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NONRADIAL TYPE II BLOW UP FOR THE ENERGY-SUPERCRITICAL SEMILINEAR HEAT EQUATION

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We consider the semilinear heat equation in large dimension $d \geq 11$

$$\partial_t u = \Delta u + |u|^{p-1} u, \quad p = 2q+1,\ q \in \mathbb{N},$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary condition. In the supercritical range $p \geq p(d) > 1 + \frac{4}{d-2}$, we prove the existence of a countable family $(u_\ell)_{\ell \in \mathbb{N}}$ of solutions blowing up at time $T > 0$ with type II blow up:

$$\|u_\ell(t)\|_{L^\infty} \sim C(T-t)^{-c_\ell}$$

with blow-up speed $c_\ell > \frac{1}{p-1}$. The blow up is caused by the concentration of a profile Q which is a radially symmetric stationary solution:

$$u(x,t) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x-x_0}{\lambda(t)}\right), \quad \lambda \sim C(u_n)(T-t)^{\frac{c_\ell(p-1)}{2}},$$

at some point $x_0 \in \Omega$. The result generalizes previous works on the existence of type II blow-up solutions, which only existed in the radial setting. The present proof uses robust nonlinear analysis tools instead, based on energy methods and modulation techniques. This is the first nonradial construction of a solution blowing up by concentration of a stationary state in the supercritical regime, and it provides a general strategy to prove similar results for dispersive equations or parabolic systems and to extend it to multiple blow ups.

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Keywords: blow up, heat, soliton, ground state, nonlinear, nonradial, supercritical.

1. Introduction

1A. *The semilinear heat equation.* We study solutions of the semilinear heat equation (NLH)

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}, \\ u(0) = u_0, \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where u is real-valued, p is such that the nonlinearity is analytic, that is $p = 2q+1$, $q \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^d$ is a smooth bounded open domain. For smooth enough initial data u_0 satisfying some compatibility conditions at the border $\partial\Omega$, the Cauchy problem is well posed and there exists a unique maximal solution $u \in C\big((0,T), L^\infty(\Omega)\big)$. If $T < +\infty$, the solution is said to blow up and necessarily

$$\lim_{t\to T} \|u(t)\|_{L^\infty(\Omega)} = +\infty.$$

This paper addresses the general issue of the asymptotic behavior as $t \to T$. In the case $\Omega = \mathbb{R}^d$, there is a natural scale invariance, namely if u is a solution then so is

$$u_\lambda(\lambda^2 t, x) := \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x). \tag{1-2}$$

The Sobolev space that has an invariant norm for this scale change is

$$\dot{H}^{s_c}(\mathbb{R}^d) := \left\{ u : \int_{\mathbb{R}^d} |\xi|^{2s_c} |\hat{u}|^2 \, d\xi < +\infty \right\}, \quad s_c := \frac{d}{2} - \frac{2}{p-1}, \tag{1-3}$$

where $\hat{u}$ stands for the Fourier transform of u. Two particular solutions arise, the constant-in-space blow-up solution

$$u(t,x) = \pm \frac{\kappa(p)}{(T-t)^{\frac{1}{p-1}}}, \quad \kappa(p) := \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}, \tag{1-4}$$

and the unique (up to translation and scale change) radially decaying stationary solution Q (see [Li 1992] and references therein) solving the stationary elliptic equation

$$\Delta Q + Q^p = 0. \tag{1-5}$$

1B. *Blow-up for* (NLH). Being one of the model nonlinear evolution equations, blow-up dynamics has attracted a great amount of work (see [Quittner and Souplet 2007] for a review). In particular, one is interested in the description of the solution near the set of blow-up points, that is, the points x for which there exists $(t_n, x_n) \to (T, x)$ such that $|u(t_n, x_n)| \to +\infty$. A comparison argument with the constant-in-space blow-up solution (1-4) implies the lower bound

$$\limsup_{t\to T} \|u(t)\|_{L^\infty} (T-t)^{\frac{1}{p-1}} \geq \kappa(p)$$

and leads to the following distinction between type I and type II blow up [Matano and Merle 2004]:

$$u \text{ blows up with type I if } \limsup_{t\to T} \|u(t)\|_{L^\infty} (T-t)^{\frac{1}{p-1}} < +\infty,$$

$$u \text{ blows up with type II if } \limsup_{t\to T} \|u(t)\|_{L^\infty} (T-t)^{\frac{1}{p-1}} = +\infty.$$

The ODE blow up (1-4) does not see the dissipative term in (1-1) whereas type II blow up involves an interplay between dissipation and nonlinearity, and therefore its existence and properties may change according to d and p. In the series of works [Giga 1986; Giga and Kohn 1985; 1987; 1989; Giga et al. 2004; Merle and Zaag 1998; 2000], the authors show that in the energy subcritical range $1 < p < \frac{d+2}{d-2}$, all blow-up solutions are of type I and match the constant-in-space solution (1-4):

$$\limsup_{t\to T} \|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} = \kappa(p).$$

In the energy critical case $p = \frac{d+2}{d-2}$, $d = 4$, Schweyer [2012] constructed a radial type II blow-up solution, following the analysis of critical problems [Merle and Raphaël 2005a; 2005b; 2006; Raphaël and Schweyer 2013; 2014; Raphaël and Rodnianski 2012; Merle et al. 2013]; see also [Filippas et al. 2000]. In that case, the scale invariance (1-2) implies that there exists a one-dimensional continuum of ground states

$$\left(\frac{1}{\lambda^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda}\right)\right)_{\lambda>0}.$$

The properties of the ground state (1-5) then allow the existence of a solution u that stays close to this manifold,

$$u = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right) + \varepsilon, \quad \|\varepsilon\| \ll 1,$$

such that $\lambda(t) \to 0$ for some time $T > 0$, which makes the solution blow up. This blow-up scenario is not always possible as it heavily relies on the asymptotic behavior of the ground state, and is impossible in dimension $d \geq 7$ [Collot et al. 2016].

In the radial energy-supercritical case $p > \frac{d+2}{d-2}$, the Joseph–Lundgren exponent [1973]

$$p_{JL} := \begin{cases} +\infty & \text{if } d \leq 10, \\ 1 + \frac{4}{d-4-2\sqrt{d-1}} & \text{if } d \geq 11 \end{cases} \tag{1-6}$$

dictates the existence of type II blow-up solutions. For $\frac{d+2}{d-2} < p < p_{JL}$, type II blow-up solutions do not exist [Matano and Merle 2004; Mizoguchi 2011b]. For $p > p_{JL}$, type II blow-up solutions are completely classified. In [Herrero and Velázquez 1994] the authors predicted the existence of a countable family of solutions u_ℓ such that

$$\|u(t)\|_{L^\infty} \sim C(u_n(0))(T-t)^{\frac{\ell}{\alpha(d,p)}\frac{2}{p-1}}, \quad \ell \in \mathbb{N},\ \ell > \tfrac{1}{2}\alpha,$$

(α is defined in (1-10)), which are the same speeds as in the present paper. The rigorous proof was first made in an unpublished paper [Herrero and Velázquez] and then in [Mizoguchi 2004]. In the series of works [Matano 2007; Matano and Merle 2009; Mizoguchi 2007; 2011a] any type II blow-up solution was proved to have one of the above blow-up rates. These works have the powerful advantage that they deal with large solutions, but strongly rely on comparison principles that are only available for radial parabolic problems.

1C. ***Outlook on blow up for other problems.*** Many model nonlinear equations share similar features with (NLH). The construction of solutions concentrating a stationary state for the energy-supercritical Schrödinger and wave equations has been done in [Collot 2014; Merle et al. 2015], and recently for the harmonic heat flow in [Biernat and Seki 2016]. These concentration scenarios happen on a central manifold near the continuum of ground states

$$\left(\frac{1}{\lambda^{\frac{2}{p-1}}}Q\left(\frac{x}{\lambda}\right)\right)_{\lambda>0},$$

whose topological and dynamical properties have been a popular subject of studies [Schlag 2009; Krieger et al. 2015]. The possibility of various blow-up speeds is linked to the regularity of the solutions, and this is why parabolic problems are more rigid, thanks to the regularizing effect, than dispersive problems, for which a wider range of concentration scenarios exists [Krieger et al. 2008].

A major goal is the study of blow up for general data, where nonradial stationary states can appear as blow-up profiles [Duyckaerts et al. 2012]. The solution may also not be a small perturbation of it. One thus needs robust tools for the perturbative study of special nonlinear profiles as well as a better understanding of the set of stationary solutions. The present work is a step toward this general aim.

1D. ***Statement of the result.*** We revisit the result of [Herrero and Velázquez 1994; Mizoguchi 2004; 2005] with the techniques employed in [Raphaël and Rodnianski 2012] to address the nonradial setting. From [Li 1992], for $p>p_{JL}$ (defined in (1-6)) the radially decaying ground state Q, solution of (1-5), admits the asymptotic

$$Q(x)=\frac{c_\infty}{|x|^{\frac{2}{p-1}}}+\frac{a_1}{|x|^\gamma}+o(|x|^{-\gamma})\quad \text{as } |x|\to+\infty,\ a_1\neq 0, \tag{1-7}$$

with

$$c_\infty:=\left[\frac{2}{p-1}\left(d-2-\frac{2}{p-1}\right)\right]^{\frac{1}{p-1}}, \tag{1-8}$$

$$\gamma:=\tfrac{1}{2}(d-2-\sqrt{\triangle}),\quad \triangle:=(d-2)^2-4pc_\infty^{p-1}\ \ (\triangle>0\iff p>p_{JL}), \tag{1-9}$$

and we define

$$\alpha:=\gamma-\frac{2}{p-1}. \tag{1-10}$$

For $n\in\mathbb{N}$ we define the following numbers ($\triangle_n>0$ if $p>p_{JL}$):

$$-\gamma_n:=\frac{-(d-2)+\sqrt{\triangle_n}}{2},\quad \triangle_n:=(d-2)^2-4pc_\infty+4n(d+n-2).$$

The above numbers are directly linked with the existence and the number of instability directions of type II blow-up solutions concentrating Q. Our result is the existence and precise description of some localized type II blow-up solutions in any domain with smooth boundary.

Theorem 1.1 (existence of nonradial type II blow up for the energy-supercritical heat equation). *Let $d\geq 11$, $p=2q+1>p_{JL}$, $q\in\mathbb{N}$, where p_{JL} is given by (1-6). Let Q, γ, α, γ_n and s_c be given by (1-7), (1-9), (1-10), (1-18) and (1-3) and $\varepsilon>0$. Let $\Omega\subset\mathbb{R}^d$ be a smooth open bounded domain. For $x_0\in\Omega$*

let $\chi(x_0)$ be a smooth cut-off function around x_0 with support in Ω. Pick $\ell \in \mathbb{N}$ satisfying $2\ell > \alpha$. Then, there exists a large enough regularity exponent

$$s_+ = s_+(\ell) \in 2\mathbb{N}, \quad s_+ \gg 1,$$

such that under the nondegeneracy condition

$$\left(\tfrac{1}{2}d - \gamma_n\right) \notin 2\mathbb{N} \quad \text{for all } n \in \mathbb{N} \text{ such that } d - 2\gamma_n \le 4s_+, \tag{1-11}$$

there exists a solution $u \in C\big([0,T), L^\infty(\Omega)\big)$ of (1-1) with $u_0 \in H^{s_+}(\Omega)$ (which can be chosen smooth and compactly supported) blowing up in finite time $0 < T < +\infty$ by concentration of the ground state at a point $x_0' \in \Omega$ with $|x_0' - x_0| \le \varepsilon$. It is given by

$$u(t,x) = \chi_{x_0}(x) \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x - x_0'}{\lambda(t)}\right) + v, \tag{1-12}$$

where:

(i) *x_0' is the only blow-up point of u.*

(ii) *Blow-up speed*:

$$\|u\|_{L^\infty(\Omega)} = c(u_0)(T-t)^{-\frac{2\ell}{\alpha(p-1)}}(1+o(1)) \quad \text{as } t \to T,\ c(u_0) > 0, \tag{1-13}$$

$$\lambda(t) = c'(u_0)(1+o(1))(T-t)^{\frac{\ell}{\alpha}} \quad \text{as } t \to T,\ c'(u_0) > 0. \tag{1-14}$$

(iii) *Asymptotic stability above scaling in renormalized variables*:

$$\lim_{t\to T} \left\|\lambda(t)^{\frac{2}{p-1}} v(t, x_0 + \lambda(t)x)\right\|_{H^s(\lambda(t)^{-1}(\Omega - \{x_0\}))} = 0 \quad \text{for all } s_c < s \le s_+. \tag{1-15}$$

(iv) *Boundedness below scaling*:

$$\limsup_{t\to T} \|u(t)\|_{H^s(\Omega)} < +\infty \quad \text{for all } 0 \le s < s_c. \tag{1-16}$$

(v) *Asymptotic of the critical norm*:

$$\|u(t)\|_{H^{s_c}(\Omega)} = c(d,p)\sqrt{\ell}\sqrt{|\log(T-t)|}(1+o(1)) \quad \text{as } t \to T,\ c(d,p) > 0. \tag{1-17}$$

Comments on Theorem 1.1:

(1) *On the assumptions*. First, the assumption $p > p_{JL}$ is not just technical as radial type II blow up is impossible for $\frac{d+2}{d-2} < p < p_{JL}$ [Matano and Merle 2004; Mizoguchi 2011b]. Nonradial type II blow up solutions in this latter range, if they exist, must have a very different dynamical description. Next, if p is not an odd integer, then the nonlinearity $x \mapsto |x|^{p-1}x$ is singular at the origin, yielding regularity issues. In that case the techniques used in the present paper could only be applied for a certain range of integers ℓ. Eventually, the condition (1-11) is purely technical, as it avoids the presence of logarithmic corrections in some inequalities that we use. It could be removed since the analysis relies on gains that are polynomial and not logarithmic, but would weigh down the already long proof. Note that a large number of couples (p,ℓ) satisfy this condition. Indeed, only finitely many integers n are concerned by (1-20), and the value of γ_n is very rarely a rational number by (1-18).

(2) *Blow-up by concentration at any point and manifold of type II blow-up solutions*. For any $x_0 \in \Omega$, Theorem 1.1 provides a solution that concentrates at a point that can be arbitrarily close to x_0. In fact there exists a solution that concentrates exactly at x_0, meaning that this blow up can happen at any point of Ω. To show that, one needs an additional continuity argument, in addition to the information contained in the proof, to be able to reason as in [Planchon and Raphaël 2007; Merle 1992], for example. This continuity property amounts to proving that the set of type II blow-up solutions that we construct is a Lipschitz manifold with exact codimension in a suitable functional space. This was proved in the radial setting in [Collot 2014] and the analysis could be adapted here using the nonradial analysis provided in the present paper. However a precise and rigorous proof of this fact would be too lengthy to be inserted in this paper. Let us stress that the solutions built here possess an explicit number of linear nonradial instabilities. An interesting question is then whether or not these new instabilities can be used, with the help of resonances through the nonlinear term, to produce new type II blow-up mechanisms around Q in the nonradial setting.

(3) *Multiple blow ups and continuation after blow up*. As in our analysis we are able to cut and localize the approximate blow-up profile, there should be no problems in constructing a solution blowing up with this mechanism at several points simultaneously, as in [Merle 1992]. Cases where the blow-up bubbles really interact can lead to very different dynamics; see [Martel and Raphaël 2015; Jendrej 2016] for recent results. From the construction, as $t \to T$, we have u admits a strong limit in $H^{s_c}_{\text{loc}}(\Omega \backslash \{x_0\})$. One could investigate the properties of this limit in order to continue the solution u beyond blow-up time, which is a relevant question for blow-up issues [Matano and Merle 2009], especially for hamiltonian equations where a subcritical norm is under control.

1E. *Notation.* In the analysis, C will stand for a constant which may vary from one line to another, whose value just depends on d and p. The notation $a \lesssim b$ means that $a \leq Cb$ for such a constant C, and $a = O(b)$ means $|a| \lesssim b$.

Supercritical numerology. For $d \geq 11$ the condition $p > p_{JL}$, where p_{JL} is defined by (1-6), is equivalent to $2 + \sqrt{d-1} < s_c < \frac{1}{2}d$. We define the sequences of numbers describing the asymptotic of particular zeros of H (defined in (1-30)) for $n \in \mathbb{N}$:

$$-\gamma_n := \frac{-(d-2) + \sqrt{\triangle_n}}{2}, \qquad \triangle_n := (d-2)^2 - 4cp_\infty + 4n(d+n-2), \tag{1-18}$$

$$\alpha_n := \gamma_n - \frac{2}{p-1}, \tag{1-19}$$

where $\triangle_n > 0$ for $p > p_{JL}$. We will use the following facts in the sequel:

$$\gamma_0 = \gamma, \qquad \gamma_1 = \frac{2}{p-1} + 1, \qquad \gamma_n < \frac{2}{p-1} \quad \text{for } n \geq 2 \text{ and } \gamma_n \sim -n; \tag{1-20}$$

see Lemma A.1 (where γ is defined in (1-9)). In particular $\alpha_0 = \alpha$, $\alpha_1 = 1$ and $\alpha_n < 0$ for $n \geq 2$. A computation yields the bound

$$2 < \alpha < \tfrac{1}{2}d - 1$$

(see [Merle et al. 2015]). We let

$$g := \min(\alpha, \triangle) - \varepsilon, \quad g' := \tfrac{1}{2}\min(g, 1, \delta_0 - \varepsilon), \tag{1-21}$$

where $0 < \varepsilon \ll 1$ is a very small constant just here to avoid keeping track of some logarithmic terms later on. For $n \in \mathbb{N}$ we define[1]

$$m_n := E\big[\tfrac{1}{2}\big(\tfrac{1}{2}d - \gamma_n\big)\big] \tag{1-22}$$

and denote by δ_n the positive real number $0 \le \delta_n < 1$ such that

$$d = 2\gamma_n + 4m_n + 4\delta_n. \tag{1-23}$$

For $1 \ll L$ a very large integer, we define the Sobolev exponent

$$s_L := m_0 + L + 1. \tag{1-24}$$

In this paper we assume the technical condition (1-11) for $s_+ = s_L$, which means

$$0 < \delta_n < 1 \tag{1-25}$$

for all integers n such that $d - 2\gamma_n \le 4s_L$ (there is only a finite number of such integers by (1-20)). We let n_0 be the last integer to satisfy the condition

$$d - 2\gamma_{n_0} \le 4s_L \quad \text{and} \quad d - 2\gamma_{n_0+1} > 4s_L \tag{1-26}$$

and we define

$$\delta_0' := \max_{0 \le n \le n_0} \delta_n \in (0, 1). \tag{1-27}$$

For all integers $n \le n_0$ we define the integers

$$L_n := s_L - m_n - 1 \tag{1-28}$$

and in particular $L_0 = L$. Given an integer $\ell > \frac{1}{2}\alpha$ (that will be fixed in the analysis later on), for $0 \le n \le n_0$ we define the real numbers

$$i_n = \ell - \frac{\gamma - \gamma_n}{2}. \tag{1-29}$$

Notations for the analysis. For $R \ge 0$, the euclidean sphere and ball are denoted by

$$\mathcal{S}^{d-1}(R) := \left\{ x \in \mathbb{R}^d, \sum_{i=1}^d x_i^2 = R^2 \right\} \quad \text{and} \quad \mathcal{B}^d(R) := \left\{ x \in \mathbb{R}^d, \sum_{i=1}^d x_i^2 \le R^2 \right\}.$$

We use the Kronecker delta notation:

$$\delta_{i,j} := \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$$

[1] $E[x]$ stands for the entire part: $x - 1 < E[x] \le x$.

for $i, j \in \mathbb{N}$. We let

$$F(u) := \Delta u + f(u), \quad f(u) := |u|^{p-1}u$$

so that (1-1) can be written as

$$\partial_t u = F(u).$$

When using the binomial expansion for the nonlinearity, we use the constants

$$f(u+v) = \sum_{l=0}^{p} C_l^p u^l v^{p-l}, \quad C_l^p := \binom{p}{l}.$$

The linearized operator close to Q (defined in (1-5)) is

$$Hu := -\Delta u - pQ^{p-1}u \tag{1-30}$$

so that $F(Q+\varepsilon) \sim -H\varepsilon$. We introduce the potential

$$V := -pQ^{p-1} \tag{1-31}$$

so that $H = -\Delta + V$. Given a strictly positive real number $\lambda > 0$ and function $u : \mathbb{R}^d \to \mathbb{R}$, we define the rescaled function

$$u_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x). \tag{1-32}$$

This semigroup has the infinitesimal generator

$$\Lambda u := \frac{\partial}{\partial \lambda}(u_\lambda)_{|\lambda=1} = \frac{2}{p-1}u + x.\nabla u.$$

The action of the scaling on (1-1) is given by the formula

$$F(u_\lambda) := \lambda^2 (F(u))_\lambda.$$

For $z \in \mathbb{R}^d$ and $u : \mathbb{R}^d \to \mathbb{R}$, the translation of vector z of u is denoted by

$$\tau_z u(x) := u(x-z). \tag{1-33}$$

This group has the infinitesimal generator

$$\left[\frac{\partial}{\partial z}(\tau_z u)\right]_{|z=0} = -\nabla u.$$

The original space variable will be denoted by $x \in \Omega$ and the renormalized one by y, related through $x = z + \lambda y$. The number of spherical harmonics of degree n is

$$k(0) := 1, \qquad k(1) := d, \qquad k(n) := \frac{2n+p-2}{n}\binom{n+p-3}{n-1} \quad \text{for } n \geq 2.$$

The Laplace–Beltrami operator on the sphere $\mathcal{S}^{d-1}(1)$ is self-adjoint with compact resolvent and its spectrum is $\{n(d+n-2) : n \in \mathbb{N}\}$. For $n \in \mathbb{N}$ the eigenvalue $n(d+2-n)$ has geometric multiplicity $k(n)$, and we denote by $(Y^{(n,k)})_{n\in\mathbb{N},\,1\le k\le k(n)}$ an associated orthonormal Hilbert basis of $L^2(\mathbb{S}^d)$:

$$L^2(\mathcal{S}^{d-1}(1))=\bigoplus_{n=0}^{+\infty}{}^{\perp}\operatorname{Span}\big(Y^{(n,k)},1\le k\le k(n)\big),$$

$$\Delta_{S^{d-1}(1)}Y^{(n,k)}=n(d+n-2)Y^{(n,k)},\quad \int_{S^{d-1}(1)}Y^{(n,k)}Y^{(n',k')}=\delta_{(n,k),(n',k')}, \tag{1-34}$$

with the special choices

$$Y^{(0,1)}(x)=C_0,\quad Y^{1,k}(x)=-C_1x_k, \tag{1-35}$$

where C_0 and C_1 are two renormalization constants. The action of H on each spherical harmonic is described by the family of operators on radial functions

$$H^{(n)}:=-\partial_{rr}-\frac{d-1}{r}\partial_r+\frac{n(d+n-2)}{r^2}-pQ^{p-1} \tag{1-36}$$

for $n\in\mathbb{N}$, as for any radial function f they produce the identity

$$H\bigg(x\mapsto f(|x|)Y^{(n,k)}\Big(\frac{x}{|x|}\Big)\bigg)=x\mapsto (H^{(n)}(f))(|x|)Y^{(n,k)}\Big(\frac{x}{|x|}\Big). \tag{1-37}$$

For two strictly positive real numbers $b_1^{(0,1)}>0$ and $\eta>0$ we define the scales

$$M\gg 1,\quad B_0=|b_1^{(0,1)}|^{-\frac{1}{2}},\quad B_1=B_0^{1+\eta}. \tag{1-38}$$

The blow-up profile of this paper is an excitation of several directions of stability and instability around the soliton Q. Each one of these directions of perturbation, denoted by $T_i^{(n,k)}$, will be associated to a triple (n,k,i), meaning that it is the i-th perturbation located on the spherical harmonics of degree (n,k). For each (n,k) with $n\le n_0$, there will be L_n+1 such perturbations for $i=0,\dots,L_n$ except for the cases $n=0$, $k=1$, and $n=1$, $k=1,\dots,d$, where there will be L_n perturbations for $i=1,\dots,L_n$ $(n=1,2)$. Hence the set of triples (n,k,i) used in the analysis is

$$\mathcal{I}:=\big\{(n,k,i)\in\mathbb{N}^3:\ 0\le n\le n_0,\ 1\le k\le k(n),\ 0\le i\le L_n\big\}\backslash\big(\{(0,1,0)\}\cup\{(1,1,0),\dots,(1,d,0)\}\big) \tag{1-39}$$

with cardinal

$$\#\mathcal{I}:=\sum_{n=0}^{n_0}k(n)(L_n+1)-d-1. \tag{1-40}$$

For $j\in\mathbb{N}$ and an n-tuple of integers $\mu=(\mu_i)_{1\le i\le j}$, the usual length is denoted by

$$|\mu|:=\sum_{i=1}^{j}\mu_i.$$

If $j=d$ and h is a smooth function on $\mathbb{R}^d$ then we use the following notation for the differentiation:

$$\partial^\mu h:=\frac{\partial^{|\mu|}}{\partial_{x_1}^{\mu_1}\cdots\partial_{x_d}^{\mu_d}}h.$$

For J an $\#\mathcal{I}$-tuple of integers, we introduce two other weighted lengths

$$|J|_2 = \sum_{n,k,i} \left(\frac{\gamma - \gamma_n}{2} + i \right) J_i^{(n,k)}, \tag{1-41}$$

$$|J|_3 = \sum_{i=1}^{L} i J_i^{(0,1)} + \sum_{\substack{1 \le i \le L_1 \\ 1 \le k \le d}} i J_i^{(1,k)} + \sum_{\substack{(n,k,i) \in \mathcal{I} \\ 2 \le n}} (i+1) J_i^{(n,k)}. \tag{1-42}$$

To localize some objects we will use a radial cut-off function $\chi \in C^\infty(\mathbb{R}^d)$:

$$0 \le \chi \le 1, \qquad \chi(|x|) = 1 \quad \text{for } |x| \le 1, \qquad \chi(|x|) = 0 \quad \text{for } |x| \ge 2, \tag{1-43}$$

and for $B > 0$, we let χ_B denote the cut-off around $\mathcal{B}^d(0, B)$:

$$\chi_B(x) := \chi\left(\frac{x}{B}\right).$$

1F. ***Strategy of the proof.*** We now describe the main ideas behind the proof of Theorem 1.1. Without loss of generality, via scale change and translation in space one can assume that $x_0 = 0$ and $\mathcal{B}^d(7) \subset \Omega$.

(i) *Linear analysis and tail computations*. The linearized operator near Q is $H = -\Delta - pQ^{p-1}$ and its generalized kernel is

$$\{f : \exists j \in \mathbb{N} \text{ such that } H^j f = 0\} = \operatorname{Span}(T_i^{(n,k)})_{(n,i) \in \mathbb{N}^2,\, 1 \le k \le k(n)},$$

where

$$T_i^{(n,k)}(x) = T_i^{(n)}(|x|) Y^{(n,k)}\left(\frac{x}{|x|}\right),$$

$T_i^{(n)}$ being radial, is located on the spherical harmonics of degree (n, k), with

$$T_0^{(0,1)} = \Lambda Q, \quad T_0^{(1,k)} = \partial_{x_k} Q, \quad H T_0^{(n,k)} = 0, \quad H T_{i+1}^{(n,k)} = -T_i^{(n,k)}. \tag{1-44}$$

For any $L \in \mathbb{N}$, defining s_L, $n_0(L)$ and $L_n(L)$ by (1-24), (1-26) and (1-28), H^{s_L} is coercive for functions that are not in the suitably truncated generalized kernel:

$$\int \varepsilon H^{s_L} \varepsilon \gtrsim \|\nabla^{s_L} \varepsilon\|_{L^2}^2 + \|\varepsilon\|_{\text{loc}}^2 \quad \text{if } \varepsilon \in \operatorname{Span}(T_i^{(n,k)})^{\perp}_{0 \le n \le n_0,\, 1 \le k \le k(n),\, 0 \le i \le L_n}, \tag{1-45}$$

where $\|\varepsilon\|_{\text{loc}}^2$ means any norm of ε on a compact set involving derivatives up to order $2s_L$. A scale change for these profiles produces the identity

$$\frac{\partial}{\partial \lambda}(T_i^{(n,k)})_{|\lambda=1}(x) = \Lambda T_i^{(n,k)}(x) \sim (2i - \alpha_n) T_i^{(n,k)}(x) \quad \text{as } |x| \to +\infty. \tag{1-46}$$

(ii) *The renormalized flow*. For u a solution, $\lambda : (0, T) \to \mathbb{R}$ and $z : (0, T) \to \mathbb{R}^d$, we define the renormalized time

$$\frac{ds}{dt} = \frac{1}{\lambda^2}, \quad s(0) = s_0. \tag{1-47}$$

Then $v = (\tau_{-z}u)_\lambda$ solves the renormalized equation

$$\partial_s v - \frac{\lambda_s}{\lambda}\Lambda v - \frac{z_s}{\lambda}.\nabla v - F(v) = 0. \tag{1-48}$$

(iii) *The dynamical system for the coordinates on the center manifold*. Let $\mathcal{I}$ be defined by (1-39). For an approximate solution of (1-1) under the form

$$u = \Big(Q + \sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} T_i^{(n,k)} \Big)_{z,\frac{1}{\lambda}} \tag{1-49}$$

described by some parameters $b_i^{(n,k)} \in \mathbb{R}$, one has the identity from (1-44) and (1-45):

$$\begin{aligned} -z_t.\nabla u - \frac{\lambda_t}{\lambda}\Lambda u + \Big(\sum_{(n,k,i)\in\mathcal{I}} b_{i,t}^{(n,k)} T_i^{(n,k)} \Big)_{z,\frac{1}{\lambda}} &\\ = \partial_t u \approx F(u) &\\ = \frac{b_1^{(1,\cdot)}}{\lambda}.\nabla u + \frac{b_1^{(0,1)}}{\lambda^2}\Lambda u + \Big(\sum_{(n,k,i)\in\mathcal{I}} \frac{b_{i+1}^{(n,k)} - (2i-\alpha_n) b_1^{(1,0)} b_i^{(n,k)}}{\lambda^2} T_i^{(n,k)} \Big)_{z,\frac{1}{\lambda}} + \psi, \end{aligned} \tag{1-50}$$

where $b_1^{(1,\cdot)} = (b_1^{(1,1)}, \dots, b_1^{(1,d)})$ and with the convention $b_{L_n+1}^{(n,k)} = 0$. The error term ψ is negligible under a size assumption on the parameters. Identifying the terms in the above identity yields the finite-dimensional dynamical system[2]

$$\begin{cases} \lambda_t = -\dfrac{b_1^{(0,1)}}{\lambda}, \quad z_t = -\dfrac{b_1^{(1,\cdot)}}{\lambda}, \\ b_{i,t}^{(n,k)} = -\dfrac{1}{\lambda^2}(2i-\alpha_n) b_1^{(0,1)} b_i^{(n,k)} + \dfrac{1}{\lambda^2} b_{i+1}^{(n,k)} \quad \forall (n,k,i)\in\mathcal{I}. \end{cases} \tag{1-51}$$

(iv) *The approximate blow-up profile*. Equation (1-51) admits for any $\ell \in \mathbb{N}$ with $2\ell > \alpha$ an explicit special solution $(\bar\lambda, \bar z, \bar b_i^{(n,k)})$ such that $\bar z = 0$ and $\bar\lambda \sim (T-t)^{\frac{\ell}{\alpha}}$ for some $T > 0$. Moreover, when linearizing (1-51) around this solution, one finds an explicit number m of directions of linear instability and $\#\mathcal{I} - m$ directions of stability. In addition, for the renormalized time s associated to $\bar\lambda$, one has

$$\lim_{t\to T} s(t) = +\infty, \quad |\bar b_k^{(i,n)}(s)| \lesssim s^{-\frac{\gamma-\gamma_n}{2}-i}. \tag{1-52}$$

Our approximate blow-up profile is then given by

$$\Big(Q + \sum_{(n,k,i)\in\mathcal{I}} \bar b_i^{(n,k)}(t) T_i^{(n,k)} \Big)_{\bar z(t),\frac{1}{\bar\lambda(t)}}.$$

(v) *The blow-up ansatz*. Following (iv), we study solutions of the form

$$u = \chi \Big(Q + \sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} T_i^{(n,k)} \Big)_{z,\frac{1}{\lambda}} + w \tag{1-53}$$

[2] Again, with the convention $b_{L_n+1}^{(n,k)} = 0$.

and decompose the remainder w according to

$$w_{\text{int}} := \chi_3 w, \quad w_{\text{ext}} := (1-\chi_3)w, \quad \varepsilon := (\tau_{-z} w_{\text{int}})_\lambda, \tag{1-54}$$

where w_{ext} is the remainder outside the blow-up zone, w_{int} the remainder inside the blow-up zone, and ε is the renormalization of the remainder inside the blow-up zone corresponding to the scale and central point of the ground state $Q_{z,1/\lambda}$. Now w is orthogonal to the suitably truncated center manifold

$$\varepsilon \in \text{Span}(T_i^{(n,k)})^{\perp}_{0\le n\le n_0,\, 1\le k\le k(n),\, 0\le i\le L_n}, \tag{1-55}$$

which fixes in a unique way the value of the parameters $b_i^{(n,k)}$, λ and z. We then define the renormalized time s associated to λ via (1-47). We take b, λ and z to be perturbations of $\bar{b}$, $\bar{\lambda}$ and $\bar{z}$ for the renormalized time:

$$b_i^{(n,k)}(s) = \bar{b}_i^{(n,k)}(s) + b_i'^{(n,k}(s),\ \lambda(s) = \bar{\lambda}(s) + \lambda'(s),\ z(s) = \bar{z}(s) + z'(s). \tag{1-56}$$

We define four norms for the remainder in (1-53) and (1-54):

$$\mathcal{E}_\sigma := \|\nabla^\sigma \varepsilon\|^2_{L^2(\mathbb{R}^d)}, \quad \mathcal{E}_{2s_L} := \int_{\mathbb{R}^d} |H^{s_L}\varepsilon|^2, \quad \|w_{\text{ext}}\|_{H^\sigma(\Omega)} \quad \text{and} \quad \|w_{\text{ext}}\|_{H^{s_L}(\Omega)},$$

where σ is a slightly supercritical regularity exponent

$$0 < \sigma - s_c \ll 1. \tag{1-57}$$

One has that $\mathcal{E}_{2s_L} \gtrsim \|\nabla^{2s_L}\varepsilon\|_{L^2}$ from (1-45).

Interpretation: We decompose a solution near the set of localized and concentrated ground states $\chi(Q_{z,1/\lambda})$ according to (1-53). A part, $\chi\big(\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} T_i^{(n,k)}\big)_{z,1/\lambda}$, is located on the truncated center manifold; it decays slowly, see (1-52), while interacting with the ground state, see (1-51), and is responsible for the blow up by concentration, and one has an explicit behavior of the coordinates, (1-51). The other part, w, is orthogonal to the truncated center manifold (1-55); it is expected to decay faster as H is more coercive, see (1-45), on this set, and not to perturb the blow-up dynamics. The change of variables (1-47) and (1-48) transforms the blow-up problem into a long-time asymptotic problem by (1-52).

Bootstrap method in a trapped regime: We study solutions that are close to the approximate blow-up profile for the renormalized time, i.e., that satisfy

$$\mathcal{E}_\sigma + \|w_{\text{ext}}\|^2_{H^\sigma(\Omega)} \lesssim 1, \quad \mathcal{E}_{2s_L} + \|w_{\text{ext}}\|_{H^{s_L}(\Omega)} \lesssim \frac{1}{\lambda^{2(2s_L-s_c)}\, s^{L+(1-\delta_0)+\nu}}, \tag{1-58}$$

$$|b_i'^{(n,k)}| \lesssim s^{-\frac{\gamma-\gamma_n}{2}-i}, \quad |\lambda| + |z| \ll 1. \tag{1-59}$$

The size of the excitation is

$$\frac{1}{\lambda^{2(2s_L-s_c)} s^{L+(1-\delta_0')}}$$

so $\chi\big(\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} T_i^{(n,k)}\big)_{z,1/\lambda}$ and $\nu > 0$ in (1-58) quantifies the fact that the remainder w is smaller than the excitation.

(v) *The bootstrap regime*. From (1-1) and (1-50), the evolution of the solution under the decomposition (1-53) and (1-54) has the form

$$\partial_t w_{\text{ext}} = \Delta w_{\text{ext}} + \Delta\chi_3 w + 2\nabla\chi_3.\nabla w + (1-\chi_3)w^p, \tag{1-60}$$

$$\begin{aligned}\partial_t w_{\text{int}} = {} & -H_{z,\frac{1}{\lambda}} w_{\text{int}} + \chi\psi + \text{NL}\\ & + \chi\biggl(\Bigl(\frac{b_1^{(1,\cdot)}}{\lambda^2}+\frac{z_t}{\lambda}\Bigr).\nabla\bigl(Q+\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)}\bigr)\biggr)_{z,\frac{1}{\lambda}}\\ & + \chi\biggl(\Bigl(\frac{b_1^{(0,1)}}{\lambda^2}+\frac{\lambda_t}{\lambda}\Bigr)\Lambda\bigl(Q+\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)}\bigr)\biggr)_{z,\frac{1}{\lambda}}\\ & + \chi\biggl(\sum_{(n,k,i)\in\mathcal{I}}\Bigl(-b_{i,t}^{(n,k)}-\frac{(2i-\alpha_n)b_1^{(0,1)}b_1^{(n,k)}+b_{i+1}^{(n,k)}}{\lambda^2}\Bigr)T_i^{(n,k)}\biggr)_{z,\frac{1}{\lambda}},\end{aligned} \tag{1-61}$$

where $H_{z,\lambda} = -\Delta - pQ_{z,1/\lambda}^{p-1}$ and NL stands for the purely nonlinear term.

<u>Modulation:</u> The evolution of the parameters is computed using the orthogonality directions related to the decomposition, i.e., by taking the scalar product between (1-61) and $(T_i^{(n,k)})_{z,1/\lambda}$ for $0\le n\le n_0$, $1\le k\le k(n)$ and $0\le i\le L_n$, yielding in renormalized time an estimate of the form[3]

$$\left|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right|+\left|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right|+\sum_{(n,k,i)\in\mathcal{I}}\left|b_{i,s}^{(n,k)}+(2i-\alpha_n)b_i^{(n,k)}b_1^{(0,1)}+b_{i+1}^{(n,k)}\right|\lesssim\sqrt{\mathcal{E}_{2s_L}}+s^{-L-3}. \tag{1-62}$$

These estimates hold because the error produced by the approximate dynamics is very small (s^{-L-3}) on compact sets, and on the other hand the remainder ε is also very small on compact sets and located far away from the origin by (1-58) and the coercivity (1-45).

<u>Lyapunov monotonicity for the remainder:</u> From the evolution equations (1-60) and (1-61), in the bootstrap regime (1-58) one performs energy estimates of the form

$$\frac{d}{dt}\left(\frac{1}{\lambda^{2(\sigma-s_c)}}\mathcal{E}_\sigma+\|w_{\text{ext}}\|_{H^\sigma(\Omega)}\right)\lesssim\frac{1}{\lambda^2 s^{1+\kappa'}}+\frac{1}{\lambda^{(\sigma-s_c)}}\sqrt{\mathcal{E}_\sigma}\|\nabla^\sigma\psi\|_{L^2}, \tag{1-63}$$

$$\frac{d}{dt}\left(\frac{1}{\lambda^{2(2s_L-s_c)}}\mathcal{E}_{2s_L}+\|w_{\text{ext}}\|_{H^{2s_L}(\Omega)}\right)\lesssim\frac{1}{\lambda^{2(2s_L-s_c)+2}s^{L+2-\delta_0+\nu+\kappa}}+\frac{1}{\lambda^{2s_L-s_c}}\sqrt{\mathcal{E}_{2s_L}}\|H_{z,\frac{1}{\lambda}}^{s_L}\psi\|_{L^2}, \tag{1-64}$$

where $\kappa>0$ represents a gain. The key properties yielding these estimates are the following. The control of a slightly supercritical norm (1-57) and another high regularity norm allows us to control precisely the energy transfer between low and high frequencies and to control the nonlinear term. The dissipation in (1-60) and (1-61) (for the second equation it is a consequence of the coercivity (1-45)) erases the border terms and smaller-order local interactions. Finally, the approximate blow-up profile is in fact a refinement of (1-49), where the error in the approximate dynamics is well localized in the self-similar

[3] With the convention $b_{L_n+1}^{(n,k)}=0$.

zone $|x-z| \sim \sqrt{T-t}$, by the addition of suitable corrections via inverting elliptic equations and by precise cuts.

(vi) *Existence via a topological argument*. In the bootstrap regime close to the approximate blow-up profile described by (1-58) and (1-59), one has precise bounds for the error term ψ. Reintegrating the energy estimates (1-63) and (1-64) then leads to the bounds

$$\mathcal{E}_\sigma + \|w_{\mathrm{ext}}\|^2_{H^\sigma(\Omega)} \ll 1, \quad \mathcal{E}_{2s_L} + \|w_{\mathrm{ext}}\|_{H^{s_L}(\Omega)} \ll \frac{1}{\lambda^{2(2s_L-s_c)} s^{L+(1-\delta_0)+\nu}},$$

which are an improvement of (1-58). Therefore, a solution ceases to be in the bootstrap regime if and only if the bound (1-59) describing the proximity of the parameters with respect to the special blow-up parameters $(\bar{b}, \bar{\lambda}, \bar{z})$ is violated. From (iv), the parameters admit $(\bar{\lambda}, \bar{z}, \bar{b})$ as a hyperbolic orbit with m directions of instability and $\#\mathcal{I}-m$ of instability. From the modulation equations (1-62), the remainder w perturbs these dynamics only at lower order. Therefore, an application of the Brouwer fixed point theorem yields the persistence of an orbit similar to $(\bar{\lambda}, \bar{z}, \bar{b})$ for the full nonlinear equation, i.e., with a perturbation along the parameters that stays small for all time. This gives the existence of a true solution of (1-1) that stays close to the approximate blow-up profile for all renormalized times, implying blow up by concentration of Q with a precise asymptotic.

The paper is organized as follows. In Section 2 we recall the known properties of the ground state in Lemma 2.1 and describe the kernel of the linearized operator H in Lemma 2.3. This provides a formula to invert elliptic equations of the form $Hu = f$, stated in Definition 2.6, and allows us to describe the generalized kernel of H in Lemma 2.10. The blow-up profile is built on functions depending polynomially on some parameters and with explicit asymptotic at infinity, and we introduce the concept of homogeneous functions in Definition 2.14 and Lemma 2.15 to track this information easily. With these tools, in Section 3 we construct a first approximate blow-up profile for which the error is localized at infinity in Proposition 3.1 and we cut it in the self-similar zone in Proposition 3.3. The evolution of the parameters describing the approximate blow-up profile is an explicit dynamical system with special solutions given in Lemma 3.4 for which the linear stability is investigated in Lemma 3.5. In Section 4 we define a bootstrap regime for solutions of the full equation close to the approximate blow-up profile. We give a suitable decomposition for such solutions, using orthogonality conditions that are provided by Definition 4.1 and Lemma 4.2, in Lemma 4.3. They must satisfy in addition some size assumption, and all the conditions describing the bootstrap regime are given in Definition 4.4. The main result of the paper is Proposition 4.6, stating the existence of a solution staying for all times in the bootstrap regime, whose proof is relegated to the next section. With this result we end the proof of Theorem 1.1 in Section 4B. To do this, the modulation equations are computed in Lemma 4.7, yielding that solutions staying in the bootstrap regime must concentrate in Lemma 4.8 with an explicit asymptotic for Sobolev norm in Lemma 4.9. In Section 5 we prove the main proposition, Proposition 4.6. For solutions in the bootstrap regime, an improved modulation equation is established in Lemma 5.1, and Lyapunov-type monotonicity formulas are established in Propositions 5.3 and 5.5 for the low regularity Sobolev norms of the remainder, and in Propositions 5.6 and 5.8 for the high regularity norms. With this analysis one can characterize the conditions under which a solution leaves the bootstrap regime in Lemma 5.9, and with a topological argument provided in Lemma 5.10, one ends the proof of Proposition 4.6.

The appendix is organized as follows. In Appendix A we give the proof of Lemma 2.3, describing the kernel of H. In Appendix B we recall some Hardy and Rellich-type estimates, among which the most useful is given in Lemma B.3. In Appendix C we investigate the coercivity of H in Lemmas C.2 and C.3. In Appendix D we prove some bounds for solutions in the bootstrap regime. In Appendix E we give the proof of the decomposition Lemma 4.3.

2. Preliminaries on Q and H

We first summarize the content and ideas of this section. The instabilities near Q underlying the blow up that we study result from the excitement of modes in the generalized kernel of H. We first describe this set. Since H is radial, we use a decomposition into spherical harmonics, restricted to spherical harmonics of degree n, see (1-37), it becomes the operator $H^{(n)}$ on radial functions defined by (1-36). Using ODE techniques, the kernel is described in Lemma 2.3 and the inversion of $H^{(n)}$ is given by Definition 2.6 and Lemma 2.13. By inverting successively the elements in the kernel of $H^{(n)}$, one obtains the generators of the generalized kernel $\bigcup_j \operatorname{Ker}((H^{(n)})^j)$ of this operator in Lemma 2.10.

To track the asymptotic behavior and the dependence in some parameters of various profiles during the construction of the approximate blow-up profile in the next section, we introduce the framework of "homogeneous" functions in Definition 2.14 and Lemma 2.15.

2A. ***Properties of the ground state and the potential.*** Any positive smooth radially symmetric solution to

$$-\Delta\phi-\phi^p=0$$

is a dilate of a given normalized ground state profile Q:

$$\phi=Q_\lambda,\quad \lambda>0,\quad \begin{cases}-\Delta Q-Q^p=0,\\ Q(0)=1.\end{cases}$$

See [Li 1992] and references therein. The following lemma describes the asymptotic behavior of Q. We refer to [Ding and Ni 1985] for earlier work.

Lemma 2.1 (asymptotics of the ground state [Li 1992, Lemma 4.3; Karageorgis and Strauss 2007, Lemma 5.4]). *Let $p>p_{JL}$ (defined in (1-6)). We recall that $g>0$, c_∞ and γ are defined in (1-9) and (1-21). One has the asymptotics*

$$Q=\frac{c_\infty}{r^{\frac{2}{p-1}}}+\frac{a_1}{r^\gamma}+O\left(\frac{1}{r^{\gamma+g}}\right)\quad \textit{as } r\to+\infty,\ a_1\neq 0, \tag{2-1}$$

$$V=-\frac{pc_\infty^{p-1}}{r^2}+O\left(\frac{1}{r^{2+\alpha}}\right)\quad \textit{as } r\to+\infty, \tag{2-2}$$

$$\frac{d}{d\lambda}[(Q_\lambda)^{p-1}]_{|\lambda=1}=O\left(\frac{1}{r^{2+\alpha}}\right)\quad \textit{as } r\to+\infty, \tag{2-3}$$

and these identities propagate to the derivatives. There exists $\delta(p)>0$ *such that the following pointwise bounds hold for all* $y\in\mathbb{R}^d$:

$$0<Q(y)<\frac{c_\infty}{|y|^{\frac{2}{p-1}}}, \tag{2-4}$$

$$-\frac{(d-2)^2}{4|y|^2}+\frac{\delta(p)}{|y|^2}\le V(y)<0. \tag{2-5}$$

Remark 2.2. The standard Hardy inequality

$$\int_{\mathbb{R}^d}|\nabla u|^2\ge\frac{(d-2)^2}{4}\int_{\mathbb{R}^d}\frac{u^2}{|y|^2}\,dy$$

and (2-4) then imply the positivity of H on $\dot{H}^1(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d}uHu\,dy\ge\int_{\mathbb{R}^d}\frac{\delta(p)u^2}{|y|^2}\,dy. \tag{2-6}$$

It is worth mentioning that the aforementioned expansion (2-1) is false for $p\le p_{JL}$. This asymptotic at infinity of Q is decisive for type II blow up via perturbation of it, as from [Matano and Merle 2004; Mizoguchi 2011b] it cannot occur for $\frac{d+2}{d-2}<p<p_{JL}$.

2B. *Kernel of H*.

Lemma 2.3 (kernel of $H^{(n)}$). *We recall that the numbers* $(\gamma_n)_{n\in\mathbb{N}}$ *and* g *are defined in* (1-18). *Let* $n\in\mathbb{N}$. *There exist* $T_0^{(n)},\Gamma^{(n)}:(0,+\infty)\to\mathbb{R}$ *two smooth functions such that if* $f:(0,+\infty)\to\mathbb{R}$ *is smooth and satisfies* $H^{(n)}f=0$, *then* $f\in\operatorname{Span}(T_0^{(n)},\Gamma^{(n)})$. *They enjoy the asymptotics*

$$\begin{cases} T_0^{(n)}(r)\underset{r\to0}{=}\sum_{j=0}^{l}c_j^{(n)}r^{n+2j}+O(r^{n+2+2l}) \quad \forall l\in\mathbb{N},\ c_0^{(n)}\neq0,\\ T_0^{(n)}\underset{r\to+\infty}{\sim}C_nr^{-\gamma_n}+O(r^{-\gamma_n-g}),\quad C_n\neq0,\\ \Gamma^{(n)}\underset{r\to0}{\sim}\frac{c_n'}{r^{d-2+n}}\quad\text{and}\quad\Gamma^{(n)}\underset{r\to+\infty}{\sim}\tilde{c}_n'r^{-\gamma_n},\quad c_n',\tilde{c}_n'\neq0.\end{cases} \tag{2-7}$$

Moreover, $T_0^{(n)}$ *is strictly positive, and for* $1\le k\le k(n)$ *the functions* $y\mapsto T_0^{(n)}(|y|)Y_{n,k}(|y|/y)$ *are smooth on* $\mathbb{R}^d$. *The first two regular and strictly positive zeros are explicit*:

$$T_0^{(0)}=\frac{1}{C_0}\Lambda Q\quad\text{and}\quad T_0^{(1)}=-\frac{1}{C_1}\partial_yQ, \tag{2-8}$$

where C_0 *and* C_1 *are the renormalized constants defined by* (1-35).

Proof. The proof of this lemma is done in Appendix A. □

Remark 2.4. The renormalized constants in (2-8) are here to produce the identities $T_0^{(0)}Y^{(0,0)}=\Lambda Q$ and $T_0^{(1)}Y^{(1,k)}=\partial_{x_k}Q$ by (1-35). For each $n\in\mathbb{N}$, only one zero, $T_0^{(n)}$, is regular at the origin. We insist on the fact that $-\gamma_n > 0$ is a positive number[4] for n large by (1-20), making these profiles grow as $r \to +\infty$.

2C. *Inversion of $H^{(n)}$*. We start by a useful factorization formula for $H^{(n)}$. Let $n \in \mathbb{N}$ and $W^{(n)}$ denote the potential

$$W^{(n)} := \partial_r(\log(T_0^{(n)})), \tag{2-9}$$

where $T_0^{(n)}$ is defined in (2-7) and define the first-order operators on radial functions

$$A^{(n)} : u \mapsto -\partial_r u + W^{(n)} u, \quad A^{(n)*} : u \mapsto \frac{1}{r^{d-1}} \partial_r (r^{d-1} u) + W^{(n)} u. \tag{2-10}$$

Lemma 2.5 (factorization of $H^{(n)}$). *The factorization*

$$H^{(n)} = A^{(n)*} A^{(n)} \tag{2-11}$$

holds. Moreover one has the adjunction formula for smooth functions with enough decay

$$\int_0^{+\infty} (A^{(n)} u) v r^{d-1}\, dr = \int_0^{+\infty} u (A^{(n)*} v) r^{d-1}\, dr.$$

Proof. As $T_0^{(n)} > 0$ by (2-7), $W^{(n)}$ is well defined. This factorization is a standard property of Schrödinger operators with a nonvanishing zero. We start by computing

$$A^{(n)*} A^{(n)} u = -\partial_{rr} u - \frac{d-1}{r} \partial_r u + \left(\frac{d-1}{r} W^{(n)} + \partial_r W^{(n)} + (W^{(n)})^2 \right) u.$$

As $W^{(n)} = \partial_r T_0^{(n)} / T_0^{(n)}$, the potential that appears is nothing but

$$\begin{aligned} \frac{d-1}{r} W^{(n)} + \partial_r W^{(n)} + (W^{(n)})^2 = \frac{\partial_{rr} T_0^{(n)} + \frac{d-1}{r} T_0^{(n)}}{T_0^{(n)}} &= \frac{-H^{(n)} T_0^{(n)} + \left(\frac{n(d+n-2)}{r^2} + V\right) T_0^{(n)}}{T_0^{(n)}} \\ &= \frac{n(d+n-2)}{r^2} + V \end{aligned}$$

as $H^{(n)} T_0^{(n)} = 0$, which proves the factorization formula (2-11). The adjunction formula comes from a direct computation using integration by parts. □

From the asymptotic behavior (2-7) of $T_0^{(n)}$ at the origin and at infinity, we deduce the asymptotic behavior of $W^{(n)}$:

$$W^{(n)} = \begin{cases} \frac{n}{r} + O(1) & \text{as } r \to 0, \\ \frac{-\gamma_n}{r} + O\left(\frac{1}{r^{1+g+j}}\right) & \text{as } r \to +\infty, \end{cases} \tag{2-12}$$

which propagates to the derivatives. Using the factorization (2-11), to define the inverse of $H^{(n)}$ we proceed in two steps: first we invert $A^{(n)*}$, then $A^{(n)}$.

[4]This notation seems unnatural but matches the standard notation in the literature.

Definition 2.6 (inverse of $H^{(n)}$). Let $f:(0,+\infty)\to\mathbb{R}$ be smooth with $f(r)=O(r^n)$ as $r\to 0$. We define[5] the inverses $(A^{(n)*})^{-1}f$ and $(H^{(n)})^{-1}f$ by

$$(A^{(n)*})^{-1}f(r)=\frac{1}{r^{d-1}T_0^{(n)}}\int_0^r fT_0^{(n)}s^{d-1}\,ds, \tag{2-13}$$

$$(H^{(n)})^{-1}f(r)=\begin{cases} T_0^{(n)}\int_r^{+\infty}(A^{(n)*})^{-1}f/T_0^{(n)}\,ds & \text{if } (A^{(n)*})^{-1}f/T_0^{(n)} \text{ is integrable on } (0,+\infty),\\ -T_0^{(n)}\int_0^r(A^{(n)*})^{-1}f/T_0^{(n)}\,ds & \text{if } (A^{(n)*})^{-1}f/T_0^{(n)} \text{ is not integrable on } (0,+\infty).\end{cases} \tag{2-14}$$

Direct computations give indeed $H^{(n)}\circ(H^{(n)})^{-1}=A^{(n)*}\circ(A^{(n)*})^{-1}=\mathrm{Id}$, and $A^{(n)}\circ(H^{(n)})^{-1}=(A^{(n)*})^{-1}$. As we do not have uniqueness for the equation $Hu=f$, one may wonder if this definition is the "right" one. The answer is yes because this inverse has the good asymptotic behavior; namely, if $f\approx r^q$ as $r\to+\infty$, one would expect $u\approx r^{q+2}$ as $r\to+\infty$, which will be proven in Lemma 2.9. To keep track of the asymptotic behaviors at the origin and at infinity, we now introduce the notion of admissible functions.

Definition 2.7 (simple admissible functions). Let n be an integer, q be a real number and $f:(0,+\infty)\to\mathbb{R}$ be smooth. We say that f is a simple admissible function of degree (n,q) if it enjoys the asymptotic behaviors

$$f=\sum_{j=0}^{l}c_jr^{n+2j}+O(r^{n+2l+2})\quad\forall l\in\mathbb{N} \tag{2-15}$$

at the origin for a sequence of numbers $(c_l)_{l\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, and at infinity

$$f=O(r^q)\quad\text{as } r\to+\infty, \tag{2-16}$$

and if the two asymptotics propagate to the derivatives of f.

Remark 2.8. Let $f:(0,+\infty)$ be smooth. We define the sequence of n-adapted derivatives of f by induction:

$$f_{[n,0]}:=f\qquad\text{and}\qquad\text{for } j\in\mathbb{N},\quad f_{[n,j+1]}:=\begin{cases}A^{(n)}f_{[n,j]} & \text{for } j \text{ even},\\ A^{(n)*}f_{[n,j]} & \text{for } j \text{ odd}.\end{cases} \tag{2-17}$$

From the definition (2-10) of $A^{(n)}$ and $A^{(n)*}$, and the asymptotic behavior (2-12) of the potential $W^{(n)}$, one notices that the condition (2-16) on the asymptotic at infinity for a simple admissible function of degree (n,q) and its derivatives is equivalent to the following condition for all $j\in\mathbb{N}$:

$$f_{[n,j]}=O(r^{q-j})\quad\text{as } r\to+\infty, \tag{2-18}$$

where the adapted derivatives $(f_{[n,j]})_{j\in\mathbb{N}}$ are defined by (2-17). We will use this fact many times in the rest of this subsection, as it is more adapted to our problem.

The operators $H^{(n)}$ and $(H^{(n)})^{-1}$ leave this class of functions invariant, and the asymptotic at infinity is increased by -2 and 2 under some conditions (that will always hold in the sequel) on the coefficient q to avoid logarithmic corrections.

[5]We know u is well defined because from the decay of f at the origin one deduces $(A^{(n)*})^{-1}f=O(r^{n+1})$ as $y\to 0$ and so u'/T_0^n is integrable at the origin from the asymptotic behavior (2-7).

