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NONLINEAR BOUNDARY LAYERS FOR ROTATING FLUIDS

ANNE-LAURE DALIBARD AND DAVID GÉRARD-VARET

We investigate the behaviour of rotating incompressible flows near a nonflat horizontal bottom. In the flat case, the velocity profile is given explicitly by a simple linear ODE. When bottom variations are taken into account, it is governed by a nonlinear PDE system, with far less obvious mathematical properties. We establish the well-posedness of this system and the asymptotic behaviour of the solution away from the boundary. In the course of the proof, we investigate in particular the action of pseudodifferential operators in nonlocalized Sobolev spaces. Our results extend an older paper of Gérard-Varet (*J. Math. Pures Appl.* (9) **82**:11 (2003), 1453–1498), restricted to periodic variations of the bottom, using the recent linear analysis of Dalibard and Prange (*Anal. & PDE* **7**:6 (2014), 1253–1315).

1. Introduction

The general concern of this paper is the effect of rough walls on fluid flows, in a context where the rough wall has very little structure. This effect is important in several problems, like transition to turbulence or drag computation. For instance, understanding the connection between roughness and drag is crucial for microfluidics, because friction at solid boundaries is a major factor of energy loss in microchannels. This issue has been much studied over recent years, through both theory and experiments [Lauga et al. 2007; Bocquet and Barrat 2007]. Conclusions are ambivalent. On the one hand, rough surfaces may increase the friction area, and thus enstrophy dissipation. On the other hand, recent experiments have shown that rough hydrophobic surfaces may lead to drag decrease: air bubbles can be trapped in the humps of the roughness, generating some slip [Vinogradova and Yakubov 2006; Ybert et al. 2007].

Mathematically, these problems are often tackled by a homogenization approach. Typically, one considers Stokes equations over a rough plate, modelled by an oscillating boundary of small wavelength and amplitude:

$$\Gamma^\varepsilon : \quad x_3 = \varepsilon\gamma(x_1/\varepsilon, x_2/\varepsilon), \quad \varepsilon \ll 1, \tag{1-1}$$

where the function $\gamma = \gamma(y_1, y_2)$ describes the roughness pattern. Within this formalism, the understanding of roughness-induced effects comes down to an asymptotic problem, as $\varepsilon \to 0$. The point is to derive effective boundary conditions at the flat plate Γ^0, retaining in this boundary condition an averaged effect of the roughness. We refer to the works [Achdou et al. 1998a; 1998b; 1998c; Amirat et al. 2001; Jäger and Mikelić 2001; 2003; Neuss et al. 2006; Bresch and Milisic 2010; Mikelić et al. 2013] on this topic. In all of these works, a restrictive hypothesis is made, namely periodicity of the roughness pattern γ. This hypothesis simplifies greatly the construction of the so-called boundary layer corrector, describing the

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small-scale variations of the flow near the boundary. This corrector is an analogue of the cell corrector in classical homogenization of heterogeneous media.

The main point and difficulty is the mathematical study of the boundary layer equations, which are satisfied formally by the boundary layer corrector. When γ is periodic in y_1, y_2, the solution of the boundary layer system is itself sought periodic, so that well-posedness and qualitative properties of the system are easy to determine. When the periodicity structure is relaxed, and replaced by general ergodicity properties, the analysis is still possible, but much more involved, as shown in [Basson and Gérard-Varet 2008; Gérard-Varet 2009; Gérard-Varet and Masmoudi 2010]. A key feature of these articles is the linearity of the boundary layer system: after the rescaling $y = x/\varepsilon$, it is governed by Stokes equations in the boundary layer domain

$$\Omega_{\mathrm{bl}} = \{y : y_3 > \gamma(y_1, y_2)\}. \tag{1-2}$$

It thus reads

$$\begin{cases} -\Delta v + \nabla p = 0 & \text{in } \Omega_{\mathrm{bl}}, \\ \operatorname{div} v = 0 & \text{in } \Omega_{\mathrm{bl}}, \\ v|_{\partial\Omega_{\mathrm{bl}}} = \phi \end{cases} \tag{1-3}$$

for some Dirichlet boundary data ϕ that has no decay as y_1, y_2 go to infinity, but no periodic structure. As a consequence, spaces of infinite energy, such as H^s_{uloc}, form a natural functional setting for such equations.

A natural challenge is to extend this type of analysis to nonlinear systems. This is the goal of the present paper. Namely, we will study a nonlinear boundary layer system that describes a rotating fluid near a rough boundary. The dynamics of rotating fluid layers are relevant in the context of geophysical flows, for which the Earth's rotation plays a dominant role. The system under consideration reads

$$\begin{cases} v \cdot \nabla v + \nabla p + e \times v - \Delta v = 0 & \text{in } \Omega_{\mathrm{bl}}, \\ \operatorname{div} v = 0 & \text{in } \Omega_{\mathrm{bl}}, \\ v|_{\partial\Omega_{\mathrm{bl}}} = \phi. \end{cases} \tag{1-4}$$

These are the incompressible Navier–Stokes equations written in a rotating frame, which is the reason for the extra Coriolis force $e \times u$, where $e = e_3 = (0, 0, 1)^t$. The equations in (1-4) can be obtained through an asymptotics of the full rotating fluid system

$$\mathrm{Ro}(\partial_t u + u \cdot \nabla u) + e \times u - \mathrm{E}\Delta u = 0, \quad \operatorname{div} u = 0, \tag{1-5}$$

where Ro and E are the so-called Rossby and Ekman numbers. These parameters are small in many applications. In the vicinity of the rough boundary (1-1), and in the special case where

$$\mathrm{E} \sim \varepsilon^2, \quad \mathrm{Ro} \sim \varepsilon, \tag{1-6}$$

it is natural to look for an asymptotic behaviour of the type

$$u^\varepsilon(t, x) \sim v(t, x_1, x_2, x/\varepsilon),$$

where $v = v(t, x_1, x_2, y)$, $y \in \Omega_{\mathrm{bl}}$. Injecting this ansatz in (1-5) yields the first two equations in the system (1-4), where the "slow variables" (t, x_1, x_2) are only parameters and are eluded.

The main goal of this paper is to construct a solution v of system (1-4), under no structural assumption on γ. We shall moreover provide information on the behaviour of v away from the boundary. We will in this way generalize [Gérard-Varet 2003] by the second author in which periodic roughness was considered. See also [Gérard-Varet and Dormy 2006]. Before stating the main difficulties and results of our study, several remarks are in order:

(1) The choice of the scaling (1-6), which leads to the derivation of the boundary layer system, may seem peculiar. It is, however, the richest possible, as it retains all terms in the equation for the boundary layer. All other scaling would provide a degeneracy of system (1-4).

(2) In the flat case, that is, for the roughness profile $\gamma = 0$, and for $\phi = (\phi_1, \phi_2, 0)$, with ϕ_1, ϕ_2 independent of y, the solution of (1-5) is explicitly given in complex form by

$$(v_1 + i v_2)(y) = (\phi_1 + i\phi_2) \exp\bigl(-(1+i) y_3/\sqrt{2}\bigr), \quad v_3 = 0. \tag{1-7}$$

This profile, sometimes called the Ekman spiral, solves the linear ODE

$$e \times v - \partial_3^2 v = 0.$$

Considering roughness turns this linear ODE into a nonlinear PDE, and as we will see, changes drastically the properties of the solution.

(3) Rather than the Dirichlet condition $v|_{\partial\Omega_{\mathrm{bl}}} = \phi$, some slightly different settings could be considered:

- One could for instance prescribe a homogeneous Dirichlet condition $v|_{\partial\Omega_{\mathrm{bl}}} = 0$, and add a source term with enough decay in y_3. This would correspond to a localized forcing of the boundary layer.
- One could replace the Dirichlet condition by a Navier condition, that is, a condition of the type

  $$D(u) n \times n|_{\partial\Omega_{\mathrm{bl}}} = f, \quad u \cdot n|_{\partial\Omega_{\mathrm{bl}}} = 0,$$

  with $D(u)$ the symmetric part of ∇u, and n the normal unit vector at the boundary. For instance, one could think of (1-1) as modelling an oscillating free surface, under the rigid lid approximation. In this context, the Navier condition would model a wind forcing, and the boundary layer domain would model the water below the free surface (changing the direction of the vertical axis). We refer to [Pedlosky 1987] for some similar modelling, and to [Casado-Díaz et al. 2003; Bucur et al. 2008; Bonnivard and Bucur 2012; Dalibard and Gérard-Varet 2011] for the treatment of such Navier condition. As shown in those papers, some hypothesis on the nondegeneracy of the roughness is necessary to the mathematical analysis.

However, our analysis does not extend to the important case of an inhomogeneous Dirichlet condition at infinity, which models a boundary layer driven by an external flow. For linear systems, one can in general lift this Dirichlet data at infinity, and recover the case of a Dirichlet data at the bottom boundary, like in (1-3). But for our nonlinear system (1-4), this lift would lead to the introduction of an additional drift term in the momentum equation, which would break down its rotational invariance.

2. Statement of the results

Our main result is a well-posedness theorem for the boundary layer system (1-4), where ϕ is a given boundary data, with no decay tangentially to the boundary, and satisfying $\phi \cdot n|_{\partial\Omega_{\mathrm{bl}}} = 0$. As usual in the theory of steady Navier–Stokes equations, the well-posedness will be obtained under a smallness hypothesis. We first introduce, for any unbounded $\Omega \subset \mathbb{R}^d$, the spaces

$$L^2_{\mathrm{uloc}}(\Omega) = \left\{ f : \sup_{k \in \mathbb{Z}^d} \int_{B(k,1)\cap\Omega} |f|^2 < +\infty \right\},$$
$$\text{and for all } m \geq 0, \quad H^m_{\mathrm{uloc}}(\Omega) = \{ f : \partial^\alpha f \in L^2_{\mathrm{uloc}}(\Omega)\ \forall \alpha \leq m \}.$$

These spaces are of course Banach spaces when endowed with their natural norms.

Theorem 1. *Let γ be bounded and Lipschitz and Ω_{bl} be defined as in* (1-2)*. There exists $\delta_0, C > 0$, such that for all $\phi \in H^2_{\mathrm{uloc}}(\partial\Omega_{\mathrm{bl}})$ satisfying $\phi \cdot n|_{\partial\Omega_{\mathrm{bl}}} = 0$ and $\|\phi\|_{H^2_{\mathrm{uloc}}} \leq \delta_0$ system* (1-4) *has a unique solution (v, p) with*

$$(1 + y_3)^{1/3} v \in H^1_{\mathrm{uloc}}(\Omega_{\mathrm{bl}}), \quad (1 + y_3)^{1/3} p \in L^2_{\mathrm{uloc}}(\Omega_{\mathrm{bl}}),$$

and

$$\|(1 + y_3)^{1/3} v\|_{H^1_{\mathrm{uloc}}} + \|(1 + y_3)^{1/3} p\|_{L^2_{\mathrm{uloc}}} \leq C \|\phi\|_{H^2_{\mathrm{uloc}}}.$$

This theorem generalizes the result of [Gérard-Varet 2003], dedicated to the case of periodic roughness pattern γ. In this case, the analysis is much easier, as the solution v of (1-4) is itself periodic in y_1, y_2. Through standard arguments, one can then build a solution v satisfying

$$\int_{\mathbb{T}^2} \int_{y_3 > \gamma(y_1, y_2)} |\nabla v|^2 < +\infty.$$

Moreover, one can establish exponential decay estimates for v as y_3 goes to infinity. This exponential decay is related to the periodicity in the horizontal variables, which provides a Poincaré inequality for functions with zero mean in x_1. When the periodicity assumption is removed, one expects the exponential convergence to be no longer true: this has been notably discussed in [Gérard-Varet and Masmoudi 2010; Prange 2013] in the context of the Laplace or the Stokes equation near a rough wall. It is worth noting that in such context, the convergence can be arbitrarily slow. In fact, there is in general no convergence when no ergodicity assumption on γ is made. A remarkable feature of our theorem for rotating flows is that decay to zero persists, despite the nonlinearity, and without any ergodicity assumption on γ. We emphasize that this decay comes from the rotation term. However, exponential decay is replaced by polynomial decay, with rate $O(y_3^{-1/3})$ for v.

Let us comment on the difficulties associated with Theorem 1. Of course, the first issue is that the data ϕ does not decay as (y_1, y_2) goes to infinity, so that the solution v is not expected to decay in the horizontal directions. If Ω_{bl} were replaced by

$$\Omega^M_{\mathrm{bl}} := \{ y : M > y_3 > \gamma(y_1, y_2) \}, \quad M > 0,$$

together with a Dirichlet condition at the upper boundary, one could build a solution v in $H^1_{\mathrm{uloc}}(\Omega^M_{\mathrm{bl}})$, adapting ideas of Ladyženskaya and Solonnikov [1980] on Navier–Stokes flows in tubes. Among those ideas, an important one is to obtain an a priori differential inequality on the local energy

$$E(t) := \int_{\{|(y_1,y_2)|\le t\}} \int_{\{M>y_3>\gamma(y_1,y_2)\}} |\nabla v|^2.$$

Such a differential inequality, known in the literature as a *Saint-Venant estimate*, appeared previously in other contexts; see for instance [Wheeler and Horgan 1976; Wheeler et al. 1975]. Namely, one shows an inequality of the type

$$E(t) \le C_M(E'(t) + E'(t)^{3/2} + t^2).$$

However, the derivation of this differential inequality relies on the Poincaré inequality between two planes, or in other words on the fact that Ω^M_{bl} has a bounded direction. For the boundary layer domain Ω_{bl}, this is no longer true, and no a priori bound can be obtained in this way. Moreover, contrary to what happens for the Laplace equation, one cannot rely on maximum principles to get an L^∞ bound.

Under a periodicity assumption on γ, one can restrict the domain to the periodic slab

$$\{y : (y_1, y_2) \in \mathbb{T}^2,\ y_3 > \gamma(y_1, y_2)\}.$$

In this manner, one has again a domain with a bounded direction (horizontal rather than vertical). One can establish again Saint-Venant estimates leading to the exponential decay mentioned above. It allows one to prove well-posedness of the boundary layer system. However, this approach does not work in our framework, where no structure is assumed on the roughness profile γ.

For the Stokes boundary layer flow

$$-\Delta v + \nabla p = 0, \qquad \operatorname{div} v = 0 \quad \text{in } \Omega_{\mathrm{bl}}, \qquad v|_{\partial\Omega_{\mathrm{bl}}} = v_0, \tag{2-1}$$

this problem is overcome in [Gérard-Varet and Masmoudi 2010] by N. Masmoudi and the second author. The main idea there is to get back to the domain Ω^M_{bl} by imposing a so-called transparent boundary condition at $y_3 = M$. This transparency condition involves the Stokes analogue of the Dirichlet-to-Neumann operator, and, despite its nonlocal nature (contrary to the Dirichlet condition), allows then to apply the method of Solonnikov. We refer to [Gérard-Varet and Masmoudi 2010] for more details.[1] Of course, the use of an explicit transparent boundary condition at $y_3 = M$ is possible because v satisfies a homogeneous Stokes equation in the half-space $\{y_3 > M\}$, which gives access to explicit formulas.

Such simplification does not occur in the context of our rotating flow system: in particular, the main issue is the quasilinear term $u \cdot \nabla u$ in system (1-4), in contrast with previous linear studies. In fact, even without this convective term, the analysis is not easy. In other words, the Coriolis–Stokes problem

$$\begin{cases} e \times v + \nabla p - \Delta v = 0 & \text{in } \Omega_{\mathrm{bl}}, \\ \operatorname{div} v = 0 & \text{in } \Omega_{\mathrm{bl}}, \\ v|_{\partial\Omega_{\mathrm{bl}}} = \phi \end{cases} \tag{2-2}$$

[1] Actually, [Gérard-Varet and Masmoudi 2010] is concerned with the 2D case. For adaptation to 3D, we refer to [Dalibard and Prange 2014].

already raises difficulties. For instance, to use a strategy based on a transparent boundary condition, one needs to construct the solution of the Dirichlet problem in a half-space for the Stokes–Coriolis operator, when the Dirichlet data has uniform local bounds. But contrary to the Stokes case, there is no easy integral representation. Still, such a linear problem was tackled in the recent paper [Dalibard and Prange 2014] by the first author and C. Prange. To solve the Dirichlet problem, they use a Fourier transform in variables y_1, y_2, leading to accurate formulas. The point is then to be able to translate information on the Fourier side to uniform local bounds on v. This requires careful estimates, as spaces like L^2_{uloc} are defined through truncations in space, which are not so suitable for a Fourier treatment. Similar difficulties arise in [Alazard et al. 2016], devoted to water waves equations in locally uniform spaces.

The linear study [Dalibard and Prange 2014] is a starting point for our study of the nonlinear system (1-4), but we will need many refined estimates, combined with a fixed point argument. More precisely, the outline of the paper is the following.

• Section 3, the main section of the paper, will be devoted to the system

$$\begin{cases} e \times v + \nabla p - \Delta v = \operatorname{div} F & \text{in } \{y_3 > M\}, \\ \operatorname{div} v = 0 & \text{in } \{y_3 > M\}, \\ v|_{y_3=M} = v_0. \end{cases} \tag{2-3}$$

The data v_0 and F will have no decay in horizontal variables (y_1, y_2). The source term F, which is reminiscent of $u \otimes u$, will decay typically like $|y_3|^{-2/3}$ as y_3 goes to infinity. This exponent is coherent with the decay of u given in Theorem 1. The point will be to establish a priori estimates on a solution v of (2-3), with no decay in (y_1, y_2), decaying like $|y_3|^{-1/3}$ at infinity. Functional spaces will be specified in due course.

• On the basis of previous a priori estimates, we will show well-posedness of the system

$$\begin{cases} v \cdot \nabla v + e \times v + \nabla p - \Delta v = 0 & \text{in } \{y_3 > M\}, \\ \operatorname{div} v = 0 & \text{in } \{y_3 > M\}, \\ v|_{y_3=M} = v_0 \end{cases} \tag{2-4}$$

for small enough boundary data v_0 (again, in a functional space to be specified). This will be done in the first subsection of Section 4.

• Finally, through the next subsections of Section 4, we will establish Theorem 1. The solution v of (1-4) will be constructed with the help of a mapping $\mathcal{F} = \mathcal{F}(\psi, \phi)$, defined in the following way:

(1) First, we will introduce the solution (v^-, p^-) of

$$\begin{cases} v^- \cdot \nabla v^- + e \times v^- + \nabla p^- - \Delta v^- = 0 & \text{in } \Omega_{\text{bl}}^M, \\ \operatorname{div} v^- = 0 & \text{in } \Omega_{\text{bl}}^M, \\ v^-|_{\partial\Omega_{\text{bl}}} = \phi, \\ \Sigma(v^-, p^-)e_3|_{y_3=M} = \psi, \end{cases} \tag{2-5}$$

where $\Sigma(v, p) = \nabla v - \big(p + \frac{1}{2}|v|^2\big)\mathrm{Id}$. Note that a quadratic term $\frac{1}{2}|v|^2$ is added to the usual Newtonian tensor in order to handle the nonlinearity.

(2) Then, we will introduce the solution (v^+, p^+) of (2-4), with $v_0 := v^-|_{y_3=M}$.

(3) Eventually, we will define $\mathcal{F}(\psi, \phi) := \Sigma(v^+, p^+)e_3|_{y_3=M} - \psi$.

The point will be to show that for small enough ϕ, the equation $\mathcal{F}(\psi, \phi) = 0$ has a solution ψ, knowing that $\mathcal{F}(0, 0) = 0$. This will be obtained via the inverse function theorem (using the linear analysis of [Dalibard and Prange 2014]). For such ψ, the field v defined by $v^\pm$ over $\{\pm y_3 > M\}$ will be a solution of (1-4). Indeed, v is always continuous at $y_3 = M$ by the definition of v^+, while the condition $\mathcal{F}(\psi, \phi) = 0$ means that the normal component of the stress tensor $\Sigma(v, p)$ is also continuous at $y_3 = M$.

3. Stokes–Coriolis equations with source

A central part of the work is the analysis of system (2-3). For simplicity, we take $M = 0$. The case without source term ($F = 0$) was partially analyzed in [Dalibard and Prange 2014], but we will establish new estimates, notably related to low frequencies. Let us emphasize that the difficulty induced by low frequencies already appeared in Proposition 2.1 on page 6 of the above work, even in the case of classical Sobolev data: in such case, some cancellation of the Fourier transform $\hat{v}_{0,3}$ at frequency $\xi = 0$ was assumed. We make a similar hypothesis here. The main theorem of the section is:

Theorem 2. *Let $m \in \mathbb{N}$, $m \gg 1$. Let $v_0 \in H^{m+1}_{\mathrm{uloc}}(\mathbb{R}^2)$ with third component satisfying $v_{0,3} = \partial_1 v_1^* + \partial_2 v_2^*$, with v_1^*, v_2^* in $L^2_{\mathrm{uloc}}(\mathbb{R}^2)$. Let $F \in H^m_{\mathrm{loc}}(\mathbb{R}^3_+)$ such that $(1 + y_3)^{2/3} F \in H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)$. There exists a unique solution v of system* (2-3) *such that*

$$\|(1 + y_3)^{1/3} v\|_{H^{m+1}_{\mathrm{uloc}}(\mathbb{R}^3_+)} \le C\big(\|v_0\|_{H^{m+1/2}_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*, v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(1 + y_3)^{2/3} F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big) \tag{3-1}$$

for a universal constant C.

Prior to the proof of the theorem, several simplifying remarks are in order:

• Obviously, uniqueness comes down to showing that if $F = 0$ and $v_0 = 0$, the only solution v of (2-3) such that $(1 + y_3)^{1/3} v \in H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)$ is $v = 0$. This result follows from [Dalibard and Prange 2014, Proposition 2.1], in which even a larger functional space was considered. Hence, the key statement our theorem is the existence of a solution satisfying the estimate (3-1).

• In order to show existence of such a solution, we can assume $v_{0,1}$, $v_{0,2}$, $v^* := (v_1^*, v_2^*)$ and F to be smooth and compactly supported (resp. in $\mathbb{R}^2$ and $\mathbb{R}^3_+$). Indeed, let us introduce

$$(v^n_{0,1}, v^n_{0,2}, v^{*,n})(y_1, y_2) := \chi((y_1, y_2)/n)\rho^n \star (v_{0,1}, v_{0,2}, v^*)(y_1, y_2),$$
$$F^n(y) := \tilde{\chi}(y/n)\tilde{\rho}^n(y) \star F(y),$$

where $\chi \in C^\infty_c(\mathbb{R}^2)$, $\tilde{\chi} \in C^\infty_c(\mathbb{R}^3)$ are 1 near the origin, and ρ^n, $\tilde{\rho}^n$ are approximations of unity. These functions are smooth, compactly supported, and satisfy

$$\|(v^n_{0,1}, v^n_{0,2})\|_{H^{m+1}_{\mathrm{uloc}}(\mathbb{R}^2)} \le C\|(v_{0,1}, v_{0,2})\|_{H^{m+1}_{\mathrm{uloc}}(\mathbb{R}^2)},$$
$$\|v^{*,n}\|_{H^{m+2}_{\mathrm{uloc}}(\mathbb{R}^2)} \le C\|v^*\|_{H^{m+2}_{\mathrm{uloc}}(\mathbb{R}^2)},$$
$$\|(1 + y_3)^{2/3} F^n\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)} \le C\|(1 + y_3)^{2/3} F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)}$$

for a universal constant C. Moreover, $(v^n_{0,1}, v^n_{0,2})$, $v^{*,n}$ and F^n converge strongly to $(v_{0,1}, v_{0,2})$, v^* and F in $H^{m+1}(K)$, $H^{m+2}(K)$ and $H^m(K')$ respectively for any compact set K of $\mathbb{R}^2$ and any compact set K' of $\mathbb{R}^3_+$. Now, assume that for all $n \in \mathbb{N}$, there exists a solution v^n corresponding to the data $v^n_{0,1}$, $v^n_{0,2}$, $v^{*,n}$, and F^n, for which we can get the estimate

$$\|(1+y_3)^{1/3} v^n\|_{H^{m+1}_{\text{uloc}}(\mathbb{R}^3_+)} \le C\big(\|(v^n_{0,1}, v^n_{0,2})\|_{H^{m+1}_{\text{uloc}}(\mathbb{R}^2)} + \|v^{*,n}\|_{H^{m+2}_{\text{uloc}}(\mathbb{R}^2)} + \|(1+y_3)^{2/3} F^n\|_{H^m_{\text{uloc}}(\mathbb{R}^3_+)}\big)$$

for a universal constant C. Then,

$$\|(1+y_3)^{1/3} v^n\|_{H^{m+1}_{\text{uloc}}(\mathbb{R}^3_+)} \le C'\big(\|(v_{0,1}, v_{0,2})\|_{H^{m+1}_{\text{uloc}}(\mathbb{R}^2)} + \|v^*\|_{H^{m+2}_{\text{uloc}}(\mathbb{R}^2)} + \|(1+y_3)^{2/3} F\|_{H^m_{\text{uloc}}(\mathbb{R}^3_+)}\big)$$

for a universal constant C'. We can then extract a subsequence weakly converging to some v, which is easily seen to satisfy (2-3) and (3-1).

- Finally, if $v_{0,1}$, $v_{0,2}$, v^* and F are smooth and compactly supported, the existence of a solution v of (2-3) can be obtained by standard variational arguments. More precisely, one can build a function v such that

$$\int_{\mathbb{R}^3_+} |\nabla v|^2 \le C\big(\|F\|_{L^2(\mathbb{R}^2)} + \|v_0\|_{H^{1/2}(\mathbb{R}^2)}\big),$$
$$\int_{\mathbb{R}^2\times\{y_3<a\}} |v|^2 \le C_a\big(\|F\|_{L^2(\mathbb{R}^2)} + \|v_0\|_{H^{1/2}(\mathbb{R}^2)}\big) \quad \forall a > 0.$$

Higher-order derivatives are then controlled by elliptic regularity. *Hence, the whole problem is to establish the estimate* (3-1) *for such a solution.*

We are now ready to tackle the proof of Theorem 2. We forget temporarily about the boundary condition and focus on the equations

$$e \times v + \nabla p - \Delta v = \operatorname{div} F, \quad \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3_+, \tag{3-2}$$

Our goal is to construct some particular solution of these equations, satisfying for some large enough m,

$$\|(1+z)^{1/3} v\|_{L^\infty} \le C\|(1+z)^{2/3} F\|_{L^\infty(H^m_{\text{uloc}})}. \tag{3-3}$$

We will turn to the solution of the whole system (2-3) in a second step.

3.1. ***Orr–Sommerfeld formulation.*** To handle (3-2), we rely on a formulation similar to Orr and Sommerfeld's rewriting of Navier–Stokes. Namely, we wish to express this system in terms of v_3 and $\omega := \partial_1 v_2 - \partial_2 v_1$. First, we apply ∂_2 to the first line, $-\partial_1$ to the second line, and combine to obtain

$$\partial_3 v_3 + \Delta\omega = s_3 := \partial_2 f_1 - \partial_1 f_2, \quad \text{with } f := \operatorname{div} F = \Big(\sum_j \partial_j F_{ij}\Big)_i. \tag{3-4}$$

Similarly, we apply $\partial_1\partial_3$ to the first line of (3-2), $\partial_2\partial_3$ to the second line, and $-(\partial_1^2 + \partial_2^2)$ to the third line. Combining the three, we are left with

$$-\partial_3\omega + \Delta^2 v_3 = s_\omega := \partial_1\partial_3 f_1 + \partial_2\partial_3 f_2 - (\partial_1^2 + \partial_2^2) f_3. \tag{3-5}$$

From ω and v_3, one recovers the horizontal velocity components v_1, v_2 using the system

$$\partial_1 v_1 + \partial_2 v_2 = -\partial_3 v_3, \quad \partial_1 v_2 - \partial_2 v_1 = \omega.$$

We are led to the (so far formal) expressions

$$\begin{aligned} v_1 &= (\partial_1^2 + \partial_2^2)^{-1}(-\partial_3\partial_1 v_3 - \partial_2\omega), \\ v_2 &= (\partial_1^2 + \partial_2^2)^{-1}(-\partial_3\partial_2 v_3 + \partial_1\omega). \end{aligned} \tag{3-6}$$

Our goal is to construct a solution (v_3, ω) of (3-4)–(3-5), by means of an integral representation. Since the vertical variable will play a special role in this construction, we will denote it by z instead of y_3: $y = (y_1, y_2, z)$. We write (3-4)–(3-5) in the compact form

$$L(D, \partial_z)V = S, \quad V := \begin{pmatrix} v_3 \\ \omega \end{pmatrix}, \quad S := \begin{pmatrix} s_3 \\ s_\omega \end{pmatrix}, \quad D := \frac{1}{i}(\partial_1, \partial_2),$$

where $L(D, \partial_z)$ is a Fourier multiplier in variables x_1, x_2 associated with

$$L(\xi, \partial_z) := \begin{pmatrix} \partial_z & (\partial_z^2 - |\xi|^2) \\ (\partial_z^2 - |\xi|^2)^2 & -\partial_z \end{pmatrix}.$$

We will look for a solution of the form

$$V(\,\cdot\,, z) = \int_0^{+\infty} G(D, z - z')S(\,\cdot\,, z')\, dz' + V_h, \tag{3-7}$$

where:

- $G(D, z)$ is a matrix Fourier multiplier, whose symbol $G(\xi, z)$ is the fundamental solution over $\mathbb{R}$ of $L(\xi, \partial_z)$ for any $\xi \in \mathbb{R}^2$:
$$L(\xi, \partial_z)G(\xi, z) = \delta_{z=0}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
- V_h is a solution of the homogeneous equation. The purpose of the addition of V_h is to ensure the decay of the solution V. More details will be given in due course.

3.1.1. *Construction of the Green function.* We start with the construction of the fundamental solution $G(\xi, z)$. Away from $z = 0$, it should satisfy the homogeneous system, which requires one to understand the kernel of the operator $L(\xi, \partial_z)$. This kernel is a combination of elements of the form $e^{\lambda z}V$, where λ is a root of the characteristic equation

$$\det L(\xi, \lambda) = 0, \quad \text{i.e.,}\ -\lambda^2 - (\lambda^2 - |\xi|^2)^3 = 0, \tag{3-8}$$

and V is an associated "eigenelement", meaning a nonzero vector in $\ker L(\xi, \lambda)$. A careful study of the characteristic equation was carried out recently in [Dalibard and Prange 2014]. Notice that (3-8) can be seen as an equation of degree three on $Y = \lambda^2 - |\xi|^2$ (with negative discriminant). Using Cardano's formula gives access to explicit expressions. The roots can be written as $\pm\lambda_1(\xi)$, $\pm\lambda_2(\xi)$ and $\pm\lambda_3(\xi)$, where $\lambda_1 \in \mathbb{R}_+$, λ_2, λ_3 have positive real parts, $\lambda_1 \in \mathbb{R}$, $\overline{\lambda_2} = \lambda_3$, $\mathrm{Im}\lambda_2 > 0$. The λ_i are continuous functions of ξ (see Remark 4 below for more). The above work also provides their asymptotic behaviour at low and high frequencies. This behaviour will be very important to establish our estimates.

Lemma 3 [Dalibard and Prange 2014, Lemma 2.4]. *As $\xi \to 0$, we have*

$$\lambda_1(\xi) = |\xi|^3 + O(|\xi|^5), \quad \lambda_2(\xi) = e^{i\pi/4} + O(|\xi|^2), \quad \lambda_3(\xi) = e^{-i\pi/4} + O(|\xi|^2).$$

As $\xi \to \infty$, we have

$$\lambda_1(\xi) = |\xi| - \tfrac{1}{2}|\xi|^{-1/3} + O(|\xi|^{-5/3}),$$
$$\lambda_2(\xi) = |\xi| - \tfrac{1}{2}j^2|\xi|^{-1/3} + O(|\xi|^{-5/3}), \quad \lambda_3(\xi) = |\xi| - \tfrac{1}{2}j|\xi|^{-1/3} + O(|\xi|^{-5/3}), \quad \textit{where } j = \exp(2i\pi/3).$$

Remark 4. We insist that λ_2 and λ_3 are distinct and have a positive real part for all values of ξ, whereas $\lambda_1 \neq 0$ for $\xi \neq 0$. Moreover, it can be easily checked that λ_i^2 is a $\mathcal{C}^\infty$ function of $|\xi|^2$ for $i = 1, \ldots, 3$. Using the fact that λ_2 and λ_3 never vanish or merge, while λ_1 vanishes for $\xi = 0$ only, we deduce that λ_2, λ_3 are $\mathcal{C}^\infty$ functions of $|\xi|^2$, and that $\lambda_1(\xi) = |\xi|^3\Lambda_1(\xi)$, where $\Lambda_1 \in \mathcal{C}^\infty(\mathbb{R}^2)$, $\Lambda_1(0) = 1$ and Λ_1 does not vanish on $\mathbb{R}^2$.

Regarding the eigenelements, an explicit computation shows that for all $i = 1, \ldots, 3$,

$$V_i^\pm := \begin{pmatrix} 1 \\ \pm\Omega_i \end{pmatrix} \quad \text{and} \quad \Omega_i := \frac{-\lambda_i}{\lambda_i^2 - |\xi|^2} \quad \text{satisfy } L(\xi, \pm\lambda_i)V_i^\pm = 0. \tag{3-9}$$

We can now determine G; our results are summarized in Lemma 5 below. We begin with its first column $G_1 = \binom{G_{11}}{G_{21}}$, a solution of $L(\xi, \partial_z)G_1 = \delta\binom{1}{0}$. As explained above, for $z \neq 0$, we know $G_1(\xi, z)$ is a linear combination of $e^{\pm\lambda_i z}V_i^\pm$. Furthermore, we want to avoid any exponential growth of G as $z \to \pm\infty$. Thus G_1 should be of the form

$$G_1 = \begin{cases} \sum_{i=1}^3 A_i^+ e^{-\lambda_i z} V_i^-, & z > 0, \\ \sum_{i=1}^3 A_i^- e^{\lambda_i z} V_i^+, & z < 0. \end{cases}$$

We now look at the jump conditions at $z = 0$. For $f = f(z)$, recall that $[f]|_{z=z'} := f(z'^+) - f(z'^-)$ denotes the jump of f at z'. Since

$$\begin{cases} (\partial_z^2 - |\xi|^2)^2 G_{11} - \partial_z G_{21} = 0, \\ \partial_z G_{11} + (\partial_z^2 - |\xi|^2) G_{21} = \delta_{z=0}, \end{cases}$$

we infer that

$$[G_{21}]|_{z=0} = 0, \qquad [\partial_z G_{21}]|_{z=0} = 1, \qquad [\partial_z^k G_{11}]|_{z=0} = 0, \quad k = 0, \ldots, 3.$$

This yields a linear system of six equations on the coefficients $A_i^\pm$. One finds $A_i := A_i^+ = -A_i^-$, and the system

$$\sum_i \lambda_i \Omega_i A_i = \tfrac{1}{2}, \quad \sum_i A_i = 0, \quad \sum_i \lambda_i^2 A_i = 0.$$

Note that

$$\sum_i \lambda_i \Omega_i A_i = -\sum_i \frac{\lambda_i^2}{\lambda_i^2 - |\xi|^2} A_i = -\sum_i \frac{|\xi|^2}{\lambda_i^2 - |\xi|^2} A_i$$

taking into account the second equality. Hence, we find

$$\begin{pmatrix} |\xi|^2/(\lambda_1^2-|\xi|^2) & |\xi|^2/(\lambda_2^2-|\xi|^2) & |\xi|^2/(\lambda_3^2-|\xi|^2) \\ 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of the matrix is

$$D_1 := |\xi|^2 D,$$

where

$$D := \begin{vmatrix} 1/(\lambda_1^2-|\xi|^2) & 1/(\lambda_2^2-|\xi|^2) & 1/(\lambda_3^2-|\xi|^2) \\ 1 & 1 & 1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix}.$$

After a few computations, we find that

$$D_1 = |\xi|^2(\lambda_2^2-\lambda_1^2)(\lambda_3^2-\lambda_1^2)\left(\frac{1}{(\lambda_1^2-|\xi|^2)(\lambda_2^2-|\xi|^2)} - \frac{1}{(\lambda_1^2-|\xi|^2)(\lambda_3^2-|\xi|^2)}\right), \tag{3-10}$$

and

$$A_1 = -\frac{1}{2D_1}(\lambda_3^2-\lambda_2^2), \quad A_2 = -\frac{1}{2D_1}(\lambda_1^2-\lambda_3^2), \quad A_3 = -\frac{1}{2D_1}(\lambda_2^2-\lambda_1^2). \tag{3-11}$$

Computations for the second column G_2 of G are similar. It is of the form

$$G_2 = \begin{cases} \sum_{i=1}^3 B_i^+ e^{-\lambda_i z} V_i^-, & z > 0, \\ \sum_{i=1}^3 B_i^- e^{\lambda_i z} V_i^+, & z < 0, \end{cases}$$

with jump conditions

$$[\partial_z^k G_{22}]|_{z=0} = 0, \quad k = 0, 1, \qquad [\partial_z^k G_{12}]|_{z=0} = 0, \quad k = 0, \dots, 2, \qquad [\partial_z^3 G_{12}]|_{z=0} = 1.$$

We find $B_i := B_i^+ = B_i^-$ and the system

$$\begin{pmatrix} \Omega_1 & \Omega_2 & \Omega_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}.$$

The determinant of the matrix is now $D_2 := -\lambda_1\lambda_2\lambda_3 D$, and

$$\begin{gathered} B_1 = \frac{\lambda_2\lambda_3}{2D_2}\left(\frac{1}{\lambda_2^2-|\xi|^2} - \frac{1}{\lambda_3^2-|\xi|^2}\right), \quad B_2 = \frac{\lambda_1\lambda_3}{2D_2}\left(\frac{1}{\lambda_3^2-|\xi|^2} - \frac{1}{\lambda_1^2-|\xi|^2}\right), \\ B_3 = \frac{\lambda_1\lambda_2}{2D_2}\left(\frac{1}{\lambda_1^2-|\xi|^2} - \frac{1}{\lambda_2^2-|\xi|^2}\right). \end{gathered} \tag{3-12}$$

This concludes the construction of the matrix G. We sum up our results in the following lemma, in which we also give the asymptotic behaviours of the coefficients A_i, B_i, $V_i^\pm$ and of G as $\xi \to 0$ and $|\xi| \to \infty$. The latter follow from Lemma 3 and Remark 4 and are left to the reader.

Lemma 5. *We have*

$$G_1 = \begin{cases} \sum_{i=1}^3 A_i e^{-\lambda_i z} V_i^-, & z > 0, \\ -\sum_{i=1}^3 A_i e^{\lambda_i z} V_i^+, & z < 0, \end{cases} \qquad G_2 = \begin{cases} \sum_{i=1}^3 B_i e^{-\lambda_i z} V_i^-, & z > 0, \\ \sum_{i=1}^3 B_i e^{\lambda_i z} V_i^+, & z < 0, \end{cases}$$

where

$$V_i^\pm = \begin{pmatrix} 1 \\ \mp\lambda_i/(\lambda_i^2 - |\xi|^2) \end{pmatrix}$$

and where A_i and B_i are defined by (3-11) *and* (3-12) *respectively.*

Asymptotic behaviour:

- *For $|\xi| \gg 1$, there exists $N > 0$ such that $A_i, B_i, \Omega_i = O(|\xi|^N)$ for $i = 1, \dots, 3$, and $|\Omega_i| \gtrsim |\xi|^{-N}$. As a consequence, $G(\xi, z) = O(|\xi|^N)$ for all z.*
- *As $\xi \to 0$, we have*

$$\begin{aligned} &A_i(\xi) \to \bar{A}_i \in \mathbb{C}^*, \quad i = 1, \dots, 3, \\ &B_1(\xi) \sim \frac{\bar{B}_1}{|\xi|}, \quad \bar{B}_1 \in \mathbb{C}^*, \quad B_i(\xi) \to \bar{B}_i \in \mathbb{C}^*, \quad i = 2, 3, \\ &\Omega_1 \sim \bar{\Omega}_1 |\xi|, \quad \bar{\Omega}_1 \in \mathbb{C}^*, \quad \Omega_i(\xi) \to \bar{\Omega}_i \in \mathbb{C}^*, \quad i = 2, 3. \end{aligned} \tag{3-13}$$

More precisely, we can write, for instance,

$$B_1(\xi) = \frac{\bar{B}_1}{|\xi|} \beta_1(\xi) \quad \forall \xi \in \mathbb{R}^2$$

for some function $\beta_1 \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that $\beta_1(0) = 1$. Similar statements hold for the other coefficients.

It follows that

$$G(\xi, z) = \begin{pmatrix} O(1) & O(|\xi|^{-1}) \\ O(1) & O(1) \end{pmatrix}$$

as $|\xi| \to 0$ for all $z \in \mathbb{R}$.

3.1.2. *Construction of the homogeneous correction.* We will see rigorously below that the field

$$V_G(\cdot, z) := \int_0^{+\infty} G(D, z - z') S(\cdot, z')\, dz' = \int_0^{+\infty} \mathcal{F}^{-1}_{\xi \to (y_1, y_2)} \big(G(\cdot, z - z') \mathcal{F}_{(y_1, y_2) \to \xi} S(\cdot, z') \big)\, dz' \tag{3-14}$$

is well-defined and satisfies (3-4)–(3-5). However, the corresponding velocity field does not have a good decay with respect to z. This is the reason for the additional field V_h in formula (3-7). To be more specific, let us split the source term S into $S(z') = S^0(z') + \partial_{z'} S^1(z') + \partial_{z'}^2 S^2(z')$, with

$$S^0(z') := \begin{pmatrix} \partial_2(\partial_1 F_{11} + \partial_2 F_{12}) - \partial_1(\partial_1 F_{21} + \partial_2 F_{22}) \\ -(\partial_1^2 + \partial_2^2)(\partial_1 F_{31} + \partial_2 F_{32}) \end{pmatrix} \tag{3-15}$$

and

$$
\begin{aligned}
S^1(z') &:= \begin{pmatrix} \partial_2 F_{13} - \partial_1 F_{23} \\ \partial_1(\partial_1 F_{11} + \partial_2 F_{12}) + \partial_2(\partial_1 F_{21} + \partial_2 F_{22}) - (\partial_1^2 + \partial_2^2) F_{33} \end{pmatrix}, \\
S^2(z') &:= \begin{pmatrix} 0 \\ \partial_1 F_{13} + \partial_2 F_{23} \end{pmatrix}.
\end{aligned} \tag{3-16}
$$

Roughly, the idea is that

$$
V(\,\cdot\,, z) := \int_0^{+\infty} \big(G(D, z - z') S^0(z') + \partial_z G(D, z - z') S^1(z') + \partial_z^2 G(D, z - z') S^2(z') \big)\, dz'
$$

has a better decay. Using the fact that

$$
\partial_z G(D, z - z') = -\partial_z' G(D, z - z'),
$$

we see that going from V_G to V is possible through integrations by parts in the variable z', which generates boundary terms. We recall that the jump of $G(D, z - z')$ at $z = z'$ is zero, and that

$$
\big[\partial_z G(D, z - z')\big]\big|_{z=z'} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
$$

On the other hand, the first component of S^2 is zero, so that the jump of $\partial_z G_{21}$ at $z = z'$ is not involved in the two integrations by parts of $\partial_z^2 G(D, z - z') S^2(z')$. Formal computations eventually lead to

$$
\begin{aligned}
V_h(\,\cdot\,, z) &:= V(\,\cdot\,, z) - V_G(\,\cdot\,, z) \\
&= -\big[G(D, z - z') \big(S^1(\,\cdot\,, z') + \partial_z S^2(\,\cdot\,, z') \big) \big]_0^{+\infty} + \big[\partial_z G(D, z - z') S^2(\,\cdot\,, z') \big]_0^{+\infty} \\
&= G(D, z) \big(S^1(\,\cdot\,, 0) + \partial_{z'} S^2(\,\cdot\,, 0) \big) - \partial_z G(D, z) S^2(\,\cdot\,, 0).
\end{aligned}
$$

Back to the expression of the Green function, we get

$$
\begin{aligned}
V_h(\,\cdot\,, z) = -\big(\textstyle\sum_i A_i e^{-\lambda_i z} V_i^- \quad \sum_i B_i e^{-\lambda_i z} V_i^- \big) \big(S^1(\,\cdot\,, 0) + \partial_{z'} S^2(\,\cdot\,, 0) \big) \\
+ \big(\textstyle\sum_i A_i \lambda_i e^{-\lambda_i z} V_i^- \quad \sum_i B_i \lambda_i e^{-\lambda_i z} V_i^- \big) S^2(\,\cdot\,, 0).
\end{aligned} \tag{3-17}
$$

It is a linear combination of terms of the form $e^{-\lambda_i z} V_i^-$, and therefore satisfies the homogeneous Orr–Sommerfeld equations. Hence, V is (still formally) a solution of (3-4)–(3-5).

We now need to put these formal arguments on rigorous grounds. As mentioned after Theorem 2, there is no loss of generality assuming that F is smooth and compactly supported.

Lemma 6. *Let F be smooth and compactly supported. The formula* (3-7), *with V_h given by* (3-17), *defines a solution $V = (v_3, \omega)^t$ of* (3-4)–(3-5) *satisfying*

$$
V \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H^m(\mathbb{R}^2)), \quad |D|^{-1}\omega \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H^m(\mathbb{R}^2)) \quad \textit{for any } m.
$$

Proof. Let us show first show that the integral term V_G (see (3-14)) satisfies the properties of the lemma. The main point is to show that for any $z, z' \geq 0$, the function

$$
J_{z,z'} : \xi \to G(\xi, z - z') \hat{S}(\xi, z') \quad \text{belongs to} \quad L^2((1 + |\xi|^2)^{m/2} d\xi) \times L^2(|\xi|^{-1}(1 + |\xi|^2)^{m/2} d\xi)
$$

for all m. Therefore, we recall that $\hat{F} = \hat{F}(\xi, z')$ is in the Schwartz class with respect to ξ, smooth and compactly supported in z'. Also, $G(\xi, z - z')$ is smooth in $\xi \neq 0$ (see Remark 4), and continuous in z, z'. It implies that $J_{z,z'}$ is smooth in $\xi \neq 0$, continuous in z, z'. It remains to check its behaviour at high and low frequencies.

- At high frequencies ($|\xi| \gg 1$), from Lemma 5, it is easily seen that $J_{z,z'}$ is bounded by
$$|J_{z,z'}(\xi)| \leq C|\xi|^N \sum_{k=0}^{2} |\partial_{z'}^k \hat{F}(\xi, z')|$$
for some N. As $\hat{F}$ and its z'-derivatives are rapidly decreasing in ξ, it will belong to any L^2 with polynomial weight.
- At low frequencies ($\xi \sim 0$), one can check that $|\hat{S}(\xi, z')| \leq C|\xi|$. Hence, using again the bounds derived in Lemma 5,
$$G(\xi, z - z')\hat{S}(\xi, z') = \begin{pmatrix} O(1) \\ O(|\xi|) \end{pmatrix}.$$
The result follows.

From there, by standard arguments, V_G defines a continuous function of z with values in $H^m(\mathbb{R}^2) \times |D|^{-1} H^m(\mathbb{R}^2)$ for all m. Moreover, a change of variable gives
$$V_G(\,\cdot\,, z) = \int_0^{+\infty} G(D, z') S(\,\cdot\,, z - z')\, dz'.$$
By the smoothness of S, we deduce that V_G is smooth in z with values in the same space. The fact that it satisfies (3-4)–(3-5) comes of course from the properties of the Green function G, and is classical. We leave it to the reader.

To conclude the proof of the lemma, we still have to consider the homogeneous correction V_h. Again, V_h is smooth in $\xi \neq 0$ and z. Thanks to the properties of F, it is decaying fast as $|\xi|$ goes to infinity. Moreover, from the asymptotics above, one can check that $V_h = \left(\begin{smallmatrix} O(1) \\ O(|\xi|) \end{smallmatrix}\right)$ for $|\xi| \ll 1$. Finally, as its Fourier transform is a linear combination of $e^{-\lambda_i(\xi) z} V_i^-(\xi)$, it satisfies (3-4)–(3-5) without source. $\square$

Let us stress that, with the same kind of arguments, one can justify the integration by parts mentioned above, and write
$$V(\,\cdot\,, z) := \int_0^{+\infty} \sum_{k=0}^{2} \partial_z^k G(D, z - z') S^k(z')\, dz'. \tag{3-18}$$
We will now try to derive the estimate (3-3), starting from this formulation.

3.1.3. *Main estimate.* By Lemma 6, we know that formula (3-7) (or equivalently (3-18)) defines a solution V of (3-4)–(3-5). Our main goal in this section is to establish that V obeys inequality (3-3). Our main ingredient will be:

Lemma 7. *Let $\chi = \chi(\xi) \in C_c^\infty(\mathbb{R}^2)$, and $P = P(\xi) \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ defined by*
$$P(\xi) = p_k(\xi)|\xi|^{\alpha - k} Q(\xi)$$

near $\xi = 0$, *with* p_k *a homogeneous polynomial in* ξ_1, ξ_2 *of degree* k, $\alpha > 0$, *and* $Q \in \mathcal{C}^\infty(\mathbb{R}^2)$. *Assume furthermore that* $\alpha - k \geq -2$. *For* $\underline{v}_0 \in L^1_{\mathrm{uloc}}(\mathbb{R}^2)$, *we define* $u^i = u^i(y_1, y_2, z)$ *by*

$$u^i(\,\cdot\,, z) := \chi(D) P(D) e^{-\lambda_i(D) z} \underline{v}_0. \tag{3-19}$$

Then, there exists C *and* $\delta > 0$ *independent of* $\underline{v}_0$ *such that*

$$\|e^{\delta z} u^2\|_{L^\infty(\mathbb{R}^3_+)} + \|e^{\delta z} u^3\|_{L^\infty(\mathbb{R}^3_+)}\| \leq C \|\underline{v}_0\|_{L^1_{\mathrm{uloc}}}.$$

Moreover, there exists C *and* $\delta > 0$ *independent of* $\underline{v}_0$ *such that*

$$\|(1+z)^{\frac{\alpha}{3}} u^1\|_{L^\infty(\mathbb{R}^3_+)}\| \leq C \|\underline{v}_0\|_{L^1_{\mathrm{uloc}}}.$$

Remark 8. Showing that the definition (3-19) makes sense is part of the proof of the lemma. Namely, it is shown that for any $z > 0$, the kernel

$$K(x_1, x_2, z) := \mathcal{F}^{-1}_{\xi \to (x_1, x_2)}\big(\chi(\xi) P(\xi) e^{-\lambda_i(\xi) z}\big)$$

defines an element of $L^1(\mathbb{R}^2)$. In particular, (3-19) is appropriate: $u^i = K(\,\cdot\,, z) \star \underline{v}_0$ defines (at least) an L^1_{uloc} function as the convolution of functions of L^1 and L^1_{uloc}.

We refer to Appendix A for a proof. Lemma 7 is the source of the asymptotic behaviour of the solution v of (1-4). As always in this type of boundary layer problem, the asymptotic behaviour is given by low frequencies, corresponding to the cut-off χ. In particular, the decay is given by the characteristic root $\lambda_1(\xi)$, which vanishes at $\xi = 0$.

Proof of estimate (3-3). We distinguish between low and high frequencies.

Low frequencies. We introduce some $\chi = \chi(\xi) \in C^\infty_c(\mathbb{R}^2)$ equal to 1 near $\xi = 0$. We consider

$$\begin{aligned} V^\flat &= \int_{\mathbb{R}^+} \sum_{k=0}^{2} I^k(\,\cdot\,, z, z')\, dz', \\ I^k(\,\cdot\,, z, z') &:= \chi(D) \partial_z^k G(D, z - z') S^k(\,\cdot\,, z'). \end{aligned} \tag{3-20}$$

In what follows, we write

$$S^k = (s^k_3, s^k_\omega)^t \quad \text{and} \quad I^k = (I^k_3, I^k_\omega)^t.$$

We will use the following fact, which is a straightforward consequence of (3-15)–(3-16): $\widehat{s^0_3}$ and $\widehat{s^1_\omega}$ are homogeneous of degree 2 and $\widehat{s^0_\omega}$ is homogeneous of degree 3, while $\widehat{s^1_3}$ and $\widehat{s^2_\omega}$ are homogeneous of degree 1.

Study of I^0. We find

$$I^0_3(\,\cdot\,, z, z') = \operatorname{sgn}(z - z') \chi(D) \sum A_i(D) e^{-\lambda_i(D)|z - z'|} s^0_3(\,\cdot\,, z') + \chi(D) \sum B_i(D) e^{-\lambda_i(D)|z - z'|} s^0_\omega(\,\cdot\,, z'),$$

$$\begin{aligned} I^0_\omega(\,\cdot\,, z, z') = &- \chi(D) \sum A_i(D) \Omega_i(D) e^{-\lambda_i(D)|z - z'|} s^0_3(\,\cdot\,, z') \\ &- \operatorname{sgn}(z - z') \chi(D) \sum B_i(D) \Omega_i(D) e^{-\lambda_i(D)|z - z'|} s^0_\omega(\,\cdot\,, z'). \end{aligned}$$

We also have

$$\partial_z I_3^0(\,\cdot\,,z,z') = -\chi(D)\sum A_i(D)\lambda_i(D)e^{-\lambda_i(D)|z-z'|}s_3^0(\,\cdot\,,z') \\ - \operatorname{sgn}(z-z')\chi(D)\sum B_i(D)\lambda_i(D)e^{-\lambda_i(D)|z-z'|}s_\omega^0(\,\cdot\,,z').$$

We note that $\widehat{s_3^0}(\xi,z')$ and $\widehat{s_\omega^0}(\xi,z')$ are products of components of $\hat{F}(\xi,z')$ by homogeneous polynomials of degrees 2 and 3 respectively in ξ. Using the asymptotic behaviours derived in Lemma 5 together with Lemma 7, we deduce

$$\begin{aligned} \|I_3^0(\,\cdot\,,z,z')\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{2/3}}\|F(\,\cdot\,,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)}, \\ \|I_\omega^0(\,\cdot\,,z,z')\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{1+|z-z'|}\|F(\,\cdot\,,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)}, \\ \left\|\frac{D}{|D|^2}I_\omega^0(\,\cdot\,,z,z')\right\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{2/3}}\|F(\,\cdot\,,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)}, \\ \left\|\frac{D}{|D|^2}\partial_z I_3^0(\,\cdot\,,z,z')\right\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{4/3}}\|F(\,\cdot\,,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)}. \end{aligned} \tag{3-21}$$

The last two bounds will be useful when estimating the horizontal velocity components through (3-6). We insist that $\partial_z I_3^0$ has a better behaviour than I_3^0, because there is an extra factor $\lambda_1(D)$ in front of A_1 and B_1, which gives a higher degree of homogeneity at low frequencies for the term in $\exp(-\lambda_1(D)z)$. This is why we can apply $D/|D|^2$ to that term. As for the terms in $\exp(-\lambda_i(D)z)$ for $i=2,3$, there is no singularity near $\xi=0$ when we apply $D/|D|^2$ because of the homogeneity of degrees 2 and 3 in $\widehat{s_3^0}(\xi,z')$ and $\widehat{s_\omega^0}(\xi,z')$ respectively.

Study of I^1. We find

$$I_3^1(\,\cdot\,,z,z') = -\chi(D)\sum A_i(D)\lambda_i(D)e^{-\lambda_i(D)|z-z'|}s_3^1(\,\cdot\,,z') \\ - \operatorname{sgn}(z-z')\chi(D)\sum B_i(D)\lambda_i(D)e^{-\lambda_i(D)|z-z'|}s_\omega^1(\,\cdot\,,z'),$$

$$I_\omega^1(\,\cdot\,,z,z') = \operatorname{sgn}(z-z')\chi(D)\sum A_i(D)\lambda_i(D)\Omega_i(D)e^{-\lambda_i(D)|z-z'|}s_3^1(\,\cdot\,,z') \\ +\chi(D)\sum B_i(D)\lambda_i(D)\Omega_i(D)e^{-\lambda_i(D)|z-z'|}s_\omega^1(\,\cdot\,,z'),$$

and also

$$\partial_z I_3^1(\,\cdot\,,z,z') = \operatorname{sgn}(z-z')\chi(D)\sum A_i(D)(\lambda_i(D))^2e^{-\lambda_i(D)|z-z'|}s_3^1(\,\cdot\,,z') \\ +\chi(D)\sum B_i(D)(\lambda_i(D))^2e^{-\lambda_i(D)|z-z'|}s_\omega^1(\,\cdot\,,z').$$

Thanks to the derivation of the Green function with respect to z, an extra factor $\lambda_1(D)$ appears together with $A_1(D)$ or $B_1(D)$. This provides a higher degree of homogeneity in $|\xi|$ at low frequencies. It compensates for the loss of homogeneity of S^1 compared to S^0. More precisely, we note that $\widehat{s_3^1}(\xi,z')$ and $\widehat{s_\omega^1}(\xi,z')$ are products of components of $\hat{F}(\xi,z')$ by homogeneous polynomials of degrees 1 and 2 respectively in ξ. We also get

$$\begin{aligned}
\|I_3^1(\cdot,z,z')\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{4/3}}\,\|F(\cdot,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)},\\
\|I_\omega^1(\cdot,z,z')\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{5/3}}\|F(\cdot,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)},\\
\left\|\frac{D}{|D|^2}I_\omega^1(\cdot,z,z')\right\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{4/3}}\|F(\cdot,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)},\\
\left\|\frac{D}{|D|^2}\partial_z I_3^1(\cdot,z,z')\right\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{2}}\|F(\cdot,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)}.
\end{aligned}\tag{3-22}$$

Study of I^2. We find

$$\begin{aligned}
I_3^2(\cdot,z,z') = \operatorname{sgn}(z-z')\chi(D)\sum A_i(D)(\lambda_i(D))^2 e^{-\lambda_i(D)|z-z'|}s_3^2(\cdot,z')\\
+\chi(D)\sum B_i(D)(\lambda_i(D))^2 e^{-\lambda_i(D)|z-z'|}s_\omega^2(\cdot,z'),
\end{aligned}$$

as well as

$$\begin{aligned}
I_\omega^2(\cdot,z,z') = -\chi(D)\sum A_i(D)(\lambda_i(D))^2\Omega_i(D) e^{-\lambda_i(D)|z-z'|}s_3^2(\cdot,z')\\
-\operatorname{sgn}(z-z')\chi(D)\sum B_i(D)(\lambda_i(D))^2\Omega_i(D) e^{-\lambda_i(D)|z-z'|}s_\omega^2(\cdot,z'),
\end{aligned}$$

and

$$\begin{aligned}
\partial_z I_3^2(\cdot,z,z') = -\chi(D)\sum A_i(D)(\lambda_i(D))^3 e^{-\lambda_i(D)|z-z'|}s_3^2(\cdot,z')\\
-\operatorname{sgn}(z-z')\chi(D)\sum B_i(D)(\lambda_i(D))^3 e^{-\lambda_i(D)|z-z'|}s_\omega^2(\cdot,z').
\end{aligned}$$

This time, $s_3^2=0$ and $\widehat{s_\omega^2}$ is homogeneous of degree 1. We get as before that

$$\begin{aligned}
\|I_3^2(\cdot,z,z')\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{2}}\,\|F(\cdot,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)},\\
\|I_\omega^2(\cdot,z,z')\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{7/3}}\|F(\cdot,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)},\\
\left\|\frac{D}{|D|^2}I_\omega^2(\cdot,z,z')\right\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{2}}\|F(\cdot,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)},\\
\left\|\frac{D}{|D|^2}\partial_z I_3^2(\cdot,z,z')\right\|_{L^\infty(\mathbb{R}^2)} &\le \frac{C}{(1+|z-z'|)^{8/3}}\|F(\cdot,z')\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)}.
\end{aligned}\tag{3-23}$$

Combining (3-21)–(3-23), we find

$$\begin{aligned}
\|v_3^\flat(\cdot,z)\|_{L^\infty(\mathbb{R}^2)} &\le C\int_0^{+\infty}\frac{1}{(1+|z-z'|)^{2/3}}\frac{1}{(1+z')^{2/3}}\,dz'\,\|(1+z^{2/3})F\|_{L^\infty(L^1_{\mathrm{uloc}}(\mathbb{R}^2))},\\
\|\omega^\flat(\cdot,z)\|_{L^\infty(\mathbb{R}^2)} &\le C\int_0^{+\infty}\frac{1}{1+|z-z'|}\frac{1}{(1+z')^{2/3}}\,dz'\,\|(1+z^{2/3})F\|_{L^\infty(L^1_{\mathrm{uloc}}(\mathbb{R}^2))},
\end{aligned}$$

and

$$\left\| \frac{D}{|D|^2}\omega^\flat(\,\cdot\,,z)\right\|_{L^\infty(\mathbb{R}^2)} \le C\int_0^{+\infty} \frac{1}{(1+|z-z'|)^{2/3}}\frac{1}{(1+z')^{2/3}}\,dz'\,\|(1+z)^{2/3}F\|_{L^\infty(L^1_{\mathrm{uloc}}(\mathbb{R}^2))},$$

$$\left\| \frac{D}{|D|^2}\partial_z v_3^\flat(\,\cdot\,,z)\right\|_{L^\infty(\mathbb{R}^2))} \le C\int_0^{+\infty} \frac{1}{(1+|z-z'|)^{4/3}}\frac{1}{(1+z')^{2/3}}\,dz'\,\|(1+z)^{2/3}F\|_{L^\infty(L^1_{\mathrm{uloc}}(\mathbb{R}^2))}.$$

We deduce that (see Lemma 16 in Appendix B)

$$\begin{aligned} &\|v_3^\flat(\,\cdot\,,z)\|_{L^\infty(\mathbb{R}^2)} \le C(1+z)^{-1/3}\|(1+z^{2/3})F\|_{L^\infty(L^1_{\mathrm{uloc}}(\mathbb{R}^2))},\\ &\|\omega^\flat(\,\cdot\,,z)\|_{L^\infty(\mathbb{R}^2)} \le C(1+z)^{-2/3}\ln(2+z)\|(1+z^{2/3})F\|_{L^\infty(L^1_{\mathrm{uloc}}(\mathbb{R}^2))},\end{aligned} \tag{3-24}$$

and

$$\begin{aligned} &\left\| \frac{D}{|D|^2}\omega^\flat(\,\cdot\,,z)\right\|_{L^\infty(\mathbb{R}^2)} \le C(1+z)^{-1/3}\|(1+z^{2/3})F\|_{L^\infty(L^1_{\mathrm{uloc}}(\mathbb{R}^2))},\\ &\left\| \frac{D}{|D|^2}\partial_z v_3^\flat(\,\cdot\,,z)\right\|_{L^\infty(\mathbb{R}^2)} \le C(1+z)^{-2/3}\ln(2+z)\|(1+z^{2/3})F\|_{L^\infty(L^1_{\mathrm{uloc}}(\mathbb{R}^2))}.\end{aligned} \tag{3-25}$$

High frequencies. To obtain the estimate (3-3), we still have to control the high frequencies

$$V^\# = \int_{\mathbb{R}^+}\sum_{k=0}^{2} J^k(\,\cdot\,,z,z')\,dz',\qquad J^k(\,\cdot\,,z,z') := (1-\chi(D))\partial_z^k G(D,z-z')S^k(\,\cdot\,,z'). \tag{3-26}$$

Instead of Lemma 7, we shall use this (see Appendix A for a proof):

Lemma 9. *Let $\chi\in C_c^\infty(\mathbb{R}^2)$, with $\chi=1$ in a ball $B_r := B(0,r)$ for some $r>0$. Let $P=P(\xi)\in C_b^3(\mathbb{R}^2\setminus B_r)$. For $\underline{v}_0=\underline{v}_0(y_1,y_2)\in H^N_{\mathrm{uloc}}(\mathbb{R}^2)$, $N\in\mathbb{N}$, we define $u^i=u^i(y_1,y_2,z)$ by*

$$u^i(\,\cdot\,,z) := (1-\chi(D))P(D)e^{-\lambda_i(D)z}\underline{v}_0. \tag{3-27}$$

Then, for N large enough and $\delta>0$ small enough,

$$\|e^{\delta z}u^1\|_{L^\infty(\mathbb{R}^3_+)} + \|e^{\delta z}u^2\|_{L^\infty(\mathbb{R}^3_+)} + \|e^{\delta z}u^3\|_{L^\infty(\mathbb{R}^3_+)}\| \le C\|\underline{v}_0\|_{H^N_{\mathrm{uloc}}(\mathbb{R}^2)}.$$

Remark 10. As in the proof of Lemma 7, part of the proof of Lemma 9 gives a meaning to (3-27). In particular, it is shown that for n large enough, and any $z>0$, the kernel

$$K_n(x_1,x_2,z) := \mathcal{F}^{-1}\big((1+|\xi|^2)^{-n}(1-\chi(\xi))P(\xi)e^{-\lambda_i(\xi)z}\big)$$

belongs to $L^1(\mathbb{R}^2)$ so that $u^i = K_n\star((1-\Delta)^n\underline{v}_0)$ defines at least an element of L^2_{uloc} as the convolution of functions in L^1 and L^2_{uloc} (assuming $N\ge 2n$).

The analysis is simpler than for low frequencies. From (3-26), (3-15)–(3-16) and Lemma 5, we decompose the components of J^k for $k=0,1,2$ into terms of the form

$$(1-\chi(D))R(D)e^{-\lambda_i(D)|z-z'|}\partial_1^{a_1}\partial_2^{a_2}F_{jl},$$

where F_{jl} are components of our source term F, $a_1,a_2=0,1,2$ with $1\le a_1+a_2\le 3$, and $R(D)$ is of the form

$$R(D) = \mathcal{R}\big(\lambda_1(D),\lambda_2(D),\lambda_3(D),D\big)$$

for some rational expression $\mathcal{R} = \mathcal{R}(\lambda_1, \lambda_2, \lambda_3, \xi)$. Considering the behaviour of $\lambda_i(\xi)$ at infinity (see Lemma 7 and Remark 4), it can be easily seen that $|\xi|^{-2n} R(\xi) \in C_b^3(\mathbb{R}^2 \setminus B_r)$ for some n large enough. Thus, we can apply Lemma 9 with

$$P(\xi) = |\xi|^{-2n} R(\xi), \quad \underline{v}_0 = (\partial_1^2 + \partial_2^2)^n \partial_1^{a_1} \partial_2^{a_2} F_{jl}(\,\cdot\,, z').$$

This shows that for m large enough ($m = N + 2n + 3$),

$$\|J^k(\,\cdot\,, z, z')\|_{L^\infty(\mathbb{R}^2)} \le C\, e^{-\delta|z-z'|} \|F(\,\cdot\,, z')\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^2)}. \tag{3-28}$$

Also, up to taking a larger m, one can check that

$$\|\partial_z J^k(\,\cdot\,, z, z')\|_{L^\infty(\mathbb{R}^2)} \le C\, e^{-\delta|z-z'|} \|F(\,\cdot\,, z')\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^2)}. \tag{3-29}$$

We deduce from (3-28)–(3-29) that for m large enough

$$\begin{aligned} \|V^\#(\cdot, z)\|_{L^\infty(\mathbb{R}^2)} + \|\partial_z V^\#(\cdot, z)\|_{L^\infty(\mathbb{R}^2)} &\le C \int_0^{+\infty} e^{-\delta|z-z'|} (1+z')^{-2/3}\, dz' \|(1+z)^{2/3} F\|_{L^\infty(H^m_{\mathrm{uloc}})} \\ &\le C (1+z)^{-2/3} \|(1+z)^{2/3} F\|_{L^\infty(H^m_{\mathrm{uloc}})}. \end{aligned} \tag{3-30}$$

Together with (3-24), this inequality implies the estimate (3-3). □

Together with (3-25), inequality (3-30) further yields

$$\left\|(1+z)^{1/3} \frac{D}{|D|^2} \partial_z v_3\right\|_{L^\infty(\mathbb{R}^3_+)} + \left\|(1+z)^{1/3} \frac{D}{|D|^2} \omega\right\|_{L^\infty(\mathbb{R}^3_+)} \le C \|(1+z)^{2/3} F\|_{L^\infty(H^m_{\mathrm{uloc}})}. \tag{3-31}$$

3.2. ***Proof of Theorem 2.*** In the last section, we have constructed a particular solution of (3-4)–(3-5) satisfying (3-3) and (3-31); in the rest of this section, we denote this particular solution as $V^p = (v_3^p, \omega^p)^t$. The bound (3-31) implies in particular that

$$\left\|(1+z)^{1/3} (v_1^p, v_2^p)\right\|_{L^\infty(\mathbb{R}^3_+)} \le C\, \|(1+z)^{2/3} F\|_{L^\infty(H^m_{\mathrm{uloc}})}, \tag{3-32}$$

where v_1^p, v_2^p are recovered from v_3^p, ω^p through formula (3-6).

We still need to make the connection with the solution of (2-3). Following the discussion after Theorem 2, for smooth and compactly supported data, such a solution exists, and the point is to establish (3-1). We introduce

$$\underline{v} := v - v^p, \quad \underline{\omega} = \omega - \omega^p.$$

Functions $\underline{v}_3$ and $\underline{\omega}$ satisfy the homogeneous version of the Orr–Sommerfeld equations:

$$\partial_3 \underline{v}_3 + \Delta \underline{\omega} = 0, \quad -\partial_3 \underline{\omega} + \Delta^2 \underline{v}_3 = 0. \tag{3-33}$$

These equations are completed by the boundary conditions

$$\begin{gathered} \underline{v}_3|_{z=0} = v_{0,3} - v_3^p|_{z=0}, \quad \partial_z \underline{v}_3|_{z=0} = -\partial_1(v_{0,1} - v_1^p) - \partial_2(v_{0,2} - v_2^p), \\ \underline{\omega}|_{z=0} = \partial_1(v_{0,2} - v_2^p) - \partial_2(v_{0,1} - v_1^p). \end{gathered} \tag{3-34}$$

System (3-33)–(3-34) is the formulation in terms of vertical velocity and vorticity of a Stokes–Coriolis system with zero source term and inhomogeneous Dirichlet data. Formal solutions are given by

$$\begin{pmatrix} \hat{\underline{v}}_3(\xi, z) \\ \hat{\underline{w}}(\xi, z) \end{pmatrix} = \sum_{i=1}^{3} e^{-\lambda_i(\xi) z} C_i(\xi) V_i^-(\xi), \tag{3-35}$$

where coefficients C_i obey the system

$$\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \Omega_1 & \Omega_2 & \Omega_3 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} \hat{\underline{v}}_3|_{z=0} \\ -\partial_z \hat{\underline{v}}_3|_{z=0} \\ -\hat{\underline{\omega}}|_{z=0} \end{pmatrix}. \tag{3-36}$$

The determinant D_3 of this system is

$$D_3 := (\lambda_2 - \lambda_1)(\Omega_3 - \Omega_1) - (\lambda_3 - \lambda_1)(\Omega_2 - \Omega_1),$$

so that $D_3 \to \bar{D}_3 \in \mathbb{C}^*$ as $\xi \to 0*$.

After tedious computation, we find

$$\begin{aligned} C_1 &= \frac{1}{D_3}\big((\lambda_2\Omega_3 - \lambda_3\Omega_2)\hat{\underline{v}}_3|_{z=0} + (\Omega_3 - \Omega_2)\partial_z \hat{\underline{v}}_3|_{z=0} + (\lambda_2 - \lambda_3)\hat{\underline{w}}|_{z=0}\big), \\ C_2 &= \frac{1}{D_3}\big((\lambda_3\Omega_1 - \lambda_1\Omega_3)\hat{\underline{v}}_3|_{z=0} + (\Omega_1 - \Omega_3)\partial_z \hat{\underline{v}}_3|_{z=0} + (\lambda_3 - \lambda_1)\hat{\underline{w}}|_{z=0}\big), \\ C_3 &= \frac{1}{D_3}\big((\lambda_1\Omega_2 - \lambda_2\Omega_1)\hat{\underline{v}}_3|_{z=0} + (\Omega_2 - \Omega_1)\partial_z \hat{\underline{v}}_3|_{z=0} + (\lambda_1 - \lambda_2)\hat{\underline{w}}|_{z=0}\big). \end{aligned} \tag{3-37}$$

Nevertheless, the expressions in (3-35) are not necessarily well-defined, due to possible singularities at $\xi = 0$. In particular, if we want to apply Lemma 7, we need the coefficient in front of $e^{-\lambda_1(\xi)z}$ to contain somehow some positive power of ξ. Using the asymptotics of Lemma 3, we compute

$$|C_1(\xi)| \le \big|\hat{\underline{v}}_3|_{z=0}\big| + \big|\partial_z \hat{\underline{v}}_3|_{z=0}\big| + \big|\hat{\underline{w}}|_{z=0}\big|, \tag{3-38}$$

$$|C_2(\xi)| \le |\xi|\,\big|\hat{\underline{v}}_3|_{z=0}\big| + \big|\partial_z \hat{\underline{v}}_3|_{z=0}\big| + \big|\hat{\underline{w}}|_{z=0}\big|, \tag{3-39}$$

$$|C_3(\xi)| \le |\xi|\,\big|\hat{\underline{v}}_3|_{z=0}\big| + \big|\partial_z \hat{\underline{v}}_3|_{z=0}\big| + \big|\hat{\underline{w}}|_{z=0}\big| \tag{3-40}$$

for small $|\xi|$. The asymptotics is given by:

Lemma 11. *The boundary data $\partial_z \hat{\underline{v}}_3|_{z=0}$, $\hat{\underline{\omega}}|_{z=0}$ in* (3-34), *as well as $\hat{v}_{0,3}|_{z=0}$ (which appears in $\hat{\underline{v}}_3|_{z=0}$) "contain a power of ξ at low frequencies". More precisely, for ξ small enough, they can all be decomposed into terms of the form $\xi \cdot \hat{f}$ for some $f \in L^2_{\mathrm{uloc}}(\mathbb{R}^2)$. As a consequence, for any function $Q \in \mathcal{C}^\infty(\mathbb{R}^2)$,*

$$\left\| \chi(D) Q(D) \exp(-\lambda_1(D) z) \begin{pmatrix} \partial_z \underline{v}_3|_{z=0} \\ \underline{\omega}|_{z=0} \\ v_{0,3}|_{z=0} \end{pmatrix} \right\|_{L^\infty(\mathbb{R}^2)}$$
$$\le C(1+z)^{-1/3} \big(\|(v_{0,1}, v_{0,2})\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*, v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(1+z)^{2/3} F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)} \big),$$

and for $j = 2, 3$,

$$\left\| \chi(D)Q(D)\exp(-\lambda_j(D)z) \begin{pmatrix} \partial_z \underline{v}_3|_{z=0} \\ \underline{\omega}|_{z=0} \\ v_{0,3}|_{z=0} \end{pmatrix} \right\|_{L^\infty(\mathbb{R}^2)} \le Ce^{-\delta z}\big(\|(v_{0,1}, v_{0,2})\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*, v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(1+z)^{2/3}F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big).$$

(2) *Concerning the boundary data* $v_3^p|_{z=0}$ (*which is the other term in* $\underline{v}_3|_{z=0}$), *we have*, *for any function* $Q \in \mathcal{C}^\infty(\mathbb{R}^2)$,

$$\big\|\big(\chi(D)Q(D)\exp(-\lambda_1(D)z)\big)v_3^p\big|_{z=0}\big\|_{L^\infty(\mathbb{R}^2)} \le C(1+z)^{-1/3}\|F\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)},$$
$$\big\|\big(\chi(D)Q(D)\exp(-\lambda_j(D)z)\big)v_3^p\big|_{z=0}\big\|_{L^\infty(\mathbb{R}^2)} \le Ce^{-\delta z}\|F\|_{L^1_{\mathrm{uloc}}(\mathbb{R}^2)}.$$

Proof. The first part of the statement is obvious for the last two boundary data, namely

$$\partial_z \underline{v}_3|_{z=0} = -\partial_1(v_{0,1} - v_1^p) - \partial_2(v_{0,2} - v_2^p), \quad \text{and} \quad \underline{\omega}|_{z=0} = \partial_1(v_{0,2} - v_2^p) - \partial_2(v_{0,1} - v_1^p).$$

It remains to consider $v_{0,3}$. This is where the assumption on $v_{0,3}$ in the theorem plays a role. Indeed, we have $v_{0,3} = \partial_1 v_1^* + \partial_2 v_2^*$, so that it satisfies the properties of the lemma. The estimate is then a straightforward consequence of Lemma 7.

The former argument does not work with the boundary data $\underline{v}_3^p|_{z=0}$: indeed, if we factor out crudely a power of ξ from the integral defining it, then the convergence of the remaining integral is no longer clear. Therefore we go back to the definition of u_3^p; we have, using the notations of (3-20),

$$\chi(D)v_3^p|_{z=0} = \int_{\mathbb{R}_+} \sum_{k=0}^{2} I_3^k(\,\cdot\,, 0, z')\,dz'.$$

It can be easily checked that the terms with I_3^k for $k = 1, 2$ do not raise any difficulty (in fact, the trace stemming from these two terms contains a power of ξ at low frequencies.) Thus we focus on

$$\int_{\mathbb{R}_+} I_3^0(\,\cdot\,, 0, z')\,dz' = \int_{\mathbb{R}_+} \left(\chi(D)\sum_{i=1}^{3} A_i(D)e^{-\lambda_i(D)z'} s_3^0(\,\cdot\,, z') + \chi(D)\sum_{i=1}^{3} B_i(D)e^{-\lambda_i(D)z'} s_\omega^0(\,\cdot\,, z')\right) dz'.$$

Applying $\exp(-\lambda_j(D)z)$, we have to estimate the $L^\infty(\mathbb{R}^2)$ norms of

$$\int_{\mathbb{R}_+} \chi(D)Q(D)\sum_{i=1}^{3} A_i(D)e^{-\lambda_i(D)z'-\lambda_j(D)z} s_3^0(\,\cdot\,, z')\,dz',$$
$$\int_{\mathbb{R}_+} \chi(D)Q(D)\sum_{i=1}^{3} B_i(D)e^{-\lambda_i(D)z'-\lambda_j(D)z} s_\omega^0(\,\cdot\,, z')\,dz'.$$

We recall that $\widehat{s_3^0}(\xi, z')$ and $\widehat{s_\omega^0}(\xi, z')$ are products of components of $\hat{F}(\xi, z')$ by homogeneous polynomials of degrees 2 and 3 respectively in ξ, and that the behaviour of A_i, B_i is given in Lemma 5. When $i = j = 1$, using Lemma 7 and Lemma 16 in Appendix B, the corresponding integral is bounded by

$$\int_{\mathbb{R}_+} \frac{1}{(1+z+z')^{2/3}} \frac{1}{(1+z')^{2/3}}\, dz' \, \|(1+z)^{2/3}F\|_{L^\infty(L^1_{\mathrm{uloc}})} \le C(1+z)^{-1/3}\|(1+z)^{2/3}F\|_{L^\infty(L^1_{\mathrm{uloc}})}.$$

When $i=2,3$, the integral is bounded by

$$\int_{\mathbb{R}_+} \frac{\exp(-\delta z')}{(1+z)^{2/3}} \frac{1}{(1+z')^{2/3}}\, dz' \, \|(1+z)^{2/3}F\|_{L^\infty(L^1_{\mathrm{uloc}})} \le C(1+z)^{-2/3}\|(1+z)^{2/3}F\|_{L^\infty(L^1_{\mathrm{uloc}})}.$$

When $j=2,3$, the integral is bounded by

$$\int_{\mathbb{R}_+} \frac{\exp(-\delta z)}{(1+z')^{2/3}} \frac{1}{(1+z')^{2/3}}\, dz' \, \|(1+z)^{2/3}F\|_{L^\infty(L^1_{\mathrm{uloc}})} \le C\exp(-\delta z)\|(1+z)^{2/3}F\|_{L^\infty(L^1_{\mathrm{uloc}})}.$$

Gathering all the terms, we obtain the estimate announced in the lemma. □

Going back to (3-35), we infer that

$$\begin{aligned}(1+z)^{1/3}\|\chi(D)\underline{v}_3(\cdot\,,z)\|_{L^\infty(\mathbb{R}^2)} + (1+z)^{2/3}\|\chi(D)\underline{\omega}(\cdot\,,z)\|_{L^\infty(\mathbb{R}^2)} \\ \le C\big(\|(v_{0,1},v_{0,2})\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*,v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(1+z)^{2/3}F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big).\end{aligned} \tag{3-41}$$

Then, for further control of the horizontal components (v_1, v_2), one would like an analogue of (3-25), namely a bound like

$$\begin{aligned}(1+z)^{1/3}\left\|\frac{D}{|D|^2}\chi(D)\partial_z\underline{v}_3(\cdot\,,z)\right\|_{L^\infty(\mathbb{R}^2)} + (1+z)^{1/3}\left\|\frac{D}{|D|^2}\chi(D)\underline{\omega}(\cdot\,,z)\right\|_{L^\infty(\mathbb{R}^2)} \\ \le C\big(\|(v_{0,1},v_{0,2})\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*,v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(1+z)^{2/3}F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big).\end{aligned}$$

However, such an estimate is not clear. Indeed, in view of (3-35), we have

$$\chi(D)\begin{pmatrix}\partial_z\underline{v}_3(\cdot\,,z)\\ \underline{\omega}(\cdot\,,z)\end{pmatrix} = \chi(D)\sum_{i=1}^{3} e^{-\lambda_i(D)z}\begin{pmatrix}-\lambda_i(D)C_i\\ -\Omega_i(D)C_i\end{pmatrix}.$$

The term with index $i=1$ does not raise any difficulty, because $\lambda_1(D)$ and $\Omega_1(D)$ bring extra powers of ξ, which are enough to apply Lemma 7. But the difficulty comes from indices 2 and 3. For instance, they involve terms of the type

$$\chi(D)P_0(D)e^{-\lambda_{2,3}(D)}\hat{v}_0, \quad \text{with } P_0 \text{ homogeneous of degree } 0,$$

and therefore are not covered by Lemma 7: with the notations of the lemma, one has $\alpha=0$, which is not enough. Typically, these homogeneous functions of degree zero involve Riesz transforms, meaning $P_0(\xi)=\xi_k\xi_l/|\xi|^2,\ k,l=1,2$.

Hence, one must use extra cancellations. We recall that in view of (3-6), we want to exhibit cancellations in $|D|^{-2}(D_1\partial_z\underline{v}_3 + D_2\underline{\omega})$ and in $|D|^{-2}(D_2\partial_z\underline{v}_3 - D_1\underline{\omega})$. Let us comment briefly on the first term. We compute $(-\xi_1\lambda_i - \xi_2\Omega_i)C_i$ for $i=2,3$ in terms of the boundary data. Setting $\underline{v}_0 = v_0 - v^p|_{z=0}$, we find that

$$C_2(\xi) = \frac{1}{D_3}(\lambda_3\Omega_1 - \lambda_1\Omega_3)\hat{\underline{v}}_{0,3} + \frac{1}{D_3}\big[\big((\Omega_3-\Omega_1)i\xi_1 - i\xi_2(\lambda_3-\lambda_1)\big)\hat{\underline{v}}_{0,1} + \big((\Omega_3-\Omega_1)i\xi_2 + i\xi_1(\lambda_3-\lambda_1)\big)\hat{\underline{v}}_{0,2}\big].$$

We then use the asymptotic formulas of Lemma 3. In particular,

$$(-\xi_1\lambda_2-\xi_2\Omega_2)\big((\Omega_3-\Omega_1)i\xi_1-i\xi_2(\lambda_3-\lambda_1)\big)=|\xi|^2+O(|\xi|^3),$$
$$(-\xi_1\lambda_2-\xi_2\Omega_2)\big((\Omega_3-\Omega_1)i\xi_2+i\xi_1(\lambda_3-\lambda_1)\big)=-i|\xi|^2+O(|\xi|)^3.$$

A similar formula holds for C_3. It follows that there exist $Q_2,\,Q_3\in\mathcal{C}^\infty(\mathbb{R}^2)^2$ such that

$$\begin{aligned}&\mathcal{F}(\chi(D)|D|^{-2}(D_1\partial_z\underline{v}_3+D_2\underline{\omega}))\\&\qquad=\chi(\xi)\frac{-\xi_1\lambda_1-\xi_2\Omega_1}{D_3|\xi|^2}e^{-\lambda_1(\xi)z}C_1(\xi)\\&\qquad\quad+\frac{1}{D_3}\big[(\lambda_3\Omega_1-\lambda_1\Omega_3)(-\xi_1\lambda_2-\xi_2\Omega_2)e^{-\lambda_2z}+(\lambda_1\Omega_2-\lambda_2\Omega_1)(-\xi_1\lambda_3-\xi_2\Omega_3)e^{-\lambda_3z}\big]\hat{\underline{v}}_{0,3}\\&\qquad\quad+\sum_{i=2,3}\chi(\xi)e^{-\lambda_iz}Q_i(\xi)\cdot\hat{\underline{v}}_{0,h}(\xi,z).\end{aligned}$$

The first two terms are treated in the same way as Lemma 11, factoring out a power of ξ when necessary, and going back to the definition of v^p. We leave the details to the reader. The inverse Fourier transform of the last term is $\mathcal{F}^{-1}(\chi Q_ie^{-\lambda_iz})*\underline{v}_{0,h}$, which is bounded in $L^\infty(\mathbb{R}^2)$ by $e^{-\delta z}\|\underline{v}_{0,h}\|_{L^2_{\mathrm{uloc}}}$. Similar statements hold for $\chi(D)|D|^{-2}(-\partial_zD_2\underline{v}_3+D_1\underline{\omega})$. It follows that

$$\begin{aligned}&(1+z)^{1/3}\|\chi(D)\underline{v}(\,\cdot\,,z)\|_{L^\infty(\mathbb{R}^2)}\\&\qquad\le C\big(\|(v_{0,1},v_{0,2})\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)}+\|(v_1^*,v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)}+\|(1+z)^{2/3}F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big).\end{aligned}\tag{3-42}$$

We now address the estimates of $\hat{\underline{v}}(\xi,z)$ for large frequencies. The arguments are very close to the ones developed after Lemma 9. Using (3-35) and (3-37), for $|\xi|\gg1$, we find that $\hat{\underline{v}}_3(\xi,z)$ and $\hat{\underline{\omega}}(\xi,z)$ can be written as linear combinations of terms of the type

$$R_{ij}(\lambda_1,\lambda_2,\lambda_3,\xi)\exp(-\lambda_i(\xi)z)\hat{g}_j(\xi),\qquad 1\le i,\,j\le3,$$

where $g_1=\underline{v}_3|_{z=0}$, $g_2=\partial_z\underline{v}_3|_{z=0}$ and $g_3=\underline{\omega}|_{z=0}$ and R_{ij} is a rational expression. Thus, using Lemmas 3 and 5, there exists $n\in\mathbb{N}$ such that $|\xi|^{-2n}R_{ij}(\lambda_1,\lambda_2,\lambda_3,\xi)$ is bounded as $|\xi|\to\infty$ for all $i,\,j$. Lemma 9 then gives that for some N sufficiently large,

$$\big\|(1-\chi)(D)\underline{v}_3(\,\cdot\,,z)\big\|_{L^\infty(\mathbb{R}^2)}\le Ce^{-\delta z}\sum_{j=1}^3\|g_j\|_{H^N_{\mathrm{uloc}}},$$
$$\big\|(1-\chi)(D)\underline{\omega}(\,\cdot\,,z)\big\|_{L^\infty(\mathbb{R}^2)}\le Ce^{-\delta z}\sum_{j=1}^3\|g_j\|_{H^N_{\mathrm{uloc}}},$$

and similar estimates hold for $(D/|D|^2)\partial_z\underline{v}_3$ and $(D/|D|^2)\underline{\omega}$. Using (3-34) and (3-28)–(3-29), we infer that for some $m\ge1$ large enough,

$$\big\|(1-\chi(D))\underline{v}(\,\cdot\,,z)\big\|_{L^\infty(\mathbb{R}^2)}\le Ce^{-\delta z}\big(\|v_0\|_{H^{m+1/2}_{\mathrm{uloc}}(\mathbb{R}^2)}+\|(1+z)^{2/3}F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big).\tag{3-43}$$

Gathering (3-42) and (3-43), we deduce that u satisfies the estimate

$$\|(1+z)^{1/3}\underline{v}\|_{L^\infty} \le C\big(\|v_0\|_{H^{m+1/2}_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*, v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(1+z)^{2/3}F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big)$$

for m large enough. Thus, in view of the estimate (3-3) satisfied by v^p, we know $v = \underline{v} + v^p$ is a solution of (2-3) satisfying

$$\|(1+z)^{1/3}v\|_{L^\infty} \le C\big(\|v_0\|_{H^{m+1/2}_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*, v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(1+z)^{2/3}F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big)$$

for m large enough. It remains to go to the higher regularity bound (3-1). First, up to taking a slightly larger m, we clearly have

$$\|(1+z)^{1/3}\nabla v\|_{L^\infty} \le C\big(\|v_0\|_{H^{m+1/2}_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*, v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(1+z)^{2/3}F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big).$$

This follows from direct differentiation of formula (3-7) satisfied by v^p and formula (3-35) satisfied by $\underline{v} = v - v^p$. Clearly, the differentiation is harmless, in particular at low frequencies where it may even add positive powers of ξ. It follows that our solution belongs to $H^1_{\mathrm{uloc}}(\mathbb{R}^3_+)$, and thus enters the framework of local elliptic regularity theory for the Stokes equation. In particular, for any $k \in \mathbb{Z}^3$ with $k_z \le 2$,

$$\begin{aligned}\|v\|_{H^{m+1}(B(k,1)\cap\Omega_{\mathrm{bl}})} &\le C\big(\|v_0\|_{H^{m+1/2}_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*, v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)} + \|v\|_{H^1(B(k,2)\cap\Omega_{\mathrm{bl}})}\big)\\ &\le C\big(\|v_0\|_{H^{m+1/2}_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*, v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)} + \|v\|_{H^1_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big)\end{aligned}$$

and for any $k \in \mathbb{Z}^3$ with $k_z > 2$,

$$\begin{aligned}\|v\|_{H^{m+1}(B(k,1)\cap\Omega_{\mathrm{bl}})} &\le C\big(\|F\|_{H^m(B(k,2)\cap\Omega_{\mathrm{bl}})} + \|v\|_{H^1(B(k,2)\cap\Omega_{\mathrm{bl}})}\big)\\ &\le C|k_z|^{-1/3}\big(\|(1+z)^{2/3}F\|_{H^m_{\mathrm{uloc}}(\mathbb{R}^3_+)} + \|(1+y)^{1/3}v\|_{H^1_{\mathrm{uloc}}(\mathbb{R}^3_+)}\big).\end{aligned}$$

The bound (3-1) follows.

4. Proof of Theorem 1

4.1. ***Navier–Stokes–Coriolis system in a half-space.*** This section is devoted to the well-posedness of system (2-4) under a smallness assumption. Once again, we can assume $M = 0$ with no loss of generality. Following the analysis of the linear case performed in the previous section, we introduce the functional spaces

$$\mathcal{H}^m := \big\{v \in H^m_{\mathrm{loc}}(\mathbb{R}^3_+) : \|(1+y_3)^{1/3}v\|_{H^m_{\mathrm{uloc}}} < +\infty\big\}, \quad m \ge 0,$$

and we set $\|v\|_{\mathcal{H}^m} = C_m\|(1+y_3)^{1/3}v\|_{H^m_{\mathrm{uloc}}}$, where the constant C_m is chosen so that if $u, v \in (\mathcal{H}^m)^3$ for some $m > \frac{3}{2}$, then

$$\|u \otimes v\|_{\mathcal{H}^m} \le \|u\|_{\mathcal{H}^m}\|v\|_{\mathcal{H}^m}.$$

Clearly $\mathcal{H}^m$ is a Banach space for all $m \ge 0$.

The result proved in this section is the following:

Proposition 12. *Let $m \in \mathbb{N}$, $m \gg 1$. There exists $\delta_0 > 0$ such that for all $v_0 \in H^{m+1}_{\mathrm{uloc}}(\mathbb{R}^2)$ such that $v_{0,3} = \partial_1 v_1^* + \partial_2 v_2^*$, with v_1^*, v_2^* in $L^2_{\mathrm{uloc}}(\mathbb{R}^2)$ and*

$$\|v_0\|_{H^{m+1}_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*, v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} \leq \delta_0, \tag{4-1}$$

the system

$$\begin{cases} v \cdot \nabla v + e \times v + \nabla p - \Delta v = 0 & \text{in } \{y_3 > 0\}, \\ \operatorname{div} v = 0 & \text{in } \{y_3 > 0\}, \\ v|_{y_3=0} = v_0 \end{cases}$$

has a unique solution in $\mathcal{H}^{m+1}$.

Remark 13. The integer m for which this result holds is the same as the one in Theorem 2.

Proof. Proposition 12 is an easy consequence of the fixed point theorem in $\mathcal{H}^{m+1}$. For any $v_0 \in H^{m+1}_{\mathrm{uloc}}(\mathbb{R}^2)$ such that $v_{0,3} = \partial_1 v_1^* + \partial_2 v_2^*$, with v_1^*, v_2^* in $L^2_{\mathrm{uloc}}(\mathbb{R}^2)$, we introduce the mapping $T_{v_0} : \mathcal{H}^{m+1} \to \mathcal{H}^{m+1}$ such that $T_{v_0}(u) = v$ is the solution of (2-3) with $F = u \otimes u$. Notice that $\|(1+z)^{2/3} F\|_{H^m_{\mathrm{uloc}}} \leq \|u\|^2_{\mathcal{H}^m}$. As a consequence, according to Theorem 2, there exists a constant C_0 such that for all $u \in \mathcal{H}^{m+1}$,

$$\|T_{v_0}(u)\|_{\mathcal{H}^{m+1}} \leq C_0\big(\|v_0\|_{H^{m+1}_{\mathrm{uloc}}(\mathbb{R}^2)} + \|(v_1^*, v_2^*)\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2)} + \|u\|^2_{\mathcal{H}^{m+1}}\big). \tag{4-2}$$

Let $\delta_0 < 1/(4C_0^2)$, and assume that (4-1) is satisfied. Thanks to the assumption on δ_0, there exists $R_0 > 0$ such that

$$C_0(\delta_0 + R_0^2) \leq R_0. \tag{4-3}$$

Moreover, $R_0 \in [R_-, R_+]$, where

$$R_\pm = \frac{1}{2C_0}\big(1 \pm \sqrt{1 - 4\delta_0 C_0^2}\big).$$

Therefore $0 < R_- < (2C_0)^{-1}$, and we can always choose R_0 so that $2R_0C_0 < 1$. Then according to (4-1), (4-2) and (4-3),

$$\|u\|_{\mathcal{H}^{m+1}} \leq R_0 \quad \Longrightarrow \quad \|T_{v_0}(u)\|_{\mathcal{H}^{m+1}} \leq R_0.$$

Moreover, if $\|u^1\|_{\mathcal{H}^{m+1}}, \|u^2\|_{\mathcal{H}^{m+1}} \leq R_0$, then setting $w = T_{v_0}(u^1) - T_{v_0}(u^2)$, we have w is a solution of (2-3) with $w|_{z=0} = 0$ and with a source term $F^1 - F^2 = u^1 \otimes u^1 - u^2 \otimes u^2$. Thus, using once again Theorem 2 and the normalization of $\|\cdot\|_{\mathcal{H}^m}$,

$$\big\|T_{v_0}(u^1) - T_{v_0}(u^2)\big\|_{\mathcal{H}^{m+1}} \leq C_0\|F^1 - F^2\|_{\mathcal{H}^m} \leq 2C_0R_0\|u^1 - u^2\|_{\mathcal{H}^{m+1}}.$$

Notice that in the inequality above, we have assumed that $\|\cdot\|_{\mathcal{H}^m} \leq \|\cdot\|_{\mathcal{H}^{m+1}}$, which is always possible if the normalization constant C_m is chosen sufficiently small (depending on C_{m+1}, m being large but fixed).

Thus, since $2C_0R_0 < 1$, we know T_{v_0} is a contraction over the ball of radius R_0 in $\mathcal{H}^{m+1}$. Using Banach's fixed point theorem, we infer that T_{v_0} has a fixed point in $\mathcal{H}^{m+1}$. This concludes the proof of Proposition 12. ☐

4.2. ***Navier–Stokes–Coriolis system over a bumped half-plane.*** We now address the study of the full system (1-4). We follow the steps outlined in the introduction, which we recall here for the reader's convenience: We first prove that there exists a solution (v^-, p^-) of the system (2-5) for ϕ, ψ in some function spaces to be specified, then construct the solution (v^+, p^+) of (2-4) with $v^+|_{y_3=M} = v^-|_{y_3=M}$. Eventually, we define a mapping $\mathcal{F}$ by $\mathcal{F}(\phi, \psi) := \Sigma(v^+, p^+)e_3|_{y_3=M} - \psi$. We recall that $v = \mathbf{1}_{y_3\geq M} v^+ + \mathbf{1}_{y_3<M} v^-$ is a solution of (1-4) if and only if $\mathcal{F}(\phi, \psi) = 0$. The goal is therefore to show that for all ϕ small enough (in a function space to be specified) the equation $\mathcal{F}(\phi, \psi) = 0$ has a unique solution.

Step 1. We study the system (2-5). We introduce the function space

$$\mathcal{V} := \left\{\phi = (\phi_h, \phi_3) : \phi_h \in H^2_{\text{uloc}}(\partial\Omega_{\text{bl}}),\ \phi_3 \in H^1_{\text{uloc}}(\partial\Omega_{\text{bl}}),\ \phi \cdot n|_{\partial\Omega_{\text{bl}}} = 0\right\} \tag{4-4}$$

for the bottom Dirichlet data, and we set

$$\|\phi\|_{\mathcal{V}} := \|\phi_h\|_{H^2_{\text{uloc}}} + \|\phi_3\|_{H^1_{\text{uloc}}}.$$

As for the stress tensor at $y_3 = M$, since we will need to construct solutions in H^{m+1}_{uloc} (see Proposition 12), we look for ψ in the space $H^{m-1/2}_{\text{uloc}}(\mathbb{R}^2)$. We then claim that the following result holds:

Lemma 14. *Let $m \geq 1$ be arbitrary. There exists $\delta > 0$ such that for all $\phi \in \mathcal{V}$ and all $\psi \in H^{m-1/2}_{\text{uloc}}(\mathbb{R}^2)$ with $\|\phi\|_{\mathcal{V}} \leq \delta$ and $\|\psi\|_{H^{m-1/2}_{\text{uloc}}(\mathbb{R}^2)} \leq \delta$, system* (2-5) *has a unique solution*

$$(v^-, p^-) \in H^1_{\text{uloc}}(\Omega^M_{\text{bl}}) \times L^2_{\text{uloc}}(\Omega^M_{\text{bl}}).$$

Moreover, it satisfies the following properties:

- H^{m+1}_{uloc} *regularity*: *for all $M' \in\,]\sup\gamma, M[$,*
$$(v^-, p^-) \in H^{m+1}_{\text{uloc}}\big(\mathbb{R}^2 \times (M', M)\big) \times H^m_{\text{uloc}}\big(\mathbb{R}^2 \times (M', M)\big),$$
with
$$\|v^-\|_{H^{m+1}_{\text{uloc}}(\mathbb{R}^2\times(M',M))} + \|p^-\|_{H^m_{\text{uloc}}(\mathbb{R}^2\times(M',M))} \leq C_{M'}\big(\|\phi\|_{\mathcal{V}} + \|\psi\|_{H^{m-1/2}_{\text{uloc}}(\mathbb{R}^2)}\big).$$
- *Compatibility condition*: *there exists $v_1^*, v_2^* \in H^{1/2}_{\text{uloc}}$ such that $v_3^-|_{y_3=M} = \nabla_h \cdot v_h^*$.*

Proof. We start with an H^1_{uloc} a priori estimate. We follow the computations of [Dalibard and Prange 2014], dedicated to the linear Stokes–Coriolis system. We first lift the boundary condition on $\partial\Omega_{\text{bl}}$, introducing

$$v_h^L := \phi_h, \quad v_3^L := \phi_3 - \nabla_h \cdot \phi_h(y_3 - \gamma(y_h)).$$

Then $\tilde{v} := v^- - v^L$ and $\tilde{p} = p^-$ satisfy

$$\begin{aligned}
-\Delta\tilde{v} + (v^L + \tilde{v})\cdot\nabla\tilde{v} + \tilde{v}\cdot\nabla v^L + e_3 \wedge \tilde{v} + \nabla\tilde{p} = f \quad &\text{in } \Omega^M_{\text{bl}},\\
\operatorname{div}\tilde{v} = 0 \quad &\text{in } \Omega^M_{\text{bl}},\\
\tilde{v}|_{\partial\Omega_{\text{bl}}} = 0,&\\
\left(\partial_3\tilde{v} - \left(\tilde{p} + \frac{|\tilde{v}+v^L|^2}{2}\right)e_3\right)\bigg|_{y_3=M} = \psi - \partial_3 v^L|_{y_3=M} := \tilde{\psi},&
\end{aligned} \tag{4-5}$$

where $f = -\Delta v^L + v^L \cdot \nabla v^L + e_3 \wedge v^L$.

Notice that thanks to the regularity assumptions on ϕ and v^*, we have $\tilde{\psi} \in L^2_{\mathrm{uloc}}(\mathbb{R}^2)$ and $f \in H^{-1}_{\mathrm{uloc}}(\mathbb{R}^2)$. We then perform energy estimates on the system (4-5), following the strategy of Gérard-Varet and Masmoudi [2010], which is inspired by the work of Ladyženskaya and Solonnikov [1980]. The idea is to work with the truncated energies

$$E_k := \int_{\Omega^M_{\mathrm{bl}} \cap \{(y_1,y_2)\in[-k,k]^2\}} \nabla\tilde{v}\cdot\nabla\tilde{v}, \tag{4-6}$$

and to derive an induction inequality on $(E_k)_{k\in\mathbb{N}}$. To that end, we consider a truncation function $\chi_k \in \mathcal{C}^\infty_0(\mathbb{R}^2)$ such that $\chi_k \equiv 1$ in $[-k,k]^2$, $\operatorname{Supp}\chi_k \subset [-k-1,k+1]^2$, and χ_k, χ_k', χ_k'' are bounded uniformly in k. Along the lines of [Dalibard and Prange 2014], we multiply (4-5) by the test function

$$\begin{aligned}\varphi = \begin{pmatrix}\varphi_h \\ \nabla\cdot\Phi_h\end{pmatrix} &:= \begin{pmatrix}\chi_k\tilde{v}_h \\ -\nabla_h\cdot\big(\chi_k\int_{\gamma(y_h)}^{y_3}\tilde{v}_h(y_h,z)\,dz\big)\end{pmatrix} \in H^1(\Omega^b)\\ &= \chi_k\tilde{v} - \begin{pmatrix}0\\ \nabla_h\chi_k(y_h)\cdot\int_{\gamma(y_h)}^{y_3}\tilde{v}_h(y_h,z)\,dz\end{pmatrix}.\end{aligned}$$

Since this test function is divergence-free, there is no commutator term stemming from the pressure. In [loc. cit.], an inequality of the following type is derived:

$$E_k \le C\big((E_{k+1}-E_k) + (\|\phi\|^2_{\mathcal{V}} + \|\psi\|^2_{H^{-1/2}_{\mathrm{uloc}}})(k+1)^2\big).$$

This discrete differential inequality is a key a priori estimate, which allows for the construction of a solution. Indeed, introducing an approximate solution $\tilde{v}^n$ for $|y_1,y_2|\le n$, say with Dirichlet boundary conditions at the lateral boundary, a standard estimate yields that $E_n \le Cn$, where this time $E_k = \int|\chi_k\nabla\tilde{v}^n|^2$. Combining this information with above induction relation allows one to obtain a uniform bound on the E_k of the type $E_k \le Ck^2$, from which we deduce a H^1_{uloc} bound on $\tilde{v}^n$ uniformly in n. From there, one obtains an exact solution by compactness. We refer to [loc. cit.] for more details.

Here, there are two noticeable differences with [loc. cit.]:

- The boundary condition at $y_3 = M$ in (4-5) does not involve a Dirichlet-to-Neumann operator, which makes things easier.
- On the other hand, one has to handle the quadratic terms $(v^L+\tilde{v})\cdot\nabla\tilde{v} + \tilde{v}\cdot\nabla v^L$, which explains the introduction of the $|v|^2$ in the stress tensor at $y_3 = M$.

Therefore we focus on the treatment of these nonlinear terms. The easiest one is

$$\left|\int_{\Omega^M_{\mathrm{bl}}}(\tilde{v}\cdot\nabla v^L)\cdot\varphi\right| \le C\|\phi\|_{\mathcal{V}}E_{k+1},$$

where the constant C depends only on M and on $\|\gamma\|_{W^{1,\infty}}$. On the other hand,

$$\begin{aligned}\int_{\Omega^M_{\mathrm{bl}}}\big((v^L+\tilde{v})\cdot\nabla\tilde{v}\big)\cdot(\chi_k\tilde{v}) &= \int_{\Omega^M_{\mathrm{bl}}}\chi_k(v^L+\tilde{v})\cdot\nabla\frac{|\tilde{v}|^2}{2}\\ &= -\int_{\Omega^M_{\mathrm{bl}}}\frac{|\tilde{v}|^2}{2}(v^L+\tilde{v})\cdot\nabla\chi_k + \int_{\mathbb{R}^2}\chi_k\left((v^L_3+\tilde{v}_3)\frac{|\tilde{v}|^2}{2}\right)\Big|_{y_3=M}.\end{aligned}$$

The first term on the right-hand side is bounded by $C(E_{k+1}-E_k)^{3/2}+C\|\phi\|_{\mathcal{V}}(E_{k+1}-E_k)$. We group the second one with the boundary terms stemming from the pressure and the Laplacian. The sum of these three boundary terms is

$$\int_{\mathbb{R}^2}\chi_k\left(-\partial_3\tilde{v}\cdot\tilde{v}+(v_3^L+\tilde{v}_3)\frac{|\tilde{v}|^2}{2}+p^-\tilde{v}_3\right)\bigg|_{y_3=M}.$$

Using the boundary condition in (4-5), the integral above is equal to

$$-\int_{\mathbb{R}^2}\chi_k\tilde{v}|_{y_3=M}\cdot\big(\tilde{\psi}+\big(\tilde{v}\cdot v^L|_{y_3=M}+\tfrac{1}{2}|v^L|_{y_3=M}|^2\big)e_3\big),$$

which is bounded for any $\delta>0$ by

$$C\|\phi\|_{\mathcal{V}}E_{k+1}+\delta E_{k+1}+C_\delta\big(\|\phi\|_{\mathcal{V}}^2+\|\phi\|_{\mathcal{V}}^4+\|\psi\|^2_{H^{m-1/2}_{\mathrm{uloc}}}\big)(k+1)^2.$$

There remains

$$\int_{\Omega_{\mathrm{bl}}^M}\big((v^L+\tilde{v})\cdot\nabla\tilde{v}\big)\cdot\begin{pmatrix}0\\ \nabla_h\chi_k(y_h)\cdot\int_{\gamma(y_h)}^{y_3}\tilde{v}_h(y_h,z)\,dz\end{pmatrix},$$

which is bounded by $C(E_{k+1}-E_k)^{3/2}+C\|\phi\|_{\mathcal{V}}(E_{k+1}-E_k)$. Gathering all the terms, we infer that for $\|\phi\|_{\mathcal{V}}\le 1$,

$$E_k\le C\big((E_{k+1}-E_k)^{3/2}+(E_{k+1}-E_k)+\|\phi\|_{\mathcal{V}}E_k+(\|\phi\|_{\mathcal{V}}^2+\|\psi\|^2_{H^{m-1/2}_{\mathrm{uloc}}})(k+1)^2\big),$$

where the constant C depends only on M and on $\|\gamma\|_{W^{1,\infty}}$. As a consequence, for $\|\phi\|_{\mathcal{V}}$ small enough, we infer that for all $k\ge 1$,

$$E_k\le C\big((E_{k+1}-E_k)^{3/2}+(E_{k+1}-E_k)+(\|\phi\|_{\mathcal{V}}^2+\|\psi\|^2_{H^{m-1/2}_{\mathrm{uloc}}})(k+1)^2\big).$$

Thanks to a backwards induction argument (again, we refer to [Gérard-Varet and Masmoudi 2010] for all details), we infer that

$$E_k\le C(\|\phi\|_{\mathcal{V}}^2+\|\psi\|^2_{H^{m-1/2}_{\mathrm{uloc}}})k^2\quad\forall k\in\mathbb{N}$$

for a possibly different constant C. It follows that

$$\|\tilde{v}\|_{H^1_{\mathrm{uloc}}(\Omega^M_{\mathrm{bl}})}\le C(\|\phi\|_{\mathcal{V}}+\|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}})$$

and therefore v^- satisfies the same estimate. From there, we can derive an L^2_{uloc} estimate for the pressure. Indeed, using the equation and the boundary condition at $y_3=M$, it follows that for all $y\in\Omega^M_{\mathrm{bl}}$,

$$p^-(y_h,y_3)=\partial_3v_3^-|_{y_3=M}-\frac{\big|v^-|_{y_3=M}\big|^2}{2}-\psi_3(y_h)-\int_{y_3}^M(\Delta v_3^--v^-\cdot\nabla v_3^-)(y_h,z)\,dz.$$

Note that by the divergence-free condition, the first-term in the right-hand side can be written as $-\operatorname{div}_h v_h^-|_{y_3=M}$. For $k\in\mathbb{Z}^2$, let $\varphi_k\in H^1_0(\Omega^M_{\mathrm{bl}})$ such that $\operatorname{Supp}\varphi_k\subset(k+[0,1]^2)\times\mathbb{R}$. We multiply the above identity by $\varphi_k(x_h, z)$ and integrate over Ω_{bl}^M. After some integrations by parts, we obtain

$$\int_{\Omega_{\mathrm{bl}}^M} p^-\varphi_k = \int_{\Omega_{\mathrm{bl}}^M} v_h^-|_{y_3=M}\cdot\nabla_h\varphi_k - \int_{\Omega_{\mathrm{bl}}^M}\frac{|v^-|_{y_3=M}|^2}{2}\varphi_k - \int_{\Omega_{\mathrm{bl}}^M}\psi_3\varphi_k \\ - \int_{\Omega_{\mathrm{bl}}^M}\left(\int_{y_3}^M(\Delta_h v_3^- + \partial_3^2 v_3^- - v^-\cdot\nabla v_3^-)(y_h, z)\,dz\right)\varphi_k(y)\,dy. \tag{4-7}$$

Using classical trace estimates and Sobolev embeddings, it follows that for all $q\in\,]1,\infty[$,

$$\left\|v^-|_{y_3=M}\right\|_{L^q_{\mathrm{uloc}}(\mathbb{R}^2)} \le C\left\|v^-|_{y_3=M}\right\|_{H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)} \le C\|v^-\|_{H^1_{\mathrm{uloc}}(\Omega_{\mathrm{bl}}^M)}. \tag{4-8}$$

Therefore the top line of the right-hand side of (4-7) is bounded by $C(\|\phi\|_{\mathcal{V}}+\|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}})\|\varphi_k\|_{H^1}$ for ϕ, ψ small enough. We now focus on the second line of (4-7). The easiest term is the advection term: we have, since φ_k has a bounded support (uniformly in k),

$$\left|\int_{\Omega_{\mathrm{bl}}^M}\int_{y_3}^M v^-\cdot\nabla v_3^-(x_h,z)\,dz\varphi_k(y)\,dy\right| \le C\|v^-\|_{L^4_{\mathrm{uloc}}}\|\nabla v^-\|_{L^2_{\mathrm{uloc}}}\|\varphi_k\|_{L^4} \le C\|\varphi_k\|_{H^1}\|v^-\|^2_{H^1_{\mathrm{uloc}}}.$$

We then treat the two terms stemming from the Laplacian separately. For the horizontal derivatives, we merely integrate by parts, recalling that $\varphi_k\in H_0^1(\Omega_{\mathrm{bl}}^M)$, so that

$$\int_{\Omega_{\mathrm{bl}}^M}\int_{y_3}^M\Delta_h v_3^-(y_h,z)dz\varphi_k(y)\,dy = -\int_{\Omega_{\mathrm{bl}}^M}\int_{y_3}^M\nabla_h v_3^-(y_h,z)\cdot\nabla_h\varphi_k(y)\,dz\,dy,$$

and the corresponding term is bounded by $C(\|\phi\|_{\mathcal{V}}+\|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}})\|\varphi_k\|_{H^1}$. As for the vertical derivatives, we have

$$\begin{aligned}\int_{\Omega_{\mathrm{bl}}^M}\left(\int_{y_3}^M\partial_3^2 v_3^-(y_h,z)dz\right)\varphi_k(y)\,dy &= \int_{\Omega_{\mathrm{bl}}^M}\big(\partial_3 v_3^-(y_h,M)-\partial_3 v_3^-(y)\big)\varphi_k(y)\,dy\\ &= -\int_{\Omega_{\mathrm{bl}}^M}\big(\nabla_h\cdot v_h^-(y_h,M)+\partial_3 v_3^-(y)\big)\varphi_k(y)\,dy\\ &= \int_{\Omega_{\mathrm{bl}}^M} v_h^-(y_h,M)\cdot\nabla_h\varphi_k(y)\,dy - \int_{\Omega_{\mathrm{bl}}^M}\partial_3 v_3^-(y)\varphi_k(y)\,dy.\end{aligned} \tag{4-9}$$

Both terms of the right-hand side are bounded by $C\|v^-\|_{H^1_{\mathrm{uloc}}}\|\varphi_k\|_{H^1}$.

Taking the estimate (4-7), we infer that there exists a constant C (independent of φ_k and of k) such that for all $\varphi_k\in H_0^1(\Omega_{\mathrm{bl}}^M)$ supported in $(k+[0,1]^2)\times\mathbb{R}$,

$$\left|\int_{\Omega_{\mathrm{bl}}^M} p^-\varphi_k\right| \le C(\|\phi\|_{\mathcal{V}}+\|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}})\|\varphi_k\|_{H_0^1(\Omega_{\mathrm{bl}}^M)}.$$

We deduce that

$$\|p^-\|_{H^{-1}_{\mathrm{uloc}}(\Omega_{\mathrm{bl}}^M)} \le C(\|\phi\|_{\mathcal{V}}+\|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}}).$$

Using the equation on (v^-, p^-), we also have

$$\|\nabla p^-\|_{H^{-1}_{\mathrm{uloc}}(\Omega_{\mathrm{bl}}^M)} \le C(\|\phi\|_{\mathcal{V}}+\|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}}).$$

It then follows from Nečas inequality (see [Boyer and Fabrie 2013, Theorem IV.1.1]) that $p^- \in L^2_{\text{uloc}}(\Omega^M_{\text{bl}})$, with

$$\|p^-\|_{L^2_{\text{uloc}}(\Omega^M_{\text{bl}})} \le C(\|\phi\|_{\mathcal{V}} + \|\psi\|_{H^{m-1/2}_{\text{uloc}}}).$$

We still have to establish the two properties itemized in Lemma 14. We focus first on the higher-order estimates. Note that using interior regularity results for the Stokes system (see [Galdi 2011]), one has $v^- \in H^N_{\text{uloc}}(\Omega')$ for all open sets $\Omega' \subset \mathbb{R}^2$ such that $\bar{\Omega}' \subset \Omega^M_{\text{bl}}$ and for all $N > 0$. In particular, for all $M_1 < M_2$ in the interval $]\sup\gamma, M[$, we have $v^- \in H^{m+1}_{\text{uloc}}(\mathbb{R}^2 \times (M_1, M_2))$ and $p^- \in H^m_{\text{uloc}}(\mathbb{R}^2 \times (M_1, M_2))$.

We now tackle the regularity for $y_3 > M'$, where $M' \in]\sup\gamma, M[$. Our arguments are somehow standard (and mainly taken from [Boyer and Fabrie 2013]), but since there are a few difficulties related to the nonlinear stress boundary condition at $y_3 = M$, we give details. The idea is to use an induction argument to show that $v^- \in H^l_{\text{uloc}}(\mathbb{R}^2 \times [M', M])$ for all $\sup\gamma < M' < M$ and for $1 \le l \le m+1$. Unfortunately, the induction only works for $l \ge 2$: indeed, the implication $h \in H^s(\mathbb{R}^2) \Rightarrow h^2 \in H^s(\mathbb{R}^2)$, which is required to handle the nonlinear boundary condition at $y_3 = M$, is true for $s > 1$ only. Therefore we treat separately the case $l = 2$. In the sequel, we write $\|\phi\| + \|\psi\|$ as a shorthand for $\|\phi\|_{\mathcal{V}} + \|\psi\|_{H^{m-1/2}_{\text{uloc}}}$.

