the index a_l with $p+p'+2$ elements. Once again, it contradicts Assumption 4.7.

The case $p, p' \in 2\mathbb{N}+1$ is quite similar to the case $p, p' \in 2\mathbb{N}$, so we omit the proof.

We now deal with the proof of the property $A_{a_l} \cap B_{b_m} = \varnothing$, the other proofs showing that the other kinds of intersections are empty are analogous and consequently they will not be treated here.

Proof of $A_{a_l} \cap B_{b_m} = \varnothing$: Once again, we argue by contradiction. Let $i \in A_{a_l} \cap B_{b_m}$. Then by the definitions of the sets A_{a_l} and B_{b_m}, we have $a_l \underset{H}{\rightarrowtail} i$ and $b_m \underset{V}{\rightarrowtail} i$. That is to say, there exists $\ell = (\ell_1, \dots, \ell_p)$ a type-H sequence linking i to a_l and a type-V sequence $\ell' = (\ell'_1, \dots, \ell'_{p'})$ linking i to b_m. We have to consider the following cases:

$\underline{p, p' \in 2\mathbb{N}}$: We have $i \in \Psi(\ell_p)$ and $i \in \Phi(\ell'_{p'})$. So it is possible to show exactly as in the proof of one of the above subcases that the sequence $(\ell, i, \overline{\ell'})$ links a_l to b_m. It follows from $a_l \in \Psi(n_2)$ and $n_4 \in \Phi(b_m)$ that the sequence $(\ell, i, \overline{\ell'}, b_m, n_4)$ is a loop for the index a_l with an odd number of elements.

$\underline{p \in 2\mathbb{N},\, p' \in 2\mathbb{N}+1}$: We can show that the sequence $(\ell, \overline{\ell'})$ links a_l to b_m. So $(\ell, \overline{\ell'}, b_m, n_4)$ is a loop for a_l with an odd number of elements, which is again a contradiction with Assumption 4.7. $\square$

We have just shown that

$$\Big(\bigcup_{i \le \complement_1} A_{a_i}\Big) \cup \Big(\bigcup_{i \le \complement_2} B_{b_i}\Big) \cup \Big(\bigcup_{i \le \complement_3} C_{c_i}\Big) \cup \Big(\bigcup_{i \le \complement_4} D_{d_i}\Big) \tag{35}$$

is a partition of $\mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}$. A consequence is that to determine all the amplitudes in the WKB expansion, it will be sufficient to construct the amplitude for the indices on the loop and then the amplitudes in each set of the partition (35).

Moreover, the construction of the amplitudes in each set of the partition (35) can be made intrinsically in this set. Indeed, the fact that (35) is a partition implies that an index i in one set of (35) is only linked, by the boundary conditions (31), with other indices in the same set.

A last consequence of the fact that (35) is a partition of the frequency set is the following refinement of Proposition 4.3:

Proposition 4.14. *Let (18) be complete for the reflections, under Assumption 4.7. Let $\mathscr{I}$ be the index set; then Φ and Ψ satisfy, in addition to the properties of Proposition 4.3, the two extra properties:*

(viii)
$$\Phi(n_1) \setminus \{n_2\} \subset \mathscr{I}_{\mathrm{io}} \cup \mathscr{I}_{\mathrm{ii}} \cup \mathscr{I}_{\mathrm{ev1}}, \quad \Psi(n_1) \setminus \{n_1\} \subset \mathscr{I}_{\mathrm{oi}} \cup \mathscr{I}_{\mathrm{ii}} \cup \mathscr{I}_{\mathrm{ev2}},$$
$$\Phi(n_4) \setminus \{n_4\} \subset \mathscr{I}_{\mathrm{io}} \cup \mathscr{I}_{\mathrm{ii}} \cup \mathscr{I}_{\mathrm{ev1}}, \quad \Psi(n_3) \setminus \{n_3\} \subset \mathscr{I}_{\mathrm{oi}} \cup \mathscr{I}_{\mathrm{ii}} \cup \mathscr{I}_{\mathrm{ev2}}.$$

(ix) *Let $i \in \mathscr{I}_{\mathrm{ii}} \cup \mathscr{I}_{\mathrm{ev1}}$ and $j \in \mathscr{I}_{\mathrm{ii}} \cup \mathscr{I}_{\mathrm{ev2}}$. Then*

$$i \in \Phi(n_1) \implies \Psi(i) = \{i\}, \quad j \in \Psi(n_1) \implies \Phi(j) = \{j\},$$
$$i \in \Phi(n_4) \implies \Psi(i) = \{i\}, \quad j \in \Psi(n_3) \implies \Phi(j) = \{j\}.$$

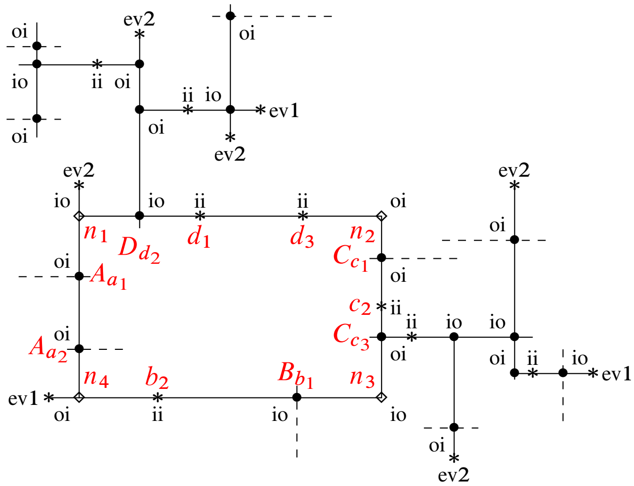

Figure 7. "Tree structure" of the frequency set $\mathscr{F}$.

Proof. (viii) We will here just show the first assertion, that is to say, $\Phi(n_1)\setminus\{n_2\}\subset\mathscr{I}_{\mathrm{io}}\cup\mathscr{I}_{\mathrm{ii}}\cup\mathscr{I}_{\mathrm{ev1}}$. By contradiction, let $i\in\Phi(n_1)\cap\mathscr{I}_{\mathrm{oi}}$, $i\neq n_2$. Then, there exists $j\in\Psi(i)\cap\mathscr{I}_{\mathrm{io}}$; otherwise, the frequency set indexed by $\mathscr{I}\setminus\{i\}$ is strictly included in $\mathscr{F}$ and satisfies (i)–(iv) of Definition 4.2.

Thanks to Lemma 4.12, we know that there exists a type-V sequence, $\ell=(\ell_1,\ell_2,\dots,\ell_p)$, with necessarily $p\in 2\mathbb{N}+1$ (because $n_1, j\in\mathscr{I}_{\mathrm{io}}$, see Definition 4.4), such that $n_1 \succ\!\!\!\!\!\!\overrightarrow{\;_V\;} j$. The sequence (ℓ, j, i) is a self-interacting loop for n_1 with an odd number of elements, but it is not the same loop as $\{n_1,n_2,n_3,n_4\}$. This is a contradiction of Assumption 4.7.

(ix) This uses exactly the same reasoning as for (viii) and we will omit it here. The only difference is that we cannot conclude that the loop is a self-interacting one because it may contain indices in $\mathscr{I}_{\mathrm{ii}}$. Then we need the uniqueness assumption of a loop and not only the uniqueness assumption of a self-interacting loop. □

Property $\Phi(n_1)\setminus\{n_2\}\subset\mathscr{I}_{\mathrm{io}}\cup\mathscr{I}_{\mathrm{ii}}\cup\mathscr{I}_{\mathrm{ev1}}$ of (viii) in Proposition 4.14 means that even if the characteristic variety contains a loop, all the frequencies (but n_2) associated with outgoing-incoming group velocity are initially discarded. We already justified this observation in the phase generation process described in Section 3. Property (ix) means that, thanks to the uniqueness assumption of a loop, an incoming-incoming phase in the direct neighborhood of the loop can only be generated by reflection on one side of $\partial\Omega$ and not on both sides.

Thanks to Proposition 4.14, partition (35) can be rewritten as

$$\begin{aligned}(\mathscr{I}\setminus\{n_1,n_2,n_3,n_4\}) &= \\ &\Bigg(\bigcup_{a_l\in\Psi(n_1)\cap\mathscr{I}_{\mathrm{io}}} A_{a_l} \bigcup_{a_l\in\Psi(n_1)\cap(\mathscr{I}_{\mathrm{ev1}}\cup\mathscr{I}_{\mathrm{ii}})}\{a_l\}\Bigg)\cup\Bigg(\bigcup_{b_m\in\Phi(n_4)\cap\mathscr{I}_{\mathrm{oi}}} B_{b_m} \bigcup_{b_m\in\Phi(n_4)\cap(\mathscr{I}_{\mathrm{ev2}}\cup\mathscr{I}_{\mathrm{ii}})}\{b_m\}\Bigg) \\ &\cup\Bigg(\bigcup_{c_q\in\Psi(n_3)\cap\mathscr{I}_{\mathrm{io}}} C_{c_q} \bigcup_{c_q\in\Psi(n_3)\cap(\mathscr{I}_{\mathrm{ev1}}\cup\mathscr{I}_{\mathrm{ii}})}\{c_q\}\Bigg)\cup\Bigg(\bigcup_{d_r\in\Phi(n_2)\cap\mathscr{I}_{\mathrm{io}}} D_{d_r} \bigcup_{d\in\Phi(n_2)\cap(\mathscr{I}_{\mathrm{ev2}}\cup\mathscr{I}_{\mathrm{ii}})}\{d_r\}\Bigg). \qquad (36)\end{aligned}$$

Let us conclude this section by noting that Figure 7 illustrates the "tree structure" of the frequency set $\mathscr{F}$.

4B2. *Determination of the amplitudes on the loop and invertibility condition.* Now that the global structure of the frequency set is described, and thanks to the new properties of functions Φ and Ψ, it is time to start the construction of the amplitudes in the WKB expansion. A good (and natural) choice to initialize this construction is to determine first the amplitudes associated with the loop indices. To do that, a new amplitude equation will be derived (see Section 3E).

The cascades of equations (26), (31) and (33) written for $n=0$ and $k=n_1$ tell us that the amplitude u_{0,n_1} satisfies

$$\begin{cases} \mathscr{L}(d\varphi_{n_1})u_{0,n_1}=0, \\ i\mathscr{L}(d\varphi_{n_1})u_{1,n_1}+L(\partial)u_{0,n_1}=0, \end{cases} \tag{37}$$

in the interior, the boundary conditions

$$\begin{cases} B_1\Big[\displaystyle\sum_{j\in\Phi(n_1)\cap\mathscr{I}_{\text{os}}} u_{0,j}+U_{0,n_1,1|X_1=0}\Big]_{|x_1=0}=g, & \text{if } n_1\in\mathfrak{R}_1, \\ \qquad\qquad B_1\Big[\displaystyle\sum_{j\in\Phi(n_1)} u_{0,j}\Big]_{|x_1=0}=g, & \text{if } n_1\in\mathscr{R}_1\setminus\mathfrak{R}_1, \end{cases} \tag{38}$$

and

$$\begin{cases} B_2\Big[\displaystyle\sum_{j\in\Psi(n_1)\cap\mathscr{I}_{\text{os}}} u_{0,j}+U_{0,n_1,1|X_1=0}\Big]_{|x_2=0}=0, & \text{if } n_1\in\mathfrak{R}_2, \\ \qquad\qquad B_2\Big[\displaystyle\sum_{j\in\Psi(n_1)} u_{0,j}\Big]_{|x_2=0}=0, & \text{if } n_1\in\mathscr{R}_2\setminus\mathfrak{R}_2, \end{cases} \tag{39}$$

and finally the initial condition

$$u_{0,n_1|t\leq 0}=0. \tag{40}$$

We will now explain the method of resolution of equations (37)–(40). The ideas described below are classical; they explain why the amplitudes associated with oscillating phases satisfy transport equations in the example of Section 3E and they will be applied to all the oscillating amplitudes.

Firstly, let us remark that the first equation of (37) tells us that the amplitude u_{0,n_1} belongs to $\ker\mathscr{L}(d\varphi_{n_1})$. In other words, we have the so-called polarization condition

$$P_1^{n_1}u_{0,n_1}=u_{0,n_1},$$

where $P_1^{n_1}$ is the projector defined in Definition 4.10. Now, composing the second equation of (37) with the projector $Q_1^{n_1}$ defined in Definition 4.10 and using the polarization condition give us

$$Q_1^{n_1}L(\partial)P_1^{n_1}u_{0,n_1}=0.$$

But Lax's lemma [1957] tells us that if the corner problem (18) is constantly hyperbolic then we have the relation

$$Q_1^{n_1}L(\partial)P_1^{n_1}=(\partial_t+v_{n_1}\cdot\nabla_x)Q_1^{n_1}P_1^{n_1},$$

where v_{n_1} is the group velocity associated with the phase φ_{n_1}. So, the amplitude u_{0,n_1} satisfies the transport equation

$$(\partial_t + v_{n_1} \cdot \nabla_x) Q_1^{n_1} u_{0,n_1} = 0.$$

We are now interested in the boundary conditions. As mentioned in Section 3, the boundary conditions needed to solve a transport equation in a quarter-space are linked with the nature of the transport velocity. Let us recall the four possible alternatives:

- The transport velocity is outgoing-outgoing; then no boundary condition has to be imposed.
- The transport velocity is incoming-outgoing; then the transport equation needs a boundary condition on $\partial\Omega_1$ only.
- The transport velocity is outgoing-incoming; then the transport equation needs a boundary condition on $\partial\Omega_2$ only.
- The transport velocity is incoming-incoming; then the transport equation needs a boundary condition on $\partial\Omega_1$ and on $\partial\Omega_2$.

Here, by assumption we have $n_1 \in \mathscr{I}_{\text{io}}$, so no boundary condition on $\partial\Omega_2$ has to be imposed and we only keep the boundary condition on $\partial\Omega_2$. Consequently, the amplitude u_{0,n_1} satisfies the transport equation

$$\begin{cases} (\partial_t + v_{n_1} \cdot \nabla_x) Q_1^{n_1} u_{0,n_1} = 0, \\ B_1 \Big[\sum_{k \in \Phi(n_1)} u_{0,k} \Big]_{|x_1=0} = g, & \text{if } n_1 \notin \mathfrak{R}_1, \\ u_{0,n_1}{}_{|t \leq 0} = 0, \end{cases} \tag{41}$$

and

$$\begin{cases} (\partial_t + v_{n_1} \cdot \nabla_x) Q_1^{n_1} u_{0,n_1} = 0, \\ B_1 \Big[\sum_{k \in \Phi(n_1)} u_{0,k} + U_{0,n_1,1}{}_{|X_1=0} \Big]_{|x_1=0} = g, & \text{if } n_1 \in \mathfrak{R}_1. \\ u_{0,n_1}{}_{|t \leq 0} = 0, \end{cases} \tag{42}$$

In both cases, using the fact that $\Phi(n_1) \cap \mathscr{I}_{\text{oi}} = \{n_2\}$ thanks to (vi) of Assumption 4.7 and (viii) of Proposition 4.14, the boundary condition of (41) reads

$$u_{0,n_1}{}_{|x_1=0} + \sum_{k \in (\Phi(n_1) \cap (\mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ii}})) \setminus \{n_1\}} u_{0,k}{}_{|x_1=0} = \phi_1^{n_1} [g - B_1 u_{0,n_2}{}_{|x_1=0}]$$

when $n_1 \notin \mathfrak{R}_1$, and

$$u_{0,n_1}{}_{|x_1=0} + \sum_{k \in (\Phi(n_1) \cap (\mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ii}})) \setminus \{n_1\}} u_{0,k}{}_{|x_1=0} + U_{0,n_1,1}{}_{|x_1=X_1=0} = \phi_1^{n_1} [g - B_1 u_{0,n_2}{}_{|x_1=0}]$$

when $n_1 \in \mathfrak{R}_1$. Multiplying these conditions by the projector $P_1^{n_1}$, using the fact that the $u_{0,k}$ are polarized on $\ker \mathscr{L}(d\varphi_k)$, we obtain, in both cases, that the trace u_{0,n_1} on $\partial\Omega_1$ is given by

$$u_{0,n_1}{}_{|x_1=0} = S_1^{n_1} [g - B_1 u_{0,n_2}{}_{|x_1=0}],$$

where we recall that the matrix $S_1^{n_1}$ has been introduced in Definition 4.10.

It is now easy to integrate (41) along the characteristics. We obtain the expression of u_{0,n_1} according to its trace on $\partial\Omega_1$; more precisely,

$$u_{0,n_1}(t,x) = S_1^{n_1}[g - B_1 u_{0,n_2}{}_{|x_1=0}]\big(t_{\mathrm{io}}^{n_1}(t,x_1), x_{\mathrm{io}}^{n_1}(x_1,x_2)\big),$$

where the new variables $t_{\mathrm{io}}^{n_1}$ and $x_{\mathrm{io}}^{n_1}$ are defined in (21). As a consequence, the trace of u_{0,n_1} on $\partial\Omega_2$ is given by

$$u_{0,n_1}(t,x_1,0) = S_1^{n_1}[g - B_1 u_{0,n_2}{}_{|x_1=0}]\left(t_{\mathrm{io}}^{n_1}(t,x_1), -\frac{v_{n_1,2}}{v_{n_1,1}}x_1\right). \tag{43}$$

Then we can repeat exactly the same reasoning for the second element n_2 of the loop. Indeed, using the fact that $n_2 \in \mathscr{I}_{\mathrm{oi}}$, u_{0,n_2} will be determined by integration along the characteristics from its trace on $\partial\Omega_2$. Thanks to Assumption 4.7 and property (viii) of Proposition 4.14, the trace $u_{0,n_2}{}_{|x_2=0}$ will depend only of the trace $u_{0,n_3}{}_{|x_2=0}$. The trace $u_{0,n_2}{}_{|x_1=0}$, which appears in (43), is consequently given by

$$u_{0,n_2}(t,0,x_2) = -S_2^{n_2} B_2 u_{0,n_3}{}_{|x_2=0}\left(t_{\mathrm{oi}}^{n_2}(t,x_2), -\frac{v_{n_2,1}}{v_{n_2,2}}x_2\right). \tag{44}$$

At last, repeating the same method, we obtain the traces of the two remaining amplitudes for the indices of the loop:

$$u_{0,n_3}(t,x_1,0) = -S_1^{n_3} B_1 u_{0,n_4}{}_{|x_1=0}\left(t_{\mathrm{io}}^{n_3}(t,x_1), -\frac{v_{n_3,2}}{v_{n_3,1}}x_1\right). \tag{45}$$

and

$$u_{0,n_4}(t,0,x_2) = -S_2^{n_4} B_2 u_{0,n_1}{}_{|x_2=0}\left(t_{\mathrm{oi}}^{n_4}(t,x_2), -\frac{v_{n_4,1}}{v_{n_4,2}}x_2\right). \tag{46}$$

An important point in this analysis is that at each step of the computation, there is one and only one outgoing phase coupled with the incoming phases in the equivalence classes, for the relations $\underset{\Phi}{\sim}$ and $\underset{\Psi}{\sim}$, of the indices n_j. This fact will, a priori, not be true if one considers a frequency set containing several loops.

Thus, combining equations (43)–(46) we obtain, after some computations, the functional equation determining the trace $u_{0,n_1}{}_{|x_2=0}$:

$$(I - \mathbb{T})u_{0,n_1}{}_{|x_2=0} = S_1^{n_1} g\left(t - \frac{1}{v_{n_1,1}}x_1, -\frac{v_{n_1,2}}{v_{n_1,1}}x_1\right), \tag{47}$$

where $\mathbb{T}$ is the operator defined by

$$(\mathbb{T}w)(t,x_1) := Sw(t+\alpha x_1, \beta x_1), \tag{48}$$

with

$$\begin{aligned} S &:= S_1^{n_1} B_1 S_2^{n_2} B_2 S_1^{n_3} B_1 S_2^{n_4} B_2, \\ \alpha &:= \frac{1}{v_{n_1,1}}\left(-1 + \frac{v_{n_1,2}}{v_{n_2,2}} - \frac{v_{n_1,2}v_{n_2,1}}{v_{n_2,2}v_{n_3,1}} + \frac{v_{n_1,2}v_{n_2,1}v_{n_3,2}}{v_{n_2,2}v_{n_3,1}v_{n_4,2}}\right) < 0, \\ \beta &:= \frac{v_{n_4,1}}{v_{n_4,2}}\frac{v_{n_3,2}}{v_{n_3,1}}\frac{v_{n_2,1}}{v_{n_2,2}}\frac{v_{n_1,2}}{v_{n_1,1}} > 0. \end{aligned} \tag{49}$$

Given (47), we make the following assumption:

Assumption 4.15. *For all $\gamma > 0$, the operator $(I - \mathbb{T})$ defined in (48) is invertible from $L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$ to $L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$ uniformly with respect with the parameter $\gamma > 0$.*

However, for $T > 0$ and for a source term in $L^2(]-\infty, T] \times \mathbb{R}_+)$ that is zero for negative times, this assumption will only give us amplitudes for indices of the loop which are $L^2(\Omega_T)$. This is not sufficient to construct the amplitudes of high order in the WKB expansion nor to make sure that the amplitudes linked with an incoming-incoming group velocity are $H^1(\Omega_T)$. We thus need to reinforce Assumption 4.15 in the following way:

Assumption 4.16. *Let $2 \leq K \leq \infty$. For all $\gamma > 0$, the operator $(I - \mathbb{T})$ defined in (48) is invertible from $H^K_{f,\gamma}$ to $H^K_{f,\gamma}$ uniformly with respect with $\gamma > 0$.*

Let us stress that Assumption 4.16 is (at this stage of the analysis) purely formal and is introduced to construct the WKB expansion. We will show in Section 4D the following proposition:

Proposition 4.17. *If $|S| < \sqrt{\beta}$ (where S and β are defined in (49)), for all $\gamma > 0$, the operator $(I - \mathbb{T})$ is uniformly invertible from $L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$ to $L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$. In particular, for all $T > 0$, equation (47) admits a unique solution $u \in L^2(]-\infty, T] \times \mathbb{R}_+)$, zero for negative times, if the source term G is in $L^2(\partial\Omega_{1,T})$ and is zero for negative times.*

If $\beta \leq 1$ and $G \in H^\infty_f$, under the assumption $|S| < \sqrt{\beta}$, the solution u of the equation $(I - \mathbb{T})u = G$ is in H^∞_f.

If $\beta > 1$ let $K \in \mathbb{N}$ and $G \in H^K_f$; then under the assumption $|S|\beta^{K-\frac{1}{2}} < 1$, the solution u of $(I - \mathbb{T})u = G$ is in H^K_f.

Assumption $|S| < \sqrt{\beta}$, or $|S|\beta^{K-\frac{1}{2}} < 1$, gives us a framework in which we can, firstly ensure enough regularity to construct (at least up to a finite order) the amplitudes in the WKB expansion, and secondly construct the incoming-incoming amplitudes. More details and comments about the condition $|S| < \sqrt{\beta}$ will be given in Section 4D.

From now on we denote by $K \in \mathbb{N} \cup \{+\infty\}$ the largest integer such that the solution u of the equation $(I - \mathbb{T})u = G$ is H^K_f for $G \in H^K_f$. In view of constructing the first corrector term and to ensure that the WKB expansion is a good approximation to the exact solution, we need $K \geq 3$.

From all these considerations about (47), it follows that the trace $u_{0,n_1}{}_{|x_2=0}$ is uniquely determined in H^K_f by the formula

$$u_{0,n_1}(t, x_1, 0) = (I - \mathbb{T})^{-1} S_1^{n_1} g\left(t - \frac{1}{v_{n_1,1}} x_1, -\frac{v_{n_1,2}}{v_{n_1,1}} x_1\right), \tag{50}$$

an equation which enables us to construct the amplitudes u_{0,n_j}, $j = 1, \dots, 4$, by using (46), (45) and (44) and integrating along the corresponding characteristics.

We summarize this construction of the amplitudes associated with loop indices by the following proposition:

Proposition 4.18. *Under Assumptions 2.1–2.2 on the complete-for-reflections corner problem (18) and under Assumptions 4.7 and 4.16, for $j = 1, \dots, 4$ and for all $T > 0$, there exist functions $u_{0,n_j} \in H^K(\Omega_T)$, with traces in H_f^K, satisfying the cascades of equations (26), (31), (32) and (33) written for $n = 0$ and $k = n_j$.*

4B3. *Determination of the amplitudes in the direct neighborhood of the loop.* In this section we will show that the knowledge of the amplitudes on the loop and the global structure of the index set $\mathscr{I}$ described in Section 4B1 are sufficient to construct the amplitudes in the direct neighborhood of the indices of the loop.

We have chosen to separate this construction from the construction of the amplitudes linked with indices in the different sets of the partition (36). This choice is motivated by the following two reasons. Firstly we think that it is important to make the computations explicit at least once (mainly because we have not yet described the construction for evanescent phases). Secondly, the construction in the close neighborhood of the loop remains unchanged, instead of the determination of the amplitudes associated with indices in the different sets of the partition (36), under a weaker uniqueness assumption of the loop (i.e., an assumption imposing the uniqueness of a self-interaction loop, but which allows other types of loops in the frequency set).

We will here describe the determination of the amplitudes in $\Phi(n_4)$; the construction for amplitudes in $\Psi(n_4)$, $\Phi(n_2)$ or $\Psi(n_2)$ is exactly the same. Using property (viii) of Proposition 4.14, we know that $\Phi(n_4) \cap \mathscr{I}_{\text{oi}} = \{n_4\}$. The boundary condition (31) written for $k = n_4$ (if we choose n_4 as a class representative of its own equivalence class for the relation $\underset{\Phi}{\sim}$) is given by

$$B_1 \Big[\sum_{j \in \Phi(n_4) \cap (\mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ii}})} u_{0,j} + U_{0,n_4,1|X_1=0} \Big]_{|x_1=0} = -B_1 u_{0,n_4|x_1=0} \qquad \text{if } n_4 \in \mathfrak{R}_1, \tag{51}$$

$$B_1 \Big[\sum_{j \in \Phi(n_4) \cap (\mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ii}})} u_{0,j} \Big]_{|x_1=0} = -B_1 u_{0,n_4|x_1=0} \qquad \text{if } n_4 \in \mathscr{R}_1 \setminus \mathfrak{R}_1, \tag{52}$$

where in both cases, the source term is a known element of H_f^K.

Applying the uniform Kreiss–Lopatinskii condition to equations (51) and (52), and composing by the projectors P_1^j for $j \in \Phi(n_4) \cap (\mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ii}})$, and/or by $P_{s,1}^{n_4}$, leads us to solve the uncoupled boundary conditions

$$\forall j \in \Phi(n_4) \cap (\mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ii}}), \qquad u_{0,j\,|x_1=0} = -S_1^j B_1 u_{0,n_4|x_1=0}, \tag{53}$$

and if, moreover, $n_4 \in \mathfrak{R}_1$,

$$U_{0,n_4,1|X_1=x_1=0} = -S_{s,1}^j B_1 u_{0,n_4|x_1=0}. \tag{54}$$

Thus, the construction of the possible evanescent amplitude for the side Ω_1 can be made independently of the construction of the amplitudes for oscillating phases.

Let us first briefly recall how to determine amplitudes for oscillating phases. We have several cases to take into account.

$j \in \mathscr{I}_{\text{io}}$: Lax's lemma and the polarization condition enable us to show that the amplitude $u_{0,j}$ satisfies a transport equation with an incoming-outgoing velocity v_j. That is why to construct this amplitude we just need to know its trace on $\partial\Omega_1$. This trace is determined by (53), so integrating along the characteristics, $u_{0,j}$ is given by

$$u_{0,j}(t,x) = S_1^j B_1 u_{0,n_4|x_1=0}\big(t_{\text{io}}^j(t,x_1), x_{\text{io}}^j(x_1,x_2)\big). \tag{55}$$

An important point for the end of the proof (more specifically for the construction of incoming-incoming amplitudes in the set B_{b_j}) is that $u_{0,j} \in H^K(\Omega_T)$ for all $T > 0$ and that its trace on the side $\partial\Omega_2$ is H_f^K. We can easily see this fact in the formula (55).

In other words, the flatness at the corner of the source term g^ε is transmitted to the amplitudes close to the loop.

$j \in \mathscr{I}_{\text{ii}}$: In this case, $u_{0,j}$ is solution to a transport equation with incoming-incoming velocity, so its determination needs the traces on both sides $\partial\Omega_1$ and $\partial\Omega_2$. The boundary condition (53) gives the trace on $\partial\Omega_1$. Concerning the trace on $\partial\Omega_2$, property (ix) of Proposition 4.14 shows that j is the only element in its equivalence class for the relation $\underset{\Psi}{\sim}$; in particular, $j \notin \mathfrak{R}_2$. So the boundary condition (31) written for $k = j$ reads

$$B_2 u_{0,j\,|x_2=0} = 0.$$

Using the uniform Kreiss–Lopatinskii condition, it follows that $u_{0,j\,|x_2=0} = 0$. So the amplitude $u_{0,j}$ satisfies the transport equation

$$u_{0,j} = P_1^j u_{0,j} = P_2^j u_{0,j}, \qquad \begin{cases} (\partial_t + v_j \cdot \nabla_x) Q_1^j u_{0,j} = 0, \\ u_{0,j\,|x_1=0} = -S_1^j B_1 u_{0,n_4|x_1=0}, \quad u_{0,j\,|x_2=0} = 0, \quad u_{0,j\,|t\le 0} = 0. \end{cases}$$

To solve this transport equation, we use the flatness at the corner of $u_{0,n_4|x_1=0}$ to extend the problem in the half-space $\{x_1 \ge 0\}$ by extending $u_{0,j}$ by zero to $\{x_2 < 0\}$, we integrate along the characteristics, and then we restrict the constructed solution to the quarter-space. The obtained solution $u_{0,j}$ is in $H^K(\Omega_T)$, thanks to the fact that $u_{0,n_4|x_1=0}$ is flat at the corner.

One can also easily check that the obtained solution $u_{0,j}$ satisfies the property: if $\underline{x}_1 \ge 0$, then $u_{0,j\,|x_1=\underline{x}_1} \in H_f^K$. This extra regularity of $u_{0,j}$ will also be needed during the construction of higher-order terms.

$n_4 \in \mathfrak{R}_1$: The determination of the amplitude associated with an evanescent index for the side $\partial\Omega_1$ (or even $\partial\Omega_2$) follows (in some sense) the same kind of ideas as the determination of amplitudes linked with oscillating indices. Indeed, it will be easy to construct the amplitude linked with an evanescent index if we know its trace (on $\partial\Omega_1$ for elements of $\mathscr{I}_{\text{ev}1}$ and the trace on $\partial\Omega_2$ for indices of $\mathscr{I}_{\text{ev}2}$).

However, we will in this proof treat the evanescent modes in only one block, as in [Lescarret 2007]; that is why the associated amplitudes will not satisfy transport equations as in the oscillating case. Thus, we first recall the evolution equations and the boundary conditions satisfied by such amplitudes and then we will give a method to solve these equations.

Plugging the ansatz (25) in the evolution equation of the corner problem (18) we have seen that the amplitude $U_{n,n_4,1}$ has to satisfy the cascade of equations

$$\begin{cases} L_{n_4}(\partial_{X_1})U_{0,n_4,1} = 0, \\ L_{n_4}(\partial_{X_1})U_{n,n_4,1} + L(\partial)U_{n-1,n_4,1} = 0, \quad \forall n \geq 1, \end{cases} \tag{56}$$

where

$$L_{n_4}(\partial_{X_1}) := A_1(\partial_{X_1} - \mathscr{A}_1(\underline{\tau}, \xi_2^{n_4})).$$

The boundary condition has also already been studied in the case $n_4 \in \mathfrak{R}_1$ and is given by (51). So $U_{0,n_4,1}$ has to satisfy the system

$$\begin{cases} L_{n_4}(\partial_{X_1})U_{0,n_4,1} = 0, \\ B_1\Big[\displaystyle\sum_{j \in \Phi(n_4) \cap (\mathscr{I}_{\mathrm{io}} \cup \mathscr{I}_{\mathrm{ii}})} u_{0,j} + U_{0,n_4,1|X_1=0}\Big]_{|x_1=0} = -B_1 u_{0,n_4|x_1=0}, \\ U_{0,n_4,1|t\leq 0} = 0, \end{cases} \tag{57}$$

and one has also to keep in mind that the boundary condition (32) will also have to be satisfied. First let us solve (57). Let us recall the following lemma from [Lescarret 2007], which permits us to solve (56) in the profile space $P_{\mathrm{ev},1}$.

Lemma 4.19. *For $i = 1, 2$, and $\underline{k} \in \mathfrak{R}_i$, let*

$$\mathbb{P}^{\underline{k}}_{\mathrm{ev},i} U(X_i) := e^{X_i \mathscr{A}_i(\underline{\tau}, \xi^{\underline{k}}_{3-i})} P^{\underline{k}}_{s,i} U(0), \tag{58}$$

$$\mathbb{Q}^{\underline{k}}_{\mathrm{ev},i} F(X_i) := \int_0^{X_i} e^{(X_i - s)\mathscr{A}_i(\underline{\tau}, \xi^{\underline{k}}_{3-i})} P^{\underline{k}}_{s,i} A_i^{-1} F(s)\, ds - \int_{X_i}^{+\infty} e^{(X_i - s)\mathscr{A}_i(\underline{\tau}, \xi^{\underline{k}}_{3-i})} P^{\underline{k}}_{u,i} A_i^{-1} F(s)\, ds. \tag{59}$$

Then, for all $F \in P_{\mathrm{ev},i}$, the equation

$$L_{\underline{k}}(\partial_{X_i})U = F$$

admits a solution in $P_{\mathrm{ev},i}$. Moreover, this solution is given by

$$U = \mathbb{P}^{\underline{k}}_{\mathrm{ev},i} U + \mathbb{Q}^{\underline{k}}_{\mathrm{ev},i} F.$$

This lemma tells us that the evanescent amplitude of leading order $U_{0,n_4,1}$ satisfies $\mathbb{P}^{n_4}_{\mathrm{ev},1} U_{0,n_4,1} = U_{0,n_4,1}$. This relation is analogous to the polarization condition for oscillating phases and thanks to the definition of $\mathbb{P}^{n_4}_{\mathrm{ev},1}$, enables us to determine $U_{0,n_4,1}$ if we know its trace on $\{X_1 = 0\}$.

Unfortunately the system (57) does not give any information about this trace but only on the "double" trace on $\{x_1 = X_1 = 0\}$. This is determined by

$$U_{0,n_4,1|X_1=x_1=0} = -S^j_{s,1} B_1 u_{0,n_4|x_1=0}.$$

It is then sufficient to lift the "double" trace in a "single" one. As in [Lescarret 2007], for example, choose

$$U_{0,n_4,1}(t, x, 0) := -\chi(x_1) S^j_{s,1} B_1 u_{0,n_4|x_1=0},$$

where $\chi \in \mathscr{C}^\infty_c(\mathbb{R})$ satisfies $\chi(0) = 1$.

Now that the trace of $U_{0,n_4,1}$ on $\{X_1=0\}$ is determined, we can apply the operator $\mathbb{P}^{n_4}_{\mathrm{ev},1}$. Thus by construction, the amplitude

$$U_{0,n_4,1}(t,x,X_1) = -\chi(x_1) e^{X_1 \mathscr{A}_1(\underline{\tau},\xi_2^{n_4})} S^{n_4}_{s,1} B_1 u_{0,n_4|x_1=0}(t,x_2) \tag{60}$$

is a solution to the system of equations (57).

Now, we consider the contribution of $U_{0,n_4,1}(t,x_1,0,x_1/\varepsilon)$ in (32). From (60), this trace is explicitly given by

$$U_{0,n_4,1}\left(t,x_1,0,\frac{x_1}{\varepsilon}\right) = -\chi(x_1) e^{\frac{x_1}{\varepsilon}\mathscr{A}_1(\underline{\tau},\xi_2^{n_4})} S^{n_4}_{s,1} B_1 u_{0,n_4|x_1=x_2=0},$$

but from the flatness of $u_{0,n_4|x_1=0}$ at the corner, it follows that $U_{0,n_4,1}(t,x_1,0,x_1/\varepsilon)$ is zero and that it does not contribute[4] to (32).

The determination of evanescent amplitudes for the side $\partial\Omega_2$ that can appear when we construct the amplitudes associated with indices in $\Psi(n_3)$ or $\Psi(n_1)$ is totally similar. For example, for the indices in $\Psi(n_3)$, we will start by determining $U_{0,n_3,2}$ on $\{x_2=X_2=0\}$ by using the boundary condition; then we lift this double trace in a single one on $\{X_2=0\}$. This trace is finally propagated in the interior of Ω by the operator $\mathbb{P}^{n_3}_{\mathrm{ev},2}$ defined in (58). We also obtain that $U_{0,n_3,2}(t,0,x_2,x_2/\varepsilon)$ is zero and as a consequence, $U_{0,n_3,2}(t,0,x_2,x_2/\varepsilon)$ does not contribute in (32).

Then we repeat this construction for all the indices in the direct neighborhood of the loop, so the indices whose amplitudes have still to be determined in partition (36) are

$$\left(\bigcup_{a_l} A_{a_l}\setminus\{a_l\}\right)\cup\left(\bigcup_{b_m} B_{b_m}\setminus\{b_m\}\right)\cup\left(\bigcup_{c_q} C_{c_q}\setminus\{c_q\}\right)\cup\left(\bigcup_{d_r} D_{d_r}\setminus\{d_r\}\right);$$

that is to say, it only remains to determine the amplitudes linked with indices in the trees of the partition (36). Before constructing these amplitudes, we will need to have a more precise description of the structure of those trees. It is the subject of the following section.

4B4. *Local structure of the trees.* Let us concentrate on the internal structure of the trees A_{a_l} appearing in the partition (36) of $\mathscr{I}$. The description for the trees B_{b_m}, C_{c_q} and D_{d_r} is, up to a few modifications, analogous and will not be given in detail here. Let us recall that a tree $A_{a_{\underline{l}}}$ has for its root an index $a_{\underline{l}} \in (\Psi(n_1)\cap\mathscr{I}_{\mathrm{oi}})\setminus\{n_4\}$ and is the set of indices j linked with $a_{\underline{l}}$ by a sequence of type H (see Definition 4.4). To simplify the future notations, we will define $A_{a_{\underline{l}}} := A_{\underline{a}}$.

The following proposition has already been mentioned in Section 4B1, and is the main proposition needed to understand the structure of $A_{\underline{a}}$.

Proposition 4.20. *Let $j\in A_{\underline{a}}$. Then there exists a unique sequence ℓ of type H linking j to $\underline{a}$.*

Proof. By contradiction, let $\ell=(\ell_1,\ell_2,\dots,\ell_p)$ and $\ell'=(\ell'_1,\ell'_2,\dots,\ell'_{p'})$, $\ell\neq\ell'$, be two sequences of type H which link j to $\underline{a}$. We will separate several cases depending on the oddness/evenness of the lengths p and p', and without loss of generality we assume that $p\le p'$.

[4]We will in fact show in the following that all the contributions of the evanescent phases for the side $\partial\Omega_1$ (resp. $\partial\Omega_2$) in (32) are zero. As a consequence, the boundary condition (32) is in fact trivially satisfied as soon as the oscillating amplitudes remain flat at the corner.

$\underline{p, p' \in 2\mathbb{N}}$: We have to distinguish two different subcases:

• If $\ell' = (\ell, \ell'_{p+1}, \dots, \ell_{p'})$, then $\ell'_{p+1} \in \Phi(\ell'_{p'})$. This allows us to show that the sequence $(\ell'_{p+2}, \dots, \ell_{p'})$ is a loop for the index ℓ'_{p+1} of length $p' - p - 1$. Thanks to Assumption 4.7, it is impossible.

• If $\ell' \neq (\ell, \ell'_{p+1}, \dots, \ell'_{p'})$, let m be the first integer such that $\ell_m \neq \ell'_m$. From the preceding subcase, we can assume that $1 \leq m < p$. We will here deal with the case $m \in 2\mathbb{N}+1$ (the case $m \in 2\mathbb{N}$ can be treated in a similar way, up to modification of the type of sequence). We have, $\ell_m \in \Phi(\ell'_m)$.

We have, once again two different possibilities:

• There exists l, where $m+1 < l \leq p$, such that $k_l = k'_l$. Then let $\underline{l}$ be the first integer l, where $m+1 < l \leq p$, such that $k_l = k'_l$. Then if $\underline{l} \in 2\mathbb{N}$ (resp. $\underline{l} \in 2\mathbb{N}+1$), we have $\ell_{\underline{l}-1} \in \Psi(\ell'_{\underline{l}-1})$ (resp. $\ell_{\underline{l}-1} \in \Phi(\ell'_{\underline{l}-1})$). Consequently the sequence $(\ell'_m, \dots, \ell'_{\underline{l}-1})$ is a sequence of type H linking $\ell_{\underline{l}-1}$ to ℓ_m, and the sequence $(\ell_{m+1}, \dots, \ell_{\underline{l}-2})$ is a sequence of type V linking $\ell_{\underline{l}-1}$ to ℓ_m. From this observation, we deduce that the sequence $(\ell'_m, \dots, \ell'_{\underline{l}-1}, \ell_{\underline{l}-1}, \ell_{l-2}, \dots, \ell_{m+1})$ is a loop for the index ℓ_m. Once again, this fact contradicts Assumption 4.7.

• If, for all $q \in \{m+2, \dots, p\}$, the indices ℓ_q and ℓ'_q are distinct, we easily see that $(\ell'_m, \dots, \ell'_p, \ell_p, \dots, \ell_{m-1})$ is a loop for ℓ_m.

We now consider the second subcase, that is to say:

$\underline{p \in 2\mathbb{N},\ p' \in 2\mathbb{N}+1}$: If $\ell' = (\ell, \ell'_{p+1}, \dots, \ell'_{p'})$, we can show that $(\ell'_{p+1}, \dots, \ell'_{p'})$ is a loop for j, whereas, if $\ell' \neq (\ell, \ell'_{p+1}, \dots, \ell'_{p'})$, we can repeat the analysis made in the subcase $p, p' \in 2\mathbb{N}$ to treat the subcase, "There exists l, where $m+1 < l \leq p$, such that $\ell_l = \ell'_l$". If, for all $q \in \{m+2, \dots, p\}$, the indices ℓ_q and ℓ'_q are distinct, we can easily show that $(\ell'_m, \dots, \ell'_p, j, \ell_p, \dots, \ell_{m-1})$ is a loop for ℓ_m.

The other case, $p, p' \in 2\mathbb{N}+1$, is analogous, up to the inversion of the role played by the functions Φ and Ψ, to the case $p, p' \in 2\mathbb{N}$. This case, is left to the reader. □

Remark. As indicated in Section 4B1, the uniqueness of the sequence linking $\underline{j} \in A_{\underline{a}}$ to the root $\underline{a}$ depends, in a nontrivial way, on Assumption 4.7.

Thanks to Proposition 4.20, it is now possible to give a more precise (and final) version of the properties satisfied by functions Φ and Ψ:

Proposition 4.21. *Let $\underline{j} \in A_{\underline{a}} \setminus \{\underline{a}\}$. We denote by $\ell = (\ell_1, \dots, \ell_p)$ the sequence of type H linking $\underline{j}$ to $\underline{a}$. Then, according to the parity of p, we have*:

(i) *If $p \in 2\mathbb{N}$, then $\underline{j} \notin \mathscr{I}_{\mathrm{oi}}$. Moreover, if $\underline{j} \in \mathscr{I}_{\mathrm{ev1}} \cup \mathscr{I}_{\mathrm{ii}}$ then $\Psi(\underline{j}) = \{\underline{j}\}$.*

(ii) *If $p \in 2\mathbb{N}+1$, then $\underline{j} \notin \mathscr{I}_{\mathrm{io}}$. Moreover, if $\underline{j} \in \mathscr{I}_{\mathrm{ev2}} \cup \mathscr{I}_{\mathrm{ii}}$ then $\Phi(\underline{j}) = \{\underline{j}\}$.*

Proof. We will consider the case $p \in 2\mathbb{N}$. Let us first show that $\underline{j} \notin \mathscr{I}_{\mathrm{oi}}$. By contradiction, we assume that $\underline{j} \in \mathscr{I}_{\mathrm{oi}}$, but using the fact that the frequency set $\mathscr{F}$ is minimal, we can assume that $\Psi(\underline{j}) \cap \mathscr{I}_{\mathrm{io}} \neq \varnothing$.

Let $\underline{i} \in \Psi(\underline{j}) \cap \mathscr{I}_{\mathrm{io}}$. According to the analysis made in Section 4B1, we have $\underline{i} \in A_{\underline{a}}$. Let $\ell' = (\ell'_1, \dots, \ell'_{p'})$ be the sequence linking $\underline{i}$ to the root $\underline{a}$. Reiterating the arguments used in the proof of Proposition 4.20, it is sufficient to study the case $\ell_i \neq \ell'_i$ for all i.

If $p' \in 2\mathbb{N}$, we can then show that $(\ell', i, j, \ell_p, \dots, \ell_2)$ is a loop with an odd number of elements for the index ℓ_1, whereas if $p' \in 2\mathbb{N}+1$, the sequence $(\ell', i, \ell_p, \dots, \ell_2)$ is a loop for ℓ_1. Both cases contradict Assumption 4.7.

The proof of the assertion "If $j \in \mathscr{I}_{\text{ii}} \cup \mathscr{I}_{\text{ev}1}$, then $\Psi(j) = \{j\}$" follows the same reasoning. □

The same proposition, up to some adaptations on the oddness/evenness according to the considered tree, is true for all trees in partition (36).

In terms of wave packet reflection, Proposition 4.21 states that, on one hand, during a reflection on the side $\partial\Omega_1$ (resp. $\partial\Omega_2$), an outgoing-incoming phase (resp. incoming-outgoing) cannot generate (resp. incoming-outgoing) outgoing-incoming phases. This "natural" idea used in [Sarason and Smoller 1974] is now rigorously justified.

4B5. *Determination of the amplitudes for indices in the trees.* Thanks to the precise description of the internal structure of the different trees in the partition (36), it is easy to determine all the remaining amplitudes in the WKB expansion, and to conclude the construction of the leading-order terms. Once again, we will here only deal with a tree $A_{\underline{a}}$. The construction is analogous for the other trees.

Let $\underline{j}$ be any index of $A_{\underline{a}}$. We will show that it is always possible to determine the amplitude $u_{0,\underline{j}}$. Thanks to Proposition 4.20, there exists a unique sequence of type H, denoted by $\ell^{\underline{j}} = \ell = (\ell_1, \dots, \ell_p)$, linking $\underline{j}$ to the root $\underline{a}$. The first step in the construction of the amplitude $u_{0,\underline{j}}$ is to remark that independently of the determination of $u_{0,\underline{j}}$, we can always first determine the amplitudes u_{0,ℓ_i}, $i = 1, \dots, p$.

Indeed, by the definition of sequences of type H (see Definition 4.4), $\ell_1 \in \Phi(\underline{a}) \cap \mathscr{I}_{\text{io}}$. So, the amplitude u_{0,ℓ_1} satisfies a transport equation given by

$$\begin{cases} (\partial_t + v_{\ell_1} \cdot \nabla_x) Q_1^{\ell_1} u_{0,\ell_1} = 0, \\ B_1 \Big[\displaystyle\sum_{i \in \Phi(\ell_1) \cap (\mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ii}})} u_{0,i\,|x_1=0} \Big] = -B_1 u_{0,\underline{a}\,|x_1=0}, & \text{if } \ell_1 \in \mathscr{R}_1 \setminus \mathfrak{R}_1, \\ u_{0,\ell_1\,|t\le 0} = 0, \end{cases}$$
$$\begin{cases} (\partial_t + v_{\ell_1} \cdot \nabla_x) Q_1^{\ell_1} u_{0,\ell_1} = 0, \\ B_1 \Big[\sum_{i \in \Phi(\ell_1) \cap (\mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{ii}})} u_{0,i} + U_{0,\ell_1,1\,|X_1=0} \Big]_{|x_1=0} = -B_1 u_{0,\underline{a}\,|x_1=0}, & \text{if } \ell_1 \in \mathfrak{R}_1, \\ u_{0,\ell_1\,|t\le 0} = 0, \end{cases} \tag{61}$$

because all the elements of $\Phi(\ell_1)$ are linked with $\underline{a}$ by a sequence of length zero. Thanks to property (i) of Proposition 4.21, it follows that $\Phi(\ell_1) \cap \mathscr{I}_{\text{oi}} = \{\underline{a}\}$. Consequently, multiplying (61) by $S_1^{\ell_1}$ (see Definition 4.10), we can write

$$u_{0,\ell_1\,|x_1=0} = -S_1^{\ell_1} B_1 u_{0,\underline{a}\,|x_1=0}.$$

This equation determines the trace of u_{0,ℓ_1} on the side $\partial\Omega_1$ because the amplitude $u_{0,\underline{a}}$ and its trace have already been determined in Section 4B3. Integrating (61) along the characteristics, we determine $u_{0,\ell_1} \in H^K(\Omega_T)$ and the trace $u_{0,\ell_1\,|x_2=0} \in H_f^K$.

Then we are interested in the construction of the amplitude u_{0,ℓ_2}, by the definition of type-H sequences, $\ell_2 \in \Psi(\ell_1) \cap \mathscr{I}_{\text{oi}}$. Once again, we can apply Proposition 4.21 to show that

$$\Psi(\ell_2) \cap \mathscr{I}_{\text{io}} = \{\ell_1\}.$$

This allows us to rewrite the transport equation on u_{0,ℓ_2} in the form

$$\begin{cases} (\partial_t + v_{\ell_2} \cdot \nabla_x) Q_2^{\ell_2} u_{0,\ell_2} = 0, \\ u_{0,\ell_2 |x_2=0} = -S_2^{\ell_2} B_2 u_{0,\ell_1 |x_2=0}, \quad u_{0,\ell_2 |t \le 0} = 0, \end{cases} \tag{62}$$

and we solve this equation by integration along the characteristics.

We can reiterate the same kind of resolutions of transport equations for all the indices of the sequence ℓ. This operation permits us to construct all the amplitudes u_{0,ℓ_l}, $l = 1, \dots, p$. The important point in these recursive resolutions is that since the first amplitude $u_{0,\underline{a}}$ has its trace on $\partial\Omega_1$ in H_f^K, this flatness at the corner is transmitted to all the amplitudes indexed by the sequence ℓ.

Indeed, integration along the characteristics gives an explicit formula and it is easy to see on this formula that the traces of the u_{0,ℓ_l} are in H_f^K for all $1 \le l \le p$.

Once we have constructed all the amplitudes associated to the indices of ℓ, it is easy to determine the amplitude $u_{0,\underline{j}}$. We distinguish the following cases depending of the nature of the index $\underline{j}$.

- $\underline{j} \in \mathscr{I}_{\text{io}}$ (resp. $\underline{j} \in \mathscr{I}_{\text{oi}}$): Proposition 4.21 tells us that an index in $\mathscr{I}_{\text{io}} \cap A_{\underline{a}}$ (resp. $\mathscr{I}_{\text{oi}}$) can appear only after an even (resp. odd) number of reflections, in other terms, the length of the sequence ℓ, $p \in 2\mathbb{N}$ (resp. $p \in 2\mathbb{N}+1$). Using the fact that $\underline{j} \in \mathscr{I}_{\text{io}}$ (resp. $\underline{j} \in \mathscr{I}_{\text{io}}$), to construct the amplitude $u_{0,\underline{j}}$ it is sufficient to know $u_{0,\underline{j}\,|x_1=0}$ (resp. $u_{0,\underline{j}\,|x_2=0}$). But, Proposition 4.21 implies that ℓ_p is the only element of $\mathscr{I}_{\text{oi}}$ (resp. $\mathscr{I}_{\text{io}}$) in $\Phi(\underline{j})$ (resp. $\Psi(\underline{j})$); multiplying by $S_1^{\underline{j}}$ (resp. $S_2^{\underline{j}}$), we can determine the trace $u_{0,\underline{j}\,|x_1=0}$ (resp. $u_{0,\underline{j}\,|x_2=0}$) as a function of the trace $u_{0,\ell_p |x_1=0}$ (resp. $u_{0,\ell_p |x_2=0}$). Consequently the amplitude $u_{0,\underline{j}}$ is constructed.

  Moreover, we can show that $u_{0,\underline{j}} \in H^K(\Omega)$, and that its traces on the sides $\partial\Omega_1$ and $\partial\Omega_2$ are in H_f^K. This fact will be crucial to construct the incoming-incoming phases as well as the evanescent phases that may appear in the WKB expansion.

- $\underline{j} \in \mathscr{I}_{\text{ii}}$: An incoming-incoming index may appear after an even number of reflections as well as after an odd number of reflections. We will here deal with the case $p \in 2\mathbb{N}$; the case $p \in 2\mathbb{N}+1$ is totally similar. Proposition 4.21 implies, on the one hand, that ℓ_p is the only index in $\Phi(\underline{j}) \cap \mathscr{I}_{\text{oi}}$ and, on the other hand, that $\Psi(\underline{j}) = \{\underline{j}\}$. So the boundary conditions for the amplitude $u_{0,\underline{j}}$ can be written in the form

$$u_{0,\underline{j}\,|x_1=0} = -S_1^{\underline{j}} B_1 u_{0,\ell_p |x_1}, \quad u_{0,\underline{j}\,|x_2=0} = 0.$$

It follows that the amplitude $u_{0,\underline{j}}$ satisfies the incoming-incoming transport equation

$$\begin{cases} (\partial_t + v_{\underline{j}} \cdot \nabla_x) Q_1^{\underline{j}} u_{0,\underline{j}} = 0, \\ u_{0,\underline{j}\,|x_1=0} = -S_1^{\underline{j}} B_1 u_{0,\ell_p |x_1=0}, \quad u_{0,\underline{j}\,|x_2=0} = 0, \quad u_{0,\underline{j}\,|t \le 0} = 0. \end{cases} \tag{63}$$

To solve this equation, we extend the source term $-S_1^{\underline{j}} B_1 u_{0,\ell_p|x_1=0}$ by zero on $\{x_2 < 0\}$ (this extension gives a regular function because $u_{0,\ell_p|x_1=0} \in H_f^K$); then we restrict to $\{x_2 \geq 0\}$ the solution to the transport equation in the half-space $\{x_1 \geq 0,\ x_2 \in \mathbb{R}\}$.

Consequently we have constructed $u_{0,\underline{j}} \in H^K(\Omega_T)$, such that for all $\underline{x}_1 \geq 0$, we have $u_{0,\underline{j}\,|x_1=\underline{x}_1} \in H_f^K$.

• $\underline{j} \in \mathfrak{R}_1$ (resp. $\underline{j} \in \mathfrak{R}_2$): As in the case $\underline{j} \in \mathscr{I}_{\mathrm{io}}$ (resp. $\underline{j} \in \mathscr{I}_{\mathrm{oi}}$), Proposition 4.21 tells us that an evanescent index for the side $\partial\Omega_1$ (resp. $\partial\Omega_2$) can only appear after an even (resp. odd) number of reflections. Moreover, Proposition 4.21 also implies that the only index of $\mathscr{I}_{\mathrm{oi}}$ (resp. $\mathscr{I}_{\mathrm{io}}$) and $\Phi(\underline{j})$ (resp. $\Psi(\underline{j})$) is ℓ_p. We are now interested in the construction of the evanescent amplitude $U_{0,\underline{j},1}$ (resp. $U_{0,\underline{j},2}$).

