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# ESTIMATES FOR RADIAL SOLUTIONS OF THE HOMOGENEOUS LANDAU EQUATION WITH COULOMB POTENTIAL

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Motivated by the question of existence of global solutions, we obtain pointwise upper bounds for radially symmetric and monotone solutions to the homogeneous Landau equation with Coulomb potential. The estimates say that blow-up in the  $L^\infty$  norm at some finite time  $T$  occurs only if a certain quotient involving  $f$  and its Newtonian potential concentrates near zero, which implies blow-up in more standard norms, such as the  $L^{3/2}$  norm. This quotient is shown to be always less than a universal constant, suggesting that the problem of regularity for the Landau equation is in some sense critical.

The bounds are obtained using the comparison principle both for the Landau equation and for the associated mass function. In particular, the method provides long-time existence results for a modified version of the Landau equation with Coulomb potential, recently introduced by Krieger and Strain.

## 1. Introduction

This manuscript is concerned with the Cauchy problem for the homogeneous Landau equation. This equation takes the general form

$$\partial_t f(v, t) = Q(f, f), \quad f(v, 0) = f_{\text{in}}(v), \quad v \in \mathbb{R}^3, \quad t > 0, \quad (1-1)$$

where  $Q(f, f)$  is a quadratic operator known as the Landau collision operator:

$$Q(f, f) = \operatorname{div} \left( \int_{\mathbb{R}^3} A(v-y) (f(y) \nabla_v f(v) - f(v) \nabla_y f(y)) dy \right). \quad (1-2)$$

The term  $A(v)$  denotes a positive and symmetric matrix

$$A(v) := C_\gamma \left( \mathbb{I} - \frac{v \otimes v}{|v|^2} \right) \varphi(|v|), \quad v \neq 0, \quad C_\gamma > 0,$$

which acts as the projection operator onto the space orthogonal to the vector  $v$ . The function  $\varphi(|v|)$  is a scalar-valued function determined from the original Boltzmann kernel describing how particles interact. If the interaction strength between particles at a distance  $r$  is proportional to  $r^{1-s}$ , then

$$\varphi(|v|) := |v|^{\gamma+2}, \quad \gamma = \frac{s-5}{s-1}. \quad (1-3)$$

Note that  $s = 2$  corresponds to the Coulomb potential, in which case we have  $\gamma = -3$  [Villani 2002, Chapter 1, Section 1.4]. Any solution to (1-1)–(1-2) is an integrable and nonnegative scalar field

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$f(v, t) : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}^+$ . Equation (1-1) describes the evolution of a plasma in spatially homogeneous regimes, which means that the density function  $f$  depends only on the velocity component  $v$ . Landau’s original intent in deriving this approximation was to make sense of the Boltzmann collision operator, which always diverges when considering purely grazing collisions.

The Cauchy problem for (1-1)–(1-3) is very well understood for the case of hard potentials, which correspond to  $\gamma \geq 0$  above. Desvillettes and Villani showed the existence of global classical solutions for hard potentials and studied its long-time behavior; see [Desvillettes and Villani 2000a; 2000b; Villani 2002] and references therein. In this case there is a unique global smooth solution, which converges exponentially to an equilibrium distribution, known as the Maxwellian function

$$\mathcal{M}(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}.$$

Analyzing the soft potentials case,  $\gamma < 0$ , has proved to be more difficult. Using a probabilistic approach, [Wu 2014; Fournier and Guérin 2009; Alexandre et al. 2015] show uniqueness and existence of weak solutions for  $\gamma \in [-2, 0]$ . For  $\gamma \in [-3, -2]$ , existence is known for small-time or global in-time with smallness assumption on initial data [Alexandre et al. 2015; Arsen’ev and Peskov 1977]. Finally, for the Coulomb case  $\gamma = -3$ , Fournier [2010] showed the uniqueness of weak solutions as long as they remain in  $L^\infty$ .

Villani [1998] introduced the so called  $H$ -solutions, which enjoy (weak) a priori bounds in a weighted Sobolev space. However, the issue of their uniqueness and regularity (i.e., no finite-time breakdown occurs) has remained open, even for smooth initial data; see [Villani 2002, Chapters 1 and 5] for further discussion.

Guo [2002] employed a completely different approach based on perturbation theory for the existence of periodic solutions to the spatially inhomogeneous Landau equation in  $\mathbb{R}^3$ . He showed that if the initial data is sufficiently close to the unique equilibrium in a certain high Sobolev norm, then a unique global solution exists. Moreover, as remarked in [loc. cit.], this approach also extends to the case of potentials (1-3), where  $\gamma$  might even take values below  $-3$ .

Due to the lack of a global well-posedness theory, several conjectures about possible finite-time blow-up for general initial data have been made throughout the years. Villani [2002] discussed the possibility that (1-1)–(1-3) could blow up for  $\gamma = -3$ . Note that for smooth solutions, (1-1)–(1-3) with  $\gamma = -3$  can be rewritten as

$$\partial_t f = \operatorname{div}(A[f]\nabla f - f\nabla a[f]) = \operatorname{Tr}(A[f]D^2 f) + f^2, \tag{1-4}$$

where

$$A[f] := A(v) * f = \frac{1}{8\pi|v|} \left( \mathbb{I} - \frac{v \otimes v}{|v|^2} \right) * f, \quad \Delta a = -f.$$

Equation (1-4) can be thought of as a quasilinear nonlocal heat equation. Support for blow-up conjectures were given by the fact that (1-4) is reminiscent of the well studied semilinear heat equation

$$\partial_t f = \Delta f + f^2. \tag{1-5}$$

Blow-up for (1-5) is known to happen for every  $L^p$  norm for  $p > \frac{3}{2}$ ; see [Giga and Kohn 1985].

However, despite the apparent similarities, (1-4) behaves differently from (1-5). The Landau equation admits a richer class of equilibrium solution: every Maxwellian  $\mathcal{M}$  solves  $Q(\mathcal{M}, \mathcal{M}) = 0$ , which holds, in particular, for those with arbitrarily large mass.

From a different perspective, Krieger and Strain [2012] considered a modified version of (1-4),

$$\partial_t f = a[f]\Delta f + \alpha f^2, \quad (1-6)$$

and showed global existence of smooth radial solutions starting from radial initial data when  $\alpha < \frac{2}{3}$ . This range for  $\alpha$  later was expanded to any  $\alpha < \frac{74}{75}$  by means of a nonlocal inequality obtained by Gressman, Krieger and Strain [Gressman et al. 2012]. Note that when  $\alpha = 1$ , the above equation can be written in divergence form,

$$\partial_t f = \operatorname{div}(a[f]\nabla f - f\nabla a[f]). \quad (1-7)$$

These results put in evidence how a nonlinear equation with a nonlocal diffusivity such as (1-7) behaves drastically differently from (and better than) (1-5).

Our main results in this manuscript are twofold. The first one gives necessary conditions for the finite-time blow-up of solutions to (1-4). The second (unconditional) result says that solutions to (1-7) do not blow up at all, and in fact become instantaneously smooth (even for initial data that might be initially unbounded). Both results deal only with radially symmetric, decreasing initial conditions; more precisely, we assume that

$$\begin{aligned} f_{\text{in}} &\geq 0, \quad f_{\text{in}} \in L^\infty(\mathbb{R}^3), \\ \int_{\mathbb{R}^3} f_{\text{in}} dv &= 1, \quad \int_{\mathbb{R}^3} f_{\text{in}} |v|^2 dv = 3, \quad \int_{\mathbb{R}^3} f_{\text{in}} \log(f_{\text{in}}) dv < \infty, \\ |v| \leq |w| &\Rightarrow f_{\text{in}}(v) \geq f_{\text{in}}(w). \end{aligned} \quad (1-8)$$

The normalization of the initial data is standard and follows a standard change of variables. The main results are the following.

**Theorem 1.1.** *Let  $f_{\text{in}}$  be as in (1-8). Then there exist  $T_0 > 0$  and  $f : \mathbb{R}^3 \times (0, T_0) \rightarrow \mathbb{R}_+$  such that  $f$  is smooth and solves (1-4) for  $t \in (0, T_0)$ , with  $f(\cdot, 0) = f_{\text{in}}$ . Moreover,  $T_0$  is maximal in the sense that either  $T_0 = \infty$  or else the  $L^{3/2}$  norm of  $f$  accumulates near  $v = 0$  as  $t \rightarrow T_0^-$ , in particular*

$$\lim_{t \rightarrow T_0^-} \|f(\cdot, t)\|_{L^p(B_1)} = \infty, \quad \forall p > \frac{3}{2}.$$

In fact, the above theorem is a consequence of the following sharper result.

**Theorem 1.2.** *There is a constant  $\varepsilon_0 \geq \frac{1}{96}$  such that if  $T_0 < \infty$ , then*

$$\limsup_{r \rightarrow 0^+} \sup_{t \in (0, T_0)} \left\{ r^2 \frac{\int_{B_r} f(v, t) dv}{\int_{B_r} a[f](v, t) dv} \right\} \geq \varepsilon_0.$$

Neither of the above theorems are enough to guarantee long-time existence of classical solutions to (1-4). However, Theorem 1.2 suggests that (1-4) is in some sense “critical” for regularity. It can be

shown (see Proposition 5.6) that for any nonnegative  $f \in L^1(\mathbb{R}^3)$ ,

$$r^2 \frac{\int_{B_r} f(v) dv}{\int_{B_r} a[f](v) dv} \leq 3, \quad \forall r > 0.$$

In particular, if the  $\varepsilon_0$  in Theorem 1.2 could be shown to be at least 3 (or in general if the upper bound in the last inequality could be improved to something less than  $\varepsilon_0$ ), it would immediately follow that solutions to the Landau equation (1-4) cannot blow up in finite time. It is not clear if this can be guaranteed for general  $f$  without at least using some partial time regularization.

On the other hand, methods used in the proof of Theorem 1.1 and Theorem 1.2 yield long-time existence for the modified Landau equation (1-7) (again, in the radial case).

**Theorem 1.3.** *Let  $f_{\text{in}}$  be as in (1-8) and such that for some  $p > 6$ ,*

$$f_{\text{in}} \in L^p_{\text{weak}}(\mathbb{R}^3).$$

*Then there exists a function  $f : \mathbb{R}^3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$ , smooth for positive times, with  $f(\cdot, 0) = f_{\text{in}}$  which solves, for  $t > 0$ ,*

$$\partial_t f = a[f]\Delta f + f^2.$$

We approach the analysis from the point of view of nonlinear parabolic equations. The nonlocal dependence of the coefficients on the solution prevents the equation from satisfying a comparison principle: if  $v_0$  is a contact point of two functions  $f$  and  $g$ , i.e.,  $f(v_0) = g(v_0)$  and everywhere else  $f(v) \leq g(v)$ , it does not follow that  $Q(f, f)(v_0) \leq Q(g, g)(v_0)$ . More precisely, for the case where  $Q(f, f)$  corresponds to (1-2) one cannot expect an inequality such as

$$\text{Tr}(A[f]D^2f)(v_0) \leq \text{Tr}(A[g]D^2g)(v_0).$$

In fact, due to the nonlocality of  $A$  one only has  $A[f](v_0) \leq A[g](v_0)$ . Equality  $A[f](v_0) = A[g](v_0)$  holds only when  $f \equiv g$  for every  $v \in \mathbb{R}^3$ . The maximum principle is not useful either, since at a maximum point for  $f$  we only obtain  $\partial_t f \leq -f \Delta a[f]$ , which does not rule out growth of the maximum of  $f$ . The same observations apply to  $Q(f, f)$  corresponding to (1-7).

On the other hand, if one could construct (using only properties of  $f$  that are independent of  $t$ ) a function  $U(v)$  such that

$$\begin{aligned} \text{Tr}(A[f]D^2U) + fU &\leq 0 \quad \text{in } \mathbb{R}^3, \\ a[f]\Delta U + fU &\leq 0 \quad \text{in } \mathbb{R}^3, \end{aligned}$$

then the comparison principle (for linear parabolic equations) would guarantee that  $f \leq cU$  for all times provided  $f(t = 0) \leq cU$ . Our main observation is that (under radial symmetry) the above can be made to work with  $U(v) = |v|^{-\gamma}$ ,  $\gamma \in (0, 1)$ . From here higher local integrability of  $f$  can be propagated, and from there higher regularity follows by standard elliptic regularization.

A previous attempt by the authors, also based on upper barrier arguments (but meant to cover any bounded, fast decaying initial data), was ultimately undone by a computational error. However, Theorems 1.1–1.3 show that the use of upper barriers to study (1-4) is fruitful at least for radially symmetric and

decreasing initial conditions. On the other hand, the authors in [Gualdani and Guillen  $\geq$  2016] show a local  $L^\infty$ -regularization estimate using the De Giorgi iteration method for  $\gamma > -2$ .

**Remark 1.4.** After the submission of this article, the authors learned of related work of Silvestre [2016] on the Boltzmann equation, covering the spatially inhomogeneous case. In that paper, a priori estimates rely on maximum principle arguments and make use of the regularity for parabolic integro-differential equations, particularly recent work of Schwab and Silvestre [2016].

**Outline.** The rest of the paper is organized as follows. After a brief review in Section 2 on nonlinear parabolic theory that will be needed to construct local solutions to the nonlinear problems, in Section 3 we outline the symmetry properties of (1-4). Section 4 deals with short-time existence. In Section 5 we present a barrier argument that allows us to prove conditional non-blow-up results for the Landau equation and global well-posedness for the modified Landau equation in Section 6.

**Notation.** Universal constants will be denoted by  $c, c_0, c_1, C_0, C_1, C$ . Vectors in  $\mathbb{R}^3$  will be denoted by  $v, w, x, y$  and so on. The inner product between  $v$  and  $w$  will be written  $(v, w)$ .  $B_R(v_0)$  will denote the closed ball of radius  $R$  centered at  $v_0$ ; if  $v_0 = 0$  we simply write  $B_R$ . The identity matrix will be denoted by  $\mathbb{I}$ , the trace of a matrix  $X$  will be denoted  $\text{Tr}(X)$ . The initial condition for the Cauchy problem will always be denoted by  $f_{\text{in}}$ .

The letter  $\Omega$  will denote a general compact subset of  $\mathbb{R}^3$ .  $Q \subset \mathbb{R}^3 \times \mathbb{R}_+$  will be a space-time cylinder of parabolic diameter  $R$  with  $R > 0$  a general constant, unless otherwise specified. Finally,  $\partial_p Q$  will denote the parabolic boundary of  $Q$ .

## 2. A rapid review of linear parabolic equations

We work with two bilinear operators, namely the one associated to (1-4),

$$Q_{\mathcal{L}}(g, f) := \text{div}(A[g]\nabla f - f\nabla a[g]) = \text{Tr}[A[g]D^2 f] + fg,$$

and the one associated to (1-7),

$$Q_{\mathcal{KS}}(g, f) := \text{div}(a[g]\nabla f - f\nabla a[g]) = a[g]\Delta f + fg.$$

As is well known, through  $Q_{\mathcal{L}}$  (and also  $Q_{\mathcal{KS}}$ ), any  $g : \mathbb{R}^3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$  gives rise to a linear elliptic operator with variable coefficients as follows:

$$\begin{aligned} \phi \rightarrow Q_{\mathcal{L}}(g, \phi) &:= \text{div}(A[g]\nabla \phi - \phi\nabla a[g]) = \text{Tr}(A[g]D^2 \phi) + \phi g, \\ \phi \rightarrow Q_{\mathcal{KS}}(g, \phi) &:= \text{div}(a[g]\nabla \phi - \phi\nabla a[g]) = a[g]\Delta \phi + \phi g. \end{aligned}$$

Accordingly, given such a  $g$  and initial data  $f_{\text{in}}$ , one considers the linear Cauchy problem,

$$\begin{cases} \partial_t f = Q(g, f) & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ f(\cdot, 0) = f_{\text{in}}, \end{cases} \quad (2-1)$$

both for  $Q = Q_{\mathcal{L}}$  and  $Q = Q_{\mathcal{KS}}$ .

