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We prove uniqueness results for a Calderón-type inverse problem for the Hodge Laplacian acting on graded forms on certain manifolds in three dimensions. In particular, we show that partial measurements of the relative-to-absolute or absolute-to-relative boundary value maps uniquely determine a zeroth-order potential. The method is based on Carleman estimates for the Hodge Laplacian with relative or absolute boundary conditions, and on the construction of complex geometrical optics solutions which reduce the Calderón-type problem to a tomography problem for 2-tensors. The arguments in this paper allow us to establish partial data results for elliptic systems that generalize the scalar results due to Kenig, Sjöstrand and Uhlmann.

1. Introduction

This article is concerned with inverse problems with partial data for elliptic systems. We first discuss the prototype for such problems, which comes from the scalar case: the inverse problem of Calderón asks to determine the electrical conductivity γ of a medium Ω from electrical measurements made on its boundary. More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $\gamma \in L^\infty(\Omega)$ satisfy $\gamma \geq c > 0$ a.e. in Ω. The full boundary measurements are given by the Dirichlet-to-Neumann map (DN map)

$$\Lambda_\gamma^{\mathrm{DN}} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \quad f \mapsto \gamma \partial_\nu u|_{\partial\Omega},$$

where $u \in H^1(\Omega)$ is the unique solution of $\operatorname{div}(\gamma \nabla u) = 0$ in Ω with $u|_{\partial\Omega} = f$, and the conormal derivative $\gamma \partial_\nu u|_{\partial\Omega}$ is defined in the weak sense. Equivalently, one can consider the Neumann-to-Dirichlet map (ND map)

$$\Lambda_\gamma^{\mathrm{ND}} : H_\diamond^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \quad g \mapsto v|_{\partial\Omega},$$

where $\operatorname{div}(\gamma \nabla v) = 0$ in Ω with $\gamma \partial_\nu v|_{\partial\Omega} = g$, and $H_\diamond^{-\frac{1}{2}}(\partial\Omega)$ consists of those elements in $H^{-\frac{1}{2}}(\partial\Omega)$ that are orthogonal to constants. The inverse problem of Calderón asks to determine the conductivity γ from the knowledge of the DN map or (equivalently) the ND map. There is a substantial literature on this problem, with pioneering works including [Faddeev 1965; Calderón 1980; Sylvester and Uhlmann 1987; Novikov and Khenkin 1987; Nachman 1988; Novikov 1988]. We refer to the surveys [Novikov 2008; Uhlmann 2014] for more information.

The Calderón problem with partial data corresponds to the case where one can only make measurements on subsets of the boundary. Let Γ_D and Γ_N be open subsets of $\partial\Omega$, and assume that we measure voltages on Γ_D and currents on Γ_N. If the potential is grounded on $\partial\Omega \setminus \Gamma_D$ but can be prescribed on Γ_D, the partial boundary measurements are given by the partial DN map

$$\Lambda_\gamma^{\mathrm{DN}} f|_{\Gamma_N} \quad \text{for all } f \in H^{\frac{1}{2}}(\partial\Omega) \text{ with } \operatorname{supp}(f) \subset \Gamma_D.$$

If instead we can freely prescribe currents on Γ_N but no current is input on $\partial\Omega \setminus \Gamma_N$, then we know the partial ND map:

$$\Lambda_\gamma^{\mathrm{ND}} g|_{\Gamma_D} \quad \text{for all } g \in H_\diamond^{-\frac{1}{2}}(\partial\Omega) \text{ with } \operatorname{supp}(g) \subset \Gamma_N.$$

The basic uniqueness question is whether a (sufficiently smooth) conductivity is determined by such boundary measurements. We remark that in the partial data case there seems to be no direct way of obtaining the partial DN map from the partial ND map or vice versa, and the two cases need to be considered separately.

By now there are many uniqueness results for the Calderón problem with partial data involving varying assumptions on the sets Γ_D and Γ_N. For further information we refer to the survey [Kenig and Salo 2014] for results in dimensions $n \geq 3$ and [Guillarmou and Tzou 2013] for the case $n = 2$. We only list here some of the main results for the partial DN map:

- When $n \geq 3$, we know Γ_D can be possibly very small but Γ_N has to be slightly larger than the complement of Γ_D [Kenig et al. 2007].

- When $n \geq 3$, we know $\Gamma_D = \Gamma_N = \Gamma$ and the complement of Γ has to be part of a hyperplane or a sphere [Isakov 2007].

- When $n = 2$, we know $\Gamma_D = \Gamma_N = \Gamma$ can be an arbitrary open set [Imanuvilov et al. 2010].

- When $n \geq 3$, we know $\Gamma_D = \Gamma_N = \Gamma$ and the complement of Γ has to be (conformally) flat in one direction and a certain ray transform needs to be injective [Kenig and Salo 2013] (a special case of this was proved independently in [Imanuvilov and Yamamoto 2013]).

The approach of [Kenig et al. 2007] is based on Carleman estimates with boundary terms and the approach of [Isakov 2007] is based on reflection arguments. The paper [Kenig and Salo 2013] combines these

two approaches and extends both. There seem to be fewer results for the partial ND map, especially in dimensions $n \geq 3$; see [Isakov 2007; Chung 2015]. In fact, in dimensions $n \geq 3$ the Carleman estimate approach for the partial ND map seems to be more involved than for the partial DN map. We remark that there are counterexamples for uniqueness when Γ_D and Γ_N are disjoint [Daudé et al. 2015].

The purpose of this paper is to consider analogous partial data results for elliptic systems. In the full data case ($\Gamma_D = \Gamma_N = \partial\Omega$), many uniqueness results are available for linear elliptic systems such as the Maxwell system [Ola et al. 1993; Kenig et al. 2011; Caro and Zhou 2014], Dirac systems [Nakamura and Tsuchida 2000; Salo and Tzou 2009], the Schrödinger equation with Yang–Mills potentials [Eskin 2001], elasticity [Nakamura and Uhlmann 1994; 2003; Eskin and Ralston 2002], and equations in fluid flow [Heck et al. 2007; Li and Wang 2007]. In contrast, the only earlier partial data results for such systems in dimensions $n \geq 3$ that we are aware of are [Caro et al. 2009] for the Maxwell system and [Salo and Tzou 2010] for the Dirac system. One reason for the lack of partial data results for systems is the fact that Carleman estimates for systems often come with boundary terms that do not seem helpful for partial data inverse problems (see [Eller 2008; Salo and Tzou 2009] for some such estimates).

In this paper we establish partial data results analogous to [Kenig et al. 2007] for systems involving the Hodge Laplacian for graded differential forms, on certain Riemannian manifolds in dimensions $n \geq 3$. These are elliptic systems that generalize the scalar Schrödinger equation $(-\triangle + q)u = 0$ and are very close to the time-harmonic Maxwell equations when $n = 3$. In fact, using the results of the present paper, we have finally been able to extend the partial data result of [Kenig et al. 2007] to the Maxwell system [Chung et al. 2015]. The main technical contribution of the present paper is a Carleman estimate for the Hodge Laplacian, with limiting Carleman weights, that has boundary terms involving the relative and absolute boundary values of graded forms. The boundary terms are of such a form that allows us to carry over the Carleman estimate approach of [Kenig et al. 2007] to the Hodge Laplace system. As far as we know, this is the first analogue of [Kenig et al. 2007] for systems besides [Salo and Tzou 2010], which considered a very special case.

In a sense, to deal with boundary terms for systems in a flexible way, one first needs a good understanding of the different splittings of Cauchy data in the scalar case. This encompasses both the scalar DN and ND maps simultaneously, since the "relative-to-absolute" map defined in Section 2 generalizes both the notion of the DN and ND maps. Therefore the methods developed in [Chung 2015] for the partial ND map, involving Fourier analysis to treat the boundary terms in Carleman estimates, will be very useful in our approach. We expect that the methods developed in this paper open the way for obtaining partial data results via Carleman estimates for various elliptic systems. This has already been achieved for Maxwell equations [Chung et al. 2015].

The plan of this document is as follows. Section 1 is the introduction, and Section 2 contains precise statements of the main results. Section 3 collects notation and identities used throughout the paper. In the interest of brevity, we have omitted the proofs of these identities and interested readers can find them in the arXiv version of this paper [Chung et al. 2013, Appendix]. Sections 4–6 will be devoted to the proofs of the Carleman estimates. In Section 4, we will give the basic integration by parts argument for k-forms and simplify the boundary terms. In Section 5, we prove the Carleman estimates for 0-forms using the arguments from [Chung 2015; Kenig and Salo 2013]. We will conclude the argument in Section 6 by

showing that the Carleman estimates for graded forms follow from an induction argument, given the corresponding result for 0-forms. In Section 7 we will construct relevant complex geometrical optics solutions, following the ideas in [Dos Santos Ferreira et al. 2009a]. In Section 8 we will present the Green's theorem argument and give the density result based on injectivity of a tensor tomography problem, which finishes the proofs of Theorems 2.1 and 2.2. Section 9 will contain the proof of Theorem 2.3 and make some remarks about the case of dimensions $n \geq 4$.

2. Statement of results

The results in this paper are new even in Euclidean space, but it will be convenient to state them on certain Riemannian manifolds following [Dos Santos Ferreira et al. 2009a; 2016; Kenig and Salo 2013]. Suppose (M_0, g_0) is a compact oriented manifold with smooth boundary, and consider a manifold $T = \mathbb{R} \times M_0$ equipped with a Riemannian metric of the form $g = c(e \oplus g_0)$, where c is a smooth conformal factor and $(\mathbb{R}, e)$ is the real line with Euclidean metric. A compact manifold (M, g) of dimension $n \geq 3$, with boundary ∂M, is said to be *CTA* (conformally transversally anisotropic) if it can be expressed as a submanifold of such a T. A CTA manifold is called *admissible* if additionally (M_0, g_0) can be chosen to be simple, meaning that ∂M_0 is strictly convex and for any point $x \in M_0$, the exponential map $\exp_x$ is a diffeomorphism from some closed neighbourhood of 0 in $T_x M_0$ onto M_0 (see [Sharafutdinov 1994]). Most of the geometric notions defined here will be from [Taylor 1996] and we refer the reader there for a more thorough treatment of the subject.

Let $\Lambda^k M$ be the k-th exterior power of the cotangent bundle on M, and let ΛM be the corresponding graded algebra. The corresponding spaces of sections (smooth differential forms) are denoted by $\Omega^k M$ and ΩM. We will define $\triangle$ to be the Hodge Laplacian on M, acting on graded forms:

$$-\triangle = d\delta + \delta d.$$

Here d is the exterior derivative and δ is the codifferential (adjoint of d in the L^2 inner product). Suppose Q is an L^∞ endomorphism of ΛM; that is, Q associates to almost every point $x \in M$ a linear map $Q(x)$ from $\Lambda_x M$ to itself, and the map $x \mapsto \|Q(x)\|$ is bounded and measurable. Later will consider continuous endomorphisms, meaning that $x \mapsto Q(x)$ is continuous in M. The continuity of Q will simplify matters since the recovery of Q from boundary measurements involves integrals over geodesics, and continuity ensures that these integrals are well defined.

We would like to consider boundary value problems for the operator $-\triangle + Q$. In order to do this, we will define the tangential trace $t : \Omega M \to \Omega \partial M$ by

$$t : \omega \mapsto i^*\omega,$$

where $i : \partial M \to M$ is the natural inclusion map. Then the first natural boundary value problem to consider for $-\triangle + Q$, acting on graded forms u, is the relative boundary problem

$$(-\triangle + Q)u = 0 \quad \text{in } M,$$
$$tu = f \quad \text{on } \partial M,$$
$$t\delta u = g \quad \text{on } \partial M.$$

If Q is such that 0 is not an eigenvalue for this problem, then this problem has a unique solution [Taylor 1996, Section 5.9] and we may define a relative-to-absolute map

$$N_Q^{\mathrm{RA}} : H^{\frac{1}{2}}(\partial M, \Lambda \partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda \partial M) \to H^{\frac{1}{2}}(\partial M, \Lambda \partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda \partial M)$$

by

$$N_Q^{\mathrm{RA}}(f, g) = (t * u, t\delta * u),$$

where $*$ is the Hodge star operator on M.

The second natural boundary value problem to consider is the absolute boundary value problem

$$(-\triangle + Q)u = 0 \quad \text{in } M,$$
$$t * u = f \quad \text{on } \partial M,$$
$$t\delta * u = g \quad \text{on } \partial M.$$

Assuming 0 is not an eigenvalue, this defines an absolute-to-relative map

$$N_Q^{\mathrm{AR}} : H^{\frac{1}{2}}(\partial M, \Lambda \partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda \partial M) \to H^{\frac{1}{2}}(\partial M, \Lambda \partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda \partial M)$$

by

$$N_Q^{\mathrm{AR}}(f, g) = (tu, t\delta u)$$

for appropriate Q. For more details on the relative and absolute boundary value problems for the Hodge Laplacian, see [Taylor 1996, Section 5.9].

These maps both give rise to a Calderón-type inverse problem which asks if knowledge of N_Q^{RA} or N_Q^{AR} suffices to determine Q. If we restrict ourselves to considering the case of 0-forms only and if Q acts on 0-forms by multiplication by a function $q \in L^\infty(M)$, then the relative-to-absolute and absolute-to-relative maps become the DN and ND maps, respectively, for the Schrödinger equation

$$(-\triangle + q)u = 0 \quad \text{in } M,$$

where u is now a function on M and $\triangle$ is the Laplace–Beltrami operator on functions. Our problem is therefore a generalization of the standard partial data problem for the scalar Schrödinger equation on a compact manifold with boundary.

Let us review some earlier results for the Schrödinger problem in the scalar case, in dimensions $n \geq 3$. If M is Euclidean, Sylvester and Uhlmann [1987] proved that knowledge of the full DN map uniquely determines the potential q. Versions of this problem on admissible and CTA manifolds as defined above have been considered in [Dos Santos Ferreira et al. 2009a; 2016]. Partial data results for the DN map have been proven in [Bukhgeim and Uhlmann 2002; Isakov 2007; Kenig et al. 2007] for the Euclidean case, and more recently in [Kenig and Salo 2013], the last of which contains the previous three results and extends them to the manifold case. Improved results in the linearized case are in [Dos Santos Ferreira et al. 2009b]. Partial data results for the ND map, analogous to the ones in [Kenig et al. 2007], were proven in [Chung 2015]. Other partial data results for scalar equations with first-order potentials as well were obtained in [Dos Santos Ferreira et al. 2007; Chung 2014], and some of those techniques will be useful to us in this paper as well.

For the Hodge Laplacian acting on graded forms, we are not aware of previous results dealing with the determination of a potential from the relative-to-absolute or absolute-to-relative maps. However, [Krupchyk et al. 2011] reconstructs a real analytic metric from these maps in the case of no potential, and [Sharafutdinov and Shonkwiler 2013; Shonkwiler 2013; Belishev and Sharafutdinov 2008; Joshi and Lionheart 2005] recover various kinds of topological information about the manifold from variants of these maps, again in the case of no potential. We remark that full data problems for the Hodge Laplacian in Euclidean space can be solved in a very similar way as in the scalar case (see Section 9), but full data problems on manifolds and partial data problems even in Euclidean space are more involved.

In order to describe the main results precisely, we will define "front" and "back" sets of the boundary ∂M as in [Kenig et al. 2007]. If $M \subset T = \mathbb{R} \times M_0$ is CTA, we can use coordinates (x_1, x'), where x_1 is the Euclidean variable, and define the function $\varphi : T \to \mathbb{R}$ by $\varphi(x_1, x') = x_1$. As discussed in [Dos Santos Ferreira et al. 2009a], φ is a natural limiting Carleman weight in M. Now define

$$\partial M_+ = \{p \in \partial M \mid \partial_\nu \varphi(p) \geq 0\},$$
$$\partial M_- = \{p \in \partial M \mid \partial_\nu \varphi(p) \leq 0\}.$$

Then the main results of this paper are the following.

Theorem 2.1. *Let $M \subset \mathbb{R} \times M_0$ be a three-dimensional admissible manifold with conformal factor $c = 1$, and let Q_1 and Q_2 be continuous endomorphisms of ΛM such that $N_{Q_1}^{\mathrm{RA}}$, $N_{Q_2}^{\mathrm{RA}}$ are defined. Let $\Gamma_+ \subset \partial M$ be a neighbourhood of ∂M_+, and let $\Gamma_- \subset \partial M$ be a neighbourhood of ∂M_-. Suppose*

$$N_{Q_1}^{\mathrm{RA}}(f, g)|_{\Gamma_+} = N_{Q_2}^{\mathrm{RA}}(f, g)|_{\Gamma_+}$$

for all $(f, g) \in H^{\frac{1}{2}}(\partial M, \Lambda \partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda \partial M)$ supported in Γ_-. Then $Q_1 = Q_2$.

Theorem 2.2. *Let M be a three-dimensional admissible manifold with conformal factor $c = 1$, and let Q_1 and Q_2 be continuous endomorphisms of ΛM such that $N_{Q_1}^{\mathrm{AR}}$, $N_{Q_2}^{\mathrm{AR}}$ are defined. Let $\Gamma_+ \subset \partial M$ be a neighbourhood of ∂M_+, and let $\Gamma_- \subset \partial M$ be a neighbourhood of ∂M_-. Suppose*

$$N_{Q_1}^{\mathrm{AR}}(f, g)|_{\Gamma_+} = N_{Q_2}^{\mathrm{AR}}(f, g)|_{\Gamma_+}$$

for all $(f, g) \in H^{\frac{1}{2}}(\partial M, \Lambda \partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda \partial M)$ supported in Γ_-. Then $Q_1 = Q_2$.

In the case that M is a domain in Euclidean space, we can also extend the results to higher dimensions.

Theorem 2.3. *Let M be a bounded smooth domain in $\mathbb{R}^n$, with $n \geq 3$, and let Q_1 and Q_2 be continuous endomorphisms of ΛM such that $N_{Q_1}^{\mathrm{RA}}$, $N_{Q_2}^{\mathrm{RA}}$ are defined. Fix a unit vector α, and let $\varphi(x) = \alpha \cdot x$. Let $\Gamma_+ \subset \partial M$ be a neighbourhood of ∂M_+, and let $\Gamma_- \subset \partial M$ be a neighbourhood of ∂M_-. Suppose*

$$N_{Q_1}^{\mathrm{RA}}(f, g)|_{\Gamma_+} = N_{Q_2}^{\mathrm{RA}}(f, g)|_{\Gamma_+}$$

for all $(f, g) \in H^{\frac{1}{2}}(\partial M, \Lambda \partial M) \times H^{-\frac{1}{2}}(\partial M, \Lambda \partial M)$ supported in Γ_-. Then $Q_1 = Q_2$. The same result holds if we replace the relative-to-absolute map with the absolute-to-relative one.

Theorem 2.1 is a generalization to certain systems of the scalar partial data result of [Kenig et al. 2007] for the DN map, and similarly Theorem 2.2 is an extension to systems of the scalar result of [Chung 2015] for the ND map. To be precise, the above theorems are stated for the linear Carleman weight and not for the logarithmic weight as in [Kenig et al. 2007; Chung 2015]. This restriction comes from the lack of conformal invariance of the full Hodge Laplacian. However, in the scalar case we could use the conformal invariance of the scalar Schrödinger operator together with a reduction from [Kenig and Salo 2013] to recover the logarithmic weight results of [Kenig et al. 2007; Chung 2015] from the above theorems.

The proofs of Theorems 2.1 and 2.2 involve three main ingredients — the construction of complex geometrical optics (CGO) solutions, a Green's theorem argument, and a density argument relating this inverse problem to a tensor tomography problem where one determines a tensor field from its integrals along geodesics (see Section 8). Both the construction of CGO solutions and the Green's theorem argument require appropriate Carleman estimates.

To describe them, we will introduce the following notation. For a CTA manifold M, let N be the inward pointing normal vector field along ∂M. We can extend N to be a vector field in a neighbourhood of ∂M by parallel transporting along normal geodesics, and then to a vector field on M by multiplying by a cutoff function. For $u \in \Omega M$ we will let

$$u_\perp = N^\flat \wedge i_N u,$$

where $N^\flat$ is the 1-form corresponding to N and i_N is the interior product, and

$$u_\parallel = u - u_\perp.$$

Let ∇ denote the Levi-Civita connection on M, and ∇' denote the pullback connection on the boundary. Let

$$\Delta_\varphi = e^{\frac{\varphi}{h}} h^2 \Delta e^{-\frac{\varphi}{h}},$$

where φ is a limiting Carleman weight as described in [Dos Santos Ferreira et al. 2009a]. Note that by [loc. cit.] such weights exist globally if M is a CTA manifold. Then the Carleman estimates are as follows.

Theorem 2.4. *Let M be a CTA manifold, and let Q be an L^∞ endomorphism of ΛM. Define $\Gamma_+ \subset \partial M$ to be a neighbourhood of ∂M_+. Suppose $u \in H^2(M, \Lambda M)$ satisfies the boundary conditions*

$$\begin{aligned}
u|_{\Gamma_+} &= 0 \quad \text{to first order}, \\
tu|_{\Gamma_+^c} &= 0, \\
th\delta e^{-\frac{\varphi}{h}} u|_{\Gamma_+^c} &= h\sigma t i_\nu e^{-\frac{\varphi}{h}} u
\end{aligned} \tag{2-1}$$

for some smooth endomorphism σ independent of h. Then there exists h_0 such that if $0 < h < h_0$,

$$\|(-\Delta_\varphi + h^2 Q)u\|_{L^2(M)} \gtrsim h\|u\|_{H^1(M)} + h^{\frac{1}{2}}\|u_\perp\|_{H^1(\Gamma_+^c)} + h^{\frac{1}{2}}\|h\nabla_N u_\parallel\|_{L^2(\Gamma_+^c)}.$$

Here H^1 signifies the semiclassical H^1 space with semiclassical parameter h, and for instance

$$\|u\|_{H^1(M)} = \|u\|_{L^2(M)} + \|h\nabla u\|_{L^2(M)}.$$

The constant implied in the $\gtrsim$ sign is meant to be independent of h. Note that the last boundary condition in (2-1) can be rewritten as

$$th\delta u|_{\partial M} = -ti_{d\varphi}u - h\sigma ti_N u.$$

Theorem 2.5. *Let M be a CTA manifold, and let Q be an L^∞ endomorphism of ΛM. Define $\Gamma_+ \subset \partial M$ to be a neighbourhood of ∂M_+. Suppose $u \in H^2(M, \Lambda M)$ satisfies the boundary conditions*

$$
\begin{aligned}
u|_{\Gamma_+} &= 0 \quad \text{to first order,} \\
t * u|_{\Gamma_+^c} &= 0, \\
th\delta * e^{-\frac{\varphi}{h}}u|_{\Gamma_+^c} &= h\sigma ti_\nu * e^{-\frac{\varphi}{h}}u
\end{aligned}
\tag{2-2}
$$

for some smooth endomorphism σ independent of h. Then there exists h_0 such that if $0 < h < h_0$,

$$\|(-\Delta_\varphi + h^2 Q)u\|_{L^2(M)} \gtrsim h\|u\|_{H^1(M)} + h^{\frac{1}{2}}\|u_\|\|_{H^1(\Gamma_+^c)} + h^{\frac{1}{2}}\|h\nabla_N u_\perp\|_{L^2(\Gamma_+^c)}.$$

Note that Theorem 2.5 is actually Theorem 2.4 with u replaced by $*u$. Therefore it suffices to prove Theorem 2.5 only. It is also worth noting that the Carleman estimates are proved for CTA manifolds in general, with no restriction on either the dimension, the conformal factor, or the transversal manifold (M_0, g_0). Theorems 2.4 and 2.5 are extensions to the Hodge Laplace system on CTA manifolds of the scalar and Euclidean Carleman estimates in [Kenig et al. 2007; Chung 2015].

