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OPTIMAL WELL-POSEDNESS FOR THE INHOMOGENEOUS INCOMPRESSIBLE NAVIER–STOKES SYSTEM WITH GENERAL VISCOSITY

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In this paper we obtain new well-posedness results concerning a linear inhomogeneous Stokes-like system. These results are used to establish local well-posedness in the critical spaces for initial density ρ_0 and velocity u_0 such that $\rho_0 - \rho \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$, $u_0 \in \dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)$, $p \in \left(\frac{6}{5}, 4\right)$ for the inhomogeneous incompressible Navier–Stokes system with variable viscosity. To the best of our knowledge, regarding the 3-dimensional case, this is the first result in a truly critical framework for which one does not assume any smallness condition on the density.

1. Introduction

In this paper we deal with the well-posedness of the inhomogeneous, incompressible Navier–Stokes system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) D(u)) + \nabla P = 0, \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{1-1}$$

In the above, $\rho > 0$ stands for the density of the fluid, $u \in \mathbb{R}^n$ is the fluid's velocity field, while P is the pressure. The viscosity coefficient μ is assumed to be a smooth, strictly positive function of the density, while

$$D(u) = \nabla u + Du$$

is the deformation tensor. This system is used to study fluids obtained as a mixture of two (or more) incompressible fluids that have different densities: fluids containing a melted substance, polluted air/water etc.

There is a very rich literature devoted to the study of the well-posedness of (1-1). Briefly, the question of existence of weak solutions with finite energy was first considered by Kažihov [1974] (see also [Antontsev et al. 1990]) in the case of constant viscosity. The case with a general viscosity law was treated in [Lions 1996]. Weak solutions for more regular data were considered in [Desjardins 1997]. Recently, weak solutions were investigated by Huang, Paicu and Zhang in [Huang et al. 2013c].

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The unique solvability of (1-1) was first addressed in the seminal work of Ladyzhenskaya and Solonnikov [1975]. More precisely, considering $u_0 \in W^{2-2/p,p}(\Omega)$, with $p > 2$, a divergence-free vector field that vanishes on $\partial\Omega$ and $\rho_0 \in C^1(\Omega)$ bounded away from zero, they construct a global strong solution in the 2-dimensional case and a local solution in the 3-dimensional case. Moreover, if u_0 is small in $W^{2-2/p,p}(\Omega)$ then global well-posedness holds true.

The question of weak-strong uniqueness was addressed in [Choe and Kim 2003] for the case of sufficiently smooth data with vanishing viscosity.

Over the last thirteen years, efforts were made to obtain well-posedness results in the so-called critical spaces, i.e., the spaces which have the same invariance with respect to time and space dilation as the system itself, namely

$$\begin{cases} (\rho_0(x), u_0(x)) \to (\rho_0(lx), lu_0(lx)), \\ (\rho(t,x), u(t,x)) \to \big(\rho(l^2t, lx), lu(l^2t, lx), l^2P(l^2t, lx)\big). \end{cases}$$

For more details and explanations for this classical approach we refer to [Danchin 2003] or [Danchin and Mucha 2015]. In the Besov space context, which includes in particular the more classical Sobolev spaces, these are

$$\rho_0 - \bar{\rho} \in \dot{B}^{n/p_1}_{p_1,r_1} \quad \text{and} \quad u_0 \in \dot{B}^{n/p_2-1}_{p_2,r_2}, \tag{1-2}$$

where $\bar{\rho}$ is some constant density state and n is the space dimension. Working with densities close (in some appropriate norm) to a constant has led to a rich literature. In [Danchin 2003] local and global existence results are obtained for the case of constant viscosity and by taking the initial data

$$\rho_0 - \bar{\rho} \in L^\infty \cap \dot{B}^{n/2}_{2,\infty} \quad \text{and} \quad u_0 \in \dot{B}^{n/2-1}_{2,1}$$

and under the assumption that $\|\rho_0 - \bar{\rho}\|_{L^\infty \cap \dot{B}^{n/2}_{2,\infty}}$ is sufficiently small. The case with variable viscosity and for initial data

$$\rho_0 - \bar{\rho} \in \dot{B}^{n/p}_{p,1} \quad \text{and} \quad u_0 \in \dot{B}^{n/p-1}_{p,1},$$

$p \in [1, 2n)$, is treated in [Abidi 2007]. However, uniqueness is guaranteed once $p \in [1, n)$. These results were further extended by H. Abidi and M. Paicu [2007] by noticing that $\rho_0 - \bar{\rho}$ can be taken in a larger Besov space. B. Haspot [2012] established results in the same spirit as those mentioned above (however, the results are obtained in the nonhomogeneous framework and thus do not fall into the critical framework) in the case where the velocity field is not Lipschitz. Using the Lagrangian formulation, R. Danchin and P. B. Mucha [2012], established local and global results for (1-1) with constant viscosity when $\rho_0 - \bar{\rho} \in \mathcal{M}(\dot{B}^{n/p-1}_{p,1})$, $u_0 \in \dot{B}^{n/p-1}_{p,1}$ and under the smallness condition

$$\|\rho_0 - \bar{\rho}\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})} \ll 1,$$

where $\mathcal{M}(\dot{B}^{n/p-1}_{p,1})$ stands for the multiplier space of $\dot{B}^{n/p-1}_{p,1}$. In particular, functions with small jumps enter this framework. Moreover, as a consequence of their approach, the range of Lebesgue exponents for which uniqueness of solutions holds is extended to $p \in [1, 2n)$. In [Paicu and Zhang 2012; Huang et al. 2013a; 2013b; 2013c] the authors improve the smallness assumptions used in order to obtain global existence. To summarize, all the previous well-posedness results in critical spaces were established assuming the density is close in some sense to a constant state.

When the latter assumption is removed, one must impose more regularity on the data. For the case of constant viscosity, R. Danchin [2004] obtained local well-posedness and global well-posedness in dimension $n = 2$ for data drawn from the nonhomogeneous Sobolev spaces:

$$(\rho_0 - \bar{\rho}, u_0) \in H^{n/2+\alpha} \times H^{n/2-1+\beta}$$

with $\alpha, \beta > 0$. The same result for the case of the general viscosity law is established in [Abidi 2007]. For data with non-Lipschitz velocity results were established in [Haspot 2012]. Concerning rougher densities, considering $\rho_0 \in L^\infty(\mathbb{R}^d)$ bounded from below and $u_0 \in H^2(\mathbb{R}^d)$, Danchin and Mucha [2013] constructed a unique local solution. Again, supposing the density is close to some constant state, they proved global well-posedness. These results are generalized in [Paicu et al. 2013]. Taking the density as above, the authors construct a global unique solution provided that $u_0 \in H^s(\mathbb{R}^2)$ for any $s > 0$ in the 2-dimensional case and a local unique solution in the 3-dimensional case considering $u_0 \in H^1(\mathbb{R}^3)$. Moreover, assuming u_0 is suitably small, the solution constructed is global even in the 3-dimensional case.

In critical spaces of the Navier–Stokes system, i.e., (1-2) there are few well-posedness results. Very recently, in the 2-dimensional case and allowing variable viscosity, H. Xu, Y. Li and X. Zhai [2016] constructed a unique local solution to (1-1) provided that the initial data satisfy $\rho_0 - \bar{\rho} \in \dot{B}^{2/p}_{p,1}(\mathbb{R}^2)$ and $u_0 \in \dot{B}^{2/p-1}_{p,1}(\mathbb{R}^2)$. Moreover, if $\rho_0 - \bar{\rho} \in L^p \cap \dot{B}^{2/p}_{p,1}(\mathbb{R}^2)$ and the viscosity is supposed constant, their solution becomes global. In the 3-dimensional situation, to the best of our knowledge, the results that are closest to the critical regularity are those presented in [Abidi et al. 2012; 2013] (for a similar result in the periodic case one can consult [Poulon 2015]). More precisely, in three dimensions, assuming

$$\rho_0 - \bar{\rho} \in L^2 \cap \dot{B}^{3/2}_{2,1} \quad \text{and} \quad u_0 \in \dot{B}^{1/2}_{2,1}$$

and taking constant viscosity, H. Abidi, G. Gui and P. Zhang [Abidi et al. 2012] show the local well-posedness of system (1-1). Moreover, if the initial velocity is small then global well-posedness holds true. In [Abidi et al. 2013] they establish the same kind of result for initial data

$$\rho_0 - \bar{\rho} \in L^\lambda \cap \dot{B}^{3/\lambda}_{\lambda,1} \quad \text{and} \quad u_0 \in \dot{B}^{3/p-1}_{p,1},$$

where $\lambda \in [1, 2]$, $p \in [3, 4]$ are such that $\frac{1}{\lambda} + \frac{1}{p} > \frac{5}{6}$ and $\frac{1}{\lambda} - \frac{1}{p} \le \frac{1}{3}$.

One of the goals of the present paper is to establish local well-posedness in the critical spaces

$$\rho_0 - \bar{\rho} \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3), \quad u_0 \in \dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3), \quad p \in \left(\tfrac{6}{5}, 4\right)$$

for system (1-1)

- with general smooth variable viscosity law,
- without any smallness assumption on the density,
- without any extra low frequencies assumption. In particular, we generalize the local existence and uniqueness result of [Abidi et al. 2012], thus achieving the critical regularity.

As in [Danchin and Mucha 2012], we will not work directly with system (1-1); instead we will use its Lagrangian formulation. By proceeding so, we are naturally led to consider the following Stokes problem with time-independent, nonconstant coefficients:

$$\begin{cases} \partial_t u - a\,\mathrm{div}(bD(u)) + a\nabla P = f, \\ \mathrm{div}\, u = \mathrm{div}\, R, \\ u_{|t=0} = u_0. \end{cases} \tag{1-3}$$

We establish global well-posedness results for system (1-3). This can be viewed as a first step towards generalizing the results of Danchin and Mucha [2015, Chapter 4] for the case of general viscosity and without assuming the density is close to a constant state. Let us mention that the estimates we obtain for system (1-3) have a wider range of applications: in a forthcoming paper we will investigate the well-posedness issue of the Navier–Stokes–Korteweg system under optimal regularity assumptions.

To summarize all the above, our main result reads:

Theorem 1.1. *Consider $p \in \left(\frac{6}{5}, 4\right)$. Assume that there exist positive constants $(\bar{\rho}, \rho_\star, \rho^\star)$ such that $\rho_0 - \bar{\rho} \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$ and $0 < \rho_\star < \rho_0 < \rho^\star$. Furthermore, consider u_0 a divergence-free vector field with coefficients in $\dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)$. Then, there exists a time $T > 0$ and a unique solution $(\rho, u, \nabla P)$ of system* (1-1) *with*

$$\rho - \bar{\rho} \in \mathcal{C}_T(\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)) \cap L^\infty_T(\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)), \quad u \in \mathcal{C}_T(\dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)), \quad (\partial_t u, \nabla^2 u, \nabla P) \in L^1_T(\dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)).$$

One salutary feature of the Lagrangian formulation is that the density becomes independent of time. More precisely, considering $(\rho, u, \nabla P)$ a solution of (1-1) and denoting by X the flow associated to the vector field u,

$$X(t, y) = y + \int_0^t u(\tau, X(\tau, y))\, dy.$$

We introduce the new Lagrangian variables

$$\bar{\rho}(t, y) = \rho(t, X(t, y)), \quad \bar{u}(t, y) = u(t, X(t, y)) \quad \text{and} \quad \overline{P}(t, y) = P(t, X(t, y)).$$

Then, using the chain rule and Proposition 3.23 we gather that $\bar{\rho}(t, \cdot) = \rho_0$ and

$$\begin{cases} \rho_0 \partial_t \bar{u} - \mathrm{div}(\mu(\rho_0) A_{\bar{u}} D_{A_{\bar{u}}}(\bar{u})) + A^T_{\bar{u}} \nabla \overline{P} = 0, \\ \mathrm{div}(A_{\bar{u}} \bar{u}) = 0, \\ \bar{u}_{|t=0} = u_0, \end{cases} \tag{1-4}$$

where $A_{\bar{u}}$ is the inverse of the differential of X, and

$$D_A(\bar{u}) = D\bar{u} A_{\bar{u}} + A^T_{\bar{u}} \nabla \bar{u}.$$

Note that we can give a meaning to (1-4) independently of the Eulerian formulation by stating

$$X(t, y) = y + \int_0^t \bar{u}(\tau, y)\, d\tau.$$

Theorem 1.1 will be a consequence of the following result:

Theorem 1.2. *Consider* $p \in \big(\frac{6}{5}, 4\big)$. *Assume there exists positive* $(\bar\rho, \rho_\star, \rho^\star)$ *such that* $\rho_0 - \bar\rho \in \dot B^{3/p}_{p,1}(\mathbb{R}^3)$ *and* $0 < \rho_\star < \rho_0 < \rho^\star$. *Furthermore, consider* u_0 *a divergence-free vector field with coefficients in* $\dot B^{3/p-1}_{p,1}(\mathbb{R}^3)$. *Then, there exists a time* $T > 0$ *and a unique solution* $(\bar u, \nabla \bar P)$ *of system* (1-4) *with*

$$\bar u \in \mathcal{C}_T(\dot B^{3/p-1}_{p,1}(\mathbb{R}^3)) \quad \text{and} \quad (\partial_t \bar u, \nabla^2 \bar u, \nabla \bar P) \in L^1_T(\dot B^{3/p-1}_{p,1}(\mathbb{R}^3)).$$

Moreover, there exists a positive constant $C = C(\rho_0)$ *such that*

$$\|u\|_{L^\infty_T(\dot B^{3/p-1}_{p,1})} + \|(\nabla^2 u, \nabla P)\|_{L^1_T(\dot B^{3/p-1}_{p,1})} \le \|u_0\|_{\dot B^{3/p-1}_{p,1}} \exp(C(T+1)).$$

The study of system (1-4) naturally leads to the Stokes-like system (1-3). In Section 2 we establish the global well-posedness of system (1-3). More precisely, we prove:

Theorem 1.3. *Consider* $n \in \{2, 3\}$ *and* $p \in (1, 4)$ *if* $n = 2$ *or* $p \in \big(\frac{6}{5}, 4\big)$ *if* $n = 3$. *Assume there exist positive constants* $(a_\star, b_\star, a^\star, b^\star, \bar a, \bar b)$ *such that* $a - \bar a \in \dot B^{n/p}_{p,1}(\mathbb{R}^n)$, $b - \bar b \in \dot B^{n/p}_{p,1}(\mathbb{R}^n)$ *and*

$$0 < a_\star \le a \le a^\star, \quad 0 < b_\star \le b \le b^\star.$$

Furthermore, consider the vector fields u_0 *and* f *with coefficients in* $\dot B^{n/p-1}_{p,1}(\mathbb{R}^n)$ *and* $L^1_{\mathrm{loc}}(\dot B^{n/p-1}_{p,1}(\mathbb{R}^n))$ *respectively. Also, consider the vector field* $R \in (\mathcal{S}'(\mathbb{R}^n))^n$ *with*[1]

$$\mathcal{Q}R \in \mathcal{C}\big([0, \infty); \dot B^{n/p-1}_{p,1}(\mathbb{R}^n)\big) \quad \text{and} \quad (\partial_t R, \nabla \operatorname{div} R) \in L^1_{\mathrm{loc}}(\dot B^{n/p-1}_{p,1}(\mathbb{R}^n))$$

such that

$$\operatorname{div} u_0 = \operatorname{div} R(0, \cdot).$$

Then, system (1-3) *has a unique global solution* $(u, \nabla P)$ *with*

$$u \in C\big([0, \infty), \dot B^{n/p-1}_{p,1}(\mathbb{R}^n)\big) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L^1_{\mathrm{loc}}(\dot B^{n/p-1}_{p,1}(\mathbb{R}^n)).$$

Moreover, there exists a constant $C = C(a, b)$ *such that*

$$\begin{aligned} &\|u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + \|(\partial_t u, \nabla^2 u, \nabla P)\|_{L^1_t(\dot B^{n/p-1}_{p,1})} \\ &\qquad \le \big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L^1_t(\dot B^{n/p-1}_{p,1})}\big) \exp(C(t+1)) \end{aligned} \tag{1-5}$$

for all $t \in [0, \infty)$.

The difficulty in establishing such a result comes from the fact that the pressure and velocity are "strongly" coupled as opposed to the case where ρ is close to a constant; see Remark 2.11 below. The key idea is to use the high-low frequency splitting technique first introduced in [Danchin 2007] combined with the particular structure of the divergence-free part of $a\nabla P$, i.e.,

$$\begin{aligned} \mathcal{P}(a\nabla P) = \mathcal{P}((a - \bar a)\nabla P) &= \mathcal{P}((a - \bar a)\nabla P) - (a - \bar a)\mathcal{P}(\nabla P) \\ &:= [\mathcal{P}, a - \bar a]\nabla P, \end{aligned}$$

[1] $\mathcal{P}$ is the Leray projector over divergence-free vector fields, $\mathcal{Q} = \mathrm{Id} - \mathcal{P}$.

which is, loosely speaking, more regular than ∇P. Let us mention that a similar principle holds for u, which is divergence free:[2] whenever we estimate some term of the form $\mathcal{Q}(bM(D)u)$, where b lies in an appropriate Besov space and $M(D)$ is some pseudodifferential operator, we may write it as

$$\mathcal{Q}(bM(D)u) = [\mathcal{Q}, b]M(D)u$$

and use the fact that the latter expression is more regular than $M(D)u$; see Proposition 3.21.

The proof of Theorem 1.3 in the 3-dimensional case is more subtle. Loosely speaking, in order to close the estimates for system (1-3) one should work in a space on which the solution operator corresponding to the elliptic equation $\operatorname{div}(a\nabla P) = \operatorname{div} f$ is continuous. It is for this reason that we first prove a more restrictive result by demanding extra low-frequency information on the initial data. Then, using a perturbative version of Danchin and Mucha's results [2015] we arrive at constructing a solution with the optimal regularity. Uniqueness is obtained by a duality method.

Once the estimates of Theorem 1.3 are established, we proceed with the proof of Theorem 1.2, which is the object of Section 3. Finally, we show the equivalence between system (1-4) and system (1-1) thus achieving the proof of Theorem 1.1. We end this paper with an Appendix where results of Littlewood–Paley theory used through the text are gathered.

We end this section with some observations regarding the global existence issue. As opposed to the case when ρ is supposed to be a small perturbation of a constant state, when considering the linearized system of the Lagrangian formulation, i.e., system (1-3), we obtain the estimates (1-5), which have a time-dependent right-hand side term. This in particular prevents us from adapting the arguments from [Danchin and Mucha 2012] to our situation and obtaining a global solution for system (1-4) and consequently for the system (1-1). In fact, even if we were able to construct such a solution for system (1-4), it is not clear how we could go back into the original formulation as passing from the Eulerian formulation to the Lagrangian one needs some smallness condition on the $\|\cdot\|_{L^1_t(\dot{B}^{n/p}_{p,1})}$-norm of the velocity.

2. The Stokes system with nonconstant coefficients

Pressure estimates. Before handling system (1-3) we shall study the elliptic equation

$$\operatorname{div}(a\nabla P) = \operatorname{div} f. \tag{2-1}$$

For the reader's convenience let us cite the following classical result, a proof of which can be found, for instance, in [Danchin 2010]:

Proposition 2.1. *Consider $a \in L^\infty(\mathbb{R}^n)$ and a constant $a_\star$ such that*

$$a \geq a_\star > 0.$$

For all vector fields f with coefficients in $L^2(\mathbb{R}^n)$, there exists a tempered distribution P unique up to constant functions such that $\nabla P \in L^2(\mathbb{R}^n)$ and equation (2-1) *is satisfied. In addition, we have*

$$a_\star \|\nabla P\|_{L^2} \leq \|\mathcal{Q} f\|_{L^2}.$$

[2] and thus $\mathcal{Q}u = 0$.

