ANALYSIS & PDE

Volume 9 No. 8 2016

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ANALYSIS AND PDE
Vol. 9, No. 8, 2016
dx.doi.org/10.2140/apde.2016.9.1811

FORWARD SELF-SIMILAR SOLUTIONS OF THE NAVIER–STOKES EQUATIONS IN THE HALF SPACE

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For the incompressible Navier–Stokes equations in the 3D half space, we show the existence of forward self-similar solutions for arbitrarily large self-similar initial data.

1. Introduction

Let $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) : x_3 > 0\}$ be a half space with boundary $\partial\mathbb{R}^3_+ = \{x = (x_1, x_2, 0)\}$. Consider the 3D incompressible Navier–Stokes equations for velocity $u : \mathbb{R}^3_+ \times [0, \infty) \to \mathbb{R}^3$ and pressure $p : \mathbb{R}^3_+ \times [0, \infty) \to \mathbb{R}$,

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \operatorname{div} u = 0, \tag{1-1}$$

in $\mathbb{R}^3_+ \times [0, \infty)$, coupled with the boundary condition

$$u|_{\partial\mathbb{R}^3_+} = 0, \tag{1-2}$$

and the initial condition

$$u|_{t=0} = a, \quad \operatorname{div} a = 0, \quad a|_{\partial\mathbb{R}^3_+} = 0. \tag{1-3}$$

The system (1-1) enjoys a scaling property: if $u(x, t)$ is a solution, then so is

$$u^{(\lambda)}(x, t) := \lambda u(\lambda x, \lambda^2 t) \tag{1-4}$$

for any $\lambda > 0$. We say that $u(x, t)$ is *self-similar* (SS) if $u = u^{(\lambda)}$ for every $\lambda > 0$. In that case,

$$u(x, t) = \frac{1}{\sqrt{2t}}\, U\left(\frac{x}{\sqrt{2t}}\right), \tag{1-5}$$

where $U(x) = u\big(x, \frac{1}{2}\big)$. It is called *discretely self-similar* (DSS) if $u = u^{(\lambda)}$ for one particular $\lambda > 1$. To get self-similar solutions $u(x, t)$ we usually assume the initial data $a(x)$ is also self-similar, i.e.,

$$a(x) = \frac{a(\hat{x})}{|x|}, \quad \hat{x} = \frac{x}{|x|}. \tag{1-6}$$

In view of the above, it is natural to look for solutions satisfying

$$|u(x, t)| \le \frac{C(C_*)}{|x|} \quad \text{or} \quad \|u(\,\cdot\,, t)\|_{L^{3,\infty}} \le C(C_*), \tag{1-7}$$

MSC2010: 35Q30, 76D05.
Keywords: Forward self-similar solutions, Navier-Stokes equations, half space.

where C_* is some norm of the initial data a. For $1 \le q, r \le \infty$, we denote the Lorentz spaces by $L^{q,r}$. In such classes, with sufficiently small C_*, the unique existence of mild solutions — solutions of the integral equation version of (1-1)–(1-3) via a contraction mapping argument — has been obtained by Giga and Miyakawa [1989] and refined by Kato [1992], Cannone, Meyer and Planchon [Cannone et al. 1994; Cannone and Planchon 1996], and Barraza [1996]. It is also obtained in the broader class BMO^{-1} in [Koch and Tataru 2001]. In the context of the half space (and smooth exterior domains), it follows from [Yamazaki 2000]. As a consequence, if $a(x)$ is SS or DSS with small norm C_* and $u(x,t)$ is a corresponding solution satisfying (1-7) with small $C(C_*)$, the uniqueness property ensures that $u(x,t)$ is also SS or DSS, because $u^{(\lambda)}$ is another solution with the same bound and same initial data $a^{(\lambda)} = a$. For large C_*, mild solutions still make sense but there is no existence theory since perturbative methods like the contraction mapping no longer work.

Alternatively, one may try to extend the concept of weak solutions (which requires $u_0 \in L^2(\mathbb{R}^3)$) to more general initial data. One such theory is local-Leray solutions in L^2_{uloc}, constructed by Lemarié-Rieusset [2002]. However, there is no uniqueness theorem for them and hence the existence of large SS or DSS solutions was unknown. Recently, Jia and Šverák [2014] constructed SS solutions for every SS u_0 which is locally Hölder continuous. Their main tool is a local Hölder estimate for local-Leray solutions near $t = 0$, assuming minimal control of the initial data in the large. This estimate enables them to prove a priori estimates of SS solutions, and then to show their existence by the Leray–Schauder degree theorem. This result is extended by Tsai [2014] to the existence of discretely self-similar solutions.

When the domain is the half space $\mathbb{R}^3_+$, however, there is so far no analogous theory of local-Leray solutions. Hence the method of [Jia and Šverák 2014; Tsai 2014] is not applicable.

In this note, our goal is to construct SS solutions in the half space for arbitrary large data. By BC_w we denote bounded and weak-* continuous functions. Our main theorem is the following.

Theorem 1.1. *Let $\Omega = \mathbb{R}^3_+$ and let A be the Stokes operator in Ω* (see (5-5)–(5-7)). *For any self-similar vector field $a \in C^1_{\mathrm{loc}}(\overline{\Omega}\backslash\{0\})$ satisfying* $\operatorname{div} a = 0$, $a|_{\partial\Omega} = 0$, *there is a smooth self-similar mild solution $u \in BC_w([0,\infty); L^{3,\infty}_\sigma(\Omega))$ of* (1-1) *with $u(0) = a$ and*

$$\|u(t) - e^{-tA}a\|_{L^2(\Omega)} = Ct^{1/4}, \quad \|\nabla(u(t) - e^{-tA}a)\|_{L^2(\Omega)} = Ct^{-1/4}, \quad \forall t > 0. \tag{1-8}$$

Comments on Theorem 1.1:

(1) There is no restriction on the size of a.

(2) It is concerned only with existence. There is no assertion on uniqueness.

(3) Our approach also gives a second construction of large self-similar solutions in the whole space $\mathbb{R}^3$, but for initial data more restrictive (C^1) than those of [Jia and Šverák 2014]. In fact, it would show the existence of self-similar solutions in the cones

$$K_\alpha = \{0 \le \phi \le \alpha\}, \quad \text{for } 0 < \alpha \le \pi,$$

(in spherical coordinates), if one could verify Assumption 3.1 for $e^{-A/2}a$. We are able to verify it only for $\alpha = \frac{\pi}{2}$ and $\alpha = \pi$.

(4) We have the uniform bound (1-7) for $u_0(t) = e^{-tA}a$ and we show $|u_0(x,t)| \lesssim (\sqrt{t} + |x|)^{-1}$ in Section 6. We expect $u_0(t) \notin L^q(\Omega)$ for any $q \le 3$, and $\|u_0(t)\|_{L^q} \to \infty$ as $t \to 0_+$ for $q > 3$. The difference $v = u - u_0$ is more localized: by interpolating (1-8), $\|v(t)\|_{L^q} \to 0$ as $t \to 0_+$ for all $q \in [2,3)$. Although $\|v(t)\|_{L^3(\Omega)} = C$ for $t > 0$, $v(t)$ weakly converges to 0 in L^3 as $t \to 0_+$, as easily shown by approximating the test function by $L^2 \cap L^{3/2}$ functions. Both $u_0(t)$ and $v(t)$ belong to $L^\infty(\mathbb{R}_+; L^{3,\infty}(\mathbb{R}^3_+))$.

