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# GLOBAL DYNAMICS BELOW THE STANDING WAVES FOR THE FOCUSING SEMILINEAR SCHRÖDINGER EQUATION WITH A REPULSIVE DIRAC DELTA POTENTIAL

MASAHIRO IKEDA AND TAKAHISA INUI

We consider the focusing mass-supercritical semilinear Schrödinger equation with a repulsive Dirac delta potential on the real line $\mathbb{R}$:

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u + \gamma\delta_0 u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}), \end{cases}$$

where $\gamma \le 0$, $\delta_0$ denotes the Dirac delta with the mass at the origin, and $p > 5$. By a result of Fukuizumi, Ohta, and Ozawa (2008), it is known that the system above is locally well-posed in the energy space $H^1(\mathbb{R})$ and there exist standing wave solutions $e^{i\omega t}Q_{\omega,\gamma}(x)$ when $\omega > \frac{1}{2}\gamma^2$, where $Q_{\omega,\gamma}$ is a unique radial positive solution to $-\frac{1}{2}\partial_x^2 Q + \omega Q - \gamma\delta_0 Q = |Q|^{p-1}Q$. Our aim in the present paper is to find a necessary and sufficient condition on the data below the standing wave $e^{i\omega t}Q_{\omega,0}$ to determine the global behavior of the solution. The similar result for NLS without potential ($\gamma = 0$) was obtained by Akahori and Nawa (2013); the scattering result was also extended by Fang, Xie, and Cazenave (2011). Our proof of the scattering result is based on the argument of Banica and Visciglia (2016), who proved all solutions scatter in the defocusing and repulsive case ($\gamma < 0$) by the Kenig–Merle method (2006). However, the method of Banica and Visciglia cannot be applicable to our problem because the energy may be negative in the focusing case. To overcome this difficulty, we use the variational argument based on the work of Ibrahim, Masmoudi, and Nakanishi (2011). Our proof of the blow-up result is based on the method of Du, Wu, and Zhang (2016). Moreover, we determine the global dynamics of the radial solution whose mass-energy is larger than that of the standing wave $e^{i\omega t}Q_{\omega,0}$. The difference comes from the existence of the potential.



## 1. Introduction

**1A.** ***Background.*** We consider the focusing mass-supercritical semilinear Schrödinger equation with a repulsive Dirac delta potential on the real line $\mathbb{R}$:

*MSC2010:* 35P25, 35Q55, 47J35.
*Keywords:* global dynamics, standing waves, nonlinear Schrödinger equation, Dirac delta potential.

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u + \gamma\delta_0 u + |u|^{p-1}u = 0, & (t,x)\in\mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x)\in H^1(\mathbb{R}), \end{cases} \tag{$\delta$NLS}$$

where $\gamma \le 0$, $\delta_0$ denotes the Dirac delta with the mass at the origin, and $p > 5$. The system ($\delta$NLS) appears in a wide variety of physical models with a point defect on the line; see [Goodman et al. 2004] and the references therein. We define the Schrödinger operator $H_\gamma$ as the formulation of a formal expression $-\frac{1}{2}\partial_x^2 - \gamma\delta_0$:

$$\begin{aligned} H_\gamma\varphi &:= -\tfrac{1}{2}\partial_x^2\varphi, \quad \varphi\in\mathcal{D}(H_\gamma), \\ \mathcal{D}(H_\gamma) &:= \bigl\{\varphi\in H^1(\mathbb{R})\cap H^2(\mathbb{R}\setminus\{0\}) : \partial_x\varphi(0+) - \partial_x\varphi(0-) = -2\gamma\varphi(0)\bigr\}. \end{aligned}$$

$H_\gamma$ is a nonnegative self-adjoint operator on $L^2(\mathbb{R})$ (see [Albeverio et al. 2005] for more details), which implies that ($\delta$NLS) is locally well-posed in the energy space $H^1(\mathbb{R})$.

**Proposition 1.1** [Fukuizumi et al. 2008, Section 2; Cazenave 2003, Theorem 3.7.1]. *For any $u_0\in H^1(\mathbb{R})$, there exist $T_\pm = T_\pm(\|u_0\|_{H^1}) > 0$ and a unique solution*

$$u\in C\bigl((-T_-,T_+);H^1(\mathbb{R})\bigr)\cap C^1\bigl((-T_-,T_+);H^{-1}(\mathbb{R})\bigr)$$

*of ($\delta$NLS). Moreover, the following statements hold*:

- (*blow-up criterion*) *$T_\pm = \infty$, or $T_\pm < \infty$ and $\lim_{t\to\pm T_\pm}\|\partial_x u(t)\|_{L^2} = \infty$, where the double-sign corresponds.*
- (*conservation laws*) *The energy $E$ and the mass $M$ are conserved by the flow*; *i.e.*,

$$E(u(t)) = E(u_0), \quad M(u(t)) = M(u_0) \quad \textit{for any } t\in(-T_-,T_+),$$

*where for $\varphi\in H^1(\mathbb{R})$, we define $E$ and $M$ as*

$$E(\varphi) = E_\gamma(\varphi) := \tfrac{1}{4}\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma|\varphi(0)|^2 - \frac{1}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}, \tag{1-1}$$

$$M(\varphi) := \tfrac{1}{2}\|\varphi\|_{L^2}^2. \tag{1-2}$$

We investigate the global behaviors of the solution. By the choice of the initial data, ($\delta$NLS) has various solutions, for example, scattering solutions, blow-up solutions, and so on. Let us recall the definitions of scattering and blow-up. Let $u$ be a solution to ($\delta$NLS) on the maximal existence time interval $(-T_-,T_+)$.

**Definition 1.1** (scattering). We say that the solution $u$ to ($\delta$NLS) scatters if and only if $T_\pm = \infty$ and there exist $u_\pm\in H^1(\mathbb{R})$ such that

$$\|u(t) - e^{-itH_\gamma}u_\pm\|_{H^1} \to 0 \quad \text{as } t\to\pm\infty,$$

where $\{e^{-itH_\gamma}\}$ denotes the evolution group of $i\partial_t u - H_\gamma u = 0$.

**Definition 1.2** (blow-up). We say that the solution $u$ to ($\delta$NLS) blows up in positive time (resp. negative time) if and only if $T_+ < \infty$ (resp. $T_- < \infty$).

Since a pioneer work by Kenig and Merle [2006], the global dynamics without assuming smallness for focusing nonlinear Schrödinger equations have been studied. For the focusing cubic semilinear Schrödinger equation in three dimensions, Holmer and Roudenko [2008] proved that $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$ implies scattering and, on the other hand, $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} > \|Q\|_{L^2}\|\nabla Q\|_{L^2}$ implies finite-time blow-up if the initial data $u_0 \in H^1(\mathbb{R}^3)$ is radially symmetric and satisfies the mass-energy condition $M(u_0)E(u_0) < M(Q)E(Q)$, where $Q$ is the ground state. For nonradial solutions, Duyckaerts, Holmer, and Roudenko [Duyckaerts et al. 2008] proved the scattering part and Holmer and Roudenko [2010] proved the solutions in the above blow-up region blow up in finite time or grow up in infinite time. Fang, Xie, and Cazenave [Fang et al. 2011] extended the scattering result and Akahori and Nawa [2013] extended both the scattering and the blow-up result to mass-supercritical and energy-subcritical Schrödinger equations in general dimensions.

Recently, Banica and Visciglia [2016] proved all solutions scatter in the defocusing case. On the other hand, in the focusing case, ($\delta$NLS) has blow-up solutions and nonscattering global solutions. Thus, their method cannot be applicable to our problem.

**1B.** ***Main results.*** To state our main result, we introduce several notations.

Let $\omega$ be a positive parameter that denotes the frequency. We define action $S_\omega$ and a functional $P$ as

$$S_\omega(\varphi) = S_{\omega,\gamma}(\varphi) := E(\varphi) + \omega M(\varphi) = \tfrac{1}{4}\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma|\varphi(0)|^2 + \tfrac{1}{2}\omega\|\varphi\|_{L^2}^2 - \frac{1}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}, \quad (1\text{-}3)$$

$$P(\varphi) = P_\gamma(\varphi) := \tfrac{1}{2}\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma|\varphi(0)|^2 - \frac{p-1}{2(p+1)}\|\varphi\|_{L^{p+1}}^{p+1}, \quad (1\text{-}4)$$

where $P$ appears in the virial identity (see [Le Coz et al. 2008]).

We often omit the index $\gamma$. We sometimes insert 0 into $\gamma$, such as $S_{\omega,0}$ and $P_0$.

We consider the three minimizing problems

$$n_\omega := \inf\{S_\omega(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ P(\varphi)=0\}, \quad (1\text{-}5)$$

$$r_\omega := \inf\{S_\omega(\varphi) : \varphi \in H^1_{\mathrm{rad}}(\mathbb{R})\setminus\{0\},\ P(\varphi)=0\}, \quad (1\text{-}6)$$

$$l_\omega := \inf\{S_{\omega,0}(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ P_0(\varphi)=0\}, \quad (1\text{-}7)$$

where $H^1_{\mathrm{rad}}(\mathbb{R}) := \{\varphi \in H^1(\mathbb{R}) : \varphi(x) = \varphi(-x)\}$.

Equation (1-7) is nothing but the minimizing problem for the nonlinear Schrödinger equation without a potential, and $l_\omega$ is positive and is attained by

$$Q_{\omega,0}(x) := \left\{\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{\sqrt{2}}|x|\right)\right\}^{\frac{1}{p-1}},$$

which is a unique positive solution of

$$-\tfrac{1}{2}\partial_x^2 Q + \omega Q = |Q|^{p-1}Q. \quad (1\text{-}8)$$

For $n_\omega$ and $r_\omega$, we prove the following statements, some of which were proved by Fukuizumi and Jeanjean [2008].

**Proposition 1.2.** *Let $\gamma$ be strictly negative. Then the following statements are true*:

(1) $n_\omega = l_\omega$ *and* $n_\omega$ *is not attained.*

(2) $n_\omega < r_\omega$ *and*

$$\begin{cases} r_\omega = 2l_\omega & \text{if } 0 < \omega \le \frac{1}{2}\gamma^2, \\ r_\omega < 2l_\omega & \text{if } \omega > \frac{1}{2}\gamma^2. \end{cases}$$

(3) *If* $\omega > \frac{1}{2}\gamma^2$, *then* $r_\omega$ *is attained by*

$$Q_\omega(x) = Q_{\omega,\gamma}(x) := \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2\left( \frac{(p-1)\sqrt{\omega}}{\sqrt{2}} |x| + \tanh^{-1}\left( \frac{\gamma}{\sqrt{2\omega}} \right) \right) \right\}^{\frac{1}{p-1}},$$

*which is a unique positive solution of* $-\frac{1}{2}\partial_x^2 Q + \omega Q - \gamma\delta_0 Q = |Q|^{p-1}Q$. *On the other hand*, $r_\omega$ *is not attained if* $0 < \omega \le \frac{1}{2}\gamma^2$.

The function $e^{i\omega t}Q_\omega$ with $\omega > \frac{1}{2}\gamma^2$ is a global nonscattering solution to ($\delta$NLS), which is called the standing wave. The fact that $n_\omega \neq r_\omega$ comes from the existence of the potential, which means that the following main result in the radial case does not follow from that in the nonradial case.

By using the minimizing problems, we define subsets in $H^1(\mathbb{R})$ for $\omega > 0$ as follows:

$$\mathcal{N}_\omega^+ := \{\varphi \in H^1(\mathbb{R}) : S_\omega(\varphi) < n_\omega,\ P(\varphi) \ge 0\},$$
$$\mathcal{N}_\omega^- := \{\varphi \in H^1(\mathbb{R}) : S_\omega(\varphi) < n_\omega,\ P(\varphi) < 0\},$$

and

$$\mathcal{R}_\omega^+ := \{\varphi \in H^1_{\mathrm{rad}}(\mathbb{R}) : S_\omega(\varphi) < r_\omega,\ P(\varphi) \ge 0\},$$
$$\mathcal{R}_\omega^- := \{\varphi \in H^1_{\mathrm{rad}}(\mathbb{R}) : S_\omega(\varphi) < r_\omega,\ P(\varphi) < 0\}.$$

We state one of our main results, which treats the nonradial case. We classify the global behavior of the solution whose action is less than $n_\omega$.

**Theorem 1.3** (nonradial case). *Let* $\omega > 0$. *Let* $u$ *be a solution to* ($\delta$NLS) *on* $(-T_-, T_+)$ *with the initial data* $u_0 \in H^1(\mathbb{R})$.

(1) *If the initial data* $u_0$ *belongs to* $\mathcal{N}_\omega^+$, *then the solution* $u$ *scatters.*

(2) *If the initial data* $u_0$ *belongs to* $\mathcal{N}_\omega^-$, *then one of the following four cases holds*:

(a) *The solution* $u$ *blows up in both time directions.*

(b) *The solution* $u$ *blows up in a positive time, and* $u$ *is global toward negative time and*

$$\limsup_{t \to -\infty} \|\partial_x u(t)\|_{L^2} = \infty.$$

(c) *The solution* $u$ *blows up in a negative time, and* $u$ *is global toward positive time and*

$$\limsup_{t \to \infty} \|\partial_x u(t)\|_{L^2} = \infty.$$

(d) *The solution* $u$ *is global in both time directions and*

$$\limsup_{t \to \pm\infty} \|\partial_x u(t)\|_{L^2} = \infty.$$

Proposition 1.2 and a direct calculation give $n_\omega = l_\omega = \omega^{\frac{p+3}{2(p-1)}} S_{1,0}(Q_{1,0})$. By these relations, we can rewrite the main theorem in the nonradial case into a version independent of the frequency $\omega$.

**Corollary 1.4.** *We define the subsets $\mathcal{N}^{\pm}$ in $H^1(\mathbb{R})$ as*

$$\mathcal{N}^{+} := \{\varphi \in H^1(\mathbb{R}) : E(\varphi)M(\varphi)^{\sigma} < E_0(Q_{1,0})M(Q_{1,0})^{\sigma},\ P(\varphi) \geq 0\},$$
$$\mathcal{N}^{-} := \{\varphi \in H^1(\mathbb{R}) : E(\varphi)M(\varphi)^{\sigma} < E_0(Q_{1,0})M(Q_{1,0})^{\sigma},\ P(\varphi) < 0\},$$

*where $\sigma := (p+3)/(p-5)$. Let $u$ be a solution to (δNLS) on $(-T_-, T_+)$ with the initial data $u_0 \in H^1(\mathbb{R})$. Then, we can prove the same conclusion as in Theorem 1.3, where $\mathcal{N}_\omega^{\pm}$ is replaced by $\mathcal{N}^{\pm}$, respective of the sign.*

The equivalency is proved in the Appendix.

Next, we state the other main result for radial solutions. If we restrict solutions to (δNLS) to radial solutions, then we can classify the global behavior of the radial solutions whose action is larger than $n_\omega$ and less than $r_\omega$.

**Theorem 1.5** (radial case). *Let $\omega > 0$ and $u$ be a solution to (δNLS) with the initial data $u_0 \in H^1_{\mathrm{rad}}(\mathbb{R})$. Then, we can prove the same conclusion as in Theorem 1.3, where $\mathcal{N}_\omega^{\pm}$ is replaced by $\mathcal{R}_\omega^{\pm}$, respective of the sign.*

**Remark 1.1.** Even if solutions to (δNLS) are restricted to radial ones, the possibility that (b)–(d) (grow-up) occurs cannot be excluded since we consider one spatial dimension. In [Le Coz et al. 2008], it was proved that if the initial data satisfies $xu_0 \in L^2$ and $P(u_0) < 0$, then the solution blows up in a finite time in both time directions.