Recently, regarding the 2-dimensional case, Xu et al. [2016], studied the elliptic equation (2-1) with the data $(a-\bar{a}, f)$ in Besov spaces. Using a different approach, we obtain estimates in both 2-dimensional and 3-dimensional situations. Let us also mention that our method allows to obtain a wider range of indices than the one of [Xu et al. 2016, Proposition 3.1(i)]. We choose to focus on the 3-dimensional case. We aim at establishing the following result:

Proposition 2.2. *Consider* $p \in \left(\frac{6}{5}, 2\right)$ *and* $q \in [1,\infty)$ *such that* $\frac{1}{p}-\frac{1}{q} \le \frac{1}{2}$. *Assume there exist positive constants* $(\bar{a}, a_\star, a^\star)$ *such that* $a-\bar{a} \in \dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$ *and* $0 < a_\star \le a \le a^\star$. *Furthermore, consider* $f \in \dot{B}^{3/p-3/2}_{p,2}(\mathbb{R}^3)$. *Then there exists a tempered distribution* P *unique up to constant functions such that* $\nabla P \in \dot{B}^{3/p-3/2}_{p,2}(\mathbb{R}^3)$ *and equation* (2-1) *is satisfied. Moreover, the following estimate holds true*:

$$\|\nabla P\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \left(\frac{1}{\bar{a}} + \left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{3/q}_{q,1}}\right)\left(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}\right)\|\mathcal{Q}f\|_{\dot{B}^{3/p-3/2}_{p,2}}. \tag{2-2}$$

Remark 2.3. Working in Besov spaces with third index $r=2$ is enough in view of the applications that we have in mind. However, similar estimates do hold true when the third index is chosen in the interval $[1,2]$.

Proof. Because $p<2$, Proposition 3.7 ensures that $\dot{B}^{3/p-3/2}_{p,2} \hookrightarrow L^2 = \dot{B}^0_{2,2}$ and owing to Proposition 2.1, we get the existence of $P \in \mathcal{S}'(\mathbb{R}^3)$ with $\nabla P \in L^2$ and

$$a_\star\|\nabla P\|_{L^2} \le \|\mathcal{Q}f\|_{L^2}. \tag{2-3}$$

Moreover, as $\mathcal{Q}$ is a continuous operator on L^2, we deduce from (2-1) that

$$\mathcal{Q}(a\nabla P) = \mathcal{Q}f. \tag{2-4}$$

Using the Bony decomposition (see Definition 3.14 and the remark that follows) and the fact that $\mathcal{P}(\nabla P)=0$, we write

$$\mathcal{P}(a\nabla P) = \mathcal{P}(\dot{T}'_{\nabla P}(a-\bar{a})) + [\mathcal{P}, \dot{T}_{a-\bar{a}}]\nabla P.$$

Using Proposition 3.16 along with Proposition 3.7 and relation (2-3), we get

$$\|\mathcal{P}(\dot{T}'_{\nabla P}(a-\bar{a}))\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \|\nabla P\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{3/p-3/2}_{p^\star,2}} \lesssim \frac{1}{a_\star}\|\mathcal{Q}f\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}, \tag{2-5}$$

where

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{p^\star}.$$

Next, proceeding as in Proposition 3.20 we get

$$\left\|[\mathcal{P}, \dot{T}_{a-\bar{a}}]\nabla P\right\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \|\nabla a\|_{\dot{B}^{3/p-5/2}_{p^\star,2}}\|\nabla P\|_{L^2} \lesssim \frac{1}{a_\star}\|\mathcal{Q}f\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}. \tag{2-6}$$

Putting together relations (2-5) and (2-6) we get

$$\|\mathcal{P}(a\nabla P)\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \frac{1}{a_\star}\|\mathcal{Q}f\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}.$$

Combining this with (2-4) and Proposition 3.7, we find

$$\|a\nabla P\|_{\dot B^{3/p-3/2}_{p,2}} \lesssim \Big(1+\frac{1}{a_\star}\|a-\bar a\|_{\dot B^{3/q}_{q,1}}\Big)\|\mathcal{Q}f\|_{\dot B^{3/p-3/2}_{p,2}}.$$

Of course, writing

$$\nabla P=\frac{1}{a}a\nabla P,$$

using product rules one gets

$$\|\nabla P\|_{\dot B^{3/p-3/2}_{p,2}} \lesssim \Big(\frac{1}{\bar a}+\Big\|\frac{1}{a}-\frac{1}{\bar a}\Big\|_{\dot B^{3/q}_{q,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar a\|_{\dot B^{3/q}_{q,1}}\Big)\|\mathcal{Q}f\|_{\dot B^{3/p-3/2}_{p,2}} \tag{2-7}$$

This concludes the proof. □

Applying the same technique as above leads to the 2-dimensional estimate:

Proposition 2.4. *Consider $p\in(1,2)$ and $q\in[1,\infty)$ such that $\frac{1}{p}-\frac{1}{q}\le\frac{1}{2}$. Assume there exists positive constants $(\bar a,a_\star,a^\star)$ such that $a-\bar a\in\dot B^{2/q}_{q,1}(\mathbb{R}^2)$ and $0<a_\star\le a\le a^\star$. Furthermore, consider $f\in\dot B^{2/p-1}_{p,2}(\mathbb{R}^2)$. Then there exists a tempered distribution P unique up to constant functions such that $\nabla P\in\dot B^{2/p-1}_{p,2}(\mathbb{R}^2)$ and equation* (2-1) *is satisfied. Moreover, the following estimate holds true*:

$$\|\nabla P\|_{\dot B^{2/p-1}_{p,2}} \lesssim \Big(\frac{1}{\bar a}+\Big\|\frac{1}{a}-\frac{1}{\bar a}\Big\|_{\dot B^{2/q}_{q,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar a\|_{\dot B^{2/q}_{q,1}}\Big)\|\mathcal{Q}f\|_{\dot B^{2/p-1}_{p,2}}. \tag{2-8}$$

Let us point out that the restriction $p>\frac{6}{5}$ comes from the fact that we need $\frac{3}{p}-\frac{5}{2}<0$ in relation (2-6). In two dimensions, instead of $\frac{3}{p}-\frac{5}{2}$ we will have $\frac{2}{p}-2$, which is negative provided $p>1$.

The next result covers the range of integrability indices larger than 2:

Proposition 2.5. *Consider $p\in(2,6)$ and $q\in[1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}\ge\frac{1}{2}$. Assume there exist positive constants $(\bar a,a_\star,a^\star)$ such that $a-\bar a\in\dot B^{3/q}_{q,1}(\mathbb{R}^3)$ and $0<a_\star\le a\le a^\star$. Furthermore, consider $f\in\dot B^{3/p-3/2}_{p,2}(\mathbb{R}^3)$ and a tempered distribution P with $\nabla P\in\dot B^{3/p-3/2}_{p,2}(\mathbb{R}^3)$ such that equation* (2-1) *is satisfied. Then, the following estimate holds true*:

$$\|\nabla P\|_{\dot B^{3/p-3/2}_{p,2}} \lesssim \Big(\frac{1}{\bar a}+\Big\|\frac{1}{a}-\frac{1}{\bar a}\Big\|_{\dot B^{3/q}_{q,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar a\|_{\dot B^{3/q}_{q,1}}\Big)\|\mathcal{Q}f\|_{\dot B^{3/p-3/2}_{p,2}}. \tag{2-9}$$

Proof. Notice that p', the conjugate Lebesgue exponent of p, satisfies $p'\in\big(\frac{6}{5},2\big)$ and $\frac{1}{p'}-\frac{1}{q}\le\frac{1}{2}$. Thus, by Proposition 2.2, for any g belonging to the unit ball of $\mathcal{S}\cap\dot B^{3/p'-3/2}_{p',2}$ there exists a $P_g\in\mathcal{S}'(\mathbb{R}^3)$ with $\nabla P_g\in\mathcal{S}\cap\,\dot B^{3/p'-3/2}_{p',2}$ such that

$$\operatorname{div}(a\nabla P_g)=\operatorname{div} g$$

and

$$\|\nabla P_g\|_{\dot B^{3/p'-3/2}_{p',2}} \lesssim \Big(\frac{1}{\bar a}+\Big\|\frac{1}{a}-\frac{1}{\bar a}\Big\|_{\dot B^{3/q}_{q,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar a\|_{\dot B^{3/q}_{q,1}}\Big).$$

We write

$$\begin{aligned}\langle\nabla P,g\rangle&=-\langle P,\operatorname{div} g\rangle=-\langle P,\operatorname{div}(a\nabla P_g)\rangle\\&=-\langle\operatorname{div}\mathcal{Q}f,P_g\rangle=\langle\mathcal{Q}f,\nabla P_g\rangle,\end{aligned}$$

and consequently

$$|\langle \nabla P, g\rangle| \lesssim \|\mathcal{Q}f\|_{\dot B^{3/p-3/2}_{p,2}} \|\nabla P_g\|_{\dot B^{3/p'-3/2}_{p',2}}$$
$$\lesssim \Big(\frac{1}{\bar a} + \Big\|\frac{1}{a} - \frac{1}{\bar a}\Big\|_{\dot B^{3/q}_{q,1}}\Big)\Big(1 + \frac{1}{a_\star}\|a-\bar a\|_{\dot B^{3/q}_{q,1}}\Big)\|\mathcal{Q}f\|_{\dot B^{3/p-3/2}_{p,2}}.$$

Using Proposition 3.8, we get that relation (2-9) holds true. □

As in the previous situation, by applying the same technique we get a similar result in two dimensions:

Proposition 2.6. *Consider $p \in (2,\infty)$ and $q \in [1,\infty)$ such that $\frac{1}{p}+\frac{1}{q} \geq \frac{1}{2}$. Assume there exist positive constants $(\bar a, a_\star, a^\star)$ such that $a-\bar a \in \dot B^{2/q}_{q,1}(\mathbb{R}^2)$ and $0 < a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot B^{2/p-1}_{p,2}(\mathbb{R}^2)$ and a tempered distribution P with $\nabla P \in \dot B^{2/p-1}_{p,2}(\mathbb{R}^2)$ such that equation* (2-1) *is satisfied. Then, following estimate holds true*:

$$\|\nabla P\|_{\dot B^{2/p-1}_{p,2}} \lesssim \Big(\frac{1}{\bar a} + \Big\|\frac{1}{a} - \frac{1}{\bar a}\Big\|_{\dot B^{2/q}_{q,1}}\Big)\Big(1 + \frac{1}{a_\star}\|a-\bar a\|_{\dot B^{2/q}_{q,1}}\Big)\|\mathcal{Q}f\|_{\dot B^{2/p-1}_{p,2}}. \tag{2-10}$$

Some preliminary results. In this section we derive estimates for a Stokes-like problem with time-independent, nonconstant coefficients. Before proceeding to the actual proof, for the reader's convenience, let us cite the following results which were established by Danchin and Mucha [2009; 2015]. These results correspond to the case where a and b are constants:

Proposition 2.7. *Consider $u_0 \in \dot B^{n/p-1}_{p,1}$ and $(f, \partial_t R, \nabla\operatorname{div} R) \in L^1_T(\dot B^{n/p-1}_{p,1})$ with $\mathcal{Q}R \in C_T(\dot B^{n/p-1}_{p,1})$ such that*

$$\operatorname{div} u_0 = \operatorname{div} R(0,\cdot).$$

Then, the system

$$\begin{cases} \partial_t u - \bar a\bar b\Delta u + \bar a\nabla P = f,\\ \operatorname{div} u = \operatorname{div} R,\\ u_{|t=0} = u_0 \end{cases}$$

has a unique solution $(u, \nabla P)$ with

$$u \in \mathcal{C}\big([0,T); \dot B^{n/p-1}_{p,1}\big) \quad \textit{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L^1_T(\dot B^{n/p-1}_{p,1})$$

and the following estimate is valid:

$$\|u\|_{L^\infty_T(\dot B^{n/p-1}_{p,1})} + \|(\partial_t u, \bar a\bar b\nabla^2 u, \bar a\nabla P)\|_{L^1_T(\dot B^{n/p-1}_{p,1})} \lesssim \|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f, \partial_t R, \bar a\bar b\nabla\operatorname{div} R)\|_{L^1_T(\dot B^{n/p-1}_{p,1})}.$$

As a consequence of the previous result, one can establish via a perturbation argument:

Proposition 2.8. *Consider $u_0 \in \dot B^{n/p-1}_{p,1}$ and $(f, \partial_t R, \nabla\operatorname{div} R) \in L^1_T(\dot B^{n/p-1}_{p,1})$ with $\mathcal{Q}R \in C_T(\dot B^{n/p-1}_{p,1})$ such that*

$$\operatorname{div} u_0 = \operatorname{div} R(0,\cdot).$$

Then, there exists an $\eta = \eta(\bar a)$ small enough such that for all $c \in \dot B^{n/p}_{p,1}$ with

$$\|c\|_{\dot B^{n/p}_{p,1}} \leq \eta,$$

the system

$$\begin{cases} \partial_t u - \bar{a}\bar{b}\Delta u + (\bar{a}+c)\nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u_{|t=0} = u_0 \end{cases}$$

has a unique solution $(u, \nabla P)$ *with*

$$u \in \mathcal{C}\big([0,T); \dot{B}^{n/p-1}_{p,1}\big) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L^1_T(\dot{B}^{n/p-1}_{p,1})$$

and the following estimate is valid:

$$\|u\|_{L^\infty_T(\dot{B}^{n/p-1}_{p,1})} + \|(\partial_t u, \bar{a}\bar{b}\nabla^2 u, \bar{a}\nabla P)\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})} \lesssim \|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|(f, \partial_t R, \bar{a}\bar{b}\nabla \operatorname{div} R)\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})}.$$

In all that follows we denote by E_{loc} the space of $(u, \nabla P)$ such that

$$u \in \mathcal{C}\big([0,\infty); \dot{B}^{n/p-1}_{p,1}\big) \quad \text{and} \quad (\nabla^2 u, \nabla P) \in L^1_{\text{loc}}(\dot{B}^{n/p-1}_{p,1}) \times L^1_{\text{loc}}(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1}).$$

Additionally, we introduce the space E_T of $u \in \mathcal{C}_T(\dot{B}^{n/p-1}_{p,1})$ with $\nabla^2 u \in L^1_T(\dot{B}^{n/p-1}_{p,1})$ and $\nabla P \in L^1_T(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})$ such that

$$\|(u,\nabla P)\|_{E_T} = \|u\|_{L^\infty_T(\dot{B}^{n/p-1}_{p,1})} + \|\nabla^2 u\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})} + \|\nabla P\|_{L^1_T(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})} < \infty.$$

The first ingredient in proving Theorem 1.3 is the following:

Proposition 2.9. *Consider* $n \in \{2,3\}$ *and* $p \in (1,4)$ *if* $n=2$ *or* $p \in \big(\frac{6}{5}, 4\big)$ *if* $n=3$. *Assume there exist positive constants* $(a_\star, b_\star, a^\star, b^\star, \bar{a}, \bar{b})$ *such that* $a - \bar{a} \in \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$, $b - \bar{b} \in \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ *and*

$$0 < a_\star \le a \le a^\star, \quad 0 < b_\star \le b \le b^\star.$$

Furthermore, consider u_0, f *vector fields with coefficients in* $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$ *and* $L^1_{\text{loc}}(\dot{B}^{n/p-n/2}_{p,2}(\mathbb{R}^n) \cap \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n))$ *respectively and a vector field* $R \in (\mathcal{S}'(\mathbb{R}^n))^n$ *with*

$$(\partial_t R, \nabla \operatorname{div} R) \in L^1_{\text{loc}}\big(\dot{B}^{n/p-n/2}_{p,2}(\mathbb{R}^n) \cap \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)\big) \quad \text{and} \quad \mathcal{Q}R \in \mathcal{C}\big([0,\infty); \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)\big)$$

such that

$$\operatorname{div} u_0 = \operatorname{div} R(0,\cdot).$$

Then, there exists a constant C_{ab} *depending on a and b such that any solution* $(u, \nabla P) \in E_T$ *of the Stokes system* (1-3) *will satisfy*

$$\|u\|_{L^\infty_t(\dot{B}^{n/p-1}_{p,1})} + \|\nabla^2 u\|_{L^1_t(\dot{B}^{n/p-1}_{p,1})} + \|\nabla P\|_{L^1_t(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})} \le \big(\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L^1_t(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})}\big) \exp(C_{ab}(t+1)) \tag{2-11}$$

for all $t \in (0,T]$.

Before proceeding with the proof, a few remarks are in order:

Remark 2.10. Proposition 2.9 is different from Theorem 1.3 when $n = 3$. Indeed, in the 3-dimensional case the theory is more subtle and thus, as a first step we construct a unique solution for the case of more regular initial data.

Remark 2.11. The difficulty when dealing with the Stokes system with nonconstant coefficients lies in the fact that the pressure and the velocity u are coupled. Indeed, in the constant coefficients case, in view of

$$\operatorname{div} u = \operatorname{div} R,$$

one can apply the divergence operator in the first equation of (1-3) in order to obtain the following elliptic equation verified by the pressure:

$$a\Delta P = \operatorname{div}(f - \partial_t R + 2ab\nabla \operatorname{div} R). \tag{2-12}$$

From (2-12) we can construct the pressure. Having built the pressure, the velocity satisfies a classical heat equation. In the nonconstant coefficient case, proceeding as above we find that

$$\operatorname{div}(a\nabla P) = \operatorname{div}\big(f - \partial_t R + a \operatorname{div}(bD(u))\big). \tag{2-13}$$

Therefore the strategy used in the previous case is not well-adapted. We will establish a priori estimates and use a continuity argument like in [Danchin 2014]. In order to be able to close the estimates on u, we have to bound $\|a\nabla P\|_{L^1_t(\dot B^{n/p-1}_{p,1})}$ in terms of

$$\|u\|^{\beta}_{L^\infty_t(\dot B^{n/p-1}_{p,1})}\|\nabla^2 u\|^{1-\beta}_{L^1_t(\dot B^{n/p-1}_{p,1})}$$

for some $\beta \in (0, 1)$. Thus, the difficulty is to find estimates for the pressure which do not feature the time derivative of the velocity.