Finally, we sketch the main ideas in the proofs of the theorems and highlight the new features in our approach. The main difficulty in proving the Carleman estimates is the fact that the standard integration by parts argument, which gives a useful Carleman estimate for scalar equations with Dirichlet boundary condition [Kenig et al. 2007], results in complicated boundary terms when one is dealing with a system of equations (see Proposition 4.1). The Fourier analytic methods of [Chung 2015] will be crucial in handling these boundary terms. We first prove Theorem 2.5 for 0-forms (i.e., scalar equations) by adapting the Euclidean arguments of [Chung 2015] to the manifold case. After an initial estimate for the vectorial boundary terms in Proposition 4.2, Theorem 2.5 is proved for k-forms by induction on k. The proof of the Carleman estimates is long and technical, due to the work required to simplify and estimate the boundary terms.

After proving the Carleman estimates, the construction of CGO solutions proceeds as in the scalar case [Kenig et al. 2007; Dos Santos Ferreira et al. 2009a] and in the full data Maxwell case [Kenig et al. 2011]. The end result is given in Lemma 7.6. There the amplitude in the solutions is vector-valued, and later one needs to use the flexibility in choosing the components of this vector. The inverse problem is solved by inserting the CGO solutions in a standard integral identity, Lemma 8.1. Here an unexpected feature appears: recovering the matrix potential reduces to inverting mixed Fourier/attenuated geodesic ray transforms as in the scalar case [Dos Santos Ferreira et al. 2009a], but the components of the matrix turn out to depend on the geodesic along which they are integrated. We resolve this difficulty when $\dim(M) = 3$ by making use of ray transforms on tensors of order ≤ 2 and using recent results on tensor tomography [Paternain et al. 2013]. When the underlying space is Euclidean, we can use classical Fourier arguments and prove the uniqueness result also when $\dim(M) \geq 4$.

3. Notation and identities

As stated before, the basic reference for the following facts on Riemannian geometry is [Taylor 1996]. Let (M, g) be a smooth ($= C^\infty$) n-dimensional Riemannian manifold with or without boundary. All manifolds will be assumed to be oriented. We write $\langle v, w \rangle$ for the g-inner product of tangent vectors, and $|v| = \langle v, v \rangle^{\frac{1}{2}}$ for the g-norm. If $x = (x_1, \dots, x_n)$ are local coordinates and ∂_j are the corresponding vector fields, we write $g_{jk} = \langle \partial_j, \partial_k \rangle$ for the metric in these coordinates. The determinant of (g_{jk}) is denoted by $|g|$, and (g^{jk}) is the matrix inverse of (g_{jk}).

We shall sometimes do computations in normal coordinates. These are coordinates x defined in a neighbourhood of a point $p \in M^{\mathrm{int}}$ such that $x(p) = 0$ and geodesics through p correspond to rays through the origin in the x-coordinates. The metric in these coordinates satisfies

$$g_{jk}(0) = \delta_{jk}, \quad \partial_l g_{jk}(0) = 0.$$

The Einstein convention of summing over repeated upper and lower indices will be used. We convert vector fields to 1-forms and vice versa by the musical isomorphisms, which are given by

$$(X^j \partial_j)^\flat = X_k \, dx^k, \quad X_k = g_{jk} X^j,$$
$$(\omega_k \, dx^k)^\sharp = \omega^j \partial_j, \quad \omega^j = g^{jk} \omega_k.$$

The set of smooth k-forms on M is denoted by $\Omega^k M$, and the graded algebra of differential forms is written as

$$\Omega M = \bigoplus_{k=0}^{n} \Omega^k M.$$

The set of k-forms with L^2 or H^s coefficients are denoted by $L^2(M, \Lambda^k M)$ and $H^s(M, \Lambda^k M)$, respectively. Here H^s for $s \in \mathbb{R}$ are the usual Sobolev spaces on M. The inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$ are extended to forms and more generally tensors on M in the usual way, and we also extend the inner product $\langle \cdot, \cdot \rangle$ to complex-valued tensors as a complex bilinear form.

Let $d : \Omega^k M \to \Omega^{k+1} M$ be the exterior derivative, and let $* : \Omega^k M \to \Omega^{n-k} M$ be the Hodge star operator. We introduce the sesquilinear inner product on $\Omega^k M$,

$$(\eta \mid \zeta) = \int_M \langle \eta, \bar{\zeta} \rangle \, dV = \int_M \eta \wedge *\bar{\zeta} = (*\eta \mid *\zeta).$$

Here $dV = *1 = |g|^{\frac{1}{2}} \, dx^1 \cdots dx^n$ is the volume form. The codifferential $\delta : \Omega^k M \to \Omega^{k-1} M$ is defined as the formal adjoint of d in the inner product on real-valued forms, so that

$$(d\eta \mid \zeta) = (\eta \mid \delta\zeta) \quad \text{for } \eta \in \Omega^{k-1} M \text{ and } \zeta \in \Omega^k M \text{ compactly supported and real.}$$

These operators satisfy the following relations on k-forms in M:

$$** = (-1)^{k(n-k)}, \quad \delta = (-1)^{k(n-k)-n+k-1} * d *.$$

If X is a vector field, the interior product $i_X : \Omega^k M \to \Omega^{k-1} M$ is defined by

$$i_X \omega(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1}).$$

If ξ is a 1-form then the interior product $i_\xi = i_{\xi^\sharp}$ is the formal adjoint of $\xi \wedge$ in the inner product on real-valued forms, and on k-forms it has the expression

$$i_\xi = (-1)^{n(k-1)} * \xi \wedge *.$$

The interior and exterior products interact by the formula

$$i_\xi \alpha \wedge \beta = (i_\xi \alpha) \wedge \beta + (-1)^k \alpha \wedge i_\xi \beta,$$

where α is a k-form and β an m-form. In particular if α and ξ are 1-forms then

$$i_\xi \alpha \wedge \beta + \alpha \wedge i_\xi \beta = \langle \alpha, \xi \rangle \beta.$$

In addition, the differential and codifferential satisfy the product rules

$$d(f\eta) = df \wedge \eta + f d\eta, \quad \delta(f\eta) = -i_{df}\eta + f\delta\eta.$$

The Hodge Laplacian on k-forms is defined by

$$-\Delta = (d + \delta)^2 = d\delta + \delta d.$$

It satisfies $\Delta* = *\Delta$. The above quantities may be naturally extended to graded forms.

We will also have to deal with forms that are not compactly supported on M. We have already introduced the tangential trace $t : \Omega M \to \Omega \partial M$ by

$$t : \omega \mapsto i^* \omega,$$

so if u is a graded form on M, then tu is a graded form on ∂M. Then

$$(tu \mid tv)_{\partial M}$$

is interpreted in the same manner as $(u \mid v)_M$ above. If u and v are graded forms on M, we will also define

$$(u \mid v)_{\partial M} = \int_{\partial M} \langle u, \bar{v} \rangle \, dS = \int_{\partial M} t i_\nu u \wedge *\bar{v} \, dS,$$

where dS is the volume form on ∂M. Now if $\eta \in \Omega^{k-1} M$ and $\zeta \in \Omega^k M$ then d and δ satisfy the integration by parts formulas

$$(d\eta \mid \zeta)_M = (\nu \wedge \eta \mid \zeta)_{\partial M} + (\eta \mid \delta\zeta)_M, \tag{3-1}$$

$$(\delta\zeta \mid \eta)_M = -(i_\nu \zeta \mid \eta)_{\partial M} + (\zeta \mid d\eta)_M. \tag{3-2}$$

Note also that

$$(i_\nu \zeta \mid \eta)_{\partial M} = (\nu \wedge \eta \mid \zeta)_{\partial M}.$$

Here ν denotes both the unit outer normal of ∂M and the corresponding 1-form.

Applying these formulas for the Hodge Laplacian gives

$$(-\Delta u \mid v)_M = (u \mid -\Delta v)_M + (v \wedge \delta u \mid v)_{\partial M} - (i_\nu du \mid v)_{\partial M} - (i_\nu u \mid dv)_{\partial M} + (v \wedge u \mid \delta v)_{\partial M},$$

where u and v are k-forms, or graded forms. We can also redo the integration by parts to write the boundary terms in terms of absolute and relative boundary conditions, so

$$(-\Delta u \mid v)_M = (u \mid -\Delta v)_M + (tu \mid ti_\nu dv)_{\partial M} + (t\delta * u \mid ti_\nu * v)_{\partial M} + (t * u \mid ti_\nu d * v)_{\partial M} + (t\delta u \mid ti_\nu v)_{\partial M}.$$

The Levi-Civita connection, defined on tensors in M, is denoted by ∇ and it satisfies $\nabla_X * = *\nabla_X$. We will sometimes write ∇f (where f is any function) for the metric gradient of f, defined by

$$\nabla f = (df)^\sharp = g^{jk} \partial_j f \partial_k.$$

If X is a vector field and η, ζ are differential forms we have

$$\nabla_X (\eta \wedge \zeta) = (\nabla_X \eta) \wedge \zeta + \eta \wedge (\nabla_X \zeta).$$

If X, Y are vector fields then

$$[\nabla_X, i_Y] = i_{\nabla_X Y}.$$

We can also express d using the ∇ operator, as follows: if ω is a k-form on M, and $X_1, \ldots, X_{k+1}$ are vector fields on M, then

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{l=1}^{k+1} (-1)^{l+1} (\nabla_{X_l} \omega)(X_1, \ldots, \hat{X}_l, \ldots, X_{k+1}),$$

where $\hat{X}_l$ means that we omit the X_l argument. Moreover if $e_1, \ldots, e_n$ are an orthonormal frame of TM defined in a neighbourhood $U \subset M$ we have

$$-\delta\omega = \sum_{j=1}^n i_{e_j} \nabla_{e_j} \omega.$$

For the statements of the Carleman estimates, we introduced the notation

$$u_\perp = N^\flat \wedge i_N u \quad \text{and} \quad u_\parallel = u - u_\perp,$$

where N is a smooth vector field which coincides with the inward pointing normal vector field at the boundary ∂M, and is extended into M by parallel transport. Note that $i_N u_\parallel = 0$, $N \wedge u_\perp = 0$, and $tu_\perp = 0$ at ∂M. In addition, if u and v are graded forms on M, then

$$(tu \mid tv)_{\partial M} = (tu_\parallel \mid tv_\parallel)_{\partial M} = (u_\parallel \mid v_\parallel)_{\partial M}$$

and

$$(ti_N u \mid ti_N v)_{\partial M} = (ti_N u_\perp \mid ti_N v_\perp)_{\partial M} = (u_\perp \mid v_\perp)_{\partial M}.$$

If X is a vector field, we can break down X into parallel and perpendicular components in the same way by using $(X_\parallel^\flat)^\sharp$ and $(X_\perp^\flat)^\sharp$. The $\perp$ and $\parallel$ signs are interchanged by the Hodge star operator:

$$*(u_\parallel) = (*u)_\perp \quad \text{and} \quad *(u_\perp) = (*u)_\parallel.$$

Note that by its definition in terms of parallel transport, $\nabla_N N = 0$. Thus ∇_N commutes with $N\wedge$ and i_N.

If we view ∂M as a submanifold embedded into M, then TM splits into $T\partial M \oplus N\partial M$, where $T\partial M$ is the tangent bundle of ∂M and $N\partial M$ is the normal bundle. Then the second fundamental form $II : T\partial M \oplus T\partial M \to N\partial M$ of ∂M relative to this embedding is defined by

$$II(X, Y) = (\nabla_X Y \mid N)N.$$

The second fundamental form can also be defined in terms of the shape operator $s : T\partial M \to T\partial M$ by

$$s(X) = \nabla_X N.$$

Then

$$II(X, Y) = (s(X) \mid Y)N.$$

These two operators carry information about the shape of the ∂M in M, and thus show up in our boundary computations.

Now we move to some more specific technical formulas used in the paper. The proofs involve routine computations and are omitted, but interested readers may find the proofs in the arXiv version of this paper [Chung et al. 2013, Appendix]. We begin with a simple computation.

Lemma 3.1. *If ξ and η are real-valued 1-forms on M and if u is a k-form, then*

$$\xi \wedge i_\eta u + i_\xi(\eta \wedge u) + \eta \wedge i_\xi u + i_\eta(\xi \wedge u) = 2\langle \xi, \eta\rangle u.$$

We also give an expression for the conjugated Laplacian.

Lemma 3.2. *Let (M, g) be an oriented Riemannian manifold, let $\rho \in C^2(M)$ be a complex-valued function, and let s be a complex number. If u is a k-form on M, then*

$$e^{s\rho}(-\Delta)(e^{-s\rho}u) = -s^2\langle d\rho, d\rho\rangle u + s[2\nabla_{\mathrm{grad}(\rho)} + \Delta\rho]u - \Delta u.$$

Next, an expansion for the expression $t\delta$.

Lemma 3.3. *Let $u \in \Omega^k(M)$. Then*

$$-t(\delta u) = -\delta' t u_\| + (S - (n-1)\kappa)t i_N u_\perp + t\nabla_N i_N u,$$

where κ is the mean curvature of ∂M, and $S : \Omega^{k-1}(\partial M) \to \Omega^{k-1}(\partial M)$ is defined by

$$S\omega(X_1, \ldots, X_{k-1}) = \sum_{\ell=1}^{k-1} \omega(X_1, \ldots, sX_\ell, \ldots, X_{k-1}),$$

with $s : T\partial M \to T\partial M$ being the shape operator of ∂M.

Now for $t i_N d$.

Lemma 3.4. *Let $u \in \Omega^k(M)$. Then on ∂M,*

$$t i_N du = t\nabla_N u_\| + Stu_\| - d' t i_N u.$$

We also need an expansion for $t\delta B$, where B is the operator

$$B = \frac{h}{i}\big[d \circ i_{d\varphi_c} + i_{d\varphi_c} \circ d - d\varphi_c \wedge \delta - \delta(d\varphi_c \wedge \cdot)\big] = \frac{h}{i}[2\nabla_{\nabla\varphi_c} + \triangle\varphi_c].$$

Lemma 3.5. *If $u \in \Omega^k(M)$ is such that $tu = 0$, then*

$$t\delta Bu = \delta' t Bu + 2ih\nabla'_{(\nabla\varphi_c)_\parallel} t\nabla_N i_N u - 2ih\partial_\nu\varphi_c t\nabla_N\nabla_N i_N u$$
$$+ ih\big(2((n-1)\kappa - S)\partial_\nu\varphi_c + 2\partial_\nu^2\varphi_c + \triangle\varphi_c\big)t\nabla_N i_N u + 2ih(S - (n-1)\kappa)t\nabla_{(\nabla\varphi_c)_\parallel} i_N u$$
$$+ ih\big((S - (n-1)\kappa)\triangle\varphi_c + \nabla_N\triangle\varphi_c\big)t i_N u$$
$$+ 2iht i_N R(N, \nabla(\varphi_c)_\parallel)u_\perp + 2iht\nabla_{[(\nabla\varphi_c)_\parallel, N]} i_N u - 2ih i_{s(\nabla\varphi_c)_\parallel} t\nabla_N u_\parallel.$$

Finally, we will need to do a computation to split the Hodge Laplacian into normal and tangential parts. To do this, we will take advantage of a Weitzenböck identity, which says

$$\triangle = \tilde{\triangle} + R,$$

where R is a zeroth-order linear operator depending only on the curvature of M, $\triangle$ is the Hodge Laplacian, and $\tilde{\triangle}$ is the connection Laplacian:

$$\tilde{\triangle}u := \nabla^*\nabla u.$$

We then have the following result for $\tilde{\triangle}$.

Lemma 3.6. *Let $u \in \Omega^k(M)$ satisfy $tu = 0$. Then*

$$t i_N \tilde{\triangle}u = \tilde{\triangle}' t i_N u + t\nabla_N\nabla_N i_N u + tr(s^2)i_N u - S_2 i_N u,$$

where $S_2\omega(X_1, \ldots, X_{k-1}) := \sum_{l=1}^{k-1}\omega(\ldots, s^2 X_l, \ldots)$.

4. Carleman estimates and boundary terms

As noted in the Introduction, Theorem 2.4 follows from Theorem 2.5, so it enough to show that we can prove Theorem 2.5.

In proving the Carleman estimates, it will suffice to work with smooth sections of ΛM and apply a density argument to get the final result. Let $\Omega^k(M)$ denote the space of smooth sections of $\Lambda^k M$, and $\Omega(M)$ denote the space of smooth sections of ΛM.

In this section we give an initial form of the Carleman estimates by using an integration by parts argument as in [Kenig et al. 2007]. To do this, we will first need to understand the relevant boundary terms. We will use the integration by parts formulas

$$(du \mid v)_M = (v \wedge u \mid v)_{\partial M} + (u \mid \delta v)_M, \tag{4-1}$$
$$(\delta u \mid v)_M = -(i_\nu u \mid v)_{\partial M} + (u \mid dv)_M \tag{4-2}$$

for $u, v \in \Omega(M)$.

As in [Kenig et al. 2007], we will need to work with the convexified weight

$$\varphi_c = \varphi + \frac{h\varphi^2}{2\varepsilon}.$$

Then

$$-\Delta_{\varphi_c} = e^{\frac{\varphi_c}{h}}(-h^2\Delta)e^{-\frac{\varphi_c}{h}}.$$

Writing

$$d_{\varphi_c} = e^{\frac{\varphi_c}{h}}hde^{-\frac{\varphi_c}{h}} = hd - d\varphi_c\wedge,$$
$$\delta_{\varphi_c} = e^{\frac{\varphi_c}{h}}h\delta e^{-\frac{\varphi_c}{h}} = h\delta + i_{d\varphi_c},$$

we have

$$-\Delta_{\varphi_c} = d_{\varphi_c}\delta_{\varphi_c} + \delta_{\varphi_c}d_{\varphi_c}.$$

By Lemma 3.2 we can write this as $A + iB$, where A and B are self-adjoint operators given by

$$A = -h^2\Delta - \left(d\varphi_c \wedge i_{d\varphi_c} + i_{d\varphi_c}(d\varphi_c \wedge \cdot)\right) = -h^2\Delta - |d\varphi_c|^2,$$
$$B = \frac{h}{i}\left[d \circ i_{d\varphi_c} + i_{d\varphi_c} \circ d - d\varphi_c \wedge \delta - \delta(d\varphi_c \wedge \cdot)\right] = \frac{h}{i}[2\nabla_{\nabla\varphi_c} + \Delta\varphi_c].$$

Let $\|\cdot\|$ indicate the L^2 norm on M, unless otherwise stated. Then, for $u \in \Omega^k(M)$,

$$\|\Delta_{\varphi_c}u\|^2 = \left((A + iB)u \mid (A + iB)u\right) = \|Au\|^2 + \|Bu\|^2 + i(Bu \mid Au) - i(Au \mid Bu).$$

Integrating by parts gives

$$(Bu \mid Au) = \left(Bu \mid h^2 d\delta u + h^2\delta du - |d\varphi_c|^2 u\right)$$
$$= (hdBu \mid hdu) + (h\delta Bu \mid h\delta u) - \left(|d\varphi_c|^2 Bu \ u\right) + h(Bu \mid v \wedge h\delta u - i_v hdu)_{\partial M}$$
$$= (ABu \mid u) + h(hdBu \mid v \wedge u)_{\partial M} - h(h\delta Bu \mid i_v u)_{\partial M} + h(Bu \mid v \wedge h\delta u - i_v hdu)_{\partial M}$$

and after a short computation

$$(Au \mid Bu) = (BAu \mid u) - \frac{2h}{i}((\partial_v\varphi_c)Au \mid u)_{\partial M}.$$

This finishes the basic integration by parts argument and shows the following:

Proposition 4.1. *If $u \in \Omega M$, then*

$$\|\Delta_{\varphi_c}u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A, B]u \mid u) + ih(hdBu \mid v \wedge u)_{\partial M}$$
$$-ih(h\delta Bu \mid i_v u)_{\partial M} + ih(Bu \mid v \wedge h\delta u - i_v hdu)_{\partial M} + 2h((\partial_v\varphi_c)Au \mid u)_{\partial M}. \quad (4\text{-}3)$$

Now we invoke the absolute boundary conditions to estimate the nonboundary terms and to simplify the boundary terms in (4-3). It is enough to consider differential forms $u \in \Omega^k(M)$ for fixed k.

Proposition 4.2. *Let $u \in \Omega^k(M)$ such that*

$$t * u = 0,$$
$$th\delta * u = -ti_{d\varphi} * u + h\sigma ti_N * u \quad (4\text{-}4)$$

for some smooth bounded endomorphism σ whose bounds are uniform in h.

Then the nonboundary terms in (4-3) *satisfy*

$$\|Au\|^2 + \|Bu\|^2 + (i[A,B]u \mid u) \gtrsim \frac{h^2}{\varepsilon}\|u\|^2_{H^1(M)} - \frac{h^3}{\varepsilon}\big(\|u_{\|}\|^2_{H^1(\partial M)} + \|h\nabla_N u_\perp\|^2_{L^2(\partial M)}\big) \qquad \text{(4-5)}$$

for $h \ll \varepsilon \ll 1$. Also, the boundary terms in (4-3) *have the form*

$$-2h^3(\partial_\nu \varphi \nabla_N u_\perp \mid \nabla_N u_\perp)_{\partial M} - 2h\big(\partial_\nu \varphi(|d\varphi|^2 + |\partial_\nu \varphi|^2)u_{\|} \mid u_{\|}\big)_{\partial M} + R, \qquad \text{(4-6)}$$

where

$$|R| \lesssim Kh^3\|\nabla' t u_{\|}\|^2_{\partial M} + \frac{h}{K}\|u_{\|}\|^2_{\partial M} + \frac{h^3}{K}\|\nabla_N u_\perp\|^2_{\partial M}$$

for any large enough K independent of h.