To prove H^2_{uloc} regularity, the first step is to prove a priori estimates for $\partial_1 v^-$, $\partial_2 v^-$ in H^1_{uloc}. To that end, we first localize the equation near a fixed $k \in \mathbb{Z}^2$, then differentiate it with respect to y_j, $j = 1, 2$. Let $\theta \in \mathcal{C}^\infty_0(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of $k \in \mathbb{Z}^2$, and such that the size of $\operatorname{Supp}\theta$ is bounded uniformly in k (we omit the k-dependence of θ and of all subsequent functions in order to alleviate the notation). It can be easily checked that the equation satisfied by $w_j := \partial_j(\theta v^-)$ is

$$\begin{gathered}
-\Delta w_j + e_3 \wedge w_j + v^- \cdot \nabla w_j + \nabla \partial_j(\theta p^-) = F_j \quad \text{in } \Omega_\theta,\\
\operatorname{div} w_j = g_j \quad \text{in } \Omega_\theta,\\
w_j|_{y_3 = M'} \in H^{1/2}(\mathbb{R}^2),\\
\big(\partial_3 w_j - \big(\partial_j(\theta p^-) + v^- \cdot w_j - \tfrac{1}{2}|v^-|^2 \partial_j \theta\big) e_3\big)\big|_{y_3 = M} = \partial_j(\theta\psi),\\
w_j = 0 \quad \text{on } \partial \operatorname{Supp}\theta \times (M', M),
\end{gathered}$$

where $\Omega_\theta := \operatorname{Supp}\theta \times (M', M)$ and

$$\begin{aligned}
F_j &= \underbrace{\partial_j\big(-2\nabla\theta \cdot \nabla v^- - v^- \Delta\theta + (v^- \cdot \nabla\theta)v^- + p^- \nabla\theta\big)}_{\|\cdot\|_{H^{-1}} \le C(\|\phi\| + \|\psi\|)} - \partial_j v^- \cdot \nabla(\theta v^-),\\
g_j &= \partial_j(v^- \cdot \nabla\theta) = O(\|\phi\| + \|\psi\|) \quad \text{in } L^2(\mathbb{R}^2 \times (M', M)).
\end{aligned}$$

By standard results, see [Galdi 2011, Section II.3], there exists $\bar{w}_j \in H^1(\Omega_\theta)$ such that

$$\begin{gathered}
\operatorname{div} \bar{w}_j = g_j, \quad \bar{w}_j = w_j \quad \text{at } \partial\Omega_\theta \setminus \{y_3 = M\},\\
\|\bar{w}_j\|_{H^1(\Omega_\theta)} \le C\big(\|g_j\|_{L^2(\Omega_\theta)} + \|w_j\|_{H^{1/2}(\{y_3 = M'\})}\big).
\end{gathered}$$

Note that we do not need to correct the trace of w_j at $\{y_3 = M\}$, as there is no Dirichlet boundary condition there. Moreover, we are not sure at this stage that this trace is an $H^{1/2}_{\text{uloc}}$ function. We rather prescribe an artificial smooth data for $\bar{w}_j$ at this boundary, chosen so that it satisfies the good compatibility condition.

Finally, $\tilde{w}_j = w_j - \bar{w}_j$ satisfies

$$\begin{aligned}
-\Delta\tilde{w}_j + e_3\wedge\tilde{w}_j + v^-\cdot\nabla\tilde{w}_j + \nabla\tilde{q}_j = \tilde{F}_j \quad \text{in } \Omega_\theta,\\
\operatorname{div}\tilde{w}_j = 0 \quad \text{in } \Omega_\theta,\\
\tilde{w}_j|_{y_3=M'} = 0, \qquad \tilde{w}_j = 0 \quad \text{on } \partial\operatorname{Supp}\theta\times(M',M),\\
\big(\partial_3\tilde{w}_j - (\tilde{q}_j + v^-\cdot\tilde{w}_j)e_3\big)\big|_{y_3=M} = \tilde{\psi}_j,
\end{aligned}$$

with $\tilde{F}_j = -\partial_j v^-\cdot\nabla(\theta v^-) + O(\|\phi\|+\|\psi\|)$ in H^{-1}, and $\|\tilde{\psi}_j\|_{H^{-1/2}} \le C(\|\phi\|+\|\psi\|)$. We obtain the estimate

$$\|\nabla\tilde{w}_j\|^2_{L^2(\Omega_\theta)} \le C(\|\phi\|^2+\|\psi\|^2) + \left|\int_{\Omega_\theta}(\partial_j v^-\cdot\nabla(\theta v^-))\cdot\tilde{w}_j\right| + 2\int_{\operatorname{Supp}\theta}|v^-|_{y_3=M}|\big|\tilde{w}_j|_{y_3=M}\big|^2.$$

We first deal with the boundary term:

$$\begin{aligned}
\int_{\operatorname{Supp}\theta}\big|v^-|_{y_3=M}\big|\big|\tilde{w}_j|_{y_3=M}\big|^2 &\le \big\|v^-|_{y_3=M}\big\|_{L^2(\operatorname{Supp}\theta)}\big\|\tilde{w}_j|_{y_3=M}\big\|^2_{L^4(\operatorname{Supp}\theta)}\\
&\le C\big\|v^-|_{y_3=M}\big\|_{H^{1/2}_{\mathrm{uloc}}}\big\|\tilde{w}_j|_{y_3=M}\big\|^2_{H^{1/2}(\operatorname{Supp}\theta)} \le C\|v^-\|_{H^1}\|\tilde{w}_j\|^2_{H^1}\\
&\le C(\|\phi\|+\|\psi\|)\|\nabla\tilde{w}_j\|^2_{L^2}.
\end{aligned}$$

Hence for ψ and ϕ small enough we can absorb this term in the left-hand side of the energy inequality. As for the quadratic source term, we write

$$\begin{aligned}
\partial_j v^-\cdot\nabla(\theta v^-) &= \partial_j v_1^- w_1 + \partial_j v_2^- w_2 + \partial_j v_3^-\theta\partial_3 v^-\\
&= \partial_j v_1^- w_1 + \partial_j v_2^- w_2 + \partial_3 v^- w_{j,3} - v_3^-\partial_j\theta\partial_3 v^-.
\end{aligned}$$

For $i = 1,\dots,3$, $j = 1,2$, $k = 1,2$, we have

$$\begin{aligned}
\int_{\Omega_\theta}|\partial_i v^-||w_j||\tilde{w}_k| &\le C\|v^-\|_{H^1_{\mathrm{uloc}}(\Omega^M_{\mathrm{bl}})}\|w_j\|_{L^4(\Omega_\theta)}\|\tilde{w}_k\|_{L^4(\Omega_\theta)}\\
&\le C(\|\phi\|+\|\psi\|)(\|\tilde{w}_1\|^2_{H^1(\Omega_\theta)}+\|\tilde{w}_2\|^2_{H^1(\Omega_\theta)}) + C(\|\phi\|+\|\psi\|)^3
\end{aligned}$$

and

$$\left|\int_{\Omega_\theta} v_3^-\partial_j\theta\partial_3 v^-\cdot\tilde{w}_j\right| \le C\|v_3^-\|_{H^1_{\mathrm{uloc}}}\|\partial_3 v^-\|_{L^2_{\mathrm{uloc}}}\|\tilde{w}_j\|_{H^1(\Omega_\theta)}.$$

Therefore, we obtain, for $\|\phi\|+\|\psi\|$ small enough,

$$\|w_1\|^2_{H^1(\Omega_\theta)} + \|w_2\|^2_{H^1(\Omega_\theta)} \le C(\|\phi\|^2+\|\psi\|^2).$$

Using the same idea as above to estimate $\partial_j(\theta p^-)$, this gives

$$\|\nabla_h v^-\|_{H^1_{\mathrm{uloc}}(\mathbb{R}^2\times(M',M))} + \|\nabla_h p^-\|_{L^2_{\mathrm{uloc}}(\mathbb{R}^2\times(M',M))} \le (\|\phi\|_\mathcal{V} + \|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}}).$$

Since v^- is divergence-free, similar estimates hold for $\partial_3 v_3^-$. Thus $v_3^- \in H^2_{\mathrm{uloc}}(\mathbb{R}^2\times(M',M))$. As for the vertical regularity of v_h^-, we observe that $\partial_3 v^-$ is a solution of the Stokes system with Dirichlet boundary conditions

$$\begin{aligned} -\Delta\partial_3 v^- + \nabla\partial_3 p^- &= F_3 \quad \text{in } \mathbb{R}^2\times(M', M),\\ \operatorname{div}\partial_3 v^- &= 0 \quad \text{in } \mathbb{R}^2\times(M', M),\\ \partial_3 v^-|_{y_3=M} &= G,\\ \partial_3 v^-|_{y_3=M'} &= G', \end{aligned}$$

where

$$F_3 = -e_3\wedge\partial_3 v^- - \partial_3(v_h^-\cdot\nabla_h v^-) - \partial_3(v_3^-\partial_3 v^-) \in H^{-1}_{\text{uloc}}(\mathbb{R}^2),\quad G_h = \psi_h \in H^{m-1/2}_{\text{uloc}}(\mathbb{R}^2),$$

and $G_3 = \partial_3 v_3^-|_{y_3=M} \in H^{1/2}_{\text{uloc}}(\mathbb{R}^2)$, $G'\in H^{m-1/2}_{\text{uloc}}(\mathbb{R}^2)$. Using the results of Chapter IV in [Galdi 2011], we infer that $\partial_3 v^- \in H^1_{\text{uloc}}(\mathbb{R}^2\times(M', M))$, $\partial_3 p^-\in L^2_{\text{uloc}}(\mathbb{R}^2\times(M', M))$, and

$$\|\partial_3 v^-\|_{H^1_{\text{uloc}}(\mathbb{R}^2\times(M',M))} + \|\partial_3 p^-\|_{L^2_{\text{uloc}}(\mathbb{R}^2\times(M',M))} \le C(\|F\|_{H^{-1}_{\text{uloc}}} + \|G\|_{H^{1/2}_{\text{uloc}}} + \|G'\|_{H^{1/2}_{\text{uloc}}}) \le C(\|\phi\|+\|\psi\|)$$

for ϕ and ψ small enough. Gathering the inequalities, we obtain

$$\|v^-\|_{H^2_{\text{uloc}}(\mathbb{R}^2\times(M',M))} + \|p^-\|_{H^1_{\text{uloc}}(\mathbb{R}^2\times(M',M))} \le C(\|\phi\|_{\mathcal{V}} + \|\psi\|_{H^{m-1/2}_{\text{uloc}}}).$$

Of course, all inequalities above are a priori estimates, but provide H^2_{uloc} regularity (and a posteriori estimates) through the usual method of translations.

We are now ready for the induction argument. Let $k\in\mathbb{Z}^2$ be fixed. Define a sequence $\vartheta_k^2, \dots, \vartheta_k^{m+1}$ such that $\vartheta_k^l := \theta_1^l(z-M)\theta_2^l(y_h-k)$, where $\theta_1^l\in\mathcal{C}_0^\infty(\mathbb{R})$, $\theta_2^l\in\mathcal{C}_0^\infty(\mathbb{R}^2)$ are equal to 1 in a neighbourhood of zero. We require furthermore that $\operatorname{Supp}\vartheta_k^{l+1}\subset(\vartheta_k^l)^{-1}(\{1\})$. We then define a $\mathcal{C}^{m+1,1}$ domain $\Omega_k\subset\Omega_{\text{bl}}^M$ such that $\operatorname{Supp}\vartheta_k^2 \Subset \overline{\Omega}_k$, and such that $\partial\Omega_k\cap\partial\Omega_{\text{bl}}=\varnothing$ (see Figure 1). Notice also that we choose Ω_k so that $\operatorname{diam}(\Omega_k)$ is bounded uniformly in k (in fact, we can always assume that $\Omega_k = (k, 0)+\Omega_0$ for some fixed domain Ω_0.)

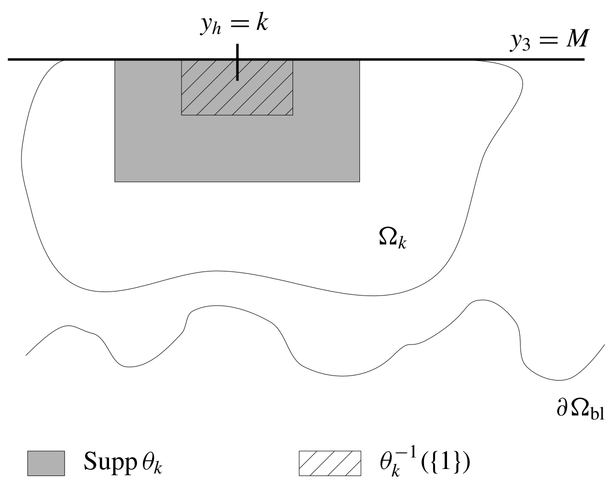

Figure 1. The domain Ω_k.

Multiplying (2-5) by ϑ_k^l and dropping the dependence with respect to k, we find that $v^l := \vartheta_k^l v^-$, $p^l := p^- \vartheta_k^l$ is a solution of

$$\begin{cases} -\Delta v^l + \nabla p^l = f^l & \text{in } \Omega_k, \\ \operatorname{div} v^l = g^l & \text{in } \Omega_k, \\ \partial_n v^l - p^l n = \Sigma^l & \text{on } \partial\Omega_k, \end{cases} \tag{4-10}$$

where

$$\begin{aligned} f^l &:= -2\nabla\vartheta_k^l \cdot \nabla v^{l-1} - \Delta\vartheta_k^l v^{l-1} - (e_3 \wedge v^{l-1} + v^{l-1} \cdot \nabla v^{l-1})\vartheta_k^l + p^{l-1} \cdot \nabla\vartheta_k, \\ g^l &= v^{l-1} \cdot \nabla\vartheta_k, \\ \Sigma^l &= \theta_2^l(y_h - k)\big(\psi + \tfrac{1}{2}|v^{l-1}|^2 e_3|_{y_3=M}\big) \quad \text{on } \partial\Omega_k \cap \{y_3 = M\}, \\ \Sigma^l &= 0 \quad \text{on } \partial\Omega_k \cap \{y_3 = M\}^c. \end{aligned} \tag{4-11}$$

Now, Theorem IV.7.1 in [Boyer and Fabrie 2013] implies that for all $l \in \{2, \dots, m\}$, for $\|\phi\|_{\mathcal{V}} + \|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}}$ small enough,

$$(v^l, p^l) \in H^l(\Omega_k) \times H^{l-1}(\Omega_k) \quad \Longrightarrow \quad (v^{l+1}, p^{l+1}) \in H^{l+1}(\Omega_k) \times H^l(\Omega_k),$$

and

$$\|v^{l+1}\|_{H^{l+1}(\Omega_k)} + \|p^{l+1}\|_{H^l(\Omega_k)} \le C\big(\|v^l\|_{H^l(\Omega_k)} + \|p^l\|_{H^{l-1}(\Omega_k)} + \|\psi\|_{H^{l-1/2}(\Omega_k)}\big).$$

Indeed, assume that $(v^l, p^l) \in H^l(\Omega_k) \times H^{l-1}(\Omega_k)$. Then $f^{l+1} \in H^{l-1}(\Omega_k)$, $g^{l+1} \in H^l(\Omega_k)$, with

$$\|f^{l+1}\|_{H^{l-1}(\Omega_k)} \le C(\|v^l\|_{H^l} + \|v^l\|^2_{H^l} + \|p^l\|_{H^{l-1}(\Omega_k)}), \quad \|g^{l+1}\|_{H^l} \le C\|v^l\|_{H^l}.$$

Moreover, $v^l \in H^{l-1/2}(\partial\Omega_k)$. Since $l \ge 2$, using product laws in fractional Sobolev spaces (see [Strichartz 1967]), we infer that $|v^l|^2|_{y_3=M} \in H^{l-1/2}(\mathbb{R}^2)$, and therefore $\Sigma^{l+1} \in H^{l-1/2}(\mathbb{R}^2)$. From [Boyer and Fabrie 2013, Theorem IV.7.1], we deduce that $(v^{l+1}, p^{l+1}) \in H^{l+1}(\Omega_k) \times H^l(\Omega_k)$, together with the announced estimate. By induction, $v^- \in H^{m+1}_{\mathrm{uloc}}(\Omega^M_{\mathrm{bl}})$ and $p^- \in H^m_{\mathrm{uloc}}(\Omega^M_{\mathrm{bl}})$.

There only remains to check the compatibility condition at $y_3 = M$. Notice that

$$v_3^-|_{|y_3=M} = \phi_3 + \int_{\gamma(y_h)}^M \partial_3 v_3^- = \phi_3 - \int_{\gamma(y_h)}^M \nabla_h \cdot v_h^- = \phi_3 - \gamma(y_h) \cdot \phi_h + \nabla_h \cdot v_h^*,$$

where

$$v_h^* = -\int_{\gamma(y_h)}^M v_h^- \in H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2).$$

Since $\phi_3 - \gamma(y_h) \cdot \phi_h = 0$ due to the nonpenetrability condition $\phi \cdot n = 0$, we obtain the desired identity. □

Step 2. Once (v^-, p^-) is defined thanks to Lemma 14, we define (v^+, p^+) in the half-space $\{y_3 > M\}$ by solving (2-4) with $v^+|_{y_3=M} = v^-|_{y_3=M}$. According to Lemma 14 and to standard trace inequalities,

$$\big\|v^-|_{y_3=M}\big\|_{H^{m+1/2}_{\mathrm{uloc}}(\mathbb{R}^2)} \le C(\|\phi\|_{\mathcal{V}} + \|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}})$$

for some constant C depending only on M and on $\|\gamma\|_{W^{1,\infty}}$. As a consequence, if $C(\|\phi\|_{\mathcal{V}} + \|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}}) + \|v_h^*\|_{L^2_{\mathrm{uloc}}} \le \delta_0$, according to Proposition 12 the system (2-4) with $v_0 = v^-|_{y_3=M}$ has a unique solution.

Additionally, $\Sigma(v^+, p^+)e_3|_{y_3=M^+}$ belongs to $H^{m-1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$. Thus the mapping

$$\begin{aligned}\mathcal{F} : \mathcal{V} \times H^{m-1/2}_{\mathrm{uloc}}(\mathbb{R}^2) &\to H^{m-1/2}_{\mathrm{uloc}}(\mathbb{R}^2),\\ (\phi, \psi) &\mapsto \Sigma(v^+, p^+)e_3|_{y_3=M^+} - \psi,\end{aligned}$$

is well-defined. Clearly, according to Lemma 14, for $\phi = 0$ and $\psi = 0$, we have $v^- = 0$, $v^+ = 0$ and therefore $\mathcal{F}(0, 0) = 0$.

The strategy is then to apply the implicit function theorem to $\mathcal{F}$ to find a solution of $\mathcal{F}(\phi, \psi) = 0$ for ϕ in a neighbourhood of zero. Therefore we check that $\mathcal{F}$ is $\mathcal{C}^1$ in a neighbourhood of zero, and that its Fréchet derivative with respect to ψ at $(0, 0)$ is an isomorphism on $H^{m-1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$.

$\mathcal{F}$ is a $\mathcal{C}^1$ mapping in a neighbourhood of zero: Let ϕ_0, ψ_0 and ϕ, ψ be in a neighbourhood of zero (in the sense of the functional norms in $\mathcal{V}$ and $H^{m-1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$). We denote by $v_0^\pm$, $p_0^\pm$, $v^\pm$, $p^\pm$ the solutions of (2-4), (2-5) associated with (ϕ_0, ψ_0) and $(\phi_0 + \phi, \psi_0 + \psi)$ respectively, and we set $w^\pm := v^\pm - v_0^\pm$, $q^\pm = p^\pm - p_0^\pm$.

On the one hand, in Ω^M_{bl}, we know w^- is a solution of the system

$$\begin{aligned}-\Delta w^- + e_3 \wedge w^- + (v_0^- + w^-) \cdot \nabla w^- + w^- \cdot \nabla v_0^- + \nabla q^- = 0,&\\ \operatorname{div} w^- = 0,&\\ w^-|_{\partial\Omega_{\mathrm{bl}}} = \phi,&\\ \left.\left(\partial_3 w^- - q^- e_3 - \frac{2v_0^- \cdot w^- + |w^-|^2}{2} e_3\right)\right|_{y_3=M} = \psi.&\end{aligned}$$

Performing estimates similar to the ones of Lemma 14, we infer that for $\|\phi_0\|_{\mathcal{V}} + \|\psi_0\|_{H^{m-1/2}_{\mathrm{uloc}}}$ and $\|\phi\|_{\mathcal{V}} + \|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}}$ small enough,

$$\|w^-\|_{H^1_{\mathrm{uloc}}(\Omega_{\mathrm{bl}})} + \|w^-|_{y_3=M}\|_{H^{m+1/2}_{\mathrm{uloc}}} \le C(\|\phi\|_{\mathcal{V}} + \|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}}).$$

It follows that

$$w^- = w_L^- + O(\|\phi\|^2_{\mathcal{V}} + \|\psi\|^2_{H^{m-1/2}_{\mathrm{uloc}}})$$

in $H^1_{\mathrm{uloc}}(\Omega^M_{\mathrm{bl}})$ and in $H^{m+1}_{\mathrm{uloc}}((M', M) \times \mathbb{R}^2)$ for all $M' > \sup\gamma$, where w_L^- solves the same system as w^- minus the quadratic terms $w^- \cdot \nabla w^-$ and $|w^-|^2|_{y_3=M}$.

On the other hand, using Theorem 2, one can show that $w^+ = w_L^+ + O(\|\phi\|^2_{\mathcal{V}} + \|\psi\|^2_{H^{m-1/2}_{\mathrm{uloc}}})$, where

$$\begin{aligned}-\Delta w_L^+ + e_3 \wedge w^+ + v_0^+ \cdot \nabla w_L^+ + w_L^+ \cdot \nabla v_0^+ + \nabla q_L^+ = 0 \quad \text{in } y_3 > M,&\\ \operatorname{div} w_L^+ = 0 \quad \text{in } y_3 > M,&\\ w_L^+|_{y_3=M} = w_L^-|_{y_3=M}.&\end{aligned}$$

Using Theorem 2, we deduce that if $\|(1 + y_3)^{1/3} v_0^+\|_{H^{m+1}_{\mathrm{uloc}}}$ is small enough (which is ensured by the smallness condition on $\|\phi\|$, $\|\psi\|$), we have

$$\left\|(1 + y_3)^{1/3} w_L^+\right\|_{H^{m+1}_{\mathrm{uloc}}(\mathbb{R}^3_+)} \le C\left\|w_L^-|_{y_3=M}\right\|_{H^{m+1/2}_{\mathrm{uloc}}} \le C(\|\phi\|_{\mathcal{V}} + \|\psi\|_{H^{m-1/2}_{\mathrm{uloc}}}).$$

Therefore, in $H^{m-1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$,

$$\mathcal{F}(\phi_0+\phi,\,\psi_0+\psi)-\mathcal{F}(\phi_0,\,\psi_0)=-\psi+\partial_3 w_L^+|_{y_3=M}-(q_L^+ + v_0^+\cdot w_L^+)|_{y_3=M}\,e_3+O(\|\phi\|_{\mathcal{V}}^2+\|\psi\|^2_{H^{m-1/2}_{\mathrm{uloc}}}).$$

It follows that the Fréchet derivative of $\mathcal{F}$ at $(\phi_0,\,\psi_0)$ is

$$\mathcal{L}_{\phi_0,\psi_0}:(\phi,\,\psi)\mapsto -\psi+\partial_3 w_L^+|_{y_3=M}-(q_L^+ + v_0^+\cdot w_L^+)|_{y_3=M}\,e_3.$$

Using the same kind of arguments as above, it is easily proved that $w_L^\pm$ depend continuously on $v_0^\pm$, and therefore on $\phi_0,\,\psi_0$. Therefore $\mathcal{F}$ is a $\mathcal{C}^1$ function in a neighbourhood of zero.

$d_\psi\mathcal{F}(0,0)$ is invertible: Since $d_\psi\mathcal{F}(0,0)=\mathcal{L}_{0,0}(0,\cdot)$, we consider the systems solved by $w_L^\pm$ with $v_0^\pm=0$ and $\phi=0$. We first notice that if $\mathcal{L}_{0,0}(0,\,\psi)=0$, then $w_L:=\mathbf{1}_{y_3\le M}\,w_L^- + \mathbf{1}_{y_3>M}\,w_L^+$ is a solution of the Stokes–Coriolis system in the whole domain Ω_{bl}, with $w_L|_{\partial\Omega_{\mathrm{bl}}}=0$. Therefore, according to [Dalibard and Prange 2014], $w_L\equiv 0$ and therefore $\psi=0$. Hence $\ker d_\psi\mathcal{F}(0,0)=\{0\}$, and $d_\psi\mathcal{F}(0,0)$ is one-to-one.

On the other hand,

$$(\partial_3 w_L^+ - q_L^+ e_3)|_{y_3=M}=\mathrm{DN}(w_L^-|_{y_3=M}),$$

where DN is the Dirichlet-to-Neumann operator for the Stokes–Coriolis system, introduced in [loc. cit.]. In particular, in order to solve the equation

$$\mathcal{L}_{0,0}(0,\,\psi_1)=\psi_2$$

for a given $\psi_2\in H^{m-1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$, we need to solve the system

$$\begin{aligned}
-\Delta w_L^- + e_3\wedge w_L^- + \nabla q_L^- &= 0,\\
\operatorname{div} w_L^- &= 0,\\
w_L^-|_{\partial\Omega_{\mathrm{bl}}} &= 0,\\
(\partial_3 w_L^- - q_L^- e_3 -)|_{y_3=M} &= -\psi_2+\mathrm{DN}(w_L^-|_{y_3=M}).
\end{aligned}$$

According to Section 3 in [loc. cit.], the above system has a unique solution $w_L^-\in H^1_{\mathrm{uloc}}(\Omega^M_{\mathrm{bl}})$. There only remains to prove that $w_L^-\in H^{m+1}_{\mathrm{uloc}}(\{M'<y_3<M\})$ for all $\sup\gamma<M'<M$. Therefore, we notice that in the domain $\mathbb{R}^2\times(M',M)$, the horizontal derivatives of w_L^- (up to order m) satisfy a Stokes–Coriolis system similar to the one above (notice that the Dirichlet-to-Neumann operator commutes with $\partial_1,\,\partial_2$). It follows that $\nabla_h^\alpha w_L^-\in H^1_{\mathrm{uloc}}(\mathbb{R}^2\times(M',M))$ for all $|\alpha|\le m$. In particular, $\nabla_h^\alpha w_L^-|_{y_3=M}\in H^{1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$ and therefore $w_L^-|_{y_3=M}\in H^{m+1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$. It can be checked that $\mathrm{DN}: H^{m+1/2}_{\mathrm{uloc}}(\mathbb{R}^2)\to H^{m-1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$. As a consequence, $\psi_1=\partial_3 w_L^- - q_L^- e_3\in H^{m-1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$. Therefore $d_\psi\mathcal{F}(0,0)$ is an isomorphism of $H^{m-1/2}(\mathbb{R}^2)$.

Using the implicit function theorem, we infer that for all $\phi\in\mathcal{V}$ in a neighbourhood of zero, there exists $\psi\in H^{m-1/2}_{\mathrm{uloc}}(\mathbb{R}^2)$ such that $\mathcal{F}(\phi,\,\psi)=0$. Let $v:=\mathbf{1}_{y_3\le M}\,v^- + \mathbf{1}_{y_3>M}\,v^+$, where v^-, v^+ are the solutions of (2-5)–(2-4) associated with $\phi,\,\psi$. By definition, the jump of v across $\{y_3=M\}$ is zero, and since $\mathcal{F}(\phi,\,\psi)=0$,

$$\Sigma(v^-,\,p^-)e_3|_{y_3=M}=\psi=\Sigma(v^+,\,p^+)e_3|_{y_3=M}.$$

Using once again the fact that $|v^+|^2|_{y_3=M} = |v^-|^2|_{y_3=M}$, we deduce that

$$(\partial_3 v^- - p^- e_3)|_{y_3=M} = (\partial_3 v^+ - p^+ e_3)|_{y_3=M}.$$

Thus there is no jump of the stress tensor across $\{y_3 = M\}$, and therefore v is a solution of the Navier–Stokes–Coriolis system in the whole domain Ω_{bl}. This concludes the proof of Theorem 1.

Appendix A: Proofs of Lemmas 7 and 9

Proof of Lemma 7. We begin with a few observations. First, replacing χ by $\chi_1 := Q\chi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, it is enough to prove the lemma with $Q = 1$. Moreover it is clearly sufficient to prove the lemma for $p_k(\xi) = \xi_1^a \xi_2^b$, with $a + b = k$. Notice also that since $\alpha - k \geq -2$, we can always write $\alpha - k = 2m + \alpha_m$, with $\alpha_m \in [-2, 0[$ and $m \in \mathbb{N}$. Then $\xi_1^a \xi_2^b |\xi|^{\alpha-k}$ is a linear combination of terms of the form $\xi_1^{a'} \xi_2^{b'} |\xi|^{\alpha_m}$, with $a' + b' + \alpha_m = \alpha$ and $a', b' \in \mathbb{N}$. Therefore, in the rest of the proof, we take

$$Q \equiv 1, \qquad P(\xi) = \xi_1^a \xi_2^b |\xi|^\beta, \quad \text{with } a, b \in \mathbb{N},\ \beta \in [-2, 0[,\ a + b + \beta = \alpha.$$

Some of the arguments of the proof are inspired by the work of Alazard, Burq and Zuily [Alazard et al. 2016] on the Cauchy problem for gravity water waves in H^s_{uloc} spaces. We introduce a partition of unity $(\varphi_q)_{q\in\mathbb{Z}^2}$, where $\operatorname{Supp}\varphi_q \subset B(q, 2)$ for $q \in \mathbb{Z}^2$ and $\sup_q \|\varphi_q\|_{W^{k,\infty}} < +\infty$ for all k. We also introduce functions $\tilde\varphi_q \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\tilde\varphi_q \equiv 1$ on $\operatorname{Supp}\varphi_q$, and, say $\operatorname{Supp}\tilde\varphi_q \subset B(q, 3)$. Then, for $j = 1, 2, 3$,

$$\begin{aligned} u^j(x_h, z) &= \sum_{q\in\mathbb{Z}^2} \chi(D) P(D) e^{-\lambda_j(D) z} (\varphi_q \underline{v}_0) \\ &= \sum_{q\in\mathbb{Z}^2} \int_{\mathbb{R}^2} K^j(x_h - y_h, z) \varphi_q(y_h) \underline{v}_0(y_h)\, dy_h = \sum_{q\in\mathbb{Z}^2} \int_{\mathbb{R}^2} K^j_q(x_h, y_h, z) \varphi_q(y_h) \underline{v}_0(y_h)\, dy_h, \end{aligned} \tag{A-1}$$

where

$$K^j(x_h, z) = \int_{\mathbb{R}^2} e^{i x_h\cdot\xi} \chi(\xi) P(\xi) e^{-\lambda_j(\xi) z}\, d\xi, \qquad K^j_q(x_h, y_h, z) = K^j(x_h - y_h, z) \tilde\varphi_q(y_h).$$

We then claim that the following estimates hold: there exists $\delta > 0$, $C \geq 0$ such that for all $x_h \in \mathbb{R}^2$, $z > 0$,

$$|K^1(x_h, z)| \leq \frac{C}{(1 + |x_h| + z^{1/3})^{2+\alpha}}, \qquad |K^j(x_h, z)| \leq C \frac{e^{-\delta z}}{(1 + |x_h|)^{2+\alpha}} \quad \text{for } j = 2, 3. \tag{A-2}$$

Let us postpone the proof of estimates (A-2) and explain why Lemma 7 follows. Going back to (A-1), we have, for $j = 2, 3$,

$$\begin{aligned} |u^j(x_h, z)| &\leq C e^{-\delta z} \sum_{q\in\mathbb{Z}^2, |q - x_h| \geq 3} \frac{1}{(|q - x_h| - 2)^{2+\alpha}} \int |\varphi_q(y_h) \underline{v}_0(y_h)|\, dy_h \\ &\qquad\qquad + C e^{-\delta z} \sum_{q\in\mathbb{Z}^2, |q - x_h| \leq 3} \int |\varphi_q(y_h) \underline{v}_0(y_h)|\, dy_h \\ &\leq C e^{-\delta z} \|\underline{v}_0\|_{L^1_{\mathrm{uloc}}}. \end{aligned}$$

In a similar fashion,

$$
\begin{aligned}
|u^1(x_h,z)| \le C \sum_{q\in\mathbb{Z}^2, |q-x_h|\ge 3} \frac{1}{(|q-x_h|-2+z^{1/3})^{2+\alpha}} \int |\varphi_q(y_h)\underline{v}_0(y_h)|\,dy_h \\
+ C \sum_{q\in\mathbb{Z}^2, |q-x_h|\le 3} \frac{1}{(1+z^{1/3})^{2+\alpha}} \int |\varphi_q(y_h)\underline{v}_0(y_h)|\,dy_h \\
\le C\|\underline{v}_0\|_{L^1_{\text{uloc}}}(1+z)^{-\alpha/3}.
\end{aligned}
$$

The estimates of Lemma 7 follow for $z \ge 1$.

We now turn to the proof of estimates (A-2). Once again we start with the estimates for K^2, K^3, which are simpler. Since λ_2, λ_3 are continuous and have nonvanishing real part on the support of χ, there exists a constant $\delta > 0$ such that $\mathrm{Re}(\lambda_j(\xi)) \ge \delta$ for all $\xi \in \operatorname{Supp}\chi$ and for $j = 2, 3$. Clearly, for $|x_h| \le 1$ we have simply

$$|K^j(x_h,z)| \le e^{-\delta z}\|\chi P\|_{L^1}.$$

We thus focus on the set $|x_h| \ge 1$. Let $\chi_j(\xi,z) := \chi(\xi)\exp(-\lambda_j(\xi)z)$. Then $\chi_j \in L^\infty(\mathbb{R}_+, \mathcal{S}(\mathbb{R}^2))$, and for all $n_1, n_2, n_3 \in \mathbb{N}$, there exists a constant $\delta_n > 0$ such that

$$\left|(1+|\xi|^{n_3})\partial_1^{n_1}\partial_2^{n_2}\chi_j(\xi,z)\right| \le C_n \exp(-\delta_n z).$$

Estimate (A-2) for K^2, K^3 then follows immediately from the following lemma (whose proof is given after the current one):

Lemma 15. *Let $P(\xi) = \xi_1^{a_1}\xi_2^{a_2}|\xi|^\beta$, with $a_1, a_2 \in \mathbb{N}$, $\beta \in [-2, 0[$, and set $\alpha := a_1 + a_2 + \beta$. Then there exists $C > 0$ such that for any $\zeta \in \mathcal{S}(\mathbb{R}^2)$, for all $x_h \in \mathbb{R}^2$, $|x_h| \ge 1$,*

$$\left|P(D)\zeta(x_h)\right| \le \frac{C}{|x_h|^{2+\alpha}}\left(\|\zeta\|_1 + \left\||y_h|^{a_1+a_2+2}\partial_1^{a_1}\partial_2^{a_2}\zeta\right\|_\infty\right).$$

We now address the estimates on K^1. When $|x_h| \le 1$, $z \le 1$, we have simply $|K^1(x_h,z)| \le \|P\chi\|_1$, and the estimate follows. When $z \le 1$ and $|x_h| \ge 1$, we apply Lemma 15 with $\zeta(\xi) = \mathcal{F}^{-1}\big(\chi(\xi)\exp(-\lambda_1(\xi)z)\big)$. Notice that

$$\|\zeta\|_1 \lesssim \left\|\chi(\xi)\exp(-\lambda_1(\xi)z)\right\|_{W^{3,1}},$$

and

$$\left\||y_h|^{a_1+a_2+2}\partial_1^{a_1}\partial_2^{a_2}\zeta\right\|_\infty \lesssim \left\|\xi_1^{a_1}\xi_2^{a_2}\chi(\xi)\exp(-\lambda_1(\xi)z)\right\|_{W^{2+a_1+a_2,1}}.$$

Since the right-hand sides of the above inequalities are bounded (recall that $\lambda_1(\xi) = |\xi|^3\Lambda_1(\xi)$ with $\Lambda_1 \in \mathcal{C}^\infty(\mathbb{R}^2)$; see Remark 4), it follows that estimate (A-2) is true for $z \le 1$ and $|x_h| \ge 1$.

We now focus on the case $z \ge 1$. We first change variables in the integral defining K^1 and we set $\xi' = z^{1/3}\xi$, $x_h' = x_h/z^{1/3}$. Since P is homogeneous, this leads to

$$K^1(x_h,z) = \frac{1}{z^{(2+\alpha)/3}}\int_{\mathbb{R}^2} e^{ix_h'\cdot\xi'}P(\xi')\chi\left(\frac{\xi'}{z^{1/3}}\right)\exp\left(-\lambda_1\left(\frac{\xi'}{z^{1/3}}\right)z\right)d\xi'.$$

Since $\lambda_1/|\xi|^3$ is continuous and does not vanish on the support of χ, there exists a positive constant δ' such that $\lambda_1(\xi) \geq \delta'|\xi|^3$ on $\operatorname{Supp}\chi$. Therefore, for $|x_h'| \leq 1$, we have

$$|K^1(x_h, z)| \leq \frac{1}{z^{(2+\alpha)/3}} \left\| \exp(-\delta'|\xi|^3) P(\xi') \right\|_{L^1},$$

and the estimate for K^1 on the set $|x_h| \leq z^{1/3}$ is proved.

For $|x_h'| \geq 1$, we split the integral in two. Let $\varphi \in \mathcal{C}_0^\infty$ such that $\varphi \equiv 1$ in a neighbourhood of zero. Then

$$\begin{aligned} K^1(x_h, z) &= \frac{1}{z^{(2+\alpha)/3}} \int_{\mathbb{R}^2} e^{i x_h' \cdot \xi'} P(\xi') \varphi(\xi') \chi\left(\frac{\xi'}{z^{1/3}}\right) \exp\left(-\lambda_1\left(\frac{\xi'}{z^{1/3}}\right) z\right) d\xi' \\ &\qquad + \frac{1}{z^{(2+\alpha)/3}} \int_{\mathbb{R}^2} e^{i x_h' \cdot \xi'} P(\xi')(1-\varphi(\xi')) \chi\left(\frac{\xi'}{z^{1/3}}\right) \exp\left(-\lambda_1\left(\frac{\xi'}{z^{1/3}}\right) z\right) d\xi' \\ &=: K_1^1 + K_2^1. \end{aligned}$$

We first consider the term K_2^1. Because of the truncation $1-\varphi$, we have removed all singularity coming from P close to $\xi = 0$. Therefore, performing integrations by parts, we have, for any $n \in \mathbb{N}$, for $j = 1, 2$,

$$x_j'^{\,n} K_2^1(x_h, z) = \frac{1}{z^{(2+\alpha)/3}} \int_{\mathbb{R}^2} e^{i x_h' \cdot \xi'} D^n_{\xi_j'} \left[P(\xi')(1-\varphi(\xi')) \chi\left(\frac{\xi'}{z^{1/3}}\right) \exp\left(-\lambda_1\left(\frac{\xi'}{z^{1/3}}\right) z\right) \right] d\xi'.$$

When the $D_{\xi_j'}$ derivative hits $P(1-\varphi)$, we end up with an integral bounded by

$$C_n \int_{\mathbb{R}^2} |\xi'|^\alpha \mathbf{1}_{\xi' \in \operatorname{Supp}(1-\varphi)} \exp(-\delta'|\xi'|^3)\, d\xi' \leq C_n.$$

When the derivative hits $\chi(\xi'/z^{1/3})$ the situation is even better, as a power of $z^{1/3}$ is gained with each derivative. Therefore the worst terms occur when the derivative hits the exponential. Remember that $\lambda_1(\xi) = |\xi|^3 \Lambda_1(\xi)$, where $\Lambda_1 \in \mathcal{C}^\infty(\mathbb{R}^2)$ with $\Lambda_1(0) = 1$ and Λ_1 does not vanish on $\mathbb{R}^2$. Therefore, for all $\xi' \in \mathbb{R}^2$, $z > 0$,

$$\exp\left(-\lambda_1\left(\frac{\xi'}{z^{1/3}}\right) z\right) = \exp\left(-|\xi'|^3 \Lambda_1\left(\frac{\xi'}{z^{1/3}}\right)\right).$$

We infer that for any $0 \leq n \leq 3 + \lfloor \alpha \rfloor$, on $\operatorname{Supp}\chi(\cdot/z^{1/3})$, we have

$$\left| P(\xi') \nabla^n_{\xi_j'} \exp\left(-\lambda_1\left(\frac{\xi'}{z^{1/3}}\right) z\right) \right| \leq C_n \exp\left(-\frac{\delta'}{2}|\xi|^3\right). \tag{A-3}$$

We deduce eventually that

$$|K_2^1(x_h, z)| \leq C \frac{1}{z^{(2+\alpha)/3}} \frac{1}{(1+|x_h'|^{2+\alpha})} \leq \frac{C}{(|x_h| + z^{1/3})^{2+\alpha}}.$$

For the term K_1^1, we use once again Lemma 15, with

$$\zeta := \mathcal{F}^{-1}\left(\varphi(\xi') \chi\left(\frac{\xi'}{z^{1/3}}\right) \exp\left(-\lambda_1\left(\frac{\xi'}{z^{1/3}}\right) z\right)\right).$$

Using the same type of estimate as (A-3) above, we obtain

$$|K_2^1(x_h, z)| \le C \frac{1}{z^{(2+\alpha)/3}} \frac{1}{|x_h'|^{2+\alpha}} \le \frac{C}{|x_h|^{2+\alpha}}.$$

This concludes the proof of Lemma 7. □

Proof of Lemma 15. We have

$$P(D)\zeta = D_1^{a_1} D_2^{a_2} \mathrm{Op}(|\xi|^\beta)\zeta.$$

Thus we first compute $\mathrm{Op}(|\xi|^\beta)\zeta$. We first focus on the case $\beta \in]-2, 0[$. We follow the ideas of Droniou and Imbert [2006, Theorem 1], recalling the main steps of the proof. The function $\xi \in \mathbb{R}^2 \mapsto |\xi|^\beta$ is radial and locally integrable, and thus belongs to $\mathcal{S}'$. Its Fourier transform in $\mathcal{S}'(\mathbb{R}^2)$ is also radial and homogeneous of degree $-\beta-2 \in]-2, 0[$. Therefore it coincides (up to a constant) with $|\cdot|^{-\beta-2}$ in $\mathcal{S}'(\mathbb{R}^2\backslash\{0\})$, and since the latter function is locally integrable, we end up with $\mathcal{F}^{-1}(|\xi|^\beta) = C|x_h|^{-\beta-2}$ in $\mathcal{S}'(\mathbb{R}^N)$. Hence

$$P(D)\zeta(x_h) = C\partial_1^{a_1}\partial_2^{a_2} \int_{\mathbb{R}^2} \frac{1}{|y_h|^{\beta+2}} \zeta(x_h - y_h)\, dy_h.$$

Notice that in the present case, we do not need to have an exact formula for $P(D)\zeta$, but merely some information about its decay at infinity. As a consequence we take a shortcut in the proof of [Droniou and Imbert 2006]. We take a cut-off function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi \equiv 1$ in a neighbourhood of zero, and we write

$$\begin{aligned} P(D)\zeta(x_h) &= C \int_{\mathbb{R}^2} \frac{\chi(y_h)}{|y_h|^{\beta+2}} \partial_1^{a_1}\partial_2^{a_2}\zeta(x_h - y_h)\, dy_h \\ &\qquad + C \sum_{\substack{0\le i_1 \le a_1 \\ 0 \le i_2 \le a_2}} C_{i_1,i_2} \int_{\mathbb{R}^2} \partial_1^{i_1}\partial_2^{i_2}(1-\chi(y_h))\partial_1^{a_1-i_1}\partial_2^{a_2-i_2}\left(\frac{1}{|y_h|^{\beta+2}}\right)\zeta(x_h-y_h)\, dy_h \\ &=: I_1 + I_2. \end{aligned}$$

We now choose χ in the following way. Let $n = \lfloor |x_h| \rfloor \in \mathbb{N}$, and take $\chi = \chi_n = \eta(\cdot/n)$, where $\mathrm{Supp}\,\eta \subset B\big(0, \frac{1}{2}\big)$ and $\eta \equiv 1$ in a neighbourhood of zero. Notice in that case that if $y_h \in \mathrm{Supp}\,\chi_n$, then $|x_h - y_h| \ge |x_h|/2$. Therefore, for the first term, we have

$$|x_h|^{2+\alpha}|I_1| \le (n+1)^\beta \left(\int_{|y_h|\le n/2} |y_h|^{-\beta-2}\, dy_h\right) \big\| |y_h|^{2+a_1+a_2}\partial_1^{a_1}\partial_2^{a_2}\zeta \big\|_{L^\infty} \le C \big\| |y_h|^{2+a_1+a_2}\partial_1^{a_1}\partial_2^{a_2}\zeta\big\|_{L^\infty}.$$

Using the assumptions on η and χ_n and the estimate

$$\left|\partial_1^{a_1-i_1}\partial_2^{a_2-i_2}\left(\frac{1}{|y_h|^{\beta+2}}\right)\right| \le \frac{C}{|y_h|^{\alpha+2-i_1-i_2}} \le \frac{C}{n^{\alpha+2-i_1-i_2}} \quad \forall y_h \in \mathrm{Supp}(1-\chi_n),$$

we infer that

$$|I_2| \le C\|\zeta\|_{L^1} n^{-\alpha-2} \le C\|\zeta\|_{L^1}|x_h|^{-\alpha-2}.$$

Gathering all the terms, we obtain the inequality announced in the lemma. To conclude the proof, we still have to consider the case $\beta = -2$: in such a case, $|\xi|^\beta$ corresponds to inverting the Laplacian over $\mathbb{R}^2$. Hence, the kernel $|x_h - y_h|^{-\beta-2}$ has to be replaced by $\frac{1}{2\pi}\ln(|x_h - y_h|)$. This does not modify the previous reasoning. □

Proof of Lemma 9. The proof is somewhat simpler than the one of Lemma 7. As indicated in the remark following Lemma 9, notice that for $n > 1$, for all $\xi \in \mathbb{R}^2$, $z > 0$,

$$\left|(1+|\xi|^2)^{-n}(1-\chi(\xi))P(\xi)e^{-\lambda_j(\xi)z}\right| \le \|P\|_{L^\infty(B_r^c)} \frac{e^{-\delta z}}{(1+|\xi|^2)^n},$$

and the right-hand side of the above inequality is in $L^1(\mathbb{R}^2)$ for all z. As a consequence, for $j = 1, \dots, 3$, $n > 1$, the kernel

$$K_{n,j}(x_h, z) := \int_{\mathbb{R}^2} e^{ix_h\cdot\xi}(1+|\xi|^2)^{-n}(1-\chi)(\xi)P(\xi)\exp(-\lambda_j(\xi)z)\,d\xi$$

is well-defined and satisfies

$$\|K_{n,j}(\,\cdot\,, z)\|_{L^\infty(\mathbb{R}^2)} \le C_n\|P\|_{L^\infty(B_r^c)}e^{-\delta z}.$$

Furthermore, if $a_1, a_2 \in \mathbb{N}$ with $a_1 + a_2 \le 3$,

$$x_1^{a_1}x_2^{a_2}K_{n,j}(x_h, z) = \int_{\mathbb{R}^2} e^{ix_h\cdot\xi}D_1^{a_1}D_2^{a_2}\big((1+|\xi|^2)^{-n}(1-\chi)(\xi)P(\xi)\exp(-\lambda_j(\xi)z)\big)\,d\xi.$$

Hence, up to taking a larger n and a smaller δ,

$$|K_{n,j}(x_h, z)| \le C_n\|P\|_{W^{3,\infty}(B_r^c)}e^{-\delta z}(1+|x_h|)^{-3},$$

and in particular, $K_{n,j} \in L^\infty_z(L^2_{x_h})$. Thus for any $f \in L^2_{\mathrm{uloc}}$,

$$\left\|(1+|D|^2)^{-n}(1-\chi(D))P(D)\exp(-\lambda_j(D)z)f\right\|_{L^\infty} = \|K_{n,j} * f\|_{L^\infty} \le Ce^{-\delta z}\|f\|_{L^2_{\mathrm{uloc}}}.$$

Taking $f = (1+|D|^2)^n\underline{v}_0 = (1-\Delta_h)^n\underline{v}_0$ for some $\underline{v}_0 \in H^{2n}_{\mathrm{uloc}}$, we obtain the result announced in Lemma 9.

Appendix B. Estimates on a few integrals

Lemma 16. *There exists a positive constant C such that for all $z \ge 0$,*

$$\int_0^\infty \frac{1}{(1+|z-z'|)^{2/3}(1+z')^{2/3}}\,dz' \le \frac{C}{(1+z)^{1/3}},$$
$$\int_0^\infty \frac{1}{(1+|z-z'|)(1+z')^{2/3}}\,dz' \le \frac{C\ln(2+z)}{(1+z)^{2/3}},$$

and for all $\gamma, \delta > 0$ such that $\delta < 1$ and $\gamma + \delta > 1$, there exists a constant $C_{\gamma,\delta}$ such that

$$\int_0^\infty \frac{1}{(1+z+z')^\gamma}\frac{1}{(1+z')^\delta}\,dz' \le \frac{C_{\gamma,\delta}}{(1+z)^{\gamma+\delta}} \quad \forall z \ge 0.$$

Proof. The first two inequalities are obvious if z is small $\left(\text{say, } z \le \frac{1}{2}\right)$, simply by writing

$$\frac{1}{1+|z-z'|} \le \frac{C}{1+z'}.$$

Hence we focus on $z' \ge \frac{1}{2}$. In that case, changing variables in the first integral, we have

$$\int_0^\infty \frac{1}{(1+|z-z'|)^{2/3}(1+z')^{2/3}}\,dz' = \frac{1}{z^{1/3}}\int_0^\infty \frac{1}{(z^{-1}+|1-t|)^{2/3}}\frac{1}{(z^{-1}+t)^{2/3}}\,dt \le \frac{1}{z^{1/3}}\int_0^\infty \frac{1}{|1-t|^{2/3}}\frac{1}{t^{2/3}}\,dt,$$

which proves the first inequality. The second one is treated in a similar fashion:

$$\int_0^\infty \frac{1}{1+|z-z'|}\frac{1}{(1+z)^{2/3}}\,dz' = z^{-1}\int_0^\infty \frac{1}{z^{-1}+|1-t|}\frac{1}{(z^{-1}+t)^{2/3}}\,dt \le z^{-1}\int_0^\infty \frac{1}{z^{-1}+|1-t|}\frac{1}{t^{2/3}}\,dt.$$

It is easily checked that

$$\int_{1/2}^{3/2} \frac{1}{z^{-1}+|1-t|}\,dt \le C\ln(2+z).$$

The second estimate follows. The last estimate is proved by similar arguments and is left to the reader. □

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PARTIAL DATA INVERSE PROBLEMS FOR THE HODGE LAPLACIAN

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We prove uniqueness results for a Calderón-type inverse problem for the Hodge Laplacian acting on graded forms on certain manifolds in three dimensions. In particular, we show that partial measurements of the relative-to-absolute or absolute-to-relative boundary value maps uniquely determine a zeroth-order potential. The method is based on Carleman estimates for the Hodge Laplacian with relative or absolute boundary conditions, and on the construction of complex geometrical optics solutions which reduce the Calderón-type problem to a tomography problem for 2-tensors. The arguments in this paper allow us to establish partial data results for elliptic systems that generalize the scalar results due to Kenig, Sjöstrand and Uhlmann.

1. Introduction

This article is concerned with inverse problems with partial data for elliptic systems. We first discuss the prototype for such problems, which comes from the scalar case: the inverse problem of Calderón asks to determine the electrical conductivity γ of a medium Ω from electrical measurements made on its boundary. More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $\gamma \in L^\infty(\Omega)$ satisfy $\gamma \geq c > 0$ a.e. in Ω. The full boundary measurements are given by the Dirichlet-to-Neumann map (DN map)

$$\Lambda_\gamma^{\mathrm{DN}} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \quad f \mapsto \gamma \partial_\nu u|_{\partial\Omega},$$

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where $u \in H^1(\Omega)$ is the unique solution of $\operatorname{div}(\gamma \nabla u) = 0$ in Ω with $u|_{\partial\Omega} = f$, and the conormal derivative $\gamma \partial_\nu u|_{\partial\Omega}$ is defined in the weak sense. Equivalently, one can consider the Neumann-to-Dirichlet map (ND map)

$$\Lambda_\gamma^{\mathrm{ND}} : H_\diamond^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \quad g \mapsto v|_{\partial\Omega},$$

where $\operatorname{div}(\gamma \nabla v) = 0$ in Ω with $\gamma \partial_\nu v|_{\partial\Omega} = g$, and $H_\diamond^{-\frac{1}{2}}(\partial\Omega)$ consists of those elements in $H^{-\frac{1}{2}}(\partial\Omega)$ that are orthogonal to constants. The inverse problem of Calderón asks to determine the conductivity γ from the knowledge of the DN map or (equivalently) the ND map. There is a substantial literature on this problem, with pioneering works including [Faddeev 1965; Calderón 1980; Sylvester and Uhlmann 1987; Novikov and Khenkin 1987; Nachman 1988; Novikov 1988]. We refer to the surveys [Novikov 2008; Uhlmann 2014] for more information.

The Calderón problem with partial data corresponds to the case where one can only make measurements on subsets of the boundary. Let Γ_{D} and Γ_{N} be open subsets of $\partial\Omega$, and assume that we measure voltages on Γ_{D} and currents on Γ_{N}. If the potential is grounded on $\partial\Omega \setminus \Gamma_{\mathrm{D}}$ but can be prescribed on Γ_{D}, the partial boundary measurements are given by the partial DN map

$$\Lambda_\gamma^{\mathrm{DN}} f|_{\Gamma_{\mathrm{N}}} \quad \text{for all } f \in H^{\frac{1}{2}}(\partial\Omega) \text{ with } \operatorname{supp}(f) \subset \Gamma_{\mathrm{D}}.$$

If instead we can freely prescribe currents on Γ_{N} but no current is input on $\partial\Omega \setminus \Gamma_{\mathrm{N}}$, then we know the partial ND map:

$$\Lambda_\gamma^{\mathrm{ND}} g|_{\Gamma_{\mathrm{D}}} \quad \text{for all } g \in H_\diamond^{-\frac{1}{2}}(\partial\Omega) \text{ with } \operatorname{supp}(g) \subset \Gamma_{\mathrm{N}}.$$

The basic uniqueness question is whether a (sufficiently smooth) conductivity is determined by such boundary measurements. We remark that in the partial data case there seems to be no direct way of obtaining the partial DN map from the partial ND map or vice versa, and the two cases need to be considered separately.

By now there are many uniqueness results for the Calderón problem with partial data involving varying assumptions on the sets Γ_{D} and Γ_{N}. For further information we refer to the survey [Kenig and Salo 2014] for results in dimensions $n \geq 3$ and [Guillarmou and Tzou 2013] for the case $n = 2$. We only list here some of the main results for the partial DN map:

- When $n \geq 3$, we know Γ_{D} can be possibly very small but Γ_{N} has to be slightly larger than the complement of Γ_{D} [Kenig et al. 2007].
- When $n \geq 3$, we know $\Gamma_{\mathrm{D}} = \Gamma_{\mathrm{N}} = \Gamma$ and the complement of Γ has to be part of a hyperplane or a sphere [Isakov 2007].
- When $n = 2$, we know $\Gamma_{\mathrm{D}} = \Gamma_{\mathrm{N}} = \Gamma$ can be an arbitrary open set [Imanuvilov et al. 2010].
- When $n \geq 3$, we know $\Gamma_{\mathrm{D}} = \Gamma_{\mathrm{N}} = \Gamma$ and the complement of Γ has to be (conformally) flat in one direction and a certain ray transform needs to be injective [Kenig and Salo 2013] (a special case of this was proved independently in [Imanuvilov and Yamamoto 2013]).

The approach of [Kenig et al. 2007] is based on Carleman estimates with boundary terms and the approach of [Isakov 2007] is based on reflection arguments. The paper [Kenig and Salo 2013] combines these two approaches and extends both. There seem to be fewer results for the partial ND map, especially in dimensions $n \geq 3$; see [Isakov 2007; Chung 2015]. In fact, in dimensions $n \geq 3$ the Carleman estimate approach for the partial ND map seems to be more involved than for the partial DN map. We remark that there are counterexamples for uniqueness when Γ_D and Γ_N are disjoint [Daudé et al. 2015].

The purpose of this paper is to consider analogous partial data results for elliptic systems. In the full data case ($\Gamma_D = \Gamma_N = \partial\Omega$), many uniqueness results are available for linear elliptic systems such as the Maxwell system [Ola et al. 1993; Kenig et al. 2011; Caro and Zhou 2014], Dirac systems [Nakamura and Tsuchida 2000; Salo and Tzou 2009], the Schrödinger equation with Yang–Mills potentials [Eskin 2001], elasticity [Nakamura and Uhlmann 1994; 2003; Eskin and Ralston 2002], and equations in fluid flow [Heck et al. 2007; Li and Wang 2007]. In contrast, the only earlier partial data results for such systems in dimensions $n \geq 3$ that we are aware of are [Caro et al. 2009] for the Maxwell system and [Salo and Tzou 2010] for the Dirac system. One reason for the lack of partial data results for systems is the fact that Carleman estimates for systems often come with boundary terms that do not seem helpful for partial data inverse problems (see [Eller 2008; Salo and Tzou 2009] for some such estimates).

In this paper we establish partial data results analogous to [Kenig et al. 2007] for systems involving the Hodge Laplacian for graded differential forms, on certain Riemannian manifolds in dimensions $n \geq 3$. These are elliptic systems that generalize the scalar Schrödinger equation $(-\Delta + q)u = 0$ and are very close to the time-harmonic Maxwell equations when $n = 3$. In fact, using the results of the present paper, we have finally been able to extend the partial data result of [Kenig et al. 2007] to the Maxwell system [Chung et al. 2015]. The main technical contribution of the present paper is a Carleman estimate for the Hodge Laplacian, with limiting Carleman weights, that has boundary terms involving the relative and absolute boundary values of graded forms. The boundary terms are of such a form that allows us to carry over the Carleman estimate approach of [Kenig et al. 2007] to the Hodge Laplace system. As far as we know, this is the first analogue of [Kenig et al. 2007] for systems besides [Salo and Tzou 2010], which considered a very special case.

In a sense, to deal with boundary terms for systems in a flexible way, one first needs a good understanding of the different splittings of Cauchy data in the scalar case. This encompasses both the scalar DN and ND maps simultaneously, since the "relative-to-absolute" map defined in Section 2 generalizes both the notion of the DN and ND maps. Therefore the methods developed in [Chung 2015] for the partial ND map, involving Fourier analysis to treat the boundary terms in Carleman estimates, will be very useful in our approach. We expect that the methods developed in this paper open the way for obtaining partial data results via Carleman estimates for various elliptic systems. This has already been achieved for Maxwell equations [Chung et al. 2015].

The plan of this document is as follows. Section 1 is the introduction, and Section 2 contains precise statements of the main results. Section 3 collects notation and identities used throughout the paper. In the interest of brevity, we have omitted the proofs of these identities and interested readers can find them in the arXiv version of this paper [Chung et al. 2013, Appendix]. Sections 4–6 will be devoted to the proofs of the Carleman estimates. In Section 4, we will give the basic integration by parts argument for k-forms and simplify the boundary terms. In Section 5, we prove the Carleman estimates for 0-forms using the arguments from [Chung 2015; Kenig and Salo 2013]. We will conclude the argument in Section 6 by showing that the Carleman estimates for graded forms follow from an induction argument, given the corresponding result for 0-forms. In Section 7 we will construct relevant complex geometrical optics solutions, following the ideas in [Dos Santos Ferreira et al. 2009a]. In Section 8 we will present the Green's theorem argument and give the density result based on injectivity of a tensor tomography problem, which finishes the proofs of Theorems 2.1 and 2.2. Section 9 will contain the proof of Theorem 2.3 and make some remarks about the case of dimensions $n \geq 4$.

2. Statement of results

The results in this paper are new even in Euclidean space, but it will be convenient to state them on certain Riemannian manifolds following [Dos Santos Ferreira et al. 2009a; 2016; Kenig and Salo 2013]. Suppose (M_0, g_0) is a compact oriented manifold with smooth boundary, and consider a manifold $T = \mathbb{R} \times M_0$ equipped with a Riemannian metric of the form $g = c(e \oplus g_0)$, where c is a smooth conformal factor and $(\mathbb{R}, e)$ is the real line with Euclidean metric. A compact manifold (M, g) of dimension $n \geq 3$, with boundary ∂M, is said to be *CTA* (conformally transversally anisotropic) if it can be expressed as a submanifold of such a T. A CTA manifold is called *admissible* if additionally (M_0, g_0) can be chosen to be simple, meaning that ∂M_0 is strictly convex and for any point $x \in M_0$, the exponential map $\exp_x$ is a diffeomorphism from some closed neighbourhood of 0 in $T_x M_0$ onto M_0 (see [Sharafutdinov 1994]). Most of the geometric notions defined here will be from [Taylor 1996] and we refer the reader there for a more thorough treatment of the subject.

Let $\Lambda^k M$ be the k-th exterior power of the cotangent bundle on M, and let ΛM be the corresponding graded algebra. The corresponding spaces of sections (smooth differential forms) are denoted by $\Omega^k M$ and ΩM. We will define $\triangle$ to be the Hodge Laplacian on M, acting on graded forms:

$$-\triangle = d\delta + \delta d.$$

Here d is the exterior derivative and δ is the codifferential (adjoint of d in the L^2 inner product). Suppose Q is an L^∞ endomorphism of ΛM; that is, Q associates to almost every point $x \in M$ a linear map $Q(x)$ from $\Lambda_x M$ to itself, and the map $x \mapsto \|Q(x)\|$ is bounded and measurable. Later will consider continuous endomorphisms, meaning that $x \mapsto Q(x)$ is continuous in M. The continuity of Q will simplify matters since the recovery of Q from boundary measurements involves integrals over geodesics, and continuity ensures that these integrals are well defined.

We would like to consider boundary value problems for the operator $-\triangle + Q$. In order to do this, we will define the tangential trace $t : \Omega M \to \Omega \partial M$ by

$$t : \omega \mapsto i^* \omega,$$

where $i : \partial M \to M$ is the natural inclusion map. Then the first natural boundary value problem to consider for $-\triangle + Q$, acting on graded forms u, is the relative boundary problem

$$\begin{aligned} (-\triangle + Q)u &= 0 \quad \text{in } M, \\ tu &= f \quad \text{on } \partial M, \\ t\delta u &= g \quad \text{on } \partial M. \end{aligned}$$

If Q is such that 0 is not an eigenvalue for this problem, then this problem has a unique solution [Taylor 1996, Section 5.9] and we may define a relative-to-absolute map

$$N_Q^{\mathrm{RA}} : H^{\frac{1}{2}}(\partial M, \Lambda\partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda\partial M) \to H^{\frac{1}{2}}(\partial M, \Lambda\partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda\partial M)$$

by

$$N_Q^{\mathrm{RA}}(f, g) = (t * u, t\delta * u),$$

where $*$ is the Hodge star operator on M.

The second natural boundary value problem to consider is the absolute boundary value problem

$$\begin{aligned} (-\triangle + Q)u &= 0 \quad \text{in } M, \\ t * u &= f \quad \text{on } \partial M, \\ t\delta * u &= g \quad \text{on } \partial M. \end{aligned}$$

Assuming 0 is not an eigenvalue, this defines an absolute-to-relative map

$$N_Q^{\mathrm{AR}} : H^{\frac{1}{2}}(\partial M, \Lambda\partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda\partial M) \to H^{\frac{1}{2}}(\partial M, \Lambda\partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda\partial M)$$

by

$$N_Q^{\mathrm{AR}}(f, g) = (tu, t\delta u)$$

for appropriate Q. For more details on the relative and absolute boundary value problems for the Hodge Laplacian, see [Taylor 1996, Section 5.9].

These maps both give rise to a Calderón-type inverse problem which asks if knowledge of N_Q^{RA} or N_Q^{AR} suffices to determine Q. If we restrict ourselves to considering the case of 0-forms only and if Q acts on 0-forms by multiplication by a function $q \in L^\infty(M)$, then the relative-to-absolute and absolute-to-relative maps become the DN and ND maps, respectively, for the Schrödinger equation

$$(-\triangle + q)u = 0 \quad \text{in } M,$$

where u is now a function on M and $\triangle$ is the Laplace–Beltrami operator on functions. Our problem is therefore a generalization of the standard partial data problem for the scalar Schrödinger equation on a compact manifold with boundary.

Let us review some earlier results for the Schrödinger problem in the scalar case, in dimensions $n \geq 3$. If M is Euclidean, Sylvester and Uhlmann [1987] proved that knowledge of the full DN map uniquely determines the potential q. Versions of this problem on admissible and CTA manifolds as defined above have been considered in [Dos Santos Ferreira et al. 2009a; 2016]. Partial data results for the DN map have been proven in [Bukhgeim and Uhlmann 2002; Isakov 2007; Kenig et al. 2007] for the Euclidean case, and more recently in [Kenig and Salo 2013], the last of which contains the previous three results and extends them to the manifold case. Improved results in the linearized case are in [Dos Santos Ferreira et al. 2009b]. Partial data results for the ND map, analogous to the ones in [Kenig et al. 2007], were proven in [Chung 2015]. Other partial data results for scalar equations with first-order potentials as well were obtained in [Dos Santos Ferreira et al. 2007; Chung 2014], and some of those techniques will be useful to us in this paper as well.

For the Hodge Laplacian acting on graded forms, we are not aware of previous results dealing with the determination of a potential from the relative-to-absolute or absolute-to-relative maps. However, [Krupchyk et al. 2011] reconstructs a real analytic metric from these maps in the case of no potential, and [Sharafutdinov and Shonkwiler 2013; Shonkwiler 2013; Belishev and Sharafutdinov 2008; Joshi and Lionheart 2005] recover various kinds of topological information about the manifold from variants of these maps, again in the case of no potential. We remark that full data problems for the Hodge Laplacian in Euclidean space can be solved in a very similar way as in the scalar case (see Section 9), but full data problems on manifolds and partial data problems even in Euclidean space are more involved.

In order to describe the main results precisely, we will define "front" and "back" sets of the boundary ∂M as in [Kenig et al. 2007]. If $M \subset T = \mathbb{R} \times M_0$ is CTA, we can use coordinates (x_1, x'), where x_1 is the Euclidean variable, and define the function $\varphi : T \to \mathbb{R}$ by $\varphi(x_1, x') = x_1$. As discussed in [Dos Santos Ferreira et al. 2009a], φ is a natural limiting Carleman weight in M. Now define

$$\partial M_+ = \{p \in \partial M \mid \partial_\nu \varphi(p) \geq 0\},$$
$$\partial M_- = \{p \in \partial M \mid \partial_\nu \varphi(p) \leq 0\}.$$

Then the main results of this paper are the following.

Theorem 2.1. *Let $M \subset \mathbb{R} \times M_0$ be a three-dimensional admissible manifold with conformal factor $c = 1$, and let Q_1 and Q_2 be continuous endomorphisms of ΛM such that $N^{\mathrm{RA}}_{Q_1}$, $N^{\mathrm{RA}}_{Q_2}$ are defined. Let $\Gamma_+ \subset \partial M$ be a neighbourhood of ∂M_+, and let $\Gamma_- \subset \partial M$ be a neighbourhood of ∂M_-. Suppose*

$$N^{\mathrm{RA}}_{Q_1}(f, g)|_{\Gamma_+} = N^{\mathrm{RA}}_{Q_2}(f, g)|_{\Gamma_+}$$

for all $(f, g) \in H^{\frac{1}{2}}(\partial M, \Lambda \partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda \partial M)$ supported in Γ_-. Then $Q_1 = Q_2$.

Theorem 2.2. *Let M be a three-dimensional admissible manifold with conformal factor $c = 1$, and let Q_1 and Q_2 be continuous endomorphisms of ΛM such that $N^{\mathrm{AR}}_{Q_1}$, $N^{\mathrm{AR}}_{Q_2}$ are defined. Let $\Gamma_+ \subset \partial M$ be a neighbourhood of ∂M_+, and let $\Gamma_- \subset \partial M$ be a neighbourhood of ∂M_-. Suppose*

$$N^{\mathrm{AR}}_{Q_1}(f, g)|_{\Gamma_+} = N^{\mathrm{AR}}_{Q_2}(f, g)|_{\Gamma_+}$$

for all $(f, g) \in H^{\frac{1}{2}}(\partial M, \Lambda \partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda \partial M)$ supported in Γ_-. Then $Q_1 = Q_2$.

In the case that M is a domain in Euclidean space, we can also extend the results to higher dimensions.

Theorem 2.3. *Let M be a bounded smooth domain in $\mathbb{R}^n$, with $n \geq 3$, and let Q_1 and Q_2 be continuous endomorphisms of ΛM such that $N^{\mathrm{RA}}_{Q_1}$, $N^{\mathrm{RA}}_{Q_2}$ are defined. Fix a unit vector α, and let $\varphi(x) = \alpha \cdot x$. Let $\Gamma_+ \subset \partial M$ be a neighbourhood of ∂M_+, and let $\Gamma_- \subset \partial M$ be a neighbourhood of ∂M_-. Suppose*

$$N^{\mathrm{RA}}_{Q_1}(f, g)|_{\Gamma_+} = N^{\mathrm{RA}}_{Q_2}(f, g)|_{\Gamma_+}$$

for all $(f, g) \in H^{\frac{1}{2}}(\partial M, \Lambda \partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda \partial M)$ supported in Γ_-. Then $Q_1 = Q_2$. The same result holds if we replace the relative-to-absolute map with the absolute-to-relative one.

Theorem 2.1 is a generalization to certain systems of the scalar partial data result of [Kenig et al. 2007] for the DN map, and similarly Theorem 2.2 is an extension to systems of the scalar result of [Chung 2015] for the ND map. To be precise, the above theorems are stated for the linear Carleman weight and not for the logarithmic weight as in [Kenig et al. 2007; Chung 2015]. This restriction comes from the lack of conformal invariance of the full Hodge Laplacian. However, in the scalar case we could use the conformal invariance of the scalar Schrödinger operator together with a reduction from [Kenig and Salo 2013] to recover the logarithmic weight results of [Kenig et al. 2007; Chung 2015] from the above theorems.

The proofs of Theorems 2.1 and 2.2 involve three main ingredients — the construction of complex geometrical optics (CGO) solutions, a Green's theorem argument, and a density argument relating this inverse problem to a tensor tomography problem where one determines a tensor field from its integrals along geodesics (see Section 8). Both the construction of CGO solutions and the Green's theorem argument require appropriate Carleman estimates.

To describe them, we will introduce the following notation. For a CTA manifold M, let N be the inward pointing normal vector field along ∂M. We can extend N to be a vector field in a neighbourhood of ∂M by parallel transporting along normal geodesics, and then to a vector field on M by multiplying by a cutoff function. For $u \in \Omega M$ we will let

$$u_\perp = N^\flat \wedge i_N u,$$

where $N^\flat$ is the 1-form corresponding to N and i_N is the interior product, and

$$u_\parallel = u - u_\perp.$$

Let ∇ denote the Levi-Civita connection on M, and ∇' denote the pullback connection on the boundary. Let

$$\Delta_\varphi = e^{\frac{\varphi}{h}} h^2 \Delta e^{-\frac{\varphi}{h}},$$

where φ is a limiting Carleman weight as described in [Dos Santos Ferreira et al. 2009a]. Note that by [loc. cit.] such weights exist globally if M is a CTA manifold. Then the Carleman estimates are as follows.

Theorem 2.4. *Let M be a CTA manifold, and let Q be an L^∞ endomorphism of ΛM. Define $\Gamma_+ \subset \partial M$ to be a neighbourhood of ∂M_+. Suppose $u \in H^2(M, \Lambda M)$ satisfies the boundary conditions*

$$\begin{aligned} u|_{\Gamma_+} &= 0 \quad \text{to first order}, \\ tu|_{\Gamma_+^c} &= 0, \\ th\delta e^{-\frac{\varphi}{h}} u|_{\Gamma_+^c} &= h\sigma t i_\nu e^{-\frac{\varphi}{h}} u \end{aligned} \tag{2-1}$$

for some smooth endomorphism σ independent of h. Then there exists h_0 such that if $0 < h < h_0$,

$$\|(-\Delta_\varphi + h^2 Q)u\|_{L^2(M)} \gtrsim h\|u\|_{H^1(M)} + h^{\frac{1}{2}} \|u_\perp\|_{H^1(\Gamma_+^c)} + h^{\frac{1}{2}} \|h\nabla_N u_\parallel\|_{L^2(\Gamma_+^c)}.$$

Here H^1 signifies the semiclassical H^1 space with semiclassical parameter h, and for instance

$$\|u\|_{H^1(M)} = \|u\|_{L^2(M)} + \|h\nabla u\|_{L^2(M)}.$$

The constant implied in the $\gtrsim$ sign is meant to be independent of h. Note that the last boundary condition in (2-1) can be rewritten as

$$th\delta u|_{\partial M} = -ti_{d\varphi}u - h\sigma ti_N u.$$

Theorem 2.5. *Let M be a CTA manifold, and let Q be an L^∞ endomorphism of ΛM. Define $\Gamma_+ \subset \partial M$ to be a neighbourhood of ∂M_+. Suppose $u \in H^2(M, \Lambda M)$ satisfies the boundary conditions*

$$\begin{aligned} u|_{\Gamma_+} &= 0 \quad \text{to first order}, \\ t * u|_{\Gamma_+^c} &= 0, \\ th\delta * e^{-\frac{\varphi}{h}} u|_{\Gamma_+^c} &= h\sigma t i_\nu * e^{-\frac{\varphi}{h}} u \end{aligned} \tag{2-2}$$

for some smooth endomorphism σ independent of h. Then there exists h_0 such that if $0 < h < h_0$,

$$\|(-\Delta_\varphi + h^2 Q)u\|_{L^2(M)} \gtrsim h\|u\|_{H^1(M)} + h^{\frac{1}{2}}\|u_\parallel\|_{H^1(\Gamma_+^c)} + h^{\frac{1}{2}}\|h\nabla_N u_\perp\|_{L^2(\Gamma_+^c)}.$$

Note that Theorem 2.5 is actually Theorem 2.4 with u replaced by $*u$. Therefore it suffices to prove Theorem 2.5 only. It is also worth noting that the Carleman estimates are proved for CTA manifolds in general, with no restriction on either the dimension, the conformal factor, or the transversal manifold (M_0, g_0). Theorems 2.4 and 2.5 are extensions to the Hodge Laplace system on CTA manifolds of the scalar and Euclidean Carleman estimates in [Kenig et al. 2007; Chung 2015].

Finally, we sketch the main ideas in the proofs of the theorems and highlight the new features in our approach. The main difficulty in proving the Carleman estimates is the fact that the standard integration by parts argument, which gives a useful Carleman estimate for scalar equations with Dirichlet boundary condition [Kenig et al. 2007], results in complicated boundary terms when one is dealing with a system of equations (see Proposition 4.1). The Fourier analytic methods of [Chung 2015] will be crucial in handling these boundary terms. We first prove Theorem 2.5 for 0-forms (i.e., scalar equations) by adapting the Euclidean arguments of [Chung 2015] to the manifold case. After an initial estimate for the vectorial boundary terms in Proposition 4.2, Theorem 2.5 is proved for k-forms by induction on k. The proof of the Carleman estimates is long and technical, due to the work required to simplify and estimate the boundary terms.

After proving the Carleman estimates, the construction of CGO solutions proceeds as in the scalar case [Kenig et al. 2007; Dos Santos Ferreira et al. 2009a] and in the full data Maxwell case [Kenig et al. 2011]. The end result is given in Lemma 7.6. There the amplitude in the solutions is vector-valued, and later one needs to use the flexibility in choosing the components of this vector. The inverse problem is solved by inserting the CGO solutions in a standard integral identity, Lemma 8.1. Here an unexpected feature appears: recovering the matrix potential reduces to inverting mixed Fourier/attenuated geodesic ray transforms as in the scalar case [Dos Santos Ferreira et al. 2009a], but the components of the matrix turn out to depend on the geodesic along which they are integrated. We resolve this difficulty when $\dim(M) = 3$ by making use of ray transforms on tensors of order ≤ 2 and using recent results on tensor tomography [Paternain et al. 2013]. When the underlying space is Euclidean, we can use classical Fourier arguments and prove the uniqueness result also when $\dim(M) \geq 4$.

3. Notation and identities

As stated before, the basic reference for the following facts on Riemannian geometry is [Taylor 1996]. Let (M, g) be a smooth ($= C^\infty$) n-dimensional Riemannian manifold with or without boundary. All manifolds will be assumed to be oriented. We write $\langle v, w\rangle$ for the g-inner product of tangent vectors, and $|v| = \langle v, v\rangle^{\frac{1}{2}}$ for the g-norm. If $x = (x_1, \dots, x_n)$ are local coordinates and ∂_j are the corresponding vector fields, we write $g_{jk} = \langle \partial_j, \partial_k\rangle$ for the metric in these coordinates. The determinant of (g_{jk}) is denoted by $|g|$, and (g^{jk}) is the matrix inverse of (g_{jk}).

We shall sometimes do computations in normal coordinates. These are coordinates x defined in a neighbourhood of a point $p \in M^{\text{int}}$ such that $x(p) = 0$ and geodesics through p correspond to rays through the origin in the x-coordinates. The metric in these coordinates satisfies

$$g_{jk}(0) = \delta_{jk}, \quad \partial_l g_{jk}(0) = 0.$$

The Einstein convention of summing over repeated upper and lower indices will be used. We convert vector fields to 1-forms and vice versa by the musical isomorphisms, which are given by

$$(X^j \partial_j)^\flat = X_k \, dx^k, \quad X_k = g_{jk} X^j,$$
$$(\omega_k \, dx^k)^\sharp = \omega^j \partial_j, \qquad \omega^j = g^{jk}\omega_k.$$

The set of smooth k-forms on M is denoted by $\Omega^k M$, and the graded algebra of differential forms is written as

$$\Omega M = \bigoplus_{k=0}^{n} \Omega^k M.$$

The set of k-forms with L^2 or H^s coefficients are denoted by $L^2(M, \Lambda^k M)$ and $H^s(M, \Lambda^k M)$, respectively. Here H^s for $s \in \mathbb{R}$ are the usual Sobolev spaces on M. The inner product $\langle \cdot, \cdot\rangle$ and norm $|\cdot|$ are extended to forms and more generally tensors on M in the usual way, and we also extend the inner product $\langle \cdot, \cdot\rangle$ to complex-valued tensors as a complex bilinear form.

Let $d : \Omega^k M \to \Omega^{k+1} M$ be the exterior derivative, and let $* : \Omega^k M \to \Omega^{n-k} M$ be the Hodge star operator. We introduce the sesquilinear inner product on $\Omega^k M$,

$$(\eta \mid \zeta) = \int_M \langle \eta, \bar{\zeta}\rangle \, dV = \int_M \eta \wedge *\bar{\zeta} = (*\eta \mid *\zeta).$$

Here $dV = *1 = |g|^{\frac{1}{2}} \, dx^1 \cdots dx^n$ is the volume form. The codifferential $\delta : \Omega^k M \to \Omega^{k-1} M$ is defined as the formal adjoint of d in the inner product on real-valued forms, so that

$$(d\eta \mid \zeta) = (\eta \mid \delta\zeta) \quad \text{for } \eta \in \Omega^{k-1} M \text{ and } \zeta \in \Omega^k M \text{ compactly supported and real.}$$

These operators satisfy the following relations on k-forms in M:

$$** = (-1)^{k(n-k)}, \quad \delta = (-1)^{k(n-k)-n+k-1} * d * .$$

If X is a vector field, the interior product $i_X : \Omega^k M \to \Omega^{k-1} M$ is defined by

$$i_X \omega(Y_1, \dots, Y_{k-1}) = \omega(X, Y_1, \dots, Y_{k-1}).$$

If ξ is a 1-form then the interior product $i_\xi = i_{\xi^\sharp}$ is the formal adjoint of $\xi \wedge$ in the inner product on real-valued forms, and on k-forms it has the expression

$$i_\xi = (-1)^{n(k-1)} * \xi \wedge *.$$

The interior and exterior products interact by the formula

$$i_\xi \alpha \wedge \beta = (i_\xi \alpha) \wedge \beta + (-1)^k \alpha \wedge i_\xi \beta,$$

where α is a k-form and β an m-form. In particular if α and ξ are 1-forms then

$$i_\xi \alpha \wedge \beta + \alpha \wedge i_\xi \beta = \langle \alpha, \xi \rangle \beta.$$

In addition, the differential and codifferential satisfy the product rules

$$d(f\eta) = df \wedge \eta + f d\eta, \quad \delta(f\eta) = -i_{df} \eta + f \delta \eta.$$

The Hodge Laplacian on k-forms is defined by

$$-\Delta = (d + \delta)^2 = d\delta + \delta d.$$

It satisfies $\Delta * = * \Delta$. The above quantities may be naturally extended to graded forms.

We will also have to deal with forms that are not compactly supported on M. We have already introduced the tangential trace $t : \Omega M \to \Omega \partial M$ by

$$t : \omega \mapsto i^* \omega,$$

so if u is a graded form on M, then tu is a graded form on ∂M. Then

$$(tu \mid tv)_{\partial M}$$

is interpreted in the same manner as $(u \mid v)_M$ above. If u and v are graded forms on M, we will also define

$$(u \mid v)_{\partial M} = \int_{\partial M} \langle u, \bar{v} \rangle \, dS = \int_{\partial M} t i_\nu u \wedge * \bar{v} \, dS,$$

where dS is the volume form on ∂M. Now if $\eta \in \Omega^{k-1} M$ and $\zeta \in \Omega^k M$ then d and δ satisfy the integration by parts formulas

$$(d\eta \mid \zeta)_M = (\nu \wedge \eta \mid \zeta)_{\partial M} + (\eta \mid \delta \zeta)_M, \tag{3-1}$$

$$(\delta \zeta \mid \eta)_M = -(i_\nu \zeta \mid \eta)_{\partial M} + (\zeta \mid d\eta)_M. \tag{3-2}$$

Note also that

$$(i_\nu \zeta \mid \eta)_{\partial M} = (\nu \wedge \eta \mid \zeta)_{\partial M}.$$

Here ν denotes both the unit outer normal of ∂M and the corresponding 1-form.

Applying these formulas for the Hodge Laplacian gives

$$(-\Delta u \mid v)_M = (u \mid -\Delta v)_M + (\nu \wedge \delta u \mid v)_{\partial M} - (i_\nu du \mid v)_{\partial M} - (i_\nu u \mid dv)_{\partial M} + (\nu \wedge u \mid \delta v)_{\partial M},$$

where u and v are k-forms, or graded forms. We can also redo the integration by parts to write the boundary terms in terms of absolute and relative boundary conditions, so

$$(-\Delta u \mid v)_M = (u \mid -\Delta v)_M + (tu \mid ti_\nu dv)_{\partial M} + (t\delta * u \mid ti_\nu * v)_{\partial M} + (t * u \mid ti_\nu d * v)_{\partial M} + (t\delta u \mid ti_\nu v)_{\partial M}.$$

The Levi-Civita connection, defined on tensors in M, is denoted by ∇ and it satisfies $\nabla_X * = *\nabla_X$. We will sometimes write ∇f (where f is any function) for the metric gradient of f, defined by

$$\nabla f = (df)^\sharp = g^{jk}\partial_j f \partial_k.$$

If X is a vector field and η, ζ are differential forms we have

$$\nabla_X(\eta \wedge \zeta) = (\nabla_X \eta) \wedge \zeta + \eta \wedge (\nabla_X \zeta).$$

If X, Y are vector fields then

$$[\nabla_X, i_Y] = i_{\nabla_X Y}.$$

We can also express d using the ∇ operator, as follows: if ω is a k-form on M, and $X_1, \dots, X_{k+1}$ are vector fields on M, then

$$d\omega(X_1, \dots, X_{k+1}) = \sum_{l=1}^{k+1} (-1)^{l+1} (\nabla_{X_l}\omega)(X_1, \dots, \widehat{X}_l, \dots, X_{k+1}),$$

where $\widehat{X}_l$ means that we omit the X_l argument. Moreover if $e_1, \dots, e_n$ are an orthonormal frame of TM defined in a neighbourhood $U \subset M$ we have

$$-\delta\omega = \sum_{j=1}^{n} i_{e_j} \nabla_{e_j} \omega.$$

For the statements of the Carleman estimates, we introduced the notation

$$u_\perp = N^\flat \wedge i_N u \quad \text{and} \quad u_\parallel = u - u_\perp,$$

where N is a smooth vector field which coincides with the inward pointing normal vector field at the boundary ∂M, and is extended into M by parallel transport. Note that $i_N u_\parallel = 0$, $N \wedge u_\perp = 0$, and $tu_\perp = 0$ at ∂M. In addition, if u and v are graded forms on M, then

$$(tu \mid tv)_{\partial M} = (tu_\parallel \mid tv_\parallel)_{\partial M} = (u_\parallel \mid v_\parallel)_{\partial M}$$

and

$$(ti_N u \mid ti_N v)_{\partial M} = (ti_N u_\perp \mid ti_N v_\perp)_{\partial M} = (u_\perp \mid v_\perp)_{\partial M}.$$

If X is a vector field, we can break down X into parallel and perpendicular components in the same way by using $(X_\parallel^\flat)^\sharp$ and $(X_\perp^\flat)^\sharp$. The $\perp$ and $\parallel$ signs are interchanged by the Hodge star operator:

$$*(u_\parallel) = (*u)_\perp \quad \text{and} \quad *(u_\perp) = (*u)_\parallel.$$

Note that by its definition in terms of parallel transport, $\nabla_N N = 0$. Thus ∇_N commutes with $N\wedge$ and i_N.

If we view ∂M as a submanifold embedded into M, then TM splits into $T\partial M \oplus N\partial M$, where $T\partial M$ is the tangent bundle of ∂M and $N\partial M$ is the normal bundle. Then the second fundamental form $II : T\partial M \oplus T\partial M \to N\partial M$ of ∂M relative to this embedding is defined by

$$II(X, Y) = (\nabla_X Y \mid N)N.$$

The second fundamental form can also be defined in terms of the shape operator $s : T\partial M \to T\partial M$ by

$$s(X) = \nabla_X N.$$

Then

$$II(X, Y) = (s(X) \mid Y)N.$$

These two operators carry information about the shape of the ∂M in M, and thus show up in our boundary computations.

Now we move to some more specific technical formulas used in the paper. The proofs involve routine computations and are omitted, but interested readers may find the proofs in the arXiv version of this paper [Chung et al. 2013, Appendix]. We begin with a simple computation.

Lemma 3.1. *If ξ and η are real-valued 1-forms on M and if u is a k-form, then*

$$\xi \wedge i_\eta u + i_\xi(\eta \wedge u) + \eta \wedge i_\xi u + i_\eta(\xi \wedge u) = 2\langle \xi, \eta\rangle u.$$

We also give an expression for the conjugated Laplacian.

Lemma 3.2. *Let (M, g) be an oriented Riemannian manifold, let $\rho \in C^2(M)$ be a complex-valued function, and let s be a complex number. If u is a k-form on M, then*

$$e^{s\rho}(-\Delta)(e^{-s\rho}u) = -s^2\langle d\rho, d\rho\rangle u + s[2\nabla_{\mathrm{grad}(\rho)} + \Delta\rho]u - \Delta u.$$

Next, an expansion for the expression $t\delta$.

Lemma 3.3. *Let $u \in \Omega^k(M)$. Then*

$$-t(\delta u) = -\delta' t u_\parallel + (S - (n-1)\kappa) t i_N u_\perp + t\nabla_N i_N u,$$

where κ is the mean curvature of ∂M, and $S : \Omega^{k-1}(\partial M) \to \Omega^{k-1}(\partial M)$ is defined by

$$S\omega(X_1, \dots, X_{k-1}) = \sum_{\ell=1}^{k-1} \omega(X_1, \dots, sX_\ell, \dots, X_{k-1}),$$

with $s : T\partial M \to T\partial M$ being the shape operator of ∂M.

Now for $t i_N d$.

Lemma 3.4. *Let $u \in \Omega^k(M)$. Then on ∂M,*

$$t i_N du = t\nabla_N u_\parallel + S t u_\parallel - d' t i_N u.$$

We also need an expansion for $t\delta B$, where B is the operator

$$B = \frac{h}{i}\big[d \circ i_{d\varphi_c} + i_{d\varphi_c} \circ d - d\varphi_c \wedge \delta - \delta(d\varphi_c \wedge \cdot)\big] = \frac{h}{i}[2\nabla_{\nabla\varphi_c} + \Delta\varphi_c].$$

Lemma 3.5. *If $u \in \Omega^k(M)$ is such that $tu = 0$, then*

$$\begin{aligned} t\delta Bu = \delta' tBu &+ 2ih\nabla'_{(\nabla\varphi_c)_\parallel} t\nabla_N i_N u - 2ih\partial_\nu\varphi_c t\nabla_N\nabla_N i_N u \\ &+ ih\big(2((n-1)\kappa - S)\partial_\nu\varphi_c + 2\partial_\nu^2\varphi_c + \triangle\varphi_c\big)t\nabla_N i_N u + 2ih(S-(n-1)\kappa)t\nabla_{(\nabla\varphi_c)_\parallel} i_N u \\ &+ ih\big((S-(n-1)\kappa)\triangle\varphi_c + \nabla_N\triangle\varphi_c\big)ti_N u \\ &+ 2ihti_N R(N, \nabla(\varphi_c)_\parallel)u_\perp + 2iht\nabla_{[(\nabla\varphi_c)_\parallel, N]} i_N u - 2ihi_{s(\nabla\varphi_c)_\parallel} t\nabla_N u_\parallel. \end{aligned}$$

Finally, we will need to do a computation to split the Hodge Laplacian into normal and tangential parts. To do this, we will take advantage of a Weitzenböck identity, which says

$$\triangle = \tilde{\triangle} + R,$$

where R is a zeroth-order linear operator depending only on the curvature of M, $\triangle$ is the Hodge Laplacian, and $\tilde{\triangle}$ is the connection Laplacian:

$$\tilde{\Delta}u := \nabla^*\nabla u.$$

We then have the following result for $\tilde{\Delta}$.

Lemma 3.6. *Let $u \in \Omega^k(M)$ satisfy $tu = 0$. Then*

$$ti_N\tilde{\Delta}u = \tilde{\Delta}' ti_N u + t\nabla_N\nabla_N i_N u + tr(s^2)i_N u - S_2 i_N u,$$

where $S_2\omega(X_1, \dots, X_{k-1}) := \sum_{l=1}^{k-1} \omega(\dots, s^2X_l, \dots)$.

4. Carleman estimates and boundary terms

As noted in the Introduction, Theorem 2.4 follows from Theorem 2.5, so it enough to show that we can prove Theorem 2.5.

In proving the Carleman estimates, it will suffice to work with smooth sections of ΛM and apply a density argument to get the final result. Let $\Omega^k(M)$ denote the space of smooth sections of $\Lambda^k M$, and $\Omega(M)$ denote the space of smooth sections of ΛM.

In this section we give an initial form of the Carleman estimates by using an integration by parts argument as in [Kenig et al. 2007]. To do this, we will first need to understand the relevant boundary terms. We will use the integration by parts formulas

$$(du \mid v)_M = (\nu \wedge u \mid v)_{\partial M} + (u \mid \delta v)_M, \tag{4-1}$$

$$(\delta u \mid v)_M = -(i_\nu u \mid v)_{\partial M} + (u \mid dv)_M \tag{4-2}$$

for $u, v \in \Omega(M)$.

As in [Kenig et al. 2007], we will need to work with the convexified weight

$$\varphi_c = \varphi + \frac{h\varphi^2}{2\varepsilon}.$$

Then

$$-\Delta_{\varphi_c} = e^{\frac{\varphi_c}{h}}(-h^2\Delta)e^{-\frac{\varphi_c}{h}}.$$

Writing

$$d_{\varphi_c} = e^{\frac{\varphi_c}{h}} h d e^{-\frac{\varphi_c}{h}} = hd - d\varphi_c \wedge,$$
$$\delta_{\varphi_c} = e^{\frac{\varphi_c}{h}} h \delta e^{-\frac{\varphi_c}{h}} = h\delta + i_{d\varphi_c},$$

we have

$$-\Delta_{\varphi_c} = d_{\varphi_c}\delta_{\varphi_c} + \delta_{\varphi_c} d_{\varphi_c}.$$

By Lemma 3.2 we can write this as $A + iB$, where A and B are self-adjoint operators given by

$$A = -h^2\Delta - \big(d\varphi_c \wedge i_{d\varphi_c} + i_{d\varphi_c}(d\varphi_c \wedge \cdot)\big) = -h^2\Delta - |d\varphi_c|^2,$$
$$B = \frac{h}{i}\big[d \circ i_{d\varphi_c} + i_{d\varphi_c} \circ d - d\varphi_c \wedge \delta - \delta(d\varphi_c \wedge \cdot)\big] = \frac{h}{i}[2\nabla_{\nabla\varphi_c} + \Delta\varphi_c].$$

Let $\|\cdot\|$ indicate the L^2 norm on M, unless otherwise stated. Then, for $u \in \Omega^k(M)$,

$$\|\Delta_{\varphi_c} u\|^2 = \big((A+iB)u \;\big|\; (A+iB)u\big) = \|Au\|^2 + \|Bu\|^2 + i(Bu \mid Au) - i(Au \mid Bu).$$

Integrating by parts gives

$$\begin{aligned}(Bu \mid Au) &= \big(Bu \;\big|\; h^2 d\delta u + h^2\delta d u - |d\varphi_c|^2 u\big)\\ &= (hdBu \mid hdu) + (h\delta Bu \mid h\delta u) - \big(|d\varphi_c|^2 Bu\ u\big) + h(Bu \mid \nu\wedge h\delta u - i_\nu h du)_{\partial M}\\ &= (ABu \mid u) + h(hdBu \mid \nu\wedge u)_{\partial M} - h(h\delta Bu \mid i_\nu u)_{\partial M} + h(Bu \mid \nu\wedge h\delta u - i_\nu h du)_{\partial M}\end{aligned}$$

and after a short computation

$$(Au \mid Bu) = (BAu \mid u) - \frac{2h}{i}((\partial_\nu\varphi_c)Au \mid u)_{\partial M}.$$

This finishes the basic integration by parts argument and shows the following:

Proposition 4.1. *If $u \in \Omega M$, then*

$$\begin{aligned}\|\Delta_{\varphi_c} u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u \mid u) + ih(hdBu \mid \nu\wedge u)_{\partial M}\\ -ih(h\delta Bu \mid i_\nu u)_{\partial M} + ih(Bu \mid \nu\wedge h\delta u - i_\nu h du)_{\partial M} + 2h((\partial_\nu\varphi_c)Au \mid u)_{\partial M}.\end{aligned} \tag{4-3}$$

Now we invoke the absolute boundary conditions to estimate the nonboundary terms and to simplify the boundary terms in (4-3). It is enough to consider differential forms $u \in \Omega^k(M)$ for fixed k.

Proposition 4.2. *Let $u \in \Omega^k(M)$ such that*

$$\begin{aligned} t * u &= 0,\\ t h\delta * u &= -t i_{d\varphi} * u + h\sigma t i_N * u\end{aligned} \tag{4-4}$$

for some smooth bounded endomorphism σ whose bounds are uniform in h.

Then the nonboundary terms in (4-3) *satisfy*

$$\|Au\|^2 + \|Bu\|^2 + (i[A,B]u \mid u) \gtrsim \frac{h^2}{\varepsilon}\|u\|^2_{H^1(M)} - \frac{h^3}{\varepsilon}\big(\|u_\parallel\|^2_{H^1(\partial M)} + \|h\nabla_N u_\perp\|^2_{L^2(\partial M)}\big) \tag{4-5}$$

for $h \ll \varepsilon \ll 1$. *Also, the boundary terms in* (4-3) *have the form*

$$-2h^3(\partial_\nu\varphi\nabla_N u_\perp \mid \nabla_N u_\perp)_{\partial M} - 2h\big(\partial_\nu\varphi(|d\varphi|^2+|\partial_\nu\varphi|^2)u_\parallel \mid u_\parallel\big)_{\partial M} + R, \tag{4-6}$$

where

$$|R| \lesssim Kh^3\|\nabla' t u_\parallel\|^2_{\partial M} + \frac{h}{K}\|u_\parallel\|^2_{\partial M} + \frac{h^3}{K}\|\nabla_N u_\perp\|^2_{\partial M}$$

for any large enough K independent of h.

Proof of Proposition 4.2. We will prove (4-5) first. The argument follows the one given in [Chung 2015] for scalar functions.

Note that A and B have the same scalar principal symbols as they do for 0-forms: that is, given a local basis $dx^1, \dots, dx^n$ for the cotangent space with $dx^I = dx^{i_1}\wedge\cdots\wedge dx^{i_k}$,

$$A = A_s + hE_1, \quad A_s(f\,dx^I) = (Af)dx^I,$$

and

$$B = B_s + hE_0, \quad B_s(f\,dx^I) = (Bf)dx^I,$$

where E_1 and E_0 are first- and zeroth-order operators, respectively, with uniform bounds in h and ε. Therefore locally

$$[A,B](f\,dx^I) = ([A,B]f)dx^I + h\big([E_1,B_s]+[A_s,E_0]+hR\big)(f\,dx^I),$$

where R is a first-order operator with uniform bounds in h and ε. Choosing a partition of unity $\chi_1,\dots,\chi_m$ of M such that this operation can be performed near each $\operatorname{supp}(\chi_j)$, the argument for scalar functions in the proof of Proposition 3.1 from [Chung 2015] implies

$$i([A,B]u \mid u) = \sum_{j=1}^m i([A,B]u \mid \chi_j u) = 4\frac{h^2}{\varepsilon}\|(1+h\varepsilon^{-1}\varphi)u\|^2_{L^2} + h(B\beta Bu \mid u) + h^2(Qu \mid u),$$

where Q is a second-order operator. Recall that

$$B = \frac{h}{i}\big(d\circ i_{d\varphi_c} + i_{d\varphi_c}\circ d - d\varphi_c\wedge\delta - \delta(d\varphi_c\wedge\cdot)\big),$$

so using integration by parts with the above formula, we get

$$\begin{aligned} h(B\beta Bu \mid u) &= h(\beta Bu \mid Bu) - ih^2(i_\nu\beta Bu \mid i_{d\varphi_c}u)_{\partial M} - ih^2(\nu\wedge i_{d\varphi_c}\beta Bu \mid u)_{\partial M} \\ &\qquad - ih^2(\nu\wedge\beta Bu \mid d\varphi_c\wedge u)_{\partial M} - ih^2(i_\nu(d\varphi_c\wedge\beta Bu) \mid u)_{\partial M} \\ &= h(\beta Bu \mid Bu) - ih^2(d\varphi_c\wedge i_\nu\beta Bu \mid u)_{\partial M} - ih^2(\nu\wedge i_{d\varphi_c}\beta Bu \mid u)_{\partial M} \\ &\qquad - ih^2(i_{d\varphi_c}\nu\wedge\beta Bu \mid u)_{\partial M} - ih^2(i_\nu(d\varphi_c\wedge\beta Bu) \mid u)_{\partial M}. \end{aligned}$$

By Lemma 3.1 we obtain

$$h(B\beta Bu \mid u) = h(\beta Bu \mid Bu) - 2ih^2(\partial_\nu \varphi_c \beta Bu \mid u)_{\partial M}.$$

The absolute boundary condition says that $t * u = 0$, so $u_\perp = 0$ at the boundary. Therefore

$$h(B\beta Bu \mid u) = h(\beta Bu \mid Bu) - 2ih^2(\partial_\nu \varphi_c \beta Bu \mid u_\parallel)_{\partial M} = h(\beta Bu \mid Bu) - 2ih^2(t\partial_\nu \varphi_c \beta Bu \mid tu_\parallel)_{\partial M}.$$

The boundary term in the last expression is bounded by

$$h^3\varepsilon^{-1}\|tBu\|^2_{L^2(\partial M)} + h^3\varepsilon^{-1}\|u_\parallel\|^2_{L^2(\partial M)}.$$

At the boundary,

$$tBu = \frac{h}{i}t[2\nabla_{\nabla\varphi_c} + \Delta\varphi_c]u = \frac{h}{i}\big[-2\partial_\nu\varphi_c t\nabla_N u_\parallel - 2\partial_\nu\varphi_c t\nabla_N u_\perp + 2t\nabla_{(\nabla\varphi_c)_\parallel}u_\parallel + \Delta\varphi_c tu_\parallel\big],$$

so

$$\begin{aligned}\|tBu\|^2_{L^2(\partial M)} &\lesssim \|th\nabla_N u_\parallel\|^2_{L^2(\partial M)} + \|th\nabla_N u_\perp\|^2_{L^2(\partial M)} + \|th\nabla_{(\nabla\varphi_c)_\parallel}u_\parallel\|^2_{L^2(\partial M)} + h^2\|tu_\parallel\|^2_{L^2(\partial M)}\\ &\lesssim \|th\nabla_N u_\parallel\|^2_{L^2(\partial M)} + \|th\nabla_N u_\perp\|^2_{L^2(\partial M)} + \|u_\parallel\|^2_{H^1(\partial M)}.\end{aligned}$$

Now by Lemma 3.4,

$$ti_N h du = th\nabla_N u_\parallel + hStu_\parallel - hd'ti_N u.$$

Since $t * u = 0$, we have $i_N u, u_\perp = 0$ at the boundary, and thus

$$ti_N h du = th\nabla_N u_\parallel + hStu_\parallel.$$

Therefore

$$\begin{aligned}\|th\nabla_N u_\parallel\|^2_{L^2(\partial M)} &\lesssim \|ti_N h du\|^2_{L^2(\partial M)} + h^2\|u_\parallel\|^2_{L^2(\partial M)}\\ &\lesssim \|ti_N * (h\delta * u)\|^2_{L^2(\partial M)} + h^2\|u_\parallel\|^2_{L^2(\partial M)}\\ &\lesssim \|th\delta * u\|^2_{L^2(\partial M)} + h^2\|u_\parallel\|^2_{L^2(\partial M)}\\ &\lesssim \|u\|^2_{L^2(\partial M)},\end{aligned}$$

where in the last step we invoked the absolute boundary condition. Therefore

$$\|tBu\|^2_{L^2(\partial M)} \lesssim \|th\nabla_N u_\perp\|^2_{L^2(\partial M)} + \|u_\parallel\|^2_{H^1(\partial M)},$$

and thus

$$h(B\beta Bu \mid u) \lesssim \frac{h^2}{\varepsilon}\|Bu\|^2_{L^2} + \frac{h^3}{\varepsilon}\|u_\parallel\|^2_{H^1(\partial M)} + \frac{h^3}{\varepsilon}\|h\nabla_N u_\perp\|^2_{L^2(\partial M)}.$$

Similarly

$$h^2(Qu \mid u) \lesssim h^2\|u\|^2_{H^1} + h^3\|u_\parallel\|^2_{H^1(\partial M)} + h^3\|h\nabla_N u_\perp\|^2_{L^2(\partial M)}.$$

Therefore

$$i([A,B]u \mid u) \gtrsim \frac{h^2}{\varepsilon}\|u\|^2_{L^2} - \frac{h^2}{\varepsilon}\|Bu\|^2_{L^2} - h^2\|u\|^2_{H^1} - h^3\varepsilon^{-1}\|u_\parallel\|^2_{H^1(\partial M)} - h^3\varepsilon^{-1}\|h\nabla_N u_\perp\|^2_{L^2(\partial M)}.$$

Meanwhile, since $t * u = 0$ on ∂M we can write

$$\begin{aligned} h^2(\|hdu\|_{L^2}^2 + \|h\delta u\|_{L^2}^2) &= h^2\big((hd*u, hd*u) + (h\delta*u, h\delta*u)\big) \\ &= h^2(-h^2\triangle{*u} \mid {*u}) - h^3(\nu\wedge h\delta{*u} \mid {*u})_{\partial M} \\ &= h^2(Au \mid u) + h^2(|d\varphi_c|^2 u \mid u) - h^3(\nu\wedge h\delta{*u} \mid {*u})_{\partial M} \\ &= h^2(Au \mid u) + h^2(|d\varphi_c|^2 u \mid u) + h^3(th\delta{*u} \mid ti_N{*u})_{\partial M}. \end{aligned}$$

Using the absolute boundary conditions again, we have

$$\begin{aligned} th\delta * u &= -ti_{d\varphi} * u + h\sigma ti_N * u \\ &= \partial_\nu\varphi ti_N * u + h\sigma ti_N * u, \end{aligned}$$

so

$$h^2(\|hdu\|_{L^2}^2 + \|h\delta u\|_{L^2}^2) \lesssim \frac{1}{K}\|Au\|_{L^2}^2 + Kh^4\|u\|_{L^2}^2 + h^2\|u\|_{L^2}^2 + h^3\|ti_N * u\|_{L^2(\partial M)}^2,$$

or

$$\|Au\|_{L^2}^2 \gtrsim Kh^2(\|hdu\|_{L^2}^2 + \|h\delta u\|_{L^2}^2) - K^2h^4\|u\|_{L^2}^2 - Kh^2\|u\|_{L^2}^2 - Kh^3\|u_\parallel\|_{L^2(\partial M)}^2.$$

We take $K \sim \frac{1}{\alpha\varepsilon}$ with α large and fixed. Putting this together with the inequality for $(i[A, B]u \mid u)$ and Gaffney's inequality $\|u\|_{H^1} \sim \|u\|_{L^2} + \|hdu\|_{L^2} + \|h\delta u\|_{L^2}$ when $t * u = 0$, we obtain

$$\|Au\|^2 + \|Bu\|^2 + (i[A, B]u \mid u) \gtrsim \frac{h^2}{\varepsilon}\|u\|_{H^1}^2 - h^3\varepsilon^{-1}\big(\|u_\parallel\|_{H^1(\partial M)}^2 + \|h\nabla_N u_\perp\|_{L^2(\partial M)}^2\big)$$

for $h \ll \varepsilon \ll 1$. This proves (4-5).

We will now show the expression (4-6) for the boundary terms in (4-3). Recall that these boundary terms are given by

$$ih(hdBu \mid \nu\wedge u)_{\partial M} - ih(h\delta Bu \mid i_\nu u)_{\partial M} + ih(Bu \mid \nu\wedge h\delta u - i_\nu hdu)_{\partial M} + 2h((\partial_\nu\varphi_c)Au \mid u)_{\partial M}. \tag{4-7}$$

Note that

$$\begin{aligned} &ih(hdB{*u} \mid \nu\wedge{*u})_{\partial M} - ih(h\delta B{*u} \mid i_\nu{*u})_{\partial M} + ih(B{*u} \mid \nu\wedge h\delta{*u} - i_\nu hd{*u})_{\partial M} + 2h((\partial_\nu\varphi_c)A{*u} \mid {*u})_{\partial M} \\ &\quad = ih(hdBu \mid \nu\wedge u)_{\partial M} - ih(h\delta Bu \mid i_\nu u)_{\partial M} + ih(Bu \mid \nu\wedge h\delta u - i_\nu hdu)_{\partial M} + 2h((\partial_\nu\varphi_c)Au \mid u)_{\partial M}. \end{aligned}$$

Moreover, if u satisfies the absolute boundary conditions (4-4), then $*u$ satisfies the relative boundary conditions

$$\begin{aligned} tu &= 0, \\ th\delta u &= -ti_{d\varphi}u + h\sigma ti_N u, \end{aligned} \tag{4-8}$$

and vice versa. Therefore it suffices to prove that if u satisfies (4-8) then the boundary terms (4-7) become

$$-2h^3(\partial_\nu\varphi\nabla_N u_\parallel \mid \nabla_N u_\parallel)_{\partial M} - 2h\big(\partial_\nu\varphi(|d\varphi|^2 + |\partial_\nu\varphi|^2)u_\perp \,\big|\, u_\perp\big)_{\partial M} + R, \tag{4-9}$$

where

$$|R| \lesssim Kh^3\|\nabla' ti_N u\|_{\partial M}^2 + \frac{h}{K}\|u_\perp\|_{\partial M}^2 + \frac{h^3}{K}\|\nabla_N u_\parallel\|_{\partial M}^2 \tag{4-10}$$

for any large enough K independent of h.

So let's return to (4-7), and assume u satisfies (4-8). The condition $tu = 0$ implies the first term $ih(hdBu \mid \nu \wedge u)_{\partial M}$ is zero. Therefore we are left with

$$-ih(h\delta Bu \mid i_\nu u)_{\partial M} + ih(Bu \mid \nu \wedge h\delta u)_{\partial M} - ih(Bu \mid i_\nu h du)_{\partial M} + 2h((\partial_\nu \varphi_c)Au \mid u)_{\partial M}.$$

We calculate each of the terms individually.

Firstly,

$$\begin{aligned} ih(Bu \mid \nu \wedge h\delta u)_{\partial M} &= ih(Bu \mid \nu \wedge h(\delta u)_{\parallel})_{\partial M} \\ &= -ih(i_N Bu \mid h(\delta u)_{\parallel})_{\partial M} \\ &= -ih(ti_N Bu \mid th\delta u)_{\partial M}. \end{aligned}$$

Now

$$Bu = \frac{h}{i}(2\nabla_{\nabla \varphi_c} + \Delta \varphi_c)u,$$

so

$$\begin{aligned} ti_N Bu &= \frac{h}{i} ti_N \big(2\nabla_{(\nabla \varphi_c)_{\parallel}} - 2\partial_\nu \varphi_c \nabla_N + \Delta \varphi_c\big)u \\ &= \frac{h}{i}\big(2\nabla_{(\nabla \varphi_c)_{\parallel}} ti_N - 2\partial_\nu \varphi_c t \nabla_N i_N + t\Delta \varphi_c i_N\big)u. \end{aligned}$$

Therefore,

$$\begin{aligned} &-ih(ti_N Bu \mid th\delta u)_{\partial M} \\ &\qquad = 2h(\partial_\nu \varphi_c th \nabla_N i_N u \mid th\delta u)_{\partial M} - 2h(h\nabla_{(\nabla \varphi_c)_{\parallel}} ti_N u \mid th\delta u)_{\partial M} - h^2(t\Delta \varphi_c i_N u \mid th\delta u)_{\partial M}. \end{aligned}$$

Now if $th\delta u|_{\partial M} = -ti_{d\varphi} u + h\sigma ti_N u$ and $tu = 0$, then

$$th\delta u|_{\partial M} = (\partial_\nu \varphi + h\sigma) ti_N u. \tag{4-11}$$

Therefore

$$\begin{aligned} -ih(ti_N Bu \mid th(\delta u))_{\partial M} &= 2h\big(\partial_\nu \varphi_c th \nabla_N i_N u \,\big|\, (\partial_\nu \varphi + h\sigma) ti_N u\big)_{\partial M} \\ &\quad - 2h\big(h\nabla_{(\nabla \varphi_c)_{\parallel}} ti_N u \,\big|\, (\partial_\nu \varphi + h\sigma) ti_N u\big)_{\partial M} \\ &\quad - h^2\big(t\Delta \varphi_c i_N u \,\big|\, (\partial_\nu \varphi + h\sigma) ti_N u\big)_{\partial M}. \end{aligned}$$

Moreover, by Lemma 3.3,

$$th(\delta u) = h\delta' tu_{\parallel} + h((n-1)\kappa - S) ti_N u_{\perp} - th\nabla_N i_N u.$$

Since $tu = 0$,

$$th(\delta u) = h((n-1)\kappa - S) ti_N u_{\perp} - th\nabla_N i_N u.$$

Substituting this into (4-11) gives

$$th\nabla_N i_N u = \big(-\partial_\nu \varphi - h\sigma + h(n-1)\kappa - hS\big) ti_N u. \tag{4-12}$$

Therefore

$$\begin{aligned} -ih(ti_N Bu \mid th(\delta u))_{\partial M} = {} & -2h\big(\partial_\nu\varphi_c(\partial_\nu\varphi+h\sigma-h(n-1)\kappa+hS)ti_N u \,\big|\, (\partial_\nu\varphi+h\sigma)ti_N u\big)_{\partial M} \\ & -2h\big(h\nabla_{(\nabla\varphi_c)_{\|}}ti_N u \,\big|\, (\partial_\nu\varphi+h\sigma)ti_N u\big)_{\partial M} \\ & -h^2\big(t\Delta\varphi_c i_N u \,\big|\, (\partial_\nu\varphi+h\sigma)ti_N u\big)_{\partial M}. \end{aligned}$$

We can write this as

$$ih(Bu \mid \nu\wedge h\delta u)_{\partial M} = -2h\big(\partial_\nu\varphi|\partial_\nu\varphi|^2 ti_N u \,\big|\, ti_N u\big)_{\partial M} + R_2, \tag{4-13}$$

where R_2 satisfies the bound on R in (4-10).

Secondly,

$$\begin{aligned} -ih(Bu \mid i_\nu h du)_{\partial M} &= ih((Bu)_{\|} \mid i_N h du)_{\partial M} \\ &= ih(t(Bu)_{\|} \mid ti_N h du)_{\partial M}. \end{aligned}$$

By Lemma 3.4,

$$ti_N h du = th\nabla_N u_{\|} + hStu_{\|} - hd'ti_N u,$$

so if $tu=0$,

$$ti_N h du = th\nabla_N u_{\|} - hd'ti_N u.$$

Therefore

$$-ih(Bu \mid i_\nu h du)_{\partial M} = ih\big(tBu \,\big|\, th\nabla_N u_{\|} - hd'ti_N u\big)_{\partial M}.$$

Expanding B, this becomes

$$h\big(th\big(-2\partial_\nu\varphi_c\nabla_N u + 2\nabla_{(\nabla\varphi_c)_{\|}}u + (\triangle\varphi_c)u\big) \,\big|\, th\nabla_N u_{\|} - hd'ti_N u\big)_{\partial M}.$$

Since $tu=0$, the last expression is equal to

$$-2h\big(\partial_\nu\varphi_c th\nabla_N u - th\nabla_{(\nabla\varphi_c)_{\|}}u \,\big|\, th\nabla_N u_{\|} - hd'ti_N u\big)_{\partial M}. \tag{4-14}$$

The

$$-2h(\partial_\nu\varphi_c th\nabla_N u_{\|} \mid -hd'ti_N u)_{\partial M}$$

part has the same type of bound as in (4-10), so

$$-ih(Bu \mid i_\nu h du)_{\partial M} = -2h(\partial_\nu\varphi_c th\nabla_N u_{\|} \mid th\nabla_N u_{\|})_{\partial M} + R_3, \tag{4-15}$$

where R_3 has the same bound as in (4-10).

Thirdly,

$$\begin{aligned} ih(h\delta Bu \mid i_\nu u)_{\partial M} &= ih(h(\delta Bu)_{\|} \mid i_\nu u)_{\partial M} \\ &= -ih(ht(\delta Bu) \mid ti_N u)_{\partial M}. \end{aligned}$$

By Lemma 3.5,

$$\begin{aligned} ht\delta Bu = {} & h\delta' tBu + 2ih^2\nabla'_{(\nabla\varphi_c)_\|} t\nabla_N i_N u - 2ih^2\partial_\nu\varphi_c t\nabla_N\nabla_N i_N u \\ & + ih^2\big(2((n-1)\kappa - S)\partial_\nu\varphi_c + 2\partial_\nu^2\varphi_c + \triangle\varphi_c\big)t\nabla_N i_N u \\ & + 2ih^2(S-(n-1)\kappa)t\nabla_{(\nabla\varphi_c)_\|} i_N u + ih^2\big((S-(n-1)\kappa)\triangle\varphi_c + \nabla_N\triangle\varphi_c\big)ti_N u \\ & + 2ih^2 ti_N R(N,\nabla(\varphi_c)_\|)u_\perp + 2ih^2 t\nabla_{[(\nabla\varphi_c)_\|,N]} i_N u - 2ih^2 i_{s(\nabla\varphi_c)_\|} t\nabla_N u_\|. \end{aligned}$$

The terms on the last two lines, when paired with $ihti_N u$, are bounded by (4-10).