**1C. *Difficulties and idea for the proofs.*** Our proof of the scattering part is based on the argument of Banica and Visciglia [2016], where they proved all solutions scatter in the defocusing case. We also use a concentration compactness argument (see Sections 3C–3E) and a rigidity argument (see Section 3E). In the focusing case, it is not clear that each profile has positive energy when we use profile decomposition. To prove this with $\gamma = 0$, the orthogonality property of the functional $P_0$ was used in [Fang et al. 2011; Akahori and Nawa 2013]. However, it is not easy to prove the orthogonality of the functional $P_\gamma$ because of the presence of the Dirac delta potential ($\gamma \neq 0$). To overcome this difficulty, we use the Nehari functional $I_{\omega,\gamma}$ (see (2-7) for the definition) instead of $P_\gamma$. Then we can prove that the subsets for the data defined by $I_\omega$ instead of $P$ are the same as the subsets $\mathcal{N}_\omega^{\pm}$ (see Proposition 2.15) using an argument similar to that of [Ibrahim et al. 2011].

Theorem 1.5 (radial case) does not follow from Theorem 1.3 (nonradial case) since we treat solutions whose action is larger than or equal to $n_\omega$ in Theorem 1.5. Recently, Killip, Murphy, Visan, and Zheng [Killip et al. 2016] also considered a similar problem and extended the region to classify solutions under radial assumption for NLS with the inverse-square potential. They used the radial Sobolev inequality, which is only effective in higher dimensions, to prove a translation parameter in the linear profile decomposition is bounded. However, this method cannot be applied to our problem. In the one-dimensional case, it is not clear whether the translation parameter is bounded or not. To avoid this difficulty, we use the fact that the translation parameter $-x_n$ appears in the profile decomposition if $x_n$ appears (see Theorem 3.5 for more detail).

Next, we explain the blow-up results. Holmer and Roudenko [2010] proved a blow-up result for the cubic Schrödinger equation without potentials in three dimensions by applying the Kenig–Merle method [2006]. Recently, Du, Wu, and Zhang [Du et al. 2016] gave a simpler proof for blow-up, in which they only used the localized virial identity. We apply their method to the equation with a potential.

**1D.** ***Construction of the paper.*** In Section 2, we consider the minimizing problems from the viewpoint of variational argument. We prove the existence and nonexistence of a minimizer for $r_\omega$ and $n_\omega$, and that the subsets for the data defined by $I_\omega$ instead of $P$ are the same as the subsets in $H^1(\mathbb{R})$ defined by $P$ in this section. In Section 3, we prove the scattering results by a concentration compactness argument and a rigidity argument. We explain the necessity of the Nehari functional $I_\omega$ instead of $P$. In Section 4, we prove the blow-up results, based on the argument of Du et al. [2016].

## 2. Minimizing problems and variational structure

**2A.** ***Minimizing problems.*** Let $(\alpha,\beta)$ satisfy the conditions

$$\alpha>0,\quad 2\alpha-\beta\ge 0,\quad 2\alpha+\beta\ge 0,\quad (\alpha,\beta)\neq(0,0). \tag{2-1}$$

We set

$$\bar{\mu}:=\max\{2\alpha-\beta,\,2\alpha+\beta\},\qquad \underline{\mu}:=\min\{2\alpha-\beta,\,2\alpha+\beta\}.$$

We define a scaling transformation and a derivative of functional as

$$\varphi_\lambda^{\alpha,\beta}(x):=e^{\alpha\lambda}\varphi(e^{-\beta\lambda}x), \tag{2-2}$$

$$\mathcal{L}_{\lambda_0}^{\alpha,\beta}S(\varphi):=\partial_\lambda S(\varphi_\lambda^{\alpha,\beta})|_{\lambda=\lambda_0}, \tag{2-3}$$

$$\mathcal{L}^{\alpha,\beta}S(\varphi):=\mathcal{L}_0^{\alpha,\beta}S(\varphi) \tag{2-4}$$

for any function $\varphi$ and any functional $S:H^1(\mathbb{R})\to\mathbb{R}$. We define functionals $K_\omega^{\alpha,\beta}$ by

$$\begin{aligned} K_\omega^{\alpha,\beta}(\varphi)&=K_{\omega,\gamma}^{\alpha,\beta}(\varphi)\\ &:=\mathcal{L}^{\alpha,\beta}S_\omega(\varphi)\\ &=\partial_\lambda S_\omega(e^{\alpha\lambda}\varphi(e^{-\beta\lambda}\,\cdot))|_{\lambda=0}\\ &=\tfrac14(2\alpha-\beta)\|\partial_x\varphi\|_{L^2}^2+\tfrac12\omega(2\alpha+\beta)\|\varphi\|_{L^2}^2-\gamma\alpha|\varphi(0)|^2-\frac{(p+1)\alpha+\beta}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{2-5}$$

We especially use the following functionals:

$$P(\varphi)=P_\gamma(\varphi):=K_\omega^{\frac12,-1}(\varphi)=\tfrac12\|\partial_x\varphi\|_{L^2}^2-\tfrac12\gamma|\varphi(0)|^2-\frac{p-1}{2(p+1)}\|\varphi\|_{L^{p+1}}^{p+1}, \tag{2-6}$$

$$I_\omega(\varphi)=I_{\omega,\gamma}(\varphi):=K_\omega^{1,0}(\varphi)=\tfrac12\|\partial_x\varphi\|_{L^2}^2-\gamma|\varphi(0)|^2+\omega\|\varphi\|_{L^2}^2-\|\varphi\|_{L^{p+1}}^{p+1}. \tag{2-7}$$

**Remark 2.1.** Both the functional $P$, which appears in the virial identity (3-2), and the Nehari functional $I_\omega$ are used to prove the scattering results. It is proved in Proposition 2.15 that $P$ and $I_\omega$ have same sign under a condition for the action. To prove this, we introduce the parameter $(\alpha,\beta)$ based on [Ibrahim et al. 2011].

We also use $J_\omega^{\alpha,\beta}$ defined by

$$J_\omega^{\alpha,\beta}(\varphi)=J_{\omega,\gamma}^{\alpha,\beta}(\varphi):=S_\omega(\varphi)-\frac{K_\omega^{\alpha,\beta}(\varphi)}{\bar{\mu}}. \tag{2-8}$$

**Lemma 2.1.** *We have the relations*

$$(\mathcal{L}^{\alpha,\beta}-\bar{\mu})\|\partial_x\varphi\|_{L^2}^2=\begin{cases}0 & \text{if } \beta\le 0,\\ -2\beta\|\partial_x\varphi\|_{L^2}^2 & \text{if } \beta>0,\end{cases}$$

$$(\mathcal{L}^{\alpha,\beta}-\bar{\mu})\|\varphi\|_{L^2}^2=\begin{cases}2\beta\|\varphi\|_{L^2}^2 & \text{if } \beta\le 0,\\ 0 & \text{if } \beta>0,\end{cases}$$

$$(\mathcal{L}^{\alpha,\beta}-\bar{\mu})|\varphi(0)|^2=\begin{cases}\beta|\varphi(0)|^2 & \text{if } \beta\le 0,\\ -\beta|\varphi(0)|^2 & \text{if } \beta>0,\end{cases}$$

$$(\mathcal{L}^{\alpha,\beta}-\bar{\mu})\|\varphi\|_{L^{p+1}}^{p+1}=\begin{cases}((p-1)\alpha+2\beta)\|\varphi\|_{L^{p+1}}^{p+1} & \text{if } \beta\le 0,\\ (p-1)\alpha\|\varphi\|_{L^{p+1}}^{p+1} & \text{if } \beta>0.\end{cases}$$

*In particular*,

$$\bar{\mu}J_\omega^{\alpha,\beta}=(\bar{\mu}-\mathcal{L}^{\alpha,\beta})S_\omega(\varphi)\ge|\beta|\min\{\tfrac{1}{2}\|\partial_x\varphi\|_{L^2}^2,\omega\|\varphi\|_{L^2}^2\}-\tfrac{1}{2}\gamma|\beta|\,|\varphi(0)|^2+\frac{(p-5)\alpha}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}.$$

*Moreover*, *we have*

$$\begin{aligned}-(\mathcal{L}^{\alpha,\beta}-\bar{\mu})(\mathcal{L}^{\alpha,\beta}-\underline{\mu})S_\omega(\varphi)&=(\mathcal{L}^{\alpha,\beta}-\bar{\mu})(\mathcal{L}^{\alpha,\beta}-\underline{\mu})\left(\tfrac{1}{2}\gamma|\varphi(0)|^2+\frac{\|\varphi\|_{L^{p+1}}^{p+1}}{p+1}\right)\\ &\ge-\tfrac{1}{2}\gamma|\beta|^2|\varphi(0)|^2+\frac{(p-5)\alpha}{p+1}\mathcal{L}^{\alpha,\beta}\|\varphi\|_{L^{p+1}}^{p+1}\ge\frac{(p-5)\alpha\bar{\mu}}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}.\end{aligned}$$

*Proof.* These relations are obtained by simple calculations. We only note that

$$(p-1)\alpha+2\beta=(p-5)\alpha+2(2\alpha+\beta)\ge(p-5)\alpha.$$ □

By this lemma and $p>5$, we find that $J_\omega^{\alpha,\beta}(\varphi)\ge 0$ for any $\varphi\in H^1(\mathbb{R})$. Next, we see that $K_\omega^{\alpha,\beta}$ is positive near the origin in $H^1(\mathbb{R})$.

**Lemma 2.2.** *Let $\{\varphi_n\}_{n\in\mathbb{N}}\subset H^1(\mathbb{R})\setminus\{0\}$ be bounded in $L^2(\mathbb{R})$ such that $\|\partial_x\varphi_n\|_{L^2}\to 0$ as $n\to\infty$. Then $K_\omega^{\alpha,\beta}(\varphi_n)>0$ for large $n\in\mathbb{N}$.*

*Proof.* By $\gamma<0$, $p>5$, and the Gagliardo–Nirenberg inequality, we have

$$K_\omega^{\alpha,\beta}(\varphi_n)\ge\tfrac{1}{4}(2\alpha-\beta)\|\partial_x\varphi_n\|_{L^2}^2-\frac{(p+1)\alpha+\beta}{p+1}C\|\partial_x\varphi_n\|_{L^2}^{\frac{1}{2}(p-1)}\|\varphi_n\|_{L^2}^{\frac{1}{2}(p+3)}>0$$

for sufficiently large $n\in\mathbb{N}$, where $C$ is a positive constant. □

We define the following minimizing problems for $\omega > 0$ and $(\alpha, \beta)$ satisfying (2-1):

$$n_\omega^{\alpha,\beta} := \inf\{S_\omega(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ K_\omega^{\alpha,\beta}(\varphi)=0\}, \tag{2-9}$$

$$r_\omega^{\alpha,\beta} := \inf\{S_\omega(\varphi) : \varphi \in H^1_{\mathrm{rad}}(\mathbb{R})\setminus\{0\},\ K_\omega^{\alpha,\beta}(\varphi)=0\}, \tag{2-10}$$

$$l_\omega^{\alpha,\beta} := \inf\{S_{\omega,0}(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ K_{\omega,0}^{\alpha,\beta}(\varphi)=0\}. \tag{2-11}$$

If $(\alpha, \beta) = \left(\frac{1}{2}, -1\right)$, these are nothing but $n_\omega$, $r_\omega$, and $l_\omega$. We prove that these minimizing problems are independent of $(\alpha, \beta)$ and Proposition 1.2 holds in the following subsections.

**2B.** ***Radial minimizing problem.*** First, we consider the radial minimizing problem $r_\omega^{\alpha,\beta}$. For $\gamma \le 0$, $S_\omega : H^1_{\mathrm{rad}}(\mathbb{R}) \to \mathbb{R}$ satisfies the following mountain pass structure:

(1) $S_\omega(0) = 0$.

(2) There exist $\delta, \rho > 0$ such that $S_\omega(\varphi) > \delta$ for all $\varphi$ with $\|\varphi\|_{H^1} = \rho$.

(3) There exists $\psi \in H^1_{\mathrm{rad}}(\mathbb{R})$ such that $S_\omega(\psi) < 0$ and $\|\psi\|_{H^1} > \rho$.

Indeed, (1) is trivial, (2) can be proved by the Gagliardo–Nirenberg inequality, and (3) is obtained by a scaling argument.

Let

$$\mathcal{C} := \{c \in C([0,1] : H^1_{\mathrm{rad}}(\mathbb{R})) : c(0)=0,\ S_\omega(c(1))<0\},$$
$$b := \inf_{c\in\mathcal{C}} \max_{t\in[0,1]} S_\omega(c(t)).$$

**Lemma 2.3.** *The identity $b = r_\omega^{\alpha,\beta}$ holds.*

*Proof.* First, we prove $b \le r_\omega^{\alpha,\beta}$. To see this, it is sufficient to prove the existence of $\{c_n\} \subset \mathcal{C}$ such that $\max_{t\in[0,1]} S_\omega(c_n(t)) \to r_\omega^{\alpha,\beta}$ as $n \to \infty$. We take a minimizing sequence $\{\varphi_n\}$ for $r_\omega^{\alpha,\beta}$, namely,

$$S_\omega(\varphi_n) \to r_\omega^{\alpha,\beta} \quad \text{as } n \to \infty \qquad \text{and} \qquad K_\omega^{\alpha,\beta}(\varphi_n) = 0 \quad \text{for all } n \in \mathbb{N}.$$

We set $\tilde{c}_n(\lambda) := \mathcal{L}_\lambda^{\alpha,\beta}\varphi_n$ for $\lambda \in \mathbb{R}$. Then, we see that $S_\omega(\tilde{c}_n(\lambda)) < 0$ for large $\lambda$. Moreover,

$$\max_{\lambda\in\mathbb{R}} S_\omega(\tilde{c}_n(\lambda)) = S_\omega(\tilde{c}_n(0)) = S_\omega(\varphi_n) \to r_\omega^{\alpha,\beta} \quad \text{as } n \to \infty$$

since $K_\omega^{\alpha,\beta}(\varphi_n) = 0$ for all $n \in \mathbb{N}$. We define $\mathscr{C}_n(t)$ for $t \in [-L, L]$ such that

$$\mathscr{C}_n(t) := \begin{cases} \tilde{c}_n(t) & \text{if } -\frac{1}{2}L \le t \le L, \\ \left(\frac{2}{L}(t+L)\right)^M \tilde{c}_n\left(-\frac{L}{2}\right) & \text{if } -L \le t < -\frac{1}{2}L. \end{cases}$$

$\mathscr{C}$ is continuous in $H^1(\mathbb{R})$ and we have $S_\omega(\mathscr{C}_n(L)) < 0$ and $\max_{t\in[-L,L]} S_\omega(\mathscr{C}_n(t)) = S_\omega(\varphi_n) \to r_\omega^{\alpha,\beta}$ when $L > 0$ and $M = M(n)$ are sufficiently large. By changing variables, we obtain a desired sequence $c_n \in \mathcal{C}$. Next, we prove $b \ge r_\omega^{\alpha,\beta}$. It is sufficient to prove

$$c([0,1]) \cap \{\varphi \in H^1_{\mathrm{rad}}(\mathbb{R})\setminus\{0\} : K_\omega^{\alpha,\beta}(\varphi)=0\} \ne \varnothing \quad \text{for all } c \in \mathcal{C}.$$

We take an arbitrary $c \in \mathcal{C}$. Now, $c(0) = 0$ and $S_\omega(c(1)) < 0$. Therefore, $K_\omega^{\alpha,\beta}(c(t)) > 0$ for some $t \in (0, 1)$ by Lemma 2.2 and $K_\omega^{\alpha,\beta}(c(1)) \le ((p+1)\alpha + \beta)S_\omega(c(1)) < 0$. By continuity, there exists $t_0 \in (0, 1)$ such that $K_\omega^{\alpha,\beta}(c(t_0)) = 0$. Thus, we get $b = r_\omega^{\alpha,\beta}$. □

Next, we prove the existence and nonexistence of a minimizer for the minimizing problem $r_\omega^{\alpha,\beta}$. See [Fukuizumi and Jeanjean 2008, Lemmas 15, 19, 20, 21, and 25] for the proofs of the following Lemmas 2.4, 2.5, 2.6, 2.7, and 2.8, respectively.

