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ISOLATED SINGULARITIES OF POSITIVE SOLUTIONS OF ELLIPTIC EQUATIONS WITH WEIGHTED GRADIENT TERM

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Let $\Omega \subset \mathbb{R}^N$ $(N > 2)$ be a C^2 bounded domain containing the origin 0. We study the behavior near 0 of positive solutions of equation (E) $-\Delta u + |x|^\alpha u^p + |x|^\beta |\nabla u|^q = 0$ in $\Omega \setminus \{0\}$, where $\alpha > -2,\ \beta > -1$, $p > 1$, and $q > 1$. When $1 < p < (N+\alpha)/(N-2)$ and $1 < q < (N+\beta)/(N-1)$, we provide a full classification of positive solutions of (E) vanishing on $\partial\Omega$. On the contrary, when $p \geq (N+\alpha)/(N-2)$ or $(N+\beta)/(N-1) \leq q \leq 2+\beta$, we show that any isolated singularity at 0 is removable.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N > 2)$ be a C^2 bounded domain containing the origin 0. In this paper, we study isolated singularities at 0 of nonnegative solutions of the quasilinear equation

$$-\Delta u + |x|^\alpha u^p + |x|^\beta |\nabla u|^q = 0 \tag{1-1}$$

in $\Omega \setminus \{0\}$ where $\alpha > -2,\ \beta > -1,\ p > 1$, and $q > 1$. By a nonnegative solution of (1-1) we mean a nonnegative function $u \in C^2(\Omega \setminus \{0\})$ which satisfies (1-1) in the classical sense.

Equation (1-1) consists of two mechanisms: the semilinear equation

$$-\Delta u + |x|^\alpha u^p = 0 \tag{1-2}$$

in $\Omega \setminus \{0\}$ and the quasilinear equation

$$-\Delta u + |x|^\beta |\nabla u|^q = 0 \tag{1-3}$$

in $\Omega \setminus \{0\}$. For the sake of simplicity, in the sequel, we use the notation

$$(F \circ u)(x) = |x|^\alpha u(x)^p + |x|^\beta |\nabla u(x)|^q. \tag{1-4}$$

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In the literature, many results concerning isolated singularities for (1-2) with $\alpha = 0$ have been published, among which we refer to [Brézis and Véron 1980/81; Vázquez and Véron 1985; Véron 1981; 1996; Baras and Pierre 1984, Marcus 2013] and references therein. Marcus and Véron [2014] provided a full description of isolated singularities of positive solutions of (1-2) (with $\alpha > -2$) when $1 < p < p_{c,\alpha}$ with

$$p_{c,\alpha} := \frac{N+\alpha}{N-2}. \tag{1-5}$$

More precisely, in this range, if v is a positive solution of (1-2) vanishing on $\partial\Omega$, then:

- either $v = v_k^\Omega$ ($k > 0$), the solution of

$$-\Delta v + |x|^\alpha v^p = k\delta_0 \qquad \text{in } \Omega, \text{ with } v = 0 \text{ on } \partial\Omega \tag{1-6}$$

  (here δ_0 is the Dirac measure concentrated at the origin) and $v(x) = kc_N(1+o(1))|x|^{2-N}$ as $|x| \to 0$ where $c_N = 1/(N(N-2)\omega_N)$ with ω_N being the volume of the unit ball in $\mathbb{R}^N$;

- or $v = v_\infty^\Omega := \lim_{k\to\infty} v_k^\Omega$ and $v(x) = \vartheta(1+o(1))|x|^{-\frac{2+\alpha}{p-1}}$ as $|x| \to 0$ with

$$\vartheta := \left[\left(\frac{2+\alpha}{p-1}\right)\left(\frac{2p+\alpha}{p-1} - N\right)\right]^{\frac{1}{p-1}}. \tag{1-7}$$

When $p \geq p_{c,\alpha}$, they showed that there is no positive solution of (1-2) vanishing on $\partial\Omega$.

Classification of interior isolated singularities in the general framework (where the nonlinearity does not depend on gradient term) was established in [Friedman and Véron 1986], in [Cîrstea and Du 2010] (for the p-laplacian), and in [Cîrstea 2014] (for elliptic equations with inverse square potentials). A deep existence and uniqueness result for a more general class of semilinear equations was given in [Marcus 2013].

Much less work concerning the behavior near the origin of positive solutions of equations with the nonlinearity depending mostly on the gradient term has been investigated. See Serrin [1965] and, more recently, Bidaut-Véron, García-Huidobro, and Véron [Bidaut-Véron et al. 2014].

Recently, boundary trace problem for semilinear equation with gradient terms were studied by P. T. Nguyen and L. Véron [2012] and by M. Marcus and Nguyen [2015].

When the nonlinearity is of the form (1-4), i.e., it depends on both u and ∇u, as well as weights, one encounters the following difficulties:

(i) The first one stems from the competition of two terms $|x|^\alpha u^p$ and $|x|^\beta |\nabla u|^q$. When $\frac{2+\alpha}{p-1} \neq \frac{2+\beta-q}{q-1}$, (1-1) admits no *similarity transformation* (see Section 2). Moreover, in this framework, the Keller–Osserman estimate is no longer a sharp upper bound for solutions of (1-1).

(ii) The second one comes from the lack of monotonicity property of the nonlinearity. Furthermore, it is noteworthy that in general the sum of two solution of (1-1) is not a supersolution.

(iii) The presence of the weights $|x|^\alpha$ and $|x|^\beta$, which may vanish or be singular at 0 according to the value of α and β, make the asymptotic behavior near 0 of solutions of (1-1) more intricate.

Fix $d_1 \in (0,1)$ such that $B_{3d_1}(0) \Subset \Omega$ and put $d_2 = \operatorname{diam}(\Omega)$. Set

$$\tau = \min\left\{\frac{2+\alpha}{p-1}, \frac{2+\beta-q}{q-1}\right\} \quad \text{with } q < 2+\beta. \tag{1-8}$$

We first give sharp estimates on solutions of (1-1) and their gradient. These estimates are obtained due to a combination of Bernstein's method, Keller–Osserman estimates, and a transformation argument.

Proposition 1.1. *Let $\alpha > -2$, $\beta > -1$, $p > 1$, and $1 < q < 2+\beta$. There exists a positive constant $c_i = c_i(\alpha, \beta, N, p, q, d_1, d_2)$ $(i = 1, 2)$ such that if u is a positive solution of* (1-1) *in $\Omega \setminus \{0\}$ vanishing on $\partial\Omega$, then*

$$u(x) \le c_1 |x|^{-\tau} \quad \textit{for all } x \in \Omega \setminus \{0\}, \tag{1-9}$$

and

$$|\nabla u(x)| \le c_2 |x|^{-\tau-1} \quad \textit{for all } x \in \overline{B_{d_1}(0)} \setminus \{0\}. \tag{1-10}$$

Estimates (1-9) and (1-10) give an upper bound of $F \circ u$ but do not ensure that $F \circ u \in L^1(\Omega)$. While investigating the integrability of $F \circ u$ we are led to the following definition.

Definition 1.2. A nonnegative solution u of (1-1) is called a *weakly singular solution* if $F \circ u \in L^1(B_\varepsilon)$ for some $\varepsilon > 0$. Otherwise, u is called a *strongly singular solution*.

We next introduce the definition of solutions to

$$\begin{cases} -\Delta u + F \circ u = k\delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-11}$$

Definition 1.3. Let $k \ge 0$. A nonnegative function u is a solution of (1-11) if $u \in L^1(\Omega)$, $F \circ u \in L^1(\Omega)$, and

$$\int_\Omega \big(-u\Delta\zeta + (F \circ u)\zeta\big)\, dx = k\zeta(0) \quad \text{for all } \zeta \in C_0^2(\overline{\Omega}). \tag{1-12}$$

Remark. Clearly, if u is a solution of (1-11) then u is a weakly singular solution of (1-1).

Let Γ_N $(N > 2)$ be the Newtonian kernel in $\mathbb{R}^N$ defined by

$$\Gamma_N(x) := c_N |x|^{2-N} = \frac{1}{N(N-2)\omega_N} |x|^{2-N}, \quad x \ne 0 \tag{1-13}$$

with ω_N the volume of the unit ball in $\mathbb{R}^N$. Denote by G^Ω the Green kernel of $(-\Delta)$ in Ω and by $\mathbb{G}^\Omega$ the corresponding operator.

The study of (1-1) is strongly linked to that of (1-3). As we will see in the sequel there exists an exponent

$$q_{c,\beta} = \frac{N+\beta}{N-1} \tag{1-14}$$

such that if $1 < q < q_{c,\beta}$, the problem (1-3) admits weakly and strongly singular solutions; while if $q_{c,\beta} < q < 2+\beta$, then such solutions don't exist. When both equations (1-2) and (1-3) are combined in (1-1), the existence result for (1-1) is valid in the range $(p, q) \in (1, p_{c,\alpha}) \times (1, q_{c,\beta})$. This is reflected in the following theorems.

Theorem A. *Assume* $\alpha > -2$, $\beta > -1$, $1 < p < p_{c,\alpha}$, *and* $1 < q < q_{c,\beta}$. *For any* $k > 0$, *there exists a unique solution* $u_k^\Omega \in C^2(\Omega \setminus \{0\}) \cap C(\bar{\Omega} \setminus \{0\})$ *of* (1-11). *Moreover*,

$$u_k^\Omega(x) = kG^\Omega(x,0) - \mathbb{G}^\Omega[F \circ u_k^\Omega](x) \quad \text{for all } x \in \Omega \setminus \{0\}, \tag{1-15}$$

$$u_k^\Omega(x) = k(1+o(1))\,\Gamma_N(x) \quad \text{as } x \to 0, \tag{1-16}$$

$$\lim_{|x|\to 0} \left(|x|^{N-1} \nabla u_k^\Omega(x) + \frac{k}{N\omega_N} \frac{x}{|x|} \right) = 0. \tag{1-17}$$

Due to (1-16) and the comparison principle [Gilbarg and Trudinger 2001, Theorem 9.2], the sequence $\{u_k^\Omega\}$ is increasing. Denote $u_\infty^\Omega := \lim_{k\to\infty} u_k^\Omega$. The asymptotic behaviors of u_∞^Ω and its gradient are given in the following theorem.

Theorem B. *Assume* $\alpha > -2$, $\beta > -1$, $1 < p < p_{c,\alpha}$, *and* $1 < q < q_{c,\beta}$. *Then* u_∞^Ω *is a strongly singular solution of* (1-1) *vanishing on* $\partial\Omega$. *Moreover*,

$$\lim_{|x|\to 0} |x|^\tau u_\infty^\Omega(x) = \Theta, \tag{1-18}$$

$$\lim_{|x|\to 0} \left(|x|^{\tau+1} \nabla u_\infty^\Omega(x) + \Theta\tau \frac{x}{|x|} \right) = 0, \tag{1-19}$$

where τ *is defined in* (1-8) *and* Θ *is a positive constant depending on* N, α, β, p, q.

