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TIME-WEIGHTED ESTIMATES IN LORENTZ SPACES AND SELF-SIMILARITY FOR WAVE EQUATIONS WITH SINGULAR POTENTIALS

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We show time-weighted estimates in Lorentz spaces for the linear wave equation with singular potential. As a consequence, assuming radial symmetry on initial data and potentials, we obtain well-posedness of global solutions in critical weak-L^p spaces for semilinear wave equations. In particular, we can consider the Hardy potential $V(x) = c|x|^{-2}$ for small $|c|$. Self-similar solutions are obtained for potentials and initial data with the right homogeneity. Our approach relies on performing estimates in the predual of weak-L^p, i.e., the Lorentz space $L^{(p',1)}$.

1. Introduction

We are concerned with the linear wave equation with potential

$$\begin{cases} \Box u + Vu = f(x,t), & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ \vec{u}(0) = (u(0,x), \partial_t u(0,x)) = (0,0), & x \in \mathbb{R}^n, \end{cases} \tag{1-1}$$

and the semilinear wave equation

$$\begin{cases} \Box u + Vu = \mu |u|^{p-1} u, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ \vec{u}(0) = (u_0, u_1), & x \in \mathbb{R}^n, \end{cases} \tag{1-2}$$

where $\Box = \partial_t^2 - \Delta_x$, $n \geq 5$ odd, $\mu \in \{+1, -1\}$ (focusing or defocusing case) and $p > (n^2+n-4)/(n(n-3))$. The problems (1-1) and (1-2) are addressed in the radial setting.

The semilinear wave equation (1-2) with $V = 0$ has three notions of critical nonlinearity, namely the Strauss critical power $p = p_{\text{str}}$, conformal critical power $p = p_{\text{conf}}$ and energy critical power $p = p_{\text{e}}$. The former p_{str} is the positive root of

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Strauss [1981] conjectured about the existence for $p > p_{\text{str}}$ or nonexistence for $1 < p \leq p_{\text{str}}$ of global solutions for (1-2) with small compact support initial data. The conjecture of Strauss has a nice history (see, e.g., [Wang and Yu 2012]) and was completed by [Yordanov and Zhang 2006; Zhou 2007] (see also [Lai and Zhou 2014]). The conformal power p_{conf} is linked to the conformal symmetry map

$$u(x,t) \mapsto u_{\text{conf}}(x,t) = (t^2 - |x|^2)^{-\frac{n-1}{2}} u\left(\frac{x}{t^2-|x|^2}, \frac{t}{t^2-|x|^2}\right) \quad \text{for } |x| < |t|.$$

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More precisely, u_{conf} solves (1-2) with $V=0$ if u does and $p=p_{\mathrm{conf}}=(n+3)/(n-1)$ for $n\geq 2$. The power $p_{\mathrm{e}}=(n+2)/(n-2)$ ($p_{\mathrm{e}}=\infty$ if $n=2$) is connected to the scaling invariance of the conserved energy. In fact, for $p=p_{\mathrm{e}}$ and $V=0$, the conserved energy

$$E(u,\partial_t u)=\frac{1}{2}\int_{\mathbb{R}^n}|\nabla_x u|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^n}|\partial_t u|^2\,dx+\frac{\mu}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}\,dx$$

is invariant by the scaling map

$$u(x,t)\mapsto u_\gamma(x,t):=\gamma^{\frac{2}{p-1}}u(\gamma x,\gamma t),\quad \gamma>0, \tag{1-3}$$

namely

$$E(u_\gamma,\partial_t u_\gamma)=\gamma^{\frac{4}{p-1}+2-n}E(u,\partial_t u)=E(u,\partial_t u).$$

We refer the reader to the classical papers [Grillakis 1990; 1992; Shatah and Struwe 1993; Struwe 1999] for results about solutions with finite energy.

A solution is called self-similar when it is invariant by (1-3), that is, $u(x,t)=u_\gamma(x,t)$. For a homogeneous function V of degree -2, equation (1-2) presents the same scaling as in the case $V=0$. Taking $t=0$, the map (1-3) induces the scaling for the initial data:

$$(u_0(x),u_1(x))\mapsto\big(\gamma^{\frac{2}{p-1}}u_0(\gamma x),\gamma^{\frac{p+1}{p-1}}u_1(\gamma x)\big). \tag{1-4}$$

In other words, self-similar solutions of (1-2) are associated to initial data u_0 and u_1 homogeneous of degrees $-\frac{2}{p-1}$ and $-\frac{p+1}{p-1}$, respectively, that is, homogeneous functions of the form

$$u_0(x)=\varepsilon c_1|x|^{-\frac{2}{p-1}}\quad\text{and}\quad u_1(x)=\varepsilon c_2|x|^{-\frac{p+1}{p-1}}, \tag{1-5}$$

where $c_1,c_2\in\mathbb{R}$ and $\varepsilon>0$.

For $V=0$, there are a number of results about self-similar solutions in different frameworks. The first work is due to Kavian and Weissler [1990], where the authors proved the nonexistence of radially symmetric self-similar solutions with finite energy $E(u,\partial_t u)$ for $n\geq 3$ and $p_{\mathrm{e}}\leq p<\infty$. Working in the infinite energy space of all Bochner-measurable functions $u:(0,\infty)\to L^r(\mathbb{R}^n)$ such that

$$\sup_{t>0}t^\beta\|u(\cdot,t)\|_{L^r(\mathbb{R}^n)}<\infty, \tag{1-6}$$

Pecher [2000a] showed the existence of self-similar solutions for $\varepsilon>0$ in (1-5) sufficiently small by considering $n=3$ and $p_1<p\leq p_{\mathrm{conf}}$, where p_1 is the larger positive root of

$$(n^2-n)p^2-(n^2+3n-2)p+2=0.$$

The parameters $\beta>0$ and $r>2$ are taken in such a way that the norm (1-6) is scaling invariant. The approach in [Pecher 2000a] is based on L^q-L^r dispersive estimates for the wave group

$$\omega(t)=(-\Delta)^{-\frac{1}{2}}\sin(t(-\Delta)^{\frac{1}{2}}). \tag{1-7}$$

In fact, the case of nonradial homogeneous data also was considered in [Pecher 2000a]. Moreover, replacing L^r by suitable homogeneous Sobolev spaces $\dot{H}^{k,l}$ with $k>0$, the upper condition $p\leq p_{\mathrm{conf}}$

was removed by him. Still for $n=3$, Pecher [2000b] obtained self-similar solutions (not necessarily radial) in the range $p_{\mathrm{str}} < p < p_{\mathrm{conf}}$ and showed that the lower bound p_{str} is sharp in the sense that in general no nontrivial self-similar solution exists even in the radial case when $p \le p_{\mathrm{str}}$. Unlike [Pecher 2000a], the paper [Pecher 2000b] developed pointwise estimates related to the weights $|x| \pm t$ and a norm due to F. John and did not employ L^p, Sobolev or Besov spaces. Hidano [2002] complemented these results by showing scattering and existence of self-similar solutions for (1-2) when $n=2,3$ and $p_{\mathrm{str}} < p < p_{\mathrm{conf}}$. The result of [Pecher 2000a] was proved to be true for $n=2,3,4,5$ by Ribaud and Youssfi [2002], recovering in particular $n=2,3$. Moreover, for $n \ge 6$ they considered $p \in (p_1, p_{\mathrm{conf}}] \cup [2,\infty)$ or $p \in (p_1, p_{\mathrm{conf}}] \cup (p_2,\infty)$, where p_2 is the larger positive root of

$$2(n+1)p^2-(n^2+3n+4)p+(n^2+5n+2)=0.$$

Note that $p_{\mathrm{str}} < p_1 < p_{\mathrm{conf}} < p_2$ for all dimensions in which these parameters are defined.