In view of Proposition 2.7, consider $(u_L, \nabla P_L)$, the unique solution of the system

$$\begin{cases} \partial_t u - \bar a \operatorname{div}(\bar b D(u)) + \bar a\nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u_{|t=0} = u_0, \end{cases} \tag{2-14}$$

with

$$u_L \in \mathcal{C}\big([0,\infty); \dot B^{n/p-1}_{p,1}\big) \quad \text{and} \quad (\partial_t u_L, \nabla^2 u_L, \nabla P_L) \in L^1_{\mathrm{loc}}(\dot B^{n/p-1}_{p,1}).$$

Recall that for any $t \in [0,\infty)$ we have

$$\|u_L\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + \|(\partial_t u_L, \bar a\bar b\nabla^2 u_L, \bar a\nabla P_L)\|_{L^1_t(\dot B^{n/p-1}_{p,1})} \le C\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f, \partial_t R, \bar a\bar b\nabla \operatorname{div} R)\|_{L^1_t(\dot B^{n/p-1}_{p,1})}\big). \tag{2-15}$$

In what follows, we will use the notation

$$\tilde u = u - u_L, \quad \nabla\widetilde{P} = \nabla P - \nabla P_L. \tag{2-16}$$

Obviously, we have

$$\operatorname{div}\tilde u = 0. \tag{2-17}$$

Thus, the system (1-3) is recast into

$$\begin{cases}\partial_t \tilde{u} - a\,\mathrm{div}(bD(\tilde{u})) + a\nabla \widetilde{P} = \tilde{f},\\ \mathrm{div}\,\tilde{u} = 0,\\ \tilde{u}_{|t=0} = 0,\end{cases} \tag{2-18}$$

where

$$\tilde{f} = a\,\mathrm{div}(bD(u_L)) - \bar{a}\,\mathrm{div}(\bar{b}D(u_L)) - (a-\bar{a})\nabla P_L.$$

Using the last equality along with Proposition 3.17, we infer

$$\begin{aligned}\|\tilde{f}\|_{\dot{B}^{n/p-1}_{p,1}} &\le \big\|a\,\mathrm{div}(bD(u_L)) - \bar{a}\,\mathrm{div}(\bar{b}D(u_L))\big\|_{\dot{B}^{n/p-1}_{p,1}} + \|(a-\bar{a})\nabla P_L\|_{\dot{B}^{n/p-1}_{p,1}}\\ &\lesssim (\bar{a}+\|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}})(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{n/p}_{p,1}})\|\nabla u_L\|_{\dot{B}^{n/p}_{p,1}} + \|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}}\|\nabla P_L\|_{\dot{B}^{n/p-1}_{p,1}}.\end{aligned} \tag{2-19}$$

Let us estimate the pressure $a\nabla\widetilde{P}$. First, we write

$$\|a\nabla\widetilde{P}\|_{\dot{B}^{n/p-1}_{p,1}} \le \|\mathcal{Q}(a\nabla\widetilde{P})\|_{\dot{B}^{n/p-1}_{p,1}} + \|\mathcal{P}(a\nabla\widetilde{P})\|_{\dot{B}^{n/p-1}_{p,1}}.$$

Applying the $\mathcal{Q}$ operator in the first equation of (2-18) we get

$$\mathcal{Q}(a\nabla\widetilde{P}) = \mathcal{Q}\tilde{f} + \mathcal{Q}(a\,\mathrm{div}(bD(\tilde{u}))).$$

Thus, we get

$$\|\mathcal{Q}(a\nabla\widetilde{P})\|_{\dot{B}^{n/p-1}_{p,1}} \le \|\mathcal{Q}\tilde{f}\|_{\dot{B}^{n/p-1}_{p,1}} + \|\mathcal{Q}(a\,\mathrm{div}(bD(\tilde{u})))\|_{\dot{B}^{n/p-1}_{p,1}}. \tag{2-20}$$

Let

$$\mathcal{Q}\big(a\,\mathrm{div}(bD(\tilde{u}))\big) = \mathcal{Q}\big(D(\tilde{u})\dot{S}_m(a\nabla b)\big) + \mathcal{Q}\big(\dot{S}_m(ab-\bar{a}\bar{b})\Delta\tilde{u}\big) \tag{2-21}$$

$$+\,\mathcal{Q}\big(D(\tilde{u})(\mathrm{Id}-\dot{S}_m)(a\nabla b)\big) \tag{2-22}$$

$$+\,\mathcal{Q}\big((\mathrm{Id}-\dot{S}_m)(ab-\bar{a}\bar{b})\Delta\tilde{u}\big), \tag{2-23}$$

where $m \in \mathbb{N}$ will be chosen later. According to Proposition 3.17 we have

$$\big\|\mathcal{Q}\big(D(\tilde{u})\dot{S}_m(a\nabla b)\big)\big\|_{\dot{B}^{n/p-1}_{p,1}} \lesssim \|\dot{S}_m(a\nabla b)\|_{\dot{B}^{n/p-1/2}_{p,1}}\|\nabla\tilde{u}\|_{\dot{B}^{n/p-1/2}_{p,1}}. \tag{2-24}$$

Owing to the fact that $\tilde{u}$ is divergence free we can write

$$\mathcal{Q}(\dot{S}_m(ab-\bar{a}\bar{b})\Delta\tilde{u}) = [\mathcal{Q}, \dot{S}_m(ab-\bar{a}\bar{b})]\Delta\tilde{u}, \tag{2-25}$$

such that applying Proposition 3.21 we get

$$\begin{aligned}\big\|\mathcal{Q}\big(\dot{S}_m(ab-\bar{a}\bar{b})\Delta\tilde{u}\big)\big\|_{\dot{B}^{n/p-1}_{p,1}} &\lesssim \big\|\big(\dot{S}_m(a\nabla b), \dot{S}_m(b\nabla a)\big)\big\|_{\dot{B}^{n/p-1/2}_{p,1}}\|\Delta\tilde{u}\|_{\dot{B}^{n/p-3/2}_{p,1}}\\ &\lesssim \big\|\big(\dot{S}_m(a\nabla b), \dot{S}_m(b\nabla a)\big)\big\|_{\dot{B}^{n/p-1/2}_{p,1}}\|\nabla\tilde{u}\|_{\dot{B}^{n/p-1/2}_{p,1}}.\end{aligned} \tag{2-26}$$

The last two terms of (2-21)–(2-23) are estimated as follows:

$$\big\|\mathcal{Q}\big((\mathrm{Id}-\dot S_m)(a\nabla b)D(\tilde u)\big)+\mathcal{Q}\big((\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\Delta\tilde u\big)\big\|_{\dot B^{n/p-1}_{p,1}} \tag{2-27}$$

$$\lesssim\big(\|(\mathrm{Id}-\dot S_m)(a\nabla b)\|_{\dot B^{n/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{n/p}_{p,1}}\big)\|\nabla\tilde u\|_{\dot B^{n/p}_{p,1}}. \tag{2-28}$$

Thus, putting together relations (2-20)–(2-28) we get

$$\|\mathcal{Q}(a\nabla\widetilde P)\|_{\dot B^{n/p-1}_{p,1}}\lesssim\|\mathcal{Q}\tilde f\|_{\dot B^{n/p-1}_{p,1}}+\big\|\big(\dot S_m(a\nabla b),\dot S_m(b\nabla a)\big)\big\|_{\dot B^{n/p-1/2}_{p,1}}\|\nabla\tilde u\|_{\dot B^{n/p-1/2}_{p,1}}$$
$$+\|\nabla\tilde u\|_{\dot B^{n/p}_{p,1}}\big(\|(\mathrm{Id}-\dot S_m)(a\nabla b,b\nabla a)\|_{\dot B^{n/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{n/p}_{p,1}}\big). \tag{2-29}$$

Next, we turn our attention towards $\mathcal{P}(a\nabla\widetilde P)$. The 2-dimensional case and the 3-dimensional case have to be treated differently.

The 3-dimensional case. Noticing that

$$\mathcal{P}(a\nabla\widetilde P)=\mathcal{P}\big((\mathrm{Id}-\dot S_m)(a-\bar a)\nabla\widetilde P\big)+[\mathcal{P},\dot S_m(a-\bar a)]\nabla\widetilde P,$$

and using again Proposition 3.21 combined with Propositions 2.2 and 2.5 we get

$$\|\nabla\widetilde P\|_{\dot B^{3/p-3/2}_{p,2}}+\|\mathcal{P}(a\nabla\widetilde P)\|_{\dot B^{3/p-1}_{p,1}}$$
$$\lesssim\|\nabla\widetilde P\|_{\dot B^{3/p-3/2}_{p,2}}+\|\mathcal{P}((\mathrm{Id}-\dot S_m)(a-\bar a)\nabla\widetilde P)\|_{\dot B^{3/p-1}_{p,1}}+\|[\mathcal{P},\dot S_m(a-\bar a)]\nabla\widetilde P\|_{\dot B^{3/p-1}_{p,1}} \tag{2-30}$$
$$\lesssim\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{3/p}_{p,1}}\|\nabla\widetilde P\|_{\dot B^{3/p-1}_{p,1}}+(1+\|\dot S_m\nabla a\|_{\dot B^{3/p-1/2}_{p,2}})\|\nabla\widetilde P\|_{\dot B^{3/p-3/2}_{p,2}} \tag{2-31}$$
$$\lesssim\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{3/p}_{p,1}}\Big(\frac{1}{\bar a}+\Big\|\frac1a-\frac1{\bar a}\Big\|_{\dot B^{3/p}_{p,1}}\Big)\|a\nabla\widetilde P\|_{\dot B^{3/p-1}_{p,1}} \tag{2-32}$$
$$+\widetilde C(a)\big(1+\|\dot S_m\nabla a\|_{\dot B^{3/p-1/2}_{p,2}}\big)\big(\|\tilde f\|_{\dot B^{3/p-3/2}_{p,2}}+\|a\,\mathrm{div}(bD(\tilde u))\|_{\dot B^{3/p-3/2}_{p,2}}\big), \tag{2-33}$$

where

$$\widetilde C(a)=\Big(\frac1{\bar a}+\Big\|\frac1a-\frac1{\bar a}\Big\|_{\dot B^{3/p}_{p,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar a\|_{\dot B^{3/p}_{p,1}}\Big).$$

We observe that

$$\|a\,\mathrm{div}(bD(\tilde u))\|_{\dot B^{3/p-3/2}_{p,2}}\lesssim(\bar a+\|a-\bar a\|_{\dot B^{3/p}_{p,1}})(\bar b+\|b-\bar b\|_{\dot B^{3/p}_{p,1}})\|\nabla\tilde u\|_{\dot B^{3/p-1/2}_{p,1}}. \tag{2-34}$$

Putting together (2-30)–(2-33) along with (2-34) we get

$$\|\nabla\widetilde P\|_{\dot B^{3/p-3/2}_{p,2}}+\|\mathcal{P}(a\nabla\widetilde P)\|_{\dot B^{3/p-1}_{p,1}}$$
$$\lesssim\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{3/p}_{p,1}}\Big(\frac1{\bar a}+\Big\|\frac1a-\frac1{\bar a}\Big\|_{\dot B^{3/p}_{p,1}}\Big)\|a\nabla\widetilde P\|_{\dot B^{3/p-1}_{p,1}}+\widetilde C(a)(1+\|\dot S_m\nabla a\|_{\dot B^{3/p-1/2}_{p,2}})$$
$$\times\big(\|\tilde f\|_{\dot B^{3/p-3/2}_{p,2}}+(\bar a+\|a-\bar a\|_{\dot B^{3/p}_{p,1}})(\bar b+\|b-\bar b\|_{\dot B^{3/p}_{p,1}})\|\nabla\tilde u\|_{\dot B^{3/p-1/2}_{p,1}}\big). \tag{2-35}$$

Combining (2-29) with (2-35) yields

$$\|\nabla\widetilde{P}\|_{\dot B^{3/p-3/2}_{p,2}}+\|a\nabla\widetilde{P}\|_{\dot B^{3/p-1}_{p,1}}\lesssim T^1_m(a,b)\|a\nabla\widetilde{P}\|_{\dot B^{3/p-1}_{p,1}}+T^2_m(a,b)\|\tilde f\|_{\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1}}\\ +T^3_m(a,b)\|\nabla\tilde u\|_{\dot B^{3/p-1/2}_{p,1}}+T^4_m(a,b)\|\nabla\tilde u\|_{\dot B^{3/p}_{p,1}},$$

where

$$\begin{aligned}
T^1_m(a,b)&=\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{3/p}_{p,1}}\Big(\frac{1}{\bar a}+\Big\|\frac1a-\frac1{\bar a}\Big\|_{\dot B^{3/p}_{p,1}}\Big),\\
T^2_m(a,b)&=\widetilde C(a)(1+\|\dot S_m\nabla a\|_{\dot B^{3/p-1/2}_{p,2}}),\\
T^3_m(a,b)&=\|(\dot S_m(a\nabla b),\dot S_m(b\nabla a))\|_{\dot B^{3/p-1/2}_{p,1}}\\
&\qquad+\widetilde C(a)(1+\|\dot S_m\nabla a\|_{\dot B^{3/p-1/2}_{p,2}})(\bar a+\|a-\bar a\|_{\dot B^{3/p}_{p,1}})(\bar b+\|b-\bar b\|_{\dot B^{3/p}_{p,1}}),\\
T^4_m(a,b)&=\|(\mathrm{Id}-\dot S_m)(a\nabla b,b\nabla a)\|_{\dot B^{3/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{3/p}_{p,1}}.
\end{aligned}$$

Observe that m could be chosen large enough such that $T^1_m(a,b)$ and $T^4_m(a,b)$ can be made arbitrarily small. Thus, there exists a constant C_{ab} depending on a and b such that

$$\|\nabla\widetilde{P}\|_{\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1}}\le C_{ab}\big(\|\tilde f\|_{\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1}}+\|\nabla\tilde u\|_{\dot B^{3/p-1/2}_{p,1}}\big)+\eta\|\nabla\tilde u\|_{\dot B^{3/p}_{p,1}},\tag{2-36}$$

where η can be made arbitrarily small (of course, with the price of increasing the constant C_{ab}). Let us take a look at the $\dot B^{3/p-3/2}_{p,2}$-norm of $\tilde f$; we get

$$\begin{aligned}
\|\tilde f\|_{\dot B^{3/p-3/2}_{p,2}}&\le\big\|a\operatorname{div}(bD(u_L))-\bar a\operatorname{div}(\bar bD(u_L))\big\|_{\dot B^{3/p-3/2}_{p,2}}+\|(a-\bar a)\nabla P_L\|_{\dot B^{3/p-3/2}_{p,2}}\\
&\lesssim(\bar a+\|a-\bar a\|_{\dot B^{3/p}_{p,1}})(\bar b+\|b-\bar b\|_{\dot B^{3/p}_{p,1}})\|\nabla u_L\|_{\dot B^{3/p-1/2}_{p,1}}+\|a-\bar a\|_{\dot B^{3/p}_{p,1}}\|\nabla P_L\|_{\dot B^{3/p-3/2}_{p,2}}.
\end{aligned}\tag{2-37}$$

As $u_L\in C\big([0,\infty),\dot B^{3/p-1}_{p,1}\big)\cap L^1\big([0,\infty),\dot B^{3/p+1}_{p,1}\big)$ and $\mathcal Q$ is a continuous operator on homogeneous Besov spaces from

$$\operatorname{div}(u_L-R)=0,$$

we deduce

$$\mathcal P(u_L-R)=u_L-R,$$

which implies

$$\mathcal Qu_L=\mathcal QR.$$

By applying the operator $\mathcal Q$ in the first equation of system (2-14) we get

$$\begin{aligned}
\bar a\nabla P_L&=\mathcal Qf-\mathcal Q\partial_tu_L+\bar a\bar b\mathcal Q\Delta u_L+\bar a\bar b\nabla\operatorname{div}R\\
&=\mathcal Qf-\mathcal Q\partial_tR+2\bar a\bar b\nabla\operatorname{div}R
\end{aligned}$$

and thus

$$\|\nabla P_L\|_{\dot B^{3/p-3/2}_{p,2}}\le\frac1{\bar a}\|\mathcal Qf\|_{\dot B^{3/p-3/2}_{p,2}}+\frac1{\bar a}\|\partial_t\mathcal QR\|_{\dot B^{3/p-3/2}_{p,2}}+2\bar b\|\nabla\operatorname{div}R\|_{\dot B^{3/p-3/2}_{p,2}}.$$

In view of (2-36), (2-19), (2-37) and interpolation we gather that there exists a constant C_{ab} such that

$$\begin{aligned}
&\|\nabla\widetilde{P}\|_{\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1}}\\
&\quad\le C_{ab}\big(\|\nabla u_L\|_{\dot B^{3/p-1/2}_{p,1}}+\|\nabla P_L\|_{\dot B^{3/p-3/2}_{p,2}}+\|(\nabla^2u_L,\nabla P_L)\|_{\dot B^{3/p-1}_{p,1}}+\|\nabla\tilde u\|_{\dot B^{3/p-1/2}_{p,1}}\big)+\eta\|\nabla\tilde u\|_{\dot B^{3/p}_{p,1}} &(2\text{-}38)\\
&\quad\le C_{ab}\|(\mathcal{Q}f,\partial_t\mathcal{Q}R,\nabla\operatorname{div}R)\|_{\dot B^{3/p-3/2}_{p,2}}+C_{ab}\|u_L\|_{\dot B^{3/p-1}_{p,1}} &(2\text{-}39)\\
&\qquad\qquad+C_{ab}\|(\nabla^2u_L,\nabla P_L)\|_{\dot B^{3/p-1}_{p,1}}+C_{ab}\|\tilde u\|_{\dot B^{3/p-1}_{p,1}}+2\eta\|\nabla\tilde u\|_{\dot B^{3/p}_{p,1}}, &(2\text{-}40)
\end{aligned}$$

where, again, at the price of increasing C_{ab}, we can make η arbitrarily small.

The 2-dimensional case. In this case, using again Proposition 3.21 combined with Propositions 2.4 and 2.6 we get

$$\begin{aligned}
&\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,2}}+\|\mathcal{P}(a\nabla\widetilde{P})\|_{\dot B^{2/p-1}_{p,1}}\\
&\qquad\lesssim\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,2}}+\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{2/p}_{p,1}}\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}+\|[\mathcal{P},\dot S_m(a-\bar a)]\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}\\
&\qquad\lesssim\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{2/p}_{p,1}}\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}+(1+\|\nabla\dot S_m a\|_{\dot B^{2/p}_{p,2}})\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,2}}\\
&\qquad\lesssim\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{2/p}_{p,1}}\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}\\
&\qquad\qquad+\widetilde{C}(a)(1+\|\nabla\dot S_m a\|_{\dot B^{2/p}_{p,2}})\big(\|\tilde f\|_{\dot B^{2/p-1}_{2,2}}+\|\mathcal{Q}(a\operatorname{div}(bD(\tilde u)))\|_{\dot B^{2/p-1}_{p,2}}\big)
\end{aligned}$$

where, as before

$$\widetilde{C}(a)=\Big(\frac{1}{\bar a}+\Big\|\frac1a-\frac1{\bar a}\Big\|_{\dot B^{2/p}_{p,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar a\|_{\dot B^{2/p}_{p,1}}\Big).$$

As we have already estimated $\|\mathcal{Q}(a\operatorname{div}(bD(\tilde u)))\|_{\dot B^{2/p-1}_{p,2}}$ in (2-29), we gather

$$\begin{aligned}
\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,2}}+\|a\nabla\widetilde{P}\|_{\dot B^{2/p}_{p,1}}&\lesssim T^1_m(a,b)\|a\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}+T^2_m(a,b)\|\tilde f\|_{\dot B^{2/p-1}_{p,1}}\\
&\qquad+T^3_{m,M}(a,b)\|\nabla\tilde u\|_{\dot B^{2/p-1/2}_{p,1}}+T^4_{m,M}(a,b)\|\nabla\tilde u\|_{\dot B^{2/p}_{p,1}}, \qquad (2\text{-}41)
\end{aligned}$$

where

$$\begin{aligned}
T^1_m(a,b)&=\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{2/p}_{p,1}}\Big(\frac1{\bar a}+\Big\|\frac1a-\frac1{\bar a}\Big\|_{\dot B^{2/p}_{p,1}}\Big),\\
T^2_m(a,b)&=\widetilde{C}(a)(1+\|\nabla\dot S_m a\|_{\dot B^{2/p}_{p,2}}),\\
T^3_{m,M}(a,b)&=\|(\dot S_m(a\nabla b),\dot S_m(b\nabla a))\|_{\dot B^{2/p}_{p,1}}+\widetilde{C}(a)(1+\|\nabla\dot S_m a\|_{\dot B^{2/p}_{p,2}})\|(\dot S_M(a\nabla b),\dot S_M(b\nabla a))\|_{\dot B^{2/p}_{p,1}},\\
T^4_{m,M}(a,b)&=\|(\mathrm{Id}-\dot S_m)(a\nabla b)\|_{\dot B^{2/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{2/p}_{p,1}}\\
&\qquad+\widetilde{C}(a)(1+\|\nabla\dot S_m a\|_{\dot B^{2/p}_{p,2}})\big(\|(\mathrm{Id}-\dot S_M)(a\nabla b)\|_{\dot B^{2/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_M)(ab-\bar a\bar b)\|_{\dot B^{2/p-1}_{p,1}}\big).
\end{aligned}$$