We now outline our proof. Unlike previous approaches based on the evolution equations, we directly prove the existence of the profile U in (1-5). It is based on the a priori estimates for U using the classical Leray–Schauder fixed point theorem and the Leray reductio ad absurdum argument (which has been fruitfully applied in recent papers of Korobkov, Pileckas and Russo [Korobkov et al. 2013; 2014a; 2014b; 2015a; 2015b] on the boundary value problem of stationary Navier–Stokes equations). Specifically, the profile $U(x)$ satisfies the Leray equations

$$-\Delta U - U - x \cdot \nabla U + (U \cdot \nabla) U + \nabla P = 0, \quad \operatorname{div} U = 0 \tag{1-9}$$

in $\mathbb{R}^3_+$ with zero boundary condition and, in a suitable sense,

$$U(x) \to U_0(x) := (e^{-A/2}a)(x) \quad \text{as } |x| \to \infty. \tag{1-10}$$

System (1-9) was proposed by Leray [1934], with the opposite sign for $U + x \cdot \nabla U$, for the study of singular *backward* self-similar solutions of (1-1) in $\mathbb{R}^3$ of the form $u(x,t) = U\big(x/\sqrt{-2t}\big)/\sqrt{-2t}$. Their triviality was first established in [Nečas et al. 1996] if $U \in L^3(\mathbb{R}^3)$, in particular if $U \in H^1(\mathbb{R}^3)$ as assumed in [Leray 1934], and then extended in [Tsai 1998] to $U \in L^q(\mathbb{R}^3)$, $3 \le q \le \infty$. In the forward case and in the whole space setting, we have

$$|U_0(x)| \sim |x|^{-1}, \quad V(x) := U(x) - U_0(x), \quad |V(x)| \lesssim |x|^{-2} \quad \text{for } |x| > 1; \tag{1-11}$$

see [Jia and Šverák 2014; Tsai 2014]. In the half space setting, it is not clear if one can show a pointwise decay bound for V. We show, however, that $V(x)$ is a priori bounded in $H^1_0(\mathbb{R}^3_+)$, and use this a priori bound to construct a solution. Due to lack of compactness of H^1_0 at spatial infinity, we use the *invading method* introduced by Leray [1933]: we approximate $\Omega = \mathbb{R}^3_+$ by $\Omega_k = \Omega \cap B_k$, $k = 1, 2, 3, \dots$, where B_k is an increasing sequence of concentric balls, construct solutions V_k in Ω_k of the difference equation (3-3) with zero boundary condition, and extract a subsequence converging to a desired solution V in $\mathbb{R}^3_+$.

Our proof is structured as follows. We first recall some properties for Euler flows in Section 2, and then use it to show that the V_k are uniformly bounded in $H^1_0(\Omega_k)$ in Section 3. In Section 4, we construct V_k using the a priori bound and a linear version of the Leray–Schauder theorem, and extract a weak limit V using the uniform bound. The arguments in Sections 2–4 are valid as long as one can show that $U_0 = e^{-A_\Omega/2}a$, A_Ω being the Stokes operator in Ω, satisfies certain decay properties to be specified in Assumption 3.1. In Section 5 we show that, for $\Omega = \mathbb{R}^3_+$ and those initial data a considered in Theorem 1.1, U_0 indeed satisfies Assumption 3.1. We finally verify that $u(x,t)$ defined by (1-5) satisfies the assertions of Theorem 1.1 in Section 6.

Because our existence proof does not use the evolution equation, we do not need the nonlinear version of the Leray–Schauder theorem as in [Jia and Šverák 2014; Tsai 2014]. As a side benefit, we do not need to check the small-large uniqueness (cf. [Tsai 2014, Lemma 4.1]).

2. Some properties of solutions to the Euler system

For $q \geq 1$, denote by $D^{1,q}(\Omega)$ the set of functions $f \in W^{1,q}_{\mathrm{loc}}(\Omega)$ such that $\|f\|_{D^{1,q}}(\Omega) = \|\nabla f\|_{L^q(\Omega)} < \infty$. Recall, that by the Sobolev embedding theorem, if $q < n$ then for any $f \in D^{1,q}(\mathbb{R}^n)$ there exists a constant $c \in \mathbb{R}$ such that $f - c \in L^p(\mathbb{R}^n)$ with $p = nq/(n-q)$. In particular,

$$f \in D^{1,2}(\mathbb{R}^3) \Rightarrow f - c \in L^6(\mathbb{R}^3), \qquad f \in D^{1,3/2}(\mathbb{R}^3) \Rightarrow f - c \in L^3(\mathbb{R}^3). \tag{2-1}$$

Further, denote by $D^{1,2}_0(\Omega)$ the closure of the set of all smooth functions having compact supports in Ω with respect to the norm $\|\cdot\|_{D^{1,2}(\Omega)}$, and $H(\Omega) = \{\boldsymbol{v} \in D^{1,2}_0(\Omega) : \operatorname{div} \boldsymbol{v} = 0\}$. In particular,

$$H(\Omega) \hookrightarrow L^6(\Omega). \tag{2-2}$$

(Recall that by the Sobolev inequality, $\|f\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^3)}$ holds for every function $f \in C^\infty_c(\mathbb{R}^3)$ having compact support in $\mathbb{R}^3$; see [Adams and Fournier 2003, Theorem 4.31].)

Assume that the following conditions are fulfilled:

(**E**) Let Ω be a domain in $\mathbb{R}^3$ with (possibly unbounded) connected Lipschitz boundary $\Gamma = \partial\Omega$, and the functions $\boldsymbol{v} \in H(\Omega)$ and $p \in D^{1,3/2}(\Omega) \cap L^3(\Omega)$ satisfy the Euler system

$$\begin{cases} (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2-3}$$

The next statement was proved in [Kapitanskiĭ and Piletskas 1983, Lemma 4] and in [Amick 1984, Theorem 2.2]; see also [Amirat et al. 1999, Lemma 4].

Theorem 2.1. *Let the conditions* (**E**) *be fulfilled. Then*

$$\exists\, \hat{p}_0 \in \mathbb{R} : \quad p(x) \equiv \hat{p}_0 \quad \textit{for } \mathfrak{H}^2\textit{-almost all } x \in \partial\Omega. \tag{2-4}$$

Here and henceforth we denote by $\mathfrak{H}^m$ the m-dimensional Hausdorff measure $\mathfrak{H}^m(F) = \lim_{t\to 0+} \mathfrak{H}^m_t(F)$, where $\mathfrak{H}^m_t(F) = \inf\{\sum_{i=1}^\infty (\operatorname{diam} F_i)^m : \operatorname{diam} F_i \leq t,\ F \subset \bigcup_{i=1}^\infty F_i\}$.