The weighted Strichartz estimate in $L^{(r,\infty)}(\mathbb{R}^{1+n}_+)$

$$\big\| |t^2-|x|^2|^a u \big\|_{L^{(r,\infty)}(\mathbb{R}^{1+n}_+)} \le C \big\| |t^2-|x|^2|^b f \big\|_{L^{(r',\infty)}(\mathbb{R}^{1+n}_+)} \tag{1-8}$$

was obtained by Kato and Ozawa [2003] for f radially symmetric in x-variables, $2<r<2(n+1)/(n-1)$ and suitable powers a, b. By using (1-8) and assuming $n \ge 3$ odd, they proved existence and uniqueness of radially symmetric self-similar solutions for (1-2) with initial data (1-5) provided that $p_{\mathrm{str}} < p < p_{\mathrm{conf}}$ and $\varepsilon > 0$ is small enough. In [Kato and Ozawa 2004], they extended their results to the case $n \ge 2$ even. By employing spherical harmonics and Sobolev spaces over the unit sphere, the condition of radial symmetry on u and f was removed in [Kato et al. 2007] for $2 \le n \le 5$. In the case $p \in \mathbb{N}$, $p > p_{\mathrm{conf}}$ and $V=0$, Planchon [2000] showed global well-posedness and existence of self-similar solutions for (1-2) in $L^\infty((0,\infty); \dot{B}^{s_p}_{2,\infty})$ for small data $(u_0,u_1) \in \dot{B}^{s_p}_{2,\infty} \times \dot{B}^{s_p-1}_{2,\infty}$ with $s_p = \frac{n}{2} - \frac{2}{p-1}$. Notice that the above results do not contradict the nonexistence result in [Kavian and Weissler 1990] because the obtained self-similar solutions have infinite energy.

Wave equations with singular potential arise in the study of stability of stationary solutions for a number of systems of PDEs, for example, wave-Schrödinger and Maxwell–Schrödinger ones (see, e.g., [D'ancona and Pierfelice 2005]). Unlike the case $p > p_{\mathrm{str}}$ and $V=0$, where no blow-up occurs for (1-2), Strauss and Tsutaya [1997] proved blow-up of solutions when $n=3$, $p>1$ and $V \in C^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ decays like $c/|x|^{(2-\varepsilon)}$ as $|x| \to \infty$ for $0<\varepsilon<2$. Also, they showed global existence for $\varepsilon<0$, $p>p_{\mathrm{str}}$ and

$$\Bigg\| (1+|x|)^{2-\varepsilon} \sum_{|\alpha|\le 2} |\partial_x^\alpha V(x)| \Bigg\|_{L^\infty}$$

small enough. Still considering small, smooth and rapidly decaying potentials, Yajima [1995] obtained $L^p - L^q$ dispersive estimates for the linear wave equation (1-1). The borderline case $\varepsilon=0$ corresponds to V homogeneous of degree $\sigma=-2$. In this case, the perturbation Vu has the same scaling of Δu and cannot be dealt with as a simple perturbation of lower order because it does not belong to the Kato class

$$\mathcal{K} = \bigg\{ V \in L^1_{\mathrm{loc}} : \|V\|_{\mathcal{K}} = \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{2-n} |V(y)|\, dy < \infty,\ n \ge 3 \bigg\}, \tag{1-9}$$

where $\|\cdot\|_{\mathcal{K}}$ is called the global Kato norm. Taking $n=3$, $\vec{u}(0)=(0,u_1)$ and $f=0$ in (1-1), Georgiev and Visciglia [2003] proved the dispersive estimate

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{n-1}{2}}\|u_1\|_{\dot{B}^1_{1,1}(\mathbb{R}^n)} \tag{1-10}$$

for potentials V that are Hölder continuous in $\mathbb{R}^3\backslash\{0\}$ satisfying

$$0 \le V(x) \le \frac{C}{|x|^{2-\varepsilon}+|x|^{2+\varepsilon}} \quad \text{for all } x\in\mathbb{R}^3.$$

D'ancona and Pierfelice [2005] improved the class of potentials to nonresonant $V\in\mathcal{K}$ and obtained, in particular, the estimate (1-10) for $V\in L^{\frac{n}{2}-\delta}\cap L^{\frac{n}{2}+\delta}\subset L^{(\frac{n}{2},1)}\subset\mathcal{K}$ with small $\delta>0$. Planchon et al. [2003a] proved (1-10) in the radial case for the critical potential $V(x)=c/|x|^2$ with $c\ge 0$. In the same work, they also proved a modified version of (1-10) for negative potentials $-((n-2)/2)^2<c<0$. Moreover, they showed that the classical $L^\infty-L^1$ estimate

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{n-1}{2}}\|(-\Delta)^{\frac{n-1}{4}}u_1\|_{L^1(\mathbb{R}^n)}$$

does not hold when $c<0$ and $V(x)=c/|x|^2$. In particular, this estimate is false for general $V\in L^{(\frac{n}{2},\infty)}$.

Burq et al. [2003] considered Strichartz estimates for (1-1) and showed

$$\|(-\Delta)^\sigma u\|_{L^p_tL^q_x} \le C(\|u_0\|_{\dot{H}^\gamma}+\|u_1\|_{\dot{H}^{\gamma-1}}) \tag{1-11}$$

for σ, p, q, γ satisfying suitable conditions. See also [Planchon et al. 2003b] for the radial case and [Burq et al. 2004] for a more general class of potentials satisfying $V\in C^1(\mathbb{R}^n\backslash\{0\})$ and $\sup_{x\in\mathbb{R}^n}|x|^2|V(x)|<\infty$, among some other conditions. Using (1-11), for $n\ge 2$, $p\ge p_{\text{conf}}$, $s_p=\frac{n}{2}-\frac{2}{p-1}$ and

$$\sqrt{c+\frac{(n-2)^2}{4}} > \frac{n-2}{2}-\frac{2}{p-1}+\max\left\{\frac{1}{2p},\ \frac{2}{(n+1)(p-1)}\right\},$$

the authors of [Burq et al. 2003] also showed global well-posedness of (1-2) provided that $(u_0,u_1)\in H^{s_p}\times H^{s_p-1}$ is small enough. This result has been extended to the range

$$1+\frac{4n}{(n-1)(n+1)} < p < p_{\text{conf}}$$

in [Miao et al. 2013] for $n\ge 3$ and small radial initial data.