Lemma 2.9 (actions of $H^{(n)}$ and $(H^{(n)})^{-1}$ on simple admissible functions). *Let $n \in \mathbb{N}$ and f be a simple admissible function of degree (n, q) in the sense of Definition 2.7, with $q > \gamma_n - d$ and $-\gamma_n - 2 - q \notin 2\mathbb{N}$. Then for all integer $i \in \mathbb{N}$:*

(i) *$(H^{(n)})^i f$ is simple admissible of degree $(n, q-2i)$.*

(ii) *$(H^{(n)})^{-i} f$ is simple admissible of degree $(n, q+2i)$.*

Proof. **Step 1:** action of $H^{(n)}$. For all integers i and j one has $((H^{(n)})^i f)_{[n,j]} = f_{[n,j+2i]}$ by (2-17) and (2-11). Using the equivalent formulation (2-18), the asymptotic at infinity (2-16) for $H^i f$ is then a straightforward consequence of the asymptotic at infinity (2-16) for f. Close to the origin, one notices that $H^{(n)} = -\Delta^{(n)} + V$ with $\Delta^{(n)} = \partial_{rr} + \frac{d-1}{r}\partial_r - n(d+n-2)$. If f satisfies (2-15) at the origin, then so does $(\Delta^{(n)})^i f$ by a direction computation. As V is smooth at the origin, $(H^{(n)})^i f$ also satisfies (2-15). Hence $(H^{(n)})^i f$ is a simple admissible function of degree $q-2i$.

Step 2: action of $(H^{(n)})^{-1}$. We will prove the property for $(H^{(n)})^{-1} f$, and the general result will follow by induction on i. Let u denote the inverse by $H^{(n)}$, that is, $u = (H^{(n)})^{-1} f$.

Asymptotic at infinity. We will prove the equivalent formulation (2-18) of the asymptotic at infinity (2-16). From (2-17), (2-13), (2-14) and (2-11), $u_{[n,j]} = f_{[n,j-2]}$ for $j \geq 2$ so the asymptotic behavior (2-18) at infinity for the n-adapted derivatives of u are true for $j \geq 2$. Therefore it remains to prove them for $j = 0, 1$.

<u>Case $j = 1$</u>. From the definition of the inverse (2-14) and of the adapted derivatives (2-17), one has

$$u_{[n,1]} = \frac{1}{r^{d-1} T_0^{(n)}} \int_0^r f T_0^{(n)} s^{d-1}\, ds.$$

From the asymptotic behaviors (2-16) and (2-7) for f and $T_0^{(n)}$ at infinity and the condition $q > \gamma_n - d$, the integral diverges and we get

$$u_{[n,1]}(r) = O(r^{q+1}) \quad \text{as } r \to +\infty, \tag{2-19}$$

which is the desired asymptotic (2-18) for $u_{[n,1]}$.

<u>Case $j = 0$</u>. Suppose $(A^{(n)*})^{-1} f / T_0^{(n)} = u_{[n,1]}/T_0^{(n)}$ is integrable on $(0, +\infty)$. In that case

$$u = T_0^{(n)} \int_r^{+\infty} \frac{u_{[n,1]}}{T_0^{(n)}}\, ds.$$

If $q > -\gamma_n - 2$, then by the integrability of the integrand and (2-7), we get the desired asymptotic $u_{[n,0]} = u = O(r^{-\gamma_n}) = O(r^{q+2})$. If $q < -\gamma_n - 2$ then from (2-19) we have $u_{[n,1]}/T_0^{(n)} = O(r^{q+1+\gamma_n})$ and then $\int_r^{+\infty} u_{[n,1]}/T_0^{(n)}\, ds = O(r^{q+2+\gamma_n})$, from which we get the desired asymptotic $u = O(r^{q+2})$. Now suppose $u_{[n,1]}/T_0^{(n)}$ is not integrable. Then we must have $q > -\gamma_n + 2$ by (2-19), and u is given by

$$u = -T_0^{(n)} \int_0^r \frac{u_{[n,1]}}{T_0^{(n)}}\, ds,$$

and the integral has asymptotic $O(r^{q+2+\gamma_n})$. We hence get $u = O(r^{q+2})$ at infinity using (2-7).

Conclusion. In both cases, we have proven that the asymptotic at infinity (2-18) holds for u.

Asymptotic at the origin. We have

$$u=-T_0^{(n)}\int_0^r \frac{u_{[n,1]}}{T_0^{(n)}}\,ds+aT_0^{(n)},$$

where $a=0$ if $u_{[n,1]}/T_0^{(n)}$ is not integrable, and $a=\int_0^{+\infty}u_{[n,1]}/T_0^{(n)}\,ds$ if it is. By (2-7), $T_0^{(n)}$ satisfies (2-15). Thus it remains to prove (2-15) for $-T_0^{(n)}\int_0^r u_{[n,1]}/T_0^{(n)}\,ds$. We proceed in two steps. First, from (2-15) for f we obtain that for all integers j, p,

$$u_{[n,1]}=\frac{1}{r^{d-1}T_0^{(n)}}\int_0^r fT_0^{(n)}s^{d-1}\,ds=\sum_{j=0}^{l}\tilde{c}_j r^{n+1+2j}+\widetilde{R}_l,$$

where $\partial_r^k\widetilde{R}_l=O(r^{\max(n+2l+3-k,0)})$ as $r\to 0$ for some coefficients $\tilde{c}_j$ depending on the c_j and the asymptotic at the origin of T_0^n. It then follows that

$$-T_0^{(n)}\int_0^r \frac{u_{[n,1]}}{T_0^{(n)}}\,ds=\sum_{j=0}^{l}\hat{c}_j r^{n+2+2j}+\widehat{R}_l,\quad \text{where } \partial_r^k\widehat{R}_l\underset{r\to 0}{=}O(r^{\max(n+2l+4-k,0)}),$$

for some coefficients $\hat{c}_l$. This implies that u satisfies (2-15) at the origin. □

We can now invert the elements in the kernel of $H^{(n)}$ and construct the generalized kernel of this operator.

Lemma 2.10 (generators of the generalized kernel of $H^{(n)}$). *Let $n\in\mathbb{N}$, γ_n, g', $(H^{(n)})^{-1}$ and $T_0^{(n)}$ be defined by* (1-18), (1-21), *Definition 2.6 and Lemma 2.3. We denote by $(T_i^{(n)})_{i\in\mathbb{N}}$ the sequence of profiles given by*

$$T_{i+1}^{(n)}:=-(H^{(n)})^{-1}T_i^{(n)},\quad i\in\mathbb{N}. \tag{2-20}$$

Let $(\Theta_i^{(n)})_{i\in\mathbb{N}}$ be the associated sequence of profiles defined by

$$\Theta_i^{(n)}:=\Lambda T_i^{(n)}-\left(2i+\frac{2}{p-1}-\gamma_n\right)T_i^{(n)},\quad i\in\mathbb{N}. \tag{2-21}$$

Then for each $i\in\mathbb{N}$,

$$T_i^{(n)}\ \textit{is simple admissible of degree}\ (n,-\gamma_n+2i), \tag{2-22}$$

$$\Theta_i^{(n)}\ \textit{is simple admissible of degree}\ (n,-\gamma_n+2i-g'), \tag{2-23}$$

where simple admissibility is defined in Definition 2.7.

Proof. **Step 1:** admissibility of $T_i^{(n)}$. From the asymptotic behaviors (2-7) at infinity and at the origin, $T_0^{(n)}$ is simple admissible of degree $(n,-\gamma_n)$ in the sense of Definition 2.7. Additionally, $-\gamma_n>\gamma_n-d$ since $-2\gamma_n+d\geq -2\gamma_0+d=2+\sqrt{\triangle}>0$ by (1-9) and since $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing by (1-18). One has also $-\gamma_n-2-(-\gamma_n)=-2\notin 2\mathbb{N}$. Therefore one can apply Lemma 2.9: for all $i\in\mathbb{N}$, $T_i^{(n)}$ given by (2-20) is an admissible profile of degree $(n,-\gamma_n+2i)$.

Step 2: admissibility of $\Theta_i^{(n)}$. We start by computing the following commutator relations using (1-36), (2-9) and (2-10):

$$\begin{aligned} A^{(n)}\Lambda &= \Lambda A^{(n)} + A^{(n)} - (W^{(n)} + y\partial_y W^{(n)}), \\ H^{(n)}\Lambda &= \Lambda H^{(n)} + 2H^{(n)} - (2V + y.\nabla V). \end{aligned} \tag{2-24}$$

We now proceed by induction. From the previous equation, and the asymptotic behaviors (2-7), (2-2) and (2-12) of the functions $T_0^{(n)}$, V and $W^{(n)}$, we get that $\Theta_0^{(n)}$ is simple admissible of degree $(n, -\gamma_n - g')$. Now let $i \geq 1$ and suppose that the property in (2-23) is true for $i-1$. Using the previous formula and (2-21) we obtain

$$H^{(n)}\Theta_i^n = -\Theta_{i-1}^{(n)} - (2V + y.\nabla V)T_i^{(n)}.$$

The asymptotic at infinity (2-2) of V yields the decay $2V + y.\nabla V = (y^{-2-\alpha})$. As $T_i^{(n)}$ is simple admissible of degree $(n, 2i - \gamma_n)$ and from the induction hypothesis, we have that $H^{(n)}\Theta_i^{(n)}$ is simple admissible of degree $(n, 2i - 2 - \gamma_n - g')$ because $g' < \alpha$ by (1-21). One has $2i - 2 - \gamma_n - g' > \gamma_n - d$ because

$$2i - 2 - 2\gamma_n - g' + d \geq -2\gamma_0 - g' + d = 2 + \sqrt{\triangle} - g' > 0$$

as $0 < g' < 1$, $i \geq 1$, and $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing by (1-18) and (1-9). Similarly

$$-\gamma_n - 2 - (2i - 2 - \gamma_n - g') = -2i + g' \notin 2\mathbb{N}.$$

Therefore we can apply Lemma 2.9 and obtain that $(H^{(n)})^{-1}H^{(n)}\Theta_i^{(n)}$ is of degree $(n, 2i - \gamma_n - g')$. From Lemma 2.3 one has $(H^{(n)})^{-1}H^{(n)}\Theta_i^{(n)} = \Theta_i^{(n)} + aT_0^{(n)} + b\Gamma^{(n)}$, for two integration constants $a, b \in \mathbb{R}$. At the origin $\Gamma^{(n)}$ is singular by (2-7); hence $b = 0$. As $T_0^{(n)}$ is of degree $(n, -\gamma_n)$ with $-\gamma_n + 2i - g' > -\gamma_n$ (because $i \geq 1$), we get that $\Theta_i^{(n)}$ is of degree $(n, 2i - \gamma_n - g')$. □

2D. ***Inversion of H on nonradial functions.*** The definition of the inverse of $H^{(n)}$, Definition 2.6, naturally extends to give an inverse of H by separately inverting the components onto each spherical harmonic. There will be no problem when summing, as for the purpose of the present paper one can restrict to the following class of functions that are located on a finite number of spherical harmonics.

Definition 2.11 (admissible functions). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function, with decomposition $f(y) = \sum_{n,k} f^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)$, and q be a real number. We say that f is admissible of degree q if there is only a finite number of couples (n, k) such that $f^{(n,k)} \neq 0$, and that for every such couple, $f^{(n,k)}$ is a simple admissible function of degree (n, q) in the sense of Definition 2.7.

For $f = \sum_{n,k} f^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)$ an admissible function, we define its inverse by H by

$$(H^{(-1)}f)(y) := \sum_{n,k} [(H^{(n)})^{-1} f^{(n,k)}(|y|)]Y^{(n,k)}\left(\frac{y}{|y|}\right) \tag{2-25}$$

(the sum being finite), where $(H^{(n)})^{-1}$ is defined by Definition 2.6. For n, k and i three integers with $1 \leq k \leq k(n)$, we define the profile $T_i^{(n,k)} : \mathbb{R}^d \to \mathbb{R}$ as

$$T_i^{(n,k)}(y) = T_i^{(n)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right), \tag{2-26}$$

where the radial function $T_i^{(n)}$ is defined by (2-20). From Lemma 2.10, $T_i^{(n,k)}$ is an admissible function of degree $(-\gamma_n+2i)$ in the sense of Definition 2.11. The class of admissible functions has some structural properties: it is stable under summation, multiplication and differentiation, and its elements are smooth with an explicit decay at infinity. This is the subject of the next lemma.

Lemma 2.12 (properties of admissible functions). *Let f and g be two admissible functions of degrees q and q' in the sense of Definition 2.11, and $\mu\in\mathbb{N}^d$. Then*:

(i) *f is smooth.*

(ii) *fg is admissible of degree $q+q'$.*

(iii) *$\partial^\mu f$ is admissible of degree $q-|\mu|$.*

(iv) *There exists a constant $C(f,\mu)$ such that for all y with $|y|\geq 1$,*

$$|\partial^\mu f(y)|\leq C(f,\mu)|y|^{q-|\mu|}.$$

Proof. From Definition 2.11, $f=\sum_{n,k} f^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)$ and $g=\sum_{n,k} g^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)$ and both sums involve finitely many nonzero terms. Therefore, without loss of generality, we will assume that f and g are each located on only one spherical harmonic: $f=f^{(n,k)}Y^{(n,k)}$ and $g=g^{(n',k')}Y^{(n',k')}$, for $f^{(n,k)}$ and $g^{(n',k')}$ simple admissible of degrees (n,q) and (n',q') in the sense of Definition 2.7. The general result will follow by a finite summation.

(i) Now $y\mapsto f^{(n,k)}(|y|)$ is smooth outside the origin since f is smooth, and $y\mapsto Y^{(n,k)}(y/|y|)$ is also smooth outside the origin; hence f is smooth outside the origin. The Laplacian on spherical harmonics is

$$(-\Delta)^i f=(-\Delta)^i\left(f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)\right)=((-\Delta^{(n)})^i f^{(n,k)})(|y|)Y^{(n,k)},$$

where $-\Delta^{(n)}=-\partial_{rr}-\frac{d-1}{r}\partial r+n(d+n-2)$. From the expansion of $f^{(n,k)}$ in (2-15), $(-\Delta^{(n)})^i f^{(n,k)}$ is bounded at the origin for each $i\in\mathbb{N}$. Therefore $(-\Delta)^i f$ is bounded at the origin for each i and f is smooth at the origin by elliptic regularity.

(ii) We treat the case where $n+n'$ is even, and the case $n+n'$ odd can be treated with exactly the same arguments. As the product of the two spherical harmonics $Y^{(n,k)}Y^{(n',k')}$ decomposes onto spherical harmonics of degree less than $n+n'$ with the same parity as $n+n'$, the product fg can be written as

$$fg=\sum_{\substack{0\leq\tilde n\leq n+n'\\ \tilde n\text{ even},\,1\leq\tilde k\leq k(\tilde n)}} a_{n,k,n',k',\tilde n,\tilde k}\, f^{(n,k)}g^{(n',k')}Y^{(\tilde n,\tilde k)}$$

with $a_{n,k,n',k',\tilde n,\tilde k}$ some fixed coefficients. Now fix $\tilde n$ and $\tilde k$ in the sum; one has $n+n'=\tilde n+2i$ for some $i\in\mathbb{N}$. Using the Leibniz rule, as $\partial_r^j f^{(n,k)}=O(r^{q-j})$ and $\partial_r^j g^{(n,k)}=O(r^{q'-j})$ at infinity, we get that $\partial_r^j(f^{(n,k)}g^{(n',k')})=O(r^{q+q'-j})$ as $y\to+\infty$, which proves that $f^{(n,k)}g^{(n',k')}$ satisfies the asymptotic at infinity (2-16) of a simple admissible function of degree $(\tilde n,q+q')$. Close to the origin, the two expansions (2-15) for $f^{(n,k)}$ and $g^{(n',k')}$, starting at r^n and $r^{n'}$ respectively, imply the same expansion (2-15) starting at $y^{n+n'}$ for the product $f^{(n,k)}g^{(n',k')}$. As $n+n'=\tilde n+2i$, we know $f^{(n,k)}g^{(n,k)}$ satisfies the expansion at the origin (2-15) of a simple admissible function of degree $(\tilde{n}, q+q')$. Therefore $f^{(n,k)}g^{(n,k)}$ is simple admissible of degree $(\tilde{n}, q+q')$ and thus fg is simple admissible of degree $q+q'$.

(iii) We treat the case where n is even, and the case n odd can be treated with exactly the same reasoning. Let $1 \le i \le d$; we just have to prove that $\partial_{y_i} f$ is admissible of degree $q-1$ and the result for higher-order derivatives will follow by induction. We recall that $Y^{(n,k)}$ is the restriction of a homogeneous harmonic polynomial of degree n to the sphere. We will still denote by $Y^{(n,k)}(y)$ this polynomial extended to the whole space $\mathbb{R}^d$ and they are related by $Y^{(n,k)}(y) = |y|^n Y^{(n,k)}(y/|y|)$. This homogeneity implies $y.\nabla(Y^{(n,k)})(y) = nY^{(n,k)}(y)$ and leads to the identity

$$\partial_{y_i}\left[f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)\right] = \left(\partial_r f^{(n,k)}(|y|) - n\frac{f(|y|)}{|y|}\right)\frac{y_i}{|y|}Y^{(n,k)}\left(\frac{y}{|y|}\right) + \frac{f(|y|)}{|y|}\partial_{y_i} Y^{(n,k)}\left(\frac{y}{|y|}\right). \tag{2-27}$$

One has now to prove that the two terms on the right-hand side are admissible of degree $q-1$. We only show it for the last term, the proof being the same for the first one. As $\partial_{y_i} Y^{(n,k)}(y/|y|)$ is a homogeneous polynomial of degree $n-1$ restricted to the sphere, it can be written as a finite sum of spherical harmonics of odd degrees (because n is even) less than $n-1$ and this gives

$$\frac{f}{|y|}\partial_{y_i} Y^{(n,k)}\left(\frac{y}{|y|}\right) = \sum_{\substack{1\le n'\le n-1\\ n' \text{ odd},\, 1\le k\le k(n')}} a_{i,n,k,n',k'}\frac{f}{|y|}Y^{(n',k')}\left(\frac{y}{|y|}\right)$$

for some coefficients $a_{i,n,k,n',k'}$. Now fix n', k' in the sum. At infinity $a_{i,n,k,n',k'} f(|y|)/|y|$ satisfies the asymptotic behavior (2-16) of a simple admissible function of degree $(n', q-1)$. Close to the origin, one has from (2-15), the fact that $n'+2j = n-1$ for some $j \in \mathbb{N}$, that for any $i \in \mathbb{N}$,

$$a_{i,n,k,n',k'}\frac{f(r)}{r} = \sum_{l=0}^{i} \tilde{c}_l r^{n-1+2l} + O(r^{n-1+2i+2}) = \sum_{l=0}^{i} \hat{c}_l r^{n'+2j+2l} + O(r^{n'+2j+2i+2}),$$

which is the asymptotic behavior (2-15) of a simple admissible function of degree $(n', q-1)$ close to the origin. Therefore, $a_{i,n,k,n',k'} f(r)/r$ is a simple admissible function of degree $(n', q-1)$. Thus $(f/|y|)\partial_{y_i} Y^{(n,k)}(y/|y|)$ is an admissible function of degree $(q-1)$. The same reasoning works for the first term on the right-hand side of (2-27), and therefore $\partial_{y_i}[f^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)]$ is admissible of degree $q-1$.

(iv) We just showed in the last step that $\partial^\mu f$ is admissible of degree $q-|\mu|$ for all $\mu \in \mathbb{N}^d$; we then only have to prove (iv) for the case $\mu = (0,\dots,0)$. This can be showed via the brute force bound for $|y| \ge 1$

$$|f(y)| = \left| f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)\right| \le \|Y^{(n,k)}\|_{L^\infty} |f^{(n,k)}(|y|)| \le C|y|^q$$

by (2-16) since f is a simple admissible function of degree (n,q). □

The next lemma extends Lemma 2.9 to admissible functions. We do not give a proof, as it is a direct consequence of the latter.

Lemma 2.13 (action of H on admissible functions). *Let f be an admissible function in the sense of Definition 2.11 written as $f(y)=\sum_{n,k} f^{(n,k)}(|y|)Y^{(n,k)}(y/|y|)$, of degree q, with $q>\gamma_n-d$. Assume that for all $n\in\mathbb{N}$ such that there exists k, $1\leq k\leq k(n)$ with $f^{(n,k)}\neq 0$, we have q satisfies $-q-\gamma_n-2\notin 2\mathbb{N}$. Then for all integers $i\in\mathbb{N}$, recalling that $H^{-1}f$ is defined by (2-25):*

(i) *$H^i f$ is admissible of degree $q-2i$.*

(ii) *$H^{-i} f$ is admissible of degree $q+2i$.*

2E. *Homogeneous functions.* The approximate blow-up profile we will build in the following subsection will look like $Q+\sum b_i^{(n,k)}T_i^{(n,k)}$ for some coefficients $b_i^{(n,k)}$ ($T_i^{(n,k)}$ being defined in (2-26)). The nonlinearity in the semilinear heat equation (1-1) will then produce terms that will be products of the profiles $T_i^{(n,k)}$ and coefficients $b_i^{(n,k)}$. Such nonlinear terms are admissible functions multiplied by monomials of the coefficients $b_i^{(n,k)}$. The set of triples (n,k,i) for which we will make a perturbation along $T_i^{(n,k)}$ is $\mathcal{I}$, defined in (1-39). Hence the vector b representing the perturbation will be

$$b=(b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}=(b_1^{(0,1)},\dots,b_L^{(0,1)},b_1^{(1,1)},\dots,b_{L_1}^{(1,1)},\dots,b_0^{(n_0,k(n_0))},\dots,b_{L_{n_0}}^{(n_0,k(n_0))}). \tag{2-28}$$

We will then represent a monomial in the coefficients $b_i^{(n,k)}$ by a tuple of $\#\mathcal{I}$ integers

$$J=(J_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}=(J_1^{(0,1)},\dots,J_L^{(0,1)},J_1^{(1,1)},\dots,J_{L_1}^{(1,1)},\dots,J_0^{(n_0,k(n_0))},\dots,J_{L_{n_0}}^{(n_0,k(n_0))})$$

through the formula

$$b^J:=(b_1^{(0,1)})^{J_1^{(0,1)}}\times\cdots\times(b_{L_{n_0}}^{(n_0,k(n_0))})^{J_{L_{n_0}}^{(n_0,k(n_0))}}. \tag{2-29}$$

We associate three different lengths to J for the analysis. The first one, $|J|:=\sum J_i^{(n,k)}$, represents the number of parameters $b_i^{(n,k)}$ that are multiplied in the above formula, counted with multiplicity, i.e., the standard degree of b^J. In the analysis, the coefficients $b_i^{(nk)}$ will have the size $|b_i^{(n,k)}|\lesssim|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$. The second length,

$$|J|_2:=\sum_{n,k,i}\left(\frac{\gamma-\gamma_n}{2}+i\right)J_i^{(n,k)},$$

is tailor-made to produce the following identity if these latter bounds hold:

$$|b^J|\lesssim(b_1^{(0,1)})^{|J|_2};$$

i.e., $|J|_2$ encodes the "size" of the real number b^J. For the construction of the approximate blow-up profile, we will invert several times some elliptic equations, and the i-th inversion will be related to the third length

$$|J|_3:=\sum_{i=1}^{L} iJ_i^{(0,1)}+\sum_{\substack{1\leq i\leq L_1\\1\leq k\leq d}} iJ_i^{(1,k)}+\sum_{\substack{(n,k,i)\in\mathcal{I}\\2\leq n}}(i+1)J_i^{(n,k)}.$$

To track information about the nonlinear terms generated by the semilinear heat equation (1-1) we eventually introduce the class of homogeneous functions.

Definition 2.14 (homogeneous functions). Let b denote a $\#\mathcal{I}$-tuple under the form (2-28), $m \in \mathbb{N}$ and $q \in \mathbb{R}$. We recall that $|J|_2$ and $|J|_3$ are defined by (1-41) (1-42) and b^J is given by (2-29). We say that a function $S : \mathbb{R}^{\mathcal{I}} \times \mathbb{R}^d \to \mathbb{R}$ is homogeneous of degree (m, q) if it can be written as a finite sum

$$S(b, y) = \sum_{J \in \mathcal{J}} b^J S_J(y),$$

$\#\mathcal{J} < +\infty$, where for each tuple $J \in \mathcal{J}$, one has that $|J|_3 = m$ and that the function S_J is admissible of degree $2|J|_2 + q$ in the sense of Definition 2.11.

As a direct consequence of Lemma 2.12, and so we do not write here the proof, we obtain the following properties for homogeneous functions.

Lemma 2.15 (calculus on homogeneous functions). *Let S and S' be two homogeneous functions of degrees (m, q) and (m', q') in the sense of Definition 2.14, and $\mu \in \mathbb{N}^d$. Then*:

(i) *$\partial^\mu S$ is homogeneous of degree $(m, q - |\mu|)$.*

(ii) *SS' is homogeneous of degree $(m + m', q + q')$.*

(iii) *If, writing $S = \sum_{J \in \mathcal{J}} b^J \sum_{n,k} S_J^{(n,k)} Y^{(n,k)}$, one has $2|J|_2 + q > \gamma_n - d$ and $-2|J|_2 - q - \gamma_n - 2 \notin 2\mathbb{N}$ for all n, J such that there exists k, $1 \le k \le k(n)$ with $S_J^{(n,k)} \neq 0$, then for all $i \in \mathbb{N}$, $H^{-i}(S)$ (given by (2-25)) is homogeneous of degree $(m, q + 2i)$.*

3. The approximate blow-up profile

3A. ***Construction.*** We first summarize the content and ideas of this section. We construct an approximate blow-up profile relying on a finite number of parameters close to the set of functions $(\tau_z(Q_\lambda))_{\lambda > 0,\ z \in \mathbb{R}^d}$. It is built on the generalized kernel of H, $\mathrm{Span}((T_i^{(n,k)})_{n,i \in \mathbb{N},\ 1 \le k \le k(n)})$ defined by (2-26), and can therefore be seen as a part of a center manifold. The profile is built on the whole space $\mathbb{R}^d$ for the moment and will be localized later.

In Proposition 3.1 we construct a first approximate blow-up profile. The procedure generates an error term ψ, and by inverting elliptic equations, i.e., adding the term $H^{-1}\psi$ to our approximate blow-up profile, one can always convert this error term into a new error term that is localized far away from the origin. We apply this procedure several times to produce an error term that is very small close to the origin. Then, in Proposition 3.3 we localize the approximate blow-up profile to eliminate the error terms that are far away from the origin. We will cut in the zone $|y| \approx B_1 = B_0^{1+\eta}$, where $\eta \ll 1$ is a very small parameter. In this zone, the perturbation in the approximate blow-up profile has the same size as ΛQ, being the reference function for scale change. It will correspond to the self-similar zone $|x| \sim \sqrt{T - t}$ for the true blow-up function, where T will be the blow-up time.

The blow-up profile is described by a finite number of parameters whose evolution is given by the explicit dynamical system (3-58). In Lemma 3.4 we show the existence of special solutions describing a type II blow up with explicit blow-up speed. The linear stability of these solutions is investigated in Lemma 3.5.

There is a natural renormalized flow linked to the invariances of the semilinear heat equations (1-1). For u a solution of (1-1), $\lambda : [0, T(u_0)) \to \mathbb{R}_+^*$ and $z : [0, T(u_0)) \to \mathbb{R}^d$ two C^1 functions, if one defines for $s_0 \in \mathbb{R}$ the renormalized time

$$s(t) := s_0 + \int_0^t \frac{1}{\lambda(t')^2}\, dt' \tag{3-1}$$

and the renormalized function

$$v(s,\cdot) := (\tau_{-z} u(t,\cdot))_\lambda,$$

then from a direct computation, v is a solution of the renormalized equation

$$\partial_s v - \frac{\lambda_s}{\lambda} \Lambda v - \frac{z_s}{\lambda} . \nabla v - F(v) = 0. \tag{3-2}$$

Our first approximate blow-up profile is adapted to this new flow and is a special perturbation of Q.

Proposition 3.1 (first approximate blow-up profile). *Let $L \in \mathbb{N}$, $L \gg 1$, and let $b = (b_i^{(n,k)})_{(n,k,i) \in \mathcal{I}}$ denote a $\#\mathcal{I}$-tuple of real numbers with $b_1^{(0,1)} > 0$. There exists a $\#\mathcal{I}$-dimensional manifold of C^∞ functions $(Q_b)_{b \in \mathbb{R}_+^* \times \mathbb{R}^{\#\mathcal{I}-1}}$ such that*

$$F(Q_b) = b_1^{(0,1)} \Lambda Q_b + b_1^{(1,\cdot)} . \nabla Q_b + \sum_{(n,k,i) \in \mathcal{I}} \big(-(2i - \alpha_n) b_1^{(0,1)} b_i^{(n,k)} + b_{i+1}^{(n,k)}\big) \frac{\partial Q_b}{\partial b_i^{(n,k)}} - \psi_b, \tag{3-3}$$

where $b_1^{(1,\cdot)}$ denotes the d-tuple of real numbers $(b_1^{(1,1)}, \dots, b_1^{(1,d)})$, where we used the convention $b_{L_n+1}^{(n,k)} = 0$, and where ψ_b is an error term. Let B_1 be defined by (1-38). If the parameters satisfy the size conditions[6] *$b_1^{(0,1)} \ll 1$ and $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ for all $(n,k,i) \in \mathcal{I}$, then ψ_b enjoys the following bounds*:

(i) Global[7] bounds. *For $0 \le j \le s_L$,*

$$\|H^j \psi_b\|^2_{L^2(|y| \le 2B_1)} \le C(L) (b_1^{(0,1)})^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}, \tag{3-4}$$

$$\|\nabla^j \psi_b\|^2_{L^2(|y| \le 2B_1)} \le C(L) (b_1^{(0,1)})^{2(\frac{j}{2}-m_0)+2(1-\delta_0)+g'-C(L)\eta}, \tag{3-5}$$

where $C(L)$ is a constant depending on L only.

(ii) Local bounds.

$$\forall j \ge 0,\ \forall B > 1, \quad \int_{|y| \le B} |\nabla^j \psi_b|^2\, dy \le C(j,L) B^{C(j,L)} (b_1^{(0,1)})^{2L+6}. \tag{3-6}$$

where $C(L,j)$ is a constant depending on L and j only.

---

[6]This means that under the bounds $|b_i^{(n,k)}| \le K |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ for some $K > 0$, there exists $b^*(K)$ such that the estimates that follow hold if $b_1^{(0,1)} \le b^*(K)$ with constants depending on K. In what follows, K will be fixed independently of the other important constants.

[7]The zone $y \le B_1$ is called global because in the next proposition we will cut the profile Q_b in the zone $|y| \sim B_1$.

The profile Q_b is of the form

$$Q_b := Q + \alpha_b, \quad \alpha_b := \sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} T_i^{(n,k)} + \sum_{i=2}^{L+2} S_i, \tag{3-7}$$

where $T_i^{(n,k)}$ is as in (2-26), and the profiles S_i are homogeneous functions in the sense of Definition 2.14 with

$$\deg(S_i) = (i, -\gamma - g') \tag{3-8}$$

and with the property that for all $2 \le j \le L+2$, we have $\partial S_j/\partial b_i^{(n,k)} = 0$ if $j \le i$ for $n = 0, 1$ and if $j \le i+1$ for $n \ge 2$.

Remark 3.2. The previous proposition is to be understood in the following way. We have a special function depending on some parameters b close to Q, that is to say, at scale 1 and with concentration point 0 for the moment. Equation (3-3) means that the force term (i.e., when applying F) generated by (NLH) makes it concentrate at speed $b_1^{(0,1)}$ and translate at speed $b_1^{(1,\cdot)}$, while the time evolution of the parameters is an explicit dynamical system given by the third term. These approximations involve an error for which we have some explicit bounds (3-4) and (3-6).

The size of this approximate profile is directly related to the size of the perturbation along $T_1^{(0,1)}$, the first term in the generalized kernel of H responsible for scale variation. Indeed we ask for $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$, and the size of the error is measured via $b_1^{(0,1)}$; see (3-4), (3-5) and (3-6). Therefore $b_1^{(0,1)}$ will be the universal order of magnitude in our problem.

Because of the shape of this approximate blow-up profile (3-7), when including the time evolution of the parameters in (3-3) we get

$$\partial_s(Q_b) - F(Q_b) + b_1^{(0,1)} \Lambda Q_b + b_1^{(1,\cdot)}.\nabla Q_b = \mathrm{Mod}(s) + \psi_b, \tag{3-9}$$

where[8]

$$\mathrm{Mod}(s) = \sum_{(n,k,i)\in\mathcal{I}} \left[b_{i,s}^{(n,k)} + (2i - \alpha_n) b_1^{(0,1)} b_i^{(n,k)} - b_{i+1}^{(n,k)}\right] \left[T_i^{(n,k)} + \sum_{j=i+1+\delta_{n\ge 2}}^{L+2} \frac{\partial S_j}{\partial b_i^{(n,k)}}\right]. \tag{3-10}$$

For all $2 \le j \le L+2$, as S_j is homogeneous of degree $(j, -\gamma - g')$ in the sense of Definition 2.14 from (3-8), and from the fact that $\partial S_j/\partial b_i^{(n,k)} = 0$ if $j \le i$ for $n = 0, 1$ and if $j \le i+1$ for $n \ge 2$, one has that for all j, n, k, i, we have $\partial S_j/\partial b_i^{(0,1)}$ is either 0 or is homogeneous of degree (a, b) with $a \ge 1$, meaning that it never contains nontrivial constant functions independent of the parameters b. Hence, if the bounds $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ hold, since $|b_1^{(0,1)}| \lesssim 1$ and $-\gamma_n \ge -\gamma$ from (1-18), one has in particular that on compact sets for any $2 \le j \le L+2$ and $(n,k,i) \in \mathcal{I}$,

$$\frac{\partial S_j}{\partial b_i^{(n,k)}} = O(|b_1^{(0,1)}|). \tag{3-11}$$

Proof of Proposition 3.1. **Step 1:** computation of ψ_b. We first find an appropriate reformulation for the error ψ_b given by (3-3) when Q_b has the form (3-7).

[8] Here $\delta_{n\ge 2} = 1$ if $n \ge 2$, and is zero otherwise.

Rewriting of $F(Q_b)$ in (3-3). We start by computing

$$
\begin{aligned}
-F(Q_b) &= H(\alpha_b)-(f(Q_b)-f(Q)-\alpha_b f'(Q)) \\
&= \sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} H T_i^{(n,k)} + \sum_{i=2}^{L+2} H(S_i) - (f(Q_b)-f(Q)-\alpha_b f'(Q)) \\
&= -b_1^{(0,1)}\Lambda Q - b_1^{(1,\cdot)}.\nabla Q - \sum_{(n,k,i)\in\mathcal{I}} b_{i+1}^{(n,k)} T_i^{(n,k)} + \sum_{i=2}^{L+2} H(S_i) - (f(Q_b)-f(Q)-\alpha_b f'(Q)),
\end{aligned}
\tag{3-12}
$$

where we used the definition of the profiles $T_i^{(n,k)}$ from (2-26), and the convention $b_{L_n+1}^{(n,k)}=0$. For $i=2,\dots,L$, we regroup the terms that involve the multiplication of i parameters $b_j^{(n,k)}$ in the nonlinear term $-(f(Q_b)-f(Q)-\alpha_b f'(Q))$. Since p is an odd integer,

$$
\begin{aligned}
\big(f(Q_b)-f(Q)-\alpha_b f'(Q)\big) &= \sum_{k=2}^{p} C_k^p Q^{p-k}\alpha_b^k \\
&= \sum_{k=2}^{p} C_k^p Q^{p-k}\Bigg[\sum_{|J|_1=k} C_J \prod_{(n,k,i)\in\mathcal{I}} (b_i^{(n,k)})^{J_i^{(n,k)}} (T_k^{(n,k)})^{J_i^{(n,k)}} \prod_{i=2}^{L+2} S_i^{J_i}\Bigg],
\end{aligned}
\tag{3-13}
$$

where $J=(J_1^{(0,1)},\dots,J_{L_{n_0}}^{(n_0,k(n_0))},J_2,\dots,J_{L+2})$ represents a $(\#\mathcal{I}+L+1)$-tuple of integers. Anticipating that the profile S_i will be a homogeneous profile of degree $(i,\gamma-g')$, we define for such tuples J,

$$
|J|_3=\sum_{i=1}^{L} iJ_i^{(0,1)} + \sum_{1\le i\le L_1,\,1\le k\le d} iJ_i^{(1,k)} + \sum_{(n,k,i)\in\mathcal{I},\,2\le n} (i+1)J_i^{(n,k)} + \sum_{i=2}^{L+2} iJ_i. \tag{3-14}
$$

We reorder the sum in the previous equation, (3-13), partitioning the $(\#\mathcal{I}+L+1)$-tuples J according to their length $|J|_3$ instead of their length J_1:

$$
\big(f(Q_b)-f(Q)-\alpha_b f'(Q)\big)=\sum_{j=2}^{L+2} P_j + R.
$$

P_j captures the terms with polynomials of the parameters $b_i^{(n,k)}$ of length $|J|_3=j$:

$$
P_j=\sum_{k=2}^{p} C_k Q^{p-k}\Bigg(\sum_{|J|=k,|J|_3=j} C_J \prod_{(n,k,i)\in\mathcal{I}} (b_i^{(n,k)})^{J_i^{(n,k)}} (T_k^{(n,k)})^{J_i^{(n,k)}} \prod_{i=2}^{L+2} S_i^{J_i}\Bigg). \tag{3-15}
$$

The remainder contains only terms involving polynomials of the parameters $b_i^{(n,k)}$ of length $|\cdot|_3$ greater than or equal to $L+3$:

$$
R=\big(f(Q_b)-f(Q)-\alpha_b f'(Q)\big)-\sum_{i=2}^{L+2} P_i. \tag{3-16}
$$

From (3-12) we end up with the final decomposition

$$-F(Q_b) = -b_1^{(0,1)}\Lambda Q - b_1^{(1,\cdot)}.\nabla Q - \sum_{(n,k,i)\in\mathcal{I}} b_{i+1}^{(n,k)} T_i^{(n,k)} + \sum_{i=2}^{L} H(S_i) - \sum_{i=2}^{L+2} P_i - R. \tag{3-17}$$

Rewriting of the other terms in (3-3). From the form of Q_b in (3-7), one has

$$b_1^{(0,1)}\Lambda Q_b = b_1^{(0,1)}\Lambda Q + \sum_{(n,k,i)\in\mathcal{I}} b_1^{(0,1)} b_i^{(n,k)} \Lambda T_i^{(n,k)} + \sum_{i=2}^{L+2} b_1^{(0,1)}\Lambda S_i, \tag{3-18}$$

$$b_1^{(1,\cdot)}.\nabla Q_b = b_1^{(1,\cdot)}.\nabla Q + \sum_{j=1}^{d}\Bigg(\sum_{(n,k,i)\in\mathcal{I}} b_1^{(1,j)} b_i^{(n,k)} \partial_{x_j} T_i^{(n,k)} + \sum_{i=2}^{L+2} b_1^{(1,j)}\partial_{x_j} S_i\Bigg), \tag{3-19}$$

$$\begin{aligned}\sum_{(n,k,i)\in\mathcal{I}} &\big(-(2i-\alpha_n) b_1^{(0,1)} b_i^{(n,k)} + b_{i+1}^{(n,k)}\big)\frac{\partial Q_b}{\partial b_i^{(n,k)}} \\ &= \sum_{(n,k,i)\in\mathcal{I}} \big(-(2i-\alpha_n) b_1^{(0,1)} b_i^{(n,k)} + b_{i+1}^{(n,k)}\big)\Bigg(T_i^{(n,k)} + \sum_{j=2}^{L+2}\frac{\partial S_j}{\partial b_i^{(n,k)}}\Bigg).\end{aligned} \tag{3-20}$$

Expression of the error term ψ_b. Using (2-21), we define

$$\Theta_i^{(n,k)}(y) := \Theta_i^{(n)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right).$$

From (3-17)–(3-20), ψ_b given by (3-3) is a sum of terms that are polynomials in b, and, denoting a monomial by b^J, we rearrange them according to the value $|J|_3$:

$$\begin{aligned}\psi_b = \sum_{i=2}^{L+2}[\Phi_i + H(S_i)] + b_1^{(0,1)}\Lambda S_{L+2} + \sum_{j=1}^{d} b_1^{(1,j)}\partial_{x_j} S_{L+2} \\ + \sum_{(n,k,i)\in\mathcal{I}} \big(-(2i-\alpha_n) b_1^{(0,1)} b_i^{(n,k)} + b_{i+1}^{(n,k)}\big)\frac{\partial S_{L+2}}{\partial b_i^{(n,k)}} - R,\end{aligned} \tag{3-21}$$

where the profiles Φ_i are given by the formulas

$$\begin{aligned}\Phi_2 := (b_1^{(0,1)})^2\Theta_1^{(0,1)} + \sum_{k=1}^{d} b_1^{(0,1)} b_1^{(1,k)}\Theta_1^{(1,k)} \\ + \sum_{j=1}^{d}\Bigg(b_1^{(1,j)} b_1^{(0,1)}\partial_{x_j} T_1^{(0,1)} + \sum_{k=1}^{d} b_1^{(1,j)} b_1^{(1,k)}\partial_{x_j} T_1^{(1,k)}\Bigg) \\ + \sum_{(n,k,0)\in\mathcal{I},\ n\geq 2}\Bigg(b_1^{(0,1)} b_0^{(n,k)}\Theta_0^{(n,k)} + \sum_{j=1}^{d} b_1^{(1,j)} b_0^{(n,k)}\partial_{x_j} T_0^{(n,k)}\Bigg) - P_2,\end{aligned} \tag{3-22}$$

and for $i=3,\dots,L+1$,

$$\begin{aligned}\Phi_i := {} & b_1^{(0,1)} b_{i-1}^{(0,1)} \Theta_{i-1}^{(0,1)} + \sum_{k=1,\,(1,k,i-1)\in\mathcal{I}}^{d} b_1^{(0,1)} b_{i-1}^{(1,k)} \Theta_{i-1}^{(1,k)} \\ & + \sum_{j=1}^{d} \Bigl(b_1^{(1,j)} b_{i-1}^{(0,1)} \partial_{x_j} T_{i-1}^{(0,1)} + \sum_{k=1,\,(1,k,i-1)\in\mathcal{I}}^{d} b_1^{(1,j)} b_{i-1}^{(1,k)} \partial_{x_j} T_1^{(1,k)} \Bigr) \\ & + \sum_{(n,k,i-2)\in\mathcal{I},\ n\geq 2} \Bigl(b_1^{(0,1)} b_{i-2}^{(n,k)} \Theta_{i-2}^{(n,k)} + \sum_{j=1}^{d} b_1^{(1,j)} b_{i-2}^{(n,k)} \partial_{x_j} T_{i-2}^{(n,k)} \Bigr) \\ & + b_1^{(0,1)} \Lambda S_{i-1} + \sum_{m=1}^{d} b_1^{(1,m)} \partial_{x_m} S_{i-1} \\ & + \sum_{(n,k,j)\in\mathcal{I}} \bigl(-(2j-\alpha_n) b_1^{(0,1)} b_j^{(n,k)} + b_{j+1}^{(n,k)} \bigr) \frac{\partial S_{i-1}}{\partial b_j^{(n,k)}} - P_i, \end{aligned} \tag{3-23}$$

$$\begin{aligned}\Phi_{L+2} := {} & b_1^{(0,1)} \Lambda S_{L+1} + \sum_{m=1}^{d} b_1^{(1,m)} \partial_{x_m} S_{L+1} \\ & + \sum_{(n,k,j)\in\mathcal{I}} \bigl(-(2j-\alpha_n) b_1^{(0,1)} b_j^{(n,k)} + b_{j+1}^{(n,k)} \bigr) \frac{\partial S_{L+1}}{\partial b_j^{(n,k)}} - P_{L+2}. \end{aligned} \tag{3-24}$$

Step 2: definition of the profiles $(S_i)_{2\leq i\leq L+2}$ and simplification of ψ_b. We define by induction a sequence of couples of profiles $(S_i)_{2\leq i\leq L+2}$ by

$$\begin{cases} S_2 := -H^{-1}(\Phi_2) \\ S_i := -H^{-1}(\Phi_i) \quad \text{for } 3\leq i\leq L+2, \quad \text{with } \Phi_i \text{ defined by (3-22), (3-23), (3-24),} \end{cases} \tag{3-25}$$

where H^{-1} is defined by (2-25). In the next step we prove that there is no problem in this construction. Since the S_i are defined in this way, by (3-21) we get the final expression for the error

$$\psi_b = b_1^{(0,1)} \Lambda S_{L+2} + \sum_{j=1}^{d} b_1^{(1,j)} \partial_{x_j} S_{L+2} + \sum_{(n,k,i)\in\mathcal{I}} \bigl(-(2i-\alpha_n) b_1^{(0,1)} b_i^{(n,k)} + b_{i+1}^{(n,k)} \bigr) \frac{\partial S_{L+2}}{\partial b_i^{(n,k)}} - R. \tag{3-26}$$

Step 3: properties of the profiles S_i. We prove by induction on $i=2,\dots,L+2$ that S_i is homogeneous of degree $(i,-\gamma-g')$ in the sense of Definition 2.14, and that for all $2\leq j\leq L+2$, we have $\partial S_j/\partial b_i^{(n,k)}=0$ if $j\leq i$ for $n=0,1$ and if $j\leq i+1$ for $n\geq 2$.

Initialization. We now prove that S_2 is homogeneous of degree $(2,-\gamma-g')$, and that $\partial S_2/\partial b_i^{(n,k)}=0$ if $2\leq i$ for $n=0,1$ and if $1\leq i$ for $n\geq 2$. We claim that Φ_2 is homogeneous of degree $(2,-\gamma-g'-2)$ and that $\partial\Phi_2/\partial b_i^{(n,k)}=0$ if $2\leq i$ for $n=0,1$ and if $1\leq i$ for $n\geq 2$. To prove this, we prove that these two properties are true for every term on the right-hand side of (3-22).

From Lemma 2.10, $\Theta_1^{(0,1)}$ is simple admissible of degree $(0,-\gamma+2-g')$ in the sense of Definition 2.11. We also know $(b_1^{(0,1)})^2$ can be written under the form $J_1^{(0,1)}=2$ and $J_i^{(n,k)}=0$ otherwise and one has $|J|_2=2$ and $|J|_3=2$. Therefore, $(b_1^{(0,1)})^2\Theta_1^{(0,1)}$ is homogeneous of degree $(|J|_3,-\gamma+2-g'-2|J|_2)=(2,-\gamma-g'-2)$. The same reasoning applies for $b_1^{(0,1)}b_1^{(1,k)}\Theta_1^{(1,k)}$ for $1\leq k\leq d$.

For $1 \le j \le d$, we know $T_1^{(0,1)}$ is admissible of degree $(0, -\gamma+2)$ by Lemma 2.12, so $\partial_{x_j} T_1^{(0,1)}$ is admissible of degree $(-\gamma+1)$ by Lemma 2.10. We also know $b_1^{(1,j)} b_1^{(0,1)}$ can be written in the form b^J with $J_1^{(0,1)} = 1$, $J_1^{(1,j)} = 1$ and $J_i^{(n,k)} = 0$ otherwise; therefore $|J|_3 = 2$ and $|J|_2 = 1 + \frac{\gamma-\gamma_1}{2} + 1 = 2 + \frac{\alpha-1}{2}$ by (1-18). Thus $b_1^{(1,j)} b_1^{(0,1)} \partial_{x_j} T_1^{(0,1)}$ is homogeneous of degree $(|J|_3, -\gamma 1 + 1 - 2|J|_2) = (2, -\gamma-2-\alpha)$. As $g' < \alpha$, it is then homogeneous of degree $(2, -\gamma-g'-2)$. The same reasoning applies for $1 \le j, k \le d$ to the term $b_1^{(1,j)} b_1^{(1,k)} \partial_{x_j} T_1^{(1,k)}$.

We now examine for $(n,k,0) \in \mathcal{I}$ the profile

$$b_1^{(0,1)} b_0^{(n,k)} \Theta_0^{(n,k)} + \sum_{j=1}^{d} b_1^{(1,j)} b_0^{(n,k)} \partial_{x_j} T_0^{(n,k)}.$$

$\Theta_0^{(n,k)}$ is simple admissible of degree $(n, -\gamma_n - g')$ by Lemma 2.10, and $b_1^{(0,1)} b_0^{(n,k)}$ can be written in the form b^J for $J_1^{(0,1)} = 1$, $J_0^{(n,k)} = 1$ and $J_i^{(n',k')} = 0$ otherwise. One then has $|J|_3 = 2$ and $|J|_2 = 1 + \frac{\gamma-\gamma_n}{2}$. Therefore, $b_1^{(0,1)} b_0^{(n,k)} \Theta_0^{(n,k)}$ is homogeneous of degree $(|J|_3, -\gamma_n - g' - 2|J|_2) = (2, -\gamma - g' - 2)$. Similarly the terms in the sum in the above identity are homogeneous of degree $(2, -\gamma - g' - 2)$.

We now look at the nonlinear term P_2. Since, for $2 \le i \le L+2$, the profile S_i involves polynomials of b in the form b^J with $|J|_3 = i$, from its definition (3-15) P_2 does not depend on the profiles S_i for $2 \le i \le L+2$ and can be written as

$$P_2 = C Q^{p-2} \biggl(b_1^{(0,1)} T_1^{(0,1)} + \sum_{k=1}^{d} b_1^{(1,k)} T_1^{(1,k)} + \sum_{(n,k,0)\in\mathcal{I}} b_0^{(n,k)} T_0^{(n,k)} \biggr)^2$$

for a constant C. We have to prove that all the mixed terms that are produced by this formula are homogeneous of degree $(2, \gamma - g' - 2)$. We write it only for one term, and apply the same reasoning to the others. For all $((n,k,0),(n',k',0)) \in \mathcal{I}^2$, by Lemmas 2.10 and 2.15 and (2-1), the profile $b_0^{(n,k)} b_0^{(n',k')} Q^{p-2} T_0^{(n,k)} T_0^{(n',k')}$ is homogeneous of degree $(2, -\gamma-2-\alpha)$ and then of degree $(2, -\gamma-2-g')$. As we said, similar considerations yield that all the other terms are homogeneous of degree $(2, \gamma - g' - 2)$. This implies that P_2 is homogeneous of degree $(2, -\gamma - g' - 2)$.

We have examined all terms in (3-22) and consequently proved that Φ_2 is homogeneous of degree $(2, -\gamma-2-g')$. By a direct check of all the terms on the right-hand side of (3-22), with P_2 given by the above identity, one has that $\partial \Phi_2 / \partial b_i^{(n,k)} = 0$ if $2 \le i$ for $n = 0,1$ and if $1 \le i$ for $n \ge 2$. We now check that we can apply Lemma 2.15(iii) to invert Φ_2 and to propagate the homogeneity. For all $\#\mathcal{I}$-tuples J with $|J|_3 = 2$, one has indeed for all integers n that $2|J|_2 - \gamma_n - 2 - g' > \gamma_n - d$ as the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing and $d - 2\gamma - 2 > 0$. For the second condition required by the lemma, we notice that g' is not a “fixed” constant in our problem, as its definition (1-21) involves a parameter ε. The purpose of the parameter ε is the following: by choosing it appropriately, we can suppose that for every $0 \le n \le n_0$ and $\#\mathcal{I}$-tuple J with $|J|_3 = 2$ we have

$$-2|J|_2 + \gamma + g' - \gamma_n \notin 2\mathbb{N}.$$

This allows us to apply Lemma 2.15(iii): S_2 is homogeneous of degree $(2, -\gamma - g')$. We also get that $\partial S_2 / \partial b_i^{(n,k)} = 0$ if $2 \le i$ for $n = 0,1$ and if $1 \le i$ for $n \ge 2$ as this is true for Φ_2. This proves the initialization of our induction.

Heredity. Suppose $3 \le i \le L+1$, and that $S_{i'}$ is homogeneous of degree $(i', -\gamma - g')$ for $2 \le i' \le i$, and that $\partial S_i' / \partial b_j^{(n,k)} = 0$ if $i' \le j$ for $n = 0, 1$ and if $i' - 1 \le j$ for $n \ge 2$. We claim that Φ_i is homogeneous of degree $(i, -\gamma - g' - 2)$ and that $\partial \Phi_i / \partial b_j^{(n,k)} = 0$ if $i \le j$ for $n = 0, 1$ and if $i - 1 \le j$ for $n \ge 2$. We prove it by looking at all the terms on the right-hand side of (3-23). With the same reasoning we used for the initialization, we prove that

$$b_1^{(0,1)} b_{i-1}^{(0,1)} \Theta_{i-1}^{(0,1)} + \sum_{k=1,\,(1,k,i-1)\in\mathcal{I}}^{d} b_1^{(0,1)} b_{i-1}^{(1,k)} \Theta_{i-1}^{(1,k)} + \sum_{j=1}^{d} \Bigl(b_1^{(1,j)} b_{i-1}^{(0,1)} \partial_{x_j} T_{i-1}^{(0,1)} + \sum_{k=1,\,(1,k,i-1)\in\mathcal{I}}^{d} b_1^{(1,j)} b_{i-1}^{(1,k)} \partial_{x_j} T_1^{(1,k)} \Bigr) + \sum_{(n,k,i-2)\in\mathcal{I},\, n\ge 2} \Bigl(b_1^{(0,1)} b_{i-2}^{(n,k)} \Theta_{i-2}^{(n,k)} + \sum_{j=1}^{d} b_1^{(1,j)} b_{i-2}^{(n,k)} \partial_{x_j} T_{i-2}^{(n,k)} \Bigr)$$

is homogeneous of degree $(i, \gamma - g' - 2)$. From the induction hypothesis, $b_1^{(0,1)} \Lambda S_{i-1}$ is homogeneous of degree $(i, -\gamma - g' - 2)$. From Lemma 2.12, for $1 \le j \le d$, we know $\partial_{x_j} S_{i-1}$ is homogeneous of degree $(i-1, -\gamma - g' - 1)$, so that $b_1^{(1,j)} \partial_{x_j} S_{i-1}$ is homogeneous of degree $(i, -\gamma - g' - 2 - \alpha)$; since α is positive, it is then homogeneous of degree $(i, -\gamma - g' - 2)$. Still from the induction hypothesis, for all $(n, k, i') \in \mathcal{I}$,

$$\bigl(-(2i' - \alpha_n) b_1^{(0,1)} b_{i'}^{(n,k)} + b_{i'+1}^{(n,k)}\bigr) \frac{\partial S_{i-1}}{\partial b_{i'}^{(n,k)}}$$

is homogeneous of degree $(i, -\gamma - g' - 2)$. The last term to be considered is P_i. Since, for $2 \le j \le L+2$, the profile S_j involves polynomials of b of the form b^J with $|J|_3 = i$, from its definition (3-15) P_i does not depend on the profiles S_j for $i \le j \le L+2$ and can be written as

$$P_i = \sum_{k=2}^{p} C_k Q^{p-k} \Bigl(\sum_{|J|=k,\, |J|_3 = i} C_J \prod_{(n,k,i)\in\mathcal{I}} (b_i^{(n,k)})^{J_i^{(n,k)}} (T_k^{(n,k)})^{J_i^{(n,k)}} \prod_{j=2}^{i-1} S_j^{J_j} \Bigr).$$

Let k be an integer $2 \le k \le p$; let J be a $\#\mathcal{I} + L$-tuple with $|J|_3 = i$. Then from the induction hypothesis,

$$Q^{p-k} \prod_{(n,k,i)\in\mathcal{I}} (b_i^{(n,k)})^{J_i^{(n,k)}} (T_k^{(n,k)})^{J_i^{(n,k)}} \prod_{j=2}^{i-1} S_j^{J_j}$$

is homogeneous of degree $\bigl(i, -\gamma - 2 - (k-1)\alpha - g' \sum_{j=2}^{i-1} J_j\bigr)$. As $k \ge 2$ and $\alpha > g'$, it is homogeneous of degree $(i, \gamma - 2 - g')$.

We just proved that Φ_i is homogeneous of degree $(i, -\gamma - 2 - g')$. By a direct check of all the terms on the right-hand side of (3-23), with P_i given by the above formula, one has that $\partial \Phi_i / \partial b_j^{(n,k)} = 0$ if $i \le j$ for $n = 0, 1$ and if $i - 1 \le j$ for $n \ge 2$. We now check that we can apply Lemma 2.15(iii) to get the desired properties for $S_i = -H^{-1} \Phi_i$. For all $\#\mathcal{I}$-tuples J with $|J|_3 = i$ and integers n, the first condition $|J|_2 - \gamma - 2 - g' > \gamma_n - d$ is fulfilled since $-2\gamma_n - d \ge -2\gamma - d > 2$. For the second condition, again as in the initialization, as g' is not a "fixed" constant in our problem (its definition (1-21) involves a parameter ε), we can choose it such that for every $0 \le n \le n_0$ and $\#\mathcal{I}$-tuple J with $|J|_3 = i$,

$$-2|J|_2 + \gamma + g' - \gamma_n \notin 2\mathbb{N}.$$

We thus can apply Lemma 2.15(iii): S_i is homogeneous of degree $(i, -\gamma - g')$. One also obtains that $\partial S_i / \partial b_j^{(n,k)} = 0$ if $i \le j$ for $n = 0, 1$ and if $i - 1 \le j$ for $n \ge 2$, as this is true for Φ_i. This proves the heredity in our induction.

The last step, that it is the heredity from $L+1$ to $L+2$, can be proved exactly the same way and we do not write it here.

Step 4: bounds for the error term. In Step 2 we computed the expression (3-26) of the error term ψ_b. In Step 3 we proved that the profiles S_i were well defined and homogeneous of degree $(i, -\gamma - g')$. We can now prove the bounds on ψ_b claimed in the proposition. In the sequel we always assume the bounds $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ and $|b_1^{(0,1)}| \ll 1$.

Homogeneity of ψ_b. We claim that ψ_b is a finite sum of homogeneous functions of degree $(i, -\gamma - g' - 2)$ for $i \ge L+3$. For this we consider all terms on the right-hand side of (3-26). As S_{L+2} is homogeneous of degree $(L+2, -\gamma - g')$ from Step 3, the function $b_1^{(0,1)} \Lambda S_{L+2}$ is homogeneous of degree $(L+3, -\gamma - g' - 2)$ by Lemma 2.15. Similarly for $1 \le j \le d$, we know $b_1^{(1,j)} \partial_{x_j} S_{L+2}$ is homogeneous of degree $(L+3, -\gamma - g' - 2 - \alpha)$ (and then homogeneous of degree $(L+3, -\gamma - g' - 2)$ as $\alpha > 0$), and for $(n,k,i) \in \mathcal{I}$,

$$(-(2i - \alpha_n) b_1^{(0,1)} b_i^{(n,k)} + b_{i+1}^{(n,k)}) \frac{\partial S_{L+2}}{\partial b_i^{(n,k)}}$$

is homogeneous of degree $(L+3, -\gamma - g' - 2)$. From its definition (3-16), and since S_i is homogeneous of degree $(i, -\gamma - g')$ for $2 \le i \le L+2$, we have R is a finite sum of homogeneous profiles of degree $(i, -\gamma - \alpha - 2)$ with $i \ge L+3$. All this implies that ψ_b is a finite sum of homogeneous functions of degree $(i, -\gamma - g' - 2)$ for $i \ge L+3$.