Moreover, using the boundary conditions in the form of equation (4-12) on the

$$h^3\big(\big(2((n-1)\kappa - S)\partial_\nu\varphi_c + 2\partial_\nu^2\varphi_c + \triangle\varphi_c\big)t\nabla_N i_N u \,\big|\, ti_N u\big)_{\partial M}$$

term shows that this too is bounded by (4-10). Therefore we need only worry about the first three terms.

For the $-ih(h\delta' tBu \mid ti_N u)$ term, we can integrate by parts to get

$$-ih(tBu \mid hd' ti_N u)_{\partial M} = -2h\big(ht\nabla_{\nabla(\varphi_c)}u + \tfrac{1}{2}h\Delta\varphi_c tu \mid hd' ti_N u\big)_{\partial M}.$$

Since $tu = 0$, we get

$$ih(tBu \mid hd' ti_N u)_{\partial M} = 2h(ht\nabla_{\nabla(\varphi_c)}u \mid hd' ti_N u)_{\partial M}.$$

Now

$$t\nabla_{\nabla(\varphi_c)}u = t\nabla_{\nabla(\varphi_c)_\|}u_\perp + t\nabla_{\nabla(\varphi_c)_\perp}u_\|$$

since $tu = 0$. Therefore

$$\big|ih(tBu \mid hd' ti_N u)_{\partial M}\big| \le Kh^3\|\nabla' ti_N u\|^2_{\partial M} + Kh^3\|u_\perp\|^2_{\partial M} + Kh^3\|\nabla_N u_\|\|^2,$$

and so this term is bounded by (4-10).

For the $2h^3\big(\nabla'_{(\nabla\varphi_c)_\|} t\nabla_N i_N u \,\big|\, ti_N u\big)_{\partial M}$ term, we can use equation (4-12) to get

$$2h^3\big(\nabla'_{(\nabla\varphi_c)_\|} t\nabla_N i_N u \,\big|\, ti_N u\big)_{\partial M} = -2h^2\big(\nabla'_{(\nabla\varphi_c)_\|}(-\partial_\nu\varphi - h\sigma + h(n-1)\kappa - hS)ti_N u \,\big|\, ti_N u\big)_{\partial M}.$$

and then use Cauchy–Schwarz, so this term is bounded by (4-10) too. Therefore

$$-ih(h\delta Bu \mid i_\nu u)_{\partial M} = 2h^3\big(\partial_\nu\varphi_c t\nabla_N\nabla_N i_N u \,\big|\, ti_N u\big)_{\partial M} + R_1, \tag{4-16}$$

where R_1 is bounded by (4-10).

Finally,

$$\begin{aligned} 2h((\partial_\nu\varphi_c)Au \mid u)_{\partial M} &= 2h\big((\partial_\nu\varphi_c)Au \,\big|\, u_\perp\big)_{\partial M} \\ &= 2h\big((\partial_\nu\varphi_c)(Au)_\perp \,\big|\, u_\perp\big)_{\partial M} \\ &= 2h\big((\partial_\nu\varphi_c)ti_N Au \,\big|\, ti_N u\big)_{\partial M} \end{aligned}$$

because of the boundary condition $tu = 0$. Now $A = -h^2\Delta - |d\varphi_c|^2$, so

$$2h\big((\partial_\nu\varphi_c)ti_N Au \,\big|\, ti_N u\big)_{\partial M} = -2h\big((\partial_\nu\varphi_c)h^2 ti_N\Delta u \,\big|\, ti_N u\big)_{\partial M} - 2h\big((\partial_\nu\varphi_c)|d\varphi_c|^2 ti_N u \,\big|\, ti_N u\big)_{\partial M}.$$

Using the Weitzenböck identity, we can write $-2h\big((\partial_\nu\varphi_c)h^2 t i_N \Delta u \mid t i_N u\big)_{\partial M}$ as

$$-2h\big((\partial_\nu\varphi_c)h^2 t i_N \tilde{\Delta} u \mid t i_N u\big)_{\partial M} + 2h\big((\partial_\nu\varphi_c)h^2 R t i_N u \mid t i_N u\big)_{\partial M}.$$

The second term is bounded by (4-10). For the first term, we can apply Lemma 3.6 to get

$$-2h\big((\partial_\nu\varphi_c)h^2 t\nabla_N\nabla_N i_N u \,|\, t i_N u\big)_{\partial M} - 2h\big((\partial_\nu\varphi_c)h^2 \tilde{\Delta}' t i_N u \,|\, t i_N u\big)_{\partial M} + h^3\big(tr(s^2) i_N u - S_2 i_N u \,|\, t i_N u\big)_{\partial M},$$

where $S_2\omega(X_1,\dots,X_{k-1}) := \sum_{l=1}^{k-1}\omega(\dots,s^2X_l,\dots)$. The last term is bounded again by (4-10) and we can integrate by parts in the $\tilde{\Delta}'$ part to get something bounded by (4-10) as well. Therefore

$$2h((\partial_\nu\varphi_c)Au \mid u)_{\partial M} = -2h\big((\partial_\nu\varphi_c)|d\varphi_c|^2 t i_N u \mid t i_N u\big)_{\partial M} - 2h\big((\partial_\nu\varphi_c)h^2 t\nabla_N\nabla_N i_N u \mid t i_N u\big)_{\partial M} + R_4,$$

where R_4 is bounded by (4-10).

Now putting this together with (4-13), (4-15), and (4-16), we get that the boundary terms in (4-3) have the form

$$\begin{aligned}-2h\big(\partial_\nu\varphi|\partial_\nu\varphi|^2 t i_N u \mid t i_N u\big)_{\partial M} - 2h\big(\partial_\nu\varphi_c t h\nabla_N u_{\parallel} \mid t h\nabla_N u_{\parallel}\big)_{\partial M} + 2h^3\big(\partial_\nu\varphi_c t\nabla_N\nabla_N i_N u \mid t i_\nu u\big)_{\partial M}\\ -2h\big((\partial_\nu\varphi_c)|d\varphi_c|^2 t i_N u \mid t i_N u\big)_{\partial M} - 2h\big((\partial_\nu\varphi_c)h^2 t\nabla_N\nabla_N i_N u \mid t i_N u\big)_{\partial M} + R.\end{aligned}$$

The $\pm 2h^3\big(\partial_\nu\varphi_c t\nabla_N\nabla_N i_N u \mid t i_\nu u\big)_{\partial M}$ terms cancel, leaving us with

$$-2h\big(\partial_\nu\varphi|\partial_\nu\varphi|^2 t i_N u \,|\, t i_N u\big)_{\partial M} - 2h\big(\partial_\nu\varphi_c t h\nabla_N u_{\parallel} \,|\, t h\nabla_N u_{\parallel}\big)_{\partial M} - 2h\big((\partial_\nu\varphi_c)|d\varphi_c|^2 t i_N u \,|\, t i_N u\big)_{\partial M} + R.$$

We can replace φ_c by φ and incorporate the error into R, without affecting the bound on R, to get

$$-2h\big(\partial_\nu\varphi|\partial_\nu\varphi|^2 t i_N u \mid t i_N u\big)_{\partial M} - 2h\big(\partial_\nu\varphi t h\nabla_N u_{\parallel} \mid t h\nabla_N u_{\parallel}\big)_{\partial M} - 2h\big(\partial_\nu\varphi|d\varphi|^2 t i_N u \mid t i_N u\big)_{\partial M} + R$$

and the proposition follows. □

5. The 0-form case

We will now prove Theorem 2.5 in the 0-form case. In the case where (M, g) is a domain in Euclidean space, Theorem 2.5 for 0-forms is the Carleman estimate given in [Chung 2015, Theorem 1.3]. In this section we will deal with the added complication of being on a CTA manifold, rather than in Euclidean space. Most of the ideas are from [Chung 2015] with necessary modifications added to adapt to the manifold case.

If u is a zero form, then $i_N u = 0$, so $u_\perp = 0$ and $u = u_\parallel$. Theorem 2.5 reduces to the estimate

$$\|(-\Delta_\varphi + h^2 Q)u\|_{L^2(M)} \gtrsim h\|u\|_{H^1(M)} + h^{\frac{1}{2}}\|u_\parallel\|_{H^1(\Gamma_+^c)}, \tag{5-1}$$

where $Q \in L^\infty(M)$ and $0 < h < h_0$, for functions $u \in H^2(M)$ with $u|_{\Gamma_+} = 0$ to first order and $h\partial_\nu(e^{-\frac{\varphi}{h}}u) = h\sigma e^{-\frac{\varphi}{h}}u$ on Γ_+^c. By arguing as in the beginning of Section 6 below, the estimate (5-1) will be a consequence of the following proposition.

Proposition 5.1. *Suppose u is a function in $H^2(M)$ which satisfies the following boundary conditions*:

$$\begin{aligned} u, \partial_\nu u &= 0 && \text{on } \Gamma_+, \\ h\partial_\nu(e^{-\frac{\varphi}{h}}u) &= h\sigma e^{-\frac{\varphi}{h}}u && \text{on } \Gamma_+^c \end{aligned} \tag{5-2}$$

for some smooth function σ independent of h.

Then

$$h^{\frac{1}{2}}\|h\nabla' u\|_{L^2(\Gamma_+^c)} \lesssim \|\Delta_{\varphi_c} u\|_{L^2(M)} + h\|u\|_{H^1(M)} + h^{\frac{3}{2}}\|u\|_{L^2(\Gamma_+^c)}.$$

We will prove this proposition in the case where the metric g has the form $g = e \oplus g_0$. However, if g were of the form $g = c(e \oplus g_0)$, we could write

$$\begin{aligned} \|\Delta_{\varphi_c} u\|_{L^2(M)} &= \|h^2 e^{\frac{\varphi_c}{h}} \triangle_{c(e\oplus g_0)} e^{-\frac{\varphi_c}{h}} u\|_{L^2(M)} \\ &\gtrsim \|h^2 e^{\frac{\varphi_c}{h}} \triangle_{e\oplus g_0} e^{-\frac{\varphi_c}{h}} u\|_{L^2(M)} - h\|u\|_{H^1(M)}. \end{aligned} \tag{5-3}$$

Therefore the proposition remains true even in the case when the conformal factor is not constant. More generally, the proofs of the Carleman estimates work for any smooth conformal factor, and thus as noted earlier, the Carleman estimates hold on CTA manifolds in general.

The operators. Here we introduce the operators we will use in the proof of Proposition 5.1. Similar operators are found in [Chung 2014; 2015]. Suppose $F(\xi)$ is a complex-valued function on $\mathbb{R}^{n-1}$, with the properties that $|F(\xi)|, \operatorname{Re} F(\xi) \simeq 1 + |\xi|$. Fix coordinates (x_1, x') on $\mathbb{R}^n$, and define $\mathbb{R}^n_+$ to be the subset of $\mathbb{R}^n$ with $x_1 > 0$. Define $\mathcal{S}(\mathbb{R}^n_+)$ as the set of restrictions to $\mathbb{R}^n_+$ of Schwartz functions on $\mathbb{R}^n$. Finally, if $u \in \mathcal{S}(\mathbb{R}^n_+)$, then define $\hat{u}(x_1, \xi)$ to be the semiclassical Fourier transform of u in the x' variables only.

Now for $u \in \mathcal{S}(\mathbb{R}^n_+)$, define J by

$$\widehat{Ju}(x_1, \xi) = (F(\xi) + h\partial_1)\hat{u}(x_1, \xi).$$

This has adjoint J^* defined by

$$\widehat{J^*u}(x_1, \xi) = (\bar{F}(\xi) - h\partial_1)\hat{u}(x_1, \xi).$$

These operators have right inverses given by

$$\begin{aligned} \widehat{J^{-1}u} &= \frac{1}{h}\int_0^{x_1} \hat{u}(t, \xi) e^{F(\xi)\frac{t - x_1}{h}}\, dt, \\ \widehat{J^{*-1}u} &= \frac{1}{h}\int_{x_1}^{\infty} \hat{u}(t, \xi) e^{\bar{F}(\xi)\frac{x_1 - t}{h}}\, dt. \end{aligned}$$

Now we have the following boundedness result, given in [Chung 2015].

Lemma 5.2. *The operators J, J^*, J^{-1}, and J^{*-1}, initially defined on $\mathcal{S}(\mathbb{R}^n_+)$, extend to bounded operators*

$$\begin{aligned} J, J^* &: H^1(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n_+), \\ J^{-1}, J^{*-1} &: L^2(\mathbb{R}^n_+) \to H^1(\mathbb{R}^n_+). \end{aligned}$$

Moreover, *these extensions for J^* and J^{*-1} are isomorphisms.*

Note that similar mapping properties hold between $H^1(\mathbb{R}^n_+)$ and $H^2(\mathbb{R}^n_+)$, by the same reasoning.

We'll record the other operator fact from [Chung 2015] here, too.

Let $m, k \in \mathbb{Z}$, with $m, k \geq 0$. Suppose $a(x, \xi, y)$ are smooth functions on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$ that satisfy the bounds

$$|\partial_x^\beta \partial_\xi^\alpha \partial_y^j a(x, \xi, y)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}$$

for all multiindices α and β, and for $0 \leq j \leq k$. In other words, each $\partial_y^j a(x, \xi, y)$ is a symbol on $\mathbb{R}^{n-1}$ of order m, with bounds uniform in y, for $0 \leq j \leq k$. Then we can define an operator A on Schwartz functions in $\mathbb{R}^n$ by applying the pseudodifferential operator on $\mathbb{R}^{n-1}$ with symbol $a(x, \xi, y)$, defined by the Kohn–Nirenberg quantization, to $f(x, y)$ for each fixed y.

Lemma 5.3. *If A is as above, then A extends to a bounded operator from $H^{k+m}(\mathbb{R}^n)$ to $H^k(\mathbb{R}^n)$.*

The graph case. Suppose $f : M_0 \to \mathbb{R}$ is smooth. In this section, we'll examine the case where M lies in the set $\{x_1 \geq f(x')\}$ and Γ_+^c lies in the graph $\{x_1 = f(x')\}$. For this section we'll make two additional assumptions on f and M_0.

First, we'll assume g_0 is nearly constant; that is, there exists a choice of coordinates on the subset $P(M)$ which consists of the projection of M onto M_0 such that when represented in these coordinates,

$$|g_0 - I| \leq \delta$$

on $P(M)$, where δ is a positive constant to be chosen later.

Second, we'll assume f is such that $\nabla_{g_0} f$ is nearly constant on $P(M)$; that is, there exists a constant vector field K on TM_0 such that

$$|\nabla_{g_0} f - K|_{g_0} \leq \delta,$$

where δ is the same constant from above. The choice of δ will depend ultimately only on K. In the next subsection we'll see how to remove these two assumptions.

Now we can do the change of variables $(x_1, x') \mapsto (x_1 - f(x'), x')$. Define $\widetilde{M}'$ and $\widetilde{\Gamma}'_+$ to be the images of M and Γ_+ respectively under this map. Note that $\{x_1 \geq f(x')\}$ maps to $(0, \infty) \times M_0$, and Γ_+^c maps to a subset of $0 \times M_0$. Observe that in the new coordinates, $\varphi(x) = x_1 + f(x')$.

Now it suffices to prove the following proposition.

Proposition 5.4. *Suppose $w \in H^2(\widetilde{M}')$, and*

$$\begin{aligned} w, \partial_\nu w = 0 \quad &\text{on } \widetilde{\Gamma}'_+, \\ h\partial_y w|_{\widetilde{\Gamma}'^c_+} &= \frac{w + \nabla_{g_0} f \cdot h\nabla_{g_0} w - h\sigma w}{1 + |\nabla_{g_0} f|^2}, \end{aligned} \tag{5-4}$$

where σ is smooth and bounded on $\widetilde{M}'$. Then

$$h^{\frac{1}{2}} \|h\nabla_{g_0} w\|_{L^2(\widetilde{\Gamma}'^c_+)} \lesssim \|\widetilde{\mathcal{L}}'_{\varphi,\varepsilon} w\|_{L^2(\widetilde{M}')} + h\|w\|_{H^1(\widetilde{M}')} + h^{\frac{3}{2}} \|w\|_{L^2(\widetilde{\Gamma}'^c_+)},$$

where

$$\widetilde{\mathcal{L}}'_{\varphi,\varepsilon} = (1 + |\nabla_{g_0} f|^2) h^2 \partial_1^2 - 2(\alpha + \nabla_{g_0} f \cdot h\nabla_{g_0}) h\partial_1 + \alpha^2 + h^2 \triangle_{g_0}$$

and $\alpha = 1 + \frac{h}{\varepsilon}(x_1 + f(x'))$. Note that on $\widetilde{M}'$, we know α is very close to 1.

This proposition implies Proposition 5.1 in the graph case described above.

Proof of Proposition 5.1 in the graph case. Suppose $u \in H^2(M)$, and u satisfies (5-2). Let w be the function on $\widetilde{M}$ defined by $w(x_1, x') = u(x_1 + f(x'), x')$. Then $w \in H^2(\widetilde{M}')$, and w satisfies (5-4). Therefore by Proposition 5.4,

$$h^{\frac{1}{2}}\|h\nabla' w\|_{L^2(\widetilde{\Gamma}'^c_+)} \lesssim \|\widetilde{\mathcal{L}}'_{\varphi,\varepsilon} w\|_{L^2(\widetilde{M}')} + h\|w\|_{H^1(\widetilde{M}')} + h^{\frac{3}{2}}\|w\|_{L^2(\widetilde{\Gamma}'^c_+)}.$$

Now by a change of variables,

$$\|u\|_{L^2(\Gamma^c_+)} \simeq \|w\|_{L^2(\widetilde{\Gamma}'^c_+)},$$

$$\|u\|_{H^1(M)} \simeq \|w\|_{H^1(\widetilde{M}')},$$

and

$$\|h\nabla' u\|_{L^2(\Gamma^c_+)} \simeq \|h\nabla_{g_0} w\|_{L^2(\widetilde{\Gamma}'^c_+)}.$$

Moreover,

$$(\widetilde{\mathcal{L}}'_{\varphi,\varepsilon} w)(x_1 - f(x'), x') = \mathcal{L}_{\varphi,\varepsilon}(u(x_1, x')) + hE_1 u(x_1, x'),$$

where E_1 is a first-order semiclassical differential operator. Therefore by a change of variables,

$$\|\widetilde{\mathcal{L}}'_{\varphi,\varepsilon} w\|_{L^2(\widetilde{M}')} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} u\|_{L^2(M)} + h\|u\|_{H^1(M)}.$$

Putting this all together gives

$$h^{\frac{1}{2}}\|h\nabla_{g_0} u\|_{L^2(\Gamma^c_+)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} u\|_{L^2(M)} + h\|u\|_{H^1(M)} + h^{\frac{3}{2}}\|u\|_{L^2(\Gamma^c_+)}. \qquad \square$$

We can do a second change of variables to move to Euclidean space. By our assumption on M_0, we can choose coordinates on $P(\widetilde{M}') = P(M)$ such that

$$|g_0 - I| \le \delta.$$

Now we have a change of variables giving a map from $P(\widetilde{M}')$ to a subset of $\mathbb{R}^{n-1}$, and hence a map from $\widetilde{M}'$ to a subset of $\mathbb{R}^n_+$, where the image of $\widetilde{\Gamma}'_+$ lies in the plane $x_1 = 0$. Let $\widetilde{M}$ and $\widetilde{\Gamma}_+$ be the images of $\widetilde{M}'$ and $\widetilde{\Gamma}'_+$ respectively under this map. We'll use the notation (x_1, x') to describe points in $\mathbb{R}^n_+$, where now x' ranges over $\mathbb{R}^{n-1}$. Now it suffices to prove the following proposition.

Proposition 5.5. *Suppose $w \in H^2(\widetilde{M})$, and*

$$\begin{aligned} w, \partial_\nu w = 0 \quad &\text{on } \widetilde{\Gamma}_+, \\ h\partial_y w|_{\widetilde{\Gamma}^c_+} &= \frac{w + \beta \cdot h\nabla_x w - h\sigma w}{1 + |\gamma|^2}, \end{aligned} \tag{5-5}$$

where σ is smooth and bounded on $\widetilde{M}$, and β and γ are a vector-valued and scalar-valued function, respectively, which coincide with the coordinate representations of $\nabla_{g_0} f$ and $|\nabla_{g_0} f|_{g_0}$. Then

$$h^{\frac{1}{2}}\|h\nabla_{x'} w\|_{L^2(\widetilde{\Gamma}^c_+)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\widetilde{M})} + h\|w\|_{H^1(\widetilde{M})} + h^{\frac{3}{2}}\|w\|_{L^2(\widetilde{\Gamma}^c_+)},$$

where

$$\widetilde{\mathcal{L}}_{\varphi,\varepsilon} = (1 + |\gamma|^2)h^2\partial_1^2 - 2(\alpha + \beta \cdot h\nabla_x)h\partial_1 + \alpha^2 + h^2\mathcal{L},$$

and $\mathcal{L}$ is the second-order differential operator in the x'-variables given by

$$\mathcal{L} = g_0^{ij}\partial_i\partial_j.$$

Proposition 5.4 can be obtained from Proposition 5.5 in the same manner as before, with errors from the change of variables being absorbed into the appropriate terms. Therefore it suffices to prove Proposition 5.5.

To do this, we'll split w into small and large frequency parts, using a Fourier transform. Recall that we are assuming

$$|\nabla_{g_0} f - K|_{g_0} \le \delta.$$

Translating down to $\widetilde{M}$, and recalling that g_0 is nearly the identity, we get that there is a constant vector field $\widetilde{K}$ on $\widetilde{M}$ such that

$$|\beta - \widetilde{K}| \le C_\delta \quad \text{and} \quad \big|\gamma - |\widetilde{K}|\big| \le C_\delta,$$

where C_δ goes to zero as δ goes to zero. Now choose $m_2 > m_1 > 0$ and μ_1 and μ_2 such that

$$\frac{|\widetilde{K}|}{\sqrt{1+|\widetilde{K}|^2}} < \mu_1 < \mu_2 < \frac{1}{2} + \frac{|\widetilde{K}|}{2\sqrt{1+|\widetilde{K}|^2}} < 1.$$

The eventual choice of μ_j and m_j will depend only on $\widetilde{K}$.

Define $\rho \in C_0^\infty(\mathbb{R}^n)$ such that $\rho(\xi) = 1$ if $|\xi| < \mu_1$ and $|\widetilde{K}\cdot\xi| < m_1$, and $\rho(\xi) = 0$ if $|\xi| > \mu_2$ or $|\widetilde{K}\cdot\xi| > m_2$.

Now suppose $w \in C^\infty(\widetilde{M})$ such that $w \equiv 0$ in a neighbourhood of $\widetilde{\Gamma}_+$, and w satisfies (5-5). We can extend w by zero to the rest of $\mathbb{R}^n_+$. Then $w \in \mathcal{S}(\mathbb{R}^n_+)$, and we can write our desired estimate as

$$h^{\frac{1}{2}}\|w\|_{\dot{H}^1(\partial\mathbb{R}^n_+)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^n_+)} + h^{\frac{3}{2}}\|w\|_{L^2(\partial\mathbb{R}^n_+)}.$$

Recall that $\hat{w}(x_1,\xi)$ is the semiclassical Fourier transform of w in the x'-directions, and define w_s and w_ℓ by $\hat{w}_s = \rho\hat{w}$ and $\hat{w}_\ell = (1-\rho)\hat{w}$, so $w = w_s + w_\ell$.

Now we can address each of these parts separately.

Proposition 5.6. *Suppose w is as above. There exist choices of m_1, m_2, μ_1, and μ_2, depending only on $\widetilde{K}$, such that if δ is small enough,*

$$h^{\frac{1}{2}}\|w_s\|_{\dot{H}^1(\partial\mathbb{R}^n_+)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^n_+)} + h^{\frac{3}{2}}\|w\|_{L^2(\partial\mathbb{R}^n_+)}.$$

Before proceeding to the proof, let's make some definitions. If $V \in \mathbb{R}^{n-1}$ and $a \in \mathbb{R}$, define $A_\pm(a,V,\xi)$ by

$$A_\pm(a,V,\xi) = \frac{1 + iV\cdot\xi \pm \sqrt{(1+iV\cdot\xi)^2 - (1+|a|^2)(1-|\xi|^2)}}{1+|a|^2}.$$

In other words, $A_\pm(a,V,\xi)$ are defined to be the roots of the polynomial

$$(1+|a|^2)X^2 - 2(1+iV\cdot\xi)X + (1-|\xi|^2).$$

In the definition, we'll choose the branch of the square root which has nonnegative real part, so the branch cut occurs on the negative real axis.

Proof. Now consider the behaviour of $A_\pm(|\widetilde{K}|, \widetilde{K}, \xi)$ on the support of ρ, or equivalently, on the support of $\hat{w}_s$. If $\eta > 0$, we can choose μ_2 such that on the support of $\hat{w}_s$,

$$1 - (1 + |\widetilde{K}|^2)(1 - |\xi|^2) < \eta.$$

Then on the support of $\hat{w}_s$, the expression

$$(1 + i\widetilde{K} \cdot \xi)^2 - (1 + |\widetilde{K}|^2)(1 - |\xi|^2)$$

has real part confined to the interval $[-\widetilde{K}^2 - m_2^2, \eta + m_2^2]$, and imaginary part confined to the interval $[-2m_2, 2m_2]$. Therefore, by correct choice of η and m_2, we can ensure

$$\mathrm{Re} A_\pm(|\widetilde{K}|, \widetilde{K}, \xi) > \frac{1}{2(1 + |\widetilde{K}|^2)}$$

on the support of $\hat{w}_s$. This allows us to fix the choice of μ_1, μ_2, m_1, and m_2. Note that the choices depend only on $\widetilde{K}$, as promised.

The bounds on $A_\pm(|\widetilde{K}|, \widetilde{K}, \xi)$ allow us to choose $F_\pm$ so that $F_\pm = A_\pm(|\widetilde{K}|, \widetilde{K}, \xi)$ on the support of $\hat{w}_s$, and $\mathrm{Re} F_\pm, |F_\pm| \simeq 1 + |\xi|$ on $\mathbb{R}^n$, with constant depending only on K. Therefore F_+ and F_- both satisfy the conditions on F in Section 2. If T_ψ represents the operator with Fourier multiplier ψ (in the x'-variables), then it follows that the operators $h\partial_y - T_{F_+}$ and $h\partial_y - T_{F_-}$ both have the properties of J^* in that section.

Up until now, the operator $\widetilde{\mathcal{L}}_{\varphi,\varepsilon}$ has only been applied to functions supported in $\widetilde{M}$. However, we can extend the coefficients of $\widetilde{\mathcal{L}}_{\varphi,\varepsilon}$ to $\mathbb{R}^n_+$ while retaining the $|\beta - \widetilde{K}| < C_\delta$ and $\big|\gamma - |\widetilde{K}|\big| \le C_\delta$ conditions. Then

$$\begin{aligned}\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} &= \big\|\big((1+|\gamma|^2)h^2\partial_y^2 - 2(\alpha + \beta \cdot h\nabla_x)h\partial_y + \alpha^2 + h^2\mathcal{L}\big)w_s\big\|_{L^2(\mathbb{R}^n_+)} \\ &\ge \big\|\big((1+|\widetilde{K}|^2)h^2\partial_y^2 - 2(1 + \widetilde{K} \cdot h\nabla_x)h\partial_y + 1 + h^2\triangle_{x'}\big)w_s\big\|_{L^2(\mathbb{R}^n_+)} - C_\delta\|w_s\|_{H^2(\mathbb{R}^n_+)}\end{aligned}$$

for sufficiently small h and some C_δ which goes to zero as δ goes to zero. Meanwhile,

$$(1 + |\widetilde{K}|^2)(h\partial_y - T_{F_+})(h\partial_y - T_{F_-})w_s = (1 + |\widetilde{K}|^2)(h^2\partial_y^2 - T_{F_+ + F_-}h\partial_y + T_{F_+F_-})w_s.$$

Since $F_\pm = A_\pm(\widetilde{K}, K, \xi)$ on the support of $\hat{w}_s$, this can be written as

$$(1 + |\widetilde{K}|^2)(h^2\partial_y^2 - T_{A_+ + A_-}h\partial_y + T_{A_+A_-})w_s = ((1 + |\widetilde{K}|^2)h^2\partial_y^2 - 2(1 + \widetilde{K} \cdot h\nabla_x)h\partial_y + 1 + h^2\triangle_x)w_s.$$

Therefore

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} \ge \big\|(h\partial_y - T_{F_+})(h\partial_y - T_{F_-})w_s\big\|_{L^2(\mathbb{R}^n_+)} - C_\delta\|w_s\|_{H^2(\mathbb{R}^n_+)}.$$

Now by the boundedness properties,

$$\big\|(h\partial_y - T_{F_+})(h\partial_y - T_{F_-})w_s\big\|_{L^2(\mathbb{R}^n_+)} \simeq \|w_s\|_{H^2(\mathbb{R}^n_+)},$$

so for small enough δ,

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} \gtrsim \|w_s\|_{H^2(\mathbb{R}^n_+)}.$$

Then by the semiclassical trace formula,

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} \gtrsim h^{\frac{1}{2}} \|w_s\|_{\dot{H}^1(\partial\mathbb{R}^n_+)}.$$

Finally, note that

$$\begin{aligned}\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} &= \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} T_\rho w\|_{L^2(\mathbb{R}^n_+)} \\ &\lesssim \|(1+|\gamma|^2)^{-1} \widetilde{\mathcal{L}}_{\varphi,\varepsilon} T_\rho w\|_{L^2(\mathbb{R}^n_+)} \\ &\lesssim \|T_\rho (1+|\gamma|^2)^{-1} \widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + \|h E_1 w\|_{L^2(\mathbb{R}^n_+)},\end{aligned}$$

where hE_1 comes from the commutator of T_ρ and $(1+|\gamma|^2)^{-1}\widetilde{\mathcal{L}}_{\varphi,\varepsilon}$. By Lemma 5.3, E_1 is bounded from $H^1(\mathbb{R}^n_+)$ to $L^2(\mathbb{R}^n_+)$, so

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^n_+)}.$$

Therefore

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^n_+)} \gtrsim h^{\frac{1}{2}} \|w_s\|_{\dot{H}^1(\partial\mathbb{R}^n_+)}$$

as desired. □

Now we have to deal with the large frequency term.

Proposition 5.7. *Suppose w is the extension by zero to $\mathbb{R}^n_+$ of a function in $C^\infty(\widetilde{M})$ which is 0 in a neighbourhood of $\widetilde{\Gamma}_+$, and satisfies* (5-5), *and let w_ℓ be defined as above. Then if δ is small enough,*

$$h^{\frac{1}{2}} \|w_\ell\|_{\dot{H}^1(\partial\mathbb{R}^n_+)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^{n+1}_+)} + h^{\frac{3}{2}} \|w\|_{L^2(\partial\mathbb{R}^n_+)}.$$

Proof. Suppose $V \in \mathbb{R}^n$. Recall that we defined

$$A_\pm(a, V, \xi) = \frac{1 + iV\cdot\xi \pm \sqrt{(1+iV\cdot\xi)^2 - (1+|a|^2)(1-|\xi|^2)}}{1+|a|^2},$$

so $A_\pm(a, V, \xi)$ are roots of the polynomial

$$(1+|a|^2)X^2 - 2(1+iV\cdot\xi)X + (1-|\xi|^2).$$

Now let's define

$$A^\varepsilon_\pm(a, V, \xi) = \frac{\alpha + iV\cdot\xi \pm \sqrt{(\alpha+iV\cdot\xi)^2 - (1+|a|^2)(\alpha^2 - g_0^{ij}\xi_i\xi_j)}}{1+|a|^2},$$

so $A^\varepsilon_\pm(V,\xi)$ are the roots of the polynomial

$$(1+|a|^2)X^2 - 2(\alpha+iV\cdot\xi)X + (\alpha^2 - g_0^{ij}\xi_i\xi_j).$$

(Recall that α is defined by $\alpha = 1 + \frac{h}{\varepsilon}(x_1 + f(x')).$) Again we'll use the branch of the square root with nonnegative real part.

Now set $\zeta \in C_0^\infty(\mathbb{R}^{n-1})$ to be a smooth cutoff function such that $\zeta = 1$ if

$$|\widetilde{K} \cdot \xi| < \frac{1}{2} m_1 \quad \text{and} \quad |\xi| < \frac{1}{2} \frac{|\widetilde{K}|}{\sqrt{1 + |\widetilde{K}|^2}} + \frac{1}{2}\mu_1,$$

and $\zeta = 0$ if $|\widetilde{K} \cdot \xi| \geq m_1$ or $|\xi| \geq \mu_1$.

Now define

$$G_\pm(a, V, \xi) = (1 - \zeta) A_\pm(a, V, \xi) + \zeta$$

and

$$G_\pm^\varepsilon(a, V, \xi) = (1 - \zeta) A_\pm^\varepsilon(a, V, \xi) + \zeta.$$

Consider the singular support of $A_\pm^\varepsilon(\gamma, \beta, \xi)$. These are smooth as functions of x and ξ except when the argument of the square root falls on the nonpositive real axis. This occurs when $\beta \cdot \xi = 0$ and

$$g_0^{ij} \xi_i \xi_j \leq \frac{\alpha^2 |\gamma|^2}{1 + |\gamma|^2}.$$

Now for δ sufficiently small, depending on $\widetilde{K}$, this does not occur on the support of $1 - \zeta$. Therefore

$$G_\pm^\varepsilon(\gamma, \beta, \xi) = (1 - \zeta) A_\pm^\varepsilon(\gamma, \beta, \xi) + \zeta$$

are smooth, and one can check that they are symbols of first order on $\mathbb{R}^n$.

Then by properties of pseudodifferential operators,

$$\begin{aligned}(1 + |\gamma|^2)&(h\partial_y - T_{G_+^\varepsilon(\gamma,\beta,\xi)})(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)}) \\ &= (1 + |\gamma|^2)\big(h^2\partial_y^2 - T_{G_+^\varepsilon(\gamma,\beta,\xi) + G_-^\varepsilon(\gamma,\beta,\xi)} h\partial_y + T_{G_+^\varepsilon(\gamma,\beta,\xi) G_-^\varepsilon(\gamma,\beta,\xi)}\big) + hE_1,\end{aligned}$$

where E_1 is bounded from $H^1(\mathbb{R}_+^{n+1})$ to $L^2(\mathbb{R}_+^{n+1})$. This last line can be written out as

$$(1 + |\gamma|^2) h^2 \partial_y^2 - 2(\alpha + \beta \cdot h\nabla_x) h\partial_y T_{1-\zeta} T_{1+\zeta} + (\alpha + h^2 \mathcal{L}) T_{(1-\zeta)^2} + hE_1 + T_{\zeta^2} - 2h\partial_y T_\zeta$$

by modifying E_1 as necessary. Now $T_\zeta w_\ell = 0$, so

$$(1 + |\gamma|^2)(h\partial_y - T_{G_+^\varepsilon(\gamma,\beta,\xi)})(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)}) w_\ell = \widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_\ell - hE_1 w_\ell.$$

Therefore

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_\ell\|_{L^2(\mathbb{R}_+^{n+1})} \gtrsim \big\|(h\partial_y - T_{G_+^\varepsilon(\gamma,\beta,\xi)})(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)}) w_\ell\big\|_{L^2(\mathbb{R}_+^{n+1})} - h\|w_\ell\|_{H^1(\mathbb{R}_+^{n+1})}.$$

Now

$$G_+^\varepsilon(\gamma, \beta, \xi) = G_+(|\widetilde{K}|, \widetilde{K}, \xi) + \big(G_+^\varepsilon(\gamma, \beta, \xi) - G_+(|\widetilde{K}|, \widetilde{K}, \xi)\big),$$

and

$$T_{G_+^\varepsilon(\gamma,\beta,\xi) - G_+(|\widetilde{K}|, \widetilde{K}, \xi)}$$

involves multiplication by functions bounded by $O(\delta)$, so

$$\|T_{G_+^\varepsilon(\gamma,\beta,\xi)-G_+(|\widetilde{K}|,\widetilde{K},\xi)}v\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|v\|_{H^1(\mathbb{R}^{n+1}_+)}.$$

Therefore

$$\begin{aligned}\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}^{n+1}_+)} \gtrsim \big\|(h\partial_y - T_{G_+(|\widetilde{K}|,\widetilde{K},\xi)})(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\big\|_{L^2(\mathbb{R}^{n+1}_+)}\\ - h\|w_\ell\|_{H^1(\mathbb{R}^{n+1}_+)} - \delta\|(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{H^1(\mathbb{R}^{n+1}_+)}.\end{aligned}$$

Now we can check that $G_+(|\widetilde{K}|,\widetilde{K},\xi)$ satisfies the necessary properties of F from this section, so

$$\begin{aligned}\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}^{n+1}_+)} \gtrsim \|(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{H^1(\mathbb{R}^{n+1}_+)} - h\|w_\ell\|_{H^1(\mathbb{R}^{n+1}_+)}\\ - \delta\|(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{H^1(\mathbb{R}^{n+1}_+)}.\end{aligned}$$

Then for small enough δ,

$$\begin{aligned}\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}^{n+1}_+)} &\gtrsim \|(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{H^1(\mathbb{R}^{n+1}_+)} - h\|w_\ell\|_{H^1(\mathbb{R}^{n+1}_+)}\\ &\gtrsim h^{\frac12}\|(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{L^2(\mathbb{R}^n_0)} - h\|w_\ell\|_{H^1(\mathbb{R}^{n+1}_+)}.\end{aligned}$$

Now by (5-5),

$$h\partial_y w = \frac{w + \beta\cdot h\nabla_x w + h\sigma w}{1+|\gamma|^2}$$

on $\partial\mathbb{R}^n_+$, so

$$h\partial_y w_\ell = \frac{w_\ell + \beta\cdot\nabla_x w_\ell}{1+|\gamma|^2} + hE_0 w$$

on $\partial\mathbb{R}^n_+$, where E_0 is bounded from $L^2(\mathbb{R}^{n-1})$ to $L^2(\mathbb{R}^{n-1})$. Therefore

$$\begin{aligned}\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}^n_+)} &\gtrsim h^{\frac12}\left\|\frac{w_\ell + \beta\cdot\nabla_x w_\ell}{1+|\gamma|^2} - T_{G_-^\varepsilon(\gamma,\beta,\xi)}w_\ell\right\|_{L^2(\partial\mathbb{R}^n_+)} - h\|w_\ell\|_{H^1(\mathbb{R}^n_+)} - h^{\frac32}\|w\|_{L^2(\partial\mathbb{R}^n)}\\ &\gtrsim h^{\frac12}\|w_\ell\|_{\dot H^1(\partial\mathbb{R}^n_+)} - h\|w_\ell\|_{H^1(\mathbb{R}^n_+)} - h^{\frac32}\|w\|_{L^2(\partial\mathbb{R}^n_+)}.\end{aligned}$$

Now

$$\|w_\ell\|_{H^1(\mathbb{R}^n_+)} \lesssim \|w\|_{H^1(\mathbb{R}^n_+)}$$

and

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}^n_+)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon}w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^n_+)}.$$

Therefore

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon}w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^n_+)} + h^{\frac32}\|w\|_{L^2(\partial\mathbb{R}^n_+)} \gtrsim h^{\frac12}\|w_\ell\|_{\dot H^1(\partial\mathbb{R}^n_+)}$$

as desired. □

Now combing the results of Propositions 5.6 and 5.7 gives

$$h^{\frac12}\|w_\ell\|_{\dot H^1(\partial\mathbb{R}^n_+)} + h^{\frac12}\|w_s\|_{\dot H^1(\partial\mathbb{R}^n_+)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon}w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^{n+1}_+)} + h^{\frac32}\|w\|_{L^2(\partial\mathbb{R}^n_+)}.$$

Since $w = w_s + w_\ell$, we get

$$h^{\frac{1}{2}}\|w\|_{\dot{H}^1(\partial\mathbb{R}^n_+)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^{n+1}_+)} + h^{\frac{3}{2}}\|w\|_{L^2(\partial\mathbb{R}^n_+)}$$

for $w \in C^\infty(\widetilde{M})$ such that $w \equiv 0$ in a neighbourhood of $\widetilde{\Gamma}_+$, and w satisfies (5-5). A density argument now proves Proposition 5.5, and hence Proposition 5.1, at least under the assumptions on g_0 and f made at the beginning of this section.

Finishing the proof. Now we need to remove the graph conditions on Γ^c_+, and the conditions on the metric g_0. Since Γ_+ is a neighbourhood of ∂M_+, in a small enough neighbourhood U around any point p on Γ^c_+, we know Γ^c_+ coincides locally with a subset of a graph of the form $x_1 = f(x')$, with $M \cap U$ lying in the set $x_1 > f(x')$. Moreover, for any $\delta > 0$, if $\nabla_{g_0} f(p) = K$, then in some small neighbourhood of p, we have $|\nabla_{g_0} f - K|_{g_0} < \delta$. Additionally, since we can choose coordinates at p such that $g_0 = I$ in those coordinates, for any $\delta > 0$ we can ensure that there are coordinates such that $|g_0 - I| \leq \delta$ in a small neighbourhood of p. We can choose δ to be small enough for Proposition 5.1 to hold, by the proof in the previous subsection.

Now we can let U_j be open sets in M such that $\{U_1, \dots, U_m\}$ is a finite open cover of M such that each $M \cap U_j$ has smooth boundary, and each $\Gamma^c_+ \cap U_j$ is represented as a graph of the form $x_1 = f_j(x')$, with $|\nabla_{g_0} f_j - K_j|_{g_0} < \delta_j$, and there is a choice of coordinates on the projection of $M \cap U_j$ in which $|g_0 - I| \leq \delta_j$, where δ_j are small enough for

$$h^{\frac{1}{2}}\|h\nabla_t v_j\|_{L^2(\Gamma^c_+\cap U_j)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} v_j\|_{L^2(M\cap U_j)} + h\|v_j\|_{H^1(M\cap U_j)} + h^{\frac{3}{2}}\|v_j\|_{L^2(\Gamma^c_+\cap U_j)}$$

to hold for all $v_j \in H^2(M \cap U_j)$ such that

$$\begin{aligned} v_j, \partial_\nu v_j &= 0 && \text{on } \partial(U_j \cap M) \setminus \Gamma^c_+, \\ h\partial_\nu(e^{-\frac{\varphi}{h}} v_j) &= h\sigma e^{-\frac{\varphi}{h}} v_j && \text{on } \Gamma^c_+ \cap U_j. \end{aligned} \tag{5-6}$$

Without loss of generality we may assume each U_j is compactly contained in $U^0_j \times (0,1)$, where U^0_j is a coordinate chart of M_0.

Now let $\chi_1, \dots, \chi_m$ be a partition of unity subordinate to $U_1, \dots, U_m$, and for $w \in H^2(M)$ satisfying (5-2), define $w_j = \chi_j w$. Then if $\Gamma^c_+ \cap U_j \neq \varnothing$, we know w_j satisfies (5-6) for some σ, and so

$$h^{\frac{1}{2}}\|h\nabla_t w_j\|_{L^2(\Gamma^c_+\cap U_j)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{L^2(M\cap U_j)} + h\|w_j\|_{H^1(M\cap U_j)} + h^{\frac{3}{2}}\|w_j\|_{L^2(\Gamma^c_+\cap U_j)}.$$

Adding together these estimates gives

$$h^{\frac{1}{2}}\|h\nabla_t w\|_{L^2(\Gamma^c_+)} \lesssim \sum_{j=1}^m \|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{L^2(M)} + h\|w\|_{H^1(M)} + h^{\frac{3}{2}}\|w\|_{L^2(\Gamma^c_+)}.$$

Each $\|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{L^2(M)} = \|\mathcal{L}_{\varphi,\varepsilon}\chi_j w\|_{L^2(M)}$ is bounded by a constant times $\|\mathcal{L}_{\varphi,\varepsilon} w\|_{L^2(M)} + h\|w\|_{H^1(M)}$, so

$$h^{\frac{1}{2}}\|h\nabla_t w\|_{L^2(\Gamma^c_+)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w\|_{L^2(M)} + h\|w\|_{H^1(M)} + h^{\frac{3}{2}}\|w\|_{L^2(\Gamma^c_+)}.$$

This finishes the proof of Proposition 5.1.

6. The k-form case

We will prove Theorem 2.5 for $u \in \Omega^k(M)$ by induction. If $k = 0$, then $i_N u = 0$, so $u_\perp = 0$ and $u = u_\parallel$. Then Theorem 2.5 for $k = 0$ becomes the Carleman estimate (5-1) that was established in Section 5.

Note that it suffices to prove Theorem 2.5 for $u \in \Omega^k(M)$, with the appropriate boundary conditions, for each k, and $Q = 0$. Then the final theorem follows by adding the resulting estimates and noting that the extra $h^2 Qu$ term on the right can be absorbed into the terms on the left for sufficiently small h.

Proof of Theorem 2.5 for k ≥ 1. Suppose $u \in \Omega^k(M)$ with $k \geq 1$. First note that if we impose the boundary conditions (2-2) of Theorem 2.5, substituting the result of Proposition 4.2 into (4-3) gives

$$\begin{aligned}\|\Delta_{\varphi_c} u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u \mid u) - 2h^3(\partial_\nu \varphi \nabla_N u_\perp \mid \nabla_N u_\perp)_{\Gamma_+^c} \\ - h\big(\partial_\nu \varphi(|d\varphi|^2 + |\partial_\nu \varphi|^2) u_\parallel \,\big|\, u_\parallel\big)_{\Gamma_+^c} + R, \quad (6\text{-}1)\end{aligned}$$

where

$$|R| \leq C\left(Kh^3 \|\nabla' t u_\parallel\|^2_{\Gamma_+^c} + \frac{h}{K}\|u_\parallel\|^2_{\Gamma_+^c} + \frac{h^3}{K}\|\nabla_N u_\perp\|^2_{\Gamma_+^c}\right).$$

Recall also from Proposition 4.2 that the nonboundary terms $\|Au\|^2 + \|Bu\|^2 + (i[A,B]u \mid u)$ satisfy

$$\|Au\|^2 + \|Bu\|^2 + (i[A,B]u \mid u) \gtrsim \frac{h^2}{\varepsilon}\|u\|^2_{H^1(M)} - \frac{h^3}{\varepsilon}\big(\|u_\parallel\|^2_{H^1(\partial M)} + \|h\nabla_N u_\perp\|^2_{L^2(\partial M)}\big) \quad (6\text{-}2)$$

for $h \ll \varepsilon \ll 1$. We now return to (6-1) and examine the boundary terms. On Γ_+^c, there exists $\varepsilon_1 > 0$ such that $\partial_\nu \varphi < -\varepsilon_1$. Using this together with (6-1) and (6-2) gives

$$\|\Delta_{\varphi_c} u\|^2 + Kh^3\|\nabla' t u_\parallel\|^2_{\Gamma_+^c} + \frac{h}{K}\|u_\parallel\|^2_{\Gamma_+^c} + \frac{h^3}{K}\|\nabla_N u_\perp\|^2_{\Gamma_+^c} \gtrsim \frac{h^2}{\varepsilon}\|u\|^2_{H^1(M)} + h^3\|\nabla_N u_\perp\|^2_{\Gamma_+^c} + h\|u_\parallel\|^2_{\Gamma_+^c}$$

for large enough K. The last two terms on the left side can be absorbed into the right side, giving

$$\|\Delta_{\varphi_c} u\|^2 + Kh^3\|\nabla' t u_\parallel\|^2_{\Gamma_+^c} \gtrsim \frac{h^2}{\varepsilon}\|u\|^2_{H^1(M)} + h^3\|\nabla_N u_\perp\|^2_{\Gamma_+^c} + h\|u_\parallel\|^2_{\Gamma_+^c}.$$

Now we want to analyze the boundary term on the left, and this is the part where we will use induction on k:

Lemma 6.1. *If $u \in \Omega^k(M)$ and u satisfies the boundary conditions* (2-2), *then*

$$h^3\|\nabla' t u_\parallel\|^2_{\Gamma_+^c} \lesssim \|\Delta_{\varphi_c} u\|^2 + h^2\|u\|^2_{H^1(M)} + h^2\|u_\parallel\|^2_{\Gamma_+^c}. \quad (6\text{-}3)$$

If (6-3) is granted, fix K sufficiently large and then take $h \ll \varepsilon \ll 1$ to obtain

$$\|\Delta_{\varphi_c} u\|^2 \gtrsim \frac{h^2}{\varepsilon}\|u\|^2_{H^1(M)} + h^3\|\nabla_N u_\perp\|^2_{\Gamma_+^c} + h\|u_\parallel\|^2_{\Gamma_+^c} + h^3\|\nabla' t u_\parallel\|^2_{\Gamma_+^c}.$$

Rewriting without the squares,

$$\|\Delta_{\varphi_c} u\| \gtrsim \frac{h}{\sqrt{\varepsilon}}\|u\|_{H^1(M)} + h^{\frac{1}{2}}\|h\nabla_N u_\perp\|_{\Gamma_+^c} + h^{\frac{1}{2}}\|u_\parallel\|_{H^1(\Gamma_+^c)}.$$

Now if u satisfies (2-2) then so does $e^{\frac{\varphi^2}{2\varepsilon}}u$ since ε is fixed. Therefore

$$\|e^{\frac{\varphi^2}{2\varepsilon}}\Delta_\varphi u\| \gtrsim \frac{h}{\sqrt{\varepsilon}}\|e^{\frac{\varphi^2}{2\varepsilon}}u\|_{H^1(M)} + h^{\frac{1}{2}}\|h\nabla_N e^{\frac{\varphi^2}{2\varepsilon}}u_\perp\|_{\Gamma_+^c} + h^{\frac{1}{2}}\|e^{\frac{\varphi^2}{2\varepsilon}}u_\parallel\|_{H^1(\Gamma_+^c)}.$$

Since $e^{\frac{\varphi^2}{2\varepsilon}}$ is smooth and bounded on M, we get

$$\|\Delta_\varphi u\| \gtrsim h\|u\|_{H^1(M)} + h^{\frac{1}{2}}\|h\nabla_N u_\perp\|_{\Gamma_+^c} + h^{\frac{1}{2}}\|u_\parallel\|_{H^1(\Gamma_+^c)}.$$

Thus Theorem 2.5 for $k \geq 1$ will follow after we have proved Lemma 6.1.

Proof of Lemma 6.1. For the 0-form case, this follows from Theorem 2.5 for 0-forms, which in this section we are assuming has been proved. Therefore we can seek to prove (6-3) for k-forms by induction on k.

Let $k > 0$, and assume (6-3) holds for $(k-1)$-forms satisfying (2-2). Now let $U_1, \dots, U_m \subset T$ be an open cover of Γ_+^c such that each $U_i \cap \Gamma_+^c$ has a coordinate patch, and let $\chi_1, \dots, \chi_m$ be a partition of unity with respect to $\{U_i\}$ such that $\sum \chi_i = 1$ near Γ_+^c and $\nabla_N \chi_i = 0$ for each i. It will suffice to show

$$h^3\|\nabla' t\chi_i u_\parallel\|^2_{\Gamma_+^c} \lesssim \|\Delta_{\varphi_c}u\|^2 + h^2\|u\|^2_{H^1(M)} + h^2\|u_\parallel\|^2_{\Gamma_+^c}.$$

Now on $U_i \cap \Gamma_+^c$, let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal frame for the tangent space, and extend these vector fields into M by parallel transport along normal geodesics.

Observe for all $\omega \in \Omega^k(U_j \cap \Gamma_+^c)$ one can write

$$\omega = \frac{1}{k}\sum_{j=1}^{n-1} e_j^\flat \wedge i_{e_j}\omega. \tag{6-4}$$

Therefore we can write

$$\nabla' t\chi_i u_\parallel = \frac{1}{k}\nabla' \sum_{j=1}^{n-1} e_j^\flat \wedge i_{e_j} t\chi_i u_\parallel = \frac{1}{k}\nabla' \sum_{j=1}^{n-1} e_j^\flat \wedge t i_{e_j}\chi_i u_\parallel.$$

Then it suffices to show

$$h^3\left\|\nabla'(e_j^\flat \wedge t i_{e_j}\chi_i u_\parallel)\right\|^2_{\Gamma_+^c} \lesssim \|\Delta_{\varphi_c}u\|^2 + h^2\|u\|^2_{H^1(M)} + h^2\|u_\parallel\|^2_{\Gamma_+^c},$$

or equivalently,

$$h^3\|\nabla' t i_{e_j}\chi_i u_\parallel\|^2_{\Gamma_+^c} \lesssim \|\Delta_{\varphi_c}u\|^2 + h^2\|u\|^2_{H^1(M)} + h^2\|u_\parallel\|^2_{\Gamma_+^c}. \tag{6-5}$$

Now we want to apply the induction hypothesis to $i_{e_j}\chi_i u_\parallel$, so we have to check that it satisfies the boundary conditions (2-2). In fact we will have to modify $i_{e_j}\chi_i u_\parallel$ slightly to achieve this. Let $\rho(x)$ be a function defined in a neighbourhood of the boundary as the distance to the boundary along a normal geodesic, and extend it to the rest of M by multiplication by a cutoff function. Then the claim is that $v = i_{e_j}\chi_i(u_\parallel + h(1 - e^{-\frac{\rho}{h}})Zu_\parallel)$ satisfies the absolute boundary conditions (2-2), where Z is an endomorphism yet to be chosen.

Since u satisfies (2-2), $i_{e_j}\chi_i u_\parallel$ and $i_{e_j}\chi_i(h(1-e^{-\frac{\rho}{h}})Zu_\parallel)$ both vanish to first order on Γ_+. Therefore v does as well.

Moreover, $t * i_{e_j}\chi_i u_\| = 0$ on Γ_+^c if $i_N i_{e_j}\chi_i u_\| = -\chi_i i_{e_j} i_N u_\| = 0$ on Γ_+^c, and this again follows from the fact that u satisfies (2-2). Note that $(1-e^{-\frac{\rho}{h}}) = 0$ at ∂M, so $t * v = 0$ on Γ_+^c.

Finally, by Lemma 3.3,

$$-t\delta * i_{e_j}\chi_i u_\| = -\delta' t(*i_{e_j}\chi_i u_\|)_\| + (S-(n-1)\kappa)t i_N(*i_{e_j}\chi_i u_\|)_\perp + t\nabla_N i_N * i_{e_j}\chi_i u_\|.$$

Since $t * i_{e_j}\chi_i u_\| = 0$ on Γ_+^c, the first term vanishes there as well. Therefore on Γ_+^c,

$$-th\delta * i_{e_j}\chi_i u_\| = h(S-(n-1)\kappa)t i_N(*i_{e_j}\chi_i u_\|)_\perp + th\nabla_N \chi_i i_N * i_{e_j} u_\|.$$

Now

$$\begin{aligned} th\nabla_N \chi_i i_N * i_{e_j} u_\| &= th\nabla_N \chi_i i_N e_j^\flat \wedge *u_\|(-1)^{k-1} \\ &= th\nabla_N \chi_i i_N e_j^\flat \wedge (*u)_\perp(-1)^{k-1} \\ &= th\nabla_N \chi_i i_N e_j^\flat \wedge *u(-1)^{k-1} \\ &= (-1)^k \chi_i e_j^\flat \wedge th\nabla_N i_N * u, \end{aligned}$$

so

$$-th\delta * i_{e_j}\chi_i u_\| = h(S-(n-1)\kappa)t i_N(*i_{e_j}\chi_i u_\|)_\perp + (-1)^k \chi_i e_j^\flat \wedge th\nabla_N i_N * u. \tag{6-6}$$

Applying the same calculation to the $i_{e_j}\chi_i h(1-e^{-\frac{\rho}{h}})Zu_\|$ term gives

$$-th\delta * i_{e_j} i_{e_j}\chi_i h(1-e^{-\frac{\rho}{h}})Zu_\| = (-1)^k \chi_i e_j^\flat \wedge th^2\nabla_N(1-e^{-\frac{\rho}{h}}) i_N * Zu_\|;$$

the other term vanishes since $(1-e^{-\frac{\rho}{h}}) = 0$ at the boundary. Thus

$$-th\delta * i_{e_j} i_{e_j}\chi_i h(1-e^{-\frac{\rho}{h}})Zu_\| = (-1)^k \chi_i e_j^\flat \wedge t i_N * hZu_\|.$$

Meanwhile, by Lemma 3.3 and by (2-2),

$$-th\delta * u = h(S-(n-1)\kappa)t i_N(*u)_\perp + th\nabla_N i_N(*u) = t i_{d\varphi} * u - h\sigma t i_N * u.$$

Viewing this as an equation for $th\nabla_N i_N(*u)$ and substituting into (6-6) gives

$$\begin{aligned} -th\delta * i_{e_j}\chi_i u_\| = h(S-(n-1)\kappa)t i_N(*i_{e_j}\chi_i u_\|)_\perp \\ + (-1)^k \chi_i e_j^\flat \wedge \big(-h(S-(n-1)\kappa)t i_N(*u)_\perp + t i_{d\varphi} * u - h\sigma t i_N * u\big). \end{aligned}$$

Therefore

$$\begin{aligned} &-th\delta * i_{e_j}\chi_i(u_\| + h(1-e^{-\frac{\rho}{h}})Zu_\|) \\ &\quad= h(S-(n-1)\kappa)t i_N(*i_{e_j}\chi_i u_\|)_\perp \\ &\qquad + (-1)^k \chi_i e_j^\flat \wedge \big(-h(S-(n-1)\kappa)t i_N(*u)_\perp + t i_{d\varphi} * u - h\sigma t i_N * u + t i_N * hZu_\|\big). \end{aligned}$$

Now if we let

$$Z = *N \wedge (S+\sigma-(n-1)\kappa) i_N *,$$

where here we identify S and σ with their extensions by parallel transport to a neighbourhood of the boundary, then

$$t i_N * Zu_\| = (S+\sigma-(n-1)\kappa)t i_N * u_\| = (S+\sigma-(n-1)\kappa)t i_N * u,$$

and

$$-th\delta * i_{e_j}\chi_i(u_\| + h(1-e^{-\frac{\rho}{h}})Zu_\|) = h(S-(n-1)\kappa)ti_N(*i_{e_j}\chi_i u_\|)_\perp + (-1)^k\chi_i e_j^\flat \wedge ti_{d\varphi} * u.$$

Since $t * u = 0$ on Γ_+^c, we can replace the $d\varphi$ in $ti_{d\varphi} * u$ with its normal component:

$$ti_{d\varphi} * u = -\partial_\nu\varphi ti_N * u.$$

Then

$$\begin{aligned}\chi_i e_j^\flat \wedge -ti_{d\varphi} * u &= \partial_\nu\varphi\chi_i e_j^\flat \wedge ti_N(*u)_\perp \\ &= \partial_\nu\varphi\chi_i e_j^\flat \wedge ti_N * u_\| \\ &= -\partial_\nu\varphi ti_N\chi_i e_j^\flat \wedge *u_\| \\ &= \partial_\nu\varphi ti_N * i_{e_j}\chi_i u_\|(-1)^k.\end{aligned}$$

Since $t * i_{e_j}\chi_i u_\| = 0$ on Γ_+^c,

$$\chi_i e_j^\flat \wedge -ti_{d\varphi} * u = -ti_{d\varphi} * i_{e_j}\chi_i u_\|(-1)^k$$

and

$$(-1)^k\chi_i e_j^\flat \wedge -ti_{d\varphi} * u = -ti_{d\varphi} * i_{e_j}\chi_i u_\|.$$

Therefore

$$-th\delta * i_{e_j}\chi_i(u_\| + h(1-e^{-\frac{\rho}{h}})Zu_\|) = ti_{d\varphi} * i_{e_j}\chi_i u_\| - h\sigma' ti_N * i_{e_j}\chi_i u_\|,$$

where σ' is a smooth bounded endomorphism. We can replace $u_\|$ on the right side by $u_\| + h(1-e^{-\frac{\rho}{h}})Zu_\|$, since $(1-e^{-\frac{\rho}{h}})$ is zero at the boundary. Therefore $v = i_{e_j}\chi_i(u_\| + h(1-e^{-\frac{\rho}{h}})Zu_\|)$ satisfies the boundary conditions (2-2), and so by the induction hypothesis,

$$h^3\|\nabla' tv\|^2_{\Gamma_+^c} \lesssim \|\Delta_{\varphi_c} v\|^2 + h^2\|v\|^2_{H^1(M)} + h^2\|v\|^2_{\Gamma_+^c}.$$

Keeping in mind that the second term of v is zero at the boundary, and $O(h)$ elsewhere, we get

$$h^3\|\nabla' ti_{e_j}\chi_i u_\|\|^2_{\Gamma_+^c} \lesssim \|\Delta_{\varphi_c} v\|^2 + h^2\|u_\|\|^2_{H^1(M)} + h^2\|u_\|\|^2_{\Gamma_+^c}. \tag{6-7}$$

Now

$$\|\Delta_{\varphi_c} v\| \lesssim \|\Delta_{\varphi_c} i_{e_j}\chi_i u_\|\| + h\|\Delta_{\varphi_c} i_{e_j}\chi_i(1-e^{-\frac{\rho}{h}})Zu_\|\|.$$

The commutators of Δ_{φ_c} with $i_{e_j}\chi_i$ and $i_{e_j}\chi_i(1-e^{-\frac{\rho}{h}})Z$ are $O(h)$ and first-order, so

$$\begin{aligned}\|\Delta_{\varphi_c} v\| &\lesssim \|i_{e_j}\chi_i\Delta_{\varphi_c} u_\|\| + h\|i_{e_j}\chi_i(1-e^{-\frac{\rho}{h}})Z\Delta_{\varphi_c} u_\|\| + h\|u_\|\|_{H^1(M)} \\ &\lesssim \|\Delta_{\varphi_c} u_\|\| + h\|u_\|\|_{H^1(M)}.\end{aligned}$$

Substituting back into (6-7) gives

$$h^3\|\nabla' ti_{e_j}\chi_i u_\|\|^2_{\Gamma_+^c} \lesssim \|\Delta_{\varphi_c} u_\|\|^2 + h^2\|u_\|\|^2_{H^1(M)} + h^2\|u_\|\|^2_{\Gamma_+^c}.$$

This proves (6-5), which finishes the proof of the lemma. □

7. Complex geometrical optics solutions

We will begin by constructing CGOs for the relative boundary case. To start, we can use the Carleman estimate from Theorem 2.4 to generate solutions via a Hahn–Banach argument. The notations are as in Section 2.

Proposition 7.1. *Let Q be an L^∞ endomorphism on ΛM, and let Γ_+ be a neighbourhood of ∂M_+. For all $v \in L^2(M, \Lambda M)$, and $f, g \in L^2(M, \Lambda\partial M)$ with support in Γ_+^c, there exists $u \in L^2(M, \Lambda M)$ such that*

$$\begin{aligned} (-\triangle_{-\varphi} + h^2 Q^*)u &= v \quad \text{on } M,\\ tu &= f \quad \text{on } \Gamma_+^c,\\ th\delta_{-\varphi}u &= g \quad \text{on } \Gamma_+^c, \end{aligned}$$

with

$$\|u\|_{L^2(M)} \lesssim h^{-1}\|v\|_{L^2(M)} + h^{\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + h^{\frac{1}{2}}\|g\|_{L^2(\Gamma_+^c)}.$$

Proof. Suppose $w \in \Omega(M)$ satisfies the relative boundary conditions (2-1) with $\sigma = 0$, and examine the expression

$$\left|(w \mid v) - (ti_\nu hd_\varphi w \mid hf)_{\Gamma_+^c} - (ti_\nu w \mid hg)_{\Gamma_+^c}\right|. \tag{7-1}$$

This is bounded above by

$$h\|w\|_{L^2(M)}h^{-1}\|v\|_{L^2(M)} + h^{\frac{1}{2}}\|ti_\nu hd_\varphi w\|_{L^2(\Gamma_+^c)}h^{\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + h^{\frac{1}{2}}\|ti_\nu w\|_{L^2(\Gamma_+^c)}\|g\|_{L^2(\Gamma_+^c)}.$$

By Lemma 3.4,

$$ti_\nu hd_\varphi w = he^{\frac{\varphi}{h}}t\nabla_N(e^{-\frac{\varphi}{h}}w)_\parallel + hStw_\parallel - he^{\frac{\varphi}{h}}d'ti_N(e^{-\frac{\varphi}{h}}w).$$

Since $tw = 0$,

$$ti_\nu hd_\varphi w = ht\nabla_N w_\parallel - he^{\frac{\varphi}{h}}d'ti_N(e^{-\frac{\varphi}{h}}w).$$

Therefore

$$\|ti_\nu hd_\varphi w\|_{L^2(\Gamma_+^c)} \le \|h\nabla_N w_\parallel\|_{L^2(\Gamma_+^c)} + \|w_\perp\|_{H^1(\Gamma_+^c)}.$$

Then by Theorem 2.4,

$$\begin{aligned} &\left|(w \mid v) + (ti_\nu hd_\varphi w \mid hf)_{\Gamma_+^c} + (ti_\nu w \mid hg)_{\Gamma_+^c}\right| \\ &\qquad \lesssim \|(-\triangle_\varphi + h^2 Q)w\|_{L^2(M)}\big(h^{-1}\|v\|_{L^2(M)} + h^{\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + h^{\frac{1}{2}}\|g\|_{L^2(\Gamma_+^c)}\big). \end{aligned}$$

Therefore on the subspace

$$\big\{(-\triangle_\varphi + h^2 Q)w \;\big|\; w \in \Omega(M) \text{ satisfies (2-1) with } \sigma = 0\big\} \subset L^2(M, \Lambda M),$$

the map

$$(-\triangle_\varphi + h^2 Q)w \mapsto (w \mid v) - (ti_\nu hd_\varphi w \mid hf)_{\Gamma_+^c} - (ti_\nu w \mid hg)_{\Gamma_+^c}$$

defines a bounded linear functional with the bound

$$h^{-1}\|v\|_{L^2(M)} + h^{\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + h^{\frac{1}{2}}\|g\|_{L^2(\Gamma_+^c)}.$$

By Hahn–Banach, this functional extends to the whole space, and thus there exists a $u \in L^2(M, \Lambda M)$ such that

$$\|u\|_{L^2(M)} \lesssim h^{-1}\|v\|_{L^2(M)} + h^{\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + h^{\frac{1}{2}}\|g\|_{L^2(\Gamma_+^c)}$$

and

$$(w \mid v) - (ti_\nu h d_\varphi w \mid hf)_{\Gamma_+^c} - (ti_\nu w \mid hg)_{\Gamma_+^c} = \big((-\Delta_\varphi + h^2 Q)w \,\big|\, u\big).$$

Integrating by parts and applying the boundary conditions (2-1) gives

$$\begin{aligned}(w \mid v) - (ti_\nu h d_\varphi w \mid hf)_{\Gamma_+^c} &- (ti_\nu w \mid hg)_{\Gamma_+^c} \\ &= \big(w \,\big|\, (-\Delta_{-\varphi} + h^2 Q^*)u\big) - h(ti_\nu h d_\varphi w \mid tu)_{\partial M} - h(ti_\nu w \mid th\delta_{-\varphi}u)_{\partial M}\end{aligned}$$

for all $w \in \Omega(M)$ satisfying the relative boundary conditions (2-1) with $\sigma = 0$. Varying w over the compactly supported elements of $\Omega(M)$ one sees that $(-\Delta_{-\varphi} + h^2 Q^*)u = v$ on M, which reduces the above relation to

$$-(ti_\nu h d_\varphi w \mid hf)_{\Gamma_+^c} - (ti_\nu w \mid hg)_{\Gamma_+^c} = -h(ti_\nu h d_\varphi w \mid tu)_{\partial M} - h(ti_\nu w \mid th\delta_{-\varphi}u)_{\partial M}$$

for all $w \in \Omega(M)$ satisfying the relative boundary conditions (2-1) with $\sigma = 0$. We now vary w satisfying condition (2-1) with $\sigma = 0$ and $i_\nu w = 0$ to obtain $tu = f$ on Γ_+^c. Finally, by varying w over all forms satisfying conditions (2-1) with $\sigma = 0$, we see that $th\delta_{-\varphi}u = g$ on Γ_+^c.

To summarize, we can see that

$$\begin{aligned}(-\Delta_{-\varphi} + h^2 Q^*)u = v &\quad \text{on } M,\\ tu = f &\quad \text{on } \Gamma_+^c,\\ th\delta_{-\varphi}u = g &\quad \text{on } \Gamma_+^c,\end{aligned}$$

as desired. ☐

To match notations with previous papers, we will begin by rewriting this result, along with the Carleman estimate, in τ notation, as follows.

Theorem 2.4 becomes the following.

Theorem 7.2. *Let Q be an L^∞ endomorphism on ΛM. Define $\Gamma_+ \subset \partial M$ to be a neighbourhood of ∂M_+. Suppose $u \in H^2(M, \Lambda M)$ satisfies the boundary conditions*

$$\begin{aligned} u|_{\Gamma_+} = 0 \quad \text{and} \quad \nabla_\nu u\, |_{\Gamma_+} &= 0,\\ tu|_{\Gamma_+^c} &= 0,\\ t\delta e^{-\tau\varphi}u|_{\Gamma_+^c} &= \sigma t i_N e^{-\tau\varphi} u\end{aligned} \tag{7-2}$$

for some smooth endomorphism σ independent of τ. Then there exists $\tau_0 > 0$ such that if $\tau > \tau_0$,

$$\begin{aligned}\|(-\Delta_\tau + Q)u\|_{L^2(M)} \gtrsim \tau\|u\|_{L^2(M)} + \|\nabla u\|_{L^2(M)} &+ \tau^{\frac{3}{2}}\|u_\perp\|_{L^2(\Gamma_+^c)} \\ &+ \tau^{\frac{1}{2}}\|\nabla' t i_N u\|_{L^2(\Gamma_+^c)} + \tau^{\frac{1}{2}}\|\nabla_N u_\parallel\|_{L^2(\Gamma_+^c)},\end{aligned}$$

where

$$\Delta_\tau = e^{\tau\varphi}\Delta e^{-\tau\varphi}.$$

By choice of coordinates, note that the same theorem holds for $\tau < 0$, with Γ_+ replaced by Γ_-. Then Proposition 7.1 becomes the following.

Proposition 7.3. *Let Q be an L^∞ endomorphism on ΛM. For all $v \in L^2(M, \Lambda M)$ and $f, g \in L^2(\Gamma_+^c, \Lambda\Gamma_+^c)$, there exists $u \in L^2(M, \Lambda M)$ such that*

$$\begin{aligned} (-\Delta_{-\tau} + Q^*)u &= v \quad \text{on } M, \\ tu &= f \quad \text{on } \Gamma_+^c, \\ t\delta_{-\tau} u &= g \quad \text{on } \Gamma_+^c, \end{aligned}$$

with

$$\|u\|_{L^2(M)} \lesssim \tau^{-1}\|v\|_{L^2(M)} + \tau^{-\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + \tau^{-\frac{3}{2}}\|g\|_{L^2(\Gamma_+^c)}.$$

Now we turn to the construction of the CGOs themselves. From now on we will invoke the assumption that the conformal factor c in the definition of M as an admissible manifold satisfies $c = 1$. Below we will consider complex-valued 1-forms, and $\langle \cdot, \cdot \rangle$ will denote the complex bilinear extension of the Riemannian inner product to complex-valued forms.

We assume

$$(M, g) \Subset (\mathbb{R} \times M_0, g), \quad g = e \oplus g_0,$$

where (M_0, g_0) is a compact $(n-1)$-dimensional manifold with smooth boundary. We write $x = (x_1, x')$ for points in $\mathbb{R} \times M_0$, where x_1 is the Euclidean coordinate and x' is a point in M_0. Let Q be an L^∞ endomorphism of ΛM. We next wish to construct solutions to the equation

$$(-\Delta + Q)Z = 0 \quad \text{in } M,$$

where Z is a graded differential form in $L^2(M, \Lambda M)$ having the form

$$Z = e^{-sx_1}(A + R).$$

Here $s = \tau + i\lambda$ is a complex parameter where $\tau, \lambda \in \mathbb{R}$ and $|\tau|$ is large, the graded form A is a smooth amplitude, and R will be a correction term obtained from the Carleman estimate. Inserting the expression for Z in the equation results in

$$e^{sx_1}(-\Delta + Q)e^{-sx_1}R = -F,$$

where

$$F = e^{sx_1}(-\Delta + Q)e^{-sx_1}A.$$

The point is to choose A so that $\|F\|_{L^2(M)} = O(1)$ as $|\tau| \to \infty$.

By Lemma 3.2, we have

$$F = (-\Delta - s^2 + 2s\nabla_{\partial_1} + Q)A.$$

We wish to choose A so that $\nabla_{\partial_1} A = 0$. The following lemma explains this condition. Below, we identify a differential form in M_0 with the corresponding differential form in $\mathbb{R} \times M_0$ which is constant in x_1.

Lemma 7.4. *If u is a k-form in $\mathbb{R}\times M_0$ with local coordinate expression $u = u_I\, dx^I$, then*

$$\nabla_{\partial_1} u = 0 \quad \Longleftrightarrow \quad u_I = u_I(x') \quad \text{for all } I.$$

If $\nabla_{\partial_1} u = 0$, then there is a unique decomposition

$$u = dx^1 \wedge u' + u'',$$

where u' is a $(k-1)$-form in M_0 and u'' is a k-form in M_0. For such a k-form u, one has

$$\Delta u = dx^1 \wedge \Delta_{x'} u' + \Delta_{x'} u'',$$

where Δ and $\Delta_{x'}$ are the Hodge Laplacians in $\mathbb{R}\times M_0$ and in M_0, respectively.

Proof. In the (x_1, x')-coordinates g has the form

$$g(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}.$$

Consequently, for any k, l the Christoffel symbols satisfy

$$\Gamma^l_{1k} = \tfrac{1}{2} g^{lm}(\partial_1 g_{km} + \partial_k g_{1m} - \partial_m g_{1k}) = 0.$$

This shows $\nabla_{\partial_1} dx^I = 0$ for all I, and therefore any k-form $u = u_I\, dx^I$ satisfies

$$\nabla_{\partial_1}(u_I\, dx^I) = \partial_1 u_I\, dx^I.$$

Thus $\nabla_{\partial_1} u = 0$ if and only if each u_I only depends on x'. In general, if u is a k-form on $\mathbb{R}\times M_0$ we have the unique decomposition

$$u = dx^1 \wedge u' + u'',$$

where $u'(x_1, \cdot)$ is a $(k-1)$-form in M_0 and $u''(x_1, \cdot)$ is a k-form in M_0, depending smoothly on the parameter x_1. If $\nabla_{\partial_1} u = 0$, then $u = dx^1 \wedge u' + u''$, where u' and u'' are differential forms in M_0.

Suppose now that $u = dx^1 \wedge u' + u''$, where u' and u'' are forms in M_0. Denote by $d_{x'}$ and $\delta_{x'}$ the exterior derivative and codifferential in x'. Clearly

$$d(dx^1 \wedge u') = -dx^1 \wedge d_{x'} u', \quad du'' = d_{x'} u''.$$

The identity $\delta = -\sum_{j=1}^n i_{e_j} \nabla_{e_j}$, where e_j is an orthonormal frame in $T(\mathbb{R}\times M_0)$ with $e_1 = \partial_1$, together with the fact that $\nabla_{\partial_1} u'' = 0$, implies

$$\delta u'' = \delta_{x'} u''.$$

Finally, computing in Riemannian normal coordinates at p gives

$$\begin{aligned} \delta(dx^1 \wedge u')|_p &= -\sum_{j=1}^n i_{\partial_j} \nabla_{\partial_j}(u'_J\, dx^1 \wedge dx^J)|_p \\ &= -\sum_{j=2}^n i_{\partial_j}(dx^1 \wedge \nabla_{\partial_j} u')|_p = -dx^1 \wedge \delta_{x'} u'|_p. \end{aligned}$$

Thus

$$\delta(dx^1 \wedge u') = -dx^1 \wedge \delta_{x'} u'.$$

It follows directly from these facts that

$$\begin{aligned}\Delta(dx^1 \wedge u' + u'') &= -(d\delta + \delta d)(dx^1 \wedge u' + u'') \\ &= dx^1 \wedge \Delta_{x'} u' + \Delta_{x'} u''.\end{aligned}$$

□

Returning to the expression for F, the assumption $\nabla_{\partial_1} A = 0$ gives

$$F = (-\Delta - s^2 + Q)A.$$

Writing Y^k for the k-form part of a graded form Y and decomposing $A^k = dx^1 \wedge (A^k)' + (A^k)''$ as in Lemma 7.4, we obtain

$$F^k = dx^1 \wedge (-\Delta_{x'} - s^2)(A^k)' + (-\Delta_{x'} - s^2)(A^k)'' + (QA)^k.$$

Thus, in order to have $\|F\|_{L^2(M)} = O(1)$ as $|\tau| \to \infty$, it is enough to find for each k a smooth $(k-1)$-form $(A^k)'$ and a smooth k-form $(A^k)''$ in M_0 such that

$$\begin{aligned}\|(-\Delta_{x'} - s^2)(A^k)'\|_{L^2(M_0)} = O(1), \quad \|(A^k)'\|_{L^2(M_0)} = O(1), \\ \|(-\Delta_{x'} - s^2)(A^k)''\|_{L^2(M_0)} = O(1), \quad \|(A^k)''\|_{L^2(M_0)} = O(1).\end{aligned}$$

If (M_0, g_0) is simple, there is a straightforward quasimode construction for achieving this.

Lemma 7.5. *Let (M_0, g_0) be a simple m-dimensional manifold, and let $0 \le k \le m$. Suppose $(\widehat{M}_0, g_0)$ is another simple manifold with $(M_0, g_0) \Subset (\widehat{M}_0, g_0)$, fix a point $\omega \in \widehat{M}_0^{\mathrm{int}} \setminus M_0$, and let (r, θ) be polar normal coordinates in $(\widehat{M}_0, g_0)$ with centre ω. Suppose $\eta^1, \dots, \eta^m$ is a global orthonormal frame of T^*M_0 with $\eta^1 = dr$ and $\nabla_{\partial_r} \eta^j = 0$ for $2 \le j \le m$, and let $\{\eta^I\}$ be a corresponding orthonormal frame of $\Lambda^k M_0$. Then for any $\lambda \in \mathbb{R}$ and for any $\binom{m}{k}$ complex functions $b_I \in C^\infty(S^{m-1})$, the smooth k-form*

$$u = e^{isr} |g_0(r, \theta)|^{-\frac{1}{4}} \sum_I b_I(\theta) \eta^I,$$

with $s = \tau + i\lambda$ for τ real, satisfies

$$\|(-\Delta_{x'} - s^2)u\|_{L^2(M_0)} = O(1), \quad \|u\|_{L^2(M_0)} = O(1)$$

as $|\tau| \to \infty$.

Proof. We first try to find the quasimode in the form $u = e^{is\psi} a$ for some smooth real-valued phase function ψ and some smooth k-form a. Lemma 3.2 implies

$$(-\Delta_{x'} - s^2)(e^{is\psi} a) = e^{is\psi} \big[s^2(|d\psi|^2 - 1)a - is[2\nabla_{\mathrm{grad}(\psi)} a + (\Delta_{x'} \psi)a] - \Delta_{x'} a \big].$$

Let (r, θ) be polar normal coordinates as in the statement of the lemma, and note that

$$g_0(r, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & h(r, \theta) \end{pmatrix}$$

globally in M_0 for some $(m-1) \times (m-1)$ symmetric positive definite matrix h.

Define

$$\psi(r,\theta) = r.$$

Then $\psi \in C^\infty(M_0)$ and $|d\psi|^2 = 1$, so that the s^2 term will be zero. We next want to choose a so that $2\nabla_{\mathrm{grad}(\psi)} a + (\Delta_{x'}\psi)a = 0$. Note that

$$\nabla_{\mathrm{grad}(\psi)} = \nabla_{\partial_r}, \quad \Delta_{x'}\psi = \frac{1}{2}\frac{\partial_r |g_0(r,\theta)|}{|g_0(r,\theta)|}.$$

Thus, choosing $a = |g_0|^{-\frac{1}{4}}\tilde{a}$ for some k-form $\tilde{a}$, it is enough to arrange that

$$\nabla_{\partial_r}\tilde{a} = 0.$$

Using the frame $\{\eta^j\}$ above, with $\eta^1 = dr$, we write

$$\tilde{a} = \eta^1 \wedge \tilde{a}' + \tilde{a}'',$$

where $\tilde{a}'$ is a $(k-1)$-form and $\tilde{a}''$ is a k-form in M_0 of the form

$$\tilde{a}' = \sum_{\substack{J\subset\{2,\dots,m\}\\ |J|=k-1}} \alpha_{1,J}\eta^J, \quad \tilde{a}'' = \sum_{\substack{J\subset\{2,\dots,m\}\\ |J|=k}} \beta_J \eta^J$$

for some functions $\alpha_{1,J}$ and β_J in M_0. Now, the form of the metric implies $\nabla_{\partial_r}\eta^1 = 0$, and by assumption $\nabla_{\partial_r}\eta^j = 0$ for $2 \le j \le m$. Therefore

$$\nabla_{\partial_r}\tilde{a} = \sum_{\substack{J\subset\{2,\dots,m\}\\ |J|=k-1}} \partial_r\alpha_{1,J}\eta^1\wedge\eta^J + \sum_{\substack{J\subset\{2,\dots,m\}\\ |J|=k}} \partial_r\beta_J\eta^J.$$

In the definitions of $\tilde{a}'$ and $\tilde{a}''$, we may now choose

$$\alpha_{1,J} = b_{\{1\}\cup J}(\theta), \quad \beta_J = b_J(\theta),$$

where b_I are the given functions in $C^\infty(S^{m-1})$. The resulting k-form $u = e^{is\psi}|g_0|^{-\frac{1}{4}}\tilde{a}$ satisfies the required conditions. ☐

The next result gives the full construction of the complex geometrical optics solutions.

Lemma 7.6. *Let $(M,g) \Subset (\mathbb{R}\times M_0, g)$, where $g = e\oplus g_0$, assume (M_0,g_0) is simple, and let Q be an L^∞ endomorphism of ΛM. Let $(\widehat{M}_0, g_0)$ be another simple manifold with $(M_0,g_0) \Subset (\widehat{M}_0, g_0)$, fix a point $\omega \in \widehat{M}_0^{\mathrm{int}} \setminus M_0$, and let (r,θ) be polar normal coordinates in $(\widehat{M}_0, g_0)$ with centre ω. Suppose $\eta^1,\dots,\eta^n$ is a global orthonormal frame of $T^*(\mathbb{R}\times M_0)$ with $\eta^1 = dx^1$, $\eta^2 = dr$, and $\nabla_{\partial_r}\eta^j = 0$ for $3 \le j \le n$, and let $\{\eta^I\}$ be a corresponding orthonormal frame of $\Lambda(\mathbb{R}\times M_0)$. Let also $\lambda \in \mathbb{R}$. If $|\tau|$ is sufficiently large and if $s = \tau + i\lambda$, then for any 2^n complex functions $b_I \in C^\infty(S^{n-2})$ there exists a solution $Z \in L^2(M,\Lambda M)$ of the equation*

$$(-\Delta + Q)Z = 0 \quad \textit{in } M$$

having the form

$$Z = e^{-sx_1}\left[e^{isr}|g_0(r,\theta)|^{-\frac{1}{4}}\left[\sum_I b_I(\theta)\eta^I\right] + R\right],$$

where $\|R\|_{L^2(M)} = O(|\tau|^{-1})$. *Further, one can arrange that the relative boundary values of* Z *vanish on* Γ_+^c *or* Γ_-^c *(depending on the sign of* τ*).*

Proof. Try first $Z = e^{-sx_1}(A+R)$, where $\nabla_{\partial_1} A = 0$. By the discussion in this section, we need to solve the equation

$$e^{sx_1}(-\Delta + Q)(e^{-sx_1}R) = -F,$$

where

$$F = (-\Delta - s^2 + Q)A.$$

Decomposing the k-form part of A as $A^k = \eta^1 \wedge (A^k)' + (A^k)''$ as in Lemma 7.4, where $\eta^1 = dx^1$, we obtain

$$F^k = \eta^1 \wedge (-\Delta_{x'} - s^2)(A^k)' + (-\Delta_{x'} - s^2)(A^k)'' + (QA)^k.$$

Let $\eta^1, \dots, \eta^n$ and $\{\eta^I\}$ be orthonormal frames as in the statement of the result. We can use Lemma 7.5 to find, for any $\binom{n-1}{k-1}$ functions $b'_J(\theta)$ and for any $\binom{n-1}{k}$ functions $b''_J(\theta)$, quasimodes

$$(A^k)' = e^{isr}|g_0|^{-\frac{1}{4}} \sum_{\substack{J\subset\{2,\dots,n\}\\ |J|=k-1}} b'_J(\theta)\eta^J,$$

$$(A^k)'' = e^{isr}|g_0|^{-\frac{1}{4}} \sum_{\substack{J\subset\{2,\dots,n\}\\ |J|=k}} b''_J(\theta)\eta^J.$$

Recalling that $A^k = \eta^1 \wedge (A^k)' + (A^k)''$ and relabeling functions, this shows that for any $\binom{n}{k}$ functions $b_I \in C^\infty(S^{n-2})$ we may find A^k of the form

$$A^k = e^{isr}|g_0|^{-\frac{1}{4}} \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=k}} b_I(\theta)\eta^I,$$

with $\|(-\Delta - s^2)A^k\|_{L^2(M)} = O(1)$ and $\|A^k\|_{L^2(M)} = O(1)$ as $|\tau| \to \infty$. Repeating this construction for all k, we obtain the amplitude

$$A = e^{isr}|g_0(r,\theta)|^{-\frac{1}{4}} \sum_I b_I(\theta)\eta^I,$$

with the same norm estimates as those for A^k. Then also $\|F\|_{L^2(M)} = O(1)$. Then Proposition 7.3 allows us to find R with the right properties. $\square$

Note that if Z is a solution to $(-\Delta + *Q*^{-1})Z = 0$ in M, and Z has relative boundary values that vanish on Γ_+^c, then $*Z$ is a solution to $(-\Delta + Q) * Z = 0$ in M, and $*Z$ has absolute boundary values that vanish on Γ_+^c. Thus this construction also gives us solutions with vanishing absolute boundary values on Γ_+^c.

8. The tensor tomography problem

Now we can begin the proof of Theorems 2.1 and 2.2. First we will use the hypotheses of Theorem 2.1 to obtain some vanishing integrals involving $(Q_2 - Q_1)$.

Lemma 8.1. *Suppose the hypotheses of Theorem 2.1 hold. Using the notation in Lemma 7.6, let* $Z_1, Z_2 \in L^2(M, \Lambda M)$ *be solutions of* $(-\Delta + Q_1)Z_1 = (-\Delta + \bar{Q}_2)Z_2 = 0$ *in* M *of the form*

$$Z_1 = e^{-sx_1}\left[e^{isr}|g_0|^{-\frac{1}{4}}\left[\sum_I c_I(\theta)\eta^I\right] + R_1\right],$$

$$Z_2 = e^{sx_1}\left[e^{isr}|g_0|^{-\frac{1}{4}}\left[\sum_I d_I(\theta)\eta^I\right] + R_2\right],$$

with vanishing relative boundary conditions on Γ_-^c *and* Γ_+^c *respectively. Then*

$$((Q_2 - Q_1)Z_1 \mid Z_2)_M = 0.$$

Note that while the orthogonality condition derived in the lemma does not use the particular form of the solution, we will only apply this identity to solutions of the given form.

Proof. Let Y be a solution of $(-\Delta + Q_2)Y = 0$ in M with the same relative boundary conditions as Z_1; such a solution exists by the assumption on Q_2. Then consider the integral

$$\big((N_{Q_1}^{\mathrm{RA}} - N_{Q_2}^{\mathrm{RA}})(tZ_1, t\delta Z_1) \,\big|\, (ti_N d * Z_2, ti_N * Z_2)\big)_{\partial M}.$$

By definition of the N^{RA} map, this is

$$\begin{aligned}\big((t*(Z_1 - Y), t\delta * (Z_1 - Y)) \,\big|\, (ti_N d * Z_2, ti_N * Z_2)\big)_{\partial M} \\ = \big(t*(Z_1 - Y) \,\big|\, ti_N d * Z_2\big)_{\partial M} + \big(t\delta * (Z_1 - Y) \,\big|\, ti_N * Z_2\big)_{\partial M}.\end{aligned}$$

Recall from the section on notation and identities that

$$(-\Delta u \mid v)_M = (u \mid -\Delta v)_M + (tu \mid ti_\nu dv)_{\partial M} + (t\delta * u \mid ti_\nu * v)_{\partial M} + (t * u \mid ti_\nu d * v)_{\partial M} + (t\delta u \mid ti_\nu v)_{\partial M}.$$

Since the relative boundary values of $(Z_1 - Y)$ vanish, by definition, the integration by parts formula above implies

$$\begin{aligned}\big(t*(Z_1 - Y) \,\big|\, ti_N d * Z_2\big)_{\partial M} + \big(t\delta*(Z_1 - Y) \,\big|\, ti_N * Z_2\big)_{\partial M} &= (-\triangle(Z_1 - Y) \mid Z_2)_M - (Z_1 - Y \mid -\triangle Z_2)_M \\ &= (Q_2 Y - Q_1 Z_1 \mid Z_2)_M - (Z_1 - Y \mid -\bar{Q}_2 Z_2)_M \\ &= ((Q_2 - Q_1)Z_1 \mid Z_2)_M.\end{aligned}$$

Meanwhile, by the hypothesis on $N_{Q_1}^{\mathrm{RA}}$ and $N_{Q_2}^{\mathrm{RA}}$, we have $N_{Q_1}^{\mathrm{RA}}(Z_1 - Y) = N_{Q_2}^{\mathrm{RA}}(Z_1 - Y)$ on Γ_+. Therefore

$$\begin{aligned}\big(t*(Z_1 - Y) \,\big|\, ti_N d * Z_2\big)_{\partial M} + (t\delta*(Z_1 - Y) \,\big|\, ti_N * Z_2)_{\partial M} \\ = \big(t*(Z_1 - Y) \,\big|\, ti_N d * Z_2\big)_{\Gamma_+^c} + \big(t\delta*(Z_1 - Y) \,\big|\, ti_N * Z_2\big)_{\Gamma_+^c}.\end{aligned}$$

Now by construction, Z_2 has relative boundary values that vanish on Γ_+^c. But

$$\begin{aligned} t i_N * Z_2|_{\Gamma_+^c} = 0 \quad &\Longleftrightarrow \quad (*Z_2)_\perp|_{\Gamma_+^c} = 0 \\ &\Longleftrightarrow \quad *(Z_2)_\parallel|_{\Gamma_+^c} = 0 \\ &\Longleftrightarrow \quad (Z_2)_\parallel|_{\Gamma_+^c} = 0 \\ &\Longleftrightarrow \quad t Z_2|_{\Gamma_+^c} = 0. \end{aligned}$$

Similarly,

$$t i_N d * Z_2|_{\Gamma_+^c} = 0 \quad \Longleftrightarrow \quad t\delta * Z_2|_{\Gamma_+^c} = 0.$$

Therefore the fact that Z_2 has relative boundary values that vanish on Γ_+^c implies

$$\big(t*(Z_1-Y) \,\big|\, t i_N d * Z_2\big)_{\Gamma_+^c} + \big(t\delta*(Z_1-Y) \,\big|\, t i_N * Z_2\big)_{\Gamma_+^c} = 0.$$

Therefore

$$((Q_2-Q_1)Z_1 \mid Z_2)_M = 0$$

for each such pair of CGO solutions Z_1 and Z_2. □

Remark. The proof of the Lemma 8.1 does not use the actual forms of the CGO solutions. The integral identity holds for all solutions Z_1 and Z_2 with vanishing relative boundary conditions on Γ_-^c and Γ_+^c respectively. However, the identity is only of interest to us for the particular forms of CGO solutions which we stated.

Working through the same argument with $*Z_1$ and $*Z_2$ gives us the following lemma as well.

Lemma 8.2. *Suppose the hypotheses of Theorem 2.2 hold. Using the notation in Lemma 7.6, let* $*Z_1, *Z_2 \in L^2(M, \Lambda M)$ *be solutions of* $(-\Delta + Q_1) * Z_1 = (-\Delta + \overline{Q}_2) * Z_2 = 0$ *in* M *of the form*

$$\begin{aligned} Z_1 &= e^{-sx_1}\bigg[e^{isr}|g_0|^{-\frac{1}{4}}\bigg[\sum_I c_I(\theta)\eta^I\bigg] + R_1\bigg], \\ Z_2 &= e^{sx_1}\bigg[e^{isr}|g_0|^{-\frac{1}{4}}\bigg[\sum_I d_I(\theta)\eta^I\bigg] + R_2\bigg]. \end{aligned}$$

Then

$$((Q_2 - Q_1)Z_1 \mid Z_2)_M = 0.$$

Therefore both of the main theorems reduce to using the condition $(QZ_1, Z_2)_{L^2(M)} = 0$ for solutions of the type given in Lemma 7.6 to show $Q = 0$.

The next result shows that from the condition $(QZ_1, Z_2)_{L^2(M)} = 0$ for solutions of the type given in Lemma 7.6, it follows that certain exponentially attenuated integrals over geodesics in (M_0, g_0) of matrix elements of Q, further Fourier transformed in x_1, must vanish.

Proposition 8.3. *Assume the hypotheses in Theorem 2.1 or 2.2, with* $Q = Q_2 - Q_1$ *extended by zero to* $\mathbb{R}\times M_0$. *Fix a geodesic* $\gamma : [0, L] \to M_0$ *with* $\gamma(0), \gamma(L) \in \partial M_0$, *let* ∂_r *be the vector field in* M_0 *tangent to geodesic rays starting at* $\gamma(0)$, *and suppose* $\{\eta^I\}$ *is an orthonormal frame of* $\Lambda(\mathbb{R}\times M_0^{\mathrm{int}})$ *with* $\eta^1 = dx^1$,

$\eta^2 = dr$, and $\nabla_{\partial_r} \eta^j = 0$ for $3 \le j \le n$. (Such a frame always exists.) Then for any $\lambda \in \mathbb{R}$ and any I, J one has
$$\int_0^L e^{-2\lambda r}\left[\int_{-\infty}^{\infty} e^{-2i\lambda x_1}\langle Q(x_1,\gamma(r))\eta^I,\eta^J\rangle\,dx_1\right]dr = 0.$$

Proof. Using the notation in Lemma 7.6, let $Z_j \in L^2(M, \Lambda M)$ be solutions of $(-\Delta + Q_1)Z_1 = (-\Delta + \bar{Q}_2)Z_2 = 0$ in M of the form
$$Z_1 = e^{-sx_1}\left[e^{isr}|g_0|^{-\frac{1}{4}}\left[\sum_I c_I(\theta)\eta^I\right] + R_1\right],$$
$$Z_2 = e^{sx_1}\left[e^{isr}|g_0|^{-\frac{1}{4}}\left[\sum_I d_I(\theta)\eta^I\right] + R_2\right],$$
where $s = \tau + i\lambda$, $\tau > 0$ is large, $\lambda \in \mathbb{R}$, and $c_I, d_I \in C^\infty(S^{n-2})$. We can assume $\|R_j\|_{L^2(M)} = O(\tau^{-1})$ as $\tau \to \infty$, and that the relative (absolute) boundary values of Z_1 are supported in $\widetilde{F}$ and the relative (absolute) boundary values of Z_2 are supported in $\widetilde{B}$. By Lemma 8.1 (Lemma 8.2), we have
$$\begin{aligned}0 &= \lim_{\tau\to\infty}(QZ_1, Z_2)_{L^2(M)}\\ &= \int_{S^{n-2}}\int_0^\infty e^{-2\lambda r}\left[\sum_{I,J}\left[\int_{-\infty}^{\infty} e^{-2i\lambda x_1}\langle Q(x_1,r,\theta)\eta^I,\eta^J\rangle\,dx_1\right]c_I(\theta)\,\overline{d_J(\theta)}\right]dr\,d\theta.\end{aligned}$$

We now extend the M_0-geodesic γ to $\widehat{M}_0$, choose $\omega = \gamma(-\varepsilon)$ for small $\varepsilon > 0$, and choose θ_0 so that $\gamma(t) = (t, \theta_0)$. The functions c_I and d_J can be chosen freely, and by varying them we obtain
$$\int_0^\infty e^{-2\lambda r}\left[\int_{-\infty}^{\infty} e^{-2i\lambda x_1}\langle Q(x_1,r,\theta_0)\eta^I,\eta^J\rangle\,dx_1\right]dr = 0$$
for each fixed I and J. Since Q is compactly supported in M_0^{int}, this implies the required result.

It remains to show that a frame $\{\eta^I\}$ with the required properties exists. Let $\omega = \gamma(0)$, and let $(\widehat{M}_0, g_0)$ be a simple manifold with $(M_0, g_0) \Subset (\widehat{M}_0, g_0)$ such that the $\widehat{M}_0$-geodesic starting at ω in direction $\nu(\omega)$ never meets M_0. (It is enough to embed (M_0, g_0) in some closed manifold and to take $\widehat{M}_0$ strictly convex and slightly larger than M_0.) Let (r, θ) be polar normal coordinates in $\widehat{M}_0$ with centre $\omega = \gamma(0)$, fix $r_0 > 0$ so that the geodesic ball $B(\omega, r_0)$ is contained in $\widehat{M}_0^{\mathrm{int}}$, and let $\hat{\theta} \in S^{n-2}$ be the direction of $\nu(\omega)$. Choose some orthonormal frame $\eta^3, \dots, \eta^n$ of the cotangent space of $\partial B(\omega, r_0) \setminus \{(r_0, \hat{\theta})\}$, and extend these as 1-forms in M_0^{int} by parallel transporting along integral curves of ∂_r. We thus obtain a global orthonormal frame $\eta^2, \dots, \eta^n$ of $T^*M_0^{\mathrm{int}}$ with $\eta^2 = dr$ and $\nabla_{\partial_r}\eta^j = 0$ for $3 \le j \le n$. Moreover, $\eta^1, \dots, \eta^n$ will be a global orthonormal frame of $T^*(\mathbb{R} \times M_0^{\mathrm{int}})$ inducing an orthonormal frame $\{\eta^I\}$ of $\Lambda(\mathbb{R} \times M_0^{\mathrm{int}})$. ☐

We will now show how the coefficients are uniquely determined by the integrals in Proposition 8.3. This follows by inverting attenuated ray transforms, a topic of considerable independent interest (see the survey [Finch 2003] for results in the Euclidean case, and the survey [Paternain et al. 2014] and references below for the manifold case). The transform in Proposition 8.3 is not exactly the same kind of attenuated ray transform/Fourier transform as in the scalar case, for instance, in [Dos Santos Ferreira et al. 2009a], since the matrix element of Q that appears in the integral may actually depend on the geodesic γ (note that the 1-forms η depend on γ). To clarify this point, we fix some global orthonormal frame $\{\varepsilon^1, \dots, \varepsilon^n\}$ of $T^*(\mathbb{R} \times M_0)$ with $\varepsilon^1 = dx^1$, and let $\{\varepsilon^I\}$ be the corresponding orthonormal frame of $\Lambda(\mathbb{R} \times M_0)$. Define the matrix elements

$$q_{I,J} = \langle Q\varepsilon^I, \varepsilon^J \rangle.$$

Define also

$$\hat{q}_{I,J}(\xi_1, x') = \int_{-\infty}^{\infty} e^{-ix_1\xi_1} q_{I,J}(x_1, x')\, dx_1.$$

Then the conclusion in Proposition 8.3 implies

$$\int_0^L e^{-2\lambda r} \hat{q}_{I',J'}(2\lambda, \gamma(r)) \langle \eta^I, \varepsilon^{I'} \rangle \langle \eta^J, \varepsilon^{J'} \rangle \, dr = 0$$

for any $\lambda \in \mathbb{R}$, for any I, J, and for any maximal geodesic γ in M_0. (Note that the inner products $\langle \eta^I, \varepsilon^{I'} \rangle$ do not depend on x_1.)

Up until now everything discussed in this paper has held for any dimension $n \geq 3$. Now, however, we will invoke the assumption that $n = 3$. Then $q_{I,J}$ is an 8×8 matrix. In this case we may choose $\eta^1 = dx^1$, $\eta^2 = dr$, and $\eta^3 = *_{g_0} dr$, where dr is the 1-form dual to $\dot{\gamma}$ on the geodesic γ. Let also $\{e_j\}$ be the orthonormal frame of vector fields dual to $\{\varepsilon^j\}$ (which is assumed to be positively oriented). It follows that

$$\begin{aligned} &\langle \eta^1, \varepsilon^1 \rangle = 1, && \langle \eta^1, \varepsilon^2 \rangle = 0, && \langle \eta^1, \varepsilon^3 \rangle = 0, \\ &\langle \eta^2, \varepsilon^1 \rangle = 0, && \langle \eta^2, \varepsilon^2 \rangle = \langle e_2, \dot{\gamma} \rangle, && \langle \eta^2, \varepsilon^3 \rangle = \langle e_3, \dot{\gamma} \rangle, \\ &\langle \eta^3, \varepsilon^1 \rangle = 0, && \langle \eta^3, \varepsilon^2 \rangle = -\langle e_3, \dot{\gamma} \rangle, && \langle \eta^3, \varepsilon^3 \rangle = \langle e_2, \dot{\gamma} \rangle. \end{aligned}$$

The relations for $\eta^{\{1,2\}} = \eta^1 \wedge \eta^2$, $\eta^{\{3,1\}}$, $\eta^{\{2,3\}}$ and $\varepsilon^{\{1,2\}}$, $\varepsilon^{\{3,1\}}$, $\varepsilon^{\{2,3\}}$ can be determined from the above relations by duality. Finally, $\langle \eta^0, \varepsilon^I \rangle = 1$ if $I = 0$ and 0 otherwise, and the other relations for η^0, ε^0, $\eta^{\{1,2,3\}}$, and $\varepsilon^{\{1,2,3\}}$ are similar.

Now choosing $I = J = 1$ (here we identify 1 with $\{1\}$) we obtain

$$\int_0^L e^{-2\lambda r} \hat{q}_{1,1}(2\lambda, \gamma(r)) \, dr = 0 \quad \text{for all } \lambda \text{ and } \gamma.$$

This means that the usual attenuated geodesic ray transform of the function $\hat{q}_{1,1}(2\lambda, \cdot)$ in M_0 vanishes for all λ. First we have $\hat{q}_{1,1}(2\lambda, \cdot) \in C^\infty(M_0)$ for all λ [Frigyik et al. 2008, Proposition 3], and then $\hat{q}_{1,1}(2\lambda, \cdot) = 0$ for all λ by the injectivity of the attenuated ray transform [Salo and Uhlmann 2011] and so $q_{1,1} = 0$. The same argument applies for all pairs (I, J) where

$$I, J \in \big\{0, 1, \{2,3\}, \{1,2,3\}\big\}.$$

Now consider the case where $I = 1$ and $J = 2$. Then

$$\int_0^L e^{-2\lambda r} \big(\hat{q}_{1,2}(2\lambda, \gamma(r)) \langle e_2, \dot{\gamma} \rangle + \hat{q}_{1,3}(2\lambda, \gamma(r)) \langle e_3, \dot{\gamma} \rangle \big) \, dr = 0.$$

Then the injectivity result for the attenuated ray transform on 1-tensors [Salo and Uhlmann 2011] together with the regularity result [Holman and Stefanov 2010, Proposition 1] says

$$\hat{q}_{1,2}(2\lambda, x)\varepsilon^2 + \hat{q}_{1,3}(2\lambda, x)\varepsilon^3 = 0$$

for all $\lambda \neq 0$, from which we can conclude

$$q_{1,2} = q_{1,3} = 0.$$

The same argument then applies for all pairs (I, J) where

$$I \in \{0, 1, \{2,3\}, \{1,2,3\}\} \quad \text{and} \quad J \in \{2, 3, \{1,2\}, \{3,1\}\},$$

or vice versa.

Finally, consider the case when $I = J = 2$. For brevity, we'll write $\langle e_j, \dot{\gamma}\rangle$ as $\dot{\gamma}_j$. Then $I = J = 2$ gives

$$\int_0^L e^{-2\lambda r}\big(\hat{q}_{2,2}\dot{\gamma}_2^2 + \hat{q}_{2,3}\dot{\gamma}_2\dot{\gamma}_3 + \hat{q}_{3,2}\dot{\gamma}_3\dot{\gamma}_2 + \hat{q}_{3,3}\dot{\gamma}_3^2\big)\, dr = 0. \tag{8-1}$$

The integrand here can be represented as the symmetric 2-tensor

$$f^{2,2} := \begin{pmatrix} \hat{q}_{2,2} & \frac{1}{2}(\hat{q}_{2,3} + \hat{q}_{3,2}) \\ \frac{1}{2}(\hat{q}_{2,3} + \hat{q}_{3,2}) & \hat{q}_{3,3} \end{pmatrix}$$

(in coordinates provided by $\{\varepsilon^2, \varepsilon^3\}$) applied to $(\dot{\gamma}, \dot{\gamma})$. This shows that the attenuated ray transform of the 2-tensor $f^{2,2}$ in (M_0, g_0), with constant attenuation -2λ, vanishes identically.

We will now make use of the methods of [Paternain et al. 2013] in this tensor tomography problem. We only give the details in the case where Q (and hence $f^{2,2}$) is C^∞. The result also holds for continuous Q by using an elliptic regularity result for the normal operator, but in the present weighted case for 2-tensors the required result may not be in the literature. We only say that such a result can be proved by adapting the methods of [Holman and Stefanov 2010] to the 2-tensor case (in particular one needs a solenoidal decomposition $f = f^s + d\beta$ of a 2-tensor f and a further solenoidal decomposition $\beta = \beta^s + d\phi$ of the 1-form β, and one then shows that the normal operator acting on "solenoidal triples" (f^s, β^s, ϕ) is elliptic because the weight comes from a nonvanishing attenuation).

Since $f^{2,2}$ is C^∞, the injectivity result for the attenuated ray transform on symmetric 2-tensors (see [Assylbekov 2012], following [Paternain et al. 2013]) says

$$f^{2,2} = -Xu + 2\lambda u,$$

where X is the geodesic vector field on (M_0, g_0), and u is a smooth function on the unit circle bundle SM_0 that corresponds to the sum of a 1-tensor and scalar function, with

$$u|_{\partial M_0} = 0.$$

Here we have identified $f^{2,2}$ and u with functions on SM_0 as in [Paternain et al. 2013]. We can also express u and $f^{2,2}$ in terms of Fourier components as in [loc. cit.],

$$u = u_{-1} + u_0 + u_1,$$
$$f^{2,2} = f^{2,2}_{-2} + f^{2,2}_0 + f^{2,2}_2.$$

Here $u_0 \in C^\infty(M_0)$, $u_1 + u_{-1}$ corresponds to a smooth 1-tensor in M_0, and u_0, u_1, u_{-1} vanish on ∂M_0. Then

$$-X(u_{-1} + u_0 + u_1) + 2\lambda(u_{-1} + u_0 + u_1) = f^{2,2}_{-2} + f^{2,2}_0 + f^{2,2}_2.$$

Now parity implies the equations

$$2\lambda(u_{-1} + u_1) = Xu_0 \quad \text{and} \quad -X(u_{-1} + u_1) + 2\lambda(u_0) = f^{2,2}_{-2} + f^{2,2}_0 + f^{2,2}_2.$$

Assume λ is nonzero. Using the first equation in the second one implies

$$-\frac{X^2(u_0)}{2\lambda} + 2\lambda u_0 = f^{2,2}, \tag{8-2}$$

where $X^2 u_0$ corresponds to the covariant Hessian $\nabla^2 u_0$ of u_0. The first equation implies u_0 vanishes to first order on ∂M_0.

Unfortunately, this is not enough to conclude that the coefficients of $f^{2,2}$ are 0. However, going back and choosing $(I, J) = (2,3), (3,2)$, and $(3,3)$ gives us three additional equations of this type with the same elements $q_{I,J}$. More specifically,

$$f^{2,3} = \begin{pmatrix} \hat{q}_{2,3} & \frac{1}{2}(\hat{q}_{3,3} - \hat{q}_{2,2}) \\ \frac{1}{2}(\hat{q}_{3,3} - \hat{q}_{2,2}) & -\hat{q}_{3,2} \end{pmatrix},$$

$$f^{3,2} = \begin{pmatrix} \hat{q}_{3,2} & \frac{1}{2}(\hat{q}_{3,3} - \hat{q}_{2,2}) \\ \frac{1}{2}(\hat{q}_{3,3} - \hat{q}_{2,2}) & -\hat{q}_{2,3} \end{pmatrix},$$

$$f^{3,3} = \begin{pmatrix} \hat{q}_{3,3} & -\frac{1}{2}(\hat{q}_{2,3} + \hat{q}_{3,2}) \\ -\frac{1}{2}(\hat{q}_{2,3} + \hat{q}_{3,2}) & \hat{q}_{2,2} \end{pmatrix}$$

are all of the same form. Therefore it follows that $f^{2,2} + f^{3,3}$ and $f^{2,3} - f^{3,2}$ are as well. But these are both scalar matrices, and if

$$-\frac{X^2(u_0)}{2\lambda} + 2\lambda u_0$$

is a scalar matrix, then also the covariant Hessian $\nabla^2 u_0$ is a scalar matrix in the $\{\varepsilon^2, \varepsilon^3\}$ basis.

To make the previous statement more explicit, identify (M_0, g_0) with the unit disk in $\mathbb{R}^2$ and choose an isothermal coordinate system (x^1, x^2) in which the metric is given by $e^{2\mu}\delta_{jk}$ for some $\mu \in C^\infty(M_0)$. Choosing $e_2 = e^{-\mu}\partial_1$ and $e_3 = e^{-\mu}\partial_2$, the condition $\nabla^2 u_0(e_2, e_2) - \nabla^2 u_0(e_3, e_3) = 0$ implies

$$\partial_1^2 u_0 - \partial_2^2 u_0 + b \cdot \nabla u_0 = 0 \quad \text{in } M_0$$

for some vector field $b \in C^\infty(M_0, \mathbb{R}^2)$ depending on μ. Since u_0 vanishes to first order on ∂M_0, extending u_0 by zero to $\mathbb{R}^2$ we have

$$\partial_1^2 u_0 - \partial_2^2 u_0 + b \cdot \nabla u_0 = 0 \quad \text{in } \mathbb{R}^2,$$

where $u_0 \in H^2(\mathbb{R}^2)$ is compactly supported and b is some smooth compactly supported vector field. Uniqueness for hyperbolic equations [Taylor 1996, Section 2.8] implies $u_0 = 0$.

The above argument shows that $f^{2,2} + f^{3,3}$ and $f^{2,3} - f^{3,2}$ are 0. Thus $\hat{q}_{2,2} + \hat{q}_{3,3} = 0$ and $\hat{q}_{2,3} - \hat{q}_{3,2} = 0$, showing that $f^{2,2}$ and $f^{2,3}$ are trace-free. Taking traces in (8-2) and using that u_0 vanishes to first order on ∂M_0 implies $u_0 = 0$ by unique continuation for elliptic equations. Thus $f^{2,2} = 0$ and similarly $f^{2,3} = 0$, which shows that $q_{2,2}$, $q_{2,3}$, $q_{3,2}$, and $q_{3,3}$ are zero as well.

The same argument now works for the remaining entries of q, and this finishes the proof.

9. Higher dimensions

In higher dimensions, $n > 3$, as noted above, everything up to and including the proof of Proposition 8.3 still holds. However, this does not reduce easily into a tensor tomography problem, as in the three-dimensional case, because we cannot choose the basis $\{\eta^i\}$ so that $\eta^3, \dots, \eta^4$ to depend on $\eta^2 = dr$ in a tensorial manner.

More precisely, in general we lack tensors T_i for which $\eta^i = T_i(\eta^2, \dots, \eta^2)$ for $i \geq 3$, as is the case in three dimensions. Moreover, even if the results of Proposition 8.3 can be reduced to a tensor tomography problem, there is no guarantee that it will be one for which there are useful injectivity results, since there are very few such results for k-tensors with $k > 2$.

However, in the Euclidean case we can do better, since we have the extra freedom to vary the Carleman weight φ. In particular, we can construct CGOs to reduce the problem in Lemmas 8.1 and 8.2 to a Fourier transform, as has been done for inverse problems for scalar functions, e.g., in [Bukhgeim and Uhlmann 2002]. Therefore we can conclude this paper by a proof for higher dimensions, in the Euclidean case.

Proof of Theorem 2.3. Fix coordinates $x_1, \dots, x_n$ on $\mathbb{R}^n$. The corresponding basis for the cotangent space is $dx^1, \dots, dx^n$, and this gives a corresponding basis $\{dx^I\}$ for ΛM.

Now note that if f is a scalar function, $\Delta(f dx^I) = (\Delta f) dx^I$. Therefore if α and β are unit vectors such that $\alpha \cdot \beta = 0$, then

$$e^{-\frac{\alpha \cdot x}{h}} h^2(-\Delta + Q)(e^{\frac{(\alpha + i\beta)\cdot x}{h}} dx^I) = O(h^2) dx^I.$$

Therefore Proposition 7.1 implies there exists $r \in L^2(M, \Lambda M)$ such that

$$(-\Delta + Q)(e^{\frac{(\alpha + i\beta)\cdot x}{h}}(dx^I + r)) = 0,$$

with $\|r\|_{L^2(M)} = O(h)$, and $Z = e^{\frac{(\alpha + i\beta)\cdot x}{h}}(dx^I + r)$ has relative boundary conditions which vanish on Γ_+^c.

Now if k and ℓ are mutually orthogonal unit vectors which are both orthogonal to α, then we can set $\beta_1 = \ell + hk$ and $\beta_2 = \ell - hk$, and create

$$Z_1 = e^{\frac{(-\alpha + i\beta_1)\cdot x}{h}}(dx^I + r_1) \quad \text{and} \quad Z_2 = e^{\frac{(\alpha + i\beta_2)\cdot x}{h}}(dx^I + r_2)$$

so that $(-\triangle + Q_1)Z_1 = (-\triangle + Q_2)Z_2 = 0$, and Z_1 and Z_2 have relative boundary conditions that vanish on Γ_-^c and Γ_+^c respectively.

Then Lemma 8.1, together with the hypotheses of Theorem 2.3, implies

$$(Q_1 - Q_2 \mid e^{-i2k\cdot x}) = 0.$$

This can be done for any k orthogonal to α. Since α can be varied slightly without preventing the relative boundary conditions of the solutions from vanishing on the correct set, this is in fact true for k in an open set, from which we can conclude that $Q_1 = Q_2$ on M.

The absolute boundary value version works similarly, with the appropriate change in the CGOs. ☐

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ON AN ISOPERIMETRIC-ISODIAMETRIC INEQUALITY

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The Euclidean mixed isoperimetric-isodiametric inequality states that the round ball maximizes the volume under constraint on the product between boundary area and radius. The goal of the paper is to investigate such mixed isoperimetric-isodiametric inequalities in Riemannian manifolds. We first prove that the same inequality, with the sharp Euclidean constants, holds on Cartan–Hadamard spaces as well as on minimal submanifolds of $\mathbb{R}^n$. The equality cases are also studied and completely characterized; in particular, the latter gives a new link with free-boundary minimal submanifolds in a Euclidean ball. We also consider the case of manifolds with nonnegative Ricci curvature and prove a new comparison result stating that metric balls in the manifold have product of boundary area and radius bounded by the Euclidean counterpart and equality holds if and only if the ball is actually Euclidean.

We then consider the problem of the existence of optimal shapes (i.e., regions minimizing the product of boundary area and radius under the constraint of having fixed enclosed volume), called here isoperimetric-isodiametric regions. While it is not difficult to show existence if the ambient manifold is compact, the situation changes dramatically if the manifold is not compact: indeed we give examples of spaces where there exists no isoperimetric-isodiametric region (e.g., minimal surfaces with planar ends and more generally C^0-locally asymptotic Euclidean Cartan–Hadamard manifolds), and we prove that on the other hand on C^0-locally asymptotic Euclidean manifolds with nonnegative Ricci curvature there exists an isoperimetric-isodiametric region for every positive volume (this class of spaces includes a large family of metrics playing a key role in general relativity and Ricci flow: the so-called Hawking gravitational instantons and the Bryant-type Ricci solitons).