Proof of Proposition 4.2. We will prove (4-5) first. The argument follows the one given in [Chung 2015] for scalar functions.

Note that A and B have the same scalar principal symbols as they do for 0-forms: that is, given a local basis $dx^1, \ldots, dx^n$ for the cotangent space with $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$,

$$A = A_s + hE_1, \quad A_s(f dx^I) = (Af)dx^I,$$

and

$$B = B_s + hE_0, \quad B_s(f dx^I) = (Bf)dx^I,$$

where E_1 and E_0 are first- and zeroth-order operators, respectively, with uniform bounds in h and ε. Therefore locally

$$[A,B](f dx^I) = ([A,B]f)dx^I + h\big([E_1, B_s] + [A_s, E_0] + hR\big)(f dx^I),$$

where R is a first-order operator with uniform bounds in h and ε. Choosing a partition of unity $\chi_1, \ldots, \chi_m$ of M such that this operation can be performed near each $\text{supp}(\chi_j)$, the argument for scalar functions in the proof of Proposition 3.1 from [Chung 2015] implies

$$i([A,B]u \mid u) = \sum_{j=1}^m i([A,B]u \mid \chi_j u) = 4\frac{h^2}{\varepsilon}\|(1 + h\varepsilon^{-1}\varphi)u\|^2_{L^2} + h(B\beta Bu \mid u) + h^2(Qu \mid u),$$

where Q is a second-order operator. Recall that

$$B = \frac{h}{i}\big(d \circ i_{d\varphi_c} + i_{d\varphi_c} \circ d - d\varphi_c \wedge \delta - \delta(d\varphi_c \wedge \cdot)\big),$$

so using integration by parts with the above formula, we get

$$h(B\beta Bu \mid u) = h(\beta Bu \mid Bu) - ih^2(i_\nu \beta Bu \mid i_{d\varphi_c}u)_{\partial M} - ih^2(\nu \wedge i_{d\varphi_c}\beta Bu \mid u)_{\partial M}$$
$$- ih^2(\nu \wedge \beta Bu \mid d\varphi_c \wedge u)_{\partial M} - ih^2(i_\nu(d\varphi_c \wedge \beta Bu) \mid u)_{\partial M}$$
$$= h(\beta Bu \mid Bu) - ih^2(d\varphi_c \wedge i_\nu \beta Bu \mid u)_{\partial M} - ih^2(\nu \wedge i_{d\varphi_c}\beta Bu \mid u)_{\partial M}$$
$$- ih^2(i_{d\varphi_c}\nu \wedge \beta Bu \mid u)_{\partial M} - ih^2(i_\nu(d\varphi_c \wedge \beta Bu) \mid u)_{\partial M}.$$

By Lemma 3.1 we obtain

$$h(B\beta Bu \mid u) = h(\beta Bu \mid Bu) - 2ih^2(\partial_\nu\varphi_c \beta Bu \mid u)_{\partial M}.$$

The absolute boundary condition says that $t * u = 0$, so $u_\perp = 0$ at the boundary. Therefore

$$h(B\beta Bu \mid u) = h(\beta Bu \mid Bu) - 2ih^2(\partial_\nu\varphi_c \beta Bu \mid u_\|)_{\partial M} = h(\beta Bu \mid Bu) - 2ih^2(t\partial_\nu\varphi_c \beta Bu \mid tu_\|)_{\partial M}.$$

The boundary term in the last expression is bounded by

$$h^3\varepsilon^{-1}\|tBu\|^2_{L^2(\partial M)} + h^3\varepsilon^{-1}\|u_\|\|^2_{L^2(\partial M)}.$$

At the boundary,

$$tBu = \frac{h}{i}t[2\nabla_{\nabla\varphi_c} + \Delta\varphi_c]u = \frac{h}{i}\big[-2\partial_\nu\varphi_c t\nabla_N u_\| - 2\partial_\nu\varphi_c t\nabla_N u_\perp + 2t\nabla_{(\nabla\varphi_c)_\|}u_\| + \Delta\varphi_c tu_\|\big],$$

so

$$\|tBu\|^2_{L^2(\partial M)} \lesssim \|th\nabla_N u_\|\|^2_{L^2(\partial M)} + \|th\nabla_N u_\perp\|^2_{L^2(\partial M)} + \|th\nabla_{(\nabla\varphi_c)_\|}u_\|\|^2_{L^2(\partial M)} + h^2\|tu_\|\|^2_{L^2(\partial M)}$$
$$\lesssim \|th\nabla_N u_\|\|^2_{L^2(\partial M)} + \|th\nabla_N u_\perp\|^2_{L^2(\partial M)} + \|u_\|\|^2_{H^1(\partial M)}.$$

Now by Lemma 3.4,

$$ti_N hdu = th\nabla_N u_\| + hStu_\| - hd'ti_N u.$$

Since $t * u = 0$, we have $i_N u, u_\perp = 0$ at the boundary, and thus

$$ti_N hdu = th\nabla_N u_\| + hStu_\|.$$

Therefore

$$\|th\nabla_N u_\|\|^2_{L^2(\partial M)} \lesssim \|ti_N hdu\|^2_{L^2(\partial M)} + h^2\|u_\|\|^2_{L^2(\partial M)}$$
$$\lesssim \|ti_N * (h\delta * u)\|^2_{L^2(\partial M)} + h^2\|u_\|\|^2_{L^2(\partial M)}$$
$$\lesssim \|th\delta * u\|^2_{L^2(\partial M)} + h^2\|u_\|\|^2_{L^2(\partial M)}$$
$$\lesssim \|u\|^2_{L^2(\partial M)},$$

where in the last step we invoked the absolute boundary condition. Therefore

$$\|tBu\|^2_{L^2(\partial M)} \lesssim \|th\nabla_N u_\perp\|^2_{L^2(\partial M)} + \|u_\|\|^2_{H^1(\partial M)},$$

and thus

$$h(B\beta Bu \mid u) \lesssim \frac{h^2}{\varepsilon}\|Bu\|^2_{L^2} + \frac{h^3}{\varepsilon}\|u_\|\|^2_{H^1(\partial M)} + \frac{h^3}{\varepsilon}\|h\nabla_N u_\perp\|^2_{L^2(\partial M)}.$$

Similarly

$$h^2(Qu \mid u) \lesssim h^2\|u\|^2_{H^1} + h^3\|u_\|\|^2_{H^1(\partial M)} + h^3\|h\nabla_N u_\perp\|^2_{L^2(\partial M)}.$$

Therefore

$$i([A, B]u \mid u) \gtrsim \frac{h^2}{\varepsilon}\|u\|^2_{L^2} - \frac{h^2}{\varepsilon}\|Bu\|^2_{L^2} - h^2\|u\|^2_{H^1} - h^3\varepsilon^{-1}\|u_\|\|^2_{H^1(\partial M)} - h^3\varepsilon^{-1}\|h\nabla_N u_\perp\|^2_{L^2(\partial M)}.$$

Meanwhile, since $t * u = 0$ on ∂M we can write

$$
\begin{aligned}
h^2(\|hdu\|_{L^2}^2 + \|h\delta u\|_{L^2}^2) &= h^2\big((hd*u, hd*u) + (h\delta*u, h\delta*u)\big) \\
&= h^2(-h^2\triangle*u \mid *u) - h^3(v \wedge h\delta*u \mid *u)_{\partial M} \\
&= h^2(Au \mid u) + h^2(|d\varphi_c|^2 u \mid u) - h^3(v \wedge h\delta*u \mid *u)_{\partial M} \\
&= h^2(Au \mid u) + h^2(|d\varphi_c|^2 u \mid u) + h^3(th\delta*u \mid ti_N*u)_{\partial M}.
\end{aligned}
$$

Using the absolute boundary conditions again, we have

$$
\begin{aligned}
th\delta * u &= -ti_{d\varphi}*u + h\sigma ti_N*u \\
&= \partial_\nu\varphi ti_N*u + h\sigma ti_N*u,
\end{aligned}
$$

so

$$
h^2(\|hdu\|_{L^2}^2 + \|h\delta u\|_{L^2}^2) \lesssim \frac{1}{K}\|Au\|_{L^2}^2 + Kh^4\|u\|_{L^2}^2 + h^2\|u\|_{L^2}^2 + h^3\|ti_N*u\|_{L^2(\partial M)}^2,
$$

or

$$
\|Au\|_{L^2}^2 \gtrsim Kh^2(\|hdu\|_{L^2}^2 + \|h\delta u\|_{L^2}^2) - K^2h^4\|u\|_{L^2}^2 - Kh^2\|u\|_{L^2}^2 - Kh^3\|u_\parallel\|_{L^2(\partial M)}^2.
$$

We take $K \sim \frac{1}{\alpha\varepsilon}$ with α large and fixed. Putting this together with the inequality for $(i[A, B]u \mid u)$ and Gaffney's inequality $\|u\|_{H^1} \sim \|u\|_{L^2} + \|hdu\|_{L^2} + \|h\delta u\|_{L^2}$ when $t*u = 0$, we obtain

$$
\|Au\|^2 + \|Bu\|^2 + (i[A, B]u \mid u) \gtrsim \frac{h^2}{\varepsilon}\|u\|_{H^1}^2 - h^3\varepsilon^{-1}\big(\|u_\parallel\|_{H^1(\partial M)}^2 + \|h\nabla_N u_\perp\|_{L^2(\partial M)}^2\big)
$$

for $h \ll \varepsilon \ll 1$. This proves (4-5).

We will now show the expression (4-6) for the boundary terms in (4-3). Recall that these boundary terms are given by

$$
ih(hdBu \mid v \wedge u)_{\partial M} - ih(h\delta Bu \mid i_\nu u)_{\partial M} + ih(Bu \mid v \wedge h\delta u - i_\nu hdu)_{\partial M} + 2h((\partial_\nu\varphi_c)Au \mid u)_{\partial M}. \quad (4\text{-}7)
$$

Note that

$$
\begin{aligned}
&ih(hdB*u \mid v \wedge *u)_{\partial M} - ih(h\delta B*u \mid i_\nu*u)_{\partial M} + ih(B*u \mid v \wedge h\delta*u - i_\nu hd*u)_{\partial M} + 2h((\partial_\nu\varphi_c)A*u \mid *u)_{\partial M} \\
&= ih(hdBu \mid v \wedge u)_{\partial M} - ih(h\delta Bu \mid i_\nu u)_{\partial M} + ih(Bu \mid v \wedge h\delta u - i_\nu hdu)_{\partial M} + 2h((\partial_\nu\varphi_c)Au \mid u)_{\partial M}.
\end{aligned}
$$

Moreover, if u satisfies the absolute boundary conditions (4-4), then $*u$ satisfies the relative boundary conditions

$$
\begin{aligned}
tu &= 0, \\
th\delta u &= -ti_{d\varphi}u + h\sigma ti_N u,
\end{aligned}
\quad (4\text{-}8)
$$

and vice versa. Therefore it suffices to prove that if u satisfies (4-8) then the boundary terms (4-7) become

$$
-2h^3(\partial_\nu\varphi\nabla_N u_\parallel \mid \nabla_N u_\parallel)_{\partial M} - 2h\big(\partial_\nu\varphi(|d\varphi|^2 + |\partial_\nu\varphi|^2)u_\perp \mid u_\perp\big)_{\partial M} + R, \quad (4\text{-}9)
$$

where

$$
|R| \lesssim Kh^3\|\nabla' ti_N u\|_{\partial M}^2 + \frac{h}{K}\|u_\perp\|_{\partial M}^2 + \frac{h^3}{K}\|\nabla_N u_\parallel\|_{\partial M}^2 \quad (4\text{-}10)
$$

for any large enough K independent of h.

So let's return to (4-7), and assume u satisfies (4-8). The condition $tu = 0$ implies the first term $ih(hdBu \mid \nu \wedge u)_{\partial M}$ is zero. Therefore we are left with

$$-ih(h\delta Bu \mid i_\nu u)_{\partial M} + ih(Bu \mid \nu \wedge h\delta u)_{\partial M} - ih(Bu \mid i_\nu hdu)_{\partial M} + 2h((\partial_\nu \varphi_c)Au \mid u)_{\partial M}.$$

We calculate each of the terms individually.

Firstly,

$$\begin{aligned}
ih(Bu \mid \nu \wedge h\delta u)_{\partial M} &= ih(Bu \mid \nu \wedge h(\delta u)_\parallel)_{\partial M} \\
&= -ih(i_N Bu \mid h(\delta u)_\parallel)_{\partial M} \\
&= -ih(ti_N Bu \mid th\delta u)_{\partial M}.
\end{aligned}$$

Now

$$Bu = \frac{h}{i}(2\nabla_{\nabla \varphi_c} + \Delta \varphi_c)u,$$

so

$$\begin{aligned}
ti_N Bu &= \frac{h}{i}ti_N\big(2\nabla_{(\nabla \varphi_c)_\parallel} - 2\partial_\nu \varphi_c \nabla_N + \Delta \varphi_c\big)u \\
&= \frac{h}{i}\big(2\nabla_{(\nabla \varphi_c)_\parallel}ti_N - 2\partial_\nu \varphi_c t\nabla_N i_N + t\Delta \varphi_c i_N\big)u.
\end{aligned}$$

Therefore,

$$\begin{aligned}
-ih(ti_N Bu \mid th\delta u)_{\partial M} \\
= 2h(\partial_\nu \varphi_c th\nabla_N i_N u \mid th\delta u)_{\partial M} - 2h(h\nabla_{(\nabla \varphi_c)_\parallel}ti_N u \mid th\delta u)_{\partial M} - h^2(t\Delta \varphi_c i_N u \mid th\delta u)_{\partial M}.
\end{aligned}$$

Now if $th\delta u|_{\partial M} = -ti_{d\varphi}u + h\sigma ti_N u$ and $tu = 0$, then

$$th\delta u|_{\partial M} = (\partial_\nu \varphi + h\sigma)ti_N u. \tag{4-11}$$

Therefore

$$\begin{aligned}
-ih(ti_N Bu \mid th(\delta u))_{\partial M} &= 2h\big(\partial_\nu \varphi_c th\nabla_N i_N u \mid (\partial_\nu \varphi + h\sigma)ti_N u\big)_{\partial M} \\
&\quad - 2h\big(h\nabla_{(\nabla \varphi_c)_\parallel}ti_N u \mid (\partial_\nu \varphi + h\sigma)ti_N u\big)_{\partial M} \\
&\quad - h^2\big(t\Delta \varphi_c i_N u \mid (\partial_\nu \varphi + h\sigma)ti_N u\big)_{\partial M}.
\end{aligned}$$

Moreover, by Lemma 3.3,

$$th(\delta u) = h\delta' tu_\parallel + h((n-1)\kappa - S)ti_N u_\perp - th\nabla_N i_N u.$$

Since $tu = 0$,

$$th(\delta u) = h((n-1)\kappa - S)ti_N u_\perp - th\nabla_N i_N u.$$

Substituting this into (4-11) gives

$$th\nabla_N i_N u = \big(-\partial_\nu \varphi - h\sigma + h(n-1)\kappa - hS\big)ti_N u. \tag{4-12}$$

Therefore

$$-ih(ti_N Bu \mid th(\delta u))_{\partial M} = -2h\big(\partial_\nu \varphi_c(\partial_\nu \varphi + h\sigma - h(n-1)\kappa + hS)ti_N u \mid (\partial_\nu \varphi + h\sigma)ti_N u\big)_{\partial M}$$
$$-2h\big(h\nabla_{(\nabla\varphi_c)_\parallel}ti_N u \mid (\partial_\nu \varphi + h\sigma)ti_N u\big)_{\partial M}$$
$$-h^2\big(t\Delta\varphi_c i_N u \mid (\partial_\nu \varphi + h\sigma)ti_N u\big)_{\partial M}.$$

We can write this as

$$ih(Bu \mid \nu \wedge h\delta u)_{\partial M} = -2h\big(\partial_\nu \varphi |\partial_\nu \varphi|^2 ti_N u \mid ti_N u\big)_{\partial M} + R_2, \tag{4-13}$$

where R_2 satisfies the bound on R in (4-10).

Secondly,

$$-ih(Bu \mid i_\nu h du)_{\partial M} = ih((Bu)_\parallel \mid i_N h du)_{\partial M}$$
$$= ih(t(Bu)_\parallel \mid ti_N h du)_{\partial M}.$$

By Lemma 3.4,

$$ti_N h du = th\nabla_N u_\parallel + hStu_\parallel - hd'ti_N u,$$

so if $tu = 0$,

$$ti_N h du = th\nabla_N u_\parallel - hd'ti_N u.$$

Therefore

$$-ih(Bu \mid i_\nu h du)_{\partial M} = ih\big(tBu \mid th\nabla_N u_\parallel - hd'ti_N u\big)_{\partial M}.$$

Expanding B, this becomes

$$h\big(th\big(-2\partial_\nu \varphi_c \nabla_N u + 2\nabla_{(\nabla\varphi_c)_\parallel}u + (\Delta\varphi_c)u\big) \mid th\nabla_N u_\parallel - hd'ti_N u\big)_{\partial M}.$$

Since $tu = 0$, the last expression is equal to

$$-2h\big(\partial_\nu \varphi_c th\nabla_N u - th\nabla_{(\nabla\varphi_c)_\parallel}u \mid th\nabla_N u_\parallel - hd'ti_N u\big)_{\partial M}. \tag{4-14}$$

The

$$-2h(\partial_\nu \varphi_c th\nabla_N u_\parallel \mid -hd'ti_N u)_{\partial M}$$

part has the same type of bound as in (4-10), so

$$-ih(Bu \mid i_\nu h du)_{\partial M} = -2h(\partial_\nu \varphi_c th\nabla_N u_\parallel \mid th\nabla_N u_\parallel)_{\partial M} + R_3, \tag{4-15}$$

where R_3 has the same bound as in (4-10).

Thirdly,

$$ih(h\delta Bu \mid i_\nu u)_{\partial M} = ih(h(\delta Bu)_\parallel \mid i_N u)_{\partial M}$$
$$= -ih(ht(\delta Bu) \mid ti_N u)_{\partial M}.$$

By Lemma 3.5,

$$ht\delta Bu = h\delta' tBu + 2ih^2\nabla'_{(\nabla\varphi_c)_\parallel}t\nabla_N i_N u - 2ih^2\partial_\nu\varphi_c t\nabla_N\nabla_N i_N u$$

$$+ ih^2\big(2((n-1)\kappa - S)\partial_\nu\varphi_c + 2\partial_\nu^2\varphi_c + \triangle\varphi_c\big)t\nabla_N i_N u$$

$$+ 2ih^2(S - (n-1)\kappa)t\nabla_{(\nabla\varphi_c)_\parallel}i_N u + ih^2\big((S-(n-1)\kappa)\triangle\varphi_c + \nabla_N\triangle\varphi_c\big)ti_N u$$

$$+ 2ih^2 ti_N R(N, \nabla(\varphi_c)_\parallel)u_\perp + 2ih^2 t\nabla_{[(\nabla\varphi_c)_\parallel, N]}i_N u - 2ih^2 i_{s(\nabla\varphi_c)_\parallel}t\nabla_N u_\parallel.$$

The terms on the last two lines, when paired with $iht i_N u$, are bounded by (4-10).

Moreover, using the boundary conditions in the form of equation (4-12) on the

$$h^3\big(\big(2((n-1)\kappa - S)\partial_\nu\varphi_c + 2\partial_\nu^2\varphi_c + \triangle\varphi_c\big)t\nabla_N i_N u \,\big|\, ti_N u\big)_{\partial M}$$

term shows that this too is bounded by (4-10). Therefore we need only worry about the first three terms.

For the $-ih(h\delta' tBu \,|\, ti_N u)$ term, we can integrate by parts to get

$$-ih(tBu \,|\, hd' ti_N u)_{\partial M} = -2h\big(ht\nabla_{\nabla(\varphi_c)}u + \tfrac{1}{2}h\triangle\varphi_c tu \,\big|\, hd' ti_N u\big)_{\partial M}.$$

Since $tu = 0$, we get

$$ih(tBu \,|\, hd' ti_N u)_{\partial M} = 2h(ht\nabla_{\nabla(\varphi_c)}u \,|\, hd' ti_N u)_{\partial M}.$$

Now

$$t\nabla_{\nabla(\varphi_c)}u = t\nabla_{\nabla(\varphi_c)_\parallel}u_\perp + t\nabla_{\nabla(\varphi_c)_\perp}u_\parallel$$

since $tu = 0$. Therefore

$$\big|ih(tBu \,|\, hd' ti_N u)_{\partial M}\big| \le Kh^3\|\nabla' ti_N u\|_{\partial M}^2 + Kh^3\|u_\perp\|_{\partial M}^2 + Kh^3\|\nabla_N u_\parallel\|^2,$$

and so this term is bounded by (4-10).

For the $2h^3\big(\nabla'_{(\nabla\varphi_c)_\parallel}t\nabla_N i_N u \,\big|\, ti_N u\big)_{\partial M}$ term, we can use equation (4-12) to get

$$2h^3\big(\nabla'_{(\nabla\varphi_c)_\parallel}t\nabla_N i_N u \,\big|\, ti_N u\big)_{\partial M} = -2h^2\big(\nabla'_{(\nabla\varphi_c)_\parallel}(-\partial_\nu\varphi - h\sigma + h(n-1)\kappa - hS)ti_N u \,\big|\, ti_N u\big)_{\partial M}.$$

and then use Cauchy–Schwarz, so this term is bounded by (4-10) too. Therefore

$$-ih(h\delta Bu \,|\, i_\nu u)_{\partial M} = 2h^3\big(\partial_\nu\varphi_c t\nabla_N\nabla_N i_N u \,\big|\, ti_N u\big)_{\partial M} + R_1, \tag{4-16}$$

where R_1 is bounded by (4-10).