First, we fix an $\eta>0$. Let us fix an $m\in\mathbb{N}$ such that $T^1_m(a,b)\|a\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}$ can be "absorbed" by the left-hand side of (2-41) and such that

$$\|(\mathrm{Id}-\dot S_m)(a\nabla b)\|_{\dot B^{2/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{2/p}_{p,1}}\le\tfrac12\eta.$$

Next, we see that by choosing M large enough we have

$$T^4_{m,M}(a,b)\le\eta.$$

Thus, using interpolation we can write

$$\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,2}}+\|a\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}\le C_{ab}\big(\|(\nabla^2u_L,\nabla P_L)\|_{\dot B^{2/p-1}_{2,1}}+\|\tilde u\|_{\dot B^{2/p-1}_{p,1}}\big)+2\eta\|\nabla^2\tilde u\|_{\dot B^{2/p-1}_{p,1}}. \quad (2\text{-}42)$$

End of the proof of Proposition 2.9. Obviously, combining the two estimates (2-38)–(2-40) and (2-42) we can continue in a unified manner the rest of the proof of Proposition 2.9. First, choose $m\in\mathbb{N}$ large enough such that

$$\bar a\bar b+\dot S_m(ab-\bar a\bar b)\ge\tfrac12 a_\star b_\star.$$

We apply $\dot\Delta_j$ to (2-18) and we write

$$\begin{aligned}\partial_t\tilde u_j&-\operatorname{div}\big((\bar a\bar b+\dot S_m(ab-\bar a\bar b))\nabla\tilde u_j\big)\\&=\tilde f_j-\dot\Delta_j(a\nabla\widetilde P)+\dot\Delta_j\operatorname{div}\big((\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\nabla\tilde u\big)+\operatorname{div}[\dot\Delta_j,\dot S_m(ab-\bar a\bar b)]\nabla\tilde u\\&\quad+\dot\Delta_j(D\tilde u\dot S_m(b\nabla a))+\dot\Delta_j\big(D\tilde u(\mathrm{Id}-\dot S_m)(b\nabla a)\big)+\dot\Delta_j(\nabla\tilde u\dot S_m(a\nabla b))+\dot\Delta_j\big(\nabla\tilde u(\mathrm{Id}-\dot S_m)(a\nabla b)\big).\end{aligned}$$

Multiplying the last relation by $|\tilde u_j|^{p-1}\operatorname{sgn}\tilde u_j$, integrating and using Lemma 8 from Appendix B of [Danchin 2010], we get

$$\begin{aligned}\|\tilde u_j\|_{L^p}+a_\star b_\star 2^{2j}C\int_0^t\|\tilde u_j\|_{L^p}\lesssim\int_0^t\|\tilde f_j\|_{L^p}+\int_0^t\|\dot\Delta_j(a\nabla\widetilde P)\|_{L^p}+\int_0^t\big\|\operatorname{div}[\dot\Delta_j,\dot S_m(ab-\bar a\bar b)]\nabla\tilde u\big\|_{L^p}\\+\int_0^t\big\|\dot\Delta_j\operatorname{div}\big((\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\nabla\tilde u\big)\big\|_{L^p}\\+\int_0^t\|\dot\Delta_j(D\tilde u\dot S_m(b\nabla a))\|_{L^p}+\int_0^t\big\|\dot\Delta_j\big(D\tilde u(\mathrm{Id}-\dot S_m)(b\nabla a)\big)\big\|_{L^p}\\+\int_0^t\|\dot\Delta_j(\nabla\tilde u\dot S_m(a\nabla b))\|_{L^p}+\int_0^t\big\|\dot\Delta_j\big(\nabla\tilde u(\mathrm{Id}-\dot S_m)(a\nabla b)\big)\big\|_{L^p}.\end{aligned}$$

Multiplying the last relation by $2^{j(n/p-1)}$, performing an $\ell^1(\mathbb{Z})$-summation and using Proposition 3.19 to deal with $\big\|\operatorname{div}[\dot\Delta_j,\dot S_m(ab-\bar a\bar b)]\nabla\tilde u\big\|_{\dot B^{n/p-1}_{p,1}}$ along with (2-38)–(2-40) and (2-41) to deal with the pressure, we get

$$\begin{aligned}&\|\tilde u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})}+a_\star b_\star C\|\nabla^2\tilde u\|_{L^1_t(\dot B^{n/p-1}_{p,1})}\\&\quad\lesssim\|\tilde f\|_{L^1_t(\dot B^{n/p-1}_{p,1})}+C\int_0^t\|a\nabla\widetilde P\|_{\dot B^{n/p-1}_{p,1}}+\int_0^t\|(\dot S_m(b\nabla a),\dot S_m(a\nabla b))\|_{\dot B^{n/p}_{p,1}}\|\nabla\tilde u\|_{\dot B^{n/p-1}_{p,1}}\\&\qquad\qquad+T_m(a,b)\|\nabla^2\tilde u\|_{L^1_t(\dot B^{n/p-1}_{p,1})}\\&\quad\le C_{ab}(1+t)\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}}+\|(f,\partial_tR,\nabla\operatorname{div}R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)+C_{ab}\int_0^t\|\tilde u\|_{\dot B^{n/p-1}_{p,1}}\\&\qquad\qquad+(T_m(a,b)+\eta)\|\nabla^2\tilde u\|_{L^1_t(\dot B^{n/p-1}_{p,1})}\end{aligned}\quad(2\text{-}43)$$

where

$$T_m(a,b)=\|(\mathrm{Id}-\dot S_m)(b\nabla a)\|_{\dot B^{n/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(a\nabla b)\|_{\dot B^{n/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{n/p-1}_{p,1}}. \quad (2\text{-}44)$$

Assuming m is large enough and η is small enough, we can "absorb" $(T_m(a,b)+\eta)\|\nabla^2 u\|_{L^1_t(\dot B^{n/p-1}_{p,1})}$ into the left-hand side of (2-43). Thus, we end up with

$$\|\tilde u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + a_\star b_\star \tfrac{1}{2} C\|\nabla^2\tilde u\|_{L^1_t(\dot B^{n/p-1}_{p,1})} \le C_{ab}(1+t)\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big) + C_{ab}\int_0^t \|\tilde u\|_{\dot B^{n/p-1}_{p,1}}$$

such that using Grönwall's lemma, (2-15) and the classical inequality

$$1+t^\alpha \le C_\alpha \exp(t)$$

yields

$$\|\tilde u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + a_\star b_\star \tfrac{1}{2} C\|\nabla^2\tilde u\|_{L^1_t(\dot B^{n/p-1}_{p,1})} \le C_{ab}\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)\exp(C_{ab}t). \quad (2\text{-}45)$$

Using the fact that $u = u_L + \tilde u$ along with (2-15) and (2-45) gives us

$$\|u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + a_\star b_\star \tfrac{1}{2} C\|\nabla^2 u\|_{L^1_t(\dot B^{n/p-1}_{p,1})} \le C_{ab}\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)\exp(C_{ab}t). \quad (2\text{-}46)$$

Next, using (2-38)–(2-40) and (2-42) combined with (2-15), we infer

$$\begin{aligned}\|\nabla P\|_{L^1_t((\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})} &\le C_a\|a\nabla P_L\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})} + C_a\|a\nabla\widetilde P\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})} & (2\text{-}47)\\ &\le C_{ab}\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)\exp(C_{ab}t). & (2\text{-}48)\end{aligned}$$

Combining (2-48) with (2-46) we finally get

$$\|u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + \|\nabla^2 u\|_{L^1_t(\dot B^{n/p-1}_{p,1})} + \|\nabla P\|_{L^1_t((\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})} \le \big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)\exp(C_{ab}(t+1)). \quad (2\text{-}49)$$

Obviously, by obtaining the last estimate we conclude the proof of Proposition 2.9.

Next, let us deal with the existence part of the Stokes problem with the coefficients having regularity as in Proposition 2.9. More precisely, we have:

Proposition 2.12. *Consider* (a,b,u_0,f,R) *as in the statement of Proposition 2.9. Then, there exists a unique solution* $(u,\nabla P)\in E_{\mathrm{loc}}$ *of the Stokes system* (1-3)*. Furthermore, there exists a constant* C_{ab} *depending on a and b such that*

$$\|u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + \|\nabla^2 u\|_{L^1_t(\dot B^{n/p-1}_{p,1})} + \|\nabla P\|_{L^1_t((\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})} \le \big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)\exp(C_{ab}(t+1)) \quad (2\text{-}50)$$

for all $t>0$.

Proof. The uniqueness property is a direct consequence of the estimates of Proposition 2.9. The proof of existence relies on Proposition 2.9 combined with a continuity argument as used in [Danchin 2014]; see also [Krylov 2008]. Let us introduce

$$(a_\theta, b_\theta) = (1-\theta)(\bar{a}, \bar{b}) + \theta(a, b)$$

and consider the Stokes system

$$\begin{cases} \partial_t u - a_\theta(\operatorname{div}(b_\theta D(u)) - \nabla P) = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u_{|t=0} = u_0. \end{cases} \tag{$\mathcal{S}_\theta$}$$

First of all, a more detailed analysis of the estimates established in Proposition 2.9 enables us to conclude that the constant $C_{a_\theta b_\theta}$ appearing in (2-49) is uniformly bounded with respect to $\theta \in [0, 1]$ by a constant $c = c_{ab}$. Indeed, repeating the estimation process carried out in Proposition 2.9 with (a_θ, b_θ) instead of (a, b) amounts to replacing $(a - \bar{a})$ and $(b - \bar{b})$ with $\theta(a - \bar{a})$ and $\theta(b - \bar{b})$. Taking into account Proposition 3.12 and the remark that follows we get that there exists

$$c := \sup_{\theta \in [0,1]} C_{a_\theta b_\theta} < \infty.$$

Let us take $T > 0$ and consider $\mathcal{E}_T$, the set of those $\theta \in [0, 1]$ such that for any (u_0, f, R) as in the statement of Proposition 2.9 problem $(\mathcal{S}_\theta)$ admits a unique solution $(u, \nabla P) \in E_T$ which satisfies

$$\begin{aligned} \|u\|_{L_t^\infty(\dot{B}^{n/p-1}_{p,1})} + \|\nabla^2 u\|_{L_t^1(\dot{B}^{n/p-1}_{p,1})} + \|\nabla P\|_{L_t^1(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})} \\ \leq \big(\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})}\big) \exp(c(t+1)) \end{aligned} \tag{2-51}$$

for all $t \in [0, T]$. According to Proposition 2.7, $0 \in \mathcal{E}_T$.

Suppose $\theta \in \mathcal{E}_T$. First we denote by $(u_\theta, \nabla P_\theta) \in E_T$ the unique solution of $(\mathcal{S}_\theta)$. We consider the space

$$E_{T,\mathrm{div}} = \{(\tilde{w}, \nabla \widetilde{Q}) \in E_T : \operatorname{div} \tilde{w} = 0\}$$

and let $S_{\theta\theta'}$ be the operator which associates to $(\tilde{w}, \nabla \widetilde{Q}) \in E_{T,\mathrm{div}}$, the unique solution $(\tilde{u}, \nabla \widetilde{P})$ of

$$\begin{cases} \partial_t \tilde{u} - a_\theta(\operatorname{div}(b_\theta D(\tilde{u})) - \nabla \widetilde{P}) = g_{\theta\theta'}(u_\theta, \nabla P_\theta) + g_{\theta\theta'}(\tilde{w}, \nabla \widetilde{Q}), \\ \operatorname{div} u = 0, \\ u_{|t=0} = 0, \end{cases} \tag{2-52}$$

where

$$g_{\theta\theta'}(u, \nabla P) = (a_\theta - a_{\theta'})\nabla P + a_{\theta'} \operatorname{div}(b_{\theta'} D(u)) - a_\theta \operatorname{div}(b_\theta D(u)). \tag{2-53}$$

Obviously, $S_{\theta\theta'}$ maps $E_{T,\mathrm{div}}$ into $E_{T,\mathrm{div}}$. We claim that there exists a positive quantity $\varepsilon = \varepsilon(T) > 0$ such that if $|\theta - \theta'| \leq \varepsilon(T)$ then $S_{\theta\theta'}$ has a fixed point $(\tilde{u}^\star, \nabla \widetilde{P}^\star)$ in a suitable ball centered at the origin of the space $E_{T,\mathrm{div}}$. Obviously,

$$(\tilde{u}^\star + u_\theta, \nabla \widetilde{P}^\star + \nabla P_\theta)$$

will solve $(\mathcal{S}_{\theta'})$ in E_T.

First, we note that, as a consequence of Proposition 2.9, we have

$$\|(\tilde{u},\nabla\widetilde{P})\|_{E_T} \le \big(\|g_{\theta\theta'}(u_\theta,\nabla P_\theta)\|_{L^1_T(\dot{B}^{n/p-n/2}_{p,2}\cap\dot{B}^{n/p-1}_{p,1})}+\|g_{\theta\theta'}(\tilde{w},\nabla\widetilde{Q})\|_{L^1_T(\dot{B}^{n/p-n/2}_{p,2}\cap\dot{B}^{n/p-1}_{p,1})}\big)\exp(c(T+1)). \tag{2-54}$$

Observe that

$$\|(a_\theta-a_{\theta'})\nabla P\|_{L^1_T(\dot{B}^{n/p-n/2}_{p,2}\cap\dot{B}^{n/p-1}_{p,1})} \le |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}}\|\nabla P\|_{L^1_T(\dot{B}^{n/p-n/2}_{p,2}\cap\dot{B}^{n/p-1}_{p,1})}. \tag{2-55}$$

Next, we write

$$a_{\theta'}\operatorname{div}(b_{\theta'}D(u))-a_\theta\operatorname{div}(b_\theta D(u)) = (a_{\theta'}-a_\theta)\operatorname{div}(b_{\theta'}D(u))+a_\theta\operatorname{div}((b_{\theta'}-b_\theta)D(u)).$$

The first term of the last identity is estimated as follows:

$$\|(a_{\theta'}-a_\theta)\operatorname{div}(b_{\theta'}D(u))\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})} \le |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}}(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{n/p}_{p,1}})\|D(u)\|_{L^1_T(\dot{B}^{n/p}_{p,1})}.$$

Regarding the second term, we have

$$\|a_\theta\operatorname{div}((b_{\theta'}-b_\theta)D(u))\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})} \le |\theta-\theta'|\,\|b-\bar{b}\|_{\dot{B}^{n/p}_{p,1}}(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}})\|D(u)\|_{L^1_T(\dot{B}^{n/p}_{p,1})}$$

and thus

$$\big\|a_{\theta'}\operatorname{div}(b_{\theta'}D(u))-a_\theta\operatorname{div}(b_\theta D(u))\big\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})} \le |\theta-\theta'|(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}})(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{n/p}_{p,1}})\|Du\|_{L^1_T(\dot{B}^{n/p}_{p,1})}. \tag{2-56}$$

The only thing left is to treat the $L^1_t(\dot{B}^{3/p-3/2}_{p,2})$-norm of $a_{\theta'}\operatorname{div}(b_{\theta'}D(u))-a_\theta\operatorname{div}(b_\theta D(u))$ in the case where $n=3$. Using the fact that $\nabla u\in L^{4/3}_T(\dot{B}^{3/p-1/2}_{p,1})$, we can write

$$\begin{aligned}\|(a_{\theta'}-a_\theta)\operatorname{div}(b_{\theta'}D(u))\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2})} &\le |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}}\|\operatorname{div}(b_{\theta'}D(u))\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,1})}\\ &\le |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}}(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{3/p}_{p,1}})\|Du\|_{L^1_T(\dot{B}^{3/p-1/2}_{p,1})} &(2\text{-}57)\\ &\le |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}}(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{3/p}_{p,1}})T^{1/4}\|u\|^{1/4}_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})}\|u\|^{3/4}_{L^1_T(\dot{B}^{3/p+1}_{p,1})} &(2\text{-}58)\\ &\le |\theta-\theta'|C(T,a,b)\big(\|u\|_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})}+\|\nabla^2u\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}\big) &(2\text{-}59)\end{aligned}$$

and, proceeding in a similar manner, we can estimate $\big\|a_\theta\operatorname{div}\big((b_{\theta'}-b_\theta)D(u)\big)\big\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2})}$.

Combining (2-55), (2-56) along with (2-59) we get

$$\|g_{\theta\theta'}(u,\nabla P)\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1})} \le |\theta-\theta'|C(T,a,b)\big(\|u\|_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})}+\|\nabla^2u\|_{L^1_T(\dot{B}^{3/p+1}_{p,1})}+\|\nabla P\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1})}\big). \tag{2-60}$$

Substituting this into (2-54), we get

$$\|(\tilde{u},\nabla\widetilde{P})\|_{E_T} \le |\theta-\theta'|C(T,a,b)\big(\|(u_\theta,\nabla P_\theta)\|_{E_T}+\|(\tilde{w},\nabla\widetilde{Q})\|_{E_T}\big),$$

and by linearity

$$\big\|(\tilde{u}^1-\tilde{u}^2,\nabla\widetilde{P}^1-\nabla\widetilde{P}^2)\big\|_{E_T}\le|\theta-\theta'|\,C(T,a,b)\big\|(\tilde{w}^1-\tilde{w}^2,\nabla\widetilde{Q}^1-\nabla\widetilde{Q}^2)\big\|_{E_T},$$

where for $k=1,2$,

$$(\tilde{u}^i,\nabla\widetilde{P}^i)=S_{\theta\theta'}((\tilde{w}^i,\nabla\widetilde{Q}^i)).$$

Thus one can choose $\varepsilon(T)$ small enough such that $|\theta-\theta'|\le\varepsilon(T)$ gives us a fixed point of the solution operator $S_{\theta\theta'}$ in $B_{E_{T,\mathrm{div}}}(0,2\|(u_\theta,\nabla P_\theta)\|_{E_T})$.

Thus, for all $T>0$, we have $E_T=[0,1]$ and owing to the uniqueness property and to Proposition 2.9, we can construct a unique solution $(u,\nabla P)\in E_{\mathrm{loc}}$ to (1-3) such that for all $t>0$ the estimate (2-11) is valid. □

The proof of Theorem 1.3 in the case n = 3. As discussed earlier, in dimension $n=3$, Proposition 2.9 is weaker than Theorem 1.3, as one requires additional low-frequency information on the data

$$(f,\partial_t R,\nabla\operatorname{div}R)\in L^1_t(\dot{B}^{3/p-3/2}_{p,2}).$$

Thus, we have to bring an extra argument in order to conclude the validity of Theorem 1.3. This is the object of interest of this section.