Repeating the construction described in Section 4B3, to determine the amplitude $U_{0,\underline{j},1}$ (resp. $U_{0,\underline{j},2}$), it is sufficient to start by determining the double trace on $\{X_1 = x_1 = 0\}$ (resp. $\{X_2 = x_2 = 0\}$). Using the fact that ℓ_p is the only element of $\mathscr{I}_{\mathrm{oi}}$ (resp. $\mathscr{I}_{\mathrm{io}}$) in $\Phi(\underline{j})$ (resp. $\Psi(\underline{j})$) allows us to show that this double trace is given by

$$U_{0,\underline{j},1}(t,0,x',0) = -S_{s,1}^{\underline{j}} B_1 u_{0,\ell_p|x_1=0}$$
$$(\text{resp. } U_{0,\underline{j},2}(t,x',0,0) = -S_{s,2}^{\underline{j}} B_2 u_{0,\ell_p|x_2=0}).$$

We then lift this double trace to a single one by setting

$$U_{0,\underline{j},1}(t,x,0) := -\chi(x_1) S_{s,1}^{\underline{j}} B_1 u_{0,\ell_p|x_1=0}$$
$$(\text{resp. } U_{0,\underline{j},2}(t,x,0) := -\chi(x_2) S_{s,2}^{\underline{j}} B_2 u_{0,\ell_p|x_2=0}),$$

where $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ is such that $\chi(0) = 1$. Finally, Lemma 4.19 shows that the function $U_{0,\underline{j},1}$ (resp. $U_{0,\underline{j},2}$) defined by

$$\begin{aligned} U_{0,\underline{j},1}(t,x,X_1) &= -\chi(x_1) e^{X_1 \mathscr{A}_1} S_{s,1}^{\underline{j}} B_1 u_{0,\ell_p|x_1=0} \\ (\text{resp. } U_{0,\underline{j},2}(t,x,X_2) &= -\chi(x_2) e^{X_2 \mathscr{A}_2} S_{s,2}^{\underline{j}} B_2 u_{0,\ell_p|x_2=0}) \end{aligned} \tag{64}$$

is a solution to the cascades of equations (26), (31) and (33) written for $n = 0$ and $k = \underline{j}$.

We are now interested in the influence of the evanescent phases previously constructed in the extra boundary condition (32). For example, let us study the trace on $\{x_2 = 0\}$ of an evanescent phase for the side $\partial\Omega_1$. From (64) this trace is explicitly given by

$$U_{0,\underline{j},1}\left(t, x_1, 0, \frac{x_1}{\varepsilon}\right) = -\chi(x_1) e^{X_1 \mathscr{A}_1} S_{s,1}^{\underline{j}} B_1 u_{0,\ell_p|x_1=0}(t,0)$$

because the only term depending on x_2 in (64) is $u_{0,\ell_p|x_1=0}$. The flatness at the corner of $u_{0,\ell_p|x_1=0}$ shows that $U_{0,\underline{j},1}(t,x_1,0,x_1/\varepsilon)$ is in fact zero and that it does not contribute in (32). The same result is also true for all the other evanescent amplitudes (for both sides) in the tree $A_{\underline{a}}$.

In this section we have shown that an arbitrary amplitude in the tree $A_{\underline{a}}$ can always be constructed. As a consequence, all the amplitudes in the tree $A_{\underline{a}}$ can be determined. Then it is sufficient to repeat the method of construction for each tree in the partition (36). So we have constructed all the amplitudes associated with indices in $\mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}$, and we have thus finished the construction of the leading term in the WKB expansion. We summarize the analysis with the following proposition:

Proposition 4.22. *Under Assumptions 2.1–2.2 on the complete-for-reflections corner problem (18) and under Assumptions 4.7–4.9 and 4.16, there exist functions $(u_{0,i})_{i\in\mathscr{I}_{\text{os}}}$, $(U_{0,i,1})_{i\in\mathfrak{R}_1}$ and $(U_{0,i,2})_{i\in\mathfrak{R}_2}$ satisfying the cascades of equations (26), (31), (32) and (33) written for $n = 0$.*

Moreover, the functions $(u_{0,i})_{i\in\mathscr{I}_{\text{os}}}$ admit the following regularity. For all $T > 0$:

- *If $i \in \mathscr{I}_{\text{io}} \cup \mathscr{I}_{\text{oi}}$ then $u_{0,i} \in H^K(\Omega_T)$ and the traces $u_{0,i}{}_{|x_1=0}$ and $u_{0,i}{}_{|x_2=0}$ are in H^K_f.*
- *If $i \in \mathscr{I}_{\text{ii}}$ then $u_{0,i} \in H^K(\Omega_T)$. Moreover, if $\Psi(i) = \{i\}$ (resp. $\Phi(i) = \{i\}$) then for all $\underline{x}_1 > 0$ (resp. $\underline{x}_2 > 0$), the trace $u_{0,i}{}_{|x_1=\underline{x}_1}$ (resp. $u_{0,i}{}_{|x_2=\underline{x}_2}$) is H^K_f.*

The functions $(U_{0,i,1})_{i\in\mathfrak{R}_1}$ (resp. $(U_{0,i,2})_{i\in\mathfrak{R}_2}$) are in $P_{\text{ev}1}$ (resp. $P_{\text{ev}2}$). Moreover, all the functions $(U_{0,i,1})_{i\in\mathfrak{R}_1}$ (resp. $(U_{0,i,2})_{i\in\mathfrak{R}_2}$) satisfy $U_{0,i,1}(t, x_1, 0, X_1) = 0$ (resp. $U_{0,i,2}(t, 0, x_2, X_2) = 0$).

4B6. *Construction of the higher-order terms in the WKB expansion.* The construction for the higher-order terms in the WKB expansion looks like the construction of the leading-order term. In particular, the order of resolution will be the same: we start with the amplitudes on the loop, and then we show that the knowledge of these amplitudes is sufficient to construct any amplitude in the trees of the partition (36). In this section we only give the main steps of the construction of the term of order ε, without all details. Let us begin with the oscillating amplitudes.

For $\underline{k} \in \mathscr{I}_{\text{os}}$, the amplitude $u_{1,\underline{k}}$ satisfies the equations

$$\begin{cases} i\mathscr{L}(d\varphi_k)u_{1,\underline{k}} + L(\partial)u_{0,\underline{k}} = 0, \\ u_{1,\underline{k}}{}_{|t\leq 0} = 0, \end{cases} \tag{65}$$

with the two boundary conditions

$$\begin{aligned} B_1\Big[\sum_{k\in\Phi(\underline{k})\cap\mathscr{I}_{\text{os}}} u_{1,k}\Big]_{|x_1=0} &= 0 \quad \text{if } \underline{k} \notin \mathfrak{R}_1, \\ B_1\Big[\sum_{k\in\Phi(\underline{k})\cap\mathscr{I}_{\text{os}}} u_{1,k} + U_{1,\underline{k},1}{}_{|X_1=0}\Big]_{|x_1=0} &= 0 \quad \text{if } \underline{k} \in \mathfrak{R}_1, \end{aligned} \tag{66}$$

and

$$\begin{aligned} B_2\Big[\sum_{k\in\Psi(\underline{k})\cap\mathscr{I}_{\text{os}}} u_{1,k}\Big]_{|x_2=0} &= 0 \quad \text{if } \underline{k} \notin \mathfrak{R}_2, \\ B_2\Big[\sum_{k\in\Psi(\underline{k})\cap\mathscr{I}_{\text{os}}} u_{1,k} + U_{1,\underline{k},2}{}_{|X_2=0}\Big]_{|x_2=0} &= 0 \quad \text{if } \underline{k} \in \mathfrak{R}_2. \end{aligned} \tag{67}$$

In a classical way, we compose the first equation of (65) by the partial inverse $R_1^{\underline{k}}$ if $\underline{k} \in \mathscr{I}_{\text{io}}$, $R_2^{\underline{k}}$ if $\underline{k} \in \mathscr{I}_{\text{oi}}$ and by $R_1^{\underline{k}}$ or $R_2^{\underline{k}}$ if $\underline{k} \in \mathscr{I}_{\text{ii}}$. Let us recall that this partial inverse satisfies: for $i = 1, 2$,

$$R_i^{\underline{k}}\mathscr{L}(d\varphi_{\underline{k}}) = I - P_i^{\underline{k}}, \quad P_i^{\underline{k}} R_i^{\underline{k}} = R_i^{\underline{k}} Q_i^{\underline{k}} = 0.$$

The first equation of (65), after this composition, reads

$$(I - P_i^{\underline{k}})u_{1,\underline{k}} = iR_i^{\underline{k}} L(\partial)u_{0,\underline{k}} \tag{68}$$

and determines in a unique way the unpolarized part of $u_{1,\underline{k}}$. Indeed, at this stage of the analysis, the term of the right-hand side of (68) has already been constructed. As $u_{0,\underline{k}} \in H^K(\Omega_T)$ and its traces on the sides $\partial\Omega_1$ and $\partial\Omega_2$ are in H_f^K, the unpolarized part of $u_{1,\underline{k}}$ belongs to $H^{K-1}(\Omega_T)$ with traces in H_f^{K-1}. To complete the construction of the oscillating amplitude $u_{1,\underline{k}}$, we just have to construct its polarized part, that is to say, $P_1^{\underline{k}} u_{1,\underline{k}}$ (or equivalently $P_2^{\underline{k}} u_{1,\underline{k}}$).

To determine the polarized part, we will repeat with some modifications, the method described for the leading order. First, we remark that the evolution equation for the amplitude $u_{2,\underline{k}}$ is

$$i\mathscr{L}(d\varphi_{\underline{k}})u_{2,\underline{k}} + L(\partial)u_{1,\underline{k}} = 0,$$

and, after composition by $Q_1^{\underline{k}}$ for $\underline{k} \in \mathscr{I}_{\text{io}}$, by $Q_2^{\underline{k}}$ for $\underline{k} \in \mathscr{I}_{\text{oi}}$ and by $Q_1^{\underline{k}}$ or $Q_2^{\underline{k}}$ for $\underline{k} \in \mathscr{I}_{\text{ii}}$, is given by

$$\begin{aligned} Q_i^{\underline{k}} L(\partial) P_i^{\underline{k}} u_{1,\underline{k}} &= -Q_i^{\underline{k}} L(\partial)(I - P_i^{\underline{k}})u_{1,\underline{k}} \\ &= -iQ_i^{\underline{k}} L(\partial) R_i^{\underline{k}} L(\partial) u_{0,\underline{k}}. \end{aligned}$$

Thanks to Lax's lemma [1957], this equation is a transport equation with speed $v_{\underline{k}}$ on the polarized part $P_i^{\underline{k}} u_{1,\underline{k}}$. As a consequence, $Q_i^{\underline{k}} P_i^{\underline{k}} u_{1,\underline{k}}$ satisfies the same transport equation (with a nonzero source term in the interior of Ω) as the transport equation satisfied by $u_{0,\underline{k}}$. This observation leads us to apply the same method of construction as in Sections 4B3 and 4B5.

More precisely, we start with the indices on the loop; to fix the ideas, we will describe the construction of u_{1,n_4}. We have already seen that its unpolarized part is known. To construct the polarized part of u_{1,n_4}, since it travels with an outgoing-incoming velocity, we need to know its trace on $\partial\Omega_2$. Repeating the computation made in Section 4B2, we obtain an invertibility condition which reads

$$(I - \mathbb{T})P_1^{n_1} u_{1,n_1|x_2=0} = G_1,$$

where $G_1 \in H_f^{K-1}$ only depends on the unpolarized traces of the amplitudes associated with the elements of the loop. Assumption 4.15 implies that $P^{n_4} u_{1,n_4}$ is solution to the transport equation

$$\begin{cases} (\partial_t + v_{n_4} \cdot \nabla_x) Q_2^{n_4} P_2^{n_4} u_{1,n_4} = -iQ_2^{n_4} L(\partial) R_2^{n_4} L(\partial) u_{0,n_4}, \\ P_2^{n_4} u_{1,n_4|x_2=0} = -S_2^{n_4} B_2\big[(I-\mathbb{T})^{-1} G_1 + (I - P_2^{n_4}) u_{1,n_4|x_2=0} + (I - P_1^{n_1}) u_{1,n_1|x_2=0}\big], \quad P_2^{n_4} u_{1,n_4|t\leq 0} = 0. \end{cases}$$

All the source terms in this equation are known, so we can integrate along the characteristics to determine $P_2^{n_4} u_{1,n_4}$. The source term in the interior is $H^{K-2}(\Omega_T)$ and the source term on the boundary is H_f^{K-1}, so the solution $P_2^{n_4} u_{1,n_4}$ is in $H^{K-2}(\Omega_T)$ with traces on $\partial\Omega_1$ and $\partial\Omega_2$ in H_f^{K-2}. The fact that the construction of the term of order one in ε needs two derivatives is classical, and more generally, the construction of the term of order N_0 needs $2N_0$ derivatives on the $u_{0,j}$.

When the amplitudes associated with indices on the loop are determined, the construction of the polarized parts of the other oscillating amplitudes follows exactly the same method. In particular the order of resolution is the same order as the order described in Section 4B5. That is why we will not give more details about this construction.

We are now interested in the construction of the evanescent amplitudes of order ε. Although these amplitudes do not satisfy transport equations, the method of construction is based on the same ideas as the method for oscillating amplitudes. Indeed, we remark that the amplitudes $U_{1,k,i}$ can be decomposed in a polarized part (whose construction will use the techniques of the construction of $U_{0,k,i}$) and an unpolarized part only depending of the known amplitude $U_{0,k,i}$.

In this section we will only consider evanescent amplitudes for the side $\partial\Omega_1$, so we let $\underline{k} \in \mathfrak{R}_1$. The amplitude $U_{1,\underline{k},1}$ satisfies the system of equations

$$\begin{cases} L_{\underline{k}}(\partial_{X_1})U_{1,\underline{k},1} + L(\partial)U_{0,\underline{k},1} = 0, \\ B_1\Big[\displaystyle\sum_{k\in\Phi(\underline{k})\cap\mathscr{I}_{\mathrm{os}}} u_{1,k} + U_{1,\underline{k},1|X_1=0}\Big]_{|x_1=0} = 0, \\ U_{1,\underline{k},1|t\leq 0} = 0, \end{cases} \tag{69}$$

but thanks to Lemma 4.19, we know that the first equation of this system has a solution reading

$$U_{1,\underline{k},1} = \mathbb{P}^{\underline{k}}_{\mathrm{ev}1} U_{1,\underline{k},1} - \mathbb{Q}^{\underline{k}}_{\mathrm{ev}1} L(\partial)U_{0,\underline{k},1}, \tag{70}$$

where we recall that the projectors $\mathbb{P}^{\underline{k}}_{\mathrm{ev}1}$ and $\mathbb{Q}^{\underline{k}}_{\mathrm{ev}1}$ are defined in (58) and (59). Using the fact that the amplitude $U_{0,\underline{k},1}$ has already been constructed, the unpolarized part of $U_{1,\underline{k},1}$, namely $\mathbb{Q}^{\underline{k}}_{\mathrm{ev}1} L(\partial)U_{0,\underline{k},1}$, is known. It is thus sufficient to construct the polarized part of $U_{1,\underline{k},1}$, namely $\mathbb{P}^{\underline{k}}_{\mathrm{ev}1} U_{1,\underline{k},1}$. To do that, we repeat the construction used for $U_{0,k,1}$. By the definition of $\mathbb{P}^{\underline{k}}_{\mathrm{ev}1}$, we know $\mathbb{P}^{\underline{k}}_{\mathrm{ev}1} U_{1,\underline{k},1}$ will be determined if we can construct the trace of $U_{1,\underline{k},1}$ on $\{X_1 = 0\}$.

Firstly, the boundary condition (69) and Proposition 4.21 give the double trace on $\{x_1 = X_1 = 0\}$. More precisely, this double trace is given by

$$U_{1,\underline{k},1|X_1=x_1=0} = -S^{\underline{k}}_{s,1} u_{1,\underline{k}|x_1=0}, \tag{71}$$

where the source term is known because we have already constructed the oscillating amplitudes of order ε. To conclude we lift this double trace on $\{x_1 = X_1 = 0\}$ in a single trace $\{X_1 = 0\}$ exactly as has been done for $U_{0,\underline{k},1}$, and then we apply the operator $\mathbb{P}^{\underline{k}}_{\mathrm{ev}1}$.

Finally we have to study the contribution of $U_{1,\underline{k},1|x_2=0}$ in the boundary condition (32). From the decomposition (70), it is in fact sufficient to study the traces on $\{x_2 = 0\}$ of $\mathbb{P}^{\underline{k}}_{\mathrm{ev}1} U_{1,\underline{k},1}$ and of $\mathbb{Q}^{\underline{k}}_{\mathrm{ev}1} L(\partial)U_{0,\underline{k},1}$. Concerning the first term, since it has been constructed exactly as lower-order evanescent amplitudes, its trace on $\{x_2 = 0\}$ will be zero if and only if (see (71)) the trace of $u_{1,\underline{k}|x_1=0}$ on $\{x_2 = 0\}$ is zero. This point is a consequence of the fact that oscillating amplitudes of order one are, as the amplitudes of order zero, flat at the corner.

Concerning the value of the trace on $\{x_2 = 0\}$ of $\mathbb{Q}^{\underline{k}}_{\mathrm{ev}1} L(\partial)U_{0,\underline{k},1}$, let us first remark that from (59), the operator $\mathbb{Q}^{\underline{k}}_{\mathrm{ev}1}$ only acts on the fast variable X_1. As a consequence it does not influence the value of a trace for the slow variable x_2. We thus have to study $L(\partial)U_{0,\underline{k},1}(t, x_1, 0, x_1/\varepsilon)$. From (64), we can compute

$$L(\partial)U_{0,\underline{k},1}\left(t,x_1,x_2,\frac{x_1}{\varepsilon}\right)=\left(\chi'(x_1)+\frac{1}{\varepsilon}\right)M_1\left(\frac{x_1}{\varepsilon}\right)u_{1,\underline{k}_{|x_1=0}}(t,x_2)+M_0\left(\frac{x_1}{\varepsilon}\right)\partial_t u_{1,\underline{k}_{|x_1=0}}(t,x_2)$$
$$+M_2\left(\frac{x_1}{\varepsilon}\right)\partial_2 u_{1,\underline{k}_{|x_1=0}}(t,x_2),$$

where $M_i(x_1/\varepsilon):=-A_i e^{\frac{x_1}{\varepsilon}\mathscr{A}_1}S^{\underline{k}}_{s,1}B_1$ (with the convention $A_0:=I$). Evaluating the previous formula at $x_2=0$, we obtain from the flatness of $u_{1,\underline{k}_{|x_1=0}}$ at the corner that the trace on $\{x_2=0\}$ of $\mathbb{Q}^{\underline{k}}_{\mathrm{ev}1}L(\partial)U_{0,\underline{k},1}$ is zero. So the construction of the WKB expansion for the corner problem (18) is complete. To summarize we give the following theorem:

Theorem 4.23. *Under Assumptions 2.1–2.2 on the complete-for-reflections corner problem (18) and under Assumptions 4.7 and 4.16, if $[\cdot]$ denotes the integer-part function, then there exist functions $(u_{n,k})_{n\le[\frac{K}{2}],k\in\mathscr{I}_{\mathrm{os}}}$, $(U_{n,k,1})_{\le[\frac{K}{2}],k\in\mathfrak{R}_1}$ and $(U_{n,k,2})_{\le[\frac{K}{2}],k\in\mathfrak{R}_2}$ satisfying the cascades of equations (26), (31), (32) and (33).*

Moreover, the functions $u_{n,k}$ admit the following regularity. For all $T>0$:

- *If $k\in\mathscr{I}_{\mathrm{io}}\cup\mathscr{I}_{\mathrm{oi}}$ then $u_{n,k}\in H^{K-2n}(\Omega_T)$ and the traces $u_{n,k_{|x_1=0}}$ and $u_{n,k_{|x_2=0}}$ are in H^{K-2n}_f.*
- *If $n\in\mathscr{I}_{\mathrm{ii}}$, then $u_{n,k}\in H^{K-2n}(\Omega_T)$. Moreover, if $\Psi(k)=\{k\}$ (resp. $\Phi(k)=\{k\}$) then for all $\underline{x}_1>0$ (resp. $\underline{x}_2>0$), the trace $u_{n,k_{|x_1=\underline{x}_1}}$ (resp. $u_{n,k_{|x_2=\underline{x}_2}}$) is H^{K-2n}_f.*

The $U_{n,k,1}$ (resp. $U_{n,k,2}$) are in $P_{\mathrm{ev}1}$ (resp. $P_{\mathrm{ev}2}$). Moreover, the $U_{n,k,1}$, $U_{n,k,2}$ satisfy for all n,k, $U_{n,k,1}(t,x_1,0,X_1)=U_{n,k,2}(t,0,x_2,X_2)=0$.

4C. *Justification of the WKB expansion.* In this section we show that, if the corner problem (18) is strongly well-posed, the truncated WKB expansion constructed in the preceding section is a good approximation to the exact solution u^ε of the corner problem (18). Let us recall what we mean by strong well-posedness:

Definition 4.24. The corner problem is said to be strongly well-posed if for all $f\in L^2(\Omega_T)$, $g_1\in L^2(\partial\Omega_{1,T})$ and $g_2\in L^2(\partial\Omega_{2,T})$ zero for negative times, the system

$$\begin{cases}\partial_t u+A_1\partial_1 u+A_2\partial_2 u=f,\\ B_1u_{|x_1=0}=g_1,\quad B_2u_{|x_2=0}=g_2,\quad u_{|t\le 0}=0,\end{cases}$$

admits a unique solution $u\in L^2(\Omega_T)$, with traces in $L^2(\partial\Omega_{1,T})$ and $L^2(\partial\Omega_{2,T})$, satisfying the energy estimate

$$\|u\|^2_{L^2(\Omega_T)}+\|u_{|x_1=0}\|^2_{L^2(\partial\Omega_{1,T})}+\|u_{|x_2=0}\|^2_{L^2(\partial\Omega_{2,T})}$$
$$\le C_T\big(\|f\|^2_{L^2(\Omega_T)}+\|g_1\|^2_{L^2(\partial\Omega_{1,T})}+\|g_2\|^2_{L^2(\partial\Omega_{2,T})}\big).\quad(72)$$

Let us recall that the strong well-posedness of the corner problem is demonstrated for the particular class of symmetric corner problems with strictly dissipative boundary conditions.

To justify the convergence of the WKB expansion, we need to be sure that the amplitudes are regular enough. The regularity has already been studied for the oscillating amplitudes. Concerning the evanescents' amplitudes, the following proposition shows that they are regular and also gives their size according to the small parameter ε.

Proposition 4.25. *Let U be an element of $P_{\mathrm{ev},1}$ (resp. $P_{\mathrm{ev},2}$). Then the functions $U(t,x,x_1/\varepsilon)$ and $(L(\partial)U(t,x,X_1))_{|X_1=\frac{x_1}{\varepsilon}}$ (resp. $U(t,x,x_2/\varepsilon)$ and $(L(\partial)U(t,x,X_2))_{|X_2=\frac{x_2}{\varepsilon}}$) are $O(\varepsilon^{\frac{1}{2}})$ in $L^2(\Omega_T)$.*

We refer to [Benoit 2014] for a proof of this result.

Before showing that the truncated WKB expansion is a good approximation to the exact solution to the corner problem (18), we have to make sure that the truncated WKB expansion makes sense. Indeed, we saw in Section 3D that when there was an infinite number of phases generated by successive reflections, it was not clear that the sum of all amplitudes defining the WKB expansion converges. That is why, to avoid this difficulty we will restrict ourselves to a finite number of phases:

Assumption 4.26. *We assume that the number of phases generated by successive reflections on the sides $\partial\Omega_1$ and $\partial\Omega_2$ is finite. That is to say, $\#\mathscr{F} < +\infty$.*

With this extra assumption, it is clear that the truncated WKB expansion makes sense. The main theorem of this article is:

Theorem 4.27. *Suppose Assumptions 2.1–2.2 for the complete-for-reflection corner problem (18) and Assumptions 4.7, 4.9, 4.16 and 4.26 hold. Then, for $N_0 \in \mathbb{N}$, with $N_0 \leq \left[\frac{K}{2} - \frac{3}{2}\right]$, we denote by $u^{\varepsilon}_{\mathrm{app},N_0}$ the geometric optics expansion truncated at order N_0 defined by*

$$u^{\varepsilon}_{\mathrm{app},N_0} := \sum_{k\in\mathscr{I}_{\mathrm{os}}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} \sum_{n=0}^{N_0} \varepsilon^n u_{n,k}(t,x) + \sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}(t,x_2)} \sum_{n=0}^{N_0} \varepsilon^n U_{n,k,1}\left(t,x,\frac{x_1}{\varepsilon}\right) + \sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}(t,x_1)} \sum_{n=0}^{N_0} \varepsilon^n U_{n,k,2}\left(t,x,\frac{x_2}{\varepsilon}\right),$$

where functions $u_{n,k}$, $U_{n,k,1}$ and $U_{n,k,2}$ are given by Proposition 4.22. Then, if the corner problem (18) is strongly well-posed, let u^{ε} be its exact solution, and the error $u^{\varepsilon} - u^{\varepsilon}_{\mathrm{app},N_0}$ is $O(\varepsilon^{N_0+1})$ in $L^2(\Omega_T)$.

Proof. Since we assumed that $N_0 \leq \left[\frac{K}{2} - \frac{3}{2}\right]$, the term of order ε^{N_0+1} of the WKB expansion makes sense and is at least in $H^1(\Omega_T)$. By construction of the $u_{n,k}$, $U_{n,k,1}$ and $U_{n,k,2}$, for $n \leq N_0+1$, the remainder $u^{\varepsilon} - u^{\varepsilon}_{\mathrm{app},N_0+1}$ satisfies the corner problem

$$\begin{cases} L(\partial)(u^{\varepsilon} - u^{\varepsilon}_{\mathrm{app},N_0+1}) = f^{\varepsilon}_{N_0+1}, \\ B_1(u^{\varepsilon} - u^{\varepsilon}_{\mathrm{app},N_0+1})_{|x_1=0} = 0, \quad B_2(u^{\varepsilon} - u^{\varepsilon}_{\mathrm{app},N_0+1})_{|x_2=0} = 0, \quad (u^{\varepsilon} - u^{\varepsilon}_{\mathrm{app},N_0+1})_{|t\leq 0} = 0, \end{cases} \tag{73}$$

with

$$f^{\varepsilon}_{N_0+1} := \varepsilon^{N_0+1}\left[\sum_{k\in\mathscr{I}_{\mathrm{os}}} e^{\frac{i}{\varepsilon}\varphi_k} L(\partial)u_{N_0+1,k} + \sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}} (L(\partial)U_{N_0+1,k,1})_{|X_1=\frac{x_1}{\varepsilon}} + \sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}} (L(\partial)U_{N_0+1,k,2})_{|X_2=\frac{x_2}{\varepsilon}}\right].$$

But the corner problem (18) is supposed to be strongly well-posed, so we can use the energy estimate (72), to obtain

$$\|u^{\varepsilon}-u^{\varepsilon}_{\mathrm{app},N_0+1}\|_{L^2(\Omega_T)} \le C_T\|f^{\varepsilon}_{N_0+1}\|_{L^2(\Omega_T)}.$$

The right-hand side can be estimated by

$$\begin{aligned}\|f^{\varepsilon}_{N_0+1}\|_{L^2(\Omega_T)} &\le \varepsilon^{N_0+1}\Bigg[\sum_{k\in\mathscr{I}_{\mathrm{os}}}\|L(\partial)u_{N_0+1,k}\|_{L^2(\Omega_T)}+\sum_{k\in\mathscr{I}_{\mathrm{ev}1}}\|L(\partial)U_{N_0+1,k,1}(\cdot,\cdot,X_1)_{|X_1=\frac{x_1}{\varepsilon}}\|_{L^2(\Omega_T)}\\ &\qquad+\sum_{k\in\mathscr{I}_{\mathrm{ev}2}}\|L(\partial)U_{N_0+1,k,2}(\cdot,\cdot,X_2)_{|X_2=\frac{x_2}{\varepsilon}}\|_{L^2(\Omega_T)}\Bigg]\\ &\le C\varepsilon^{N_0+1}\end{aligned}$$

because, according to Proposition 4.25, $(L(\partial)U_{N_0,k,1})_{|X_1=\frac{x_1}{\varepsilon}}$ and $(L(\partial)U_{N_0+1,k,2})_{|X_2=\frac{x_2}{\varepsilon}}$ are $O(\varepsilon^{\frac{1}{2}})$ in $L^2(\Omega_T)$, whereas $L(\partial)u_{N_0+1,k}$ are $O(1)$ in $L^2(\Omega_T)$, because $u_{N_0+1,k}$ is at least in $H^1(\Omega_T)$.

We thus have shown that

$$\|u^{\varepsilon}-u^{\varepsilon}_{\mathrm{app},N_0+1}\|_{L^2(\Omega_T)} \le C_T\varepsilon^{N_0+1},$$

and we conclude by the triangle inequality. □

4D. *Study of the invertibility condition* (47). In this section we will give a sufficient (and also necessary in several relevant cases) condition ensuring that the invertibility condition (47) is satisfied. Let us recall that this condition reads

$$u_{0,n_1|x_2=0}(t,x_1)-Su_{0,n_1|x_2=0}(t-\alpha x_1,\beta x_1)=S_1^{n_1}g(t+\delta x_1,\kappa x_1), \tag{74}$$

with $\alpha,\beta>0$, and $\delta<0$, $\kappa>0$. The exact expressions of these parameters are given by

$$\begin{aligned}S&:=S_1^{n_1}B_1S_2^{n_2}B_2S_1^{n_3}B_1S_2^{n_4}B_2,\\ \alpha&:=-\frac{1}{v_{n_1,1}}\left(-1+\frac{v_{n_1,2}}{v_{n_2,2}}-\frac{v_{n_1,2}v_{n_2,1}}{v_{n_2,2}v_{n_3,1}}+\frac{v_{n_1,2}v_{n_2,1}v_{n_3,2}}{v_{n_2,2}v_{n_3,1}v_{n_4,2}}\right),\\ \beta&:=\frac{v_{n_4,1}}{v_{n_4,2}}\frac{v_{n_3,2}}{v_{n_3,1}}\frac{v_{n_2,1}}{v_{n_2,2}}\frac{v_{n_1,2}}{v_{n_1,1}}.\end{aligned}$$

If we assume that $\dim\ker\mathscr{L}(i\underline{\tau},\xi_1^{n_1},\xi_2^{n_1})=1$ (an assumption which is automatically satisfied in the strictly hyperbolic framework), then (74) is in fact a scalar equation:

$$u(t,x_1)-\widetilde{S}u(t-\alpha x_1,\beta x_1)=G(t,x_1), \tag{75}$$

where, thanks to the polarization condition, we write $u_{0,n_1|x_2=0}(t,x_1)=u(t,x_1)e_{n_1}$, with e_{n_1} chosen such that $\ker(\mathscr{L}(i\underline{\tau},\xi_1^{n_1},\xi_2^{n_1}))=\operatorname{Span}e_{n_1}$. The scalar $\widetilde{S}$ is defined by the equality

$$Se_{n_1}=\widetilde{S}e_{n_1},$$

and without loss of generality we can assume that $\widetilde{S}\neq 0$.

In all that follows, it will be more convenient to rewrite (75) as

$$(I-\mathbb{T})u = G, \tag{76}$$

where

$$(\mathbb{T}u)(t,x_1) := \widetilde{S}u(t-\alpha x_1, \beta x_1). \tag{77}$$

A sufficient condition for (74) (and thus also (75)) to have a unique solution in the profiles space $L^2(]-\infty,T]\times\mathbb{R}_+)$ is given by the following theorem:

Theorem 4.28. *If*

$$|S| < \sqrt{\beta},$$

then for all $\gamma > 0$, for all $G \in L^2_\gamma(\mathbb{R}_+\times\mathbb{R})$, the functional equation (74) admits a unique solution $u \in L^2_\gamma(\mathbb{R}_+\times\mathbb{R})$, polarized on $\ker\mathscr{L}(i\underline{\tau},\xi_1^{n_1},\xi_2^{n_1})$, *and satisfying*

$$\|u\|_{L^2_\gamma(\mathbb{R}_+\times\mathbb{R})} \le C\|G\|_{L^2_\gamma(\mathbb{R}_+\times\mathbb{R})},$$

where C does not depend on the parameter γ.

In particular, for all $T>0$, if G is in $L^2(\partial\Omega_{2,T})$ and is zero for negative times, then (74) has a unique solution $u \in L^2(\partial\Omega_{2,T})$, polarized on $\ker(\mathscr{L}(i\underline{\tau},\xi_1^{n_1},\xi_2^{n_1}))$, *and satisfying*

$$\|u\|_{L^2(\partial\Omega_{2,T})} \le C_T\|G\|_{L^2(\partial\Omega_{2,T})}.$$

Proof. To solve (76) in a unique way, it is sufficient that $\mathbb{T}$ is a contraction on $L^2_\gamma(\mathbb{R}_+\times\mathbb{R})$ (or equivalently on $L^2(]-\infty,T]\times\mathbb{R}_+)$). A simple computation shows that is it effectively the case under the assumption $|S| < \sqrt{\beta}$. □

Remark. The fact that we are interested in uniform bounds (compared with the parameter γ) of the operator $\mathbb{T}$ is motivated by the following fact. In the analysis of the initial boundary value problem in the half-space, one starts to deal with global problems in time. Then from the uniformity of the energy estimate compared to γ follows a principle of causality which allows for the restriction to problems where the time variable lies in $]-\infty,T]$. We refer to [Benzoni-Gavage and Serre 2007; Chazarain and Piriou 1981] for more details about this proof.

To fully understand the condition $|S| < \sqrt{\beta}$, it is important to note the following fact: if one considers a point $(0,L)\in\partial\Omega_1$ and follows the characteristic curves associated with the indices on the loop, then after three reflections, this traveling point goes back to $\partial\Omega_1$ in a new position $(0,L')\in\partial\Omega_1$. Some basic computations show that

$$\beta = \frac{L}{L'}. \tag{78}$$

So, we have three possible behaviors depending of the value of β:

- If $\beta > 1$, then traveling along the bicharacteristics, the information approaches the corner.
- If $\beta < 1$, then traveling along the bicharacteristics, the information goes away from the corner.
- If $\beta = 1$, then the travel along the bicharacteristics is periodic.

The condition $|S| < \sqrt{\beta}$ imposes that after one turn along the bicharacteristics associated with the loop, the L^2-norm of the trace has decreased.

In the scalar case, that is, when the matrix S can be replaced by the scalar $\widetilde{S}$,[5] that is to say, when the rank of the projector $P_1^{n_1}$ is one, we can show that the condition $|\widetilde{S}| < \sqrt{\beta}$ is also necessary for the well-posedness of (75). The idea of the proof is to use Laplace transformation in the time variable to reduce to a situation already studied by Osher [1974b].

Theorem 4.29. *Let $\alpha, \beta > 0$ and $\widetilde{S} \in \mathbb{R} \setminus \{0\}$ be such that $\widetilde{S} > \sqrt{\beta}$. Then the equation*

$$u(t,x) - \widetilde{S} u(t - \alpha x, \beta x) = G \tag{79}$$

satisfies one of the alternatives:

(i) *If $\beta < 1$, then (79) written for $G = 0$ admits a nonzero solution in $L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$, for all $\gamma > 0$.*

(ii) *If $\beta > 1$, then there exists $g \in L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$ such that (79) does not have any solution.*

Proof. We begin with the proof of (i). We are looking for a nonzero solution u written in the form $u(t,x) = H(t)v(t,x)$, where H is the Heaviside function. Applying the Laplace transform in the time variable to (79) leads us to solve

$$\hat{v}(\sigma, x) - \widetilde{S} e^{-\alpha\sigma x} \hat{v}(\sigma, \beta x) = 0, \tag{80}$$

where $\sigma \in \mathbb{C}$, $\operatorname{Re} \sigma > 0$, is the dual variable of t. Following [Osher 1974b], let

$$\hat{\hat{v}}(\sigma, x) = e^{\frac{\alpha\sigma x}{1-\beta}} x^{-\frac{\ln \widetilde{S}}{\ln \beta}}.$$

It easy to check that this function is a solution to (80). But $\operatorname{Re}(\alpha\sigma/(1-\beta)) < 0$, so using the assumption $\widetilde{S} > \sqrt{\beta}$, it follows that $\hat{\hat{v}} \in L^2_x(\mathbb{R}_+)$.

However, in order to come back to the time variable, we want to apply the Paley–Wiener theorem to $\hat{\hat{v}}$. That is why we denote by $\hat{v}(\sigma, x)$ the following modification of $\hat{\hat{v}}$:

$$\hat{v}(\sigma, x) = \frac{1}{(1+\sigma)} \hat{\hat{v}}(\sigma, x).$$

It is easy to see that $\hat{v}$ is still a solution to (80). Moreover,

$$\sup_{\gamma > 0} \int_{\mathbb{R}} \int_{\mathbb{R}_+} |v(\gamma + i\eta, x)|^2 \, dx \, d\eta \le \sup_{\gamma > 0} \left(\int_{\mathbb{R}_+} x^{-2\frac{\ln \widetilde{S}}{\ln \beta}} e^{\frac{2x\alpha\gamma}{1-\beta}} \, dx \right) \int_{\mathbb{R}} \frac{1}{1+\eta^2} \, d\eta \le C.$$

We can thus apply the Paley–Wiener theorem, so there exists $v \in \bigcap_{\gamma > 0} L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$ such that $\hat{v}$ is the Laplace transform. As a consequence we have constructed a nonzero solution to (79).

To show (ii), using the same proof as in [Osher 1974b], it is sufficient to remark that the adjoint of $\mathbb{T}$ is given by

$$\mathbb{T}^* v = -\frac{\widetilde{S}}{\beta} v\left(t - \frac{\alpha}{\beta} x, \frac{1}{\beta} x\right).$$

[5] When S is a matrix and not a number, it seems more difficult to show the analog of Theorem 4.29. That is why the restriction to strictly hyperbolic operators is an easy way to obtain sharp results.

So if $\beta > 1$, the operator $\mathbb{T}^*$ is as in (i), so

$$\frac{\widetilde{S}}{\beta} > \frac{1}{\sqrt{\beta}} \iff \widetilde{S} > \sqrt{\beta},$$

and $\mathbb{T}^*$ is not injective. As a consequence, $\mathbb{T}$ is not surjective. □

During the construction of the geometric optics expansion we saw that the invertibility of the operator $I - \mathbb{T}$ on the weighted space $L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$ was not sufficient to construct the term of order ε which is, however, necessary if we want to show that the truncated expansion approximates the exact solution. More precisely, to construct the first corrector term, it is necessary that $I - \mathbb{T}$ is (at least) invertible from H^3_f to H^3_f, and more generally if one wants the remainder $u^\varepsilon - u^\varepsilon_{\mathrm{app},N_0}$ to be $O(\varepsilon^{N_0+1})$, it is needed that $I - \mathbb{T}$ is invertible from $H^{N_0+3}_f$ in $H^{N_0+3}_f$ (to ensure that the term of order $N_0 + 1$ is at least in $H^1(\Omega_T)$). The following theorem shows that the solution to the functional equation (75) given by Theorem 4.28 inherits (under some restrictions) the regularity H^K_f of the source term. There are two different cases to handle:

Theorem 4.30. (i) *If $0 < \beta \leq 1$ and if $|S| < \sqrt{\beta}$, then $I - \mathbb{T}$ is invertible from H^∞_f to H^∞_f.*

(ii) *Let $K \in \mathbb{N}$, if $\beta > 1$ and if $|S|\beta^{K-\frac{1}{2}} < 1$, then $I - \mathbb{T}$ is invertible from H^K_f to H^K_f.*

Proof. Let

$$u(t, x_1) = G(t, x_1) + \sum_{j=1}^{+\infty} S^j G(t + X^j_{\alpha,\beta} x_1, \beta^j x_1), \tag{81}$$

where

$$X^j_{\alpha,\beta} := \sum_{k=0}^{j-1} \alpha\beta^k + \beta^j \alpha.$$

It is easy to check that, under the assumption $|S| < \sqrt{\beta}$, we have u is a solution to (75) which belongs to $L^2(\partial\Omega_{2,T})$. According to Theorem 4.28, it is unique.

Then, we show that the solution u defined by (81) inherits the regularity of G. Firstly, according to the particular form of (75), it is clear that independently of β, for all $n \in \mathbb{N}$,

$$\|\partial^n_t u\|_{L^2(\partial\Omega_{2,T})} \leq C \|\partial^n_t g\|_{L^2(\partial\Omega_{2,T})}, \tag{82}$$

so we only have to deal with the derivatives in the spatial variable, because it will also permit us to deal with the cross-derivatives by using (82). For $n \in \mathbb{N}$, a simple computation gives

$$\partial^n_{x_1} u = \partial^n_{x_1} G + \sum_{j=1}^{+\infty} S^j \left[\sum_{l=0}^{n} \binom{n}{l} (X^j_{\alpha,\beta})^l (\beta^j \kappa)^{n-l} \partial^{n-l}_t \partial^l_{x_1} G \right] (t + X^j_{\alpha,\beta} x_1, \beta^j x_1) \tag{83}$$

and leads us to a distinction depending on the value of β.

If $\beta \le 1$, then all the constants appearing during the derivation can be abruptly bounded from above and we obtain

$$\|\partial_{x_1}^n u\|_{L^2(\partial\Omega_{2,T})} \le \|\partial_{x_1}^n G\|_{L^2(\partial\Omega_{2,T})} + C_{n,\alpha} \sum_{j=1}^{+\infty} \left(\frac{|S|}{\sqrt{\beta}}\right)^j \sum_{l=0}^{n} \|\partial_t^{n-l} \partial_{x_1}^l G\|_{L^2(\partial\Omega_{2,T})},$$

where we used the change of variable

$$\begin{bmatrix} s \\ y \end{bmatrix} = \begin{bmatrix} 1 & X_{\alpha,\beta}^j \\ 0 & \beta^j \end{bmatrix} \begin{bmatrix} t \\ x_1 \end{bmatrix}$$

to force the appearance of the factor $\sqrt{\beta}$. As a consequence, under the assumption $|S| < \sqrt{\beta}$, the solution u given by (81) is in $H^\infty(\partial\Omega_{2,T})$ and we can check by (83) that its trace is also in H_f^∞.

If $\beta > 1$, we have

$$\|\partial_{x_1}^n u\|_{L^2(\partial\Omega_{2,T})} \le \|\partial_{x_1}^n G\|_{L^2(\partial\Omega_{2,T})} + C_{n,\alpha} \sum_{j=1}^{+\infty} (|S|\beta^{n-\frac{1}{2}})^j \sum_{l=0}^{n} \|\partial_t^{n-l} \partial_{x_1}^l G\|_{L^2(\partial\Omega_{2,T})}$$

for $0 \le n \le K$, this sum is finite thanks to the assumption $|S|\beta^{K-\frac{1}{2}} < 1$. We have thus shown that $u \in H^K(\partial\Omega_{2,T})$. Once again, the flatness at the corner is given by computing the trace in (83), so we have $u \in H_f^K$. □

Remark. As in the situation where an infinite number of phases was present in the WKB expansion (see the example in Section 3D), we can remark that when the source term g is in $\mathscr{C}_c^\infty$ with its support away from the corner, if we restrict ourselves to the construction of the WKB expansion for a finite time $T < +\infty$, then the number of nonzero terms in the sum (81) is finite. Thus, in this framework the operator $(I - \mathbb{T})$ is automatically invertible (independently of the parameters β and S). Its inverse is given by (81).

Theorem 4.30 seems to indicate that a corner concentration phenomenon is more difficult to handle with than a separation from the corner phenomenon. Indeed, if $\beta < 1$, the error between the exact solution and the truncated WKB expansion is $O(\varepsilon^N)$ with N arbitrarily large, whereas if $\beta > 1$, the norms of the derivatives of the solution to (75) seem to get larger and larger. To prevent this "blow up", we have made the assumption $|S|\beta^{K-\frac{1}{2}} < 1$ which "implies" that there exists a maximal N_0 such that the error is $O(\varepsilon^{N_0})$. We do not claim that, for $\beta > 1$, Theorem 4.30 is sharp. But it is sufficient to treat the example of Section 3E.

4E. *Examples for which the invertibility condition (75) is satisfied.*

4E1. *The example of Section 3E.* We come back to the corner problem (16) and more precisely to the resolution of the amplitude equation (75) for this problem.

First of all we have to specify the chosen boundary conditions. We set for B_1 and B_2 in (16)

$$B_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 & -\delta & 1 \end{bmatrix}, \tag{84}$$

where $\delta \in \mathbb{R}$, $\delta \neq 0$, is a fixed parameter.

It is easy to see that the boundary condition defined by B_1 on the side $\partial\Omega_1$ is strictly dissipative; in particular, it satisfies the uniform Kreiss–Lopatinskii condition (see [Benzoni-Gavage and Serre 2007, Proposition 4.4]). The real parameter $\delta \neq 0$ encodes the dissipativity on the side $\partial\Omega_2$ in the following way:

- If $|\delta| > 2^{-\frac{1}{4}}$, the boundary condition defined by B_2 is strictly dissipative.
- If $|\delta| = 2^{-\frac{1}{4}}$, the boundary condition defined by B_2 is maximal dissipative.
- If $0 < |\delta| < 2^{-\frac{1}{4}}$, the boundary condition defined by B_2 is not dissipative but it satisfies the uniform Kreiss–Lopatinskii condition.

Reiterating the same kind of computations as those described in Section 4B2, using the fact that $\dim \mathscr{L}(d\varphi_1) = 1$, shows that the amplitude equation for the amplitude associated with the phase φ_1 is scalar and is given by

$$u(t,0,x_2) = -\frac{\frac{1}{5} - 6\sqrt{2}}{(\sqrt{2}+1)\left(\frac{1}{5} - \sqrt{2}\right)\delta^2} u(t - \alpha x_2, 0, 28x_2) + G(t,x_2), \tag{85}$$

where u is a real-valued function, G is an explicitly computable, but nonrelevant, source term and α is also not relevant.

According to Theorem 4.28, if

$$|\delta| > \sqrt{\frac{\frac{1}{5} - 6\sqrt{2}}{(\sqrt{2}+1)\left(\frac{1}{5} - \sqrt{2}\right)\sqrt{28}}} := \delta_0 \approx 0.73$$

then the functional equation (85) admits a unique solution. We are thus able to construct the leading-order term of the geometric optics expansion for more parameters than those leading to strictly dissipative boundary conditions.

On the contrary, if

$$0 < |\delta| < \delta_0,$$

we are in a nondissipative framework, and according to Theorem 4.29, equation (85) admits a nonzero solution for $G = 0$, so the leading-order term in the WKB expansion is not determined in a unique way. This example shows that imposing the uniform Kreiss–Lopatinskii condition on each side of the boundary is not sufficient to construct the geometric optics expansion in a unique way. It seems to be a good argument in favor of the fact that the same situation is true for the strong well-posedness of the corner problem (18).

4E2. *The example of Sarason and Smoller* [1974]. In [Sarason and Smoller 1974], the authors construct an example of 4×4 a strictly hyperbolic operator whose section of the characteristic variety contains a loop. This example, with the example of Section 3E, constitute, to our knowledge, the only two examples of corner problems with a loop in the literature.

The main idea of the construction is a perturbation argument: we first choose a centered ellipse and we fix three points P_2, P_3, P_4 on this ellipse such that angle $\widehat{P_2 P_3 P_4}$ is a perpendicular angle and the group velocities are incoming-outgoing in P_3 and outgoing-incoming in P_2 and P_4. This choice determines

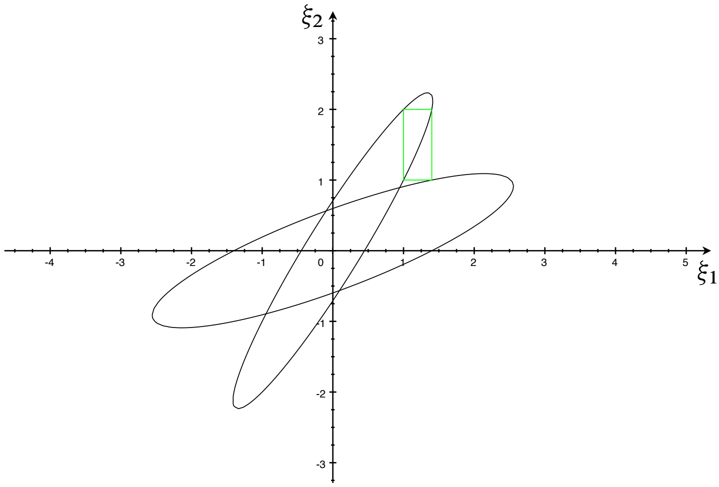

Figure 8. The construction of Sarason and Smoller for $p_1 := 5\xi_1^2 + 2\xi_2^2 - 6\xi_1\xi_2 - 1$ and $p_2 := \frac{25}{49}\xi_1^2 + \frac{14}{5}\xi_2^2 - 2\xi_1\xi_2 - 1$.

in a unique way a point P_1 such that $P_1P_2P_3P_4$ is a rectangle. Then we construct a second ellipse meeting P_1 with an incoming-outgoing group velocity at this point (see Figure 8).

The variety constructed can be written in the form

$$p_1(\tau, \xi_1, \xi_2)p_2(\tau, \xi_1, \xi_2) = 0,$$

where the polynomials p_1 and p_2 are homogeneous of degree two. This variety contains the loop (P_1, P_2, P_3, P_4), but it cannot represent the section at $\tau = 1$ of the characteristic variety of a strictly hyperbolic operator. Indeed, the two ellipses constructed previously intersect in four points, namely Q_1, Q_2, Q_3 and Q_4. However, it can be shown that it is the section at $\tau = 1$ of the characteristic variety of a geometrically regular hyperbolic operator with A_1 and A_2 of block diagonal form

$$A_1 := \begin{bmatrix} -a_1 & a_2 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ 0 & 0 & -\tilde{a}_1 & \tilde{a}_2 \\ 0 & 0 & \tilde{a}_2 & \tilde{a}_1 \end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix} -b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -\tilde{b} & 0 \\ 0 & 0 & 0 & \tilde{b} \end{bmatrix}$$

for suitable real parameters $a_1, a_2, \tilde{a}_1, \tilde{a}_2, b$ and $\tilde{b}$ (we refer to [Benoit 2015, paragraphe 6.9.6] or [Sarason and Smoller 1974] for more details about the construction of A_1 and A_2).

Once the operator $L(\partial)$ is constructed, we add the boundary conditions

$$B_1 u_{|x_1=0} := g^{\varepsilon}, \quad B_2 u_{|x_2=0} := 0,$$

where B_1 and B_2 are defined by

$$B_1 := \begin{bmatrix} 1 & 0 & 0 & -\delta \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 0 & 1 \end{bmatrix}. \tag{86}$$

It is easy to check, using the particular form of A_1 and A_2 and the fact that the boundary conditions (86) written for $\delta = 0$ are strictly dissipative, that B_1 and B_2 satisfy the uniform Kreiss–Lopatinskii condition for all δ. It is also easy to check that the boundary conditions B_1 and B_2 are strictly dissipative if δ is sufficiently small.

Now that boundary conditions are fixed, we want to study the invertibility condition $|S| < \sqrt{\beta}$, which appears when we construct the WKB expansion of the corner problem[6]

$$\begin{cases} L(\partial)u^\varepsilon = 0, \\ B_1 u^\varepsilon{}_{|x_1=0} = g^\varepsilon, \quad B_2 u^\varepsilon{}_{|x_2=0} = 0, \quad u^\varepsilon{}_{|t\leq 0} = 0, \end{cases} \tag{87}$$

where the source term is

$$g^\varepsilon = e^{\frac{i}{\varepsilon}(t+P_{1,2}x_2)} g,$$

with $g \in H_f^\infty$, zero for negative times, and where $P_1 := (P_{1,1}, P_{1,2})$.

The factor β only depends of the coefficients of the operator $L(\partial)$. In particular, it does not depend on δ and can be explicitly computed. The term S can be considered as a scalar (see Section 4D) and it is given by

$$S := S_1^{P_1} B_1 S_2^{P_2} B_2 S_1^{P_3} B_1 S_2^{P_4} B_2,$$

where we have made the amalgam between the indices of the loop and the associated frequencies.

Once again, using the fact that the operator $L(\partial)$ defined two 2×2 uncoupled systems, the stable subspace $E_2^s(i, P_{4,1})$ is given by $E_2^s(i, P_{4,1}) = \operatorname{vect}\{(0,0,p_4,q_4), (\tilde{p},\tilde{q},0,0)\}$. Thus, we can easily compute

$$S_2^{P_4} B_2 \begin{bmatrix} p_1 \\ q_1 \\ 0 \\ 0 \end{bmatrix} = \mathbb{P}^{P_4} \left(\begin{bmatrix} \frac{\tilde{p}}{\tilde{q}} q_1 \\ q_1 \\ 0 \\ 0 \end{bmatrix} + \delta \begin{bmatrix} 0 \\ 0 \\ \frac{p_4}{q_4}\left(q_1 \frac{\tilde{p}}{\tilde{q}} - p_1\right) \\ q_1 \frac{\tilde{p}}{\tilde{q}} - p_1 \end{bmatrix} \right) := \tilde{c}\delta \begin{bmatrix} 0 \\ 0 \\ p_4 \\ q_4 \end{bmatrix},$$

where $\tilde{c}$ is not zero and only depends of the projector $\mathbb{P}^{P_4}$ on $\ker(\mathscr{L}(\underline{\tau}, P_4))$ and we set $\ker(\mathscr{L}(\underline{\tau}, P_1)) = \operatorname{vect}\{(p_1, q_1, 0, 0)\}$. Repeating exactly the same arguments for the other terms composing S, it is easy to show that the invertibility condition (75) is equivalent to

$$c\delta^2 < \sqrt{\beta}, \quad c > 0.$$

This condition is not satisfied for large values of δ. Let us remark on the following points: firstly, the blow up phenomenon is in δ^2 and not in δ^4 as predicted in [Sarason and Smoller 1974]. Secondly, for $|\delta|$ small enough, the invertibility condition is satisfied and we are in the strictly dissipative framework.

Moreover, the condition $c\delta^2 < \sqrt{\beta}$ is more precise than the analogous condition of [Sarason and Smoller 1974]. Indeed it says that since we are working with L^2-norms, to prevent the signal from

[6] We do not try here to determine all the phases in the WKB expansion because, due to a concentration of the phases in the neighborhood of the intersection points of the ellipses (see Section 3E), the number of expected phases in the expansion will be infinite.

increasing in strength after a complete circuit, we must have that the amplification caused by the boundary condition is less than the contraction of the support of the source term after a complete circuit.

Finally, since we have $\beta > 1$ and S "scalar", Theorem 4.29 tells us that if the condition $c\delta^2 < \sqrt{\beta}$ is not satisfied then the WKB expansion cannot be constructed for any source term on the boundary. This conclusion is, in fact, worse than the conclusion of Sarason and Smoller, which says that in this case the corner problem is poorly posed (see [Sarason and Smoller 1974, Definition 4.3]).

To construct a strictly hyperbolic corner problem whose characteristic variety contains a loop, Sarason and Smoller use a perturbation argument. More precisely, they introduce a small coupling between the two 2×2 uncoupled systems defining $L(\partial)$ constructed in such a way to "pull apart" the two intersecting ellipses of the characteristic variety; see Sections 7 and 9 of [Sarason and Smoller 1974]. They show that if the perturbation is small enough then the obtained corner problem is strictly hyperbolic and the boundary conditions defined by B_1 and B_2 satisfy the uniform Kreiss–Lopatinskii condition, and finally that the perturbed system admits a loop in the section of its characteristic variety.

5. Conclusion

In this article we have shown a theorem which gives a rigorous geometric optics expansion for a hyperbolic corner problem when the number of phases generated by reflections is finite, but our theorem is general enough to apply to problems involving self-interacting phases. For such problems, a sequence of propagation of, at least, four fixed group velocities is repeated ad vitam aeternam. The construction of the geometric optics expansion then needs the solvability of a new amplitude equation, which is an invertibility condition in the spirit of Osher's condition [1973](see Assumption 4.15).