3. A priori bound for Leray equations

Recall that the profile $U(x)$ in (1-5) satisfies Leray equations (1-9) with zero boundary condition and $U(x) \to U_0(x)$ at spatial infinity. Decompose

$$U = U_0 + V, \quad U_0 = e^{-A/2}a. \tag{3-1}$$

Because a is self-similar, $u_0(\cdot\,,t)=e^{-tA}a$ is also self-similar, i.e., $u_0(x,t)=\lambda u_0(\lambda x,\lambda^2 t)$ for all $\lambda>0$. Differentiating in λ and evaluating at $\lambda=1$ and $t=\frac{1}{2}$, we get

$$0=U_0+x\cdot\nabla U_0+\partial_t u_0\big(x,\tfrac{1}{2}\big)=U_0+x\cdot\nabla U_0+\Delta U_0-\nabla P_0 \tag{3-2}$$

for some scalar P_0. Thus, the difference $V(x)$ satisfies

$$-\Delta V-V-x\cdot\nabla V+\nabla P=F_0+F_1(V),\quad \operatorname{div} V=0 \tag{3-3}$$

for some scalar P, where

$$F_0=-U_0\cdot\nabla U_0, \tag{3-4}$$

$$F_1(V)=-(U_0+V)\cdot\nabla V-V\cdot\nabla U_0, \tag{3-5}$$

and V vanishes at the boundary and the spatial infinity.

For a Sobolev function $f\in W^{1,2}(\Omega)$, set

$$\|f\|_{H^1(\Omega)}:=\left(\int_\Omega |\nabla f|^2+\tfrac{1}{2}|f|^2\right)^{1/2}. \tag{3-6}$$

Denote by $H_0^1(\Omega)$ the closure of the set of all smooth functions having compact supports in Ω with respect to the norm $\|\cdot\|_{H^1(\Omega)}$, and

$$H^1_{0,\sigma}(\Omega)=\{f\in H_0^1(\Omega):\operatorname{div} f=0\}.$$

Note that $H_0^1(\Omega)=\{f\in W^{1,2}(\Omega):f|_{\partial\Omega}=0,\ \|f\|_{H^1(\Omega)}<\infty\}$ for bounded Lipschitz domains.

We assume the following.

Assumption 3.1 (boundary data at infinity). Let Ω be a domain in $\mathbb{R}^3$. The vector field $U_0:\Omega\to\mathbb{R}^3$ satisfies $\operatorname{div} U_0=0$ and

$$\|U_0\|_{L^6(\Omega)}<\infty,\qquad \|\nabla U_0\|_{L^2(\Omega)}<\infty. \tag{3-7}$$

Note that from Assumption 3.1 and (3-4) it follows, in particular, that

$$\left|\int_\Omega F_0\cdot\eta\right|\le C,\qquad \left|\int_\Omega(\eta\cdot\nabla)U_0\cdot\eta\right|\le C \tag{3-8}$$

for any $\eta\in H^1_{0,\sigma}(\Omega)$ with $\|\eta\|_{H^1_{0,\sigma}(\Omega)}\le 1$ (by virtue of the evident imbedding $H^1_{0,\sigma}(\Omega)\hookrightarrow L^p$ for all $p\in[2,6]$).

If it is valid in Ω, it is also valid in any subdomain of Ω with the same constant C. We show in Section 5 that for $\Omega=\mathbb{R}^3_+$ and a satisfying (5-1), $U_0=e^{-A/2}a$ satisfies (5-3) and hence Assumption 3.1. This is also true if $\Omega=\mathbb{R}^3$ and a is self-similar, divergence free, and locally Hölder continuous.

Theorem 3.2 (a priori estimate for bounded domain). *Let Ω be a bounded domain in $\mathbb{R}^3$ with connected Lipschitz boundary $\partial\Omega$, and assume Assumption 3.1 for U_0. Then for any function $V\in H_0^1(\Omega)$ satisfying*

$$-\Delta V+\nabla P=\lambda(V+x\cdot\nabla V+F_0+F_1(V)),\quad \operatorname{div} V=0 \tag{3-9}$$

for some $\lambda \in [0, 1]$, *we have the a priori bound*

$$\|V\|^2_{H^1(\Omega)} = \int_\Omega \left(|\nabla V|^2 + \tfrac{1}{2}|V|^2\right) \le C(U_0, \Omega).$$

Remark. Note that $C(U_0, \Omega)$ is independent of $\lambda \in [0, 1]$.

Proof. Let the assumptions of the theorem be fulfilled. Suppose that its assertion is not true. Then there exists a sequence of numbers $\lambda_k \in [0, 1]$ and functions $V_k \in H^1_0(\Omega)$ such that

$$-\Delta V_k - \lambda_k V_k - \lambda_k x \cdot \nabla V_k + \nabla P_k = \lambda_k (F_0 + F_1(V_k)), \quad \operatorname{div} V_k = 0, \tag{3-10}$$

and moreover,

$$J_k^2 := \int_\Omega |\nabla V_k|^2 \to \infty. \tag{3-11}$$

Multiplying (3-10) by V_k and integrating by parts in Ω, we obtain the identity

$$J_k^2 + \frac{\lambda_k}{2} \int_\Omega |V_k|^2 = \lambda_k \int_\Omega (F_0 - V_k \cdot \nabla U_0) V_k. \tag{3-12}$$

Consider the normalized sequence of functions

$$\widehat{V}_k = \frac{1}{J_k} V_k, \quad \widehat{P}_k = \frac{1}{\lambda_k J_k^2} P_k. \tag{3-13}$$

Since

$$\int_\Omega |\nabla \widehat{V}_k|^2 \equiv 1,$$

we could extract a subsequence, still denoted by $\widehat{V}_k$, which converges weakly in $W^{1,2}(\Omega)$ to some function $V \in H^1_0(\Omega)$, and strongly in $L^3(\Omega)$. Also we could assume without loss of generality that $\lambda_k \to \lambda_0 \in [0, 1]$.

Multiplying the identity (3-12) by $1/J_k^2$ and taking a limit as $k \to \infty$, we have

$$1 + \frac{\lambda_0}{2} \int_\Omega |V|^2 = -\lambda_0 \int_\Omega (V \cdot \nabla U_0) V = \lambda_0 \int_\Omega (V \cdot \nabla V) U_0. \tag{3-14}$$

In particular, λ_k is separated from zero for large k.

Multiplying (3-10) by $1/(\lambda_k J_k^2)$, we see that the pairs $(\widehat{V}_k, \widehat{P}_k)$ satisfy the equation

$$\widehat{V}_k \cdot \nabla \widehat{V}_k + \nabla \widehat{P}_k = \frac{1}{J_k} \Big(\frac{1}{\lambda_k} \Delta \widehat{V}_k + \widehat{V}_k + x \cdot \nabla \widehat{V}_k + \frac{1}{J_k} F_0 - U_0 \cdot \nabla \widehat{V}_k - \widehat{V}_k \cdot \nabla U_0 \Big). \tag{3-15}$$

Take an arbitrary function $\eta \in C^\infty_{c,\sigma}(\Omega)$. Multiplying (3-15) by η, integrating by parts and taking a limit, we obtain finally

$$\int_\Omega (V \cdot \nabla V) \cdot \eta = 0. \tag{3-16}$$

Since $\eta \in C^\infty_{c,\sigma}(\Omega)$ is arbitrary, we see that V is a weak solution to the Euler equation

$$\begin{cases} (V \cdot \nabla) V + \nabla P = 0 & \text{in } \Omega, \\ \operatorname{div} V = 0 & \text{in } \Omega, \\ V = 0 & \text{on } \partial\Omega, \end{cases} \tag{3-17}$$

for some $P \in D^{1,3/2}(\Omega) \cap L^3(\Omega)$. By Theorem 2.1, there exists a constant $\hat{p}_0 \in \mathbb{R}$ such that $P(x) \equiv \hat{p}_0$ on $\partial\Omega$. Of course, we can assume without loss of generality that $\hat{p}_0 = 0$, i.e., $P(x) \equiv 0$ on $\partial\Omega$. Then by (3-14) and the first line of (3-17), we get

$$1 + \frac{\lambda_0}{2} \int_\Omega |V|^2 = -\lambda_0 \int_\Omega U_0 \cdot \nabla P = -\lambda_0 \int_\Omega \operatorname{div}(P \cdot U_0) = 0.$$