Existence. We begin by taking $m\in\mathbb{N}$ large enough and owing to Proposition 2.8 we can consider $(u^1,\nabla P^1)$, the unique solution with $u^1\in\mathcal{C}(\mathbb{R}^+;\dot{B}^{3/p-1}_{p,1})$ and $(\partial_t u^1,\nabla^2u^1,\nabla P^1)\in L^1_{\mathrm{loc}}(\dot{B}^{3/p-1}_{p,1})$ of the system

$$\begin{cases}\partial_t u-\bar{a}\bar{b}\operatorname{div}D(u)+(\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P=f,\\ \operatorname{div}u=\operatorname{div}R,\\ u_{|t=0}=u_0,\end{cases}$$

which also satisfies

$$\begin{aligned}\|u^1\|_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})}+\|(\partial_t u^1,\bar{a}\bar{b}\nabla^2u^1,\bar{a}\nabla P^1)\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}&\\ \le C\big(\|u_0\|_{\dot{B}^{3/p-1}_{p,1}}+\|(f,\partial_t R,\bar{a}\bar{b}\nabla\operatorname{div}R)\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}\big)&\end{aligned}$$

for all $T>0$. Let us consider

$$G(u^1,\nabla P^1)=a\operatorname{div}(bD(u^1))-\bar{a}\operatorname{div}(\bar{b}D(u^1))-((\mathrm{Id}-\dot{S}_{-m})(a-\bar{a}))\nabla P^1.$$

We claim $G(u^1,\nabla P^1)\in L^1_{\mathrm{loc}}(\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1})$. Indeed

$$a\operatorname{div}(bD(u^1))-\bar{a}\operatorname{div}(\bar{b}D(u^1))=(a-\bar{a})\operatorname{div}(bD(u^1))+\bar{a}\operatorname{div}((b-\bar{b})D(u^1))$$

and proceeding as in (2-56) and (2-58) we get

$$\begin{aligned}\big\|a\operatorname{div}(bD(u^1))-\bar{a}\operatorname{div}(\bar{b}D(u^1))\big\|_{L^1_t(\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1})}&\\ \le C_{ab}(1+t^{1/4})\big(\|u^1\|_{L^\infty_t(\dot{B}^{3/p-1}_{p,1})}+\|u^1\|_{L^1_t(\dot{B}^{3/p+1}_{p,1})}\big)&\\ \le\exp(C_{ab}(t+1))\big(\|u_0\|_{\dot{B}^{3/p-1}_{p,1}}+\|(f,\partial_t R,\nabla\operatorname{div}R)\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}\big).&\qquad(2\text{-}61)\end{aligned}$$

Next, we obviously have

$$\big\|\big((\mathrm{Id}-\dot S_{-m})(a-\bar a)\big)\nabla P^1\big\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\le C\|(a-\bar a)\|_{\dot B^{3/p}_{p,1}}\|\nabla P^1\|_{L^1_t(\dot B^{3/p-1}_{p,1})}. \tag{2-62}$$

Using the fact that the product maps $\dot B^{3/p-1/2}_{p,1}\times\dot B^{3/p-1}_{p,1}\to\dot B^{3/p-3/2}_{p,2}$, we get

$$\big\|\big((\mathrm{Id}-\dot S_{-m})(a-\bar a)\big)\nabla P^1\big\|_{L^1_t(\dot B^{3/p-3/2}_{p,2})}\le C\|(\mathrm{Id}-\dot S_{-m})(a-\bar a)\|_{\dot B^{3/p-1/2}_{p,1}}\|\nabla P^1\|_{L^1_t(\dot B^{3/p-1}_{p,1})}. \tag{2-63}$$

Of course

$$\begin{aligned}\|(\mathrm{Id}-\dot S_{-m})(a-\bar a)\|_{\dot B^{3/p-1/2}_{p,1}}&\le C\sum_{j\ge -m}2^{j(3/p-1/2)}\|\dot\Delta_j(a-\bar a)\|_{L^2}\le C2^{m/2}\sum_{j\ge -m}2^{3/pj}\|\dot\Delta_j(a-\bar a)\|_{L^2}.\\&\le C2^{m/2}\|a-\bar a\|_{\dot B^{3/p}_{p,1}}\end{aligned}$$

so that the first term on the right-hand side of (2-63) is finite. We thus gather from (2-61), (2-62) and (2-63) that $G(u^1,\nabla P^1)\in L^1_{\mathrm{loc}}(\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1})$ and that for all $t>0$ there exists a constant C_{ab} such that

$$\|G(u^1,\nabla P^1)\|_{L^1_t(\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1})}\le(\|u_0\|_{\dot B^{3/p-1}_{p,1}}+\|(f,\partial_t R,\nabla\operatorname{div}R)\|_{L^1_t(\dot B^{3/p-1}_{p,1})})\exp(C_{ab}(t+1)).$$

According to Proposition 2.12, there exists a unique solution $(u^2,\nabla P^2)\in E_{\mathrm{loc}}$ of the system

$$\begin{cases}\partial_t u-a\operatorname{div}(bD(u))+a\nabla P=G(u^1,\nabla P^1),\\ \operatorname{div}u=0,\\ u_{|t=0}=0,\end{cases}$$

which satisfies the estimate

$$\begin{aligned}\|u^2\|_{L^\infty_t(\dot B^{3/p-1}_{p,1})}&+\|(\nabla^2u^2,\nabla P^2)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\\&\le\|G(u^1,\nabla P^1)\|_{L^1_t(\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1})}\exp(C_{ab}(t+1))\\&\le\big(\|u_0\|_{\dot B^{3/p-1}_{p,1}}+\|(f,\partial_t R,\nabla\operatorname{div}R)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\big)\exp(C_{ab}(t+1)).\end{aligned}$$

We observe that

$$(u,\nabla P):=(u^1+u^2,\nabla P^1+\nabla P^2)$$

is a solution of (1-3) which satisfies

$$\begin{aligned}\|u\|_{L^\infty_t(\dot B^{3/p-1}_{p,1})}&+\|(\nabla^2u,\nabla P)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\\&\le\big(\|u_0\|_{\dot B^{3/p-1}_{p,1}}+\|(f,\partial_t R,\nabla\operatorname{div}R)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\big)\exp(C_{ab}(t+1)).\end{aligned}\tag{2-64}$$

Of course, using again the first equation of (1-3) we get

$$\|\partial_t u\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\le C_{ab}\|(f,\nabla^2u,\nabla P)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}$$

and thus, we get the estimate

$$\begin{aligned}\|u\|_{L^\infty_t(\dot B^{3/p-1}_{p,1})}&+\|(\partial_t u,\nabla^2u,\nabla P)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\\&\le\big(\|u_0\|_{\dot B^{3/p-1}_{p,1}}+\|(f,\partial_t R,\nabla\operatorname{div}R)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\big)\exp(C_{ab}(t+1)).\end{aligned}\tag{2-65}$$

Uniqueness. Next, let us prove the uniqueness property. Let us suppose there exists a $T > 0$ and a pair $(u, \nabla P)$ that solves

$$\begin{cases} \partial_t u - a\,\mathrm{div}(bD(u)) + a\nabla P = 0, \\ \mathrm{div}\, u = 0, \\ u_{|t=0} = 0, \end{cases} \tag{2-66}$$

with

$$u \in C_T(\dot{B}^{3/p-1}_{p,1}) \text{ and } (\partial_t u, \nabla^2 u, \nabla P) \in L^1_T(\dot{B}^{3/p-1}_{p,1}).$$

Observe that we cannot directly conclude to the uniqueness property by appealing to Proposition 2.12 because the pressure does not belong (a priori) to $L^1_T(\dot{B}^{3/p-3/2}_{p,2})$. Recovering this low-frequency information is done in the following lines. Suppose $3 < p < 4$. Applying the operator $\mathcal{Q}$ in the first equation of (2-66) we can write

$$\mathcal{Q}\big((\bar{a} + \dot{S}_{-m}(a-\bar{a}))\nabla P\big) = \mathcal{Q}\big(a\,\mathrm{div}(bD(u))\big) - \mathcal{Q}\big((\mathrm{Id} - \dot{S}_{-m})(a-\bar{a})\nabla P\big),$$

where $m \in \mathbb{N}$ will be fixed later. We observe that

$$\begin{aligned} &\big\|\mathcal{Q}\big((\bar{a} + \dot{S}_{-m}(a-\bar{a}))\nabla P\big)\big\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,1})} \\ &\quad\lesssim \big\|\mathcal{Q}\big(a\,\mathrm{div}(bD(u))\big)\big\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,1})} + \big\|\mathcal{Q}\big((\mathrm{Id} - \dot{S}_{-m})(a-\bar{a})\nabla P\big)\big\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,1})} \\ &\quad\lesssim T^{1/4}(\bar{a} + \|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}})(\bar{b} + \|b-\bar{b}\|_{\dot{B}^{3/p}_{p,1}})\|\nabla u\|_{L^{4/3}_T(\dot{B}^{3/p-1/2}_{p,1})} \\ &\qquad\qquad + \|(\mathrm{Id} - \dot{S}_{-m})(a-\bar{a})\|_{\dot{B}^{3/p-1/2}_{p,1}}\|\nabla P\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}. \end{aligned}$$

Consequently, we get

$$\mathcal{Q}\big((\bar{a} + \dot{S}_{-m}(a-\bar{a}))\nabla P\big) \in L^1_T(\dot{B}^{3/p-3/2}_{p,1}). \tag{2-67}$$

Let us observe that the condition $p \in (3,4)$ ensures that $\dot{B}^{3/p}_{p,1}$ is contained in the multiplier space of $\dot{B}^{-3/p+1}_{p',2} = \dot{B}^{3/p'-2}_{p',2}$. More precisely, we get:

Proposition 2.13. *Consider $p \in (3,4)$ and $(u, v) \in \dot{B}^{3/p}_{p,1} \times \dot{B}^{-3/p+1}_{p',2}$. Then $uv \in \dot{B}^{-3/p+1}_{p',2}$ and*

$$\|uv\|_{\dot{B}^{-3/p+1}_{p',2}} \lesssim \|u\|_{\dot{B}^{3/p}_{p,1}}\|v\|_{\dot{B}^{-3/p+1}_{p',2}}.$$

Proof. Indeed, considering $(u, v) \in \dot{B}^{3/p}_{p,1} \times \dot{B}^{-3/p+1}_{p',2}$ and using the Bony decomposition we get

$$\|\dot{T}_u v\|_{\dot{B}^{-3/p+1}_{p',2}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^{-3/p+1}_{p',2}}.$$

Next, considering

$$\frac{1}{p'} = \frac{1}{2} + \frac{1}{p^\star},$$

we see

$$\begin{aligned} 2^{j(-3/p+1)}\|\dot{\Delta}_j \dot{T}'_v u\|_{L^{p'}} &\lesssim \sum_{\ell \geq j-3} 2^{(-3/p+1)(j-\ell)} 2^{(-3/p+1)\ell}\|S_{\ell+1} v\|_{L^2}\|\dot{\Delta}_\ell u\|_{L^{p^\star}} \\ &= \sum_{\ell \geq j-3} 2^{(-3/p+1)(j-\ell)} 2^{-1/2\ell}\|S_{\ell+1} v\|_{L^2} 2^{3/p^\star \ell}\|\dot{\Delta}_\ell u\|_{L^{p^\star}} \end{aligned}$$

such that, with the help of Proposition 3.10, we get

$$\|\dot T'_v u\|_{\dot B^{-3/p+1}_{p',2}} \lesssim \|v\|_{\dot H^{-1/2}}\|u\|_{\dot B^{3/p^\star}_{p^\star,1}} \lesssim \|v\|_{\dot B^{-3/p+1}_{p',2}}\|u\|_{\dot B^{3/p}_{p,1}}. \qquad \square$$

Proposition 2.14. *Consider $p \in (3,4)$. Furthermore, consider a constant $\bar c > 0$ and $c \in \dot B^{3/p}_{p,1}$. Then there exists a universal constant $\eta > 0$ such that if*

$$\|c\|_{\dot B^{3/p}_{p,1}} \le \eta,$$

then for any $\psi \in \dot B^{3/p'-3/2}_{p',2} \cap \dot B^{3/p'-2}_{p',2}$ there exists a unique solution $\nabla P \in \dot B^{3/p'-3/2}_{p',2} \cap \dot B^{3/p'-2}_{p',2}$ of the elliptic equation

$$\operatorname{div}((\bar c + c)\nabla P) = \operatorname{div}\psi.$$

Moreover, the following estimate holds true:

$$\|\nabla P\|_{\dot B^{3/p'-\sigma}_{p',2}} \lesssim \|\mathcal{Q}\psi\|_{\dot B^{3/p'-\sigma}_{p',2}},$$

where $\sigma \in \left\{\frac{3}{2}, 2\right\}$.

Proof. The proof is standard. Under some smallness condition on $c \in \dot B^{3/p}_{p,1}$, the operator

$$\nabla R \to \nabla P = \frac{1}{\bar c}\mathcal{Q}(\psi - c\nabla R)$$

has a fixed point in a suitable chosen ball of the space $\dot B^{3/p'-3/2}_{p',2} \cap \dot B^{3/p'-2}_{p',2}$. $\square$

Choose $m \in \mathbb{N}$ such that $\|\dot S_{-m}(a-\bar a)\|_{\dot B^{3/p}_{p,1}}$ is small enough that we can apply Proposition 2.14 with $\bar a$ and $\dot S_{-m}(a-\bar a)$ instead of $\bar c$ and c, and we consider ψ a vector field with coefficients in $\mathcal{S}$. As the Schwartz class is included in $\dot B^{3/p'-3/2}_{p',2} \cap \dot B^{3/p'-2}_{p',2}$, let us consider $\nabla P_\psi \in \dot B^{3/p'-3/2}_{p',2} \cap \dot B^{3/p'-2}_{p',2}$, the solution of the equation

$$\operatorname{div}\big((\bar a + \dot S_{-m}(a-\bar a))\nabla P_\psi\big) = \operatorname{div}\psi,$$

the existence of which is granted by Proposition 2.14. Then, using Propositions 3.8 and 3.9, we can write[3]

$$\begin{aligned}
&\langle \nabla P, \psi\rangle_{\mathcal{S}'\times\mathcal{S}} \\
&\quad= \sum_j \langle \dot\Delta_j \nabla P, \tilde\Delta_j \psi\rangle = \sum_j -\langle \dot\Delta_j P, \tilde\Delta_j \operatorname{div}\psi\rangle && \text{(2-68)}\\
&\quad= \sum_j -\big\langle \dot\Delta_j P, \tilde\Delta_j \operatorname{div}\big((\bar a + \dot S_{-m}(a-\bar a))\nabla P_\psi\big)\big\rangle = \sum_j \big\langle \dot\Delta_j \nabla P, \tilde\Delta_j \big((\bar a + \dot S_{-m}(a-\bar a))\nabla P_\psi\big)\big\rangle && \text{(2-69)}\\
&\quad= \sum_j \big\langle \dot\Delta_j \big(\bar a + \dot S_{-m}(a-\bar a)\big)\nabla P, \tilde\Delta_j \nabla P_\psi\big\rangle = \sum_j \big\langle \dot\Delta_j \mathcal{Q}\big((\bar a + \dot S_{-m}(a-\bar a))\nabla P\big), \tilde\Delta_j \nabla P_\psi\big\rangle && \text{(2-70)}\\
&\quad\lesssim \big\|\mathcal{Q}\big((\bar a + \dot S_{-m}(a-\bar a))\nabla P\big)\big\|_{\dot B^{3/p-3/2}_{p,2}} \|\nabla P_\psi\|_{\dot B^{3/p'-3/2}_{p',1}} && \text{(2-71)}\\
&\quad\lesssim \big\|\mathcal{Q}\big((\bar a + \dot S_{-m}(a-\bar a))\nabla P\big)\big\|_{\dot B^{3/p-3/2}_{p,2}} \|\psi\|_{\dot B^{3/p'-3/2}_{p',1}}. && \text{(2-72)}
\end{aligned}$$

[3] We define $\tilde\Delta_j := \dot\Delta_{j-1} + \dot\Delta_j + \dot\Delta_{j+1}$.

Taking the supremum over all $\psi \in \mathcal{S}$ with $\|\psi\|_{\dot{B}^{3/p'-3/2}_{p',2}} \le 1$, by (2-67) and Proposition 3.8, it follows that $\nabla P \in L^1_T(\dot{B}^{3/p-3/2}_{p,2})$ and that

$$\|\nabla P\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2})} \lesssim \big\|\mathcal{Q}\big((\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P\big)\big\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2})}.$$

According to the uniqueness property of Proposition 2.12 we conclude that $(u,\nabla P)=(0,0)$.

Observe that in the case $p \in \big(\frac{6}{5},3\big]$, owing to the fact that $\dot{B}^{3/p-1}_{p,1} \hookrightarrow \dot{B}^{3/q-1}_{q,1}$ for any $q \in (3,4)$ and $u \in C_T(\dot{B}^{3/p-1}_{p,1})$ along with $(\partial_t u, \nabla^2 u, \nabla P) \in L^1_T(\dot{B}^{3/p-1}_{p,1})$, we get $u \in C_T(\dot{B}^{3/q-1}_{q,1})$ along with $(\partial_t u, \nabla^2 u, \nabla P) \in L^1_T(\dot{B}^{3/q-1}_{q,1})$. Thus, by the uniqueness property for the case $q \in (3,4)$, we conclude that $(u,\nabla P)$ is identically null for $p \in \big(\frac{6}{5},3\big]$.

3. Proof of Theorem 1.2

In the rest of the paper we aim to prove Theorem 1.2. Thus, from now on we will work in a 3-dimensional framework.

The linear theory. Let us fix some notation. The space $\widetilde{F}_T$ consists of $(\tilde{w},\nabla\widetilde{Q})$ with $\tilde{w} \in \mathcal{C}_T(\dot{B}^{3/p-1}_{p,1})$ and $(\partial_t \tilde{w}, \nabla^2\tilde{w}, \nabla\widetilde{Q}) \in L^1_T(\dot{B}^{3/p-1}_{p,1})$ with the norm

$$\|(\tilde{w},\nabla\widetilde{Q})\|_{\widetilde{F}_T} = \|\tilde{w}\|_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})} + \|(\partial_t\tilde{w},\nabla^2\tilde{w},\nabla\widetilde{Q})\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}. \tag{3-1}$$

For any time-dependent vector field $\bar{v}$ we define

$$X_{\bar{v}}(t,x) = x + \int_0^t \bar{v}(\tau,x)\,d\tau,$$

and $A_{\bar{v}} = (DX_{\bar{v}})^{-1}$. Also, let us denote by $\operatorname{adj}(DX_{\bar{v}})$ the adjugate matrix (i.e., the transpose of the cofactor matrix) of $DX_{\bar{v}}$ and $J_{\bar{v}} = \det(DX_{\bar{v}})$.