Finally,

$$2h((\partial_\nu\varphi_c)Au \,|\, u)_{\partial M} = 2h\big((\partial_\nu\varphi_c)Au \,\big|\, u_\perp\big)_{\partial M}$$

$$= 2h\big((\partial_\nu\varphi_c)(Au)_\perp \,\big|\, u_\perp\big)_{\partial M}$$

$$= 2h\big((\partial_\nu\varphi_c)ti_N Au \,\big|\, ti_N u\big)_{\partial M}$$

because of the boundary condition $tu = 0$. Now $A = -h^2\triangle - |d\varphi_c|^2$, so

$$2h\big((\partial_\nu\varphi_c)ti_N Au \,\big|\, ti_N u\big)_{\partial M} = -2h\big((\partial_\nu\varphi_c)h^2 ti_N\triangle u \,\big|\, ti_N u\big)_{\partial M} - 2h\big((\partial_\nu\varphi_c)|d\varphi_c|^2 ti_N u \,\big|\, ti_N u\big)_{\partial M}.$$

Using the Weitzenböck identity, we can write $-2h\big((\partial_\nu\varphi_c)h^2 t i_N \Delta u \mid t i_N u\big)_{\partial M}$ as

$$-2h\big((\partial_\nu\varphi_c)h^2 t i_N \tilde{\Delta} u \mid t i_N u\big)_{\partial M} + 2h\big((\partial_\nu\varphi_c)h^2 R t i_N u \mid t i_N u\big)_{\partial M}.$$

The second term is bounded by (4-10). For the first term, we can apply Lemma 3.6 to get

$$-2h\big((\partial_\nu\varphi_c)h^2 t \nabla_N \nabla_N i_N u \mid t i_N u\big)_{\partial M} - 2h\big((\partial_\nu\varphi_c)h^2 \tilde{\Delta}' t i_N u \mid t i_N u\big)_{\partial M} + h^3\big(\mathrm{tr}(s^2) i_N u - S_2 i_N u \mid t i_N u\big)_{\partial M},$$

where $S_2\omega(X_1,\ldots,X_{k-1}) := \sum_{l=1}^{k-1} \omega(\ldots, s^2 X_l, \ldots)$. The last term is bounded again by (4-10) and we can integrate by parts in the $\tilde{\Delta}'$ part to get something bounded by (4-10) as well. Therefore

$$2h\big((\partial_\nu\varphi_c) A u \mid u\big)_{\partial M} = -2h\big((\partial_\nu\varphi_c)|d\varphi_c|^2 t i_N u \mid t i_N u\big)_{\partial M} - 2h\big((\partial_\nu\varphi_c)h^2 t \nabla_N \nabla_N i_N u \mid t i_N u\big)_{\partial M} + R_4,$$

where R_4 is bounded by (4-10).

Now putting this together with (4-13), (4-15), and (4-16), we get that the boundary terms in (4-3) have the form

$$-2h\big(\partial_\nu\varphi|\partial_\nu\varphi|^2 t i_N u \mid t i_N u\big)_{\partial M} - 2h\big(\partial_\nu\varphi_c t h \nabla_N u_\parallel \mid t h \nabla_N u_\parallel\big)_{\partial M} + 2h^3\big(\partial_\nu\varphi_c t \nabla_N \nabla_N i_N u \mid t i_\nu u\big)_{\partial M}$$
$$-2h\big((\partial_\nu\varphi_c)|d\varphi_c|^2 t i_N u \mid t i_N u\big)_{\partial M} - 2h\big((\partial_\nu\varphi_c)h^2 t \nabla_N \nabla_N i_N u \mid t i_N u\big)_{\partial M} + R.$$

The $\pm 2h^3\big(\partial_\nu\varphi_c t \nabla_N \nabla_N i_N u \mid t i_\nu u\big)_{\partial M}$ terms cancel, leaving us with

$$-2h\big(\partial_\nu\varphi|\partial_\nu\varphi|^2 t i_N u \mid t i_N u\big)_{\partial M} - 2h\big(\partial_\nu\varphi_c t h \nabla_N u_\parallel \mid t h \nabla_N u_\parallel\big)_{\partial M} - 2h\big((\partial_\nu\varphi_c)|d\varphi_c|^2 t i_N u \mid t i_N u\big)_{\partial M} + R.$$

We can replace φ_c by φ and incorporate the error into R, without affecting the bound on R, to get

$$-2h\big(\partial_\nu\varphi|\partial_\nu\varphi|^2 t i_N u \mid t i_N u\big)_{\partial M} - 2h\big(\partial_\nu\varphi t h \nabla_N u_\parallel \mid t h \nabla_N u_\parallel\big)_{\partial M} - 2h\big(\partial_\nu\varphi|d\varphi|^2 t i_N u \mid t i_N u\big)_{\partial M} + R$$

and the proposition follows. $\qquad\square$

5. The 0-form case

We will now prove Theorem 2.5 in the 0-form case. In the case where (M, g) is a domain in Euclidean space, Theorem 2.5 for 0-forms is the Carleman estimate given in [Chung 2015, Theorem 1.3]. In this section we will deal with the added complication of being on a CTA manifold, rather than in Euclidean space. Most of the ideas are from [Chung 2015] with necessary modifications added to adapt to the manifold case.

If u is a zero form, then $i_N u = 0$, so $u_\perp = 0$ and $u = u_\parallel$. Theorem 2.5 reduces to the estimate

$$\|(-\Delta_\varphi + h^2 Q)u\|_{L^2(M)} \gtrsim h\|u\|_{H^1(M)} + h^{\frac{1}{2}}\|u\|_{H^1(\Gamma_+^c)}, \tag{5-1}$$

where $Q \in L^\infty(M)$ and $0 < h < h_0$, for functions $u \in H^2(M)$ with $u|_{\Gamma_+} = 0$ to first order and $h\partial_\nu(e^{-\frac{\varphi}{h}}u) = h\sigma e^{-\frac{\varphi}{h}}u$ on Γ_+^c. By arguing as in the beginning of Section 6 below, the estimate (5-1) will be a consequence of the following proposition.

Proposition 5.1. *Suppose u is a function in $H^2(M)$ which satisfies the following boundary conditions*:

$$u, \partial_\nu u = 0 \qquad \qquad on\ \Gamma_+,$$
$$h\partial_\nu(e^{-\frac{\varphi}{h}}u) = h\sigma e^{-\frac{\varphi}{h}}u \quad on\ \Gamma_+^c \tag{5-2}$$

for some smooth function σ independent of h.

   Then

$$h^{\frac{1}{2}}\|h\nabla'u\|_{L^2(\Gamma_+^c)} \lesssim \|\Delta_{\varphi_c}u\|_{L^2(M)} + h\|u\|_{H^1(M)} + h^{\frac{3}{2}}\|u\|_{L^2(\Gamma_+^c)}.$$

   We will prove this proposition in the case where the metric g has the form $g = e \oplus g_0$. However, if g were of the form $g = c(e \oplus g_0)$, we could write

$$\|\Delta_{\varphi_c}u\|_{L^2(M)} = \|h^2 e^{\frac{\varphi_c}{h}}\Delta_{c(e\oplus g_0)}e^{-\frac{\varphi_c}{h}}u\|_{L^2(M)}$$
$$\gtrsim \|h^2 e^{\frac{\varphi_c}{h}}\Delta_{e\oplus g_0}e^{-\frac{\varphi_c}{h}}u\|_{L^2(M)} - h\|u\|_{H^1(M)}. \tag{5-3}$$

Therefore the proposition remains true even in the case when the conformal factor is not constant. More generally, the proofs of the Carleman estimates work for any smooth conformal factor, and thus as noted earlier, the Carleman estimates hold on CTA manifolds in general.

The operators. Here we introduce the operators we will use in the proof of Proposition 5.1. Similar operators are found in [Chung 2014; 2015]. Suppose $F(\xi)$ is a complex-valued function on $\mathbb{R}^{n-1}$, with the properties that $|F(\xi)|, \operatorname{Re}F(\xi) \simeq 1 + |\xi|$. Fix coordinates (x_1, x') on $\mathbb{R}^n$, and define $\mathbb{R}^n_+$ to be the subset of $\mathbb{R}^n$ with $x_1 > 0$. Define $\mathcal{S}(\mathbb{R}^n_+)$ as the set of restrictions to $\mathbb{R}^n_+$ of Schwartz functions on $\mathbb{R}^n$. Finally, if $u \in \mathcal{S}(\mathbb{R}^n_+)$, then define $\hat{u}(x_1, \xi)$ to be the semiclassical Fourier transform of u in the x' variables only.

   Now for $u \in \mathcal{S}(\mathbb{R}^n_+)$, define J by

$$\widehat{Ju}(x_1, \xi) = (F(\xi) + h\partial_1)\hat{u}(x_1, \xi).$$

This has adjoint J^* defined by

$$\widehat{J^*u}(x_1, \xi) = (\overline{F}(\xi) - h\partial_1)\hat{u}(x_1, \xi).$$

These operators have right inverses given by

$$\widehat{J^{-1}u} = \frac{1}{h}\int_0^{x_1} \hat{u}(t, \xi)e^{F(\xi)\frac{t-x_1}{h}}\,dt,$$
$$\widehat{J^{*-1}u} = \frac{1}{h}\int_{x_1}^{\infty} \hat{u}(t, \xi)e^{\overline{F}(\xi)\frac{x_1-t}{h}}\,dt.$$

Now we have the following boundedness result, given in [Chung 2015].

Lemma 5.2. *The operators J, J^*, J^{-1}, and J^{*-1}, initially defined on $\mathcal{S}(\mathbb{R}^n_+)$, extend to bounded operators*

$$J, J^* : H^1(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n_+),$$
$$J^{-1}, J^{*-1} : L^2(\mathbb{R}^n_+) \to H^1(\mathbb{R}^n_+).$$

Moreover, these extensions for J^* and J^{*-1} are isomorphisms.

Note that similar mapping properties hold between $H^1(\mathbb{R}^n_+)$ and $H^2(\mathbb{R}^n_+)$, by the same reasoning. We'll record the other operator fact from [Chung 2015] here, too.

Let $m, k \in \mathbb{Z}$, with $m, k \geq 0$. Suppose $a(x, \xi, y)$ are smooth functions on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$ that satisfy the bounds

$$|\partial_x^\beta \partial_\xi^\alpha \partial_y^j a(x, \xi, y)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}$$

for all multiindices α and β, and for $0 \leq j \leq k$. In other words, each $\partial_y^j a(x, \xi, y)$ is a symbol on $\mathbb{R}^{n-1}$ of order m, with bounds uniform in y, for $0 \leq j \leq k$. Then we can define an operator A on Schwartz functions in $\mathbb{R}^n$ by applying the pseudodifferential operator on $\mathbb{R}^{n-1}$ with symbol $a(x, \xi, y)$, defined by the Kohn–Nirenberg quantization, to $f(x, y)$ for each fixed y.

Lemma 5.3. *If A is as above, then A extends to a bounded operator from $H^{k+m}(\mathbb{R}^n)$ to $H^k(\mathbb{R}^n)$.*

The graph case. Suppose $f : M_0 \to \mathbb{R}$ is smooth. In this section, we'll examine the case where M lies in the set $\{x_1 \geq f(x')\}$ and Γ^c_+ lies in the graph $\{x_1 = f(x')\}$. For this section we'll make two additional assumptions on f and M_0.

First, we'll assume g_0 is nearly constant; that is, there exists a choice of coordinates on the subset $P(M)$ which consists of the projection of M onto M_0 such that when represented in these coordinates,

$$|g_0 - I| \leq \delta$$

on $P(M)$, where δ is a positive constant to be chosen later.

Second, we'll assume f is such that $\nabla_{g_0} f$ is nearly constant on $P(M)$; that is, there exists a constant vector field K on TM_0 such that

$$|\nabla_{g_0} f - K|_{g_0} \leq \delta,$$

where δ is the same constant from above. The choice of δ will depend ultimately only on K. In the next subsection we'll see how to remove these two assumptions.

Now we can do the change of variables $(x_1, x') \mapsto (x_1 - f(x'), x')$. Define $\tilde{M}'$ and $\tilde{\Gamma}'_+$ to be the images of M and Γ_+ respectively under this map. Note that $\{x_1 \geq f(x')\}$ maps to $(0, \infty) \times M_0$, and Γ^c_+ maps to a subset of $0 \times M_0$. Observe that in the new coordinates, $\varphi(x) = x_1 + f(x')$.

Now it suffices to prove the following proposition.

Proposition 5.4. *Suppose $w \in H^2(\tilde{M}')$, and*

$$w, \partial_\nu w = 0 \quad on \ \tilde{\Gamma}'_+,$$

$$h\partial_y w|_{\tilde{\Gamma}'^c_+} = \frac{w + \nabla_{g_0} f \cdot h\nabla_{g_0} w - h\sigma w}{1 + |\nabla_{g_0} f|^2}, \tag{5-4}$$

where σ is smooth and bounded on $\tilde{M}'$. Then

$$h^{\frac{1}{2}} \|h\nabla_{g_0} w\|_{L^2(\tilde{\Gamma}'^c_+)} \lesssim \|\tilde{\mathcal{L}}'_{\varphi,\varepsilon} w\|_{L^2(\tilde{M}')} + h\|w\|_{H^1(\tilde{M}')} + h^{\frac{3}{2}} \|w\|_{L^2(\tilde{\Gamma}'^c_+)},$$

where

$$\tilde{\mathcal{L}}'_{\varphi,\varepsilon} = (1 + |\nabla_{g_0} f|^2)h^2\partial_1^2 - 2(\alpha + \nabla_{g_0} f \cdot h\nabla_{g_0})h\partial_1 + \alpha^2 + h^2\triangle_{g_0}$$

and $\alpha = 1 + \frac{h}{\varepsilon}(x_1 + f(x'))$. Note that on $\tilde{M}'$, we know α is very close to 1.

This proposition implies Proposition 5.1 in the graph case described above.

Proof of Proposition 5.1 in the graph case. Suppose $u \in H^2(M)$, and u satisfies (5-2). Let w be the function on $\tilde{M}$ defined by $w(x_1, x') = u(x_1 + f(x'), x')$. Then $w \in H^2(\tilde{M}')$, and w satisfies (5-4). Therefore by Proposition 5.4,

$$h^{\frac{1}{2}} \|h\nabla' w\|_{L^2(\tilde{\Gamma}'^c_+)} \lesssim \|\tilde{\mathcal{L}}'_{\varphi,\varepsilon} w\|_{L^2(\tilde{M}')} + h\|w\|_{H^1(\tilde{M}')} + h^{\frac{3}{2}} \|w\|_{L^2(\tilde{\Gamma}'^c_+)}.$$

Now by a change of variables,

$$\|u\|_{L^2(\Gamma^c_+)} \simeq \|w\|_{L^2(\tilde{\Gamma}'^c_+)},$$

$$\|u\|_{H^1(M)} \simeq \|w\|_{H^1(\tilde{M}')},$$

and

$$\|h\nabla' u\|_{L^2(\Gamma^c_+)} \simeq \|h\nabla_{g_0} w\|_{L^2(\tilde{\Gamma}'^c_+)}.$$

Moreover,

$$(\tilde{\mathcal{L}}'_{\varphi,\varepsilon} w)(x_1 - f(x'), x') = \mathcal{L}_{\varphi,\varepsilon}(u(x_1, x')) + hE_1 u(x_1, x'),$$

where E_1 is a first-order semiclassical differential operator. Therefore by a change of variables,

$$\|\tilde{\mathcal{L}}'_{\varphi,\varepsilon} w\|_{L^2(\tilde{M}')} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} u\|_{L^2(M)} + h\|u\|_{H^1(M)}.$$

Putting this all together gives

$$h^{\frac{1}{2}} \|h\nabla_{g_0} u\|_{L^2(\Gamma^c_+)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} u\|_{L^2(M)} + h\|u\|_{H^1(M)} + h^{\frac{3}{2}} \|u\|_{L^2(\Gamma^c_+)}. \qquad \square$$

We can do a second change of variables to move to Euclidean space. By our assumption on M_0, we can choose coordinates on $P(\tilde{M}') = P(M)$ such that

$$|g_0 - I| \le \delta.$$

Now we have a change of variables giving a map from $P(\tilde{M}')$ to a subset of $\mathbb{R}^{n-1}$, and hence a map from $\tilde{M}'$ to a subset of $\mathbb{R}^n_+$, where the image of $\tilde{\Gamma}'_+$ lies in the plane $x_1 = 0$. Let $\tilde{M}$ and $\tilde{\Gamma}_+$ be the images of $\tilde{M}'$ and $\tilde{\Gamma}'_+$ respectively under this map. We'll use the notation (x_1, x') to describe points in $\mathbb{R}^n_+$, where now x' ranges over $\mathbb{R}^{n-1}$. Now it suffices to prove the following proposition.

Proposition 5.5. *Suppose $w \in H^2(\tilde{M})$, and*

$$w, \partial_\nu w = 0 \quad \text{on } \tilde{\Gamma}_+,$$

$$h\partial_y w|_{\tilde{\Gamma}^c_+} = \frac{w + \beta \cdot h\nabla_x w - h\sigma w}{1 + |\gamma|^2}, \tag{5-5}$$

where σ is smooth and bounded on $\tilde{M}$, and β and γ are a vector-valued and scalar-valued function, respectively, which coincide with the coordinate representations of $\nabla_{g_0} f$ and $|\nabla_{g_0} f|_{g_0}$. Then

$$h^{\frac{1}{2}} \|h\nabla_{x'} w\|_{L^2(\tilde{\Gamma}^c_+)} \lesssim \|\tilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\tilde{M})} + h\|w\|_{H^1(\tilde{M})} + h^{\frac{3}{2}} \|w\|_{L^2(\tilde{\Gamma}^c_+)},$$

where

$$\tilde{\mathcal{L}}_{\varphi,\varepsilon} = (1 + |\gamma|^2)h^2\partial_1^2 - 2(\alpha + \beta \cdot h\nabla_x)h\partial_1 + \alpha^2 + h^2\mathcal{L},$$

and $\mathcal{L}$ is the second-order differential operator in the x'-variables given by

$$\mathcal{L} = g_0^{ij} \partial_i \partial_j.$$

Proposition 5.4 can be obtained from Proposition 5.5 in the same manner as before, with errors from the change of variables being absorbed into the appropriate terms. Therefore it suffices to prove Proposition 5.5.

To do this, we'll split w into small and large frequency parts, using a Fourier transform. Recall that we are assuming

$$|\nabla_{g_0} f - K|_{g_0} \leq \delta.$$

Translating down to $\widetilde{M}$, and recalling that g_0 is nearly the identity, we get that there is a constant vector field $\widetilde{K}$ on $\widetilde{M}$ such that

$$|\beta - \widetilde{K}| \leq C_\delta \quad \text{and} \quad \left| \gamma - |\widetilde{K}| \right| \leq C_\delta,$$

where C_δ goes to zero as δ goes to zero. Now choose $m_2 > m_1 > 0$ and μ_1 and μ_2 such that

$$\frac{|\widetilde{K}|}{\sqrt{1 + |\widetilde{K}|^2}} < \mu_1 < \mu_2 < \frac{1}{2} + \frac{|\widetilde{K}|}{2\sqrt{1 + |\widetilde{K}|^2}} < 1.$$

The eventual choice of μ_j and m_j will depend only on $\widetilde{K}$.

Define $\rho \in C_0^\infty(\mathbb{R}^n)$ such that $\rho(\xi) = 1$ if $|\xi| < \mu_1$ and $|\widetilde{K} \cdot \xi| < m_1$, and $\rho(\xi) = 0$ if $|\xi| > \mu_2$ or $|\widetilde{K} \cdot \xi| > m_2$.

Now suppose $w \in C^\infty(\widetilde{M})$ such that $w \equiv 0$ in a neighbourhood of $\widetilde{\Gamma}_+$, and w satisfies (5-5). We can extend w by zero to the rest of $\mathbb{R}_+^n$. Then $w \in \mathcal{S}(\mathbb{R}_+^n)$, and we can write our desired estimate as

$$h^{\frac{1}{2}} \|w\|_{\dot{H}^1(\partial \mathbb{R}_+^n)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi, \varepsilon} w\|_{L^2(\mathbb{R}_+^n)} + h \|w\|_{H^1(\mathbb{R}_+^n)} + h^{\frac{3}{2}} \|w\|_{L^2(\partial \mathbb{R}_+^n)}.$$

Recall that $\hat{w}(x_1, \xi)$ is the semiclassical Fourier transform of w in the x'-directions, and define w_s and w_ℓ by $\hat{w}_s = \rho \hat{w}$ and $\hat{w}_\ell = (1 - \rho)\hat{w}$, so $w = w_s + w_\ell$.

Now we can address each of these parts separately.

Proposition 5.6. *Suppose w is as above. There exist choices of m_1, m_2, μ_1, and μ_2, depending only on $\widetilde{K}$, such that if δ is small enough,*

$$h^{\frac{1}{2}} \|w_s\|_{\dot{H}^1(\partial \mathbb{R}_+^n)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi, \varepsilon} w\|_{L^2(\mathbb{R}_+^n)} + h \|w\|_{H^1(\mathbb{R}_+^n)} + h^{\frac{3}{2}} \|w\|_{L^2(\partial \mathbb{R}_+^n)}.$$

Before proceeding to the proof, let's make some definitions. If $V \in \mathbb{R}^{n-1}$ and $a \in \mathbb{R}$, define $A_\pm(a, V, \xi)$ by

$$A_\pm(a, V, \xi) = \frac{1 + iV \cdot \xi \pm \sqrt{(1 + iV \cdot \xi)^2 - (1 + |a|^2)(1 - |\xi|^2)}}{1 + |a|^2}.$$

In other words, $A_\pm(a, V, \xi)$ are defined to be the roots of the polynomial

$$(1 + |a|^2)X^2 - 2(1 + iV \cdot \xi)X + (1 - |\xi|^2).$$

In the definition, we'll choose the branch of the square root which has nonnegative real part, so the branch cut occurs on the negative real axis.

Proof. Now consider the behaviour of $A_\pm(|\widetilde{K}|, \widetilde{K}, \xi)$ on the support of ρ, or equivalently, on the support of $\hat{w}_s$. If $\eta > 0$, we can choose μ_2 such that on the support of $\hat{w}_s$,

$$1 - (1 + |\widetilde{K}|^2)(1 - |\xi|^2) < \eta.$$

Then on the support of $\hat{w}_s$, the expression

$$(1 + i\,\widetilde{K}\cdot\xi)^2 - (1 + |\widetilde{K}|^2)(1 - |\xi|^2)$$

has real part confined to the interval $[-\widetilde{K}^2 - m_2^2, \eta + m_2^2]$, and imaginary part confined to the interval $[-2m_2, 2m_2]$. Therefore, by correct choice of η and m_2, we can ensure

$$\mathrm{Re}\,A_\pm(|\widetilde{K}|, \widetilde{K}, \xi) > \frac{1}{2(1 + |\widetilde{K}|^2)}$$

on the support of $\hat{w}_s$. This allows us to fix the choice of μ_1, μ_2, m_1, and m_2. Note that the choices depend only on $\widetilde{K}$, as promised.