Of course, the construction given in this article can also be made if the section of characteristic variety does not contain any loop. Without any surprise, in that framework the construction is much simpler because the results in Sections 4B1 and 4B4 are more or less immediate and we can construct the expansion as it has been done in [Benoit 2015, paragraphes 6.4–6.6]. Moreover, the construction can also be adapted if the source term on the boundary does not turn on a self-interacting phase but if this phase appears after several reflections.

We also think that it should be possible to show a version of Theorem 4.27 without the assumption of the nonappearance of glancing modes during the phase generation process. Indeed, if one starts with a hyperbolic frequency, then nothing ensures that after several reflections a glancing mode will not appear in the phase generation process. However, thanks to [Williams 1996; 2000], we think that, after the suitable modifications of the oscillating scales in the ansatz, glancing modes will, more or less, behave like evanescent modes in the sense that they will create boundary layers in the expansion and that they will not be reflected from one side to the other.

The proof of Theorem 4.27 should also work when there are several loops. There are two cases to distinguish: first, if there is still a unique loop of interaction but other loops that are not interaction loops and, second, the case where there is more than one interaction loop. In the first case, the proof of Theorem 4.27 just has to be a bit adapted in a more technical way. In the second case, we think that the proof of Theorem 4.27 can also be adapted as long as the interaction loops do not intersect themselves, but this is left for future work.

Finally, let us give some comments about Assumption 4.9. On one hand, this restriction is really anecdotal because in the phase generation process described in Section 3 there is no mathematical reason to systematically meet one of the axes $\{\xi_1 = 0\}$ or $\{\xi_2 = 0\}$. The process depends on the geometry of the characteristic variety and on the phase of the source term. Moreover, it is easy to see that Assumption 4.9 is clearly not sharp (but it has the advantage of simplicity). Indeed what we really need in our construction is the following sharper version of Assumption 4.9: Let $\underline{i} \in \mathscr{I}_{\text{os}}$ be the index associated to a frequency given by $(\underline{\tau}, 0, \xi_2^k)$ (resp. $(\underline{\tau}, \xi_1^k, 0)$) and let Γ be defined by

$$\Gamma := \{i \in \mathscr{I} \setminus \underline{i} \underset{V}{\rightarrowtail} i\},$$
$$(\text{resp. } \Gamma := \{i \in \mathscr{I} \setminus \underline{i} \underset{H}{\rightarrowtail} i\}).$$

Then $\Gamma \cap \mathscr{I}_{\text{ev}1}$ (resp. $\Gamma \cap \mathscr{I}_{\text{ev}2}$) is empty. However, this new version seems not to be really less restrictive than Assumption 4.9 as soon as we have some kinds of symmetries in the characteristic variety.

But on the other hand, and in the author's opinion, an interesting question may be to consider WKB expansions without Assumption 4.9. Indeed in such a framework one of the boundary conditions is given by, for example,

$$B_1\Bigg[\sum_{k \in \mathscr{I}_{\text{os}}} u_{0,k} + \sum_{k \in \mathfrak{R}_2} U_{0,k,2}\Bigg]_{|x_1=0} = \mathscr{G},$$

and it seems to be not so easy to adapt the previous construction when such a boundary condition occurs because the resolution described in this paper is mainly based on the fact that we are able to determine oscillating phases before evanescent ones. That is why the construction without Assumption 4.9 is left for future studies.

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THE INTERIOR C^2 ESTIMATE FOR THE MONGE–AMPÈRE EQUATION IN DIMENSION $n = 2$

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We introduce a new auxiliary function, and establish the interior C^2 estimate for the Monge–Ampère equation in dimension $n = 2$, which was first proved by Heinz by a geometric method.

1. Introduction

We consider the convex solution of the Monge–Ampère equation

$$\det D^2 u = f(x) \quad \text{in } B_R(0) \subset \mathbb{R}^2. \tag{1-1}$$

When the solution u is convex, (1-1) is elliptic. It is well known that the interior C^2 estimate is an important problem for elliptic equations. For the Monge–Ampère equation in dimension $n = 2$, the corresponding interior C^2 estimate was established by Heinz [1959], and for higher dimensions $n \geq 3$ Pogorelov [1978] constructed his famous counterexample, namely irregular solutions to Monge–Ampère equations.

Later, Urbas [1990] generalized the counterexample for σ_k Hessian equations with $k \geq 3$. So the interior C^2 estimate of the σ_2 Hessian equation

$$\sigma_2(D^2 u) = f \quad \text{in } B_R(0) \subset \mathbb{R}^n \tag{1-2}$$

is an interesting problem, where $\sigma_2(D^2u) = \sigma_2(\lambda(D^2u)) = \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1}\lambda_{i_2}$, the eigenvalues of D^2u are $\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)$, and $f > 0$. For $n = 2$, (1-2) is the Monge–Ampère equation (1-1). For $n = 3$ and $f \equiv 1$, (1-2) can be viewed as a special Lagrangian equation, and Warren and Yuan [2009] obtained the corresponding interior C^2 estimate in their celebrated paper. Moreover, the problem is still open for general f with $n \geq 4$ and nonconstant f with $n = 3$.

Moreover, Pogorelov-type estimates for the Monge–Ampère equations and the σ_k Hessian equation ($k \geq 2$) were derived by Pogorelov [1978] and Chou and Wang [2001], respectively, and see [Guan et al. 2015; Li et al. 2016] for some generalizations.

In this paper, we introduce a new auxiliary function, and establish the interior C^2 estimate as follows:

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MSC2010: 35B45, 35B65, 35J96.
Keywords: interior C^2 a priori estimate, Monge–Ampère equation, σ_2 Hessian equation, optimal concavity.

Theorem 1.1. *Let $u \in C^4(B_R(0))$ be a convex solution of the Monge–Ampère equation (1-1) in dimension $n = 2$, where $0 < m \le f \le M$ in $B_R(0)$. Then*

$$|D^2u(0)| \le C_1 e^{C_2 \sup |Du|^2/R^2}, \tag{1-3}$$

where C_1 is a positive constant depending only on m, M, $R \sup |\nabla f|$ and $R^2 \sup |\nabla^2 f|$, and C_2 is a positive constant depending only on m and M.

Remark 1.2. By Trudinger's gradient estimates [1997], we can bound $|D^2u(0)|$ in terms of u. In fact, we get from the convexity of u that

$$\sup_{B_{R/2}(0)} |Du| \le \frac{\operatorname{osc}_{B_R(0)} u}{R/2} \le \frac{4 \sup_{B_R(0)} |u|}{R}$$

and

$$|D^2u(0)| \le C_1 e^{C_2 \sup_{B_{R/2}(0)} |Du|^2/(R/2)^2} \le C_1 e^{16 C_2 \sup_{B_R(0)} |u|^2/R^4}. \tag{1-4}$$

Remark 1.3. The result was first proved by Heinz [1959]. In fact, Heinz's proof depends on the strict convexity of solutions and the geometry of convex hypersurfaces in dimension two. Our proof, which is based on a suitable choice of auxiliary functions, is elementary and avoids geometric computations on the graph of solutions.

Remark 1.4. The interior C^2 estimate of the σ_2 Hessian equation (1-2) in higher dimensions is a longstanding problem. As we all know, it is hard to find a corresponding geometry in higher dimensions, so we cannot generalize Heinz's proof or Warren and Yuan's proof to higher dimensions. But the method in this paper and the optimal concavity in [Chen 2013] is helpful for this problem.

The rest of the paper is organized as follows. In Section 2, we give the calculations of the derivatives of eigenvalues and eigenvectors with respect to the matrix. In Section 3, we introduce a new auxiliary function, and prove Theorem 1.1.

2. Derivatives of eigenvalues and eigenvectors

In this section, we give the calculations of the derivatives of eigenvalues and eigenvectors with respect to the matrix. We expect the following result is known to many people; for example see [Andrews 2007] for a similar result. For completeness, we give the result and a detailed proof.

Proposition 2.1. *Let $W = \{W_{ij}\}$ be an $n \times n$ symmetric matrix with eigenvalues $\lambda(W) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and with corresponding continuous eigenvector field $\tau^i = (\tau^i_1, \dots, \tau^i_n) \in \mathbb{S}^{n-1}$. Suppose that $W = \{W_{ij}\}$ is diagonal with $\lambda_i = W_{ii}$ and corresponding eigenvector $\tau^i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{S}^{n-1}$ with the 1 in the i-th slot. If λ_k is distinct from the other eigenvalues, then we have, at the diagonal matrix W,*

$$\frac{\partial \tau^k_i}{\partial W_{pq}} = \begin{cases} 0 & \text{if } i = k \text{ for all } p, q, \\ \dfrac{1}{\lambda_k - \lambda_i} & \text{if } i \ne k,\ p = i,\ q = k, \\ 0 & \text{otherwise}; \end{cases} \tag{2-1}$$

$$\frac{\partial^2 \tau_k^k}{\partial W_{pk}\partial W_{pk}} = -\frac{1}{(\lambda_k-\lambda_p)^2} \quad \textit{if } p \neq k; \tag{2-2}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{ik}\partial W_{ii}} = \frac{1}{(\lambda_k-\lambda_i)^2} \quad \textit{if } i \neq k; \qquad \frac{\partial^2 \tau_i^k}{\partial W_{ik}\partial W_{kk}} = -\frac{1}{(\lambda_k-\lambda_i)^2} \quad \textit{if } i \neq k; \tag{2-3}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{iq}\partial W_{qk}} = \frac{1}{\lambda_k-\lambda_i}\frac{1}{\lambda_k-\lambda_q} \quad \textit{if } i \neq k,\ i \neq q,\ q \neq k; \tag{2-4}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{pq}\partial W_{rs}} = 0 \quad \textit{otherwise}. \tag{2-5}$$

Proof. From the definitions of eigenvalue and eigenvector of a matrix W, we have

$$(W-\lambda_k I)\tau^k \equiv 0,$$

where τ^k is the eigenvector of W corresponding to the eigenvalue λ_k. That is, for $i=1,\dots,n$,

$$(W_{ii}-\lambda_k)\tau_i^k + \sum_{j\neq i} W_{ij}\tau_j^k = 0. \tag{2-6}$$

When $W=\{W_{ij}\}$ is diagonal and λ_k is distinct from the other eigenvalues, λ_k and τ^k are C^2 at the matrix W. In fact,

$$\tau_k^k = 1 \quad \text{and} \quad \tau_i^k = 0 \ \text{ for } \ i \neq k \quad \text{at } \ W. \tag{2-7}$$

Taking the first derivative of (2-6), we have

$$\left(\frac{\partial W_{ii}}{\partial W_{pq}} - \frac{\partial \lambda_k}{\partial W_{pq}}\right)\tau_i^k + (W_{ii}-\lambda_k)\frac{\partial \tau_i^k}{\partial W_{pq}} + \sum_{j\neq i}\left(\frac{\partial W_{ij}}{\partial W_{pq}}\tau_j^k + W_{ij}\frac{\partial \tau_j^k}{\partial W_{pq}}\right) = 0.$$

Hence, for $i=k$, we get, from (2-7),

$$\frac{\partial \lambda_k}{\partial W_{pq}} = \frac{\partial W_{kk}}{\partial W_{pq}} = \begin{cases} 1 & \text{if } \ p=k,\ q=k, \\ 0 & \text{otherwise,} \end{cases} \tag{2-8}$$

and, for $i \neq k$,

$$(W_{ii}-\lambda_k)\frac{\partial \tau_i^k}{\partial W_{pq}} + \sum_{j\neq i}\frac{\partial W_{ij}}{\partial W_{pq}}\tau_j^k = 0,$$

then

$$\frac{\partial \tau_i^k}{\partial W_{pq}} = \frac{1}{\lambda_k-\lambda_i}\frac{\partial W_{ik}}{\partial W_{pq}} = \begin{cases} \dfrac{1}{\lambda_k-\lambda_i} & \text{if } \ p=i,\ q=k, \\ 0 & \text{otherwise.} \end{cases} \tag{2-9}$$

Since $\tau^k \in \mathbb{S}^{n-1}$, we have

$$1 = |\tau^k|^2 = (\tau_1^k)^2 + \cdots + (\tau_k^k)^2 + \cdots + (\tau_n^k)^2. \tag{2-10}$$

Taking the first derivative of (2-10), and using (2-7),

$$\frac{\partial \tau_k^k}{\partial W_{pq}} = 0 \quad \text{for all } (p,q). \tag{2-11}$$

For $i = k$, taking the second derivative of (2-6), and using (2-7),

$$\left(\frac{\partial^2 W_{kk}}{\partial W_{pq}\partial W_{rs}} - \frac{\partial^2 \lambda_k}{\partial W_{pq}\partial W_{rs}}\right)\tau_k^k + \sum_{j\neq k}\left(\frac{\partial W_{kj}}{\partial W_{pq}}\frac{\partial \tau_j^k}{\partial W_{rs}} + \frac{\partial W_{kj}}{\partial W_{rs}}\frac{\partial \tau_j^k}{\partial W_{pq}}\right) = 0,$$

hence

$$\frac{\partial^2 \lambda_k}{\partial W_{pq}\partial W_{rs}} = \sum_{j\neq k}\left(\frac{\partial W_{kj}}{\partial W_{pq}}\frac{\partial \tau_j^k}{\partial W_{rs}} + \frac{\partial W_{kj}}{\partial W_{rs}}\frac{\partial \tau_j^k}{\partial W_{pq}}\right) = \begin{cases} \dfrac{1}{\lambda_k - \lambda_q} & \text{if } p = k,\ q \neq k,\ r = q,\ s = k, \\ \dfrac{1}{\lambda_k - \lambda_s} & \text{if } r = k,\ s \neq k,\ p = s,\ q = k, \\ 0 & \text{if otherwise.} \end{cases} \tag{2-12}$$

For $i \neq k$,

$$\left(\frac{\partial W_{ii}}{\partial W_{pq}} - \frac{\partial \lambda_k}{\partial W_{pq}}\right)\frac{\partial \tau_i^k}{\partial W_{rs}} + \left(\frac{\partial W_{ii}}{\partial W_{rs}} - \frac{\partial \lambda_k}{\partial W_{rs}}\right)\frac{\partial \tau_i^k}{\partial W_{pq}} + (W_{ii} - \lambda_k)\frac{\partial^2 \tau_i^k}{\partial W_{pq}\partial W_{rs}} + \sum_{j\neq i}\left(\frac{\partial W_{ij}}{\partial W_{pq}}\frac{\partial \tau_j^k}{\partial W_{rs}} + \frac{\partial W_{ij}}{\partial W_{rs}}\frac{\partial \tau_j^k}{\partial W_{pq}}\right) = 0,$$

then

$$\frac{\partial^2 \tau_i^k}{\partial W_{ik}\partial W_{ii}} = \frac{1}{\lambda_k - \lambda_i}\frac{\partial \tau_i^k}{\partial W_{ik}} = \frac{1}{\lambda_k - \lambda_i}\frac{1}{\lambda_k - \lambda_i} \qquad \text{if } i \neq k; \tag{2-13}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{iq}\partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i}\frac{\partial W_{iq}}{\partial W_{iq}}\frac{\partial \tau_q^k}{\partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i}\frac{1}{\lambda_k - \lambda_q} \qquad \text{if } i \neq k,\ i \neq q,\ q \neq k; \tag{2-14}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{ik}\partial W_{kk}} = \frac{1}{\lambda_k - \lambda_i}\left(-\frac{\partial \lambda_k}{\partial W_{kk}}\frac{\partial \tau_i^k}{\partial W_{ik}}\right) = -\frac{1}{\lambda_k - \lambda_i}\frac{1}{\lambda_k - \lambda_i} \qquad \text{if } i \neq k; \tag{2-15}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{pq}\partial W_{rs}} = 0 \qquad \text{otherwise.} \tag{2-16}$$

From (2-10), we have

$$2\tau_k^k \frac{\partial^2 \tau_k^k}{\partial W_{pq}\partial W_{rs}} + 2\sum_{i\neq k}\frac{\partial \tau_i^k}{\partial W_{pq}}\frac{\partial \tau_i^k}{\partial W_{rs}} = 0,$$

then

$$\frac{\partial^2 \tau_k^k}{\partial W_{pq}\partial W_{rs}} = -\sum_{i\neq k}\frac{\partial \tau_i^k}{\partial W_{pq}}\frac{\partial \tau_i^k}{\partial W_{rs}} = \begin{cases} -\dfrac{1}{\lambda_k - \lambda_p}\dfrac{1}{\lambda_k - \lambda_p} & \text{if } p \neq k,\ q = k,\ r = p,\ s = q, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Proposition 2.1 is finished. $\square$

Example 2.2. When $n=2$, the matrix $\left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{smallmatrix}\right)$ has two eigenvalues

$$\lambda_1 = \frac{(u_{11}+u_{22}) + \sqrt{(u_{11}-u_{22})^2 + 4u_{12}u_{21}}}{2} \quad \text{and} \quad \lambda_2 = \frac{(u_{11}+u_{22}) - \sqrt{(u_{11}-u_{22})^2 + 4u_{12}u_{21}}}{2}$$

with $\lambda_1 \geq \lambda_2$. If $\lambda_1 > \lambda_2$,

$$\left(\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} - \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0,$$

we get

$$\xi_1 = \frac{(u_{22}-u_{11}) - \sqrt{(u_{11}-u_{22})^2 + 4u_{12}u_{21}}}{2} \quad \text{and} \quad \xi_2 = -u_{21}.$$

Then the eigenvector τ corresponding to λ_1 is

$$\tau = -\frac{(\xi_1, \xi_2)}{\sqrt{{\xi_1}^2 + {\xi_2}^2}}.$$

By direct calcutions, we can get the derivatives of eigenvalues and the eigenvector τ with respect to the matrix, and this verifies Proposition 2.1.

3. Proof of Theorem 1.1

Now we start to prove Theorem 1.1.

Let $\tau(x) = \tau(D^2u(x)) = (\tau_1, \tau_2) \in \mathbb{S}^1$ be the continuous eigenvector field of $D^2u(x)$ corresponding to the largest eigenvalue. Let

$$\Sigma := \left\{ x \in B_R(0) : r^2 - |x|^2 + \langle x, \tau(x)\rangle^2 > 0,\ r^2 - \langle x, \tau(x)\rangle^2 > 0 \right\}, \tag{3-1}$$

where $r = R/\sqrt{2}$. It is easy to show that Σ is an open set and $B_r(0) \subset \Sigma \subset B_R(0)$. We introduce a new auxiliary function in Σ as follows:

$$\phi(x) = \eta(x)^\beta g\left(\tfrac{1}{2}|Du|^2\right) u_{\tau\tau}, \tag{3-2}$$

where $\eta(x) = \left(r^2 - |x|^2 + \langle x, \tau(x)\rangle^2\right)\left(r^2 - \langle x, \tau(x)\rangle^2\right)$ with $\beta = 4$ and $g(t) = e^{c_0 t/r^2}$ with $c_0 = 32/m$. In fact, $\langle x, \tau(x)\rangle$ is invariant under rotations of the coordinates, and so is $\eta(x)$.

From the definition of Σ, we know $\eta(x) > 0$ in Σ, and $\eta = 0$ on $\partial\Sigma$. Assume the maximum of $\phi(x)$ in Σ is attained at $x_0 \in \Sigma$. By rotating the coordinates, we can assume $D^2u(x_0)$ is diagonal. In the following, we let $\lambda_i = u_{ii}(x_0)$, $\lambda = (\lambda_1, \lambda_2)$. Without loss of generality, we can assume $\lambda_1 \geq \lambda_2$ and $\tau(x_0) = (1, 0)$.

If

$$\eta\lambda_1 \leq 10^3 \left(1 + M + r \sup|\nabla f| + \frac{M}{m} \frac{\sup|Du|}{r} \right) r^4 =: \Theta,$$

then we easily get

$$\begin{aligned} u_{\tau(0)\tau(0)}(0) &\leq \frac{1}{r^{4\beta}} \phi(0) \leq \frac{1}{r^{4\beta}} \phi(x_0) \\ &\leq \Theta e^{c_0 \sup|Du|^2/r^2} \\ &\leq 10^3 (1 + M + r \sup|\nabla f|) e^{(c_0 + 2M/m) \sup|Du|^2/r^2}. \end{aligned}$$

Hence we get

$$|u_{\xi\xi}(0)| \le u_{\tau(0)\tau(0)}(0) \le 10^3(1+M+r\sup|\nabla f|)e^{(c_0+2M/m)\sup|Du|^2/r^2} \quad \text{for all } \xi \in \mathbb{S}^1.$$

This completes the proof of Theorem 1.1 under the condition $\eta\lambda_1 \le \Theta$.

Now, we assume $\eta\lambda_1 \ge \Theta$. Then we have

$$\lambda_1 = \frac{\eta\lambda_1}{\eta} \ge \frac{\Theta}{r^4} = 10^3\left(1+M+r\sup|\nabla f| + \frac{M}{m}\frac{\sup|Du|}{r}\right). \tag{3-3}$$

From (1-1), we have

$$\lambda_2 = \frac{f}{\lambda_1} \le \frac{M}{\lambda_1} < \lambda_1.$$

Hence λ_1 is distinct from the other eigenvalue, and $\tau(x)$ is C^2 at x_0. Moreover, the test function

$$\varphi = \beta\log\eta + \log g\left(\tfrac{1}{2}|Du|^2\right) + \log u_{11} \tag{3-4}$$

attains the local maximum at x_0. In the following, all the calculations are at x_0.

Then, we get

$$0 = \varphi_i = \beta\frac{\eta_i}{\eta} + \frac{g'}{g}\sum_k u_k u_{ki} + \frac{u_{11i}}{u_{11}},$$

so we have

$$\frac{u_{11i}}{u_{11}} = -\beta\frac{\eta_i}{\eta} - \frac{g'}{g}u_i u_{ii} \quad \text{if } i = 1, 2. \tag{3-5}$$

At x_0, we also have

$$\begin{aligned} 0 \ge \varphi_{ii} &= \beta\left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\right) + \frac{g''g - g'^2}{g^2}\sum_k u_k u_{ki}\sum_l u_l u_{li} + \frac{g'}{g}\sum_k (u_{ki}u_{ki} + u_k u_{kii}) + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} \\ &= \beta\left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\right) + \frac{g'}{g}\left(u_{ii}^2 + \sum_k u_k u_{kii}\right) + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2}, \end{aligned}$$

since $g''g - g'^2 = 0$. Let

$$F^{11} = \frac{\partial \det D^2 u}{\partial u_{11}} = \lambda_2, \quad F^{22} = \frac{\partial \det D^2 u}{\partial u_{22}} = \lambda_1,$$
$$F^{12} = \frac{\partial \det D^2 u}{\partial u_{12}} = 0, \quad F^{21} = \frac{\partial \det D^2 u}{\partial u_{21}} = 0.$$

Then from (1-1) we get

$$\lambda_2 = \frac{f}{\lambda_1}. \tag{3-6}$$

Differentiating (1-1) once, we get

$$F^{11}u_{11i} + F^{22}u_{22i} = f_i,$$

then

$$u_{22i} = \frac{1}{F^{22}}(f_i - F^{11}u_{11i}) = \frac{f_i}{\lambda_1} - \frac{f}{\lambda_1}\frac{u_{11i}}{u_{11}}. \tag{3-7}$$

Differentiating (1-1) twice, we get

$$\begin{aligned} F^{11}u_{1111}+F^{22}u_{2211} &= f_{11}-2\frac{\partial^2 \det D^2u}{\partial u_{11}\partial u_{22}}u_{111}u_{221}-2\frac{\partial^2 \det D^2u}{\partial u_{12}\partial u_{21}}u_{112}^2 \\ &= f_{11}-2u_{111}u_{221}+2u_{112}^2 \\ &= f_{11}+2u_{112}^2-2u_{111}\left(\frac{f_1}{\lambda_1}-\frac{f}{\lambda_1}\frac{u_{111}}{u_{11}}\right) \\ &= f_{11}+2u_{112}^2-2f_1\frac{u_{111}}{u_{11}}+2f\left(\frac{u_{111}}{u_{11}}\right)^2 \end{aligned} \tag{3-8}$$

and

$$\begin{aligned} F^{11}u_{1112}+F^{22}u_{2212} &= f_{12}-\frac{\partial^2 \det D^2u}{\partial u_{11}\partial u_{22}}u_{111}u_{222}-\frac{\partial^2 \det D^2u}{\partial u_{22}\partial u_{11}}u_{221}u_{112}-2\frac{\partial^2 \det D^2u}{\partial u_{12}\partial u_{21}}u_{121}u_{212} \\ &= f_{12}-u_{111}u_{222}-u_{112}u_{221}+2u_{112}u_{221} \\ &= f_{12}-u_{111}\left(\frac{f_2}{\lambda_1}-\frac{f}{\lambda_1}\frac{u_{112}}{u_{11}}\right)+u_{112}\left(\frac{f_1}{\lambda_1}-\frac{f}{\lambda_1}\frac{u_{111}}{u_{11}}\right) \\ &= f_{12}+f_1\frac{u_{112}}{u_{11}}-f_2\frac{u_{111}}{u_{11}}. \end{aligned} \tag{3-9}$$

Hence

$$\begin{aligned} 0 &\ge \sum_{i=1}^{2}F^{ii}\varphi_{ii} \\ &= \beta\sum_i F^{ii}\left(\frac{\eta_{ii}}{\eta}-\frac{\eta_i^2}{\eta^2}\right)+\frac{g'}{g}\sum_i F^{ii}u_{ii}^2+\frac{g'}{g}\sum_k u_k f_k+\frac{1}{u_{11}}\sum_i F^{ii}u_{11ii}-\sum_i F^{ii}\left(\frac{u_{11i}}{u_{11}}\right)^2 \\ &= \beta\lambda_2\left(\frac{\eta_{11}}{\eta}-\frac{\eta_1^2}{\eta^2}\right)+\beta\lambda_1\left(\frac{\eta_{22}}{\eta}-\frac{\eta_2^2}{\eta^2}\right)+\frac{g'}{g}(\lambda_1+\lambda_2)f+\frac{g'}{g}(u_1f_1+u_2f_2) \\ &\qquad +\frac{1}{u_{11}}\left(f_{11}+2u_{112}^2-2f_1\frac{u_{111}}{u_{11}}+2f\left(\frac{u_{111}}{u_{11}}\right)^2\right)-\lambda_2\left(\frac{u_{111}}{u_{11}}\right)^2-\lambda_1\left(\frac{u_{112}}{u_{11}}\right)^2 \\ &\ge \beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta}+\lambda_1\frac{\eta_{22}}{\eta}\right)-\beta\frac{f}{\lambda_1}\frac{\eta_1^2}{\eta^2}-\beta\lambda_1\frac{\eta_2^2}{\eta^2}+\frac{f}{\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2-2\frac{f_1}{u_{11}}\frac{u_{111}}{u_{11}} \\ &\qquad +\frac{\lambda_1}{2}\left(\frac{u_{112}}{u_{11}}\right)^2+\frac{\lambda_1}{2}\left(\beta\frac{\eta_2}{\eta}+\frac{g'}{g}u_2u_{22}\right)^2+\frac{g'}{g}f\lambda_1-\frac{g'}{g}|\nabla u||\nabla f|-\frac{|f_{11}|}{\lambda_1} \\ &\ge \beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta}+\lambda_1\frac{\eta_{22}}{\eta}\right)-\beta\frac{f}{\lambda_1}\frac{\eta_1^2}{\eta^2}-\beta\lambda_1\frac{\eta_2^2}{\eta^2}+\frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2 \\ &\qquad +\frac{\lambda_1}{2}\left(\frac{u_{112}}{u_{11}}\right)^2+\frac{\beta^2}{2}\lambda_1\frac{\eta_2^2}{\eta^2}+\beta f\frac{g'}{g}\frac{\eta_2}{\eta}u_2+\frac{g'}{g}f\lambda_1-\frac{g'}{g}|\nabla u||\nabla f|-\frac{|f_{11}|}{\lambda_1}-2\frac{f_1^2}{f\lambda_1}. \end{aligned} \tag{3-10}$$

Lemma 3.1. *Under the condition $\eta\lambda_1\ge\Theta$ we have, at x_0,*

$$\beta\frac{f}{\lambda_1}\frac{\eta_1^2}{\eta^2}\le\frac{8\beta f r^6}{\eta^2\lambda_1}+\frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2 \quad \textit{and} \quad \beta f\frac{g'}{g}\frac{\eta_2}{\eta}u_2\ge -4\beta f\frac{g'}{g}\frac{r^4}{\eta}\frac{|u_2|}{r}, \tag{3-11}$$

and

$$\beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta}+\lambda_1\frac{\eta_{22}}{\eta}\right)\geq -\frac{1}{2}\frac{g'}{g}f\lambda_1-\beta\lambda_1\left(\frac{\eta_2}{\eta}\right)^2-\frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2-\frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2-2\beta f\frac{r^2}{\eta\lambda_1}$$
$$-4\beta|f_{12}|\frac{r^4}{\eta\lambda_1}-32\beta\frac{f_1^2r^4}{\eta\lambda_1^3}-8\beta\frac{|f_1f_2|r^4}{\eta\lambda_1^3}-24\beta\frac{|f_1|r^3}{\eta\lambda_1}-\left(\frac{6\beta|f_1|^2r^4}{\eta\lambda_1}+\frac{12\beta f r^2}{\eta\lambda_1}\right)$$
$$-\left(\frac{8\beta f r^4|f_2|^2}{\eta\lambda_1^3}+\frac{(48\beta)^2fr^6}{\eta^2\lambda_1}+\frac{32\beta^2|f_2|^2r^8}{\eta^2\lambda_1 f}\right). \tag{3-12}$$

Proof. At $x_0,\ \tau=(\tau_1,\tau_2)=(1,0)$. Then from Proposition 2.1 we get

$$\langle x,\partial_i\tau\rangle=\sum_{m=1}^{2}x_m\frac{\partial\tau_m}{\partial x_i}=\sum_{m=1}^{2}x_m\frac{\partial\tau_m}{\partial u_{pq}}u_{pqi}=x_2\frac{\partial\tau_2}{\partial u_{pq}}u_{pqi}=x_2\frac{u_{12i}}{\lambda_1-\lambda_2}\quad\text{if } i=1,2. \tag{3-13}$$

From the definition of η, then we have, at x_0,

$$\eta=(r^2-|x|^2+\langle x,\tau\rangle^2)(r^2-\langle x,\tau\rangle^2)=(r^2-x_2^2)(r^2-x_1^2). \tag{3-14}$$

Taking the first derivative of η, we get

$$\begin{aligned}\eta_i&=(-2x_i+2\langle x,\tau\rangle\langle x,\tau\rangle_i)(r^2-\langle x,\tau\rangle^2)+(r^2-|x|^2+\langle x,\tau\rangle^2)(-2\langle x,\tau\rangle\langle x,\tau\rangle_i)\\ &=\big(-2x_i+2x_1(\delta_{i1}+\langle x,\partial_i\tau\rangle)\big)(r^2-x_1^2)+(r^2-x_2^2)\big(-2x_1(\delta_{i1}+\langle x,\partial_i\tau\rangle)\big)\\ &=\begin{cases}-2x_1(r^2-x_2^2)+2x_1\langle x,\partial_1\tau\rangle(x_2^2-x_1^2) & \text{if } i=1,\\ -2x_2(r^2-x_1^2)+2x_1\langle x,\partial_2\tau\rangle(x_2^2-x_1^2) & \text{if } i=2.\end{cases}\end{aligned}$$

Hence

$$\begin{aligned}\beta\frac{f}{\lambda_1}\frac{\eta_1^2}{\eta^2}&=\beta\frac{f}{\lambda_1}\left(\frac{-2x_1(r^2-x_2^2)}{\eta}+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{u_{112}}{\lambda_1-\lambda_2}\right)^2\\ &\leq\beta\frac{f}{\lambda_1}\left(\frac{8r^6}{\eta^2}+\frac{8r^8}{\eta^2}\left(\frac{u_{112}}{u_{11}}\right)^2\right)\leq\frac{8\beta f r^6}{\eta^2\lambda_1}+\frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2.\end{aligned} \tag{3-15}$$

Also we have

$$\begin{aligned}\frac{\eta_2}{\eta}&=\frac{-2x_2(r^2-x_1^2)}{\eta}+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{u_{221}}{\lambda_1-\lambda_2}\\ &=\frac{-2x_2(r^2-x_1^2)}{\eta}+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{1}{\lambda_1-\lambda_2}\left(\frac{f_1}{\lambda_1}-\frac{f}{\lambda_1}\frac{u_{111}}{u_{11}}\right)\\ &=\frac{-2x_2(r^2-x_1^2)}{\eta}\left(1-x_1\frac{(x_2^2-x_1^2)(r^2-x_2^2)}{\eta}\frac{1}{\lambda_1-\lambda_2}\frac{f_1}{\lambda_1}\right)+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{1}{\lambda_1-\lambda_2}\frac{f}{\lambda_1}\left(\beta\frac{\eta_1}{\eta}+\frac{g'}{g}u_1u_{11}\right)\\ &=\frac{-2x_2(r^2-x_1^2)}{\eta}\left(1-x_1\frac{(x_2^2-x_1^2)(r^2-x_2^2)}{\eta}\frac{1}{\lambda_1-\lambda_2}\left(\frac{f_1}{\lambda_1}+\frac{g'}{g}u_1f\right)\right)\\ &\qquad+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{1}{\lambda_1-\lambda_2}\frac{f}{\lambda_1}\beta\left(\frac{-2x_1(r^2-x_2^2)}{\eta}+2x_1x_2\frac{x_2^2-x_1^2}{\eta}\frac{u_{112}}{\lambda_1-\lambda_2}\right)\end{aligned}$$

$$= \frac{-2x_2(r^2-x_1^2)}{\eta}\left(1 - x_1\frac{(x_2^2-x_1^2)(r^2-x_2^2)}{\eta}\frac{1}{\lambda_1-\lambda_2}\left(\frac{f_1}{\lambda_1}+\frac{g'}{g}u_1 f - \frac{f}{\lambda_1}\beta\frac{2x_1(r^2-x_2^2)}{\eta}\right)\right) + \left(2x_1x_2\frac{x_2^2-x_1^2}{\eta}\right)^2\frac{f}{(\lambda_1-\lambda_2)^2}\beta\left(-\beta\frac{\eta_2}{\eta}-\frac{g'}{g}u_2u_{22}\right),$$

then we get

$$\begin{aligned}&\left(1+\beta^2\left(2x_1x_2\frac{x_2^2-x_1^2}{\eta}\right)^2\frac{f}{(\lambda_1-\lambda_2)^2}\right)\frac{\eta_2}{\eta}\\ &\quad= \frac{-2x_2(r^2-x_1^2)}{\eta}\left(1 - x_1\frac{(x_2^2-x_1^2)(r^2-x_2^2)}{\eta}\frac{1}{\lambda_1-\lambda_2}\left(\frac{f_1}{\lambda_1}+\frac{g'}{g}u_1 f - \frac{f}{\lambda_1}\beta\frac{2x_1(r^2-x_2^2)}{\eta}\right)\right)\\ &\qquad + \frac{-2x_2(r^2-x_1^2)}{\eta}\frac{2x_1^2x_2(x_2^2-x_1^2)^2(r^2-x_2^2)}{\eta^2}\frac{f}{(\lambda_1-\lambda_2)^2}\beta\frac{g'}{g}u_2\frac{f}{\lambda_1}.\end{aligned} \tag{3-16}$$

Hence

$$\begin{aligned}\beta f\frac{g'}{g}\frac{\eta_2}{\eta}u_2 &\ge -\beta f\frac{g'}{g}\frac{2r^3}{\eta}\left(1+\frac{2r^5}{\eta\lambda_1}\left(\frac{|f_1|}{\lambda_1}+\frac{g'}{g}|u_1|f+\frac{2\beta f r^3}{\eta\lambda_1}\right)+\frac{16\beta r^9}{\eta^2}\frac{f^2}{\lambda_1^3}\frac{g'}{g}|u_2|\right)|u_2|\\ &\ge -\beta f\frac{g'}{g}\frac{2r^3}{\eta}\left(1+\frac14+\frac14+\frac14+\frac14\right)|u_2|\\ &= -4\beta f\frac{g'}{g}\frac{r^4}{\eta}\frac{|u_2|}{r}.\end{aligned} \tag{3-17}$$

In fact, $\eta_2/\eta \approx -2x_2(r^2-x_1^2)/\eta$ if $\eta\lambda_1$ is big enough, and we get from (3-16)

$$\begin{aligned}\frac{\eta_2^2}{\eta^2} &\ge \left(\left(1+\beta^2\left(2x_1x_2\frac{x_2^2-x_1^2}{\eta}\right)^2\frac{f}{(\lambda_1-\lambda_2)^2}\right)\frac{\eta_2}{\eta}\right)^2\left(1-\beta^2\left(2x_1x_2\frac{x_2^2-x_1^2}{\eta}\right)^2\frac{f}{(\lambda_1-\lambda_2)^2}\right)^2\\ &\ge \left(\frac{-2x_2(r^2-x_1^2)}{\eta}\right)^2\left(1-\frac{2r^5}{\eta\lambda_1}\left(\frac{|f_1|}{\lambda_1}+\frac{g'}{g}|u_1|f+\frac{2\beta f r^3}{\eta\lambda_1}\right)-\frac{8\beta r^9}{\eta^2\lambda_1^2}\frac{g'}{g}|u_2|\frac{f^2}{\lambda_1}\right)^2\left(1-\beta^2\frac{16r^8}{\eta^2}\frac{f}{\lambda_1^2}\right)^2\\ &\ge \left(\frac{-2x_2(r^2-x_1^2)}{\eta}\right)^2\left(1-\frac{1}{10^3}-\frac{64}{10^3}-\frac{1}{10}-\frac{1}{10^3}\right)^2\left(1-\frac{1}{10^3}\right)^2\\ &\ge \frac12\left(\frac{-2x_2(r^2-x_1^2)}{\eta}\right)^2.\end{aligned}$$

Taking second derivatives of η, we get

$$\begin{aligned}\eta_{ii} &= (-2+2\langle x,\tau\rangle\langle x,\tau\rangle_{ii}+2\langle x,\tau\rangle_i\langle x,\tau\rangle_i)(r^2-\langle x,\tau\rangle^2)\\ &\quad + 2(-2x_i+2\langle x,\tau\rangle\langle x,\tau\rangle_i)(-2\langle x,\tau\rangle\langle x,\tau\rangle_i)\\ &\quad + (r^2-|x|^2+\langle x,\tau\rangle^2)(-2\langle x,\tau\rangle\langle x,\tau\rangle_{ii}-2\langle x,\tau\rangle_i\langle x,\tau\rangle_i)\\ &= \big(-2+2x_1\langle x,\tau\rangle_{ii}+2(\delta_{i1}+\langle x,\partial_i\tau\rangle)^2\big)(r^2-x_1^2)\\ &\quad + 2\big(-2x_i+2x_1(\delta_{i1}+\langle x,\partial_i\tau\rangle)\big)\big(-2x_1(\delta_{i1}+\langle x,\partial_i\tau\rangle)\big)\\ &\quad + (r^2-x_2^2)\big(-2x_1\langle x,\tau\rangle_{ii}-2(\delta_{i1}+\langle x,\partial_i\tau\rangle)^2\big),\end{aligned}$$

so

$$\eta_{11} = -2(r^2 - x_2^2) - 2x_1(x_1^2 - x_2^2)\langle x, \tau\rangle_{11} + (4x_2^2 - 12x_1^2)\langle x, \partial_1\tau\rangle + (2x_2^2 - 10x_1^2)\langle x, \partial_1\tau\rangle^2, \quad (3\text{-}18)$$

$$\eta_{22} = -2(r^2 - x_1^2) - 2x_1(x_1^2 - x_2^2)\langle x, \tau\rangle_{22} + 8x_1x_2\langle x, \partial_2\tau\rangle + (2x_2^2 - 10x_1^2)\langle x, \partial_2\tau\rangle^2. \quad (3\text{-}19)$$

Hence

$$\begin{aligned}\beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta} + \lambda_1\frac{\eta_{22}}{\eta}\right) = &-2\beta\left(\frac{f}{\lambda_1}\frac{r^2 - x_2^2}{\eta} + \lambda_1\frac{r^2 - x_1^2}{\eta}\right) - 2\beta\frac{x_1(x_1^2 - x_2^2)}{\eta}\left(\frac{f}{\lambda_1}\langle x, \tau\rangle_{11} + \lambda_1\langle x, \tau\rangle_{22}\right)\\ &+\beta\frac{f}{\lambda_1}\left(\frac{x_2(4x_2^2 - 12x_1^2)}{\eta}\frac{u_{112}}{\lambda_1 - \lambda_2} + \frac{x_2^2(2x_2^2 - 10x_1^2)}{\eta}\left(\frac{u_{112}}{\lambda_1 - \lambda_2}\right)^2\right)\\ &+\beta\lambda_1\left(\frac{8x_1x_2^2}{\eta}\frac{u_{221}}{\lambda_1 - \lambda_2} + \frac{x_2^2(2x_2^2 - 10x_1^2)}{\eta}\left(\frac{u_{221}}{\lambda_1 - \lambda_2}\right)^2\right). \end{aligned} \quad (3\text{-}20)$$

Direct calculations yield

$$\begin{aligned}\langle x, \tau\rangle_{11} &= \frac{\partial^2}{\partial x_1^2}\left(\sum_{m=1}^{2} x_m\tau_m\right) = 2\frac{\partial\tau_1}{\partial x_1} + \sum_{m=1}^{2} x_m\frac{\partial^2\tau_m}{\partial x_1^2}\\ &= 2\frac{\partial\tau_1}{\partial u_{pq}}u_{pq1} + \sum_{m=1}^{2} x_m\left(\frac{\partial\tau_m}{\partial u_{pq}}u_{pq11} + \frac{\partial^2\tau_m}{\partial u_{pq}\partial u_{rs}}u_{pq1}u_{rs1}\right)\\ &= 0 + x_1\frac{\partial^2\tau_1}{\partial u_{pq}\partial u_{rs}}u_{pq1}u_{rs1} + x_2\left(\frac{\partial\tau_2}{\partial u_{pq}}u_{pq11} + \frac{\partial^2\tau_2}{\partial u_{pq}\partial u_{rs}}u_{pq1}u_{rs1}\right)\\ &= -x_1\left(\frac{u_{112}}{\lambda_1 - \lambda_2}\right)^2 + x_2\left(\frac{1}{\lambda_1 - \lambda_2}\right)u_{1211} + 2x_2\left(-\frac{u_{112}u_{111}}{(\lambda_1 - \lambda_2)^2} + \frac{u_{112}u_{221}}{(\lambda_1 - \lambda_2)^2}\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\langle x, \tau\rangle_{22} &= \frac{\partial^2}{\partial x_2^2}\left(\sum_{m=1}^{2} x_m\tau_m\right) = 2\frac{\partial\tau_2}{\partial x_2} + \sum_{m=1}^{2} x_m\frac{\partial^2\tau_m}{\partial x_1^2}\\ &= 2\frac{\partial\tau_2}{\partial u_{pq}}u_{pq2} + \sum_{m=1}^{2} x_m\left(\frac{\partial\tau_m}{\partial u_{pq}}u_{pq22} + \frac{\partial^2\tau_m}{\partial u_{pq}\partial u_{rs}}u_{pq2}u_{rs2}\right)\\ &= 2\left(\frac{1}{\lambda_1 - \lambda_2}\right)u_{221} - x_1\left(\frac{u_{221}}{\lambda_1 - \lambda_2}\right)^2 + x_2\left(\frac{1}{\lambda_1 - \lambda_2}\right)u_{1222} + 2x_2\left(-\frac{u_{112}u_{221}}{(\lambda_1 - \lambda_2)^2} + \frac{u_{222}u_{221}}{(\lambda_1 - \lambda_2)^2}\right),\end{aligned}$$

then

$$\begin{aligned}\frac{f}{\lambda_1}\langle x, \tau\rangle_{11} + \lambda_1\langle x, \tau\rangle_{22} = &-x_1\frac{f}{\lambda_1}\left(\frac{u_{112}}{\lambda_1 - \lambda_2}\right)^2 + 2\lambda_1\left(\frac{u_{221}}{\lambda_1 - \lambda_2}\right) - x_1\lambda_1\left(\frac{u_{221}}{\lambda_1 - \lambda_2}\right)^2\\ &+ x_2\left(\frac{1}{\lambda_1 - \lambda_2}\right)\left(f_{12} + f_1\frac{u_{112}}{u_{11}} - f_2\left(\frac{u_{111}}{u_{11}}\right)\right)\\ &+ 2x_2\left(-\frac{u_{112}}{(\lambda_1 - \lambda_2)^2}f_1 + \frac{u_{221}}{(\lambda_1 - \lambda_2)^2}f_2\right).\end{aligned} \quad (3\text{-}21)$$

From (3-20) and (3-21), we get

$$
\begin{aligned}
&\beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta}+\lambda_1\frac{\eta_{22}}{\eta}\right)\\
&\quad= -2\beta\left(\frac{f}{\lambda_1}\frac{r^2-x_2^2}{\eta}+\lambda_1\frac{r^2-x_1^2}{\eta}\right)-2\beta\frac{x_1x_2(x_1^2-x_2^2)}{\eta}\left(\frac{1}{\lambda_1-\lambda_2}\right)\left(f_{12}-f_2\left(\frac{u_{111}}{u_{11}}\right)\right)\\
&\qquad+\left(\frac{u_{112}}{\lambda_1-\lambda_2}\right)^2\left(2\beta\frac{f}{\lambda_1}\frac{x_1^2(x_1^2-x_2^2)}{\eta}+\beta\frac{f}{\lambda_1}\frac{x_2^2(2x_2^2-10x_1^2)}{\eta}\right)\\
&\qquad+\frac{u_{112}}{\lambda_1-\lambda_2}\left(-2\beta\frac{x_1(x_1^2-x_2^2)}{\eta}\left(x_2\frac{f_1}{\lambda_1}-2x_2\frac{f_1}{\lambda_1-\lambda_2}\right)+\beta\frac{f}{\lambda_1}\frac{x_2(4x_2^2-12x_1^2)}{\eta}\right)\\
&\qquad+\left(\frac{u_{221}}{\lambda_1-\lambda_2}\right)^2\left(2\beta\lambda_1\frac{x_1^2(x_1^2-x_2^2)}{\eta}+\beta\lambda_1\frac{x_2^2(2x_2^2-10x_1^2)}{\eta}\right)\\
&\qquad+\frac{u_{221}}{\lambda_1-\lambda_2}\left(-2\beta\frac{x_1(x_1^2-x_2^2)}{\eta}\left(2\lambda_1+2x_2\frac{f_2}{\lambda_1-\lambda_2}\right)+\beta\lambda_1\frac{8x_1x_2^2}{\eta}\right)\\
&\quad\geq -2\beta\lambda_1\frac{r^2-x_1^2}{\eta}-2\beta f\frac{r^2}{\eta\lambda_1}-2\beta|f_{12}|\frac{r^4}{\eta(\lambda_1-\lambda_2)}-2\beta|f_2|\frac{r^4}{\eta(\lambda_1-\lambda_2)}\left|\frac{u_{111}}{u_{11}}\right|\\
&\qquad-\left(\frac{u_{112}}{(\lambda_1-\lambda_2)}\right)^2\beta\frac{f}{\lambda_1}\frac{8r^4}{\eta}-\left|\frac{u_{112}}{\lambda_1-\lambda_2}\right|\left(6\beta\frac{|f_1|}{\lambda_1-\lambda_2}\frac{r^4}{\eta}+\beta\frac{f}{\lambda_1}\frac{12r^3}{\eta}\right)\\
&\qquad-\frac{1}{(\lambda_1-\lambda_2)^2}(u_{221})^2\beta\lambda_1\frac{8r^4}{\eta}-\frac{1}{\lambda_1-\lambda_2}|u_{221}|\left(4\beta\frac{r^4}{\eta}\frac{|f_2|}{\lambda_1}+\beta\lambda_1\frac{12r^3}{\eta}\right)\\
&\quad\geq -2\beta\lambda_1\frac{r^2-x_1^2}{\eta}-2\beta f\frac{r^2}{\eta\lambda_1}-4\beta|f_{12}|\frac{r^4}{\eta\lambda_1}-4\beta|f_2|\frac{r^4}{\eta\lambda_1}\left|\frac{u_{111}}{u_{11}}\right|\\
&\qquad-\left(\frac{u_{112}}{u_{11}}\right)^2\beta\frac{f}{\lambda_1}\frac{16r^4}{\eta}-\left|\frac{u_{112}}{u_{11}}\right|\left(12\beta\frac{|f_1|}{\lambda_1}\frac{r^4}{\eta}+\beta\frac{f}{\lambda_1}\frac{24r^3}{\eta}\right)\\
&\qquad-2\left(\frac{f_1^2}{\lambda_1^4}+\frac{f^2}{\lambda_1^4}\left(\frac{u_{111}}{u_{11}}\right)^2\right)\beta\lambda_1\frac{16r^4}{\eta}-\left(\frac{|f_1|}{\lambda_1^2}+\frac{f}{\lambda_1^2}\left|\frac{u_{111}}{u_{11}}\right|\right)\left(8\beta\frac{r^4}{\eta}\frac{|f_2|}{\lambda_1}+\beta\lambda_1\frac{24r^3}{\eta}\right)\\
&\quad\geq -2\beta\lambda_1\frac{r^2-x_1^2}{\eta}-2\beta f\frac{r^2}{\eta\lambda_1}-4\beta|f_{12}|\frac{r^4}{\eta\lambda_1}-32\beta\frac{f_1^2r^4}{\eta\lambda_1^3}-8\beta\frac{|f_1f_2|r^4}{\eta\lambda_1^3}-24\beta\frac{|f_1|r^3}{\eta\lambda_1}\\
&\qquad-\frac{\lambda_1}{2}\left(\frac{u_{112}}{u_{11}}\right)^2\left(\frac{32\beta fr^4}{\eta\lambda_1^2}+\frac{12\beta r^4}{\eta\lambda_1^2}+\frac{24\beta fr^4}{\eta\lambda_1^2}\right)-\frac{\lambda_1}{2}\left(\frac{12\beta|f_1|^2r^4}{\eta\lambda_1^2}+\frac{24\beta fr^2}{\eta\lambda_1^2}\right)\\
&\qquad-\frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2\left(\frac{64\beta fr^4}{\eta\lambda_1^2}+\frac{16\beta r^4}{\eta\lambda_1^2}+\frac{1}{4}+\frac{1}{4}\right)\\
&\qquad-\frac{f}{2\lambda_1}\left(\frac{16\beta r^4|f_2|^2}{\eta\lambda_1^2}+\left(\frac{48\beta r^3}{\eta}\right)^2+\left(\frac{8\beta r^4}{\eta}\frac{|f_2|}{f}\right)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\geq -2\beta\lambda_1\frac{r^2-x_1^2}{\eta}-2\beta f\frac{r^2}{\eta\lambda_1}-4\beta|f_{12}|\frac{r^4}{\eta\lambda_1}-32\beta\frac{f_1^2r^4}{\eta\lambda_1^3}-8\beta\frac{|f_1f_2|r^4}{\eta\lambda_1^3}-24\beta\frac{|f_1|r^3}{\eta\lambda_1}\\
&\qquad -\frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2-\left(\frac{6\beta|f_1|^2r^4}{\eta\lambda_1}+\frac{12\beta fr^2}{\eta\lambda_1}\right)\\
&\qquad\quad -\frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2-\left(\frac{8\beta fr^4|f_2|^2}{\eta\lambda_1^3}+\frac{(48\beta)^2fr^6}{\eta^2\lambda_1}+\frac{32\beta^2|f_2|^2r^8}{\eta^2\lambda_1 f}\right).
\end{aligned}
$$

Now we just need to estimate $-2\beta\lambda_1(r^2-x_1^2)/\eta$. If $x_2^2\leq\frac{1}{2}r^2$, we get

$$
-2\beta\lambda_1\frac{r^2-x_1^2}{\eta}=-\frac{8}{r^2-x_2^2}\lambda_1\geq-\frac{16}{r^2}\lambda_1\geq-\frac{1}{2}\frac{c_0}{r^2}f\lambda_1=-\frac{1}{2}\frac{g'}{g}f\lambda_1.
$$

If $x_2^2\geq\frac{1}{2}r^2$, we get

$$
-2\beta\lambda_1\frac{r^2-x_1^2}{\eta}=-\frac{8}{r^2-x_2^2}\lambda_1\geq-\frac{x_2^2}{r^2-x_2^2}\frac{8}{r^2-x_2^2}\lambda_1=-\beta\lambda_1\frac{1}{2}\left(\frac{2x_2}{r^2-x_2^2}\right)^2\geq-\beta\lambda_1\left(\frac{\eta_2}{\eta}\right)^2.
$$

Hence

$$
-2\beta\lambda_1\frac{r^2-x_1^2}{\eta}\geq-\frac{1}{2}\frac{g'}{g}f\lambda_1-\beta\lambda_1\left(\frac{\eta_2}{\eta}\right)^2 \tag{3-22}
$$

and

$$
\begin{aligned}
\beta\left(\frac{f}{\lambda_1}\frac{\eta_{11}}{\eta}+\lambda_1\frac{\eta_{22}}{\eta}\right)\geq &-\frac{1}{2}\frac{g'}{g}f\lambda_1-\beta\lambda_1\left(\frac{\eta_2}{\eta}\right)^2-\frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2-\frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2\\
&-2\beta f\frac{r^2}{\eta\lambda_1}-4\beta|f_{12}|\frac{r^4}{\eta\lambda_1}-32\beta\frac{f_1^2r^4}{\eta\lambda_1^3}-8\beta\frac{|f_1f_2|r^4}{\eta\lambda_1^3}-24\beta\frac{|f_1|r^3}{\eta\lambda_1}\\
&\quad-\left(\frac{6\beta|f_1|^2r^4}{\eta\lambda_1}+\frac{12\beta fr^2}{\eta\lambda_1}\right)\\
&\qquad-\left(\frac{8\beta fr^4|f_2|^2}{\eta\lambda_1^3}+\frac{(48\beta)^2fr^6}{\eta^2\lambda_1}+\frac{32\beta^2|f_2|^2r^8}{\eta^2\lambda_1 f}\right).
\end{aligned} \tag{3-23}
$$

This concludes the proof of Lemma 3.1. □

Now we continue to prove Theorem 1.1. From (3-10) and Lemma 3.1, we get

$$
\begin{aligned}
0&\geq\sum_{i=1}^2F^{ii}\varphi_{ii}\\
&\geq\frac{1}{2}\frac{g'}{g}f\lambda_1-2\beta f\frac{r^2}{\eta\lambda_1}-4\beta|f_{12}|\frac{r^4}{\eta\lambda_1}-32\beta\frac{f_1^2r^4}{\eta\lambda_1^3}-8\beta\frac{|f_1f_2|r^4}{\eta\lambda_1^3}-24\beta\frac{|f_1|r^3}{\eta\lambda_1}\\
&\qquad-\left(\frac{6\beta|f_1|^2r^4}{\eta\lambda_1}+\frac{12\beta fr^2}{\eta\lambda_1}\right)-\left(\frac{8\beta fr^4|f_2|^2}{\eta\lambda_1^3}+\frac{(48\beta)^2fr^6}{\eta^2\lambda_1}+\frac{32\beta^2|f_2|^2r^8}{\eta^2\lambda_1 f}\right)\\
&\qquad\quad-4\beta f\frac{g'}{g}\frac{r^4}{\eta}\frac{|u_2|}{r}-\frac{g'}{g}|\nabla u||\nabla f|-\frac{|f_{11}|}{\lambda_1}-2\frac{f_1^2}{f\lambda_1}-\frac{8\beta fr^6}{\eta^2\lambda_1}
\end{aligned}
$$

$$\begin{aligned}\geq \frac{c_0 m}{2r^2}\lambda_1 &- \frac{8r^2\cdot M}{\eta\lambda_1} - \frac{32r^2\cdot r^2|f_{12}|}{\eta\lambda_1} - \frac{128r^2\cdot r^2 f_1^2}{\eta\lambda_1^3} - \frac{32r^2\cdot r^2|f_1 f_2|}{\eta\lambda_1^3} - \frac{96r^2\cdot r|f_1|}{\eta\lambda_1}\\ &- \frac{24r^2\cdot r^2|f_1|^2}{\eta\lambda_1} - \frac{48r^2\cdot M}{\eta\lambda_1} - \frac{32r^2\cdot M\cdot r^2|f_2|^2}{\eta\lambda_1^3} - \frac{192^2\cdot M\cdot r^6}{\eta^2\lambda_1} - \frac{512r^6\cdot r^2|f_2|^2\cdot 1/m}{\eta^2\lambda_1}\\ &- \frac{16r^2\cdot c_0 m}{\eta}\frac{|u_2|}{r} - \frac{c_0}{r^2}\cdot r|\nabla f|\cdot\frac{|\nabla u|}{r} - \frac{|f_{11}|}{\lambda_1} - 2\frac{f_1^2}{m\lambda_1} - \frac{32Mr^6}{\eta^2\lambda_1}.\end{aligned}$$

So we get

$$\eta\lambda_1 \leq C\left(1+\frac{|\nabla u|}{r}\right)r^4, \tag{3-24}$$

where C is a positive constant depending only on c_0, m, M, $r|\nabla f|$ and $r^2|\nabla^2 f|$. So we easily get

$$u_{\tau(0)\tau(0)}(0) \leq \frac{1}{r^{4\beta}}\phi(0) \leq \frac{1}{r^{4\beta}}\phi(x_0) \leq C\left(1+\frac{\sup|Du|}{r}\right)e^{c_0\sup|Du|^2/r^2} \leq Ce^{(c_0+2)\sup|Du|^2/r^2}$$

and

$$|u_{\xi\xi}(0)| \leq u_{\tau(0)\tau(0)}(0) \leq Ce^{(c_0+2)\sup|Du|^2/r^2} \quad \text{for all } \xi\in\mathbb{S}^1. \tag{3-25}$$

This completes the proof of Theorem 1.1 under the condition $\eta\lambda_1\geq\Theta$. Hence Theorem 1.1 holds.

