Analysis & PDE

Volume 10 No. 2 2017

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Analysis & PDE

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS AND PDE
Vol. 10, No. 2, 2017
dx.doi.org/10.2140/apde.2017.10.253

SOME ENERGY INEQUALITIES INVOLVING FRACTIONAL GJMS OPERATORS

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Under a spectral assumption on the Laplacian of a Poincaré–Einstein manifold, we establish an energy inequality relating the energy of a fractional GJMS operator of order $2\gamma \in (0, 2)$ or $2\gamma \in (2, 4)$ and the energy of the weighted conformal Laplacian or weighted Paneitz operator, respectively. This spectral assumption is necessary and sufficient for such an inequality to hold. We prove the energy inequalities by introducing conformally covariant boundary operators associated to the weighted conformal Laplacian and weighted Paneitz operator which generalize the Robin operator. As an application, we establish a new sharp weighted Sobolev trace inequality on the upper hemisphere.

1. Introduction

Fractional GJMS operators are conformally covariant pseudodifferential operators defined on the boundary of a Poincaré–Einstein manifold via scattering theory which have principal symbol equal to that of the fractional powers of the Laplacian [Graham and Zworski 2003]. Fractional GJMS operators can also be understood as generalized Dirichlet-to-Neumann operators associated to weighted GJMS operators of a suitable order defined in the interior [Branson and Gover 2001; Caffarelli and Silvestre 2007; Case and Chang 2016; Chang and González 2011; Yang 2013]. In particular, one can identify the energy associated to a fractional GJMS operator with the energy associated to a suitable weighted GJMS operator when restricted to canonical extensions; see [Caffarelli and Silvestre 2007; Yang 2013] for the flat case and [Case and Chang 2016; Chang and González 2011] for the curved case.

In this article, we are interested in obtaining, as a generalization of known results in the flat case [Yang 2013], a general relationship between the energy associated to a fractional GJMS operator and the energy associated to a suitable weighted GJMS operator for arbitrary extensions. One reason for this interest is the role of such relationships in establishing sharp Sobolev trace inequalities (see [Ache and Chang 2015; Escobar 1988]) and in studying the fractional Yamabe problem (see [Escobar 1992; González and Qing 2013]). Indeed, this article is partly motivated by a subtle issue which arises in the works of Escobar [1992; 1994] and González and Qing [2013] on the fractional Yamabe problem of order $\gamma \in (0, 1)$. In both works, one tries to find a metric on a compact manifold with boundary which is scalar flat in the interior and for which the boundary has constant mean curvature (in a sense made precise in Section 3) by minimizing an energy functional in the interior subject to a volume-normalization

MSC2010: primary 58J32; secondary 53A30, 58J40.
Keywords: fractional Laplacian, fractional GJMS operator, Poincaré–Einstein manifold, Robin operator, smooth metric measure space.

on the boundary. However, there is no guarantee that the energy functional is bounded below within this class, an issue overlooked in [Escobar 1992; González and Qing 2013] and corrected in the special case $\gamma = \frac{1}{2}$ in [Escobar 1994]. Proposition 5.1 corrects this issue by giving a spectral condition under which the energy functional is bounded below.

The main results of this article are the following two theorems. These results establish, under spectral assumptions on a Poincaré–Einstein manifold, energy inequalities on suitable compactifications of the Poincaré–Einstein manifold which relate the energy of the weighted conformal Laplacian and the weighted Paneitz operator to the energy of the fractional GJMS operators $P_{2\gamma}$ in the cases $\gamma \in (0,1)$ and $\gamma \in (1,2)$, respectively. That equality holds for the special extensions U was established in [Case and Chang 2016].

Theorem 1.1. *Fix $\gamma \in (0,1)$ and set $m = 1-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold satisfying $\lambda_1(-\Delta_{g_+}) > \frac{1}{4}n^2 - \gamma^2$. Let r be a geodesic defining function for M and let ρ be a defining function such that, asymptotically near M,*

$$\rho = r + \Phi r^{1+2\gamma} + o(r^{1+2\gamma})$$

for some $\Phi \in C^\infty(M)$. Fix $f \in C^\infty(M)$ and denote by $\mathcal{D}_f^\gamma$ the set of functions $U \in C^\infty(X) \cap C^0(\overline{X})$ such that, asymptotically near M,

$$U = f + \psi\rho^{2\gamma} + o(\rho^{2\gamma})$$

for some $\psi \in C^\infty(M)$. Set $g = \rho^2 g_+$ and $h = g|_{TM}$. Then

$$\int_X \left(|\nabla U|^2 + \frac{m+n-1}{2} J_\phi^m U^2\right)\rho^m \,\mathrm{dvol}_g \geq -\frac{2\gamma}{d_\gamma}\left[\oint_M f P_{2\gamma} f \,\mathrm{dvol}_h - \frac{n-2\gamma}{2} d_\gamma \oint_M \Phi f^2 \,\mathrm{dvol}_h\right] \tag{1-1}$$

for all $U \in \mathcal{D}_f^\gamma$, where J_ϕ^m is the weighted scalar curvature of $(\overline{X}, g, \rho, m, 1)$. Moreover, equality holds if and only if $L_{2,\phi}^m U = 0$.

Note that the left-hand side of (1-1) is the Dirichlet energy of the weighted conformal Laplacian $L_{2,\phi}^m$ of $(\overline{X}, g, \rho, m, 1)$. See Section 2 for a detailed explanation of the terminology and notation used in Theorem 1.1. The spectral condition in Theorem 1.1 holds for Poincaré–Einstein manifolds for which the conformal infinity $(M^n, [h])$ has nonnegative Yamabe constant [Lee 1995].

A key point is that the spectral assumption $\lambda_1(-\Delta_{g_+}) > \frac{1}{4}n^2 - \gamma^2$ is necessary; see Proposition 5.1. This corrects the aforementioned mistake in [González and Qing 2013]. Observe also that the left-hand side of (1-1) involves the interior L^2-norm of U. This contrasts with the sharp Sobolev trace inequalities of Jin and Xiong [2013] which instead involve a boundary L^2-norm of $f = U|_M$: Given a Poincaré–Einstein manifold (X^{n+1}, M^n, g_+), a constant $\gamma \in (0,1)$, and a defining function ρ as in Theorem 1.1, there is a constant A such that

$$\int_X |\nabla U|^2 \rho^{1-2\gamma} \,\mathrm{dvol}_g + A\oint_M f^2 \,\mathrm{dvol} \geq S(n,\gamma)\left(\oint_M |f|^{\frac{2n}{n-2\gamma}}\right)^{\frac{n-2\gamma}{n}} \tag{1-2}$$

for any $U \in \mathcal{D}^\gamma := \bigcup_f \mathcal{D}_f^\gamma$, where $g = \rho^2 g_+$, $f = U|_M$, and $S(n,\gamma)$ is the corresponding constant in the upper half space [González and Qing 2013; Jin and Xiong 2013]. Under the spectral assumption $\lambda_1(-\Delta_{g_+}) > \frac{1}{4}n^2 - \gamma^2$, one can use the adapted defining function [Case and Chang 2016, Subsection 6.1] in Theorem 1.1 to eliminate the interior L^2-norm of U; indeed, combining this with (1-2) yields the sharp fractional Sobolev inequality

$$\oint_M f P_{2\gamma} f + A \oint_M f^2 \geq -\frac{d_\gamma}{2\gamma} S(n,\gamma) \left(\oint_M |f|^{\frac{2n}{n-2\gamma}} \right)^{\frac{n-2\gamma}{n}}$$

for all $f \in C^\infty(M)$ (see [Hebey and Vaugon 1996; Jin and Xiong 2013]).

Theorem 1.2. *Fix $\gamma \in (1,2)$ and set $m = 3 - 2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold satisfying $\lambda_1(-\Delta_{g_+}) > \frac{1}{4}n^2 - (2-\gamma)^2$. Let r be a geodesic defining function for M and let ρ be a defining function such that, asymptotically near M,*

$$\rho = r + \rho_2 r^3 + \Phi r^{1+2\gamma} + o(r^{1+2\gamma})$$

for some $\rho_2, \Phi \in C^\infty(M)$. Fix $f \in C^\infty(M)$ and denote by $\mathcal{D}_f^\gamma$ the set of functions $U \in C^\infty(X) \cap C^0(\overline{X})$ such that, asymptotically near M,

$$U = f + f_2 \rho^2 + \psi \rho^{2\gamma} + o(\rho^{2\gamma})$$

for some $f_2, \psi \in C^\infty(M)$. Set $g = \rho^2 g_+$ and $h = g|_{TM}$. Then for any $U \in \mathcal{D}_f^\gamma$ it holds that

$$\begin{aligned}\int_X \Big((\Delta_\phi U)^2 - (4P - (n-2\gamma+2) J_\phi^m g)(\nabla U, \nabla U) + \frac{n-2\gamma}{2} Q_\phi^m U^2 \Big) \\ \geq \frac{8\gamma(\gamma-1)}{d_\gamma} \Big(\oint_M f P_{2\gamma} f - \frac{n-2\gamma}{2} d_\gamma \oint_M \Phi f^2 \Big), \quad (1\text{-}3)\end{aligned}$$

where P is the Schouten tensor of g, J_ϕ^m and Q_ϕ^m are the weighted scalar curvature and the weighted Q-curvature, respectively, of $(\overline{X}, g, \rho, m, 1)$, and integrals on X and M are evaluated with respect to $\rho^m \operatorname{dvol}_g$ and dvol_h, respectively. Moreover, equality holds if and only if $L_{4,\phi}^m U = 0$.

Note that the left-hand side of (1-3) is the Dirichlet energy of the weighted Paneitz operator $L_{4,\phi}^m$ of $(\overline{X}, g, \rho, m, 1)$. See Section 2 for a detailed explanation of the terminology and notation used in Theorem 1.1. The spectral condition in Theorem 1.2 holds for Poincaré–Einstein manifolds for which the conformal infinity $(M^n, [h])$ has nonnegative Yamabe constant [Lee 1995].

The proofs of Theorem 1.1 and Theorem 1.2 rely on three observations. First, we introduce conformally boundary-covariant operators associated to the weighted conformal Laplacian and the weighted Paneitz operator in the same sense as the trace and Robin operators act as boundary operators associated to the conformal Laplacian (cf. [Branson 1997; Branson and Gover 2001; Escobar 1990; 1992]). Second, we show that our conformally covariant operators recover certain scattering operators when acting on functions which lie in the kernel of the corresponding weighted GJMS operator on a Poincaré–Einstein manifold; this yields another approach to defining the fractional GJMS operators via extensions (cf. [Ache and Chang 2015; Case and Chang 2016; Chang and González 2011; Graham and Zworski 2003; Guillarmou and Guillopé 2007]). Third, using conformal covariance, we characterize when the left-hand sides of (1-1) and (1-3) are uniformly bounded below in terms of spectral data for the metric g_+. When these spectral conditions are met, the left-hand sides of (1-1) and (1-3) can be minimized, and the identification of the minimizers follows from our extension theorem.

The second step in the above outline is a refinement of previous work in [Case and Chang 2016]. In that work, it was shown that the fractional GJMS operators are generalized Dirichlet-to-Neumann operators for the weighted GJMS operators. For example, under the assumptions of Theorem 1.1, it was shown that if $L_{2,\phi}^m U = 0$ and $U|_M = f$, then

$$P_{2\gamma} f = -\frac{d_\gamma}{2\gamma} \lim_{\rho\to 0} \bigl(\rho^m \eta U - \gamma(n-2\gamma)\Phi U\bigr);$$

see [Case and Chang 2016, Theorem 4.1]. In particular, equality holds in (1-1). The novelty introduced in this article is to realize the right-hand side of the above display as the evaluation of a conformally covariant boundary operator. This also allows us to establish the energy inequality of Theorem 1.1. A similar comparison of our results to those in [Case and Chang 2016] holds in the case $\gamma \in (1,2)$.

As an application of our results, we establish a sharp Sobolev trace inequality on the standard upper hemisphere

$$S_+^{n+1} := \bigl\{x = (x_0, \dots, x_{n+1}) \in \mathbb{R}^{n+2} \bigm| x_{n+1} > 0,\ |x| = 1\bigr\}$$

with the metric induced by the Euclidean metric. To that end, let $\gamma \in (1,2)$ and set

$$\mathcal{D}^\gamma := \bigcup_{f \in C^\infty(S^n)} \mathcal{D}_f^\gamma$$

for $\mathcal{D}_f^\gamma$ determined by the defining function x_{n+1} for $S^n = \partial S_+^{n+1}$ as in Theorem 1.2.

Theorem 1.3. *Fix $\gamma \in (1,2)$, choose $2\gamma < n \in \mathbb{N}$, and let $(S_+^{n+1}, d\theta^2)$ be the standard upper hemisphere. Then*

$$\begin{aligned} c_{n,\gamma}^{(2)} &\left(\oint_{S^n} |f|^{\frac{2n}{n-2\gamma}} \,\mathrm{dvol}\right)^{\frac{n-2\gamma}{n}} \\ &\leq \int_{S_+^{n+1}} \left[(\Delta_\phi U)^2 + \frac{(n+3-2\gamma)^2 - 5}{2} |\nabla U|^2 + \frac{\Gamma\bigl(\frac12(n+8-2\gamma)\bigr)}{\Gamma\bigl(\frac12(n-2\gamma)\bigr)} U^2\right] x_{n+1}^{3-2\gamma} \,\mathrm{dvol} \end{aligned} \tag{1-4}$$

for all $U \in \mathcal{D}^\gamma$, where $f = U|_{S^n}$ and

$$c_{n,\gamma}^{(2)} = 8\pi^\gamma \frac{\Gamma(2-\gamma)}{\Gamma(\gamma)} \frac{\Gamma\bigl(\frac12(n+2\gamma)\bigr)}{\Gamma\bigl(\frac12(n-2\gamma)\bigr)} \left(\frac{\Gamma\bigl(\frac12 n\bigr)}{\Gamma(n)}\right)^{\frac{2\gamma}{n}}.$$

Moreover, equality holds if and only if

$$\bigl(\Delta_\phi - \tfrac14((n+3-2\gamma)^2 - 1)\bigr)\bigl(\Delta_\phi - \tfrac14((n+3-2\gamma)^2 - 9)\bigr) U = 0 \tag{1-5}$$

and $f(x) = c(1 + a\cdot x)^{-\frac{n-2\gamma}{2}}$ for some $c \in \mathbb{R}$ and $a \in \mathbb{R}^{n+1}$ with $|a| < 1$.

The corresponding result when $\gamma \in (0,1)$ is that

$$c_{n,\gamma}^{(1)}\left(\oint_{S^n} |f|^{\frac{2n}{n-2\gamma}}\,\mathrm{dvol}\right)^{\frac{n-2\gamma}{n}} \leq \int_{S_+^{n+1}} \left[|\nabla U|^2 + \frac{\Gamma\big(\frac{1}{2}(n+4-2\gamma)\big)}{\Gamma\big(\frac{1}{2}(n-2\gamma)\big)}U^2\right] x_{n+1}^{1-2\gamma}\,\mathrm{dvol}$$

for all $U \in \mathcal{D}^\gamma$ with trace $f = U|_{S^n}$, where

$$c_{n,\gamma}^{(1)} = 2\pi^\gamma \frac{\Gamma(1-\gamma)}{\Gamma(\gamma)} \frac{\Gamma\big(\frac{1}{2}(n+2\gamma)\big)}{\Gamma\big(\frac{1}{2}(n-2\gamma)\big)} \left(\frac{\Gamma(\frac{1}{2}n)}{\Gamma(n)}\right)^{\frac{2\gamma}{n}}.$$

This follows easily from [González and Qing 2013, Corollary 5.3] and conformal covariance.

The key observation in the proof of Theorem 1.3 is that the right-hand side of (1-4) is the energy of the weighted Paneitz operator on $(S_+^{n+1}, d\theta^2, x_{n+1}, m, 1)$. The relation to the $L^{\frac{2n}{n-2\gamma}}$-norm of the trace then follows from Theorem 1.2 and the sharp fractional Sobolev inequality [Beckner 1993; Cotsiolis and Tavoularis 2004; Frank and Lieb 2012; Lieb 1983]. In fact, Theorem 1.3 can be extended to a much more general class of functions U and a large class of conformally flat metrics on the upper hemisphere; see Theorem 6.1.

This article is organized as follows:

In Section 2 we recall some facts about both the fractional GJMS operators as defined via scattering theory [Graham and Zworski 2003] and smooth metric measure spaces as used to study fractional GJMS operators via extensions [Case and Chang 2016].

In Section 3 we introduce conformally covariant boundary operators which, when coupled with the weighted conformal Laplacian and weighted Paneitz operator, are formally self-adjoint.

In Section 4 we give formulae for our conformally covariant operators in terms of the asymptotics of compactifications of Poincaré–Einstein manifolds and thereby obtain new interpretations of the fractional GJMS operators via extensions.

In Section 5 we give characterizations for when the left-hand sides of (1-1) and (1-3) are uniformly bounded below and also state and prove more refined versions of Theorem 1.1 and Theorem 1.2.

In Section 6 we prove the more general version of Theorem 1.3.

In the Appendix we prove a family of Sobolev trace theorems which are relevant to this article and slightly different from the usual ones.

2. Background

Scattering theory. A *Poincaré–Einstein manifold* is a triple (X^{n+1}, M^n, g_+) consisting of a complete Einstein manifold (X^{n+1}, g_+) with $\mathrm{Ric}(g_+) = -ng_+$ and $n \geq 3$ such that X is diffeomorphic to the interior of a compact manifold $\overline{X}$ with boundary $M = \partial\overline{X}$. We further require the existence of a defining function for M; i.e., a smooth nonnegative function $\rho : \overline{X} \to \mathbb{R}$ such that $\rho^{-1}(0) = M$, the metric $g := \rho^2 g_+$ extends to a $C^{n-1,\alpha}$ metric on $\overline{X}$, and $|d\rho|_g^2 = 1$ on M. If ρ is a defining function for M, then so too is $e^\sigma\rho$ for any $\sigma \in C^\infty(\overline{X})$, and hence only the conformal class $[g|_{TM}]$ on M is well-defined. An element $h \in [g|_{TM}]$ is a *representative of the conformal boundary*, and to each such representative there is a defining function r, unique in a neighborhood of M and called the *geodesic defining function*, such that $g_+ = r^{-2}(dr^2 + h_r)$ near M for h_r a one-parameter family of Riemannian metrics on M with

$$\begin{aligned} h_r &= h + h_{(2)}r^2 + \cdots + h_{(n-1)}r^{n-1} + kr^n + o(r^n) && \text{if } n \text{ is odd,} \\ h_r &= h + h_{(2)}r^2 + \cdots + h_{(n-2)}r^{n-2} + h_{(n)}r^n \log r + kr^n + o(r^n) && \text{if } n \text{ is even,} \end{aligned}$$

where the terms $h_{(\ell)}$ for $\ell \leq n$ even are locally determined by h while the term k is nonlocal. For example,

$$h_{(2)} = -\frac{1}{n-2}\left(\mathrm{Ric}_h - \frac{1}{2(n-1)}R_h h\right)$$

is the negative of the *Schouten tensor* of h. For further details, including a discussion of optimal regularity, see [Chruściel et al. 2005] and the references therein.

Given a Poincaré–Einstein manifold (X^{n+1}, M^n, g_+), a representative h of the conformal boundary, and a parameter $\gamma \in \left(0, \frac{1}{2}n\right) \setminus \mathbb{N}$ such that $\frac{1}{4}n^2 - \gamma^2$ does not lie in the L^2-spectrum of $-\Delta_{g_+}$, we define the *fractional GJMS operator* $P_{2\gamma}$ as follows: Let $s = \frac{1}{2}n + \gamma$. For any $f \in C^\infty(M)$, there exists a unique solution v, denoted $\mathcal{P}\left(\frac{1}{2}n + \gamma\right)f$, of the generalized eigenvalue problem

$$-\Delta_{g_+} v - s(n-s)v = 0 \tag{2-1a}$$

such that, asymptotically near M,

$$v = Fr^{n-s} + Gr^s \tag{2-1b}$$

for $F, G \in C^\infty(\overline{X})$ and $F|_M = f$. Then

$$P_{2\gamma} f := d_\gamma G|_M \quad \text{for } d_\gamma = 2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)}. \tag{2-2}$$

Among the key properties of the fractional GJMS operator $P_{2\gamma} : C^\infty(M) \to C^\infty(M)$ are that it is formally self-adjoint, that its principal symbol is that of $(-\Delta)^\gamma$, and that it is conformally covariant; indeed, if $\hat{h} = e^{2\sigma}h$ is another representative of the conformal boundary, then

$$\widehat{P}_{2\gamma}(f) = e^{-\frac{n+2\gamma}{2}\sigma} P_{2\gamma}(e^{\frac{n-2\gamma}{2}\sigma} f)$$

for all $f \in C^\infty(M)$. In fact, this definition extends to the cases $\gamma \in \mathbb{N}$ by analytic continuation, and in these cases the operators $P_{2\gamma}$ recover the GJMS operators. For further details, see [Graham and Zworski 2003].

A useful fact about the solution v of (2-1) is that, up to order $r^{\frac{n}{2}}$, the Taylor series expansion of F (resp. G) is even in r and depends only on h and $F|_M$ (resp. $G|_M$). For example,

$$F = f + \frac{1}{4(1-\gamma)}\left(-\overline{\Delta} f + \tfrac{1}{2}(n-2\gamma)\overline{J} f\right)r^2 + o(r^2), \tag{2-3}$$

where $\overline{J}$ is the trace (with respect to h) of the Schouten tensor $\overline{P}$ and we adopt the convention that barred operators are defined with respect to the boundary (M^n, h).

The fractional GJMS operators $P_{2\gamma}$ can be interpreted as generalized Dirichlet-to-Neumann operators associated to weighted GJMS operators. To state this precisely and in the widest generality in which we are interested requires a discussion of smooth metric measure spaces.

Smooth metric measure spaces. A *smooth metric measure space* is a five-tuple $(\overline{X}^{n+1}, g, \rho, m, 1)$ formed from a smooth manifold $\overline{X}^{n+1}$ with (possibly empty) boundary $M^n = \partial\overline{X}$, a Riemannian metric g on $\overline{X}$, a nonnegative function $\rho \in C^\infty(\overline{X})$ with $\rho^{-1}(0) = M$, and a dimensional constant $m \in (1-n, \infty)$. Given such a smooth metric measure space, we always denote by X the interior of $\overline{X}$. Heuristically, the interior of a smooth metric measure space represents the base of a warped product

$$(X^{n+1} \times S^m,\ g \oplus \rho^2 d\theta^2) \tag{2-4}$$

for $(S^m, d\theta^2)$ the m-sphere with a metric of constant sectional curvature one; this is the meaning of the 1 as the fifth element of the five-tuple defining a smooth metric measure space. The choice of the standard m-sphere allows us to partially compactify (2-4), though not necessarily smoothly, by adding the boundary M of X. The model case is the upper half space $(\mathbb{R}_+ \times \mathbb{R}^n, dy^2 \oplus dx^2, y, m, 1)$ for y the coordinate on $\mathbb{R}_+ := (0, \infty)$; in this case the warped product (2-4) is the flat metric on $\mathbb{R}^{n+m+1} \setminus \{0\}$, and the partial compactification obtained from $[0, \infty) \times \mathbb{R}^n$ is the whole of $\mathbb{R}^{n+m+1}$.

The heuristic of passing through the warped product (2-4) is useful in that most geometric invariants defined on a smooth metric measure space — and all which are considered in this article — can be formally obtained by considering their Riemannian counterparts on (2-4) while restricting to the base X. More precisely, when $m \in \mathbb{N}$, the warped product (2-4) makes sense and one can define invariants on X in terms of Riemannian invariants on (2-4) by means of the canonical projection $\pi : X^{n+1} \times S^m \to X^{n+1}$. Invariants obtained in this way are polynomial in m, and can be extended to general $m \in (1-n, \infty)$ by treating m as a formal variable. This is illustrated by means of specific examples below.

The *weighted Laplacian* $\Delta_\phi : C^\infty(X) \to C^\infty(X)$ is defined by

$$\Delta_\phi U := \Delta U + m\rho^{-1}\langle \nabla\rho, \nabla U\rangle.$$

This operator is formally self-adjoint with respect to the measure $\rho^m\ \mathrm{dvol}_g$; the notation Δ_ϕ is used for consistency with the literature on smooth metric measure spaces, where one usually writes $\rho^m = e^{-\phi}$ and allows m to become infinite. In terms of (2-4), one readily checks that $\pi^*\Delta_\phi U = \boldsymbol{\Delta}(\pi^* U)$ for $\boldsymbol{\Delta}$ the Laplacian of (2-4). The *weighted Schouten scalar* J_ϕ^m and the *weighted Schouten tensor* P_ϕ^m are the tensors

$$\begin{aligned} J_\phi^m &:= \frac{1}{2(m+n)}\bigl(R - 2m\rho^{-1}\Delta\rho - m(m-1)\rho^{-2}(|\nabla\rho|^2 - 1)\bigr), \\ P_\phi^m &:= \frac{1}{m+n-1}\bigl(\mathrm{Ric} - m\rho^{-1}\nabla^2\rho - J_\phi^m\bigr). \end{aligned}$$

Denoting by $\boldsymbol{P}$ the Schouten tensor of (2-4) and by $\boldsymbol{J}$ its trace, one readily checks that $\boldsymbol{J} = \pi^* J_\phi^m$ and that $P_\phi^m(Z, Z) = \boldsymbol{P}(\tilde{Z}, \tilde{Z})$ for all $Z \in TX$, where $\tilde{Z}$ is the horizontal lift of Z to $X \times S^m$. The *weighted conformal Laplacian* $L_{2,\phi}^m : C^\infty(X) \to C^\infty(X)$ and the *weighted Paneitz operator* $L_{4,\phi}^m : C^\infty(X) \to C^\infty(X)$ are defined by

$$\begin{aligned} L_{2,\phi}^m U &:= -\Delta_\phi U + \tfrac{1}{2}(m+n-1)J_\phi^m U, \\ L_{4,\phi}^m U &:= (-\Delta_\phi)^2 U + \delta_\phi\bigl((4P_\phi^m - (m+n-1)J_\phi^m g)(\nabla U)\bigr) + \tfrac{1}{2}(m+n-3)Q_\phi^m U, \end{aligned}$$

where $\delta_\phi X = \operatorname{tr}_g \nabla X + m\rho^{-1}\langle X, \nabla\rho\rangle$ is the negative of the formal adjoint of the gradient with respect to ρ^m dvol,

$$Q_\phi^m := -\Delta_\phi J_\phi^m - 2|P_\phi^m|^2 - \frac{2}{m}(Y_\phi^m)^2 + \frac{m+n-1}{2}(J_\phi^m)^2$$

is the *weighted Q-curvature*, and $Y_\phi^m = J_\phi^m - \operatorname{tr}_g P_\phi^m$. Observe that the weighted conformal Laplacian and the weighted Paneitz operator are both formally self-adjoint with respect to ρ^m dvol. These definitions recover the conformal Laplacian and the Paneitz operator, respectively, of (2-4) when restricted to the base.

An important property of the weighted conformal Laplacian and the weighted Paneitz operator is that they are both conformally covariant. Two smooth metric measure spaces $(\overline{X}^{n+1}, g, \rho, m, 1)$ and $(\overline{X}^{n+1}, \hat{g}, \hat{\rho}, m, 1)$ are *pointwise conformally equivalent* if there is a function $\sigma \in C^\infty(\overline{X})$ such that $\hat{g} = e^{2\sigma} g$ and $\hat{\rho} = e^\sigma \rho$. This is equivalent to requiring that the respective warped products (2-4) are pointwise conformally equivalent with conformal factor independent of S^m. Under this assumption, it holds that

$$\widehat{L_{2,\phi}^m}(U) = e^{-\frac{m+n+3}{2}\sigma} L_{2,\phi}^m (e^{\frac{m+n-1}{2}\sigma} U), \tag{2-5}$$

$$\widehat{L_{4,\phi}^m}(U) = e^{-\frac{m+n+5}{2}\sigma} L_{2,\phi}^m (e^{\frac{m+n-3}{2}\sigma} U) \tag{2-6}$$

for all $U \in C^\infty(X)$.

As defined above, the weighted conformal Laplacian and the weighted Paneitz operator are defined only in the interior of a smooth metric measure space. The purpose of this article is to introduce and study boundary operators associated to the weighted conformal Laplacian and the Paneitz operator, respectively, which share their conformal covariance and formal self-adjointness properties. To do this in such a way as to meaningfully study Poincaré–Einstein manifolds and the fractional GJMS operators requires us to allow weaker-than-C^∞ regularity for both the metric g and the function ρ at the boundary of our smooth metric measure spaces. This requires some definitions.

Definition 2.1. Let $(\overline{X}^{n+1}, g)$ be a Riemannian manifold with nonempty boundary $M = \partial\overline{X}$. Let $\gamma \in \left(0, \frac{n}{2}\right) \setminus \mathbb{N}$ and set $k = \lfloor \gamma \rfloor$ and $m = 1 + 2k - 2\gamma$. The smooth metric measure space $(\overline{X}, g, r, m, 1)$ is *geodesic* if $|\nabla r|^2 = 1$ in a neighborhood of M and if

$$g = dr^2 + \sum_{j=0}^{k} h_{(2j)} r^{2j} + o(r^{2\gamma}) \tag{2-7}$$

for sections $h_{(0)}, \dots, h_{(2k)}$ of S^2T^*M.

The asymptotic expansion (2-7) is to be understood in the following way: Each point $p \in M$ admits an open neighborhood $U \subset \overline{X}$ and a constant $\varepsilon > 0$ such that the map

$$[0, \varepsilon) \times V \to U, \quad (t, q) \mapsto \gamma_q(t), \tag{2-8}$$

is a diffeomorphism with image U, where $V := U \cap M$ and γ_q is the integral curve in the direction ∇r originating at q. By shrinking U if necessary, we may assume that $|\nabla r|^2 = 1$ in U, and hence $r(\gamma_q(t)) = t$; note that if M is compact, then we may take U to be a neighborhood of M. The composition of the canonical projection $[0,\varepsilon)\times V \to V$ with the inverse of the diffeomorphism (2-8) gives a map $\pi : U \to V$. We then consider covariant tensor fields on V as covariant tensor fields in U by pulling them back by π. Finally, since $|\nabla r|^2 = 1$ in a neighborhood of M, it is straightforward to check that there is a one-parameter family h_r of sections of S^2T^*M such that $g = dr^2 + h_r$ near M. The assumption (2-7) imposes the additional requirement that h_r is even in r to order $o(r^{2\gamma})$. In particular, if $\gamma > \frac{1}{2}$, then M is totally geodesic with respect to g; if also $\gamma > \frac{3}{2}$, then the scalar curvature R of g satisfies $\partial_r R = 0$ along M.

Note that if r is a geodesic defining function for a Poincaré–Einstein manifold (X^{n+1}, M^n, g_+) and if m, γ are as in Definition 2.1, then $(\overline{X}, r^2 g_+, r, m, 1)$ is a geodesic smooth metric measure space.

Definition 2.2. Let $\overline{X}^{n+1}$ be a smooth manifold with boundary $M = \partial\overline{X}$ and let $\gamma \in \left(0, \frac{n}{2}\right) \setminus \mathbb{N}$. Set $k = \lfloor \gamma \rfloor$ and $m = 1 + 2k - 2\gamma$. A smooth metric measure space $(\overline{X}^{n+1}, g, \rho, m, 1)$ is *γ-admissible* if it is pointwise conformally equivalent to a geodesic smooth metric measure space $(\overline{X}, g_0, r, m, 1)$ such that

$$\frac{\rho}{r} = \sum_{j=0}^{k} \rho_{(2j)} r^{2j} + \Phi r^{2\gamma} + o(r^{2\gamma}) \tag{2-9}$$

for $\rho_{(0)}, \dots, \rho_{(2k)}, \Phi \in C^\infty(M)$ and $\rho_{(0)} = 1$.

Note that if $(\overline{X}, g, \rho, m, 1)$ is a γ-admissible smooth metric measure space and there are two geodesic smooth metric measure spaces $(\overline{X}, g_i, r_i, m, 1)$, $i \in \{1, 2\}$, as in Definition 2.2, then $r_2 = r_1$ near M (see [Graham and Lee 1991, Lemma 5.2] or [Lee 1995, Lemma 5.1]); in particular, all asymptotic statements about γ-admissible smooth metric measure spaces (e.g., (2-9)) are independent of the choice of geodesic smooth metric measure space in Definition 2.2. Combining the expansions (2-7) and (2-9), we see that if $(\overline{X}, g, \rho, m, 1)$ is a γ-admissible smooth metric measure space with $\gamma > \frac{1}{2}$, then M is totally geodesic (with respect to g); if also $\gamma > \frac{3}{2}$, then $\partial_\rho R = 0$ along M.

Given a Poincaré–Einstein manifold (X^{n+1}, M^n, g_+) and $\gamma \in \left(0, \frac{n}{2}\right) \setminus \mathbb{N}$, a defining function ρ is *γ-admissible* if $(\overline{X}, \rho^2 g_+, \rho, m, 1)$, $m = 1 - 2\lfloor\gamma\rfloor - 2\gamma$, is a γ-admissible smooth metric measure space. In particular, the extension theorems established in [Case and Chang 2016, Theorems 4.1 and 4.4] are all stated in terms of γ-admissible smooth metric measure spaces. An important example of γ-admissible smooth metric measure spaces which arise as compactifications of Poincaré–Einstein manifolds and for which the function Φ in (2-9) is not necessarily zero are obtained from the adapted defining function [Case and Chang 2016, Section 6.1].

In light of both our weakened regularity hypotheses and the asymptotics of solutions to the Poisson equation (2-1), it is natural to introduce the following function spaces.

Definition 2.3. Fix $\gamma \in (0, 1)$, set $m = 1 - 2\gamma$, and let $(\overline{X}^{n+1}, g, \rho, m, 1)$ be a γ-admissible smooth metric measure space. Given $f \in C^\infty(M)$, denote by $\mathcal{C}^\gamma_f$ the set of all $U \in C^\infty(X) \cap C^0(\overline{X})$ such that, asymptotically near M,

$$U = f + \psi \rho^{2\gamma} + o(\rho^{2\gamma}) \tag{2-10}$$

for some $\psi \in C^\infty(M)$. Set

$$\mathcal{C}^\gamma := \bigcup_{f \in C^\infty(M)} \mathcal{C}^\gamma_f. \tag{2-11}$$

The Sobolev spaces $W_0^{1,2}(\overline{X}, \rho^m \text{ dvol})$ and $W^{1,2}(\overline{X}, \rho^m \text{ dvol})$ are the completions of $\mathcal{C}_0^\gamma$ and $\mathcal{C}^\gamma$, respectively, with respect to the norm

$$\|U\|_{W^{1,2}}^2 := \int_X (|\nabla U|^2 + U^2)\rho^m \text{ dvol}.$$

For notational convenience, in the case $\gamma \in (0,1)$ we sometimes denote by $\mathcal{D}^\gamma$ the space $\mathcal{C}^\gamma$ and by $\mathcal{H}^\gamma$ the space $W^{1,2}(\overline{X}, \rho^m \text{ dvol})$.

When $\gamma \in (0,1)$, the Sobolev trace theorem (e.g., [Triebel 1978]) states that there is a surjective bounded linear operator $\operatorname{Tr}: W^{1,2}(\overline{X}, \rho^m \text{ dvol}) \to H^\gamma(M)$ such that $\operatorname{Tr} U = f$ for every $U \in \mathcal{C}_f^\gamma$, where $H^\gamma(M)$ denotes the completion of $C^\infty(M)$ with respect to the norm obtained by pulling back

$$\|f\|_{H^\gamma(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} f^2 dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\gamma}}\, dx\, dy$$

to M via coordinate charts.

Definition 2.4. Fix $\gamma \in (1,2)$, set $m = 3-2\gamma$, and let $(\overline{X}^{n+1}, g, \rho, m, 1)$ be a γ-admissible smooth metric measure space. Given $f, \psi \in C^\infty(M)$, denote by $\mathcal{C}_{f,\psi}^\gamma$ the set of all $U \in C^\infty(X) \cap C^0(\overline{X})$ such that, asymptotically near M,

$$U = f + \psi\rho^{2\gamma-2} + f_2\rho^2 + \psi_2\rho^{2\gamma} + o(\rho^{2\gamma}) \tag{2-12}$$

for some $f_2, \psi_2 \in C^\infty(M)$. Set

$$\mathcal{C}^\gamma := \bigcup_{f,\psi \in C^\infty(M)} \mathcal{C}_{f,\psi}^\gamma, \tag{2-13}$$

$$\mathcal{D}^\gamma := \bigcup_{f \in C^\infty(M)} \mathcal{C}_{f,0}^\gamma. \tag{2-14}$$

The Sobolev spaces $W_0^{2,2}(\overline{X}, \rho^m \text{ dvol})$, $W^{2,2}(\overline{X}, \rho^m \text{ dvol})$, and $\mathcal{H}^\gamma$ are the completions of $\mathcal{C}_{0,0}^\gamma$, $\mathcal{C}^\gamma$, and $\mathcal{D}^\gamma$, respectively, with respect to the norm

$$\|U\|_{W^{2,2}}^2 := \int_X \left(\left|\nabla^2 U + m\rho^{-1}(\partial_\rho U)^2 d\rho \otimes d\rho\right|^2 + |\nabla U|^2 + U^2\right)\rho^m \text{ dvol}. \tag{2-15}$$

The particular modification of the Hessian used in (2-15) ensures that the integral is finite for all $U \in \mathcal{C}^\gamma$. Given $U \in \mathcal{C}_{f,\psi}^\gamma$, the weighted Bochner formula (see the Appendix) allows one to rewrite this Hessian term in terms of the L^2-norm of $\Delta_\phi U$, lower-order interior terms depending on curvature, and boundary terms involving only f and ψ.

When $\gamma \in (1,2)$, the Sobolev trace theorem (see the Appendix) states that there is a surjective bounded linear operator $\operatorname{Tr}: W^{2,2}(\overline{X}, \rho^m \text{ dvol}) \to H^\gamma(M) \oplus H^{2-\gamma}(M)$ such that $\operatorname{Tr}(U) = (f, \psi)$ for every $U \in \mathcal{C}_{f,\psi}^\gamma$, where $H^\gamma(M)$ denotes the completion of $C^\infty(M)$ with respect to the norm obtained by pulling back

$$\|f\|_{H^\gamma(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} (f^2 + |\nabla f|^2)\, dx + \sum_{j=1}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\partial_j f(x) - \partial_j f(y)|^2}{|x-y|^{n+2\gamma-2}}\, dx\, dy$$

to M via coordinate charts.

We conclude with two useful observations. The first is the following relationship between a defining function for a Poincaré–Einstein manifold and certain weighted geometric invariants of the induced compactification.

Lemma 2.5 [Case and Chang 2016, Lemma 3.2]. *Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold and ρ be a defining function. Fix $m > 1-n$. The smooth metric measure space $(X^{n+1}, g := \rho^2 g_+, \rho, m, 1)$ has*

$$J + \rho^{-1}\Delta\rho = \tfrac{1}{2}(n+1)\rho^{-2}(|\nabla\rho|^2 - 1), \tag{2-16}$$

$$J_\phi^m = J - \frac{m}{n+1}(J + \rho^{-1}\Delta\rho), \tag{2-17}$$

$$P_\phi^m = P. \tag{2-18}$$

Here P and J are the Schouten tensor of g and its trace, respectively.

The second is the following characterization of pointwise conformally equivalent γ-admissible smooth metric measure spaces in terms of the conformal factors.

Lemma 2.6. *Fix $\gamma \in (0,2) \setminus \{1\}$ and let $(\overline{X}^{n+1}, g, \rho, m, 1)$ be a γ-admissible smooth metric measure space with $m = 1 + 2\lfloor\gamma\rfloor - 2\gamma$. Let $\sigma \in C^\infty(X) \cap C^0(\overline{X})$ and set $\hat{g} = e^{2\sigma} g$ and $\hat{\rho} = e^\sigma \rho$. Then $(\overline{X}^{n+1}, \hat{g}, \hat{\rho}, m, 1)$ is a γ-admissible smooth metric measure space if and only if $\sigma \in \mathcal{D}^\gamma$.*

Proof. Let $(\overline{X}, g_0, r, m, 1)$ and $(\overline{X}, \hat{g}_0, \hat{r}, m, 1)$ be geodesic smooth metric measure spaces associated to $(\overline{X}, g, \rho, m, 1)$ and $(\overline{X}, \hat{g}, \hat{r}, m, 1)$, respectively, as in Definition 2.2. Suppose first that $(\overline{X}, \hat{g}, \hat{\rho}, m, 1)$ is γ-admissible. We readily check that

$$e^\sigma = \frac{\hat{\rho}}{\hat{r}} \cdot \frac{\hat{r}}{r} \cdot \frac{r}{\rho} \in \mathcal{D}^\gamma,$$

whence $\sigma \in \mathcal{D}^\gamma$. Conversely, if $\sigma \in \mathcal{D}^\gamma$, we readily check that $\hat{r}/r \in \mathcal{D}^\gamma$, whence $(\overline{X}^{n+1}, \hat{g}, \hat{\rho}, m, 1)$ is γ-admissible. ☐

For the remainder of this article, unless otherwise specified, the measure with respect to which an integral is evaluated is specified by context: if a smooth metric measure space $(\overline{X}^{n+1}, g, \rho, m, 1)$ with boundary $M = \partial\overline{X}$ is given, all integrals over X are evaluated with respect to $\rho^m \, \mathrm{dvol}_g$ and all integrals over M are evaluated with respect to the Riemannian volume element of $g|_{TM}$.

3. The conformally covariant boundary operators

In order to study boundary value problems associated to the weighted conformal Laplacian and the weighted Paneitz operator — for instance, to study the fractional GJMS operators as in [Case and Chang 2016] — it is useful to find conformally covariant boundary operators associated to these respective operators. In the case of the weighted conformal Laplacian $L_{2,\phi}^m$ with $m = 1-2\gamma$, this means finding conformally covariant operators $B_0^{2\gamma}$ and $B_{2\gamma}^{2\gamma}$ such that $(L_{2,\phi}^m U, V) = (L_{2,\phi}^m V, U)$ for all $U, V \in \ker B_0^{2\gamma}$ or for all $U, V \in \ker B_{2\gamma}^{2\gamma}$. That is, the boundary value problems $(L_{2,\phi}^m; B_0^{2\gamma})$ and $(L_{2,\phi}^m; B_{2\gamma}^{2\gamma})$ are formally self-adjoint. In the case of the weighted Paneitz operator $L_{4,\phi}^m$ with $m = 3-2\gamma$, this means defining conformally covariant operators $B_0^{2\gamma}, B_{2\gamma-2}^{2\gamma}, B_2^{2\gamma}, B_{2\gamma}^{2\gamma}$ such that $(L_{4,\phi}^m U, V) = (L_{4,\phi}^m V, U)$ for all U, V in the kernel of one of the pairs

$$\mathfrak{B}_1 = (B_0^{2\gamma}, B_2^{2\gamma}), \quad \mathfrak{B}_2 = (B_0^{2\gamma}, B_{2\gamma-2}^{2\gamma}), \quad \text{or} \quad \mathfrak{B}_3 = (B_{2\gamma-2}^{2\gamma}, B_{2\gamma}^{2\gamma}).$$

That is, the boundary value problems $(L_{4,\phi}^m; \mathfrak{B}_j)$ for $j \in \{1,2,3\}$ are all formally self-adjoint. These boundary value problems are all elliptic, as is apparent from the definitions of the operators given below, and our definitions are such that the formal self-adjointness follows from simple integration-by-parts identities; see Theorem 3.2 for the case of the weighted conformal Laplacian and Theorem 3.4 and Theorem 3.7 for the case of the weighted Paneitz operator.

The existence of such operators when $m = 0$ is already known: $B = \eta + \frac{n-1}{2n} H$ is a boundary operator for the conformal Laplacian (see [Branson 1997; Escobar 1990]), while Branson and Gover [2001] have constructed via the tractor calculus conformally covariant boundary operators associated to the noncritical GJMS operators and Grant [2003] derived the third-order boundary operator associated to the Paneitz operator (see also [Chang and Qing 1997; Juhl 2009] for the case of critical dimension). As is apparent from Definition 3.1, $B_1^1 = B$, while the operators B_k^3 for $k \in \{0,1,2,3\}$ give explicit formulae for the boundary operators associated to the Paneitz operator in the case of manifolds with totally geodesic boundary; see [Case 2015] for the general case.

The case $\gamma \in (0,1)$. The conformally covariant boundary operators associated to the weighted conformal Laplacian are defined as follows.

Definition 3.1. Fix $\gamma \in (0,1)$ and set $m = 1 - 2\gamma$. Let $(\overline{X}^{n+1}, g, \rho, m, 1)$ be a γ-admissible smooth metric measure space with boundary $M = \partial\overline{X}$ and let $(\overline{X}, g_0, r, m, 1)$ be the geodesic smooth metric measure space as in Definition 2.2. Set $\eta = -\frac{\rho}{r}\nabla^g r$. As operators mapping $\mathcal{C}^\gamma$ to $C^\infty(M)$,

$$B_0^{2\gamma} U := U|_M, \quad B_{2\gamma}^{2\gamma} U := \lim_{\rho\to 0} \rho^m \left(\eta U + \frac{n-2\gamma}{2n} U \delta\eta \right).$$

where $\delta\eta := \operatorname{tr}_g \nabla^g \eta$.

Note that η is the outward-pointing unit normal (with respect to g) vector field along the level sets of r in a neighborhood of M. In particular, if $\gamma = \frac{1}{2}$, then $\delta\eta|_M = H$ is the mean curvature of M with respect to g. For this reason, we call

$$H_{2\gamma} := \lim_{\rho\to 0} \rho^m \delta\eta$$

the *γ-mean curvature* of M. Since $(\overline{X}^{n+1}, g_0, r, m, 1)$ is uniquely determined near M by $(\overline{X}, g, \rho, m, 1)$, the asymptotic assumptions of Definition 3.1 guarantee that the γ-mean curvature and the operators $B_0^{2\gamma}$ and $B_{2\gamma}^{2\gamma}$ are well-defined; indeed,

$$\begin{aligned} H_{2\gamma} &= -2n\gamma\Phi, \\ B_0^{2\gamma} U &= f, \\ B_{2\gamma}^{2\gamma} U &= -2\gamma\big(\psi + \tfrac{1}{2}(n-2\gamma)\Phi f\big), \end{aligned}$$

where ρ and U satisfy (2-9) and (2-10), respectively, near M.

That the operators $B_0^{2\gamma}$ and $B_{2\gamma}^{2\gamma}$ are the conformally covariant boundary operators associated to the weighted conformal Laplacian is a consequence of the following result.

Theorem 3.2. *Fix $\gamma \in (0,1)$ and set $m = 1-2\gamma$. Let $(\overline{X}^{n+1}, g, \rho, m, 1)$ and $(\overline{X}^{n+1}, \hat{g}, \hat{\rho}, m, 1)$ be two pointwise conformally equivalent γ-admissible smooth metric measure spaces with $\hat{g} = e^{2\sigma} g$ and $\hat{\rho} = e^{\sigma}\rho$. Then for any $U \in \mathcal{C}^\gamma$ it holds that*

$$\widehat{B}_0^{2\gamma}(U) = e^{-\frac{n-2\gamma}{2}\sigma|_M} B_0^{2\gamma}(e^{\frac{n-2\gamma}{2}\sigma} U), \tag{3-1}$$

$$\widehat{B}_{2\gamma}^{2\gamma}(U) = e^{-\frac{n+2\gamma}{2}\sigma|_M} B_{2\gamma}^{2\gamma}(e^{\frac{n-2\gamma}{2}\sigma} U). \tag{3-2}$$

Moreover, given $U, V \in \mathcal{C}^\gamma$, it holds that

$$\int_X V L_{2,\phi}^m U + \oint_M B_0^{2\gamma}(V) B_{2\gamma}^{2\gamma}(U) = \mathcal{Q}_{2\gamma}(U,V) \tag{3-3}$$

for $\mathcal{Q}_{2\gamma}$ the symmetric bilinear form

$$\mathcal{Q}_{2\gamma}(U,V) = \int_X \left(\langle \nabla U, \nabla V \rangle + \frac{n-2\gamma}{2} J_\phi^m U V \right) + \frac{n-2\gamma}{2n} \oint_M H_{2\gamma} B_0^{2\gamma}(U) B_0^{2\gamma}(V).$$

In particular, $\mathcal{Q}_{2\gamma}$ is conformally covariant.

Proof. Equation (3-1) follows immediately from the definition of $B_0^{2\gamma}$.

By Lemma 2.6, we have that $\sigma \in \mathcal{C}^\gamma$, and in particular $\rho^m \eta\sigma$ is well-defined. On the other hand, if η and $\hat{\eta}$ are as in Definition 3.1, then $\hat{\eta} = e^{-\sigma}\eta$. Hence

$$\hat{\rho}^m \hat{\eta}(e^{-\frac{n-2\gamma}{2}\sigma} U) = e^{-\frac{n+2\gamma}{2}\sigma} \rho^m \big(\eta U - \tfrac{1}{2}(n-2\gamma) U \eta\sigma \big),$$
$$\hat{\rho}^m \hat{\delta}\hat{\eta} = e^{-2\gamma\sigma} \rho^m (\delta\eta + n\eta\sigma).$$

Combining these two equations yields (3-2).

Finally, integration by parts yields (3-3). Combining (3-1) and (3-2) with (3-3) yields the conformal covariance of $\mathcal{Q}_{2\gamma}$. $\square$

The case $\gamma \in (1,2)$. The conformally covariant boundary operators associated to the weighted Paneitz operator are defined as follows.

Definition 3.3. Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let $(\overline{X}^{n+1}, g, \rho, m, 1)$ be a γ-admissible smooth metric measure space with boundary $M = \partial\overline{X}$ and let η be as in Definition 3.1. As operators mapping $\mathcal{C}^\gamma$ to $C^\infty(M)$,

$$\begin{aligned}
B_0^{2\gamma} U &:= U, \\
B_{2\gamma-2}^{2\gamma} U &:= \rho^m \eta U, \\
B_2^{2\gamma} U &:= -\frac{2-\gamma}{\gamma-1} \overline{\Delta} U + (\nabla^2 U(\eta,\eta) + m\rho^{-1}\partial_\rho U) + \frac{n-2\gamma}{2} T_2^{2\gamma} U, \\
B_{2\gamma}^{2\gamma} U &:= -\rho^m \eta \Delta_\phi U - \frac{1}{\gamma-1} \overline{\Delta} \rho^m \eta U + S_2^{2\gamma} \rho^m \eta U + \frac{n-2\gamma}{2} (\rho^m \eta J_\phi^m) U,
\end{aligned}$$

where

$$T_2^{2\gamma} := \frac{2-\gamma}{\gamma-1}\bar{J} - \left(P(\eta,\eta) - \frac{3-2\gamma}{n+1}(\bar{J}+\rho^{-1}\Delta\rho + P(\eta,\eta)) \right), \tag{3-4}$$

$$S_2^{2\gamma} := \left(\frac{n-2\gamma}{2} + \frac{n+2\gamma-4}{2(\gamma-1)} \right)\bar{J} + \frac{n-2\gamma-4}{2}P(\eta,\eta) - \frac{(3-2\gamma)(n-2\gamma+4)}{2(n+1)}(\bar{J}+\rho^{-1}\Delta\rho+P(\eta,\eta)) \tag{3-5}$$

and we understand the right-hand sides to all be evaluated in the limit $\rho \to 0$.

Due to the length of the computations, we break the proof that the operators given in Definition 3.3 are conformally covariant boundary operators associated to the weighted Paneitz operator on γ-admissible smooth metric measure spaces into two parts. First, we show that they are conformally covariant of the correct weight.

Theorem 3.4. *Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let $(\bar{X}^{n+1}, g, \rho, m, 1)$ and $(\bar{X}^{n+1}, \hat{g}, \hat{\rho}, m, 1)$ be two pointwise conformally equivalent γ-admissible smooth metric measure spaces with $\hat{g} = e^{2\sigma}g$ and $\hat{\rho} = e^{\sigma}\rho$. Then for any $U \in \mathcal{C}^\gamma$ it holds that*

$$\hat{B}_0^{2\gamma}U = e^{-\frac{n-2\gamma}{2}\sigma|_M} B_0^{2\gamma}(e^{\frac{n-2\gamma}{2}\sigma}U), \tag{3-6}$$

$$\hat{B}_{2\gamma-2}^{2\gamma}U = e^{-\frac{n+2\gamma-4}{2}\sigma|_M} B_{2\gamma-2}^{2\gamma}(e^{\frac{n-2\gamma}{2}\sigma}U), \tag{3-7}$$

$$\hat{B}_2^{2\gamma}U = e^{-\frac{n-2\gamma+4}{2}\sigma|_M} B_2^{2\gamma}(e^{\frac{n-2\gamma}{2}\sigma}U), \tag{3-8}$$

$$\hat{B}_{2\gamma}^{2\gamma}U = e^{-\frac{n+2\gamma}{2}\sigma|_M} B_{2\gamma}^{2\gamma}(e^{\frac{n-2\gamma}{2}\sigma}U). \tag{3-9}$$

The proof of Theorem 3.4 is a somewhat lengthy computation. While such computations are routine in conformal geometry (see [Branson 1985; Chang and Qing 1997]), they have not been carried out in this form in the literature for smooth metric measure spaces, and so we sketch the details here.

Fix $\gamma \in (1,2)$. An operator $T : \mathcal{C}^\gamma \to C^\infty(M)$ defined on a γ-admissible smooth metric measure space $(\bar{X}^{n+1}, g, \rho, m, 1)$ with boundary $M = \partial\bar{X}$ is *natural* if it can be expressed as a polynomial involving the Levi-Civita connection and the Riemann curvature tensor of g, powers of ρ, the outward-pointing normal η along $M = \partial\bar{X}$, and contractions thereof. A natural operator T is said to be *homogeneous of degree* $k \in \mathbb{R}$ if for any positive constant $c \in \mathbb{R}$, the operators T and $\hat{T}$ defined on $(\bar{X}^{n+1}, g, \rho, m, 1)$ and $(\bar{X}^{n+1}, \hat{g}, \hat{\rho}, m, 1)$, respectively, for $\hat{g} = c^2 g$ and $\hat{\rho} = c\rho$, are related by

$$\hat{T}(U) = c^k T(U)$$

for all U in the domain $\mathrm{Dom}(T)$ of T. Given a homogeneous operator T of degree k, a function $\sigma \in \mathcal{D}^\gamma$, and a fixed weight $w \in \mathbb{R}$, we define

$$(T(U))' := \frac{\partial}{\partial t}\bigg|_{t=0} \left(e^{-(w+k)t\sigma|_M} T_{e^{2t\sigma}g}(e^{wt\sigma}U) \right), \tag{3-10}$$

where $T_{e^{2\sigma}g}$ denotes the operator T as defined with respect to the smooth metric measure space $(\bar{X}^{n+1}, e^{2\sigma}g, e^{\sigma}\rho, m, 1)$. One readily shows (see [Branson 1985, Corollary 1.14]) that, given a natural operator T which is homogeneous of degree k and a fixed weight w, it holds that

$$T_{e^{2\sigma}g}(U) = e^{(w+k)\sigma|_M} T(e^{-w\sigma}U)$$

for all $\sigma \in \mathcal{D}^\gamma$ and all $U \in \operatorname{Dom}(T)$ if and only if $(T(U))' = 0$ for all $\sigma \in \mathcal{D}^\gamma$ and all $U \in \operatorname{Dom}(T)$.

To prove Theorem 3.4, it thus suffices to compute the linearizations (3-10) of the operators given in Definition 3.3 — which are all natural and homogeneous — with the fixed weight $w = -\frac{1}{2}(n-2\gamma)$. We accomplish this through a pair of lemmas. We first consider operators which are homogeneous of degree -2.

Lemma 3.5. *Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let $(\overline{X}^{n+1}, g, \rho, m, 1)$ be a γ-admissible smooth metric measure space with boundary $M = \partial\overline{X}$. Let $\sigma \in \mathcal{D}^\gamma$ and let $U \in \mathcal{C}^\gamma$. Fix a weight $w \in \mathbb{R}$. Then*

$$\begin{aligned}
(\bar{\Delta}U)' &= (n+2w-2)\langle\overline{\nabla}U, \overline{\nabla}\sigma\rangle + wU\,\bar{\Delta}\sigma,\\
(\nabla^2U(\eta,\eta) + m\rho^{-1}\partial_\rho U)' &= (m+1)\langle\overline{\nabla}U, \overline{\nabla}\sigma\rangle + wU(\nabla^2\sigma(\eta,\eta) + m\rho^{-1}\partial_\rho\sigma),\\
(\bar{J}U)' &= -U\,\bar{\Delta}\sigma,\\
(UP(\eta,\eta))' &= -U\nabla^2\sigma(\eta,\eta),\\
(\rho^{-1}U\Delta\rho)' &= (\bar{\Delta}\sigma + \nabla^2\sigma(\eta,\eta) + (n+1)\rho^{-1}\partial_\rho\sigma)U.
\end{aligned}$$

Proof. Let $\hat{g} = e^{2t\sigma}g$. It is well-known that

$$\begin{aligned}
\widehat{P} &= P - t\nabla^2\sigma + O(t^2),\\
\widehat{\nabla}^2U &= \nabla^2U - t\,dU\otimes d\sigma - t\,d\sigma\otimes dU + t\langle\nabla U, \nabla\sigma\rangle_g\, g,
\end{aligned}$$

and similarly for quantities defined in terms of the induced metric on M. The conclusion readily follows. □

We next consider operators which are homogeneous of degree -2γ.

Lemma 3.6. *Under the same hypotheses as Lemma 3.5, it holds that*

$$\begin{aligned}
(\bar{\Delta}\rho^m\eta U)' &= (2m+n+2w-4)\langle\overline{\nabla}\rho^m\eta U, \overline{\nabla}\sigma\rangle + (m+w-1)(\rho^m\eta U)\bar{\Delta}\sigma,\\
(\rho^m\eta\Delta_\phi U)' &= (m+n+2w-1)\langle\overline{\nabla}\rho^m\eta U, \overline{\nabla}\sigma\rangle + wU\rho^m\eta\Delta_\phi\sigma\\
&\qquad + \big((m+n+2w-1)(\nabla^2\sigma(\eta,\eta) - m\rho^{-1}\partial_\rho\sigma) + w\Delta_\phi\sigma\big)\rho^m\eta U,\\
(U\rho^m\eta J_\phi^m)' &= -U\rho^m\eta\Delta_\phi\sigma.
\end{aligned}$$

Proof. The first identity follows immediately from Lemma 3.5 and the homogeneity of $\rho^m\eta$.

The conformal transformation formula for the weighted Laplacian [Case 2012] yields

$$(\rho^m\eta\Delta_\phi U)' = \rho^m\eta\big((m+n+2w-1)\langle\nabla U, \nabla\sigma\rangle + wU\Delta_\phi\sigma\big).$$

A straightforward computation shows that

$$\rho^m\eta(\langle\nabla U, \nabla\sigma\rangle) = \langle\overline{\nabla}\sigma, \overline{\nabla}\rho^m\eta U\rangle + (\nabla^2\sigma(\eta,\eta) - m\rho^{-1}\partial_\rho\sigma)\rho^m\eta U,$$

from which the second identity follows.

Finally, the conformal transformation formula for the weighted scalar curvature (see [Case 2012, Proposition 4.4]) yields the last identity. □

Proof of Theorem 3.4. It is clear that (3-6) and (3-7) hold.