The obtained contradiction finishes the proof of the theorem. □

Theorem 3.3 (a priori bound for invading method). *Let $\Omega = \mathbb{R}^3_+$, and assume Assumption 3.1 for U_0. Take a sequence of balls $B_k = B(0, R_k) \subset \mathbb{R}^3$ with $R_k \to \infty$, and consider half-balls $\Omega_k = \Omega \cap B_k$. Then for functions $V_k \in H^1_0(\Omega_k)$ satisfying*

$$-\Delta V_k - V_k - x \cdot \nabla V_k + \nabla P_k = F_0 + F_1(V_k), \quad \operatorname{div} V_k = 0, \tag{3-18}$$

we have the a priori bound

$$\int_{\Omega_k} \big(|\nabla V_k|^2 + \tfrac{1}{2}|V_k|^2\big) \le C(U_0),$$

where the constant $C(U_0)$ is independent of k.

Proof. Let the assumptions of the theorem be fulfilled. Suppose that its assertion is not true. Then there exists a sequence of domains Ω_k and a sequence of solutions $V_k \in H^1_0(\Omega_k)$ of (3-18) such that

$$J_k^2 := \|V_k\|^2_{H^1(\Omega_k)} = \int_{\Omega_k} \big(|\nabla V_k|^2 + \tfrac{1}{2}|V_k|^2\big) \to \infty. \tag{3-19}$$

Multiplying (3-18) by V_k and integrating by parts in Ω_k, we obtain the identity

$$J_k^2 = \int_{\Omega_k} (F_0 - V_k \cdot \nabla U_0) V_k. \tag{3-20}$$

Consider the normalized sequence of functions

$$\widehat{V}_k = \frac{1}{J_k} V_k, \qquad \widehat{P}_k = \frac{1}{J_k^2} P_k. \tag{3-21}$$

Multiplying (3-18) by $1/J_k^2$, we see that the pairs $(\widehat{V}_k, \widehat{P}_k)$ satisfy the equation

$$\widehat{V}_k \cdot \nabla \widehat{V}_k + \nabla \widehat{P}_k = \frac{1}{J_k}(\Delta \widehat{V}_k + \widehat{V}_k + x \cdot \nabla \widehat{V}_k + F_0 - U_0 \cdot \nabla \widehat{V}_k - \widehat{V}_k \cdot \nabla U_0). \tag{3-22}$$

Since

$$\int_{\Omega_k} \big(|\nabla \widehat{V}_k|^2 + \tfrac{1}{2}|\widehat{V}_k|^2\big) \equiv 1,$$

we could extract a subsequence, still denoted by $\widehat{V}_k$, which converges weakly in $W^{1,2}(\Omega)$ to some function $V \in H^1_0(\Omega)$, and strongly in $L^2(E)$ for any $E \Subset \overline{\Omega}$.

Multiplying the identity (3-20) by $1/J_k^2$ and taking a limit as $k \to \infty$, we have

$$1 = \int_\Omega (-V \cdot \nabla U_0) V. \tag{3-23}$$

Take an arbitrary function $\eta \in C^{\infty}_{c,\sigma}(\Omega)$. Multiplying (3-22) by η, integrating by parts and taking a limit, we obtain finally

$$\int_{\Omega} (V \cdot \nabla V) \cdot \eta = 0. \tag{3-24}$$

Since $\eta \in C^{\infty}_{c,\sigma}(\Omega)$ is arbitrary, we see that V is a weak solution to the Euler equation

$$\begin{cases} (V \cdot \nabla)V + \nabla P = 0 & \text{in } \Omega, \\ \operatorname{div} V = 0 & \text{in } \Omega, \\ V = 0 & \text{on } \partial\Omega, \end{cases} \tag{3-25}$$

with some $P \in D^{1,3/2}(\Omega) \cap L^3(\Omega)$. More precisely, since $V, \nabla V \in L^2(\Omega)$, we have $P \in D^{1,q}(\Omega)$ for every $q \in \left[1, \frac{3}{2}\right]$. Consequently, $P \in L^s(\Omega)$ for each $s \in \left[\frac{3}{2}, 3\right]$. In particular, $P \in L^3(\Omega)$ and $\nabla P \in L^{9/8}(\Omega)$. Furthermore,

$$\begin{aligned} \int_{S_R^+} |P|^{4/3} &= -R^2 \int_R^{\infty} \int_{S_1^+} \frac{d}{dr}\big(|P(r\omega)|^{4/3}\big)\, d\omega\, dr \\ &\lesssim \int_{|x|>R} |P|^{1/3} |\nabla P| \le \left(\int_{|x|>R} |P|^3\right)^{1/9} \left(\int_{|x|>R} |\nabla P|^{9/8}\right)^{8/9}, \end{aligned}$$

where $S_R^+ = \{x \in \Omega : |x| = R\}$ is the corresponding half-sphere. Hence, we conclude that

$$\int_{S_R^+} |P|^{4/3} \to 0 \quad \text{as } R \to \infty. \tag{3-26}$$

Analogously, from the assumption $U_0 \in L^6(\Omega)$, $\nabla U \in L^2(\Omega)$, it is very easy to deduce that

$$\int_{S_R^+} |U_0|^4 \to 0 \quad \text{as } R \to \infty. \tag{3-27}$$

On the other hand, by (3-23) and the first line of (3-25) we obtain

$$1 = \int_{\Omega} (V \cdot \nabla)V \cdot U_0 = -\int_{\Omega} \nabla P \cdot U_0 = -\lim_{R\to\infty} \int_{\Omega_R} \operatorname{div}(P \cdot U_0) = -\lim_{R\to\infty} \int_{S_R^+} P(U_0 \cdot \boldsymbol{n}) = 0, \tag{3-28}$$

where $\Omega_R = \Omega \cap B(0, R)$ and the last equality follows from (3-26)–(3-27). The obtained contradiction finishes the proof of the theorem. □

4. Existence for Leray equations

The proof of the existence theorem for the system of equations (3-3)–(3-5) in bounded domains Ω is based on the following fundamental fact.

Theorem 4.1 (Leray–Schauder theorem). *Let $S : X \to X$ be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x \in X : x = \lambda S x \text{ for some } \lambda \in [0, 1]\}$$

is bounded. Then S has a fixed point $x_* = S x_*$.

Let Ω be a domain in $\mathbb{R}^3$ with connected Lipschitz boundary $\Gamma = \partial\Omega$, and set $X = H^1_{0,\sigma}(\Omega)$.