In this paper we obtain estimates for solutions of (1-1) in weak-L^r ($L^{(r,\infty)}$) spaces for the case of small radial singular potentials $V\in L^{(\frac{n}{2},\infty)}$. Examples of those are Hardy potentials $V=c|x|^{-2}$ with $|c|$ small enough (see Remark 3.2(II)). More precisely, for certain conditions on r,s, we prove the estimate

$$\|u\|_{L^\infty(\mathbb{R};L^{(r,\infty)})} \le \frac{K}{1-KC_0\|V\|_{L^{(n/2,\infty)}}}\|f\|_{L^\infty(\mathbb{R};L^{(s,\infty)})}, \tag{1-12}$$

where C_0 and K are positive constants and V, f and u are radially symmetric in the x-variable. In our results, the potential V can have indefinite sign. Notice that taking $V=0$ in (1-12), one also obtains, in particular, an estimate for the linear wave equation $\Box u=f$. The estimate (1-12) can be regarded as an endpoint-inhomogeneous Strichartz-type estimate in weak-L^p spaces, specifically, from $L^{l_1}_tL^{(l_2,\infty)}_x$ to

$L_t^{m_1} L_x^{(m_2,\infty)}$ in the case $(l_1, m_1) = (\infty, \infty)$, which is important because it corresponds to the natural persistence space in existence results. Even when $V = 0$, notice that (1-12) cannot be obtained as a consequence of the inhomogeneous Strichartz estimates by Keel and Tao [1998] and Taggart [2010].

In order to obtain (1-12), we need to show a time-weighted estimate for the wave group (1-7) in the predual of $L^{(r,\infty)}$, i.e., the Lorentz space $L^{(r',1)}$, which is of its own interest (see Lemma 4.1). As will be seen below, this estimate will lead us to global well-posedness results for (1-2) in critical spaces. We denote the solution of the Cauchy problem for the linear homogeneous wave equation by

$$L_{\vec{u}(0)}(t) = \omega(t)u_1 + \dot{\omega}(t)u_0, \quad \text{where } \dot{\omega}(t) = \partial_t \omega(t), \tag{1-13}$$

and consider the space of initial data

$$\mathcal{I}_{\mathrm{rad}} = \{(u_0, u_1) \in \mathcal{S}'_{\mathrm{rad}} \times \mathcal{S}'_{\mathrm{rad}} : L_{\vec{u}(0)}(t) \in L^\infty(\mathbb{R}; L^{(r_0,\infty)}_{\mathrm{rad}}(\mathbb{R}^n))\}, \tag{1-14}$$

where $r_0 = \frac{n(p-1)}{2}$ and the subindex "rad" means space of radial distributions. The norm $\|\cdot\|_{\mathcal{I}_{\mathrm{rad}}}$ is defined as

$$\|(u_0, u_1)\|_{\mathcal{I}_{\mathrm{rad}}} = \sup_{t\in\mathbb{R}} \|L_{\vec{u}(0)}(t)\|_{L^{(r_0,\infty)}_{\mathrm{rad}}}. \tag{1-15}$$

Applying the estimate (1-12), we obtain global well-posedness for (1-2) in the scaling-invariant space $E = L^\infty(\mathbb{R}; L^{(r_0,\infty)}_{\mathrm{rad}}(\mathbb{R}^n))$ provided that $n \geq 5$ odd, $p > (n^2+n-4)/(n(n-3))$ and $\|(u_0,u_1)\|_{\mathcal{I}_{\mathrm{rad}}}$ is small enough (see Theorem 3.3(I)). The continuous inclusion $(\dot{B}^{s_p}_{2,\infty} \times \dot{B}^{s_p-1}_{2,\infty})_{\mathrm{rad}} \subset \mathcal{I}_{\mathrm{rad}}$ holds true and so, in the radial case, our result extends the initial data class in [Planchon 2000]. In fact, we have $\dot{B}^{s_p}_{2,\infty} \subset L^{(\frac{n(p-1)}{2},\infty)}$ (see Remark 3.4(I)) and

$$\sup_{t\in\mathbb{R}} \|L_{\vec{u}(0)}(t)\|_{L^{(\frac{n(p-1)}{2},\infty)}} \leq C \sup_{t\in\mathbb{R}} \|L_{\vec{u}(0)}(t)\|_{\dot{B}^{s_p}_{2,\infty}} \leq C\|(u_0,u_1)\|_{\dot{B}^{s_p}_{2,\infty}\times\dot{B}^{s_p-1}_{2,\infty}}, \tag{1-16}$$

where the second inequality in (1-16) can be found in [Planchon 2000, estimate (29), p. 815]. Also, we have $\mathcal{K} \subsetneq L^{(\frac{n}{2},\infty)}$ and then our class of potentials is larger than the Kato one in the radial setting (see Remark 3.4(II)). Note that

$$p_{\mathrm{str}} < p_{\mathrm{conf}} < p_{\mathrm{e}} < \frac{n^2+n-4}{n(n-3)}$$

and our range of admissible powers p differs from those of [Kato and Ozawa 2003; 2004; Planchon 2000; Ribaud and Youssfi 2002]. Finally, as a byproduct, we obtain the existence of radially symmetric self-similar solutions when u_0, u_1 and V are homogeneous of degrees $-\frac{2}{p-1}$, $-\frac{p+1}{p-1}$ and -2, respectively (see Theorem 3.3(II)).

This paper is organized as follows. In Section 2, we recall the definition of Lorentz spaces and some of their properties. Our results are stated in Section 3 and proved in Section 4.

2. Lorentz spaces

We start by recalling the decreasing rearrangement of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$,

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\} \quad \text{for } t > 0,$$

where $d_f(s) = \left|\{x \in \mathbb{R}^n : |f(x)| > s\}\right|$ is the distribution function of f. The Lorentz space $L^{(p,z)} = L^{(p,z)}(\mathbb{R}^n)$ is the vector space of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|f\|^*_{(p,z)} = \begin{cases} \left[\int_0^\infty \left(t^{\frac{1}{p}}[f^*(t)]\right)^z \frac{dt}{t}\right]^{\frac{1}{z}} < \infty & \text{for } 0 < p \le \infty,\ 1 \le z < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}}[f^*(t)] < \infty & \text{for } 0 < p \le \infty,\ z = \infty. \end{cases} \tag{2-1}$$

The space $L^{(\infty,z)}$ is trivial for $1 \le z < \infty$. Also, $L^{(p,p)}$ is the Lebesgue space L^p with $\|f\|^*_{(p,p)} = \|\cdot\|_{L^p}$ and $L^{(p,\infty)}$ is the so-called weak-L^p. The quantity $\|f\|^*_{(p,z)}$ defines a complete quasinorm on $L^{(p,z)}$ that in general is not a norm. Considering the double rearrangement