It follows immediately from Lemma 3.5 that the operator $B_{2,w}$ defined by

$$B_{2,w}U := -\bar{\Delta}U + \frac{n+2w-2}{m+1}(\nabla^2 U(\eta,\eta) + m\rho^{-1}\partial_\rho U) \\ -w\left(\bar{J} - \frac{n+2w-2}{m+1}\left(P(\eta,\eta) - \frac{m}{n+1}(\bar{J} + P(\eta,\eta) + \rho^{-1}\Delta\rho)\right)\right)U$$

satisfies $(B_{2,w}U)' = 0$ for any $w \in \mathbb{R}$. This yields (3-8) upon observing that

$$B_2^{2\gamma} = \frac{2-\gamma}{\gamma-1}B_{2,-\frac{n-2\gamma}{2}}.$$

It follows immediately from Lemma 3.5 and Lemma 3.6 that the operator $B_{2\gamma,w}$ defined by

$$B_{2\gamma,w}U := -\rho^m\eta\Delta_\phi U + \frac{m+n+2w-1}{2m+n+2w-4}\bar{\Delta}\rho^m\eta U + T_{2,w}\rho^m\eta U - w(\rho^m\eta J_\phi^m)U,$$

$$T_{2,w} := \left(\frac{(m+w-1)(m+n+2w-1)}{2m+n+2w-4} - w\right)\bar{J} - (m+n+3w-1)P(\eta,\eta) \\ - \frac{m(m+n+w-1)}{n+1}(\bar{J} + P(\eta,\eta) + \rho^{-1}\Delta\rho)$$

satisfies $(B_{2\gamma,w}U)' = 0$ for any $w \in \mathbb{R}$. This yields (3-9) upon observing that $B_{2\gamma}^{2\gamma} = B_{2\gamma,-\frac{n-2\gamma}{2}}$. □

We next show that the operators given in Definition 3.3 are boundary operators associated to the weighted Paneitz operator, in the sense that the pairing

$$\mathcal{C}^\gamma \times \mathcal{C}^\gamma \ni (U,V) \mapsto (L_{4,\phi}^m U, V) + (B_{2\gamma}^{2\gamma}U, B_0^{2\gamma}V) + (B_2^{2\gamma}U, B_{2\gamma-2}^{2\gamma}V)$$

is a symmetric bilinear form. Indeed, this form can be written explicitly, and is the polarization of the energy associated to the weighted Paneitz operator on a γ-admissible smooth metric measure space with boundary.

Theorem 3.7. *Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let $(\bar{X}^{n+1}, g, \rho, m, 1)$ be a γ-admissible compact smooth metric measure space with boundary $M = \partial\bar{X}$. Given $U, V \in \mathcal{C}^\gamma$, it holds that*

$$\int_X VL_{4,\phi}^m U + \oint_M \left(B_0^{2\gamma}(V)B_{2\gamma}^{2\gamma}(U) + B_{2\gamma-2}^{2\gamma}(V)B_2^{2\gamma}(U)\right) = \mathcal{Q}_{2\gamma}(U,V) \tag{3-11}$$

for $\mathcal{Q}_{2\gamma}$ the symmetric bilinear form

$$\mathcal{Q}_{2\gamma}(U,V) \\ = \int_X \left[(\Delta_\phi U)(\Delta_\phi V) - (4P-(n-2\gamma+2)J_\phi^m g)(\nabla U,\nabla V) + \frac{n-2\gamma}{2}Q_\phi^m UV\right] \\ + \oint_M \left[\frac{1}{\gamma-1}\left(\langle\bar{\nabla}B_0^{2\gamma}(U),\bar{\nabla}B_{2\gamma-2}^{2\gamma}(V)\rangle + \langle\bar{\nabla}B_0^{2\gamma}(V),\bar{\nabla}B_{2\gamma-2}^{2\gamma}(U)\rangle\right) \right. \\ \left. + \frac{n-2\gamma}{2}T_2^{2\gamma}\left(B_0^{2\gamma}(U)B_{2\gamma-2}^{2\gamma}(V) + B_0^{2\gamma}(V)B_{2\gamma-2}^{2\gamma}(U)\right) + \frac{n-2\gamma}{2}(\rho^m\eta J_\phi^m)B_0^{2\gamma}(U)B_0^{2\gamma}(V)\right].$$

In particular, $\mathcal{Q}_{2\gamma}$ is conformally invariant.

Proof. Equation (3-11) follows from a straightforward computation using integration by parts, the consequences

$$\begin{aligned} J &= \bar{J} + P(\eta,\eta), \\ P(\eta,\nabla U) &= P(\eta,\eta)\eta U \end{aligned} \tag{3-12}$$

of the Gauss–Codazzi equations, and (2-17).

It follows immediately from Theorem 3.4 and (3-11) that $\mathcal{Q}_{2\gamma}$ is conformally invariant. □

4. Some asymptotic expansions

For Poincaré–Einstein manifolds, the fractional GJMS operator P_1 can be interpreted as the Dirichlet-to-Neumann operator for functions in the kernel of the conformal Laplacian [Chang and González 2011]. There is another conformally covariant operator defined on the boundary with the same principal symbol as P_1, namely $f \mapsto B_1^1 U$ for U the unique extension of f in the kernel of the conformal Laplacian [Branson 1997]. In fact, these two operators are the same [Guillarmou and Guillopé 2007].

The boundary operators introduced in Section 3 all give conformally covariant operators as follows: Fix $\gamma \in (0,2)\setminus\{1\}$ and set $k = \lfloor\gamma\rfloor + 1$ and $m = 2k-1-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold with $n > 2\gamma$, let ρ be a γ-admissible defining function, and consider $(\bar{X}, \rho^2 g_+, \rho, m, 1)$. Given a function $f \in C^\infty(M)$, let U be the unique extension of f in $\mathcal{C}^\gamma_{f,0}$ such that $L^m_{2k,\phi} U = 0$. Then the map $\mathcal{B}_{2\gamma}(f) := B^{2\gamma}_{2\gamma} U$ is conformally covariant in the sense that

$$\widehat{\mathcal{B}}_{2\gamma}(f) = e^{-\frac{n+2\gamma}{2}\sigma|_M}\mathcal{B}_{2\gamma}(e^{\frac{n-2\gamma}{2}\sigma|_M} f)$$

for all $\sigma \in \mathcal{D}^\gamma$, where $\widehat{\mathcal{B}}$ is defined in terms of $(\bar{X}^{n+1}, \hat{g}, \hat{\rho}, m, 1)$ for $\hat{g} = e^{2\sigma} g$ and $\hat{\rho} = e^\sigma \rho$. The fractional GJMS operators can also be regarded as generalized Dirichlet-to-Neumann operators associated to the kernel of the weighted conformal Laplacian and the weighted Paneitz operator in the cases $\gamma \in (0,1)$ and $\gamma \in (1,2)$, respectively [Case and Chang 2016]. We show that, as in the case $\gamma = \frac{1}{2}$, the operators $\mathcal{B}_{2\gamma}$ and $P_{2\gamma}$ are the same.

The case $\gamma \in (0,1)$. A direct computation using the definition of $B^{2\gamma}_{2\gamma}$ and [Case and Chang 2016, Theorem 4.1] readily shows that the fractional GJMS operator $P_{2\gamma}$ and the operator $\mathcal{B}_{2\gamma}$ defined above are the same when $\gamma \in (0,1)$.

Proposition 4.1. *Fix $\gamma \in (0,1)$ and set $m = 1-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold such that $\frac{1}{4}n^2 - \gamma^2 \notin \sigma_{pp}(-\Delta_{g_+})$. Let ρ be a γ-admissible defining function. Given $f \in C^\infty(M)$, let U be the solution to the boundary value problem*

$$\begin{cases} L^m_{2,\phi} U = 0 & \text{in } (X^{n+1}, \rho^2 g_+, \rho, m, 1), \\ U = f & \text{on } M. \end{cases} \tag{4-1}$$

Then

$$P_{2\gamma} f = -\frac{d_\gamma}{2\gamma} B^{2\gamma}_{2\gamma} U. \tag{4-2}$$

Proof. By conformal covariance, we may assume that $\rho = r$ is the geodesic defining function associated to $g|_{TM}$. From [Case and Chang 2016, Theorem 4.1], we see that $U = r^{-\frac{n-2\gamma}{2}}\mathcal{P}\big(\frac{1}{2}n+\gamma\big)f$. In particular, $U \in \mathcal{C}_f^\gamma$ and

$$P_{2\gamma}f = -\frac{d_\gamma}{2\gamma}\lim_{\rho\to 0}\rho^m\eta U$$

for $\eta = -\partial_r$ the outward-pointing normal along M. It is straightforward to check that $r^m\delta\eta \to 0$ as $r \to 0$, and hence (4-2) holds. □

The case $\gamma \in (1,2)$. An argument similar to the one given in the proof of Proposition 4.1 shows that $B_{2\gamma}^{2\gamma}U$ is proportional to $P_{2\gamma}f$ if $U \in \mathcal{C}_{f,0}^\gamma$ satisfies $L_{4,\phi}^m U = 0$. However, much more can be said. Indeed, this statement remains true if $U \in \mathcal{C}_{f,\psi}^\gamma$ for any $\psi \in C^\infty(M)$! To make this more precise, we shall evaluate the operators $B_s^{2\gamma}$ for $s \in \{0, 2\gamma-2, 2, 2\gamma\}$ when acting on elements of the kernel of the weighted Paneitz operator in terms of scattering operators.

We begin by investigating the asymptotic behavior of the summands which appear in the definitions of the operators $B_s^{2\gamma}$. For our intended applications, it suffices to compute with respect to the compactification of a Poincaré–Einstein manifold by a geodesic defining function. We first observe the following simple asymptotic behavior of the interior scalar curvature.

Lemma 4.2. *Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold and let r be a geodesic defining function. Then, in terms of $(\overline{X}, r^2g_+, r, m, 1)$, it holds that, asymptotically near M,*

$$J = \bar{J} + O(r^2). \tag{4-3}$$

Proof. A straightforward computation shows that

$$r^{-1}\Delta r = -\bar{J} + O(r^2).$$

The conclusion now follows from (2-16). □

We next compute certain derivatives of elements of $\mathcal{C}^\gamma$.

Lemma 4.3. *Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold and let r be a geodesic defining function. Let $U \in \mathcal{C}^\gamma$ have the expansion (2-12) asymptotically near M. Then, in terms of $(\overline{X}, r^2g_+, r, m, 1)$, it holds that*

$$\lim_{r\to 0}[r^m\eta U] = 2(1-\gamma)\psi, \tag{4-4}$$

$$\lim_{r\to 0}[\nabla^2U(\eta,\eta) + mr^{-1}\partial_r U] = 4(2-\gamma)f_2, \tag{4-5}$$

$$\lim_{r\to 0}[-r^m\eta\Delta_\phi U] = 2(\gamma-1)[4\gamma\psi_2 + \bar{\Delta}\psi - 2(\gamma-1)\bar{J}\psi]. \tag{4-6}$$

Proof. Equation (4-4) is an immediate consequence of (2-12).

We next compute that

$$r^{-m}\partial_r(r^m\partial_r U) = 4(2-\gamma)f_2 + 4\gamma\psi_2 r^{2\gamma-2} + o(r^{2\gamma-2}),$$

from which (4-5) immediately follows.

Finally,

$$\Delta_\phi U = \bar{\Delta} f + 4(2-\gamma) f_2 + \bigl(4\gamma\psi_2 + \bar{\Delta}\psi - 2(\gamma-1)\bar{J}\psi\bigr) r^{2\gamma-2} + o(r^{2\gamma-2}).$$

Differentiating yields (4-6). ☐

Combining Lemmas 4.2 and 4.3 yields the following evaluations of the operators $B_s^{2\gamma}$ in terms of the asymptotic expansion (2-12).

Proposition 4.4. *Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold and let r be a geodesic defining function. Let $U \in \mathcal{C}^\gamma$ be such that (2-12) holds near M. In terms of $(\bar{X}, r^2 g_+, r, m, 1)$, it holds that*

$$B_0^{2\gamma} U = f, \tag{4-7}$$

$$B_{2\gamma-2}^{2\gamma} U = 2(1-\gamma)\psi, \tag{4-8}$$

$$B_2^{2\gamma} U = \frac{2-\gamma}{\gamma-1}\bigl(-\bar{\Delta} f + \tfrac{1}{2}(n-2\gamma)\bar{J} f\bigr) + 4(2-\gamma) f_2, \tag{4-9}$$

$$B_{2\gamma}^{2\gamma} U = 8\gamma(\gamma-1)\psi_2 - 2\gamma\bigl(-\bar{\Delta}\psi + \tfrac{1}{2}(n+2\gamma-4)\bar{J}\psi\bigr). \tag{4-10}$$

Proof. Equation (4-7) is obvious, while (4-8) is (4-4). Combining (3-12) and (4-3) yields that $P(\eta,\eta) = 0$. Hence, by (3-12) and Lemma 4.2,

$$T_2^{2\gamma} = \frac{2-\gamma}{\gamma-1}\bar{J}. \tag{4-11}$$

It then follows from (4-5) that (4-9) holds. Finally, Lemma 4.2 implies that

$$S_2^{2\gamma} = \left(\frac{n-2\gamma}{2} + \frac{n+2\gamma-4}{2(\gamma-1)}\right)\bar{J}.$$

Combining this and Lemma 4.3 yields (4-10). ☐

Applying Proposition 4.4 to solutions of the Poisson equation (2-1) yields the following interpretation of the operators $B_s^{2\gamma}$.

Corollary 4.5. *Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold such that $\frac{1}{4}n^2 - \gamma^2, \frac{1}{4}n^2 - (2-\gamma)^2 \notin \sigma_{pp}(-\Delta_{g_+})$. Let ρ be a γ-admissible defining function with expansion (2-9) near M. Fix $f, \psi \in C^\infty(M)$ and set $u_1 = \mathcal{P}\bigl(\frac{1}{2}n+\gamma\bigr) f$ and $u_2 = \mathcal{P}\bigl(\frac{1}{2}n+2-\gamma\bigr)\psi$. Set $U = \rho^{-\frac{n-2\gamma}{2}}(u_1+u_2)$. In terms of $(\bar{X}, \rho^2 g_+, \rho, m, 1)$, it holds that*

$$L_{4,\phi}^m U = 0,$$

$$B_0^{2\gamma} U = f,$$

$$B_{2\gamma-2}^{2\gamma} U = 2(1-\gamma)\psi,$$

$$B_2^{2\gamma} U = \frac{4(2-\gamma)}{d_{2-\gamma}} P_{4-2\gamma}\psi,$$

$$B_{2\gamma}^{2\gamma} U = \frac{8\gamma(\gamma-1)}{d_\gamma} P_{2\gamma} f.$$

Proof. By conformal covariance, we may assume that $\rho = r$ is the geodesic defining function associated to $g|_{TM}$. That $L_{4,\phi}^m U = 0$ follows from conformal covariance and the factorization

$$(L_{4,\phi}^m)_{g_+} = \left(-\Delta_{g_+} - \tfrac{1}{4}n^2 + (2-\gamma)^2\right) \circ \left(-\Delta_{g_+} - \tfrac{1}{4}n^2 + \gamma^2\right) \tag{4-12}$$

of the weighted Paneitz operator of $(X^{n+1}, g_+, 1, m, 1)$; see [Case and Chang 2016, Theorem 3.1].

Using the asymptotic expansion (2-3), we observe that

$$u_1 \cong r^{\frac{n-2\gamma}{2}} \left[f + \frac{1}{4(1-\gamma)} \left(-\bar{\Delta} f + \frac{n-2\gamma}{2} \bar{J} f \right) r^2 + d_\gamma^{-1} P_{2\gamma} f r^{2\gamma} \right],$$

$$u_2 \cong r^{\frac{n-4+2\gamma}{2}} \left[\psi + d_{2-\gamma}^{-1} P_{4-2\gamma}(\psi) r^{4-2\gamma} + \frac{1}{4(\gamma-1)} \left(-\bar{\Delta}\psi + \frac{n+2\gamma-4}{2} \bar{J}\psi \right) r^2 \right],$$

where we write $A \cong B$ if $A - B = o(r^{\frac{n+2\gamma}{2}})$. Set $U_j = r^{-\frac{n-2\gamma}{2}} u_j$ for $j \in \{1, 2\}$. Applying Proposition 4.4 to U_1 and U_2 and using the linearity of the operators $B_s^{2\gamma}$ for $s \in \{0, 2\gamma-2, 2, 2\gamma\}$ yields the result. □

By appealing to the uniqueness of solutions to $L_{4,\phi}^m U = 0$ with suitable boundary conditions, we obtain two extension theorems relating the fractional GJMS operator $P_{2\gamma}$ to the operator $B_{2\gamma}^{2\gamma}$. The first result is a reformulation of [Case and Chang 2016, Theorem 4.4].

Proposition 4.6. *Fix $\gamma \in (1, 2)$ and set $m = 3 - 2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold such that $\frac{1}{4}n^2 - \gamma^2$, $\frac{1}{4}n^2 - (2-\gamma)^2 \notin \sigma_{pp}(-\Delta_{g_+})$. Let ρ be a γ-admissible defining function. Given $f \in C^\infty(M)$, let U be the unique solution to the boundary value problem*

$$\begin{cases} L_{4,\phi}^m U = 0 & \textit{in } (\bar{X}^{n+1}, \rho^2 g_+, \rho, m, 1), \\ B_0^{2\gamma} U = f & \textit{on } M, \\ B_{2\gamma-2}^{2\gamma} U = 0 & \textit{on } M. \end{cases} \tag{4-13}$$

Then

$$P_{2\gamma} f = \frac{d_\gamma}{8\gamma(\gamma-1)} B_{2\gamma}^{2\gamma} U. \tag{4-14}$$

Proof. Let $u = \mathcal{P}\left(\frac{1}{2}n + \gamma\right) f$ and set $\tilde{U} = \rho^{-\frac{n-2\gamma}{2}} u$. It follows from Proposition 4.4 and conformal covariance that $\tilde{U}$ satisfies (4-13). Hence, by uniqueness of solutions of (4-13), it holds that $U = \tilde{U}$. Equation (4-14) now follows from Corollary 4.5. □

The second result is an analogous extension theorem formulated in terms of the iterated Dirichlet data of a fourth-order boundary value problem.

Proposition 4.7. *Fix $\gamma \in (1, 2)$ and set $m = 3 - 2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold such that $\frac{1}{4}n^2 - \gamma^2$, $\frac{1}{4}n^2 - (2-\gamma)^2 \notin \sigma_{pp}(-\Delta_{g_+})$. Let ρ be a γ-admissible defining function. Given $f \in C^\infty(M)$, let U be the unique solution to the boundary value problem*

$$\begin{cases} L_{4,\phi}^m U = 0 & \textit{in } (\bar{X}^{n+1}, \rho^2 g_+, \rho, m, 1), \\ B_0^{2\gamma} U = f & \textit{on } M, \\ B_2^{2\gamma} U = 0 & \textit{on } M. \end{cases} \tag{4-15}$$

Then

$$P_{2\gamma} f = \frac{d_\gamma}{8\gamma(\gamma-1)} B_{2\gamma}^{2\gamma} U. \tag{4-16}$$

Proof. Let $u = \mathcal{P}\big(\frac{1}{2}n+\gamma\big) f$ and set $\tilde{U} = \rho^{-\frac{n-2\gamma}{2}} u$. It follows from Proposition 4.4 and conformal covariance that $\tilde{U}$ satisfies (4-15). Hence, by uniqueness of solutions of (4-15), it holds that $U = \tilde{U}$. Equation (4-16) now follows from Corollary 4.5. □

Surprisingly, the fractional GJMS operator $P_{2\gamma}$ can be recovered from the boundary operator $B_{2\gamma}^{2\gamma}$ without finding a unique extension.

Proposition 4.8. *Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold such that $\frac{1}{4}n^2-\gamma^2, \frac{1}{4}n^2-(2-\gamma)^2 \notin \sigma_{pp}(-\Delta_{g_+})$. Let ρ be a γ-admissible defining function. Suppose that $U \in \mathcal{C}^\gamma$ satisfies*

$$\begin{cases} L_{4,\phi}^m U = 0 & \text{in } (\overline{X}^{n+1}, \rho^2 g_+, \rho, m, 1), \\ B_0^{2\gamma} U = f & \text{on } M. \end{cases}$$

Then

$$P_{2\gamma} f = \frac{d_\gamma}{8\gamma(\gamma-1)} B_{2\gamma}^{2\gamma} U. \tag{4-17}$$

Proof. Set

$$\psi = \frac{1}{2(1-\gamma)} B_{2\gamma-2}^{2\gamma} U.$$

Let $u_1 = \mathcal{P}\big(\frac{1}{2}n+\gamma\big) f$ and $u_2 = \mathcal{P}\big(\frac{1}{2}n+2-\gamma\big)\psi$ and set $\tilde{U} = \rho^{-\frac{n-2\gamma}{2}}(u_1+u_2)$. It follows from Corollary 4.5 that $\tilde{U}$ satisfies

$$\begin{cases} L_{4,\phi}^m \tilde{U} = 0 & \text{in } X, \\ B_0^{2\gamma} \tilde{U} = f & \text{on } M, \\ B_{2\gamma-2}^{2\gamma} \tilde{U} = 2(1-\gamma)\psi & \text{on } M. \end{cases} \tag{4-18}$$

By uniqueness of solutions of (4-18), we have $U = \tilde{U}$. Equation (4-17) now follows from Corollary 4.5. □

5. The energy inequality

We are now ready to prove the energy inequalities stated in the Introduction. As throughout this article, we consider the cases $\gamma \in (0,1)$ and $\gamma \in (1,2)$ separately. Nevertheless, the basic ideas are the same. We start by considering the energy functional $\mathcal{E}_{2\gamma} : C^\gamma \to \mathbb{R}$ given by $\mathcal{E}_{2\gamma}(U) = \mathcal{Q}_{2\gamma}(U,U)$, where $\mathcal{Q}_{2\gamma}$ is as in Theorem 3.2 and Theorem 3.7. Concretely, if $\gamma \in (0,1)$, then

$$\mathcal{E}_{2\gamma}(U) = \int_X \left(|\nabla U|^2 + \frac{n-2\gamma}{2} J_\phi^m U^2 \right) + \frac{n-2\gamma}{2n} \oint_M H_{2\gamma} (B_0^{2\gamma} U)^2, \tag{5-1}$$

while if $\gamma \in (1,2)$, then

$$\begin{aligned}\mathcal{E}_{2\gamma}(U) = \int_X &\Big((\Delta_\phi U)^2 - (4P - (n-2\gamma+2)J_\phi^m g)(\nabla U, \nabla U) + \frac{n-2\gamma}{2} Q_\phi^m U^2 \Big) \\ &+ \oint_M \bigg(\frac{2}{\gamma-1} \langle \overline{\nabla} B_0^{2\gamma}(U), \overline{\nabla} B_{2\gamma-2}^{2\gamma}(U) \rangle \\ &\qquad + (n-2\gamma) T_2^{2\gamma} B_0^{2\gamma}(U) B_{2\gamma-2}^{2\gamma}(U) + \frac{n-2\gamma}{2} (\rho^m \eta J_\phi^m)(B_0^{2\gamma} U)^2 \bigg). \end{aligned} \tag{5-2}$$

Using the fact that $\mathcal{C}^\gamma_{f,\psi} = U + \mathcal{C}^\gamma_{0,0}$ for some, and hence any, fixed $U \in \mathcal{C}^\gamma_{f,\psi}$, we obtain a necessary and sufficient condition for $\mathcal{E}_{2\gamma}$ to be uniformly bounded below in $\mathcal{C}^\gamma_{f,\psi}$ in terms of the bottom of the L^2-spectrum of $L^m_{2k,\phi}$ for $k = \lfloor \gamma \rfloor + 1$. When $\mathcal{E}_{2\gamma}$ is uniformly bounded below, we construct a minimizer which necessarily solves (4-1) or (4-13). This construction together with Proposition 4.1 or Proposition 4.6, respectively, yields the result.

The case $\gamma \in (0,1)$. Following the outline above, we start by characterizing when $\mathcal{E}_{2\gamma}$ is uniformly bounded below on $\mathcal{C}^\gamma_f$ in terms of the bottom of the spectrum of $-\Delta_{g_+}$ on a Poincaré–Einstein manifold (X^{n+1}, M^n, g_+). This result generalizes an observation of Escobar [1994] in the case $\gamma = \frac{1}{2}$, and corrects an error in the remark following the statement of [González and Qing 2013, Theorem 1.4].

Proposition 5.1. *Fix $\gamma \in (0,1)$ and set $m = 1-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold such that $\frac{1}{4}n^2 - \gamma^2 \notin \sigma_{pp}(-\Delta_{g_+})$. Let ρ be a γ-admissible defining function and consider $(\overline{X}, \rho^2 g_+, \rho, m, 1)$. Fix $f \in C^\infty(M)$. Then*

$$\inf_{U \in \mathcal{C}^\gamma_f} \mathcal{E}_{2\gamma}(U) > -\infty \tag{5-3}$$

if and only if

$$\lambda_1(-\Delta_{g_+}) > \tfrac{1}{4}n^2 - \gamma^2. \tag{5-4}$$

Proof. Fix $U \in \mathcal{C}^\gamma_f$, so that $\mathcal{C}^\gamma_f = U + \mathcal{C}^\gamma_0$. By definition,

$$\lambda_1(L^m_{2,\phi}) = \inf\{\mathcal{E}_{2\gamma}(V) \mid V \in \mathcal{C}^\gamma_0, \textstyle\int_X V^2 = 1\}.$$

Since $(L^m_{2,\phi})_+ = -\Delta_{g_+} - \frac{1}{4}n^2 + \gamma^2$, we have that $\lambda_1(L^m_{2,\phi}) > 0$ if and only if (5-4) holds.

Next, given any $V \in \mathcal{C}^\gamma_0$ and $t \in \mathbb{R}$, we compute that

$$\mathcal{E}_{2\gamma}(U + tV) = t^2 \mathcal{E}_{2\gamma}(V) + 2t\, \mathcal{Q}_{2\gamma}(U,V) + \mathcal{E}_{2\gamma}(U). \tag{5-5}$$

If (5-4) does not hold, then there is a $V \in \mathcal{C}^\gamma_0$ such that $\mathcal{E}_{2\gamma}(V) < 0$. In particular, inserting this V into (5-5) and letting $t \to \infty$ shows that (5-3) does not hold. If (5-4) holds, then Theorem 3.2 and (5-5) imply that

$$\mathcal{E}_{2\gamma}(U+V) \geq \lambda_1(L^m_{2,\phi}) \int_X V^2 + 2\Big(\int_X (L^m_{2,\phi} U)^2\Big)^{\frac{1}{2}} \Big(\int_X V^2\Big)^{\frac{1}{2}} + \mathcal{E}_{2\gamma}(U)$$

for any $V \in \mathcal{C}^\gamma_0$. An application of the Cauchy–Schwarz inequality yields a lower bound for $\mathcal{E}_{2\gamma}(U+V)$ which depends only on U. In particular, (5-3) holds. ∎

We next implement the minimization scheme described at the beginning of this section to prove Theorem 1.1. For convenience, we restate the result here.

Theorem 5.2. *Fix $\gamma \in (0,1)$ and set $m = 1-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold which satisfies (5-4). Let ρ be a γ-admissible defining function with expansion (2-9) and consider $(\overline{X}, \rho^2 g_+, \rho, m, 1)$. For any $f \in C^\infty(M)$, it holds that*

$$\int_X \left(|\nabla U|^2 + \frac{m+n-1}{2} J_\phi^m U^2 \right) \rho^m \,\mathrm{dvol} \geq -\frac{2\gamma}{d_\gamma} \left[\oint_M f P_{2\gamma} f \,\mathrm{dvol} - \frac{n-2\gamma}{2} d_\gamma \oint_M \Phi f^2 \,\mathrm{dvol} \right] \quad (5\text{-}6)$$

for all $U \in W^{1,2}(\overline{X}, \rho^m \,\mathrm{dvol})$ with $\operatorname{Tr} U = f$. Moreover, equality holds if and only if $L_{2,\phi}^m U = 0$.

Proof. From (5-1) and Proposition 5.1 we observe that the left-hand side of (5-6) is uniformly bounded below in $\mathcal{C}_f^\gamma$. Let $U \in W^{1,2}(\overline{X}, \rho^m \,\mathrm{dvol})$ be a minimizer. Necessarily U solves (4-1). Theorem 3.2 and Proposition 4.1 then imply that

$$\mathcal{E}_{2\gamma}(U) = -\frac{2\gamma}{d_\gamma} \oint_M f P_{2\gamma} f,$$

from which (5-6) immediately follows. □

The case $\gamma \in (1,2)$. We again start by finding a necessary and sufficient condition for the energy $\mathcal{E}_{2\gamma}$ to be uniformly bounded below on $\mathcal{C}_{f,\psi}^\gamma$.

Proposition 5.3. *Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold such that $\frac{1}{4}n^2 - \gamma^2, \frac{1}{4}n^2 - (2-\gamma)^2 \notin \sigma_{pp}(-\Delta_{g_+})$. Let ρ be a γ-admissible defining function and consider $(\overline{X}, \rho^2 g_+, \rho, m, 1)$. Fix $f, \psi \in C^\infty(M)$. Then*

$$\inf_{U \in \mathcal{C}_{f,\psi}^\gamma} \mathcal{E}_\gamma(U) > -\infty \quad (5\text{-}7)$$

if and only if

$$\lambda_1(L_{4,\phi}^m) > 0. \quad (5\text{-}8)$$

Moreover, if $\lambda_1(-\Delta_{g_+}) > \frac{1}{4}n^2 - (2-\gamma)^2$, then (5-8) holds.

Proof. Arguing as in the proof of Proposition 5.1, but using Theorem 3.7 instead of Theorem 3.2, yields the equivalence of (5-7) and (5-8).

Suppose now that $\lambda_1(-\Delta_{g_+}) > \frac{1}{4}n^2 - (2-\gamma)^2$. It follows immediately from (4-12) that (5-8) holds. □

We next implement the minimization scheme to derive the following improvement of Theorem 1.2.

Theorem 5.4. *Fix $\gamma \in (1,2)$ and set $m = 3-2\gamma$. Let (X^{n+1}, M^n, g_+) be a Poincaré–Einstein manifold which satisfies (5-8). Let ρ be a γ-admissible defining function with expansion (2-9) and consider $(\overline{X}, \rho^2 g_+, \rho, m, 1)$. For any $f, \psi \in C^\infty(M)$, it holds that*

$$\int_X \left((\Delta_\phi U)^2 - (4P - (n-2\gamma+2)J_\phi^m g)(\nabla U, \nabla U) + \frac{n-2\gamma}{2} Q_\phi^m U^2 \right)$$
$$\geq \frac{8\gamma(\gamma-1)}{d_\gamma} \left(\oint_M f P_{2\gamma} f - \frac{n-2\gamma}{2} d_\gamma \oint_M \Phi f^2 \right) + \frac{d_\gamma}{2\gamma(\gamma-1)} \oint_M \psi P_{4-2\gamma} \psi$$
$$+ 4 \oint_M \left(\langle \overline{\nabla} f, \overline{\nabla} \psi \rangle + \frac{(n-2\gamma)(2-\gamma)}{2} (\bar{J} + 4(\gamma-1)\rho_2) f \psi \right) \tag{5-9}$$

for all $U \in W^{2,2}(\overline{X}, \rho^m \,\mathrm{dvol})$ *with* $\operatorname{Tr} U = (f, \psi)$. *Moreover, equality holds if and only if* $L_{4,\phi}^m U = 0$.

Proof. From (4-8), (5-2) and Proposition 5.3, we observe that the left-hand side of (5-9) is uniformly bounded below in $\mathcal{C}_{f,\psi}^\gamma$. Let $U \in W^{2,2}(\overline{X}, \rho^m \,\mathrm{dvol})$ be a minimizer. Necessarily U solves (4-18). Note that the factorization (4-12) and the assumption (5-8) together imply that $\frac{1}{4}n^2 - \gamma^2, \frac{1}{4}n^2 - (2-\gamma)^2 \notin \sigma_{pp}(-\Delta_{g_+})$. Therefore, by Theorem 3.7 and Corollary 4.5,

$$\mathcal{E}_{2\gamma}(U) = \frac{8\gamma(\gamma-1)}{d_\gamma} \oint_M f P_{2\gamma} f + 8(1-\gamma)(2-\gamma)\, d_{2-\gamma}^{-1} \oint_M \psi P_{4-2\gamma} \psi.$$

The conclusion follows from (2-2). □

6. A sharp Sobolev inequality

As an application of Theorem 5.4, we prove the following sharp Sobolev trace inequality for γ-admissible compactifications of hyperbolic space with $\gamma \in (1,2)$. This statement is more general than Theorem 1.3 in that it involves the full trace on $\mathcal{C}^\gamma$ and it allows for arbitrary γ-admissible compactifications.

Theorem 6.1. *Fix* $\gamma \in (1,2)$ *and set* $m = 3 - 2\gamma$. *Choose* $n \in \mathbb{N}$ *such that* $n > 2\gamma$ *and let* ρ *be a* γ*-admissible defining function on hyperbolic space* (H^{n+1}, S^n, g_+). *Then, in terms of* $(\overline{H}, \rho^2 g_+, \rho, m, 1)$,

$$\mathcal{E}_{2\gamma}(U) \geq c_{n,\gamma}^{(2)} \left(\oint_{S^n} |f|^{\frac{2n}{n-2\gamma}} \right)^{\frac{n-2\gamma}{n}} + \tfrac{1}{4} c_{n,2-\gamma}^{(2)} \left(\oint_{S^n} |\psi|^{\frac{2n}{n-4+2\gamma}} \right)^{\frac{n-4+2\gamma}{n}}$$

for all $U \in \mathcal{C}^\gamma$, *where* f *is the trace of* U, Φ *is as in* (2-9), *and*

$$c_{n,\gamma}^{(2)} = 8\pi^\gamma \frac{\Gamma(2-\gamma)}{\Gamma(\gamma)} \frac{\Gamma\left(\frac{1}{2}(n+2\gamma)\right)}{\Gamma\left(\frac{1}{2}(n-2\gamma)\right)} \left(\frac{\Gamma\left(\frac{1}{2}n\right)}{\Gamma(n)} \right)^{\frac{2\gamma}{n}}.$$

Moreover, equality holds if and only if $L_{4,\phi}^m U = 0$ *and both* $f^{\frac{4}{n-2\gamma}} (\rho^2 g_+)|_{TS^n}$ *and* $\psi^{\frac{4}{n-4+2\gamma}} (\rho^2 g_+)|_{TS^n}$ *are Einstein with positive scalar curvature.*

Proof. Since $\lambda_1(-\Delta_{g_+}) = \frac{1}{4}n^2$, Theorem 5.4 implies that

$$\mathcal{E}_{2\gamma}(U) \geq \frac{8\gamma(\gamma-1)}{d_\gamma} \oint_{S^n} f P_{2\gamma} f + \frac{d_\gamma}{2\gamma(\gamma-1)} \oint_{S^n} \psi P_{4-2\gamma} \psi \tag{6-1}$$

for all $U \in \mathcal{C}^\gamma$, where f and ψ are as in (2-12). The conformal covariance of the fractional GJMS operators and the sharp fractional Sobolev inequality [Beckner 1993; Cotsiolis and Tavoularis 2004; Frank and

Lieb 2012; Lieb 1983] together imply that

$$\oint_{S^n} f P_{2\gamma} f \geq 2^{2\gamma}\pi^{\gamma}\frac{\Gamma\left(\frac{1}{2}(n+2\gamma)\right)}{\left(\frac{1}{2}(n-2\gamma)\right)}\left(\frac{\Gamma\left(\frac{1}{2}n\right)}{\Gamma(n)}\right)^{\frac{2\gamma}{n}}\left(\oint_{S^n}|f|^{\frac{2n}{n-2\gamma}}\right)^{\frac{n-2\gamma}{n}} \tag{6-2}$$

with equality if and only if $f^{\frac{4n}{n-2\gamma}}(\rho^2 g_+)|_{TS^n}$ is Einstein with positive scalar curvature. Combining (6-1), (6-2) and the corresponding result for $P_{4-2\gamma}$ yields the desired result. □

Proof of Theorem 1.3. It is straightforward to check that $(S^{n+1}_+, d\theta^2, x_{n+1}, m, 1)$ satisfies

$$L^m_{4,\phi} = \left(-\Delta_\phi + \tfrac{1}{4}((m+n)^2-1)\right)\left(-\Delta_\phi + \tfrac{1}{4}((m+n)^2-9)\right).$$

Since $g_+ = x_{n+1}^{-2}d\theta^2$ satisfies $\mathrm{Ric}_{g_+} = -ng_+$, we see find that x_{n+1} is a γ-admissible defining function for hyperbolic space. Applying Theorem 6.1 then yields the desired conclusion. □

Appendix: Proof of the Sobolev trace theorem

The proof of Theorem 5.4 requires the Sobolev space $W^{2,2}(\overline{X}^{n+1}, \rho^m\ \mathrm{dvol})$ and its trace onto $H^\gamma(M)\oplus H^{2-\gamma}(M)$. Since our definitions of $W^{2,2}(\overline{X}^{n+1}, \rho^m\ \mathrm{dvol})$ and the trace via the space $\mathcal{C}^\gamma$ are nonstandard — the usual approach is via completions of $C^\infty(\overline{X})$ [Triebel 1978] — we prove the existence of the trace map.

Theorem A.1. *Fix $\gamma\in(1,2)$ and set $m = 3-2\gamma$. Let $(\overline{X}^{n+1}, g, \rho, m, 1)$ be a γ-admissible smooth metric measure space with nonempty boundary $M = \partial X$. There is a unique bounded linear operator*

$$\mathrm{Tr} : W^{2,2}(\overline{X}, \rho^m\ \mathrm{dvol}) \to H^\gamma(M)\oplus H^{2-\gamma}(M)$$

such that $\mathrm{Tr}(U) = (f,\psi)$ *for all $U\in\mathcal{C}^\gamma_{f,\psi}$. Moreover, there is a continuous mapping*

$$E : H^\gamma(M)\oplus H^{2-\gamma}(M) \to W^{2,2}(\overline{X}, \rho^m\ \mathrm{dvol})$$

such that $\mathrm{Tr}\circ E$ *is the identity map on $H^\gamma(M)\oplus H^{2-\gamma}(M)$.*

The proof of Theorem A.1 involves two steps. First, we prove the corresponding result in upper half space $(\overline{\mathbb{R}^{n+1}_+}, dy^2\oplus dx^2, y, m, 1)$ for $\mathbb{R}^{n+1}_+ = (0,\infty)\times\mathbb{R}^n$ and y the standard coordinate on $(0,\infty)$. Second, we use coordinate charts to pull the Euclidean result back to γ-admissible smooth metric measure spaces. Indeed, the second step is routine, and the proof will be omitted. The first step is carried out via the extension theorems for the fractional Laplacian [Caffarelli and Silvestre 2007; Yang 2013]. To that end, define $\mathcal{C}^\gamma_{f,\psi}$ and $\mathcal{C}^\gamma$ as in Section 2 using Schwartz functions to obtain the completion $W^{2,2}(\mathbb{R}^{n+1}_+, y^m\ \mathrm{dvol})$, and recall that the H^γ- and $H^{2-\gamma}$-norms defined in Section 2 are equivalent to the ones defined via Fourier transform.

Theorem A.2. *Fix $\gamma\in(1,2)$ and set $m = 3-2\gamma$. There is a unique bounded linear operator*

$$\mathrm{Tr} : W^{2,2}(\mathbb{R}^{n+1}_+, y^m\ \mathrm{dvol}) \to H^\gamma(\mathbb{R}^n)\oplus H^{2-\gamma}(\mathbb{R}^n) \tag{A-1}$$

such that $\operatorname{Tr}(U) = (f, \psi)$ *for any* $U \in \mathcal{C}^{\gamma}_{f,\psi}$. *Moreover, there is a bounded linear operator*

$$E : H^{\gamma}(\mathbb{R}^n) \oplus H^{2-\gamma}(\mathbb{R}^n) \to W^{2,2}(\mathbb{R}^{n+1}_+, y^m \,\mathrm{dvol}) \tag{A-2}$$

such that $\operatorname{Tr} \circ E$ *is the identity map on* $H^{\gamma}(\mathbb{R}^n) \oplus H^{2-\gamma}(\mathbb{R}^n)$.

Proof. Combining Theorem 5.4 and the proof of [Yang 2013, Theorem 3.1], we find that for any $U \in \mathcal{C}^{\gamma}_{f,\psi}$, it holds that

$$\int_{\mathbb{R}^{n+1}_+} (\Delta_\phi U)^2 \geq \frac{8\gamma(\gamma-1)}{d_\gamma} \|(-\Delta)^{\frac{\gamma}{2}} f\|_2^2 + 4 \oint_{\mathbb{R}^n} \langle \overline{\nabla} f, \overline{\nabla} \psi \rangle + \frac{d_\gamma}{2\gamma(\gamma-1)} \|(-\Delta)^{\frac{2-\gamma}{2}} \psi\|_2^2 \tag{A-3}$$

with equality if and only if U is the unique solution to

$$\begin{cases} \Delta_\phi^2 U = 0 & \text{in } (0,\infty) \times \mathbb{R}^n, \\ U(0,x) = f(x) & \text{for all } x \in \mathbb{R}^n, \\ \lim_{y \to 0} y^m \partial_y U(y,x) = 2(\gamma-1)\psi & \text{for all } x \in \mathbb{R}^n \end{cases} \tag{A-4}$$

with $\Delta_\phi = \Delta + m y^{-1} \partial_y$. Using integration by parts, it is straightforward to check that

$$\int_{\mathbb{R}^{n+1}_+} \left| \nabla^2 U + m y^{-1} (\partial_y U)^2 \, dy \otimes dy \right|^2 = \int_{\mathbb{R}^{n+1}_+} (\Delta_\phi U)^2 - 4(\gamma-1) \oint_{\mathbb{R}^n} \langle \overline{\nabla} f, \overline{\nabla} \psi \rangle$$

for all $U \in \mathcal{C}^{\gamma}_{f,\psi}$. Combining this with (A-3) yields

$$\int_{\mathbb{R}^{n+1}_+} \left| \nabla^2 U + m y^{-1} (\partial_y U)^2 \, dy \otimes \, dy \right|^2 \geq \frac{8\gamma(\gamma-1)^2}{d_\gamma} \|(-\Delta)^{\frac{\gamma}{2}} f\|_2^2 + \frac{d_\gamma}{2\gamma} \|(-\Delta)^{\frac{2-\gamma}{2}} \psi\|_2^2 \tag{A-5}$$

with the same characterization of the equality case. It follows from (A-5) that $\operatorname{Tr} : \mathcal{C}^{\gamma} \to H^{\gamma} \oplus H^{2-\gamma}$ is a bounded linear operator, and hence can be extended uniquely to a bounded linear operator as in (A-1). Moreover, the map $E(f,\psi) = U$ obtained by solving (A-4) is linear and, by (A-5), bounded, whence can be extended to a bounded linear operator as in (A-2). □

Acknowledgments

I would like to thank Antonio Ache and Alice Chang for discussions relating to their article [Ache and Chang 2015] which helped shape the investigations of Section 4. I would also like to thank Rod Gover for pointing out the reference [Grant 2003], María del Mar González for comments on an early draft of this article, and the referees for helpful comments which helped improve the clarity of the exposition.

References

[Ache and Chang 2015] A. Ache and S.-Y. A. Chang, "Sobolev-Trace inequalities of order four", preprint, 2015. arXiv

[Beckner 1993] W. Beckner, "Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality", *Ann. of Math.* (2) **138**:1 (1993), 213–242. MR Zbl

[Branson 1985] T. P. Branson, "Differential operators canonically associated to a conformal structure", *Math. Scand.* **57**:2 (1985), 293–345. MR Zbl

[Branson 1997] T. Branson, "Spectral theory of invariant operators, sharp inequalities, and representation theory", *Rend. Circ. Mat. Palermo* (2) *Suppl.* 46 (1997), 29–54. MR Zbl

[Branson and Gover 2001] T. Branson and A. R. Gover, "Conformally invariant non-local operators", *Pacific J. Math.* **201**:1 (2001), 19–60. MR Zbl

[Caffarelli and Silvestre 2007] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian", *Comm. Partial Differential Equations* **32**:7–9 (2007), 1245–1260. MR Zbl

[Case 2012] J. S. Case, "Smooth metric measure spaces and quasi-Einstein metrics", *Internat. J. Math.* **23**:10 (2012), art. id. 1250110. MR Zbl

[Case 2015] J. S. Case, "Boundary operators associated to the Paneitz operator", preprint, 2015. arXiv

[Case and Chang 2016] J. S. Case and S.-Y. A. Chang, "On fractional GJMS operators", *Comm. Pure Appl. Math.* **69**:6 (2016), 1017–1061. MR Zbl

[Chang and González 2011] S.-Y. A. Chang and M. d. M. González, "Fractional Laplacian in conformal geometry", *Adv. Math.* **226**:2 (2011), 1410–1432. MR Zbl

[Chang and Qing 1997] S.-Y. A. Chang and J. Qing, "The zeta functional determinants on manifolds with boundary, I: The formula", *J. Funct. Anal.* **147**:2 (1997), 327–362. MR Zbl

[Chruściel et al. 2005] P. T. Chruściel, E. Delay, J. M. Lee, and D. N. Skinner, "Boundary regularity of conformally compact Einstein metrics", *J. Differential Geom.* **69**:1 (2005), 111–136. MR Zbl

[Cotsiolis and Tavoularis 2004] A. Cotsiolis and N. K. Tavoularis, "Best constants for Sobolev inequalities for higher order fractional derivatives", *J. Math. Anal. Appl.* **295**:1 (2004), 225–236. MR Zbl

[Escobar 1988] J. F. Escobar, "Sharp constant in a Sobolev trace inequality", *Indiana Univ. Math. J.* **37**:3 (1988), 687–698. MR Zbl

[Escobar 1990] J. F. Escobar, "Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate", *Comm. Pure Appl. Math.* **43**:7 (1990), 857–883. MR Zbl

[Escobar 1992] J. F. Escobar, "Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary", *Ann. of Math.* (2) **136**:1 (1992), 1–50. MR Zbl

[Escobar 1994] J. F. Escobar, "Addendum: 'Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary' [*Ann. of Math.* (2) **136**:1 (1992), 1–50]", *Ann. of Math.* (2) **139**:3 (1994), 749–750. MR Zbl

[Frank and Lieb 2012] R. L. Frank and E. H. Lieb, "A new, rearrangement-free proof of the sharp Hardy–Littlewood–Sobolev inequality", pp. 55–67 in *Spectral theory, function spaces and inequalities*, edited by B. M. Brown et al., Oper. Theory Adv. Appl. **219**, Springer, Basel, 2012. MR Zbl

[González and Qing 2013] M. d. M. González and J. Qing, "Fractional conformal Laplacians and fractional Yamabe problems", *Anal. PDE* **6**:7 (2013), 1535–1576. MR Zbl

[Graham and Lee 1991] C. R. Graham and J. M. Lee, "Einstein metrics with prescribed conformal infinity on the ball", *Adv. Math.* **87**:2 (1991), 186–225. MR Zbl

[Graham and Zworski 2003] C. R. Graham and M. Zworski, "Scattering matrix in conformal geometry", *Invent. Math.* **152**:1 (2003), 89–118. MR Zbl

[Grant 2003] D. H. Grant, *A conformally invariant third order Neumann-type operator for hypersurfaces*, master's thesis, The University of Auckland, 2003, available at https://www.math.auckland.ac.nz/mathwiki/images/5/51/GrantMSc.pdf.

[Guillarmou and Guillopé 2007] C. Guillarmou and L. Guillopé, "The determinant of the Dirichlet-to-Neumann map for surfaces with boundary", *Int. Math. Res. Not.* **2007**:22 (2007), art. id. rnm099. MR Zbl

[Hebey and Vaugon 1996] E. Hebey and M. Vaugon, "Meilleures constantes dans le théorème d'inclusion de Sobolev", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **13**:1 (1996), 57–93. MR Zbl

[Jin and Xiong 2013] T. Jin and J. Xiong, "Sharp constants in weighted trace inequalities on Riemannian manifolds", *Calc. Var. Partial Differential Equations* **48**:3–4 (2013), 555–585. MR Zbl

[Juhl 2009] A. Juhl, *Families of conformally covariant differential operators, Q-curvature and holography*, Progress in Mathematics **275**, Birkhäuser, Basel, 2009. MR Zbl

[Lee 1995] J. M. Lee, "The spectrum of an asymptotically hyperbolic Einstein manifold", *Comm. Anal. Geom.* **3**:1–2 (1995), 253–271. MR Zbl

[Lieb 1983] E. H. Lieb, "Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities", *Ann. of Math.* (2) **118**:2 (1983), 349–374. MR Zbl

[Triebel 1978] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library **18**, North-Holland, Amsterdam, 1978. MR Zbl

[Yang 2013] R. Yang, "On higher order extensions for the fractional Laplacian", preprint, 2013. arXiv

Received 6 Oct 2015. Revised 6 Oct 2016. Accepted 28 Nov 2016.

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ANALYSIS AND PDE
Vol. 10, No. 2, 2017
dx.doi.org/10.2140/apde.2017.10.281

msp

EXACT CONTROLLABILITY FOR QUASILINEAR PERTURBATIONS OF KDV

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We prove that the KdV equation on the circle remains exactly controllable in arbitrary time with localized control, for sufficiently small data, also in the presence of quasilinear perturbations, namely nonlinearities containing up to three space derivatives, having a Hamiltonian structure at the highest orders. We use a procedure of reduction to constant coefficients up to order zero (adapting a result of Baldi, Berti and Montalto (2014)), the classical Ingham inequality and the Hilbert uniqueness method to prove the controllability of the linearized operator. Then we prove and apply a modified version of the Nash–Moser implicit function theorems by Hörmander (1976, 1985).

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1. Introduction

A question in control theory for PDEs regards the persistence of controllability under perturbations. In this paper we study the effect of *quasilinear* perturbations (namely nonlinearities containing derivatives of the highest order) on the controllability of the KdV equation. We consider equations of the form

$$u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = 0 \tag{1-1}$$

on the circle $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, with $t \in \mathbb{R}$, where $u = u(t, x)$ is real-valued, and $\mathcal{N}$ is a given real-valued nonlinear function which is at least quadratic around $u = 0$. For solutions of small amplitude, (1-1) is a quasilinear perturbation of the Airy equation $u_t + u_{xxx} = 0$, which is the linear part of KdV; then the KdV nonlinear term uu_x can be included in $\mathcal{N}$.

Motivated by a question, which was posed in [Kappeler and Pöschel 2003], about the possibility of including the dependence on higher derivatives in nonlinear perturbations of KdV, equations of the form

MSC2010: 35Q53, 35Q93.
Keywords: control of PDEs, exact controllability, internal controllability, KdV equation, quasilinear PDEs, observability of PDEs, HUM, Nash–Moser theorem.

(1-1) have recently been studied in [Baldi, Berti, and Montalto 2014; 2016a; 2016b] in the context of KAM theory. In this paper we study (1-1) from the point of view of control theory, proving its exact controllability by means of an internal control, in arbitrary time, for sufficiently small data (Theorem 1.1).

Most of the known results about controllability of quasilinear PDEs deal with first-order quasilinear hyperbolic systems of the form $u_t + A(u)u_x = 0$ (including quasilinear wave, shallow water, and Euler equations); see, for example, [Li and Zhang 1998; Coron 2007, Chapter 6.2; Li and Rao 2003; Coron, Glass, and Wang 2010; Alabau-Boussouira, Coron and Olive 2015]. Recent results for different kinds of quasilinear PDEs are contained in [Alazard, Baldi, and Han-Kwan 2015] about the internal controllability of 2-dimensional gravity-capillary water waves equations, and in [Alazard 2015] about the boundary observability of 2- and 3-dimensional (fully nonlinear) gravity water waves. For a general introduction to the theory of control for PDEs, see, for example, [Lions 1988; Micu and Zuazua 2005; Coron 2007], while for important results in control for hyperbolic PDEs, see, for example, [Bardos, Lebeau, and Rauch 1992; Burq and Gérard 1997; Burq and Zworski 2012].

Regarding the KdV equation, the first controllability results are due to Zhang [1990] and Russell [1991]. Among recent results, we mention the work by Laurent, Rosier and Zhang [2010] for large data. A beautiful review on the literature on control for KdV can be found in [Rosier and Zhang 2009]. For more on KdV, see the rich survey [Guan and Kuksin 2014], and the many references therein.

1A. *Main result.* We assume that the nonlinearity $\mathcal{N}(x, u, u_x, u_{xx}, u_{xxx})$ is at least quadratic around $u = 0$; namely the real-valued function $\mathcal{N} : \mathbb{T} \times \mathbb{R}^4 \to \mathbb{R}$ satisfies

$$|\mathcal{N}(x, z_0, z_1, z_2, z_3)| \le C|z|^2 \quad \forall z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4,\ |z| \le 1. \tag{1-2}$$

We assume that the dependence of $\mathcal{N}$ on u_{xx}, u_{xxx} is Hamiltonian, while no structure is required on its dependence on u, u_x. More precisely, we assume that

$$\mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}) = \mathcal{N}_1(x, u, u_x, u_{xx}, u_{xxx}) + \mathcal{N}_0(x, u, u_x), \tag{1-3}$$

where

$$\mathcal{N}_1(x, u, u_x, u_{xx}, u_{xxx}) = \partial_x\{(\partial_u \mathcal{F})(x, u, u_x)\} - \partial_{xx}\{(\partial_{u_x} \mathcal{F})(x, u, u_x)\}$$
$$\text{for some function } \mathcal{F} : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}. \tag{1-4}$$

Note that the case $\mathcal{N}=\mathcal{N}_1$, $\mathcal{N}_0=0$ corresponds to the Hamiltonian equation $\partial_t u = \partial_x \nabla H(u)$, where the Hamiltonian is

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} u_x^2\, dx + \int_{\mathbb{T}} \mathcal{F}(x, u, u_x)\, dx \tag{1-5}$$

and ∇ denotes the $L^2(\mathbb{T})$-gradient. The unperturbed KdV is the case $\mathcal{F} = -\frac{1}{6}u^3$.

Notation. For periodic functions $u(x)$, $x \in \mathbb{T}$, we expand $u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx}$, and, for $s \in \mathbb{R}$, we consider the standard Sobolev space of periodic functions

$$H^s_x := H^s(\mathbb{T}, \mathbb{R}) := \{u : \mathbb{T} \to \mathbb{R} : \|u\|_s < \infty\}, \quad \|u\|_s^2 := \sum_{n\in\mathbb{Z}} |u_n|^2 \langle n \rangle^{2s}, \tag{1-6}$$

where $\langle n\rangle := (1+n^2)^{1/2}$. We consider the space $C([0,T],H^s_x)$ of functions $u(t,x)$ that are continuous in time with values in H^s_x. We will use the following notation for the standard norm in $C([0,T],H^s_x)$:

$$\|u\|_{T,s} := \|u\|_{C([0,T],H^s_x)} := \sup_{t\in[0,T]} \|u(t)\|_s. \tag{1-7}$$

For continuous functions $a:[0,T]\to\mathbb{R}$, we will define

$$|a|_T := \sup\{|a(t)| : t\in[0,T]\}. \tag{1-8}$$

Theorem 1.1 (exact controllability). *Let $T>0$, and let $\omega\subset\mathbb{T}$ be a nonempty open set. There exist positive universal constants r, s_1 such that, if $\mathcal{N}$ in (1-1) is of class C^r in its arguments and satisfies (1-2), (1-3), (1-4), then there exists a positive constant δ_* depending on T, ω, $\mathcal{N}$ with the following property.*

Let $u_{\text{in}}, u_{\text{end}} \in H^{s_1}(\mathbb{T},\mathbb{R})$ with

$$\|u_{\text{in}}\|_{s_1} + \|u_{\text{end}}\|_{s_1} \le \delta_*.$$

Then there exists a function $f(t,x)$ satisfying

$$f(t,x)=0 \quad \text{for all } x\notin\omega, \ \text{ for all } t\in[0,T],$$

belonging to $C([0,T],H^s_x)\cap C^1([0,T],H^{s-3}_x)\cap C^2([0,T],H^{s-6}_x)$ for all $s<s_1$, such that the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x,u,u_x,u_{xx},u_{xxx}) = f & \forall (t,x)\in[0,T]\times\mathbb{T}, \\ u(0,x) = u_{\text{in}}(x) \end{cases} \tag{1-9}$$

has a unique solution $u(t,x)$ belonging to $C([0,T],H^s_x)\cap C^1([0,T],H^{s-3}_x)\cap C^2([0,T],H^{s-6}_x)$ for all $s<s_1$ which satisfies

$$u(T,x) = u_{\text{end}}(x). \tag{1-10}$$

Moreover, for all $s<s_1$,

$$\|u,f\|_{C([0,T],H^s_x)} + \|\partial_t u, \partial_t f\|_{C([0,T],H^{s-3}_x)} + \|\partial_{tt} u, \partial_{tt} f\|_{C([0,T],H^{s-6}_x)} \le C_s(\|u_{\text{in}}\|_{s_1} + \|u_{\text{end}}\|_{s_1}) \tag{1-11}$$

for some $C_s>0$ depending on s, T, ω, $\mathcal{N}$.

Remark 1.2. In Theorem 1.1 there is an arbitrarily small loss of regularity: if the initial and final data $u_{\text{in}}, u_{\text{end}}$ have Sobolev regularity $H^{s_1}_x$, then the control f and the solution u are continuous in time with values in H^s_x for all $s<s_1$. Such loss of regularity is in some sense fictitious: it is due to our choice of working with standard Sobolev spaces, but it could be avoided by working with the (slightly "worse-looking") weak spaces E'_a introduced by Hörmander [1985] (see Appendix B). What we actually prove is that, if the initial and final data are in the weak space $(H^{s_1}_x)'$ (i.e., the weak version à la Hörmander [1985] of the Sobolev space $H^{s_1}_x$), then f and u are continuous in time with values in *the same* space $(H^{s_1}_x)'$. □

Remark 1.3. Our proof of Theorem 1.1 does not use results of existence and uniqueness for the Cauchy problem (1-9). On the contrary, our method directly proves local existence and uniqueness for (1-9) (see Theorem 1.4). This situation occurs quite often in control problems (see Remark 4.12 of [Coron 2007]). □

1B. ***Description of the proof.*** It would be natural to try to solve the control problem (1-9)–(1-10) using a fixed point argument or the usual implicit function theorem. However, this seems to be impossible because of the presence of three derivatives in the nonlinear term. A similar difficulty was overcome in [Alazard, Baldi, and Han-Kwan 2015] by using a suitable nonlinear iteration scheme adapted to quasilinear problems. Such a nonlinear scheme requires solving a linear control problem with variable coefficients at each step of the iteration, with no loss of regularity with respect to the coefficients (i.e., the solution must have the same regularity as the coefficients). In [Alazard, Baldi, and Han-Kwan 2015] this is achieved by means of paradifferential calculus, together with linear transformations, Ingham-type inequalities and the Hilbert uniqueness method.

As an alternative method, in this paper we use a Nash–Moser implicit function theorem. The Nash–Moser approach also demands the solving of a linear control problem with variable coefficients, but it has the advantage of requiring weaker estimates, allowing losses of regularity. The proof of such weaker estimates is easier to obtain, and it does not require the use of powerful techniques like paradifferential calculus. In this sense our Nash–Moser method is alternative to the method in [Alazard, Baldi, and Han-Kwan 2015] (for a discussion about pseudo- and paradifferential calculus in connection with the Nash–Moser theorem, see, for example, [Hörmander 1990; Alinhac and Gérard 2007]). On the other hand, the result that we obtain with the Nash–Moser method is slightly weaker than the one in [Alazard, Baldi, and Han-Kwan 2015] regarding the regularity of the solution of the nonlinear control problem with respect to the regularity of the data: the arbitrarily small loss of regularity in Theorem 1.1 is discussed in Remark 1.2, while Theorem 1.1 of [Alazard, Baldi, and Han-Kwan 2015] has no loss of regularity also in the standard Sobolev spaces.

Nash–Moser schemes in control problems for PDEs have been used in [Beauchard 2005; 2008; Beauchard and Coron 2006; Alabau-Boussouira, Coron and Olive 2015]. A discussion about Nash–Moser as a method to overcome the problem of the loss of derivatives in the context of controllability for PDEs can be found in [Coron 2007, Section 4.2.2]. Beauchard and Laurent [2010] were able to avoid the use of the Nash–Moser theorem in semilinear control problems thanks to a regularizing effect. We remark that Theorem 1.1 could also be proved without Nash–Moser (for example, by adapting the method of [Alazard, Baldi, and Han-Kwan 2015]).

Now we describe our method in more detail. Given a nonempty open set $\omega \subset \mathbb{T}$, we first fix a C^∞ function $\chi_\omega(x)$ with values in the interval $[0, 1]$ which vanishes outside ω, and takes value $\chi_\omega = 1$ on a nonempty open subset of ω. Thus, given initial and final data u_{in}, u_{end}, we look for u, f that solve

$$\begin{cases} P(u) = \chi_\omega f, \\ u(0) = u_{\text{in}}, \\ u(T) = u_{\text{end}}, \end{cases} \tag{1-12}$$

where

$$P(u) := u_t + u_{xxx} + \mathcal{N}(x, u, u_x, u_{xx}, u_{xxx}). \tag{1-13}$$

We define

$$\Phi(u, f) := \begin{pmatrix} P(u) - \chi_\omega f \\ u(0) \\ u(T) \end{pmatrix} \tag{1-14}$$

so that problem (1-12) is written as

$$\Phi(u, f) = (0, u_{\rm in}, u_{\rm end}).$$

The crucial assumption to verify in order to apply any Nash–Moser theorem is the existence of a right inverse of the linearized operator. The linearized operator $\Phi'(u, f)[h, \varphi]$ at the point (u, f) in the direction (h, φ) is

$$\Phi'(u, f)[h, \varphi] := \begin{pmatrix} P'(u)[h] - \chi_\omega \varphi \\ h(0) \\ h(T) \end{pmatrix}. \tag{1-15}$$

Thus we have to prove that, given any (u, f) and any $g := (g_1, g_2, g_3)$ in suitable function spaces, there exists (h, φ) such that

$$\Phi'(u, f)[h, \varphi] = g. \tag{1-16}$$

Moreover we have to estimate (h, φ) in terms of u, f, g in a "tame" way (an estimate is said to be tame when it is linear in the highest norms; see (B-13) and (4-41)).