For functions $V_1, V_2 \in H^1_{0,\sigma}(\Omega)$, write $\langle V_1, V_2\rangle_H = \int_\Omega \nabla V_1 \cdot \nabla V_2$. Then the system (3-3)–(3-5) is equivalent to the following identities:

$$\langle V, \zeta\rangle_H = \int_\Omega G(V)\cdot\zeta, \quad \forall \zeta \in C^\infty_{c,\sigma}(\Omega), \tag{4-1}$$

where $G(V) = V + x\cdot\nabla V + F(V)$, $F(V) = F_0 + F_1(V)$,

$$F_0(x) = -U_0\cdot\nabla U_0, \tag{4-2}$$

$$F_1(V) = -(U_0 + V)\cdot\nabla V - V\cdot\nabla U_0. \tag{4-3}$$

Since $H^1_{0,\sigma}(\Omega) \hookrightarrow L^6(\Omega)$, by the Riesz representation theorem, for any $f \in L^{6/5}(\Omega)$ there exists a unique mapping $T(f) \in H^1_{0,\sigma}(\Omega)$ such that

$$\langle T(f), \zeta\rangle_H = \int_\Omega f\cdot\zeta, \quad \forall \zeta \in C^\infty_{c,\sigma}(\Omega), \tag{4-4}$$

and moreover,

$$\|T(f)\|_H \le \|f\|_{X'},$$

where

$$\|f\|_{X'} = \sup_{\zeta\in C^\infty_{c,\sigma}(\Omega),\ \|\zeta\|_H\le 1} \int_\Omega f\cdot\zeta.$$

Then the system (3-3)–(3-5)$\sim$(4-1) is equivalent to the equality

$$V = T(G(V)). \tag{4-5}$$

Theorem 4.2 (compactness). *If Ω is a bounded domain in $\mathbb{R}^3$ with connected Lipschitz boundary $\Gamma = \partial\Omega$, and Assumption 3.1 holds for U_0, then for $X = H^1_{0,\sigma}(\Omega)$ the operator $S : X \ni V \mapsto T(G(V)) \in X$ is continuous and compact.*

Proof. (i) For $V, \tilde V \in X$, setting $v = \tilde V - V$,

$$F(\tilde V) - F(V) = -(U_0 + V + v)\cdot\nabla v - v\cdot\nabla(U_0 + V).$$

Thus we have

$$\begin{aligned}
&\|S(\tilde V) - S(V)\|_X \\
&\quad\lesssim \|v\|_{L^2} + \|\nabla v\|_{L^2} + \|F(\tilde V) - F(V)\|_{L^{6/5}} \\
&\quad\lesssim \|v\|_{L^2} + \|\nabla v\|_{L^2} + \|U_0\|_{L^3}\|\nabla v\|_{L^2} + \|V+v\|_{L^3}\|\nabla v\|_{L^2} + \|\nabla U_0\|_{L^2}\|v\|_{L^3} + \|v\|_{L^3}\|\nabla V\|_{L^2} \\
&\quad\lesssim (1 + \|V\|_X + \|v\|_X)\|v\|_X.
\end{aligned} \tag{4-6}$$

(ii) By the Sobolev theorems, we have the compact embedding $X \hookrightarrow L^r(\Omega)$ for all $r \in [1, 6)$. Thus if a sequence $V_k \in X$ is bounded in X, i.e., $\|V_k\|_{L^2(\Omega)} + \|\nabla V_k\|_{L^2(\Omega)} \le C$, then we can extract a subsequence V_{k_l} which converges to some $V \in X$ in $L^3(\Omega)$ norm: $\|V_{k_l} - V\|_{L^3(\Omega)} \to 0$ as $l \to \infty$. Then using the condition $V_{k_l} \equiv V \equiv 0$ on $\partial\Omega$ and integration by parts, it is easy to see that $\|F(V_{k_l}) - F(V)\|_{X'} \to 0$ and, consequently, $\|G(V_{k_l}) - G(V)\|_{X'} \to 0$ as $l \to \infty$. □

Corollary 4.3 (existence in bounded domains). *Let Ω be a bounded domain in $\mathbb{R}^3$ with connected Lipschitz boundary $\partial\Omega$, and assume Assumption 3.1 for U_0. Then the system* (3-3)–(3-5) *has a solution $V \in H^1_{0,\sigma}(\Omega)$.*

Proof. This is a direct consequence of Theorems 4.1–4.2 and 3.2. □

Theorem 4.4 (existence in unbounded domains). *Let $\Omega = \mathbb{R}^3_+$, and assume Assumption 3.1 for U_0. Then the system* (3-3)–(3-5) *has a solution $V \in H^1_{0,\sigma}(\Omega)$.*

Proof. Take balls $B_k = B(0,k)$ and consider the increasing sequence of domains $\Omega_k = \Omega \cap B_k$ from Theorem 3.3. By Corollary 4.3 there exists a sequence of solutions $V_k \in H^1_{0,\sigma}(\Omega_k)$ of the system (3-3)–(3-5) in Ω_k. By Theorem 3.3, the norms $\|V_k\|_{H^1_{0,\sigma}(\Omega)}$ are uniformly bounded, thus we can extract a subsequence V_{k_l} such that the weak convergence $V_{k_l} \rightharpoonup V$ in $W^{1,2}(\Omega')$ holds for any bounded subdomain $\Omega' \subset \Omega$. It is easy to check that the limit function V is a solution of the system (3-3)–(3-5) in Ω. □

5. Boundary data at infinity in the half space

In this section we restrict ourselves to the half space $\Omega = \mathbb{R}^3_+$ with boundary $\Sigma = \partial\mathbb{R}^3_+$ and study the decay property of $U_0 = e^{-A/2}a$. Our goal is to prove the following lemma, which ensures Assumption 3.1 under the conditions of Theorem 1.1.

Write $x^* = (x', -x_3)$ given $x = (x', x_3) \in \mathbb{R}^3$, and $\langle z\rangle = (1+|z|^2)^{1/2}$ for $z \in \mathbb{R}^m$.

Lemma 5.1. *Suppose a is a vector field in $\Omega = \mathbb{R}^3_+$ satisfying*

$$\begin{gathered} a \in C^1_{\mathrm{loc}}(\overline{\Omega}\backslash\{0\}; \mathbb{R}^3), \quad \operatorname{div} a = 0, \quad a|_{\partial\Omega} = 0, \\ a(x) = \lambda a(\lambda x), \quad \forall x \in \Omega,\ \forall \lambda > 0. \end{gathered} \tag{5-1}$$

Let $U_0 = e^{-A/2}a$, where A is the Stokes operator in Ω. Then

$$|\nabla^k U_0(x)| \le c_k[a]_1(1+x_3)^{-\min(1,k)}(1+|x|)^{-1}, \quad \forall k \in \mathbb{Z}_+ = \{0,1,2,\dots\}, \tag{5-2}$$

and, for any $0 < \delta \ll 1$,

$$|\nabla U_0(x)| \le c_\delta [a]_1 x_3^{-\delta}\langle x\rangle^{2\delta-2}, \tag{5-3}$$

where $[a]_m = \sup_{k\le m,\,|x|=1} |\nabla^k a(x)|$.

If we further assume $a \in C^m_{\mathrm{loc}}$, $m \ge 2$, and $\partial_3^k a|_\Sigma = 0$ for $k < m$, then $|\nabla^k U_0(x)| \le c_k[a]_m\langle x_3\rangle^{-k}\langle x\rangle^{-1}$ for $k \le m$.

Estimates (5-2) and (5-3) imply, in particular,

$$U_0 \in L^4(\Omega) \cap L^\infty(\Omega), \quad \nabla U_0 \in L^2(\Omega), \tag{5-4}$$

and hence Assumption 3.1 for U_0 is satisfied.