Proof of an intermediate estimate. We claim that there exists an integer $A \ge L+3$ such that for μ a d-tuple of integers, $j \in \mathbb{N}$ and $B > 1$ we have

$$\int_{|y| \le B} \frac{|\partial^\mu \psi_b|^2}{1 + |y|^{2j}} \, dy \le C(L) \sum_{i=L+3}^{A} |b_1^{(0,1)}|^{2i} B^{\max\left(4i + 4\left(m_0 - \frac{|\mu| + j}{2}\right) + 4(\delta_0 - 1) - 2g', 0\right)}. \tag{3-27}$$

We now prove this bound. We proved earlier that ψ_b is a finite sum of homogeneous functions of degree $(i, -\gamma - g' - 2)$ for $i \ge L+3$. Consequently, it suffices to prove this bound for a homogeneous function $b^J f(y)$ of degree $(|J|_3, -\gamma - g' - 2)$ with $|J|_3 \ge L+3$. As f is admissible of degree $(2|J|_2 - \gamma - g' - 2)$, one then computes

$$\begin{aligned}\int_{|y| \le B} \frac{|b^J \partial^\mu f|^2}{1 + |y|^{2j}} &\le C(f) |b_1^{(0,1)}|^{2|J|_2} \int_0^B (1+r)^{4|J|_2 - 2\gamma - 2g' - 4 - 2j - 2|\mu|} r^{d-1} \, dr \\ &\le C(f) |b_1^{(0,1)}|^{2|J|_2} B^{\max\left(4|J|_2 + 4\left(m_0 + \frac{j + |\mu|}{2}\right) + 4(\delta_0 - 1) - 2g', 0\right)}\end{aligned}$$

(we avoid the logarithmic case in the integral by changing a bit the value of g' defined in (1-21), by changing a bit the value of ε). This concludes the proof of (3-27).

Proof of the local bounds for the error. Let j be an integer, and $\mu\in\mathbb{N}^d$ with $|\mu|=j$. From (3-27), $|b_1^{(0,1)}|\ll 1$ and $B>1$, we obtain, by (3-27),

$$\int_{|y|\le B}|\partial^\mu\psi_b|^2\,dy\le C(L)|b_1^{(0,1)}|^{2L+6}B^{\max\left(4A+4\left(m_0-\frac{|\mu|+j}{2}\right)+4(\delta_0-1)-2g',0\right)},$$

which gives the desired bound (3-6).

Proof of the global bounds for the error. Let $j\le 2s_L$, and $\mu\in\mathbb{N}^d$ with $|\mu|=j$. Using (3-27), we notice that for $L+3\le i\le A$ one has

$$\max\left(4i+4\left(m_0-\frac{|\mu|+j}{2}\right)+4(\delta_0-1)-2g',0\right)=4i+4\left(m_0-\frac{|\mu|+j}{2}\right)+4(\delta_0-1)-2g'.$$

This implies

$$\begin{aligned}\int_{|y|\le B_1}\frac{|\partial^\mu\psi_b|^2}{1+|y|^{2j}}\,dy&\le C(L)\sum_{i=L+3}^{A}|b_1^{(0,1)}|^{2i}B_1^{4i+4\left(m_0-\frac{|\mu|+j}{2}\right)+4(\delta_0-1)-2g'}\\&\le C(L)|b_1^{(0,1)}|^{2\left(\frac{j}{2}-m_0\right)+2(1-\delta_0)+g'-C(L)\eta},\end{aligned}$$

which is the desired bound (3-5). Let j be an integer, $j\le s_L$. Now, as $H=-\Delta+V$, where V is a smooth potential satisfying $|\partial^\mu V|\le C(\mu)(1+|y|)^{-2-|\mu|}$, by (2-2) one obtains

$$\begin{aligned}\int_{|y|\le B_1}|H^j\psi_b|^2\,dy&\le C(L)\sum_{j'+|\mu|_1=2j}\int_{|y|\le B_1}\frac{|\partial^\mu\psi_b|^2}{1+|y|^{2j'}}\,dy\\&\le C(L)\sum_{j'+|\mu|=2j}\sum_{i=L+3}^{A}|b_1^{(0,1)}|^{2i}B_1^{\max(4i+4(m_0-j)+4(\delta_0-1)-2g',0)}\\&\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}\end{aligned}$$

using (3-27) (because again $4i+4(m_0-j)+4(\delta_0-1)-2g'>0$ as $i\ge L+3$ and $j\le s_L$). This proves the last estimate (3-4). ☐

We now localize the perturbation built in Proposition 3.1 in the zone $|y|\le B_1$ and estimate error generated by the cut. We also include the time-dependence of the parameters following Remark 3.2. We recall that s_L is defined by (1-24).

Proposition 3.3 (localization of the perturbation). *The function χ is a cut-off defined by (1-43). We keep the notations from Proposition 3.1. $I=(s_0,s_1)$ is an interval, and*

$$b:I\to\mathbb{R}^{\#\mathcal{I}},\quad s\mapsto(b_i^{(n,k)}(s))_{(n,k,i)\in\mathcal{I}},$$

is a C^1 function with the a priori bounds[9]

$$|b_i^{(n,k)}|\lesssim|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i},\quad 0<b_1^{(0,1)}\ll 1,\quad |b_{1,s}^{(0,1)}|\lesssim|b_1^{(0,1)}|^2. \tag{3-28}$$

[9]This means that under the bounds $|b_i^{(n,k)}|\le K|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ for some $K>0$, there exists $b^*(K)$ such that the estimates that follow hold if $b_1^{(0,1)}\le b^*(K)$ with constants depending on K. In what follows, K will be fixed independently of the other important constants.

We define the profile $\tilde{Q}_b$ *as*

$$\tilde{Q}_b := Q + \tilde{\alpha}_b = Q + \chi_{B_1}\alpha_b, \quad \tilde{\alpha}_b := \chi_{B_1}\alpha_b. \tag{3-29}$$

Then one has the identity (Mod(s) *being defined by* (3-10))

$$\partial_s \tilde{Q}_b - F(\tilde{Q}_b) + b_1^{(0,1)}\Lambda\tilde{Q}_b + b_1^{(1,\cdot)}.\nabla\tilde{Q}_b = \tilde{\psi}_b + \chi_{B_1}\operatorname{Mod}(s) \tag{3-30}$$

with, for $0<\eta\ll 1$ *small enough, an error term* $\tilde{\psi}_b$ *satisfying the following bounds*:

(1) Global bounds. *For any integer* j *with* $1\le j\le s_L-1$ *we have*

$$\int_{\mathbb{R}^d} |H^j\tilde{\psi}_b|^2\,dy \le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C_j\eta}. \tag{3-31}$$

For any real number $s_c\le j<2s_L-2$,

$$\int_{\mathbb{R}^d} |\nabla^j\tilde{\psi}_b|^2\,dy \le C(L)|b_1^{(0,1)}|^{2(\frac{j}{2}-m_0)+2(1-\delta_0)-C_j\eta}, \tag{3-32}$$

and for $j=s_L$, *one has the improved bound*

$$\int_{\mathbb{R}^d} |H^{s_L}\tilde{\psi}_b|^2\,dy \le C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)+2\eta(1-\delta_0')}. \tag{3-33}$$

(2) Local bounds. *One has that* (ψ_b *being defined by* (3-3))

$$\forall |y|<B_1, \quad \tilde{\psi}_b(y)=\psi_b, \tag{3-34}$$

and for any $1\le B\le B_1$ *and* $j\in\mathbb{N}$,

$$\int_{|y|\le B} |\nabla^j\tilde{\psi}_b|^2\,dy \le C(L,j)B^{C(L,j)}|b_1^{(0,1)}|^{2L+6}. \tag{3-35}$$

Proof. First, we compute the expression of the new error term by rewriting the left-hand side of (3-30) using (3-9) and the fact that $F(Q)=0$:

$$\begin{aligned}\tilde{\psi}_b = {}& \chi_{B_1}\psi_b + \partial_s(\chi_{B_1})\tilde{\alpha_b} - \big[F(Q+\chi_{B_1}\alpha_b) - F(Q) - \chi_{B_1}\big(F(Q+\alpha_b)-F(Q)\big)\big] \\ &+ b_1^{(0,1)}(\Lambda Q - \chi_{B_1}\Lambda Q) + b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b) \\ &\quad + b_1^{(1,\cdot)}.(\nabla Q - \chi_{B_1}\nabla Q) + b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b) - \chi_{B_1}\nabla\alpha_b).\end{aligned} \tag{3-36}$$

Local bounds. In the previous identity, one clearly sees that all the terms, except $\chi_{B_1}\psi_b$, have their support in $B_1\le|y|$. Thus, for $B\le B_1$, the bound (3-35) is a direct consequence of the local bound (3-6) for ψ_b.

Global bounds. Let $m_1+1\le j\le s_L$. We will prove the bounds (3-31) and (3-33) by proving that this estimate holds for all terms on the right-hand side of (3-36). The reasoning to prove the estimates will be similar from one term to another. For this reason, we shall go quickly whenever an argument has already been used earlier.

The $\chi_{B_1}\psi_b$ term. As $H=-\Delta+V$ for V a smooth potential with $\partial^\mu V\lesssim(1+|y|)^{-2-|\mu|}$ by (2-2), and as $(\partial_r^k(\chi_{B_1}))(r)=B_1^{-k}\partial_r^k\chi(r/B_1)$, we have the identity

$$H^j(\chi_{B_1}\psi_b)=\chi_{B_1}H^j\psi_b+\sum_{\substack{\mu\in\mathbb{N}^d\\ 0\le|\mu|\le 2j-1}}^{j} f_\mu\partial^\mu\psi_b,$$

where for each $\mu\in\mathbb{N}^d$, with $0\le|\mu|\le j-1$, we have f_μ has its support in $B_1\le|x|\le 2B_1$ and satisfies $|f_\mu|\le C(L)B_1^{-(2j-|\mu|)}$. Using (3-4) and (3-5) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}&|H^j(\chi_{B_1}\psi_b)|^2\,dy\\ &\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}+\sum_{\substack{\mu\in\mathbb{N}^d\\ 0\le|\mu|\le 2j-1}}^{j} B_1^{-(4j-2|\mu|)}b_1^{2(\frac{|\mu|}{2}-m_0+2(1-\delta_0)+g'-C(L)\eta)}\\ &\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}.\end{aligned}\tag{3-37}$$

Similarly, one obtains, for any integer j' with $0\le j'\le 2s_L-2$,

$$\int_{\mathbb{R}^d}|\nabla^{j'}(\chi_{B_1}\psi_b)|^2\le C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)+g'-C(L)\eta}.\tag{3-38}$$

Using interpolation, this estimate remains true for any real number j' with $0\le j'\le 2s_L-2$.

The $\partial_s(\chi_{B_1})\alpha_b$ term. We first split using (3-7):

$$\partial_s(\chi_{B_1})\alpha_b=\partial_s(\chi_{B_1})\bigg(\sum_{(n,k,i)\in\mathcal{I}}b_i^{(n,k)}T_i^{(n,k)}+\sum_{i=2}^{L+2}S_i\bigg).\tag{3-39}$$

We compute

$$\partial_s(\chi_{B_1})=(b_1^{(0,1)})^{-1}b_{1,s}^{(0,1)}\frac{|y|}{B_1}(\partial_r\chi_{B_1})\Big(\frac{y}{B_1}\Big).$$

We first treat the S_i terms. As we already explained in the study of the $\chi_{B_1}\psi_b$ term, one has

$$H^j(\partial_s(\chi_{B_1})S_i)=\sum_{\mu\in\mathbb{N}^d,\,|\mu|\le 2j}f_\mu\partial^\mu S_i$$

with f_μ a smooth function, with support in $B_1\le|x|\le 2B_1$ and satisfying $|f_\mu|\le C(L)b_1^{(0,1)}B_1^{-(2j-|\mu|_1)}$ (because $|b_{1,s}^{(0,1)}|\lesssim|b_1^{(0,1)}|^2$ by (3-28)). As S_i is homogeneous of degree $(i,-\gamma-g')$ in the sense of Definition 2.14, from (3-8) and $|b_i^{(n,k)}|\lesssim|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ we get

$$\int_{\mathbb{R}^d}|H^j(\partial_s(\chi_{B_1})S_i)|^2\,dy\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}\tag{3-40}$$

using Lemma 2.15. Now we treat the $T_i^{(n,k)}$ terms in the identity (3-39). Let $(i,n,k)\in\mathcal{I}$. Then again one has the decomposition

$$H^j[\partial_s(\chi_{B_1})b_i^{(n,k)}T_i^{(n,k)}]=b_i^{(n,k)}\sum_{\mu\in\mathbb{N}^d,\,|\mu|\le 2j}f_\mu\partial^\mu T_i^{n,k}$$

with f_μ a smooth function, with support in $B_1\le|y|\le 2B_1$ and satisfying $|f_\mu|\le C(L)b_1^{(0,1)}B_1^{-(2j-|\mu|)}$. As $T_i^{(n,k)}$ is an admissible profile of degree $(-\gamma_n+2i)$ in the sense of Definition 2.11 by (2-26) and Lemma 2.10, $\partial^\mu T_i^{n,k}$ is admissible of degree $(-\gamma_n+2i-|\mu|)$ by Lemma 2.12 and we compute

$$\begin{aligned}\int_{\mathbb{R}^d}|b_i^{(n,k)}f_\mu\partial^\mu T_i^{n,k}|^2\,dy&\le\frac{C(L)|b_1^{(0,1)}|^{\gamma-\gamma_n+2i+2}}{B_1^{2(2j-|\mu_|1)}}\int_{B_1}^{2B_1}r^{-2\gamma_n+4i-2|\mu|_1}r^{d-1}\,dr\\&\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\eta(2j-2i-2\delta_n-2m_n)}.\end{aligned}$$

As $(i,n,k)\in\mathcal{I}$, we know $i\le L_n$ so if $j=s_L$ one has $2j-2i-2\delta_n-2m_n\ge 2-2\delta_n$. Therefore we have proved the bound (we recall that $\delta_0'=\max_{0\le n\le n_0}\delta_n\in(0,1)$)

$$\int_{\mathbb{R}^d}|H^j(\partial_s(\chi_{B_1})b_i^{(n,k)}T_i^{(n,k)})|^2\,dy\le\begin{cases}C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta}&\text{if } m_0+1\le j<s_L,\\C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)+\eta(1-\delta_0')}&\text{if } j=s_L.\end{cases}\tag{3-41}$$

From the decomposition (3-39), the bounds (3-40) and (3-41), we deduce the bound

$$\begin{aligned}&\int_{\mathbb{R}^d}|H^j(\partial_s(\chi_{B_1})\alpha_b|^2\,dy\\&\quad\le\begin{cases}C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta}&\text{if } 0\le j<s_L,\\C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}\big(|b_1^{(0,1)}|^{2\eta(1-\delta_0')}+|b_1^{(0,1)}|^{g'-C(L)\eta}\big)&\text{if } j=s_L.\end{cases}\end{aligned}\tag{3-42}$$

Using verbatim the same arguments, one gets that for any integer $0\le j'\le 2s_L-2$,

$$\int_{\mathbb{R}^d}|\nabla^{j'}(\partial_s(\chi_{B_1})\alpha_b|^2\,dy\le C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(L)\eta},\tag{3-43}$$

which remains true for any real number j' with $0\le j'\le 2s_L-2$ by interpolation.

The $F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))$ *term.* It can be written as

$$\begin{aligned}&F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))\\&\qquad=\Delta(\chi_{B_1}\alpha_b)-\chi_{B_1}\Delta\alpha_b+(Q+\chi_{B_1}\alpha_b)^p-Q^p-\chi_{B_1}((Q+\alpha_b)^p-Q^p).\end{aligned}\tag{3-44}$$

We now prove the bound for the two terms that have appeared. From the identity

$$\Delta(\chi_{B_1}\alpha_b)-\chi_{B_1}\Delta\alpha_b=\Delta(\chi_{B_1})\alpha_b+2\nabla\chi_{B_1}.\nabla\alpha_b,$$

as χ is radial and as $(\partial_r^k(\chi_{B_1}))(r)=B_1^{-k}\partial_r^k\chi(r/B_1)$, one sees that this term can be treated exactly the same way we treated the previous term: $\partial_s(\chi_{B_1})\alpha_b$. This is why we claim the following estimates that can be proved using exactly the same arguments:

$$\int_{\mathbb{R}^d} |H^j(\Delta(\chi_{B_1}\alpha_b)-\chi_{B_1}\Delta\alpha_b)|^2\,dy \le \begin{cases} C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } m_0+1\le j<s_L, \\ C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}\big(|b_1^{(0,1)}|^{2\eta(1-\delta_0')}+|b_1^{(0,1)}|^{g'-C(L)\eta}\big) & \text{if } j=s_L. \end{cases} \tag{3-45}$$

We now turn to the other term in (3-44), which can be rewritten as

$$(Q+\chi_{B_1}\alpha_b)^p-Q^p-\chi_{B_1}((Q+\alpha_b)^p-Q^p)=\sum_{k=2}^{p} C_k^p Q^{p-k}\chi_{B_1}(\chi_{B_1}^{k-1}-1)\alpha_b^k.$$

All the terms are localized in the zone $B_1\le|y|\le 2B_1$. From the definition (3-7) of α_b, (3-8), (2-1) and Lemma 2.15, for each $2\le k\le p$ one has that $Q^{p-k}\alpha_b^k$ is a finite sum of homogeneous profiles of degree $(i,-\gamma-\alpha-2)$ for $i\ge k$, yielding

$$\int_{\mathbb{R}^d} \big|H^j\big((Q+\chi_{B_1}\alpha_b)^p-Q^p-\chi_{B_1}((Q+\alpha_b)^p-Q^p)\big)\big|^2\,dy \le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}. \tag{3-46}$$

From the decomposition (3-44) and the estimates (3-45) and (3-46) one gets

$$\int_{\mathbb{R}^d} \big|H^j\big(F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))\big)\big|^2\,dy \le C(L)\begin{cases} |b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } m_0+1\le j<s_L, \\ |b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}\big(|b_1^{(0,1)}|^{2\eta(1-\delta_0')}+|b_1^{(0,1)}|^{\alpha-C(L)\eta}\big) & \text{if } j=s_L. \end{cases} \tag{3-47}$$

The same methods used for the two previous terms yield the analogue estimate for

$$\nabla^{j'}\big[F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))\big]$$

for any integer $0\le j'\le 2s_L-2$, and by interpolation, we obtain, for any real number j' with $0\le j'\le 2s_L-2$,

$$\int_{\mathbb{R}^d} \big|\nabla^{j'}\big(F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))\big)\big|^2\,dy \le C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(L)\eta}. \tag{3-48}$$

The $b_1^{(0,1)}(\Lambda Q-\chi_{B_1}\Lambda Q)$ term. As $\partial^\mu(\Lambda Q)\le C(\mu)(1+|y|)^{-\gamma-|\mu|}$ for all $\mu\in\mathbb{N}^d$ by (2-7) and $H\Lambda Q=0$, one computes

$$\int_{\mathbb{R}^d} \big|H^j(b_1^{(0,1)}(\Lambda Q-\chi_{B_1}\Lambda Q))\big|^2\,dy \le C(j)|b_1^{(0,1)}|^2\int_{B_1}^{2B_1} r^{-2\gamma-4j}r^{d-1}\,dr \le C(j)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+2\eta(j-m_0-\delta_0)} \tag{3-49}$$

with $s_L - m_0 - \delta_0 = L + 1 - \delta_0 > 1 - \delta_0$ for $j = s_L$. For any integer j' with $E[s_c] \le j' \le 2s_L - 2$, similar reasoning yields the estimate

$$\int_{\mathbb{R}^d} |\nabla^{j'}(b_1^{(0,1)}(\Lambda Q - \chi_{B_1}\Lambda Q))|^2\,dy \le C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}.$$

By interpolation, one has for any real number j' with $E[s_c] \le j' \le 2s_L - 2$,

$$\int_{\mathbb{R}^d} \big|\nabla^{j'}(b_1^{(0,1)}(\Lambda Q - \chi_{B_1}\Lambda Q))\big|^2\,dy \le C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}. \tag{3-50}$$

The $b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b)$ term. First we write this term as

$$b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b = b_1^{(0,1)}(y.\nabla\chi_{B_1})\alpha_b.$$

Now, we notice that

$$b_1^{(0,1)}(y.\nabla\chi_{B_1}) = b_1^{(0,1)}\frac{|y|}{B_1}(\partial_r\chi)\left(\frac{|y|}{B_1}\right)$$

is very similar to

$$\partial_s(\chi_{B_1}) = (b_1^{(0,1)})^{-1}b_{1,s}^{(0,1)}\frac{|y|}{B_1}(\partial_r\chi_{B_1})\left(\frac{y}{B_1}\right)$$

in the sense that it enjoys the same estimates, as $|b_{1,s}^{(0,1)}| \lesssim (b_1^{(0,1)})^2$ by (3-28). Thus, we can get exactly the same estimates for the term $b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b)$ that we obtained previously for the term $\partial_s(\chi_{B_1})\alpha_b$ with the exact same methodology, yielding

$$\begin{aligned}\int_{\mathbb{R}^d} &\big|H^j\big(b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b)\big)\big|^2\,dy \\ &\le \begin{cases} C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } 0 \le j < s_L, \\ C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}\big(|b_1^{(0,1)}|^{2\eta(1-\delta_0')} + |b_1^{(0,1)}|^{g'-C(L)\eta}\big) & \text{if } j = s_L,\end{cases}\end{aligned} \tag{3-51}$$

and for any integer j' with $0 \le j' \le 2s_L - 2$,

$$\int_{\mathbb{R}^d} |\nabla^{j'}(b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b) - \chi_{B_1}\Lambda\alpha_b))|^2\,dy \le C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(L)\eta}. \tag{3-52}$$

The $b_1^{(1,\cdot)}.(\nabla Q - \chi_{B_1}\nabla Q)$ term. First we rewrite

$$b_1^{(1,\cdot)}.(\nabla Q - \chi_{B_1}\nabla Q) = \sum_{i=1}^d b_1^{(1,i)}(1-\chi_{B_1})\partial_{y_i}Q. \tag{3-53}$$

Now let i be an integer, $1 \le i \le d$. From the asymptotic (2-1) of the ground state

$$|\partial^\mu Q| \le C(\mu)(1+|y|)^{-\frac{2}{p-1}-|\mu|}$$

and the fact that $H\partial_{x_i}Q=0$, we deduce

$$\int_{\mathbb{R}^d}\left|H^j\left(b_1^{(1,i)}((1-\chi_{B_1})\partial_{y_i}Q)\right)\right|^2dy\le C(j)|b_1^{(0,1)}|^{\gamma-\gamma_1+2}\int_{B_1}^{2B_1}r^{-2\gamma_1-4j}r^{d-1}\,dr$$
$$\le C(j)|b_1^{(0,1)}|^{2(j-m_0)-2(1-\delta_0)+2\eta(j-m_1-\delta_1)}$$

with $s_L-m_1-\delta_1=L+m_0-m_1+1-\delta_1>1-\delta_1$ for $j=s_L$. So we finally get, putting together the two previous equations,

$$\int_{\mathbb{R}^d}\left|H^j\left(b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q)\right)\right|^2dy\le C(j)|b_1^{(0,1)}|^2\int_{B_1}^{+\infty}r^{-2\gamma-4j}r^{d-1}\,dr$$
$$\le C(j)|b_1^{(0,1)}|^{2(j-m_0)-2(1-\delta_0)+2\eta(1-\delta_1)}. \tag{3-54}$$

Now, for any integer j' with $E[s_c]\le j'\le 2s_L-2$, as $E[s_c]>s_c-1$, similar reasoning yields the estimate

$$\int_{\mathbb{R}^d}\left|\nabla^{j'}(b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q))\right|^2dy\le C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}.$$

By interpolation, one has for any real number j' with $E[s_c]\le j'\le 2s_L-2$,

$$\int_{\mathbb{R}^d}\left|\nabla^{j'}(b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q))\right|^2dy\le C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}. \tag{3-55}$$

The $b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b)$ *term.* We first rewrite

$$b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b)=\sum_{i=1}^{d}b_1^{(1,i)}\partial_{y_i}(\chi_{B_1})\alpha_b.$$

Let i be an integer, $1\le i\le d$. For all $\mu\in\mathbb{N}^d$, we know $\partial^\mu(\chi_{B_1})\le C(\mu)B_1^{-|\mu|}$. From (3-7) and (3-8), α_b is a sum of homogeneous profiles of degree $(i,-\gamma)$. Using Lemma 2.15, one computes

$$\int_{\mathbb{R}^d}\left|H^j(b_1^{(1,i)}\partial_{y_i}(\chi_{B_1})\alpha_b)\right|^2dy\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}.$$

With the two previous equations, one has proved that

$$\int_{\mathbb{R}^d}\left|H^j\left(b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b)\right)\right|^2dy\le C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}. \tag{3-56}$$

Using exactly the same arguments, one can prove that for any integer $0\le j'\le 2s_L-2$, the analogue estimate for $\nabla^{j'}(b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b))$ holds. By interpolation, it gives that for any real number $0\le j'\le 2s_L-2$ we have

$$\int_{\mathbb{R}^d}\left|\nabla^{j'}\left(b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b)\right)\right|^2dy\le C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}. \tag{3-57}$$

End of the proof. For the estimate concerning the operator H (resp. the operator ∇), we have estimated all terms on the right-hand side of (3-36) in (3-37), (3-42), (3-47), (3-49), (3-51), (3-54) and (3-56) (resp. the right-hand side of (3-36) in (3-38), (3-43), (3-48), (3-50), (3-52), (3-55) and (3-57)). Adding all these estimates, as $0 < b_1^{(0,1)} \ll 1$ is a very small parameter, one sees that there exists $\eta_0 := \eta_0(L)$ such that for $0 < \eta < \eta_0$, the bounds (3-31) and (3-33) hold (resp. the bound (3-32) holds). □

3B. ***Study of the approximate dynamics for the parameters.*** In Proposition 3.3 we stated the existence of a profile $\tilde{Q}_b$ such that the force term $F(\tilde{Q}_b)$ generated by (NLH) has an almost explicit formulation in terms of the parameters $b = (b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ up to an error term $\tilde{\psi}_b$. Suppose that for some time, the solution that started at $\tilde{Q}_{b(0)}$ stays close to this family of approximate solutions, up to scaling and translation invariances, meaning that it can be written approximately as $\tau_{z(t)}(\tilde{Q}_{b(t),1/\lambda(t)})$. Then $\tilde{Q}_{b(s)}$ is almost a solution of the renormalized flow (3-2) associated to the functions of time $\lambda(t)$ and $z(t)$, meaning that

$$\partial_s(\tilde{Q}_b) - \frac{\lambda_s}{\lambda}\Lambda\tilde{Q}_b - \frac{z_s}{\lambda}.\nabla\tilde{Q}_b - F(\tilde{Q}_b) \approx 0.$$

Using the identity (3-30), this means

$$-\left(b_1^{(0,1)} + \frac{\lambda_s}{\lambda}\right)\Lambda\tilde{Q}_b - \left(b_1^{(1,\cdot)} + \frac{z_s}{\lambda}\right).\nabla\tilde{Q}_b + \chi_{B_1}\operatorname{Mod}(s) \approx 0.$$

From the very definition (3-10) of the modulation term $\operatorname{Mod}(s)$, projecting the previous relation onto the different modes that appeared[10] yields

$$\begin{cases} \dfrac{\lambda_s}{\lambda} = -b_1^{(0,1)}, \\ \dfrac{z_s}{\lambda} = -b_1^{(1,\cdot)}, \\ b_{i,s}^{(n,k)} = -(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)} + b_{i+1}^{(n,k)} \quad \forall (n,k,i)\in\mathcal{I} \end{cases} \tag{3-58}$$

with the convention $b_{L_n+1}^{(n,k)} = 0$. The understanding of a solution starting at $\tilde{Q}_{b(0)}$ then relies on the understanding of the solutions of the finite-dimensional dynamical system (3-58) driving the evolution of the parameters $b_i^{(n,k)}$. First we derive some explicit solutions such that $\lambda(t)$ touches 0 in finite time, signifying concentration in finite time.

Lemma 3.4 (special solutions for the dynamical system of the parameters). *We recall that the renormalized time s is defined by (3-1). Let $\ell \le L$ be an integer such that $2\alpha < \ell$. We define the functions*

$$\begin{cases} \bar{b}_i^{(0,1)}(s) = \dfrac{c_i}{s^i} & \text{for } 1 \le i \le \ell, \\ \bar{b}_i^{(0,1)} = 0 & \text{for } \ell < i \le L, \\ \bar{b}_i^{(n,k)} = 0 & \text{for } (n,k,i)\in\mathcal{I} \text{ with } n \ge 1, \end{cases} \tag{3-59}$$

with $(c_i)_{1\le i\le\ell}$ being ℓ constants defined by induction as

$$c_1 = \frac{\ell}{2\ell-\alpha} \qquad \text{and} \qquad c_{i+1} = -\frac{\alpha(\ell-i)}{2\ell-\alpha}c_i \quad \text{for } 1 \le i \le \ell-1. \tag{3-60}$$

[10] This will be done rigorously in the next section.

Then $\bar b = (\bar b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ *is a solution of the last equation in* (3-58). *Moreover, the solutions* $\lambda(s)$ *and* $z(s)$ *of the first two equations in* (3-58) *starting at* $\lambda(0)=1$ *and* $z(0)=0$, *taken in original time variable* t, *are* $z(t)=0$ *and*

$$\lambda(t)=\left(\frac{\alpha}{(2\ell-\alpha)s_0}\right)^{\frac{\ell}{\alpha}}\left(\frac{(2\ell-\alpha)}{\alpha}s_0-t\right)^{\frac{\ell}{\alpha}}. \tag{3-61}$$

Proof. It is a direct computation that can safely be left to the reader. ☐

As $s_0>0$ and $2\ell>\alpha$, (3-61) can be interpreted as: there exists $T>0$ with $\lambda(t)\approx(T-t)^{\frac{\ell}{\alpha}}$ as $t\to T$. Now, given $\frac12\alpha<\ell\le L$, we want to know the exact number of instabilities of the particular solution $\bar b$. In addition, in Propositions 3.1 and 3.3, we needed the a priori bounds

$$|b_i^{(n,k)}|\lesssim|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$$

to show sufficient estimates for the errors ψ_b and $\tilde\psi_b$. Around the solution $\bar b$ defined by (3-59), $b_1^{(0,1)}$ is of order s^{-1}, and so the a priori bounds we need become[11]

$$b_i^{(n,k)}\lesssim s^{\frac{\gamma_n-\gamma}{2}-i}.$$

Therefore, by "stability" of $\bar b$ we mean stability with respect to this size and introduce the following renormalization for a solution of (3-58) close to $\bar b$:

$$b_i^{(n,k)}=\bar b_i^{(n,k)}+\frac{U_i^{(n,k)}}{s^{\frac{\gamma-\gamma_n}{2}+i}}. \tag{3-62}$$

It defines a $\#\mathcal{I}$-tuple of real numbers $U=(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$, and we order the parameters as in (2-28) by

$$U=\big(U_1^{(0,1)},\dots,U_L^{(0,1)},U_1^{(1,1)},\dots,U_{L_1}^{(1,1)},\dots,U_0^{(n_0,k(n_0))},\dots,U_{L_{n_0}}^{(n_0,k(n_0))}\big). \tag{3-63}$$

In the next lemma we state the linear stability result for the renormalized perturbation $(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$.

Lemma 3.5 (linear stability of special solutions). *Suppose* b *is a solution of the last equation in* (3-58). *Define* $U=(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ *by* (3-62) *and order it as in* (3-63).

(i) Linearized dynamics. *The time evolution of* U *is given by*

$$\partial_s U=\frac1s AU+O\left(\frac{|U|^2}{s}\right), \tag{3-64}$$

where A *is the block diagonal matrix*

$$A=\begin{pmatrix}A_\ell & & & (0)\\ & \tilde A_1 & & \\ & & \ddots & \\ (0) & & & \tilde A_{n_0}\end{pmatrix}.$$

[11] One notices that this bound holds for $\bar b_i^{(n,k)}$.

The matrix A_ℓ is defined by

$$A_\ell=\begin{pmatrix}
-(2-\alpha)c_1+\alpha\frac{\ell-1}{2\ell-\alpha} & 1 & & & & & & & \\
\vdots & \ddots & \ddots & & & & & & \\
-(2i-\alpha)c_i & & \alpha\frac{\ell-i}{2\ell-\alpha} & 1 & & & (0) & & \\
\vdots & & & \ddots & \ddots & & & & \\
-(2\ell-\alpha)c_\ell & & & & 0 & 1 & & & \\
0 & & & & & -\alpha\frac{1}{2\ell-\alpha} & 1 & & \\
\vdots & & & & & & \ddots & \ddots & \\
0 & & & & & & -\alpha\frac{i-\ell}{2\ell-\alpha} & 1 & \\
\vdots & & (0) & & & & & \ddots & \ddots \\
\vdots & & & & & & & \ddots & 1 \\
0 & & & & & & & & -\alpha\frac{(L-\ell)}{2\ell-\alpha}
\end{pmatrix}. \tag{3-65}$$

The matrix $\widetilde{A}_1$ is a block diagonal matrix constituted of d matrices $\widetilde{A}'_1$:

$$\widetilde{A}_1=\begin{pmatrix}\widetilde{A}'_1 & & (0)\\ & \ddots & \\ (0) & & \widetilde{A}'_1\end{pmatrix},\quad \widetilde{A}'_1=\begin{pmatrix}
\alpha\frac{\ell-\frac{\alpha-1}{2}-1}{2\ell-\alpha} & 1 & & & \\
& \ddots & \ddots & & (0) \\
& & \alpha\frac{\ell-\frac{\alpha-1}{2}-i}{2\ell-\alpha} & 1 & \\
& & & \ddots & \ddots \\
& (0) & & \ddots & 1 \\
& & & & \alpha\frac{\ell-\frac{\alpha-1}{2}-L_1}{2\ell-\alpha}
\end{pmatrix}. \tag{3-66}$$

For $2\le n\le n_0$ the matrix $\widetilde{A}_n$ is a block diagonal matrix constituted of $k(n)$ times the matrix $\widetilde{A}'_n$:

$$\widetilde{A}_n=\begin{pmatrix}\widetilde{A}'_n & & (0)\\ & \ddots & \\ (0) & & \widetilde{A}'_n\end{pmatrix},\quad \widetilde{A}'_n=\begin{pmatrix}
\alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}}{2\ell-\alpha} & 1 & & & \\
& \ddots & \ddots & & (0) \\
& & \alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-i}{2\ell-\alpha} & 1 & \\
& & & \ddots & \ddots \\
& (0) & & \ddots & 1 \\
& & & & \alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-L_n}{2\ell-\alpha}
\end{pmatrix}. \tag{3-67}$$

(ii) Diagonalization, stability and instability. *A is diagonalizable because A_ℓ and $\widetilde{A}_n$ for $1\le n\le n_0$ are. A_ℓ is diagonalizable into the matrix*

$$\operatorname{diag}\left(-1,\frac{2\alpha}{2\ell-\alpha},\dots,\frac{i\alpha}{2\ell-\alpha},\dots,\frac{\ell\alpha}{2\ell-\alpha},\frac{-1}{2\ell-\alpha},\dots,\frac{\ell-L}{2\ell-\alpha}\right).$$

We denote the eigenvector of A associated to the eigenvalue -1 by v_1 and the eigenvectors associated to the unstable modes $2\alpha/(\ell-\alpha),\dots,\ell\alpha/(\ell-\alpha)$ of A by $v_2,\dots,v_\ell$. They are a linear combination of the

ℓ first components only. That is to say, there exists a $\#\mathcal{I}\times\#\mathcal{I}$ matrix coding a change of variables:

$$P_\ell := \begin{pmatrix} P'_\ell & 0 \\ 0 & \mathrm{Id}_{\#\mathcal{I}-\ell} \end{pmatrix}, \tag{3-68}$$

with P'_ℓ an invertible $\ell\times\ell$ matrix and $\mathrm{Id}_{\#\mathcal{I}-\ell}$ the $(\#\mathcal{I}-\ell)\times(\#\mathcal{I}-\ell)$ identity matrix such that

$$P_\ell A P_\ell^{-1} = \begin{pmatrix} A'_\ell & & & (0) \\ & \widetilde{A}_1 & & \\ & & \ddots & \\ (0) & & & \widetilde{A}_{n_0} \end{pmatrix}, \tag{3-69}$$

$$A'_\ell = \begin{pmatrix}
-1 & & & & q_1 & & & \\
& \frac{2\alpha}{2\ell-\alpha} & & (0) & q_2 & & & \\
& & \ddots & & \vdots & & (0) & \\
& & & \frac{\ell\alpha}{2\ell-\alpha} & q_\ell & & & \\
& & & & \frac{-\alpha}{2\ell-\alpha} & 1 & & \\
& & & & & \ddots & \ddots & \\
& (0) & & & & & \ddots & 1 \\
& & & & & & & \alpha\frac{\ell-L}{2\ell-\alpha}
\end{pmatrix} \tag{3-70}$$

with $(q_i)_{1\le i\le \ell}\in\mathbb{R}^\ell$ being some fixed coefficients. $\widetilde{A}'_1$ has $\max(E[i_1],0)$ nonnegative eigenvalues and $L_1-\max(E[i_1],0)$ strictly negative eigenvalues (i_n being defined by (1-29)). For $2\le n\le n_0$, we know $\widetilde{A}'_n$ has $\max(E[i_n]+1,0)$ nonnegative eigenvalues and $L_n+1-\max(E[i_n]+1,0)$ strictly negative eigenvalues.

Proof. (i) As b and $\bar b$ are solutions of (3-58), we compute (with the convention $\bar b^{(n,k)}_{L_n+1}=0$ and $U^{(n,k)}_{L_n+1}=0$)

$$U^{(n,k)}_{i,s} = \frac{1}{s}\biggl[\biggl(\frac{\gamma-\gamma_n}{2}+i-(2i-\alpha_n)\bar b_1^{(0,1)}s\biggr)U_i^{(n,k)} - (2i-\alpha_n)\bar b_i^{(n,k)} s^{\frac{\gamma-\gamma_n}{2}+i}U_1^{(0,1)} \\ -(2k-\alpha_n)U_1^{(0,1)}U_i^{(n,k)}+U_{i+1}^{(n,k)}\biggr].$$

As $\bar b_1^{(0,1)}=\ell/(2\ell-\alpha)$, we obtain

$$\frac{\gamma-\gamma_n}{2}+i-(2i-\alpha_n)\bar b_1^{(0,1)} = \alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-i}{2\ell-\alpha}.$$

We then get (3-65) by noticing that $\bar b_i^{(0,1)}=0$ for $i\ge \ell+1$ and because by definition $\gamma=\gamma_0$. We get (3-66) and (3-67) by noticing that $\bar b_i^{(n,k)}=0$ for $i\ge 1$.

(ii) $\widetilde{A}_n$ for $1\le n\le n_0$ is diagonalizable because it is upper triangular. Their eigenvalues are then the values on the diagonal, and the last statement in (ii), about the stability and instability directions comes from the very definition (1-29) of the real number i_n for $1\le n\le n_0$. It remains to prove that A_ℓ is diagonalizable. We will do it by calculating its characteristic polynomial.

Computation of the characteristic polynomial for the top left corner matrix. We let A'_ℓ be the $\ell\times\ell$ matrix

$$A'_\ell=\begin{pmatrix} -(2-\alpha)c_1+\alpha\frac{\ell-1}{2\ell-\alpha} & 1 & & & (0) \\ \vdots & \ddots & \ddots & & \\ -(2i-\alpha)c_i & & \alpha\frac{\ell-i}{2\ell-\alpha} & 1 & \\ \vdots & & & \ddots & \ddots \\ \vdots & & (0) & & \ddots\ 1 \\ -(2\ell-\alpha)c_l & & & & 0 \end{pmatrix}.$$

We recall that as $\alpha>2$, we have $\ell\geq 2$ so A'_ℓ has at least 2 rows and 2 columns. We let

$$\mathcal{P}_\ell(X)=\det(A'_\ell-X\,\mathrm{Id}).$$

We compute this determinant by expanding with respect to the last row and iterating by doing that again for the subdeterminant appearing in the process. Eventually we obtain an expression of the form

$$\mathcal{P}_\ell=(-1)^\ell(2\ell-\alpha)c_\ell+(-X)\Bigg[(-1)^{\ell+1}(2\ell-2-\alpha)c_{\ell-1}+\left(\frac{\alpha}{2\ell-\alpha}-X\right)\\ \Bigg[(-1)^\ell(2\ell-4-\alpha)c_{\ell-2}+\left(\frac{2\alpha}{2\ell-\alpha}-X\right)[\cdots]\Bigg]\Bigg]. \tag{3-71}$$

We define the polynomials $(A_i)_{1\leq i\leq\ell}$ and $(B_i)_{1\leq i\leq\ell}$ and $(C_i)_{1\leq i\leq\ell-1}$ as

$$\begin{aligned} A_i &:=(-1)^{\ell-i+1}(2\ell+2-2i-\alpha)c_{\ell+1-i},\\ B_i &:=(i-1)\frac{\alpha}{2\ell-\alpha}-X,\\ C_i &:=(-1)^{\ell+1-i}\Big(X(2\ell-2i-\alpha)c_{\ell-i}+\frac{2\ell-\alpha}{i}c_{\ell-i+1}\Big). \end{aligned} \tag{3-72}$$

This way, the determinant $\mathcal{P}_\ell$ given by (3-71) can be rewritten as

$$\mathcal{P}_\ell=A_1+B_1\big(A_2+B_2[A_3+B_3[\cdots]]\big). \tag{3-73}$$

We notice by a direct computation from (3-72) that

$$A_1+B_1A_2=C_1.$$

Moreover, this identity propagates by induction and we claim that for $1\leq i\leq\ell-2$,

$$C_i+B_1B_2A_{i+2}=B_{i+2}C_{i+1}. \tag{3-74}$$

Indeed, from (3-60) one has

$$\frac{2\ell-\alpha}{i+1}c_{\ell-i}=-\alpha c_{\ell-i-1},$$

and from (3-72)

$$\begin{aligned} B_{i+2}C_{i+1}-C_i &= \left((i+1)\frac{\alpha}{2\ell-\alpha}-X\right)(-1)^{\ell-i}\left(X(2\ell-2i-2-\alpha)c_{\ell-i-1}+\frac{2\ell-\alpha}{i+1}c_{\ell-i}\right) \\ &\qquad -(-1)^{\ell+1-i}\left(X(2\ell-2i-\alpha)c_{\ell-i}+\frac{2\ell-\alpha}{i}c_{\ell-i+1}\right) \\ &= (-1)^{\ell-i}\Biggl(\left((i+1)\frac{\alpha}{2\ell-\alpha}-X\right)\bigl(X(2\ell-2i-2-\alpha)c_{\ell-i-1}-\alpha c_{\ell-i-1}\bigr) \\ &\qquad -X(2\ell-2i-\alpha)\alpha\frac{i+1}{2\ell-\alpha}c_{\ell-i-1}+\alpha^2\frac{i+1}{2\ell-\alpha}c_{\ell-i-1}\Biggr) \\ &= (-1)^{\ell-i}c_{\ell-i-1}X\Biggl(\alpha\frac{i+1}{2\ell-\alpha}(2\ell-2i-2-\alpha)+\alpha-X(2\ell-2i-2-\alpha) \\ &\qquad -\frac{2\ell-2i-\alpha}{2\ell-\alpha}\alpha(i+1)\Biggr) \\ &= (-1)^{\ell-i}c_{\ell-i-1}X(2\ell-2i-2-\alpha)\left(\frac{\alpha}{2\ell-\alpha}-X\right) \\ &= A_{i+2}B_1B_i. \end{aligned}$$

From the above identity we can rewrite $\mathcal{P}_\ell$ given by (3-73) as

$$\begin{aligned} \mathcal{P}_\ell &= A_1+B_1A_2+B_1B_2A_3+B_1B_2B_3(A_4+B_4(\cdots)) \\ &= C_1+B_1B_2A_3+B_1B_2B_3(A_4+B_4(\cdots)) \\ &= B_3\bigl(C_2+B_1B_2(A_4+B_4(\cdots))\bigr) \qquad = B_3B_4\bigl(C_3+B_1B_2(A_5+B_5(\cdots))\bigr) \\ &\;\vdots \\ &= B_3\cdots B_\ell(C_{\ell-1}+B_1B_2). \end{aligned} \tag{3-75}$$

The last polynomial that appeared is, by (3-72),

$$C_{\ell-1}+B_1B_2 = X(2-\alpha)c_1+\frac{2\ell-\alpha}{\ell-1}c_2-X\left(\frac{\alpha}{2\ell-\alpha}-X\right) = (X+1)\left(X-\frac{\alpha\ell}{2\ell-\alpha}\right)$$

and so we end up from (3-75) with the final identity for $\mathcal{P}_\ell$:

$$\mathcal{P}_\ell = (X+1)\prod_{i=2}^{\ell}\left(\frac{i\alpha}{2\ell-\alpha}-X\right).$$

This means that A'_ℓ is diagonalizable with eigenvalues $(1,-2\alpha/(2\ell-\alpha),\dots,\ell/(2\ell-\alpha))$: there exists an invertible $\ell\times\ell$ matrix $\widetilde{P}_\ell$ such that $\widetilde{P}_\ell A_\ell\widetilde{P}_\ell^{-1}=\operatorname{diag}(-1,2/(2\ell-\alpha),\dots,\ell/(2\ell-\alpha))$. We denote by P'_ℓ the matrix

$$P'_\ell := \begin{pmatrix}\widetilde{P}_\ell & \\ & \mathrm{Id}_{L-\ell}\end{pmatrix}.$$

Then, from (3-65), there exists ℓ real numbers $(q_i)_{1\le i\le n}\in\mathbb{R}^\ell$ such that

$$P'_\ell A_\ell (P'_\ell)^{-1}=\begin{pmatrix} -(2-\alpha)c_1+\alpha\frac{\ell-1}{2\ell-\alpha} & 1 & & & & (0) \\ \vdots & & \ddots & \ddots & & \\ -(2i-\alpha)c_i & & & \alpha\frac{\ell-i}{2\ell-\alpha} & 1 & \\ \vdots & & & & \ddots & \ddots \\ \vdots & & (0) & & & \ddots\ 1 \\ -(2\ell-\alpha)c_l & & & & & 0 \end{pmatrix}.$$

This implies that A_ℓ can be diagonalized and that its eigenvalues are of simple multiplicity given by $\big(-1, 2\alpha/(2\ell-\alpha),\dots,\alpha\ell/(2\ell-\alpha),-\alpha/(2\ell-\alpha),\dots,-\alpha L-\ell/(2\ell-\alpha)\big)$, and that the eigenvectors associated to the eigenvalues -1, and $2\alpha/(2\ell-\alpha),\dots,\alpha\ell/(2\ell-\alpha)$ are linear combinations of the ℓ first components only. This concludes the proof of the lemma. □

4. Main proposition and proof of Theorem 1.1

We recall that the approximate blow-up profile $\tau_z(\tilde{Q}_{\bar{b},1/\lambda})$ was designed for a blow up on the whole space $\mathbb{R}^d$. In this section, we state in the main proposition of this paper, Proposition 4.6, the existence of solutions staying in a trapped regime (defined in Definition 4.4) close to the cut approximate blow-up profile $\chi\tau_z(\tilde{Q}_{\bar{b},1/\lambda})$. We then end the proof of Theorem 1.1 by proving that such a solution will blow up as described in the theorem.

4A. *The trapped regime and the main proposition.*

4A1. *Projection of the solution on the manifold of approximate blow-up profiles.* The following reasoning is made for a blow up on the whole space $\mathbb{R}^d$. As in this case our blow-up solution should stay close to the manifold of approximate blow-up profiles $(\tau_z(\tilde{Q}_{b,\lambda}))_{b,z,\lambda}$, we want to decompose it as a sum $\tau_z(\tilde{Q}_{b,\lambda}+\varepsilon_\lambda)$ for some parameters b,z,λ such that ε has "minimal" size. The tangent space of $(\tau_z(\tilde{Q}_{b,\lambda}))_{b,z,\lambda}$ at the point Q is $\operatorname{Span}(T_i^{(n,k)})_{(n,k,i)\in\mathcal{I}\cup\{(0,1,0),(1,1,0),\dots,(1,d,0)\}}$. One could then think of an orthogonal projection at the linear level, i.e., $\langle T_i^{(n,k)},\varepsilon\rangle=0$. The profiles $T_i^{(n,k)}$ are, however, not decaying quickly enough at infinity so that this duality bracket would make sense in the functional space where ε lies. For these grounds we will approximate such orthogonality conditions by smooth profiles that are compactly supported.

Definition 4.1 (generators of orthogonality conditions). For a very large scale $M\gg 1$, for $n\le n_0$ and $1\le k\le k(n)$ we define

$$\Phi_M^{(n,k)}=\sum_{i=0}^{L_n} c_{i,n,M}(-H)^i(\chi_M T_0^{(n,k)})=\sum_{i=0}^{L_n} c_{i,n,M}(-H^{(n)})^i(\chi_M T_0^{(n)})Y^{(n,k)} \tag{4-1}$$

(L_n and $T_0^{(n,k)}$ being defined by (1-28) and (2-26)), where

$$c_{0,n,M}=1 \quad\text{and}\quad c_{i,n,M}=-\frac{\sum_{j=0}^{i-1}c_{j,n,M}\big\langle(-H)^j(\chi_M T_0^{(n,k)}),T_i^{(n,k)}\big\rangle}{\langle\chi_M T_0^{(n)},T_0^{(n)}\rangle}. \tag{4-2}$$

Lemma 4.2 (generation of orthogonality conditions). *For $n \le n_0$, $1 \le k \le k(n)$, $0 \le i \le L_n$, $j \in \mathbb{N}$, $n' \in \mathbb{N}$ and $1 \le k' \le k(n')$, the following holds for $c > 0$:*

$$\langle (-H)^j \Phi_M^{(n,k)}, T_i^{(n',k')} \rangle = \delta_{(n,k,i),(n',k',j)} \int_0^{+\infty} \chi_M |T_0^{(n)}|^2 r^{d-1} \sim c M^{4m_n+4\delta_n} \delta_{(n,k,i),(n',k',j)}. \tag{4-3}$$

Proof. The scalar product is zero if $(n,k) \neq (n',k')$ because by construction $\Phi_M^{(n,k)}$ (resp. $T_i^{(n',k')}$) lives on the spherical harmonic $Y^{(n,k)}$ (resp. $Y^{(n',k')}$). We now suppose $(n,k) = (n',k')$ and compute using (4-1):

$$\langle (-H)^j \Phi_M^{(n,k)}, T_i^{(n,k)} \rangle = \sum_{l=0}^{L_n} c_{l,n,M} \langle T_0^{(n)} \chi_M, (-H^{(n)})^{l+j} T_i^{(n)} \rangle.$$

If $j > i$ for all l, then $(H^{(n)})^{l+j} T_i^{(n)} = 0$ and $\langle (-H)^j \Phi_M^{(n,k)}, T_i^{(n,k)} \rangle = 0$. If $j = i$ then only the first term in the sum is not zero since $(-H^{(n)})^i T_i^{(n)} = T_0^{(n,k)}$ and

$$\sum_{l=0}^{L_n} c_{l,n,M} \langle T_0^{(n)} \chi_M, (-H^{(n)})^{l+j} T_i^{(n)} \rangle = \langle T_0^{(n)} \chi_M, T_0^{(n)} \rangle \sim c M^{4m_n+4\delta_n}$$

from the asymptotic behavior (2-7) of $T_0^{(n)}$. If $j < i$ then

$$\sum_{l=0}^{L_n} c_{l,n,M} \langle T_0^{(n)} \chi_M, (-H^{(n)})^{l+j} T_i^{n} \rangle = c_{i-j,n,M} \langle T_0^{(n)} \chi_M, T_0^{(n)} \rangle + \sum_{l=0}^{i-j-1} c_{l,n,M} \langle T_0^{(n)} \chi_M, (-H^{(n)})^{l+j} T_i^{(n)} \rangle = 0$$

from the definition (4-2) of the constant $c_{i-j,n,M}$. □

4A2. *Geometrical decomposition.* First we describe here how we decompose a solution of (1-1) on the unit ball $\mathcal{B}^d(1)$ onto the set $(\tau_z(\tilde{Q}_{b,\lambda}))_{b,|z|\le\frac{1}{8},0<\lambda<\frac{1}{8M}}$ of concentrated ground states, using the orthogonality conditions provided by Lemma 4.2. This provides a decomposition for any domain containing $\mathcal{B}^d(1)$. Let $0 < \kappa \ll 1$ to be fixed later on. We study the set of functions close to $(\tau_z(\tilde{Q}_{b,\lambda}))_{b,|z|\le\frac{1}{8},0<\lambda<\frac{1}{8M}}$ such that the projection onto the first element in the generalized kernel dominates:[12]

$$u : \exists (\tilde{\lambda}, \tilde{z}) \in \left(0, \tfrac{1}{8M}\right) \times \mathcal{B}^d\left(\tfrac{1}{8}\right) \text{ such that}$$
$$\|u - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} < \frac{\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}} \quad \text{and} \quad \|(\tau_{-\tilde{z}} u)_{\tilde{\lambda}} - Q\|_{L^\infty(\mathcal{B}^d(3M))} < \langle (\tau_{-\tilde{z}} u)_{\tilde{\lambda}} - Q, H\Phi_M^{(0,1)} \rangle. \tag{4-4}$$

Lemma 4.3 (decomposition). *There exist $\kappa, K > 0$ such that for any solution $u \in \mathcal{C}^1([0,T), \times\mathcal{B}^d(1))$ of (1-1) satisfying (4-4) for all $t \in [0,T)$, there exists a unique choice of the parameters $\lambda : [0,T) \to \left(0, \frac{1}{4M}\right)$,*

[12] Note that $(\tau_{-\tilde{z}} u)_{\tilde{\lambda}}$ is defined on $\frac{1}{\tilde{\lambda}}(\mathcal{B}^d(1) - \tilde{z})$, which contains $\mathcal{B}^d(7M)$ as $|\tilde{z}| < \frac{1}{8}$ and $0 < |\tilde{\lambda}| < \frac{1}{8M}$; thus the second estimate makes sense.

$z : [0, T) \to \mathcal{B}^d\left(\frac{1}{4}\right)$ and $b : [0, T) \to \mathbb{R}^{\mathcal{I}}$ such that $b_1^{(0,1)} > 0$ and

$$u = (\tilde{Q}_b + v)_{z,\lambda} \quad \text{on } \mathcal{B}^d(1), \qquad \sum_{(n,k,i)\in\mathcal{I}} |b_i^{(n,k)}| + \|v\|_{L^\infty(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\}))} \le K\kappa$$

with $v = (\tau_{-z}u)_\lambda - \tilde{Q}_b$ satisfying the orthogonality conditions

$$\langle v, H^i \Phi_M^{(n,k)} \rangle = 0 \quad \text{for } 0 \le n \le n_0,\ 1 \le k \le k(n),\ 0 \le i \le L_n.$$

Moreover, λ, b and z are $\mathcal{C}^1$ functions.

Proof. It is a direct consequence of Lemma E.2 from the appendix. □

Decomposition and adapted norms for the remainder inside a bounded domain. Let u be a solution of (NLH) in $C^1([0,T),\Omega)$ with Dirichlet boundary condition such that the restriction[13] of u to $\mathcal{B}^d(1)$ satisfies the conditions of Lemma 4.3. Then from this lemma, for all $t \in [0, T)$ we can decompose u according to

$$u := \chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}}) + w, \tag{4-5}$$

cutting the approximate blow-up profile in the zone $1 \le |x| \le 2$, and w is a remainder term satisfying $w_{|\partial\Omega} = 0$ as $\mathcal{B}^d(7) \subset \Omega$ and $u_{|\partial\Omega} = 0$. To study w inside and outside the blow-up zone, we decompose it according to

$$w_{\text{int}} := \chi_3 w, \quad w_{\text{ext}} := (1-\chi_3)w, \quad \varepsilon := (\tau_{-z(t)} w_{\text{int}})_{\lambda(t)}, \tag{4-6}$$

where w_{int} and w_{ext} are the remainder cut in the zone $3 \le |x| \le 6$, ε is the renormalized remainder at the blow-up area, and is adapted to the renormalized flow. We notice that the support of w_{ext} does not intersect the support of the approximate blow-up profile $\chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})$, that the supports of w_{int} and w_{ext} overlap, and that $(w_{\text{ext}})_{|\partial\Omega} = 0$. From Lemma 4.3 and its definition, ε is compactly supported and satisfies the orthogonality conditions (4-11). We measure ε through the following norms:

(i) *High-order Sobolev norm adapted to the linearized flow.* We define

$$\mathcal{E}_{2s_L} := \int_{\mathbb{R}^d} |H^{s_L}\varepsilon|^2. \tag{4-7}$$

This norm controls the L^2 norms of all smaller-order derivatives with appropriate weight from Lemma C.3 since ε satisfies the orthogonality conditions (4-11), and the standard $\dot{H}^{2s_L}$ Sobolev norm

$$\mathcal{E}_{2s_L} \ge C \sum_{|\mu|\le 2s_L} \int_{\mathbb{R}^d} \frac{|\partial^\mu \varepsilon|^2}{1+|x|^{4i-2\mu+}} + C\|\varepsilon\|^2_{\dot{H}^{2s_L}}.$$

(ii) *Low-order slightly supercritical Sobolev norm.* Let σ be a slightly supercritical regularity:

$$0 < \sigma - s_c \ll 1. \tag{4-8}$$

[13] We recall that Ω contains $\mathcal{B}^d(7)$.