**Remark.** Note that  $Q_{\mathcal{L}}(g, f)$  and  $Q_{\mathcal{KS}}(g, f)$  can both be expressed as a divergence, so any solution to (2-1) preserves its mass over time, i.e.,

$$\|f(\cdot, t)\|_{L^1(\mathbb{R}^3)} = \|f_{\text{in}}(\cdot)\|_{L^1(\mathbb{R}^3)} =: M_{\text{in}}, \quad \forall t > 0.$$

**Lemma 2.1** (see [Ladyženskaja et al. 1968, Theorem 5.1, page 320]). *Let  $f_{\text{in}} : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^3 \times \mathbb{R}_+ \rightarrow \mathbb{R}$  be nonnegative functions such that for some  $\beta \in (0, 1)$  we have*

$$\begin{aligned} f_{\text{in}} &\in L^1(\mathbb{R}^3) \cap C^{2+\beta}(\mathbb{R}^3), \\ A[g], \nabla a[g] &\in C^{\beta, \beta/2}(\mathbb{R}^3 \times \mathbb{R}_+). \end{aligned} \tag{2-2}$$

*Then for every  $\delta > 0$ , there exists a unique  $f : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$  with  $f \in C^{2+\beta, 1+\beta/2}(\mathbb{R}^3 \times \mathbb{R}_+)$  which is a classical solution of*

$$\begin{cases} \partial_t f = \delta \Delta f + Q(g, f) & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ f(\cdot, 0) = f_{\text{in}}, \end{cases} \tag{2-3}$$

where  $Q(\cdot, \cdot)$  denotes either  $Q = Q_{\mathcal{L}}$  or  $Q = Q_{\mathcal{KS}}$ .

Next we summarize in three theorems several classical local regularity estimates for parabolic equations of the form

$$\partial_t f = \text{div}(B \nabla f + fb),$$

where  $f : Q \rightarrow \mathbb{R}$  and  $Q = B_R(v_0) \times (t_0 - R^2, t_0) \subset \mathbb{R}^d \times \mathbb{R}_+$  is the parabolic cylinder of radius  $R$  centered at some points  $x_0, t_0$ . The first two theorems are, respectively, a local Hölder estimate (from De Giorgi–Nash–Moser theory) and an  $L^\infty$  estimate for  $f$  in terms of its boundary data (Stampacchia estimate); see [Ladyženskaja et al. 1968, Chapter III, Theorem 10.1, page 204 and Chapter IV, Theorem 10.1, page 351 of] as well as [Lieberman 1996, Chapter VI, Theorem 6.29, page 131] for the respective proofs. The main point of these theorems is that they do not require any regularity assumption on the diffusion matrix  $B$  (beyond ellipticity and boundedness).

**Theorem 2.2** (De Giorgi–Nash–Moser estimate). *Suppose  $f$  is a weak solution of the equation*

$$\partial_t f = \text{div}(B \nabla f + fb),$$

where  $b$  is a vector field and  $B$  is a symmetric matrix such that

$$\lambda \mathbb{I} \leq B(v, t) \leq \Lambda \mathbb{I} \quad \text{a.e. in } Q.$$

Then there is some  $\alpha \in (0, 1)$  and  $C > 0$  such that the estimate

$$[f]_{C^{\alpha, \alpha/2}(Q_{1/2})} \leq C(\|f\|_{L^\infty(Q)} + R^2 \|b\|_{L^\infty(Q)}) \tag{2-4}$$

holds, where  $Q_{1/2} := B_{R/2}(x_0) \times (t_0 - (R/2)^2, t_0)$  and  $\alpha$  and  $C$  are determined by  $\lambda, \Lambda, R$  and  $d$ .

**Theorem 2.3** (Stampacchia estimate). *If  $f$  is a weak solution of*

$$\partial_t f \leq \text{div}(B \nabla f + fb),$$



with  $B$  and  $b$  as in the previous theorem, then there exists a constant  $C > 0$  such that

$$\|f\|_{L^\infty(Q)} \leq C(\|f\|_{L^2(Q)} + \|b\|_{L^\infty(Q)}). \quad (2-5)$$

As before,  $C$  is determined by  $\lambda$ ,  $\Lambda$ ,  $d$  and  $R$ .

The last theorem recalls interior classical regularity estimates when the coefficients are Hölder continuous in time and space. See [Ladyženskaja et al. 1968, Chapter IV] or also [Lieberman 1996, Chapter III, Theorem 6.17] for a proof.

**Theorem 2.4** (Schauder estimates). *If  $B, b \in C^{\beta, \beta/2}(Q)$ , then there is a finite  $C$  such that*

$$[D^2 f]_{C^{\beta, \beta/2}(Q_{1/2})} + [\partial_t f]_{C^{\beta, \beta/2}(Q_{1/2})} \leq C(\lambda, \Lambda, R, \|B\|_{C^{\beta, \beta/2}(Q)}, \|b\|_{C^{\beta, \beta/2}(Q)}, \|f\|_{L^\infty(Q)}).$$

### 3. Radial symmetry

This section is devoted to some technical lemmas. The proofs of the first two propositions are rather technical and can be found in the Appendix.

**Proposition 3.1.** *Suppose  $f_{\text{in}}$  and  $g(\cdot, t)$  are both radially symmetric, and let  $Q(\cdot, \cdot)$  denote either  $Q_{\mathcal{L}}$  or  $Q_{\mathcal{KS}}$ . Then any solution of the linear Cauchy problem*

$$\partial_t f = Q(g, f), \quad f(v, 0) = f_{\text{in}}(v),$$

*is radially symmetric for all  $t$ . Furthermore, if  $f_{\text{in}}$  and  $g$  are radially decreasing, then so is  $f$ .*

Let  $h : \mathbb{R}^3 \rightarrow \mathbb{R}_+$ . Define

$$A^*[h](v) := (A[h](v)\hat{v}, \hat{v}), \quad v \neq 0, \quad \hat{v} := v|v|^{-1}. \quad (3-1)$$

There are two useful expressions for  $A^*[h]$  and  $a[h]$  when  $h$  is radially symmetric.

**Proposition 3.2.** *Let  $h \in L^1(\mathbb{R}^3)$  be radially symmetric and nonnegative. Then*

$$A^*[h](v) = \frac{1}{12\pi|v|^3} \int_{B_{|v|}} h(w)|w|^2 dw + \frac{1}{12\pi} \int_{B_{|v|}^c} \frac{h(w)}{|w|} dw, \quad (3-2)$$

$$a[h](v) = \frac{1}{4\pi|v|} \int_{B_{|v|}} h(w) dw + \frac{1}{4\pi} \int_{B_{|v|}^c} \frac{h(w)}{|w|} dw. \quad (3-3)$$

The second formula above is simply the classical formula for the Newtonian potential in the case of radial symmetry; the formula for  $A^*[h]$  is new and the proof can be found in the Appendix.

**Lemma 3.3.** *Let  $h \in L^1(\mathbb{R}^3)$  be a nonnegative, spherically symmetric function.*

(1) *If  $h$  is monotone decreasing with  $|v|$ , and*

$$\int_{B_{R_1} \setminus B_{R_0}} h dv \geq \theta > 0$$

for some  $\delta > 0$  and  $0 < R_0 < R_1$ , then

$$A[h](v) \geq \frac{\theta R_0^2}{12\pi(1 + R_1^3)} \frac{1}{1 + |v|^3} \mathbb{I}. \tag{3-4}$$

(2) If  $h$  is bounded, i.e., if  $\|h\|_{L^\infty(\mathbb{R}^3)} = h(0) < +\infty$ , then

$$A[h](v) \leq a[h] \mathbb{I} \leq 2 \left( \frac{\|h\|_{L^\infty(\mathbb{R}^3)} + \|h\|_{L^1(\mathbb{R}^3)}}{1 + |v|} \right) \mathbb{I}, \quad \forall v \in \mathbb{R}^3. \tag{3-5}$$

*Proof.* (1) Let  $A^*[h]$  be as in (3-2). If  $|v| \geq R_1$ , then

$$\begin{aligned} A^*[h](v) &\geq \frac{1}{12\pi|v|^3} \int_{B_{R_1}} h(w)|w|^2 dw \geq \frac{1}{12\pi|v|^3} \int_{B_{R_1} \setminus B_{R_0}} h(w)|w|^2 dw \\ &\geq \frac{R_0^2}{12\pi|v|^3} \int_{B_{R_1} \setminus B_{R_0}} h(w, t) dw \geq \frac{\theta R_0^2}{12\pi|v|^3}. \end{aligned}$$

Note that Proposition 3.2 guarantees that  $A^*[h]$  is radially decreasing. Thus,

$$A^*[h](v) \geq \frac{\theta R_0^2}{12\pi R_1^3}, \quad \forall v \in B_{R_1}.$$

Combining both estimates, we conclude that

$$A^*[h](v) \geq \frac{\theta R_0^2}{12\pi(1 + R_1^3)} \frac{1}{1 + |v|^3}.$$

(2) If  $h \in L^\infty$ , then we may use (3-3) to obtain the estimate

$$\begin{aligned} [h] &\leq a[h](v) \mathbb{I} \leq \left( \frac{h(0)}{4\pi|v|} \int_{B_{|v|}} dw + \frac{1}{4\pi} \int_{B_{|v|}^c} h(w) dw \right) \mathbb{I} \\ &\leq (\|h\|_{L^\infty(\mathbb{R}^3)} + \|h\|_{L^1(\mathbb{R}^3)}) \mathbb{I}, \quad \text{if } |v| \leq 1, \end{aligned}$$

and

$$A[h] \leq a[h](v) \mathbb{I} \leq \left( \frac{\|h\|_{L^1(\mathbb{R})}}{2\pi|v|} \right) \mathbb{I} \leq \left( \frac{\|h\|_{L^1(\mathbb{R})}}{1 + |v|} \right) \mathbb{I}, \quad \text{if } |v| \geq 1. \quad \square$$

**Proposition 3.4.** *Let  $h$  be a positive and radially symmetric and decreasing function. For any  $\gamma \in (0, 1)$ , define  $U_\gamma(v)$  as*

$$U_\gamma(v) := |v|^{-\gamma}.$$

Then for  $Q = Q_{\mathcal{L}}$  or  $Q = Q_{\mathcal{KS}}$ ,

$$Q(h, U_\gamma) \leq U_\gamma \left( -\frac{1}{3}\gamma(1 - \gamma)a[h]|v|^{-2} + h \right).$$

*Proof.* As  $U_\gamma$  is radial

$$\nabla U_\gamma(v) = U'_\gamma(v) \frac{v}{|v|}, \quad D^2 U_\gamma(v) = U''_\gamma(v) \frac{v}{|v|} \otimes \frac{v}{|v|} + U'_\gamma(v) \frac{1}{|v|} \left( \mathbb{I} - \frac{v}{|v|} \otimes \frac{v}{|v|} \right).$$

Thus, in the case  $Q = Q_{\mathcal{L}}$ ,

$$Q(h, U_\gamma) = \text{Tr}(A[h]D^2U_\gamma) + hU_\gamma = A^*[h]U_\gamma'' + \frac{a[h] - A^*[h]}{|v|}U_\gamma' + hU_\gamma.$$

In particular, since  $U_\gamma' = -\gamma r^{-1}U_\gamma$ ,  $U_\gamma'' = \gamma(\gamma + 1)|v|^{-2}U_\gamma$ , it follows that

$$Q_{\mathcal{L}}(h, U_\gamma) = U_\gamma(\gamma(\gamma + 1)A^*[h]|v|^{-2} - \gamma(a[h] - A^*[h])|v|^{-2} + h).$$

The thesis follows by noticing that  $A^*[h] \leq \frac{1}{3}a[h]$ .

For the case  $Q = Q_{\mathcal{KS}}$ , an analogous computation shows that

$$\begin{aligned} Q(h, U_\gamma) &= U_\gamma(-\gamma(1 - \gamma)a[h]|v|^{-2} + h) \\ &\leq U_\gamma(-\frac{1}{3}\gamma(1 - \gamma)a[h]|v|^{-2} + h), \end{aligned}$$

where in the last inequality we use  $\gamma \in (0, 1)$  and  $a[h] \geq 0$ . □

#### 4. Short-time existence

In this section,  $Q$  denotes either  $Q_{\mathcal{L}}$  or  $Q_{\mathcal{KS}}$ . For some nontrivial interval of existence  $[0, T)$ , a smooth solution to

$$\begin{cases} \partial_t f = Q(f, f) & \text{in } \mathbb{R}^3 \times [0, T), \\ f(\cdot, 0) = f_{\text{in}}, \end{cases}$$

will be obtained by taking the limit of a sequence of functions  $\{f_k\}_{k \geq 0}$  constructed recursively (as explained further below). The interval of existence  $[0, T)$  is maximal in the sense that either  $T = \infty$  or else the  $L^\infty$  norm of  $f(\cdot, t)$  blows up as  $t$  approaches  $T$ , so the classical solution cannot be extended to a longer time interval.

**Remark 4.1.** As mentioned in the introduction, existence and uniqueness of bounded weak solutions to (1-4) have been obtained, respectively, by Arsen'ev and Peskov [1977] and by Fournier [2010]. It is likely (but not at all obvious) that the method used in [Fournier 2010] will carry over to the case of the isotropic equation (1-7). Thus, for the sake of completeness, we provide in this section a detailed proof of existence (but not uniqueness) of a classical solution for the nonlinear problem that covers the isotropic equation. For completely classical solutions this is certainly new for the isotropic equation (1-7) with  $\alpha = 1$ , although the methods used in the proof—a priori estimates for linear equations, which yield compactness for a sequence of approximate solutions to the nonlinear problem—are fairly well known, but still somewhat different from the approach used in [Krieger and Strain 2012] for the case  $\alpha < \frac{3}{4}$ . Uniqueness for classical solutions of (1-4) is contained in Fournier's [2010] result, since classical solutions are in particular weak solutions, and as it was just mentioned above, it is likely that this result can be expanded to cover (1-7).

For technical reasons we first assume that  $f_{\text{in}}$  satisfies (1-8) and for some  $c > 0$ ,

$$f_{\text{in}} \in C^{2+\beta}(\mathbb{R}^3), \quad \|f_{\text{in}}\|_{C^{2+\beta}(B_1(v))} \leq \frac{c}{1 + |v|^5}, \quad \forall v \in \mathbb{R}^3. \quad (4-1)$$

The inequality yields a rate of decay for the second derivatives of  $f_{\text{in}}$  which somewhat simplifies the existence proof. The assumptions (4-1) are auxiliary, and will be removed (by an approximation argument) in the proof of Theorem 4.14 at the end of this section.

Fix  $\delta > 0$ . A sequence  $\{f_k^\delta\}_{k \geq 0}$  will be constructed recursively, so that for every  $k$ ,

$$f_k^\delta \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)) \cap C^{2+\beta, 1+\beta/2}(\mathbb{R}^3 \times \mathbb{R}_+) \tag{4-2}$$

for some  $\alpha \in (0, 1)$  independent of  $k$ . The construction is done as follows: First, we set  $f_0(v, t) := f_{\text{in}}(v)$  for all  $v$  and  $t > 0$ . Next, given  $f_{k-1}^\delta \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)) \cap C^{2+\beta, 1+\beta/2}(\mathbb{R}^3 \times \mathbb{R}_+)$ , define  $f_k^\delta$  as the unique classical solution to the linear Cauchy problem

$$\begin{cases} \partial_t f = \delta \Delta f + Q(f_{k-1}^\delta, f) & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \\ f(\cdot, 0) = f_{\text{in}}. \end{cases} \tag{4-3}$$

The fact that the sequence  $f_k^\delta$  is well defined and satisfies (4-2) follows by repeatedly applying Lemma 2.1, making use of the fact that for every  $k \geq 1$ ,  $\beta' \in (0, 1)$ ,

$$f_k^\delta \text{ satisfies (4-2) and solves (4-3)} \Rightarrow A[f_k^\delta], \nabla a[f_k^\delta] \in C^{\beta', \beta'/2}(\mathbb{R}^3 \times [0, \infty)). \tag{4-4}$$

That this is so is essentially a consequence of the fact that  $A[f_k^\delta]$  and  $\nabla a[f_k^\delta]$  are convolutions of  $f_k^\delta$  with relatively nice kernels; we do not write out the explicit proof of the above fact here, as the proof is essentially the same as that of Lemma 4.7, where a quantified version of the assertion (4-4) is proved. Thus, we have entirely constructed the sequence  $\{f_k^\delta\}_{k \geq 0}$ , each  $f_k^\delta$  being also radially symmetric and monotone, thanks to Proposition 3.1 and (1-8).

**Remark 4.2.** Note that, for the purpose of iteration in  $k$ , the coefficients  $A[f_{\text{in}}]$  and  $\nabla a[f_{\text{in}}]$  (which are independent of time) are Hölder continuous in space thanks to (4-1).