Green tensor for the nonstationary Stokes system in the half space. Consider the nonstationary Stokes system in the half space $\mathbb{R}^3_+$,

$$\partial_t v - \Delta v + \nabla p = 0, \quad \operatorname{div} v = 0, \quad \text{for } x \in \mathbb{R}^3_+,\ t > 0, \tag{5-5}$$

$$v|_{x_3=0} = 0, \quad v|_{t=0} = a. \tag{5-6}$$

In our notation,

$$v(t) = e^{-tA} a. \tag{5-7}$$

It is shown by Solonnikov [2003, §2] that, if $a = \breve{a}$ satisfies

$$\operatorname{div} \breve{a} = 0, \quad \breve{a}_3|_{x_3=0} = 0, \tag{5-8}$$

then

$$v_i(x,t) = \int_{\mathbb{R}^3_+} \breve{G}_{ij}(x,y,t)\breve{a}_j(y)\,dy \tag{5-9}$$

with

$$\breve{G}_{ij}(x,y,t) = \delta_{ij}\Gamma(x-y,t) + G^*_{ij}(x,y,t), \tag{5-10}$$
$$G^*_{ij}(x,y,t) = -\delta_{ij}\Gamma(x-y^*,t) - 4(1-\delta_{j3})\frac{\partial}{\partial x_j}\int_{\mathbb{R}^2\times[0,x_3]} \frac{\partial}{\partial x_i} E(x-z)\Gamma(z-y^*,t)\,dz,$$

where $E(x) = 1/(4\pi|x|)$ and $\Gamma(x,t) = (4\pi t)^{-3/2}e^{-|x|^2/(4t)}$ are the fundamental solutions of the Laplace and heat equations in $\mathbb{R}^3$. (A sign difference occurs since $E(x) = -1/(4\pi|x|)$ in [Solonnikov 2003].) Moreover, G^*_{ij} satisfies the pointwise bound

$$|\partial_t^s D_x^k D_y^\ell G^*_{ij}(x,y,t)| \lesssim t^{-s-\ell_3/2}\big(\sqrt{t}+x_3\big)^{-k_3}\big(\sqrt{t}+|x-y^*|\big)^{-3-|k'|-|\ell'|}e^{-cy_3^2/t} \tag{5-11}$$

for all $s \in \mathbb{N} = \{0,1,2,\dots\}$ and $k, \ell \in \mathbb{N}^3$ [Solonnikov 2003, (2.38)].

Note that $\breve{G}_{ij}$ is not the Green tensor in the strict sense since it requires (5-8). There is no known pointwise estimate for the Green tensor; cf. [Solonnikov 1964; Kang 2004].

We now estimate $U_0 = e^{-A/2}a$ for a satisfying (5-1). By (5-9) and (5-10),

$$U_{0,i}(x) = \int_{\mathbb{R}^3_+} \Gamma\big(x-y,\tfrac12\big)a_i(y)\,dy + \int_{\mathbb{R}^3_+} G^*_{ij}\big(x,y,\tfrac12\big)a_j(y)\,dy =: U_{1,i}(x) + U_{2,i}(x). \tag{5-12}$$

By (5-11), for $k \in \mathbb{Z}_+$ and using only $|a(y)| \lesssim 1/|y'|$,

$$\begin{aligned}
|\nabla^k U_2(x)| &\lesssim \int_{\mathbb{R}^3_+} (1+x_3)^{-k}(1+x_3+|x'-y'|)^{-3}e^{-cy_3^2}\frac{1}{|y'|}\,dy \\
&\lesssim (1+x_3)^{-k}\int_{\mathbb{R}^2}(1+x_3+|x'-y'|)^{-3}\frac{1}{|y'|}\,dy' \\
&= (1+x_3)^{-k-2}\int_{\mathbb{R}^2}(1+|\bar{x}-z'|)^{-3}\frac{1}{|z'|}\,dz' \\
&\lesssim (1+x_3)^{-k-2}(1+|\bar{x}|)^{-1} \\
&= (1+x_3)^{-k-1}(1+x_3+|x'|)^{-1},
\end{aligned} \tag{5-13}$$

where $\bar{x} = x'/(1+x_3)$. To estimate U_1, fix a cut-off function $\zeta(x) \in C_c^\infty(\mathbb{R}^3)$ with $\zeta(x) = 1$ for $|x| < 1$. We have

$$\nabla^k U_{1,i}(x) = \int_{\mathbb{R}^3_+} \Gamma\big(x-y,\tfrac12\big)\nabla_y^k\big((1-\zeta(y))a_i(y)\big)\,dy + \int_{\mathbb{R}^3_+}\nabla_x^k\Gamma\big(x-y,\tfrac12\big)(\zeta(y)a_i(y))\,dy, \tag{5-14}$$

using $a|_\Sigma = 0$. Hence, for $k \le 1$,

$$|\nabla^k U_1(x)| \lesssim \int_{\mathbb{R}^3} e^{-|x-y|^2/2} \langle y\rangle^{-1-k}\, dy + e^{-x^2/4} \lesssim \langle x\rangle^{-1-k}. \tag{5-15}$$

We can get the same estimate for $k \ge 2$ if we assume $\nabla^k a$ is defined and has the same decay. On the other hand, we can show $|\nabla_x^k U_1(x)| \lesssim \langle x\rangle^{-2}$ for $k \ge 2$ if we place the extra derivatives on Γ in the first integral of (5-14).

Combining (5-13) and (5-15), we get (5-2) and the last statement of Lemma 5.1.

Write

$$\Omega_- = \{x \in \Omega : 1 + x_3 > |x'|\}, \quad \Omega_+ = \{x \in \Omega : 1 + x_3 \le |x'|\}. \tag{5-16}$$

By (5-13) and (5-15), we have shown (5-3) in Ω_- (with $\delta = 0$).

It remains to show (5-3) in Ω_+.

Estimates using boundary layer integrals. Set $\varepsilon_j = 1$ for $j < 3$ and $\varepsilon_3 = -1$. Thus $x_j^* = \varepsilon_j x_j$. Let $\bar{a}(x)$ be an extension of $a(x)$ to $x \in \mathbb{R}^3$ with

$$\bar{a}_j(x) = \varepsilon_j a_j(x^*), \quad \text{if } x_3 < 0.$$

Since $\operatorname{div} a = 0$ in $\mathbb{R}^3_+$ and $a|_\Sigma = 0$, it follows that $\operatorname{div} \bar{a} = 0$ in $\mathbb{R}^3$. Let $u(x,t)$ be the solution of the nonstationary Stokes system in $\mathbb{R}^3$ with initial data $\bar{a}$, given simply by

$$u_i(x,t) = \int_{\mathbb{R}^3} \Gamma(y,t)\bar{a}_i(x-y)\, dy.$$

It follows that $u_i(x,t) = \varepsilon_i u_i(x^*,t)$. Thus

$$\partial_3 u_i(x,t)|_\Sigma = 0, \quad \text{for } i < 3; \qquad u_3(x,t)|_\Sigma = 0. \tag{5-17}$$

We have $|\nabla^k a(y)| \lesssim |y|^{-1-k}$ for $k \le 1$. By the same estimates leading to (5-15) for U_1, we have

$$\left|\nabla_x^k u_i\left(x, \tfrac12\right)\right| \lesssim \langle x\rangle^{-1-\min(1,k)}, \quad \text{for } k \le 2. \tag{5-18}$$

Thus $u\left(x, \frac12\right)$ satisfies (5-3).