We then define the following second norm for the remainder:

$$\mathcal{E}_\sigma := \|\varepsilon\|^2_{\dot{H}^\sigma}. \tag{4-9}$$

Existence of a solution staying in a trapped regime close to the approximate blow-up solution. From now on we focus on solutions that are close to an approximate blow-up profile in the sense of the following definition.

Definition 4.4 (solutions in the trapped regime). We say that a solution u of (1-1) in $C^1([0,T),\Omega)$ is trapped on $[0,T)$ if it satisfies all of the following. First, it satisfies the condition (4-4) and then can be decomposed via Lemma 4.3 according to (4-5) and (4-6):

$$u := \chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}}) + w,\ w_{\text{int}} := \chi_3 w, \quad w_{\text{ext}} := (1-\chi_3)w, \quad \varepsilon := (\tau_{-z(t)}w_{\text{int}})_{\lambda(t)} \tag{4-10}$$

with ε satisfying the orthogonality conditions

$$\langle \varepsilon, H^i\Phi_M^{(n,k)}\rangle = 0 \quad \text{for } 0 \le n \le n_0,\ 1 \le k \le k(n),\ 0 \le i \le L_n. \tag{4-11}$$

To the scale λ given by this decomposition, we associate the renormalized time s defined by (3-1) with $s_0 > 0$. The $\#\mathcal{I}$-tuple of parameters b is represented as a perturbation of the solution $\bar{b}$ of the dynamical system (3-58) given by (3-59):

$$b_i^{(n,k)}(s) = \bar{b}_i^{(n,k)}(s) + \frac{U_i^{(n,k)}(s)}{s^{\frac{\gamma-\gamma_n}{2}+i}}. \tag{4-12}$$

We let $U := (U_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$. To use the eigenvectors of the linearized dynamics, Lemma 3.5, we define

$$V_i := (P_\ell U)_i \quad \text{for } 1 \le i \le \ell, \tag{4-13}$$

where P_ℓ is defined by (3-68). All these parameters must satisfy the following estimates, where $0<\tilde{\eta}\ll 1$, $0<\varepsilon_i^{(n,k)} \ll 1$ for $(n,k,i)\in\mathcal{I}$ with $(n,k,i)\notin\{1,\dots,\ell\}\times\{0\}\times\{1\}$; K_1 and K_2 will be fixed later on.

Initial conditions. At time $t=0$ (or equivalently $s=s_0$):

(i) Control of the unstable modes on the radial component:

$$|V_i(0)| \le s_0^{-\tilde{\eta}} \quad \text{for } 2 \le i \le \ell. \tag{4-14}$$

(ii) Control of the unstable modes on the other spherical harmonics:

$$|(U_i^{(n,k)}(0))| \le \varepsilon_i^{(n,k)} \quad \text{for } (n,k,i)\in\mathcal{I} \text{ with } 1 \le n \text{ and } 0 \le i < i_n. \tag{4-15}$$

(iii) Control of the stable modes:

$$V_1(0) \le \frac{1}{10 s_0^{\tilde{\eta}}}, \qquad |U_i^{(0,1)}(0)| \le \frac{\varepsilon_i^{(0,1)}}{10 s_0^{\tilde{\eta}}} \quad \text{for } \ell+1 \le i \le L, \tag{4-16}$$

$$|U_i^{(n,k)}(0)| \le \frac{\varepsilon_i^{(n,k)}}{10 s_0^{\tilde{\eta}}} \quad \text{for } (n,k,i)\in\mathcal{I} \text{ with } 1 \le n \text{ and } i_n < i \le L_n, \tag{4-17}$$

$$|U_i^{(n,k)}(0)| \le \frac{\varepsilon_i^{(n,k)}}{10} \quad \text{for } (n,k,i)\in\mathcal{I} \text{ with } 1 \le n \text{ and } i = i_n. \tag{4-18}$$

(iv) Smallness of the remainder:

$$\|w\|^2_{H^{2s_L}} < \frac{1}{s_0^{\frac{2\ell}{2\ell-\alpha}(2s_L-s_c)}}. \tag{4-19}$$

(v) Compatibility conditions at the border:[14]

$$\begin{cases} \tilde{w}_0 := w(0) \in H^1_0(\Omega), \\ \tilde{w}_1 := \partial_t w(0) = \Delta w(0) + w(0)^p \in H^1_0(\Omega), \\ \tilde{w}_2 := \partial_t^2 w(0) = \Delta^2 w(0) + \Delta(w(0)^p) + pw(0)^{p-1}(\Delta w(0) + w(0)^p) \in H^1_0(\Omega), \\ \quad\vdots \\ \tilde{w}_{s_L-1} := \partial_t^{s_L-1} w(0) \in H^1_0(\Omega). \end{cases} \tag{4-20}$$

(vi) Initial scale and initial blow-up point:

$$\lambda(0) = s_0^{-\frac{\ell}{2\ell-\alpha}} \quad \text{and} \quad z(0) = 0. \tag{4-21}$$

Pointwise in time estimates. The following bounds hold on $(0, T)$:

(i) Parameters on the first spherical harmonics:

$$|V_i(s)| \le s^{-\tilde{\eta}} \quad \text{for } 1 \le i \le \ell, \qquad |U_i^{(0,1)}(s)| \le \varepsilon_i^{(0,1)} s^{-\tilde{\eta}} \quad \text{for } \ell+1 \le i \le L. \tag{4-22}$$

(ii) Parameters on the other spherical harmonics: for $(n, k, i) \in \mathcal{I}$ with $n \ge 1$,

$$|(U_i^{(n,k)}(s))| \le 1 \quad \text{if } 0 \le i < i_n, \tag{4-23}$$

$$|U_i^{(n,k)}(s)| \le \frac{\varepsilon_i^{(n,k)}}{s^{\tilde{\eta}}} \quad \text{if } i_n < i \le L_n \qquad \text{and} \qquad |U_i^{(n,k)}(s)| \le \varepsilon_i^{(n,k)} \quad \text{if } i = i_n. \tag{4-24}$$

(iii) Control of the remainder:

$$\begin{aligned} &\mathcal{E}_{s_L}(s) \le \frac{K_2}{s^{2L+2(1-\delta_0)+2(1-\delta_0')\eta}}, \qquad \mathcal{E}_\sigma(s) \le \frac{K_1}{s^{2(\sigma-s_c)\frac{\ell}{2\ell-\alpha}}}, \\ &\|w_{\text{ext}}\|^2_{H^{2s_L}} \le \frac{K_2}{\lambda^{2(2s_L-s_c)} s^{2L+2(1-\delta_0)+2(1-\delta_0')\eta}}, \qquad \|w_{\text{ext}}\|^2_{H^\sigma} \le K_1. \end{aligned} \tag{4-25}$$

(iv) Estimates on the scale and the blow-up point:

$$\lambda \le 2s^{-\frac{\ell}{2\ell-\alpha}} \quad \text{and} \quad |z| \le \tfrac{1}{10}. \tag{4-26}$$

Remark 4.5. For a trapped solution one has the above estimates on the parameters from (3-59), (4-12), (4-13), (4-22), (4-23) and (4-24),

$$|b_i^{(n,k)}| \le \frac{C}{s^{\frac{\gamma-\gamma_n}{2}+i}}, \quad b_1^{(0,1)} = \frac{\ell}{2\ell-\alpha}\frac{1}{s} + O(s^{-1-\tilde{\eta}}) \tag{4-27}$$

[14] We make an abuse of notations here. The identities given for the time derivatives of w are only true close to the border of Ω, but which is enough as the required conditions are trace-type conditions; see [Evans 2010].

for C independent of the other constants. The bounds (4-25) on the remainders for the solution described by Proposition 4.6, because of the coercivity estimate Lemma C.3 implies that

$$\|w\|_{H^\sigma(\Omega)} \leq CK_1, \quad \|w\|_{H^{2s_L}(\Omega)} \leq \frac{C(K_1,K_2,M)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}}. \tag{4-28}$$

A trapped solution must first satisfy the condition (4-4) in order to apply the decomposition in Lemma E.1, and then the variables of this decomposition must satisfy suitable bounds. However, these additional bounds in turn provide a much stronger estimate than (4-4). Indeed, one has, from (4-10), (3-29), (3-7), (4-27), (D-2),

$$\begin{aligned}
\inf_{(\tilde\lambda,\tilde z)\in(0,\frac{1}{8M})\times\mathcal{B}^d(\frac18)} & \tilde\lambda^{\frac{2}{p-1}} \|u-Q_{\tilde z,\frac{1}{\tilde\lambda}}\|_{L^\infty(\mathcal{B}^d(1))} \\
&\leq \lambda^{\frac{2}{p-1}}\|u-Q_{z,\frac{1}{\lambda}}\|_{L^\infty(\mathcal{B}^d(1))} \\
&= \|\tilde Q_b+\varepsilon-Q\|_{L^\infty(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\}))} = \|\chi_{B_1}\alpha_b+\varepsilon\|_{L^\infty(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\}))} \\
&\leq \|\chi_{B_1}\alpha_b\|_{L^\infty(\mathbb{R}^d)}+\|\varepsilon\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{s}+\frac{C}{s^{\frac d4-\frac\sigma2}} \ll \kappa,
\end{aligned}$$

$$\|(\tau_{-z})u_\lambda-Q\|_{L^\infty(\mathcal{B}^d(3M))} \leq \|\alpha_b\|_{L^\infty(\mathcal{B}^d(3M))}+\|\varepsilon\|_{L^\infty(\mathcal{B}^d(3M))} \leq \frac{C}{s}+\frac{C}{s^2}. \tag{4-29}$$

Using (4-10), (4-11), (3-29), (3-7), (4-27), (4-3) and (2-7) one gets

$$\begin{aligned}
\langle(\tau_{-z})u_\lambda - Q, H\Phi_M^{(0,1)}\rangle &= \langle \alpha_b, H\Phi_M^{(0,1)}\rangle \\
&= b_1^{(0,1)}\langle T_0^{(0,1)},\chi_M T_0^{(0,1)}\rangle + O(s^{-2}) \sim \frac{c}{s} = \frac{c_1}{s}cM^{d-2\gamma}+O(s^{-2})
\end{aligned}$$

for some $c>0$, which, combined with the above estimate gives

$$\|(\tau_{-z})u_\lambda - Q\|_{L^\infty(\mathcal{B}^d(3M))} \ll \big\langle(\tau_{-z})u_\lambda - Q, H\Phi_M^{(0,1)}\big\rangle$$

for M large enough as $d-2\gamma>0$. Therefore, a solution cannot exit the trapped regime because the condition (4-4) fails: the estimates on the parameters and the remainder have to be violated first. We thus forget about this condition in the following.

The key result of this paper is the existence of solutions that are trapped on their whole lifespan.

Proposition 4.6 (existence of fully trapped solutions). *There exists a choice of universal constants for the analysis*[15]

$$\begin{gathered}
L=L(\ell,d,p)\gg 1, \quad 0<\eta=\eta(d,p,L)\ll 1, \quad M=M(d,p,L)\gg 1, \\
\sigma=\sigma(L,d,p), \quad K_1=K_1(d,p,L)\gg 1, \quad K_2=K_2(d,p,L)\gg 1, \\
0<\varepsilon_i^{(0,1)}=\varepsilon_i^{(0,1)}(L,d)\ll 1 \quad \textit{for } \ell+1\leq i\leq L, \ 0<\varepsilon_1=\varepsilon_1(L,d)\ll 1, \\
0<\varepsilon_i^{(n,k)}=\varepsilon_i^{(n,k)}(L,d)\ll 1 \quad \textit{for } (n,k,i)\in\mathcal{I} \textit{ with } 1\leq n, \ i_n+1\leq i\leq L_n, \\
0<\tilde\eta=\tilde\eta(\ell,L,d,p,\eta)\ll 1 \quad \textit{and} \quad s_0=s_0(\ell,d,p,L,M,K_1,K_2,\varepsilon_i^{(n,k)},\tilde\eta)\gg 1
\end{gathered} \tag{4-30}$$

[15]The interdependence of the constants is written here so that the reader knows, for example, that s_0 is chosen after all the other constants.

such that the following fact holds close to $\chi\tilde{Q}_{\bar{b}(s_0),1/\lambda(s_0)}$, *where* $\bar{b}$ *is given by* (3-59) *and* $\lambda(s_0)$ *satisfies* (4-21). *Given a perturbation along the stable directions, represented by* $w(s_0)$, *decomposed in* (4-5), *satisfying* (4-19) *and* (4-11), *and* $V_1(s_0)$, $(U^{(0,1)}_{\ell+1}(s_0),\dots,U^{(0,1)}_{L}(s_0))$, $(U^{(n,k)}_i(s_0))_{(n,k,i)\in\mathcal{I},\, n\geq 1,\, i_n\leq i}$ *satisfying* (4-16), (4-17) *and* ((iii)), *there exists a correction along the unstable directions represented by* $(V_2(s_0),\dots,V_\ell(s_0))$ *and* $(U^{(n,k)}_i(s_0))_{(n,k,i)\in\mathcal{I},1\leq n,\, i<i_n}$ *satisfying* (4-14) *and* (4-15) *such that the solution* $u(t)$ *of* (1-1) *with initial datum* $u(0)=\chi\tilde{Q}_{b(s_0),1/\lambda(s_0)}+w(s_0)$ *with*

$$b(s_0)=\left(\bar{b}^{(n,k)}_i+\frac{U^{(n,k)}_i(s_0)}{s_0^{\frac{\gamma-\gamma_n}{2}+i}}\right)_{(n,k,i)\in\mathcal{I}} \tag{4-31}$$

is trapped until its maximal time of existence in the sense of Definition 4.4.

Proof. The proof is relegated to Section 5. □

4B. ***End of the proof of Theorem 1.1 using Proposition 4.6.*** In this subsection we end the proof of the main theorem, Theorem 1.1, by proving that the solutions given by Proposition 4.6 lead to a finite-time blow up with the properties described in Theorem 1.1. The proof of Theorem 1.1 is a direct consequence of Proposition 4.6 and Lemmas 4.8, 4.9 and 4.10. Until the end of this subsection, u will denote a solution that is trapped in the sense of Definition 4.4 on its maximal interval of existence. First, we describe the time evolution equation for ε. It then allows us to compute how the time evolution law for the parameters λ and z related to the decomposition (4-5) depends on the other parameters. The bounds on the parameters and the remainder for a trapped solution then imply that λ goes to zero with explicit asymptotic in finite time, that z converges, and that the solution undergoes blow up by concentration with a control on the asymptotic behavior for Sobolev norms.

4B1. *Time evolution for the error.* Let u be a trapped solution. From the decomposition (4-5) we compute that the time evolution of the remainder is

$$w_t=-\frac{1}{\lambda^2}\chi\tau_z(\widetilde{\mathrm{Mod}}(t)_{\frac{1}{\lambda}}+\tilde{\psi}_{b,\frac{1}{\lambda}})+\Delta w+\sum_{k=1}^{p}C^p_k(\chi\tau_z\tilde{Q}_{b,\frac{1}{\lambda}})^{p-k}w^k$$
$$+\Delta\chi\tau_zQ_{\frac{1}{\lambda}}+2\nabla\chi.\nabla\tau_zQ_{\frac{1}{\lambda}}+\chi\tau_zQ^p_{\frac{1}{\lambda}}(\chi^{p-1}-1) \tag{4-32}$$

with the new modulation term being defined as

$$\widetilde{\mathrm{Mod}}(t):=\chi_{B_1}\,\mathrm{Mod}(t)-\left(\frac{\lambda_s}{\lambda}+b^{(0,1)}_1\right)\Lambda\tilde{Q}_b-\left(\frac{z_s}{\lambda}+b^{(1,\cdot)}_1\right).\nabla\tilde{Q}_b. \tag{4-33}$$

From (4-32) and (4-6), as the support of w_{ext} is outside $\mathcal{B}^d(2)$ and as $\tau_z(\tilde{Q}_{b,\lambda})$ is cut in the zone $1\leq|x|\leq 2$, the time evolution of w_{ext} is

$$\partial_t w_{\text{ext}}=\Delta w_{\text{ext}}+\Delta\chi_3 w+2\nabla\chi_3.\nabla w+(1-\chi_3)w^p.$$

The excitation of the solitary wave $\tau_z(\tilde{\alpha}_{b,1/\lambda})$ has support in the zone $|x-z|\leq 2\lambda B_1$ and from (4-26), $|z|+\lambda B_1\ll 1$, so it does not see the cut by χ of the approximate blow-up profile. From this, (4-32) and

(4-6), the time evolution of w_{int} is therefore given by

$$\partial_t w_{\text{int}} + H_{z,\frac{1}{\lambda}} w_{\text{int}} = -\frac{1}{\lambda^2}\chi\tau_z(\widetilde{\text{Mod}(t)}_{\frac{1}{\lambda}} + \tilde{\psi}_{b,\frac{1}{\lambda}}) + L(w_{\text{int}}) + NL(w_{\text{int}}) + \tilde{L} + \widetilde{\text{NL}} + \tilde{R}, \tag{4-34}$$

where $H_{z,1/\lambda}$, $\text{NL}(w_{\text{int}})$, $L(w_{\text{int}})$ are the linearized operator, the nonlinear term and the small linear term resulting from the interaction between w_{int} and a noncut approximate blow-up profile $\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})$:

$$H_{z,\frac{1}{\lambda}} := -\Delta - p(\tau_z(\tilde{Q}_{\frac{1}{\lambda}}))^{p-1}, \quad H_{b,z,\frac{1}{\lambda}} := -\Delta - p(\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}}))^{p-1} \tag{4-35}$$

$$\text{NL}(w_{\text{int}}) := F(\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}}) + w_{\text{int}}) - F(\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})) + H_{b,\frac{1}{\lambda}}(w_{\text{int}}), \tag{4-36}$$

$$L(w_{\text{int}}) := H_{z,\frac{1}{\lambda}} w_{\text{int}} - H_{b,z,\frac{1}{\lambda}} w_{\text{int}} = \frac{p}{\lambda^2}\tau_z(\chi_{B_1}^{p-1}\alpha_b^{p-1})_{\frac{1}{\lambda}}. \tag{4-37}$$

The last terms in (4-34) are the corrective terms induced by the cut of the approximate blow-up profile and the cut of the error term:[16]

$$\tilde{L} := -\Delta\chi_3 w - 2\nabla\chi_3.\nabla w + p\tau_z Q_{\frac{1}{\lambda}}^{p-1}(\chi^{p-1} - \chi_3)w, \tag{4-38}$$

$$\widetilde{\text{NL}} := \sum_{k=2}^{p} C_k^p \tau_z Q_{\frac{1}{\lambda}}^{p-k}(\chi^{p-k} - \chi_3^{k-1})\chi_3 w^k, \tag{4-39}$$

$$\tilde{R} := \Delta\chi\tau_z Q_{\frac{1}{\lambda}} + 2\nabla\chi\nabla\tau_z Q_{\frac{1}{\lambda}} + \chi\tau_z Q_{\frac{1}{\lambda}}^p(\chi^{p-1} - 1), \tag{4-40}$$

and one notices that their support is in the zone $1 \le |x| \le 6$. Using the definition of the renormalized flow (3-2) and the decomposition (4-5) we compute, using (4-32),

$$\partial_s\varepsilon - \frac{\lambda_s}{\lambda}\Lambda\varepsilon - \frac{z_s}{\lambda}.\nabla\varepsilon + H\varepsilon = -\chi(\lambda y + z)(\tilde{\text{Mod}}(s) + \tilde{\psi}_b) + \text{NL}(\varepsilon) + L(\varepsilon) + \lambda^2[\tau_{-z}(\tilde{L} + \tilde{R} + \widetilde{\text{NL}})]_\lambda, \tag{4-41}$$

with the purely nonlinear term and the small linear term in adapted renormalized variables being defined as

$$\text{NL}(\varepsilon) := F(\tilde{Q}_b + \varepsilon) - F(\tilde{Q}_b) + H_b(\varepsilon), \quad L(\varepsilon) := H\varepsilon - H_b\varepsilon, \tag{4-42}$$

where $H_b := -\Delta - p\tilde{Q}_b^{p-1}$ is the linearized operator near $\tilde{Q}_b$. One notices that the extra terms induced by the cut, $\lambda^2[\tau_{-z}(\tilde{L} + \tilde{R} + \widetilde{\text{NL}})]_\lambda$, have support in the zone $\frac{1}{2\lambda} \le |y| \le \frac{7}{\lambda}$ (by(4-26)).

4B2. *Modulation equations.* We now quantify how the evolution of one parameter $b_i^{(n,k)}$, λ or z depends on all the parameters $(b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ and the remainder ε.

Lemma 4.7 (modulation). *Let all the constants of the analysis described in Proposition 4.6 be fixed except s_0. Then for s_0 large enough, for any solution u that is trapped on $[s_0, s')$ in the sense of Definition 4.4 the following holds for $s_0 \le s < s'$:*

$$\left|\frac{\lambda_s}{\lambda} + b_1^{(0,1)}\right| + \left|\frac{z_s}{\lambda} + b_1^{(1,\cdot)}\right| + \sum_{(n,k,i)\in\mathcal{I}, i\neq L_n} \left|b_{i,s}^{(n,k)} + (2i - \alpha_n)b_1^{(0,1)}b_i^{(n,k)} + b_{i+1}^{(n,k)}\right| \le \frac{C(L,M)}{s^{L+3}} + \frac{C(L,M)}{s}\sqrt{\mathcal{E}_{2s_L}}, \tag{4-43}$$

[16] Again, the excitation of the solitary wave $\tau_z(\tilde{\alpha}_{b,1/\lambda})$ is not present here as its support is in the zone $|x| \ll 1$; see (4-26).

$$\sum_{(n,k,i)\in\mathcal{I},\, i=L_n} \left|b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}\right| \le \frac{C(M,L)}{s^{L+3}}+C(M,L)\sqrt{\mathcal{E}_{2s_L}}. \tag{4-44}$$

Proof. We let

$$D(s)=\left|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right|+\left|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right|+\sum_{(n,k,i)\in\mathcal{I}}\left|b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}\right| \tag{4-45}$$

with the convention $b_{L_n+1}^{(n,k)}=0$. Taking the scalar product of (4-41) with $(-H)^i\,\Phi_M^{(n,k)}$, using (4-3), gives[17]

$$\begin{aligned}\langle\widetilde{\mathrm{Mod}}(s),(-H)^i\,\Phi_M^{(n,k)}\rangle &= \langle -H\varepsilon,(-H)^i\,\Phi_M^{(n,k)}\rangle-\langle\tilde{\psi}_b,(-H)^i\,\Phi_M^{(n,k)}\rangle\\ &\quad+\left\langle\frac{\lambda_s}{\lambda}\Lambda\varepsilon+\frac{z_s}{\lambda}.\nabla\varepsilon+\mathrm{NL}(\varepsilon)+L(\varepsilon),(-H)^i\,\Phi_M^{(n,k)}\right\rangle.\end{aligned} \tag{4-46}$$

Now we look closely at each one of the terms of this identity.

The modulation term. From the expression (3-29) of $\tilde{Q}_b$, the bound (3-11) on $\partial S_j/\partial b_i^{(n,k)}$, and the bounds (4-27) on the parameters, one has

$$\tilde{Q}_b=Q+\chi_{B_1}\alpha_b=Q+O(s^{-1})\qquad\text{and}\qquad\frac{\partial S_j}{\partial b_i^{(n,k)}}=O(s^{-1})\quad\text{on } \mathcal{B}^d(0,2M).$$

From (3-10), (4-33) and (4-45), the modulation term can then be rewritten as

$$\begin{aligned}\mathrm{Mod}(s)&=\chi_{B_1}\sum_{(n,k,i)\in\mathcal{I}}\left[b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}\right]\left[T_i^{(n,k)}+\sum_{j=i+1+\delta_{n\ge2}}^{L+2}\frac{\partial S_j}{\partial b_i^{(n,k)}}\right]\\ &\qquad-\left(\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right)\Lambda\tilde{Q}_b-\left(\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right).\nabla\tilde{Q}_b\\ &=\chi_{B_1}\sum_{(n,k,i)\in\mathcal{I}}\left[b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}\right]T_i^{(n,k)}\\ &\qquad-\left(\frac{\lambda_s}{a}+b_1^{(0,1)}\right)\Lambda Q-\left(\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right).\nabla Q+O\left(\frac{|D(s)|}{s}\right),\end{aligned}$$

where the $O(|D(s)|/s)$ is valid in the zone $|y|\le 2M$. From the orthogonality relations (4-3), we then get

$$\begin{aligned}&\langle\widetilde{\mathrm{Mod}}(s),(-H)^i\,\Phi_M^{(n,k)}\rangle+O\left(\frac{|D(s)|}{s}\right)\\ &=\begin{cases}-C\langle\chi_M\Lambda Q,\Lambda Q\rangle\left(\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right) & \text{for } (n,k,i)=(0,1,0),\\ -C'\langle\chi_M\nabla Q,\nabla Q\rangle\left(\frac{z_{j,s}}{\lambda}+b_1^{(1,k)}\right) & \text{for } (n,i)=(1,0),\ 1\le k\le d,\\ \langle\chi_M T_0^{(n,k)},T_0^{(n,k)}\rangle\left(b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}\right) & \text{otherwise,}\end{cases}\end{aligned} \tag{4-47}$$

where C and C' are two positive renormalization constants.

[17] We do not see the extra terms $\tilde{L}$, $\tilde{R}$ and $\widetilde{\mathrm{NL}}$ because their support is in the zone $\frac{1}{2\lambda}\le|y|$ (from (4-26)) which is very far away from the support of $\Phi_M^{(n,k)}$, in the zone $|y|\le 2M$ (s_0 being chosen large enough so that this statement holds).

The main linear term. The coercivity estimate (C-16) and the Hölder inequality imply

$$\int_{|y|\leq 2M} |\varepsilon|\, dy \lesssim C(M)\sqrt{\mathcal{E}_{2s_L}}.$$

Hence, from the orthogonality (4-11) for ε, we obtain, for $0\leq n\leq n_0$, $1\leq k\leq k(n)$,

$$\left|\langle H\varepsilon, H^i\Phi_M^{(n,k)}\rangle\right| = \begin{cases} 0 & \text{for } i<L_n,\\ \left|\langle \varepsilon,(-H)^{i+1}\Phi_M^{(n,k)}\rangle\right| = O(\sqrt{\mathcal{E}_{2s_L}}) & \text{for } i=L_n.\end{cases} \tag{4-48}$$

The error term. Using the local bound (3-35) for $\tilde{\psi}_b$ and (4-27),

$$\left|\langle \tilde{\psi}_b, H^i\Phi_M^{(n,k)}\rangle\right| \leq \frac{C(L,M)}{s^{L+3}}. \tag{4-49}$$

The extra terms. From (4-27), the coercivity estimate (C-16), the bound (4-25) on $\mathcal{E}_{2s_L}$ and (4-45), one obtains

$$\left|\left\langle \frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon, H^i\Phi_M^{(n,k)}\right\rangle\right| \leq \frac{C(L,M)}{s}\sqrt{\mathcal{E}_{2s_L}} + \frac{|D(s)|}{s^{L+1-\delta_0+\eta(1-\delta_0')}}.$$

Now, as $Q^{p-1}-\tilde{Q}_b^{p-1}=O(s^{-1})$ on the set $|y|\leq 2M$ from (3-7) and (4-27), using the estimate (D-2) on $\|\varepsilon\|_{L^\infty}$, from the definition (4-42) of $NL(\varepsilon)$ and $L(\varepsilon)$ and the coercivity (C-16), one gets, for s_0 large enough,

$$\left|\langle \mathrm{NL}(\varepsilon)+L(\varepsilon), H^i\Phi_M^{(n,k)}\rangle\right| \leq C(L,M)\mathcal{E}_{2s_L} + C(L,M)\frac{\sqrt{\mathcal{E}_{2s_L}}}{s} \leq C(L,M)\frac{\sqrt{\mathcal{E}_{2s_L}}}{s}.$$

Putting together the last two estimates yields

$$\left|\left\langle \frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon + \mathrm{NL}(\varepsilon)+L(\varepsilon), H^i\Phi_M^{(n,k)}\right\rangle\right| \leq \frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s} + \frac{C(L,M)|D(s)|}{s^{L+1-\delta_0+\eta(1-\delta_0')}}. \tag{4-50}$$

Final bound on $|D(s)|$. Summing the previous estimates we performed on each term of (4-46) in (4-47)–(4-50) yields

$$|D(s)| \leq C(L,M)\sqrt{\mathcal{E}_{s_L}} + \frac{C(L,M)}{s^{L+3}}.$$

We now come back to (4-46), combine again (4-47) with the above bound on $|D|$, (4-48), (4-49) and (4-50), yielding the desired bounds (4-43) and (4-44) of the lemma. □

4B3. *Finite-time blow up*. We now reintegrate in time the time evolution of λ and z we found in Lemma 4.7 to obtain their behavior and show the blow up.

Lemma 4.8 (concentration and asymptotic of the blow-up point). *Let u be a solution that is trapped on its maximal interval of existence. Then it blows up in finite time $T>0$ with $s(t)\to+\infty$ as $t\to T$ and we have the following*:

(1) Concentration speed. *We have* $\lambda \underset{t\to T}{\sim} C(u(0))(T-t)^{\frac{\ell}{\alpha}}$, *with* $C(u(0))>0$.

(2) Behavior of the blow-up point. *There exists z_0 such that* $\lim_{t\to T} z(t)=z_0$ *and for all times* $s\geq s_0$,

$$|z(s)| = O(s_0^{-\tilde{\eta}}). \tag{4-51}$$

Proof. From the Cauchy theory in L^∞, (3-1) and (4-26), if $T \in (0, +\infty]$ denotes the maximal time of existence of u, one necessarily has $\lim_{s\to T} s(t) = +\infty$. From the estimate (4-27) on $b_1^{(0,1)}$, the modulation (4-43) and (4-25), one has

$$\frac{\lambda_s}{\lambda} = -\frac{c_1}{s} + O(s^{-1-\tilde{\eta}}).$$

We reintegrate using (4-21) (we recall that $c_1 = \ell/(2\ell - \alpha)$ from (3-59)):

$$\lambda = \frac{(1 + O(s_0^{-\tilde{\eta}}))}{s^{\frac{\ell}{2\ell-\alpha}}}, \tag{4-52}$$

which is valid as long as the solution u is trapped. In addition, if the solution is trapped on its maximal interval of existence, then the function represented by $O(\cdot)$ admits a limit as $s \to +\infty$. In turn, from $\frac{ds}{dt} = \frac{1}{\lambda^2}$ we obtain

$$s = s_0 \left(1 - \frac{\alpha s_0^{\frac{\alpha}{2\ell-\alpha}}}{2\ell - \alpha} \int_0^t (1 + O(s_0^{-\tilde{\eta}}))\, dt'\right)^{-\frac{2\ell-\alpha}{\alpha}}.$$

Hence there exists $T > 0$ with

$$s \underset{t\to T}{\sim} C(u(0))(T - t)^{-\frac{2\ell-\alpha}{\alpha}}. \tag{4-53}$$

Injecting this identity in (4-52) then gives $\lambda \sim C(u(0))(T - t)^{\frac{\ell}{\alpha}}$ as $t \to T$. Now we turn to the asymptotic behavior of the point of concentration z. From (4-43), using $b_1^{(1,i)} = O(s^{-\frac{\alpha+1}{2}})$ from (4-23) for $1 \le i \le d$, one gets

$$|z_{i,s}| = O(s^{-c_1-\frac{\alpha+1}{2}}) = O(s^{-1-\frac{\alpha}{2}(1+\frac{1}{2\ell-\alpha})}). \tag{4-54}$$

As $\alpha > 0$, this implies the convergence and the estimate of z claimed in the lemma. □

4B4. *Behavior of Sobolev norms near blow-up time.* From Lemma 4.8, the L^∞ bound on the error (D-2) and the bounds on the parameters (4-27), any solution that is trapped on its maximal interval of existence indeed blows up at the time T given by Lemma 4.8 because $\lim_{t\to T} \|u\|_{L^\infty} = +\infty$. The behavior of the Sobolev norms is the following.

Lemma 4.9 (asymptotic behavior for subcritical norms). *Let u be a solution that is trapped for all times $s \ge s_0$ and T be its finite maximal lifespan.*[18] *Then*

(i) Behavior of subcritical norms.

$$\limsup_{t\to T} \|u\|_{H^m(\Omega)} < +\infty \quad \textit{for } 0 \le m < s_c.$$

(ii) Behavior of the critical norm.

$$\|u\|_{H^{s_c}(\Omega)} \underset{t\to T}{=} C(d, p)\sqrt{\ell}\sqrt{|\log(T - t)|}(1 + o(1)).$$

[18] T is finite by Lemma 4.8.

(iii) Boundedness of the perturbation in slightly supercritical norms.

$$\limsup_{t\to T} \|u-\chi\tau_z(Q_{\frac{1}{\lambda}})\|_{H^m(\Omega)} < +\infty \quad \textit{for } s_c < m \le \sigma. \tag{4-55}$$

Proof. The trapped solution u can be written as

$$u = \chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}}) + w = \chi\tau_z(Q_{\frac{1}{\lambda}}) + \tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}}) + w.$$

We first look at the second term $\tau_z(\tilde{\alpha}_{b,1/\lambda})$, being the excitation of the ground state. It has compact support in the zone $|x| \le 2B_1\lambda$. From (1-38) and (4-52), one gets $2B_1\lambda \ll 1$ as $s_0 \gg 1$, so that $\tau_z(\tilde{\alpha}_{b,1/\lambda})$ has compact support inside $\mathcal{B}^d(1)$. This implies that

$$\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{H^\sigma(\Omega)} \le C\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{\dot{H}^\sigma(\mathbb{R}^d)},$$

the latter norm being easier to compute. Indeed by renormalizing one has

$$\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{\dot{H}^\sigma(\mathbb{R}^d)} = \frac{1}{\lambda^{\sigma-s_c}}\|\tilde{\alpha}_b\|_{\dot{H}^\sigma(\mathbb{R}^d)}.$$

As

$$\tilde{\alpha}_b = \chi_{B_1}\left(\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)} + \sum_{i=2}^{L+2} S_i\right)$$

from (3-29) and (3-7), the bounds (4-27) on the parameters $b_i^{(n,k)}$, together with the asymptotic at infinity of the profiles $T_i^{(n,k)}$ and S_i described in Lemma 2.10 and Proposition 3.3 imply that $\|\tilde{\alpha}_b\|_{\dot{H}^\sigma} \le C/s$. Hence

$$\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{H^\sigma} \le \frac{C}{s^{1-\frac{\ell(\sigma-s_c)}{2\ell-\alpha}}} \to 0$$

as $t\to T$ as $\sigma - s_c \ll 1$.

Now, following the second paragraph of Remark 4.5, we get that $\|w\|_{H^\sigma} \le CK_1$ is uniformly bounded until the blow-up time. Combined with what was just said about the boundedness of $\tau_z(\tilde{\alpha}_{b,1/\lambda})$, we get that (iii) holds for all $0 \le m \le \sigma$. This, together with the asymptotic of the ground state (2-1) then gives (i) and (ii). □

4B5. *The blow-up set.* We recall that $x\in\Omega$ is a blow-up point of u if there exists $(t_n,x_n)\to(T,x)$ such that $|u(t_n,x_n)|\to+\infty$. For trapped solutions one has the following result.

Lemma 4.10 (description of the blow-up set). *Let u be a solution that is trapped for all times $s\ge s_0$ and T be its finite maximal lifespan.*[19] *Then z_0 given by Lemma 4.8 is a blow-up point of u, and it is the only one.*

Proof. From the L^∞ bound (4-29) and the fact that $\lim_{t\to T} s(t) = +\infty$ from Lemma 4.8, $u(s,z(s)) \sim \lambda(s)^{-\frac{2}{p-1}}Q(0)$ as $s\to+\infty$. From Lemma 4.8, this implies that $u(t,z(t))\to+\infty$ as $t\to T$ and that $z_0 = \lim_{t\to T} z(t)$ is indeed a blow-up point.

[19] T is finite by Lemma 4.8.

Now take another point $x \in \Omega$, $x \neq z_0$. From (4-55), the asymptotic of Q (Lemma 2.1), and Lemma 4.8, there exists $R > 0$ such that

$$\sup_{0 \le t < T} \|u(t)\|_{H^\sigma(\mathcal{B}^d(x,R))} < +\infty.$$

This local boundedness, by Sobolev embedding and Hölder, implies that

$$\sup_{0 \le t < T} \|u(t)\|_{W^{1,q}(\mathcal{B}^d(x,R))} < +\infty, \qquad q = \frac{2d}{d+2-2\sigma} > \frac{2d}{d+2-2s_c} = d\,\frac{p-1}{p+1}.$$

The above inequality, after applying Lemma 4.11 several times and using Sobolev embedding, implies that there exists $r > 0$ such that

$$\sup_{0 \le t < T} \|u(t)\|_{L^\infty(\mathcal{B}^d(x,r))} < +\infty.$$

Therefore, x is not a blow-up point of u. □

In the proof of the previous lemma, we used the following result.

Lemma 4.11 (parabolic bootstrap). *Let $R > 0$ and $x \in \Omega$ such that $B(x,R) \subset \Omega$. Let $q_0 > \frac{p-1}{p+1}d$. There exists $\kappa(q_0) > 0$ such that for any $q > q_0$, if $u \in C\big([0,T), W^{1,\infty}(\Omega)\big)$ is a solution of (1-1) satisfying*

$$\sup_{0 \le t < T} \|u(t)\|_{W^{1,q}(\mathcal{B}^d(x,R))} < +\infty \tag{4-56}$$

then

$$\sup_{0 \le t < T} \|u(t)\|_{W^{1,q(1+\kappa)}(\mathcal{B}^d(x,\frac{R}{2}))} < +\infty. \tag{4-57}$$

Proof. The proof relies on a classical use of estimates for the heat kernel. Without loss of generality we assume $q_0 < d$. If u solves (1-1) and satisfies (4-56) then the localisation $v = \chi_{R/2} u$ solves

$$v_t = \Delta v - 2\nabla.\chi_{\frac{R}{2}}.\nabla u - \Delta \chi_{\frac{R}{2}} u + \chi_{\frac{R}{2}} |u|^{p-1} u$$

and using the Duhamel formula can then be written as

$$v(t) = K_t * v(0) + \int_0^t K_{t-s} * \big[-2\nabla.\chi_{\frac{R}{2}}.\nabla u - \Delta\chi_{\frac{R}{2}} u + \chi_{\frac{R}{2}} |u|^{p-1} u\big]\, ds,$$

where the heat kernel is $K_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$. One then has the formula

$$\nabla v(t) = \nabla K_t * v(0) + \int_0^t \nabla K_{t-s} * [-2\nabla.\chi_{\frac{R}{2}}.\nabla u - \Delta\chi_{\frac{R}{2}} u]\, ds + \int_0^t K_{t-s} * \big[\nabla\chi_{\frac{R}{2}} |u|^{p-1} u + \chi_{\frac{R}{2}} \nabla u |u|^{p-1}\big]\, ds. \tag{4-58}$$

We estimate the last term using the Hölder, Sobolev and Young inequalities:[20]

$$\begin{aligned}\left\|\int_0^t K_{t-s}*[\chi_{\frac{R}{2}}\nabla u|u|^{p-1}]\,ds\right\|_{L^{q(1+\kappa)}} &\le \int_0^t \left\|K_{t-s}*[\chi_{\frac{R}{2}}\nabla u|u|^{p-1}]\right\|_{L^{q(1+\kappa)}}ds\\ &\lesssim \int_0^t \|K_{t-s}\|_{L^{\left(1+\frac{1}{q(1+\kappa)}-(\omega-\frac{1}{q})\right)^{-1}}}\left\|\nabla u|u|^{p-1}\right\|_{L^{(\omega+\frac{1}{q})^{-1}}}ds\\ &\lesssim \int_0^t \|K_{t-s}\|_{L^{\left(1-\omega-\frac{\kappa}{q(1+\kappa)}\right)^{-1}}}\|\nabla u\|_{L^q}\left\||u|^{p-1}\right\|_{L^{\omega^{-1}}}ds\\ &\lesssim \int_0^t \frac{1}{(t-s)^{\theta(\kappa,q)}}\|\nabla u\|_{L^q}\|\nabla u\|_{L^{q_0}}^{p-1}\,ds \lesssim \int_0^T \frac{ds}{(t-s)^{\theta(\kappa,q)}},\end{aligned}$$

where

$$\omega=\frac{(d-q_0)(p-1)}{dq_0}\quad\text{and}\quad \theta(\kappa,q)=\frac{(d-q_0)(p-1)}{2q_0}+\frac{\kappa d}{2q(1+\kappa)}$$

(note $\theta\ge 0$ as $q_0<d$). For $\kappa\ge 0$ and $\frac{p-1}{p+1}d\le q\le d$, if κ is fixed, θ is strictly decreasing with respect to q, and if q is fixed, θ is strictly increasing with respect to κ. As $\theta(0,q_0)<1$ since $q_0>\frac{p-1}{p+1}d$, this implies that there exists $\kappa(q_0)>0$ such that for all $q_0\le q\le d$, and $0<\kappa\le\kappa(q_0)$, we have $\theta(\kappa,q)<1$. The above inequality then implies that in that range,

$$\left\|\int_0^t K_{t-s}*[\chi_{\frac{R}{2}}\nabla u|u|^{p-1}]\,ds\right\|_{L^{q(1+\kappa)}}<+\infty.$$

We claim that this term was the "worst" to be estimated in (4-58) and that using the very same techniques, one can estimate similarly all the other terms on the right-hand side in the same range $0<\kappa\le\kappa(q_0)$ leading to

$$\sup_{0\le t<T}\|\nabla v(t)\|_{L^{(1+\kappa)q}}<+\infty,$$

which implies that $\sup_{0\le t<T}\|v(t)\|_{W^{1,(1+\kappa)q}}<+\infty$ by Sobolev embedding and the Hölder inequality. This concludes the proof, as $v=u$ on $B\big(x,\frac{R}{2}\big)$. □

5. Proof of Proposition 4.6

This section is devoted to the proof of this latter proposition, which will then end the proof of the main theorem. For all trapped solutions u in the sense of Definition 4.4, we let $s^*=s^*(u(0))$ be the exit time from the trapped regime:

$$s^*=\sup\big\{s\ge s_0 \text{ such that (4-22), (4-23), (4-24), (4-25) and (4-26) hold on } [s_0,s)\big\}. \tag{5-1}$$

If $s^*<+\infty$, after s^*, one of the bounds (4-22), (4-23), (4-24), (4-25) or (4-26) must then be violated. The result of the first part of this section is a refinement of this exit condition. In Lemma 5.1 and Propositions 5.3, 5.5, 5.6 and 5.8 we quantify accurately the time evolution of the parameters and the remainder in the trapped regime. Combined with the modulation equations of Lemma 4.7, this allows us to show that in

[20] As $q\ge q_0>\frac{p-1}{p+1}d$, $p>\frac{d+2}{d-2}$, and $d\ge 11$ all the computations below are rigorous.

the trapped regime, all the components of the solution along the stable directions of perturbation are under control; see Lemma 5.9. Moreover, from (4-52), (4-26) is always fulfilled as long as the other bounds hold. As a consequence, the exit time of the trapped regime is in fact characterized by the following condition: just after s^*, one of the bounds in (4-22) and (4-23) regarding the unstable parameters is violated.

We prove Proposition 4.6 by contradiction. Suppose that given a stable perturbation of $\chi\tilde{Q}_{\bar{b}(s_0),1/\lambda(s_0)}$ as described in Proposition 4.6, the solution starting from $\chi\tilde{Q}_{b(s_0),1/\lambda(s_0)}+w(s_0)$ leaves the trapped regime in finite time for all initial corrections $(V_2(s_0),\dots,V_\ell(s_0))$ and $(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\le n,\ i<i_n}$ along the unstable directions. This means from the previous paragraph that the trajectory of

$$\big(V_2(s),\dots,V_\ell(s),(U_i^{(n,k)}(s))_{(n,k,i)\in\mathcal{I},1\le n,\ i<i_n}\big)$$

leaves the set[21] $\mathcal{B}_\infty^{\ell-1}(s^{-\tilde{\eta}})\times\mathcal{B}_\infty^K(1)$ in finite time. But at the leading order, the dynamics of this trajectory are linear repulsive. In Lemma 5.10 we show how the fact that all the trajectories leave this ball is a contradiction to Brouwer's fixed point theorem.

5A. ***Improved modulation for the last parameters*** $b_{L_n}^{(n,k)}$**.** In Lemma 4.7, the modulation estimates (4-43) for the first parameters are better than the ones for the last parameters $b_{L_n}^{(n,k)}$, (4-44). When looking at the proof of Lemma 4.7, we see that this is a consequence of the fact that the projection of the linearized dynamics onto the profile generating the orthogonality conditions, $\langle H\varepsilon, H^i\Phi_M^{(n,k)}\rangle$, cancels only for $i<L_n$. However, as we explained in the introduction of Lemma 4.2, $H^i\Phi_M^{(n,k)}$ has to be thought as an approximation of $T_i^{(n,k)}$, and in that case the previous term would cancel also for $i=L_n$. It is therefore natural to look for a better modulation estimate for $b_{L_n}^{(n,k)}$. In the next lemma we find a better bound by, roughly speaking, integrating by parts in time the projection of ε onto $T_{L_n}^{(n,k)}$ in the self-similar zone.

Lemma 5.1 (improved modulation equation for $b_{L_n}^{(n,k)}$). *Suppose all the constants in Proposition 4.6 are fixed except s_0. Then for s_0 large enough, for any solution that is trapped on $[s_0,s')$, for $0\le n\le n_0$, $1\le k\le k(n)$, the following holds for $s\in[s_0,s')$:*

$$\left|b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}-\frac{d}{ds}\left[\frac{\langle H^{L_n}(\varepsilon-\sum_{i=2}^{L+2}S_i),\chi_{B_0}T_0^{(n,k)}\rangle}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{n,k}\rangle}\right]\right| \le\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{\delta_n}}+\frac{C(L,M)}{s^{L+\frac{g'}{2}+\delta_n-\delta_0+1}}. \quad (5\text{-}2)$$

Remark 5.2. From (5-19), we see that the denominator is not zero. From (5-19) and (5-20), one has the following bound for the new quantity that appeared when comparing this new modulation estimate to the former one (4-44):

$$\left|\frac{\langle H^{L_n}(\varepsilon-\sum_{i=2}^{L+2}S_i),\chi_{B_0}T_0^{(n,k)}\rangle}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{n,k}\rangle}\right|\le C(L,M)s^{-L-\frac{g'}{2}+\delta_0-\delta_n}+C(L,M,K_2)s^{-L+\delta_0-\delta_n+\eta(1-\delta_0')}. \quad (5\text{-}3)$$

[21]Here K is the number of directions of instabilities on the spherical harmonics of degree greater than 0, that is, $K=d(E[i_1]-\delta_{i_1\in\mathbb{N}})+\sum_{2\le n\le n_0}k(n)(E[i_n]+1-\delta_{i_n\in\mathbb{N}})$, and $\mathcal{B}_\infty^a(r)$ is the ball of radius r of $\mathbb{R}^a$ for the usual $|\cdot|_\infty$ norm.

This is a better bound compared to the required bound (4-24) on $b_{L_n}^{(n,k)}$ in the trapped regime, that is, $|b_{L_n}^{(n,k)}|\leq Cs^{-\frac{\gamma-\gamma_n}{2}-L_n}=Cs^{-L-\delta_n+\delta_0}$.

Proof of Lemma 5.1. First, from the fact that $HT_0^{(n,k)}=0$, the asymptotic (2-7) of $T_0^{(n,k)}$ and (4-27), we obtain

$$\operatorname{supp}[H^{L_n}(\chi_{B_0}T_0^{(n,k)})]\subset\{B_0\leq|y|\leq 2B_0\}\quad\text{and}\quad|H^{L_n}(\chi_{B_0}T_0^{(n,k)})|\leq\frac{C(L)}{s^{\frac{\gamma_n}{2}+L_n}}. \tag{5-4}$$

Step 1: computation of a first identity. We will now prove the identity

$$\begin{aligned}\frac{d}{ds}\big(\langle H^{L_n}\varepsilon,\chi_{B_0}T_0^{(n,k)}\rangle\big)&=\big(b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}\big)\langle T_0^{(n,k)},\chi_{B_0}T_0^{(n,k)}\rangle\\&\quad+\frac{d}{ds}\bigg(\sum_{j=2}^{L+2}\langle S_j,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\bigg)\\&\quad+O(\sqrt{\mathcal{E}_{2s_L}}B_0^{4m_n+2\delta_n})+O\bigg(\frac{C(L)}{s^{L+1+\frac{g'}{2}-\delta_0-\delta_n-2m_n}}\bigg).\end{aligned} \tag{5-5}$$

From the evolution equation (4-41) and the fact that H is self-adjoint we obtain

$$\begin{aligned}\frac{d}{ds}\big(\langle H^{L_n}\varepsilon,\chi_{B_0}T_0^{(n,k)}\rangle\big)&=\langle\varepsilon,H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})\rangle\\&\quad+\Big\langle-\widetilde{\mathrm{Mod}}(s)-\tilde\psi_b+\frac{\lambda_s}{\lambda}\Lambda\varepsilon+\frac{z_s}{\lambda}.\nabla\varepsilon-H\varepsilon+\mathrm{NL}(\varepsilon)+L(\varepsilon),H^{L_n}(\chi_{B_0}T_0^{(n,k)})\Big\rangle.\end{aligned} \tag{5-6}$$

The terms created by the cut of the solitary wave $\lambda^2\tau_{-z}[(\tilde L+\tilde R+\widetilde{\mathrm{NL}})_\lambda]$ do not appear because they have their support in the zone $\frac{1}{2\lambda}\leq|y|$, which is far away from the zone $|y|\leq 2B_0$ as $B_0\ll\frac{1}{\lambda}$ in the trapped regime by (4-52). We now look at all the terms in the above equation.

The $\partial_s(\chi_{B_0})$ term. From the modulation equation (4-43) and the bound (4-25), one has $|b_{1,s}^{(0,1)}|\leq Cs^{-2}$. Hence, using the asymptotic (2-7) of $T_0^{(n,k)}$ and the fact that $HT_0^{(n,k)}=0$ and (4-27), we get that $H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})$ has support in $B_0\leq|y|\leq 2B_0$ and satisfies the bound

$$|H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})|\leq\frac{C(L)}{s^{\frac{\gamma_n}{2}+L_n+1}}.$$

Using the coercivity estimate (C-16), we obtain

$$\big|\langle\varepsilon,H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})\rangle\big|\leq C(L)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n}. \tag{5-7}$$

The error term. For $|y|\leq 2B_0$, one has $\tilde\psi_b=\psi_b$ by (3-34). As ψ_b is a finite sum of homogeneous profiles of degree $(i,-\gamma-2-g')$ for some $i\in\mathbb{N}$ (which was proved in Step 4 of the proof of Proposition 3.1), the bounds on the parameters (4-27) imply that $|\psi_b(y)|\leq C(L)s^{-\frac{\gamma+2+g}{2}}$ for $B_0\leq|y|\leq 2B_0$. Combined with (5-4), this yields

$$\big|\langle\tilde\psi_b,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\big|\leq C(L)B_0^{d-\gamma_n-2L_n-\gamma-g'-2}\leq\frac{C(L)}{s^{L+1+\frac{g'}{2}-\delta_0-\delta_n-2m_n}}. \tag{5-8}$$

The remainder's contribution. Using (5-4), the bounds $|\frac{\lambda_s}{\lambda}| \le Cs^{-1}$ and $|\frac{z_s}{\lambda}| \le Cs^{-\frac{\alpha+1}{2}}$ (which are consequences of the modulation estimate (4-43) and (4-25)) and the coercivity estimate Lemma C.3, one gets

$$\left|\left\langle \frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon - H\varepsilon, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right| \le C(L)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n}. \tag{5-9}$$

The small linear term can be written as $L(\varepsilon) = (pQ^{p-1} - p\tilde{Q}_b^{p-1})$; hence from the form of $\tilde{Q}_b$, see (3-29), one has $|(pQ^{p-1} - p\tilde{Q}_b^{p-1})| \le C(L)s^{-1-\frac{\alpha}{2}}$. Its contribution is then of smaller order using (5-4):

$$\left|\langle L(\varepsilon), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right| \le C(L)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n-\frac{\alpha}{2}}. \tag{5-10}$$

The nonlinear term can be written as $\mathrm{NL}(\varepsilon) = \sum_{k=2}^{p} C_k^p \varepsilon^k \tilde{Q}_b^{p-k}$. From the coercivity estimate Lemma C.3, we get

$$\int_{B_0\le|y|\le 2B_0} \frac{\varepsilon^2}{|y|^{\gamma_n+2L_n}}\,dy \le C(L,M)\mathcal{E}_{2s_L}s^{2s_L-\frac{\gamma_n}{2}-L_n}.$$

Using the bootstrap bounds (4-25) and (4-27), one computes

$$\sqrt{\mathcal{E}_{2s_L}}s^{2s_L-\frac{\gamma_n}{2}-L_n} \le K_2 s^{\delta_n+2m_n-(\frac{\gamma-2}{4}+\frac{\eta(1-\delta_0')}{2})} \le B_0^{\delta_n+2m_n}$$

for s_0 large enough (because $\gamma > 2$). For $2 \le k \le p$, we know $|\varepsilon^{k-2}\tilde{Q}_b^{p-k}| \le C$ is bounded by (D-2), so using the two previous equations and (5-4), one gets

$$\left|\langle \mathrm{NL}(\varepsilon), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right| \le \sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n} \tag{5-11}$$

for s_0 large enough. Combining (5-9), (5-10) and (5-11), we have the following upper bound for the remainder's contribution:

$$\left|\left\langle \frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon - H\varepsilon + \mathrm{NL}(\varepsilon) + L(\varepsilon), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right| \le C(L,M)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n}. \tag{5-12}$$

The modulation term. For $(n',k',i) \in \mathcal{I}$, one has

$$\langle T_i^{(n,k)}, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle = \langle H^{L_n}T_i^{(n,k)}, \chi_{B_0}T_0^{(n,k)}\rangle = 0$$

if $(n',k',i) \ne (n,k,L_n)$. Indeed, if $(n',k') \ne (n,k)$ then the two functions are located on different spherical harmonics and their scalar product is 0. If $i \ne L_n$ then $i < L_n$ and $H^{L_n}T_i^{(n,k)} = 0$. This implies the identity from (4-33) since $B_1 \gg B_0$:

$$\begin{aligned}
&\langle \widetilde{\mathrm{Mod}}(s), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle \\
&\quad = \big(b_{L_n,s}^{(n,k)} + (2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}\big)\langle T_0^{(n,k)}, \chi_{B_0}T_0^{(n,k)}\rangle \\
&\qquad + \sum_{j=2}^{L+2}\sum_{(n',k',i)\in\mathcal{I}} \big(b_{i,s}^{(n',k')} + (2i-\alpha_{n'})b_1^{(0,1)}b_i^{(n',k')}\big)\left\langle \frac{\partial S_j}{\partial b_i^{(n',k')}}, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle \\
&\qquad - \left(\frac{\lambda_s}{\lambda} + b_1^{(1,0)}\right)\langle \Lambda\tilde{Q}_b, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle - \left\langle \left(\frac{z_s}{\lambda} + b_1^{(1,\cdot)}\right).\nabla\tilde{Q}_b, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle.
\end{aligned} \tag{5-13}$$

For $2\leq j\leq L+2$ and $(n',k',i)\in\mathcal{I}$, as S_i is homogeneous of degree $(i,-\gamma-g')$, using (4-27) and (5-4), we have

$$\left|(2i-\alpha_{n'})b_1^{(0,1)}b_i^{(n',k')}\left\langle\frac{\partial S_j}{\partial b_i^{(n',k')}},H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right|\leq\frac{C(L,M)}{s^{L-\delta_0-\delta_n+2m_n+1+\frac{g'}{2}}}. \tag{5-14}$$

Using the modulation bound (4-43), the asymptotics (2-1) and (2-7) of Q and ΛQ, (4-27) and (5-4), we find

$$\left|\left(\frac{\lambda_s}{\lambda}+b_1^{(1,0)}\right)\langle\Lambda\tilde{Q}_b,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle-\left\langle\left(\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right).\nabla\tilde{Q}_b,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right|$$
$$\leq\frac{C(L,M)}{s^{2L+\frac{3-\alpha}{2}-2m_n-\delta_n}} \tag{5-15}$$

is very small as $L\gg 1$. Moreover for $2\leq j\leq L+2$, one has

$$\sum_{(n',k',i)\in\mathcal{I}}b_{i,s}^{(n',k')}\left\langle\frac{\partial S_j}{\partial b_i^{(n',k')}},H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle=\frac{d}{ds}(\langle S_j,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle)-\langle S_j,H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})\rangle.$$

From similar arguments we used to derive (5-14), one has the similar bound for the last term, yielding

$$\sum_{(n',k',i)\in\mathcal{I}}b_{i,s}^{(n',k')}\left\langle\frac{\partial S_j}{\partial b_i^{(n',k')}},H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle$$
$$=\frac{d}{ds}(\langle S_j,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle)+O(s^{-L+\delta_0+\delta_n+2m_n-1-\frac{g'}{2}}). \tag{5-16}$$

Coming back to the decomposition (5-13), and applying (5-14) and (5-16) gives

$$\langle\widetilde{\text{Mod}}(s),H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle=\big(b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}\big)\langle T_0^{(n,k)},\chi_{B_0}T_0^{(n,k)}\rangle$$
$$+\frac{d}{ds}\left(\sum_{j=2}^{L+2}\langle S_j,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right)+O(s^{-L+\delta_0+\delta_n+2m_n-1-\frac{g'}{2}}). \tag{5-17}$$

In the decomposition (5-6), we examined each term in (5-7), (5-8), (5-12) and (5-17), yielding the identity (5-5) we claimed in this first step.