The bounds on $A_\pm(|\widetilde{K}|, \widetilde{K}, \xi)$ allow us to choose $F_\pm$ so that $F_\pm = A_\pm(|\widetilde{K}|, \widetilde{K}, \xi)$ on the support of $\hat{w}_s$, and $\mathrm{Re}\,F_\pm, |F_\pm| \simeq 1 + |\xi|$ on $\mathbb{R}^n$, with constant depending only on K. Therefore F_+ and F_- both satisfy the conditions on F in Section 2. If T_ψ represents the operator with Fourier multiplier ψ (in the x'-variables), then it follows that the operators $h\partial_y - T_{F_+}$ and $h\partial_y - T_{F_-}$ both have the properties of J^* in that section.

Up until now, the operator $\widetilde{\mathcal{L}}_{\varphi,\varepsilon}$ has only been applied to functions supported in $\widetilde{M}$. However, we can extend the coefficients of $\widetilde{\mathcal{L}}_{\varphi,\varepsilon}$ to $\mathbb{R}^n_+$ while retaining the $|\beta - \widetilde{K}| < C_\delta$ and $|\gamma - |\widetilde{K}|| \le C_\delta$ conditions. Then

$$\begin{aligned}
\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} &= \left\|\left((1+|\gamma|^2)h^2\partial_y^2 - 2(\alpha + \beta\cdot h\nabla_x)h\partial_y + \alpha^2 + h^2\mathcal{L}\right)w_s\right\|_{L^2(\mathbb{R}^n_+)} \\
&\ge \left\|\left((1+|\widetilde{K}|^2)h^2\partial_y^2 - 2(1 + \widetilde{K}\cdot h\nabla_x)h\partial_y + 1 + h^2\triangle_{x'}\right)w_s\right\|_{L^2(\mathbb{R}^n_+)} - C_\delta\|w_s\|_{H^2(\mathbb{R}^n_+)}
\end{aligned}$$

for sufficiently small h and some C_δ which goes to zero as δ goes to zero. Meanwhile,

$$(1 + |\widetilde{K}|^2)(h\partial_y - T_{F_+})(h\partial_y - T_{F_-})w_s = (1 + |\widetilde{K}|^2)(h^2\partial_y^2 - T_{F_+ + F_-}h\partial_y + T_{F_+ F_-})w_s.$$

Since $F_\pm = A_\pm(\widetilde{K}, K, \xi)$ on the support of $\hat{w}_s$, this can be written as

$$(1 + |\widetilde{K}|^2)(h^2\partial_y^2 - T_{A_+ + A_-}h\partial_y + T_{A_+ A_-})w_s = ((1 + |\widetilde{K}|^2)h^2\partial_y^2 - 2(1 + \widetilde{K}\cdot h\nabla_x)h\partial_y + 1 + h^2\triangle_x)w_s.$$

Therefore

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} \ge \left\|(h\partial_y - T_{F_+})(h\partial_y - T_{F_-})w_s\right\|_{L^2(\mathbb{R}^n_+)} - C_\delta\|w_s\|_{H^2(\mathbb{R}^n_+)}.$$

Now by the boundedness properties,

$$\left\|(h\partial_y - T_{F_+})(h\partial_y - T_{F_-})w_s\right\|_{L^2(\mathbb{R}^n_+)} \simeq \|w_s\|_{H^2(\mathbb{R}^n_+)},$$

so for small enough δ,

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} \gtrsim \|w_s\|_{H^2(\mathbb{R}^n_+)}.$$

Then by the semiclassical trace formula,

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} \gtrsim h^{\frac{1}{2}} \|w_s\|_{\dot{H}^1(\partial\mathbb{R}^n_+)}.$$

Finally, note that

$$
\begin{aligned}
\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} &= \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} T_\rho w\|_{L^2(\mathbb{R}^n_+)} \\
&\lesssim \|(1+|\gamma|^2)^{-1} \widetilde{\mathcal{L}}_{\varphi,\varepsilon} T_\rho w\|_{L^2(\mathbb{R}^n_+)} \\
&\lesssim \|T_\rho (1+|\gamma|^2)^{-1} \widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + \|h E_1 w\|_{L^2(\mathbb{R}^n_+)},
\end{aligned}
$$

where $h E_1$ comes from the commutator of T_ρ and $(1+|\gamma|^2)^{-1} \widetilde{\mathcal{L}}_{\varphi,\varepsilon}$. By Lemma 5.3, E_1 is bounded from $H^1(\mathbb{R}^n_+)$ to $L^2(\mathbb{R}^n_+)$, so

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w_s\|_{L^2(\mathbb{R}^n_+)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^n_+)}.$$

Therefore

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^n_+)} \gtrsim h^{\frac{1}{2}} \|w_s\|_{\dot{H}^1(\partial\mathbb{R}^n_+)}$$

as desired. $\qquad\square$

Now we have to deal with the large frequency term.

Proposition 5.7. *Suppose w is the extension by zero to $\mathbb{R}^n_+$ of a function in $C^\infty(\widetilde{M})$ which is 0 in a neighbourhood of $\widetilde{\Gamma}_+$, and satisfies (5-5), and let w_ℓ be defined as above. Then if δ is small enough,*

$$h^{\frac{1}{2}} \|w_\ell\|_{\dot{H}^1(\partial\mathbb{R}^n_+)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}^n_+)} + h\|w\|_{H^1(\mathbb{R}^{n+1}_+)} + h^{\frac{3}{2}} \|w\|_{L^2(\partial\mathbb{R}^n_+)}.$$

Proof. Suppose $V \in \mathbb{R}^n$. Recall that we defined

$$A_\pm(a, V, \xi) = \frac{1 + iV\cdot\xi \pm \sqrt{(1 + iV\cdot\xi)^2 - (1+|a|^2)(1-|\xi|^2)}}{1+|a|^2},$$

so $A_\pm(a, V, \xi)$ are roots of the polynomial

$$(1+|a|^2)X^2 - 2(1 + iV\cdot\xi)X + (1 - |\xi|^2).$$

Now let's define

$$A^\varepsilon_\pm(a, V, \xi) = \frac{\alpha + iV\cdot\xi \pm \sqrt{(\alpha + iV\cdot\xi)^2 - (1+|a|^2)(\alpha^2 - g_0^{ij}\xi_i\xi_j)}}{1+|a|^2},$$

so $A^\varepsilon_\pm(V, \xi)$ are the roots of the polynomial

$$(1+|a|^2)X^2 - 2(\alpha + iV\cdot\xi)X + (\alpha^2 - g_0^{ij}\xi_i\xi_j).$$

(Recall that α is defined by $\alpha = 1 + \frac{h}{\varepsilon}(x_1 + f(x'))$.) Again we'll use the branch of the square root with nonnegative real part.

Now set $\zeta \in C_0^\infty(\mathbb{R}^{n-1})$ to be a smooth cutoff function such that $\zeta = 1$ if

$$|\widetilde{K}\cdot\xi| < \frac{1}{2}m_1 \quad \text{and} \quad |\xi| < \frac{1}{2}\frac{|\widetilde{K}|}{\sqrt{1+|\widetilde{K}|^2}} + \frac{1}{2}\mu_1,$$

and $\zeta = 0$ if $|\widetilde{K}\cdot\xi| \geq m_1$ or $|\xi| \geq \mu_1$.

Now define

$$G_\pm(a, V, \xi) = (1-\zeta)A_\pm(a, V, \xi) + \zeta$$

and

$$G_\pm^\varepsilon(a, V, \xi) = (1-\zeta)A_\pm^\varepsilon(a, V, \xi) + \zeta.$$

Consider the singular support of $A_\pm^\varepsilon(\gamma, \beta, \xi)$. These are smooth as functions of x and ξ except when the argument of the square root falls on the nonpositive real axis. This occurs when $\beta \cdot \xi = 0$ and

$$g_0^{ij}\xi_i\xi_j \leq \frac{\alpha^2|\gamma|^2}{1+|\gamma|^2}.$$

Now for δ sufficiently small, depending on $\widetilde{K}$, this does not occur on the support of $1 - \zeta$. Therefore

$$G_\pm^\varepsilon(\gamma, \beta, \xi) = (1-\zeta)A_\pm^\varepsilon(\gamma, \beta, \xi) + \zeta$$

are smooth, and one can check that they are symbols of first order on $\mathbb{R}^n$.

Then by properties of pseudodifferential operators,

$$(1+|\gamma|^2)(h\partial_y - T_{G_+^\varepsilon(\gamma,\beta,\xi)})(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})$$
$$= (1+|\gamma|^2)\big(h^2\partial_y^2 - T_{G_+^\varepsilon(\gamma,\beta,\xi)+G_-^\varepsilon(\gamma,\beta,\xi)}h\partial_y + T_{G_+^\varepsilon(\gamma,\beta,\xi)G_-^\varepsilon(\gamma,\beta,\xi)}\big) + hE_1,$$

where E_1 is bounded from $H^1(\mathbb{R}_+^{n+1})$ to $L^2(\mathbb{R}_+^{n+1})$. This last line can be written out as

$$(1+|\gamma|^2)h^2\partial_y^2 - 2(\alpha + \beta\cdot h\nabla_x)h\partial_y T_{1-\zeta}T_{1+\zeta} + (\alpha + h^2\mathcal{L})T_{(1-\zeta)^2} + hE_1 + T_{\zeta^2} - 2h\partial_y T_\zeta$$

by modifying E_1 as necessary. Now $T_\zeta w_\ell = 0$, so

$$(1+|\gamma|^2)(h\partial_y - T_{G_+^\varepsilon(\gamma,\beta,\xi)})(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell = \widetilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell - hE_1w_\ell.$$

Therefore

$$\|\widetilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}_+^{n+1})} \gtrsim \|(h\partial_y - T_{G_+^\varepsilon(\gamma,\beta,\xi)})(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{L^2(\mathbb{R}_+^{n+1})} - h\|w_\ell\|_{H^1(\mathbb{R}_+^{n+1})}.$$

Now

$$G_+^\varepsilon(\gamma, \beta, \xi) = G_+(|\widetilde{K}|, \widetilde{K}, \xi) + \big(G_+^\varepsilon(\gamma, \beta, \xi) - G_+(|\widetilde{K}|, \widetilde{K}, \xi)\big),$$

and

$$T_{G_+^\varepsilon(\gamma,\beta,\xi)-G_+(|\widetilde{K}|,\widetilde{K},\xi)}$$

involves multiplication by functions bounded by $O(\delta)$, so

$$\|T_{G_+^\varepsilon(\gamma,\beta,\xi)-G_+(|\tilde{K}|,\tilde{K},\xi)}v\|_{L^2(\mathbb{R}_+^{n+1})} \lesssim \delta\|v\|_{H^1(\mathbb{R}_+^{n+1})}.$$

Therefore

$$\|\tilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}_+^{n+1})} \gtrsim \|(h\partial_y - T_{G_+(|\tilde{K}|,\tilde{K},\xi)})(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{L^2(\mathbb{R}_+^{n+1})}$$
$$- h\|w_\ell\|_{H^1(\mathbb{R}_+^{n+1})} - \delta\|(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{H^1(\mathbb{R}_+^{n+1})}.$$

Now we can check that $G_+(|\tilde{K}|,\tilde{K},\xi)$ satisfies the necessary properties of F from this section, so

$$\|\tilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}_+^{n+1})} \gtrsim \|(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{H^1(\mathbb{R}_+^{n+1})} - h\|w_\ell\|_{H^1(\mathbb{R}_+^{n+1})}$$
$$- \delta\|(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{H^1(\mathbb{R}_+^{n+1})}.$$

Then for small enough δ,

$$\|\tilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}_+^{n+1})} \gtrsim \|(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{H^1(\mathbb{R}_+^{n+1})} - h\|w_\ell\|_{H^1(\mathbb{R}_+^{n+1})}$$
$$\gtrsim h^{\frac{1}{2}}\|(h\partial_y - T_{G_-^\varepsilon(\gamma,\beta,\xi)})w_\ell\|_{L^2(\mathbb{R}_0^n)} - h\|w_\ell\|_{H^1(\mathbb{R}_+^{n+1})}.$$

Now by (5-5),

$$h\partial_y w = \frac{w + \beta \cdot h\nabla_x w + h\sigma w}{1 + |\gamma|^2}$$

on $\partial\mathbb{R}_+^n$, so

$$h\partial_y w_\ell = \frac{w_\ell + \beta \cdot \nabla_x w_\ell}{1 + |\gamma|^2} + hE_0 w$$

on $\partial\mathbb{R}_+^n$, where E_0 is bounded from $L^2(\mathbb{R}^{n-1})$ to $L^2(\mathbb{R}^{n-1})$. Therefore

$$\|\tilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}_+^n)} \gtrsim h^{\frac{1}{2}}\left\|\frac{w_\ell + \beta \cdot \nabla_x w_\ell}{1 + |\gamma|^2} - T_{G_-^\varepsilon(\gamma,\beta,\xi)}w_\ell\right\|_{L^2(\partial\mathbb{R}_+^n)} - h\|w_\ell\|_{H^1(\mathbb{R}_+^n)} - h^{\frac{3}{2}}\|w\|_{L^2(\partial\mathbb{R}^n)}$$
$$\gtrsim h^{\frac{1}{2}}\|w_\ell\|_{\dot{H}^1(\partial\mathbb{R}_+^n)} - h\|w_\ell\|_{H^1(\mathbb{R}_+^n)} - h^{\frac{3}{2}}\|w\|_{L^2(\partial\mathbb{R}_+^n)}.$$

Now

$$\|w_\ell\|_{H^1(\mathbb{R}_+^n)} \lesssim \|w\|_{H^1(\mathbb{R}_+^n)}$$

and

$$\|\tilde{\mathcal{L}}_{\varphi,\varepsilon}w_\ell\|_{L^2(\mathbb{R}_+^n)} \lesssim \|\tilde{\mathcal{L}}_{\varphi,\varepsilon}w\|_{L^2(\mathbb{R}_+^n)} + h\|w\|_{H^1(\mathbb{R}_+^n)}.$$

Therefore

$$\|\tilde{\mathcal{L}}_{\varphi,\varepsilon}w\|_{L^2(\mathbb{R}_+^n)} + h\|w\|_{H^1(\mathbb{R}_+^n)} + h^{\frac{3}{2}}\|w\|_{L^2(\partial\mathbb{R}_+^n)} \gtrsim h^{\frac{1}{2}}\|w_\ell\|_{\dot{H}^1(\partial\mathbb{R}_+^n)}$$

as desired. $\qquad\square$

Now combing the results of Propositions 5.6 and 5.7 gives

$$h^{\frac{1}{2}}\|w_\ell\|_{\dot{H}^1(\partial\mathbb{R}_+^n)} + h^{\frac{1}{2}}\|w_s\|_{\dot{H}^1(\partial\mathbb{R}_+^n)} \lesssim \|\tilde{\mathcal{L}}_{\varphi,\varepsilon}w\|_{L^2(\mathbb{R}_+^n)} + h\|w\|_{H^1(\mathbb{R}_+^{n+1})} + h^{\frac{3}{2}}\|w\|_{L^2(\partial\mathbb{R}_+^n)}.$$

Since $w = w_s + w_\ell$, we get

$$h^{\frac{1}{2}} \|w\|_{\dot{H}^1(\partial \mathbb{R}_+^n)} \lesssim \|\widetilde{\mathcal{L}}_{\varphi,\varepsilon} w\|_{L^2(\mathbb{R}_+^n)} + h\|w\|_{H^1(\mathbb{R}_+^{n+1})} + h^{\frac{3}{2}} \|w\|_{L^2(\partial \mathbb{R}_+^n)}$$

for $w \in C^\infty(\widetilde{M})$ such that $w \equiv 0$ in a neighbourhood of $\widetilde{\Gamma}_+$, and w satisfies (5-5). A density argument now proves Proposition 5.5, and hence Proposition 5.1, at least under the assumptions on g_0 and f made at the beginning of this section.

Finishing the proof. Now we need to remove the graph conditions on Γ_+^c, and the conditions on the metric g_0. Since Γ_+ is a neighbourhood of ∂M_+, in a small enough neighbourhood U around any point p on Γ_+^c, we know Γ_+^c coincides locally with a subset of a graph of the form $x_1 = f(x')$, with $M \cap U$ lying in the set $x_1 > f(x')$. Moreover, for any $\delta > 0$, if $\nabla_{g_0} f(p) = K$, then in some small neighbourhood of p, we have $|\nabla_{g_0} f - K|_{g_0} < \delta$. Additionally, since we can choose coordinates at p such that $g_0 = I$ in those coordinates, for any $\delta > 0$ we can ensure that there are coordinates such that $|g_0 - I| \leq \delta$ in a small neighbourhood of p. We can choose δ to be small enough for Proposition 5.1 to hold, by the proof in the previous subsection.

Now we can let U_j be open sets in M such that $\{U_1, \ldots, U_m\}$ is a finite open cover of M such that each $M \cap U_j$ has smooth boundary, and each $\Gamma_+^c \cap U_j$ is represented as a graph of the form $x_1 = f_j(x')$, with $|\nabla_{g_0} f_j - K_j|_{g_0} < \delta_j$, and there is a choice of coordinates on the projection of $M \cap U_j$ in which $|g_0 - I| \leq \delta_j$, where δ_j are small enough for

$$h^{\frac{1}{2}} \|h\nabla_t v_j\|_{L^2(\Gamma_+^c \cap U_j)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} v_j\|_{L^2(M \cap U_j)} + h\|v_j\|_{H^1(M \cap U_j)} + h^{\frac{3}{2}} \|v_j\|_{L^2(\Gamma_+^c \cap U_j)}$$

to hold for all $v_j \in H^2(M \cap U_j)$ such that

$$\begin{aligned}
v_j, \partial_\nu v_j &= 0 && \text{on } \partial(U_j \cap M) \setminus \Gamma_+^c, \\
h\partial_\nu(e^{-\frac{\varphi}{h}} v_j) &= h\sigma e^{-\frac{\varphi}{h}} v_j && \text{on } \Gamma_+^c \cap U_j.
\end{aligned} \tag{5-6}$$

Without loss of generality we may assume each U_j is compactly contained in $U_j^0 \times (0, 1)$, where U_j^0 is a coordinate chart of M_0.

Now let $\chi_1, \ldots, \chi_m$ be a partition of unity subordinate to $U_1, \ldots, U_m$, and for $w \in H^2(M)$ satisfying (5-2), define $w_j = \chi_j w$. Then if $\Gamma_+^c \cap U_j \neq \varnothing$, we know w_j satisfies (5-6) for some σ, and so

$$h^{\frac{1}{2}} \|h\nabla_t w_j\|_{L^2(\Gamma_+^c \cap U_j)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{L^2(M \cap U_j)} + h\|w_j\|_{H^1(M \cap U_j)} + h^{\frac{3}{2}} \|w_j\|_{L^2(\Gamma_+^c \cap U_j)}.$$

Adding together these estimates gives

$$h^{\frac{1}{2}} \|h\nabla_t w\|_{L^2(\Gamma_+^c)} \lesssim \sum_{j=1}^m \|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{L^2(M)} + h\|w\|_{H^1(M)} + h^{\frac{3}{2}} \|w\|_{L^2(\Gamma_+^c)}.$$

Each $\|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{L^2(M)} = \|\mathcal{L}_{\varphi,\varepsilon} \chi_j w\|_{L^2(M)}$ is bounded by a constant times $\|\mathcal{L}_{\varphi,\varepsilon} w\|_{L^2(M)} + h\|w\|_{H^1(M)}$, so

$$h^{\frac{1}{2}} \|h\nabla_t w\|_{L^2(\Gamma_+^c)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w\|_{L^2(M)} + h\|w\|_{H^1(M)} + h^{\frac{3}{2}} \|w\|_{L^2(\Gamma_+^c)}.$$

This finishes the proof of Proposition 5.1.

6. The k-form case

We will prove Theorem 2.5 for $u \in \Omega^k(M)$ by induction. If $k = 0$, then $i_N u = 0$, so $u_\perp = 0$ and $u = u_\parallel$. Then Theorem 2.5 for $k = 0$ becomes the Carleman estimate (5-1) that was established in Section 5.

Note that it suffices to prove Theorem 2.5 for $u \in \Omega^k(M)$, with the appropriate boundary conditions, for each k, and $Q = 0$. Then the final theorem follows by adding the resulting estimates and noting that the extra $h^2 Q u$ term on the right can be absorbed into the terms on the left for sufficiently small h.