Once we have constructed the sequence  $\{f_k^\delta\}_k$ , we focus on showing that it converges locally uniformly in  $\mathbb{R}^3 \times [0, T_*^\delta)$  ( $\delta$  fixed,  $k \rightarrow \infty$ ) to some function  $f^\delta$  in  $\mathbb{R}^3 \times [0, T_*^\delta)$ , where  $f^\delta$  is a classical solution of

$$\partial_t f^\delta = \delta \Delta f^\delta + Q(f^\delta, f^\delta), \quad f^\delta = f_{\text{in}}.$$

The proof of this fact will take most of this section, and is achieved in Theorem 4.12. The selection of  $T_*^\delta$  will guarantee that either  $T_*^\delta = \infty$  or else  $\|f^\delta(\cdot, t)\|_\infty$  blows up as  $t \rightarrow T_*^\delta$ . Then, we take the limit  $\delta \rightarrow 0$  along a subsequence, making sure  $f^\delta$  and its derivatives converge locally uniformly to a solution of the original nonlinear problem. This is done in Theorem 4.14, where the auxiliary assumption (4-1) is also removed.

We start by using a differential inequality argument to control the  $L^\infty$  norm of the  $f_k^\delta$  uniformly in  $k$  and  $\delta$  for at least some time interval depending only on  $\|f_{\text{in}}\|_{L^\infty(\mathbb{R}^3)}$ .

**Lemma 4.3.** *Let  $\{f_k^\delta\}_k$  be the sequence defined above. Then for every  $k \in \mathbb{N}$  we have*

$$f_k^\delta(0, t) \leq \frac{f_{\text{in}}(0)}{1 - f_{\text{in}}(0)t}, \quad \forall t \in \left[0, \frac{1}{f_{\text{in}}(0)}\right).$$



*Proof.* Since  $f_{\text{in}}(0) > 0$ , it is immediate that the estimate holds for  $k = 0$ . Arguing by induction, suppose that

$$f_{k-1}^\delta(0, t) \leq \frac{f_{\text{in}}(0)}{1 - f_{\text{in}}(0)t}, \quad \forall t \in \left[0, \frac{1}{f_{\text{in}}(0)}\right).$$

Let us prove the corresponding inequality for  $f_k^\delta$ . By virtue of  $f_k^\delta$  being smooth, radially symmetric and monotone decaying, it follows that  $f_k^\delta(0, t) \geq f_k^\delta(v, t)$  for all  $v$  and  $t$  and  $D^2 f_k^\delta(0, t) \leq 0$  for all  $t$ . Plugging this information into the equation solved by  $f_k^\delta$ , we obtain

$$\begin{aligned} \partial_t f_k^\delta(0, t) &= 2^{-k} \Delta f_k^\delta(0, t) + \text{Tr}(A[f_{k-1}^\delta](0, t) D^2 f_k^\delta(0, t)) + f_{k-1}^\delta(0, t) f_k^\delta(0, t) \\ &\leq f_{k-1}^\delta(0, t) f_k^\delta(0, t). \end{aligned}$$

Then we may integrate the differential inequality

$$\partial_t f_k^\delta(0, t) \leq f_{k-1}^\delta(0, t) f_k^\delta(0, t)$$

in time, and it follows that

$$f_k^\delta(0, t) \leq f_{\text{in}}(0) e^{\int_0^t f_{k-1}^\delta(0, s) ds} \leq f_{\text{in}}(0) e^{\int_0^t f_{\text{in}}(0)/(1-f_{\text{in}}(0)s) ds}, \quad \forall t \in \left[0, \frac{1}{f_{\text{in}}(0)}\right),$$

where the last inequality was due to the inductive hypothesis. Since

$$\int_0^t \frac{f_{\text{in}}(0)}{1 - f_{\text{in}}(0)s} ds = -\log(1 - f_{\text{in}}(0)t),$$

it follows, as desired, that

$$f_k^\delta(0, t) \leq \frac{f_{\text{in}}(0)}{1 - f_{\text{in}}(0)t}, \quad \forall t \in \left[0, \frac{1}{f_{\text{in}}(0)}\right). \quad \square$$

Continuing with our analysis of the sequence  $\{f_k^\delta\}_k$ , we introduce a quantity that will play a crucial role in what follows: for every  $T > 0$ ,  $\delta > 0$ , let

$$M(f_{\text{in}}, T, \delta) := \sup_k \|f_k^\delta\|_{L^\infty(\mathbb{R}^3 \times [0, T])} = \sup_k \sup_{0 \leq t \leq T} f_k^\delta(0, t). \quad (4-5)$$

Lemma 4.3 shows that  $M(f_{\text{in}}, T, \delta) < \infty$  for at least every  $T < f_{\text{in}}(0)^{-1}$  and any  $\delta > 0$ . For the rest of this section, we will be concerned only with those  $T$  such that

$$M(f_{\text{in}}, T, \delta) < \infty. \quad (4-6)$$

**Remark 4.4.** In the following series of lemmas and propositions, culminating with Theorem 4.12, we use a series of estimates that depend on  $f_{\text{in}}$ ,  $T$ ,  $\delta$  and the function  $M(f_{\text{in}}, T, \delta)$ . For the sake of brevity, throughout this section we write  $C(f_{\text{in}}, T, \delta)$ ,  $C_0(f_{\text{in}}, T, \delta)$ ,  $C_1(f_{\text{in}}, T, \delta)$ ,  $C'(f_{\text{in}}, T, \delta)$  (as well as  $c(f_{\text{in}}, T, \delta)$  et cetera) to denote constants that depend solely on  $f_{\text{in}}$ ,  $T$ ,  $\delta$  and  $M(f_{\text{in}}, T, \delta)$ , with the understanding that the constants may change from one line to the next.

The next proposition says that we can control the  $L^\infty$  norm of the coefficients of (4-3) uniformly in  $k$  and  $\delta$ , as long as (4-6) holds.

**Proposition 4.5.** *Let  $\delta, k$  be arbitrary and  $M(f_{\text{in}}, T, \delta)$  as in (4-5). For any  $t \leq T$  and  $v \in \mathbb{R}^3$  we have the pointwise bounds*

$$A[f_k^\delta](v, t) \leq a[f_k^\delta](v, t)\mathbb{I} \leq \frac{2(M(f_{\text{in}}, T, \delta) + 1)}{1 + |v|} \mathbb{I}, \tag{4-7}$$

$$|\nabla a[f_k^\delta](v, t)| \leq \frac{M(f_{\text{in}}, T, \delta) + 1}{1 + |v|^2}. \tag{4-8}$$

*Proof.* The bound (4-7) follows immediately from (3-2) in Lemma 3.3 applied to  $h = f_k^\delta$ . On the other hand, from Newton’s formula (3-3) one sees immediately that

$$\nabla a[f_k^\delta] = -\frac{v}{4\pi|v|^3} \int_{B_{|v|}} f_k^\delta(w, t) dw. \tag{4-9}$$

Therefore,

$$|\nabla a[f_k^\delta](v, t)| = \frac{1}{4\pi|v|^2} \int_{B_{|v|}} f_k^\delta(w, t) dw.$$

Using the fact that  $\|f_k^\delta(\cdot, t)\|_{L^1} = 1$  yields

$$|\nabla a[f_k^\delta](v, t)| \leq \frac{1}{4\pi|v|^2}, \quad \forall (v, t),$$

while

$$\begin{aligned} |\nabla a[f_k^\delta](v, t)| &\leq \frac{1}{4\pi|v|^2} \frac{4\pi}{3} |v|^3 \|f_k^\delta(\cdot, t)\|_{L^\infty} \\ &\leq \frac{1}{3} M(f_{\text{in}}, T, \delta), \quad \forall (v, t) \in B_1(0) \times [0, T]. \end{aligned}$$

Using that  $4\pi|v|^2 \geq 1 + |v|^2$  if  $|v| \geq 1$ , we combine the previous inequalities to obtain the bound

$$|\nabla a[f_k^\delta](v, t)| \leq \frac{M(f_{\text{in}}, T, \delta) + 1}{1 + |v|^2}, \quad \forall (v, t) \in \mathbb{R}^3 \times [0, T],$$

which proves (4-8). □

For the purpose of controlling the size of  $f_k^\delta(v, t)$  for large  $v$ , it is necessary to bound the second moment of  $f_k^\delta$ , in a manner which is uniform in  $k$ .

**Proposition 4.6.** *Let  $T > 0$  and  $\delta \in (0, \frac{1}{10})$ . For any  $k \in \mathbb{N}$ ,  $f_k^\delta$  satisfies the bound*

$$\int_{\mathbb{R}^3} f_k^\delta(v, t) |v|^2 dv \leq 3 + 10(1 + M(f_{\text{in}}, T, \delta))T, \quad \forall t \in [0, T]. \tag{4-10}$$

*Proof.* Let  $\phi(v)$  be a smooth function with compact support. Using the equation solved by  $f_k^\delta$ , and integrating by parts, we obtain for every  $t > 0$

$$\frac{d}{dt} \int f_k^\delta(v, t) \phi(v) dv = \int f_k^\delta (\delta \Delta \phi + \text{Tr}(B[f_{k-1}^\delta] D^2 \phi) + 2(\nabla a[f_{k-1}^\delta], \nabla \phi)) dv.$$

Above,  $B[f_{k-1}^\delta]$  denotes  $a[f_{k-1}^\delta]\mathbb{I}$  or  $A[f_{k-1}^\delta]$  depending on whether  $Q = Q_{\mathcal{K}S}$  or  $Q = Q_{\mathcal{L}}$ . Integrating in time, it follows that

$$\begin{aligned} \int f_k^\delta(v, t_2)\phi(v) dv - \int f_k^\delta(v, t_1)\phi(v) dv \\ = \int_{t_1}^{t_2} \int f_k^\delta(\delta\Delta\phi + \text{Tr}(B[f_{k-1}^\delta]D^2\phi) + 2(\nabla a[f_{k-1}^\delta], \nabla\phi)) dv dt, \end{aligned}$$

for all  $0 \leq t_1 < t_2$ . Next, we apply this identity to the sequence  $\phi_j(v) = |v|^2\eta_j(v)$ , where  $\eta_j \in C_c^\infty(\mathbb{R}^3)$ , and  $\eta_j(v) \rightarrow 1$  locally uniformly. Due to the integrability of  $f_k^\delta$  and the bounds (4-7)–(4-8), we have enough decay at infinity to pass to the limit  $j \rightarrow \infty$  in the integral and conclude that the identity also holds for the function  $\phi(v) = |v|^2$ . Therefore, given  $0 \leq t_1 < t_2$ , we have the identity

$$\int f_k^\delta(v, t_2)|v|^2 dv - \int f_k^\delta(v, t_1)|v|^2 dv = \int_{t_1}^{t_2} \int f_k^\delta(\delta 6 + 2 \text{Tr}(B[f_{k-1}^\delta]) + 4(\nabla a[f_{k-1}^\delta], v)) dv dt.$$

Now, the bounds (3-2)–(3-3) guarantee that in  $\mathbb{R}^3 \times [0, T]$  we have

$$\begin{aligned} \text{Tr}(B[f_{k-1}^\delta]) &\leq 2M(f_{\text{in}}, T, \delta) + 2, \\ |(\nabla a[f_{k-1}^\delta], v)| &\leq \frac{(M(f_{\text{in}}, T, \delta) + 1)|v|}{1 + |v|^2} \leq M(f_{\text{in}}, T, \delta) + 1. \end{aligned}$$

Therefore, as long as  $t \in [0, T]$ ,

$$\begin{aligned} \left| \int_{t_1}^{t_2} \int f_k^\delta(\delta 6 + 2 \text{Tr}(B[f_{k-1}^\delta]) + 4(\nabla a[f_{k-1}^\delta], v)) dv dt \right| &\leq \int_{t_1}^{t_2} \int f_k^\delta(\delta 6 + 8M(f_{\text{in}}, T, \delta) + 8) dv dt \\ &\leq (6\delta + 8 + 8M(f_{\text{in}}, T, \delta))(t_2 - t_1). \end{aligned}$$

Taking  $t_1 = 0$  it follows that for  $\delta \in (0, \frac{1}{10})$ ,

$$\int f_k^\delta(v, t_2)|v|^2 dv \leq \int f_{\text{in}}|v|^2 dv + 10(1 + M(f_{\text{in}}, T, \delta))T, \quad \forall t \in [0, T].$$

Since  $\int f_{\text{in}}|v|^2 dv = 3$  by assumption (1-8), this proves the proposition.  $\square$

Next, we show how  $f_{k-1}^\delta \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)) \cap C^{\alpha, \alpha/2}(\mathbb{R}^3 \times \mathbb{R}_+)$  implies Hölder continuity of the coefficients appearing in  $Q(f_{k-1}^\delta, f)$ , emphasizing that the estimate is uniform in  $k$  for  $\delta > 0$  fixed whenever  $T$  is such that (4-6) holds.

**Lemma 4.7.** *Let  $\delta \in (0, \frac{1}{10})$  and  $T > 0$  be such that (4-6) holds. Then there is an absolute constant  $C > 0$  such that for any  $\alpha \in (0, 1)$  we have, for every  $k \geq 1$ , the bound*

$$\begin{aligned} [A[f_k^\delta]]_{C^{\alpha, \alpha/2}(\mathbb{R}^3 \times [0, T])} &\leq C([f_k^\delta]_{C^{\alpha, \alpha/2}(\mathbb{R}^3 \times [0, T])} + M(f_{\text{in}}, T, \delta) + 1), \\ [\nabla a[f_k^\delta]]_{C^{\alpha, \alpha/2}(\mathbb{R}^3 \times [0, T])} &\leq C([f_k^\delta]_{C^{\alpha, \alpha/2}(\mathbb{R}^3 \times [0, T])} + M(f_{\text{in}}, T, \delta) + 1). \end{aligned}$$

*Proof.* Let  $\eta \in C^\infty(\mathbb{R}^3)$  be an even function such that  $\eta \equiv 1$  in  $B_1(0)$  and  $\eta \equiv 0$  outside  $B_2$ . Let us write

$$A[f_k^\delta] = A_1[f_k^\delta] + A_2[f_k^\delta].$$

Each  $A_i$  ( $i = 1, 2$ ) is given by convolutions  $A_i[f_k^\delta] = K_i * f_k^\delta$  with the respective kernels

$$K_1(v) := \frac{1}{8\pi|v|} \left( \mathbb{I} - \frac{v \otimes v}{|v|^2} \right) \eta(v), \quad K_2(v) := \frac{1}{8\pi|v|} \left( \mathbb{I} - \frac{v \otimes v}{|v|^2} \right) (1 - \eta(v)).$$

Let us show that  $A_1, A_2$  are Hölder continuous in  $v$  and  $t$ . We make use of the fact that there is a constant  $C(\eta)$  such that

$$\int_{\mathbb{R}^3} |K_1(v)| dv + \sup_v |K_2(v)| + \sum_{i=1}^3 \sup_v |\partial_i K_2(v)| + \sum_{i,j=1}^3 \sup_v |\partial_{ij} K_2(v)| \leq C(\eta),$$

where the matrix norm used is the standard  $L^2$  norm  $|A| = \text{Tr}(AA^*)^{1/2}$ . For  $A_1$  it is straightforward that

$$\begin{aligned} |A_1(v_1, t_1) - A_1(v_2, t_2)| &\leq \int_{B_2} |K_1(w)| |f_k^\delta(v_1 - w, t_1) - f_k^\delta(v_2 - w, t_2)| dw \\ &\leq \left( \int_{B_2} |K_1(w)| dw \right) \sup_{w \in B_2(0)} |f_k^\delta(v_1 - w, t_1) - f_k^\delta(v_2 - w, t_2)|, \end{aligned}$$

the above holding for any  $(v_i, t_i)$ , so that

$$[A_1]_{C^{\alpha, \alpha/2}} \leq C(\eta) [f_k^\delta]_{C^{\alpha, \alpha/2}}.$$

Next we deal with  $A_2$ , which in fact will be Lipschitz continuous. Fix  $e \in \mathbb{S}^2$  and set  $K_{2,e}(v) := (K_2(v)e, e)$ . Using the equation for  $f_k^\delta$  and integration by parts,

$$\begin{aligned} \partial_t(A_2[f_k^\delta](v)e, e) &= \int_{B_1^c} K_{2,e}(w - v) \partial_t f_k^\delta dw \\ &= - \int_{B_1^c} (\nabla_w K_{2,e}(w - v), (A[f_{k-1}^\delta] + \delta \mathbb{I}) \nabla_w f_k^\delta) dw + \int_{B_1^c} f_k^\delta (\nabla_w a[f_{k-1}^\delta], \nabla_w K_{2,e}(w - v)) dw. \end{aligned}$$