Remark 3.2. The eigenvector field τ is important. In fact, it is well-defined when the largest eigenvalue is distinct from the others, and τ depends only on the adjoint matrix. For the Monge–Ampère equation in dimension $n\geq 3$, we do not know whether the largest eigenvalue is distinct from the others, so our method is not suitable for this case.

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BOUNDED SOLUTIONS TO THE ALLEN–CAHN EQUATION WITH LEVEL SETS OF ANY COMPACT TOPOLOGY

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We make use of the flexibility of infinite-index solutions to the Allen–Cahn equation to show that, given any compact hypersurface Σ of $\mathbb{R}^d$ with $d \geqslant 3$, there is a bounded entire solution of the Allen–Cahn equation on $\mathbb{R}^d$ whose zero level set has a connected component diffeomorphic (and arbitrarily close) to a rescaling of Σ. More generally, we prove the existence of solutions with a finite number of compact connected components of prescribed topology in their zero level sets.

1. Introduction

The study of the analogies between the level sets of the solutions to the Allen–Cahn equation

$$\Delta u + u - u^3 = 0$$

in $\mathbb{R}^d$ and minimal hypersurfaces in $\mathbb{R}^d$ was greatly fostered by De Giorgi's 1978 conjecture that all the level sets of any entire solution to the Allen–Cahn equation that is monotone in one direction have to be hyperplanes for $d \leqslant 8$. This is a natural counterpart of the Bernstein problem for minimal hypersurfaces, which asserts that any minimal graph in $\mathbb{R}^d$ must be a hyperplane provided that $d \leqslant 8$. Ghoussoub and Gui [1998] and Ambrosio and Cabré [2000] proved De Giorgi's conjecture for $d = 2$, 3, and the work of Savin [2009] showed that it is also true for $4 \leqslant d \leqslant 8$ under a weak additional technical assumption. Del Pino, Kowalczyk and Wei [del Pino et al. 2011] employed the Bombieri–De Giorgi–Giusti hypersurface to show that the statement of De Giorgi's conjecture does not hold for $d \geqslant 9$.

In dimension 2, it is well known [Dancer 2005] that the monotonicity hypothesis can be relaxed to the assumption that the solution u is *stable*, i.e., that its Morse index is 0. Let us recall that the *Morse index* of u is the maximal dimension of a vector space $V \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (|\nabla v|^2 - v^2 + 3u^2v^2)\, dx < 0$$

for all nonzero $v \in V$. Remarkably, it has been shown recently [Pacard and Wei 2013] that in dimension 8 (actually, in any even dimension $d \geqslant 8$) there are bounded stable solutions to the Allen–Cahn equation whose level sets are not hyperplanes, but rather they are asymptotic to a minimal cone. For the role of minimal cones in the Allen–Cahn equation, see also [Cabré and Terra 2009] and references therein. In dimensions $d \leqslant 7$, the level sets of stable solutions to the Allen–Cahn equation are conjectured to all be hyperplanes [Pacard and Wei 2013].

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The analysis and possible classification of bounded entire solutions to the Allen–Cahn equation is an important open problem, where the Morse index of the solutions plays a key role. Unlike the stable case [Dancer 2005], the structure of solutions with finite Morse index can be very complex; in fact, in dimension 3 a result of del Pino, Kowalczyk and Wei [del Pino et al. 2013] ensures that, under mild technical assumptions, given any embedded complete minimal surface in $\mathbb{R}^3$ with finite total curvature there is a bounded entire solution to the Allen–Cahn equation with a level set that is close to a large rescaling of this minimal surface, and that the Morse index of this solution coincides with the genus of the surface. Also in this direction, the existence of solutions to the Allen–Cahn equation with a level set close to a nondegenerate minimal hypersurface was proved by Pacard and Ritoré [2003] provided that the ambient space is a compact Riemannian manifold (instead of $\mathbb{R}^d$). Furthermore, Agudelo, del Pino and Wei [Agudelo et al. 2015] have recently constructed bounded entire axisymmetric solutions on $\mathbb{R}^3$ of arbitrarily large index that have multiple catenoidal ends.

Generally speaking, it is expected [del Pino et al. 2012; 2013] that the condition that the Morse index of the solution be finite should play a similar role to the finite total curvature assumption in the study of minimal hypersurfaces in Euclidean spaces. In particular, it is well known that there are many infinite-index solutions to the Allen–Cahn equation [Dancer 2001; Cabré and Terra 2009; Kowalczyk et al. 2015], and this abundance of solutions should translate into a wealth of possible level sets.

Our objective in this paper is to explore the flexibility of bounded entire solutions to the Allen–Cahn equation of infinite index by showing that there are bounded solutions to the Allen–Cahn equation on $\mathbb{R}^d$ with level sets of any compact topology. Specifically, given a compact hypersurface Σ without boundary of $\mathbb{R}^d$, we will show that there is a rescaling of Σ that is arbitrarily close to a connected component of the nodal set of a bounded entire solution of the Allen–Cahn equation. Furthermore, this level set is *structurally stable*, in the sense that any function on $\mathbb{R}^d$ which is sufficiently close to u in the C^1 norm in a neighborhood of this set will also have a zero level set of the same topology. In view of the existing literature, we are particularly interested in the case of high-dimension d.

To present a precise statement, let us agree to say that an ϵ-*rescaling* is a diffeomorphism of $\mathbb{R}^d$ that can be written as $\Phi = \Phi_1 \circ \Phi_2$, where Φ_2 is a rescaling and $\|\Phi_1 - \mathrm{id}\|_{C^1(\mathbb{R}^d)} < \epsilon$ (here we could have taken any other fixed C^k norm, though). By a *hypersurface* we will refer to a smoothly embedded codimension 1 submanifold of $\mathbb{R}^d$, so self-intersections will not be allowed. Furthermore, in what follows we will use the notation $\langle x\rangle := (1+|x|^2)^{1/2}$ for the Japanese bracket.

Theorem 1.1. *Let Σ be any compact orientable hypersurface without boundary of $\mathbb{R}^d$ with $d \geqslant 3$, and take any $\epsilon > 0$. Then there is a bounded entire solution u of the Allen–Cahn equation in $\mathbb{R}^d$ such that its zero level set $u^{-1}(0)$ has a connected component given by $\Phi(\Sigma)$, where Φ is an ϵ-rescaling. This set is structurally stable. Furthermore, u falls off at infinity as $|u(x)| < C\langle x\rangle^{(1-d)/2}$ if $d \geqslant 4$ and is in $L^4(\mathbb{R}^3)$ if $d = 3$.*

It is worth mentioning that the result that we will actually prove (Theorem 4.1) is in fact stronger, in the sense that given any finite number of hypersurfaces $\Sigma_1, \dots, \Sigma_N$ that are not linked (see Definition 2.1) we will show that there is a diffeomorphism Φ such that $\Phi(\Sigma_1) \cup \cdots \cup \Phi(\Sigma_N)$ is a union of connected components of the nodal set of a bounded entire solution to the Allen–Cahn equation. The diffeomorphism Φ is not an ϵ-rescaling, although it does act on each hypersurface Σ_j as an ϵ-rescaling composed with a rigid motion.

The idea of the proof of the theorem is that, when u is small in a suitable sense, solutions to the Allen–Cahn equation behave as solutions to the Helmholtz equation

$$\Delta w + w = 0.$$

Hence, a key step of the proof is to establish an analog of Theorem 1.1 for solutions to the Helmholtz equations with a sharp fall-off rate at infinity, which is as $\langle x\rangle^{(1-d)/2}$ (Theorem 2.2). For this we combine a construction using the first eigenfunction of the domain bounded by Σ with a Runge-type theorem with decay conditions at infinity that generalizes the results that we proved in [Enciso and Peralta-Salas 2012; 2015] for Beltrami fields on $\mathbb{R}^3$. Using suitable weighted estimates for a convolution operator associated with the Helmholtz equation (Theorem 3.1), we then promote these solutions of the Helmholtz equation to solutions of the Allen–Cahn equation and show that the latter still possess a nodal set of the desired topology. From the method of proof it stems that the statement of Theorem 1.1 remains valid for much more general nonlinearities. (More precisely, one can replace u^3 by a smooth enough function $F(u)$ that behaves as $u^{1+\alpha}$ as $u \to 0$ with $\alpha > (d+1)/(d-1)$.) Observe that solutions of the Helmholtz equation were also used in [Gutiérrez 2004] to construct nontrivial solutions in some L^q space to the closely related Ginzburg–Landau equation on $\mathbb{R}^3$ and $\mathbb{R}^4$, and that in fact for $d = 3$ our proof will borrow estimates from this work.

2. Bounded solutions to the Helmholtz equation

In this section we will prove an analog of Theorem 1.1 for solutions to the Helmholtz equation on $\mathbb{R}^d$. We shall begin by introducing some notation.

Let us consider the function

$$G(x) := \beta |x|^{1-d/2} Y_{d/2-1}(|x|), \tag{2-1}$$

where $Y_{d/2-1}$ denotes the Bessel function of the second kind and we have set

$$\beta := \frac{2^{1-d/2}\pi}{|\mathbb{S}^{d-1}|\Gamma(d/2-1)},$$

with $|\mathbb{S}^{d-1}|$ the area of the unit $(d-1)$-sphere and Γ the Gamma function. A simple computation in spherical coordinates shows that $\Delta G + G = 0$ everywhere but at the origin and the asymptotics for Bessel functions show that

$$G(x) = -\frac{1}{|\mathbb{S}^{d-1}|\,|x|^{d-2}} + O(|x|^{3-d})$$

as $x \to 0$. It then follows that G is a fundamental solution for the Helmholtz equation, so, if v is, say, a Schwartz function on $\mathbb{R}^d$, the convolution $G * v$ satisfies

$$\Delta(G * v) + G * v = v. \tag{2-2}$$

As we discussed in the introduction, we will prove a result that is considerably more general than Theorem 1.1, as it applies to an arbitrary number of hypersurfaces. There is, however, a topological condition that we must impose on these hypersurfaces, which is described in the following:

Definition 2.1. Let $\Sigma_1, \dots, \Sigma_N$ be compact orientable hypersurfaces without boundary of $\mathbb{R}^d$. We will say that they are *not linked* if there are N pairwise disjoint contractible sets $S_1, \dots, S_N$ such that each hypersurface Σ_j is contained in S_j.

We are now ready to state and prove the main result of this section. Notice that the proof of the theorem provides a satisfactory description of the structure of the diffeomorphism Ψ, as noted in Remark 2.3. The proof makes use of some techniques we introduced in [Enciso and Peralta-Salas 2013] to study the level sets of harmonic functions and in [Enciso and Peralta-Salas 2015] to construct Beltrami fields with prescribed vortex tubes. Throughout, diffeomorphisms are assumed to be of class C^∞ and connected with the identity, and B_R denotes the ball centered at the origin of radius R. Observe that, of course, for $N = 1$ the condition that the hypersurface be not linked is satisfied trivially.

Theorem 2.2. *Let $\Sigma_1, \dots, \Sigma_N$ be compact orientable hypersurfaces without boundary of $\mathbb{R}^d$ that are not linked with $d \geqslant 3$. Then there is a function w satisfying the Helmholtz equation*

$$\Delta w + w = 0$$

in $\mathbb{R}^d$ and a diffeomorphism Ψ of $\mathbb{R}^d$ such that $\Psi(\Sigma_1), \dots, \Psi(\Sigma_N)$ are structurally stable connected components of the zero set $w^{-1}(0)$. Furthermore, w falls off at infinity as $|\partial^\alpha w(x)| < C_\alpha \langle x\rangle^{(1-d)/2}$ for any multiindex α.

Proof. An easy application of Whitney's approximation theorem ensures that, by perturbing the hypersurfaces a little if necessary, we can assume that Σ_j is a real analytic hypersurface of $\mathbb{R}^d$. The fact that the hypersurfaces are not linked allow us now to rescale and translate them so that the (unique) precompact domains Ω_j that are bounded by each rescaled and translated real-analytic hypersurface, which we will call $\Sigma'_j := \partial\Omega_j$, are pairwise disjoint and their first Dirichlet eigenvalue $\lambda_1(\Omega_j)$ is 1. The first eigenvalue is always simple, so there is a unique eigenfunction ψ_j, modulo a multiplicative constant, that satisfies the eigenvalue equation

$$\Delta\psi_j + \psi_j = 0 \quad \text{in } \Omega_j, \qquad \psi_j|_{\Sigma'_j} = 0.$$

We can choose ψ_j so that it is positive in Ω_j.

Hopf's boundary point lemma shows that the gradient of ψ_j does not vanish on Σ_j:

$$\min_{x\in\Sigma_j} |\nabla\psi_j(x)| > 0. \tag{2-3}$$

Furthermore, as the hypersurface Σ_j is analytic, it is standard that ψ_j is analytic in an open neighborhood $\widetilde{\Omega}_j$ of the closure of Ω_j.

Our goal is to construct a solution w of the Helmholtz equation in $\mathbb{R}^d$ that approximates each function ψ_j in the set Ω_j. To this end, let us take a smooth function $\chi : \mathbb{R}^d \to \mathbb{R}$ that is equal to 1 in a narrow neighborhood of the closure $\overline{\Omega}$ and is identically zero outside $\widetilde{\Omega}$, with

$$\widetilde{\Omega} := \bigcup_{j=1}^{N} \widetilde{\Omega}_j, \quad \Omega := \bigcup_{j=1}^{N} \Omega_j.$$

We can now define a smooth function w_1 on $\mathbb{R}^d$ by setting

$$w_1 := \sum_{j=1}^{N} \chi \psi_j.$$

Here we are assuming that $w_1 := 0$ outside $\widetilde{\Omega}$.

Let us now write

$$w_1 = w_1' + h$$

with

$$w_1'(x) = \int_{\mathbb{R}^d} G(x-y) f(y)\, dy, \tag{2-4}$$

where f is the compactly supported function $f := \Delta w_1 + w_1$ and G is the fundamental solution (2-1). By construction, h satisfies the homogeneous Helmholtz equation

$$\Delta h + h = 0.$$

The support of the function f is obviously contained in the open set $\widetilde{\Omega} \setminus \overline{\Omega}$. Therefore, an easy continuity argument ensures that one can approximate the integral (2-4) uniformly in the compact set $\overline{\Omega}$ by a finite Riemann sum of the form

$$w_2(x) := \sum_{n=1}^{M} c_n G(x - x_n). \tag{2-5}$$

Specifically, it is standard that for any $\delta > 0$ there is a large integer M, real numbers c_n and points $x_n \in \widetilde{\Omega} \setminus \overline{\Omega}$ such that the finite sum (2-5) satisfies

$$\|w_1' - w_2\|_{C^0(\Omega)} < \delta. \tag{2-6}$$

Let us now take a large ball B_R containing the closure of the set $\widetilde{\Omega}$. We shall next show that there is a finite number of points $\{x_n'\}_{n=1}^{M'}$ in $\mathbb{R}^d \setminus \bar{B}_R$ and constants c_n' such that the finite linear combination

$$w_3(x) := \sum_{n=1}^{M'} c_n' G(x - x_n') \tag{2-7}$$

approximates the function w_2 uniformly in Ω:

$$\|w_2 - w_3\|_{C^0(\Omega)} < \delta. \tag{2-8}$$

Here δ is the same arbitrarily small constant as above.

Consider the space $\mathcal{V}$ of all finite linear combinations of the form (2-7), where x_n' can be any point in $\mathbb{R}^d \setminus \bar{B}_R$ and the constants c_n' take arbitrary values. Restricting these functions to the set Ω, we can regard $\mathcal{V}$ as a subspace of the Banach space $C^0(\Omega)$ of continuous functions on Ω.

By the Riesz–Markov theorem, the dual of $C^0(\Omega)$ is the space $\mathcal{M}(\Omega)$ of the finite signed Borel measures on $\mathbb{R}^d$ whose support is contained in the set Ω. Let us take any measure $\mu \in \mathcal{M}(\Omega)$ such that $\int_{\mathbb{R}^d} f\, d\mu = 0$ for all $f \in \mathcal{V}$. We now define a function $F \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ as

$$F(x) := \int_{\mathbb{R}^d} G(x - x')\, d\mu(x'),$$

so that F satisfies the equation

$$\Delta F + F = \mu.$$

Notice that F is identically zero on $\mathbb{R}^d \setminus \bar{B}_R$ by the definition of the measure μ and that F satisfies the elliptic equation

$$\Delta F + F = 0$$

in $\mathbb{R}^d \setminus \bar{\Omega}$, so F is analytic in this set. Hence, since $\mathbb{R}^d \setminus \bar{\Omega}$ is connected and contains the set $\mathbb{R}^d \setminus B_R$, by analyticity the function F must vanish on the complement of Ω. It then follows that the measure μ also annihilates the function $G(\cdot - y)$ with $y \notin \bar{\Omega}$ because

$$0 = F(y) = \int_{\mathbb{R}^d} G(y - x')\, d\mu(x').$$

Therefore

$$\int_{\mathbb{R}^d} w_2\, d\mu = 0,$$

which implies that w_2 can be uniformly approximated on Ω by elements of the subspace $\mathcal{V}$, due to the Hahn–Banach theorem. Accordingly, there is a finite set of points $\{x_n'\}_{n=1}^{M'}$ in $\mathbb{R}^d \setminus \bar{B}_R$ and reals c_n' such that the function w_3 defined by (2-7) satisfies the estimate (2-8).

To complete the proof of the theorem, notice that the function

$$w_4 := w_3 + h$$

satisfies the equation

$$\Delta w_4 + w_4 = 0 \tag{2-9}$$

in the ball B_R, whose interior contains Ω. Let us take hyperspherical coordinates $r := |x|$ and $\omega := x/|x| \in \mathbb{S}^{d-1}$ in B_R. Expanding the function w_4 (with respect to the angular variables) in a series of spherical harmonics and using (2-9), we immediately obtain that w_4 can be written in the ball as a Fourier–Bessel series of the form

$$w_4 = \sum_{l=0}^{\infty} \sum_{m \in I_l} c_{lm}\, j_l(r) Y_{lm}(\omega),$$

where j_l denotes a d-dimensional hyperspherical Bessel function, Y_{lm} are spherical harmonics on $\mathbb{S}^{d-1}$ and I_l is a finite set that depends on l and whose explicit expression will not be needed here.

Since the above series converges in $L^2(B_R)$, for any $\delta > 0$ there is an integer l_0 such that the finite sum

$$w := \sum_{l=0}^{l_0} \sum_{m \in I_l} c_{lm}\, j_l(r)\, Y_{lm}(\omega)$$

approximates the function w_4 in an L^2 sense:

$$\|w - w_4\|_{L^2(B_R)} < \delta. \tag{2-10}$$

By the properties of Bessel functions, w is smooth in $\mathbb{R}^d$, falls off as

$$|\partial^\alpha w(x)| \leqslant C_\alpha \langle x \rangle^{(1-d)/2}$$

at infinity for any multiindex α and satisfies the equation

$$\Delta w + w = 0 \tag{2-11}$$

in the whole space.

Given any $R' < R$ large enough for the set Ω to be contained in the ball $B_{R'}$, standard elliptic estimates allow us to pass from the L^2 bound (2-10) to a uniform estimate

$$\|w - w_4\|_{C^0(B_{R'})} < C\delta.$$

From this inequality and the bounds (2-6) and (2-8) we infer

$$\|w - w_1\|_{C^0(\Omega)} < C\delta. \tag{2-12}$$

Moreover, since w_1 also satisfies the Helmholtz equation in a neighborhood of the compact set $\overline{\Omega}$, standard elliptic estimates again imply that the uniform estimate (2-12) can be promoted to the C^1 bound

$$\|w - w_1\|_{C^1(\Omega)} < C\delta. \tag{2-13}$$

Finally, since $\Sigma_1 \cup \cdots \cup \Sigma_N$ is a union of components of the nodal set of w_1 and, by (2-3), the gradient of w_1 does not vanish on these hypersurfaces, the estimate (2-13) and a direct application of Thom's isotopy theorem [Abraham and Robbin 1967, Theorem 20.2] imply that there is a diffeomorphism Ψ of $\mathbb{R}^d$ such that

$$\Psi(\Sigma_1 \cup \cdots \cup \Sigma_N) \tag{2-14}$$

is a union of components of the zero set $w^{-1}(0)$. Moreover, the diffeomorphism Ψ is C^1-close to the identity. The structural stability of the set (2-14) for the function w also follows from Thom's isotopy theorem and the lower bound

$$\min_{x \in \Psi(\Sigma_1 \cup \cdots \cup \Sigma_N)} |\nabla w(x)| > 0,$$

as a consequence of the C^1 estimate (2-13) and the fact that the function w_1 satisfies the gradient condition (2-3). □

Remark 2.3. It follows from the proof that there are rescalings Ψ_j^2, translations Ψ_j^3 and diffeomorphisms Ψ_j^1 with $\|\Psi_j^1 - \mathrm{id}\,\|_{C^1(\mathbb{R}^d)}$ arbitrarily small such that

$$\Psi(\Sigma_j) = (\Psi_j^1 \circ \Psi_j^2 \circ \Psi_j^3)(\Sigma_j).$$

In particular, if $N = 1$ the diffeomorphism Ψ can be assumed to be an ϵ-rescaling. A minor modification of the argument would have allowed us to take $\|\Psi_j^1 - \mathrm{id}\,\|_{C^k(\mathbb{R}^d)}$ arbitrarily small, with k any fixed number.

3. A weighted estimate for a convolution operator

In promoting solutions to the Helmholtz equation with sharp decay at infinity to solutions to the Allen–Cahn equation, the estimates that we establish in this section will play a key role.

Specifically, we will be interested in the convolution of the fundamental solution G, introduced in (2-1), with functions with certain decay rate at infinity. To quantify this, for any nonnegative integer k and any positive real ν let us denote by $C_\nu^k(\mathbb{R}^d)$ the closure of the space of Schwartz functions on $\mathbb{R}^d$ with respect to the metric

$$\|v\|_{k,\nu} := \max_{|\alpha|\leqslant k} \sup_{x\in\mathbb{R}^d} |\langle x\rangle^\nu \partial^\alpha v(x)|.$$

Clearly

$$\|vw\|_{k,\nu+\nu'} \leqslant C\|v\|_{k,\nu}\|w\|_{k,\nu'}$$

whenever $v \in C_\nu^k(\mathbb{R}^d)$ and $w \in C_{\nu'}^k(\mathbb{R}^d)$, where C is a constant that only depends on k. In particular,

$$\|v^s\|_{k,\nu} \leqslant C\|v\|_{k,\nu/s}^s. \tag{3-1}$$

The following theorem, which asserts that the convolution with G defines a bounded map

$$C_\nu^k(\mathbb{R}^d) \to C_{(d-1)/2}^k(\mathbb{R}^d)$$

for any $\nu > d$, provides the estimates that we need:

Theorem 3.1. *Suppose that $d \geqslant 3$. Then, for any $v \in C_\nu^k(\mathbb{R}^d)$ with $k \geqslant 0$ and $\nu > d$, one has*

$$\|G * v\|_{k,(d-1)/2} \leqslant C\|v\|_{k,\nu},$$

with a constant that depends on d and ν but not on v or k.

Proof. In view of the well-known asymptotics for Bessel functions when $d \geqslant 3$, there is a positive constant C such that G is bounded by

$$|G(x)| \leqslant \begin{cases} C|x|^{2-d} & \text{if } |x| < 1, \\ C|x|^{(1-d)/2} & \text{if } |x| > 1. \end{cases}$$

It then follows that

$$\begin{aligned} |G * v(x)| &\leqslant \int_{\mathbb{R}^d} |G(z)|\,|v(x-z)|\,dz \\ &\leqslant C\|v\|_{0,\nu}\left(\int_{B_1} |z|^{2-d}\langle x-z\rangle^{-\nu}\,dz + \int_{\mathbb{R}^d\setminus B_1} |z|^{(1-d)/2}\langle x-z\rangle^{-\nu}\,dz\right). \end{aligned} \tag{3-2}$$

For any fixed x, the first integral is convergent for any value of ν, while the second converges provided that $\nu > \frac{1}{2}(d+1)$. Since $\nu > d > \frac{1}{2}(d+1)$, we infer that $G * v(x)$ is well defined as a convergent integral for any $v \in C^0_\nu(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$, and it only remains to analyze its behavior for large $|x|$.

For concreteness, let us assume that $|x| > 2$. We shall next show that the integrals

$$\begin{aligned}
I_1 &:= \int_{B_1} |z|^{2-d} \langle x-z\rangle^{-\nu}\, dz,\\
I_2 &:= \int_{B_{|x|/2}} |z|^{(1-d)/2} \langle x-z\rangle^{-\nu}\, dz,\\
I_3 &:= \int_{B_{2|x|}\setminus B_{|x|/2}} |z|^{(1-d)/2} \langle x-z\rangle^{-\nu}\, dz,\\
I_4 &:= \int_{\mathbb{R}^d\setminus B_{2|x|}} |z|^{(1-d)/2} \langle x-z\rangle^{-\nu}\, dz,
\end{aligned}$$

are then bounded as

$$I_j < C|x|^{(1-d)/2}, \tag{3-3}$$

where C does not depend on v. In view of the inequality (3-2) and the fact that

$$\int_{\mathbb{R}^d\setminus B_1} |z|^{(1-d)/2} \langle x-z\rangle^{-\nu} \leqslant I_2 + I_3 + I_4,$$

this shows that the convolution with G is a bounded map $C^0_\nu(\mathbb{R}^d) \to C^0_{(d-1)/2}(\mathbb{R}^d)$. Since for any multiindex α we have

$$\partial^\alpha (G * v) = G * (\partial^\alpha v),$$

this immediately implies that the convolution with G is also a bounded map $C^k_\nu(\mathbb{R}^d) \to C^k_{(d-1)/2}(\mathbb{R}^d)$, thereby proving the theorem.

So it only remains to prove the estimate (3-3) for $1 \leqslant j \leqslant 4$. For this we start by using the elementary inequality

$$\langle x-z\rangle > \begin{cases} \frac{1}{2}|x| & \text{if } |z| < \frac{1}{2}|x|,\\ \frac{1}{2}|z| & \text{if } |z| > 2|x|,\end{cases}$$

to obtain, for $|x| > 2$,

$$\begin{aligned}
I_1 &< C|x|^{-\nu} \int_{B_1} |z|^{2-d}\, dz = C|x|^{-\nu} < C|x|^{-d} < C|x|^{(1-d)/2},\\
I_2 &< C|x|^{-\nu} \int_{B_{|x|/2}} |z|^{(1-d)/2}\, dz = C|x|^{(d+1)/2-\nu} < C|x|^{(1-d)/2},\\
I_4 &< C \int_{\mathbb{R}^d\setminus B_{2|x|}} |z|^{(1-d)/2-\nu}\, dz = C|x|^{(d+1)/2-\nu} < C|x|^{(1-d)/2}.
\end{aligned}$$

To obtain these bounds we have used that $\nu > d$ by assumption.

To estimate I_3 we choose a Cartesian basis such that $x = |x|e_1$ and then use the rescaled variable $\bar{z} := z/|x|$ to write

$$I_3 = \int_{B_{2|x|}\setminus B_{|x|/2}} |z|^{(1-d)/2}\langle x-z\rangle^{-\nu}\, dz = |x|^{(d+1)/2-\nu}\int_{B_2\setminus B_{1/2}} \frac{|\bar{z}|^{(1-d)/2}\, d\bar{z}}{(1/|x|^2+|e_1-\bar{z}|^2)^{\nu/2}}. \tag{3-4}$$

Denoting by B' the ball centered at e_1 of radius $\frac{1}{4}$, one can check that

$$\begin{aligned}\int_{B'} \frac{|\bar{z}|^{(1-d)/2}\, d\bar{z}}{(1/|x|^2+|e_1-\bar{z}|^2)^{\nu/2}} &< C\int_0^{1/4} \frac{\rho^{d-1}\, d\rho}{(1/|x|^2+\rho^2)^{\nu/2}}\\ &= C|x|^{\nu-d}\int_0^{|x|/4} \frac{\bar{\rho}^{d-1}\, d\bar{\rho}}{(1+\bar{\rho}^2)^{\nu/2}}\\ &< C|x|^{\nu-d}\int_0^{\infty} \frac{\bar{\rho}^{d-1}\, d\bar{\rho}}{(1+\bar{\rho}^2)^{\nu/2}}\\ &< C|x|^{\nu-d},\end{aligned}$$

where we have defined $\bar{\rho} := |x|\rho$, and we have used that the integral in $\bar{\rho}$ is convergent for any $\nu > d$. Plugging this into (3-4), one gets

$$\begin{aligned}I_3 &= |x|^{(d+1)/2-\nu}\left(\int_{B'} \frac{|\bar{z}|^{(1-d)/2}\, d\bar{z}}{(1/|x|^2+|e_1-\bar{z}|^2)^{\nu/2}} + \int_{B_2\setminus(B_{1/2}\cup B')} \frac{|\bar{z}|^{(1-d)/2}\, d\bar{z}}{(1/|x|^2+|e_1-\bar{z}|^2)^{\nu/2}}\right)\\ &< C|x|^{(d+1)/2-\nu}(|x|^{\nu-d}+C) < C|x|^{(1-d)/2}.\end{aligned}$$

To obtain the first inequality we have used that $|e_1-\bar{z}|^2 \geqslant \frac{1}{16}$ for all $\bar{z} \in B_2\setminus(B_{1/2}\cup B')$, and the second inequality follows from the assumption $\nu > d$. This is the last estimate that we needed in (3-3) and thus the theorem follows. □

In view of the structure of the nonlinearity of the Allen–Cahn equation, the following corollary will be useful:

Corollary 3.2. *For any $v \in C^k_{(d-1)/2}(\mathbb{R}^d)$ with $k \geqslant 0$ and $d \geqslant 4$, one has the estimate*

$$\|G * (v^3)\|_{k,(d-1)/2} \leqslant C\|v\|^3_{k,(d-1)/2}.$$

Proof. We can apply Theorem 3.1 with $\nu := \frac{1}{2}(3d-3)$ because $\nu > d$ for all dimensions $d \geqslant 4$, thus implying that

$$\|G * (v^3)\|_{k,(d-1)/2} \leqslant C\|v^3\|_{k,(3d-3)/2} \leqslant C\|v\|^3_{k,(d-1)/2},$$

where we have used the relation (3-1). □

4. Proof of Theorem 1.1

We are now ready to prove the main result of this paper, which reduces to Theorem 1.1 when $N = 1$.

Theorem 4.1. *Let $\Sigma_1, \dots, \Sigma_N$ be compact orientable hypersurfaces without boundary of $\mathbb{R}^d$ that are not linked with $d \geqslant 3$, and let us take any positive integer k. Then there is a diffeomorphism Φ of $\mathbb{R}^d$ such* *that $\Phi(\Sigma_1),\dots,\Phi(\Sigma_N)$ are connected components of the level set $u^{-1}(0)$ of a smooth solution to the Allen–Cahn equation in $\mathbb{R}^d$, which is bounded as $|\partial^\alpha u(x)| < C_\alpha \langle x\rangle^{(1-d)/2}$ for any multiindex with $|\alpha| < k$ if $d \geqslant 4$, and if $d = 3$ then u is in $L^4(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Furthermore, these level sets are structurally stable and the diffeomorphism Φ can be assumed to have the same structure as in Remark 2.3.*

Proof. Let us first assume that $d \geqslant 4$, since this will allow us to apply Corollary 3.2. By Theorem 2.2 there is a solution w to the Helmholtz equation

$$\Delta w + w = 0$$

on $\mathbb{R}^d$ such that $\Psi(\Sigma_1),\dots,\Psi(\Sigma_N)$ are connected components of its zero set $w^{-1}(0)$, where Ψ is a diffeomorphism of $\mathbb{R}^d$. Moreover, $\|w\|_{k,(d-1)/2} < C$ and the above hypersurfaces are structurally stable, in the sense that there exist a large ball B_R and a positive constant η such that, if w' is any function with

$$\|w - w'\|_{C^1(B_R)} < \eta, \tag{4-1}$$

then there is a diffeomorphism Φ of $\mathbb{R}^d$ such that

$$\Phi(\Sigma_1)\cup\cdots\cup\Phi(\Sigma_N) \tag{4-2}$$

are structurally stable connected components of the level set $w'^{-1}(0)$. Furthermore, Φ is close to Ψ in the norm $C^1(\mathbb{R}^d)$.

Let us take a small positive constant ϵ that will be fixed later and consider the iterative scheme

$$u_0 := \delta w, \quad u_{n+1} := \delta w + G * (u_n^3), \tag{4-3}$$

where we have set

$$\delta := \frac{\epsilon}{2\|w\|_{k,(d-1)/2}}.$$

Our goal is to show that, if ϵ is small enough, u_n converges in $C^k_{(d-1)/2}(\mathbb{R}^d)$ to a function u that satisfies the Allen–Cahn equation

$$\Delta u + u - u^3 = 0$$

and is close to δw in a suitable norm.

A first observation is that, if $\|u_n\|_{k,(d-1)/2} < \epsilon$ and ϵ is small enough, by the definition of δ we automatically have

$$\begin{aligned}\|u_{n+1}\|_{k,(d-1)/2} &\leqslant \delta\|w\|_{k,(d-1)/2} + \|G*(u_n^3)\|_{k,(d-1)/2}\\ &\leqslant \delta\|w\|_{k,(d-1)/2} + C\|u_n\|^3_{k,(d-1)/2} \leqslant \tfrac{1}{2}\epsilon + C\epsilon^3 < \epsilon.\end{aligned} \tag{4-4}$$

Here we have used Corollary 3.2 to estimate $G*(u_n^3)$. Notice that the smallness that we have to impose on ϵ only depends on the constant that appears in Corollary 3.2. In particular, since the first function u_0 of the iteration satisfies

$$\|u_0\|_{k,(d-1)/2} = \tfrac{1}{2}\epsilon,$$

the induction property (4-4) then implies that

$$\|u_n\|_{k,(d-1)/2} < \epsilon \tag{4-5}$$

for all n.

To estimate the difference $u_{n+1} - u_n$, let us start by noticing that for any functions v, v' we have

$$\begin{aligned}\|v^3 - v'^3\|_{k,(3d-3)/2} &= \max_{|\alpha|\leqslant k} \sup_{x\in\mathbb{R}^d} \langle x\rangle^{(3d-3)/2} \big|\partial^\alpha(v^2(v-v') + vv'(v-v') + v'^2(v-v'))\big| \\ &\leqslant C\big(\|v\|^2_{k,(d-1)/2} + \|v'\|^2_{k,(d-1)/2}\big)\|v-v'\|_{k,(d-1)/2}.\end{aligned}$$

It then follows from Theorem 3.1, the fact that $\frac{1}{2}(3d-3) > d$ when $d \geqslant 4$, and (4-5) that we can write

$$\begin{aligned}\|u_{n+1} - u_n\|_{k,(d-1)/2} &= \|G * (u_n^3 - u_{n-1}^3)\|_{k,(d-1)/2} \\ &\leqslant C\|u_n^3 - u_{n-1}^3\|_{k,(3d-3)/2} \leqslant C\epsilon^2 \|u_n - u_{n-1}\|_{k,(d-1)/2}.\end{aligned} \tag{4-6}$$

If ϵ is small enough that $C\epsilon^2 < \frac{1}{2}$, it is standard that (4-4) and (4-6) imply that u_n converges in $C^k_{(d-1)/2}(\mathbb{R}^d)$ to some function u with

$$\|u\|_{k,(d-1)/2} \leqslant \epsilon. \tag{4-7}$$

Since the map $v \mapsto G * (v^3)$ is continuous in $C^k_{(d-1)/2}(\mathbb{R}^d)$, from (4-3) we infer that u satisfies the integral equation

$$u = \delta w + G * (u^3). \tag{4-8}$$

As w is a solution of the Helmholtz equation and G is a fundamental solution satisfying (2-2), it then follows that

$$\Delta u + u = u^3,$$

so u is a solution of the Allen–Cahn equation, which is smooth by elliptic regularity.

One can now use the bound (4-7), the relation (4-8) and the definition of δ to write

$$\Big\| w - \frac{u}{\delta}\Big\|_{k,(d-1)/2} = \frac{1}{\delta}\|\delta w - u\|_{k,(d-1)/2} = \frac{1}{\delta}\|G * (u^3)\|_{k,(d-1)/2} \leqslant \frac{C}{\delta}\|u\|^3_{k,(d-1)/2} \leqslant C\epsilon^2. \tag{4-9}$$

In view of the stability estimate (4-1), if ϵ is small enough (namely, $C\epsilon^2 < \eta$), we infer that there is a diffeomorphism Φ, close to the diffeomorphism Ψ in the norm $C^1(\mathbb{R}^d)$, such that the hypersurfaces (4-2) are structurally stable connected components of the level set $u^{-1}(0)$. The theorem then follows for $d \geqslant 4$.

For $d = 3$ the proof is similar but employs the $L^4(\mathbb{R}^3)$ estimates for the inverse of the Helmholtz equation proved in [Gutiérrez 2004]. More precisely, as G is the real part of the fundamental solution of the Helmholtz equation that satisfies the Sommerfeld outgoing radiation condition (namely, $-e^{i|x|}/4\pi|x|$), the iteration (4-3) converges in $L^4(\mathbb{R}^3)$ by [Gutiérrez 2004]. The rest of the proof is just as in higher dimensions, replacing the estimate (4-9) in the ball B_R by the standard elliptic estimate

$$\Big\| w - \frac{u}{\delta}\Big\|_{C^k(B_R)} \leqslant C\Big\| w - \frac{u}{\delta}\Big\|_{L^4(\mathbb{R}^3)} \leqslant C\epsilon^2,$$

where the last bound is what one obtains directly from the iteration. Moreover by [Gutiérrez 2004, Proposition 1], it follows that any $L^4(\mathbb{R}^3)$ solution to the Allen–Cahn equation is bounded. $\square$

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HÖLDER ESTIMATES AND LARGE TIME BEHAVIOR FOR A NONLOCAL DOUBLY NONLINEAR EVOLUTION

RYAN HYND AND ERIK LINDGREN

The nonlinear and nonlocal PDE

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v = 0,$$

where

$$(-\Delta_p)^s v(x,t) = 2\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{|v(x,t)-v(x+y,t)|^{p-2}(v(x,t)-v(x+y,t))}{|y|^{n+sp}}\,dy,$$

has the interesting feature that an associated Rayleigh quotient is nonincreasing in time along solutions. We prove the existence of a weak solution of the corresponding initial value problem which is also unique as a viscosity solution. Moreover, we provide Hölder estimates for viscosity solutions and relate the asymptotic behavior of solutions to the eigenvalue problem for the fractional p-Laplacian.

1. Introduction

We study the nonlinear and nonlocal PDE

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v = 0, \tag{1-1}$$

where $p \in (1,\infty)$, $s \in (0,1)$ and $(-\Delta_p)^s$ is the fractional p-Laplacian

$$(-\Delta_p)^s u\,(x) := 2\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{|u(x)-u(x+y)|^{p-2}(u(x)-u(x+y))}{|y|^{n+sp}}\,dy. \tag{1-2}$$

Here and throughout P.V. denotes principal value. The main reason of our interest in solutions of (1-1) is the connection with ground states for $(-\Delta_p)^s$, i.e., extremals of the nonlocal Rayleigh quotient

$$\lambda_{s,p} = \inf_{u \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy}{\int_\Omega |u(x)|^p\,dx}. \tag{1-3}$$

Here and throughout $\Omega \subset \mathbb{R}^n$ is a bounded domain. Clearly, $1/\lambda_{s,p}$ is the optimal constant in the Poincaré inequality in the fractional Sobolev space $W_0^{s,p}(\Omega)$.

In recent years there has been a surge of interest around this nonlinear and nonlocal eigenvalue problem; see [Lindgren and Lindqvist 2014; Brasco and Franzina 2014; Brasco and Parini 2015; Brasco et al. 2016; Del Pezzo and Salort 2015; Franzina and Palatucci 2014; Iannizzotto and Squassina 2014]. In particular, it is known that ground states (or first eigenfunctions) are unique up to a multiplicative constant and have

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a definite sign (see Theorem 14 in [Lindgren and Lindqvist 2014] together with Corollary 3.14 in [Brasco and Parini 2015]). The corresponding local problem (formally $s = 1$), i.e., the eigenvalue problem for the p-Laplacian, has been extensively studied throughout the years. See, for instance, [Lieb 1983; Lindqvist 1990; 2008].

The first of our main results is a local Hölder estimate for *viscosity solutions* of (1-1). This is one of the first continuity estimates for parabolic equations involving the fractional p-Laplacian.

Theorem 1.1. *Let $p \geq 2$, $s \in (0,1)$ and $v \in L^\infty(\mathbb{R}^n \times (-2,0])$ be a viscosity solution of*

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v = 0 \quad \textit{in } B_2 \times (-2,0].$$

Then v is Hölder continuous in $B_1 \times (-1,0]$ and in particular there exist α and C depending on p and s such that

$$\|v\|_{C^\alpha(B_1\times(-1,0])} \leq C\|v\|_{L^\infty(\mathbb{R}^n\times(-2,0])}.$$

We also study the initial value problem

$$\begin{cases} |v_t|^{p-2}v_t + (-\Delta_p)^s v = 0 & \text{in } \Omega \times (0,\infty), \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \times [0,\infty), \\ v = g & \text{in } \Omega \times \{0\}, \end{cases} \tag{1-4}$$

and show that (1-4) has a *weak solution* in the sense of a doubly nonlinear evolution and a unique *viscosity solution*. In addition, we relate the long time behavior of solutions to the eigenvalue problem for the fractional p-Laplacian. These results are presented in the two theorems below.

Theorem 1.2. *Let $p \in (1,\infty)$ and $s \in (0,1)$. Assume $g \in W_0^{s,p}(\Omega)$ and define*

$$\mu_{s,p} := \lambda_{s,p}^{\frac{1}{p-1}}.$$

Then for any weak solution v of (1-4),

$$w(x) := \lim_{t\to\infty} e^{\mu_{s,p}t} v(x,t) \tag{1-5}$$

exists in $W^{s,p}(\mathbb{R}^n)$ and is a ground state for $(-\Delta_p)^s$, provided it is not identically zero. In this case, $v(\,\cdot\,,t) \neq 0$ for $t \geq 0$ and

$$\lambda_{s,p} = \lim_{t\to\infty} \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{n+sp}}\,dx\,dy}{\int_\Omega |v(x,t)|^p\,dx}.$$

Theorem 1.3. *Let $p \geq 2$, $s \in (0,1)$, Ω be a $C^{1,1}$ domain and assume that $g \in W_0^{s,p}(\Omega) \cap C(\overline{\Omega})$ satisfies $|g| \leq \Psi$, where Ψ is a ground state for $(-\Delta_p)^s$. Then there is a unique viscosity solution of (1-4) that is also a weak solution. In addition, the convergence in (1-5) is uniform in $\overline{\Omega}$.*

In our previous work [Hynd and Lindgren 2016], we studied the large time behavior of the doubly nonlinear, local equation

$$|v_t|^{p-2}v_t = \Delta_p v. \tag{1-6}$$

One of the novelties of the present paper in comparison with the above-mentioned work is that we obtain *uniform* convergence to a ground state and a uniform Hölder estimate for the doubly nonlinear, nonlocal equation (1-1). No such results are known for (1-6). Related to this is also the work for more general systems in [Hynd 2016]. The method in these papers, as the method in the present paper, differs substantially from most of the other methods used in the literature to study asymptotic behavior of nonlinear and possibly degenerate flows, as in [Agueh et al. 2010; Aronson and Peletier 1981; Armstrong and Trokhimtchouk 2010; Kamin and Vázquez 1988; Kim and Lee 2013; Stan and Vázquez 2013]. Our methods are based on energy and compactness in Sobolev spaces, while most of the earlier work is based on comparison principles. This allows us, in contrast to most earlier work, to treat initial data without any assumption on the sign.

In the case of a linear equation, i.e., when $p = 2$, the large time behavior of solutions is especially well understood. Due to the theory of eigenfunctions in Hilbert spaces, one can then recover our result (and more) using the eigenfunction expansion. When $p \neq 2$, this expansion is not available.

The literature on equations of the type (1-1) is very limited. Equations of type (1-6) appear in [Kilpeläinen and Lindqvist 1996] and in the theory of doubly nonlinear flows. In the case of linear nonlocal equations, i.e., when $p = 2$, the literature on regularity is vast. We mention only a fraction; see [Silvestre 2010; 2012; Caffarelli and Vasseur 2010; Lara and Dávila 2014]. Neither of these results apply to our setting. However, our proof of the Hölder regularity is very much inspired by the work of Luis Silvestre [2010; 2012]. We also seize the opportunity to mention the recent papers [Puhst 2015; Mazón et al. 2016; Vázquez 2016; Warma 2016] where the corresponding heat flow is studied, i.e., the equation

$$v_t + (-\Delta_p)^s v = 0.$$

The stationary equation, i.e.,

$$(-\Delta_p)^s v = 0,$$

has in recent years attracted a lot of attention; see [Ishii and Nakamura 2010; Brasco and Lindgren 2017; Chasseigne and Jakobsen 2015; Di Castro et al. 2014; 2016; Chambolle et al. 2012; Korvenpää et al. 2016; Iannizzotto et al. 2016; Kuusi et al. 2015a; 2015b; Gal and Warma 2016; Lindgren 2016]. In [Bjorland et al. 2012], a different nonlocal version of the p-Laplacian is studied.

The plan of the paper is as follows. In Section 2, we introduce the fractional Sobolev spaces $W^{s,p}$, the fractional p-Laplacian $(-\Delta_p)^s$ and additional notation used in this paper. In Section 3, we define weak solutions and derive several of their important properties. The section ends with a key compactness result and some brief explanations on how to construct weak solutions. This is followed by Section 4, where we introduce viscosity solutions and prove that the weak solution constructed in Section 3 is also the unique viscosity solution. In Section 5, we verify Hölder estimates for viscosity solutions. Finally, in Section 6, we prove Theorem 1.2 and Theorem 1.3, which involves the large time behavior of weak solutions.

2. Notation and prerequisites

The fractional Rayleigh quotient (1-3) naturally relates to the so-called fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$. If $1 < p < \infty$ and $s \in (0, 1)$, the norm is given by

$$\|u\|^p_{W^{s,p}(\mathbb{R}^n)} = [u]^p_{W^{s,p}(\mathbb{R}^n)} + \|u\|^p_{L^p(\mathbb{R}^n)},$$

where the Gagliardo seminorm is

$$[u]^p_{W^{s,p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^p}{|y-x|^{sp+n}}\,dx\,dy = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(y)-u(x)|^p\,d\mu(x,y).$$

Here and throughout, we will use the notation

$$d\mu(x,y) := |x-y|^{-n-sp}\,dx\,dy.$$

The space $W^{s,p}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,p}(\mathbb{R}^n)}$. Many properties that are known for the more common Sobolev spaces $W^{1,p}$, also hold for $W^{s,p}$ and can be found in [Di Nezza et al. 2012]. In particular, we have the compact embedding of $W^{s,p}_0(\Omega)$ in $L^q(\Omega)$ for $q \in [1,p]$. This result can be found in Theorem 2.7 in [Brasco et al. 2014] (see also Theorem 7.1 on page 33 in [Di Nezza et al. 2012]).

The operator $(-\Delta_p)^s$ arises as the first variation of the functional

$$[u]^p_{W^{s,p}(\mathbb{R}^n)}.$$

More specifically, minimizers satisfy

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))\,d\mu(x,y) = 0$$

for each $\phi \in W^{s,p}_0(\Omega)$. If the solution is regular enough, one can split this into two equal terms, make a change of variables and write the equation in the sense of the principal value, as in (1-2). Note that the notation $(-\Delta_p)^s$ is slightly abusive; this operator is not the s-th power of $-\Delta_p$ unless $p = 2$. See Section 3 in [Di Nezza et al. 2012].

Ground states of $(-\Delta_p)^s$ are minimizers of the Rayleigh quotient

$$\lambda_{s,p} = \inf_{u\in W^{s,p}_0(\Omega)} \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x)-u(y)|^p\,d\mu(x,y)}{\int_\Omega |u(x)|^p\,dx},$$

and they are signed solutions of

$$\begin{cases} (-\Delta_p)^s u = \lambda_{s,p}|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The notation

$$\mathcal{J}_p(t) = |t|^{p-2}t$$

will also come in handy. With this notation, equation (1-1) can be written as

$$\mathcal{J}_p(v_t) + (-\Delta_p)^s v = 0$$

and the operator $(-\Delta_p)^s$ can be written as

$$(-\Delta_p)^s u(x) = 2\,\text{P.V.}\int_{\mathbb{R}^n} \frac{\mathcal{J}_p(u(x)-u(x+y))}{|y|^{n+sp}}\,dy.$$

Due to the scaling of the equation, we introduce the following notation for parabolic cylinders

$$Q_r(x_0,t_0) = B_r(x_0)\times(t_0 - r^{\frac{sp}{p-1}}, t_0 + r^{\frac{sp}{p-1}}), \quad Q_r^-(x_0,t_0) = B_r(x_0)\times(t_0 - r^{\frac{sp}{p-1}}, t_0),$$

where $B_r(x_0)$ is the ball of radius r centered at x_0. When $x_0 = 0$ and $t_0 = 0$, we will simply write B_r, Q_r and Q_r^-.

3. Weak solutions

In this section, we present our theory of weak solutions of (1-4). The main results are that the Rayleigh quotient is monotone along the flow (Proposition 3.6) and that "bounded" sequences of weak solutions are compact (Theorem 3.8). The interested reader could also consult [Hynd and Lindgren 2016], where a similar theory is built for equation (1-6).

Definition 3.1. Let $g \in W_0^{s,p}(\Omega)$. We say that v is a *weak solution* of (1-4) if

$$v \in L^\infty([0,\infty); W_0^{s,p}(\Omega)), \quad v_t \in L^p(\Omega\times[0,\infty)), \tag{3-1}$$

and

$$\int_\Omega |v_t(x,t)|^{p-2}v_t(x,t)\phi(x)\,dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{J}_p(v(x,t)-v(y,t))(\phi(x)-\phi(y))\,d\mu(x,y) = 0 \tag{3-2}$$

for each $\phi \in W_0^{s,p}(\Omega)$ and for a.e. $t>0$, and

$$v(x,0) = g(x). \tag{3-3}$$

Remark 3.2. We note that if v satisfies (3-1), the L^p norm of v is absolutely continuous in time (one can for instance adapt the proof of Theorem 3 on page 287 of [Evans 2010]), so that it makes sense to assign values in $L^p(\Omega)$ at $t = 0$, as in (3-3).

In the rest of this section, we derive various identities and estimates for weak solutions.

Lemma 3.3. *Assume v is a weak solution of (1-4). Then*

$$[v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}$$

is absolutely continuous in t and

$$\frac{d}{dt}\frac{1}{p}[v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} = -\int_\Omega |v_t(x,t)|^p\,dx \tag{3-4}$$

holds for almost every $t>0$.

Proof. Define

$$\Phi(w) := \begin{cases} \frac{1}{p}[w]^p_{W^{s,p}(\mathbb{R}^n)}, & w \in W_0^{s,p}(\Omega),\\ +\infty & \text{otherwise}\end{cases}$$

for each $w \in L^p(\Omega)$. Then Φ is convex, proper, and lower-semicontinuous. In addition, (3-2) implies

$$\partial\Phi(v(\cdot,t)) = \{-|v_t(\cdot,t)|^{p-2}v_t(\cdot,t)\}$$

for almost every $t > 0$. Since $t \mapsto v(\cdot\,,t)$ is absolutely continuous in $L^p(\Omega)$ (see Remark 3.2) and since $|\partial\Phi(v)||v_t| \in L^1(\Omega\times[0,T))$ for any $T > 0$, Remark 1.4.6 in [Ambrosio et al. 2008] implies that $t \mapsto \Phi(v(\cdot\,,t))$ is absolutely continuous and that identity (3-4) holds for a.e. $t > 0$. □

Lemma 3.4. *Assume v is a weak solution of (1-4). Then*

$$\frac{d}{dt}[v(\cdot\,,t)]^p_{W^{s,p}(\mathbb{R}^n)} \le -p\mu_{s,p}[v(\cdot\,,t)]^p_{W^{s,p}(\mathbb{R}^n)} \tag{3-5}$$

for a.e. $t > 0$, and

$$[v(\cdot\,,t)]^p_{W^{s,p}(\mathbb{R}^n)} \le e^{-(p\mu_{s,p})t}[g]_{W^{s,p}(\mathbb{R}^n)} \tag{3-6}$$

for each $t > 0$.