Proof of Theorem 2.5 for $k \geq 1$. Suppose $u \in \Omega^k(M)$ with $k \geq 1$. First note that if we impose the boundary conditions (2-2) of Theorem 2.5, substituting the result of Proposition 4.2 into (4-3) gives

$$\|\Delta_{\varphi_c} u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A, B]u \mid u) - 2h^3(\partial_\nu \varphi \nabla_N u_\perp \mid \nabla_N u_\perp)_{\Gamma_+^c}$$
$$- h(\partial_\nu \varphi(|d\varphi|^2 + |\partial_\nu \varphi|^2)u_\parallel \mid u_\parallel)_{\Gamma_+^c} + R, \quad (6\text{-}1)$$

where

$$|R| \leq C\left(Kh^3 \|\nabla' t u_\parallel\|_{\Gamma_+^c}^2 + \frac{h}{K}\|u_\parallel\|_{\Gamma_+^c}^2 + \frac{h^3}{K}\|\nabla_N u_\perp\|_{\Gamma_+^c}^2\right).$$

Recall also from Proposition 4.2 that the nonboundary terms $\|Au\|^2 + \|Bu\|^2 + (i[A, B]u \mid u)$ satisfy

$$\|Au\|^2 + \|Bu\|^2 + (i[A, B]u \mid u) \gtrsim \frac{h^2}{\varepsilon}\|u\|_{H^1(M)}^2 - \frac{h^3}{\varepsilon}\left(\|u\|_{H^1(\partial M)}^2 + \|h\nabla_N u_\perp\|_{L^2(\partial M)}^2\right) \quad (6\text{-}2)$$

for $h \ll \varepsilon \ll 1$. We now return to (6-1) and examine the boundary terms. On Γ_+^c, there exists $\varepsilon_1 > 0$ such that $\partial_\nu \varphi < -\varepsilon_1$. Using this together with (6-1) and (6-2) gives

$$\|\Delta_{\varphi_c} u\|^2 + Kh^3\|\nabla' t u_\parallel\|_{\Gamma_+^c}^2 + \frac{h}{K}\|u_\parallel\|_{\Gamma_+^c}^2 + \frac{h^3}{K}\|\nabla_N u_\perp\|_{\Gamma_+^c}^2 \gtrsim \frac{h^2}{\varepsilon}\|u\|_{H^1(M)}^2 + h^3\|\nabla_N u_\perp\|_{\Gamma_+^c}^2 + h\|u_\parallel\|_{\Gamma_+^c}^2$$

for large enough K. The last two terms on the left side can be absorbed into the right side, giving

$$\|\Delta_{\varphi_c} u\|^2 + Kh^3\|\nabla' t u_\parallel\|_{\Gamma_+^c}^2 \gtrsim \frac{h^2}{\varepsilon}\|u\|_{H^1(M)}^2 + h^3\|\nabla_N u_\perp\|_{\Gamma_+^c}^2 + h\|u_\parallel\|_{\Gamma_+^c}^2.$$

Now we want to analyze the boundary term on the left, and this is the part where we will use induction on k:

Lemma 6.1. *If $u \in \Omega^k(M)$ and u satisfies the boundary conditions (2-2), then*

$$h^3\|\nabla' t u_\parallel\|_{\Gamma_+^c}^2 \lesssim \|\Delta_{\varphi_c} u\|^2 + h^2\|u\|_{H^1(M)}^2 + h^2\|u\|_{\Gamma_+^c}^2. \quad (6\text{-}3)$$

If (6-3) is granted, fix K sufficiently large and then take $h \ll \varepsilon \ll 1$ to obtain

$$\|\Delta_{\varphi_c} u\|^2 \gtrsim \frac{h^2}{\varepsilon}\|u\|_{H^1(M)}^2 + h^3\|\nabla_N u_\perp\|_{\Gamma_+^c}^2 + h\|u_\parallel\|_{\Gamma_+^c}^2 + h^3\|\nabla' t u_\parallel\|_{\Gamma_+^c}^2.$$

Rewriting without the squares,

$$\|\Delta_{\varphi_c} u\| \gtrsim \frac{h}{\sqrt{\varepsilon}}\|u\|_{H^1(M)} + h^{\frac{1}{2}}\|h\nabla_N u_\perp\|_{\Gamma_+^c} + h^{\frac{1}{2}}\|u_\parallel\|_{H^1(\Gamma_+^c)}.$$

Now if u satisfies (2-2) then so does $e^{\frac{\varphi^2}{2\varepsilon}}u$ since ε is fixed. Therefore

$$\|e^{\frac{\varphi^2}{2\varepsilon}}\Delta_\varphi u\| \gtrsim \frac{h}{\sqrt{\varepsilon}}\|e^{\frac{\varphi^2}{2\varepsilon}}u\|_{H^1(M)} + h^{\frac{1}{2}}\|h\nabla_N e^{\frac{\varphi^2}{2\varepsilon}}u_\perp\|_{\Gamma_+^c} + h^{\frac{1}{2}}\|e^{\frac{\varphi^2}{2\varepsilon}}u_\|\|_{H^1(\Gamma_+^c)}.$$

Since $e^{\frac{\varphi^2}{2\varepsilon}}$ is smooth and bounded on M, we get

$$\|\Delta_\varphi u\| \gtrsim h\|u\|_{H^1(M)} + h^{\frac{1}{2}}\|h\nabla_N u_\perp\|_{\Gamma_+^c} + h^{\frac{1}{2}}\|u_\|\|_{H^1(\Gamma_+^c)}.$$

Thus Theorem 2.5 for $k \geq 1$ will follow after we have proved Lemma 6.1.

Proof of Lemma 6.1. For the 0-form case, this follows from Theorem 2.5 for 0-forms, which in this section we are assuming has been proved. Therefore we can seek to prove (6-3) for k-forms by induction on k.

Let $k > 0$, and assume (6-3) holds for $(k-1)$-forms satisfying (2-2). Now let $U_1, \ldots, U_m \subset T$ be an open cover of Γ_+^c such that each $U_i \cap \Gamma_+^c$ has a coordinate patch, and let $\chi_1, \ldots, \chi_m$ be a partition of unity with respect to $\{U_i\}$ such that $\sum \chi_i = 1$ near Γ_+^c and $\nabla_N \chi_i = 0$ for each i. It will suffice to show

$$h^3\|\nabla' t\chi_i u_\|\|_{\Gamma_+^c}^2 \lesssim \|\Delta_{\varphi_c} u\|^2 + h^2\|u\|_{H^1(M)}^2 + h^2\|u_\|\|_{\Gamma_+^c}^2.$$

Now on $U_i \cap \Gamma_+^c$, let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal frame for the tangent space, and extend these vector fields into M by parallel transport along normal geodesics.

Observe for all $\omega \in \Omega^k(U_j \cap \Gamma_+^c)$ one can write

$$\omega = \frac{1}{k}\sum_{j=1}^{n-1} e_j^\flat \wedge i_{e_j}\omega. \tag{6-4}$$

Therefore we can write

$$\nabla' t\chi_i u_\| = \frac{1}{k}\nabla' \sum_{j=1}^{n-1} e_j^\flat \wedge i_{e_j} t\chi_i u_\| = \frac{1}{k}\nabla' \sum_{j=1}^{n-1} e_j^\flat \wedge ti_{e_j}\chi_i u_\|.$$

Then it suffices to show

$$h^3\|\nabla'(e_j^\flat \wedge ti_{e_j}\chi_i u_\|)\|_{\Gamma_+^c}^2 \lesssim \|\Delta_{\varphi_c} u\|^2 + h^2\|u\|_{H^1(M)}^2 + h^2\|u_\|\|_{\Gamma_+^c}^2,$$

or equivalently,

$$h^3\|\nabla' ti_{e_j}\chi_i u_\|\|_{\Gamma_+^c}^2 \lesssim \|\Delta_{\varphi_c} u\|^2 + h^2\|u\|_{H^1(M)}^2 + h^2\|u_\|\|_{\Gamma_+^c}^2. \tag{6-5}$$

Now we want to apply the induction hypothesis to $i_{e_j}\chi_i u_\|$, so we have to check that it satisfies the boundary conditions (2-2). In fact we will have to modify $i_{e_j}\chi_i u_\|$ slightly to achieve this. Let $\rho(x)$ be a function defined in a neighbourhood of the boundary as the distance to the boundary along a normal geodesic, and extend it to the rest of M by multiplication by a cutoff function. Then the claim is that $v = i_{e_j}\chi_i(u_\| + h(1 - e^{-\frac{\rho}{h}})Zu_\|)$ satisfies the absolute boundary conditions (2-2), where Z is an endomorphism yet to be chosen.

Since u satisfies (2-2), $i_{e_j}\chi_i u_\|$ and $i_{e_j}\chi_i(h(1 - e^{-\frac{\rho}{h}})Zu_\|)$ both vanish to first order on Γ_+. Therefore v does as well.

Moreover, $t * i_{e_j} \chi_i u_\| = 0$ on Γ_+^c if $i_N i_{e_j} \chi_i u_\| = -\chi_i i_{e_j} i_N u_\| = 0$ on Γ_+^c, and this again follows from the fact that u satisfies (2-2). Note that $(1 - e^{-\frac{\rho}{h}}) = 0$ at ∂M, so $t * v = 0$ on Γ_+^c.

Finally, by Lemma 3.3,

$$-t\delta * i_{e_j} \chi_i u_\| = -\delta' t (* i_{e_j} \chi_i u_\|)_\| + (S - (n-1)\kappa) t i_N (* i_{e_j} \chi_i u_\|)_\perp + t \nabla_N i_N * i_{e_j} \chi_i u_\|.$$

Since $t * i_{e_j} \chi_i u_\| = 0$ on Γ_+^c, the first term vanishes there as well. Therefore on Γ_+^c,

$$-th\delta * i_{e_j} \chi_i u_\| = h(S - (n-1)\kappa) t i_N (* i_{e_j} \chi_i u_\|)_\perp + th\nabla_N \chi_i i_N * i_{e_j} u_\|.$$

Now

$$\begin{aligned}
th\nabla_N \chi_i i_N * i_{e_j} u_\| &= th\nabla_N \chi_i i_N e_j^\flat \wedge * u_\| (-1)^{k-1} \\
&= th\nabla_N \chi_i i_N e_j^\flat \wedge (*u)_\perp (-1)^{k-1} \\
&= th\nabla_N \chi_i i_N e_j^\flat \wedge *u (-1)^{k-1} \\
&= (-1)^k \chi_i e_j^\flat \wedge th\nabla_N i_N * u,
\end{aligned}$$

so

$$-th\delta * i_{e_j} \chi_i u_\| = h(S - (n-1)\kappa) t i_N (* i_{e_j} \chi_i u_\|)_\perp + (-1)^k \chi_i e_j^\flat \wedge th\nabla_N i_N * u. \tag{6-6}$$

Applying the same calculation to the $i_{e_j} \chi_i h(1 - e^{-\frac{\rho}{h}}) Z u_\|$ term gives

$$-th\delta * i_{e_j} i_{e_j} \chi_i h(1 - e^{-\frac{\rho}{h}}) Z u_\| = (-1)^k \chi_i e_j^\flat \wedge th^2 \nabla_N (1 - e^{-\frac{\rho}{h}}) i_N * Z u_\|;$$

the other term vanishes since $(1 - e^{-\frac{\rho}{h}}) = 0$ at the boundary. Thus

$$-th\delta * i_{e_j} i_{e_j} \chi_i h(1 - e^{-\frac{\rho}{h}}) Z u_\| = (-1)^k \chi_i e_j^\flat \wedge t i_N * h Z u_\|.$$

Meanwhile, by Lemma 3.3 and by (2-2),

$$-th\delta * u = h(S - (n-1)\kappa) t i_N (*u)_\perp + th\nabla_N i_N (*u) = t i_{d\varphi} * u - h\sigma t i_N * u.$$

Viewing this as an equation for $th\nabla_N i_N (*u)$ and substituting into (6-6) gives

$$\begin{aligned}
-th\delta * i_{e_j} \chi_i u_\| = {}&h(S - (n-1)\kappa) t i_N (* i_{e_j} \chi_i u_\|)_\perp \\
&+ (-1)^k \chi_i e_j^\flat \wedge \big(-h(S - (n-1)\kappa) t i_N (*u)_\perp + t i_{d\varphi} * u - h\sigma t i_N * u \big).
\end{aligned}$$

Therefore

$$\begin{aligned}
-th\delta * i_{e_j} \chi_i (u_\| {}&+ h(1 - e^{-\frac{\rho}{h}}) Z u_\|) \\
= {}&h(S - (n-1)\kappa) t i_N (* i_{e_j} \chi_i u_\|)_\perp \\
&+ (-1)^k \chi_i e_j^\flat \wedge \big(-h(S - (n-1)\kappa) t i_N (*u)_\perp + t i_{d\varphi} * u - h\sigma t i_N * u + t i_N * h Z u_\| \big).
\end{aligned}$$

Now if we let

$$Z = *N \wedge (S + \sigma - (n-1)\kappa) i_N *,$$

where here we identify S and σ with their extensions by parallel transport to a neighbourhood of the boundary, then

$$t i_N * Z u_\| = (S + \sigma - (n-1)\kappa) t i_N * u_\| = (S + \sigma - (n-1)\kappa) t i_N * u,$$

and

$$-th\delta * i_{e_j}\chi_i(u_\| + h(1 - e^{-\frac{\rho}{h}})Zu_\|) = h(S - (n-1)\kappa)ti_N(*i_{e_j}\chi_i u_\|)_\perp + (-1)^k\chi_i e_j^\flat \wedge ti_{d\varphi} * u.$$

Since $t * u = 0$ on Γ_+^c, we can replace the $d\varphi$ in $ti_{d\varphi} * u$ with its normal component:

$$ti_{d\varphi} * u = -\partial_\nu\varphi ti_N * u.$$

Then

$$\begin{aligned}
\chi_i e_j^\flat \wedge -ti_{d\varphi} * u &= \partial_\nu\varphi\chi_i e_j^\flat \wedge ti_N(*u)_\perp \\
&= \partial_\nu\varphi\chi_i e_j^\flat \wedge ti_N * u_\| \\
&= -\partial_\nu\varphi ti_N \chi_i e_j^\flat \wedge *u_\| \\
&= \partial_\nu\varphi ti_N * i_{e_j}\chi_i u_\|(-1)^k.
\end{aligned}$$

Since $t * i_{e_j}\chi_i u_\| = 0$ on Γ_+^c,

$$\chi_i e_j^\flat \wedge -ti_{d\varphi} * u = -ti_{d\varphi} * i_{e_j}\chi_i u_\|(-1)^k$$

and

$$(-1)^k\chi_i e_j^\flat \wedge -ti_{d\varphi} * u = -ti_{d\varphi} * i_{e_j}\chi_i u_\|.$$

Therefore

$$-th\delta * i_{e_j}\chi_i(u_\| + h(1 - e^{-\frac{\rho}{h}})Zu_\|) = ti_{d\varphi} * i_{e_j}\chi_i u_\| - h\sigma' ti_N * i_{e_j}\chi_i u_\|,$$

where σ' is a smooth bounded endomorphism. We can replace $u_\|$ on the right side by $u_\| + h(1 - e^{-\frac{\rho}{h}})Zu_\|$, since $(1 - e^{-\frac{\rho}{h}})$ is zero at the boundary. Therefore $v = i_{e_j}\chi_i(u_\| + h(1 - e^{-\frac{\rho}{h}})Zu_\|)$ satisfies the boundary conditions (2-2), and so by the induction hypothesis,

$$h^3\|\nabla' tv\|_{\Gamma_+^c}^2 \lesssim \|\Delta_{\varphi_c}v\|^2 + h^2\|v\|_{H^1(M)}^2 + h^2\|v\|_{\Gamma_+^c}^2.$$

Keeping in mind that the second term of v is zero at the boundary, and $O(h)$ elsewhere, we get

$$h^3\|\nabla' ti_{e_j}\chi_i u_\|\|_{\Gamma_+^c}^2 \lesssim \|\Delta_{\varphi_c}v\|^2 + h^2\|u_\|\|_{H^1(M)}^2 + h^2\|u_\|\|_{\Gamma_+^c}^2. \tag{6-7}$$

Now

$$\|\Delta_{\varphi_c}v\| \lesssim \|\Delta_{\varphi_c}i_{e_j}\chi_i u_\|\| + h\|\Delta_{\varphi_c}i_{e_j}\chi_i(1 - e^{-\frac{\rho}{h}})Zu_\|\|.$$

The commutators of Δ_{φ_c} with $i_{e_j}\chi_i$ and $i_{e_j}\chi_i(1 - e^{-\frac{\rho}{h}})Z$ are $O(h)$ and first-order, so

$$\begin{aligned}
\|\Delta_{\varphi_c}v\| &\lesssim \|i_{e_j}\chi_i\Delta_{\varphi_c}u_\|\| + h\|i_{e_j}\chi_i(1 - e^{-\frac{\rho}{h}})Z\Delta_{\varphi_c}u_\|\| + h\|u_\|\|_{H^1(M)} \\
&\lesssim \|\Delta_{\varphi_c}u_\|\| + h\|u_\|\|_{H^1(M)}.
\end{aligned}$$

Substituting back into (6-7) gives

$$h^3\|\nabla' ti_{e_j}\chi_i u_\|\|_{\Gamma_+^c}^2 \lesssim \|\Delta_{\varphi_c}u_\|\|^2 + h^2\|u_\|\|_{H^1(M)}^2 + h^2\|u_\|\|_{\Gamma_+^c}^2.$$

This proves (6-5), which finishes the proof of the lemma.   □

7. Complex geometrical optics solutions

We will begin by constructing CGOs for the relative boundary case. To start, we can use the Carleman estimate from Theorem 2.4 to generate solutions via a Hahn–Banach argument. The notations are as in Section 2.

Proposition 7.1. *Let Q be an L^∞ endomorphism on ΛM, and let Γ_+ be a neighbourhood of ∂M_+. For all $v \in L^2(M, \Lambda M)$, and $f, g \in L^2(M, \Lambda \partial M)$ with support in Γ_+^c, there exists $u \in L^2(M, \Lambda M)$ such that*

$$
\begin{aligned}
(-\triangle_{-\varphi} + h^2 Q^*)u &= v &&\text{on } M, \\
tu &= f &&\text{on } \Gamma_+^c, \\
th\delta_{-\varphi}u &= g &&\text{on } \Gamma_+^c,
\end{aligned}
$$

with

$$
\|u\|_{L^2(M)} \lesssim h^{-1}\|v\|_{L^2(M)} + h^{\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + h^{\frac{1}{2}}\|g\|_{L^2(\Gamma_+^c)}.
$$

Proof. Suppose $w \in \Omega(M)$ satisfies the relative boundary conditions (2-1) with $\sigma = 0$, and examine the expression

$$
\left|(w \mid v) - (ti_\nu h d_\varphi w \mid hf)_{\Gamma_+^c} - (ti_\nu w \mid hg)_{\Gamma_+^c}\right|. \tag{7-1}
$$

This is bounded above by

$$
h\|w\|_{L^2(M)}h^{-1}\|v\|_{L^2(M)} + h^{\frac{1}{2}}\|ti_\nu h d_\varphi w\|_{L^2(\Gamma_+^c)}h^{\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + h^{\frac{1}{2}}\|ti_\nu w\|_{L^2(\Gamma_+^c)}\|g\|_{L^2(\Gamma_+^c)}.
$$

By Lemma 3.4,

$$
ti_\nu h d_\varphi w = h e^{\frac{\varphi}{h}} t\nabla_N (e^{-\frac{\varphi}{h}}w)_\| + hStw_\| - h e^{\frac{\varphi}{h}} d' t i_N (e^{-\frac{\varphi}{h}}w).
$$

Since $tw = 0$,

$$
ti_\nu h d_\varphi w = ht\nabla_N w_\| - h e^{\frac{\varphi}{h}} d' t i_N (e^{-\frac{\varphi}{h}}w).
$$

Therefore

$$
\|ti_\nu h d_\varphi w\|_{L^2(\Gamma_+^c)} \leq \|h\nabla_N w_\|\|_{L^2(\Gamma_+^c)} + \|w_\perp\|_{H^1(\Gamma_+^c)}.
$$

Then by Theorem 2.4,

$$
\begin{aligned}
\left|(w \mid v) + (ti_\nu h d_\varphi w \mid hf)_{\Gamma_+^c} + (ti_\nu w \mid hg)_{\Gamma_+^c}\right| \\
\lesssim \|(-\triangle_\varphi + h^2 Q)w\|_{L^2(M)}\left(h^{-1}\|v\|_{L^2(M)} + h^{\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + h^{\frac{1}{2}}\|g\|_{L^2(\Gamma_+^c)}\right).
\end{aligned}
$$

Therefore on the subspace

$$
\{(-\triangle_\varphi + h^2 Q)w \mid w \in \Omega(M) \text{ satisfies (2-1) with } \sigma = 0\} \subset L^2(M, \Lambda M),
$$

the map

$$
(-\triangle_\varphi + h^2 Q)w \mapsto (w \mid v) - (ti_\nu h d_\varphi w \mid hf)_{\Gamma_+^c} - (ti_\nu w \mid hg)_{\Gamma_+^c}
$$

defines a bounded linear functional with the bound

$$
h^{-1}\|v\|_{L^2(M)} + h^{\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + h^{\frac{1}{2}}\|g\|_{L^2(\Gamma_+^c)}.
$$

By Hahn–Banach, this functional extends to the whole space, and thus there exists a $u \in L^2(M, \Lambda M)$ such that

$$\|u\|_{L^2(M)} \lesssim h^{-1}\|v\|_{L^2(M)} + h^{\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + h^{\frac{1}{2}}\|g\|_{L^2(\Gamma_+^c)}$$

and

$$(w \mid v) - (t i_\nu h d_\varphi w \mid hf)_{\Gamma_+^c} - (t i_\nu w \mid hg)_{\Gamma_+^c} = ((-\triangle_\varphi + h^2 Q)w \mid u).$$

Integrating by parts and applying the boundary conditions (2-1) gives

$$(w \mid v) - (t i_\nu h d_\varphi w \mid hf)_{\Gamma_+^c} - (t i_\nu w \mid hg)_{\Gamma_+^c}$$
$$= (w \mid (-\triangle_{-\varphi} + h^2 Q^*)u) - h(t i_\nu h d_\varphi w \mid tu)_{\partial M} - h(t i_\nu w \mid t h \delta_{-\varphi}u)_{\partial M}$$

for all $w \in \Omega(M)$ satisfying the relative boundary conditions (2-1) with $\sigma = 0$. Varying w over the compactly supported elements of $\Omega(M)$ one sees that $(-\triangle_{-\varphi} + h^2 Q^*)u = v$ on M, which reduces the above relation to

$$-(t i_\nu h d_\varphi w \mid hf)_{\Gamma_+^c} - (t i_\nu w \mid hg)_{\Gamma_+^c} = -h(t i_\nu h d_\varphi w \mid tu)_{\partial M} - h(t i_\nu w \mid t h \delta_{-\varphi}u)_{\partial M}$$

for all $w \in \Omega(M)$ satisfying the relative boundary conditions (2-1) with $\sigma = 0$. We now vary w satisfying condition (2-1) with $\sigma = 0$ and $i_\nu w = 0$ to obtain $tu = f$ on Γ_+^c. Finally, by varying w over all forms satisfying conditions (2-1) with $\sigma = 0$, we see that $t h \delta_{-\varphi}u = g$ on Γ_+^c.

To summarize, we can see that

$$(-\triangle_{-\varphi} + h^2 Q^*)u = v \quad \text{on } M,$$
$$tu = f \quad \text{on } \Gamma_+^c,$$
$$t h \delta_{-\varphi}u = g \quad \text{on } \Gamma_+^c,$$

as desired. $\qquad \square$

To match notations with previous papers, we will begin by rewriting this result, along with the Carleman estimate, in τ notation, as follows.

Theorem 2.4 becomes the following.

Theorem 7.2. *Let Q be an L^∞ endomorphism on ΛM. Define $\Gamma_+ \subset \partial M$ to be a neighbourhood of ∂M_+. Suppose $u \in H^2(M, \Lambda M)$ satisfies the boundary conditions*

$$u|_{\Gamma_+} = 0 \quad \text{and} \quad \nabla_\nu u \mid_{\Gamma_+} = 0,$$
$$tu|_{\Gamma_+^c} = 0, \tag{7-2}$$
$$t\delta e^{-\tau\varphi}u|_{\Gamma_+^c} = \sigma t i_N e^{-\tau\varphi}u$$

for some smooth endomorphism σ independent of τ. Then there exists $\tau_0 > 0$ such that if $\tau > \tau_0$,

$$\|(-\Delta_\tau + Q)u\|_{L^2(M)} \gtrsim \tau\|u\|_{L^2(M)} + \|\nabla u\|_{L^2(M)} + \tau^{\frac{3}{2}}\|u_\perp\|_{L^2(\Gamma_+^c)}$$
$$+ \tau^{\frac{1}{2}}\|\nabla' t i_N u\|_{L^2(\Gamma_+^c)} + \tau^{\frac{1}{2}}\|\nabla_N u_\|\|_{L^2(\Gamma_+^c)},$$

where

$$\Delta_\tau = e^{\tau\varphi}\Delta e^{-\tau\varphi}.$$

By choice of coordinates, note that the same theorem holds for $\tau < 0$, with Γ_+ replaced by Γ_-. Then Proposition 7.1 becomes the following.