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds,$$

one can define the norm $\|f\|_{(p,z)}$ on $L^{(p,z)}$ by replacing f^* by f^{**} in (2-1). For $1 < p \le \infty$, we have the inequality

$$\|\cdot\|^*_{(p,z)} \le \|\cdot\|_{(p,z)} \le \frac{p}{p-1}\|\cdot\|^*_{(p,z)},$$

and then $\|\cdot\|_{(p,z)}$ and $\|\cdot\|^*_{(p,z)}$ are topologically equivalent. The pair $(L^{(p,z)}, \|\cdot\|_{(p,z)})$ is a Banach space. From now on, we consider $L^{(p,z)}$ endowed with $\|\cdot\|_{(p,z)}$ and $\|\cdot\|^*_{(p,z)}$ when $1 < p \le \infty$ and $0 < p \le 1$, respectively. The continuous inclusions

$$L^{(p,1)} \subset L^{(p,z_1)} \subset L^p \subset L^{(p,z_2)} \subset L^{(p,\infty)} \tag{2-2}$$

hold true for $1 \le z_1 \le p \le z_2 \le \infty$ and $1 \le p \le \infty$. Lorentz spaces have the same scaling as L^p-spaces, namely

$$\|\delta_c(f)\|_{(p,z)} = c^{-\frac{n}{p}}\|f\|_{(p,z)},$$

where δ_λ stands for the operator $\delta_c(f)(x) = f(cx)$.

Let $0 < \theta < 1$ and $1 \le z \le \infty$. Consider the interpolation functor $(\cdot,\cdot)_{\theta,z}$ constructed via the $K_{\theta,z}$-method and defined on the categories of quasinormed and normed spaces. For $0 < p_1 < p_2 \le \infty$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $1 \le z_1, z_2 \le \infty$, we have (see [Bergh and Löfström 1976, Theorems 5.3.1 and 5.3.2])

$$(L^{(p_1,z_1)}, L^{(p_2,z_2)})_{\theta,z} = L^{(p,z)}. \tag{2-3}$$

Moreover, $(\cdot,\cdot)_{\theta,z}$ is exact of exponent θ.

The pointwise product operator works well in Lorentz spaces; i.e., Hölder inequality is verified in this setting (see [Hunt 1966; O'Neil 1963]). Let $1 < p_1, p_2, p_3 \le \infty$ and $1 \le z_1, z_2, z_3 \le \infty$ be such that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{z_1} + \frac{1}{z_2} \ge \frac{1}{z_3}$. Then

$$\|fg\|_{(p_3,z_3)} \le C\|f\|_{(p_1,z_1)}\|g\|_{(p_2,z_2)}, \tag{2-4}$$

where the constant $C > 0$ is independent of f and g.

Finally, we recall that the dual space of $L^{(p,z)}$ is $L^{(p',z')}$ for $1 \le p, z < \infty$ (see [Grafakos 2004, p. 52]). Taking $z = 1$, we have $(L^{(p,1)})' = L^{(p',\infty)}$ for $1 \le p < \infty$.

3. Main results

Throughout the paper, the subindex "rad" means space of radial functions or distributions. For instance,

$$L^{(r,z)}_{\mathrm{rad}} = \{u \in L^{(r,z)} : u \text{ is radially symmetric}\}. \tag{3-1}$$

We define the open triangles $\Delta_{P_1P_2P_3}$ and $\Delta_{P_2P_4P_5}$ whose vertices P_i are

$$\begin{aligned} P_1 &= \Big(\frac{1}{2}+\frac{1}{n+1}, \frac{1}{2}-\frac{1}{n+1}\Big), \quad P_2 = \Big(\frac{1}{2}-\frac{1}{n-1}, \frac{1}{2}-\frac{1}{n-1}\Big), \\ P_3 &= \Big(\frac{1}{2}+\frac{1}{n-1}, \frac{1}{2}+\frac{1}{n-1}\Big), \quad P_4 = \Big(1, \frac{n-1}{2n}\Big) \quad \text{and} \quad P_5 = (1,1) \end{aligned} \tag{3-2}$$

(see Figure 1). The vertices P_2 and P_3 are defined as $(0,0)$ and $(1,1)$, respectively, when $n = 1, 2$.

Our first result consists in linear estimates in weak-L^p for the linear wave equation with singular potential.

Theorem 3.1. *Let $n \geq 5$ be odd and $\Delta_{P_2P_4P_5}$ be the open triangle defined by the points P_2, P_4 and P_5 in (3-2). If*

$$1 < r', s' < \frac{2(n-1)}{n-3} \quad \text{with} \ \Big(1-\frac{1}{r}, 1-\frac{1}{s}\Big) \in \Delta_{P_2P_4P_5} \ \text{and} \ \frac{1}{s}-\frac{1}{r} = \frac{2}{n}, \tag{3-3}$$

then there are $K, C_0 > 0$ such that the solution u of (1-1) satisfies

$$\sup_{t\in\mathbb{R}} \|u(\cdot,t)\|_{(r,\infty)} \leq \frac{K}{1-C_0K\|V\|_{(\frac{n}{2},\infty)}} \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{(s,\infty)} \tag{3-4}$$

for all $f \in L^\infty(\mathbb{R}; L^{(s,\infty)}_{\mathrm{rad}}(\mathbb{R}^n))$, provided that $V \in L^{(\frac{n}{2},\infty)}_{\mathrm{rad}}$ and $C_0K\|V\|_{(\frac{n}{2},\infty)} < 1$. The supremum in (3-4) is taken in the essential sense.

Some comments on Theorem 3.1 are in order.

Remark 3.2. (I) Let us point out that the range in Theorem 3.1 is not empty. In order to see this, set $w = 1-\frac{1}{r}$ and $h = \frac{1}{s}-\frac{1}{r}$. Now notice that $\big(1-\frac{1}{r}, 1-\frac{1}{s}\big) \in \Delta_{P_2P_4P_5}$ is equivalent to

$$\Big(1-\frac{1}{r}, 1-\frac{1}{s}\Big) = (w, w-h) \in \Delta_{P_2P_4P_5}. \tag{3-5}$$

In turn, for (3-5) we need only that $0 < h < \frac{n+1}{2n}$ holds true when $h = \frac{2}{n}$ and $n \geq 5$.

(II) The critical Hardy potential $V(x) = c_0|x|^{-2} \in L^{(\frac{n}{2},\infty)}_{\mathrm{rad}}(\mathbb{R}^n)$ is covered by Theorem 3.1 with $|c_0| < \big(C_0K\big\||x|^{-2}\big\|_{(\frac{n}{2},\infty)}\big)^{-1}$. The constant C_0 in (3-4) is that of the Hölder inequality

$$\|Vu\|_{(s,\infty)} \leq C_0\|V\|_{(\frac{n}{2},\infty)}\|u\|_{(r,\infty)} \quad \text{with} \ \frac{1}{s} = \frac{2}{n}+\frac{1}{r}.$$

(III) Taking $V = 0$, Theorem 3.1 also provides an estimate for the linear wave equation $\square u = f$.