Using the self-similarity condition

$$u(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad \forall \lambda > 0, \tag{5-19}$$

from (5-18) we get

$$|\nabla_x^m u_i(x,t)| \lesssim \begin{cases} \left(|x| + \sqrt{t}\right)^{-1-m}, & m \le 1, \\ t^{-1/2}\left(|x| + \sqrt{t}\right)^{-2}, & m = 2. \end{cases} \tag{5-20}$$

Now decompose

$$v = u - w.$$

Then w satisfies the nonstationary Stokes system in $\mathbb{R}^3_+$ with zero force, zero initial data, and has boundary value

$$w_j(x,t)|_{x_3=0} = u_j(x',0,t), \quad \text{if } j < 3; \qquad w_3(x,t)|_{x_3=0} = 0. \tag{5-21}$$

Using (5-21), it is given by the boundary layer integral

$$w_i(x,t) = \sum_{j=1,2} \int_0^t \int_\Sigma K_{ij}(x-z',s)\, u_j(z',0,t-s)\, dz'\, ds, \tag{5-22}$$

where, for $j < 3$,

$$K_{ij}(x,t) = -2\delta_{ij}\partial_3\Gamma - \frac{1}{\pi}\partial_j \mathcal{C}_i, \tag{5-23}$$

$$\mathcal{C}_i(x,t) = \int_{\Sigma\times[0,x_3]} \partial_3\Gamma(y,t)\frac{y_i - x_i}{|y-x|^3}\, dy \tag{5-24}$$

[Solonnikov 1964, pp. 40, 48]. (Note that the K_{i3} $(j=3)$ have extra terms.) They satisfy for $j < 3$

$$|\partial_t^m D_{x'}^\ell \partial_{x_3}^k \mathcal{C}_i(x,t)| \le c t^{-m-(1/2)}\big(x_3 + \sqrt{t}\big)^{-k}\big(|x| + \sqrt{t}\big)^{-2-\ell} \tag{5-25}$$

[Solonnikov 1964, pp. 41, 48].

We now show (5-3) for $w\big(x, \tfrac12\big)$ in the region $\Omega_+ : 1 + x_3 \le |x'|$.

For $t = \frac12$ and $i, k \in \{1,2,3\}$,

$$\begin{aligned} \partial_{x_k} w_i\big(x, \tfrac12\big) &= -\sum_{j=1,2} \int_0^{1/2}\int_\Sigma \frac{1}{\pi}\partial_k \mathcal{C}_i(x-z',s)\,\partial_{z_j} u_j\big(z',0,\tfrac12 - s\big)\, dz'\, ds \\ &\qquad - \mathbf{1}_{i<3}\int_0^{1/2}\int_\Sigma 2\partial_k\partial_3\Gamma(x-z',s)\, u_i\big(z',0,\tfrac12 - s\big)\, dz'\, ds \\ &= I_1 + I_2. \end{aligned} \tag{5-26}$$

Above, we have integrated by parts in tangential directions x_j in I_1.

By (5-20) and (5-25),

$$|I_1| \lesssim \int_0^{1/2}\int_\Sigma s^{-1/2}\big(x_3 + \sqrt{s}\big)^{-1}\big(|x-z'| + \sqrt{s}\big)^{-2}\Big(|z'| + \sqrt{\tfrac12 - s}\Big)^{-2}\, dz'\, ds.$$

Fix $0 < \varepsilon \le \frac12$. Splitting $\big(0, \frac12\big)$ as $\big(0, \frac14\big] \cup \big(\frac14, \frac12\big)$, and making the change of variable $s \to \frac12 - s$ in $\big(\frac14, \frac12\big)$, we get

$$\begin{aligned} |I_1| \lesssim \int_0^{1/4}\int_\Sigma x_3^{-2\varepsilon} s^{-1+\varepsilon}\big(|x'-z'| + x_3 + \sqrt{s}\big)^{-2}(|z'|+1)^{-2}\, dz'\, ds \\ + \int_0^{1/4}\int_\Sigma (x_3+1)^{-1}(|x'-z'| + x_3 + 1)^{-2}\big(|z'| + \sqrt{s}\big)^{-2}\, dz'\, ds. \end{aligned}$$

Integrating first in time and using, for $0 < b < \infty$, $0 \le a < 1 < a+b$, and $0 < N < \infty$, that

$$\int_0^1 \frac{ds}{s^a (N+s)^b} \le \frac{C}{N^{a+b-1}(N+1)^{1-a}}, \tag{5-27}$$

$$\int_0^1 \frac{ds}{s^a (N+s)^{1-a}} \le C \min\Big(\frac{1}{N^{1-a}}, \log\frac{2N+2}{N}\Big), \tag{5-28}$$

where the constant C is independent of N, we get

$$|I_1| \lesssim \int_\Sigma x_3^{-2\varepsilon}(|x'-z'|+x_3)^{-2+2\varepsilon}(|x'-z'|+x_3+1)^{-2\varepsilon}(|z'|+1)^{-2}\,dz' \\ + \int_\Sigma (x_3+1)^{-1}(|x'-z'|+x_3+1)^{-2}\min\biggl(\frac{1}{|z'|^2},\log\frac{2|z'|^2+2}{|z'|^2}\biggr)\,dz'.$$

Dividing the integration domain into $|z'|<\frac{1}{2}|x'|$, $\frac{1}{2}|x'|<|z'|<2|x'|$, and $|z'|>2|x'|$, we get

$$|I_1| \lesssim x_3^{-2\varepsilon}\langle x\rangle^{-2+\delta}, \quad \text{for } x\in\Omega_+ \tag{5-29}$$

for any $0<\delta\ll 1$. Taking $\varepsilon=\frac{1}{2}\delta$ and $\varepsilon=\frac{1}{2}$, we get

$$(1+x_3)|I_1| \lesssim x_3^{-\delta}\langle x\rangle^{-2+2\delta}, \quad \text{for } x\in\Omega_+. \tag{5-30}$$

To estimate I_2 for $i<3$ (note $I_2=0$ if $i=3$), we separate two cases. If $k<3$, integration by parts gives

$$I_2 = -\int_0^{1/2}\int_\Sigma 2\partial_3\Gamma(x-z',s)\,\partial_{z_k}u_i\bigl(z',0,\tfrac{1}{2}-s\bigr)\,dz'\,ds.$$

Using $ue^{-u^2}\le C_\ell(1+u)^{-\ell}$ for $u>0$ and any $\ell>0$,

$$\partial_3\Gamma(x,s) = cs^{-2}\frac{x_3}{\sqrt{s}}e^{-x^2/4s} \le cs^{-2}\biggl(1+\frac{|x|}{\sqrt{s}}\biggr)^{-3} = cs^{-1/2}\bigl(|x|+\sqrt{s}\bigr)^{-3}. \tag{5-31}$$

Hence I_2 can be estimated in the same way as I_1, and (5-30) is valid if I_1 is replaced by I_2 and $k<3$.