Proposition 7.3. *Let Q be an L^∞ endomorphism on ΛM. For all $v \in L^2(M, \Lambda M)$ and $f, g \in L^2(\Gamma_+^c, \Lambda \Gamma_+^c)$, there exists $u \in L^2(M, \Lambda M)$ such that*

$$
\begin{aligned}
(-\Delta_{-\tau} + Q^*)u &= v &&\text{on } M, \\
tu &= f &&\text{on } \Gamma_+^c, \\
t\delta_{-\tau}u &= g &&\text{on } \Gamma_+^c,
\end{aligned}
$$

with

$$
\|u\|_{L^2(M)} \lesssim \tau^{-1}\|v\|_{L^2(M)} + \tau^{-\frac{1}{2}}\|f\|_{L^2(\Gamma_+^c)} + \tau^{-\frac{3}{2}}\|g\|_{L^2(\Gamma_+^c)}.
$$

Now we turn to the construction of the CGOs themselves. From now on we will invoke the assumption that the conformal factor c in the definition of M as an admissible manifold satisfies $c = 1$. Below we will consider complex-valued 1-forms, and $\langle \cdot, \cdot \rangle$ will denote the complex bilinear extension of the Riemannian inner product to complex-valued forms.

We assume

$$
(M, g) \Subset (\mathbb{R} \times M_0, g), \quad g = e \oplus g_0,
$$

where (M_0, g_0) is a compact $(n-1)$-dimensional manifold with smooth boundary. We write $x = (x_1, x')$ for points in $\mathbb{R} \times M_0$, where x_1 is the Euclidean coordinate and x' is a point in M_0. Let Q be an L^∞ endomorphism of ΛM. We next wish to construct solutions to the equation

$$
(-\Delta + Q)Z = 0 \quad \text{in } M,
$$

where Z is a graded differential form in $L^2(M, \Lambda M)$ having the form

$$
Z = e^{-sx_1}(A + R).
$$

Here $s = \tau + i\lambda$ is a complex parameter where $\tau, \lambda \in \mathbb{R}$ and $|\tau|$ is large, the graded form A is a smooth amplitude, and R will be a correction term obtained from the Carleman estimate. Inserting the expression for Z in the equation results in

$$
e^{sx_1}(-\Delta + Q)e^{-sx_1}R = -F,
$$

where

$$
F = e^{sx_1}(-\Delta + Q)e^{-sx_1}A.
$$

The point is to choose A so that $\|F\|_{L^2(M)} = O(1)$ as $|\tau| \to \infty$.

By Lemma 3.2, we have

$$
F = (-\Delta - s^2 + 2s\nabla_{\partial_1} + Q)A.
$$

We wish to choose A so that $\nabla_{\partial_1}A = 0$. The following lemma explains this condition. Below, we identify a differential form in M_0 with the corresponding differential form in $\mathbb{R} \times M_0$ which is constant in x_1.

Lemma 7.4. *If u is a k-form in $\mathbb{R} \times M_0$ with local coordinate expression $u = u_I \, dx^I$, then*

$$\nabla_{\partial_1} u = 0 \quad \Longleftrightarrow \quad u_I = u_I(x') \quad \text{for all } I.$$

If $\nabla_{\partial_1} u = 0$, then there is a unique decomposition

$$u = dx^1 \wedge u' + u'',$$

where u' is a $(k-1)$-form in M_0 and u'' is a k-form in M_0. For such a k-form u, one has

$$\Delta u = dx^1 \wedge \Delta_{x'} u' + \Delta_{x'} u'',$$

where Δ and $\Delta_{x'}$ are the Hodge Laplacians in $\mathbb{R} \times M_0$ and in M_0, respectively.

Proof. In the (x_1, x')-coordinates g has the form

$$g(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}.$$

Consequently, for any k, l the Christoffel symbols satisfy

$$\Gamma^l_{1k} = \tfrac{1}{2} g^{lm} (\partial_1 g_{km} + \partial_k g_{1m} - \partial_m g_{1k}) = 0.$$

This shows $\nabla_{\partial_1} dx^I = 0$ for all I, and therefore any k-form $u = u_I \, dx^I$ satisfies

$$\nabla_{\partial_1} (u_I \, dx^I) = \partial_1 u_I \, dx^I.$$

Thus $\nabla_{\partial_1} u = 0$ if and only if each u_I only depends on x'. In general, if u is a k-form on $\mathbb{R} \times M_0$ we have the unique decomposition

$$u = dx^1 \wedge u' + u'',$$

where $u'(x_1, \cdot)$ is a $(k-1)$-form in M_0 and $u''(x_1, \cdot)$ is a k-form in M_0, depending smoothly on the parameter x_1. If $\nabla_{\partial_1} u = 0$, then $u = dx^1 \wedge u' + u''$, where u' and u'' are differential forms in M_0.

Suppose now that $u = dx^1 \wedge u' + u''$, where u' and u'' are forms in M_0. Denote by $d_{x'}$ and $\delta_{x'}$ the exterior derivative and codifferential in x'. Clearly

$$d(dx^1 \wedge u') = -dx^1 \wedge d_{x'} u', \quad du'' = d_{x'} u''.$$

The identity $\delta = -\sum_{j=1}^{n} i_{e_j} \nabla_{e_j}$, where e_j is an orthonormal frame in $T(\mathbb{R} \times M_0)$ with $e_1 = \partial_1$, together with the fact that $\nabla_{\partial_1} u'' = 0$, implies

$$\delta u'' = \delta_{x'} u''.$$

Finally, computing in Riemannian normal coordinates at p gives

$$\delta(dx^1 \wedge u')|_p = -\sum_{j=1}^{n} i_{\partial_j} \nabla_{\partial_j} (u'_J \, dx^1 \wedge dx^J)|_p$$

$$= -\sum_{j=2}^{n} i_{\partial_j} (dx^1 \wedge \nabla_{\partial_j} u')|_p = -dx^1 \wedge \delta_{x'} u'|_p.$$

Thus

$$\delta(dx^1 \wedge u') = -dx^1 \wedge \delta_{x'} u'.$$

It follows directly from these facts that

$$\Delta(dx^1 \wedge u' + u'') = -(d\delta + \delta d)(dx^1 \wedge u' + u'')$$
$$= dx^1 \wedge \Delta_{x'} u' + \Delta_{x'} u''. \qquad \square$$

Returning to the expression for F, the assumption $\nabla_{\partial_1} A = 0$ gives

$$F = (-\Delta - s^2 + Q)A.$$

Writing Y^k for the k-form part of a graded form Y and decomposing $A^k = dx^1 \wedge (A^k)' + (A^k)''$ as in Lemma 7.4, we obtain

$$F^k = dx^1 \wedge (-\Delta_{x'} - s^2)(A^k)' + (-\Delta_{x'} - s^2)(A^k)'' + (QA)^k.$$

Thus, in order to have $\|F\|_{L^2(M)} = O(1)$ as $|\tau| \to \infty$, it is enough to find for each k a smooth $(k-1)$-form $(A^k)'$ and a smooth k-form $(A^k)''$ in M_0 such that

$$\|(-\Delta_{x'} - s^2)(A^k)'\|_{L^2(M_0)} = O(1), \quad \|(A^k)'\|_{L^2(M_0)} = O(1),$$
$$\|(-\Delta_{x'} - s^2)(A^k)''\|_{L^2(M_0)} = O(1), \quad \|(A^k)''\|_{L^2(M_0)} = O(1).$$

If (M_0, g_0) is simple, there is a straightforward quasimode construction for achieving this.

Lemma 7.5. *Let (M_0, g_0) be a simple m-dimensional manifold, and let $0 \leq k \leq m$. Suppose $(\hat{M}_0, g_0)$ is another simple manifold with $(M_0, g_0) \Subset (\hat{M}_0, g_0)$, fix a point $\omega \in \hat{M}_0^{\text{int}} \setminus M_0$, and let (r, θ) be polar normal coordinates in $(\hat{M}_0, g_0)$ with centre ω. Suppose $\eta^1, \ldots, \eta^m$ is a global orthonormal frame of $T^* M_0$ with $\eta^1 = dr$ and $\nabla_{\partial_r} \eta^j = 0$ for $2 \leq j \leq m$, and let $\{\eta^I\}$ be a corresponding orthonormal frame of $\Lambda^k M_0$. Then for any $\lambda \in \mathbb{R}$ and for any $\binom{m}{k}$ complex functions $b_I \in C^\infty(S^{m-1})$, the smooth k-form*

$$u = e^{isr} |g_0(r, \theta)|^{-\frac{1}{4}} \sum_I b_I(\theta) \eta^I,$$

with $s = \tau + i\lambda$ for τ real, satisfies

$$\|(-\Delta_{x'} - s^2)u\|_{L^2(M_0)} = O(1), \quad \|u\|_{L^2(M_0)} = O(1)$$

as $|\tau| \to \infty$.

Proof. We first try to find the quasimode in the form $u = e^{is\psi} a$ for some smooth real-valued phase function ψ and some smooth k-form a. Lemma 3.2 implies

$$(-\Delta_{x'} - s^2)(e^{is\psi} a) = e^{is\psi} \left[s^2(|d\psi|^2 - 1)a - is[2\nabla_{\text{grad}(\psi)} a + (\Delta_{x'}\psi)a] - \Delta_{x'} a \right].$$

Let (r, θ) be polar normal coordinates as in the statement of the lemma, and note that

$$g_0(r, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & h(r, \theta) \end{pmatrix}$$

globally in M_0 for some $(m-1) \times (m-1)$ symmetric positive definite matrix h.

Define

$$\psi(r, \theta) = r.$$

Then $\psi \in C^\infty(M_0)$ and $|d\psi|^2 = 1$, so that the s^2 term will be zero. We next want to choose a so that $2\nabla_{\mathrm{grad}(\psi)}a + (\Delta_{x'}\psi)a = 0$. Note that

$$\nabla_{\mathrm{grad}(\psi)} = \nabla_{\partial_r}, \quad \Delta_{x'}\psi = \frac{1}{2}\frac{\partial_r|g_0(r, \theta)|}{|g_0(r, \theta)|}.$$

Thus, choosing $a = |g_0|^{-\frac{1}{4}}\tilde{a}$ for some k-form $\tilde{a}$, it is enough to arrange that

$$\nabla_{\partial_r}\tilde{a} = 0.$$

Using the frame $\{\eta^j\}$ above, with $\eta^1 = dr$, we write

$$\tilde{a} = \eta^1 \wedge \tilde{a}' + \tilde{a}'',$$

where $\tilde{a}'$ is a $(k-1)$-form and $\tilde{a}''$ is a k-form in M_0 of the form

$$\tilde{a}' = \sum_{\substack{J \subset \{2,\ldots,m\} \\ |J|=k-1}} \alpha_{1,J}\eta^J, \quad \tilde{a}'' = \sum_{\substack{J \subset \{2,\ldots,m\} \\ |J|=k}} \beta_J\eta^J$$

for some functions $\alpha_{1,J}$ and β_J in M_0. Now, the form of the metric implies $\nabla_{\partial_r}\eta^1 = 0$, and by assumption $\nabla_{\partial_r}\eta^j = 0$ for $2 \le j \le m$. Therefore

$$\nabla_{\partial_r}\tilde{a} = \sum_{\substack{J \subset \{2,\ldots,m\} \\ |J|=k-1}} \partial_r\alpha_{1,J}\eta^1 \wedge \eta^J + \sum_{\substack{J \subset \{2,\ldots,m\} \\ |J|=k}} \partial_r\beta_J\eta^J.$$

In the definitions of $\tilde{a}'$ and $\tilde{a}''$, we may now choose

$$\alpha_{1,J} = b_{\{1\}\cup J}(\theta), \quad \beta_J = b_J(\theta),$$

where b_I are the given functions in $C^\infty(S^{m-1})$. The resulting k-form $u = e^{is\psi}|g_0|^{-\frac{1}{4}}\tilde{a}$ satisfies the required conditions. $\square$

The next result gives the full construction of the complex geometrical optics solutions.

Lemma 7.6. *Let $(M, g) \Subset (\mathbb{R} \times M_0, g)$, where $g = e \oplus g_0$, assume (M_0, g_0) is simple, and let Q be an L^∞ endomorphism of ΛM. Let $(\hat{M}_0, g_0)$ be another simple manifold with $(M_0, g_0) \Subset (\hat{M}_0, g_0)$, fix a point $\omega \in \hat{M}_0^{\mathrm{int}} \setminus M_0$, and let (r, θ) be polar normal coordinates in $(\hat{M}_0, g_0)$ with centre ω. Suppose $\eta^1, \ldots, \eta^n$ is a global orthonormal frame of $T^*(\mathbb{R} \times M_0)$ with $\eta^1 = dx^1$, $\eta^2 = dr$, and $\nabla_{\partial_r}\eta^j = 0$ for $3 \le j \le n$, and let $\{\eta^I\}$ be a corresponding orthonormal frame of $\Lambda(\mathbb{R} \times M_0)$. Let also $\lambda \in \mathbb{R}$. If $|\tau|$ is sufficiently large and if $s = \tau + i\lambda$, then for any 2^n complex functions $b_I \in C^\infty(S^{n-2})$ there exists a solution $Z \in L^2(M, \Lambda M)$ of the equation*

$$(-\Delta + Q)Z = 0 \quad in \ M$$

having the form

$$Z = e^{-sx_1}\left[e^{isr}|g_0(r,\theta)|^{-\frac{1}{4}}\left[\sum_I b_I(\theta)\eta^I\right] + R\right],$$

where $\|R\|_{L^2(M)} = O(|\tau|^{-1})$. *Further, one can arrange that the relative boundary values of Z vanish on* Γ_+^c *or* Γ_-^c *(depending on the sign of τ).*

Proof. Try first $Z = e^{-sx_1}(A + R)$, where $\nabla_{\partial_1} A = 0$. By the discussion in this section, we need to solve the equation

$$e^{sx_1}(-\Delta + Q)(e^{-sx_1}R) = -F,$$

where

$$F = (-\Delta - s^2 + Q)A.$$

Decomposing the k-form part of A as $A^k = \eta^1 \wedge (A^k)' + (A^k)''$ as in Lemma 7.4, where $\eta^1 = dx^1$, we obtain

$$F^k = \eta^1 \wedge (-\Delta_{x'} - s^2)(A^k)' + (-\Delta_{x'} - s^2)(A^k)'' + (QA)^k.$$

Let $\eta^1, \ldots, \eta^n$ and $\{\eta^I\}$ be orthonormal frames as in the statement of the result. We can use Lemma 7.5 to find, for any $\binom{n-1}{k-1}$ functions $b'_J(\theta)$ and for any $\binom{n-1}{k}$ functions $b''_J(\theta)$, quasimodes

$$(A^k)' = e^{isr}|g_0|^{-\frac{1}{4}} \sum_{\substack{J \subset \{2,\ldots,n\} \\ |J|=k-1}} b'_J(\theta)\eta^J,$$

$$(A^k)'' = e^{isr}|g_0|^{-\frac{1}{4}} \sum_{\substack{J \subset \{2,\ldots,n\} \\ |J|=k}} b''_J(\theta)\eta^J.$$

Recalling that $A^k = \eta^1 \wedge (A^k)' + (A^k)''$ and relabeling functions, this shows that for any $\binom{n}{k}$ functions $b_I \in C^\infty(S^{n-2})$ we may find A^k of the form

$$A^k = e^{isr}|g_0|^{-\frac{1}{4}} \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I|=k}} b_I(\theta)\eta^I,$$

with $\|(-\Delta - s^2)A^k\|_{L^2(M)} = O(1)$ and $\|A^k\|_{L^2(M)} = O(1)$ as $|\tau| \to \infty$. Repeating this construction for all k, we obtain the amplitude

$$A = e^{isr}|g_0(r,\theta)|^{-\frac{1}{4}} \sum_I b_I(\theta)\eta^I,$$

with the same norm estimates as those for A^k. Then also $\|F\|_{L^2(M)} = O(1)$. Then Proposition 7.3 allows us to find R with the right properties. □

Note that if Z is a solution to $(-\Delta + *Q*^{-1})Z = 0$ in M, and Z has relative boundary values that vanish on Γ_+^c, then $*Z$ is a solution to $(-\Delta + Q) * Z = 0$ in M, and $*Z$ has absolute boundary values that vanish on Γ_+^c. Thus this construction also gives us solutions with vanishing absolute boundary values on Γ_+^c.

8. The tensor tomography problem

Now we can begin the proof of Theorems 2.1 and 2.2. First we will use the hypotheses of Theorem 2.1 to obtain some vanishing integrals involving $(Q_2 - Q_1)$.

Lemma 8.1. *Suppose the hypotheses of Theorem 2.1 hold. Using the notation in Lemma 7.6, let* $Z_1, Z_2 \in L^2(M, \Lambda M)$ *be solutions of* $(-\Delta + Q_1)Z_1 = (-\Delta + \bar{Q}_2)Z_2 = 0$ *in* M *of the form*

$$Z_1 = e^{-sx_1}\left[e^{isr}|g_0|^{-\frac{1}{4}}\left[\sum_I c_I(\theta)\eta^I \right] + R_1 \right],$$

$$Z_2 = e^{sx_1}\left[e^{isr}|g_0|^{-\frac{1}{4}}\left[\sum_I d_I(\theta)\eta^I \right] + R_2 \right],$$

with vanishing relative boundary conditions on Γ_-^c *and* Γ_+^c *respectively. Then*

$$((Q_2 - Q_1)Z_1 \mid Z_2)_M = 0.$$

Note that while the orthogonality condition derived in the lemma does not use the particular form of the solution, we will only apply this identity to solutions of the given form.

Proof. Let Y be a solution of $(-\Delta + Q_2)Y = 0$ in M with the same relative boundary conditions as Z_1; such a solution exists by the assumption on Q_2. Then consider the integral

$$\left((N_{Q_1}^{\mathrm{RA}} - N_{Q_2}^{\mathrm{RA}})(tZ_1, t\delta Z_1) \mid (ti_N d * Z_2, ti_N * Z_2) \right)_{\partial M}.$$

By definition of the N^{RA} map, this is

$$\left((t*(Z_1-Y), t\delta*(Z_1-Y)) \mid (ti_N d * Z_2, ti_N * Z_2) \right)_{\partial_M}$$
$$= \left(t*(Z_1-Y) \mid ti_N d * Z_2 \right)_{\partial M} + \left(t\delta*(Z_1-Y) \mid ti_N * Z_2 \right)_{\partial M}.$$

Recall from the section on notation and identities that

$$(-\Delta u \mid v)_M = (u \mid -\Delta v)_M + (tu \mid ti_\nu dv)_{\partial M} + (t\delta * u \mid ti_\nu * v)_{\partial M} + (t * u \mid ti_\nu d * v)_{\partial M} + (t\delta u \mid ti_\nu v)_{\partial M}.$$

Since the relative boundary values of $(Z_1 - Y)$ vanish, by definition, the integration by parts formula above implies

$$\left(t*(Z_1-Y) \mid ti_N d*Z_2 \right)_{\partial M} + \left(t\delta*(Z_1-Y) \mid ti_N*Z_2 \right)_{\partial M} = (-\Delta(Z_1-Y) \mid Z_2)_M - (Z_1-Y \mid -\Delta Z_2)_M$$
$$= (Q_2Y - Q_1Z_1 \mid Z_2)_M - (Z_1-Y \mid -\bar{Q}_2Z_2)_M$$
$$= ((Q_2-Q_1)Z_1 \mid Z_2)_M.$$

Meanwhile, by the hypothesis on $N_{Q_1}^{\mathrm{RA}}$ and $N_{Q_2}^{\mathrm{RA}}$, we have $N_{Q_1}^{\mathrm{RA}}(Z_1 - Y) = N_{Q_2}^{\mathrm{RA}}(Z_1 - Y)$ on Γ_+. Therefore

$$\left(t*(Z_1-Y) \mid ti_N d*Z_2 \right)_{\partial M} + (t\delta*(Z_1-Y) \mid ti_N*Z_2)_{\partial M}$$
$$= \left(t*(Z_1-Y) \mid ti_N d*Z_2 \right)_{\Gamma_+^c} + (t\delta*(Z_1-Y) \mid ti_N*Z_2)_{\Gamma_+^c}.$$

Now by construction, Z_2 has relative boundary values that vanish on Γ_+^c. But

$$
\begin{aligned}
t i_N * Z_2|_{\Gamma_+^c} = 0 \quad &\Longleftrightarrow \quad (*Z_2)_\perp|_{\Gamma_+^c} = 0 \\
&\Longleftrightarrow \quad *(Z_2)_\parallel|_{\Gamma_+^c} = 0 \\
&\Longleftrightarrow \quad (Z_2)_\parallel|_{\Gamma_+^c} = 0 \\
&\Longleftrightarrow \quad t Z_2|_{\Gamma_+^c} = 0.
\end{aligned}
$$

Similarly,

$$
t i_N d * Z_2|_{\Gamma_+^c} = 0 \quad \Longleftrightarrow \quad t\delta * Z_2|_{\Gamma_+^c} = 0.
$$

Therefore the fact that Z_2 has relative boundary values that vanish on Γ_+^c implies

$$
\left(t*(Z_1-Y) \mid t i_N d * Z_2\right)_{\Gamma_+^c} + \left(t\delta *(Z_1-Y) \mid t i_N * Z_2\right)_{\Gamma_+^c} = 0.
$$

Therefore

$$
((Q_2-Q_1)Z_1 \mid Z_2)_M = 0
$$

for each such pair of CGO solutions Z_1 and Z_2. $\qquad\square$

Remark. The proof of the Lemma 8.1 does not use the actual forms of the CGO solutions. The integral identity holds for all solutions Z_1 and Z_2 with vanishing relative boundary conditions on Γ_-^c and Γ_+^c respectively. However, the identity is only of interest to us for the particular forms of CGO solutions which we stated.

Working through the same argument with $*Z_1$ and $*Z_2$ gives us the following lemma as well.