Remark. The value of Θ varies according to the relationship between the parameters α, β, p, and q. For simplicity, set

$$D := \frac{2+\alpha}{p-1} \times \frac{q-1}{2+\beta-q} \quad \text{with } q < 2+\beta. \tag{1-20}$$

In Theorem B, Θ is the unique solution of

$$\lambda t^{p-1} + j\tau^q t^{q-1} - \tau(\tau+2-N) = 0, \tag{1-21}$$

where j and λ are given by

$$\begin{cases} j = 0 \text{ and } \lambda = 1 & \text{if } D < 1 \text{ (hence } \Theta = \vartheta \text{ defined in (1-7));} \\ j = 1 \text{ and } \lambda = 0 & \text{if } D > 1 \text{ (hence } \Theta = \theta_0 \text{ defined in (4-3));} \\ j = \lambda = 1 & \text{if } D = 1 \text{ (hence } \Theta = \theta_1 \text{, the solution of } g_1(t) = 0, \\ & \qquad \text{where } g_\lambda \text{ is defined defined in (4-2)).} \end{cases} \tag{1-22}$$

Theorem B shows the competition between two terms $|x|^\alpha u^p$ and $|x|^\beta |\nabla u|^q$: if $D < 1$ then $|x|^\alpha u^p$ plays a dominant role, otherwise $|x|^\beta |\nabla u|^q$ plays a dominant role.

As a consequence of Theorems A and B, we obtain a description of nonnegative singular solutions of (1-1) vanishing on $\partial\Omega$.

Theorem C. *Assume* $\alpha > -2$, $\beta > -1$, $1 < p < p_{c,\alpha}$, *and* $1 < q < q_{c,\beta}$. *Let* $u \in C^2(\Omega \setminus \{0\}) \cap C(\bar{\Omega} \setminus \{0\})$ *be a nonnegative solution of* (1-1) *in* $\Omega \setminus \{0\}$ *vanishing on* $\partial\Omega$. *Then either* $u \equiv 0$, *or* $u \equiv u_k^\Omega$ *for some* $k > 0$, *or* $u \equiv u_\infty^\Omega$.

On the contrary, the next theorem states that when $p \geq p_{c,\alpha}$ or $q_{c,\beta} \leq q < 2+\beta$ there exists no positive singular solution.

Theorem D. *Assume $\alpha > -2$, $\beta > -1$, $p > 1$, and $1 < q \leq 2+\beta$. If $p \geq p_{c,\alpha}$ or $q \geq q_{c,\beta}$ then any nonnegative solution $u \in C^2(\Omega \setminus \{0\}) \cap C(\bar{\Omega} \setminus \{0\})$ of* (1-1) *in $\Omega \setminus \{0\}$ vanishing on $\partial\Omega$ must be zero.*

The paper is organized as follows. In Section 2, we prove Proposition 1.1 by treating successively the equations (1-3) and (1-1). Section 3 is devoted to the proof of Theorem A. Construction of weakly singular solutions u_k^Ω is based on an approximation method and delicate estimates on approximating solutions and on their gradient. In Section 4, the existence of a strongly singular solution u_∞^Ω (Theorem B) is obtained due to the monotonicity of the sequence $\{u_k^\Omega\}$ and a priori estimates established in Section 2. In Section 5, by combining Harnack's inequality, a scaling argument, and the asymptotic behavior of weakly singular solutions and a strongly singular solution, we obtain a complete description of isolated singularities (Theorem C). Finally, Theorem D is proved thanks to a nonexistence result for suitable equations on the unit sphere S^{N-1}.

Notation and terminology. Denote by $B_r(x_0)$ the ball of center $x_0 \in \mathbb{R}^N$ and radius r. Henceforth, we simply write B_r for $B_r(0)$. Unless otherwise stated, Ω is a C^2 bounded domain containing the origin 0. Fix $d_1 \in (0,1)$ such that $B_{3d_1} \Subset \Omega$ and put $d_2 = \operatorname{diam}(\Omega)$.

Define, for $\ell > 0$ and $x \in \Omega_\ell := \ell^{-1}\Omega$,

$$R_\ell[u](x) = \ell^{N-2}u(\ell x), \quad S_\ell[u](x) = \ell^{\frac{2+\alpha}{p-1}}u(\ell x), \quad T_\ell[u](x) = \ell^{\frac{2+\beta-q}{q-1}}u(\ell x). \tag{1-23}$$

If u is a solution of (1-2) (resp., (1-3)) in $\Omega \setminus \{0\}$ then $S_\ell[u]$ (resp., $T_\ell[u]$) is a solution of (1-2) (resp., (1-3)) in $\Omega_\ell \setminus \{0\}$. If $\Omega = \Omega_\ell$ and $u = S_\ell[u]$ (resp., $u = T_\ell[u]$) for every $\ell > 0$, we say that S_ℓ (resp., T_ℓ) is a *similarity transformation* and u is a *self-similar solution* of (1-2) (resp., (1-3)).

2. A priori estimates

2.1. ***A priori estimates on solutions of* (1-3).** Let us start this section by recalling the comparison principle [Gilbarg and Trudinger 2001, Theorem 10.1].

Proposition 2.1. *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^N$. Assume $H : \mathcal{O} \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+$ is nondecreasing with respect to u for any $(x,\xi) \in \mathcal{O} \times \mathbb{R}^N$, continuously differentiable with respect to ξ, and $H(x,0,0) = 0$. Let $u_1, u_2 \in C^2(\mathcal{O}) \cap C(\bar{\mathcal{O}})$ be two nonnegative functions satisfying*

$$-\Delta u_1 + H(x,u_1,\nabla u_1) \leq -\Delta u_2 + H(x,u_2,\nabla u_2) \quad \text{in } \mathcal{O}$$

and $u_1 \leq u_2$ on $\partial\mathcal{O}$. Then $u_1 \leq u_2$ in $\mathcal{O}$.

We shall establish a priori estimates on solutions of (1-3) and on their gradients. By using Bernstein's method (see [Lasry and Lions 1989; Lions 1985]), we derive estimates on the gradients of solutions of (1-3).

Lemma 2.2. *Assume $\beta > -1$ and $q > 1$. There exists $c_3 = c_3(N,q,\beta)$ such that if $u \in C^2(\Omega \setminus \{0\})$ is a solution of* (1-3) *in $\Omega \setminus \{0\}$ then*

$$|\nabla u(x)| \leq c_3|x|^{-\frac{1+\beta}{q-1}} \quad \text{for all } x \in \bar{B}_{d_1} \setminus \{0\}. \tag{2-1}$$

Proof. Pick an arbitrary point $x_0 \in \overline{B_{d_1}} \setminus \{0\}$ and denote $\rho_0 = |x_0|$. Take $\eta \in C^\infty(\mathbb{R}^N)$ such that $0 \le \eta \le 1$, $\operatorname{supp} \eta \subset B_{1/2}$ and $\eta \equiv 1$ in $B_{1/3}$. Put $\phi(x) = \eta(\rho_0^{-1}(x - x_0))$; then $|D^2\phi| \le c_3' \rho_0^{-2}$ and $|\nabla\phi| \le c_3' \rho_0^{-1} \phi^{\frac{1}{2}}$ with $c_3' = c_3'(N)$. Set $w = \phi^{2m} |\nabla u|^2$ with $m = \frac{1}{2(q-1)}$ and define the operator

$$\mathcal{L}[w] := -\Delta w + q|x|^\beta |\nabla u|^{q-2} \nabla u \cdot \nabla w.$$

Due to (1-3) we get

$$\mathcal{L}[w] = -2m(2m-1)\phi^{2(m-1)} |\nabla\phi|^2 |\nabla u|^2 - 2m\phi^{2m-1} \Delta\phi |\nabla u|^2 - 8m\phi^{2m-1} \sum_{i,j} \partial_i \phi \, \partial_j u \, \partial_{ij} u$$
$$- 2\phi^{2m} |D^2 u|^2 - 2\beta |x|^{\beta-2} \phi^{2m} |\nabla u|^q x \nabla u + 2mq|x|^\beta \phi^{2m-1} |\nabla u|^q \nabla\phi \nabla u.$$

By virtue of the inequality $N|D^2u|^2 \ge (\Delta u)^2$ and the inequality $2ab \le a^2 + b^2$ for any $a, b \in \mathbb{R}$, we obtain, in $B_{\rho_0/2}(x_0)$,

$$\mathcal{L}[w] \le c_4 \Big(\rho_0^{-2} \phi^{2m-1} |\nabla u|^2 + \rho_0^{\beta-1} \phi^{2m} |\nabla u|^{q+1} + \rho_0^{\beta-1} \phi^{2m-\frac{1}{2}} |\nabla u|^{q+1} \Big) - \frac{\phi^{2m} |x|^{2\beta} |\nabla u|^{2q}}{N} \tag{2-2}$$

where $c_4 = c_4(\beta, q, N)$. Denote by x^* a maximizer of w then $\mathcal{L}[w](x^*) \ge 0$. In light of the relation $|\nabla u| = \phi^{-m} w^{\frac{1}{2}}$, the fact that $\frac{1}{2}\rho_0 \le |x| \le \frac{3}{2}\rho_0$ with $x \in B_{\rho_0/2}(x_0)$ and (2-2), we deduce

$$w(x^*)^{q-1} \le c_5 \big(\rho_0^{-2(\beta+1)} + \rho_0^{-(\beta+1)} w(x^*)^{\frac{q-1}{2}} \big),$$

where $c_5 = c_5(\beta, q, N)$. Consequently,

$$\max_{x \in B_{\rho_0/2}(x_0)} (\phi^{2m} |\nabla u|^2) \le w(x^*) \le c_5' \rho_0^{-\frac{2(1+\beta)}{q-1}}.$$

Therefore, $|\nabla u(x_0)| \le c_6 |x_0|^{-\frac{1+\beta}{q-1}}$, where c_6 depends on N, q, and β. □

Remark. From Lemma 2.2, one can verify that if $u \in C^2(\Omega \setminus \{0\})$ is a positive solution of (1-3) then, for every $x \in B_{d_1} \setminus \{0\}$,

$$u(x) \le \max\big\{u(x) : x \in \partial B_{d_1}\big\} + c_3 \frac{q-1}{2+\beta-q} \Big(|x|^{-\frac{2+\beta-q}{q-1}} - d_1^{-\frac{2+\beta-q}{q-1}} \Big)$$

if $q \ne 2 + \beta$, and

$$u(x) \le \max\{u(x) : x \in \partial B_{d_1}\} + c_3 (\ln d_1 - \ln|x|) \tag{2-3}$$

if $q = 2 + \beta$. Consequently, when $q > 2 + \beta$, we can conclude that u remains bounded. Therefore, in the sequel, we consider the case $q \le 2 + \beta$.