The following lemma means that it is sufficient to find a nonnegative minimizer.

**Lemma 2.4.** *If $\varphi \in H^1(\mathbb{R})$ is a minimizer of $r_\omega^{\alpha,\beta}$, then $|\varphi| \in H^1(\mathbb{R})$ is also a minimizer.*

**Definition 2.1** (Palais–Smale sequence). We say that $\{\varphi_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$ is a Palais–Smale sequence for $S_\omega$ at the level $c$ if and only if the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfies

$$S_\omega(\varphi_n) \to c \qquad \text{and} \qquad S'_\omega(\varphi_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}) \qquad \text{as } n \to \infty.$$

By the mountain pass theorem, we obtain a Palais–Smale sequence at the level $b = r_\omega^{\alpha,\beta}$. We may assume that the sequence is bounded.

**Lemma 2.5.** *Any Palais–Smale sequence of $S_\omega$ considered on $H^1_{\mathrm{rad}}(\mathbb{R})$ is also a Palais–Smale sequence of $S_\omega$ considered on $H^1(\mathbb{R})$. In particular, a critical point of $S_\omega$ considered on $H^1_{\mathrm{rad}}(\mathbb{R})$ is also a critical point of $S_\omega$ considered on $H^1(\mathbb{R})$.*

**Lemma 2.6.** *Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$ be a bounded Palais–Smale sequence at the level $c$ for $S_\omega$. Then there exists a subsequence still denoted by $\{\varphi_n\}$ for which the following holds: there exist a critical point $\varphi_0$ of $S_\omega$, an integer $k \ge 0$, for $j = 1, \dots, k$, a sequence of points $\{x_n^j\} \subset \mathbb{R}$, and nontrivial solutions $v^j(x)$ of the equation (1-8) satisfying*

$$\varphi_n \rightharpoonup \varphi_0 \quad \text{weakly in } H^1(\mathbb{R}),$$
$$S_\omega(\varphi_n) \to c = S_\omega(\varphi_0) + \sum_{j=1}^{k} S_{\omega,0}(v^j),$$
$$\varphi_n - \Big(\varphi_0 + \sum_{j=1}^{k} v^j(x - x_n^j)\Big) \to 0 \quad \text{strongly in } H^1(\mathbb{R}),$$
$$|x_n^j| \to \infty, \quad |x_n^j - x_n^i| \to \infty \quad \text{for } 1 \le j \ne i \le k$$

*as $n \to \infty$, where we agree that in the case $k = 0$, the above holds without $v^j$ and $x_n^j$.*

**Lemma 2.7.** *Assume that*

$$r_\omega^{\alpha,\beta} < 2l_\omega^{\alpha,\beta}.$$

*Then the bounded Palais–Smale sequence at the level $r_\omega^{\alpha,\beta}$ admits a strongly convergent subsequence.*

**Lemma 2.8.** *If $\varphi \in H^1(\mathbb{R})\setminus\{0\}$ is a critical point of $S_\omega$, that is, $\varphi$ satisfies*

$$-\tfrac{1}{2}\partial_x^2\varphi + \omega\varphi - \gamma\delta_0\varphi = |\varphi|^{p-1}\varphi \tag{2-12}$$

*in the distribution sense, then it satisfies*

$$\begin{gathered}\varphi \in C^j(\mathbb{R}\setminus\{0\})\cap C(\mathbb{R}), \quad j=1,2,\\ -\tfrac{1}{2}\partial_x^2\varphi+\omega\varphi=|\varphi|^{p-1}\varphi, \quad x\neq 0,\\ \partial_x\varphi(0+)-\partial_x\varphi(0-)=-2\gamma\varphi(0),\\ \partial_x\varphi(x),\varphi(x)\to 0 \quad \textit{as } |x|\to\infty.\end{gathered}$$

**Lemma 2.9.** *There exists a unique positive classical solution $\varphi$ of (2-12) if and only if $\omega > \frac{1}{2}\gamma^2$. It is nothing but $Q_\omega$. If $0<\omega\leq\frac{1}{2}\gamma^2$, then the classical solution does not exist.*

*Proof.* We have a unique positive classical solution $Q_{\omega,0}$ of (1-8). If $\omega>\frac{1}{2}\gamma^2$, then we get a classical solution $\varphi$ of (2-12) by the translation of $Q_{\omega,0}$. See [Fukuizumi and Jeanjean 2008] for more detail. □

**Lemma 2.10.** *The inequality $r_\omega^{\alpha,\beta}<2l_\omega^{\alpha,\beta}$ holds when $\omega>\frac{1}{2}\gamma^2$.*

*Proof.* When $\omega>\frac{1}{2}\gamma^2$, we know $Q_\omega$ is well defined. We find that $Q_\omega$ satisfies $K_\omega^{\alpha,\beta}(Q_\omega)=0$ and $S_\omega(Q_\omega)<2l_\omega^{\alpha,\beta}$ by direct calculations. □

By Lemmas 2.7 and 2.10, we find that when $\omega>\frac{1}{2}\gamma^2$, the function $Q_\omega$ attains $r_\omega^{\alpha,\beta}$.

**Lemma 2.11.** *If $0<\omega\leq\frac{1}{2}\gamma^2$, then $r_\omega^{\alpha,\beta}=2l_\omega^{\alpha,\beta}$ holds.*

*Proof.* Suppose that $r_\omega^{\alpha,\beta}<2l_\omega^{\alpha,\beta}$. By Lemmas 2.7 and 2.8, we have a unique positive classical solution of (2-12), which contradicts Lemma 2.9. Thus, it suffices to show $r_\omega^{\alpha,\beta}\leq 2l_\omega^{\alpha,\beta}$ for all $\omega>0$. Let

$$\varphi_n(x):=Q_{\omega,0}(x-n)+Q_{\omega,0}(x+n).$$

Then, $S_\omega(\varphi_n)\to 2l_\omega$ and $K_\omega^{\alpha,\beta}(\varphi_n)\to 0$ as $n\to\infty$. Thus, there exists a sequence $\{\lambda_n\}$ such that $K_\omega^{\alpha,\beta}(\lambda_n\varphi_n)=0$ and $\lambda_n\to 1$ as $n\to\infty$. Therefore, we have $S_\omega(\lambda_n\varphi_n)\to 2l_\omega$ as $n\to\infty$ and $K_\omega^{\alpha,\beta}(\lambda_n\varphi_n)=0$ for all $n\in\mathbb{N}$. This means that $r_\omega^{\alpha,\beta}\leq 2l_\omega^{\alpha,\beta}$. □

**Remark 2.2.** The rearrangement argument implies

$$l_\omega^{\alpha,\beta}=\inf\{S_{\omega,0}(\varphi):\varphi\in H^1_{\mathrm{rad}}(\mathbb{R})\setminus\{0\},\ K_{\omega,0}^{\alpha,\beta}(\varphi)=0\}.$$

Therefore, the arguments in Section 2B do work for $l_\omega^{\alpha,\beta}$.

**2C. *Nonradial minimizing problem.*** In this subsection, we prove $n_\omega^{\alpha,\beta}=l_\omega^{\alpha,\beta}$ and $n_\omega^{\alpha,\beta}$ is not attained.

**Lemma 2.12.** *We have*

$$l_\omega^{\alpha,\beta}=j_\omega^{\alpha,\beta}:=\inf\{J_{\omega,0}^{\alpha,\beta}(\varphi):\varphi\in H^1(\mathbb{R})\setminus\{0\},\ K_{\omega,0}^{\alpha,\beta}(\varphi)\leq 0\}.$$

*Proof.* First, we prove $j_\omega^{\alpha,\beta}\leq l_\omega^{\alpha,\beta}$:

$$\begin{aligned} j_\omega^{\alpha,\beta} &\leq \inf\{J_{\omega,0}^{\alpha,\beta}(\varphi):\varphi\in H^1(\mathbb{R})\setminus\{0\},\ K_{\omega,0}^{\alpha,\beta}(\varphi)=0\}\\ &=\inf\{S_{\omega,0}(\varphi):\varphi\in H^1(\mathbb{R})\setminus\{0\},\ K_{\omega,0}^{\alpha,\beta}(\varphi)=0\}\\ &=l_\omega^{\alpha,\beta}.\end{aligned}$$

Next, we prove $l_\omega^{\alpha,\beta} \le j_\omega^{\alpha,\beta}$. We take $\varphi \in H^1(\mathbb{R})\setminus\{0\}$ such that $K_{\omega,0}^{\alpha,\beta}(\varphi) \le 0$. If $K_{\omega,0}^{\alpha,\beta}(\varphi) = 0$, then

$$l_\omega^{\alpha,\beta} \le S_{\omega,0}(\varphi) = J_{\omega,0}^{\alpha,\beta}(\varphi).$$

If $K_{\omega,0}^{\alpha,\beta}(\varphi) < 0$, then there exists $\lambda_* \in (0,1)$ such that $K_{\omega,0}^{\alpha,\beta}(\lambda_*\varphi) = 0$. Indeed, this follows from continuity and the fact that $K_{\omega,0}^{\alpha,\beta}(\lambda\varphi) > 0$ holds for small $\lambda \in (0,1)$ by Lemma 2.2. By $\lambda_* < 1$,

$$l_\omega^{\alpha,\beta} \le S_{\omega,0}(\lambda_*\varphi) = J_{\omega,0}^{\alpha,\beta}(\lambda_*\varphi) \le J_{\omega,0}^{\alpha,\beta}(\varphi).$$

Therefore, we have $l_\omega^{\alpha,\beta} \le J_{\omega,0}^{\alpha,\beta}(\varphi)$ for any $\varphi \in H^1(\mathbb{R})\setminus\{0\}$ such that $K_{\omega,0}^{\alpha,\beta}(\varphi) \le 0$. This implies $l_\omega^{\alpha,\beta} \le j_\omega^{\alpha,\beta}$. Hence, we get $l_\omega^{\alpha,\beta} = j_\omega^{\alpha,\beta}$. □

Let $\tau_y\varphi(x) := \varphi(x-y)$ throughout this paper.

**Proposition 2.13.** *The identity $n_\omega^{\alpha,\beta} = l_\omega^{\alpha,\beta}$ holds.*

*Proof.* First, we prove $n_\omega^{\alpha,\beta} \ge l_\omega^{\alpha,\beta}$. We take an arbitrary $\varphi \in H^1(\mathbb{R})\setminus\{0\}$ such that $K_\omega^{\alpha,\beta}(\varphi) = 0$. Since $K_{\omega,0}^{\alpha,\beta}(\varphi) \le K_\omega^{\alpha,\beta}(\varphi) = 0$ due to $\gamma \le 0$, by Lemma 2.12, we then have

$$l_\omega^{\alpha,\beta} \le J_{\omega,0}^{\alpha,\beta}(\varphi) \le J_\omega^{\alpha,\beta}(\varphi),$$

which implies

$$\begin{aligned} l_\omega^{\alpha,\beta} &\le \inf\{J_\omega^{\alpha,\beta}(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ K_\omega^{\alpha,\beta}(\varphi) = 0\} \\ &= \inf\{S_\omega(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ K_\omega^{\alpha,\beta}(\varphi) = 0\} = n_\omega^{\alpha,\beta}. \end{aligned}$$

Next, we prove $n_\omega^{\alpha,\beta} \le l_\omega^{\alpha,\beta}$. We note that $Q_{\omega,0}$ attains $l_\omega^{\alpha,\beta}$. Then, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ with $y_n \to \infty$ as $n \to \infty$ such that $S_\omega(\tau_{y_n}Q_{\omega,0}) \to S_{\omega,0}(Q_{\omega,0}) = l_\omega^{\alpha,\beta}$ as $n \to \infty$. For this $\{y_n\}$,

$$K_\omega^{\alpha,\beta}(\tau_{y_n}Q_{\omega,0}) \ge K_{\omega,0}^{\alpha,\beta}(\tau_{y_n}Q_{\omega,0}) = K_{\omega,0}^{\alpha,\beta}(Q_{\omega,0}) = 0$$

holds for all $n \in \mathbb{N}$. Since $K_\omega^{\alpha,\beta}(\lambda\tau_{y_n}Q_{\omega,0}) < 0$ for large $\lambda > 1$ and $K_\omega^{\alpha,\beta}(\tau_{y_n}Q_{\omega,0}) > 0$, there exists $\lambda_n > 1$ such that $K_\omega^{\alpha,\beta}(\lambda_n\tau_{y_n}Q_{\omega,0}) = 0$ by continuity. For this $\{\lambda_n\}$, we have $\lambda_n \to 1$ as $n \to \infty$. Indeed, since

$$\begin{aligned} 0 &= K_\omega^{\alpha,\beta}(\lambda_n\tau_{y_n}Q_{\omega,0}) \\ &= \lambda_n^2\big(\tfrac{1}{4}(2\alpha-\beta)\|\partial_x\tau_{y_n}Q_{\omega,0}\|_{L^2}^2 + \tfrac{1}{2}\omega(2\alpha+\beta)\|\tau_{y_n}Q_{\omega,0}\|_{L^2}^2 - \gamma\alpha|\tau_{y_n}Q_{\omega,0}(0)|^2\big) \\ &\qquad - \lambda_n^{p+1}\frac{(p+1)\alpha+\beta}{p+1}\|\tau_{y_n}Q_{\omega,0}\|_{L^{p+1}}^{p+1}, \end{aligned}$$

and $K_{\omega,0}^{\alpha,\beta}(\tau_{y_n}Q_{\omega,0}) = 0$, we have

$$\begin{aligned} 0 &= \tfrac{1}{4}(2\alpha-\beta)\|\partial_x\tau_{y_n}Q_{\omega,0}\|_{L^2}^2 + \tfrac{1}{2}\omega(2\alpha+\beta)\|\tau_{y_n}Q_{\omega,0}\|_{L^2}^2 - \gamma\alpha|\tau_{y_n}Q_{\omega,0}(0)|^2 \\ &\qquad - \lambda_n^{p-1}\frac{(p+1)\alpha+\beta}{p+1}\|\tau_{y_n}Q_{\omega,0}\|_{L^{p+1}}^{p+1} \\ &= (1-\lambda_n^{p-1})\frac{(p+1)\alpha+\beta}{p+1}\|\tau_{y_n}Q_{\omega,0}\|_{L^{p+1}}^{p+1} - \gamma\alpha|\tau_{y_n}Q_{\omega,0}(0)|^2 \\ &= (1-\lambda_n^{p-1})\frac{(p+1)\alpha+\beta}{p+1}\|Q_{\omega,0}\|_{L^{p+1}}^{p+1} - \gamma\alpha|\tau_{y_n}Q_{\omega,0}(0)|^2. \end{aligned}$$