Before attacking the well-posedness of (1-4), we first have to solve the linear system

$$\begin{cases} \rho_0\partial_t\bar{u} - \operatorname{div}\big(\mu(\rho_0)A_{\bar{v}}D_{A_{\bar{v}}}(\bar{u})\big) + A^T_{\bar{v}}\nabla\bar{P} = 0, \\ \operatorname{div}(\operatorname{adj}(DX_{\bar{v}})\bar{u}) = 0, \\ \bar{u}_{|t=0} = u_0, \end{cases} \tag{3-2}$$

where $\bar{v} \in \mathcal{C}_T(\dot{B}^{3/p-1}_{p,1})$ with $\nabla\bar{v} \in L^1_T(\dot{B}^{3/p}_{p,1}) \cap L^2_T(\dot{B}^{3/p-1}_{p,1})$ is such that

$$\|\nabla\bar{v}\|_{L^2_T(\dot{B}^{3/p-1}_{p,1})} + \|\nabla\bar{v}\|_{L^1_T(\dot{B}^{3/p}_{p,1})} \le 2\alpha \tag{3-3}$$

for a suitably small α. Obviously, this will be achieved using the estimates of the Stokes system established in the previous section; see Theorem 1.3. Let us write (3-2) in the form

$$\begin{cases} \partial_t\bar{u} - \frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0)D(\bar{u})\big) + \frac{1}{\rho_0}\nabla\bar{P} = \frac{1}{\rho_0}F_{\bar{v}}(\bar{u},\nabla\bar{P}), \\ \operatorname{div}\bar{u} = \operatorname{div}\big((\mathrm{Id}-\operatorname{adj}(DX_{\bar{v}}))\bar{u}\big), \\ \bar{u}_{t=0} = u_0, \end{cases}$$

with

$$F_{\bar v}(\bar w, \nabla \overline{Q}) := \operatorname{div}\big(\mu(\rho_0) A_{\bar v} D_{A_{\bar v}}(\bar w) - \mu(\rho_0) D(\bar w)\big) + (\mathrm{Id} - A_{\bar v}^T)\nabla \overline{Q}.$$

Consider $(u_L, \nabla P_L)$ with $u_L \in \mathcal{C}(\mathbb{R}^+, \dot B^{3/p-1}_{p,1})$ and $(\partial_t u_L, \nabla^2 u_L, \nabla P_L) \in L^1_{\mathrm{loc}}(\dot B^{3/p-1}_{p,1})$, the unique solution of

$$\begin{cases} \partial_t u_L - \frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0) D(u_L)\big) + \frac{1}{\rho_0}\nabla P_L = 0, \\ \operatorname{div} u_L = 0, \\ u_{L|t=0} = u_0, \end{cases} \tag{3-4}$$

for which we know that

$$\|(u_L, \nabla P_L)\|_{E_T} \le \|u_0\|_{\dot B^{3/p-1}_{p,1}} \exp(C_{\rho_0}(T+1)).$$

Moreover, T can be chosen small enough such that

$$\|\nabla u_L\|_{L^2_T(\dot B^{3/p-1}_{p,1})} + \|(\partial_t u_L, \nabla^2 u_L, \nabla P_L)\|_{L^1_T(\dot B^{3/p-1}_{p,1})} \le \alpha. \tag{3-5}$$

Following the idea in [Danchin and Mucha 2012], and owing to Theorem 1.3, we consider the operator

$$\Phi(\tilde w, \nabla \widetilde{Q}) = (\tilde u, \nabla \widetilde{P}), \tag{3-6}$$

which associates to $(\tilde w, \nabla \widetilde{Q}) \in \widetilde{F}_T$ the unique solution $(\tilde u, \nabla \widetilde{P}) \in \widetilde{F}_T$ of

$$\begin{cases} \partial_t \tilde u - \frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0) D(\tilde u)\big) + \frac{1}{\rho_0}\nabla \widetilde{P} = \frac{1}{\rho_0} F_{\bar v}(u_L + \tilde w, \nabla P_L + \nabla \widetilde{Q}), \\ \operatorname{div} \tilde u = \operatorname{div}\big((\mathrm{Id} - \operatorname{adj}(DX_{\bar v}))(u_L + \tilde w)\big), \\ \tilde u_{|t=0} = 0. \end{cases}$$

We will show in the following that for any $R > 0$ there exists a sufficiently small $T > 0$ such that there exists a fixed point for Φ in the ball of radius R centered at the origin of $\widetilde{F}_T$. More precisely, according to Theorem 1.3 we get

$$\begin{aligned} \|\Phi(\tilde w, \nabla \widetilde{Q})\|_{\widetilde{F}_T} \le{}& \Big\|\frac{1}{\rho_0} F_{\bar v}(u_L + \tilde w, \nabla P_L + \nabla \widetilde{Q})\Big\|_{L^1_T(\dot B^{3/p-1}_{p,1})} \\ &+ \|\partial_t (\mathrm{Id} - \operatorname{adj}(DX_{\bar v}))(u_L + \tilde w)\|_{L^1_T(\dot B^{3/p-1}_{p,1})} \\ &\quad + \big\|\nabla \operatorname{div}\big((\mathrm{Id} - \operatorname{adj}(DX_{\bar v}))(u_L + \tilde w)\big)\big\|_{L^1_T(\dot B^{3/p-1}_{p,1})}. \end{aligned} \tag{3-7}$$

We begin by treating the first term:

$$\begin{aligned} &\Big\|\frac{1}{\rho_0} F_{\bar v}(u_L + \tilde w, \nabla P_L + \nabla \widetilde{Q})\Big\|_{L^1_T(\dot B^{3/p-1}_{p,1})} \\ &\qquad\qquad \lesssim \Big(\frac{1}{\bar\rho} + \Big\|\frac{1}{\rho_0} - \frac{1}{\bar\rho}\Big\|_{\dot B^{3/p}_{p,1}}\Big) \|F_{\bar v}(u_L + \tilde w, \nabla P_L + \nabla \widetilde{Q})\|_{L^1_T(\dot B^{3/p-1}_{p,1})}. \end{aligned} \tag{3-8}$$

We write

$$\begin{aligned}T_1&=\operatorname{div}\big(\mu(\rho_0)A_{\bar v}D_{A_{\bar v}}(u_L+\tilde w)\big)-\operatorname{div}\big(\mu(\rho_0)D(u_L+\tilde w)\big)\\&=\operatorname{div}\big(\mu(\rho_0)(A_{\bar v}-\mathrm{Id})D_{A_{\bar v}}(u_L+\tilde w)\big)+\operatorname{div}\big(\mu(\rho_0)D_{A_{\bar v}-\mathrm{Id}}(u_L+\tilde w)\big)\\&=\operatorname{div}\big(\mu(\rho_0)(A_{\bar v}-\mathrm{Id})D_{A_{\bar v}-\mathrm{Id}}(u_L+\tilde w)\big)+\operatorname{div}\big(\mu(\rho_0)(A_{\bar v}-\mathrm{Id})D(u_L+\tilde w)\big)+\operatorname{div}\big(\mu(\rho_0)D_{A_{\bar v}-\mathrm{Id}}(u_L+\tilde w)\big).\end{aligned}$$

Thus, using (3-22) of Proposition 3.27 along with product laws in Besov spaces (see Proposition 3.17) we get the following bound for T_1:

$$\begin{aligned}\|T_1\|_{L^1_T(\dot B^{3/p-1}_{p,1})}&\lesssim C_{\rho_0}\|A_{\bar v}-\mathrm{Id}\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\big(1+\|A_{\bar v}-\mathrm{Id}\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\big)\big(\|\nabla u_L\|_{L^1_T(\dot B^{3/p}_{p,1})}+\|\nabla\tilde w\|_{L^1_T(\dot B^{3/p}_{p,1})}\big)\\&\lesssim C_{\rho_0}\|\nabla\bar v\|_{L^1_T(\dot B^{3/p}_{p,1})}\big(1+\|\nabla\bar v\|_{L^1_T(\dot B^{3/p}_{p,1})}\big)\big(\|\nabla u_L\|_{L^1_T(\dot B^{3/p}_{p,1})}+\|\nabla\tilde w\|_{L^1_T(\dot B^{3/p}_{p,1})}\big)\\&\lesssim C_{\rho_0}\alpha\big(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}\big).\end{aligned}\tag{3-9}$$

The second term is estimated as

$$\begin{aligned}\|(\mathrm{Id}-A^T_{\bar v})(\nabla P_L+\nabla\widetilde Q)\|_{L^1_T(\dot B^{3/p-1}_{p,1})}&\lesssim\|\nabla\bar v\|_{L^1_T(\dot B^{3/p}_{p,1})}\big(\|\nabla P_L\|_{L^1_T(\dot B^{3/p-1}_{p,1})}+\|\nabla\widetilde Q\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\big)\\&\lesssim\alpha(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T})\end{aligned}\tag{3-10}$$

so that combining (3-8), (3-9) and (3-10) we get

$$\Big\|\frac{1}{\rho_0}F_{\bar v}(u_L+\tilde w,\nabla P_L+\nabla\widetilde Q)\Big\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\lesssim C_{\rho_0}\alpha(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}).\tag{3-11}$$

In order to treat the second term of (3-7) we use the estimates (3-23) and (3-24) of Proposition 3.27 along with Hölder's inequality in order to obtain

$$\begin{aligned}&\|\partial_t(\mathrm{Id}-\mathrm{adj}(DX_{\bar v}))(u_L+\tilde w)\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\&\quad\lesssim\|(u_L+\tilde w)\partial_t\,\mathrm{adj}(DX_{\bar v})\|_{L^1_T(\dot B^{3/p-1}_{p,1})}+\|(\mathrm{Id}-\mathrm{adj}(DX_{\bar v}))(\partial_t u_L+\partial_t\tilde w)\|_{L^1_T(\dot B^{3/p-1}_{p,1})}&&\text{(3-12)}\\&\quad\lesssim\|\partial_t\,\mathrm{adj}(DX_{\bar v})\|_{L^2_T(\dot B^{3/p-1}_{p,1})}\|u_L+\tilde w\|_{L^2_T(\dot B^{3/p}_{p,1})}+\|\mathrm{Id}-\mathrm{adj}(DX_{\bar v})\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\|\partial_t u_L+\partial_t\tilde w\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\&\quad\lesssim\|\nabla\bar v\|_{L^2_T(\dot B^{3/p-1}_{p,1})}\big(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}\big)+\alpha(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T})\\&\quad\lesssim\alpha(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}).&&\text{(3-13)}\end{aligned}$$

Treating the last term of (3-7) is done with the aid of Corollary 3.24:

$$\begin{aligned}\operatorname{div}((\mathrm{Id}-\mathrm{adj}(DX_{\bar v}))(u_L+\tilde w))&=(Du_L+D\tilde w):(\mathrm{Id}-J_{\bar v}A_{\bar v})\\&=J_{\bar v}(Du_L+D\tilde w):(\mathrm{Id}-A_{\bar v})+(1-J_{\bar v})(\operatorname{div}u_L+\operatorname{div}\tilde w).\end{aligned}$$

Thus, using the estimates (3-22) and (3-26) of Proposition 3.27, we may write

$$\begin{aligned}
&\big\|\nabla \operatorname{div}\big((\mathrm{Id}-\operatorname{adj}(DX_{\bar v}))(u_L+\tilde w)\big)\big\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\
&\qquad\lesssim \big\|J_{\bar v}(Du_L+D\tilde w):(\mathrm{Id}-A_{\bar v})\big\|_{L^1_T(\dot B^{3/p}_{p,1})}+\big\|(1-J_{\bar v})(\operatorname{div}u_L+\operatorname{div}\tilde w)\big\|_{L^1_T(\dot B^{3/p}_{p,1})}\\
&\qquad\lesssim \big(1+\|J_{\bar v}-1\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\big)\|\mathrm{Id}-A_{\bar v}\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\|Du_L+D\tilde w\|_{L^1_T(\dot B^{3/p}_{p,1})}\\
&\qquad+\|(J_{\bar v}-1)\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\|\operatorname{div}u_L+\operatorname{div}\tilde w\|_{L^1_T(\dot B^{3/p}_{p,1})}\\
&\qquad\lesssim \alpha(1+\alpha)(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}).
\end{aligned}\tag{3-14}$$

Combining the estimates (3-11), (3-13) and (3-14) we get

$$\|\Phi(\tilde w,\nabla\widetilde Q)\|_{\widetilde F_T}\lesssim \alpha(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}).\tag{3-15}$$

Thus, for a suitably small α the operator Φ maps the ball of radius R centered at the origin of $\widetilde F_T$ into itself. Due to the linearity of Φ, one can repeat the above arguments in order to show that Φ is a contraction for small values of α. This concludes the existence of a fixed point of Φ, say $(\tilde u^\star,\nabla\widetilde P^\star)\in\widetilde F_T$. Of course,

$$(\bar u,\nabla\bar P)=(\tilde u^\star,\nabla\widetilde P^\star)+(u_L,\nabla P_L)$$

is a solution of (3-2).

Proof of Theorem 1.2. Consider T small enough such that $(u_L,\nabla P_L)$, the solution of (3-4), satisfies

$$\|\nabla u_L\|_{L^2_T(\dot B^{3/p-1}_{p,1})}+\|\nabla u_L\|_{L^1_T(\dot B^{3/p}_{p,1})}\le\alpha,$$

and consider the closed set

$$\widetilde F_T(\alpha)=\big\{(\tilde v,\nabla\widetilde Q)\in F_T:\tilde v_{|t=0}=0,\ \|(\tilde v,\nabla\widetilde Q)\|_{F_T}\le R\alpha\big\}$$

with R sufficiently small such that

$$\|\nabla\tilde v\|_{L^2_T(\dot B^{3/p-1}_{p,1})}+\|\nabla\tilde v\|_{L^1_T(\dot B^{3/p}_{p,1})}\le\alpha.\tag{3-16}$$

Let us consider the operator S which associates to $(\tilde v,\nabla\widetilde Q)\in\widetilde F_T(\alpha)$, the solution of

$$\begin{cases}\partial_t\tilde u-\frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0)D(\tilde u)\big)+\frac{1}{\rho_0}\nabla\widetilde P=\frac{1}{\rho_0}F_{(u_L+\tilde v)}(u_L+\tilde u,\nabla P_L+\nabla\widetilde P),\\ \operatorname{div}\big(\operatorname{adj}(DX_{u_L+\tilde v})(u_L+\tilde u)\big)=0,\\ \tilde u_{|t=0}=0\end{cases}$$

constructed in the previous section. We will show that for suitably small T and α, the operator S maps the closed set $\widetilde F_T(\alpha)$ into itself and that S is a contraction. First of all, recalling that $(\tilde u,\nabla\widetilde P)$ is in fact the fixed point of the operator Φ defined in (3-6) and using the estimates established in the last section, we conclude that

$$\|(\tilde u,\nabla\widetilde P)\|_{\widetilde F_T}=\|S(\tilde v,\nabla\widetilde Q)\|_{F_T}\le R\alpha\tag{3-17}$$

for some small enough T.

Next, we will deal with the stability estimates. For $i=1,2$, let us consider $(\tilde v_i,\nabla\widetilde{Q}_i)\in\widetilde{F}_T(\alpha)$ and $(\tilde u_i,\nabla\widetilde{P}_i)=S(\tilde v_i,\nabla\widetilde{Q}_i)\in\widetilde{F}_T(\alpha)$. Defining

$$(\delta\tilde v,\nabla\delta\widetilde{Q})=(\tilde v_1-\tilde v_2,\nabla\widetilde{Q}_1-\nabla\widetilde{Q}_2),$$
$$(\delta\tilde u,\nabla\delta\widetilde{P})=(\tilde u_1-\tilde u_2,\nabla\widetilde{P}_1-\nabla\widetilde{P}_2),$$

we see

$$\begin{cases}\partial_t\delta\tilde u-\frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0)D(\delta\tilde u)\big)+\frac{1}{\rho_0}\nabla\delta\widetilde{P}=\frac{1}{\rho_0}\widetilde{F},\\ \operatorname{div}\delta\tilde u=\operatorname{div}\widetilde{G},\\ \delta\tilde u_{|t=0}=0,\end{cases}$$

where

$$\widetilde{F}=F_1(\delta\tilde v,u_L+\tilde u_1)+F_1(u_L+\tilde v_2,\delta\tilde u)+F_2(\delta\tilde v,\nabla P_L+\nabla\widetilde{P}_1)+F_2(u_L+\tilde v_2,\nabla\delta\widetilde{P}),$$
$$\widetilde{G}=-\big(\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\mathrm{Id}\big)\delta\tilde u-\big(\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\operatorname{adj}(DX_{(u_L+\tilde v_2)})\big)(u_L+\tilde u_2):=\widetilde{G}_1+\widetilde{G}_2,$$

and

$$F_1(\bar v,\bar w)=\operatorname{div}\big(\mu(\rho_0)A_{\bar v}D_{A_{\bar v}}(\bar w)-\mu(\rho_0)D(\bar w)\big),$$
$$F_2(\bar v,\nabla\overline{Q})=(\mathrm{Id}-A^T_{\bar v})\nabla\overline{Q}.$$

According to Theorem 1.3 we get

$$\|(\delta\tilde u,\nabla\delta\widetilde{P})\|_{\widetilde{F}_T}\lesssim C_{\rho_0}\big(\|\widetilde{F}\|_{L^1_T(\dot B^{3/p-1}_{p,1})}+\|\nabla\operatorname{div}\widetilde{G}\|_{L^1_T(\dot B^{3/p-1}_{p,1})}+\|\partial_t\widetilde{G}\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\big). \tag{3-18}$$

Proceeding as in relations (3-8) and (3-9) we get

$$\begin{aligned}\|\widetilde{F}\|_{L^1_T(\dot B^{3/p-1}_{p,1})}&\lesssim\|\nabla\delta\tilde v\|_{L^1_T(\dot B^{3/p}_{p,1})}\|\nabla u_L+\nabla\tilde u_1\|_{L^1_T(\dot B^{3/p}_{p,1})}+\|\nabla u_L+\nabla\tilde v_2\|_{L^1_T(\dot B^{3/p}_{p,1})}\|\nabla\delta\tilde u\|_{L^1_T(\dot B^{3/p}_{p,1})}\\&\quad+\|\nabla\delta\tilde v\|_{L^1_T(\dot B^{3/p}_{p,1})}\|\nabla P_L+\nabla\widetilde{P}_1\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\&\quad+\|\nabla u_L+\nabla\tilde v_2\|_{L^1_T(\dot B^{3/p}_{p,1})}\|\nabla\delta\widetilde{P}\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\&\lesssim\alpha\|(\nabla\delta\tilde v,\nabla\delta\widetilde{Q})\|_{L^1_T(\dot B^{3/p}_{p,1})}+\alpha\|(\delta\tilde u,\nabla\delta\widetilde{P})\|_{\widetilde{F}_T}.\end{aligned} \tag{3-19}$$

Of course, we will use the smallness of α to absorb $\alpha\|(\nabla\delta\tilde u,\nabla\delta\widetilde{P})\|_{L^1_T(\dot B^{3/p}_{p,1})}$ into the left-hand side of (3-18).

Next, we treat $\|\nabla\operatorname{div}\widetilde{G}\|_{L^1_T(\dot B^{3/p-1}_{p,1})}$. Using Proposition 3.23, we can write

$$\begin{aligned}-\operatorname{div}\widetilde{G}_2&=\operatorname{div}\big(\big(\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\operatorname{adj}(DX_{(u_L+\tilde v_2)})\big)(u_L+\tilde u_2)\big)\\&=\operatorname{div}\big(\operatorname{adj}(DX_{(u_L+\tilde v_1)})(u_L+\tilde u_2)\big)-\operatorname{div}\big(\operatorname{adj}(DX_{(u_L+\tilde v_2)})(u_L+\tilde u_2)\big)\\&=J_{u_L+\tilde v_1}D(u_L+\tilde u_2):A_{(u_L+\tilde v_1)}-J_{(u_L+\tilde v_2)}D(u_L+\tilde u_2):A_{(u_L+\tilde v_2)}\\&=(J_{u_L+\tilde v_1}-J_{(u_L+\tilde v_2)})D(u_L+\tilde u_2):A_{(u_L+\tilde v_1)}+J_{(u_L+\tilde v_2)}D(u_L+\tilde u_2):(A_{(u_L+\tilde v_1)}-A_{(u_L+\tilde v_2)}),\end{aligned}$$

and thus, using Propositions 3.27 and 3.28 we get

$$\begin{aligned}
&\|\nabla \operatorname{div} \widetilde{G}_2\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\
&\quad\lesssim \|(J_{u_L+\tilde v_1}-J_{(u_L+\tilde v_2)})\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\|(Du_L, D\tilde u_2)\|_{L^1_T(\dot B^{3/p}_{p,1})}(1+\|\mathrm{Id}-A_{(u_L+\tilde v_1)}\|_{L^\infty_T(\dot B^{3/p}_{p,1})})\\
&\qquad\quad+(1+\|(J_{u_L+\tilde v_2}-1)\|_{L^\infty_T(\dot B^{3/p}_{p,1})})\|(Du_L, D\tilde u_2)\|_{L^1_T(\dot B^{3/p}_{p,1})}\|A_{(u_L+\tilde v_1)}-A_{(u_L+\tilde v_2)}\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\\
&\quad\lesssim \alpha\|\nabla\delta\tilde v\|_{L^1_T(\dot B^{3/p}_{p,1})}.
\end{aligned}\tag{3-20}$$

Next, using again Proposition 3.23 we see

$$\begin{aligned}
-\operatorname{div}\widetilde{G}_1 &= \operatorname{div}\big((\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\mathrm{Id})\delta\tilde u\big) = D\delta\tilde u : (J_{u_L+\tilde v_1}A_{(u_L+\tilde v_1)}-\mathrm{Id})\\
&= J_{u_L+\tilde v_1} D\delta\tilde u : (A_{(u_L+\tilde v_1)}-\mathrm{Id}) + (J_{u_L+\tilde v_1}-1)\operatorname{div}\delta\tilde u
\end{aligned}$$

and consequently

$$\begin{aligned}
&\|\nabla \operatorname{div}\widetilde{G}_1\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\
&\quad\lesssim \big\|J_{u_L+\tilde v_1}D\delta\tilde u:(A_{(u_L+\tilde v_1)}-\mathrm{Id})\big\|_{L^1_T(\dot B^{3/p}_{p,1})}+\big\|(J_{u_L+\tilde v_1}-1)\operatorname{div}\delta\tilde u\big\|_{L^1_T(\dot B^{3/p}_{p,1})}\\
&\quad\lesssim \big(1+\|J_{u_L+\tilde v_1}-1\|_{L^1_T(\dot B^{3/p}_{p,1})}\big)\|D\delta\tilde u:(A_{(u_L+\tilde v_1)}-\mathrm{Id})\|_{L^1_T(\dot B^{3/p}_{p,1})}+\|J_{u_L+\tilde v_1}-1\|_{L^1_T(\dot B^{3/p}_{p,1})}\|\operatorname{div}\delta\tilde u\|_{L^1_T(\dot B^{3/p}_{p,1})}\\
&\quad\lesssim \alpha\|(\delta\tilde u,\nabla\delta P)\|_{\widetilde{F}_T}.
\end{aligned}\tag{3-21}$$

Combining (3-20) with (3-21) yields

$$\|\nabla\operatorname{div}\widetilde{G}\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\lesssim \alpha\|\nabla\delta\tilde v\|_{L^1_T(\dot B^{3/p}_{p,1})}+\alpha\|(\delta\tilde u,\nabla\delta P)\|_{\widetilde{F}_T}.\tag{3-22}$$

Again, we will use the smallness of α to absorb $\alpha\|(\nabla\delta\tilde u,\nabla\delta\widetilde{P})\|_{L^1_T(\dot B^{3/p}_{p,1})}$ into the left-hand side of (3-22).