Integrating by parts once again,

$$\begin{aligned} - \int_{B_1^c} (\nabla_w K_{2,e}(w - v), (A[f_{k-1}^\delta] + \delta \mathbb{I}) \nabla_w f_k^\delta) dw &= \int_{B_1^c} \text{div}_w ((A[f_{k-1}^\delta] + \delta \mathbb{I}) \cdot \nabla_w K_{2,e}(w - v)) f_k^\delta dw \\ &= \int_{B_1^c} f_k^\delta \text{Tr}(A[f_{k-1}^\delta] D_w^2 K_{2,e}(w - v)) dw + \int_{B_1^c} f_k^\delta \nabla_w a[f_{k-1}^\delta] \cdot \nabla_w K_{2,e}(w - v) dw \\ &\quad + \delta \int_{B_1^c} f_k^\delta \Delta_w K_{2,e}(w - v) dw. \end{aligned}$$

Gathering all of the above, it follows that

$$\begin{aligned} \partial_t(A_2[f_k^\delta]e, e) &= \int_{B_1^c} f_k^\delta \text{Tr}(A[f_{k-1}^\delta] D_w^2 K_{2,e}(w - v)) dw \\ &\quad + 2 \int_{B_1^c} f_k^\delta (\nabla_w a[f_{k-1}^\delta], \nabla_w K_{2,e}(w - v)) dw + \delta \int_{B_1^c} f_k^\delta \Delta_w K_{2,e}(w - v) dw. \end{aligned}$$



Therefore, we have the bound

$$\begin{aligned} & |\partial_t(A_2[f_k^\delta])(v)e, e)| \\ & \leq \|D^2 K_{2,e}\|_{L^\infty} \|A[f_{k-1}^\delta]\|_{L^\infty} \|f\|_{L^1} + 2\|\nabla K_{2,e}\|_{L^\infty} \|\nabla a[f_{k-1}^\delta]\|_{L^\infty} \|f\|_{L^1} + \delta \|\Delta K_{2,e}\|_{L^\infty} \|f_k^\delta\|_{L^1} \\ & \leq \|K_{2,e}\|_{C^2} (\|A[f_{k-1}^\delta]\|_{L^\infty} + \|\nabla a[f_{k-1}^\delta]\|_{L^\infty} + \delta) \\ & \leq \|K_{2,e}\|_{C^2} (3M(f_{\text{in}}, T, \delta) + 4), \end{aligned}$$

where we used (4-7)–(4-8) and  $\delta \in (0, \frac{1}{10})$  in the last inequality. Since  $\|K_{2,e}\| \leq C(\eta)$  for all  $e$ ,

$$|\partial_t(A_2[f_k^\delta])(v)e, e)| \leq 4C(\eta)(M(f_{\text{in}}, T, \delta) + 1).$$

This immediately implies a Lipschitz bound in time for  $A_2$ , namely

$$|A_2(v, t_1) - A_2(v, t_2)| \leq 12\|K_2\|_{C^2}(M(f_{\text{in}}, T, \delta) + 1)|t_1 - t_2|, \quad \forall v \in \mathbb{R}^3, t_1, t_2 \geq 0.$$

For the spatial regularity, from the definition of  $A_2$  and the triangle inequality it follows that

$$\begin{aligned} |A_2(v_1, t) - A_2(v_2, t)| & \leq \int |K_2(w - v_1) - K_2(w - v_2)| f_k^\delta(w, t) dw \\ & \leq C(\eta)|v_1 - v_2| \int f_k^\delta(w, t) dw \quad \forall v_1, v_2 \in \mathbb{R}^3, t \geq 0. \end{aligned}$$

Then, thanks to  $\|f_k^\delta(\cdot, t)\|_{L^1} = 1$ , it follows that

$$|A_2(v_1, t) - A_2(v_2, t)| \leq C(\eta)|v_1 - v_2|, \quad \forall v_1, v_2 \in \mathbb{R}^3, t > 0.$$

Finally, we combine the estimates in time and space to see that

$$\begin{aligned} |A_2(v_1, t_1) - A_2(v_2, t_2)| & \leq |A_2(v_1, t_1) - A_2(v_2, t_1)| + |A_2(v_2, t_1) - A_2(v_2, t_2)| \\ & \leq 15C(\eta)(M(f_{\text{in}}, T, \delta) + 1)(|v_1 - v_2| + |t_1 - t_2|), \quad \forall (v_i, t_i), i = 1, 2. \end{aligned}$$

Since  $|v_1 - v_2| + |t_1 - t_2| \leq |v_1 - v_2|^\alpha + |t_1 - t_2|^{\alpha/2}$  when  $|v_1 - v_2|, |t_1 - t_2| \leq 1$ , we conclude that

$$[A_2]_{C^{\alpha, \alpha/2}(\mathbb{R}^3 \times [0, T])} \leq 15C(\eta)(M(f_{\text{in}}, T, \delta) + 1).$$

The proof of Hölder regularity for  $\nabla a[f_k^\delta](v, t)$  can be done in an entirely analogous manner, writing the kernel as the sum of integrable and  $C^2$  parts. One may also make a slightly different argument, using the fact that since  $f_k^\delta$  is spherically symmetric, we have the identity (4-9), which yields a similar bound.  $\square$

For the purposes of the proof of existence of solutions, we require several parabolic estimates that are local in space but uniform up to  $t = 0$ . Notice these are different to the interior estimates stated in Section 2, namely Theorems 2.2, 2.3 and 2.4, which will be of chief importance in later sections. The parabolic estimates hold in a space-time cylinder, which starts at time  $t = 0$ , and are in terms of norms of the initial data. They guarantee in particular that under the auxiliary assumptions (4-1) on  $f_{\text{in}}$  the functions  $f_k^\delta$  have spatial decay on their second derivatives.

**Lemma 4.8** (Hölder estimate for regular initial data). *There exists some  $\alpha \in (0, 1)$  and constant  $c$ , which only depends on  $\delta, f_{\text{in}}, T$  and  $[f_{\text{in}}]_{C^{2+\beta}(\mathbb{R}^3)}$ , such that for any  $v \in \mathbb{R}^3$  and  $k \geq 1$ ,*

$$[f_k^\delta]_{C^{\alpha,\alpha/2}(B_1(v) \times [0, T])} \leq c(\delta, M(f_{\text{in}}, T, \delta), [f_{\text{in}}]_{C^1(B_2(v))}). \tag{4-11}$$

(Schauder estimate up to the initial time). *Let  $\beta \in (0, 1)$ . Then for any  $v \in \mathbb{R}^3, k \geq 1$ ,*

$$[f_k^\delta]_{C^{2+\alpha, 1+\alpha/2}(B_1(v) \times [0, T])} \leq C(\|f_k^\delta\|_{L^\infty(B_2(v) \times [0, T])} + [f_{\text{in}}]_{C^{2+\beta}(B_2(v))}), \tag{4-12}$$

where  $C = C(f_{\text{in}}, T, \delta)$ .

*Proof.* For the proof of the first estimate we refer to [Ladyženskaja et al. 1968, Theorem 10.1, page 204]. Note that the constant does not depend in any way on the regularity of the coefficients in the equation solved by  $f_k^\delta$ , and depends only on the ellipticity constants and the regularity of  $f_{\text{in}}$ . The second estimate follows from [Ladyženskaja et al. 1968, Theorem 10.1, page 351], noting that the space-time Hölder norm of the coefficients  $A[f_{k-1}^\delta], \nabla a[f_{k-1}^\delta]$  is bounded by a constant  $C(f_{\text{in}}, T, \delta)$ , thanks to Lemma 4.7 and the first estimate (4-11) applied to  $f_{k-1}^\delta$  (when  $k > 1$ ;  $f_0^\delta \equiv f_{\text{in}}$  for  $k = 1$ , which is regular in space and constant in time). □

Next we show that the diffusion matrices  $A[f_k^\delta] + \delta \mathbb{I}$  are Hölder continuous in a manner which is uniform in  $k$  (but possibly depending on  $\delta$ ). In this case, standard estimates for linear parabolic equations yield Hölder bounds on the second-order spatial derivatives and first-order temporal derivatives for  $f_k^\delta$ , these being uniform in  $k$ . Particularly, since we are assuming a spatial decay for the second derivatives of  $f_{\text{in}}$  (see (2-2)), the same holds for  $f_k^\delta$ .

**Proposition 4.9.** *Let  $\delta \in (0, \frac{1}{10})$  and  $0 < T < \infty$  be such that (4-6) holds. Then there is a  $C$  depending only on  $f_{\text{in}}, \delta, T, M(f_{\text{in}}, T, \delta)$  such that*

$$\|D^2 f_k^\delta\|_{C^\alpha(B_1(v) \times [0, T])} \leq C(1 + |v|^5)^{-1}, \quad \forall v \in \mathbb{R}^3. \tag{4-13}$$

*Proof.* We first show that  $f_k^\delta(v, t)$  decays as  $(1 + |v|^5)^{-1}$  for  $v$  large. Fixing  $v \in \mathbb{R}^3$ , the spherical symmetry and radial monotonicity of  $f_k^\delta$  implies that

$$\frac{7}{6}\pi |v|^3 f_k^\delta(v, t) \leq \int_{B_{|v|} \setminus B_{|v|/2}} f_k^\delta(w, t) dw \leq \frac{4}{|v|^2} \int_{\mathbb{R}^3} f_k^\delta(w, t) |w|^2 dw.$$

Using the second moment bound (4-10), we arrive at the estimate

$$f_k^\delta(v, t) \leq \frac{4}{\pi |v|^5} (3 + 10(1 + M(f_{\text{in}}, T, \delta))T)$$

for all  $|v| \geq 1$  and  $t \in [0, T]$ . Since  $f_k^\delta(v, t) \leq M(f_{\text{in}}, T, \delta)$  as long as  $t \leq T$ , we conclude that

$$f_k^\delta(v, t) \leq \frac{C'(f_{\text{in}}, T, \delta)}{1 + |v|^5}, \quad \forall (v, t) \in \mathbb{R}^3 \times [0, T], \tag{4-14}$$

with  $C'(f_{\text{in}}, T, \delta) := \max\{M, \frac{4}{\pi}(3 + 10(1 + M(f_{\text{in}}, T, \delta))T)\}$ . The bound follows, combining the initial bound (4-1), the decay estimate(4-14) and the estimate (4-12) from Lemma 4.8. □

So far we have shown the existence of the sequence  $\{f_k^\delta\}$ , and proven several uniform estimates which are uniform in  $k$  for times  $T < T_*^\delta$ . Moving towards obtaining a limit from this sequence, we prove an iterative estimate on the size of the functions  $\{f_k^\delta - f_{k-1}^\delta\}_k$  in  $\mathbb{R}^3 \times [0, T]$ , for  $\delta > 0$  fixed and  $T$  such that  $C(f_{in}, T, \delta) < \infty$ .

**Lemma 4.10.** *Let  $\delta \in (0, \frac{1}{10})$  and  $T > 0$  be such that (4-6) holds, and let  $w_k^\delta := f_{k-1}^\delta - f_k^\delta$  for each  $k \geq 1$ . There is a number  $0 < T_0 < T$ ,  $T_0 = T_0(f_{in}, T, \delta)$  with the following properties:*

(1) For each  $k \geq 2$ ,

$$\|w_k(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [0, T_0])} \leq \frac{1}{4} \|w_{k-1}(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [0, T_0])}.$$

(2) For each  $k \geq 2$  and  $l = 1, \dots, l_0$ , we have

$$\|w_k^\delta(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [t_{l-1}, t_l])} \leq \frac{1}{4} \|w_{k-1}^\delta(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [t_{l-1}, t_l])} + 2 \|w_k^\delta(t_{l-1})\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)}.$$

Here  $l_0 \in \mathbb{N}$  is the largest such that  $(l_0 - 1)T_0 \leq T$ , and  $t_l := \min\{lT_0, T\}$ .

*Proof.* We drop the superscript  $\delta$  for convenience. Using the equations for  $f_{k-1}$  and  $f_k$  we get that  $w_k = f_{k-1} - f_k$  satisfies

$$\begin{cases} \partial_t w_k = \delta \Delta w_k + \text{Tr}(A[f_{k-2}]D^2 w_k) + f_{k-2} w_k + \text{Tr}(A[w_{k-1}]D^2 f_k) + f_k w_{k-1}, & \text{for } t > 0, \\ w_k = 0, & \text{for } t = 0. \end{cases} \tag{4-15}$$

*Step 1.* According to Proposition 4.9, there is a positive constant  $C(f_{in}, T, \delta)$  such that

$$|D^2 f_k^\delta(v, t)| \leq C(f_{in}, T, \delta)(1 + |v|^5)^{-1}, \quad \forall v \in \mathbb{R}^3, t \in [0, T]. \tag{4-16}$$

The estimate (4-16) and the estimate (3-5) applied to  $w_{k-1}$  imply the inequality

$$|\text{Tr}(A[w_{k-1}]D^2 f_k(v, t))| \leq C(f_{in}, T, \delta) \left( \frac{\|w_{k-1}(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} + \|w_{k-1}(\cdot, t)\|_{L^1(\mathbb{R}^3)}}{1 + |v|^5} \right),$$

which holds for any  $(v, t) \in \mathbb{R}^3 \times [0, T]$ . On the other hand,  $\langle v \rangle^{-4} \in L^1(\mathbb{R}^3)$ . Therefore,

$$\|w_k(t)\|_{L^1} = \int_{\mathbb{R}^3} |w_k(v, t)| \langle v \rangle^4 \langle v \rangle^{-4} dv \leq \|w_k(t)\langle v \rangle^4\|_{L^\infty} \|\langle v \rangle^{-4}\|_{L^1(\mathbb{R}^3)}.$$

Substituting this in the last estimate, we arrive at the bound,

$$|\text{Tr}(A[w_{k-1}]D^2 f_k(v, t))| \leq C(f_{in}, T, \delta) \|w_{k-1}(t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)} (1 + |v|^5)^{-1}.$$

*Step 2.* Consider the function  $h_0(v) := \langle v \rangle^{-4} = (1 + |v|^2)^{-2}$ . We have

$$\begin{aligned} Dh_0(v) &= -4(1 + |v|^2)^{-3}v, \\ D^2 h_0(v) &= -4(1 + |v|^2)^{-3}\mathbb{I} + 24(1 + |v|^2)^{-4}v \otimes v. \end{aligned}$$

In particular,

$$\begin{aligned} \Delta h_0 &= 12(|v|^2 - 1)\langle v \rangle^{-8}, \\ \text{Tr}(A[f_{k-2}]D^2 h_0) &= -4\langle v \rangle^{-6}a[f_{k-2}] + 24\langle v \rangle^{-8}(A[f_{k-2}]v, v). \end{aligned}$$

Using the inequalities  $||v|^2 - 1|, |v|^2 \leq \langle v \rangle^2$ , the above leads to

$$|\delta \Delta h_0| \leq 12\delta \langle v \rangle^{-6},$$

$$|\operatorname{Tr}(A[f_{k-2}]D^2 h_0)| \leq 4\langle v \rangle^{-6} a[f_{k-2}] + 24\langle v \rangle^{-6} a[f_{k-2}].$$

Then, recalling that  $\delta \in (0, \frac{1}{10}) \Rightarrow 12\delta < \frac{3}{2}$ , we combine the above inequalities into one,

$$|\delta \Delta h_0 + \operatorname{Tr}(A[f_{k-2}]D^2 h_0)| \leq 28(1 + a[f_{k-2}])\langle v \rangle^{-6} \leq 56(1 + C(f_{\text{in}}, T, \delta))h_0,$$

where we have used (4-7) to bound  $a[f_{k-2}]$ .