Proof. By Lemma 3.3, $v(\cdot\,,t)$ is an admissible test function in (3-2), which yields

$$\begin{aligned}
[v(\cdot\,,t)]^p_{W^{s,p}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{J}_p(v(x,t)-v(y,t))(v(x,t)-v(y,t))\,d\mu(x,y)\\
&= -\int_\Omega |v_t(x,t)|^{p-2}v_t(x,t)\cdot v(x,t)\,dx\\
&\le \left(\int_\Omega |v_t(x,t)|^p\,dx\right)^{1-\frac{1}{p}}\left(\int_\Omega |v(x,t)|^p\,dx\right)^{\frac{1}{p}}\\
&\le \lambda_{s,p}^{-\frac{1}{p}}\left(\int_\Omega |v_t(x,t)|^p\,dx\right)^{1-\frac{1}{p}}[v(\cdot\,,t)]_{W^{s,p}(\mathbb{R}^n)}.
\end{aligned} \tag{3-7}$$

Hence,

$$[v(\cdot\,,t)]^p_{W^{s,p}(\mathbb{R}^n)} \le \frac{1}{\mu_{s,p}}\int_\Omega |v_t(x,t)|^p\,dx. \tag{3-8}$$

Identity (3-4) together with (3-8) implies (3-5). From Grönwall's inequality, we can now deduce inequality (3-6). □

Corollary 3.5. *Let v be a weak solution of (1-4). Then the function*

$$e^{(\mu_{s,p}p)t}[v(\cdot\,,t)]^p_{W^{s,p}(\mathbb{R}^n)}$$

is nonincreasing in t and

$$\frac{1}{p}\frac{d}{dt}\left(e^{(\mu_{s,p}p)t}[v(\cdot\,,t)]^p_{W^{s,p}(\mathbb{R}^n)}\right) = e^{(\mu_{s,p}p)t}\left(\mu_{s,p}[v(\cdot\,,t)]^p_{W^{s,p}(\mathbb{R}^n)} - \int_\Omega |v_t(x,t)|^p\,dx\right) \le 0 \tag{3-9}$$

for a.e. $t \ge 0$.

Proof. The monotonicity is a consequence of (3-5). The identity (3-9) follows from (3-4). □

Proposition 3.6. *Assume that v is a weak solution of (1-4) such that $v(\cdot\,,t) \ne 0 \in L^p(\Omega)$ for each $t \ge 0$. Then the Rayleigh quotient*

$$\frac{[v(\cdot\,,t)]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |v(x,t)|^p\,dx}$$

is nonincreasing in t.

Proof. By (3-1),

$$\frac{d}{dt}\int_\Omega \frac{1}{p}|v(x,t)|^p\,dx = \int_\Omega |v(x,t)|^{p-2}v(x,t)v_t(x,t)\,dx \tag{3-10}$$

for a.e. $t>0$. Suppressing the (x,t)-dependence, we compute, using (3-4) in Lemma 3.3 and (3-10), to find

$$\begin{aligned}\frac{d}{dt}\frac{[v]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |v|^p\,dx} &= -p\frac{\int_\Omega |v_t|^p\,dx}{\int_\Omega |v|^p\,dx} - p\frac{[v]^p_{W^{s,p}(\mathbb{R}^n)}}{\left(\int_\Omega |v|^p\,dx\right)^2}\int_\Omega |v|^{p-2}vv_t\,dx \\ &= \frac{p}{\left(\int_\Omega |v|^p\,dx\right)^2}\left([v]^p_{W^{s,p}(\mathbb{R}^n)}\int_\Omega |v|^{p-2}v(-v_t)\,dx - \int_\Omega |v|^p\,dx\int_\Omega |v_t|^p\,dx\right)\end{aligned} \tag{3-11}$$

for a.e. $t>0$. By Hölder's inequality,

$$\int_\Omega |v|^{p-2}v(-v_t)\,dx \le \left(\int_\Omega |v|^p\,dx\right)^{1-\frac{1}{p}}\left(\int_\Omega |v_t|^p\,dx\right)^{\frac{1}{p}},$$

which together with (3-7) gives

$$[v]^p_{W^{s,p}(\mathbb{R}^n)}\int_\Omega |v|^{p-2}v(-v_t)\,dx \le \int_\Omega |v|^p\,dx\int_\Omega |v_t|^p\,dx.$$

Inserted into (3-11), this yields

$$\frac{d}{dt}\frac{[v]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |v|^p\,dx} \le 0. \qquad \square$$

As a corollary, we obtain that any weak solution with a ground state as initial data can be written explicitly. Since the proof is exactly the same as the proof of the corresponding result in [Hynd and Lindgren 2016], Corollary 2.5, we have chosen to omit it.

Corollary 3.7. *Suppose that v is a weak solution of (1-4) and that g is a ground state of $(-\Delta_p)^s$. Then*

$$v(x,t) = e^{-\mu_{s,p}t}g(x).$$

The following compactness result is the key both to the long time behavior and to the construction of weak solutions, as we will see. The proof is based on the compact embedding of $W_0^{s,p}(\Omega)$ into $L^p(\Omega)$ and it is fairly similar to the proof of Theorem 2.6 in [Hynd and Lindgren 2016].

Theorem 3.8. *Assume $\{g^k\}_{k\in\mathbb{N}} \in W_0^{s,p}(\Omega)$ is uniformly bounded in $W_0^{s,p}(\Omega)$ and that v^k is a weak solution of (1-4) with $v^k(x,0)=g^k(x)$. Then there is a subsequence $\{v^{k_j}\}_{j\in\mathbb{N}} \subset \{v^k\}_{k\in\mathbb{N}}$ and v satisfying (3-1) such that*

$$v^{k_j} \to v \quad \text{in } \begin{cases} C([0,T];L^p(\Omega)) & \text{for all } T>0, \\ L^r_{\text{loc}}([0,\infty);W^{s,p}(\mathbb{R}^n)) & \text{for all } 1\le r<\infty \end{cases} \tag{3-12}$$

and

$$v_t^{k_j} \to v_t \quad \text{in } L^p_{\text{loc}}([0,\infty);L^p(\Omega)) \tag{3-13}$$

as $j\to\infty$. Moreover, v is a weak solution of (1-4), where g is a weak limit of $\{g^{k_j}\}_{k\in\mathbb{N}}$ in $W_0^{s,p}(\Omega)$.

Proof. As in (3-4),

$$\frac{d}{dt}\frac{1}{p}[v^k(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} = -\int_\Omega |v^k_t(x,t)|^p\,dx \tag{3-14}$$

for almost every $t > 0$. After integration, we obtain

$$p\int_0^\infty\int_\Omega |v^k_t(x,t)|^p\,dx\,dt + \sup_{t\geq 0}[v_k(\cdot,t)]_{W^{s,p}(\mathbb{R}^n)} \leq 2[g^k]^p_{W^{s,p}(\mathbb{R}^n)}. \tag{3-15}$$

By assumption, the right-hand side above is uniformly bounded. It follows that the sequence $\{v^k\}_{k\in\mathbb{N}} \in C_{\text{loc}}([0,\infty), L^p(\Omega))$ is equicontinuous, and $\{v^k(\cdot,t)\}_{k\in\mathbb{N}} \in W^{s,p}_0(\Omega)$ is uniformly bounded for each $t \geq 0$. By Theorem 1 in [Simon 1987], we can conclude that there is a subsequence $\{v^{k_j}\}_{j\in\mathbb{N}} \subset \{v^k\}_{k\in\mathbb{N}}$ converging in $C_{\text{loc}}([0,\infty), L^p(\Omega))$ to some v satisfying (3-1). Passing to a further subsequence, we may also assume that $v^{k_j} \rightharpoonup v$ in $L^p_{\text{loc}}([0,\infty); W^{s,p}(\mathbb{R}^n))$.

Since v^k_t is bounded in $L^p(\Omega\times[0,\infty))$, we may also assume

$$\begin{cases} v^{k_j}_t \rightharpoonup v_t & \text{in } L^p(\Omega\times[0,\infty)), \\ \mathcal{J}_p(v^{k_j}_t) \rightharpoonup \xi & \text{in } L^q(\Omega\times[0,\infty)), \end{cases}$$

where $1/p + 1/q = 1$. We will prove below that

$$\xi = \mathcal{J}_p(v_t). \tag{3-16}$$

Let us assume for the moment that (3-16) holds. Note that since the function $|z|^p$ is convex,

$$\frac{1}{p}[w]^p_{W^{s,p}(\mathbb{R}^n)} \geq \frac{1}{p}[v^{k_j}(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} - \int_\Omega \mathcal{J}_p(v^{k_j}_t(x,t))(w(x)-v^{k_j}(x,t))\,dx$$

for any $w \in W^{s,p}_0(\Omega)$. Integrating over the interval $[t_0,t_1]$ and passing to the limit, we obtain

$$\int_{t_0}^{t_1}\frac{1}{p}[w]^p_{W^{s,p}(\mathbb{R}^n)}\,dt \geq \int_{t_0}^{t_1}\left(\frac{1}{p}[v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} - \int_\Omega \xi(x,t)(w(x)-v(x,t))\,dx\right)dt.$$

Here we made use of Fatou's lemma, the weak convergence of $\mathcal{J}_p(v^{k_j}_t)$ and the strong convergence of v^{k_j}.

Therefore,

$$\frac{1}{p}[w]^p_{W^{s,p}(\mathbb{R}^n)} \geq \frac{1}{p}[v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} - \int_\Omega \xi(x,t)(w(x)-v(x,t))\,dx$$

for a.e. $t \geq 0$. In particular, for each $\phi \in W^{s,p}_0(\Omega)$,

$$\int_\Omega \xi(x,t)\phi(x)\,dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{J}_p(v(x,t)-v(y,t))(\phi(x)-\phi(y))\,d\mu(x,y) = 0 \tag{3-17}$$

for a.e. $t \geq 0$. Thus, once we verify (3-16), v is then a weak solution of (1-4).

For each interval $[t_0, t_1]$,

$$\begin{aligned}
\lim_{j\to\infty}\int_{t_0}^{t_1}[v^{k_j}(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}\,dt
&= \lim_{j\to\infty}\int_{t_0}^{t_1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathcal{J}_p(v^{k_j}(x,t)-v^{k_j}(y,t))(v^{k_j}(x,t)-v^{k_j}(y,t))\,d\mu(x,y)\\
&= -\lim_{j\to\infty}\int_{t_0}^{t_1}\int_\Omega \mathcal{J}_p(v_t^{k_j}(x,t))v^{k_j}(x,t)\,dx\,dt\\
&= -\int_{t_0}^{t_1}\int_\Omega \xi(x,t)v(x,t)\,dx\,dt\\
&= \int_{t_0}^{t_1}[v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}\,dt,
\end{aligned}$$

where the last equality is a consequence of (3-17). Since weak convergence together with convergence of the norm implies strong convergence, we have

$$v^{k_j}\to v \quad \text{in } L^p_{\mathrm{loc}}([0,\infty);W^{s,p}(\mathbb{R}^n)).$$

It is now routine to combine the interpolation of L^p spaces with the uniform bound (3-15) to obtain the stronger convergence $v^{k_j}\to v$ in $L^r_{\mathrm{loc}}([0,\infty);W^{s,p}(\mathbb{R}^n))$ for each $1\le r<\infty$. Further, upon extracting yet another subsequence, we can assume that

$$[v^{k_j}(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}\to[v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} \tag{3-18}$$

as $j\to\infty$ for a.e. $t\ge 0$.

We will now verify (3-16). As in the proof of Lemma 3.3, (3-17) implies

$$\frac{d}{dt}\frac{1}{p}[v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}=-\int_\Omega \xi(x,t)v_t(x,t)\,dx$$

for a.e. $t\ge 0$. Therefore, for each $t_1>t_0$,

$$\int_{t_0}^{t_1}\int_\Omega \xi(x,s)v_t(x,s)\,dx\,ds+\frac{1}{p}[v(\cdot,t_1)]^p_{W^{s,p}(\mathbb{R}^n)}=\frac{1}{p}[v(\cdot,t_0)]^p_{W^{s,p}(\mathbb{R}^n)}. \tag{3-19}$$

In addition, integrating (3-14) yields

$$\int_{t_0}^{t_1}\int_\Omega \frac{1}{p}|v_t^{k_j}(x,s)|^p+\frac{1}{q}|\mathcal{J}_p(v_t^{k_j}(x,s))|^q\,dx\,ds+\frac{1}{p}[v^{k_j}(\cdot,t_1)]^p_{W^{s,p}(\mathbb{R}^n)} = \frac{1}{p}[v^{k_j}(\cdot,t_0)]^p_{W^{s,p}(\mathbb{R}^n)}. \tag{3-20}$$

Let now t_0 and t_1 be times for which (3-18) holds and pass to the limit to obtain

$$\int_{t_0}^{t_1}\int_\Omega \frac{1}{p}|v_t(x,s)|^p+\frac{1}{q}|\xi(x,s)|^q\,dx\,ds+\frac{1}{p}[v(\cdot,t_1)]^p_{W^{s,p}(\mathbb{R}^n)}\le\frac{1}{p}[v(\cdot,t_0)]^p_{W^{s,p}(\mathbb{R}^n)}$$

by weak convergence. Together with (3-19) this implies

$$\int_{t_0}^{t_1}\int_\Omega\Big(\frac{1}{p}|v_t(x,s)|^p+\frac{1}{q}|\xi(x,s)|^q-\xi(x,s)v_t(x,s)\Big)\,dx\,ds\le 0.$$

Identity (3-16) now follows from the case of equality in Young's inequality.

Substituting $\xi=\mathcal{J}_p(v_t)$ into (3-19) and passing to the limit as $j\to\infty$ in (3-20) also gives

$$\lim_{j\to\infty}\int_{t_0}^{t_1}\int_\Omega |v_t^{k_j}(x,s)|^p\,dx\,ds=\int_{t_0}^{t_1}\int_\Omega |v_t(x,s)|^p\,dx\,ds.$$

Again, since weak convergence together with convergence of the norm implies strong convergence, we obtain (3-13). ☐

Let us now discuss how the ideas above can be used to construct weak solutions. As in [Hynd and Lindgren 2016], we aim to build weak solutions (1-4) by using the implicit time scheme for $\tau>0$: $v^0:=g$,

$$\begin{cases}\mathcal{J}_p\Big(\dfrac{v^k-v^{k-1}}{\tau}\Big)+(-\Delta_p)^s v^k=0 & \text{for } x\in\Omega,\\ v^k=0 & \text{for } x\in\mathbb{R}^n\setminus\Omega,\end{cases}\qquad \text{where } k=1,2,\dots,N. \tag{3-21}$$

The direct methods in the calculus of variations can be used to show that this scheme has a unique weak solution sequence $\{v^1,\dots,v^N\}\subset W_0^{s,p}(\Omega)$ for each $\tau>0$ and N, in the sense that

$$\int_\Omega \mathcal{J}_p\Big(\frac{v^k(x)-v^{k-1}(x)}{\tau}\Big)\phi(x)\,dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathcal{J}_p(v^k(x)-v^k(y))(\phi(x)-\phi(y))\,d\mu(x,y)=0$$

for any $\phi\in W_0^{s,p}(\Omega)$. Our candidate for a solution $v(x,t)$ of (1-4) is the limit of $v^N(x)$, when N tends to infinity with $\tau=t/N$.

Choosing $\phi=v^k-v^{k-1}$ as test function, we obtain

$$\int_\Omega\frac{|v^k-v^{k-1}|^p}{\tau^{p-1}}\,dx+\frac{1}{p}[v^k]^p_{W^{s,p}(\mathbb{R}^n)}\le\frac{1}{p}[v^{k-1}]^p_{W^{s,p}(\mathbb{R}^n)},\qquad\text{where } k=1,\dots,N.$$

Summing over $k=1,\dots,j\le N$ yields

$$\sum_{k=1}^{j}\int_\Omega\frac{|v^k-v^{k-1}|^p}{\tau^{p-1}}\,dx+\frac{1}{p}[v^j]^p_{W^{s,p}(\mathbb{R}^n)}\le\frac{1}{p}[g]^p_{W^{s,p}(\mathbb{R}^n)}, \tag{3-22}$$

which is a discrete analogue of the energy identity (3-4).

Let $\tau=T/N$ and $\tau_k=k\tau$, and define the "linear interpolation" of the solution sequence as

$$w_N(x,t):=v^{k-1}(x)+\Big(\frac{t-\tau_{k-1}}{\tau}\Big)(v^k(x)-v^{k-1}(x)),\quad\text{where } \tau_{k-1}\le t\le\tau_k,\ k=1,\dots,N. \tag{3-23}$$

From (3-22) we conclude

$$p\int_0^T\int_\Omega|\partial_t w_N(x,t)|^p\,dx\,dt+\sup_{0\le t\le T}[w_N(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}\le 2[g]^p_{W^{s,p}(\mathbb{R}^n)}. \tag{3-24}$$

Arguing as in the proof of Theorem 3.8, we can obtain a subsequence w_{N_j} and a weak solution w of (1-1) on $\Omega \times (0,T)$ such that

$$w_{N_j} \to w \quad \text{in} \begin{cases} C([0,T]; L^p(\Omega)), \\ L^p([0,T]; W^{s,p}(\mathbb{R}^n)) \end{cases}$$

and

$$\begin{cases} \partial_t w_{N_j} \to w_t & \text{in } L^p(\Omega \times [0,T]), \\ \mathcal{J}_p(\partial_t w_{N_j}) \rightharpoonup \mathcal{J}_p(w_t) & \text{in } L^q(\Omega \times [0,T]). \end{cases}$$

It remains to construct a global-in-time solution. This can be accomplished as follows: Let $k \in \mathbb{N}$ and let w^k be the weak solution of (1-1) above for $T = k$. Define

$$z^k(\cdot,t) = \begin{cases} w^k(\cdot,t), & t \in [0,k], \\ w^k(\cdot,k), & t \in [k,\infty). \end{cases}$$

One readily verifies that z^k satisfies (3-1). In addition, the proof of Theorem 3.8 can easily be adapted to give that z^k has a subsequence converging as in (3-12) and (3-13) to a global weak solution of (1-4). We omit the details.

Remark 3.9. At this point, we seize the opportunity to mention that the "step function" approximation

$$v_N(\cdot,t) := \begin{cases} g, & t = 0, \\ v^k, & t \in (\tau_{k-1}, \tau_k],\ k = 1,\dots,N, \end{cases} \tag{3-25}$$

converges in $C([0,T]; L^p(\Omega))$ to the same weak solution v as the linear interpolating sequence (3-23). Indeed, by (3-22),

$$\begin{aligned} \int_\Omega |w_N(x,t) - v_N(x,t)|^p\,dx &\le \max_{1\le k\le N} \int_\Omega |v^k(x) - v^{k-1}(x)|^p\,dx \\ &\le \frac{1}{p}\tau^{p-1}[g]^p_{W^{s,p}(\mathbb{R}^n)} \\ &= \frac{1}{p}\Big(\frac{T}{N}\Big)^{p-1}[g]^p_{W^{s,p}(\mathbb{R}^n)}. \end{aligned}$$

This fact will be used in Section 4, where we verify that the viscosity solution we construct is also a weak solution.

4. Viscosity solutions

Throughout this section we assume that $\partial\Omega$ is $C^{1,1}$, $p \ge 2$, $g \in W^{s,p}_0(\Omega) \cap C(\overline{\Omega})$ and that there is a ground state Ψ such that

$$|g| \le \Psi.$$

Our main result in this section is:

Proposition 4.1. *There is a unique viscosity solution v of the initial value problem (1-4) which is also a weak solution.*

It is not known whether or not uniqueness holds for weak solutions of (1-4), even in the local case. However, quite standard methods for viscosity solutions apply to (1-4). The key here is that the term $|v_t|^{p-2}v_t$ is strictly monotone with respect to v_t. In what follows, we will prove that the discrete scheme (3-21) converges both to the unique viscosity solution and to a weak solution.

We first define *viscosity solutions* of the relevant equations.

Definition 4.2. Let Ω be an open set in $\mathbb{R}^n$ and $f(x,u)$ a continuous function. A function $u \in L^\infty(\mathbb{R}^n)$ which is upper semicontinuous in Ω is a *subsolution* of

$$(-\Delta_p)^s u \le f(x,u) \quad \text{in } \Omega$$

if the following holds: whenever $x_0 \in \Omega$ and $\phi \in C^2(B_r(x_0))$ for some $r > 0$ are such that

$$\phi(x_0) = u(x_0), \qquad \phi(x) \ge u(x) \quad \text{for } x \in B_r(x_0) \subset \Omega,$$

then we have

$$(-\Delta_p)^s \phi^r\,(x_0) \le f(x_0, \phi(x_0)),$$

where

$$\phi^r = \begin{cases} \phi & \text{in } B_r(x_0), \\ u & \text{in } \mathbb{R}^n \setminus B_r(x_0). \end{cases}$$

A *supersolution* is defined similarly and a *solution* is a function which is both a sub- and a supersolution.

Remark 4.3. For a bounded function f which is C^2 in a neighborhood of x_0, we know $(-\Delta_p)^s f\,(x_0)$ is well defined. Indeed, we may split the operator into one integral over $B_1(x_0)$ and another over $\mathbb{R}^n \setminus B_1(x_0)$. The latter is well-defined since f is bounded. For the former, we may write, for $\varepsilon > 0$,

$$\begin{aligned} 2\text{P.V.}\int_{B_1(x_0)} & \mathcal{J}_p(f(x_0)-f(y))|x_0-y|^{-n-sp}\,dy \\ &= 2\lim_{\varepsilon\to 0}\int_{B_1\setminus B_\varepsilon} \mathcal{J}_p(f(x_0)-f(x_0+y))|y|^{-n-sp}\,dy \\ &= \lim_{\varepsilon\to 0}\int_{B_1\setminus B_\varepsilon} \big(\mathcal{J}_p(f(x_0)-f(x_0+y)) + \mathcal{J}_p(f(x_0)-f(x_0-y))\big)|y|^{-n-sp}\,dy. \end{aligned}$$

Since f is a C^2 function and $p \ge 2$, we have the estimate

$$|f(x)-f(x+y)|^{p-2}(f(x)-f(x+y)) + |f(x)-f(x-y)|^{p-2}(f(x)-f(x-y)) \le C|y|^p.$$

This is due to the elementary inequality

$$\big||a+b|^{p-2}(a+b) - |a|^{p-2}a\big| \le C|b|(|a|+|b|)^{p-2}.$$

Therefore, the integral

$$\int_{B_1} \big(\mathcal{J}_p(f(x_0)-f(x_0+y)) + \mathcal{J}_p(f(x_0)-f(x_0-y))\big)|y|^{-n-sp}\,dy$$

is absolutely convergent, so that the principal value exists.

We define viscosity solutions of the evolutionary equation (1-1). We introduce the class of test functions

$$C^{2,1}_{x,t}(D\times I), \quad \text{where } D\times I\subset\mathbb{R}^n\times\mathbb{R},$$

consisting of functions that are C^2 in the spatial variables and C^1 in t, in the set $D\times I$.

Definition 4.4. Let $\Omega\in\mathbb{R}^n$ be an open set, $I\in\mathbb{R}$ be an open interval. A function $v\in L^\infty(\mathbb{R}^n\times I)$ which is upper semicontinuous in $\Omega\times I$ is a *subsolution* of

$$|v_t|^{p-2}v_t+(-\Delta_p)^s v\le C \quad \text{in } \Omega\times I$$

if the following holds: whenever $(x_0,t_0)\in\Omega\times I$ and $\phi\in C^{2,1}_{x,t}\big(B_r(x_0)\times(t_0-r,t_0+r)\big)$ for some $r>0$ are such that

$$\phi(x_0,t_0)=v(x_0,t_0), \qquad \phi(x,t)\ge v(x,t) \quad \text{for } (x,t)\in B_r(x_0)\times(t_0-r,t_0+r)$$

then

$$|\phi_t(x_0,t_0)|^{p-2}\phi_t(x_0,t_0)+(-\Delta_p)^s\phi^r\,(x_0,t_0)\le C,$$

where

$$\phi^r(x,t)=\begin{cases}\phi & \text{in } B_r(x_0)\times(t_0-r,t_0+r),\\ v & \text{in } \mathbb{R}^n\setminus B_r(x_0)\times(t_0-r,t_0+r).\end{cases}$$

A *supersolution* is defined similarly and a *solution* is a function which is both a sub- and a supersolution.

Remark 4.5. In both of the definitions above, it is obvious that we can replace the condition that ϕ touches v from above at a point with the condition that $v-\phi$ has a maximum at a point. In addition, as is standard when dealing with viscosity solutions, it is enough to ask that ϕ touches v strictly at a point or equivalently that $v-\phi$ has a strict maximum at a point.

Now we are ready to treat the implicit scheme (3-21). We first construct viscosity solutions $v^1,\dots,v^N$.

Lemma 4.6. *For each N and τ, the implicit scheme (3-21) generates viscosity solutions $v^k\in C(\overline{\Omega})$ for $k=1,\dots,N$. Moreover,*

$$\max_{1\le k\le N}\sup_{\mathbb{R}^n}|v^k|\le\sup_{\mathbb{R}^n}|g|.$$

Proof. Consider the implicit scheme (3-21) for $k=1$:

$$\mathcal{J}_p\Big(\frac{v^1-g}{\tau}\Big)+(-\Delta_p)^s v^1=0, \quad \text{where } x\in\Omega.$$

This means that

$$\int_\Omega\mathcal{J}_p\Big(\frac{v^1-g}{\tau}\Big)\phi\,dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathcal{J}_p(v^1(x)-v^1(y))(\phi(x)-\phi(y))\,d\mu(x,y)=0$$

for any $\phi\in W^{s,p}_0(\Omega)$. The existence of such a weak solution follows from the direct methods of calculus of variations. Since $\mathcal{J}_p$ is strictly increasing, it is standard to prove a comparison principle for weak sub- and supersolutions; see, for instance, Lemma 9 in [Lindgren and Lindqvist 2014] for a proof. Clearly, the constant function $\sup_{\mathbb{R}^n}|g|$ is a supersolution; hence $v^1\le\sup_{\mathbb{R}^n}|g|$. Similarly, $v^1\ge-\sup_{\mathbb{R}^n}|g|$, and thus $|v^1|\le\sup_{\mathbb{R}^n}|g|$. By induction, $|v^k|\le\sup_{\mathbb{R}^n}|g|$ for $k=2,\dots,N$. As the left-hand side of the PDE

(3-21) is bounded it follows by Theorem 1.1 in [Iannizzotto et al. 2016], that v^k is continuous in $\overline{\Omega}$ for $k = 1, \dots, N$.

That each v^k is a viscosity solution can be verified by following the proof of Proposition 11 in [Lindgren and Lindqvist 2014] line by line. We omit the details. $\square$

The natural candidate for a viscosity solution of (1-4) is $\lim_{N\to\infty} v_N$, where v_N is defined in (3-25). Before proving this, we present some technical lemmas.

Lemma 4.7. *Let $N \in \mathbb{N}$. Further assume $\{\psi^0, \psi^1, \dots, \psi^N\} \subset C^2(\Omega)$ and $(x_0, k_0) \in \Omega \times \{1, \dots, N\}$ are such that*

$$v^k(x) - \psi^k(x) \le v^{k_0}(x_0) - \psi^{k_0}(x_0) \tag{4-1}$$

for x in $B_r(x_0)$ and $k \in \{k_0 - 1, k_0\}$. Then

$$\mathcal{J}_p\left(\frac{\psi^{k_0}(x_0) - \psi^{k_0-1}(x_0)}{\tau}\right) + (-\Delta)_p^s(\psi^{k_0})^r(x_0) \le 0.$$

Proof. Evaluating the left-hand side of (4-1) at $k = k_0$ gives

$$\mathcal{J}_p\left(\frac{v^{k_0}(x_0) - v^{k_0-1}(x_0)}{\tau}\right) + (-\Delta_p)^s(\psi^{k_0})^r(x_0) \le 0,$$

as v^k is a viscosity solution of (3-21). Evaluating the left-hand side of (4-1) at $x = x_0$ and $k = k_0 - 1$ gives $\psi^{k_0}(x_0) - \psi^{k_0-1}(x_0) \le v^{k_0}(x_0) - v^{k_0-1}(x_0)$. The claim follows from the above inequality and the monotonicity of $\mathcal{J}_p$. $\square$

Let $\bar{v}$ and $\underline{v}$ denote the weak upper and lower limits respectively of v_N defined in (3-25), i.e.,

$$\bar{v}(x,t) := \limsup_{\substack{N\to\infty \\ (y,s)\to(x,t)}} v_N(y,s), \qquad \underline{v}(x,t) := \liminf_{\substack{N\to\infty \\ (v,s)\to(x,t)}} v_N(y,s).$$

By Lemma 4.6, the sequence $\{v_N\}_{N\in\mathbb{N}}$ is bounded independently of $N \in \mathbb{N}$. As a result, $\bar{v}$ and $\underline{v}$ are well defined and finite. In addition, $\bar{v}$ and $-\underline{v}$ are upper semicontinuous. We recall the following result, which is Lemma 4.4 in [Hynd and Lindgren 2016]. Its statement there is for smooth ϕ, but the proof holds also for ϕ as below.

Lemma 4.8. *Assume $\phi \in C^{2,1}_{x,t}((\Omega \times (0,T))$. For $N \in \mathbb{N}$, define*

$$\phi_N(x,t) := \begin{cases} \phi(x,0), & t = 0, \\ \phi(x,\tau_k), & t \in (\tau_{k-1}, \tau_k],\ k = 1, \dots, N. \end{cases}$$

Suppose $\bar{v} - \phi$ has a strict local maximum at $(x_0, t_0) \in \Omega \times (0,T)$. Then there exist $(x_j, t_j) \to (x_0, t_0)$ and $N_j \to \infty$, as $j \to \infty$, such that $v_{N_j} - \phi_{N_j}$ has local maximum at (x_j, t_j). A corresponding result holds in the case of a strict local minimum.

Before proving the uniqueness of viscosity solutions, we need the following result, which verifies that whenever we can touch a subsolution from above with a $C^{2,1}_{x,t}$ function, we can treat the subsolution as a classical subsolution in space. The proof is almost identical to the one of Theorem 2.2 in [Caffarelli and Silvestre 2009] or the one of Proposition 1 in [Lindgren 2016].

Proposition 4.9. *Suppose*

$$|v_t|^{p-2}v_t+(-\Delta_p)^s v\le C \quad \text{in } B_1\times I$$

in the viscosity sense. Further assume that $(x_0,t_0)\in B_1\times I$ *and* $\phi\in C^{2,1}_{x,t}\big(B_r(x_0)\times(t_0-r,t_0+r)\big)$ *are such that*

$$\phi(x_0,t_0)=v(x_0,t_0),\qquad \phi(x,t)\ge v(x,t)\quad \text{for } (x,t)\in B_r(x_0)\times(t_0-r,t_0+r)$$

for some $r>0$. *Then* $(-\Delta_p)^s v$ *is defined pointwise at* (x_0,t_0) *and*

$$|\phi_t(x_0,t_0)|^{p-2}\phi_t(x_0,t_0)+(-\Delta_p)^s v\,(x_0,t_0)\le C.$$

Proof. For $0<\rho\le r$, let

$$\phi^\rho=\begin{cases}\phi & \text{in } B_\rho(x_0)\times(t_0-r,t_0+r),\\ v & \text{in } \mathbb{R}^n\setminus B_\rho(x_0)\times(t_0-r,t_0+r).\end{cases}$$

Since v is a viscosity subsolution,

$$|\phi_t(x_0,t_0)|^{p-2}\phi_t(x_0,t_0)+(-\Delta_p)^s\phi^\rho\,(x_0,t_0)\le C.$$

Now introduce the notation

$$\begin{aligned}\delta(\phi^\rho,x,y,t):=\tfrac12|\phi^\rho(x,t)-\phi^\rho(x+y,t)|^{p-2}(\phi^\rho(x,t)-\phi^\rho(x+y,t))\\ +\tfrac12|\phi^\rho(x,t)-\phi^\rho(x-y,t)|^{p-2}(\phi^\rho(x,t)-\phi^\rho(x-y,t)),\end{aligned}$$

$$\delta^\pm(\phi^\rho,x,y,t)=\max(\pm\delta(\phi^\rho,x,y,t),0).$$

Since ϕ^ρ is C^2 in space near x_0, we can substitute $-y$ for y in the integral and obtain the convergent integral

$$2\int_{\mathbb{R}^n}\delta(\phi^\rho,x_0,y,t_0)|y|^{-n-sp}\,dy\le C-|\phi_t(x_0,t_0)|^{p-2}\phi_t(x_0,t_0):=D. \tag{4-2}$$

See Remark 4.3 for more details.

We note that

$$\delta(\phi^{\rho_2},x_0,y,t_0)\le\delta(\phi^{\rho_1},x_0,y,t_0)\le\delta(v,x_0,y,t_0)\quad\text{for } \rho_1<\rho_2<r,$$

so that

$$\delta^-(\phi^{\rho_2},x_0,y,t_0)\ge\delta^-(\phi^{\rho_1},x_0,y,t_0)\ge\delta^-(v,x_0,y,t_0)\quad\text{for } \rho_1<\rho_2<r. \tag{4-3}$$

In particular,

$$\delta^-(v,x_0,y,t_0)\le|\delta(\phi^r,x_0,y,t_0)|.$$

Since $\big|\delta(\phi^r,x_0,y,t_0)|y|^{-n-sp}\big|$ is integrable, so is $\delta^-(v,x_0,y,t_0)|y|^{-n-sp}$. In addition, by (4-2),

$$2\int_{\mathbb{R}^n}\delta^+(\phi^\rho,x_0,y,t_0)|y|^{-n-sp}\,dy\le 2\int_{\mathbb{R}^n}\delta^-(\phi^\rho,x_0,y,t_0)|y|^{-n-sp}\,dy+D.$$

Thus, for $\rho_1 < \rho_2$,

$$2\int_{\mathbb{R}^n} \delta^+(\phi^{\rho_1}, x_0, y, t_0)|y|^{-n-sp}\,dy \le 2\int_{\mathbb{R}^n} \delta^-(\phi^{\rho_1}, x_0, y, t_0)|y|^{-n-sp}\,dy + D$$
$$\le 2\int_{\mathbb{R}^n} \delta^-(\phi^{\rho_2}, x_0, y, t_0)|y|^{-n-sp}\,dy + D < \infty, \tag{4-4}$$

where we have used (4-3).

Since $\delta^+(\phi^\rho, x_0, y, t_0) \nearrow \delta^+(v, x_0, y, t_0)$, the monotone convergence theorem implies

$$\int_{\mathbb{R}^n} \delta^+(\phi^\rho, x_0, y, t_0)|y|^{-n-sp}\,dy \to \int_{\mathbb{R}^n} \delta^+(v, x_0, y, t_0)|y|^{-n-sp}\,dy.$$

By (4-4),

$$2\int_{\mathbb{R}^n} \delta^+(v, x_0, y, t_0)|y|^{-n-sp}\,dy \le 2\int_{\mathbb{R}^n} \delta^-(\phi^\rho, x_0, y, t_0)|y|^{-n-sp}\,dy + D < \infty \tag{4-5}$$

for any $0<\rho<r$. We conclude that $\delta^+(v, x_0, y, t_0)|y|^{-n-sp}$ is integrable. By the dominated convergence theorem, we can pass to the limit in the right-hand side of (4-5) and obtain

$$2\int_{\mathbb{R}^n} \delta^+(v, x_0, y, t_0)|y|^{-n-sp}\,dy \le 2\int_{\mathbb{R}^n} \delta^-(v, x_0, y, t_0)|y|^{-n-sp}\,dy + D < \infty.$$

This is simply another way of writing

$$2\int_{\mathbb{R}^n} \delta(v, x_0, y, t_0)|y|^{-n-sp}\,dy \le D.$$

Therefore $(-\Delta_p)^s v\,(x_0, t_0)$ exists in the pointwise sense and $(-\Delta_p)^s v\,(x_0, t_0) \le D$, which concludes the proof as $D = C - |\phi_t(x_0, t_0)|^{p-2}\phi_t(x_0, t_0)$. □

Proposition 4.10. *Assume that u is a viscosity subsolution and that v is a viscosity supersolution of*

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v = 0 \quad \textit{in } \Omega \times (0, T).$$

Suppose $u, v \in L^\infty(\mathbb{R}^n \times [0, T])$, $u \le v$ in $\mathbb{R}^n \setminus \Omega \times [0, T]$ and

$$\limsup_{(x,t)\to(x_0,t_0)} u(x_0, t_0) \le \liminf_{(x,t)\to(x_0,t_0)} v(x_0, t_0) \quad \textit{for } (x_0, t_0) \in \partial\Omega \times (0, T) \cup \Omega \times \{0\}.$$

Then $u \le v$.

Proof. We employ the usual trick of adding a term $\delta/(t-T)$; let $\tilde{u} = u + \delta/(t-T)$. Then v is a supersolution, $\tilde{u}$ is a subsolution of

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v = -\frac{\delta}{(t-T)^2},$$

$\tilde{u} < v$ in $\mathbb{R}^n \setminus \Omega \times [0, T]$ and

$$\limsup_{(x,t)\to(x_0,t_0)} \tilde{u}(x_0, t_0) < \liminf_{(x,t)\to(x_0,t_0)} v(x_0, t_0) \tag{4-6}$$

for $(x_0,t_0)\in\partial\Omega\times(0,T)\cup\Omega\times\{0\}$. Moreover, $\tilde u(x,t)-v(x,t)\to-\infty$ as $t\to T$. It is now sufficient to prove that $\tilde u\le v$ for any $\delta>0$ since we can then let $\delta\to 0$. We argue by contradiction and assume that

$$\sup_{\mathbb{R}^n\times[0,T]}(\tilde u-v)>0.$$

Fix $\varepsilon>0$ and define

$$M_\varepsilon:=\sup_{\mathbb{R}^n\times[0,T]\times\mathbb{R}^n\times[0,T]}\left(\tilde u(x,t)-v(y,\tau)-\frac{|x-y|^2+|t-\tau|^2}{\varepsilon}\right).$$

Note $M_\varepsilon\ge\sup_{\mathbb{R}^n\times[0,T]}(\tilde u-v)>0$ and select $x_\varepsilon,y_\varepsilon\in\mathbb{R}^n$ and $t_\varepsilon,\tau_\varepsilon\in[0,T]$ for which

$$M_\varepsilon<\tilde u(x_\varepsilon,t_\varepsilon)-v(y_\varepsilon,\tau_\varepsilon)-\frac{|x_\varepsilon-y_\varepsilon|^2+|\tau_\varepsilon-t_\varepsilon|^2}{\varepsilon}+\varepsilon.$$

By Proposition 3.7 in [Crandall et al. 1992], $(x_\varepsilon,t_\varepsilon)$ and $(y_\varepsilon,\tau_\varepsilon)$ each have subsequences converging to $(\hat x,\hat t)\in\Omega\times(0,T)$ as $\varepsilon\to 0$ for which

$$\sup_{\mathbb{R}^n\times[0,T]}(\tilde u-v)=(\tilde u-v)(\hat x,\hat t).$$

As a result, there is ε small enough such that $x_\varepsilon,y_\varepsilon\in\Omega$ and $t_\varepsilon,\tau_\varepsilon\in(0,T)$. For this ε, it also follows that the maximum M_ε is attained in $\Omega\times(0,T)\times\Omega\times(0,T)$. For convenience, we will again call this point $(x_\varepsilon,t_\varepsilon,y_\varepsilon,\tau_\varepsilon)$.

Observe that the function

$$\frac{|x-y_\varepsilon|^2+|t-\tau_\varepsilon|^2}{\varepsilon}+\tilde u(x_\varepsilon,t_\varepsilon)-\frac{|x_\varepsilon-y_\varepsilon|^2+|t_\varepsilon-\tau_\varepsilon|^2}{\varepsilon}$$

touches $\tilde u$ from above at $(x_\varepsilon,t_\varepsilon)$ and

$$\frac{|x_\varepsilon-y|^2+|t_\varepsilon-\tau|^2}{\varepsilon}-v(y_\varepsilon,\tau_\varepsilon)-\frac{|x_\varepsilon-y_\varepsilon|^2+|t_\varepsilon-\tau_\varepsilon|^2}{\varepsilon}$$

touches $-v$ from above at $(y_\varepsilon,\tau_\varepsilon)$. From Proposition 4.9, we can conclude that $(-\Delta_p)^s\tilde u(x_\varepsilon,t_\varepsilon)$ and $(-\Delta_p)^s v(y_\varepsilon,\tau_\varepsilon)$ exist pointwise and satisfy

$$(-\Delta_p)^s\tilde u\,(x_\varepsilon,t_\varepsilon)<(\Delta_p)^s v\,(y_\varepsilon,\tau_\varepsilon).$$

In addition, since the function

$$\tilde u(x,t)-v(y,\tau)-\frac{|x-y|^2+|t-\tau|^2}{\varepsilon}$$

is larger at $(x_\varepsilon,y_\varepsilon,t_\varepsilon,\tau_\varepsilon)$ than at $(x_\varepsilon+y,y_\varepsilon+y,t_\varepsilon,\tau_\varepsilon)$ for any y, we obtain

$$\tilde u(x_\varepsilon,t_\varepsilon)-\tilde u(x_\varepsilon+y,t_\varepsilon)\ge v(y_\varepsilon,\tau_\varepsilon)-v(y_\varepsilon+y,\tau_\varepsilon).$$

This implies

$$(-\Delta_p)^s\tilde u\,(x_\varepsilon,t_\varepsilon)\ge(-\Delta_p)^s\,v(y_\varepsilon,\tau_\varepsilon),$$

which is a contradiction. Therefore, we must have $\tilde u\le v$. $\square$

Now we present a general result for nonlocal parabolic equations, inspired by previous work of Petri Juutinen [2001, Theorem 1]. This fact will be important in the proof of Hölder regularity of solutions of (1-1).

Proposition 4.11. *Suppose that v is a viscosity subsolution of*

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v \leq 0$$

in $B_r(x_0) \times (t_0 - r, t_0 + r)$ and $\phi \in C^{2,1}_{x,t}(B_r(x_0) \times (t_0 - r, t_0 + r))$. If

$$\phi(x_0, t_0) = v(x_0, t_0), \qquad \phi(x, t) \geq v(x, t) \quad \textit{for } (x, t) \in B_r(x_0) \times (t_0 - r, t_0], \tag{4-7}$$

then

$$|\phi_t(x_0, t_0)|^{p-2}\phi_t(x_0, t_0) + (-\Delta_p)^s \phi^r\,(x_0, t_0) \leq 0.$$

Proof. We argue by contradiction. If the assertion is not true then

$$|\phi_t(x_0, t_0)|^{p-2}\phi_t(x_0, t_0) + (-\Delta_p)^s \phi^r\,(x_0, t_0) \geq \varepsilon > 0$$

for some ε. Recall ϕ^r is defined in Definition 4.4. By continuity, we have

$$|\phi_t|^{p-2}\phi_t + (-\Delta_p)^s \phi^r \geq \tfrac{1}{2}\varepsilon > 0$$

in $B_\rho(x_0) \times (t_0 - \rho, t_0)$ for ρ small enough.

Let $\eta : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function satisfying

$$\begin{cases} 0 \leq \eta \leq 1, \\ \eta(x_0, t_0) = 0, \\ \eta(x, t) > 0 & \text{if } (x, t) \neq (x_0, t_0), \\ \eta(x, t) \geq 1 & \text{if } (x, t) \notin B_\rho(x_0) \times (t_0 - \rho, t_0). \end{cases}$$

Also define

$$\phi_\delta(x, t) = \phi(x, t) + \delta\eta(x, t) - \delta,$$

where $\delta > 0$ is considered small. By continuity,

$$|(\phi_\delta)_t|^{p-2}(\phi_\delta)_t + (-\Delta_p)^s(\phi_\delta)^r \geq \tfrac{1}{4}\varepsilon > 0$$

in $B_\rho(x_0) \times (t_0 - \rho, t_0)$, provided δ is small enough.

This means that $(\phi_\delta)^r$ is a supersolution in the pointwise classical sense in $B_\rho(x_0) \times (t_0 - \rho, t_0)$, and in particular it means that $(\phi_\delta)^r$ is a viscosity supersolution in this region. Moreover, $(\phi_\delta)^r \geq \phi^r \geq v$ in the complement of

$$\mathbb{R}^n \setminus B_\rho(x_0) \times (t_0 - \rho, t_0) \cup B_\rho(x_0) \times \{t_0 - \rho\}.$$

By Proposition 4.10, $(\phi_\delta)^r \geq v$ in $\mathbb{R}^n \times [t_0 - \rho, t_0]$. Furthermore, $(\phi_\delta)^r(x_0, t_0) \geq v(x_0, t_0)$ which is a contradiction since $(\phi_\delta)^r(x_0, t_0) = \phi(x_0, t_0) - \delta = v(x_0, t_0) - \delta$. □

Let us now return to our study of the implicit time scheme. We are now in position to construct barriers that assure that $\bar{v}$ and $\underline{v}$ satisfy the correct boundary and initial conditions.

Lemma 4.12. *Assume that $-\Psi \leq g \leq \Psi$, where Ψ is a nonnegative ground state of $(-\Delta_p)^s$. Then $\bar{v}$ and $\underline{v}$ satisfy the boundary condition in the classical sense*; *i.e.*,

$$\lim_{y\to x} \bar{v}(y,t) = \lim_{y\to x} \underline{v}(y,t) = 0 \quad \textit{for any } x \in \partial\Omega \textit{ and any } t \geq 0.$$

Proof. We observe that

$$-(-\Delta_p)^s\Psi - \frac{|\Psi-g|^{p-2}(\Psi-g)}{\tau^{p-1}} = -\lambda_{s,p}\Psi^{p-1} - \frac{(\Psi-g)^{p-1}}{\tau^{p-1}} \leq 0.$$

Hence Ψ is a supersolution of (3-21). Since $\Psi = v^1 = 0$ in $\mathbb{R}^n \setminus \Omega$, the comparison principle implies

$$v^1 \leq \Psi.$$

We can argue similarly to obtain

$$|v^1| \leq \Psi.$$

Iterating this method for each v^k yields $|v^k| \leq \Psi$ for any $k = 1, \dots, N$. By the definition of v_N in (3-25),

$$|v_N| \leq \Psi. \tag{4-8}$$

By inequality (4-8), the assertion would follow as long as Ψ is continuous up to the boundary. To establish this continuity, we first note that Ψ is globally bounded. This fact is due to Theorem 3.2 in [Franzina and Palatucci 2014], Theorem 3.3 in [Brasco et al. 2014] or Theorem 3.1 together with Remark 3.2 in [Brasco and Parini 2015]. Theorem 1.1 in [Iannizzotto et al. 2016] can now be used to establish the desired continuity of Ψ. □

Lemma 4.13. *Assume g is continuous. Then $\bar{v}$ and $\underline{v}$ satisfy the initial condition in the classical sense*; *i.e.*,

$$\lim_{t\to 0} \bar{v}(x,t) = \lim_{t\to 0} \underline{v}(x,t) = g(x) \quad \textit{for any } x \in \Omega.$$

Proof. Take η to be a bounded, smooth and strictly increasing radial function such that $\eta(0) = 0$. Let $d = \operatorname{diam}\Omega$ and

$$\alpha^{p-1} = \sup_{x\in B_d} \left|(-\Delta_p)^s\eta(x)\right|.$$

Clearly α is finite. Now we fix $x_0 \in \Omega$. We first prove that given $\varepsilon > 0$ there is $C = C(x_0, \varepsilon)$ such that

$$u(x) = g(x_0) + \varepsilon + C(\alpha\tau + \eta(x - x_0))$$

lies above v^1.

As g is continuous, for each $\varepsilon > 0$ there is $\delta > 0$ and $C > 0$ so that

$$|g(x) - g(x_0)| < \varepsilon \quad \text{if } |x - x_0| < \delta$$

and

$$\sup|g| \leq C\eta(x - x_0) \quad \text{if } |x - x_0| \geq \delta.$$

Upon choosing C even larger, we may also assume that $u \geq 0$ in $\mathbb{R}^n \setminus \Omega$. In addition

$$-(-\Delta_p)^s u - \frac{|u-g|^{p-2}(u-g)}{\tau^{p-1}} \leq C^{p-1}\alpha^{p-1} - \frac{\big(g(x_0)+\varepsilon+C\eta(\cdot-x_0)-g+C\alpha\tau\big)^{p-1}}{\tau^{p-1}}$$
$$\leq C^{p-1}\alpha^{p-1} - C^{p-1}\alpha^{p-1} = 0,$$

since $g(x_0)+\varepsilon+C\eta(\cdot - x_0)-g(\cdot) \geq 0$ by construction. Now it follows from the comparison principle that

$$v^1(x) \leq u(x) = g(x_0)+\varepsilon+C(\alpha\tau+\eta(x-x_0)).$$

Arguing in the same fashion, we have

$$v^1(x) \geq g(x_0)-\varepsilon-C(\alpha\tau+\eta(x-x_0)).$$

Similarly we can obtain the bounds

$$g(x_0)-\varepsilon-C(k\alpha\tau+\eta(x-x_0)) \leq v^k(x) \leq g(x_0)+\varepsilon+C(k\alpha\tau+\eta(x-x_0))$$

for each $k = 1, \dots, N$. Using the definition (3-25) of v_N, we also have

$$v_N(x,t) \leq v^k(x) \leq g(x_0)+\varepsilon+C(\alpha k\tau+\eta(x-x_0)) \leq g(x_0)+\varepsilon+C\Big(\alpha t+\alpha\frac{T}{N}+\eta(x-x_0)\Big)$$

for $t \in ((k-1)\tau, k\tau)$ as $\tau = T/N$. A similar estimate from below holds as well. In total,

$$g(x_0)-\varepsilon-C\Big(\alpha t+\alpha\frac{T}{N}+\eta(x-x_0)\Big) \leq v_N(x,t) \leq g(x_0)+\varepsilon+C\Big(\alpha t+\alpha\frac{T}{N}+\eta(x-x_0)\Big).$$

Passing to the liminf and limsup in the above inequalities, we find

$$g(x_0)-\varepsilon-C(\alpha t+\eta(x-x_0)) \leq \underline{v}(x,t) \leq \bar{v}(x,t) \leq g(x_0)+\varepsilon+C(\alpha t+\eta(x-x_0)).$$

And after letting $x = x_0$ and $t \to 0$,

$$g(x_0)-\varepsilon \leq \liminf_{t\to 0} \underline{v}(x_0,t) \leq \limsup_{t\to 0} \bar{v}(x_0,t) \leq g(x_0)+\varepsilon.$$

Since both ε and $x_0 \in \Omega$ are arbitrary, the desired result follows. $\square$

Proof of Proposition 4.1. It is enough to show that $\bar{v}$ is a viscosity subsolution of (1-1). The same argument (applied to $-\bar{v}$) yields that $\underline{v}$ is a supersolution. Combining Lemma 4.12, Lemma 4.13, and Proposition 4.10, would then imply $\bar{v} \leq \underline{v}$. Hence, $v := \bar{v} = \underline{v}$ is continuous and v_N converges to v locally uniformly. The claim would then follow as v_N has a subsequence converging to a weak solution of (1-1) in $C([0,T], L^p(\Omega))$; see Remark 3.9.

We now prove that $\bar{v}$ is a viscosity subsolution of (1-1). Assume that $\phi \in C^{2,1}_{x,t}(B_r(x_0)\times(t_0-r,t_0+r))$ and $\bar{v}-\phi$ has a strict maximum in $B_r(x_0)\times(t_0-r,t_0+r)$ at $(x_0,t_0) \in \Omega\times(0,T)$. By Lemma 4.8, there are points (x_j,t_j) converging to (x_0,t_0) and $N_j \in \mathbb{N}$ tending to $+\infty$, as $j \to \infty$, such that $v_{N_j}-\phi_{N_j}$ has a maximum in $B_r(x_0)\times(t_0-r,t_0+r)$ at (x_j,t_j). Observe that for each $j \in \mathbb{N}$, we have $t_j \in (\tau_{k_j-1}, \tau_{k_j}]$ for some $k_j \in \{0, 1, \dots, N_j\}$. Hence, by the definition of v_{N_j} and ϕ_{N_j},

$$\Omega \times \{0, 1, \dots, N_j\} \ni (x,k) \mapsto v^k(x) - \phi(x,\tau_k)$$

has a local maximum in $B_r(x_0) \times \{0, 1, \dots, N_j\}$ at $(x, k) = (x_j, k_j)$. By Lemma 4.7,

$$\mathcal{J}_p\left(\frac{\phi(x_j, \tau_{k_j}) - \phi(x_j, \tau_{k_j - 1})}{T/N_j}\right) + (-\Delta_p)^s \phi^r(x_j, \tau_{k_j}) \le 0.$$

As $\tau_{k_j - 1} = \tau_{k_j} - T/N_j$ and $|t_j - \tau_{k_j}| \le T/N_j$ for $j \in \mathbb{N}$, we can send $j \to \infty$ above by appealing to the smoothness of ϕ and arrive at

$$\mathcal{J}_p(\phi_t(x_0, t_0)) + (-\Delta_p)^s \phi^r(x_0, t_0) \le 0.$$

It follows that $\bar{v}$ is a viscosity subsolution. □

5. Hölder estimates for viscosity solutions

In this section we prove Theorem 1.1. The proof of this regularity result is based on Lemma 5.1 below. We start by noting an elementary inequality that will come in handy:

$$|a+b|^{p-2}(a+b) \le 2^{p-2}(|a|^{p-2}a + |b|^{p-2}b), \quad a+b \ge 0, \;\; p \ge 2. \tag{5-1}$$

Lemma 5.1. *Fix $\delta > 0$. Suppose v is continuous in Q_1^- and satisfies (in the viscosity sense)*

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v \le 0 \quad \text{in } Q_1^-,$$
$$v \le 1 \quad \text{in } Q_1^-,$$
$$v(x,t) \le 2|2x|^\eta - 1 \quad \text{in } \mathbb{R}^n \setminus B_1 \times (-1, 0),$$
$$\left|\left\{B_1 \times \left[-1, -\frac{1}{2^{\frac{sp}{p-1}}}\right]\right\} \cap \{v \le 0\}\right| > \delta.$$

Then for η small enough, $v \le 1 - \theta < 1$ in $Q_{1/2}^-$, where $\theta = \theta(\delta, p, s) > 0$.

Recall that the parabolic cylinders Q_1^- and $Q_{\frac{1}{2}}^-$ have been defined on page 1. Before proving this lemma, we will first need to gain control of a certain function.

Lemma 5.2. *Fix $\delta > 0$, let $\varepsilon > 0$ and assume the following:*

$$m(t) = e^{-c_1 t} \int_{-1}^{t} c_0 e^{c_1 s} |G(s)| \, ds,$$
$$G(t) = \{x \in B_1 : v(x,t) \le 0\},$$
$$b(x,t) = 1 + \varepsilon - m(t)\rho(x),$$
$$0 \le \rho \le 1, \qquad \rho = 1 \quad \text{in } B_{\frac{1}{2}}, \qquad \rho \in C_0^\infty(B_{\frac{3}{4}}),$$
$$\left|\left\{B_1 \times \left[-1, -\frac{1}{2^{\frac{sp}{p-1}}}\right]\right\} \cap \{v \le 0\}\right| > \delta.$$

If $c_0|B_1| \le \frac{1}{2}c_1$ then

$$b(x,t) \ge \tfrac{1}{2},$$

and for $0 \ge t \ge -1/2^{\frac{sp}{p-1}}$,

$$m(t) \ge c_0 e^{-c_1} \delta.$$

Remark 5.3. Note that m solves the equation

$$m'(t) = c_0|G(t)| - c_1 m(t)$$

for a.e. $t \in [-1, 0]$.

Proof. As $|G(t)| \le |B_1|$, it follows that $m(t) \le c_0/c_1|B_1|$. And since $c_0|B_1| \le \frac{1}{2}c_1$,

$$b(x,t) \ge 1 + \varepsilon - \frac{c_0}{c_1}|B_1|\rho(x) \ge 1 + \varepsilon - \frac{c_0}{c_1}|B_1| \ge \tfrac{1}{2}.$$

Moreover,

$$m(t) = e^{-c_1 t}\int_{-1}^{t} c_0 e^{c_1 s}|G(s)|\,ds \ge e^{c_1(-1-t)}\int_{-1}^{t} c_0|G(s)|\,ds.$$

From our hypotheses,

$$\int_{-1}^{-1/2^{\frac{sp}{p-1}}} |G(s)|\,ds \ge \delta.$$

Therefore,

$$m(t) \ge c_0 e^{-c_1}\delta$$

for $0 \ge t \ge -1/2^{\frac{sp}{p-1}}$. □

Proof of Lemma 5.1. Assume the hypotheses of Lemma 5.2. Choose c_0, c_1 and ε so that

$$c_0^{p-1} < 1/(2^{2p-4+n+sp}|B_1|^{p-2}), \quad c_0|B_1| \le \tfrac{1}{2}c_1, \quad 2^{2-p}(c_1)^{p-1} > 2\sup_{x_0 \in B_{\frac{3}{4}}}(-\Delta_p)^s\left(\frac{\rho}{\rho(x_0)}\right)(x_0),$$

and

$$2\varepsilon < e^{-c_1}c_0\delta.$$

Note that the quantity

$$2\sup_{x_0 \in B_{\frac{3}{4}}}(-\Delta_p)^s\left(\frac{\rho}{\rho(x_0)}\right)(x_0)$$

is finite, since the only way it could be infinite is if there is maximizing sequence of points x_j where $\rho(x_j) \to 0$. But then

$$(-\Delta_p)^s\left(\frac{\rho}{\rho(x_j)}\right)(x_j)$$

would be negative for j large enough.