Therefore, $\lambda_n \to 1$, since $|\tau_{y_n} Q_{\omega,0}(0)| \to 0$ as $n \to \infty$. Hence, $S_\omega(\lambda_n \tau_{y_n} Q_{\omega,0}) \to S_{\omega,0}(Q_{\omega,0}) = l_\omega^{\alpha,\beta}$ as $n \to \infty$ and $K_\omega^{\alpha,\beta}(\lambda_n \tau_{y_n} Q_{\omega,0}) = 0$ for all $n \in \mathbb{N}$. This implies $n_\omega^{\alpha,\beta} \le l_\omega^{\alpha,\beta}$. □

**Proposition 2.14.** *For any $\omega > 0$, the minimizing problem $n_\omega^{\alpha,\beta}$ is not attained; namely, there does not exist $\varphi \in H^1(\mathbb{R})$ such that $K_\omega^{\alpha,\beta}(\varphi) = 0$ and $S_\omega(\varphi) = n_\omega^{\alpha,\beta}$.*

*Proof.* We assume that $\varphi$ attains $n_\omega^{\alpha,\beta}$. If $\varphi(0) = 0$, then $S_{\omega,0}(\varphi) = S_\omega(\varphi) = n_\omega^{\alpha,\beta} = l_\omega^{\alpha,\beta}$ and $K_{\omega,0}^{\alpha,\beta}(\varphi) = K_\omega^{\alpha,\beta}(\varphi) = 0$ holds; that is, $\varphi$ also attains $l_\omega^{\alpha,\beta}$. By the uniqueness of the ground state for $l_\omega^{\alpha,\beta}$, we know $\varphi = Q_{\omega,0}$. However, $Q_{\omega,0}(0) \ne 0$. Therefore, $\varphi(0) \ne 0$. Now, $|\varphi(x)| \to 0$ as $x \to \infty$ since $\varphi \in H^1(\mathbb{R})$. Hence, $|\varphi(0)| > |\varphi(y)|$ for sufficiently large $|y|$. Thus,

$$K_\omega^{\alpha,\beta}(\tau_y \varphi) < K_\omega^{\alpha,\beta}(\varphi) = 0.$$

Since $K_\omega^{\alpha,\beta}(\lambda \tau_y \varphi) > 0$ for small $\lambda \in (0,1)$ by Lemma 2.2 and $K_\omega^{\alpha,\beta}(\tau_y \varphi) \le 0$, there exists $\lambda_* \in (0,1)$ such that $K_\omega^{\alpha,\beta}(\lambda_* \tau_y \varphi) = 0$ by continuity. By the definition of $n_\omega^{\alpha,\beta}$,

$$n_\omega^{\alpha,\beta} \le J_\omega^{\alpha,\beta}(\lambda_* \tau_y \varphi) < J_\omega^{\alpha,\beta}(\tau_y \varphi) < J_\omega^{\alpha,\beta}(\varphi) \le n_\omega^{\alpha,\beta}.$$

This is a contradiction. □

Since $S_{\omega,0}(Q_{\omega,0}) = l_\omega^{\alpha,\beta} = n_\omega^{\alpha,\beta}$ and $S_{\omega,\gamma}(Q_{\omega,\gamma}) = r_\omega^{\alpha,\beta}$ if $\omega > \frac{1}{2}\gamma^2$, and $2l_\omega^{\alpha,\beta} = r_\omega^{\alpha,\beta}$ if $\omega \le \frac{1}{2}\gamma^2$ hold, we find that $r_\omega^{\alpha,\beta}$, $l_\omega^{\alpha,\beta}$ and $n_\omega^{\alpha,\beta}$ are independent of $(\alpha,\beta)$ and so we denote $r_\omega^{\alpha,\beta}$, $l_\omega^{\alpha,\beta}$ and $n_\omega^{\alpha,\beta}$ by $r_\omega$, $l_\omega$ and $n_\omega$ respectively and obtain Proposition 1.2.

## 2D. *Variational structure.*

We define subsets $\mathcal{N}_\omega^{\alpha,\beta,\pm}$ and $\mathcal{R}_\omega^{\alpha,\beta,\pm}$ in $H^1(\mathbb{R})$ such that

$$\begin{aligned}
\mathcal{N}_\omega^{\alpha,\beta,+} &:= \{\varphi \in H^1(\mathbb{R}) : S_\omega(\varphi) < n_\omega,\ K_\omega^{\alpha,\beta}(\varphi) \ge 0\},\\
\mathcal{N}_\omega^{\alpha,\beta,-} &:= \{\varphi \in H^1(\mathbb{R}) : S_\omega(\varphi) < n_\omega,\ K_\omega^{\alpha,\beta}(\varphi) < 0\},\\
\mathcal{R}_\omega^{\alpha,\beta,+} &:= \{\varphi \in H^1_{\mathrm{rad}}(\mathbb{R}) : S_\omega(\varphi) < r_\omega,\ K_\omega^{\alpha,\beta}(\varphi) \ge 0\},\\
\mathcal{R}_\omega^{\alpha,\beta,-} &:= \{\varphi \in H^1_{\mathrm{rad}}(\mathbb{R}) : S_\omega(\varphi) < r_\omega,\ K_\omega^{\alpha,\beta}(\varphi) < 0\}.
\end{aligned}$$

We note that $\mathcal{N}_\omega^{\pm} = \mathcal{N}_\omega^{\frac{1}{2},-1,\pm}$ and $\mathcal{R}_\omega^{\pm} = \mathcal{R}_\omega^{\frac{1}{2},-1,\pm}$. From now on, let $(m_\omega, \mathcal{M}_\omega^{\alpha,\beta,\pm})$ denote either $(n_\omega, \mathcal{N}_\omega^{\alpha,\beta,\pm})$ or $(r_\omega, \mathcal{R}_\omega^{\alpha,\beta,\pm})$. The following proposition implies that $P$ and $I_\omega$ have same sign if $S_\omega < m_\omega$.

**Proposition 2.15.** *For any $(\alpha,\beta)$ satisfying (2-1), $\mathcal{M}_\omega^{\pm} = \mathcal{M}_\omega^{\alpha,\beta,\pm}$.*

*Proof.* It is easy to check that $\mathcal{M}_\omega^{\alpha,\beta,\pm}$ are open subsets in $H^1(\mathbb{R})$ because of Lemma 2.2. Moreover, we have $0 \in \mathcal{M}_\omega^{\alpha,\beta,+}$ and $\mathcal{M}_\omega^{\alpha,\beta,+} \cup \mathcal{M}_\omega^{\alpha,\beta,-}$ is independent of $(\alpha,\beta)$. And $\mathcal{M}_\omega^{\alpha,\beta,+}$ are connected if $\mu > 0$ by the scaling contraction argument (see the proof of Lemma 2.9 in [Ibrahim et al. 2011]). Then $\mathcal{M}_\omega^{\alpha,\beta,+} = \mathcal{M}_\omega^{\alpha',\beta',+}$ for $(\alpha,\beta) \ne (\alpha',\beta')$ such that $2\alpha-\beta > 0,\ 2\alpha+\beta > 0$ and $2\alpha'-\beta' > 0,\ 2\alpha'+\beta' > 0$. Of course, then $\mathcal{M}_\omega^{\alpha,\beta,-} = \mathcal{M}_\omega^{\alpha',\beta',-}$.

We take $\{(\alpha_n, \beta_n)\}$ satisfying $2\alpha_n - \beta_n > 0$ and $2\alpha_n + \beta_n > 0$ for all $n \in \mathbb{N}$ and $(\alpha_n, \beta_n)$ converges to some $(\alpha, \beta)$ such that $\underline{\mu} = 0$. Then $K_\omega^{\alpha_n,\beta_n} \to K_\omega^{\alpha,\beta}$, and so

$$\mathcal{M}_\omega^{\alpha,\beta,\pm} \subset \bigcup_{n\in\mathbb{N}} \mathcal{M}_\omega^{\alpha_n,\beta_n,\pm}.$$

Since each set in the right-hand side is independent of $(\alpha, \beta)$, so is the left. □

Let $\|\varphi\|_{\mathcal{H}}^2 := \frac{1}{4}\|\partial_x\varphi\|_{L^2}^2 + \frac{1}{2}\omega\|\varphi\|_{L^2}^2 - \frac{1}{2}\gamma|\varphi(0)|^2$.

**Lemma 2.16.** *If $P(\varphi) \geq 0$, then*

$$S_\omega(\varphi) \leq \|\varphi\|_{\mathcal{H}}^2 \leq \frac{p-1}{p-5} S_\omega(\varphi),$$

*which means that $S_\omega(\varphi)$ is equivalent to $\|\varphi\|_{H^1}^2$.*

*Proof.* The left inequality is trivial. We consider the right inequality:

$$\begin{aligned}
0 &\leq 2P(\varphi) \\
&\leq \|\partial_x\varphi\|_{L^2}^2 - \gamma|\varphi(0)|^2 - \frac{p-1}{p+1}\|\varphi\|_{L^{p+1}}^{p+1} \\
&= -\tfrac{1}{4}(p-5)\|\partial_x\varphi\|_{L^2}^2 + \tfrac{1}{2}(\gamma(p-3))|\varphi(0)|^2 + (p-1)E(\varphi) \\
&\leq -\tfrac{1}{4}(p-5)\|\partial_x\varphi\|_{L^2}^2 + \tfrac{1}{2}(\gamma(p-5))|\varphi(0)|^2 + (p-1)E(\varphi).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\tfrac{1}{4}(p-5)\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma(p-5)|\varphi(0)|^2 + \tfrac{1}{2}(p-5)\omega\|\varphi\|_{L^2}^2 &\leq (p-1)E(\varphi) + (p-5)\omega M(\varphi) \\
&\leq (p-1)(E(\varphi) + \omega M(\varphi)).
\end{aligned}$$

Hence, we obtain

$$\|\varphi\|_{\mathcal{H}}^2 \leq \frac{p-1}{p-5} S_\omega(\varphi).$$ □

**Lemma 2.17.** *If $u_0 \in \mathcal{M}_\omega^+$, then the corresponding solution $u$ stays in $\mathcal{M}_\omega^+$ for all $t \in (-T_-, T_+)$. Moreover, if $u_0 \in \mathcal{M}_\omega^-$, then the corresponding solution $u$ stays in $\mathcal{M}_\omega^-$ for all $t \in (-T_-, T_+)$.*

*Proof.* Let $u_0 \in \mathcal{M}_\omega^+$. Since the energy and the mass are conserved, $u(t) \in \mathcal{M}_\omega^+ \cup \mathcal{M}_\omega^-$ for all $t \in (-T_-, T_+)$. We assume that there exists $t_{**} > 0$ such that $u(t_{**}) \in \mathcal{M}_\omega^-$. By continuity, there exists $t_* \in (0, t_{**})$ such that $P(u(t_*)) = 0$ and $P(u(t)) < 0$ for $t \in (t_*, t_{**}]$. By the definition of $m_\omega$, if $u(t_*) \neq 0$, then

$$m_\omega > E(u_0) + \omega M(u_0) = E(u(t_*)) + \omega M(u(t_*)) \geq m_\omega.$$

This is a contradiction. Thus, $u(t_*) = 0$. By the uniqueness of solution, $u = 0$ for all time. This contradicts $u(t_{**}) \in \mathcal{M}_\omega^-$. By the same argument, the second statement can be proved. □

Lemmas 2.16 and 2.17 imply that all the solutions in $\mathcal{M}_\omega^+$ are global in both time directions.

**Proposition 2.18** (uniform bounds on $P$). *There exists $\delta > 0$ such that for any $\varphi \in H^1(\mathbb{R})$ with $S_\omega(\varphi) < m_\omega$, we have*

$$P(\varphi) \geq \min\{2(m_\omega - S_\omega(\varphi), \delta\|\varphi\|_{\mathcal{H}}^2\} \quad \text{or} \quad P(\varphi) \leq -2(m_\omega - S_\omega(\varphi)).$$

*Proof.* We may assume $\varphi \neq 0$. Now $s(\lambda) := S_\omega(\varphi^\lambda)$ and $n(\lambda) := \|\varphi^\lambda\|_{L^{p+1}}^{p+1}$, where $\varphi^\lambda(x) := e^{\frac{1}{2}\lambda}\varphi(e^\lambda x)$ for $\lambda \in \mathbb{R}$. Then $s(0) = S_\omega(\varphi)$ and $s'(0) = P(\varphi)$, and we have

$$s(\lambda) = \tfrac{1}{4}e^{2\lambda}\|\partial_x\varphi\|_{L^2}^2 + \tfrac{1}{2}\omega\|\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma e^\lambda|\varphi(0)|^2 - \frac{e^{\frac{p-1}{2}\lambda}}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}, \quad n(\lambda) = e^{\frac{p-1}{2}\lambda}\|\varphi\|_{L^{p+1}}^{p+1},$$

$$s'(\lambda) = \tfrac{1}{2}e^{2\lambda}\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma e^\lambda|\varphi(0)|^2 - \frac{e^{\frac{p-1}{2}\lambda}(p-1)}{2(p+1)}\|\varphi\|_{L^{p+1}}^{p+1}, \qquad n'(\lambda) = \tfrac{1}{2}e^{\frac{p-1}{2}\lambda}(p-1)\|\varphi\|_{L^{p+1}}^{p+1},$$

$$s''(\lambda) = e^{2\lambda}\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma e^\lambda|\varphi(0)|^2 - \frac{e^{\frac{p-1}{2}\lambda}(p-1)^2}{4(p+1)}\|\varphi\|_{L^{p+1}}^{p+1}, \qquad n''(\lambda) = \tfrac{1}{4}e^{\frac{p-1}{2}\lambda}(p-1)^2\|\varphi\|_{L^{p+1}}^{p+1}.$$

By an easy calculation, we have

$$s'' = 2s' + \tfrac{1}{2}\gamma|\varphi(0)|^2 - \frac{p-5}{2(p+1)}n' \le 2s' - \frac{p-5}{2(p+1)}n' \le 2s'.$$

First, we consider $P < 0$. We have $s'(\lambda) > 0$ for sufficiently small $\lambda < 0$. Therefore, by continuity, there exists $\lambda_0 < 0$ such that $s'(\lambda) < 0$ for $\lambda_0 < \lambda \le 0$ and $s'(\lambda_0) = 0$. Integrating the inequality on $[\lambda_0, 0]$, we have

$$s'(0) - s'(\lambda_0) \le 2(s(0) - s(\lambda_0)).$$

Therefore, we obtain

$$P(\varphi) \le -2(m_\omega - S_\omega(\varphi)).$$

Next, we consider $P \ge 0$. If

$$4P(\varphi) \ge \frac{p-5}{2(p+1)}\mathcal{L}^{\frac{1}{2},-1}\|\varphi\|_{L^{p+1}}^{p+1},$$

then, by adding

$$\frac{p-5}{2}P(\varphi) \ge \frac{p-5}{2}\|\varphi\|_{\mathcal{H}}^2 - \frac{p-5}{2(p+1)}\mathcal{L}^{\frac{1}{2},-1}\|\varphi\|_{L^{p+1}}^{p+1}$$

to both sides, we get

$$\left\{4 + \tfrac{1}{2}(p-5)\right\}P(\varphi) \ge \tfrac{1}{2}(p-5)\|\varphi\|_{\mathcal{H}}^2.$$

Thus, we get $P(\varphi) \ge \delta\|\varphi\|_{\mathcal{H}}^2$. If

$$4P(\varphi) < \frac{p-5}{2(p+1)}\mathcal{L}^{\frac{1}{2},-1}\|\varphi\|_{L^{p+1}}^{p+1},$$

then

$$0 < 4s' < \frac{p-5}{2(p+1)}n' \tag{2-13}$$

at $\lambda = 0$. Moreover,

$$s'' \le 4s' - 2s' - \frac{p-5}{2(p+1)}n' < -2s'$$

holds at $\lambda = 0$. Now let $\lambda$ increase. As long as (2-13) holds and $s' > 0$, we have $s'' < 0$ and so $s'$ decreases and $s$ increases. Since $p > 5$, we also have

$$n'' \geq 2n' \geq 4n > 0$$

for all $\lambda \geq 0$ Hence, (2-13) is preserved until $s'$ reaches 0, which it does at finite $\lambda_1 > 0$. Integrating $s'' < -2s'$ on $[0, \lambda_1]$, we obtain

$$s'(\lambda_1) - s'(0) < -2(s(\lambda_1) - s(0)).$$

Therefore, by the definition of $m_\omega$,

$$P(\varphi) > 2(m_\omega - S_\omega(\varphi)).$$

□

## 3. Proof of the scattering part

**3A.** ***Strichartz estimates and small data scattering.*** We recall the Strichartz estimates and a small data scattering result in this subsection. See [Banica and Visciglia 2016, Sections 3.1 and 3.2] for the proofs. We define the exponents $r$, $a$, and $b$ as

$$r = p+1, \quad a := \frac{2(p-1)(p+1)}{p+3}, \quad b := \frac{2(p-1)(p+1)}{(p-1)^2 - (p-1) - 4}.$$

Then we have the following estimates.