Finally we prove the optimal regularity of the boundary of isoperimetric-isodiametric regions: in the part which does not touch a minimal enclosing ball, the boundary is a smooth hypersurface outside of a closed subset of Hausdorff codimension 8, and in a neighborhood of the contact region, the boundary is a $C^{1,1}$ hypersurface with explicit estimates on the L^∞ norm of the mean curvature.

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1. Introduction

One of the oldest questions of mathematics is the *isoperimetric* problem: *what is the largest amount of volume that can be enclosed by a given amount of area?* A related classical question is the *isodiametric* problem: *what is the largest amount of volume that can be enclosed by a domain having a fixed diameter?*

In this paper we address a mix of the previous two questions, namely we investigate the following mixed isoperimetric-isodiametric problem: *what is the largest amount of volume that can be enclosed by a domain having a fixed product of diameter and boundary area?*

Of course, if we ask the three above questions in the Euclidean space, the answer is given by round balls of suitable radius, but, of course, the situation in nonflat geometries is much more subtle. We start by recalling classical material on the isoperimetric problem which motivated our investigation on the mixed isoperimetric-isodiametric one.

The solution of the isoperimetric problem in the Euclidean space $\mathbb{R}^n$ can be summarized by the classical isoperimetric inequality

$$n\omega_n^{1/n}\,\mathrm{Vol}(\Omega)^{(n-1)/n} \le \mathcal{A}(\partial\Omega) \quad \text{for every } \Omega \subset \mathbb{R}^n \text{ open subset with smooth boundary,} \tag{1-1}$$

where $\mathrm{Vol}(\Omega)$ is the n-dimensional Hausdorff measure of Ω (i.e., the “volume” of Ω), $\mathcal{A}(\partial\Omega)$ is the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega$ (i.e., the “area” of $\partial\Omega$), and $\omega_n := \mathrm{Vol}(B^n)$ is the volume of the unit ball in $\mathbb{R}^n$. As is well known, the regularity assumption on Ω can be relaxed a lot (for instance (1-1) holds for every set Ω of finite perimeter), but let us not enter into technicalities here since we are just motivating our problem.

As anticipated above, in the present paper we will not deal with the isoperimetric problem itself but we will focus on a mixed isoperimetric-isodiametric problem. Let us start by stating the Euclidean mixed isoperimetric-isodiametric inequality, which will act as model for this paper. Given a bounded open subset $\Omega \subset \mathbb{R}^n$ with smooth boundary, by the divergence theorem in $\mathbb{R}^n$ (see Section 2 for the easy proof), we have

$$n\,\mathrm{Vol}(\Omega) \le \mathrm{rad}(\Omega)\mathcal{A}(\partial\Omega), \tag{1-2}$$

where $\mathrm{rad}(\Omega)$ is the radius of the smallest ball of $\mathbb{R}^n$ containing Ω (see (2-1) for the precise definition). As observed in Remark 2.1, inequality (1-2) is *sharp* and *rigid*; indeed, equality occurs if and only if Ω is a round ball in $\mathbb{R}^n$.

In sharp contrast with the classical isoperimetric problem, where both problems are still open in the general case, it is not difficult to show that the inequality (1-2) holds in Cartan–Hadamard spaces (i.e., simply connected Riemannian manifolds with nonpositive sectional curvature) and on minimal submanifolds of $\mathbb{R}^n$; see Propositions 3.1, 3.3 and 3.7. Even if the validity of inequality (1-2) in such spaces is probably known to experts, we included it here in order to motivate the reader and also because the equality case for minimal submanifolds presents an interesting link with free-boundary minimal surfaces: equality is attained in (1-2) if and only if the minimal submanifold is a free-boundary minimal surface in a Euclidean ball (see Proposition 3.3 for the precise statement and Remarks 3.5–3.6 for more information about free-boundary minimal surfaces).

If on one hand the negative curvature gives a stronger isoperimetric-isodiametric inequality, on the other hand we show that nonnegative Ricci curvature forces metric balls to satisfy a weaker isoperimetric-isodiametric inequality. The precise statement is the following.

Theorem 1.1 (Theorem 4.1). *Let* (M^n, g) *be a complete (possibly noncompact) Riemannian n-manifold with nonnegative Ricci curvature. Let* $B_r \subset M$ *be a metric ball of volume* $V = \mathrm{Vol}_g(B_r)$, *and denote by* $B^{\mathbb{R}^n}(V)$ *the round ball in* $\mathbb{R}^n$ *having volume V. Then*

$$\mathrm{rad}(B_r)\mathcal{A}(\partial B_r) = r\mathcal{A}(\partial B_r) \leq n\,\mathrm{Vol}_g(B_r) = \mathrm{rad}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V))\mathcal{A}_{\mathbb{R}^n}(\partial B^{\mathbb{R}^n}(V)). \tag{1-3}$$

Moreover equality holds if and only if B_r *is* isometric *to a round ball in the Euclidean space* $\mathbb{R}^n$. *In particular, for every* $V \in (0, \mathrm{Vol}_g(M))$,

$$\inf\bigl\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) : \Omega \subset M, \mathrm{Vol}_g(\Omega) = V\bigr\} \leq nV = \inf\bigl\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) : \Omega \subset \mathbb{R}^n, \mathrm{Vol}_{\mathbb{R}^n}(\Omega) = V\bigr\}, \tag{1-4}$$

with equality for some $V \in (0, \mathrm{Vol}_g(M))$ *if and only if every metric ball in* M *of volume* V *is isometric to a round ball in* $\mathbb{R}^n$. *In particular if equality occurs for some* $V \in (0, \mathrm{Vol}_g(M))$ *then* (M, g) *is flat, i.e., it has identically zero sectional curvature.*

Remark 1.2. Since by Bishop–Gromov volume comparison, we know that if $\mathrm{Ric}_g \geq 0$ then for every metric ball $B_r(x_0) \subset M$,

$$\mathrm{Vol}_g(B_r(x_0)) \leq \omega_n r^n = \mathrm{Vol}_{\mathbb{R}^n}(B_r^{\mathbb{R}^n}).$$

It follows that

$$\mathrm{rad}(B_r(x_0)) \geq \mathrm{rad}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V)),$$

where $B^{\mathbb{R}^n}(V)$ is a Euclidean ball of volume $V = \mathrm{Vol}_g(B_r(x_0))$. Therefore Theorem 1.1 in particular implies $\mathcal{P}(B_r(x_0)) \leq \mathcal{P}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V))$, but is a strictly stronger statement, which to the best of our knowledge is original. The aforementioned counterpart of Theorem 1.1 for the isoperimetric problem was proved instead by Morgan and Johnson [2000, Theorem 3.5] for compact manifolds and extended to noncompact manifolds in [Mondino and Nardulli 2016, Proposition 3.2].

In Section 5 we investigate the existence of optimal shapes in a general Riemannian manifold (M, g). More precisely, given a measurable subset $E \subset M$ we denote by $\mathcal{P}(E)$ its perimeter and define its extrinsic radius as

$$\mathrm{rad}(E) := \inf\bigl\{r > 0 : \mathrm{Vol}_g(E \setminus B_r(z_0)) = 0 \text{ for some } z_0 \in M\bigr\},$$

where $B_r(z_0)$ denotes the open metric ball with center z_0 and radius $r > 0$. We consider the following minimization problem: for every fixed $V \in (0, \mathrm{Vol}_g(M))$, find

$$\min\{\mathrm{rad}(E)\mathcal{P}(E) : E \subset M, \mathrm{Vol}_g(E) = V\}, \tag{1-5}$$

and call the minimizers of (1-5) isoperimetric-isodiametric sets (or regions). To best of our knowledge this is first time such a problem is considered in the literature.

As it happens also for the isoperimetric problem, we will find that if the ambient manifold is compact then for every volume there exists an isoperimetric-isodiametric region (see Theorem 5.2 and Corollary 5.3) but if the ambient space is noncompact the situation changes dramatically. Indeed in Examples 5.6–5.7 we show that in complete minimal submanifolds with planar ends (like the helicoid) and in asymptotically locally Euclidean Cartan–Hadamard manifolds, there exists no isoperimetric-isodiametric region of positive volume. On the other hand, we show that in C^0-locally asymptotically Euclidean manifolds (see Definition 5.4 for the precise notion) with nonnegative Ricci curvature for every volume there exists an isoperimetric-isodiametric region:

Theorem 1.3 (Theorem 5.5). *Let (M, g) be a complete Riemannian n-manifold with nonnegative Ricci curvature and fix any reference point $\bar{x} \in M$. Assume that for any diverging sequence of points $(x_k)_{k\in N} \subset M$, i.e.,* $\mathsf{d}(x_k, \bar{x}) \to \infty$, *the sequence of pointed manifolds (M, g, x_k) converges in the pointed C^0 topology to the Euclidean space $(\mathbb{R}^n, g_{\mathbb{R}^n}, 0)$.*

Then for every $V \in (0, \mathrm{Vol}_g(M))$ there exists a minimizer of the problem (1-5); *in other words, there exists an isoperimetric-isodiametric region of volume V.*

Let us mention that the counterpart of Theorem 1.3 for the isoperimetric problem was proved in [Mondino and Nardulli 2016] capitalizing on the work by Nardulli [2014].

Remark 1.4. It is well known that the only manifold with nonnegative Ricci curvature and C^0-*globally* asymptotic to $\mathbb{R}^n$ is $\mathbb{R}^n$ itself. Indeed if M is C^0-globally asymptotic to $\mathbb{R}^n$ then

$$\lim_{R\to\infty} \frac{\mathrm{Vol}_g(B_R(\bar{x}))}{\omega_n R^n} = 1,$$

which by the rigidity statement associated to the Bishop–Gromov inequality implies that (M, g) is globally isometric to $\mathbb{R}^n$. On the other hand, the assumption of Theorem 1.3 is much weaker as it asks (M, g) to be just *locally* asymptotic to $\mathbb{R}^n$ in the C^0 topology and many important examples enter in this framework, as explained in Example 1.5.

Example 1.5. The class of manifolds satisfying the assumptions of Theorem 1.3 contains many geometrically and physically relevant examples.

• *Eguchi–Hanson and, more generally, ALE gravitational instantons.* These are 4-manifolds, solutions of the Einstein vacuum equations with null cosmological constant (i.e., they are Ricci flat, $\mathrm{Ric}_g \equiv 0$), they are noncompact with just one end which is topologically a quotient of $\mathbb{R}^4$ by a finite subgroup of $O(4)$, and the Riemannian metric g on this end is asymptotic to the Euclidean metric up to terms of order $O(r^{-4})$,

$$g_{ij} = \delta_{ij} + O(r^{-4}),$$

with appropriate decay in the derivatives of g_{ij} (in particular, such metrics are C^0-locally asymptotic, in the sense of Definition 5.4, to the Euclidean 4-dimensional space). The first example of such manifolds was discovered by Eguchi and Hanson [1978]; inspired by the discovery of self-dual instantons in Yang–Mills theory, they found a self-dual ALE instanton metric. The Eguchi–Hanson example was then generalized by Gibbons and Hawking [1978]; see also the work by Hitchin [1979]. These metrics constitute the building blocks of the Euclidean quantum gravity theory of Hawking (see [Hawking 1977; 1979]). The ALE gravitational instantons were classified by Kronheimer [1989a; 1989b].

• *Bryant-type solitons.* The Bryant solitons, discovered by R. Bryant [2005], are special but fundamental solutions to the Ricci flow (see, for instance, the work of Brendle [2013; 2014] for higher dimensions). Such metrics are complete, have nonnegative Ricci curvature (they actually satisfy the stronger condition of having nonnegative curvature operator) and are locally C^0-asymptotically Euclidean. Other soliton examples fitting our assumptions are given by Catino and Mazzieri [2016].

Section 6 is then devoted to establishing the optimal regularity for isoperimetric-isodiametric regions under suitable assumptions on regularity of the enclosing ball. We first observe that outside of the contact region with the minimal enclosing ball B, such sets are locally minimizers of the perimeter under volume constraint. Therefore by classical results (see, for example, [Morgan 2003, Corollary 3.8]) in the interior of B the boundary of the region is a smooth hypersurface outside a singular set of Hausdorff codimension at least 8.

The rest of the paper is devoted to proving the optimal regularity at the contact region. We first show in Section 6A that isoperimetric-isodiametric regions are almost-minimizers for the perimeter (see Lemma 6.3) and therefore, by a result of Tamanini [1982] their boundaries are $C^{1,1/2}$ regular (see Proposition 6.1). In Section 6B, by means of geometric comparisons and sharp first-variation arguments, we show that the mean curvature of the boundary of an isoperimetric-isodiametric region is in L^∞ with explicit estimates. Finally in Section 6C we establish the optimal $C^{1,1}$ regularity. We mention that, strictly speaking, Section 6B is not needed to prove the optimal regularity; in any case we included such a section since it provides an explicit sharp L^∞ estimate on the mean curvature and is of independent interest. Now let us state the main regularity result.

Theorem 1.6 (Theorem 6.11). *Let $E \subset M$ be an isoperimetric-isodiametric set and $x_0 \in M$ be such that* $\mathrm{Vol}_g(E \setminus B_{\mathrm{rad}(E)}(x_0)) = 0$. *Assume $B := B_{\mathrm{rad}(E)}(x_0)$ has smooth boundary. Then, there exists $\delta > 0$ such that $\partial E \setminus B_{\mathrm{rad}(E)-\delta}(x_0)$ is $C^{1,1}$ regular.*

An essential ingredient in the proof of Theorem 1.6 is Proposition 6.12, which roughly tells that the boundary of E leaves the obstacle at most quadratically. Then the conclusion will follow by combining Schauder estimates outside of the contact region (see Lemma 6.13) with the general fact that functions which leave the first-order approximation quadratically are $C^{1,1}$ — see Lemma 6.14. Although the techniques exploited for this part of the paper are inspired by the ones introduced in the study of the classical obstacle problem (see, for example, [Caffarelli 1998]), here we treat the geometric case of the area functional in a Riemannian manifold with volume constraints and we take several short-cuts thanks to some specifically geometric arguments, such as the theory of almost minimizers. In particular, such a geometric situation doesn't seem to be trivially covered by the regularity results for nonlinear variational inequalities, as developed, for example, by Gerhardt [1973] — see Remark 6.16.

Remark 1.7. Note that the $C^{1,1}$ regularity is optimal, because in general one cannot expect to have continuity of the second fundamental form of ∂E across the free boundary of ∂E, i.e., the points on the relative (with respect to ∂B) boundary of $\partial E \cap \partial B$. The same is indeed true for the simplest case of the classical obstacle problem.

2. Notation, preliminaries and the Euclidean case

Let (Z, d) be a metric space. Given an open subset $\Omega \subset Z$, we define its *extrinsic radius* as

$$\operatorname{rad}(\Omega) := \inf\{r > 0 : \Omega \subset B_r(z_0) \text{ for some } z_0 \in Z\}, \tag{2-1}$$

where $B_r(z_0)$ denotes the open metric ball of center z_0 and radius $r > 0$.

The model inequality for the first part of the paper is the Euclidean mixed isoperimetric-isodiametric inequality obtained by the following integration by parts. Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary and let $x_0 \in \mathbb{R}^n$ be a point such that

$$\max_{x \in \overline{\Omega}} |x - x_0| = \operatorname{rad}(\Omega). \tag{2-2}$$

Denoting by X the vector field $X(x) := x - x_0$, by the divergence theorem in $\mathbb{R}^n$ we then get

$$n \operatorname{Vol}(\Omega) = \int_\Omega \operatorname{div} X \, d\mathcal{H}^n = -\int_{\partial\Omega} X \cdot \nu \, d\mathcal{H}^{n-1} \le \operatorname{rad}(\Omega)\mathcal{A}(\partial\Omega), \tag{2-3}$$

where $\operatorname{Vol}(\Omega)$ denotes the Euclidean n-dimensional volume of Ω, ν is the inward-pointing unit normal vector and $\mathcal{A}(\partial\Omega)$ is the Euclidean $(n-1)$-dimensional area of $\partial\Omega$, which here is assumed to be smooth. Notice that, analogously, if $\Omega \subset \mathbb{R}^n$ is a finite-perimeter set, one gets the inequality

$$\operatorname{Vol}(\Omega) \le \frac{\operatorname{rad}(\Omega)}{n} \mathcal{P}(\Omega), \tag{2-4}$$

where, of course, $\mathcal{P}(\Omega)$ denotes the perimeter of Ω (see Section 5A for the definitions of $\mathcal{P}(\Omega)$ and $\operatorname{rad}(\Omega)$ for finite-perimeter sets).

Remark 2.1. The inequalities (2-3) and (2-4) are *sharp* and *rigid*: indeed equality occurs if and only if Ω is a round ball.

3. Euclidean isoperimetric-isodiametric inequality in Cartan–Hadamard manifolds and minimal submanifolds

In order to motivate and gently introduce the reader to the topic, in this section we will prove that the Euclidean isoperimetric-isodiametric inequality holds with the same constant in Cartan–Hadamard spaces and in minimal submanifolds. Possibly apart from the rigidity statements, here we do not claim originality since such inequalities are probably well known to experts (see [Burago and Zalgaller 1988; Hoffman and Spruck 1974; Michael and Simon 1973]). However we included this section for the following reasons:

- While for the isoperimetric-isodiametric inequality the proofs are a consequence of a nondifficult integration by parts argument, the corresponding statements for the classical isoperimetric inequality are still open problems (see Remarks 3.2 and 3.4). This suggests that possibly in other situations isoperimetric-isodiametric inequalities may behave better than the classical isoperimetric ones.
- The rigidity statements, in the case of minimal submanifolds, show interesting connections between the isoperimetric-isodiametric inequality and free-boundary minimal surfaces, a topic which recently has received a lot of attention (for more details, see Remarks 3.5 and 3.6).

3A. ***The case of Cartan–Hadamard manifolds.*** Recall that a Cartan–Hadamard n-manifold is a complete simply connected Riemannian n-dimensional manifold with nonpositive sectional curvature. By a classical theorem of Cartan and Hadamard (see, for instance, [do Carmo 1992]) such manifolds are diffeomorphic to $\mathbb{R}^n$ via the exponential map. The next result is a sharp and rigid mixed isoperimetric-isodiametric inequality in such spaces. For this section, without losing much, the nonexpert reader may assume the region $\Omega \subset M$ has smooth boundary; in this case the perimeter is just the standard $(n-1)$-volume of the boundary (the perimeter will instead play a role in the next sections about existence and regularity of optimal sets).

Proposition 3.1. *Let* (M^n, g) *be a Cartan–Hadamard manifold. Then for every smooth open subset* (*or more generally for every finite-perimeter set*) $\Omega \subset M^n$,

$$n \operatorname{Vol}(\Omega) \le \operatorname{rad}(\Omega) \mathcal{A}(\partial\Omega), \tag{3-1}$$

where $\operatorname{Vol}(\Omega)$ *denotes the n-dimensional Riemannian volume of* Ω *and* $\mathcal{A}(\partial\Omega)$ *the* $(n-1)$*-dimensional area of the smooth boundary* $\partial\Omega$ (*in the case where* Ω *is a finite-perimeter set*, *just replace* $\mathcal{A}(\partial\Omega)$ *with* $\mathcal{P}(\Omega)$, *the perimeter of* Ω, *on the right-hand side*, *and* $\operatorname{rad}(\Omega)$ *is as in Section 5A*).[1] *Moreover*, *if for some* Ω *the equality is achieved*, *then* Ω *is isometric to* a Euclidean ball.

Proof. Let $\Omega \subset M^n$ be a subset with finite perimeter; without loss of generality we can assume that Ω is bounded (otherwise $\operatorname{rad}(\Omega) = +\infty$ and the inequality is trivial). Let $x_0 \in M^n$ be such that

$$\max_{x \in \overline{\Omega}} \mathsf{d}(x, x_0) = \operatorname{rad}(\Omega),$$

where d is the Riemannian distance on (M^n, g); for convenience we will also define $\mathsf{d}_{x_0}(\,\cdot\,) := \mathsf{d}(x_0, \cdot\,)$. Let $u := \frac{1}{2}\mathsf{d}_{x_0}^2$; by the aforementioned Cartan–Hadamard theorem (see, for instance, [do Carmo 1992]), we know that $u : M^n \to \mathbb{R}^+$ is smooth and by the Hessian comparison theorem, one has $(D^2 u)_{ij} \ge g_{ij}$; in particular, by tracing, we get $\Delta u \ge n$. Therefore, by the divergence theorem, we get

$$\begin{aligned} n \operatorname{Vol}(\Omega) &\le \int_\Omega \Delta u \, d\mu_g = -\int_{\partial^*\Omega} g(\nabla u, \nu)\, d\mathcal{H}^{n-1} = -\int_{\partial^*\Omega} \mathsf{d}(x, x_0) g(\nabla \mathsf{d}_{x_0}, \nu)\, d\mathcal{H}^{n-1} \\ &\le \operatorname{rad}(\Omega)\mathcal{H}^{n-1}(\partial^*\Omega) = \operatorname{rad}(\Omega)\mathcal{P}(\Omega), \end{aligned} \tag{3-2}$$

where μ_g is the measure associated to the Riemannian volume form, $\partial^*\Omega$ is the reduced boundary of Ω (of course, in the case where Ω is a smooth open subset, one has $\partial^*\Omega = \partial\Omega$), ν is the inward-pointing unit normal vector (recall that it is $\mathcal{H}^{n-1}$-a.e. well-defined on $\partial^*\Omega$), and we used that d_{x_0} is 1-Lipschitz. Of course (3-2) implies (3-1). Notice that if equality holds in the second line, then Ω is a metric ball of center x_0 and radius $\operatorname{rad}(\Omega)$. Moreover if equality occurs in the first inequality of the first line then we must have $(D^2 \mathsf{d}_{x_0}^2)_{ij} \equiv 2 g_{ij}$ on Ω, and by standard comparison (see, for instance, [Ritoré 2005, Section 4.1]) it follows that Ω is flat. But since the exponential map in M is a global diffeomorphism, it follows that Ω is isometric to a Euclidean ball. □

[1] For the readers' convenience we recall here the definition of $\operatorname{rad}(\Omega)$ for a finite-perimeter set $\Omega \subset M$ such that $\operatorname{rad}(\Omega) := \inf\{r > 0 : \operatorname{Vol}(\Omega \setminus B_r) = 0,\ B_r \subset M \text{ metric ball}\}$.

Remark 3.2 (Euclidean isoperimetric inequality on Cartan–Hadamard spaces). The statement corresponding to Proposition 3.1 for the isoperimetric problem is the following celebrated conjecture: *Let* (M^n, g) *be a Cartan–Hadamard space, i.e., a complete simply connected Riemannian n-manifold with nonpositive sectional curvature. Then every smooth open subset* $\Omega \subset M^n$ *satisfies the Euclidean isoperimetric inequality.*

This conjecture is generally attributed to Aubin [1976, Conjecture 1] but has its roots in earlier work by Weil [1926], as we are going to explain. The problem has been solved affirmatively in the following cases: in dimension 2 by Weil [1926] (Beckenbach and Radó [1933] gave an independent proof in 1933, capitalizing on a result of Carleman [1921] for minimal surfaces), in dimension 3 by Kleiner [1992] (see also the survey paper by Ritoré [2005] for a variant of Kleiner's arguments), and in dimension 4 by Croke [1984]. An interesting feature of this problem is that the above proofs have nothing to do with each other and that they work only for one specific dimension; probably also for this reason such a problem is still open in the general case.

3B. ***The case of minimal submanifolds.*** Given a smoothly immersed submanifold $M^n \hookrightarrow \mathbb{R}^{n+k}$, by the first variation formula for the area functional we know that for every $\Omega \subset M^n$ open bounded subset with smooth boundary and every smooth vector field X along Ω,

$$\int_\Omega \operatorname{div}_M X \, d\mathcal{H}^n = -\int_\Omega H \cdot X \, d\mathcal{H}^n - \int_{\partial\Omega} X \cdot \nu \, d\mathcal{H}^{n-1}, \tag{3-3}$$

where H is the mean curvature vector of M and ν is the inward-pointing conormal to Ω (i.e., ν is the unit vector tangent to M, normal to $\partial\Omega$ and pointing inside Ω).

We are interested in the case where $M^n \hookrightarrow \mathbb{R}^{n+k}$ is a minimal submanifold, i.e., $H \equiv 0$, and $\Omega \subset M^n$ is a bounded open subset with smooth boundary $\partial\Omega$. Let $x_0 \in \mathbb{R}^{n+k}$ be such that

$$\max_{x \in \overline{\Omega}} |x - x_0|_{\mathbb{R}^{n+k}} = \operatorname{rad}_{\mathbb{R}^{n+k}}(\Omega),$$

and observe that, defining $X(x) := x - x_0$, one has $\operatorname{div}_M X \equiv n$. By applying (3-3), we then get

$$n\mathcal{H}^n(\Omega) = \int_\Omega \operatorname{div}_M X \, d\mathcal{H}^n = -\int_{\partial\Omega} X \cdot \nu \, d\mathcal{H}^{n-1} \le \operatorname{rad}_{\mathbb{R}^{n+k}}(\Omega)\, \mathcal{H}^{n-1}(\partial\Omega). \tag{3-4}$$

Notice that equality is achieved if and only if Ω is the intersection of M with a round ball in $\mathbb{R}^{n+k}$ centered at x_0 and $\nu(x)$ is parallel to $x - x_0$, or in other words if and only if Ω is a free-boundary minimal n-submanifold in a ball of $\mathbb{R}^{n+k}$. So we have just proved the following result.

Proposition 3.3. *Let* $M^n \hookrightarrow \mathbb{R}^{n+k}$ *be a minimal submanifold and* $\Omega \subset M^n$ *a bounded open subset with smooth boundary* $\partial\Omega$. *Then*

$$n\mathcal{H}^n(\Omega) \le \operatorname{rad}_{\mathbb{R}^{n+k}}(\Omega)\, \mathcal{H}^{n-1}(\partial\Omega)$$

with equality if and only if Ω *is a free-boundary minimal n-submanifold in a ball of* $\mathbb{R}^{n+k}$.

Remark 3.4 (Euclidean isoperimetric inequality on minimal submanifolds). The statement corresponding to Proposition 3.3 for the isoperimetric problem is the following celebrated conjecture: *Let* $M^n \subset \mathbb{R}^m$

be a minimal n-dimensional submanifold and let $\Omega \subset M^n$ be a smooth open subset. Then Ω satisfies the Euclidean isoperimetric inequality (1-1)*, and equality holds if and only if Ω is a ball in an affine n-plane of $\mathbb{R}^m$.*

To our knowledge the only two solved cases are (i) when $\partial\Omega$ lies on an $(m-1)$-dimensional Euclidean sphere centered at a point of Ω (the argument is by monotonicity; see, for instance, [Choe 2005, Section 8.1]) and (ii) when Ω is area-minimizing with respect to its boundary $\partial\Omega$ by Almgren [1986]. Let us mention that a complete solution of the above conjecture is still not available even for minimal surfaces in $\mathbb{R}^m$, i.e., for $n = 2$; however, in the latter situation, the statement is known to be true in many cases (let us just mention that in the case where Ω is a topological disk, the problem was solved by Carleman [1921], and the case $m = 3$ and $\partial\Omega$ has two connected components was settled much later by Li, Schoen and Yau [Li et al. 1984]; for more results in this direction and for a comprehensive overview, see the beautiful survey paper [Choe 2005]). Let us finally observe that, when $n = 2$ and $m = 3$, the above conjecture is a special case of the Aubin conjecture recalled in Remark 3.2, since of course the induced metric on a immersed minimal surface in $\mathbb{R}^3$ has nonpositive Gauss curvature; this case was settled in the pioneering work by Weil [1926].

Remark 3.5 (free-boundary minimal submanifolds and critical metrics). After a classical work of Nitsche [1985], recent years have seen an increasing interest in free-boundary submanifolds, also thanks to works of Fraser and Schoen [2011; 2012] on the topic. By definition, a *free-boundary submanifold M^n* of the unit ball B^{n+k} is a proper submanifold which is critical for the area functional with respect to variations of M^n that are allowed to move also the boundary ∂M^n, but under the constraint $\partial M^n \subset \partial B^{n+k}$. As a consequence of the first variational formula, such a definition forces on one hand the mean curvature to vanish on $M^n \cap B^{n+k}$ and on the other hand the submanifold to the meet the ambient boundary ∂B^{n+k} orthogonally. These are characterized by the condition that the coordinate functions are Steklov eigenfunctions with eigenvalue 1 [Fraser and Schoen 2011, Lemma 2.2]; that is,

$$\Delta x_i = 0 \quad \text{on } M \qquad \text{and} \qquad \nabla_\nu x_i = -x_i \quad \text{on } \partial M.$$

It turns out that surfaces of this type arise naturally as extremal metrics for the Steklov eigenvalues (see [Fraser and Schoen 2012] for more details); Steklov eigenvalues are eigenvalues of the Dirichlet-to-Neumann map, which sends a given smooth function on the boundary to the normal derivative of its harmonic extension to the interior.

Remark 3.6 (examples of free-boundary minimal submanifolds). Let us recall here some well known examples of free-boundary minimal submanifolds in the unit ball $B^{n+k} \subset \mathbb{R}^{n+k}$; for a deeper discussion on the examples below, see [Fraser and Schoen 2012].

• *Equatorial disk.* Equatorial n-disks $D^n \subset B^{n+k}$ are the simplest examples of free-boundary minimal submanifolds. By a result of Nitsche [1985], any simply connected free-boundary minimal surface in B^3 must be a flat equatorial disk. However, if we admit minimal surfaces of a different topological type, there are other examples, such as the critical catenoid described below.

• *Critical Catenoid.* Consider the catenoid parametrized on $\mathbb{R} \times S^1$ by the function

$$\varphi(t, \theta) = (\cosh t \cos\theta, \cosh t \sin\theta, t).$$

For a unique choice of $T_0 > 0$, the restriction of φ to $[-T_0, T_0] \times S^1$ defines a minimal embedding into a ball meeting the boundary of the ball orthogonally. By rescaling the radius of the ball to 1 we get the critical catenoid in B^3. Explicitly, T_0 is the unique positive solution of $t = \coth t$.

• *Critical Möbius band.* We think of the Möbius band M^2 as $\mathbb{R} \times S^1$ with the identification $(t, \theta) \sim (-t, \theta + \pi)$. There is a minimal embedding of M^2 into $\mathbb{R}^4$ given by

$$\varphi(t, \theta) = (2 \sinh t \cos\theta, 2 \sinh t \sin\theta, \cosh 2t \cos 2\theta, \cosh 2t \sin 2\theta).$$

For a unique choice of $T_0 > 0$, the restriction of φ to $[-T_0, T_0] \times S^1$ defines a minimal embedding into a ball meeting the boundary of the ball orthogonally. By rescaling the radius of the ball to 1 we get the critical Möbius band in B^4. Explicitly T_0 is the unique positive solution of $\coth t = 2 \tanh 2t$.

• A consequence of the results of [Fraser and Schoen 2012] is that for every $k \geq 1$ there exists an embedded free-boundary minimal surface in B^3 of genus 0 with k boundary components.

Since of course $\mathrm{rad}_{\mathbb{R}^{n+k}}(\Omega) \leq \mathrm{rad}_M(\Omega)$, where $\mathrm{rad}_M(\cdot)$ is the extrinsic radius in the metric space (M, d_g), we have a fortiori that

$$n\mathcal{H}^n(\Omega) \leq \mathrm{rad}_M(\Omega)\mathcal{H}^{n-1}(\partial\Omega). \tag{3-5}$$

But in this case the rigidity statement is much stronger, indeed in the case of equality, the center of the ball x_0 must be a point of M. Moreover, for every $x \in \partial\Omega$ the segment $\overline{x, x_0}$ must be contained in M; therefore M contains a portion of a minimal cone $\mathcal{C}$ centered at x_0. But since by assumption M is a smooth submanifold and since the only cone smooth at its origin is an affine subspace, it must be that M contains a portion of an affine subspace. By the classical weak unique continuation property for solutions to the minimal submanifold system, we conclude that M is an affine subspace of $\mathbb{R}^{n+k}$. Therefore we have just proven the next result.

Proposition 3.7. *Let $M^n \hookrightarrow \mathbb{R}^{n+k}$ be a connected smooth minimal submanifold and $\Omega \subset M^n$ a bounded open subset with smooth boundary $\partial\Omega$. Then*

$$n\mathcal{H}^n(\Omega) \leq \mathrm{rad}_M(\Omega)\mathcal{H}^{n-1}(\partial\Omega) \tag{3-6}$$

with equality if and only if M is an affine subspace and Ω is the intersection of M with a round ball in $\mathbb{R}^{n+k}$ centered at a point of M.

Remark 3.8. If we allow M to have conical singularities, then (3-6) still holds with equality if and only if M is a minimal cone and Ω is the intersection of M with a round ball in $\mathbb{R}^{n+k}$ centered at a point of M.

Concerning this, recall that in the case where $n = 2$ and $k = 1$ every minimal cone smooth away from the vertex is totally geodesic; indeed one of the principal curvatures is always null for cones and so the mean curvature vanishes if and only if all of the second fundamental form is null. Therefore equality in (3-6) is attained if and only if M^2 is an affine plane and Ω is a flat 2-disk. The analogous result for $n = 3$ and $k = 1$ is due to Almgren [1966] (see also the work of Calabi [1967]).

For the general case of higher dimensions and codimensions, a minimal submanifold Σ^k in S^n is naturally the boundary of a minimal submanifold of the ball, the cone $C(\Sigma)$ over Σ. Using this correspondence it is possible to construct many nontrivial minimal cones: Hsiang [1983a; 1983b] gave infinitely many codimension-1 examples for $n \geq 4$, the higher-codimensional problem was investigated in the celebrated paper of Simons [1968] and the related work of Bombieri, De Giorgi and Giusti [Bombieri et al. 1969].

4. The isoperimetric-isodiametric inequality in manifolds with nonnegative Ricci curvature

In this section we show a comparison result for manifolds with nonnegative Ricci curvature which will be used in Section 5 to get existence of isoperimetric-isodiametric regions in manifolds which are asymptotically locally Euclidean and have nonnegative Ricci (the so-called ALE spaces).

Theorem 4.1. *Let (M^n, g) be a complete (possibly noncompact) Riemannian n-manifold with nonnegative Ricci curvature. Let $B_r \subset M$ be a metric ball of volume $V = \mathrm{Vol}(B_r)$, and denote by $B^{\mathbb{R}^n}(V)$ the round ball in $\mathbb{R}^n$ having volume V. Then*

$$\mathrm{rad}(B_r)\mathcal{P}(B_r) = r\mathcal{P}(B_r) \leq nV = \mathrm{rad}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V))\mathcal{P}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V)). \tag{4-1}$$

Moreover equality holds if and only if B_r is isometric *to a round ball in the Euclidean space $\mathbb{R}^n$. In particular, for every $V \in (0, \mathrm{Vol}(M))$,*

$$\inf\bigl\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) : \Omega \subset M,\, \mathrm{Vol}(\Omega)=V\bigr\} \leq nV = \inf\bigl\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) : \Omega \subset \mathbb{R}^n,\, \mathrm{Vol}_{\mathbb{R}^n}(\Omega)=V\bigr\}, \tag{4-2}$$

with equality for some $V \in (0, \mathrm{Vol}(M))$ if and only if every metric ball in M of volume V is isometric to a round ball in $\mathbb{R}^n$. In particular, if equality occurs for some $V \in (0, \mathrm{Vol}(M))$ then (M, g) is flat, i.e., it has identically zero sectional curvature.

Proof. Let us fix an arbitrary $x_0 \in M$ and let $B_r = B_r(x_0)$ be the metric ball in M centered at x_0 of radius $r > 0$. It is well known that the distance function $\mathsf{d}_{x_0}(\,\cdot\,) := \mathsf{d}(x_0, \cdot\,)$ is smooth outside the cut locus $\mathcal{C}_{x_0}$ of x_0 and that $\mu_g(\mathcal{C}_{x_0}) = 0$. From the coarea formula it follows that for $\mathcal{L}^1$-a.e. $r \geq 0$ one has $\mathcal{H}^{n-1}(\mathcal{C}_{x_0} \cap \partial B_r(x_0)) = 0$ and, since the cut locus is closed by definition, we get that for $\mathcal{L}^1$-a.e. $r \geq 0$ the distance function $\mathsf{d}_{x_0}(\,\cdot\,)$ is smooth on an open subset of full $\mathcal{H}^{n-1}$ measure on $\partial B_r(x_0)$.

Let us first assume that $r > 0$ is one of these regular radii; the general case will be settled in the end by an approximation argument. It is immediate to see that on $\partial B_r(x_0) \setminus \mathcal{C}_{x_0}$ we have $|\nabla \mathsf{d}_{x_0}| = 1$ and thus $\partial B_r(x_0) \setminus \mathcal{C}_{x_0}$ is a smooth hypersurface. In particular, since $\mathcal{H}^{n-1}(\partial B_r(x_0) \cap \mathcal{C}_{x_0}) = 0$, we have that $B_r(x_0)$ is a finite-perimeter set whose reduced boundary is contained in $\partial B_r(x_0) \setminus \mathcal{C}_{x_0}$. Letting ν be the inward-pointing unit normal to $\partial B_r(x_0)$ on the regular part $\partial B_r(x_0) \setminus \mathcal{C}_{x_0}$, from the Gauss lemma we have

$$\nu = -\nabla \mathsf{d}_{x_0} \quad \text{on } \partial B_r(x_0) \setminus \mathcal{C}_{x_0}. \tag{4-3}$$

Therefore, setting $u := \frac{1}{2}\mathsf{d}^2_{x_0}$, we get

$$\begin{aligned} r\mathcal{P}(B_r(x_0)) &= -\int_{\partial B_r(x_0)\setminus \mathcal{C}_{x_0}} \mathsf{d}_{x_0}(x) g(\nabla \mathsf{d}_{x_0}(x), \nu(x))\, d\mathcal{H}^{n-1}(x) = -\int_{\partial B_r(x_0)\setminus \mathcal{C}_{x_0}} g(\nabla u, \nu)\, d\mathcal{H}^{n-1} \\ &= -\lim_{\varepsilon \downarrow 0} \int_{\partial B_r(x_0)\setminus \mathcal{C}_{x_0}} g(\nabla u_\varepsilon, \nu)\, d\mathcal{H}^{n-1}, \end{aligned}$$

where $u_\varepsilon \in C^2(M)$ is an approximation by convolution of u such that $\|\nabla u_\varepsilon - \nabla u\|_{L^\infty(\partial B_r(x_0), \mathcal{H}^{n-1})} \to 0$, $\Delta u_\varepsilon \to \Delta u$ in $C^0_{\mathrm{loc}}(M \setminus \mathcal{C}_{x_0})$ and $\Delta u_\varepsilon \leq n$, where in the last estimate we used the global Laplacian comparison stating that Δu is a Radon measure with $\Delta u \le n\mu_g$. More precisely, one has that $\Delta u \llcorner M \setminus \mathcal{C}_{x_0}$ is given by μ_g multiplied by a smooth function bounded above by n, and the singular part $(\Delta u)^s$ of Δu is a nonpositive measure concentrated on $\mathcal{C}_{x_0}$. Now ∇u_ε is a C^1 vector field and we can apply the Gauss–Green formula for finite perimeter sets [Ambrosio et al. 2000, Theorem 3.36] to get

$$\begin{aligned} r\mathcal{P}(B_r(x_0)) &= \lim_{\varepsilon\downarrow 0}\int_{B_r(x_0)} \Delta u_\varepsilon\, d\mu_g = \lim_{\varepsilon\downarrow 0}\int_{B_r(x_0)\setminus\mathcal{C}_{x_0}} \Delta u_\varepsilon\, d\mu_g \le \int_{B_r(x_0)\setminus\mathcal{C}_{x_0}} \limsup_{\varepsilon\downarrow 0} \Delta u_\varepsilon\, d\mu_g \\ &= \int_{B_r(x_0)\setminus\mathcal{C}_{x_0}} \Delta u\, d\mu_g \le n\,\mathrm{Vol}(B_r), \end{aligned} \tag{4-4}$$

where in the first inequality we used Fatou's lemma combined with the upper bound $\Delta u_\varepsilon \le n$ and the last inequality is ensured by the local Laplacian comparison theorem. Notice that if equality occurs then $\Delta u = n\mu_g$ on $B_r(x_0)\setminus\mathcal{C}_{x_0}$ and, by analyzing the equality in Riccati equations, it is well known that this implies $B_r(x_0)$ is isometric to the round ball in $\mathbb{R}^n$.

If $r > 0$ is a singular radius, in the sense that $\mathcal{H}^{n-1}(\partial B_r(x_0)\cap\mathcal{C}_{x_0}) > 0$, then by the above discussion we can find a sequence of regular radii $r_n \to r$ and, by the lower semicontinuity of the perimeter under L^1_{loc} convergence [Ambrosio et al. 2000, Proposition 3.38] combined with (4-4), which is valid for $B_{r_n}(x_0)$, we get

$$\begin{aligned} r\mathcal{P}(B_r(x_0)) &\le \liminf_{n\to\infty} r_n\mathcal{P}(B_{r_n}(x_0)) \le \liminf_{n\to\infty}\int_{B_{r_n}(x_0)\setminus\mathcal{C}_{x_0}} \Delta u\, d\mu_g \le \limsup_{n\to\infty}\int_{M\setminus\mathcal{C}_{x_0}} \chi_{B_{r_n}(x_0)}\Delta u\, d\mu_g \\ &\le \int_{M\setminus\mathcal{C}_{x_0}} \limsup_{n\to\infty}\chi_{B_{r_n}(x_0)}\Delta u\, d\mu_g = \int_{B_r(x_0)\setminus\mathcal{C}_{x_0}} \Delta u\, d\mu_g \le n\,\mathrm{Vol}(B_r), \end{aligned} \tag{4-5}$$

where in the first inequality of the second line we used Fatou's lemma (we are allowed since $\chi_{B_{r_n}(x_0)}\Delta u \le n$ on $M\setminus\mathcal{C}_{x_0}$), and the last inequality follows again by local Laplacian comparison. Notice that, as before, equality in (4-5) forces $\Delta u = n\mu_g$ on $B_r(x_0)\setminus\mathcal{C}_{x_0}$ and then $B_r(x_0)$ is isometric to a Euclidean ball.

The second part of the statement clearly follows from the first part combined with the Euclidean isoperimetric-isodiametric inequality (2-3). □

5. Existence of isoperimetric-isodiametric regions

In Section 3 we saw explicit isoperimetric-inequalities in some special situations: Cartan–Hadamard spaces and minimal submanifolds. In the present section we investigate the existence of optimal shapes: as it happens also for the isoperimetric problem, we will find that if the ambient manifold is compact, an optimal set always exists but if the ambient space is noncompact the situation changes dramatically. The subsequent sections will be devoted to establishing the sharp regularity for the optimal sets.

5A. ***Notation.*** Let (M^n, g) be a complete Riemannian manifold and denote by d_g the geodesic distance, by μ_g the measure associated to the Riemannian volume form and by $\mathfrak{X}(M)$ the smooth vector fields. Given a measurable subset $E \subset M$, the perimeter of E is denoted by $\mathcal{P}(E)$ and is given by the formula

$$\mathcal{P}(E) := \sup\left\{\int_E \operatorname{div} X\, \mathrm{d}\mu_g : X \in \mathfrak{X}(M),\ \operatorname{spt}(X) \Subset M,\ \|X\|_{L^\infty(M,g)} \le 1\right\},$$

and, for any open subset $\Omega \subset M$, we write $\mathcal{P}(E, \Omega)$ when the fields X are restricted to having compact support in Ω. It is out of the scope of this paper to discuss the theory of finite-perimeter sets; standard references are [Ambrosio et al. 2000; Evans and Gariepy 1992; Maggi 2012].

Since from now on we will work with sets of finite perimeter, which are well defined up to subsets of measure zero, we will adopt the following definition of extrinsic radius of a measurable subset $E \subset M$:

$$\mathrm{rad}(E) := \inf\{r > 0 : \mu_g(E \setminus B_r(z_0)) = 0 \ \text{ for some } z_0 \in M\},$$

where $B_r(z_0)$ denotes the open metric ball with center z_0 and radius $r > 0$. A metric ball $B_r(z_0)$ satisfying $\mu_g(E \setminus B_r(z_0)) = 0$, is called an *enclosing ball* for E.

We consider the following minimization problem: for every fixed $V \in (0, \mu_g(M))$, find

$$\min\{\mathrm{rad}(E)\mathcal{P}(E) : E \subset M,\ \mu_g(E) = V\}, \tag{5-1}$$

and call the minimizers of (5-1) isoperimetric-isodiametric sets (or regions).

5B. ***Existence of isoperimetric-isodiametric regions in compact manifolds.*** Let us start with the following lemma, stating the lower semicontinuity of the extrinsic radius under L^1_{loc} convergence.

Lemma 5.1 (lower semicontinuity of extrinsic radius under L^1_{loc} convergence). *Let (M, g) be a (not necessarily compact) Riemannian manifold and let $(E_k)_{k\in\mathbb{N}\cup\{\infty\}}$ be a sequence of measurable subsets such that $\chi_{E_k} \to \chi_{E_\infty}$ in $L^1_{\mathrm{loc}}(M, \mu_g)$. Then*

$$\mathrm{rad}(E_\infty) \le \liminf_{k\in\mathbb{N}} \mathrm{rad}(E_k).$$

Proof. Without loss of generality we can assume $\liminf_{k\in\mathbb{N}} \mathrm{rad}(E_k) < \infty$ so, up to selecting a subsequence, we can assume $\chi_{E_k} \to \chi_{E_\infty}$ a.e. and $\lim_{k\uparrow+\infty} \mathrm{rad}(E_k) = \ell < \infty$. Let $B_k := B_{\mathrm{rad}(E_k)}(x_k)$ be enclosing balls for E_k. Then two cases can occur. Either x_k is unbounded, i.e., $\sup_k d_g(x_k, \bar{x}) = \infty$ for any $\bar{x} \in M$, in which case it follows that $E_\infty = \varnothing$ and the conclusion of the lemma is proved, or there exists $x_\infty \in M$ such that, up to passing to a subsequence, $x_k \to x_\infty$. In this case it is readily verified that

$$\mu_g\big(E_k \setminus B_{\mathrm{rad}(E_k)+|x_k - x_\infty|}(x_\infty)\big) = 0,$$

from which it follows, by taking the limit as $k \to +\infty$, that $\mu_g(E_\infty \setminus B_\ell(x_\infty)) = 0$, which by definition implies $\mathrm{rad}(E_\infty) \le \ell$. □

The next theorem is a general existence result for minimizers of the problem (5-1), as a special case it will be applied in Corollary 5.3 to compact manifolds and in Theorem 5.5 for asymptotically locally Euclidean manifolds (ALE for short) having nonnegative Ricci curvature. Let us observe that the existence of a minimizer in a noncompact manifold for the classical isoperimetric problem is much harder due to the possibility of "small tentacles" going to infinity in a minimizing sequence; this difficulty is simply not there in the isoperimetric-isodiametric problem we are considering, since it would imply the radius goes to infinity. We believe that this simplification, together with sharp inequalities obtained in the previous section, is another motivation to look at the isoperimetric-isoperimetric inequality since it appears more manageable in many situations than the classical isoperimetric one.

Theorem 5.2 (sufficient conditions for existence of isoperimetric-isodiametric regions). *Let* (M^n, g) *be a possibly noncompact Riemannian n-manifold satisfying the following two conditions*:

(1) $\liminf_{r\to 0^+} \sup_{x\in M} \mu_g(B_r(x)) = 0$.

(2) *There exists* $\varepsilon_0 > 0$ *and a function*

$$\Phi_{\mathrm{Isop}} : [0, \varepsilon_0) \to \mathbb{R}^+, \quad \textit{with } \lim_{t\downarrow 0}\Phi_{\mathrm{Isop}}(t) = 0,$$

such that for every finite-perimeter set $E \subset M$ *with* $\mathcal{P}(E) < \varepsilon_0$ *the weak isoperimetric inequality* $\mu_g(E) \le \Phi_{\mathrm{Isop}}(\mathcal{P}(E))$ *holds.*

Let $V \in (0, \mu_g(M))$ *be fixed and let* $(E_k)_{k\in\mathbb{N}} \subset M$ *be a sequence of finite-perimeter sets satisfying*

$$\mu_g(E_k) = V \quad \forall k \in \mathbb{N} \qquad \textit{and} \qquad \sup_{k\in\mathbb{N}}\big(\mathrm{rad}(E_k)\mathcal{P}(E_k)\big) < \infty. \tag{5-2}$$

Then there exist $R > 0$ *and a sequence* $(x_k)_{k\in N}$ *of points in* M *such that* $\mu_g(E_k \setminus B_R(x_k)) = 0$, *i.e.*, $B_R(x_k)$ *are enclosing balls for* E_k.

In particular, *if there exists a minimizing sequence* $(E_k)_{k\in\mathbb{N}}$ *for the problem* (5-1) *relative to some fixed* $V \in (0, \mu_g(M))$ *such that* $\mu_g(E_k \cap K) > 0$ *for infinitely many k and a fixed compact subset* $K \subset M$, *then there exists an isoperimetric-isodiametric region of volume* V.

Proof. We start the proof with the following two claims.

Claim 1: $\inf_k \mathrm{rad}(E_k) > 0$. Otherwise, up to subsequences in k, there exist $r_k \downarrow 0$ and $x_k \in M$ such that $\mu_g(E_k \setminus B_{r_k}(x_k)) = 0$. But then the assumption (1) implies $\mu_g(E_k) \le \mu_g(B_{r_k}(x_k)) = 0$, contradicting (5-2).

Claim 2: $\inf_k \mathcal{P}(E_k) > 0$. Otherwise, by the assumption (2) we get $\mu_g(E_k) \le \Phi_{\mathrm{Isop}}(\mathcal{P}(E_k)) \to 0$, contradicting again (5-2).

Combining the two claims with (5-2), we have that there exists $C > 1$ such that

$$\frac{1}{C} \le \mathcal{P}(E_k) \le C \quad \text{and} \quad \frac{1}{C} \le \mathrm{rad}(E_k) \le C, \tag{5-3}$$

so that the first part of the proposition is proved.

If now there exists a compact subset $K \subset M$ such that $\mu_g(E_k \cap K) > 0$ for infinitely many k then by (5-3), up to enlarging K and selecting a subsequence in k, we can assume $\mu_g(E_k \setminus K) = 0$. But then the characteristic functions $(\chi_{E_k})_{k\in\mathbb{N}}$ are precompact in $L^1(K, \mu_g)$ since the total variations of χ_{E_k} are equibounded by (5-3) (see [Ambrosio et al. 2000, Theorem 3.23]). The thesis then follows by the lower semicontinuity of the perimeter under L^1_{loc} convergence (see [loc. cit., Proposition 3.38]) combined with Lemma 5.1. □

Clearly if the manifold is compact all the assumptions of Theorem 5.2 are satisfied and we can state the following corollary.

Corollary 5.3 (existence of isoperimetric-isodiametric regions in compact manifolds). *Let* (M^n, g) *be a compact Riemannian manifold. Then for every* $V \in (0, \mu_g(M))$ *there exists a minimizer of the problem* (5-1); *in other words*, *there exists an isoperimetric-isodiametric region of volume* V.

5C. ***Existence of isoperimetric-isodiametric regions in noncompact ALE spaces with nonnegative Ricci curvature.*** Let us start by recalling the notion of pointed C^0 convergence of metrics.

Definition 5.4. Let (M^n, g) be a smooth complete Riemannian manifold and fix $\bar{x} \in M$. A sequence of pointed smooth complete Riemannian n-manifolds (M_k, g_k, x_k) is said to converge in the pointed C^0 topology to the manifold $(M, g, \bar{x})$, and we write $(M_k, g_k, x_k) \to (M, g, \bar{x})$, if for every $R > 0$ we can find a domain Ω_R with $B_R(\bar{x}) \subseteq \Omega_R \subseteq M$, a natural number $N_R \in \mathbb{N}$, and C^1 embeddings $F_{k,R} : \Omega_R \to M_k$ for large $k \geq N_R$ such that $B_R(x_k) \subseteq F_{k,R}(\Omega_R)$ and $F^*_{k,R}(g_k) \to g$ on Ω_R in the C^0 topology.

Theorem 5.5. *Let (M, g) be a complete Riemannian n-manifold with nonnegative Ricci curvature and fix any reference point $\bar{x} \in M$. Assume that for any diverging sequence of points $(x_k)_{k \in N} \subset M$, i.e., $\mathsf{d}(x_k, \bar{x}) \to \infty$, the sequence of pointed manifolds (M, g, x_k) converges in the pointed C^0 topology to the Euclidean space $(\mathbb{R}^n, g_{\mathbb{R}^n}, 0)$.*

Then for every $V \in [0, \mu_g(M))$ there exists a minimizer of the problem (5-1); *in other words, there exists an isoperimetric-isodiametric region of volume V.*

Proof. Since volume and perimeter involve only the metric tensor g and not its derivatives, the hypothesis on the manifold (M, g) of being C^0-locally asymptotic to $\mathbb{R}^n$ implies directly that assumptions (1) and (2) of Theorem 5.2 are satisfied. Therefore the thesis will be a consequence of Theorem 5.2 once we show the following: given $E_k \subset M$ a minimizing sequence of the problem (5-1) for some fixed volume $V \in [0, \mu_g(M))$, there exists a compact subset $K \subset M$ such that $\mu_g(E_k \cap K) > 0$ for infinitely many k. We will show that if this last statement is violated then (M, g) is flat and minimizers are metric balls of volume V.

By the first part of Theorem 5.2 we know that there exist $R > 0$ and a sequence $(x_k)_{k \in \mathbb{N}}$ of points in M such that $\mu_g(E_k \setminus B_R(x_k)) = 0$, i.e., $B_R(x_k)$ are enclosing balls for E_k.

Fixing any reference point $\bar{x} \in M$, if $\liminf_k \mathsf{d}(x_k, \bar{x})$ then clearly we can find a compact subset $K \subset M$ such that $\mu_g(E_k \cap K) > 0$ for infinitely many k and the conclusion follows from the last part of Theorem 5.2. So assume $\mathsf{d}(\bar{x}, x_k) \to \infty$. Since M is C^0-locally asymptotic to $\mathbb{R}^n$, combining Definition 5.4 with the Euclidean isoperimetric-isodiametric inequality (2-3), we get

$$\liminf_{k \to \infty} \operatorname{rad}(E_k)\mathcal{P}(E_k) \geq nV. \tag{5-4}$$

But since (M, g) has nonnegative Ricci curvature, the comparison estimate (4-2) yields

$$\lim_{k \to \infty} \operatorname{rad}(E_k)\mathcal{P}(E_k) = \inf\big\{\operatorname{rad}(\Omega)\mathcal{P}(\Omega) : \Omega \subset M, \operatorname{Vol}(\Omega) = V\big\} \leq nV. \tag{5-5}$$

The combination of (5-4) with (5-5) clearly implies

$$\inf\big\{\operatorname{rad}(\Omega)\mathcal{P}(\Omega) : \Omega \subset M, \operatorname{Vol}(\Omega) = V\big\} = nV.$$

The rigidity statement of Theorem 4.1 then gives that any metric ball in (M, g) of volume V is isometric to a round ball in $\mathbb{R}^n$, and therefore in particular is a minimizer of the problem (5-1). $\square$

5D. ***Examples of noncompact spaces where existence of isoperimetric-isodiametric regions fails.***

Example 5.6 (minimal surfaces with planar ends). If $M \subset \mathbb{R}^3$ is a helicoid, or more generally a minimal surface with planar ends, then it is in particular C^0-locally asymptotic to $\mathbb{R}^2$ in the sense of Definition 5.4.

Then, if we consider a sequence of metric balls $B_{r_k}(x_k) \subset M$ of fixed volume $V > 0$ such that $x_k \to \infty$, we get $\lim_{k\to\infty} \mathrm{rad}(B_{r_k}(x_k)) \,\mathrm{Vol}(B_{r_k}(x_k)) = 2V$. In particular, for every $V > 0$ we have

$$\inf\big\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) : \Omega \subset M,\ \mathrm{Vol}(\Omega) = V\big\} \le 2V.$$

But then Proposition 3.7 implies the infimum is never achieved, or more precisely it is achieved if and only if M is an affine subspace.

The same argument holds for any minimal n-dimensional submanifold in $\mathbb{R}^m$ with ends which are C^0-locally asymptotic to $\mathbb{R}^n$.

Example 5.7 (ALE spaces of negative sectional curvature). Let (M^n, g) be a simply connected noncompact Riemannian manifold with negative sectional curvature and assume that (M, g) is C^0-locally asymptotic to $\mathbb{R}^n$ in the sense of Definition 5.4. Then, if we consider a sequence of metric balls $B_{r_k}(x_k) \subset M$ of fixed volume $V > 0$ such that $x_k \to \infty$, we get $\lim_{k\to\infty} \mathrm{rad}(B_{r_k}(x_k)) \,\mathrm{Vol}(B_{r_k}(x_k)) = nV$. In particular, for every $V > 0$ we have

$$\inf\big\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) : \Omega \subset M,\ \mathrm{Vol}(\Omega) = V\big\} \le nV.$$

But then Proposition 3.1 implies the infimum is never achieved, or more precisely it is achieved by a region Ω if and only if Ω is isometric to a Euclidean region, which is forbidden since M has negative sectional curvature.

6. Optimal regularity of isoperimetric-isodiametric regions

In this last section we establish the optimal regularity for the isoperimetric-isodiametric regions, i.e., the minimizers of problem (5-1), under the assumption that the enclosing ball is regular.

6A. $C^{1,\frac{1}{2}}$ *regularity.*

6A1. *First properties.* Let E be a minimizer of the isoperimetric–isodiametric problem in (M, g) with volume $\mu_g(E) = V > 0$. Let $x_0 \in M$ satisfy $\mu_g(E \setminus B_{\mathrm{rad}(E)}(x_0)) = 0$ and, for the sake of simplicity, we fix the notation $B := B_{\mathrm{rad}(E)}(x_0)$ for an enclosing ball. In the sequel, we always assume that B has regular boundary and we assume to be in the nontrivial case $\mu_g(B \setminus E) > 0$.

By the very definition of isoperimetric-isodiametric sets, we have

$$\mathcal{P}(E) \le \mathcal{P}(F) \quad \forall\, F \,\triangle\, E \Subset B \text{ such that } \mu_g(F) = V. \tag{6-1}$$

In particular, E is a minimizer of the perimeter with constrained volume in B, and therefore we can apply the classical regularity results (see, for example, [Morgan 2003, Corollary 3.8]) in order to deduce that there exists a relatively closed set $\mathrm{Sing}(E) \subset B$ such that $\dim_{\mathcal{H}}(\mathrm{Sing}(E)) \le n-8$ and $\partial E \cap B \setminus \mathrm{Sing}(E)$ is a smooth $(n-1)$-dimensional hypersurface.

Moreover, by the first variations of the area functional under volume constraint, one deduces that the mean curvature is constant on the regular part of the boundary: i.e., there exits $H_0 \in \mathbb{R}$ such that

$$\vec{H}_E(x) = H_0 \nu_E \quad \forall\, x \in \partial E \cap B \setminus \mathrm{Sing}(E), \tag{6-2}$$

where

$$\vec{H}_E(x) := \sum_{i=1}^{n-1} \nabla_{\tau_i} \tau_i,$$

for $\{\tau_1, \dots, \tau_{n-1}\}$ a local orthonormal frame of ∂E around $x \in \partial E \cap B \setminus \mathrm{Sing}(E)$, ν_E the interior normal to E and ∇ the Riemannian connection on (M, g).

In this section we prove the following.

Proposition 6.1. *Let $E \subset M$ be an isoperimetric-isodiametric set and $x_0 \in M$ be such that*

$$\mu_g(E \setminus B_{\mathrm{rad}(E)}(x_0)) = 0.$$

Assume that $B := B_{\mathrm{rad}(E)}(x_0)$ has smooth boundary. Then, there exists $\delta > 0$ such that $\partial E \setminus B_{\mathrm{rad}(E)-\delta}(x_0)$ is $C^{1,\frac{1}{2}}$ regular.

Remark 6.2. In particular, given the partial regularity in B as explained in Section 6A1, we conclude that E is a closed set whose boundary is $C^{1,\frac{1}{2}}$ regular except on at most a closed singular set $\mathrm{Sing}(E)$ of dimension less than or equal to $n-8$.

6A2. *Almost-minimizing property.* The main ingredient of the proof of Proposition 6.1 is the following almost-minimizing property.

Lemma 6.3. *Let E be an isoperimetric-isodiametric set in M and let B denote an enclosing ball as above. There exist constants $C, r_0 > 0$ such that, for every $x \in B$ and for every $0 < r < r_0$,*

$$\mathcal{P}(E) \le \mathcal{P}(F) + Cr^n \quad \forall\, F \triangle E \Subset B_r(x). \tag{6-3}$$

Remark 6.4. Note that $B_r(x)$ is not necessarily contained in B.

Proof. We start by fixing parameters $\eta, c_1 > 0$ and two points $y_1, y_2 \in B$ such that $d_g(y_1, y_2) > 4\eta$, $B_{4\eta}(y_1) \subset B$, $B_{4\eta}(y_2) \subset B$ and

$$\mathcal{P}(E, B_\eta(y_i)) > c_1, \quad i = 1, 2. \tag{6-4}$$

Note that the possibility of such a choice is easily deduced from the regularity of the previous subsection, or more simply from the density estimates for sets of finite perimeter in points of the reduced boundary. For simplicity of notation, set $D_i := B_\eta(y_i)$. By a result by Giusti [1981, Lemma 2.1], there exist $v_0, C_1 > 0$ such that, for every $v \in \mathbb{R}$ with $|v| < v_0$ and for every $i = 1, 2$, there exists F_i which satisfies

$$\begin{cases} F_i \triangle E \subset D_i, \\ \mu_g(F_i) = \mu_g(E) + v, \\ \mathcal{P}(F_i) \le \mathcal{P}(E) + C_1 v. \end{cases} \tag{6-5}$$

Note that in [Giusti 1981, Lemma 2.1] the property (6-5) is proven in the Euclidean space with the flat metric, but the proof remains unchanged in a Riemannian manifold (up to a suitable choice of the constants v_0, C_1).

Next, let $r_0 > 0$ be a constant to be fixed momentarily such that $r_0 < \eta$ and

$$\sup_{x\in B} \mu_g(B_r(x)) \le C_2 r^n < v_0 \quad \forall r \in [0, r_0] \tag{6-6}$$

for some $C_2 > 0$ depending just on B and r_0. Since $d_g(y_1, y_2) > 4\eta$, for every $x \in B$, we know $B_{r_0}(x)$ cannot intersect both D_1 and D_2: therefore, without loss of generality, we can assume $B_{r_0}(x) \cap D_1 = \varnothing$. If $r < r_0$ and $F \subset M$ is any set such that $F \,\triangle\, E \Subset B_r(x)$, we consider $F' := F \cap B$. Note that $F' \subset B$ and moreover

$$|\mu_g(F') - \mu_g(E)| \le \mu_g(B_r(x)) \le C_2 r^n < v_0.$$

According to (6-5) we can then find $F'' \subset B$ such that $\mu_g(F'') = \mu_g(E)$, $F'' \,\triangle\, F' \Subset D_1$ and

$$\mathcal{P}(F'') \le \mathcal{P}(F') + C_1 |\mu_g(F') - \mu_g(E)|. \tag{6-7}$$

Using the fact that E minimizes the perimeter among compactly supported perturbation in $\bar{B}$, we deduce that

$$\mathcal{P}(E) \le \mathcal{P}(F'') \overset{(6\text{-}7)}{\le} \mathcal{P}(F') + C_1 |\mu_g(F') - \mu_g(E)| \le \mathcal{P}(F) + \mathcal{P}(B) - \mathcal{P}(F \cup B) + C_2 r^n. \tag{6-8}$$

Next note that, if ∂B is $C^{1,1}$ regular, then one can choose $r_0 > 0$ such that the following holds: there exists a constant $C_3 > 0$ such that, for every $x \in B$ and for every $r \in (0, r_0)$,

$$\mathcal{P}(B) \le \mathcal{P}(G) + C_3 r^n \quad \forall\, G \,\triangle\, B \Subset B_r(x). \tag{6-9}$$

In order to show this claim, it enough to take r_0 small enough (in particular smaller than half the injectivity radius) in such a way that, for every $p \in \partial B$, there exists a coordinate chart $\xi : B_{2r_0}(p) \to \mathbb{R}^n$ such that $\xi(\partial B) \subset \{x_n = 0\}$ and ξ is a $C^{1,1}$ diffeomorphism with $\mathrm{d}\xi(p) \in \mathrm{SO}(n)$, $\xi(p) = 0$ and $g(0) = \mathrm{Id}$, where g is the metric tensor in the coordinates induced by ξ. Indeed, in this case we have $\mathcal{P}(B, B_r(p)) \le (1 + Cr)\omega_{n-1} r^{n-1}$ for every $r < r_0$ and, for every G such that $G \,\triangle\, B \Subset B_r(p)$,

$$\mathcal{P}(G, B_r(p)) \ge (1 - Cr)\mathcal{P}\big(\mathrm{proj}(\xi(G)), \xi(B_r(p))\big) \ge (1 - Cr)\omega_{n-1} r^{n-1},$$

where proj denotes the orthogonal Euclidean projection on $\{x_n = 0\}$ and we have used the regularity of ξ.

Applying (6-9) to $G = F \cup B$ and using (6-8), we conclude the proof. □

6A3. *Proof of Proposition 6.1.* Now we are in the position to apply a result by Tamanini [1982, Theorem 1] (the result is proved in $\mathbb{R}^n$ with a flat metric, but the proof is unchanged in a Riemannian manifold) in order to give a proof of the above proposition.

To this aim, we start by considering any point $p \in \partial B \cap \partial E$; we denote by $\mathrm{Exp}_p : T_p M \to M$ the exponential map and we let $r_0 > 0$ be less than the injectivity radius. Since by Lemma 6.3 the set E is an almost minimizer of the perimeter, the rescaled sets

$$E_{p,r} := \frac{\mathrm{Exp}_p^{-1}(E \cap B_{r_0}(p))}{r} \subset T_p M \simeq \mathbb{R}^n \tag{6-10}$$

converge, up to passing to a suitable subsequence, to a minimizing cone C_∞ in the Euclidean space (see [Maggi 2012, Theorem 28.6]). Moreover, since E is enclosed by B and ∂B is $C^{1,1}$, it is immediate to check that if $r_0 > 0$ is chosen small enough in (6-10), then $C_\infty \subset \{x : g(\nu_B(p), x) \geq 0\}$; we deduce that every tangent cone to E at p needs to be contained in a half-space, and therefore by the Bernstein theorem is flat (see [Giusti 1984, Theorem 17.4]). This implies that every such point p is a point of the reduced boundary of the set (see [Ambrosio et al. 2000, Definition 3.54]) and therefore we can apply the aforementioned result by Tamanini to conclude that ∂E is a $C^{1,1/2}$ regular hypersurface in $B_r(p)$ for every $p \in \partial B \cap \partial E$ and for every $r < \frac{1}{2} r_0$. By a simple covering argument, the conclusion of the corollary follows.

6B. ***L^∞ estimates on the mean curvature of the minimizer.*** In this section we prove that the boundary of E has generalized mean curvature, in the sense of varifolds, which is bounded in L^∞. To this aim, we compute the first variations of the perimeter of E along suitable diffeomorphisms.

6B1. *First variations.* We start by fixing two points $y_1, y_2 \in \partial E \cap B \setminus \mathrm{Sing}(E)$ and a real number $\eta > 0$ such that $B_{4\eta}(y_1) \subset B$, $B_{4\eta}(y_2) \subset B$ and

$$B_{4\eta}(y_1) \cap B_{4\eta}(y_2) = B_{4\eta}(y_1) \cap \mathrm{Sing}(E) = B_{4\eta}(y_2) \cap \mathrm{Sing}(E) = \varnothing.$$

Note that such a choice is possible under the hypothesis that $\mu_g(B \setminus E) > 0$ because of the partial regularity in Section 6A1. Let $X \in \mathfrak{X}(M)$ be a vector field with support contained in a metric ball $B_\eta(y)$ for some $y \in M$. Clearly, $B_\eta(y)$ cannot intersect both $B_{2\eta}(y_1)$ and $B_{2\eta}(y_2)$, because $d_g(y_1, y_2) \geq 8\eta$; therefore, without loss of generality let us assume $B_\eta(y) \cap B_{2\eta}(y_1) = \varnothing$. It is not difficult to construct a smooth vector field Y supported in $B_\eta(y_1)$ such that the generated flow $\{\Phi_t^Y\}$ satisfies the following property for small $|t|$:

$$\mu_g(\Phi_t^Y \circ \Phi_t^X(E)) = \mu_g(E). \tag{6-11}$$

Note that the generated flows $\{\Phi_t^X\}_{t\in\mathbb{R}}$ and $\{\Phi_t^Y\}_{t\in\mathbb{R}}$ are well defined and for $|t|$ sufficiently small are diffeomorphisms of M. Moreover, $\Phi_t^Y \circ \Phi_t^X(E) \subset B_{\mathrm{rad}(E)+|t|\,\|X\|_\infty}$. We can then deduce that

$$\mathrm{rad}(E)\mathcal{P}(E) \leq \mathrm{rad}\big(\Phi_t^Y \circ \Phi_t^X(E)\big)\mathcal{P}\big(\Phi_t^Y \circ \Phi_t^X(E)\big) \leq \big(\mathrm{rad}(E)+|t|\,\|X\|_\infty\big)\mathcal{P}\big(\Phi_t^Y \circ \Phi_t^X(E)\big) =: f(t). \tag{6-12}$$

Taking the derivative of the last functional as $t \downarrow 0^+$ and as $t \uparrow 0^-$, by the well-known computation of the first variations of the area we get that

$$0 \leq \lim_{t\downarrow 0^+} \frac{f(t)-f(0)}{t} = \|X\|_\infty \mathcal{P}(E) + \mathrm{rad}(E)\int_{\partial E} \mathrm{div}_{\partial E} X \,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial E} g(\vec{H}_E, Y)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{6-13}$$

$$0 \geq \lim_{t\uparrow 0^-} \frac{f(t)-f(0)}{t} = -\|X\|_\infty \mathcal{P}(E) + \mathrm{rad}(E)\int_{\partial E} \mathrm{div}_{\partial E} X \,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial E} g(\vec{H}_E, Y)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{6-14}$$

where $\mathrm{div}_{\partial E} X := \sum_{i=1}^{n-1} g(\nabla_{\tau_i} X, \tau_i)$ for a (measurable) local orthonormal frame $\{\tau_1, \dots, \tau_{n-1}\}$ of ∂E. (Note that in writing (6-13) and (6-14) we have used that ∂E is a $C^{1,1/2}$ regular submanifold up to singular set of dimension at most $n-8$ and that Y is supported in $B_\eta(y)$ where ∂E is smooth in order to make the integration by parts.) In the case $V \in (0, \mu_g(M))$, we have $\mathrm{rad}(E) > 0$ and thus $\mathcal{P}(E) < \infty$. Moreover, from (6-11) we deduce that

$$0 = \frac{d}{dt}_{|t=0} \mu_g(\Phi_t^Y \circ \Phi_t^X(E)) = -\int_{\partial E} g(X, \nu_E)\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\partial E} g(Y, \nu_E)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{6-15}$$

Therefore, from (6-2) and (6-13)–(6-15) we conclude

$$\begin{aligned}\left|\int_{\partial E} \operatorname{div}_{\partial E} X \, d\mathcal{H}^{n-1}\right| &\le \frac{1}{\operatorname{rad}(E)}\left(\mathcal{P}(E)\|X\|_\infty + \left|\int_{\partial E} g\big(\vec{H}_E, Y\big)\, d\mathcal{H}^{n-1}\right|\right)\\ &\le \frac{1}{\operatorname{rad}(E)}\left(\mathcal{P}(E)\|X\|_\infty + |H_0|\left|\int_{\partial E} g(Y, \nu_E)\, d\mathcal{H}^{n-1}\right|\right)\\ &= \frac{1}{\operatorname{rad}(E)}\left(\mathcal{P}(E)\|X\|_\infty + |H_0|\left|\int_{\partial E} g(X, \nu_E)\, d\mathcal{H}^{n-1}\right|\right) \le C\|X\|_\infty \qquad (6\text{-}16)\end{aligned}$$

for some $C = C(\operatorname{rad}(E), \mathcal{P}(E), |H_0|) > 0$, for every vector field X with support contained in a metric ball $B_\eta(y)$ for some $y \in M$. By a simple partition of unity argument, (6-16) holds for every $X \in \mathfrak{X}(M)$. In particular, by the use of Riesz representation theorem we have proved the following lemma. To this aim, we denote by $\mathcal{M}(M, TM)$ the vectorial Radon measures $\vec{\mu}$ on M with values in the tangent bundle TM.

Lemma 6.5 (the mean curvature is represented by a vectorial Radon measure). *Let $E \subset M$ be an isoperimetric-isodiametric region for some $V \in (0, \mu_g(M))$ and denote by B an enclosing ball. If ∂B is smooth, then there exists a vectorial Radon measure $\vec{\boldsymbol{H}}_E \in \mathcal{M}(M, TM)$ concentrated on ∂E such that for every C^1 vector field X on M with compact support, letting $\Phi_t^X : M \to M$ be the corresponding one-parameter family of diffeomorphisms for $t \in \mathbb{R}$,*

$$\delta E(X) := \frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^X(E)) = -\int_M g(X, \vec{\boldsymbol{H}}_E). \qquad (6\text{-}17)$$

Moreover, the total variation of $\vec{\boldsymbol{H}}_E$ is finite; i.e.,

$$|\vec{\boldsymbol{H}}_E|(M) \le C = C\big(\mathcal{P}(E), \operatorname{rad}(E), |H_0|\big) \in [0, \infty).$$

Remark 6.6. Note that

$$\vec{\boldsymbol{H}}_E \llcorner B := \vec{H}_E \mathcal{H}^{n-1} \llcorner (\partial E \cap B), \qquad (6\text{-}18)$$

where $\vec{H}_E$ is the mean curvature vector on the smooth part of ∂E as defined in (6-2).

We close this subsection by noting that if

$$g(X(x), \nu_B(x)) \ge 0 \quad \forall\, x \in \partial B \cap B_\eta(y), \qquad (6\text{-}19)$$

where ν_B is the interior normal to ∂B (note that $\partial B \cap B_\eta(y)$ can also be empty), then $\Phi_t^Y \circ \Phi_t^X(E) \subset B$ for $t \ge 0$. In particular, the minimizing property of E gives

$$\mathcal{P}(\Phi_t^Y \circ \Phi_t^X(E)) \ge \mathcal{P}(E) \quad \forall\, t \ge 0, \qquad (6\text{-}20)$$

which combined with (6-2) and (6-15) implies

$$\begin{aligned}0 \le \frac{d}{dt}_{|t=0^+} \mathcal{P}(\Phi_t^Y \circ \Phi_t^X(E)) &= \int_{\partial E} \operatorname{div}_{\partial E} X \, d\mathcal{H}^{n-1} - \int_{\partial E} g(\vec{H}_E, Y)\\ &= \int_{\partial E} \operatorname{div}_{\partial E} X \, d\mathcal{H}^{n-1} + H_0 \int_{\partial E} g(\nu_E, X), \qquad (6\text{-}21)\end{aligned}$$

which in view of (6-17) gives

$$g(\nu_B, \vec{\boldsymbol{H}}_E) \llcorner (\partial E \cap \partial B) \leq H_0 \mathcal{H}^{n-1} \llcorner (\partial E \cap \partial B), \tag{6-22}$$

where the inequality is intended in the sense of measures, i.e., $\int_A g(\nu_B, \vec{\boldsymbol{H}}_E) \leq H_0 \mathcal{H}^{n-1}(A)$ for every measurable set $A \subset \partial E \cap \partial B$.

6B2. *Orthogonality of $\vec{\boldsymbol{H}}_E$.* We have seen in the previous section that $\vec{\boldsymbol{H}}_E$ is well defined as a measure on all ∂E. Translated into the language of varifolds, we have shown that the integral varifold associated to ∂E has finite first variation. A classical result due to Brakke [1978, Section 5.8] (see also [Menne 2013] for an alternative proof and for fine structural properties of varifolds with locally finite first variation) implies that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$ it holds that $\vec{\boldsymbol{H}}_E(x) \in (T_x \partial E)^{\perp}$. This is not quite enough for our purposes; indeed in the next lemma we will show that $\vec{\boldsymbol{H}}_E$ is *normal to ∂E as measure*, which is a strictly stronger statement. Note that the proof is based on the fact that E is a minimizer for the problem (5-1), and will not make use of the aforementioned structural result by Brakke.

Lemma 6.7 (the mean curvature measure is orthogonal to ∂E). *Let E, B, M, V, $\vec{\boldsymbol{H}}_E$ be as in Lemma 6.5. Then $\vec{\boldsymbol{H}}_E(x) \in (T_x \partial E)^{\perp}$ for $|\vec{\boldsymbol{H}}_E|$-a.e. $x \in \partial E$; i.e., the mean curvature is orthogonal to ∂E as a measure.*

Remark 6.8. In other words, there exists an $\mathbb{R}$-valued finite Radon measure $\boldsymbol{H}_E$ on M concentrated on ∂E such that $\vec{\boldsymbol{H}}_E = \boldsymbol{H}_E \nu_E$; moreover, by (6-2), $\boldsymbol{H}_E \llcorner (B \cap \partial E) = H_0 \mathcal{H}^{n-1} \llcorner (\partial E \cap B)$.

Proof. In view of (6-2) we only need to prove the claim for $\vec{\boldsymbol{H}}_E \llcorner \partial B$. Assume by contradiction that there exists a compact subset $K \subset \partial B \cap \partial E$ such that

$$|\vec{\boldsymbol{H}}_E^T|(K) > 0, \tag{6-23}$$

where $\vec{\boldsymbol{H}}_E^T := P_{T\partial E}(\vec{\boldsymbol{H}}_E)$ is the projection of $\vec{\boldsymbol{H}}_E$ onto the tangent space of ∂E (or, equivalently, onto $T\partial B$, because ∂E and ∂B are C^1 and $T_x \partial E = T_x \partial B$ for every $x \in \partial B \cap \partial E$).

The geometric idea of the proof is very neat: if the mean curvature along $K \subset \partial E \cap \partial B$ has a nontrivial tangential part, then deforming infinitesimally E along this tangential direction will not increase the extrinsic radius (since the deformation of E will stay in the ball B), will not increase the volume (because the deformation is tangential to ∂E) but will strictly decrease the perimeter; so, after adjusting the volume in a smooth portion of ∂E, this procedure builds an infinitesimal deformation of E which preserves the volume, does not increase the extrinsic radius but strictly decreases the perimeter, contradicting that E is a minimizer of the problem (5-1). The rest of the proof is a technical implementation of this neat geometric idea.

For every $\varepsilon > 0$ we construct a suitable C^1 regular tangential vector field. To this aim, we consider the polar decomposition of the measure $\vec{\boldsymbol{H}}_E^T = v|\vec{\boldsymbol{H}}_E^T|$, where v is a Borel vector field such that $v(x) \in T\partial B$ and $g(v(x), v(x)) = 1$ for $|\vec{\boldsymbol{H}}_E^T|$-a.e. $x \in M$. By the Lusin theorem we can find a continuous vector field w such that $|\vec{\boldsymbol{H}}_E^T|(\{v \neq w\}) \leq \varepsilon$ and $\operatorname{spt}(w) \subset K_\varepsilon := \{x \in \partial E \cap \partial B : d_g(x, K) < \varepsilon\}$. Moreover, by a standard regularization procedure via mollification and projection on $T\partial B$, we find a vector field X_ε such that $X_\varepsilon(x) \in T\partial B$ for every $x \in \partial B \cap K_{2\varepsilon}$, $\|X_\varepsilon - w\|_\infty \leq \varepsilon$ and $\operatorname{spt}(X_\varepsilon) \subset K_{2\varepsilon}$. Note that

$$\int_M g(X_\varepsilon, \vec{\boldsymbol{H}}_E) = \int_M g(X_\varepsilon - w, \vec{\boldsymbol{H}}_E) + \int_{\{w=v\}} g(v, \vec{\boldsymbol{H}}_E) + \int_{\{w \neq v\}} g(w, \vec{\boldsymbol{H}}_E) \to |\vec{\boldsymbol{H}}_E^T|(K) \quad \text{as } \varepsilon \to 0. \tag{6-24}$$

Since X_ε is a smooth vector field compactly supported in M and tangent to ∂B, the generated flow $\Phi_t^{X_\varepsilon}$ is well defined and maps B into B for every $t \in \mathbb{R}$ and by (6-24)

$$\frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{X_\varepsilon}(E)) = -\int_{\partial E} g(X_\varepsilon, \vec{\boldsymbol{H}}_E) \le -\tfrac{1}{2}|\vec{\boldsymbol{H}}_E^T|(K) < 0 \tag{6-25}$$

for $\varepsilon > 0$ small enough. Moreover, since X_ε is supported in $K_{2\varepsilon}$ and $K \subset \partial B$ and X_ε is tangent to $\partial B = \partial E$ in K, we have

$$\frac{d}{dt}_{|t=0} \mu_g(\Phi_t^{X_\varepsilon}(E)) = -\int_{\partial E} g(\nu_E, X_\varepsilon)\, d\mathcal{H}^{n-1} \to 0 \quad \text{as } \varepsilon \to 0. \tag{6-26}$$

Up to choosing a smaller compact set, we can suppose that K is contained in a small ball $B_{r_0}(x)$ with $x \in \partial E \cap \partial B$ such that $(\partial E \setminus \partial B) \cap (M \setminus B_{4r_0}(x)) \neq \varnothing$. Now fix $y \in \partial E \setminus \big(\partial B \cup B_{4r_0}(x) \cup \operatorname{Sing}(E)\big)$ and let $r \in (0, r_0)$ be such that $B_{2r}(y) \cap \big(\partial B \cup B_{4r_0}(x) \cup \operatorname{Sing}(E)\big) = \varnothing$. For $\varepsilon > 0$ small enough it is not difficult to construct a smooth vector field Y_ε supported in $B_r(y)$ such that the generated flow $\Phi_t^{Y_\varepsilon}$ satisfies the following properties ((6-28) is intended for small t):

$$\frac{d}{dt}_{|t=0} \mu_g(\Phi_t^{Y_\varepsilon} \circ \Phi_t^{X_\varepsilon}(E)) = 0, \tag{6-27}$$

$$\big|\mathcal{P}(\Phi_t^{Y_\varepsilon}(E), B_{2r}(y)) - \mathcal{P}(E, B_{2r}(y))\big| \le C \mu_g(\Phi_t^{Y_\varepsilon}(E) \Delta E). \tag{6-28}$$

Notice that the combination of (6-26), (6-27) and (6-28) gives

$$\Big|\frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{Y_\varepsilon}(E))\Big| \le C \Big|\frac{d}{dt}_{|t=0} \mu_g(\Phi_t^{Y_\varepsilon}(E))\Big| = C \Big|\frac{d}{dt}_{|t=0} \mu_g(\Phi_t^{X_\varepsilon}(E))\Big| \to 0 \quad \text{as } \varepsilon \to 0. \tag{6-29}$$

Moreover, since for small $t > 0$ we have $\Phi_t^{Y_\varepsilon}(E) \Delta E \subset B_{2r}(y)$, which is disjoint from ∂B, and since by construction $\Phi_t^{X_\varepsilon}$ maps B into B, it is clear that

$$\Phi_t^{Y_\varepsilon} \circ \Phi_t^{X_\varepsilon}(E) \subset B \quad \text{for } t > 0 \text{ sufficiently small.}$$

Therefore, since by assumption E is a minimizer for the problem (5-1), we get

$$\frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{Y_\varepsilon} \circ \Phi_t^{X_\varepsilon}(E)) \ge 0. \tag{6-30}$$

But on the other hand, combining (6-25) and (6-29) we get

$$\begin{aligned}\frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{Y_\varepsilon} \circ \Phi_t^{X_\varepsilon}(E)) &= \frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{Y_\varepsilon}(E)) + \frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{X_\varepsilon}(E)) \\ &\le -\tfrac{1}{4}|\vec{\boldsymbol{H}}_E^T|(K) < 0 \quad \text{for } \varepsilon > 0 \text{ small enough.}\end{aligned}$$

Clearly the last inequality contradicts (6-30). We conclude that it is not possible to find a compact subset $K \subset \partial B \cap \partial E$ satisfying (6-23); therefore the measure $|\vec{\boldsymbol{H}}_E^T|$ vanishes identically and the proof is complete. □

6B3. *L^∞ estimate.* The next step is to show that the signed measure $\boldsymbol{H}_E$ is actually absolutely continuous with respect to $\mathcal{H}^{n-1}\llcorner\partial E$ with L^∞ bounds on the density. The upper bound follows from (6-22). For the lower bound we use the following lemma, which is an adaptation of [White 2010, Theorem 2] to our setting (notice that the statement of White's theorem is more general as includes higher codimensions and arbitrary varifolds, but let us state below just the result we will use in the sequel).

Lemma 6.9. *Let $N^n \subset M^n$ be an n-dimensional submanifold with C^2 boundary ∂N and denote by ν_N the inward-pointing unit normal to ∂N. Fix a compact subset $K \subset \partial N$ and assume that, denoting by $\vec{H}_N$ the mean curvature of ∂N, we have*

$$g(\vec{H}_N, \nu_N) \geq \eta \quad \text{on } K.$$

Then, for every $\varepsilon > 0$ there exists a C^1 vector field X_ε on M with the following properties:

$$X_\varepsilon(x) = \nu_N \quad \forall x \in K, \tag{6-31}$$

$$|X_\varepsilon|(x) \leq 1 \quad \forall x \in M, \tag{6-32}$$

$$\operatorname{spt}(X_\varepsilon) \subset K_\varepsilon := \{x \in M : d(x, K) \leq \varepsilon\}, \tag{6-33}$$

$$g(X_\varepsilon, \nu_N)(x) \geq 0 \quad \forall x \in \partial N, \tag{6-34}$$

$$\frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{X_\varepsilon}(E)) \leq -\eta \int_{\partial E} |X_\varepsilon| \, d\mathcal{H}^{n-1} \tag{6-35}$$

for every subset $E \subset N$ with C^1 boundary ∂E, where $\Phi_t^{X_\varepsilon}$ denotes the flow generated by the vector field X_ε.

Lemma 6.9 will be used to prove the following lower bound on the mean curvature measure $\boldsymbol{H}_E$ of ∂E.

Lemma 6.10 (lower bound on $\boldsymbol{H}_E$). *Let E, B, M, V, $\vec{\boldsymbol{H}}_E$, $\boldsymbol{H}_E$ be as in Lemma 6.7. Assume $\eta := \inf_{\partial B} H_B > -\infty$, where $H_B := g(\vec{H}_B, \nu_B)$ and $\vec{H}_B$ is the mean curvature vector of ∂B. Then*

$$\boldsymbol{H}_E \llcorner (\partial E \cap \partial B) \geq \eta \mathcal{H}^{n-1} \llcorner (\partial E \cap \partial B). \tag{6-36}$$

Proof. Fix any $K \subset \partial E \cap \partial B$. For every $\varepsilon \in (0, 1)$ let X_ε be the C^1 vector field obtained by applying Lemma 6.9 with $N = B$; then by (6-35) and (6-33) we get

$$\begin{aligned} -\eta \int_{\partial E} |X_\varepsilon| \, d\mathcal{H}^{n-1} &\geq \frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{X_\varepsilon}(E)) = -\int_{K_\varepsilon} g(X_\varepsilon, \nu_E) \, d\boldsymbol{H}_E \\ &= -\int_K g(X_\varepsilon, \nu_B) \, d\boldsymbol{H}_E - \int_{K_\varepsilon \setminus K} g(X_\varepsilon, \nu_E) \, d\boldsymbol{H}_E \to -\boldsymbol{H}_E(K) \quad \text{as } \varepsilon \to 0, \end{aligned} \tag{6-37}$$

where in the second identity we used that $\nu_B = \nu_E$ on $K \subset \partial E \cap \partial B$. Using (6-31) and (6-32), we have

$$-\eta \int_{\partial E} |X_\varepsilon| \, d\mathcal{H}^{n-1} = -\eta \int_K |X_\varepsilon| \, d\mathcal{H}^{n-1} - \eta \int_{\partial E \cap (K_\varepsilon \setminus K)} |X_\varepsilon| \, d\mathcal{H}^{n-1} \to -\eta \mathcal{H}^{n-1}(K) \quad \text{as } \varepsilon \to 0. \tag{6-38}$$

In particular, in the limit as $\varepsilon \to 0$ we deduce from (6-37) that

$$\eta \mathcal{H}^{n-1}(K) \leq \boldsymbol{H}_E(K). \tag{6-39}$$

Since this holds for every $K \subset \partial E \cap \partial B$, it is easily recognized that (6-36) follows. □

6C. ***Optimal regularity.*** In this section we prove that the boundary of an isoperimetric-isodiametric set E is $C^{1,1}$ regular away from the singular set.

Theorem 6.11. *Let $E \subset M$ be an isoperimetric-isodiametric set and $x_0 \in M$ be such that*

$$\mu_g(E \setminus B_{\mathrm{rad}(E)}(x_0)) = 0.$$

Assume $B := B_{\mathrm{rad}(E)}(x_0)$ has smooth boundary. Then, there exists $\delta > 0$ such that $\partial E \setminus B_{\mathrm{rad}(E)-\delta}(x_0)$ is $C^{1,1}$ regular.

Note that the $C^{1,1}$ regularity is optimal, because in general one cannot expect to have continuity of the second fundamental form of ∂E across the free boundary of ∂E, i.e., the points on the relative (with respect to ∂B) boundary of $\partial E \cap \partial B$.

6C1. *Coordinate charts.* We start by fixing suitable coordinate charts. Since E is bounded, there exists $r_0 > 0$ such that for every $x_0 \in \partial E$ there is a normal coordinate chart (Ω, φ) with $x_0 \in \Omega$ and

$$\varphi : \Omega \subset M \to B^{n-1}_{r_0} \times (-r_0, r_0) \subset \mathbb{R}^{n-1} \times \mathbb{R}$$

such that $\varphi(x_0) = 0$, $g(0) = \mathrm{Id}$ and $\nabla g(0) = 0$, where g denotes the metric tensor in these coordinates. Moreover, by the $C^{1,1/2}$ regularity of ∂E established in Section 6A, up to rotating these coordinate charts and eventually changing r_0, we can also assume that for every point $x_0 \in \partial B \cap \partial E$ the following also holds:

- ∂E and ∂B are, respectively, $C^{1,1/2}$ and C^∞ regular submanifolds, given in this chart as graphs of functions $u, \psi : B^{n-1}_{r_0} \to \left(-\frac{1}{2}r_0, \frac{1}{2}r_0\right)$ with $u \in C^{1,1/2}$ and $\psi \in C^\infty$.
- The functions u and ψ satisfy $\psi(x) \le u(x)$ for every $x \in B^{n-1}_{r_0}$,

  $$u(0) = \psi(0) = |\nabla u(0)| = |\nabla \psi(0)| = 0,$$

  and $\|u\|_{C^1} \le \delta_0$ and $\|\psi\|_{C^1} \le \delta_0$ for a fixed $\delta_0 > 0$, which will be later assumed to be suitably small.

On every such a chart, the $C^{1,1/2}$ regular submanifold $\partial E \cap \Omega$ is given as the set $\{(x, u(x)) : x \in B^{n-1}_{r_0}\}$. We can consider the natural coordinate chart on it given by $(x, u(x)) \mapsto x \in B^{n-1}_r$ with induced metric tensor given by $h_{ij} := g(E_i, E_j)$, where $E_i := e_i + \partial_i u\, e_n$ for $i = 1, \dots, n-1$. In particular,

$$h_{ij} = g_{ij} + \partial_i u\, g_{nj} + \partial_j u\, g_{ni} + \partial_i u\, \partial_j u\, g_{nn}, \tag{6-40}$$

where $\partial_i u = \partial_i u(x)$ and $g_{ij} = g_{ij}(x, u(x))$. We will use the notation $\tilde{h}$ for the function

$$\tilde{h} : B^{n-1}_{r_0} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times n},$$
$$\tilde{h}_{ij}(x, z, p) = g_{ij}(x, z) + p_i g_{jn}(x, z) + p_j g_{ni}(x, z) + p_i p_j g_{nn}(x, z),$$

with the obvious relation $h_{ij} = \tilde{h}_{ij}(x, u(x), \nabla u(x))$. Note that $\tilde{h}$ is smooth as a function in (x, z, p).

6C2. *First variation formula in local coordinates.* We consider next functions $\xi \in C_c^\infty(B_{r_0}^{n-1})$ and $\chi \in C_c^\infty(-r_0, r_0)$, and we assume $\chi|_{(-r_0/2, r_0/2)} \equiv 1$ in such a way to ensure that $\chi \circ u(x) = 1$ for every $x \in B_{r_0}^{n-1}$ (by the assumptions made on u). Consider the associated vector field $X(x, y) := \xi(x)\chi(y)e_n$ and note that $X \in C_c^\infty(\Omega, \mathbb{R}^n)$ and $X|_{\partial E} = \xi(x)e_n$. Setting $F(t, p) := p + tX(p)$, there exists $\varepsilon_0 > 0$ such that $F_t := F(t, \cdot)$ is a diffeomorphism of Ω into itself for every $|t| \le \varepsilon_0$.

Consider now the variations of the area along this one-parameter family of diffeomorphisms under the assumption $\xi \ge 0$ on $\Lambda(u) := \{x \in B_{r_0}^{n-1} : u(x) = \psi(x)\}$. Arguing as in (6-21), we get

$$
\begin{aligned}
0 &\le \int_{\partial E} \operatorname{div}_{\partial E} X \, d\mathcal{H}^{n-1} - H_0 \int_{\partial E} g(X, \nu_E) \, d\mathcal{H}^{n-1} \\
&= \int_\Sigma h^{ij} g(\nabla_{E_i} X, E_j) \, d\mathcal{H}^{n-1} - H_0 \int g(X, \nu_E) \, d\mathcal{H}^{n-1},
\end{aligned}
\tag{6-41}
$$

where in the second line we have used a simple computation for the tangential divergence of X. Noting that

$$
\begin{aligned}
\nabla_{E_i} X = \nabla_{e_i + \partial_i u \, e_n} X &= \nabla_{e_i} X + \partial_i u \, \nabla_{e_n} X \\
&= \partial_i \xi \, e_n + \xi \nabla_{e_i} e_n + \partial_i u \, \xi \nabla_{e_n} e_n = \partial_i \xi \, e_n + \xi \Gamma_{in}^k e_k + \partial_i u \, \xi \Gamma_{nn}^k e_k,
\end{aligned}
$$

we get

$$
\begin{aligned}
h^{ij} g(\nabla_{E_i} X, E_j) &= h^{ij}\big(\partial_i \xi \, g_{jn} + \xi \Gamma_{in}^k g_{jk} + \partial_i u \, \xi \Gamma_{nn}^k g_{jk}\big) + h^{ij}\big(\partial_j u \, \partial_i \xi \, g_{nn} + \xi \partial_j u \, \Gamma_{in}^k g_{kn} + \partial_j u \, \partial_i u \, \xi \Gamma_{nn}^k g_{kn}\big) \\
&= \partial_i \xi \big(h^{ij} g_{jn} + h^{ij} \partial_j u \, g_{nn}\big) \xi \big(h^{ij} \partial_i u \, \Gamma_{nn}^k g_{jk} + h^{ij} \partial_j u \, \partial_i u \, \Gamma_{nn}^k g_{kn}\big) \\
&\qquad + \xi \big(h^{ij} \Gamma_{in}^k g_{jk} + h^{ij} \partial_j u \, \Gamma_{in}^k g_{kn}\big).
\end{aligned}
\tag{6-42}
$$

In particular, by a simple integration by parts, (6-41) reads as

$$
\int_{B_r^{n-1}} \xi L u \sqrt{\det(h_{ij})} \, dx \le 0 \quad \forall \xi \in C_c^1(B_r^{n-1}), \ \xi|_{\Lambda(u)} \ge 0,
\tag{6-43}
$$

where $\Lambda(u) := \{x \in B_r^{n-1} : u(x) = \psi(x)\}$ and

$$
Lu(x) := \operatorname{div}\big(A(x, u(x), \nabla u(x)) \nabla u(x) + b(x, u(x), \nabla u(x))\big) - f(x)
\tag{6-44}
$$

with

- $A = (a^{ij})_{i,j=1,\dots,n-1} : B_r^{n-1} \times (-r, r) \times \mathbb{R}^{n-1} \to \mathbb{R}^{(n-1)\times(n-1)}$ is a smooth function given by
$$
a^{ij}(x, z, p) := g_{nn}(x, z)\tilde{h}^{ij}(x, z, p);
$$
- $b : B_r^{n-1} \times (-r, r) \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is a smooth regular function given by
$$
b^i(x, z, p) := \tilde{h}^{ij}(x, z, p) g_{jn}(x, z);
$$
- $f : B_r^{n-1} \to \mathbb{R}$ is a $C^{0,\alpha}$ regular function given by
$$
f(x) := h^{ij} \partial_i u \, \Gamma_{nn}^k g_{jk} + h^{ij} \partial_j u \, \partial_i u \, \Gamma_{nn}^k g_{kn} + h^{ij} \Gamma_{in}^k g_{jk} + h^{ij} \partial_j u \, \Gamma_{in}^k g_{kn} - H_0 g(e_n, \nu_E),
$$
where $h^{ij} = \tilde{h}^{ij}(x, u(x), \nabla u(x))$, $g_{ij} = g_{ij}(x, u(x))$, $\Gamma_{ij}^k = \Gamma_{ij}^k(x, u(x))$ and $\nu_E = \nu_E(x, u(x))$.

Explicitly expanding the divergence term in Lu we deduce that

$$Lu(x) = c^{ij}\partial_{ij}u + d, \tag{6-45}$$

where

$$c^{ij} = a^{ij} + g_{nn}\partial_l u\,\partial_{p^j}h^{il} + g_{ln}\partial_{p_j}h^{il}, \tag{6-46}$$

with $\partial_{p^j}h^{il} = \partial_{p^j}\tilde{h}^{il}(x, u(x), \nabla u(x))$, $g_{ij} = g_{ij}(x, u(x))$ and $d \in C^{0,\alpha}(B^{n-1}_r)$ is given by

$$\begin{aligned} d = g_{nn}\partial_i h^{ij}\partial_j u + g_{nn}\partial_z h^{ij}\partial_i u\,\partial_j u + \partial_i g_{nn}h^{ij}\partial_j u + \partial_n g_{nn}h^{ij}\partial_i u\,\partial_j u \\ + g_{jn}\partial_i h^{ij} + g_{jn}\partial_z h^{ij}\partial_i u + \partial_i g_{jn}h^{ij} + \partial_n g_{jn}h^{ij}\partial_i u - f \end{aligned} \tag{6-47}$$

where the entries of h and of its derivatives are computed in $(x, u(x), \nabla u(x))$, while those of g and the derivatives of the metric are computed in $(x, u(x))$.

Note that (6-43) is equivalent to the pair of differential relations

$$\begin{cases} Lu \le 0 & \text{in } B^{n-1}_r, \\ Lu = 0 & \text{in } B^{n-1}_r \setminus \Lambda(u), \end{cases} \tag{6-48}$$

where the first inequality is meant in the sense of distribution, while the second equation is pointwise (also recalling that u is smooth outside the contact set $\Lambda(u)$).

6C3. *Quadratic growth.* Note that by the explicit expressions of the previous subsection it turns out that $c^{ij}, d \in C^{0,\alpha}(B^{n-1}_{r_0})$ with uniform estimates (by the assumptions in Section 6C1):

$$\|c^{ij}\|_{C^{0,\alpha}(B^{n-1}_{r_0})} + \|d\|_{C^{0,\alpha}(B^{n-1}_{r_0})} \le C. \tag{6-49}$$

Since $c(0) = \mathrm{Id}$ and c^{ij} are Hölder continuous, up to choosing a smaller $\delta_0 > 0$ (and consistently a smaller $r_0 > 0$), we can also ensure that c^{ij} is uniformly elliptic with bounds

$$\tfrac{1}{2}\,\mathrm{Id} \le c \le 2\,\mathrm{Id}.$$

The next lemma shows that u leaves the obstacle ψ at most as a quadratic function of the distance to the free-boundary point.

Proposition 6.12. *Let $E \subset M$ be an isoperimetric-isodiametric set. Then, there exists a constant $C > 0$ such that, for every $x_0 \in \partial E \cap \partial B$, setting coordinates as in Section 6C1, we have*

$$u(x) - \psi(x) \le C|x|^2 \quad \forall\, x \in B^{n-1}_{r_0/2}. \tag{6-50}$$

Proof. Let us consider the homogeneous part of the operator L, i.e., $\mathcal{L}w := c^{ij}\partial_{ij}w$. Since $\mathcal{L}(u - \psi) = Lu - \mathcal{L}\psi - d$, for every $r \le r_0$ we can write $(u - \psi)|_{B^{n-1}_r} = w_1 + w_2$ with

$$\begin{cases} \mathcal{L}w_1 = 0 & \text{in } B^{n-1}_r, \\ w_1 = u - \psi & \text{on } \partial B^{n-1}_r, \end{cases} \tag{6-51}$$

and

$$\begin{cases} \mathcal{L}w_2 = Lu - \mathcal{L}\psi - d & \text{in } B^{n-1}_r, \\ w_2 = 0 & \text{on } \partial B^{n-1}_r. \end{cases} \tag{6-52}$$

We start by estimating w_2 from below. Considering that $\mathcal{L}w_2+\mathcal{L}\psi+d=Lu\le 0$, we can apply the L^∞ estimate for elliptic equations [Gilbarg and Trudinger 1983, Theorem 8.16]. In order to understand the dependence of the constant on the domain, we can rescale the variables in this way: $v:B_1^{n-1}\to\mathbb{R}$ given by $v(y):=r^{-2}w_2(ry)$. Then, the equation satisfied by v is

$$\mathcal{L}v(y)+\mathcal{L}\psi(ry)+d(ry)=Lu(ry)\le 0.$$

We can then conclude using [loc. cit., (8.39)] that

$$\sup_{B_1^{n-1}}(-v)\le C\|\mathcal{L}\psi(ry)+d(ry)\|_{L^{q/2}(B_1^{n-1})}\le C,$$

where now C is a dimensional constant (only depending on $q>n-1$, which for us is any fixed exponent — note that the hypothesis (8.8) in [loc. cit., Theorem 8.16] is satisfied because we are considering the operator $\mathcal{L}$ which has no lower-order terms). In particular, scaling back to w_2 we deduce that

$$w_2(x)\ge -Cr^2\quad \forall\, x\in B_r^{n-1}. \tag{6-53}$$

This clearly implies $w_1(0)=u(0)-\psi(0)-w_2(0)\le Cr^2$. We can then use the Harnack inequality for w_1 (see [loc. cit., Theorem 8.20]) and conclude

$$w_1(x)\le C\inf_{B_{r/2}^{n-1}} w_1\le Cw_1(0)\le Cr^2\quad \forall\, x\in B_{r/2}^{n-1}. \tag{6-54}$$

Finally note that in $B_r^{n-1}\setminus\Lambda(u)$ we have the equality $\mathcal{L}w_2=-\mathcal{L}\psi-d$. Therefore, the function $z:=w_2+C|x|^2$ satisfies $\mathcal{L}z\ge 0$ for a suitably chosen constant $C=C(\|\mathcal{L}\psi\|_{L^\infty},\|d\|_{L^\infty})$. By the strong maximum principle [loc. cit., Theorem 8.19] we deduce that

$$\max_{B_r^{n-1}\setminus\Lambda(u)} z\le \max_{\partial(B_r^{n-1}\setminus\Lambda(u))} z\le Cr^2,$$

where we used that $z|_{\partial B_r^{n-1}}=Cr^2$ and that for every $x\in\Lambda(u)\cap B_r^{n-1}$ we have $z(x)=-w_1(x)+C|x|^2\le Cr^2$ by the positivity of w_1. In conclusion, we have

$$u(x)-\psi(x)\le |w_1(x)|+|w_2(x)|\le Cr^2$$

for every $x\in B_{r/2}^{n-1}$. Since $r\le r_0$ is arbitrary, by eventually changing the constant C, we conclude the proof of the proposition. □

6C4. *Curvature bounds away from the contact set.* Next we analyze the points $p\in\partial E\setminus\partial B$ which are close to ∂B. To this aim we fix a constant $s_0>0$ such that the following holds: if $\operatorname{dist}(p,\partial E\cap\partial B)=\operatorname{dist}(p,x_0)<s_0$, then p belongs to the coordinate chart Ω around x_0 as fixed in Section 6C1 and moreover, in these coordinates, $p=(x,z)\in B_{r_0}^{n-1}\times(-r_0,r_0)$ (necessarily with $x\notin\Lambda(u)$) satisfies

$$B_{4\delta}^{n-1}(x)\subset B_{r_0}^{n-1}\quad\text{with }\ \delta:=\tfrac12\operatorname{dist}(x,\Lambda(u)).$$

Note that the existence of such a constant $s_0>0$ is ensured by a simple compactness argument. Recall also that by the quadratic growth proved in the previous section we know

$$\|u\|_{L^\infty(B_{2\delta}^{n-1}(x))}\le C\delta^2.$$

The following lemma gives a curvature bound for ∂E in points p as above.

Lemma 6.13. *Let $p \in \partial E \setminus \partial B$ satisfy* $\operatorname{dist}(p, \partial E \cap \partial B) < s_0$. *Fixing $x_0 \in \partial E \cap \partial B$ and the corresponding coordinate chart as in Section 6C1 with the notation fixed above, we then conclude*

$$\|D^2 u\|_{L^\infty(B^{n-1}_\delta(x))} \le C, \tag{6-55}$$

where $C > 0$ is a dimensional constant.

Proof. Since on $B^{n-1}_{4\delta} \subset B^{n-1}_{r_0} \setminus \Lambda(u)$ the *equation* $Lu = 0$ is satisfied, the proof is a consequence of the basic interior Schauder estimates for second-order elliptic equations (see [Gilbarg and Trudinger 1983, Theorem 6.2]). More precisely we write the equation as $\mathcal{L}u = -d$, where $d \in C^{0,\alpha}$ is defined as in (6-47) and satisfies (6-49), and we apply [loc. cit., Theorem 6.2]) to such an equation. Indeed, by simply recalling the definition of the norms in [loc. cit., Theorem 6.2] we have, setting $\mathsf{d}_y := \operatorname{dist}(y, \partial B^{n-1}_{2\delta}(x))$,

$$\begin{aligned}\delta^2\|D^2u\|_{L^\infty(B^{n-1}_\delta(x))} &\le C\Big(\|u\|_{L^\infty(B^{n-1}_{2\delta}(x))} + \sup_{y\in B^{n-1}_{2\delta}(x)} \mathsf{d}_y^2|d(y)|\Big) + C\sup_{y,z\in B^{n-1}_{2\delta}(x)} \min\{\mathsf{d}_y,\mathsf{d}_z\}^{2+\alpha}\frac{|d(y)-d(z)|}{|y-z|^\alpha}\\ &\le C\Big(\|u\|_{L^\infty(B^{n-1}_{2\delta}(x))} + \delta^2\|d\|_{L^\infty(B^{n-1}_{2\delta}(x))}\Big) + C\delta^{2+\alpha}[d]_{C^{0,\alpha}(B^{n-1}_{2\delta}(x))} \le C\delta^2. \qquad\square\end{aligned}$$

6C5. *$C^{1,1}$-regularity.* In this section we finally prove Theorem 6.11. The proof is based on the following property: by Proposition 6.12 and Lemma 6.13, there exists $\delta > 0$ such that for every $x_0 \in \partial B \cap \partial E$ there exists $r_0 > 0$ satisfying, fixing coordinates as in Section 6C1,

$$|u(y) - u(x) - \nabla u(x)\cdot(y-x)| \le \tfrac{1}{2}\bar{C}\,|x-y|^2 \quad \forall\, x, y \in B_{r_0}(x_0). \tag{6-56}$$

Indeed, if $x \in \partial E \cap \partial B$, then centering the coordinates at x, we have $0 = u(0) = |\nabla u(0)|$, and (6-56) is a direct consequence of (6-50). On the other hand, if $x \notin \partial E \cap \partial B$, then setting the coordinates as in Lemma 6.13, we deduce (6-56) from (6-55).

The conclusion of Theorem 6.11 is then a direct consequence of the following lemma combined with a standard partition of unity argument.

Lemma 6.14. *Let $\Omega \subset \mathbb{R}^n$ be an open subset and let $u : \Omega \to \mathbb{R}$ be a C^1 function. Assume there exist $\bar{C} > 0$ and a countable covering $\{B_i\}_{i\in\mathbb{N}}$ of Ω made by open balls $B_i \subset \Omega$ such that for every $x, y \in B_i$,*

$$\big|u(y) - u(x) - \nabla u(x)\cdot(y-x)\big| \le \tfrac{1}{2}\bar{C}\,|x-y|^2. \tag{6-57}$$

Then the distribution $\partial^2_{ij}u \in \mathcal{D}'(\Omega)$ is represented by an $L^\infty(\Omega)$ function, and

$$\|\partial^2_{ij}u\|_{L^\infty(\Omega)} \le \bar{C}.$$

Proof. By a standard partition of unity argument it is enough to prove that for every ball B_i the restriction of the distribution $\partial^2_{ij}u \llcorner B_i$ is represented by an $L^\infty(B_i)$ function, and $\|\partial^2_{ij}u\|_{L^\infty(B_i)} \le \bar{C}$. In order to simplify the notation, let us fix $i \in \mathbb{N}$ and set $B := B_i$. For every fixed $\varphi \in C^\infty_c(B)$ let $Q^\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$Q^\varphi(v_1, v_2) := \int_B u\,\frac{\partial^2\varphi}{\partial v_1 \partial v_2}. \tag{6-58}$$

We first claim

$$|Q^{\varphi}(v,v)| \le \bar{C}|v|^2\,\|\varphi\|_{L^1(B)} \quad \forall\,\varphi \in C_c^\infty(B),\ \forall\, v \in \mathbb{R}^n, \tag{6-59}$$

where $\bar{C}$ is given by (6-57). To prove (6-59), we write (6-57) exchanging x and y and sum up to get

$$\big|(\nabla u(x) - \nabla u(y))\cdot(x-y)\big| \le \bar{C}|x-y|^2.$$

Choosing $y = x + tv$ in the last estimate, we get

$$\frac{|(\nabla u(x+tv) - \nabla u(x))\cdot v|}{t} \le \bar{C} \quad \forall\, v \in S^{n-1},\ \forall\, t \in (0, 1-|x|). \tag{6-60}$$

Now using that u is C^1 and $\varphi \in C_c^\infty(B)$, we can integrate by parts to get

$$\begin{aligned}\left|\int_B u\,\frac{\partial^2\varphi}{\partial v\partial v}\right| &= \left|\int_B \frac{\partial u}{\partial v}\,\frac{\partial\varphi}{\partial v}\right| = \left|\int_B (\nabla u(x)\cdot v)\lim_{t\downarrow 0}\frac{\varphi(x+tv)-\varphi(x)}{t}\,dx\right| \\ &= \left|\lim_{t\downarrow 0}\int_B\left(\frac{\nabla u(x-tv)-\nabla u(x)}{t}\cdot v\right)\varphi(x)\,dx\right| \le \bar{C}\|\varphi\|_{L^1(B)} \quad \forall v \in S^{n-1},\end{aligned} \tag{6-61}$$

where in the second line we used the change of variable $x \mapsto x + tv$, and the last inequality follows from (6-60). The inequality (6-61) proves our claim (6-59).

We now show (6-59) implies that the distribution $\partial^2_{ij}u$ is represented by an $L^\infty(B)$ function and $\|\partial^2_{ij}u\|_{L^\infty(B)} \le \bar{C}$. To this aim, observe that for every $\varphi \in C_c^\infty(B)$, by the Schwartz lemma, the map $Q^\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined in (6-58) is a symmetric bilinear form. Using (6-59), by polarization of Q^φ we get

$$|Q^\varphi(\partial_i,\partial_j)| = \tfrac{1}{4}\big|Q^\varphi(\partial_i+\partial_j,\partial_i+\partial_j) - Q^\varphi(\partial_i-\partial_j,\partial_i-\partial_j)\big| \le \bar{C}\|\varphi\|_{L^1(B)} \tag{6-62}$$

for every $i, j = 1, \dots, n$. But now

$$Q^\varphi(\partial_i,\partial_j) = \langle \partial^2_{ij}u, \varphi\rangle_{\mathcal{D}',\mathcal{D}},$$

where $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{D}',\mathcal{D}}$ denotes the pairing between distributions and C_c^∞ test functions. Therefore (6-62) combined with the Riesz representation theorem concludes the proof. □

The arguments above prove also the following slightly more general regularity result for isoperimetric regions inside a C^2 domain. In order to state it, for a subset $A \subset M$ and for some $\delta > 0$, let us denote by $B_\delta(A) = \{x \in M : \inf_{y\in A} \mathsf{d}(x,y) \le \delta\}$ the δ-tubular neighborhood of A.

Theorem 6.15 ($C^{1,1}$ regularity of isoperimetric regions inside a C^2 domain). *Let (M, g) be a Riemannian manifold, let $\Omega \subset M$ be an open subset with C^2 boundary $\partial\Omega$ and fix $v \in (0, \mu_g(\Omega))$. Let $E \subset \Omega$ be a finite-perimeter set with $\mu_g(E) = v$ and minimizing the perimeter among regions contained in Ω, i.e.,*

$$\mathcal{P}(E) = \inf\{\mathcal{P}(F) : F \subset \Omega,\ \mu_g(F) = v\}.$$

Then, there exists $\delta > 0$ such that $\partial E \cap B_\delta(\partial\Omega)$ is $C^{1,1}$ regular.

Remark 6.16. Theorem 6.15 already appeared in [White 1991, Proposition, p. 418], though the arguments in the proof are very concise (line 7, p. 419 in [White 1991]) and basically consist of referring to the work of Gerhardt [1973]. Nevertheless, it seems that one of the hypotheses of [Gerhardt 1973] is not met for the operator H in [White 1991]. Indeed, H is the Euler–Lagrange operator of the functional

$$\Phi(u) = \int L(x, u(x), \nabla u(x))\, dx,$$

and a simple computation shows

$$H(u) = \frac{\partial L}{\partial z}(x, u(x), \nabla u(x)) - \operatorname{div}\left(\frac{\partial L}{\partial p}(x, u(x), \nabla u(x))\right),$$

where we named the variables as $L = L(x, z, p)$. Now the operator H is of the form considered in [Gerhardt 1973] (here there is a conflict of notation between the two papers, therefore we put a bar for the notation in [loc. cit.]),

$$\bar{A}u + \overline{H} = -\operatorname{div}\big(\bar{a}(x, u(x), \nabla u(x))\big) + \overline{H}.$$

In our case the vector field $\bar{a}$ is given by $\partial L/\partial p$ and the forcing term $\overline{H}$ is given by $(\partial L/\partial z)(x, u(x), \nabla u(x))$. In [loc. cit.] the forcing term $\overline{H}$ is assumed to be $W^{1,\infty}$ (see equation (5) in [loc. cit.]), which in the present situation would be verified only knowing already that $u \in W^{2,\infty}$, which is, however, what one wants to deduce.

We do not exclude that going through the proofs of [loc. cit.] one could overcome such a difficulty; however, we think the approach of the present paper could be of independent interest, especially because it is self-contained and based on an elementary use of Schauder estimates.

6D. ***Further comments.*** We have proven the above regularity of the isoperimetric-isodiametric sets $E \subset M$ under the assumptions that the enclosing ball $B = B_{\mathrm{rad}(E)}(x_0)$ has smooth boundary. Actually, the following is true and is a direct consequence of the argument used above.

(A) If $\partial B \in C^{1,\alpha}$ for some $\alpha \in (0, 1]$, then in a neighborhood of ∂B the isoperimetric-isodiametric sets have the boundary ∂E, which is $C^{1,\alpha}$ regular.

Indeed, under the assumption in (A), the arguments in Lemma 6.3 show that ∂E is $C^{1,\kappa}$ regular in a neighborhood of ∂B for $k = \min\{\alpha, \frac{1}{2}\}$. Moreover, a careful inspection of the proof of the optimal regularity in Theorem 6.11 shows that the conclusion of (A) holds true with the right Hölder exponent (in the case $\alpha = 1$ the proof is a straightforward generalization; for $\alpha \in \left(\frac{1}{2}, 1\right)$ more details need to be checked). Nevertheless, we do not do it here.