Problem (1-16) is a linear control problem. We observe that the linearized operator $P'(u)[h]$ is a differential operator having variable coefficients also at the highest order (which is a consequence of linearizing a *quasilinear* PDE). Explicitly, it has the form

$$P'(u)[h] = \partial_t h + (1 + a_3(t, x))\, \partial_{xxx} h + a_2(t, x)\, \partial_{xx} h + a_1(t, x)\, \partial_x h + a_0(t, x) h.$$

We solve (1-16) in Theorem 4.5. Note that the choice of the function spaces is not given a priori: to fix a suitable functional setting is part of the problem.

Theorem 4.5 is proved by adapting a procedure of reduction to constant coefficients developed in [Baldi, Berti, and Montalto 2014; 2016a]. Such a procedure conjugates $P'(u)$ to an operator $\mathcal{L}_5$ (see (2-57)) having constant coefficients up to a bounded remainder. This conjugation is achieved by means of changes of the space variable, reparametrization of time, multiplication operators, and Fourier multipliers. Using the Ingham inequality and a perturbation argument we prove the observability of $\mathcal{L}_5$. Then we prove the observability of $P'(u)$, exploiting the explicit formulas of the transformations that conjugate $P'(u)$ to $\mathcal{L}_5$. The linear control problem (1-16) is solved in L^2_x by the HUM (Hilbert uniqueness method). Then further regularity of the solution (h, φ) of (1-16) is proved by adapting an argument used by Dehman and Lebeau [2009], Laurent [2010], and Alazard, Baldi, and Han-Kwan [2015].

To conclude the proof of Theorem 1.1 we apply Theorem B.1, which is a modified version of two Nash–Moser implicit function theorems (Theorem 2.2.2 in [Hörmander 1976] and the main theorem in [Hörmander 1985]; see also [Alinhac and Gérard 2007]). With respect to the abstract theorem in [Hörmander 1985], our Theorem B.1 assumes slightly stronger hypotheses on the nonlinear operator, and it removes two conditions that are assumed in [Hörmander 1985], which are the compact embeddings in the codomain scale of Banach spaces and the continuity of the approximate right inverse of the linearized operator with respect to the approximate linearization point. This improvement is obtained by adapting the iteration scheme introduced in [Hörmander 1976]. On the other hand, the Nash–Moser implicit function theorem in that work holds for Hölder spaces with noninteger indices, and it does not apply to Sobolev spaces (in particular, Theorem A.11 of [Hörmander 1976] does not hold for Sobolev spaces).

This method is not confined to KdV, and it could be applied to prove controllability of other quasilinear evolution PDEs.

The use of Ingham-type inequalities and the HUM is classical in control theory (see, for example, [Haraux 1989; Micu and Zuazua 2005; Komornik and Loreti 2005; Kahane 1962] for Ingham-type inequalities and [Lions 1988; Micu and Zuazua 2005; Coron 2007; Komornik 1994] for the HUM). As mentioned above, the Nash–Moser theorem has also been used in control theory (see, for example, [Beauchard 2005; 2008; Beauchard and Coron 2006; Alabau-Boussouira, Coron and Olive 2015]). It was first introduced by Nash [1956], and then several refinements were developed afterwards; see, for example, [Moser 1961; Zehnder 1975; 1976; Hamilton 1982; Gromov 1972; Hörmander 1976; 1985; 1990; Berti, Bolle, and Procesi 2010; Berti, Corsi, and Procesi 2015; Ekeland 2011; Ekeland and Séré 2015]. For our problem, Hörmander's versions [1976; 1985] seem to be the best ones concerning the loss of regularity of the solution with respect to the regularity of the data (see also Remark 1.2). As already said, the theorems in [Hörmander 1976; 1985] cannot be applied directly, but they can be adapted to our goal. This is the content of Appendix B.

1C. ***Byproduct: a local existence and uniqueness result.*** As a byproduct, with the same technique and no extra work, we have the following existence and uniqueness theorem for the Cauchy problem of the quasilinear PDE (1-1).

Theorem 1.4 (local existence and uniqueness). *There exist positive universal constants r, s_0 such that, if $\mathcal{N}$ in (1-1) is of class C^r in its arguments and satisfies (1-2), (1-3), (1-4), then the following property holds. For all $T > 0$ there exists $\delta_* > 0$ such that for all $u_{\text{in}} \in H_x^{s_0}$ and $f \in C([0,T], H_x^{s_0}) \cap C^1([0,T], H_x^{s_0-6})$ (possibly $f = 0$) satisfying*

$$\|u_{\text{in}}\|_{s_0} + \|f\|_{T,s_0} + \|\partial_t f\|_{T,s_0-6} \le \delta_*, \tag{1-17}$$

the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \mathcal{N}(x,u,u_x,u_{xx},u_{xxx}) = f, & (t,x) \in [0,T] \times \mathbb{T}, \\ u(0,x) = u_{\text{in}}(x) \end{cases} \tag{1-18}$$

has one and only one solution $u \in C([0,T], H_x^s) \cap C^1([0,T], H_x^{s-3}) \cap C^2([0,T], H_x^{s-6})$ for all $s < s_0$. Moreover, for all $s < s_0$,

$$\|u\|_{C([0,T],H_x^s)} + \|\partial_t u\|_{C([0,T],H_x^{s-3})} + \|\partial_{tt} u\|_{C([0,T],H_x^{s-6})} \\ \le C_s \big(\|u_{\text{in}}\|_{s_0} + \|f\|_{C([0,T],H_x^{s_0})} + \|\partial_t f\|_{C([0,T],H_x^{s_0-6})} \big) \tag{1-19}$$

for some $C_s > 0$ depending on s, T, $\mathcal{N}$.

Remark 1.5. Theorem 1.4 is not sharp: we expect that better results for the Cauchy problem (1-18) can be proved by using a paradifferential approach. □

Remark 1.6. The loss of regularity in Theorem 1.4 is of the same type as the one in Theorem 1.1; see the discussion in Remark 1.2. □

1D. ***Organization of the paper.*** In Section 2 we describe the transformations that conjugate the linearized operator $P'(u)$ to constant coefficients up to a bounded remainder, and we give quantitative estimates on these transformations. In Section 3 we exploit these results to prove the observability of $P'(u)$. In Section 4 we use observability to solve the linear control problem (1-16) via the HUM (Theorem 4.5) and we fix suitable function spaces (4-36)–(4-37). In Section 5 we prove Theorems 1.1 and 1.4 by applying Theorem B.1. In Appendix A we prove well-posedness with tame estimates for all the linear operators involved in the reduction procedure. These well-posedness results are used many times in Sections 3, 4, and 5. In Appendix B we prove our Nash–Moser theorem, Theorem B.1. In Appendix C we recall standard tame estimates that are used in the rest of the paper.

2. Reduction of the linearized operator to constant coefficients

In this section we consider some changes of variables that conjugate the linearized operator to constant coefficients up to a bounded remainder. This reduction procedure closely follows the analysis in [Baldi, Berti, and Montalto 2014; 2016a], with some adaptations.

The linearized operator $P'(u)$ is

$$P'(u)[h] = \partial_t h + (1+a_3)\,\partial_{xxx} h + a_2\,\partial_{xx} h + a_1\,\partial_x h + a_0 h, \tag{2-1}$$

where the coefficients $a_i = a_i(t,x)$, $i = 0, \dots, 3$, are real-valued functions of $(t,x) \in [0,T] \times \mathbb{T}$, depending on u by

$$a_i = a_i(u) := (\partial_{z_i} \mathcal{N})(x, u, u_x, u_{xx}, u_{xxx}), \quad i = 0, \dots, 3 \tag{2-2}$$

(recall the notation $\mathcal{N} = \mathcal{N}(x, z_0, z_1, z_2, z_3)$). Note that $a_2 = 2\partial_x a_3$ because of the Hamiltonian structure of the component $\mathcal{N}_1$ of the nonlinearity; see (1-3)–(1-4).

Lemma 2.1. *Let $\mathcal{N} \in C^r(\mathbb{T} \times \mathbb{R}^4, \mathbb{R})$ satisfy (1-2). For all $1 \le s \le r-3$, and for all $u \in C^2([0,T], H^{s+3}_x)$ such that $\|u, \partial_t u, \partial_{tt} u\|_{T,4} \le 1$, the coefficients $a_i(u)$ satisfy*

$$\|a_i(u), \partial_t a_i(u), \partial_{tt} a_i(u)\|_{T,s} \le C \|u, \partial_t u, \partial_{tt} u\|_{T,s+3}, \qquad i = 0,1,2,3. \tag{2-3}$$

Proof. Apply standard tame estimates for composition of functions; see Lemma C.2. □

Now we apply the reduction procedure to any linear operator of the form (2-1) where

$$a_2(t,x) = c\,\partial_x a_3(t,x) \tag{2-4}$$

for some constant $c \in \mathbb{R}$ (note that $P'(u)$ has $c = 2$ because of the Hamiltonian structure of $\mathcal{N}_1$). Regarding the loss of regularity with respect to the space variable x, the estimates in the sequel will be not sharp. In the whole section we consider $T > 0$ fixed, and, unless otherwise specified, all the constants may depend on T.

Remark 2.2. Given a linear operator $\mathcal{L}_0$ of the form (2-1), define the operator $\mathcal{L}_0^*$ as

$$\mathcal{L}_0^* h := -\partial_t h - \partial_{xxx}\{(1+a_3)h\} + \partial_{xx}(a_2 h) - \partial_x(a_1 h) + a_0 h. \tag{2-5}$$

Note that $-\mathcal{L}_0^*$ is still an operator of the form (2-1), namely

$$-\mathcal{L}_0^* = \partial_t + (1 + a_3^*)\,\partial_{xxx} + a_2^*\,\partial_{xx} + a_1^*\,\partial_x + a_0^*, \tag{2-6}$$

with

$$\begin{aligned} a_3^* &:= a_3, & a_2^* &:= 3(a_3)_x - a_2, \\ a_1^* &:= 3(a_3)_{xx} - 2(a_2)_x + a_1, & a_0^* &:= (a_3)_{xxx} - (a_2)_{xx} + (a_1)_x - a_0. \end{aligned} \tag{2-7}$$

It follows from (2-6), (2-7) that if $\mathcal{L}_0$ satisfies (2-4), then also $-\mathcal{L}_0^*$ satisfies (2-4) (with a different constant), namely $a_2^* = (3-c)\,\partial_x a_3^*$. In particular, if $\mathcal{L}_0$ satisfies (2-4) with $c = 2$ (which is the case if $\mathcal{L}_0 = P'(u)$), then $-\mathcal{L}_0^*$ satisfies (2-4) with $c = 1$. □

2A. ***Step 1: change of the space variable.*** We consider a t-dependent family of diffeomorphisms of the circle $\mathbb{T}$ of the form

$$y = x + \beta(t, x), \tag{2-8}$$

where β is a real-valued function, 2π periodic in x, defined for $t \in [0, T]$, with $|\beta_x(t, x)| \le \frac{1}{2}$ for all $(t, x) \in [0, T] \times \mathbb{T}$. We define the linear operator

$$(\mathcal{A}h)(t, x) := h(t, x + \beta(t, x)). \tag{2-9}$$

The operator $\mathcal{A}$ is invertible, with inverse $\mathcal{A}^{-1}$, transpose $\mathcal{A}^T$ (transpose with respect to the usual L^2_x-scalar product) and inverse transpose $\mathcal{A}^{-T}$ given by

$$\begin{aligned} (\mathcal{A}^{-1}v)(t, y) &= v(t, y + \tilde\beta(t, y)), \\ (\mathcal{A}^T v)(t, y) &= (1 + \tilde\beta_y(t, y))\, v(t, y + \tilde\beta(t, y)), \\ (\mathcal{A}^{-T} h)(t, x) &= (1 + \beta_x(t, x))\, h(t, x + \beta(t, x)), \end{aligned} \tag{2-10}$$

where $y \mapsto y + \tilde\beta(t, y)$ is the inverse diffeomorphism of (2-8), namely

$$x = y + \tilde\beta(t, y) \quad \Longleftrightarrow \quad y = x + \beta(t, x). \tag{2-11}$$

Given the operator

$$\mathcal{L}_0 := \partial_t + (1 + a_3(t, x))\,\partial_{xxx} + a_2(t, x)\,\partial_{xx} + a_1(t, x)\,\partial_x + a_0(t, x), \tag{2-12}$$

with $a_2(t, x) = c\,\partial_x a_3(t, x)$, we calculate the conjugate $\mathcal{A}^{-1}\mathcal{L}_0\mathcal{A}$. The conjugate $\mathcal{A}^{-1}a\mathcal{A}$ of any multiplication operator $a : h(t, x) \mapsto a(t, x)h(t, x)$ is the multiplication operator $(\mathcal{A}^{-1}a)$ that maps $v(t, y)$ to $(\mathcal{A}^{-1}a)(t, y)\, v(t, y)$. By conjugation, the differential operators become

$$\mathcal{A}^{-1}\,\partial_t\mathcal{A} = \partial_t + (\mathcal{A}^{-1}\beta_t)\,\partial_y, \quad \mathcal{A}^{-1}\,\partial_x\mathcal{A} = \{\mathcal{A}^{-1}(1 + \beta_x)\}\,\partial_y.$$

Then $\mathcal{A}^{-1}\,\partial_{xx}\mathcal{A} = (\mathcal{A}^{-1}\,\partial_x\mathcal{A})(\mathcal{A}^{-1}\,\partial_x\mathcal{A})$, and similarly for the conjugate of ∂_{xxx}. We calculate

$$\mathcal{L}_1 := \mathcal{A}^{-1}\mathcal{L}_0\mathcal{A} = \partial_t + a_4(t, y)\,\partial_{yyy} + a_5(t, y)\,\partial_{yy} + a_6(t, y)\,\partial_y + a_7(t, y), \tag{2-13}$$

where

$$\begin{aligned} a_4 &= \mathcal{A}^{-1}\{(1+a_3)(1+\beta_x)^3\}, & a_5 &= \mathcal{A}^{-1}\{a_2(1+\beta_x)^2 + 3(1+a_3)\beta_{xx}(1+\beta_x)\}, \\ a_6 &= \mathcal{A}^{-1}\{\beta_t + (1+a_3)\beta_{xxx} + a_2\beta_{xx} + a_1(1+\beta_x)\}, & a_7 &= \mathcal{A}^{-1}a_0. \end{aligned} \tag{2-14}$$

We look for $\beta(t,x)$ such that the coefficient $a_4(t,y)$ of the highest-order derivative ∂_{yyy} in (2-13) does not depend on y; namely $a_4(t,y)=b(t)$ for some function $b(t)$ of t only. This is equivalent to

$$(1+a_3(t,x))(1+\beta_x(t,x))^3=b(t); \tag{2-15}$$

namely

$$\beta_x=\rho_0,\qquad \rho_0(t,x):=b(t)^{1/3}(1+a_3(t,x))^{-1/3}-1. \tag{2-16}$$

Equation (2-16) has a solution β, periodic in x, if and only if $\int_{\mathbb{T}}\rho_0(t,x)\,dx=0$ for all t. This condition uniquely determines

$$b(t)=\left(\frac{1}{2\pi}\int_{\mathbb{T}}(1+a_3(t,x))^{-1/3}\,dx\right)^{-3}. \tag{2-17}$$

Then we fix the solution (with zero average) of (2-16),

$$\beta(t,x):=(\partial_x^{-1}\rho_0)(t,x), \tag{2-18}$$

where $\partial_x^{-1}h$ is the primitive of h with zero average in x (defined in Fourier). We have conjugated $\mathcal{L}_0$ to

$$\mathcal{L}_1=\mathcal{A}^{-1}\mathcal{L}_0\mathcal{A}=\partial_t+a_4(t)\,\partial_{yyy}+a_5(t,y)\,\partial_{yy}+a_6(t,y)\,\partial_y+a_7(t,y), \tag{2-19}$$

where $a_4(t):=b(t)$ is defined in (2-17).

We prove here some bounds that will be used later.

Lemma 2.3. *There exist positive constants σ, δ_* with the following properties. Let $s\ge 0$, and let $a_3(t,x), a_2(t,x), a_1(t,x), a_0(t,x)$ be four functions with $a_2=c\,\partial_x a_3$ for some $c\in\mathbb{R}$. Moreover, assume $\partial_{tt}a_3, \partial_t a_3, a_3, \partial_t a_1, a_1, a_0\in C([0,T],H_x^{s+\sigma})$. Let*

$$\delta(\mu):=\|\partial_{tt}a_3,\partial_t a_3,a_3,\partial_t a_1,a_1,a_0\|_{T,\mu+\sigma}\quad\forall\mu\in[0,s]. \tag{2-20}$$

If $\delta(0)\le\delta_*$, then the operator $\mathcal{A}$ defined in (2-9), (2-18), (2-16), (2-17) belongs to $C([0,T],\mathcal{L}(H_x^\mu))$ for all $\mu\in[0,s]$ and satisfies

$$\|\mathcal{A}h\|_{T,\mu}\le C_\mu\big(\|h\|_{T,\mu}+\delta(\mu)\|h\|_{T,0}\big)\quad\forall h\in C([0,T],H_x^\mu), \tag{2-21}$$

for some positive C_μ depending on μ. The inverse operator $\mathcal{A}^{-1}$, the transpose $\mathcal{A}^T$ and the inverse transpose $\mathcal{A}^{-T}$ all satisfy the same estimate (2-21) as $\mathcal{A}$.

The functions $a_4(t)=b(t)$, $a_5(t,y)$, $a_6(t,y)$, $a_7(t,y)$, $\beta(t,x)$, $\tilde\beta(t,y)$ defined in (2-17), (2-16), (2-18), (2-14), (2-11) belong to $C([0,T],H_x^\mu)$ for all $\mu\in[0,s]$ and satisfy

$$\|\beta,\tilde\beta,a_5,\partial_t a_5,a_6,\partial_t a_6,a_7\|_{T,\mu}+|a_4-1,a_4'|_T\le C_\mu\delta(\mu). \tag{2-22}$$

Finally, the coefficient $a_5(t,y)$ satisfies

$$\int_{\mathbb{T}}a_5(t,y)\,dy=0\quad\forall t\in[0,T]. \tag{2-23}$$

Proof. The proof of (2-21) and (2-22) is a straightforward application of the standard tame estimates for products, composition of functions and changes of variable; see Appendix C.

To prove (2-23), we use the definition of $b(t)$ in (2-17), the equality $a_2 = c\,\partial_x a_3$, and the change of variables (2-11), and we compute

$$\begin{aligned}\int_{\mathbb{T}} a_5(t,y)\,dy &= \int_{\mathbb{T}} \big[a_2(1+\beta_x)^2 + 3(1+a_3)\beta_{xx}(1+\beta_x)\big](1+\beta_x)\,dx \\ &= b(t)\bigg\{c\int_{\mathbb{T}} \frac{\partial_x a_3(t,x)}{1+a_3(t,x)}\,dx + 3\int_{\mathbb{T}} \frac{\beta_{xx}(t,x)}{1+\beta_x(t,x)}\,dx\bigg\} \\ &= b(t)\bigg\{c\int_{\mathbb{T}} \partial_x \log(1+a_3(t,x))\,dx + 3\int_{\mathbb{T}} \partial_x \log(1+\beta_x(t,x))\,dx\bigg\} = 0. \qquad \square\end{aligned}$$

2B. ***Step 2: time reparametrization.*** The goal of this section is to obtain a constant coefficient instead of $a_4(t)$. We consider a diffeomorphism $\psi : [0,T] \to [0,T]$ which gives the change of the time variable

$$\psi(t) = \tau \quad \Longleftrightarrow \quad t = \psi^{-1}(\tau), \tag{2-24}$$

with $\psi(0) = 0$ and $\psi(T) = T$. We define

$$(\mathcal{B}h)(t,y) := h(\psi(t),y), \quad (\mathcal{B}^{-1}v)(\tau,y) := v(\psi^{-1}(\tau),y). \tag{2-25}$$

By conjugation, the differential operators become

$$\mathcal{B}^{-1}\,\partial_t\mathcal{B} = \rho(\tau)\partial_\tau, \quad \mathcal{B}^{-1}\,\partial_y\mathcal{B} = \partial_y, \quad \rho := \mathcal{B}^{-1}(\psi'), \tag{2-26}$$

and therefore (2-19) is conjugated to

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\,\partial_\tau + (\mathcal{B}^{-1}a_4)\,\partial_{yyy} + (\mathcal{B}^{-1}a_5)\,\partial_{yy} + (\mathcal{B}^{-1}a_6)\,\partial_y + (\mathcal{B}^{-1}a_7). \tag{2-27}$$

We look for ψ such that the (variable) coefficients of the highest-order derivatives (∂_τ and ∂_{yyy}) are proportional; namely

$$(\mathcal{B}^{-1}a_4)(\tau) = m\rho(\tau) = m(\mathcal{B}^{-1}(\psi'))(\tau) \tag{2-28}$$

for some constant $m \in \mathbb{R}$. Since $\mathcal{B}$ is invertible, this is equivalent to requiring that

$$a_4(t) = m\psi'(t). \tag{2-29}$$

Integrating on $[0,T]$ determines the value of the constant m, and then we fix ψ:

$$m := \frac{1}{T}\int_0^T a_4(t)\,dt, \quad \psi(t) := \frac{1}{m}\int_0^t a_4(s)\,ds. \tag{2-30}$$

With this choice of ψ we get

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\,\mathcal{L}_2, \quad \mathcal{L}_2 := \partial_\tau + m\partial_{yyy} + a_8(\tau,y)\,\partial_{yy} + a_9(\tau,y)\,\partial_y + a_{10}(\tau,y), \tag{2-31}$$

where

$$a_8(\tau, y) := \frac{1}{\rho(\tau)} (\mathcal{B}^{-1} a_5)(\tau, y), \quad a_9(\tau, y) := \frac{1}{\rho(\tau)} (\mathcal{B}^{-1} a_6)(\tau, y), \quad a_{10}(\tau, y) := \frac{1}{\rho(\tau)} (\mathcal{B}^{-1} a_7)(\tau, y). \tag{2-32}$$

Note that for all $\tau \in [0, T]$ one has

$$\int_{\mathbb{T}} a_8(\tau, y)\, dy = \frac{1}{(\mathcal{B}^{-1}\psi')(\tau)} \int_{\mathbb{T}} (\mathcal{B}^{-1} a_5)(\tau, y)\, dy = \frac{1}{\psi'(t)} \int_{\mathbb{T}} a_5(t, y)\, dy = 0. \tag{2-33}$$

By straightforward calculations, we prove the following lemma.

Lemma 2.4. *There exists $\delta_* > 0$ with the following properties. Let $a_4 \in C([0, T], \mathbb{R})$ with $|a_4(t) - 1| \le \delta_*$ for all $t \in [0, T]$. Then the operator $\mathcal{B}$ defined in (2-25), (2-30) is an invertible isometry of $C([0, T], H^s_x)$ for all $s \ge 0$; namely,*

$$\|\mathcal{B}h\|_{T,s} = \|h\|_{T,s} \quad \forall h \in C([0, T], H^s_x),\ s \ge 0. \tag{2-34}$$

Moreover there exists a positive constant σ with the following property. Let $a_4 \in C^1([0, T], \mathbb{R})$, with $|a_4(t) - 1| \le \delta_*$ and $|a_4'(t)| \le 1$ for all $t \in [0, T]$. Let $s \ge 0$, and $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0, T], H^s_x)$ with $\int_{\mathbb{T}} a_5(t, y)\, dy = 0$ for all $t \in [0, T]$. Then the functions $a_8(t, x), a_9(t, x), a_{10}(t, x), \psi(t), \rho(t)$ and the constant m defined in (2-32), (2-30), (2-26) satisfy

$$|m - 1| + |\psi' - 1, \rho - 1|_T + \|a_8, \partial_\tau a_8, a_9, \partial_\tau a_9, a_{10}\|_{T,s} \le C \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,s}, \tag{2-35}$$

where C is independent of s. Moreover one has

$$\int_{\mathbb{T}} a_8(\tau, y)\, dy = 0 \quad \forall \tau \in [0, T]. \tag{2-36}$$

2C. *Step 3: multiplication.* In this section we eliminate the term $a_8(\tau, y)\, \partial_{yy}$ from the operator $\mathcal{L}_2$ defined in (2-31). To this end, we consider the multiplication operator $\mathcal{M}$ defined as

$$\mathcal{M}h(\tau, y) := q(\tau, y) h(\tau, y), \tag{2-37}$$

with $q : [0, T] \times \mathbb{T} \to \mathbb{R}$. We compute

$$\mathcal{M}^{-1} \mathcal{L}_2 \mathcal{M} = \partial_\tau + m\, \partial_{yyy} + a_{11}(\tau, y)\, \partial_{yy} + a_{12}(\tau, y)\, \partial_y + a_{13}(\tau, y), \tag{2-38}$$

with

$$a_{11} := a_8 + \frac{3m q_y}{q}, \quad a_{12} := a_9 + \frac{2a_8 q_y + 3m q_{yy}}{q}, \quad a_{13} := \frac{\mathcal{L}_2 q}{q}. \tag{2-39}$$

We want to choose q such that $a_{11} = 0$, which is equivalent to

$$3m q_y + a_8 q = 0. \tag{2-40}$$

Thanks to (2-36), equation (2-40) admits the space-periodic solution

$$q(\tau, y) := \exp\Big\{ -\frac{1}{3m} (\partial_y^{-1} a_8)(\tau, y) \Big\}. \tag{2-41}$$

As a consequence, we get

$$\mathcal{L}_3 := \mathcal{M}^{-1} \mathcal{L}_2 \mathcal{M} = \partial_\tau + m\, \partial_{yyy} + a_{12}(\tau, y)\, \partial_y + a_{13}(\tau, y). \tag{2-42}$$

The proof of the following lemma is straightforward.

Lemma 2.5. *Let $s \geq 0$ and let $a_8 \in C([0,T], H^s_x)$ with $\int_{\mathbb{T}} a_8(\tau, y)\,dy = 0$ for all $\tau \in [0,T]$. Then for all $\mu \in [0,s]$, the operator $\mathcal{M}$ defined in (2-37), (2-41) and its inverse $\mathcal{M}^{-1}$ belong to $C([0,T], \mathcal{L}(H^\mu_x))$. Note that $\mathcal{M} = \mathcal{M}^T$.*

Furthermore, there exist two positive constants δ_*, σ with the following properties. Assume that $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0,T], H^{s+\sigma}_x)$ and let

$$\delta(\mu) := \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T,\mu+\sigma}. \tag{2-43}$$

Then if $\delta(0) \leq \delta_*$, for all $\mu \in [0,s]$ the operator $\mathcal{M}$ and its inverse $\mathcal{M}^{-1}$ satisfy

$$\|\mathcal{M}^{\pm 1} h\|_{T,\mu} \leq C_\mu \big(\|h\|_{T,\mu} + \delta(\mu)\|h\|_{T,0}\big) \quad \forall h \in C([0,T], H^\mu_x), \tag{2-44}$$

for some positive C_μ depending on μ. Moreover, the functions $a_{12}(\tau,y)$, $a_{13}(\tau,y)$, $q(\tau,y)$ defined in (2-39), (2-41) satisfy

$$\|q-1, a_{12}, \partial_t a_{12}, a_{13}\|_{T,\mu} \leq C_\mu \delta(\mu). \tag{2-45}$$

2D. *Step 4: translation of the space variable.* We consider the change of the space variable $z = y + p(\tau)$ and the operators

$$\mathcal{T} h(\tau, y) := h(\tau, y + p(\tau)), \quad \mathcal{T}^{-1} v(\tau, z) := v(\tau, z - p(\tau)), \tag{2-46}$$

where p is a function $p : [0,T] \to \mathbb{R}$. The differential operators become $\mathcal{T}^{-1}\partial_y \mathcal{T} = \partial_z$ and $\mathcal{T}^{-1}\partial_\tau \mathcal{T} = \partial_\tau + p'(\tau)\partial_z$. This is a special, simple case of the transformation $\mathcal{A}$ of Section 2A. Thus

$$\mathcal{L}_4 := \mathcal{T}^{-1}\mathcal{L}_3\mathcal{T} = \partial_\tau + m\,\partial_{zzz} + a_{14}(\tau,z)\,\partial_z + a_{15}(\tau,z), \tag{2-47}$$

where

$$a_{14}(\tau,z) := p'(\tau) + (\mathcal{T}^{-1}a_{12})(\tau,z), \quad a_{15}(\tau,z) := (\mathcal{T}^{-1}a_{13})(\tau,z). \tag{2-48}$$

Now we look for $p(\tau)$ such that a_{14} has zero space average. We fix

$$p(\tau) := -\frac{1}{2\pi}\int_0^\tau \int_{\mathbb{T}} a_{12}(s,y)\,dy\,ds. \tag{2-49}$$

With this choice of p, after renaming the space-time variables $z = x$ and $\tau = t$, we have

$$\mathcal{L}_4 = \partial_t + m\,\partial_{xxx} + a_{14}(t,x)\,\partial_x + a_{15}(t,x), \qquad \int_{\mathbb{T}} a_{14}(t,x)\,dx = 0 \quad \forall t \in [0,T]. \tag{2-50}$$

With direct calculations we prove the following estimates.

Lemma 2.6. *Let $a_{12} \in C([0,T], L^2_x)$. Then the operator $\mathcal{T}$ defined in (2-46) and (2-49) belongs to $C([0,T], \mathcal{L}(H^s_x))$ for all $s \in [0,+\infty)$. In fact $\mathcal{T}$ is an isometry, namely*

$$\|\mathcal{T}h\|_{T,s} = \|h\|_{T,s} \quad \forall h \in C([0,T], H^s_x). \tag{2-51}$$

Moreover, $\mathcal{T}$ is invertible and its transpose is $\mathcal{T}^T = \mathcal{T}^{-1}$.

Let $s \geq 0$, and let $a_{12}, \partial_t a_{12}, a_{13} \in C([0,T], H_x^{s+1})$ with $\|a_{12}\|_{T,0} \leq 1$. Then the functions a_{14}, a_{15}, p defined in (2-48) and (2-49) satisfy

$$\sup_{t\in[0,T]} |p(t)| + \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,s} \leq C \|a_{12}, \partial_t a_{12}, a_{13}\|_{T,s+1}, \tag{2-52}$$

where C is independent of s.

2E. *Step 5: elimination of the order one.* The goal of this section is to eliminate the term $a_{14}(t,x)\,\partial_x$. Consider an operator $\mathcal{S}$ of the form

$$\mathcal{S}h := h + \gamma(t,x)\,\partial_x^{-1} h, \tag{2-53}$$

where $\gamma(t,x)$ is a function to be determined. Note $\partial_x^{-1}\partial_x = \partial_x \partial_x^{-1} = \pi_0$, where $\pi_0 h := h - \frac{1}{2\pi}\int_{\mathbb{T}} h\,dx$. We directly calculate

$$\mathcal{L}_4 \mathcal{S} - \mathcal{S}(\partial_t + m\partial_{xxx}) = a_{16}\,\partial_x + a_{17} + a_{18}\,\partial_x^{-1}, \tag{2-54}$$

where

$$a_{16} := 3m\gamma_x + a_{14}, \quad a_{17} := a_{15} + (3m\gamma_{xx} + a_{14}\gamma)\pi_0, \quad a_{18} := \gamma_t + m\gamma_{xxx} + a_{14}\gamma_x + a_{15}\gamma. \tag{2-55}$$

We fix γ as

$$\gamma := -\frac{1}{3m}\,\partial_x^{-1} a_{14}, \tag{2-56}$$

so that $a_{16} = 0$. By the following Lemma 2.7, $\mathcal{S}$ is invertible, and we obtain

$$\mathcal{L}_5 := \mathcal{S}^{-1}\mathcal{L}_4\mathcal{S} = \partial_t + m\,\partial_{xxx} + \mathcal{R}, \quad \mathcal{R} := \mathcal{S}^{-1}(a_{17} + a_{18}\,\partial_x^{-1}). \tag{2-57}$$

Lemma 2.7. *There exist positive constants σ, δ_* with the following properties. Let $s \geq 0$, let a_{14}, a_{15} be two functions with $a_{14}, \partial_t a_{14}, a_{15} \in C([0,T], H_x^{s+\sigma})$ and $\int_{\mathbb{T}} a_{14}(t,x)\,dx = 0$. Let*

$$\delta(\mu) := \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s]. \tag{2-58}$$

If $\delta(0) \leq \delta_*$, then the operator $\mathcal{S}$ defined in (2-53), (2-56) belongs to $C([0,T], \mathcal{L}(H_x^\mu))$ for all $\mu \in [0,s]$ and satisfies

$$\|\mathcal{S}h\|_{T,\mu} \leq C_\mu \big(\|h\|_{T,\mu} + \delta(\mu)\|h\|_{T,0}\big) \quad \forall h \in C([0,T], H_x^\mu), \tag{2-59}$$

for some positive C_μ depending on μ. The operator $\mathcal{S}$ is invertible, and its inverse $\mathcal{S}^{-1}$, its transpose $\mathcal{S}^T$ and its inverse transpose $\mathcal{S}^{-T}$ all satisfy the same estimate (2-59) as $\mathcal{S}$.

The operator $\mathcal{R}$ defined in (2-57) belongs to $C([0,T], \mathcal{L}(H_x^\mu))$ for all $\mu \in [0,s]$ and it satisfies

$$\|\mathcal{R}h\|_{T,\mu} \leq C_\mu \big(\delta(0)\|h\|_{T,\mu} + \delta(\mu)\|h\|_{T,0}\big) \quad \forall h \in C([0,T], H_x^\mu). \tag{2-60}$$

The transpose $\mathcal{R}^T$ belongs to $C([0,T], \mathcal{L}(H_x^\mu))$ and satisfies the same estimate (2-60) as $\mathcal{R}$.

Proof. Estimate $\|\gamma\,\partial_x^{-1} h\|_{T,\mu}$ by the usual tame estimates for the product of two functions (Lemma C.1), then use Neumann series in its tame version. □

3. Observability

In this section we prove the observability of linear operators of the form (2-12). Such an observability property will be used in Section 4 in order to prove controllability of the linearized problem. We split the proof into several simple lemmas, starting with a direct consequence of the Ingham inequality. Since we actually need observability of a Cauchy problem flowing backwards in time (see Lemma 4.2) with datum at time T, we will accordingly state our lemmas.

Lemma 3.1 (Ingham inequality for $\partial_t + m\,\partial_{xxx}$). *For every $T > 0$ there exists a positive constant $C_1(T)$ such that, for all $(w_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{C})$, all $m \geq \frac{1}{2}$,*

$$\int_0^T \Big|\sum_{n\in\mathbb{Z}} w_n e^{imn^3 t}\Big|^2 \, dt \geq C_1(T) \sum_{n\in\mathbb{Z}} |w_n|^2.$$

Proof. See, for example, Theorem 4.3 in Section 4.1 of [Micu and Zuazua 2005]. The fact that the constant $C_1(T)$ does not depend on m is obtained by closely following the proof in the above-mentioned work, and taking into account the lower bound for the distance between two different eigenvalues $|mn^3 - mk^3| \geq m \geq \frac{1}{2}$ for all $n, k \in \mathbb{Z}$, $n \neq k$. $\square$

The following observability result is classical (see, e.g., [Russell and Zhang 1993] for a closely related result); for completeness, we also give here its proof.

Lemma 3.2 (observability for $\partial_t + m\,\partial_{xxx}$). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let $v_T \in L^2(\mathbb{T})$, $m \geq \frac{1}{2}$, and let v satisfy*

$$\partial_t v + m\,\partial_{xxx} v = 0, \qquad v(T) = v_T. \tag{3-1}$$

Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx\, dt \geq C_2 \|v_T\|^2_{L^2_x}, \tag{3-2}$$

with $C_2 := C_1(T)|\omega|$, where $C_1(T)$ is the constant of Lemma 3.1, and $|\omega|$ is the Lebesgue measure of ω.

Proof. Let $v_T(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$, so that $v(t,x) = \sum_{n\in\mathbb{Z}} w_n(x) e^{imn^3 t}$, where $w_n(x) := a_n e^{i(nx - mn^3 T)}$. By Lemma 3.1, for each $x \in \mathbb{T}$ we have

$$\int_0^T \Big|\sum_{n\in\mathbb{Z}} w_n(x) e^{imn^3 t}\Big|^2 \, dt \geq C_1(T) \sum_{n\in\mathbb{Z}} |w_n(x)|^2 = C_1(T) \sum_{n\in\mathbb{Z}} |a_n|^2 = C_1(T) \|v_T\|^2_{L^2(\mathbb{T})}.$$

Then we integrate over $x \in \omega$. $\square$

Lemma 3.3 (observability of $\mathcal{L}_5 := \partial_t + m\,\partial_{xxx} + \mathcal{R}$). *Let $T > 0$, let $\omega \subset \mathbb{T}$ be an open set and let $m \geq \frac{1}{2}$. Let $\mathcal{R} \in C([0,T], \mathcal{L}(L^2_x))$, with $\|\mathcal{R}(t)h\|_0 \leq r_0 \|h\|_0$ for all $h \in L^2_x$, all $t \in [0,T]$, where r_0 is a positive constant. Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem*

$$\partial_t v + m\,\partial_{xxx} v + \mathcal{R} v = 0, \qquad v(T) = v_T, \tag{3-3}$$

which is globally well-posed by Lemma A.2(iii). *Then*

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \geq C_3 \|v_T\|_{L^2_x}^2,$$

with $C_3 := \frac{1}{4} C_2$, *provided that* r_0 *is small enough* (*more precisely*, r_0 *is smaller than a constant depending only on* T, C_2, *where* C_2 *is the constant in Lemma 3.2*).

Proof. Let v_1 be the solution of $\partial_t v_1 + m\,\partial_{xxx} v_1 = 0$, $v_1(T) = v_T$, and let $v_2 := v - v_1$. Then v_2 solves

$$(\partial_t + m\,\partial_{xxx} + \mathcal{R}) v_2 = -\mathcal{R} v_1, \qquad v_2(T) = 0. \tag{3-4}$$

By (A-10), applied for $s=0$, $\alpha=0$, $f = -\mathcal{R} v_1$, we get

$$\|v_2\|_{T,0} \leq 2^{4Tr_0} 4T \|\mathcal{R} v_1\|_{T,0} \leq 2^{4Tr_0} 4T r_0 \|v_T\|_0. \tag{3-5}$$

Using the elementary inequality $(a+b)^2 \geq \frac{1}{2} a^2 - b^2$ for all $a, b \in \mathbb{R}$,

$$\int_0^T \int_\omega |v|^2 \, dx \, dt \geq \frac{1}{2} \int_0^T \int_\omega |v_1|^2 \, dx \, dt - \int_0^T \int_\omega |v_2|^2 \, dx \, dt.$$

The integral of $|v_1|^2$ is estimated from below by (3-2). The integral of $|v_2|^2$ is bounded by $T\|v_2\|_{T,0}^2$; then use (3-5). □

Lemma 3.4 (observability of $\mathcal{L}_4 := \partial_t + m\,\partial_{xxx} + a_{14}(t,x)\,\partial_x + a_{15}(t,x)$, a_{14} with zero mean). *There exists a universal constant* $\sigma > 0$ *with the following property. Let* $T > 0$, *and let* $\omega \subset \mathbb{T}$ *be an open set. Let* $m \geq \frac{1}{2}$ *and let* $a_{14}(t,x)$, $a_{15}(t,x)$ *be two functions, with* $a_{14}, \partial_t a_{14}, a_{15} \in C([0,T], H^\sigma_x)$,

$$\int_{\mathbb{T}} a_{14}(t,x) \, dx = 0 \quad \forall t \in [0,T], \qquad \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,\sigma} \leq \delta. \tag{3-6}$$

Let $v_T \in L^2(\mathbb{T})$ *and let* $v \in C([0,T], L^2_x)$ *be the solution of the Cauchy problem*

$$\mathcal{L}_4 v = 0, \qquad v(T) = v_T, \tag{3-7}$$

which is globally well-posed by Lemma A.3. Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \geq C_4 \|v_T\|_{L^2_x}^2,$$

with $C_4 := \frac{1}{16} C_3$, *provided that* δ *is small enough* (*more precisely*, δ *is smaller than a constant depending only on* T, C_3).

Proof. Following the procedure of Section 2E, we consider the transformation $\mathcal{S}$ in (2-53), (2-56), which conjugates $\mathcal{L}_4$ to

$$\mathcal{L}_5 := \mathcal{S}^{-1} \mathcal{L}_4 \mathcal{S} = \partial_t + m\,\partial_{xxx} + \mathcal{R},$$

where the operator $\mathcal{R}$ is defined in (2-57), (2-55); it belongs to $C([0,T], \mathcal{L}(L^2_x))$, and satisfies the bounds in Lemma 2.7. Let v be the solution of (3-7), and define $\tilde{v} := \mathcal{S}^{-1} v$. Then $\tilde{v}$ solves $\mathcal{L}_5 \tilde{v} = 0$, $\tilde{v}(T) = \tilde{v}_T$, where $\tilde v_T := \mathcal{S}^{-1}(T) v_T$, and therefore Lemma 3.3 applies to $\tilde v$ if δ is sufficiently small. By Lemmas 2.7 and A.3 and Remark A.8 we get

$$\int_0^T \int_\omega |(\mathcal{S}^{-1} - I)v|^2 \, dx \, dt \le T \|(\mathcal{S}^{-1} - I)v\|_{T,0}^2 \le C\delta^2 \|v\|_{T,0}^2 \le C'\delta^2 \|v_T\|_0^2$$

for some constant C' depending on T. We split $\tilde v = v + (\mathcal{S}^{-1} - I)v$, and we get

$$\int_0^T \int_\omega |\tilde v|^2 \, dx \, dt \le 2 \int_0^T \int_\omega |v|^2 \, dx \, dt + 2C'\delta^2 \|v_T\|_0^2.$$

Moreover $\|v_T\|_0 = \|\mathcal{S}(T) v_T\|_0 \le 2 \|\tilde v_T\|_0$, and the thesis follows for δ small enough. □

Lemma 3.5 (observability of $\mathcal{L}_3 := \partial_t + m\,\partial_{xxx} + a_{12}(t,x)\,\partial_x + a_{13}(t,x)$). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set and let $m \ge \frac{1}{2}$. Let $a_{12}(t,x)$, $a_{13}(t,x)$ be two functions, with $a_{12}, \partial_t a_{12}, a_{13} \in C([0,T], H^\sigma_x)$,*

$$\|a_{12}, \partial_t a_{12}, a_{13}\|_{T,\sigma} \le \delta. \tag{3-8}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_3 v = 0, \quad v(T) = v_T, \tag{3-9}$$

which is globally well-posed by Lemma A.4. Then

$$\int_0^T \int_\omega |v(t,x)|^2 \, dx \, dt \ge C_5 \|v_T\|_{L^2_x}^2 \tag{3-10}$$

for some $C_5 > 0$ depending on T, ω, provided that δ in (3-8) is sufficiently small (more precisely, δ is smaller than a constant depending on T, ω, C_4).

Proof. Following the procedure of Section 2D, we consider the transformation $\mathcal{T}$ defined in (2-46), (2-49), which conjugates $\mathcal{L}_3$ to

$$\mathcal{L}_4 := \mathcal{T}^{-1} \mathcal{L}_3 \mathcal{T} = \partial_t + m\,\partial_{xxx} + a_{14}(t,x)\,\partial_x + a_{15}(t,x),$$

where a_{14}, a_{15} are defined in (2-48), and $\int_{\mathbb{T}} a_{14}(t,x)\,dx = 0$. By (2-52), the function p defined in (2-49) satisfies $|p(t)| \le C\delta$ for all $t \in [0,T]$. Let v be the solution of the Cauchy problem (3-9). Then $\tilde v := \mathcal{T}^{-1} v$ solves $\mathcal{L}_4 \tilde v = 0$, $\tilde v(T) = \mathcal{T}^{-1}(T) v_T$. Let $\omega_1 = [\alpha_1, \beta_1]$ be an interval contained in ω. For δ small enough, one has

$$[\alpha_1 - p(t), \beta_1 - p(t)] \subseteq [\alpha_1 - \delta, \beta_1 + \delta] \subset \omega \quad \forall t \in [0,T].$$

The change of variable $x - p(t) = y$, $dx = dy$ gives

$$\int_0^T \int_{\omega_1} |\tilde v(t,x)|^2 \, dx \, dt = \int_0^T \int_{\alpha_1 - p(t)}^{\beta_1 - p(t)} |v(t,y)|^2 \, dy \, dt \le \int_0^T \int_\omega |v(t,y)|^2 \, dy \, dt.$$

By (2-52), for δ small enough, Lemma 3.4 can be applied to $\tilde v$ on the interval ω_1 and the thesis follows, since $\|\tilde v(T)\|_0 = \|\mathcal{T}^{-1}(T) v_T\|_0 = \|v_T\|_0$. □

Lemma 3.6 (observability of $\mathcal{L}_2 := \partial_t + m\,\partial_{xxx} + a_8(t,x)\,\partial_{xx} + a_9(t,x)\,\partial_x + a_{10}(t,x)$). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set and let $m \geq \frac{1}{2}$. Let $a_8(t,x)$, $a_9(t,x)$, $a_{10}(t,x)$ be three functions, with $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0,T], H^\sigma_x)$,*

$$\int_{\mathbb{T}} a_8(t,x)\,dx = 0 \quad \forall t \in [0,T], \qquad \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T,\sigma} \leq \delta. \tag{3-11}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_2 v = 0, \quad v(T) = v_T, \tag{3-12}$$

which is globally well-posed by Lemma A.5. Then

$$\int_0^T \int_\omega |v(t,x)|^2\,dx\,dt \geq C_6 \|v_T\|^2_{L^2_x} \tag{3-13}$$

for some $C_6 > 0$ depending on T, ω, provided that δ in (3-11) is sufficiently small (more precisely, δ is smaller than a constant depending on T, ω, C_5).

Proof. Following the procedure of Section 2C, we consider the multiplication operator $\mathcal{M}$ defined in (2-37), (2-41), which conjugates $\mathcal{L}_2$ to

$$\mathcal{M}^{-1}\mathcal{L}_2\mathcal{M} = \mathcal{L}_3, \quad \mathcal{L}_3 = \partial_t + m\,\partial_{xxx} + a_{12}(t,x)\,\partial_x + a_{13}(t,x),$$

where a_{12}, a_{13} are defined in (2-39). Let v be the solution of the Cauchy problem (3-12). Then $\tilde{v} := \mathcal{M}^{-1}v$ solves $\mathcal{L}_3\tilde{v} = 0$, $\tilde{v}(T) = \mathcal{M}^{-1}(T)v_T$. Using (2-45), we have

$$\int_0^T \int_\omega |v(t,x)|^2\,dx\,dt = \int_0^T \int_\omega |\tilde{v}|^2\,dx\,dt + \int_0^T \int_\omega |\tilde{v}|^2(|q|^2 - 1)\,dx\,dt \geq (C_5 - C\delta)\|v_T\|_0^2.$$

The first of the two integrals has been estimated from below by applying Lemma 3.5 to $\mathcal{L}_3$ (by Lemma 2.5, this can be done provided that δ is sufficiently small). The second integral has been estimated using the bound (2-45), since $|q(t) - 1| \leq C\|q - 1\|_{T,1} \leq C'\delta$. Moreover, we have used the inequality $\|\tilde{v}\|_{T,0} \leq C\|\tilde{v}_T\|_0$ from Lemma A.4. The thesis follows with $C_6 := \frac{1}{2}C_5$ by choosing δ small enough. □

Lemma 3.7 (observability of $\mathcal{L}_1 := \partial_t + a_4(t)\,\partial_{xxx} + a_5(t,x)\,\partial_{xx} + a_6(t,x)\,\partial_x + a_7(t,x)$). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let a_4, a_5, a_6, a_7 be four functions, with $a_4 \in C^1([0,T], \mathbb{R})$ and $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0,T], H^\sigma_x)$, satisfying*

$$\int_{\mathbb{T}} a_5(t,x)\,dx = 0 \quad \forall t \in [0,T], \qquad \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,\sigma} + |a_4 - 1, a_4'|_T \leq \delta. \tag{3-14}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_1 v = 0, \quad v(T) = v_T, \tag{3-15}$$

which is globally well-posed by Lemma A.6. Then

$$\int_0^T \int_\omega |v(t,x)|^2\,dx\,dt \geq C_7 \|v_T\|^2_{L^2_x} \tag{3-16}$$

for some $C_7 > 0$ depending on T, ω, provided that δ in (3-14) is sufficiently small (more precisely, δ is smaller than a constant depending on T, ω, C_6).

Proof. Following the procedure of Section 2B, we consider the reparametrization of time $\mathcal{B}$ defined in (2-25), (2-30), which conjugates $\mathcal{L}_1$ to

$$\mathcal{B}^{-1}\mathcal{L}_1\mathcal{B} = \rho\mathcal{L}_2, \quad \mathcal{L}_2 = \partial_\tau + m\,\partial_{xxx} + a_8(\tau,x)\,\partial_{xx} + a_9(\tau,x)\,\partial_x + a_{10}(\tau,x),$$

where ρ, a_8, a_9, a_{10} are defined in (2-28), (2-32) and $\int_{\mathbb{T}} a_8(\tau,x) = 0$ for all $\tau \in [0,T]$. Let v be the solution of the Cauchy problem (3-15). Then $\tilde v := \mathcal{B}^{-1}v$ solves $\mathcal{L}_2\tilde v = 0$, $\tilde v(T) = \mathcal{B}^{-1}(T)v_T$. Using (2-35), we have

$$\begin{aligned}\int_0^T\int_\omega |v(t,x)|^2\,dx\,dt &= \int_0^T\int_\omega |\tilde v(\psi(t),x)|^2\,dx\,dt\\ &= \int_0^T\int_\omega |\tilde v(\psi(t),x)|^2[\psi'(t) + (1-\psi'(t))]\,dx\,dt\\ &= \int_0^T\int_\omega |\tilde v(\tau,x)|^2\,dx\,d\tau + \int_0^T\int_\omega |\tilde v(\psi(t),x)|^2(1-\psi'(t))\,dx\,dt\\ &\geq (C_6 - C\delta)\|v_T\|_0^2.\end{aligned}$$

The first of the two integrals has been estimated from below by applying Lemma 3.6 to $\mathcal{L}_2$ (by Lemma 2.4, this can be done provided that δ is sufficiently small). The second integral has been estimated using the bound (2-35) for $|\psi'(t)-1|$ and also the inequality $\|\tilde v\|_{T,0} \leq C\|\tilde v_T\|_0$ from Lemma A.5. The thesis follows with $C_7 := \frac{1}{2}C_6$ by choosing δ small enough, since $\|\tilde v_T\|_0 = \|\mathcal{B}^{-1}(T)v_T\|_0 = \|v_T\|_0$. □

Lemma 3.8 (observability of $\mathcal{L}_0 := \partial_t + (1+a_3)\,\partial_{xxx} + a_2\,\partial_{xx} + a_1\,\partial_x + a_0$). *There exists a universal constant $\sigma > 0$ with the following property. Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let $c \in \mathbb{R}$ and $a_3(t,x)$, $a_2(t,x)$, $a_1(t,x)$, $a_0(t,x)$ be four functions with $a_2 = c\,\partial_x a_3$,*

$$\|\partial_{tt}a_3, \partial_t a_3, a_3, \partial_t a_1, a_1, a_0\|_{T,\sigma} \leq \delta. \tag{3-17}$$

Let $v_T \in L^2(\mathbb{T})$ and let $v \in C([0,T], L^2_x)$ be the solution of the Cauchy problem

$$\mathcal{L}_0 v = 0, \quad v(T) = v_T, \tag{3-18}$$

which is globally well-posed by Lemma A.7. Then

$$\int_0^T\int_\omega |v(t,x)|^2\,dx\,dt \geq C_8\|v_T\|^2_{L^2_x} \tag{3-19}$$

for some $C_8 > 0$ depending on T, ω, provided that δ in (3-17) is sufficiently small (more precisely, δ is smaller than a constant depending on T, ω, C_7).

Proof. Following the procedure of Section 2A, we consider the transformation $\mathcal{A}$ defined in (2-9), (2-16), (2-17), (2-18), which conjugates $\mathcal{L}_0$ to

$$\mathcal{A}^{-1}\mathcal{L}_0\mathcal{A} = \mathcal{L}_1 = \partial_t + a_4(t)\,\partial_{xxx} + a_5(t,x)\,\partial_{xx} + a_6(t,x)\,\partial_x + a_7(t,x)$$

(see (2-19)), where a_4, a_5, a_6, a_7 are defined in (2-14) and $\int_{\mathbb{T}} a_5(t,x) = 0$ for all $t \in [0,T]$. Let v be the solution of the Cauchy problem (3-18). Then $\tilde{v} := \mathcal{A}^{-1} v$ solves $\mathcal{L}_1 \tilde{v} = 0$, $\tilde{v}(T) = \tilde{v}_T$, where $\tilde{v}_0 := \mathcal{A}^{-1}(0) v_0$. Let $\omega_1 = [\alpha_1, \beta_1] \subset \omega$. By (2-22) in Lemma 2.3, for δ sufficiently small Lemma 3.7 applies to $\tilde{v}$ on ω_1, and

$$\int_0^T \int_{\omega_1} |\tilde{v}|^2 \, dy \, dt \geq C_7 \|\tilde{v}_T\|_0^2.$$

By Lemma 2.3, $\|v_T\|_0 = \|\mathcal{A}(T)\tilde{v}_T\|_0 \leq C\|\tilde{v}_T\|_0$. The change of integration variable $y = x + \beta(t,x)$, $dy = (1 + \beta_x(t,x)) dx$ gives

$$\begin{aligned}\int_0^T \int_{\omega_1} |\tilde{v}|^2 \, dy \, dt &= \int_0^T \int_{\omega_1} |(\mathcal{A}^{-1} v)(t,y)|^2 \, dy \, dt \\ &= \int_0^T \int_{\omega_2(t)} \frac{|v(t,x)|^2}{1 + \beta_x(t,x)} \, dx \, dt \leq 2 \int_0^T \int_{\omega} |v(t,x)|^2 \, dx \, dt,\end{aligned}$$

where $\omega_2(t) := \{x : x + \beta(t,x) \in \omega_1\}$. We have used the fact that, for δ small enough, $\omega_2(t) \subset \omega$, and the bound (2-22) for $|\beta_x(t,x)| \leq C\|\beta\|_{T,2} \leq C'\delta$. □

4. Controllability

In this section we prove the controllability of the linearized operator $\mathcal{L}_0$, using its observability (Lemma 3.8), by means of the HUM. We also prove higher regularity of the control.

Lemma 4.1 (controllability of $\mathcal{L}_0$). *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. Let a_3, a_2, a_1, a_0 be four functions of (t,x) with $a_2 = 2\partial_x a_3$ satisfying (3-17). Let $\mathcal{L}_0$ be the linear operator*

$$\mathcal{L}_0 := \partial_t + (1 + a_3)\, \partial_{xxx} + a_2\, \partial_{xx} + a_1\, \partial_x + a_0. \tag{4-1}$$

(i) Existence. *There exist constants δ_0, C such that, if δ in (3-17) is smaller than δ_0, then the following property holds. Given any three functions $g_1(t,x)$, $g_2(x)$, $g_3(x)$, with $g_1 \in C([0,T], L^2_x)$ and $g_2, g_3 \in L^2_x$, there exists a function $\varphi \in C([0,T], L^2_x)$ such that the solution h of the Cauchy problem*

$$\mathcal{L}_0 h = g_1 + \chi_\omega \varphi, \qquad h(0) = g_2 \tag{4-2}$$

satisfies $h(T) = g_3$. (Note that the Cauchy problem (4-2) is globally well-posed by Lemma A.7). Moreover

$$\|\varphi\|_{T,0} \leq C(\|g_1\|_{T,0} + \|g_2\|_0 + \|g_3\|_0). \tag{4-3}$$

(ii) Uniqueness. *Let $\mathcal{L}_0^*$ be the linear operator*

$$\mathcal{L}_0^* \psi := -\partial_t \psi - \partial_{xxx}\{(1 + a_3)\psi\} + \partial_{xx}(a_2 \psi) - \partial_x(a_1 \psi) + a_0 \psi. \tag{4-4}$$

The control φ in (i) *is the unique solution of the equation $\mathcal{L}_0^* \varphi = 0$ such that the solution h of the Cauchy problem (4-2) satisfies $h(T) = g_3$.*

The proof of Lemma 4.1 is given below, and it is based on the following classical lemma. In this section we use the standard notation $\langle u, v \rangle := \int_{\mathbb{T}} uv \, dx$.

Lemma 4.2. *Let a_3, a_2, a_1, a_0 be functions satisfying (3-17) and $a_2 = 2\partial_x a_3$. Let $\mathcal{L}_0^*$ be the operator defined in (4-4). For every (g_1, g_2, g_3), with $g_1 \in C([0,T], L^2_x)$ and $g_2, g_3 \in L^2_x$, there exists a unique $\varphi_1 \in L^2_x$ such that for all $\psi_1 \in L^2_x$, the solutions $\varphi, \psi \in C([0,T], L^2_x)$ of the Cauchy problems*

$$\begin{cases} \mathcal{L}_0^* \varphi = 0, \\ \varphi(T) = \varphi_1 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}_0^* \psi = 0, \\ \psi(T) = \psi_1 \end{cases} \tag{4-5}$$

satisfy

$$\int_0^T \langle g_1 + \chi_\omega \varphi, \psi \rangle \, dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle = 0 \tag{4-6}$$

(*note that the global well-posedness of the Cauchy problems (4-5) follows by Lemma A.7 and Remark A.8*). *Moreover φ satisfies (4-3).*

Proof. Given $\varphi_1, \psi_1 \in L^2_x$, let φ, ψ be the solutions of the Cauchy problems (4-5), and define

$$B(\varphi_1, \psi_1) := \int_0^T \langle \chi_\omega \varphi, \psi \rangle \, dt, \quad \Lambda(\psi_1) := \langle g_3, \psi(T) \rangle - \langle g_2, \psi(0) \rangle - \int_0^T \langle g_1, \psi \rangle \, dt. \tag{4-7}$$

The bilinear map $B : L^2_x \times L^2_x \to \mathbb{R}$ is well-defined and continuous because $|\chi_\omega(x)| \le 1$ and, by Lemma A.7 and Remark A.8, $\|\varphi\|_{T,0} \le C\|\varphi_1\|_0$, and similarly for ψ. Moreover B is coercive by Lemma 3.8 and Remark 2.2. The linear functional Λ is bounded, with

$$|\Lambda(\psi_1)| \le C\|g\|_{T,0}\|\psi_1\|_0 \quad \forall \psi_1 \in L^2_x, \qquad \|g\|_{T,0} := \|g_1\|_{T,0} + \|g_2\|_0 + \|g_3\|_0.$$

Thus, by Riesz representation theorem (or Lax–Milgram), there exists a unique $\varphi_1 \in L^2_x$ such that

$$B(\varphi_1, \psi_1) = \Lambda(\psi_1) \quad \forall \psi_1 \in L^2_x. \tag{4-8}$$

Moreover $\|\varphi_1\|_0 \le C\|\Lambda\|_{\mathcal{L}(L^2_x, \mathbb{R})} \le C'\|g\|_{T,0}$. Since $\|\varphi\|_{T,0} \le C\|\varphi_1\|_0$, we get (4-3). □

Proof of Lemma 4.1. (i) Let $\varphi_1 \in L^2_x$ be the unique solution of (4-8) given by Lemma 4.2. Consider any $\psi_1 \in L^2_x$, and let $\varphi, \psi \in C([0,T], L^2_x)$ be the unique solutions of the Cauchy problems (4-5). Recalling (4-6), (4-2) and integrating by parts, we have

$$\begin{aligned} 0 &= \int_0^T \langle g_1 + \chi_\omega \varphi, \psi \rangle \, dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle \\ &= \int_0^T \langle \mathcal{L}_0 h, \psi \rangle \, dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle \\ &= \langle h(T), \psi(T) \rangle - \langle h(0), \psi(0) \rangle + \int_0^T \langle h, \mathcal{L}_0^* \psi \rangle \, dt + \langle g_2, \psi(0) \rangle - \langle g_3, \psi(T) \rangle \\ &= \langle h(T), \psi(T) \rangle - \langle g_3, \psi(T) \rangle \\ &= \langle h(T) - g_3, \psi_1 \rangle, \end{aligned}$$

from which it follows that $h(T) = g_3$.