We next derive an upper bound for subsolutions of (1-3) with $\beta \ge 0$.

Lemma 2.3. *Assume $K > 0$, $\beta \ge 0$, and $1 < q < 2 + \beta$. If $u \in C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega} \setminus \{0\})$ is a positive function such that*

$$-\Delta u + K|x|^\beta |\nabla u|^q \le 0 \tag{2-4}$$

in $\Omega \setminus \{0\}$ *and vanishing on* $\partial\Omega$, *then*

$$u(x) \le c_7 |x|^{-\frac{2+\beta-q}{q-1}} \tag{2-5}$$

for every $x \in \Omega \setminus \{0\}$, *where* $c_7 = K^{-\frac{1}{q-1}}(1+\beta)^{\frac{1}{q-1}}(q-1)^{\frac{q-2}{q-1}}(2+\beta-q)^{-1}$.

Proof. Let $\epsilon > 0$ be small, and put $\Phi_\epsilon(x) = c_7(|x|-\epsilon)^{-\frac{2+\beta-q}{q-1}} + \epsilon$ with $x \in B_\epsilon^c$. By a simple computation, we get $-\Delta\Phi_\epsilon + K|x|^\beta|\nabla\Phi_\epsilon|^q \ge 0$ in $\Omega \setminus \overline{B}_\epsilon$. Since Φ_ϵ dominates u on $\partial\Omega \cup \partial B_\epsilon$, it follows from Proposition 2.1 that $\Phi_\epsilon \ge u$ in $\Omega \setminus B_\epsilon$. Letting $\epsilon \to 0$ leads to (2-5). □

Combining Lemmas 2.2 and 2.3 we get:

Lemma 2.4. *Let* $\beta > -1$ *and* $1 < q < 2+\beta$. *There exists a constant* $c_8 = c_8(N, q, \beta, d_1, d_2)$ *such that if* $u \in C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega} \setminus \{0\})$ *is a solution of* (1-3) *vanishing on* $\partial\Omega$ *then*

$$u(x) \le c_8|x|^{-\frac{2+\beta-q}{q-1}} \quad \textit{for all } x \in \Omega \setminus \{0\}. \tag{2-6}$$

Proof. If $\beta \ge 0$ then (2-6) follows from (2-5). Next we consider $\beta \in (-1, 0)$. Fix $x \in B_{d_1} \setminus \{0\}$ and pick $z \in \partial B_{d_1}$ such that $|z - x| = d_1 - |x|$. By Lemmas 2.2 and 2.3,

$$u(x) \le c_7 d_1^{-\frac{2+\beta-q}{q-1}} + c_3 \tfrac{q-1}{2+\beta-q}|x|^{-\frac{2+\beta-q}{q-1}} \le c_9|x|^{-\frac{2+\beta-q}{q-1}} \quad \text{for all } x \in B_{d_1} \setminus \{0\}, \tag{2-7}$$

where $c_9 = c_9(N, q, \beta, d_1, d_2)$. Next put $c_9' > \max\{c_9, c_7\}$ so that the function $x \mapsto c_9'|x|^{-\frac{2+\beta-q}{q-1}}$ is a supersolution of (1-3) in $\Omega \setminus B_{d_1/2}$ which dominates u on $\partial\Omega \cup \partial B_{d_1/2}$. By Proposition 2.1, $u(x) \le c_9'|x|^{-\frac{2+\beta-q}{q-1}}$ for every $x \in \Omega \setminus B_{d_1/2}$. This, together with (2-7), implies (2-6). □

By a similar argument, we obtain the following result.

Lemma 2.5. *Let* $\beta > -1$ *and* $1 < q < 2+\beta$. *There exist* $c_i = c_i(N, q, \beta)$ *with* $i = 10, 11$ *such that if* $u \in C^2(\mathbb{R}^N \setminus \{0\})$ *is a solution of* (1-3) *in* $\mathbb{R}^N \setminus \{0\}$ *satisfying* $\lim_{|x|\to\infty} u(x) = 0$ *then*

$$u(x) \le c_{10}|x|^{-\frac{2+\beta-q}{q-1}} \quad \textit{and} \quad |\nabla u(x)| \le c_{11}|x|^{-\frac{1+\beta}{q-1}} \quad \textit{for all } x \in \mathbb{R}^N \setminus \{0\}. \tag{2-8}$$

2.2. *A priori estimates on solutions of* (1-1). We recall that τ is defined in (1-8). Due to the Keller–Osserman estimate and the above result, we obtain a sharp upper bound for solutions of (1-1).

Lemma 2.6. *Let* $\alpha > -2$, $\beta > -1$, $p > 1$, *and* $1 < q < 2+\beta$. *There exists* $c_{12} = c_{12}(\alpha, \beta, N, p, q, d_1, d_2)$ *such that if* u *is a positive solution of* (1-1) *in* $\Omega \setminus \{0\}$ *vanishing on* $\partial\Omega$ *then*

$$u(x) \le c_{12}|x|^{-\tau} \quad \textit{for all } x \in \Omega \setminus \{0\}. \tag{2-9}$$

Proof. Since u is a positive subsolution of (1-2), due to Keller–Osserman estimate, there exists a constant $c_{13} = c_{13}(N, p, \alpha)$ such that

$$u(x) \le c_{13}|x|^{-\frac{2+\alpha}{p-1}} \quad \text{for all } x \in \Omega \setminus \{0\}.$$

We consider two cases: $D \le 1$ and $D > 1$ where D is defined in (1-20).

Case 1: $D \le 1$. In this case, $\tau = \frac{2+\alpha}{p-1}$ and hence we obtain (2-9).

Case 2: $D > 1$. Notice that in this case $\tau = \frac{2+\beta-q}{q-1}$. For $\epsilon \in (0, d_1)$, let w_ϵ be the solution of

$$-\Delta w + |x|^\beta |\nabla w|^q = 0 \quad \text{in } \Omega \setminus \overline{B}_\epsilon, \quad \text{such that } w = \begin{cases} u & \text{on } \partial B_\epsilon, \\ 0 & \text{on } \partial\Omega. \end{cases} \tag{2-10}$$

By Proposition 2.1, $u \le w_\epsilon$ in $\Omega \setminus B_\epsilon$. Therefore, $u \le w_{\epsilon'} \le w_\epsilon$ in $\Omega \setminus B_{\epsilon'}$ for $0 < \epsilon < \epsilon'$. It can be checked that the function $x \mapsto c_{14}|x|^{-\frac{2+\alpha}{p-1}}$ (with $c_{14} > c_{13}$ large, depending on N, p, q, α, β, and d_2) is a supersolution of (1-3) which dominates w_ϵ on $\partial\Omega \cup \partial B_\epsilon$. By the comparison principle, $w_\epsilon(x) \le c_{14}|x|^{-\frac{2+\alpha}{p-1}}$ for $x \in \Omega \setminus B_\epsilon$. Consequently, the sequence $\{w_\epsilon\}$ is locally uniformly bounded in $\Omega \setminus \{0\}$. In light of local regularity results for elliptic equations [DiBenedetto 1983], for every compact subset $\mathcal{O} \Subset \Omega \setminus \{0\}$, there exist constants $M > 0$ and $\mu \in (0, 1)$ depending on N, p, q, α, β, d_2, and $\operatorname{dist}(0, \mathcal{O})$ such that $\|w_\epsilon\|_{C^{1,\mu}(\mathcal{O})} \le M$. Therefore, $\{w_\epsilon\}$ converges to a function $\tilde{w}$ in $C^1_{\text{loc}}(\Omega \setminus \{0\})$ which is a solution of (1-3) in $\Omega \setminus \{0\}$, vanishing on $\partial\Omega$, and satisfying $\tilde{w} \ge u$ in $\Omega \setminus \{0\}$. By virtue of Lemma 2.4, $\tilde{w} \le c_8|x|^{-\frac{2+\beta-q}{q-1}}$ for every $x \in \Omega \setminus \{0\}$. Consequently, $u \le c_8|x|^{-\frac{2+\beta-q}{q-1}}$ for every $x \in \Omega \setminus \{0\}$. This completes the proof. □

We next establish a sharp estimate from above for the gradient of solutions of (1-1).

Proposition 2.7. *Let $\alpha > -2$, $\beta > -1$, $p > 1$, and $1 < q < 2+\beta$. There exists $c_{15} = c_{15}(\alpha, \beta, N, p, q, d_1, d_2)$ such that if u is a nonnegative solution of* (1-1) *in $\Omega \setminus \{0\}$ vanishing on $\partial\Omega$ then*

$$|\nabla u(x)| \le c_{15}|x|^{-(\tau+1)} \quad \textit{for all } x \in B_{d_1} \setminus \{0\}. \tag{2-11}$$

Proof. Let x_0, ρ_0, η, ϕ, w, m, $\mathcal{L}[w]$, and x^* as in the proof of Lemma 2.2. Then we get

$$\begin{aligned} \mathcal{L}[w] = &-2m(2m-1)\phi^{2(m-1)}|\nabla\phi|^2|\nabla u|^2 - 2m\phi^{2m-1}\Delta\phi|\nabla u|^2 - 8m\phi^{2m-1}\sum_{i,j}\partial_i\phi\,\partial_j u\,\partial_{ij}u \\ &- 2\phi^{2m}|D^2u|^2 - 2\alpha|x|^{\alpha-2}\phi^{2m}u^p\, x\nabla u - 2p|x|^\alpha\phi^{2m}u^{p-1}|\nabla u|^2 \\ &- 2\beta|x|^{\beta-2}\phi^{2m}|\nabla u|^q\, x\nabla u + 2mq|x|^\beta\phi^{2m-1}|\nabla u|^q\nabla\phi\nabla u. \end{aligned}$$

Case 1: $D \ge 1$. In this case, we have

$$\frac{(\beta+1)(1-2q)}{q-1} \le \alpha - 2\beta - 1 - \tau p, \tag{2-12}$$

where τ is defined in (1-8). By Lemma 2.6 and Young's inequality, proceeding as in the proof of Lemma 2.2, we obtain in $B_{\rho_0/2}(x_0)$

$$w(x^*)^{q-\frac{1}{2}} \le c_{16}\big(\rho_0^{-2(\beta+1)}w(x^*)^{\frac{1}{2}} + \rho_0^{\alpha-2\beta-1-\tau p} + \rho_0^{-(\beta+1)}w(x^*)^{\frac{q}{2}}\big), \tag{2-13}$$

where $c_{16} = c_{16}(\alpha, \beta, p, q, N, d_1, d_2)$. By Young's inequality, we get

$$\rho_0^{-2(\beta+1)}w(x^*)^{\frac{1}{2}} \le \frac{1}{q}\rho_0^{-(\beta+1)}w(x^*)^{\frac{q}{2}} + \frac{q-1}{q}\rho_0^{\frac{(\beta+1)(1-2q)}{q-1}}. \tag{2-14}$$