Step 2: end of the proof. From (5-5) one obtains

$$\frac{d}{ds}\left(\frac{(\langle H^{L_n}(\varepsilon-\sum_{i=2}^{L+2}S_i),\chi_{B_0}T_0^{(n,k)}\rangle)}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{(n,k)}\rangle}\right)$$
$$=b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}+\frac{O\big(\sqrt{\mathcal{E}_{2s_L}}B_0^{4m_n+2\delta_n}\big)+O\left(\frac{C(L)}{s^{L+1+\frac{g'}{2}-\delta_0-\delta_n-2m_n}}\right)}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{(n,k)}\rangle}$$
$$+\left\langle H^{L_n}\left(\varepsilon-\sum_{i=2}^{L+2}S_i\right),\chi_{B_0}T_0^{(n,k)}\right\rangle\frac{d}{ds}\left(\frac{1}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{(n,k)}\rangle}\right). \tag{5-18}$$

The size of the denominator is, from the asymptotic (2-7) of $T_0^{(n,k)}$ and (4-27),

$$\langle\chi_{B_0}T_0^{(n,k)},T_0^{(n,k)}\rangle\sim cs^{2m_n+2\delta_n} \tag{5-19}$$

for some constant $c>0$. As the denominator just depends on $b_1^{(0,1)}$, using the bound $|b_{1,s}^{(0,1)}|\le Cs^{-2}$ and the asymptotics (2-7) of $T_0^{(n,k)}$, we obtain

$$\left|\frac{d}{ds}\left(\frac{1}{\langle \chi_{B_0}T_0^{(n,k)},T_0^{(n,k)}\rangle}\right)\right|\le \frac{C(L,M)}{s^{2m_n+2\delta_n+1}}.$$

Also, using again the coercivity estimate Lemma C.3, (5-4) and the fact that for $2\le j\le L+2$, we know S_j is homogeneous of degree $(j,-\gamma-g')$, we obtain

$$\left|\left\langle H^{L_n}\left(\varepsilon-\sum_{i=2}^{L+2}S_i\right),\chi_{B_0}T_0^{(n,k)}\right\rangle\right|\le C(L,M)\left(\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n}+s^{-L-\frac{g'}{2}+\delta_0+\delta_n+2m_n}\right). \tag{5-20}$$

Hence, plugging the three previous identities in (5-18) gives the identity (5-3) claimed in the lemma. □

5B. ***Lyapunov monotonicity for low regularity norms of the remainder.*** The key estimate concerning the remainder w is the bound on the high regularity adapted Sobolev norm at the blow-up area: $\mathcal{E}_{2s_L}$. However, the nonlinearity can transfer energy from low to high frequencies, and consequently to control $\mathcal{E}_{2s_L}$ we need to control the low frequencies. This is the purpose of Propositions 5.3 and 5.5, where we find an upper bound for the time evolution of $\|w_{\text{int}}\|_{\dot H^\sigma(\mathbb{R}^d)}$ and $\|w_{\text{ext}}\|_{H^\sigma(\Omega)}$.

Proposition 5.3 (Lyapunov monotonicity for the low Sobolev norm of the remainder in the blow-up zone). *Suppose all the constants involved in Proposition 4.6 are fixed except s_0 and η. Then for s_0 large enough and η small enough, for any solution u that is trapped on $[s_0,s')$ the following holds for $0\le t<t(s')$:*

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\}\le \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\frac{1}{s^{\frac{\alpha}{4L}}}\left[1+\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\right], \tag{5-21}$$

where the norm $\mathcal{E}_\sigma$ is defined in (4-9).

Remark 5.4. Equation (5-21) should be interpreted as follows. The term

$$\frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}$$

is from (4-25) and (4-52) of order $\frac{1}{s}\frac{ds}{dt}$ (as $\frac{ds}{dt}=\lambda^{-2}$). The $1/s^{\frac{\alpha}{4L}}$ then represents a gain: it gives that the right-hand side of (5-21) is of order $(1/s^{1+\frac{\alpha}{4L}})\frac{ds}{dt}$, which when reintegrated in time is convergent and arbitrarily small for s_0 large enough. The third term shows that one needs to have $\sqrt{\mathcal{E}_\sigma}\lesssim s^{-\frac{\sigma-s_c}{2}}$ to control the nonlinear terms, which holds because of the bootstrap bound (4-25).

Proof of Proposition 5.3. To show this result, we compute the left-hand side of (5-21) and we bound it above it using all the bounds that hold in the trapped regime. The time evolution w_{int} given by (4-34) yields

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\}&=\frac{d}{dt}\left\{\int|\nabla^\sigma w_{\text{int}}|^2\right\}\\&=\int\nabla^\sigma w_{\text{int}}.\nabla^\sigma\left(-H_{z,\frac{1}{\lambda}}w_{\text{int}}-\frac{1}{\lambda^2}\chi\tau_z(\widetilde{\text{Mod}}(t)_{\frac{1}{\lambda}}+\tilde\psi_{b\frac{1}{\lambda}})+\text{NL}(w_{\text{int}})+L(w_{\text{int}})+\tilde L+\widetilde{\text{NL}}+\tilde R\right).\end{aligned} \tag{5-22}$$

We now give an upper bound for each term in (5-22). As all the terms involve functions that are compactly supported in Ω since w_{int} is, all integrations by parts are legitimate and all computations and integrations are performed in $\mathbb{R}^d$ (e.g., L^2 denotes $L^2(\mathbb{R}^d)$).

Step 1: inside the blow-up zone (all terms except the three last ones in (5-22)).

The linear term. By (4-35) using dissipation, we first compute

$$\begin{aligned}\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(-H_{z,\frac{1}{\lambda}}w_{\text{int}}) &= \int \nabla^\sigma w_{\text{int}}.\nabla^\sigma\big(\Delta w_{\text{int}}+p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1}w_{\text{int}}\big)\\ &\le \int \nabla^\sigma w_{\text{int}}.\nabla^\sigma\big(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1}w_{\text{int}}\big),\end{aligned}$$

which becomes after an integration by parts and using the Cauchy–Schwarz inequality

$$\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma\big(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1}w_{\text{int}}\big) \le \|\nabla^{\sigma+2}w_{\text{int}}\|_{L^2}\big\|\nabla^{\sigma-2}\big(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1}w_{\text{int}}\big)\big\|_{L^2}.$$

Using interpolation, the coercivity estimate (C-16) and the bounds of the trapped regime (4-25) on ε, one has for the first term (performing a change of variables to go back to renormalized variables)

$$\begin{aligned}\|\nabla^{\sigma+2}w_{\text{int}}\|_{L^2} &= \frac{1}{\lambda^{\sigma+2-s_c}}\|\nabla^{\sigma+2}\varepsilon\|_{L^2}\\ &\le \frac{C}{\lambda^{\sigma+2-s_c}}\|\nabla^\sigma\varepsilon\|_{L^2}^{1-\frac{2}{2s_L-\sigma}}\|\varepsilon\|_{\dot H^{2s_L}}^{\frac{2}{2s_L-\sigma}}\\ &\le \frac{C(L,M)}{\lambda^{\sigma+2-s_c}}\sqrt{\mathcal{E}_\sigma}^{1-\frac{2}{2s_L-\sigma}}\sqrt{\mathcal{E}_{2s_L}}^{\frac{2}{2s_L-\sigma}}\\ &\le \frac{C(L,M,K_1,K_2)}{\lambda^{\sigma+2-s_c}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+\frac{2}{2s_L-\sigma}(L+1-\delta_0+\eta(1-\delta_0')-\frac{(\sigma-s_c)\ell}{2\ell-\alpha})}}\\ &= \frac{C(L,M,K_1,K_2)}{\lambda^{\sigma+2-s_c}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}.\end{aligned}$$

As $Q^{p-1}=O((1+|y|)^{-2})$ from (2-2), using the Hardy inequality (B-7) we get for the second term after a change of variables

$$\begin{aligned}\big\|\nabla^{\sigma-2}\big(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1}w\big)\big\|_{L^2} &= \frac{p}{\lambda^{\sigma-s_c}}\big\|\nabla^{\sigma-2}(Q^{p-1}\varepsilon)\big\|_{L^2}\\ &\le \frac{C}{\lambda^{\sigma-s_c}}\|\nabla^\sigma\varepsilon\|_{L^2} = \frac{C}{\lambda^{\sigma-s_c}}\sqrt{\mathcal{E}_\sigma}.\end{aligned}$$

Combining the four above identities we obtain

$$\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(-H_{z,\frac{1}{\lambda}}w_{\text{int}}) \le \frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}. \tag{5-23}$$

The modulation term. To treat the error induced by the cut separately, we decompose as follows, going back to renormalized variables using Cauchy–Schwarz:

$$\begin{aligned}\left|\int \nabla^\sigma w.\nabla^\sigma\left(\frac{1}{\lambda^2}\chi\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}})\right)\right| \\ &\le \left|\int \nabla^\sigma w.\nabla^\sigma\left(\frac{1}{\lambda^2}(1+(\chi-1))\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}})\right)\right| \\ &\le \frac{1}{\lambda^{2(\sigma-s_c)+2}}\sqrt{\mathcal{E}_\sigma}\left[\left\|\nabla^\sigma \tilde{\text{Mod}}(s)\right\|_{L^2}+\left\|\nabla^\sigma\left(\frac{1}{\lambda^2}(\chi-1)\tau_z(\widetilde{\text{Mod}}(t)_{\frac{1}{\lambda}})\right)\right\|_{L^2}\right].\end{aligned} \tag{5-24}$$

For the first term in the above equation, using (4-33) and the modulation estimates (4-43) and (4-44), we get

$$\begin{aligned}&\|\nabla^\sigma \tilde{\text{Mod}}(s)\|_{L^2} \\ &\le \sum_{(n,k,i)\in\mathcal{I}}\left|b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}\right|\left\|\nabla^\sigma\left(\chi_{B_1}\left(T_i^{(n,k)}+\sum_{j=2}^{L+2}\frac{\partial S_j}{\partial b_i^{(n,k)}}\right)\right)\right\|_{L^2} \\ &\qquad +\left|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right|\|\nabla^\sigma(\Lambda\tilde{Q}_b)\|_{L^2}+\left|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right|\|\nabla^{\sigma+1}(\tilde{Q}_b)\|_{L^2} \\ &\le C(L,M)(\sqrt{\mathcal{E}_{2s_L}}+s^{-L-3})\Big[\|\nabla^\sigma(\Lambda\tilde{Q}_b)\|_{L^2}+\|\nabla^{\sigma+1}(\tilde{Q}_b)\|_{L^2} \\ &\qquad +\sum_{(n,k,i)\in\mathcal{I}}\|\nabla^\sigma(\chi_{B_1}T_i^{(n,k)})\|_{L^2}+\sum_{j=2}^{L+2}\left\|\nabla^\sigma\left(\chi_{B_1}\frac{\partial S_j}{\partial b_i^{(n,k)}}\right)\right\|_{L^2}\Big].\end{aligned}$$

Under the trapped regime bound (4-25), one has $\sqrt{\mathcal{E}_{2s_L}}+s^{-L-3}\le s^{-L-1+\delta_0-\eta(1-\delta_0')}$. Moreover, from the asymptotics of Q, ΛQ, $T_i^{(n,k)}$ and S_j ((2-1), (2-7), Lemma 2.10 and (3-8)), and the bounds on the parameters (4-27), one has

$$\|\nabla^\sigma(\Lambda\tilde{Q}_b)\|_{L^2}\le C,\quad \|\nabla^{\sigma+1}(\tilde{Q}_b)\|_{L^2}\le C,$$

$$\begin{aligned}&\sum_{(n,k,i)\in\mathcal{I}}\|\nabla^\sigma(\chi_{B_1}T_i^{(n,k)})\|_{L^2}+\sum_{j=2}^{L+2}\left\|\nabla^\sigma\left(\chi_{B_1}\frac{\partial S_j}{\partial b_i^{(n,k)}}\right)\right\|_{L^2} \\ &\le C(L)\le C(L)s^{L+\sup_{0\le n\le n_0}\delta_n-\delta_0-\frac{\alpha}{2}-\frac{(\sigma-s_c)}{2}+C(L)\eta}+C(L)s^{L+\sup_{0\le n\le n_0}\delta_n-\delta_0-\frac{\alpha}{2}-\frac{(\sigma-s_c)}{2}+C(L)\eta-\frac{g'}{2}}.\end{aligned}$$

All these bounds then imply that for the modulation term that is located at the blow-up zone in (5-24), we have

$$\begin{aligned}\frac{1}{\lambda^{2(\sigma-s_c)+2}}\sqrt{\mathcal{E}_\sigma}\|\nabla^\sigma\tilde{\text{Mod}}(s)\|_{L^2} &\le \frac{C(L,M)\sqrt{\mathcal{E}_\sigma}s^{L+\sup_{0\le n\le n_0}\delta_n-\delta_0-\frac{\alpha}{2}-\frac{(\sigma-s_c)}{2}+C(L)\eta}}{\lambda^{2(\sigma-s_L)+2}s^{L+1-\delta_0+(1-\delta_0')\eta}} \\ &\le \frac{C(L,M)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{1+(\frac{\alpha}{2}-\sup_{0\le n\le n_0}\delta_n)+\frac{\sigma-s_c}{2}-C(L)\eta}}.\end{aligned}$$

We now turn to the second term in (5-24). The blow-up point z is arbitrarily close to 0 by (4-51) and from the expression of the modulation term (4-33), all the terms except $\tau_z\big([\frac{\lambda_s}{\lambda}+b_1^{(0,1)}]\Lambda Q+$

$[b_1^{(1,\cdot)}+\frac{z_s}{\lambda}].\nabla Q)_{1/\lambda}$ have support in the zone $\{|x-z|\leq 2B_1\lambda\}\subset B\big(0,\frac12\big)$ because $B_1\lambda\ll 1$. This means that from the modulation estimates (4-43)

$$\left\|\nabla^\sigma\left(\frac{1}{\lambda^2}(\chi-1)\tau_z(\widetilde{\mathrm{Mod}}(t)_{\frac{1}{\lambda}})\right)\right\|_{L^2}=\left\|\nabla^\sigma\left(\frac{1}{\lambda^2}(\chi-1)\tau_z\left(\left[\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right]\Lambda Q+\left[b_1^{(1,\cdot)}+\frac{z_s}{\lambda}\right].\nabla Q)_{\frac{1}{\lambda}}\right)\right)\right\|_{L^2}$$
$$\leq\frac{C\big[\big|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\big|+\big|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\big|\big]}{\lambda^2}\leq\frac{C}{\lambda^2 s^{L+1}}.$$

We insert the two previous equations into the expression (5-24), yielding

$$\left|\int\nabla^\sigma w_{\mathrm{int}}.\nabla^\sigma\left(\frac{1}{\lambda^2}\chi\tau_z(\tilde{\mathrm{Mod}}(t)_{\frac{1}{\lambda}})\right)\right|\leq\frac{C(L,M)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{1+(\frac{\alpha}{2}-\sup_{0\leq n\leq n_0}\delta_n)+\frac{\sigma-s_c}{2}-C(L)\eta}}.\tag{5-25}$$

The error term. As $|z|\ll 1$ by (4-51) and $B_1\lambda\ll 1$ by (4-27) and (4-52), from the expression of the error term (3-36), all the terms except $\tau_z(b_1^{(0,1)}\Lambda Q+b_1^{(1,\cdot)}.\nabla Q)_{1/\lambda}$ have support in the zone $\{|x-z|\leq 2B_1\lambda\}\subset B\big(0,\frac12\big)$. Therefore, making the following decomposition and coming back to renormalized variables, using the estimates (3-32) and (4-43), one computes

$$\begin{aligned}
&\left|\int\nabla^\sigma w_{\mathrm{int}}.\nabla^\sigma\left(\frac{1}{\lambda^2}\chi\tau_z(\tilde{\psi}_{b\,\frac{1}{\lambda}})\right)\right|\\
&\leq\frac{\|\nabla^\sigma\varepsilon\|_{L^2}}{\lambda^{\sigma-s_c+2}}\left(\frac{\|\nabla^\sigma\tilde{\psi}_b\|_{L^2}}{\lambda^{2(\sigma-s_c)+2}}+\big\|\nabla^\sigma((\chi-1)\tau_z(\tilde{\psi}_{b\,\frac{1}{\lambda}}))\big\|_{L^2}\right)\\
&\leq\frac{C(L)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{1+\frac{\alpha}{2}+\frac{\sigma-s_c}{2}-C(L)\eta}}+\frac{\|\nabla^\sigma\varepsilon\|_{L^2}}{\lambda^{\sigma-s_c+2}}\big\|\nabla^\sigma(\chi-1)\big(\tau_z(b_1^{(0,1)}\Lambda Q+b_1^{(1,\cdot)}.\nabla Q)_{\frac{1}{\lambda}}\big)\big\|_{L^2}\\
&\leq\frac{C(L)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{1+\frac{\alpha}{2}+\frac{\sigma-s_c}{2}-C(L)\eta}}+C\frac{\|\nabla^\sigma\varepsilon\|_{L^2}}{\lambda^{2(\sigma-s_c)+2}}\big(|b_1^{(0,1)}|\lambda^{\alpha+\sigma-s_c}+|b_1^{(1,\cdot)}|\lambda^{1+\sigma-s_c}\big)\\
&\leq\frac{C(L)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{1+\frac{\alpha}{2}+\frac{\sigma-s_c}{2}-C(L)\eta}}.
\end{aligned}\tag{5-26}$$

The nonlinear term. First, coming back to renormalized variables, as $\mathrm{NL}(\varepsilon)=\sum_{k=2}^pC_k^p\tilde{Q}_b^{p-k}\varepsilon^k$, and performing an integration by parts we write

$$\left|\int\nabla^\sigma w_{\mathrm{int}}.\nabla^\sigma(\mathrm{NL}(w_{\mathrm{int}}))\right|\leq C\sum_{k=2}^p\frac{\|\nabla^{\sigma+2-(k-1)(\sigma-s_c)}\varepsilon\|_{L^2}\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)}(\tilde{Q}_b^{p-k}\varepsilon^k)\|_{L^2}}{\lambda^{2(\sigma-s_c)+2}}.\tag{5-27}$$

We fix k, $2\leq k\leq p$, and focus on the k-th term in the sum. The first term is estimated using interpolation, the coercivity estimate (C-16) and the bound (4-25):

$$\begin{aligned}
\|\nabla^{\sigma+2-(k-1)(\sigma-s_c)}\varepsilon\|_{L^2}&\leq C\|\nabla^\sigma\varepsilon\|_{L^2}^{1-\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}}\|\nabla^{2s_L}\varepsilon\|_{L^2}^{\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}}\\
&\leq C(L,M)\sqrt{\mathcal{E}_\sigma}^{1-\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}}\sqrt{\mathcal{E}_{2s_L}}^{\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}}\\
&\leq\frac{C(L,M,K_1,K_2)}{s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1-\frac{(k-1)(\sigma-s_c)}{2}+\frac{\alpha}{2L}+O\left(\frac{|\sigma-s_c|+|\eta|}{L}\right)}}.
\end{aligned}\tag{5-28}$$

For the second term in (5-27), as $\tilde{Q}_b = O((1+|y|)^{-2})$ by (3-29) and (4-27), we first use the Hardy inequality (B-7):

$$\big\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)}(\tilde{Q}_b^{p-k}\varepsilon^k)\big\|_{L^2} \le C\big\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1}}(\varepsilon^k)\big\|_{L^2}. \tag{5-29}$$

We write

$$\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1} = \sigma(n,k)+\delta(n,k),$$

where $\sigma(n,k) := E\big[\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1}\big] \in \mathbb{N}$ and $0 \le \delta(n,k) < 1$. Developing the entire part of the derivative yields

$$\big\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1}}(\varepsilon^k)\big\|_{L^2} \le \sum_{\substack{(\mu_i)_{1\le i\le k}\in\mathbb{N}^{kd}\\ \sum_i|\mu_i|=\sigma(n,k)}} \bigg\|\nabla^{\delta(\sigma,k)}\bigg(\prod_{i=1}^k \partial^{\mu_i}\varepsilon\bigg)\bigg\|_{L^2}. \tag{5-30}$$

Fix $(\mu_i)_{1\le i\le k}\in\mathbb{N}^{kd}$ satisfying $\sum_{i=1}^k|\mu_i| = \sigma(n,k)$ in the above sum. We define the following family of Lebesgue exponents (that are well defined since $\sigma < \frac{d}{2}$):

$$\frac{1}{p_i} := \frac{1}{2}-\frac{\sigma-|\mu_i|_1}{d}, \qquad \frac{1}{p_i'} := \frac{1}{2}-\frac{\sigma-|\mu_i|-\delta(\sigma,k)}{d} \quad \text{for } 1\le i\le k.$$

One has $p_i > 2$ and a direct computation shows that

$$\frac{1}{p_j'}+\sum_{i\ne j}\frac{1}{p_i} = \frac{1}{2}.$$

We now recall the commutator estimate

$$\|\nabla^{\delta_\sigma}(uv)\|_{L^q} \le C\|\nabla^{\delta_\sigma}u\|_{L^{p_1}}\|v\|_{L^{p_2}} + C\|\nabla^{\delta_\sigma}v\|_{L^{p_1'}}\|u\|_{L^{p_2'}}$$

for

$$\frac{1}{p_1}+\frac{1}{p_2} = \frac{1}{p_1'}+\frac{1}{p_2'} = \frac{1}{q},$$

provided $1 < q,\, p_1,\, p_1' < +\infty$ and $1 \le p_2,\, p_2' \le +\infty$. This estimate, combined with the Hölder inequality allows us to compute by iteration:

$$\begin{aligned}
&\bigg\|\nabla^{\delta(\sigma,k)}\bigg(\prod_{i=1}^k \partial^{\mu_i}\varepsilon\bigg)\bigg\|_{L^2}\\
&\quad\le C\|\partial^{\mu_1+\delta(\sigma,k)}\varepsilon\|_{L^{p_1'}}\bigg\|\prod_{i=2}^k\partial^{\mu_i}\varepsilon\bigg\|_{L^{(\sum_{i=2}^k\frac{1}{p_i})^{-1}}} + C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\bigg\|\nabla^{\delta(\sigma,k)}\bigg(\prod_{i=2}^k\partial^{\mu_i}\varepsilon\bigg)\bigg\|_{L^{(\frac{1}{2}-\frac{1}{p_1})^{-1}}}\\
&\quad\le C\|\partial^{\mu_1+\delta(\sigma,k)}\varepsilon\|_{L^{p_1'}}\prod_{i=2}^k\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}} + C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\|\partial^{\mu_2+\delta(\sigma,k)}\varepsilon\|_{L^{p_2'}}\bigg\|\prod_{i=3}^k\partial^{\mu_i}\varepsilon\bigg\|_{L^{(\sum_{i=3}^k\frac{1}{p_i})^{-1}}}\\
&\qquad\qquad + C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\|\partial^{\mu_2}\varepsilon\|_{L^{p_2}}\bigg\|\nabla^{\delta(\sigma,k)}\bigg(\prod_{i=3}^k\partial^{\mu_i}\varepsilon\bigg)\bigg\|_{L^{(\frac{1}{2}-\frac{1}{p_1}-\frac{1}{p_2})^{-1}}}
\end{aligned}$$

$$\le C\|\partial^{\mu_1+\delta(\sigma,k)}\varepsilon\|_{L^{p_1'}}\prod_{i=2}^{k}\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}}+C\|\partial^{\mu_2+\delta(\sigma,k)}\varepsilon\|_{L^{p_2'}}\prod_{i\neq 2}\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}}$$

$$+C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\|\partial^{\mu_2}\varepsilon\|_{L^{p_2}}\left\|\nabla^{\delta(\sigma,k)}\left(\prod_{i=3}^{k}\partial^{\mu_i}\varepsilon\right)\right\|_{L^{(\frac12-\frac{1}{p_1}-\frac{1}{p_2})^{-1}}}$$

$$\vdots$$

$$\le C\sum_{i=1}^{k}\|\partial^{\mu_i+\delta(\sigma,k)}\varepsilon\|_{L^{p_i'}}\prod_{j=1,\,j\neq i}^{k}\|\partial^{\mu_i}\varepsilon\|_{L^{p_j}}.$$

From Sobolev embedding, one has on the other hand that

$$\|\partial^{\mu_i+\delta(\sigma,k)}\varepsilon\|_{L^{p_i'}}+\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}}\le C\|\nabla^{\sigma}\varepsilon\|_{L^2}=C\sqrt{\mathcal{E}_\sigma}.$$

Therefore (the strategy was designed to obtain this),

$$\left\|\nabla^{\delta(\sigma,k)}\left(\prod_{i=1}^{k}\partial^{\mu_i}\varepsilon\right)\right\|_{L^2}\le\sqrt{\mathcal{E}_\sigma}^{\,k}.$$

Plugging this estimate in (5-29) using (5-30) gives

$$\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)}(\tilde{Q}_b^{p-k}\varepsilon^k)\|_{L^2}\le C\sqrt{\mathcal{E}_\sigma}^{\,k}.$$

Injecting this bound and the bound (5-28) in the decomposition (5-27) yields

$$\left|\int\nabla^\sigma w_{\text{int}}.\nabla^\sigma(\text{NL}(w_{\text{int}}))\right|\le\frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O\left(\frac{|\eta|+|\sigma-s_c|}{L}\right)}}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}. \tag{5-31}$$

The small linear term. One has $L(\varepsilon)=-p(Q^{p-1}-\tilde{Q}^{p-1})\varepsilon$. The potential here admits the asymptotic $Q^{p-1}-\tilde{Q}^{p-1}\lesssim|y|^{-2-\alpha}$ at infinity, which is better than the asymptotic of the potential appearing in the linear term $Q^{p-1}\sim|y|^{-2}$ we used previously to estimate it. Hence using exactly the same techniques one can prove the same estimate

$$\left|\int\nabla^\sigma w_{\text{int}}.\nabla^\sigma(L(w_{\text{int}}))\right|\le\frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O\left(\frac{|\eta|+|\sigma-s_c|}{L}\right)}}. \tag{5-32}$$

End of Step 1. We come back to the first identity we derived, (5-22), and insert the bounds we found for each term in (5-23), (5-25), (5-26), (5-31) and (5-32) to obtain

$$\left|\int\nabla^\sigma w_{\text{int}}.\nabla^\sigma\left(-H_{z,\frac{1}{\lambda}}w_{\text{int}}-\frac{1}{\lambda^2}\chi\tau_z(\widetilde{\text{Mod}}(t)_{\frac{1}{\lambda}}+\tilde{\psi}_{b\,\frac{1}{\lambda}})+\text{NL}(w_{\text{int}})+L(w_{\text{int}})\right)\right|$$

$$\le\frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\Bigg[\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}+\frac{C(L,M,K_2)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+\left(\frac{\alpha}{2}-\sup_{0\le n\le n_0}\delta_n\right)-C(L)\eta}}$$

$$+\frac{C(L)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+\frac{\alpha}{2}-C(L)\eta}}+\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\Bigg]. \tag{5-33}$$

Step 2: the last three terms outside the blow-up zone in (5-22). By a change of variables, we see that the extra error term (4-40) is bounded:

$$\|\nabla^\sigma \widetilde{R}\|_{L^2(\mathbb{R}^d)} \le C.$$

Then, the extra linear term in (5-22) is estimated directly via interpolation using the bound (4-28):

$$\begin{aligned}\big\|\nabla^\sigma\big(-\Delta\chi_{B(0,3)}w-2\nabla\chi_{B(0,3)}.\nabla w+p\tau_z Q_{\frac{1}{\lambda}}^{p-1}(\chi_{B(0,1)}^{p-1}-\chi_{B(0,3)})w\big)\big\|_{L^2(\mathbb{R}^d)}\\ \le \|w\|_{H^{\sigma+1}} \le \|w\|_{H^\sigma}^{1-\frac{1}{2s_L-\sigma}}\|w\|_{H^{2s_L}}^{\frac{1}{2s_L-\sigma}}\\ \le C(K_1,K_2)\bigg(\frac{1}{\lambda^{2s_L-\sigma}s^{L+1-\delta_0+\eta(1-\delta_0')}}\bigg)^{\frac{1}{2s_L-\sigma}}\\ \le C(K_1,K_2)\bigg(\frac{1}{\lambda^{2s_L-\sigma}s^{L+1-\delta_0+\eta(1-\delta_0')}}\bigg)^{\frac{2}{2s_L-\sigma}} = \frac{C(K_1,K_2)}{\lambda^2 s^{1+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}\end{aligned}$$

because $1/\lambda^{2s_L-\sigma}s^{L+1-\delta_0+\eta(1-\delta_0')} \gg 1$ in the trapped regime. For the last nonlinear in (5-22), one has, using (D-4) and (4-28),

$$\begin{aligned}\|\widetilde{\mathrm{NL}}\|_{H^\sigma} \le C\|w\|_{H^\sigma}\|w\|_{H^{\frac{d}{2}+\sigma-s_c}}^{p-1} \le C(K_1)\|w\|_{H^{2s_L}}^{(p-1)(\frac{d}{2}+\sigma-s_c-\sigma)/(2s_L-\sigma)}\\ \le C(K_1,K_2)\bigg(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\bigg)^{\frac{2}{2s_L-\sigma}} \le C(K_1,K_2)\frac{1}{\lambda^2 s^{1+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}.\end{aligned}$$

The three previous estimates imply that for the terms created by the cut in (5-22), we have the estimate (we recall that $\lambda^{\sigma-s_c}/s^{\frac{\ell(\sigma-s_c)}{2\ell-\alpha}} = 1+O(s_0^{-\tilde{\eta}})$ from (4-52))

$$\bigg|\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(\tilde{L}+\widetilde{R}+\widetilde{\mathrm{NL}})\bigg| \le \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}. \tag{5-34}$$

Step 3: conclusion. We now come back to the first identity we derived, (5-22), and insert the bounds (5-33) and (5-34), yielding

$$\begin{aligned}\frac{d}{dt}\bigg\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\bigg\}\\ \le \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\bigg[\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}+\frac{C(L,M,K_2)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+(\frac{\alpha}{2}-\sup_{0\le n\le n_0}\delta_n)-C(L)\eta}}\\ +\frac{C(L)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+\frac{\alpha}{2}-C(L)\eta}}+\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}\sum_{k=2}^{p}\bigg(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\bigg)^{k-1}\bigg].\end{aligned}$$

As the constants never depend on s_0 or on η, as $L \gg 1$ is an arbitrary large integer, $0 < \sigma - s_c \ll 1$, $\frac{\alpha}{2} - \sup_{0\le n\le n_0}\delta_n > 0$, we see that for s_0 sufficiently large and η sufficiently small, the terms on the right-hand side of the previous equation can be as small as we want, and (5-21) is obtained. ☐

Proposition 5.5 (Lyapunov monotonicity for the low Sobolev norm of the remainder outside the blow-up area). *Suppose all the constants involved in Proposition 4.6 are fixed except s_0 and η. Then for s_0 large enough and η small enough, for any solution u that is trapped on $[s_0, s')$ the following holds for $t \in [0, t(s'))$:*

$$\frac{d}{dt}\big[\|w_{\text{ext}}\|^2_{H^\sigma}\big] \leq \frac{C(K_1,K_2)}{s^{1+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}\lambda^2}\|w_{\text{ext}}\|_{H^\sigma}. \tag{5-35}$$

Proof. From the evolution equation of w_{ext}, given in Section 4B1, we deduce

$$\frac{d}{dt}\|w_{\text{ext}}\|^2_{H^\sigma(\Omega)} \leq C\|w_{\text{ext}}\|_{H^\sigma(\Omega)}\big\|\Delta w_{\text{ext}} + \Delta\chi_3 w + 2\nabla\chi_3.\nabla w + (1-\chi_3)w^p\big\|_{H^\sigma(\Omega)}. \tag{5-36}$$

For the linear terms, using interpolation and the bounds (4-25) and (4-28) one finds

$$\begin{aligned}
&\|\Delta w_{\text{ext}} + \Delta\chi_3 w + 2\nabla\chi_3.\nabla w\|_{H^\sigma(\Omega)}\\
&\quad\leq C\|w_{\text{ext}}\|_{H^{\sigma+2}(\Omega)} + C\|w\|_{H^{\sigma+1}(\Omega)}\\
&\quad\leq C\|w_{\text{ext}}\|^{1-\frac{2}{2s_L-\sigma}}_{H^\sigma(\Omega)}\|w_{\text{ext}}\|^{\frac{2}{2s_L-\sigma}}_{H^{2s_L}(\Omega)} + C\|w\|^{1-\frac{1}{2s_L-\sigma}}_{H^\sigma(\Omega)}\|w\|^{\frac{1}{2s_L-\sigma}}_{H^{2s_L}(\Omega)}\\
&\quad\leq C(K_1,K_2)\bigg[\bigg(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\bigg)^{\frac{1}{2s_L-\sigma}} + \bigg(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\bigg)^{\frac{2}{2s_L-\sigma}}\bigg]\\
&\quad\leq C(K_1,K_2)\bigg(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\bigg)^{\frac{2}{2s_L-\sigma}} \leq C(K_1,K_2)\frac{1}{\lambda^2 s^{1+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}
\end{aligned}$$

because $1/\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')} \gg 1$ in the trapped regime from (4-52). For the nonlinear term, using (D-4), interpolation and then the bootstrap bound (4-28),

$$\begin{aligned}
\|(1-\chi_3)w^p\|_{H^\sigma} &\leq C\|w^p\|_{H^\sigma(\Omega)} \leq C\|w\|_{H^\sigma(\Omega)}\|w\|^{p-1}_{H^{\frac{d}{2}+\sigma-s_c}(\Omega)}\\
&\leq C(K_1)\|w\|^{(p-1)\frac{\frac{d}{2}+\sigma-s_c-\sigma}{2s_L-\sigma}}_{H^{2s_L}(\Omega)} \leq C(K_1)\|w\|^{\frac{2}{2s_L-\sigma}}_{H^{2s_L}(\Omega)} \leq \frac{C(K_1,K_2)}{s^{1+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}\lambda^2}.
\end{aligned}$$

Injecting the two above estimates in (5-36) yields the desired identity (5-35). □

5C. ***Lyapunov monotonicity for high regularity norms of the remainder.*** We derive Lyapunov-type monotonicity formulas for the high regularity norms of the remainder inside and outside the blow-up zone, $\mathcal{E}_{2s_L}$ and $\|w_{\text{ext}}\|_{H^{2s_L}}$, in Propositions 5.6 and 5.8. In our general strategy, we have to find a way to say that w is of smaller order compared to the excitation $\chi\tau_z(\tilde{\alpha}_{b,1/\lambda})$ and does not affect the blow-up dynamics induced by the latter. This is why we study the quantity $\mathcal{E}_{2s_L}$: it controls the usual Sobolev norm H^{2s_L} and any local norm of lower-order derivative, which is useful for estimates, and is it adapted to the linear dynamics as it undergoes dissipation. Finally, for this norm one sees that the error $\tilde{\psi}_b$ is of smaller order compared to the main dynamics of $\chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})$ (this is the $\eta(1-\delta_0')$ gain in (3-33)).

Proposition 5.6 (Lyapunov monotonicity for the high regularity adapted Sobolev norm of the remainder inside the blow-up area). *Suppose all the constants of Proposition 4.6 are fixed, except s_0 and η. Then* *there exists a constant $\delta > 0$ such that for any constant $N \gg 1$, for s_0 large enough and η small enough, for any solution u that is trapped on $[s_0, s')$, the following holds for $0 \le t < t(s')$:*

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}+O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}\left(\sqrt{\mathcal{E}_{2s_L}}+\frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)\right)\right\}$$
$$\le \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\Bigg[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')}}+\frac{C(L,M)}{N^{2\delta}}\mathcal{E}_{2s_L}$$
$$+\mathcal{E}_{2s_L}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O(\frac{1}{L})}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c}{L})}}+\frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}\Bigg], \tag{5-37}$$

where $O_{L,M}(f)$ denotes a function depending on time such that $|O_{L,M}(f)(t)| \le C(L,M)f$ for a constant $C(L,M) > 0$, and where $\mathcal{E}_\sigma$ and $\mathcal{E}_{2s_L}$ are defined in (4-9) and (4-7).

Remark 5.7. Equation (5-37) has to be understood the following way. The $O(\cdot)$ in the time derivative is a corrective term coming from the refinement of the last modulation equations; see (4-44) and (5-2). It is of smaller order for our purpose so one can "forget" it. On the right-hand side of (5-37), the first two terms come from the error $\tilde{\psi}_b$ made in the approximate dynamics. The third one results from the competition of the dissipative linear dynamics and the lower-order linear terms that are of smaller order (the motion of the potential in the operator $H_{z,1/\lambda}$ involved in $\mathcal{E}_{2s_L}$, and the difference between the potentials $\tau_z(\tilde{Q}_{b,1/\lambda})^{p-1}$ and $\tau_z(Q_{1/\lambda})^{p-1}$). The penultimate represents the effect of the main nonlinear term, and shows that one needs $\mathcal{E}_\sigma$ smaller than $s^{s_c-\sigma}$ to control the energy transfer from low to high frequencies. The last one results from the cut of w at the border of the blow-up zone.

Proof of Proposition 5.6. From (4-41) one has the identity

$$\frac{d}{dt}\left(\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s_L-s_c)}}\right)=\frac{d}{dt}\left(\int |H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}|^2\right)$$
$$=-2\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H^{s_L+1}_{z,\frac{1}{\lambda}}w_{\text{int}}+\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H^{s_L}_{z,\frac{1}{\lambda}}\left(\frac{1}{\lambda^2}\chi\tau_z(-\widetilde{\text{Mod}}(t)_{\frac{1}{\lambda}})\right)$$
$$+2\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}\left[H^{s_L}_{z,\frac{1}{\lambda}}\left[\frac{1}{\lambda^2}\chi\tau_z(-\tilde{\psi}_{b\frac{1}{\lambda}})+\text{NL}(w_{\text{int}})+L(w_{\text{int}})\right]+\frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w_{\text{int}}\right]$$
$$+2\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H_{z,\frac{1}{\lambda}}(\tilde{L}+\widetilde{\text{NL}}+\tilde{R}). \tag{5-38}$$

The proof is organized as follows. For the terms appearing in this identity: for some (those on the second line), we find direct upper bounds (Step 1), then we integrate by parts in time some modulation terms that are problematic to treat the second term on the right-hand side (Step 2), and eventually we prove that the terms created by the cut of the solitary wave (the last line) are harmless and use a dissipation property at the linear level (produced by the first term on the right-hand side) to improve the result (Step 3). Throughout the proof, the estimates are performed on $\mathbb{R}^d$, as w_{int} has compact support inside Ω, and we omit it in the notations.

Step 1: brute force upper bounds. We claim that the nonlinear term, the error term, the small linear term and the term involving the time derivative of the linearized operator in (5-38) can be directly bounded above, yielding

$$\begin{aligned}\Big\| H^{s_L}_{z,\frac{1}{\lambda}}\Big[\mathrm{NL}(w_{\mathrm{int}}) - \frac{1}{\lambda^2}\chi\tau_z(\tilde{\psi}_{b,\frac{1}{\lambda}}) + L(w_{\mathrm{int}})\Big] + \frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w_{\mathrm{int}}\Big\|_{L^2} \\ \leq \frac{1}{\lambda^{(2s_L-s_c)+2}s}\Bigg[\sqrt{\mathcal{E}_{2s_L}}\sum_{k=2}^{p}\Big(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\Big)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}} \\ + \frac{C(L)}{s^{L+1-\delta_0+\eta(1-\delta_0)'}} + C(L,M)\Big(\int \frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\Big)^{\frac12}\Bigg] \end{aligned} \tag{5-39}$$

for some constant $\delta > 0$. We now analyse these four terms separately.

The error term. We decompose between the main terms and the terms created by the cut. The cut induced by $\tilde{\chi} := \chi(\lambda y + z)$ only sees the terms $b_1^{(0,1)}\Lambda Q + b_1^{(1,\cdot)}.\nabla Q$ because all the other terms in the expression (3-36) of $\tilde{\psi}_b$ have support inside $\mathcal{B}^d(2B_1)$ and because $|z| \ll 1$ by (4-51) and $B_1 \ll \frac{1}{\lambda}$ by (4-52). For the main term we use the estimate (3-33), and for the second the bound on the parameters (4-27) and the asymptotics (2-7) and (2-1) of ΛQ and ∂Q,

$$\begin{aligned}\Big\| H^{s_L}_{z,\frac{1}{\lambda}}\Big(\frac{1}{\lambda^2}\chi\tau_z\tilde{\psi}_{b,\frac{1}{\lambda}}\Big)\Big\|_{L^2} &\leq C\Big\| H^{s_L}_{z,\frac{1}{\lambda}}\Big(\frac{1}{\lambda^2}\tau_z\tilde{\psi}_{b,\frac{1}{\lambda}}\Big)\Big\|_{L^2} + C\Big\| H^{s_L}_{z,\frac{1}{\lambda}}\Big(\frac{1}{\lambda^2}(1-\chi)\tau_z\tilde{\psi}_{b,\frac{1}{\lambda}}\Big)\Big\|_{L^2} \\ &\leq \frac{\|H^{s_L}\tilde{\psi}_b\|_{L^2}}{\lambda^{2s_L-s_c}} + \frac{1}{\lambda^{2(2s_L-s_c)+4}}\int \big|H^{s_L}[(1-\tilde{\chi})(b_1^{(0,1)}\Lambda Q + b_1^{(1,\cdot)}.\nabla Q]\big|^2 \\ &\leq \frac{C(L)}{\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')}} + \frac{C\lambda^{2(\alpha-1)}}{s} + \frac{C}{s^{\frac{\alpha+1}{2}}} \\ &\leq \frac{C(L)}{\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')}} \end{aligned} \tag{5-40}$$

since $\alpha > 1$; hence

$$\frac{\lambda^{2(\alpha-1)}}{s} + \frac{1}{s^{\frac{\alpha+1}{2}}} \ll 1,$$

since $1/\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')} \gg 1$ in the trapped regime from (4-52).

The nonlinear term. We begin by coming back to renormalized variables:

$$\|H^{s_L}_{z,\frac{1}{\lambda}}(\mathrm{NL}(w_{\mathrm{int}}))\|_{L^2} \leq \frac{\|H^{s_L}(\mathrm{NL}(\varepsilon))\|_{L^2}}{\lambda^{(2s_L-s_c)+2}} \leq C\sum_{k=2}^{p}\frac{\|H^{s_L}(\widetilde{Q}_b^{p-k}\varepsilon^k)\|_{L^2}}{\lambda^{(2s_L-s_c)+2}} \tag{5-41}$$

because $\mathrm{NL}(\varepsilon) = \sum_{k=2}^{p} C_k^p\widetilde{Q}_b^{p-k}\varepsilon^k$. We fix k with $2 \leq k \leq p$ and study the corresponding term in the above sum. One has $H = -\Delta - pQ^{p-1}$, and Q is a smooth profile satisfying the estimate $Q = O((1+|y|)^{-\frac{2}{p-1}})$, which propagates to its derivatives from (2-1). Similarly, from (4-27) and (3-29), one has $\widetilde{Q}_b = O((1+|y|)^{-\frac{2}{p-1}})$ and it propagates to the derivatives. The Leibniz rule for derivation then yields

$$\begin{aligned}\|H^{s_L}(\widetilde{Q}_b^{p-k}\varepsilon^k)\|_{L^2}^2 &\le C(L)\sum_{\substack{\mu\in\mathbb{N}^d\\ 0\le|\mu|\le 2s_L}}\int\frac{|\partial^\mu(\varepsilon^k)|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2|\mu|}}\\ &\le C(L)\sum_{\substack{(\mu_i)_{1\le i\le k}\in\mathbb{N}^{kd}\\ \sum_{i=1}^k|\mu_i|\le 2s_L}}\int\frac{\prod_{i=1}^k|\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|}}.\end{aligned}\tag{5-42}$$

We fix $\mu_i\in\mathbb{N}^{kd}$ with $\sum|\mu_i|_1\le 2s_L$ and focus on the corresponding term in the above equation. Without loss of generality we order by increasing length: $|\mu_1|\le\dots\le|\mu_k|$. We now distinguish between two cases.

Case 1: $|\mu_k|+\frac{2(p-k)}{p-1}+2s_L-\sum_{i=1}^k|\mu_i|\le 2s_L$. As one has

$$|\mu_k|_1+\frac{(p-k)}{p-1}+2s_L-\sum_{i=1}^k|\mu_i|_1\ge\sigma$$

because the $|\mu_i|_1$ are increasing and $\sum|\mu_i|_1\le 2s_L$, using (D-1)

$$\int\frac{|\partial^{\mu_k}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|_1}}\le C(M)\mathcal{E}_\sigma^{\frac{\sum|\mu_i|-|\mu_k|_1-\frac{2(p-k)}{p-1}}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{2s_L-\sigma-\sum|\mu_i|+|\mu_k|_1+\frac{2(p-k)}{p-1}}{2s_L-\sigma}}.$$

As the coefficients are in increasing order and L is arbitrarily very large, for $1\le j<k$ we have $|\mu_i|+\frac{d}{2}\le 2s_L$. We then recall the L^∞ estimate (D-3):

$$\|\partial^{\mu_i}\varepsilon\|_{L^\infty}\le\sqrt{\mathcal{E}_\sigma}^{\frac{2s_L-|\mu_i|_1-\frac{d}{2}}{2s_L-\sigma}+O(\frac{1}{L}^2)}\sqrt{\mathcal{E}_{2s_L}}^{\frac{|\mu_i|_1+\frac{d}{2}-\sigma}{2s_L-\sigma}+O(\frac{1}{L^2})}.$$

The two previous estimates imply that

$$\begin{aligned}\int\frac{\prod_{i=1}^k|\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|_1}}&\le\int\frac{|\partial^{\mu_k}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|_1}}\prod_{i=1}^{k-1}\|\partial^{\mu_i}\varepsilon\|_{L^\infty}^2\\ &\le\mathcal{E}_\sigma^{\frac{2(k-1)s_L-(k-1)\frac{d}{2}-2\frac{p-k}{p-1}}{2s_L-\sigma}+O(\frac{1}{L^2})}\mathcal{E}_{2s_L}^{\frac{(k-1)\frac{d}{2}-k\sigma+2s_L+2\frac{p-k}{p-1}}{2s_L-\sigma}+O(\frac{1}{L^2})}\\ &\le\mathcal{E}_\sigma^{k-1+\frac{-2+(k-1)(\sigma-s_c)}{2s_L-\sigma}+O(\frac{1}{L^2})}\mathcal{E}_{2s_L}^{1+\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}+O(\frac{1}{L^2})}\\ &\le\mathcal{E}_{2s_L}\left(\frac{\mathcal{E}_\sigma^{1+O(\frac{1}{L})}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{1+\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}.\end{aligned}\tag{5-43}$$

Case 2: $|\mu_k|+\frac{2(p-k)}{p-1}+2s_L-\sum_{i=1}^k|\mu_i|>2s_L$. This means $\frac{2(p-k)}{p-1}-\sum_{i=1}^{k-1}|\mu_i|>0$. Hence, there are two subcases: the subcase $|\mu_i|=0$ for $1\le i\le k-1$ and the subcase $|\mu_{k-1}|=1$ (because the μ_i are ordered by increasing size $|\mu_i|$). If $|\mu_i|=0$ for $1\le i\le k-1$, then, using the weighted L^∞ estimate

(D-2), the coercivity estimate (C-16) and the bound (4-25), we obtain

$$\int \frac{\prod_{i=1}^k |\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|}} = \int \frac{|\varepsilon|^{2(k-1)}}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2|\mu_k|}}$$
$$\leq \left\| \frac{\varepsilon}{1+|y|^{\frac{2(p-k)}{p-1}}} \right\|_{L^\infty}^2 \|\varepsilon\|_{L^\infty}^{2(k-2)} \mathcal{E}_{s_L}$$
$$\leq \left(\frac{\mathcal{E}_\sigma^{1+O(\frac{1}{L})}}{s^{-(\sigma-s_c)}} \right)^{k-1} \frac{C(L,M,K_1,K_2)\mathcal{E}_{s_L}}{s^{1+\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}.$$

If $|\mu_{k-1}|=1$, then, using the weighted L^∞ estimate (D-2) for $\nabla\varepsilon$, the coercivity estimate (C-16) and the bound (4-25), we obtain

$$\int \frac{\prod_{i=1}^k |\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|}} = \int \frac{|\partial^{\mu_{k-1}}\varepsilon|^2|\varepsilon|^{2(k-2)}}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2|\mu_k|-2}}$$
$$\leq \left\| \frac{\partial^{\mu_{k-1}}\varepsilon}{1+|y|^{\frac{2(p-k)}{p-1}-1}} \right\|_{L^\infty}^2 \|\varepsilon\|_{L^\infty}^{2(k-2)} \mathcal{E}_{s_L}$$
$$\leq \left(\frac{\mathcal{E}_\sigma^{1+O(\frac{1}{L})}}{s^{-(\sigma-s_c)}} \right)^{k-1} \frac{C(L,M,K_1,K_2)\mathcal{E}_{s_L}}{s^{1+\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}.$$

In both subcases, we have

$$\int \frac{\prod_{i=1}^k |\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_{i=1}^k|\mu_i|}} \leq \left(\frac{\mathcal{E}_\sigma^{1+O(\frac{1}{L})}}{s^{-(\sigma-s_c)}} \right)^{k-1} \frac{C(L,M,K_1,K_2)\mathcal{E}_{s_L}}{s^{1+\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}. \tag{5-44}$$

Now we come back to (5-41), which we reformulated in (5-42) where we estimated the terms appearing in the sum in (5-43) and (5-44), obtaining the following bound for the nonlinear term's contribution in (5-38):

$$\|H^{s_L}_{z,\frac{1}{\lambda}}(\mathrm{NL}(w_{\mathrm{int}}))\|_{L^2} \leq \frac{\sqrt{\mathcal{E}_{2s_L}}}{\lambda^{(2s_L-s_c)+2}} \sum_{k=2}^p \left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O(\frac{1}{L})}}{s^{-\frac{\sigma-s_c}{2}}} \right)^{k-1} \frac{C(L,M,K_1,K_2)}{s^{1+\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c+L^{-1}}{L})}}. \tag{5-45}$$

The small linear term and the term involving the time derivative of the linearized operator. We claim that there exists a constant $\delta := \delta(d,L,p) > 0$ such that

$$\left\| H^{s_L}_{z,\frac{1}{\lambda}}(L(w_{\mathrm{int}})) + \frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}}) w_{\mathrm{int}} \right\|_{L^2} \leq \frac{C(L,M)}{\lambda^{2s_L-s_c+2}s} \left(\int \frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}} \right)^{\frac{1}{2}}. \tag{5-46}$$

We now prove this estimate. The small linear term is in renormalized variables by (4-36) and (4-37):

$$\int \left|H^{s_L}_{z,\frac{1}{\lambda}}(L(w_{\mathrm{int}}))\right|^2 = \frac{p^2}{\lambda^{2(2s_L-s_c)+4}} \int \left(H^{s_L}((Q^{p-1}-\tilde{Q}_b^{p-1})\varepsilon)\right)^2.$$

For $\mu \in \mathbb{N}^s$, one has the following asymptotic behavior for the potential that appeared, from the bounds on the parameters (4-27) and the expression of $\tilde{Q}_b$ (3-29):

$$\left|\partial^\mu (Q^{p-1} - \tilde{Q}_b^{p-1})\right| \le \frac{1}{s} \frac{C(\mu)}{1+|y|^{\alpha - C(L)\eta + |\mu|}} \le \frac{1}{s} \frac{C(\mu)}{1+|y|^{\delta+|\mu|}}$$

for η small enough, because $\alpha > 2$, and for some constant δ that can be chosen small enough so that

$$0 < \delta \ll 1, \quad \text{with } \delta < \sup_{0 \le n \le n_0} \delta_n \ \text{ and } \ \delta < \tfrac{1}{4} d - \tfrac{1}{2}\gamma_{n_0+1} - s_L. \tag{5-47}$$

(This technical condition is useful to apply a coercivity estimate for the next equation; all the terms appearing are indeed strictly positive by (1-25).) We recall that $H = -\Delta - pQ^{p-1}$, where Q is a smooth potential satisfying

$$|\partial^\mu Q| \le \frac{C(\mu)}{1+|y|^{\frac{2}{p-1}+|\mu|}}.$$

Using the Leibniz rule, this implies

$$\int \left(H^{s_L}((Q^{p-1} - \tilde{Q}_b^{p-1})\varepsilon)\right)^2 \le \frac{C(L)}{s^2} \sum_{\substack{\mu_i \in \mathbb{N}^d \\ |\mu_i| \le 2s_L,\, i=1,2}} \int \frac{|\partial^{\mu_1}\varepsilon||\partial^{\mu_2}\varepsilon|}{1+|y|^{4s_L + 2\delta - 2|\mu_1| - 2|\mu_2|}} \le \frac{C(L)}{s^2} \int \frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}, \tag{5-48}$$

where we used for the last line the weighted coercivity estimate (C-16), which we could apply because δ satisfies the technical condition (5-47). We now turn to the term involving the time derivative of the linearized operator in (5-38). Going back to renormalized variables, it can be written as

$$\int \left|\frac{d}{dt} H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}}\right|^2 = \frac{p^2(p-1)^2}{\lambda^{2(2s_L - s_c)+4}} \sum_{i=1}^{s_L} \int \left(H^{i-1}\left[\left(Q^{p-2}\frac{z_s}{\lambda}.\nabla Q + \frac{\lambda_s}{\lambda} Q^{p-2}\Lambda Q\right) H^{s_L - i}\varepsilon\right]\right)^2.$$

For $\mu \in \mathbb{N}^d$, one has the following asymptotic behavior for the two potentials that appeared (from the asymptotic (2-1) and (2-7) of Q and ΛQ):

$$|\partial^\mu (Q^{p-2}\partial_{y_i} Q)| \le \frac{C(\mu)}{1+|y|^{2+1+|\mu|}} \quad \text{for } 1 \le i \le d, \qquad \text{and} \qquad |\partial^\mu (Q^{p-2}\Lambda Q)| \le \frac{C(\mu)}{1+|y|^{2+\alpha}}.$$

Therefore, as $H = -\Delta - pQ^{p-1}$, where Q is a smooth potential satisfying

$$|\partial^\mu Q| \le \frac{C(\mu)}{1+|y|^{\frac{2}{p-1}+|\mu|_1}},$$

using the Leibniz rule and the two above identities,

$$\left|\int H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}} \frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}}) w_{\text{int}}\right| \leq \frac{C(L)\left(\left|\frac{\lambda_s}{\lambda}\right|^2+\left|\frac{z_s}{\lambda}\right|^2\right)}{\lambda^{2(2s_L-s_c)+4}} \sum_{\substack{\mu_i\in\mathbb{N}^d\\ |\mu_i|_1\leq 2s_L,\, i=1,2}} \int \frac{|\partial^{\mu_1}\varepsilon||\partial^{\mu_2}\varepsilon|}{1+|y|^{4s_L+2-2|\mu_1|-2|\mu_2|}}$$
$$\leq \frac{C(L)}{\lambda^{2(2s_L-s_c)+4}s^2} \sum_{\substack{\mu_i\in\mathbb{N}^d\\ |\mu_i|_1\leq 2s_L,\, i=1,2}} \int \frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}} \tag{5-49}$$

for $\delta<\alpha,1$ being defined by (5-47), where we used the weighted coercivity estimate (C-16) and the fact that $\left|\frac{\lambda_s}{\lambda}\right|\sim s^{-1}$ and $\left|\frac{z_s}{\lambda}\right|\sim s^{-1-\frac{\alpha-1}{2}}$ by (4-43) and (4-27). We now combine the estimates we have proved, (5-48) and (5-49), to obtain the estimate (5-46) we claimed.

End of the proof of Step 1. We now gather the brute force upper bounds we have found for the terms we had to treat in (5-40), (5-45) and (5-46), yielding the bound (5-39) we claimed in this first step.

Step 2: integration by parts in time to treat the modulation term. We now focus on the modulation term in (5-38) which requires a careful treatment. Indeed, the brute force upper bounds on the modulation (4-43) are not sufficient and we need to make an integration by parts in time to treat the problematic term $b^{(n,k)}_{L_n,s}$. We do this in two steps. First we define a radiation term. Next we use it to prove a modified energy estimate.