Finally, we write

$$\begin{aligned}
&\partial_t\big[\big(\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\operatorname{adj}(DX_{(u_L+\tilde v_2)})\big)(u_L+\tilde u_2)\big]\\
&\qquad= \big(\partial_t\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\partial_t\operatorname{adj}(DX_{(u_L+\tilde v_2)})\big)(u_L+\tilde u_2)\\
&\qquad\quad+\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\operatorname{adj}(DX_{(u_L+\tilde v_2)})(\partial_t u_L+\partial_t\tilde u_2).
\end{aligned}$$

Using Proposition 3.28 gives us

$$\begin{aligned}
&\big\|\partial_t\big[(\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\operatorname{adj}(DX_{(u_L+\tilde v_2)}))(u_L+\tilde u_2)\big]\big\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\
&\quad\lesssim \big\|\partial_t\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\partial_t\operatorname{adj}(DX_{(u_L+\tilde v_2)})\big\|_{L^2_T(\dot B^{3/p-1}_{p,1})}\|u_L\|_{L^2_T(\dot B^{3/p}_{p,1})}\\
&\qquad+\big\|\partial_t\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\partial_t\operatorname{adj}(DX_{(u_L+\tilde v_2)})\big\|_{L^1_T(\dot B^{3/p}_{p,1})}\|\tilde u_2\|_{L^\infty_T(\dot B^{3/p-1}_{p,1})}\\
&\qquad\quad+\big\|\operatorname{adj}(DX_{(u_L+\tilde v_1)})-\operatorname{adj}(DX_{(u_L+\tilde v_2)})\big\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\|\partial_t u_L+\partial_t\tilde u_2\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\
&\quad\lesssim\alpha\|\delta\tilde v\|_{L^2_T(\dot B^{3/p}_{p,1})}.
\end{aligned}$$

The conclusion is

$$\|\partial_t\widetilde{G}\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\lesssim\alpha\|\delta\tilde v\|_{L^2_T(\dot B^{3/p}_{p,1})}.\tag{3-23}$$

Combining (3-19), (3-22) and (3-23) we get if α is chosen sufficiently small then

$$\|((\delta\tilde{u}, \nabla\delta\widetilde{P}))\|_{\widetilde{F}_T} \le \tfrac{1}{2}\|((\delta\tilde{v}, \nabla\delta\widetilde{Q}))\|_{F_T} \tag{3-24}$$

and the operator S is also a contraction over $\widetilde{F}_T(\alpha)$. Thus, according to Banach's theorem there exists a fixed point $(\bar{u}^\star, \nabla\bar{P}^\star)$ of S. Obviously,

$$(\bar{u}, \nabla\bar{P}) = (u_L, \nabla P_L) + (\bar{u}^\star, \nabla\bar{P}^\star)$$

is a solution of

$$\begin{cases} \rho_0\partial_t\bar{u} - \operatorname{div}\big(\mu(\rho_0)A_{\bar{u}}D_{A_{\bar{u}}}(\bar{u})\big) + A_{\bar{u}}^T\nabla\bar{P} = 0, \\ \operatorname{div}(\operatorname{adj}(DX_{\bar{u}})\bar{u}) = 0, \\ \bar{u}_{|t=0} = u_0. \end{cases} \tag{3-25}$$

In view of Proposition 3.26 we also get $J_{\bar{u}} = 1$. Thus, the second equation of (3-25) becomes

$$\operatorname{div}(A_{\bar{u}}\bar{u}) = 0.$$

The only thing left to prove is the uniqueness property. Consider $(\bar{u}^1, \nabla\bar{P}^1)$, $(\bar{u}^2, \nabla\bar{P}^2) \in F_T$, two solutions of (3-25) with the same initial data $u_0 \in \dot{B}^{3/p-1}_{p,1}$. With $(u_L, \nabla P_L)$ defined above, we let

$$(\tilde{u}^i, \nabla\widetilde{P}^i) = (\bar{u}^i, \nabla\bar{P}^i) - (u_L, \nabla P_L) \quad \text{for } i = 1, 2$$

such that the system verified by $(\tilde{u}^i, \nabla\widetilde{P}^i)$ is

$$\begin{cases} \partial_t\tilde{u}^i - \frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0)D(\tilde{u}^i)\big) + \frac{1}{\rho_0}\nabla\widetilde{P}^i = \frac{1}{\rho_0}F_{(u_L+\tilde{u}^i)}(u_L+\tilde{u}^i, \nabla P_L + \nabla\widetilde{P}^i), \\ \operatorname{div}\big(A_{(u_L+\tilde{u}^i)}(u_L+\tilde{u}^i)\big) = 0, \\ \tilde{u}_{|t=0} = 0. \end{cases}$$

We are now in the position of performing exactly the same computations as above so that we obtain a time T' sufficiently small such that

$$(\bar{u}^1, \nabla\bar{P}^1) = (\bar{u}^2, \nabla\bar{P}^2) \quad \text{on } [0, T'].$$

It is classical that the above local uniqueness property extends to all of $[0, T]$. □

Proof of Theorem 1.1. Considering $(\rho_0, u_0) \in \dot{B}^{3/p}_{p,1} \times \dot{B}^{3/p-1}_{p,1}$ and applying Theorem 1.2, there exists a positive $T > 0$ such that we may construct a solution $(\bar{u}, \nabla\bar{P})$ to the system (1-4) in F_T. Then, working with a smaller T if needed and considering $X_{\bar{u}}$, the "flow" of $\bar{u}$ defined by (3-17), by using Proposition 3.26 from the Appendix, one obtains that $X_{\bar{u}}$ is a measure preserving C^1-diffeomorphism over $\mathbb{R}^n$ for all $t \in [0, T]$. Thus we may introduce the Eulerian variable:

$$\rho(t,x) = \rho_0(X_{\bar{u}}^{-1}(t,x)), \quad u(t,x) = \bar{u}(t, X_{\bar{u}}^{-1}(t,x)) \quad \text{and} \quad P(t,x) = \bar{P}(t, X_{\bar{u}}^{-1}(t,x)).$$

Then, Proposition 3.23 ensures that $(\rho, u, \nabla P)$ is a solution of (1-1). As $DX_{\bar{u}} - \mathrm{Id}$ belongs to $\dot{B}^{3/p}_{p,1}$, using Proposition 3.22, we may conclude that $(\rho, u, \nabla P)$ has the announced regularity.

The uniqueness property comes from the fact that considering two solutions $(\rho^i, u^i, \nabla P^i)$ of (1-1), $i = 1, 2$, and considering Y_{u^i}, the flow of u^i, we find that $\big(u^i(t, Y_{u^i}(t,y)), \nabla P^i(t, Y_{u^i}(t,y))\big)$ are solutions of the system (1-4) with the same data. Thus, they are equal according to the uniqueness property announced in Theorem 1.2. Thus, on some nontrivial interval $[0, T'] \subset [0, T]$ (chosen such that condition (3-19) holds), the solutions $(\rho^i, u^i, \nabla P^i)$ are equal. This local uniqueness property obviously entails uniqueness on all of $[0, T]$. □

Appendix

We present here a few results of Fourier analysis used through the text. The full proofs along with other complementary results can be found in [Bahouri et al. 2011, Chapter 2].

Let us introduce the dyadic partition of the space:

Proposition A.1. *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. There exists a radial function $\varphi \in \mathcal{D}(\mathcal{C})$ valued in the interval* $[0, 1]$ *such that*

$$\textit{for all } \xi \in \mathbb{R}^n \backslash \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{A-1}$$

$$2 \le |j - j'| \quad \Rightarrow \quad \operatorname{Supp}(\varphi(2^{-j}\,\cdot)) \cap \operatorname{Supp}(\varphi(2^{-j'}\,\cdot)) = \varnothing. \tag{A-2}$$

Also, the following inequality holds:

$$\textit{for all } \xi \in \mathbb{R}^n \backslash \{0\}, \quad \frac{1}{2} \le \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \le 1. \tag{A-3}$$

From now on we fix functions χ and φ satisfying the assertions of the above proposition and denote by $\tilde{h}$ and h their Fourier inverses.

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\dot{S}_j$ are

$$\dot{\Delta}_j u = \varphi(2^{-j} D) u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y) u(x - y)\, dy,$$

$$\dot{S}_j u = \chi(2^{-j} D) u = 2^{jn} \int_{\mathbb{R}^n} \tilde{h}(2^j y) u(x - y)\, dy$$

for all $j \in \mathbb{Z}$.

Definition A.2. We denote by $\mathcal{S}_h'$ the space of tempered distributions such that

$$\lim_{j \to -\infty} \|\dot{S}_j u\|_{L^\infty} = 0.$$

Let us now define the homogeneous Besov spaces:

Definition A.3. Let s be a real number and $(p, r) \in [1, \infty]$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is the subset of tempered distributions $u \in S_h'$ such that

$$\|u\|_{\dot{B}^s_{p,r}} := \big\| (2^{js} \|\dot{\Delta}_j u\|_{L^2})_{j \in \mathbb{Z}} \big\|_{\ell^r(\mathbb{Z})} < \infty.$$

The next propositions gather some basic properties of Besov spaces.

Proposition A.4. *Let us consider $s \in \mathbb{R}$ and $p, r \in [1, \infty]$ such that*

$$s < \frac{n}{p} \quad \textit{or} \quad s = \frac{n}{p} \ \textit{ and } \ r = 1. \tag{A-4}$$

Then $(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}})$ is a Banach space.

Proposition 3.5. *A tempered distribution $u \in S'_h$ belongs to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ if and only if there exists a sequence $(c_j)_j$ such that $(2^{js}c_j)_j \in \ell^r(\mathbb{Z})$ with norm 1 and a constant $C = C(u) > 0$ such that for any $j \in \mathbb{Z}$ we have*

$$\|\dot{\Delta}_j u\|_{L^p} \leq C c_j.$$

Proposition 3.6. *Consider s_1 and s_2 two real numbers such that $s_1 < s_2$ and $\theta \in (0,1)$. Then, there exists a constant $C > 0$ such that for all $r \in [1,\infty]$ we have*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}},$$
$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_2-s_1}\left(\frac{1}{\theta}+\frac{1}{1-\theta}\right)\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}.$$

Proposition 3.7. (1) *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then, for any real number s, the space $\dot{B}^s_{p_1,r_1}$ is continuously embedded in $\dot{B}^{s-n(1/p_1-1/p_2)}_{p_2,r_2}$.*

(2) *Let $1 \leq p < \infty$. Then, $\dot{B}^{n/p}_{p,1}$ is continuously embedded in $(C_0(\mathbb{R}^n), \|\cdot\|_{L^\infty})$, the space of continuous functions vanishing at infinity.*

Proposition 3.8. *For all $1 \leq p, r \leq \infty$ and $s \in \mathbb{R}$,*

$$\begin{cases} \dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'} \to \mathbb{R}, \\ (u,v) \to \sum_j \langle \dot{\Delta}_j u, \tilde{\Delta}_j v\rangle, \end{cases} \tag{3-5}$$

where $\tilde{\Delta}_j := \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}$, defines a continuous bilinear functional on $\dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'}$. Denote by $Q^{-s}_{p',r'}$ the set of functions $\varphi \in \mathcal{S} \cap \dot{B}^{-s}_{p',r'}$ such that $\|\varphi\|_{\dot{B}^{-s}_{p',r'}} \leq 1$. If $u \in \mathcal{S}'_h$, then we have

$$\|u\|_{\dot{B}^s_{p,r}} \lesssim \sup_{\varphi \in Q^{-s}_{p',r'}} \langle u, \varphi\rangle_{\mathcal{S}' \times \mathcal{S}}.$$

Proposition 3.9. *Consider $1 < p, r < \infty$ and $s \in \mathbb{R}$. Furthermore, let $u \in \dot{B}^s_{p,r}$, $v \in \dot{B}^{-s}_{p',r'}$ and $\rho \in L^\infty \cap \mathcal{M}(\dot{B}^s_{p,r}) \cap \mathcal{M}(\dot{B}^{-s}_{p',r'})$. Then, we have*

$$(\rho u, v) = \sum_j \sum_j \langle \dot{\Delta}_j u, \tilde{\Delta}_j(\rho v)\rangle = (u, \rho v). \tag{3-6}$$

The proof of Proposition 3.9 follows from a density argument. Relation (3-6) clearly holds for functions from the Schwartz class: then we may write

$$\int_{\mathbb{R}^n} \rho u v = (\rho u, v) = (u, \rho v).$$

The conditions $1 < p, r < \infty$ and $s \in \mathbb{R}$ ensure that u and v may be approximated by Schwartz functions.

An important feature of Besov spaces with negative index of regularity is the following:

Proposition 3.10. *Let $s < 0$ and $1 \leq p, r \leq \infty$. Let u be a distribution in S'_h. Then, u belongs to $\dot{B}^s_{p,r}$ if and only if*

$$(2^{js}\|\dot{S}_j u\|_{L^p})_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z}).$$

Moreover, there exists a constant C depending only on the dimension n such that

$$C^{-|s|+1}\|u\|_{\dot B^s_{p,r}} \le \big\|(2^{js}\|\dot S_j u\|_{L^p})_{j\in\mathbb{Z}}\big\|_{\ell^r(\mathbb{Z})} \le C\Big(1+\frac{1}{|s|}\Big)\|u\|_{\dot B^s_{p,r}}.$$

The next proposition tells us how certain multipliers act on Besov spaces.

Proposition 3.11. *Consider A a smooth function on $\mathbb{R}^n\backslash\{0\}$ which is homogeneous of degree m. Then, for any $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ such that*

$$s-m<\frac{n}{p} \quad \text{or} \quad s-m=\frac{n}{p} \ \text{ and } \ r=1,$$

the operator[4] $A(D)$ maps $\dot B^s_{p,r}$ continuously into $\dot B^{s-m}_{p,r}$.

The next proposition describes how smooth functions act on homogeneous Besov spaces.

Proposition 3.12. *Let f be a smooth function on $\mathbb{R}$ which vanishes at 0. Consider $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ such that*

$$0<s<\frac{n}{p} \quad \text{or} \quad s=\frac{n}{p} \ \text{ and } \ r=1.$$

Then for any real-valued function $u\in\dot B^s_{p,r}\cap L^\infty$, the function $f\circ u$ is in $\dot B^s_{p,r}\cap L^\infty$ and we have

$$\|f\circ u\|_{\dot B^s_{p,r}} \le C(f',\|u\|_{L^\infty})\|u\|_{\dot B^s_{p,r}}.$$

Remark 3.13. The constant $C(f',\|u\|_{L^\infty})$ appearing above can be taken to be

$$\sup_{i\in\overline{1,[s]+1}} \|f^{(i)}\|_{L^\infty([-M\|u\|_{L^\infty},-M\|u\|_{L^\infty}])},$$

where M is a constant depending only on the dimension n.

Commutator and product estimates. Next, we want to see how the product acts in Besov spaces. The Bony decomposition, introduced in [Bony 1981], offers a mathematical framework to obtain estimates of the product of two distributions, when the latter is defined.

Definition 3.14. Given two tempered distributions $u,v\in S'_h$, the homogeneous paraproduct of v by u is defined as

$$\dot T_u v=\sum_{j\in\mathbb{Z}}\dot S_{j-1}u\dot\Delta_j v. \tag{3-7}$$

The homogeneous remainder of u and v is defined by

$$\dot R(u,v)=\sum_{j\in\mathbb{Z}}\dot\Delta_j u\dot\Delta'_j v, \tag{3-8}$$

where

$$\dot\Delta'_j=\dot\Delta_{j-1}+\dot\Delta_j+\dot\Delta_{j+1}.$$

[4] $A(D)w=\mathcal{F}^{-1}(A\mathcal{F}w)$.

Remark 3.15. Notice that at a formal level, one has the following decomposition of the product of two (sufficiently well-behaved) distributions:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v) = \dot{T}_u v + \dot{T}'_v u.$$

The next result describes how the paraproduct and remainder behave.