From (2-12), (2-13), and (2-14), we deduce

$$w(x^*)^{q-\frac{1}{2}} \le c_{17}\big(\rho_0^{-(\beta+1)}w(x^*)^{\frac{q}{2}} + \rho_0^{\frac{(\beta+1)(1-2q)}{q-1}}\big), \tag{2-15}$$

which implies

$$\rho_0^{\beta+1} w(x^*)^{\frac{q-1}{2}} \le c_{17}\big(\rho_0^{-\frac{(\beta+1)q}{q-1}} w(x^*)^{-\frac{q}{2}} + 1\big), \tag{2-16}$$

where $c_{17} = c_{17}(\alpha, \beta, p, q, N, d_1, d_2)$. Consequently, $w(x^*) \le c_{18}\rho_0^{-\frac{2(1+\beta)}{q-1}}$, and therefore

$$|\nabla u(x)| \le c_{19}|x|^{-\frac{1+\beta}{q-1}} \quad \text{for all } x \in B_{d_1} \setminus \{0\}, \tag{2-17}$$

where $c_i = c_i(\alpha, \beta, N, p, q, d_1, d_2)$ with $i = 18, 19$. Notice that $\dfrac{1+\beta}{q-1} = \tau + 1$; hence we obtain (2-11).

Case 2: $D < 1$. Take $x_0 \in B_{d_1} \setminus \{0\}$. Put $\ell = |x_0| \in (0, d_1)$ then $S_\ell[u]$ is a solution of

$$-\Delta v + |x|^\alpha v^p + \ell^{\frac{p(2+\beta-q)-\alpha(q-1)-q-\beta}{p-1}} |x|^\beta |\nabla v|^q = 0 \quad \text{in } \Omega_\ell \setminus \{0\}. \tag{2-18}$$

By the regularity result in [DiBenedetto 1983], there exists $c_{20} = c_{20}(\alpha, \beta, p, q)$ such that

$$\sup\big\{|\nabla S_\ell[u](x)| : x \in B_{3/2} \setminus B_{3/4}\big\} \le c_{20}.$$

Consequently,

$$\ell^{\frac{1+p+\alpha}{p-1}} |\nabla u(\ell x)| \le c_{21} \quad \text{for all } x \in B_{3/2} \setminus B_{3/4}.$$

By choosing $x = \ell^{-1}x_0$, we derive $|\nabla u(x_0)| \le c_{22}|x_0|^{-\frac{1+p+\alpha}{p-1}}$. This completes the proof since

$$\frac{1+p+\alpha}{p-1} = \tau + 1. \qquad \square$$

Proof of Proposition 1.1. Estimates (1-9) and (1-10) follow directly from Lemmas 2.2, 2.4, and 2.6, as well as Proposition 2.7. □

3. Weakly singular solutions

We start with the existence of weakly singular solutions of (1-1). The construction is based on approximation method.

Proof of Theorem A. We prove the theorem in five steps.

Step 1: Construction of solutions. Let $k > 0$. For every $\epsilon > 0$, let $u_{k,\epsilon}^\Omega$ be the unique solution of

$$\begin{cases} -\Delta u + |x|^\alpha u^p + |x|^\beta |\nabla u|^q = 0 & \text{in } \Omega \setminus \overline{B}_\epsilon, \\ u = 0 & \text{on } \partial\Omega, \\ u = k\Gamma_N(\epsilon) & \text{on } \partial B_\epsilon. \end{cases} \tag{3-1}$$

The existence of $u_{k,\epsilon}^\Omega$ can be obtained by using an argument similar to the proof of [Gilbarg and Trudinger 2001, Theorem 11.4] and the uniqueness follows from the comparison principle Proposition 2.1. Moreover, by the comparison principle, $0 \le u_{k,\epsilon}^\Omega \le k\Gamma_N$ in $\overline{\Omega} \setminus B_\epsilon$ and $u_{k,\epsilon}^\Omega \le u_{k,\epsilon'}^\Omega$ in $\overline{\Omega} \setminus B_{\epsilon'}$ for every $0 < \epsilon < \epsilon'$. Therefore, $u_k^\Omega := \lim_{\epsilon\to 0} u_{k,\epsilon}^\Omega$ satisfies

$$u_k^\Omega(x) \le k\Gamma_N(x) \quad \text{for all } x \in \Omega \setminus \{0\}. \tag{3-2}$$

By regularity results for elliptic equations, u_k^Ω is a solution of (1-1) in $\Omega \setminus \{0\}$ vanishing on $\partial\Omega$.

Fix an arbitrary point $x_0 \in \overline{B_{d_1}} \setminus \overline{B_\epsilon}$ and put $\ell = |x_0| \in (\epsilon, d_1]$. Note that $R_\ell[u_{k,\epsilon}^\Omega]$ solves

$$\begin{cases} -\Delta v + \ell^{N+\alpha-p(N-2)}|x|^\alpha v^p + \ell^{N+\beta-q(N-1)}|x|^\beta |\nabla v|^q = 0 & \text{in } \Omega_\ell \setminus \overline{B_{\epsilon/\ell}}, \\ v = 0 & \text{on } \partial\Omega_\ell, \\ v = k\Gamma_N\left(\frac{\epsilon}{\ell}\right) & \text{on } \partial B_{\epsilon/\ell}. \end{cases} \tag{3-3}$$

Since $1 < p < p_{c,\alpha}$ and $1 < q < q_{c,\beta}$, it follows that

$$\ell^{N+\alpha-p(N-2)}|x|^\alpha < \max\{1, 3^\alpha\} \quad \text{and} \quad \ell^{N+\beta-q(N-1)}|x|^\beta < \max\{1, 3^\beta\} \quad \text{for all } x \in B_3 \setminus B_1.$$

By the maximum principle, $R_\ell[u_{k,\epsilon}^\Omega] \le k\Gamma_N$ in $\Omega_\ell \setminus \overline{B_{\epsilon/\ell}}$, which implies $R_\ell[u_{k,\epsilon}^\Omega] \le k\Gamma_N(1)$ in $B_3 \setminus B_1$. Due to local regularity for elliptic equations (see, e.g., [DiBenedetto 1983]), there exist constants $c_{23} = c_{23}(N, \alpha, \beta, p, q, k)$ and $\mu = \mu(N, \alpha, \beta, p, q, k) \in (0, 1)$ such that

$$\| R_\ell[u_{k,\epsilon}^\Omega] \|_{C^{1,\mu}(B_{5/2} \setminus \overline{B_{3/2}})} \le c_{23}.$$

Again by the regularity results (see [Lieberman 1988, Theorem 1] and [DiBenedetto 1983]), there exists $c_{24} = c_{24}(\alpha, \beta, N, p, q, k)$ such that

$$\ell^{N-1} \sup\{|\nabla u_{k,\epsilon}^\Omega(\ell x)| : |x| = 1\} \le c_{24}.$$

By choosing $x = \ell^{-1} x_0$, we deduce $|\nabla u_{k,\epsilon}^\Omega(x_0)| \le c_{24}|x_0|^{1-N}$. Thus

$$|\nabla u_{k,\epsilon}^\Omega(x)| \le c_{25}|x|^{1-N} \quad \text{for all } x \in \Omega \setminus B_\epsilon \tag{3-4}$$

with $c_{25} = c_{25}(\alpha, \beta, N, p, q, k, d_1, d_2)$.

Step 2: Proof of (1-16). The solution $u_{k,\epsilon}^\Omega$ can be written in the form

$$u_{k,\epsilon}^\Omega(x) = k\Gamma_N(\epsilon) - \mathbb{G}^{\Omega \setminus \overline{B_\epsilon}}[F \circ u_{k,\epsilon}^\Omega](x),$$

where $\mathbb{G}^{\Omega \setminus \overline{B_\epsilon}}$ is the Green operator in $\Omega \setminus \overline{B_\epsilon}$ [Marcus and Véron 2014, Theorem 1.2.2]. Hence, by (3-4),

$$k\Gamma_N(x) \ge u_{k,\epsilon}^\Omega(x) \ge k\Gamma_N(x) - c_{26}\mathbb{G}^\Omega[|\cdot|^{\alpha+p(2-N)} + |\cdot|^{\beta+q(1-N)}](x) \quad \text{for all } x \in \Omega \setminus \overline{B_\epsilon}.$$

By letting $\epsilon \to 0$, we get

$$k\Gamma_N(x) \ge u_k^\Omega(x) \ge k\Gamma_N(x) - c\mathbb{G}^\Omega[|\cdot|^{\alpha+p(2-N)} + |\cdot|^{\beta+q(1-N)}](x) \quad \text{for all } x \in \Omega \setminus \{0\}. \tag{3-5}$$

It is classical (see [op. cit.]) that

$$G^\Omega(x, y) \sim \min\{|x-y|^{2-N}, \rho(x)\rho(y)|x-y|^{-N}\}$$

for every $x, y \in \Omega, x \neq y$, where $\rho(x) = \text{dist}(x, \partial\Omega)$. Therefore there exists $c_{27} = c_{27}(N, \Omega)$ such that, for x near 0,

$$\frac{\mathbb{G}^\Omega[|\cdot|^{\alpha+p(2-N)} + |\cdot|^{\beta+q(1-N)}](x)}{\Gamma_N(x)} \le c_{27}|x|^{N-2} \int_\Omega |x-y|^{2-N}\left(|y|^{\alpha-p(N-2)} + |y|^{\beta-q(N-1)}\right) dy. \tag{3-6}$$

Choose α' and β' such that $p(N-2)-N<\alpha'<\min\{\alpha, p(N-2)-2\}$ and $q(N-1)-N<\beta'<\min\{\beta, q(N-1)-2\}$. Then by [Lieb and Loss 1997, Corollary 5.10],

$$\begin{aligned}\int_\Omega |x-y|^{2-N}|y|^{\alpha-p(N-2)}\,dy &\le c_{28}\,d_2^{\alpha-\alpha'}|x|^{2+\alpha'-p(N-2)},\\ \int_\Omega |x-y|^{2-N}|y|^{\beta-q(N-1)}\,dy &\le c_{28}\,d_2^{\beta-\beta'}|x|^{2+\beta'-q(N-1)}.\end{aligned}\tag{3-7}$$

Combining (3-6) and (3-7) yields

$$\lim_{|x|\to 0}\frac{\mathbb{G}^\Omega[|\cdot|^{\alpha+p(2-N)}+|\cdot|^{\beta+q(1-N)}](x)}{\Gamma_N(x)}=0.\tag{3-8}$$

From (3-5) and (3-8), we obtain (1-16).