Definition of the radiation. We recall that $\alpha_b=\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)}+\sum_{i=2}^{L+2}S_i$, where $T_i^{(n,k)}$ is defined by (2-26) and S_i is homogeneous of degree $(i,-\gamma-g')$ in the sense of Definition 2.14; see (3-8). We want to split α_b in two parts to distinguish the problematic terms involving the parameters $b^{(n,k)}_{L_n}$. For $i=2,\dots,L+2$, as S_i is homogeneous of degree $(i,-\gamma-g')$, it is a finite sum

$$S_i=\sum_{J\in\mathcal{J}(i)} b^J f_J, \quad \text{with } b^J=\prod_{(n,k,i)\in\mathcal{I}} (b_i^{(n,k)})^{J_i^{(n,k)}}, \tag{5-50}$$

where $\mathcal{J}(i)$ is a finite subset of $\mathbb{N}^{\#\mathcal{I}}$ and for all $J\in\mathcal{J}(i)$, $|J|_3=i$ and f_J is admissible of degree $(2|J|_2-\gamma-g')$ in the sense of Definition 2.11. We then define the following partition of $\mathcal{J}(i)$:

$$\begin{gathered}\mathcal{J}_1(i):=\{J\in\mathcal{J}(i),\ J^{(n,k)}_{L_n}=0 \text{ for all } 0\leq n\leq n_0,\ 1\leq k\leq k(n)\},\\ \mathcal{J}_2(i):=\{J\in\mathcal{J}(i),\ |J|=2 \text{ and } \exists (n,k,L_n)\in\mathcal{I},\ J^{(n,k)}_{L_n}\geq 1\},\\ \mathcal{J}_3(i):=\mathcal{J}(i)\backslash[\mathcal{J}_1(i)\cup\mathcal{J}_2(i)],\\ \bar{S}_i:=\sum_{J\in\mathcal{J}_2(i)} b^J f_J,\quad \bar{S}'_i:=\sum_{J\in\mathcal{J}_3(i)} b^J f_J,\end{gathered} \tag{5-51}$$

and the following radiation term:

$$\xi:=H^{s_L}\left(\chi_{B_1}\left[\sum_{\substack{0\leq n\leq n_0\\ 1\leq k\leq k(n)}} b^{(n,k)}_{L_n}T^{(n,k)}_{L_n}+\sum_{i=2}^{L+2}\bar{S}'_i\right]\right)+\sum_{i=2}^{L+2}H^{s_L}(\chi_{B_1}\bar{S}_i)-\chi_{B_1}H^{s_L}\bar{S}_i. \tag{5-52}$$

From (5-51), for all $J \in \mathcal{J}_3(i)$ there exists n with $0 \le n \le n_0$ such that $J_{L_n}^{(n,k)} \ge 1$ and $|J| \ge 3$. As $\delta_{n'} > 0$, this implies

$$\forall J \in \mathcal{J}_3(i), \quad |J|_2 > L + 2 - \delta_0. \tag{5-53}$$

Using this fact, (2-7), the fact that $H^{s_L} T_{L_n}^{(n,k)} = 0$ since $s_L > L_n$ for all $0 \le n \le n_0$, (5-51) and (4-27), the radiation satisfies

$$\|\xi\|_{L^2} \le \frac{C(L,M)}{s^{L+1-\delta_0+\eta(1-\delta_0')}}, \qquad \|H\xi\|_{L^2} \le \frac{C(L,M)}{s^{L+2-\delta_0+\eta(2-\delta_0')}}, \tag{5-54}$$

$$\|\nabla\xi\|_{L^2} \le \frac{C(L,M)}{s^{L+\frac{3}{2}-\delta_0+\eta(\frac{3}{2}-\delta_0')}}, \qquad \|\Lambda\xi\|_{L^2} \le \frac{C(L,M)}{s^{L+1-\delta_0+\eta(1-\delta_0')}}. \tag{5-55}$$

We eventually introduce the remainders

$$\begin{aligned}
R_1 := {} & H^{s_L}\Biggl(\chi_{B_1} \sum_{(n,k,i)\in\mathcal{I},\, i\neq L_n} \bigl(b_{i,s}^{(n,k)} + (2i-\alpha_n) b_i^{(n,k)} b_1^{(0,1)} - b_{i+1}^{(n,k)}\bigr)\Biggl(T_i^{(n,k)} + \sum_{j=2}^{L+2} \frac{\partial S_j}{\partial b_i^{(n,k)}}\Biggr)\Biggr) \\
& - \left(\frac{\lambda_s}{\lambda} + b_1^{(0,1)}\right) H^{s_L} \Lambda \tilde{Q}_b - \left(\frac{z_s}{\lambda} + b_1^{(1,\cdot)}\right) . H^{s_L} \nabla \tilde{Q}_b \\
& + H^{s_L}\Biggl(\chi_{B_1} \sum_{(n,k,L_n)\in\mathcal{I}} (2L_n - \alpha_n) b_{L_n}^{(n,k)} b_1^{(0,1)} \Biggl(T_{L_n}^{(n,k)} + \sum_{j=2}^{L+2} \frac{\partial \bar{S}_j'}{\partial b_{L_n}^{(n,k)}}\Biggr)\Biggr) \\
& + \sum_{(n,k,L_n)\in\mathcal{I}} (2L_n - \alpha_n) b_{L_n}^{(n,k)} b_1^{(0,1)} \Biggl(\sum_{j=2}^{L+2} H^{s_L}\Bigl(\chi_{B_1} \frac{\partial \bar{S}_j}{\partial b_{L_n}^{(n,k)}}\Bigr) - \chi_{B_1} H^{s_L} \frac{\partial \bar{S}_j}{\partial b_{L_n}^{(n,k)}}\Biggr) \\
R_2 := {} & \sum_{(n,k,L_n)\in\mathcal{I}} \bigl(b_{L_n,s}^{(n,k)} + (2L_n - \alpha_n) b_{L_n}^{(n,k)} b_1^{(0,1)}\bigr) \Biggl(\sum_{j=2}^{L+2} \chi_{B_1} H^{s_L} \frac{\partial \bar{S}_j}{\partial b_{L_n}^{(n,k)}}\Biggr), \\
R_3 := {} & \sum_{(n,k,i)\in\mathcal{I},\, i\neq L_n} b_{i,s}^{(n,k)} \frac{\partial}{\partial_{b_i^{(n,k)}}} \xi,
\end{aligned}$$

so that they produce, by (5-52) and (4-33), the identity

$$H^{s_L}(\widetilde{\mathrm{Mod}}(s)) = \partial_s \xi + R_1 + R_2 + R_3. \tag{5-56}$$

The remainder R_1 enjoys the following bounds by (4-43), (2-22), (3-8), (5-51), (5-53) and (4-27):

$$\|R_1\|_{L^2} \le \frac{C(L,M)}{s^{L+2-\delta_0+(1-\delta_0')\eta}} + \frac{C(L,M)\mathcal{E}_{2s_L}}{s^2}. \tag{5-57}$$

From the definition (5-51) of $\mathcal{S}_j$ and the construction (3-25) of S_j, one has

$$\begin{aligned}
\sum_{j=2}^{L+2} H\bar{S}_j = {} & - \sum_{(n,k,L_n)\in\mathcal{I}} b_1^{(0,1)} b_{L_n}^{(n,k)} \bigl(\Lambda T_{L_n}^{(n,k)} - (2L_n - \alpha_n) T_{L_n}^{(n,k)}\bigr) - \sum_{(n,k,L_n)\in\mathcal{I}} b_{L_n}^{(n,k)} b_1^{(1,\cdot)} . \nabla \Lambda T_{L_n}^{(n,k)} \\
& + p(p-1) Q^{p-2} \Biggl(\sum_{(n,k,L_n)\in\mathcal{I}} b_{L_n}^{(n,k)} T_{L_n}^{(n,k)}\Biggr)\Biggl(\sum_{(n',k',i)\in\mathcal{I}} b_i^{(n',k')} T_i^{(n',k')}\Biggr).
\end{aligned}$$

As $H^{s_L}T^{(n,k)}_{L_n}=0$ since $s_L>L_n$ for all $0\leq n\leq n_0$, using the commutator identity (2-24), the asymptotic (2-22) of $T^{(n,k)}_i$, (4-27) and (2-2) (as $\alpha>2$), one has

$$\int(1+|y|^{4+2\delta})\left(\chi_{B_1}H^{s_L+1}\sum_{j=2}^{L+2}\frac{\partial \bar{S}_j}{\partial_{b^{(n,k)}_{Ln}}}\right)^2\leq\frac{C(L)}{s},$$

where δ is defined by (5-47), from which we deduce, using (4-44),

$$\|(1+|y|)^{2+\delta}HR_2\|_{L^2}\leq\frac{C(L,M)}{s^{L+4}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s}. \tag{5-58}$$

Finally for the last remainder, from (5-52), (4-43), (4-27), (4-25), (2-22) and (5-51), for s_0 large enough one has the estimate

$$\|R_3\|_{L^2}\leq\frac{C(L,M)}{s^{L+2-\delta_0+\eta(1-\delta_0')}}. \tag{5-59}$$

Modified energy estimate. We now prove the modified energy estimate (compared to (5-38))

$$\begin{aligned}\frac{d}{dt}&\left\{\int(H^{s_L}_{z,\frac1\lambda}w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac1\lambda}))^2\right\}\\&\leq\frac{1}{\lambda^{2(2s_L-s_c)+2}s}\left[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}}+C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac12}\right.\\&\qquad\left.+\mathcal{E}_{2s_L}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O(\frac1L)}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c}{L})}}\right]-2\int H^{s_L}_{z,\frac1\lambda}w_{\text{int}}H^{s_L+1}_{z,\frac1\lambda}w_{\text{int}}\\&\qquad+2\int H^{s_L}_{z,\frac1\lambda}w_{\text{int}}H^{s_L}_{z,\frac1\lambda}(\tilde{L}+\tilde{R}+\widetilde{\text{NL}}).\end{aligned} \tag{5-60}$$

From the time evolution (5-56), (4-32) of ξ and w and because the support of $\tau_z(\xi_{1/\lambda})$ is disjoint from the one of $\tilde{L}$, $\tilde{R}$, and $\widetilde{\text{NL}}$, one gets the following expression for the left-hand side of (5-60):

$$\begin{aligned}\frac{d}{dt}&\left\{\int\left(H^{s_L}_{z,\frac1\lambda}w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac1\lambda})\right)^2\right\}\\&=-2\int H^{s_L}_{z,\frac1\lambda}w_{\text{int}}H^{s_L+1}_{z,\frac1\lambda}w_{\text{int}}-\frac{2}{\lambda^{2s_L+2}}\int H^{s_L}_{z,\frac1\lambda}w_{\text{int}}\tau_z(R_{2,\frac1\lambda})-\frac{2}{\lambda^{2s_L}}\int\tau_z(\xi_{\frac1\lambda})H^{s_L+1}_{z,\frac1\lambda}w_{\text{int}}\\&\quad+2\int\left[H^{s_L}_{z,\frac1\lambda}w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac1\lambda})\right]\left[H^{s_L}_{z,\frac1\lambda}\left(\text{NL}(w_{\text{int}})-\frac{1}{\lambda^2}\tau_z(\tilde{\psi}_{b,\frac1\lambda}+(\chi-1)\widetilde{\text{Mod}}(t)_{\frac1\lambda})+L(w_{\text{int}})\right)\right.\\&\qquad\left.+\frac{d}{dt}(H^{s_L}_{z,\frac1\lambda})w_{\text{int}}-\frac{1}{\lambda^{2+2s_L}}\tau_z\left(\left(R_1+R_3+\frac{\lambda_s}{\lambda}\Lambda\xi+2s_L\frac{\lambda_s}{\lambda}\xi-\frac{z_s}{\lambda}.\nabla\xi\right)_{\frac1\lambda}\right)\right]\\&\quad-\frac{2}{\lambda^{4s_L+2}}\int\tau_z(\xi_{\frac1\lambda})\tau_z(R_{2,\frac1\lambda})+2\int H^{s_L}_{z,\frac1\lambda}w_{\text{int}}H^{s_L}_{z,\frac1\lambda}(\tilde{L}+\widetilde{\text{NL}}+\tilde{R}).\end{aligned} \tag{5-61}$$

We now analyse all the terms in this identity, except the first one and the last one, which we will study in the next step. Using the estimate (5-58) on the remainder R_2, going back in renormalized variables and using the coercivity (C-16), one gets for the second term in (5-61)

$$\left|\frac{2}{\lambda^{2s_L+2}}\int H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}}\tau_z(R_{2,\frac{1}{\lambda}})\right| \le C\int \frac{|H^{s_L-1}\varepsilon|}{1+|y|^{2+\delta}}(1+|y|^{2+\delta})|HR_2|$$
$$\le \frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{\lambda^{2(2s_L-s_c)+2}s}\left(\left(\int \frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}}+\frac{1}{s^{L+3}}\right).$$

Going back to renormalized variables, integrating by parts and using the estimate (5-54) on $H\xi$ gives for the third term in (5-61)

$$\left|\frac{2}{\lambda^{2s_L}}\int \tau_z(\xi_{\frac{1}{\lambda}})H^{s_L+1}_{z,\frac{1}{\lambda}} w_{\text{int}}\right| \le \frac{C(L,M)}{\lambda^{2(2s_L-s_c)+2}}\frac{\sqrt{\mathcal{E}_{2s_L}}}{s^{L+2-\delta_0+\eta(2-\delta_0')}}.$$

To bound the fourth and the fifth terms in (5-61) from above, we go back to renormalized variables and use the bound (5-39) on the error, the nonlinear term, the small linear term and the term involving the time derivative of the linearized operator we derived in Step 1, together with the bounds (5-54) and (5-55) on ξ, $\Lambda\xi$, $\nabla\xi$ and the fact that

$$\left|\frac{\lambda_s}{\lambda}\right| \le Cs^{-1} \quad\text{and}\quad \left|\frac{z_s}{\lambda}\right| \le Cs^{-1-\frac{\alpha-1}{2}}$$

in the trapped regime, and the bound (5-57) and (5-59) on the remainders R_1 and R_3, yielding

$$\Bigg|\int\left[H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}})\right]\bigg[H^{s_L}_{z,\frac{1}{\lambda}}\big(\text{NL}(w_{\text{int}})-\frac{1}{\lambda^2}\tau_z(\tilde{\psi}_{b,\frac{1}{\lambda}}+(\chi-1)\widetilde{\text{Mod}}(t)_{\frac{1}{\lambda}})+L(w_{\text{int}})\big)$$
$$+\frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w-\frac{1}{\lambda^{2+2s_L}}\tau_z\bigg(\bigg(R_1+R_3+\frac{\lambda_s}{\lambda}\Lambda\xi+2s_L\frac{\lambda_s}{\lambda}\xi-\frac{z_s}{\lambda}.\nabla\xi\bigg)_{\frac{1}{\lambda}}\bigg)\bigg]-\frac{2}{\lambda^{4s_L+2}}\int\tau_z(\xi_{\frac{1}{\lambda}})\tau_z(R_{1,\frac{1}{\lambda}})\Bigg|$$
$$\le \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\Bigg[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}}+C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|x|^{2\delta}}\right)^{\frac{1}{2}}$$
$$+\mathcal{E}_{2s_L}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O(\frac{1}{L})}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c}{L})}}\Bigg].$$

We finish the proof of the bound (5-60) by inserting into the identity (5-61) the three previous bounds we proved on the second, third, fourth and fifth terms.

Step 3: use of dissipation. We find an upper bound for the last terms in (5-60) and improve the energy estimate using the coercivity of the quantity $-\int H^{s_L+1}\varepsilon H^{s_L}\varepsilon$.

The dissipation estimate. We recall that $H=-\Delta-pQ^{p-1}$, the potential $-pQ^{p-1}$ being the Hardy potential

$$pQ^{p-1} < \frac{(d-2)^2-4\delta(p)}{4|y|^2}$$

for some constant $\delta(p)>0$ by (2-5). Hence, using the standard Hardy inequality one gets for the linear term

$$
\begin{aligned}
-\int H^{s_L}_{z,\frac{1}{\lambda}} w_{\mathrm{int}} H_{z,\frac{1}{\lambda}} H^{s_L}_{z,\frac{1}{\lambda}} w_{\mathrm{int}}
&= -\frac{1}{\lambda^{2(2s-L-s_c)+2}} \int H^{s_L}\varepsilon H H^{s_L}\varepsilon \\
&= \frac{1}{\lambda^{2(2s-L-s_c)+2}} \left(-\int |\nabla H^{s_L}\varepsilon|^2 + \int pQ^{p-1}|H^{s_L}\varepsilon|^2 \right) \\
&= \frac{1}{\lambda^{2(2s-L-s_c)+2}} \left(\left[\frac{(d-2)^2-\frac{1}{2}\delta(p)}{(d-2)^2} + \frac{\delta(p)}{2(d-2)^2} \right] \int |\nabla H^{s_L}\varepsilon|^2 + \int pQ^{p-1}|H^{s_L}\varepsilon|^2 \right) \\
&\leq \frac{1}{\lambda^{2(2s-L-s_c)+2}} \left(-\frac{(d-2)^2-\frac{1}{2}\delta(p)}{4} \int \frac{|H^{s_L}\varepsilon|^2}{|y|^2} - \frac{\delta(p)}{2(d-2)^2} \int |\nabla H^{s_L}\varepsilon|^2 \right. \\
&\qquad\qquad \left. + \frac{(d-2)^2-\delta(p)}{4} \int \frac{|H^{s_L}\varepsilon|^2}{|y|^2} \right) \\
&= -\frac{\delta(p)}{8\lambda^{2(2s-L-s_c)+2}} \int \frac{|H^{s_L}\varepsilon|^2}{|y|^2} - \frac{\delta(p)}{2(d-2)^2\lambda^{2(2s-L-s_c)+2}} \int |\nabla H^{s_L}\varepsilon|^2. \qquad (5\text{-}62)
\end{aligned}
$$

Bounds for the terms created by the cut. We study the last terms in (5-60). From its definition (4-40), and as $\lambda+|z|\ll 1$ by (4-52) and (4-51), the remainder $\widetilde{R}$ is bounded by a constant independent of the others:

$$
\|H^{s_L}_{z,\frac{1}{\lambda}} \widetilde{R}\|_{L^2} \leq C. \qquad (5\text{-}63)
$$

For the nonlinear term, for any very small $\kappa>0$, by (D-4), (4-39) and (4-28),

$$
\begin{aligned}
\|H^{s_L}_{z,\frac{1}{\lambda}} \widetilde{\mathrm{NL}}\|_{L^2} &\leq C \sum_{k=2}^{p} \|w^k\|_{H^{2s_L}} \\
&\leq C\|w\|_{H^{2s_L}} \sum_{k=2}^{p} \|w\|^{k-1}_{H^{d/2+\kappa}} \\
&\leq C\|w\|_{H^{2s_L}} \sum_{k=2}^{p} \|w\|_{H^{\sigma}}^{(k-1)\left(1-\frac{d/2+\kappa-\sigma}{2s_L-\sigma}\right)} \|w\|_{H^{2s_L}}^{(k-1)\left(\frac{d/2+\kappa-\sigma}{2s_L-\sigma}\right)} \\
&\leq C(K_1,K_2) \left(\frac{1}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}} \right)^{1+(p-1)\frac{d/2+\kappa-\sigma}{2s_L-\sigma}} \\
&= C(K_1,K_2) \left(\frac{1}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}} \right)^{1+(p-1)\frac{2/(p-1)-\sigma-s_c+\kappa}{2s_L-\sigma}} \\
&\leq C(K_1,K_2) \left(\frac{1}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}} \right)^{1+\frac{2}{2s_L-\sigma}} \\
&= \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c+2} s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}} \qquad (5\text{-}64)
\end{aligned}
$$

because $1/\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')} \gg 1$ by (4-52), if κ has been chosen small enough. For the extra linear term in (5-60), performing an integration by parts, using Young's inequality for any $\varepsilon > 0$, (4-25) and (4-28) give

$$\begin{aligned}\left|\int H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}} H^{s_L}_{z,\frac{1}{\lambda}} \widetilde{L}\right| &= \left|\int H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}} H^{s_L}_{z,\frac{1}{\lambda}} \big[-\Delta\chi_3 w - 2\nabla\chi_3.\nabla w + p\tau_z Q^{p-1}_{\frac{1}{\lambda}}(\chi_1^{p-1}-\chi_3)w\big]\right| \\ &\le C\|H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}}\|_{L^2}\|w\|_{H^{2s_L}} + C\varepsilon\|\nabla H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}}\|^2_{L^2} + \frac{C}{\varepsilon}\|w_{\text{int}}\|^2_{H^{2s_L}} \\ &\le C\varepsilon\|\nabla H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}}\|^2_{L^2} + \frac{C(K_1,K_2,\varepsilon)}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}} \\ &\le \frac{C\varepsilon}{\lambda^{2(2s-L-s_c)+2}}\int|\nabla H^{s_L}\varepsilon|^2 + \frac{C(K_1,K_2,\varepsilon)}{\lambda^{2(2s_L-s_c)+2}s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}} \qquad (5\text{-}65)\end{aligned}$$

because in the trapped regime $\lambda^2 s \sim s^{-\frac{\alpha}{2\ell-\alpha}}$ by (4-52).

Conclusion. We insert into the modified energy estimate (5-60) the bounds (5-62), (5-63), (5-64) and (5-65), yielding

$$\begin{aligned}&\frac{d}{dt}\left\{\int\left(H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}} + \frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}})\right)^2\right\} \\ &\quad\le \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\Bigg[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}} + \frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}} + C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}} \\ &\qquad + \mathcal{E}_{2s_L}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c}{L})}} - \frac{s\delta(p)}{8}\int\frac{|H^{s_L}\varepsilon|^2}{|y|^2} - \frac{s\delta(p)}{2(d-2)^2}\int|\nabla H^{s_L}\varepsilon|^2 \\ &\qquad + C\varepsilon s\int|\nabla H^{s_L}\varepsilon|^2 + \frac{C(K_1,K_2,M,L)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}\Bigg]. \qquad (5\text{-}66)\end{aligned}$$

For any $N \gg 1$, using Young's inequality and splitting the weighted integrals in the zone $|y| \le N^2$ and $|y| \ge N^2$ gives for ε small enough and s_0 large enough,

$$\begin{aligned}&C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}} - \frac{s\delta(p)-sC\varepsilon}{8}\int\frac{|H^{s_L}}{|y|^2} \\ &\qquad \le \frac{C(L,M)\mathcal{E}_{2s_L}}{N^{2\delta}} + C(L,M)N^{2\delta}\int_{|y|\le N^2}\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}} - \frac{s\delta(p)}{16}\int\frac{|H^{s_L}\varepsilon|^2}{|y|^2} \le \frac{C(L,M)\mathcal{E}_{2s_L}}{N^{2\delta}}.\end{aligned}$$

Finally, from the bound (5-54) on the size of ξ, one has

$$\begin{aligned}&\frac{d}{dt}\left\{\int\left(H^{s_L}_{z,\frac{1}{\lambda}} w + \frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}})\right)^2\right\} \\ &\quad = \frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}\right\} + \frac{d}{dt}\left\{\int\frac{2}{\lambda^{2s_L}}H^{s_L}_{z,\frac{1}{\lambda}} w\tau_z(\xi_{\frac{1}{\lambda}}) + \frac{1}{\lambda^{4s_L}}(\tau_z(\xi_{\frac{1}{\lambda}}))^2\right\} \\ &\quad = \frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}\right\} + \frac{d}{dt}\left\{O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}\left(\sqrt{\mathcal{E}_{2s_L}} + \frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)\right)\right\},\end{aligned}$$

where $O_{L,M}(\cdot)$ denotes the usual $O(\cdot)$ for a constant in the upper bound that depends only on L and M. Plugging the two previous identities into the modified energy estimate (5-66) yields the bound (5-37) we claimed in this proposition. $\square$

Proposition 5.8 (Lyapunov monotonicity for the high regularity Sobolev norm of the remainder outside the blow-up zone). *Suppose all the constants of Proposition 4.6 are fixed except s_0. Then for s_0 large enough, for any solution u that is trapped on $[s_0, s')$ the following holds for $0 \le t < t(s')$:*

$$\|w_{\text{ext}}\|^2_{H^{2s_L}} \le \|\partial_t^{s_L} w_{\text{ext}}(0)\|^2_{L^2} + \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}} dt' + \int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}} dt' + \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}. \tag{5-67}$$

Proof. From the time evolution of w_{ext}, given in Section 4B1, we get

$$\partial_t^{k+1} w_{\text{ext}} = \Delta \partial_t^k w_{\text{ext}} + (1-\chi_3)\partial_t^k(w^p) + \Delta\chi_3 \partial_t^k w + 2\nabla\chi_3.\nabla\partial_t^k w. \tag{5-68}$$

We make an energy estimate for $\partial_t^{s_L} w_{\text{ext}}$ and propagate this bound via elliptic regularity by iterations, which is standard in the study of parabolic problems. All computations, unless mentioned, are performed on Ω, and we omit this in the notation for simplicity.

Step 1: estimate on the force terms. We first prove some estimates on the force terms on the right-hand side of (5-68). From the decomposition (4-10) and the evolution (4-32) of w, in the exterior zone $\Omega\backslash\mathcal{B}^d(2)$, $\partial_t^k w$ can be written as

$$\partial_t^k w = \sum_{j=0}^{k} \sum C(\mu) \prod_{i=1}^{1+j(p-1)} \partial^{\mu_i} w \tag{5-69}$$

for some constants $C(\mu)$, where the inner sum is over $\mu = (\mu_i)_{1\le i\le 1+j(p-1)} \in \mathbb{N}^{dk(p-1)}$ with $\sum_{i=1}^{1+j(p-1)} |\mu_i|_1 = 2(k-j)$. Fix $k \le s_L$, an integer j with $0 \le j \le k$, and a sequence of d-tuples $(\mu_i)_{1\le i\le 1+k(p-1)} \in \mathbb{N}^{dk(p-1)}$ satisfying $\sum_{i=1}^{1+j(p-1)} |\mu_i| = 2(k-j)$. One can assume that the d-tuples μ_i are ordered by decreasing length: $|\mu_1| \ge |\mu_2| \ge \cdots$.

The case $k = s_L$. We want to estimate the above term in the zone $\Omega\backslash\mathcal{B}^d(2)$.

<u>Subcase 1:</u> $|\mu_1| \ge \sigma$. Using Hölder, Sobolev embedding (since in that case $\mu_i < 2s_L - \frac{d}{2}$ for $2 \le i \le 1+j(p-1)$), interpolation and (4-28), for $\kappa > 0$ small enough,

$$\left\| \prod_{i=1}^{1+j(p-1)} \partial^{\mu_i} w \right\|_{L^2} \le \|\partial^{\mu_1} w\|_{L^2} \prod_{i=2}^{1+j(p-1)} \|\partial^{\mu_i} w\|_{L^\infty} \le \|w\|_{H^{|\mu_1|}} \prod_{i=2}^{1+j(p-1)} \|w\|_{H^{d/2+\kappa+|\mu_i|}}$$

$$
\begin{aligned}
&\leq C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{|\mu_1|-\sigma+\sum_{i=2}^{1+j(p-1)}|\mu_i|+d/2+\kappa-\sigma}{2s_L-\sigma}}\\
&= C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{1-\frac{(j(p-1)-1)(\sigma-s_c-\kappa)}{2s_L-\sigma}}\\
&\leq \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}},
\end{aligned}
\tag{5-70}
$$

as $1/\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}\gg 1$ by (4-52).

Subcase 2: $|\mu_1|<\sigma$. Then $\mu_i<\sigma$ for all $1\leq i\leq j(p-1)$ and $\partial^{\mu_i}w\in L^{p_i}$ with p_i given by

$$\frac{1}{p_i}=\frac{1}{2}-\frac{\sigma-|\mu_i|}{d}$$

by Sobolev embedding. We define i_0 as the integer $2\leq i_0\leq 1+j(p-1)$ such that $\sum_{i=1}^{i_0-1}\frac{1}{p_i}<\frac{1}{2}$ and $\sum_{i=1}^{i_0}\frac{1}{p_i}\geq\frac{1}{2}$. We know i_0 exists because $\frac{1}{p_1}<\frac{1}{2}$ and $\sum_{i=1}^{1+j(p-1)}\frac{1}{p_i}\gg\frac{1}{2}$. We define $\tilde{p}_{i_0}>2$ by $\frac{1}{\tilde{p}_{i_0}}=\frac{1}{2}-\sum_{i=1}^{i_0-1}\frac{1}{p_i}$ and $\tilde{s}\geq\sigma$ as the regularity giving the Sobolev embedding $H^{\tilde{s}-|\mu_{i_0}|}\to L^{\tilde{p}_{i_0}}$:

$$\tilde{s}=\sum_{i=1}^{i_0}|\mu_i|+(i_0-1)\left(\frac{d}{2}-\sigma\right).$$

This implies that $\prod_{i=1}^{i_0}\partial^{\mu_i}w\in L^2$ with the estimate (from Hölder inequality)

$$\left\|\prod_{i=1}^{i_0}\partial^{\mu_i}w\right\|_{L^2}\leq C\|\partial^{\mu_{i_0}}w\|_{L^{\tilde{p}_{i_0}}}\prod_{i=1}^{i_0-1}\|\partial^{\mu_i}w\|_{L^{p_i}}\leq\|w\|_{H^{\tilde{s}}}\prod_{i=1}^{i_0-1}\|w\|_{H^\sigma}\leq C(K_1)\|w\|_{H^{2s_L}}^{\frac{\tilde{s}-\sigma}{2s_L-\sigma}},$$

where we used interpolation and (4-25). Therefore, for $\kappa>0$ small enough, using Sobolev embedding, the above estimate, interpolation and (4-25),

$$
\begin{aligned}
\left\|\prod_{i=1}^{1+j(p-1)}\partial^{\mu_i}w\right\|_{L^2}&\leq\left\|\prod_{i=1}^{i_0}\partial^{\mu_i}w\right\|_{L^2}\prod_{i=i_0+1}^{1+j(p-1)}\|w\|_{H^{\frac{d}{2}+\kappa+|\mu_i|}}\\
&\leq C(K_1)\|w\|_{H^{2s_L}}^{\frac{\tilde{s}-\sigma}{2s_L-\sigma}}\prod_{i=i_0+1}^{1+j(p-1)}\|w\|_{H^\sigma}^{1-\frac{d/2+\kappa+|\mu_i|-\sigma}{2s_L-\sigma}}\|w\|_{H^{2s_L}}^{\frac{d/2+\kappa+|\mu_i|-\sigma}{2s_L-\sigma}}\\
&\leq C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{2s_L-\sigma-j(p-1)(\sigma-s_c)+(j(p-1)-i_0+1)\kappa}{2s_L-\sigma}}\\
&\leq C(K_1,K_2)\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}
\end{aligned}
\tag{5-71}
$$

as $1/\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}\gg 1$ by (4-52).

End of substep 1. Inserting (5-70) and (5-71) into the identity we obtain

$$\|\partial_t^{s_L} w\|_{L^2(\Omega\setminus\mathcal{B}^d(2))} \leq \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}}. \tag{5-72}$$

Estimate for the nonlinear term in (5-68). With the very same arguments used in the first substep, one obtains the bound

$$\|\partial_t^{s_L} w^p\|_{L^2(\Omega\setminus\mathcal{B}^d(2))} \leq \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c+2} s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}. \tag{5-73}$$

The case $k<s_L$. Again, for $0\leq k<s_L$, the same method yields

$$\|\partial_t^k w\|_{H^{2(s_L-1-k)}(\Omega\setminus\mathcal{B}^d(2))} \leq \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}+O(\frac{1}{L})}}, \tag{5-74}$$

$$\|\nabla\partial_t^k w\|_{H^{2(s_L-1-k)}(\Omega\setminus\mathcal{B}^d(2))} \leq \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2(2\ell-\alpha)}+O(\frac{1}{L})}}, \tag{5-75}$$

$$\|\partial_t^k w^p\|_{H^{2(s_L-1-k)}(\Omega\setminus\mathcal{B}^d(2))} \leq \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O(\frac{\sigma-s_c+\eta}{L})}}. \tag{5-76}$$

Step 2: energy estimate for $\partial_t^{s_L} w_{\mathrm{ext}}$. We claim that for $0\leq t<t'$,

$$\begin{aligned}\|\partial_t^{s_L} w_{\mathrm{ext}}\|_{L^2}^2 \leq \|\partial_t^{s_L} w_{\mathrm{ext}}(0)\|_{L^2}^2 &+ \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}\,dt' \\ &+ \int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\mathrm{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}\,dt' \end{aligned} \tag{5-77}$$

and we now prove this estimate. From (5-68) one has the identity

$$\begin{aligned}&\partial_t(\|\partial_t^{s_L} w_{\mathrm{ext}}\|_{L^2}^2) \\ &\quad = -2\int |\nabla\partial_t^{s_L} w_{\mathrm{ext}}|^2 + 4\int \partial_t^{s_L} w_{\mathrm{ext}}\nabla\chi_3.\nabla\partial_t^{s_L} w + 2\int \partial_t^{s_L} w_{\mathrm{ext}}\partial_t^{s_L}((1-\chi_3)w^p+\Delta\chi_3 w)\end{aligned} \tag{5-78}$$

and we are now going to study the right-hand side of this equation.

Use of dissipation. We study all the terms except the nonlinear one in (5-78). After an integration by parts, using Cauchy–Schwarz, Young's and Poincare's inequalities,

$$\begin{aligned}&\left|\int \partial_t^{s_L} w_{\mathrm{ext}}\nabla\chi_3.\nabla\partial_t^{s_L} w + \int \partial_t^{s_L} w_{\mathrm{ext}}\partial_t^{s_L}(\Delta\chi_3 w)\right| \\ &\qquad = \left|-\int \Delta\chi_3\partial_t^{s_L} w\partial_t^{s_L} w_{\mathrm{ext}} - \nabla\chi_3.\nabla\partial_t^{s_L} w_{\mathrm{ext}}\partial_t^{s_L} w + \int \partial_t^{s_L} w_{\mathrm{ext}}\partial_t^{s_L}(\Delta\chi_3 w)\right| \\ &\qquad \leq C\big[\|(1-\chi_2)\partial_t^{s_L} w\|_{L^2}\|\partial_t^{s_L} w_{\mathrm{ext}}\|_{L^2} + \|(1-\chi_2)\partial_t^{s_L} w\|_{L^2}\|\nabla\partial_t^{s_L} w_{\mathrm{ext}}\|_{L^2}\big] \\ &\qquad \leq C(\varepsilon)\|(1-\chi_2)\partial_t^{s_L} w\|_{L^2} + \varepsilon\|\nabla\partial_t^{s_L} w\|_{H^1}^2\end{aligned}$$

for any $\varepsilon > 0$. Adding the dissipation term in (5-78), taking ε small enough and using the bound (5-72) on the force term $\partial_t^{s_L} w$ gives

$$-\int |\nabla \partial_t^{s_L} w_{\text{ext}}|^2 + 4\int \nabla\chi_3.\nabla\partial_t^{s_L} w \partial_t^{s_L} w_{\text{ext}} + \int \partial_t^{s_L} w_{\text{ext}} \partial_t^{s_L}(\Delta\chi_{B(0,3)} w)$$
$$\le C\|(1-\chi_2)\partial_t^{s_L} w\|_{L^2}^2 \le C\|\partial_t^{s_L} w\|_{L^2}^2 \le \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')}}$$
$$\le \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}} \tag{5-79}$$

because in the trapped regime, $\lambda^2 s \sim s^{-\frac{\alpha}{2\ell-\alpha}}$.

Estimate for the nonlinear term. We now turn to the nonlinear term in (5-78), and use the estimate (5-73) for $\partial_t^{s_L} w^p$ we found in the first step, yielding

$$\left| \int \partial_t^{s_L} w_{\text{ext}} \partial_t^{s_L}((1-\chi_3)w^p \right| \le \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}. \tag{5-80}$$

End of Step 2. We collect the estimates (5-79) and (5-80) found in the previous substeps, which gives the desired bound (5-77) we claimed in this step.

Step 3: iteration of elliptic regularity. We claim that for $i = 0, \dots, s_L$,

$$\|\partial_t^i w_{\text{ext}}\|_{H^{2(s_L-i)}}^2 \le \|\partial_t^{s_L} w_{\text{ext}}(0)\|_{L^2}^2 + \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}\,dt'$$
$$+\int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}\,dt'$$
$$+\frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O(\frac{\sigma-s_c+\eta}{L})}}. \tag{5-81}$$

We are going to show this estimate by induction. This is true for $i = s_L$ from the result (5-77) of the last step, and because of the compatibility conditions (4-20) at the border. Now suppose it is true for i, with $1 \le i \le s_L$. Then as $\partial_t^{i-1} w_{\text{ext}}$ solves (5-68), from elliptic regularity one gets (again because of the compatibility conditions (4-20) at the border), from the induction hypothesis and the bounds (5-76), (5-76) and (5-76) on the force terms

$$\|\partial_t^{i-1} w_{\text{ext}}\|_{H^{2(s_L-i)+2}}^2 \le \|(1-\chi_{B(0,4)})\partial_t^{i-1}(w^p) + \Delta\chi_{B(0,4)}\partial_t^{i-1} w$$
$$+ 2\nabla\chi_{B(0,4)}.\nabla\partial_t^{i-1} w\|_{H^{2(s_L-i)}}^2 + \|\partial_t^i w_{\text{ext}}\|_{H^{2(s_L-i)}}^2$$
$$\le \|\partial_t^{s_L} w_{\text{ext}}(0)\|_{L^2}^2 + \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}\,dt'$$
$$+\int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}}\,dt'$$
$$+\frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O(\frac{\sigma-s_c+\eta}{L})}}.$$

This shows that the inequality (5-81) is true for $i-1$. Hence, by iterations, the inequality (5-81) is true for $i=0$, which gives the estimate (5-67) we had to prove. □

5D. ***End of the proof of Proposition 4.6.*** Proposition 4.6 states that, once the constants involved in the analysis, which are listed at its beginning, are well chosen, given an initial data of (1-1) that is a perturbation of the approximate blow-up profile along the stable directions of perturbation, there is a way to perturb it along the unstable directions of perturbation to produce a solution that stays trapped for all time in the sense of Definition 4.4. The strategy of the proof is the following. We argue by contradiction and suppose that for all perturbations along the unstable directions, the corresponding solution will eventually escape from the trapped regime. First, we characterize the exit of the trapped regime through a condition on the size of the unstable parameters, and then we show that arguing by contradiction would amount to go against Brouwer's fixed point theorem.

We fix $\lambda(s_0)$ satisfying (4-21), $w(s_0)$ decomposed in (4-5) satisfying (4-19) and (4-11), $V_1(s_0)$, $(U_{\ell+1}^{(0,1)}(s_0),\dots,U_L^{(0,1)}(s_0))$ and $(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I}\text{ with }1\le n,\ i_n\le i}$ satisfying (4-16), (4-17) and ((iii)). For any $(V_2(s_0),\dots,V_\ell(s_0))$ and $(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\le n,\ i<i_n}$ satisfying (4-14) and (4-15), let u denote the solution of (1-1) with initial datum $u(0)=\chi\widetilde{Q}_{b(s_0),1/\lambda(s_0)}+w(s_0)$ with $b(s_0)$ given by (4-31). We define the renormalized exit time $s^*=s^*\big((V_2(s_0),\dots,V_\ell(s_0)),(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\le n,\ i<i_n}\big)$:

$$s^*:=\sup\big\{s\ge s_0,\ u\text{ is trapped in the sense of Definition 4.4 on }[s_0,s)\big\}. \tag{5-82}$$

By a continuity argument, one always has $s^*>s_0$.

Lemma 5.9 (characterization of the exit of the trapped regime). *For L and M large enough and σ close enough to s_c, there exists a choice of the other constants in (4-30), except s_0 and η, such that for any s_0 large enough and η small enough, if $s^*<+\infty$, at least one of the following two scenarios hold*:

(i) Exit via instabilities on the first spherical harmonics.

$$V_i(s^*)=(s^*)^{-\tilde{\eta}}\ \textit{for some } 1\le i\le\ell.$$

(ii) Exit via instabilities on the other spherical harmonics.

$$U_i^{(n,k)}(s^*)=1\quad \textit{for some } (n,k,i)\in\mathcal{I},\ \textit{with } 1\le n \textit{ and } i<i_n.$$

Proof. A solution u is trapped if the parameters and the error involved in its decomposition (4-10) satisfy the bounds (4-22), (4-23), (4-24), (4-25) and (4-52). At time s^*, the bound (4-52) is strict by (4-51) and (4-52), and we are going to prove that (4-25) is strict in Step 1 and that (4-24) is strict in Step 2. Thus, (4-22) or (4-23) must be violated at the time s^* and the lemma is proved.

Step 1: improved bounds for the remainder w. We will now prove the estimates

$$\begin{gathered}\mathcal{E}_\sigma(s^*)\le\frac{K_1}{2(s^*)^{\frac{2(\sigma-s_c)\ell}{2\ell-\alpha}}},\quad \mathcal{E}_{2s_L}(s^*)\le\frac{K_2}{2(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}},\\ \|w_{\text{ext}}(s^*)\|_{H^\sigma}^2\le\frac{K_1}{2}\quad\text{and}\quad \|w_{\text{ext}}(s^*)\|_{H^{2s_L}}^2\le\frac{K_2}{2\lambda^{2(2s_L-s_c)}s^{2L+2(1-\delta_0)+2\eta(1-\delta_0')}}.\end{gathered} \tag{5-83}$$

Bound on $\mathcal{E}_\sigma$. Let K_1 and K_2 be any strictly positive real numbers. Then from Proposition 5.3, for s_0 and η large enough, we have

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\}\le\frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\frac{1}{s^{\frac{\alpha}{4L}}}\left[1+\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\right].$$

On $[s_0,s^*]$, one has

$$\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\le K_1 s^{-\frac{\alpha(\sigma-s_c)}{4\ell-2\alpha}}$$

by (4-25); hence for s_0 large enough,

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\}\le\frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\frac{1}{s^{\frac{\alpha}{8L}}}.$$

One has $\lambda=\left(\frac{s_0}{s}\right)^{\frac{\ell}{2\ell-\alpha}}(1+O(s_0^{-\tilde\eta}))$ by (4-52) and we assume that $|O(s_0^{-\tilde\eta})|\le\frac{1}{2}$. We reintegrate the above equation using (4-25) and (4-19):

$$\mathcal{E}_\sigma(s^*)\le\frac{1}{(s^*)^{\frac{2\ell(\sigma-s_c)}{2\ell-\sigma}}}\left(\left(\frac{3}{2}\right)^{2\sigma-s_c}+s_0^{\frac{2\ell(\sigma-s_c)}{2\ell-\alpha}}\frac{2^{2(\sigma-s_c)+3}L}{\alpha s_0^{\frac{\alpha}{8L}}}\sqrt{K_1}\right).$$

Therefore, once L is fixed we choose σ close enough to s_c so that

$$\frac{\alpha}{8L}>\frac{2\ell(\sigma-s_c)}{2\ell-\alpha}$$

and then for s_0 large enough one has

$$s_0^{\frac{2\ell(\sigma-s_c)}{2\ell-\alpha}}\frac{2^{2(\sigma-s_c)+3}L}{\alpha s_0^{\frac{\alpha}{8L}}}\le 1.$$

For any choice of the constants $K_1>10$, we then have

$$\mathcal{E}_\sigma(s^*)\le\frac{1}{(s^*)^{\frac{2\ell(\sigma-s_c)}{2\ell-\sigma}}}\left(\left(\frac{3}{2}\right)^{2\sigma-s_c}+\sqrt{K_1}\right)\le\frac{K_1}{2(s^*)^{\frac{2\ell(\sigma-s_c)}{2\ell-\sigma}}}. \tag{5-84}$$

Bound on $\mathcal{E}_{2s_L}$. Let K_1 and K_2 be any strictly positive real numbers. By Proposition 5.6, for any $N\gg 1$ the following holds for s_0 and η large enough:

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}+O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}\left(\sqrt{\mathcal{E}_{2s_L}}+\frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)\right)\right\}\\
&\quad\le\frac{1}{\lambda^{2(2s_L-s_c)+2}s}\left[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')}}+\frac{C(L,M)}{N^{2\delta}}\mathcal{E}_{2s_L}\right.\\
&\qquad\left.+\mathcal{E}_{2s_L}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O(\frac{1}{L})}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O(\frac{\eta+\sigma-s_c}{L})}}+\frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O(\frac{\sigma-s_c+\eta}{L})}}\right].\end{aligned}$$

In the trapped regime, from (4-25) one has

$$\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\le K_1 s^{-\frac{\alpha(\sigma-s_c)}{4\ell-2\alpha}}.$$

Consequently, for N and s_0 large enough the previous identity becomes

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}+O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}\left(\sqrt{\mathcal{E}_{2s_L}}+\frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)\right)\right\}$$
$$\leq \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\left[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}}+\frac{1}{N^{2\delta}}\mathcal{E}_{2s_L}\right].$$

Since from (4-52) we have

$$\lambda=\left(\frac{s_0}{s}\right)^{\frac{\ell}{2\ell-\alpha}}(1+O(s_0^{-\tilde{\eta}})),$$

when reintegrating in time the previous equation using the trapped regime bounds (4-25) and (4-19), one gets

$$\begin{aligned}\mathcal{E}_{2s_L}(s^*)&\leq \lambda(s^*)^{2(2s_L-s_c)}\Bigg[O_{(L,M)}\left(\frac{1}{\lambda(s^*)^{2(2s_L-s_c)}(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}}(\sqrt{K_1}+1)\right)\\&\quad+\mathcal{E}_{2s_L}(s_0)+O_{L,M}\left(\frac{1}{s_0^{L+1-\delta_0+\eta(1-\delta_0')}}\left(\sqrt{\mathcal{E}_{2s_L}(s_0)}+\frac{1}{s_0^{L+1-\delta_0+\eta(1-\delta_0')}}\right)\right)\\&\quad+\int_{s_0}^{s^*}\frac{1}{\lambda^{2(2s_L-s_c)}s^{2L+3-2\delta_0+\eta(1-\delta_0')}}\left(C(L,M)\sqrt{K_2}+C(L,M)+\frac{K_2}{N^{2\delta}}\right)\Bigg]\\&\leq \frac{1}{(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}}\left[C(L,M)(1+\sqrt{K_2})+C(L)\frac{K_2}{N^{2\delta}}\right]\\&\leq \frac{1}{K_2(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}}\end{aligned}\tag{5-85}$$

if N and K_1 have been chosen large enough.

Bound on $\|w_{\text{ext}}\|_{H^\sigma}$. We recall the estimate (5-35):

$$\frac{d}{dt}\left[\|w_{\text{ext}}\|_{H^\sigma}^2\right]\leq \frac{C(K_1,K_2)}{s^{1+\frac{\alpha}{2L}+O(\frac{\eta+\sigma-s_c}{L})}\lambda^2}\|w_{\text{ext}}\|_{H^\sigma}.$$

For any choice of the constants of the analysis in Proposition 4.6 such that all the previous propositions and lemmas hold, for s_0 large enough,

$$\frac{d}{dt}\left[\|w_{\text{ext}}\|_{H^\sigma}^2\right]\leq \frac{1}{s^{\frac{\alpha}{4L}}\lambda^2}\|w_{\text{ext}}\|_{H^\sigma}.$$

We reintegrate this equation in the bootstrap regime, by applying the bounds (4-25) and (4-19) on $\|w_{\text{ext}}\|_{H^\sigma}$ (using the relation $\frac{ds}{dt}=\frac{1}{\lambda^2}$):

$$\|w_{\text{ext}}(s^*)\|_{H^\sigma}\leq \sqrt{K_2}\frac{C(L)}{s_0^{\frac{\alpha}{4L}}}+\frac{C}{s_0^{\frac{2\ell}{2\ell-\alpha}(2s_L-s_c)}}\leq \frac{K_2}{2}\tag{5-86}$$

for K_2 chosen large enough.

Bound on $\|w_{\text{ext}}\|_{H^{2s_L}}$. We recall the estimate (5-67):

$$\|w_{\text{ext}}\|^2_{H^{2s_L}} \le \|\partial_t^{s_L} w_{\text{ext}}(0)\|^2_{L^2} + \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2}\, s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}\,dt' + \int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2}\, s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}\,dt' + \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)}\, s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}.$$

One has $w_{\text{ext}}=(1-\chi_3)w$, so $\partial_t^{s_L} w_{\text{ext}}=(1-\chi_3)\partial_t^{s_L} w$. Recall that we proved the bound (5-72) in the trapped regime for $\partial_t^{s_L} w(t)$ outside the blow-up zone in the proof of Proposition 5.8. The same proof gives for s_0 large enough, taking in account the bound (4-19) on w at initial time,

$$\|\partial_t^{s_L} w_{\text{ext}}(0)\|_{L^2} \le 1.$$

Inserting this estimate and (5-72) into the previous identity gives, for s_0 large enough,

$$\begin{aligned}
\|w_{\text{ext}}\|^2_{H^{2s_L}} &\le 1 + \int_0^t \frac{dt'}{\lambda^{2(2s_L-s_c)+2}\, s^{2L+3-2\delta_0+2\eta(1-\delta_0')}} + \frac{1}{\lambda^{2(2s_L-s_c)}\, s^{2L+2-2\delta_0+2\eta(1-\delta_0')}}\\
&\le \frac{2}{\lambda^{2(2s_L-s_c)}\, s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} + \int_0^t \frac{C\,dt'}{s^{-\frac{\ell[2(2s_L-s_c)+2]}{2\ell-\alpha}}\, s^{2L+3-2\delta_0+2\eta(1-\delta_0')}}\\
&\le \frac{2}{\lambda^{2(2s_L-s_c)}\, s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} + \frac{C(L)}{s^{-\frac{\ell 2(2s_L-s_c)}{2\ell-\alpha}}\, s^{2L+2-2\delta_0+2\eta(1-\delta_0')}}\\
&\le \frac{2}{\lambda^{2(2s_L-s_c)}\, s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} + \frac{C(L)}{\lambda^{2(2s_L-s_c)}\, s^{2L+2-2\delta_0+2\eta(1-\delta_0')}}\\
&\le \frac{K_2}{2\lambda^{2(2s_L-s_c)}\, s^{2L+2-2\delta_0+2\eta(1-\delta_0')}},
\end{aligned} \tag{5-87}$$

where we used the equivalence $\lambda \sim s^{-\frac{\ell}{2\ell-\alpha}}$ from (4-52), and where the last line holds for K_2 large enough.

End of step 1. We have proven (5-84), (5-85), (5-86) and (5-87), yielding the estimate we claimed, (5-83).

Step 2: improved bounds for the stable parameters. We claim that once L, M, η, K_1 and K_2 have been chosen so that the result of Step 1 holds, there exist $\tilde\eta>0$ and strictly positive constants $(\varepsilon_i^{(0,1)})_{\ell+1\le i\le L}$, $(\varepsilon_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i_n\le i}$ such that

$$|V_1(s^*)| \le \frac{1}{2(s^*)^{-\tilde\eta}}, \qquad |U_i^{(0,1)}(s^*)| \le \frac{\varepsilon_i^{(0,1)}}{2(s^*)^{\tilde\eta}} \quad \text{for } \ell+1\le i\le L, \tag{5-88}$$

and for $(n,k,i)\in\mathcal{I}$, $n\ge 1$,

$$|U_i^{(n,k)}(s^*)| \le \frac{\varepsilon_i^{(n,k)}}{2(s^*)^{\tilde\eta}} \quad \text{if } i_n<i, \qquad |U_i^{(n,k)}(s^*)| \le \frac{\varepsilon_i^{(n,k)}}{2} \quad \text{if } i_n=i. \tag{5-89}$$

We now prove all these improved bounds: first we prove the one for $b_{L_n}^{(n,k)}$, then the one for the $U_i^{(n,k)}$, $i \neq L_n$, and finally the one for V_1. For technical reasons, we introduce for $(n,k,i) \in \mathcal{I}$ the function $g_i^{(n,k)}$, a solution of the ODE

$$\frac{\frac{d}{ds} g_i^{(n,k)}}{g_i^{(n,k)}} = (2i-\alpha_n) b_1^{(0,1)}, \quad g(s_0) = s_0^{\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}}. \tag{5-90}$$

As $b_1^{(0,1)} = \frac{\ell}{s(2\ell-\alpha)} + O(s^{-1-\tilde{\eta}})$, for $\tilde{\eta}$ small enough and s_0 large enough one has

$$g_i^{(n,k)}(s) = s^{\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}} (1 + O(s_0^{-\tilde{\eta}})) \quad \text{with } |O(s_0^{-\tilde{\eta}})| \leq \tfrac{1}{2}. \tag{5-91}$$

Improved bound for $b_{L_n}^{(n,k)}$. First we notice that since L is chosen after ℓ, one can assume that for all $0 \leq n \leq n_0$, we have $i_n < L$. We rewrite the improved modulation equation (5-2) for $b_{L_n}^{(n,k)}$, using the estimate (5-3) for the extra term in the time derivative and the function $g_{L_n}^{(n,k)}$ (satisfying (5-90) and (5-91)), yielding

$$\left| \frac{d}{ds} \left[g_{L_n}^{(n,k)} b_{L_n}^{(n,k)} + O_{L,M,K_2}\big(s^{-L-\eta(1-\delta_0')+\delta_0-\delta_n+\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}}\big) \right] \right| \leq C(L,M,K_2) s^{-1-L-\eta(1-\delta_0')+\delta_0-\delta_n+\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}}$$

as $\eta(1-\delta_0') < \frac{g'}{2}$ for η small enough (g' being fixed). The notation $O_{L,M,K_2}(\cdot)$ is the usual $O(\cdot)$ notation with a constant depending on L, M and K_2. One has $2L_n - \alpha_n = 2L - \frac{d}{2} - 2\delta_n + 2m_0 + \frac{2}{p-1}$. Hence for L large enough, the quantity $-L-\eta(1-\delta_0')+\delta_0-\delta_n+\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}$ is strictly positive for all $0 \leq n \leq n_0$. Therefore, reintegrating in time the previous identity yields, using (4-16) and (4-17),

$$\begin{aligned} |b_{L_n}^{(n,k)}(s^*)| &\leq \frac{C(L,M,K_2)}{(s^*)^{L+\eta(1-\delta_0')+\delta_0-\delta_n}} + \frac{1}{s^{L+\delta_0-\delta_n+\tilde{\eta}}} \frac{s_0^{\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}-L-\delta_0+\delta_n-\tilde{\eta}}}{(s^*)^{\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}-L-\delta_0+\delta_n-\tilde{\eta}}} \frac{3}{2} s_0^{L+\delta_0-\delta_n+\tilde{\eta}} |b_{L_n}^{(n,k)}(s_0)| \\ &\leq \frac{C(L,M,K_2)}{(s^*)^{L+\eta(1-\delta_0')+\delta_0-\delta_n}} + \frac{3\varepsilon_{L_n}^{(n,k)}}{20} \frac{1}{(s^*)^{L+\delta_0-\delta_n+\tilde{\eta}}}. \end{aligned}$$

Therefore, if $\tilde{\eta} < \eta(1-\delta_0')$, for any $0 < \varepsilon_{L_n}^{(n,k)} < 1$, for s_0 large enough, we have

$$|b_{L_n}^{(n,k)}(s^*)| \leq \frac{\varepsilon_{L_n}^{(n,k)}}{2(s^*)^{L+\delta_0-\delta_n+\tilde{\eta}}}. \tag{5-92}$$

Improved bound for $b_i^{(n,k)}$, $i_n < i < L_n$. Using the same methodology we used to study the parameter $b_{L_n}^{(n,k)}$, we take the modulation equation (4-43), we integrate it in time, applying the bounds (4-22), (4-23), (4-24) and (4-25), yielding

$$\left| \frac{d}{ds} (g_i^{(n,k)} b_i^{(n,k)}) \right| \leq \frac{3\varepsilon_{i+1}^{(n,k)} s^{\frac{\ell}{2\ell-\alpha}(2i-\alpha_n)-\frac{\gamma-\gamma_n}{2}-i-\tilde{\eta}-1}}{2} + C(L,M,K_1) s^{-L-1+\delta_0-\eta(1-\delta_0')+\frac{\ell}{2\ell-\alpha}(2i-\alpha_n)}.$$

The condition $i_n < i$ ensures that $\frac{\ell}{2\ell-\alpha}(2i-\alpha_n) - \frac{\gamma-\gamma_n}{2} - i > 0$. For $\tilde{\eta}$ small enough, we can then integrate in time the previous equation, the first term on the right-hand side giving then a divergent integral. Then applying the bound (5-91) on $g_i^{(n,k)}$ and the initial bound (4-17) on $b_i^{(n,k)}$, one obtains

$$|b_i^{(n,k)}(s^*)| \le \frac{1}{(s^*)^{\frac{\gamma-\gamma_n}{2}+i+\tilde{\eta}}}\Bigg(\frac{3\varepsilon_i^{(n,k)}}{20}+C(L)\varepsilon_{i+1}^{(n,k)} + \frac{C(L,M)}{(s^*)^{\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}-\frac{\gamma-\gamma_n}{2}-i-\tilde{\eta}}}\int_{s_0}^{s^*} s^{-L-1+\delta_0-\eta(1-\delta_0')+\frac{\ell(2i-\alpha)}{2\ell-\alpha}}\,ds\Bigg) \le \frac{\varepsilon_i^{(n,k)}}{2(s^*)^{\frac{\gamma-\gamma_n}{2}+i}} \tag{5-93}$$

if s_0 is large enough and $\varepsilon_{i+1}^{(n,k)}$ is small enough, because $L-\delta_0 > \frac{\gamma-\gamma_n}{2}+i$.

Improved bound for $b_i^{(n,k)}$ if $i_n = i$ and $1 \le n$. In that case, $\frac{\ell}{2\ell-\alpha}(2i-\alpha_n) = \frac{\gamma-\gamma_n}{2}+i$. Hence one has

$$\frac{1}{2} \le \frac{g_i^{(n,k)}}{s^{\frac{\gamma-\gamma_n}{2}+i}} \le \frac{3}{2}.$$

Integrating the modulation equation and making the same manipulations we made for $i_n < i$ then yields

$$|b_i^{(n,k)}(s^*)| \le \frac{1}{(s^*)^{\frac{\gamma-\gamma_n}{2}+i}}\left(\frac{3\varepsilon_i^{(n,k)}}{20}+C(L)\varepsilon_{i+1}^{(n,k)}+\frac{C(L,M)}{s_0^{L-\delta_0-\frac{\gamma-\gamma_n}{2}-i}}\right) \le \frac{\varepsilon_i^{(n,k)}}{2(s^*)^{\frac{\gamma-\gamma_n}{2}+i}} \tag{5-94}$$

if $\varepsilon_{i+1}^{(n,k)}$ is small enough and s_0 is large enough.