We claim that $v \le b$ in Q_1^-. Let us describe how the lemma follows once this claim is proved. By the lower bound on m in Lemma 5.2, we have

$$b(x,t) \le 1 + \varepsilon - e^{-c_1}c_0\delta$$

for $0 \ge t \ge -1/2^{\frac{sp}{p-1}}$. Since $2\varepsilon < e^{-c_1}c_0\delta$,

$$b(x,t) \le 1 - \tfrac{1}{2}e^{-c_1}c_0\delta.$$

Therefore,

$$v \le b \le 1 - \theta$$

in $Q^-_{\frac{1}{2}}$ as long as we choose

$$\theta = \tfrac{1}{2} e^{-c_1} c_0 \delta.$$

Let us now prove that $v \le b$ in Q^-_1. We argue by contradiction. Assume that, starting from $t=-1$, the first time v touches b at some point in Q^-_1 is at the point (x_0,t_0). Since $\rho = 0$ outside $B_{\frac{3}{4}}$ and $v \le 1$ in Q^-_1, we know that $x_0 \in B_{\frac{3}{4}}$. In addition, since $m(-1)=0$, we know $t_0 > -1$. It is not difficult to see b touches v from above at (x_0,t_0) in the sense of (4-7). In order to simplify the presentation, we first assume that b is C^1 at (x_0,t_0) and explain in the last paragraph of this proof how to relax this assumption.

By Proposition 4.9, $(-\Delta_p)^s v(x_0,t_0)$ is well defined and

$$|b_t(x_0,t_0)|^{p-2} b_t(x_0,t_0) + (-\Delta_p)^s v(x_0,t_0) \le 0. \tag{5-2}$$

Note that $b_t(x_0,t_0) = -m'(t_0)\rho(x_0)$. We will now estimate $(-\Delta_p)^s(b-v)\,(x_0,t_0)$ from above and from below and arrive at a contradiction. This part of the proof will be divided into four steps. Along the way, we will use the notation

$$L_D w\,(x,t) := 2\,\text{P.V.}\int_{y\in D} \frac{\mathcal{J}_p(w(y,t)-w(x,t))}{|x-y|^{n+ps}}\,dy$$

for a measurable function w and an open or closed set $D \subset \mathbb{R}^n$. Notice that

$$(-\Delta_p)^s w = -L_{\mathbb{R}^n} w.$$

<u>Step 1</u>: Estimate L_{B_1}. Since $b(\cdot,t_0) \ge v(\cdot,t_0)$ in B_1, (5-1) implies

$$\begin{aligned} L_{B_1}(b-v)\,(x_0,t_0) &= 2\,\text{P.V.}\int_{B_1} \frac{\mathcal{J}_p\big((b-v)(y,t_0)-(b-v)(x_0,t_0)\big)}{|x_0-y|^{n+ps}}\,dy \\ &\le 2^{p-1}\text{P.V.}\int_{B_1} \frac{\mathcal{J}_p(b(y,t_0)-b(x_0,t_0)) - \mathcal{J}_p(v(y,t_0)-v(x_0,t_0))}{|x_0-y|^{n+ps}}\,dy \\ &= 2^{p-2}\big(L_{B_1} b\,(x_0,t_0) - L_{B_1} v\,(x_0,t_0)\big). \end{aligned}$$

In addition, since $v(x_0,t_0) = b(x_0,t_0)$,

$$\begin{aligned} L_{B_1}(b-v)\,(x_0,t_0) &= 2\,\text{P.V.}\int_{B_1} \frac{\mathcal{J}_p((b-v)(y,t_0))}{|x_0-y|^{n+ps}}\,dy \\ &\ge 2\int_{G(t_0)} \frac{|b(y,t_0)|^{p-2} b(y,t_0)}{|x_0-y|^{n+ps}}\,dy \\ &\ge 2\big(\tfrac{1}{2}\big)^{n+sp} \inf_{y\in B_1} |b(y,t_0)|^{p-1} |G(t_0)| \\ &\ge \big(\tfrac{1}{2}\big)^{p-2+n+sp} |G(t_0)|, \end{aligned}$$

from Lemma 5.2.

<u>Step 2</u>: Estimate $L_{\mathbb{R}^n\setminus B_1}$. By our hypotheses,

$$v(y,t_0) \le 2|2y|^{\eta} - 1, \quad b = 1+\varepsilon > 1$$

whenever $|y|>1$. Hence, $b(y,t_0)-v(y,t_0)\geq 2(1-|2y|^\eta)$ so that

$$b(y,t_0)-v(y,t_0)\leq b(y,t_0)-v(y,t_0)+2(|2y|^\eta-1) \tag{5-3}$$

and

$$b(y,t_0)-v(y,t_0)+2(|2y|^\eta-1)\geq 0. \tag{5-4}$$

By (5-1), (5-3) and (5-4),

$$\begin{aligned} L_{\mathbb{R}^n\setminus B_1}(b-v)\,(x_0,t_0) &\leq 2\int_{\mathbb{R}^n\setminus B_1}\frac{\mathcal{J}_p(b(y,t_0)-v(y,t_0)+2(|2y|^\eta-1)-(b(x_0,t_0)-v(x_0,t_0)))}{|x_0-y|^{n+ps}}\,dy \\ &\leq 2^{p-2}\biggl(-L_{\mathbb{R}^n\setminus B_1}v(x_0,t_0)+2\int_{\mathbb{R}^n\setminus B_1}\frac{\mathcal{J}_p(b(y,t_0)+2(|2y|^\eta-1)-b(x_0,t_0))}{|x_0-y|^{n+ps}}\,dy\biggr). \end{aligned}$$

Using (5-4), we obtain the estimate from below

$$\begin{aligned} L_{\mathbb{R}^n\setminus B_1}(b-v)\,(x_0,t_0) &= 2\int_{\mathbb{R}^n\setminus B_1}\frac{\mathcal{J}_p(b(y,t_0)-v(y,t_0))}{|x_0-y|^{n+ps}}\,dy \\ &\geq -2\int_{\mathbb{R}^n\setminus B_1}\bigl(2(|2y|^\eta-1)\bigr)^{p-1}\frac{dy}{|x_0-y|^{n+sp}} \\ &:= -c_\eta. \end{aligned}$$

We note that $\lim_{\eta\to 0+}c_\eta=0$ by an application of the dominated convergence theorem.

Step 3: Use the equation. The two steps above together imply

$$\begin{aligned} \left(\tfrac{1}{2}\right)^{p-2+n+sp}|G(t_0)|-c_\eta &\leq -(-\Delta_p)^s(b-v)\,(x_0,t_0) \\ &\leq 2^{p-2}(-\Delta_p)^s v\,(x_0,t_0)+2^{p-2}L_{B_1}b(x_0,t_0) \\ &\quad +2^{p-1}\int_{\mathbb{R}^n\setminus B_1}\frac{\mathcal{J}_p(b(y,t_0)+2(|2y|^\eta-1)-b(x_0,t_0))}{|x_0-y|^{n+ps}}\,dy. \end{aligned} \tag{5-5}$$

From inequality (5-2), it follows that

$$\begin{aligned} (-\Delta_p)^s v\,(x_0,t_0) &\leq -|b_t|^{p-2}b_t(x_0,t_0) \\ &= |m'(t_0)\rho(x_0)|^{p-2}m'(t_0)\rho(x_0) \\ &= \bigl|\rho(x_0)c_0|G(t_0)|-\rho(x_0)c_1m(t_0)\bigr|^{p-2}\bigl(\rho(x_0)c_0|G(t_0)|-\rho(x_0)c_1m(t_0)\bigr) \\ &\leq \bigl|c_0|G(t_0)|-\rho(x_0)c_1m(t_0)\bigr|^{p-2}\bigl(c_0|G(t_0)|-\rho(x_0)c_1m(t_0)\bigr). \end{aligned} \tag{5-6}$$

Using (5-1), with $a=c_1\rho(x_0)m(t_0)-c_0|G(t_0)|$ and $b=c_0|G(t_0)|$, we then obtain

$$\begin{aligned} &(c_1\rho(x_0)m(t_0))^{p-1} \\ &\qquad\leq 2^{p-2}\bigl|\rho(x_0)c_1m(t_0)-c_0|G(t_0)|\bigr|^{p-2}\bigl(\rho(x_0)c_1m(t_0)-c_0|G(t_0)|\bigr)+2^{p-2}(c_0|G(t_0)|)^{p-1}. \end{aligned}$$

After rearranging

$$2^{p-2}\big|c_0|G(t_0)|-\rho(x_0)c_1m(t_0)\big|^{p-2}\big(c_0|G(t_0)|-\rho(x_0)c_1m(t_0)\big) \le 2^{p-2}(c_0|G(t_0)|)^{p-1}-(c_1\rho(x_0)m(t_0))^{p-1}. \quad (5\text{-}7)$$

Combining (5-5), (5-6) and (5-7) yields

$$(c_1\rho(x_0)m(t_0))^{p-1}-2^{p-2}(c_0|G(t_0)|)^{p-1}+\left(\tfrac{1}{2}\right)^{p-2+n+sp}|G(t_0)|-c_\eta \le 2^{p-2}L_{B_1}b(x_0,t_0)+2^{p-1}\int_{\mathbb{R}^n\setminus B_1}\frac{\mathcal{J}_p(b(y,t_0)+2(|2y|^\eta-1)-b(x_0,t_0))}{|x_0-y|^{n+ps}}\,dy.$$

Since we assumed at the outset that $c_0^{p-1}<1/(2^{2p-4+n+sp}|G(t_0)|^{p-2})$, we have, by the definition of b and L_{B_1},

$$2^{2-p}(c_1\rho(x_0)m(t_0))^{p-1} \le 2^{2-p}c_\eta+2\,\mathrm{P.V.}\int_{B_1}\frac{\mathcal{J}_p(m(t_0)(\rho(x_0)-\rho(y)))}{|x_0-y|^{n+ps}}\,dy+2\int_{\mathbb{R}^n\setminus B_1}\frac{\mathcal{J}_p(m(t_0)\rho(x_0)+2(|2y|^\eta-1))}{|x_0-y|^{n+ps}}\,dy. \quad (5\text{-}8)$$

Here we also used that $\rho(y)=0$ whenever $y\notin B_1$.

Step 4: Arrive at a contradiction. It follows from the proof of Lemma 5.2 that m is uniformly bounded with respect to η. Consequently, the second integral on the right-hand side of (5-8) is uniformly bounded for all small η. We can again apply the dominated convergence theorem to show that the right-hand side of (5-8) converges to the quantity

$$-(-\Delta_p)^s b\,(x_0,t_0)=(m(t_0))^{p-1}(-\Delta_p)^s\rho\,(x_0)$$

as $\eta\to 0$. As m is bounded from below by $\frac{1}{2}$ (by Lemma 5.2), there is $\gamma_\eta\searrow 0$ as $\eta\to 0$ such that

$$2^{2-p}(c_1\rho(x_0))^{p-1}\le \gamma_\eta+(-\Delta_p)^s\rho\,(x_0). \quad (5\text{-}9)$$

In general, x_0 will depend on η. Let us now consider two cases depending on the size of $(-\Delta_p)^s\rho\,(x_0)$ for η small.

For the first case, we suppose $\limsup_{\eta\to 0^+}(-\Delta_p)^s\rho\,(x_0)\le 0$. Then (5-9) forces $\lim_{\eta\to 0^+}\rho(x_0)=0$ as $\eta\to 0$. It would then follow that $(-\Delta_p)^s\rho\,(x_0)<-\gamma_\eta$ for all small $\eta>0$. Together with (5-9), this would in turn would force $\rho(x_0)<0$ for η small enough, which is a contradiction.

Alternatively if $\limsup_{\eta\to 0^+}(-\Delta_p)^s\rho\,(x_0)>0$, then for some sequence of $\eta\to 0$, we have that $(-\Delta_p)^s\rho\,(x_0)\ge\gamma_\eta$. By (5-9),

$$2^{2-p}(c_1\rho(x_0))^{p-1}\le 2(-\Delta_p)^s\rho\,(x_0)$$

along this sequence. Also note that $(-\Delta_p)^s\rho\,(x_0)>0$ implies that $x_0\in B_{\frac{3}{4}}$. After dividing by $(\rho(x_0))^{p-1}$, we have

$$2^{2-p}(c_1)^{p-1}\le 2(-\Delta_p)^s\left(\frac{\rho}{\rho(x_0)}\right)(x_0)\le 2\sup_{x_0\in B_{\frac{3}{4}}}(-\Delta_p)^s\left(\frac{\rho}{\rho(x_0)}\right)(x_0).$$

However, by our hypotheses on c_1,

$$2^{2-p}(c_1)^{p-1} > 2 \sup_{x_0 \in B_{\frac{3}{4}}} (-\Delta_p)^s \left(\frac{\rho}{\rho(x_0)} \right)(x_0),$$

which is a contradiction.

Step 5: Relax the C^1 assumption on b. As mentioned above, m is not necessarily C^1 since $|G(t)|$ is not necessarily continuous. We have chosen to ignore this fact in the reasoning above in order to make the proof more accessible. This issue can be handled as follows.

First, set

$$\chi_k(x,t) := \int_{-1}^{0} \phi_k(t-s)\chi_{\{v\le 0\}}(x,s)\,ds$$

for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, where ϕ_k is a standard mollifier. Also define

$$g_k(t) = \int_{B_1} \chi_k(x,t)\,dx.$$

Observe $g_k(t) \to |G(t)|$ a.e. and in $L^1(\mathbb{R})$ as $k \to \infty$.

Now set

$$m_k(t) := e^{-c_1 t} \int_{-1}^{t} c_0 e^{c_1 s} g_k(s)\,ds$$

for $t \in [-1,0]$ and

$$b_k(x,t) := 1 + \varepsilon - m_k(t)\rho(x)$$

for $(x,t) \in Q_1^-$. It is evident that $m_k \to m$ and $b_k \to b$ uniformly as $k \to \infty$.

Recall that $b - v \ge \varepsilon$ and $b_k - v \ge \varepsilon$ on $\partial B_1 \times [-1,0] \cup B_1 \times \{-1\}$. These facts combined with the above uniform convergence imply that v touches b_k from below at some $(x_k,t_k) \to (x_0,t_0)$, where v touches b from below at (x_0,t_0). Without loss of generality, we may assume that $t_k < 0$ for all $k \in \mathbb{N}$ large enough. Moreover, as in Step 1 we find

$$\begin{aligned} 2\,\mathrm{P.V.}\int_{B_1} \frac{\mathcal{J}_p((b_k - v)(y,t_k))}{|x_k - y|^{n+ps}}\,dy \ge \left(\tfrac{1}{2}\right)^{p-2+n+sp} \inf_{y\in B_1} \left(\frac{|b_k(y,t_k)|}{1/2} \right)^{p-1} g_k(t_k) \\ -2\int_{B_1} (\chi_k - \chi_{\{v\le 0\}})(y,t_k) \frac{|b_k(y,t_k)|^{p-2} b_k(y,t_k)}{|x_k-y|^{n+ps}}\,dy. \end{aligned} \tag{5-10}$$

Notice that as $k \to \infty$,

$$\inf_{y\in B_1} \left(\frac{|b_k(y,t_k)|}{1/2} \right)^{p-1} \to \inf_{y\in B_1} \left(\frac{|b(y,t_0)|}{1/2} \right)^{p-1} \ge 1.$$

Let us now argue that the second term on the right-hand side of (5-10) goes to zero as $k \to \infty$.

By Lemma 5.2, $b > 0$ and so $b_k > 0$ for all k large enough. Hence, $v(x_k,t_k) = b(x_k,t_k) > 0$. Since v is continuous, $v > 0$ in a neighborhood of (x_k,t_k) for k large. This means that $\chi_k = \chi_{\{v\le 0\}} = 0$ in

$B_\tau(x_k)\times\{t_k\}$ if τ is small enough and k large enough. Hence,

$$\int_{B_1}(\chi_k-\chi_{\{v\le 0\}})(y,t_k)\frac{|b_k(y,t_k)|^{p-2}b_k(y,t_k)}{|x_k-y|^{n+ps}}\,dy \\ =\int_{B_1\setminus B_\tau(x_k)}(\chi_k-\chi_{\{v\le 0\}})(y,t_k)\frac{|b_k(y,t_k)|^{p-2}b_k(y,t_k)}{|x_k-y|^{n+ps}}\,dy.$$

As a result, the integrand is uniformly bounded and converges to zero almost everywhere. By Lebesgue's dominated convergence theorem, we can conclude

$$c_k:=2\int_{B_1}(\chi_k-\chi_{\{v\le 0\}})(y,t_k)\frac{|b_k(y,t_k)|^{p-2}b_k(y,t_k)}{|x_k-y|^{n+ps}}\,dy\to 0$$

as $k\to\infty$.

Steps 2 and 3 go through with minor modifications, so that we can obtain the following analog of (5-8)

$$2^{2-p}(c_1\rho(x_k)m_k(t_k))^{p-1} \\ \le 2^{2-p}(c_\eta+c_k)+2\mathrm{P.V.}\int_{B_1}\frac{\mathcal{J}_p\big(m_k(t_k)(\rho(x_k)-\rho(y))\big)}{|x_k-y|^{n+ps}}\,dy+2\int_{\mathbb{R}^n\setminus B_1}\frac{\mathcal{J}_p\big(m_k(t_k)\rho(x_k)+2(|2y|^\eta-1)\big)}{|x_k-y|^{n+ps}}\,dy$$

for all k sufficiently large. We can then send $k\to\infty$ and recover (5-8). At this point, we can repeat Step 4 to complete this proof. $\square$

We are now in a position to verify Theorem 1.1 and prove that solutions of equation (1-1) are Hölder continuous.

Proof of Theorem 1.1. Upon rescaling v by the factor

$$\frac{1}{2\|v\|_{L^\infty(\mathbb{R}^n\times[-2,0))}},$$

we may assume that v satisfies

$$|v_t|^{p-2}v_t+(-\Delta_p)^s v=0\quad\text{in } Q_2^-,\qquad \operatorname{osc}_{\mathbb{R}^n\times[-2,0)} v\le 1.$$

We will now show that for any $(x_0,t_0)\in Q_1^-$,

$$\operatorname{osc}_{Q^-_{2^{-j}}(x_0,t_0)} v\le 2^{-j\alpha},\quad\text{where } j=0,1,\ldots.$$

Here α is chosen so that

$$\frac{2-\theta}{2}\le 2^{-\alpha}\text{ and }\alpha\le\eta, \tag{5-11}$$

where $\theta=\theta(\delta(p,s),p,s)$ and η are from Lemma 5.1 with $\delta(p,s):=\frac{1}{2}\big(1-1/2^{\frac{ps}{p-1}}\big)|B_1|$. This will imply the desired result with $C=2^\alpha$.

To this end, we will find constants a_j and b_j so that

$$b_j\le v\le a_j\quad\text{in } Q^-_{2^{-j}}(x_0,t_0),\qquad |a_j-b_j|\le 2^{-j\alpha} \tag{5-12}$$

for $j \in \mathbb{Z}$. We construct these constants by induction on j. For $j \le 0$, (5-12) holds true with $b_j = \inf_{\mathbb{R}^n \times [-2,0)} v$ and $a_j = b_j + 1$.

Now assume (5-12) holds for all $j \le k$. We need to construct a_{k+1} and b_{k+1}. Put $m_k = \frac{1}{2}(a_k + b_k)$. Then

$$|v - m_k| \le 2^{-k\alpha-1} \quad \text{in } Q^-_{2^{-k}}(x_0, t_0).$$

Let

$$w(x,t) = 2^{\alpha k+1}\big(v(2^{-k}x + x_0, 2^{-k\gamma}t + t_0) - m_k\big), \qquad \gamma = \frac{sp}{p-1}.$$

Then

$$|w_t|^{p-2}w_t + (-\Delta_p)^s w = 0 \quad \text{in } Q^-_1$$

and

$$|w| \le 1 \quad \text{in } Q^-_1.$$

It also follows for $|y| > 1$, such that $2^\ell \le |y| \le 2^{\ell+1}$, and $t \ge -2^{\gamma(\ell+1)}$ that

$$\begin{aligned} w(y,t) = 2^{\alpha k+1}\big(v(2^{-k}y + x_0, 2^{-k\gamma}t + t_0) - m_k\big) &\le 2^{\alpha k+1}(a_{k-\ell-1} - m_k) \\ &\le 2^{\alpha k+1}(a_{k-\ell-1} - b_{k-\ell-1} + b_k - m_k) \\ &\le 2^{\alpha k+1}\big(2^{-\alpha(k-\ell-1)} - \tfrac{1}{2}2^{-k\alpha}\big) \\ &\le 2^{1+\alpha(\ell+1)} - 1 \le 2|2y|^\alpha - 1 \\ &\le 2|2y|^\eta - 1. \end{aligned}$$

Here we used that (5-12) holds for $j \le k$.

Suppose now that

$$\Big|\{(x,t) : w(x,t) \le 0\} \cap \big\{B_1 \times \big[-1, -\tfrac{1}{2}\big]\big\}\Big| \ge \tfrac{1}{2}(1 - 1/2^{\frac{ps}{p-1}})|B_1| = \delta(p,s).$$

If not we would apply the same procedure to $-w$. Then w satisfies all the assumptions of Lemma 5.1 with $\delta = \delta(p,s)$, and so

$$w \le 1 - \theta \text{ in } Q^-_{\frac{1}{2}}.$$

Scaling back to v yields

$$\begin{aligned} v(x,t) \le 2^{-1-\alpha k}(1-\theta) + m_k &\le 2^{-1-k\alpha}(1-\theta) + \tfrac{1}{2}a_k + b_k \\ &\le b_k + 2^{-1-\alpha k}(1-\theta) + 2^{-1-\alpha k} \\ &\le b_k + 2^{-\alpha(k+1)} \end{aligned}$$

for $(x,t) \in Q_{2^{-k-1}}(x_0, t_0)$, by (5-11). Hence, if we let $b_{k+1} = b_k$ and $a_{k+1} = b_k + 2^{-\alpha(k+1)}$, we obtain (5-12) for the step $j = k+1$ and the induction is complete. □

6. Large time limit

In this section, we prove Theorem 1.2 and Theorem 1.3. The main tools are the monotonicity of the Rayleigh quotient and the $W^{s,p}$ seminorm (equation (3-11) and Proposition 3.6), the compactness of weak solutions (Theorem 3.8) and the Hölder estimates (Theorem 1.1). In order to control the sign of the limiting ground state, we also need the following lemma.

Lemma 6.1. *Assume that v is a weak solution of (1-4). For any positive ground state w for $(-\Delta_p)^s$ and any constant $C > 0$, there is a $\delta = \delta(w, C) > 0$ such that if*

(1) $[v(\cdot, 0)]^p_{W^{s,p}(\mathbb{R}^n)} \geq \lambda_{s,p} \int_\Omega |w|^p \, dx$,

(2) $\int_\Omega |v(x,0)|^p \, dx \leq C$,

(3) $\dfrac{[v(\cdot, 0)]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |v(x,0)|^p \, dx} \leq \lambda_{s,p} + \delta$,

(4) $\int_\Omega |v^+(x,0)|^p \, dx \geq \dfrac{1}{2} \int_\Omega |w|^p \, dx$,

then

$$\int_\Omega |e^{\mu_{s,p} t} v^+(x,t)|^p \, dx \geq \frac{1}{2} \int_\Omega |w|^p \, dx \tag{6-1}$$

for $t \in [0, 1]$.

Proof. We argue towards a contradiction. If the result fails, then there are w and C such that for every $\delta > 0$, there is a weak solution v that satisfies (1)–(4) while (6-1) fails. Therefore, associated to $\delta_j := 1/j$ $(j \in \mathbb{N})$, there is a weak solution v_j that satisfies

(1) $[v_j(\cdot, 0)]^p_{W^{s,p}(\mathbb{R}^n)} \geq \lambda_{s,p} \int_\Omega |w|^p \, dx$,

(2) $\int_\Omega |v_j(x,0)|^p \, dx \leq C$,

(3) $\dfrac{[v_j(\cdot, 0)]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |v_j(x,0)|^p \, dx} \leq \lambda_{s,p} + \dfrac{1}{j}$,

(4) $\int_\Omega |v_j^+(x,0)|^p \, dx \geq \dfrac{1}{2} \int_\Omega |w|^p \, dx$,

while

$$\int_\Omega |e^{\mu_{s,p} t_j} v_j^+(x,t_j)|^p \, dx < \frac{1}{2} \int_\Omega |w|^p \, dx \tag{6-2}$$

for some $t_j \in [0, 1]$.

Consequently, the sequence of initial conditions $(v_j(\cdot, 0))_{j \in \mathbb{N}}$ is bounded in $W_0^{s,p}(\Omega)$ and has a subsequence (not relabeled) that converges to a positive ground state g of $(-\Delta_p)^s$ in $W^{s,p}(\mathbb{R}^n)$. By Theorem 3.8, it also follows that (a subsequence of) the sequence of weak solutions $(v_j)_{j \in \mathbb{N}}$ converges to a weak solution $\tilde{w}$ in $C([0,2], L^p(\Omega)) \cap L^p([0,2]; W^{s,p}(\mathbb{R}^n))$ with $\tilde{w}(\cdot,0) = g$. By Corollary 3.7, $\tilde{w}(\cdot,t) = e^{-\mu_{s,p}t} g$.

In addition, by (1) and since g is a ground state,

$$\int_\Omega |g|^p\,dx = \frac{1}{\lambda_{s,p}}[g]^p_{W^{s,p}(\mathbb{R}^n)} = \frac{1}{\lambda_{s,p}} \lim_{j\to\infty} [v_j(x,0)]^p_{W^{s,p}(\mathbb{R}^n)} \geq \int_\Omega |w|^p\,dx.$$

However, sending $j \to \infty$ in (6-2) gives

$$\int_\Omega |g|^p\,dx = \int_\Omega |g^+|^p\,dx \leq \frac{1}{2}\int_\Omega |w|^p\,dx.$$

This is a contradiction as $w \not\equiv 0$. □

Corollary 6.2. *Assume that v is a weak solution of (1-4). For any positive ground state w for $(-\Delta_p)^s$ and any constant $C > 0$, there is a $\tilde{\delta} = \tilde{\delta}(w, C) > 0$ such that if*

(1) $e^{\mu_{s,p}pt}[v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} \geq \lambda_{s,p} \int_\Omega |w|^p$ *for all* $t \geq 0$,

(2) $\int_\Omega |v(x,0)|^p\,dx \leq C$,

(3) $\dfrac{[v(\cdot,0)]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |v(x,0)|^p\,dx} \leq \lambda_{s,p} + \tilde{\delta}$,

(4) $\int_\Omega |v^+(x,0)|^p\,dx \geq \frac{1}{2}\int_\Omega |w|^p\,dx$,

then

$$\int_\Omega |e^{\mu_{s,p}t} v^+(x,t)|^p\,dx \geq \frac{1}{2}\int_\Omega |w|^p\,dx$$

for all $t \geq 0$.

Proof. Choose

$$\tilde{\delta} = \min\big(\lambda_{s,p}, \delta(w, 2C)\big),$$

where $\delta(w,2C)$ is from Lemma 6.1. It is clear that v satisfies the assumptions of Lemma 6.1, so that in particular

$$\int_\Omega |e^{\mu_{s,p}} v^+(x,1)|^p\,dx \geq \frac{1}{2}\int_\Omega |w|^p\,dx.$$

By Proposition 3.6 combined with (2) and (3)

$$\int_\Omega |e^{\mu_{s,p}t} v(x,t)|^p\,dx \leq \frac{1}{\lambda_{s,p}}[e^{\mu_{s,p}t} v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} \leq \frac{1}{\lambda_{s,p}}[v(\cdot,0)]^p_{W^{s,p}(\mathbb{R}^n)} \leq C\left(1 + \frac{\tilde{\delta}}{\lambda_{s,p}}\right) \leq 2C$$

for any $t > 0$.

The above inequality, together with (1) and the monotonicity of the Rayleigh quotient, implies that $e^{\mu_{s,p}k} v(x, t+k)$ satisfies properties (1)–(3) in Lemma 6.1 with C replaced by $2C$. In particular, Lemma 6.1 applied to $e^{\mu_{s,p}}v(x,t+1)$ yields

$$\int_\Omega |e^{2\mu_{s,p}}v^+(x,2)|^p\,dx \geq \frac{1}{2}\int_\Omega |w|^p\,dx.$$

Now we can apply Lemma 6.1 repeatedly to $e^{\mu_{s,p}k}v(x,t+k)$ for $k=2,3,\ldots$ in order to obtain the desired result. □

We are now ready to treat the large time behavior of solutions of equation (1-4).

Proof of Theorem 1.2. Let v be a weak solution of (1-4). By (3-9) in Corollary 3.5,

$$\frac{d}{dt}[e^{\mu_{s,p}t}v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} \leq 0 \tag{6-3}$$

for almost every $t \geq 0$. Consequently, the limit

$$S := \lim_{t\to\infty}[e^{\mu_{s,p}t}v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}$$

exists. If $S=0$, there is nothing else to prove. Let us assume otherwise.

Let τ_k be an increasing sequence of positive numbers such that $\tau_k \to \infty$ as $k \to \infty$, and define, for $k=1,2,3,\ldots$,

$$v^k(x,t) = e^{\mu_{s,p}\tau_k}v(x,t+\tau_k). \tag{6-4}$$

Then v^k is a weak solution of (1-4) with initial data

$$g^k(x) := e^{\mu_{s,p}\tau_k}v(x,\tau_k).$$

By (6-3), $g^k \in W^{s,p}_0(\Omega)$ is uniformly bounded in $W^{s,p}(\mathbb{R}^n)$. Hence, it is clear that v^k satisfies the hypotheses of Theorem 3.8. Therefore, we can extract a subsequence $\{v^{k_j}\}_{j\in\mathbb{N}}$ converging to a weak solution w as detailed in Theorem 3.8. We may also assume that $v^{k_j}(\cdot,t)$ converges to $w(\cdot,t)$ in $W^{s,p}(\mathbb{R}^n)$ for almost every time $t \geq 0$ since this occurs for a subsequence.

We now observe that by (6-3)

$$S = e^{\mu_{s,p}pt}\lim_{j\to\infty}[v^{k_j}(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} = e^{\mu_{s,p}tp}[w(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} \tag{6-5}$$

for almost every time $t \geq 0$. Since $[0,\infty) \ni t \mapsto [w(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}$ is absolutely continuous (by Lemma 3.3),

$$S = e^{\mu_{s,p}tp}[w(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}$$

holds for all $t \geq 0$. As w is a solution of (1-4), (3-9) in Corollary 3.5 implies

$$0 = \frac{1}{p}\frac{d}{dt}\big(e^{\mu_{s,p}tp}[w(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}\big) = e^{(\mu_{s,p}p)t}\left(\mu_{s,p}[w(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} - \int_\Omega |w_t(x,t)|^p\,dx\right)$$

for almost every $t \geq 0$.

A more careful inspection of the proof of (3-8) reveals that if

$$\mu_{s,p}[w(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} = \int_\Omega |w_t(x,t)|^p\,dx$$

then we must have $[w(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} = \lambda_{s,p}\int_\Omega |w(x,t)|^p\,dx$. Therefore $w(\cdot,t)$ is a ground state for almost every $t>0$. The absolute continuity of $[w(\cdot,t)]_{W^{s,p}(\mathbb{R}^n)}$ and $\|w(\cdot,t)\|_{L^p(\Omega)}$ then implies that $w(\cdot,t)$ is a ground state for all $t\ge 0$. By Corollary 3.7,

$$w(x,t)=e^{-\mu_{s,p}t}w_0,$$

where $w_0(x)=w(x,0)$ is a ground state.

For any $t_0\in[0,T]$ such that the limit (6-5) holds, we have, by Proposition 3.6,

$$\begin{aligned}\lim_{t\to\infty}\frac{[v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |v(x,t)|^p\,dx} &= \lim_{j\to\infty}\frac{[v(\cdot,\tau_{k_j}+t_0)]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |v(x,\tau_{k_j}+t_0)|^p\,dx} = \lim_{j\to\infty}\frac{[v^{k_j}(\cdot,t_0)]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |v^{k_j}(x,t_0)|^p\,dx}\\ &= \frac{[w(\cdot,t_0)]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |w(x,t_0)|^p\,dx} = \frac{[w_0]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |w_0|^p\,dx} = \lambda_{s,p}.\end{aligned}$$

Since weak convergence together with the convergence of the norm implies strong convergence, the limit $w_0=\lim_{j\to\infty}e^{\mu_{s,p}\tau_{k_j}}v(\cdot,\tau_{k_j})$ holds in $W^{s,p}(\mathbb{R}^n)$. We conclude that $\{e^{\mu_{s,p}\tau_k}v(\cdot,\tau_k)\}_{k\in\mathbb{N}}$ has a subsequence converging in $W^{s,p}(\mathbb{R}^n)$ to a ground state w_0.

Recall that, due to the simplicity of the ground states, w_0 is determined completely by its sign and the constant

$$S=[w_0]^p_{W^{s,p}(\mathbb{R}^n)}.$$

We may assume $w_0\ge 0$; if not we can consider $-v$ instead.

Now we note that for any $t\ge 0$,

$$e^{\mu_{s,p}pt}[v(\cdot,t)]^p_{W^{s,p}(\mathbb{R}^n)} \ge \lim_{j\to\infty} e^{\mu_{s,p}p\tau_{k_j}}[v(\cdot,\tau_{k_j})]^p_{W^{s,p}(\mathbb{R}^n)} = [w_0]^p_{W^{s,p}(\mathbb{R}^n)} = \lambda_{s,p}\int_\Omega |w_0|^p. \tag{6-6}$$

Since $v^{k_j}(x,0)=g^{k_j}(x)$ converges in $W^{s,p}(\mathbb{R}^n)$ to ground state w_0, we also have

$$\lim_{j\to\infty}\frac{[v^{k_j}(\cdot,0)]^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega |v^{k_j}(x,0)|^p\,dx}=\lambda_{s,p} \tag{6-7}$$

and

$$\int_\Omega |(v^{k_j}(x,0))^+|^p\,dx \ge \frac{1}{2}\int_\Omega |w_0|^p\,dx \tag{6-8}$$

whenever j is large enough. In addition, due to the monotonicity of the $W^{s,p}$ norm,

$$\int_\Omega |v^{k_j}(x,0)|^p\,dx \le \frac{1}{\lambda_{s,p}}[v^{k_j}(\cdot,0)]^p_{W^{s,p}(\mathbb{R}^n)} \le \frac{1}{\lambda_{s,p}}[g]^p_{W^{s,p}(\mathbb{R}^n)}, \tag{6-9}$$

where $g=v(\cdot,0)$. From (6-6)–(6-9), it is clear that for j large enough, v^{k_j} satisfies assumptions (1)–(4) in Corollary 6.2, with $w=w_0$ and $C=[g]^p_{W^{s,p}(\mathbb{R}^n)}/\lambda_{s,p}$. As a result,

$$\int_\Omega |e^{\mu_{s,p}t}(v^{k_j})^+(x,t)|^p\,dx \ge \frac{1}{2}\int_\Omega |w_0|^p\,dx$$

for all $t \geq 0$, which implies

$$\int_\Omega |e^{\mu_{s,p}t}(v)^+(x,t)|^p\,dx \geq \frac{1}{2}\int_\Omega |w_0|^p\,dx \tag{6-10}$$

for t large enough.

Suppose now that we pick another convergent subsequence of $\{e^{\mu_{s,p}\tau_k}v(\cdot,\tau_k)\}_{k\in\mathbb{N}}$. Then arguing as above, the sequence converges in $W^{s,p}(\mathbb{R}^n)$ to a ground state w_1 and by (6-5),

$$[w_1]^p_{W^{s,p}(\mathbb{R}^n)} = S.$$

By the simplicity of ground states, $w_1 = w_0$ or $w_1 = -w_0$. Passing $t \to \infty$ in (6-10), we obtain

$$\int_\Omega (w_1^+)^p\,dx \geq \frac{1}{2}\int_\Omega |w_0|^p\,dx,$$

forcing w_1 to be positive and hence $w_1 = w_0$. As the sequence $\{\tau_k\}_{k\in\mathbb{N}}$ was chosen arbitrarily, it follows that $e^{\mu_{s,p}t}v(\cdot,t) \to w_0$ as $t \to \infty$ in $W^{s,p}(\mathbb{R}^n)$. □

We are finally in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Let τ_k be an increasing sequence of positive numbers such that $\tau_k \to \infty$ as $k \to \infty$. In Theorem 1.2, we established that $\lim_{k\to\infty} e^{\mu_{s,p}\tau_k}v(\cdot,\tau_k) = w$ in $W^{s,p}(\mathbb{R}^n)$. In view of Proposition 4.1, it suffices to show that this convergence occurs uniformly on $\overline{\Omega}$.

To this end, define v^k as in (6-4). We also remark that $e^{-\mu_{s,p}t}\Psi$ is a solution of equation (1-1). By the comparison principle,

$$|v^k(x,t)| \leq \Psi(x) \tag{6-11}$$

for $(x,t) \in \mathbb{R}^n \times [-1,1]$ for all $k \in \mathbb{N}$ large enough. These bounds with Theorem 1.1 give that v^k is uniformly bounded in $C^\alpha(B \times [0,1])$ for any ball $B \Subset \Omega$. By a routine covering argument, v^k is then uniformly bounded in $C^\alpha(K \times [0,1])$ for any compact $K \Subset \Omega$. We now claim that the sequence v^k is equicontinuous in $\overline{\Omega} \times [0,1]$.

Fix $\varepsilon > 0$. Recall that Ψ is continuous up to the boundary of Ω. By (6-11), it follows that there is δ so that $|v^k(x,t)| \leq \frac{1}{2}\varepsilon$ whenever $d(x) := d(x,\partial\Omega) < \delta$ and $t \in [0,1]$. Now we will show that if $|x-y|+|t-\tau|$ is small enough, then $|v^k(x,t)-v^k(y,\tau)| \leq \varepsilon$. We treat three cases differently as follows.

(1) $d(x) < \frac{1}{2}\delta$ and $d(y) < \frac{1}{2}\delta$: Then

$$|v^k(x,t)-v^k(y,\tau)| \leq |v^k(x,t)| + |v^k(y,\tau)| \leq \varepsilon.$$

(2) $d(x) < \frac{1}{2}\delta$ and $d(y) > \frac{1}{2}\delta$: Then $|x-y| < \frac{1}{2}\delta$ implies $d(y) < \delta$ so that again

$$|v^k(x,t)-v^k(y,\tau)| \leq |v^k(x,t)| + |v^k(y,\tau)| \leq \varepsilon.$$

(3) $d(x) > \frac{1}{2}\delta$ and $d(y) > \frac{1}{2}\delta$: Then by the local Hölder estimates,

$$|v^k(x,t)-v^k(y,\tau)| \leq C_\delta(|x-y|+|t-\tau|)^\alpha \leq \varepsilon$$

if we choose $|x-y|+|t-\tau| \leq (\varepsilon/C_\delta)^{1/\alpha}$.

Hence, if

$$|x-y|+|t-\tau| \le \min\left(\tfrac{1}{2}\delta, (\varepsilon/C_\delta)^{1/\alpha}\right)$$

then $|v^k(x,t)-v^k(y,\tau)| \le \varepsilon$. Therefore, the sequence v^k is equicontinuous on $\overline{\Omega}\times[0,1]$. By the Arzelà–Ascoli theorem, we can extract a subsequence v^{k_j} converging to $e^{-\mu_{s,p}t}w$, the limit in (1-5), uniformly in $\overline{\Omega}\times[0,1]$. $\square$

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BOUNDARY $C^{1,\alpha}$ REGULARITY OF POTENTIAL FUNCTIONS IN OPTIMAL TRANSPORTATION WITH QUADRATIC COST

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We provide a different proof for the global $C^{1,\alpha}$ regularity of potential functions in the optimal transport problem, which was originally proved by Caffarelli. Our method applies to a more general class of domains.

1. Introduction

We study the global $C^{1,\alpha}$ regularity of potential functions in optimal transportation with quadratic cost. Let Ω and Ω^* be the source and target domains associated with densities $1/C < f < C$ and $1/C < g < C$, respectively, where C is a positive constant. The optimal transport problem with quadratic cost is about finding a map $T : \Omega \to \Omega^*$ among all measure-preserving maps minimizing the transportation cost

$$\int_\Omega |x - Tx|^2 \, dx.$$

Here the term "measure-preserving" means that $\int_{T^{-1}(B)} f = \int_B g$ for any Borel set $B \subset \Omega^*$. Brenier [1991] proved that one can find a convex function u such that

$$T(x) = Du(x) \quad \text{for a.e. } x \in \Omega.$$

Indeed, the convex function u satisfies $\int_{(\partial u)^{-1}B} f = \int_B g$ for any Borel set $B \subset \Omega$, where ∂u is the standard subgradient map of the convex function u. We call u a *Brenier solution* of the optimal transport problem if it satisfies the property above. When the target domain Ω^* is convex, Caffarelli proved that $\partial u(\Omega) = \Omega^*$ and that u is an *Alexandrov solution*, namely u satisfies $(1/C)|A \cap \Omega| \le |\partial u(A)| \le C|A \cap \Omega|$ for any Borel set $A \subset \Omega$. Moreover, if we extend u to $\mathbb{R}^n$ via

$$\tilde{u} := \sup\{L \mid L \text{ is linear, } L|_\Omega \le u, \ L(z) = u(z) \text{ for some } z \in \Omega\},$$

then $\tilde{u}$ is a globally Lipschitz convex solution of

$$C^{-1}\chi_\Omega \le \det \tilde{u}_{ij} \le C\chi_\Omega.$$

We will still use u to denote this extended function. Caffarelli [1992b] proved interior $C^{1,\alpha}$ regularity by using his techniques for studying the standard Monge–Ampère-type equation; see [Caffarelli 1990a; 1990b; 1991].

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Keywords: optimal transport, quadratic cost, boundary regularity.

Then, Caffarelli [1992a] proved the boundary $C^{1,\alpha}$ regularity result under the condition that both Ω and Ω^* are convex. Below we will briefly discuss the main ideas involved in his proof. First, Caffarelli established a fundamental property of convex functions, namely the existence of sections centred at a given point (see the statement of Lemma 2.5). Then, he proved that such sections are decaying geometrically, namely there exists a constant δ such that

$$S_{\delta h}(y) \subset \tfrac{3}{4} S_h(x) \quad \text{for any } \ y \in \tfrac{1}{2} S_h(x). \tag{1-1}$$

Here $S_h(x)$ denotes the section of u centred at x with height h. From (1-1) we obtain the quantitative strict convexity estimate

$$u(z) \geq u(x) + Du(x) \cdot (z - x) + C|z - x|^{\beta} \quad \text{for any } \ x, z \in \overline{\Omega}, \tag{1-2}$$

for some $\beta > 1$. From (1-2), it is easy to check that u^*, the standard Legendre transform of u, is $C^{1,\alpha}$ on $\overline{\Omega}^*$. Recall the well-known fact that u^* is indeed the potential function of the optimal transport problem from Ω^* to Ω. Therefore, by switching the role of u and u^* one can show the global $C^{1,\alpha}$ regularity of u.

The convexity of domains is crucial in Caffarelli's approach. Indeed, the convexity of Ω ensures that u^* is an Alexandrov solution, while the convexity of Ω^* ensures that the sections of u^*, centred at some point in $\overline{\Omega}^*$, have some doubling property. Here we provide a different proof of the global $C^{1,\alpha}$ result. Instead of deducing the $C^{1,\alpha}$ regularity of u from the strict convexity of u^*, we prove the $C^{1,\alpha}$ regularity of u directly. Moreover, our method works for a slightly more general class of domains, namely we allow the source to be a domain obtained by removing finitely many disjoint convex subsets from a convex domain.

We would like to mention that in recent years the regularity of optimal transport maps has attracted much interest and there are many important works related to it; to cite a few, see [Figalli and Loeper 2009; Liu 2009; Trudinger and Wang 2009b; 2009a; Figalli and Rifford 2009; Loeper 2011; Loeper and Villani 2010; Liu et al. 2010; Kim and McCann 2010; Figalli et al. 2010; 2011; 2012; 2013a; 2013b].

The rest of the paper is organized as follows. In Section 2 we introduce some notations and preliminaries, and state the main results. Section 3 is devoted to the proof of global C^1 regularity. In the last section we complete the proof of the main results.

2. Preliminaries and main result

The main result of this paper is the following theorem:

Theorem 2.1. *Let Ω and Ω^* be two bounded domains in $\mathbb{R}^n$, $n \geq 2$, and f and g be densities of two positive probability measures defined in Ω and Ω^*, respectively, satisfying $C^{-1} \leq f, g \leq C$ for a positive constant C. Assume that Ω^* is convex and Ω is Lipschitz.*

(i) *If, for any given $x \in \overline{\Omega}$, there exists a small ball $B_{r_x}(x)$ such that, for any convex set $\omega \subset B_{r_x}(x)$ centred in Ω, we have $\int_{\omega} f \leq C \int_{\omega/2} f$ for some constant C independent of ω, then the potential function u is $C^1(\overline{\Omega})$.* (Here f is defined to be 0 outside Ω.)

(ii) *If Ω is a domain obtained by removing finitely many disjoint convex subsets from a convex set, then the potential function u is $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

Remark 2.2. (a) It is easy to see that in Theorem 2.1(i) we allow Ω to be any polytope (not necessarily convex). We also note that the C^1 regularity always holds in dimension two without any condition on Ω. This is a classical result of Alexandrov; see also [Figalli and Loeper 2009].

(b) One may want to prove higher regularity when the densities are smooth; however, in view of the following simple example we see that this is impossible. Let the dimension be $n = 2$. Let $\Omega := B_2 - B_1$, with uniform probability density, and let $\Omega^* := B_{\sqrt{3}}$, with uniform probability density. Then by symmetry it is easy to compute that the optimal transport map is $T(x) = \sqrt{|x|^2 - 1}\, x/|x|$, which is only $C^{1/2}$ on $\partial B_1 \subset \partial\Omega$.

In the following we will use $S_h(x_0)$ to denote a section of u with height h, namely

$$S_h(x_0) := \{x \mid u < p \cdot (x - x_0) + h\},$$

where p is chosen so that x_0 is the centre of mass of $S_h(x_0)$. We say a point $x_0 \in \overline{\Omega}$ is localized (with respect to u) if, for any sequences $h_k \to 0$ and $x_k \to x_0$ satisfying $x_0 \in S_{h_k}(x_k)$, we have that $S_{h_k}(x_k)$ shrinks to the point $x_0 \in \overline{\Omega}$.

Now we record a fundamental property of convex sets.

Lemma 2.3 (John's lemma). *Let $U \subset \mathbb{R}^n$ be a bounded, convex domain with its centre of mass at the origin. There exists an ellipsoid E, also centred at the origin, such that*

$$E \subset U \subset n^{3/2} E.$$

The original John's lemma does not require that the ellipsoid is centered at the origin, and the constant $n^{3/2}$ can be replaced by n. We refer the reader to [Liu and Wang 2015] for a simple proof of the existence and uniqueness of such an ellipsoid.

By John's lemma we can show the following property of convex functions:

Lemma 2.4. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Let L be a supporting function of u. Then any extreme point of $\{u = L\}$ is localized.*

Proof. Suppose to the contrary that there exists an extreme point x_0 of $\{u = L\}$ which is not localized. Then there exist sequences $x_k \to x_0$ and $h_k \to 0$ such that $x_0 \in S_{h_k}(x_k)$, and that $S_{h_k}(x_k)$ contains a segment of length greater than or equal to some positive constant δ. Since $S_{h_k}(x_k)$ is convex and centred at x_k, by John's lemma there exists a unit vector ξ_k such that I_k, the segment connecting $x_k - \delta/(2n^{3/2})\,\xi_k$ and $x_k + \delta/(2n^{3/2})\,\xi_k$, is contained in $S_{h_k}(x_k)$. Denote by L_k the defining function of $S_{h_k}(x_k)$, namely $S_{h_k}(x_k) = \{u \le L_k\}$. Then it is easy to see that DL_k is bounded; hence, by passing to a subsequence, $L_k \to L_\infty$ for some linear function L_∞. Also by passing to a subsequence we may assume $\xi_k \to \xi_\infty$ for some unit vector ξ_∞. Then u is linear on I_∞, which is the segment connecting $x_0 - \delta/(2n^{3/2})\,\xi_\infty$ and $x_0 + \delta/(2n^{3/2})\,\xi_\infty$. Hence $I_\infty \subset \{u = L\}$, which contradicts the assumption that x_0 is an extreme point of $\{u = L\}$. □

The following property of sections of convex functions was proved by Caffarelli [1992a]. Here we provide a different proof by using a well-known fact that if a continuous map from a ball to itself fixes the boundary then it must be surjective. We learned this method from Wang; see [Sheng et al. 2004, Section 4].

Lemma 2.5. *Let $u : \mathbb{R}^n \to [0, \infty]$ be a convex function. Assume that*:

(1) $u(0) = 0$, $u \geq 0$.

(2) *u is finite in a neighbourhood of* 0.

(3) *The graph of u contains no complete lines.*

Then for $h > 0$ there exists a slope p such that the centre of mass of the section

$$S_{h,p} := \{x \mid u \leq x \cdot p + h\}$$

is defined and equal to 0.

Proof. Let

$$\begin{cases} u_k(x) = u(x) & \text{in } B_k, \\ u_k = \infty & \text{in } \mathbb{R}^n - B_k. \end{cases} \tag{2-1}$$

We only need to show the existence of sections $S^k := \{x \mid u_k \leq x \cdot p_k + h\}$ centred at 0 with bounded p_k. Then $S_{h,p} = \lim_{k\to\infty} S^k$ is the desired section in the lemma.

Take a large ball B_r. For any $p \in B_r$, let z_p be the centre of mass of the section $S_p := \{x \mid u_k(x) \leq x \cdot p + h\}$. Then we obtain a mapping $M_1 : p \to z_p$ from B_r to $\mathbb{R}^n$. If $p \in \partial B_r$, it is easy to see that $p \cdot z_p > 0$ provided r is sufficiently large.

If there is no $p \in B_r$ such that $z_p = 0$, then we can define a mapping $M_2 : z_p \to t_p z_p$, where $t_p > 0$ is a constant such that $t_p z_p \in \partial B_r$. We then obtain a continuous mapping $M = M_2 \circ M_1$ from B_r to ∂B_r with the property that

$$p \cdot M(p) > 0 \quad \text{on } \partial B_r. \tag{2-2}$$

To get a contradiction, we extend the mapping M to B_{2r} as follows. For any point $p \in \partial B_{2r}$, let $p_1 = p$, $p_0 = \frac{1}{2}p \in \partial B_r$ and $p_t = (1-t)p_0 + p_1$. We extend the mapping M to B_{2r} by letting $M(p_t) = (1-t)M(p_0) + tp_1$. Then, by (2-2), $M(p) \neq 0$ on B_{2r} and M is the identity mapping on ∂B_{2r}. This is a contradiction.

Hence, for each $k > 0$, there exists a $p_k \in \mathbb{R}^n$ such that $S^k := \{x \mid u_k \leq x \cdot p_k + h\}$ is centred at 0. Moreover, $|p_k| \leq C$ for some constant independent of k. Indeed, we can argue as follows: By rotating the coordinates we may assume $p_k = (a, 0, \dots, 0)$ with $a > 0$. Let $\alpha^+ = \sup\{x_1 \mid (x_1, 0, \dots, 0) \in S^k\}$ and $\alpha^- = -\inf\{x_1 \mid (x_1, 0, \dots, 0) \in S^k\}$. Then $\alpha^+/\alpha^- \to \infty$ as $a \to \infty$. Since S_k is centred at 0, a cannot be too large. □

The following Alexandrov-type estimates were proved by Caffarelli [1996]:

Lemma 2.6. *Let u be a convex solution of*

$$\det D^2 u = d\mu$$

in the convex domain S with $u = 0$ on ∂S. Assume S is normalized, namely $B_1 \subset S \subset n^{3/2} B_1$. Assume $d\mu(S) \leq \theta\, d\mu\big(\frac{1}{2}S\big)$ for some constant θ, where $\frac{1}{2}S$ is a dilation of S with respect to the origin.

(a) $(1/C)|\inf_S u|^n \leq d\mu(S) \leq C|\inf_S u|^n$, *where C is a constant depending only on θ.*

(b) $|u(x)|^n \leq C\, d\mu(S)\, d(x, \partial S)$.

3. Global C^1 regularity

In this section, we prove Theorem 2.1(i).

Lemma 3.1. *Suppose u is a globally Lipschitz convex function. Assume that u is C^1 at all of the extreme points of a convex set $K = \{u = L\}$, where L is a linear function satisfying $u \geq L$ and $u(y) = L(y)$ for some $y \in \mathbb{R}^n$. Then u is C^1 on K.*

Proof. By subtracting L we may assume $K = \{u = 0\}$. If K is a bounded convex set, then for any $x \in K$ we have

$$x = \sum_{i=1}^{k} \lambda_i x_i,$$

where x_i, $i = 1, \dots, k$, are extreme points of K, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Since u is C^1 at x_i, $i = 1, \dots, k$, we have $0 \leq u(z) = o(z - x_i)$, $i = 1, \dots, k$. Now, by convexity we have

$$0 \leq u(z) = u\biggl(\sum_{i=1}^{k} \lambda_i (z - x + x_i)\biggr) \leq \sum_{i=1}^{k} \lambda_i u(z - x + x_i) = \sum_{i=1}^{k} \lambda_i o(z - x) = o(z - x).$$

Hence, u is C^1 at x.

If K is unbounded, it is well-known that $K = \operatorname{covext}[K] + \operatorname{rc}[K]$, where $\operatorname{covext}[K]$ is the convex hull of the extreme points of K, and $\operatorname{rc}[K] := \lim_{\lambda \downarrow 0} \lambda K$ is the recession cone of K. Hence we need only to show that u is C^1 at points represented by $x = x_0 + q$, where x_0 is an extreme point of K and $q \in \operatorname{rc}[K]$. For any $M \geq 0$, by using the facts that u is Lipschitz and $x_1 := x_0 + Mq \in K$ we have that $u(z - x + x_1) \leq C|z - x|$. By convexity we have

$$u(z) = u\biggl(\frac{M-1}{M}(z - x + x_0) + \frac{1}{M}(z - x + x_1)\biggr) \leq \frac{M-1}{M} o(|z - x|) + \frac{C}{M}|z - x|.$$

By letting $M \to \infty$ we have $0 \leq u(z) \leq o(|z - x|)$. Hence u is C^1 at x. □

Since u is convex, for any unit vector γ the lateral derivatives

$$\partial_\gamma^+ u(x) =: \lim_{t \searrow 0} t^{-1}(u(x + t\gamma) - u(x)) \quad \text{and} \quad \partial_\gamma^- u(x) =: \lim_{t \searrow 0} t^{-1}(u(x) - u(x - t\gamma))$$

exist. To prove that $u \in C^1(\overline{\Omega})$, it suffices to prove that

$$\partial_\gamma^+ u(x_0) = \partial_\gamma^- u(x_0) \tag{3-1}$$

at any point $x_0 \in \partial\Omega$ for any unit vector γ. By convexity, it suffices to prove this for $\xi = \xi_k$ for all $k = 1, 2, \dots, n$, where ξ_k, $k = 1, \dots, n$, are any n linearly independent unit vectors.