Step 3: Proof of (1-17). For $\ell\in(0,1)$, put $v_\ell=R_\ell[u_k^\Omega]$ then v_ℓ is the solution of

$$\begin{cases}-\Delta v+\ell^{N+\alpha-p(N-2)}|x|^\alpha v^p+\ell^{N+\beta-q(N-1)}|x|^\beta|\nabla v|^q=0, & \text{in } \Omega_\ell\setminus\{0\}\\ v=0 & \text{on } \partial\Omega_\ell.\end{cases}\tag{3-9}$$

Since $0<u_k^\Omega<k\Gamma_N$ in $\Omega\setminus\{0\}$, it follows that $0<v_\ell<k\Gamma_N$ in $\Omega_\ell\setminus\{0\}$.

Since $1<p<p_{c,\alpha}$ and $1<q<q_{c,\beta}$, by local regularity for elliptic equations [DiBenedetto 1983], the Arzelà–Ascoli theorem, and a standard diagonalization argument, there exists a subsequence $\{v_{\ell_n}\}$ converging to a positive harmonic function in $C^1_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$ as $\ell_n\to 0$. On the other hand, from (1-16), we deduce that $\{v_\ell\}$ converges to $k\Gamma_N$ uniformly in $B_2\setminus B_{1/2}$ as $\ell\to 0$. Therefore, the whole sequence $\{v_\ell\}$ converges to $k\Gamma_N$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$ as $\ell\to 0$. In particular, $\nabla v_\ell\to k\nabla\Gamma_N$ in $B_2\setminus B_{1/2}$, which implies (1-17).

Step 4: u_k^Ω is a weak solution of (1-11). By a similar argument as in Step 1, we derive

$$|\nabla u_k^\Omega(x)|\le c_{29}k|x|^{1-N}\quad\text{for all } x\in\Omega\setminus\{0\}\tag{3-10}$$

where $c_{29}=c_{29}(\alpha,\beta,N,p,q,d_1,d_2)$. This, together with (3-2), implies $u_k^\Omega\in L^1(\Omega)$ and $F\circ u_k^\Omega\in L^1(\Omega)$.

For every $\epsilon>0$, by Green's formula, one gets

$$\int_{\Omega\setminus\overline{B}_\epsilon}\big(-u_k^\Omega\Delta\zeta+(F\circ u_k^\Omega)\zeta\big)\,dx=-\int_{\partial B_\epsilon}\frac{\partial u_k^\Omega}{\partial\boldsymbol{n}}\zeta\,dS+\int_{\partial\overline{B}_\epsilon}u_k^\Omega\frac{\partial\zeta}{\partial\boldsymbol{n}}\,dS,\tag{3-11}$$

where $\boldsymbol{n}$ is the outward normal unit vector on ∂B_ϵ. Due to (1-17), the right-hand side of (3-11) converges to $k\zeta(0)$. Therefore, thanks to dominated convergence theorem, by letting $\epsilon\to 0$, we obtain (1-12). Finally, by [Marcus and Véron 2014, Theorem 1.2.2], we get (1-15).

Step 5: Uniqueness. Assume u' is a positive solutions of (1-1) satisfying (1-16); then

$$\lim_{|x|\to 0}\frac{u_k^\Omega(x)}{u'(x)}=1.$$

Hence, for every $\delta>0$, there exists $r(\delta)>0$ such that $(1+\delta)u_k^\Omega+\delta\ge u'$ on $\partial B_{r(\delta)}$. The function $(1+\delta)u_k^\Omega+\delta$ is a supersolution of (1-1) which dominates u' on $\partial\Omega\cup\partial B_{r(\delta)}$; therefore, by the comparison principle, $(1+\delta)u_k^\Omega + \delta \geq u'$ in $\Omega \setminus B_{r(\delta)}$. Letting $\delta \to 0$ yields $u_k^\Omega \geq u'$ in $\Omega \setminus \{0\}$. By permuting u_k^Ω and u', we derive $u' = u_k^\Omega$. □

If Ω is replaced by $\mathbb{R}^N$, we have the following variant of Theorem A.

Proposition 3.1. *Assume $\alpha > -2,\ \beta > -1,\ 1 < p < p_{c,\alpha}$, and $1 < q < q_{c,\beta}$. Then for any $k > 0$, there exists a unique solution $u_k^{\mathbb{R}^N} \in C^2(\mathbb{R}^N \setminus \{0\})$ of (1-1) in $\mathbb{R}^N \setminus \{0\}$ satisfying*

$$\lim_{|x|\to\infty} u_k^{\mathbb{R}^N}(x) = 0 \quad \text{and} \quad u_k^{\mathbb{R}^N}(x) = k\,(1+o(1))\Gamma_N(x) \ \text{ as } |x| \to 0. \tag{3-12}$$

Moreover, $u_k^{\mathbb{R}^N} \in L^1_{\text{loc}}(\mathbb{R}^N),\ F \circ u_k^{\mathbb{R}^N} \in L^1_{\text{loc}}(\mathbb{R}^N)$, and there holds

$$\int_{\mathbb{R}^N} \big(-u_k^{\mathbb{R}^N}\Delta\zeta + (F \circ u_k^{\mathbb{R}^N})\zeta\big)\,dx = k\zeta(0) \quad \text{for all } \zeta \in C_c^2(\mathbb{R}^N). \tag{3-13}$$

Proof. For each $R > 0$, let $u_k^{B_R}$ be the unique solution of (1-1) in $B_R \setminus \{0\}$, vanishing on ∂B_R and satisfying

$$\lim_{|x|\to 0} \frac{u_k^{B_R}(x)}{\Gamma_N(x)} = k. \tag{3-14}$$

By the comparison principle, $u_k^{B_R} \leq u_k^{B_{R'}} \leq k\Gamma_N$ in $B_R \setminus \{0\}$ for every $R < R'$. In light of local regularity [DiBenedetto 1983] and a standard argument,

$$u_k^{\mathbb{R}^N} := \lim_{R\to\infty} u_k^{B_R} \in C^2(\mathbb{R}^N \setminus \{0\})$$

is a solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$. By combining (3-14) and the fact that $u_k^{B_R} \leq u_k^{\mathbb{R}^N} \leq k\Gamma_N$ in $B_R \setminus \{0\}$ for every $R > 0$, we derive (3-12). Uniqueness follows from the comparison principle. By proceeding as in the proof of Theorem A, one can verify (3-13). □

By a similar, and more simpler, argument as in the proof of Theorem A, one can easily obtain the existence of weakly singular solutions of (1-3).

Proposition 3.2. *Assume $\beta > -1$ and $1 < q < q_{c,\beta}$ with $q_{c,\beta}$ defined in* (1-14). *For any $k > 0$, there exists a unique solution $w_k^\Omega \in C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega} \setminus \{0\})$ of*

$$-\Delta w + |x|^\beta |\nabla w|^q = k\delta_0 \quad \text{in} \ \ \Omega, \text{ with } w = 0 \text{ on } \partial\Omega. \tag{3-15}$$

Moreover,

$$w_k^\Omega = kG^\Omega(\cdot\,,0) - \mathbb{G}^\Omega[|\cdot|^\beta |\nabla w_k^\Omega|^q]; \tag{3-16}$$

$$w_k^\Omega(x) = k(1+o(1))\,\Gamma_N(x) \quad \text{as } |x| \to 0; \tag{3-17}$$

$$\lim_{|x|\to 0}\left(|x|^{N-1}\,\nabla w_k^\Omega(x) + \frac{k}{N\omega_N}\frac{x}{|x|}\right) = 0. \tag{3-18}$$

Remark. In addition, by proceeding as in the proof of Proposition 3.1, we obtain the existence of the weak singular solutions $w_k^{\mathbb{R}^N}$ of (1-3) in $\mathbb{R}^N \setminus \{0\}$.

4. Strongly singular solutions

Denote by S^{N-1} the unit sphere in $\mathbb{R}^N$ and let $(r,\sigma)\in(0,\infty)\times S^{N-1}$ be the spherical coordinates in $\mathbb{R}^N\setminus\{0\}$. Let ∇' and Δ' denote respectively the covariant gradient and the Laplace–Beltrami operator on S^{N-1}. In order to characterize strongly singular solutions of (1-1), we study the following quasilinear equation on S^{N-1}:

$$-\Delta'\omega+\lambda\omega^{\frac{\alpha(q-1)+q+\beta}{2+\beta-q}}+\left(\left(\frac{2+\beta-q}{q-1}\right)^2\omega^2+|\nabla'\omega|^2\right)^{\frac{q}{2}}-\Lambda\omega=0, \tag{4-1}$$

where

$$\lambda\geq 0,\quad \text{and}\quad \Lambda=\Lambda(N,q,\beta):=\frac{2+\beta-q}{q-1}\left(\frac{q+\beta}{q-1}-N\right).$$

We introduce an auxiliary function

$$g_\lambda(t)=\lambda t^{\frac{(2+\alpha)(q-1)}{2+\beta-q}}+\left(\frac{2+\beta-q}{q-1}\right)^q t^{q-1}-\Lambda,\quad t\in(0,\infty),\quad \lambda\geq 0. \tag{4-2}$$

It is easy to see that if $1<q<q_{c,\beta}$ then $\Lambda>0$; therefore, the algebraic equation $g_\lambda(t)=0$ admits a unique positive solution θ_λ. Obviously, θ_λ is a positive solution of (4-1), and θ_0 is explicitly given by

$$\theta_0=\frac{q-1}{2+\beta-q}\left(\frac{q+\beta}{q-1}-N\right)^{\frac{1}{q-1}}. \tag{4-3}$$

Proposition 4.1. *Let $\alpha>-2,\ \beta>-1,\ 1<q<2+\beta$, and $\lambda\geq 0$. Denote by $\mathcal{E}_\lambda$ the set of C^2 positive solutions of* (4-1) *in S^{N-1}.*

(i) *If $q\geq q_{c,\beta}$, then $\mathcal{E}_\lambda=\varnothing$.*

(ii) *If $1<q<q_{c,\beta}$, then $\mathcal{E}_\lambda=\{\theta_\lambda\}$.*

Proof. (i) Suppose by contradiction that ω is a positive solution of (4-1) and $\omega(\sigma_{\max})=\max_{S^{N-1}}\omega>0$ with $\sigma_{\max}\in S^{N-1}$. From (4-1), we get

$$\lambda\omega(\sigma_{\max})^{\frac{\alpha(q-1)+q+\beta}{2+\beta-q}}+\left(\frac{2+\beta-q}{q-1}\right)^q\omega(\sigma_{\max})^q-\Lambda\omega(\sigma_{\max})\leq 0. \tag{4-4}$$

Since $q\geq q_{c,\beta}$, we know $\Lambda\leq 0$. Therefore, the left hand side is positive, which is a contradiction.