*Step 3.* Next, let

$$H_0(v, t) := RA^{-1}(e^{At} - 1)h_0(v),$$

for  $A, R > 0$  to be determined. It is immediate that

$$\partial_t H_0 = AH_0 + Rh_0.$$

The last inequality in Step 2 implies that

$$|\delta \Delta H_0 + \operatorname{Tr}(A[f_{k-2}]D^2 H_0)| + f_{k-2}H_0 \leq 60(1 + C(f_{\text{in}}, T, \delta))H_0.$$

The estimates from Step 1, the definition of  $h_0(v)$  and (4-14) yield

$$\operatorname{Tr}(A[w_{k-1}]D^2 f_k) + f_k w_{k-1} \leq C_0 \|w_{k-1}(t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)} h_0(v),$$

with  $C_0 = C_0(f_{\text{in}}, T, \delta)$ . In light of this, for any  $T_0 \in (0, T)$ , we choose  $A$  and  $R$  to be

$$A = 60(1 + C(f_{\text{in}}, T, \delta)),$$

$$R = C_0 \sup_{0 \leq t \leq T_0} \|w_{k-1}(t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)},$$

in which case we have, for any  $(v, t) \in \mathbb{R}^3 \times [0, T_0]$ ,

$$\begin{aligned} \partial_t H_0 &\geq 60(1 + C(f_{\text{in}}, T, \delta))H_0 + C_0(\|w_{k-1}(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} + \|w_{k-1}(\cdot, t)\|_{L^1(\mathbb{R}^3)})h_0 \\ &\geq \delta \Delta H_0 + \operatorname{Tr}(A[f_{k-2}]D^2 H_0) + f_{k-2}H_0 + (\operatorname{Tr}(A[w_{k-1}]D^2 f_k) + f_k w_{k-1}). \end{aligned}$$

This means that  $H_0$  is a supersolution of (4-15), the parabolic equation solved by  $w_k$ . Furthermore,  $H_0(\cdot, 0) = w_k(\cdot, 0) = 0$ . Then, thanks to the comparison principle,

$$w_k \leq H_0 \quad \text{in } \mathbb{R}^3 \times [0, T_0].$$

The same argument applied to  $-w_k$  yields

$$\eta_k \leq H_0 \quad \text{in } \mathbb{R}^3 \times [0, T_0].$$

We have shown that there are constants  $C_0(f_{\text{in}}, T, \delta)$  and  $C_1(f_{\text{in}}, T, \delta)$  such that

$$|w_k(v, t)| \leq C_0 \|w_{k-1}(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [0, T_0])} (e^{C_1(f_{\text{in}}, T, \delta)t} - 1)\langle v \rangle^{-4} \quad \text{in } \mathbb{R}^3 \times [0, T_0].$$



In particular, there is a  $T_0$ , depending only on  $T$  and  $C_0(f_{\text{in}}, T, \delta)$ , such that

$$T_0 \in (0, T) \quad \text{and} \quad C_0(e^{C_1(f_{\text{in}}, T, \delta)T_0} - 1) \leq \frac{1}{4}.$$

This results in the estimate

$$\|w_k(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [0, T_0])} \leq \frac{1}{4} \|w_{k-1}(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [0, T_0])},$$

and the first part of the lemma is proved.

*Step 4.* Fix  $k \geq 2$ . Assume for now that  $2T_0 < T$  — the same  $T_0$  as in Step 3 — and define the function  $H_1 : \mathbb{R}^3 \times [T_0, \infty) \rightarrow \mathbb{R}$  by

$$H_1(v, t) := RA^{-1}(e^{A(t-T_0)} - 1)h_0(v) + \|w_k(T_0)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)}h_0(v),$$

where  $A$  and  $R$  are to be determined. A straightforward computation yields

$$\begin{aligned} \partial_t H_1 &= Re^{A(t-T_0)}h_0(v) \\ &= A(RA^{-1}(e^{A(t-T_0)} - 1)h_0(v) + \|w_k(T_0)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)}h_0(v)) + Rh_0(v) - \|w_k(T_0)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)}h_0(v) \\ &= AH_1 + Rh_0(v) - \|w_k(T_0)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)}h_0(v). \end{aligned}$$

As in the previous step, we have

$$\begin{aligned} \delta \Delta H_1 + \text{Tr}(A[f_{k-2}]D^2 H_1) + f_{k-2}H_1 + \text{Tr}(A[w_{k-1}]D^2 f_k) + f_k w_{k-1} + 2^{-k} \Delta f_k \\ \leq 60(1 + C(f_{\text{in}}, T, \delta))H_1 + h_0 C_0 \|w_{k-1}(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [T_0, 2T_0])} \\ = AH_1 + h_0(R - A\|w_k(v, T_0)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)}) = \partial_t H_1 \end{aligned}$$

by choosing

$$\begin{aligned} A &= 60(1 + C(f_{\text{in}}, T, \delta)), \\ R &= C_0 \|w_{k-1}(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [T_0, 2T_0])} + 60(1 + C(f_{\text{in}}, T, \delta))\|w_k(v, T_0)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)}. \end{aligned}$$

Likewise,  $H_1(\cdot, T_0) \geq w_k(\cdot, T_0)$ . Then, just as before, the comparison principle says that  $H_1(\cdot, t) \geq w_k(\cdot, t)$  for  $t \in [T_0, 2T_0]$ ,

$$\begin{aligned} |w_k(v, t)| &\leq C_0(e^{C_1(f_{\text{in}}, T, \delta)t} - 1)\|w_{k-1}(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [T_0, 2T_0])}\langle v \rangle^{-4} \\ &\quad + \|w_k(v, T_0)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)}(e^{C_1(f_{\text{in}}, T, \delta)t} - 1)\langle v \rangle^{-4} + \|w_k(v, T_0)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3)}\langle v \rangle^{-4}. \end{aligned}$$

Hence for  $t \in [T_0, 2T_0]$  we get

$$\|w_k(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [T_0, 2T_0])} \leq \frac{1}{4} \|w_{k-1}(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [T_0, 2T_0])} + 2\|w_k(v, T_0)(1 + |v|^2)^2\|_{L^\infty(\mathbb{R}^3)}.$$

This yields the second estimate in the case  $l = 2$ . The above argument can be repeated to obtain a further estimate in the interval  $[2T_0, 3T_0]$ , and so on. After a finite number of iterations we will reach some  $l_0 \in \mathbb{N}$  such that  $(l_0 - 1)T_0 \leq T$  and  $l_0 T_0 > T$ . In that case we repeat the above argument on the interval  $[(l_0 - 1)T_0, T]$ , yielding the respective bound and completing the proof of the second estimate.  $\square$

The next lemma shows that if  $\delta \in (0, \frac{1}{10})$  and  $T$  is a time for which (4-6) holds, the sequence  $f_k^\delta$  converges uniformly in  $\mathbb{R}^3 \times [0, T]$  to a continuous limit  $f^\delta$ .

**Lemma 4.11.** *Let  $\{f_k^\delta\}_k$ ,  $\delta \in (0, \frac{1}{10})$ , and  $T > 0$  be such that (4-6) holds. Then there is a continuous function  $f^\delta : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}$  such that*

$$\begin{aligned} \lim_k \|f^\delta - f_k^\delta\|_{L^\infty(\mathbb{R}^3 \times [0, T])} &= 0, \\ \lim_k \|f^\delta - f_k^\delta\|_{L^\infty(0, T; L^1(\mathbb{R}^3))} &= 0. \end{aligned}$$

*Proof.* Let  $T_0 > 0$  and  $l_0$  and  $t_l$  be as in Lemma 4.10. Define, for  $l = 0, 1, \dots, l_0$  and  $k \in \mathbb{N}$ ,

$$E_{k,l} := \|w_k(v, t)\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [t_{l-1}, t_l])}.$$

Then Lemma 4.10 says that the recursive relations

$$\begin{aligned} E_{k,1} &\leq \frac{1}{4} E_{k-1,1}, \\ E_{k,l} &\leq 4E_{k,l-1} + \frac{1}{2} E_{k-1,l} \end{aligned}$$

hold for  $k \geq 2$  and  $l = 0, \dots, l_0$ . We claim that these recurrence relations guarantee the summability in  $k$  of the sequence  $\{E_{k,l}\}_k$  for any fixed  $l = 1, \dots, l_0$ . The first recurrence relation implies that  $E_{k,1}$  decays geometrically, thus we immediately have

$$\sum_{k=3}^\infty E_{k,1} < \infty.$$

Next, suppose that for some  $1 < l < l_0$  we have

$$\sum_{k=3}^\infty E_{k,l} < \infty.$$

Taking the sum for  $k$  from 3 to  $N$  of the second recursive relation, we get

$$\sum_{k=3}^N \frac{1}{2} E_{k,l+1} \leq 4 \sum_{k=3}^N E_{k,l} + \frac{1}{2} E_{2,l+1}.$$

We can then pass to the limit  $N \rightarrow +\infty$ , and use the summability for  $E_{k,l}$  to obtain

$$\sum_{k=3}^N \frac{1}{2} E_{k,l+1} < +\infty.$$

Combining the summability of the sequences  $\{E_{k,l}\}_k$  for every  $l \leq l_0$ , we conclude that

$$\sum_k \|(f_k(v, t) - f_{k-1}(v, t))\langle v \rangle^4\|_{L^\infty(\mathbb{R}^3 \times [0, T])} < \infty.$$

Since  $\langle v \rangle \geq 1$  for all  $v$ , and  $\langle v \rangle^{-4} = (1 + |v|^2)^{-2} \in L^1(\mathbb{R}^3)$ , this implies that

$$\sum_k \|f_k - f_{k-1}\|_{L^\infty(\mathbb{R}^3 \times [0, T])} < \infty,$$

$$\sum_k \|f_k - f_{k-1}\|_{L^\infty(0, T; L^1(\mathbb{R}^3))} < \infty.$$

This summability implies  $\{f_k\}$  is a Cauchy sequence in each norm, proving the lemma. □

**Theorem 4.12.** *For each  $\delta \in (0, \frac{1}{10})$ , there is a time  $T_*^\delta = T_*^\delta(f_{\text{in}})$  with  $0 < T_*^\delta \leq \infty$  and a function  $f^\delta$  in  $C_{\text{loc}}^{2,1}(\mathbb{R}^3 \times [0, T_*^\delta))$  such that*

$$\begin{cases} \partial_t f^\delta = \delta \Delta f^\delta + Q(f^\delta, f^\delta) & \text{in } \mathbb{R}^3 \times [0, T_*^\delta), \\ f^\delta(\cdot, 0) = f_{\text{in}}. \end{cases}$$

Moreover, either  $T_*^\delta = \infty$  or

$$\limsup_{T \rightarrow T_*^\delta-} \|f^\delta\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} = \infty.$$

*Proof. Step 1.* Let

$$T_*^\delta := \sup\{T > 0 \mid M(f_{\text{in}}, T, \delta) < \infty\}.$$

By Lemma 4.3 we have  $T_*^\delta \geq (2f_{\text{in}}(0))^{-1}$ , thus  $T_*^\delta > 0$ . It may certainly be that  $T_*^\delta = \infty$ . Now, we may apply Lemma 4.11 to  $f_k^\delta$  and any fixed  $T < T_*^\delta$ , resulting in a continuous function  $f^\delta : \mathbb{R}^3 \times [0, T_*^\delta) \rightarrow \mathbb{R}$  such that

$$f_k^\delta \rightarrow f^\delta \text{ uniformly in } \mathbb{R}^3 \times [0, T), \quad \forall T < T_*^\delta.$$

On the other hand, we have the estimates from Lemma 4.8, which guarantee, by the Arzelà–Ascoli theorem, that for any subsequence  $k_n \rightarrow \infty$  there is a subsequence  $k'_n$  such that  $\partial_t f_{k'_n}^\delta$  and  $D^2 f_{k'_n}^\delta$  converge locally uniformly in  $\mathbb{R}^3 \times [0, T_*)$  as  $n \rightarrow \infty$ . Since  $f_k^\delta \rightarrow f$  locally uniformly and  $\{k_n\}$  was arbitrary, it follows that (i)  $f^\delta \in C_{\text{loc}}^{2,1}(\mathbb{R}^3 \times [0, T_*^\delta))$ , and (ii) the sequences  $D^2 f_k^\delta$  and  $\partial_t f_k^\delta$  converge locally uniformly to  $D^2 f^\delta$  and  $\partial_t f^\delta$  as  $k \rightarrow \infty$ , respectively.

*Step 2.* Let us show the matrices  $\{A[f_k^\delta]\}_k$  converge locally uniformly in  $\mathbb{R}^3 \times [0, T_*^\delta)$  to  $A[f^\delta]$ . Indeed, let  $t \in [0, T_*^\delta)$  and apply the estimate (3-5) to  $g = |f_k(\cdot, t) - f_k^\delta(\cdot, t)|$  (which is a nonnegative, bounded, spherically symmetric function), which leads to the bound

$$|A[f_k^\delta](v, t) - A[f^\delta](v, t)| \leq 2(\|f_k(\cdot, t) - f_k^\delta(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} + \|f_k(\cdot, t) - f_k^\delta(\cdot, t)\|_{L^1(\mathbb{R}^3)})$$

for all  $v$  and  $t < T_*^\delta$ . Then Lemma 4.11 shows that  $A[f_k^\delta]$  converges uniformly to  $A[f^\delta]$  uniformly in  $\mathbb{R}^3 \times [0, T]$  for every  $T < T_*^\delta$ .

*Step 3.* Thanks to the local uniform convergence of  $f_k^\delta$ ,  $D^2 f_k^\delta$ ,  $\partial_t f_k^\delta$  and  $A[f_k^\delta]$  proved in the previous two steps, we can pass to the limit in the equation for  $f_k^\delta$  and conclude that

$$\partial_t f^\delta = \delta \Delta f^\delta + Q(f^\delta, f^\delta) \quad \text{in } \mathbb{R}^3 \times [0, T_*^\delta).$$

Step 4. We show here that if  $T_*^\delta$  is finite, then the  $L^\infty$  norm of  $f^\delta(\cdot, t)$  goes to infinity as  $t$  approaches  $T_*^\delta$ . Arguing by contradiction, suppose that  $T_*^\delta$  is finite, and

$$\limsup_{T \rightarrow T_*^\delta} f^\delta(0, t) < +\infty.$$

Since  $f^\delta$  is continuous and bounded for any  $t < T_*^\delta$ , then  $f^\delta$  is bounded for any  $t \leq T_*^\delta$  and in particular,

$$f^\delta(0, T_*^\delta - \varepsilon) \leq C, \quad \varepsilon > 0.$$

The uniform convergence of  $f_k^\delta \rightarrow f^\delta$  for all  $t < T_*^\delta$  shows that for any small enough  $\varepsilon > 0$  there is some  $k_0$  such that

$$f_k^\delta(0, T_*^\delta - \varepsilon) < 2C, \quad \forall k > k_0. \tag{4-17}$$

Since  $\sup_k f_k^\delta(0, T_*^\delta - \varepsilon) < +\infty$ , we have that (4-17) implies

$$f_k^\delta(0, T_*^\delta - \varepsilon) < \tilde{C}, \quad \forall k \geq 1.$$

Then the differential inequality argument from Lemma 4.3, applied with starting time shifted to  $T_*^\delta - \varepsilon$ , proves that

$$f_k^\delta(0, T_*^\delta - \varepsilon + t) \leq \frac{\tilde{C}}{1 - \tilde{C}t}, \quad \forall k \geq 1, \quad 0 < t < \frac{1}{\tilde{C}}.$$

Taking now  $t = 1/(2\tilde{C})$  and  $\varepsilon = 1/(4\tilde{C})$  yields

$$f_k^\delta(0, T_*^\delta + \varepsilon) < 2\tilde{C},$$

which contradicts the maximality of  $T_*^\delta$  and the theorem is proved. □

Next, we show that as long as  $f^\delta(v, t)$  is bounded in a time interval  $[0, T]$ , the mass of  $f^\delta(v, t)$  cannot escape to infinity nor concentrate at the origin. The bound is independent of  $\delta$ . A consequence of this result is a local lower bound for  $A[f^\delta]$  along radial directions.

**Proposition 4.13.** *Let  $\delta \in (0, \frac{1}{10})$ , let  $f^\delta$  be a function given by Theorem 4.14, let  $T < T_*^\delta$  and let  $M > 0$  be such that*

$$\|f^\delta\|_{L^\infty \times [0, T]} < M.$$

*Then there are radii  $r(f_{\text{in}}, T, M)$  and  $R(f_{\text{in}}, T, M)$  such that  $0 < r < R < \infty$  and*

$$\int_{B_R \setminus B_r} f^\delta(v, t) \, dv \geq \frac{1}{2}, \quad \forall t \in [0, T]. \tag{4-18}$$

*As a consequence, there is a positive constant  $c_0 = c_0(f_{\text{in}}, T, M)$  such that*

$$A^*[f^\delta](v, t) \geq \frac{c_0}{1 + |v|^3}, \quad \forall v \in \mathbb{R}^3, \quad t \in [0, T], \quad k \in \mathbb{N}, \tag{4-19}$$

where  $A^*[\cdot]$  is as defined in (3-2).