Let $\{\omega(t)\}_{t\in\mathbb{R}}$ be the wave group $\omega(t) = (-\Delta)^{-\frac{1}{2}}\sin(t(-\Delta)^{\frac{1}{2}})$ and define $\zeta(f)$ by

$$\zeta(f)(x,t) = \int_0^t \omega(t-s)f(s)\,ds. \tag{3-6}$$

Formally, the IVP (1-2) is equivalent to the integral equation

$$u = L_{\vec{u}(0)}(t) + \mathcal{N}(u) + \mathcal{T}(u), \tag{3-7}$$

where

$$\mathcal{N}(u) = \mu\zeta(|u|^{p-1}u) \quad \text{and} \quad \mathcal{T}(u) = -\zeta(Vu).$$

Solutions of (3-7) are called mild solutions for the Cauchy problem (1-2).

We will look for solutions of (3-7) in the Banach space $E = L^\infty(\mathbb{R}; L^{(r_0,\infty)}_{\text{rad}})$ whose norm is

$$\|u\|_E = \sup_{t\in\mathbb{R}} \|u(\cdot,t)\|_{(r_0,\infty)}. \tag{3-8}$$

The supremum in (3-8) is taken in the essential sense. This space is invariant by the scaling (1-3) and allows the existence of self-similar solutions (i.e., $u = u_\gamma$).

Consider

$$A_1 = \left(\frac{n+1}{2(n-1)}, \frac{n+1}{2(n-1)} - \frac{2}{n}\right) \quad \text{and} \quad A_2 = \left(1, \frac{n-2}{n}\right).$$

Let $]A_1, A_2[$ be the open segment line. Notice that $]A_1, A_2[\in \Delta_{P_2P_4P_5} \backslash \Delta_{P_1P_2P_3}$ for all $n \geq 4$.

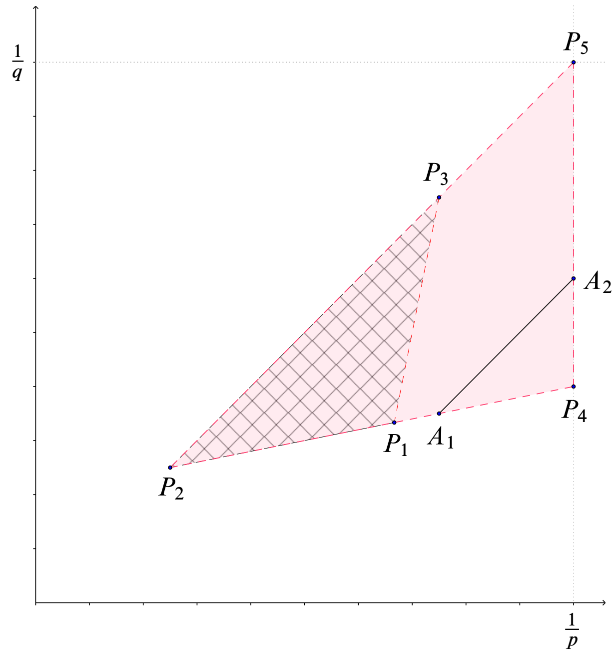

Figure 1. $]A_1, A_2[\in \Delta_{P_2P_4P_5} \backslash \Delta_{P_1P_2P_3}$.

Observe that $p > (n^2+n-4)/(n(n-3))$ is equivalent to

$$\left(1 - \frac{2}{n(p-1)}, 1 - \frac{2p}{n(p-1)}\right) \in]A_1, A_2[. \tag{3-9}$$

Our well-posedness and self-similarity results for (1-2) are stated below.

Theorem 3.3. *Let $n \geq 5$ be odd, $p > (n^2+n-4)/(n(n-3))$ and $r_0 = n(p-1)/2$. Suppose $(u_0, u_1) \in \mathcal{I}_{\mathrm{rad}}$ and $V \in L_{\mathrm{rad}}^{(\frac{n}{2},\infty)}$.*

(I) (***Global well-posedness***) *There are $\varepsilon, C_1 > 0$ such that if $\|(u_0,u_1)\|_{\mathcal{I}_{\mathrm{rad}}} \leq \varepsilon$, then the IVP (1-2) has a unique mild solution $u \in L^\infty(\mathbb{R}; L_{\mathrm{rad}}^{(r_0,\infty)})$ satisfying*

$$\sup_{t\in\mathbb{R}} \|u(\cdot,t)\|_{(r_0,\infty)} \leq \frac{2\varepsilon}{1-\eta}$$

provided that $\eta = C_1\|V\|_{(\frac{n}{2},\infty)} < 1$. Moreover, the solution u depends continuously on data (u_0,u_1) and potential V.

(II) (***Self-similarity***) *Under the hypotheses of item* (I), *the solution u is self-similar provided that u_0, u_1, V are homogeneous of degrees $-\frac{2}{p-1}, -\frac{p+1}{p-1}$ and -2, respectively.*

In what follows, we make some comments on Theorem 3.3.

Remark 3.4. (I) Taking $V = 0$, Theorem 3.3 provides a well-posedness result for semilinear wave equations in odd dimensions $n \geq 5$. Moreover, we have the continuous inclusions (see [Bergh and Löfström 1976, p. 154])

$$\dot{H}^s_{r_1} \hookrightarrow \dot{B}^s_{r_1,\infty} \hookrightarrow L^{(r_2,\infty)}, \tag{3-10}$$

where $\frac{1}{r_1} - \frac{s}{n} = \frac{1}{r_2}$ and $r_2 \geq r_1$. In particular, for $s_p = \frac{n}{2} - \frac{2}{p-1}$ we obtain $\dot{H}^{s_p} \hookrightarrow \dot{B}^{s_p}_{2,\infty} \hookrightarrow L^{(\frac{n(p-1)}{2},\infty)}$. In fact, the inclusions in (3-10) are strict and then the space $L^{(\frac{n(p-1)}{2},\infty)}$ is larger than $\dot{B}^{s_p}_{2,\infty}$, i.e., the one considered by Planchon [2000]. So, Theorem 3.3 extends the existence result of [Planchon 2000] in the case of radial solutions and $n \geq 5$ odd.