Proof of Theorem 2.1(i). By Lemmas 3.1 and 2.4 we only need to show that u is C^1 at localized points. Assume to the contrary that u is not C^1 at $x_0 \in \partial\Omega$. Let us assume that $x_0 = 0$, $u(0) = 0$, $u \geq 0$ and $\partial_1^+ u(0) > \partial_1^- u(0) = 0$. Since $\partial\Omega$ is Lipschitz, we may also assume that $-te_1 \in \Omega$ for $t \in (0, 1)$, where e_1 is the first coordinate direction.

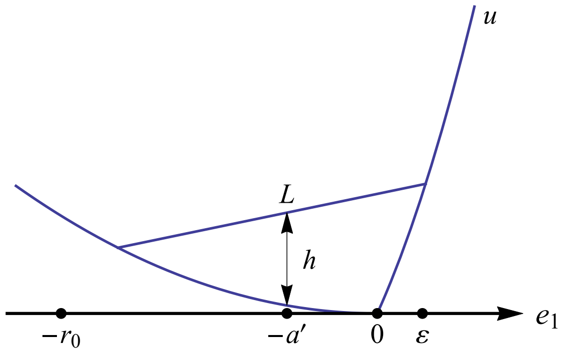

Figure 1. Centred section.

Now we consider a section $S_h(x')$, where $x' = (-a', 0, \dots, 0)$ for some small constant $0 < a' < \frac{1}{2}r_0$, where $r_0 := r_{x_0}$ is the radius in the condition of Theorem 2.1(i). Note that by John's lemma there exists an ellipsoid E with centre x' such that $E \subset S_h(x') \subset n^{3/2}E$. Since u is Lipschitz and $\partial_1^+ u(0) > 0$, we have that $C^{-1}\varepsilon \le u(\varepsilon e_1) \le C\varepsilon$ for any small positive ε, where C is a positive constant. Since $\partial_1^- u(0) = 0$, we have $u(-Ma'e_1) = o(a')$, where $M = 2n^{3/2}$. Hence, we can choose small ε and a' so that the following properties hold:

(1) $o(a') = u(-Ma'e_1) \le C^{-1}\varepsilon \ll a'$,

(2) εe_1 is on the boundary of some section $S_h(x')$, and

(3) $S_h(x') \subset B_{r_0}(0)$.

The existence of such a section $S_h(x')$ in (2) follows from the property that a centred section, say $S_h(x)$, various continuously with respect to the height h; see [Caffarelli and McCann 2010, Lemma A.8], and (3) follows from the assumption that $x_0 = 0$ is localized.

Let L be the defining linear function of $S_h(x')$; by (1) it is easy to see that L is increasing in the e_1 direction (see Figure 1); hence,

$$(L - u)(0) \ge (L - u)(x') = h. \tag{3-2}$$

Since $\int_{S_h(x')} f \le C \int_{\frac{1}{2}S_h(x')} f$, we have that

$$(L - u)(0) \le C\left(\frac{\varepsilon}{a'}\right)^{\frac{1}{n}} h, \tag{3-3}$$

contradicting (3-2), since $a' \gg \varepsilon$. Here we have followed the argument of [Caffarelli 1996]. Indeed, let A be an affine transform normalizing $S_h(x')$; then $v := (u - L)(A^{-1}x)/h$ satisfies $\det D^2 v = f(A^{-1}x)/h^n$ in $A(S_h(x'))$ and $v = 0$ on $\partial S_h(x')$. Hence, by applying Lemma 2.6 to v and translating back to u we get (3-3).

Hence, u must be C^1 at any localized point x_0. Therefore $u \in C^1(\mathbb{R}^n)$. $\square$

Remark 3.2. The proof of Theorem 2.1(i) shares some similarities with the proof of C^1 regularity for the obstacle problem in [Savin 2005] (see Proposition 2.8 in that paper).

4. Global $C^{1,\alpha}$ regularity

In this section, we prove Theorem 2.1(ii). First we point out that to prove $u \in C^{1,\alpha}(\overline{\Omega})$, it suffices to prove that there exist constants $C > 0$, $\alpha \in (0, 1)$ and $r > 0$ such that, for any point $x_0 \in \overline{\Omega}$,

$$u(x) - \ell_{x_0}(x) \leq C|x - x_0|^{1+\alpha} \tag{4-1}$$

for every $x \in B_r(x_0) \cap \overline{\Omega}$. From (4-1) one can prove that $u \in C^{1,\alpha}(\overline{\Omega})$, using the convexity of u. In the following we will show that a relaxed version of (4-1) is enough to show $u \in C^{1,\alpha}(\overline{\Omega})$, and it has the advantage of avoiding some annoying limiting picture.

By the assumption of Theorem 2.1(ii) we write $\Omega = U - \sum_{i=1}^{k} C_i$, where U is an open convex set, and C_i, $i = 1, \dots, k$, are closed disjoint convex subsets of U; see Figure 2. Given any $x \in \overline{\Omega}$, we introduce the function

$$\rho_x(t) := \sup\left\{ u(z) - u(x) - Du(x) \cdot (z - x) \;\middle|\; |z - x| = t,\ x + s\frac{z - x}{|z - x|} \in \overline{\Omega} \text{ for any } s \in [0, r_0] \right\}, \tag{4-2}$$

where r_0 is a fixed small positive constant depending on Ω, and its smallness will be clear in the proof of Lemma 4.1. Indeed, we need to take r_0 small enough that $B_{r_0}(x) \cap \partial U$ can be represented as the graph of some Lipschitz function for any $x \in \partial U$ with the Lipschitz constant independent of x, and that

$$r_0 \ll \min\left\{ \operatorname{dist}(\partial U, \partial C_i),\ \operatorname{dist}(\partial C_j, \partial C_l) \mid i = 1, \dots, k,\ 1 \leq j \neq l \leq k \right\}.$$

Lemma 4.1. *Suppose that there exist $r > 0$ and $\delta \in (0, 1)$ such that for any $x \in \overline{\Omega}$ we have*

$$\rho_x\left(\tfrac{1}{2}t\right) \leq \tfrac{1}{2}(1 - \delta)\rho_x(t) \tag{4-3}$$

whenever $t \leq r$. Then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

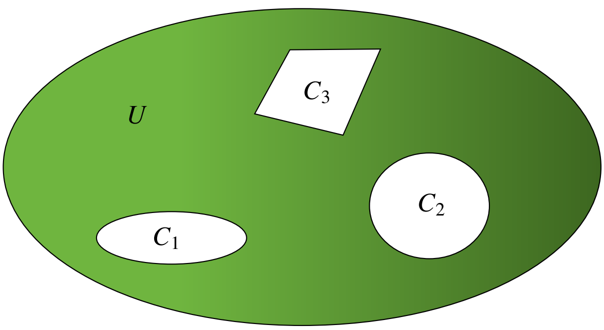

Figure 2. Domain Ω.

Proof. For $t = r/2^k$, we have

$$\rho_x(t) \le \frac{(1-\delta)^k}{2^k}\rho_x(r) \le \frac{t}{r}(1-\delta)^{\log(r/t)/\log 2}\rho_x(r) \le Ct^{1+\alpha}, \tag{4-4}$$

where C depends on r, δ and $\rho_x(r)$, and $\alpha = -\log(1-\delta)/\log 2$.

Suppose $x, y \in \overline{\Omega}$ and $|x-y| \ll r \ll r_0$. We need to consider two cases:

(a) x, y are close to ∂U.

(b) x, y are close to ∂C_i for some $1 \le i \le k$.

We will deal with case (a) first; case (b) follows from a similar argument. Without loss of generality we may assume that $B_{3r_1} \subset U$ for some small fixed r_1, that $r_0 \ll r_1$, and that $\operatorname{dist}(\partial B_{3r_1}, \partial U) \gg r_1$. Denote by $\mathscr{C}_{x,r_1}$ the convex hull of x and B_{r_1}. By convexity, $\mathscr{C}_{x,3r_1} \subset U$. Then we prove the following claim:

Claim 1. For any $z \in B_{r/2}(x) \cap \mathscr{C}_{x,2r_1}$, we have $|Du(x) - Du(z)| \le C|x-z|^\alpha$.

Proof of Claim 1. Observe that $\operatorname{dist}(z, \partial\mathscr{C}_{x,3r_1}) \ge (1/C)|x-z|$ for some large constant C. Hence, $B_{(1/C)|x-z|}(z) \subset B_r \cap \mathscr{C}_{x,3r_1}$. Now, for any $\tilde{z} \in \partial B_{(1/C)|x-z|}(z)$, by (4-4) we have that

$$u(\tilde{z}) \le u(x) + Du(x)\cdot(\tilde{z}-x) + C|\tilde{z}-x|^{1+\alpha}. \tag{4-5}$$

By convexity we also have

$$u(\tilde{z}) \ge u(z) + Du(z)\cdot(\tilde{z}-z) \tag{4-6}$$

and

$$u(z) \ge u(x) + Du(x)\cdot(z-x). \tag{4-7}$$

By (4-5), (4-6) and (4-7) we have

$$(Du(z) - Du(x))\cdot(\tilde{z}-z) \le C|\tilde{z}-x|^{1+\alpha}. \tag{4-8}$$

Note that $|\tilde{z}-z| \approx |\tilde{z}-x| \approx |z-x|$ provided $\tilde{z} \in \partial B_{(1/C)|x-z|}(z)$ and C is sufficiently large. Since (4-8) holds for any $\tilde{z} \in \partial B_{(1/C)|x-z|}(z)$, it follows that $|Du(x) - Du(z)| \le C|x-z|^\alpha$. □

Now suppose $|x-y| \ll r$. If either $y \in \mathscr{C}_{x,2r_1}$ or $x \in \mathscr{C}_{y,2r_1}$ holds, then by Claim 1 we have $|Du(x) - Du(y)| \le C|x-y|^\alpha$. Otherwise one may find a point $z \in \mathscr{C}_{x,r_1} \cap \mathscr{C}_{y,r_1}$ such that $|z-x| \approx |z-y| \approx |x-y|$. Then by applying the estimate in Claim 1 we have

$$|Du(x) - Du(y)| \le |Du(x) - Du(z)| + |Du(y) - Du(z)| \le C(|x-z|^\alpha + |y-z|^\alpha) \le C|x-y|^\alpha.$$

We can prove case (b) by a similar argument. Indeed, $\partial C_1 \cap B_r(x)$ can be represented as the graph of some Lipschitz function for any fixed $x \in \partial C_1$ provided $r \ll r_0$. Then, by the assumption that the C_i are disjoint, it is easy to find a small ball $B_{3r_1} \subset \Omega$ such that $\mathscr{C}_{z,3r_1} \subset \Omega$ for any $z \in B_r(x) \cap \overline{\Omega}$. Then, by a similar argument to the proof of case (a), we can show that $|Du(x) - Du(y)| \le C|x-y|^\alpha$ provided $|x-y| \ll r$. □

The following lemma shows that the centred sections are well-localized provided the heights are sufficiently small.

Lemma 4.2. *There exists a height $h_0 > 0$ such that, for any $x \in \overline{\Omega}$, the section $S_h(x)$ intersects at most one of ∂U, ∂C_i, $i = 1, \dots, m$, provided $h \le h_0$.*

Proof. Suppose to the contrary there exist sequences $x_k \in \overline{\Omega}$ and $h_k \to 0$, such that $S_{h_k}(x_k)$ intersects at least two of ∂U, ∂C_i, $i = 1, \dots, m$. Passing to a subsequence we may assume $x_k \to y \in \overline{\Omega}$. Since u is strictly convex in the interior of Ω, we have either $y \in \partial U$ or $y \in \partial C_i$ for some i. Denote by L_k the defining function of $S_{h_k}(x_k)$, namely $S_{h_k}(x_k) = \{u \le L_k\}$. Then, passing to a subsequence we may assume $L_k \to L$ for some affine function L, and $S_{h_k}(x_k) \to S \subset \{u \le L\}$. It follows from the properties of $S_{h_k}(x_k)$ that:

(i) S is centred at y.

(ii) S intersects at least two of ∂U, ∂C_i, $i = 1, \dots, m$.

(iii) $L(y) = \lim_{k\to\infty} L_k(x_k) = \lim_{k\to\infty} u(x_k) + h_k = u(y)$.

By (i) and (iii) we have that $S \subset \{u = L\}$. Then by (ii) we see that S passes through the interior of Ω, which contradicts the fact that u is strictly convex in the interior of Ω. □

Proof of Theorem 2.1(ii). *Step 1.* The main observation in this step is that if (4-3) is violated for small δ, then u is close to a linear function on a segment connecting x and some point $z_\delta \in \overline{\Omega}$. Hence, if (4-3) is violated for arbitrary r, δ, then one can find a sequence of points x_k such that u is more and more linear around x_k in some direction as $k \to \infty$. The "almost linearity" will be clear if we perform blow-up and an affine transform on u properly restricted to some carefully chosen section around x_k, and a line segment will appear on the graph of the limiting function. The detailed argument goes as follows.

To prove $\rho_x(t) \le Ct^{1+\alpha}$ for any $x \in \overline{\Omega}$ and any $t \le r$, by Lemma 4.1 we assume to the contrary that there exist sequences $t_k \le 1/k$, $\delta_k = 1/k$ and $x_k \in \overline{\Omega}$ such that

$$\rho_{x_k}\left(\tfrac{1}{2}t_k\right) \ge \tfrac{1}{2}(1 - 1/k)\rho_{x_k}(t_k). \tag{4-9}$$

Suppose the supremum in (4-2) $\big($when $x = x_k$ and $t = \frac{1}{2}t_k\big)$ is attained at $\frac{1}{2}(x_k + z_k) \in \overline{\Omega}$; by the definition of ρ_x we see that $\overline{z_k x_k} \subset \overline{\Omega}$, where $\overline{z_k x_k}$ denotes the segment connecting z_k and x_k. By passing to a subsequence, we may assume $x_k \to x_\infty \in \partial\Omega$.

Choosing sections. For each k, let $S_{h_k}(x_k)$ be a section of u with centre x_k, where h_k is chosen so that $z_k \in \partial S_{h_k}(x_k)$. Similar to the proof of Theorem 2.1(i), the existence of such a section follows from the property that a centred section, say $S_h(x)$, varies continuously with respect to the height h; see [Caffarelli and McCann 2010, Lemma A.8] for a proof. It is easy to see that $h_k \to 0$.

Normalization. Let L_k be the defining function of $S_{h_k}(x_k)$. We normalize the section $S_{h_k}(x_k)$ by a linear transformation T_k, and let $S_k = T_k(S_{h_k}(x_k))$. Note that $T_k(x_k) = 0$ and $B_1 \subset S_k \subset n^{3/2}B_1$. Also we let $u_k = (u - L_k)(T_k^{-1}x)/h_k$. Then u_k solves

$$\begin{cases} \det D^2 u_k = f_k & \text{in } S_k, \\ u_k = 0 & \text{on } \partial S_k, \end{cases} \tag{4-10}$$

where $f_k = h_k^{-n}(\det T_k)^{-1} f(T_k^{-1}x)/g(Du(T_k^{-1}x))$. After a rotation of coordinates, we may assume $T_k(z_k)$ is on the x_1-axis.

Linearity estimate. Let

$$v_k(x) := u(x) - Du(x_k)\cdot(x - x_k) - u(x_k);$$

from (4-9) we have that $v_k\big(\frac{1}{2}(x_k + z_k)\big) \ge \frac{1}{2}(1 - 1/k)v_k(z_k)$. Let

$$\tilde{L}_k(x) := L_k(x) - Du(x_k)\cdot(x - x_k) - u(x_k).$$

Then we have that $S_{h_k}(x_k) = \{v_k \le \tilde{L}_k\}$. Since $S_{h_k}(x_k)$ is centred at x_k, $z_k \in \partial S_{h_k}(x_k)$, $v_k \ge 0$ and $\tilde{L}_k(x_k) = h_k$, by John's lemma we have that $0 \le \tilde{L}_k(z_k) \le 2n^{3/2}h_k$. Now,

$$(v_k - \tilde{L}_k)\big(\tfrac{1}{2}(x_k + z_k)\big) - \tfrac{1}{2}\left(1 - \frac{1}{k}\right)((v_k - \tilde{L}_k)(x_k) + (v_k - \tilde{L}_k)(z_k)) \ge -\frac{1}{2k}(\tilde{L}_k(x_k) + \tilde{L}_k(z_k)) \ge -\frac{3n^{3/2}}{2k}h_k.$$

Since $v_k - \tilde{L}_k = u - L_k$, from the above estimate and the definition of u_k we have

$$u_k\big(\tfrac{1}{2}T_k z_k\big) \ge \tfrac{1}{2}\left(1 - \frac{1}{k}\right)(u_k(0) + u_k(T_k z_k)) - \frac{3n^{3/2}}{2k}. \tag{4-11}$$

Limiting problem. Now, by convexity we may take limits $S_k \to S_\infty$ and $u_k \to u_\infty$. Let f_∞ be the weak limit of f_k. Then u_∞ satisfies $\det D^2 u_\infty = f_\infty$ in the Alexandrov sense. Let $z_\infty := \lim_{k\to\infty} T_k(z_k)$. By (4-11) we have

$$u_\infty = L \text{ on the segment connecting } 0 \text{ and } z_\infty, \tag{4-12}$$

where L is a supporting function of u_∞ at 0.

Step 2. In this step, we need to consider two situations:

(a) $x_\infty \in \partial C_i$ for some $1 \le i \le k$.

(b) $x_\infty \in \partial U$.

In each case, a contradiction is obtained at some carefully chosen extreme point (denoted by y) of $\{u_\infty = L\}$. Heuristically, we can choose a section of u_∞ (denoted by S) around y such that y is much closer to ∂S in one direction than in the opposite direction. Hence, on one hand the Alexandrov-type estimate Lemma 2.6(a) shows that h, the height of the section S, should not be too small. On the other hand, Lemma 2.6(b) shows that h is very small, which is a contradiction.

We deal with case (a) first.

Proof in case (a). Note that since $x_\infty \in \partial C_i$ for some $1 \le i \le k$ and $h_k \to 0$ as $k \to \infty$, by Lemma 4.2 we have that the support of f_k can be represented by $S_k - A_k$ when k is large, where A_k is an open convex subset of S_k. Let the convex set A_∞ be the limit of the A_k. Then $S_\infty - A_\infty$ is the support of f. Since the centre of mass of S_∞ is 0 and $0 \in S_\infty - A_\infty$, we have that the volume of $S_\infty - A_\infty$ is positive. Hence, it is easy to see that there exists a constant C such that $C^{-1}\chi_{S_\infty - A_\infty} \le f_\infty \le C\chi_{S_\infty - A_\infty}$.

Since $\overline{z_k x_k} \subset \bar{\Omega}$, we have $\overline{0 z_\infty} \cap A_\infty = \varnothing$.

Subcase 1: $\{u_\infty = L\}$ contains an interior point of $S_\infty - \bar{A}_\infty$.

Subcase 2: $\{u_\infty = L\} \cap S_\infty \subset \bar{A}_\infty$.

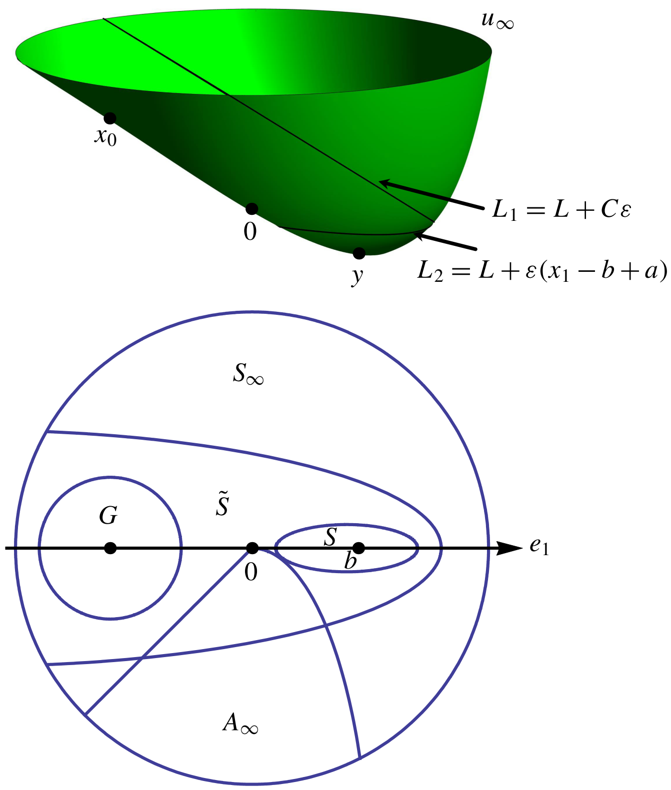

Figure 3. Two related sections.

For subcase 1, take $x_0 \in (S_\infty - \bar{A}_\infty) \cap \{u = L\}$. Take δ sufficiently small that $B_\delta(x_0) \subset S_\infty - \bar{A}_\infty$.

Choosing an extreme point. Let $y \in \{u = L\}$ be the point such that:

(1) $u_\infty(y) = \inf_{\{u=L\}} u_\infty$.

(2) y is an extreme point of the convex set $\{u_\infty = L\} \cap \{u_\infty = u(y)\}$.

It is easy to see that y is an extreme point of $\{u_\infty = L\}$.

Cutting a suitable section. By rotating the coordinates we may assume that $\{u_\infty = L\} \subset \{x_1 \le b\}$ for some constant $b > 0$, and that $\{u_\infty = L\} \cap \{x_1 = b\} = \{y\}$. Then we consider the section $S = \{u_\infty < L + \varepsilon(x_1 - b + a)\}$ (see Figure 3), where we fix a sufficiently small and then take $\varepsilon \ll a$, so that $S \Subset S_\infty$ and $a \gg d := \max\{x_1 \mid (x_1, 0, \dots, 0) \in S\} - b$.

Using Alexandrov estimates to obtain a contradiction. On one hand, by the Alexandrov estimate we have

$$|S|^2 > C\frac{a}{d}\varepsilon^n. \tag{4-13}$$

On the other hand, we consider another section $\tilde{S} = \{u_\infty < L + C\varepsilon\}$. Since u is Lipschitz, it is easy to see that $S \subset \tilde{S}$ provided C (independent of ε) is sufficiently large. By convexity we have $|B_\delta(x_0) \cap \tilde{S}| \ge C|\tilde{S}|$

for some constant C. We claim

$$|S|^2 \le C\varepsilon^n, \tag{4-14}$$

where the constant C is independent of d. The claim follows from the following argument. Let $v = u_\infty - L - C\varepsilon$. Let $G := \tilde{S} \cap B_\delta(x_0)$. By John's lemma, there exists an affine transformation A with $\det A = 1$ such that

$$B_{\bar r} \subset A(G) \subset n^{3/2} B_{\bar r}$$

for some $\bar r$. Now $\bar v = v(A^{-1}x)$ satisfies $\det D^2 \bar v = f_\infty(A^{-1}x) \ge C^{-1}$ in $A(G)$ and $|v| \le C\varepsilon$ in $A(G)$. Then we have

$$C^{-1}|G| \le \int_{G/2} f_\infty = \left|\partial \bar v\left(A\left(\tfrac{1}{2}G\right)\right)\right| \le C\frac{\varepsilon^n}{\bar r^n}. \tag{4-15}$$

Equation (4-14) follows from (4-15) and the fact that $|\tilde S| \approx |G| \approx \bar r^n$. Since $d \ll a$, it is easy to see that (4-14) contradicts (4-13).

For subcase 2, we need to choose the extreme point more carefully.

Choosing an extreme point. Let $\tilde K \subset \mathbb{R}^n$ be a supporting plane of the convex set A_∞ at 0. If A_∞ is not C^1 at 0 we choose $\tilde K$ to be the one containing $\overline{z_\infty 0}$. Let y' be the point where u_∞ attains its minimum on $D := \{u = L\} \cap \tilde K \cap \bar S_\infty$. It is easy to check that D is a convex set, and the set $D \cap \{x \mid u(x) = u(y')\}$ is also convex. Let y be an extreme point of $D \cap \{x \mid u(x) = u(y')\}$. We claim that y is an extreme point of $\{u = L\}$. Indeed, suppose not; then there exist $y_1, y_2 \in \{u = L\} \cap S_\infty \subset \bar A_\infty$ such that $y = \frac{1}{2}(y_1 + y_2)$. Since $\tilde K$ is a supporting plane of A_∞ and $y \in \bar A_\infty$, we have that $y_1, y_2 \in D$. However, since $u(y) = \min\{u(x) \mid x \in D\}$, we have $y_1, y_2 \in D \cap \{x \mid u(x) = u(y')\}$, which contradicts the choice of y as an extreme point of $D \cap \{x \mid u(x) = u(y')\}$.

Cutting a suitable section. By subtracting L and translating the coordinates we may assume that $y = 0$, that $u_\infty \ge 0$, that $u_\infty(te_1) = 0$ for $t \in (0, 1)$, and that $u_\infty(te_1) > 0$ for $t < 0$. Let $0 < \varepsilon \ll a$ be small positive numbers. Let $S_h(ae_1)$ be a section of u_∞ with centre ae_1, where h is chosen so that $-\varepsilon e_1 \in \partial S_h(ae_1)$. Since y is an extreme point of $\{u = L\}$, we have that $S_h(ae_1) \Subset S_\infty$ provided h is sufficiently small. Note that $h \to 0$ as $\varepsilon \to 0$.

Using Alexandrov estimates to obtain a contradiction. Since A_∞ is convex, it is easy to see that

$$\int_{S_h(ae_1)} f_\infty \le C \int_{\frac{1}{2}S_h(ae_1)} f_\infty$$

for some constant C. Let L_1 be the defining function of the section $S_h(ae_1)$, which is obviously decreasing in the e_1 direction. Hence $(L_1 - u_\infty)(0) \ge h$. Then by Lemma 2.6 we also have

$$(L_1 - u_\infty)(0) \le C\left(\frac{\varepsilon}{a}\right)^{1/n} h,$$

which contradicts the previous estimate. □

Proof in case (b). The proof in case (b) follows from a similar argument to the proof of [Caffarelli 1992a, Lemma 4]; we sketch the argument here. Note that f_k is now supported in a convex domain $D_k \subset \bar S_k$.

Let $D_\infty := \lim_{k\to\infty} D_k$. We have $z_\infty \in D_\infty$. Let L be the supporting function of u_∞ at 0 such that $\overline{0z_\infty} \subset \{u_\infty = L\}$. Similarly to the proof of subcase 1 of case (a), let $y \in \{u_\infty = L\}$ be the point such that:

(1) $u_\infty(y) = \inf_{\{u_\infty = L\}} u_\infty$.

(2) y is an extreme point of the convex set $\{u_\infty = L\} \cap \{u_\infty = u(y)\}$.

It is easy to see that y is an extreme point of $\{u_\infty = L\}$. Observe that $y \in D_\infty$, since otherwise u_k has positive Monge–Ampère measure outside D_k for large k. Let $z = (1-\sigma)y + \sigma z_\infty$ for some small positive σ; we may also find a section satisfying $S_h(z) := \{u_\infty < L\} \Subset S_\infty$ and $y + \varepsilon(y - z_\infty)/|y - z_\infty| \in \partial S_h(z)$ for small $\varepsilon \ll \sigma$. Since $y \in D_\infty$, there exists a sequence $y_k \in D_k$ such that $y_k \to y$ as $k \to \infty$. Let

$$\tilde{z}_k := (1-\sigma)y_k + \sigma T(z_k);$$

it is easy to see that $\tilde{z}_k \to z$ as $k \to \infty$. Recall that $z_\infty := \lim_{k\to\infty} T(z_k)$ with $T(z_k) \in D_k$. Let $\tilde{S}_k := \{u_k \le L_k\}$ be a section of u_k centred at $\tilde{z}_k$ with height h. Then, passing to a subsequence, $\tilde{S}_k \to S_h(z)$ in Hausdorff distance. In particular, $\tilde{S}_k \Subset S_k$ provided k is sufficiently large. Then, by Lemma 2.6, we have that

$$Ch \le (L_k - u_k)(y_k) \le \left(\frac{\varepsilon}{\sigma}\right)^{1/n} h$$

for large k, which is a contradiction because $\varepsilon \ll \sigma$. □

Theorem 2.1(ii) follows from the above discussions. □

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COMMUTATORS WITH FRACTIONAL DIFFERENTIATION AND NEW CHARACTERIZATIONS OF BMO-SOBOLEV SPACES

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For $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\alpha \in (0, 1)$, let D^α be the fractional differential operator and T be the singular integral operator. We obtain a necessary and sufficient condition on the function b to guarantee that $[b, D^\alpha T]$ is a bounded operator on a function space such as $L^p(\mathbb{R}^n)$ and $L^{p,\lambda}(\mathbb{R}^n)$ for any $1 < p < \infty$. Furthermore, we establish a necessary and sufficient condition on the function b to guarantee that $[b, D^\alpha T]$ is a bounded operator from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. This is a new theory. Finally, we apply our general theory to the Hilbert and Riesz transforms.

1. Introduction

For $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, denote by B the multiplication operator defined by $Bf(x) = b(x)f(x)$ for any measurable function f. If T is a linear operator on some measurable function space, then the commutator formed by B and T is defined by $[b, T]f(x) := (BT - TB)f(x)$. Let $0 \le \alpha \le 1$. The commutators we are interested in here are of the form

$$[b, T_\alpha]f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+\alpha}}(b(x) - b(y))f(y)\,dy,$$

where Ω is homogeneous of degree zero, integrable on S^{n-1}.

The case $\alpha = 1$ was first investigated by Calderón [1965] and now is well known as Calderón's first-order commutator. Calderón proved that $b \in \mathrm{Lip}(\mathbb{R}^n)$ (Lipschitz space) is a sufficient condition for the L^p-boundedness of $[b, T_1]$ when Ω satisfies some assumptions but may fail to have any regularity. However, this result has inspired many mathematicians to find new proofs, to make generalizations and to find further applications. We refer the reader to [Calderón 1980; Coifman and Meyer 1975; 1978; Cohen 1981; Hofmann 1994; 1998], among numerous references, for its development and applications. We would like to single out the work by Coifman and Meyer [1975], who found a new proof of Calderón's first-order commutator by reducing the commutator estimates to continuity of multilinear operators, which was used to deal with higher-order commutators in the same paper and has since been widely developed.

Let us comment on the main idea of Calderón's proof for future convenience. Firstly, the special properties such as locality of Lipschitz functions enable Calderón to use a rotation method to reduce

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commutator estimates in the higher-dimensional cases to the one-dimensional case. Secondly, the one-dimension commutator is just the commutator formed by b and dH/dx, the derivative of the Hilbert transform. Then Calderón exploited the special properties of the Hilbert transform as being closely related to analytic functions and used a characterization of the Hardy space $H^1(\mathbb{R})$ in terms of the Lusin square function to prove his theorem. It is the second part that has been reproved by Coifman and Meyer using techniques from multilinear analysis.

The case $\alpha = 0$ was first studied by Coifman, Rochberg and Weiss [Coifman et al. 1976], who showed that $b \in \mathrm{BMO}(\mathbb{R}^n)$, the bounded mean oscillation space, is a sufficient and necessary condition for the L^p-boundedness of $[b, T_0]$ when $\Omega \in \mathrm{Lip}(S^{n-1})$ (see also [Janson 1978; Uchiyama 1978]). For rough Ω, similar results have also been obtained in [Álvarez et al. 1993; Hu 2003; Chen and Ding 2015]. It is worth mentioning that the operator $[b, T_0]$ has a different character from $[b, T_1]$, whose research actually was inspired by the factorization of Hardy spaces.

The case $0 < \alpha < 1$ was first investigated by Segovia and Wheeden [1971], who obtained an analogue for differentiation of a fractional order of the one-dimensional version of Calderón's result [1965]. Murray [1985] improved the results of [Stein and Weiss 1971], more or less along the research line initiated by Calderón, by extending the commutator with derivatives of the Hilbert transform to those with fractional derivatives of the Hilbert transform. It turns out that these commutators with fractional differentiation are closely related to BMO-Sobolev spaces. Let $0 < \alpha \le 1$, and consider the fractional differentiation operators defined for f by

$$\widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

The fractional Laplacian can be defined in a distributional sense for functions that are not differentiable as long as $\hat{f}$ is not too singular at the origin or, in terms of the variable x, as long as

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(|1+|x|)^\alpha}\, dx < \infty.$$

For a function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the extension $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that satisfies the equation

$$u(x, 0) = f(x), \qquad \triangle_x u + \frac{1-\alpha}{y} u_y + u_{yy} = 0.$$

Caffarelli and Silvestre [2007] showed that

$$C D^\alpha f = \lim_{y \to 0^+} -y^{1-\alpha} u_y = \frac{1}{\alpha} \lim_{y \to 0} \frac{u(x, y) - u(x, 0)}{y^\alpha}$$

for some C depending on n and α.

Let I_α be the Riesz potential operator of order α. The Sobolev space $I_\alpha(\mathrm{BMO})$ is the image of BMO under I_α. Equivalently, a locally integrable function b is in $I_\alpha(\mathrm{BMO})$ if and only if $D^\alpha b \in \mathrm{BMO}$. Strichartz [1980] showed that, for $\alpha \in (0, 1)$, $I_\alpha(\mathrm{BMO})$ is a space of functions modulo constants that is properly contained in Lip_α, while Lip_1 is properly contained in $I_1(\mathrm{BMO})$.

Murray studied it only in the one-dimensional case, the commutators $[b, T_\alpha]$ formed by b and $D^\alpha H$ or D^α, and showed that $b \in I_\alpha(\mathrm{BMO})(\mathbb{R})$ is equivalent to the L^p-boundedness of $[b, T_\alpha]$. Calderón's original proof did not work well in this new situation. Instead, Murray used special properties of one-dimensional commutators to represent them in a way that techniques of multilinear analysis developed in [Coifman and Meyer 1975] could come into play. In the meantime, she showed that $b \in \mathrm{Lip}(\mathbb{R})$ is also a necessary condition for L^p-boundedness of $[b, T_1]$, thus providing a converse of Calderón's results on $\mathbb{R}$. In the review of [Murray 1985] in Math Reviews, Y. Meyer indicates that the results there apply to functions on $\mathbb{R}^n$. However, a direct perusal of [Murray 1985] reveals that the paper only tackles the case $n = 1$. (Meyer might have known how to treat $n > 1$.) Maybe, it can in particular be applied to $[b, D^\alpha]$ on $\mathbb{R}^n$ for $n > 1$. But we think the techniques may fail for $[b, T_\alpha]$ on $\mathbb{R}^n$ for $n > 1$. The reason is that the higher-dimensional commutators are much more complicated due to the presence of Ω, which cannot be represented easily.

In the case of $0 < \alpha < 1$, by applying an off-diagonal $T1$ theorem (see [Hofmann 1998]), Q. Chen and Z. Zhang [2004] obtained the (L^p, L^q) bounds for the operator $[b, T_\alpha]$ with $\Omega \in \mathrm{Lip}(S^{n-1})$ and $D^\alpha b \in L^r(\mathbb{R}^n)$, where $1 < r < \infty$ and $1/p + 1/r = 1/q$. However, they pointed out that they do not know whether the off-diagonal $T1$ theorem is true for $r = \infty$, so the (L^p, L^p)-boundedness of the operator $[b, T_\alpha]$ cannot be obtained in [Chen and Zhang 2004]. We think there are two reasons that the (L^p, L^p)-boundedness of the operator $[b, T_\alpha]$ cannot be obtained in [Chen and Zhang 2004]. Firstly, Calderón's rotation method is of no use, since the elements in $I_\alpha(\mathrm{BMO})(\mathbb{R}^n)$ are not local and do not enjoy the properties of Lipschitz functions. Secondly, the $T1$ theorem developed by David and Journé [1984], which is a powerful tool for the commutators $[b, dH/dx]$ and $[b, D^\alpha]$ to give an alternate proof, does not work well in general situations, such as the cases where the operators are rough or the cases where the weak-boundedness property (WBP) is not easy to verify.

Here we use Fourier transform estimates and Littlewood–Paley theory developed in [Duoandikoetxea and Rubio de Francia 1986] to get the L^p-boundedness of $[b, T_\alpha]$ with rough kernel for all $1 < p < \infty$, which can be stated as follows.

Theorem 1.1. *Suppose $\alpha \in (0, 1)$ and $b \in I_\alpha(\mathrm{BMO})$. If $\Omega \in L \log^+ L(S^{n-1})$ having the mean value zero property, that is,*

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0, \tag{1-1}$$

then there is a constant C such that, for $1 < p < \infty$,

$$\|[b, T_\alpha] f\|_{L^p} \le C \|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^p}.$$

We will prove this result in Section 2.

Remark 1.2. Our arguments depend heavily on the Fourier transform estimates, which is not a surprise from the historical point of view of techniques in handling rough operators [Duoandikoetxea and Rubio de Francia 1986]. But, as Murray has pointed out, the cases $0 < \alpha < 1$ are fundamentally different: the underlying details turn out to be very subtle and quite different from the cases of $\alpha = 0$ and $\alpha = 1$. Furthermore, we believe some modifications of the method in the present paper should provide an alternate proof of Calderón's first-order commutator estimate.

As applications to partial differential equations have been found in the cases $\alpha = 0$, 1 and Murray's one-dimensional result in the case $0 < \alpha < 1$ (see [Calderón 1980; Chiarenza et al. 1991; Di Fazio and Ragusa 1991; 1993; Murray 1987; Lewis and Silver 1988; Lewis and Murray 1991; 1995; Taylor 1991; 1997; 2015]), we also expect applications of our results to fractional-order partial differential equations (see for instance [Silvestre 2007; Caffarelli and Silvestre 2007; Caffarelli and Stinga 2016] on fractional elliptic equations).

Definition 1.3. A measurable function $f \in L^p(\mathbb{R}^n)$, $p \in (1, \infty)$, belongs to the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ with $\lambda \in [0, n)$ if the following norm is finite:

$$\|f\|_{L^{p,\lambda}} = \left(\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^\lambda} \int_{Q(x,r)} |f(y)|^p \, dy \right)^{1/p},$$

where $Q(x, r)$stands for any cube of radius r and centered at x_0. When $\lambda = 0$, $L^{p,\lambda}(\mathbb{R}^n)$ coincides with the Lebesgue space $L^p(\mathbb{R}^n)$.

It is well known that the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ [1938] is connected to certain problems in elliptic PDEs. Later, the Morrey spaces were found to have many important applications to the Navier–Stokes equations, the Schrödinger equations, elliptic equations and potential analysis (see [Chiarenza and Frasca 1987; Kato 1992; Taylor 1992; Ruiz and Vega 1991; Shen 2003; Di Fazio et al. 1999; Palagachev and Softova 2004; Deng et al. 2005; Adams and Xiao 2004; 2011; 2012]).

Recently, Chen, Ding and Wang gave a criterion of the boundedness of a general linear or sublinear operator on Morrey spaces:

Theorem A [Chen et al. 2012]. *Let $0 < \lambda < n$. Suppose that $\Omega \in L^q(S^{n-1})$ for $q > n/(n - \lambda)$ and S is a sublinear operator satisfying $|Sf(x)| \le C \int_{\mathbb{R}^n} |\Omega(x - y)| \, |f(y)|/|x - y|^n \, dy$. Let $1 < p < \infty$. If the operator S is bounded on $L^p(\mathbb{R}^n)$, then S is bounded on $L^{p,\lambda}(\mathbb{R}^n)$.*

Clearly, $b \in I_\alpha(\mathrm{BMO}) \subset \mathrm{Lip}_\alpha$ for $0<\alpha<1$ implies $|[b, T_\alpha]f(x)| \le C\int_{\mathbb{R}^n} |\Omega(x-y)| \, |f(y)|/|x - y|^n \, dy$. Since $\Omega \in L^q(S^{n-1}) \subset L \log^+ L(S^{n-1})$ for $q > n/(n - \lambda)$, applying Theorem A and Theorem 1.1, we get:

Corollary 1.4. *Let $0 < \lambda < n$. Suppose $\alpha \in (0, 1)$ and $b \in I_\alpha(\mathrm{BMO})$. If $\Omega \in L^q(S^{n-1})$ for $q > n/(n - \lambda)$ and satisfies (1-1), then there is a constant C such that, for $1 < p < \infty$,*

$$\|[b, T_\alpha] f\|_{L^{p,\lambda}} \le C \|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^{p,\lambda}}.$$

Pérez [1995] gave a simple example to show that the commutator $[b, T_0]$ is not of weak type $(1, 1)$ when $b \in \mathrm{BMO}$. However, if $0 < \alpha < 1$, $b \in I_\alpha(\mathrm{BMO})$ and $\Omega \in \mathrm{Lip}(S^{n-1})$, it is easy to verify that $k(x, y) = (\Omega(x - y)/|x - y|^{n+\alpha})(b(x) - b(y))$ is a standard kernel. Moreover, $\Omega \in \mathrm{Lip}(S^{n-1}) \subset L \log^+ L(S^{n-1})$, we apply Theorem 1.1 (the L^2-boundedness of $[b, T_\alpha]$) to see $[b, T_\alpha]$ is a generalized Calderón–Zygmund operator. So the weak type $(1, 1)$-boundedness of $[b, T_\alpha]$ is a natural consequence. Therefore, it will be interesting to give a necessary condition for the $L^1 \to L^{1,\infty}$ bounds of $[b, T_\alpha]$, which is our main aim in this part. Moreover, we will also give the necessity of the $L^{p,\lambda}$-boundedness of the commutator $[b, T_\alpha]$.

The following useful characterization of $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ is due to Meyers [1964]:

Theorem B. *Let $\alpha \in (0, 1]$. A locally integrable function b is in $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ if and only if there is a constant C such that, for any cube Q,*

$$\frac{1}{|Q|^{1+\alpha/n}} \int_Q |b(x) - b_Q|\, dx \le C.$$

We first give a necessary condition for the $L^{p,\lambda}$ bounds of $[b, T_\alpha]$.

Theorem 1.5. *Suppose $\alpha \in (0, 1]$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ satisfies (1-1) or*

$$\int_{S^{n-1}} \Omega(x')x'_j \, d\sigma(x') = 0, \tag{1-2}$$

for $j = 1, 2, \ldots, n$. Assume Ω is not identically zero. If $[b, T_\alpha]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$ for some $1 < p < \infty$ and $0 \le \lambda < n$, then $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

Remark 1.6. In particular, if $[b, T_\alpha]$ is a bounded on $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, then $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

Remark 1.7. Since the structure of Ω is complicated and cannot be represented easily, the idea of proving Theorem 1.5 is very different from Murray's method [1985], where the proof depends on a special property of the Hilbert transform H.

Theorem 1.8. *Suppose $\alpha \in (0, 1]$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ satisfies (1-1) or (1-2). Assume Ω is not identically zero. If $[b, T_\alpha]$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, then $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.*

Remark 1.9. As far as we know, this is the first example of a necessary condition for the $L^1 \to L^{1,\infty}$-boundedness of an operator.

The proof of Theorems 1.5 and 1.8 will be given in Sections 3 and 4, respectively.

Moreover, in the course of showing the main result, in conjunction with Calderón's first-order estimates, we obtain the characterizations of $\mathrm{Lip}(\mathbb{R}^n)$ in terms of the L^p-, $(L^1, L^{1,\infty})$- and $L^{p,\lambda}$-boundedness of commutators. If $b \in \mathrm{Lip}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$, then by Theorem 2 in [Calderón 1965] it is easy to check that $[b, T_1]$ is a Calderón–Zygmund operator, so the weak type $(1, 1)$-boundedness of $[b, T_1]$ is a natural consequence. Applying Calderón's conclusion [1965, Theorem 2] and Theorems A, 1.5 and 1.8 for the case of $\alpha = 1$, we give the characterizations for the Calderón commutator $[b, T_1]$ as follows.

Corollary 1.10. *Let $1 < p < \infty$ and $0 < \lambda < n$. Suppose that $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ satisfy (1-2); then the following four statements are equivalent:*

(i) $b \in \mathrm{Lip}(\mathbb{R}^n)$;

(ii) *$[b, T_1]$ is bounded on $L^p(\mathbb{R}^n)$;*

(iii) *$[b, T_1]$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$;*

(iv) *$[b, T_1]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$.*

For the case of $\alpha \in (0, 1)$, in conjunction with Theorems 1.1, 1.5 and 1.8, we get:

Theorem 1.11. *Suppose $\alpha \in (0, 1)$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega \in \mathrm{Lip}(S^{n-1})$ satisfy the mean value zero property. Let $1 < p < \infty$ and $0 < \lambda < n$. Then the implications* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) *hold for the following four statements:*

(i) $[b, T_\alpha]$ *is bounded on* $L^p(\mathbb{R}^n)$;

(ii) $[b, T_\alpha]$ *is bounded from* $L^1(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$;

(iii) $[b, T_\alpha]$ *is bounded on* $L^{p,\lambda}(\mathbb{R}^n)$;

(iv) $[b, T_\alpha]$ *is bounded from* $L^\infty(\mathbb{R}^n)$ *to* $\mathrm{BMO}(\mathbb{R}^n)$.

We will prove Theorem 1.11 in Section 5.

Let T_α and T be the operators which are defined respectively by

$$T_\alpha f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega((x-y)')}{|x-y|^{n+\alpha}} f(y)\, dy, \quad 0 < \alpha < 1, \tag{1-3}$$

and

$$T f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\widetilde{\Omega}((x-y)')}{|x-y|^{n}} f(y)\, dy. \tag{1-4}$$

We will give a relation between $[b, T_\alpha]$ and $[b, D^\alpha T]$ for the case of $0 < \alpha < 1$.

Proposition 1.12. *Let* $0 < \alpha < 1$. *For any fixed operator* T_α *defined by* (1-3) *with* $\Omega \in L^2(S^{n-1})$ *satisfying* (1-1), *there exists a singular integral operator* T *defined by* (1-4) *with* $\widetilde{\Omega} \in L^2_\alpha(S^{n-1})$ *satisfying* (1-1) *such that* $T_\alpha = D^\alpha T$. *Conversely, for any fixed singular integral operator* T *with* $\widetilde{\Omega} \in L^2_\alpha(S^{n-1})$ *satisfying* (1-1), *there exists an operator* T_α *with* $\Omega \in L^2(S^{n-1})$ *satisfying* (1-1) *such that* $T_\alpha = D^\alpha T$.

We will prove Proposition 1.12 in Section 6.

In particular, for any fixed singular integral operator T with $\widetilde{\Omega} \in C^2(S^{n-1})$ satisfying (1-1), there exists an operator T_α with $\Omega \in C^1(S^{n-1})$ satisfying (1-1) such that $T_\alpha = D^\alpha T$. Then, applying the result of Proposition 1.12, we get:

Corollary 1.13. *Suppose* $\alpha \in (0, 1)$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ *and* $\widetilde{\Omega} \in C^2(S^{n-1})$ *satisfy* (1-1). *Let* $1 < p < \infty$ *and* $0 < \lambda < n$. *Then the implications* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) *hold for the following four statements*:

(i) $[b, D^\alpha T]$ *is bounded on* $L^p(\mathbb{R}^n)$;

(ii) $[b, D^\alpha T]$ *is bounded from* $L^1(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$;

(iii) $[b, D^\alpha T]$ *is bounded on* $L^{p,\lambda}(\mathbb{R}^n)$;

(iv) $[b, D^\alpha T]$ *is bounded from* $L^\infty(\mathbb{R}^n)$ *to* $\mathrm{BMO}(\mathbb{R}^n)$.

Remark 1.14. We will give an application of Theorem 1.1 and Corollary 1.13 to Riesz transforms. In fact, for $0 < \alpha < 1$, since $\widehat{D^\alpha R_j f}(\xi) = -i\xi_j |\xi|^{\alpha-1} \hat{f}(\xi)$ a trivial computation gives

$$\eta(\alpha)\Big(\text{p.v.} \frac{x_j}{|x|^{n+1+\alpha}}\Big)^{\wedge}(\xi) = i\xi_j|\xi|^{\alpha-1}, \quad \text{where } \eta(\alpha) = \frac{1-n-\alpha}{2\pi} \frac{\Gamma\big(\frac{1}{2}(n+\alpha-1)\big)}{\pi^{n/2+\alpha-1}\Gamma\big(\frac{1}{2}(1-\alpha)\big)}.$$

From the above facts, we get

$$[b, D^\alpha R_j] f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega_j(x-y)}{|x-y|^{n+\alpha}} (b(x) - b(y)) f(y)\, dy,$$

where $\Omega_j(x') = \eta(\alpha)x_j/|x|$, $j = 1, 2, \dots, n$. Since $\Omega_j(x')$ is in $L\log^+ L(S^{n-1})$ and satisfies the mean value zero property, by Theorem 1.1 we get, for $1 < p < \infty$,

$$\|[b, D^\alpha R_j]\|_{L^p} \le C\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}, \qquad j = 1, 2, \dots, n.$$

Now suppose that $[b, D^\alpha R_j]$ are bounded operators from L^∞ to BMO for $j = 1, 2, \dots, n$. The vanishing moment of Ω_j gives $[b, D^\alpha R_j](1)(x) = -D^\alpha R_j b(x) = -R_j D^\alpha(b)(x) \in \mathrm{BMO}$, $j = 1, 2, \dots, n$. Since $R_j : \mathrm{BMO} \to \mathrm{BMO}$ and $\sum_{j=1}^n R_j^2 f = -f$, we get

$$\|D^\alpha b\|_{\mathrm{BMO}} = \left\|\sum_{j=1}^n R_j^2 D^\alpha b\right\|_{\mathrm{BMO}} \le C\sum_{j=1}^n \left\|(R_j D^\alpha b)\right\|_{\mathrm{BMO}} \le C.$$

This gives that $D^\alpha b \in \mathrm{BMO}$. Then, applying Corollary 1.13, for $\alpha \in (0, 1)$, $1 < p < \infty$ and $0 < \lambda < n$ the following five statements are equivalent:

(i) $b \in I_\alpha(\mathrm{BMO})$;

(ii) $[b, D^\alpha R_j]$, $j = 1, \dots, n$, are bounded on $L^p(\mathbb{R}^n)$;

(iii) $[b, D^\alpha R_j]$, $j = 1, \dots, n$, are bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$;

(iv) $[b, D^\alpha R_j]$, $j = 1, \dots, n$, are bounded on $L^{p,\lambda}(\mathbb{R}^n)$;

(v) $[b, D^\alpha R_j]$, $j = 1, \dots, n$, are bounded from $L^\infty(\mathbb{R})$ to $\mathrm{BMO}(\mathbb{R}^n)$.

The following results show that if we assume some conditions on T, then we may characterize the commutator $[b, D^\alpha T]$ directly.

Corollary 1.15. *Suppose $\alpha \in (0, 1)$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If T is a bounded, invertible operator on* BMO, *then when $\widetilde{\Omega} \in C^2(S^{n-1})$ satisfies (1-1), for $1 < p < \infty$ and $0 < \lambda < n$ the following five statements are equivalent*:

(i) $b \in I_\alpha(\mathrm{BMO})$;

(ii) $[b, D^\alpha T]$ *is bounded on* $L^p(\mathbb{R}^n)$;

(iii) $[b, D^\alpha T]$ *is bounded from* $L^1(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$;

(iv) $[b, D^\alpha T]$ *is bounded on* $L^{p,\lambda}(\mathbb{R}^n)$;

(v) $[b, D^\alpha T]$ *is bounded from* $L^\infty(\mathbb{R}^n)$ *to* $\mathrm{BMO}(\mathbb{R}^n)$.

Proof. (i) $\Longrightarrow$ (ii) follows from Theorem 1.1 and (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) follows from Corollary 1.13, so it remains to prove (v) $\Longrightarrow$ (i). If $[b, T_\alpha]$ is bounded from L^∞ to BMO, the vanishing moment of Ω gives $[b, D^\alpha T](1)(x) = -T D^\alpha b(x) \in \mathrm{BMO}$. Since T is a bounded, invertible operator on BMO, we get $D^\alpha b \in \mathrm{BMO}$. □

Remark 1.16. Since H is a bounded, invertible operator on $\mathrm{BMO}(\mathbb{R})$, by Corollary 1.15 we have for $\alpha \in (0, 1)$, $1 < p < \infty$ and $0 < \lambda < n$ that the following five statements are equivalent:

(i) $b \in I_\alpha(\mathrm{BMO})$;

(ii) $[b, D^\alpha H]$ is bounded on $L^p(\mathbb{R})$;

(iii) $[b, D^\alpha H]$ is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$;

(iv) $[b, D^\alpha H]$ is bounded on $L^{p,\lambda}(\mathbb{R})$;

(v) $[b, D^\alpha H]$ is bounded from $L^\infty(\mathbb{R})$ to $\mathrm{BMO}(\mathbb{R})$.

2. Proof of Theorem 1.1

We first prove Theorem 1.1 by a key lemma, whose proof will be given below. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be a radial function such that $\operatorname{supp}\phi \subset \left\{\frac{1}{2} \le |\xi| \le 2\right\}$ and

$$\sum_{l\in\mathbb{Z}} \phi^3(2^{-l}\xi) = 1 \quad \text{for any } |\xi| > 0.$$

Define the multiplier S_l by $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\hat{f}(\xi)$ for all $l \in \mathbb{Z}$.

Lemma 2.1. *Suppose that $\Omega(x')$ satisfies (1-1). Let*

$$K_j(x) = \frac{\Omega(x')}{|x|^{n+\alpha}}\chi_{\{2^j \le |x| < 2^{j+1}\}}(x)$$

for $j \in \mathbb{Z}$. Define the multiplier T_j^l ($l \in \mathbb{Z}$) by $\widehat{T_j^l f}(\xi) = \phi(2^{j-l}\xi)\widehat{K_j}(\xi)\hat{f}(\xi)$. Set

$$V_l f(x) = \sum_{j\in\mathbb{Z}} [b, S_{l-j}T_j^l S_{l-j}](f)(x).$$

Let $0 < \alpha < 1$. For $b \in I_\alpha(\mathrm{BMO})(\mathbb{R}^n)$, the following conclusions hold:

(i) *If $\Omega \in L^\infty(S^{n-1})$, then there exists $0 < \tau < 1$ such that*

$$\|V_l f\|_{L^2} \le C\|\Omega\|_{L^\infty} 2^{-\tau|l|}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^2} \quad \text{for } l \in \mathbb{Z}. \tag{2-1}$$

(ii) *If $\Omega \in L^1(S^{n-1})$ then, for $1 < p < \infty$,*

$$\|V_l f\|_{L^p} \le C\|\Omega\|_{L^1}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p} \quad \text{for } l \in \mathbb{Z}. \tag{2-2}$$

The constants C in (2-1) and (2-2) are independent of l.

Proof of Theorem 1.1. Let us now finish the proof of Theorem 1.1 by using Lemma 2.1.