(ii) Assume that $\tilde{\varphi} \in C([0, T], L^2_x)$ satisfies $\mathcal{L}_0^* \tilde{\varphi} = 0$ and it has the property that the solution h of the Cauchy problem (4-2) satisfies $h(T) = g_3$. Let $\tilde{\varphi}_1 := \tilde{\varphi}(T)$. The same integration by parts as above shows that $B(\tilde{\varphi}_1, \psi_1) = \Lambda(\psi_1)$ for all $\psi_1 \in L^2_x$. By the uniqueness in Lemma 4.2, $\tilde{\varphi}_1 = \varphi_1$. □

Lemma 4.3 (higher regularity). *Let T, ω, a_3, a_2, a_1, a_0, $\mathcal{L}_0$, g_1, g_2, g_3 be as in Lemma 4.1. There exist two positive constants δ_*, σ with the following property. Let $s > 0$ be given. Assume that $a_0, a_1, a_2, a_3 \in C^2([0, T], H^{s+\sigma}_x)$. Let*

$$\delta(\mu) := \sum_{k=0,1,2,\ i=0,1,2,3} \|\partial_t^k a_i\|_{T,\mu+\sigma}, \qquad \mu \in [0, s].$$

Let $\|g\|_{T,s} := \|g_1\|_{T,s} + \|g_2\|_s + \|g_3\|_s < \infty$. If $\delta(0) \le \delta_*$, then the control φ constructed in Lemma 4.1 and the solution h of (4-2) satisfy

$$\|\varphi, h\|_{T,s} \le C_s(\|g\|_{T,s} + \delta(s)\|g\|_{T,0}) \tag{4-9}$$

for some positive C_s depending on s, T, ω. Moreover, if $g_1 \in C^1([0, T], H^s_x)$, then

$$\|\partial_t\varphi, \partial_t h\|_{T,s+3} + \|\partial_{tt}\varphi, \partial_{tt}h\|_{T,s} \le C_s\big\{\|g\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \delta(s)\|g\|_{T,6}\big\}. \tag{4-10}$$

Proof. Let $g_1 \in C([0, T], H^s_x)$ and $g_2, g_3 \in H^s_x$. Let $\varphi, h \in C([0, T], L^2_x)$ be the solution of the control problem constructed in Lemma 4.1, namely

$$\mathcal{L}_0^*\varphi = 0, \quad \mathcal{L}_0 h = \chi_\omega\varphi + g_1, \quad h(0) = g_2, \quad h(T) = g_3. \tag{4-11}$$

To prove that $h, \varphi \in C([0, T], H^s_x)$, it is convenient to use the transformations of Section 2, to prove higher regularity for the solution $\tilde{h}$, $\tilde{\varphi}$ of the transformed control problem, and then to go back to h, φ proving their higher regularity. Recall that

$$\mathcal{L}_0 = \mathcal{A}\mathcal{B}\rho\mathcal{M}\mathcal{T}\mathcal{S}\mathcal{L}_5\mathcal{S}^{-1}\mathcal{T}^{-1}\mathcal{M}^{-1}\mathcal{B}^{-1}\mathcal{A}^{-1}, \tag{4-12}$$

where $\mathcal{L}_5 = \partial_t + m\,\partial_{xxx} + \mathcal{R}$ and $\mathcal{A}$, $\mathcal{B}$, ρ, $\mathcal{M}$, $\mathcal{T}$, $\mathcal{S}$ are defined in Section 2. In particular,

- $\mathcal{A}$ is the change of the space variable $(\mathcal{A}h)(t, x) = h(t, x + \beta(t, x))$ (see (2-9)), where β is defined in (2-18), (2-16), (2-17);
- $\mathcal{B}$ is the reparametrization of time $(\mathcal{B}h)(t, x) = h(\psi(t), x)$ (see (2-25)), where ψ is defined in (2-30);
- $\rho(t)$ is the function defined in (2-26);
- $\mathcal{M}$ is the multiplication operator $(\mathcal{M}h)(t, x) = q(t, x)h(t, x)$ (see (2-37)), where q is defined in (2-41);
- $\mathcal{T}$ is the translation of the space variable $(\mathcal{T}h)(t, x) = h(t, x + p(t))$ (see (2-46)), where p is defined in (2-49);
- $\mathcal{S}$ is the pseudodifferential operator $(\mathcal{S}h)(t, x) = h(t, x) + \gamma(t, x)\partial_x^{-1}h(t, x)$ (see (2-53)), where γ is defined in (2-56) and $\partial_x^{-1}h$ is the primitive of h with zero average in x (defined in Fourier);
- $\mathcal{R}$ is the bounded operator defined in (2-57).

Let

$$\mathcal{L}_5^* := -\partial_t - m\,\partial_{xxx} + \mathcal{R}^T, \tag{4-13}$$

where $\mathcal{R}^T$ is the L^2_x-adjoint of $\mathcal{R}$. Let

$$\begin{aligned} \tilde h &:= (\mathcal{A}\mathcal{B}\mathcal{M}\mathcal{T}\mathcal{S})^{-1} h, & \tilde g_1 &:= (\mathcal{A}\mathcal{B}\rho\mathcal{M}\mathcal{T}\mathcal{S})^{-1} g_1, \\ \tilde g_2 &:= (\mathcal{A}\mathcal{B}\mathcal{M}\mathcal{T}\mathcal{S})^{-1}|_{t=0}\, g_2, & \tilde g_3 &:= (\mathcal{A}\mathcal{B}\mathcal{M}\mathcal{T}\mathcal{S})^{-1}|_{t=T}\, g_3, \\ \tilde\varphi &:= \mathcal{S}^T\mathcal{T}^T\mathcal{M}^T\mathcal{B}^{-1}\mathcal{A}^T\varphi, & K\tilde\varphi &:= (\mathcal{A}\mathcal{B}\rho\mathcal{M}\mathcal{T}\mathcal{S})^{-1}(\chi_\omega(\mathcal{S}^T\mathcal{T}^T\mathcal{M}^T\mathcal{B}^{-1}\mathcal{A}^T)^{-1}\tilde\varphi). \end{aligned} \tag{4-14}$$

Note that, except for $\mathcal{S}^{-1}$, $\mathcal{S}^{-T}$, the operator K is a multiplication operator; namely

$$K\tilde\varphi = \mathcal{S}^{-1}(\zeta\mathcal{S}^{-T}\tilde\varphi), \quad \text{where } \zeta(t,x) := \rho^{-1}\mathcal{T}^{-1}\mathcal{M}^{-2}\mathcal{B}^{-1}\mathcal{A}^{-1}[(1+\beta_x)\chi_\omega]. \tag{4-15}$$

Since $h, \varphi \in C([0,T], L^2_x)$, and $g_1 \in C([0,T], H^s_x)$ and $g_2, g_3 \in H^s_x$, by (4-14) and the estimates for $\mathcal{A}, \mathcal{B}, \rho, \mathcal{M}, \mathcal{T}, \mathcal{S}$ in Section 2, one has

$$\tilde h, \tilde\varphi, K\tilde\varphi \in C([0,T], L^2_x), \quad \tilde g_1 \in C([0,T], H^s_x), \quad \tilde g_2, \tilde g_3 \in H^s_x.$$

Since h, φ satisfy (4-11), one proves that $\tilde h, \tilde\varphi$ satisfy

$$\mathcal{L}_5^*\tilde\varphi = 0, \quad \mathcal{L}_5\tilde h = K\tilde\varphi + \tilde g_1, \quad \tilde h(0) = \tilde g_2, \quad \tilde h(T) = \tilde g_3. \tag{4-16}$$

The last three equations in (4-16) are straightforward. To prove that $\mathcal{L}_5^*\tilde\varphi = 0$, we start from the equality

$$\langle\varphi(T), v(T)\rangle - \langle\varphi(0), v(0)\rangle = \int_0^T \langle\varphi, \mathcal{L}_0 v\rangle\, dt \quad \forall v \in C^\infty([0,T]\times\mathbb{T})$$

(which is a weak form of $\mathcal{L}_0^*\varphi = 0$), we recall (4-12), and apply all the changes of variables $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{T}, \mathcal{S}$ in the integral. Thus $\tilde h, \tilde\varphi$ solve this control problem:

$$\begin{gathered} \text{Given } \tilde g_1, \tilde g_2, \tilde g_3, \text{ find } \tilde\varphi \text{ such that the solution } \tilde h \text{ of the Cauchy problem} \\ \mathcal{L}_5\tilde h = K\tilde\varphi + \tilde g_1,\ \tilde h(0) = \tilde g_2 \text{ satisfies } \tilde h(T) = \tilde g_3, \text{ and moreover } \tilde\varphi \text{ solves } \mathcal{L}_5^*\tilde\varphi = 0. \end{gathered} \tag{4-17}$$

The function $\tilde\varphi$ is the unique solution of (4-17). To prove it, assume that $\tilde\varphi_{\rm bis} \in C([0,T], L^2_x)$ solves (4-17), and let $\tilde h_{\rm bis}$ be the solution of the corresponding Cauchy problem $\mathcal{L}_5\tilde h_{\rm bis} = K\tilde\varphi_{\rm bis} + \tilde g_1$, $\tilde h_{\rm bis}(0) = \tilde g_2$. Define

$$h_{\rm bis} := \mathcal{A}\mathcal{B}\mathcal{M}\mathcal{T}\mathcal{S}\tilde h_{\rm bis}, \quad \varphi_{\rm bis} := \mathcal{A}^{-T}\mathcal{B}\mathcal{M}^{-T}\mathcal{T}^{-T}\mathcal{S}^{-T}\tilde\varphi_{\rm bis}.$$

Then $h_{\rm bis}, \varphi_{\rm bis}$ solve (4-11). By the uniqueness in Lemma 4.1(ii) it follows that $\varphi_{\rm bis} = \varphi$, $h_{\rm bis} = h$. Therefore $\tilde\varphi_{\rm bis} = \tilde\varphi$ and $\tilde h_{\rm bis} = \tilde h$.

Now we prove that $\tilde h, \tilde\varphi \in C([0,T], H^s_x)$. We follow an argument used by Dehman and Lebeau [2009, Lemma 4.2], Laurent [2010, Lemma 3.1], and Alazard, Baldi, and Han-Kwan [2015, Proposition 8.1]. First, we prove the thesis for $\tilde g_1 = 0$, $\tilde g_3 = 0$. Consider the map

$$S\colon L^2_x \to L^2_x, \quad S\tilde\varphi_1 = \tilde h(0), \tag{4-18}$$

obtained by the composition $\tilde\varphi_1 \mapsto \tilde\varphi \mapsto \tilde h \mapsto \tilde h(0)$, where $\tilde\varphi$, $\tilde h$ are the solutions of the Cauchy problems

$$\begin{cases} \mathcal{L}_5^* \tilde\varphi = 0, \\ \tilde\varphi(T) = \tilde\varphi_1, \end{cases} \qquad \begin{cases} \mathcal{L}_5 \tilde h = K\tilde\varphi, \\ \tilde h(T) = 0. \end{cases} \tag{4-19}$$

From the existence and uniqueness of $\tilde\varphi_1 \in L^2_x$ such that $\tilde\varphi$ solves (4-17), it follows that S is an isomorphism of L^2_x. The initial datum $\tilde g_2$ is given, so we fix $\tilde\varphi_1 \in L^2_x$ such that $S\tilde\varphi_1 = \tilde g_2$. We have to estimate $\|\Lambda^s \tilde\varphi_1\|_0 \le C\|S\Lambda^s\tilde\varphi_1\|_0$, where Λ^s is the Fourier multiplier of symbol $\langle\xi\rangle^s := (1+\xi^2)^{s/2}$, $s > 0$. To study the commutator $[S, \Lambda^s]$, we compare $(\Lambda^s\tilde\varphi, \Lambda^s\tilde h)$ with $(\bar\varphi, \bar h)$ defined by

$$\begin{cases} \mathcal{L}_5^* \bar\varphi = 0, \\ \bar\varphi(T) = \Lambda^s\varphi_1, \end{cases} \qquad \begin{cases} \mathcal{L}_5 \bar h = K\bar\varphi, \\ \bar h(T) = 0. \end{cases} \tag{4-20}$$

The difference $\Lambda^s\tilde\varphi - \bar\varphi$ satisfies

$$\begin{cases} \mathcal{L}_5^*(\Lambda^s\tilde\varphi - \bar\varphi) = \mathcal{F}_1, \\ (\Lambda^s\tilde\varphi - \bar\varphi)(T) = 0, \end{cases} \qquad \text{where } \mathcal{F}_1 := [\mathcal{L}_5^*, \Lambda^s]\tilde\varphi = [\mathcal{R}^T, \Lambda^s]\tilde\varphi. \tag{4-21}$$

From Lemma A.2 and Remark A.8, $\|\Lambda^s\tilde\varphi - \bar\varphi\|_{T,0} \le C\|\mathcal{F}_1\|_{T,0}$. We recall the classical estimate for the commutator of Λ^s and any multiplication operator $h \mapsto ah$:

$$\|[\Lambda^s, a]h\|_0 \le C_s\big(\|a\|_2\|h\|_{s-1} + \|a\|_{s+1}\|h\|_0\big). \tag{4-22}$$

By (4-22) and formulas (2-53), (2-56), (2-57), the commutator $\mathcal{F}_1 = [\mathcal{R}^T, \Lambda^s]\tilde\varphi$ satisfies

$$\begin{aligned} \|\mathcal{F}_1\|_{T,0} &\le C_s\big(\|a_{14}, a_{17}, a_{18}\|_{T,\sigma}\|\tilde\varphi\|_{T,s-1} + \|a_{14}, a_{17}, a_{18}\|_{T,s+\sigma}\|\tilde\varphi\|_{T,0}\big) \\ &\le C_s\big(\delta(0)\|\tilde\varphi\|_{T,s-1} + \delta(s)\|\tilde\varphi\|_{T,0}\big). \end{aligned} \tag{4-23}$$

The difference $\Lambda^s\tilde h - \bar h$ satisfies

$$\begin{cases} \mathcal{L}_5(\Lambda^s\tilde h - \bar h) = K(\Lambda^s\tilde\varphi - \bar\varphi) + \mathcal{F}_2, \\ (\Lambda^s\tilde h - \bar h)(T) = 0, \end{cases} \qquad \text{where } \mathcal{F}_2 := [\mathcal{R}^T, \Lambda^s]\tilde h + [\Lambda^s, K]\tilde\varphi. \tag{4-24}$$

We have $\|K(\Lambda^s\tilde\varphi - \bar\varphi)\|_{T,0} \le C\|\Lambda^s\tilde\varphi - \bar\varphi\|_{T,0} \le C\|\mathcal{F}_1\|_{T,0}$, and therefore, by Lemma A.2,

$$\|\Lambda^s\tilde h - \bar h\|_{T,0} \le C\big(\|\mathcal{F}_1\|_{T,0} + \|\mathcal{F}_2\|_{T,0}\big). \tag{4-25}$$

Using (4-22) and (4-15), we get

$$\|\mathcal{F}_2\|_{T,0} \le C_s\big(\|\tilde h, \tilde\varphi\|_{T,s-1} + \delta(s)\|\tilde h, \tilde\varphi\|_{T,0}\big). \tag{4-26}$$

By (4-23), (4-25) and (4-26) we deduce that

$$\|\Lambda^s\tilde h - \bar h\|_{T,0} \le C_s\big(\|\tilde h, \tilde\varphi\|_{T,s-1} + \delta(s)\|\tilde h, \tilde\varphi\|_{T,0}\big).$$

By (4-19), Lemma A.2 and Remark A.8,

$$\|\tilde h, \tilde\varphi\|_{T,\mu} \le C_\mu\big(\|\tilde\varphi\|_{T,\mu} + \delta(\mu)\|\tilde\varphi\|_{T,0}\big) \le C_\mu\big(\|\tilde\varphi_1\|_\mu + \delta(\mu)\|\tilde\varphi_1\|_0\big), \quad \mu \ge 0. \tag{4-27}$$

Therefore

$$\|(\Lambda^s\tilde h - \bar h)(0)\|_0 \le \|\Lambda^s\tilde h - \bar h\|_{T,0} \le C_s\big(\|\tilde\varphi_1\|_{s-1} + \delta(s)\|\tilde\varphi_1\|_0\big). \tag{4-28}$$

Since $S\tilde{\varphi}_1 = \tilde{h}(0) = \tilde{g}_2$, we have $\Lambda^s \tilde{h}(0) = \Lambda^s g_2$. Moreover, by the definition of S in (4-18)–(4-19), $\bar{h}(0) = S\Lambda^s \tilde{\varphi}_1$. Thus

$$\|S\Lambda^s \tilde{\varphi}_1\|_0 \le \|(\Lambda^s \tilde{h} - \bar{h})(0)\|_0 + \|\Lambda^s \tilde{h}(0)\|_0 \le C_s\big(\|\tilde{\varphi}_1\|_{s-1} + \delta(s)\|\tilde{\varphi}_1\|_0\big) + \|\tilde{g}_2\|_s. \tag{4-29}$$

Since S is an isomorphism of L^2_x, we have $\|\Lambda^s \tilde{\varphi}_1\|_0 \le C\|S\Lambda^s \tilde{\varphi}_1\|_0$, whence

$$\|\tilde{\varphi}_1\|_s \le C_s\big(\|\tilde{g}_2\|_s + \|\tilde{\varphi}_1\|_{s-1} + \delta(s)\|\tilde{\varphi}_1\|_0\big). \tag{4-30}$$

Since $\|\tilde{\varphi}_1\|_0 \le C\|\tilde{g}_2\|_0$, by induction we deduce that

$$\|\tilde{\varphi}_1\|_s \le C_s\big(\|\tilde{g}_2\|_s + \delta(s)\|\tilde{g}_2\|_0\big). \tag{4-31}$$

By (4-27), we obtain

$$\|\tilde{h}, \tilde{\varphi}\|_{T,s} \le C_s\big(\|\tilde{g}_2\|_s + \delta(s)\|\tilde{g}_2\|_0\big), \tag{4-32}$$

which is the thesis in the case $\tilde{g}_1 = 0,\ \tilde{g}_3 = 0$.

Now we prove the higher regularity of $\tilde{h}, \tilde{\varphi}$ removing the assumption $\tilde{g}_1 = 0,\ \tilde{g}_3 = 0$. Let $\tilde{g}_1 \in C([0, T], H^s_x)$ and $\tilde{g}_2, \tilde{g}_3 \in H^s_x$, and let $\tilde{h}, \tilde{\varphi}$ be the solution of (4-17). Let w be the solution of the problem

$$\mathcal{L}_5 w = \tilde{g}_1, \quad w(T) = \tilde{g}_3.$$

By Lemma A.2, $w \in C([0, T], H^s_x)$, with

$$\|w\|_{T,s} \le C_s\big\{\|\tilde{g}_1\|_{T,s} + \|\tilde{g}_3\|_s + \delta(s)(\|\tilde{g}_1\|_{T,0} + \|\tilde{g}_3\|_0)\big\}. \tag{4-33}$$

Let $v := \tilde{h} - w$. Then

$$\mathcal{L}_5 v = K\tilde{\varphi}, \quad v(0) = \tilde{g}_2 - w(0), \quad v(T) = 0.$$

This means that $v, \tilde{\varphi}$ solve (4-17) where $(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3)$ are replaced by $(0, \tilde{g}_2 - w(0), 0)$. Hence (4-32) applies to $v, \tilde{\varphi}$, and we get

$$\|v, \tilde{\varphi}\|_{T,s} \le C_s\big(\|\tilde{g}_2 - w(0)\|_s + \delta(s)\|\tilde{g}_2 - w(0)\|_0\big). \tag{4-34}$$

We estimate $\|\tilde{g}_2 - w(0)\|_s \le \|\tilde{g}_2\|_s + \|w\|_{T,s}$; we use (4-33) and $\|\tilde{h}\|_{T,s} \le \|v\|_{T,s} + \|w\|_{T,s}$ to conclude

$$\|\tilde{h}, \tilde{\varphi}\|_{T,s} \le C_s\big\{\|\tilde{g}\|_{T,s} + \delta(s)\|\tilde{g}\|_{T,0}\big\}, \tag{4-35}$$

where we have denoted, in short, $\|\tilde{g}\|_{T,s} := \|\tilde{g}_1\|_{T,s} + \|\tilde{g}_2\|_s + \|\tilde{g}_3\|_s$. This proves the higher regularity for the transformed control problem (4-17). By the definitions in (4-14),

$$\|\varphi\|_{T,s} \le C_s\big(\|\tilde{\varphi}\|_{T,s} + \delta(s)\|\tilde{\varphi}\|_{T,0}\big), \quad \|h\|_{T,s} \le C_s\big(\|\tilde{h}\|_{T,s} + \delta(s)\|\tilde{h}\|_{T,0}\big),$$
$$\|\tilde{g}\|_{T,s} \le C_s\big(\|g\|_{T,s} + \delta(s)\|g\|_{T,0}\big),$$

and the proof of (4-9) is complete.

The bound (4-10) is deduced in a classical way from the fact that h, φ solve the equations $\mathcal{L}_0^* \varphi = 0$, $\mathcal{L}_0 h = \chi_\omega \varphi + g_1$. $\square$

Remark 4.4. Another possible way to prove higher regularity for h, φ is to apply the argument of [Dehman and Lebeau 2009; Laurent 2010; Alazard, Baldi, and Han-Kwan 2015] directly to the control problem for $\mathcal{L}_0$, instead of passing to the transformed problem (4-17), applying that argument, and then going back to h, φ. Such a more direct method adapted to the present case would require the construction of two operators A_s, B_s such that

(1) $C_1\|v\|_s \le \|A_s v\|_0 \le C_2\|v\|_s$ (equivalent norm in H^s),

(2) the commutator $[\mathcal{L}_0, A_s]$ is an operator of order $s-1$,

(3) the difference $B_s\mathcal{L}_0^* - \mathcal{L}_0^* A_s$ is also of order $s-1$.

The construction of such A_s, B_s is possible, but probably the proof given above is more straightforward, and it fully exploits the advantages of conjugating $\mathcal{L}_0$ to $\mathcal{L}_5$ (Section 2). The main point is that the commutator $[\mathcal{L}_5, \Lambda^s]$ is of order $s-1$ (because $\mathcal{L}_5$ has constant coefficients up to a *bounded* remainder), while $[\mathcal{L}_0, \Lambda^s]$ is of order $s+2$ (because $\mathcal{L}_0$, which was obtained by linearizing a *quasilinear* PDE, has variable coefficients also at the highest order), so that a modified version A_s of Λ^s is needed. □

In view of the application of the Nash–Moser theorem in Section 5, we define the spaces

$$E_s := X_s \times X_s, \quad X_s := C([0,T], H_x^{s+6}) \cap C^1([0,T], H_x^{s+3}) \cap C^2([0,T], H_x^s), \tag{4-36}$$

$$F_s := \{g = (g_1, g_2, g_3) : g_1 \in C([0,T], H_x^{s+6}) \cap C^1([0,T], H_x^s),\, g_2, g_3 \in H_x^{s+6}\} \tag{4-37}$$

equipped with the norms

$$\|u, f\|_{E_s} := \|u\|_{X_s} + \|f\|_{X_s}, \quad \|u\|_{X_s} := \|u\|_{T,s+6} + \|\partial_t u\|_{T,s+3} + \|\partial_{tt} u\|_{T,s}, \tag{4-38}$$

$$\|g\|_{F_s} := \|g_1\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \|g_2, g_3\|_{s+6}. \tag{4-39}$$

With this notation, we have proved the following linear inversion result.

Theorem 4.5 (right inverse of the linearized operator). *Let $T > 0$ and $\omega \subset \mathbb{T}$ be an open set. There exist two universal constants $\tau, \sigma \ge 3$ and a positive constant δ_* depending on T, ω with the following property.*

Let $s \in [0, r-\tau]$, where r is the regularity of the nonlinearity $\mathcal{N}$ (see Lemma 2.1). *Let $g = (g_1, g_2, g_3) \in F_s$ and let $(u, f) \in E_{s+\sigma}$, with $\|u\|_{X_\sigma} \le \delta_*$. Then there exists $(h, \varphi) := \Psi(u, f)[g] \in E_s$ such that*

$$P'(u)[h] - \chi_\omega \varphi = g_1, \quad h(0) = g_2, \quad h(T) = g_3, \tag{4-40}$$

and

$$\|h, \varphi\|_{E_s} \le C_s\big(\|g\|_{F_s} + \|u\|_{X_{s+\sigma}}\|g\|_{F_0}\big), \tag{4-41}$$

where C_s depends on s, T, ω.

5. Proofs

In this section we prove Theorems 1.1 and 1.4.

5A. ***Proof of Theorem 1.1.*** The spaces defined in (4-36)–(4-39), with $s \ge 0$, form scales of Banach spaces. We define smoothing operators S_θ in the following way. We fix a C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}$ with $0 \le \varphi \le 1$,

$$\varphi(\xi) = 1 \quad \forall |\xi| \le 1 \qquad \text{and} \qquad \varphi(\xi) = 0 \quad \forall |\xi| \ge 2.$$

For any real number $\theta \geq 1$, let S_θ be the Fourier multiplier with symbol $\varphi(\xi/\theta)$, namely

$$S_\theta u(x) := \sum_{k\in\mathbb{Z}} \hat{u}_k\, \varphi(k/\theta)\, e^{ikx}, \quad \text{where } u(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx} \in L^2(\mathbb{T}). \tag{5-1}$$

The definition of S_θ extends to functions $u(t,x) = \sum_{k\in\mathbb{Z}} \hat{u}_k(t)\, e^{ikx}$ depending on time in the obvious way. Since S_θ and ∂_t commute, the smoothing operators S_θ are defined on the spaces E_s, F_s defined in (4-36)–(4-37) by setting $S_\theta(u, f) := (S_\theta u, S_\theta f)$ and similarly on $g = (g_1, g_2, g_3)$. One easily verifies that S_θ satisfies (B-1)–(B-4) on E_s and F_s. We define the spaces E'_a with norm $\|\cdot\|'_a$ and F'_b with $\|\cdot\|'_b$ as constructed in Appendix B.

We observe that $\Phi(u, f) := (P(u) - \chi_\omega f,\ u(0),\ u(T))$ defined in (1-13)–(1-14) belongs to F_s when $(u, f) \in E_{s+3}$, $s \in [0, r-6]$, with $\|u\|_{T,4} \leq 1$. Its second derivative is

$$\Phi''(u, f)[(h_1, \varphi_1), (h_2, \varphi_2)] = \begin{pmatrix} P''(u)[h_1, h_2] \\ 0 \\ 0 \end{pmatrix}.$$

For u in a fixed ball $\|u\|_{X_1} \leq \delta_0$, with δ_0 small enough, we estimate

$$\|P''(u)[h, w]\|_{F_s} \leq C_s\big(\|h\|_{X_1}\|w\|_{X_{s+3}} + \|h\|_{X_{s+3}}\|w\|_{X_1} + \|u\|_{X_{s+3}}\|h\|_{X_1}\|w\|_{X_1}\big) \tag{5-2}$$

for all $s \in [0, r-6]$. We fix $V = \{(u, f) \in E_3 : \|(u, f)\|_{E_3} \leq \delta_0\}$, $\delta_1 = \delta_*$,

$$a_0 = 1, \quad \mu = 3, \quad a_1 = \sigma, \quad \alpha = \beta = 2\sigma, \quad a_2 \in (3\sigma, r - \tau], \tag{5-3}$$

where δ_*, σ, τ are given by Theorem 4.5, and r is the regularity of $\mathcal{N}$ in Theorem 1.1. The right inverse Ψ in Theorem 4.5 satisfies the assumptions of Theorem B.1. Thus by Theorem B.1 we obtain that, if $g = (0, u_{\text{in}}, u_{\text{end}}) \in F'_\beta$ with $\|g\|'_{F_\beta} \leq \delta$, then there exists a solution $(u, f) \in E'_\alpha$ of the equation $\Phi(u, f) = g$, with $\|u, f\|'_{E_\alpha} \leq C\|g\|'_{F_\beta}$ (and recall that $\beta = \alpha$). We fix $s_1 := \alpha + 6$, and (1-11) is proved. In fact, we have proved slightly more than (1-11), because $\|g\|'_{F_\beta} \leq C\|g\|_{F_\beta}$ and $\|u, f\|_{E_a} \leq C_a\|u, f\|'_{E_\alpha}$ for all $a < \alpha$.

We have found a solution (u, f) of the control problem (1-9)–(1-10). Now we prove that u is the unique solution of the Cauchy problem (1-9), with that given f. Let u, v be two solutions of (1-9) in E_{s-6} for all $s < s_1$. We calculate

$$P(u) - P(v) = \int_0^1 P'(v + \lambda(u-v))[u-v]\, d\lambda =: \widetilde{\mathcal{L}}_0[u-v],$$

where

$$\widetilde{\mathcal{L}}_0 := \partial_t + (1 + \tilde{a}_3(t,x))\partial_{xxx} + \tilde{a}_2(t,x)\partial_{xx} + \tilde{a}_1(t,x)\partial_x + \tilde{a}_0(t,x),$$

$$\tilde{a}_i(t,x) := \int_0^1 a_i(v + \lambda(u-v))(t,x)\, d\lambda, \quad i = 0, 1, 2, 3,$$

and $a_i(u)$ is defined in (2-2). Note that $\tilde{a}_2 = 2\partial_x \tilde{a}_3$ because $a_2(v + \lambda(u-v)) = 2\partial_x a_3(v + \lambda(u-v))$ for all $\lambda \in [0,1]$. The difference $u - v$ satisfies $\widetilde{\mathcal{L}}_0(u-v) = 0$, $(u-v)(0) = 0$. Hence, by Lemma A.7, $u - v = 0$. The proof of Theorem 1.1 is complete. □

5B. ***Proof of Theorem 1.4.*** We define

$$E_s := C([0,T], H_x^{s+6}) \cap C^1([0,T], H_x^{s+3}) \cap C^2([0,T], H_x^s), \tag{5-4}$$

$$F_s := \{ g = (g_1, g_2) : g_1 \in C([0,T], H_x^{s+6}) \cap C^1([0,T], H_x^s),\ g_2 \in H_x^{s+6} \} \tag{5-5}$$

equipped with norms

$$\|u\|_{E_s} := \|u\|_{T,s+6} + \|\partial_t u\|_{T,s+3} + \|\partial_{tt} u\|_{T,s}, \tag{5-6}$$

$$\|g\|_{F_s} := \|g_1\|_{T,s+6} + \|\partial_t g_1\|_{T,s} + \|g_2\|_{s+6}, \tag{5-7}$$

and $\Phi(u) := (P(u), u(0))$. Given $g := (f, u_{\mathrm{in}}) \in F_{s_0}$, the Cauchy problem (1-18) becomes $\Phi(u) = g$. We fix $V, \delta_1, a_0, \mu, a_1, \alpha, \beta, a_2$ as in (5-3), where the constants σ, δ_* are now given in Lemma A.7 and $\tau = \sigma + 9$ by Lemma 2.1 combined with Lemma A.7 and the definition of the spaces E_s, F_s. Assumption (B-13) about the right inverse of the linearized operator is satisfied by Lemmas A.7 and 2.1. We fix $s_0 := \alpha + 6$. Then Theorem B.1 applies, giving the existence part of Theorem 1.4. The uniqueness of the solution is proved exactly as in the proof of Theorem 1.1. □

Appendix A: Well-posedness of linear operators

Lemma A.1. *Let $T > 0$, $m \in \mathbb{R}$, $s \in \mathbb{R}$, $f \in C([0,T], H_x^s)$, with $f(t,x) = \sum_{n\in\mathbb{Z}} f_n(t) e^{inx}$. Let A be the linear operator defined by $Af := v$, where v is the solution of*

$$\begin{cases} \partial_t v + m\partial_{xxx} v = f & \forall (t,x) \in [0,T] \times \mathbb{T}, \\ v(0,x) = 0. \end{cases} \tag{A-1}$$

Then

$$Af(t,x) = \sum_{n\in\mathbb{Z}} (Af)_n(t) e^{inx}, \qquad (Af)_n(t) = \int_0^t e^{imn^3(\tau - t)} f_n(\tau)\, d\tau, \tag{A-2}$$

Af belongs to $C([0,T], H_x^s) \cap C^1([0,T], H_x^{s-3})$, and

$$\|Af\|_{T,s} \le T \|f\|_{T,s}. \tag{A-3}$$

Proof. Formula (A-2) simply comes from variation of constants. By Hölder's inequality,

$$|(Af)_n(t)| \le \sqrt{t} \left(\int_0^t |f_n(\tau)|^2 \, d\tau \right)^{1/2} \quad \forall t \in [0,T]$$

and therefore, for each $t \in [0,T]$,

$$\begin{aligned} \|Af(t)\|_{H_x^s}^2 &= \sum_{n\in\mathbb{Z}} |(Af)_n(t)|^2 \langle n\rangle^{2s} \le \sum_{n\in\mathbb{Z}} t \int_0^t |f_n(\tau)|^2 \, d\tau \, \langle n\rangle^{2s} \\ &\le t \int_0^t \sum_{n\in\mathbb{Z}} |f_n(\tau)|^2 \langle n\rangle^{2s} \, d\tau = t \int_0^t \|f(\tau)\|_{H_x^s}^2 \, d\tau \le t^2 \|f\|_{C([0,t],H_x^s)}^2. \end{aligned}$$

Taking the sup over $t \in [0,T]$ we get the thesis. □

We remark that for $s \le 3$ the operator A is well-defined in the sense of distributions. We also recall that $\mathcal{L}(H^s_x)$ is the space of linear bounded operators of H^s_x into itself, with operator norm $\|L\|_{\mathcal{L}(H^s_x)} := \sup\{\|Lh\|_s : h \in H^s_x,\ \|h\|_s = 1\}$.

Lemma A.2. (i) (LWP). *Let* $T > 0$, $s \in \mathbb{R}$, $\mathcal{R} \in C([0,T], \mathcal{L}(H^s_x))$, *and let*

$$r_s := \|\mathcal{R}\|_{C([0,T],\mathcal{L}(H^s_x))} = \sup_{t\in[0,T]} \|\mathcal{R}(t)\|_{\mathcal{L}(H^s_x)}, \qquad \mathcal{L}_5 := \partial_t + m\partial_{xxx} + \mathcal{R}. \tag{A-4}$$

Let $\alpha \in H^s_x$ *and* $f \in C([0,T], H^s_x)$. *If* $Tr_s \le \frac{1}{2}$, *then the Cauchy problem*

$$\begin{cases} \mathcal{L}_5 u = f, \\ u(0,x) = \alpha(x) \end{cases} \tag{A-5}$$

has a unique solution $u \in C([0,T], H^s_x)$. *The solution* u *satisfies*

$$\|u\|_{T,s} \le (1 + 2Tr_s)\|\alpha\|_s + 2T\|f\|_{T,s} \le 2(\|\alpha\|_s + T\|f\|_{T,s}). \tag{A-6}$$

(ii) (tame LWP). *Let* $T > 0$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ *with* $s \ge s_1$, *and let* $\mathcal{R} \in C([0,T], \mathcal{L}(H^s_x)) \cap C([0,T], \mathcal{L}(H^{s_1}_x))$. *Assume that*

$$\|\mathcal{R}(t)h\|_s \le c_1\|h\|_s + c_s\|h\|_{s_1}, \quad \|\mathcal{R}(t)h\|_{s_1} \le c_1\|h\|_{s_1} \quad \forall h \in H^s_x, \tag{A-7}$$

for all $t \in [0,T]$, *where* c_1, c_s *are positive constants. Let* $\alpha \in H^s_x$. *If*

$$Tc_1 \le \tfrac{1}{2}, \tag{A-8}$$

then the solution $u \in C([0,T], H^{s_1}_x)$ *of the Cauchy problem* (A-5) *given in* (i) *belongs to* $C([0,T], H^s_x)$, *with*

$$\|u\|_{T,s} \le 2T\|f\|_{T,s} + (1 + 2Tc_1)\|\alpha\|_s + 4Tc_s(T\|f\|_{T,s_1} + \|\alpha\|_{s_1}). \tag{A-9}$$

(iii) (GWP). *Let* $T > 0$, $s \in \mathbb{R}$, $\mathcal{R} \in C([0,T], \mathcal{L}(H^s_x))$, *and let* r_s *be defined in* (A-4). *Let* $\alpha \in H^s_x$. *Then the Cauchy problem* (A-5) *has a unique global solution* $u \in C([0,T], H^s_x)$, *with*

$$\|u\|_{T,s} \le 2^{4Tr_s}(\|\alpha\|_s + 4T\|f\|_{T,s}). \tag{A-10}$$

(iv) (tame GWP). *Let* $T > 0$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ *with* $s \ge s_1$, *and let* $\mathcal{R} \in C([0,T], \mathcal{L}(H^s_x)) \cap C([0,T], \mathcal{L}(H^{s_1}_x))$. *Assume that* (A-7) *holds for all* $t \in [0,T]$, *where* c_1, c_s *are positive constants. Let* $\alpha \in H^s_x$. *Then the global solution* $u \in C([0,T], H^s_x)$ *of the Cauchy problem* (A-5) *given in* (iii) *satisfies*

$$\|u\|_{T,s} \le 2^{4Tc_1}\big(\|\alpha\|_s + 4Tc_s\|\alpha\|_{s_1} + 2T\|f\|_{T,s} + 4T^2c_s\|f\|_{T,s_1}\big). \tag{A-11}$$

Proof. (i) Write $u = v + w$, where $v(t,x)$ is the solution of

$$\partial_t v + m\,\partial_{xxx} v = 0, \qquad v(0,x) = \alpha(x). \tag{A-12}$$

Hence u solves (A-5) if and only if $w(t,x)$ solves

$$\partial_t w + m\,\partial_{xxx} w + \mathcal{R}w = -\mathcal{R}v + f, \qquad w(0,x) = 0. \tag{A-13}$$

By Lemma A.1, (A-13) is the fixed point problem

$$w = \Psi(w), \tag{A-14}$$

where $\Psi(w) := A[f - \mathcal{R}(v+w)]$. Let $B_\rho := \{w \in C([0,T], H^s_x) : \|u\|_{T,s} \le \rho\}$, $\rho \ge 0$. Then

$$\|\Psi(w)\|_{T,s} \le T(\|f\|_{T,s} + r_s\|\alpha\|_s + r_s\rho), \qquad \|\Psi(w_1) - \Psi(w_2)\|_{T,s} \le T\, r_s \|w_1 - w_2\|_{T,s} \tag{A-15}$$

for all $w, w_1, w_2 \in B_\rho$. By assumption, $Tr_s \le \frac{1}{2}$. Therefore, for any $\rho \ge 2T(\|f\|_{T,s} + r_s\|\alpha\|_s)$, Ψ is a contraction in B_ρ. In particular, we fix $\rho = \rho_0 := 2T(\|f\|_{T,s} + r_s\|\alpha\|_s)$. Hence there exists a fixed point $w \in B_{\rho_0}$ of Ψ, with $\|w\|_{T,s} \le \rho_0 \le 2T\|f\|_{T,s} + \|\alpha\|_s$. As a consequence, there exists a solution $u \in C([0,T], H^s_x)$ of (A-5) with $\|u\|_{T,s} \le 2(T\|f\|_{T,s} + \|\alpha\|_s)$. By the contraction lemma, the solution u is unique in any ball B_ρ, $\rho \ge \rho_0$, and therefore it is unique in $C([0,T], H^s_x)$.

(ii) By assumption, $Tc_1 \le \frac{1}{2}$, and therefore, by (i), there exists a unique solution $u \in C([0,T], H^{s_1}_x)$. It remains to prove that u satisfies (A-9). By construction, $u = v + w$, where $v \in C([0,T], H^s_x)$ is the solution of (A-12), with $\|v(t)\|_s = \|\alpha\|_s$ for all $t \in [0,T]$, and $w \in C([0,T], H^{s_1}_x)$ solves (A-14). By the iterative scheme of the contraction lemma, w is the limit in $C([0,T], H^{s_1}_x)$ of the sequence (w_n), where $w_0 := 0$, and $w_{n+1} := \Psi(w_n)$ for all $n \in \mathbb{N}$. By (A-7) and (A-3), Ψ maps $C([0,T], H^s_x)$ into itself; therefore $w_n \in C([0,T], H^s_x)$ for all $n \ge 0$. Let $h_n := w_n - w_{n-1}$, $n \ge 1$, so that $w_n = \sum_{k=1}^n h_k$. One has $h_{n+1} = -A\mathcal{R}h_n$ for all $n \ge 1$, and

$$\|h_{n+1}\|_{T,s} \le Tc_1\|h_n\|_{T,s} + Tc_s\|h_n\|_{T,s_1}, \qquad \|h_{n+1}\|_{T,s_1} \le Tc_1\|h_n\|_{T,s_1} \quad \forall n \ge 1.$$

Hence, by induction, for all $n \ge 1$ we have

$$\|h_n\|_{T,s} \le (Tc_1)^{n-1}\|h_1\|_{T,s} + (n-1)(Tc_1)^{n-2}Tc_s\|h_1\|_{T,s_1}, \qquad \|h_n\|_{T,s_1} \le (Tc_1)^{n-1}\|h_1\|_{T,s_1}. \tag{A-16}$$

Also, $\|h_1\|_{T,s} \le T\|f\|_{T,s} + Tc_1\|\alpha\|_s + Tc_s\|\alpha\|_{s_1}$ and $\|h_1\|_{T,s_1} \le T\|f\|_{T,s_1} + Tc_1\|\alpha\|_{s_1}$. Therefore

$$\begin{aligned} \|h_n\|_{T,s} &\le (Tc_1)^{n-1}T\|f\|_{T,s} + (Tc_1)^n\|\alpha\|_s + (n-1)(Tc_1)^{n-2}Tc_sT\|f\|_{T,s_1} \\ &\qquad + n(Tc_1)^{n-1}Tc_s\|\alpha\|_{s_1}, \qquad \text{(A-17)} \\ \|h_n\|_{T,s_1} &\le (Tc_1)^{n-1}T\|f\|_{T,s_1} + (Tc_1)^n\|\alpha\|_{s_1} \qquad \forall n \ge 1. \end{aligned}$$

Since $Tc_1 \le \frac{1}{2}$, the sequence $w_n = \sum_{k=1}^n h_k$ converges in $C([0,T], H^s_x)$ to some limit $\tilde{w} \in C([0,T], H^s_x)$. Since w_n converges to w in $C([0,T], H^{s_1}_x)$, the two limits coincide, and $w \in C([0,T], H^s_x)$. Since $\|w\|_{T,s} \le \sum_{k=1}^\infty \|h_k\|_{T,s}$, we get

$$\|w\|_{T,s} \le 2T(\|f\|_{T,s} + c_1\|\alpha\|_s) + 4Tc_s(T\|f\|_{T,s_1} + \|\alpha\|_{s_1}). \tag{A-18}$$

Since $u = v + w$, we deduce (A-9).

(iii) If $Tr_s \le \frac{1}{2}$, the result is given by (i). Let $Tr_s > \frac{1}{2}$, and fix $N \in \mathbb{N}$ such that $2Tr_s \le N \le 4Tr_s$. Let $T_0 := T/N$, so that $\frac{1}{4} \le T_0 r_s \le \frac{1}{2}$. Divide the interval $[0,T]$ into the union $I_1 \cup \dots \cup I_N$, where $I_n := [(n-1)T_0, nT_0]$. Applying (i) on the time interval $I_1 = [0, T_0]$ gives the solution $u_1 \in C(I_1, H^s_x)$, with $\|u_1\|_{C(I_1, H^s_x)} \le b\|\alpha\|_s + 2T_0\|f\|_{T,s}$, where $b := 1 + 2T_0 r_s$. Now consider the Cauchy problem on I_2

with initial datum $u(T_0) = u_1(T_0)$. Applying (i) on I_2 gives the solution $u_2 \in C(I_2, H^s_x)$, with

$$\|u_2\|_{C(I_2,H^s_x)} \le b\|u_1(T_0)\|_s + 2T_0\|f\|_{T,s} \le b^2\|\alpha\|_s + (1+b)2T_0\|f\|_{T,s}.$$

We iterate the procedure N times. At the last step, we find the solution u_N defined on I_N, with

$$\|u_N\|_{C(I_N,H^s_x)} \le b^N\|\alpha\|_s + (b^N - 1)\frac{1}{b-1}2T_0\|f\|_{T,s}.$$

We define $u(t) := u_n(t)$ for $t \in I_n$, and the thesis follows, using that $b \le 2$.

(iv) If $Tc_1 \le \frac{1}{2}$, the result is given by (ii). Let $Tc_1 > \frac{1}{2}$, and fix $N \in \mathbb{N}$ such that $2Tc_1 \le N \le 4Tc_1$. Let $T_0 := T/N$, so that $\frac{1}{4} \le T_0 c_1 \le \frac{1}{2}$. Split $[0, T] = I_1 \cup \cdots \cup I_N$, where $I_n := [(n-1)T_0, nT_0]$. Perform the same procedure as above. Using (A-9), and $1 + 2T_0c_1 \le 2$, by induction we get

$$\|u_n\|_{C(I_n,H^s_x)} \le 2^n\|\alpha\|_s + (2^n-1)2T_0\|f\|_{T,s} + n2^{n-1}4T_0c_s\|\alpha\|_{s_1} + [2^n(n-1)+1]4T_0c_sT_0\|f\|_{T,s_1},$$
$$\|u_n\|_{C(I_n,H^{s_1}_x)} \le 2^n\|\alpha\|_{s_1} + (2^n-1)2T_0\|f\|_{T,s_1}.$$

This implies (A-11), recalling that $T_0c_1 \le \frac{1}{2}$ and also $NT_0 = T,\ N \ge 1$. □

Lemma A.3. *There exist universal positive constants σ, δ_* with the following properties. Let $s \ge 0$, let $m \ge \frac{1}{2}$, and let $a_{14}(t,x)$, $a_{15}(t,x)$ be two functions with $a_{14}, \partial_t a_{14}, a_{15} \in C([0,T], H^{s+\sigma}_x)$ and $\int_{\mathbb{T}} a_{14}(t,x)\,dx = 0$, and let $\mathcal{L}_4 := \partial_t + m\,\partial_{xxx} + a_{14}\,\partial_x + a_{15}$. Let*

$$\delta(\mu) := \|a_{14}, \partial_t a_{14}, a_{15}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s].$$

Assume $\delta(0) \le \delta_*$. Let $f \in C([0,T], H^s_x)$ and $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_4 u = f, \quad u(0) = \alpha \tag{A-19}$$

admits a unique solution $u \in C([0,T], H^s_x)$, with

$$\|u\|_{T,s} \le C_s\big\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\big\}. \tag{A-20}$$

Proof. Following the procedure given in Section 2E, we define $\mathcal{S} := I + \gamma(t,x)\partial_x^{-1}$ (see (2-53)) with $\gamma(t,x) := -\frac{1}{3m}\partial_x^{-1}a_{14}(t,x)$. We have that u solves (A-19) if and only if $\tilde{u} := \mathcal{S}^{-1}u$ satisfies

$$\mathcal{L}_5\tilde{u} = \tilde{f}, \quad \tilde{u}(0) = \tilde{\alpha},$$

where $\tilde{f} := \mathcal{S}^{-1}f$, $\tilde{\alpha} := \mathcal{S}^{-1}(0)\alpha$ and $\mathcal{L}_5 = \partial_t + m\,\partial_{xxx} + \mathcal{R}$, with

$$\mathcal{R} = \mathcal{S}^{-1}\big\{a_{15} + (a_{14}\gamma - (a_{14})_x)\pi_0 + (\mathcal{L}_4\gamma)\,\partial_x^{-1}\big\}.$$

Then the thesis follows by Lemmas A.2 and 2.7. □

Lemma A.4. *There exist universal positive constants σ, δ_* with the following properties. Let $s \ge 0$, let $m \ge \frac{1}{2}$, and let $a_{12}(t,x)$, $a_{13}(t,x)$ be two functions with $a_{12}, \partial_t a_{12}, a_{13} \in C([0,T], H^{s+\sigma}_x)$, and let $\mathcal{L}_3 := \partial_t + m\,\partial_{xxx} + a_{12}\,\partial_x + a_{13}$. Let*

$$\delta(\mu) := \|a_{12}, \partial_t a_{12}, a_{13}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s].$$

Assume $\delta(0) \le \delta_*$. Let $f \in C([0,T], H^s_x)$ and $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_3 u = f, \quad u(0) = \alpha \tag{A-21}$$

admits a unique solution $u \in C([0,T], H^s_x)$, with

$$\|u\|_{T,s} \le C_s \bigl\{ \|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0) \bigr\}. \tag{A-22}$$

Proof. Following the procedure given in Section 2D, we define $\mathcal{T}h(t,x) := h(t, x + p(t))$ (see (2-46)), with $p(t) := -\frac{1}{2\pi}\int_0^t \int_{\mathbb{T}} a_{12}(s,x)\,dx\,ds$. We have that u solves (A-21) if and only if $\tilde{u} := \mathcal{T}^{-1}u$ satisfies

$$\mathcal{L}_4 \tilde{u} = \tilde{f}, \quad \tilde{u}(0) = \alpha$$

(note that $\mathcal{T}(0)$ is the identity), where $\tilde{f} := \mathcal{T}^{-1} f$, and $\mathcal{L}_4 = \partial_t + m\,\partial_{xxx} + a_{14}\,\partial_x + a_{15}$, with a_{14}, a_{15} given by formula (2-48). Then the thesis follows by Lemmas A.3 and 2.6. $\square$

Lemma A.5. *There exist universal positive constants σ, δ_* with the following properties. Let $s \ge 0$, let $m \ge \frac{1}{2}$, and let $a_8(t,x)$, $a_9(t,x)$, $a_{10}(t,x)$ be three functions with $a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10} \in C([0,T], H^{s+\sigma}_x)$ and $\int_{\mathbb{T}} a_8(t,x)\,dx = 0$, and let $\mathcal{L}_2 := \partial_t + m\,\partial_{xxx} + a_8\,\partial_{xx} + a_9\,\partial_x + a_{10}$. Let*

$$\delta(\mu) := \|a_8, \partial_t a_8, a_9, \partial_t a_9, a_{10}\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s].$$

Assume $\delta(0) \le \delta_*$. Let $f \in C([0,T], H^s_x)$ and $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_2 u = f, \quad u(0) = \alpha \tag{A-23}$$

admits a unique solution $u \in C([0,T], H^s_x)$, with

$$\|u\|_{T,s} \le C_s \bigl\{ \|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0) \bigr\}. \tag{A-24}$$

Proof. Following the procedure given in Section 2C, we define $\mathcal{M}h(t,x) := q(t,x)h(t,x)$ (see (2-37)), with $q(t,x) := \exp\{-\frac{1}{3m}(\partial_x^{-1} a_8)(t,x)\}$. We have that u solves (A-23) if and only if $\tilde{u} := \mathcal{M}^{-1}u$ satisfies

$$\mathcal{L}_3 \tilde{u} = \tilde{f}, \quad \tilde{u}(0) = \tilde{\alpha},$$

where $\tilde{f} := \mathcal{M}^{-1} f$, $\tilde{\alpha} := \mathcal{M}^{-1}(0)\alpha$, and $\mathcal{L}_3 = \partial_t + m\,\partial_{xxx} + a_{12}\,\partial_x + a_{13}$, with a_{12}, a_{13} given by formula (2-39). Then the thesis follows by Lemmas A.4 and 2.5. $\square$

Lemma A.6. *There exist universal positive constants σ, δ_* with the following properties. Let $s \ge 0$ and let $a_4(t)$, $a_5(t,x)$, $a_6(t,x)$, $a_7(t,x)$ be four functions with $a_4 \in C^1([0,T], \mathbb{R})$ and $a_5, \partial_t a_5, a_6, \partial_t a_6, a_7 \in C([0,T], H^{s+\sigma}_x)$ and $\int_{\mathbb{T}} a_5(t,x)\,dx = 0$, and let $\mathcal{L}_1 := \partial_t + a_4\partial_{xxx} + a_5\partial_{xx} + a_6\partial_x + a_7$. Let*

$$\delta(\mu) := \sup_{t\in[0,T]} |a_4(t) - 1| + \sup_{t\in(0,T)} |a_4'(t)| + \|a_5, \partial_t a_5, a_6, \partial_t a_6, a_7\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s]. \tag{A-25}$$

Assume $\delta(0) \le \delta_*$. Let $f \in C([0,T], H^s_x)$ and $\alpha \in H^s_x$. Then the Cauchy problem

$$\mathcal{L}_1 u = f, \quad u(0) = \alpha \tag{A-26}$$

admits a unique solution $u \in C([0,T], H^s_x)$, *with*

$$\|u\|_{T,s} \le C_s\big\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\big\}. \tag{A-27}$$

Proof. Following the procedure given in Section 2B, we define $\mathcal{B}h(t,x) := h(\psi(t), x)$ (see (2-25)), with $\psi(t) := \frac{1}{m}\int_0^t a_4(s)\,ds$, where $m := \frac{1}{T}\int_0^T a_4(t)\,dt$. We have that u solves (A-26) if and only if $\tilde u := \mathcal{B}^{-1}u$ satisfies

$$\mathcal{L}_2 \tilde u = \tilde f, \quad \tilde u(0) = \alpha$$

(note that $\mathcal{B}(0)$ is the identity), where $\tilde f := \mathcal{B}^{-1} f$, and $\mathcal{L}_2 = \partial_t + m\,\partial_{xxx} + a_8\,\partial_{xx} + a_9\,\partial_x + a_{10}$, with a_8, a_9, a_{10} given by formula (2-32) (see also (2-26)). Then the thesis follows by Lemma A.5 and 2.4. □

Lemma A.7. *There exist universal positive constants* σ, δ_* *with the following properties. Let* $s \ge 0$ *and let* $a_3(t,x)$, $a_2(t,x)$, $a_1(t,x)$, $a_0(t,x)$ *be four functions with* $a_3, \partial_t a_3, \partial_{tt} a_3, a_1, \partial_t a_1, a_0 \in C([0,T], H^{s+\sigma}_x)$ *and* $a_2 = c\,\partial_x a_3$ *for some* $c \in \mathbb{R}$. *Let*

$$\delta(\mu) := \|a_3, \partial_t a_3, \partial_{tt} a_3, a_1, \partial_t a_1, a_0\|_{T,\mu+\sigma} \quad \forall \mu \in [0,s]. \tag{A-28}$$

Assume $\delta(0) \le \delta_*$. *Let* $\mathcal{L}_0 := \partial_t + (1+a_3)\,\partial_{xxx} + a_2\,\partial_{xx} + a_1\,\partial_x + a_0$. *Let* $f \in C([0,T], H^s_x)$ *and* $\alpha \in H^s_x$. *Then the Cauchy problem*

$$\mathcal{L}_0 u = f, \quad u(0) = \alpha \tag{A-29}$$

admits a unique solution $u \in C([0,T], H^s_x)$, *with*

$$\|u\|_{T,s} \le C_s\big\{\|f\|_{T,s} + \|\alpha\|_s + \delta(s)(\|f\|_{T,0} + \|\alpha\|_0)\big\}. \tag{A-30}$$

Proof. Following the procedure given in Section 2A, we define $(\mathcal{A}h)(t,x) := h(t, x+\beta(t,x))$ (see (2-9)), with $\beta(t,x) := (\partial_x^{-1}\rho_0)(t,x)$, where ρ_0 is defined in (2-16)–(2-17). We have that u solves (A-29) if and only if $\tilde u := \mathcal{A}^{-1}u$ satisfies

$$\mathcal{L}_1 \tilde u = \tilde f, \quad \tilde u(0) = \tilde\alpha,$$

where $\tilde f := \mathcal{A}^{-1} f$, $\tilde\alpha := \mathcal{A}^{-1}(0)\alpha$, and $\mathcal{L}_1 = \partial_t + a_4\,\partial_{xxx} + a_5\,\partial_{xx} + a_6\,\partial_x + a_7$, with a_4 not depending on the space variable x and with a_4, a_5, a_6, a_7 given by formula (2-14). Then the thesis follows by Lemmas A.6 and 2.3. □

Remark A.8. Consider the operators $\mathcal{L}_0, \dots, \mathcal{L}_5$ defined in Lemmas A.2–A.7. Define

$$\begin{aligned}
\mathcal{L}_0^* h &:= -\partial_t h - \partial_{xxx}[(1+a_3)h] + \partial_{xx}(a_2 h) - \partial_x(a_1 h) + a_0 h,\\
\mathcal{L}_1^* h &:= -\partial_t h - a_4 \partial_{xxx} h + \partial_{xx}(a_5 h) - \partial_x(a_6 h) + a_7 h,\\
\mathcal{L}_2^* h &:= -\partial_t h - m \partial_{xxx} h + \partial_{xx}(a_8 h) - \partial_x(a_9 h) + a_{10} h,\\
\mathcal{L}_3^* h &:= -\partial_t h - m \partial_{xxx} h - \partial_x(a_{12} h) + a_{13} h,\\
\mathcal{L}_4^* h &:= -\partial_t h - m \partial_{xxx} h - \partial_x(a_{14} h) + a_{15} h,\\
\mathcal{L}_5^* h &:= -\partial_t h - m \partial_{xxx} h + \mathcal{R}^T h.
\end{aligned}$$

It is straightforward to check that Lemmas A.2–A.7 also hold when the operator $\mathcal{L}_k$ ($k = 0, \dots, 5$) is replaced by $\mathcal{L}_k^*$. The crucial observation is that for all $k = 0, \dots, 5$ (see Remark 2.2 for the case $k = 0$) the operator $-\mathcal{L}_k^*$ has the same structure as $\mathcal{L}_k$ (one might need to worsen the constants σ since the coefficients of $-\mathcal{L}_k^*$ involve space derivatives of the coefficients of $\mathcal{L}_k$). It is also immediate to verify that the same estimates also hold for the backward Cauchy problems

$$\begin{cases} \mathcal{L}_k u = f, \\ u(T) = \alpha, \end{cases} \qquad \begin{cases} \mathcal{L}_k^* u = f, \\ u(T) = \alpha, \end{cases} \qquad k = 0, \dots, 5. \tag{A-31}$$

Appendix B: Nash–Moser theorem

In this section we prove a Nash–Moser implicit function theorem that is a modified version of the theorem in [Hörmander 1985]. With respect to that paper, here (Theorem B.1) we assume slightly stronger hypotheses on the nonlinear operator Φ and its second derivative. These hypotheses are naturally verified in applications to PDEs. We use the iteration scheme of [Hörmander 1976] (called the *discrete Nash method* by Hörmander), which is neither the Newton scheme with smoothings used in [Berti, Bolle, and Procesi 2010; Berti, Corsi, and Procesi 2015; Baldi, Berti, and Montalto 2016a], nor the scheme in [Hörmander 1985; Alinhac and Gérard 2007]. The scheme of [Hörmander 1976] is based on a telescoping series like in [Hörmander 1985], but some corrections y_n (see (B-15)) are also introduced. In this way the scheme converges directly to a solution of the equation $\Phi(u) = \Phi(0) + g$, avoiding the intermediate step in [Hörmander 1985] where the Leray–Schauder theorem is applied. This makes it possible to remove two assumptions of Hörmander's theorem [1985], which are the compact embeddings $F_b \hookrightarrow F_a$ in the codomain scale of Banach spaces $(F_a)_{a \geq 0}$, and the continuity of the approximate right inverse $\Psi(v)$ with respect to the approximate linearization point v. We point out that, unlike Theorem 2.2.2 of [Hörmander 1976], our Theorem B.1 also applies to the case of Sobolev spaces.

Let us begin with recalling the construction of "weak" spaces in [Hörmander 1985].