(II) Let $\mathcal{K}$ be the Kato class of potentials defined in (1-9). In view of the continuous strict inclusions

$$L^{\frac{n}{2}-\delta} \cap L^{\frac{n}{2}+\delta} \hookrightarrow L^{(\frac{n}{2},1)} \hookrightarrow \mathcal{K} \hookrightarrow L^{(\frac{n}{2},\infty)}, \quad \delta > 0, \tag{3-11}$$

our class for V is larger than $\mathcal{K}$ in the radial setting. For $L^{(\frac{n}{2},1)} \hookrightarrow \mathcal{K}$, we can use Hölder inequality (2-4) to obtain

$$\|V\|_{\mathcal{K}} \leq C \left\| \frac{1}{|x-y|^{(n-2)}} \right\|_{(\frac{n}{n-2},\infty)} \|V\|_{(\frac{n}{2},1)} \leq L\|V\|_{(\frac{n}{2},1)},$$

where

$$L = C\big\| |x-y|^{-(n-2)} \big\|_{(\frac{n}{n-2},\infty)}$$

is a positive constant. Next recall that $f \in L^{(p,\infty)}$ if and only if there is a constant $C > 0$ such that

$$|E|^{\frac{1}{p}-1} \int_E |f(y)|\, dy \leq C \tag{3-12}$$

for every Borel set E. The supremum of the left-hand side of (3-12) over all Borel sets gives an equivalent norm in $L^{(p,\infty)}$. It follows from (3-12) that $\mathcal{K} \hookrightarrow L^{(\frac{n}{2},\infty)}$. In fact, it is sufficient to check (3-12) for every open ball $E = \mathcal{B}_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$. For that, we estimate

$$\int_{\mathcal{B}_r(x)} |V(y)|\, dy \leq \int_{\mathcal{B}_r(x)} \frac{r^{n-2}}{|x-y|^{n-2}} |V(y)|\, dy \leq C\|V\|_{\mathcal{K}} |\mathcal{B}_r(x)|^{1-\frac{2}{n}},$$

where $|\mathcal{B}_r(x)| = \big(\pi^{\frac{n}{2}} / \Gamma\big(\frac{n}{2}+1\big)\big) r^n$ is the volume of $\mathcal{B}_r(x)$.

4. Proofs

Collecting estimates in [Brenner 1975; Peral 1980; Strichartz 1970], we have that the wave group $\{\omega(t)\}_{t\in\mathbb{R}}$ is bounded from L^{l_1} to L^{l_2} at $t=1$, i.e.,

$$\|\omega(1)h\|_{l_2} \le M_1\|h\|_{l_1}, \tag{4-1}$$

provided that $\left(\frac{1}{l_1},\frac{1}{l_2}\right)\in\overline{\Delta_{P_1P_2P_3}}$, where $\overline{X}$ stands for the closure of X. It follows from scaling properties of $\omega(t)$ and L^p-spaces that

$$\|\omega(t)h\|_{l_2} \le M_1|t|^{-n\left(\frac{1}{l_1}-\frac{1}{l_2}\right)+1}\|h\|_{l_1}. \tag{4-2}$$

Interpolating the estimate (4-2) (see, e.g., [Bergh and Löfström 1976]), we get

$$\|\omega(t)h\|_{(l_2,z)} \le M_2|t|^{-n\left(\frac{1}{l_1}-\frac{1}{l_2}\right)+1}\|h\|_{(l_1,z)}, \tag{4-3}$$

where $1\le z\le\infty$. Assuming radial symmetry for h, the authors of [Ebert et al. 2016] extended the range of (4-2) to the closed triangle $\overline{\Delta_{P_2P_4P_5}}$ except for the semiopen segment line $]P_1,P_4]$ (see Figure 1). Thus, again by interpolation, for $\left(\frac{1}{l_1},\frac{1}{l_2}\right)$ belonging to the open triangle $\Delta_{P_2P_4P_5}$ and $1\le z\le\infty$, we obtain the estimate

$$\|\omega(t)h\|_{(l_2,z)} \le M_3|t|^{-n\left(\frac{1}{l_1}-\frac{1}{l_2}\right)+1}\|h\|_{(l_1,z)} \tag{4-4}$$

for all $h\in L^{(l_1,z)}_{\mathrm{rad}}(\mathbb{R}^n)$.

Yamazaki [2000] dealt with Navier–Stokes equations and Stokes and heat semigroups in weak-L^p spaces. The next estimate could be seen as a version of the Yamazaki estimate [2000, Corollary 2.3] for the wave group $\{\omega(t)\}_{t\in\mathbb{R}}$. Notice that it consists in a time-weighted estimate in preduals of weak-L^p spaces.

Lemma 4.1. *Let f be radially symmetric, $n\ge 3$ odd, and let $\Delta_{P_2P_4P_5}$ be the open triangle defined by the points P_2,P_4,P_5 in (3-2). If $1<d_1,d_2<2(n-1)/(n-3)$ (∞ if $n=3$) with $\left(\frac{1}{d_1},\frac{1}{d_2}\right)\in\Delta_{P_2P_4P_5}$ then $|t|^{n\left(\frac{1}{d_1}-\frac{1}{d_2}\right)-2}\omega(t)f\in L^1(\mathbb{R};L^{(d_2,1)}(\mathbb{R}^n))$ and there is $C>0$ such that*

$$\int_{\mathbb{R}}|t|^{\frac{n}{d_1}-\frac{n}{d_2}-2}\|\omega(t)f\|_{(d_2,1)}\,dt \le C\|f\|_{(d_1,1)} \tag{4-5}$$

for all $f\in L^{(d_1,1)}_{\mathrm{rad}}(\mathbb{R}^n)$.

Proof. Let p_1 and p_2 be such that $p_1<d_1<p_2$, $\frac{1}{d_1}-\frac{1}{p_2}<\frac{1}{n}$ and

$$\left(\frac{1}{p_j},\frac{1}{d_2}\right)\in\Delta_{P_2P_4P_5}\quad\text{for } j=1,2.$$

Using the estimate (4-4) with $(l_1,l_2,z)=(p_1,d_2,1)$ and $(l_1,l_2,z)=(p_2,d_2,1)$, we obtain

$$\|\omega(t)f\|_{(d_2,1)} \le C_k|t|^{1+\frac{n}{d_2}-\frac{n}{p_k}}\|f\|_{(p_k,1)}\quad\text{for } k=1,2. \tag{4-6}$$

Next consider the sublinear operator Ξ as a map from $L^{(p_1,1)}_{\mathrm{rad}}\cap L^{(p_2,1)}_{\mathrm{rad}}$ to a function $\Xi(f)(t)$ in $\mathbb{R}$ defined by

$$\Xi(f)(t)=|t|^{\frac{n}{d_1}-\frac{n}{d_2}-2}\|\omega(t)f\|_{(d_2,1)}.$$

In view of (4-6), we can estimate

$$\Xi(f)(t) \le C_k |t|^{\frac{n}{d_1}-\frac{n}{d_2}-2} |t|^{1+\frac{n}{d_2}-\frac{n}{p_k}} \|f\|_{(p_k,1)} = C_k |t|^{\frac{n}{d_1}-\frac{n}{p_k}-1} \|f\|_{(p_k,1)} \quad \text{for } k=1,2. \tag{4-7}$$

Hence, the operator Ξ is bounded from $L^{(p_k,1)}_{\mathrm{rad}}(\mathbb{R}^n)$ to $L^{(s_k,\infty)}(\mathbb{R})$, where $\frac{1}{s_k} = 1-\left(\frac{n}{d_1}-\frac{n}{p_k}\right)$. Indeed, it follows from (4-7) that

$$\|\Xi(f)(t)\|_{L^{(s_k,\infty)}(\mathbb{R})} \le C_k \left\| |t|^{-\frac{1}{s_k}} \right\|_{L^{(s_k,\infty)}(\mathbb{R})} \|f\|_{(p_k,1)} \le L_k \|f\|_{(p_k,1)}, \tag{4-8}$$

where $L_k = C_k \left\| |t|^{-1/s_k} \right\|_{L^{(s_k,\infty)}(\mathbb{R})}$, $k=1,2$.