*Proof.* Given  $R > 0$ , the mass of  $f^\delta$  outside  $B_R(0)$  may be estimated via its second moment

$$\int_{B_R^c} f^\delta dv \leq \int_{B_R^c} f^\delta \frac{|v|^2}{R^2} dv \leq \frac{1}{R^2} \int_{\mathbb{R}^3} f^\delta(v, t) |v|^2 dv.$$

Moreover, for any  $r, R$  with  $R > r > 0$  there is the obvious lower bound

$$\begin{aligned} \int_{B_R \setminus B_r} f^\delta(v, t) dv &= 1 - \int_{B_R^c} f^\delta(v, t) dv - \int_{B_r} f^\delta(v, t) dv \\ &\geq 1 - \frac{1}{R^2} \int_{\mathbb{R}^3} f^\delta(v, t) |v|^2 dv - \frac{4\pi}{3} r^3 M, \end{aligned} \tag{4-20}$$

using the fact that  $\|f^\delta(\cdot, t)\|_{L^1} = 1$ . Following exactly the same steps as in the proof of Proposition 4.6, one can show

$$\int_{\mathbb{R}^3} f^\delta(v, t) |v|^2 dv \leq 3 + 10(1 + M)T, \quad \forall t \in [0, T]. \tag{4-21}$$

Hence (4-18) follows from (4-20) and (4-21) by choosing

$$\begin{aligned} R &:= 2(3 + 10(1 + M)T)^{1/2}, \\ r &:= (8\pi M)^{-1/3}. \end{aligned}$$

Finally, (4-19) follows from (4-18), the selection of  $R$  and  $r$  above and Lemma 3.3. □

**Theorem 4.14.** *Given  $f_{\text{in}}$  as in (1-8), there is a time  $T_*$  and a function  $f \in C_{\text{loc}}^{2;1}(\mathbb{R}^3 \times (0, T_*))$  with initial data  $f_{\text{in}}$ , which solves (1-4) or (1-7). Moreover, either  $T_* = \infty$  or*

$$\limsup_{t \rightarrow T_*^-} \|f\|_{L^\infty(\mathbb{R}^3 \times [0, t])} = \infty.$$

*The initial data is achieved in the sense that for any  $\phi \in C_c^\infty(\mathbb{R}^3)$  and any  $t \in (0, T_*)$  we have*

$$\int_{\mathbb{R}^3} f(v, t) \phi(v) dv - \int_{\mathbb{R}^3} f_{\text{in}}(v) \phi(v) dv = - \int_0^t \int_{\mathbb{R}^3} (B[f] \nabla f - f \nabla a[f], \nabla \phi) dv dt.$$

*Here  $B[f]$  denotes  $A[f]$  or  $a[f]\mathbb{I}$  depending on whether we are dealing with (1-4) or (1-7).*

*Proof. Step 1.* Let us assume first that  $f_{\text{in}}$  satisfies the additional assumptions (4-1); this assumption will be removed in the final step. For each  $n \in \mathbb{N}$ , let  $f_n := f^{\delta_n}$  and  $T_n := T_*^{\delta_n}$  correspond to  $f^\delta$  with  $\delta = 10^{-n}$ , as constructed in Theorem 4.12. Then each  $f_n$  is a spherically symmetric, monotone solution to

$$\partial_t f_n = \frac{1}{10^n} \Delta f_n + Q(f_n, f_n) \quad \text{in } \mathbb{R}^3 \times [0, T_n), \quad f_n(v, 0) = f_{\text{in}}(v).$$

Moreover, for each  $n$ , we have that either  $T_n = \infty$  or else  $\|f_n(\cdot, t)\|_\infty \rightarrow \infty$  as  $t \rightarrow T_n$ .

We define  $T_*$  by

$$T_* := \inf\{T \mid \liminf_n M(f_{\text{in}}, T, 10^{-n}) = \infty\}, \tag{4-22}$$

with the understanding that  $T_* = \infty$  if the set above is empty. As before, it is not difficult to see that  $T_* \geq (2f_{\text{in}}(0))^{-1}$ . See Remark 4.15 for further discussion about the definition of  $T_*$ .

Step 2. Let us show, then, that there exists a solution in  $\mathbb{R}^3 \times (0, T_*)$ . Let  $T_j$  be a strictly increasing sequence of times, with  $\lim T_j = T_*$ . Fix  $j$ , then since  $T_j < T_*$  there is a subsequence  $\{n_{j,k}\}$ ,  $n_{j,k} \rightarrow \infty$  as  $k \rightarrow \infty$ , such that

$$\sup_k M(f_{\text{in}}, T, 10^{-n_{j,k}}) < \infty.$$

The above combined with Proposition 4.13 implies there is a constant  $c = c(f_{\text{in}}, T_j)$  such that for all  $k \in \mathbb{N}$  we have

$$A[f_{n_{j,k}}](v, t) \geq \frac{c(f_{\text{in}}, T_j)}{1 + |v|^3} \mathbb{I}, \quad \forall (v, t) \in \mathbb{R}^3 \times (0, T_j).$$

The interior Hölder estimate (Theorem 2.2) then says that for any cylinder  $Q \Subset \mathbb{R}^3 \times (0, T)$  we have

$$[f_{n_{j,k}}]_{C^{\alpha, \alpha/2}(Q)} \leq C(Q, T_j), \quad \forall k.$$

From here, the same argument as in Lemma 4.7 shows that  $A[f_{n_{j,k}}]$  and  $\nabla a[f_{n_{j,k}}]$  are  $C^{\alpha, \alpha/2}$  uniformly in  $k$  in compact subsets of  $\mathbb{R}^3 \times (0, T_j)$ . Accordingly, the uniform regularity of these coefficients together with the Schauder estimates (Theorem 2.4) guarantee that for every cylinder  $Q \Subset \mathbb{R}^3 \times (0, T_j)$  we have a constant  $C(Q, T_j)$  independent of  $k$  such that

$$[f_{n_{j,k}}]_{C^{2+\alpha, 1+\alpha/2}(Q)} \leq C(Q, T_j).$$

Then, the Arzelà–Ascoli theorem and a Cantor diagonalization argument yield local uniform convergence of  $f_n$  to a function  $f$  in  $\mathbb{R}^3 \times (0, T)$  which will be differentiable in time and second-order differentiable in space. In particular,  $\tilde{f}_j$  is a spherically symmetric, monotone solution to

$$\partial_t \tilde{f}_j = Q(\tilde{f}_j, \tilde{f}_j) \quad \text{in } \mathbb{R}^3 \times (0, T_j), \quad \tilde{f}_j(\cdot, 0) = f_{\text{in}},$$

with  $f_{\text{in}}$  as in (1-8). We can take this argument one step further and apply the Arzelà–Ascoli theorem one more time to the sequence  $\{\tilde{f}_j\}_j$  and conclude that along a subsequence they (along with their derivatives) converge uniformly in compact subsets of  $\mathbb{R}^3 \times (0, T_*)$  to a function

$$f : \mathbb{R}^3 \times (0, T_*) \rightarrow \mathbb{R}$$

which is again a solution. In summary, we have constructed a function  $f : \mathbb{R}^3 \times (0, T_*)$  which is differentiable in time and second-order differentiable in space, such that

$$\partial_t f = Q(f, f)$$

and

$$\int_{\mathbb{R}^3} f(v, t)\phi(v) dv - \int_{\mathbb{R}^3} f_{\text{in}}(v)\phi(v) dv = - \int_0^t \int_{\mathbb{R}^3} (B[f]\nabla f - f\nabla a[f], \nabla \phi) dv dt, \quad \forall \phi \in C_c^\infty(\mathbb{R}^3), t \in (0, T_*). \quad (4-23)$$

Moreover, the function  $f$  has the property that for every  $T < T_*$ , there is a sequence  $n_k \rightarrow \infty$  such that the functions  $f_{n_k}$  defined in Step 1 converge to  $f$  locally uniformly in  $\mathbb{R}^3 \times [0, T]$ .

*Step 3.* It remains to show that if  $T_* < \infty$ , then the solution built in Step 2 blows up in  $L^\infty$  as time approaches  $T_*$ . We argue by contradiction, similarly to the proof of Theorem 4.12, but with slight modifications accounting for the fact that we do not know whether the functions  $f_n$  have a unique limit as  $n \rightarrow \infty$  (see Remark 4.15 for further discussion). Suppose  $C > 0$  is a constant such that

$$\lim_{T \rightarrow T_*} \|f\|_{L^\infty(\mathbb{R}^3 \times [0, T])} < C.$$

Let  $\varepsilon > 0$  be a small number (to be determined). According to Step 2, there is a sequence  $n_k \rightarrow \infty$  such that  $f_{n_k} \rightarrow f$  locally uniformly in  $\mathbb{R}^3 \times [0, T_* - \varepsilon/2]$ . In particular, there must be some  $k_0 > 0$  such that

$$\|f_{n_k}\|_{L^\infty(B_1 \times [0, T_* - \varepsilon])} < 2C, \quad \forall k > k_0.$$

As in the proof of Theorem 4.14, choosing  $\varepsilon$  such that  $2\varepsilon(2C) < \frac{1}{2}$ , the differential inequality argument guarantees that

$$\|f_{n_k}\|_{L^\infty(\mathbb{R}^3 \times [0, T_* + \varepsilon])} \leq 4C, \quad \forall k > k_0.$$

This shows there is a positive  $\varepsilon > 0$  such that

$$\liminf_n M(f_{\text{in}}, T, 10^{-n}) < \infty, \quad \forall T < T_* + \varepsilon.$$

This is impossible, since  $T_*$  is the infimum of  $\{T \mid \liminf_n M(f_{\text{in}}, T, 10^{-n}) = \infty\}$ . This contradiction shows that

$$\lim_{T \rightarrow T_*} \|f\|_{L^\infty(\mathbb{R}^3 \times [0, T])} = \infty,$$

and the theorem is proved at least for  $f_{\text{in}}$ , for which (4-1) holds.

*Step 4.* In order to remove (4-1), given  $f_{\text{in}}$  for which only (1-8) holds, let  $f_{\text{in}}^{(n)}$  be a sequence of functions such that (4-1) holds for each  $f_{\text{in}}^{(n)}$  (with a constant  $c$  that may depend on  $n$ ) and such that

$$\lim_n \|f_{\text{in}} - f_{\text{in}}^{(n)}\|_{L^\infty} = \lim_n \|f_{\text{in}} - f_{\text{in}}^{(n)}\|_{L^1} = 0.$$

Let  $f^{(n)}$  be a corresponding sequence of solutions as constructed in Steps 1–4 above. Then each  $f^{(n)}$  is defined up to some time  $T_{*,n}$ . The times  $T_{*,n}$  are bounded uniformly away from 0 since  $f_{\text{in}} \in L^\infty$ . The functions  $f^{(n)}$  enjoy uniform local a priori estimates, therefore the same compactness argument from Step 2 allows us to pick a subsequence  $n_k \rightarrow \infty$  and a time  $T_*$  such that the functions  $f^{(n_k)}$  and their derivatives have a local uniform limit as  $k \rightarrow \infty$  to a function  $f : \mathbb{R}^3 \times (0, T_*) \rightarrow \mathbb{R}$  which is a smooth solution to the nonlinear equation and which blows up in  $L^\infty$  as time approaches  $T_*$ . Finally, fixing a test function  $\phi$  and  $t \in (0, T_*)$ , we may apply (4-23) to each  $f^{(n_k)}$  and conclude that  $f$  satisfies the respective relation in the limit, proving the theorem.  $\square$

**Remark 4.15.** It is worth comparing the definition of  $T_*^\delta$  in Theorem 4.12 with that of  $T_*$  in Theorem 4.14. In the present situation, a priori it is unclear whether the sequence  $f_n$  has a unique limit as  $n \rightarrow \infty$ . Hence, if we define

$$T_* := \sup\{T \mid \sup_n M(f_{\text{in}}, T, 10^{-n}) < \infty\},$$

the existence of a subsequence bounded for times strictly greater than  $T^*$  does not contradict the definition of  $T^*$ . However, the contradiction holds if  $T^*$  is defined via the  $\liminf$  as in (4-22). In the proof of the

former theorem, matters were simplified by the fact that  $\{f_k^\delta\}_k$  was a Cauchy sequence (for  $\delta$  fixed), meaning in particular that if it is shown that a subsequence of  $f_k^\delta$  remains bounded in  $[0, T]$ , then the entire sequence remains bounded. This was key in proving the maximality of the interval of existence  $(0, T_*^\delta)$ .

### 5. Pointwise bounds and proof of Theorem 1.1

**Conditional pointwise bound.** The first lemma of this section (Lemma 5.2) is the key argument for the proofs of Theorem 1.1 and Theorem 1.3. It consists of a barrier argument based on the observation that the function  $U(v) = |v|^{-\gamma}$  is a supersolution for the elliptic operator  $Q(f, \cdot)$  under certain assumptions on  $f$  (this is where the radial symmetry and monotonicity is needed). It affords control of certain spatial  $L^p$  norms of the solution, and from these higher regularity will follow by standard elliptic estimates (Lemma 5.5).

First, we prove an elementary proposition that will be of use in proving the key lemma.

**Proposition 5.1.** *If  $h$  is a nonnegative, radially symmetric and decreasing function, then*

$$\frac{h(v)}{a[h](v)} \leq 8 \sup_{r \leq |v|} \left\{ r^2 \frac{\int_{B_r} h(w) \, dw}{\int_{B_r} a[h](w) \, dw} \right\} |v|^{-2}, \quad \forall v \in \mathbb{R}^3.$$

*Proof.* First of all, since  $h$  is radially symmetric and decreasing,

$$\frac{1}{|B_{|v|}(0)|} \int_{B_{|v|}(0)} h(w) \, dw \geq h(v).$$

On the other hand, since  $h \geq 0$  and (in particular)  $a[h]$  is superharmonic,

$$a[h](v) \geq \frac{1}{|B_{2|v|}(v)|} \int_{B_{2|v|}(v)} a[h](w) \, dw = \frac{2^{-3}}{|B_{|v|}(0)|} \int_{B_{|v|}} a[h](w) \, dw, \quad \forall v \in \mathbb{R}^3.$$

Therefore,

$$\frac{h(v)}{a[h](v)} \leq 8 \frac{\int_{B_{|v|}} h(w) \, dw}{\int_{B_{|v|}(0)} a[h](w) \, dw},$$

which implies that

$$\frac{h(v)}{a[h](v)} \leq 8|v|^{-2} \sup_{r \leq |v|} \left\{ r^2 \frac{\int_{B_r} h(w) \, dw}{\int_{B_r} a[h](w) \, dw} \right\}. \quad \square$$

**Lemma 5.2.** *Suppose  $f : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}_+$  is a classical solution of (2-1). Let  $\gamma \in (0, 1)$  and suppose there exists some  $R_0 > 0$  such that*

$$r^2 \frac{\int_{B_r} g(w, t) \, dw}{\int_{B_r} a[g](w, t) \, dw} \leq \frac{1}{24} \gamma (1 - \gamma), \quad \forall r \leq R_0, t \leq T. \tag{5-1}$$

Then

$$f(v, t) \leq \max \left\{ \frac{3}{4\pi} R_0^{\gamma-3}, \left( \frac{3}{4\pi} \right)^{\gamma/3} \|f_{in}\|_{L^{3/\gamma}_{weak}} \right\} |v|^{-\gamma}, \quad \text{in } B_{R_0} \times [0, T].$$

In particular, the conclusion of the lemma holds for some  $R_0 > 0$  whenever there is a modulus of continuity  $\omega(r)$  and some  $R_1 > 0$  such that

$$\sup_{r < |v|} \sup_{t \in [0, T]} \left\{ r^2 \frac{\int_{B_r} g(w, t) dw}{\int_{B_r} a[g](w, t) dw} \right\} \leq \omega(|v|), \quad \forall 0 < |v| \leq R_1. \quad (5-2)$$

**Remark 5.3.** It is easy to see that for any radially decreasing function  $h(v)$ , the condition that  $h$  belongs to  $L^p_{\text{weak}}(\mathbb{R}^3)$  implies that  $h$  lies below a power function of the form  $1/|v|^{3/p}$ , and vice versa. More precisely,

$$\|h(v)\|_{L^p_{\text{weak}}} \leq C \Leftrightarrow h(v) \leq C \left( \frac{3}{4\pi} \right)^{1/p} |v|^{-3/p}. \quad (5-3)$$

*Proof of Lemma 5.2.* Let  $U_\gamma = |v|^{-\gamma}$ . Then Proposition 3.4 says that

$$Q(g, U_\gamma) \leq U_\gamma a[g] \left( -\frac{1}{3} \gamma (1 - \gamma) |v|^{-2} + \frac{g}{a[g]} \right).$$

Applying Proposition 5.1 with  $h = g(\cdot, t)$ ,

$$\frac{g}{a[g]}(v, t) \leq 8|v|^{-2} \sup_{r \leq |v|} \left\{ r^2 \frac{\int_{B_r} g(w, t) dw}{\int_{B_r} a[g](w, t) dw} \right\} \leq \frac{1}{3} \gamma (1 - \gamma) |v|^{-2},$$

where we used (5-1) to get the last inequality. It follows that

$$Q(g, U_\gamma) \leq 0, \quad \text{in } B_{R_0} \times [0, T]. \quad (5-4)$$

In particular, if there is a modulus of continuity as in (5-2), then  $Q(g, U_\gamma) \leq 0$  in  $B_{R_0} \times [0, T]$  provided  $R_0$  is chosen so that  $\omega(R_0) \leq \frac{1}{24}$ .