Proposition 3.16. (1) *Assume* $(s, p, p_1, p_2, r) \in \mathbb{R}\times[1,\infty]^4$ *such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad s < \frac{n}{p} \qquad \text{or} \qquad s = \frac{n}{p} \text{ and } r = 1.$$

Then, the paraproduct maps $L^{p_1} \times \dot{B}^s_{p_2,r}$ *into* $\dot{B}^s_{p,r}$ *and the following estimate holds*:

$$\|\dot{T}_f g\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^{p_1}} \|g\|_{\dot{B}^s_{p_2,r}}.$$

(2) *Assume* $(s, p, p_1, p_2, r, r_1, r_2) \in \mathbb{R}\times[1,\infty]^6$ *and* $\nu > 0$ *such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$s < \frac{n}{p} - \nu \qquad \text{or} \qquad s = \frac{n}{p} - \nu \text{ and } r = 1.$$

Then, the paraproduct maps $\dot{B}^{-\nu}_{p_1,r_1} \times \dot{B}^{s+\nu}_{p_2,r_2}$ *into* $\dot{B}^s_{p,r}$ *and the following estimate holds*:

$$\|\dot{T}_f g\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{\dot{B}^{-\nu}_{p_1,r_1}} \|g\|_{\dot{B}^{s+\nu}_{p_2,r_2}}.$$

(3) *Consider* $(s_1, s_2, p, p_1, p_2, r, r_1, r_2) \in \mathbb{R}^2 \times [1,\infty]^6$ *such that*

$$0 < s_1 + s_2 < \frac{n}{p} \qquad \text{or} \qquad s_1 + s_2 = \frac{n}{p} \text{ and } r = 1.$$

Then, the remainder maps $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ *into* $\dot{B}^{s_1+s_2}_{p,r}$ *and*

$$\|\dot{R}(f,g)\|_{\dot{B}^{s_1+s_2}_{p,r}} \le \|f\|_{\dot{B}^{s_1}_{p_1,r_1}} \|g\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

As a consequence we obtain the following product rules in Besov space:

Proposition 3.17. *Consider* $p \in [1,\infty]$ *and the real numbers* $\nu_1 \ge 0$ *and* $\nu_2 \ge 0$ *with*

$$\nu_1 + \nu_2 < \frac{n}{p} + \min\left\{\frac{n}{p}, \frac{n}{p'}\right\}.$$

Then, the following estimate holds:

$$\|fg\|_{\dot{B}^{n/p-\nu_1-\nu_2}_{p,1}} \lesssim \|f\|_{\dot{B}^{n/p-\nu_1}_{p,1}} \|g\|_{\dot{B}^{n/p-\nu_2}_{p,1}}.$$

Proposition 3.18. *Consider* θ *a* $\mathcal{C}^1$ *function on* $\mathbb{R}^n$ *such that* $(1+|\cdot|)\hat{\theta} \in L^1$. *Let us also consider* $p, q \in [1,\infty]$ *such that*

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} \le 1.$$

Then, there exists a constant C such that for any Lipschitz function a with gradient in L^p, any function $b \in L^q$ and any positive λ,

$$\big\|[\theta(\lambda^{-1}D), a]b\big\|_{L^r} \le C\lambda^{-1}\|\nabla a\|_{L^p}\|b\|_{L^q}.$$

In particular, when $\theta = \varphi$ and $\lambda = 2^j$ we get

$$\big\|[\dot\Delta_j, a]b\big\|_{L^r} \le C2^{-j}\|\nabla a\|_{L^p}\|b\|_{L^q}.$$

Proposition 3.19. *Assume s, ν and $p \in [1,\infty]$ are such that*

$$0 \le \nu \le \frac{n}{p} \quad \text{and} \quad -1-\min\left\{\frac{n}{p}, \frac{n}{p'}\right\} < s \le \frac{n}{p} - \nu.$$

Then, there exists a constant C depending only on s, ν, p and n such that for all $l \in \overline{1,n}$ we have for some sequence $(c_j)_{j\in\mathbb{Z}}$ with $\|(c_j)_{j\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})} = 1$,

$$\big\|\partial_l[a, \dot\Delta_j]w\big\|_{L^p} \le Cc_j 2^{-js}\|\nabla a\|_{\dot B^{n/p-\nu}_{p,1}}\|w\|_{\dot B^{s+\nu}_{p,1}}$$

for all $j \in \mathbb{Z}$.

For a proof of the above results we refer the reader to the Appendix of [Danchin 2014, Lemmas A.5 and A.6].

Proposition 3.20. *Consider a homogeneous function $A : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ of degree 0. Let us consider $s \in \mathbb{R}$, $0 < \nu \le 1$ and $p, r, r_1, r_2 \in [1,\infty]$ such that*

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$s < \frac{n}{p} - \nu \quad \text{or} \quad s = \frac{n}{p} - \nu \ \text{ and } \ r_2 = 1. \tag{3-9}$$

Moreover, assume $w \in \dot B^{s+\nu}_{p,r_2}$ and $a \in L^\infty$ with $\nabla a \in \dot B^{-\nu}_{\infty,r_1}$. Then, the following estimate holds:

$$\big\|[A(D), \dot T_a]w\big\|_{\dot B^{s+1}_{p,r}} \lesssim \|\nabla a\|_{\dot B^{-\nu}_{\infty,r_1}}\|w\|_{\dot B^{s+\nu}_{p,r_2}}. \tag{3-10}$$

As this result is of great importance in the analysis of the pressure term, we present a sketched proof below (see also [Bahouri et al. 2011, Chapter 2, Lemma 2.99]).

Proof. The fact that $a \in L^\infty$, along with relation (3-9), guarantees that $A(D)w \in \dot B^{s+\nu}_{p,r}$ and that the paraproducts $\dot T_a w$ and $\dot T_a A(D)w$ are well-defined. We observe that there exists a function $\tilde\varphi$ supported in some annulus which equals 1 on the support of φ such that one may write (of course it is here that we use the homogeneity of A)

$$[A(D), \dot T_a]w = \sum_j \big[(A\tilde\varphi)(2^{-j}D), \dot S_{j-1}a\big]\dot\Delta_j w.$$

But according to Proposition 3.18 we have

$$2^{j(s+1)}\big\|\big[(A\tilde\varphi)(2^{-j}D), \dot S_{j-1}a\big]\dot\Delta_j w\big\|_{L^p} \lesssim 2^{-j\nu}\|\nabla \dot S_{j-1}a\|_{L^\infty}2^{j(s+\nu)}\|\dot\Delta_j w\|_{L^p}.$$

The last relation obviously implies (3-10). □

As a consequence of the above proposition and Proposition 3.16 we get the following:

Proposition 3.21. *Let us consider a homogeneous function* $A:\mathbb{R}^n\backslash\{0\}\to\mathbb{R}$ *of degree* 0, $s\in\mathbb{R}$, $0<\nu\leq 1$ *and* $p,r,r_1,r_2\in[1,\infty]$ *such that*

$$\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$$

and

$$-1-\min\left\{\frac{n}{p},\frac{n}{p'}\right\}<s<\frac{n}{p}-\nu \quad \text{or} \quad s=\frac{n}{p}-\nu \text{ and } r=r_2=1. \tag{3-11}$$

assume $w\in\dot{B}^{s+\nu}_{p,r_2}$ *and* $a\in L^\infty$ *with* $\nabla a\in\dot{B}^{-\nu}_{\infty,r_1}$. *Then, the following estimate holds*:

$$\|[A(D),a]w\|_{\dot{B}^{s+1}_{p,r}}\lesssim\|\nabla a\|_{\dot{B}^{n/p-\nu}_{p,r_1}}\|w\|_{\dot{B}^{s+\nu}_{p,r_2}}.$$

Properties of Lagrangian coordinates. The following results are gathered from [Danchin 2014] and [Danchin and Mucha 2012]. More precisely, proofs of Propositions 3.22, 3.23, the estimate (3-26) of Proposition 3.27 and the estimate (3-31) of Proposition 3.28 can be found in [Danchin 2014, pp. 782–786]. Propositions 3.27 and 3.28 can be found in the Appendix of [Danchin and Mucha 2012]. Proposition 3.26 is inspired by [Danchin and Mucha 2012].

Proposition 3.22. *Let* X *be a globally defined bi-Lipschitz diffeomorphism of* $\mathbb{R}^3$ *and* $-\frac{3}{p'}<s\leq\frac{3}{p}$. *Then* $a\to a\circ X$ *is a self-map over* $\dot{B}^s_{p,1}$ *whenever*

(1) $s\in(0,1)$;

(2) $s\geq 1$ *and* $(DX-\mathrm{Id})\in\dot{B}^{3/p}_{p,1}$.

The following result interferes in a crucial manner in the proof of the well-posedness result for the inhomogeneous incompressible Navier–Stokes system.

Proposition 3.23. *Let* m *be a* C^1 *scalar function over* $\mathbb{R}^3$ *and* $u\in\mathbb{R}^3$ *a* C^1 *vector field. Let* X *be a* C^1 *diffeomorphism and we define* $J:=\det(DX)$. *Suppose* $J>0$. *Then, the following relations hold*:

$$(\nabla m)\circ X=J^{-1}\operatorname{div}(\operatorname{adj}(DX)m\circ X), \tag{3-12}$$

$$(\operatorname{div}u)\circ X=J^{-1}\operatorname{div}(\operatorname{adj}(DX)u\circ X). \tag{3-13}$$

Corollary 3.24. *Let* m *be a* C^1 *scalar function over* $\mathbb{R}^3$ *and* $u\in\mathbb{R}^3$ *be a* C^1 *vector field. Let* X *be a* C^1 *diffeomorphism and* $J:=\det(DX)$. *Suppose* $J>0$. *Then, we have*

$$J^{-1}\operatorname{div}(\operatorname{adj}(DX)u)=Du:(DX)^{-1}, \tag{3-14}$$

$$J^{-1}\operatorname{div}(\operatorname{adj}(DX)m)=[(DX)^{-1}]^T\nabla m. \tag{3-15}$$

Proof. In order to ease reading, we define $F_{|x}:=F(x)$. Writing u as $u\circ X\circ X^{-1}$, using the chain rule and Einstein convention over repeated index, we write

$$\begin{aligned}(\operatorname{div}u)_{|x}&=\partial_k(u^i\circ X)_{|X^{-1}(x)}\partial_i(X^{-1})^k_{|x}\\&=D(u\circ X)_{|X^{-1}(x)}:D(X^{-1})_{|x}\\&=D(u\circ X)_{|X^{-1}(x)}:(DX)^{-1}_{|X^{-1}(x)},\end{aligned}$$

and thus, we get

$$(\operatorname{div} u)\circ X = D(u\circ X) : (DX)^{-1}. \tag{3-16}$$

Then, using (3-13) and (3-16) we get

$$\begin{aligned} J^{-1}\operatorname{div}(\operatorname{adj}(DX)u) &= J^{-1}\operatorname{div}\big(\operatorname{adj}(DX)u\circ X^{-1}\circ X\big) = (\operatorname{div} u\circ X^{-1})\circ X \\ &= D(u\circ X^{-1}\circ X) : (DX)^{-1} = Du : (DX)^{-1}. \end{aligned}$$

In a similar manner we prove (3-15). □

For any $\bar v$ a time-dependent vector field we set

$$X_{\bar v}(t,x) = x + \int_0^t \bar v(\tau,x)\,d\tau \tag{3-17}$$

and we define

$$A_{\bar v} = (DX_{\bar v})^{-1}. \tag{3-18}$$

It is crucial to know (in order to pass back in Eulerian coordinates, for instance) when $X_{\bar v}$ is a global diffeomorphism. In order to achieve this, we will use the following theorem due to Hadamard:

Theorem 3.25 (Hadamard). *Let $X:\mathbb{R}^n\to\mathbb{R}^n$ a function of class C^1. Then, the following are equivalent:*

(1) *X is a local diffeomorphism and* $\lim_{|x|\to\infty}|X(x)| = \infty$.

(2) *X is a global C^1-diffeomorphism over $\mathbb{R}^n$.*

For a proof of this result one can consult, for instance, [Katriel 1994].

Proposition 3.26. *Let us consider $\bar v\in C_b\big([0,T],\dot B^{3/p-1}_{p,1}\big)$ with $\partial_t\bar v,\ \nabla^2\bar v\in L^1_T(\dot B^{3/p-1}_{p,1})$. Then, there exists a positive α such that if*

$$\|\nabla\bar v\|_{L^1_T(\dot B^{3/p}_{p,1})}\le\alpha, \tag{3-19}$$

then, for any $t\in[0,T]$, we have $X_{\bar v}(t,\cdot)$ introduced in (3-17) *is a global C^1-diffeomorphism over $\mathbb{R}^3$ and* $\det(DX_{\bar v})>0$. *Moreover, if*

$$\operatorname{div}(\operatorname{adj}(DX_{\bar v})\bar v) = 0 \tag{3-20}$$

then, $X_{\bar v}$ is measure-preserving, i.e.,

$$\det DX_{\bar v} = 1. \tag{3-21}$$

Proof. Differentiating $X_{\bar v}$, we obtain

$$DX_{\bar v}(t,\cdot) = \mathrm{Id} + \int_0^t D\bar v(\tau,\cdot)\,d\tau$$

and because of the embedding of $\dot B^{3/p}_{p,1}$ into the space of continuous functions, see Proposition 3.7, we conclude $X_{\bar v}\in C^1([0,T]\times\mathbb{R}^3)$. We observe that

$$\begin{aligned} \|DX_{\bar v}(t,\cdot)-\mathrm{Id}\|_{L^\infty(\mathbb{R}^3)} &\le \int_0^t \|D\bar v(\tau,\cdot)\|_{L^\infty}\,d\tau \\ &\le C\|\nabla\bar v\|_{L^1_t(\dot B^{3/p}_{p,1})}\le \alpha C. \end{aligned}$$

Thus choosing α sufficiently small ensures that $X_{\bar{v}}(t,\cdot)$ is a local C^1-diffeomorphism over $\mathbb{R}^3$. The second condition of Hadamard's theorem is verified in the following lines. Using the triangle inequality we get

$$\begin{aligned}|X_{\bar{v}}(t,x)| &\geq |x| - \int_0^t |\bar{v}(\tau,x)|d\tau \geq |x| - \int_0^t \|\bar{v}(\tau,\cdot)\|_{L^\infty} d\tau \\ &\geq |x| - C\int_0^t \|\bar{v}(\tau,\cdot)\|_{\dot{B}^{3/p}_{p,1}}\, d\tau \\ &\geq |x| - C\sqrt{t}\|\bar{v}\|_{L^2_t(\dot{B}^{3/p}_{p,1})}.\end{aligned}$$

The conclusion is that $X_{\bar{v}}(t,\cdot)$ is a global C^1-diffeomorphism over $\mathbb{R}^3$. Let us define $J_{\bar{v}} := \det DX_{\bar{v}} \neq 0$. Using Jacobi's formula we get

$$J_{\bar{v}}(t,x) = 1 + \int_0^t \operatorname{tr}\big(D\bar{v}(\tau,x)\operatorname{adj}(DX_{\bar{v}})(\tau,x)\big)\, d\tau.$$

Recall that according to [Danchin and Mucha 2012, Lemma A.4] we may write

$$\operatorname{Id} - \operatorname{adj}(DX_{\bar{v}}) = \int_0^t (D\bar{v} - \operatorname{div}\bar{v}\operatorname{Id})\, d\tau + P_2\bigg(\int_0^t D\bar{v}\, d\tau\bigg),$$

where the coefficients of the matrix $P_2 : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ are at least quadratic polynomial functions of degree $n-1$. Using this identity combined with the embedding $L^\infty \hookrightarrow \dot{B}^{3/p}_{p,1}$ and the smallness condition (3-19) we get $J_{\bar{v}} > 0$. In order to prove the second part of the proposition, let us define

$$v(t,x) := \bar{v}(t, X^{-1}_{\bar{v}}(t,x)).$$

Using relation (3-20) combined with (3-13) we get

$$0 = J^{-1}_{\bar{v}} \operatorname{div}(\operatorname{adj}(DX_{\bar{v}})\bar{v}) = \operatorname{div}(\bar{v}\circ X^{-1}_{\bar{v}})\circ X_{\bar{v}},$$

which implies

$$\operatorname{div} v = \operatorname{div}(\bar{v}\circ X^{-1}_{\bar{v}}) = 0.$$

Since $X_{\bar{v}}$ can be viewed as being the flow of v, using Jacobi's formula we can conclude the validity of (3-21). Indeed, we have

$$\begin{aligned}X_{\bar{v}}(t,x) &= x + \int_0^t \bar{v}(\tau,x)d\tau \\ &= x + \int_0^t \bar{v}(\tau, X^{-1}_{\bar{v}}(\tau, X_{\bar{v}}(\tau,x)))\, d\tau \\ &= x + \int_0^t v(\tau, X_{\bar{v}}(\tau,x))\, d\tau.\end{aligned}$$

Then, Jacobi's formula implies

$$\det(DX_{\bar{v}})(t,x) = \exp\bigg(\int_0^t (\operatorname{div} v)(\tau, X_{\bar{v}}(\tau,x))\bigg) = 1. \qquad \square$$

Proposition 3.27. *Consider* $\bar{v} \in C_b\big([0,T], \dot{B}^{3/p-1}_{p,1}\big)$ *with* $\partial_t \bar{v}$, $\nabla^2 \bar{v} \in L^1_T(\dot{B}^{3/p-1}_{p,1})$ *satisfying the smallness condition* (3-3). *Let* $X_{\bar{v}}$ *be defined by* (3-17) *and* $J_{\bar{v}} = \det DX_{\bar{v}}$. *Then for all* $t \in [0,T]$,

$$\|\mathrm{Id} - A_{\bar{v}}(t)\|_{\dot{B}^{3/p}_{p,1}} \lesssim \|\nabla \bar{v}\|_{L^1_t(\dot{B}^{3/p}_{p,1})}, \tag{3-22}$$

$$\|\mathrm{Id} - \mathrm{adj}(DX_{\bar{v}})(t)\|_{\dot{B}^{3/p}_{p,1}} \lesssim \|\nabla \bar{v}\|_{L^1_t(\dot{B}^{3/p}_{p,1})}, \tag{3-23}$$

$$\|\partial_t\, \mathrm{adj}(DX_{\bar{v}})(t)\|_{\dot{B}^{3/p-1}_{p,1}} \lesssim \|\nabla \bar{v}(t)\|_{\dot{B}^{3/p-1}_{p,1}}, \quad \text{if } p < 6, \tag{3-24}$$

$$\|\partial_t\, \mathrm{adj}(DX_{\bar{v}})(t)\|_{\dot{B}^{3/p}_{p,1}} \lesssim \|\nabla \bar{v}(t)\|_{\dot{B}^{3/p}_{p,1}}, \tag{3-25}$$

$$\|J^{\pm}_{\bar{v}}(t) - 1\|_{\dot{B}^{3/p}_{p,1}} \lesssim \|\nabla \bar{v}\|_{L^1_t(\dot{B}^{3/p}_{p,1})}. \tag{3-26}$$

In order to establish stability estimates we use the following:

Proposition 3.28. *Let* $\bar{v}_1, \bar{v}_2 \in C_b\big([0,T], \dot{B}^{3/p-1}_{p,1}\big)$ *with* $\partial_t \bar{v}_1, \partial_t \bar{v}_2, \nabla^2 \bar{v}_1, \nabla^2 \bar{v}_2 \in L^1_T(\dot{B}^{3/p-1}_{p,1})$, *both satisfying the smallness condition* (3-19) *and* $\delta v = \bar{v}_2 - \bar{v}_1$. *Then we have*

$$\|A_{\bar{v}_1} - A_{\bar{v}_2}\|_{L^\infty_T(\dot{B}^{3/p}_{p,1})} \lesssim \|\nabla \delta v\|_{L^1_T(\dot{B}^{3/p}_{p,1})}, \tag{3-27}$$

$$\|\,\mathrm{adj}(DX_{\bar{v}_1}) - \mathrm{adj}(DX_{\bar{v}_2})\|_{L^\infty_T(\dot{B}^{3/p}_{p,1})} \lesssim \|\nabla \delta v\|_{L^1_T(\dot{B}^{3/p}_{p,1})}, \tag{3-28}$$

$$\|\partial_t\, \mathrm{adj}(DX_{\bar{v}_1}) - \partial_t\, \mathrm{adj}(DX_{\bar{v}_2})\|_{L^1_T(\dot{B}^{3/p}_{p,1})} \lesssim \|\nabla \delta v\|_{L^1_T(\dot{B}^{3/p}_{p,1})}, \tag{3-29}$$

$$\|\partial_t\, \mathrm{adj}(DX_{\bar{v}_1}) - \partial_t\, \mathrm{adj}(DX_{\bar{v}_2})\|_{L^2_T(\dot{B}^{3/p-1}_{p,1})} \lesssim \|\nabla \delta v\|_{L^2_T(\dot{B}^{3/p-1}_{p,1})}, \quad \text{if } p < 6, \tag{3-30}$$

$$\|J^{\pm}_{\bar{v}_1}(t) - J^{\pm}_{\bar{v}_2}(t)\|_{\dot{B}^{3/p}_{p,1}} \lesssim \|\nabla \delta v\|_{L^1_t(\dot{B}^{3/p}_{p,1})}. \tag{3-31}$$

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After submitting this work, we were informed of the existence of [Zhai and Yin 2017], which contains similar results obtained using another method.

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