(ii) If ω is a positive solution of (4-1), let $\sigma_{\max},\sigma_{\min}\in S^{N-1}$ such that

$$\omega(\sigma_{\max})=\max_{S^{N-1}}\omega\geq\min_{S^{N-1}}\omega=\omega(\sigma_{\min})>0.$$

Clearly, $\sigma_{\max}$ satisfies (4-4) and $\sigma_{\min}$ satisfies

$$\lambda\omega(\sigma_{\min})^{\frac{\alpha(q-1)+q+\beta}{2+\beta-q}}+\left(\frac{2+\beta-q}{q-1}\right)^q\omega(\sigma_{\min})^q-\Lambda\omega(\sigma_{\min})\geq 0. \tag{4-5}$$

Consequently, $g_\lambda(\omega(\sigma_{\max}))\leq 0\leq g_\lambda(\omega(\sigma_{\min}))$. Since g_λ is strictly increasing in $(0,\infty)$, it follows that $\omega(\sigma_{\max})\leq\theta_\lambda\leq\omega(\sigma_{\min})$. Thus, $\omega\equiv\theta_\lambda$. This completes the proof. □

The next lemma states existence result for both equations (1-3) and (1-1).

Lemma 4.2. *Let Ω be either a smooth bounded domain containing the origin* 0 *or* $\mathbb{R}^N$.

(i) *Assume $\beta > -1$ and $1 < q < q_{c,\beta}$. Then $w_\infty^\Omega := \lim_{k\to\infty} w_k^\Omega$ is a nonnegative solution of* (1-3) *in* $\Omega \setminus \{0\}$ *satisfying either $w_\infty^\Omega = 0$ on $\partial\Omega$ if Ω is bounded or* $\lim_{|x|\to\infty} w_\infty^\Omega(x) = 0$ *if* $\Omega = \mathbb{R}^N$.

(ii) *Assume $\alpha > -2$, $\beta > -1$, $1 < p < p_{c,\alpha}$, and $1 < q < q_{c,\beta}$. Then $u_\infty^\Omega := \lim_{k\to\infty} u_k^\Omega$ is a nonnegative solution of* (1-1) *in* $\Omega \setminus \{0\}$ *satisfying either $u_\infty^\Omega = 0$ on $\partial\Omega$ if Ω is bounded or* $\lim_{|x|\to\infty} u_\infty^\Omega(x) = 0$ *if* $\Omega = \mathbb{R}^N$.

Proof. We only demonstrate (ii) since the proof of (i) is similar and simpler. It follows from Theorem A and Proposition 3.1 that $\{u_k^\Omega\}$ is increasing and bounded from above by the function $\overline{U}(x) = c_{30}\,|x|^{-\frac{2+\alpha}{p-1}}$ where c_{30} is a large positive constant depending on N, p, and α. Therefore, $u_\infty^\Omega := \lim_{k\to\infty} u_k^\Omega$ is a solution of (1-1) in $\Omega \setminus \{0\}$ and $u_\infty^\Omega \le \overline{U}$ in $\Omega \setminus \{0\}$. □

The asymptotic behavior of w_∞^Ω near the origin 0 is analyzed in the following result.

Proposition 4.3. *Assume $\beta > -1$, $1 < q < q_{c,\beta}$, and Ω is either a smooth bounded domain containing the origin* 0 *or* $\mathbb{R}^N$. *Let w_∞^Ω be the solution in* Lemma 4.2(i). *Then w_∞^Ω is a strongly singular solution of* (1-3). *Moreover*, *with θ_0 as in* (4-3),

$$\lim_{|x|\to 0} |x|^{\frac{2+\beta-q}{q-1}}\, w_\infty^\Omega(x) = \theta_0 \tag{4-6}$$

$$\lim_{|x|\to 0}\left(|x|^{\frac{1+\beta}{q-1}}\, \nabla w_\infty^\Omega(x) + \left(\frac{q+\beta}{q-1} - N\right)^{\frac{1}{q-1}} \frac{x}{|x|}\right) = 0. \tag{4-7}$$

Proof. The proof is based upon the similarity argument.

Case 1: $\Omega = \mathbb{R}^N$. For $k > 0$, recall that w_k^Ω is the weakly singular solution of (1-3) in $\mathbb{R}^N$. For every $\ell > 0$, $T_\ell[w_k^{\mathbb{R}^N}]$ is a solution of (1-3) in $\mathbb{R}^N \setminus \{0\}$ which satisfies

$$\lim_{|x|\to 0} \frac{T_\ell[w_k^{\mathbb{R}^N}](x)}{\Gamma_N(x)} = \ell^{\frac{2+\beta-q}{q-1}+2-N} k.$$

Due to the uniqueness,

$$T_\ell[w_k^{\mathbb{R}^N}] = w^{\mathbb{R}^N}_{\ell^{(2+\beta-q)/(q-1)+2-N} k}.$$

By letting $k \to \infty$, we deduce that $T_\ell[w_\infty^{\mathbb{R}^N}] = w_\infty^{\mathbb{R}^N}$, i.e., $w_\infty^{\mathbb{R}^N}$ is self-similar. Consequently, $w_\infty^{\mathbb{R}^N}$ can be written in the form

$$w_\infty^{\mathbb{R}^N}(x) = |x|^{-\frac{2+\beta-q}{q-1}}\, \omega(x/|x|) \quad \text{for all } x \ne 0, \tag{4-8}$$

where ω is a positive solution of (4-1) with $\lambda = 0$. Since $1 < q < q_{c,\beta}$, by Proposition 4.1, $\omega \equiv \theta_0$. Therefore,

$$w_\infty^{\mathbb{R}^N}(x) = \theta_0\,|x|^{-\frac{2+\beta-q}{q-1}} =: W_0(x) \quad \text{for all } x \ne 0,$$

the unique self-similar solution of (1-3) in $\mathbb{R}^N \setminus \{0\}$.

Case 2: Ω is a bounded smooth domain. Since $T_\ell[w_k^\Omega] = w^{\Omega_\ell}_{\ell^{(2+\beta-q)/(q-1)+2-N}k}$ by uniqueness, it follows that

$$T_\ell[w_\infty^\Omega] = w_\infty^{\Omega_\ell}. \tag{4-9}$$

Since $w_\infty^\Omega(x) \le c_8|x|^{-\frac{2+\beta-q}{q-1}}$ in $\Omega\setminus\{0\}$, $w_\infty^{\Omega_\ell}$ satisfies the same estimate in $\Omega_\ell\setminus\{0\}$ for every $\ell\in(0,1)$. By local regularity for elliptic equations and Arzelà–Ascoli theorem, there exists a subsequence $\{w_\infty^{\Omega_{\ell_n}}\}$ converging in $C^1_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$ to a function w_0 which is a solution of (1-3) in $\mathbb{R}^N\setminus\{0\}$.

If Ω is star-shaped with respect to the origin 0 then we get $w_k^{\Omega_\ell} \le w_k^{\Omega_{\ell'}}$ for every $k>0$ and $0<\ell'<\ell<1$, which implies that $w_\infty^{\Omega_\ell} \le w_\infty^{\Omega_{\ell'}}$ for every $0<\ell'<\ell<1$. Therefore, the whole sequence $\{w_\infty^{\Omega_\ell}\}$ converges to w_0 in $C^1_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$ as $\ell\to 0$. By (4-9), for any $\ell,\ell'>0$,

$$T_\ell[T_{\ell'}[w_\infty^\Omega]] = T_\ell[w_\infty^{\Omega_{\ell'}}] = w_\infty^{\Omega_{\ell'\ell}}.$$

By letting $\ell'\to 0$, we obtain $T_\ell[w_0]=w_0$ for every $\ell>0$, namely w_0 is a self-similar solution of (1-3) in $\mathbb{R}^N\setminus\{0\}$. Therefore, $w_0 = w_\infty^{\mathbb{R}^N} = W_0$ and consequently,

$$\lim_{\ell\to 0} \ell^{\frac{2+\beta-q}{q-1}} w_\infty^\Omega(\ell x) = \theta_0|x|^{-\frac{2+\beta-q}{q-1}}.$$

By putting $y=\ell x$ with $|x|=1$, we get (4-6).

In general, if Ω is not necessarily star-shaped with respect to the origin 0, since $\overline{B_{3d_1}}\subset\Omega\subset B_{d_2}$, it follows that $w_\infty^{B_{3d_1}} \le w_\infty^\Omega \le w_\infty^{B_{d_2}}$. As (4-6) holds for $w_\infty^{B_{3d_1}}$ (i.e., Ω is replaced by B_{3d_1}) and $w_\infty^{B_{d_2}}$, we derive (4-6). Consequently, for every $x\ne 0$,

$$w_0(x) = \lim_{n\to\infty} w_\infty^{\Omega_{\ell_n}}(x) = \lim_{n\to\infty} \ell_n^{\frac{2+\beta-q}{q-1}} w_\infty^\Omega(\ell_n x) = \theta_0|x|^{-\frac{2+\beta-q}{q-1}} = W_0(x).$$

Hence the whole sequence $\{w_\infty^{\Omega_\ell}\}_\ell$ converges to W_0 in $C^1_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$ as $\ell\to 0$. By using a similar argument as in Step 3 of the proof of Theorem A, we obtain (4-7). This implies $|x|^\beta|\nabla w_\infty^\Omega|^q \notin L^1(B_\epsilon)$ for every $\epsilon>0$. Thus w_∞^Ω is a strongly singular solution of (1-3). □

Note that (1-1) does not admit any similarity transformation, except when $D=1$. However, due to the asymptotic behavior of v_∞^Ω (the strongly singular solution of (1-2)) and of w_∞^Ω near 0, we can establish the asymptotic behavior of u_∞^Ω. Put

$$\Theta = \begin{cases} \vartheta & \text{if } D<1,\\ \theta_1 & \text{if } D=1,\\ \theta_0 & \text{if } D>1,\end{cases} \tag{4-10}$$

where ϑ is as in (1-7) and θ_λ ($\lambda=0,1$) is given in (4-2).

Now we are ready to deal with strongly singular solution of (1-1).

Proposition 4.4. *Assume $\alpha>-2$, $\beta>-1$, $1<p<p_{c,\alpha}$, and $1<q<q_{c,\beta}$. Let Ω be either a smooth bounded domain containing the origin* 0 *or $\mathbb{R}^N$ and u_∞^Ω be the solution of* (1-1) *defined in Lemma 4.2. Then u_∞^Ω is a strongly singular solution of* (1-1). *Moreover* (1-18) *and* (1-19) *hold.*

Proof. We consider three cases.