Improved bound for V_1. We recall that from (4-13), V_1 denotes the stable direction of perturbation for the dynamical system (3-58) contained in $\operatorname{Span}((U_i^{(0,1)})_{1\le i\le \ell})$. From the quasidiagonalization (3-69) of the linearized matrix A_ℓ, under the bootstrap bounds (4-22), (4-23), (4-24) and (4-25), its time evolution is given by

$$\begin{aligned} V_{1,s} &= -\frac{V_1}{s} + O\left(\frac{|(V_i)_{1\le i\le \ell}|^2}{s}\right) + O(C(L,M,K_2)s^{-L-\ell}) + \frac{q_1}{s}U_{i+1}^{(0,1)} \\ &= -\frac{V_1}{s} + O\left(\frac{1}{s^{1+2\tilde{\eta}}} + s^{-L-\ell} + \frac{\varepsilon_{\ell+1}^{(0,1)}}{s^{1+\tilde{\eta}}}\right), \end{aligned}$$

which when reintegrated in time gives, if $\varepsilon_{\ell+1}^{(0,1)}$ is small enough, s_0 is large enough, and using (4-16),

$$|V_1(s^*)| \le \frac{s_0 V_1(s_0)}{s^*} + \frac{C(L,M,K_1)}{(s^*)^{2\tilde{\eta}}} + \frac{C(L)\varepsilon_{\ell+1}^{(0,1)}}{(s^*)^{\tilde{\eta}}} \le \frac{1}{2s^{\tilde{\eta}}}. \tag{5-95}$$

End of Step 2. We choose the constants of smallness in the following order so that all the improved bounds we proved, (5-92), (5-93), (5-94), (5-95), hold together. For any choice of K_1, K_2, L, M, η in their ranges, there exists $\tilde{\eta} > 0$ such that $\tilde{\eta} < \eta(1-\delta_0')$ and $\frac{\gamma-\gamma_n}{2}+i+\tilde{\eta} < \frac{\ell}{2\ell-\alpha}(2i-\alpha_n)$ for all $(n,k,i) \in \mathcal{I}$ with $i_n < i$. First choose the constant $\varepsilon_{\ell+1}^{(0,1)}$ small enough so that the improved bound (5-95) for V_1 holds for s_0 large enough. Next choose $\varepsilon_{\ell+2}^{(0,1)}$ such that the improved bound (5-93) for $U_{\ell+1}^{(0,1)}$ holds for s_0 large enough. By iteration we then choose $\varepsilon_{\ell+3}^{(0,1)}, \dots, \varepsilon_L^{(0,1)}$ to make all the bounds (5-93) hold until the one for $U_{L-1}^{(0,1)}$. Then the final one, (5-92), for $U_L^{(0,1)}$, holds for s_0 large enough without any conditions on $\varepsilon_i^{(0,1)}$ for $\ell+1\le i\le L-1$. The same reasoning applies for the stable parameters on the spherical harmonics of higher degree ($1\le n\le n_0$). We have proved (5-88). □

We fix all the constants of the analysis so that Lemma 5.9 holds, and we will just possibly increase the initial renormalized time s_0, which does not change its validity. The number of instability directions is

$$m=\ell-1+d(E[i_1]-\delta_{i_1\in\mathbb{N}})+\sum_{2\le n\le n_0}k(n)(E[i_n]+1-\delta_{i_n\in\mathbb{N}}).$$

To prove Proposition 4.6, we have to prove that there exists an additional perturbation along the unstable directions of perturbations such that the solution stays forever trapped. We prove it via a topological argument, by looking at all the solutions associated to the possible perturbations along the unstable directions of perturbation. For this purpose, we introduce the set

$$\mathcal{B}:=\big\{\big(V_2(s_0),\dots,V_\ell(s_0),(U^{(n,k)}(s_0)_i)_{(n,k,i)\in\mathcal{I},\,1\le n,\,i<i_n}\big)\in\mathbb{R}^m:|V_i(s_0)|\le s_0^{-\tilde\eta}\text{ for }2\le i\le\ell,$$
$$|U^{(n,k)}(s_0)_i|\le\varepsilon_i^{(n,k)}\text{ for }(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n\big\},$$

which represents all the possible values of the unstable parameters so that the solution to (1-1) with initial data given by (4-5) and (4-31) starts in the trapped regime. We then define the following application $f:\mathcal{D}(f)\subset\mathcal{B}\to\partial\mathcal{B}$ that gives the last value taken by the unstable parameters before the solution leaves the trapped regime (when it does):

$$f\big(V_2(s_0),\dots,V_\ell(s_0),(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\,1\le n,\,i<i_n}\big)=\left(\frac{(s^*)^{\tilde\eta}}{s_0^{\tilde\eta}}V_2(s^*),\dots,\frac{(s^*)^{\tilde\eta}}{(s_0)^{\tilde\eta}}V_\ell(s^*),(U_i^{(n,k)}(s^*))_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n}\right).\tag{5-96}$$

The domain $\mathcal{D}(f)$ of the application f is the set of the m-tuples of real numbers

$$\big(V_2(s_0),\dots,V_\ell(s_0),(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n}\big)$$

in $\mathcal{B}$ such that the solution starting initially with a decomposition given by (4-5) and (4-31) leaves the trapped regime in finite time s^*. The following lemma describes the topological properties of f.

Lemma 5.10 (topological properties of the exit application). *There exists a choice of smallness constants $(\varepsilon_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\,1\le n,\,i<i_n+1}$ such that the following properties hold for s_0 large enough*:

(i) *$\mathcal{D}(f)$ is nonempty and open, and the inclusion $\partial\mathcal{B}\subset\mathcal{D}(f)$ holds.*

(ii) *f is continuous and is the identity on the boundary $\partial\mathcal{B}$.*

Proof. **Step 1:** the outgoing flux property. We prove in this step that one can choose the smallness constants $(\varepsilon_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n+1}$ such that for any $\big(V_2(s_0),\dots,V_\ell(s_0),(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\le n,\ i<i_n}\big)$ in $\mathcal{B}$ such that the solution starting initially with the decomposition given by (4-5) and (4-31) is in the trapped regime on $[s_0,s]$ and satisfies at time s

$$\left(\frac{(s)^{\tilde\eta}}{s_0^{\tilde\eta}}V_2(s),\dots,\frac{(s)^{\tilde\eta}}{(s_0)^{\tilde\eta}}V_\ell(s),(U_i^{(n,k)}(s))_{(n,k,i)\in\mathcal{I},\,1\le n,\,i<i_n}\right)\in\partial\mathcal{B},$$

the exit time from the trapped regime is s. To prove this we compute the time derivative of the unstable parameters when they are on $\partial\mathcal{B}$, and show that it points toward the exterior. Indeed from the modulation equation (4-43) and (3-69) (where we injected the bounds of the trapped regime (4-22), (4-23), (4-24) and (4-25)),

$$V_{i,s}=\frac{i\alpha}{2\ell-\alpha}\frac{V_i}{s}+O\left(\frac{|(V_1(s),\dots,V_\ell(s))|^2}{s}\right)+\frac{q_iU^{(0,1)}_{\ell+1}}{s}+O(s^{-L+\ell})=\frac{i\alpha}{2\ell-\alpha}\frac{V_i}{s}+O\left(s^{-1-2\tilde\eta}+\frac{\varepsilon^{(0,1)}_{\ell+1}}{s^{1+\tilde\eta}}\right),$$

$$U^{(n,k)}_{i,s}=\alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-i}{(2\ell-\alpha)s}U^{(n,k)}_i+\frac{U^{(n,k)}_{i+1}}{s}+O(s^{-1-\tilde\eta})=\alpha\frac{i_n-i}{(2\ell-\alpha)s}U^{(n,k)}_i+O\left(\frac{\varepsilon^{(n,k)}_{i+1}}{s}+s^{-1-\tilde\eta}\right).$$

Therefore, as $i<i_n$, by iterations (i.e., by choosing first $\varepsilon^{(n,k)}_0$, then $\varepsilon^{(n,k)}_1$, and so on until choosing $\varepsilon^{(n,k)}_{\ell+1}$) we can choose all the smallness constants and s_0 large enough so that

$$\frac{i\alpha}{2\ell-\alpha}\frac{(-1)^j}{s^{1+\tilde\eta}}+O\left(s^{-1-2\tilde\eta}+\frac{\varepsilon^{(0,1)}_{\ell+1}}{s^{1+\tilde\eta}}\right)>0 \text{ (resp. } <0) \quad \text{if } j=0 \text{ (resp. } j=1),$$

$$\alpha\frac{i_n-i}{(2\ell-\alpha)s}(-1)^j\varepsilon^{(n,k)}_i+O\left(\frac{\varepsilon^{(n,k)}_{i+1}}{s}+s^{-L+\ell}\right)>0 \text{ (resp. } <0) \quad \text{if } j=0 \text{ (resp. } j=1).$$

Consequently, any solution that is trapped until s such that at time s,

$$\left(\frac{(s)^{\tilde\eta}}{s_0^{\tilde\eta}}V_2(s),\dots,\frac{(s)^{\tilde\eta}}{(s_0)^{\tilde\eta}}V_\ell(s),(U^{(n,k)}_i(s))_{(n,k,i)\in\mathcal{I},\,1\le n,\,i<i_n}\right)\in\partial\mathcal{B}$$

leaves the trapped regime after s.

Step 2: end of the proof of the lemma. Step 1 directly implies that $\mathcal{D}(f)$ contains $\partial\mathcal{B}$, and that f is the identity on $\partial\mathcal{B}$. If a solution u leaves at time s^*, it also implies that it never hit the boundary before s^*. Consequently, as the trapped regime is characterized by nonstrict inequalities, and because everything in the dynamics of (1-1) is continuous with respect to variation on these unstable parameters, we get that $\mathcal{D}(f)$ is open, and that the exit time s^* and f are continuous on $\mathcal{D}(f)$. □

We can now end the proof of Proposition 4.6.

Proof of Proposition 4.6. We argue by contradiction. If for any choice of initial perturbation along the unstable directions of perturbation, the solution leaves the trapped regime, then it means that the domain of the exit application f defined by (5-96) is $\mathcal{D}(f)=\mathcal{B}$. But then from Lemma 5.10, f would be a continuous application from $\mathcal{B}$ towards its boundary, being the identity on the boundary, which is impossible thanks to Brouwer's theorem, and the contradiction is obtained. □

Appendix A: Properties of the zeros of H

This section is devoted to the proof of Lemma 2.3.

Proof of Lemma 2.3. The proof relies solely on ODE techniques (in the same spirit as [Gui et al. 1992; Li 1992]) and is as follows. First, we describe the asymptotics of the equation $H^{(n)}f=0$ at the origin and at infinity in Lemma A.1. Then we construct the special zeroes $T_0^{(n)}$ and $\Gamma^{(n)}$ in these asymptotic regimes using a perturbative argument and obtain their asymptotic behavior in Lemma A.2. Finally we show that they are not equal via global invariance properties of the ODE in the phase space $(f, \partial_r f)$ in Lemma A.3, yielding that they form indeed a basis of the set of solutions.

Let $f : (0, +\infty)$ be smooth such that $H^{(n)} f = 0$. First we make the change of variables $f(r) = w(t)$ with $t = \ln(r) \in (-\infty, +\infty)$. Then w solves

$$w'' + (d-2)w' - [e^{2t} V(e^t) + n(d+n-2)]w = 0, \tag{A-1}$$

where V is defined by (1-31) and satisfies $e^{2t} V(e^t) = O(e^{2t}) \to 0$ as $t \to -\infty$, and $e^{2t} V(e^t) = -pc_\infty^{p-1} + O(e^{-t\alpha})$ as $t \to +\infty$, by (2-2). Hence (A-1) is similar to the following ODEs as $t \to \pm\infty$:

$$w'' + (d-2)w' + (pc_\infty^{p-1} - n(d+n-2))w = 0, \tag{A-2}$$

$$w'' + (d-2)w' - n(d+n-2)w = 0. \tag{A-3}$$

The first step in the proof of Lemma 2.3 is to describe their solutions.

Lemma A.1. $\mathrm{Span}(e^{-\gamma_n t}, e^{-\gamma_n' t})$ *(resp.* $\mathrm{Span}(e^{nt}, e^{(-n-d+2)t})$*) is the set of solutions of* (A-2) *(resp.* (A-3)*), where* γ_n *is defined in* (1-18) *and*

$$\gamma_n' := \frac{d-2+\sqrt{\triangle_n}}{2}, \tag{A-4}$$

where $\triangle_n > 0$ *is defined in* (1-18)*. These numbers satisfy*

$$\gamma_0 = \gamma, \quad \gamma_1 = \frac{2}{p-1} + 1 \quad \text{and} \quad \forall n \geq 2, \quad \gamma_n < \frac{2}{p-1}, \quad \gamma_n' > \frac{(d-2)}{2}, \tag{A-5}$$

where γ *is defined in* (1-9)*.*

Proof. From the standard theory of second-order differential equations with constant coefficients, the set of solutions of (A-2) (resp. (A-3)) is $\mathrm{Span}(e^{-\gamma_n t}, e^{-\gamma_n' t})$ (resp. $\mathrm{Span}(e^{nt}, e^{(-n-d+2)t})$), where γ_n and γ_n' are defined by (1-18) and (A-4). For any $n \in \mathbb{N}$, one computes from its definition in (1-18) that the number $\triangle_n$ used in the definitions (1-18) and (A-4) of γ_n and γ_n' is strictly positive: $\triangle_n > 0$. Indeed, $\triangle_n \geq \triangle_0$ by (1-18), and $\triangle_0 > 0$ if and only if $p > p_{JL}$, where p_{JL} is defined in (1-6), and the present paper is concerned with the case $p > p_{JL}$.

From the formula (1-18), one computes that $\gamma_0 = \gamma$ and $\gamma_1 = \frac{2}{p-1} + 1$, where γ is defined in (1-9). For all $n \in \mathbb{N}$, from the definition (A-4) of γ_n' and since $\triangle_n > 0$, one gets that $\gamma_n' > \frac{d-2}{2}$. Eventually we compute from (1-18) that

$$\triangle_1 = \left(d - 4 - \frac{4}{p-1}\right)^2, \quad \triangle_2 = \left(d - 4 - \frac{4}{p-1}\right)^2 + 4d + 4,$$

which implies in particular that

$$\triangle_2 - \triangle_1 - 4\sqrt{\triangle_1} - 4 = 4d + 4 - 4\left(d - 4 - \frac{4}{p-1}\right) - 4 = 16 + \frac{16}{p-1} > 0,$$

giving $\sqrt{\triangle_2} > \sqrt{\triangle_1} + 2$. This, by (1-18), implies

$$\gamma_2 = \frac{d-2-\sqrt{\triangle_2}}{2} < \frac{d-2-\sqrt{\triangle_1}-2}{2} = \gamma_1 - 1 = \frac{2}{p-1} + 1 - 1 = \frac{2}{p-1}.$$

This implies $\gamma_n < \frac{2}{p-1}$ for all $n \geq 2$ because the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing by its definition (1-18). □

Lemma A.2. *There exist $w_1^{(n)}$, $w_2^{(n)}$, $w_3^{(n)}$ and $w_4^{(n)}$ solving (A-1) such that*

$$w_1^{(n)} \underset{t\to-\infty}{=} \sum_{i=0}^{q} c_i e^{(n+2i)t} + O(e^{(n+2q+2)t}), \quad w_2^{(n)} \underset{t\to-\infty}{\sim} \tilde{c}_1 e^{(-n-d+2)t}, \tag{A-6}$$

$$w_3^{(n)} \underset{t\to+\infty}{=} \tilde{c}_2 e^{-\gamma_n t} + O(e^{(-\gamma_n-g)t}) \quad \text{and} \quad w_4^{(n)} \underset{t\to+\infty}{\sim} \tilde{c}_3 e^{-\gamma_n' t} = O(e^{(-\gamma_n-g)t}), \tag{A-7}$$

with constants $c_1, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \neq 0$. Moreover the asymptotics hold for the derivatives.

Proof. **Step 1:** existence of $w_1^{(n)}$. For $n = 0$, we take the explicit solution $w_1^{(0)} = \Lambda Q(e^t)$, which satisfies (A-6) by (2-1). Now let $n \geq 1$. Using the Duhamel formula for solutions of (A-1), the fundamental set of solutions for the constant coefficient ODE (A-3) begin provided by Lemma A.1, a solution of (A-1) satisfying the condition on the left in (A-6) with $c_0 = 1$ can be written as

$$w_1^{(n)}(t) = e^{nt} + \frac{1}{2n+d-2} \int_{-\infty}^{t} (e^{n(t-t')} - e^{(-n-d+2)(t-t')}) w_1^{(n)}(t') e^{2t'} V(e^{t'})\, dt'. \tag{A-8}$$

We now use a standard contraction argument. For $t_0 \in \mathbb{R}$ we endow the space

$$X := \left\{ u \in C((-\infty, t_0], \mathbb{R}) : \sum_{t\leq t_0} |u(t)|e^{-t} < +\infty \right\}$$

with the norm

$$\|u\|_X := \sup_{t\leq t_0} |u(t)|e^{-(n+1)t}. \tag{A-9}$$

For $u \in X$ we define the function $\Phi u : (-\infty, t_0] \to \mathbb{R}$ by

$$(\Phi u)(t) := \frac{1}{2n+d-2} \int_{-\infty}^{t} (e^{n(t-t')} - e^{(-n-d+2)(t-t')})[e^{nt'} + u(t')] e^{2t'} V(e^{t'})\, dt'. \tag{A-10}$$

Φ maps X into itself. Indeed as the potential V is bounded from (2-2), a brute force bound on the above equation yields that

$$|(\Phi u)(t)| \leq C\|V\|_{L^\infty}(e^t + \|u\|_X e^{2t}) e^{(n+1)t},$$

and therefore $\|\Phi u\|_X \leq C\|V\|_{L^\infty}(e^{t_0} + \|u\|_X e^{2t_0})$. The same brute force bound for the difference of two images under Φ of two elements gives

$$\left|(\Phi u)(t) - (\Phi v)(t)\right| \leq C\|V\|_{L^\infty} e^{2t} \|u-v\|_X e^{(n+1)t}.$$

Hence $\|\Phi u - \Phi v\|_X \leq C\|V\|_{L^\infty} e^{2t_0}\|u-v\|_X$ and Φ is a contraction for $t_0 \ll 0$ small enough. Therefore, Φ admits a fixed point in X, denoted by u_1. From the Duhamel formula (A-8) and the definition (A-10) of Φ, we know $w_1^{(n)} := e^{nt} + u_1(t)$ is then a solution of (A-1) on $(-\infty, t_0]$, which, from the definition

(A-9) of X, satisfies

$$w_1^{(n)} = e^{nt} + O(e^{(n+1)t}) \quad \text{as } t \to -\infty. \tag{A-11}$$

We extend it to a solution of (A-1) on $\mathbb{R}$ ((A-1) being linear with smooth coefficients), still naming it $w_0^{(n)}$.
Step 2: asymptotics of $w_1^{(n)}$. At present, we will refine the asymptotics (A-11). We reason by induction. We claim that if for $k \in \mathbb{N}$ and $(c_i)_{0 \le i \le k} \in \mathbb{R}^{k+1}$ one has

$$w_1^{(n)} = \sum_{i=0}^{k} c_i e^{(n+2i)t} + O(e^{(n+2k+2)t}) \quad \text{as } t \to -\infty \tag{A-12}$$

then there exists $c_{k+1} \in \mathbb{R}$ such that

$$w_1^{(n)} = \sum_{i=0}^{k+1} c_i e^{(n+2i)t} + O(e^{(n+2k+4)t}) \quad \text{as } t \to -\infty. \tag{A-13}$$

We now prove this fact. Fix $k \ge 1$ and assume that $w_1^{(n)}$ satisfies (A-12). As V is a smooth radial profile, one has that $\partial_r^{2q+1} V(0) = 0$ for any $q \in \mathbb{N}$, implying that there exists $(d_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that

$$V(e^t) = \sum_{i=0}^{k} d_i e^{2it} + O(e^{(2k+2)t}) \quad \text{as } t \to -\infty. \tag{A-14}$$

We insert this and (A-12) into (A-8) and integrate to find

$$\begin{aligned} w_1^{(n)} &= e^{nt} + \frac{1}{2n+d-2} \int_{-\infty}^{t} (e^{n(t-t')} - e^{(2-n-d)(t-t')}) \left[\sum_{i=0}^{k} \sum_{j=0}^{i} c_j d_{i-j} e^{(n+2i+2)t'} + O(e^{(n+2k+4)t'}) \right] dt' \\ &= e^{nt} + \sum_{i=0}^{k} \frac{e^{(n+2i+2)t}}{2n+d-2} \left(\frac{1}{2i+2} - \frac{1}{2n+d+2i} \right) \sum_{j=0}^{i} c_j d_{i-j} + O(e^{(2+2k+4)t}). \end{aligned}$$

This asymptotic has to be coherent with the assumption (A-12); hence for all $0 \le i \le k-1$ one has

$$\left(\frac{1}{2i+2} - \frac{1}{2n+d+2i} \right) \sum_{j=0}^{i} \frac{c_j d_{i-j}}{2n+d-2} = c_{i+1}.$$

The above identity is then the formula (A-13) one has to prove.

Thus, one has proven that the asymptotic on the left of (A-6) holds for $w_1^{(n)}$. It remains to show that it also holds for the derivatives. Differentiating (A-8) gives

$$(w_1^{(n)})'(t) = n e^{nt} + \frac{1}{2n+d-2} \int_{-\infty}^{t} [n e^{n(t-t')} + (n+d-2) e^{(2-n-d)(t-t')}] w_1^{(n)} e^{2t'} V.$$

We use the same reasoning we did for $w_1^{(n)}$: we insert the asymptotic (A-12) at any order for $w_1^{(n)}$ we just showed and (A-14) into the above formula, integrate in time and match the coefficients we find with

(A-12), yielding that

$$(w_1^{(n)})'(t) = \sum_{i=0}^{k} (n+2i)c_i e^{(n+2i)t} + O(e^{(n+2k+2)t})$$

for any $k \in \mathbb{N}$. Therefore, one has proven that the asymptotic on the left of (A-6) holds for $w_1^{(n)}$ and $(w_1^{(n)})'$. As $w_1^{(n)}$ solves (A-1), its second derivative is given by

$$(w_1^{(n)})'' = -(d-2)(w_1^{(n)})' + [e^{2t}V(e^t) + n(d+n-2)]w_1^{(n)},$$

and therefore by (A-14) the expansion also holds for $(w_1^{(n)})''$. Differentiating the above equation, using again (A-14) and the expansions for $w_1^{(n)}$, $(w_1^{(n)})'$ and $(w_1^{(n)})''$, one obtains the expansion for $(w_1^{(n)})'''$. By iterating this procedure we obtain the expansion on the left of (A-6) for all derivatives of $w_1^{(n)}$.

Step 3: existence and asymptotics of $w_2^{(n)}$. Let $t_0 \in \mathbb{R}$. We use the Duhamel formula for (A-1), the solutions of the underlying constant coefficient ODE (A-3) being provided by Lemma A.1. For $t \le t_0$, the solution of (A-1) starting from $w_2^{(n)}(t_0) = e^{(2-d-n)t_0}$, $(w_2^{(n)})'(t_0) = (2-d-n)e^{(2-d-n)t_0}$ can be written as

$$w_2^{(n)} = e^{(2-d-n)t} - \frac{1}{2n+d-2} \int_t^{t_0} (e^{n(t-t')} - e^{(2-n-d)(t-t')}) V(e^{t'}) e^{2t'} w_2^{(n)}(t')\, dt'. \tag{A-15}$$

We claim that for $t_0 \ll 0$ small enough, we have

$$|w_2^{(n)} - e^{(2-d-n)t}| \le \frac{e^{(2-d-n)}}{2} \tag{A-16}$$

for all $t \le t_0$. To show that, let $\mathcal{T}$ be the set of times $t \le t_0$ such that this inequality holds. $\mathcal{T}$ is closed via a continuity argument, and is nonempty as it contains t_0. For $t \in \mathcal{T}$ we compute by brute force on the above identity:

$$|w_2^{(n)} - e^{(2-d-n)t}| \le C\|V\|_{L^\infty} e^{(2-n-d)t} e^{2t_0}.$$

Hence, for $t_0 \ll 0$ small enough, $|w_2^{(n)} - e^{(2-d-n)t}| \le e^{(2-n-d)t}/3$, implying that $\mathcal{T}$ is open. Therefore, $\mathcal{T} = (-\infty, t_0]$ by a connectedness argument and $w_2^{(n)}$ satisfies (A-16) for all $t \le t_0$. We insert (A-16) into (A-15) to refine the asymptotics (the constant in the $O(\cdot)$ depends on $\|V\|_{L^\infty}$):

$$\begin{aligned}
w_2^{(n)} &= e^{(2-d-n)t} + \int_t^{t_0} (e^{n(t-t')} - e^{(2-d-n)(t-t')}) O(e^{(4-n-d)(t-t')})\, dt' \\
&= e^{(2-d-n)t} + e^{nt} \int_t^{t_0} O(e^{(4-2n-d)t'})\, dt' + e^{(2-n-d)t} \int_t^{t_0} O(e^{2t'})\, dt' \\
&= e^{(2-d-n)t} + O(e^{(4-n-d)t}) + e^{(2-n-d)t} \left(\int_{-\infty}^{t_0} O(e^{2t'})\, dt' - \int_{-\infty}^{t} O(e^{2t'})\, dt' \right) \\
&= e^{(2-d-n)t} \left(1 + \int_{-\infty}^{t_0} O(e^{2t'})\, dt' \right) + O(e^{(4-n-d)t}) \\
&= \tilde{c}_1 e^{(2-d-n)t} + O(e^{(4-n-d)t})
\end{aligned}$$

with $\tilde{c}_1 \neq 0$ if $t_0 \ll 0$ is chosen small enough. We just showed the asymptotic on the right of (A-6).

Step 4: existence and asymptotics of $w_3^{(n)}$ and $w_4^{(n)}$. Using exactly the same techniques we used at $-\infty$ to construct $w_1^{(n)}$ and $w_2^{(n)}$ as perturbations of the solutions described by Lemma A.1 of the asymptotic constant coefficients ODE (A-3), we can construct two solutions of (A-1), $w_3^{(n)}$ and $w_4^{(n)}$, satisfying

$$w_3^{(n)} \sim \tilde{c}_2 e^{-\gamma_n t}, \quad w_4^{(n)} \sim \tilde{c}_3 e^{-\gamma_n' t} \quad \text{as } t \to +\infty \tag{A-17}$$

with $\tilde{c}_2, \tilde{c}_3 \neq 0$, as perturbations of the solutions $e^{-\gamma_n t}$ and $e^{-\gamma_n' t}$ of the asymptotic ODE (A-2) at $+\infty$. We leave safely the proof of this fact to the reader. We now show why the second term in the asymptotic of $w_3^{(n)}$ is $O(e^{(-\gamma_n - g)t})$, where g is defined in (1-21). Using Duhamel's formula for (A-1), with the set of fundamental solutions of the asymptotic equation (A-2) described in Lemma A.1, $w_3^{(n)}$ can be written as

$$w_3^{(n)} = a_1 e^{-\gamma_n t} + b_1 e^{-\gamma_n' t} - \frac{1}{-\gamma_n + \gamma_n'} \int_0^t (e^{-\gamma_n(t-t')} - e^{-\gamma_n'(t-t')}) e^{2t'} (V(e^{t'}) + p c_\infty^{p-1} e^{-2t'}) w_3^{(n)}(t')\, dt'$$

for a_1 and b_1 two coefficients. We use the bounds $V(e^{t'}) + p c_\infty^{p-1} e^{-2t'} = O(e^{-\alpha t'})$ from (2-2) and (A-17) to find

$$w_3^{(n)}(t) = a_1 e^{-\gamma_n t} + b_1 e^{-\gamma_n' t} - \frac{1}{-\gamma_n + \gamma_n'} \int_0^t (e^{-\gamma_n(t-t')} - e^{-\gamma_n'(t-t')}) O(e^{(-\gamma_n - \alpha)t'}).$$

After few computations, we obtain two new coefficients $\tilde{a}_1$ and $\tilde{a}_2$ such that

$$w_3^{(n)}(t) = \tilde{a}_1 e^{-\gamma_n t} + \tilde{b}_1 e^{-\gamma_n' t} + O(e^{(-\gamma_n - \alpha)t}).$$

As $-\gamma_n' < -\gamma_n$ by (1-18), the asymptotic (A-17) implies $\tilde{a}_1 = \tilde{c}_2 \neq 0$. From the definition (1-21) of g, this parameter is tailor-made to produce $-\gamma_0 - g > -\gamma_0'$ (by (1-9) and (1-18)). By (1-18), one then has $-\gamma_n - g + \gamma_n' \geq -\gamma_0 - g + \gamma_0' > 0$. As g satisfies also $g < \alpha$, the above identity then yields

$$w_3^{(n)}(t) = \tilde{c}_2 e^{-\gamma_n t} + O(e^{(-\gamma_n - g)t}).$$

Using exactly the same methods we use to propagate the asymptotic of $w_1^{(n)}$ to its derivatives in Step 2, the above identity propagates to the derivatives of $w_3^{(n)}$. ☐

Lemma A.3. *The solutions $w_1^{(n)}$ and $w_4^{(n)}$ given by Lemma A.2 are not collinear. Moreover, $w_1^{(n)}$ has constant sign.*

Proof. We formulate (ODE_n) as a planar dynamical system:

$$\frac{d}{dt}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ n(d+n-2) + e^{2t} V(e^t) & -(d-2) \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix},$$

with $w^1 = w$ and $w^2 = w'$. By their asymptotics from Lemma A.1,

$$\begin{pmatrix} w_1^{(n)}(t) \\ (w_1^{(n)})'(t) \end{pmatrix} = c_1 e^{nt} \begin{pmatrix} 1 \\ n \end{pmatrix} + O(e^{(n+2)t}) \quad \text{as } t \to -\infty,$$

$$\begin{pmatrix} w_4^{(n)}(t) \\ (w_4^{(n)})'(t) \end{pmatrix} \sim \tilde{c}_3 e^{-\gamma_n' t} \begin{pmatrix} 1 \\ -\gamma_n' \end{pmatrix} \quad \text{as } t \to -\infty,$$

and we may take $c_1, \tilde{c}_3 > 0$ without loss of generality. Thus, close to $-\infty$, we know $(w_1^{(n)}(t), (w_1^{(n)})'(t))$ is in the top right corner of the plane. It cannot cross the ray $\{0\} \times (0, +\infty)$ because there the vector field $\begin{pmatrix} w^2 \\ -(d-2)w^2 \end{pmatrix}$ points toward the right. Neither can it go below the ray $\left(x, -\frac{d-2}{2}x\right)_{x \geq 0}$. To see that, we compute the scalar product between the vector field and a vector that is orthogonal to this ray and that points toward north at any time $t \in \mathbb{R}$:

$$\left(\begin{pmatrix} 0 & 1 \\ n(d+n-2)+e^{2t}V(e^t) & -(d-2) \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{d-2}{2} \end{pmatrix}\right) \cdot \begin{pmatrix} \frac{d-2}{2} \\ 1 \end{pmatrix} = \frac{(d-2)^2}{4} + e^{2t}V(e^t) + n(d+n-2) > 0$$

because $e^{2t}V(e^t) > \frac{(d-2)^2}{4}$, where the potential $-V$ is below the Hardy potential (see (2-5)). Hence $(w_1^{(n)}(t), (w_1^{(n)})'(t))$ stays in the top right zone whose border is

$$\{0\} \times (0, +\infty) \cup \left(x, -\frac{d-2}{2}x\right)_{x \geq 0}.$$

In particular, $w_1^{(n)} > 0$ for all times, which proves the positivity of $w_1^{(n)}$. Since the trajectory $(w_4^{(n)}(t), (w_4^{(n)})'(t))$ is asymptotically collinear to the vector $\begin{pmatrix} 1 \\ -\gamma_n' \end{pmatrix}$, which does not belong to this zone (from Lemma A.1) nor its opposite, one obtains that $w_1^{(n)}$ and $w_4^{(n)}$ are not collinear. □

We now end the proof of Lemma 2.3. The fundamental set of solutions of (A-1) is provided by Lemma A.2. As $w_1^{(n)}$ is not collinear to $w_4^{(n)}$, there exists $a_1 \neq 0$ and a_2 such that $w_1^{(n)} = a_1 w_3^{(n)} + a_2 w_4^{(n)}$. From the asymptotics (A-7) and the positivity of $w_1^{(n)}$ shown in Lemma A.3, one then has

$$w_1^{(n)} = be^{-\gamma_n t} + O(e^{(-\gamma_n - g)t}) \quad \text{as } t \to +\infty,\ b > 0.$$

We call T_0^n the profile associated to $w_1^{(n)}$ in the original space variable r: $T_0^n(r) = w_1^{(n)}(\ln(r))$, which solves $H^{(n)}T_0^{(n)} = 0$. The above identity means $T_0^n = a_1 r^{-\gamma_n} + O(r^{(-\gamma_n - g)})$ as $r \to +\infty$, and (A-6) implies $T_0^n(r) = \sum_{i=0}^q b_i^n r^{n+2l} + O(r^{n+2+2q})$ as $r \to 0$, for some coefficients $(b_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, for any $q \in \mathbb{N}$. These asymptotics propagate to the derivatives. This is the identity (2-7) we had to prove.

Let us denote by w another solution of (A-1) that is not collinear to $w_1^{(n)}$ and $w_4^{(n)}$. Now (A-6) and (A-7) imply that $w \sim ce^{(2-n-d)t}$ as $t \to -\infty$ and $w = de^{-\gamma_n t} + O(e^{(-\gamma_n - g)t})$ as $t \to +\infty$ with $c, d \neq 0$. These asymptotics propagate to higher derivatives. The solution of $H^{(n)}\Gamma^{(n)} = 0$ given by $\Gamma^{(n)}(r) = w(\ln(r))$ then satisfies the desired asymptotics (2-7). Eventually, the Laplacian on spherical harmonics of degree n is (for f radial)

$$\Delta(f Y_{n,k}) = \left(\left(\partial_{rr} + \frac{d-1}{r}\partial_r - \frac{n(d+n-2)}{r^2}\right) f\right) Y_{n,k},$$

meaning, by the asymptotics (2-7), that for any $j \in \mathbb{N}$, we know $\Delta^j(T_0^n(|x|)Y_{n,k}(x/|x|))$ is a continuous function near the origin. Therefore, $T_0^n Y_{n,k}$ is smooth close to the origin by elliptic regularity. It is also smooth outside as a product of smooth functions, and thus smooth everywhere, ending the proof of Lemma 2.3. □

Appendix B: Hardy- and Rellich-type inequalities

We recall in this section the Hardy and Rellich estimates, to make this paper self-contained. They are used throughout the paper, and especially to derive a fundamental coercivity property of the adapted high Sobolev norm in Appendix C. We now state a useful and very general Hardy inequality with possibly fractional weights and derivatives. A proof can be found in [Merle et al. 2015, Lemma B.2].

Lemma B.1 (Hardy-type inequalities). *Let $\delta > 0$, $q \geq 0$ satisfy $\left|q - \left(\frac{d}{2} - 1\right)\right| \geq \delta$ and $u : [1, +\infty) \to \mathbb{R}$ be smooth and satisfy*

$$\int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q}} y^{d-1}\, dy + \int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy < +\infty.$$

(i) *If $q > \frac{d}{2} - 1 + \delta$, then*

$$C(d,\delta) \int_{y \geq 1} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy - C'(d,\delta) u^2(1) \leq \int_{y\geq 1} \frac{|\partial_y u|^2}{y^{2q}} y^{d-1}\, dy. \tag{B-1}$$

(ii) *If $q < \frac{d}{2} - 1 - \delta$, then*

$$C(d,\delta) \int_{y \geq 1} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy \leq \int_{y\geq 1} \frac{|\partial_y u|^2}{y^{2q}} y^{d-1}\, dy. \tag{B-2}$$

Proof. Let $R > 1$. The fundamental theorem of calculus gives

$$\frac{u^2(R)}{R^{2q+2-d}} - u^2(1) = 2\int_1^R \frac{u \partial_y u}{y^{2q+2-d}}\, dy - (2q+2-d) \int_1^R \frac{u^2}{y^{2q+2-d}}\, dy.$$

The integrability of u^2/y^{2q+3-d} over $[1, +\infty)$ implies that $u^2(R_n)/R_n^{2q+2-d} \to 0$ along a sequence of radii $R_n \to +\infty$. Passing to the limit through this sequence we get

$$(2q+2-d) \int_1^{+\infty} \frac{u^2}{y^{2q+2-d}}\, dy - u^2(1) = 2 \int_1^{+\infty} \frac{u \partial_y u}{y^{2q+2-d}}\, dy.$$

We apply the Cauchy–Schwarz and Young inequalities to find

$$\begin{aligned}\left| 2 \int_1^{+\infty} \frac{u \partial_y u}{y^{2q+2-d}}\, dy \right| &\leq 2 \left(\int_1^{+\infty} \frac{u^2}{y^{2q+3-d}}\, dy \right)^{\frac12} \left(\int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q+1-d}}\, dy \right)^{\frac12} \\ &\leq \varepsilon \int_1^{+\infty} \frac{u^2}{y^{2q+3-d}}\, dy + \frac{1}{\varepsilon} \int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q+3-d}}\, dy\end{aligned}$$

for any $\varepsilon > 0$. If $q > \frac{d}{2} - 1 + \delta$, then the two above identities give

$$\int_1^{+\infty} \frac{u^2}{y^{2q+2-d}}\, dy \leq \frac{u^2(1)}{2\delta} + \frac{\varepsilon}{2\delta} \int_1^{+\infty} \frac{u^2}{y^{2q+3-d}}\, dy + \frac{1}{2\delta\varepsilon} \int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q+3-d}}\, dy.$$

Taking $\varepsilon=\delta$, one gets

$$\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}\,dy\le\frac{u^2(1)}{\delta}+\frac{1}{\delta^2}\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}\,dy,$$

which is precisely the identity (B-1) we had to prove. If $q<\frac{d}{2}-1-\delta$ then one obtains

$$\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}\,dy\le-\frac{u^2(1)}{2(\frac{d}{2}-1-q)}+\frac{\varepsilon}{2\delta}\int_1^{+\infty}\frac{u^2}{y^{2q+3-d}}\,dy+\frac{1}{2\delta\varepsilon}\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}\,dy.$$

Taking $\varepsilon=\delta$, one gets

$$\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}\,dy\le\frac{1}{\delta^2}\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}\,dy,$$

which is precisely the second identity (B-2) we had to prove. □

Lemma B.2 (Rellich-type inequalities). *For any $u\in H^2(\mathbb{R}^d)$,*

$$\left(\frac{(d-4)d}{4}\right)^2\int_{\mathbb{R}^d}\frac{u^2}{|x|^4}\,dx\le\int_{\mathbb{R}^d}|\Delta u|^2\,dx,\quad \frac{d^2}{4}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^2}\,dx\le\int_{\mathbb{R}^d}|\Delta u|^2\,dx. \tag{B-3}$$

If $q\ge 0$ and $u:\mathbb{R}^d\to\mathbb{R}$ is a smooth function satisfying

$$\int_{\mathbb{R}^d}\left(\frac{|\Delta u|^2}{1+|x|^{2q}}+\frac{|\nabla u|^2}{1+|x|^{2q+2}}+\frac{u^2}{1+|x|^{2q+4}}\right)dx<+\infty,$$

then

$$C(d,q)\sum_{1\le|\mu|\le 2}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{2q+4-2\mu}}\,dx-C'(d,q)\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4}}\,dx\le\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}\,dx. \tag{B-4}$$

Proof. The inequality (B-3) is standard and we omit its proof. To prove (B-4) we reason with smooth and compactly supported functions, and then conclude by a density argument.

Step 1: control of the first derivatives. Using integration by parts we compute

$$\int_{\mathbb{R}^d}\frac{u\Delta u}{1+|x|^{2q+2}}\,dx=-\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}\,dx+\frac12\int_{\mathbb{R}^d}u^2\Delta\left(\frac{1}{1+|x|^{2q+2}}\right)dx.$$

We then use the Cauchy–Schwarz and Young inequalities to obtain

$$C\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}\,dx-C'\int_{\mathbb{R}^d}u^2\left(\Delta\left(\frac{1}{1+|x|^{2q+2}}\right)-\frac{1}{(1+|x|^{2q+2})(1+|x|)^2}\right)dx$$
$$\le\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{(1+|x|^{2q+2})(1+|x|)^{-2}}\,dx.$$

Noticing that $(1+|x|^{2q+2})(1+|x|)^{-1}\sim(1+|x|^{2q})$ and that

$$\left|\Delta\left(\frac{1}{1+|x|^{2q+2}}\right)-\frac{1}{(1+|x|^{2q+2})(1+|x|)^2}\right|\le\frac{C}{1+|x|^{2q+4}}$$

leads to the estimate

$$C(d,p)\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}\,dx - C'(d,q)\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4}}\,dx \le \int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}\,dx. \tag{B-5}$$

Step 2: control of the second order derivatives. Again using integrations by parts one finds

$$\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}} = \int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^{2q}} + \sum_{i=1}^{n}\partial_{x_i}u\nabla\partial_{x_i}u.\nabla\Big(\frac{1}{1+|x|^{2q}}\Big) - \Delta u\nabla u.\nabla\Big(\frac{1}{1+|x|^{2q}}\Big),$$

in which by using the Cauchy–Schwarz and Young inequalities, for any $\varepsilon>0$, we can control the last two terms by

$$\Big|\int_{\mathbb{R}^d}\sum_{i=1}^{n}\partial_{x_i}u\nabla\partial_{x_i}u.\nabla\Big(\frac{1}{1+|x|^{2q}}\Big) - \Delta u\nabla u.\nabla\Big(\frac{1}{1+|x|^{2q}}\Big)\Big| \le C\varepsilon\int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^{2q}}\,dx + \frac{C}{\varepsilon}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}\,dx.$$

Therefore for ε small enough the two above identities yield

$$\int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^{2q}}\,dx \le C\Big(\int_{\mathbb{R}^d}\Big(\frac{|\Delta u|^2}{1+|x|^{2q}} + \frac{|\nabla u|^2}{1+|x|^{2q+2}} + \frac{u^2}{1+|x|^{2q+4}}\Big)\,dx\Big).$$

Combining this identity and (B-5), one obtains the desired identity (B-4). □

Lemma B.3 (weighted and fractional Hardy inequality). *Let*

$$0<\nu<1,\ k\in\mathbb{N}\ \textit{and}\ 0<\mu\ \textit{satisfying}\ \mu+\nu+k<\tfrac{1}{2}d,$$

and let f be a smooth function satisfying the decay estimates

$$|\partial^\kappa f(x)| \le \frac{C(f)}{1+|x|^{\mu+i}}\quad \textit{for}\ \kappa\in\mathbb{N}^d,\ |\kappa|_1=i,\ i=0,1,\dots,k+1. \tag{B-6}$$

Then for $\varepsilon\in\dot H^{\mu+k+\nu}$, we have $\varepsilon f\in\dot H^{\nu+k}$ with

$$\|\nabla^{\nu+k}(\varepsilon f)\|_{L^2} \le C(C(f),\nu,k,\mu,d)\,\|\nabla^{\mu+k+\nu}\varepsilon\|_{L^2}. \tag{B-7}$$

If f is smooth and radial then (B-6) is equivalent to

$$|\partial_r^i f(r)| \le \frac{C(f)}{1+r^{\mu+i}},\quad i=0,1,\dots,k+1. \tag{B-8}$$

Proof. **Step 1:** the case $k=0$. A proof of the case $k=0$ can be found in [Merle et al. 2015], for example.

Step 2: the case $k\ge 1$. Let f, ε, μ, ν and k satisfy the conditions of the lemma, with $k\ge 1$. Using the Leibniz rule for the entire part of the derivation,

$$\|\nabla^{\nu+k}(\varepsilon f)\|_{L^2}^2 \le C\sum_{\substack{(\kappa,\tilde\kappa)\in\mathbb{N}^{2d}\\ |\kappa|_1+|\tilde\kappa|_1=k}}\|\nabla^\nu(\partial^\kappa\varepsilon\partial^{\tilde\kappa}f\|_{L^2}^2. \tag{B-9}$$

We can now apply the result obtained for $k=0$ to the norms $\|\nabla^{\nu}(\partial^{\kappa_k}\varepsilon\partial^{\tilde{\kappa}_k}f\|_{L^2}^2$ in (B-9). We have indeed that $\partial^{\kappa}\varepsilon\in\dot{H}^{\mu+k_2+\nu}$, and that $\partial^{\tilde{\kappa}}f$ satisfies the appropriate decay condition by (B-6). It implies that for all $(\kappa,\tilde{\kappa})\in\mathbb{N}^{2d}$ with $|\kappa|_1+|\tilde{\kappa}|_1=k$,

$$\|\nabla^{\nu}(\partial^{\kappa_k}\varepsilon\partial^{\tilde{\kappa}_k}f\|_{L^2}^2\le C\|\nabla^{\nu+\mu+k}\varepsilon\|_{L^2}^2$$

which implies the result: $\|\nabla^{\nu+k}(\varepsilon f)\|_{L^2}^2\le C(C(f),\nu,d,k,\alpha)\|\nabla^{\nu+\mu+k}\varepsilon\|_{L^2}^2$.

Step 3: equivalence between the decay properties. We want to show that (B-6) and (B-8) are equivalent for radial smooth functions. Suppose that f is smooth, radial, and satisfies (B-6). Then one has

$$\partial_y^i f(y)=\frac{\partial f}{\partial_{x_1}^i}(|y|e_1),$$

where e_1 stands for the unit vector $(1,\dots,0)$ of $\mathbb{R}^d$. From this formula, we see that the condition (B-6) on $(\partial f/\partial_{x_1}^i)(|y|e_1)$ implies the radial condition (B-8). We now suppose that f is a smooth radial function satisfying the radial condition (B-8). Then there exists a smooth radial function ϕ such that

$$f(y)=\phi(y^2).$$

With a proof by induction that can be left to the reader, one has that the decay property (B-8) for f implies the following decay property for ϕ:

$$|\partial_y^i\phi(y)|\le\frac{C(f)}{1+y^{\frac{\mu}{2}+i}},\qquad i=0,1,\dots,k+1.$$

Now the standard derivatives of f are easier to compute with ϕ. We claim that for all $\kappa\in\mathbb{N}^d$ there exists a finite number of polynomials $P_i(x):=C_i x_1^{i_1}\cdots x_d^{i_d}$, for $1\le i\le l(\kappa)$, such that

$$\partial^{\kappa}f(x)=\sum_{i=1}^{l(\kappa)}P_i(x)\partial_{|x|}^{q(i)}\phi(|x|^2),$$

with $2q(i)-\sum_{j=1}^d i_j=|\kappa|_1$ for all i. The proof by induction of this fact can also be left to the reader. The decay property for ϕ then implies

$$\left|P_i(x)\partial_{|x|}^{q(i)}\phi(|x|^2)\right|\le\frac{C}{1+y^{\alpha+2q(i)-\sum_{j=1}^d i_j}}=\frac{C}{1+y^{\alpha+|\kappa|_1}},$$

which in turn implies the property (B-6). □

Appendix C: Coercivity of the adapted norms

Here we prove coercivity estimates for the operator H under suitable orthogonality conditions, following the techniques of [Raphaël and Rodnianski 2012]. We recall that the profiles used as orthogonality directions, $\Phi_M^{(n,k)}$, are defined by (4-1). To perform an analysis on each spherical harmonic and to be able to track the constants, we will not study directly $A^{(n)}$ and $A^{(n)*}$, but the asymptotically equivalent operators

$$\widetilde{A}^{(n)}: u \mapsto -\partial_y u + \widetilde{W}^{(n)} u, \quad A^{(n)*}: u \mapsto \frac{1}{y^{d-1}} \partial_y (y^{d-1} u) + \widetilde{W}^{(n)} u, \tag{C-1}$$

where

$$\widetilde{W}^{(n)} = -\frac{\gamma_n}{y}. \tag{C-2}$$

By the definition (1-18) of γ_n, they factorize the operator

$$\widetilde{H}^{(n)} := -\partial_{yy} - \frac{d-1}{y}\partial_y - \frac{pc_\infty^{p-1}}{y^2} + \frac{n(d+n-2)}{y^2} = \widetilde{A}^{(n)*}\widetilde{A}^{(n)}. \tag{C-3}$$

The strategy is the following. First we derive subcoercivity estimates for $\widetilde{A}^{(n)*}$, $\widetilde{A}^{(n)}$ and $H^{(n)}$. A summation yields subcoercivity for $-\Delta - pc_\infty^{p-1}/|x|^2$, and hence for H as they are asymptotically equivalent. Roughly, this subcoercivity implies that minimizing sequences of the functional $I(u) = \int uH^s u$ are "almost compact" on the unit ball of $\dot{H}^s \cap (\mathrm{Span}(\Phi_M^{(n,k)}))^\perp$. In particular if the infimum of I on this set was 0, it would be attained, which is impossible from the orthogonality conditions, yielding the coercivity $\int uH^s u \gtrsim \|u\|^2_{\dot{H}^s}$ via homogeneity.

Lemma C.1. *Let n be an integer, $q \geq 0$ and $u : [1, +\infty) \to \mathbb{R}$ be smooth satisfying*

$$\int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q}} y^{d-1}\, dy + \int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy < +\infty. \tag{C-4}$$

(i) *There exist two constants $c, c' > 0$ independent of n and q such that*

$$c\int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy - c'u^2(1) \leq \int_1^{+\infty} \frac{|\widetilde{A}^{(n)*}u|^2}{y^{2q}} y^{d-1}\, dy. \tag{C-5}$$

(ii) *Let $\delta > 0$ and suppose $\left|q - \left(\frac{d}{2} - 1 - \gamma_n\right)\right| > \delta$. Then there exist two constants $c(\delta), c'(\delta) > 0$ depending only on δ such that*

$$c(\delta)\int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1}\, dy - c'(\delta)u^2(1) \leq \int_1^{+\infty} \frac{|\widetilde{A}^{(n)}u|^2}{y^{2q}} y^{d-1}\, dy. \tag{C-6}$$

Proof. **Coercivity for $\widetilde{A}^{(n)*}$.** We first compute

$$\int_1^{+\infty} \frac{|\widetilde{A}^{(n)*}u|^2}{y^{2q}} y^{d-1}\, dy = \int_1^{+\infty} \frac{|\partial_y u + y^{-1}(d-1-\gamma_n)u|^2}{y^{2q}} y^{d-1}\, dy.$$

We make the change of variable $u = vy^{\gamma_n+1-d}$. By (C-4), $v^2/y^{2q-2\gamma_n+d+1}$ and $|\partial_y v|^2/y^{2q-2\gamma_n+d-1}$ are integrable on $[1, +\infty)$. As $q + \frac{d}{2} - \gamma_n \geq \frac{d}{2} - \gamma > 1$ by (1-9) and (1-18), we can apply (B-2) to the above identity and obtain (C-5) via

$$\int_1^{+\infty} \frac{|\widetilde{A}^{(n)*}u|^2}{y^{2q}} y^{d-1}\,dy = \int_1^{+\infty} \frac{|\partial_y v|^2}{y^{2q-2\gamma_n+2d-2}} y^{d-1}\,dy$$
$$\geq C \int_1^{+\infty} \frac{v^2}{y^{2q-2\gamma_n+2d-2}} y^{d-1}\,dy - C' v^2(1)$$
$$= C \int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1}\,dy - C' u^2(1).$$

Coercivity for $\widetilde{A}^{(n)}$. This time the integral we have to estimate is

$$\int_1^{+\infty} \frac{|\widetilde{A}^{(n)}u|^2}{y^{2q}} y^{d-1}\,dy = \int_1^{+\infty} \frac{|\partial_y u + y^{-1}\gamma_n u|}{y^{2p}} y^{d-1}\,dy.$$

We make the change of variable $u = v y^{-\gamma_n}$. By (C-4), $v^2/y^{2p+2\gamma_n-d+1}$ and $|\partial_y v|^2/y^{2p+2\gamma_n+3-d}$ are integrable on $[1,+\infty)$. As $\left|q - \left(\frac{d}{2} - 1 - \gamma_n\right)\right| > \delta$, one can apply (B-1) or (B-2) to the above identity: there exists $c = c(\delta)$ and $c' = c'(\delta)$ such that

$$\int_1^{+\infty} \frac{|\widetilde{A}^{(n)}u|^2}{y^{2q}} y^{d-1}\,dy = \int_1^{+\infty} \frac{|\partial_y v|^2}{y^{2q+2\gamma_n}} y^{d-1}$$
$$\geq c \int_1^{+\infty} \frac{v^2}{y^{2q+2\gamma_n+2}} y^{d-1}\,dy - c' v^2(1)$$
$$= c \int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1}\,dy - c' u^2(1),$$

which is precisely the identity (C-6). □

Lemma C.2 (coercivity of H under suitable orthogonality conditions). *Let $\delta > 0$ and $q \geq 0$ such that*[22] $\left|q - \left(\frac{d}{2} - 2 - \gamma_n\right)\right| \geq \delta$ *for all $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N} \cup \{-1\}$ be the lowest number such that $q - \left(\frac{d}{2} - 2 - \gamma_{n_0+1}\right) < 0$. Then there exists a constant $c(\delta) > 0$ such that for all $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ satisfying the integrability condition*

$$\int_{\mathbb{R}^d} \frac{|\Delta u|^2}{1+|x|^{2q}} + \frac{|\nabla u|^2}{1+|x|^{2q+2}} + \int \frac{u^2}{1+|x|^{2q+4}} < +\infty$$

and the orthogonality conditions[23] ($\Phi_M^{(n,k)}$ *being defined in* (4-1))

$$\langle u, \Phi_M^{(n,k)} \rangle = 0 \quad \text{for } 0 \leq n \leq n_0,\ 1 \leq k \leq k(n), \tag{C-7}$$

one has the inequality

$$c(\delta)\left(\int_{\mathbb{R}^d} \frac{|\Delta u|^2}{1+|x|^{2q}} + \frac{|\nabla u|^2}{|x|^2(1+|x|^{2q})} + \frac{u^2}{|x|^4(1+|x|^{2q})}\right) \leq \int_{\mathbb{R}^d} \frac{|Hu|^2}{1+|x|^{2q}}. \tag{C-8}$$

[22] We recall that $\gamma_n \to -\infty$; hence for δ small enough many q satisfy this condition.
[23] With the convention that there are no orthogonality conditions required if $n_0 = -1$.

Proof. In what follows, $C(\delta)$ and $C'(\delta)$ denote strictly positive constants that may vary but only depend on δ, d and p.

Step 1: We claim the following subcoercivity estimate for $\tilde{H} := -\Delta - pc_\infty^{p-1}/|x|^2$:

$$\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)} \frac{|\tilde{H}u|^2}{|x|^{2q}}\,dx \geq C(\delta)\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)} \frac{u^2}{|x|^{2q+4}}\,dx - C'(\delta)\big(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2 + \|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2\big), \quad \text{(C-9)}$$

where $f_{|\mathcal{S}^{d-1}(1)}$ denotes the restriction of f to the sphere. We now prove this inequality. We start with the decomposition

$$u(x) = \sum_{n,\,1\leq k\leq k(n)} u^{(n,k)}(|x|)Y^{(n,k)}\Big(\frac{x}{|x|}\Big).$$

We recall the link between u and its decomposition ($\tilde{H}^{(n)}$ being defined by (C-3)):

$$\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)} \frac{|\tilde{H}u|^2}{|x|^{2q}}\,dx = \sum_{n,\,1\leq k\leq k(n)} \int_1^{+\infty} \frac{|\tilde{H}^{(n)}u^{(n,k)}|^2}{y^{2q}} y^{d-1}\,dy, \quad \text{(C-10)}$$

$$\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)} \frac{u^2}{|x|^{2q+4}}\,dx = \sum_{n,\,1\leq k\leq k(n)} \int_1^{+\infty} \frac{|u^{(n,k)}|^2}{y^{2q+4}} y^{d-1}\,dy. \quad \text{(C-11)}$$

As $\tilde{H}^{(n)} = \tilde{A}^{(n)*}\tilde{A}^{(n)}$ and $\big|q - \big(\frac{d}{2} - 2 - \gamma_n\big)\big| > \delta$ for all $n \in \mathbb{N}$, we apply (C-5) and (C-6) to obtain for each $n \in \mathbb{N}$,

$$\begin{aligned}\int_1^{+\infty} &\frac{|\tilde{H}^{(n)}u^{(n,k)}|^2}{y^{2q}} y^{d-1}\,dy \\ &\geq C(\delta)\int_1^{+\infty} \frac{|u^{(n,k)}|^2}{y^{2q+4}} y^{d-1}\,dy - C'(\delta)\big((u^{(n,k)})^2(1) + \tilde{A}^{(n)}(u^{(n,k)})^2(1)\big). \quad \text{(C-12)}\end{aligned}$$

We now sum this identity over n and k. The second term on the right-hand side is

$$\sum_{n,1\leq k\leq k(n)} (u^{(n,k)})^2(1) = \int_{\mathcal{S}^{d-1}} \Big(\sum_{n,\,1\leq k\leq k(n)} u^{(n,k)}(1)Y^{(n,k)}(x)\Big)^2\,dx = \int_{\mathcal{S}^{d-1}} u^2(x)\,dx$$

because $(Y^{(n,k)})_{n,1\leq k\leq n}$ is an orthonormal basis of $L^2(\mathcal{S}^{d-1})$. From (C-1), and as $\gamma_n \sim -n$ as $n \to +\infty$ by (1-18), the last term on the right-hand side of (C-12) is

$$\begin{aligned}\sum_{n,\,1\leq k\leq n} |\tilde{A}^{(n)}u^{(n,k)}|^2(1) &\leq C\sum_{n,\,1\leq k\leq k(n)} (1+n^2)|u^{(n,k)}|^2(1) + |\partial_y u^{(n,k)}|^2 \\ &\leq C\big(\|u_{|\mathcal{S}^{d-1}(1)}\|_{H^1}^2 + \|\nabla u_{|\mathcal{S}^{d-1}(1)}\cdot\vec{n}\|_{L^2}^2\big) \\ &\leq C\big(\|u_{|\mathcal{S}^{d-1}}\|_{L^2}^2 + \|\nabla u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2\big).\end{aligned}$$

We insert the two above equations into (C-12) and obtain

$$\sum_{n,\,1\le k\le n}\int_1^{+\infty}\frac{|\widetilde{H}^{(n)}u^{(n,k)}|^2}{y^{2q}}y^{d-1}\,dy \ge C(\delta)\sum_{n,\,1\le k\le n}\int_1^{+\infty}\frac{|u^{(n,k)}|^2}{y^{2q+4}}y^{d-1}\,dy - C'(\delta)\big(\|u_{|\mathcal{S}^{d-1}}\|_{L^2}^2+\|\nabla u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2\big).$$

In turn, we insert this identity into (C-10) using (C-11) to obtain the desired estimate (C-9).