Take now $\theta \in (0,1)$ such that $\frac{1}{d_1} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. So, we have $\frac{1-\theta}{s_1} + \frac{\theta}{s_2} = 1$ and $0 < s_1 < 1 < s_2$. By interpolation, it follows that

$$L^{(d_1,1)}_{\mathrm{rad}} = (L^{(p_1,1)}_{\mathrm{rad}}, L^{(p_2,1)}_{\mathrm{rad}})_{\theta,1} \quad \text{and} \quad L^1(\mathbb{R}) = \big(L^{(s_1,\infty)}(\mathbb{R}), L^{(s_2,\infty)}(\mathbb{R})\big)_{\theta,1}$$

and then

$$\begin{aligned} \|\Xi(f)(t)\|_{L^1(\mathbb{R})} &\le m_1^{1-\theta} m_2^{\theta} \|f\|_{(d_1,1)} \\ &\le L_1^{1-\theta} L_2^{\theta} \|f\|_{(d_1,1)}, \end{aligned}$$

where $m_k = \|\Xi(f)\|_{L^{(p_k,1)}_{\mathrm{rad}} \to L^{(s_k,\infty)}} \le L_k$. This gives us the estimate (4-5). ☐

Proof of Theorem 3.1. Let us rewrite $\zeta(f)$ in (3-6) as

$$\zeta(f)(x,t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} W(x-y, t-s) f(y,s)\, dy\, ds,$$

where the kernel W is given by

$$\widehat{W}(\xi, t-s) = \begin{cases} \sin((t-s)|\xi|)/|\xi|, & 0 < s < t, \\ 0, & \text{otherwise}. \end{cases}$$

Given a suitable function $\phi \in C^\infty(\mathbb{R}^n)$, we set

$$\langle \zeta(f), \phi \rangle = \int_{\mathbb{R}^n} \zeta(f)(x,t) \phi(x)\, dx.$$

Here all functions are considered to be radially symmetric. Using Tonelli's theorem and the Hölder inequality (2-4), we obtain

$$\begin{aligned} |\langle \zeta(f), \phi \rangle| &\le \int_{-\infty}^{\infty} \big\langle |f(\cdot,\tau)|, |\omega(t-\tau)\phi| \big\rangle\, d\tau \\ &\le C \int_{-\infty}^{\infty} \|f(\cdot,\tau)\|_{(s,\infty)} \|\omega(t-\tau)\phi\|_{(s',1)}\, d\tau. \end{aligned} \tag{4-9}$$

In (4-9) we have proceeded somewhat formally but we are going to see that its right-hand side is indeed finite, which justifies the above computations. Take $(d_1, d_2) = (r', s')$ and note that

$$\frac{n}{d_1} - \frac{n}{d_2} - 2 = \frac{n}{r'} - \frac{n}{s'} - 2 = 0.$$

Using duality in $L^{(p,z)}$, the inequality (4-9) and Lemma 4.1 with $(d_1, d_2) = (s', r')$, it follows that

$$\begin{aligned}\|\zeta(f)(\cdot,t)\|_{(r,\infty)} &= \sup_{\|\phi\|_{(r',1)}=1} |\langle \zeta(f), \phi\rangle| \\ &\leq C \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{(s,\infty)} \sup_{\|\phi\|_{(r',1)}=1} \int_{-\infty}^{\infty} \|\omega(t-\tau)\phi\|_{(s',1)}\, d\tau \\ &\leq K \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{(s,\infty)} \sup_{\|\phi\|_{(r',1)}=1} \{\|\phi\|_{(r',1)}\} \\ &\leq K \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{(s,\infty)} \qquad (4\text{-}10)\end{aligned}$$

for a.e. $t \in \mathbb{R}$. Next let $\tilde{f} = f + Vu$ and $u = \zeta(\tilde{f})$ be the mild solution of (1-1). Since $\frac{1}{s} = \frac{1}{n/2} + \frac{1}{r}$, the Hölder inequality (2-4) gives

$$\|Vu\|_{(s,\infty)} \leq C_0 \|V\|_{(\frac{n}{2},\infty)} \|u(\cdot,t)\|_{(r,\infty)}. \qquad (4\text{-}11)$$

Thus, in view of (4-10), we get

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|u(\cdot,t)\|_{(r,\infty)} &\leq K \sup_{t\in\mathbb{R}} \|\tilde{f}(\cdot,t)\|_{(s,\infty)} \\ &\leq K \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{(s,\infty)} + KC_0 \|V\|_{(\frac{n}{2},\infty)} \sup_{t\in\mathbb{R}} \|u(\cdot,t)\|_{(r,\infty)},\end{aligned}$$

which implies the desired estimate because $KC_0\|V\|_{(\frac{n}{2},\infty)} < 1$. □

Proof of Theorem 3.3. **Part (I).** (Well-posedness) Take $r = r_0 = \frac{n(p-1)}{2}$ and $s = \frac{r_0}{p}$, and note that

$$\frac{1}{s} - \frac{1}{r} = (p-1)\frac{1}{r_0} = \frac{2}{n}.$$

In view of (3-9), we have $\left(1 - \frac{1}{r}, 1 - \frac{1}{s}\right) \in \Delta_{P_2 P_4 P_5}$ and then we can employ Theorem 3.1 with $V = 0$ in order to obtain

$$\|\zeta(f)\|_E \leq K \|f\|_{L^\infty(\mathbb{R}; L^{(r_0/p,\infty)}_{\mathrm{rad}})}. \qquad (4\text{-}12)$$

Since $\frac{2}{n} + \frac{1}{r_0} = \frac{p}{r_0}$, it follows from (4-12) and the Hölder inequality (2-4) that

$$\begin{aligned}\|\mathcal{T}(u) - \mathcal{T}(v)\|_E = \|\mathcal{T}(u-v)\|_E &= \sup_{t\in\mathbb{R}} \|\zeta(V(u-v))(\cdot,t)\|_{(r_0,\infty)} \\ &\leq K \sup_{t\in\mathbb{R}} \|V(u-v)(\cdot,t)\|_{(\frac{r_0}{p},\infty)} \\ &\leq KC_0 \|V\|_{(\frac{n}{2},\infty)} \sup_{t\in\mathbb{R}} \|u(\cdot,t) - v(\cdot,t)\|_{(r_0,\infty)} \\ &\leq \eta \|u - v\|_E, \qquad (4\text{-}13)\end{aligned}$$

where $\eta = C_1\|V\|_{(\frac{n}{2},\infty)}$, $C_1 = KC_0$, and C_0 is the constant in the Hölder inequality $\|Vh\|_{(\frac{r_0}{p},\infty)} \leq C_0\|V\|_{(\frac{n}{2},\infty)}\|h\|_{(r_0,\infty)}$. Next recall the inequality

$$\left||u|^{p-1}u - |v|^{p-1}v\right| \leq C|u-v|(|u|^{p-1} + |v|^{p-1})$$

and let $\frac{p}{r_0} = \frac{1}{r_0} + \frac{p-1}{r_0}$. Using the Hölder inequality (2-4), we can estimate

$$\begin{aligned}\big\| |u|^{p-1}u - |v|^{p-1}v \big\|_{(\frac{r_0}{p},\infty)} &\le C \big\| |u-v|(|u|^{p-1} + |v|^{p-1}) \big\|_{(\frac{r_0}{p},\infty)} \\ &\le C \|u-v\|_{(r_0,\infty)} \big\| (|u|^{p-1} + |v|^{p-1}) \big\|_{(\frac{r_0}{p-1},\infty)} \\ &\le C \|u-v\|_{(r_0,\infty)} \big(\|u\|^{p-1}_{(r_0,\infty)} + \|v\|^{p-1}_{(r_0,\infty)} \big). \end{aligned} \tag{4-14}$$