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NONRADIAL TYPE II BLOW UP FOR THE ENERGY-SUPERCRITICAL SEMILINEAR HEAT EQUATION

CHARLES COLLOT

We consider the semilinear heat equation in large dimension $d \geq 11$

$$\partial_t u = \Delta u + |u|^{p-1}u, \quad p = 2q+1,\ q \in \mathbb{N},$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary condition. In the supercritical range $p \geq p(d) > 1 + \frac{4}{d-2}$, we prove the existence of a countable family $(u_\ell)_{\ell \in \mathbb{N}}$ of solutions blowing up at time $T > 0$ with type II blow up:

$$\|u_\ell(t)\|_{L^\infty} \sim C(T-t)^{-c_\ell}$$

with blow-up speed $c_\ell > \frac{1}{p-1}$. The blow up is caused by the concentration of a profile Q which is a radially symmetric stationary solution:

$$u(x,t) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x-x_0}{\lambda(t)}\right), \quad \lambda \sim C(u_n)(T-t)^{\frac{c_\ell(p-1)}{2}},$$

at some point $x_0 \in \Omega$. The result generalizes previous works on the existence of type II blow-up solutions, which only existed in the radial setting. The present proof uses robust nonlinear analysis tools instead, based on energy methods and modulation techniques. This is the first nonradial construction of a solution blowing up by concentration of a stationary state in the supercritical regime, and it provides a general strategy to prove similar results for dispersive equations or parabolic systems and to extend it to multiple blow ups.

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Keywords: blow up, heat, soliton, ground state, nonlinear, nonradial, supercritical.

1. Introduction

1A. ***The semilinear heat equation.*** We study solutions of the semilinear heat equation (NLH)

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}, \\ u(0) = u_0, \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where u is real-valued, p is such that the nonlinearity is analytic, that is $p = 2q+1$, $q \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^d$ is a smooth bounded open domain. For smooth enough initial data u_0 satisfying some compatibility conditions at the border $\partial\Omega$, the Cauchy problem is well posed and there exists a unique maximal solution $u \in C\big((0,T), L^\infty(\Omega)\big)$. If $T < +\infty$, the solution is said to blow up and necessarily

$$\lim_{t\to T} \|u(t)\|_{L^\infty(\Omega)} = +\infty.$$

This paper addresses the general issue of the asymptotic behavior as $t \to T$. In the case $\Omega = \mathbb{R}^d$, there is a natural scale invariance, namely if u is a solution then so is

$$u_\lambda(\lambda^2 t, x) := \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x). \tag{1-2}$$

The Sobolev space that has an invariant norm for this scale change is

$$\dot H^{s_c}(\mathbb{R}^d) := \left\{ u : \int_{\mathbb{R}^d} |\xi|^{2s_c} |\hat u|^2 \, d\xi < +\infty \right\}, \quad s_c := \frac{d}{2} - \frac{2}{p-1}, \tag{1-3}$$

where $\hat u$ stands for the Fourier transform of u. Two particular solutions arise, the constant-in-space blow-up solution

$$u(t,x) = \pm \frac{\kappa(p)}{(T-t)^{\frac{1}{p-1}}}, \quad \kappa(p) := \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}, \tag{1-4}$$

and the unique (up to translation and scale change) radially decaying stationary solution Q (see [Li 1992] and references therein) solving the stationary elliptic equation

$$\Delta Q + Q^p = 0. \tag{1-5}$$

1B. ***Blow-up for*** **(NLH).** Being one of the model nonlinear evolution equations, blow-up dynamics has attracted a great amount of work (see [Quittner and Souplet 2007] for a review). In particular, one is interested in the description of the solution near the set of blow-up points, that is, the points x for which there exists $(t_n, x_n) \to (T, x)$ such that $|u(t_n, x_n)| \to +\infty$. A comparison argument with the constant-in-space blow-up solution (1-4) implies the lower bound

$$\limsup_{t\to T} \|u(t)\|_{L^\infty} (T-t)^{\frac{1}{p-1}} \geq \kappa(p)$$

and leads to the following distinction between type I and type II blow up [Matano and Merle 2004]:

$$u \text{ blows up with type I if } \limsup_{t\to T} \|u(t)\|_{L^\infty} (T-t)^{\frac{1}{p-1}} < +\infty,$$

$$u \text{ blows up with type II if } \limsup_{t\to T} \|u(t)\|_{L^\infty} (T-t)^{\frac{1}{p-1}} = +\infty.$$

The ODE blow up (1-4) does not see the dissipative term in (1-1) whereas type II blow up involves an interplay between dissipation and nonlinearity, and therefore its existence and properties may change according to d and p. In the series of works [Giga 1986; Giga and Kohn 1985; 1987; 1989; Giga et al. 2004; Merle and Zaag 1998; 2000], the authors show that in the energy subcritical range $1 < p < \frac{d+2}{d-2}$, all blow-up solutions are of type I and match the constant-in-space solution (1-4):

$$\limsup_{t\to T} \|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} = \kappa(p).$$

In the energy critical case $p = \frac{d+2}{d-2}$, $d = 4$, Schweyer [2012] constructed a radial type II blow-up solution, following the analysis of critical problems [Merle and Raphaël 2005a; 2005b; 2006; Raphaël and Schweyer 2013; 2014; Raphaël and Rodnianski 2012; Merle et al. 2013]; see also [Filippas et al. 2000]. In that case, the scale invariance (1-2) implies that there exists a one-dimensional continuum of ground states

$$\left(\frac{1}{\lambda^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda}\right)\right)_{\lambda>0}.$$

The properties of the ground state (1-5) then allow the existence of a solution u that stays close to this manifold,

$$u = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right) + \varepsilon, \quad \|\varepsilon\| \ll 1,$$

such that $\lambda(t) \to 0$ for some time $T > 0$, which makes the solution blow up. This blow-up scenario is not always possible as it heavily relies on the asymptotic behavior of the ground state, and is impossible in dimension $d \geq 7$ [Collot et al. 2016].

In the radial energy-supercritical case $p > \frac{d+2}{d-2}$, the Joseph–Lundgren exponent [1973]

$$p_{JL} := \begin{cases} +\infty & \text{if } d \leq 10, \\ 1 + \frac{4}{d-4-2\sqrt{d-1}} & \text{if } d \geq 11 \end{cases} \tag{1-6}$$

dictates the existence of type II blow-up solutions. For $\frac{d+2}{d-2} < p < p_{JL}$, type II blow-up solutions do not exist [Matano and Merle 2004; Mizoguchi 2011b]. For $p > p_{JL}$, type II blow-up solutions are completely classified. In [Herrero and Velázquez 1994] the authors predicted the existence of a countable family of solutions u_ℓ such that

$$\|u(t)\|_{L^\infty} \sim C(u_n(0))(T-t)^{\frac{\ell}{\alpha(d,p)}\frac{2}{p-1}}, \quad \ell \in \mathbb{N},\ \ell > \tfrac{1}{2}\alpha,$$

(α is defined in (1-10)), which are the same speeds as in the present paper. The rigorous proof was first made in an unpublished paper [Herrero and Velázquez] and then in [Mizoguchi 2004]. In the series of works [Matano 2007; Matano and Merle 2009; Mizoguchi 2007; 2011a] any type II blow-up solution was proved to have one of the above blow-up rates. These works have the powerful advantage that they deal with large solutions, but strongly rely on comparison principles that are only available for radial parabolic problems.

1C. ***Outlook on blow up for other problems.*** Many model nonlinear equations share similar features with (NLH). The construction of solutions concentrating a stationary state for the energy-supercritical Schrödinger and wave equations has been done in [Collot 2014; Merle et al. 2015], and recently for the harmonic heat flow in [Biernat and Seki 2016]. These concentration scenarios happen on a central manifold near the continuum of ground states

$$\left(\frac{1}{\lambda^{\frac{2}{p-1}}}Q\left(\frac{x}{\lambda}\right)\right)_{\lambda>0},$$

whose topological and dynamical properties have been a popular subject of studies [Schlag 2009; Krieger et al. 2015]. The possibility of various blow-up speeds is linked to the regularity of the solutions, and this is why parabolic problems are more rigid, thanks to the regularizing effect, than dispersive problems, for which a wider range of concentration scenarios exists [Krieger et al. 2008].

A major goal is the study of blow up for general data, where nonradial stationary states can appear as blow-up profiles [Duyckaerts et al. 2012]. The solution may also not be a small perturbation of it. One thus needs robust tools for the perturbative study of special nonlinear profiles as well as a better understanding of the set of stationary solutions. The present work is a step toward this general aim.

1D. ***Statement of the result.*** We revisit the result of [Herrero and Velázquez 1994; Mizoguchi 2004; 2005] with the techniques employed in [Raphaël and Rodnianski 2012] to address the nonradial setting. From [Li 1992], for $p > p_{JL}$ (defined in (1-6)) the radially decaying ground state Q, solution of (1-5), admits the asymptotic

$$Q(x) = \frac{c_\infty}{|x|^{\frac{2}{p-1}}} + \frac{a_1}{|x|^\gamma} + o(|x|^{-\gamma}) \quad \text{as } |x| \to +\infty,\ a_1 \neq 0, \tag{1-7}$$

with

$$c_\infty := \left[\frac{2}{p-1}\left(d-2-\frac{2}{p-1}\right)\right]^{\frac{1}{p-1}}, \tag{1-8}$$

$$\gamma := \tfrac{1}{2}(d-2-\sqrt{\triangle}), \quad \triangle := (d-2)^2 - 4pc_\infty^{p-1} \ \ (\triangle > 0 \iff p > p_{JL}), \tag{1-9}$$

and we define

$$\alpha := \gamma - \frac{2}{p-1}. \tag{1-10}$$

For $n \in \mathbb{N}$ we define the following numbers ($\triangle_n > 0$ if $p > p_{JL}$):

$$-\gamma_n := \frac{-(d-2)+\sqrt{\triangle_n}}{2}, \quad \triangle_n := (d-2)^2 - 4pc_\infty + 4n(d+n-2).$$

The above numbers are directly linked with the existence and the number of instability directions of type II blow-up solutions concentrating Q. Our result is the existence and precise description of some localized type II blow-up solutions in any domain with smooth boundary.

Theorem 1.1 (existence of nonradial type II blow up for the energy-supercritical heat equation). *Let $d \geq 11$, $p = 2q+1 > p_{JL}$, $q \in \mathbb{N}$, where p_{JL} is given by* (1-6). *Let Q, γ, α, γ_n and s_c be given by* (1-7), (1-9), (1-10), (1-18) *and* (1-3) *and $\varepsilon > 0$. Let $\Omega \subset \mathbb{R}^d$ be a smooth open bounded domain. For $x_0 \in \Omega$*

let $\chi(x_0)$ be a smooth cut-off function around x_0 with support in Ω. Pick $\ell \in \mathbb{N}$ satisfying $2\ell > \alpha$. Then, there exists a large enough regularity exponent

$$s_+ = s_+(\ell) \in 2\mathbb{N}, \quad s_+ \gg 1,$$

such that under the nondegeneracy condition

$$\left(\tfrac{1}{2}d - \gamma_n\right) \notin 2\mathbb{N} \quad \textit{for all } n \in \mathbb{N} \textit{ such that } d - 2\gamma_n \leq 4s_+, \tag{1-11}$$

there exists a solution $u \in C\big([0,T), L^\infty(\Omega)\big)$ of (1-1) *with $u_0 \in H^{s_+}(\Omega)$* (*which can be chosen smooth and compactly supported*) *blowing up in finite time $0 < T < +\infty$ by concentration of the ground state at a point $x_0' \in \Omega$ with $|x_0' - x_0| \leq \varepsilon$. It is given by*

$$u(t,x) = \chi_{x_0}(x)\frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x - x_0'}{\lambda(t)}\right) + v, \tag{1-12}$$

where:

(i) *x_0' is the only blow-up point of u.*

(ii) *Blow-up speed*:

$$\|u\|_{L^\infty(\Omega)} = c(u_0)(T-t)^{-\frac{2\ell}{\alpha(p-1)}}(1+o(1)) \quad \textit{as } t \to T,\ c(u_0) > 0, \tag{1-13}$$

$$\lambda(t) = c'(u_0)(1+o(1))(T-t)^{\frac{\ell}{\alpha}} \quad \textit{as } t \to T,\ c'(u_0) > 0. \tag{1-14}$$

(iii) *Asymptotic stability above scaling in renormalized variables*:

$$\lim_{t\to T} \left\| \lambda(t)^{\frac{2}{p-1}} v(t, x_0 + \lambda(t)x) \right\|_{H^s(\lambda(t)^{-1}(\Omega - \{x_0\}))} = 0 \quad \textit{for all } \ s_c < s \leq s_+. \tag{1-15}$$

(iv) *Boundedness below scaling*:

$$\limsup_{t\to T} \|u(t)\|_{H^s(\Omega)} < +\infty \quad \textit{for all } \ 0 \leq s < s_c. \tag{1-16}$$

(v) *Asymptotic of the critical norm*:

$$\|u(t)\|_{H^{s_c}(\Omega)} = c(d,p)\sqrt{\ell}\sqrt{|\log(T-t)|}(1+o(1)) \quad \textit{as } t \to T,\ c(d,p) > 0. \tag{1-17}$$

Comments on Theorem 1.1:

(1) *On the assumptions*. First, the assumption $p > p_{JL}$ is not just technical as radial type II blow up is impossible for $\frac{d+2}{d-2} < p < p_{JL}$ [Matano and Merle 2004; Mizoguchi 2011b]. Nonradial type II blow up solutions in this latter range, if they exist, must have a very different dynamical description. Next, if p is not an odd integer, then the nonlinearity $x \mapsto |x|^{p-1}x$ is singular at the origin, yielding regularity issues. In that case the techniques used in the present paper could only be applied for a certain range of integers ℓ. Eventually, the condition (1-11) is purely technical, as it avoids the presence of logarithmic corrections in some inequalities that we use. It could be removed since the analysis relies on gains that are polynomial and not logarithmic, but would weigh down the already long proof. Note that a large number of couples (p,ℓ) satisfy this condition. Indeed, only finitely many integers n are concerned by (1-20), and the value of γ_n is very rarely a rational number by (1-18).

(2) *Blow-up by concentration at any point and manifold of type II blow-up solutions*. For any $x_0 \in \Omega$, Theorem 1.1 provides a solution that concentrates at a point that can be arbitrarily close to x_0. In fact there exists a solution that concentrates exactly at x_0, meaning that this blow up can happen at any point of Ω. To show that, one needs an additional continuity argument, in addition to the information contained in the proof, to be able to reason as in [Planchon and Raphaël 2007; Merle 1992], for example. This continuity property amounts to proving that the set of type II blow-up solutions that we construct is a Lipschitz manifold with exact codimension in a suitable functional space. This was proved in the radial setting in [Collot 2014] and the analysis could be adapted here using the nonradial analysis provided in the present paper. However a precise and rigorous proof of this fact would be too lengthy to be inserted in this paper. Let us stress that the solutions built here possess an explicit number of linear nonradial instabilities. An interesting question is then whether or not these new instabilities can be used, with the help of resonances through the nonlinear term, to produce new type II blow-up mechanisms around Q in the nonradial setting.

(3) *Multiple blow ups and continuation after blow up*. As in our analysis we are able to cut and localize the approximate blow-up profile, there should be no problems in constructing a solution blowing up with this mechanism at several points simultaneously, as in [Merle 1992]. Cases where the blow-up bubbles really interact can lead to very different dynamics; see [Martel and Raphaël 2015; Jendrej 2016] for recent results. From the construction, as $t \to T$, we have u admits a strong limit in $H^{s_c}_{\text{loc}}(\Omega \backslash \{x_0\})$. One could investigate the properties of this limit in order to continue the solution u beyond blow-up time, which is a relevant question for blow-up issues [Matano and Merle 2009], especially for hamiltonian equations where a subcritical norm is under control.

1E. *Notation.* In the analysis, C will stand for a constant which may vary from one line to another, whose value just depends on d and p. The notation $a \lesssim b$ means that $a \leq Cb$ for such a constant C, and $a = O(b)$ means $|a| \lesssim b$.

Supercritical numerology. For $d \geq 11$ the condition $p > p_{JL}$, where p_{JL} is defined by (1-6), is equivalent to $2 + \sqrt{d-1} < s_c < \frac{1}{2}d$. We define the sequences of numbers describing the asymptotic of particular zeros of H (defined in (1-30)) for $n \in \mathbb{N}$:

$$-\gamma_n := \frac{-(d-2) + \sqrt{\triangle_n}}{2}, \quad \triangle_n := (d-2)^2 - 4cp_\infty + 4n(d+n-2), \tag{1-18}$$

$$\alpha_n := \gamma_n - \frac{2}{p-1}, \tag{1-19}$$

where $\triangle_n > 0$ for $p > p_{JL}$. We will use the following facts in the sequel:

$$\gamma_0 = \gamma, \qquad \gamma_1 = \frac{2}{p-1} + 1, \qquad \gamma_n < \frac{2}{p-1} \quad \text{for } n \geq 2 \text{ and } \gamma_n \sim -n; \tag{1-20}$$

see Lemma A.1 (where γ is defined in (1-9)). In particular $\alpha_0 = \alpha$, $\alpha_1 = 1$ and $\alpha_n < 0$ for $n \geq 2$. A computation yields the bound

$$2 < \alpha < \tfrac{1}{2}d - 1$$

(see [Merle et al. 2015]). We let

$$g := \min(\alpha, \triangle) - \varepsilon, \quad g' := \tfrac{1}{2}\min(g, 1, \delta_0 - \varepsilon), \tag{1-21}$$

where $0 < \varepsilon \ll 1$ is a very small constant just here to avoid keeping track of some logarithmic terms later on. For $n \in \mathbb{N}$ we define[1]

$$m_n := E\left[\tfrac{1}{2}\left(\tfrac{1}{2}d - \gamma_n\right)\right] \tag{1-22}$$

and denote by δ_n the positive real number $0 \le \delta_n < 1$ such that

$$d = 2\gamma_n + 4m_n + 4\delta_n. \tag{1-23}$$

For $1 \ll L$ a very large integer, we define the Sobolev exponent

$$s_L := m_0 + L + 1. \tag{1-24}$$

In this paper we assume the technical condition (1-11) for $s_+ = s_L$, which means

$$0 < \delta_n < 1 \tag{1-25}$$

for all integers n such that $d - 2\gamma_n \le 4s_L$ (there is only a finite number of such integers by (1-20)). We let n_0 be the last integer to satisfy the condition

$$d - 2\gamma_{n_0} \le 4s_L \quad \text{and} \quad d - 2\gamma_{n_0+1} > 4s_L \tag{1-26}$$

and we define

$$\delta_0' := \max_{0 \le n \le n_0} \delta_n \in (0, 1). \tag{1-27}$$

For all integers $n \le n_0$ we define the integers

$$L_n := s_L - m_n - 1 \tag{1-28}$$

and in particular $L_0 = L$. Given an integer $\ell > \frac{1}{2}\alpha$ (that will be fixed in the analysis later on), for $0 \le n \le n_0$ we define the real numbers

$$i_n = \ell - \frac{\gamma - \gamma_n}{2}. \tag{1-29}$$

Notations for the analysis. For $R \ge 0$, the euclidean sphere and ball are denoted by

$$\mathcal{S}^{d-1}(R) := \left\{ x \in \mathbb{R}^d, \sum_{i=1}^{d} x_i^2 = R^2 \right\} \quad \text{and} \quad \mathcal{B}^d(R) := \left\{ x \in \mathbb{R}^d, \sum_{i=1}^{d} x_i^2 \le R^2 \right\}.$$

We use the Kronecker delta notation:

$$\delta_{i,j} := \begin{cases} 0 & \text{if } i \ne j, \\ 1 & \text{if } i = j \end{cases}$$

[1] $E[x]$ stands for the entire part: $x - 1 < E[x] \le x$.

for $i, j \in \mathbb{N}$. We let

$$F(u) := \Delta u + f(u), \quad f(u) := |u|^{p-1}u$$

so that (1-1) can be written as

$$\partial_t u = F(u).$$

When using the binomial expansion for the nonlinearity, we use the constants

$$f(u+v) = \sum_{l=0}^{p} C_l^p u^l v^{p-l}, \quad C_l^p := \binom{p}{l}.$$

The linearized operator close to Q (defined in (1-5)) is

$$Hu := -\Delta u - pQ^{p-1}u \tag{1-30}$$

so that $F(Q+\varepsilon) \sim -H\varepsilon$. We introduce the potential

$$V := -pQ^{p-1} \tag{1-31}$$

so that $H = -\Delta + V$. Given a strictly positive real number $\lambda > 0$ and function $u : \mathbb{R}^d \to \mathbb{R}$, we define the rescaled function

$$u_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x). \tag{1-32}$$

This semigroup has the infinitesimal generator

$$\Lambda u := \frac{\partial}{\partial \lambda}(u_\lambda)_{|\lambda=1} = \frac{2}{p-1}u + x.\nabla u.$$

The action of the scaling on (1-1) is given by the formula

$$F(u_\lambda) := \lambda^2 (F(u))_\lambda.$$

For $z \in \mathbb{R}^d$ and $u : \mathbb{R}^d \to \mathbb{R}$, the translation of vector z of u is denoted by

$$\tau_z u(x) := u(x-z). \tag{1-33}$$

This group has the infinitesimal generator

$$\left[\frac{\partial}{\partial z}(\tau_z u)\right]_{|z=0} = -\nabla u.$$

The original space variable will be denoted by $x \in \Omega$ and the renormalized one by y, related through $x = z + \lambda y$. The number of spherical harmonics of degree n is

$$k(0) := 1, \qquad k(1) := d, \qquad k(n) := \frac{2n+p-2}{n}\binom{n+p-3}{n-1} \quad \text{for } n \geq 2.$$

The Laplace–Beltrami operator on the sphere $\mathcal{S}^{d-1}(1)$ is self-adjoint with compact resolvent and its spectrum is $\{n(d+n-2) : n \in \mathbb{N}\}$. For $n \in \mathbb{N}$ the eigenvalue $n(d+2-n)$ has geometric multiplicity $k(n)$, and we denote by $(Y^{(n,k)})_{n\in\mathbb{N},\,1\le k\le k(n)}$ an associated orthonormal Hilbert basis of $L^2(\mathbb{S}^d)$:

$$L^2(\mathcal{S}^{d-1}(1)) = \bigoplus_{n=0}^{+\infty}{}^{\perp}\,\mathrm{Span}\big(Y^{(n,k)}, 1\le k\le k(n)\big),$$

$$\Delta_{\mathcal{S}^{d-1}(1)}Y^{(n,k)} = n(d+n-2)Y^{(n,k)}, \quad \int_{\mathcal{S}^{d-1}(1)} Y^{(n,k)}Y^{(n',k')} = \delta_{(n,k),(n',k')}, \tag{1-34}$$

with the special choices

$$Y^{(0,1)}(x) = C_0, \quad Y^{1,k}(x) = -C_1x_k, \tag{1-35}$$

where C_0 and C_1 are two renormalization constants. The action of H on each spherical harmonic is described by the family of operators on radial functions

$$H^{(n)} := -\partial_{rr} - \frac{d-1}{r}\partial_r + \frac{n(d+n-2)}{r^2} - pQ^{p-1} \tag{1-36}$$

for $n\in\mathbb{N}$, as for any radial function f they produce the identity

$$H\left(x\mapsto f(|x|)Y^{(n,k)}\left(\frac{x}{|x|}\right)\right) = x\mapsto (H^{(n)}(f))(|x|)Y^{(n,k)}\left(\frac{x}{|x|}\right). \tag{1-37}$$

For two strictly positive real numbers $b_1^{(0,1)}>0$ and $\eta>0$ we define the scales

$$M\gg 1, \quad B_0 = |b_1^{(0,1)}|^{-\frac{1}{2}}, \quad B_1 = B_0^{1+\eta}. \tag{1-38}$$

The blow-up profile of this paper is an excitation of several directions of stability and instability around the soliton Q. Each one of these directions of perturbation, denoted by $T_i^{(n,k)}$, will be associated to a triple (n,k,i), meaning that it is the i-th perturbation located on the spherical harmonics of degree (n,k). For each (n,k) with $n\le n_0$, there will be L_n+1 such perturbations for $i=0,\dots,L_n$ except for the cases $n=0$, $k=1$, and $n=1$, $k=1,\dots,d$, where there will be L_n perturbations for $i=1,\dots,L_n$ $(n=1,2)$. Hence the set of triples (n,k,i) used in the analysis is

$$\mathcal{I} := \big\{(n,k,i)\in\mathbb{N}^3:\ 0\le n\le n_0,\ 1\le k\le k(n),\ 0\le i\le L_n\big\}\backslash\big(\{(0,1,0)\}\cup\{(1,1,0),\dots,(1,d,0)\}\big) \tag{1-39}$$

with cardinal

$$\#\mathcal{I} := \sum_{n=0}^{n_0} k(n)(L_n+1) - d - 1. \tag{1-40}$$

For $j\in\mathbb{N}$ and an n-tuple of integers $\mu=(\mu_i)_{1\le i\le j}$, the usual length is denoted by

$$|\mu| := \sum_{i=1}^{j}\mu_i.$$

If $j=d$ and h is a smooth function on $\mathbb{R}^d$ then we use the following notation for the differentiation:

$$\partial^{\mu}h := \frac{\partial^{|\mu|}}{\partial_{x_1}^{\mu_1}\cdots\partial_{x_d}^{\mu_d}}h.$$

For J an $\#\mathcal{I}$-tuple of integers, we introduce two other weighted lengths

$$|J|_2=\sum_{n,k,i}\left(\frac{\gamma-\gamma_n}{2}+i\right)J_i^{(n,k)}, \tag{1-41}$$

$$|J|_3=\sum_{i=1}^{L} iJ_i^{(0,1)}+\sum_{\substack{1\le i\le L_1\\ 1\le k\le d}} iJ_i^{(1,k)}+\sum_{\substack{(n,k,i)\in\mathcal{I}\\ 2\le n}}(i+1)J_i^{(n,k)}. \tag{1-42}$$

To localize some objects we will use a radial cut-off function $\chi\in C^\infty(\mathbb{R}^d)$:

$$0\le\chi\le 1,\qquad \chi(|x|)=1\quad\text{for } |x|\le 1,\qquad \chi(|x|)=0\quad\text{for } |x|\ge 2, \tag{1-43}$$

and for $B>0$, we let χ_B denote the cut-off around $\mathcal{B}^d(0,B)$:

$$\chi_B(x):=\chi\left(\frac{x}{B}\right).$$

1F. ***Strategy of the proof.*** We now describe the main ideas behind the proof of Theorem 1.1. Without loss of generality, via scale change and translation in space one can assume that $x_0=0$ and $\mathcal{B}^d(7)\subset\Omega$.

(i) *Linear analysis and tail computations*. The linearized operator near Q is $H=-\Delta-pQ^{p-1}$ and its generalized kernel is

$$\{f:\exists j\in\mathbb{N}\text{ such that } H^jf=0\}=\operatorname{Span}(T_i^{(n,k)})_{(n,i)\in\mathbb{N}^2,\,1\le k\le k(n)},$$

where

$$T_i^{(n,k)}(x)=T_i^{(n)}(|x|)Y^{(n,k)}\left(\frac{x}{|x|}\right),$$

$T_i^{(n)}$ being radial, is located on the spherical harmonics of degree (n,k), with

$$T_0^{(0,1)}=\Lambda Q,\quad T_0^{(1,k)}=\partial_{x_k}Q,\quad HT_0^{(n,k)}=0,\quad HT_{i+1}^{(n,k)}=-T_i^{(n,k)}. \tag{1-44}$$

For any $L\in\mathbb{N}$, defining s_L, $n_0(L)$ and $L_n(L)$ by (1-24), (1-26) and (1-28), H^{s_L} is coercive for functions that are not in the suitably truncated generalized kernel:

$$\int\varepsilon H^{s_L}\varepsilon\gtrsim\|\nabla^{s_L}\varepsilon\|_{L^2}^2+\|\varepsilon\|_{\mathrm{loc}}^2\quad\text{if } \varepsilon\in\operatorname{Span}(T_i^{(n,k)})^{\perp}_{0\le n\le n_0,\,1\le k\le k(n),\,0\le i\le L_n}, \tag{1-45}$$

where $\|\varepsilon\|_{\mathrm{loc}}^2$ means any norm of ε on a compact set involving derivatives up to order $2s_L$. A scale change for these profiles produces the identity

$$\frac{\partial}{\partial\lambda}(T_i^{(n,k)})_{|\lambda=1}(x)=\Lambda T_i^{(n,k)}(x)\sim(2i-\alpha_n)T_i^{(n,k)}(x)\quad\text{as } |x|\to+\infty. \tag{1-46}$$

(ii) *The renormalized flow*. For u a solution, $\lambda:(0,T)\to\mathbb{R}$ and $z:(0,T)\to\mathbb{R}^d$, we define the renormalized time

$$\frac{ds}{dt}=\frac{1}{\lambda^2},\quad s(0)=s_0. \tag{1-47}$$

Then $v=(\tau_{-z}u)_\lambda$ solves the renormalized equation

$$\partial_s v-\frac{\lambda_s}{\lambda}\Lambda v-\frac{z_s}{\lambda}.\nabla v-F(v)=0. \tag{1-48}$$

(iii) *The dynamical system for the coordinates on the center manifold.* Let $\mathcal{I}$ be defined by (1-39). For an approximate solution of (1-1) under the form

$$u=\Bigg(Q+\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)}\Bigg)_{z,\frac{1}{\lambda}} \tag{1-49}$$

described by some parameters $b_i^{(n,k)}\in\mathbb{R}$, one has the identity from (1-44) and (1-45):

$$\begin{aligned}-z_t.\nabla u-\frac{\lambda_t}{\lambda}\Lambda u+\Bigg(\sum_{(n,k,i)\in\mathcal{I}} b_{i,t}^{(n,k)}T_i^{(n,k)}\Bigg)_{z,\frac{1}{\lambda}}\\
=\partial_t u\approx F(u)\\
=\frac{b_1^{(1,\cdot)}}{\lambda}.\nabla u+\frac{b_1^{(0,1)}}{\lambda^2}\Lambda u+\Bigg(\sum_{(n,k,i)\in\mathcal{I}}\frac{b_{i+1}^{(n,k)}-(2i-\alpha_n)b_1^{(1,0)}b_i^{(n,k)}}{\lambda^2}T_i^{(n,k)}\Bigg)_{z,\frac{1}{\lambda}}+\psi,\end{aligned} \tag{1-50}$$

where $b_1^{(1,\cdot)}=(b_1^{(1,1)},\dots,b_1^{(1,d)})$ and with the convention $b_{L_n+1}^{(n,k)}=0$. The error term ψ is negligible under a size assumption on the parameters. Identifying the terms in the above identity yields the finite-dimensional dynamical system[2]

$$\begin{cases}\lambda_t=-\dfrac{b_1^{(0,1)}}{\lambda},\quad z_t=-\dfrac{b_1^{(1,\cdot)}}{\lambda},\\ b_{i,t}^{(n,k)}=-\dfrac{1}{\lambda^2}(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+\dfrac{1}{\lambda^2}b_{i+1}^{(n,k)}\quad\forall(n,k,i)\in\mathcal{I}.\end{cases} \tag{1-51}$$

(iv) *The approximate blow-up profile.* Equation (1-51) admits for any $\ell\in\mathbb{N}$ with $2\ell>\alpha$ an explicit special solution $(\bar\lambda,\bar z,\bar b_i^{(n,k)})$ such that $\bar z=0$ and $\bar\lambda\sim(T-t)^{\frac{\ell}{\alpha}}$ for some $T>0$. Moreover, when linearizing (1-51) around this solution, one finds an explicit number m of directions of linear instability and $\#\mathcal{I}-m$ directions of stability. In addition, for the renormalized time s associated to $\bar\lambda$, one has

$$\lim_{t\to T}s(t)=+\infty,\quad |\bar b_k^{(i,n)}(s)|\lesssim s^{-\frac{\gamma-\gamma_n}{2}-i}. \tag{1-52}$$

Our approximate blow-up profile is then given by

$$\Bigg(Q+\sum_{(n,k,i)\in\mathcal{I}}\bar b_i^{(n,k)}(t)T_i^{(n,k)}\Bigg)_{\bar z(t),\frac{1}{\bar\lambda(t)}}.$$

(v) *The blow-up ansatz.* Following (iv), we study solutions of the form

$$u=\chi\Bigg(Q+\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)}\Bigg)_{z,\frac{1}{\lambda}}+w \tag{1-53}$$

[2] Again, with the convention $b_{L_n+1}^{(n,k)}=0$.

and decompose the remainder w according to

$$w_{\text{int}} := \chi_3 w, \quad w_{\text{ext}} := (1-\chi_3)w, \quad \varepsilon := (\tau_{-z} w_{\text{int}})_\lambda, \tag{1-54}$$

where w_{ext} is the remainder outside the blow-up zone, w_{int} the remainder inside the blow-up zone, and ε is the renormalization of the remainder inside the blow-up zone corresponding to the scale and central point of the ground state $Q_{z,1/\lambda}$. Now w is orthogonal to the suitably truncated center manifold

$$\varepsilon \in \text{Span}(T_i^{(n,k)})^{\perp}_{0\le n\le n_0,\, 1\le k\le k(n),\, 0\le i\le L_n}, \tag{1-55}$$

which fixes in a unique way the value of the parameters $b_i^{(n,k)}$, λ and z. We then define the renormalized time s associated to λ via (1-47). We take b, λ and z to be perturbations of $\bar{b}$, $\bar{\lambda}$ and $\bar{z}$ for the renormalized time:

$$b_i^{(n,k)}(s) = \bar{b}_i^{(n,k)}(s) + b_i^{\prime(n,k}(s),\ \lambda(s) = \bar{\lambda}(s) + \lambda'(s),\ z(s) = \bar{z}(s) + z'(s). \tag{1-56}$$

We define four norms for the remainder in (1-53) and (1-54):

$$\mathcal{E}_\sigma := \|\nabla^\sigma \varepsilon\|^2_{L^2(\mathbb{R}^d)}, \quad \mathcal{E}_{2s_L} := \int_{\mathbb{R}^d} |H^{s_L}\varepsilon|^2, \quad \|w_{\text{ext}}\|_{H^\sigma(\Omega)} \quad \text{and} \quad \|w_{\text{ext}}\|_{H^{s_L}(\Omega)},$$

where σ is a slightly supercritical regularity exponent

$$0 < \sigma - s_c \ll 1. \tag{1-57}$$

One has that $\mathcal{E}_{2s_L} \gtrsim \|\nabla^{2s_L}\varepsilon\|_{L^2}$ from (1-45).

Interpretation: We decompose a solution near the set of localized and concentrated ground states $\chi(Q_{z,1/\lambda})$ according to (1-53). A part, $\chi\big(\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} T_i^{(n,k)}\big)_{z,1/\lambda}$, is located on the truncated center manifold; it decays slowly, see (1-52), while interacting with the ground state, see (1-51), and is responsible for the blow up by concentration, and one has an explicit behavior of the coordinates, (1-51). The other part, w, is orthogonal to the truncated center manifold (1-55); it is expected to decay faster as H is more coercive, see (1-45), on this set, and not to perturb the blow-up dynamics. The change of variables (1-47) and (1-48) transforms the blow-up problem into a long-time asymptotic problem by (1-52).

Bootstrap method in a trapped regime: We study solutions that are close to the approximate blow-up profile for the renormalized time, i.e., that satisfy

$$\mathcal{E}_\sigma + \|w_{\text{ext}}\|^2_{H^\sigma(\Omega)} \lesssim 1, \quad \mathcal{E}_{2s_L} + \|w_{\text{ext}}\|_{H^{s_L}(\Omega)} \lesssim \frac{1}{\lambda^{2(2s_L-s_c)}\, s^{L+(1-\delta_0)+\nu}}, \tag{1-58}$$

$$|b_i^{\prime(n,k)}| \lesssim s^{-\frac{\gamma-\gamma_n}{2}-i}, \quad |\lambda| + |z| \ll 1. \tag{1-59}$$

The size of the excitation is

$$\frac{1}{\lambda^{2(2s_L-s_c)}\, s^{L+(1-\delta_0')}}$$

so $\chi\big(\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} T_i^{(n,k)}\big)_{z,1/\lambda}$ and $\nu > 0$ in (1-58) quantifies the fact that the remainder w is smaller than the excitation.

(v) *The bootstrap regime*. From (1-1) and (1-50), the evolution of the solution under the decomposition (1-53) and (1-54) has the form

$$\partial_t w_{\mathrm{ext}} = \Delta w_{\mathrm{ext}} + \Delta\chi_3 w + 2\nabla\chi_3.\nabla w + (1-\chi_3)w^p, \tag{1-60}$$

$$\begin{aligned}\partial_t w_{\mathrm{int}} = {} & -H_{z,\frac{1}{\lambda}} w_{\mathrm{int}} + \chi\psi + \mathrm{NL} \\ & + \chi\biggl(\biggl(\frac{b_1^{(1,\cdot)}}{\lambda^2}+\frac{z_t}{\lambda}\biggr).\nabla\Bigl(Q+\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)}\Bigr)\biggr)_{z,\frac{1}{\lambda}} \\ & + \chi\biggl(\biggl(\frac{b_1^{(0,1)}}{\lambda^2}+\frac{\lambda_t}{\lambda}\biggr)\Lambda\Bigl(Q+\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)}\Bigr)\biggr)_{z,\frac{1}{\lambda}} \\ & + \chi\biggl(\sum_{(n,k,i)\in\mathcal{I}}\biggl(-b_{i,t}^{(n,k)} - \frac{(2i-\alpha_n)b_1^{(0,1)}b_1^{(n,k)} + b_{i+1}^{(n,k)}}{\lambda^2}\biggr)T_i^{(n,k)}\biggr)_{z,\frac{1}{\lambda}},\end{aligned} \tag{1-61}$$

where $H_{z,\lambda} = -\Delta - pQ_{z,1/\lambda}^{p-1}$ and NL stands for the purely nonlinear term.

Modulation: The evolution of the parameters is computed using the orthogonality directions related to the decomposition, i.e., by taking the scalar product between (1-61) and $(T_i^{(n,k)})_{z,1/\lambda}$ for $0 \le n \le n_0$, $1 \le k \le k(n)$ and $0 \le i \le L_n$, yielding in renormalized time an estimate of the form[3]

$$\biggl|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\biggr| + \biggl|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\biggr| + \sum_{(n,k,i)\in\mathcal{I}} \bigl|b_{i,s}^{(n,k)}+(2i-\alpha_n)b_i^{(n,k)}b_1^{(0,1)}+b_{i+1}^{(n,k)}\bigr| \lesssim \sqrt{\mathcal{E}_{2s_L}}+s^{-L-3}. \tag{1-62}$$

These estimates hold because the error produced by the approximate dynamics is very small (s^{-L-3}) on compact sets, and on the other hand the remainder ε is also very small on compact sets and located far away from the origin by (1-58) and the coercivity (1-45).

Lyapunov monotonicity for the remainder: From the evolution equations (1-60) and (1-61), in the bootstrap regime (1-58) one performs energy estimates of the form

$$\frac{d}{dt}\biggl(\frac{1}{\lambda^{2(\sigma-s_c)}}\mathcal{E}_\sigma + \|w_{\mathrm{ext}}\|_{H^\sigma(\Omega)}\biggr) \lesssim \frac{1}{\lambda^2 s^{1+\kappa'}} + \frac{1}{\lambda^{(\sigma-s_c)}}\sqrt{\mathcal{E}_\sigma}\|\nabla^\sigma\psi\|_{L^2}, \tag{1-63}$$

$$\frac{d}{dt}\biggl(\frac{1}{\lambda^{2(2s_L-s_c)}}\mathcal{E}_{2s_L} + \|w_{\mathrm{ext}}\|_{H^{2s_L}(\Omega)}\biggr) \lesssim \frac{1}{\lambda^{2(2s_L-s_c)+2} s^{L+2-\delta_0+\nu+\kappa}} + \frac{1}{\lambda^{2s_L-s_c}}\sqrt{\mathcal{E}_{2s_L}}\|H_{z,\frac{1}{\lambda}}^{s_L}\psi\|_{L^2}, \tag{1-64}$$

where $\kappa > 0$ represents a gain. The key properties yielding these estimates are the following. The control of a slightly supercritical norm (1-57) and another high regularity norm allows us to control precisely the energy transfer between low and high frequencies and to control the nonlinear term. The dissipation in (1-60) and (1-61) (for the second equation it is a consequence of the coercivity (1-45)) erases the border terms and smaller-order local interactions. Finally, the approximate blow-up profile is in fact a refinement of (1-49), where the error in the approximate dynamics is well localized in the self-similar

[3] With the convention $b_{L_n+1}^{(n,k)} = 0$.

zone $|x-z|\sim\sqrt{T-t}$, by the addition of suitable corrections via inverting elliptic equations and by precise cuts.

(vi) *Existence via a topological argument.* In the bootstrap regime close to the approximate blow-up profile described by (1-58) and (1-59), one has precise bounds for the error term ψ. Reintegrating the energy estimates (1-63) and (1-64) then leads to the bounds

$$\mathcal{E}_\sigma+\|w_{\mathrm{ext}}\|^2_{H^\sigma(\Omega)}\ll 1,\quad \mathcal{E}_{2s_L}+\|w_{\mathrm{ext}}\|_{H^{s_L}(\Omega)}\ll\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+(1-\delta_0)+\nu}},$$

which are an improvement of (1-58). Therefore, a solution ceases to be in the bootstrap regime if and only if the bound (1-59) describing the proximity of the parameters with respect to the special blow-up parameters $(\bar b,\bar\lambda,\bar z)$ is violated. From (iv), the parameters admit $(\bar\lambda,\bar z,\bar b)$ as a hyperbolic orbit with m directions of instability and $\#\mathcal{I}-m$ of instability. From the modulation equations (1-62), the remainder w perturbs these dynamics only at lower order. Therefore, an application of the Brouwer fixed point theorem yields the persistence of an orbit similar to $(\bar\lambda,\bar z,\bar b)$ for the full nonlinear equation, i.e., with a perturbation along the parameters that stays small for all time. This gives the existence of a true solution of (1-1) that stays close to the approximate blow-up profile for all renormalized times, implying blow up by concentration of Q with a precise asymptotic.

The paper is organized as follows. In Section 2 we recall the known properties of the ground state in Lemma 2.1 and describe the kernel of the linearized operator H in Lemma 2.3. This provides a formula to invert elliptic equations of the form $Hu=f$, stated in Definition 2.6, and allows us to describe the generalized kernel of H in Lemma 2.10. The blow-up profile is built on functions depending polynomially on some parameters and with explicit asymptotic at infinity, and we introduce the concept of homogeneous functions in Definition 2.14 and Lemma 2.15 to track this information easily. With these tools, in Section 3 we construct a first approximate blow-up profile for which the error is localized at infinity in Proposition 3.1 and we cut it in the self-similar zone in Proposition 3.3. The evolution of the parameters describing the approximate blow-up profile is an explicit dynamical system with special solutions given in Lemma 3.4 for which the linear stability is investigated in Lemma 3.5. In Section 4 we define a bootstrap regime for solutions of the full equation close to the approximate blow-up profile. We give a suitable decomposition for such solutions, using orthogonality conditions that are provided by Definition 4.1 and Lemma 4.2, in Lemma 4.3. They must satisfy in addition some size assumption, and all the conditions describing the bootstrap regime are given in Definition 4.4. The main result of the paper is Proposition 4.6, stating the existence of a solution staying for all times in the bootstrap regime, whose proof is relegated to the next section. With this result we end the proof of Theorem 1.1 in Section 4B. To do this, the modulation equations are computed in Lemma 4.7, yielding that solutions staying in the bootstrap regime must concentrate in Lemma 4.8 with an explicit asymptotic for Sobolev norm in Lemma 4.9. In Section 5 we prove the main proposition, Proposition 4.6. For solutions in the bootstrap regime, an improved modulation equation is established in Lemma 5.1, and Lyapunov-type monotonicity formulas are established in Propositions 5.3 and 5.5 for the low regularity Sobolev norms of the remainder, and in Propositions 5.6 and 5.8 for the high regularity norms. With this analysis one can characterize the conditions under which a solution leaves the bootstrap regime in Lemma 5.9, and with a topological argument provided in Lemma 5.10, one ends the proof of Proposition 4.6.

The appendix is organized as follows. In Appendix A we give the proof of Lemma 2.3, describing the kernel of H. In Appendix B we recall some Hardy and Rellich-type estimates, among which the most useful is given in Lemma B.3. In Appendix C we investigate the coercivity of H in Lemmas C.2 and C.3. In Appendix D we prove some bounds for solutions in the bootstrap regime. In Appendix E we give the proof of the decomposition Lemma 4.3.

2. Preliminaries on Q and H

We first summarize the content and ideas of this section. The instabilities near Q underlying the blow up that we study result from the excitement of modes in the generalized kernel of H. We first describe this set. Since H is radial, we use a decomposition into spherical harmonics, restricted to spherical harmonics of degree n, see (1-37), it becomes the operator $H^{(n)}$ on radial functions defined by (1-36). Using ODE techniques, the kernel is described in Lemma 2.3 and the inversion of $H^{(n)}$ is given by Definition 2.6 and Lemma 2.13. By inverting successively the elements in the kernel of $H^{(n)}$, one obtains the generators of the generalized kernel $\bigcup_j \operatorname{Ker}((H^{(n)})^j)$ of this operator in Lemma 2.10.

To track the asymptotic behavior and the dependence in some parameters of various profiles during the construction of the approximate blow-up profile in the next section, we introduce the framework of "homogeneous" functions in Definition 2.14 and Lemma 2.15.

2A. ***Properties of the ground state and the potential.*** Any positive smooth radially symmetric solution to

$$-\Delta\phi - \phi^p = 0$$

is a dilate of a given normalized ground state profile Q:

$$\phi = Q_\lambda, \quad \lambda > 0, \quad \begin{cases} -\Delta Q - Q^p = 0, \\ Q(0) = 1. \end{cases}$$

See [Li 1992] and references therein. The following lemma describes the asymptotic behavior of Q. We refer to [Ding and Ni 1985] for earlier work.

Lemma 2.1 (asymptotics of the ground state [Li 1992, Lemma 4.3; Karageorgis and Strauss 2007, Lemma 5.4]). *Let $p > p_{JL}$ (defined in (1-6)). We recall that $g > 0$, c_∞ and γ are defined in (1-9) and (1-21). One has the asymptotics*

$$Q = \frac{c_\infty}{r^{\frac{2}{p-1}}} + \frac{a_1}{r^\gamma} + O\left(\frac{1}{r^{\gamma+g}}\right) \quad \text{as } r \to +\infty,\ a_1 \neq 0, \tag{2-1}$$

$$V = -\frac{pc_\infty^{p-1}}{r^2} + O\left(\frac{1}{r^{2+\alpha}}\right) \quad \text{as } r \to +\infty, \tag{2-2}$$

$$\frac{d}{d\lambda}[(Q_\lambda)^{p-1}]_{|\lambda=1} = O\left(\frac{1}{r^{2+\alpha}}\right) \quad \text{as } r \to +\infty, \tag{2-3}$$

and these identities propagate to the derivatives. There exists $\delta(p) > 0$ *such that the following pointwise bounds hold for all* $y \in \mathbb{R}^d$:

$$0 < Q(y) < \frac{c_\infty}{|y|^{\frac{2}{p-1}}}, \tag{2-4}$$

$$-\frac{(d-2)^2}{4|y|^2} + \frac{\delta(p)}{|y|^2} \le V(y) < 0. \tag{2-5}$$

Remark 2.2. The standard Hardy inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2 \ge \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{u^2}{|y|^2}\, dy$$

and (2-4) then imply the positivity of H on $\dot{H}^1(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} uHu\, dy \ge \int_{\mathbb{R}^d} \frac{\delta(p)u^2}{|y|^2}\, dy. \tag{2-6}$$

It is worth mentioning that the aforementioned expansion (2-1) is false for $p \le p_{JL}$. This asymptotic at infinity of Q is decisive for type II blow up via perturbation of it, as from [Matano and Merle 2004; Mizoguchi 2011b] it cannot occur for $\frac{d+2}{d-2} < p < p_{JL}$.

2B. *Kernel of H.*

Lemma 2.3 (kernel of $H^{(n)}$). *We recall that the numbers* $(\gamma_n)_{n\in\mathbb{N}}$ *and* g *are defined in* (1-18). *Let* $n \in \mathbb{N}$. *There exist* $T_0^{(n)}, \Gamma^{(n)} : (0,+\infty) \to \mathbb{R}$ *two smooth functions such that if* $f : (0,+\infty) \to \mathbb{R}$ *is smooth and satisfies* $H^{(n)} f = 0$, *then* $f \in \operatorname{Span}(T_0^{(n)}, \Gamma^{(n)})$. *They enjoy the asymptotics*

$$\begin{cases} T_0^{(n)}(r) \underset{r\to 0}{=} \sum_{j=0}^{l} c_j^{(n)} r^{n+2j} + O(r^{n+2+2l}) \quad \forall l \in \mathbb{N},\ c_0^{(n)} \neq 0, \\ T_0^{(n)} \underset{r\to+\infty}{\sim} C_n r^{-\gamma_n} + O(r^{-\gamma_n - g}), \quad C_n \neq 0, \\ \Gamma^{(n)} \underset{r\to 0}{\sim} \frac{c_n'}{r^{d-2+n}} \quad \textit{and} \quad \Gamma^{(n)} \underset{r\to+\infty}{\sim} \tilde{c}_n' r^{-\gamma_n}, \quad c_n', \tilde{c}_n' \neq 0. \end{cases} \tag{2-7}$$

Moreover, $T_0^{(n)}$ *is strictly positive, and for* $1 \le k \le k(n)$ *the functions* $y \mapsto T_0^{(n)}(|y|) Y_{n,k}(|y|/y)$ *are smooth on* $\mathbb{R}^d$. *The first two regular and strictly positive zeros are explicit*:

$$T_0^{(0)} = \frac{1}{C_0} \Lambda Q \quad \textit{and} \quad T_0^{(1)} = -\frac{1}{C_1} \partial_y Q, \tag{2-8}$$

where C_0 *and* C_1 *are the renormalized constants defined by* (1-35).

Proof. The proof of this lemma is done in Appendix A. □

Remark 2.4. The renormalized constants in (2-8) are here to produce the identities $T_0^{(0)} Y^{(0,0)} = \Lambda Q$ and $T_0^{(1)} Y^{(1,k)} = \partial_{x_k} Q$ by (1-35). For each $n \in \mathbb{N}$, only one zero, $T_0^{(n)}$, is regular at the origin. We insist on the fact that $-\gamma_n > 0$ is a positive number[4] for n large by (1-20), making these profiles grow as $r \to +\infty$.

2C. *Inversion of $H^{(n)}$*. We start by a useful factorization formula for $H^{(n)}$. Let $n \in \mathbb{N}$ and $W^{(n)}$ denote the potential

$$W^{(n)} := \partial_r(\log(T_0^{(n)})), \tag{2-9}$$

where $T_0^{(n)}$ is defined in (2-7) and define the first-order operators on radial functions

$$A^{(n)} : u \mapsto -\partial_r u + W^{(n)} u, \quad A^{(n)*} : u \mapsto \frac{1}{r^{d-1}} \partial_r (r^{d-1} u) + W^{(n)} u. \tag{2-10}$$

Lemma 2.5 (factorization of $H^{(n)}$). *The factorization*

$$H^{(n)} = A^{(n)*} A^{(n)} \tag{2-11}$$

holds. Moreover one has the adjunction formula for smooth functions with enough decay

$$\int_0^{+\infty} (A^{(n)} u) v r^{d-1}\, dr = \int_0^{+\infty} u (A^{(n)*} v) r^{d-1}\, dr.$$

Proof. As $T_0^{(n)} > 0$ by (2-7), $W^{(n)}$ is well defined. This factorization is a standard property of Schrödinger operators with a nonvanishing zero. We start by computing

$$A^{(n)*} A^{(n)} u = -\partial_{rr} u - \frac{d-1}{r} \partial_r u + \left(\frac{d-1}{r} W^{(n)} + \partial_r W^{(n)} + (W^{(n)})^2 \right) u.$$

As $W^{(n)} = \partial_r T_0^{(n)} / T_0^{(n)}$, the potential that appears is nothing but

$$\begin{aligned} \frac{d-1}{r} W^{(n)} + \partial_r W^{(n)} + (W^{(n)})^2 &= \frac{\partial_{rr} T_0^{(n)} + \frac{d-1}{r} T_0^{(n)}}{T_0^{(n)}} = \frac{-H^{(n)} T_0^{(n)} + \left(\frac{n(d+n-2)}{r^2} + V\right) T_0^{(n)}}{T_0^{(n)}} \\ &= \frac{n(d+n-2)}{r^2} + V \end{aligned}$$

as $H^{(n)} T_0^{(n)} = 0$, which proves the factorization formula (2-11). The adjunction formula comes from a direct computation using integration by parts. □

From the asymptotic behavior (2-7) of $T_0^{(n)}$ at the origin and at infinity, we deduce the asymptotic behavior of $W^{(n)}$:

$$W^{(n)} = \begin{cases} \frac{n}{r} + O(1) & \text{as } r \to 0, \\ \frac{-\gamma_n}{r} + O\left(\frac{1}{r^{1+g+j}}\right) & \text{as } r \to +\infty, \end{cases} \tag{2-12}$$

which propagates to the derivatives. Using the factorization (2-11), to define the inverse of $H^{(n)}$ we proceed in two steps: first we invert $A^{(n)*}$, then $A^{(n)}$.

[4]This notation seems unnatural but matches the standard notation in the literature.

Definition 2.6 (inverse of $H^{(n)}$). Let $f:(0,+\infty)\to\mathbb{R}$ be smooth with $f(r)=O(r^n)$ as $r\to 0$. We define[5] the inverses $(A^{(n)*})^{-1}f$ and $(H^{(n)})^{-1}f$ by

$$(A^{(n)*})^{-1}f(r)=\frac{1}{r^{d-1}T_0^{(n)}}\int_0^r fT_0^{(n)}s^{d-1}\,ds, \tag{2-13}$$

$$(H^{(n)})^{-1}f(r)=\begin{cases} T_0^{(n)}\int_r^{+\infty}(A^{(n)*})^{-1}f/T_0^{(n)}\,ds & \text{if } (A^{(n)*})^{-1}f/T_0^{(n)} \text{ is integrable on } (0,+\infty),\\ -T_0^{(n)}\int_0^r(A^{(n)*})^{-1}f/T_0^{(n)}\,ds & \text{if } (A^{(n)*})^{-1}f/T_0^{(n)} \text{ is not integrable on } (0,+\infty).\end{cases} \tag{2-14}$$

Direct computations give indeed $H^{(n)}\circ(H^{(n)})^{-1}=A^{(n)*}\circ(A^{(n)*})^{-1}=\mathrm{Id}$, and $A^{(n)}\circ(H^{(n)})^{-1}=(A^{(n)*})^{-1}$. As we do not have uniqueness for the equation $Hu=f$, one may wonder if this definition is the "right" one. The answer is yes because this inverse has the good asymptotic behavior; namely, if $f\approx r^q$ as $r\to+\infty$, one would expect $u\approx r^{q+2}$ as $r\to+\infty$, which will be proven in Lemma 2.9. To keep track of the asymptotic behaviors at the origin and at infinity, we now introduce the notion of admissible functions.

Definition 2.7 (simple admissible functions). Let n be an integer, q be a real number and $f:(0,+\infty)\to\mathbb{R}$ be smooth. We say that f is a simple admissible function of degree (n,q) if it enjoys the asymptotic behaviors

$$f=\sum_{j=0}^{l}c_jr^{n+2j}+O(r^{n+2l+2})\quad \forall l\in\mathbb{N} \tag{2-15}$$

at the origin for a sequence of numbers $(c_l)_{l\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, and at infinity

$$f=O(r^q)\quad \text{as } r\to+\infty, \tag{2-16}$$

and if the two asymptotics propagate to the derivatives of f.

Remark 2.8. Let $f:(0,+\infty)$ be smooth. We define the sequence of n-adapted derivatives of f by induction:

$$f_{[n,0]}:=f\qquad\text{and}\qquad\text{for } j\in\mathbb{N},\quad f_{[n,j+1]}:=\begin{cases}A^{(n)}f_{[n,j]} & \text{for } j \text{ even},\\ A^{(n)*}f_{[n,j]} & \text{for } j \text{ odd}.\end{cases} \tag{2-17}$$

From the definition (2-10) of $A^{(n)}$ and $A^{(n)*}$, and the asymptotic behavior (2-12) of the potential $W^{(n)}$, one notices that the condition (2-16) on the asymptotic at infinity for a simple admissible function of degree (n,q) and its derivatives is equivalent to the following condition for all $j\in\mathbb{N}$:

$$f_{[n,j]}=O(r^{q-j})\quad\text{as } r\to+\infty, \tag{2-18}$$

where the adapted derivatives $(f_{[n,j]})_{j\in\mathbb{N}}$ are defined by (2-17). We will use this fact many times in the rest of this subsection, as it is more adapted to our problem.

The operators $H^{(n)}$ and $(H^{(n)})^{-1}$ leave this class of functions invariant, and the asymptotic at infinity is increased by -2 and 2 under some conditions (that will always hold in the sequel) on the coefficient q to avoid logarithmic corrections.

[5] We know u is well defined because from the decay of f at the origin one deduces $(A^{(n)*})^{-1}f=O(r^{n+1})$ as $y\to 0$ and so u'/T_0^n is integrable at the origin from the asymptotic behavior (2-7).

Lemma 2.9 (actions of $H^{(n)}$ and $(H^{(n)})^{-1}$ on simple admissible functions). *Let $n \in \mathbb{N}$ and f be a simple admissible function of degree (n, q) in the sense of Definition 2.7, with $q > \gamma_n - d$ and $-\gamma_n - 2 - q \notin 2\mathbb{N}$. Then for all integer $i \in \mathbb{N}$:*

(i) *$(H^{(n)})^i f$ is simple admissible of degree $(n, q - 2i)$.*

(ii) *$(H^{(n)})^{-i} f$ is simple admissible of degree $(n, q + 2i)$.*

Proof. **Step 1:** action of $H^{(n)}$. For all integers i and j one has $((H^{(n)})^i f)_{[n,j]} = f_{[n,j+2i]}$ by (2-17) and (2-11). Using the equivalent formulation (2-18), the asymptotic at infinity (2-16) for $H^i f$ is then a straightforward consequence of the asymptotic at infinity (2-16) for f. Close to the origin, one notices that $H^{(n)} = -\Delta^{(n)} + V$ with $\Delta^{(n)} = \partial_{rr} + \frac{d-1}{r}\partial_r - n(d + n - 2)$. If f satisfies (2-15) at the origin, then so does $(\Delta^{(n)})^i f$ by a direction computation. As V is smooth at the origin, $(H^{(n)})^i f$ also satisfies (2-15). Hence $(H^{(n)})^i f$ is a simple admissible function of degree $q - 2i$.

Step 2: action of $(H^{(n)})^{-1}$. We will prove the property for $(H^{(n)})^{-1} f$, and the general result will follow by induction on i. Let u denote the inverse by $H^{(n)}$, that is, $u = (H^{(n)})^{-1} f$.

Asymptotic at infinity. We will prove the equivalent formulation (2-18) of the asymptotic at infinity (2-16). From (2-17), (2-13), (2-14) and (2-11), $u_{[n,j]} = f_{[n,j-2]}$ for $j \geq 2$ so the asymptotic behavior (2-18) at infinity for the n-adapted derivatives of u are true for $j \geq 2$. Therefore it remains to prove them for $j = 0, 1$.

Case $j = 1$. From the definition of the inverse (2-14) and of the adapted derivatives (2-17), one has

$$u_{[n,1]} = \frac{1}{r^{d-1} T_0^{(n)}} \int_0^r f T_0^{(n)} s^{d-1} \, ds.$$

From the asymptotic behaviors (2-16) and (2-7) for f and $T_0^{(n)}$ at infinity and the condition $q > \gamma_n - d$, the integral diverges and we get

$$u_{[n,1]}(r) = O(r^{q+1}) \quad \text{as } r \to +\infty, \tag{2-19}$$

which is the desired asymptotic (2-18) for $u_{[n,1]}$.

Case $j = 0$. Suppose $(A^{(n)*})^{-1} f / T_0^{(n)} = u_{[n,1]} / T_0^{(n)}$ is integrable on $(0, +\infty)$. In that case

$$u = T_0^{(n)} \int_r^{+\infty} \frac{u_{[n,1]}}{T_0^{(n)}} \, ds.$$

If $q > -\gamma_n - 2$, then by the integrability of the integrand and (2-7), we get the desired asymptotic $u_{[n,0]} = u = O(r^{-\gamma_n}) = O(r^{q+2})$. If $q < -\gamma_n - 2$ then from (2-19) we have $u_{[n,1]} / T_0^{(n)} = O(r^{q+1+\gamma_n})$ and then $\int_r^{+\infty} u_{[n,1]} / T_0^{(n)} \, ds = O(r^{q+2+\gamma_n})$, from which we get the desired asymptotic $u = O(r^{q+2})$. Now suppose $u_{[n,1]} / T_0^{(n)}$ is not integrable. Then we must have $q > -\gamma_n + 2$ by (2-19), and u is given by

$$u = -T_0^{(n)} \int_0^r \frac{u_{[n,1]}}{T_0^{(n)}} \, ds,$$

and the integral has asymptotic $O(r^{q+2+\gamma_n})$. We hence get $u = O(r^{q+2})$ at infinity using (2-7).

Conclusion. In both cases, we have proven that the asymptotic at infinity (2-18) holds for u.

Asymptotic at the origin. We have

$$u=-T_0^{(n)}\int_0^r \frac{u_{[n,1]}}{T_0^{(n)}}\,ds+aT_0^{(n)},$$

where $a=0$ if $u_{[n,1]}/T_0^{(n)}$ is not integrable, and $a=\int_0^{+\infty}u_{[n,1]}/T_0^{(n)}\,ds$ if it is. By (2-7), $T_0^{(n)}$ satisfies (2-15). Thus it remains to prove (2-15) for $-T_0^{(n)}\int_0^r u_{[n,1]}/T_0^{(n)}\,ds$. We proceed in two steps. First, from (2-15) for f we obtain that for all integers j, p,

$$u_{[n,1]}=\frac{1}{r^{d-1}T_0^{(n)}}\int_0^r fT_0^{(n)}s^{d-1}\,ds=\sum_{j=0}^{l}\tilde{c}_j r^{n+1+2j}+\widetilde{R}_l,$$

where $\partial_r^k\widetilde{R}_l=O(r^{\max(n+2l+3-k,0)})$ as $r\to 0$ for some coefficients $\tilde{c}_j$ depending on the c_j and the asymptotic at the origin of T_0^n. It then follows that

$$-T_0^{(n)}\int_0^r\frac{u_{[n,1]}}{T_0^{(n)}}\,ds=\sum_{j=0}^{l}\hat{c}_j r^{n+2+2j}+\widehat{R}_l,\quad\text{where } \partial_r^k\widehat{R}_l\underset{r\to 0}{=}O(r^{\max(n+2l+4-k,0)}),$$

for some coefficients $\hat{c}_l$. This implies that u satisfies (2-15) at the origin. □

We can now invert the elements in the kernel of $H^{(n)}$ and construct the generalized kernel of this operator.

Lemma 2.10 (generators of the generalized kernel of $H^{(n)}$). *Let $n\in\mathbb{N}$, γ_n, g', $(H^{(n)})^{-1}$ and $T_0^{(n)}$ be defined by* (1-18), (1-21), *Definition 2.6 and Lemma 2.3. We denote by $(T_i^{(n)})_{i\in\mathbb{N}}$ the sequence of profiles given by*

$$T_{i+1}^{(n)}:=-(H^{(n)})^{-1}T_i^{(n)},\quad i\in\mathbb{N}. \tag{2-20}$$

Let $(\Theta_i^{(n)})_{i\in\mathbb{N}}$ be the associated sequence of profiles defined by

$$\Theta_i^{(n)}:=\Lambda T_i^{(n)}-\left(2i+\frac{2}{p-1}-\gamma_n\right)T_i^{(n)},\quad i\in\mathbb{N}. \tag{2-21}$$

Then for each $i\in\mathbb{N}$,

$$T_i^{(n)} \textit{ is simple admissible of degree } (n,-\gamma_n+2i), \tag{2-22}$$

$$\Theta_i^{(n)} \textit{ is simple admissible of degree } (n,-\gamma_n+2i-g'), \tag{2-23}$$

where simple admissibility is defined in Definition 2.7.

Proof. **Step 1:** admissibility of $T_i^{(n)}$. From the asymptotic behaviors (2-7) at infinity and at the origin, $T_0^{(n)}$ is simple admissible of degree $(n,-\gamma_n)$ in the sense of Definition 2.7. Additionally, $-\gamma_n>\gamma_n-d$ since $-2\gamma_n+d\ge -2\gamma_0+d=2+\sqrt{\triangle}>0$ by (1-9) and since $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing by (1-18). One has also $-\gamma_n-2-(-\gamma_n)=-2\notin 2\mathbb{N}$. Therefore one can apply Lemma 2.9: for all $i\in\mathbb{N}$, $T_i^{(n)}$ given by (2-20) is an admissible profile of degree $(n,-\gamma_n+2i)$.

Step 2: admissibility of $\Theta_i^{(n)}$. We start by computing the following commutator relations using (1-36), (2-9) and (2-10):

$$\begin{aligned} A^{(n)}\Lambda &= \Lambda A^{(n)} + A^{(n)} - (W^{(n)} + y\partial_y W^{(n)}), \\ H^{(n)}\Lambda &= \Lambda H^{(n)} + 2H^{(n)} - (2V + y.\nabla V). \end{aligned} \tag{2-24}$$

We now proceed by induction. From the previous equation, and the asymptotic behaviors (2-7), (2-2) and (2-12) of the functions $T_0^{(n)}$, V and $W^{(n)}$, we get that $\Theta_0^{(n)}$ is simple admissible of degree $(n, -\gamma_n - g')$. Now let $i \geq 1$ and suppose that the property in (2-23) is true for $i-1$. Using the previous formula and (2-21) we obtain

$$H^{(n)}\Theta_i^n = -\Theta_{i-1}^{(n)} - (2V + y.\nabla V)T_i^{(n)}.$$

The asymptotic at infinity (2-2) of V yields the decay $2V + y.\nabla V = (y^{-2-\alpha})$. As $T_i^{(n)}$ is simple admissible of degree $(n, 2i-\gamma_n)$ and from the induction hypothesis, we have that $H^{(n)}\Theta_i^{(n)}$ is simple admissible of degree $(n, 2i-2-\gamma_n-g')$ because $g' < \alpha$ by (1-21). One has $2i-2-\gamma_n-g' > \gamma_n - d$ because

$$2i - 2 - 2\gamma_n - g' + d \geq -2\gamma_0 - g' + d = 2 + \sqrt{\triangle} - g' > 0$$

as $0<g'<1$, $i \geq 1$, and $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing by (1-18) and (1-9). Similarly

$$-\gamma_n - 2 - (2i - 2 - \gamma_n - g') = -2i + g' \notin 2\mathbb{N}.$$

Therefore we can apply Lemma 2.9 and obtain that $(H^{(n)})^{-1}H^{(n)}\Theta_i^{(n)}$ is of degree $(n, 2i-\gamma_n-g')$. From Lemma 2.3 one has $(H^{(n)})^{-1}H^{(n)}\Theta_i^{(n)} = \Theta_i^{(n)} + aT_0^{(n)} + b\Gamma^{(n)}$, for two integration constants $a, b \in \mathbb{R}$. At the origin $\Gamma^{(n)}$ is singular by (2-7); hence $b = 0$. As $T_0^{(n)}$ is of degree $(n, -\gamma_n)$ with $-\gamma_n + 2i - g' > -\gamma_n$ (because $i \geq 1$), we get that $\Theta_i^{(n)}$ is of degree $(n, 2i-\gamma_n-g')$. □

2D. ***Inversion of H on nonradial functions.*** The definition of the inverse of $H^{(n)}$, Definition 2.6, naturally extends to give an inverse of H by separately inverting the components onto each spherical harmonic. There will be no problem when summing, as for the purpose of the present paper one can restrict to the following class of functions that are located on a finite number of spherical harmonics.

Definition 2.11 (admissible functions). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function, with decomposition $f(y) = \sum_{n,k} f^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)$, and q be a real number. We say that f is admissible of degree q if there is only a finite number of couples (n,k) such that $f^{(n,k)} \neq 0$, and that for every such couple, $f^{(n,k)}$ is a simple admissible function of degree (n,q) in the sense of Definition 2.7.

For $f = \sum_{n,k} f^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)$ an admissible function, we define its inverse by H by

$$(H^{(-1)}f)(y) := \sum_{n,k} [(H^{(n)})^{-1}f^{(n,k)}(|y|)]Y^{(n,k)}\left(\frac{y}{|y|}\right) \tag{2-25}$$

(the sum being finite), where $(H^{(n)})^{-1}$ is defined by Definition 2.6. For n, k and i three integers with $1 \leq k \leq k(n)$, we define the profile $T_i^{(n,k)} : \mathbb{R}^d \to \mathbb{R}$ as

$$T_i^{(n,k)}(y) = T_i^{(n)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right), \tag{2-26}$$

where the radial function $T_i^{(n)}$ is defined by (2-20). From Lemma 2.10, $T_i^{(n,k)}$ is an admissible function of degree $(-\gamma_n+2i)$ in the sense of Definition 2.11. The class of admissible functions has some structural properties: it is stable under summation, multiplication and differentiation, and its elements are smooth with an explicit decay at infinity. This is the subject of the next lemma.

Lemma 2.12 (properties of admissible functions). *Let f and g be two admissible functions of degrees q and q' in the sense of Definition 2.11, and $\mu \in \mathbb{N}^d$. Then*:

(i) *f is smooth.*

(ii) *fg is admissible of degree $q+q'$.*

(iii) *$\partial^\mu f$ is admissible of degree $q-|\mu|$.*

(iv) *There exists a constant $C(f,\mu)$ such that for all y with $|y|\geq 1$,*

$$|\partial^\mu f(y)| \leq C(f,\mu)|y|^{q-|\mu|}.$$

Proof. From Definition 2.11, $f=\sum_{n,k} f^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)$ and $g=\sum_{n,k} g^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)$ and both sums involve finitely many nonzero terms. Therefore, without loss of generality, we will assume that f and g are each located on only one spherical harmonic: $f=f^{(n,k)}Y^{(n,k)}$ and $g=g^{(n',k')}Y^{(n',k')}$, for $f^{(n,k)}$ and $g^{(n',k')}$ simple admissible of degrees (n,q) and (n',q') in the sense of Definition 2.7. The general result will follow by a finite summation.

(i) Now $y\mapsto f^{(n,k)}(|y|)$ is smooth outside the origin since f is smooth, and $y\mapsto Y^{(n,k)}(y/|y|)$ is also smooth outside the origin; hence f is smooth outside the origin. The Laplacian on spherical harmonics is

$$(-\Delta)^i f = (-\Delta)^i \left(f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)\right) = ((-\Delta^{(n)})^i f^{(n,k)})(|y|)Y^{(n,k)},$$

where $-\Delta^{(n)} = -\partial_{rr} - \frac{d-1}{r}\partial r + n(d+n-2)$. From the expansion of $f^{(n,k)}$ in (2-15), $(-\Delta^{(n)})^i f^{(n,k)}$ is bounded at the origin for each $i\in\mathbb{N}$. Therefore $(-\Delta)^i f$ is bounded at the origin for each i and f is smooth at the origin by elliptic regularity.

(ii) We treat the case where $n+n'$ is even, and the case $n+n'$ odd can be treated with exactly the same arguments. As the product of the two spherical harmonics $Y^{(n,k)}Y^{(n',k')}$ decomposes onto spherical harmonics of degree less than $n+n'$ with the same parity as $n+n'$, the product fg can be written as

$$fg = \sum_{\substack{0\leq \tilde n\leq n+n' \\ \tilde n \text{ even},\, 1\leq \tilde k\leq k(\tilde n)}} a_{n,k,n',k',\tilde n,\tilde k} f^{(n,k)} g^{(n',k')} Y^{(\tilde n,\tilde k)}$$

with $a_{n,k,n',k',\tilde n,\tilde k}$ some fixed coefficients. Now fix $\tilde n$ and $\tilde k$ in the sum; one has $n+n'=\tilde n+2i$ for some $i\in\mathbb{N}$. Using the Leibniz rule, as $\partial_r^j f^{(n,k)} = O(r^{q-j})$ and $\partial_r^j g^{(n,k)} = O(r^{q'-j})$ at infinity, we get that $\partial_r^j(f^{(n,k)}g^{(n',k')}) = O(r^{q+q'-j})$ as $y\to+\infty$, which proves that $f^{(n,k)}g^{(n',k')}$ satisfies the asymptotic at infinity (2-16) of a simple admissible function of degree $(\tilde n, q+q')$. Close to the origin, the two expansions (2-15) for $f^{(n,k)}$ and $g^{(n',k')}$, starting at r^n and $r^{n'}$ respectively, imply the same expansion (2-15) starting at $y^{n+n'}$ for the product $f^{(n,k)}g^{(n',k')}$. As $n+n'=\tilde n+2i$, we know $f^{(n,k)}g^{(n,k)}$ satisfies the expansion at the origin (2-15) of a simple admissible function of degree $(\tilde{n}, q+q')$. Therefore $f^{(n,k)}g^{(n,k)}$ is simple admissible of degree $(\tilde{n}, q+q')$ and thus fg is simple admissible of degree $q+q'$.

(iii) We treat the case where n is even, and the case n odd can be treated with exactly the same reasoning. Let $1 \le i \le d$; we just have to prove that $\partial_{y_i} f$ is admissible of degree $q-1$ and the result for higher-order derivatives will follow by induction. We recall that $Y^{(n,k)}$ is the restriction of a homogeneous harmonic polynomial of degree n to the sphere. We will still denote by $Y^{(n,k)}(y)$ this polynomial extended to the whole space $\mathbb{R}^d$ and they are related by $Y^{(n,k)}(y) = |y|^n Y^{(n,k)}(y/|y|)$. This homogeneity implies $y.\nabla(Y^{(n,k)})(y) = nY^{(n,k)}(y)$ and leads to the identity

$$\partial_{y_i}\left[f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)\right] = \left(\partial_r f^{(n,k)}(|y|) - n\frac{f(|y|)}{|y|}\right)\frac{y_i}{|y|}Y^{(n,k)}\left(\frac{y}{|y|}\right) + \frac{f(|y|)}{|y|}\partial_{y_i}Y^{(n,k)}\left(\frac{y}{|y|}\right). \quad (2\text{-}27)$$

One has now to prove that the two terms on the right-hand side are admissible of degree $q-1$. We only show it for the last term, the proof being the same for the first one. As $\partial_{y_i}Y^{(n,k)}(y/|y|)$ is a homogeneous polynomial of degree $n-1$ restricted to the sphere, it can be written as a finite sum of spherical harmonics of odd degrees (because n is even) less than $n-1$ and this gives

$$\frac{f}{|y|}\partial_{y_i}Y^{(n,k)}\left(\frac{y}{|y|}\right) = \sum_{\substack{1\le n'\le n-1 \\ n' \text{ odd},\, 1\le k\le k(n')}} a_{i,n,k,n',k'}\frac{f}{|y|}Y^{(n',k')}\left(\frac{y}{|y|}\right)$$

for some coefficients $a_{i,n,k,n',k'}$. Now fix n', k' in the sum. At infinity $a_{i,n,k,n',k'}f(|y|)/|y|$ satisfies the asymptotic behavior (2-16) of a simple admissible function of degree $(n', q-1)$. Close to the origin, one has from (2-15), the fact that $n'+2j = n-1$ for some $j\in\mathbb{N}$, that for any $i\in\mathbb{N}$,

$$a_{i,n,k,n',k'}\frac{f(r)}{r} = \sum_{l=0}^{i}\tilde{c}_l r^{n-1+2l} + O(r^{n-1+2i+2}) = \sum_{l=0}^{i}\hat{c}_l r^{n'+2j+2l} + O(r^{n'+2j+2i+2}),$$

which is the asymptotic behavior (2-15) of a simple admissible function of degree $(n', q-1)$ close to the origin. Therefore, $a_{i,n,k,n',k'}f(r)/r$ is a simple admissible function of degree $(n', q-1)$. Thus $(f/|y|)\partial_{y_i}Y^{(n,k)}(y/|y|)$ is an admissible function of degree $(q-1)$. The same reasoning works for the first term on the right-hand side of (2-27), and therefore $\partial_{y_i}[f^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)]$ is admissible of degree $q-1$.

(iv) We just showed in the last step that $\partial^\mu f$ is admissible of degree $q-|\mu|$ for all $\mu\in\mathbb{N}^d$; we then only have to prove (iv) for the case $\mu=(0,\dots,0)$. This can be showed via the brute force bound for $|y|\ge 1$

$$|f(y)| = \left|f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)\right| \le \|Y^{(n,k)}\|_{L^\infty}|f^{(n,k)}(|y|)| \le C|y|^q$$

by (2-16) since f is a simple admissible function of degree (n,q). □

The next lemma extends Lemma 2.9 to admissible functions. We do not give a proof, as it is a direct consequence of the latter.

Lemma 2.13 (action of H on admissible functions). *Let f be an admissible function in the sense of Definition 2.11 written as $f(y)=\sum_{n,k} f^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)$, of degree q, with $q>\gamma_n-d$. Assume that for all $n\in\mathbb{N}$ such that there exists k, $1\le k\le k(n)$ with $f^{(n,k)}\neq 0$, we have q satisfies $-q-\gamma_n-2\notin 2\mathbb{N}$. Then for all integers $i\in\mathbb{N}$, recalling that $H^{-1}f$ is defined by* (2-25):

(i) *$H^i f$ is admissible of degree $q-2i$.*

(ii) *$H^{-i} f$ is admissible of degree $q+2i$.*

2E. *Homogeneous functions.* The approximate blow-up profile we will build in the following subsection will look like $Q+\sum b_i^{(n,k)}T_i^{(n,k)}$ for some coefficients $b_i^{(n,k)}$ ($T_i^{(n,k)}$ being defined in (2-26)). The nonlinearity in the semilinear heat equation (1-1) will then produce terms that will be products of the profiles $T_i^{(n,k)}$ and coefficients $b_i^{(n,k)}$. Such nonlinear terms are admissible functions multiplied by monomials of the coefficients $b_i^{(n,k)}$. The set of triples (n,k,i) for which we will make a perturbation along $T_i^{(n,k)}$ is $\mathcal{I}$, defined in (1-39). Hence the vector b representing the perturbation will be

$$b=(b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}=(b_1^{(0,1)},\dots,b_L^{(0,1)},b_1^{(1,1)},\dots,b_{L_1}^{(1,1)},\dots,b_0^{(n_0,k(n_0))},\dots,b_{L_{n_0}}^{(n_0,k(n_0))}). \tag{2-28}$$

We will then represent a monomial in the coefficients $b_i^{(n,k)}$ by a tuple of $\#\mathcal{I}$ integers

$$J=(J_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}=(J_1^{(0,1)},\dots,J_L^{(0,1)},J_1^{(1,1)},\dots,J_{L_1}^{(1,1)},\dots,J_0^{(n_0,k(n_0))},\dots,J_{L_{n_0}}^{(n_0,k(n_0))})$$

through the formula

$$b^J:=(b_1^{(0,1)})^{J_1^{(0,1)}}\times\cdots\times(b_{L_{n_0}}^{(n_0,k(n_0))})^{J_{L_{n_0}}^{(n_0,k(n_0))}}. \tag{2-29}$$

We associate three different lengths to J for the analysis. The first one, $|J|:=\sum J_i^{(n,k)}$, represents the number of parameters $b_i^{(n,k)}$ that are multiplied in the above formula, counted with multiplicity, i.e., the standard degree of b^J. In the analysis, the coefficients $b_i^{(nk)}$ will have the size $|b_i^{(n,k)}|\lesssim|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$. The second length,

$$|J|_2:=\sum_{n,k,i}\left(\frac{\gamma-\gamma_n}{2}+i\right)J_i^{(n,k)},$$

is tailor-made to produce the following identity if these latter bounds hold:

$$|b^J|\lesssim(b_1^{(0,1)})^{|J|_2};$$

i.e., $|J|_2$ encodes the "size" of the real number b^J. For the construction of the approximate blow-up profile, we will invert several times some elliptic equations, and the i-th inversion will be related to the third length

$$|J|_3:=\sum_{i=1}^{L} iJ_i^{(0,1)}+\sum_{\substack{1\le i\le L_1\\ 1\le k\le d}} iJ_i^{(1,k)}+\sum_{\substack{(n,k,i)\in\mathcal{I}\\ 2\le n}}(i+1)J_i^{(n,k)}.$$

To track information about the nonlinear terms generated by the semilinear heat equation (1-1) we eventually introduce the class of homogeneous functions.

Definition 2.14 (homogeneous functions). Let b denote a $\#\mathcal{I}$-tuple under the form (2-28), $m \in \mathbb{N}$ and $q \in \mathbb{R}$. We recall that $|J|_2$ and $|J|_3$ are defined by (1-41) (1-42) and b^J is given by (2-29). We say that a function $S : \mathbb{R}^{\mathcal{I}} \times \mathbb{R}^d \to \mathbb{R}$ is homogeneous of degree (m, q) if it can be written as a finite sum

$$S(b, y) = \sum_{J \in \mathcal{J}} b^J S_J(y),$$

$\#\mathcal{J} < +\infty$, where for each tuple $J \in \mathcal{J}$, one has that $|J|_3 = m$ and that the function S_J is admissible of degree $2|J|_2 + q$ in the sense of Definition 2.11.

As a direct consequence of Lemma 2.12, and so we do not write here the proof, we obtain the following properties for homogeneous functions.

Lemma 2.15 (calculus on homogeneous functions). *Let S and S' be two homogeneous functions of degrees (m, q) and (m', q') in the sense of Definition 2.14, and $\mu \in \mathbb{N}^d$. Then*:

(i) *$\partial^\mu S$ is homogeneous of degree $(m, q - |\mu|)$.*

(ii) *SS' is homogeneous of degree $(m + m', q + q')$.*

(iii) *If, writing $S = \sum_{J \in \mathcal{J}} b^J \sum_{n,k} S_J^{(n,k)} Y^{(n,k)}$, one has $2|J|_2 + q > \gamma_n - d$ and $-2|J|_2 - q - \gamma_n - 2 \notin 2\mathbb{N}$ for all n, J such that there exists k, $1 \le k \le k(n)$ with $S_J^{(n,k)} \neq 0$, then for all $i \in \mathbb{N}$, $H^{-i}(S)$ (given by (2-25)) is homogeneous of degree $(m, q + 2i)$.*

3. The approximate blow-up profile

3A. ***Construction.*** We first summarize the content and ideas of this section. We construct an approximate blow-up profile relying on a finite number of parameters close to the set of functions $(\tau_z(Q_\lambda))_{\lambda > 0,\, z \in \mathbb{R}^d}$. It is built on the generalized kernel of H, $\operatorname{Span}((T_i^{(n,k)})_{n,i \in \mathbb{N},\, 1 \le k \le k(n)})$ defined by (2-26), and can therefore be seen as a part of a center manifold. The profile is built on the whole space $\mathbb{R}^d$ for the moment and will be localized later.

In Proposition 3.1 we construct a first approximate blow-up profile. The procedure generates an error term ψ, and by inverting elliptic equations, i.e., adding the term $H^{-1}\psi$ to our approximate blow-up profile, one can always convert this error term into a new error term that is localized far away from the origin. We apply this procedure several times to produce an error term that is very small close to the origin. Then, in Proposition 3.3 we localize the approximate blow-up profile to eliminate the error terms that are far away from the origin. We will cut in the zone $|y| \approx B_1 = B_0^{1+\eta}$, where $\eta \ll 1$ is a very small parameter. In this zone, the perturbation in the approximate blow-up profile has the same size as ΛQ, being the reference function for scale change. It will correspond to the self-similar zone $|x| \sim \sqrt{T - t}$ for the true blow-up function, where T will be the blow-up time.

The blow-up profile is described by a finite number of parameters whose evolution is given by the explicit dynamical system (3-58). In Lemma 3.4 we show the existence of special solutions describing a type II blow up with explicit blow-up speed. The linear stability of these solutions is investigated in Lemma 3.5.

There is a natural renormalized flow linked to the invariances of the semilinear heat equations (1-1). For u a solution of (1-1), $\lambda : [0, T(u_0)) \to \mathbb{R}_+^*$ and $z : [0, T(u_0)) \to \mathbb{R}^d$ two C^1 functions, if one defines for $s_0 \in \mathbb{R}$ the renormalized time

$$s(t) := s_0 + \int_0^t \frac{1}{\lambda(t')^2}\, dt' \tag{3-1}$$

and the renormalized function

$$v(s, \cdot) := (\tau_{-z} u(t, \cdot))_\lambda,$$

then from a direct computation, v is a solution of the renormalized equation

$$\partial_s v - \frac{\lambda_s}{\lambda} \Lambda v - \frac{z_s}{\lambda}.\nabla v - F(v) = 0. \tag{3-2}$$

Our first approximate blow-up profile is adapted to this new flow and is a special perturbation of Q.

Proposition 3.1 (first approximate blow-up profile). *Let $L \in \mathbb{N}$, $L \gg 1$, and let $b = (b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ denote a $\#\mathcal{I}$-tuple of real numbers with $b_1^{(0,1)} > 0$. There exists a $\#\mathcal{I}$-dimensional manifold of C^∞ functions $(Q_b)_{b\in\mathbb{R}_+^*\times\mathbb{R}^{\#\mathcal{I}-1}}$ such that*

$$F(Q_b) = b_1^{(0,1)} \Lambda Q_b + b_1^{(1,\cdot)}.\nabla Q_b + \sum_{(n,k,i)\in\mathcal{I}} \bigl(-(2i - \alpha_n) b_1^{(0,1)} b_i^{(n,k)} + b_{i+1}^{(n,k)}\bigr) \frac{\partial Q_b}{\partial b_i^{(n,k)}} - \psi_b, \tag{3-3}$$

where $b_1^{(1,\cdot)}$ denotes the d-tuple of real numbers $(b_1^{(1,1)}, \dots, b_1^{(1,d)})$, where we used the convention $b_{L_n+1}^{(n,k)} = 0$, and where ψ_b is an error term. Let B_1 be defined by (1-38). *If the parameters satisfy the size conditions*[6] $b_1^{(0,1)} \ll 1$ *and* $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ *for all* $(n, k, i) \in \mathcal{I}$, *then* ψ_b *enjoys the following bounds*:

(i) Global[7] bounds. *For* $0 \le j \le s_L$,

$$\|H^j \psi_b\|^2_{L^2(|y|\le 2B_1)} \le C(L) (b_1^{(0,1)})^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}, \tag{3-4}$$

$$\|\nabla^j \psi_b\|^2_{L^2(|y|\le 2B_1)} \le C(L) (b_1^{(0,1)})^{2(\frac{j}{2}-m_0)+2(1-\delta_0)+g'-C(L)\eta}, \tag{3-5}$$

where $C(L)$ is a constant depending on L only.

(ii) Local bounds.

$$\forall j \ge 0,\ \forall B > 1, \quad \int_{|y|\le B} |\nabla^j \psi_b|^2\, dy \le C(j, L) B^{C(j,L)} (b_1^{(0,1)})^{2L+6}. \tag{3-6}$$

where $C(L, j)$ is a constant depending on L and j only.

[6] This means that under the bounds $|b_i^{(n,k)}| \le K |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ for some $K > 0$, there exists $b^*(K)$ such that the estimates that follow hold if $b_1^{(0,1)} \le b^*(K)$ with constants depending on K. In what follows, K will be fixed independently of the other important constants.

[7] The zone $y \le B_1$ is called global because in the next proposition we will cut the profile Q_b in the zone $|y| \sim B_1$.

The profile Q_b is of the form

$$Q_b := Q + \alpha_b, \quad \alpha_b := \sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} T_i^{(n,k)} + \sum_{i=2}^{L+2} S_i, \tag{3-7}$$

where $T_i^{(n,k)}$ is as in (2-26), *and the profiles S_i are homogeneous functions in the sense of Definition 2.14 with*

$$\deg(S_i) = (i, -\gamma - g') \tag{3-8}$$

and with the property that for all $2 \le j \le L+2$, we have $\partial S_j / \partial b_i^{(n,k)} = 0$ if $j \le i$ for $n = 0, 1$ and if $j \le i+1$ for $n \ge 2$.

Remark 3.2. The previous proposition is to be understood in the following way. We have a special function depending on some parameters b close to Q, that is to say, at scale 1 and with concentration point 0 for the moment. Equation (3-3) means that the force term (i.e., when applying F) generated by (NLH) makes it concentrate at speed $b_1^{(0,1)}$ and translate at speed $b_1^{(1,\cdot)}$, while the time evolution of the parameters is an explicit dynamical system given by the third term. These approximations involve an error for which we have some explicit bounds (3-4) and (3-6).

The size of this approximate profile is directly related to the size of the perturbation along $T_1^{(0,1)}$, the first term in the generalized kernel of H responsible for scale variation. Indeed we ask for $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$, and the size of the error is measured via $b_1^{(0,1)}$; see (3-4), (3-5) and (3-6). Therefore $b_1^{(0,1)}$ will be the universal order of magnitude in our problem.

Because of the shape of this approximate blow-up profile (3-7), when including the time evolution of the parameters in (3-3) we get

$$\partial_s(Q_b) - F(Q_b) + b_1^{(0,1)} \Lambda Q_b + b_1^{(1,\cdot)}.\nabla Q_b = \mathrm{Mod}(s) + \psi_b, \tag{3-9}$$

where[8]

$$\mathrm{Mod}(s) = \sum_{(n,k,i)\in\mathcal{I}} \left[b_{i,s}^{(n,k)} + (2i - \alpha_n) b_1^{(0,1)} b_i^{(n,k)} - b_{i+1}^{(n,k)}\right] \left[T_i^{(n,k)} + \sum_{j=i+1+\delta_{n\ge 2}}^{L+2} \frac{\partial S_j}{\partial b_i^{(n,k)}}\right]. \tag{3-10}$$

For all $2 \le j \le L+2$, as S_j is homogeneous of degree $(j, -\gamma - g')$ in the sense of Definition 2.14 from (3-8), and from the fact that $\partial S_j / \partial b_i^{(n,k)} = 0$ if $j \le i$ for $n = 0, 1$ and if $j \le i+1$ for $n \ge 2$, one has that for all j, n, k, i, we have $\partial S_j / \partial b_i^{(0,1)}$ is either 0 or is homogeneous of degree (a, b) with $a \ge 1$, meaning that it never contains nontrivial constant functions independent of the parameters b. Hence, if the bounds $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ hold, since $|b_1^{(0,1)}| \lesssim 1$ and $-\gamma_n \ge -\gamma$ from (1-18), one has in particular that on compact sets for any $2 \le j \le L+2$ and $(n,k,i) \in \mathcal{I}$,

$$\frac{\partial S_j}{\partial b_i^{(n,k)}} = O(|b_1^{(0,1)}|). \tag{3-11}$$

Proof of Proposition 3.1. **Step 1:** computation of ψ_b. We first find an appropriate reformulation for the error ψ_b given by (3-3) when Q_b has the form (3-7).

[8] Here $\delta_{n\ge 2} = 1$ if $n \ge 2$, and is zero otherwise.

Rewriting of $F(Q_b)$ in (3-3). We start by computing

$$
\begin{aligned}
-F(Q_b) &= H(\alpha_b) - (f(Q_b) - f(Q) - \alpha_b f'(Q)) \\
&= \sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} H T_i^{(n,k)} + \sum_{i=2}^{L+2} H(S_i) - (f(Q_b) - f(Q) - \alpha_b f'(Q)) \\
&= -b_1^{(0,1)} \Lambda Q - b_1^{(1,\cdot)}.\nabla Q - \sum_{(n,k,i)\in\mathcal{I}} b_{i+1}^{(n,k)} T_i^{(n,k)} + \sum_{i=2}^{L+2} H(S_i) - (f(Q_b) - f(Q) - \alpha_b f'(Q)),
\end{aligned} \tag{3-12}
$$

where we used the definition of the profiles $T_i^{(n,k)}$ from (2-26), and the convention $b_{L_n+1}^{(n,k)} = 0$. For $i = 2, \dots, L$, we regroup the terms that involve the multiplication of i parameters $b_j^{(n,k)}$ in the nonlinear term $-(f(Q_b) - f(Q) - \alpha_b f'(Q))$. Since p is an odd integer,

$$
\begin{aligned}
\big(f(Q_b) - f(Q) - \alpha_b f'(Q)\big) &= \sum_{k=2}^{p} C_k^p Q^{p-k} \alpha_b^k \\
&= \sum_{k=2}^{p} C_k^p Q^{p-k} \Bigg[\sum_{|J|_1 = k} C_J \prod_{(n,k,i)\in\mathcal{I}} (b_i^{(n,k)})^{J_i^{(n,k)}} (T_k^{(n,k)})^{J_i^{(n,k)}} \prod_{i=2}^{L+2} S_i^{J_i} \Bigg],
\end{aligned} \tag{3-13}
$$

where $J = (J_1^{(0,1)}, \dots, J_{L_{n_0}}^{(n_0,k(n_0))}, J_2, \dots, J_{L+2})$ represents a $(\#\mathcal{I}+L+1)$-tuple of integers. Anticipating that the profile S_i will be a homogeneous profile of degree $(i, \gamma - g')$, we define for such tuples J,

$$
|J|_3 = \sum_{i=1}^{L} i J_i^{(0,1)} + \sum_{1\le i\le L_1,\, 1\le k\le d} i J_i^{(1,k)} + \sum_{(n,k,i)\in\mathcal{I},\, 2\le n} (i+1) J_i^{(n,k)} + \sum_{i=2}^{L+2} i J_i. \tag{3-14}
$$

We reorder the sum in the previous equation, (3-13), partitioning the $(\#\mathcal{I} + L + 1)$-tuples J according to their length $|J|_3$ instead of their length J_1:

$$
\big(f(Q_b) - f(Q) - \alpha_b f'(Q)\big) = \sum_{j=2}^{L+2} P_j + R.
$$

P_j captures the terms with polynomials of the parameters $b_i^{(n,k)}$ of length $|J|_3 = j$:

$$
P_j = \sum_{k=2}^{p} C_k Q^{p-k} \Bigg(\sum_{|J|=k,\, |J|_3 = j} C_J \prod_{(n,k,i)\in\mathcal{I}} (b_i^{(n,k)})^{J_i^{(n,k)}} (T_k^{(n,k)})^{J_i^{(n,k)}} \prod_{i=2}^{L+2} S_i^{J_i} \Bigg). \tag{3-15}
$$

The remainder contains only terms involving polynomials of the parameters $b_i^{(n,k)}$ of length $|\cdot|_3$ greater than or equal to $L+3$:

$$
R = \big(f(Q_b) - f(Q) - \alpha_b f'(Q)\big) - \sum_{i=2}^{L+2} P_i. \tag{3-16}
$$

From (3-12) we end up with the final decomposition

$$-F(Q_b)=-b_1^{(0,1)}\Lambda Q-b_1^{(1,\cdot)}.\nabla Q-\sum_{(n,k,i)\in\mathcal{I}} b_{i+1}^{(n,k)}T_i^{(n,k)}+\sum_{i=2}^{L} H(S_i)-\sum_{i=2}^{L+2} P_i-R. \tag{3-17}$$

Rewriting of the other terms in (3-3). From the form of Q_b in (3-7), one has

$$b_1^{(0,1)}\Lambda Q_b=b_1^{(0,1)}\Lambda Q+\sum_{(n,k,i)\in\mathcal{I}} b_1^{(0,1)}b_i^{(n,k)}\Lambda T_i^{(n,k)}+\sum_{i=2}^{L+2} b_1^{(0,1)}\Lambda S_i, \tag{3-18}$$

$$b_1^{(1,\cdot)}.\nabla Q_b=b_1^{(1,\cdot)}.\nabla Q+\sum_{j=1}^{d}\biggl(\sum_{(n,k,i)\in\mathcal{I}} b_1^{(1,j)}b_i^{(n,k)}\partial_{x_j}T_i^{(n,k)}+\sum_{i=2}^{L+2} b_1^{(1,j)}\partial_{x_j}S_i\biggr), \tag{3-19}$$

$$\begin{aligned}\sum_{(n,k,i)\in\mathcal{I}}&\bigl(-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)}\bigr)\frac{\partial Q_b}{\partial b_i^{(n,k)}}\\&=\sum_{(n,k,i)\in\mathcal{I}}\bigl(-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)}\bigr)\biggl(T_i^{(n,k)}+\sum_{j=2}^{L+2}\frac{\partial S_j}{\partial b_i^{(n,k)}}\biggr).\end{aligned} \tag{3-20}$$

Expression of the error term ψ_b. Using (2-21), we define

$$\Theta_i^{(n,k)}(y):=\Theta_i^{(n)}(|y|)Y^{(n,k)}\biggl(\frac{y}{|y|}\biggr).$$

From (3-17)–(3-20), ψ_b given by (3-3) is a sum of terms that are polynomials in b, and, denoting a monomial by b^J, we rearrange them according to the value $|J|_3$:

$$\begin{aligned}\psi_b=\sum_{i=2}^{L+2}[\Phi_i+H(S_i)]+b_1^{(0,1)}\Lambda S_{L+2}+\sum_{j=1}^{d} b_1^{(1,j)}\partial_{x_j}S_{L+2}&\\+\sum_{(n,k,i)\in\mathcal{I}}(-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)})\frac{\partial S_{L+2}}{\partial b_i^{(n,k)}}-R,&\end{aligned} \tag{3-21}$$

where the profiles Φ_i are given by the formulas

$$\begin{aligned}\Phi_2:=(b_1^{(0,1)})^2\Theta_1^{(0,1)}&+\sum_{k=1}^{d} b_1^{(0,1)}b_1^{(1,k)}\Theta_1^{(1,k)}\\&+\sum_{j=1}^{d}\biggl(b_1^{(1,j)}b_1^{(0,1)}\partial_{x_j}T_1^{(0,1)}+\sum_{k=1}^{d} b_1^{(1,j)}b_1^{(1,k)}\partial_{x_j}T_1^{(1,k)}\biggr)\\&\quad+\sum_{(n,k,0)\in\mathcal{I},\ n\geq 2}\biggl(b_1^{(0,1)}b_0^{(n,k)}\Theta_0^{(n,k)}+\sum_{j=1}^{d} b_1^{(1,j)}b_0^{(n,k)}\partial_{x_j}T_0^{(n,k)}\biggr)-P_2,\end{aligned} \tag{3-22}$$

and for $i = 3, \dots, L+1$,

$$\begin{aligned}
\Phi_i := {} & b_1^{(0,1)} b_{i-1}^{(0,1)} \Theta_{i-1}^{(0,1)} + \sum_{k=1,\,(1,k,i-1)\in\mathcal{I}}^{d} b_1^{(0,1)} b_{i-1}^{(1,k)} \Theta_{i-1}^{(1,k)} \\
& + \sum_{j=1}^{d} \Bigl(b_1^{(1,j)} b_{i-1}^{(0,1)} \partial_{x_j} T_{i-1}^{(0,1)} + \sum_{k=1,\,(1,k,i-1)\in\mathcal{I}}^{d} b_1^{(1,j)} b_{i-1}^{(1,k)} \partial_{x_j} T_1^{(1,k)} \Bigr) \\
& + \sum_{(n,k,i-2)\in\mathcal{I},\ n\geq 2} \Bigl(b_1^{(0,1)} b_{i-2}^{(n,k)} \Theta_{i-2}^{(n,k)} + \sum_{j=1}^{d} b_1^{(1,j)} b_{i-2}^{(n,k)} \partial_{x_j} T_{i-2}^{(n,k)} \Bigr) \\
& + b_1^{(0,1)} \Lambda S_{i-1} + \sum_{m=1}^{d} b_1^{(1,m)} \partial_{x_m} S_{i-1} \\
& + \sum_{(n,k,j)\in\mathcal{I}} \bigl(-(2j-\alpha_n) b_1^{(0,1)} b_j^{(n,k)} + b_{j+1}^{(n,k)} \bigr) \frac{\partial S_{i-1}}{\partial b_j^{(n,k)}} - P_i, \qquad (3\text{-}23)
\end{aligned}$$

$$\begin{aligned}
\Phi_{L+2} := {} & b_1^{(0,1)} \Lambda S_{L+1} + \sum_{m=1}^{d} b_1^{(1,m)} \partial_{x_m} S_{L+1} \\
& + \sum_{(n,k,j)\in\mathcal{I}} \bigl(-(2j-\alpha_n) b_1^{(0,1)} b_j^{(n,k)} + b_{j+1}^{(n,k)} \bigr) \frac{\partial S_{L+1}}{\partial b_j^{(n,k)}} - P_{L+2}. \qquad (3\text{-}24)
\end{aligned}$$

Step 2: definition of the profiles $(S_i)_{2\leq i\leq L+2}$ and simplification of ψ_b. We define by induction a sequence of couples of profiles $(S_i)_{2\leq i\leq L+2}$ by

$$\begin{cases} S_2 := -H^{-1}(\Phi_2) \\ S_i := -H^{-1}(\Phi_i) \quad \text{for } 3\leq i\leq L+2, \quad \text{with } \Phi_i \text{ defined by (3-22), (3-23), (3-24),} \end{cases} \qquad (3\text{-}25)$$

where H^{-1} is defined by (2-25). In the next step we prove that there is no problem in this construction. Since the S_i are defined in this way, by (3-21) we get the final expression for the error

$$\psi_b = b_1^{(0,1)} \Lambda S_{L+2} + \sum_{j=1}^{d} b_1^{(1,j)} \partial_{x_j} S_{L+2} + \sum_{(n,k,i)\in\mathcal{I}} \bigl(-(2i-\alpha_n) b_1^{(0,1)} b_i^{(n,k)} + b_{i+1}^{(n,k)} \bigr) \frac{\partial S_{L+2}}{\partial b_i^{(n,k)}} - R. \qquad (3\text{-}26)$$

Step 3: properties of the profiles S_i. We prove by induction on $i = 2, \dots, L+2$ that S_i is homogeneous of degree $(i, -\gamma-g')$ in the sense of Definition 2.14, and that for all $2\leq j\leq L+2$, we have $\partial S_j/\partial b_i^{(n,k)} = 0$ if $j\leq i$ for $n = 0, 1$ and if $j\leq i+1$ for $n\geq 2$.

Initialization. We now prove that S_2 is homogeneous of degree $(2, -\gamma-g')$, and that $\partial S_2/\partial b_i^{(n,k)} = 0$ if $2\leq i$ for $n = 0, 1$ and if $1\leq i$ for $n\geq 2$. We claim that Φ_2 is homogeneous of degree $(2, -\gamma-g'-2)$ and that $\partial \Phi_2/\partial b_i^{(n,k)} = 0$ if $2\leq i$ for $n = 0, 1$ and if $1\leq i$ for $n\geq 2$. To prove this, we prove that these two properties are true for every term on the right-hand side of (3-22).

From Lemma 2.10, $\Theta_1^{(0,1)}$ is simple admissible of degree $(0, -\gamma+2-g')$ in the sense of Definition 2.11. We also know $(b_1^{(0,1)})^2$ can be written under the form $J_1^{(0,1)} = 2$ and $J_i^{(n,k)} = 0$ otherwise and one has $|J|_2 = 2$ and $|J|_3 = 2$. Therefore, $(b_1^{(0,1)})^2 \Theta_1^{(0,1)}$ is homogeneous of degree $(|J|_3, -\gamma+2-g'-2|J|_2) = (2, -\gamma-g'-2)$. The same reasoning applies for $b_1^{(0,1)} b_1^{(1,k)} \Theta_1^{(1,k)}$ for $1\leq k\leq d$.

For $1 \le j \le d$, we know $T_1^{(0,1)}$ is admissible of degree $(0, -\gamma+2)$ by Lemma 2.12, so $\partial_{x_j} T_1^{(0,1)}$ is admissible of degree $(-\gamma+1)$ by Lemma 2.10. We also know $b_1^{(1,j)} b_1^{(0,1)}$ can be written in the form b^J with $J_1^{(0,1)} = 1$, $J_1^{(1,j)} = 1$ and $J_i^{(n,k)} = 0$ otherwise; therefore $|J|_3 = 2$ and $|J|_2 = 1 + \frac{\gamma-\gamma_1}{2} + 1 = 2 + \frac{\alpha-1}{2}$ by (1-18). Thus $b_1^{(1,j)} b_1^{(0,1)} \partial_{x_j} T_1^{(0,1)}$ is homogeneous of degree $(|J|_3, -\gamma 1 + 1 - 2|J|_2) = (2, -\gamma-2-\alpha)$. As $g' < \alpha$, it is then homogeneous of degree $(2, -\gamma-g'-2)$. The same reasoning applies for $1 \le j,k \le d$ to the term $b_1^{(1,j)} b_1^{(1,k)} \partial_{x_j} T_1^{(1,k)}$.

We now examine for $(n,k,0) \in \mathcal{I}$ the profile

$$b_1^{(0,1)} b_0^{(n,k)} \Theta_0^{(n,k)} + \sum_{j=1}^{d} b_1^{(1,j)} b_0^{(n,k)} \partial_{x_j} T_0^{(n,k)}.$$

$\Theta_0^{(n,k)}$ is simple admissible of degree $(n, -\gamma_n - g')$ by Lemma 2.10, and $b_1^{(0,1)} b_0^{(n,k)}$ can be written in the form b^J for $J_1^{(0,1)} = 1$, $J_0^{(n,k)} = 1$ and $J_i^{(n',k')} = 0$ otherwise. One then has $|J|_3 = 2$ and $|J|_2 = 1 + \frac{\gamma-\gamma_n}{2}$. Therefore, $b_1^{(0,1)} b_0^{(n,k)} \Theta_0^{(n,k)}$ is homogeneous of degree $(|J|_3, -\gamma_n - g' - 2|J|_2) = (2, -\gamma - g' - 2)$. Similarly the terms in the sum in the above identity are homogeneous of degree $(2, -\gamma - g' - 2)$.

We now look at the nonlinear term P_2. Since, for $2 \le i \le L+2$, the profile S_i involves polynomials of b in the form b^J with $|J|_3 = i$, from its definition (3-15) P_2 does not depend on the profiles S_i for $2 \le i \le L+2$ and can be written as

$$P_2 = C Q^{p-2} \Bigg(b_1^{(0,1)} T_1^{(0,1)} + \sum_{k=1}^{d} b_1^{(1,k)} T_1^{(1,k)} + \sum_{(n,k,0)\in\mathcal{I}} b_0^{(n,k)} T_0^{(n,k)} \Bigg)^2$$

for a constant C. We have to prove that all the mixed terms that are produced by this formula are homogeneous of degree $(2, \gamma - g' - 2)$. We write it only for one term, and apply the same reasoning to the others. For all $((n,k,0),(n',k',0)) \in \mathcal{I}^2$, by Lemmas 2.10 and 2.15 and (2-1), the profile $b_0^{(n,k)} b_0^{(n',k')} Q^{p-2} T_0^{(n,k)} T_0^{(n',k')}$ is homogeneous of degree $(2, -\gamma-2-\alpha)$ and then of degree $(2, -\gamma-2-g')$. As we said, similar considerations yield that all the other terms are homogeneous of degree $(2, \gamma-g'-2)$. This implies that P_2 is homogeneous of degree $(2, -\gamma-g'-2)$.

We have examined all terms in (3-22) and consequently proved that Φ_2 is homogeneous of degree $(2, -\gamma-2-g')$. By a direct check of all the terms on the right-hand side of (3-22), with P_2 given by the above identity, one has that $\partial \Phi_2 / \partial b_i^{(n,k)} = 0$ if $2 \le i$ for $n = 0,1$ and if $1 \le i$ for $n \ge 2$. We now check that we can apply Lemma 2.15(iii) to invert Φ_2 and to propagate the homogeneity. For all $\#\mathcal{I}$-tuples J with $|J|_3 = 2$, one has indeed for all integers n that $2|J|_2 - \gamma_n - 2 - g' > \gamma_n - d$ as the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing and $d - 2\gamma - 2 > 0$. For the second condition required by the lemma, we notice that g' is not a “fixed” constant in our problem, as its definition (1-21) involves a parameter ε. The purpose of the parameter ε is the following: by choosing it appropriately, we can suppose that for every $0 \le n \le n_0$ and $\#\mathcal{I}$-tuple J with $|J|_3 = 2$ we have

$$-2|J|_2 + \gamma + g' - \gamma_n \notin 2\mathbb{N}.$$

This allows us to apply Lemma 2.15(iii): S_2 is homogeneous of degree $(2, -\gamma - g')$. We also get that $\partial S_2 / \partial b_i^{(n,k)} = 0$ if $2 \le i$ for $n = 0,1$ and if $1 \le i$ for $n \ge 2$ as this is true for Φ_2. This proves the initialization of our induction.

Heredity. Suppose $3\le i\le L+1$, and that $S_{i'}$ is homogeneous of degree $(i',-\gamma-g')$ for $2\le i'\le i$, and that $\partial S_i'/\partial b_j^{(n,k)}=0$ if $i'\le j$ for $n=0,1$ and if $i'-1\le j$ for $n\ge 2$. We claim that Φ_i is homogeneous of degree $(i,-\gamma-g'-2)$ and that $\partial\Phi_i/\partial b_j^{(n,k)}=0$ if $i\le j$ for $n=0,1$ and if $i-1\le j$ for $n\ge 2$. We prove it by looking at all the terms on the right-hand side of (3-23). With the same reasoning we used for the initialization, we prove that

$$\begin{aligned}b_1^{(0,1)}b_{i-1}^{(0,1)}\Theta_{i-1}^{(0,1)}&+\sum_{k=1,\,(1,k,i-1)\in\mathcal{I}}^{d}b_1^{(0,1)}b_{i-1}^{(1,k)}\Theta_{i-1}^{(1,k)}\\&+\sum_{j=1}^{d}\Bigl(b_1^{(1,j)}b_{i-1}^{(0,1)}\partial_{x_j}T_{i-1}^{(0,1)}+\sum_{k=1,\,(1,k,i-1)\in\mathcal{I}}^{d}b_1^{(1,j)}b_{i-1}^{(1,k)}\partial_{x_j}T_1^{(1,k)}\Bigr)\\&+\sum_{(n,k,i-2)\in\mathcal{I},\,n\ge 2}\Bigl(b_1^{(0,1)}b_{i-2}^{(n,k)}\Theta_{i-2}^{(n,k)}+\sum_{j=1}^{d}b_1^{(1,j)}b_{i-2}^{(n,k)}\partial_{x_j}T_{i-2}^{(n,k)}\Bigr)\end{aligned}$$

is homogeneous of degree $(i,\gamma-g'-2)$. From the induction hypothesis, $b_1^{(0,1)}\Lambda S_{i-1}$ is homogeneous of degree $(i,-\gamma-g'-2)$. From Lemma 2.12, for $1\le j\le d$, we know $\partial_{x_j}S_{i-1}$ is homogeneous of degree $(i-1,-\gamma-g'-1)$, so that $b_1^{(1,j)}\partial_{x_j}S_{i-1}$ is homogeneous of degree $(i,-\gamma-g'-2-\alpha)$; since α is positive, it is then homogeneous of degree $(i,-\gamma-g'-2)$. Still from the induction hypothesis, for all $(n,k,i')\in\mathcal{I}$,

$$\bigl(-(2i'-\alpha_n)b_1^{(0,1)}b_{i'}^{(n,k)}+b_{i'+1}^{(n,k)}\bigr)\frac{\partial S_{i-1}}{\partial b_{i'}^{(n,k)}}$$

is homogeneous of degree $(i,-\gamma-g'-2)$. The last term to be considered is P_i. Since, for $2\le j\le L+2$, the profile S_j involves polynomials of b of the form b^J with $|J|_3=i$, from its definition (3-15) P_i does not depend on the profiles S_j for $i\le j\le L+2$ and can be written as

$$P_i=\sum_{k=2}^{p}C_kQ^{p-k}\Bigl(\sum_{|J|=k,\,|J|_3=i}C_J\prod_{(n,k,i)\in\mathcal{I}}(b_i^{(n,k)})^{J_i^{(n,k)}}(T_k^{(n,k)})^{J_i^{(n,k)}}\prod_{j=2}^{i-1}S_j^{J_j}\Bigr).$$

Let k be an integer $2\le k\le p$; let J be a $\#\mathcal{I}+L$-tuple with $|J|_3=i$. Then from the induction hypothesis,

$$Q^{p-k}\prod_{(n,k,i)\in\mathcal{I}}(b_i^{(n,k)})^{J_i^{(n,k)}}(T_k^{(n,k)})^{J_i^{(n,k)}}\prod_{j=2}^{i-1}S_j^{J_j}$$

is homogeneous of degree $\bigl(i,-\gamma-2-(k-1)\alpha-g'\sum_{j=2}^{i-1}J_j\bigr)$. As $k\ge 2$ and $\alpha>g'$, it is homogeneous of degree $(i,\gamma-2-g')$.

We just proved that Φ_i is homogeneous of degree $(i,-\gamma-2-g')$. By a direct check of all the terms on the right-hand side of (3-23), with P_i given by the above formula, one has that $\partial\Phi_i/\partial b_j^{(n,k)}=0$ if $i\le j$ for $n=0,1$ and if $i-1\le j$ for $n\ge 2$. We now check that we can apply Lemma 2.15(iii) to get the desired properties for $S_i=-H^{-1}\Phi_i$. For all $\#\mathcal{I}$-tuples J with $|J|_3=i$ and integers n, the first condition $|J|_2-\gamma-2-g'>\gamma_n-d$ is fulfilled since $-2\gamma_n-d\ge-2\gamma-d>2$. For the second condition, again as in the initialization, as g' is not a "fixed" constant in our problem (its definition (1-21) involves a parameter ε), we can choose it such that for every $0 \le n \le n_0$ and $\#\mathcal{I}$-tuple J with $|J|_3 = i$,

$$-2|J|_2 + \gamma + g' - \gamma_n \notin 2\mathbb{N}.$$

We thus can apply Lemma 2.15(iii): S_i is homogeneous of degree $(i, -\gamma - g')$. One also obtains that $\partial S_i / \partial b_j^{(n,k)} = 0$ if $i \le j$ for $n = 0, 1$ and if $i - 1 \le j$ for $n \ge 2$, as this is true for Φ_i. This proves the heredity in our induction.

The last step, that it is the heredity from $L+1$ to $L+2$, can be proved exactly the same way and we do not write it here.

Step 4: bounds for the error term. In Step 2 we computed the expression (3-26) of the error term ψ_b. In Step 3 we proved that the profiles S_i were well defined and homogeneous of degree $(i, -\gamma - g')$. We can now prove the bounds on ψ_b claimed in the proposition. In the sequel we always assume the bounds $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ and $|b_1^{(0,1)}| \ll 1$.

Homogeneity of ψ_b. We claim that ψ_b is a finite sum of homogeneous functions of degree $(i, -\gamma - g' - 2)$ for $i \ge L+3$. For this we consider all terms on the right-hand side of (3-26). As S_{L+2} is homogeneous of degree $(L+2, -\gamma - g')$ from Step 3, the function $b_1^{(0,1)} \Lambda S_{L+2}$ is homogeneous of degree $(L+3, -\gamma - g' - 2)$ by Lemma 2.15. Similarly for $1 \le j \le d$, we know $b_1^{(1,j)} \partial_{x_j} S_{L+2}$ is homogeneous of degree $(L+3, -\gamma - g' - 2 - \alpha)$ (and then homogeneous of degree $(L+3, -\gamma - g' - 2)$ as $\alpha > 0$), and for $(n,k,i) \in \mathcal{I}$,

$$(-(2i - \alpha_n) b_1^{(0,1)} b_i^{(n,k)} + b_{i+1}^{(n,k)}) \frac{\partial S_{L+2}}{\partial b_i^{(n,k)}}$$

is homogeneous of degree $(L+3, -\gamma - g' - 2)$. From its definition (3-16), and since S_i is homogeneous of degree $(i, -\gamma - g')$ for $2 \le i \le L+2$, we have R is a finite sum of homogeneous profiles of degree $(i, -\gamma - \alpha - 2)$ with $i \ge L+3$. All this implies that ψ_b is a finite sum of homogeneous functions of degree $(i, -\gamma - g' - 2)$ for $i \ge L+3$.