Case 1: $D = 1$. In this case, S_ℓ is a similarity transformation for (1-1). Therefore, (1-18) and (1-19) can be obtained by proceeding as in the proof of Proposition 4.3 and consequently u_∞^Ω is a strongly singular solution of (1-1). Notice that if $\Omega = \mathbb{R}^N$ then $\Omega_\ell = \mathbb{R}^N$ and $u = 0$ on $\partial\Omega_\ell$ is understood as $u(x) \to 0$ as $|x| \to \infty$.

Case 2: $D > 1$. For every $\ell \in (0, 1)$, put $W_\ell = T_\ell[u_\infty^\Omega]$. Then W_ℓ is a solution of

$$-\Delta u + \ell^{\frac{\alpha(q-1)+q+\beta-p(2+\beta-q)}{q-1}} |x|^\alpha u^p + |x|^\beta |\nabla u|^q = 0 \quad \text{in } \Omega_\ell \setminus \{0\}, \text{ with } u = 0 \text{ on } \partial\Omega_\ell. \tag{4-11}$$

By the regularity result [DiBenedetto 1983], for every $R > 1$ there exist $M = M(\alpha, \beta, p, q, N, R, d_1, d_2)$ and $\mu = \mu(\alpha, \beta, p, q, N, d_1, d_2) \in (0, 1)$ such that

$$\|W_\ell\|_{C^{1,\mu}(B_R \setminus B_{R^{-1}})} < M.$$

Consequently, there exists a subsequence $\{W_{\ell_n}\}$ which converges to a function $\widetilde{W}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $\ell_n \to 0$. The function $\widetilde{W}$ is a solution of (1-3) in $\mathbb{R}^N \setminus \{0\}$ satisfying $\lim_{|x|\to\infty} \widetilde{W}(x) = 0$. By Proposition 2.1, $w_\infty^{\mathbb{R}^N} \geq \widetilde{W} \geq u_k^{\mathbb{R}^N}$ for every $k > 0$. Therefore, thanks to (3-12), we get

$$\liminf_{x\to 0} \frac{\widetilde{W}(x)}{w_k^{\mathbb{R}^N}(x)} = \liminf_{x\to 0} \frac{\widetilde{W}(x)}{k\Gamma_N(x)} = \liminf_{x\to 0} \frac{\widetilde{W}(x)}{u_k^{\mathbb{R}^N}(x)} \geq 1.$$

By using a similar argument as in the proof Proposition 3.1, together with the comparison principle, we deduce that $\widetilde{W} \geq w_k^{\mathbb{R}^N}$ in $\mathbb{R}^N \setminus \{0\}$ for every $k > 0$. It follows that $\widetilde{W} \geq w_\infty^{\mathbb{R}^N}$ in $\mathbb{R}^N \setminus \{0\}$ and hence $\tilde{W} = w_\infty^{\mathbb{R}^N}$ in $\mathbb{R}^N \setminus \{0\}$. Thus the whole sequence $\{W_\ell\}$ converges to $w_\infty^{\mathbb{R}^N}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $\ell \to 0$. This leads to (1-18) and (1-19). Consequently u_∞^Ω is a strongly singular solution.

Case 3: $D < 1$. For every $\ell \in (0, 1)$, put $V_\ell = S_\ell[u_\infty^\Omega]$. Similarly, we can show that the sequence $\{V_\ell\}$ converges to $v_\infty^{\mathbb{R}^N}$ (the strongly singular solution of (1-2)) in $C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ as $\ell \to 0$. This yields the desired result. □

Proof of Theorem B. The theorem follows from Lemma 4.2 and Proposition 4.4. □

5. Classification and removability of isolated singularities

5.1. ***Classification of isolated singularities.*** The following lemma plays an important role in proving the classification result.

Lemma 5.1. *Assume Ω is a bounded domain containing the origin* 0, $\alpha > -2$, $\beta > -1$, $1 < p < p_{c,\alpha}$, *and* $1 < q < q_{c,\beta}$. *Let $u \in C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega} \setminus \{0\})$ be a nonnegative solution of* (1-1) *in $\Omega \setminus \{0\}$ vanishing on $\partial\Omega$. Then there exists $c_{31} = c_{31}(N, \alpha, \beta, p, q, d_1, d_2)$ such that for any $\delta \in \left(0, \frac{1}{4}d_1\right)$, there holds*

$$\sup\{u(x) : x \in \partial B_\delta\} \leq c_{31} \inf\{u(x) : x \in \partial B_\delta\}. \tag{5-1}$$

Proof. Fix $\delta \in \left(0, \frac{1}{4}d_1\right)$ and take $x_0 \in \partial B_\delta \setminus \{0\}$. Put $r_0 = |x_0|$, $y_0 = r_0^{-1} x_0 \in \partial B_1$,

$$\varphi_{r_0} = \begin{cases} S_{r_0}[u] & \text{if } D \leq 1, \\ T_{r_0}[u] & \text{if } D > 1. \end{cases}$$

It is easy to see that φ_{r_0} is a nonnegative solution of one of the following equations

$$\begin{cases} -\Delta\varphi + |x|^\alpha \varphi^p + r_0^{\frac{p(2+\beta-q)-\alpha(q-1)-q-\beta}{p-1}} |x|^\beta |\nabla\varphi|^q = 0 & \text{if } D < 1, \\ -\Delta\varphi + r_0^{\frac{\alpha(q-1)+q+\beta-p(2+\beta-q)}{q-1}} |x|^\alpha \varphi^p + |x|^\beta |\nabla\varphi|^q = 0 & \text{if } D > 1, \\ -\Delta\varphi + |x|\alpha\varphi^p + |x|^\beta |\nabla\varphi|^q = 0 & \text{if } D = 1. \end{cases}$$

in $\Omega_{r_0} = r_0^{-1}\Omega$. By Lemma 2.6, for every $y \in B_{1/4}(y_0)$,

$$\varphi_{r_0}(y) = r_0^\tau u(r_0 y) \le c_{12}|y|^{-\tau} < c_{12}2^\tau.$$

By Harnack's inequality (see, e.g., [Trudinger 1980; 1967]) there exists $c_{32} = c_{32}(\alpha, \beta, p, q, N, d_1, d_2)$ such that

$$\sup\{\varphi_{r_0}(y) : y \in B_{1/8}(y_0)\} \le c_{32} \inf\{\varphi_{r_0}(y) : y \in B_{1/8}(y_0)\}.$$

As ∂B_δ can be covered by a finite number (depending only on N) of balls of center on ∂B_δ and of radius $\frac{1}{4}\delta$, we obtain (5-1). □

Proof of Theorem C. The proof is based on Lemma 5.1, scaling argument and asymptotic behavior of weakly singular solutions and strongly singular solutions. Put

$$L := \limsup_{|x|\to 0} \frac{u(x)}{\Gamma_N(x)} \ge 0. \tag{5-2}$$

Case 1: $L = 0$. Then for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\delta \to 0$ as $\epsilon \to 0$ and $u \le \epsilon\Gamma_N$ on ∂B_δ. Thanks to Proposition 2.1, $u \le \epsilon\Gamma_N$ in $\Omega \setminus B_\delta$. Letting $\epsilon \to 0$ yields $u \equiv 0$.

Case 2: $L = \infty$. By (5-1),

$$\liminf_{|x|\to 0} \frac{u(x)}{\Gamma_N(x)} = \infty,$$

which along with (1-16) implies

$$\liminf_{|x|\to 0} \frac{u(x)}{u_k^\Omega(x)} = \infty \quad \text{for all } k > 0.$$

By the comparison principle, $u \ge u_k^\Omega$ in $\Omega \setminus \{0\}$ for every $k > 0$. Hence $u \ge u_\infty^\Omega$ in $\Omega \setminus \{0\}$. Consequently, by Theorem B, we derive

$$\liminf_{|x|\to 0} |x|^\tau u(x) \ge \lim_{|x|\to 0} |x|^\tau u_\infty^\Omega(x) = \Theta. \tag{5-3}$$

We next prove that[1]

$$\limsup_{|x|\to 0} |x|^\tau u(x) \le \Theta. \tag{5-4}$$

For any $\epsilon > 0$, it can be checked that there exists $\Theta_\epsilon > 0$ with $\Theta_\epsilon \to \Theta$ as $\epsilon \to 0$ such that $\Theta_\epsilon |x|^{-\tau-\epsilon}$ is a supersolution of (1-1) in $B_{d_1} \setminus \{0\}$ when $D = 1$ (respectively, of (1-2) in $B_{d_1} \setminus \{0\}$ when $D < 1$ and of

[1] The proof of (5-4) was proposed by a referee.

(1-3) in $B_{d_1} \setminus \{0\}$ when $D > 1$). Then by (2-9) and the comparison principle, we find that

$$u(x) \leq \Theta_\epsilon |x|^{-\tau-\epsilon} + \max\nolimits_{\partial B_{d_1}} u$$

in $B_{d_1} \setminus \{0\}$ for every $\epsilon > 0$. Letting $\epsilon \to 0$ for fixed $x \in B_{d_1} \setminus \{0\}$, then $|x| \to 0$, we obtain (5-4).

Case 3: $0 < L < \infty$. In light of (5-1), there is a positive number k such that

$$\liminf_{|x|\to 0} \frac{u(x)}{\Gamma_N(x)} = k > c_{34}^{-1} L, \tag{5-5}$$

here $c_{34} = c_{34}(N, \alpha, \beta, p, q, d_1, d_2) > 1$, which implies

$$\liminf_{|x|\to 0} \frac{u(x)}{u_k^\Omega(x)} = 1. \tag{5-6}$$

By Proposition 2.1, $u \geq u_k^\Omega$ in $\Omega \setminus \{0\}$. From (5-6), there exists a sequence $\{x_n\}$ converging to 0 such that

$$\lim_{n\to\infty} \frac{u(x_n)}{u_k^\Omega(x_n)} = 1.$$

Put $r_n = |x_n|$, $v_{k,n} = R_{r_n}[u_k^\Omega]$ and $v_n = R_{r_n}[u]$ in $\Omega_{r_n} = r_n^{-1}\Omega$. Then both $v_{k,n}$ and v_n are solutions of

$$-\Delta v + r_n^{N+\alpha-p(N-2)} |x|^\alpha v^p + r_n^{N+\beta-q(N-1)} |x|^\beta |\nabla v|^q = 0 \quad \text{in } \Omega_{r_n} \setminus \{0\}.$$

By the Arzelà–Ascoli theorem, regularity theory of elliptic equations and a standard diagonalization argument, up to subsequences, $\{v_{k,n}\}$ and $\{v_n\}$ converge respectively in $C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ to nonnegative harmonic functions V_k^* and V^* in $\mathbb{R}^N \setminus \{0\}$. Since $u \geq u_k^\Omega$, it follows that $V^* \geq V_k^*$. Put

$$\kappa_n = \sup\Big\{ \frac{u(x)}{u_k^\Omega(x)} : x \in \partial B_{r_n} \Big\} \in [1, c_{34}]$$

and $y_n = r_n^{-1} x_n \in \partial B_1$. Therefore, up to subsequences, $\kappa_n \to \kappa \in [1, c_{34}]$ and $y_n \to y^* \in \partial B_1$. Consequently, $V^*(y^*) = V_k^*(y^*)$. By the strong maximum principle, we deduce that $V^* = V_k^*$ in $\mathbb{R}^N \setminus \{0\}$, which implies $\kappa = 1$. Thus, for every $\epsilon > 0$, there exists $n_\epsilon > 0$ such that

$$n \geq n_\epsilon \implies u_k^\Omega \leq u \leq (1+\epsilon) u_k^\Omega \quad \text{in } \partial B_{r_n}.$$

The comparison principle implies $u \leq (1+\epsilon) u_k^\Omega$ in $\Omega \setminus B_{r_n}$. Letting $\epsilon \to 0$ yields $u \leq u_k^\Omega$ in $\Omega \setminus \{0\}$. Thus $u \equiv u_k^\Omega$. □

5.2. *Removability.* We shall treat successively two cases: $q_{c,\beta} \leq q < 2+\beta$ and $q = 2+\beta$.