Let E_a, $a \geq 0$, be a decreasing family of Banach spaces with injections $E_b \hookrightarrow E_a$ of norm ≤ 1 when $b \geq a$. Set $E_\infty = \bigcap_{a \geq 0} E_a$ with the weakest topology making the injections $E_\infty \hookrightarrow E_a$ continuous. Assume that $S_\theta : E_0 \to E_\infty$ for $\theta \geq 1$ are linear operators such that, with constants C bounded when a and b are bounded,

$$\|S_\theta u\|_b \leq C \|u\|_a \qquad \text{if } b \leq a, \tag{B-1}$$

$$\|S_\theta u\|_b \leq C \theta^{b-a} \|u\|_a \qquad \text{if } a < b, \tag{B-2}$$

$$\|u - S_\theta u\|_b \leq C \theta^{b-a} \|u\|_a \qquad \text{if } a > b, \tag{B-3}$$

$$\left\| \frac{d}{d\theta} S_\theta u \right\|_b \leq C \theta^{b-a-1} \|u\|_a. \tag{B-4}$$

From (B-2)–(B-3) one can obtain the logarithmic convexity of the norms

$$\|u\|_{\lambda a + (1-\lambda) b} \leq C \|u\|_a^\lambda \|u\|_b^{1-\lambda} \qquad \text{if } 0 < \lambda < 1. \tag{B-5}$$

Consider the sequence $\{\theta_j\}_{j\in\mathbb{N}}$, with $1=\theta_0<\theta_1<\cdots\to\infty$, such that θ_{j+1}/θ_j is bounded. Set $\Delta_j:=\theta_{j+1}-\theta_j$ and

$$R_0u:=\frac{S_{\theta_1}u}{\Delta_0},\qquad R_ju:=\frac{S_{\theta_{j+1}}u-S_{\theta_j}u}{\Delta_j},\qquad j\geq 1. \tag{B-6}$$

By (B-3) we deduce that, if $u\in E_b$ for some $b>a$, then

$$u=\sum_{j=0}^{\infty}\Delta_jR_ju \tag{B-7}$$

with convergence in E_a. Moreover, (B-4) implies that, for all b,

$$\|R_ju\|_b\leq C_{a,b}\theta_j^{b-a-1}\|u\|_a. \tag{B-8}$$

Conversely, assume that $a_1<a<a_2$, that $u_j\in E_{a_2}$ and that

$$\|u_j\|_b\leq M\theta_j^{b-a-1}\quad\text{if } b=a_1 \text{ or } b=a_2. \tag{B-9}$$

By (B-5) this remains true with a constant factor on the right-hand side if $a_1<b<a_2$, so that $u=\sum\Delta_ju_j$ converges in E_b if $b<a$.

Let E'_a be the set of all sums $u=\sum\Delta_ju_j$ with u_j satisfying (B-9) and introduce the norm $\|u\|'_a$ as the infimum of M over all such decompositions. It follows that $\|\cdot\|'_a$ is stronger than $\|\cdot\|_b$ if $a>b$, while (B-7) and (B-8) show that $\|\cdot\|'_a$ is weaker than $\|\cdot\|_a$. Moreover (i) the space E'_a and, up to equivalence, its norm are independent of the choice of a_1 and a_2; (ii) E'_a is defined by (B-8) for any values of b to the left and to the right of a; (iii) E'_a does not depend on the smoothing operators; (iv) in (B-3) we can replace $\|u\|_a$ by $\|u\|'_a$, namely,

$$\|u-S_\theta u\|_b\leq C'_{a,b}\theta^{b-a}\|u\|'_a\quad\text{if } a>b, \tag{B-10}$$

if we take another constant $C'_{a,b}$, which may tend to ∞ as b approaches a. These four statements (i)–(iv) are proved in [Hörmander 1985].

Now let us suppose that we have another family F_a of decreasing Banach spaces with smoothing operators having the same properties as above. We use the same notation also for the smoothing operators. Unlike [Hörmander 1985], here we do not need to assume that the embedding $F_b\hookrightarrow F_a$ is compact for $b>a$.

Theorem B.1. *Let $a_1, a_2, \alpha, \beta, a_0, \mu$ be real numbers with*

$$0\leq a_0\leq\mu\leq a_1,\qquad a_1+\tfrac{1}{2}\beta\leq\alpha<a_1+\beta\leq a_2,\qquad 2\alpha<a_1+a_2. \tag{B-11}$$

Let V be a convex neighborhood of 0 in E_μ. Let Φ be a map from V to F_0 such that $\Phi: V\cap E_{a+\mu}\to F_a$ is of class C^2 for all $a\in[0,a_2-\mu]$, with

$$\|\Phi''(u)[v,w]\|_a\leq C\big(\|v\|_{a+\mu}\|w\|_{a_0}+\|v\|_{a_0}\|w\|_{a+\mu}+\|u\|_{a+\mu}\|v\|_{a_0}\|w\|_{a_0}\big) \tag{B-12}$$

for all $u \in V \cap E_{a+\mu}$, $v, w \in E_{a+\mu}$. Also assume that $\Phi'(v)$ for $v \in E_\infty \cap V$ belonging to some ball $\|v\|_{a_1} \le \delta_1$ has a right inverse $\Psi(v)$ mapping F_∞ to E_{a_2}, and that

$$\|\Psi(v)g\|_a \le C\big(\|g\|_{a+\beta-\alpha} + \|g\|_0\|v\|_{a+\beta}\big) \quad \forall a \in [a_1, a_2]. \tag{B-13}$$

There exists $\delta > 0$ such that, for every $g \in F'_\beta$ in the ball $\|g\|'_\beta \le \delta$, there exists $u \in E'_\alpha$, with $\|u\|'_\alpha \le C\|g\|'_\beta$, solving $\Phi(u) = \Phi(0) + g$.

Proof. We follow the proof in [Hörmander 1985] where possible, but we use a different iteration scheme. Let $\theta_j := j+1$, so that $\Delta_j = 1$ for all j. Let $g \in F'_\beta$ and $g_j := R_j g$. Thus

$$g = \sum_{j=0}^{\infty} g_j, \quad \|g_j\|_b \le C_b \theta_j^{b-\beta-1}\|g\|'_\beta \quad \forall b \in [0, +\infty). \tag{B-14}$$

We claim that if $\|g\|'_\beta$ is small enough, then we can define a sequence $u_j \in V \cap E_{a_2}$ with $u_0 := 0$ by the recursion formula

$$u_{j+1} := u_j + h_j, \quad v_j := S_{\theta_j} u_j, \quad h_j := \Psi(v_j)(g_j + y_j) \quad \forall j \ge 0, \tag{B-15}$$

where $y_0 := 0$,

$$y_1 := -S_{\theta_1} e_0, \qquad y_j := -S_{\theta_j} e_{j-1} - R_{j-1}\sum_{i=0}^{j-2} e_i \quad \forall j \ge 2, \tag{B-16}$$

and $e_j := e'_j + e''_j$,

$$e'_j := \Phi(u_j + h_j) - \Phi(u_j) - \Phi'(u_j)h_j, \quad e''_j := (\Phi'(u_j) - \Phi'(v_j))h_j. \tag{B-17}$$

We prove that for all $j \ge 0$,

$$\|h_j\|_a \le K_1\|g\|'_\beta\,\theta_j^{a-\alpha-1} \quad \forall a \in [a_1, a_2], \tag{B-18}$$

$$\|v_j\|_a \le K_2\|g\|'_\beta\,\theta_j^{a-\alpha} \quad \forall a \in [a_1+\beta, a_2+\beta], \tag{B-19}$$

$$\|u_j - v_j\|_a \le K_3\|g\|'_\beta\,\theta_j^{a-\alpha} \quad \forall a \in [0, a_2]. \tag{B-20}$$

For $j = 0$, (B-19) and (B-20) are trivially satisfied, and (B-18) follows from (B-14) because $h_0 = \Psi(0)g_0$ and $\theta_0 = 1$.

Now assume that (B-18), (B-19), (B-20) hold for $j = 0, \dots, k$, for some $k \ge 0$. First we prove (B-20) for $j = k+1$. Since $u_{k+1} = \sum_{j=0}^{k} h_j$, the definition of the norm of E'_α and (B-18) for $j = 0, \dots, k$ imply that $\|u_{k+1}\|'_\alpha \le K_1\|g\|'_\beta$. By (B-10) one has

$$\|u_{k+1} - v_{k+1}\|_0 \le CK_1\|g\|'_\beta\theta_{k+1}^{-\alpha}, \tag{B-21}$$

where the constant C depends on α. From now until the end of this proof we denote by C any constant (possibly different from line to line) depending only on a_1, a_2, α, β, μ, a_0, which are fixed parameters.

From (B-18) with $j = 0, \dots, k$ we get

$$\|u_{k+1}\|_a \le K_1 \|g\|'_\beta \sum_{j=0}^{k} \theta_j^{a-\alpha-1} \quad \forall a \in [a_1, a_2]. \tag{B-22}$$

We note that

$$\sum_{j=0}^{k} \theta_j^{p-1} \le \frac{2}{p} \theta_{k+1}^p \quad \forall p > 0. \tag{B-23}$$

For $a = a_2$, by (B-1) one gets $\|v_{k+1}\|_{a_2} \le C \|u_{k+1}\|_{a_2}$. Thus, using (B-23) at $p = a_2 - \alpha$,

$$\|u_{k+1} - v_{k+1}\|_{a_2} \le C \|u_{k+1}\|_{a_2} \le C K_1 \|g\|'_\beta \theta_{k+1}^{a_2-\alpha}. \tag{B-24}$$

Using (B-5) to interpolate between (B-21) and (B-24), we get (B-20) for $j = k+1$, for all $a \in [0, a_2]$, provided that $K_3 \ge C K_1$.

To prove (B-19) for $j = k+1$, we use (B-2), (B-22) and (B-23) and we get

$$\|v_{k+1}\|_a \le C \theta_{k+1}^{a-a_1-\beta} \|u_{k+1}\|_{a_1+\beta} \le C \theta_{k+1}^{a-a_1-\beta} K_1 \|g\|'_\beta \sum_{j=0}^{k} \theta_j^{a_1+\beta-\alpha-1} \le C K_1 \|g\|'_\beta \, \theta_{k+1}^{a-\alpha}$$

for all $a \in [a_1 + \beta, a_2 + \beta]$. This gives (B-19) for $j = k+1$ provided that $K_2 \ge C K_1$.

To prove (B-18) for $j = k+1$, we begin with proving that

$$\|y_{k+1}\|_b \le C K_1 (K_1 + K_3) \|g\|'^2_\beta \theta_{k+1}^{b-\beta-1} \quad \forall b \in [0, a_2 + \beta - \alpha]. \tag{B-25}$$

Since $u_j, v_j, u_j + h_j$ belong to V for all $j = 0, \dots, k$, we use Taylor's formula and (B-12) to deduce that, for $j = 0, \dots, k$ and $a \in [0, a_2 - \mu]$,

$$\begin{aligned} \|e_j\|_a \le C\big(&\|h_j\|_{a_0}\|h_j\|_{a+\mu} + \|u_j\|_{a+\mu}\|h_j\|_{a_0}^2 + \|h_j\|_{a_0}\|v_j - u_j\|_{a+\mu} \\ &+ \|h_j\|_{a+\mu}\|v_j - u_j\|_{a_0} + \|u_j\|_{a+\mu}\|h_j\|_{a_0}\|v_j - u_j\|_{a_0}\big). \end{aligned} \tag{B-26}$$

Hence at $j = k$, using (B-2) and then (B-26), we have

$$\begin{aligned} \|S_{\theta_{k+1}} e_k\|_{a_2+\beta-\alpha} &\le C\theta_{k+1}^p \|e_k\|_{a_2+\beta-\alpha-p} \\ &\le C\theta_{k+1}^p \big(\|h_k\|_{a_0}\|h_k\|_q + \|u_k\|_q\|h_k\|_{a_0}^2 + \|h_k\|_{a_0}\|v_k - u_k\|_q \\ &\qquad + \|h_k\|_q\|v_k - u_k\|_{a_0} + \|u_k\|_q\|h_k\|_{a_0}\|v_k - u_k\|_{a_0}\big), \end{aligned} \tag{B-27}$$

where $p := \max\{0, \beta - \alpha + \mu\}$ and $q := a_2 + \beta - \alpha - p + \mu$. Note that $a_2 + \beta - \alpha - p \ge 0$ because $a_2 \ge \mu$. Since $q \le a_2$, using also (B-23) we have

$$\|u_k\|_q \le \|u_k\|_{a_2} \le \sum_{j=0}^{k-1} \|h_j\|_{a_2} \le K_1 \|g\|'_\beta \sum_{j=0}^{k-1} \theta_j^{a_2-\alpha-1} \le C K_1 \|g\|'_\beta \theta_k^{a_2-\alpha}. \tag{B-28}$$

By (B-28), (B-18), (B-20), and since $a_0 \le a_1$, the bound (B-27) implies that

$$\|S_{\theta_{k+1}} e_k\|_{a_2+\beta-\alpha} \le C K_1 (K_1 + K_3) \|g\|'^2_\beta \theta_{k+1}^p (\theta_k^{a_1+q-2\alpha-1} + \theta_k^{a_2+2a_1-3\alpha-1})$$

provided that $K_1\|g\|'_\beta \le 1$. We assume that

$$K_1\|g\|'_\beta \le 1. \tag{B-29}$$

Both the exponents $a_1+q-2\alpha-1$ and $a_2+2a_1-3\alpha-1$ are $\le a_2-\alpha-1-p$ because $a_1 < \alpha$ and $a_1+\beta+\mu \le 2\alpha$. Thus

$$\|S_{\theta_{k+1}}e_k\|_{a_2+\beta-\alpha} \le CK_1(K_1+K_3)\|g\|'^2_\beta\theta_{k+1}^{a_2-\alpha-1}. \tag{B-30}$$

Now we estimate $\|S_{\theta_{k+1}}e_k\|_0$. Since $a_0,\mu \le a_1$, by (B-1) and (B-26) we get

$$\|S_{\theta_{k+1}}e_k\|_0 \le C\|e_k\|_0 \le C(1+\|u_k\|_\mu)(\|h_k\|^2_{a_1}+\|h_k\|_{a_1}\|v_k-u_k\|_{a_1}). \tag{B-31}$$

By (B-18) and (B-29),

$$\|u_k\|_\mu \le \|u_k\|_{a_1} \le \sum_{j=0}^{k-1}\|h_j\|_{a_1} \le K_1\|g\|'_\beta\sum_{j=0}^{\infty}\theta_j^{a_1-\alpha-1} = CK_1\|g\|'_\beta \le C. \tag{B-32}$$

We use (B-18), (B-20) and (B-32) in (B-31), and the bound $\theta_{k+1}^{2a_1-2\alpha-1} \le \theta_{k+1}^{-\beta-1}$, to deduce that

$$\|S_{\theta_{k+1}}e_k\|_0 \le CK_1(K_1+K_3)\|g\|'^2_\beta\theta_{k+1}^{-\beta-1}. \tag{B-33}$$

Using (B-5) to interpolate between (B-30) and (B-33), we obtain

$$\|S_{\theta_{k+1}}e_k\|_b \le CK_1(K_1+K_3)\|g\|'^2_\beta\theta_{k+1}^{b-\beta-1} \quad \forall b\in[0,a_2+\beta-\alpha]. \tag{B-34}$$

Now we estimate the other terms in y_{k+1} (see (B-16)). By (B-8), (B-26), (B-18), (B-20) and (B-23),

$$\begin{aligned}\sum_{i=0}^{k-1}\|R_ke_i\|_b &\le \sum_{i=0}^{k-1}C\theta_k^{b-a_2+\mu-1}\|e_i\|_{a_2-\mu}\\ &\le CK_1(K_1+K_3)\|g\|'^2_\beta\theta_k^{b-a_2+\mu-1}\sum_{i=0}^{k-1}\theta_i^{a_1+a_2-2\alpha-1}\end{aligned} \tag{B-35}$$

for all $b\in[0,a_2+\beta-\alpha]$. Since $a_1+a_2-2\alpha>0$, we apply (B-23) to the last sum in (B-35). Then, recalling that $\theta_k/\theta_{k+1}\in\left[\frac{1}{2},1\right]$, and using the bound $a_1+\beta+\mu\le 2\alpha$, we deduce that

$$\sum_{i=0}^{k-1}\|R_ke_i\|_b \le CK_1(K_1+K_3)\|g\|'^2_\beta\theta_{k+1}^{b-\beta-1} \quad \forall b\in[0,a_2+\beta-\alpha]. \tag{B-36}$$

The sum of (B-34) and (B-36) completes the proof of (B-25).

Now we are ready to prove (B-18) at $j=k+1$. By (B-1) and (B-22) we have $\|v_{k+1}\|_{a_1} \le C\|u_{k+1}\|_{a_1} \le CK_1\|g\|'_\beta$, and we assume that $CK_1\|g\|'_\beta \le \delta_1$, so that $\Psi(v_{k+1})$ is defined. By (B-15), (B-13), (B-14), (B-25), (B-19) one has, for all $a\in[a_1,a_2]$,

$$\|h_{k+1}\|_a \le C\|g\|'_\beta\left\{1+(K_1+K_3)K_1\|g\|'_\beta\right\}\theta_{k+1}^{a-\alpha-1} \tag{B-37}$$

provided that $K_2\|g\|'_\beta \le 1$. Bound (B-37) implies (B-18) provided that $C\left\{1+(K_1+K_3)K_1\|g\|'_\beta\right\} \le K_1$.

The induction proof of (B-18), (B-19), (B-20) is complete if K_1, K_2, K_3, $\|g\|'_\beta$ satisfy

$$K_3 \geq C_0 K_1, \quad K_2 \geq C_0 K_1, \quad C_0 K_1 \|g\|'_\beta \leq 1, \quad K_2 \|g\|'_\beta \leq 1, \quad C_0\big\{1 + (K_1 + K_3) K_1 \|g\|'_\beta\big\} \leq K_1,$$

where C_0 is the largest of the constants appearing above. First we fix $K_1 \geq 2C_0$. Then we fix K_2 and K_3 larger than $C_0 K_1$, and finally we fix $\delta_0 > 0$ such that the last three inequalities hold for all $\|g\|'_\beta \leq \delta_0$. This completes the proof of (B-18), (B-19), (B-20).

Bound (B-18) implies that the sequence (u_k) converges in E_a for all $a \in [0, \alpha)$. We call u its limit. Since $u = \sum_{j=0}^{\infty} h_j$ and each term h_j satisfies (B-18), it follows that $u \in E'_\alpha$ and $\|u\|'_\alpha \leq K_1 \|g\|'_\beta$ by the definition of the norm in E'_α.

Finally, we prove the convergence of the Nash–Moser scheme. By (B-16) and (B-6) one proves by induction that

$$\sum_{j=0}^{k} (e_j + y_j) = e_k + r_k, \quad \text{where } r_k := (I - S_{\theta_k}) \sum_{j=0}^{k-1} e_j, \quad \forall k \geq 1.$$

Hence, by (B-15) and (B-17), recalling that $\Phi'(v_j)\Psi(v_j)$ is the identity map, one has

$$\Phi(u_{k+1}) - \Phi(u_0) = \sum_{j=0}^{k} [\Phi(u_{j+1}) - \Phi(u_j)] = \sum_{j=0}^{k} (e_j + g_j + y_j) = G_k + e_k + r_k,$$

where $G_k := \sum_{j=0}^{k} g_j$. By (B-14), $\|G_k - g\|_b \to 0$ as $k \to \infty$ for all $b \in [0, \beta)$. Let $a \in [a_1 - \mu, \alpha - \mu)$. By (B-22) and (B-29) we get $\|u_j\|_{a+\mu} \leq C$. By (B-26), (B-18) and (B-20) we deduce that

$$\|e_j\|_a \leq C K_1 (K_1 + K_3) \|g\|'^2_\beta \, \theta_j^{a_1 + a + \mu - 2\alpha - 1}. \tag{B-38}$$

Hence $\|e_k\|_a \to 0$ as $k \to \infty$ because $a_1 + a + \mu - 2\alpha < 0$, and, moreover, $\sum_{j=0}^{\infty} \|e_j\|_a$ converges. By (B-3) and (B-38), for all $\rho \in [0, a)$ we have

$$\|r_k\|_\rho \leq \sum_{j=0}^{k-1} \|(I - S_{\theta_k}) e_j\|_\rho \leq C \sum_{j=0}^{k-1} \theta_k^{\rho - a} \|e_j\|_a \leq C \theta_k^{\rho - a}, \tag{B-39}$$

so that $\|r_k\|_\rho \to 0$ as $k \to \infty$. We have proved that $\|\Phi(u_k) - \Phi(u_0) - g\|_\rho \to 0$ as $k \to \infty$ for all ρ in the interval $0 \leq \rho < \min\{\alpha - \mu, \beta\}$. Since $u_k \to u$ in E_a for all $a \in [0, \alpha)$, it follows that $\Phi(u_k) \to \Phi(u)$ in F_b for all $b \in [0, \alpha - \mu)$. $\square$

Appendix C: Tame estimates

In this appendix we recall classical tame estimates for products, compositions of functions and changes of variables which are repeatedly used in the paper. Recall the notation (1-6) for functions $u(x)$, $x \in \mathbb{T}$, in the Sobolev space $H^s := H^s(\mathbb{T}, \mathbb{R})$.

Lemma C.1. *Let s_0, s_1, s_2, s denote nonnegative real numbers, with $s_0 > \frac{1}{2}$. There exist positive constants C_s, $s \geq s_0$, with the following properties.*

- (embedding and algebra) *For all* $u, v \in H^{s_0}$,

$$\|u\|_{L^\infty} \le C_{s_0} \|u\|_{s_0}, \quad \|uv\|_{s_0} \le C_{s_0} \|u\|_{s_0} \|v\|_{s_0}. \tag{C-1}$$

- (interpolation) *For* $0 \le s_1 \le s \le s_2$ *and* $s = \lambda s_1 + (1-\lambda)s_2$, *for all* $u \in H^{s_2}$,

$$\|u\|_s \le \|u\|_{s_1}^{\lambda} \|u\|_{s_2}^{1-\lambda}. \tag{C-2}$$

- (tame product) *For* $s \ge s_0$, *for all* $u, v \in H^s$,

$$\|uv\|_s \le C_{s_0} \|u\|_s \|v\|_{s_0} + C_s \|u\|_{s_0} \|v\|_s, \tag{C-3}$$

and for $s \in [0, s_0]$, *for all* $u \in H^{s_0}$ *and* $v \in H^s$,

$$\|uv\|_s \le C_{s_0} \|u\|_{s_0} \|v\|_s. \tag{C-4}$$

Proof. The lemma can be proved by using Fourier series and the Hölder inequality. Otherwise, for (C-2) see, e.g., [Alinhac and Gérard 2007, p. 82] or [Moser 1966, p. 269]; for (C-3) adapt [Berti, Bolle, and Procesi 2010, Appendix] or [Alinhac and Gérard 2007, p. 84]. For (C-4) use the bound $\sum_{j\in\mathbb{Z}} \langle n\rangle^{2s} \langle j\rangle^{-2s} \langle n-j\rangle^{-2s_0} \le C_{s_0}$ for all $n \in \mathbb{Z}$, all $0 \le s \le s_0$, which can be proved by splitting the two cases $2|j| \le |n|$ and $2|j| > |n|$. □

A function $f : \mathbb{T} \times B \to \mathbb{R}$, where $B := \{y \in \mathbb{R}^{p+1} : |y| < R\}$, induces the composition operator

$$\tilde{f}(u)(x) := f\big(x, u(x), u'(x), u''(x), \dots, u^{(p)}(x)\big), \tag{C-5}$$

where $u^{(k)}(x)$ denotes the k-th derivative of $u(x)$. Let B_p be a ball in $W^{p,\infty}(\mathbb{T}, \mathbb{R})$ such that, if $u \in B_p$, then the vector $\big(u(x), u'(x), \dots, u^{(p)}(x)\big)$ belongs to B for all $x \in \mathbb{T}$.

Lemma C.2 (composition of functions). *Assume* $f \in C^r(\mathbb{T} \times B)$. *Then, for all* $u \in H^{s+p} \cap B_p$, $s \in [0, r]$, *the composition operator* (C-5) *is well-defined and*

$$\|\tilde{f}(u)\|_s \le C \|f\|_{C^r} (\|u\|_{s+p} + 1),$$

where C *depends on* r, p. *If, in addition,* $f \in C^{r+2}$, *then, for* $u, h \in H^{s+p}$ *with* $u, u+h \in B_p$, *one has*

$$\|\tilde{f}(u+h) - \tilde{f}(u)\|_s \le C \|f\|_{C^{r+1}} \big(\|h\|_{s+p} + \|h\|_{W^{p,\infty}} \|u\|_{s+p}\big),$$
$$\|\tilde{f}(u+h) - \tilde{f}(u) - \tilde{f}'(u)[h]\|_s \le C \|f\|_{C^{r+2}} \|h\|_{W^{p,\infty}} \big(\|h\|_{s+p} + \|h\|_{W^{p,\infty}} \|u\|_{s+p}\big).$$

Proof. For $s \in \mathbb{N}$ see [Moser 1966, pp. 272–275] and [Rabinowitz 1967, Lemma 7, pp. 202–203]. For $s \notin \mathbb{N}$ see [Alinhac and Gérard 2007, Proposition 2.2, p. 87]. □

Lemma C.3 (change of variable). *Let* $p \in W^{s,\infty}(\mathbb{T}, \mathbb{R})$, $s \ge 1$, *with* $\|p\|_{W^{1,\infty}} \le \frac{1}{2}$. *Let* $f(x) = x + p(x)$. *Then* f *is invertible, its inverse is* $f^{-1}(y) = g(y) = y + q(y)$, *where* q *is* 2π*-periodic,* $q \in W^{s,\infty}(\mathbb{T}, \mathbb{R})$, *and* $\|q\|_{W^{s,\infty}} \le C \|p\|_{W^{s,\infty}}$, *where* C *depends on* d, s.

Moreover, if $u \in H^s(\mathbb{T}, \mathbb{R})$, *then* $u \circ f(x) = u(x + p(x))$ *also belongs to* H^s, *and*

$$\|u \circ f\|_s + \|u \circ g\|_s \le C(\|u\|_s + \|p\|_{W^{s,\infty}} \|u\|_1). \tag{C-6}$$

Proof. For $s \in \mathbb{N}$ see, e.g., [Baldi 2013, Lemma B.4], where this lemma is proved by adapting [Hamilton 1982, Lemma 2.3.6, p. 149]. For $s \notin \mathbb{N}$ the lemma can be proved by studying the conjugate of the pseudodifferential operator $|D_x|^s$ by a change of variable, either by Egorov's theorem, see [Taylor 1981, Chapter VIII, Section 1, p. 150] and [Alazard, Baldi, and Han-Kwan 2015, Appendix C, Section C.1], or by an asymptotic formula, see [Alinhac and Gérard 2007, Proposition 7.1, p. 37]. □

Remark C.4. For time-dependent functions $u(t,x)$, $u \in C([0,T], H^s(\mathbb{T},\mathbb{R}))$, all the estimates of the present appendix hold with $\|u\|_s$ replaced by $\|u\|_{T,s} := \sup_{t\in[0,T]} \|u(t)\|_s$. □

Acknowledgements

We thank Thomas Alazard and Daniel Han-Kwan for useful comments and discussions. We thank the anonymous referee for pointing out a defect, which now has been corrected, and for other useful suggestions.

This research was carried out in the frame of Programma STAR, financially supported by UniNA and Compagnia di San Paolo. This research was supported by the European Research Council under FP7 (ERC Project 306414), by PRIN 2012 "Variational and perturbative aspects of nonlinear differential problems", by INDAM-GNAMPA Research Project 2015 "Analisi e controllo di equazioni a derivate parziali nonlineari", by GDRE CONEDP "Control of Partial Differential Equations" issued by CNRS, INDAM and Université de Provence.

References

[Alabau-Boussouira, Coron and Olive 2015] F. Alabau-Boussouira, J.-M. Coron, and G. Olive, "Internal controllability of first order quasilinear hyperbolic systems with a reduced number of controls", preprint, 2015, https://hal.archives-ouvertes.fr/hal-01139980.

[Alazard 2015] T. Alazard, "Boundary observability of gravity water waves", preprint, 2015. arXiv

[Alazard, Baldi, and Han-Kwan 2015] T. Alazard, P. Baldi, and D. Han-Kwan, "Control of water waves", 2015. To appear in *J. Eur. Math. Soc.* arXiv

[Alinhac and Gérard 2007] S. Alinhac and P. Gérard, *Pseudo-differential operators and the Nash–Moser theorem*, Graduate Studies in Mathematics **82**, American Mathematical Society, Providence, RI, 2007. MR Zbl

[Baldi 2013] P. Baldi, "Periodic solutions of fully nonlinear autonomous equations of Benjamin–Ono type", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30**:1 (2013), 33–77. MR Zbl

[Baldi, Berti, and Montalto 2014] P. Baldi, M. Berti, and R. Montalto, "KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation", *Math. Ann.* **359**:1–2 (2014), 471–536. MR Zbl

[Baldi, Berti, and Montalto 2016a] P. Baldi, M. Berti, and R. Montalto, "KAM for autonomous quasi-linear perturbations of KdV", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33**:6 (2016), 1589–1638. MR Zbl

[Baldi, Berti, and Montalto 2016b] P. Baldi, M. Berti, and R. Montalto, "KAM for autonomous quasi-linear perturbations of mKdV", *Boll. Unione Mat. Ital.* **9**:2 (2016), 143–188. MR Zbl

[Bardos, Lebeau, and Rauch 1992] C. Bardos, G. Lebeau, and J. Rauch, "Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary", *SIAM J. Control Optim.* **30**:5 (1992), 1024–1065. MR Zbl

[Beauchard 2005] K. Beauchard, "Local controllability of a 1-D Schrödinger equation", *J. Math. Pures Appl.* (9) **84**:7 (2005), 851–956. MR Zbl

[Beauchard 2008] K. Beauchard, "Local controllability of a one-dimensional beam equation", *SIAM J. Control Optim.* **47**:3 (2008), 1219–1273. MR Zbl

[Beauchard and Coron 2006] K. Beauchard and J.-M. Coron, "Controllability of a quantum particle in a moving potential well", *J. Funct. Anal.* **232**:2 (2006), 328–389. MR Zbl

[Beauchard and Laurent 2010] K. Beauchard and C. Laurent, "Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control", *J. Math. Pures Appl.* (9) **94**:5 (2010), 520–554. MR Zbl

[Berti, Bolle, and Procesi 2010] M. Berti, P. Bolle, and M. Procesi, "An abstract Nash–Moser theorem with parameters and applications to PDEs", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27**:1 (2010), 377–399. MR Zbl

[Berti, Corsi, and Procesi 2015] M. Berti, L. Corsi, and M. Procesi, "An abstract Nash–Moser theorem and quasi-periodic solutions for NLW and NLS on compact Lie groups and homogeneous manifolds", *Comm. Math. Phys.* **334**:3 (2015), 1413–1454. MR Zbl

[Burq and Gérard 1997] N. Burq and P. Gérard, "Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes", *C. R. Acad. Sci. Paris Sér. I Math.* **325**:7 (1997), 749–752. MR Zbl

[Burq and Zworski 2012] N. Burq and M. Zworski, "Control for Schrödinger operators on tori", *Math. Res. Lett.* **19**:2 (2012), 309–324. MR Zbl

[Coron 2007] J.-M. Coron, *Control and nonlinearity*, Mathematical Surveys and Monographs **136**, American Mathematical Society, Providence, RI, 2007. MR Zbl

[Coron, Glass, and Wang 2010] J.-M. Coron, O. Glass, and Z. Wang, "Exact boundary controllability for 1-D quasilinear hyperbolic systems with a vanishing characteristic speed", *SIAM J. Control Optim.* **48**:5 (2010), 3105–3122. MR Zbl

[Dehman and Lebeau 2009] B. Dehman and G. Lebeau, "Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time", *SIAM J. Control Optim.* **48**:2 (2009), 521–550. MR Zbl

[Ekeland 2011] I. Ekeland, "An inverse function theorem in Fréchet spaces", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28**:1 (2011), 91–105. MR Zbl

[Ekeland and Séré 2015] I. Ekeland and E. Séré, "An implicit function theorem for non-smooth maps between Fréchet spaces", preprint, 2015. arXiv

[Gromov 1972] M. L. Gromov, "Smoothing and inversion of differential operators", *Mat. Sb.* (*N.S.*) **88**:3 (1972), 382–441. In Russian; translated in *Math. USSR Sb.* **17**:3 (1972), 381–435. MR Zbl

[Guan and Kuksin 2014] H. Guan and S. Kuksin, "The KdV equation under periodic boundary conditions and its perturbations", *Nonlinearity* **27**:9 (2014), R61–R88. MR Zbl

[Hamilton 1982] R. S. Hamilton, "The inverse function theorem of Nash and Moser", *Bull. Amer. Math. Soc.* (*N.S.*) **7**:1 (1982), 65–222. MR Zbl

[Haraux 1989] A. Haraux, "Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire", *J. Math. Pures Appl.* (9) **68**:4 (1989), 457–465. MR Zbl

[Hörmander 1976] L. Hörmander, "The boundary problems of physical geodesy", *Arch. Rational Mech. Anal.* **62**:1 (1976), 1–52. MR Zbl

[Hörmander 1985] L. Hörmander, "On the Nash–Moser implicit function theorem", *Ann. Acad. Sci. Fenn. Ser. A I Math.* **10** (1985), 255–259. MR Zbl

[Hörmander 1990] L. Hörmander, "The Nash–Moser theorem and paradifferential operators", pp. 429–449 in *Analysis, et cetera*, edited by P. H. Rabinowitz and E. Zehnder, Academic Press, Boston, 1990. MR Zbl

[Kahane 1962] J.-P. Kahane, "Pseudo-périodicité et séries de Fourier lacunaires", *Ann. Sci. École Norm. Sup.* (3) **79** (1962), 93–150. MR Zbl

[Kappeler and Pöschel 2003] T. Kappeler and J. Pöschel, *KdV & KAM*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **45**, Springer, 2003. MR Zbl

[Komornik 1994] V. Komornik, *Exact controllability and stabilization: the multiplier method*, Research in Applied Mathematics **36**, Wiley, Chichester, 1994. MR Zbl

[Komornik and Loreti 2005] V. Komornik and P. Loreti, *Fourier series in control theory*, Springer, 2005. MR Zbl

[Laurent 2010] C. Laurent, "Global controllability and stabilization for the nonlinear Schrödinger equation on an interval", *ESAIM Control Optim. Calc. Var.* **16**:2 (2010), 356–379. MR Zbl

[Laurent, Rosier, and Zhang 2010] C. Laurent, L. Rosier, and B.-Y. Zhang, "Control and stabilization of the Korteweg–de Vries equation on a periodic domain", *Comm. Partial Differential Equations* **35**:4 (2010), 707–744. MR Zbl

[Li and Rao 2003] T.-t. Li and B.-P. Rao, "Exact boundary controllability for quasi-linear hyperbolic systems", *SIAM J. Control Optim.* **41**:6 (2003), 1748–1755. MR Zbl

[Li and Zhang 1998] T.-t. Li and B.-Y. Zhang, "Global exact controllability of a class of quasilinear hyperbolic systems", *J. Math. Anal. Appl.* **225**:1 (1998), 289–311. MR Zbl

[Lions 1988] J.-L. Lions, "Exact controllability, stabilization and perturbations for distributed systems", *SIAM Rev.* **30**:1 (1988), 1–68. MR Zbl

[Micu and Zuazua 2005] S. Micu and E. Zuazua, "An introduction to the controllability of partial differential equations", pp. 67–150 in *Contrôle non linéair et applications: cours de l'école d'été du CIMPA de l'Université de Tlemcen* (Algérie, 2003), edited by T. Sari, Travaux en Cours **64**, Hermann, Paris, 2005. Zbl

[Moser 1961] J. Moser, "A new technique for the construction of solutions of nonlinear differential equations", *Proc. Nat. Acad. Sci. U.S.A.* **47** (1961), 1824–1831. MR Zbl

[Moser 1966] J. Moser, "A rapidly convergent iteration method and non-linear partial differential equations, I", *Ann. Scuola Norm. Sup. Pisa* (3) **20** (1966), 265–315. MR Zbl

[Nash 1956] J. Nash, "The imbedding problem for Riemannian manifolds", *Ann. of Math.* (2) **63** (1956), 20–63. MR Zbl

[Rabinowitz 1967] P. H. Rabinowitz, "Periodic solutions of nonlinear hyperbolic partial differential equations", *Comm. Pure Appl. Math.* **20** (1967), 145–205. MR Zbl

[Rosier and Zhang 2009] L. Rosier and B.-Y. Zhang, "Control and stabilization of the Korteweg–de Vries equation: recent progresses", *J. Syst. Sci. Complex.* **22**:4 (2009), 647–682. MR Zbl

[Russell 1991] D. L. Russell, "Computational study of the Korteweg–de Vries equation with localized control action", pp. 195–203 in *Distributed parameter control systems* (Minneapolis, MN, 1989), edited by G. Chen et al., Lecture Notes in Pure and Appl. Math. **128**, Dekker, New York, 1991. MR Zbl

[Russell and Zhang 1993] D. L. Russell and B. Y. Zhang, "Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain", *SIAM J. Control Optim.* **31**:3 (1993), 659–676. MR Zbl

[Taylor 1981] M. E. Taylor, *Pseudodifferential operators*, Princeton Mathematical Series **34**, Princeton University Press, 1981. MR Zbl

[Zehnder 1975] E. Zehnder, "Generalized implicit function theorems with applications to some small divisor problems, I", *Comm. Pure Appl. Math.* **28**:1 (1975), 91–140. MR Zbl

[Zehnder 1976] E. Zehnder, "Generalized implicit function theorems with applications to some small divisor problems, II", *Comm. Pure Appl. Math.* **29**:1 (1976), 49–111. MR Zbl

[Zhang 1990] B. Zhang, *Some results for nonlinear dispersive wave equations with applications to control*, Ph.D. thesis, University of Wisconsin–Madison, 1990, http://search.proquest.com/docview/303921081. MR

Received 23 Nov 2015. Revised 20 Sep 2016. Accepted 12 Dec 2016.

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ANALYSIS AND PDE
Vol. 10, No. 2, 2017
dx.doi.org/10.2140/apde.2017.10.323

OPERATORS OF SUBPRINCIPAL TYPE

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In this paper we consider the solvability of pseudodifferential operators when the principal symbol vanishes of at least second order at a nonradial involutive manifold Σ_2. We shall assume that the subprincipal symbol is of principal type with Hamilton vector field tangent to Σ_2 at the characteristics, but transversal to the symplectic leaves of Σ_2. We shall also assume that the subprincipal symbol is essentially constant on the leaves of Σ_2 and does not satisfying the Nirenberg–Trèves condition (Ψ) on Σ_2. In the case when the sign change is of infinite order, we also need a condition on the rate of vanishing of both the Hessian of the principal symbol and the complex part of the gradient of the subprincipal symbol compared with the subprincipal symbol. Under these conditions, we prove that P is not solvable.

1. Introduction

We will consider the solvability for a classical pseudodifferential operator $P \in \Psi^m_{\mathrm{cl}}(M)$ on a C^∞ manifold M. This means that P has an expansion $p_m + p_{m-1} + \cdots$, where $p_k \in S^k_{\mathrm{hom}}$ is homogeneous of degree k for all k, and $p_m = \sigma(P)$ is the principal symbol of the operator. A pseudodifferential operator is said to be of principal type if the Hamilton vector field H_{p_m} of the principal symbol does not have the radial direction $\xi \cdot \partial_\xi$ on $p_m^{-1}(0)$, in particular $H_{p_m} \neq 0$. We shall consider the case when the principal symbol vanishes of at least second order at an involutive manifold Σ_2; then P is not of principal type.

P is locally solvable at a compact set $K \subseteq M$ if the equation

$$Pu = v \tag{1-1}$$

has a local solution $u \in \mathcal{D}'(M)$ in a neighborhood of K for any $v \in C^\infty(M)$ in a set of finite codimension. We can also define microlocal solvability of P at any compactly based cone $K \subset T^*M$; see Definition 2.6.

For pseudodifferential operators of principal type, it is known [Dencker 2006; Hörmander 1981] that local solvability is equivalent to condition (Ψ) on the principal symbol, which means that

$$\operatorname{Im} ap_m \text{ does not change sign from } - \text{ to } + \text{ along the oriented bicharacteristics of } \operatorname{Re} ap_m \tag{1-2}$$

for any $0 \neq a \in C^\infty(T^*M)$. The oriented bicharacteristics are the positive flow-out of the Hamilton vector field $H_{\operatorname{Re} ap_m} \neq 0$ on which $\operatorname{Re} ap_m = 0$; these are also called semibicharacteristics of p_m. Condition (1-2) is invariant under multiplication of p_m with nonvanishing factors, and symplectic changes of variables; thus it is invariant under conjugation of P with elliptic Fourier integral operators. Observe that the sign changes in (1-2) are reversed when taking adjoints, and that it suffices to check (1-2) for some $a \neq 0$ for which $H_{\operatorname{Re} ap} \neq 0$, according to [Hörmander 1985b, Theorem 26.4.12].

MSC2010: primary 35S05; secondary 35A01, 58J40, 47G30.
Keywords: solvability, pseudodifferential operator, subprincipal symbol.

For operators which are not of principal type, the situation is more complicated and the solvability may depend on the lower-order terms. When the set Σ_2, where the principal symbol vanishes of second order, is involutive, the subprincipal symbol $\sigma_{\mathrm{sub}}(P) = p_{m-1}$ is invariantly defined at Σ_2. In fact, on Σ_2 it is equal to the *refined principal symbol*; see [Hörmander 1985a, Theorem 18.1.33].

In the case where the principal symbol is real and vanishes of at least second order at the involutive manifold, there are several results, mostly in the case when the principal symbol is a product of real symbols of principal type. Then the operator is not solvable if the imaginary part of the subprincipal symbol has a sign change of finite order on a bicharacteristic of one the factors of the principal symbol; see [Egorov 1977; Popivanov 1974; Wenston 1977; 1978].

This necessary condition for solvability has been extended to some cases when the principal symbol is real and vanishes of second order at the involutive manifold. The conditions for solvability then involve the sign changes of the imaginary part of the subprincipal symbol on the limits of bicharacteristics from outside the manifold, thus on the leaves of the symplectic foliation of the manifold; see [Mendoza and Uhlmann 1983; 1984; Mendoza 1984; Yamasaki 1983]. This has been extended to more general limit bicharacteristics of real principal symbols in [Dencker 2016].

When Σ_2 is not involutive, there are examples where the operator is solvable for any lower-order terms. For example when P is effectively hyperbolic, then even the Cauchy problem is solvable for any lower-order term; see [Hörmander 1977; Nishitani 2004]. There are also results in the cases when the principal symbol is a product of principal-type symbols not satisfying condition (Ψ); see [Cardoso and Trèves 1974; Gilioli and Trèves 1974; Goldman 1975; Trèves 1973; Yamasaki 1980].

In the present paper, we shall consider the case when the principal symbol (not necessarily real-valued) vanishes of at least second order at a nonradial involutive manifold Σ_2. We shall assume that the subprincipal symbol is of principal type with Hamilton vector field tangent to Σ_2 at the characteristics, but transversal to the symplectic leaves of Σ_2. We shall also assume that the subprincipal symbol is essentially constant on the symplectic leaves of Σ_2 by (2-8), and does not satisfy condition (Ψ); see Definition 2.4. In the case when the sign change is of infinite order, we will need a condition on the rate of vanishing of both the Hessian of the principal symbol and the complex part of the gradient of the subprincipal symbol on the semibicharacteristic of the subprincipal symbol; see condition (2-11). Under these conditions, P is not solvable in a neighborhood of the semibicharacteristic; see Theorem 2.7, which is the main result of the paper. In this case P is an evolution operator; see [Colombini et al. 2003; 2010] for some earlier results on the solvability of evolution operators.

2. Statement of results

Let $\sigma(P) = p \in S^m_{\mathrm{hom}}$ be the homogeneous principal symbol. We shall assume that

$$\sigma(P) \text{ vanishes of at least second order at } \Sigma_2, \tag{2-1}$$

where

$$\Sigma_2 \text{ is a nonradial involutive manifold.} \tag{2-2}$$

Here nonradial means that the radial direction $\langle \xi, \partial_\xi \rangle$ is not in the span of the Hamilton vector fields of the manifold, i.e., not equal to H_f on Σ_2 for some $f \in C^1$ vanishing at Σ_2. Then by a change of homogeneous symplectic coordinates we may assume that

$$\Sigma_2 = \{\xi' = 0\}, \quad \xi = (\xi', \xi'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} \tag{2-3}$$

for some $k > 0$; this can be achieved by conjugation by elliptic Fourier integral operators. If P is of principal type near Σ_2 then, since solvability is an open property, we find that a necessary condition for P to be solvable at Σ_2 is that condition (Ψ) for the principal symbol is satisfied in some neighborhood of Σ_2. Naturally, this condition is empty on Σ_2, where we instead need some conditions on the subprincipal symbol

$$p_s = p_{m-1} + \frac{i}{2}\sum_j \partial_{x_j}\partial_{\xi_j} p, \tag{2-4}$$

which is equal to p_{m-1} on Σ_2 and invariantly defined as a function on Σ_2 under symplectic changes of coordinates and conjugation with elliptic pseudodifferential operators. (In the Weyl quantization, the subprincipal symbol is equal to p_{m-1}.) When composing P with an elliptic pseudodifferential operator C, the value of the subprincipal symbol of CP is equal to $cp_s + \frac{1}{2} i H_p c = cp_s$ at Σ_2, where $c = \sigma(C)$. Observe that the subprincipal symbol is complexly conjugated when taking the adjoint of the operator.

Let $T\Sigma_2^\sigma$ be the symplectic polar to $T\Sigma_2$, which spans the symplectic leaves of Σ_2. If $\Sigma_2 = \{\xi' = 0\}$ and $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, then the leaves are spanned by $\partial_{x'}$. Let

$$T^\sigma \Sigma_2 = T\Sigma_2 / T\Sigma_2^\sigma, \tag{2-5}$$

which is a symplectic space over Σ_2, which in these coordinates is given by

$$T^\sigma \Sigma_2 = \big\{\big((x_0, 0, \xi_0''); (0, y'', 0, \eta'')\big) : (y'', \eta'') \in T^*\mathbb{R}^{n-k}\big\}. \tag{2-6}$$

Next, we are going to study the Hamilton vector field $H_{p_{m-1}}$ at Σ_2. If $H_{p_{m-1}} \subseteq T\Sigma_2$ at Σ_2 then we find that dp_s vanishes on $T\Sigma_2^\sigma$ so dp_s is well defined on $T^\sigma \Sigma_2$. In fact, $p_s = p_{m-1}$ on Σ_2 so if we choose coordinates so that (2-3) holds, then $H_{p_{m-1}} \subseteq T\Sigma_2$ is equivalent to

$$H_{p_{m-1}}\xi' = -\partial_{x'} p_{m-1} = -\partial_{x'} p_s = 0 \quad \text{when } \xi' = 0, \tag{2-7}$$

which is invariant under multiplication with nonvanishing factors when $p_s = 0$. Let H_{p_s} be the Hamilton vector field of p_s with respect the symplectic structure on the symplectic manifold $T^\sigma \Sigma_2$. In the chosen coordinates we have

$$H_{p_s} = \partial_{\xi''} p_s \partial_{x''} - \partial_{x''} p_s \partial_{\xi''}$$

modulo $\partial_{x'}$, which is nonvanishing if $\partial_{x''\xi''} p_s \neq 0$. Since $p_s = p_{m-1}$ on Σ_2, the difference between $H_{p_{m-1}}$ and H_{p_s} is tangent to the leaves of Σ_2. Actually, since the subprincipal symbol is only well defined on Σ_2, the vector field H_{p_s} is only well defined up to terms tangent to the leaves.

Because of that, we would need that the subprincipal symbol p_s is constant on the leaves of Σ_2, but that condition is *not* invariant under multiplication with nonvanishing factors when $p_s \neq 0$. Instead we shall use the invariant condition

$$\left|dp_s|_{TL}\right| \le C_0|p_s| \tag{2-8}$$

for any leaf L of Σ_2. Then p_s is constant on the leaves modulo nonvanishing factors, according to the following lemma.

Lemma 2.1. *If $dp_s|_{T^\sigma\Sigma_2} \ne 0$, then condition (2-8) is equivalent to the fact that p_s is constant on the leaves of Σ_2 after multiplication with a smooth nonvanishing factor. Thus, if $\Sigma_2 = \{\xi' = 0\}$ then (2-8) gives $p_s(x, 0, \xi'') = c(x, \xi'')q(x'', \xi'')$ with $0 \ne c \in C^\infty$.*

Proof. Choose coordinates so that $\Sigma_2 = \{\xi' = 0\}$. If $p_s \ne 0$ at a point $w_0 \in \Sigma_2$ then (2-8) gives that $\partial_{x'} \log p_s$ is uniformly bounded near w_0, where $\log p_s$ is a branch of the complex logarithm. Thus, by integrating with respect to x' in a simply connected neighborhood starting at $x' = x'_0$, we find that

$$p_s(x, 0, \xi'') = c(x, \xi'')q(x'', \xi''), \tag{2-9}$$

where $q(x'', \xi'') = p_s(x'_0, x'', 0, \xi'') \in C^\infty$, so $0 \ne c \in C^\infty$ satisfies $c(x'_0, x'', \xi'') = 1$. When $p_s = 0$ we find that $d\,\mathrm{Re}\, zp_s|_{T^\sigma\Sigma_2} \ne 0$ for some $z \in \mathbb{C}\setminus\{0\}$ by assumption. Thus we obtain locally that

$$p_s(x, 0, \xi'') = c_\pm(x, \xi'')q_\pm(x'', \xi'') \quad \text{on } S_\pm = \{\pm\,\mathrm{Re}\, zp_s(x, 0, \xi'') > 0\},$$

where $q_\pm(x'', \xi'') = p_s(x'_0, x'', 0, \xi'')$, $0 \ne c_\pm \in C^\infty$ and $c_\pm(x'_0, x'', \xi'') = 1$ on $S_\pm$. Then we find that $p_s^{-1}(0)$ is independent of x' and

$$\partial^\alpha_{x''}\partial^\beta_{\xi''} q_\pm(x'', \xi'') = \partial^\alpha_{x''}\partial^\beta_{\xi''} p_s(x'_0, x'', 0, \xi'') \quad \forall\,\alpha, \beta, \quad \text{on } S_\pm,$$

so by taking the limit at $S = \{\mathrm{Re}\, zp_s = 0\}$, we find that the functions $q_\pm$ extend to $q \in C^\infty$. Since $c_+q = c_-q = p_s$ at S, we find that $c_+ = c_-$ at S when $q \ne 0$. When $q = 0$ at S, we may differentiate in the normal direction of S to obtain $c_-\partial_\nu q = c_+\partial_\nu q$, and since $\partial_\nu q \ne 0$, the functions $c_\pm$ extend to a continuous function c. By differentiating and taking the limit, we find that

$$\nabla c_- q + c\nabla q = \nabla p_s = \nabla c_+ q + c\nabla q \quad \text{at } S,$$

which similarly gives that $\nabla c_- = \nabla c_+$ at S, so $c \in C^1$. By repeatedly differentiating $c_\pm q$, we find by induction that $c \in C^\infty$, so we get smooth quotients c and q in (2-9). □

Now, a semibicharacteristic of p_s will be a bicharacteristic of $\mathrm{Re}\, ap_s$ on $T^\sigma\Sigma_2$, where $C^\infty \ni a \ne 0$, with the natural orientation. Observe that condition (2-7) is only invariant under multiplication with nonvanishing factors when $p_s = 0$.

Definition 2.2. We say that the operator P is of *subprincipal type* if, when $p_s = 0$ on Σ_2, the following hold: $H_{p_{m-1}}|_{\Sigma_2} \subseteq T\Sigma_2$,

$$dp_s|_{T^\sigma\Sigma_2} \ne 0, \tag{2-10}$$

and the corresponding Hamilton vector field H_{p_s} of (2-10) does not have the radial direction. We call H_{p_s} the *subprincipal Hamilton vector field* and the (semi)bicharacteristics are called the *subprincipal (semi)bicharacteristics* on Σ_2.

Clearly, if (2-3) holds, then the condition that the Hamilton vector field does not have the radial direction means that $\partial_{\xi''} p_s \neq 0$ or $\partial_{x''} p_s \not\parallel \xi''$ when $p_s = 0$ on $\Sigma_2 = \{\xi' = 0\}$.

In the case when the principal symbol p is real, a necessary condition for solvability of the operator is that the imaginary part of the subprincipal symbol does not change sign from $-$ to $+$ when going in the positive direction on a C^∞ limit of normalized bicharacteristics of the principal symbol p at Σ_2; see [Dencker 2016]. When p vanishes of exactly second order on $\Sigma_2 = \{\xi' = 0\}$, such limit bicharacteristics are tangent to the leaves of Σ_2. In fact, then Taylor's formula gives $H_p = \langle B\xi', \partial_{x'}\rangle + \mathcal{O}(|\xi'|^2)$, where $B \neq 0$, so the normalized Hamilton vector fields have limits that are tangent to the leaves. When the principal symbol is proportional to a real-valued symbol, this gives examples of nonsolvability when the subprincipal symbol is not constant on the leaves of Σ_2. Thus condition (2-8) is natural if there are no other conditions on the principal symbol.

Remark 2.3. If p_s is real-valued, then by the proof of Lemma 2.1 it follows from (2-8) that p_s has constant sign on the leaves of Σ_2, since then $c > 0$ in (2-9).

Definition 2.4. We say that P satisfies condition Sub(Ψ) if $\operatorname{Im} ap_s$ does not change sign from $-$ to $+$ when going in the positive direction on the subprincipal bicharacteristics of $\operatorname{Re} ap_s$ for any $0 \neq a \in C^\infty$.

Thus, condition Sub(Ψ) is condition (Ψ) given by (1-2) on the subprincipal symbol p_s. Observe that since p_s is only defined on Σ_2, the Hamilton vector field H_{p_s} is only well defined in $T^\sigma\Sigma_2 = T\Sigma_2/T\Sigma_2^\sigma$; thus it is well defined modulo $\partial_{x'}$. But if (2-8) holds then we find that Sub(Ψ) is a condition on p_s with respect to the symplectic structure of $T^\sigma\Sigma_2$. In fact, by the invariance of condition (Ψ) given by [Hörmander 1985b, Lemma 26.4.10], condition Sub(Ψ) holds for any $a \neq 0$ such that $H_{\operatorname{Re} ap_s} \neq 0$, so we may assume by Lemma 2.1 that p_s is constant on the leaves of Σ_2.

Since condition Sub(Ψ) is invariant under symplectic changes of variables and multiplication with nonvanishing functions, it is invariant under conjugation of the operator by elliptic Fourier integral operators. Observe that the sign change is reversed when taking the adjoint of the operator.

Recall that the Hessian of the principal symbol Hess p is the quadratic form given by $\partial^2 p$ at Σ_2, which is defined on the normal bundle $N\Sigma_2$ since it vanishes on $T\Sigma_2$. By the calculus, Hess p is invariant, modulo nonvanishing smooth factors, under symplectic changes of variables and multiplication of P with elliptic pseudodifferential operators.

Next, we assume that condition Sub(Ψ) is not satisfied on a semibicharacteristic Γ of p_s; that is, $\operatorname{Im} ap_s$ changes sign from $-$ to $+$ on the positive flow-out of $H_{\operatorname{Re} ap_s} \neq 0$ for some $0 \neq a \in C^\infty$. Now if the sign change is *not* of finite order, we shall also need an extra condition on the rate of vanishing of both the Hessian of the principal symbol and the complex part of the gradient of the subprincipal symbol on the subprincipal semibicharacteristic. Then, we shall assume that there exists $C > 0$, $\varepsilon > 0$ and $0 \neq a \in C^\infty$ so that $d \operatorname{Re} ap_s|_{T\Sigma_2} \neq 0$ and

$$\|\text{Hess}\, p\| + |dp_s \wedge d\overline{p_s}| \leq C|p_s|^\varepsilon \quad \text{when } \operatorname{Re} ap_s = 0 \text{ on } \Sigma_2 \tag{2-11}$$

near Γ. Since (2-11) also holds for smaller ε and larger C, it is no restriction to assume $\varepsilon \leq 1$. The motivation for (2-11) is to prevent the transport equation (6-1) from dispersing the support of the solution before the sign change of the imaginary part of the subprincipal symbol localizes it; see Remark 3.1. We also find that ∇p_s is proportional to a real vector when $p_s = 0$ since then $dp_s \wedge d\overline{p_s} = 0$.

Remark 2.5. Condition (2-11) is invariant under multiplication of P with elliptic pseudodifferential operators, and symplectic changes of coordinates. If (2-8) also holds, then we obtain

$$\left\| d \operatorname{Hess} p|_{TL} \right\| \leq C_1 |p_s|^{\varepsilon/2} \tag{2-12}$$

for any leaf L of Σ_2 when $\operatorname{Re} ap_s = 0$ near Γ.

In fact, multiplication with an elliptic pseudodifferential operator with principal symbol c changes the principal symbol into cp, the Hessian of the principal symbol into $c \operatorname{Hess} p$ and the subprincipal symbol into

$$cp_s + \frac{i}{2} H_p c \quad \text{at } \Sigma_2,$$

where the last term vanishes at Σ_2 and contains the factor $\operatorname{Hess} p$, modulo terms vanishing of second order at Σ_2. Now we have

$$|dcp_s \wedge d\overline{cp_s}| \leq |c|^2 |dp_s \wedge d\overline{p_s}| + C|p_s|.$$

Thus we find that (2-11) holds with p replaced by cp, p_s replaced by cp_s and a replaced with a/c. If (2-8) also holds and we choose coordinates so that $\Sigma_2 = \{\xi' = 0\}$, then we obtain from Lemma 2.1 that $|p_s(x', x'', 0, \xi'')| \cong |p_s(x_0', x'', 0, \xi'')|$ when $|x' - x_0'| \leq c$. Thus (2-11) gives

$$\|\operatorname{Hess} p(x', x'', 0, \xi'')\| \leq C_2 |p_s(x_0', x'', 0, \xi'')|^{\varepsilon} \quad \text{when } |x' - x_0'| \leq c.$$

To show (2-12) it suffices to consider an element $b_{jk}(x', x'', 0, \xi'')$ of $\operatorname{Hess} p$. Clearly $|b_{jk}| \leq \|\operatorname{Hess} p\|$, so by adding $C_2|p_s(x_0, x'', 0, \xi'')|^{\varepsilon}$, we obtain

$$0 \leq b_{jk}(x', x'', 0, \xi'') \leq 2C_2 |p_s(x_0, x'', 0, \xi'')|^{\varepsilon} \quad \text{when } |x' - x_0'| \leq c.$$

Then we find that

$$|\partial_{x'} b_{jk}(x_0, x'', 0, \xi'')| \leq C\sqrt{b_{jk}(x_0, x'', 0, \xi'')} \leq C' |p_s(x_0, x'', 0, \xi'')|^{\varepsilon/2}$$

by [Hörmander 1983, Lemma 7.7.2].

We shall study the microlocal solvability of the operator, which is given by the following definition. Recall that $H_{(s)}^{\mathrm{loc}}(X)$ is the set of distributions that are locally in the L^2 Sobolev space $H_{(s)}(X)$.

Definition 2.6. If $K \subset S^*X$ is a compact set, then we say that P is microlocally solvable at K if there exists an integer N so that for every $f \in H_{(N)}^{\mathrm{loc}}(X)$ there exists $u \in \mathcal{D}'(X)$ such that $K \cap \mathrm{WF}(Pu - f) = \varnothing$.

Observe that solvability at a compact set $M \subset X$ is equivalent to solvability at $S^*X|_M$ by [Hörmander 1985b, Theorem 26.4.2], and that solvability at a set implies solvability at a subset. Also, by [Hörmander 1985b, Proposition 26.4.4] the microlocal solvability is invariant under conjugation by elliptic Fourier integral operators and multiplication by elliptic pseudodifferential operators. We can now state the main result of the paper.

Theorem 2.7. *Assume that $P \in \Psi^m_{\mathrm{cl}}(X)$ has principal symbol that vanishes of at least second order at a nonradial involutive manifold Σ_2, is of subprincipal type, does not satisfy condition* Sub(Ψ) *on the subprincipal semibicharacteristic $\Gamma \subset \Sigma_2$, and satisfies (2-8) near Γ. In the case the sign change in* Sub(Ψ) *is of infinite order, we also assume condition (2-11) near Γ. Then P is not locally solvable at Γ.*

Example 2.8. Let

$$P = D_1 D_2 + B(x, D_x) \tag{2-13}$$

with $B \in \Psi^1_{\mathrm{cl}}$. Then $\sigma(B)$ is the subprincipal symbol on $\Sigma_2 = \{\xi_1 = \xi_2 = 0\}$. Mendoza and Uhlmann [1983] proved that P is not solvable if $\operatorname{Im}\sigma(B)$ changes sign as x_1 or x_2 increases on Σ_2, and they proved in [Mendoza and Uhlmann 1984] that P is solvable if $\operatorname{Im}\sigma(B) \neq 0$ on Σ_2. From this it is natural to conjecture that the condition for solvability of P is that $\operatorname{Im}\sigma(B)$ does not change sign on the leaves of Σ, which are foliated by ∂_{x_1} and ∂_{x_2}. But the following is a counterexample to that conjecture. Let

$$P = D_1 D_2 + D_t + if(t, x, D_x) \tag{2-14}$$

with real and homogeneous $f(t, x, \xi) \in S^1_{\mathrm{hom}}$ satisfying $\partial_{x_j} f = \mathcal{O}(|f|)$ for $j = 1, 2$. This operator is of subprincipal type and satisfies (2-8). Then Theorem 2.7 gives that P is not solvable if $t \mapsto f(t, x, \xi)$ changes sign of finite order from $-$ to $+$, but observe that f has constant sign on the leaves of Σ_2 by Remark 2.3. Thus the solvability of the operator P in (2-13) also depends on the real part of the subprincipal symbol at Σ_2. In fact, with the above conditions one can prove that $D_1 D_2 + if(t, x, D_x)$ is solvable.