**Lemma 3.1** (Strichartz estimates). *We have*

$$\|e^{-itH_\gamma}\varphi\|_{L_t^a L_x^r} \lesssim \|\varphi\|_{H^1},$$

$$\|e^{-itH_\gamma}\varphi\|_{L_t^{p-1} L_x^\infty} \lesssim \|\varphi\|_{H^1},$$

$$\left\|\int_0^t e^{-i(t-s)H_\gamma} F(s)\,ds\right\|_{L_t^a L_x^r} \lesssim \|F\|_{L_t^{b'} L_x^{r'}},$$

$$\left\|\int_0^t e^{-i(t-s)H_\gamma} F(s)\,ds\right\|_{L_t^{p-1} L_x^\infty} \lesssim \|F\|_{L_t^{b'} L_x^{r'}},$$

*where $b'$ denotes the Hölder conjugate of $b$, namely, $1/b' + 1/b = 1$.*

**Proposition 3.2.** *Let the solution $u \in C(\mathbb{R} : H^1(\mathbb{R}))$ to ($\delta$NLS) satisfy $u \in L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. Then the solution $u$ scatters.*

For the proof of Proposition 3.2, see [Banica and Visciglia 2016, Proposition 3.1].

The analogous statement to Proposition 3.2 for the following semilinear Schrödinger equation without potentials is well known:

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \tag{NLS}$$

where $p > 5$.

**Proposition 3.3** (small data scattering). *Let $\varphi \in H^1(\mathbb{R})$ and $u, v$ denote the solutions to ($\delta$NLS), (NLS), respectively, with the initial data $\varphi$. Then, there exist $\varepsilon > 0$ and $C > 0$ independent of $\varepsilon$ such that $u$ and $v$ are global and they satisfy $\|u\|_{L_t^q L_x^r(\mathbb{R})} < C\|\varphi\|_{H^1}$ and $\|v\|_{L_t^q L_x^r(\mathbb{R})} < C\|\varphi\|_{H^1}$, if $\|\varphi\|_{H^1} < \varepsilon$.*

For the proof of Proposition 3.3, see [Banica and Visciglia 2016, Proposition 3.2].

**3B. *Linear profile decomposition and its radial version.*** To prove the scattering results, we introduce the linear profile decomposition theorems. The linear profile decomposition for nonradial data, Proposition 3.4, is obtained in [Banica and Visciglia 2016].

**Proposition 3.4** (linear profile decomposition). *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R})$. Then, up to subsequence, we can write*

$$\varphi_n = \sum_{j=1}^{J} e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j + W_n^J \quad \forall J \in \mathbb{N},$$

*where $t_n^j \in \mathbb{R}$, $x_n^j \in \mathbb{R}$, $\psi^j \in H^1(\mathbb{R})$, and the following hold:*

- *For any fixed $j$, we have*

$$\begin{array}{llll} \text{either} & t_n^j = 0 \ \text{ for any } n \in \mathbb{N}, & \text{or} & t_n^j \to \pm\infty \ \text{ as } n \to \infty, \\ \text{either} & x_n^j = 0 \ \text{ for any } n \in \mathbb{N}, & \text{or} & x_n^j \to \pm\infty \ \text{ as } n \to \infty. \end{array}$$

- *Orthogonality of the parameters:*

$$|t_n^j - t_n^k| + |x_n^j - x_n^k| \to \infty \quad \text{as } n \to \infty, \ \forall j \neq k.$$

- *Smallness of the reminder:*

$$\forall \varepsilon > 0, \exists J = J(\varepsilon) \in \mathbb{N} \quad \text{such that} \quad \limsup_{n\to\infty} \|e^{-itH_\gamma} W_n^J\|_{L_t^\infty L_x^\infty} < \varepsilon.$$

- *Orthogonality in norms: for any $J \in \mathbb{N}$,*

$$\|\varphi_n\|_{L^2}^2 = \sum_{j=1}^{J} \|\psi^j\|_{L^2}^2 + \|W_n^J\|_{L^2}^2 + o_n(1), \quad \|\varphi_n\|_H^2 = \sum_{j=1}^{J} \|\tau_{x_n^j}\psi^j\|_H^2 + \|W_n^J\|_H^2 + o_n(1),$$

*where $\|v\|_H^2 := \frac{1}{2}\|\partial_x v\|_{L^2}^2 - \gamma|v(0)|^2$. Moreover, we have*

$$\|\varphi_n\|_{L^q}^q = \sum_{j=1}^{J} \|e^{it_n^j H_\gamma} \tau_{x_n^j}\psi^j\|_{L^q}^q + \|W_n^J\|_{L^q}^q + o_n(1), \quad q \in (2,\infty), \ \forall J \in \mathbb{N},$$

*and in particular, for any $J \in \mathbb{N}$,*

$$S_\omega(\varphi_n) = \sum_{j=1}^{J} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j}\psi^j) + S_\omega(W_n^J) + o_n(1),$$

$$I_\omega(\varphi_n) = \sum_{j=1}^{J} I_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j}\psi^j) + I_\omega(W_n^J) + o_n(1).$$

*Proof.* See [Banica and Visciglia 2016, Theorem 2.1 and Section 2.2]. □

**Remark 3.1.** It is not clear whether

$$P(\varphi_n) = \sum_{j=1}^{J} P(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + P(W_n^J) + o_n(1) \quad \forall J \in \mathbb{N}$$

holds or not. That is why we use the Nehari functional $I_\omega$ to prove the scattering results.

We introduce the reflection operator $\mathcal{R}$ such that $\mathcal{R}\varphi(x) := \varphi(-x)$.

Proposition 3.4 is insufficient to prove the scattering result for radial data. We need the following linear profile decomposition for radial solutions, which is a key ingredient.

**Theorem 3.5** (linear profile decomposition for radial data). *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1_{\text{rad}}(\mathbb{R})$. Then, up to subsequence, we can write*

$$\varphi_n = \frac{1}{2}\sum_{j=1}^{J}\bigl(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + e^{it_n^j H_\gamma}\tau_{-x_n^j}\mathcal{R}\psi^j\bigr) + \frac{1}{2}(W_n^J + \mathcal{R}W_n^J) \quad \forall J \in \mathbb{N}, \tag{3-1}$$

*where $t_n^j \in \mathbb{R}$, $x_n^j \in \mathbb{R}$, $\psi^j \in H^1(\mathbb{R})$, and the following hold*:

- *For any fixed $j$, we have*

$$\begin{array}{llllll} \textit{either} & t_n^j = 0 & \textit{for any } n \in \mathbb{N}, & \textit{or} & t_n^j \to \pm\infty & \textit{as } n \to \infty, \\ \textit{either} & x_n^j = 0 & \textit{for any } n \in \mathbb{N}, & \textit{or} & x_n^j \to \pm\infty & \textit{as } n \to \infty. \end{array}$$

- *Orthogonality of the parameters*:

$$|t_n^j - t_n^k| \to \infty, \quad \textit{or} \quad |x_n^j - x_n^k| \to \infty \;\; \textit{and} \;\; |x_n^j + x_n^k| \to \infty \quad \textit{as } n \to \infty, \; \forall j \neq k.$$

- *Smallness of the reminder*:

$$\forall \varepsilon > 0, \quad \exists J = J(\varepsilon) \in \mathbb{N} \quad \textit{such that} \quad \limsup_{n\to\infty} \|e^{-itH_\gamma} W_n^J\|_{L_t^\infty L_x^\infty} < \varepsilon.$$

- *Orthogonality in norms*: *for any $J \in \mathbb{N}$*,

$$\|\varphi_n\|_{L^2}^2 = \sum_{j=1}^{J} \bigl\|\tfrac{1}{2}(\tau_{x_n^j}\psi^j + \tau_{-x_n^j}\mathcal{R}\psi^j)\bigr\|_{L^2}^2 + \bigl\|\tfrac{1}{2}(W_n^J + \mathcal{R}W_n^J)\bigr\|_{L^2}^2 + o_n(1),$$

$$\|\varphi_n\|_{H}^2 = \sum_{j=1}^{J} \bigl\|\tfrac{1}{2}(\tau_{x_n^j}\psi^j + \tau_{-x_n^j}\mathcal{R}\psi^j)\bigr\|_{H}^2 + \bigl\|\tfrac{1}{2}(W_n^J + \mathcal{R}W_n^J)\bigr\|_{H}^2 + o_n(1).$$

*Moreover, for any $q \in (2,\infty)$, we have*

$$\|\varphi_n\|_{L^q}^q = \sum_{j=1}^{J} \bigl\|\tfrac{1}{2}e^{it_n^j H_\gamma}(\tau_{x_n^j}\psi^j + \tau_{-x_n^j}\mathcal{R}\psi^j)\bigr\|_{L^q}^q + \bigl\|\tfrac{1}{2}(W_n^J + \mathcal{R}W_n^J)\bigr\|_{L^q}^q + o_n(1) \quad \forall J \in \mathbb{N},$$

*and in particular, for any* $J \in \mathbb{N}$,

$$S_\omega(\varphi_n) = \sum_{j=1}^{J} S_\omega\big(\tfrac{1}{2}e^{it_n^j H_\gamma}(\tau_{x_n^j}\psi^j + \tau_{-x_n^j}\mathcal{R}\psi^j)\big) + S_\omega\big(\tfrac{1}{2}(W_n^J + \mathcal{R}W_n^J)\big) + o_n(1),$$

$$I_\omega(\varphi_n) = \sum_{j=1}^{J} I_\omega\big(\tfrac{1}{2}e^{it_n^j H_\gamma}(\tau_{x_n^j}\psi^j + \tau_{-x_n^j}\mathcal{R}\psi^j)\big) + I_\omega\big(\tfrac{1}{2}(W_n^J + \mathcal{R}W_n^J)\big) + o_n(1).$$

*Proof.* Since $\{\varphi_n\}$ is bounded in $H^1(\mathbb{R})$, we can apply the linear profile decomposition without the radial assumption, Proposition 3.4, and obtain the following: for any $J \in \mathbb{N}$ and $j \in \{1, 2, \ldots, J\}$, up to subsequence, there exist $\{t_n^j\}_{n\in\mathbb{N}}$, $\{x_n^j\}_{n\in\mathbb{N}}$, and $\psi^j \in H^1(\mathbb{R})$ such that we can write

$$\varphi_n = \sum_{j=1}^{J} e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + W_n^J.$$

Since $\varphi_n$ is radial,

$$2\varphi_n(x) = \varphi_n(x) + \varphi_n(x) = \varphi_n(x) + \varphi_n(-x) = \varphi_n(x) + \mathcal{R}\varphi_n(x).$$

By combining the identities, we get

$$\begin{aligned} 2\varphi_n(x) &= \sum_{j=1}^{J} e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + W_n^J + \mathcal{R}\bigg(\sum_{j=1}^{J} e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + W_n^J\bigg) \\ &= \sum_{j=1}^{J}\big(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + e^{it_n^j H_\gamma}\tau_{-x_n^j}\mathcal{R}\psi^j\big) + W_n^J + \mathcal{R}W_n^J, \end{aligned}$$

where we have used $\mathcal{R}e^{it_n^j H_\gamma} = e^{it_n^j H_\gamma}\mathcal{R}$ and $\mathcal{R}\tau_y = \tau_{-y}\mathcal{R}$, which gives (3-1).