On the other hand, given that  $f(v, t)$  is radially decreasing and lies in  $L^1$  (see (5-3)),

$$f(v, t) \leq \frac{3}{4\pi |v|^3} \|f\|_{L^1(\mathbb{R}^3)} = \frac{3}{4\pi |v|^3}, \quad \forall v \in \mathbb{R}^3, t \in [0, T], \quad (5-5)$$

where we used that  $\|f(\cdot, t)\|_{L^1(\mathbb{R}^3)} = 1$  for all  $t$ . Finally, the function  $\tilde{U}_\gamma(v)$  defined by

$$\tilde{U}_\gamma(v) := \max \left\{ \frac{3}{4\pi} R_0^{\gamma-3}, \left( \frac{3}{4\pi} \right)^{\gamma/3} \|f_{\text{in}}\|_{L^{3/\gamma}_{\text{weak}}} \right\} |v|^{-\gamma}$$

is a supersolution for the equation solved by  $f$  in  $B_{R_0} \times [0, T]$ . Moreover, clearly  $\tilde{U}_\gamma$  lies above  $f_{\text{in}}$  in  $B_{R_0}$ , while by (5-5),  $\tilde{U}_\gamma$  lies above  $f$  in  $\partial B_{R_0} \times [0, T]$ . Then the comparison principle implies that  $f \leq \tilde{U}_\gamma$  in  $B_{R_0} \times [0, T]$ , and the lemma is proved.  $\square$

The next lemma deals specifically with solutions to the nonlinear equations (1-4) or (1-7). It controls from below the integral of a solution in some ball  $B_R$ . For the case of the Landau equation (1-4), the constant is independent of time (by conservation of mass and second moment), while for the Krieger–Strain equation (1-7) the bound decays exponentially in time.

**Lemma 5.4.** For  $f$  solving (1-4), there is a constant  $R > 0$  such that

$$\int_{B_R} f(v, t) dv \geq \frac{1}{2}, \quad t > 0. \quad (5-6)$$

For  $f$  solving (1-7) and any radii  $R > r > 0$ , there are  $\beta > 0$  and  $C_0 > 0$  such that

$$\int_{B_R \setminus B_r} f(v, t) \, dv \geq C_0 e^{-\beta t} \int_{B_{4R} \setminus B_{r/4}} f_{\text{in}}(v) \, dv, \quad t > 0. \tag{5-7}$$

*Proof.* If  $f$  solves (1-4), then

$$\int_{B_R(0)^c} f(v, t) \, dv \leq R^{-2} \int_{B_R(0)^c} f(v, t) |v|^2 \, dv \leq 3R^{-2}.$$

Thus

$$\int_{B_R(0)} f(v, t) \, dv = 1 - \int_{B_R(0)^c} f(v, t) \, dv \geq 1 - 3R^{-2}.$$

Estimate (5-6) follows by choosing  $R$  large enough. The corresponding estimate (5-7) for  $f$  solving (1-7) follows a similar argument used in [Krieger and Strain 2012], and the derivation of the estimate is done in detail in the Appendix.  $\square$

The next lemma says that any solution  $f$  to (1-4) or (1-7) is a bounded function for all times, provided that  $f$  satisfies (5-2).

**Lemma 5.5.** *Let  $f : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}$  be a radially symmetric, radially decreasing solution to (1-4) (or (1-7)) with initial data as in (1-8) and such that for some  $R_0 > 0$ , we have*

$$r^2 \frac{\int_{B_r} f(w, t) \, dw}{\int_{B_r} a[f](w, t) \, dw} \leq \frac{1}{24} \gamma (1 - \gamma), \quad \forall r \leq R_0, \, t \leq T.$$

*Or, assume that there is some modulus of continuity  $\omega(r)$  such that*

$$\sup_{r < |v|} \sup_{t \in [0, T]} \left\{ r^2 \frac{\int_{B_r} f(w, t) \, dw}{\int_{B_r} a[f](w, t) \, dw} \right\} \leq \omega(|v|), \quad \forall 0 < r \leq R_0. \tag{5-8}$$

*Then*

$$\sup_{t \in [T/2, T]} \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq C_0 \tag{5-9}$$

*for some constant  $C_0$  depending only on  $f_{\text{in}}$ ,  $T$  and  $R_0$ .*

*Proof.* The assumptions of the lemma are simply the same as those of Lemma 5.2 with  $g(v, t) = f(v, t)$ , from which it follows, using also (5-3), that

$$\begin{aligned} \sup_{t \in [0, T]} \|f(\cdot, t)\|_{L^p_{\text{weak}}(B_{R_0})} &\leq \max \left\{ \frac{3}{4\pi} R_0^{-3(1-1/p)}, \left( \frac{3}{4\pi} \right)^{1/p} \|f_{\text{in}}\|_{L^p_{\text{weak}}} \right\} \| |v|^{-3/p} \|_{L^p_{\text{weak}}} \\ &=: C_0(f_{\text{in}}, R_0, p) \end{aligned}$$

for some  $p > 6$ . By interpolation and the Sobolev embedding, it follows that  $\|f(\cdot, t)\|_{L^6(\mathbb{R}^3)}$  and  $\|\nabla a[f(\cdot, t)]\|_{L^\infty(\mathbb{R}^3)}$  are bounded by some constant  $C$  determined by  $C_0(f_{\text{in}}, R_0, p)$ . Then, applying (2-5) from Theorem 2.3 with  $Q = B_{R_0} \times [0, T]$ , we arrive at

$$\|f\|_{L^\infty(B_{R_0/2} \times [T/2, T])} \leq C \left\{ \|f\|_{L^2(Q)} + R_0^2 \|\nabla a[f]\|_{L^\infty(Q)} \right\} < \infty$$

for some  $C = C(f_{\text{in}}, R_0, T)$ , and the lemma is proved.  $\square$

*Proof of Theorem 1.2.* According to Theorem 4.12, for  $f_{\text{in}} \in L^\infty$ , there exists a time  $T_0 > 0$  and a solution  $f(v, t)$  to (1-4) defined in  $\mathbb{R}^3 \times [0, T_0)$  and with initial values  $f_{\text{in}}$ .

The time  $T_0$  is maximal, in the sense that  $T_0 = \infty$  or else

$$\lim_{t \rightarrow T_0^-} \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} = \infty. \tag{5-10}$$

Moreover, since  $f \in L^\infty$  in  $\mathbb{R}^3 \times [0, t]$  for every  $t < T_0$ , interior regularity estimates (see Theorem 2.2 and Theorem 2.4) show that  $f$  must be twice differentiable in  $v$  and differentiable in  $t$  as long as  $t \in (0, T)$ .

Finally, arguing by contradiction, let us assume that

$$\limsup_{r \rightarrow 0^+} \sup_{t \in (0, T_0)} \left\{ r^2 \frac{\int_{B_r} f(v, t) \, dv}{\int_{B_r} a[f](v, t) \, dv} \right\} < \frac{1}{96}.$$

In this case, there must be some  $R_0 > 0$  such that

$$\sup_{t \in (0, T_0)} \left\{ r^2 \frac{\int_{B_r} f(v, t) \, dv}{\int_{B_r} a[f](v, t) \, dv} \right\} \leq \frac{1}{96}, \quad \forall r \leq R_0.$$

This means Lemma 5.5 can be applied with  $T = T_0$ , and it follows that

$$\sup_{t \in [T_0/2, T_0]} \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} < \infty,$$

which contradicts (5-10), and the theorem is proved. □

*Proof of Theorem 1.1.* As in the proof of Theorem 1.2, we have a solution  $f(v, t)$  defined up to some maximal time  $T_0$ . In case  $T_0 < \infty$ , we know that  $\|f(\cdot, t)\|_{L^\infty}$  goes to infinity as  $t \rightarrow T_0^-$ . As before, this  $f(v, t)$  is twice differentiable in  $v$  and differentiable in  $t$  for  $t \in (0, T)$ .

Now assume the  $L^{3/2}$  norm of  $f(\cdot, t)$  does not concentrate at 0 as  $t \rightarrow T^-$ . That is, suppose there is a modulus of continuity  $\omega(\cdot)$  such that

$$\sup_{t \in (0, T_0)} \|f(\cdot, t)\|_{L^{3/2}(B_r)} \leq \omega(r).$$

Then there is some  $C > 0$  such that

$$r^2 \frac{\int_{B_r} f(v, t) \, dv}{\int_{B_r} a[f](v, t) \, dv} = \frac{\frac{4\pi}{3r} \int_{B_r} f(v, t) \, dv}{\frac{1}{|B_r|} \int_{B_r} a[f](v, t) \, dv} \leq C \frac{1}{r} \int_{B_r} f(v, t) \, dv, \quad \forall r > 0, t \in (0, T_0).$$

Then Hölder's inequality says that

$$r^2 \frac{\int_{B_r} f(v, t) \, dv}{\int_{B_r} a[f](v, t) \, dv} \leq C' \|f(\cdot, t)\|_{L^{3/2}(B_r)} \leq C' \omega(r).$$

It follows that if  $R_0 > 0$  is chosen so that  $C' \omega(R_0) < \frac{1}{96}$ , then Lemma 5.5 can be applied to conclude again that

$$\sup_{t \in [T_0/2, T_0]} \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} < \infty,$$

which as before directly contradicts  $\lim_{t \rightarrow T_0^-} \|f(\cdot, t)\|_{L^\infty} = \infty$ , and the theorem is proved. □

To end this section, we present a computation indicating that for an arbitrary function  $f$  the quotient appearing in the assumption of Theorem 1.2 is always smaller than or equal to 3.

**Proposition 5.6.** *Let  $h \in L^1(\mathbb{R}^3)$  be a nonnegative function. Then*

$$r^2 \frac{\int_{B_r} h(v) \, dv}{\int_{B_r} a[h](v) \, dv} \leq 3, \quad \forall r > 0.$$

**Remark 5.7.** It could be of use in understanding the blow-up or (non-blow-up) of (1-4) to characterize those  $h$  for which the above quotient goes to 0 as  $r$  approaches 0. In particular, it would be useful to understand this when  $h$  is not necessarily in a regular enough  $L^p$  space or Morrey space, namely when  $h$  is such that

$$h \notin L^{3/2}_{\text{loc}} \quad \text{or} \quad \sup_{r>0} \frac{1}{r} \int_{B_r} h \, dv = \infty.$$

*Proof of Proposition 5.6.* Let us write  $a(v) = a[h](v)$  for the sake of brevity. Note that

$$\int_{B_r} a(v) \, dx = \int_{\mathbb{R}^3} a(v) \chi_{B_r}(v) \, dv = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} h(w) |v - w|^{-1} \chi_{B_r}(v) \, dw \, dv.$$

The goal is to compare the two integrals

$$\frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} h(w) |v - w|^{-1} \chi_{B_r}(v) \, dw \, dv \quad \text{and} \quad r^2 \int_{\mathbb{R}^3} h(v) \chi_{B_r}(v) \, dv.$$

Note that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} h(w) |v - w|^{-1} \chi_{B_r}(v) \, dw \, dv = \int_{\mathbb{R}^3} h(v) (\chi_{B_r} * \Phi)(v) \, dv, \quad \Phi(v) = (4\pi |v|)^{-1}.$$

It is not hard to compute  $\Phi_B := \chi_{B_r} * \Phi$  directly. Indeed, it is the unique  $C^{1,1}$  solution of

$$\Delta \Phi_{B_r} = -\chi_{B_r}, \quad \Phi_{B_r} \rightarrow 0 \text{ at } \infty,$$

which has the simple expression

$$\Phi_{B_r}(x) = \begin{cases} -\frac{1}{6}|v|^2 + \frac{1}{2}r^2 & \text{in } B_r, \\ \frac{1}{3}r^3|v|^{-1} & \text{in } B_r^c. \end{cases}$$

It follows that

$$\int_{B_r} a(v) \, dv = \int_{B_r} \left(\frac{1}{2}r^2 - \frac{1}{6}|v|^2\right)h(v) \, dv + \frac{r^3}{3} \int_{B_r^c} h(v)|v|^{-1} \, dv \geq \int_{B_r} \left(\frac{1}{2}r^2 - \frac{1}{6}|v|^2\right)h(v) \, dv.$$

This proves the stated bound, since the last inequality guarantees that

$$\int_{B_r} a(v) \, dv \geq \frac{r^2}{3} \int_{B_r} h(v) \, dv. \quad \square$$



### 6. Mass comparison and proof of Theorem 1.3

In this section we apply the ideas from previous sections to construct global solutions (in the radial, monotone case) for (1-7), namely

$$\partial_t f = a[f]\Delta f + f^2.$$

In view of Lemma 5.5, the fact that  $T_0 = \infty$  in Theorem 1.1 results from a bound of any  $L^p(\mathbb{R}^3)$  norm of  $f$  with  $p > \frac{3}{2}$ . For (1-7), the bound of any  $L^p(\mathbb{R}^3)$  norm of  $f$  with  $p > \frac{3}{2}$  will be proven by a barrier argument done at the level of the *mass function* of  $f(v, t)$ , which is defined by

$$M_f(r, t) = \int_{B_r} f(v, t) dv, \quad (r, t) \in \mathbb{R}_+ \times (0, T_0).$$

Depending on which problem  $f$  solves, the associated function  $M_f(r, t)$  solves a one-dimensional parabolic equation with diffusivity given by  $A^*[f]$  or  $a[f]$ .

**Proposition 6.1.** *Let  $f$  be a solution of (1-4) or (1-7) in  $\mathbb{R}^3 \times [0, T_0]$ . Then  $M(r, t)$  solves, respectively,*

$$\partial_t M_f = A^* \partial_{rr} M_f + \frac{2}{r} \left( \frac{M_f}{8\pi r} - A^* \right) \partial_r M_f \quad \text{in } \mathbb{R}_+ \times (0, T_0) \quad (6-1)$$

$$\partial_t M_f = a \partial_{rr} M_f + \frac{2}{r} \left( \frac{M_f}{8\pi r} - a \right) \partial_r M_f \quad \text{in } \mathbb{R}_+ \times (0, T_0). \quad (6-2)$$

*Proof.* We briefly show how to obtain (6-2); for (6-1), the calculations are identical. Using the divergence theorem and the divergence expression in (1-7), we get

$$\partial_t M_f = \int_{\partial B_r} (a[f]\nabla f - f\nabla a[f], n) d\sigma = 4\pi r^2 (a[f]\partial_r f - f\partial_r a[f]).$$

Furthermore, straightforward differentiation yields the formulas

$$4\pi r^2 \partial_r f = r^2 \partial_r (r^{-2} \partial_r M_f), \quad \partial_r a[f] = -(4\pi r^2)^{-1} M_f.$$

Substituting these in the expression for  $\partial_t M_f$  above, we get

$$\partial_t M_f = a[f] r^2 \partial_r \left( \frac{1}{r^2} \partial_r M_f \right) + \frac{1}{4\pi r^2} M_f \partial_r M_f.$$

Expansion and rearrangement of the terms result in

$$\partial_t M_f = a \left( -\frac{2}{r} \partial_r M_f + \partial_{rr} M_f \right) + \frac{M_f}{4\pi r^2} \partial_r M_f = a \partial_{rr} M_f + \frac{2}{r} \left( \frac{M_f}{8\pi r} - a \right) \partial_r M_f,$$

and the conclusion follows.  $\square$

Define the linear parabolic operator  $L$  in  $\mathbb{R}_+ \times (0, T)$  as

$$Lh := \partial_t h - a \partial_{rr} h - \frac{2}{r} \left( \frac{M_f}{8\pi r} - a[f] \right) \partial_r h.$$

The above proposition simply says that  $LM_f = 0$  in  $\mathbb{R}_+ \times (0, T)$ . The next proposition identifies suitable supersolutions for  $L$ .