Example 2.9. The linearized Navier–Stokes equation

$$\partial_t u + \sum_j a_j(t, x)\partial_{x_j} u + \Delta_x u = f, \quad a_j(x) \in C^\infty, \tag{2-15}$$

is of subprincipal type. The symbol is

$$i\tau + i\sum_j a_j(t, x)\xi_j - |\xi|^2, \tag{2-16}$$

so the subprincipal symbol is proportional to a real symbol on $\Sigma_2 = \{\xi = 0\}$. Thus condition Sub(Ψ) is satisfied.

Now let $S^*M \subset T^*M$ be the cosphere bundle where $|\xi| = 1$, and let $\|u\|_{(k)}$ be the L^2 Sobolev norm of order k, $u \in C_0^\infty$. In the following, P^* will be the L^2 adjoint of P. To prove Theorem 2.7 we shall use the following result.

Remark 2.10. If P is microlocally solvable at $\Gamma \subset S^*\mathbb{R}^n$, then [Hörmander 1985b, Lemma 26.4.5] gives that for any $Y \Subset \mathbb{R}^n$ such that $\Gamma \subset S^*Y$, there exists an integer ν and a pseudodifferential operator A so that $\mathrm{WF}(A) \cap \Gamma = \emptyset$ and

$$\|u\|_{(-N)} \le C\big(\|P^*u\|_{(\nu)} + \|u\|_{(-N-n)} + \|Au\|_{(0)}\big), \quad u \in C_0^\infty(Y), \tag{2-17}$$

where N is given by Definition 2.6.

We shall prove Theorem 2.7 in Section 8 by constructing localized approximate solutions to $P^*u \cong 0$ and use (2-17) to show that P is not microlocally solvable at Γ. We shall first find a normal form for the adjoint operator.

3. The normal form

Assume that P^* has the symbol expansion $p_m + p_{m-1} + \cdots$, where $p_j \in S^j_{\mathrm{hom}}$ is homogeneous of degree j. By multiplying P^* with an elliptic pseudodifferential operator, we may assume that $m = 2$. Choose local symplectic coordinates $(t, x, y, \tau, \xi, \eta)$ so that $\Sigma_2 = \{\eta = 0\}$, which is foliated by leaves spanned by ∂_y. Since p_2 vanishes of at least second order at Σ_2, we find that

$$p_2(t, x, y, \tau, \xi, \eta) = \sum_{jk} B_{jk}(t, x, y, \tau, \xi, \eta)\eta_j\eta_k,$$

where B_{jk} is homogeneous of degree 0 for all j, k.

The differential inequality (2-8) in these coordinates means that $|\partial_y p_1| \le C|p_1|$ when $\eta = 0$, which by Lemma 2.1 gives that

$$p_1(t, x, y, \tau, \xi, 0) = q(t, x, y, \tau, \xi)r_1(t, x, \tau, \xi)$$

near Γ, where q is a nonvanishing smooth homogeneous function. By multiplying with pseudodifferential operators with principal symbol equal to q^{-1} on Σ_2, we may assume that $q \equiv 1$ and that p_1 is constant on the leaves of Σ_2. The Hamilton vector field of p_1 is then tangent to Σ_2 by (2-7).

We have assumed that P does not satisfy condition $\mathrm{Sub}(\Psi)$ on a semibicharacteristic Γ of p_1 on Σ_2. Since we are now considering the adjoint P^* this means that $\mathrm{Im}\, ap_1$ changes sign from $+$ to $-$ on the flow-out Γ of $H_{\mathrm{Re}\, ap_1}$ on $\mathrm{Re}\, ap_1^{-1}(0)$ for some $0 \ne a \in C^\infty$. By the invariance of condition $\mathrm{Sub}(\Psi)$ given by [Hörmander 1985b, Lemma 26.4.10], it is no restriction to assume that a is homogeneous and constant in y. By multiplication with an elliptic pseudodifferential operator having principal symbol a^{-1}, we may assume that $a \equiv 1$. Since $\mathrm{Im}\, p_1$ changes sign on Γ, there is a maximal semibicharacteristic $\Gamma' \subset \Gamma$ on which $\mathrm{Im}\, p_1 = 0$. Here Γ' could be a point, which is always the case if the sign change is of finite order.

Since P is of subprincipal type, we find that $\partial_{t,x,\tau,\xi} \mathrm{Re}\, p_1 \ne 0$ on Γ' by (2-10), so Γ' is transversal to the leaves of Σ_2. Since $\mathrm{Im}\, p_1|_\Gamma$ has opposite signs near the boundary of Γ', we may shrink Γ so that it is not a closed curve. Since $H_{\mathrm{Re}\, p_1}$ is tangent to Σ_2, we can complete $\tau = \mathrm{Re}\, p_1$ to a symplectic coordinate system in a convex neighborhood of Γ' so that $\eta = 0$ on Σ_2, which preserves the leaves. In fact, this is obtained by solving the equation $H_\tau \eta = 0$ with initial value on a submanifold transversal to H_τ. The change of variables can be then done by conjugation with suitable elliptic Fourier integral operators.

Now, using Malgrange's preparation theorem in a neighborhood of Γ' in Σ_2, we find that

$$p_1(t, x, y, \tau, \xi, 0) = q(t, x, \tau, \xi)(\tau + r(t, x, \xi)), \qquad q \ne 0,$$

near Γ, since p_1 is constant on the leaves of Σ_2. In fact, on Γ' we have $p_1 = 0$ and $dp_1 \ne 0$, so the division can be done locally and by a partition of unity globally near Γ after possibly shrinking Γ. Then using Taylor's formula on p_1, we find since $q \neq 0$ that

$$p_1(t, x, y, \tau, \xi, \eta) = q(t, x, \tau, \xi)\big(\tau + r(t, x, \xi) + A(t, x, y, \tau, \xi, \eta) \cdot \eta\big). \tag{3-1}$$

By multiplying P with an elliptic pseudodifferential operator, we may again assume $q \equiv 1$. Since p_2 vanishes of second order at Σ_2, this only changes A with terms which have Hess p_2 as a factor and terms that vanish at Σ_2.

We can write $r = r_1 + ir_2$ and $A = A_1 + iA_2$ with real-valued r_j and A_j, $j = 1, 2$. Now we may complete

$$\operatorname{Re} p_1 = \tau + r_1(t, x, \xi) + A_1(t, x, y, \tau, \xi, \eta) \cdot \eta$$

to a symplectic coordinate system in a convex neighborhood of Γ'. Since $H_{\operatorname{Re} p_1} \in T\Sigma_2$ at Σ_2, we may keep $\Sigma_2 = \{\eta = 0\}$, which preserves the leaves of Σ_2 on which p_1 is constant. Thus, we find that

$$p_1 = \tau + if(t, x, \xi) + iA(t, x, y, \tau, \xi, \eta) \cdot \eta, \tag{3-2}$$

where $f = r_2$ and $A = A_2$ are real-valued. We also find that

$$\Gamma = \{(t, x_0, y_0, 0, \xi_0, 0)\}, \quad t \in I, \tag{3-3}$$

where I is an interval in $\mathbb{R}$. The symplectic change of coordinates can be made by conjugation with elliptic Fourier integral operators, which only changes A with terms having Hess p_2 as a factor and terms that vanish at Σ_2. Observe that A need not be real-valued after these changes.

We have assumed that condition $\operatorname{Sub}(\Psi)$ is *not* satisfied for P on the subprincipal semibicharacteristic Γ. Thus the imaginary part of the subprincipal symbol of P^* on Σ_2

$$t \mapsto f(t, x_0, \xi_0) \tag{3-4}$$

changes sign from $+$ to $-$ as t increases on $I \subset \mathbb{R}$. Similarly, we have $f = 0$ on Γ', where Γ' is given by (3-3) with I replaced by $I' \subset I$. By reducing to *minimal bicharacteristics* on which $t \mapsto f(t, x, \xi)$ changes sign as in [Hörmander 1981, p. 75], we may assume that f vanishes of infinite order on a bicharacteristic Γ' arbitrarily close to the original bicharacteristic, if Γ' is not a point (see [Wittsten 2012, Section 2] for a more refined analysis). If Γ' is not a point then it is a *one-dimensional bicharacteristic* by [Hörmander 1981, Definition 3.5], which means that the Hamilton vector field on Γ' is proportional to a real vector.

In fact, if $f(a, x_0, \xi_0) > 0 > f(b, x_0, \xi_0)$ for some $a < b$, then we can define

$$L(x, \xi) = \inf\big\{t - s : a < s < t < b \text{ such that } f(s, x, \xi) > 0 > f(t, x, \xi)\big\}$$

when (x, ξ) is close to (x_0, ξ_0), and we put $L_0 = \liminf_{(x,\xi)\to(x_0,\xi_0)} L(x, \xi)$. Then for every $\varepsilon > 0$ there exists an open neighborhood V_ε of (x_0, ξ_0) such that the diameter of V_ε is less than ε and $L(x, \xi) > L_0 - \varepsilon/2$ when $(x, \xi) \in V_\varepsilon$. By definition, there exists $(x_\varepsilon, \xi_\varepsilon) \in V_\varepsilon$ and $a < s_\varepsilon < t_\varepsilon < b$ so that $t_\varepsilon - s_\varepsilon < L_0 + \varepsilon/2$ and $f(s_\varepsilon, x_\varepsilon, \xi_\varepsilon) > 0 > f(t_\varepsilon, x_\varepsilon, \xi_\varepsilon)$. Then it is easy to see that

$$\partial_x^\alpha \partial_\xi^\beta f(t, x_\varepsilon, \xi_\varepsilon) = 0 \quad \forall \alpha, \beta \text{ when } s_\varepsilon + \varepsilon < t < t_\varepsilon - \varepsilon \tag{3-5}$$

since else we would have a sign change in an interval with length less than $L_0 - \varepsilon/2$ in V_ε. We may choose a sequence $\varepsilon_j \to 0$ so that $s_{\varepsilon_j} \to s_0$ and $t_{\varepsilon_j} \to t_0$. Then $L_0 = t_0 - s_0$ and (3-5) holds at (x_0, ξ_0) for $s_0 < t < t_0$.

We also obtain the following condition from (2-11).

Remark 3.1. If the sign change of $t \mapsto f(t, x, \xi)$ is of infinite order on Γ, then we find from assumption (2-11) that

$$\|\{B_{jk}\}_{jk}\| + |A| + |df| \lesssim |f|^\varepsilon \quad \text{near } \Gamma \text{ on } \Sigma_2 \tag{3-6}$$

for some $\varepsilon > 0$. Here $a \lesssim b$ (and $b \gtrsim a$) means that $a \le Cb$ for some $C > 0$.

In fact, terms having Hess $p_2|_{\Sigma_2} = \{B_{jk}\}_{jk}$ as a factor can be estimated by (2-11), so we may assume that (3-2) holds with real A. The subprincipal symbol is equal to $p_s = p_1 + i \sum_{jk} \partial_{y_j} B_{jk}\eta_k$ modulo terms that are $\mathcal{O}(|\eta|^2)$, so $p_s = p_1$ on Σ_2. By Remark 2.5 and (2-8), we can estimate the terms $\partial_{y_j} B_{jk} d\eta_k$ in dp_s by replacing ε with $\varepsilon/2$ in (2-11), so we may replace p_s by p_1 in the estimate. Let $0 \ne a = a_1 + ia_2$ with real-valued a_j in (2-11) so that $d\,\mathrm{Re}\, ap_1|_{T\Sigma_2} \ne 0$. We have $dp_1 = d\tau + i(df + Ad\eta)$ on Σ_2, so

$$|dp_1 \wedge d\overline{p_1}| \cong |df| + |A| \quad \text{on } \Sigma_2.$$

Thus we find from (2-11) that $|df| + |A| = 0$ on Γ'. Since $d\,\mathrm{Re}\, ap_1|_{T\Sigma_2} \ne 0$, we find that $a_1 \ne 0$ on Γ'. On Σ_2 we have $\mathrm{Re}\, ap_1 = a_1\tau - a_2 f = 0$ when $\tau = a_2 f/a_1$. We obtain

$$\mathrm{Im}\, ap_1 = a_2\tau + a_1 f = |a|^2 f/a_1 \quad \text{when } \mathrm{Re}\, ap_1 = 0 \text{ on } \Sigma_2 \text{ near } \Gamma',$$

which gives (3-6) from (2-11).

We obtain the following normal form for these operators of subprincipal type:

$$P^* = D_t + F(t, x, y, D_t, D_x, D_y), \tag{3-7}$$

where $F \sim F_2 + F_1 + \cdots$ with homogeneous $F_j \in C^\infty(\mathbb{R}, S^j_{\mathrm{hom}})$. Here F_2 vanishes of at least second order on $\Sigma_2 = \{\eta = 0\}$, so we find by Taylor's formula that

$$F_2(t, x, y, \tau, \xi, \eta) = B(t, x, y, \tau, \xi, \eta) = \sum_{jk} B_{jk}(t, x, y, \tau, \xi, \eta)\eta_j\eta_k \tag{3-8}$$

with homogeneous B_{jk}. Then $\{B_{jk}\}_{jk}|_{\Sigma_2} = \mathrm{Hess}\, F_2(t, x, y, \tau, \xi, 0)$. Also we have that F_1 vanishes on the semibicharacteristic Γ' and

$$F_1(t, x, y, \tau, \xi, \eta) = if(t, x, \xi) + A(t, x, y, \tau, \xi, \eta) \cdot \eta. \tag{3-9}$$

Here f is real and homogeneous of degree 1 and $A|_{\Sigma_2} = \partial_\eta F_1|_{\Sigma_2}$. We have that the principal symbol $\sigma(P^*)$ is equal to F_2, and the subprincipal symbol $\sigma_{\mathrm{sub}}(P^*)$ is equal to $\tau + if$ on Σ_2. Thus we obtain the following result.

Proposition 3.2. *Assume that P satisfies the conditions in Theorem 2.7. Then by conjugation with elliptic Fourier integral operators and multiplication with an elliptic pseudodifferential operator, we may assume*

$$P^* = D_t + F(t, x, y, D_t, D_x, D_y) \tag{3-10}$$

microlocally near $\Gamma = \{(t, x_0, y_0, 0, \xi_0, 0) : t \in I\} \subset \Sigma_2$, *where* $S^2_{\mathrm{cl}} \ni F \cong F_2 + F_1 + \cdots$ *with* $F_j \in S^j_{\mathrm{hom}}$ *is homogeneous of degree* j *and*

$$F_2(t, x, y, \tau, \xi, \eta) = \sum_{jk} B_{jk}(t, x, y, \tau, \xi, \eta)\eta_j\eta_k \in S^2_{\mathrm{hom}}$$

vanishes of second order on Σ_2. *We may also assume*

$$F_1(t, x, y, \tau, \xi, \eta) = if(t, x, \xi) + A(t, x, y, \tau, \xi, \eta) \cdot \eta$$

is homogeneous of degree 1 *and* f *is real-valued such that* $t \mapsto f(t, x_0, \xi_0)$ *changes sign from* $+$ *to* $-$ *as* t *increases on* $I \subset \mathbb{R}$. *If* $f(t, x_0, \xi_0) = 0$ *on a subinterval* $I' \subseteq I$ *such that* $|I'| \neq 0$, *then we may assume that* $\partial_t^k \partial_x^\alpha \partial_\xi^\beta f(t, x_0, \xi_0) = 0$ *for all* k, α, β, *for* $t \in I'$. *If the sign change of* f *is of infinite order then* (3-6) *is satisfied near* Γ.

For the proof of Theorem 2.7, we shall modify the Moyer–Hörmander construction of approximate solutions of the type

$$u_\lambda(t, x, y) = e^{i\lambda\omega(t,x,y)} \sum_{j\geq 0} \phi_j(t, x, y)\lambda^{-j/N}, \quad \lambda \geq 1, \tag{3-11}$$

with N to be determined later. Here the phase function $\omega(t, x)$ will be complex-valued, but $\operatorname{Im}\omega \geq 0$ and $\partial \operatorname{Re}\omega \neq 0$ when $\operatorname{Im}\omega = 0$. Letting $z = (t, x, y)$, we therefore have the formal expansion

$$p(z, D)(\exp(i\lambda\omega)\phi) \sim \exp(i\lambda\omega) \sum_\alpha \partial_\zeta^\alpha p(z, \lambda\partial_z\omega(z))\mathcal{R}_\alpha(\omega, \lambda, D)\phi(z)/\alpha!, \tag{3-12}$$

where $\mathcal{R}_\alpha(\omega, \lambda, D)\phi(z) = D_w^\alpha\big(\exp(i\lambda\tilde{\omega}(z, w))\phi(w)\big)\big|_{w=z}$ and

$$\tilde{\omega}(z, w) = \omega(w) - \omega(z) + (z - w)\partial\omega(z).$$

Observe that the values of the symbol are given by an almost analytic extension; see Theorem 3.1 in Chapter VI and Chapter X:4 in [Trèves 1980]. This gives

$$\begin{aligned} e^{-i\lambda\omega}P^*e^{i\lambda\omega}\phi = \big(&\lambda\partial_t\omega + \lambda^2 B(t, x, y, \partial_{t,x,y}\omega) + i\lambda f(t, x, \partial_x\omega) - \lambda\partial_\eta^2 B(t, x, y, \partial_{t,x,y}\omega)\partial_y^2\omega/2\big)\phi \\ &+ D_t\phi + \lambda\partial_\eta B(t, x, y, \partial_{t,x,y}\omega)D_y\phi + \partial_\eta^2 B(t, x, y, \partial_{t,x,y}\omega)D_y^2\phi/2 \\ &+ i\partial_\xi f(t, x, \partial_x\omega)D_x\phi + A(t, x, y, \partial_{t,x,y}\omega)D_y\phi + \sum_{j\geq 0}\lambda^{-j}R_j(t, x, y, D_{t,x,y})\phi, \end{aligned} \tag{3-13}$$

where $R_0(t, x, y) = F_0(t, x, y, \partial_{t,x,y}\omega)$. Here the values of the symbols at $(t, x, y, \partial_{t,x,y}\omega)$ will be replaced by finite Taylor expansions at $(t, x, y, \partial_{t,x,y}\operatorname{Re}\omega)$. In fact, the almost analytic extensions are determined by these Taylor expansions.

Because of the inhomogeneity coming from the terms of B, we shall use a phase function $\omega(t, x)$ which is constant in y so that

$$u_\lambda(t, x, y) = e^{i\lambda\omega(t,x)} \sum_{j\geq 0} \phi_j(t, x, y)\lambda^{-j/N}, \quad \lambda \geq 1. \tag{3-14}$$

When $\partial_y \omega \equiv 0$ the expansion (3-13) becomes

$$e^{-i\lambda\omega} P^* e^{i\lambda\omega}\phi = \lambda(\partial_t\omega + i f(t,x,\partial_x\omega))\phi + D_t\phi + \partial_\eta^2 B(t,x,y,\partial_{t,x}\omega,0) D_y^2\phi/2 \\ + A(t,x,y,\partial_{t,x}\omega,0) D_y\phi + i\partial_\xi f(t,x,\partial_x\omega) D_x\phi + \sum_{j\ge 0} \lambda^{-j} R_j(t,x,y,D_{t,x,y})\phi, \quad (3\text{-}15)$$

where $R_0(t,x,y) = F_0(t,x,y,\partial_{t,x}\omega,0)$, and $R_m(t,x,y,D_{t,x,y})$ are differential operators of order j in t, order k in x and order ℓ in y, where $j+k+\ell \le m+2$ for $m>0$. In fact, this follows since $\partial_\tau^j \partial_\xi^\alpha \partial_\eta^\beta F_k \in S^{k-j-|\alpha|-|\beta|}$ by homogeneity.

4. The eikonal equation

We shall first solve the eikonal equation approximately, which is given by the highest-order term of (3-15)

$$\partial_t\omega + i f(t,x,\partial_x\omega) = 0, \quad (4\text{-}1)$$

where $t \mapsto f(t,x,\xi)$ changes sign from $+$ to $-$ for some (x,ξ) as t increases in a neighborhood of $\Gamma = \{(t,x_0,\xi_0) : t \in I\}$ on which $f(t,x,\xi)$ vanishes. If $|I| \ne 0$ then by reducing to minimal bicharacteristics as in Section 3, we may assume that f vanishes of infinite order at Γ. We shall choose the phase function so that $\operatorname{Im}\omega \ge 0$ and $\partial_x^2 \operatorname{Im}\omega > 0$ near the interval. By changing coordinates, it is no restriction to assume $0 \in I$. We shall use the approach by Hörmander [1981] in the principal-type case and use the phase function to localize in t and x. Observe that since ω does not depend on y, the localization in the y-variables will be done in the amplitude ϕ.

We shall take the Taylor expansion of ω in x:

$$\omega(t,x) = w_0(t) + \langle x - x_0(t), \xi_0(t)\rangle + \sum_{2\le|\alpha|\le K} w_\alpha(t)(x-x_0(t))^\alpha/\alpha!. \quad (4\text{-}2)$$

Here $\alpha = (\alpha_1,\alpha_2,\dots)$, with $\alpha_j \in \mathbb{N}$, $\alpha! = \prod_j \alpha_j!$ and $|\alpha| = \alpha_1+\alpha_2+\cdots$. Then we find that

$$\partial_t\omega(t,x) = w_0'(t) - \langle x_0'(t), \xi_0(t)\rangle + \langle x - x_0(t), \xi_0'(t)\rangle \\ + \sum_{2\le|\alpha|\le K} w_\alpha'(t)(x-x_0(t))^\alpha/\alpha! - \sum_{\substack{1\le|\alpha|\le K-1 \\ k}} w_{\alpha+e_k}(t)(x-x_0(t))^\alpha x_{0,k}'(t)/\alpha!, \quad (4\text{-}3)$$

where $e_k = (0,\dots,0,1,0,\dots,0)$ is the k-th unit vector. We also find

$$\partial_{x_j}\omega(t,x) = \xi_{0j}(t) + \sum_{1\le|\alpha|\le K-1} w_{\alpha+e_j}(t)(x-x_0(t))^\alpha/\alpha! = \xi_{0j}(t) + \sigma_j(t,x). \quad (4\text{-}4)$$

Here $\xi_0(t) = (\xi_{0,1}(t),\dots)$ and $\sigma = \{\sigma_j\}_j$ is a finite expansion in powers of $\Delta x = x - x_0$. We define the value of $f(t,x,\partial_x\omega)$ by the Taylor expansion

$$f(t,x,\partial_x\omega) = f(t,x,\xi_0+\sigma) \\ = f(t,x,\xi_0) + \sum_j \partial_{\xi_j} f(t,x,\xi_0)\sigma_j + \sum_{jk} \partial_{\xi_j}\partial_{\xi_k} f(t,x,\xi_0)\sigma_j\sigma_k/2 + \cdots. \quad (4\text{-}5)$$

Now the value at $x = x_0$ of (4-1) is equal to $w_0'(t) - \langle x_0'(t), \xi_0(t)\rangle + if(t, x_0(t), \xi_0(t))$. This vanishes if

$$\begin{cases} \operatorname{Re} w_0'(t) = \langle x_0'(t), \xi_0(t)\rangle, \\ \operatorname{Im} w_0'(t) = -f(t, x_0(t), \xi_0(t)), \end{cases} \tag{4-6}$$

so by putting $w_0(0) = 0$, this will determine w_0 once we have $(x_0(t), \xi_0(t))$.

We shall simplify the notation and put $w_k = \{w_\alpha k!/\alpha!\}_{|\alpha|=k}$ so that w_k is a multilinear form. The first-order terms in $x - x_0$ of (4-1) vanish if

$$\xi_0'(t) - w_2(t)x_0'(t) + i\big(\partial_x f(t, x_0(t), \xi_0(t)) + \partial_\xi f(t, x_0(t), \xi_0(t))w_2(t)\big) = 0.$$

We find by taking real and imaginary parts that

$$\begin{cases} \xi_0' = \operatorname{Re} w_2 x_0' + \partial_\xi f(t, x_0, \xi_0) \operatorname{Im} w_2, \\ x_0' = (\operatorname{Im} w_2)^{-1}(\partial_x f(t, x_0, \xi_0) + \partial_\xi f(t, x_0, \xi_0) \operatorname{Re} w_2) \end{cases} \tag{4-7}$$

with $(x_0(0), \xi_0(0)) = (x_0, \xi_0)$, which will determine $x_0(t)$ and $\xi_0(t)$ if $|\operatorname{Im} w_2| \neq 0$.

The second-order terms in $x - x_0$ vanish if

$$w_2'/2 - w_3 x_0'/2 + i\big(\partial_\xi f w_3/2 + \partial_x^2 f/2 + \partial_x \partial_\xi f w_2 + w_2 \partial_\xi^2 f w_2/2\big) = 0,$$

which gives

$$w_2' = w_3 x_0' - i\big(\partial_\xi f w_3 + \partial_x^2 f + 2\partial_x \partial_\xi f w_2 + w_2 \partial_\xi^2 f w_2\big) \tag{4-8}$$

with initial data $w_2(0)$ such that $\operatorname{Im} w_2(0) > 0$.

We find that the terms of order $k > 2$ vanish if

$$w_k' - w_{k+1} x_0' = F_k(t, x_0, \xi_0, \{w_j\}), \tag{4-9}$$

where we may choose $w_k(0) = 0$. Here F_k is a linear combination of the derivatives of f of order $\le k$ multiplied by polynomials in w_j with $2 \le j \le k+1$. When $k = K$ we get $w_K' = F_K(t, x_0, \xi_0, \{w_j\})$, where $j \le K$. The equations (4-7)–(4-9) form a quasilinear system of differential equations, which can be solved in a convex neighborhood of 0. In the case when $|I| \neq 0$, we have assumed that $\partial_{t,x,\xi}^\alpha f(t, x_0, \xi_0) \equiv 0$ for all α, for $t \in I$. Then we find from (4-7)–(4-9) that x_0, ξ_0 and w_k are constant in $t \in I$, so we may solve (4-7)–(4-9) in a convex neighborhood of I. Observe that the higher-order terms cannot change the condition that $\operatorname{Im} \partial_x^2 \omega \ge c > 0$ and $\operatorname{Im} \omega(t, x) \ge 0$ if $|x - x_0(t)| \ll 1$. Summing up, we have proved the following result.

Proposition 4.1. *Let $\Gamma = \{(t, x_0, \xi_0) : t \in I\}$ and assume that $\partial_t^k \partial_x^\alpha \partial_\xi^\beta f(t, x_0, \xi_0) = 0$ for all $t \in I$ in the case $|I| \neq 0$. Then we may solve (4-1) with $\omega(t, x)$ given by (4-2) in a convex neighborhood Ω of Γ modulo $\mathcal{O}(|x - x_0(t)|^M)$ for any M such that $(x_0(t), \xi_0(t)) = (x_0, \xi_0)$ when $t \in I$ and $w_k(t) \in C^\infty$ such that $w_0(t) = 0$, $\operatorname{Im} w_2(t) > 0$ and $w_k(t) = 0$, $k > 2$, when $t \in I$.*

Then we obtain $\operatorname{Im} \omega(t, x) \ge c|x - x_0(t)|^2$ near Γ, with $c > 0$, so the errors that are $\mathcal{O}(|x - x_0|^M)$ in the eikonal equation will give terms that are bounded by $C_M \lambda^{-M/2}$. But we have to show that $t \mapsto f(t, x_0(t), \xi_0(t))$ also changes sign from $+$ to $-$ as t increases for some choice of (x_0, ξ_0). This problem will be studied in the next section, with a special emphasis on the finite vanishing case. By (4-6) we then obtain that $t \mapsto \operatorname{Im} w_0(t)$ has a local minimum on I which can be equal to 0 by subtracting a constant.

5. The change of sign

We have assumed that condition $\operatorname{Sub}(\Psi)$ for P is *not* satisfied near the subprincipal semibicharacteristic $\Gamma = \{(t, x_0, \xi_0) : t \in I\}$, so that $t \mapsto f(t, x, \xi)$ changes sign from $+$ to $-$ for some (x, ξ) as t increases near Γ. But after solving the eikonal equation, we have to know that $t \mapsto f(t, x_0(t), \xi_0(t))$ has the same sign change, possibly after changing the starting point (x_0, ξ_0). In order to do so, we shall use the invariance of condition $\operatorname{Sub}(\Psi)$, but note that condition (3-6) is only assumed when the change of sign is of infinite order. Therefore we shall first consider the case when the sign change is of finite order and show that this condition is preserved after solving the eikonal equation. Thus assume that

$$\partial_t^k f(t_0, x_0, \xi_0) < 0 \quad \text{and} \quad \partial_t^j f(t_0, x_0, \xi_0) = 0 \quad \text{for } j < k \tag{5-1}$$

for some odd integer k, where we may assume $t_0 = 0$. Now, if the order of the zero is not constant in a neighborhood of (x_0, ξ_0) then in any neighborhood the mapping $t \mapsto f(t, x, \xi)$ must have a zero of odd order with sign change from $+$ to $-$, and the order of vanishing is constant almost everywhere on $f^{-1}(0)$. We obtain this because $\partial_t^k f \neq 0$, $t \mapsto f(t, x, \xi)$ goes from $+$ to $-$ and the set where the order of the zero changes is nowhere dense in $f^{-1}(0)$ since it is the union of boundaries of closed sets in the relative topology. By possibly changing (t_0, x_0, ξ_0), we may assume that (5-1) holds with $t_0 = 0$, and that the order of the zero is odd and constant near (x_0, ξ_0). Then the zeros form a smooth manifold by the implicit function theorem. Using Taylor's formula, we find that $f(t, w) = a(t, w)(t - t_0(w))^k$, where $k \geq 1$ is odd, $w = (x, \xi)$, $t_0(w_0) = 0$ and $a < 0$ in a neighborhood of $(0, w_0) = (0, x_0, \xi_0)$. Then we find

$$\partial_w f = \partial_w a (t - t_0)^k - ak(t - t_0)^{k-1} \partial_w t_0, \tag{5-2}$$

which vanishes of at least order $k - 1$ in t at $f^{-1}(0)$. Let $w(t) = (x_0(t), \xi_0(t))$. Then

$$f(t, w(t)) = f(t, w_0) + \partial_w f(t, w_0) \Delta w(t) + \mathcal{O}(|\Delta w(t)|^2),$$

where $\Delta w(t) = w(t) - w_0$. Now $t \mapsto f(t, w_0)$ vanishes of order k in t at 0 and $t \mapsto \partial_w f(t, w_0)$ vanishes of at least order $k - 1$, so if $t \mapsto \Delta w(t)$ vanishes of at least order $k > 1$ then by (5-2) we find that $t \mapsto f(t, w(t))$ vanishes of order k. Since $(d/dt)\Delta w(t) = w'(t)$, we will need the following result.

Lemma 5.1. *Let $(x_0(t), \xi_0(t))$ be the solution to equation (4-7) with $\operatorname{Im} w_2(0) \neq 0$ and assume that $t \mapsto \partial_w f(t, x_0, \xi_0)$ vanishes of order $r \geq 1$ at $t = 0$. Then $(x_0'(t), \xi_0'(t))$ vanishes of order r and $\Delta w(t)$ vanishes of order $r+1$ at $t = 0$.*

Proof. By (4-7) we have

$$w'(t) = (x_0'(t), \xi_0'(t)) = A(t) \partial_w f(t, w(t)), \qquad w(0) = w_0. \tag{5-3}$$

Here we have $|A(0)| \neq 0$ if $\operatorname{Im} w_2(0) \neq 0$; in fact $w'(0) = 0$ then gives $\partial_\xi f(0, w_0) = 0$ and $\partial_x f(0, w_0) = 0$ by (4-7).

Now we define $\phi_0(t) = \partial_w f(t, w_0)$ and $\phi_1(t) = \partial_w f(t, w(t))$. Then we have $w'(t) = A(t)\phi_1(t)$ and the condition is that $\phi_0(t)$ vanishes of order $r \geq 1$ at 0. We shall proceed by induction, and first assume that $r = 1$. Since $w(0) = w_0$ we find $\phi_1(0) = \phi_0(0) = 0$ and thus $w'(0) = 0$.

Next, for $r > 1$ we assume by induction that $w'(t)$ vanishes of order $r-1$ at 0 so $w^{(k)}(0) = 0$ for $k < r$, and then we shall show that $w^{(r)}(0) = 0$ so that w' vanishes of order r. Using the chain rule we obtain

$$\partial_t^r(g(t, w(t))) = \sum_{\substack{0\leq j\leq r\\ \sum_i r_i + j = r}} c_{j,\alpha}\partial_t^j \partial_w^\alpha g(t, w(t)) \prod_{i=1}^{|\alpha|} w^{(r_i)}(t) \tag{5-4}$$

for any $g(t, w) \in C^\infty$. Thus, for $g = \partial_w f$ we find that

$$\phi_1^{(k)}(0) = \phi_0^{(k)}(0) + \partial_t^{k-1}\partial_w^2 f(0, x_0, \xi_0)w'(0) + \cdots + \partial_w^2 f(0, x_0, \xi_0)w^{(k)}(0) = \phi_0^{(k)}(0) = 0$$

for $k < r$, since the other terms have some factor $w^{(j)}(0) = 0$, $j \leq k$, which implies that $\phi_1(t)$ vanishes of order r. Since $w' = A\phi_1$ we find that $w'(t)$ vanishes of order r, which gives the induction step and the proof. ☐

Now, if $f(t, w_0)$ vanishes of order k then $\partial_w f(t, w_0)$ vanishes of order $k-1$. Thus $w'(t)$ vanishes of order $k-1$ by Lemma 5.1, and since $w(0) = w_0$ we find that $\Delta w(t)$ vanishes of order k. Thus, we find that $f(t, w(t)) - f(t, w_0)$ vanishes of order $2k-1$, so these terms vanish of same order if $k > 1$. In the case $k = 1$, we shall use an argument of Hörmander [1981] for the principal-type case. We obtain from (4-6) that $\partial_t(f(t, w(t))) = -\operatorname{Im} w_0''(t)$; thus

$$\operatorname{Im} w_0''(0) = -\partial_t f(0, w_0) - \partial_\xi f(0, w_0)\cdot\xi_0' - \partial_x f(0, w_0)\cdot x_0', \tag{5-5}$$

where $\partial_t f(0, w_0) = -c < 0$. We find from (4-7) that

$$\begin{cases} \xi_0'(0) = \operatorname{Re} w_2(0)x_0'(0) + \partial_\xi f(0, w_0)\operatorname{Im} w_2(0), \\ x_0'(0) = (\operatorname{Im} w_2(0))^{-1}\big(\partial_x f(0, w_0) + \partial_\xi f(0, w_0)\operatorname{Re} w_2(0)\big). \end{cases} \tag{5-6}$$

If $\partial_\xi f(0, w_0) = 0$ then we find that $x_0'(0) = (\operatorname{Im} w_2(0))^{-1}\partial_x f(0, w_0)$ and obtain

$$\operatorname{Im} w_0''(0) = c - \partial_x f(0, w_0)(\operatorname{Im} w_2(0))^{-1}\partial_x f(0, w_0) > c/2 > 0 \tag{5-7}$$

by choosing $\operatorname{Im} w_2(0) = \kappa \operatorname{Id}$ with $\kappa \gg 1$. If $\partial_\xi f(0, w_0) \neq 0$ then we may choose $\operatorname{Re} w_2(0)$ so that

$$\partial_x f(0, w_0) + \partial_\xi f(0, w_0)\operatorname{Re} w_2(0) = 0. \tag{5-8}$$

Then we find $x_0'(0) = 0$ and we obtain

$$\operatorname{Im} w_0''(0) = c - \partial_\xi f(0, w_0)\operatorname{Im} w_2(0)\partial_\xi f(0, w_0) > c/2 > 0 \tag{5-9}$$

by choosing $\operatorname{Im} w_2(0) = \kappa \operatorname{Id}$ with $0 < \kappa \ll 1$. Thus in both cases we find that $\partial_t f(t, w(t)) = \operatorname{Im} w_0'(t) < 0$ at $t = 0$.

We find that $t \mapsto f(t, w(t))$ changes sign from $+$ to $-$ of order k as t increases at $t = 0$. We may then rewrite the equation as

$$\operatorname{Im} w_0'(t) = t^k c(t), \tag{5-10}$$

where $c(t) > 0$ in a neighborhood of the origin. Since $\operatorname{Im} w_2(0) > 0$ we find that

$$e^{i\lambda\omega(t,x)} \le e^{-c_0\lambda(t^{k+1}+|x-x_0|^2)}, \quad |x - x_0| \ll 1, \quad |t| \ll 1, \quad c_0 > 0. \tag{5-11}$$

Thus the errors that are $\mathcal{O}(|x-x_0|^M)$ in the eikonal equation will give terms that are bounded by $C_M\lambda^{-M/2}$.

We shall also consider the case when $t \mapsto f(t, x, \xi)$ changes sign from $+$ to $-$ of infinite order near Γ. If Γ is not a point, then by reducing to a minimal bicharacteristic as in Section 3, we may assume that $f(t, x, \xi)$ vanishes of infinite order at Γ. We then obtain an approximate solution to the eikonal equation by solving (4-7)–(4-9) with initial data $w = (x, \xi)$ and $w_k(0)$, $k \ge 2$, which gives a change of coordinates $(t, w) \mapsto (t, w(t))$. If in any neighborhood of $\Gamma = \{(t, x_0, \xi_0) : t \in I\}$ there exist points in $f^{-1}(0)$ where $\partial_t f < 0$, then as before we can construct approximate solutions in any neighborhood of Γ satisfying (5-11) with $k = 1$. If $\partial_t f \ge 0$ on $f^{-1}(0)$ in some neighborhood of Γ, then by the invariance of condition (Ψ) there will still exist a change of sign of $t \mapsto f(t, w(t))$ from $+$ to $-$ in any neighborhood of Γ after the change of coordinates; see [Hörmander 1985b, Lemma 26.4.11]. (Recall that conditions (2-8) and (2-11) hold in some neighborhood of Γ.) Thus if $F'(t) = -\operatorname{Im} w_0'(t) = f(t, w(t))$ then $t \mapsto F(t)$ has a local maximum at some $t = t_0$, and after subtraction the maximum can be assumed to be equal to 0. By choosing suitable initial value (x_0, ξ_0) for (4-7) at $t = t_0$, we obtain

$$e^{i\lambda\omega(t,x)} \le e^{\lambda(F(t)-c|x-x_0|^2)}, \quad |x - x_0| \ll 1, \tag{5-12}$$

where $F'(t) = f(t, w(t))$ so that $\max_I F(t) = 0$ with $F(t) < 0$ for some $t \notin I$ near ∂I.

Proposition 5.2. *Assume that $t \mapsto f(t, x_0, \xi_0)$ changes sign from $+$ to $-$ as t increases near I and that $\partial_t^k \partial_x^\alpha \partial_\xi^\beta f(t, x_0, \xi_0) = 0$ for all $t \in I$ when $|I| \ne 0$. Then we may solve (4-1) in a neighborhood Ω of $\Gamma = \{(t, x_0, \xi_0) : t \in I\}$ modulo $\mathcal{O}(|x - x_0(t)|^M)$ for any M, with $\omega(t, x)$ given by (4-2) such that the curve $t \mapsto (x_0(t), \xi_0(t))$, $t \in (t_1, t_2)$, is arbitrarily close to Γ, $w_k(t) \in C^\infty$, $\operatorname{Im} w_2(t) \ge c > 0$ when $t \in (t_1, t_2)$, $\min_{(t_1,t_2)} \operatorname{Im} w_0(t) = 0$ and $\operatorname{Im} w_0(t_j) = c > 0$, $j = 1, 2$.*

Observe that since $\operatorname{Im} w_0 \ge 0$ we find that $f(t_0, x_0(t_0), \xi_0(t_0)) = -\operatorname{Im} w_0'(t) = 0$ at a minimum $t_0 \in (t_1, t_2)$. As before, the errors that are $\mathcal{O}(|x-x_0|^M)$ in the eikonal equation will give terms that are bounded by $C_M\lambda^{-M/2}$ for all M. Observe that cutting off where $\operatorname{Im} w_0 > 0$ will give errors that are $\mathcal{O}(\lambda^{-M})$ for all M.

6. The transport equations

Next, we shall solve the transport equations given by the following terms in (3-15):

$$D_t\phi + \partial_\eta^2 B(t, x, y, \partial_{t,x}\omega, 0) D_y^2\phi/2 + A(t, x, y, \partial_{t,x}\omega, 0) D_y\phi \\ + i\partial_\xi f(t, x, \partial_x\omega) D_x\phi + \sum_{j\ge 0} \lambda^{-j} R_j(t, x, y, D_{t,x,y})\phi \tag{6-1}$$

near $\Gamma = \{(t, x_0, y_0, 0, \xi_0, 0) : t \in I\}$. Here $R_0(t, x, y) = F_0(t, x, y, \partial_{t,x}\omega, 0)$ and when $m > 0$ we have that $R_m(t, x, y, D_{t,x,y})$ are differential operators of order j in t, order k in x and order ℓ in y, where $j + k + \ell \le m + 2$. Assuming the conclusions in Proposition 5.2 hold, we shall choose suitable initial values of the amplitude ϕ at $t = t_0$, which is chosen so that $\operatorname{Im} w_0(t_0) = 0$. Observe that the second-order differential operator given by the first four terms in (6-1) need not be solvable in general. Instead, by Lemma 6.1 we can treat the D_x and D_y terms as perturbations, using condition (3-6) in the infinite vanishing case.

Since the phase function $\omega(t, x)$ is complex-valued, we will replace the values of the symbols at $(\tau, \xi) = \partial_{t,x}\omega(t, x)$ by finite Taylor expansions at $(\operatorname{Re} w_0'(t), \xi_0(t))$. By (4-3) and (4-4) this will give expansions in powers of $x - x_0(t)$ and $\operatorname{Im} w_0'(t) = -f(t, x_0(t), \xi_0(t))$. Then, we shall solve the transport equations up to arbitrarily high powers of $x - x_0(t)$ and f. Since the imaginary part of the phase function $\operatorname{Im}\omega \ge 0$ vanishes of second order at $x = x_0(t)$, we will obtain by Lemma 6.1 below that this will give a solution modulo any negative power of λ.

We shall use the amplitude expansion

$$\phi(t, x, y) = \sum_{k\ge 0} \varrho^{-k}\phi_k(t, x, y) \tag{6-2}$$

and solve the transport equation recursively in k. Here ϕ_k depends on ϱ but with uniform bounds in a suitable symbol class, and $\varrho = \lambda^{1/N}$ with N to be determined later. By doing the change of variables $(t, x, y) \mapsto (t - t_0, x - x_0(t), y - y_0)$, we find that D_t changes into $D_t - x_0'(t)D_x$, which does not change the order of R_j as differential operator. Thus we may assume $t_0 = 0$, $x_0(t) \equiv 0$ and $y_0 = 0$.

Next, we apply (6-1) on ϕ given by (6-2). Since $\varrho = \lambda^{1/N}$, we obtain the terms

$$D_t\phi + A_0(t, x)D_x\phi + A_1(t, x, y)D_y\phi + A_2(t, x, y)D_y^2\phi + \sum_{j\ge 0} \varrho^{-jN} R_j(t, x, y, D_{t,x,y})\phi, \tag{6-3}$$

where

$$\begin{aligned} A_0(t, x) &= i\partial_\xi f(t, x, \xi_0(t) + \sigma(t, x)) - x_0'(t), \\ A_1(t, x, y) &= A\big(t, x, y, \partial_t\omega(t, x), \xi_0(t) + \sigma(t, x), 0\big), && (6\text{-}4) \\ A_2(t, x, y) &= \partial_\eta^2 B_2\big(t, x, y, \partial_t\omega(t, x), \xi_0(t) + \sigma(t, x), 0\big)/2. && (6\text{-}5) \end{aligned}$$

Here $\sigma(t, x)$ is given by (4-4) and $\partial_t\omega(t, x)$ by (4-3), where the expansion will be up to a sufficiently high order in x. Observe that after the change of variables we have $\sigma(t, 0) \equiv 0$. The values of the symbols will as before be defined by finite Taylor expansions in the τ- and ξ-variables, which gives expansions in powers of x and $f(t, 0, \xi(t))$.

We are going to construct solutions $\phi_k(t, x, y) = \phi_k(t, x, \varrho y)$ so that $y \mapsto \phi_k(t, x, y) \in C_0^\infty$ uniformly in ϱ, which gives localization in $|y| \lesssim \varrho^{-1}$. Therefore we shall choose ϱy as new y-coordinates. Then (6-3) becomes

$$D_t\phi + A_0(t, x)D_x\phi + \varrho A_1(t, x, y/\varrho)D_y\phi + \varrho^2 A_2(t, x, y/\varrho)D_y^2\phi + \sum_{j\ge 0} \varrho^{-jN} R_j(t, y/\varrho, x, D_t, D_x, \varrho D_y)\phi. \tag{6-6}$$

By Proposition 5.2 the phase function $e^{i\lambda w(t,x)}$ gives the cut-off in x, and we shall expand the symbols in powers of x. Now the Taylor expansion of $x \mapsto \varrho^2 A_2(t,x,y/\varrho)$ will give terms that are $\mathcal{O}(\varrho^2 x)$. Therefore we take $\varrho^2 x$ as new x-coordinates, which gives

$$D_t\phi + \varrho^2 A_0(t,x/\varrho^2)D_x\phi + \varrho A_1(t,x/\varrho^2,y/\varrho)D_y\phi + \varrho^2 A_2(t,x/\varrho^2,y/\varrho)D_y^2\phi \\ + \sum_{j\ge 0}\varrho^{-jN}R_j(t,y/\varrho,x/\varrho^2,D_t,\varrho^2D_x,\varrho D_y)\phi. \quad (6\text{-}7)$$

Now the phase function $e^{i\lambda w(t,x)}$ is $\mathcal{O}(e^{-c\varrho^{N-4}|x|^2})$ in the new coordinates. So if we take $N > 4$, it suffices to solve the transport equation up to a sufficiently high order of x; then we may cut off where $|x| \lesssim 1$, which corresponds to $|x| \lesssim \varrho^{-2}$ in the original coordinates. Thus we expand in x:

$$\phi_k(t,x,y) = \sum_{k,\alpha}\phi_{k,\alpha}(t,y)x^\alpha \quad \phi_{k,\alpha}(t,y)\in C_0^\infty, \quad (6\text{-}8)$$

$$A_0(t,x/\varrho^2)D_x = \sum_{\alpha,j}A_{0,\alpha,j}(t)\varrho^{-2|\alpha|}x^\alpha D_{x_j},$$

$$A_j(t,x/\varrho^2,y/\varrho) = \sum_{\alpha}A_{j,\alpha}(t,y/\varrho)\varrho^{-2|\alpha|}x^\alpha, \quad j>0,$$

$$R_k(t,x/\varrho^2,y/\varrho,\varrho D_y,\varrho^2 D_x) = \sum_{\alpha,\ell,\nu,\mu}R_{k,\alpha,\ell,\nu,\mu}(t,y/\varrho)\varrho^{-2|\alpha|+2|\nu|+|\mu|}x^\alpha D_t^\ell D_x^\nu D_y^\mu. \quad (6\text{-}9)$$

Here $\ell+|\nu|+|\mu| \le k+2$ so we have at most the factor $\varrho^{2|\nu|+|\mu|} \le \varrho^{2k+4}$ in (6-9). When $k=0$ we have $\ell+|\nu|+|\mu| = 0$ and

$$R_0(t,x/\varrho^2,y/\varrho) = \sum_\alpha R_{0,\alpha}(t,y/\varrho)\varrho^{-2|\alpha|}x^\alpha.$$

Observe that the coefficients in the expansions are given by expansions in powers of $f(t,0,\xi(t))$. After cut-off in x we find in the original coordinates that $\phi_k(t,x,y) = \varphi_k(t,\varrho^2 x,\varrho y)$, where φ_k for any t is uniformly bounded in C_0^∞.

We shall first apply (6-7) on ϕ_0 and expand in x. Then we find that the terms that are independent of x are

$$D_t\phi_{0,0} - i\varrho^2\sum_j A_{0,0,j}(t)\phi_{0,e_j} + \varrho A_{1,0}(t,y/\varrho)D_y\phi_{0,0} + \varrho^2 A_{2,0}(t,y/\varrho)D_y^2\phi_{0,0} + R_{0,0}(t,y/\varrho)\phi_{0,0}. \quad (6\text{-}10)$$

We shall need the following result, which gives estimates on f and A_j on the interval of integration. It will be proved in the next section. In the following, we shall denote $f(t) = f(t,0,\xi_0(t))$ and $F(t) = \int_0^t f(s)\,ds$. Observe that $f(0) = 0$ since $\operatorname{Im} w_0'(0) = 0$.

Lemma 6.1. *Assume that the conclusions in Proposition 5.2 hold and that (3-6) holds if $t \mapsto f(t)$ vanishes of infinite order at 0. Then there exists ε and $C \ge 1$ with the property that if $N \ge C$, $\varrho = \lambda^{1/N} \ge C$ and*

$$|f(t)| + \left|\int_0^t |A_0(s,0)| + |A_1(s,0,y/\varrho)| + \|A_2(s,0,y/\varrho)\|\,ds\right| \ge C/\varrho^3 \quad (6\text{-}11)$$

holds for some $|y| \le \varrho/C$, then $\lambda F(s) \le -\lambda^\varepsilon/C$ for some s in the interval connecting 0 and t.

Observe that if Lemma 6.1 holds for some ε and C, then it trivially holds for smaller ε and larger C. We shall assume that $\varepsilon < 1$ and that both N and λ are large enough so that the conclusion in Lemma 6.1 holds. Since (6-11) does not hold when $t = 0$, we can choose the maximal interval I containing 0 such that (6-11) does not hold in I; thus

$$|f(t)| + \left| \int_0^t |A_0(s,0)| + |A_1(s,0,y/\varrho)| + \|A_2(s,0,y/\varrho)\| \, ds \right| < C/\varrho^3, \quad t \in I, \tag{6-12}$$

when $|y| \le \varrho/C$. By definition we obtain that (6-11) holds for some $|y| \le \varrho/C$ when $t \in \partial I$, so Lemma 6.1 gives that $\lambda F \lesssim -\lambda^\varepsilon$ at ∂I_0 for some open interval $I_0 \subseteq I$ that contains 0. This means that $e^{i\lambda\omega(t,0)} = e^{\lambda F(t)} \le C_N \lambda^{-N}$ for any N at ∂I_0 when $\lambda \gg 1$. Since $F' = f$ is uniformly bounded and the left-hand side of (6-12) is Lipschitz continuous, we may cut off near I_0 with $\chi(t) \in S(1, \lambda^{6/N} dt^2) \subset S(1, \lambda^{2-2\varepsilon} dt^2)$ for $N \gg 1$ so that $\chi(0) \ne 0$, $\lambda F(t) \lesssim -\lambda^\varepsilon$ in $\operatorname{supp} \chi'$ and (6-12) holds with some C when $t \in \operatorname{supp} \chi$ and $|y| \le \varrho/C$. Then as before, the cut-off errors can be absorbed by the exponential and the expansion in powers of $f(t, 0, \xi(0)) = f(t)$ is justified. In fact, $f(t) = \mathcal{O}(\varrho^{-3})$ in $\operatorname{supp} \chi$, which gives errors of any negative power of $\varrho = \lambda^{1/N}$. The bound on the integral in (6-12) means that we can ignore the A_j terms in (6-10) in $\operatorname{supp} \chi$ modulo lower-order terms in ϱ. In the following we shall change the notation and let $I = \operatorname{supp} \chi$. We need to measure the error terms in the following way.

Definition 6.2. For $a(t) \in L^\infty(\mathbb{R})$ and $\kappa > 0$, we say that $a(t) \in I(\kappa)$ if $\int_0^t a(s)\, ds = \mathcal{O}(\kappa)$ for all $t \in I$.

For example, $f(t) \in I(\varrho^{-3})$, and since the integral in (6-12) is $\mathcal{O}(\varrho^{-3})$ in I, the integrand is in $I(\varrho^{-3})$. Then according to (6-12) it suffices to solve

$$D_t \phi_{0,0} = -R_{0,0} \phi_{0,0}, \quad t \in I, \tag{6-13}$$

to obtain that the terms in (6-10) are in $I(\varrho^{-1})$; here $R_{0,0}(t, y/\varrho) \in C^\infty$ uniformly since $\varrho \ge 1$. Now we can solve (6-13) with $\phi_{0,0}(0, y) = \phi(y) \in C_0^\infty$ uniformly with support where $|y| \ll 1$ such that $\phi(0) = 1$. In fact, the solution is $\phi_{0,0}(t, y) = E(t, y)\phi(y)$, where

$$E(t,y) = \exp\left(-i \int_0^t R_{0,0}(s, y/\varrho)\, ds\right), \quad t \in I,$$

is uniformly bounded in C^∞. Thus $\phi_{0,0}(t, y) \in C^\infty$ uniformly and by choosing $\phi(y)$ with sufficiently small support, we obtain for any $t \in I$ that $\phi_{0,0}(t, \cdot)$ has support in a sufficiently small compact set in which (6-12) holds.

The coefficients of the terms in (6-7) which are homogeneous of degree $\alpha \ne 0$ in x are

$$D_t \phi_{0,\alpha} + R_{0,0}(t, y/\varrho)\phi_{0,\alpha} - i \sum_{\substack{|\beta|=1 \\ j}} A_{0,\beta,j}(t)(\alpha_j + 1 - \beta_j)\phi_{0,\alpha+e_j-\beta} + \sum_{|\beta|=1} A_{2,\beta}(t, y/\varrho) D_y^2 \phi_{0,\alpha-\beta} \tag{6-14}$$

modulo $I(\varrho^{-1})$. Letting $\Phi_{k,j} = \{\phi_{k,\alpha}\}_{|\alpha|=j}$ and $\Phi_k = \{\Phi_{k,j}\}_j$ for $k, j \ge 0$, we find that (6-14) vanishes if Φ_0 satisfies the system

$$D_t \Phi_{0,k} = S_{0,0}^k \Phi_{0,k} + S_{0,1}^k \Phi_{0,k-1}, \tag{6-15}$$

where $S^k_{0,0}(t)$ is a uniformly bounded matrix depending on t, and $S^k_{0,1}(t, y/\varrho, D_y)$ is a system of uniformly bounded differential operators of order 2 in y when $|y| \lesssim \varrho$. Let $E_{0,k}(t)$ be the fundamental solution to $D_t E_{0,k} = S^k_{0,0} E_{0,k}$ so that $E_{0,k}(0) = \mathrm{Id}$. Then letting

$$\Phi_{0,k}(t, y) = E_{0,k}(t)\Psi_{0,k}(t, y),$$

the system (6-15) reduces to

$$D_t \Psi_{0,k}(t, y) = E^{-1}_{0,k} S^k_{0,1} E_{0,k} \Psi_{0,k-1}(t, y).$$

This is a recursion equation which we can solve uniformly in I with $\Psi_{0,k}(t, y)$ having initial values $\Psi_{0,k}(0, y) \equiv 0$ for $0 < k \le M$. Observe that since the initial data $\Phi_{0,k}(0, y)$ has compact support, we find that $\Phi_{0,k}(t, y) \in C^\infty$ uniformly. For any t we find that $\Phi_{0,k}(t, y)$ has support in a sufficiently small compact set so that (6-12) holds for any $t \in I$.

We shall now apply (6-7) to ϕ given by the full expansion (6-8). We find that the coefficients of the terms in (6-7) which are homogeneous of degree $\alpha \ne 0$ in x are equal to

$$\begin{aligned}\varrho^{-1}\Big(D_t\phi_{1,\alpha} &+ R_{0,0}(t, y/\varrho)\phi_{1,\alpha} - i\sum_{\substack{|\beta|=1\\ j}} A_{0,\beta,j}(t)(\alpha_j+1-\beta_j)\phi_{1,\alpha+e_j-\beta} + \sum_{|\beta|=1} A_{2,\beta}(t, y/\varrho)D_y^2\phi_{1,\alpha-\beta}\\ &+ \sum_{|\beta|=1} A_{1,\beta}(t, y/\varrho)D_y^2\phi_{0,\alpha-\beta} - i\varrho^3\sum_j A_{0,0,j}(t)(\alpha_j+1)\phi_{0,\alpha+e_j} + \varrho^3 A_{2,0}(t, y/\varrho)D_y^2\phi_{0,\alpha}\Big)\end{aligned} \tag{6-16}$$

modulo $I(\varrho^{-2})$. We find that (6-16) vanishes if Φ_1 satisfies the system

$$D_t\Phi_{1,k} = S^k_{1,0}\Phi_{1,k} + S^k_{1,1}\Phi_{1,k-1} + \mathbb{A}^0_1\Phi_0, \tag{6-17}$$

where $S^k_{1,0}(t)$ is a uniformly bounded matrix depending on t, $S^k_{1,1}(t, y/\varrho, D_y)$ is a system of uniformly bounded differential operators of order 2 when $|y| \lesssim \varrho$ and $\mathbb{A}^0_1$ is a differential operator in y of order 2 with coefficients in $I(1)$ because of (6-12). By letting $\Phi_{1,k} = E_{1,k}\Psi_{1,k}$ with the fundamental solution $E_{1,k}$ to $D_t E_{1,k} = S^k_{1,0}E_{1,k}$, $E_{1,k}(0) = \mathrm{Id}$, this reduces to the equation

$$D_t\Psi_{1,k} = E^{-1}_{1,k}S^k_{1,1}E_{1,k-1}\Psi_{1,k-1} + E^{-1}_{1,k}\mathbb{A}^0_1\Phi_0.$$

Thus we can solve (6-17) in I recursively with uniformly bounded $\Phi_{1,k}$ having initial values $\Phi_{1,k}(0, y) \equiv 0$, $k \ge 0$. But observe that Φ_1 is not in C^∞ uniformly; instead we have $D^j_t\Phi_1 = \mathcal{O}(\varrho^3)$ if $j \ge 1$, since $|\partial^j_t\mathbb{A}^0_1| \le C_j\varrho^3$ for all j by (6-16). For that reason, we shall define $S^3_\varrho \subset C^\infty$ by

$$|\partial^j_t\partial^\alpha_y\phi(t, y)| \le C_{j,\alpha}\varrho^{3j} \quad \forall\, j, \alpha \tag{6-18}$$

when $\phi \in S^3_\varrho$. Observe that $\phi \in S^3_\varrho$ if and only if $\phi(t, y) = \chi(\varrho^3 t, y)$, where $\chi \in C^\infty$ uniformly, and that the operator $\varrho^{-3}D_t$ maps $S^3_\varrho \mapsto S^3_\varrho$. Note that the expansion of the symbols also contains terms with factors $\varrho^3 f^k$, $k \ge 1$, which are uniformly bounded in S^3_ϱ for $t \in I$ by (6-12). Since $\int_0^t \mathbb{A}^0_1\,dt \in S^3_\varrho$ in I, we find that $\Phi_1 \in S^3_\varrho$ in I.

Recursively, the coefficients of the terms in (6-7) that are homogeneous in x of degree α are

$$\begin{aligned}\varrho^{-k}\Big(& D_t\phi_{k,\alpha} - i\sum_{\beta\neq 0} A_{0,\beta,j}(t)(\alpha_j+1-\beta_j)\phi_{k+2-2|\beta|,\alpha+e_j-\beta} \\ & + \sum_{\beta\neq 0} A_{1,\beta}(t,y/\varrho)D_y\phi_{k+1-2|\beta|,\alpha-\beta} + \sum_{\beta\neq 0} A_{2,\beta}(t,y/\varrho)D_y^2\phi_{k+2-2|\beta|,\alpha-\beta} \\ & - i\varrho^3\sum_j A_{0,0,j}(t)(\alpha_j+1)\phi_{k-1,\alpha+e_j} + \varrho^3 A_{1,0}(t,y/\varrho)D_y\phi_{k-2,\alpha} + \varrho^3 A_{2,0}(t,y/\varrho)D_y^2\phi_{k-1,\alpha} \\ & + \sum_{\ell+|\nu|+|\mu|\le j+2} \varrho^{-jN} R_{j,\beta,\ell,\nu,\mu}(t,y/\varrho)c_{\alpha,\beta,\nu}\varrho^{-2|\beta|+2|\nu|+|\mu|+i+3\ell}(\varrho^{-3}D_t)^\ell D_y^\mu \phi_{k-i,\alpha+\nu-\beta}\Big)\end{aligned} \tag{6-19}$$

modulo $I(\varrho^{-k-1})$. Here the last sum has $\ell+|\nu|+|\mu|=0$ when $j=0$, $(\varrho^{-3}D_t)^\ell D_y^\mu$ maps $S^3_\varrho \mapsto S^3_\varrho$ and the values of the symbols are given by a finite expansion in powers of $f(t)$.