Proof of an intermediate estimate. We claim that there exists an integer $A \ge L+3$ such that for μ a d-tuple of integers, $j \in \mathbb{N}$ and $B > 1$ we have

$$\int_{|y| \le B} \frac{|\partial^\mu \psi_b|^2}{1 + |y|^{2j}} \, dy \le C(L) \sum_{i=L+3}^{A} |b_1^{(0,1)}|^{2i} B^{\max\left(4i + 4\left(m_0 - \frac{|\mu| + j}{2}\right) + 4(\delta_0 - 1) - 2g', 0\right)}. \tag{3-27}$$

We now prove this bound. We proved earlier that ψ_b is a finite sum of homogeneous functions of degree $(i, -\gamma - g' - 2)$ for $i \ge L+3$. Consequently, it suffices to prove this bound for a homogeneous function $b^J f(y)$ of degree $(|J|_3, -\gamma - g' - 2)$ with $|J|_3 \ge L+3$. As f is admissible of degree $(2|J|_2 - \gamma - g' - 2)$, one then computes

$$\begin{aligned}\int_{|y| \le B} \frac{|b^J \partial^\mu f|^2}{1 + |y|^{2j}} &\le C(f) |b_1^{(0,1)}|^{2|J|_2} \int_0^B (1+r)^{4|J|_2 - 2\gamma - 2g' - 4 - 2j - 2|\mu|} r^{d-1} \, dr \\ &\le C(f) |b_1^{(0,1)}|^{2|J|_2} B^{\max\left(4|J|_2 + 4\left(m_0 + \frac{j + |\mu|}{2}\right) + 4(\delta_0 - 1) - 2g', 0\right)}\end{aligned}$$

(we avoid the logarithmic case in the integral by changing a bit the value of g' defined in (1-21), by changing a bit the value of ε). This concludes the proof of (3-27).

Proof of the local bounds for the error. Let j be an integer, and $\mu \in \mathbb{N}^d$ with $|\mu| = j$. From (3-27), $|b_1^{(0,1)}| \ll 1$ and $B > 1$, we obtain, by (3-27),

$$\int_{|y|\le B} |\partial^\mu \psi_b|^2 \, dy \le C(L)|b_1^{(0,1)}|^{2L+6} B^{\max\left(4A+4\left(m_0-\frac{|\mu|+j}{2}\right)+4(\delta_0-1)-2g',0\right)},$$

which gives the desired bound (3-6).

Proof of the global bounds for the error. Let $j \le 2s_L$, and $\mu \in \mathbb{N}^d$ with $|\mu| = j$. Using (3-27), we notice that for $L+3 \le i \le A$ one has

$$\max\left(4i+4\left(m_0-\frac{|\mu|+j}{2}\right)+4(\delta_0-1)-2g',0\right) = 4i+4\left(m_0-\frac{|\mu|+j}{2}\right)+4(\delta_0-1)-2g'.$$

This implies

$$\begin{aligned}\int_{|y|\le B_1} \frac{|\partial^\mu \psi_b|^2}{1+|y|^{2j}} \, dy &\le C(L) \sum_{i=L+3}^{A} |b_1^{(0,1)}|^{2i} B_1^{4i+4\left(m_0-\frac{|\mu|+j}{2}\right)+4(\delta_0-1)-2g'} \\ &\le C(L)|b_1^{(0,1)}|^{2\left(\frac{j}{2}-m_0\right)+2(1-\delta_0)+g'-C(L)\eta},\end{aligned}$$

which is the desired bound (3-5). Let j be an integer, $j \le s_L$. Now, as $H = -\Delta + V$, where V is a smooth potential satisfying $|\partial^\mu V| \le C(\mu)(1+|y|)^{-2-|\mu|}$, by (2-2) one obtains

$$\begin{aligned}\int_{|y|\le B_1} |H^j \psi_b|^2 \, dy &\le C(L) \sum_{j'+|\mu|_1=2j} \int_{|y|\le B_1} \frac{|\partial^\mu \psi_b|^2}{1+|y|^{2j'}} \, dy \\ &\le C(L) \sum_{j'+|\mu|=2j} \sum_{i=L+3}^{A} |b_1^{(0,1)}|^{2i} B_1^{\max(4i+4(m_0-j)+4(\delta_0-1)-2g',0)} \\ &\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}\end{aligned}$$

using (3-27) (because again $4i+4(m_0-j)+4(\delta_0-1)-2g' > 0$ as $i \ge L+3$ and $j \le s_L$). This proves the last estimate (3-4). □

We now localize the perturbation built in Proposition 3.1 in the zone $|y| \le B_1$ and estimate error generated by the cut. We also include the time-dependence of the parameters following Remark 3.2. We recall that s_L is defined by (1-24).

Proposition 3.3 (localization of the perturbation). *The function χ is a cut-off defined by* (1-43). *We keep the notations from Proposition 3.1. $I = (s_0, s_1)$ is an interval, and*

$$b : I \to \mathbb{R}^{\#\mathcal{I}}, \quad s \mapsto (b_i^{(n,k)}(s))_{(n,k,i)\in\mathcal{I}},$$

is a C^1 function with the a priori bounds[9]

$$|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}, \quad 0 < b_1^{(0,1)} \ll 1, \quad |b_{1,s}^{(0,1)}| \lesssim |b_1^{(0,1)}|^2. \tag{3-28}$$

[9]This means that under the bounds $|b_i^{(n,k)}| \le K|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ for some $K > 0$, there exists $b^*(K)$ such that the estimates that follow hold if $b_1^{(0,1)} \le b^*(K)$ with constants depending on K. In what follows, K will be fixed independently of the other important constants.

We define the profile $\widetilde{Q}_b$ *as*

$$\widetilde{Q}_b := Q + \tilde{\alpha}_b = Q + \chi_{B_1}\alpha_b, \quad \tilde{\alpha}_b := \chi_{B_1}\alpha_b. \tag{3-29}$$

Then one has the identity (Mod(s) *being defined by* (3-10))

$$\partial_s \widetilde{Q}_b - F(\widetilde{Q}_b) + b_1^{(0,1)}\Lambda \widetilde{Q}_b + b_1^{(1,\cdot)}.\nabla \widetilde{Q}_b = \tilde{\psi}_b + \chi_{B_1}\operatorname{Mod}(s) \tag{3-30}$$

with, for $0 < \eta \ll 1$ *small enough, an error term* $\tilde{\psi}_b$ *satisfying the following bounds*:

(1) Global bounds. *For any integer* j *with* $1 \le j \le s_L - 1$ *we have*

$$\int_{\mathbb{R}^d} |H^j \tilde{\psi}_b|^2\, dy \le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C_j\eta}. \tag{3-31}$$

For any real number $s_c \le j < 2s_L - 2$,

$$\int_{\mathbb{R}^d} |\nabla^j \tilde{\psi}_b|^2\, dy \le C(L)|b_1^{(0,1)}|^{2(\frac{j}{2}-m_0)+2(1-\delta_0)-C_j\eta}, \tag{3-32}$$

and for $j = s_L$, *one has the improved bound*

$$\int_{\mathbb{R}^d} |H^{s_L} \tilde{\psi}_b|^2\, dy \le C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)+2\eta(1-\delta_0')}. \tag{3-33}$$

(2) Local bounds. *One has that* (ψ_b *being defined by* (3-3))

$$\forall |y| < B_1, \quad \tilde{\psi}_b(y) = \psi_b, \tag{3-34}$$

and for any $1 \le B \le B_1$ *and* $j \in \mathbb{N}$,

$$\int_{|y|\le B} |\nabla^j \tilde{\psi}_b|^2\, dy \le C(L,j)B^{C(L,j)}|b_1^{(0,1)}|^{2L+6}. \tag{3-35}$$

Proof. First, we compute the expression of the new error term by rewriting the left-hand side of (3-30) using (3-9) and the fact that $F(Q) = 0$:

$$\begin{aligned}\tilde{\psi}_b = \chi_{B_1}\psi_b + \partial_s(\chi_{B_1})\tilde{\alpha_b} - \big[F(Q+\chi_{B_1}\alpha_b) - F(Q) - \chi_{B_1}\big(F(Q+\alpha_b) - F(Q)\big)\big] \\ + b_1^{(0,1)}(\Lambda Q - \chi_{B_1}\Lambda Q) + b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b) \\ + b_1^{(1,\cdot)}.(\nabla Q - \chi_{B_1}\nabla Q) + b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b) - \chi_{B_1}\nabla\alpha_b).\end{aligned} \tag{3-36}$$

Local bounds. In the previous identity, one clearly sees that all the terms, except $\chi_{B_1}\psi_b$, have their support in $B_1 \le |y|$. Thus, for $B \le B_1$, the bound (3-35) is a direct consequence of the local bound (3-6) for ψ_b.

Global bounds. Let $m_1 + 1 \le j \le s_L$. We will prove the bounds (3-31) and (3-33) by proving that this estimate holds for all terms on the right-hand side of (3-36). The reasoning to prove the estimates will be similar from one term to another. For this reason, we shall go quickly whenever an argument has already been used earlier.

The $\chi_{B_1}\psi_b$ term. As $H=-\Delta+V$ for V a smooth potential with $\partial^\mu V\lesssim(1+|y|)^{-2-|\mu|}$ by (2-2), and as $(\partial_r^k(\chi_{B_1}))(r)=B_1^{-k}\partial_r^k\chi(r/B_1)$, we have the identity

$$H^j(\chi_{B_1}\psi_b)=\chi_{B_1}H^j\psi_b+\sum_{\substack{\mu\in\mathbb{N}^d\\ 0\le|\mu|\le 2j-1}}^{j} f_\mu\partial^\mu\psi_b,$$

where for each $\mu\in\mathbb{N}^d$, with $0\le|\mu|\le j-1$, we have f_μ has its support in $B_1\le|x|\le 2B_1$ and satisfies $|f_\mu|\le C(L)B_1^{-(2j-|\mu|)}$. Using (3-4) and (3-5) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}&|H^j(\chi_{B_1}\psi_b)|^2\,dy\\ &\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}+\sum_{\substack{\mu\in\mathbb{N}^d\\ 0\le|\mu|\le 2j-1}}^{j} B_1^{-(4j-2|\mu|)}b_1^{2(\frac{|\mu|}{2}-m_0+2(1-\delta_0)+g'-C(L)\eta)}\\ &\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}.\end{aligned}\tag{3-37}$$

Similarly, one obtains, for any integer j' with $0\le j'\le 2s_L-2$,

$$\int_{\mathbb{R}^d}|\nabla^{j'}(\chi_{B_1}\psi_b)|^2\le C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)+g'-C(L)\eta}.\tag{3-38}$$

Using interpolation, this estimate remains true for any real number j' with $0\le j'\le 2s_L-2$.

The $\partial_s(\chi_{B_1})\alpha_b$ term. We first split using (3-7):

$$\partial_s(\chi_{B_1})\alpha_b=\partial_s(\chi_{B_1})\biggl(\sum_{(n,k,i)\in\mathcal{I}}b_i^{(n,k)}T_i^{(n,k)}+\sum_{i=2}^{L+2}S_i\biggr).\tag{3-39}$$

We compute

$$\partial_s(\chi_{B_1})=(b_1^{(0,1)})^{-1}b_{1,s}^{(0,1)}\frac{|y|}{B_1}(\partial_r\chi_{B_1})\biggl(\frac{y}{B_1}\biggr).$$

We first treat the S_i terms. As we already explained in the study of the $\chi_{B_1}\psi_b$ term, one has

$$H^j(\partial_s(\chi_{B_1})S_i)=\sum_{\mu\in\mathbb{N}^d,\,|\mu|\le 2j}f_\mu\partial^\mu S_i$$

with f_μ a smooth function, with support in $B_1\le|x|\le 2B_1$ and satisfying $|f_\mu|\le C(L)b_1^{(0,1)}B_1^{-(2j-|\mu|_1)}$ (because $|b_{1,s}^{(0,1)}|\lesssim|b_1^{(0,1)}|^2$ by (3-28)). As S_i is homogeneous of degree $(i,-\gamma-g')$ in the sense of Definition 2.14, from (3-8) and $|b_i^{(n,k)}|\lesssim|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ we get

$$\int_{\mathbb{R}^d}|H^j(\partial_s(\chi_{B_1})S_i)|^2\,dy\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}\tag{3-40}$$

using Lemma 2.15. Now we treat the $T_i^{(n,k)}$ terms in the identity (3-39). Let $(i,n,k)\in\mathcal{I}$. Then again one has the decomposition

$$H^j[\partial_s(\chi_{B_1})b_i^{(n,k)}T_i^{(n,k)}]=b_i^{(n,k)}\sum_{\mu\in\mathbb{N}^d,\,|\mu|\le 2j}f_\mu\partial^\mu T_i^{n,k}$$

with f_μ a smooth function, with support in $B_1\le|y|\le 2B_1$ and satisfying $|f_\mu|\le C(L)b_1^{(0,1)}B_1^{-(2j-|\mu|)}$. As $T_i^{(n,k)}$ is an admissible profile of degree $(-\gamma_n+2i)$ in the sense of Definition 2.11 by (2-26) and Lemma 2.10, $\partial^\mu T_i^{n,k}$ is admissible of degree $(-\gamma_n+2i-|\mu|)$ by Lemma 2.12 and we compute

$$\begin{aligned}\int_{\mathbb{R}^d}|b_i^{(n,k)}f_\mu\partial^\mu T_i^{n,k}|^2\,dy&\le\frac{C(L)|b_1^{(0,1)}|^{\gamma-\gamma_n+2i+2}}{B_1^{2(2j-|\mu_|1)}}\int_{B_1}^{2B_1}r^{-2\gamma_n+4i-2|\mu|_1}r^{d-1}\,dr\\&\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\eta(2j-2i-2\delta_n-2m_n)}.\end{aligned}$$

As $(i,n,k)\in\mathcal{I}$, we know $i\le L_n$ so if $j=s_L$ one has $2j-2i-2\delta_n-2m_n\ge 2-2\delta_n$. Therefore we have proved the bound (we recall that $\delta_0'=\max_{0\le n\le n_0}\delta_n\in(0,1)$)

$$\int_{\mathbb{R}^d}|H^j(\partial_s(\chi_{B_1})b_i^{(n,k)}T_i^{(n,k)})|^2\,dy\le\begin{cases}C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta}&\text{if } m_0+1\le j<s_L,\\ C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)+\eta(1-\delta_0')}&\text{if } j=s_L.\end{cases}\tag{3-41}$$

From the decomposition (3-39), the bounds (3-40) and (3-41), we deduce the bound

$$\begin{aligned}&\int_{\mathbb{R}^d}|H^j(\partial_s(\chi_{B_1})\alpha_b|^2\,dy\\&\quad\le\begin{cases}C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta}&\text{if } 0\le j<s_L,\\ C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}\big(|b_1^{(0,1)}|^{2\eta(1-\delta_0')}+|b_1^{(0,1)}|^{g'-C(L)\eta}\big)&\text{if } j=s_L.\end{cases}\end{aligned}\tag{3-42}$$

Using verbatim the same arguments, one gets that for any integer $0\le j'\le 2s_L-2$,

$$\int_{\mathbb{R}^d}|\nabla^{j'}(\partial_s(\chi_{B_1})\alpha_b|^2\,dy\le C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(L)\eta},\tag{3-43}$$

which remains true for any real number j' with $0\le j'\le 2s_L-2$ by interpolation.

The $F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))$ term. It can be written as

$$\begin{aligned}&F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))\\&\qquad=\Delta(\chi_{B_1}\alpha_b)-\chi_{B_1}\Delta\alpha_b+(Q+\chi_{B_1}\alpha_b)^p-Q^p-\chi_{B_1}((Q+\alpha_b)^p-Q^p).\end{aligned}\tag{3-44}$$

We now prove the bound for the two terms that have appeared. From the identity

$$\Delta(\chi_{B_1}\alpha_b)-\chi_{B_1}\Delta\alpha_b=\Delta(\chi_{B_1})\alpha_b+2\nabla\chi_{B_1}.\nabla\alpha_b,$$

as χ is radial and as $(\partial_r^k(\chi_{B_1}))(r)=B_1^{-k}\partial_r^k\chi(r/B_1)$, one sees that this term can be treated exactly the same way we treated the previous term: $\partial_s(\chi_{B_1})\alpha_b$. This is why we claim the following estimates that can be proved using exactly the same arguments:

$$\int_{\mathbb{R}^d} |H^j(\Delta(\chi_{B_1}\alpha_b)-\chi_{B_1}\Delta\alpha_b)|^2\,dy \\ \leq \begin{cases} C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } m_0+1\leq j<s_L, \\ C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}\big(|b_1^{(0,1)}|^{2\eta(1-\delta_0')}+|b_1^{(0,1)}|^{g'-C(L)\eta}\big) & \text{if } j=s_L. \end{cases} \tag{3-45}$$

We now turn to the other term in (3-44), which can be rewritten as

$$(Q+\chi_{B_1}\alpha_b)^p-Q^p-\chi_{B_1}((Q+\alpha_b)^p-Q^p)=\sum_{k=2}^{p} C_k^p Q^{p-k}\chi_{B_1}(\chi_{B_1}^{k-1}-1)\alpha_b^k.$$

All the terms are localized in the zone $B_1\leq|y|\leq 2B_1$. From the definition (3-7) of α_b, (3-8), (2-1) and Lemma 2.15, for each $2\leq k\leq p$ one has that $Q^{p-k}\alpha_b^k$ is a finite sum of homogeneous profiles of degree $(i,-\gamma-\alpha-2)$ for $i\geq k$, yielding

$$\int_{\mathbb{R}^d}\big|H^j\big((Q+\chi_{B_1}\alpha_b)^p-Q^p-\chi_{B_1}((Q+\alpha_b)^p-Q^p)\big)\big|^2\,dy \\ \leq C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}. \tag{3-46}$$

From the decomposition (3-44) and the estimates (3-45) and (3-46) one gets

$$\int_{\mathbb{R}^d}\big|H^j\big(F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))\big)\big|^2\,dy \\ \leq C(L)\begin{cases} |b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } m_0+1\leq j<s_L, \\ |b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}\big(|b_1^{(0,1)}|^{2\eta(1-\delta_0')}+|b_1^{(0,1)}|^{\alpha-C(L)\eta}\big) & \text{if } j=s_L. \end{cases} \tag{3-47}$$

The same methods used for the two previous terms yield the analogue estimate for

$$\nabla^{j'}\big[F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))\big]$$

for any integer $0\leq j'\leq 2s_L-2$, and by interpolation, we obtain, for any real number j' with $0\leq j'\leq 2s_L-2$,

$$\int_{\mathbb{R}^d}\big|\nabla^{j'}\big(F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))\big)\big|^2\,dy \\ \leq C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(L)\eta}. \tag{3-48}$$

The $b_1^{(0,1)}(\Lambda Q-\chi_{B_1}\Lambda Q)$ term. As $\partial^\mu(\Lambda Q)\leq C(\mu)(1+|y|)^{-\gamma-|\mu|}$ for all $\mu\in\mathbb{N}^d$ by (2-7) and $H\Lambda Q=0$, one computes

$$\int_{\mathbb{R}^d}\big|H^j(b_1^{(0,1)}(\Lambda Q-\chi_{B_1}\Lambda Q))\big|^2\,dy\leq C(j)|b_1^{(0,1)}|^2\int_{B_1}^{2B_1} r^{-2\gamma-4j}r^{d-1}\,dr \\ \leq C(j)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+2\eta(j-m_0-\delta_0)} \tag{3-49}$$

with $s_L - m_0 - \delta_0 = L + 1 - \delta_0 > 1 - \delta_0$ for $j = s_L$. For any integer j' with $E[s_c] \le j' \le 2s_L - 2$, similar reasoning yields the estimate

$$\int_{\mathbb{R}^d} |\nabla^{j'}(b_1^{(0,1)}(\Lambda Q - \chi_{B_1}\Lambda Q))|^2\,dy \le C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}.$$

By interpolation, one has for any real number j' with $E[s_c] \le j' \le 2s_L - 2$,

$$\int_{\mathbb{R}^d} \big|\nabla^{j'}(b_1^{(0,1)}(\Lambda Q - \chi_{B_1}\Lambda Q))\big|^2\,dy \le C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}. \tag{3-50}$$

The $b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b)$ term. First we write this term as

$$b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b = b_1^{(0,1)}(y.\nabla\chi_{B_1})\alpha_b.$$

Now, we notice that

$$b_1^{(0,1)}(y.\nabla\chi_{B_1}) = b_1^{(0,1)}\frac{|y|}{B_1}(\partial_r\chi)\left(\frac{|y|}{B_1}\right)$$

is very similar to

$$\partial_s(\chi_{B_1}) = (b_1^{(0,1)})^{-1}b_{1,s}^{(0,1)}\frac{|y|}{B_1}(\partial_r\chi_{B_1})\left(\frac{y}{B_1}\right)$$

in the sense that it enjoys the same estimates, as $|b_{1,s}^{(0,1)}| \lesssim (b_1^{(0,1)})^2$ by (3-28). Thus, we can get exactly the same estimates for the term $b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b)$ that we obtained previously for the term $\partial_s(\chi_{B_1})\alpha_b$ with the exact same methodology, yielding

$$\begin{aligned}\int_{\mathbb{R}^d} &\big|H^j\big(b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b)\big)\big|^2\,dy \\ &\le \begin{cases} C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } 0 \le j < s_L, \\ C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}\big(|b_1^{(0,1)}|^{2\eta(1-\delta_0')} + |b_1^{(0,1)}|^{g'-C(L)\eta}\big) & \text{if } j = s_L, \end{cases}\end{aligned} \tag{3-51}$$

and for any integer j' with $0 \le j' \le 2s_L - 2$,

$$\int_{\mathbb{R}^d} |\nabla^{j'}(b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b))|^2\,dy \le C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(L)\eta}. \tag{3-52}$$

The $b_1^{(1,\cdot)}.(\nabla Q - \chi_{B_1}\nabla Q)$ term. First we rewrite

$$b_1^{(1,\cdot)}.(\nabla Q - \chi_{B_1}\nabla Q) = \sum_{i=1}^{d} b_1^{(1,i)}(1-\chi_{B_1})\partial_{y_i}Q. \tag{3-53}$$

Now let i be an integer, $1 \le i \le d$. From the asymptotic (2-1) of the ground state

$$|\partial^\mu Q| \le C(\mu)(1+|y|)^{-\frac{2}{p-1}-|\mu|}$$

and the fact that $H\partial_{x_i}Q=0$, we deduce

$$\int_{\mathbb{R}^d}\bigl|H^j\bigl(b_1^{(1,i)}((1-\chi_{B_1})\partial_{y_i}Q)\bigr)\bigr|^2\,dy \le C(j)|b_1^{(0,1)}|^{\gamma-\gamma_1+2}\int_{B_1}^{2B_1} r^{-2\gamma_1-4j}r^{d-1}\,dr$$
$$\le C(j)|b_1^{(0,1)}|^{2(j-m_0)-2(1-\delta_0)+2\eta(j-m_1-\delta_1)}$$

with $s_L-m_1-\delta_1=L+m_0-m_1+1-\delta_1>1-\delta_1$ for $j=s_L$. So we finally get, putting together the two previous equations,

$$\int_{\mathbb{R}^d}\bigl|H^j\bigl(b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q)\bigr)\bigr|^2\,dy \le C(j)|b_1^{(0,1)}|^2\int_{B_1}^{+\infty} r^{-2\gamma-4j}r^{d-1}\,dr$$
$$\le C(j)|b_1^{(0,1)}|^{2(j-m_0)-2(1-\delta_0)+2\eta(1-\delta_1)}. \tag{3-54}$$

Now, for any integer j' with $E[s_c]\le j'\le 2s_L-2$, as $E[s_c]>s_c-1$, similar reasoning yields the estimate

$$\int_{\mathbb{R}^d}\bigl|\nabla^{j'}(b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q))\bigr|^2\,dy \le C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}.$$

By interpolation, one has for any real number j' with $E[s_c]\le j'\le 2s_L-2$,

$$\int_{\mathbb{R}^d}\bigl|\nabla^{j'}(b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q))\bigr|^2\,dy \le C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}. \tag{3-55}$$

The $b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b)$ term. We first rewrite

$$b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b)=\sum_{i=1}^d b_1^{(1,i)}\partial_{y_i}(\chi_{B_1})\alpha_b.$$

Let i be an integer, $1\le i\le d$. For all $\mu\in\mathbb{N}^d$, we know $\partial^\mu(\chi_{B_1})\le C(\mu)B_1^{-|\mu|}$. From (3-7) and (3-8), α_b is a sum of homogeneous profiles of degree $(i,-\gamma)$. Using Lemma 2.15, one computes

$$\int_{\mathbb{R}^d}\bigl|H^j(b_1^{(1,i)}\partial_{y_i}(\chi_{B_1})\alpha_b)\bigr|^2\,dy \le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}.$$

With the two previous equations, one has proved that

$$\int_{\mathbb{R}^d}\bigl|H^j\bigl(b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b)\bigr)\bigr|^2\,dy \le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}. \tag{3-56}$$

Using exactly the same arguments, one can prove that for any integer $0\le j'\le 2s_L-2$, the analogue estimate for $\nabla^{j'}(b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b))$ holds. By interpolation, it gives that for any real number $0\le j'\le 2s_L-2$ we have

$$\int_{\mathbb{R}^d}\bigl|\nabla^{j'}\bigl(b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b)\bigr)\bigr|^2\,dy \le C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}. \tag{3-57}$$

End of the proof. For the estimate concerning the operator H (resp. the operator ∇), we have estimated all terms on the right-hand side of (3-36) in (3-37), (3-42), (3-47), (3-49), (3-51), (3-54) and (3-56) (resp. the right-hand side of (3-36) in (3-38), (3-43), (3-48), (3-50), (3-52), (3-55) and (3-57)). Adding all these estimates, as $0 < b_1^{(0,1)} \ll 1$ is a very small parameter, one sees that there exists $\eta_0 := \eta_0(L)$ such that for $0 < \eta < \eta_0$, the bounds (3-31) and (3-33) hold (resp. the bound (3-32) holds). □

3B. ***Study of the approximate dynamics for the parameters.*** In Proposition 3.3 we stated the existence of a profile $\tilde{Q}_b$ such that the force term $F(\tilde{Q}_b)$ generated by (NLH) has an almost explicit formulation in terms of the parameters $b = (b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ up to an error term $\tilde{\psi}_b$. Suppose that for some time, the solution that started at $\tilde{Q}_{b(0)}$ stays close to this family of approximate solutions, up to scaling and translation invariances, meaning that it can be written approximately as $\tau_{z(t)}(\tilde{Q}_{b(t),1/\lambda(t)})$. Then $\tilde{Q}_{b(s)}$ is almost a solution of the renormalized flow (3-2) associated to the functions of time $\lambda(t)$ and $z(t)$, meaning that

$$\partial_s(\tilde{Q}_b) - \frac{\lambda_s}{\lambda}\Lambda\tilde{Q}_b - \frac{z_s}{\lambda}.\nabla\tilde{Q}_b - F(\tilde{Q}_b) \approx 0.$$

Using the identity (3-30), this means

$$-\left(b_1^{(0,1)} + \frac{\lambda_s}{\lambda}\right)\Lambda\tilde{Q}_b - \left(b_1^{(1,\cdot)} + \frac{z_s}{\lambda}\right).\nabla\tilde{Q}_b + \chi_{B_1}\operatorname{Mod}(s) \approx 0.$$

From the very definition (3-10) of the modulation term $\operatorname{Mod}(s)$, projecting the previous relation onto the different modes that appeared[10] yields

$$\begin{cases} \dfrac{\lambda_s}{\lambda} = -b_1^{(0,1)}, \\ \dfrac{z_s}{\lambda} = -b_1^{(1,\cdot)}, \\ b_{i,s}^{(n,k)} = -(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)} + b_{i+1}^{(n,k)} \quad \forall (n,k,i)\in\mathcal{I} \end{cases} \tag{3-58}$$

with the convention $b_{L_n+1}^{(n,k)} = 0$. The understanding of a solution starting at $\tilde{Q}_{b(0)}$ then relies on the understanding of the solutions of the finite-dimensional dynamical system (3-58) driving the evolution of the parameters $b_i^{(n,k)}$. First we derive some explicit solutions such that $\lambda(t)$ touches 0 in finite time, signifying concentration in finite time.

Lemma 3.4 (special solutions for the dynamical system of the parameters). *We recall that the renormalized time s is defined by* (3-1). *Let $\ell \le L$ be an integer such that $2\alpha < \ell$. We define the functions*

$$\begin{cases} \bar{b}_i^{(0,1)}(s) = \dfrac{c_i}{s^i} & \text{for } 1 \le i \le \ell, \\ \bar{b}_i^{(0,1)} = 0 & \text{for } \ell < i \le L, \\ \bar{b}_i^{(n,k)} = 0 & \text{for } (n,k,i)\in\mathcal{I} \text{ with } n \ge 1, \end{cases} \tag{3-59}$$

with $(c_i)_{1\le i\le\ell}$ being ℓ constants defined by induction as

$$c_1 = \frac{\ell}{2\ell-\alpha} \qquad \text{and} \qquad c_{i+1} = -\frac{\alpha(\ell-i)}{2\ell-\alpha}c_i \quad \text{for } 1 \le i \le \ell-1. \tag{3-60}$$

[10] This will be done rigorously in the next section.

Then $\bar{b}=(\bar{b}_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ *is a solution of the last equation in* (3-58). *Moreover*, *the solutions* $\lambda(s)$ *and* $z(s)$ *of the first two equations in* (3-58) *starting at* $\lambda(0)=1$ *and* $z(0)=0$, *taken in original time variable* t, *are* $z(t)=0$ *and*

$$\lambda(t)=\left(\frac{\alpha}{(2\ell-\alpha)s_0}\right)^{\frac{\ell}{\alpha}}\left(\frac{(2\ell-\alpha)}{\alpha}s_0-t\right)^{\frac{\ell}{\alpha}}. \tag{3-61}$$

Proof. It is a direct computation that can safely be left to the reader. □

As $s_0>0$ and $2\ell>\alpha$, (3-61) can be interpreted as: there exists $T>0$ with $\lambda(t)\approx(T-t)^{\frac{\ell}{\alpha}}$ as $t\to T$. Now, given $\frac{1}{2}\alpha<\ell\le L$, we want to know the exact number of instabilities of the particular solution $\bar{b}$. In addition, in Propositions 3.1 and 3.3, we needed the a priori bounds

$$|b_i^{(n,k)}|\lesssim|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$$

to show sufficient estimates for the errors ψ_b and $\tilde{\psi}_b$. Around the solution $\bar{b}$ defined by (3-59), $b_1^{(0,1)}$ is of order s^{-1}, and so the a priori bounds we need become[11]

$$b_i^{(n,k)}\lesssim s^{\frac{\gamma_n-\gamma}{2}-i}.$$

Therefore, by "stability" of $\bar{b}$ we mean stability with respect to this size and introduce the following renormalization for a solution of (3-58) close to $\bar{b}$:

$$b_i^{(n,k)}=\bar{b}_i^{(n,k)}+\frac{U_i^{(n,k)}}{s^{\frac{\gamma-\gamma_n}{2}+i}}. \tag{3-62}$$

It defines a $\#\mathcal{I}$-tuple of real numbers $U=(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$, and we order the parameters as in (2-28) by

$$U=\big(U_1^{(0,1)},\dots,U_L^{(0,1)},U_1^{(1,1)},\dots,U_{L_1}^{(1,1)},\dots,U_0^{(n_0,k(n_0))},\dots,U_{L_{n_0}}^{(n_0,k(n_0))}\big). \tag{3-63}$$

In the next lemma we state the linear stability result for the renormalized perturbation $(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$.

Lemma 3.5 (linear stability of special solutions). *Suppose* b *is a solution of the last equation in* (3-58). *Define* $U=(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ *by* (3-62) *and order it as in* (3-63).

(i) Linearized dynamics. *The time evolution of* U *is given by*

$$\partial_s U=\frac{1}{s}AU+O\left(\frac{|U|^2}{s}\right), \tag{3-64}$$

where A *is the block diagonal matrix*

$$A=\begin{pmatrix} A_\ell & & & (0)\\ & \widetilde{A}_1 & & \\ & & \ddots & \\ (0) & & & \widetilde{A}_{n_0}\end{pmatrix}.$$

[11] One notices that this bound holds for $\bar{b}_i^{(n,k)}$.

The matrix A_ℓ is defined by

$$A_\ell=\left(\begin{array}{lcccccccccc}
-(2-\alpha)c_1+\alpha\frac{\ell-1}{2\ell-\alpha} & 1 & & & & & & & & & \\
\vdots & \ddots & \ddots & & & & & & & & \\
-(2i-\alpha)c_i & & \alpha\frac{\ell-i}{2\ell-\alpha} & 1 & & & & (0) & & & \\
\vdots & & & \ddots & \ddots & & & & & & \\
-(2\ell-\alpha)c_\ell & & & & 0 & 1 & & & & & \\
0 & & & & & -\alpha\frac{1}{2\ell-\alpha} & 1 & & & & \\
\vdots & & & & & & \ddots & \ddots & & & \\
0 & & & & & & & -\alpha\frac{i-\ell}{2\ell-\alpha} & 1 & & \\
\vdots & & (0) & & & & & & \ddots & \ddots & \\
\vdots & & & & & & & & & \ddots & 1 \\
0 & & & & & & & & & & -\alpha\frac{(L-\ell)}{2\ell-\alpha}
\end{array}\right). \tag{3-65}$$

The matrix $\widetilde{A}_1$ is a block diagonal matrix constituted of d matrices $\widetilde{A}'_1$:

$$\widetilde{A}_1=\begin{pmatrix}\widetilde{A}'_1 & & (0)\\ & \ddots & \\ (0) & & \widetilde{A}'_1\end{pmatrix},\quad \widetilde{A}'_1=\begin{pmatrix}
\alpha\frac{\ell-\frac{\alpha-1}{2}-1}{2\ell-\alpha} & 1 & & & (0)\\
 & \ddots & \ddots & & \\
 & & \alpha\frac{\ell-\frac{\alpha-1}{2}-i}{2\ell-\alpha} & 1 & \\
 & & & \ddots & \ddots \\
 & (0) & & \ddots & 1\\
 & & & & \alpha\frac{\ell-\frac{\alpha-1}{2}-L_1}{2\ell-\alpha}
\end{pmatrix}. \tag{3-66}$$

For $2\le n\le n_0$ the matrix $\widetilde{A}_n$ is a block diagonal matrix constituted of $k(n)$ times the matrix $\widetilde{A}'_n$:

$$\widetilde{A}_n=\begin{pmatrix}\widetilde{A}'_n & & (0)\\ & \ddots & \\ (0) & & \widetilde{A}'_n\end{pmatrix},\quad \widetilde{A}'_n=\begin{pmatrix}
\alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}}{2\ell-\alpha} & 1 & & & (0)\\
 & \ddots & \ddots & & \\
 & & \alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-i}{2\ell-\alpha} & 1 & \\
 & & & \ddots & \ddots \\
 & (0) & & \ddots & 1\\
 & & & & \alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-L_n}{2\ell-\alpha}
\end{pmatrix}. \tag{3-67}$$

(ii) Diagonalization, stability and instability. *A is diagonalizable because A_ℓ and $\widetilde{A}_n$ for $1\le n\le n_0$ are. A_ℓ is diagonalizable into the matrix*

$$\operatorname{diag}\left(-1,\frac{2\alpha}{2\ell-\alpha},\dots,\frac{i\alpha}{2\ell-\alpha},\dots,\frac{\ell\alpha}{2\ell-\alpha},\frac{-1}{2\ell-\alpha},\dots,\frac{\ell-L}{2\ell-\alpha}\right).$$

We denote the eigenvector of A associated to the eigenvalue -1 by v_1 and the eigenvectors associated to the unstable modes $2\alpha/(\ell-\alpha),\dots,\ell\alpha/(\ell-\alpha)$ of A by $v_2,\dots,v_\ell$. They are a linear combination of the

ℓ first components only. That is to say, there exists a $\#\mathcal{I}\times\#\mathcal{I}$ matrix coding a change of variables:

$$P_\ell := \begin{pmatrix} P'_\ell & 0 \\ 0 & \mathrm{Id}_{\#\mathcal{I}-\ell} \end{pmatrix}, \tag{3-68}$$

with P'_ℓ an invertible $\ell\times\ell$ matrix and $\mathrm{Id}_{\#\mathcal{I}-\ell}$ the $(\#\mathcal{I}-\ell)\times(\#\mathcal{I}-\ell)$ identity matrix such that

$$P_\ell A P_\ell^{-1} = \begin{pmatrix} A'_\ell & & & (0) \\ & \tilde{A}_1 & & \\ & & \ddots & \\ (0) & & & \tilde{A}_{n_0} \end{pmatrix}, \tag{3-69}$$

$$A'_\ell = \begin{pmatrix} -1 & & & & q_1 & & & & \\ & \frac{2\alpha}{2\ell-\alpha} & (0) & & q_2 & & & & \\ & & \ddots & & \vdots & & (0) & & \\ & & & \frac{\ell\alpha}{2\ell-\alpha} & q_\ell & & & & \\ & & & & \frac{-\alpha}{2\ell-\alpha} & 1 & & & \\ & & & & & \ddots & \ddots & & \\ & (0) & & & & & \ddots & 1 & \\ & & & & & & & \alpha\frac{\ell-L}{2\ell-\alpha} \end{pmatrix} \tag{3-70}$$

with $(q_i)_{1\le i\le \ell}\in\mathbb{R}^\ell$ being some fixed coefficients. $\tilde{A}'_1$ has $\max(E[i_1],0)$ nonnegative eigenvalues and $L_1-\max(E[i_1],0)$ strictly negative eigenvalues (i_n being defined by (1-29)). For $2\le n\le n_0$, we know $\tilde{A}'_n$ has $\max(E[i_n]+1,0)$ nonnegative eigenvalues and $L_n+1-\max(E[i_n]+1,0)$ strictly negative eigenvalues.

Proof. (i) As b and $\bar{b}$ are solutions of (3-58), we compute (with the convention $\bar{b}^{(n,k)}_{L_n+1}=0$ and $U^{(n,k)}_{L_n+1}=0$)

$$U^{(n,k)}_{i,s} = \frac{1}{s}\biggl[\biggl(\frac{\gamma-\gamma_n}{2}+i-(2i-\alpha_n)\bar{b}_1^{(0,1)}s\biggr)U_i^{(n,k)} - (2i-\alpha_n)\bar{b}_i^{(n,k)}s^{\frac{\gamma-\gamma_n}{2}+i}U_1^{(0,1)} \\ -(2k-\alpha_n)U_1^{(0,1)}U_i^{(n,k)}+U_{i+1}^{(n,k)}\biggr].$$

As $\bar{b}_1^{(0,1)}=\ell/(2\ell-\alpha)$, we obtain

$$\frac{\gamma-\gamma_n}{2}+i-(2i-\alpha_n)\bar{b}_1^{(0,1)} = \alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-i}{2\ell-\alpha}.$$

We then get (3-65) by noticing that $\bar{b}_i^{(0,1)}=0$ for $i\ge\ell+1$ and because by definition $\gamma=\gamma_0$. We get (3-66) and (3-67) by noticing that $\bar{b}_i^{(n,k)}=0$ for $i\ge1$.

(ii) $\tilde{A}_n$ for $1\le n\le n_0$ is diagonalizable because it is upper triangular. Their eigenvalues are then the values on the diagonal, and the last statement in (ii), about the stability and instability directions comes from the very definition (1-29) of the real number i_n for $1\le n\le n_0$. It remains to prove that A_ℓ is diagonalizable. We will do it by calculating its characteristic polynomial.

Computation of the characteristic polynomial for the top left corner matrix. We let A'_ℓ be the $\ell\times\ell$ matrix

$$A'_\ell=\begin{pmatrix} -(2-\alpha)c_1+\alpha\frac{\ell-1}{2\ell-\alpha} & 1 & & & & \\ \vdots & \ddots & \ddots & & (0) & \\ -(2i-\alpha)c_i & & & \alpha\frac{\ell-i}{2\ell-\alpha} & 1 & \\ \vdots & & & & \ddots & \ddots \\ \vdots & & (0) & & & \ddots\ 1 \\ -(2\ell-\alpha)c_l & & & & & 0 \end{pmatrix}.$$

We recall that as $\alpha>2$, we have $\ell\geq 2$ so A'_ℓ has at least 2 rows and 2 columns. We let

$$\mathcal{P}_\ell(X)=\det(A'_\ell-X\,\mathrm{Id}).$$

We compute this determinant by expanding with respect to the last row and iterating by doing that again for the subdeterminant appearing in the process. Eventually we obtain an expression of the form

$$\mathcal{P}_\ell=(-1)^\ell(2\ell-\alpha)c_\ell+(-X)\Bigg[(-1)^{\ell+1}(2\ell-2-\alpha)c_{\ell-1}+\left(\frac{\alpha}{2\ell-\alpha}-X\right)\\ \Bigg[(-1)^\ell(2\ell-4-\alpha)c_{\ell-2}+\left(\frac{2\alpha}{2\ell-\alpha}-X\right)[\cdots]\Bigg]\Bigg]. \tag{3-71}$$

We define the polynomials $(A_i)_{1\leq i\leq\ell}$ and $(B_i)_{1\leq i\leq\ell}$ and $(C_i)_{1\leq i\leq\ell-1}$ as

$$\begin{aligned} A_i&:=(-1)^{\ell-i+1}(2\ell+2-2i-\alpha)c_{\ell+1-i},\\ B_i&:=(i-1)\frac{\alpha}{2\ell-\alpha}-X,\\ C_i&:=(-1)^{\ell+1-i}\Big(X(2\ell-2i-\alpha)c_{\ell-i}+\frac{2\ell-\alpha}{i}c_{\ell-i+1}\Big). \end{aligned} \tag{3-72}$$

This way, the determinant $\mathcal{P}_\ell$ given by (3-71) can be rewritten as

$$\mathcal{P}_\ell=A_1+B_1\big(A_2+B_2[A_3+B_3[\cdots]]\big). \tag{3-73}$$

We notice by a direct computation from (3-72) that

$$A_1+B_1A_2=C_1.$$

Moreover, this identity propagates by induction and we claim that for $1\leq i\leq\ell-2$,

$$C_i+B_1B_2A_{i+2}=B_{i+2}C_{i+1}. \tag{3-74}$$

Indeed, from (3-60) one has

$$\frac{2\ell-\alpha}{i+1}c_{\ell-i}=-\alpha c_{\ell-i-1},$$

and from (3-72)

$$\begin{aligned}
B_{i+2}C_{i+1}-C_i &= \left((i+1)\frac{\alpha}{2\ell-\alpha}-X\right)(-1)^{\ell-i}\left(X(2\ell-2i-2-\alpha)c_{\ell-i-1}+\frac{2\ell-\alpha}{i+1}c_{\ell-i}\right)\\
&\qquad\qquad -(-1)^{\ell+1-i}\left(X(2\ell-2i-\alpha)c_{\ell-i}+\frac{2\ell-\alpha}{i}c_{\ell-i+1}\right)\\
&= (-1)^{\ell-i}\left(\left((i+1)\frac{\alpha}{2\ell-\alpha}-X\right)\big(X(2\ell-2i-2-\alpha)c_{\ell-i-1}-\alpha c_{\ell-i-1}\big)\right.\\
&\qquad\qquad \left.-X(2\ell-2i-\alpha)\alpha\frac{i+1}{2\ell-\alpha}c_{\ell-i-1}+\alpha^2\frac{i+1}{2\ell-\alpha}c_{\ell-i-1}\right)\\
&= (-1)^{\ell-i}c_{\ell-i-1}X\left(\alpha\frac{i+1}{2\ell-\alpha}(2\ell-2i-2-\alpha)+\alpha-X(2\ell-2i-2-\alpha)\right.\\
&\qquad\qquad \left.-\frac{2\ell-2i-\alpha}{2\ell-\alpha}\alpha(i+1)\right)\\
&= (-1)^{\ell-i}c_{\ell-i-1}X(2\ell-2i-2-\alpha)\left(\frac{\alpha}{2\ell-\alpha}-X\right)\\
&= A_{i+2}B_1B_i.
\end{aligned}$$

From the above identity we can rewrite $\mathcal{P}_\ell$ given by (3-73) as

$$\begin{aligned}
\mathcal{P}_\ell &= A_1+B_1A_2+B_1B_2A_3+B_1B_2B_3(A_4+B_4(\cdots))\\
&= C_1+B_1B_2A_3+B_1B_2B_3(A_4+B_4(\cdots))\\
&= B_3\big(C_2+B_1B_2(A_4+B_4(\cdots))\big) \qquad\qquad = B_3B_4\big(C_3+B_1B_2(A_5+B_5(\cdots))\big)\\
&\;\vdots\\
&= B_3\cdots B_\ell(C_{\ell-1}+B_1B_2).
\end{aligned}\tag{3-75}$$

The last polynomial that appeared is, by (3-72),

$$C_{\ell-1}+B_1B_2=X(2-\alpha)c_1+\frac{2\ell-\alpha}{\ell-1}c_2-X\left(\frac{\alpha}{2\ell-\alpha}-X\right)=(X+1)\left(X-\frac{\alpha\ell}{2\ell-\alpha}\right)$$

and so we end up from (3-75) with the final identity for $\mathcal{P}_\ell$:

$$\mathcal{P}_\ell=(X+1)\prod_{i=2}^{\ell}\left(\frac{i\alpha}{2\ell-\alpha}-X\right).$$

This means that A'_ℓ is diagonalizable with eigenvalues $(1,-2\alpha/(2\ell-\alpha),\dots,\ell/(2\ell-\alpha))$: there exists an invertible $\ell\times\ell$ matrix $\widetilde{P}_\ell$ such that $\widetilde{P}_\ell A_\ell \widetilde{P}_\ell^{-1}=\operatorname{diag}(-1,2/(2\ell-\alpha),\dots,\ell/(2\ell-\alpha))$. We denote by P'_ℓ the matrix

$$P'_\ell:=\begin{pmatrix}\widetilde{P}_\ell & \\ & \mathrm{Id}_{L-\ell}\end{pmatrix}.$$

Then, from (3-65), there exists ℓ real numbers $(q_i)_{1\leq i\leq n}\in\mathbb{R}^\ell$ such that

$$P'_\ell A_\ell (P'_\ell)^{-1} = \begin{pmatrix} -(2-\alpha)c_1+\alpha\frac{\ell-1}{2\ell-\alpha} & 1 & & & & \\ \vdots & & \ddots & \ddots & & (0) \\ -(2i-\alpha)c_i & & & \alpha\frac{\ell-i}{2\ell-\alpha} & 1 & \\ \vdots & & & & \ddots & \ddots \\ \vdots & & (0) & & & \ddots\ 1 \\ -(2\ell-\alpha)c_l & & & & & 0 \end{pmatrix}.$$

This implies that A_ℓ can be diagonalized and that its eigenvalues are of simple multiplicity given by $\big(-1, 2\alpha/(2\ell-\alpha),\dots,\alpha\ell/(2\ell-\alpha), -\alpha/(2\ell-\alpha),\dots,-\alpha L-\ell/(2\ell-\alpha)\big)$, and that the eigenvectors associated to the eigenvalues -1, and $2\alpha/(2\ell-\alpha),\dots,\alpha\ell/(2\ell-\alpha)$ are linear combinations of the ℓ first components only. This concludes the proof of the lemma. □

4. Main proposition and proof of Theorem 1.1

We recall that the approximate blow-up profile $\tau_z(\tilde{Q}_{\bar b,1/\lambda})$ was designed for a blow up on the whole space $\mathbb{R}^d$. In this section, we state in the main proposition of this paper, Proposition 4.6, the existence of solutions staying in a trapped regime (defined in Definition 4.4) close to the cut approximate blow-up profile $\chi\tau_z(\tilde{Q}_{\bar b,1/\lambda})$. We then end the proof of Theorem 1.1 by proving that such a solution will blow up as described in the theorem.

4A. *The trapped regime and the main proposition.*

4A1. *Projection of the solution on the manifold of approximate blow-up profiles.* The following reasoning is made for a blow up on the whole space $\mathbb{R}^d$. As in this case our blow-up solution should stay close to the manifold of approximate blow-up profiles $(\tau_z(\tilde{Q}_{b,\lambda}))_{b,z,\lambda}$, we want to decompose it as a sum $\tau_z(\tilde{Q}_{b,\lambda}+\varepsilon_\lambda)$ for some parameters b,z,λ such that ε has "minimal" size. The tangent space of $(\tau_z(\tilde{Q}_{b,\lambda}))_{b,z,\lambda}$ at the point Q is $\operatorname{Span}(T_i^{(n,k)})_{(n,k,i)\in\mathcal{I}\cup\{(0,1,0),(1,1,0),\dots,(1,d,0)\}}$. One could then think of an orthogonal projection at the linear level, i.e., $\langle T_i^{(n,k)},\varepsilon\rangle=0$. The profiles $T_i^{(n,k)}$ are, however, not decaying quickly enough at infinity so that this duality bracket would make sense in the functional space where ε lies. For these grounds we will approximate such orthogonality conditions by smooth profiles that are compactly supported.

Definition 4.1 (generators of orthogonality conditions). For a very large scale $M\gg1$, for $n\leq n_0$ and $1\leq k\leq k(n)$ we define

$$\Phi_M^{(n,k)}=\sum_{i=0}^{L_n} c_{i,n,M}(-H)^i(\chi_M T_0^{(n,k)})=\sum_{i=0}^{L_n} c_{i,n,M}(-H^{(n)})^i(\chi_M T_0^{(n)})Y^{(n,k)} \tag{4-1}$$

(L_n and $T_0^{(n,k)}$ being defined by (1-28) and (2-26)), where

$$c_{0,n,M}=1 \quad\text{and}\quad c_{i,n,M}=-\frac{\sum_{j=0}^{i-1}c_{j,n,M}\big\langle(-H)^j(\chi_M T_0^{(n,k)}),T_i^{(n,k)}\big\rangle}{\langle\chi_M T_0^{(n)},T_0^{(n)}\rangle}. \tag{4-2}$$

Lemma 4.2 (generation of orthogonality conditions). *For $n \le n_0$, $1 \le k \le k(n)$, $0 \le i \le L_n$, $j \in \mathbb{N}$, $n' \in \mathbb{N}$ and $1 \le k' \le k(n')$, the following holds for $c > 0$:*

$$\langle (-H)^j \Phi_M^{(n,k)}, T_i^{(n',k')} \rangle = \delta_{(n,k,i),(n',k',j)} \int_0^{+\infty} \chi_M |T_0^{(n)}|^2 r^{d-1} \sim c M^{4m_n+4\delta_n} \delta_{(n,k,i),(n',k',j)}. \tag{4-3}$$

Proof. The scalar product is zero if $(n,k) \neq (n',k')$ because by construction $\Phi_M^{(n,k)}$ (resp. $T_i^{(n',k')}$) lives on the spherical harmonic $Y^{(n,k)}$ (resp. $Y^{(n',k')}$). We now suppose $(n,k) = (n',k')$ and compute using (4-1):

$$\big\langle (-H)^j \Phi_M^{(n,k)}, T_i^{(n,k)} \big\rangle = \sum_{l=0}^{L_n} c_{l,n,M} \big\langle T_0^{(n)} \chi_M, (-H^{(n)})^{l+j} T_i^{(n)} \big\rangle.$$

If $j > i$ for all l, then $(H^{(n)})^{l+j} T_i^{(n)} = 0$ and $\big\langle (-H)^j \Phi_M^{(n,k)}, T_i^{(n,k)} \big\rangle = 0$. If $j = i$ then only the first term in the sum is not zero since $(-H^{(n)})^i T_i^{(n)} = T_0^{(n,k)}$ and

$$\sum_{l=0}^{L_n} c_{l,n,M} \big\langle T_0^{(n)} \chi_M, (-H^{(n)})^{l+j} T_i^{(n)} \big\rangle = \langle T_0^{(n)} \chi_M, T_0^{(n)} \rangle \sim c M^{4m_n+4\delta_n}$$

from the asymptotic behavior (2-7) of $T_0^{(n)}$. If $j < i$ then

$$\sum_{l=0}^{L_n} c_{l,n,M} \big\langle T_0^{(n)} \chi_M, (-H^{(n)})^{l+j} T_i^{n} \big\rangle = c_{i-j,n,M} \langle T_0^{(n)} \chi_M, T_0^{(n)} \rangle + \sum_{l=0}^{i-j-1} c_{l,n,M} \big\langle T_0^{(n)} \chi_M, (-H^{(n)})^{l+j} T_i^{(n)} \big\rangle = 0$$

from the definition (4-2) of the constant $c_{i-j,n,M}$. □

4A2. *Geometrical decomposition.* First we describe here how we decompose a solution of (1-1) on the unit ball $\mathcal{B}^d(1)$ onto the set $(\tau_z(\widetilde{Q}_{b,\lambda}))_{b,|z| \le \frac{1}{8}, 0<\lambda<\frac{1}{8M}}$ of concentrated ground states, using the orthogonality conditions provided by Lemma 4.2. This provides a decomposition for any domain containing $\mathcal{B}^d(1)$. Let $0 < \kappa \ll 1$ to be fixed later on. We study the set of functions close to $(\tau_z(\widetilde{Q}_{b,\lambda}))_{b,|z| \le \frac{1}{8}, 0<\lambda<\frac{1}{8M}}$ such that the projection onto the first element in the generalized kernel dominates:[12]

$$u : \exists (\tilde{\lambda}, \tilde{z}) \in \big(0, \tfrac{1}{8M}\big) \times \mathcal{B}^d\big(\tfrac{1}{8}\big) \text{ such that}$$
$$\|u - Q_{\tilde{z}, \frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} < \frac{\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}} \quad \text{and} \quad \|(\tau_{-\tilde{z}} u)_{\tilde{\lambda}} - Q\|_{L^\infty(\mathcal{B}^d(3M))} < \big\langle (\tau_{-\tilde{z}} u)_{\tilde{\lambda}} - Q, H \Phi_M^{(0,1)} \big\rangle. \tag{4-4}$$

Lemma 4.3 (decomposition). *There exist $\kappa, K > 0$ such that for any solution $u \in \mathcal{C}^1([0,T), \times \mathcal{B}^d(1))$ of* (1-1) *satisfying* (4-4) *for all $t \in [0,T)$, there exists a unique choice of the parameters $\lambda : [0,T) \to \big(0, \frac{1}{4M}\big)$,*

[12]Note that $(\tau_{-\tilde{z}} u)_{\tilde{\lambda}}$ is defined on $\frac{1}{\tilde{\lambda}}(\mathcal{B}^d(1) - \tilde{z})$, which contains $\mathcal{B}^d(7M)$ as $|\tilde{z}| < \frac{1}{8}$ and $0 < |\tilde{\lambda}| < \frac{1}{8M}$; thus the second estimate makes sense.

$z:[0,T)\to\mathcal{B}^d\left(\frac{1}{4}\right)$ and $b:[0,T)\to\mathbb{R}^{\mathcal{I}}$ such that $b_1^{(0,1)}>0$ and

$$u=(\tilde{Q}_b+v)_{z,\lambda}\quad\text{on } \mathcal{B}^d(1),\qquad \sum_{(n,k,i)\in\mathcal{I}}|b_i^{(n,k)}|+\|v\|_{L^\infty\left(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\})\right)}\le K\kappa$$

with $v=(\tau_{-z}u)_\lambda-\tilde{Q}_b$ satisfying the orthogonality conditions

$$\langle v,H^i\Phi_M^{(n,k)}\rangle=0\quad\text{for } 0\le n\le n_0,\ 1\le k\le k(n),\ 0\le i\le L_n.$$

Moreover, λ, b and z are $\mathcal{C}^1$ functions.

Proof. It is a direct consequence of Lemma E.2 from the appendix. $\square$

Decomposition and adapted norms for the remainder inside a bounded domain. Let u be a solution of (NLH) in $C^1([0,T),\Omega)$ with Dirichlet boundary condition such that the restriction[13] of u to $\mathcal{B}^d(1)$ satisfies the conditions of Lemma 4.3. Then from this lemma, for all $t\in[0,T)$ we can decompose u according to

$$u:=\chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})+w, \tag{4-5}$$

cutting the approximate blow-up profile in the zone $1\le|x|\le 2$, and w is a remainder term satisfying $w_{|\partial\Omega}=0$ as $\mathcal{B}^d(7)\subset\Omega$ and $u_{|\partial\Omega}=0$. To study w inside and outside the blow-up zone, we decompose it according to

$$w_{\text{int}}:=\chi_3 w,\quad w_{\text{ext}}:=(1-\chi_3)w,\quad \varepsilon:=(\tau_{-z(t)}w_{\text{int}})_{\lambda(t)}, \tag{4-6}$$

where w_{int} and w_{ext} are the remainder cut in the zone $3\le|x|\le 6$, ε is the renormalized remainder at the blow-up area, and is adapted to the renormalized flow. We notice that the support of w_{ext} does not intersect the support of the approximate blow-up profile $\chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})$, that the supports of w_{int} and w_{ext} overlap, and that $(w_{\text{ext}})_{|\partial\Omega}=0$. From Lemma 4.3 and its definition, ε is compactly supported and satisfies the orthogonality conditions (4-11). We measure ε through the following norms:

(i) *High-order Sobolev norm adapted to the linearized flow.* We define

$$\mathcal{E}_{2s_L}:=\int_{\mathbb{R}^d}|H^{s_L}\varepsilon|^2. \tag{4-7}$$

This norm controls the L^2 norms of all smaller-order derivatives with appropriate weight from Lemma C.3 since ε satisfies the orthogonality conditions (4-11), and the standard $\dot{H}^{2s_L}$ Sobolev norm

$$\mathcal{E}_{2s_L}\ge C\sum_{|\mu|\le 2s_L}\int_{\mathbb{R}^d}\frac{|\partial^\mu\varepsilon|^2}{1+|x|^{4i-2\mu+}}+C\|\varepsilon\|^2_{\dot{H}^{2s_L}}.$$

(ii) *Low-order slightly supercritical Sobolev norm.* Let σ be a slightly supercritical regularity:

$$0<\sigma-s_c\ll 1. \tag{4-8}$$

[13] We recall that Ω contains $\mathcal{B}^d(7)$.

We then define the following second norm for the remainder:

$$\mathcal{E}_\sigma := \|\varepsilon\|^2_{\dot{H}^\sigma}. \tag{4-9}$$

Existence of a solution staying in a trapped regime close to the approximate blow-up solution. From now on we focus on solutions that are close to an approximate blow-up profile in the sense of the following definition.

Definition 4.4 (solutions in the trapped regime). We say that a solution u of (1-1) in $C^1([0,T),\Omega)$ is trapped on $[0,T)$ if it satisfies all of the following. First, it satisfies the condition (4-4) and then can be decomposed via Lemma 4.3 according to (4-5) and (4-6):

$$u := \chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}}) + w,\ w_{\text{int}} := \chi_3 w, \quad w_{\text{ext}} := (1-\chi_3)w, \quad \varepsilon := (\tau_{-z(t)}w_{\text{int}})_{\lambda(t)} \tag{4-10}$$

with ε satisfying the orthogonality conditions

$$\langle \varepsilon, H^i\Phi_M^{(n,k)}\rangle = 0 \quad \text{for } 0\le n\le n_0,\ 1\le k\le k(n),\ 0\le i\le L_n. \tag{4-11}$$

To the scale λ given by this decomposition, we associate the renormalized time s defined by (3-1) with $s_0>0$. The $\#\mathcal{I}$-tuple of parameters b is represented as a perturbation of the solution $\bar{b}$ of the dynamical system (3-58) given by (3-59):

$$b_i^{(n,k)}(s) = \bar{b}_i^{(n,k)}(s) + \frac{U_i^{(n,k)}(s)}{s^{\frac{\gamma-\gamma_n}{2}+i}}. \tag{4-12}$$

We let $U := (U_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$. To use the eigenvectors of the linearized dynamics, Lemma 3.5, we define

$$V_i := (P_\ell U)_i \quad \text{for } 1\le i\le \ell, \tag{4-13}$$

where P_ℓ is defined by (3-68). All these parameters must satisfy the following estimates, where $0<\tilde{\eta}\ll 1$, $0<\varepsilon_i^{(n,k)}\ll 1$ for $(n,k,i)\in\mathcal{I}$ with $(n,k,i)\notin\{1,\dots,\ell\}\times\{0\}\times\{1\}$; K_1 and K_2 will be fixed later on.

Initial conditions. At time $t=0$ (or equivalently $s=s_0$):

(i) Control of the unstable modes on the radial component:

$$|V_i(0)| \le s_0^{-\tilde{\eta}} \quad \text{for } 2\le i\le \ell. \tag{4-14}$$

(ii) Control of the unstable modes on the other spherical harmonics:

$$|(U_i^{(n,k)}(0))| \le \varepsilon_i^{(n,k)} \quad \text{for } (n,k,i)\in\mathcal{I} \text{ with } 1\le n \text{ and } 0\le i< i_n. \tag{4-15}$$

(iii) Control of the stable modes:

$$V_1(0) \le \frac{1}{10 s_0^{\tilde{\eta}}}, \qquad |U_i^{(0,1)}(0)| \le \frac{\varepsilon_i^{(0,1)}}{10 s_0^{\tilde{\eta}}} \quad \text{for } \ell+1\le i\le L, \tag{4-16}$$

$$|U_i^{(n,k)}(0)| \le \frac{\varepsilon_i^{(n,k)}}{10 s_0^{\tilde{\eta}}} \quad \text{for } (n,k,i)\in\mathcal{I} \text{ with } 1\le n \text{ and } i_n< i\le L_n, \tag{4-17}$$

$$|U_i^{(n,k)}(0)| \le \frac{\varepsilon_i^{(n,k)}}{10} \quad \text{for } (n,k,i)\in\mathcal{I} \text{ with } 1\le n \text{ and } i = i_n. \tag{4-18}$$

(iv) Smallness of the remainder:

$$\|w\|_{H^{2s_L}}^2 < \frac{1}{s_0^{\frac{2\ell}{2\ell-\alpha}(2s_L-s_c)}}. \tag{4-19}$$

(v) Compatibility conditions at the border:[14]

$$\begin{cases} \tilde{w}_0 := w(0) \in H_0^1(\Omega), \\ \tilde{w}_1 := \partial_t w(0) = \Delta w(0) + w(0)^p \in H_0^1(\Omega), \\ \tilde{w}_2 := \partial_t^2 w(0) = \Delta^2 w(0) + \Delta(w(0)^p) + p w(0)^{p-1}(\Delta w(0) + w(0)^p) \in H_0^1(\Omega), \\ \qquad \vdots \\ \tilde{w}_{s_L-1} := \partial_t^{s_L-1} w(0) \in H_0^1(\Omega). \end{cases} \tag{4-20}$$

(vi) Initial scale and initial blow-up point:

$$\lambda(0) = s_0^{-\frac{\ell}{2\ell-\alpha}} \quad \text{and} \quad z(0) = 0. \tag{4-21}$$

Pointwise in time estimates. The following bounds hold on $(0, T)$:

(i) Parameters on the first spherical harmonics:

$$|V_i(s)| \le s^{-\tilde{\eta}} \quad \text{for } 1 \le i \le \ell, \qquad |U_i^{(0,1)}(s)| \le \varepsilon_i^{(0,1)} s^{-\tilde{\eta}} \quad \text{for } \ell+1 \le i \le L. \tag{4-22}$$

(ii) Parameters on the other spherical harmonics: for $(n, k, i) \in \mathcal{I}$ with $n \ge 1$,

$$|(U_i^{(n,k)}(s))| \le 1 \quad \text{if } 0 \le i < i_n, \tag{4-23}$$

$$|U_i^{(n,k)}(s)| \le \frac{\varepsilon_i^{(n,k)}}{s^{\tilde{\eta}}} \quad \text{if } i_n < i \le L_n \qquad \text{and} \qquad |U_i^{(n,k)}(s)| \le \varepsilon_i^{(n,k)} \quad \text{if } i = i_n. \tag{4-24}$$

(iii) Control of the remainder:

$$\begin{aligned} \mathcal{E}_{s_L}(s) \le \frac{K_2}{s^{2L+2(1-\delta_0)+2(1-\delta_0')\eta}}, \quad \mathcal{E}_\sigma(s) \le \frac{K_1}{s^{2(\sigma-s_c)\frac{\ell}{2\ell-\alpha}}}, \\ \|w_{\text{ext}}\|_{H^{2s_L}}^2 \le \frac{K_2}{\lambda^{2(2s_L-s_c)} s^{2L+2(1-\delta_0)+2(1-\delta_0')\eta}}, \quad \|w_{\text{ext}}\|_{H^\sigma}^2 \le K_1. \end{aligned} \tag{4-25}$$

(iv) Estimates on the scale and the blow-up point:

$$\lambda \le 2 s^{-\frac{\ell}{2\ell-\alpha}} \quad \text{and} \quad |z| \le \tfrac{1}{10}. \tag{4-26}$$

Remark 4.5. For a trapped solution one has the above estimates on the parameters from (3-59), (4-12), (4-13), (4-22), (4-23) and (4-24),

$$|b_i^{(n,k)}| \le \frac{C}{s^{\frac{\gamma-\gamma_n}{2}+i}}, \quad b_1^{(0,1)} = \frac{\ell}{2\ell-\alpha}\frac{1}{s} + O(s^{-1-\tilde{\eta}}) \tag{4-27}$$

[14] We make an abuse of notations here. The identities given for the time derivatives of w are only true close to the border of Ω, but which is enough as the required conditions are trace-type conditions; see [Evans 2010].

for C independent of the other constants. The bounds (4-25) on the remainders for the solution described by Proposition 4.6, because of the coercivity estimate Lemma C.3 implies that

$$\|w\|_{H^\sigma(\Omega)} \le CK_1, \quad \|w\|_{H^{2s_L}(\Omega)} \le \frac{C(K_1,K_2,M)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}}. \tag{4-28}$$

A trapped solution must first satisfy the condition (4-4) in order to apply the decomposition in Lemma E.1, and then the variables of this decomposition must satisfy suitable bounds. However, these additional bounds in turn provide a much stronger estimate than (4-4). Indeed, one has, from (4-10), (3-29), (3-7), (4-27), (D-2),

$$\begin{aligned}
\inf_{(\tilde\lambda,\tilde z)\in(0,\frac{1}{8M})\times\mathcal{B}^d(\frac18)} & \tilde\lambda^{\frac{2}{p-1}}\|u-Q_{\tilde z,\frac{1}{\tilde\lambda}}\|_{L^\infty(\mathcal{B}^d(1))} \\
& \le \lambda^{\frac{2}{p-1}}\|u-Q_{z,\frac{1}{\lambda}}\|_{L^\infty(\mathcal{B}^d(1))} \\
& = \|\tilde Q_b+\varepsilon-Q\|_{L^\infty(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\}))} = \|\chi_{B_1}\alpha_b+\varepsilon\|_{L^\infty(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\}))} \\
& \le \|\chi_{B_1}\alpha_b\|_{L^\infty(\mathbb{R}^d)}+\|\varepsilon\|_{L^\infty(\mathbb{R}^d)} \le \frac{C}{s}+\frac{C}{s^{\frac d4-\frac\sigma2}} \ll \kappa,
\end{aligned}$$

$$\|(\tau_{-z})u_\lambda-Q\|_{L^\infty(\mathcal{B}^d(3M))} \le \|\alpha_b\|_{L^\infty(\mathcal{B}^d(3M))}+\|\varepsilon\|_{L^\infty(\mathcal{B}^d(3M))} \le \frac{C}{s}+\frac{C}{s^2}. \tag{4-29}$$

Using (4-10), (4-11), (3-29), (3-7), (4-27), (4-3) and (2-7) one gets

$$\begin{aligned}
\langle (\tau_{-z})u_\lambda - Q, H\Phi_M^{(0,1)}\rangle &= \langle \alpha_b, H\Phi_M^{(0,1)}\rangle \\
&= b_1^{(0,1)}\langle T_0^{(0,1)}, \chi_M T_0^{(0,1)}\rangle + O(s^{-2}) \sim \frac{c}{s} = \frac{c_1}{s} c M^{d-2\gamma} + O(s^{-2})
\end{aligned}$$

for some $c>0$, which, combined with the above estimate gives

$$\|(\tau_{-z})u_\lambda - Q\|_{L^\infty(\mathcal{B}^d(3M))} \ll \langle (\tau_{-z})u_\lambda - Q, H\Phi_M^{(0,1)}\rangle$$

for M large enough as $d-2\gamma>0$. Therefore, a solution cannot exit the trapped regime because the condition (4-4) fails: the estimates on the parameters and the remainder have to be violated first. We thus forget about this condition in the following.

The key result of this paper is the existence of solutions that are trapped on their whole lifespan.

Proposition 4.6 (existence of fully trapped solutions). *There exists a choice of universal constants for the analysis*[15]

$$\begin{gathered}
L = L(\ell,d,p) \gg 1, \quad 0<\eta=\eta(d,p,L)\ll 1, \quad M=M(d,p,L)\gg 1, \\
\sigma=\sigma(L,d,p), \quad K_1=K_1(d,p,L)\gg 1, \quad K_2=K_2(d,p,L)\gg 1, \\
0<\varepsilon_i^{(0,1)}=\varepsilon_i^{(0,1)}(L,d)\ll 1 \quad \textit{for } \ell+1\le i\le L,\ 0<\varepsilon_1=\varepsilon_1(L,d)\ll 1, \\
0<\varepsilon_i^{(n,k)}=\varepsilon_i^{(n,k)}(L,d)\ll 1 \quad \textit{for } (n,k,i)\in\mathcal{I} \textit{ with } 1\le n,\ i_n+1\le i\le L_n, \\
0<\tilde\eta=\tilde\eta(\ell,L,d,p,\eta)\ll 1 \quad \textit{and} \quad s_0=s_0(\ell,d,p,L,M,K_1,K_2,\varepsilon_i^{(n,k)},\tilde\eta)\gg 1
\end{gathered} \tag{4-30}$$

[15]The interdependence of the constants is written here so that the reader knows, for example, that s_0 is chosen after all the other constants.

such that the following fact holds close to $\chi\tilde{Q}_{\bar{b}(s_0),1/\lambda(s_0)}$, *where* $\bar{b}$ *is given by* (3-59) *and* $\lambda(s_0)$ *satisfies* (4-21). *Given a perturbation along the stable directions, represented by* $w(s_0)$, *decomposed in* (4-5), *satisfying* (4-19) *and* (4-11), *and* $V_1(s_0)$, $(U_{\ell+1}^{(0,1)}(s_0),\dots,U_L^{(0,1)}(s_0))$, $(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},\, n\geq 1,\, i_n\leq i}$ *satisfying* (4-16), (4-17) *and* ((iii)), *there exists a correction along the unstable directions represented by* $(V_2(s_0),\dots,V_\ell(s_0))$ *and* $(\tilde{U}_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\leq n,\, i<i_n}$ *satisfying* (4-14) *and* (4-15) *such that the solution* $u(t)$ *of* (1-1) *with initial datum* $u(0)=\chi\tilde{Q}_{b(s_0),1/\lambda(s_0)}+w(s_0)$ *with*

$$b(s_0)=\left(\bar{b}_i^{(n,k)}+\frac{U_i^{(n,k)}(s_0)}{s_0^{\frac{\gamma-\gamma_n}{2}+i}}\right)_{(n,k,i)\in\mathcal{I}} \tag{4-31}$$

is trapped until its maximal time of existence in the sense of Definition 4.4.

Proof. The proof is relegated to Section 5. □

4B. *End of the proof of Theorem 1.1 using Proposition 4.6.* In this subsection we end the proof of the main theorem, Theorem 1.1, by proving that the solutions given by Proposition 4.6 lead to a finite-time blow up with the properties described in Theorem 1.1. The proof of Theorem 1.1 is a direct consequence of Proposition 4.6 and Lemmas 4.8, 4.9 and 4.10. Until the end of this subsection, u will denote a solution that is trapped in the sense of Definition 4.4 on its maximal interval of existence. First, we describe the time evolution equation for ε. It then allows us to compute how the time evolution law for the parameters λ and z related to the decomposition (4-5) depends on the other parameters. The bounds on the parameters and the remainder for a trapped solution then imply that λ goes to zero with explicit asymptotic in finite time, that z converges, and that the solution undergoes blow up by concentration with a control on the asymptotic behavior for Sobolev norms.

4B1. *Time evolution for the error.* Let u be a trapped solution. From the decomposition (4-5) we compute that the time evolution of the remainder is

$$\begin{aligned} w_t=-\frac{1}{\lambda^2}\chi\tau_z(\widetilde{\mathrm{Mod}}(t)_{\frac{1}{\lambda}}+\tilde{\psi}_{b,\frac{1}{\lambda}})+\Delta w+\sum_{k=1}^{p}C_k^p(\chi\tau_z\tilde{Q}_{b,\frac{1}{\lambda}})^{p-k}w^k \\ +\Delta\chi\tau_z Q_{\frac{1}{\lambda}}+2\nabla\chi.\nabla\tau_z Q_{\frac{1}{\lambda}}+\chi\tau_z Q_{\frac{1}{\lambda}}^p(\chi^{p-1}-1) \end{aligned} \tag{4-32}$$

with the new modulation term being defined as

$$\widetilde{\mathrm{Mod}}(t):=\chi_{B_1}\mathrm{Mod}(t)-\left(\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right)\Lambda\tilde{Q}_b-\left(\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right).\nabla\tilde{Q}_b. \tag{4-33}$$

From (4-32) and (4-6), as the support of w_{ext} is outside $\mathcal{B}^d(2)$ and as $\tau_z(\tilde{Q}_{b,\lambda})$ is cut in the zone $1\leq|x|\leq 2$, the time evolution of w_{ext} is

$$\partial_t w_{\text{ext}}=\Delta w_{\text{ext}}+\Delta\chi_3 w+2\nabla\chi_3.\nabla w+(1-\chi_3)w^p.$$

The excitation of the solitary wave $\tau_z(\tilde{\alpha}_{b,1/\lambda})$ has support in the zone $|x-z|\leq 2\lambda B_1$ and from (4-26), $|z|+\lambda B_1\ll 1$, so it does not see the cut by χ of the approximate blow-up profile. From this, (4-32) and

(4-6), the time evolution of w_{int} is therefore given by

$$\partial_t w_{\text{int}} + H_{z,\frac{1}{\lambda}} w_{\text{int}} = -\frac{1}{\lambda^2}\chi\tau_z(\widetilde{\text{Mod}(t)}_{\frac{1}{\lambda}} + \tilde{\psi}_{b,\frac{1}{\lambda}}) + L(w_{\text{int}}) + NL(w_{\text{int}}) + \widetilde{L} + \widetilde{\text{NL}} + \widetilde{R}, \tag{4-34}$$

where $H_{z,1/\lambda}$, $\text{NL}(w_{\text{int}})$, $L(w_{\text{int}})$ are the linearized operator, the nonlinear term and the small linear term resulting from the interaction between w_{int} and a noncut approximate blow-up profile $\tau_z(\widetilde{Q}_{b,\frac{1}{\lambda}})$:

$$H_{z,\frac{1}{\lambda}} := -\Delta - p(\tau_z(\widetilde{Q}_{\frac{1}{\lambda}}))^{p-1}, \quad H_{b,z,\frac{1}{\lambda}} := -\Delta - p(\tau_z(\widetilde{Q}_{b,\frac{1}{\lambda}}))^{p-1} \tag{4-35}$$

$$\text{NL}(w_{\text{int}}) := F(\tau_z(\widetilde{Q}_{b,\frac{1}{\lambda}}) + w_{\text{int}}) - F(\tau_z(\widetilde{Q}_{b,\frac{1}{\lambda}})) + H_{b,\frac{1}{\lambda}}(w_{\text{int}}), \tag{4-36}$$

$$L(w_{\text{int}}) := H_{z,\frac{1}{\lambda}} w_{\text{int}} - H_{b,z,\frac{1}{\lambda}} w_{\text{int}} = \frac{p}{\lambda^2}\tau_z(\chi_{B_1}^{p-1}\alpha_b^{p-1})_{\frac{1}{\lambda}}. \tag{4-37}$$

The last terms in (4-34) are the corrective terms induced by the cut of the approximate blow-up profile and the cut of the error term:[16]

$$\widetilde{L} := -\Delta\chi_3 w - 2\nabla\chi_3.\nabla w + p\tau_z Q_{\frac{1}{\lambda}}^{p-1}(\chi^{p-1} - \chi_3)w, \tag{4-38}$$

$$\widetilde{\text{NL}} := \sum_{k=2}^{p} C_k^p \tau_z Q_{\frac{1}{\lambda}}^{p-k}(\chi^{p-k} - \chi_3^{k-1})\chi_3 w^k, \tag{4-39}$$

$$\widetilde{R} := \Delta\chi\tau_z Q_{\frac{1}{\lambda}} + 2\nabla\chi\nabla\tau_z Q_{\frac{1}{\lambda}} + \chi\tau_z Q_{\frac{1}{\lambda}}^p(\chi^{p-1} - 1), \tag{4-40}$$

and one notices that their support is in the zone $1 \le |x| \le 6$. Using the definition of the renormalized flow (3-2) and the decomposition (4-5) we compute, using (4-32),

$$\partial_s\varepsilon - \frac{\lambda_s}{\lambda}\Lambda\varepsilon - \frac{z_s}{\lambda}.\nabla\varepsilon + H\varepsilon = -\chi(\lambda y + z)(\tilde{\text{Mod}}(s) + \tilde{\psi}_b) + \text{NL}(\varepsilon) + L(\varepsilon) + \lambda^2[\tau_{-z}(\widetilde{L} + \widetilde{R} + \widetilde{\text{NL}})]_\lambda, \tag{4-41}$$

with the purely nonlinear term and the small linear term in adapted renormalized variables being defined as

$$\text{NL}(\varepsilon) := F(\widetilde{Q}_b + \varepsilon) - F(\widetilde{Q}_b) + H_b(\varepsilon), \quad L(\varepsilon) := H\varepsilon - H_b\varepsilon, \tag{4-42}$$

where $H_b := -\Delta - p\widetilde{Q}_b^{p-1}$ is the linearized operator near $\widetilde{Q}_b$. One notices that the extra terms induced by the cut, $\lambda^2[\tau_{-z}(\widetilde{L} + \widetilde{R} + \widetilde{\text{NL}})]_\lambda$, have support in the zone $\frac{1}{2\lambda} \le |y| \le \frac{7}{\lambda}$ (by(4-26)).

4B2. *Modulation equations.* We now quantify how the evolution of one parameter $b_i^{(n,k)}$, λ or z depends on all the parameters $(b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ and the remainder ε.

Lemma 4.7 (modulation). *Let all the constants of the analysis described in Proposition 4.6 be fixed except s_0. Then for s_0 large enough, for any solution u that is trapped on $[s_0, s')$ in the sense of Definition 4.4 the following holds for $s_0 \le s < s'$:*

$$\left|\frac{\lambda_s}{\lambda} + b_1^{(0,1)}\right| + \left|\frac{z_s}{\lambda} + b_1^{(1,\cdot)}\right| + \sum_{(n,k,i)\in\mathcal{I}, i\neq L_n} \left|b_{i,s}^{(n,k)} + (2i - \alpha_n)b_1^{(0,1)}b_i^{(n,k)} + b_{i+1}^{(n,k)}\right|$$
$$\le \frac{C(L,M)}{s^{L+3}} + \frac{C(L,M)}{s}\sqrt{\mathcal{E}_{2s_L}}, \tag{4-43}$$

[16] Again, the excitation of the solitary wave $\tau_z(\tilde{\alpha}_{b,1/\lambda})$ is not present here as its support is in the zone $|x| \ll 1$; see (4-26).

$$\sum_{(n,k,i)\in\mathcal{I},\, i=L_n} \left|b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}\right| \le \frac{C(M,L)}{s^{L+3}}+C(M,L)\sqrt{\mathcal{E}_{2s_L}}. \tag{4-44}$$

Proof. We let

$$D(s)=\left|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right|+\left|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right|+\sum_{(n,k,i)\in\mathcal{I}}\left|b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}\right| \tag{4-45}$$

with the convention $b_{L_n+1}^{(n,k)}=0$. Taking the scalar product of (4-41) with $(-H)^i\,\Phi_M^{(n,k)}$, using (4-3), gives[17]

$$\begin{aligned}\langle\widetilde{\mathrm{Mod}}(s),(-H)^i\,\Phi_M^{(n,k)}\rangle &= \langle -H\varepsilon,(-H)^i\,\Phi_M^{(n,k)}\rangle-\langle\tilde{\psi}_b,(-H)^i\,\Phi_M^{(n,k)}\rangle\\ &\quad+\left\langle\frac{\lambda_s}{\lambda}\Lambda\varepsilon+\frac{z_s}{\lambda}.\nabla\varepsilon+\mathrm{NL}(\varepsilon)+L(\varepsilon),(-H)^i\,\Phi_M^{(n,k)}\right\rangle.\end{aligned} \tag{4-46}$$

Now we look closely at each one of the terms of this identity.

The modulation term. From the expression (3-29) of $\tilde{Q}_b$, the bound (3-11) on $\partial S_j/\partial b_i^{(n,k)}$, and the bounds (4-27) on the parameters, one has

$$\tilde{Q}_b=Q+\chi_{B_1}\alpha_b=Q+O(s^{-1}) \qquad \text{and} \qquad \frac{\partial S_j}{\partial b_i^{(n,k)}}=O(s^{-1}) \quad \text{on } \mathcal{B}^d(0,2M).$$

From (3-10), (4-33) and (4-45), the modulation term can then be rewritten as

$$\begin{aligned}\mathrm{Mod}(s)&=\chi_{B_1}\sum_{(n,k,i)\in\mathcal{I}}\left[b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}\right]\left[T_i^{(n,k)}+\sum_{j=i+1+\delta_{n\ge2}}^{L+2}\frac{\partial S_j}{\partial b_i^{(n,k)}}\right]\\ &\qquad-\left(\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right)\Lambda\tilde{Q}_b-\left(\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right).\nabla\tilde{Q}_b\\ &=\chi_{B_1}\sum_{(n,k,i)\in\mathcal{I}}\left[b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}\right]T_i^{(n,k)}\\ &\qquad-\left(\frac{\lambda_s}{a}+b_1^{(0,1)}\right)\Lambda Q-\left(\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right).\nabla Q+O\left(\frac{|D(s)|}{s}\right),\end{aligned}$$

where the $O(|D(s)|/s)$ is valid in the zone $|y|\le 2M$. From the orthogonality relations (4-3), we then get

$$\begin{aligned}&\langle\widetilde{\mathrm{Mod}}(s),(-H)^i\,\Phi_M^{(n,k)}\rangle+O\left(\frac{|D(s)|}{s}\right)\\ &=\begin{cases}-C\,\langle\chi_M\Lambda Q,\Lambda Q\rangle\left(\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right) & \text{for } (n,k,i)=(0,1,0),\\ -C'\langle\chi_M\nabla Q,\nabla Q\rangle\left(\frac{z_{j,s}}{\lambda}+b_1^{(1,k)}\right) & \text{for } (n,i)=(1,0),\ 1\le k\le d,\\ \langle\chi_M T_0^{(n,k)},T_0^{(n,k)}\rangle\left(b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}\right) & \text{otherwise,}\end{cases}\end{aligned} \tag{4-47}$$

where C and C' are two positive renormalization constants.

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[17] We do not see the extra terms $\tilde{L}$, $\tilde{R}$ and $\widetilde{\mathrm{NL}}$ because their support is in the zone $\frac{1}{2\lambda}\le|y|$ (from (4-26)) which is very far away from the support of $\Phi_M^{(n,k)}$, in the zone $|y|\le 2M$ (s_0 being chosen large enough so that this statement holds).

The main linear term. The coercivity estimate (C-16) and the Hölder inequality imply

$$\int_{|y|\leq 2M} |\varepsilon|\, dy \lesssim C(M)\sqrt{\mathcal{E}_{2s_L}}.$$

Hence, from the orthogonality (4-11) for ε, we obtain, for $0 \leq n \leq n_0$, $1 \leq k \leq k(n)$,

$$\big|\langle H\varepsilon, H^i \Phi_M^{(n,k)}\rangle\big| = \begin{cases} 0 & \text{for } i < L_n, \\ \big|\langle \varepsilon, (-H)^{i+1}\Phi_M^{(n,k)}\rangle\big| = O(\sqrt{\mathcal{E}_{2s_L}}) & \text{for } i = L_n. \end{cases} \tag{4-48}$$

The error term. Using the local bound (3-35) for $\tilde{\psi}_b$ and (4-27),

$$\big|\langle \tilde{\psi}_b, H^i \Phi_M^{(n,k)}\rangle\big| \leq \frac{C(L,M)}{s^{L+3}}. \tag{4-49}$$

The extra terms. From (4-27), the coercivity estimate (C-16), the bound (4-25) on $\mathcal{E}_{2s_L}$ and (4-45), one obtains

$$\left|\left\langle \frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon, H^i\Phi_M^{(n,k)}\right\rangle\right| \leq \frac{C(L,M)}{s}\sqrt{\mathcal{E}_{2s_L}} + \frac{|D(s)|}{s^{L+1-\delta_0+\eta(1-\delta_0')}}.$$

Now, as $Q^{p-1} - \tilde{Q}_b^{p-1} = O(s^{-1})$ on the set $|y| \leq 2M$ from (3-7) and (4-27), using the estimate (D-2) on $\|\varepsilon\|_{L^\infty}$, from the definition (4-42) of $NL(\varepsilon)$ and $L(\varepsilon)$ and the coercivity (C-16), one gets, for s_0 large enough,

$$\big|\langle \mathrm{NL}(\varepsilon) + L(\varepsilon), H^i\Phi_M^{(n,k)}\rangle\big| \leq C(L,M)\mathcal{E}_{2s_L} + C(L,M)\frac{\sqrt{\mathcal{E}_{2s_L}}}{s} \leq C(L,M)\frac{\sqrt{\mathcal{E}_{2s_L}}}{s}.$$

Putting together the last two estimates yields

$$\left|\left\langle \frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon + \mathrm{NL}(\varepsilon) + L(\varepsilon), H^i\Phi_M^{(n,k)}\right\rangle\right| \leq \frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s} + \frac{C(L,M)|D(s)|}{s^{L+1-\delta_0+\eta(1-\delta_0')}}. \tag{4-50}$$

Final bound on $|D(s)|$. Summing the previous estimates we performed on each term of (4-46) in (4-47)–(4-50) yields

$$|D(s)| \leq C(L,M)\sqrt{\mathcal{E}_{s_L}} + \frac{C(L,M)}{s^{L+3}}.$$

We now come back to (4-46), combine again (4-47) with the above bound on $|D|$, (4-48), (4-49) and (4-50), yielding the desired bounds (4-43) and (4-44) of the lemma. □

4B3. *Finite-time blow up.* We now reintegrate in time the time evolution of λ and z we found in Lemma 4.7 to obtain their behavior and show the blow up.

Lemma 4.8 (concentration and asymptotic of the blow-up point). *Let u be a solution that is trapped on its maximal interval of existence. Then it blows up in finite time $T > 0$ with $s(t) \to +\infty$ as $t \to T$ and we have the following*:

(1) Concentration speed. *We have* $\lambda \underset{t\to T}{\sim} C(u(0))(T-t)^{\frac{\ell}{\alpha}}$, *with* $C(u(0)) > 0$.

(2) Behavior of the blow-up point. *There exists z_0 such that* $\lim_{t\to T} z(t) = z_0$ *and for all times* $s \geq s_0$,

$$|z(s)| = O(s_0^{-\tilde{\eta}}). \tag{4-51}$$

Proof. From the Cauchy theory in L^∞, (3-1) and (4-26), if $T \in (0, +\infty]$ denotes the maximal time of existence of u, one necessarily has $\lim_{s\to T} s(t) = +\infty$. From the estimate (4-27) on $b_1^{(0,1)}$, the modulation (4-43) and (4-25), one has

$$\frac{\lambda_s}{\lambda} = -\frac{c_1}{s} + O(s^{-1-\tilde\eta}).$$

We reintegrate using (4-21) (we recall that $c_1 = \ell/(2\ell-\alpha)$ from (3-59)):

$$\lambda = \frac{(1 + O(s_0^{-\tilde\eta}))}{s^{\frac{\ell}{2\ell-\alpha}}}, \tag{4-52}$$

which is valid as long as the solution u is trapped. In addition, if the solution is trapped on its maximal interval of existence, then the function represented by $O(\cdot)$ admits a limit as $s \to +\infty$. In turn, from $\frac{ds}{dt} = \frac{1}{\lambda^2}$ we obtain

$$s = s_0\biggl(1 - \frac{\alpha s_0^{\frac{\alpha}{2\ell-\alpha}}}{2\ell-\alpha}\int_0^t (1 + O(s_0^{-\tilde\eta}))\,dt'\biggr)^{-\frac{2\ell-\alpha}{\alpha}}.$$

Hence there exists $T > 0$ with

$$s \underset{t\to T}{\sim} C(u(0))(T-t)^{-\frac{2\ell-\alpha}{\alpha}}. \tag{4-53}$$

Injecting this identity in (4-52) then gives $\lambda \sim C(u(0))(T-t)^{\frac{\ell}{\alpha}}$ as $t \to T$. Now we turn to the asymptotic behavior of the point of concentration z. From (4-43), using $b_1^{(1,i)} = O(s^{-\frac{\alpha+1}{2}})$ from (4-23) for $1 \le i \le d$, one gets

$$|z_{i,s}| = O(s^{-c_1-\frac{\alpha+1}{2}}) = O(s^{-1-\frac{\alpha}{2}(1+\frac{1}{2\ell-\alpha})}). \tag{4-54}$$

As $\alpha > 0$, this implies the convergence and the estimate of z claimed in the lemma. □

4B4. *Behavior of Sobolev norms near blow-up time.* From Lemma 4.8, the L^∞ bound on the error (D-2) and the bounds on the parameters (4-27), any solution that is trapped on its maximal interval of existence indeed blows up at the time T given by Lemma 4.8 because $\lim_{t\to T}\|u\|_{L^\infty} = +\infty$. The behavior of the Sobolev norms is the following.

Lemma 4.9 (asymptotic behavior for subcritical norms). *Let u be a solution that is trapped for all times $s \ge s_0$ and T be its finite maximal lifespan.*[18] *Then*

(i) Behavior of subcritical norms.

$$\limsup_{t\to T}\|u\|_{H^m(\Omega)} < +\infty \quad \textit{for } 0 \le m < s_c.$$

(ii) Behavior of the critical norm.

$$\|u\|_{H^{s_c}(\Omega)} \underset{t\to T}{=} C(d,p)\sqrt{\ell}\sqrt{|\log(T-t)|}(1+o(1)).$$

[18] T is finite by Lemma 4.8.

(iii) Boundedness of the perturbation in slightly supercritical norms.

$$\limsup_{t\to T} \|u-\chi\tau_z(Q_{\frac{1}{\lambda}})\|_{H^m(\Omega)} < +\infty \quad \textit{for } s_c < m \le \sigma. \tag{4-55}$$

Proof. The trapped solution u can be written as

$$u = \chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}}) + w = \chi\tau_z(Q_{\frac{1}{\lambda}}) + \tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}}) + w.$$

We first look at the second term $\tau_z(\tilde{\alpha}_{b,1/\lambda})$, being the excitation of the ground state. It has compact support in the zone $|x|\le 2B_1\lambda$. From (1-38) and (4-52), one gets $2B_1\lambda \ll 1$ as $s_0 \gg 1$, so that $\tau_z(\tilde{\alpha}_{b,1/\lambda})$ has compact support inside $\mathcal{B}^d(1)$. This implies that

$$\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{H^\sigma(\Omega)} \le C\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{\dot{H}^\sigma(\mathbb{R}^d)},$$

the latter norm being easier to compute. Indeed by renormalizing one has

$$\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{\dot{H}^\sigma(\mathbb{R}^d)} = \frac{1}{\lambda^{\sigma-s_c}}\|\tilde{\alpha}_b\|_{\dot{H}^\sigma(\mathbb{R}^d)}.$$

As

$$\tilde{\alpha}_b = \chi_{B_1}\Bigg(\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)} + \sum_{i=2}^{L+2} S_i\Bigg)$$

from (3-29) and (3-7), the bounds (4-27) on the parameters $b_i^{(n,k)}$, together with the asymptotic at infinity of the profiles $T_i^{(n,k)}$ and S_i described in Lemma 2.10 and Proposition 3.3 imply that $\|\tilde{\alpha}_b\|_{\dot{H}^\sigma} \le C/s$. Hence

$$\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{H^\sigma} \le \frac{C}{s^{1-\frac{\ell(\sigma-s_c)}{2\ell-\alpha}}} \to 0$$

as $t\to T$ as $\sigma - s_c \ll 1$.

Now, following the second paragraph of Remark 4.5, we get that $\|w\|_{H^\sigma} \le CK_1$ is uniformly bounded until the blow-up time. Combined with what was just said about the boundedness of $\tau_z(\tilde{\alpha}_{b,1/\lambda})$, we get that (iii) holds for all $0\le m\le\sigma$. This, together with the asymptotic of the ground state (2-1) then gives (i) and (ii). □

4B5. *The blow-up set.* We recall that $x\in\Omega$ is a blow-up point of u if there exists $(t_n,x_n)\to(T,x)$ such that $|u(t_n,x_n)|\to+\infty$. For trapped solutions one has the following result.

Lemma 4.10 (description of the blow-up set). *Let u be a solution that is trapped for all times $s\ge s_0$ and T be its finite maximal lifespan.*[19] *Then z_0 given by Lemma 4.8 is a blow-up point of u, and it is the only one.*

Proof. From the L^∞ bound (4-29) and the fact that $\lim_{t\to T} s(t) = +\infty$ from Lemma 4.8, $u(s,z(s)) \sim \lambda(s)^{-\frac{2}{p-1}}Q(0)$ as $s\to+\infty$. From Lemma 4.8, this implies that $u(t,z(t))\to+\infty$ as $t\to T$ and that $z_0 = \lim_{t\to T} z(t)$ is indeed a blow-up point.

[19] T is finite by Lemma 4.8.

Now take another point $x \in \Omega$, $x \neq z_0$. From (4-55), the asymptotic of Q (Lemma 2.1), and Lemma 4.8, there exists $R > 0$ such that

$$\sup_{0 \le t < T} \|u(t)\|_{H^\sigma(\mathcal{B}^d(x,R))} < +\infty.$$

This local boundedness, by Sobolev embedding and Hölder, implies that

$$\sup_{0 \le t < T} \|u(t)\|_{W^{1,q}(\mathcal{B}^d(x,R))} < +\infty, \quad q = \frac{2d}{d+2-2\sigma} > \frac{2d}{d+2-2s_c} = d\,\frac{p-1}{p+1}.$$

The above inequality, after applying Lemma 4.11 several times and using Sobolev embedding, implies that there exists $r > 0$ such that

$$\sup_{0 \le t < T} \|u(t)\|_{L^\infty(\mathcal{B}^d(x,r))} < +\infty.$$

Therefore, x is not a blow-up point of u. □

In the proof of the previous lemma, we used the following result.

Lemma 4.11 (parabolic bootstrap). *Let $R > 0$ and $x \in \Omega$ such that $B(x, R) \subset \Omega$. Let $q_0 > \frac{p-1}{p+1} d$. There exists $\kappa(q_0) > 0$ such that for any $q > q_0$, if $u \in C\big([0, T), W^{1,\infty}(\Omega)\big)$ is a solution of* (1-1) *satisfying*

$$\sup_{0 \le t < T} \|u(t)\|_{W^{1,q}(\mathcal{B}^d(x,R))} < +\infty \tag{4-56}$$

then

$$\sup_{0 \le t < T} \|u(t)\|_{W^{1,q(1+\kappa)}(\mathcal{B}^d(x,\frac{R}{2}))} < +\infty. \tag{4-57}$$

Proof. The proof relies on a classical use of estimates for the heat kernel. Without loss of generality we assume $q_0 < d$. If u solves (1-1) and satisfies (4-56) then the localisation $v = \chi_{R/2} u$ solves

$$v_t = \Delta v - 2\nabla.\chi_{\frac{R}{2}}.\nabla u - \Delta\chi_{\frac{R}{2}} u + \chi_{\frac{R}{2}} |u|^{p-1} u$$

and using the Duhamel formula can then be written as

$$v(t) = K_t * v(0) + \int_0^t K_{t-s} * \big[-2\nabla.\chi_{\frac{R}{2}}.\nabla u - \Delta\chi_{\frac{R}{2}} u + \chi_{\frac{R}{2}} |u|^{p-1} u\big]\, ds,$$

where the heat kernel is $K_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$. One then has the formula

$$\begin{aligned}\nabla v(t) = \nabla K_t * v(0) + \int_0^t \nabla K_{t-s} * [-2\nabla.\chi_{\frac{R}{2}}.\nabla u - \Delta\chi_{\frac{R}{2}} u]\, ds \\ + \int_0^t K_{t-s} * \big[\nabla\chi_{\frac{R}{2}} |u|^{p-1} u + \chi_{\frac{R}{2}} \nabla u |u|^{p-1}\big]\, ds.\end{aligned} \tag{4-58}$$

We estimate the last term using the Hölder, Sobolev and Young inequalities:[20]

$$\begin{aligned}\left\| \int_0^t K_{t-s} * [\chi_{\frac{R}{2}} \nabla u |u|^{p-1}]\, ds \right\|_{L^{q(1+\kappa)}} &\leq \int_0^t \left\| K_{t-s} * [\chi_{\frac{R}{2}} \nabla u |u|^{p-1}] \right\|_{L^{q(1+\kappa)}} ds \\ &\lesssim \int_0^t \| K_{t-s} \|_{L^{\left(1+\frac{1}{q(1+\kappa)} - (\omega - \frac{1}{q})\right)^{-1}}} \left\| \nabla u |u|^{p-1} \right\|_{L^{\left(\omega + \frac{1}{q}\right)^{-1}}} ds \\ &\lesssim \int_0^t \| K_{t-s} \|_{L^{\left(1-\omega - \frac{\kappa}{q(1+\kappa)}\right)^{-1}}} \| \nabla u \|_{L^q} \left\| |u|^{p-1} \right\|_{L^{\omega^{-1}}} ds \\ &\lesssim \int_0^t \frac{1}{(t-s)^{\theta(\kappa,q)}} \| \nabla u \|_{L^q} \| \nabla u \|_{L^{q_0}}^{p-1} ds \lesssim \int_0^T \frac{ds}{(t-s)^{\theta(\kappa,q)}},\end{aligned}$$

where

$$\omega = \frac{(d-q_0)(p-1)}{dq_0} \quad \text{and} \quad \theta(\kappa,q) = \frac{(d-q_0)(p-1)}{2q_0} + \frac{\kappa d}{2q(1+\kappa)}$$

(note $\theta \geq 0$ as $q_0 < d$). For $\kappa \geq 0$ and $\frac{p-1}{p+1} d \leq q \leq d$, if κ is fixed, θ is strictly decreasing with respect to q, and if q is fixed, θ is strictly increasing with respect to κ. As $\theta(0, q_0) < 1$ since $q_0 > \frac{p-1}{p+1} d$, this implies that there exists $\kappa(q_0) > 0$ such that for all $q_0 \leq q \leq d$, and $0 < \kappa \leq \kappa(q_0)$, we have $\theta(\kappa, q) < 1$. The above inequality then implies that in that range,

$$\left\| \int_0^t K_{t-s} * [\chi_{\frac{R}{2}} \nabla u |u|^{p-1}]\, ds \right\|_{L^{q(1+\kappa)}} < +\infty.$$

We claim that this term was the "worst" to be estimated in (4-58) and that using the very same techniques, one can estimate similarly all the other terms on the right-hand side in the same range $0 < \kappa \leq \kappa(q_0)$ leading to

$$\sup_{0 \leq t < T} \| \nabla v(t) \|_{L^{(1+\kappa)q}} < +\infty,$$

which implies that $\sup_{0 \leq t < T} \| v(t) \|_{W^{1,(1+\kappa)q}} < +\infty$ by Sobolev embedding and the Hölder inequality. This concludes the proof, as $v = u$ on $B\left(x, \frac{R}{2}\right)$. □

5. Proof of Proposition 4.6

This section is devoted to the proof of this latter proposition, which will then end the proof of the main theorem. For all trapped solutions u in the sense of Definition 4.4, we let $s^* = s^*(u(0))$ be the exit time from the trapped regime:

$$s^* = \sup\left\{ s \geq s_0 \text{ such that (4-22), (4-23), (4-24), (4-25) and (4-26) hold on } [s_0, s) \right\}. \tag{5-1}$$

If $s^* < +\infty$, after s^*, one of the bounds (4-22), (4-23), (4-24), (4-25) or (4-26) must then be violated. The result of the first part of this section is a refinement of this exit condition. In Lemma 5.1 and Propositions 5.3, 5.5, 5.6 and 5.8 we quantify accurately the time evolution of the parameters and the remainder in the trapped regime. Combined with the modulation equations of Lemma 4.7, this allows us to show that in

[20] As $q \geq q_0 > \frac{p-1}{p+1} d$, $p > \frac{d+2}{d-2}$, and $d \geq 11$ all the computations below are rigorous.

the trapped regime, all the components of the solution along the stable directions of perturbation are under control; see Lemma 5.9. Moreover, from (4-52), (4-26) is always fulfilled as long as the other bounds hold. As a consequence, the exit time of the trapped regime is in fact characterized by the following condition: just after s^*, one of the bounds in (4-22) and (4-23) regarding the unstable parameters is violated.

We prove Proposition 4.6 by contradiction. Suppose that given a stable perturbation of $\chi \tilde{Q}_{\bar{b}(s_0),1/\lambda(s_0)}$ as described in Proposition 4.6, the solution starting from $\chi \tilde{Q}_{b(s_0),1/\lambda(s_0)} + w(s_0)$ leaves the trapped regime in finite time for all initial corrections $(V_2(s_0),\dots,V_\ell(s_0))$ and $(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\le n,\ i<i_n}$ along the unstable directions. This means from the previous paragraph that the trajectory of

$$\big(V_2(s),\dots,V_\ell(s),(U_i^{(n,k)}(s))_{(n,k,i)\in\mathcal{I},1\le n,\ i<i_n}\big)$$

leaves the set[21] $\mathcal{B}_\infty^{\ell-1}(s^{-\tilde{\eta}})\times\mathcal{B}_\infty^{K}(1)$ in finite time. But at the leading order, the dynamics of this trajectory are linear repulsive. In Lemma 5.10 we show how the fact that all the trajectories leave this ball is a contradiction to Brouwer's fixed point theorem.

5A. ***Improved modulation for the last parameters $b_{L_n}^{(n,k)}$.*** In Lemma 4.7, the modulation estimates (4-43) for the first parameters are better than the ones for the last parameters $b_{L_n}^{(n,k)}$, (4-44). When looking at the proof of Lemma 4.7, we see that this is a consequence of the fact that the projection of the linearized dynamics onto the profile generating the orthogonality conditions, $\langle H\varepsilon, H^i\Phi_M^{(n,k)}\rangle$, cancels only for $i<L_n$. However, as we explained in the introduction of Lemma 4.2, $H^i\Phi_M^{(n,k)}$ has to be thought as an approximation of $T_i^{(n,k)}$, and in that case the previous term would cancel also for $i=L_n$. It is therefore natural to look for a better modulation estimate for $b_{L_n}^{(n,k)}$. In the next lemma we find a better bound by, roughly speaking, integrating by parts in time the projection of ε onto $T_{L_n}^{(n,k)}$ in the self-similar zone.

Lemma 5.1 (improved modulation equation for $b_{L_n}^{(n,k)}$). *Suppose all the constants in Proposition 4.6 are fixed except s_0. Then for s_0 large enough, for any solution that is trapped on $[s_0,s')$, for $0\le n\le n_0$, $1\le k\le k(n)$, the following holds for $s\in[s_0,s')$:*

$$\left|b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}-\frac{d}{ds}\left[\frac{\langle H^{L_n}(\varepsilon-\sum_{i=2}^{L+2}S_i),\chi_{B_0}T_0^{(n,k)}\rangle}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{n,k}\rangle}\right]\right| \le\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{\delta_n}}+\frac{C(L,M)}{s^{L+\frac{g'}{2}+\delta_n-\delta_0+1}}. \tag{5-2}$$

Remark 5.2. From (5-19), we see that the denominator is not zero. From (5-19) and (5-20), one has the following bound for the new quantity that appeared when comparing this new modulation estimate to the former one (4-44):

$$\left|\frac{\langle H^{L_n}(\varepsilon-\sum_{i=2}^{L+2}S_i),\chi_{B_0}T_0^{(n,k)}\rangle}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{n,k}\rangle}\right|\le C(L,M)s^{-L-\frac{g'}{2}+\delta_0-\delta_n}+C(L,M,K_2)s^{-L+\delta_0-\delta_n+\eta(1-\delta_0')}. \tag{5-3}$$

[21]Here K is the number of directions of instabilities on the spherical harmonics of degree greater than 0, that is, $K=d(E[i_1]-\delta_{i_1\in\mathbb{N}})+\sum_{2\le n\le n_0}k(n)(E[i_n]+1-\delta_{i_n\in\mathbb{N}})$, and $\mathcal{B}_\infty^a(r)$ is the ball of radius r of $\mathbb{R}^a$ for the usual $|\cdot|_\infty$ norm.

This is a better bound compared to the required bound (4-24) on $b_{L_n}^{(n,k)}$ in the trapped regime, that is, $|b_{L_n}^{(n,k)}|\le Cs^{-\frac{\gamma-\gamma_n}{2}-L_n}=Cs^{-L-\delta_n+\delta_0}$.

Proof of Lemma 5.1. First, from the fact that $HT_0^{(n,k)}=0$, the asymptotic (2-7) of $T_0^{(n,k)}$ and (4-27), we obtain

$$\operatorname{supp}[H^{L_n}(\chi_{B_0}T_0^{(n,k)})]\subset\{B_0\le|y|\le 2B_0\}\quad\text{and}\quad |H^{L_n}(\chi_{B_0}T_0^{(n,k)})|\le\frac{C(L)}{s^{\frac{\gamma_n}{2}+L_n}}. \tag{5-4}$$

Step 1: computation of a first identity. We will now prove the identity

$$\begin{aligned}\frac{d}{ds}\big(\langle H^{L_n}\varepsilon,\chi_{B_0}T_0^{(n,k)}\rangle\big)&=\big(b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}\big)\langle T_0^{(n,k)},\chi_{B_0}T_0^{(n,k)}\rangle\\&\quad+\frac{d}{ds}\bigg(\sum_{j=2}^{L+2}\big\langle S_j,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\big\rangle\bigg)\\&\quad+O\big(\sqrt{\mathcal{E}_{2s_L}}B_0^{4m_n+2\delta_n}\big)+O\bigg(\frac{C(L)}{s^{L+1+\frac{g'}{2}-\delta_0-\delta_n-2m_n}}\bigg).\end{aligned} \tag{5-5}$$

From the evolution equation (4-41) and the fact that H is self-adjoint we obtain

$$\begin{aligned}\frac{d}{ds}\big(\langle H^{L_n}\varepsilon,\chi_{B_0}T_0^{(n,k)}\rangle\big)&=\langle\varepsilon,H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})\rangle\\&\quad+\Big\langle-\widetilde{\mathrm{Mod}}(s)-\tilde{\psi}_b+\frac{\lambda_s}{\lambda}\Lambda\varepsilon+\frac{z_s}{\lambda}.\nabla\varepsilon-H\varepsilon+\mathrm{NL}(\varepsilon)+L(\varepsilon),H^{L_n}(\chi_{B_0}T_0^{(n,k)})\Big\rangle.\end{aligned} \tag{5-6}$$

The terms created by the cut of the solitary wave $\lambda^2\tau_{-z}[(\tilde{L}+\tilde{R}+\widetilde{\mathrm{NL}})_\lambda]$ do not appear because they have their support in the zone $\frac{1}{2\lambda}\le|y|$, which is far away from the zone $|y|\le 2B_0$ as $B_0\ll\frac{1}{\lambda}$ in the trapped regime by (4-52). We now look at all the terms in the above equation.

The $\partial_s(\chi_{B_0})$ term. From the modulation equation (4-43) and the bound (4-25), one has $|b_{1,s}^{(0,1)}|\le Cs^{-2}$. Hence, using the asymptotic (2-7) of $T_0^{(n,k)}$ and the fact that $HT_0^{(n,k)}=0$ and (4-27), we get that $H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})$ has support in $B_0\le|y|\le 2B_0$ and satisfies the bound

$$|H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})|\le\frac{C(L)}{s^{\frac{\gamma_n}{2}+L_n+1}}.$$

Using the coercivity estimate (C-16), we obtain

$$\big|\langle\varepsilon,H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})\rangle\big|\le C(L)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n}. \tag{5-7}$$

The error term. For $|y|\le 2B_0$, one has $\tilde{\psi}_b=\psi_b$ by (3-34). As ψ_b is a finite sum of homogeneous profiles of degree $(i,-\gamma-2-g')$ for some $i\in\mathbb{N}$ (which was proved in Step 4 of the proof of Proposition 3.1), the bounds on the parameters (4-27) imply that $|\psi_b(y)|\le C(L)s^{-\frac{\gamma+2+g}{2}}$ for $B_0\le|y|\le 2B_0$. Combined with (5-4), this yields

$$\big|\langle\tilde{\psi}_b,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\big|\le C(L)B_0^{d-\gamma_n-2L_n-\gamma-g'-2}\le\frac{C(L)}{s^{L+1+\frac{g'}{2}-\delta_0-\delta_n-2m_n}}. \tag{5-8}$$

The remainder's contribution. Using (5-4), the bounds $|\frac{\lambda_s}{\lambda}|\le Cs^{-1}$ and $|\frac{z_s}{\lambda}|\le Cs^{-\frac{\alpha+1}{2}}$ (which are consequences of the modulation estimate (4-43) and (4-25)) and the coercivity estimate Lemma C.3, one gets

$$\left|\left\langle \frac{\lambda_s}{\lambda}\Lambda\varepsilon+\frac{z_s}{\lambda}.\nabla\varepsilon-H\varepsilon, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right|\le C(L)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n}. \tag{5-9}$$

The small linear term can be written as $L(\varepsilon)=(pQ^{p-1}-p\widetilde{Q}_b^{p-1})$; hence from the form of $\widetilde{Q}_b$, see (3-29), one has $|(pQ^{p-1}-p\widetilde{Q}_b^{p-1})|\le C(L)s^{-1-\frac{\alpha}{2}}$. Its contribution is then of smaller order using (5-4):

$$\left|\langle L(\varepsilon), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right|\le C(L)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n-\frac{\alpha}{2}}. \tag{5-10}$$

The nonlinear term can be written as $\mathrm{NL}(\varepsilon)=\sum_{k=2}^{p}C_k^p\varepsilon^k\widetilde{Q}_b^{p-k}$. From the coercivity estimate Lemma C.3, we get

$$\int_{B_0\le|y|\le 2B_0}\frac{\varepsilon^2}{|y|^{\gamma_n+2L_n}}\,dy\le C(L,M)\mathcal{E}_{2s_L}s^{2s_L-\frac{\gamma_n}{2}-L_n}.$$

Using the bootstrap bounds (4-25) and (4-27), one computes

$$\sqrt{\mathcal{E}_{2s_L}}s^{2s_L-\frac{\gamma_n}{2}-L_n}\le K_2 s^{\delta_n+2m_n-(\frac{\gamma-2}{4}+\frac{\eta(1-\delta_0')}{2})}\le B_0^{\delta_n+2m_n}$$

for s_0 large enough (because $\gamma>2$). For $2\le k\le p$, we know $|\varepsilon^{k-2}\widetilde{Q}_b^{p-k}|\le C$ is bounded by (D-2), so using the two previous equations and (5-4), one gets

$$\left|\langle \mathrm{NL}(\varepsilon), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right|\le\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n} \tag{5-11}$$

for s_0 large enough. Combining (5-9), (5-10) and (5-11), we have the following upper bound for the remainder's contribution:

$$\left|\left\langle \frac{\lambda_s}{\lambda}\Lambda\varepsilon+\frac{z_s}{\lambda}.\nabla\varepsilon-H\varepsilon+\mathrm{NL}(\varepsilon)+L(\varepsilon), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right|\le C(L,M)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n}. \tag{5-12}$$

The modulation term. For $(n',k',i)\in\mathcal{I}$, one has

$$\langle T_i^{(n,k)}, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle=\langle H^{L_n}T_i^{(n,k)}, \chi_{B_0}T_0^{(n,k)}\rangle=0$$

if $(n',k',i)\neq(n,k,L_n)$. Indeed, if $(n',k')\neq(n,k)$ then the two functions are located on different spherical harmonics and their scalar product is 0. If $i\neq L_n$ then $i<L_n$ and $H^{L_n}T_i^{(n,k)}=0$. This implies the identity from (4-33) since $B_1\gg B_0$:

$$\begin{aligned}
&\langle\widetilde{\mathrm{Mod}}(s), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\\
&\quad=\big(b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}\big)\langle T_0^{(n,k)}, \chi_{B_0}T_0^{(n,k)}\rangle\\
&\qquad+\sum_{j=2}^{L+2}\sum_{(n',k',i)\in\mathcal{I}}\big(b_{i,s}^{(n',k')}+(2i-\alpha_{n'})b_1^{(0,1)}b_i^{(n',k')}\big)\left\langle\frac{\partial S_j}{\partial b_i^{(n',k')}}, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\\
&\qquad-\left(\frac{\lambda_s}{\lambda}+b_1^{(1,0)}\right)\langle\Lambda\widetilde{Q}_b, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle-\left\langle\left(\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right).\nabla\widetilde{Q}_b, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle.
\end{aligned} \tag{5-13}$$

For $2 \le j \le L+2$ and $(n',k',i) \in \mathcal{I}$, as S_i is homogeneous of degree $(i, -\gamma - g')$, using (4-27) and (5-4), we have

$$\left|(2i-\alpha_{n'})b_1^{(0,1)}b_i^{(n',k')}\left\langle \frac{\partial S_j}{\partial b_i^{(n',k')}}, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right| \le \frac{C(L,M)}{s^{L-\delta_0-\delta_n+2m_n+1+\frac{g'}{2}}}. \tag{5-14}$$

Using the modulation bound (4-43), the asymptotics (2-1) and (2-7) of Q and ΛQ, (4-27) and (5-4), we find

$$\left|\left(\frac{\lambda_s}{\lambda}+b_1^{(1,0)}\right)\langle \Lambda \widetilde{Q}_b, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle - \left\langle \left(\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right).\nabla \widetilde{Q}_b, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right| \le \frac{C(L,M)}{s^{2L+\frac{3-\alpha}{2}-2m_n-\delta_n}} \tag{5-15}$$

is very small as $L \gg 1$. Moreover for $2 \le j \le L+2$, one has

$$\sum_{(n',k',i)\in\mathcal{I}} b_{i,s}^{(n',k')}\left\langle \frac{\partial S_j}{\partial b_i^{(n',k')}}, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle = \frac{d}{ds}\big(\langle S_j, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\big) - \langle S_j, H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})\rangle.$$

From similar arguments we used to derive (5-14), one has the similar bound for the last term, yielding

$$\sum_{(n',k',i)\in\mathcal{I}} b_{i,s}^{(n',k')}\left\langle \frac{\partial S_j}{\partial b_i^{(n',k')}}, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle = \frac{d}{ds}\big(\langle S_j, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\big) + O(s^{-L+\delta_0+\delta_n+2m_n-1-\frac{g'}{2}}). \tag{5-16}$$

Coming back to the decomposition (5-13), and applying (5-14) and (5-16) gives

$$\langle \widetilde{\mathrm{Mod}}(s), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle = \big(b_{L_n,s}^{(n,k)} + (2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}\big)\langle T_0^{(n,k)}, \chi_{B_0}T_0^{(n,k)}\rangle + \frac{d}{ds}\left(\sum_{j=2}^{L+2}\langle S_j, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right) + O(s^{-L+\delta_0+\delta_n+2m_n-1-\frac{g'}{2}}). \tag{5-17}$$

In the decomposition (5-6), we examined each term in (5-7), (5-8), (5-12) and (5-17), yielding the identity (5-5) we claimed in this first step.