**Proposition 6.2.** *If  $m \in [0, 2]$  and  $h(r, t) = r^m$ , then  $Lh \geq 0$  in  $\mathbb{R}_+ \times (0, T)$ .*

*Proof.* By direct computation we see that

$$Lh = -mr^{m-2} \left( (m-1)a + 2 \left( \frac{M_f}{8\pi r} - a[f] \right) \right).$$

On the other hand,

$$a[f](r) = \frac{1}{4\pi r} \int_{B_r} f \, dv + \int_{B_r^c} \frac{f}{4\pi|v|} \, dv \geq \frac{M_f}{4\pi r},$$

which guarantees that  $\frac{1}{2}a[f](r) \geq \frac{M_f}{8\pi r}$ . Thus,

$$Lh = mr^{m-2} \left( (1-m)a[f] + 2 \left( a[f] - \frac{M_f}{8\pi r} \right) \right) \geq mr^{m-2}(2-m)a[f] \geq 0,$$

the last inequality being true for  $m \leq 2$ . □

*Proof of Theorem 1.3.* Assume  $f_{in} \in L^\infty$ , in which case Theorem 4.12 yields a solution  $f(v, t)$  that exists for some time  $T_0 > 0$  (possibly infinite). As the bound for  $f(v, t)$  will not rely on the  $L^\infty$  norm of  $f_{in}$  but an  $L^p_{\text{weak}}$  norm of  $f_{in}$ , the existence of a solution for unbounded initial data in  $L^p$  ( $p > 6$ ) will follow by a standard density argument.

Since  $p > 6$ , there is some  $\alpha > 0$  and some  $C_0 > 0$  (depending only on  $\|f\|_{L^p_{\text{weak}}}$ ) such that

$$M_{f_{in}}(r, 0) = \int_{B_r} f_{in} \, dv \leq C_0 r^{1+\alpha}.$$

Moreover, since  $f(\cdot, t)$  has total mass 1 for every  $t > 0$ , we also have

$$M_f(r, t) \leq 1, \quad \forall r > 0, t \in (0, T).$$

Proposition 6.2 says that  $h = Cr^{1+\alpha}$  is a supersolution of the parabolic equation solved by  $M_f$  in  $\mathbb{R}_+ \times (0, T)$ . Then, choosing  $C := \max\{C_0, 1\}$ , the comparison principle yields

$$M_f(r, t) \leq h(r) = Cr^{1+\alpha}, \quad \forall r \in (0, 1), t \in (0, T). \tag{6-3}$$

Since  $f(v, t)$  is radially symmetric and decreasing, bound (6-3) implies that  $f(|v|, t) \leq \frac{3C}{4\pi} \frac{1}{|v|^{2-\alpha}}$  for  $v \in B_1$  and  $t \in (0, T)$ ; hence there is some  $p' > \frac{3}{2}$  and some  $C_{p'} > 0$  such that

$$\|f(\cdot, t)\|_{L^{p'}(B_1)} \leq C_{p'}, \quad \forall t \in (0, T).$$

Then Lemma 5.2 says that  $f(v, t)$  is bounded in  $\mathbb{R}^3 \times (0, T_0)$ . By Lemma 5.5 and the characterization of  $T_0$  in Theorem 4.12, it follows that  $T_0 = +\infty$ , so the solution is global in time. □

The method of the proof for Theorem 1.3 falls short in preventing finite time blow-up for (1-4). In any case, it at least gives another criterion for blow-up, the proof of which is essentially the same as that of Theorem 1.3.

**Corollary 6.3.** *Suppose that for all  $t \in [0, T_0]$  there is some  $r_0 > 0$  and  $0 < \lambda < 8\pi$  such that*

$$M_f(r, t) \leq \lambda r A^*(r, t), \quad \forall r < r_0.$$

*Then any solution to (1-4) is bounded for any  $t > 0$ .*

### Appendix

*Proof of Proposition 3.1.* The radial symmetry of any solution  $f$  to (2-1) follows by the uniqueness property of (2-1) and by the fact that  $Q(g, f)$  commutes with rotations, as shown below. We first rewrite the collision operator as

$$Q(g, f) = \operatorname{div}(A[g]\nabla f - f\nabla a[g]) = a[g]\Delta f - \operatorname{div}(\tilde{A}[g]\nabla f) + fg,$$

with

$$\tilde{A}[g]\nabla f := \int \frac{g(|v-y|)}{|y|^3} \langle \nabla f(v), y \rangle y dy.$$

Let  $\mathbb{T}$  be a rotation operator. Since  $g$  is radially symmetric, so is  $a[g]$ . Hence

$$a[g]\Delta(f \circ \mathbb{T}) = a[g \circ \mathbb{T}]\Delta(f \circ \mathbb{T}) = (a[g] \circ \mathbb{T})(\Delta f \circ \mathbb{T}) = (a[g]\Delta f) \circ \mathbb{T},$$

taking into account that the Laplacian operator commutes with rotations. Moreover,

$$\begin{aligned} \operatorname{div}(\tilde{A}[g]\nabla f(\mathbb{T}v)) &= \operatorname{div}\left(\int \frac{g(|v-y|)}{|y|^3} \langle \nabla f(\mathbb{T}v), y \rangle y dy\right) \\ &= \operatorname{div}\left(\int \frac{g(|v-y|)}{|y|^3} \langle \mathbb{T}^* \nabla_z f(z)|_{z=\mathbb{T}v}, y \rangle y dy\right) \\ &= \operatorname{div}\left(\int \frac{g(|\mathbb{T}(v-y)|)}{|y|^3} \langle \nabla_z f(z)|_{z=\mathbb{T}v}, \mathbb{T}y \rangle \mathbb{T}^* \mathbb{T} y dy\right) \\ &= \operatorname{div}\left(\underbrace{\mathbb{T}^* \int \frac{g(|\mathbb{T}v-y|)}{|y|^3} \langle \nabla_z f(z)|_{z=\mathbb{T}v}, y \rangle y dy}_{=: V(\mathbb{T}v)}\right) \\ &= \operatorname{Tr}(\mathbb{T}^* \operatorname{Jac}(V)|_{z=\mathbb{T}v} \mathbb{T}) + \underbrace{\nabla(\operatorname{Tr}(\mathbb{T}^*))}_{=0} \cdot V(\mathbb{T}v) \\ &= \operatorname{Tr}(\mathbb{T} \mathbb{T}^* \operatorname{Jac}(V)|_{z=\mathbb{T}v}) \\ &= \operatorname{Tr}(\mathbb{I} \operatorname{Jac}(V)|_{z=\mathbb{T}v}) \\ &= \operatorname{div}\left(\int \frac{g(|z-y|)}{|y|^3} \langle \nabla_z f(z), y \rangle y dy\right) \circ \mathbb{T}. \end{aligned}$$

Hence  $Q(g, f(\mathbb{T}v)) = Q(g, f) \circ \mathbb{T}$ . Now we rewrite the linear equation (2-1) in spherical coordinates:

$$\partial_t f = A^* \partial_{rr} f + \frac{a-A^*}{r} \partial_r f + fg, \quad (6-4)$$

with  $A^*[g](v) := (A[g](v)\hat{v}, \hat{v})$ ,  $\hat{v} := v/|v|$  and differentiate (6-4) with respect to  $r$ . The function  $w := \partial_r f$  satisfies the inequality

$$\partial_t w \leq A^* \partial_{rr} w + \frac{a - A^*}{r} \partial_r w + wg + \partial_r A^* \partial_r w + \partial_r \left( \frac{a - A^*}{r} \right) w.$$

If  $w(\cdot, 0) \leq 0$  it follows from the maximum principle that  $w(\cdot, t) \leq 0$  for all  $t \geq 0$ . In other words, the (negative) sign of  $\partial_r f$  is preserved in time. □

*Proof of Proposition 3.2.* The identity (3-3) is classical and a proof can be found in [Lieb and Loss 2001, Section 9.7]. To prove (3-2), let  $v \in \mathbb{R}^3$  be nonzero and  $r := |v|$ . Then

$$(A[g](v)\hat{v}, \hat{v}) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{1}{|v - w|} g(w) \left( \left( \mathbb{I} - \frac{v - w}{|v - w|} \otimes \frac{v - w}{|v - w|} \right) \hat{v}, \hat{v} \right) dw.$$

Note that

$$\left( \left( \mathbb{I} - \frac{v - w}{|v - w|} \otimes \frac{v - w}{|v - w|} \right) \hat{v}, \hat{v} \right) = 1 - \cos(\hat{\theta}(w))^2,$$

where  $\hat{\theta}$  denotes the angle between  $w - v$  and  $v$ . Consider, for  $0 \leq t, r$ , the function

$$I(r, t) := \int_{\partial B_t} \frac{1 - \cos(\hat{\theta})^2}{|v - w|} dw.$$

The function  $I(r, t)$  encodes all the information about  $A^*$ . In particular, integration in spherical coordinates yields the expression

$$A^*[h](v) = \frac{1}{8\pi} \int_0^\infty f(t) I(|v|, t) dt.$$

As it turns out,  $I(r, t)$  has rather different behavior according to whether  $r < t$  or not. By averaging in the  $v$  variable, it is not hard to see that

$$I(r, t) = \frac{t^2}{r^4} I(t, r), \quad \forall r < t.$$

Accordingly, we focus on  $I(r, t)$  when  $r > t$ . To do so, denote by  $\theta$  the angle between  $w$  and  $v$  and observe that

$$1 - \cos(\hat{\theta})^2 = \sin(\hat{\theta})^2 = \frac{t^2 - t^2 \cos(\theta)^2}{|v - w|^2} = \frac{t^2 - w_1^2}{|v - w|^2},$$

where  $w_1 = (w, \hat{v})$ . Thus,

$$\begin{aligned} I(r, t) &= \int_{\partial B_t} \frac{t^2 - w_1^2}{|v - w|^3} dw = \int_{\partial B_t} \frac{t^2 - w_1^2}{(t^2 - w_1^2 + (r - w_1)^2)^{3/2}} dw \\ &= \int_{\partial B_t} \frac{t^2 - w_1^2}{(t^2 - 2r w_1 + r^2)^{3/2}} dw = \int_{\partial B_1} \frac{t^2(1 - z_1^2)}{t^3(1 - 2(\frac{r}{t})z_1 + (\frac{r}{t})^2)^{3/2}} t^2 dz \\ &= \int_{\partial B_1} \frac{1 - z_1^2}{(1 - 2(\frac{r}{t})z_1 + (\frac{r}{t})^2)^{3/2}} t dz. \end{aligned}$$

This surface integral can be written entirely as an integral in terms of the variable  $z_1 \in (-1, 1)$ :

$$I(r, t) = 2\pi t \int_{-1}^1 \frac{1 - z_1^2}{\left(1 - 2\left(\frac{r}{t}\right)z_1 + \left(\frac{r}{t}\right)^2\right)^{3/2}} dz_1.$$

For brevity, set for now  $s = r/t$ . Then

$$\begin{aligned} \int_{-1}^1 \frac{1 - z_1^2}{(1 - 2sz_1 + s^2)^{3/2}} dz_1 &= \frac{-2s^4 + 2s^3 + 2s - 2}{3s^3\sqrt{s^2 - 2s + 1}} - \frac{-2s^4 - 2s^3 - 2s - 2}{3s^3\sqrt{s^2 + 2s + 1}} \\ &= \frac{-2s^4 + 2s^3 + 2s - 2}{3s^3(s - 1)} - \frac{-2s^4 - 2s^3 - 2s - 2}{3s^3(s + 1)} \\ &= \frac{-2s^4 + 2s^3 + 2s - 2}{3s^3(s - 1)} + \frac{2s^4 + 2s^3 + 2s + 2}{3s^3(s + 1)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{-2s^4 + 2s^3 + 2s - 2}{3s^3(s - 1)} + \frac{2s^4 + 2s^3 + 2s + 2}{3s^3(s + 1)} &= \frac{2}{3s^3} \left( \frac{-s^4 + s^3 + s - 1}{s - 1} + \frac{s^4 + s^3 + s + 1}{s + 1} \right) \\ &= \frac{2}{3s^3} \frac{(-s^4 + s^3 + s - 1)(s + 1) + (s^4 + s^3 + s + 1)(s - 1)}{s^2 - 1} \\ &= \frac{2}{3s^3} \frac{2s^2 - 2}{s^2 - 1} = \frac{4}{3s^3}. \end{aligned}$$

Then, since  $s = r/t$ , we conclude that

$$\begin{aligned} I(r, t) &= 8\pi \frac{t^4}{3r^3}, \quad \text{for } t < r, \\ I(r, t) &= 8\pi \frac{1}{3t}, \quad \text{for } t > r. \end{aligned}$$

Going back to  $A^*[h]$ , the above leads to

$$\begin{aligned} A^*[h](v) &= \int_0^r h(t)I(r, t) dt + \int_r^\infty h(t)I(r, t) dt \\ &= \frac{1}{3r^3} \int_0^r h(t)t^4 dt + \frac{1}{3} \int_r^\infty h(t)t dt. \end{aligned} \quad \square$$

*Proof of Lemma 5.4.* This argument is inspired by the one in [Krieger and Strain 2012, Section 2.6]. For  $\beta, R, r$  (with  $0 < r < R, 0 < \beta$ ), consider the function

$$\Phi(v, t) := e^{-\beta t} (|v| - R)^2 (|v| - r)^2.$$

Since  $\Phi$  is a  $C^{1,1}$  function with compact support, we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} f(v, t) \Phi(v) dv = - \int_{\mathbb{R}^3} (a \nabla f - f \nabla a, \nabla \Phi) dv = \int_{\mathbb{R}^3} f \operatorname{div}(a \nabla \Phi) dv + \int_{\mathbb{R}^3} f (\nabla a, \nabla \Phi) dv.$$

Hence,

$$\begin{aligned} \operatorname{div}(a\nabla\Phi) + (\nabla a, \nabla\Phi) &= a\Delta\Phi + 2(\nabla a, \nabla\Phi) \\ &= a\Phi'' + \frac{2}{|v|}(a + |v|a')\Phi' = a\Phi'' + \frac{2}{|v|}\Phi' \int_{|v|}^{+\infty} sf(s, t) ds. \end{aligned}$$

We have

$$\begin{aligned} \Phi'(s) &= 2(R - s)(s - r)(-(s - r) + R - s) = 2(R - s)(s - r)(R + r - 2s), \\ \Phi''(s) &= 2(R - s)(r + R - 2s) - 2(s - r)(r + R - 2s) - 4(R - s)(s - r), \\ \Phi'(r) = \Phi'(R) &= 0, \quad \Phi''(r) = \Phi''(R) = 2(R - r)^2, \\ |\Phi''|, |\Phi'| &\leq C_{\delta,r,R}\Phi, \quad |v| \in ((1 + \delta)r, (1 - \delta)R). \end{aligned}$$

Hence in a small neighborhood of  $|v| = R$  and  $|v| = r$  one can show that  $\frac{d}{dt} \int_{\mathbb{R}^3} f(v, t)\Phi(v) dv \geq 0$ ; more precisely,

$$\operatorname{div}(a\nabla\Phi) + (\nabla a, \nabla\Phi) \geq 0 \quad \text{in } B_R \setminus B_{(1-\delta)R} \cup B_{(1+\delta)r} \setminus B_r.$$

Since  $a[g](v) \leq \frac{\|g\|_{L^1(\mathbb{R}^3)}}{|v|}$ , it follows that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} f(v, t)\Phi(v) dv &\geq -C_{\delta,r,R} \frac{\|g\|_{L^1(\mathbb{R}^3)}}{r} \int_{B_{(1-\delta)R} \setminus B_{(1+\delta)r}} f(v, t)\Phi(v) dv \\ &\geq -\frac{\|g\|_{L^1(\mathbb{R}^3)}}{r} C_{\delta,r,R} \int_{\mathbb{R}^3} f(v, t)\Phi(v) dv. \end{aligned}$$

This above differential inequality implies

$$\int_{\mathbb{R}^3} f(v, t)\Phi(v) dv \geq e^{-\beta t} \int_{\mathbb{R}^3} f_{\text{in}}\Phi(v) dv, \quad \forall t < T,$$

where  $\beta = C_{r,R,\alpha}\|g\|_{L^1}$ . Finally, since

$$\Phi(v) \leq \frac{1}{4}(R - r)^2 \quad \text{in } B_R \setminus B_r, \quad \Phi(v) \geq \frac{1}{4}R^2r^2,$$

we conclude that

$$\int_{B_R \setminus B_r} f(v, t) dv \geq \frac{R^2r^2}{(R - r)^4} e^{-\beta t} \int_{B_{R/2} \setminus B_{2r}} f_{\text{in}}(v)\Phi(v) dv, \quad \forall t < T. \quad \square$$

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
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