Since $\phi_j \in S^3_\varrho$ we obtain that the terms in (6-19) are in $I(\varrho^{-k-1})$ if

$$\begin{aligned} & D_t\phi_{k,\alpha} - i\sum_{\beta\neq 0} A_{0,\beta,j}(t)(\alpha_j+1-\beta_j)\phi_{k+2-2|\beta|,\alpha+e_j-\beta} + \sum_{\beta\neq 0} A_{1,\beta}(t,y/\varrho)D_y\phi_{k+1-2|\beta|,\alpha-\beta} \\ & \quad + \sum_{\beta\neq 0} A_{2,\beta}(t,y/\varrho)D_y^2\phi_{k+2-2|\beta|,\alpha-\beta} - i\varrho^3\sum_j A_{0,0,j}(t)(\alpha_j+1)\phi_{k-1,\alpha+e_j} \\ & \quad + \varrho^3 A_{1,0}(t,y/\varrho)D_y\phi_{k-2,\alpha} + \varrho^3 A_{2,0}(t,y/\varrho)D_y^2\phi_{k-1,\alpha} \\ & \qquad = -\sum_{\substack{i+3\ell+2|\nu|+|\mu|=jN+2|\beta| \\ \ell+|\nu|+|\mu|\le j+2}} R_{j,\beta,\ell,\nu,\mu}(t,y/\varrho)c_{\alpha,\beta,\nu}(\varrho^{-3}D_t)^\ell D_y^\mu\phi_{k-i,\alpha+\nu-\beta}. \end{aligned} \tag{6-20}$$

When $j=0$ we find that $\ell+|\nu|+|\mu|=0$, $i=2|\beta|$ and we only have an expansion in β in the last sum. Now if $j>0$, $\ell+|\nu|+|\mu|\le j+2$ and $i+3\ell+2|\nu|+|\mu|=jN+2|\beta|$ then we find that

$$jN \le i+3\ell+2|\nu|+|\mu| < i+3(j+2),$$

which gives $i \ge j(N-3)-6 \ge N-9 \ge 1$ if $N\ge 10$. Thus we find that (6-20) can be written as

$$D_t\Phi_k = \mathbb{A}_0^k\Phi_k + \mathbb{A}_1^k\Phi_{k-1} + \mathbb{A}_2^k\Phi_{k-2} + \cdots, \tag{6-21}$$

where $\int_0^t \mathbb{A}_j^k\,dt$ is a uniformly bounded differential operator on S^3_ϱ for $t\in I$ and $j>0$. We have

$$\{\mathbb{A}_0^k\Phi_k\}_j = S^j_{k,0}\Phi_{k,j} + S^j_{k,1}\Phi_{k,j-1},$$

where $S^j_{k,0}(t)$ is a uniformly bounded matrix depending on t, and $S^j_{k,1}(t, y/\varrho, D_y)$ is a system of uniformly bounded differential operators of order 2 when $|y|\lesssim\varrho$. By letting $\Phi_{k,j}=E_{k,j}\Psi_{k,j}$ with the fundamental solution $E_{k,j}$ to $D_tE_{k,j}=S^j_{k,0}E_{k,j}$, $E_{k,j}(0)=\mathrm{Id}$, (6-21) becomes a system of recursion equations in j and k. Thus (6-21) can be solved in I with $\Phi_k\in S^3_\varrho$ having initial values $\Phi_k(0)\equiv 0$, $k>0$. We find from (6-8) and the definition of S^3_ϱ that $\phi_k(t,x,y)=\phi_k(\varrho^3 t,\varrho^2 x,\varrho y)$, where $\phi_k\in C^\infty$ uniformly when $t\in I$. Thus we can solve the transport equation (6-1) up to any negative power of λ. Observe that by cutting off in t and x, we may assume that $\phi_k \in C_0^\infty$ has fixed compact support in (x, y) and support where $|t| \lesssim \varrho^3$. It follows that the support of ϕ_k can be chosen in an arbitrarily small neighborhood of Γ for large enough λ. Changing to the original coordinates, we obtain the following result.

Proposition 6.3. *Assume that the conclusions in Proposition 5.2 hold, and that (3-6) is satisfied near Γ when the sign change of $t \mapsto f(t, x_0, \xi_0)$ is of infinite order. If $\varrho = \lambda^{1/N}$ for sufficiently large N, then for any K and M we can solve the transport equations (6-20) for $k \le K$ and $|\alpha| \le M$ near $\{(t, x_0(t), y_0) : t \in [t_1, t_2]\}$. By (6-8) this gives*

$$\phi_k(t, x, y) = \phi_k\big(\varrho^3(t - t_0), \varrho^2(x - x_0(t)), \varrho(y - y_0)\big), \quad k \le K,$$

where $\phi_k(t, x, y) \in C^\infty$ uniformly, has support where $|x| + |y| \lesssim 1$ and $|t| \lesssim \varrho^3$, and $\phi_0(0, 0, 0) = 1$ for some $t_0 \in (t_1, t_2)$ such that $\operatorname{Im} w_0(t_0) = 0$.

7. The rate of change of sign

We have showed that $t \mapsto f(t, x, \xi)$ changes sign from $+$ to $-$ on an interval I. Then

$$F(t) = \int^t f(s, x_0(s), \xi_0(s))\, ds = \int^t f(s)\, ds \tag{7-1}$$

has a local maximum in the interval. By choosing that maximum as the starting point, we may assume it is equal to 0 so that $F(t) \le 0$. By changing t-coordinate, we may assume $F(0) = 0$. We shall study how the size of the derivative f affects the size of the function F.

Lemma 7.1. *Assume that $0 \ge F(t) \in C^\infty$ has local maximum at $t = 0$, and let I_{t_0} be the closed interval joining 0 and $t_0 \in \mathbb{R}$. If*

$$\max_{I_{t_0}} |F'(t)| = |F'(t_0)| = \kappa \le 1$$

with $|t_0| \ge \kappa^\varrho$ for some $\varrho > 0$, then we have $\min_{I_{t_0}} F(t) \le -C_\varrho \kappa^{1+\varrho}$. The constant $C_\varrho > 0$ only depends on ϱ and the bounds on F in C^∞.

Proof. Let $f = F'$. Then since $F(t) = F(0) + \int_0^t f(s)\, ds \le \int_0^t f(s)\, ds$, it is no restriction to assume the maximum $F(0) = 0$. By switching t to $-t$, we may assume $t_0 \le -\kappa^\varrho < 0$. Let

$$g(t) = \kappa^{-1} f(t_0 + t\kappa^\varrho). \tag{7-2}$$

Then $|g(0)| = 1,\ |g(t)| \le 1$ for $0 \le t \le 1$ and

$$|g^{(N)}(t)| = \kappa^{\varrho N - 1} |f^{(N)}(t_0 + t\kappa^\varrho)| \le C_N$$

when $N \ge 1/\varrho$ for $0 \le t \le 1$. Using the Taylor expansion at $t = 0$ for $N \ge 1/\varrho$, we find

$$g(t) = p(t) + r(t), \tag{7-3}$$

where p is the Taylor polynomial of order $N - 1$ of g at 0, and

$$r(t) = t^N \int_0^1 g^{(N)}(ts)(1 - s)^{N-1}\, ds/(N - 1)! \tag{7-4}$$

is uniformly bounded in C^∞ for $0 \le t \le 1$ and $r(0) = 0$. Since g also is bounded on the interval, we find that $p(t)$ is uniformly bounded in $0 \le t \le 1$. Since all norms on the finite-dimensional space of polynomials of fixed degree are equivalent, we find that $p^{(k)}(0) = g^{(k)}(0)$ are uniformly bounded for $0 \le k < N$ which implies that $g(t)$ is uniformly bounded in C^∞ for $0 \le t \le 1$. Since $|g(0)| = 1$, there exists a uniformly bounded $\delta^{-1} \ge 1$ such that $|g(t)| \ge \frac{1}{2}$ when $0 \le t \le \delta$; thus g has the same sign in that interval. Since $g(s) = \kappa^{-1} f(t_0 + s\kappa^\varrho)$, we find

$$\delta/2 \le \left| \int_0^\delta g(s)\, ds \right| = \left| \kappa^{-\varrho} \int_{t_0}^{t_0+\delta\kappa^\varrho} \kappa^{-1} f(t)\, dt \right|. \tag{7-5}$$

Since $t_0 + \delta\kappa^\varrho \le 0$, we find that the variation of $F(t)$ on $[t_0, 0]$ is greater than $\delta\kappa^{1+\varrho}/2$ and since $F \le 0$, we find that the minimum of F on I_{t_0} is smaller than $-\delta\kappa^{1+\varrho}/2$. □

Proof of Lemma 6.1. As before we let $F(t)$ satisfy $F(0) = 0$ and $F'(t) = f(t)$, where $f(t) = f(t, 0, \xi_0(t))$ satisfies $f(0) = 0$. We have assumed that the estimate (3-6) holds near Γ if $f(t)$ vanishes of infinite order at $t = 0$. Observe that the term $x_0'(t)$ in A_0 can be estimated by $|\partial_w f(t, 0, \xi_0(t))|$ by (4-7), which gives that $|A_0(t, 0)| \lesssim |\partial_w f(t, 0, \xi_0(t))|$. We find from (4-3), (4-6) and (4-7) that

$$|\partial_t \omega(t, 0)| \lesssim |f(t)| + |\partial_w f(t, 0, \xi_0(t))|.$$

Thus (6-11) follows if

$$|f(t)| + \left| \int_0^t |f(s)| + A_0(s, 0) + A_1(s, 0, y/\varrho) + A_2(s, 0, y/\varrho)\, ds \right| \gtrsim \varrho^{-3}, \tag{7-6}$$

where

$$A_0(t) = |\partial_w f(t, 0, \xi_0(t))|,$$

$$A_1(t, y/\varrho) = |A(t, 0, y/\varrho, 0, \xi_0(t), 0)|, \tag{7-7}$$

$$A_2(t, y/\varrho) = \|\partial_\eta^2 B(t, 0, y/\varrho, 0, \xi_0(t), 0)\|. \tag{7-8}$$

In the following we shall suppress the y-variables in (7-6); the results will be uniform when $|y| \le c\varrho$ for some $c > 0$ since (3-6) holds near Γ. Observe that if $|f(s)|$ and $A_j(s)$ are $\ll \varrho^{-3}$ for $0 \le j \le 2$ when s is between 0 and t, then (7-6) does not hold.

We shall first consider the case when $|f(t)| \cong |t|^m$ vanishes of finite order at $t = 0$. Then the order must be odd so we find $F(t) = \int_0^t f(s)\, ds \le 0$ and $c \le |F(t)|/t^{2k} \le C < 0$ for some $k > 0$. Thus we find

$$\varrho^{-3} \lesssim \left| \int_0^t |f(s)| + A_0(s) + A_1(s) + A_2(s)\, ds \right| \lesssim |t| \lesssim |F(t)|^{1/2k} \tag{7-9}$$

implies that $|F(t)| \gtrsim \varrho^{-6k}$. Since $\lambda = \varrho^N$, we then obtain $\lambda F(t) \lesssim -\varrho^{N-6k} \le -\varrho = \lambda^{1/N}$ if $N > 6k$. The case when $|t|^{2k-1} \cong |f(t)| \gtrsim \varrho^{-3}$ gives that $|t| \gtrsim \varrho^{-3/(2k-1)}$ so $\lambda F(t) \lesssim -\varrho^{N-6k/(2k-1)} \le -\varrho$ if $N > 6$. Now one of these cases must hold if (6-11) holds, so we get the result in the finite vanishing case.

Next, we consider the infinite vanishing case. Then we have assumed that condition (3-6) holds, which means that

$$\sum_{j=0}^{2} A_j(t) \lesssim |f(t)|^\varepsilon,$$

which implies that $A_j(0) = 0$ for all j. Now we assume that (7-6) holds at t; by switching t and $-t$, we may assume $t > 0$. Then we obtain for some $s \in [0, t]$ that $|f(s)| \geq c\varrho^{-3}$ or $A_j(s) \geq c\varrho^{-3}$ for some $c > 0$ and j. Now we define t_0 as the smallest $t_0 > 0$ such that $|f(t_0)| = c\varrho^{-3}$ or $A_j(t_0) = c\varrho^{-3}$ for some j; then $t_0 \leq t$. Then we obtain from condition (3-6) in the first case that $c\varrho^{-3} = |f(t_0)| \lesssim |f(t_0)|^\varepsilon$ and in the second case that

$$c\varrho^{-3} = A_j(t_0) \lesssim |f(t_0)|^\varepsilon. \tag{7-10}$$

Since $\varrho = \lambda^{1/N}$, we find in both cases that

$$\lambda^{-3/\varepsilon N} = \kappa \leq c|f(t_0)|, \quad c > 0, \tag{7-11}$$

where $\lambda \gg 1$ if and only if $\kappa \ll 1$. By taking the smallest $t_0 > 0$ such that (7-11) is satisfied, we find that $|f(t)| \leq |f(t_0)|$ for $0 \leq t \leq t_0$. Since $f(t)$ vanishes of infinite order at $t = 0$, we find using Taylor's formula that $|f(t)| \leq C_M|t|^M$ for any positive integer M. (Actually, it suffices to take $M = 1$.) Condition (7-11) then gives

$$\kappa^{1/M} \lesssim |f(t_0)|^{1/M} \lesssim |t_0|, \tag{7-12}$$

so using Lemma 7.1 with $\varrho = 1/M$, we find that

$$\min_{0 \leq s \leq t_0} F(s) \lesssim -\kappa^{1+1/M} = -\lambda^{-3(1+1/M)/\varepsilon N}, \quad \lambda \gg 1. \tag{7-13}$$

Thus we find that $\min_{0 \leq s \leq t_0} F(s) \lesssim -\lambda^{c-1}$ for some $c > 0$ if $3(1+1/M)/\varepsilon N < 1$, that is, $N > 3(1+1/M)/\varepsilon$, which gives Lemma 6.1. ☐

8. The proof of Theorem 2.7

We shall use the following modification of [Hörmander 1985b, Lemma 26.4.15]. Recall that $\|u\|_{(k)}$ is the L^2 Sobolev norm of order k of $u \in C_0^\infty$ and let $\mathcal{D}'_\Gamma = \{u \in \mathcal{D}' : \mathrm{WF}(u) \subset \Gamma\}$ for $\Gamma \subseteq T^*\mathbb{R}^n$.

Lemma 8.1. *Let*

$$u_\lambda(x) = \lambda^{(n-1)\delta/2} \exp(i\lambda\omega(x)) \sum_{j=0}^{M} \phi_j(\lambda^\delta x)\lambda^{-j\varrho}, \quad \lambda \geq 1, \tag{8-1}$$

with $\varrho > 0$, $0 < \delta < 1$, $\omega \in C^\infty(\mathbb{R}^n)$ satisfying $\operatorname{Im}\omega \geq 0$, $|d\operatorname{Re}\omega| \geq c > 0$, and $\phi_j \in C_0^\infty(\mathbb{R}^n)$. Here ω and ϕ_j may depend on λ but uniformly, and ϕ_j has fixed compact support in all but one of the variables, for which the support is bounded by $C\lambda^\delta$. Then for any integer N we have

$$\|u_\lambda\|_{(-N)} \leq C\lambda^{-N}, \quad \lambda \geq 1. \tag{8-2}$$

If $\phi_0(x_0) \neq 0$ and $\operatorname{Im}\omega(x_0) = 0$ for some x_0 then there exists $c > 0$ so that

$$\|u_\lambda\|_{(-N)} \geq c\lambda^{-N-n/2+(n-1)\delta/2}, \quad \lambda \geq 1, \quad \forall N. \tag{8-3}$$

Let $\Sigma = \bigcap_{\lambda \geq 1} \bigcup_j \operatorname{supp}\phi_j(\lambda^\delta\,\cdot\,)$ and let Γ be the cone generated by

$$\big\{(x, \partial\omega(x)) : x \in \Sigma,\ \operatorname{Im}\omega(x) = 0\big\}. \tag{8-4}$$

Then for any k we find $\lambda^k u_\lambda \to 0$ in $\mathcal{D}'_\Gamma$, so $\lambda^k A u_\lambda \to 0$ in C^∞ if A is a pseudodifferential operator such that $\mathrm{WF}(A) \cap \Gamma = \emptyset$. *The estimates are uniform if $\omega \in C^\infty$ uniformly with fixed lower bound on $|d \operatorname{Re} \omega|$, and $\phi_j \in C_0^\infty$ uniformly with the support condition.*

In the expansion (8-1), we shall take $\varrho = 1/N$ and $\delta = 3/N$ with $N > 3$, and the cone Γ will be generated by

$$\big\{\big(t, x_0(t), y_0, 0, \xi_0(t), 0\big) : t \in I\big\}, \tag{8-5}$$

where $I = \{t : \operatorname{Im} w_0(t) = 0\}$. Observe that the phase function in (4-2) will satisfy the conditions in Lemma 8.1 near $\{(t, x_0(t), y_0) : t \in I\}$ since $\xi_0(t) \neq 0$ and $\operatorname{Im} \omega(t, x) \geq 0$ by Proposition 5.2. Also, we find from Proposition 6.3 that the functions ϕ_k will satisfy the conditions in Lemma 8.1 with $\delta = 3/N$ after making the change of variables $(t, x, y) \mapsto (t - t_0, x - x_0(t), y - y_0)$ since $\phi_0(t_0, x_0(t_0), y_0) = 1$. Observe that the conclusions of Lemma 8.1 are invariant under uniform changes of coordinates.

Proof of Lemma 8.1. We shall modify the proof of [Hörmander 1985b, Lemma 26.4.15] to this case. We have

$$\hat{u}_\lambda(\xi) = \lambda^{(n-1)\delta/2} \sum_{j=0}^{M} \lambda^{-j\delta} \int e^{i\lambda\omega(x) - i\langle x, \xi\rangle} \phi_j(\lambda^\delta x)\, dx. \tag{8-6}$$

Let U be a neighborhood of the projection on the second component of the set in (8-4). When $\xi/\lambda \notin U$, for $\lambda \gg 1$ we find that

$$\bigcup_j \operatorname{supp} \phi_j(\lambda^\delta \cdot) \ni x \mapsto (\lambda\omega(x) - \langle x, \xi\rangle)/(\lambda + |\xi|)$$

is in a compact set of functions with nonnegative imaginary part with a fixed lower bound on the gradient of the real part. Thus, by integrating by parts we find for any positive integer k that

$$|\hat{u}_\lambda(\xi)| \leq C_k \lambda^{((n-1)/2+k)\delta} (\lambda + |\xi|)^{-k}, \quad \xi/\lambda \notin U, \quad \lambda \gg 1, \tag{8-7}$$

which gives any negative power of λ for k large enough, since $\delta < 1$. If V is bounded and $0 \notin \overline{V}$ then since u_λ is uniformly bounded in L^2, we find

$$\int_{\lambda V} |\hat{u}_\lambda(\xi)|^2 (1 + |\xi|^2)^{-N}\, d\xi \leq C_V \lambda^{-2N}, \tag{8-8}$$

which together with (8-7) gives (8-2). If $\chi \in C_0^\infty$ then we may apply (8-7) to χu_λ; thus we find for any positive integer k that

$$|\widehat{\chi u}_\lambda(\xi)| \leq C \lambda^{((n-1)/2+k)\delta} (\lambda + |\xi|)^{-k}, \quad \xi \in W, \quad \lambda \gg 1, \tag{8-9}$$

if W is any closed cone with $(\operatorname{supp} \chi \times W) \cap \Gamma = \emptyset$. Thus we find that $\lambda^k u_\lambda \to 0$ in $\mathcal{D}'_\Gamma$ for every k. To prove (8-3) we may assume that $x_0 = 0$ and take $\psi \in C_0^\infty$. If $\operatorname{Im} \omega(0) = 0$ and $\phi_0(0) \neq 0$ then since $\delta < 1$, we obtain

$$\lambda^{n-(n-1)\delta/2}e^{-i\lambda\,\mathrm{Re}\,w(0)}\langle u_\lambda, \psi(\lambda\,\cdot)\rangle = \int e^{i\lambda(w(x/\lambda)-\mathrm{Re}\,w(0))}\psi(x)\sum_j \phi_j(\lambda^{\delta-1}x)\lambda^{-j\delta}\,dx$$
$$\to \int e^{i\langle \mathrm{Re}\,\partial_x\omega(0),x\rangle}\psi(x)\phi_0(0)\,dx, \quad \lambda\to\infty, \tag{8-10}$$

which is not equal to zero for some suitable $\psi \in C_0^\infty$. Since

$$\|\psi(\lambda\,\cdot)\|_{(N)} \le C_N\lambda^{N-n/2}, \tag{8-11}$$

we obtain from (8-10) that $0 < c \le \lambda^{N+n/2-(n-1)\delta/2}\|u_\lambda\|_{(-N)}$, which gives (8-3) and the lemma. $\square$

Proof of Theorem 2.7. By conjugating with elliptic Fourier integral operators and multiplying with pseudodifferential operators, we obtain that $P^* \in \Psi^2_{\mathrm{cl}}$ is of the form given by Proposition 3.2 microlocally near $\Gamma = \{(t, x_0, y_0, 0, \xi_0, 0) : t \in I\}$. Thus we may assume

$$P^* = D_t + F(t, x, y, D_t, D_x, D_y) + R, \tag{8-12}$$

where $R \in \Psi^2_{\mathrm{cl}}$ satisfies $\mathrm{WF}(R)\cap\Gamma = \varnothing$.

Now we can construct approximate solutions u_λ of the form (3-14) by using the expansion (3-15). By reducing to minimal bicharacteristics, we may solve first the eikonal equation by using Proposition 5.2 and then the transport equations (6-20) by using Proposition 6.3 with $\varrho = \lambda^{1/N}$ for $N > 3$. Thus after making the change of coordinates $(t, x, y) \mapsto (t - t_0, x - x_0(t), y - y_0)$, we obtain approximate solutions u_λ of the form (8-1) in Lemma 8.1 with $\varrho = 1/N$ and $\delta = 3/N$. For N large enough, we may choose K and M in Proposition 6.3 so that $|(D_t + F)u_\lambda| \lesssim \lambda^{-k}$ for any k. Now differentiation of $(D_t + F)u_\lambda$ can at most give a factor λ since $\delta < 1$, and a loss of a factor $x - x_0(t)$ gives at most a factor $\lambda^{1/2}$. Because of the bounds on the support of u_λ, we may obtain

$$\|(D_t + F)u_\lambda\|_{(\nu)} = \mathcal{O}(\lambda^{-N-n}) \tag{8-13}$$

for any chosen ν. Since $\phi_0(t_0, x_0(t_0), y_0) = 1$ by Proposition 6.3 and $\mathrm{Im}\,w(t_0, x_0(t_0)) = 0$ by Proposition 5.2, we find by (8-2)–(8-3) that

$$\lambda^{-N-n/2} \ll \lambda^{-N-n/2+(n-1)\delta/2} \lesssim \|u\|_{(-N)} \lesssim \lambda^{-N} \quad \forall N, \quad \lambda \gg 1. \tag{8-14}$$

Since u_λ has support in a fixed compact set that shrinks towards $\{(t, x_0(t), y_0) : t \in I\}$ as $\lambda \to \infty$, we find from Lemma 8.1 that $\|Ru\|_{(\nu)}$ and $\|Au\|_{(0)}$ are $\mathcal{O}(\lambda^{-N-n})$ if $\mathrm{WF}(A)$ does not intersect Γ. Thus we find from (8-13) and (8-14) that (2-17) does not hold when $\lambda \to \infty$, so P is not solvable at Γ by Remark 2.10. $\square$

References

[Cardoso and Trèves 1974] F. Cardoso and F. Trèves, "A necessary condition of local solvability for pseudo-differential equations with double characteristics", *Ann. Inst. Fourier (Grenoble)* **24**:1 (1974), 225–292. MR Zbl

[Colombini et al. 2003] F. Colombini, L. Pernazza, and F. Trèves, "Solvability and nonsolvability of second-order evolution equations", pp. 111–120 in *Hyperbolic problems and related topics*, edited by F. Colombini and T. Nishitani, International Press, Somerville, MA, 2003. MR

[Colombini et al. 2010] F. Colombini, P. D. Cordaro, and L. Pernazza, "Local solvability for a class of evolution equations", *J. Funct. Anal.* **258**:10 (2010), 3469–3491. MR Zbl

[Dencker 2006] N. Dencker, "The resolution of the Nirenberg–Treves conjecture", *Ann. of Math.* (2) **163**:2 (2006), 405–444. MR Zbl

[Dencker 2016] N. Dencker, "Solvability and limit bicharacteristics", *J. Pseudo-Differ. Oper. Appl.* **7**:3 (2016), 295–320. MR Zbl

[Egorov 1977] J. V. Egorov, "Solvability conditions for equations with double characteristics", *Dokl. Akad. Nauk SSSR* **234**:2 (1977), 280–282. In Russian; translated in *Sov. Math. Dokl.* **18** (1977), 632–635. MR Zbl

[Gilioli and Trèves 1974] A. Gilioli and F. Trèves, "An example in the solvability theory of linear PDE's", *Amer. J. Math.* **96** (1974), 367–385. MR Zbl

[Goldman 1975] R. Goldman, "A necessary condition for the local solvability of a pseudodifferential equation having multiple characteristics", *J. Differential Equations* **19**:1 (1975), 176–200. MR Zbl

[Hörmander 1977] L. Hörmander, "The Cauchy problem for differential equations with double characteristics", *J. Analyse Math.* **32** (1977), 118–196. MR Zbl

[Hörmander 1981] L. Hörmander, "Pseudodifferential operators of principal type", pp. 69–96 in *Singularities in boundary value problems* (Maratea, Italy, 1980), edited by H. G. Garnir, NATO Adv. Study Inst. Ser. C: Math. Phys. Sci. **65**, Reidel, Dordrecht, 1981. MR

[Hörmander 1983] L. Hörmander, *The analysis of linear partial differential operators, I: Distribution theory and Fourier analysis*, Grundlehren der Mathematischen Wissenschaften **256**, Springer, Berlin, 1983. MR Zbl

[Hörmander 1985a] L. Hörmander, *The analysis of linear partial differential operators, III: Pseudodifferential operators*, Grundlehren der Mathematischen Wissenschaften **274**, Springer, Berlin, 1985. MR Zbl

[Hörmander 1985b] L. Hörmander, *The analysis of linear partial differential operators, IV: Fourier integral operators*, Grundlehren der Mathematischen Wissenschaften **275**, Springer, Berlin, 1985. MR Zbl

[Mendoza 1984] G. A. Mendoza, "A necessary condition for solvability for a class of operators with involutive double characteristics", pp. 193–197 in *Microlocal analysis* (Boulder, CO, 1983), Contemp. Math. **27**, Amer. Math. Soc., Providence, RI, 1984. MR Zbl

[Mendoza and Uhlmann 1983] G. A. Mendoza and G. A. Uhlmann, "A necessary condition for local solvability for a class of operators with double characteristics", *J. Funct. Anal.* **52**:2 (1983), 252–256. MR Zbl

[Mendoza and Uhlmann 1984] G. A. Mendoza and G. A. Uhlmann, "A sufficient condition for local solvability for a class of operators with double characteristics", *Amer. J. Math.* **106**:1 (1984), 187–217. MR Zbl

[Nishitani 2004] T. Nishitani, "Effectively hyperbolic Cauchy problem", pp. 363–449 in *Phase space analysis of partial differential equations*, vol. 2, Scuola Norm. Sup., Pisa, 2004. MR Zbl

[Popivanov 1974] P. R. Popivanov, "The local solvability of a certain class of pseudodifferential equations with double characteristics", *C. R. Acad. Bulgare Sci.* **27** (1974), 607–609. In Russian. MR

[Trèves 1973] F. Trèves, "Concatenations of second-order evolution equations applied to local solvability and hypoellipticity", *Comm. Pure Appl. Math.* **26** (1973), 201–250. MR Zbl

[Trèves 1980] F. Trèves, *Introduction to pseudodifferential and Fourier integral operators*, vol. 2: Fourier integral operators, Plenum Press, New York, 1980. MR Zbl

[Wenston 1977] P. R. Wenston, "A necessary condition for the local solvability of the operator $P_m^2(x, D) + P_{2m-1}(x, D)$", *J. Differential Equations* **25**:1 (1977), 90–95. MR Zbl

[Wenston 1978] P. R. Wenston, "A local solvability result for operators with characteristics having odd order multiplicity", *J. Differential Equations* **28**:3 (1978), 369–380. MR Zbl

[Wittsten 2012] J. Wittsten, "On some microlocal properties of the range of a pseudodifferential operator of principal type", *Anal. PDE* **5**:3 (2012), 423–474. MR Zbl

[Yamasaki 1980] A. Yamasaki, "On a necessary condition for the local solvability of pseudodifferential operators with double characteristics", *Comm. Partial Differential Equations* **5**:3 (1980), 209–224. MR Zbl

[Yamasaki 1983] A. Yamasaki, "On the local solvability of $D_1^2 + A(x_2,\ D_2)$", *Math. Japon.* **28**:4 (1983), 479–485. MR Zbl

Received 23 Nov 2015. Revised 18 Sep 2016. Accepted 1 Nov 2016.

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ANALYSIS AND PDE
Vol. 10, No. 2, 2017
dx.doi.org/10.2140/apde.2017.10.351

ANISOTROPIC ORNSTEIN NONINEQUALITIES

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We investigate the existence of a priori estimates for differential operators in the L^1 norm: for anisotropic homogeneous differential operators $T_1, \dots, T_\ell$, we study the conditions under which the inequality

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}$$

holds true. Properties of homogeneous rank-one convex functions play the major role in the subject. We generalize the notions of quasi- and rank-one convexity to fit the anisotropic situation. We also discuss a similar problem for martingale transforms and provide various conjectures.

1. Introduction

In his seminal paper, Ornstein [1962] proved the following: let $\{T_j\}_{j=1}^{\ell}$ be homogeneous differential operators of the same order in d variables (with constant coefficients); if the inequality

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}$$

holds true for any $f \in C_0^\infty(\mathbb{R}^d)$, then T_1 can be expressed as a linear combination of the other T_j. Here and in what follows "$a \lesssim b$" means "there exists a constant c such that $a \leqslant cb$ uniformly"; the meaning of the word "uniformly" will be clear from the context. For example, in the statement above, the constant should be uniform with respect to all functions f. The aim of the present paper is to extend this theorem to the case where the differential operators are anisotropic homogeneous; see also [Kazaniecki and Wojciechowski 2014], where partial progress in this direction was obtained by a simple Riesz product technique.

To formulate the results, we have to introduce a few notions. Each differential polynomial $P(\partial)$ in d variables has a Newton diagram which matches a set of integral points in $\mathbb{R}^d$ to each such polynomial. The monomial $a\partial_1^{m_1}\partial_2^{m_2}\cdots\partial_d^{m_d}$ corresponds to the point $m = (m_1, m_2, \dots, m_d)$; for an arbitrary polynomial, its Newton diagram is the union of the Newton diagrams of its monomials.

Let Λ be an affine hyperplane in $\mathbb{R}^d$ that intersects all the positive semiaxes. We call such a plane a *pattern of homogeneity*. We say that a differential polynomial is homogeneous with respect to Λ (or simply Λ-homogeneous) if its Newton diagram lies on Λ.

Stolyarov was supported by RFBR grant no. 14-01-00198.
MSC2010: 26B35, 26B25.
Keywords: Ornstein noninequalities, Bellman function, martingale transform.

Conjecture 1. *Let Λ be a pattern of homogeneity in $\mathbb{R}^d$ and let $\{T_j\}_{j=1}^{\ell}$ be a collection of Λ-homogeneous differential operators. If the inequality*

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}$$

holds true for any $f \in C_0^\infty(\mathbb{R}^d)$, then T_1 can be expressed as a linear combination of the other T_j.

This conjecture may seem to be a simple generalization of Ornstein's theorem. We warn the reader that sometimes the anisotropic character of homogeneity brings new difficulties to inequalities for differential operators (the main one being the lack of geometric tools such as the isoperimetric inequality, or the coarea formula, etc.). For example, the classical embedding

$$W_1^1(\mathbb{R}^d) \hookrightarrow L_{d/(d-1)}$$

due to Gagliardo and Nirenberg was generalized to the anisotropic case only in [Solonnikov 1972] and additionally considered in [Kolyada 1993]; if one deals with similar embeddings for vector fields, the isotropic case was successfully considered in [Van Schaftingen 2013] (see also the survey [Van Schaftingen 2014]), but there is almost no progress for the anisotropic case (however, see [Kislyakov et al. 2013; 2015]).

The method we use to attack the conjecture differs from that of Ornstein (though there are some similarities). However, it is not new. It was noticed in [Conti et al. 2005] that Ornstein's theorem is related to the behavior of certain rank-one convex functions (for some special operators this link had already been known; see [Iwaniec 2002]). The case $d = 2$ was considered there. As for the general case of Ornstein's (isotropic) theorem, its proof via rank-one convexity was announced in [Kirchheim and Kristensen 2011] (and the proof is now available in the very recent preprint [Kirchheim and Kristensen 2016]). In a sense, we follow the plan suggested in [Kirchheim and Kristensen 2011]. However, the notions of quasiconvexity, rank-one convexity and others should be properly adjusted to the anisotropic world; we have not seen such an adjustment anywhere. For all these notions in the classical setting of the first gradient, their relationship with each other, properties, etc., we refer the reader to the book [Dacorogna 2008]. There are certain problems in the general anisotropic case that are not present in the classical setting. For example, the existence of the elementary laminate is not quite clear; at least, the classical reasoning does not work. Quasiconvexity still implies the rank-one convexity, but this requires a new proof. The approach of rank-one convexity reduces Conjecture 1 to a certain geometric problem about separately convex functions (Theorem 14) that is covered by Theorem 1 in [Kirchheim and Kristensen 2011] (Theorem 1.1 in [Kirchheim and Kristensen 2016]). We give a simple proof of this fact, which is the second advantage of our paper (though our proof does not give the more advanced Theorem 1 of [Kirchheim and Kristensen 2011]). We did not know of the preprint [Kirchheim and Kristensen 2016] until shortly before the publication of the present text, and did our work independently. Discussion with the authors of this preprint has shown that though the spirit of our approach in the geometric part is similar to theirs, the presentation and details appear to be different.

We will prove a particular case of Conjecture 1, which still seems to be rather general (in particular, it covers the classical isotropic case).

Theorem 2. *Let Λ be a pattern of homogeneity in $\mathbb{R}^d$ and let $\{T_j\}_{j=1}^{\ell}$ be Λ-homogeneous differential operators. Suppose all the monomials present in the T_j have the same parity of degree. If the inequality*

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)} \tag{1}$$

holds true for any $f \in C_0^\infty(\mathbb{R}^d)$, then T_1 can be expressed as a linear combination of the other T_j.

We note that the differential operators here are not necessarily scalar; i.e., one can prove the same theorem for the case where operators act on vector fields. It is one of the advantages of the general rank-one convexity approach. However, to facilitate the notation, we work on the scalar case.

We outline the structure of the paper. We begin with restating inequality (1) as an extremal problem described by a certain Bellman function (if inequality (1) holds, then the corresponding Bellman function is nonnegative). We also study the properties of our Bellman function (they are gathered in Theorem 6), the most important of which is the quasiconvexity. All this material constitutes Section 2. It turns out that quasiconvexity leads to a softer, but easier to work with, property of rank-one convexity. The proof of this fact is given in Section 3; see Theorem 9. So, the Bellman function in question is rank-one convex. In Section 4, we prove that rank-one convex functions homogeneous of order one are nonnegative, which gives us Theorem 2. In fact, it suffices to show a similar principle for separately convex functions on $\mathbb{R}^d$, which is formalized in Theorem 14. This theorem is purely convex geometric. Finally, we discuss related questions in Section 5.

2. Bellman function and its properties

Inequality (1) can be rewritten as

$$\inf_{\varphi \in C_0^\infty([0,1]^d)} \left(\sum_{j=2}^{\ell} \|T_j \varphi\|_{L_1(\mathbb{R}^d)} - c\|T_1 \varphi\|_{L_1(\mathbb{R}^d)} \right) = 0, \tag{2}$$

where c is a sufficiently small positive constant.

Definition 3. Suppose ∂^α, $\alpha \in A$, are all the partial derivatives that are present in the T_j (thus A is a subset of $\Lambda \cap \mathbb{Z}^d$). Consider the Hilbert space E with an orthonormal basis e_α indexed with the set A. For each function φ and each point x, we have a mapping

$$[0, 1]^d \ni x \mapsto \nabla[\varphi](x) = \sum_{\alpha \in A} \partial^\alpha[\varphi](x) e_\alpha \in E.$$

We call the function $\nabla[\varphi]$ the generalized gradient of φ.

The operator $\nabla[\,\cdot\,]$ is an analogue of the usual gradient suitable for our problem.

Example 4. Let $T_j = \partial_{x_j}$ for $j = 1, \dots, d$. In this case the generalized gradient turns out to be the usual gradient on the Euclidean space $\mathbb{R}^d$.

Example 5. Let us take the differential operators

$$T_1[\varphi] = \partial^{(2,0,1)}[\varphi] - \partial^{(0,3,1)}[\varphi], \quad T_2[\varphi] = \partial^{(4,0,0)}\varphi, \quad T_3[\varphi] = \partial^{(0,6,0)}[\varphi], \quad T_4[\varphi] = \partial^{(0,0,2)}[\varphi]. \tag{3}$$

We can list all the partial derivatives present in the operators:

$$A = \left\{\partial^{(0,0,2)}, \partial^{(0,6,0)}, \partial^{(4,0,0)}, \partial^{(0,3,1)}, \partial^{(2,0,1)}\right\}.$$

All the operators T_j are Λ-homogeneous, where $\Lambda = \{x \in R^3 : \langle x, (3, 2, 6)\rangle = 12\}$. In this case the generalized gradient is of the form

$$\nabla[\varphi] = \left(\partial^{(0,0,2)}[\varphi], \partial^{(0,6,0)}[\varphi], \partial^{(4,0,0)}[\varphi], \partial^{(0,3,1)}[\varphi], \partial^{(2,0,1)}[\varphi]\right) \in \mathbb{R}^5.$$

We also consider the function $V : E \to \mathbb{R}$ given by the rule

$$V(e) = \left(\sum_{j=2}^{\ell} |\tilde{T}_j e| - c|\tilde{T}_1 e|\right), \tag{4}$$

where the $\tilde{T}_j$ are the linear functionals on E such that $\tilde{T}_j(e) = \sum_A c_{\alpha,j} e_\alpha$ if $T_j = \sum_A c_{\alpha,j}\partial^\alpha$. With this bit of abstract linear algebra, we rewrite formula (2) as

$$\inf_{\varphi \in C_0^\infty([0,1]^d)} \int_{[0,1]^d} V(\nabla[\varphi](x))\,dx = 0.$$

The main idea is to consider a perturbation of this extremal problem, i.e., the function $\boldsymbol{B} : E \to \mathbb{R}$ given by the formula

$$\boldsymbol{B}(e) = \inf_{\varphi \in C_0^\infty([0,1]^d)} \int_{[0,1]^d} V(e + \nabla[\varphi](x))\,dx. \tag{5}$$

Theorem 6. *Suppose that inequality (2) holds true. Then, the function $\boldsymbol{B}$ possesses the properties listed below.*

(1) *It satisfies the inequalities $-\|e\| \lesssim \boldsymbol{B}(e) \lesssim \|e\|$ and $\boldsymbol{B} \leqslant V$.*

(2) *It is one-homogeneous; i.e., $\boldsymbol{B}(\lambda e) = |\lambda|\boldsymbol{B}(e)$.*

(3) *It is a Lipschitz function.*

(4) *It is a generalized quasiconvex function; i.e., for any $\varphi \in C_0^\infty([0, 1]^d)$ and any $e \in E$ the inequality*

$$\boldsymbol{B}(e) \leqslant \int_{[0,1]^d} \boldsymbol{B}(e + \nabla[\varphi](x))\,dx \tag{6}$$

holds true.

Proof. (1) We get the upper estimates on the function $\boldsymbol{B}$ by plugging $\varphi \equiv 0$ in the formula for it:

$$\boldsymbol{B}(e) \leqslant \int_{[0,1]^d} V(e + \nabla[\varphi]) = V(e) \lesssim \|e\|.$$

We obtain the lower bounds on the function $\boldsymbol{B}$ from inequality (2) and the triangle inequality:

$$\begin{aligned}\int_{[0,1]^d}\Big(\sum_{j=2}^{\ell}|\tilde{T}_j(e+\nabla[\varphi])|-c|\tilde{T}_1(e+\nabla[\varphi])|\Big) &\geqslant \int_{[0,1]^d}\Big(\sum_{j=2}^{\ell}|\tilde{T}_j(e+\nabla[\varphi])|-c|\tilde{T}_1(\nabla[\varphi])|-c|\tilde{T}_1e|\Big)\\ &\geqslant \int_{[0,1]^d}\Big(\sum_{j=2}^{\ell}|\tilde{T}_j(e+\nabla[\varphi])|-\sum_{j=2}^{\ell}|\tilde{T}_j(\nabla[\varphi])|-c|\tilde{T}_1e|\Big)\\ &= \int_{[0,1]^d}\Big(\sum_{j=2}^{\ell}\big(|\tilde{T}_j(e+\nabla[\varphi])|-|\tilde{T}_j(\nabla[\varphi])|\big)-c|\tilde{T}_1e|\Big)\\ &\geqslant -\sum_{j=2}^{\ell}|\tilde{T}_je|-c|\tilde{T}_1e|,\end{aligned}$$

where $\varphi \in C_0^\infty([0,1]^d)$ is an arbitrary function. We take infimum of the above inequality over all admissible φ:

$$-\|e\| \lesssim -\sum_{j=2}^{\ell}|\tilde{T}_je|-c|\tilde{T}_1e| \leqslant \boldsymbol{B}(e).$$

(2) Since V is a one-homogeneous function, the following equality holds for every $\lambda \neq 0$:

$$\boldsymbol{B}(\lambda e) = \inf_{\varphi\in C_0^\infty([0,1]^d)}\int_{[0,1]^d} V(\lambda e+\nabla[\varphi]) = \inf_{\varphi\in C_0^\infty([0,1]^d)}\int_{[0,1]^d}|\lambda|\,V(e+\nabla[\lambda^{-1}\varphi]).$$

We know that $\lambda^{-1}C_0^\infty([0,1]^d) = C_0^\infty([0,1]^d)$ for every $\lambda \neq 0$; therefore

$$\boldsymbol{B}(\lambda e) = \inf_{\varphi\in C_0^\infty([0,1]^d)}\int_{[0,1]^d}|\lambda|\,V(e+\nabla[\lambda^{-1}\varphi]) = |\lambda|\inf_{\varphi\in C_0^\infty([0,1]^d)}\int_{[0,1]^d}V(e+\nabla[\varphi]) = |\lambda|\boldsymbol{B}(e).$$

(3) In order to get the Lipschitz continuity of $\boldsymbol{B}$, we rewrite the formula for it:

$$\text{for all } e\in E,\qquad \boldsymbol{B}(e) = \inf_{\varphi\in C_0^\infty([0,1]^d)} V_\varphi(e),$$

where

$$V_\varphi(e) = \int_{[0,1]^d} V(e+\nabla[\varphi](x))\,dx.$$

It follows from the Lipschitz continuity of V that every function V_φ is a Lipschitz function with the Lipschitz constant bounded by L, where L is the Lipschitz constant of the function V. For every two points $v_1, v_2 \in E$, we can find a sequence of functions V_{φ_n} such that $\boldsymbol{B}(v_j) = \inf_{n\in\mathbb{N}} V_{\varphi_n}(v_j)$ for $j\in\{1,2\}$. We define

$$f_k(e) = \min_{n=1,2,\dots,k} V_{\varphi_n}(e).$$

For every $k\in\mathbb{N}$ the function f_k is the Lipschitz function with the Lipschitz constant bounded by L. Hence

$$|\boldsymbol{B}(v_1)-\boldsymbol{B}(v_2)| = \lim_{k\to\infty}|f_k(v_1)-f_k(v_2)| \leqslant L\|v_1-v_2\|.$$

(4) Before we prove the generalized quasiconvexity of this function, we need to introduce some notation. We know that all $\alpha \in A$ have common pattern of homogeneity Λ; thus we can find a vector $\gamma \in \mathbb{N}^d$ and a number $k \in \mathbb{N}$ such that $\langle \alpha, \gamma \rangle = k$ for every $\alpha \in A$.

For every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^d$, we define

$$x_\lambda = (\lambda^{\gamma_1} x_1, \lambda^{\gamma_2} x_2, \dots, \lambda^{\gamma_d} x_d).$$

For every $\lambda \in \mathbb{N}$ we define the partition of the unit cube $[0, 1]^d$ into small parallelepipeds:

$$Q_y = y + \prod_{j=1}^{d} [0, \lambda^{-\gamma_j}] \quad \text{for every } y \in Y,$$

where

$$Y = \left\{ y \in [0, 1]^d : y = \left(\frac{\kappa_1}{\lambda^{\gamma_1}}, \frac{\kappa_2}{\lambda^{\gamma_2}}, \dots, \frac{\kappa_d}{\lambda^{\gamma_d}} \right) \text{ for } \kappa_j \in \mathbb{N} \cup \{0\} \text{ and } \kappa_j < \lambda^{\gamma_j} \right\}.$$

Here Y is the set of "leftmost lowest" vertices of the parallelepipeds Q_y. The parallelepipeds Q_y are disjoint up to sets of measure zero and $\bigcup_{y \in Y} Q_y = [0, 1]^d$. Let us fix $\varphi \in C_0^\infty([0, 1]^d)$. Since $\nabla[\varphi]$ is a uniformly continuous function on $[0, 1]^d$ and the diameter of the parallelepipeds Q_y tends to zero uniformly with the growth of λ, we can choose λ sufficiently large to obtain

$$\text{for all } y \in Y, \text{ for all } z, v \in Q_y, \quad \left| \nabla[\varphi](z) - \nabla[\varphi](v) \right| \leqslant \frac{\varepsilon}{L}, \tag{7}$$

where L is the Lipschitz constant of the function V. Let $\{\psi_y\}_{y \in Y}$ be a family of functions in $C_0^\infty([0, 1]^d)$. For these functions, we use the rescaling

$$\psi_{y,\lambda}(x) = \lambda^{-k} \psi_y((x - y)_\lambda).$$

Let us observe that the rescaling $(x - y)_\lambda$ transforms the cube $[0, 1]^d$ into Q_y; thus $\operatorname{supp} \psi_{y,\lambda} \subset Q_y$. Moreover, we know that

$$\partial^\alpha[\psi_{y,\lambda}](x) = \lambda^{-k} \lambda^{\left(\sum_{j=1}^d \alpha_j \gamma_j\right)} \partial^\alpha[\psi_y]((x - y)_\lambda) = \partial^\alpha[\psi_y]((x - y)_\lambda)$$

for every $\alpha \in A$. By (5), we have

$$\boldsymbol{B}(e) \leqslant \int_{[0,1]^d} V\Big(e + \sum_{y \in Y} \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](x)\Big)\, dx = \sum_{y \in Y} \int_{Q_y} V\big(e + \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](x)\big)\, dx.$$

We assumed that (7) holds; therefore, for arbitrary $v_y \in Q_y$ we have the estimate

$$\begin{aligned} \int_{Q_y} V\big(e + \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](x)\big)\, dx &\leqslant \int_{Q_y} V\big(e + \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](v_y)\big)\, dx + \varepsilon |Q_y| \\ &= \int_{Q_y} V\big(e + \nabla[\psi_y]((x - y)_\lambda) + \nabla[\varphi](v_y)\big)\, dx + \varepsilon |Q_y|. \end{aligned}$$

Since $\lambda^{-\left(\sum_{j=1}^d \gamma_j\right)} = |Q_y|$, we have

$$\int_{Q_y} V\big(e + \nabla[\psi_y]((x - y)_\lambda) + \nabla[\varphi](v_y)\big)\, dx = |Q_y| \int_{[0,1]^d} V\big(e + \nabla[\psi_y](z) + \nabla[\varphi](v_y)\big)\, dz$$

for $z = (x - y)_\lambda$. Now for every $y \in Y$ and $v_y \in Q_y$ we can choose ψ_y such that

$$\int_{[0,1]^d} V\big(e + \nabla[\psi_y](z) + \nabla[\varphi](v_y)\big)\, dz \leqslant \boldsymbol{B}(e + \nabla[\varphi](v_y)) + \varepsilon$$

(this choice depends on v_y, however, we treat v_y as of a fixed parameter). We obtain

$$\boldsymbol{B}(e) \leqslant \sum_{y \in Y} |Q_y|\, \boldsymbol{B}(e + \nabla[\varphi](v_y)) + 2\varepsilon$$

from the above inequalities. We take mean integrals of this inequality over each cube Q_y with respect to v_y, which gives us

$$\boldsymbol{B}(e) \leqslant \sum_{y \in Y} \int_{Q_y} \boldsymbol{B}(e + \nabla[\varphi](v_y))\, dv_y + 2\varepsilon = \int_{[0,1]^d} \boldsymbol{B}(e + \nabla[\varphi](x))\, dx + 2\varepsilon.$$

Since ε was an arbitrary positive number, we have proved the generalized quasiconvexity of $\boldsymbol{B}$. □

The proof of the fourth point seems very similar to the standard *Bellman induction step* (see [Nazarov et al. 2001; Osękowski 2012; Stolyarov and Zatitskiy 2016; Volberg 2011] or any other paper on the Bellman function method in probability or harmonic analysis); moreover, the function $\boldsymbol{B}$ itself is, in a sense, a Bellman function and inequality (6) is a Bellman inequality. We suspect that this "similarity" should be more well-studied.

3. Rank-one convexity

Inequality (6) looks like a convexity inequality. Sometimes that is really the case.

Definition 7. We call a vector $e_x \in E$ a generalized rank-one vector if it is of the form

$$\sum_{\alpha \in A} i^{|\alpha|+|\alpha_0|} x^\alpha e_\alpha, \qquad x \in \mathbb{R}^d,\ \alpha_0 \in A.$$

Remark 8. In Theorem 2, we only consider the case where every $\alpha \in A$ has the same parity as the other elements of A. Therefore, $i^{|\alpha|+|\alpha_0|} \in \mathbb{R}$ for every $\alpha_0, \alpha \in A$. Hence the coefficients of the generalized rank-one vector are real.

Theorem 9. *The function $\boldsymbol{B}$ is a generalized rank-one convex function*; *i.e., it is convex in the directions of generalized rank-one vectors.*

To prove the theorem, we need two auxiliary lemmas.

Lemma 10. *For every $x \in \mathbb{R}^d$ and every $\varepsilon, \delta > 0$, there exists a function $l_{x,\varepsilon,\delta} \in C_0^\infty([0,1]^d)$ and a set $B \subset [0,1]^d$ such that the following hold.*

(1) $\|\nabla[l_{x,\varepsilon,\delta}]\| \leqslant \|e_x\| + \varepsilon$.

(2) $|B| \geqslant 1 - \delta$.

(3) *The function* $\nabla[l_{x,\varepsilon,\delta}]|_B$ *with respect to the measure* $\mu = |B|^{-1}\,dx|_B$ *is equimeasurable with the function* $\cos(2\pi t)e_x,\ t\in[0,1]$; *i.e.*,

$$\mu\big(\{\nabla[l_{x,\varepsilon,\delta}]\in W\}\big) = \Big|\Big\{t\in[0,1] : \cos(2\pi t)e_x \in W\Big\}\Big|$$

for every Borel set W *in* E.

Proof. For a given $x\in\mathbb{R}^d$ we take the same γ and k as in the proof of the fourth point of Theorem 6. We consider the function

$$l_{x,\varepsilon,\delta}(\xi) = t^{-k}\cos\Big(\sum_{j=1}^{d} t^{\gamma_j}x_j\xi_j\Big)\Phi(\xi),$$

where Φ is the smooth hat function:

$$\Phi(\xi) = \begin{cases} 1, & \xi\in[2\delta', 1-2\delta']^d,\\ 0, & \xi\in[0,1]^d\setminus[\delta', 1-\delta']^d,\\ \Theta(\xi)\in[0,1], & \text{otherwise}\end{cases}$$

for δ' sufficiently small (in particular, we need $2(2\delta')^d<\delta$). Similarly to the fourth point of Theorem 6, we define the set of proper parallelepipeds

$$Y_t = \Big\{Q\colon\ Q = (k_jv_j)_{j=1,\dots,d} + \prod_{j=1}^{d}[0,w_j],\ \ k_j\in\{1\}\cup\Big\{k_j\in\mathbb{N}\colon k_j < \frac{t^{\gamma_j}x_j}{2\pi} - 1\Big\}\Big\},$$

where $v_j = w_j = 2\pi t^{-\gamma_j}x_j^{-1}$ if $x_j\neq 0$ and $v_j=\delta',\ w_j=(1-2\delta')$ otherwise. For any δ', we can choose t to be so large that

$$\Big|\bigcup_{\substack{Q\in Y_t\\ Q\subset[2\delta',1-2\delta']^d}} Q\Big| \geqslant 1-\delta.$$

We put B to be this union, i.e., the union of the parallelepipeds Q from the family Y_t that belong to $[2\delta', 1-2\delta']^d$ entirely.

If t is sufficiently large, then for every $\beta\in\mathbb{N}^d$ satisfying $0\leqslant\langle\beta,\gamma\rangle<k$, we have

$$\sup_{\xi\in[0,1]^d} |t^{-1}\partial^\beta[\Phi](\xi)| \leqslant \varepsilon'. \tag{8}$$

For any $\beta\in\mathbb{N}^d$,

$$\partial^\beta\Big[\cos\Big(\sum_{j=1}^{d}t^{\gamma_j}x_j\xi_j\Big)\Big] = t^{\langle\beta,\gamma\rangle}x^\beta\partial^\beta[\cos]\Big(\sum_{j=1}^{d}t^{\gamma_j}x_j\xi_j\Big).$$

Since all $\alpha\in A$ have the same parity, we either have $\partial^\alpha[\cos](\xi) = (-1)^{|\alpha|/2}\cos(\xi)$ for every $\alpha\in A$ or $\partial^\alpha[\cos](x) = (-1)^{(|\alpha|+1)/2}\sin(\xi)$ for every $\alpha\in A$. Without loss of generality we may assume $2\,\big|\,|\alpha|$, because the functions sine and cosine are equimeasurable on their periodic domains. Therefore, for every

$\xi \in [0,1]^d$ and $\alpha \in A$ we have

$$\begin{aligned}
\partial^\alpha[l_{\xi,\varepsilon,\delta}](\xi) &= \Phi(\xi)\partial^\alpha\Bigg[t^{-k}\cos\Bigg(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Bigg)\Bigg] + \sum_{\substack{\alpha'+\beta=\alpha\\ \beta\neq(0,0,\dots,0)}} c_{\alpha',\beta}\, t^{-k}\partial^{\alpha'}\Bigg[\cos\Bigg(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j(x)\Bigg)\Bigg]\partial^\beta[\Phi] \\
&= \Phi(\xi)x^\alpha\partial^\alpha[\cos]\Bigg(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Bigg) + \sum_{\substack{\alpha'+\beta=\alpha\\ \beta\neq(0,0,\dots,0)}} c_{\alpha',\beta}\, t^{\langle\alpha',\gamma\rangle-k}\partial^{\alpha'}[\cos]\Bigg(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j(x)\Bigg)\partial^\beta[\Phi] \\
&= (-1)^{|\alpha|/2}x^\alpha\cos\Bigg(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Bigg) + \text{error}, \qquad (9)
\end{aligned}$$

where the coefficients $c_{\alpha',\beta}$ come from the Leibniz formula. The error is $O(\varepsilon')$ in absolute value by (8) and is equal to zero on the set $[2\delta', 1-2\delta']^d$ (because the function Φ is constant there). For every $\xi \in [0,1]^d$ we have

$$\begin{aligned}
\nabla[l_{\xi,\varepsilon,\delta}](\xi) = \sum_{\alpha\in A}\partial^\alpha[l_{\xi,\varepsilon,\delta}](\xi)e_\alpha &= \sum_{\alpha\in A}\Bigg((-1)^{|\alpha|/2}x^\alpha\cos\Bigg(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Bigg) + \text{error}\Bigg)e_\alpha \\
&= e_x\cos\Bigg(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Bigg) + \text{error}.
\end{aligned}$$

Thus, for every $\xi \in [0,1]^d$ and ε' sufficiently small, we obtain

$$\|\nabla[l_{\xi,\varepsilon,\delta}](\xi)\| \leqslant \|e_x\| + \|\text{error}\| \leqslant \|e_x\| + \varepsilon.$$

Since the error is equal to zero on the set $[2\delta', 1-2\delta']^d$, it follows from (9) that for every $\xi \in B$ we have

$$\nabla[l_{\xi,\varepsilon,\delta}](\xi) = \cos\Bigg(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\Bigg)e_x.$$

We note that the function $\cos\big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\big)e_x$ restricted to any $Q \in Y_t$ is equimeasurable (with respect to the measure $dx/|Q|$ on Q) with the function $\cos(2\pi t)e_x$, $t \in [0,1]$ (one can verify this fact using an appropriate dilation). Since B is a union of several parallelepipeds Q, the same holds with Q replaced by B. □

Lemma 11. *Suppose $v : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that*

$$v(x) \leqslant \int_0^1 v(x + \lambda\cos(2\pi t))\,dt \qquad (10)$$

for any $x, \lambda \in \mathbb{R}$. Then v is convex.