We only prove the orthogonality of the parameters. If $x_n^j + x_n^k \to \bar{x} \in \mathbb{R}$ and $t_n^j = t_n^k$ for $j < k$, then we replace $\psi^j + \tau_{-\bar{x}}\mathcal{R}\psi^k$ by $\psi^j$ and 0 by $\psi^k$ and regard the remainder terms as $W_n^J$. By this replacement, we have $|x_n^j - x_n^k| \to \infty$ and $|x_n^j + x_n^k| \to \infty$ as $n \to \infty$ when $t_n^j = t_n^k$. The orthogonality in norms follows from the orthogonality of the parameters by a standard argument. □

**Lemma 3.6.** *Let $k$ be a nonnegative integer and, for $l \in \{0, 1, 2, \ldots, k\}$, we have $\varphi_l \in H^1(\mathbb{R})$ (or $\varphi_l \in H^1_{\mathrm{rad}}(\mathbb{R})$) satisfying*

$$S_\omega\bigg(\sum_{l=0}^{k}\varphi_l\bigg) \le m_\omega - \delta, \quad S_\omega\bigg(\sum_{l=0}^{k}\varphi_l\bigg) \ge \sum_{l=0}^{k} S_\omega(\varphi_l) - \varepsilon,$$

$$I_\omega\bigg(\sum_{l=0}^{k}\varphi_l\bigg) \ge -\varepsilon, \qquad I_\omega\bigg(\sum_{l=0}^{k}\varphi_l\bigg) \le \sum_{l=0}^{k} I_\omega(\varphi_l) + \varepsilon$$

*for $\delta, \varepsilon$ satisfying $2\varepsilon < \delta$. Then $\varphi_l \in \mathcal{M}_\omega^+$ for all $l \in \{0, 1, 2, \ldots, k\}$. Namely, we have $0 \le S_\omega(\varphi_l) < m_\omega$ and $I_\omega(\varphi_l) \ge 0$ for all $l \in \{0, 1, 2, \ldots, k\}$.*

*Proof.* We assume there exists an $l \in \{0,1,2,\dots,k\}$ such that $I_\omega(\varphi_l) < 0$. Then, we have $J_\omega^{1,0}(\varphi_l) \geq m_\omega$. Indeed, there exists $\lambda_* \in (0,1)$ such that $I_\omega(\lambda_*\varphi_l) = 0$ since $I_\omega(\varphi_l) < 0$ and $I_\omega(\lambda\varphi_l) > 0$ for small $\lambda \in (0,1)$ by Lemma 2.2. Thus, we obtain

$$m_\omega \leq S_\omega(\lambda_*\varphi_l) = J_\omega^{1,0}(\lambda_*\varphi_l) \leq J_\omega^{1,0}(\varphi_l).$$

By the positivity of $J_\omega = J_\omega^{1,0}$ and the assumptions, we obtain

$$\begin{aligned} m_\omega \leq J_\omega(\varphi_l) \leq \sum_{l=0}^{k} J_\omega(\varphi_l) &= \sum_{l=0}^{k}\Big(S_\omega(\varphi_l) - \frac{1}{2}I_\omega(\varphi_l)\Big) \\ &= \sum_{l=0}^{k} S_\omega(\varphi_l) - \frac{1}{2}\sum_{l=0}^{k} I_\omega(\varphi_l) \\ &\leq S_\omega\Big(\sum_{l=0}^{k}\varphi_l\Big) + \varepsilon - \frac{1}{2}\Big(I_\omega\Big(\sum_{l=0}^{k}\varphi_l\Big) - \varepsilon\Big) \leq m_\omega - \delta + \varepsilon + \varepsilon < m_\omega. \end{aligned}$$

This is a contradiction. So, $I_\omega(\varphi_l) \geq 0$ for all $l \in \{0,1,2,\dots,k\}$. Moreover, for any $l \in \{0,1,2,\dots,k\}$, we have

$$S_\omega(\varphi_l) = J_\omega(\varphi_l) + \tfrac{1}{2}I_\omega(\varphi_l) \geq 0,$$

and

$$S_\omega(\varphi_l) \leq \sum_{l=0}^{k} S_\omega(\varphi_l) \leq S_\omega\Big(\sum_{l=0}^{k}\varphi_l\Big) + \varepsilon \leq m_\omega - \delta + \varepsilon < m_\omega.$$

Therefore, we get $\varphi_l \in \mathcal{M}_\omega^+$ for all $l \in \{0,1,2,\dots,k\}$. □

**3C.** ***Perturbation lemma and nonlinear profile decomposition.*** We use a perturbation lemma and lemmas for nonlinear profiles. The proofs of these results are the same as in the defocusing case (see [Banica and Visciglia 2016]).

**Lemma 3.7.** *For any $M > 0$, there exist $\varepsilon = \varepsilon(M) > 0$ and $C = C(M) > 0$ such that the following occurs. Let $v \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$ be a solution of the integral equation with source term $e$:*

$$v(t) = e^{-itH_\gamma}\varphi + i\int_0^t e^{-i(t-s)H_\gamma}\big(|v(s)|^{p-1}v(s)\big)\,ds + e(t)$$

*with $\|v\|_{L_t^aL_x^r} < M$ and $\|e\|_{L_t^aL_x^r} < \varepsilon$. Moreover assume $\varphi_0 \in H^1(\mathbb{R})$ is such that $\|e^{-itH_\gamma}\varphi_0\|_{L_t^aL_x^r} < \varepsilon$. Then the solution $u(t,x)$ to ($\delta$NLS) with initial condition $\varphi + \varphi_0$,*

$$u(t) = e^{-itH_\gamma}(\varphi + \varphi_0) + i\int_0^t e^{-i(t-s)H_\gamma}\big(|u(s)|^{p-1}u(s)\big)\,ds,$$

*satisfies $u \in L_t^aL_x^r$ and moreover $\|u - v\|_{L_t^aL_x^r} < C\varepsilon$.*

See [Fang et al. 2011, Proposition 4.7] and [Banica and Visciglia 2016, Proposition 3.3] for the proof.

Following Lemmas 3.8, 3.9, and 3.10 can be proved in the same manner as [Banica and Visciglia 2016, Propositions 3.4, 3.5, and 3.6], respectively.

**Lemma 3.8.** *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $|x_n| \to \infty$ as $n \to \infty$, $u_0 \in H^1(\mathbb{R})$ and $U \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$ be a solution of (NLS) with the initial data $u_0$. Then we have*

$$U_n(t) = e^{-itH_\gamma}\tau_{x_n}u_0 + i\int_0^t e^{-i(t-s)H_\gamma}\big(|U_n(s)|^{p-1}U_n(s)\big)\,ds + g_n(t),$$

*where $U_n(t,x) = U(t, x - x_n)$ and $\|g_n\|_{L_t^a L_x^r} \to 0$ as $n \to \infty$.*

**Lemma 3.9.** *Let $\varphi \in H^1(\mathbb{R})$. Then there exist solutions $W_\pm \in C(\mathbb{R}_\pm : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_\pm : L_x^r(\mathbb{R}))$ to ($\delta$NLS) such that*

$$\|W_\pm(t,\cdot) - e^{-itH_\gamma}\varphi\|_{H^1} \to 0 \quad \text{as } t \to \pm\infty.$$

*Moreover, if $\{t_n\}_{n\in\mathbb{N}}$ is such that $t_n \to \mp\infty$ as $n \to \infty$ and $W_\pm$ is global, then*

$$W_{\pm,n}(t) = e^{-itH_\gamma}\varphi_n + i\int_0^t e^{-i(t-s)H_\gamma}\big(|W_{\pm,n}(s)|^{p-1}W_{\pm,n}(s)\big)\,ds + f_{\pm,n}(t),$$

*where $\varphi_n = e^{it_nH_\gamma}\varphi$, $W_{\pm,n}(t,x) = W_\pm(t - t_n, x)$, $\|f_{\pm,n}\|_{L_t^a L_x^r} \to 0$ as $n \to \infty$, and the double-sign corresponds.*

**Lemma 3.10.** *Let $\{t_n\}_{n\in\mathbb{N}}$, $\{x_n\}_{n\in\mathbb{N}}$ be sequences of real numbers such that $t_n \to \mp\infty$ and $|x_n| \to \infty$ as $n \to \infty$, $\varphi \in H^1(\mathbb{R})$ and $V_\pm \in C(\mathbb{R}_\pm : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_\pm : L_x^r(\mathbb{R}))$ be solutions of (NLS) such that*

$$\|V_\pm(t,\cdot) - e^{-itH_0}\varphi\|_{H^1} \to 0 \quad \text{as } t \to \pm\infty.$$

*Then we have*

$$V_{\pm,n}(t,x) = e^{-itH_\gamma}\varphi_n + i\int_0^t e^{-i(t-s)H_\gamma}\big(|V_{\pm,n}(s)|^{p-1}V_{\pm,n}(s)\big)\,ds + e_{\pm,n}(t,x),$$

*where $\varphi_n = e^{it_nH_\gamma}\tau_{x_n}\varphi$, $V_{\pm,n}(t,x) = V_\pm(t - t_n, x - x_n)$, $\|e_{\pm,n}\|_{L_t^a L_x^r} \to 0$ as $n \to \infty$, and the double-sign corresponds.*

## 3D. *Construction of a critical element.*

We define the critical action level $S_\omega^c$ for fixed $\omega$ as

$$S_\omega^c := \sup\{S : S_\omega(\varphi) < S \text{ for any } \varphi \in \mathcal{M}_\omega^+ \text{ implies } u \in L_t^a L_x^r\}.$$

By the small data scattering result Proposition 3.3, we obtain $S_\omega^c > 0$. We prove $S_\omega^c = m_\omega$ by contradiction.

We assume $S_\omega^c < m_\omega$. By this assumption, we can take a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_\omega^+$ such that $S_\omega(\varphi_n) \to S_\omega^c$ as $n \to \infty$, and $\|u_n\|_{L_t^a L_x^r(\mathbb{R})} = \infty$ for all $n \in \mathbb{N}$, where $u_n$ is a global solution to ($\delta$NLS) with the initial data $\varphi_n$. Then, we obtain the following lemma.

**Lemma 3.11** (critical element). *We assume $S_\omega^c < m_\omega$. Then we find a global solution $u^c \in C(\mathbb{R} : H^1(\mathbb{R}))$ of ($\delta$NLS) which satisfies $u^c(t) \in \mathcal{M}_\omega^+$ for any $t \in \mathbb{R}$ and*

$$S_\omega(u^c) = S_\omega^c, \quad \|u^c\|_{L_t^a L_x^r(\mathbb{R})} = \infty.$$

This $u^c$ is called a critical element.

*Proof.* First, we consider the nonradial case.

**Case 1:** nonradial data. By $\varphi_n \in \mathcal{N}_\omega^+$ and Lemma 2.16, we have

$$\|\varphi_n\|_{H^1}^2 \lesssim \|\varphi_n\|_{\mathcal{H}}^2 \lesssim E(\varphi_n) + \omega M(\varphi_n) < n_\omega$$

for all $n \in \mathbb{N}$. Since $\{\varphi_n\}$ is a bounded sequence in $H^1(\mathbb{R})$, we apply the linear profile decomposition, Proposition 3.4, and then obtain

$$\varphi_n = \sum_{j=1}^{J} e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j + W_n^J \quad \forall J \in \mathbb{N}.$$

By the orthogonality of the functionals in Proposition 3.4, we have

$$S_\omega(\varphi_n) = \sum_{j=1}^{J} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + S_\omega(W_n^J) + o_n(1),$$

$$I_\omega(\varphi_n) = \sum_{j=1}^{J} I_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + I_\omega(W_n^J) + o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$.

By these decompositions and $S_\omega(\varphi_n) < n_\omega$, we can find $\delta, \varepsilon > 0$ satisfying $2\varepsilon < \delta$ and

$$S_\omega(\varphi_n) \leq n_\omega - \delta, \quad S_\omega(\varphi_n) \geq \sum_{j=0}^{J} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + S_\omega(W_n^J) - \varepsilon,$$

$$I_\omega(\varphi_n) \geq -\varepsilon, \qquad I_\omega(\varphi_n) \leq \sum_{j=0}^{J} I_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + I_\omega(W_n^J) + \varepsilon$$

for large $n$. Therefore, by Lemma 3.6, we see that

$$e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j \in \mathcal{N}_\omega^+ \quad \text{and} \quad W_n^J \in \mathcal{N}_\omega^+ \quad \text{for large } n,$$

which means that, by Lemma 2.16,

$$S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) \geq 0 \quad \text{and} \quad S_\omega(W_n^J) \geq 0 \quad \text{for large } n.$$

So, we have

$$S_\omega^c = \limsup_{n\to\infty} S_\omega(\varphi_n) \geq \limsup_{n\to\infty} \sum_{j=1}^{J} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j)$$

for any $J$. We prove $S_\omega^c = \limsup_{n\to\infty} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j)$ for some $j$. We may assume $j = 1$ by reordering. If this is proved, then we find that $J = 1$ and $W_n^J \to 0$ in $L_t^\infty H_x^1$ as $n \to \infty$. Indeed, $\limsup_{n\to\infty} S_\omega(W_n^1) = 0$ holds and thus $\limsup_{n\to\infty} \|W_n^1\|_{H^1} = 0$ holds by $\|W_n^1\|_{H^1} \approx S_\omega(W_n^1)$ since $W_n^1$ belongs to $\mathcal{N}_\omega^+$ for large $n \in \mathbb{N}$. On the contrary, we assume $S_\omega^c = \limsup_{n\to\infty} S_\omega(e^{-it_n^j H_\gamma} \tau_{x_n^j} \psi^j)$

fails for all $j$. Then, for all $j$, there exists $\delta = \delta_j > 0$ such that

$$\limsup_{n\to\infty} S_\omega(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j) < S_\omega^c - \delta.$$

By reordering, we can choose $0 \le J_1 \le J_2 \le J_3 \le J_4 \le J_5 \le J$ such that

$$\begin{aligned}
1 \le j \le J_1: &\quad t_n^j = 0 \quad \forall n, & x_n^j = 0 \quad \forall n,\\
J_1+1 \le j \le J_2: &\quad t_n^j = 0 \quad \forall n, & \lim_{n\to\infty}|x_n^j| = \infty,\\
J_2+1 \le j \le J_3: &\quad \lim_{n\to\infty} t_n^j = +\infty, & x_n^j = 0 \quad \forall n,\\
J_3+1 \le j \le J_4: &\quad \lim_{n\to\infty} t_n^j = -\infty, & x_n^j = 0 \quad \forall n,\\
J_4+1 \le j \le J_5: &\quad \lim_{n\to\infty} t_n^j = +\infty, & \lim_{n\to\infty}|x_n^j| = \infty,\\
J_5+1 \le j \le J: &\quad \lim_{n\to\infty} t_n^j = -\infty, & \lim_{n\to\infty}|x_n^j| = \infty.
\end{aligned}$$

Above we are assuming that if $a > b$, then there is no $j$ such that $a \le j \le b$. Notice that $J_1 \in \{0, 1\}$ by the orthogonality of the parameters. We may treat only the case $J_1 = 1$ here. The case $J_1 = 0$ is easier. We have $0 < S_\omega(\psi^1) < S_\omega^c - \delta$ by $(t_n^j, x_n^j) = (0, 0)$ and the assumption. Hence, by the definition of $S_\omega^c$, we can find $N \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$ such that

$$N(t,x) = e^{-itH_\gamma}\psi^1 + i\int_0^t e^{-i(t-s)H_\gamma}(|N(s)|^{p-1}N(s))\,ds.$$

For every $j$ such that $J_1+1 \le j \le J_2$, let $U^j$ be the solution of (NLS) with the initial data $\psi^j$. Since we have $\tau_{x_n^j}\psi^j \in \mathcal{N}_\omega^+$, we know $\psi^j$ satisfies

$$S_{\omega,0}(\psi^j) \le S_\omega(\tau_{x_n^j}\psi^j) \le S_\omega^c < n_\omega = l_\omega$$

and $P_0(\psi^j) \ge 0$. (since $0 > P_0(\psi^j) = \lim_{n\to\infty} P(\tau_{x_n^j}\psi^j) \ge 0$ if we assume $P_0(\psi^j) < 0$.) Therefore, we see that the solution $U^j$ scatters by [Fang et al. 2011; Akahori and Nawa 2013]; that is, $U^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $U_n^j(t,x) := U^j(t, x - x_n^j)$.

For every $j$ such that $J_2+1 \le j \le J_3$, we associate with profile $\psi^j$ the function

$$W_-^j(t,x) \in C(\mathbb{R}_- : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_- : L_x^r(\mathbb{R}))$$

by Lemma 3.9. We claim that $W_-^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. Indeed, by the assumption, we see that $S_\omega(W_-^j) = \lim_{n\to\infty} S_\omega(e^{it_n^j H_\gamma}\psi^j) < S_\omega^c$, since $e^{it_n^j H_\gamma}\psi^j \to W_-^j$ in $H^1(\mathbb{R})$ with $t_n^j \to \infty$ as $n \to \infty$. Therefore, by the definition of $S_\omega^c$, we obtain $W_-^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $W_{-,n}^j(t,x) := W_-^j(t - t_n^j, x)$.