Step 2: subcoercivity for H. We will prove the estimate

$$\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2q}}\,dx \ge C(\delta)\Big(\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}\,dx+\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^2(1+|x|^{2q})}\,dx+\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}\,dx\Big) -C'(\delta)\Big(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+\alpha}}+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}^2\Big). \tag{C-13}$$

Away from the origin, the Cauchy–Schwarz and Young inequalities, the bound $V+pc_\infty^{p-1}|x|^{-2}=O(|x|^{-2-\alpha})$ from (2-2) and (C-9) give (for $C>0$)

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{|Hu|^2}{|x|^{2q}}\,dx &= \int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{|\widetilde{H}u+(V+pc_\infty^{p-1}|x|^{-2})u|^2}{|x|^{2q}}\,dx\\ &\ge C\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{|\widetilde{H}u|^2}{|x|^{2q}}\,dx - C'\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{|u|^2}{|x|^{2q+4+2\alpha}}\,dx\\ &\ge C(\delta)\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{u^2}{1+|x|^{2q+4}}\\ &\qquad -C'(\delta)\Big(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d\setminus\mathcal{B}^d(1)}\frac{|u|^2}{1+|x|^{2q+4+2\alpha}}\Big).\end{aligned}$$

Close to the origin, using Rellich's inequality (B-3),

$$\begin{aligned}\int_{\mathcal{B}^d(1)}|Hu|^2\,dx &\ge C\int_{\mathcal{B}^d(1)}|\Delta u|^2\,dx-\frac{1}{C}\int_{\mathcal{B}^d(1)}|u|^2\,dx\\ &\ge C\int_{\mathcal{B}^d(1)}\frac{|u|^2}{|x|^4}\,dx-\frac{1}{C}\|u\|_{H^1(\mathcal{B}^{d-1}(1))}.\end{aligned}$$

Combining the two previous estimates we obtain the intermediate identity

$$\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2q}}\,dx\ge C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}\,dx-C'(\delta)\Big(\|u_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|\mathcal{S}^{d-1}(1)}\|_{L^2}^2 +\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}\,dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}^2\Big).$$

Now, as $H=-\Delta+V$ with $V=O((1+|x|)^{-2})$, using Young's inequality, the above identity and (B-4), for $\varepsilon>0$ small enough (depending on δ) one has

$$\begin{aligned}
&\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2p}}\,dx\\
&\quad=(1-\varepsilon)\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2p}}\,dx|Hu|^2dx+\varepsilon\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2p}}\,dx\\
&\quad\geq(1-\varepsilon)C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}\,dx\\
&\qquad-C'(\delta)\biggl(\|u_{|S^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|S^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}\,dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}\biggr)\\
&\qquad+\frac{\varepsilon}{2}\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}\,dx-\varepsilon\int_{\mathbb{R}^d}\frac{|Vu|^2}{1+|x|^{2q}}\,dx\\
&\quad\geq(1-\varepsilon)C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}\,dx\\
&\qquad-C'(\delta)\biggl(\|u_{|S^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|S^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}\,dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}\biggr)\\
&\qquad+C(q)\frac{\varepsilon}{2}\sum_{1\leq|\mu|\leq 2}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{2q+4-2\mu}}\,dx-\varepsilon C'(q)\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4}}\,dx\\
&\quad\geq C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}+\frac{C(q)\varepsilon}{2}\sum_{1\leq|\mu|\leq 2}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{2q+4-2\mu}}\\
&\qquad-C'(\delta)\biggl(\|u_{|S^{d-1}(1)}\|_{L^2}^2+\|(\nabla u)_{|S^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}\,dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}\biggr),
\end{aligned}$$

which is the identity (C-13) we claimed.

Step 3: coercivity for H. We now argue by contradiction. Suppose that (C-8) does not hold. Up to a renormalization, this means that there exists a sequence of functions $(u_n)_{n\in\mathbb{N}}$ such that, for all n,

$$\int_{\mathbb{R}^d}\frac{|Hu_n|^2}{1+|x|^{2q}}\to 0,\qquad \int_{\mathbb{R}^d}\frac{|\Delta u_n|^2}{1+|x|^{2q}}+\frac{|\nabla u_n|^2}{|x|^2(1+|x|^{2q})}+\frac{|u_n|^2}{|x|^4(1+|x|^{2q})}=1. \tag{C-14}$$

Up to a subsequence, we can suppose that $u_n\to u_\infty\in H^2_{\mathrm{loc}}(\mathbb{R}^d)$, the local convergence in L^2 being strong for $(u_n)_{n\in\mathbb{N}}$ and $(\nabla u_n)_{n\in\mathbb{N}}$, and weak for $(\nabla^2 u_n)_{n\in\mathbb{N}}$. Then (C-14) implies

$$\|u_n\|_{H^1(\mathcal{B}^{d-1}(1))}^2+\int_{\mathbb{R}^d}\frac{|u_n|^2}{1+|x|^{2q+4+\alpha}}\to\|u_\infty\|_{H^1(\mathcal{B}^{d-1}(1))}^2+\int_{\mathbb{R}^d}\frac{|u_\infty|^2}{1+|x|^{2q+4+\alpha}}.$$

Now u_n converges strongly to u_∞ in $H^s(\mathcal{B}^d(0,1))$ for any $0\leq s<2$. The trace theorem for Sobolev spaces ensures that

$$\|(u_n)_{|S^{d-1}(1)}\|_{L^2}^2+\|(\nabla u_n)_{|S^{d-1}(1)}\|_{L^2}^2\to\|(u_\infty)_{|S^{d-1}(1)}\|_{L^2}^2+\|(\nabla u_\infty)_{|S^{d-1}(1)}\|_{L^2}^2.$$

We insert the three previous identities into the subcoercivity estimate (C-13) yielding

$$\|(u_\infty)_{|S^{d-1}(1)}\|_{L^2}^2+\|(\nabla u_\infty)_{|S^{d-1}(1)}\|_{L^2}^2+\int_{\mathbb{R}^d}\frac{|u_\infty|^2}{1+|x|^{2q+4+\alpha}}+\|u_\infty\|_{H^1(\mathcal{B}^d(1))}^2\neq 0,$$

which means that $u_\infty\neq 0$. On the other hand, the lower semicontinuity of norms for the weak topology and (C-14) imply

$$Hu_\infty=0.$$

Hence u_∞ is a nontrivial function in the kernel of H, and is smooth from elliptic regularity. It satisfies the integrability condition (still from lower semicontinuity)

$$\int_{\mathbb{R}^d}\frac{|\Delta u_\infty|^2}{1+|x|^{2q}}\,dx+\frac{|\nabla u_\infty|^2}{1+|x|^{2q+2}}\,dx+\int\frac{|u_\infty|^2}{1+|x|^{2q+4}}\,dx<+\infty.$$

We now decompose u_∞ into spherical harmonics, $u_\infty=\sum_{n,\,1\leq k\leq k(n)}u_\infty^{(n,k)}Y_{(n,k)}$, and will show that for each n,k one must have $u_\infty^{(n,k)}=0$, which will give a contradiction. For each n,k, the nullity $Hu_\infty=0$ implies $H^{(n)}u_\infty^{(n,k)}$, where $H^{(n)}$ is defined in (1-36). By Lemma 2.3 this means $u_\infty=aT_0^{(n)}+b\Gamma^{(n)}$ for a and b two real numbers. The previous equation implies the following integrability for $u_\infty^{(n,k)}$:

$$\int\frac{|u_\infty^{(n,k)}|^2}{1+y^{2q+4}}y^{d-1}\,dy<+\infty.$$

By (2-7), as $\Gamma^{(n)}\sim y^{-d-n+2}$ does not satisfy this integrability at the origin whereas $T_0^{(n)}$ is regular, one must have $b=0$. Then, if $n\geq n_0+1$,

$$\frac{|T_0^{(n)}|^2}{1+y^{2q+4}}y^{d-1}\sim y^{-2\gamma_n-2q-5+d}.$$

From the assumption on n_0 and (1-18), one has

$$-2\gamma_n-2q-5+d=-1-2\big(q+2+\gamma_{n_0+1}-\tfrac{d}{2}\big)+2(\gamma_{n_0+1}-\gamma_n)>-1,$$

implying that $|T_0^{(n)}|^2/(1+y^{2q+4})y^{d-1}$ is not integrable on $[0,+\infty)$; hence $a=0$. If $n\leq n_0$ then the orthogonality condition (C-7) goes to the limit as $\Phi_M^{(n,k)}$ is compactly supported and implies

$$\langle u^\infty,\Phi_M^{(n,k)}\rangle=0,$$

which, in spherical harmonics, can be rewritten as

$$0=\langle u_\infty^{(n,k)},\Phi_M^{(n,k)}\rangle=a\langle T_0^{(n)},\Phi_M^{(n,k)}\rangle.$$

However, from (4-3) this in turn implies $a=0$. We have proven that for all n,k $u_\infty^{(n,k)}=0$; hence $u_\infty=0$, which is the desired contradiction, as we proved earlier that u_∞ is nontrivial. The coercivity (C-8) must then be true. □

If one adds analogous orthogonality conditions for the derivatives of u and uses a bit more the structure of the Laplacian, one gets that the weighted norm $\|H^i/(1+|x|^p)u\|_{L^2}$ controls all derivatives of lower order with corresponding weights.

Lemma C.3 (coercivity of the iterates of H). *Let i be an integer with $2i > \sigma$ such that for all $n \in \mathbb{N}$ satisfying $m_n + \delta_n \le i$ one has $\delta_n \neq 0$. Let n_0 be the lowest integer such that $m_{n_0+1} + \delta_{n_0+1} > i$. Let $u \in \dot H^{2i} \cap \dot H^\sigma(\mathbb{R}^d)$ satisfy (where $\Phi_M^{(n,k)}$ is defined in (4-1))*

$$\langle u, H^j \Phi_M^{n,k}\rangle = 0 \quad \textit{for } 0 \le n \le n_0,\ 0 \le j \le i - m_n - 1,\ 1 \le k \le k(n). \tag{C-15}$$

Then there exists a constant $\delta > 0$ such that for all $0 \le \delta' \le \delta$,

$$C(\delta, i) \sum_{|\mu| \le 2i} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1 + |x|^{4i - 2\mu + 2\delta'}}\, dx \le \int_{\mathbb{R}^d} \frac{|H^i u|^2}{1 + |x|^{2\delta'}}\, dx, \tag{C-16}$$

which in particular implies that

$$\|u\|_{\dot H^{2i}} \le C(\delta, i) \left(\int_{\mathbb{R}^d} |H^i u|^2\, dx \right)^{\frac12}. \tag{C-17}$$

Proof. **Step 1:** equivalence of weighted norms. We claim that for all integers j,

$$H^j u = (-\Delta)^j u + \sum_{|\mu| \le 2j-2} f_{j,\mu} \partial^\mu u \tag{C-18}$$

for some smooth functions f_μ having the decay $|\partial^{\mu'} f_{j,\mu}| \le C(1 + |x|^{2j - |\mu| + |\mu'|})^{-1}$. This identity is true for $j = 1$ because $Hu = -\Delta u + Vu$ with the potential V being smooth and having the required decay by (2-2). If the aforementioned identity holds true for $j \ge 1$ then

$$\begin{aligned} H^{j+1} u &= (-\Delta + V)\Big((-\Delta)^j u + \sum_{|\mu| \le 2j-2} f_{j,\mu} \partial^\mu u \Big) \\ &= (-\Delta)^{j+1} u + V(-\Delta)^j u + \sum_{|\mu| \le 2j-2} (-\Delta + V)(f_{j,\mu} \partial^\mu u), \end{aligned}$$

and hence it is true for $j+1$ since V is smooth and satisfies the decay (2-2). By induction it is true for all $j \in \mathbb{N}$ and (C-18) is proven. Then (C-18) implies that

$$\int_{\mathbb{R}^d} \frac{|H^i u|^2}{1 + |x|^{2\delta}}\, dx \le C \sum_{|\mu| \le 2i} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1 + |x|^{4i - 2|\mu| + 2\delta'}}\, dx. \tag{C-19}$$

Step 2: weighted integrability in $\dot H^{2i} \cap \dot H^\sigma$. We claim that for all functions $u \in \dot H^{2i} \cap \dot H^\sigma(\mathbb{R}^d)$ and $\delta' > 0$,

$$\sum_{|\mu| \le 2i} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1 + |x|^{4i - 2|\mu| + 2\delta'}}\, dx < +\infty. \tag{C-20}$$

Indeed, let μ be a $|\mu|$-tuple with $|\mu| \le 2i$. We split into two cases. First if $|\mu| \le \sigma$, as $\sigma < \frac d2$ and $2i > \sigma$, the Hardy inequality B.3 yields

$$\int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1 + |x|^{4i - 2|\mu| + 2\delta'}}\, dx \le \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1 + |x|^{2(\sigma - |\mu|)}}\, dx \le C \|u\|^2_{\dot H^\sigma} < +\infty$$

and we are done. If $\sigma < \mu \le 2i$ then by interpolation $u \in \dot H^{|\mu|}(\mathbb{R}^d)$ and then

$$\int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1+|x|^{4i-2|\mu|+2\delta'}}\,dx \le \int |\partial^\mu u|^2\,dx < +\infty.$$

Thus (C-20) holds, which together with (C-19), implies, for all $\delta' \ge 0$,

$$\sum_{j=0}^{i} \int_{\mathbb{R}^d} \frac{|H^j u|^2}{1+|x|^{4i-4j+2\delta'}}\,dx + \frac{|\nabla H^{j-1} u|^2}{1+|x|^{4i-4j+2+2\delta'}}\,dx < +\infty. \tag{C-21}$$

Step 3: intermediate coercivity. Let $\delta = \min\big(\delta_0,\dots,\delta_{n_0+1},\frac12\big)$ if $\delta_{n_0+1} \ne 0$ and $\delta = \min\big(\delta_0,\dots,\delta_{n_0},\frac12\big)$ if $\delta_{n_0+1} = 0$. The conditions on the δ_n of the lemma imply $\delta > 0$. We claim that for all integers $1 \le l \le i$,

$$C(\delta)\int_{\mathbb{R}^d} \frac{|H^{l-1}u|^2}{1+|x|^{4i-4(l-1)+2\delta'}} + C(\delta)\int_{\mathbb{R}^d} \frac{|\nabla H^{l-1}u|^2}{1+|x|^{4i-4l+2+2\delta'}} \le \int_{\mathbb{R}^d} \frac{|H^l u|^2}{1+|x|^{4i-4l+2\delta'}}. \tag{C-22}$$

We now prove this estimate. We want to apply Lemma C.2 to the function $H^{l-1}u$ with weight $q = \delta' + 2(i-l)$. To use it, we have to check the orthogonality and integrability conditions that are required, and the conditions on the weight.

Integrability condition. It is true because of (C-21).

Condition on the weight. For the case $n \ge n_0+1$, by (1-23) one computes

$$\big|\delta'+2(i-l)-\big(\tfrac d2-\gamma_n-2\big)\big| = \big|\delta'-2\delta_{n_0+1}-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1})\big|. \tag{C-23}$$

One has $2(l-1) \ge 0$ as $l \ge 1$ and $2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1}) \ge 0$ because $(m_n+\delta_n)_n$ is an increasing sequence from (1-22) and (1-18). For the subcase $\delta_{n_0+1} = 0$, as $m_{n_0+1} > i$ and m_{n_0+1} is an integer, $2(m_{n_0+1}-i) > 2$. Therefore $-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1}) = -a$ for $a \ge 2$, and inserting it into the above identity as $0 < \delta' < 1$ gives

$$\big|\delta'+2(i-l)-\big(\tfrac d2-\gamma_n-2\big)\big| = |\delta'-a| \ge \delta' \ge \delta.$$

For the subcase $\delta_{n_0+1} \ne 0$, we have $\delta'-2\delta_{n_0+1} \le \delta-2\delta_{n_0+1} \le -\delta_{n_0+1} \le -\delta$. Moreover, $m_{n_0+1} \ge i$ and $-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1}) \le 0$, implying

$$\delta'-2\delta_{n_0+1}-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1}) \le \delta'-2\delta_{n_0+1} \le -\delta,$$

and therefore by (C-23) this yields in that case

$$\big|\delta'+2(i-l)-\big(\tfrac d2-\gamma_n-2\big)\big| \ge \delta.$$

In both subcases one has $\big|\delta'+2(i-l)-\big(\frac d2-\gamma_n-2\big)\big| \ge \delta$. For the case $n \le n_0$,

$$\big|\delta'+2(i-l)-\big(\tfrac d2-\gamma_n-2\big)\big| = \big|\delta'-2\delta_n+2(i-l+1-m_n)\big|.$$

In the above identity, $2(i-l+1-m_n)$ is an even integer, and $\delta'-2\delta_n$ is a number satisfying $\delta'-2\delta_n \le \delta-2\delta_n \le -\delta$ and we recall that $\delta<1$, and $\delta'-2\delta_n \ge -2\delta_n \ge -1$. Therefore $|\delta'-2\delta_n+2(i-l+1-m_n)| \ge \delta$, yielding

$$\left|\delta' + 2(i-l) - \left(\tfrac{d}{2} - \gamma_n - 2\right)\right| \geq \delta.$$

Therefore, for each $n \in \mathbb{N}$, we have $\left|\delta' + 2(i-l) - \left(\frac{d}{2} - \gamma_n - 2\right)\right| \geq \delta$.

Orthogonality conditions. Let $n_0' = n_0'(l) \in \mathbb{N} \cup \{-1\}$ be the lowest number such that

$$2(i-l+1) + \delta' - 2(m_{n_0'+1} + \delta_{n_0'+1}) < 0.$$

By construction one has $n_0' \leq n_0$. If $n_0' = -1$ then we are done because no orthogonality condition is required. If $n_0' \neq -1$, let n be an integer, $0 \leq n \leq n_0'$. By the definition of n_0',

$$2(i-l+1) + \delta' - 2(m_n + \delta_n) > 0,$$

which implies $0 \leq l-1 \leq i - m_n - 1$ as $\delta' - 2\delta_n \leq \delta - 2\delta_n \leq -\delta_n \leq 0$. The orthogonality condition (C-15) then gives, for any $1 \leq k \leq k(n)$,

$$\langle u, H^{l-1}\Phi_M^{(n,k)}\rangle = 0.$$

We have then proved that for all $0 \leq n \leq n_0'$, $1 \leq k \leq k(n)$,

$$\langle H^{l-1}u, \Phi_M^{(n,)k}\rangle = 0,$$

which are the required orthogonality conditions.

Conclusion. One can apply Lemma C.2 to $H^{l-1}u$ with weight $q = 2i - 2l + \delta'$, giving the desired coercivity estimate (C-22).

Step 4: iterations of coercivity estimates. We show the following bound by induction on $l = 0, \dots, i$:

$$\int_{\mathbb{R}^d} \frac{|H^l u|^2}{1+|x|^{2\delta'}}\,dx \geq c(\delta, i) \sum_{0 \leq |\mu| \leq 2l} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1+|x|^{4i-2\mu+2\delta'}}\,dx. \tag{C-24}$$

This property is naturally true for $l = 0$. We now suppose it is true for $l-1$ with $0 \leq l-1 \leq i-1$. From the formula (C-18) relating Δ^l to H^l, we see that (using the Cauchy–Schwarz and Young inequalities)

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{|H^l u|^2}{1+|x|^{4(i-l)+2\delta'}} &\geq C(i)\int_{\mathbb{R}^d} \frac{|\Delta^l u|^2}{1+|x|^{4(i-l)+2\delta'}} - C'(i) \sum_{0 \leq |\mu| \leq 2l-2} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1+|x|^{4i-2|\mu|+2\delta'}} \\ &\geq C(i)\int_{\mathbb{R}^d} \frac{|\Delta^l u|^2}{1+|x|^{4(i-l)+2\delta'}} - C'(i)\int_{\mathbb{R}^d} \frac{|H^i u|^2}{1+|x|^{2\delta'}},\end{aligned}$$

where we used the induction hypothesis (C-24) for $l-1$ for the second line. We now use (C-24) and (B-4) to recover a control over all derivatives:

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{|\Delta^l u|^2}{1+|x|^{4(i-l)+2\delta'}} &\geq C(i)\sum_{1\leq|\mu|\leq 2}\int_{\mathbb{R}^d}\frac{|\partial^\mu\Delta^{l-1}u|^2}{1+|x|^{4(i-l)+4-2|\mu|}}-C'(i)\int_{\mathbb{R}^d}\frac{|\Delta^{l-1}u|^2}{1+|x|^{4(i-l)+4}}\\
&\geq C(i)\sum_{0\leq|\mu|\leq 2}\int_{\mathbb{R}^d}\frac{|\Delta^{l-1}\partial^\mu u|^2}{1+|x|^{4(i-(l-1))-2|\mu|}}-C'(\delta,i)\int_{\mathbb{R}^d}\frac{|H^{l-1}u|^2}{1+|x|^{2\delta'}}\\
&\geq C(i)\sum_{0\leq|\mu|\leq 2}\sum_{1\leq|\mu'|\leq 2}\int_{\mathbb{R}^d}\frac{|\partial^{\mu'}\Delta^{l-2}\partial^\mu u|^2}{1+|x|^{4(i-(l-1))+4-2|\mu|-2|\mu'|}}\\
&\qquad -C'(i)\int_{\mathbb{R}^d}\frac{|\Delta^{l-2}u|^2}{1+|x|^{4(i-l)+8}}-C'(\delta,i)\int_{\mathbb{R}^d}\frac{|H^{l-1}u|^2}{1+|x|^{2\delta'}}\\
&\geq C(i)\sum_{0\leq|\mu|\leq 4}\int_{\mathbb{R}^d}\frac{|\Delta^{l-2}\partial^\mu u|^2}{1+|x|^{2p+4(i-(l-2))-2\mu}}-C'(i,\delta)\int_{\mathbb{R}^d}\frac{|H^{l-1}u|^2}{1+|x|^{2\delta'}}\\
&\;\;\vdots\\
&\geq C(i)\sum_{0\leq|\mu|\leq 2l}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{2p+4-2\mu+2\delta'}}-C'(\delta,i)\int_{\mathbb{R}^d}\frac{|H^{l-1}u|^2}{1+|x|^{2\delta'}}.\end{aligned}$$

Inserting this last equation into the previous one we obtain

$$\int_{\mathbb{R}^d}\frac{|H^l u|^2}{1+|x|^{4(i-l)+2\delta'}}\geq C(\delta,i)\sum_{0\leq|\mu|\leq 2l}\int_{\mathbb{R}^d}\frac{|\Delta^{l-2}\partial^\mu u|^2}{1+|x|^{2p+4-2\mu}}-C'(\delta,i)\int_{\mathbb{R}^d}\frac{|H^{l-1}u|^2}{1+|x|^{2\delta'}}.$$

This, together with (C-22), gives that (C-24) is true for l. Hence by induction it is true for i, which is precisely the estimate (C-16) we had to show and ends the proof of the lemma. □

Appendix D: Specific bounds for the analysis

This section is dedicated to the statement and the proof of several estimates used in the analysis.

Lemma D.1 (specific bounds for the error in the trapped regime). *Let ε be a function satisfying (4-25) and (4-11). We recall that $\mathcal{E}_\sigma$ and $\mathcal{E}_{2s_L}$ are defined by (4-9) and (4-7). Then the following bounds hold*:

(i) Interpolated Hardy-type inequality. *For $\mu\in\mathbb{N}^d$ and $q>0$ satisfying $\sigma\leq|\mu|+q\leq 2s_L$*

$$\int\frac{|\partial^\mu\varepsilon|^2}{1+|y|^{2q}}\,dy\leq C(M)\mathcal{E}_\sigma^{\frac{2s_L-(|\mu|+q)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{|\mu|+q-\sigma}{2s_L-\sigma}}. \tag{D-1}$$

(ii) Weighted L^∞ bound for low order derivative. *For $0\leq a\leq 2$ and $\mu\in\mathbb{N}^d$ with $|\mu|\leq 1$,*

$$\left\|\frac{\partial^\mu\varepsilon}{1+|y|^a}\right\|_{L^\infty}\leq C(K_1,K_2,M)\sqrt{\mathcal{E}_\sigma}^{1+O(\frac{1}{L^2})}\frac{1}{s^{a+|\mu|_1+(\frac{d}{2}-\sigma)+(\frac{2}{p-1}+a+|\mu|_1)\alpha/L+O(\frac{\sigma-s_c}{L})}}. \tag{D-2}$$

(iii) L^∞ bound for high order derivative. *For $\mu\in\mathbb{N}^d$ with $|\mu|\leq s_L$,*

$$\|\partial^\mu\varepsilon\|_{L^\infty}^2\leq C(M)\mathcal{E}_\sigma^{\frac{2s_L-|\mu|_1-d/2}{2s_L-\sigma}+O(\frac{1}{L^2})}\mathcal{E}_{2s_L}^{\frac{|\mu|_1+d/2-\sigma}{2s_L-\sigma}+O(\frac{1}{L^2})}. \tag{D-3}$$

Proof. (i) We first recall that from the coercivity estimate (C-16) one has

$$\|\nabla^\sigma \varepsilon\|_{L^2}^2 = \mathcal{E}_\sigma, \quad \|\nabla^{2s_L}\varepsilon\|_{L^2}^2 \le C(M)\|H^{s_L}\varepsilon\|_{L^2}^2 = C(M)\mathcal{E}_{2s_L}.$$

If the weight satisfies $q < \frac{d}{2}$, then the inequality (D-1) claimed in the lemma is a consequence of the standard Hardy inequality, followed by an interpolation:

$$\begin{aligned}\left\|\frac{\partial^\mu \varepsilon}{1+|x|^q}\right\|_{L^2}^2 &\le C\|\nabla^{|\mu|_1+q}\varepsilon\|_{L^2}^2 \le C\|\nabla^\sigma\varepsilon\|_{L^2}^{2\frac{2s_L-(|\mu|_1+q)}{2s_L-\sigma}}\|\nabla^{2s_L}\varepsilon\|_{L^2}^{2\frac{|\mu|_1+q-\sigma}{2s_L-\sigma}} \\ &\le C(M)\mathcal{E}_\sigma^{\frac{2s_L-(|\mu|_1+q)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{|\mu|_1+q-\sigma}{2s_L-\sigma}}.\end{aligned}$$

If the potential satisfies $q = 2s_L - |\mu|$, then the inequality (D-1) claimed in the lemma is a consequence of the coercivity estimate (C-16):

$$\left\|\frac{\partial^\mu \varepsilon}{1+|x|^q}\right\|_{L^2}^2 \le C(M)\mathcal{E}_{2s_L}.$$

For a weight that is in-between, i.e., $\frac{d}{2} \le q < 2s_L - |\mu|_1$, the inequality (D-1) is then obtained by interpolating the two previous ones, as

$$\frac{|\varepsilon|^2}{1+|x|^{2b}} \sim \left(\frac{|\varepsilon|^2}{1+|x|^{2a}}\right)^{\frac{c-b}{c-a}}\left(\frac{|\varepsilon|^2}{1+|x|^{2c}}\right)^{\frac{b-a}{c-a}}.$$

(ii) As the dimension is $d \ge 11$ and $L \gg 1$ is big, one has $\partial^\mu\varepsilon/(1+|x|^a) \in L^\infty$ with the following bound (using the bound (i) we just derived):

$$\begin{aligned}\left\|\frac{\partial^\mu\varepsilon}{1+|x|^a}\right\|_{L^\infty} &\le C(z)\left(\left\|\nabla^{\frac{d}{2}-z}\left(\frac{\partial^\mu\varepsilon}{1+|x|^a}\right)\right\|_{L^2} + \left\|\nabla^{\frac{d}{2}+z}\left(\frac{\partial^\mu\varepsilon}{1+|x|^a}\right)\right\|_{L^2}\right) \\ &\le C(z)\big(\|\nabla^{\frac{d}{2}-z+a+|\mu|_1}\varepsilon\|_{L^2} + \|\nabla^{\frac{d}{2}+a+|\mu|_1+z}\varepsilon\|_{L^2}\big) \\ &\le C(M,z)\big(\mathcal{E}_\sigma^{\frac{2s_L-(a+|\mu|_1+d/2-z)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{a+|\mu|_1+d/2-z-\sigma}{2s_L-\sigma}} + \mathcal{E}_\sigma^{\frac{2s_L-(a+|\mu|_1+d/2+z)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{a+|\mu|_1+d/2+z-\sigma}{2s_L-\sigma}}\big)\end{aligned}$$

for $z > 0$ small enough. We then let z_1 be so close to 0 (of order L^{-1}) that its impact when using the bootstrap bounds (4-25) is of order $s^{-\frac{1}{L^2}}$ (since the constant $C(M, z_1)$ explodes as z_1 approaches 0, we cannot take $z_1 = 0$, but z_1 very close to $\frac{d}{2}$ is enough for our purpose). Inserting the bootstrap bounds (4-25) then yields the desired result (D-2).

(iii) It can be proved exactly the same way we did for (ii). □

Lemma D.2 (a nonlinear estimate). *Let $d \in \mathbb{N}$, $a \ge 0$ and $b > \frac{d}{2}$. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain. There exists a constant $C > 0$ such that for any $u, v \in H^{\max(a,b)}(\Omega)$,*[24]

$$\|uv\|_{H^a(\Omega)} \le C\big(\|u\|_{H^a(\Omega)}\|v\|_{H^b(\Omega)} + \|u\|_{H^b(\Omega)}\|v\|_{H^a(\Omega)}\big). \tag{D-4}$$

[24]The product uv indeed belongs to $H^a(\Omega)$ as $H^{\max(a,b)}(\Omega)$ is an algebra since $b > \frac{d}{2}$.

Proof. Without loss of generality one assumes $\frac{d}{2} < b \leq \frac{d}{2} + \frac{1}{4}$,

$$b := \frac{d}{2} + \delta_b, \quad \text{with } 0 < \delta_b \leq \tfrac{1}{4}. \tag{D-5}$$

Indeed, if (D-4) holds for all $b \in \left(\frac{d}{2}, \frac{d}{2} + \frac{1}{4}\right]$ then for any $b' > \frac{d}{2} + \frac{1}{4}$, applying (D-4) to the pair of parameters $\left(a, \frac{d}{2} + \frac{1}{4}\right)$ and using the fact that $\|f\|_{H^{d/2+1/4}(\Omega)} \leq \|f\|_{H^b(\Omega)}$ for any $f \in H^b(\Omega)$ gives that (D-4) holds for the pair of parameters (a, b').

Step 1: a scalar inequality. We claim that for all $(\nu_1, \nu_2) \in [0,1]^2$ with $\nu_1 + \nu_2 \geq 1$ and for all $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, +\infty)$ satisfying $\lambda_1 \leq \lambda_2$ and $\lambda_3 \leq \lambda_4$,

$$\lambda_1^{\nu_1} \lambda_2^{1-\nu_1} \lambda_3^{\nu_2} \lambda_4^{1-\nu_2} \leq \lambda_1 \lambda_4 + \lambda_2 \lambda_3. \tag{D-6}$$

We now prove this estimate. Since $1 - \nu_1 - \nu_2 \leq 0$ and $0 \leq 1 - \nu_2 \leq 1$, one has

$$\forall (x, z) \in [1, +\infty) \times [0, +\infty), \quad x^{1-\nu_1-\nu_2} z^{1-\nu_2} \leq z^{1-\nu_2} \leq 1 + z.$$

Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, +\infty)$ satisfying $0 < \lambda_1 \leq \lambda_2$ and $0 < \lambda_3 \leq \lambda_4$. We apply the above estimate to $x = \frac{\lambda_2}{\lambda_1} \geq 1$ and $z = \frac{\lambda_1 \lambda_4}{\lambda_2 \lambda_3}$, and multiply both sides by $\lambda_2 \lambda_3$, yielding the desired estimate (D-6) after simplifications. If $\lambda_1 = 0$ or $\lambda_3 = 0$, (D-6) always holds. Consequently, (D-6) holds for all $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, +\infty)$ satisfying $0 < \lambda_1 \leq \lambda_2$ and $0 < \lambda_3 \leq \lambda_4$.

Step 2: proof in the case $\Omega = \mathbb{R}^d$ and $a \geq b$. We claim that for $u, v \in H^a(\mathbb{R}^d)$,

$$\|uv\|_{H^a(\mathbb{R}^d)} \leq C\left(\|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)} + \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)}\right). \tag{D-7}$$

We now show the above estimate. Let $u, v \in H^{s_2}(\mathbb{R}^d)$. First, one obtains an L^2 bound using Hölder and Sobolev embedding $\left(\text{as } b > \frac{d}{2}\right)$:

$$\|uv\|_{L^2(\mathbb{R}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)} \|v\|_{L^\infty(\mathbb{R}^d)} \leq C \|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)}. \tag{D-8}$$

Secondly, one decomposes $a = A + \delta_a$, where $A := E[a] \in \mathbb{N}$ is the entire part of a and $0 \leq \delta_a < 1$. Using the Leibniz rule one has the identity

$$\|\nabla^a (uv)\|_{L^2(\mathbb{R}^d)}^2 \leq C \sum_{\substack{(\mu_1, \mu_2) \in \mathbb{N}^{2d} \\ |\mu_1| + |\mu_2| = A}} \|\nabla^{\delta_a} (\partial^{\mu_1} u \partial^{\mu_2} v)\|_{L^2(\mathbb{R}^d)}^2. \tag{D-9}$$

We fix $(\mu_1, \mu_2) \in \mathbb{N}^{2d}$ with $|\mu_1| + |\mu_2| = A$ in the sum and aim at estimating the corresponding term. We recall the commutator estimate

$$\|\nabla^{\delta_a} (\partial^{\mu_1} u \partial^{\mu_2} v)\|_{L^2} \lesssim \|\nabla^{|\mu_1| + \delta_a} u\|_{L^{p_1}} \|\partial^{\mu_2} v\|_{L^{q_1}} + \|\nabla^{|\mu_2| + \delta_a} v\|_{L^{p_2}} \|\partial^{\mu_1} u\|_{L^{q_2}} \tag{D-10}$$

for $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{2}$, provided $2 \leq p_1, p_2 < +\infty$ and $2 \leq q_1, q_2 \leq +\infty$. We now chose appropriate exponents p_1 and p_2 in several cases.

Case 1. $|\mu_2| = 0$. Then $|\mu_1| + \delta_a = a$ and using Sobolev embedding $\left(\text{as } b > \frac{d}{2}\right)$,

$$\|\nabla^{|\mu_1| + \delta_a} u\|_{L^2(\mathbb{R}^d)} \|\partial^{\mu_2} v\|_{L^\infty(\mathbb{R}^d)} \leq C \|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)}. \tag{D-11}$$

Case 2. $1 \leq |\mu_2| < a - \frac{d}{2}$ and $|\mu_1| + \delta_a < b$. Then $b < |\mu_2| + \frac{d}{2} < a$ by (D-5) and using Sobolev embedding, one computes

$$\|\nabla^{|\mu_1|+\delta_a} u\|_{L^2(\mathbb{R}^d)} \|\partial^{\mu_2} v\|_{L^\infty(\mathbb{R}^d)} \leq C \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)}. \tag{D-12}$$

Case 3. $1 \leq |\mu_2| < a - \frac{d}{2}$ and $b \leq |\mu_1| + \delta_a$. Then $b < |\mu_2| + \frac{d}{2} < a$ by (D-5) and $b \leq |\mu_1| + \delta_a \leq a$. We let $x := \min\big(\frac{\delta_b}{2}, a - |\mu_2| - \frac{d}{2}\big) > 0$. Using Sobolev embedding, interpolation and (D-6) $\big($since $b > \frac{d}{2} + x$ and $|\mu_1| + |\mu_2| + \delta_a = a\big)$, one computes

$$\begin{aligned} \|\nabla^{|\mu_1|+\delta_a} u\|_{L^2(\mathbb{R}^d)} \|\partial^{\mu_2} v\|_{L^\infty(\mathbb{R}^d)} &\leq C \|u\|_{H^{|\mu_1|+\delta_a}(\mathbb{R}^d)} \|v\|_{H^{|\mu_2|+d/2+x}(\mathbb{R}^d)} \\ &\leq C \|u\|_{H^b(\mathbb{R}^d)}^{\frac{a-|\mu_1|-\delta_a}{a-b}} \|u\|_{H^a(\mathbb{R}^d)}^{\frac{|\mu_1|+\delta_a-b}{a-b}} \|v\|_{H^b(\mathbb{R}^d)}^{\frac{a-|\mu_2|-d/2-x}{a-b}} \|v\|_{H^a(\mathbb{R}^d)}^{\frac{|\mu_2|+d/2+x-b}{a-b}} \\ &\leq C\big(\|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)} + \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)}\big). \end{aligned} \tag{D-13}$$

Case 4. $a - \frac{d}{2} \leq |\mu_2| < a$. Let $x := \frac{1}{2}\min(a - |\mu|_2, \delta_b) > 0$. We define p_1, q_1 and s by

$$\frac{1}{q_1} := \frac{1}{2} - \frac{a - x - |\mu_2|}{d}, \qquad \frac{1}{p_1} = \frac{1}{2} - \frac{1}{q_1} \quad \text{and} \quad s = \frac{d}{q_1}.$$

One has $|\mu_1| + \delta_a + s = \frac{d}{2} + x < b$, and, using Sobolev embedding,

$$\|\nabla^{|\mu_1|+\delta_a} u\|_{L^{p_1}} \|\partial^{\mu_2} v\|_{L^{q_1}} \leq C \|u\|_{H^{|\mu_1|+\delta_a+s}} \|v\|_{H^{a-x}} \leq C \|u\|_{H^b} \|v\|_{H^a} \tag{D-14}$$

and $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}$, $p_1 \neq +\infty$.

Case 5. $|\mu_2| = a$. Then $|\mu_1| + \delta_a = 0$ and using Sobolev embedding $\big($as $b > \frac{d}{2}\big)$,

$$\|\nabla^{|\mu_1|+\delta_a} u\|_{L^\infty(\mathbb{R}^d)} \|\partial^{\mu_2} v\|_{L^2(\mathbb{R}^d)} \leq C \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)}. \tag{D-15}$$

Conclusion. In all possible cases, by (D-11)–(D-15) there always exist $p_1, q_1, p_2, q_2 \in [2, +\infty)$ with $p_1, p_2 \neq +\infty$, $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}$ and

$$\begin{aligned} \|\nabla^{|\mu_1|+\delta_a} u\|_{L^{p_1}(\mathbb{R}^d)} \|\partial^{\mu_2} v\|_{L^{q_1}(\mathbb{R}^d)} + \|\nabla^{|\mu_1|} u\|_{L^{q_2}} v \|\nabla^{|\mu_2|+\delta_a} v\|_{L^{p_2}(\mathbb{R}^d)} \\ \leq C \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)} + C \|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)}, \end{aligned}$$

where the estimate for the second term on the left-hand side of the above equation comes from symmetric reasoning. We now come back to (D-9), and apply (D-10) and the above identity to obtain

$$\|\nabla^a (uv)\|_{L^2(\mathbb{R}^d)} \leq C \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)} + C \|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)}.$$

The above estimate and (D-8) imply the desired estimate (D-7) by interpolation.

Step 3: proof in the case $\Omega = \mathbb{R}^d$ and $a \leq b$. The proof is similar and simpler and we do not write it here. Therefore, (D-7) holds for all $a \geq 0$ and $b > \frac{d}{2}$.

Step 4: proof in the case of a smooth bounded domain Ω. There exists $\widetilde{C} > 0$ such that for any $f \in H^{\max(a,b)}(\Omega)$ there exists an extension $\tilde{f} \in H^{\max(a,b)}(\mathbb{R}^d)$ with compact support, satisfying $\tilde{f} = f$

on Ω and
$$\frac{1}{\widetilde{C}}\|\tilde{f}\|_{H^c(\mathbb{R}^d)} \le \|f\|_{H^c(\Omega)} \le \widetilde{C}\|\tilde{f}\|_{H^c(\mathbb{R}^d)}, \quad c=a,b;$$

see [Adams and Fournier 2003]. Let $u, v \in H^{\max(a,b)}(\Omega)$ and denote by $\tilde{u}$ and $\tilde{v}$ their respective extensions. Using (D-7) and the above estimate then yields

$$\begin{aligned}\|uv\|_{H^a(\Omega)} &\le \|\tilde{u}\tilde{v}\|_{H^a(\mathbb{R}^d)}\\ &\le C\big(\|\tilde{u}\|_{H^a(\mathbb{R}^d)}\|\tilde{v}\|_{H^b(\mathbb{R}^d)} + \|\tilde{u}\|_{H^b(\mathbb{R}^d)}\|\tilde{v}\|_{H^a(\mathbb{R}^d)}\big)\\ &\le C\widetilde{C}^2\big(\|u\|_{H^a(\Omega)}\|v\|_{H^b(\Omega)} + \|u\|_{H^b(\Omega)}\|v\|_{H^a(\Omega)}\big)\end{aligned}$$

and (D-4) is obtained. □

Appendix E: Geometrical decomposition

This section is devoted to the proof of Lemma 4.3.

Lemma E.1. *Let X denote the functional space*
$$X := \big\{u \in L^\infty(\mathcal{B}^d(0,4M)) : \langle u - Q, H\Phi_M^{(0,1)}\rangle > \|u-Q\|_{L^\infty(\mathcal{B}^d(0,3M))}\big\}. \tag{E-1}$$
There exists $\kappa, K > 0$ such that for all $u \in X \cap \{\|u-Q\|_{L^\infty(\mathcal{B}^d(0,4M)))} < \kappa\}$, there exists a unique choice of parameters $b \in \mathbb{R}^{\mathcal{I}}$ with $b_1^{(0,1)} > 0$, $\lambda > 0$ and $z \in \mathbb{R}^d$ such that the function $v := (\tau_{-z}u)_\lambda - \widetilde{Q}_b$ satisfies
$$\langle v, H^i\Phi_M^{(n,k)}\rangle = 0 \quad \text{for } 0 \le n \le n_0,\ 1 \le k \le k(n),\ 0 \le i \le L_n \tag{E-2}$$
and such that
$$|\lambda - 1| + |z| + \sum_{(n,k,i)\in\mathcal{I}} |b_i^{(n,k)}| \le K. \tag{E-3}$$
Moreover, b, λ and z are Fréchet differentiable[25] *and satisfy*
$$|\lambda - 1| + |z| + \sum_{(n,k,i)\in\mathcal{I}} |b_i^{(n,k)}| \le K\|u-Q\|_{L^\infty(\mathcal{B}^d(0,3M))}. \tag{E-4}$$

Proof. We first define the application ξ as
$$\begin{aligned}&\xi : L^\infty(\mathcal{B}^d(0,3M)) \times (0,+\infty) \times \mathbb{R}^{d+\#\mathcal{I}} \to \mathbb{R}^{1+d+\#\mathcal{I}},\\ &(u,\tilde{\lambda},\tilde{z},\tilde{b}) \mapsto \big(\langle(\tau_{\tilde{z}}u)_{\frac{1}{\tilde{\lambda}}} - Q - \alpha_{\tilde{b}}, H^i\Phi_M^{(n,k)}\rangle\big), \quad \text{where } 1 \le k \le k(n),\ 0 \le n \le n_0,\ 0 \le i \le L_n.\end{aligned} \tag{E-5}$$

Then ξ is $\mathcal{C}^\infty$. From the definition (3-7) of α_b, and the orthogonality conditions (4-3), the differential of ξ with respect to the second variable at the point $(Q,1,0,\dots,0)$ is the diagonal matrix
$$D^{(2)}\xi(Q,1,0,\dots,0) = -\begin{pmatrix}\langle T_0^{(0)}, \chi_M T_0^{(0)}\rangle \,\mathrm{Id}_{L+1} & & \\ & \ddots & \\ & & \langle T_0^{(n_0)}, \chi_M T_0^{(n_0)}\rangle\,\mathrm{Id}_{L_{n_0}}\end{pmatrix}, \tag{E-6}$$

[25] For the ambient Banach space $L^\infty(\mathcal{B}^d(0,3M))$.

where Id_{L_n} is the $L_n\times L_n$ identity matrix. $D^{(2)}\xi(Q,1,0,\dots,0)$ is invertible for M large by (4-3). Consequently, from the implicit functions theorem, there exist κ, $K>0$, such that for all

$$u\in X\cap\{\|u-Q\|_{L^\infty(\mathcal{B}^d(0,3M))}<\kappa\},$$

there exists a choice of the parameters $\tilde\lambda=\tilde\lambda(u)$, $\tilde z=\tilde z(u)$ and $\tilde b=\tilde b(u)$ such that

$$\xi(u,\tilde\lambda,\tilde z,\tilde b)=0,\quad |\tilde\lambda-1|+|\tilde z|+\sum_{(n,k,i)\in\mathcal{I}}|\tilde b_i^{(n,k)}|\le K\|u-Q\|_{L^\infty(\mathcal{B}^d(3M))} \tag{E-7}$$

and it is the unique solution of $\xi(u,\tilde\lambda,\tilde z,\tilde b)=0$ in the range

$$|\tilde\lambda-1|+|\tilde z|+\sum_{(n,k,i)\in\mathcal{I}}|\tilde b_i^{(n,k)}|\le K.$$

Moreover, they are Fréchet differentiable, again from the implicit function theorem. Now, defining $\lambda=1/\tilde\lambda$, $b=\tilde b$ and $z=-\tilde z$, this means by (E-5) that the function $w:=(\tau_{-z}u)_\lambda-Q-\alpha_b$ satisfies

$$\langle w,H^i\Phi_M^{(n,k)}\rangle=0,\quad \text{for } 0\le n\le n_0,\ 1\le k\le k(n),\ 0\le i\le L_n.$$

Finally, still from the implicit function theorem, from the identity for the differential (E-6), the definition (E-1) of X and (4-3),

$$\begin{aligned} b_1^{(0,1)}&=-[D^{(2)}\xi(Q,1,0,\dots,0)]^{-1}(\xi(u,1,0,\dots,0))+o(\|u-Q\|_{L^\infty(\mathcal{B}^d(3M))})\\ &=\frac{\langle u-Q,H^1\Phi_M^{(0,1)}\rangle}{\langle T_0^{(0)},\chi_M T_0^{(0)}\rangle}+o\big(\langle u-Q,H^1\Phi_M^{(0,1)}\rangle\big)>0,\end{aligned}$$

where the $o(\cdot)$ is as $\kappa\to0$, and the strict positivity is then for κ small enough. Consequently, in that case $\tilde Q_b=Q+\chi_{(b_1^{(0,1)})^{-(1+\eta)/2}}\alpha_b$ is well defined, and one has $(b_1^{(0,1)})^{-\frac{1+\eta}{2}}\gg 2M$ for κ small enough. Thus, for $v:=(\tau_{-z}u)_\lambda-\tilde Q_b$,

$$\langle v,H^i\Phi_M^{(n,k)}\rangle=\langle\tilde v,H^i\Phi_M^{(n,k)}\rangle=0\ \text{ for } 0\le n\le n_0,\ 1\le k\le k(n),\ 0\le i\le L_n$$

because the support of $v-\tilde v$ is outside $\mathcal{B}^d(0,2M)$. One has found a choice of the parameters λ, b and z such that $b_1^{(0,1)}>0$ and (E-2) and (E-3) hold. This choice is unique in the range (E-3) and the parameters are Fréchet differentiable since under (E-3), they are equal to the parameters given by the above inversion of ξ. □

Lemma E.2. *There exist κ^*, $\tilde K>0$ such that the following holds for all $0<\kappa<\kappa^*$. Let $\mathcal{O}$ be the open set of $L^\infty(\mathcal{B}^d(0,1))$ of functions u satisfying (4-4). For each $u\in\mathcal{O}$ there exists a unique choice of the parameters $\lambda\in(0,\frac{1}{4M})$, $z\in\mathcal{B}^d(0,\frac14)$ and $b\in\mathbb{R}^{\mathcal{I}}$ such that $b_1^{(0,1)}>0$ and $v=(\tau_{-z}u)_\lambda-\tilde Q_b\in L^\infty\big(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\})\big)$ satisfies*[26]

$$\langle v,H^i\Phi_M^{(n,k)}\rangle=0\quad \textit{for } 0\le n\le n_0,\ 1\le k\le k(n),\ 0\le i\le L_n \tag{E-8}$$

[26]The following assertions make sense as v is defined on $\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\})$, which indeed contains $\mathcal{B}^d(0,2M)$ since $0<\lambda<\frac{1}{4M}$ and $|z|\le\frac14$, and as $\Phi_M^{(n,k)}$ is compactly supported in $\mathcal{B}^d(0,2M)$ by (4-1).

and

$$\sum_{(n,k,i)\in\mathcal{I}} |b_i^{(n,k)}| + \|v\|_{L^\infty(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\}))} \le \tilde{K}\kappa. \tag{E-9}$$

Moreover, the functions λ, z *and* b *defined this way are Fréchet differentiable on* $\mathcal{O}$.

Proof. Let K and κ_0 be the numbers associated to Lemma E.1.

Step 1: existence. Let

$$(\tilde{\lambda},\tilde{z}) \in \big(0,\tfrac{1}{8M}\big)\times\mathcal{B}^d\big(0,\tfrac{1}{8}\big) \tag{E-10}$$

be such that

$$\|u - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} < \frac{\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}},$$

$$\|(\tau_{-\tilde{z}}u)_{\tilde{\lambda}} - Q\|_{L^\infty(\mathcal{B}^d(4M))} < \langle(\tau_{-\tilde{z}}u)_{\tilde{\lambda}} - Q, H\Phi_M^{(0,1)}\rangle,$$

which exists by (4-4). We define $w := (\tau_{-\tilde{z}}u)_{\tilde{\lambda}}$. It is defined on the set $(1/\tilde{\lambda})(\mathcal{B}(1)-\tilde{z})$, which contains $\mathcal{B}^d(7M)$ as $0<\tilde{\lambda}<\frac{1}{8M}$ and $|z|\le\frac{1}{8}$. From this fact and the above estimates, w satisfies

$$\|w-Q\|_{L^\infty(\mathcal{B}(7M))} < \kappa, \quad \|w-Q\|_{L^\infty(\mathcal{B}^d(3M))} < \langle w-Q, H\Phi_M^{(0,1)}\rangle. \tag{E-11}$$

Thus for κ small enough, one can apply Lemma E.1: there exists a choice of the parameters z', b' and λ' such that $v' = (\tau_{-z'}w)_{\lambda'} - \tilde{Q}_{b'}$ satisfies (E-8) and $b_1^{\prime(0,1)} > 0$. This choice is unique in the range

$$|\lambda'-1| + |z'| + \sum_{(n,k,i)\in\mathcal{I}} |b_i^{\prime(n,k)}| \le K. \tag{E-12}$$

Moreover, the estimate

$$|\lambda'-1| + |z'| + \sum_{(n,k,i)\in\mathcal{I}} |b_i^{\prime(n,k)}| \le K\|w-Q\|_{L^\infty(\mathcal{B}^d(0,3M))} \le K\kappa.$$

holds. Now we define

$$b = b', \quad z = \tilde{z} + \tilde{\lambda}z', \quad \lambda = \tilde{\lambda}\lambda' \tag{E-13}$$

and $v = v'$. One has then $b_1^{(0,1)} > 0$, and from (E-10) and the above estimate,

$$\sum_{(n,k,i)\in\mathcal{I}} |b_i^{(n,k)}| \le K\kappa, \quad |z| \le \tfrac{1}{4}, \quad 0 < \lambda < \frac{1}{4M}$$

for κ small enough. From the definitions of w, v' and v one has the identity

$$u = (v + \tilde{Q}_b)_{z,\frac{1}{\lambda}}, \quad \text{with } v \text{ satisfying (E-8).}$$

From (3-7), (3-29) and the above estimate,

$$\|v\|_{L^\infty(\frac{1}{\lambda}(\mathcal{B}^d(1)-z))} = \lambda^{\frac{2}{p-1}}\|u - \tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})\|_{L^\infty(\mathcal{B}^d(1))}$$
$$\le \lambda^{\frac{2}{p-1}}\|u - \tau_{\tilde{z}}(Q_{\frac{1}{\tilde{\lambda}}})\|_{L^\infty(\mathcal{B}^d(1))} + \lambda^{\frac{2}{p-1}}\|\tau_{\tilde{z}}(Q_{\frac{1}{\tilde{\lambda}}}) - \tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})\|_{L^\infty(\mathcal{B}^d(1))} \le CK\kappa$$

for some constant $C>1$ independent of the others. Therefore, one takes $\widetilde{K}=CK$, and the choice of parameters λ, z and b that we just found provides the decomposition claimed by the lemma and the existence is proven.

Step 2: differentiability. We claim that the parameters λ, b and z found in Step 1 are unique; this will be proven in the next step. Therefore, from their construction using the auxiliary variables $\tilde\lambda$ and $\tilde z$ in Step 1, and since the parameters λ', z' and b' provided by Lemma E.1 are Fréchet differentiable, λ, b and z are Fréchet differentiable.

Step 3: uniqueness. Let $\hat b$, $\hat\lambda$, $\hat z$ be another choice of parameters with $\hat b_1^{(0,1)}>0$, $0<\lambda<\frac{1}{4M}$ and $|z|\leq\frac14$ such that (E-8) and (E-9) hold for $\hat v=(\tau_{-\hat z}u)_{\hat\lambda}-\widetilde{Q}_b$. The function $(\tau_{-\tilde z}u)_{\tilde\lambda}$, where $\tilde\lambda$ and $\tilde z$ were defined in (E-10) in the first step, then satisfies the bound

$$\|(\tau_{-\tilde z}u)_{\tilde\lambda}-Q\|_{L^\infty(\mathcal{B}(3M))}<\kappa_0$$

for κ small enough by (E-11), and admits two decompositions

$$(\tau_{-\tilde z}u)_{\tilde\lambda}=(\widetilde{Q}_{b'}+v')_{z',\frac{1}{\lambda'}}=(\widetilde{Q}_{\hat b}+\hat v)_{\frac{\hat z-\tilde z}{\tilde\lambda},\frac{\tilde\lambda}{\hat\lambda}}$$

such that v and v' satisfy (E-8). By (E-12), the first parameters satisfy

$$|\lambda'-1|+|z'|+\sum_{(n,k,i)\in\mathcal{I}}|b_i'^{(n,k)}|\leq K\kappa_0.$$

We claim that the second parameters satisfy

$$\left|\frac{\tilde\lambda}{\hat\lambda}-1\right|+\left|\frac{\hat z-\tilde z}{\tilde\lambda}\right|+\sum_{(n,k,i)\in\mathcal{I}}|\hat b_i^{(n,k)}|\leq K\kappa_0, \tag{E-14}$$

which will be proven hereafter. Then, as such parameters are unique under the above bound by Lemma E.1, one obtains

$$\frac{\tilde\lambda}{\hat\lambda}=\frac{1}{\lambda'},\quad \frac{\hat z-\tilde z}{\tilde\lambda}=z',\quad \hat b=b',$$

implying that $\hat\lambda=\lambda$, $\hat z=z$ and $\hat b=b$, where λ, z and b are the choice of the parameters given by the first step defined by (E-13). The uniqueness is obtained.

Proof of (E-14). From the assumptions on $\hat b$, $\hat\lambda$ and $\hat z$, the definition of $\widetilde{Q}_b$ (3-29) and (E-9), for κ small enough we have

$$\|u-Q_{\hat z,\frac{1}{\hat\lambda}}\|_{L^\infty(\mathcal{B}^d(1))}\leq\frac{C\widetilde{K}\kappa}{\hat\lambda^{\frac{2}{p-1}}}.$$

From (E-10) one also has

$$\|u-Q_{\tilde z,\frac{1}{\tilde\lambda}}\|_{L^\infty(\mathcal{B}^d(1))}\leq\frac{\kappa}{\tilde\lambda^{\frac{2}{p-1}}}.$$

From the two above estimates, one deduces that

$$\|Q_{\hat z,\frac{1}{\hat\lambda}}-Q_{\tilde z,\frac{1}{\tilde\lambda}}\|_{L^\infty(\mathcal{B}^d(1))}\leq\frac{\kappa}{\tilde\lambda^{\frac{2}{p-1}}}+\frac{C\widetilde{K}\kappa}{\hat\lambda^{\frac{2}{p-1}}}. \tag{E-15}$$

Assume that $\hat{\lambda} \le \tilde{\lambda}$. Then, since Q is radially symmetric and attains its maximum at the origin, and $\hat{z} \in \mathcal{B}^d(0,1)$ because $|\hat{z}| \le \frac{1}{4}$, the above inequality at $x = \hat{z}$ implies

$$\begin{aligned} Q(0)\bigg(\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}} - \frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}\bigg) &= Q_{\hat{z},\frac{1}{\hat{\lambda}}}(\hat{z}) - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}(\tilde{z}) \\ &\le Q_{\hat{z},\frac{1}{\hat{\lambda}}}(\hat{z}) - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}(\hat{z}) \\ &= |Q_{\hat{z},\frac{1}{\hat{\lambda}}}(\hat{z}) - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}(\hat{z})| \\ &\le C\tilde{K}\kappa\bigg(\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}} + \frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}\bigg), \end{aligned}$$

which gives

$$\bigg|\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}} - \frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}\bigg| \le C\tilde{K}\kappa\bigg(\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}} + \frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}\bigg).$$

The symmetric reasoning works in the case $\hat{\lambda} \ge \tilde{\lambda}$ and one obtains that in both cases

$$\bigg|\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}} - \frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}\bigg| \le C\tilde{K}\kappa\bigg(\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}} + \frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}\bigg).$$

Basic computations show that for κ small enough the above identity implies

$$\bigg|1 - \frac{\hat{\lambda}}{\tilde{\lambda}}\bigg| \le C\tilde{K}\kappa \quad \text{or} \quad \hat{\lambda} = \tilde{\lambda}(1 + O(\kappa)),$$

obtaining the first bound in (E-14) for κ small enough. We insert the above estimate into (E-15), yielding

$$\|Q_{\hat{z},\frac{1}{\hat{\lambda}}} - Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} \le \|Q_{\hat{z},\frac{1}{\hat{\lambda}}} - Q_{\hat{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))}\| + \|Q_{\tilde{z},\frac{1}{\tilde{\lambda}}} - Q_{\hat{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))}\| \le \frac{C\tilde{K}\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}},$$

which implies in renormalized variables $\big($as $|\hat{z}| \le \frac{1}{8}$ and $\tilde{\lambda} \le \frac{1}{8M}\big)$,

$$\|Q - \tau_{\frac{\hat{z}-\tilde{z}}{\tilde{\lambda}}} Q\|_{L^\infty(\mathcal{B}^d(0,2M))} \le C\tilde{K}\kappa.$$

As Q is smooth, radially symmetric and radially decreasing this implies

$$\bigg|\frac{\hat{z} - \tilde{z}}{\tilde{\lambda}}\bigg| \le C\tilde{K}\kappa \quad \text{or} \quad \hat{z} = \tilde{z} + \tilde{\lambda}O(\kappa)$$

and the second bound in (E-14) is obtained. □

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