Step 2: end of the proof. From (5-5) one obtains

$$\begin{aligned}\frac{d}{ds}&\left(\frac{\big(\langle H^{L_n}(\varepsilon - \sum_{i=2}^{L+2} S_i), \chi_{B_0}T_0^{(n,k)}\rangle\big)}{\langle \chi_{B_0}T_0^{(n,k)}, T_0^{(n,k)}\rangle}\right) \\ &= b_{L_n,s}^{(n,k)} + (2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)} + \frac{O\big(\sqrt{\mathcal{E}_{2s_L}}B_0^{4m_n+2\delta_n}\big) + O\left(\frac{C(L)}{s^{L+1+\frac{g'}{2}-\delta_0-\delta_n-2m_n}}\right)}{\langle \chi_{B_0}T_0^{(n,k)}, T_0^{(n,k)}\rangle} \\ &\quad + \left\langle H^{L_n}\left(\varepsilon - \sum_{i=2}^{L+2} S_i\right), \chi_{B_0}T_0^{(n,k)}\right\rangle \frac{d}{ds}\left(\frac{1}{\langle \chi_{B_0}T_0^{(n,k)}, T_0^{(n,k)}\rangle}\right).\end{aligned} \tag{5-18}$$

The size of the denominator is, from the asymptotic (2-7) of $T_0^{(n,k)}$ and (4-27),

$$\langle \chi_{B_0}T_0^{(n,k)}, T_0^{(n,k)}\rangle \sim c s^{2m_n+2\delta_n} \tag{5-19}$$

for some constant $c > 0$. As the denominator just depends on $b_1^{(0,1)}$, using the bound $|b_{1,s}^{(0,1)}| \le C s^{-2}$ and the asymptotics (2-7) of $T_0^{(n,k)}$, we obtain

$$\left| \frac{d}{ds} \left(\frac{1}{\langle \chi_{B_0} T_0^{(n,k)}, T_0^{(n,k)} \rangle} \right) \right| \le \frac{C(L,M)}{s^{2m_n+2\delta_n+1}}.$$

Also, using again the coercivity estimate Lemma C.3, (5-4) and the fact that for $2 \le j \le L+2$, we know S_j is homogeneous of degree $(j, -\gamma - g')$, we obtain

$$\left| \left\langle H^{L_n} \left(\varepsilon - \sum_{i=2}^{L+2} S_i \right), \chi_{B_0} T_0^{(n,k)} \right\rangle \right| \le C(L,M) \big(\sqrt{\mathcal{E}_{2s_L}} s^{2m_n+\delta_n} + s^{-L-\frac{g'}{2}+\delta_0+\delta_n+2m_n} \big). \tag{5-20}$$

Hence, plugging the three previous identities in (5-18) gives the identity (5-3) claimed in the lemma. □

5B. ***Lyapunov monotonicity for low regularity norms of the remainder.*** The key estimate concerning the remainder w is the bound on the high regularity adapted Sobolev norm at the blow-up area: $\mathcal{E}_{2s_L}$. However, the nonlinearity can transfer energy from low to high frequencies, and consequently to control $\mathcal{E}_{2s_L}$ we need to control the low frequencies. This is the purpose of Propositions 5.3 and 5.5, where we find an upper bound for the time evolution of $\|w_{\text{int}}\|_{\dot{H}^\sigma(\mathbb{R}^d)}$ and $\|w_{\text{ext}}\|_{H^\sigma(\Omega)}$.

Proposition 5.3 (Lyapunov monotonicity for the low Sobolev norm of the remainder in the blow-up zone). *Suppose all the constants involved in Proposition 4.6 are fixed except s_0 and η. Then for s_0 large enough and η small enough, for any solution u that is trapped on $[s_0, s')$ the following holds for $0 \le t < t(s')$:*

$$\frac{d}{dt} \left\{ \frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}} \right\} \le \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}} \frac{1}{s^{\frac{\alpha}{4L}}} \left[1 + \sum_{k=2}^{p} \left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}} \right)^{k-1} \right], \tag{5-21}$$

where the norm $\mathcal{E}_\sigma$ is defined in (4-9).

Remark 5.4. Equation (5-21) should be interpreted as follows. The term

$$\frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}$$

is from (4-25) and (4-52) of order $\frac{1}{s}\frac{ds}{dt}$ (as $\frac{ds}{dt} = \lambda^{-2}$). The $1/s^{\frac{\alpha}{4L}}$ then represents a gain: it gives that the right-hand side of (5-21) is of order $(1/s^{1+\frac{\alpha}{4L}})\frac{ds}{dt}$, which when reintegrated in time is convergent and arbitrarily small for s_0 large enough. The third term shows that one needs to have $\sqrt{\mathcal{E}_\sigma} \lesssim s^{-\frac{\sigma-s_c}{2}}$ to control the nonlinear terms, which holds because of the bootstrap bound (4-25).

Proof of Proposition 5.3. To show this result, we compute the left-hand side of (5-21) and we bound it above it using all the bounds that hold in the trapped regime. The time evolution w_{int} given by (4-34) yields

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}} \right\} &= \frac{d}{dt} \left\{ \int |\nabla^\sigma w_{\text{int}}|^2 \right\} \\ &= \int \nabla^\sigma w_{\text{int}} . \nabla^\sigma \left(-H_{z,\frac{1}{\lambda}} w_{\text{int}} - \frac{1}{\lambda^2} \chi \tau_z (\widetilde{\text{Mod}}(t)_{\frac{1}{\lambda}} + \tilde{\psi}_{b\frac{1}{\lambda}}) + \text{NL}(w_{\text{int}}) + L(w_{\text{int}}) + \tilde{L} + \widetilde{\text{NL}} + \tilde{R} \right). \end{aligned} \tag{5-22}$$

We now give an upper bound for each term in (5-22). As all the terms involve functions that are compactly supported in Ω since w_{int} is, all integrations by parts are legitimate and all computations and integrations are performed in $\mathbb{R}^d$ (e.g., L^2 denotes $L^2(\mathbb{R}^d)$).

Step 1: inside the blow-up zone (all terms except the three last ones in (5-22)).

The linear term. By (4-35) using dissipation, we first compute

$$\begin{aligned}\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(-H_{z,\frac{1}{\lambda}} w_{\text{int}}) &= \int \nabla^\sigma w_{\text{int}}.\nabla^\sigma\big(\Delta w_{\text{int}} + p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1} w_{\text{int}}\big)\\ &\le \int \nabla^\sigma w_{\text{int}}.\nabla^\sigma\big(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1} w_{\text{int}}\big),\end{aligned}$$

which becomes after an integration by parts and using the Cauchy–Schwarz inequality

$$\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma\big(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1} w_{\text{int}}\big) \le \|\nabla^{\sigma+2} w_{\text{int}}\|_{L^2}\big\|\nabla^{\sigma-2}\big(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1} w_{\text{int}}\big)\big\|_{L^2}.$$

Using interpolation, the coercivity estimate (C-16) and the bounds of the trapped regime (4-25) on ε, one has for the first term (performing a change of variables to go back to renormalized variables)

$$\begin{aligned}\|\nabla^{\sigma+2} w_{\text{int}}\|_{L^2} &= \frac{1}{\lambda^{\sigma+2-s_c}}\|\nabla^{\sigma+2}\varepsilon\|_{L^2}\\ &\le \frac{C}{\lambda^{\sigma+2-s_c}}\|\nabla^\sigma\varepsilon\|_{L^2}^{1-\frac{2}{2s_L-\sigma}}\|\varepsilon\|_{\dot H^{2s_L}}^{\frac{2}{2s_L-\sigma}}\\ &\le \frac{C(L,M)}{\lambda^{\sigma+2-s_c}}\sqrt{\mathcal{E}_\sigma}^{1-\frac{2}{2s_L-\sigma}}\sqrt{\mathcal{E}_{2s_L}}^{\frac{2}{2s_L-\sigma}}\\ &\le \frac{C(L,M,K_1,K_2)}{\lambda^{\sigma+2-s_c}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+\frac{2}{2s_L-\sigma}\left(L+1-\delta_0+\eta(1-\delta_0')-\frac{(\sigma-s_c)\ell}{2\ell-\alpha}\right)}}\\ &= \frac{C(L,M,K_1,K_2)}{\lambda^{\sigma+2-s_c}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}.\end{aligned}$$

As $Q^{p-1}=O((1+|y|)^{-2})$ from (2-2), using the Hardy inequality (B-7) we get for the second term after a change of variables

$$\begin{aligned}\big\|\nabla^{\sigma-2}\big(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1} w\big)\big\|_{L^2} &= \frac{p}{\lambda^{\sigma-s_c}}\big\|\nabla^{\sigma-2}(Q^{p-1}\varepsilon)\big\|_{L^2}\\ &\le \frac{C}{\lambda^{\sigma-s_c}}\|\nabla^\sigma\varepsilon\|_{L^2} = \frac{C}{\lambda^{\sigma-s_c}}\sqrt{\mathcal{E}_\sigma}.\end{aligned}$$

Combining the four above identities we obtain

$$\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(-H_{z,\frac{1}{\lambda}} w_{\text{int}}) \le \frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}. \tag{5-23}$$

The modulation term. To treat the error induced by the cut separately, we decompose as follows, going back to renormalized variables using Cauchy–Schwarz:

$$\begin{aligned}
&\left|\int \nabla^\sigma w.\nabla^\sigma\left(\frac{1}{\lambda^2}\chi\tau_z(\tilde{\mathrm{Mod}}(t)_{\frac{1}{\lambda}})\right)\right| \\
&\qquad \le \left|\int \nabla^\sigma w.\nabla^\sigma\left(\frac{1}{\lambda^2}(1+(\chi-1))\tau_z(\tilde{\mathrm{Mod}}(t)_{\frac{1}{\lambda}})\right)\right| \\
&\qquad \le \frac{1}{\lambda^{2(\sigma-s_c)+2}}\sqrt{\mathcal{E}_\sigma}\left[\left\|\nabla^\sigma \tilde{\mathrm{Mod}}(s)\|_{L^2}+\left\|\nabla^\sigma\left(\frac{1}{\lambda^2}(\chi-1)\tau_z(\widetilde{\mathrm{Mod}}(t)_{\frac{1}{\lambda}})\right)\right\|_{L^2}\right]. \qquad (5\text{-}24)
\end{aligned}$$

For the first term in the above equation, using (4-33) and the modulation estimates (4-43) and (4-44), we get

$$\begin{aligned}
&\|\nabla^\sigma \tilde{\mathrm{Mod}}(s)\|_{L^2} \\
&\quad\le \sum_{(n,k,i)\in\mathcal{I}} \left|b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}\right|\left\|\nabla^\sigma\left(\chi_{B_1}\left(T_i^{(n,k)}+\sum_{j=2}^{L+2}\frac{\partial S_j}{\partial b_i^{(n,k)}}\right)\right)\right\|_{L^2} \\
&\qquad\qquad + \left|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right|\|\nabla^\sigma(\Lambda\tilde{Q}_b)\|_{L^2}+\left|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right|\|\nabla^{\sigma+1}(\tilde{Q}_b)\|_{L^2} \\
&\quad\le C(L,M)(\sqrt{\mathcal{E}_{2s_L}}+s^{-L-3})\Bigg[\|\nabla^\sigma(\Lambda\tilde{Q}_b)\|_{L^2}+\|\nabla^{\sigma+1}(\tilde{Q}_b)\|_{L^2} \\
&\qquad\qquad + \sum_{(n,k,i)\in\mathcal{I}}\|\nabla^\sigma(\chi_{B_1}T_i^{(n,k)})\|_{L^2}+\sum_{j=2}^{L+2}\left\|\nabla^\sigma\left(\chi_{B_1}\frac{\partial S_j}{\partial b_i^{(n,k)}}\right)\right\|_{L^2}\Bigg].
\end{aligned}$$

Under the trapped regime bound (4-25), one has $\sqrt{\mathcal{E}_{2s_L}}+s^{-L-3}\le s^{-L-1+\delta_0-\eta(1-\delta_0')}$. Moreover, from the asymptotics of Q, ΛQ, $T_i^{(n,k)}$ and S_j ((2-1), (2-7), Lemma 2.10 and (3-8)), and the bounds on the parameters (4-27), one has

$$\|\nabla^\sigma(\Lambda\tilde{Q}_b)\|_{L^2}\le C,\quad \|\nabla^{\sigma+1}(\tilde{Q}_b)\|_{L^2}\le C,$$

$$\begin{aligned}
&\sum_{(n,k,i)\in\mathcal{I}}\|\nabla^\sigma(\chi_{B_1}T_i^{(n,k)})\|_{L^2}+\sum_{j=2}^{L+2}\left\|\nabla^\sigma\left(\chi_{B_1}\frac{\partial S_j}{\partial b_i^{(n,k)}}\right)\right\|_{L^2} \\
&\quad \le C(L)\le C(L)s^{L+\sup_{0\le n\le n_0}\delta_n-\delta_0-\frac{\alpha}{2}-\frac{(\sigma-s_c)}{2}+C(L)\eta}+C(L)s^{L+\sup_{0\le n\le n_0}\delta_n-\delta_0-\frac{\alpha}{2}-\frac{(\sigma-s_c)}{2}+C(L)\eta-\frac{g'}{2}}.
\end{aligned}$$

All these bounds then imply that for the modulation term that is located at the blow-up zone in (5-24), we have

$$\begin{aligned}
\frac{1}{\lambda^{2(\sigma-s_c)+2}}\sqrt{\mathcal{E}_\sigma}\|\nabla^\sigma\tilde{\mathrm{Mod}}(s)\|_{L^2} &\le \frac{C(L,M)\sqrt{\mathcal{E}_\sigma}s^{L+\sup_{0\le n\le n_0}\delta_n-\delta_0-\frac{\alpha}{2}-\frac{(\sigma-s_c)}{2}+C(L)\eta}}{\lambda^{2(\sigma-s_L)+2}s^{L+1-\delta_0+(1-\delta_0')\eta}} \\
&\le \frac{C(L,M)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{1+(\frac{\alpha}{2}-\sup_{0\le n\le n_0}\delta_n)+\frac{\sigma-s_c}{2}-C(L)\eta}}.
\end{aligned}$$

We now turn to the second term in (5-24). The blow-up point z is arbitrarily close to 0 by (4-51) and from the expression of the modulation term (4-33), all the terms except $\tau_z\big(\big[\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\big]\Lambda Q+$

$[b_1^{(1,\cdot)}+\frac{z_s}{\lambda}].\nabla Q\big)_{1/\lambda}$ have support in the zone $\{|x-z|\leq 2B_1\lambda\}\subset B\big(0,\frac12\big)$ because $B_1\lambda\ll 1$. This means that from the modulation estimates (4-43)

$$\left\|\nabla^\sigma\left(\frac{1}{\lambda^2}(\chi-1)\tau_z(\widetilde{\mathrm{Mod}}(t)_{\frac{1}{\lambda}})\right)\right\|_{L^2}=\left\|\nabla^\sigma\left(\frac{1}{\lambda^2}(\chi-1)\tau_z\left(\left[\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right]\Lambda Q+\left[b_1^{(1,\cdot)}+\frac{z_s}{\lambda}\right].\nabla Q)_{\frac{1}{\lambda}}\right)\right)\right\|_{L^2}$$
$$\leq\frac{C\big[\big|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\big|+\big|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\big|\big]}{\lambda^2}\leq\frac{C}{\lambda^2 s^{L+1}}.$$

We insert the two previous equations into the expression (5-24), yielding

$$\left|\int\nabla^\sigma w_{\mathrm{int}}.\nabla^\sigma\left(\frac{1}{\lambda^2}\chi\tau_z(\tilde{\mathrm{Mod}}(t)_{\frac{1}{\lambda}})\right)\right|\leq\frac{C(L,M)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{1+(\frac{\alpha}{2}-\sup_{0\leq n\leq n_0}\delta_n)+\frac{\sigma-s_c}{2}-C(L)\eta}}.\tag{5-25}$$

The error term. As $|z|\ll 1$ by (4-51) and $B_1\lambda\ll 1$ by (4-27) and (4-52), from the expression of the error term (3-36), all the terms except $\tau_z(b_1^{(0,1)}\Lambda Q+b_1^{(1,\cdot)}.\nabla Q)_{1/\lambda}$ have support in the zone $\{|x-z|\leq 2B_1\lambda\}\subset B\big(0,\frac12\big)$. Therefore, making the following decomposition and coming back to renormalized variables, using the estimates (3-32) and (4-43), one computes

$$\begin{aligned}
&\left|\int\nabla^\sigma w_{\mathrm{int}}.\nabla^\sigma\left(\frac{1}{\lambda^2}\chi\tau_z(\tilde{\psi}_{b\frac{1}{\lambda}})\right)\right|\\
&\quad\leq\frac{\|\nabla^\sigma\varepsilon\|_{L^2}}{\lambda^{\sigma-s_c+2}}\left(\frac{\|\nabla^\sigma\tilde{\psi}_b\|_{L^2}}{\lambda^{2(\sigma-s_c)+2}}+\|\nabla^\sigma((\chi-1)\tau_z(\tilde{\psi}_{b\frac{1}{\lambda}}))\|_{L^2}\right)\\
&\quad\leq\frac{C(L)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{1+\frac{\alpha}{2}+\frac{\sigma-s_c}{2}-C(L)\eta}}+\frac{\|\nabla^\sigma\varepsilon\|_{L^2}}{\lambda^{\sigma-s_c+2}}\big\|\nabla^\sigma(\chi-1)\big(\tau_z(b_1^{(0,1)}\Lambda Q+b_1^{(1,\cdot)}.\nabla Q)_{\frac{1}{\lambda}}\big)\big\|_{L^2}\\
&\quad\leq\frac{C(L)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{1+\frac{\alpha}{2}+\frac{\sigma-s_c}{2}-C(L)\eta}}+C\frac{\|\nabla^\sigma\varepsilon\|_{L^2}}{\lambda^{2(\sigma-s_c)+2}}\big(|b_1^{(0,1)}|\lambda^{\alpha+\sigma-s_c}+|b_1^{(1,\cdot)}|\lambda^{1+\sigma-s_c}\big)\\
&\quad\leq\frac{C(L)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{1+\frac{\alpha}{2}+\frac{\sigma-s_c}{2}-C(L)\eta}}.
\end{aligned}\tag{5-26}$$

The nonlinear term. First, coming back to renormalized variables, as $\mathrm{NL}(\varepsilon)=\sum_{k=2}^p C_k^p\tilde{Q}_b^{p-k}\varepsilon^k$, and performing an integration by parts we write

$$\left|\int\nabla^\sigma w_{\mathrm{int}}.\nabla^\sigma(\mathrm{NL}(w_{\mathrm{int}}))\right|\leq C\sum_{k=2}^p\frac{\big\|\nabla^{\sigma+2-(k-1)(\sigma-s_c)}\varepsilon\big\|_{L^2}\big\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)}(\tilde{Q}_b^{p-k}\varepsilon^k)\big\|_{L^2}}{\lambda^{2(\sigma-s_c)+2}}.\tag{5-27}$$

We fix k, $2\leq k\leq p$, and focus on the k-th term in the sum. The first term is estimated using interpolation, the coercivity estimate (C-16) and the bound (4-25):

$$\begin{aligned}
\|\nabla^{\sigma+2-(k-1)(\sigma-s_c)}\varepsilon\|_{L^2}&\leq C\|\nabla^\sigma\varepsilon\|_{L^2}^{1-\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}}\|\nabla^{2s_L}\varepsilon\|_{L^2}^{\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}}\\
&\leq C(L,M)\sqrt{\mathcal{E}_\sigma}^{1-\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}}\sqrt{\mathcal{E}_{2s_L}}^{\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}}\\
&\leq\frac{C(L,M,K_1,K_2)}{s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1-\frac{(k-1)(\sigma-s_c)}{2}+\frac{\alpha}{2L}+O\left(\frac{|\sigma-s_c|+|\eta|}{L}\right)}}.
\end{aligned}\tag{5-28}$$

For the second term in (5-27), as $\tilde{Q}_b = O((1+|y|)^{-2})$ by (3-29) and (4-27), we first use the Hardy inequality (B-7):

$$\left\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)}(\tilde{Q}_b^{p-k}\varepsilon^k)\right\|_{L^2} \le C\left\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1}}(\varepsilon^k)\right\|_{L^2}. \tag{5-29}$$

We write

$$\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1} = \sigma(n,k)+\delta(n,k),$$

where $\sigma(n,k) := E\big[\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1}\big] \in \mathbb{N}$ and $0 \le \delta(n,k) < 1$. Developing the entire part of the derivative yields

$$\left\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1}}(\varepsilon^k)\right\|_{L^2} \le \sum_{\substack{(\mu_i)_{1\le i\le k}\in\mathbb{N}^{kd}\\ \sum_i |\mu_i|=\sigma(n,k)}} \left\|\nabla^{\delta(\sigma,k)}\left(\prod_{i=1}^k \partial^{\mu_i}\varepsilon\right)\right\|_{L^2}. \tag{5-30}$$

Fix $(\mu_i)_{1\le i\le k} \in \mathbb{N}^{kd}$ satisfying $\sum_{i=1}^k |\mu_i| = \sigma(n,k)$ in the above sum. We define the following family of Lebesgue exponents $\big($that are well defined since $\sigma < \frac{d}{2}\big)$:

$$\frac{1}{p_i} := \frac{1}{2} - \frac{\sigma-|\mu_i|_1}{d}, \quad \frac{1}{p_i'} := \frac{1}{2} - \frac{\sigma-|\mu_i|-\delta(\sigma,k)}{d} \quad \text{for } 1\le i\le k.$$

One has $p_i > 2$ and a direct computation shows that

$$\frac{1}{p_j'} + \sum_{i\ne j}\frac{1}{p_i} = \frac{1}{2}.$$

We now recall the commutator estimate

$$\|\nabla^{\delta_\sigma}(uv)\|_{L^q} \le C\|\nabla^{\delta_\sigma}u\|_{L^{p_1}}\|v\|_{L^{p_2}} + C\|\nabla^{\delta_\sigma}v\|_{L^{p_1'}}\|u\|_{L^{p_2'}}$$

for

$$\frac{1}{p_1}+\frac{1}{p_2} = \frac{1}{p_1'}+\frac{1}{p_2'} = \frac{1}{q},$$

provided $1 < q, p_1, p_1' < +\infty$ and $1 \le p_2, p_2' \le +\infty$. This estimate, combined with the Hölder inequality allows us to compute by iteration:

$$\begin{aligned}
&\left\|\nabla^{\delta(\sigma,k)}\left(\prod_{i=1}^k \partial^{\mu_i}\varepsilon\right)\right\|_{L^2}\\
&\quad\le C\|\partial^{\mu_1+\delta(\sigma,k)}\varepsilon\|_{L^{p_1'}}\left\|\prod_{i=2}^k\partial^{\mu_i}\varepsilon\right\|_{L^{(\sum_{i=2}^k \frac{1}{p_i})^{-1}}} + C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\left\|\nabla^{\delta(\sigma,k)}\left(\prod_{i=2}^k\partial^{\mu_i}\varepsilon\right)\right\|_{L^{(\frac{1}{2}-\frac{1}{p_1})^{-1}}}\\
&\quad\le C\|\partial^{\mu_1+\delta(\sigma,k)}\varepsilon\|_{L^{p_1'}}\prod_{i=2}^k\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}} + C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\|\partial^{\mu_2+\delta(\sigma,k)}\varepsilon\|_{L^{p_2'}}\left\|\prod_{i=3}^k\partial^{\mu_i}\varepsilon\right\|_{L^{(\sum_{i=3}^k\frac{1}{p_i})^{-1}}}\\
&\qquad + C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\|\partial^{\mu_2}\varepsilon\|_{L^{p_2}}\left\|\nabla^{\delta(\sigma,k)}\left(\prod_{i=3}^k\partial^{\mu_i}\varepsilon\right)\right\|_{L^{(\frac{1}{2}-\frac{1}{p_1}-\frac{1}{p_2})^{-1}}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\partial^{\mu_1+\delta(\sigma,k)}\varepsilon\|_{L^{p_1'}}\prod_{i=2}^k\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}}+C\|\partial^{\mu_2+\delta(\sigma,k)}\varepsilon\|_{L^{p_2'}}\prod_{i\neq 2}\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}}\\
&\qquad\qquad+C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\|\partial^{\mu_2}\varepsilon\|_{L^{p_2}}\left\|\nabla^{\delta(\sigma,k)}\Big(\prod_{i=3}^k\partial^{\mu_i}\varepsilon\Big)\right\|_{L^{(\frac12-\frac{1}{p_1}-\frac{1}{p_2})^{-1}}}\\
&\ \vdots\\
&\le C\sum_{i=1}^k\|\partial^{\mu_i+\delta(\sigma,k)}\varepsilon\|_{L^{p_i'}}\prod_{j=1,\,j\neq i}^k\|\partial^{\mu_i}\varepsilon\|_{L^{p_j}}.
\end{aligned}
$$

From Sobolev embedding, one has on the other hand that

$$\|\partial^{\mu_i+\delta(\sigma,k)}\varepsilon\|_{L^{p_i'}}+\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}}\le C\|\nabla^\sigma\varepsilon\|_{L^2}=C\sqrt{\mathcal{E}_\sigma}.$$

Therefore (the strategy was designed to obtain this),

$$\left\|\nabla^{\delta(\sigma,k)}\Big(\prod_{i=1}^k\partial^{\mu_i}\varepsilon\Big)\right\|_{L^2}\le\sqrt{\mathcal{E}_\sigma}^{\,k}.$$

Plugging this estimate in (5-29) using (5-30) gives

$$\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)}(\widetilde{Q}_b^{p-k}\varepsilon^k)\|_{L^2}\le C\sqrt{\mathcal{E}_\sigma}^{\,k}.$$

Injecting this bound and the bound (5-28) in the decomposition (5-27) yields

$$\left|\int\nabla^\sigma w_{\text{int}}.\nabla^\sigma(\text{NL}(w_{\text{int}}))\right|\le\frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O\left(\frac{|\eta|+|\sigma-s_c|}{L}\right)}}\sum_{k=2}^p\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}.\tag{5-31}$$

The small linear term. One has $L(\varepsilon)=-p(Q^{p-1}-\widetilde{Q}^{p-1})\varepsilon$. The potential here admits the asymptotic $Q^{p-1}-\widetilde{Q}^{p-1}\lesssim|y|^{-2-\alpha}$ at infinity, which is better than the asymptotic of the potential appearing in the linear term $Q^{p-1}\sim|y|^{-2}$ we used previously to estimate it. Hence using exactly the same techniques one can prove the same estimate

$$\left|\int\nabla^\sigma w_{\text{int}}.\nabla^\sigma(L(w_{\text{int}}))\right|\le\frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O\left(\frac{|\eta|+|\sigma-s_c|}{L}\right)}}.\tag{5-32}$$

End of Step 1. We come back to the first identity we derived, (5-22), and insert the bounds we found for each term in (5-23), (5-25), (5-26), (5-31) and (5-32) to obtain

$$
\begin{aligned}
&\left|\int\nabla^\sigma w_{\text{int}}.\nabla^\sigma\Big(-H_{z,\frac{1}{\lambda}}w_{\text{int}}-\frac{1}{\lambda^2}\chi\tau_z(\widetilde{\text{Mod}}(t)_{\frac{1}{\lambda}}+\tilde{\psi}_{b\,\frac{1}{\lambda}})+\text{NL}(w_{\text{int}})+L(w_{\text{int}})\Big)\right|\\
&\le\frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\left[\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}+\frac{C(L,M,K_2)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+\left(\frac{\alpha}{2}-\sup_{0\le n\le n_0}\delta_n\right)-C(L)\eta}}\right.\\
&\qquad\qquad\left.+\frac{C(L)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+\frac{\alpha}{2}-C(L)\eta}}+\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}\sum_{k=2}^p\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\right].
\end{aligned}\tag{5-33}
$$

Step 2: the last three terms outside the blow-up zone in (5-22). By a change of variables, we see that the extra error term (4-40) is bounded:

$$\|\nabla^\sigma \widetilde{R}\|_{L^2(\mathbb{R}^d)} \le C.$$

Then, the extra linear term in (5-22) is estimated directly via interpolation using the bound (4-28):

$$\begin{aligned}
&\big\|\nabla^\sigma\big(-\Delta\chi_{B(0,3)}w - 2\nabla\chi_{B(0,3)}.\nabla w + p\tau_z Q_{\frac{1}{\lambda}}^{p-1}(\chi_{B(0,1)}^{p-1} - \chi_{B(0,3)})w\big)\big\|_{L^2(\mathbb{R}^d)} \\
&\qquad \le \|w\|_{H^{\sigma+1}} \le \|w\|_{H^\sigma}^{1-\frac{1}{2s_L-\sigma}} \|w\|_{H^{2s_L}}^{\frac{1}{2s_L-\sigma}} \\
&\qquad \le C(K_1,K_2)\bigg(\frac{1}{\lambda^{2s_L-\sigma} s^{L+1-\delta_0+\eta(1-\delta_0')}}\bigg)^{\frac{1}{2s_L-\sigma}} \\
&\qquad \le C(K_1,K_2)\bigg(\frac{1}{\lambda^{2s_L-\sigma} s^{L+1-\delta_0+\eta(1-\delta_0')}}\bigg)^{\frac{2}{2s_L-\sigma}} = \frac{C(K_1,K_2)}{\lambda^2 s^{1+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}
\end{aligned}$$

because $1/\lambda^{2s_L-\sigma}s^{L+1-\delta_0+\eta(1-\delta_0')} \gg 1$ in the trapped regime. For the last nonlinear in (5-22), one has, using (D-4) and (4-28),

$$\begin{aligned}
\|\widetilde{\mathrm{NL}}\|_{H^\sigma} &\le C\|w\|_{H^\sigma}\|w\|_{H^{\frac{d}{2}+\sigma-s_c}}^{p-1} \le C(K_1)\|w\|_{H^{2s_L}}^{(p-1)(\frac{d}{2}+\sigma-s_c-\sigma)/(2s_L-\sigma)} \\
&\le C(K_1,K_2)\bigg(\frac{1}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}}\bigg)^{\frac{2}{2s_L-\sigma}} \le C(K_1,K_2)\frac{1}{\lambda^2 s^{1+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}.
\end{aligned}$$

The three previous estimates imply that for the terms created by the cut in (5-22), we have the estimate (we recall that $\lambda^{\sigma-s_c}/s^{\frac{\ell(\sigma-s_c)}{2\ell-\alpha}} = 1+O(s_0^{-\tilde{\eta}})$ from (4-52))

$$\bigg|\int \nabla^\sigma w_{\mathrm{int}}.\nabla^\sigma(\tilde{L}+\widetilde{R}+\widetilde{\mathrm{NL}})\bigg| \le \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}. \tag{5-34}$$

Step 3: conclusion. We now come back to the first identity we derived, (5-22), and insert the bounds (5-33) and (5-34), yielding

$$\begin{aligned}
&\frac{d}{dt}\bigg\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\bigg\} \\
&\quad \le \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\bigg[\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}} + \frac{C(L,M,K_2)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+(\frac{\alpha}{2}-\sup_{0\le n\le n_0}\delta_n)-C(L)\eta}} \\
&\qquad\qquad + \frac{C(L)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+\frac{\alpha}{2}-C(L)\eta}} + \frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}\sum_{k=2}^{p}\bigg(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\bigg)^{k-1}\bigg].
\end{aligned}$$

As the constants never depend on s_0 or on η, as $L \gg 1$ is an arbitrary large integer, $0 < \sigma - s_c \ll 1$, $\frac{\alpha}{2} - \sup_{0\le n\le n_0}\delta_n > 0$, we see that for s_0 sufficiently large and η sufficiently small, the terms on the right-hand side of the previous equation can be as small as we want, and (5-21) is obtained. □

Proposition 5.5 (Lyapunov monotonicity for the low Sobolev norm of the remainder outside the blow-up area). *Suppose all the constants involved in Proposition 4.6 are fixed except s_0 and η. Then for s_0 large enough and η small enough, for any solution u that is trapped on $[s_0, s')$ the following holds for $t \in [0, t(s'))$:*

$$\frac{d}{dt}\left[\|w_{\text{ext}}\|^2_{H^\sigma}\right] \le \frac{C(K_1,K_2)}{s^{1+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}\lambda^2}\|w_{\text{ext}}\|_{H^\sigma}. \tag{5-35}$$

Proof. From the evolution equation of w_{ext}, given in Section 4B1, we deduce

$$\frac{d}{dt}\|w_{\text{ext}}\|^2_{H^\sigma(\Omega)} \le C\|w_{\text{ext}}\|_{H^\sigma(\Omega)}\left\|\Delta w_{\text{ext}} + \Delta\chi_3 w + 2\nabla\chi_3.\nabla w + (1-\chi_3)w^p\right\|_{H^\sigma(\Omega)}. \tag{5-36}$$

For the linear terms, using interpolation and the bounds (4-25) and (4-28) one finds

$$\begin{aligned}
&\|\Delta w_{\text{ext}} + \Delta\chi_3 w + 2\nabla\chi_3.\nabla w\|_{H^\sigma(\Omega)} \\
&\qquad \le C\|w_{\text{ext}}\|_{H^{\sigma+2}(\Omega)} + C\|w\|_{H^{\sigma+1}(\Omega)} \\
&\qquad \le C\|w_{\text{ext}}\|^{1-\frac{2}{2s_L-\sigma}}_{H^\sigma(\Omega)}\|w_{\text{ext}}\|^{\frac{2}{2s_L-\sigma}}_{H^{2s_L}(\Omega)} + C\|w\|^{1-\frac{1}{2s_L-\sigma}}_{H^\sigma(\Omega)}\|w\|^{\frac{1}{2s_L-\sigma}}_{H^{2s_L}(\Omega)} \\
&\qquad \le C(K_1,K_2)\left[\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{1}{2s_L-\sigma}} + \left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{2}{2s_L-\sigma}}\right] \\
&\qquad \le C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{2}{2s_L-\sigma}} \le C(K_1,K_2)\frac{1}{\lambda^2 s^{1+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}
\end{aligned}$$

because $1/\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')} \gg 1$ in the trapped regime from (4-52). For the nonlinear term, using (D-4), interpolation and then the bootstrap bound (4-28),

$$\begin{aligned}
\|(1-\chi_3)w^p\|_{H^\sigma} &\le C\|w^p\|_{H^\sigma(\Omega)} \le C\|w\|_{H^\sigma(\Omega)}\|w\|^{p-1}_{H^{\frac{d}{2}+\sigma-s_c}(\Omega)} \\
&\le C(K_1)\|w\|^{(p-1)\frac{\frac{d}{2}+\sigma-s_c-\sigma}{2s_L-\sigma}}_{H^{2s_L}(\Omega)} \le C(K_1)\|w\|^{\frac{2}{2s_L-\sigma}}_{H^{2s_L}(\Omega)} \le \frac{C(K_1,K_2)}{s^{1+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}\lambda^2}.
\end{aligned}$$

Injecting the two above estimates in (5-36) yields the desired identity (5-35). □

5C. ***Lyapunov monotonicity for high regularity norms of the remainder.*** We derive Lyapunov-type monotonicity formulas for the high regularity norms of the remainder inside and outside the blow-up zone, $\mathcal{E}_{2s_L}$ and $\|w_{\text{ext}}\|_{H^{2s_L}}$, in Propositions 5.6 and 5.8. In our general strategy, we have to find a way to say that w is of smaller order compared to the excitation $\chi\tau_z(\tilde{\alpha}_{b,1/\lambda})$ and does not affect the blow-up dynamics induced by the latter. This is why we study the quantity $\mathcal{E}_{2s_L}$: it controls the usual Sobolev norm H^{2s_L} and any local norm of lower-order derivative, which is useful for estimates, and is it adapted to the linear dynamics as it undergoes dissipation. Finally, for this norm one sees that the error $\tilde{\psi}_b$ is of smaller order compared to the main dynamics of $\chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})$ (this is the $\eta(1-\delta_0')$ gain in (3-33)).

Proposition 5.6 (Lyapunov monotonicity for the high regularity adapted Sobolev norm of the remainder inside the blow-up area). *Suppose all the constants of Proposition 4.6 are fixed, except s_0 and η. Then* *there exists a constant $\delta > 0$ such that for any constant $N \gg 1$, for s_0 large enough and η small enough, for any solution u that is trapped on $[s_0, s')$, the following holds for $0 \le t < t(s')$:*

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}+O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}\left(\sqrt{\mathcal{E}_{2s_L}}+\frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)\right)\right\}\\
&\le \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\left[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')}}+\frac{C(L,M)}{N^{2\delta}}\mathcal{E}_{2s_L}\right.\\
&\qquad\left.+\mathcal{E}_{2s_L}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O(\frac{1}{L})}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c}{L})}}+\frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}\right], \qquad (5\text{-}37)
\end{aligned}$$

where $O_{L,M}(f)$ denotes a function depending on time such that $|O_{L,M}(f)(t)| \le C(L,M)f$ for a constant $C(L,M) > 0$, and where $\mathcal{E}_\sigma$ and $\mathcal{E}_{2s_L}$ are defined in (4-9) *and* (4-7).

Remark 5.7. Equation (5-37) has to be understood the following way. The $O(\cdot)$ in the time derivative is a corrective term coming from the refinement of the last modulation equations; see (4-44) and (5-2). It is of smaller order for our purpose so one can "forget" it. On the right-hand side of (5-37), the first two terms come from the error $\tilde{\psi}_b$ made in the approximate dynamics. The third one results from the competition of the dissipative linear dynamics and the lower-order linear terms that are of smaller order (the motion of the potential in the operator $H_{z,1/\lambda}$ involved in $\mathcal{E}_{2s_L}$, and the difference between the potentials $\tau_z(\widetilde{Q}_{b,1/\lambda})^{p-1}$ and $\tau_z(Q_{1/\lambda})^{p-1}$). The penultimate represents the effect of the main nonlinear term, and shows that one needs $\mathcal{E}_\sigma$ smaller than $s^{s_c-\sigma}$ to control the energy transfer from low to high frequencies. The last one results from the cut of w at the border of the blow-up zone.

Proof of Proposition 5.6. From (4-41) one has the identity

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s_L-s_c)}}\right)&=\frac{d}{dt}\left(\int|H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}|^2\right)\\
&=-2\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H^{s_L+1}_{z,\frac{1}{\lambda}}w_{\text{int}}+\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H^{s_L}_{z,\frac{1}{\lambda}}\left(\frac{1}{\lambda^2}\chi\tau_z(-\widetilde{\text{Mod}}(t)_{\frac{1}{\lambda}})\right)\\
&\quad+2\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}\left[H^{s_L}_{z,\frac{1}{\lambda}}\left[\frac{1}{\lambda^2}\chi\tau_z(-\tilde{\psi}_{b\frac{1}{\lambda}})+\text{NL}(w_{\text{int}})+L(w_{\text{int}})\right]+\frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w_{\text{int}}\right]\\
&\quad+2\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H_{z,\frac{1}{\lambda}}(\tilde{L}+\widetilde{\text{NL}}+\tilde{R}). \qquad (5\text{-}38)
\end{aligned}$$

The proof is organized as follows. For the terms appearing in this identity: for some (those on the second line), we find direct upper bounds (Step 1), then we integrate by parts in time some modulation terms that are problematic to treat the second term on the right-hand side (Step 2), and eventually we prove that the terms created by the cut of the solitary wave (the last line) are harmless and use a dissipation property at the linear level (produced by the first term on the right-hand side) to improve the result (Step 3). Throughout the proof, the estimates are performed on $\mathbb{R}^d$, as w_{int} has compact support inside Ω, and we omit it in the notations.

Step 1: brute force upper bounds. We claim that the nonlinear term, the error term, the small linear term and the term involving the time derivative of the linearized operator in (5-38) can be directly bounded above, yielding

$$\begin{aligned}\Big\| H^{s_L}_{z,\frac{1}{\lambda}}\Big[\mathrm{NL}(w_{\mathrm{int}})-\frac{1}{\lambda^2}\chi\tau_z(\tilde\psi_{b,\frac{1}{\lambda}})+L(w_{\mathrm{int}})\Big]+\frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w_{\mathrm{int}}\Big\|_{L^2}\\ \leq \frac{1}{\lambda^{(2s_L-s_c)+2}s}\Bigg[\sqrt{\mathcal{E}_{2s_L}}\sum_{k=2}^{p}\Bigg(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\Bigg)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}\\ +\frac{C(L)}{s^{L+1-\delta_0+\eta(1-\delta_0)'}}+C(L,M)\Bigg(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\Bigg)^{\frac{1}{2}}\Bigg]\end{aligned} \tag{5-39}$$

for some constant $\delta>0$. We now analyse these four terms separately.

The error term. We decompose between the main terms and the terms created by the cut. The cut induced by $\tilde\chi:=\chi(\lambda y+z)$ only sees the terms $b_1^{(0,1)}\Lambda Q+b_1^{(1,\cdot)}.\nabla Q$ because all the other terms in the expression (3-36) of $\tilde\psi_b$ have support inside $\mathcal{B}^d(2B_1)$ and because $|z|\ll 1$ by (4-51) and $B_1\ll\frac{1}{\lambda}$ by (4-52). For the main term we use the estimate (3-33), and for the second the bound on the parameters (4-27) and the asymptotics (2-7) and (2-1) of ΛQ and ∂Q,

$$\begin{aligned}\Big\|H^{s_L}_{z,\frac{1}{\lambda}}\Big(\frac{1}{\lambda^2}\chi\tau_z\tilde\psi_{b,\frac{1}{\lambda}}\Big)\Big\|_{L^2} &\leq C\Big\|H^{s_L}_{z,\frac{1}{\lambda}}\Big(\frac{1}{\lambda^2}\tau_z\tilde\psi_{b,\frac{1}{\lambda}}\Big)\Big\|_{L^2}+C\Big\|H^{s_L}_{z,\frac{1}{\lambda}}\Big(\frac{1}{\lambda^2}(1-\chi)\tau_z\tilde\psi_{b,\frac{1}{\lambda}}\Big)\Big\|_{L^2}\\ &\leq \frac{\|H^{s_L}\tilde\psi_b\|_{L^2}}{\lambda^{2s_L-s_c}}+\frac{1}{\lambda^{2(2s_L-s_c)+4}}\int\big|H^{s_L}[(1-\tilde\chi)(b_1^{(0,1)}\Lambda Q+b_1^{(1,\cdot)}.\nabla Q]\big|^2\\ &\leq \frac{C(L)}{\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')}}+\frac{C\lambda^{2(\alpha-1)}}{s}+\frac{C}{s^{\frac{\alpha+1}{2}}}\\ &\leq \frac{C(L)}{\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')}}\end{aligned} \tag{5-40}$$

since $\alpha>1$; hence

$$\frac{\lambda^{2(\alpha-1)}}{s}+\frac{1}{s^{\frac{\alpha+1}{2}}}\ll 1,$$

since $1/\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')}\gg 1$ in the trapped regime from (4-52).

The nonlinear term. We begin by coming back to renormalized variables:

$$\|H^{s_L}_{z,\frac{1}{\lambda}}(\mathrm{NL}(w_{\mathrm{int}}))\|_{L^2}\leq\frac{\|H^{s_L}(\mathrm{NL}(\varepsilon))\|_{L^2}}{\lambda^{(2s_L-s_c)+2}}\leq C\sum_{k=2}^{p}\frac{\|H^{s_L}(\tilde Q_b^{p-k}\varepsilon^k)\|_{L^2}}{\lambda^{(2s_L-s_c)+2}} \tag{5-41}$$

because $\mathrm{NL}(\varepsilon)=\sum_{k=2}^{p}C_k^p\tilde Q_b^{p-k}\varepsilon^k$. We fix k with $2\leq k\leq p$ and study the corresponding term in the above sum. One has $H=-\Delta-pQ^{p-1}$, and Q is a smooth profile satisfying the estimate $Q=O((1+|y|)^{-\frac{2}{p-1}})$, which propagates to its derivatives from (2-1). Similarly, from (4-27) and (3-29), one has $\tilde Q_b=O((1+|y|)^{-\frac{2}{p-1}})$ and it propagates to the derivatives. The Leibniz rule for derivation then yields

$$\begin{aligned}\|H^{s_L}(\tilde{Q}_b^{p-k}\varepsilon^k)\|_{L^2}^2 &\le C(L)\sum_{\substack{\mu\in\mathbb{N}^d\\ 0\le|\mu|\le 2s_L}}\int\frac{|\partial^\mu(\varepsilon^k)|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2|\mu|}}\\ &\le C(L)\sum_{\substack{(\mu_i)_{1\le i\le k}\in\mathbb{N}^{kd}\\ \sum_{i=1}^k|\mu_i|\le 2s_L}}\int\frac{\prod_{i=1}^k|\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|}}.\end{aligned}\tag{5-42}$$

We fix $\mu_i\in\mathbb{N}^{kd}$ with $\sum|\mu_i|_1\le 2s_L$ and focus on the corresponding term in the above equation. Without loss of generality we order by increasing length: $|\mu_1|\le\cdots\le|\mu_k|$. We now distinguish between two cases.

Case 1: $|\mu_k|+\frac{2(p-k)}{p-1}+2s_L-\sum_{i=1}^k|\mu_i|\le 2s_L$. As one has

$$|\mu_k|_1+\frac{(p-k)}{p-1}+2s_L-\sum_{i=1}^k|\mu_i|_1\ge\sigma$$

because the $|\mu_i|_1$ are increasing and $\sum|\mu_i|_1\le 2s_L$, using (D-1)

$$\int\frac{|\partial^{\mu_k}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|_1}}\le C(M)\mathcal{E}_\sigma^{\frac{\sum|\mu_i|-|\mu_k|_1-\frac{2(p-k)}{p-1}}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{2s_L-\sigma-\sum|\mu_i|+|\mu_k|_1+\frac{2(p-k)}{p-1}}{2s_L-\sigma}}.$$

As the coefficients are in increasing order and L is arbitrarily very large, for $1\le j<k$ we have $|\mu_i|+\frac{d}{2}\le 2s_L$. We then recall the L^∞ estimate (D-3):

$$\|\partial^{\mu_i}\varepsilon\|_{L^\infty}\le\sqrt{\mathcal{E}_\sigma}^{\frac{2s_L-|\mu_i|_1-\frac{d}{2}}{2s_L-\sigma}+O(\frac{1}{L}^2)}\sqrt{\mathcal{E}_{2s_L}}^{\frac{|\mu_i|_1+\frac{d}{2}-\sigma}{2s_L-\sigma}+O(\frac{1}{L^2})}.$$

The two previous estimates imply that

$$\begin{aligned}\int\frac{\prod_{i=1}^k|\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|_1}} &\le\int\frac{|\partial^{\mu_k}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|_1}}\prod_{i=1}^{k-1}\|\partial^{\mu_i}\varepsilon\|_{L^\infty}^2\\ &\le\mathcal{E}_\sigma^{\frac{2(k-1)s_L-(k-1)\frac{d}{2}-2\frac{p-k}{p-1}}{2s_L-\sigma}+O(\frac{1}{L^2})}\mathcal{E}_{2s_L}^{\frac{(k-1)\frac{d}{2}-k\sigma+2s_L+2\frac{p-k}{p-1}}{2s_L-\sigma}+O(\frac{1}{L^2})}\\ &\le\mathcal{E}_\sigma^{k-1+\frac{-2+(k-1)(\sigma-s_c)}{2s_L-\sigma}+O(\frac{1}{L^2})}\mathcal{E}_{2s_L}^{1+\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}+O(\frac{1}{L^2})}\\ &\le\mathcal{E}_{2s_L}\left(\frac{\mathcal{E}_\sigma^{1+O(\frac{1}{L})}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{1+\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}.\end{aligned}\tag{5-43}$$

Case 2: $|\mu_k|+\frac{2(p-k)}{p-1}+2s_L-\sum_{i=1}^k|\mu_i|>2s_L$. This means $\frac{2(p-k)}{p-1}-\sum_{i=1}^{k-1}|\mu_i|>0$. Hence, there are two subcases: the subcase $|\mu_i|=0$ for $1\le i\le k-1$ and the subcase $|\mu_{k-1}|=1$ (because the μ_i are ordered by increasing size $|\mu_i|$). If $|\mu_i|=0$ for $1\le i\le k-1$, then, using the weighted L^∞ estimate

(D-2), the coercivity estimate (C-16) and the bound (4-25), we obtain

$$\begin{aligned}\int \frac{\prod_{i=1}^{k}|\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^{k}|\mu_i|}} &= \int \frac{|\varepsilon|^{2(k-1)}}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2|\mu_k|}} \\ &\leq \left\| \frac{\varepsilon}{1+|y|^{\frac{2(p-k)}{p-1}}}\right\|_{L^\infty}^2 \|\varepsilon\|_{L^\infty}^{2(k-2)}\mathcal{E}_{s_L} \\ &\leq \left(\frac{\mathcal{E}_\sigma^{1+O(\frac{1}{L})}}{s^{-(\sigma-s_c)}}\right)^{k-1}\frac{C(L,M,K_1,K_2)\mathcal{E}_{s_L}}{s^{1+\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}.\end{aligned}$$

If $|\mu_{k-1}|=1$, then, using the weighted L^∞ estimate (D-2) for $\nabla\varepsilon$, the coercivity estimate (C-16) and the bound (4-25), we obtain

$$\begin{aligned}\int \frac{\prod_{i=1}^{k}|\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^{k}|\mu_i|}} &= \int \frac{|\partial^{\mu_{k-1}}\varepsilon|^2|\varepsilon|^{2(k-2)}}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2|\mu_k|-2}} \\ &\leq \left\| \frac{\partial^{\mu_{k-1}}\varepsilon}{1+|y|^{\frac{2(p-k)}{p-1}-1}}\right\|_{L^\infty}^2 \|\varepsilon\|_{L^\infty}^{2(k-2)}\mathcal{E}_{s_L} \\ &\leq \left(\frac{\mathcal{E}_\sigma^{1+O(\frac{1}{L})}}{s^{-(\sigma-s_c)}}\right)^{k-1}\frac{C(L,M,K_1,K_2)\mathcal{E}_{s_L}}{s^{1+\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}.\end{aligned}$$

In both subcases, we have

$$\int \frac{\prod_{i=1}^{k}|\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^{k}|\mu_i|}} \leq \left(\frac{\mathcal{E}_\sigma^{1+O(\frac{1}{L})}}{s^{-(\sigma-s_c)}}\right)^{k-1}\frac{C(L,M,K_1,K_2)\mathcal{E}_{s_L}}{s^{1+\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}. \tag{5-44}$$

Now we come back to (5-41), which we reformulated in (5-42) where we estimated the terms appearing in the sum in (5-43) and (5-44), obtaining the following bound for the nonlinear term's contribution in (5-38):

$$\|H^{s_L}_{z,\frac{1}{\lambda}}(\mathrm{NL}(w_{\mathrm{int}}))\|_{L^2} \leq \frac{\sqrt{\mathcal{E}_{2s_L}}}{\lambda^{(2s_L-s_c)+2}}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O(\frac{1}{L})}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{1+\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}. \tag{5-45}$$

The small linear term and the term involving the time derivative of the linearized operator. We claim that there exists a constant $\delta := \delta(d,L,p)>0$ such that

$$\left\| H^{s_L}_{z,\frac{1}{\lambda}}(L(w_{\mathrm{int}})) + \frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w_{\mathrm{int}}\right\|_{L^2} \leq \frac{C(L,M)}{\lambda^{2s_L-s_c+2}s}\left(\int \frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}}. \tag{5-46}$$

We now prove this estimate. The small linear term is in renormalized variables by (4-36) and (4-37):

$$\int \left|H^{s_L}_{z,\frac{1}{\lambda}}(L(w_{\mathrm{int}}))\right|^2 = \frac{p^2}{\lambda^{2(2s_L-s_c)+4}}\int \left(H^{s_L}((Q^{p-1}-\tilde{Q}_b^{p-1})\varepsilon)\right)^2.$$

For $\mu \in \mathbb{N}^s$, one has the following asymptotic behavior for the potential that appeared, from the bounds on the parameters (4-27) and the expression of $\widetilde{Q}_b$ (3-29):

$$\left|\partial^\mu (Q^{p-1} - \widetilde{Q}_b^{p-1})\right| \leq \frac{1}{s} \frac{C(\mu)}{1+|y|^{\alpha - C(L)\eta + |\mu|}} \leq \frac{1}{s} \frac{C(\mu)}{1+|y|^{\delta+|\mu|}}$$

for η small enough, because $\alpha > 2$, and for some constant δ that can be chosen small enough so that

$$0 < \delta \ll 1, \quad \text{with } \delta < \sup_{0 \leq n \leq n_0} \delta_n \text{ and } \delta < \tfrac{1}{4}d - \tfrac{1}{2}\gamma_{n_0+1} - s_L. \tag{5-47}$$

(This technical condition is useful to apply a coercivity estimate for the next equation; all the terms appearing are indeed strictly positive by (1-25).) We recall that $H = -\Delta - pQ^{p-1}$, where Q is a smooth potential satisfying

$$|\partial^\mu Q| \leq \frac{C(\mu)}{1+|y|^{\frac{2}{p-1}+|\mu|}}.$$

Using the Leibniz rule, this implies

$$\int \left(H^{s_L}((Q^{p-1} - \widetilde{Q}_b^{p-1})\varepsilon)\right)^2 \leq \frac{C(L)}{s^2} \sum_{\substack{\mu_i \in \mathbb{N}^d \\ |\mu_i| \leq 2s_L,\, i=1,2}} \int \frac{|\partial^{\mu_1}\varepsilon||\partial^{\mu_2}\varepsilon|}{1+|y|^{4s_L+2\delta-2|\mu_1|-2|\mu_2|}} \leq \frac{C(L)}{s^2} \int \frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}, \tag{5-48}$$

where we used for the last line the weighted coercivity estimate (C-16), which we could apply because δ satisfies the technical condition (5-47). We now turn to the term involving the time derivative of the linearized operator in (5-38). Going back to renormalized variables, it can be written as

$$\int \left|\frac{d}{dt} H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}}\right|^2 = \frac{p^2(p-1)^2}{\lambda^{2(2s_L-s_c)+4}} \sum_{i=1}^{s_L} \int \left(H^{i-1}\left[\left(Q^{p-2}\frac{z_s}{\lambda}.\nabla Q + \frac{\lambda_s}{\lambda} Q^{p-2}\Lambda Q\right) H^{s_L-i}\varepsilon\right]\right)^2.$$

For $\mu \in \mathbb{N}^d$, one has the following asymptotic behavior for the two potentials that appeared (from the asymptotic (2-1) and (2-7) of Q and ΛQ):

$$|\partial^\mu (Q^{p-2}\partial_{y_i} Q)| \leq \frac{C(\mu)}{1+|y|^{2+1+|\mu|}} \quad \text{for } 1 \leq i \leq d, \qquad \text{and} \qquad |\partial^\mu (Q^{p-2}\Lambda Q)| \leq \frac{C(\mu)}{1+|y|^{2+\alpha}}.$$

Therefore, as $H = -\Delta - pQ^{p-1}$, where Q is a smooth potential satisfying

$$|\partial^\mu Q| \leq \frac{C(\mu)}{1+|y|^{\frac{2}{p-1}+|\mu|_1}},$$

using the Leibniz rule and the two above identities,

$$\left|\int H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}} \frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}}) w_{\text{int}}\right| \leq \frac{C(L)\left(\left|\frac{\lambda_s}{\lambda}\right|^2+\left|\frac{z_s}{\lambda}\right|^2\right)}{\lambda^{2(2s_L-s_c)+4}} \sum_{\substack{\mu_i\in\mathbb{N}^d\\ |\mu_i|_1\leq 2s_L,\, i=1,2}} \int \frac{|\partial^{\mu_1}\varepsilon||\partial^{\mu_2}\varepsilon|}{1+|y|^{4s_L+2-2|\mu_1|-2|\mu_2|}}$$
$$\leq \frac{C(L)}{\lambda^{2(2s_L-s_c)+4}\, s^2} \sum_{\substack{\mu_i\in\mathbb{N}^d\\ |\mu_i|_1\leq 2s_L,\, i=1,2}} \int \frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}} \tag{5-49}$$

for $\delta<\alpha$, 1 being defined by (5-47), where we used the weighted coercivity estimate (C-16) and the fact that $\left|\frac{\lambda_s}{\lambda}\right|\sim s^{-1}$ and $\left|\frac{z_s}{\lambda}\right|\sim s^{-1-\frac{\alpha-1}{2}}$ by (4-43) and (4-27). We now combine the estimates we have proved, (5-48) and (5-49), to obtain the estimate (5-46) we claimed.

End of the proof of Step 1. We now gather the brute force upper bounds we have found for the terms we had to treat in (5-40), (5-45) and (5-46), yielding the bound (5-39) we claimed in this first step.

Step 2: integration by parts in time to treat the modulation term. We now focus on the modulation term in (5-38) which requires a careful treatment. Indeed, the brute force upper bounds on the modulation (4-43) are not sufficient and we need to make an integration by parts in time to treat the problematic term $b^{(n,k)}_{L_n,s}$. We do this in two steps. First we define a radiation term. Next we use it to prove a modified energy estimate.

Definition of the radiation. We recall that $\alpha_b=\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)}+\sum_{i=2}^{L+2}S_i$, where $T_i^{(n,k)}$ is defined by (2-26) and S_i is homogeneous of degree $(i,-\gamma-g')$ in the sense of Definition 2.14; see (3-8). We want to split α_b in two parts to distinguish the problematic terms involving the parameters $b^{(n,k)}_{L_n}$. For $i=2,\dots,L+2$, as S_i is homogeneous of degree $(i,-\gamma-g')$, it is a finite sum

$$S_i=\sum_{J\in\mathcal{J}(i)} b^J f_J, \quad \text{with } b^J=\prod_{(n,k,i)\in\mathcal{I}} (b_i^{(n,k)})^{J_i^{(n,k)}}, \tag{5-50}$$

where $\mathcal{J}(i)$ is a finite subset of $\mathbb{N}^{\#\mathcal{I}}$ and for all $J\in\mathcal{J}(i)$, $|J|_3=i$ and f_J is admissible of degree $(2|J|_2-\gamma-g')$ in the sense of Definition 2.11. We then define the following partition of $\mathcal{J}(i)$:

$$\begin{gathered}
\mathcal{J}_1(i):=\left\{J\in\mathcal{J}(i),\ J^{(n,k)}_{L_n}=0 \text{ for all } 0\leq n\leq n_0,\ 1\leq k\leq k(n)\right\},\\
\mathcal{J}_2(i):=\left\{J\in\mathcal{J}(i),\ |J|=2 \text{ and } \exists (n,k,L_n)\in\mathcal{I},\ J^{(n,k)}_{L_n}\geq 1\right\},\\
\mathcal{J}_3(i):=\mathcal{J}(i)\backslash[\mathcal{J}_1(i)\cup\mathcal{J}_2(i)],\\
\bar{S}_i:=\sum_{J\in\mathcal{J}_2(i)} b^J f_J, \quad \bar{S}'_i:=\sum_{J\in\mathcal{J}_3(i)} b^J f_J,
\end{gathered} \tag{5-51}$$

and the following radiation term:

$$\xi:=H^{s_L}\left(\chi_{B_1}\left[\sum_{\substack{0\leq n\leq n_0\\ 1\leq k\leq k(n)}} b^{(n,k)}_{L_n}T^{(n,k)}_{L_n}+\sum_{i=2}^{L+2}\bar{S}'_i\right]\right)+\sum_{i=2}^{L+2} H^{s_L}(\chi_{B_1}\bar{S}_i)-\chi_{B_1}H^{s_L}\bar{S}_i. \tag{5-52}$$

From (5-51), for all $J \in \mathcal{J}_3(i)$ there exists n with $0 \le n \le n_0$ such that $J_{L_n}^{(n,k)} \ge 1$ and $|J| \ge 3$. As $\delta_{n'} > 0$, this implies

$$\forall J \in \mathcal{J}_3(i), \quad |J|_2 > L + 2 - \delta_0. \tag{5-53}$$

Using this fact, (2-7), the fact that $H^{s_L} T_{L_n}^{(n,k)} = 0$ since $s_L > L_n$ for all $0 \le n \le n_0$, (5-51) and (4-27), the radiation satisfies

$$\|\xi\|_{L^2} \le \frac{C(L,M)}{s^{L+1-\delta_0+\eta(1-\delta_0')}}, \qquad \|H\xi\|_{L^2} \le \frac{C(L,M)}{s^{L+2-\delta_0+\eta(2-\delta_0')}}, \tag{5-54}$$

$$\|\nabla\xi\|_{L^2} \le \frac{C(L,M)}{s^{L+\frac{3}{2}-\delta_0+\eta(\frac{3}{2}-\delta_0')}}, \qquad \|\Lambda\xi\|_{L^2} \le \frac{C(L,M)}{s^{L+1-\delta_0+\eta(1-\delta_0')}}. \tag{5-55}$$

We eventually introduce the remainders

$$\begin{aligned}
R_1 := {} & H^{s_L}\Bigg(\chi_{B_1} \sum_{(n,k,i)\in\mathcal{I},\, i\neq L_n} \big(b_{i,s}^{(n,k)} + (2i-\alpha_n) b_i^{(n,k)} b_1^{(0,1)} - b_{i+1}^{(n,k)}\big)\Bigg(T_i^{(n,k)} + \sum_{j=2}^{L+2} \frac{\partial S_j}{\partial b_i^{(n,k)}}\Bigg)\Bigg) \\
& - \Big(\frac{\lambda_s}{\lambda} + b_1^{(0,1)}\Big) H^{s_L} \Lambda \tilde{Q}_b - \Big(\frac{z_s}{\lambda} + b_1^{(1,\cdot)}\Big).H^{s_L}\nabla\tilde{Q}_b \\
& + H^{s_L}\Bigg(\chi_{B_1} \sum_{(n,k,L_n)\in\mathcal{I}} (2L_n-\alpha_n) b_{L_n}^{(n,k)} b_1^{(0,1)} \Bigg(T_{L_n}^{(n,k)} + \sum_{j=2}^{L+2} \frac{\partial \bar{S}_j'}{\partial b_{L_n}^{(n,k)}}\Bigg)\Bigg) \\
& + \sum_{(n,k,L_n)\in\mathcal{I}} (2L_n-\alpha_n) b_{L_n}^{(n,k)} b_1^{(0,1)} \Bigg(\sum_{j=2}^{L+2} H^{s_L}\Big(\chi_{B_1} \frac{\partial \bar{S}_j}{\partial b_{L_n}^{(n,k)}}\Big) - \chi_{B_1} H^{s_L} \frac{\partial \bar{S}_j}{\partial b_{L_n}^{(n,k)}}\Bigg) \\
R_2 := {} & \sum_{(n,k,L_n)\in\mathcal{I}} \big(b_{L_n,s}^{(n,k)} + (2L_n-\alpha_n) b_{L_n}^{(n,k)} b_1^{(0,1)}\big)\Bigg(\sum_{j=2}^{L+2} \chi_{B_1} H^{s_L} \frac{\partial \bar{S}_j}{\partial b_{L_n}^{(n,k)}}\Bigg), \\
R_3 := {} & \sum_{(n,k,i)\in\mathcal{I},\, i\neq L_n} b_{i,s}^{(n,k)} \frac{\partial}{\partial_{b_i^{(n,k)}}} \xi,
\end{aligned}$$

so that they produce, by (5-52) and (4-33), the identity

$$H^{s_L}(\widetilde{\mathrm{Mod}}(s)) = \partial_s \xi + R_1 + R_2 + R_3. \tag{5-56}$$

The remainder R_1 enjoys the following bounds by (4-43), (2-22), (3-8), (5-51), (5-53) and (4-27):

$$\|R_1\|_{L^2} \le \frac{C(L,M)}{s^{L+2-\delta_0+(1-\delta_0')\eta}} + \frac{C(L,M)\mathcal{E}_{2s_L}}{s^2}. \tag{5-57}$$

From the definition (5-51) of $\mathcal{S}_j$ and the construction (3-25) of S_j, one has

$$\begin{aligned}
\sum_{j=2}^{L+2} H\bar{S}_j = {} & -\sum_{(n,k,L_n)\in\mathcal{I}} b_1^{(0,1)} b_{L_n}^{(n,k)} \big(\Lambda T_{L_n}^{(n,k)} - (2L_n-\alpha_n) T_{L_n}^{(n,k)}\big) - \sum_{(n,k,L_n)\in\mathcal{I}} b_{L_n}^{(n,k)} b_1^{(1,\cdot)}.\nabla\Lambda T_{L_n}^{(n,k)} \\
& + p(p-1)Q^{p-2}\Bigg(\sum_{(n,k,L_n)\in\mathcal{I}} b_{L_n}^{(n,k)} T_{L_n}^{(n,k)}\Bigg)\Bigg(\sum_{(n',k',i)\in\mathcal{I}} b_i^{(n',k')} T_i^{(n',k')}\Bigg).
\end{aligned}$$

As $H^{s_L}T^{(n,k)}_{L_n}=0$ since $s_L>L_n$ for all $0\le n\le n_0$, using the commutator identity (2-24), the asymptotic (2-22) of $T_i^{(n,k)}$, (4-27) and (2-2) (as $\alpha>2$), one has

$$\int (1+|y|^{4+2\delta})\left(\chi_{B_1}H^{s_L+1}\sum_{j=2}^{L+2}\frac{\partial \bar S_j}{\partial_{b^{(n,k)}_{L_n}}}\right)^2\le \frac{C(L)}{s},$$

where δ is defined by (5-47), from which we deduce, using (4-44),

$$\left\|(1+|y|)^{2+\delta}HR_2\right\|_{L^2}\le \frac{C(L,M)}{s^{L+4}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s}. \tag{5-58}$$

Finally for the last remainder, from (5-52), (4-43), (4-27), (4-25), (2-22) and (5-51), for s_0 large enough one has the estimate

$$\|R_3\|_{L^2}\le \frac{C(L,M)}{s^{L+2-\delta_0+\eta(1-\delta_0')}}. \tag{5-59}$$

Modified energy estimate. We now prove the modified energy estimate (compared to (5-38))

$$\begin{aligned}
&\frac{d}{dt}\left\{\int (H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}}))^2\right\}\\
&\quad\le \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\Bigg[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}}+C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac12}\\
&\qquad\qquad+\mathcal{E}_{2s_L}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O(\frac1L)}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c}{L})}}\Bigg]-2\int H^{s_L}_{z,\frac1\lambda}w_{\text{int}}H^{s_L+1}_{z,\frac1\lambda}w_{\text{int}}\\
&\qquad\qquad+2\int H^{s_L}_{z,\frac1\lambda}w_{\text{int}}H^{s_L}_{z,\frac1\lambda}(\widetilde{L}+\widetilde{R}+\widetilde{\text{NL}}).
\end{aligned} \tag{5-60}$$

From the time evolution (5-56), (4-32) of ξ and w and because the support of $\tau_z(\xi_{1/\lambda})$ is disjoint from the one of $\widetilde{L}$, $\widetilde{R}$, and $\widetilde{\text{NL}}$, one gets the following expression for the left-hand side of (5-60):

$$\begin{aligned}
&\frac{d}{dt}\left\{\int\left(H^{s_L}_{z,\frac1\lambda}w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac1\lambda})\right)^2\right\}\\
&\quad=-2\int H^{s_L}_{z,\frac1\lambda}w_{\text{int}}H^{s_L+1}_{z,\frac1\lambda}w_{\text{int}}-\frac{2}{\lambda^{2s_L+2}}\int H^{s_L}_{z,\frac1\lambda}w_{\text{int}}\tau_z(R_{2,\frac1\lambda})-\frac{2}{\lambda^{2s_L}}\int\tau_z(\xi_{\frac1\lambda})H^{s_L+1}_{z,\frac1\lambda}w_{\text{int}}\\
&\qquad+2\int\left[H^{s_L}_{z,\frac1\lambda}w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac1\lambda})\right]\Bigg[H^{s_L}_{z,\frac1\lambda}\left(\text{NL}(w_{\text{int}})-\frac{1}{\lambda^2}\tau_z(\tilde\psi_{b,\frac1\lambda}+(\chi-1)\widetilde{\text{Mod}}(t)_{\frac1\lambda})+L(w_{\text{int}})\right)\\
&\qquad\qquad+\frac{d}{dt}(H^{s_L}_{z,\frac1\lambda})w_{\text{int}}-\frac{1}{\lambda^{2+2s_L}}\tau_z\left(\left(R_1+R_3+\frac{\lambda_s}{\lambda}\Lambda\xi+2s_L\frac{\lambda_s}{\lambda}\xi-\frac{z_s}{\lambda}.\nabla\xi\right)_{\frac1\lambda}\right)\Bigg]\\
&\qquad-\frac{2}{\lambda^{4s_L+2}}\int\tau_z(\xi_{\frac1\lambda})\tau_z(R_{2,\frac1\lambda})+2\int H^{s_L}_{z,\frac1\lambda}w_{\text{int}}H^{s_L}_{z,\frac1\lambda}(\tilde L+\widetilde{\text{NL}}+\widetilde{R}).
\end{aligned} \tag{5-61}$$

We now analyse all the terms in this identity, except the first one and the last one, which we will study in the next step. Using the estimate (5-58) on the remainder R_2, going back in renormalized variables and using the coercivity (C-16), one gets for the second term in (5-61)

$$\begin{aligned}\left|\frac{2}{\lambda^{2s_L+2}}\int H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}}\tau_z(R_{2,\frac{1}{\lambda}})\right| &\leq C\int \frac{|H^{s_L-1}\varepsilon|}{1+|y|^{2+\delta}}(1+|y|^{2+\delta})|HR_2|\\ &\leq \frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{\lambda^{2(2s_L-s_c)+2}s}\left(\left(\int \frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}}+\frac{1}{s^{L+3}}\right).\end{aligned}$$

Going back to renormalized variables, integrating by parts and using the estimate (5-54) on $H\xi$ gives for the third term in (5-61)

$$\left|\frac{2}{\lambda^{2s_L}}\int \tau_z(\xi_{\frac{1}{\lambda}})H^{s_L+1}_{z,\frac{1}{\lambda}} w_{\text{int}}\right| \leq \frac{C(L,M)}{\lambda^{2(2s_L-s_c)+2}}\frac{\sqrt{\mathcal{E}_{2s_L}}}{s^{L+2-\delta_0+\eta(2-\delta_0')}}.$$

To bound the fourth and the fifth terms in (5-61) from above, we go back to renormalized variables and use the bound (5-39) on the error, the nonlinear term, the small linear term and the term involving the time derivative of the linearized operator we derived in Step 1, together with the bounds (5-54) and (5-55) on ξ, $\Lambda\xi$, $\nabla\xi$ and the fact that

$$\left|\frac{\lambda_s}{\lambda}\right| \leq Cs^{-1} \quad \text{and} \quad \left|\frac{z_s}{\lambda}\right| \leq Cs^{-1-\frac{\alpha-1}{2}}$$

in the trapped regime, and the bound (5-57) and (5-59) on the remainders R_1 and R_3, yielding

$$\begin{aligned}&\left|\int\left[H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}})\right]\left[H^{s_L}_{z,\frac{1}{\lambda}}\big(\text{NL}(w_{\text{int}})-\frac{1}{\lambda^2}\tau_z(\tilde{\psi}_{b,\frac{1}{\lambda}}+(\chi-1)\widetilde{\text{Mod}}(t)_{\frac{1}{\lambda}})+L(w_{\text{int}})\big)\right.\right.\\ &\quad\left.\left.+\frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w-\frac{1}{\lambda^{2+2s_L}}\tau_z\left(\left(R_1+R_3+\frac{\lambda_s}{\lambda}\Lambda\xi+2s_L\frac{\lambda_s}{\lambda}\xi-\frac{z_s}{\lambda}.\nabla\xi\right)_{\frac{1}{\lambda}}\right)\right]-\frac{2}{\lambda^{4s_L+2}}\int \tau_z(\xi_{\frac{1}{\lambda}})\tau_z(R_{1,\frac{1}{\lambda}})\right|\\ &\leq \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\left[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}}+C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|x|^{2\delta}}\right)^{\frac{1}{2}}\right.\\ &\qquad\left.+\mathcal{E}_{2s_L}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O(\frac{1}{L})}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c}{L})}}\right].\end{aligned}$$

We finish the proof of the bound (5-60) by inserting into the identity (5-61) the three previous bounds we proved on the second, third, fourth and fifth terms.

Step 3: use of dissipation. We find an upper bound for the last terms in (5-60) and improve the energy estimate using the coercivity of the quantity $-\int H^{s_L+1}\varepsilon H^{s_L}\varepsilon$.

The dissipation estimate. We recall that $H=-\Delta-pQ^{p-1}$, the potential $-pQ^{p-1}$ being the Hardy potential

$$pQ^{p-1}<\frac{(d-2)^2-4\delta(p)}{4|y|^2}$$

for some constant $\delta(p)>0$ by (2-5). Hence, using the standard Hardy inequality one gets for the linear term

$$
\begin{aligned}
-\int H^{s_L}_{z,\frac{1}{\lambda}} w_{\mathrm{int}} H_{z,\frac{1}{\lambda}} H^{s_L}_{z,\frac{1}{\lambda}} w_{\mathrm{int}}
&= -\frac{1}{\lambda^{2(2s-L-s_c)+2}} \int H^{s_L}\varepsilon H H^{s_L}\varepsilon \\
&= \frac{1}{\lambda^{2(2s-L-s_c)+2}} \left(-\int |\nabla H^{s_L}\varepsilon|^2 + \int pQ^{p-1}|H^{s_L}\varepsilon|^2 \right) \\
&= \frac{1}{\lambda^{2(2s-L-s_c)+2}} \left(\left[\frac{(d-2)^2-\frac{1}{2}\delta(p)}{(d-2)^2} + \frac{\delta(p)}{2(d-2)^2} \right] \int |\nabla H^{s_L}\varepsilon|^2 + \int pQ^{p-1}|H^{s_L}\varepsilon|^2 \right) \\
&\le \frac{1}{\lambda^{2(2s-L-s_c)+2}} \left(-\frac{(d-2)^2-\frac{1}{2}\delta(p)}{4} \int \frac{|H^{s_L}\varepsilon|^2}{|y|^2} - \frac{\delta(p)}{2(d-2)^2} \int |\nabla H^{s_L}\varepsilon|^2 \right. \\
&\qquad\qquad \left. + \frac{(d-2)^2-\delta(p)}{4} \int \frac{|H^{s_L}\varepsilon|^2}{|y|^2} \right) \\
&= -\frac{\delta(p)}{8\lambda^{2(2s-L-s_c)+2}} \int \frac{|H^{s_L}\varepsilon|^2}{|y|^2} - \frac{\delta(p)}{2(d-2)^2\lambda^{2(2s-L-s_c)+2}} \int |\nabla H^{s_L}\varepsilon|^2. \qquad (5\text{-}62)
\end{aligned}
$$

Bounds for the terms created by the cut. We study the last terms in (5-60). From its definition (4-40), and as $\lambda+|z|\ll 1$ by (4-52) and (4-51), the remainder $\widetilde{R}$ is bounded by a constant independent of the others:

$$
\|H^{s_L}_{z,\frac{1}{\lambda}} \widetilde{R}\|_{L^2} \le C. \qquad (5\text{-}63)
$$

For the nonlinear term, for any very small $\kappa>0$, by (D-4), (4-39) and (4-28),

$$
\begin{aligned}
\|H^{s_L}_{z,\frac{1}{\lambda}} \widetilde{\mathrm{NL}}\|_{L^2} &\le C \sum_{k=2}^{p} \|w^k\|_{H^{2s_L}} \\
&\le C\|w\|_{H^{2s_L}} \sum_{k=2}^{p} \|w\|^{k-1}_{H^{d/2+\kappa}} \\
&\le C\|w\|_{H^{2s_L}} \sum_{k=2}^{p} \|w\|_{H^\sigma}^{(k-1)\left(1-\frac{d/2+\kappa-\sigma}{2s_L-\sigma}\right)} \|w\|_{H^{2s_L}}^{(k-1)\left(\frac{d/2+\kappa-\sigma}{2s_L-\sigma}\right)} \\
&\le C(K_1,K_2) \left(\frac{1}{\lambda^{2s_L-s_c}\, s^{L+1-\delta_0+\eta(1-\delta_0')}} \right)^{1+(p-1)\frac{d/2+\kappa-\sigma}{2s_L-\sigma}} \\
&= C(K_1,K_2) \left(\frac{1}{\lambda^{2s_L-s_c}\, s^{L+1-\delta_0+\eta(1-\delta_0')}} \right)^{1+(p-1)\frac{2/(p-1)-\sigma-s_c+\kappa}{2s_L-\sigma}} \\
&\le C(K_1,K_2) \left(\frac{1}{\lambda^{2s_L-s_c}\, s^{L+1-\delta_0+\eta(1-\delta_0')}} \right)^{1+\frac{2}{2s_L-\sigma}} \\
&= \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c+2}\, s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}} \qquad (5\text{-}64)
\end{aligned}
$$

because $1/\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}\gg 1$ by (4-52), if κ has been chosen small enough. For the extra linear term in (5-60), performing an integration by parts, using Young's inequality for any $\varepsilon>0$, (4-25) and (4-28) give

$$\begin{aligned}\left|\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\mathrm{int}}H^{s_L}_{z,\frac{1}{\lambda}}\widetilde{L}\right| &= \left|\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\mathrm{int}}H^{s_L}_{z,\frac{1}{\lambda}}\bigl[-\Delta\chi_3 w-2\nabla\chi_3.\nabla w+p\tau_z Q^{p-1}_{\frac{1}{\lambda}}(\chi_1^{p-1}-\chi_3)w\bigr]\right|\\ &\le C\,\|H^{s_L}_{z,\frac{1}{\lambda}}w_{\mathrm{int}}\|_{L^2}\,\|w\|_{H^{2s_L}}+C\varepsilon\|\nabla H^{s_L}_{z,\frac{1}{\lambda}}w_{\mathrm{int}}\|^2_{L^2}+\frac{C}{\varepsilon}\|w_{\mathrm{int}}\|^2_{H^{2s_L}}\\ &\le C\varepsilon\|\nabla H^{s_L}_{z,\frac{1}{\lambda}}w_{\mathrm{int}}\|^2_{L^2}+\frac{C(K_1,K_2,\varepsilon)}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}\\ &\le \frac{C\varepsilon}{\lambda^{2(2s-L-s_c)+2}}\int|\nabla H^{s_L}\varepsilon|^2+\frac{C(K_1,K_2,\varepsilon)}{\lambda^{2(2s_L-s_c)+2}s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}\end{aligned}\tag{5-65}$$

because in the trapped regime $\lambda^2 s\sim s^{-\frac{\alpha}{2\ell-\alpha}}$ by (4-52).

Conclusion. We insert into the modified energy estimate (5-60) the bounds (5-62), (5-63), (5-64) and (5-65), yielding

$$\begin{aligned}&\frac{d}{dt}\left\{\int\left(H^{s_L}_{z,\frac{1}{\lambda}}w_{\mathrm{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}})\right)^2\right\}\\ &\le\frac{1}{\lambda^{2(2s_L-s_c)+2}s}\Biggl[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}}+C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}}\\ &\qquad+\mathcal{E}_{2s_L}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c}{L})}}-\frac{s\delta(p)}{8}\int\frac{|H^{s_L}\varepsilon|^2}{|y|^2}-\frac{s\delta(p)}{2(d-2)^2}\int|\nabla H^{s_L}\varepsilon|^2\\ &\qquad+C\varepsilon s\int|\nabla H^{s_L}\varepsilon|^2+\frac{C(K_1,K_2,M,L)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}\Biggr].\end{aligned}\tag{5-66}$$

For any $N\gg 1$, using Young's inequality and splitting the weighted integrals in the zone $|y|\le N^2$ and $|y|\ge N^2$ gives for ε small enough and s_0 large enough,

$$\begin{aligned}&C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}}-\frac{s\delta(p)-sC\varepsilon}{8}\int\frac{|H^{s_L}}{|y|^2}\\ &\qquad\le\frac{C(L,M)\mathcal{E}_{2s_L}}{N^{2\delta}}+C(L,M)N^{2\delta}\int_{|y|\le N^2}\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}-\frac{s\delta(p)}{16}\int\frac{|H^{s_L}\varepsilon|^2}{|y|^2}\le\frac{C(L,M)\mathcal{E}_{2s_L}}{N^{2\delta}}.\end{aligned}$$

Finally, from the bound (5-54) on the size of ξ, one has

$$\begin{aligned}&\frac{d}{dt}\left\{\int\left(H^{s_L}_{z,\frac{1}{\lambda}}w+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}})\right)^2\right\}\\ &=\frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}\right\}+\frac{d}{dt}\left\{\int\frac{2}{\lambda^{2s_L}}H^{s_L}_{z,\frac{1}{\lambda}}w\tau_z(\xi_{\frac{1}{\lambda}})+\frac{1}{\lambda^{4s_L}}(\tau_z(\xi_{\frac{1}{\lambda}}))^2\right\}\\ &=\frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}\right\}+\frac{d}{dt}\left\{O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}\left(\sqrt{\mathcal{E}_{2s_L}}+\frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)\right)\right\},\end{aligned}$$

where $O_{L,M}(\cdot)$ denotes the usual $O(\cdot)$ for a constant in the upper bound that depends only on L and M. Plugging the two previous identities into the modified energy estimate (5-66) yields the bound (5-37) we claimed in this proposition. ☐

Proposition 5.8 (Lyapunov monotonicity for the high regularity Sobolev norm of the remainder outside the blow-up zone). *Suppose all the constants of Proposition 4.6 are fixed except s_0. Then for s_0 large enough, for any solution u that is trapped on $[s_0, s')$ the following holds for $0 \le t < t(s')$:*

$$\begin{aligned}\|w_{\text{ext}}\|^2_{H^{2s_L}} \le \|\partial_t^{s_L} w_{\text{ext}}(0)\|^2_{L^2} &+ \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}} dt' \\ &+ \int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}} dt' \\ &+ \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}.\end{aligned} \tag{5-67}$$

Proof. From the time evolution of w_{ext}, given in Section 4B1, we get

$$\partial_t^{k+1} w_{\text{ext}} = \Delta \partial_t^k w_{\text{ext}} + (1-\chi_3)\partial_t^k(w^p) + \Delta\chi_3 \partial_t^k w + 2\nabla\chi_3.\nabla\partial_t^k w. \tag{5-68}$$

We make an energy estimate for $\partial_t^{s_L} w_{\text{ext}}$ and propagate this bound via elliptic regularity by iterations, which is standard in the study of parabolic problems. All computations, unless mentioned, are performed on Ω, and we omit this in the notation for simplicity.

Step 1: estimate on the force terms. We first prove some estimates on the force terms on the right-hand side of (5-68). From the decomposition (4-10) and the evolution (4-32) of w, in the exterior zone $\Omega\backslash\mathcal{B}^d(2)$, $\partial_t^k w$ can be written as

$$\partial_t^k w = \sum_{j=0}^{k} \sum C(\mu) \prod_{i=1}^{1+j(p-1)} \partial^{\mu_i} w \tag{5-69}$$

for some constants $C(\mu)$, where the inner sum is over $\mu = (\mu_i)_{1\le i\le 1+j(p-1)} \in \mathbb{N}^{dk(p-1)}$ with $\sum_{i=1}^{1+j(p-1)} |\mu_i|_1 = 2(k-j)$. Fix $k \le s_L$, an integer j with $0 \le j \le k$, and a sequence of d-tuples $(\mu_i)_{1\le i\le 1+k(p-1)} \in \mathbb{N}^{dk(p-1)}$ satisfying $\sum_{i=1}^{1+j(p-1)} |\mu_i| = 2(k-j)$. One can assume that the d-tuples μ_i are ordered by decreasing length: $|\mu_1| \ge |\mu_2| \ge \cdots$.

The case $k = s_L$. We want to estimate the above term in the zone $\Omega\backslash\mathcal{B}^d(2)$.

<u>Subcase 1:</u> $|\mu_1| \ge \sigma$. Using Hölder, Sobolev embedding (since in that case $\mu_i < 2s_L - \frac{d}{2}$ for $2 \le i \le 1+j(p-1)$), interpolation and (4-28), for $\kappa > 0$ small enough,

$$\begin{aligned}\left\| \prod_{i=1}^{1+j(p-1)} \partial^{\mu_i} w \right\|_{L^2} &\le \|\partial^{\mu_1} w\|_{L^2} \prod_{i=2}^{1+j(p-1)} \|\partial^{\mu_i} w\|_{L^\infty} \\ &\le \|w\|_{H^{|\mu_1|}} \prod_{i=2}^{1+j(p-1)} \|w\|_{H^{d/2+\kappa+|\mu_i|}}\end{aligned}$$

$$
\begin{aligned}
&\le C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{|\mu_1|-\sigma+\sum_{i=2}^{1+j(p-1)}|\mu_i|+d/2+\kappa-\sigma}{2s_L-\sigma}}\\
&= C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{1-\frac{(j(p-1)-1)(\sigma-s_c-\kappa)}{2s_L-\sigma}}\\
&\le \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}},
\end{aligned}
\tag{5-70}
$$

as $1/\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}\gg 1$ by (4-52).

Subcase 2: $|\mu_1|<\sigma$. Then $\mu_i<\sigma$ for all $1\le i\le j(p-1)$ and $\partial^{\mu_i}w\in L^{p_i}$ with p_i given by

$$\frac{1}{p_i}=\frac{1}{2}-\frac{\sigma-|\mu_i|}{d}$$

by Sobolev embedding. We define i_0 as the integer $2\le i_0\le 1+j(p-1)$ such that $\sum_{i=1}^{i_0-1}\frac{1}{p_i}<\frac{1}{2}$ and $\sum_{i=1}^{i_0}\frac{1}{p_i}\ge\frac{1}{2}$. We know i_0 exists because $\frac{1}{p_1}<\frac{1}{2}$ and $\sum_{i=1}^{1+j(p-1)}\frac{1}{p_i}\gg\frac{1}{2}$. We define $\tilde p_{i_0}>2$ by $\frac{1}{\tilde p_{i_0}}=\frac{1}{2}-\sum_{i=1}^{i_0-1}\frac{1}{p_i}$ and $\tilde s\ge\sigma$ as the regularity giving the Sobolev embedding $H^{\tilde s-|\mu_{i_0}|}\to L^{\tilde p_{i_0}}$:

$$\tilde s=\sum_{i=1}^{i_0}|\mu_i|+(i_0-1)\left(\frac{d}{2}-\sigma\right).$$

This implies that $\prod_{i=1}^{i_0}\partial^{\mu_i}w\in L^2$ with the estimate (from Hölder inequality)

$$\left\|\prod_{i=1}^{i_0}\partial^{\mu_i}w\right\|_{L^2}\le C\|\partial^{\mu_{i_0}}w\|_{L^{\tilde p_{i_0}}}\prod_{i=1}^{i_0-1}\|\partial^{\mu_i}w\|_{L^{p_i}}\le\|w\|_{H^{\tilde s}}\prod_{i=1}^{i_0-1}\|w\|_{H^\sigma}\le C(K_1)\|w\|_{H^{2s_L}}^{\frac{\tilde s-\sigma}{2s_L-\sigma}},$$

where we used interpolation and (4-25). Therefore, for $\kappa>0$ small enough, using Sobolev embedding, the above estimate, interpolation and (4-25),

$$
\begin{aligned}
\left\|\prod_{i=1}^{1+j(p-1)}\partial^{\mu_i}w\right\|_{L^2}&\le\left\|\prod_{i=1}^{i_0}\partial^{\mu_i}w\right\|_{L^2}\prod_{i=i_0+1}^{1+j(p-1)}\|w\|_{H^{\frac{d}{2}+\kappa+|\mu_i|}}\\
&\le C(K_1)\|w\|_{H^{2s_L}}^{\frac{\tilde s-\sigma}{2s_L-\sigma}}\prod_{i=i_0+1}^{1+j(p-1)}\|w\|_{H^\sigma}^{1-\frac{d/2+\kappa+|\mu_i|-\sigma}{2s_L-\sigma}}\|w\|_{H^{2s_L}}^{\frac{d/2+\kappa+|\mu_i|-\sigma}{2s_L-\sigma}}\\
&\le C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{2s_L-\sigma-j(p-1)(\sigma-s_c)+(j(p-1)-i_0+1)\kappa}{2s_L-\sigma}}\\
&\le C(K_1,K_2)\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}
\end{aligned}
\tag{5-71}
$$

as $1/\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}\gg 1$ by (4-52).

End of substep 1. Inserting (5-70) and (5-71) into the identity we obtain

$$\|\partial_t^{s_L} w\|_{L^2(\Omega\backslash\mathcal{B}^d(2))} \le \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}}. \tag{5-72}$$

Estimate for the nonlinear term in (5-68). With the very same arguments used in the first substep, one obtains the bound

$$\|\partial_t^{s_L} w^p\|_{L^2(\Omega\backslash\mathcal{B}^d(2))} \le \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c+2} s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}. \tag{5-73}$$

The case $k<s_L$. Again, for $0\le k<s_L$, the same method yields

$$\|\partial_t^k w\|_{H^{2(s_L-1-k)}(\Omega\backslash\mathcal{B}^d(2))} \le \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}+O\left(\frac{1}{L}\right)}}, \tag{5-74}$$

$$\|\nabla\partial_t^k w\|_{H^{2(s_L-1-k)}(\Omega\backslash\mathcal{B}^d(2))} \le \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2(2\ell-\alpha)}+O\left(\frac{1}{L}\right)}}, \tag{5-75}$$

$$\|\partial_t^k w^p\|_{H^{2(s_L-1-k)}(\Omega\backslash\mathcal{B}^d(2))} \le \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}. \tag{5-76}$$

Step 2: energy estimate for $\partial_t^{s_L} w_{\text{ext}}$. We claim that for $0\le t<t'$,

$$\begin{aligned}\|\partial_t^{s_L} w_{\text{ext}}\|_{L^2}^2 \le \|\partial_t^{s_L} w_{\text{ext}}(0)\|_{L^2}^2 &+ \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}\,dt' \\ &+ \int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}\,dt' \end{aligned} \tag{5-77}$$

and we now prove this estimate. From (5-68) one has the identity

$$\begin{aligned}&\partial_t(\|\partial_t^{s_L} w_{\text{ext}}\|_{L^2}^2) \\ &\quad= -2\int |\nabla\partial_t^{s_L} w_{\text{ext}}|^2 + 4\int \partial_t^{s_L} w_{\text{ext}}\nabla\chi_3.\nabla\partial_t^{s_L} w + 2\int \partial_t^{s_L} w_{\text{ext}}\partial_t^{s_L}((1-\chi_3)w^p+\Delta\chi_3 w)\end{aligned} \tag{5-78}$$

and we are now going to study the right-hand side of this equation.

Use of dissipation. We study all the terms except the nonlinear one in (5-78). After an integration by parts, using Cauchy–Schwarz, Young's and Poincare's inequalities,

$$\begin{aligned}&\left|\int \partial_t^{s_L} w_{\text{ext}}\nabla\chi_3.\nabla\partial_t^{s_L} w + \int \partial_t^{s_L} w_{\text{ext}}\partial_t^{s_L}(\Delta\chi_3 w)\right| \\ &\qquad= \left|-\int \Delta\chi_3\partial_t^{s_L} w\partial_t^{s_L} w_{\text{ext}} - \nabla\chi_3.\nabla\partial_t^{s_L} w_{\text{ext}}\partial_t^{s_L} w + \int \partial_t^{s_L} w_{\text{ext}}\partial_t^{s_L}(\Delta\chi_3 w)\right| \\ &\qquad\le C\left[\|(1-\chi_2)\partial_t^{s_L} w\|_{L^2}\|\partial_t^{s_L} w_{\text{ext}}\|_{L^2} + \|(1-\chi_2)\partial_t^{s_L} w\|_{L^2}\|\nabla\partial_t^{s_L} w_{\text{ext}}\|_{L^2}\right] \\ &\qquad\le C(\varepsilon)\|(1-\chi_2)\partial_t^{s_L} w\|_{L^2} + \varepsilon\|\nabla\partial_t^{s_L} w\|_{H^1}^2\end{aligned}$$

for any $\varepsilon > 0$. Adding the dissipation term in (5-78), taking ε small enough and using the bound (5-72) on the force term $\partial_t^{s_L} w$ gives

$$
\begin{aligned}
-\int |\nabla \partial_t^{s_L} w_{\text{ext}}|^2 + 4\int \nabla\chi_3.\nabla\partial_t^{s_L} w \partial_t^{s_L} w_{\text{ext}} + \int \partial_t^{s_L} w_{\text{ext}} \partial_t^{s_L}(\Delta\chi_{B(0,3)} w)& \\
\le C\|(1-\chi_2)\partial_t^{s_L} w\|_{L^2}^2 \le C\|\partial_t^{s_L} w\|_{L^2}^2 \le \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')}}& \\
\le \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}& \qquad (5\text{-}79)
\end{aligned}
$$

because in the trapped regime, $\lambda^2 s \sim s^{-\frac{\alpha}{2\ell-\alpha}}$.

Estimate for the nonlinear term. We now turn to the nonlinear term in (5-78), and use the estimate (5-73) for $\partial_t^{s_L} w^p$ we found in the first step, yielding

$$
\left|\int \partial_t^{s_L} w_{\text{ext}} \partial_t^{s_L}((1-\chi_3)w^p\right| \le \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}. \qquad (5\text{-}80)
$$

End of Step 2. We collect the estimates (5-79) and (5-80) found in the previous substeps, which gives the desired bound (5-77) we claimed in this step.

Step 3: iteration of elliptic regularity. We claim that for $i = 0, \dots, s_L$,

$$
\begin{aligned}
\|\partial_t^i w_{\text{ext}}\|_{H^{2(s_L-i)}}^2 \le \|\partial_t^{s_L} w_{\text{ext}}(0)\|_{L^2}^2 + \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}\,dt'& \\
+\int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}\,dt'& \\
+\frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}.& \qquad (5\text{-}81)
\end{aligned}
$$

We are going to show this estimate by induction. This is true for $i = s_L$ from the result (5-77) of the last step, and because of the compatibility conditions (4-20) at the border. Now suppose it is true for i, with $1 \le i \le s_L$. Then as $\partial_t^{i-1} w_{\text{ext}}$ solves (5-68), from elliptic regularity one gets (again because of the compatibility conditions (4-20) at the border), from the induction hypothesis and the bounds (5-76), (5-76) and (5-76) on the force terms

$$
\begin{aligned}
\|\partial_t^{i-1} w_{\text{ext}}\|_{H^{2(s_L-i)+2}}^2 \le \|(1-\chi_{B(0,4)})\partial_t^{i-1}(w^p) + \Delta\chi_{B(0,4)}\partial_t^{i-1} w& \\
+2\nabla\chi_{B(0,4)}.\nabla\partial_t^{i-1} w\|_{H^{2(s_L-i)}}^2 + \|\partial_t^i w_{\text{ext}}\|_{H^{2(s_L-i)}}^2& \\
\le \|\partial_t^{s_L} w_{\text{ext}}(0)\|_{L^2}^2 + \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}\,dt'& \\
+\int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}\,dt'& \\
+\frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}.&
\end{aligned}
$$

This shows that the inequality (5-81) is true for $i-1$. Hence, by iterations, the inequality (5-81) is true for $i=0$, which gives the estimate (5-67) we had to prove. $\square$

5D. ***End of the proof of Proposition 4.6.*** Proposition 4.6 states that, once the constants involved in the analysis, which are listed at its beginning, are well chosen, given an initial data of (1-1) that is a perturbation of the approximate blow-up profile along the stable directions of perturbation, there is a way to perturb it along the unstable directions of perturbation to produce a solution that stays trapped for all time in the sense of Definition 4.4. The strategy of the proof is the following. We argue by contradiction and suppose that for all perturbations along the unstable directions, the corresponding solution will eventually escape from the trapped regime. First, we characterize the exit of the trapped regime through a condition on the size of the unstable parameters, and then we show that arguing by contradiction would amount to go against Brouwer's fixed point theorem.

We fix $\lambda(s_0)$ satisfying (4-21), $w(s_0)$ decomposed in (4-5) satisfying (4-19) and (4-11), $V_1(s_0)$, $(U_{\ell+1}^{(0,1)}(s_0),\dots,U_L^{(0,1)}(s_0))$ and $(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I}\text{ with }1\leq n,\ i_n\leq i}$ satisfying (4-16), (4-17) and ((iii)). For any $(V_2(s_0),\dots,V_\ell(s_0))$ and $(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\leq n,\ i<i_n}$ satisfying (4-14) and (4-15), let u denote the solution of (1-1) with initial datum $u(0)=\chi\tilde{Q}_{b(s_0),1/\lambda(s_0)}+w(s_0)$ with $b(s_0)$ given by (4-31). We define the renormalized exit time $s^*=s^*\big((V_2(s_0),\dots,V_\ell(s_0)),(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\leq n,\ i<i_n}\big)$:

$$s^*:=\sup\big\{s\geq s_0,\ u\text{ is trapped in the sense of Definition 4.4 on }[s_0,s)\big\}. \tag{5-82}$$

By a continuity argument, one always has $s^*>s_0$.

Lemma 5.9 (characterization of the exit of the trapped regime). *For L and M large enough and σ close enough to s_c, there exists a choice of the other constants in* (4-30), *except s_0 and η, such that for any s_0 large enough and η small enough, if $s^*<+\infty$, at least one of the following two scenarios hold*:

(i) Exit via instabilities on the first spherical harmonics.

$$V_i(s^*)=(s^*)^{-\tilde{\eta}}\quad\textit{for some } 1\leq i\leq\ell.$$

(ii) Exit via instabilities on the other spherical harmonics.

$$U_i^{(n,k)}(s^*)=1\qquad\textit{for some } (n,k,i)\in\mathcal{I},\ \textit{with } 1\leq n \textit{ and } i<i_n.$$

Proof. A solution u is trapped if the parameters and the error involved in its decomposition (4-10) satisfy the bounds (4-22), (4-23), (4-24), (4-25) and (4-52). At time s^*, the bound (4-52) is strict by (4-51) and (4-52), and we are going to prove that (4-25) is strict in Step 1 and that (4-24) is strict in Step 2. Thus, (4-22) or (4-23) must be violated at the time s^* and the lemma is proved.

Step 1: improved bounds for the remainder w. We will now prove the estimates

$$\begin{gathered}\mathcal{E}_\sigma(s^*)\leq\frac{K_1}{2(s^*)^{\frac{2(\sigma-s_c)\ell}{2\ell-\alpha}}},\quad \mathcal{E}_{2s_L}(s^*)\leq\frac{K_2}{2(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}},\\ \|w_{\text{ext}}(s^*)\|_{H^\sigma}^2\leq\frac{K_1}{2}\quad\text{and}\quad\|w_{\text{ext}}(s^*)\|_{H^{2s_L}}^2\leq\frac{K_2}{2\lambda^{2(2s_L-s_c)}s^{2L+2(1-\delta_0)+2\eta(1-\delta_0')}}.\end{gathered}\tag{5-83}$$

Bound on $\mathcal{E}_\sigma$. Let K_1 and K_2 be any strictly positive real numbers. Then from Proposition 5.3, for s_0 and η large enough, we have

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\} \leq \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}\, s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}} \frac{1}{s^{\frac{\alpha}{4L}}}\left[1+\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\right].$$

On $[s_0, s^*]$, one has

$$\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}} \leq K_1 s^{-\frac{\alpha(\sigma-s_c)}{4\ell-2\alpha}}$$

by (4-25); hence for s_0 large enough,

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\} \leq \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}\, s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}} \frac{1}{s^{\frac{\alpha}{8L}}}.$$

One has $\lambda = \left(\frac{s_0}{s}\right)^{\frac{\ell}{2\ell-\alpha}}(1+O(s_0^{-\tilde{\eta}}))$ by (4-52) and we assume that $|O(s_0^{-\tilde{\eta}})| \leq \frac{1}{2}$. We reintegrate the above equation using (4-25) and (4-19):

$$\mathcal{E}_\sigma(s^*) \leq \frac{1}{(s^*)^{\frac{2\ell(\sigma-s_c)}{2\ell-\sigma}}}\left(\left(\frac{3}{2}\right)^{2\sigma-s_c} + s_0^{\frac{2\ell(\sigma-s_c)}{2\ell-\alpha}} \frac{2^{2(\sigma-s_c)+3}L}{\alpha s_0^{\frac{\alpha}{8L}}}\sqrt{K_1}\right).$$

Therefore, once L is fixed we choose σ close enough to s_c so that

$$\frac{\alpha}{8L} > \frac{2\ell(\sigma-s_c)}{2\ell-\alpha}$$

and then for s_0 large enough one has

$$s_0^{\frac{2\ell(\sigma-s_c)}{2\ell-\alpha}} \frac{2^{2(\sigma-s_c)+3}L}{\alpha s_0^{\frac{\alpha}{8L}}} \leq 1.$$

For any choice of the constants $K_1 > 10$, we then have

$$\mathcal{E}_\sigma(s^*) \leq \frac{1}{(s^*)^{\frac{2\ell(\sigma-s_c)}{2\ell-\sigma}}}\left(\left(\frac{3}{2}\right)^{2\sigma-s_c} + \sqrt{K_1}\right) \leq \frac{K_1}{2(s^*)^{\frac{2\ell(\sigma-s_c)}{2\ell-\sigma}}}. \tag{5-84}$$

Bound on $\mathcal{E}_{2s_L}$. Let K_1 and K_2 be any strictly positive real numbers. By Proposition 5.6, for any $N \gg 1$ the following holds for s_0 and η large enough:

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}} + O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}\, s^{L+1-\delta_0+\eta(1-\delta_0')}}\left(\sqrt{\mathcal{E}_{2s_L}} + \frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)\right)\right\} \\
&\leq \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\left[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}} + \frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')}} + \frac{C(L,M)}{N^{2\delta}}\mathcal{E}_{2s_L}\right. \\
&\qquad \left. + \mathcal{E}_{2s_L}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O(\frac{1}{L})}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1} \frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c}{L})}} + \frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}\right].
\end{aligned}$$

In the trapped regime, from (4-25) one has

$$\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}} \leq K_1 s^{-\frac{\alpha(\sigma-s_c)}{4\ell-2\alpha}}.$$

Consequently, for N and s_0 large enough the previous identity becomes

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}+O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}\left(\sqrt{\mathcal{E}_{2s_L}}+\frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)\right)\right\}$$
$$\leq \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\left[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}}+\frac{1}{N^{2\delta}}\mathcal{E}_{2s_L}\right].$$

Since from (4-52) we have

$$\lambda=\left(\frac{s_0}{s}\right)^{\frac{\ell}{2\ell-\alpha}}(1+O(s_0^{-\tilde{\eta}})),$$

when reintegrating in time the previous equation using the trapped regime bounds (4-25) and (4-19), one gets

$$\begin{aligned}
\mathcal{E}_{2s_L}(s^*)\leq \lambda(s^*)^{2(2s_L-s_c)}\Bigg[&O_{(L,M)}\left(\frac{1}{\lambda(s^*)^{2(2s_L-s_c)}(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}}(\sqrt{K_1}+1)\right)\\
&+\mathcal{E}_{2s_L}(s_0)+O_{L,M}\left(\frac{1}{s_0^{L+1-\delta_0+\eta(1-\delta_0')}}\left(\sqrt{\mathcal{E}_{2s_L}(s_0)}+\frac{1}{s_0^{L+1-\delta_0+\eta(1-\delta_0')}}\right)\right)\\
&+\int_{s_0}^{s^*}\frac{1}{\lambda^{2(2s_L-s_c)}s^{2L+3-2\delta_0+\eta(1-\delta_0')}}\left(C(L,M)\sqrt{K_2}+C(L,M)+\frac{K_2}{N^{2\delta}}\right)\Bigg]\\
\leq{}& \frac{1}{(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}}\left[C(L,M)(1+\sqrt{K_2})+C(L)\frac{K_2}{N^{2\delta}}\right]\\
\leq{}& \frac{1}{K_2(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}}
\end{aligned}\tag{5-85}$$

if N and K_1 have been chosen large enough.

Bound on $\|w_{\text{ext}}\|_{H^\sigma}$. We recall the estimate (5-35):

$$\frac{d}{dt}\left[\|w_{\text{ext}}\|^2_{H^\sigma}\right]\leq\frac{C(K_1,K_2)}{s^{1+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}\lambda^2}\|w_{\text{ext}}\|_{H^\sigma}.$$

For any choice of the constants of the analysis in Proposition 4.6 such that all the previous propositions and lemmas hold, for s_0 large enough,

$$\frac{d}{dt}\left[\|w_{\text{ext}}\|^2_{H^\sigma}\right]\leq\frac{1}{s^{\frac{\alpha}{4L}}\lambda^2}\|w_{\text{ext}}\|_{H^\sigma}.$$

We reintegrate this equation in the bootstrap regime, by applying the bounds (4-25) and (4-19) on $\|w_{\text{ext}}\|_{H^\sigma}$ $\left(\text{using the relation } \frac{ds}{dt}=\frac{1}{\lambda^2}\right)$:

$$\|w_{\text{ext}}(s^*)\|_{H^\sigma}\leq\sqrt{K_2}\frac{C(L)}{s_0^{\frac{\alpha}{4L}}}+\frac{C}{s_0^{\frac{2\ell}{2\ell-\alpha}(2s_L-s_c)}}\leq\frac{K_2}{2}\tag{5-86}$$

for K_2 chosen large enough.

Bound on $\|w_{\text{ext}}\|_{H^{2s_L}}$. We recall the estimate (5-67):

$$\begin{aligned}\|w_{\text{ext}}\|^2_{H^{2s_L}} \le \|\partial_t^{s_L} w_{\text{ext}}(0)\|^2_{L^2} &+ \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}\, dt' \\ &+ \int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}\, dt' \\ &+ \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O(\frac{\sigma-s_c+\eta}{L})}}.\end{aligned}$$

One has $w_{\text{ext}}=(1-\chi_3)w$, so $\partial_t^{s_L} w_{\text{ext}}=(1-\chi_3)\partial_t^{s_L} w$. Recall that we proved the bound (5-72) in the trapped regime for $\partial_t^{s_L} w(t)$ outside the blow-up zone in the proof of Proposition 5.8. The same proof gives for s_0 large enough, taking in account the bound (4-19) on w at initial time,

$$\|\partial_t^{s_L} w_{\text{ext}}(0)\|_{L^2} \le 1.$$

Inserting this estimate and (5-72) into the previous identity gives, for s_0 large enough,

$$\begin{aligned}\|w_{\text{ext}}\|^2_{H^{2s_L}} &\le 1 + \int_0^t \frac{dt'}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')}} + \frac{1}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} \\ &\le \frac{2}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} + \int_0^t \frac{C\,dt'}{s^{-\frac{\ell[2(2s_L-s_c)+2]}{2\ell-\alpha}} s^{2L+3-2\delta_0+2\eta(1-\delta_0')}} \\ &\le \frac{2}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} + \frac{C(L)}{s^{-\frac{\ell 2(2s_L-s_c)}{2\ell-\alpha}} s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} \\ &\le \frac{2}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} + \frac{C(L)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} \\ &\le \frac{K_2}{2\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')}}, \end{aligned} \tag{5-87}$$

where we used the equivalence $\lambda \sim s^{-\frac{\ell}{2\ell-\alpha}}$ from (4-52), and where the last line holds for K_2 large enough.

End of step 1. We have proven (5-84), (5-85), (5-86) and (5-87), yielding the estimate we claimed, (5-83).

Step 2: improved bounds for the stable parameters. We claim that once L, M, η, K_1 and K_2 have been chosen so that the result of Step 1 holds, there exist $\tilde\eta>0$ and strictly positive constants $(\varepsilon_i^{(0,1)})_{\ell+1\le i\le L}$, $(\varepsilon_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i_n\le i}$ such that

$$|V_1(s^*)| \le \frac{1}{2(s^*)^{-\tilde\eta}}, \qquad |U_i^{(0,1)}(s^*)| \le \frac{\varepsilon_i^{(0,1)}}{2(s^*)^{\tilde\eta}} \quad \text{for } \ell+1\le i\le L, \tag{5-88}$$

and for $(n,k,i)\in\mathcal{I}$, $n\ge 1$,

$$|U_i^{(n,k)}(s^*)| \le \frac{\varepsilon_i^{(n,k)}}{2(s^*)^{\tilde\eta}} \quad \text{if } i_n<i, \qquad |U_i^{(n,k)}(s^*)| \le \frac{\varepsilon_i^{(n,k)}}{2} \quad \text{if } i_n=i. \tag{5-89}$$

We now prove all these improved bounds: first we prove the one for $b_{L_n}^{(n,k)}$, then the one for the $U_i^{(n,k)}$, $i\neq L_n$, and finally the one for V_1. For technical reasons, we introduce for $(n,k,i)\in\mathcal{I}$ the function $g_i^{(n,k)}$, a solution of the ODE

$$\frac{\frac{d}{ds}g_i^{(n,k)}}{g_i^{(n,k)}}=(2i-\alpha_n)b_1^{(0,1)},\quad g(s_0)=s_0^{\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}}. \tag{5-90}$$

As $b_1^{(0,1)}=\frac{\ell}{s(2\ell-\alpha)}+O(s^{-1-\tilde{\eta}})$, for $\tilde{\eta}$ small enough and s_0 large enough one has

$$g_i^{(n,k)}(s)=s^{\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}}(1+O(s_0^{-\tilde{\eta}}))\quad\text{with }|O(s_0^{-\tilde{\eta}})|\le\tfrac12. \tag{5-91}$$

Improved bound for $b_{L_n}^{(n,k)}$. First we notice that since L is chosen after ℓ, one can assume that for all $0\le n\le n_0$, we have $i_n<L$. We rewrite the improved modulation equation (5-2) for $b_{L_n}^{(n,k)}$, using the estimate (5-3) for the extra term in the time derivative and the function $g_{L_n}^{(n,k)}$ (satisfying (5-90) and (5-91)), yielding

$$\left|\frac{d}{ds}\left[g_{L_n}^{(n,k)}b_{L_n}^{(n,k)}+O_{L,M,K_2}\big(s^{-L-\eta(1-\delta_0')+\delta_0-\delta_n+\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}}\big)\right]\right|\\ \le C(L,M,K_2)s^{-1-L-\eta(1-\delta_0')+\delta_0-\delta_n+\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}}$$

as $\eta(1-\delta_0')<\frac{g'}{2}$ for η small enough (g' being fixed). The notation $O_{L,M,K_2}(\cdot)$ is the usual $O(\cdot)$ notation with a constant depending on L, M and K_2. One has $2L_n-\alpha_n=2L-\frac{d}{2}-2\delta_n+2m_0+\frac{2}{p-1}$. Hence for L large enough, the quantity $-L-\eta(1-\delta_0')+\delta_0-\delta_n+\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}$ is strictly positive for all $0\le n\le n_0$. Therefore, reintegrating in time the previous identity yields, using (4-16) and (4-17),

$$\begin{aligned}|b_{L_n}^{(n,k)}(s^*)|&\le\frac{C(L,M,K_2)}{(s^*)^{L+\eta(1-\delta_0')+\delta_0-\delta_n}}+\frac{1}{s^{L+\delta_0-\delta_n+\tilde{\eta}}}\frac{s_0^{\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}-L-\delta_0+\delta_n-\tilde{\eta}}}{(s^*)^{\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}-L-\delta_0+\delta_n-\tilde{\eta}}}\frac32 s_0^{L+\delta_0-\delta_n+\tilde{\eta}}|b_{L_n}^{(n,k)}(s_0)|\\ &\le\frac{C(L,M,K_2)}{(s^*)^{L+\eta(1-\delta_0')+\delta_0-\delta_n}}+\frac{3\varepsilon_{L_n}^{(n,k)}}{20}\frac{1}{(s^*)^{L+\delta_0-\delta_n+\tilde{\eta}}}.\end{aligned}$$

Therefore, if $\tilde{\eta}<\eta(1-\delta_0')$, for any $0<\varepsilon_{L_n}^{(n,k)}<1$, for s_0 large enough, we have

$$|b_{L_n}^{(n,k)}(s^*)|\le\frac{\varepsilon_{L_n}^{(n,k)}}{2(s^*)^{L+\delta_0-\delta_n+\tilde{\eta}}}. \tag{5-92}$$

Improved bound for $b_i^{(n,k)}$, $i_n<i<L_n$. Using the same methodology we used to study the parameter $b_{L_n}^{(n,k)}$, we take the modulation equation (4-43), we integrate it in time, applying the bounds (4-22), (4-23), (4-24) and (4-25), yielding

$$\left|\frac{d}{ds}(g_i^{(n,k)}b_i^{(n,k)})\right|\le\frac{3\varepsilon_{i+1}^{(n,k)}s^{\frac{\ell}{2\ell-\alpha}(2i-\alpha_n)-\frac{\gamma-\gamma_n}{2}-i-\tilde{\eta}-1}}{2}+C(L,M,K_1)s^{-L-1+\delta_0-\eta(1-\delta_0')+\frac{\ell}{2\ell-\alpha}(2i-\alpha_n)}.$$

The condition $i_n<i$ ensures that $\frac{\ell}{2\ell-\alpha}(2i-\alpha_n)-\frac{\gamma-\gamma_n}{2}-i>0$. For $\tilde{\eta}$ small enough, we can then integrate in time the previous equation, the first term on the right-hand side giving then a divergent integral. Then applying the bound (5-91) on $g_i^{(n,k)}$ and the initial bound (4-17) on $b_i^{(n,k)}$, one obtains

$$\begin{aligned}|b_i^{(n,k)}(s^*)| &\le \frac{1}{(s^*)^{\frac{\gamma-\gamma_n}{2}+i+\tilde\eta}}\biggl(\frac{3\varepsilon_i^{(n,k)}}{20}+C(L)\varepsilon_{i+1}^{(n,k)}\\ &\qquad\qquad + \frac{C(L,M)}{(s^*)^{\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}-\frac{\gamma-\gamma_n}{2}-i-\tilde\eta}}\int_{s_0}^{s^*} s^{-L-1+\delta_0-\eta(1-\delta_0')+\frac{\ell(2i-\alpha)}{2\ell-\alpha}}\,ds\biggr)\\ &\le \frac{\varepsilon_i^{(n,k)}}{2(s^*)^{\frac{\gamma-\gamma_n}{2}+i}} \end{aligned}\tag{5-93}$$

if s_0 is large enough and $\varepsilon_{i+1}^{(n,k)}$ is small enough, because $L-\delta_0>\frac{\gamma-\gamma_n}{2}+i$.

Improved bound for $b_i^{(n,k)}$ if $i_n=i$ and $1\le n$. In that case, $\frac{\ell}{2\ell-\alpha}(2i-\alpha_n)=\frac{\gamma-\gamma_n}{2}+i$. Hence one has

$$\frac{1}{2}\le\frac{g_i^{(n,k)}}{s^{\frac{\gamma-\gamma_n}{2}+i}}\le\frac{3}{2}.$$

Integrating the modulation equation and making the same manipulations we made for $i_n<i$ then yields

$$|b_i^{(n,k)}(s^*)|\le\frac{1}{(s^*)^{\frac{\gamma-\gamma_n}{2}+i}}\biggl(\frac{3\varepsilon_i^{(n,k)}}{20}+C(L)\varepsilon_{i+1}^{(n,k)}+\frac{C(L,M)}{s_0^{L-\delta_0-\frac{\gamma-\gamma_n}{2}-i}}\biggr)\le\frac{\varepsilon_i^{(n,k)}}{2(s^*)^{\frac{\gamma-\gamma_n}{2}}+i}\tag{5-94}$$

if $\varepsilon_{i+1}^{(n,k)}$ is small enough and s_0 is large enough.