Lemma 8.2. *Suppose the hypotheses of Theorem 2.2 hold. Using the notation in Lemma 7.6, let $*Z_1, *Z_2 \in L^2(M, \Lambda M)$ be solutions of $(-\Delta + Q_1) * Z_1 = (-\Delta + \overline{Q}_2) * Z_2 = 0$ in M of the form*

$$
Z_1 = e^{-sx_1}\left[e^{isr}|g_0|^{-\frac{1}{4}}\left[\sum_I c_I(\theta)\eta^I\right] + R_1\right],
$$

$$
Z_2 = e^{sx_1}\left[e^{isr}|g_0|^{-\frac{1}{4}}\left[\sum_I d_I(\theta)\eta^I\right] + R_2\right].
$$

Then

$$
((Q_2 - Q_1)Z_1 \mid Z_2)_M = 0.
$$

Therefore both of the main theorems reduce to using the condition $(QZ_1, Z_2)_{L^2(M)} = 0$ for solutions of the type given in Lemma 7.6 to show $Q = 0$.

The next result shows that from the condition $(QZ_1, Z_2)_{L^2(M)} = 0$ for solutions of the type given in Lemma 7.6, it follows that certain exponentially attenuated integrals over geodesics in (M_0, g_0) of matrix elements of Q, further Fourier transformed in x_1, must vanish.

Proposition 8.3. *Assume the hypotheses in Theorem 2.1 or 2.2, with $Q = Q_2 - Q_1$ extended by zero to $\mathbb{R} \times M_0$. Fix a geodesic $\gamma : [0, L] \to M_0$ with $\gamma(0), \gamma(L) \in \partial M_0$, let ∂_r be the vector field in M_0 tangent to geodesic rays starting at $\gamma(0)$, and suppose $\{\eta^I\}$ is an orthonormal frame of $\Lambda(\mathbb{R} \times M_0^{\text{int}})$ with $\eta^1 = dx^1$,*

$\eta^2 = dr$, and $\nabla_{\partial_r} \eta^j = 0$ for $3 \leq j \leq n$. (*Such a frame always exists.*) *Then for any* $\lambda \in \mathbb{R}$ *and any* I, J
one has

$$\int_0^L e^{-2\lambda r} \left[\int_{-\infty}^{\infty} e^{-2i\lambda x_1} \langle Q(x_1, \gamma(r)) \eta^I, \eta^J \rangle \, dx_1 \right] dr = 0.$$

Proof. Using the notation in Lemma 7.6, let $Z_j \in L^2(M, \Lambda M)$ be solutions of $(-\Delta + Q_1)Z_1 = (-\Delta + \overline{Q}_2)Z_2 = 0$ in M of the form

$$Z_1 = e^{-sx_1} \left[e^{isr} |g_0|^{-\frac{1}{4}} \left[\sum_I c_I(\theta) \eta^I \right] + R_1 \right],$$

$$Z_2 = e^{sx_1} \left[e^{isr} |g_0|^{-\frac{1}{4}} \left[\sum_I d_I(\theta) \eta^I \right] + R_2 \right],$$

where $s = \tau + i\lambda$, $\tau > 0$ is large, $\lambda \in \mathbb{R}$, and $c_I, d_I \in C^{\infty}(S^{n-2})$. We can assume $\|R_j\|_{L^2(M)} = O(\tau^{-1})$
as $\tau \to \infty$, and that the relative (absolute) boundary values of Z_1 are supported in $\widetilde{F}$ and the relative
(absolute) boundary values of Z_2 are supported in $\widetilde{B}$. By Lemma 8.1 (Lemma 8.2), we have

$$0 = \lim_{\tau \to \infty} (QZ_1, Z_2)_{L^2(M)}$$
$$= \int_{S^{n-2}} \int_0^{\infty} e^{-2\lambda r} \left[\sum_{I,J} \left[\int_{-\infty}^{\infty} e^{-2i\lambda x_1} \langle Q(x_1, r, \theta) \eta^I, \eta^J \rangle \, dx_1 \right] c_I(\theta) \, \overline{d_J(\theta)} \right] dr \, d\theta.$$

We now extend the M_0-geodesic γ to $\widehat{M}_0$, choose $\omega = \gamma(-\varepsilon)$ for small $\varepsilon > 0$, and choose θ_0 so that
$\gamma(t) = (t, \theta_0)$. The functions c_I and d_J can be chosen freely, and by varying them we obtain

$$\int_0^{\infty} e^{-2\lambda r} \left[\int_{-\infty}^{\infty} e^{-2i\lambda x_1} \langle Q(x_1, r, \theta_0) \eta^I, \eta^J \rangle \, dx_1 \right] dr = 0$$

for each fixed I and J. Since Q is compactly supported in M_0^{int}, this implies the required result.

It remains to show that a frame $\{\eta^I\}$ with the required properties exists. Let $\omega = \gamma(0)$, and let $(\widehat{M}_0, g_0)$
be a simple manifold with $(M_0, g_0) \Subset (\widehat{M}_0, g_0)$ such that the $\widehat{M}_0$-geodesic starting at ω in direction $\nu(\omega)$
never meets M_0. (It is enough to embed (M_0, g_0) in some closed manifold and to take $\widehat{M}_0$ strictly convex
and slightly larger than M_0.) Let (r, θ) be polar normal coordinates in $\widehat{M}_0$ with centre $\omega = \gamma(0)$, fix $r_0 > 0$
so that the geodesic ball $B(\omega, r_0)$ is contained in $\widehat{M}_0^{\text{int}}$, and let $\hat{\theta} \in S^{n-2}$ be the direction of $\nu(\omega)$. Choose
some orthonormal frame $\eta^3, \ldots, \eta^n$ of the cotangent space of $\partial B(\omega, r_0) \setminus \{(r_0, \hat{\theta})\}$, and extend these as
1-forms in M_0^{int} by parallel transporting along integral curves of ∂_r. We thus obtain a global orthonormal
frame $\eta^2, \ldots, \eta^n$ of $T^* M_0^{\text{int}}$ with $\eta^2 = dr$ and $\nabla_{\partial_r} \eta^j = 0$ for $3 \leq j \leq n$. Moreover, $\eta^1, \ldots, \eta^n$ will be a
global orthonormal frame of $T^*(\mathbb{R} \times M_0^{\text{int}})$ inducing an orthonormal frame $\{\eta^I\}$ of $\Lambda(\mathbb{R} \times M_0^{\text{int}})$. $\square$

We will now show how the coefficients are uniquely determined by the integrals in Proposition 8.3.
This follows by inverting attenuated ray transforms, a topic of considerable independent interest (see the
survey [Finch 2003] for results in the Euclidean case, and the survey [Paternain et al. 2014] and references
below for the manifold case). The transform in Proposition 8.3 is not exactly the same kind of attenuated
ray transform/Fourier transform as in the scalar case, for instance, in [Dos Santos Ferreira et al. 2009a],
since the matrix element of Q that appears in the integral may actually depend on the geodesic γ (note

that the 1-forms η depend on γ). To clarify this point, we fix some global orthonormal frame $\{\varepsilon^1, \ldots, \varepsilon^n\}$ of $T^*(\mathbb{R} \times M_0)$ with $\varepsilon^1 = dx^1$, and let $\{\varepsilon^I\}$ be the corresponding orthonormal frame of $\Lambda(\mathbb{R} \times M_0)$. Define the matrix elements

$$q_{I,J} = \langle Q\varepsilon^I, \varepsilon^J \rangle.$$

Define also

$$\hat{q}_{I,J}(\xi_1, x') = \int_{-\infty}^{\infty} e^{-ix_1\xi_1} q_{I,J}(x_1, x') \, dx_1.$$

Then the conclusion in [Proposition 8.3](#) implies

$$\int_0^L e^{-2\lambda r} \hat{q}_{I',J'}(2\lambda, \gamma(r)) \langle \eta^I, \varepsilon^{I'} \rangle \langle \eta^J, \varepsilon^{J'} \rangle \, dr = 0$$

for any $\lambda \in \mathbb{R}$, for any I, J, and for any maximal geodesic γ in M_0. (Note that the inner products $\langle \eta^I, \varepsilon^{I'} \rangle$ do not depend on x_1.)

Up until now everything discussed in this paper has held for any dimension $n \geq 3$. Now, however, we will invoke the assumption that $n = 3$. Then $q_{I,J}$ is an 8×8 matrix. In this case we may choose $\eta^1 = dx^1$, $\eta^2 = dr$, and $\eta^3 = *_{g_0} dr$, where dr is the 1-form dual to $\dot{\gamma}$ on the geodesic γ. Let also $\{e_j\}$ be the orthonormal frame of vector fields dual to $\{\varepsilon^j\}$ (which is assumed to be positively oriented). It follows that

$$\langle \eta^1, \varepsilon^1 \rangle = 1, \quad \langle \eta^1, \varepsilon^2 \rangle = 0, \quad \langle \eta^1, \varepsilon^3 \rangle = 0,$$
$$\langle \eta^2, \varepsilon^1 \rangle = 0, \quad \langle \eta^2, \varepsilon^2 \rangle = \langle e_2, \dot{\gamma} \rangle, \quad \langle \eta^2, \varepsilon^3 \rangle = \langle e_3, \dot{\gamma} \rangle,$$
$$\langle \eta^3, \varepsilon^1 \rangle = 0, \quad \langle \eta^3, \varepsilon^2 \rangle = -\langle e_3, \dot{\gamma} \rangle, \quad \langle \eta^3, \varepsilon^3 \rangle = \langle e_2, \dot{\gamma} \rangle.$$

The relations for $\eta^{\{1,2\}} = \eta^1 \wedge \eta^2$, $\eta^{\{3,1\}}$, $\eta^{\{2,3\}}$ and $\varepsilon^{\{1,2\}}$, $\varepsilon^{\{3,1\}}$, $\varepsilon^{\{2,3\}}$ can be determined from the above relations by duality. Finally, $\langle \eta^0, \varepsilon^I \rangle = 1$ if $I = 0$ and 0 otherwise, and the other relations for η^0, ε^0, $\eta^{\{1,2,3\}}$, and $\varepsilon^{\{1,2,3\}}$ are similar.

Now choosing $I = J = 1$ (here we identify 1 with $\{1\}$) we obtain

$$\int_0^L e^{-2\lambda r} \hat{q}_{1,1}(2\lambda, \gamma(r)) \, dr = 0 \quad \text{for all } \lambda \text{ and } \gamma.$$

This means that the usual attenuated geodesic ray transform of the function $\hat{q}_{1,1}(2\lambda, \cdot)$ in M_0 vanishes for all λ. First we have $\hat{q}_{1,1}(2\lambda, \cdot) \in C^\infty(M_0)$ for all λ [Frigyik et al. 2008, Proposition 3], and then $\hat{q}_{1,1}(2\lambda, \cdot) = 0$ for all λ by the injectivity of the attenuated ray transform [Salo and Uhlmann 2011] and so $q_{1,1} = 0$. The same argument applies for all pairs (I, J) where

$$I, J \in \{0, 1, \{2, 3\}, \{1, 2, 3\}\}.$$

Now consider the case where $I = 1$ and $J = 2$. Then

$$\int_0^L e^{-2\lambda r} \left(\hat{q}_{1,2}(2\lambda, \gamma(r)) \langle e_2, \dot{\gamma} \rangle + \hat{q}_{1,3}(2\lambda, \gamma(r)) \langle e_3, \dot{\gamma} \rangle \right) dr = 0.$$

Then the injectivity result for the attenuated ray transform on 1-tensors [Salo and Uhlmann 2011] together with the regularity result [Holman and Stefanov 2010, Proposition 1] says

$$\hat{q}_{1,2}(2\lambda, x)\varepsilon^2 + \hat{q}_{1,3}(2\lambda, x)\varepsilon^3 = 0$$

for all $\lambda \neq 0$, from which we can conclude

$$q_{1,2} = q_{1,3} = 0.$$

The same argument then applies for all pairs (I, J) where

$$I \in \{0, 1, \{2, 3\}, \{1, 2, 3\}\} \quad \text{and} \quad J \in \{2, 3, \{1, 2\}, \{3, 1\}\},$$

or vice versa.

Finally, consider the case when $I = J = 2$. For brevity, we'll write $\langle e_j, \dot{\gamma} \rangle$ as $\dot{\gamma}_j$. Then $I = J = 2$ gives

$$\int_0^L e^{-2\lambda r} \left(\hat{q}_{2,2}\dot{\gamma}_2^2 + \hat{q}_{2,3}\dot{\gamma}_2\dot{\gamma}_3 + \hat{q}_{3,2}\dot{\gamma}_3\dot{\gamma}_2 + \hat{q}_{3,3}\dot{\gamma}_3^2 \right) dr = 0. \tag{8-1}$$

The integrand here can be represented as the symmetric 2-tensor

$$f^{2,2} := \begin{pmatrix} \hat{q}_{2,2} & \frac{1}{2}(\hat{q}_{2,3} + \hat{q}_{3,2}) \\ \frac{1}{2}(\hat{q}_{2,3} + \hat{q}_{3,2}) & \hat{q}_{3,3} \end{pmatrix}$$

(in coordinates provided by $\{\varepsilon^2, \varepsilon^3\}$) applied to $(\dot{\gamma}, \dot{\gamma})$. This shows that the attenuated ray transform of the 2-tensor $f^{2,2}$ in (M_0, g_0), with constant attenuation -2λ, vanishes identically.

We will now make use of the methods of [Paternain et al. 2013] in this tensor tomography problem. We only give the details in the case where Q (and hence $f^{2,2}$) is C^∞. The result also holds for continuous Q by using an elliptic regularity result for the normal operator, but in the present weighted case for 2-tensors the required result may not be in the literature. We only say that such a result can be proved by adapting the methods of [Holman and Stefanov 2010] to the 2-tensor case (in particular one needs a solenoidal decomposition $f = f^s + d\beta$ of a 2-tensor f and a further solenoidal decomposition $\beta = \beta^s + d\phi$ of the 1-form β, and one then shows that the normal operator acting on "solenoidal triples" (f^s, β^s, ϕ) is elliptic because the weight comes from a nonvanishing attenuation).

Since $f^{2,2}$ is C^∞, the injectivity result for the attenuated ray transform on symmetric 2-tensors (see [Assylbekov 2012], following [Paternain et al. 2013]) says

$$f^{2,2} = -Xu + 2\lambda u,$$

where X is the geodesic vector field on (M_0, g_0), and u is a smooth function on the unit circle bundle SM_0 that corresponds to the sum of a 1-tensor and scalar function, with

$$u|_{\partial M_0} = 0.$$

Here we have identified $f^{2,2}$ and u with functions on SM_0 as in [Paternain et al. 2013]. We can also express u and $f^{2,2}$ in terms of Fourier components as in [loc. cit.],

$$u = u_{-1} + u_0 + u_1,$$
$$f^{2,2} = f_{-2}^{2,2} + f_0^{2,2} + f_2^{2,2}.$$

Here $u_0 \in C^\infty(M_0)$, $u_1 + u_{-1}$ corresponds to a smooth 1-tensor in M_0, and u_0, u_1, u_{-1} vanish on ∂M_0. Then

$$-X(u_{-1} + u_0 + u_1) + 2\lambda(u_{-1} + u_0 + u_1) = f_{-2}^{2,2} + f_0^{2,2} + f_2^{2,2}.$$

Now parity implies the equations

$$2\lambda(u_{-1} + u_1) = Xu_0 \quad \text{and} \quad -X(u_{-1} + u_1) + 2\lambda(u_0) = f_{-2}^{2,2} + f_0^{2,2} + f_2^{2,2}.$$

Assume λ is nonzero. Using the first equation in the second one implies

$$-\frac{X^2(u_0)}{2\lambda} + 2\lambda u_0 = f^{2,2}, \tag{8-2}$$

where $X^2 u_0$ corresponds to the covariant Hessian $\nabla^2 u_0$ of u_0. The first equation implies u_0 vanishes to first order on ∂M_0.

Unfortunately, this is not enough to conclude that the coefficients of $f^{2,2}$ are 0. However, going back and choosing $(I, J) = (2, 3), (3, 2)$, and $(3, 3)$ gives us three additional equations of this type with the same elements $q_{I,J}$. More specifically,

$$f^{2,3} = \begin{pmatrix} \hat{q}_{2,3} & \frac{1}{2}(\hat{q}_{3,3} - \hat{q}_{2,2}) \\ \frac{1}{2}(\hat{q}_{3,3} - \hat{q}_{2,2}) & -\hat{q}_{3,2} \end{pmatrix},$$

$$f^{3,2} = \begin{pmatrix} \hat{q}_{3,2} & \frac{1}{2}(\hat{q}_{3,3} - \hat{q}_{2,2}) \\ \frac{1}{2}(\hat{q}_{3,3} - \hat{q}_{2,2}) & -\hat{q}_{2,3} \end{pmatrix},$$

$$f^{3,3} = \begin{pmatrix} \hat{q}_{3,3} & -\frac{1}{2}(\hat{q}_{2,3} + \hat{q}_{3,2}) \\ -\frac{1}{2}(\hat{q}_{2,3} + \hat{q}_{3,2}) & \hat{q}_{2,2} \end{pmatrix}$$

are all of the same form. Therefore it follows that $f^{2,2} + f^{3,3}$ and $f^{2,3} - f^{3,2}$ are as well. But these are both scalar matrices, and if

$$-\frac{X^2(u_0)}{2\lambda} + 2\lambda u_0$$

is a scalar matrix, then also the covariant Hessian $\nabla^2 u_0$ is a scalar matrix in the $\{\varepsilon^2, \varepsilon^3\}$ basis.

To make the previous statement more explicit, identify (M_0, g_0) with the unit disk in $\mathbb{R}^2$ and choose an isothermal coordinate system (x^1, x^2) in which the metric is given by $e^{2\mu}\delta_{jk}$ for some $\mu \in C^\infty(M_0)$. Choosing $e_2 = e^{-\mu}\partial_1$ and $e_3 = e^{-\mu}\partial_2$, the condition $\nabla^2 u_0(e_2, e_2) - \nabla^2 u_0(e_3, e_3) = 0$ implies

$$\partial_1^2 u_0 - \partial_2^2 u_0 + b \cdot \nabla u_0 = 0 \quad \text{in } M_0$$

for some vector field $b \in C^\infty(M_0, \mathbb{R}^2)$ depending on μ. Since u_0 vanishes to first order on ∂M_0, extending u_0 by zero to $\mathbb{R}^2$ we have

$$\partial_1^2 u_0 - \partial_2^2 u_0 + b \cdot \nabla u_0 = 0 \quad \text{in } \mathbb{R}^2,$$

where $u_0 \in H^2(\mathbb{R}^2)$ is compactly supported and b is some smooth compactly supported vector field. Uniqueness for hyperbolic equations [Taylor 1996, Section 2.8] implies $u_0 = 0$.

The above argument shows that $f^{2,2} + f^{3,3}$ and $f^{2,3} - f^{3,2}$ are 0. Thus $\hat{q}_{2,2} + \hat{q}_{3,3} = 0$ and $\hat{q}_{2,3} - \hat{q}_{3,2} = 0$, showing that $f^{2,2}$ and $f^{2,3}$ are trace-free. Taking traces in (8-2) and using that u_0 vanishes to first order on ∂M_0 implies $u_0 = 0$ by unique continuation for elliptic equations. Thus $f^{2,2} = 0$ and similarly $f^{2,3} = 0$, which shows that $q_{2,2}, q_{2,3}, q_{3,2}$, and $q_{3,3}$ are zero as well.

The same argument now works for the remaining entries of q, and this finishes the proof.

9. Higher dimensions

In higher dimensions, $n > 3$, as noted above, everything up to and including the proof of Proposition 8.3 still holds. However, this does not reduce easily into a tensor tomography problem, as in the three-dimensional case, because we cannot choose the basis $\{\eta^i\}$ so that $\eta^3, \ldots, \eta^4$ to depend on $\eta^2 = dr$ in a tensorial manner.

More precisely, in general we lack tensors T_i for which $\eta^i = T_i(\eta^2, \ldots, \eta^2)$ for $i \geq 3$, as is the case in three dimensions. Moreover, even if the results of Proposition 8.3 can be reduced to a tensor tomography problem, there is no guarantee that it will be one for which there are useful injectivity results, since there are very few such results for k-tensors with $k > 2$.

However, in the Euclidean case we can do better, since we have the extra freedom to vary the Carleman weight φ. In particular, we can construct CGOs to reduce the problem in Lemmas 8.1 and 8.2 to a Fourier transform, as has been done for inverse problems for scalar functions, e.g., in [Bukhgeim and Uhlmann 2002]. Therefore we can conclude this paper by a proof for higher dimensions, in the Euclidean case.

Proof of Theorem 2.3. Fix coordinates $x_1, \ldots, x_n$ on $\mathbb{R}^n$. The corresponding basis for the cotangent space is $dx^1, \ldots, dx^n$, and this gives a corresponding basis $\{dx^I\}$ for ΛM.

Now note that if f is a scalar function, $\triangle(f dx^I) = (\triangle f) dx^I$. Therefore if α and β are unit vectors such that $\alpha \cdot \beta = 0$, then

$$e^{-\frac{\alpha \cdot x}{h}} h^2(-\triangle + Q)(e^{\frac{(\alpha + i\beta) \cdot x}{h}} dx^I) = O(h^2) dx^I.$$

Therefore Proposition 7.1 implies there exists $r \in L^2(M, \Lambda M)$ such that

$$(-\triangle + Q)(e^{\frac{(\alpha + i\beta) \cdot x}{h}}(dx^I + r)) = 0,$$

with $\|r\|_{L^2(M)} = O(h)$, and $Z = e^{\frac{(\alpha + i\beta) \cdot x}{h}}(dx^I + r)$ has relative boundary conditions which vanish on Γ_+^c.

Now if k and ℓ are mutually orthogonal unit vectors which are both orthogonal to α, then we can set $\beta_1 = \ell + hk$ and $\beta_2 = \ell - hk$, and create

$$Z_1 = e^{\frac{(-\alpha + i\beta_1) \cdot x}{h}}(dx^I + r_1) \quad \text{and} \quad Z_2 = e^{\frac{(\alpha + i\beta_2) \cdot x}{h}}(dx^I + r_2)$$

so that $(-\triangle + Q_1)Z_1 = (-\triangle + Q_2)Z_2 = 0$, and Z_1 and Z_2 have relative boundary conditions that vanish on Γ_-^c and Γ_+^c respectively.

Then Lemma 8.1, together with the hypotheses of Theorem 2.3, implies

$$(Q_1 - Q_2 \mid e^{-i2k \cdot x}) = 0.$$

This can be done for any k orthogonal to α. Since α can be varied slightly without preventing the relative boundary conditions of the solutions from vanishing on the correct set, this is in fact true for k in an open set, from which we can conclude that $Q_1 = Q_2$ on M.

The absolute boundary value version works similarly, with the appropriate change in the CGOs. $\square$

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