Let $E_0 = \{x' \in S^{n-1} : |\Omega(x')| < 2\}$ and $E_d = \{x' \in S^{n-1} : 2^d \le |\Omega(x')| < 2^{d+1}\}$ for $d \in \mathbb{N}$. For $d \ge 0$, let

$$\Omega_d(y') = \Omega(y')\chi_{E_d}(y') - \frac{1}{|S^{n-1}|}\int_{E_d} \Omega(x')\,d\sigma(x'),$$

Then $\Omega(y') = \sum_{d\ge0} \Omega_d(y')$. Since Ω satisfies (1-1),

$$\int_{S^{n-1}} \Omega_d(y')\,d\sigma(y') = 0 \quad \text{for all } d \ge 0.$$

Set

$$K_{j,d}(x) = \frac{\Omega_d(x)}{|x|^{n+\alpha}}\chi_{\{2^j \le |x| < 2^{j+1}\}}(x)$$

and define $T^l_{j,d}$ and $V_{l,d}$ in the same way as T^l_j and V_l are defined in Lemma 2.1, replacing K_j by $K_{j,d}$. With the notations above, it is easy to see that

$$[b, T_\alpha]f(x) = \sum_{d\geq 0}\sum_{l\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}[b, S_{l-j}T^l_{j,d}S_{l-j}]f(x) = \sum_{d\geq 0}\sum_{l\in\mathbb{Z}} V_{l,d}f(x).$$

By interpolating between (2-1) and (2-2), there exists $0 < \theta < 1$ such that

$$\|V_{l,d}f\|_{L^p} \leq C\|\Omega_d\|_\infty 2^{-\theta|l|}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p} \quad \text{for } l\in\mathbb{Z}. \tag{2-3}$$

Taking a large positive integer N such that $N > 2\theta^{-1}$,

$$\|[b, T_\alpha]f\|_{L^p} \leq \sum_{d\geq 0}\sum_{Nd<|l|}\|V_{l,d}f\|_{L^p} + \sum_{d\geq 0}\sum_{0\leq|l|\leq Nd}\|V_{l,d}f\|_{L^p} =: J_1 + J_2.$$

For J_1, using (2-3) we get

$$J_1 \leq C\|D^\alpha b\|_{\mathrm{BMO}}\sum_{d\geq 0}2^d\sum_{|l|>Nd}2^{-\theta|l|}\|f\|_{L^p} \leq C\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}.$$

Finally, by (2-2) we get

$$\begin{aligned} J_2 &\leq C\|D^\alpha b\|_{\mathrm{BMO}}\sum_{d\geq 0}\sum_{0\leq|l|<Nd}2^d\sigma(E_d)\|f\|_{L^p} \\ &\leq C\|D^\alpha b\|_{\mathrm{BMO}}\sum_{d\geq 0}2^d\sigma(E_d)\|f\|_{L^p} \\ &\leq C\|\Omega\|_{L\log^+ L}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}. \end{aligned}$$

Combining the estimates of J_1 and J_2, we get

$$\|[b, T_\alpha]f\|_{L^p} \leq C(1+\|\Omega\|_{L\log^+ L})\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p},$$

which is exactly the required conclusion of Theorem 1.1. □

Proof of Lemma 2.1. Hence, to finish the proof of Theorem 1.1, it remains to prove Lemma 2.1. Let us begin by giving some lemmas and their proofs, which will play a key role in the proof.

Lemma 2.2 [Christ and Journé 1987]. *Let $\Theta_j f(x) := \int_{\mathbb{R}^n}\psi_j(x,y)f(y)\,dy$, where $\psi_j(x,y)$ satisfies the standard kernel conditions, i.e., for some $\gamma > 0$ and $C > 0$,*

$$|\psi_j(x,y)| \leq C\frac{2^{j\gamma}}{(2^{-j}+|x-y|)^{n+\gamma}} \tag{2-4}$$

and

$$|\psi_j(x,y+h)-\psi_j(x,y)| \leq C\frac{|h|^\gamma}{(2^{-j}+|x-y|)^{n+\gamma}}, \quad |h|\leq 2^j, \tag{2-5}$$

for all $x, y\in\mathbb{R}^n$ and $j\in\mathbb{Z}$. Suppose that $du(x,t) = \sum_{j\in\mathbb{Z}}|\Theta_j 1(x)|^2\,dx\,\delta_{2^{-j}}(t)$ is a Carleson measure, where $\delta_{2^{-j}}(t)$ is Dirac mass at the point $t = 2^{-j}$. Then $\sum_{j\in\mathbb{Z}}\|\Theta_j f\|^2_{L^2} \leq C\|f\|^2_{L^2}$.

Lemma 2.3. *Let $\alpha \in (0,1)$ and $b \in I_\alpha(\mathrm{BMO})(\mathbb{R}^n)$. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be a radial function such that* $\operatorname{supp}\phi \subset \{\frac{1}{2} \le |\xi| \le 2\}$. *Define the multiplier operator S_l by $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\hat{f}(\xi)$ for $l \in \mathbb{Z}$. Then for* $1 < p < \infty$ *we have*

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{2j\alpha} |[b, S_j]f|^2 \Big)^{1/2} \Big\|_{L^p} \le C \|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^p}.$$

Proof. Let $\widehat{\Phi} = \phi$ and $\Phi_{2^{-j}}(x) = 2^{jn}\Phi(2^j x)$; then $S_j f = \Phi_{2^{-j}} * f$. Let

$$k_j(x,y) = 2^{j\alpha}(b(x) - b(y))\Phi_{2^{-j}}(x-y);$$

then

$$2^{j\alpha}[b, S_j]f(x) = \int_{\mathbb{R}^n} k_j(x,y) f(y)\,dy.$$

It is easy to verify that $k_j(x,y)$ satisfies (2-4) and (2-5). Since

$$2^{j\alpha}[b, S_j]1 = 2^{j\alpha} S_j b = 2^{j\alpha}(|\xi|^\alpha |\xi|^{-\alpha}\phi(2^{-j}\xi)\hat{b})^\vee = (\hat{\sigma}(2^{-j}\xi)\widehat{D^\alpha b})^\vee =: S_j^\alpha(D^\alpha b),$$

where $\hat{\sigma}(\xi) = \phi(\xi)|\xi|^{-\alpha}$ and S_j^α is a multiplier defined by $S_j^\alpha f(x) = \sigma_{2^{-j}} * f(x)$, by $D^\alpha b \in \mathrm{BMO}$ we know

$$du(x,t) = \sum_{j\in\mathbb{Z}} |2^{j\alpha}[b, S_j]1(x)|^2 \, dx\, \delta_{2^{-j}}(t)$$

is a Carleson measure. Thus, by Lemma 2.2 we get

$$\sum_{j\in\mathbb{Z}} \|2^{j\alpha}[b, S_j]f\|_{L^2}^2 \le C\|f\|_{L^2}^2. \tag{2-6}$$

Define the operator $\mathbb{T}$ by

$$\mathbb{T}f(x) = \int_{\mathbb{R}^n} \mathbb{K}(x,y) f(y)\,dy,$$

where $\mathbb{K} : (x,y) \mapsto \{k_j(x,y)\}_{j\in\mathbb{Z}}$ with $\|\mathbb{K}(x,y)\|_{\mathbf{C}\mapsto \ell^2} := \big(\sum_{j\in\mathbb{Z}} |k_j(x,y)|^2\big)^{1/2}$. Thus, (2-6) says that

$$\|\mathbb{T}f\|_{L^2(\ell^2)} \le C\|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^2}.$$

On the other hand, for $b \in I_\alpha(\mathrm{BMO})$, it is easy to verify that, for $2|x - x_0| \le |x-y|$,

$$\Big(\sum_{j\in\mathbb{Z}} |k_j(x,y) - k_j(x_0,y)|^2 \Big)^{1/2} \le C\|D^\alpha b\|_{\mathrm{BMO}} \frac{|x-x_0|^\alpha}{|x-y|^{n+\alpha}},$$

since $I_\alpha(\mathrm{BMO}) \subset \mathrm{Lip}_\alpha$ for $0 < \alpha < 1$. Then, by the result in [Grafakos 2004], we prove Lemma 2.3. □

Lemma 2.4. *Let $m_{\delta,j} \in C_0^\infty(\mathbb{R}^n)$, $0 < \delta < \infty$, for any fixed $j \in \mathbb{Z}$ and let $T_{\delta,j}$ be the multiplier operator defined by $\widehat{T_{\delta,j}f}(\xi) = m_{\delta,j}(\xi)\hat{f}(\xi)$. For $0 < \alpha < 1$, let $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ and let $[b, T_{\delta,j}]$ be the commutator of $T_{\delta,j}$. If, for some constants $A > 0$ and $0 < \beta < 1$,*

$$\|m_{\delta,j}\|_{L^\infty} \le CA2^{-j\alpha}\min\{\delta, \delta^{-\beta}\} \quad \textit{and} \quad \|\nabla m_{\delta,j}\|_{L^\infty} \le CA2^j 2^{-j\alpha},$$

then there exists a constant $0<\lambda<1$ *such that*

$$\|[b,T_{\delta,j}]f\|_{L^2}\le CA\min\{\delta^{\lambda},\delta^{-\beta\lambda}\}\|b\|_{\mathrm{Lip}_\alpha}\|f\|_{L^2},$$

where C *is independent of* δ *and* j.

Proof. Without loss of generality, we may assume that $\|b\|_{\mathrm{Lip}_\alpha}=1$. Taking a $C_0^\infty(\mathbb{R}^n)$ radial function ϕ with $\operatorname{supp}\phi\subset\left\{\frac{1}{2}\le|x|\le 2\right\}$ and $\sum_{l\in\mathbb{Z}}\phi(2^{-l}x)=1$ for any $|x|>0$. Let $\phi_0(x)=\sum_{l=-\infty}^{0}\phi(2^{-l}x)$ and $\phi_l(x)=\phi(2^{-l}x)$ for positive integers l. Let $K_{\delta,j}(x)=m_{\delta,j}^{\vee}(x)$, the inverse Fourier transform of $m_{\delta,j}$. Split $K_{\delta,j}$ into

$$K_{\delta,j}(x)=K_{\delta,j}(x)\phi_0(x)+\sum_{l=1}^{\infty}K_{\delta,j}(x)\phi_l(x)=:\sum_{l=0}^{\infty}K_{\delta,j}^{l}(x).$$

Note that $\int_{\mathbb{R}^n}\widehat{\phi}(\eta)\,d\eta=0$ and

$$\widehat{K_{\delta,j}^{l}}(x)=2^{ln}\int_{\mathbb{R}^n}m_{\delta,j}(x-y)\widehat{\phi}(2^{l}y)\,dy=\int_{\mathbb{R}^n}m_{\delta,j}(x-2^{-l}y)\widehat{\phi}(y)\,dy.$$

Thus,

$$\begin{aligned}\|\widehat{K_{\delta,j}^{l}}\|_{L^\infty}&\le\left\|\int_{\mathbb{R}^n}(m_{\delta,j}(x-2^{-l}y)-m_{\delta,j}(x))\widehat{\phi}(y)\,dy\right\|_{L^\infty}\\&\le CA2^{-l}\|\nabla m_{\delta,j}\|_{L^\infty}\int_{\mathbb{R}^n}|y|\,|\widehat{\phi}(y)|\,dy\\&\le CA2^{-l}2^{j}2^{-j\alpha}.\end{aligned}\tag{2-7}$$

On the other hand, by the Young inequality,

$$\|\widehat{K_{\delta,j}^{l}}\|_{L^\infty}=\|\widehat{K_{\delta,j}}*\widehat{\phi_l}\|_{L^\infty}\le\|\widehat{K_{\delta,j}}\|_{L^\infty}\|\widehat{\phi_l}\|_{L^1}\le CA2^{-j\alpha}\min\{\delta,\delta^{-\beta}\}.\tag{2-8}$$

Therefore, by (2-7) and (2-8), for each $0<\theta<1$,

$$\|\widehat{K_{\delta,j}^{l}}\|_{L^\infty}\le CA2^{-\theta l}2^{(\theta-\alpha)j}\min\{\delta^{1-\theta},\delta^{-(1-\theta)\beta}\}.\tag{2-9}$$

Then, from (2-8), (2-9) and the Plancherel theorem, we get

$$\|T_{\delta,j}^{l}f\|_{L^2}\le CA2^{-j\alpha}\min\{\delta,\delta^{-\beta}\}\|f\|_{L^2}\tag{2-10}$$

and

$$\|T_{\delta,j}^{l}f\|_{L^2}\le CA2^{-\theta l}2^{(\theta-\alpha)j}\min\{\delta^{1-\theta},\delta^{-(1-\theta)\beta}\}\|f\|_{L^2}.\tag{2-11}$$

Now we turn our attention to $[b,T_{\delta,j}^{l}]$, the commutator of the operator $T_{\delta,j}^{l}$. Decompose $\mathbb{R}^n$ into a grid of nonoverlapping cubes with side length 2^{l}. That is, $\mathbb{R}^n=\bigcup_{d=-\infty}^{\infty}Q_d$. Set $f_d=f\chi_{Q_d}$; then

$$f(x)=\sum_{d=-\infty}^{\infty}f_d(x)\quad\text{for a.e. } x\in\mathbb{R}^n.$$

It is obvious that $\operatorname{supp}([b, T^l_{\delta,j}]f_d) \subset 2nQ_d$ and that the supports of $\{[b, T^l_{\delta,j}]f_d\}_{d=-\infty}^{+\infty}$ have bounded overlaps. So we have the almost orthogonality property

$$\|[b, T^l_{\delta,j}]f\|^2_{L^2} \le C \sum_{d=-\infty}^{\infty} \|[b, T^l_{\delta,j}]f_d\|^2_{L^2}.$$

Thus, we may assume that $\operatorname{supp} f \subset Q$ for some cube with side length 2^l. Choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $0 \le \varphi \le 1$, $\operatorname{supp}\varphi \subset 100nQ$ and $\varphi = 1$ when $x \in 30nQ$. Set $\widetilde{Q} = 200nQ$ and $\tilde{b} = (b(x) - b_{\widetilde{Q}})\varphi(x)$; then

$$\|[b, T_{\delta,j}]f\|_{L^2} \le \sum_{l\ge 0} \|[b, T^l_{\delta,j}]f\|_{L^2} \le \sum_{l\ge 0} \|\tilde{b}T^l_{\delta,j}f\|_{L^2} + \sum_{l\ge 0} \|T^l_{\delta,j}(\tilde{b}f)\|_{L^2} =: I_1 + I_2.$$

For I_1, we have

$$I_1 \le \sum_{l\ge 0} \|\tilde{b}\|_{L^\infty} \|T^l_{\delta,j}f\|_{L^2} \le C \sum_{l\ge 0} 2^{l\alpha} \|b\|_{\mathrm{Lip}_\alpha} \|T^l_{\delta,j}f\|_{L^2}.$$

Take θ such that $\alpha < \theta < 1$ in (2-11); then, by (2-10) and (2-11),

$$\begin{aligned}
I_1 &\le C\biggl(\sum_{l<j} 2^{l\alpha} \|T^l_{\delta,j}f\|_{L^2} + \sum_{l\ge j} 2^{l\alpha} \|T^l_{\delta,j}f\|_{L^2}\biggr)\\
&\le CA\biggl(\sum_{l<j} 2^{(l-j)\alpha} \min\{\delta, \ \delta^{-\beta}\} + \sum_{l\ge j} 2^{(l-j)(\alpha-\theta)} \min\{\delta^{1-\theta}, \ \delta^{-\beta(1-\theta)}\}\biggr)\|f\|_{L^2}\\
&\le CA \min\{\delta^{1-\theta}, \ \delta^{-\beta(1-\theta)}\}\|f\|_{L^2},
\end{aligned}$$

where C is independent of δ. Similarly, we can get

$$I_2 \le CA \min\{\delta^{1-\theta}, \ \delta^{-\beta(1-\theta)}\}\|f\|_{L^2}.$$

Thus

$$\|[b, T_{\delta,j}]f\|_{L^2} \le CA \min\{\delta^{\lambda}, \ \delta^{-\beta\lambda}\}\|f\|_{L^2}$$

with $0 < \lambda = 1 - \theta < 1$ and C independent of δ. □

Proof of (2-1) in Lemma 2.1. For $j \in \mathbb{Z}$, define the operator T_j by $T_j f = K_j * f$, where $K_j(x) = (\Omega(x')/|x|^{n+\alpha})\chi_{\{2^j \le |x| < 2^{j+1}\}}(x)$. Since $\Omega \in L^\infty(S^{n-1})$, for some $0 < \beta < 1$ we have

$$|\widehat{K_j}(\xi)| \le C\|\Omega\|_{L^\infty} 2^{-j\alpha} \min\{|2^j\xi|^{-\beta}, \ |2^j\xi|\}$$

(see [Duoandikoetxea and Rubio de Francia 1986, pp. 551–552]). A trivial computation shows that $|\nabla\widehat{K_j}(\xi)| \le C\|\Omega\|_{L^1} 2^{(1-\alpha)j}$. Set

$$m_j(\xi) = \widehat{K_j}(\xi), \quad m^l_j(\xi) = m_j(\xi)\phi(2^{j-l}\xi).$$

Define the operator T^l_j by $\widehat{T^l_j f}(\xi) = m^l_j(\xi)\hat{f}(\xi)$. Thus $m^l_j \in C_0^\infty(\mathbb{R}^n)$ with

$$\|m^l_j\|_{L^\infty} \le C\|\Omega\|_{L^\infty} 2^{-j\alpha} \min\{2^{-\beta l}, \ 2^l\} \quad \text{and} \quad \|\nabla m^l_j\|_{L^\infty} \le C\|\Omega\|_{L^\infty} 2^{(1-\alpha)j}. \tag{2-12}$$

Thus Lemma 2.4 with $\delta = 2^l$ and $I_\alpha(\mathrm{BMO}) \subset \mathrm{Lip}_\alpha$ for $0 < \alpha < 1$ says that, for some constant $0 < \lambda < 1$,

$$\|[b, T_j^l]f\|_{L^2} \le C\|\Omega\|_{L^\infty}\|D^\alpha b\|_{\mathrm{BMO}} \min\{2^{-\beta\lambda l},\ 2^{\lambda l}\}\|f\|_{L^2}, \quad l \in \mathbb{Z}. \tag{2-13}$$

By the Plancherel theorem, we get

$$\|T_j^l f\|_{L^2} \le C\|\Omega\|_{L^\infty} 2^{-j\alpha} \min\{2^{-\beta l},\ 2^l\}\|f\|_{L^2}. \tag{2-14}$$

For any $j, l \in \mathbb{Z}$ we may write

$$[b, S_{l-j}T_j^l S_{l-j}]f = [b, S_{l-j}](T_j^l S_{l-j} f) + S_{l-j}([b, T_j^l]S_{l-j} f) + S_{l-j}T_j^l([b, S_{l-j}]f).$$

Then

$$\begin{aligned}\|V_l f\|_{L^2} &\le \Big\|\sum_{j\in\mathbb{Z}} S_{l-j}([b, T_j^l]S_{l-j} f)\Big\|_{L^2} + \Big\|\sum_{j\in\mathbb{Z}} S_{l-j}T_j^l([b, S_{l-j}]f)\Big\|_{L^2} + \Big\|\sum_{j\in\mathbb{Z}}[b, S_{l-j}](T_j^l S_{l-j}]f)\Big\|_{L^2} \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Below we shall estimate I_i for $i = 1, 2, 3$. By Littlewood–Paley theory and (2-13), we get

$$\begin{aligned} I_1 &\le \Big(\sum_{j\in\mathbb{Z}} \|[b, T_j^l](S_{l-j} f)\|_{L^2}^2\Big)^{1/2} \\ &\le C\|\Omega\|_{L^\infty} \min\{2^{-\beta\lambda l},\ 2^{\lambda l}\}\|D^\alpha b\|_{\mathrm{BMO}}\Big(\sum_{j\in\mathbb{Z}} \|S_{l-j} f\|_{L^2}^2\Big)^{1/2} \\ &\le C\|\Omega\|_{L^\infty} \min\{2^{-\beta\lambda l},\ 2^{\lambda l}\}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^2}. \end{aligned} \tag{2-15}$$

Now we estimate I_2. By (2-14) and Lemma 2.3, we get

$$\begin{aligned} I_2 &\le \Big(\sum_{j\in\mathbb{Z}} \|T_j^l([b, S_{l-j}]f)\|_{L^2}^2\Big)^{1/2} \\ &\le C\|\Omega\|_{L^\infty} \min\{2^{-(\beta+\alpha)l},\ 2^{(1-\alpha)l}\}\Big(\sum_{j\in\mathbb{Z}} 2^{2j\alpha}\|[b, S_j]f\|_{L^2}^2\Big)^{1/2} \\ &\le C\|\Omega\|_{L^\infty} \min\{2^{-(\beta+\alpha)l},\ 2^{(1-\alpha)l}\}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^2}. \end{aligned} \tag{2-16}$$

Finally, by duality and (2-16) we get

$$I_3 \le C\|\Omega\|_{L^\infty} \min\{2^{-(\beta+\alpha)l},\ 2^{(1-\alpha)l}\}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^2}. \tag{2-17}$$

It follows from (2-15)–(2-17) that, for some constant $0 < \tau < 1$,

$$\|V_l f\|_{L^2} \le C\|\Omega\|_{L^\infty} 2^{-\tau|l|}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^2} \quad \text{for } l \in \mathbb{Z}.$$

This completes the proof of (2-1). $\square$

Proof of (2-2) in Lemma 2.1. Since $T_j^l = T_j S_{l-j}$ for any $j, l \in \mathbb{Z}$, we may write

$$[b, S_{l-j} T_j^l S_{l-j}] f = S_{l-j}([b, T_j] S_{l-j}^2 f) + S_{l-j} T_j([b, S_{l-j}^2] f) + [b, S_{l-j}](T_j S_{l-j}^2 f).$$

Thus,

$$\begin{aligned} \|V_l f\|_{L^p} &\le \Big\| \sum_{j\in\mathbb{Z}} S_{l-j}([b, T_j] S_{l-j}^2 f) \Big\|_{L^p} + \Big\| \sum_{j\in\mathbb{Z}} S_{l-j} T_j([b, S_{l-j}^2] f) \Big\|_{L^p} + \Big\| \sum_{j\in\mathbb{Z}} [b, S_{l-j}](T_j S_{l-j}^2 f) \Big\|_{L^p} \\ &=: L_1 + L_2 + L_3. \end{aligned}$$

Below we shall estimate L_i, $i = 1, 2, 3$. It is well known that, for any $g \in L^p(\mathbb{R}^n)$,

$$|[b, T_j] g(x)| \le C \|b\|_{\mathrm{Lip}_\alpha} M_\Omega g(x),$$

where

$$M_\Omega g(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)| |g(y)| \, dy.$$

From this we get, for $1 < p < \infty$,

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |[b, T_j] g_j|^2 \Big)^{1/2} \Big\|_{L^p} \le C \|\Omega\|_{L^1} \|b\|_{\mathrm{Lip}_\alpha} \Big\| \Big(\sum_{j\in\mathbb{Z}} |g_j|^2 \Big)^{1/2} \Big\|_{L^p}.$$

Then, by Littlewood–Paley theory and since $I_\alpha(\mathrm{BMO}) \subset \mathrm{Lip}_\alpha$ for $0 < \alpha < 1$, we have

$$L_1 \le C \Big\| \Big(\sum_{j\in\mathbb{Z}} |[b, T_j](S_{l-j}^2 f)|^2 \Big)^{1/2} \Big\|_{L^p} \le C \|\Omega\|_{L^1} \|D^\alpha b\|_{\mathrm{BMO}} \|f\|_{L^p}.$$

For L_2, by a similar proof to that of [Chen and Zhang 2004, (1.13)], we get

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |T_j f_j|^2 \Big)^{1/2} \Big\|_{L^p} \le C \|\Omega\|_{L^1} \Big\| \Big(\sum_{j\in\mathbb{Z}} |D^\alpha f_j|^2 \Big)^{1/2} \Big\|_{L^p}.$$

Then, by Littlewood–Paley theory and the above inequality, we get

$$\begin{aligned} L_2 &\le C \|\Omega\|_{L^1} \Big\| \Big(\sum_{j\in\mathbb{Z}} |D^\alpha [b, S_{l-j}^2] f|^2 \Big)^{1/2} \Big\|_{L^p} \\ &\le C \|\Omega\|_{L^1} \Big\| \Big(\sum_{j\in\mathbb{Z}} |[b, D^\alpha] S_{l-j}^2 f|^2 \Big)^{1/2} \Big\|_{L^p} + C \|\Omega\|_{L^1} \Big\| \Big(\sum_{j\in\mathbb{Z}} |[b, D^\alpha S_{l-j}^2] f|^2 \Big)^{1/2} \Big\|_{L^p}. \end{aligned}$$

Note that the kernel of $[b, D^\alpha]$ is

$$K(x, y) = \eta(\alpha) \frac{b(x) - b(y)}{|x-y|^{n+\alpha}},$$

where $\eta(\alpha)$ is some normalization constant (see [Stein 1970]). Since $K(x, y)$ is antisymmetric, WBP is satisfied automatically. Also $[b, D^\alpha] 1 = D^\alpha b \in \mathrm{BMO}$ so, by the $T1$ theorem (see [David and Journé 1984]), $[b, D^\alpha]$ is bounded on $L^2(\mathbb{R}^n)$. It is easy to verify that $K(x, y)$ is a standard kernel; then, by the Calderón–Zygmund theorem (see [Grafakos 2004]), we get that $[b, D^\alpha]$ is bounded on $L^p(\ell^2(\mathbb{R}^n))$. Combining this with Lemma 2.3, we get

$$\begin{aligned} L_2 &\le C\|\Omega\|_{L^1}\|D^\alpha b\|_{\mathrm{BMO}}\bigg\|\bigg(\sum_{j\in\mathbb{Z}}|S_j f|^2\bigg)^{1/2}\bigg\|_{L^p} + C\|\Omega\|_{L^1}\bigg\|\bigg(\sum_{j\in\mathbb{Z}}|2^{j\alpha}[b, \bar{S}_j]f|^2\bigg)^{1/2}\bigg\|_{L^p} \\ &\le C\|\Omega\|_{L^1}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}, \end{aligned}$$

where $\bar{S}_j$ is the Littlewood–Paley operator given in the transform by multiplication with the function $|2^{-j}\xi|^\alpha\phi^2(2^{-j}\xi)$. By duality and the estimate of L_2, we get

$$L_3 \le C2^{-l\alpha}\|\Omega\|_{L^1}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p}.$$

Combining the estimates of L_1, L_2 and L_3, we get

$$\|V_l f\|_{L^p} \le C\|\Omega\|_{L^1}\|D^\alpha b\|_{\mathrm{BMO}}\|f\|_{L^p} \quad \text{for } l \in \mathbb{Z}.$$

This completes the proof of (2-2). □

3. Proof of Theorem 1.5

In the proof of Theorem 1.5, for $j = 1, \dots, 15$, A_j is a positive constant depending only on Ω, n, p, α, λ and A_i, $1 \le i < j$. We may assume $\|[b, T_\alpha]\|_{L^{p,\lambda}\to L^{p,\lambda}} = 1$. We want to prove that, for any fixed $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$,

$$M := \frac{1}{|B(x_0, r)|^{1+\alpha/n}}\int_{B(x_0,r)}|b(y) - a_0|\,dy \le A(p, \Omega, \alpha, \lambda), \tag{3-1}$$

where $a_0 = |B(x_0, r)|^{-1}\int_{B(x_0,r)} b(y)\,dy$. Since $[b - a_0, T_\alpha] = [b, T_\alpha]$, we may assume $a_0 = 0$. Let

$$f(y) = (\operatorname{sgn} b(y) - c_0)\chi_{B(x_0,r)}(y), \tag{3-2}$$

where $c_0 = (1/|B(x_0, r)|)\int_{B(x_0,r)} \operatorname{sgn} b(y)\,dy$. Then f has the following properties:

$$\int_{\mathbb{R}^n} f(y)\,dy = 0, \tag{3-3}$$

$$f(y)b(y) > 0, \tag{3-4}$$

$$\frac{1}{|B(x_0, r)|^{1+\alpha/n}}\int_{\mathbb{R}^n} f(y)b(y)\,dy = M. \tag{3-5}$$

Without loss of generality, we may assume that $|\Omega(x') - \Omega(y')| \le |x' - y'|$ for all $x', y' \in S^{n-1}$. Since Ω satisfies (1-1) or (1-2), there exists a positive number $A_1 < 1$ such that

$$\sigma(\Lambda) := \sigma\big(\{x' \in S^{n-1} : \Omega(x') \ge 2A_1\}\big) > 0, \tag{3-6}$$

where σ is the measure on S^{n-1} which is induced from the Lebesgue measure on $\mathbb{R}^n$. Then, for $x \in G = \{x \in \mathbb{R}^n : |x - x_0| > A_2 r = (2A_1^{-1} + 1)r \text{ and } (x - x_0)' \in \Lambda\}$,

$$\begin{aligned}|[b, T_\alpha] f(x)| &\geq \left| \int_{\mathbb{R}^n} \Omega((x-y)')|x-y|^{-n-\alpha} b(y) f(y)\, dy \right| - |b(x)| \left| \int_{\mathbb{R}^n} \Omega((x-y)')|x-y|^{-n-\alpha} f(y)\, dy \right| \\ &=: I_1(x) - I_2(x).\end{aligned}$$

For $I_1(x)$, noting that if $|y - x_0| < r$, we get $|(x-x_0)' - (x-y)'| \leq 2|y-x_0|/|x-x_0| \leq A_1$, then, since $\Omega \in \mathrm{Lip}(S^{n-1})$, we get $\Omega((x-y)') \geq A_1$. Thus it follows from (3-4) and (3-5) that

$$I_1(x) \geq A_1 \int_{B(x_0,r)} b(y) f(y) |y-x|^{-n-\alpha}\, dy \geq A_3 r^{n+\alpha} M |x-x_0|^{-n-\alpha}.$$

Since $\Omega \in \mathrm{Lip}(S^{n-1})$ and by (3-3), we have

$$I_2(x) \leq |b(x)| \int_{B(x_0,r)} |f(y)| \left| \frac{\Omega((x-y)')}{|x-y|^{n+\alpha}} - \frac{\Omega((x-x_0)')}{|x-x_0|^{n+\alpha}} \right| dy \leq A_4 r^{n+1} |b(x)|\, |x-x_0|^{-n-\alpha-1}.$$

Let $\theta = p/(n(p-1) + p\alpha + \lambda)$ and

$$F = \left\{ x \in G : |b(x)| > \frac{A_3 M r^{\alpha-1}}{2A_4} |x - x_0| \text{ and } |x-x_0| < M^\theta r \right\}.$$

This gives that $I_1(x) \geq 2I_2(x)$ when $x \in (G \setminus F) \cap \{x : |x-x_0| < M^\theta r\}$. Then we have

$$|[b, T_\alpha] f(x)| \geq I_1(x) - I_2(x) \geq \tfrac{1}{2} I_1(x) \quad \text{for } x \in (G \setminus F) \cap \{x : |x-x_0| < M^\theta r\}.$$

Hence,

$$\begin{aligned}\|f\|_{L^{p,\lambda}}^p &\geq \|[b, T_\alpha] f\|_{L^{p,\lambda}}^p \\ &\geq \frac{1}{M^{\theta\lambda} r^\lambda} \int_{|x-x_0|<M^\theta r} |[b, T_\alpha] f(x)|^p\, dx \\ &\geq \frac{1}{M^{\theta\lambda} r^\lambda} \int_{(G\setminus F)\cap\{|x-x_0|<M^\theta r\}} \left(\tfrac{1}{2} A_3 M r^{\alpha+n} |x-x_0|^{-n-\alpha}\right)^p dx \\ &\geq \frac{1}{M^{\theta\lambda} r^\lambda} \int_{\{A_5(|F|+(B_2 r)^n)^{1/n} < |x-x_0| < M^\theta r\}\cap G} \left(\tfrac{1}{2} A_3 M r^{\alpha+n} |x-x_0|^{-n-\alpha}\right)^p dx \\ &= \frac{\sigma(\Lambda)}{M^{\theta\lambda} r^\lambda} \left(\frac{A_3 M r^{\alpha+n}}{2} \right)^p \int_{A_5(|F|+(A_2 r)^n)^{1/n}}^{M^\theta r} t^{-n(p-1)-p\alpha-1}\, dt \\ &= \frac{\sigma(\Lambda)}{M^{\theta\lambda} r^\lambda} \frac{\left(\tfrac{1}{2} B_3 M r^{\alpha+n}\right)^p}{(-n(p-1)-p\alpha)} \left((M^\theta r)^{-n(p-1)-p\alpha} - A_6 (|F| + (A_2 r)^n)^{(-n(p-1)-p\alpha)/n} \right).\end{aligned}$$

Then, by $\|f\|_{L^{p,\lambda}} \leq C r^{(n-\lambda)/p}$ and an elementary computation, we have

$$(|F| + (A_2 r)^n)^{-(p-1)-p\alpha/n} \leq A_7 (M^{\theta(-n(p-1)-p\alpha)} + M^{\theta\lambda - p}) r^{-n(p-1)-p\alpha}.$$

Since $\lambda = p/\theta - n(p-1) - p\alpha$, we get

$$(|F| + (A_2 r)^n)^{-(p-1)-p\alpha/n} \leq A_8 M^{\theta(-n(p-1)-p\alpha)} r^{-n(p-1)-p\alpha}.$$

Then we have

$$|F| \geq A_9 M^{\theta n} r^n - (A_2 r)^n.$$

If $M \leq (2A_9^{-1} A_2^n)^{1/(\theta n)}$, then Theorem 1.5 is proved. If $M > (2A_9^{-1} A_2^n)^{1/(\theta n)}$, then

$$|F| \geq \tfrac{1}{2} A_9 M^{\theta n} r^n. \tag{3-7}$$

Now let $g(y) = \chi_{B(x_0,r)}(y)$. For $x \in F$,

$$\begin{aligned}|[b, T_\alpha]g(x)| &\geq |b(x)| \left| \int_{B(x_0,r)} \frac{\Omega((x-y)')}{|x-y|^{n+\alpha}} g(y)\,dy \right| - \int_{B(x_0,r)} |\Omega((x-y)')|\,|x-y|^{-n-\alpha} |b(y)|\,dy \\ &=: K_1 - K_2. \end{aligned} \tag{3-8}$$

For $y \in B(x_0, r)$ and $x \in F$ we have that $|x - x_0| \simeq |x - y|$ and $\Omega((x-y)') \geq A_1$. Now, regarding K_1, it follows that

$$K_1 \geq C|b(x)| \int_{B(x_0,r)} |x-y|^{-n-\alpha}\,dy \geq A_{10} |b(x)|\,|x-x_0|^{-n-\alpha} r^n. \tag{3-9}$$

For K_2, since $\Omega \in L^\infty(S^{n-1})$, we have

$$K_2 \leq C|x-x_0|^{-n-\alpha} \int_{B(x_0,r)} |b(y)|\,dy \leq A_{11} |x-x_0|^{-n-\alpha} r^{n+\alpha} M. \tag{3-10}$$

So, by (3-8)–(3-10) and since $|b(x)| > (A_3 M r^\alpha/(2A_4))|x-x_0|/r$ when $x \in F$, we get, for $x \in F$,

$$|[b, T_\alpha]g(x)| \geq A_{12} |x-x_0|^{1-n-\alpha} r^{n+\alpha-1} M - A_{11} |x-x_0|^{-n-\alpha} r^{n+\alpha} M. \tag{3-11}$$

Since $\|g\|_{L^{p,\lambda}} \leq C r^{(n-\lambda)/p}$, by (3-11) and Hölder's inequality we have

$$\begin{aligned} A_{13} r^{(n-\lambda)/p} &\geq \|[b, T_\alpha]g\|_{L^{p,\lambda}} \\ &\geq \left(\frac{1}{(M^\theta r)^\lambda} \int_{\{\frac14 A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r\}} |[b, T_\alpha]g(x)|^p\,dx \right)^{1/p} \\ &\geq \frac{1}{(M^\theta r)^{\lambda/p+n/p'}} \int_{F \cap \{\frac14 A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r\}} |[b, T_\alpha]g(x)|\,dx \\ &\geq A_{12} \frac{M r^{n+\alpha-1}}{(M^\theta r)^{\lambda/p+n/p'}} \int_{F \cap \{\frac14 A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r\}} |x-x_0|^{1-n-\alpha}\,dx \\ &\qquad - A_{11} \frac{r^{n+\alpha} M}{(M^\theta r)^{\lambda/p+n/p'}} \int_{F \cap \{\frac14 A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r\}} |x-x_0|^{-n-\alpha}\,dx \\ &=: L_1 - L_2. \end{aligned} \tag{3-12}$$

To estimate L_1 and L_2, we first prove that

$$\left| F \cap \left\{ \tfrac14 A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r \right\} \right| \geq \tfrac14 A_9 M^{\theta n} r^n. \tag{3-13}$$

Let

$$F = \left(F \cap \left\{\tfrac14 A_9^{1/n} M^\theta r < |x-x_0| < M^\theta r\right\}\right) \cup \left(F \cap \left\{|x-x_0| < \tfrac14 A_9^{1/n} M^\theta r\right\}\right) =: E_1 \cup E_2.$$

Notice that

$$|E_2| \le \left|\left\{x : |x-x_0| < \tfrac{1}{4}A_9^{1/n}M^\theta r\right\}\right| \le \left(\tfrac{1}{4}\right)^n A_9 M^{\theta n} r^n.$$

If $|E_1| < \frac{1}{4}A_9 M^{\theta n} r^n$, then

$$|F| = |E_1| + |E_2| < \tfrac{1}{4}A_9 M^{\theta n} r^n + \left(\tfrac{1}{4}\right)^n A_9 M^{\theta n} r^n < \tfrac{1}{2}A_9 M^{\theta n} r^n.$$

This contradicts $|F| \ge \frac{1}{2}A_9 M^{\theta n} r^n$. This proves (3-13). Now we turn to give the estimates of L_1 and L_2. Since $|x-x_0| < M^\theta r$ and by (3-13),

$$\begin{aligned} L_1 &\ge A_{12}\left|F \cap \left\{\tfrac{1}{2}A_9^{1/n}M^\theta r < |x-x_0| < M^\theta r\right\}\right| \frac{M r^{n+\alpha-1}}{(M^\theta r)^{\lambda/p+n/p'}}(M^\theta r)^{1-n-\alpha} \\ &\ge A_{14}\frac{M^{\theta(1-\alpha)+1} r^{(n-\lambda)/p}}{M^{\theta(\lambda/p+n/p')}}. \end{aligned} \tag{3-14}$$

For L_2, we have

$$\begin{aligned} L_2 &\le A_{11}\frac{r^{n+\alpha}M}{(M^\theta r)^{\lambda/p+n/p'}}\int_{F\cap\{\frac{1}{2}A_9^{1/n}M^\theta r<|x-x_0|<M^\theta r\}} |x-x_0|^{-n-\alpha}\,dx \\ &\le A_{11}\frac{r^{n+\alpha}M}{(M^\theta r)^{\lambda/p+n/p'}}\int_{\{\frac{1}{2}A_9^{1/n}M^\theta r<|x-x_0|<M^\theta r\}} |x-x_0|^{-n-\alpha}\,dx \\ &\le A_{15}\frac{r^{(n-\lambda)/p}M^{1-\alpha\theta}}{M^{\theta(\lambda/p+n/p')}}. \end{aligned} \tag{3-15}$$

Now (3-12) and (3-14)–(3-15) show that

$$A_{13} \ge (A_{14}M^{\theta(1-\alpha)} - A_{15}M^{-\alpha\theta})\frac{M}{M^{\theta(\lambda/p+n/p')}}.$$

Since $\theta = p/(n(p-1)+p\alpha+\lambda)$,

$$M^{\theta(\lambda/p+n/p')} = M^{1-p\alpha/(n(p-1)+p\alpha+\lambda)} = M^{1-\alpha\theta}.$$

Thus, we get

$$A_{13} \ge A_{14}M^\theta - A_{15}.$$

Therefore, $M \le A(p,\Omega,\alpha,\lambda)$ and we complete the proof of Theorem 1.5. □

4. Proof of Theorem 1.8

As in the proof of Theorem 1.8, let A_j, $j = 1,\dots,14$, be positive constants depending only on Ω, n, α and A_i, $1 \le i < j$. Without loss of generality, we may assume that $\|[b, T_\alpha]\|_{L^1\to L^{1,\infty}} = 1$. For any fixed $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, we also set $a_0 := |B(x_0,r)|^{-1}\int_{B(x_0,r)} b(y)\,dy = 0$ since $[b-a_0, T_\alpha] = [b, T_\alpha]$. It is our aim to prove the inequality

$$M = \frac{1}{|B(x_0,r)|^{1+\alpha/n}}\int_{B(x_0,r)} |b(y)|\,dy \le A(n,\Omega,\alpha).$$

Let f be as defined in (3-2) and Λ be as defined in (3-6). Take

$$G=\{x\in\mathbb{R}^n : |x-x_0|>A_2r=(2A_1^{-1}+1)r \ \text{ and } \ (x-x_0)'\in\Lambda\}.$$

Then for $x\in G$ we have

$$\begin{aligned}|[b,T_\alpha]f(x)| &\geq |T_\alpha(bf)(x)|-|b(x)|\,|T_\alpha f(x)|\\ &=\left|\int_{\mathbb{R}^n}\Omega((x-y)')|x-y|^{-n-\alpha}b(y)f(y)\,dy\right|-|b(x)|\left|\int_{\mathbb{R}^n}\Omega((x-y)')|x-y|^{-n-\alpha}f(y)\,dy\right|\\ &=: I_1(x)-I_2(x).\end{aligned}$$

Similar to the proof of Theorem 1.8, we get

$$I_1(x)\geq A_3r^{n+\alpha}M|x-x_0|^{-n-\alpha}$$

and

$$I_2(x)\leq A_4r^{n+1}|b(x)|\,|x-x_0|^{-n-\alpha-1}.$$

Let

$$F=\left\{x\in G : |b(x)|>\frac{A_3Mr^{\alpha-1}}{2A_4}|x-x_0| \ \text{ and } \ |x-x_0|<M^{1/(n+\alpha)}r\right\}.$$

Then we have $|[b,T_\alpha]f(x)|\geq \frac{1}{2}I_1(x)$ when $x\in(G\setminus F)\cap\{x:|x-x_0|<M^{1/(n+\alpha)}r\}$. Thus,

$$\begin{aligned}\|f\|_{L^1} &\geq \int_{\{x\in\mathbb{R}^n:|[b,T_\alpha]f(x)|>1\}}dx\\ &\geq \int_{(G\setminus F)\cap\{|x-x_0|<M^{1/(n+\alpha)}r\}\cap\{|[b,T_\alpha]f(x)|>1\}}dx\\ &\geq \int_{(G\setminus F)\cap\{|x-x_0|<M^{1/(n+\alpha)}r\}\cap\{A_3Mr^{\alpha+n}|x-x_0|^{-n-\alpha}>2\}}dx\\ &\geq \int_{\{A_6(|F|+(A_2r)^n)^{1/n}<|x-x_0|<A_5M^{1/(n+\alpha)}r\}\cap G}dx\\ &=\int_{A_6(|F|+(A_2r)^n)^{1/n}}^{A_5M^{1/(n+\alpha)}r}t^{n-1}\,dt\int_\Lambda d\sigma(x').\end{aligned}$$

Since $\|f\|_{L^1}\leq r^n$, we then have

$$|F|\geq A_7M^{n/(n+\alpha)}r^n-A_8r^n.$$

If $M\leq(2A_8A_7^{-1})^{(n+\alpha)/n}$, then Theorem 1.8 is proved. If $M>(2A_8A_7^{-1})^{(n+\alpha)/n}$, then

$$|F|\geq \tfrac{1}{2}A_7M^{n/(n+\alpha)}r^n. \tag{4-1}$$

Now, let $g(y)=\chi_{B(x_0,r)}(y)$. Similar to (3-11) in the proof of Theorem 1.5, for $x\in F$ we have

$$|[b,T_\alpha]g(x)|\geq A_9|x-x_0|^{1-n-\alpha}r^{n+\alpha-1}M-A_{10}|x-x_0|^{-n-\alpha}r^{n+\alpha}M.$$

Since $\|g\|_{L^1} \le Cr^n$, we have

$$A_{11}r^n \ge \|g\|_{L^1} \ge \int_{\{x\in\mathbb{R}^n:|[b,T_\alpha]g(x)|>1\}} dx \ge \int_{F\cap\{x:|x-x_0|\ge(2A_{10}/A_9)r\}\cap\{x\in\mathbb{R}^n:|[b,T_\alpha]g(x)|>1\}} dx.$$

For $|x-x_0| \ge (2A_{10}/A_9)r$,

$$|[b, T_\alpha]g(x)| \ge \tfrac{1}{2}A_9|x-x_0|^{1-n-\alpha}r^{n+\alpha-1}M.$$

Thus,

$$\begin{aligned} A_{11}r^n &\ge \int_{F\cap\{x:|x-x_0|\ge(2A_{10}/A_9)r\}\cap\{x\in\mathbb{R}^n:(A_9/2)|x-x_0|^{1-n-\alpha}r^{n+\alpha-1}M>1\}} dx \\ &= \int_{F\cap\{x:|x-x_0|\ge(2A_{10}/A_9)r\}\cap\{x\in\mathbb{R}^n:|x-x_0|\le A_{12}M^{1/(n+\alpha-1)}r\}} dx \\ &= \int_{\{x\in F:A_{13}r\le|x-x_0|\le A_{12}M^{1/(n+\alpha-1)}r\}} dx. \end{aligned} \tag{4-2}$$

If $M \le (A_{13}/A_{12})^{n+\alpha-1}$, then we have proved Theorem 1.8. If $M > (A_{13}/A_{12})^{n+\alpha-1}$, then

$$\begin{aligned} \int_{\{x\in F:A_{13}r\le|x-x_0|\le A_{12}M^{1/(n+\alpha-1)}r\}} dx &= \int_{\{x\in F:|x-x_0|\le A_{12}M^{1/(n+\alpha-1)}r\}} dx - \int_{\{x\in F:|x-x_0|\le A_{13}r\}} dx \\ &=: K_1 - K_2. \end{aligned} \tag{4-3}$$

If $M \le A_{12}^{-(n+\alpha)(n+\alpha-1)}$, then Theorem 1.8 is proved. If $M > A_{12}^{-(n+\alpha)(n+\alpha-1)}$, we have

$$A_{12}M^{1/(n+\alpha-1)} \ge M^{1/(n+\alpha)}.$$

By (4-1), we get

$$K_1 \ge \int_{\{x\in F:|x-x_0|\le M^{1/(n+\alpha)}r\}} dx = \int_F dx = |F| \ge \tfrac{1}{2}A_7M^{n/(n+\alpha)}r^n$$

and

$$K_2 \le \int_{\{x\in F:|x-x_0|\le A_{13}r\}} dx \le A_{14}r^n.$$

Combining these estimates, from (4-2) and (4-3) we get

$$A_{11} \ge \tfrac{1}{2}A_7M^{n/(n+\alpha)} - A_{14}.$$

Then $M \le A(n, \Omega, \alpha)$. □

5. Proof of Theorem 1.11

Let

$$k(x, y) = \frac{\Omega(x-y)}{|x-y|^{n+\alpha}}(b(x) - b(y)).$$

Proof of (i) $\Longrightarrow$ (ii). Suppose that, for some $1 < p < \infty$,

$$\|[b, T_\alpha]f\|_{L^p} \le C\|f\|_{L^p}; \tag{5-1}$$

then, by Theorem 1.5 for $\lambda = 0$, we must have $b \in \mathrm{Lip}_\alpha$. If $\Omega \in \mathrm{Lip}(S^{n-1})$ and $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, there is a constant $C > 0$ such that, for all $x, x_0, y \in \mathbb{R}^n$ with $2|x - x_0| \le |y - x|$, the kernel $k(x, y)$ satisfies the inequality

$$|k(x, y) - k(x_0, y)| \le C|x - x_0|^\alpha |y - x|^{-n-\alpha}. \tag{5-2}$$

Applying (5-1) and (5-2), by using a Calderón–Zygmund decomposition and a trivial computation, we get

$$\|[b, T_\alpha] f\|_{L^{1,\infty}} \le C\|f\|_{L^1}. \qquad \square$$

Proof of (ii) $\Longrightarrow$ (iii). Suppose that $[b, T_\alpha]$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$; then by Theorem 1.8 we must have $b \in \mathrm{Lip}_\alpha$. So $k(x, y)$ satisfies (5-2). For fixed $x \in \mathbb{R}^n$, pick a cube $Q = Q(x_0, r)$ that contains x. Let $f = f_1 + f_2$, with $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$. We select $a = [b, T_\alpha] f(x_0)$ and let $0 < \delta < 1$; then

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q \big||[b, T_\alpha] f(y)|^\delta - |a|^\delta\big|\,dy\right)^{1/\delta} &\le \left(\frac{1}{|Q|}\int_Q |[b, T_\alpha] f(y) - a|^\delta\,dy\right)^{1/\delta}\\ &\le \left(\frac{1}{|Q|}\int_Q |[b, T_\alpha] f_1(y)|^\delta\,dy\right)^{1/\delta} + \frac{1}{|Q|}\int_Q |[b, T_\alpha] f_2(y) - a|\,dy.\end{aligned}$$

Since $[b, T_\alpha] : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ and $0 < \delta < 1$, Kolmogorov's inequality [García-Cuerva and Rubio de Francia 1985, p. 485] yields

$$\left(\frac{1}{|Q|}\int_Q |[b, T_\alpha] f_1(y)|^\delta\,dy\right)^{1/\delta} \le \frac{1}{|Q|}\int_{\mathbb{R}^n} |f_1(y)|\,dy \le CMf(x).$$

By (5-2), it is easy to get

$$\frac{1}{|Q|}\int_Q |[b, T_\alpha] f_2(y) - a|\,dy \le CMf(x).$$

Combining these estimates, we get, for any fixed $x \in \mathbb{R}^n$,

$$\big(M^\sharp(|[b, T_\alpha] f|^\delta)\big)^{1/\delta}(x) \le CMf(x).$$

Applying this inequality we get, for $1 < p < \infty$ and $0 < \lambda < n$,

$$\big\|\big(M^\sharp(|[b, T_\alpha] f|^\delta)\big)\big\|^{1/\delta}_{L^{p/\delta,\lambda}} = \big\|\big(M^\sharp(|[b, T_\alpha] f|^\delta)\big)^{1/\delta}\big\|_{L^{p,\lambda}} \le C\|Mf\|_{L^{p,\lambda}} \le C\|f\|_{L^{p,\lambda}}.$$

(see [Chiarenza and Frasca 1987]). On the other hand,

$$\|[b, T_\alpha] f\|_{L^{p,\lambda}} = \||[b, T_\alpha] f|^\delta\|^{1/\delta}_{L^{p/\delta,\lambda}} \le \big\|M^\sharp\big(|[b, T_\alpha] f|^\delta\big)\big\|^{1/\delta}_{L^{p/\delta,\lambda}}.$$

Combining these estimates, we get

$$\|[b, T_\alpha] f\|_{L^{p,\lambda}} \le C\|f\|_{L^{p,\lambda}}. \qquad \square$$

Proof of (iii) $\Longrightarrow$ (iv). Suppose that $[b, T_\alpha]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$ for some $1 < p < \infty$ and $0 < \lambda < n$; then, by Theorem 1.5, we must have $b \in \mathrm{Lip}_\alpha$. So $k(x, y)$ satisfies (5-2). Let $f = f_1 + f_2$, with $f_1 = f\chi_{2Q}$

and $f_2 = f\chi_{(2Q)^c}$. For any cube $Q = Q(x_0, r)$,

$$\frac{1}{|Q|}\int_Q |[b,T_\alpha]f(y) - [b,T_\alpha]f(x_0)|\,dy = \frac{1}{|Q|}\int_Q |[b,T_\alpha]f_1(y)|\,dy + \frac{1}{|Q|}\int_Q |[b,T_\alpha]f_2(y) - [b,T_\alpha]f(x_0)|\,dy.$$

By Hölder's inequality and since $[b, T_\alpha]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$, we get

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |[b,T_\alpha]f_1(y)|^p\,dy\right)^{1/p} &\le \frac{1}{r^{(n-\lambda)/p}}\sup_{t>0,x\in\mathbb{R}^n}\left(\frac{1}{t^\lambda}\int_{Q(x,t)\cap 2Q(x_0,r)}|f(y)|^p\,dy\right)^{1/p}\\ &\le \frac{C}{r^{(n-\lambda)/p}}r^{(n-\lambda)/p}\|f\|_{L^\infty} \le C\|f\|_{L^\infty}.\end{aligned}$$

By (5-2), it is easy to get

$$\frac{1}{|Q|}\int_Q |[b,T_\alpha]f_2(y) - [b,T_\alpha]f_2(x_0)|\,dy \le C\|f\|_{L^\infty}.$$

Combining these estimates, we get

$$\|[b,T_\alpha]f\|_{\mathrm{BMO}} \le C\|f\|_{L^\infty}. \qquad \square$$

6. Proof of Proposition 1.12

Denote by $\mathscr{H}_m$ the spaces of spherical harmonics of degree m and let $d_m = \dim \mathscr{H}_m$. If $\Omega \in L^2(S^{n-1})$ satisfies (1-1), then we can write

$$\Omega(x') = \sum_{m\ge 1}\sum_{j=1}^{d_m} a_{m,j}Y_{m,j}(x'),$$

where $\{Y_{m,j}\}_{j=1}^{d_m}$ denotes the normalized orthonormal basis of $\mathscr{H}_m$ (see [Calderón and Zygmund 1978] or [Stein and Weiss 1971]). Then

$$\sum_{m\ge 1}\sum_{j=1}^{d_m} a_{m,j}^2 < \infty.$$

By [Chen et al. 2003, p. 528], we have

$$(Y_{m,j}(x')|x|^{-n-\alpha})^\wedge(\xi) \backsimeq m^{-n/2-\alpha}|\xi|^\alpha Y_{m,j}(\xi').$$

Then we get

$$\widehat{T_\alpha f}(\xi) \backsimeq |\xi|^\alpha \sum_{m\ge 1}\sum_{j=1}^{d_m} m^{-n/2-\alpha}a_{m,j}Y_{m,j}(\xi')\hat{f}(\xi).$$

Using this, we get

$$\widehat{I_\alpha T_\alpha f}(\xi) \simeq \sum_{m\ge 1}\sum_{j=1}^{d_m} m^{-n/2-\alpha}a_{m,j}Y_{m,j}(\xi')\hat{f}(\xi).$$

Let

$$\Omega_0(\xi') = \sum_{m\geq 1}\sum_{j=1}^{d_m} m^{-n/2-\alpha} a_{m,j} Y_{m,j}(\xi').$$

It is easy to verify that Ω_0 satisfies (1-1) and

$$\sum_{m\geq 1}\sum_{j=1}^{d_m} m^n \|m^{-n/2-\alpha} a_{m,j} Y_{m,j}\|^2_{L^2(S^{n-1})} < \infty.$$

Then by [Stein and Weiss 1971, Theorem 4.7, p. 165] there exists a function $K(x) = \widetilde{\Omega}(x')/|x|^n$ such that $\widehat{K}(\xi) = \Omega_0(\xi')$ in the sense of principal value, where $\widetilde{\Omega}$ satisfies (1-1). Therefore, we get that

$$Tf(x) = I_\alpha T_\alpha f(x) = \text{p.v.}(K * f(x))$$

is a singular integral operator. In fact,

$$\widetilde{\Omega}(x') \simeq \sum_{m\geq 1}\sum_{j=1}^{d_m} m^{-\alpha} a_{m,j} Y_{m,j}(x'),$$

and

$$\|\widetilde{\Omega}\|^2_{L^2_\alpha(S^{n-1})} = \sum_{m\geq 1}\sum_{j=1}^{d_m} m^{2\alpha}(m^{-\alpha} a_{m,j})^2 < \infty.$$

This says that, for $0 < \alpha < 1$ and any operator T_α defined by (1-3) with $\Omega \in L^2(S^{n-1})$ satisfying (1-1), there exists a singular integral operator T defined by (1-4) with $\widetilde{\Omega} \in L^2_\alpha(S^{n-1})$ satisfying (1-1) such that $T_\alpha = D^\alpha T$. Conversely, for any fixed singular integral operator T with $\widetilde{\Omega} \in L^2_\alpha(S^{n-1})$ satisfying (1-1), there exists an operator T_α with $\Omega \in L^2(S^{n-1})$ satisfying (1-1) such that $T_\alpha = D^\alpha T$. □

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