For every $j$ such that $J_3+1 \le j \le J_4$, we associate with profile $\psi^j$ the function

$$W_+^j(t,x) \in C(\mathbb{R}_+ : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_+ : L_x^r(\mathbb{R}))$$

by Lemma 3.9. And the same argument as above gives us that $W_+^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $W_{+,n}^j(t,x) := W_+^j(t - t_n^j, x)$.

For every $j$ such that $J_4+1 \le j \le J_5$, we associate with profile $\psi^j$ the function

$$V_-^j(t,x) \in C(\mathbb{R}_- : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_- : L_x^r(\mathbb{R}))$$

by Lemma 3.10. We will prove $V_-^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. Now,

$$\limsup_{n\to\infty} S_\omega(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j) < S_\omega^c - \delta$$

holds by the assumption. Here, since $e^{-itH_\gamma}$ is unitary in $L^2(\mathbb{R})$ and conserves the linear energy, and $\gamma \le 0$, we have

$$\begin{aligned} S_\omega(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j) &= E(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j) + \omega M(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j) \\ &= \|\tau_{x_n^j}\psi^j\|_{\mathcal{H}}^2 - \frac{1}{p+1}\|e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j\|_{L^{p+1}}^{p+1} \\ &\ge \tfrac14\|\partial_x(\tau_{x_n^j}\psi^j)\|_{L^2}^2 + \tfrac12\omega\|\tau_{x_n^j}\psi^j\|_{L^2}^2 - \frac{1}{p+1}\|e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j\|_{L^{p+1}}^{p+1} \\ &= \tfrac14\|\partial_x\psi^j\|_{L^2}^2 + \tfrac12\omega\|\psi^j\|_{L^2}^2 - \frac{1}{p+1}\|e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Since $t_n^j \to \infty$, we have $\|e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j\|_{L^{p+1}}^{p+1} \to 0$ as $n\to\infty$ by [Banica and Visciglia 2016, Section 2, (2.4)]. Therefore, we obtain

$$\tfrac14\|\partial_x\psi^j\|_{L^2}^2 + \tfrac12\omega\|\psi^j\|_{L^2}^2 \le S_\omega^c - \delta.$$

Since $\psi^j$ is the final state of $V_-^j$, we have

$$S_{\omega,0}(V_-^j) = \tfrac14\|\partial_x\psi^j\|_{L^2}^2 + \tfrac12\omega\|\psi^j\|_{L^2}^2 \le S_\omega^c - \delta < n_\omega = l_\omega.$$

By [Fang et al. 2011; Akahori and Nawa 2013], we have $V_-^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $V_{-,n}^j(t,x) := V_-^j(t-t_n^j, x-x_n^j)$.

For every $j$ such that $J_5+1 \le j \le J$, we associate with profile $\psi^j$ the function

$$V_+^j(t,x) \in C(\mathbb{R}_+ : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_+ : L_x^r(\mathbb{R}))$$

by Lemma 3.10. And the same argument as above gives us that $V_+^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $V_{+,n}^j(t,x) := V_+^j(t-t_n^j, x-x_n^j)$.

We define the nonlinear profile as

$$Z_n^J := N + \sum_{j=J_1+1}^{J_2} U_n^j + \sum_{j=J_2+1}^{J_3} W_{-,n}^j + \sum_{j=J_3+1}^{J_4} W_{+,n}^j + \sum_{j=J_4+1}^{J_5} V_{-,n}^j + \sum_{j=J_5+1}^{J_6} V_{+,n}^j.$$

By Lemmas 3.8, 3.9, and 3.10, we have

$$Z_n^J = e^{-itH_\gamma}(\varphi_n - W_n^J) + iz_n^J + r_n^J,$$

where $\|r_n^J\|_{L_t^a L_x^r} \to 0$ as $n \to \infty$ and

$$
\begin{aligned}
z_n^J(t) := \int_0^t & e^{-i(t-s)H_\gamma}\big(|N(s)|^{p-1}N(s)\big)\,ds \\
&+ \sum_{j=J_1+1}^{J_2} \int_0^t e^{-i(t-s)H_\gamma}\big(|U_n^j(s)|^{p-1}U_n^j(s)\big)\,ds \\
&+ \sum_{j=J_2+1}^{J_3} \int_0^t e^{-i(t-s)H_\gamma}\big(|W_{-,n}^j(s)|^{p-1}W_{-,n}^j(s)\big)\,ds \\
&+ \sum_{j=J_3+1}^{J_4} \int_0^t e^{-i(t-s)H_\gamma}\big(|W_{+,n}^j(s)|^{p-1}W_{+,n}^j(s)\big)\,ds \\
&+ \sum_{j=J_4+1}^{J_5} \int_0^t e^{-i(t-s)H_\gamma}\big(|V_{-,n}^j(s)|^{p-1}V_{-,n}^j(s)\big)\,ds \\
&+ \sum_{j=J_5+1}^{J} \int_0^t e^{-i(t-s)H_\gamma}\big(|V_{+,n}^j(s)|^{p-1}V_{+,n}^j(s)\big)\,ds.
\end{aligned}
$$

We also have

$$
\left\| z_n^J - \int_o^t e^{-i(t-s)H_\gamma}\big(|Z_n^J(s)|^{p-1}Z_n^J(s)\big)\,ds \right\|_{L_t^a L_x^r} \to 0 \quad \text{as } n \to \infty.
$$

Therefore, we get

$$
Z_n^J = e^{-itH_\gamma}(\varphi_n - W_n^J) + i\int_o^t e^{-i(t-s)H_\gamma}\big(|Z_n^J(s)|^{p-1}Z_n^J(s)\big)\,ds + s_n^J,
$$

with $\|s_n^J\|_{L_t^a L_x^r} \to 0$ as $n \to \infty$. In order to apply the perturbation lemma, Lemma 3.7, we need a bound on $\sup_J(\limsup_{n\to\infty}\|Z_n^J\|_{L_t^a L_x^r})$. We have

$$
\begin{aligned}
\limsup_{n\to\infty}(\|Z_n^J\|_{L_t^a L_x^r})^p \le 2\|N\|_{L_t^a L_x^r}^p &+ 2\sum_{j=J_1+1}^{J_2}\|U^j\|_{L_t^a L_x^r}^p + 2\sum_{j=J_2+1}^{J_3}\|W_-^j\|_{L_t^a L_x^r}^p \\
&+2\sum_{j=J_3+1}^{J_4}\|W_+^j\|_{L_t^a L_x^r}^p + 2\sum_{j=J_4+1}^{J_5}\|V_-^j\|_{L_t^a L_x^r}^p + 2\sum_{j=J_5+1}^{J}\|V_+^j\|_{L_t^a L_x^r}^p,
\end{aligned}
$$

where we have used Corollary A.2 in [Banica and Visciglia 2016]. For simplicity, $a_n^j$ denotes $2\|N\|_{L_t^a L_x^r}^p$ if $1 \le j \le J_1$, $2\|U_n^j\|_{L_t^a L_x^r}^p = 2\|U^j\|_{L_t^a L_x^r}^p$ if $J_1+1 \le j \le J_2$, and so on. Then, the above inequality means

$$
\limsup_{n\to\infty}(\|Z_n^J\|_{L_t^a L_x^r})^p \le \sum_{j=1}^{J} a_n^j.
$$

There exists a finite set $\mathcal{J}$ such that $\|\psi^j\|_{H^1} < \varepsilon_0$ for any $j \notin \mathcal{J}$, where $\varepsilon_0$ is the universal constant in the small data scattering result, Proposition 3.3. By Proposition 3.3 and the orthogonalities in $H$-norm and $L^2$-norm,

$$\begin{aligned}\limsup_{n\to\infty}(\|Z_n^J\|_{L_t^a L_x^r})^p &\le \limsup_{n\to\infty}\sum_{j=1}^J a_n^j = \limsup_{n\to\infty}\sum_{j\in\mathcal{J}} a_n^j + \limsup_{n\to\infty}\sum_{j\notin\mathcal{J}} a_n^j \\ &\lesssim \limsup_{n\to\infty}\sum_{j\in\mathcal{J}} a_n^j + \limsup_{n\to\infty}\sum_{j\notin\mathcal{J}} \|e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j\|_{\mathcal{H}} \\ &\lesssim \limsup_{n\to\infty}\sum_{j\in\mathcal{J}} a_n^j + \limsup_{n\to\infty}\|\varphi_n\|_{\mathcal{H}} \\ &\lesssim \limsup_{n\to\infty}\sum_{j\in\mathcal{J}} a_n^j + n_\omega \lesssim \sum_{j\in\mathcal{J}} a^j + n_\omega \le M,\end{aligned}$$

where $M$ is independent of $J$.

By Lemma 3.7 and Proposition 3.4, we can choose $J$ large enough in such a way that

$$\limsup_{n\to\infty}\|e^{-itH_\gamma}W_n^J\|_{L_t^a L_x^r} < \varepsilon,$$

where $\varepsilon = \varepsilon(M) > 0$. Then, we get the fact that $u_n$ scatters for large $n$, and this contradicts $\|u_n\|_{L_t^a L_x^r} = \infty$.

Therefore, we obtain $J = 1$ and

$$\varphi_n = e^{it_n^1 H_\gamma}\tau_{x_n^1}\psi^1 + W_n^1, \qquad S_\omega^c = \limsup_{n\to\infty} S_\omega(e^{it_n^1 H_\gamma}\tau_{x_n^1}\psi^1), \qquad W_n^1 \to 0 \quad \text{in } L_t^\infty H_x^1.$$

By the same argument as [Banica and Visciglia 2016], we get $x_n^1 = 0$. Let $u^c$ be the nonlinear profile associated with $\psi^1$. Then, $S_\omega^c = S_\omega(u^s)$ and the global solution $u^c$ does not scatter by a contradiction argument and the perturbation lemma (see the proof of Proposition 6.1 in [Fang et al. 2011] for more detail).

**Case 2:** radial data. We only focus on the difference of the proof between the radial case and the nonradial case, which is in the profiles. By the linear profile decomposition for the radial data Theorem 3.5, we have

$$\varphi_n = \frac{1}{2}\sum_{j=1}^J\bigl(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + e^{it_n^j H_\gamma}\tau_{-x_n^j}\mathcal{R}\psi^j\bigr) + \frac{1}{2}(W_n^J + \mathcal{R}W_n^J) \quad \forall J \in \mathbb{N}.$$

For every $j$ such that $J_1 + 1 \le j \le J_2$, let $U^j$ be the solution to (NLS) with the initial data $\frac{1}{2}\psi^j$. Since we have

$$\tfrac{1}{2}\tau_{x_n^j}\psi^j + \tfrac{1}{2}\tau_{-x_n^j}\mathcal{R}\psi^j \in \mathcal{R}_\omega^+,$$

$\psi^j$ satisfies $S_{\omega,0}\bigl(\frac{1}{2}\psi^j\bigr) < l_\omega$ and $P_0\bigl(\frac{1}{2}\psi^j\bigr) \ge 0$. Indeed, if we assume $S_{\omega,0}\bigl(\frac{1}{2}\psi^j\bigr) \ge l_\omega$, then by Theorem 3.5 and $\gamma \le 0$,

$$\begin{aligned} r_\omega > S_\omega^c &\ge \limsup_{n\to\infty} S_\omega(\varphi_n) \ge \limsup_{n\to\infty}\bigl(S_\omega\bigl(\tau_{x_n^j}\tfrac{1}{2}\psi^j\bigr) + S_\omega(\tau_{-x_n^j}\mathcal{R}\tfrac{1}{2}\psi^j)\bigr) \\ &\ge \limsup_{n\to\infty}\bigl(S_{\omega,0}\bigl(\tau_{x_n^j}\tfrac{1}{2}\psi^j\bigr) + S_{\omega,0}\bigl(\tau_{-x_n^j}\mathcal{R}\tfrac{1}{2}\psi^j\bigr)\bigr) \\ &= S_{\omega,0}\bigl(\tfrac{1}{2}\psi^j\bigr) + S_{\omega,0}\bigl(\tfrac{1}{2}\psi^j\bigr) \ge 2l_\omega.\end{aligned}$$

This contradicts $r_\omega \le 2l_\omega$. Moreover, we see that

$$2P_0\big(\tfrac{1}{2}\psi^j\big) = \limsup_{n\to\infty}\big(P_0\big(\tau_{x_n^j}\tfrac{1}{2}\psi^j\big) + P_0\big(\tau_{-x_n^j}\mathcal{R}\tfrac{1}{2}\psi^j\big)\big) = \limsup_{n\to\infty} P\big(\tau_{x_n^j}\tfrac{1}{2}\psi^j + \tau_{-x_n^j}\mathcal{R}\tfrac{1}{2}\psi^j\big) \ge 0.$$

Therefore, by [Fang et al. 2011; Akahori and Nawa 2013], we have

$$U^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R})).$$

We set $U_n^j(t,x) := U^j(t, x - x_n^j)$.

For every $j$ such that $J_4 + 1 \le j \le J_5$, we associate with profile $\psi^j$ the function

$$V_-^j(t,x) \in C(\mathbb{R}_- : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_- : L_x^r(\mathbb{R}))$$

by Lemma 3.10. We prove $V_-^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. Now, by the assumption, we have

$$\limsup_{n\to\infty} 2S_\omega\big(\tfrac{1}{2}e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j\big) = \limsup_{n\to\infty}\big\{S_\omega(e^{it_n^j H_\gamma}\tau_{x_n^j}\tfrac{1}{2}\psi^j) + S_\omega\big(\mathcal{R}e^{it_n^j H_\gamma}\tau_{x_n^j}\tfrac{1}{2}\psi^j\big)\big\} < S_\omega^c - \delta.$$

In the same argument as that for $V_-^j$ in the nonradial case, we obtain

$$\tfrac{1}{4}\big\|\partial_x \tfrac{1}{2}1\psi^j\big\|_{L^2}^2 + \tfrac{1}{2}\omega\big\|\tfrac{1}{2}\psi^j\big\|_{L^2}^2 \le \tfrac{1}{2}(S_\omega^c - \delta).$$

Now, since $\psi^j$ is the final state of $V_-^j$, we have

$$S_{\omega,0}(V_-^j) = \tfrac{1}{4}\big\|\partial_x \tfrac{1}{2}\psi^j\big\|_{L^2}^2 + \tfrac{1}{2}\omega\big\|\tfrac{1}{2}\psi^j\big\|_{L^2}^2 \le \tfrac{1}{2}(S_\omega^c - \delta) < \tfrac{1}{2}r_\omega \le l_\omega.$$

By [Fang et al. 2011; Akahori and Nawa 2013], we have $V_-^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $V_{-,n}^j(t,x) := V_-^j(t - t_n^j, x - x_n^j)$.

Other statements are the same as in the nonradial case. □

**3E. *Extinction of the critical element.*** We assume that $\|u^c\|_{L_t^a((0,\infty):L_x^r)} = \infty$, where such $u^c$ is called a forward critical element, and we prove $u^c = 0$. In the case of $\|u^c\|_{L_t^a((-\infty,0):L_x^r)} = \infty$, the same argument as below does work.

**Lemma 3.12.** *Let $u$ be a forward critical element. Then the orbit of $u$, $\{u(t,x) : t > 0\}$, is precompact in $H^1(\mathbb{R})$. And then, for any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{|x|>R} |\partial_x u(t,x)|^2\,dx + \int_{|x|>R} |u(t,x)|^2\,dx + \int_{|x|>R} |u(t,x)|^{p+1}\,dx < \varepsilon \quad \textit{for any } t \in \mathbb{R}_+.$$

This lemma is obtained in the same way as the defocusing case (see [Banica and Visciglia 2016]).