Estimates (4-12) and (4-14) yield

$$\begin{aligned}\|\mathcal{N}(u) - \mathcal{N}(v)\|_E &= \big\| \zeta(|u|^{p-1}u - |v|^{p-1}v)(\cdot,t) \big\|_E \\ &\le K \big\| (|u|^{p-1}u - |v|^{p-1}v)(\cdot,t) \big\|_{L^\infty(\mathbb{R};L^{(r_0/p,\infty)}_{\mathrm{rad}})} \\ &\le C_2 \|u-v\|_E (\|u\|_E^{p-1} + \|v\|_E^{p-1}). \end{aligned} \tag{4-15}$$

Let $\Psi(u) = L_{\vec{u}(0)}(t) + \mathcal{N}(u) + \mathcal{T}(u)$ be defined in the closed ball $B_\varepsilon = \{u \in E : \|u\|_E \le 2\varepsilon/(1-\eta)\}$, where $\varepsilon > 0$ and $\eta = C_1 \|V\|_{(n/2,\infty)} < 1$. We are going to show that Ψ is a contraction in B_ε for ε small enough. For $u, v \in B_\varepsilon$, we have

$$\begin{aligned}\|\Psi(u) - \Psi(v)\|_E &\le \|\mathcal{N}(u) - \mathcal{N}(v)\|_E + \|\mathcal{T}u - \mathcal{T}v\|_E \\ &\le C_2 \|u-v\|_E (\|u\|_E^{p-1} + \|v\|_E^{p-1}) + \eta \|u-v\|_E \\ &\le \|u-v\|_E \left(C_2 \left(\frac{2\varepsilon}{1-\eta} \right)^{p-1} + C_2 \left(\frac{2\varepsilon}{1-\eta} \right)^{p-1} + \eta \right) \\ &\le \left(C_2 \frac{2^p \varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta \right) \|u-v\|_E. \end{aligned} \tag{4-16}$$

Choose $\varepsilon > 0$ in such a way that

$$\left(C_2 \frac{2^p \varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta \right) < 1. \tag{4-17}$$

Moreover, taking $v = 0$ in (4-16), we arrive at

$$\|\Psi(u)\|_E \le \|L_{\vec{u}(0)}\|_E + \|\Psi(u) - \Psi(0)\|_E \le \varepsilon + \left(C_2 \frac{2^{p-1}\varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta \right) \frac{2\varepsilon}{1-\eta} < \frac{2\varepsilon}{1-\eta}$$

for all $u \in B_\varepsilon$. Hence, the map $\Psi : B_\varepsilon \to B_\varepsilon$ is a contraction in E. It follows that its fixed point in B_ε is the unique solution for (3-7) such that

$$\|u\|_E \le \frac{2\varepsilon}{1-\eta}.$$

The continuous dependence follows naturally from the above estimates and fixed point argument. We include its proof for the sake of completeness. Let $u, v \in B_\varepsilon$ be the unique mild solutions associated to data (u_0, u_1, V) and (v_0, v_1, U), respectively. Then, defining $L_{\vec{u}(0)}(t) = \omega(t)u_1 + \dot{\omega}(t)u_0$ and

$L_{\vec{v}(0)}(t) = \omega(t)v_1 + \dot{\omega}(t)v_0$, we have

$$\begin{aligned}\|u-v\|_E &\le \|L_{\vec{u}(0)} - L_{\vec{v}(0)}\|_E + \|\mathcal{N}(u)-\mathcal{N}(v)\|_E + \|\mathcal{T}(u)-\mathcal{T}(v)\|_E \\ &= \|L_{\vec{u}(0)} - L_{\vec{v}(0)}\|_E + \|\mathcal{N}(u)-\mathcal{N}(v)\|_E + \big\|\zeta\big((V-U)v + V(u-v)\big)\big\|_E \\ &\le \|L_{\vec{u}(0)} - L_{\vec{v}(0)}\|_E + C_2\frac{2^p\varepsilon^{p-1}}{(1-\eta)^{p-1}}\|u-v\|_E + KC_0\|V-U\|_{(\frac{n}{2},\infty)}\|v\|_E + \eta\|u-v\|_E \\ &\le \|(u_0-v_0, u_1-v_1)\|_{\mathcal{I}_{\mathrm{rad}}} + \left(C_2\frac{2^p\varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta\right)\|u-v\|_E + \frac{2KC_0\varepsilon}{1-\eta}\|V-U\|_{(\frac{n}{2},\infty)},\end{aligned}$$

which yields the desired continuity because of (4-17).

Part (II). (Self-similarity) First note that the homogeneous pair (u_0, u_1) is in $\mathcal{I}_{\mathrm{rad}}$. Due to the fixed point argument, the solution u in item (I) is the limit in $E = L^\infty(\mathbb{R}; L^{(r_0,\infty)}_{\mathrm{rad}})$ of the sequence

$$u^{(1)} = L_{\vec{u}(0)}(t) \quad \text{and} \quad u^{(k+1)} = L_{\vec{u}(0)}(t) + \mathcal{N}(u^{(k)}) + \mathcal{T}(u^{(k)}) \quad \text{for } k \in \mathbb{N}. \tag{4-18}$$

Using the homogeneity properties of u_0, u_1 and V, one can show that $u^{(k)}$ is invariant by (1-3), that is,

$$u^{(k)} = (u^{(k)})_\gamma := \gamma^{\frac{2}{p-1}} u^{(k)}(\gamma x, \gamma t).$$

Now, since $(E, \|\cdot\|_E)$ is invariant by (1-3), a change of variable gives

$$\|(u^{(k)})_\gamma - (u)_\gamma\|_E = \|(u^{(k)}-u)_\gamma\|_E = \|u^{(k)}-u\|_E \to 0, \quad \text{as } k \to \infty. \tag{4-19}$$

Since $(u^{(k)})_\gamma = u^{(k)}$, it follows that $u^{(k)}$ also converges to $(u)_\gamma$. Then, $u \equiv (u)_\gamma$ for each $\gamma > 0$, as required. □

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