Proof of Theorem D with $q_{c,\beta} \leq q < 2+\beta$. The proof is divided into three cases and strongly based upon Proposition 4.1 and self-similarity arguments.

Case 1: If $D = 1$ then $p \geq p_{c,\alpha}$ and $q \geq q_{c,\beta}$. For $0 < \delta < \frac{1}{2}d_1$ and $R > d_2 = \operatorname{diam}(\Omega)$, let $u_{\delta,R}$ be the solution of

$$\begin{cases} -\Delta u + F \circ u = 0 & \text{in } B_R \setminus \overline{B_\delta}, \\ \qquad u = c_{33}\delta^{-\tau} & \text{on } \partial B_\delta, \\ \qquad u = 0 & \text{on } \partial B_R, \end{cases} \tag{5-7}$$

where $c_{33} = \max\{c_8, c_{12}, \Theta\}$. By the comparison principle, $u \leq u_{\delta,R} \leq u_{\delta',R'}$ in $\Omega \setminus B_{\delta'}$ for every $0 < \delta \leq \delta'$ and $0 < R \leq R'$. Put $\tilde{u} := \lim_{R\to\infty} \lim_{\delta\to 0} u_{\delta,R}$; then $\tilde{u}$ is a solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$ and $u \leq \tilde{u}$ in $\Omega \setminus \{0\}$. By uniqueness, $T_\ell[u_{\delta,R}] = u_{\delta/\ell,R/\ell}$ for every $\ell > 0$. Letting $\delta \to 0$ and $R \to \infty$ successively implies $T_\ell[\tilde{u}] = \tilde{u}$ for every $\ell > 0$. Hence $\tilde{u}$ is a self-similar solution of (1-1) in $\mathbb{R}^N \setminus \{0\}$ and can be represented in the form

$$\tilde{u}(x) = |x|^{-\frac{2+\beta-q}{q-1}} \omega(x/|x|) \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\},$$

where ω is a solution of (4-1). Since $q_{c,\beta} \leq q < 2 + \beta$, from Proposition 4.1 we deduce that $\omega \equiv 0$. It follows that $\tilde{u} \equiv 0$ and thus $u \equiv 0$.

Case 2: If $D > 1$ then we must have $q \geq q_{c,\beta}$. For any $0 < \delta < R$, let $w_{\delta,R}$ be the solution of

$$\begin{cases} -\Delta w + |x|^\beta |\nabla w|^q = 0 & \text{in } B_R \setminus \overline{B_\delta}, \\ \qquad w = c_{33}\delta^{-\frac{2+\beta-q}{q-1}} & \text{on } \partial B_\delta, \\ \qquad w = & \text{on } \partial B_R. \end{cases} \tag{5-8}$$

By the comparison principle, $u \leq w_{\delta,R} \leq w_{\delta',R'}$ in $\Omega \setminus B_{\delta'}$ for every $0 < \delta \leq \delta'$ and $0 < R \leq R'$. Put $\tilde{w} := \lim_{R\to\infty} \lim_{\delta\to 0} w_{\delta,R}$ then $\tilde{w}$ is a solution of (1-3) in $\mathbb{R}^N \setminus \{0\}$ and $u \leq \tilde{w}$ in $\Omega \setminus \{0\}$. By uniqueness, $T_\ell[w_{\delta,R}] = w_{\delta/\ell,R/\ell}$ for every $\ell > 0$. Letting $\delta \to 0$ and $R \to \infty$ successively implies $T_\ell[\tilde{w}] = \tilde{w}$ for every $\ell > 0$. Hence $\tilde{w}$ is a self-similar solution of (1-3) in $\mathbb{R}^N \setminus \{0\}$ and can be represented in the form

$$\tilde{w}(x) = |x|^{-\frac{2+\beta-q}{q-1}} \omega(x/|x|) \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\},$$

where ω is a solution of (4-1) with $\lambda = 0$. Since $q_{c,\beta} \leq q < 2 + \beta$, from Proposition 4.1 we deduce that $\omega \equiv 0$. It follows that $\tilde{w} \equiv 0$ and thus $u \equiv 0$.

Case 3: If $D < 1$ then we must have $p \geq p_{c,\alpha}$. One can use an argument similar to the proof in Case 2 to obtain $u \equiv 0$. □

Remark. Theorem D with $q < 2 + \beta$ can be obtained by a different way which is suggested by the referee. The proof, that we present below, is more direct, independent of Proposition 4.1 and does not require any self-similarity arguments.

Assume that either $p \geq p_{c,\alpha}$ or $q \geq q_{c,\beta}$. We distinguish two cases:

Case 1: If $D \geq 1$ then we must have $q \geq q_{c,\beta}$.

Case 2: If $D < 1$ then we must have $p \geq p_{c,\alpha}$.

If $q > q_{c,\beta}$ in Case 1 or $p > p_{c,\alpha}$ in Case 2, then by (1-13) and (2-9), we deduce that

$$\lim_{|x|\to 0} \frac{u(x)}{\Gamma_N(x)} = 0.$$

Since $u = 0$ on $\partial\Omega$, the comparison principle gives that $u \equiv 0$ in $\Omega \setminus \{0\}$.

If $q = q_{c,\beta}$ in Case 1 or $p = p_{c,\alpha}$ in Case 2 then by (1-13) and (2-9), we deduce that

$$\lim_{|x|\to 0} \frac{u(x)}{\Gamma_N(x)} < \infty.$$

For every $\epsilon > 0$ small, it can be easily checked that there exists $C_\epsilon > 0$ with $C_\epsilon \to 0$ as $\epsilon \to 0$ such that $S_\epsilon(x) := C_\epsilon |x|^{2-N-\epsilon}$ is a supersolution of (1-3) in $B_1 \setminus \{0\}$ when $q = q_{c,\beta}$ in Case 1 (respectively, a supersolution of (1-2) in $B_1 \setminus \{0\}$ when $p = p_{c,\alpha}$ in Case 2). Since

$$\lim_{|x|\to 0} \frac{u(x)}{S_\epsilon(x)} = 0,$$

by the comparison principle, $u(x) \le S_\epsilon(x) + \max_{\partial B_{d_1}} u$ in $B_{d_1} \setminus \{0\}$. Letting $\epsilon \to 0$, we get $u \le \max_{\partial B_{d_1}} u$. Since $u = 0$ on $\partial\Omega \setminus \{0\}$, we find that $u \equiv 0$ in $\Omega \setminus \{0\}$.

In order to prove Theorem D in the case $q = 2 + \beta$ we need the following lemma.

Lemma 5.2. *Let $\beta > -1$. If $w \in C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega} \setminus \{0\})$ is a nonnegative solution of*

$$-\Delta w + |x|^\beta |\nabla w|^{2+\beta} = 0 \quad \textit{in } \Omega \setminus \{0\}, \tag{5-9}$$

which vanishes on $\partial\Omega$ then $w \equiv 0$.

Proof. By (2-3), there exists a positive constant $c_{35} = c_{35}(N, q, \beta, d_1, d_2, \|w\|_{L^\infty(\partial B_{d_1})})$ such that $w(x) \le c_{35} - c_3 \ln|x|$ in $B_{d_1} \setminus \{0\}$. The constant c_{35} can be chosen such that $\Phi(x) := c_{35} - c_3 \ln|x|$ is a positive superharmonic function in $\Omega \setminus \{0\}$.

For $\epsilon \in (0, d_1)$, let h_ϵ be the harmonic function in $\Omega \setminus B_\epsilon$ such that $h_\epsilon = w$ on ∂B_ϵ and $h_\epsilon = 0$ on $\partial\Omega$. By the comparison principle, $w \le h_\epsilon$ in $\Omega \setminus B_\epsilon$ for every $\epsilon \in (0, d_1)$. Consequently, $h_\epsilon \le h_{\epsilon'}$ for $0 < \epsilon' < \epsilon$. On the other hand, since Φ is a positive superharmonic function in $\Omega \setminus B_\epsilon$ which dominates h_ϵ on $\partial\Omega \cup \partial B_\epsilon$, by the comparison principle, $h_\epsilon \le \Phi$ in $\Omega \setminus B_\epsilon$. Therefore, $\{h_\epsilon\}$ converges, as $\epsilon \to 0$, to a harmonic function $\hat{h}$ in $\Omega \setminus \{0\}$ which vanishes on $\partial\Omega$ and satisfies $w \le \hat{h} \le \Phi$ in $\Omega \setminus \{0\}$. Since $N > 2$, we deduce that $\hat{h}(x) = o(\Gamma_N(x))$ as $|x| \to 0$. Therefore $\hat{h} \equiv 0$. Thus $w \equiv 0$. □

Proof of Theorem D with $q = 2 + \beta$.

For $\epsilon \in (0, d_1)$, let w_ϵ be the solution of (2-10) with $q = 2 + \beta$. The sequence $\{w_\epsilon\}$ converges, as $\epsilon \to 0$, to a solution $\hat{w}$ of (5-9) in $\Omega \setminus \{0\}$ which vanishes on $\partial\Omega$. Since $u \le w_\epsilon$ for every $\epsilon \in (0, d_1)$, it follows that $u \le \hat{w}$. By Lemma 5.2, $\hat{w} \equiv 0$ and thus $u \equiv 0$. □

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