Now, we prove $u = 0$ by the localized virial identity and contradiction. Let $u \ne 0$. For $\varphi : \mathbb{R}_+ \to \mathbb{R}$, we define a function $I$ by

$$I(t) := \int_{\mathbb{R}} \varphi(|x|)|u(t,x)|^2\,dx.$$

Then, by a direct calculation and using ($\delta$NLS), we have

$$\begin{aligned}
I'(t) &= \operatorname{Im}\int_{\mathbb{R}} \partial_x(\varphi(|x|))\overline{u(t,x)}\partial_x u(t,x)\,dx,\\
I''(t) &= \operatorname{Re}\int_{\mathbb{R}} \partial_x^2(\varphi(|x|))|\partial_x u(t,x)|^2\,dx - \gamma\partial_x^2(\varphi(|x|))|_{x=0}|u(t,0)|^2\\
&\quad -\frac{p-1}{p+1}\operatorname{Re}\int_{\mathbb{R}} \partial_x^2(\varphi(|x|))|u(t,x)|^{p+1}\,dx - \frac{1}{4}\operatorname{Re}\int_{\mathbb{R}} \partial_x^4(\varphi(|x|))|u(t,x)|^2\,dx\\
&\quad -2\gamma\operatorname{Re}\{\partial_x(\varphi(|x|))|_{x=0}u(t,0)\overline{\partial_x u(t,0)}\}.
\end{aligned}$$

Taking $\varphi=\varphi(r)$ such that, for $R>0$,

$$0\le\varphi\le r^2,\quad |\varphi'|\lesssim r,\quad |\varphi''|\le 2,\quad |\varphi^{(4)}|\le\frac{4}{R^2},$$

and

$$\varphi(r)=\begin{cases} r^2, & 0\le r\le R,\\ 0, & r\ge 2R,\end{cases}$$

we obtain

$$\begin{aligned}
I''(t) &= 4P(u(t))+\operatorname{Re}\int_{\mathbb{R}}(\partial_x^2(\varphi(|x|))-2)|\partial_x u(t,x)|^2\,dx-\frac{p-1}{p+1}\operatorname{Re}\int_{\mathbb{R}}(\partial_x^2(\varphi(|x|))-2)|u(t,x)|^{p+1}\,dx\\
&\quad -\tfrac{1}{4}\operatorname{Re}\int_{\mathbb{R}}\partial_x^4(\varphi(|x|))|u(t,x)|^2\,dx = 4P(u(t))+R_1+R_2+R_3,
\end{aligned}\tag{3-2}$$

where

$$\begin{aligned}
R_1 &:= \operatorname{Re}\int_{\mathbb{R}}(\partial_x^2(\varphi(|x|))-2)|\partial_x u(t,x)|^2\,dx,\\
R_2 &:= -\frac{p-1}{p+1}\operatorname{Re}\int_{\mathbb{R}}(\partial_x^2(\varphi(|x|))-2)|u(t,x)|^{p+1}\,dx,\\
R_3 &:= -\tfrac{1}{4}\operatorname{Re}\int_{\mathbb{R}}\partial_x^4(\varphi(|x|))|u(t,x)|^2\,dx.
\end{aligned}$$

By the property of $\varphi$, we have

$$\begin{aligned}
|R_1| &= \left|\operatorname{Re}\int_{\mathbb{R}}\{\partial_x^2(\varphi(|x|))-2\}|\partial_x u(t,x)|^2\,dx\right| \le C\int_{|x|>R}|\partial_x u(t,x)|^2\,dx,\\
|R_2| &= \left|\frac{p-1}{p+1}\operatorname{Re}\int_{\mathbb{R}}\{\partial_x^2(\varphi(|x|))-2\}|u(t,x)|^{p+1}\,dx\right| \le C\int_{|x|>R}|u(t,x)|^{p+1}\,dx,\\
|R_3| &= \left|\tfrac{1}{4}\operatorname{Re}\int_{\mathbb{R}}\partial_x^4(\varphi(|x|))|u(t,x)|^2\,dx\right| \le C\int_{|x|>R}|u(t,x)|^2\,dx.
\end{aligned}$$

Therefore, we obtain

$$I''(t)=4P(u(t))-C\left(\int_{|x|>R}|\partial_x u(t,x)|^2\,dx+\int_{|x|>R}|u(t,x)|^2\,dx+\int_{|x|>R}|u(t,x)|^{p+1}\,dx\right).$$

We note that there exists $\delta > 0$ independent of $t$ such that $P(u(t)) > \delta$ by Proposition 2.18 since $u$ belongs to $\mathcal{M}_\omega^+$. Therefore, by Lemma 3.12, if we take $\varepsilon \in (0, 3\delta)$, then there exists $R > 0$ such that $I''(t) \geq \delta$ for any $t \in \mathbb{R}_+$. On the other hand, the mass conservation law gives $I(t) \leq R^2 \|u(t)\|_{L^2}^2 < C$, where $C$ is independent of $t$, for any $t \in \mathbb{R}_+$. Hence, we obtain a contradiction.

## 4. Proof of the blow-up part

To prove the blow-up results, we use the method of Du et al. [2016]. On the contrary, we assume that the solution $u$ to ($\delta$NLS) with $u_0 \in \mathcal{M}_\omega^-$ is global in the positive time direction and $\sup_{t\in\mathbb{R}_+} \|\partial_x u(t)\|_{L^2}^2 < C_0 < \infty$. Then, we have $\sup_{t\in\mathbb{R}_+} \|u(t)\|_{L^q} < \infty$ for any $q > p+1$ by energy conservation and the Sobolev embedding.

For $R > 0$, we take $\varphi$ such that

$$\varphi(r) = \begin{cases} 0, & 0 < r < \frac{1}{2}R, \\ 1, & r \geq R, \end{cases}$$

$$0 \leq \varphi \leq 1, \quad \varphi' \leq \frac{4}{R}.$$

By the fundamental formula and the Hölder inequality, we have

$$\begin{aligned} I(t) = I(0) + \int_0^t I'(s)\,ds &\leq I(0) + \int_0^t |I'(s)|\,ds \\ &\leq I(0) + t\|\varphi'\|_{L^\infty} \|u(t)\|_{L^2}^2 \|\partial_x u(t)\|_{L^2}^2 \\ &\leq I(0) + \frac{8M(u)C_0 t}{R}. \end{aligned}$$

Here, we note that $I(0) \leq \int_{|x|>R/2} |u(0,x)|^2\,dx = o_R(1)$ and $\int_{|x|>R} |u(t,x)|^2\,dx \leq I(t)$. Therefore, we obtain the following lemma.

**Lemma 4.1.** *Let $\eta_0 > 0$ be fixed. Then, for any $t \leq \eta_0 R/(8M(u)C_0)$, we have*

$$\int_{|x|>R} |u(t,x)|^2\,dx \leq o_R(1) + \eta_0.$$

We take another $\varphi$ such that

$$0 \leq \varphi \leq r^2, \quad |\varphi'| \lesssim r, \quad |\varphi''| \leq 2, \quad |\varphi^{(4)}| \leq \frac{4}{R^2},$$

and

$$\varphi(r) = \begin{cases} r^2, & 0 \leq r \leq R, \\ 0, & r \geq 2R. \end{cases}$$

Then we have the following lemma.

**Lemma 4.2.** *There exist two constants $C = C(p, M(u), C_0) > 0$ and $\theta_q > 0$ such that*

$$I''(t) \leq 4P(u(t)) + C\|u\|_{L^2(|x|>R)}^{\theta_q} + CR^{-2}\|u\|_{L^2(|x|>R)}^2.$$

*Proof.* By (3-2), we have

$$I''(t) = 4P(u(t)) + R_1 + R_2 + R_3.$$

First, we prove $R_1 \le 0$. By the definition of $\varphi$, we see that

$$R_1 = \operatorname{Re}\int_{\mathbb{R}} (\partial_x^2(\varphi(|x|)) - 2)|\partial_x u(t,x)|^2\,dx = \operatorname{Re}\int_{\mathbb{R}} (\varphi''(|x|) - 2)|\partial_x u(t,x)|^2\,dx \le 0.$$

Next, we consider $R_2$. By the Hölder inequality, we have

$$\begin{aligned} R_2 &= -\frac{p-1}{p+1}\operatorname{Re}\int_{\mathbb{R}} (\partial_x^2(\varphi(|x|)) - 2)|u(t,x)|^{p+1}\,dx \\ &\le C\int_{|x|>R} |u(t,x)|^{p+1}\,dx \\ &\le C\|u\|_{L^q(|x|>R)}^{1-\theta_q}\|u\|_{L^2(|x|>R)}^{\theta_q} \\ &\le C\|u\|_{L^2(|x|>R)}^{\theta_q}, \end{aligned}$$

where $q > p+1$ and $0 < \theta_q \le 1$, since $\sup_{t\in\mathbb{R}_+}\|u(t)\|_{L^q} < \infty$. Finally, we consider $R_3$:

$$R_3 = -\tfrac{1}{4}\operatorname{Re}\int_{\mathbb{R}} \partial_x^4(\varphi(|x|))|u(t,x)|^2\,dx \le CR^{-2}\int_{|x|>R} |u(t,x)|^2\,dx = CR^{-2}\|u\|_{L^2(|x|>R)}^2. \qquad \square$$

*Proof of Theorem 1.3(2) (and Theorem 1.5(2)).* Since $u(t)$ belongs to $\mathcal{M}_\omega^-$, there exists $\delta > 0$ independent of $t$ such that $P(u(t)) < -\delta$ for all $t \in \mathbb{R}_+$ by Proposition 2.18. Therefore, we obtain

$$I''(t) \le -4\delta + C\|u\|_{L^2(|x|>R)}^{\theta_q} + CR^{-2}\|u\|_{L^2(|x|>R)}^2.$$

We take $\eta_0 > 0$ such that $C\eta_0^{\theta_q} + C\eta_0^2 < \delta$. By Lemma 4.1, for $t \in [0, \eta_0 R/(8M(u)C_0)]$, we have

$$I''(t) \le -3\delta + o_R(1).$$

Let $T := \eta_0 R/(8M(u)C_0)$. Integrating the above inequality from 0 to $T$, we get

$$I(T) \le I(0) + I'(0)T + \tfrac{1}{2}(-3\delta + o_R(1))T^2.$$

For sufficiently large $R > 0$, we have $-3\delta + o_R(1) < -2\delta$. Thus, we get

$$I(T) \le I(0) + I'(0)\eta_0 R/(8M(u)C_0) - \alpha_0 R^2,$$

where $\alpha_0 := \delta\eta_0^2/(8M(u)C_0)^2 > 0$, and we can prove $I(0) = o_R(1)R^2$ and $I'(0) = o_R(1)R$. Indeed,

$$\begin{aligned} I(0) &\le \int_{|x|<\sqrt{R}} |x|^2|u_0(x)|^2\,dx + \int_{\sqrt{R}<|x|<2R} |x|^2|u_0(x)|^2\,dx \\ &\lesssim M(u)R + R^2\int_{\sqrt{R}<|x|} |u_0(x)|^2\,dx \\ &= o_R(1)R^2, \end{aligned}$$

and

$$
\begin{aligned}
I'(0) &\leq \int_{|x|<\sqrt{R}} |\varphi'(|x|)||u_0(x)||\partial_x u_0(x)|\,dx + \int_{\sqrt{R}<|x|<2R} |\varphi'(|x|)||u_0(x)||\partial_x u_0(x)|\,dx \\
&\leq \int_{|x|<\sqrt{R}} |x||u_0(x)||\partial_x u_0(x)|\,dx + \int_{\sqrt{R}<|x|<2R} |x||u_0(x)||\partial_x u_0(x)|\,dx \\
&\lesssim \|u_0\|_{H^1}^2 \sqrt{R} + R \int_{\sqrt{R}<|x|} |u_0(x)||\partial_x u_0(x)|\,dx \\
&= o_R(1) R.
\end{aligned}
$$

Therefore, we see that

$$
I(T) \leq o_R(1) R^2 - \alpha_0 R^2.
$$

For sufficiently large $R > 0$, we have $o_R(1) - \alpha_0 < 0$. However, this contradicts

$$
I(T) = \int_{\mathbb{R}} \varphi(|x|)|u(T,x)|^2\,dx > 0.
$$

This argument can be applied in the negative time direction. □

## Appendix: Rewriting the main theorem into a version independent of the frequency

We prove Corollary 1.4. To see this, it is sufficient to prove the following lemma.

**Lemma A.1.** *Let $\varphi \in H^1(\mathbb{R})$. The following statements are equivalent*:

(1) *There exists $\omega > 0$ such that $S_\omega(\varphi) < l_\omega = n_\omega$.*

(2) *$\varphi$ satisfies $E(\varphi)M(\varphi)^\sigma < E_0(Q_{1,0})M(Q_{1,0})^\sigma$.*

*Proof.* If $\varphi = 0$, the statement holds. Let $\varphi \in H^1(\mathbb{R}) \setminus \{0\}$ be fixed. We define $f(\omega) := l_\omega - S_\omega(\varphi)$. Then, (1) is true if and only if $\sup_{\omega>0} f(\omega) > 0$. Noting that $l_\omega = \omega^{\frac{p+3}{2(p-1)}} S_{1,0}(Q_{1,0})$, we know $f$ has a maximum at $\omega = \omega_0$, where

$$
\omega_0 := \left( \frac{M(\varphi)}{\frac{p+3}{2(p-1)} S_{1,0}(Q_{1,0})} \right)^{-\frac{2(p-1)}{p-5}} > 0.
$$

Therefore, (1) is equivalent to $f(\omega_0) > 0$. Now, since

$$
\begin{aligned}
f(\omega_0) &= \left( \frac{M(\varphi)}{\frac{p+3}{2(p-1)} S_{1,0}(Q_{1,0})} \right)^{-\frac{p+3}{p-5}} S_{1,0}(Q_{1,0}) - \left( \frac{M(\varphi)}{\frac{p+3}{2(p-1)} S_{1,0}(Q_{1,0})} \right)^{-\frac{2(p-1)}{p-5}} M(\varphi) - E(\varphi) \\
&= \frac{\left(\frac{p+3}{2(p-1)} S_{1,0}(Q_{1,0})\right)^{\frac{2(p-1)}{p-5}}}{M(\varphi)^{\frac{p+3}{p-5}}} - E(\varphi) > 0,
\end{aligned}
$$

we have

$$
\left( \frac{p+3}{2(p-1)} S_{1,0}(Q_{1,0}) \right)^{\frac{2(p-1)}{p-5}} > E(\varphi) M(\varphi)^{\frac{p+3}{p-5}}.
$$

Noting $Q_{1,0}$ satisfies

$$\|Q_{1,0}\|_{L^2}^2 = \frac{p+3}{2(p-1)}\|\partial_x Q_{1,0}\|_{L^2}^2 = \frac{p+3}{2(p+1)}\|Q_{1,0}\|_{L^{p+1}}^{p+1},$$

we have

$$\left(\frac{p+3}{2(p-1)}S_{1,0}(Q_{1,0})\right)^{\frac{2(p-1)}{p-5}} = E_0(Q_{1,0})M(Q_{1,0})^{\frac{p+3}{p-5}}. \qquad \square$$

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