When $k=3$, by $\partial_t\Gamma=\Delta\Gamma$ and integration by parts,

$$\begin{aligned} I_2 &= \int_0^{1/2}\int_\Sigma 2\biggl(\sum_{j<3}\partial_j^2-\partial_t\biggr)\Gamma(x-z',s)\,u_i\bigl(z',0,\tfrac{1}{2}-s\bigr)\,dz'ds \\ &= \sum_{j<3}\int_0^{1/2}\int_\Sigma 2\partial_j\Gamma(x-z',s)\,\partial_{z_j}u_i\bigl(z',0,\tfrac{1}{2}-s\bigr)\,dz'\,ds \\ &\qquad + \int_0^{1/2}\int_\Sigma 2\Gamma(x-z',s)\,\partial_t u_i\bigl(z',0,\tfrac{1}{2}-s\bigr)\,dz'\,ds \\ &\qquad - \lim_{\mu\to 0_+}\biggl(\int_\Sigma 2\Gamma\bigl(x-z',\tfrac{1}{2}-\mu\bigr)u_i(z',0,\mu)\,dz - \int_\Sigma 2\Gamma(x-z',\mu)\,u_i\bigl(z',0,\tfrac{1}{2}-\mu\bigr)\,dz\biggr) \\ &= I_3+I_4+\lim_{\mu\to 0_+}(I_{5,\mu}+I_{6,\mu}). \end{aligned}$$

Here I_3 can be estimated in the same way as I_1, and (5-30) is valid if I_1 is replaced by I_3. For I_4, since $\partial_t u_i=\Delta u_i$, by estimate (5-20) for $\nabla^2 u$,

$$|I_4| \lesssim \int_0^{1/2}\int_\Sigma s^{-3/2}\biggl(1+\frac{|x-z'|^2}{4s}\biggr)^{-3/2}\biggl(\frac{1}{2}-s\biggr)^{-1/2}\biggl(|z'|+\sqrt{\frac{1}{2}-s}\biggr)^{-2}\,dz'\,ds. \tag{5-32}$$

We have a similar estimate as I_1 with the following difference: we have to use the estimate (5-27) during the integration over each subinterval $s \in \left[0, \frac{1}{4}\right]$ and $s \in \left[\frac{1}{4}, \frac{1}{2}\right]$; for the second subinterval we apply (5-27) with $a = \frac{1}{2}$, $b = 1$, $N = |z'|^2$.

For the boundary terms, the integrand of $I_{5,\mu}$ is bounded by $e^{-|x-z'|^2/2}|z'|^{-1}$ and converges to 0 as $\mu \to 0_+$ for each $z' \in \Sigma$. Thus $\lim I_{5,\mu} = 0$ by the Lebesgue dominated convergence theorem. For $I_{6,\mu}$,

$$|I_{6,\mu}| \lesssim \mu^{-1/2} e^{-x_3^2/(4\mu)} \int_\Sigma \Gamma_{\mathbb{R}^2}(x' - z', \mu) \frac{1}{\langle z' \rangle}\, dz' \lesssim \mu^{-1/2} e^{-x_3^2/(4\mu)} \frac{1}{\langle x' \rangle}, \tag{5-33}$$

which converges to 0 as $\mu \to 0_+$ for any $x \in \Omega$.

We conclude that, for either $k < 3$ or $k = 3$, (5-30) is valid if I_1 is replaced by I_2 and hence, for any $0 < \delta \ll 1$,

$$(1 + x_3)\left|\partial_k w_i\left(x, \tfrac{1}{2}\right)\right| \lesssim x_3^{-\delta} \langle x \rangle^{-2+2\delta}, \quad \forall x \in \Omega_+,\ \forall i, k \le 3. \tag{5-34}$$

Combining (5-18) and (5-34), we have shown (5-3) in Ω_+, concluding the proof of Lemma 5.1. □

6. Self-similar solutions in the half space

In this section we first complete the proof of Theorem 1.1, and then give a few comments.

Proof of Theorem 1.1. By Lemma 5.1, for those a satisfying the assumptions of Theorem 1.1, $U_0 = e^{-A/2}a$ satisfies (5-2) and (5-3), and hence Assumption 3.1 is satisfied. By Theorem 4.4, there is a solution $V \in H^1_{0,\sigma}(\mathbb{R}^3_+)$ of the system (3-3)–(3-5).

Noting $U_0 \in C^\infty(\mathbb{R}^3_+)$ by (5-2), the system (3-3)–(3-5) is a perturbation of the stationary Navier–Stokes system with smooth coefficients. The regularity theory for the Navier–Stokes system implies that $V \in C^\infty_{\mathrm{loc}}\left(\overline{\mathbb{R}^3_+}\right)$. The vector field $U = U_0 + V$ is thus a smooth solution of the Leray equations (1-9) in $\mathbb{R}^3_+$.

The vector field $u(x,t)$ defined by (1-5), $u(x,t) = U\left(x/\sqrt{2t}\right)/\sqrt{2t}$, is thus smooth and self-similar. Moreover,

$$v(x,t) = u(x,t) - e^{-tA}a = \frac{1}{\sqrt{2t}}\, V\left(\frac{x}{\sqrt{2t}}\right)$$

satisfies

$$\|v(t)\|_{L^q(\mathbb{R}^3_+)} = \|V\|_{L^q(\mathbb{R}^3_+)}(2t)^{(3/2q)-(1/2)} \quad \text{and} \quad \|\nabla v(t)\|_{L^2(\mathbb{R}^3_+)} = \|\nabla V\|_{L^2(\mathbb{R}^3_+)}(2t)^{-1/4}.$$

This finishes the proof of Theorem 1.1. □

Remark. Let $u_0(x,t) = (e^{-tA}a)(x) = U_0\left(x/\sqrt{2t}\right)/\sqrt{2t}$. We have $u_0(\cdot\,, t) \to a$ as $t \to 0_+$ in $L^{3,\infty}(\mathbb{R}^3_+)$. Indeed, by (5-2), $|U_0(x)| \lesssim \langle x \rangle^{-1} \in L^{3,\infty} \cap L^q$, $q > 3$. We have $\|u_0(t)\|_{L^q(\mathbb{R}^3_+)} = \|U_0\|_{L^q(\mathbb{R}^3_+)}(2t)^{(3/2q)-(1/2)}$, which remains finite as $t \to 0_+$ only if $q = (3, \infty)$, and

$$|u_0(x,t)| \lesssim \frac{1}{\sqrt{t}} \cdot \frac{1}{1 + |x|/\sqrt{t}} = \frac{1}{\sqrt{t} + |x|}. \tag{6-1}$$

This is consistent with the whole space case $\Omega = \mathbb{R}^3$.

For the difference $V(x)$, we only have its $L^q(\mathbb{R}^3_+)$ bounds, and not pointwise bounds as (1-11) in [Jia and Šverák 2014; Tsai 2014].

Acknowledgments

The authors are deeply indebted to K. Pileckas and R. Russo for valuable discussions. In particular, they pointed out the way to simplify the proofs for a stronger result.

This work was initiated when Korobkov visited the Center for Advanced Study in Theoretical Sciences (CASTS) of the National Taiwan University. The research of Korobkov was partially supported by the Russian Foundation for Basic Research (projects 14-01-00768-a and 15-01-08275-a) and by the Dynasty Foundation. The research of Tsai was partially supported by the Natural Sciences and Engineering Research Council of Canada grant 261356-13.

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Received 16 Aug 2015. Revised 20 Jun 2016. Accepted 13 Sep 2016.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 9 No. 8 2016

2157-5045(2016)9:8;1-4