Improved bound for V_1. We recall that from (4-13), V_1 denotes the stable direction of perturbation for the dynamical system (3-58) contained in $\mathrm{Span}((U_i^{(0,1)})_{1\le i\le\ell})$. From the quasidiagonalization (3-69) of the linearized matrix A_ℓ, under the bootstrap bounds (4-22), (4-23), (4-24) and (4-25), its time evolution is given by

$$\begin{aligned}V_{1,s}&=-\frac{V_1}{s}+O\biggl(\frac{|(V_i)_{1\le i\le\ell}|^2}{s}\biggr)+O(C(L,M,K_2)s^{-L-\ell})+\frac{q_1}{s}U_{i+1}^{(0,1)}\\ &=-\frac{V_1}{s}+O\biggl(\frac{1}{s^{1+2\tilde\eta}}+s^{-L-\ell}+\frac{\varepsilon_{\ell+1}^{(0,1)}}{s^{1+\tilde\eta}}\biggr),\end{aligned}$$

which when reintegrated in time gives, if $\varepsilon_{\ell+1}^{(0,1)}$ is small enough, s_0 is large enough, and using (4-16),

$$|V_1(s^*)|\le\frac{s_0V_1(s_0)}{s^*}+\frac{C(L,M,K_1)}{(s^*)^{2\tilde\eta}}+\frac{C(L)\varepsilon_{\ell+1}^{(0,1)}}{(s^*)^{\tilde\eta}}\le\frac{1}{2s^{\tilde\eta}}.\tag{5-95}$$

End of Step 2. We choose the constants of smallness in the following order so that all the improved bounds we proved, (5-92), (5-93), (5-94), (5-95), hold together. For any choice of K_1, K_2, L, M, η in their ranges, there exists $\tilde\eta>0$ such that $\tilde\eta<\eta(1-\delta_0')$ and $\frac{\gamma-\gamma_n}{2}+i+\tilde\eta<\frac{\ell}{2\ell-\alpha}(2i-\alpha_n)$ for all $(n,k,i)\in\mathcal{I}$ with $i_n<i$. First choose the constant $\varepsilon_{\ell+1}^{(0,1)}$ small enough so that the improved bound (5-95) for V_1 holds for s_0 large enough. Next choose $\varepsilon_{\ell+2}^{(0,1)}$ such that the improved bound (5-93) for $U_{\ell+1}^{(0,1)}$ holds for s_0 large enough. By iteration we then choose $\varepsilon_{\ell+3}^{(0,1)},\dots,\varepsilon_L^{(0,1)}$ to make all the bounds (5-93) hold until the one for $U_{L-1}^{(0,1)}$. Then the final one, (5-92), for $U_L^{(0,1)}$, holds for s_0 large enough without any

conditions on $\varepsilon_i^{(0,1)}$ for $\ell+1\le i\le L-1$. The same reasoning applies for the stable parameters on the spherical harmonics of higher degree ($1\le n\le n_0$). We have proved (5-88). □

We fix all the constants of the analysis so that Lemma 5.9 holds, and we will just possibly increase the initial renormalized time s_0, which does not change its validity. The number of instability directions is

$$m=\ell-1+d(E[i_1]-\delta_{i_1\in\mathbb{N}})+\sum_{2\le n\le n_0}k(n)(E[i_n]+1-\delta_{i_n\in\mathbb{N}}).$$

To prove Proposition 4.6, we have to prove that there exists an additional perturbation along the unstable directions of perturbations such that the solution stays forever trapped. We prove it via a topological argument, by looking at all the solutions associated to the possible perturbations along the unstable directions of perturbation. For this purpose, we introduce the set

$$\begin{aligned}\mathcal{B}:=\big\{\big(V_2(s_0),\dots,V_\ell(s_0),(U^{(n,k)}(s_0)_i)_{(n,k,i)\in\mathcal{I},\,1\le n,\,i<i_n}\big)\in\mathbb{R}^m: |V_i(s_0)|\le s_0^{-\tilde\eta}\text{ for }2\le i\le\ell,\\ |U^{(n,k)}(s_0)_i|\le\varepsilon_i^{(n,k)}\text{ for }(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n\big\},\end{aligned}$$

which represents all the possible values of the unstable parameters so that the solution to (1-1) with initial data given by (4-5) and (4-31) starts in the trapped regime. We then define the following application $f:\mathcal{D}(f)\subset\mathcal{B}\to\partial\mathcal{B}$ that gives the last value taken by the unstable parameters before the solution leaves the trapped regime (when it does):

$$\begin{aligned}&f\big(V_2(s_0),\dots,V_\ell(s_0),(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\,1\le n,\,i<i_n}\big)\\&\qquad=\bigg(\frac{(s^*)^{\tilde\eta}}{s_0^{\tilde\eta}}V_2(s^*),\dots,\frac{(s^*)^{\tilde\eta}}{(s_0)^{\tilde\eta}}V_\ell(s^*),(U_i^{(n,k)}(s^*))_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n}\bigg).\end{aligned}\tag{5-96}$$

The domain $\mathcal{D}(f)$ of the application f is the set of the m-tuples of real numbers

$$\big(V_2(s_0),\dots,V_\ell(s_0),(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n}\big)$$

in $\mathcal{B}$ such that the solution starting initially with a decomposition given by (4-5) and (4-31) leaves the trapped regime in finite time s^*. The following lemma describes the topological properties of f.

Lemma 5.10 (topological properties of the exit application). *There exists a choice of smallness constants $(\varepsilon_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\,1\le n,\,i<i_n+1}$ such that the following properties hold for s_0 large enough*:

(i) *$\mathcal{D}(f)$ is nonempty and open, and the inclusion $\partial\mathcal{B}\subset\mathcal{D}(f)$ holds.*

(ii) *f is continuous and is the identity on the boundary $\partial\mathcal{B}$.*

Proof. **Step 1:** the outgoing flux property. We prove in this step that one can choose the smallness constants $(\varepsilon_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n+1}$ such that for any $\big(V_2(s_0),\dots,V_\ell(s_0),(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n}\big)$ in $\mathcal{B}$ such that the solution starting initially with the decomposition given by (4-5) and (4-31) is in the trapped regime on $[s_0,s]$ and satisfies at time s

$$\bigg(\frac{(s)^{\tilde\eta}}{s_0^{\tilde\eta}}V_2(s),\dots,\frac{(s)^{\tilde\eta}}{(s_0)^{\tilde\eta}}V_\ell(s),(U_i^{(n,k)}(s))_{(n,k,i)\in\mathcal{I},\,1\le n,\,i<i_n}\bigg)\in\partial\mathcal{B},$$

the exit time from the trapped regime is s. To prove this we compute the time derivative of the unstable parameters when they are on $\partial\mathcal{B}$, and show that it points toward the exterior. Indeed from the modulation equation (4-43) and (3-69) (where we injected the bounds of the trapped regime (4-22), (4-23), (4-24) and (4-25)),

$$V_{i,s}=\frac{i\alpha}{2\ell-\alpha}\frac{V_i}{s}+O\left(\frac{|(V_1(s),\dots,V_\ell(s))|^2}{s}\right)+\frac{q_iU^{(0,1)}_{\ell+1}}{s}+O(s^{-L+\ell})=\frac{i\alpha}{2\ell-\alpha}\frac{V_i}{s}+O\left(s^{-1-2\tilde\eta}+\frac{\varepsilon^{(0,1)}_{\ell+1}}{s^{1+\tilde\eta}}\right),$$

$$U^{(n,k)}_{i,s}=\alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-i}{(2\ell-\alpha)s}U^{(n,k)}_i+\frac{U^{(n,k)}_{i+1}}{s}+O(s^{-1-\tilde\eta})=\alpha\frac{i_n-i}{(2\ell-\alpha)s}U^{(n,k)}_i+O\left(\frac{\varepsilon^{(n,k)}_{i+1}}{s}+s^{-1-\tilde\eta}\right).$$

Therefore, as $i<i_n$, by iterations (i.e., by choosing first $\varepsilon^{(n,k)}_0$, then $\varepsilon^{(n,k)}_1$, and so on until choosing $\varepsilon^{(n,k)}_{\ell+1}$) we can choose all the smallness constants and s_0 large enough so that

$$\frac{i\alpha}{2\ell-\alpha}\frac{(-1)^j}{s^{1+\tilde\eta}}+O\left(s^{-1-2\tilde\eta}+\frac{\varepsilon^{(0,1)}_{\ell+1}}{s^{1+\tilde\eta}}\right)>0 \text{ (resp. } <0) \quad \text{if } j=0 \text{ (resp. } j=1),$$

$$\alpha\frac{i_n-i}{(2\ell-\alpha)s}(-1)^j\varepsilon^{(n,k)}_i+O\left(\frac{\varepsilon^{(n,k)}_{i+1}}{s}+s^{-L+\ell}\right)>0 \text{ (resp. } <0) \quad \text{if } j=0 \text{ (resp. } j=1).$$

Consequently, any solution that is trapped until s such that at time s,

$$\left(\frac{(s)^{\tilde\eta}}{s_0^{\tilde\eta}}V_2(s),\dots,\frac{(s)^{\tilde\eta}}{(s_0)^{\tilde\eta}}V_\ell(s),(U^{(n,k)}_i(s))_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n}\right)\in\partial\mathcal{B}$$

leaves the trapped regime after s.

Step 2: end of the proof of the lemma. Step 1 directly implies that $\mathcal{D}(f)$ contains $\partial\mathcal{B}$, and that f is the identity on $\partial\mathcal{B}$. If a solution u leaves at time s^*, it also implies that it never hit the boundary before s^*. Consequently, as the trapped regime is characterized by nonstrict inequalities, and because everything in the dynamics of (1-1) is continuous with respect to variation on these unstable parameters, we get that $\mathcal{D}(f)$ is open, and that the exit time s^* and f are continuous on $\mathcal{D}(f)$. □

We can now end the proof of Proposition 4.6.

Proof of Proposition 4.6. We argue by contradiction. If for any choice of initial perturbation along the unstable directions of perturbation, the solution leaves the trapped regime, then it means that the domain of the exit application f defined by (5-96) is $\mathcal{D}(f)=\mathcal{B}$. But then from Lemma 5.10, f would be a continuous application from $\mathcal{B}$ towards its boundary, being the identity on the boundary, which is impossible thanks to Brouwer's theorem, and the contradiction is obtained. □

Appendix A: Properties of the zeros of *H*

This section is devoted to the proof of Lemma 2.3.

Proof of Lemma 2.3. The proof relies solely on ODE techniques (in the same spirit as [Gui et al. 1992; Li 1992]) and is as follows. First, we describe the asymptotics of the equation $H^{(n)}f=0$ at the origin and at infinity in Lemma A.1. Then we construct the special zeroes $T_0^{(n)}$ and $\Gamma^{(n)}$ in these asymptotic regimes using a perturbative argument and obtain their asymptotic behavior in Lemma A.2. Finally we show that they are not equal via global invariance properties of the ODE in the phase space $(f, \partial_r f)$ in Lemma A.3, yielding that they form indeed a basis of the set of solutions.

Let $f : (0, +\infty)$ be smooth such that $H^{(n)} f = 0$. First we make the change of variables $f(r) = w(t)$ with $t = \ln(r) \in (-\infty, +\infty)$. Then w solves

$$w'' + (d-2)w' - [e^{2t} V(e^t) + n(d+n-2)]w = 0, \tag{A-1}$$

where V is defined by (1-31) and satisfies $e^{2t} V(e^t) = O(e^{2t}) \to 0$ as $t \to -\infty$, and $e^{2t} V(e^t) = -pc_\infty^{p-1} + O(e^{-t\alpha})$ as $t \to +\infty$, by (2-2). Hence (A-1) is similar to the following ODEs as $t \to \pm\infty$:

$$w'' + (d-2)w' + (pc_\infty^{p-1} - n(d+n-2))w = 0, \tag{A-2}$$

$$w'' + (d-2)w' - n(d+n-2)w = 0. \tag{A-3}$$

The first step in the proof of Lemma 2.3 is to describe their solutions.

Lemma A.1. $\mathrm{Span}(e^{-\gamma_n t}, e^{-\gamma_n' t})$ *(resp.* $\mathrm{Span}(e^{nt}, e^{(-n-d+2)t})$*) is the set of solutions of* (A-2) *(resp.* (A-3)*), where γ_n is defined in* (1-18) *and*

$$\gamma_n' := \frac{d-2+\sqrt{\triangle_n}}{2}, \tag{A-4}$$

where $\triangle_n > 0$ is defined in (1-18)*. These numbers satisfy*

$$\gamma_0 = \gamma, \quad \gamma_1 = \frac{2}{p-1} + 1 \quad \text{and} \quad \forall n \geq 2, \quad \gamma_n < \frac{2}{p-1}, \quad \gamma_n' > \frac{(d-2)}{2}, \tag{A-5}$$

where γ is defined in (1-9)*.*

Proof. From the standard theory of second-order differential equations with constant coefficients, the set of solutions of (A-2) (resp. (A-3)) is $\mathrm{Span}(e^{-\gamma_n t}, e^{-\gamma_n' t})$ (resp. $\mathrm{Span}(e^{nt}, e^{(-n-d+2)t})$), where γ_n and γ_n' are defined by (1-18) and (A-4). For any $n \in \mathbb{N}$, one computes from its definition in (1-18) that the number $\triangle_n$ used in the definitions (1-18) and (A-4) of γ_n and γ_n' is strictly positive: $\triangle_n > 0$. Indeed, $\triangle_n \geq \triangle_0$ by (1-18), and $\triangle_0 > 0$ if and only if $p > p_{JL}$, where p_{JL} is defined in (1-6), and the present paper is concerned with the case $p > p_{JL}$.

From the formula (1-18), one computes that $\gamma_0 = \gamma$ and $\gamma_1 = \frac{2}{p-1} + 1$, where γ is defined in (1-9). For all $n \in \mathbb{N}$, from the definition (A-4) of γ_n' and since $\triangle_n > 0$, one gets that $\gamma_n' > \frac{d-2}{2}$. Eventually we compute from (1-18) that

$$\triangle_1 = \left(d - 4 - \frac{4}{p-1}\right)^2, \quad \triangle_2 = \left(d - 4 - \frac{4}{p-1}\right)^2 + 4d + 4,$$

which implies in particular that

$$\triangle_2 - \triangle_1 - 4\sqrt{\triangle_1} - 4 = 4d + 4 - 4\left(d - 4 - \frac{4}{p-1}\right) - 4 = 16 + \frac{16}{p-1} > 0,$$

giving $\sqrt{\triangle_2} > \sqrt{\triangle_1} + 2$. This, by (1-18), implies

$$\gamma_2 = \frac{d-2-\sqrt{\triangle_2}}{2} < \frac{d-2-\sqrt{\triangle_1}-2}{2} = \gamma_1 - 1 = \frac{2}{p-1} + 1 - 1 = \frac{2}{p-1}.$$

This implies $\gamma_n < \frac{2}{p-1}$ for all $n \geq 2$ because the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing by its definition (1-18). $\square$

Lemma A.2. *There exist $w_1^{(n)}$, $w_2^{(n)}$, $w_3^{(n)}$ and $w_4^{(n)}$ solving* (A-1) *such that*

$$w_1^{(n)} \underset{t\to-\infty}{=} \sum_{i=0}^{q} c_i e^{(n+2i)t} + O(e^{(n+2q+2)t}), \quad w_2^{(n)} \underset{t\to-\infty}{\sim} \tilde{c}_1 e^{(-n-d+2)t}, \tag{A-6}$$

$$w_3^{(n)} \underset{t\to+\infty}{=} \tilde{c}_2 e^{-\gamma_n t} + O(e^{(-\gamma_n - g)t}) \quad \text{and} \quad w_4^{(n)} \underset{t\to+\infty}{\sim} \tilde{c}_3 e^{-\gamma_n' t} = O(e^{(-\gamma_n - g)t}), \tag{A-7}$$

with constants $c_1, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \neq 0$. Moreover the asymptotics hold for the derivatives.

Proof. **Step 1:** existence of $w_1^{(n)}$. For $n = 0$, we take the explicit solution $w_1^{(0)} = \Lambda Q(e^t)$, which satisfies (A-6) by (2-1). Now let $n \geq 1$. Using the Duhamel formula for solutions of (A-1), the fundamental set of solutions for the constant coefficient ODE (A-3) begin provided by Lemma A.1, a solution of (A-1) satisfying the condition on the left in (A-6) with $c_0 = 1$ can be written as

$$w_1^{(n)}(t) = e^{nt} + \frac{1}{2n+d-2} \int_{-\infty}^{t} (e^{n(t-t')} - e^{(-n-d+2)(t-t')}) w_1^{(n)}(t') e^{2t'} V(e^{t'})\, dt'. \tag{A-8}$$

We now use a standard contraction argument. For $t_0 \in \mathbb{R}$ we endow the space

$$X := \left\{ u \in C((-\infty, t_0], \mathbb{R}) : \sum_{t \leq t_0} |u(t)| e^{-t} < +\infty \right\}$$

with the norm

$$\|u\|_X := \sup_{t \leq t_0} |u(t)| e^{-(n+1)t}. \tag{A-9}$$

For $u \in X$ we define the function $\Phi u : (-\infty, t_0] \to \mathbb{R}$ by

$$(\Phi u)(t) := \frac{1}{2n+d-2} \int_{-\infty}^{t} (e^{n(t-t')} - e^{(-n-d+2)(t-t')})[e^{nt'} + u(t')] e^{2t'} V(e^{t'})\, dt'. \tag{A-10}$$

Φ maps X into itself. Indeed as the potential V is bounded from (2-2), a brute force bound on the above equation yields that

$$|(\Phi u)(t)| \leq C \|V\|_{L^\infty} (e^t + \|u\|_X e^{2t}) e^{(n+1)t},$$

and therefore $\|\Phi u\|_X \leq C \|V\|_{L^\infty} (e^{t_0} + \|u\|_X e^{2t_0})$. The same brute force bound for the difference of two images under Φ of two elements gives

$$\left|(\Phi u)(t) - (\Phi v)(t)\right| \leq C \|V\|_{L^\infty} e^{2t} \|u - v\|_X e^{(n+1)t}.$$

Hence $\|\Phi u - \Phi v\|_X \leq C \|V\|_{L^\infty} e^{2t_0} \|u - v\|_X$ and Φ is a contraction for $t_0 \ll 0$ small enough. Therefore, Φ admits a fixed point in X, denoted by u_1. From the Duhamel formula (A-8) and the definition (A-10) of Φ, we know $w_1^{(n)} := e^{nt} + u_1(t)$ is then a solution of (A-1) on $(-\infty, t_0]$, which, from the definition

(A-9) of X, satisfies

$$w_1^{(n)} = e^{nt} + O(e^{(n+1)t}) \quad \text{as } t \to -\infty. \tag{A-11}$$

We extend it to a solution of (A-1) on $\mathbb{R}$ ((A-1) being linear with smooth coefficients), still naming it $w_0^{(n)}$.
Step 2: asymptotics of $w_1^{(n)}$. At present, we will refine the asymptotics (A-11). We reason by induction. We claim that if for $k \in \mathbb{N}$ and $(c_i)_{0\le i\le k} \in \mathbb{R}^{k+1}$ one has

$$w_1^{(n)} = \sum_{i=0}^{k} c_i e^{(n+2i)t} + O(e^{(n+2k+2)t}) \quad \text{as } t \to -\infty \tag{A-12}$$

then there exists $c_{k+1} \in \mathbb{R}$ such that

$$w_1^{(n)} = \sum_{i=0}^{k+1} c_i e^{(n+2i)t} + O(e^{(n+2k+4)t}) \quad \text{as } t \to -\infty. \tag{A-13}$$

We now prove this fact. Fix $k \ge 1$ and assume that $w_1^{(n)}$ satisfies (A-12). As V is a smooth radial profile, one has that $\partial_r^{2q+1} V(0) = 0$ for any $q \in \mathbb{N}$, implying that there exists $(d_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that

$$V(e^t) = \sum_{i=0}^{k} d_i e^{2it} + O(e^{(2k+2)t}) \quad \text{as } t \to -\infty. \tag{A-14}$$

We insert this and (A-12) into (A-8) and integrate to find

$$\begin{aligned} w_1^{(n)} &= e^{nt} + \frac{1}{2n+d-2}\int_{-\infty}^{t} \bigl(e^{n(t-t')} - e^{(2-n-d)(t-t')}\bigr)\Bigl[\sum_{i=0}^{k}\sum_{j=0}^{i} c_j d_{i-j} e^{(n+2i+2)t'} + O(e^{(n+2k+4)t'})\Bigr]\,dt' \\ &= e^{nt} + \sum_{i=0}^{k} \frac{e^{(n+2i+2)t}}{2n+d-2}\Bigl(\frac{1}{2i+2} - \frac{1}{2n+d+2i}\Bigr)\sum_{j=0}^{i} c_j d_{i-j} + O(e^{(2+2k+4)t}). \end{aligned}$$

This asymptotic has to be coherent with the assumption (A-12); hence for all $0 \le i \le k-1$ one has

$$\Bigl(\frac{1}{2i+2} - \frac{1}{2n+d+2i}\Bigr)\sum_{j=0}^{i} \frac{c_j d_{i-j}}{2n+d-2} = c_{i+1}.$$

The above identity is then the formula (A-13) one has to prove.

Thus, one has proven that the asymptotic on the left of (A-6) holds for $w_1^{(n)}$. It remains to show that it also holds for the derivatives. Differentiating (A-8) gives

$$(w_1^{(n)})'(t) = ne^{nt} + \frac{1}{2n+d-2}\int_{-\infty}^{t} \bigl[ne^{n(t-t')} + (n+d-2)e^{(2-n-d)(t-t')}\bigr] w_1^{(n)} e^{2t'} V.$$

We use the same reasoning we did for $w_1^{(n)}$: we insert the asymptotic (A-12) at any order for $w_1^{(n)}$ we just showed and (A-14) into the above formula, integrate in time and match the coefficients we find with

(A-12), yielding that

$$(w_1^{(n)})'(t) = \sum_{i=0}^{k} (n+2i) c_i e^{(n+2i)t} + O(e^{(n+2k+2)t})$$

for any $k \in \mathbb{N}$. Therefore, one has proven that the asymptotic on the left of (A-6) holds for $w_1^{(n)}$ and $(w_1^{(n)})'$. As $w_1^{(n)}$ solves (A-1), its second derivative is given by

$$(w_1^{(n)})'' = -(d-2)(w_1^{(n)})' + [e^{2t} V(e^t) + n(d+n-2)] w_1^{(n)},$$

and therefore by (A-14) the expansion also holds for $(w_1^{(n)})''$. Differentiating the above equation, using again (A-14) and the expansions for $w_1^{(n)}$, $(w_1^{(n)})'$ and $(w_1^{(n)})''$, one obtains the expansion for $(w_1^{(n)})'''$. By iterating this procedure we obtain the expansion on the left of (A-6) for all derivatives of $w_1^{(n)}$.

Step 3: existence and asymptotics of $w_2^{(n)}$. Let $t_0 \in \mathbb{R}$. We use the Duhamel formula for (A-1), the solutions of the underlying constant coefficient ODE (A-3) being provided by Lemma A.1. For $t \le t_0$, the solution of (A-1) starting from $w_2^{(n)}(t_0) = e^{(2-d-n)t_0}$, $(w_2^{(n)})'(t_0) = (2-d-n)e^{(2-d-n)t_0}$ can be written as

$$w_2^{(n)} = e^{(2-d-n)t} - \frac{1}{2n+d-2} \int_t^{t_0} (e^{n(t-t')} - e^{(2-n-d)(t-t')}) V(e^{t'}) e^{2t'} w_2^{(n)}(t')\, dt'. \tag{A-15}$$

We claim that for $t_0 \ll 0$ small enough, we have

$$|w_2^{(n)} - e^{(2-d-n)t}| \le \frac{e^{(2-d-n)}}{2} \tag{A-16}$$

for all $t \le t_0$. To show that, let $\mathcal{T}$ be the set of times $t \le t_0$ such that this inequality holds. $\mathcal{T}$ is closed via a continuity argument, and is nonempty as it contains t_0. For $t \in \mathcal{T}$ we compute by brute force on the above identity:

$$|w_2^{(n)} - e^{(2-d-n)t}| \le C \|V\|_{L^\infty} e^{(2-n-d)t} e^{2t_0}.$$

Hence, for $t_0 \ll 0$ small enough, $|w_2^{(n)} - e^{(2-d-n)t}| \le e^{(2-n-d)t}/3$, implying that $\mathcal{T}$ is open. Therefore, $\mathcal{T} = (-\infty, t_0]$ by a connectedness argument and $w_2^{(n)}$ satisfies (A-16) for all $t \le t_0$. We insert (A-16) into (A-15) to refine the asymptotics (the constant in the $O(\cdot)$ depends on $\|V\|_{L^\infty}$):

$$\begin{aligned}
w_2^{(n)} &= e^{(2-d-n)t} + \int_t^{t_0} (e^{n(t-t')} - e^{(2-d-n)(t-t')}) O(e^{(4-n-d)(t-t')})\, dt' \\
&= e^{(2-d-n)t} + e^{nt} \int_t^{t_0} O(e^{(4-2n-d)t'})\, dt' + e^{(2-n-d)t} \int_t^{t_0} O(e^{2t'})\, dt' \\
&= e^{(2-d-n)t} + O(e^{(4-n-d)t}) + e^{(2-n-d)t} \left(\int_{-\infty}^{t_0} O(e^{2t'})\, dt' - \int_{-\infty}^{t} O(e^{2t'})\, dt' \right) \\
&= e^{(2-d-n)t} \left(1 + \int_{-\infty}^{t_0} O(e^{2t'})\, dt' \right) + O(e^{(4-n-d)t}) \\
&= \tilde{c}_1 e^{(2-d-n)t} + O(e^{(4-n-d)t})
\end{aligned}$$

with $\tilde{c}_1 \neq 0$ if $t_0 \ll 0$ is chosen small enough. We just showed the asymptotic on the right of (A-6).

Step 4: existence and asymptotics of $w_3^{(n)}$ and $w_4^{(n)}$. Using exactly the same techniques we used at $-\infty$ to construct $w_1^{(n)}$ and $w_2^{(n)}$ as perturbations of the solutions described by Lemma A.1 of the asymptotic constant coefficients ODE (A-3), we can construct two solutions of (A-1), $w_3^{(n)}$ and $w_4^{(n)}$, satisfying

$$w_3^{(n)} \sim \tilde{c}_2 e^{-\gamma_n t}, \quad w_4^{(n)} \sim \tilde{c}_3 e^{-\gamma_n' t} \quad \text{as } t \to +\infty \tag{A-17}$$

with $\tilde{c}_2, \tilde{c}_3 \neq 0$, as perturbations of the solutions $e^{-\gamma_n t}$ and $e^{-\gamma_n' t}$ of the asymptotic ODE (A-2) at $+\infty$. We leave safely the proof of this fact to the reader. We now show why the second term in the asymptotic of $w_3^{(n)}$ is $O(e^{(-\gamma_n - g)t})$, where g is defined in (1-21). Using Duhamel's formula for (A-1), with the set of fundamental solutions of the asymptotic equation (A-2) described in Lemma A.1, $w_3^{(n)}$ can be written as

$$w_3^{(n)} = a_1 e^{-\gamma_n t} + b_1 e^{-\gamma_n' t} - \frac{1}{-\gamma_n + \gamma_n'} \int_0^t (e^{-\gamma_n (t-t')} - e^{-\gamma_n' (t-t')}) e^{2t'} (V(e^{t'}) + p c_\infty^{p-1} e^{-2t'}) w_3^{(n)}(t')\, dt'$$

for a_1 and b_1 two coefficients. We use the bounds $V(e^{t'}) + p c_\infty^{p-1} e^{-2t'} = O(e^{-\alpha t'})$ from (2-2) and (A-17) to find

$$w_3^{(n)}(t) = a_1 e^{-\gamma_n t} + b_1 e^{-\gamma_n' t} - \frac{1}{-\gamma_n + \gamma_n'} \int_0^t (e^{-\gamma_n (t-t')} - e^{-\gamma_n' (t-t')}) O(e^{(-\gamma_n - \alpha)t'}).$$

After few computations, we obtain two new coefficients $\tilde{a}_1$ and $\tilde{a}_2$ such that

$$w_3^{(n)}(t) = \tilde{a}_1 e^{-\gamma_n t} + \tilde{b}_1 e^{-\gamma_n' t} + O(e^{(-\gamma_n - \alpha)t}).$$

As $-\gamma_n' < -\gamma_n$ by (1-18), the asymptotic (A-17) implies $\tilde{a}_1 = \tilde{c}_2 \neq 0$. From the definition (1-21) of g, this parameter is tailor-made to produce $-\gamma_0 - g > -\gamma_0'$ (by (1-9) and (1-18)). By (1-18), one then has $-\gamma_n - g + \gamma_n' \geq -\gamma_0 - g + \gamma_0' > 0$. As g satisfies also $g < \alpha$, the above identity then yields

$$w_3^{(n)}(t) = \tilde{c}_2 e^{-\gamma_n t} + O(e^{(-\gamma_n - g)t}).$$

Using exactly the same methods we use to propagate the asymptotic of $w_1^{(n)}$ to its derivatives in Step 2, the above identity propagates to the derivatives of $w_3^{(n)}$. □

Lemma A.3. *The solutions $w_1^{(n)}$ and $w_4^{(n)}$ given by Lemma A.2 are not collinear. Moreover, $w_1^{(n)}$ has constant sign.*

Proof. We formulate (ODE$_n$) as a planar dynamical system:

$$\frac{d}{dt}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ n(d+n-2) + e^{2t} V(e^t) & -(d-2) \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix},$$

with $w^1 = w$ and $w^2 = w'$. By their asymptotics from Lemma A.1,

$$\begin{pmatrix} w_1^{(n)}(t) \\ (w_1^{(n)})'(t) \end{pmatrix} = c_1 e^{nt} \begin{pmatrix} 1 \\ n \end{pmatrix} + O(e^{(n+2)t}) \quad \text{as } t \to -\infty,$$

$$\begin{pmatrix} w_4^{(n)}(t) \\ (w_4^{(n)})'(t) \end{pmatrix} \sim \tilde{c}_3 e^{-\gamma_n' t} \begin{pmatrix} 1 \\ -\gamma_n' \end{pmatrix} \quad \text{as } t \to -\infty,$$

and we may take $c_1, \tilde{c}_3 > 0$ without loss of generality. Thus, close to $-\infty$, we know $(w_1^{(n)}(t), (w_1^{(n)})'(t))$ is in the top right corner of the plane. It cannot cross the ray $\{0\} \times (0, +\infty)$ because there the vector field $\left(\begin{smallmatrix} w^2 \\ -(d-2)w^2 \end{smallmatrix}\right)$ points toward the right. Neither can it go below the ray $\left(x, -\frac{d-2}{2}x\right)_{x \geq 0}$. To see that, we compute the scalar product between the vector field and a vector that is orthogonal to this ray and that points toward north at any time $t \in \mathbb{R}$:

$$\left(\begin{pmatrix} 0 & 1 \\ n(d+n-2)+e^{2t}V(e^t) & -(d-2) \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{d-2}{2} \end{pmatrix}\right) \cdot \begin{pmatrix} \frac{d-2}{2} \\ 1 \end{pmatrix} = \frac{(d-2)^2}{4} + e^{2t}V(e^t) + n(d+n-2) > 0$$

because $e^{2t}V(e^t) > \frac{(d-2)^2}{4}$, where the potential $-V$ is below the Hardy potential (see (2-5)). Hence $(w_1^{(n)}(t), (w_1^{(n)})'(t))$ stays in the top right zone whose border is

$$\{0\} \times (0, +\infty) \cup \left(x, -\frac{d-2}{2}x\right)_{x \geq 0}.$$

In particular, $w_1^{(n)} > 0$ for all times, which proves the positivity of $w_1^{(n)}$. Since the trajectory $(w_4^{(n)}(t), (w_4^{(n)})'(t))$ is asymptotically collinear to the vector $\left(\begin{smallmatrix} 1 \\ -\gamma_n' \end{smallmatrix}\right)$, which does not belong to this zone (from Lemma A.1) nor its opposite, one obtains that $w_1^{(n)}$ and $w_4^{(n)}$ are not collinear. □

We now end the proof of Lemma 2.3. The fundamental set of solutions of (A-1) is provided by Lemma A.2. As $w_1^{(n)}$ is not collinear to $w_4^{(n)}$, there exists $a_1 \neq 0$ and a_2 such that $w_1^{(n)} = a_1 w_3^{(n)} + a_2 w_4^{(n)}$. From the asymptotics (A-7) and the positivity of $w_1^{(n)}$ shown in Lemma A.3, one then has

$$w_1^{(n)} = be^{-\gamma_n t} + O(e^{(-\gamma_n - g)t}) \quad \text{as } t \to +\infty, \ b > 0.$$

We call T_0^n the profile associated to $w_1^{(n)}$ in the original space variable r: $T_0^n(r) = w_1^{(n)}(\ln(r))$, which solves $H^{(n)}T_0^{(n)} = 0$. The above identity means $T_0^n = a_1 r^{-\gamma_n} + O(r^{(-\gamma_n - g)})$ as $r \to +\infty$, and (A-6) implies $T_0^n(r) = \sum_{i=0}^q b_i^n r^{n+2l} + O(r^{n+2+2q})$ as $r \to 0$, for some coefficients $(b_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, for any $q \in \mathbb{N}$. These asymptotics propagate to the derivatives. This is the identity (2-7) we had to prove.

Let us denote by w another solution of (A-1) that is not collinear to $w_1^{(n)}$ and $w_4^{(n)}$. Now (A-6) and (A-7) imply that $w \sim ce^{(2-n-d)t}$ as $t \to -\infty$ and $w = de^{-\gamma_n t} + O(e^{(-\gamma_n - g)t})$ as $t \to +\infty$ with $c, d \neq 0$. These asymptotics propagate to higher derivatives. The solution of $H^{(n)}\Gamma^{(n)} = 0$ given by $\Gamma^{(n)}(r) = w(\ln(r))$ then satisfies the desired asymptotics (2-7). Eventually, the Laplacian on spherical harmonics of degree n is (for f radial)

$$\Delta(f Y_{n,k}) = \left(\left(\partial_{rr} + \frac{d-1}{r}\partial_r - \frac{n(d+n-2)}{r^2}\right) f\right) Y_{n,k},$$

meaning, by the asymptotics (2-7), that for any $j \in \mathbb{N}$, we know $\Delta^j(T_0^n(|x|)Y_{n,k}(x/|x|))$ is a continuous function near the origin. Therefore, $T_0^n Y_{n,k}$ is smooth close to the origin by elliptic regularity. It is also smooth outside as a product of smooth functions, and thus smooth everywhere, ending the proof of Lemma 2.3. □

Appendix B: Hardy- and Rellich-type inequalities

We recall in this section the Hardy and Rellich estimates, to make this paper self-contained. They are used throughout the paper, and especially to derive a fundamental coercivity property of the adapted high Sobolev norm in Appendix C. We now state a useful and very general Hardy inequality with possibly fractional weights and derivatives. A proof can be found in [Merle et al. 2015, Lemma B.2].

Lemma B.1 (Hardy-type inequalities). *Let $\delta > 0$, $q \geq 0$ satisfy $\left|q - \left(\frac{d}{2} - 1\right)\right| \geq \delta$ and $u : [1, +\infty) \to \mathbb{R}$ be smooth and satisfy*

$$\int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q}} y^{d-1}\, dy + \int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy < +\infty.$$

(i) *If $q > \frac{d}{2} - 1 + \delta$, then*

$$C(d, \delta) \int_{y \geq 1} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy - C'(d, \delta) u^2(1) \leq \int_{y \geq 1} \frac{|\partial_y u|^2}{y^{2q}} y^{d-1}\, dy. \tag{B-1}$$

(ii) *If $q < \frac{d}{2} - 1 - \delta$, then*

$$C(d, \delta) \int_{y \geq 1} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy \leq \int_{y \geq 1} \frac{|\partial_y u|^2}{y^{2q}} y^{d-1}\, dy. \tag{B-2}$$

Proof. Let $R > 1$. The fundamental theorem of calculus gives

$$\frac{u^2(R)}{R^{2q+2-d}} - u^2(1) = 2 \int_1^R \frac{u \partial_y u}{y^{2q+2-d}}\, dy - (2q + 2 - d) \int_1^R \frac{u^2}{y^{2q+2-d}}\, dy.$$

The integrability of u^2 / y^{2q+3-d} over $[1, +\infty)$ implies that $u^2(R_n) / R_n^{2q+2-d} \to 0$ along a sequence of radii $R_n \to +\infty$. Passing to the limit through this sequence we get

$$(2q + 2 - d) \int_1^{+\infty} \frac{u^2}{y^{2q+2-d}}\, dy - u^2(1) = 2 \int_1^{+\infty} \frac{u \partial_y u}{y^{2q+2-d}}\, dy.$$

We apply the Cauchy–Schwarz and Young inequalities to find

$$\begin{aligned} \left| 2 \int_1^{+\infty} \frac{u \partial_y u}{y^{2q+2-d}}\, dy \right| &\leq 2 \left(\int_1^{+\infty} \frac{u^2}{y^{2q+3-d}}\, dy \right)^{\frac{1}{2}} \left(\int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q+1-d}}\, dy \right)^{\frac{1}{2}} \\ &\leq \varepsilon \int_1^{+\infty} \frac{u^2}{y^{2q+3-d}}\, dy + \frac{1}{\varepsilon} \int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q+3-d}}\, dy \end{aligned}$$

for any $\varepsilon > 0$. If $q > \frac{d}{2} - 1 + \delta$, then the two above identities give

$$\int_1^{+\infty} \frac{u^2}{y^{2q+2-d}}\, dy \leq \frac{u^2(1)}{2\delta} + \frac{\varepsilon}{2\delta} \int_1^{+\infty} \frac{u^2}{y^{2q+3-d}}\, dy + \frac{1}{2\delta\varepsilon} \int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q+3-d}}\, dy.$$

Taking $\varepsilon=\delta$, one gets

$$\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}\,dy\le\frac{u^2(1)}{\delta}+\frac{1}{\delta^2}\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}\,dy,$$

which is precisely the identity (B-1) we had to prove. If $q<\frac{d}{2}-1-\delta$ then one obtains

$$\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}\,dy\le-\frac{u^2(1)}{2\left(\frac{d}{2}-1-q\right)}+\frac{\varepsilon}{2\delta}\int_1^{+\infty}\frac{u^2}{y^{2q+3-d}}\,dy+\frac{1}{2\delta\varepsilon}\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}\,dy.$$

Taking $\varepsilon=\delta$, one gets

$$\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}\,dy\le\frac{1}{\delta^2}\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}\,dy,$$

which is precisely the second identity (B-2) we had to prove. □

Lemma B.2 (Rellich-type inequalities). *For any $u\in H^2(\mathbb{R}^d)$,*

$$\left(\frac{(d-4)d}{4}\right)^2\int_{\mathbb{R}^d}\frac{u^2}{|x|^4}\,dx\le\int_{\mathbb{R}^d}|\Delta u|^2\,dx,\quad \frac{d^2}{4}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^2}\,dx\le\int_{\mathbb{R}^d}|\Delta u|^2\,dx. \tag{B-3}$$

If $q\ge 0$ and $u:\mathbb{R}^d\to\mathbb{R}$ is a smooth function satisfying

$$\int_{\mathbb{R}^d}\left(\frac{|\Delta u|^2}{1+|x|^{2q}}+\frac{|\nabla u|^2}{1+|x|^{2q+2}}+\frac{u^2}{1+|x|^{2q+4}}\right)dx<+\infty,$$

then

$$C(d,q)\sum_{1\le|\mu|\le 2}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{2q+4-2\mu}}\,dx-C'(d,q)\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4}}\,dx\le\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}\,dx. \tag{B-4}$$

Proof. The inequality (B-3) is standard and we omit its proof. To prove (B-4) we reason with smooth and compactly supported functions, and then conclude by a density argument.

Step 1: control of the first derivatives. Using integration by parts we compute

$$\int_{\mathbb{R}^d}\frac{u\Delta u}{1+|x|^{2q+2}}\,dx=-\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}\,dx+\frac12\int_{\mathbb{R}^d}u^2\Delta\left(\frac{1}{1+|x|^{2q+2}}\right)dx.$$

We then use the Cauchy–Schwarz and Young inequalities to obtain

$$C\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}\,dx-C'\int_{\mathbb{R}^d}u^2\left(\Delta\left(\frac{1}{1+|x|^{2q+2}}\right)-\frac{1}{(1+|x|^{2q+2})(1+|x|)^2}\right)dx$$
$$\le\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{(1+|x|^{2q+2})(1+|x|)^{-2}}\,dx.$$

Noticing that $(1+|x|^{2q+2})(1+|x|)^{-1}\sim(1+|x|^{2q})$ and that

$$\left|\Delta\left(\frac{1}{1+|x|^{2q+2}}\right)-\frac{1}{(1+|x|^{2q+2})(1+|x|)^2}\right|\le\frac{C}{1+|x|^{2q+4}}$$

leads to the estimate

$$C(d,p)\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}\,dx - C'(d,q)\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4}}\,dx \le \int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}\,dx. \tag{B-5}$$

Step 2: control of the second order derivatives. Again using integrations by parts one finds

$$\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}} = \int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^{2q}} + \sum_{i=1}^{n}\partial_{x_i}u\nabla\partial_{x_i}u.\nabla\left(\frac{1}{1+|x|^{2q}}\right) - \Delta u\nabla u.\nabla\left(\frac{1}{1+|x|^{2q}}\right),$$

in which by using the Cauchy–Schwarz and Young inequalities, for any $\varepsilon > 0$, we can control the last two terms by

$$\left|\int_{\mathbb{R}^d}\sum_{i=1}^{n}\partial_{x_i}u\nabla\partial_{x_i}u.\nabla\left(\frac{1}{1+|x|^{2q}}\right) - \Delta u\nabla u.\nabla\left(\frac{1}{1+|x|^{2q}}\right)\right| \le C\varepsilon\int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^{2q}}\,dx + \frac{C}{\varepsilon}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}\,dx.$$

Therefore for ε small enough the two above identities yield

$$\int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^{2q}}\,dx \le C\left(\int_{\mathbb{R}^d}\left(\frac{|\Delta u|^2}{1+|x|^{2q}} + \frac{|\nabla u|^2}{1+|x|^{2q+2}} + \frac{u^2}{1+|x|^{2q+4}}\right)dx\right).$$

Combining this identity and (B-5), one obtains the desired identity (B-4). □

Lemma B.3 (weighted and fractional Hardy inequality). *Let*

$$0 < \nu < 1,\ k \in \mathbb{N} \ \text{ and } \ 0 < \mu \ \text{ satisfying } \ \mu + \nu + k < \tfrac{1}{2}d,$$

and let f be a smooth function satisfying the decay estimates

$$|\partial^\kappa f(x)| \le \frac{C(f)}{1+|x|^{\mu+i}} \quad \text{for } \kappa \in \mathbb{N}^d,\ |\kappa|_1 = i,\ i = 0, 1, \dots, k+1. \tag{B-6}$$

Then for $\varepsilon \in \dot{H}^{\mu+k+\nu}$, we have $\varepsilon f \in \dot{H}^{\nu+k}$ with

$$\|\nabla^{\nu+k}(\varepsilon f)\|_{L^2} \le C(C(f),\nu,k,\mu,d)\,\|\nabla^{\mu+k+\nu}\varepsilon\|_{L^2}. \tag{B-7}$$

If f is smooth and radial then (B-6) *is equivalent to*

$$|\partial_r^i f(r)| \le \frac{C(f)}{1+r^{\mu+i}}, \quad i = 0, 1, \dots, k+1. \tag{B-8}$$

Proof. **Step 1:** the case $k = 0$. A proof of the case $k = 0$ can be found in [Merle et al. 2015], for example.

Step 2: the case $k \ge 1$. Let f, ε, μ, ν and k satisfy the conditions of the lemma, with $k \ge 1$. Using the Leibniz rule for the entire part of the derivation,

$$\|\nabla^{\nu+k}(\varepsilon f)\|_{L^2}^2 \le C\sum_{\substack{(\kappa,\tilde{\kappa})\in\mathbb{N}^{2d} \\ |\kappa|_1+|\tilde{\kappa}|_1=k}}\|\nabla^\nu(\partial^\kappa\varepsilon\partial^{\tilde{\kappa}}f\|_{L^2}^2. \tag{B-9}$$

We can now apply the result obtained for $k=0$ to the norms $\|\nabla^\nu(\partial^{\kappa_k}\varepsilon\partial^{\tilde{\kappa}_k}f\|_{L^2}^2$ in (B-9). We have indeed that $\partial^\kappa\varepsilon\in\dot{H}^{\mu+k_2+\nu}$, and that $\partial^{\tilde{\kappa}}f$ satisfies the appropriate decay condition by (B-6). It implies that for all $(\kappa,\tilde{\kappa})\in\mathbb{N}^{2d}$ with $|\kappa|_1+|\tilde{\kappa}|_1=k$,

$$\|\nabla^\nu(\partial^{\kappa_k}\varepsilon\partial^{\tilde{\kappa}_k}f\|_{L^2}^2\leq C\|\nabla^{\nu+\mu+k}\varepsilon\|_{L^2}^2$$

which implies the result: $\|\nabla^{\nu+k}(\varepsilon f)\|_{L^2}^2\leq C(C(f),\nu,d,k,\alpha)\|\nabla^{\nu+\mu+k}\varepsilon\|_{L^2}^2$.

Step 3: equivalence between the decay properties. We want to show that (B-6) and (B-8) are equivalent for radial smooth functions. Suppose that f is smooth, radial, and satisfies (B-6). Then one has

$$\partial_y^i f(y)=\frac{\partial f}{\partial_{x_1}^i}(|y|e_1),$$

where e_1 stands for the unit vector $(1,\dots,0)$ of $\mathbb{R}^d$. From this formula, we see that the condition (B-6) on $(\partial f/\partial_{x_1}^i)(|y|e_1)$ implies the radial condition (B-8). We now suppose that f is a smooth radial function satisfying the radial condition (B-8). Then there exists a smooth radial function ϕ such that

$$f(y)=\phi(y^2).$$

With a proof by induction that can be left to the reader, one has that the decay property (B-8) for f implies the following decay property for ϕ:

$$|\partial_y^i\phi(y)|\leq\frac{C(f)}{1+y^{\frac{\mu}{2}+i}},\quad i=0,1,\dots,k+1.$$

Now the standard derivatives of f are easier to compute with ϕ. We claim that for all $\kappa\in\mathbb{N}^d$ there exists a finite number of polynomials $P_i(x):=C_ix_1^{i_1}\cdots x_d^{i_d}$, for $1\leq i\leq l(\kappa)$, such that

$$\partial^\kappa f(x)=\sum_{i=1}^{l(\kappa)}P_i(x)\partial_{|x|}^{q(i)}\phi(|x|^2),$$

with $2q(i)-\sum_{j=1}^d i_j=|\kappa|_1$ for all i. The proof by induction of this fact can also be left to the reader. The decay property for ϕ then implies

$$\left|P_i(x)\partial_{|x|}^{q(i)}\phi(|x|^2)\right|\leq\frac{C}{1+y^{\alpha+2q(i)-\sum_{j=1}^d i_j}}=\frac{C}{1+y^{\alpha+|\kappa|_1}},$$

which in turn implies the property (B-6). □

Appendix C: Coercivity of the adapted norms

Here we prove coercivity estimates for the operator H under suitable orthogonality conditions, following the techniques of [Raphaël and Rodnianski 2012]. We recall that the profiles used as orthogonality directions, $\Phi_M^{(n,k)}$, are defined by (4-1). To perform an analysis on each spherical harmonic and to be able to track the constants, we will not study directly $A^{(n)}$ and $A^{(n)*}$, but the asymptotically equivalent operators

$$\widetilde{A}^{(n)} : u \mapsto -\partial_y u + \widetilde{W}^{(n)} u, \quad A^{(n)*} : u \mapsto \frac{1}{y^{d-1}} \partial_y (y^{d-1} u) + \widetilde{W}^{(n)} u, \tag{C-1}$$

where

$$\widetilde{W}^{(n)} = -\frac{\gamma_n}{y}. \tag{C-2}$$

By the definition (1-18) of γ_n, they factorize the operator

$$\widetilde{H}^{(n)} := -\partial_{yy} - \frac{d-1}{y}\partial_y - \frac{pc_\infty^{p-1}}{y^2} + \frac{n(d+n-2)}{y^2} = \widetilde{A}^{(n)*}\widetilde{A}^{(n)}. \tag{C-3}$$

The strategy is the following. First we derive subcoercivity estimates for $\widetilde{A}^{(n)*}$, $\widetilde{A}^{(n)}$ and $H^{(n)}$. A summation yields subcoercivity for $-\Delta - pc_\infty^{p-1}/|x|^2$, and hence for H as they are asymptotically equivalent. Roughly, this subcoercivity implies that minimizing sequences of the functional $I(u) = \int uH^s u$ are "almost compact" on the unit ball of $\dot{H}^s \cap (\mathrm{Span}(\Phi_M^{(n,k)}))^\perp$. In particular if the infimum of I on this set was 0, it would be attained, which is impossible from the orthogonality conditions, yielding the coercivity $\int uH^s u \gtrsim \|u\|_{\dot{H}^s}^2$ via homogeneity.

Lemma C.1. *Let n be an integer, $q \geq 0$ and $u : [1, +\infty) \to \mathbb{R}$ be smooth satisfying*

$$\int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q}} y^{d-1}\, dy + \int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy < +\infty. \tag{C-4}$$

(i) *There exist two constants $c, c' > 0$ independent of n and q such that*

$$c \int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy - c' u^2(1) \leq \int_1^{+\infty} \frac{|\widetilde{A}^{(n)*} u|^2}{y^{2q}} y^{d-1}\, dy. \tag{C-5}$$

(ii) *Let $\delta > 0$ and suppose $\left|q - \left(\frac{d}{2} - 1 - \gamma_n\right)\right| > \delta$. Then there exist two constants $c(\delta), c'(\delta) > 0$ depending only on δ such that*

$$c(\delta) \int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy - c'(\delta) u^2(1) \leq \int_1^{+\infty} \frac{|\widetilde{A}^{(n)} u|^2}{y^{2q}} y^{d-1}\, dy. \tag{C-6}$$

Proof. **Coercivity for $\widetilde{A}^{(n)*}$.** We first compute

$$\int_1^{+\infty} \frac{|\widetilde{A}^{(n)*} u|^2}{y^{2q}} y^{d-1}\, dy = \int_1^{+\infty} \frac{|\partial_y u + y^{-1}(d-1-\gamma_n)u|^2}{y^{2q}} y^{d-1}\, dy.$$

We make the change of variable $u = v y^{\gamma_n + 1 - d}$. By (C-4), $v^2/y^{2q-2\gamma_n+d+1}$ and $|\partial_y v|^2/y^{2q-2\gamma_n+d-1}$ are integrable on $[1, +\infty)$. As $q + \frac{d}{2} - \gamma_n \geq \frac{d}{2} - \gamma > 1$ by (1-9) and (1-18), we can apply (B-2) to the above identity and obtain (C-5) via

$$\begin{aligned}\int_1^{+\infty}\frac{|\widetilde{A}^{(n)*}u|^2}{y^{2q}}y^{d-1}\,dy&=\int_1^{+\infty}\frac{|\partial_y v|^2}{y^{2q-2\gamma_n+2d-2}}y^{d-1}\,dy\\&\geq C\int_1^{+\infty}\frac{v^2}{y^{2q-2\gamma_n+2d-2}}y^{d-1}\,dy-C'v^2(1)\\&=C\int_1^{+\infty}\frac{u^2}{y^{2q+2}}y^{d-1}\,dy-C'u^2(1).\end{aligned}$$

Coercivity for $\widetilde{A}^{(n)}$. This time the integral we have to estimate is

$$\int_1^{+\infty}\frac{|\widetilde{A}^{(n)}u|^2}{y^{2q}}y^{d-1}\,dy=\int_1^{+\infty}\frac{|\partial_y u+y^{-1}\gamma_n u|}{y^{2p}}y^{d-1}\,dy.$$

We make the change of variable $u=vy^{-\gamma_n}$. By (C-4), $v^2/y^{2p+2\gamma_n-d+1}$ and $|\partial_y v|^2/y^{2p+2\gamma_n+3-d}$ are integrable on $[1,+\infty)$. As $\left|q-\left(\frac{d}{2}-1-\gamma_n\right)\right|>\delta$, one can apply (B-1) or (B-2) to the above identity: there exists $c=c(\delta)$ and $c'=c'(\delta)$ such that

$$\begin{aligned}\int_1^{+\infty}\frac{|\widetilde{A}^{(n)}u|^2}{y^{2q}}y^{d-1}\,dy&=\int_1^{+\infty}\frac{|\partial_y v|^2}{y^{2q+2\gamma_n}}y^{d-1}\\&\geq c\int_1^{+\infty}\frac{v^2}{y^{2q+2\gamma_n+2}}y^{d-1}\,dy-c'v^2(1)\\&=c\int_1^{+\infty}\frac{u^2}{y^{2q+2}}y^{d-1}\,dy-c'u^2(1),\end{aligned}$$

which is precisely the identity (C-6). □

Lemma C.2 (coercivity of H under suitable orthogonality conditions). *Let $\delta>0$ and $q\geq 0$ such that*[22] $\left|q-\left(\frac{d}{2}-2-\gamma_n\right)\right|\geq\delta$ *for all $n\in\mathbb{N}$. Let $n_0\in\mathbb{N}\cup\{-1\}$ be the lowest number such that $q-\left(\frac{d}{2}-2-\gamma_{n_0+1}\right)<0$. Then there exists a constant $c(\delta)>0$ such that for all $u\in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying the integrability condition*

$$\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}+\frac{|\nabla u|^2}{1+|x|^{2q+2}}+\int\frac{u^2}{1+|x|^{2q+4}}<+\infty$$

and the orthogonality conditions[23] ($\Phi_M^{(n,k)}$ *being defined in* (4-1))

$$\langle u,\Phi_M^{(n,k)}\rangle=0\quad\textit{for } 0\leq n\leq n_0,\ 1\leq k\leq k(n),\tag{C-7}$$

one has the inequality

$$c(\delta)\left(\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}+\frac{|\nabla u|^2}{|x|^2(1+|x|^{2q})}+\frac{u^2}{|x|^4(1+|x|^{2q})}\right)\leq\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2q}}.\tag{C-8}$$

[22] We recall that $\gamma_n\to-\infty$; hence for δ small enough many q satisfy this condition.

[23] With the convention that there are no orthogonality conditions required if $n_0=-1$.

Proof. In what follows, $C(\delta)$ and $C'(\delta)$ denote strictly positive constants that may vary but only depend on δ, d and p.

Step 1: We claim the following subcoercivity estimate for $\widetilde{H} := -\Delta - pc_\infty^{p-1}/|x|^2$:

$$\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)} \frac{|\widetilde{H}u|^2}{|x|^{2q}}\,dx \geq C(\delta)\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)} \frac{u^2}{|x|^{2q+4}}\,dx - C'(\delta)\big(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2 + \|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2\big), \quad \text{(C-9)}$$

where $f_{|\mathcal{S}^{d-1}(1)}$ denotes the restriction of f to the sphere. We now prove this inequality. We start with the decomposition

$$u(x) = \sum_{n,\,1\leq k\leq k(n)} u^{(n,k)}(|x|)Y^{(n,k)}\Big(\frac{x}{|x|}\Big).$$

We recall the link between u and its decomposition ($\widetilde{H}^{(n)}$ being defined by (C-3)):

$$\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)} \frac{|\widetilde{H}u|^2}{|x|^{2q}}\,dx = \sum_{n,\,1\leq k\leq k(n)} \int_1^{+\infty} \frac{|\widetilde{H}^{(n)}u^{(n,k)}|^2}{y^{2q}}y^{d-1}\,dy, \quad \text{(C-10)}$$

$$\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)} \frac{u^2}{|x|^{2q+4}}\,dx = \sum_{n,\,1\leq k\leq k(n)} \int_1^{+\infty} \frac{|u^{(n,k)}|^2}{y^{2q+4}}y^{d-1}\,dy. \quad \text{(C-11)}$$

As $\widetilde{H}^{(n)} = \widetilde{A}^{(n)*}\widetilde{A}^{(n)}$ and $\big|q-\big(\frac{d}{2}-2-\gamma_n\big)\big| > \delta$ for all $n\in\mathbb{N}$, we apply (C-5) and (C-6) to obtain for each $n\in\mathbb{N}$,

$$\int_1^{+\infty} \frac{|\widetilde{H}^{(n)}u^{(n,k)}|^2}{y^{2q}}y^{d-1}\,dy \geq C(\delta)\int_1^{+\infty} \frac{|u^{(n,k)}|^2}{y^{2q+4}}y^{d-1}\,dy - C'(\delta)\big((u^{(n,k)})^2(1) + \widetilde{A}^{(n)}(u^{(n,k)})^2(1)\big). \quad \text{(C-12)}$$

We now sum this identity over n and k. The second term on the right-hand side is

$$\sum_{n,1\leq k\leq k(n)} (u^{(n,k)})^2(1) = \int_{\mathcal{S}^{d-1}}\Big(\sum_{n,\,1\leq k\leq k(n)} u^{(n,k)}(1)Y^{(n,k)}(x)\Big)^2\,dx = \int_{\mathcal{S}^{d-1}} u^2(x)\,dx$$

because $(Y^{(n,k)})_{n,1\leq k\leq n}$ is an orthonormal basis of $L^2(\mathcal{S}^{d-1})$. From (C-1), and as $\gamma_n \sim -n$ as $n\to+\infty$ by (1-18), the last term on the right-hand side of (C-12) is

$$\begin{aligned}\sum_{n,\,1\leq k\leq n} |\widetilde{A}^{(n)}u^{(n,k)}|^2(1) &\leq C\sum_{n,\,1\leq k\leq k(n)} (1+n^2)|u^{(n,k)}|^2(1) + |\partial_y u^{(n,k)}|^2\\ &\leq C\big(\|u_{|\mathcal{S}^{d-1}(1)}\|_{H^1}^2 + \|\nabla u_{|\mathcal{S}^{d-1}(1)}\cdot\vec{n}\|_{L^2}^2\big)\\ &\leq C\big(\|u_{|\mathcal{S}^{d-1}}\|_{L^2}^2 + \|\nabla u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2\big).\end{aligned}$$

We insert the two above equations into (C-12) and obtain

$$\sum_{n,\,1\le k\le n}\int_1^{+\infty}\frac{|\widetilde{H}^{(n)}u^{(n,k)}|^2}{y^{2q}}y^{d-1}\,dy \\ \ge C(\delta)\sum_{n,\,1\le k\le n}\int_1^{+\infty}\frac{|u^{(n,k)}|^2}{y^{2q+4}}y^{d-1}\,dy - C'(\delta)\big(\|u_{|\mathcal{S}^{d-1}}\|_{L^2}^2+\|\nabla u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2\big).$$

In turn, we insert this identity into (C-10) using (C-11) to obtain the desired estimate (C-9).

Step 2: subcoercivity for H. We will prove the estimate

$$\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2q}}\,dx \\ \ge C(\delta)\bigg(\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}\,dx+\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^2(1+|x|^{2q})}\,dx+\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}\,dx\bigg) \\ -C'(\delta)\bigg(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+\alpha}}+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}^2\bigg). \tag{C-13}$$

Away from the origin, the Cauchy–Schwarz and Young inequalities, the bound $V+pc_\infty^{p-1}|x|^{-2}=O(|x|^{-2-\alpha})$ from (2-2) and (C-9) give (for $C>0$)

$$\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{|Hu|^2}{|x|^{2q}}\,dx=\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{|\widetilde{H}u+(V+pc_\infty^{p-1}|x|^{-2})u|^2}{|x|^{2q}}\,dx \\ \ge C\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{|\widetilde{H}u|^2}{|x|^{2q}}\,dx-C'\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{|u|^2}{|x|^{2q+4+2\alpha}}\,dx \\ \ge C(\delta)\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{u^2}{1+|x|^{2q+4}} \\ -C'(\delta)\bigg(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{|u|^2}{1+|x|^{2q+4+2\alpha}}\bigg).$$

Close to the origin, using Rellich's inequality (B-3),

$$\int_{\mathcal{B}^d(1)}|Hu|^2\,dx\ge C\int_{\mathcal{B}^d(1)}|\Delta u|^2\,dx-\frac{1}{C}\int_{\mathcal{B}^d(1)}|u|^2\,dx \\ \ge C\int_{\mathcal{B}^d(1)}\frac{|u|^2}{|x|^4}\,dx-\frac{1}{C}\|u\|_{H^1(\mathcal{B}^{d-1}(1))}.$$

Combining the two previous estimates we obtain the intermediate identity

$$\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2q}}\,dx\ge C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}\,dx-C'(\delta)\bigg(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2 \\ +\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}\,dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}^2\bigg).$$

Now, as $H=-\Delta+V$ with $V=O((1+|x|)^{-2})$, using Young's inequality, the above identity and (B-4), for $\varepsilon>0$ small enough (depending on δ) one has

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2p}}\,dx\\
&\quad=(1-\varepsilon)\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2p}}\,dx|Hu|^2dx+\varepsilon\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2p}}\,dx\\
&\quad\geq(1-\varepsilon)C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}\,dx\\
&\qquad-C'(\delta)\biggl(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}\,dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}\biggr)\\
&\qquad+\frac{\varepsilon}{2}\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}\,dx-\varepsilon\int_{\mathbb{R}^d}\frac{|Vu|^2}{1+|x|^{2q}}\,dx\\
&\quad\geq(1-\varepsilon)C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}\,dx\\
&\qquad-C'(\delta)\biggl(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}\,dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}\biggr)\\
&\qquad+C(q)\frac{\varepsilon}{2}\sum_{1\leq|\mu|\leq 2}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{2q+4-2\mu}}\,dx-\varepsilon C'(q)\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4}}\,dx\\
&\quad\geq C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}+\frac{C(q)\varepsilon}{2}\sum_{1\leq|\mu|\leq 2}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{2q+4-2\mu}}\\
&\qquad-C'(\delta)\biggl(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}\,dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}\biggr),
\end{aligned}
$$

which is the identity (C-13) we claimed.

Step 3: coercivity for H. We now argue by contradiction. Suppose that (C-8) does not hold. Up to a renormalization, this means that there exists a sequence of functions $(u_n)_{n\in\mathbb{N}}$ such that, for all n,

$$\int_{\mathbb{R}^d}\frac{|Hu_n|^2}{1+|x|^{2q}}\to 0,\qquad\int_{\mathbb{R}^d}\frac{|\Delta u_n|^2}{1+|x|^{2q}}+\frac{|\nabla u_n|^2}{|x|^2(1+|x|^{2q})}+\frac{|u_n|^2}{|x|^4(1+|x|^{2q})}=1.\tag{C-14}$$

Up to a subsequence, we can suppose that $u_n\to u_\infty\in H^2_{\mathrm{loc}}(\mathbb{R}^d)$, the local convergence in L^2 being strong for $(u_n)_{n\in\mathbb{N}}$ and $(\nabla u_n)_{n\in\mathbb{N}}$, and weak for $(\nabla^2u_n)_{n\in\mathbb{N}}$. Then (C-14) implies

$$\|u_n\|_{H^1(\mathcal{B}^{d-1}(1))}^2+\int_{\mathbb{R}^d}\frac{|u_n|^2}{1+|x|^{2q+4+\alpha}}\ \to\|u_\infty\|_{H^1(\mathcal{B}^{d-1}(1))}^2+\int_{\mathbb{R}^d}\frac{|u_\infty|^2}{1+|x|^{2q+4+\alpha}}.$$

Now u_n converges strongly to u_∞ in $H^s(\mathcal{B}^d(0,1))$ for any $0\leq s<2$. The trace theorem for Sobolev spaces ensures that

$$\|(u_n)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u_n)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2\to\|(u_\infty)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u_\infty)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2.$$

We insert the three previous identities into the subcoercivity estimate (C-13) yielding

$$\|(u_\infty)_{|S^{d-1}(1)}\|_{L^2}^2+\|(\nabla u_\infty)_{|S^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d}\frac{|u_\infty|^2}{1+|x|^{2q+4+\alpha}}+\|u_\infty\|_{H^1(\mathcal{B}^d(1))}^2\neq 0,$$

which means that $u_\infty\neq 0$. On the other hand, the lower semicontinuity of norms for the weak topology and (C-14) imply

$$Hu_\infty=0.$$

Hence u_∞ is a nontrivial function in the kernel of H, and is smooth from elliptic regularity. It satisfies the integrability condition (still from lower semicontinuity)

$$\int_{\mathbb{R}^d}\frac{|\Delta u_\infty|^2}{1+|x|^{2q}}\,dx+\frac{|\nabla u_\infty|^2}{1+|x|^{2q+2}}\,dx+\int\frac{|u_\infty|^2}{1+|x|^{2q+4}}\,dx<+\infty.$$

We now decompose u_∞ into spherical harmonics, $u_\infty=\sum_{n,\,1\le k\le k(n)}u_\infty^{(n,k)}Y_{(n,k)}$, and will show that for each n,k one must have $u_\infty^{(n,k)}=0$, which will give a contradiction. For each n,k, the nullity $Hu_\infty=0$ implies $H^{(n)}u_\infty^{(n,k)}$, where $H^{(n)}$ is defined in (1-36). By Lemma 2.3 this means $u_\infty=aT_0^{(n)}+b\Gamma^{(n)}$ for a and b two real numbers. The previous equation implies the following integrability for $u_\infty^{(n,k)}$:

$$\int\frac{|u_\infty^{(n,k)}|^2}{1+y^{2q+4}}y^{d-1}\,dy<+\infty.$$

By (2-7), as $\Gamma^{(n)}\sim y^{-d-n+2}$ does not satisfy this integrability at the origin whereas $T_0^{(n)}$ is regular, one must have $b=0$. Then, if $n\ge n_0+1$,

$$\frac{|T_0^{(n)}|^2}{1+y^{2q+4}}y^{d-1}\sim y^{-2\gamma_n-2q-5+d}.$$

From the assumption on n_0 and (1-18), one has

$$-2\gamma_n-2q-5+d=-1-2\big(q+2+\gamma_{n_0+1}-\tfrac{d}{2}\big)+2(\gamma_{n_0+1}-\gamma_n)>-1,$$

implying that $|T_0^{(n)}|^2/(1+y^{2q+4})y^{d-1}$ is not integrable on $[0,+\infty)$; hence $a=0$. If $n\le n_0$ then the orthogonality condition (C-7) goes to the limit as $\Phi_M^{(n,k)}$ is compactly supported and implies

$$\langle u^\infty,\Phi_M^{(n,k)}\rangle=0,$$

which, in spherical harmonics, can be rewritten as

$$0=\langle u_\infty^{(n,k)},\Phi_M^{(n,k)}\rangle=a\langle T_0^{(n)},\Phi_M^{(n,k)}\rangle.$$

However, from (4-3) this in turn implies $a=0$. We have proven that for all n,k $u_\infty^{(n,k)}=0$; hence $u_\infty=0$, which is the desired contradiction, as we proved earlier that u_∞ is nontrivial. The coercivity (C-8) must then be true. $\square$

If one adds analogous orthogonality conditions for the derivatives of u and uses a bit more the structure of the Laplacian, one gets that the weighted norm $\|H^i/(1+|x|^p)u\|_{L^2}$ controls all derivatives of lower order with corresponding weights.

Lemma C.3 (coercivity of the iterates of H). *Let i be an integer with $2i > \sigma$ such that for all $n \in \mathbb{N}$ satisfying $m_n + \delta_n \le i$ one has $\delta_n \neq 0$. Let n_0 be the lowest integer such that $m_{n_0+1} + \delta_{n_0+1} > i$. Let $u \in \dot{H}^{2i} \cap \dot{H}^{\sigma}(\mathbb{R}^d)$ satisfy (where $\Phi_M^{(n,k)}$ is defined in* (4-1)*)*

$$\langle u, H^j \Phi_M^{n,k} \rangle = 0 \quad \textit{for } 0 \le n \le n_0,\ 0 \le j \le i - m_n - 1,\ 1 \le k \le k(n). \tag{C-15}$$

Then there exists a constant $\delta > 0$ such that for all $0 \le \delta' \le \delta$,

$$C(\delta, i) \sum_{|\mu| \le 2i} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1 + |x|^{4i - 2\mu + 2\delta'}}\, dx \le \int_{\mathbb{R}^d} \frac{|H^i u|^2}{1 + |x|^{2\delta'}}\, dx, \tag{C-16}$$

which in particular implies that

$$\|u\|_{\dot{H}^{2i}} \le C(\delta, i) \left(\int_{\mathbb{R}^d} |H^i u|^2\, dx \right)^{\frac{1}{2}}. \tag{C-17}$$

Proof. **Step 1:** equivalence of weighted norms. We claim that for all integers j,

$$H^j u = (-\Delta)^j u + \sum_{|\mu| \le 2j-2} f_{j,\mu} \partial^\mu u \tag{C-18}$$

for some smooth functions f_μ having the decay $|\partial^{\mu'} f_{j,\mu}| \le C(1 + |x|^{2j - |\mu| + |\mu'|})^{-1}$. This identity is true for $j = 1$ because $Hu = -\Delta u + Vu$ with the potential V being smooth and having the required decay by (2-2). If the aforementioned identity holds true for $j \ge 1$ then

$$\begin{aligned} H^{j+1} u &= (-\Delta + V)\left((-\Delta)^j u + \sum_{|\mu| \le 2j-2} f_{j,\mu} \partial^\mu u \right) \\ &= (-\Delta)^{j+1} u + V(-\Delta)^j u + \sum_{|\mu| \le 2j-2} (-\Delta + V)(f_{j,\mu} \partial^\mu u), \end{aligned}$$

and hence it is true for $j + 1$ since V is smooth and satisfies the decay (2-2). By induction it is true for all $j \in \mathbb{N}$ and (C-18) is proven. Then (C-18) implies that

$$\int_{\mathbb{R}^d} \frac{|H^i u|^2}{1 + |x|^{2\delta}}\, dx \le C \sum_{|\mu| \le 2i} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1 + |x|^{4i - 2|\mu| + 2\delta'}}\, dx. \tag{C-19}$$

Step 2: weighted integrability in $\dot{H}^{2i} \cap \dot{H}^{\sigma}$. We claim that for all functions $u \in \dot{H}^{2i} \cap \dot{H}^{\sigma}(\mathbb{R}^d)$ and $\delta' > 0$,

$$\sum_{|\mu| \le 2i} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1 + |x|^{4i - 2|\mu| + 2\delta'}}\, dx < +\infty. \tag{C-20}$$

Indeed, let μ be a $|\mu|$-tuple with $|\mu| \le 2i$. We split into two cases. First if $|\mu| \le \sigma$, as $\sigma < \frac{d}{2}$ and $2i > \sigma$, the Hardy inequality B.3 yields

$$\int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1 + |x|^{4i - 2|\mu| + 2\delta'}}\, dx \le \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1 + |x|^{2(\sigma - |\mu|)}}\, dx \le C \|u\|_{\dot{H}^\sigma}^2 < +\infty$$

and we are done. If $\sigma < \mu \le 2i$ then by interpolation $u \in \dot H^{|\mu|}(\mathbb{R}^d)$ and then

$$\int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1+|x|^{4i-2|\mu|+2\delta'}}\,dx \le \int |\partial^\mu u|^2\,dx < +\infty.$$

Thus (C-20) holds, which together with (C-19), implies, for all $\delta' \ge 0$,

$$\sum_{j=0}^{i} \int_{\mathbb{R}^d} \frac{|H^j u|^2}{1+|x|^{4i-4j+2\delta'}}\,dx + \frac{|\nabla H^{j-1} u|^2}{1+|x|^{4i-4j+2+2\delta'}}\,dx < +\infty. \tag{C-21}$$

Step 3: intermediate coercivity. Let $\delta = \min\big(\delta_0,\dots,\delta_{n_0+1},\frac12\big)$ if $\delta_{n_0+1} \ne 0$ and $\delta = \min\big(\delta_0,\dots,\delta_{n_0},\frac12\big)$ if $\delta_{n_0+1} = 0$. The conditions on the δ_n of the lemma imply $\delta > 0$. We claim that for all integers $1 \le l \le i$,

$$C(\delta)\int_{\mathbb{R}^d} \frac{|H^{l-1}u|^2}{1+|x|^{4i-4(l-1)+2\delta'}} + C(\delta)\int_{\mathbb{R}^d} \frac{|\nabla H^{l-1}u|^2}{1+|x|^{4i-4l+2+2\delta'}} \le \int_{\mathbb{R}^d} \frac{|H^l u|^2}{1+|x|^{4i-4l+2\delta'}}. \tag{C-22}$$

We now prove this estimate. We want to apply Lemma C.2 to the function $H^{l-1}u$ with weight $q = \delta' + 2(i-l)$. To use it, we have to check the orthogonality and integrability conditions that are required, and the conditions on the weight.

Integrability condition. It is true because of (C-21).

Condition on the weight. For the case $n \ge n_0+1$, by (1-23) one computes

$$\big|\delta'+2(i-l)-\big(\tfrac d2-\gamma_n-2\big)\big| = \big|\delta'-2\delta_{n_0+1}-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1})\big|. \tag{C-23}$$

One has $2(l-1) \ge 0$ as $l \ge 1$ and $2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1}) \ge 0$ because $(m_n+\delta_n)_n$ is an increasing sequence from (1-22) and (1-18). For the subcase $\delta_{n_0+1} = 0$, as $m_{n_0+1} > i$ and m_{n_0+1} is an integer, $2(m_{n_0+1}-i) > 2$. Therefore $-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1}) = -a$ for $a \ge 2$, and inserting it into the above identity as $0 < \delta' < 1$ gives

$$\big|\delta'+2(i-l)-\big(\tfrac d2-\gamma_n-2\big)\big| = |\delta'-a| \ge \delta' \ge \delta.$$

For the subcase $\delta_{n_0+1} \ne 0$, we have $\delta'-2\delta_{n_0+1} \le \delta-2\delta_{n_0+1} \le -\delta_{n_0+1} \le -\delta$. Moreover, $m_{n_0+1} \ge i$ and $-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1}) \le 0$, implying

$$\delta'-2\delta_{n_0+1}-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1}) \le \delta'-2\delta_{n_0+1} \le -\delta,$$

and therefore by (C-23) this yields in that case

$$\big|\delta'+2(i-l)-\big(\tfrac d2-\gamma_n-2\big)\big| \ge \delta.$$

In both subcases one has $\big|\delta'+2(i-l)-\big(\tfrac d2-\gamma_n-2\big)\big| \ge \delta$. For the case $n \le n_0$,

$$\big|\delta'+2(i-l)-\big(\tfrac d2-\gamma_n-2\big)\big| = \big|\delta'-2\delta_n+2(i-l+1-m_n)\big|.$$

In the above identity, $2(i-l+1-m_n)$ is an even integer, and $\delta'-2\delta_n$ is a number satisfying $\delta'-2\delta_n \le \delta-2\delta_n \le -\delta$ and we recall that $\delta < 1$, and $\delta'-2\delta_n \ge -2\delta_n \ge -1$. Therefore $|\delta'-2\delta_n+2(i-l+1-m_n)| \ge \delta$, yielding

$$\left|\delta' + 2(i-l) - \left(\tfrac{d}{2} - \gamma_n - 2\right)\right| \geq \delta.$$

Therefore, for each $n \in \mathbb{N}$, we have $\left|\delta' + 2(i-l) - \left(\tfrac{d}{2} - \gamma_n - 2\right)\right| \geq \delta$.

Orthogonality conditions. Let $n_0' = n_0'(l) \in \mathbb{N} \cup \{-1\}$ be the lowest number such that

$$2(i-l+1) + \delta' - 2(m_{n_0'+1} + \delta_{n_0'+1}) < 0.$$

By construction one has $n_0' \leq n_0$. If $n_0' = -1$ then we are done because no orthogonality condition is required. If $n_0' \neq -1$, let n be an integer, $0 \leq n \leq n_0'$. By the definition of n_0',

$$2(i-l+1) + \delta' - 2(m_n + \delta_n) > 0,$$

which implies $0 \leq l-1 \leq i - m_n - 1$ as $\delta' - 2\delta_n \leq \delta - 2\delta_n \leq -\delta_n \leq 0$. The orthogonality condition (C-15) then gives, for any $1 \leq k \leq k(n)$,

$$\langle u, H^{l-1}\Phi_M^{(n,k)}\rangle = 0.$$

We have then proved that for all $0 \leq n \leq n_0'$, $1 \leq k \leq k(n)$,

$$\langle H^{l-1}u, \Phi_M^{(n,)k}\rangle = 0,$$

which are the required orthogonality conditions.

Conclusion. One can apply Lemma C.2 to $H^{l-1}u$ with weight $q = 2i - 2l + \delta'$, giving the desired coercivity estimate (C-22).

Step 4: iterations of coercivity estimates. We show the following bound by induction on $l = 0, \dots, i$:

$$\int_{\mathbb{R}^d} \frac{|H^l u|^2}{1+|x|^{2\delta'}}\,dx \geq c(\delta, i) \sum_{0 \leq |\mu| \leq 2l} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1+|x|^{4i-2\mu+2\delta'}}\,dx. \tag{C-24}$$

This property is naturally true for $l = 0$. We now suppose it is true for $l-1$ with $0 \leq l-1 \leq i-1$. From the formula (C-18) relating Δ^l to H^l, we see that (using the Cauchy–Schwarz and Young inequalities)

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{|H^l u|^2}{1+|x|^{4(i-l)+2\delta'}} &\geq C(i)\int_{\mathbb{R}^d} \frac{|\Delta^l u|^2}{1+|x|^{4(i-l)+2\delta'}} - C'(i) \sum_{0 \leq |\mu| \leq 2l-2} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1+|x|^{4i-2|\mu|+2\delta'}} \\ &\geq C(i)\int_{\mathbb{R}^d} \frac{|\Delta^l u|^2}{1+|x|^{4(i-l)+2\delta'}} - C'(i)\int_{\mathbb{R}^d} \frac{|H^i u|^2}{1+|x|^{2\delta'}},\end{aligned}$$

where we used the induction hypothesis (C-24) for $l-1$ for the second line. We now use (C-24) and (B-4) to recover a control over all derivatives:

$$
\begin{aligned}
\int_{\mathbb{R}^d} \frac{|\Delta^l u|^2}{1+|x|^{4(i-l)+2\delta'}} &\geq C(i) \sum_{1\leq|\mu|\leq 2} \int_{\mathbb{R}^d} \frac{|\partial^\mu \Delta^{l-1} u|^2}{1+|x|^{4(i-l)+4-2|\mu|}} - C'(i) \int_{\mathbb{R}^d} \frac{|\Delta^{l-1} u|^2}{1+|x|^{4(i-l)+4}} \\
&\geq C(i) \sum_{0\leq|\mu|\leq 2} \int_{\mathbb{R}^d} \frac{|\Delta^{l-1} \partial^\mu u|^2}{1+|x|^{4(i-(l-1))-2|\mu|}} - C'(\delta, i) \int_{\mathbb{R}^d} \frac{|H^{l-1} u|^2}{1+|x|^{2\delta'}} \\
&\geq C(i) \sum_{0\leq|\mu|\leq 2} \sum_{1\leq|\mu'|\leq 2} \int_{\mathbb{R}^d} \frac{|\partial^{\mu'} \Delta^{l-2} \partial^\mu u|^2}{1+|x|^{4(i-(l-1))+4-2|\mu|-2|\mu'|}} \\
&\qquad\qquad - C'(i) \int_{\mathbb{R}^d} \frac{|\Delta^{l-2} u|^2}{1+|x|^{4(i-l)+8}} - C'(\delta, i) \int_{\mathbb{R}^d} \frac{|H^{l-1} u|^2}{1+|x|^{2\delta'}} \\
&\geq C(i) \sum_{0\leq|\mu|\leq 4} \int_{\mathbb{R}^d} \frac{|\Delta^{l-2} \partial^\mu u|^2}{1+|x|^{2p+4(i-(l-2))-2\mu}} - C'(i, \delta) \int_{\mathbb{R}^d} \frac{|H^{l-1} u|^2}{1+|x|^{2\delta'}} \\
&\ \ \vdots \\
&\geq C(i) \sum_{0\leq|\mu|\leq 2l} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1+|x|^{2p+4-2\mu+2\delta'}} - C'(\delta, i) \int_{\mathbb{R}^d} \frac{|H^{l-1} u|^2}{1+|x|^{2\delta'}}.
\end{aligned}
$$

Inserting this last equation into the previous one we obtain

$$
\int_{\mathbb{R}^d} \frac{|H^l u|^2}{1+|x|^{4(i-l)+2\delta'}} \geq C(\delta, i) \sum_{0\leq|\mu|\leq 2l} \int_{\mathbb{R}^d} \frac{|\Delta^{l-2} \partial^\mu u|^2}{1+|x|^{2p+4-2\mu}} - C'(\delta, i) \int_{\mathbb{R}^d} \frac{|H^{l-1} u|^2}{1+|x|^{2\delta'}}.
$$

This, together with (C-22), gives that (C-24) is true for l. Hence by induction it is true for i, which is precisely the estimate (C-16) we had to show and ends the proof of the lemma. ☐

Appendix D: Specific bounds for the analysis

This section is dedicated to the statement and the proof of several estimates used in the analysis.

Lemma D.1 (specific bounds for the error in the trapped regime). *Let ε be a function satisfying* (4-25) *and* (4-11). *We recall that $\mathcal{E}_\sigma$ and $\mathcal{E}_{2s_L}$ are defined by* (4-9) *and* (4-7). *Then the following bounds hold*:

(i) Interpolated Hardy-type inequality. *For $\mu \in \mathbb{N}^d$ and $q > 0$ satisfying $\sigma \leq |\mu| + q \leq 2s_L$*

$$
\int \frac{|\partial^\mu \varepsilon|^2}{1+|y|^{2q}}\, dy \leq C(M) \mathcal{E}_\sigma^{\frac{2s_L-(|\mu|+q)}{2s_L-\sigma}} \mathcal{E}_{2s_L}^{\frac{|\mu|+q-\sigma}{2s_L-\sigma}}. \tag{D-1}
$$

(ii) Weighted L^∞ bound for low order derivative. *For $0 \leq a \leq 2$ and $\mu \in \mathbb{N}^d$ with $|\mu| \leq 1$,*

$$
\left\| \frac{\partial^\mu \varepsilon}{1+|y|^a} \right\|_{L^\infty} \leq C(K_1, K_2, M) \sqrt{\mathcal{E}_\sigma}^{1+O(\frac{1}{L^2})} \frac{1}{s^{a+|\mu|_1+(\frac{d}{2}-\sigma)+(\frac{2}{p-1}+a+|\mu|_1)\alpha/L+O(\frac{\sigma-s_c}{L})}}. \tag{D-2}
$$

(iii) L^∞ bound for high order derivative. *For $\mu \in \mathbb{N}^d$ with $|\mu| \leq s_L$,*

$$
\|\partial^\mu \varepsilon\|_{L^\infty}^2 \leq C(M) \mathcal{E}_\sigma^{\frac{2s_L-|\mu|_1-d/2}{2s_L-\sigma}+O(\frac{1}{L^2})} \mathcal{E}_{2s_L}^{\frac{|\mu|_1+d/2-\sigma}{2s_L-\sigma}+O(\frac{1}{L^2})}. \tag{D-3}
$$

Proof. (i) We first recall that from the coercivity estimate (C-16) one has

$$\|\nabla^\sigma \varepsilon\|_{L^2}^2 = \mathcal{E}_\sigma, \quad \|\nabla^{2s_L}\varepsilon\|_{L^2}^2 \le C(M)\|H^{s_L}\varepsilon\|_{L^2}^2 = C(M)\mathcal{E}_{2s_L}.$$

If the weight satisfies $q < \frac{d}{2}$, then the inequality (D-1) claimed in the lemma is a consequence of the standard Hardy inequality, followed by an interpolation:

$$\begin{aligned}\left\|\frac{\partial^\mu \varepsilon}{1+|x|^q}\right\|_{L^2}^2 &\le C\|\nabla^{|\mu|_1+q}\varepsilon\|_{L^2}^2 \le C\|\nabla^\sigma\varepsilon\|_{L^2}^{2\frac{2s_L-(|\mu|_1+q)}{2s_L-\sigma}}\|\nabla^{2s_L}\varepsilon\|_{L^2}^{2\frac{|\mu|_1+q-\sigma}{2s_L-\sigma}} \\ &\le C(M)\mathcal{E}_\sigma^{\frac{2s_L-(|\mu|_1+q)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{|\mu|_1+q-\sigma}{2s_L-\sigma}}.\end{aligned}$$

If the potential satisfies $q = 2s_L - |\mu|$, then the inequality (D-1) claimed in the lemma is a consequence of the coercivity estimate (C-16):

$$\left\|\frac{\partial^\mu \varepsilon}{1+|x|^q}\right\|_{L^2}^2 \le C(M)\mathcal{E}_{2s_L}.$$

For a weight that is in-between, i.e., $\frac{d}{2} \le q < 2s_L - |\mu|_1$, the inequality (D-1) is then obtained by interpolating the two previous ones, as

$$\frac{|\varepsilon|^2}{1+|x|^{2b}} \sim \left(\frac{|\varepsilon|^2}{1+|x|^{2a}}\right)^{\frac{c-b}{c-a}}\left(\frac{|\varepsilon|^2}{1+|x|^{2c}}\right)^{\frac{b-a}{c-a}}.$$

(ii) As the dimension is $d \ge 11$ and $L \gg 1$ is big, one has $\partial^\mu\varepsilon/(1+|x|^a) \in L^\infty$ with the following bound (using the bound (i) we just derived):

$$\begin{aligned}\left\|\frac{\partial^\mu\varepsilon}{1+|x|^a}\right\|_{L^\infty} &\le C(z)\left(\left\|\nabla^{\frac{d}{2}-z}\left(\frac{\partial^\mu\varepsilon}{1+|x|^a}\right)\right\|_{L^2}+\left\|\nabla^{\frac{d}{2}+z}\left(\frac{\partial^\mu\varepsilon}{1+|x|^a}\right)\right\|_{L^2}\right) \\ &\le C(z)\big(\|\nabla^{\frac{d}{2}-z+a+|\mu|_1}\varepsilon\|_{L^2}+\|\nabla^{\frac{d}{2}+a+|\mu|_1+z}\varepsilon\|_{L^2}\big) \\ &\le C(M,z)\big(\mathcal{E}_\sigma^{\frac{2s_L-(a+|\mu|_1+d/2-z)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{a+|\mu|_1+d/2-z-\sigma}{2s_L-\sigma}}+\mathcal{E}_\sigma^{\frac{2s_L-(a+|\mu|_1+d/2+z)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{a+|\mu|_1+d/2+z-\sigma}{2s_L-\sigma}}\big)\end{aligned}$$

for $z > 0$ small enough. We then let z_1 be so close to 0 (of order L^{-1}) that its impact when using the bootstrap bounds (4-25) is of order $s^{-\frac{1}{L^2}}$ (since the constant $C(M, z_1)$ explodes as z_1 approaches 0, we cannot take $z_1 = 0$, but z_1 very close to $\frac{d}{2}$ is enough for our purpose). Inserting the bootstrap bounds (4-25) then yields the desired result (D-2).

(iii) It can be proved exactly the same way we did for (ii). □

Lemma D.2 (a nonlinear estimate). *Let $d \in \mathbb{N}$, $a \ge 0$ and $b > \frac{d}{2}$. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain. There exists a constant $C > 0$ such that for any $u, v \in H^{\max(a,b)}(\Omega)$,*[24]

$$\|uv\|_{H^a(\Omega)} \le C\big(\|u\|_{H^a(\Omega)}\|v\|_{H^b(\Omega)} + \|u\|_{H^b(\Omega)}\|v\|_{H^a(\Omega)}\big). \tag{D-4}$$

[24] The product uv indeed belongs to $H^a(\Omega)$ as $H^{\max(a,b)}(\Omega)$ is an algebra since $b > \frac{d}{2}$.

Proof. Without loss of generality one assumes $\frac{d}{2} < b \le \frac{d}{2} + \frac{1}{4}$,

$$b := \frac{d}{2} + \delta_b, \quad \text{with } 0 < \delta_b \le \frac{1}{4}. \tag{D-5}$$

Indeed, if (D-4) holds for all $b \in \left(\frac{d}{2}, \frac{d}{2} + \frac{1}{4}\right]$ then for any $b' > \frac{d}{2} + \frac{1}{4}$, applying (D-4) to the pair of parameters $\left(a, \frac{d}{2} + \frac{1}{4}\right)$ and using the fact that $\|f\|_{H^{d/2+1/4}(\Omega)} \le \|f\|_{H^b(\Omega)}$ for any $f \in H^b(\Omega)$ gives that (D-4) holds for the pair of parameters (a, b').

Step 1: a scalar inequality. We claim that for all $(\nu_1, \nu_2) \in [0,1]^2$ with $\nu_1 + \nu_2 \ge 1$ and for all $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, +\infty)$ satisfying $\lambda_1 \le \lambda_2$ and $\lambda_3 \le \lambda_4$,

$$\lambda_1^{\nu_1} \lambda_2^{1-\nu_1} \lambda_3^{\nu_2} \lambda_4^{1-\nu_2} \le \lambda_1 \lambda_4 + \lambda_2 \lambda_3. \tag{D-6}$$

We now prove this estimate. Since $1 - \nu_1 - \nu_2 \le 0$ and $0 \le 1 - \nu_2 \le 1$, one has

$$\forall (x, z) \in [1, +\infty) \times [0, +\infty), \quad x^{1-\nu_1-\nu_2} z^{1-\nu_2} \le z^{1-\nu_2} \le 1 + z.$$

Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, +\infty)$ satisfying $0 < \lambda_1 \le \lambda_2$ and $0 < \lambda_3 \le \lambda_4$. We apply the above estimate to $x = \frac{\lambda_2}{\lambda_1} \ge 1$ and $z = \frac{\lambda_1 \lambda_4}{\lambda_2 \lambda_3}$, and multiply both sides by $\lambda_2 \lambda_3$, yielding the desired estimate (D-6) after simplifications. If $\lambda_1 = 0$ or $\lambda_3 = 0$, (D-6) always holds. Consequently, (D-6) holds for all $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, +\infty)$ satisfying $0 < \lambda_1 \le \lambda_2$ and $0 < \lambda_3 \le \lambda_4$.

Step 2: proof in the case $\Omega = \mathbb{R}^d$ and $a \ge b$. We claim that for $u, v \in H^a(\mathbb{R}^d)$,

$$\|uv\|_{H^a(\mathbb{R}^d)} \le C \left(\|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)} + \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)} \right). \tag{D-7}$$

We now show the above estimate. Let $u, v \in H^{s_2}(\mathbb{R}^d)$. First, one obtains an L^2 bound using Hölder and Sobolev embedding $\left(\text{as } b > \frac{d}{2}\right)$:

$$\|uv\|_{L^2(\mathbb{R}^d)} \le \|u\|_{L^2(\mathbb{R}^d)} \|v\|_{L^\infty(\mathbb{R}^d)} \le C \|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)}. \tag{D-8}$$

Secondly, one decomposes $a = A + \delta_a$, where $A := E[a] \in \mathbb{N}$ is the entire part of a and $0 \le \delta_a < 1$. Using the Leibniz rule one has the identity

$$\|\nabla^a (uv)\|_{L^2(\mathbb{R}^d)}^2 \le C \sum_{\substack{(\mu_1, \mu_2) \in \mathbb{N}^{2d} \\ |\mu_1| + |\mu_2| = A}} \|\nabla^{\delta_a} (\partial^{\mu_1} u \partial^{\mu_2} v)\|_{L^2(\mathbb{R}^d)}^2. \tag{D-9}$$

We fix $(\mu_1, \mu_2) \in \mathbb{N}^{2d}$ with $|\mu_1| + |\mu_2| = A$ in the sum and aim at estimating the corresponding term. We recall the commutator estimate

$$\|\nabla^{\delta_a} (\partial^{\mu_1} u \partial^{\mu_2} v)\|_{L^2} \lesssim \|\nabla^{|\mu_1| + \delta_a} u\|_{L^{p_1}} \|\partial^{\mu_2} v\|_{L^{q_1}} + \|\nabla^{|\mu_2| + \delta_a} v\|_{L^{p_2}} \|\partial^{\mu_1} u\|_{L^{q_2}} \tag{D-10}$$

for $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{2}$, provided $2 \le p_1, p_2 < +\infty$ and $2 \le q_1, q_2 \le +\infty$. We now chose appropriate exponents p_1 and p_2 in several cases.

Case 1. $|\mu_2| = 0$. Then $|\mu_1| + \delta_a = a$ and using Sobolev embedding $\left(\text{as } b > \frac{d}{2}\right)$,

$$\|\nabla^{|\mu_1| + \delta_a} u\|_{L^2(\mathbb{R}^d)} \|\partial^{\mu_2} v\|_{L^\infty(\mathbb{R}^d)} \le C \|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)}. \tag{D-11}$$

Case 2. $1 \le |\mu_2| < a - \frac{d}{2}$ and $|\mu_1| + \delta_a < b$. Then $b < |\mu_2| + \frac{d}{2} < a$ by (D-5) and using Sobolev embedding, one computes

$$\|\nabla^{|\mu_1|+\delta_a} u\|_{L^2(\mathbb{R}^d)} \|\partial^{\mu_2} v\|_{L^\infty(\mathbb{R}^d)} \le C \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)}. \tag{D-12}$$

Case 3. $1 \le |\mu_2| < a - \frac{d}{2}$ and $b \le |\mu_1| + \delta_a$. Then $b < |\mu_2| + \frac{d}{2} < a$ by (D-5) and $b \le |\mu_1| + \delta_a \le a$. We let $x := \min\big(\frac{\delta_b}{2}, a - |\mu_2| - \frac{d}{2}\big) > 0$. Using Sobolev embedding, interpolation and (D-6) $\big($since $b > \frac{d}{2} + x$ and $|\mu_1| + |\mu_2| + \delta_a = a\big)$, one computes

$$\begin{aligned}
\|\nabla^{|\mu_1|+\delta_a} u\|_{L^2(\mathbb{R}^d)} \|\partial^{\mu_2} v\|_{L^\infty(\mathbb{R}^d)} &\le C \|u\|_{H^{|\mu_1|+\delta_a}(\mathbb{R}^d)} \|v\|_{H^{|\mu_2|+d/2+x}(\mathbb{R}^d)} \\
&\le C \|u\|_{H^b(\mathbb{R}^d)}^{\frac{a-|\mu_1|-\delta_a}{a-b}} \|u\|_{H^a(\mathbb{R}^d)}^{\frac{|\mu_1|+\delta_a-b}{a-b}} \|v\|_{H^b(\mathbb{R}^d)}^{\frac{a-|\mu_2|-d/2-x}{a-b}} \|v\|_{H^a(\mathbb{R}^d)}^{\frac{|\mu_2|+d/2+x-b}{a-b}} \\
&\le C \big(\|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)} + \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)} \big).
\end{aligned} \tag{D-13}$$

Case 4. $a - \frac{d}{2} \le |\mu_2| < a$. Let $x := \frac{1}{2} \min(a - |\mu|_2, \delta_b) > 0$. We define p_1, q_1 and s by

$$\frac{1}{q_1} := \frac{1}{2} - \frac{a - x - |\mu_2|}{d}, \quad \frac{1}{p_1} = \frac{1}{2} - \frac{1}{q_1} \quad \text{and} \quad s = \frac{d}{q_1}.$$

One has $|\mu_1| + \delta_a + s = \frac{d}{2} + x < b$, and, using Sobolev embedding,

$$\|\nabla^{|\mu_1|+\delta_a} u\|_{L^{p_1}} \|\partial^{\mu_2} v\|_{L^{q_1}} \le C \|u\|_{H^{|\mu_1|+\delta_a+s}} \|v\|_{H^{a-x}} \le C \|u\|_{H^b} \|v\|_{H^a} \tag{D-14}$$

and $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}$, $p_1 \ne +\infty$.

Case 5. $|\mu_2| = a$. Then $|\mu_1| + \delta_a = 0$ and using Sobolev embedding $\big($as $b > \frac{d}{2}\big)$,

$$\|\nabla^{|\mu_1|+\delta_a} u\|_{L^\infty(\mathbb{R}^d)} \|\partial^{\mu_2} v\|_{L^2(\mathbb{R}^d)} \le C \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)}. \tag{D-15}$$

Conclusion. In all possible cases, by (D-11)–(D-15) there always exist $p_1, q_1, p_2, q_2 \in [2, +\infty)$ with $p_1, p_2 \ne +\infty$, $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}$ and

$$\begin{aligned}
\|\nabla^{|\mu_1|+\delta_a} u\|_{L^{p_1}(\mathbb{R}^d)} \|\partial^{\mu_2} v\|_{L^{q_1}(\mathbb{R}^d)} + \|\nabla^{|\mu_1|} u\|_{L^{q_2} v} \|\nabla^{|\mu_2|+\delta_a} v\|_{L^{p_2}(\mathbb{R}^d)} \\
\le C \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)} + C \|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)},
\end{aligned}$$

where the estimate for the second term on the left-hand side of the above equation comes from symmetric reasoning. We now come back to (D-9), and apply (D-10) and the above identity to obtain

$$\|\nabla^a (uv)\|_{L^2(\mathbb{R}^d)} \le C \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)} + C \|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)}.$$

The above estimate and (D-8) imply the desired estimate (D-7) by interpolation.

Step 3: proof in the case $\Omega = \mathbb{R}^d$ and $a \le b$. The proof is similar and simpler and we do not write it here. Therefore, (D-7) holds for all $a \ge 0$ and $b > \frac{d}{2}$.

Step 4: proof in the case of a smooth bounded domain Ω. There exists $\widetilde{C} > 0$ such that for any $f \in H^{\max(a,b)}(\Omega)$ there exists an extension $\tilde{f} \in H^{\max(a,b)}(\mathbb{R}^d)$ with compact support, satisfying $\tilde{f} = f$

on Ω and

$$\frac{1}{\widetilde{C}}\|\tilde f\|_{H^c(\mathbb{R}^d)} \le \|f\|_{H^c(\Omega)} \le \widetilde{C}\|\tilde f\|_{H^c(\mathbb{R}^d)}, \quad c=a,b;$$

see [Adams and Fournier 2003]. Let $u, v \in H^{\max(a,b)}(\Omega)$ and denote by $\tilde u$ and $\tilde v$ their respective extensions. Using (D-7) and the above estimate then yields

$$\begin{aligned}\|uv\|_{H^a(\Omega)} &\le \|\tilde u\tilde v\|_{H^a(\mathbb{R}^d)}\\ &\le C\big(\|\tilde u\|_{H^a(\mathbb{R}^d)}\|\tilde v\|_{H^b(\mathbb{R}^d)} + \|\tilde u\|_{H^b(\mathbb{R}^d)}\|\tilde v\|_{H^a(\mathbb{R}^d)}\big)\\ &\le C\widetilde{C}^2\big(\|u\|_{H^a(\Omega)}\|v\|_{H^b(\Omega)} + \|u\|_{H^b(\Omega)}\|v\|_{H^a(\Omega)}\big)\end{aligned}$$

and (D-4) is obtained. □

Appendix E: Geometrical decomposition

This section is devoted to the proof of Lemma 4.3.

Lemma E.1. *Let X denote the functional space*

$$X := \big\{u \in L^\infty(\mathcal{B}^d(0,4M)) : \langle u-Q, H\Phi_M^{(0,1)}\rangle > \|u-Q\|_{L^\infty(\mathcal{B}^d(0,3M))}\big\}. \tag{E-1}$$

There exists $\kappa, K > 0$ such that for all $u \in X \cap \{\|u-Q\|_{L^\infty(\mathcal{B}^d(0,4M))} < \kappa\}$, there exists a unique choice of parameters $b \in \mathbb{R}^{\mathcal{I}}$ with $b_1^{(0,1)} > 0$, $\lambda > 0$ and $z \in \mathbb{R}^d$ such that the function $v := (\tau_{-z}u)_\lambda - \widetilde{Q}_b$ satisfies

$$\langle v, H^i\Phi_M^{(n,k)}\rangle = 0 \quad \textit{for } 0 \le n \le n_0,\ 1 \le k \le k(n),\ 0 \le i \le L_n \tag{E-2}$$

and such that

$$|\lambda-1| + |z| + \sum_{(n,k,i)\in\mathcal{I}} |b_i^{(n,k)}| \le K. \tag{E-3}$$

Moreover, b, λ *and* z *are Fréchet differentiable*[25] *and satisfy*

$$|\lambda-1| + |z| + \sum_{(n,k,i)\in\mathcal{I}} |b_i^{(n,k)}| \le K\|u-Q\|_{L^\infty(\mathcal{B}^d(0,3M))}. \tag{E-4}$$

Proof. We first define the application ξ as

$$\begin{aligned}&\xi : L^\infty(\mathcal{B}^d(0,3M)) \times (0,+\infty) \times \mathbb{R}^{d+\#\mathcal{I}} \to \mathbb{R}^{1+d+\#\mathcal{I}},\\ &(u,\tilde\lambda,\tilde z,\tilde b) \mapsto \big(\langle (\tau_{\tilde z}u)_{\frac{1}{\tilde\lambda}} - Q - \alpha_{\tilde b}, H^i\Phi_M^{(n,k)}\rangle\big), \quad \text{where } 1 \le k \le k(n),\ 0 \le n \le n_0,\ 0 \le i \le L_n.\end{aligned} \tag{E-5}$$

Then ξ is $\mathcal{C}^\infty$. From the definition (3-7) of α_b, and the orthogonality conditions (4-3), the differential of ξ with respect to the second variable at the point $(Q,1,0,\dots,0)$ is the diagonal matrix

$$D^{(2)}\xi(Q,1,0,\dots,0) = -\begin{pmatrix}\langle T_0^{(0)}, \chi_M T_0^{(0)}\rangle \operatorname{Id}_{L+1} & & \\ & \ddots & \\ & & \langle T_0^{(n_0)}, \chi_M T_0^{(n_0)}\rangle \operatorname{Id}_{L_{n_0}}\end{pmatrix}, \tag{E-6}$$

[25] For the ambient Banach space $L^\infty(\mathcal{B}^d(0,3M))$.

where Id_{L_n} is the $L_n \times L_n$ identity matrix. $D^{(2)}\xi(Q,1,0,\dots,0)$ is invertible for M large by (4-3). Consequently, from the implicit functions theorem, there exist $\kappa, K > 0$, such that for all

$$u \in X \cap \left\{ \|u-Q\|_{L^\infty(\mathcal{B}^d(0,3M))} < \kappa \right\},$$

there exists a choice of the parameters $\tilde\lambda = \tilde\lambda(u)$, $\tilde z = \tilde z(u)$ and $\tilde b = \tilde b(u)$ such that

$$\xi(u,\tilde\lambda,\tilde z,\tilde b) = 0, \quad |\tilde\lambda - 1| + |\tilde z| + \sum_{(n,k,i)\in\mathcal{I}} |\tilde b_i^{(n,k)}| \le K \|u-Q\|_{L^\infty(\mathcal{B}^d(3M))} \tag{E-7}$$

and it is the unique solution of $\xi(u,\tilde\lambda,\tilde z,\tilde b) = 0$ in the range

$$|\tilde\lambda - 1| + |\tilde z| + \sum_{(n,k,i)\in\mathcal{I}} |\tilde b_i^{(n,k)}| \le K.$$

Moreover, they are Fréchet differentiable, again from the implicit function theorem. Now, defining $\lambda = 1/\tilde\lambda$, $b = \tilde b$ and $z = -\tilde z$, this means by (E-5) that the function $w := (\tau_{-z}u)_\lambda - Q - \alpha_b$ satisfies

$$\langle w, H^i \Phi_M^{(n,k)} \rangle = 0, \quad \text{for } 0 \le n \le n_0,\ 1 \le k \le k(n),\ 0 \le i \le L_n.$$

Finally, still from the implicit function theorem, from the identity for the differential (E-6), the definition (E-1) of X and (4-3),

$$\begin{aligned} b_1^{(0,1)} &= -[D^{(2)}\xi(Q,1,0,\dots,0)]^{-1}(\xi(u,1,0,\dots,0)) + o(\|u-Q\|_{L^\infty(\mathcal{B}^d(3M))}) \\ &= \frac{\langle u-Q, H^1\Phi_M^{(0,1)}\rangle}{\langle T_0^{(0)}, \chi_M T_0^{(0)}\rangle} + o\big(\langle u-Q, H^1\Phi_M^{(0,1)}\rangle\big) > 0, \end{aligned}$$

where the $o(\cdot)$ is as $\kappa \to 0$, and the strict positivity is then for κ small enough. Consequently, in that case $\widetilde Q_b = Q + \chi_{(b_1^{(0,1)})^{-(1+\eta)/2}}\alpha_b$ is well defined, and one has $(b_1^{(0,1)})^{-\frac{1+\eta}{2}} \gg 2M$ for κ small enough. Thus, for $v := (\tau_{-z}u)_\lambda - \widetilde Q_b$,

$$\langle v, H^i\Phi_M^{(n,k)}\rangle = \langle \tilde v, H^i\Phi_M^{(n,k)}\rangle = 0 \ \text{ for } 0 \le n \le n_0,\ 1 \le k \le k(n),\ 0 \le i \le L_n$$

because the support of $v - \tilde v$ is outside $\mathcal{B}^d(0,2M)$. One has found a choice of the parameters λ, b and z such that $b_1^{(0,1)} > 0$ and (E-2) and (E-3) hold. This choice is unique in the range (E-3) and the parameters are Fréchet differentiable since under (E-3), they are equal to the parameters given by the above inversion of ξ. □

Lemma E.2. *There exist $\kappa^*, \widetilde K > 0$ such that the following holds for all $0 < \kappa < \kappa^*$. Let $\mathcal{O}$ be the open set of $L^\infty(\mathcal{B}^d(0,1))$ of functions u satisfying* (4-4). *For each $u \in \mathcal{O}$ there exists a unique choice of the parameters $\lambda \in \left(0, \frac{1}{4M}\right)$, $z \in \mathcal{B}^d\left(0,\frac14\right)$ and $b \in \mathbb{R}^{\mathcal{I}}$ such that $b_1^{(0,1)} > 0$ and $v = (\tau_{-z}u)_\lambda - \widetilde Q_b \in L^\infty\big(\frac{1}{\lambda}(\mathcal{B}^d(0,1) - \{z\})\big)$ satisfies*[26]

$$\langle v, H^i\Phi_M^{(n,k)}\rangle = 0 \quad \textit{for } 0 \le n \le n_0,\ 1 \le k \le k(n),\ 0 \le i \le L_n \tag{E-8}$$

[26] The following assertions make sense as v is defined on $\frac{1}{\lambda}(\mathcal{B}^d(0,1) - \{z\})$, which indeed contains $\mathcal{B}^d(0,2M)$ since $0 < \lambda < \frac{1}{4M}$ and $|z| \le \frac14$, and as $\Phi_M^{(n,k)}$ is compactly supported in $\mathcal{B}^d(0,2M)$ by (4-1).

and

$$\sum_{(n,k,i)\in\mathcal{I}} |b_i^{(n,k)}| + \|v\|_{L^\infty(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\}))} \le \widetilde{K}\kappa. \tag{E-9}$$

Moreover, the functions λ, z and b defined this way are Fréchet differentiable on $\mathcal{O}$.

Proof. Let K and κ_0 be the numbers associated to Lemma E.1.

Step 1: existence. Let

$$(\tilde{\lambda}, \tilde{z}) \in \left(0, \tfrac{1}{8M}\right) \times \mathcal{B}^d\left(0, \tfrac{1}{8}\right) \tag{E-10}$$

be such that

$$\|u - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} < \frac{\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}},$$
$$\|(\tau_{-\tilde{z}}u)_{\tilde{\lambda}} - Q\|_{L^\infty(\mathcal{B}^d(4M))} < \langle (\tau_{-\tilde{z}}u)_{\tilde{\lambda}} - Q, H\Phi_M^{(0,1)}\rangle,$$

which exists by (4-4). We define $w := (\tau_{-\tilde{z}}u)_{\tilde{\lambda}}$. It is defined on the set $(1/\tilde{\lambda})(\mathcal{B}(1)-\tilde{z})$, which contains $\mathcal{B}^d(7M)$ as $0 < \tilde{\lambda} < \frac{1}{8M}$ and $|z| \le \frac{1}{8}$. From this fact and the above estimates, w satisfies

$$\|w - Q\|_{L^\infty(\mathcal{B}(7M))} < \kappa, \quad \|w - Q\|_{L^\infty(\mathcal{B}^d(3M))} < \langle w - Q, H\Phi_M^{(0,1)}\rangle. \tag{E-11}$$

Thus for κ small enough, one can apply Lemma E.1: there exists a choice of the parameters z', b' and λ' such that $v' = (\tau_{-z'}w)_{\lambda'} - \widetilde{Q}_{b'}$ satisfies (E-8) and $b_1^{'(0,1)} > 0$. This choice is unique in the range

$$|\lambda' - 1| + |z'| + \sum_{(n,k,i)\in\mathcal{I}} |b_i^{\prime(n,k)}| \le K. \tag{E-12}$$

Moreover, the estimate

$$|\lambda' - 1| + |z'| + \sum_{(n,k,i)\in\mathcal{I}} |b_i^{\prime(n,k)}| \le K\|w - Q\|_{L^\infty(\mathcal{B}^d(0,3M)))} \le K\kappa.$$

holds. Now we define

$$b = b', \quad z = \tilde{z} + \tilde{\lambda}z', \quad \lambda = \tilde{\lambda}\lambda' \tag{E-13}$$

and $v = v'$. One has then $b_1^{(0,1)} > 0$, and from (E-10) and the above estimate,

$$\sum_{(n,k,i)\in\mathcal{I}} |b_i^{(n,k)}| \le K\kappa, \quad |z| \le \tfrac{1}{4}, \quad 0 < \lambda < \frac{1}{4M}$$

for κ small enough. From the definitions of w, v' and v one has the identity

$$u = (v + \widetilde{Q}_b)_{z,\frac{1}{\lambda}}, \quad \text{with } v \text{ satisfying (E-8)}.$$

From (3-7), (3-29) and the above estimate,

$$\begin{aligned}\|v\|_{L^\infty(\frac{1}{\lambda}(\mathcal{B}^d(1)-z))} &= \lambda^{\frac{2}{p-1}}\|u - \tau_z(\widetilde{Q}_{b,\frac{1}{\lambda}})\|_{L^\infty(\mathcal{B}^d(1))} \\ &\le \lambda^{\frac{2}{p-1}}\|u - \tau_{\tilde{z}}(Q_{\frac{1}{\tilde{\lambda}}})\|_{L^\infty(\mathcal{B}^d(1))} + \lambda^{\frac{2}{p-1}}\|\tau_{\tilde{z}}(Q_{\frac{1}{\tilde{\lambda}}}) - \tau_z(\widetilde{Q}_{b,\frac{1}{\lambda}})\|_{L^\infty(\mathcal{B}^d(1))} \le CK\kappa\end{aligned}$$

for some constant $C>1$ independent of the others. Therefore, one takes $\widetilde{K}=CK$, and the choice of parameters λ, z and b that we just found provides the decomposition claimed by the lemma and the existence is proven.

Step 2: differentiability. We claim that the parameters λ, b and z found in Step 1 are unique; this will be proven in the next step. Therefore, from their construction using the auxiliary variables $\tilde{\lambda}$ and $\tilde{z}$ in Step 1, and since the parameters λ', z' and b' provided by Lemma E.1 are Fréchet differentiable, λ, b and z are Fréchet differentiable.

Step 3: uniqueness. Let $\hat{b}$, $\hat{\lambda}$, $\hat{z}$ be another choice of parameters with $\hat{b}_1^{(0,1)}>0,\ 0<\lambda<\frac{1}{4M}$ and $|z|\le\frac14$ such that (E-8) and (E-9) hold for $\hat{v}=(\tau_{-\hat{z}}u)_{\hat{\lambda}}-\widetilde{Q}_b$. The function $(\tau_{-\tilde{z}}u)_{\tilde{\lambda}}$, where $\tilde{\lambda}$ and $\tilde{z}$ were defined in (E-10) in the first step, then satisfies the bound

$$\|(\tau_{-\tilde{z}}u)_{\tilde{\lambda}}-Q\|_{L^\infty(\mathcal{B}(3M))}<\kappa_0$$

for κ small enough by (E-11), and admits two decompositions

$$(\tau_{-\tilde{z}}u)_{\tilde{\lambda}}=(\widetilde{Q}_{b'}+v')_{z',\frac{1}{\lambda'}}=(\widetilde{Q}_{\hat{b}}+\hat{v})_{\frac{\hat{z}-\tilde{z}}{\tilde{\lambda}},\frac{\tilde{\lambda}}{\hat{\lambda}}}$$

such that v and v' satisfy (E-8). By (E-12), the first parameters satisfy

$$|\lambda'-1|+|z'|+\sum_{(n,k,i)\in\mathcal{I}}|b_i'^{(n,k)}|\le K\kappa_0.$$

We claim that the second parameters satisfy

$$\left|\frac{\tilde{\lambda}}{\hat{\lambda}}-1\right|+\left|\frac{\hat{z}-\tilde{z}}{\tilde{\lambda}}\right|+\sum_{(n,k,i)\in\mathcal{I}}|\hat{b}_i^{(n,k)}|\le K\kappa_0, \tag{E-14}$$

which will be proven hereafter. Then, as such parameters are unique under the above bound by Lemma E.1, one obtains

$$\frac{\tilde{\lambda}}{\hat{\lambda}}=\frac{1}{\lambda'},\quad \frac{\hat{z}-\tilde{z}}{\tilde{\lambda}}=z',\quad \hat{b}=b',$$

implying that $\hat{\lambda}=\lambda$, $\hat{z}=z$ and $\hat{b}=b$, where λ, z and b are the choice of the parameters given by the first step defined by (E-13). The uniqueness is obtained.

Proof of (E-14). From the assumptions on $\hat{b}$, $\hat{\lambda}$ and $\hat{z}$, the definition of $\widetilde{Q}_b$ (3-29) and (E-9), for κ small enough we have

$$\|u-Q_{\hat{z},\frac{1}{\hat{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))}\le\frac{C\widetilde{K}\kappa}{\hat{\lambda}^{\frac{2}{p-1}}}.$$

From (E-10) one also has

$$\|u-Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))}\le\frac{\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}}.$$

From the two above estimates, one deduces that

$$\|Q_{\hat{z},\frac{1}{\hat{\lambda}}}-Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))}\le\frac{\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}}+\frac{C\widetilde{K}\kappa}{\hat{\lambda}^{\frac{2}{p-1}}}. \tag{E-15}$$

Assume that $\hat{\lambda} \leq \tilde{\lambda}$. Then, since Q is radially symmetric and attains its maximum at the origin, and $\hat{z} \in \mathcal{B}^d(0,1)$ because $|\hat{z}| \leq \frac{1}{4}$, the above inequality at $x = \hat{z}$ implies

$$\begin{aligned} Q(0)\biggl(\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}} - \frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}\biggr) &= Q_{\hat{z},\frac{1}{\hat{\lambda}}}(\hat{z}) - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}(\tilde{z}) \\ &\leq Q_{\hat{z},\frac{1}{\hat{\lambda}}}(\hat{z}) - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}(\hat{z}) \\ &= |Q_{\hat{z},\frac{1}{\hat{\lambda}}}(\hat{z}) - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}(\hat{z})| \\ &\leq C\widetilde{K}\kappa\biggl(\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}} + \frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}\biggr), \end{aligned}$$

which gives

$$\left|\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}} - \frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}\right| \leq C\widetilde{K}\kappa\biggl(\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}} + \frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}\biggr).$$

The symmetric reasoning works in the case $\hat{\lambda} \geq \tilde{\lambda}$ and one obtains that in both cases

$$\left|\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}} - \frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}\right| \leq C\widetilde{K}\kappa\biggl(\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}} + \frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}\biggr).$$

Basic computations show that for κ small enough the above identity implies

$$\left|1 - \frac{\hat{\lambda}}{\tilde{\lambda}}\right| \leq C\widetilde{K}\kappa \quad \text{or} \quad \hat{\lambda} = \tilde{\lambda}(1 + O(\kappa)),$$

obtaining the first bound in (E-14) for κ small enough. We insert the above estimate into (E-15), yielding

$$\|Q_{\hat{z},\frac{1}{\hat{\lambda}}} - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} \leq \|Q_{\hat{z},\frac{1}{\hat{\lambda}}} - Q_{\hat{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))}\| + \|Q_{\hat{z},\frac{1}{\tilde{\lambda}}} - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))}\| \leq \frac{C\widetilde{K}\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}},$$

which implies in renormalized variables $\bigl($as $|\hat{z}| \leq \frac{1}{8}$ and $\tilde{\lambda} \leq \frac{1}{8M}\bigr)$,

$$\|Q - \tau_{\frac{\hat{z}-\tilde{z}}{\tilde{\lambda}}} Q\|_{L^\infty(\mathcal{B}^d(0,2M))} \leq C\widetilde{K}\kappa.$$

As Q is smooth, radially symmetric and radially decreasing this implies

$$\left|\frac{\hat{z} - \tilde{z}}{\tilde{\lambda}}\right| \leq C\widetilde{K}\kappa \quad \text{or} \quad \hat{z} = \tilde{z} + \tilde{\lambda}O(\kappa)$$

and the second bound in (E-14) is obtained. □

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