Proof. We are going to verify that v is convex as a distribution, or equivalently, that the distribution v'' is nonnegative. For that, we multiply inequality (10) by a positive function $\varphi \in C_0^\infty(\mathbb{R})$. Since v is a

Lipschitz function, we can integrate it over $\mathbb{R}$:

$$\begin{aligned}\int_{\mathbb{R}} v(x)\varphi(x)\,dx &\leqslant \int_{\mathbb{R}}\int_0^1 v(x+\lambda\cos(2\pi t))\varphi(x)\,dt\,dx = \int_{\mathbb{R}}\int_0^1 v(x)\varphi(x-\lambda\cos(2\pi t))\,dt\,dx\\ &= \int_{\mathbb{R}} v(x)\int_0^1 \bigl(\varphi(x)-\lambda\cos(2\pi t)\varphi^{'}(x)+\tfrac{1}{2}\lambda^2\cos^2(2\pi t)\varphi''(x)+o(\lambda^2)\bigr)\,dt\,dx\\ &= \int_{\mathbb{R}}\biggl(v(x)\varphi(x)+v(x)\varphi''(x)\tfrac{1}{2}\lambda^2\biggl(\int_0^1\cos^2(2\pi t)\biggr)+o(\lambda^2)\biggr)\,dx.\end{aligned}$$

Therefore,

$$0\leqslant \frac{1}{2}\biggl(\int_0^1\cos^2(2\pi t)\,dt\biggr)\int_{\mathbb{R}} v(x)\varphi''(x)\,dx+\frac{o(\lambda^2)}{\lambda^2}.$$

Letting $\lambda\to 0$, we show that v'' as a distribution satisfies $v''(\varphi)\geqslant 0$ for all $\varphi\in C_0^\infty(\mathbb{R})$ and $\varphi\geqslant 0$. From the Schwartz theorem it follows that v'' is a nonnegative measure of locally finite variation. Thus v' is an increasing function and therefore v is convex. □

Proof of Theorem 9. The function $\boldsymbol{B}$ is a generalized quasiconvex function; hence it satisfies (6) for every $\varphi\in C_0^\infty([0,1]^d)$. Let us fix $x\in\mathbb{R}^d$, $\lambda\in\mathbb{R}$. We plug $\lambda l_{x,\varepsilon,\delta}$ into (6). We get (for every $e\in E$)

$$\begin{aligned}\boldsymbol{B}(e)\leqslant\int_{[0,1]^d}\boldsymbol{B}(e+\nabla[\lambda l_{x,\varepsilon,\delta}]) &= \int_B\boldsymbol{B}(e+\nabla[\lambda l_{x,\varepsilon,\delta}])+\int_{[0,1]^d\setminus B}\boldsymbol{B}(e+\nabla[\lambda l_{x,\varepsilon,\delta}])\\ &\leqslant\int_B\boldsymbol{B}(e+\nabla[\lambda l_{x,\varepsilon,\delta}])+O\bigl(\lambda(\|e\|+\|e_x\|+\varepsilon)\delta\bigr)\end{aligned}$$

by Lemma 10. Since $\nabla[l_{x,\varepsilon,\delta}]|_B$ is equimeasurable (B equipped with the measure $dx/|B|$) with $\cos(2\pi t)e_x$,

$$\int_B\boldsymbol{B}(e+\nabla[\lambda l_{x,\varepsilon,\delta}])\,\frac{dx}{|B|}=\int_{[0,1]}\boldsymbol{B}(e+\lambda\cos(2\pi t)e_x)\,dt.$$

Therefore,

$$\boldsymbol{B}(e)\leqslant|B|\int_{[0,1]}\boldsymbol{B}(e+\lambda\cos(2\pi t)e_x)\,dt+O\bigl(\lambda(\|e\|+\|e_x\|+\varepsilon)\delta\bigr).$$

Since for $\delta\to 0$, we have $|B|\to 1$, we get

$$\boldsymbol{B}(e)\leqslant\int_{[0,1]}\boldsymbol{B}(e+\lambda\cos(2\pi t)e_x)\,dt. \tag{11}$$

For a fixed $e\in E$, consider the function $\mathbb{R}\ni s\mapsto\boldsymbol{B}(e+se_x)$. By (11),

$$\boldsymbol{B}(e+se_x)\leqslant\int_{[0,1]}\boldsymbol{B}(e+se_x+\lambda\cos(2\pi t)e_x)\,dt.$$

Thus, by Lemma 11, the function $\mathbb{R}\ni s\mapsto\boldsymbol{B}(e+se_x)$ is convex (one simply applies the lemma to this function). Since $e\in E$ and $x\in\mathbb{R}^d$, $\lambda\in\mathbb{R}$ were arbitrary, this proves the generalized rank-one convexity of the function $\boldsymbol{B}$. □

4. Separately convex homogeneous functions and proof of Theorem 2

Lemma 12. *Generalized rank-one vectors span E.*

Proof. Since E is a finite-dimensional Hilbert space, every functional on E is of the form $\varphi^*(\cdot) = \big\langle \sum_{\alpha\in A} a_\alpha e_\alpha, \cdot\big\rangle$. We get

$$\varphi^*(e_x) = \sum_{\alpha\in A} a_\alpha x^\alpha i^{|\alpha|+|\alpha_0|}$$

for every $x \in \mathbb{R}^d$. If E is not a span of generalized rank-one vectors, then there exists a nontrivial φ^* such that

$$0 = \varphi^*(e_x) = \sum_{\alpha\in A} a_\alpha x^\alpha i^{|\alpha|+|\alpha_0|}$$

for every $x \in \mathbb{R}^d$. However, x^α are linearly independent monomials. Therefore, $a_\alpha = 0$ for every $\alpha \in A$. Hence $\varphi^* \equiv 0$ and the generalized rank-one vectors span E. $\square$

We recall that our aim was to show that T_1 is a linear combination of the other T_j. By comparing the kernels of the $\tilde{T}_j$, it is equivalent to the fact that $V \geqslant 0$ everywhere. By the evident inequality $B \leqslant V$, it suffices to prove that B is nonnegative. By Lemma 12 and Theorem 9, this will follow from the theorem below. Hence it suffices to prove Theorem 14 to get Theorem 2.

Definition 13. A function $F : \mathbb{R}^d \to \mathbb{R}$ is separately convex if it is convex with respect to each variable.

Theorem 14. *A function $F : \mathbb{R}^d \to \mathbb{R}$ that is separately convex and homogeneous of order one is nonnegative.*

Before moving to the proof, we cite [Dacorogna 2008, Theorem 2.31], which says that a separately convex function is continuous. This fact will be implicitly used several times in the reasoning below.

Proof. We proceed by induction. Suppose the statement of the theorem holds true for the dimension $d-1$. We then prove it for the dimension d. Construct the function $G : \mathbb{R}^{d-1} \to \mathbb{R}$ by the formula

$$G(x) = F(x, 1), \quad x \in \mathbb{R}^{d-1}.$$

This function is separately convex and convex with respect to radius, i.e., for every $x \in \mathbb{R}^{d-1}$ the function $\mathbb{R}_+ \ni t \mapsto G(tx)$ is a convex function. Indeed, the function F is one-homogeneous and separately convex; thus for $t, r > 0$ and $\tau \in (0, 1)$ we have

$$\begin{aligned}
\tau G(tx) + (1-\tau)G(rx) &= \tau F(tx, 1) + (1-\tau)F(rx, 1)\\
&= (\tau t + (1-\tau)r)\left(\frac{\tau t F\big(x, \frac{1}{t}\big) + (1-\tau)r F\big(x, \frac{1}{r}\big)}{\tau t + (1-\tau)r}\right)\\
&\geqslant (\tau t + (1-\tau)r)\, F\left(x, \frac{1}{\tau t + (1-\tau)r}\right)\\
&= F\big((\tau t + (1-\tau)r)x, 1\big) = G\big((\tau t + (1-\tau)r)x\big).
\end{aligned}$$

We claim that for each $x \in \mathbb{R}^{d-1}$, the function $\mathbb{R} \ni t \mapsto G(tx)$ is convex. Since the function G is continuous, it suffices to prove that $G(tx) + G(-tx) \geqslant G(0)$ for all $t \in \mathbb{R}$. Consider another function V:

$$V(x) = \lim_{t \to 0+} \frac{G(tx) + G(-tx) - 2G(0)}{t}, \quad x \in \mathbb{R}^{d-1}.$$

The limit exists due to the convexity with respect to radius. This function V is one-homogeneous and separately convex. However, it may have attained the value $-\infty$. Fortunately, this is not the case. If there exists $x \in \mathbb{R}^d$ such that $V(x) = -\infty$ then

$$2V(0, x_2, \dots, x_d) \leqslant V(x_1, \dots, x_d) + V(-x_1, \dots, x_d) = -\infty.$$

Therefore $V(0, x_2, \dots, x_d) = -\infty$. We repeat the above reasoning with $x_2, \dots, x_d$ instead of x_1 and we get $V(0) = -\infty$, but from the definition of V we know that

$$V(0) = \lim_{t \to 0+} \frac{G(0) + G(0) - 2G(0)}{t} = 0.$$

Hence $V(x)$ is finite for every $x \in \mathbb{R}^{d-1}$. Thus, by the induction hypothesis, V is nonnegative. So, $\mathbb{R} \ni t \mapsto G(tx)$ is a convex function.

By symmetry, $G(x) + G(-x) \geqslant 2F(x, 0)$. On the other hand, $\lim_{t \to \pm\infty} G(tx)/t = F(x, 0)$. So, the convexity of $t \mapsto G(tx)$ gives the inequality $|G(x) - G(-x)| \leqslant 2F(x, 0)$. Adding these two inequalities, we get $F(x, 1) \geqslant 0$. □

Proof of Theorem 2. Assume that inequality (1) holds. Then, by Theorem 6, the function $\boldsymbol{B}$ given by (5) is Lipschitz, one-homogeneous, generalized quasiconvex, and satisfies the inequality $B \leqslant V$, where the function V is given by formula (4). Then, by Theorem 9, $\boldsymbol{B}$ is a generalized rank-one convex function.

Let $e \in E$ be an arbitrary point. By Lemma 12, e is a linear combination of generalized rank-one vectors $e_{x_1}, e_{x_2}, \dots, e_{x_k}$. We may assume that they are linearly independent. Consider the function $F : \mathbb{R}^k \to \mathbb{R}$ given by the rule

$$F(z_1, z_2, \dots, z_k) = \boldsymbol{B}(z_1 e_{x_1} + z_2 e_{x_2} + \cdots + z_k e_{x_k}).$$

By the generalized rank-one convexity of $\boldsymbol{B}$, we see that F is separately convex. It is also one-homogeneous; thus $F \geqslant 0$ by Theorem 14. Therefore, $\boldsymbol{B}(e)$ is also nonnegative for arbitrary $e \in E$.

Since $\boldsymbol{B} \geqslant 0$, we have $V \geqslant 0$. In such a case, it follows from formula (4) that $\operatorname{Ker} \tilde{T}_1 \supset \bigcap_{j=2}^{\ell} \operatorname{Ker} \tilde{T}_j$. Therefore, T_1 is a linear combination of the other T_j. □

5. Related questions

5.1. *Towards Conjecture 1.* The following statement plays the same role in view of Conjecture 1 as Theorem 14 plays in the proof of Theorem 2.

Conjecture 15. *Let $F : \mathbb{R}^{2d} \to \mathbb{R}$ be a Lipschitz homogeneous function of order one. Suppose that for any $j = 1, 2, \dots, d$ the function F is subharmonic with respect to the variables (x_j, x_{j+d}). Then, F is nonnegative.*

Indeed, plugging the cosine function into (6) as we did in the proof of Theorem 9 leads to "subharmonicity"[1] of the function $\boldsymbol{B}$ in the directions of projections of a generalized rank-one vector onto subspaces generated by odd and even monomials in A correspondingly. Therefore, Conjecture 1 follows from Conjecture 15.

We are not able to prove Conjecture 15. However, we know the following: in the case $d = 1$, the function F is not only nonnegative, but, in fact, convex (i.e., a one-homogeneous subharmonic function is convex). On the other hand, there is not much hope for simplifications: a subharmonic one-homogeneous function in $\mathbb{R}^3$ (and thus in $\mathbb{R}^d$, $d \geqslant 3$) can attain negative values; e.g., in $\mathbb{R}^4$ one may take the function

$$\frac{x_1^2+x_2^2+x_3^2-x_4^2}{\sqrt{x_1^2+x_2^2+x_3^2}}.$$

There are also reasons that differ from the ones discussed in the present paper that may "break" inequality (1). One of them is a certain geometric property of the spaces generated by the operators T_j. Not stating any general theorem or conjecture, we treat an instructive example. Consider the noninequality

$$\|\partial_1^2\partial_2 f\|_{L_1} \lesssim \|\partial_1^4 f\|_{L_1} + \|\partial_2^2 f\|_{L_1}. \tag{12}$$

Conjecture 1 suggests that it cannot be true. We will disprove it on the torus $\mathbb{T}^2$ and leave to the reader the rigorous formulation and proof of the corresponding transference principle, whose heuristic form is "inequalities of the sort (1) are true or untrue simultaneously on the torus and the Euclidean space". Consider two anisotropic homogeneous Sobolev spaces W_1 and W_2, which are obtained from the set of trigonometric polynomials by completion and factorization over the null-space with respect to the seminorms

$$\|f\|_{W_1} = \|\partial_1^4 f\|_{L_1} + \|\partial_2^2 f\|_{L_1}, \quad \|f\|_{W_2} = \|\partial_1^2\partial_2 f\|_{L_1} + \|\partial_1^4 f\|_{L_1} + \|\partial_2^2 f\|_{L_1}.$$

If inequality (12) holds true, then these two spaces are, in fact, equal (the identity operator is a Banach-space isomorphism between these spaces). However, it follows from the results of [Pełczyński and Wojciechowski 1992] (see [Wojciechowski 1991; 1993] as well) that W_2 has a complemented translation-invariant Hilbert subspace,[2] whereas W_1 does not, a contradiction.

Martingale transforms. Let $S = \{S_n\}_n$, $n \in \{0\} \cup \mathbb{N}$, be an increasing filtration of finite algebras on the standard probability space. We suppose that it differentiates L_1 (i.e., for any $f \in L_1(\Omega)$ the sequence $\mathbb{E}(f \mid S_n)$ tends to f almost surely). We will be working with martingales adapted to this filtration.

Definition 16. Let $\alpha = \{\alpha_n\}_n$ be a bounded sequence. The linear operator

$$T_\alpha[f] = \sum_{j=1}^{\infty} \alpha_{j-1}(f_j - f_{j-1}), \quad f = \{f_n\}_n \text{ is an } L_1 \text{ martingale},$$

is called a martingale transform.

[1] The "subharmonicity" means that $D\boldsymbol{B} \geqslant 0$ as a distribution, where D is an elliptic symmetric differential operator of second order (with constant real coefficients); one can then pass to usual subharmonicity by an appropriate change of variable.

[2] That means that there exists a subspace $X \subset W_2$ such that $g \in X$ whenever $g(\cdot + t) \in X$, $t \in \mathbb{T}^2$, X is isomorphic to an infinite-dimensional Hilbert space, and there exists a continuous projector $P : W_2 \to X$.

Our definition is not as general as the usual one, and we refer the reader to the book [Osękowski 2012] for the information about such operators. We only mention that martingale transforms serve as a probabilistic analogue for the Calderón–Zygmund operators. Here is the probabilistic version of Conjecture 1.

Conjecture 17. *Suppose $\alpha^1, \alpha^2, \dots, \alpha^\ell$ are bounded sequences. Suppose that the algebras S_n uniformly grow*; *i.e., there exists $\gamma < 1$ such that each atom a of S_n is split in S_{n+1} into atoms of probability not greater than $\gamma|a|$ each. The inequality*

$$\|T_{\alpha^1} f\|_{L_1} \lesssim \sum_{j=2}^{\ell} \|T_{\alpha^j} f\|_{L_1} \tag{13}$$

holds for any martingale f adapted to $\{S_n\}_n$ if and only if α^1 is a sum of a linear combination of the α^j and an ℓ_1 sequence.

We do not know whether the condition of uniform growth fits this conjecture. Anyway, it is clear that one should require some condition of this sort (otherwise one may take $S_n = S_{n+1} = \cdots = S_{n+k}$ very often and lose all the control of the sequences α^j on these time intervals). Again, we are not able to prove the conjecture in the full generality, but will deal with an important particular case.

Theorem 18. *Suppose $\alpha^1, \alpha^2, \dots, \alpha^\ell$ to be bounded periodic sequences. The inequality*

$$\|T_{\alpha^1} f\|_{L_1} \lesssim \sum_{j=2}^{\ell} \|T_{\alpha^j} f\|_{L_1}$$

holds if and only if α^1 is a linear combination of the other α^j.

Proof. To avoid technicalities, we will be working with finite martingales (denote the class of such martingales by $\mathcal{M}$). The general case can be derived by stopping time. Assume that inequality (13) holds true. Consider the Bellman function $\boldsymbol{B} : \mathbb{R}^\ell \to \mathbb{R}$ given by the formula

$$\boldsymbol{B}(x) = \inf_{f \in \mathcal{M}} \left(\sum_{j=2}^{\ell} \bigl\| x_j + T_{\alpha^j}[f] \bigr\|_{L_1} - c \bigl\| x_1 + T_{\alpha^1}[f] \bigr\|_{L_1} \right).$$

It is easy to verify that this function is one-homogeneous and Lipschitz. Moreover, $\boldsymbol{B}$ is convex in the direction of $(\alpha_n^1, \alpha_n^2, \dots, \alpha_n^\ell)$ for each n (by the assumption of periodicity, there is only a finite number of these vectors); the proof of this assertion is a simplification of Theorem 9 (here we do not have to make additional approximations; however, see [Stolyarov and Zatitskiy 2016, Lemma 2.17] for a very similar reasoning). Thus, by Theorem 14, $\boldsymbol{B}$ is nonnegative on the span of $\{(\alpha_n^1, \alpha_n^2, \dots, \alpha_n^\ell)\}_n$. Since $\boldsymbol{B}(x) \leqslant \sum_{j \geqslant 2} |x_j| - c|x_1|$, the aforementioned span does not contain the x_1-axis. Therefore, α^1 is a linear combination of the other α^j. □

Case $p > 1$. Inequality (1) may become valid provided one replaces the L_1 norm with the L_p one, $1 < p < \infty$. Let c_p be the best possible constant in the inequality

$$\|T_1 f\|_{L_p(\mathbb{R}^d)}^p \leqslant c_p \sum_{j=2}^{\ell} \|T_j f\|_{L_p(\mathbb{R}^d)}^p. \tag{14}$$

It is interesting to compute the asymptotics of c_p as $p \to 1$. Some particular cases have been considered in [Berkson et al. 2001]; we also refer the reader there for a discussion of similar questions.

Conjecture 19. *Let Λ be a pattern of homogeneity in $\mathbb{R}^d$ and let $\{T_j\}_{j=1}^{\ell}$ be a collection of Λ-homogeneous differential operators. If T_1 cannot be expressed as a linear combination of the other T_j, then $c_p \gtrsim \frac{1}{p-1}$.*

The conjecture claims that if there is no continuity at the endpoint, then the inequality behaves at least as if it had a weak type $(1, 1)$ there (it is also interesting to study when there is a weak type $(1, 1)$ indeed). First, we note that this question is interesting even when there are only two polynomials. Second, this is only a bound from below for c_p. Even in the case of two polynomials, c_p can be as big as $(p-1)^{1-d}$ (and thus the endpoint inequality may not be of weak type $(1, 1)$, at least when $d \geqslant 3$); see [Berkson et al. 2001] for the example.

Conjecture 19 will follow from the corresponding geometric statement in the spirit of Theorem 14.

Conjecture 20. *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a separately convex p-homogeneous function* (*i.e.*, $F(\lambda x) = |\lambda|^p F(x)$). *Suppose $F(x) \leqslant |x|^p$. Then, $F(x) \gtrsim (1-p)|x|^p$.*

Conjecture 19 is derived from Conjecture 20 in the same way as Theorem 2 is derived from Theorem 14: one considers the Bellman function (5) with the function V given by the formula

$$V(e) = \left(c_p \sum_{j=2}^{\ell} |\tilde{T}_j e|^p - |\tilde{T}_1 e|^p \right),$$

proves its generalized quasiconvexity, which leads to the generalized rank-one convexity, and then uses Conjecture 20 to estimate c_p from below.

It is not difficult to verify the case $d = 2$ of Conjecture 20. Therefore, there exists a C_0^∞-function f_p such that

$$(p-1)\|\partial_1\partial_2 f_p\|_{L_p(\mathbb{R}^2)} \gtrsim \left(\|\partial_1^2 f_p\|_{L_p(\mathbb{R}^2)} + \|\partial_2^2 f_p\|_{L_p(\mathbb{R}^2)}\right).$$

Acknowledgements

We thank Bernd Kirchheim, Fedor Nazarov, and Pavel Zatitskiy for fruitful discussions on the topic, and Alexander Logunov for his help with subharmonic functions.

References

[Berkson et al. 2001] E. Berkson, J. Bourgain, A. Pełczyński, and M. Wojciechowski, *Canonical Sobolev projections of weak type* $(1, 1)$, Mem. Amer. Math. Soc. **714**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl

[Conti et al. 2005] S. Conti, D. Faraco, and F. Maggi, "A new approach to counterexamples to L^1 estimates: Korn's inequality, geometric rigidity, and regularity for gradients of separately convex functions", *Arch. Ration. Mech. Anal.* **175**:2 (2005), 287–300. MR Zbl

[Dacorogna 2008] B. Dacorogna, *Direct methods in the calculus of variations*, 2nd ed., Applied Mathematical Sciences **78**, Springer, New York, 2008. MR Zbl

[Iwaniec 2002] T. Iwaniec, "Nonlinear Cauchy–Riemann operators in $\mathbb{R}^n$", *Trans. Amer. Math. Soc.* **354**:5 (2002), 1961–1995. MR Zbl

[Kazaniecki and Wojciechowski 2014] K. Kazaniecki and M. Wojciechowski, "Ornstein's non-inequalities Riesz product approach", preprint, 2014. arXiv

[Kirchheim and Kristensen 2011] B. Kirchheim and J. Kristensen, "Automatic convexity of rank-1 convex functions", *C. R. Math. Acad. Sci. Paris* **349**:7–8 (2011), 407–409. MR Zbl

[Kirchheim and Kristensen 2016] B. Kirchheim and J. Kristensen, "On rank one convex functions that are homogeneous of degree one", *Arch. Ration. Mech. Anal.* **221**:1 (2016), 527–558. MR Zbl

[Kislyakov et al. 2013] S. V. Kislyakov, D. V. Maksimov, and D. M. Stolyarov, "Spaces of smooth functions generated by nonhomogeneous differential expressions", *Funktsional. Anal. i Prilozhen.* **47**:2 (2013), 89–92. In Russian; translated in *Funct. Anal. Its Appl.* **47**:2 (2013), 157–159. MR Zbl

[Kislyakov et al. 2015] S. V. Kislyakov, D. V. Maksimov, and D. M. Stolyarov, "Differential expressions with mixed homogeneity and spaces of smooth functions they generate in arbitrary dimension", *J. Funct. Anal.* **269**:10 (2015), 3220–3263. MR Zbl

[Kolyada 1993] V. I. Kolyada, "On the embedding of Sobolev spaces", *Mat. Zametki* **54**:3 (1993), 48–71, 158. In Russian; translated in *Math Notes* **54**:3 (1993), 908–922. MR Zbl

[Nazarov et al. 2001] F. Nazarov, S. Treil, and A. Volberg, "Bellman function in stochastic control and harmonic analysis", pp. 393–423 in *Systems, approximation, singular integral operators, and related topics* (Bordeaux, 2000), edited by A. A. Borichev and N. K. Nikolski, Oper. Theory Adv. Appl. **129**, Birkhäuser, Basel, 2001. MR Zbl

[Ornstein 1962] D. Ornstein, "A non-equality for differential operators in the L_1 norm", *Arch. Rational Mech. Anal.* **11** (1962), 40–49. MR Zbl

[Osękowski 2012] A. Osękowski, *Sharp martingale and semimartingale inequalities*, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) **72**, Springer, Basel, 2012. MR Zbl

[Pełczyński and Wojciechowski 1992] A. Pełczyński and M. Wojciechowski, "Paley projections on anisotropic Sobolev spaces on tori", *Proc. London Math. Soc.* (3) **65**:2 (1992), 405–422. MR Zbl

[Solonnikov 1972] V. A. Solonnikov, "Certain inequalities for functions from the classes $\vec{W}_p(R^n)$", *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **27** (1972), 194–210. In Russian. MR

[Stolyarov and Zatitskiy 2016] D. M. Stolyarov and P. B. Zatitskiy, "Theory of locally concave functions and its applications to sharp estimates of integral functionals", *Adv. Math.* **291** (2016), 228–273. MR Zbl

[Van Schaftingen 2013] J. Van Schaftingen, "Limiting Sobolev inequalities for vector fields and canceling linear differential operators", *J. Eur. Math. Soc.* (*JEMS*) **15**:3 (2013), 877–921. MR Zbl

[Van Schaftingen 2014] J. Van Schaftingen, "Limiting Bourgain–Brezis estimates for systems of linear differential equations: theme and variations", *J. Fixed Point Theory Appl.* **15**:2 (2014), 273–297. MR Zbl

[Volberg 2011] A. Volberg, "Bellman function technique in harmonic analysis", Lectures of INRIA Summer School, Antibes, 2011. arXiv

[Wojciechowski 1991] M. Wojciechowski, "Translation invariant projections in Sobolev spaces on tori in the L^1 and uniform norms", *Studia Math.* **100**:2 (1991), 149–167. MR Zbl

[Wojciechowski 1993] M. Wojciechowski, "Characterizing translation invariant projections on Sobolev spaces on tori by the coset ring and Paley projections", *Studia Math.* **104**:2 (1993), 181–193. MR Zbl

Received 26 Jan 2016. Revised 6 Oct 2016. Accepted 14 Nov 2016.

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ANALYSIS AND PDE
Vol. 10, No. 2, 2017
dx.doi.org/10.2140/apde.2017.10.367

A NOTE ON STABILITY SHIFTING FOR THE MUSKAT PROBLEM, II: FROM STABLE TO UNSTABLE AND BACK TO STABLE

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In this note, we show that there exist solutions of the Muskat problem which shift stability regimes in the following sense: they start stable, then become unstable, and finally return back to the stable regime. This proves the existence of double stability shifting in the direction opposite to (as well as more difficult and more surprising than) the one shown by Cordoba et al. (*Philos. Trans. Roy. Soc. A* **373**:2050 (2015), art. id. 20140278).

1. Introduction

In this paper, we study two incompressible fluids with the same viscosity but different densities, ρ^+ and ρ^-, evolving in a two-dimensional porous medium with constant permeability κ. The velocity v is determined by Darcy's law

$$\mu\frac{v}{\kappa} = -\nabla p - g\begin{pmatrix}0\\ \rho\end{pmatrix}, \tag{1-1}$$

where p is the pressure, $\mu > 0$ viscosity, and $g > 0$ gravitational acceleration. In addition, v is incompressible:

$$\nabla\cdot v = 0. \tag{1-2}$$

By rescaling properly, we can assume $\kappa = \mu = g = 1$. The fluids also satisfy the conservation of mass equation

$$\partial_t\rho + v\cdot\nabla\rho = 0. \tag{1-3}$$

This is known as the Muskat problem [1937]. We denote by Ω^+ the region occupied by the fluid with density ρ^+ and by Ω^- the region occupied by the fluid with density $\rho^- \neq \rho^+$. The point $(0,\infty)$ belongs to Ω^+, whereas the point $(0,-\infty)$ belongs to Ω^-. All quantities with superindex $\pm$ will refer to $\Omega^\pm$ respectively. The interface between both fluids at any time t is a planar curve $z(\,\cdot\,,t)$. We will work in the setting of horizontally periodic interfaces, although our results can be extended to the flat-at-infinity case.

A quantity that will play a major role in this paper is the Rayleigh–Taylor condition, which is defined as

$$\mathrm{RT}(x,t) = -\big[\nabla p^-(z(x,t)) - \nabla p^+(z(x,t))\big]\cdot\partial_x^\perp z(x,t),$$

where we use the convention $(u,v)^\perp = (-v,u)$. If $\mathrm{RT}(x,t) > 0$ for all $x\in\mathbb{R}$, then we say that the curve is in the Rayleigh–Taylor stable regime at time t, and if $\mathrm{RT}(x,t)\le 0$ for some $x\in\mathbb{R}$, then we say that the curve is in the Rayleigh–Taylor unstable regime.

MSC2010: primary 35Q35; secondary 35R35, 65G30, 76B03.
Keywords: Muskat problem, interface, incompressible fluid, porous media, Rayleigh–Taylor, computer-assisted.

One can rewrite the system (1-1)–(1-3) in terms of the curve $z = (z^1, z^2)$, obtaining

$$\partial_t z(x,t) = \frac{\rho^- - \rho^+}{2\pi} \,\text{P.V.}\int_{\mathbb{R}} \frac{z^1(x,t) - z^1(y,t)}{|z(x,t) - z(y,t)|^2}(\partial_x z(x,t) - \partial_y z(y,t))\,dy. \tag{1-4}$$

In the horizontally periodic case with $z(x+2\pi,t) = z(x,t) + (2\pi, 0)$, the formula

$$\frac{1}{2}\cot\frac{y}{2} = \frac{1}{y} + \sum_{n=1}^{\infty} \frac{2y}{y^2 - (2\pi n)^2}$$

can be used to show [Castro et al. 2012] that the velocity satisfies

$$\partial_t z(x,t) = \frac{\rho^- - \rho^+}{4\pi} \int_{\mathbb{T}} \frac{\sin(z^1(x,t) - z^1(y,t))(\partial_x z(x,t) - \partial_y z(y,t))}{\cosh(z^2(x,t) - z^2(y,t)) - \cos(z^1(x,t) - z^1(y,t))}\,dy. \tag{1-5}$$

A simple calculation of the Rayleigh–Taylor condition in terms of z yields

$$\text{RT}(x,t) = (\rho^- - \rho^+)\partial_x z^1(x,t).$$

When the interface is a graph, parametrized as $z(x,t) = (x, f(x,t))$, equation (1-4) becomes

$$\partial_t f(x,t) = \frac{\rho^- - \rho^+}{4\pi} \int_{\mathbb{T}} \frac{\sin(x-y)(\partial_x f(x,t) - \partial_y f(y,t))}{\cosh(f(x,t) - f(y,t)) - \cos(x-y)}\,dy \tag{1-6}$$

and the Rayleigh–Taylor condition simplifies to

$$\text{RT}(x,t) = \rho^- - \rho^+.$$

The curve is now in the RT stable regime whenever $\rho^+ < \rho^-$; that is, the denser fluid is at the bottom. From now on, we assume that $\rho^- - \rho^+ = 4\pi$, which can be done after an appropriate scaling in time.

The Muskat problem has been studied in many works. A proof of local existence of classical solutions in the Rayleigh–Taylor stable regime in H^3 and ill-posedness in the unstable regime appears in [Córdoba and Gancedo 2007]. See also [Constantin et al. 2016b] for an improvement on the regularity (to $W^{2,p}$ spaces). In the one-phase case (i.e., when one of the densities and permeabilities is zero) local existence in H^2 was proved in [Cheng et al. 2016].

A maximum principle for $\|\partial_x f(\cdot,t)\|_{L^\infty}$ can be found in [Córdoba and Gancedo 2009]. Moreover, the authors showed in [Córdoba and Gancedo 2009] that if $\|\partial_x f_0\|_{L^\infty} < 1$, then $\|\partial_x f(\cdot,t)\|_{L^\infty} \le \|\partial_x f_0\|_{L^\infty}$ for all $t > 0$. Further work has shown existence of finite-time turning [Castro et al. 2012] (i.e., the curve ceases to be a graph in finite time and the Rayleigh–Taylor condition changes sign to negative somewhere along the curve). The gap between these two results (i.e., the question whether the constant 1 is sharp or not for guaranteeing global existence) is still an open question, and there is numerical evidence of data with $\|\partial_x f_0\|_{L^\infty} = 50$ which turns over [Córdoba et al. 2015].

As was demonstrated in [Castro et al. 2013], the curve may lose regularity after shifting from the stable regime to the unstable regime. However, the possibility of it recoiling and returning to the stable regime has not been excluded. The occurrence of this phenomenon is the main result of this note, Theorem 2.1. (In Theorem 2.3 we also extend this to a proof of existence of the quadruple stability shift scenario unstable → stable → unstable → stable → unstable.) In [Córdoba et al. 2015] we showed that there exist curves which undergo the unstable → stable → unstable transition, so this settles the question about existence of double stability shift scenarios in both directions. We stress that existence of the stable → unstable → stable scenario is in fact by no means expected, as well as considerably more challenging to establish than the unstable → stable → unstable one.

More general models, which take into account finite depth or nonconstant permeability, and which also exhibit (single) turning were studied in [Berselli et al. 2014; Gómez-Serrano and Granero-Belinchón 2014]. The estimates in [Gómez-Serrano and Granero-Belinchón 2014] were carried out by rigorous computer-assisted methods, as opposed to the traditional pencil and paper ones in [Berselli et al. 2014].

Concerning global existence, the first proof for small initial data was carried out in [Siegel et al. 2004] in the case where the fluids have different viscosities and the same densities (see also [Córdoba and Gancedo 2007] for the setting of the present paper — different densities and the same viscosities — and also [Cheng et al. 2016] for the general case). Global existence for medium-sized initial data was established in [Constantin et al. 2013; 2016a]. In the case where surface tension is taken into account, global existence was shown in [Escher and Matioc 2011; Friedman and Tao 2003]. Global existence for the confined case was treated in [Granero-Belinchón 2014]. A blow-up criterion was found in [Constantin et al. 2016b].

Recent advances in computing power have made possible rigorous computer-assisted proofs. Of course, floating-point operations can result in numerical errors, and we will employ interval arithmetics to deal with this issue. Instead of working with arbitrary real numbers, we perform computations over intervals which have representable numbers as endpoints. On these objects, an arithmetic is defined in such a way that we are guaranteed that for every $x \in X$, $y \in Y$,

$$x \star y \in X \star Y$$

for any operation $\star$. For example,

$$\begin{aligned}[]
[\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] &= [\underline{x} + \underline{y}, \bar{x} + \bar{y}], \\
[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] &= \big[\min\{\underline{x}\,\underline{y}, \underline{x}\,\bar{y}, \bar{x}\,\underline{y}, \bar{x}\,\bar{y}\}, \max\{\underline{x}\,\underline{y}, \underline{x}\,\bar{y}, \bar{x}\,\underline{y}, \bar{x}\,\bar{y}\}\big].
\end{aligned}$$

We can also define the interval version of a function $f(X)$ as an interval I that satisfies that for every $x \in X$, we have $f(x) \in I$. Rigorous computation of integrals has been theoretically developed since the seminal works of Moore and many others (see [Berz and Makino 1999; Krämer and Wedner 1996; Lang 2001; Moore 1979; Tucker 2011] for just a small sample). For readability purposes, instead of writing the intervals as, for instance, [123456, 123789], we will sometimes instead refer to them as 123^{456}_{789}.

This note is organized as follows. In Section 2 we prove Theorems 2.1 and 2.3, and in Section 3 we provide technical details regarding the performance and implementation of the computations. The appendix, found in an online supplement, contains a detailed derivation and enumeration of all the necessary integrals which have to be rigorously computed, their enclosures, and the performance of the computations.

2. The main result

The following theorem is the main result of this paper (see also Theorem 2.3 below).

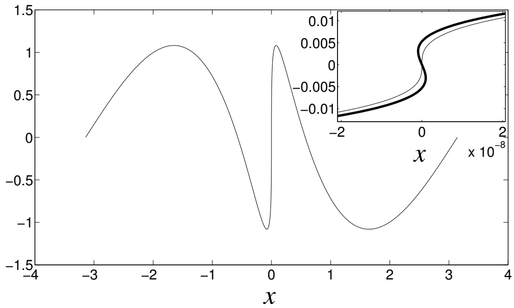

Figure 1. The curve $z_\varepsilon(x,0)$ from Lemma 2.2 with $A(\varepsilon) = 1.08050$. Inset: closeup around $x = 0$. Thick curve: $\varepsilon = 10^{-6}$, thin curve: $\varepsilon = 0$. We remark that both curves are indistinguishable at the larger scale.

Theorem 2.1. *There exist $T > \gamma > 0$ and a spatially analytic solution z to (1-5) on the time interval $[-T, T]$ such that $z(\cdot\,, t)$ is a graph of a smooth function of x when $|t| \in [T-\gamma, T]$ (i.e., z is in the stable regime near $t = \pm T$) but $z(\cdot\,, t)$ is not a graph of a function of x when $|t| \le \gamma$ (i.e., z is in the unstable regime near $t = 0$).*

The intuition behind this result comes from the numerical experiments which were started in [Córdoba et al. 2015]. These suggested existence of curves which are (barely) in the unstable regime, and such that the evolution both forward and backwards in time transports them into the stable regime. (We note that neither the velocity nor any other quantity was observed to become degenerate in these experiments). We remark that this behavior is purely nonlinear and thus nonlinear effects may dominate the linear ones under certain conditions. The following lemma constructs a family of such curves (see Figure 1).

Lemma 2.2. *Let $\varepsilon \ge 0$ and consider the initial curve $z_\varepsilon(x,0) = (z^1_\varepsilon(x,0), z^2_\varepsilon(x,0))$, with*

$$z^1_\varepsilon(x,0) = x - \sin(x) - \varepsilon \sin(x) \quad \textit{and} \quad z^2_\varepsilon(x,0) = A(\varepsilon)\sin(2x).$$

(1) *For any $\varepsilon \in [0, 10^{-6}]$, there exists $A(\varepsilon) \in (1.08050, 1.08055)$ such that if z_ε solves (1-5) with initial data $z_\varepsilon(x,0)$, then*

$$\partial_t \partial_x z^1_\varepsilon(0,0) = 0.$$

(2) *There are $T > 0$ and $C \ge 1$, independent of ε, such that for any $\varepsilon \in [0, 10^{-6}]$ and $A(\varepsilon)$ from (1), there is a unique analytic solution z_ε of (1-5) on the time interval $(-T, T)$ with initial data $z_\varepsilon(x,0)$, and it satisfies*

$$\partial_{tt} \partial_x z^1_\varepsilon(0,0) \ge 30, \tag{2-1}$$

as well as

$$|\partial_t \partial_x z^1_\varepsilon(x,t)| + |\partial_t \partial_x^3 z^1_\varepsilon(x,t)| + |\partial_t^2 \partial_x^2 z^1_\varepsilon(x,t)| + |\partial_t^3 \partial_x z^1_\varepsilon(x,t)| \le C \tag{2-2}$$

for each $(x,t) \in \mathbb{T} \times (-T, T)$.

Proof. The proofs of (1) and (2-1) are computer-assisted, and the codes can be found in the online supplement.

Let us start with (1). Since $\partial_t\partial_x z_\varepsilon^1(0,0)$ (i.e., the spatial derivative of the first coordinate of the right-hand side of (1-5) at $(x,t)=(0,0)$) is a continuous function of $A(\varepsilon)$, it suffices to show that the signs of $\partial_t\partial_x z_\varepsilon^1(0,0)$ for $A(\varepsilon)=1.08050$ and for $A(\varepsilon)=1.08055$ are different for each $\varepsilon\in[0,10^{-6}]$. This holds because for each such ε we obtain the bounds

$$\begin{aligned} \partial_t\partial_x z_\varepsilon^1(0,0) &\in 0.000_{27}^{01} && \text{for } A(\varepsilon)=1.08050,\\ \partial_t\partial_x z_\varepsilon^1(0,0) &\in -0.000_{02}^{28} && \text{for } A(\varepsilon)=1.08055. \end{aligned} \tag{2-3}$$

Existence and uniqueness of the solution z_ε in (2) follows directly from the proof of Theorem 5.1 in [Castro et al. 2012], which proves local well-posedness for (1-5) in the class of analytic functions of x. The time $T>0$ is then uniform in all small ε (and $\sup_{|t|<T}\|\partial_x^k z_\varepsilon(\cdot,t)\|_{L^\infty}$ is also uniformly bounded for each k) because the same is true for all the estimates in that proof.

Then (2-1) follows by taking $A(\varepsilon)=[1.08050,1.08055]$ (the full interval, since we do not know $A(\varepsilon)$ explicitly) and propagating this interval in the relevant computations. Specifically, we obtain

$$\partial_{tt}\partial_x z_\varepsilon^1(0,0)\in[38.706,48.787].$$

The proof of Theorem 5.1 in [Castro et al. 2012] shows that the chord-arc constant

$$\sup_{x,y\in\mathbb{T},\,y\neq 0}\frac{|y|}{|z_\varepsilon(x,t)-z_\varepsilon(x-y,t)|}$$

(where $\mathbb{T}=[-\pi,\pi]$ with $\pm\pi$ identified) is bounded uniformly in all ε,t under consideration (provided $T>0$ is small enough). Thus there is $D<\infty$ such that

$$\left|\frac{1}{\cosh(z_\varepsilon^2(x,t)-z_\varepsilon^2(x-y,t))-\cos(z_\varepsilon^1(x,t)-z_\varepsilon^1(x-y,t))}\right|\leq\frac{D}{y^2}$$

for all these ε,t and all $x,y\in\mathbb{T}$. This allows us to obtain (2-2) by brute force, differentiating and estimating all the resulting terms separately. The most singular term in $\partial_t\partial_x^j z_\varepsilon^1(x,t)$ ($j=1,3$) is

$$\int_{\mathbb{T}}\frac{\sin(z_\varepsilon^1(x,t)-z_\varepsilon^1(x-y,t))(\partial_x^{j+1}z_\varepsilon^1(x,t)-\partial_x^{j+1}z_\varepsilon^1(x-y,t))}{\cosh(z_\varepsilon^2(x,t)-z_\varepsilon^2(x-y,t))-\cos(z_\varepsilon^1(x,t)-z_\varepsilon^1(x-y,t))}\,dy\leq 2\pi D\|\partial_x^{j+2}z_\varepsilon^1(\cdot,t)\|_{L^\infty}\leq C$$

for some C which is uniform in ε due to the above-mentioned uniform bounds on $\|\partial_x^k z_\varepsilon(\cdot,t)\|_{L^\infty}$. Analogously, the most singular term in $\partial_t^2\partial_x^2 z_\varepsilon^1(x,t)$ is given by

$$\int_{\mathbb{T}}\frac{\sin(z_\varepsilon^1(x,t)-z_\varepsilon^1(x-y,0))(\partial_t\partial_x^3 z_\varepsilon^1(x,t)-\partial_t\partial_x^3 z_\varepsilon^1(x-y,t))}{\cosh(z_\varepsilon^2(x,t)-z_\varepsilon^2(x-y,t))-\cos(z_\varepsilon^1(x,t)-z_\varepsilon^1(x-y,t))}\,dy\leq 2\pi D\|\partial_t\partial_x^4 z_\varepsilon^1(\cdot,t)\|_{L^\infty},$$

and the last term can be bounded by a uniform C in the same way as $\partial_t\partial_x^3 z_\varepsilon^1(x,t)$. Finally, the most singular term in $\partial_t^3\partial_x z_\varepsilon^1(x,t)$ is

$$\int_{\mathbb{T}}\frac{\sin(z_\varepsilon^1(x,t)-z_\varepsilon^1(x-y,t))(\partial_t^2\partial_x^2 z_\varepsilon^1(x,t)-\partial_t^2\partial_x^2 z_\varepsilon^1(x-y,t))}{\cosh(z_\varepsilon^2(x,t)-z_\varepsilon^2(x-y,t))-\cos(z_\varepsilon^1(x,t)-z_\varepsilon^1(x-y,t))}\,dy\leq 2\pi D\|\partial_t^2\partial_x^3 z_\varepsilon^1(\cdot,t)\|_{L^\infty},$$

which is bounded by a uniform C in the same way as $\partial_t^2\partial_x^2 z_\varepsilon^1(x,t)$ (with the bound this time involving the uniformly bounded quantity $\|\partial_x^7 z_\varepsilon^1(\cdot,t)\|_{L^\infty}$). $\square$

Proof of Theorem 2.1. Let T, C be from Lemma 2.2. Then (2-2) shows that for any small enough $\varepsilon > 0$ and any $|t| \le \sqrt{\varepsilon}$ and $|x| \in [2C^{1/2}\varepsilon^{1/4}, \pi]$ we have $\partial_x z_\varepsilon^1(x,t) > 0$. Indeed, this is because

$$1-(1+\varepsilon)\cos(2C^{1/2}\varepsilon^{1/4}) - C\sqrt{\varepsilon} > 0$$

when $\varepsilon > 0$ is small (since C is fixed).

Next let $|t| \le \sqrt{\varepsilon}$ and $|x| \le 2C^{1/2}\varepsilon^{1/4}$. Then there are $|x^\sharp|, |x^{\sharp\sharp}| \le |x|$ and $|t^\sharp| \le \sqrt{\varepsilon}$ such that

$$\begin{aligned}\partial_x z_\varepsilon^1(x,t) =& \partial_x z_\varepsilon^1(x,0) + t\partial_t\partial_x z_\varepsilon^1(x,0) + \tfrac{1}{2}t^2\partial_t^2\partial_x z_\varepsilon^1(x,0) + \tfrac{1}{6}t^3\partial_t^3\partial_x z_\varepsilon^1(x,t^\sharp)\\ =& -\varepsilon\cos(x) + [1-\cos(x)] + t\big[\partial_t\partial_x z_\varepsilon^1(0,0) + x\partial_t\partial_x^2 z_\varepsilon^1(0,0) + \tfrac{1}{2}x^2\partial_t\partial_x^3 z_\varepsilon^1(x^\sharp,0)\big]\\ &+ \tfrac{1}{2}t^2\big[\partial_t^2\partial_x z_\varepsilon^1(0,0) + x\partial_t^2\partial_x^2 z_\varepsilon^1(x^{\sharp\sharp},0)\big] + \tfrac{1}{6}t^3\partial_t^3\partial_x z_\varepsilon^1(x,t^\sharp)\\ \ge& -\varepsilon + x^2\big(\tfrac{1}{4} - \tfrac{1}{2}C|t|\big) + t^2\big(15 - \tfrac{1}{2}C|x| - \tfrac{1}{6}C|t|\big),\end{aligned}$$

where we used the estimates from Lemma 2.2 and also that $\partial_t\partial_x^2 z_\varepsilon^1(0,0) = 0$ by oddness of z_ε. Since $|t| \le \sqrt{\varepsilon}$ and $|x| \le 2C^{1/2}\varepsilon^{1/4}$, taking small enough $\varepsilon > 0$ now yields $\partial_x z_\varepsilon^1(x,t) \ge 14t^2 - \varepsilon > 0$ for all $|t| \in \big[\frac{1}{2}\sqrt{\varepsilon}, \sqrt{\varepsilon}\big]$ and $|x| \le 2C^{1/2}\varepsilon^{1/4}$. The theorem then follows with $z = z_\varepsilon$, $T = \sqrt{\varepsilon}$, and $\gamma = \varepsilon/(2C)$ for such ε (here we also used (2-2) and $\partial_x z_\varepsilon^1(0,0) = -\varepsilon$). □

We next show that our approach allows for the proof of existence of solutions which exhibit even more complicated stability shifting. We will construct a solution with an unstable → stable → unstable → stable → unstable stability regime profile.

We start by noticing that it suffices to consider solutions to (1-4) with periodicity of the form

$$z(x+8N\pi, t) = z(x,t) + (8N\pi, 0)$$

for some integer $N \ge 1$, because then $\tilde{z}(x,t) = 1/(4N)z(4Nx, 4Nt)$ also solves (1-4) and $\tilde{z}(x+2\pi,t) = \tilde{z}(x,t) + (2\pi,0)$. Our initial data will be a perturbation of the $8N\pi$-periodic extension of the odd function

$$z(x,0) = \bar{z}_{A(0)}(x)\chi_{[0,N\pi]}(|x|) + \bar{z}_{1.08055}(x)\chi_{(N\pi,3N\pi]}(|x|) + \bar{z}_{1.08050}(x)\chi_{(3N\pi,4N\pi]}(|x|), \tag{2-4}$$

with $\bar{z}_B(x) = (x - \sin x, B\sin(2x))$ and $A(0) \in (1.08050, 1.08055)$ from Lemma 2.2. If N is large, the estimates from the lemma and its proof show that at time $t = 0$, the corresponding solution wants to make the shifts stable → unstable → stable at $x = 0$, stable → unstable at $|x| = 2N\pi$, and unstable → stable at $|x| = 4N\pi$. An appropriate perturbation of this initial data, which makes the unstable phase of the first shift last a positive length of time, delays the second shift, and brings the third shift forward in time, would then achieve our goal.

We will also need this perturbation to resolve some other issues. Specifically, the initial condition must be analytic so that we can solve the PDE both forward and backward in time, and the solution must remain stable near $x = 2n\pi$ for any integer n with $|n| \in (0, 2N) \setminus \{N\}$ (note that the tangent to $z(\cdot, 0)$ is vertical at these points). For any large N we therefore let

$$\begin{aligned}&B_{N,A}(x)\\ &= \big[A + (1.08055 - A)\phi(|x| - N\pi)\big]\chi_{[0,3N\pi]}(|x|) + \big[1.08050 + 0.00005\phi(3N\pi + 1 - |x|)\big]\chi_{(3N\pi,4N\pi]}(|x|),\end{aligned}$$

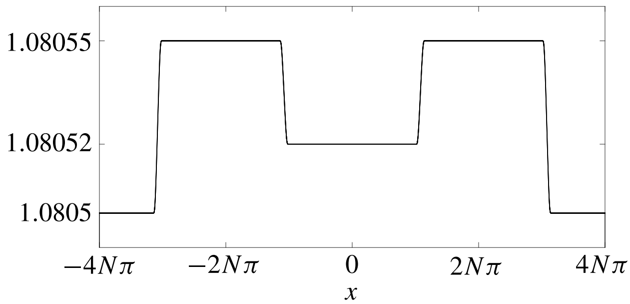

Figure 2. $B_{N,A}(x)$ for $A = 1.08052$.

with $A \in [1.08050, 1.08055]$ and $0 \le \phi \le 1$ smooth such that $\phi(y) = 0$ for $y \le 0$ and $\phi(y) = 1$ for $y \ge 1$, and we extend $B_{N,A}$ to $\mathbb{R}$ periodically (with period $8N$). See Figure 2 for a depiction of $B_{N,A}(x)$. Next we let δ_y be the delta function at $y \in \mathbb{R}$, and define the $8N\pi$-periodic odd functions

$$\bar{z}_{N,A}(x) = (x - \sin x,\, B_{N,A}(x)\sin(2x))$$

and

$$z_{N,A,r,a,c}(\,\cdot\,,0) = P_r * \bar{z}_{N,A} + P_1 * (a\beta_{N,0} + c\beta_{N,2N\pi} + c\beta_{N,-2N\pi} + c\beta_{N,4N\pi}, 0),$$

with

$$P_r(x) = \frac{1}{\pi}\frac{r}{x^2 + r^2}$$

the Poisson kernel for the half-plane (note that $P_r * \mathrm{Id}_{\mathbb{R}} = \mathrm{Id}_{\mathbb{R}}$) and $\beta_{N,y}(x) = \beta_N(x-y)$, where β_N is the (unique and $8N\pi$-periodic) primitive of

$$\frac{1}{8N\pi} - \sum_{n\in\mathbb{Z}} \delta_{8N\pi n}$$

which has $\int_{-4N\pi}^{4N\pi} \beta_N(x)\,dx = 0$. This and the smoothness of ϕ means $z_{N,A,r,a,c}(\,\cdot\,,0)$ can be extended analytically to the strip $S_r = \mathbb{R} \times [-r, r]$ and this extension satisfies for each $k \ge 0$,

$$\sup_{\substack{N\ge 1,\ A\in[1.08050,1.08055]\\ r,a,c\in[0,1/2],\ |\zeta|\le r}} \|\partial_x^k z_{N,A,r,a,c}(\,\cdot + i\zeta, 0)\|_{L^\infty} < \infty.$$

Before we continue, let us discuss the different components of the function $z_{N,A,r,a,c}(\,\cdot\,,0)$. First, $\bar{z}_{N,A}$ is just a smooth version of the function from (2-4), and we convolve it with P_r because we need the initial condition to be analytic. Since $\partial_x \bar{z}^1_{N,A} \ge 0$, this yields $\partial_x z_{N,A,r,a,c}(\,\cdot\,,0) > 0$ for any $r > 0$. That would mean that for a short (positive and negative) time, the solution would remain stable everywhere — in particular, near $x = 2n\pi$ for any integer n with $|n| \in (0, 2N) \setminus \{N\}$ as we want (see above). However, we do not want this to be the case near $x \in 2N\pi\mathbb{Z}$, which is where the term $P_1 * (\cdots)$ comes in. It is analytic and we will choose a, c to be close to the unique $a_{N,r} > 0$ such that (2-9) and (2-10) below hold. In fact, we will have $a = a_{N,r} - \delta$ and $c = a_{N,r} - \varepsilon$ for some small $0 < \delta \approx 3\varepsilon/(8N-1) \ll 1$, chosen so that $\partial_x z^1_{N,A,r,a,c}(x,0) > 0$ for $x \in 2N\pi\mathbb{Z} \setminus 8N\pi\mathbb{Z}$ and (2-12) holds. We will then finally choose $A = A_{N,r,\delta,\varepsilon} \in (1.08050, 1.08055)$ so that (2-11) also holds, and all this will ensure that $z_{N,A,r,a,c}$

undergoes the stability shifts described after (2-4). We note that we will first have to choose N large and $r > 0$ small so that (2-5), (2-6), and (2-8) below hold for all $a, c \approx a_{N,r}$ (which is small if r is). This will then specify $a_{N,r}$, after which sufficiently small ε, δ will be chosen and they will determine $A_{N,r,\delta,\varepsilon}$.

The proof of Theorem 5.1 in [Castro et al. 2012] shows that for each $r > 0$ there is T_r (depending only on r) such that (1-4) has a unique analytic solution $z_{N,A,r,a,c}$ on the time interval $(-T_r, T_r)$ with initial condition $z_{N,A,r,a,c}(\cdot, 0)$ (moreover, $\partial_t z_{N,A,r,a,c}$ is also analytic), and this satisfies for each $k \geq 0$,

$$\sup_{\substack{N\geq 1,\ A\in[1.08050,1.08055]\\ r,a,c\in[0,1/2],\ |t|<T_r}} \left(\|\partial_x^k z_{N,A,r,a,c}(\,\cdot\,,t)\|_{L^\infty} + \|\partial_t\partial_x^k z_{N,A,r,a,c}(\,\cdot\,,t)\|_{L^\infty}\right) < \infty.$$

Below we always consider $A \in [1.08050, 1.08055]$ and $r, a, c \in \left[0, \frac{1}{2}\right]$.

This means that the bound (2-2) extends to each $z_{N,A,r,a,c}$ and $(x,t) \in \mathbb{R} \times (-T_r, T_r)$ (where $T_0 = 0$), with a uniform C. We also have

$$\partial_t\partial_x z^1_{N,A,r,a,c}(4N\pi, 0) \geq 10^{-6} \quad \text{and} \quad \partial_t\partial_x z^1_{N,A,r,a,c}(\pm 2N\pi, 0) \leq -10^{-6}, \tag{2-5}$$

as well as

$$\partial_t\partial_x z^1_{N,1.08050,r,a,c}(0, 0) \geq 10^{-6} \quad \text{and} \quad \partial_t\partial_x z^1_{N,1.08055,r,a,c}(0, 0) \leq -10^{-6}, \tag{2-6}$$

both when $N^{-1} + r + a + c$ is small enough. This follows from the bounds (2-3) and from

$$\|\partial_x^k z_{N,A,r,a,c}(\,\cdot\,, 0) - \partial_x^k \bar{z}_{N,A}\|_{L^\infty(I_N)} \to 0 \qquad \text{as } N^{-1} + r + a + c \to 0 \tag{2-7}$$

for each k, where $I_N = \bigcup_{n\in\mathbb{Z}}(2N\pi n - N, 2N\pi n + N)$. Similarly, (2-2) and (2-7) also prove

$$\partial_{tt}\partial_x z^1_{N,A,r,a,c}(0, 0) \geq 20 \tag{2-8}$$

for small enough $N^{-1} + r + a + c$. Fix now N so that (2-5), (2-6), and (2-8) hold for all small enough $r + a + c$.

We next notice that for each $r > 0$ we have $\partial_x z^1_{N,A,r,0,0}(x) = 1 - (P_r * \cos)(x)$, which is a 2π-periodic function with a positive minimum at $x = 0$ (independent of N, A). Thus there is a unique $a_{N,r} > 0$ (small if $r > 0$ is small) such that

$$\partial_x z^1_{N,A,r,a_{N,r},a_{N,r}}(2N\pi n, 0) = 0 \tag{2-9}$$

for each $n \in \mathbb{Z}$, and

$$\partial_x z^1_{N,A,r,a_{N,r},a_{N,r}}(x, 0) > 0 \tag{2-10}$$

for $x \notin 2N\pi\mathbb{Z}$. Finally, for any $\delta, \varepsilon \in [0, a_{N,r})$ let $A_{N,r,\delta,\varepsilon} \in (1.08050, 1.08055)$ be such that

$$\partial_t\partial_x z^1_{N,A_{N,r,\delta,\varepsilon},r,a_{N,r}-\delta,a_{N,r}-\varepsilon}(0, 0) = 0, \tag{2-11}$$

which exists due to (2-6) and the continuity of $\partial_t\partial_x z^1_{N,A,r,a_{N,r}-\delta,a_{N,r}-\varepsilon}(0, 0)$ in A.

For the sake of simplicity of notation, define $z = z_{N,A_{N,r,\delta,\varepsilon},r,a_{N,r}-\delta,a_{N,r}-\varepsilon}$. We now let $r > 0$ be small enough, and then pick $\delta, \varepsilon \in (0, a_{N,r})$ small enough (we will need $\varepsilon \ll 10^{-6} T_r$, see below) such that

$$0 < -\partial_x z^1(0, 0) \ll \left[\frac{1}{C} \min_{n\in\{-1,1,2\}} \partial_x z^1(2N\pi n, 0)\right]^2, \tag{2-12}$$

with C the constant from (2-2) for $z_{N,A,r,a,c}$. (That is, $\varepsilon > 0$ is chosen small and δ slightly smaller than the value that makes $\partial_x z^1(0,0) = 0$ for this ε, which means that

$$\delta = \frac{3\varepsilon}{8N-1} - O\left(\frac{\varepsilon}{N^2}\right);$$

moreover, then all three values inside the min are $\approx \varepsilon/\pi$.) Then we claim that this z is the desired solution. Indeed, $\partial_x z^1(0,t) < 0$ for all small enough $|t|$ and the argument from the proof of Theorem 2.1 shows that $\partial_x z^1(x,t) > 0$ for all $x \in \mathbb{R}$ when

$$|\partial_x z^1(0,0)|^{1/2} \ll |t| \ll \frac{\varepsilon}{C\pi}.$$

Finally, (2-5) and a uniform bound on $\partial_t^2 \partial_x z^1_{N,A,r,a,c}$ (obtained similarly to (2-2)) show that

$$\partial_x z^1(4N\pi, -t) < 0 \quad \text{and} \quad \partial_x z^1(\pm 2N\pi, t) < 0$$

for all $t \approx 10^6\varepsilon$ if $\varepsilon \ll 10^{-6} T_r$ is small enough (because $\partial_x z^1(4N\pi, 0) \approx \varepsilon/\pi \approx \partial_x z^1(\pm 2N\pi, 0)$).

We thus proved the following result.

Theorem 2.3. *There exist $T > T' > \gamma > 0$ and a spatially analytic solution z to (1-5) on the time interval $[-T, T]$ such that $z(\cdot, t)$ is a graph of a smooth function of x when $|t| \in [T'-\gamma, T'+\gamma]$ but $z(\cdot, t)$ is not a graph of a function of x when $|t| \in [0,\gamma] \cup [T-\gamma, T]$.*

3. Technical details of the numerical implementation

In this section, we give some technical details of the implementation of the computer-assisted part of the proof of Lemma 2.2. In order to perform the rigorous computations we used the C-XSC library [Hofschuster and Krämer 2004]. We refer the reader to the appendices, found in the online supplement, to see a detailed expression of the integral terms involved in the calculations. For the sake of readability, we kept the same names for the integrals in the paper and in the code. The code can also be found in the online supplement.

The implementation is split into several files, and many of the headers of the functions (such as the integration methods) contain pointers to functions (the integrands) so that they can be reused for an arbitrary number of integrals with minimal changes and easy and safe debugging. For the sake of clarity, and at the cost of numerical performance and duplicity in the code, we decided to treat many simple integrals instead of a single big one.

We start discussing the details of the first part of Lemma 2.2, corresponding to the one-dimensional integrals. The three integrals can be found in Appendix A. We split them into two parts: a nonsingular one over the interval $[\delta, \pi]$ and a singular one over the interval $[0, \delta]$. The nonsingular part is calculated using a Gauss–Legendre quadrature of order 2, given by

$$\int_a^b f(\eta)\,d\eta \in \frac{b-a}{2}\left(f\left(\frac{b-a}{2}\frac{\sqrt{3}}{3} + \frac{b+a}{2}\right) + f\left(-\frac{b-a}{2}\frac{\sqrt{3}}{3} + \frac{b+a}{2}\right)\right) + \frac{1}{4320}(b-a)^5 f^4([a,b]).$$

Moreover, the integration is done in an adaptive way. For each region, we accepted or rejected the result depending on the width in an absolute and a relative way. It is important to notice that because of the uncertainty in ε and/or overestimation, division by zero might occur, even in small integration intervals. We used $\delta = 2^{-9}$ and tolerances `AbsTol` and `RelTol` equal to 10^{-6}.

In the singular region, the singularity around $y = 0$ is integrable; hence the integral is finite. We performed a Taylor expansion around $y = 0$ in both the numerator and denominator (resp. of orders 2, 2 and 4 for A_1, A_2 and A_3), simplified the powers of y and then integrated. Potentially this could fail because the uncertainty in the parameters or overestimation could yield a Taylor series in which 0 belongs to the coefficient of the first nonsimplified power of the denominator. Whenever this happens, we try to integrate instead using a Gauss–Legendre quadrature of order 2.

The maximum number of subdivision levels was 18 (2^{18} intervals) for the bounded region and 12 (2^{12} intervals) for the singular one. The splitting of the intervals is done in an arithmetic way; i.e, we split an integration interval $[a, b]$ into $\left[a, \frac{1}{2}(a+b)\right]$ and $\left[\frac{1}{2}(a+b), b\right]$.

In the second part of Lemma 2.2 we have to deal with 41 two-dimensional integrals (see Appendix B for a detailed list of them and their derivation). The first step is to exploit the symmetry of the integrands in (y, z)-variables to integrate only over the region $[0, \pi] \times [-\pi, \pi]$. We will distinguish four different regions labeled in the following way: nonsingular $([\delta, \pi] \times [\delta, \pi]) \cup ([\delta, \pi] \times [-\pi, -\delta])$, singular-first-coordinate $([0, \delta] \times [\delta, \pi]) \cup ([0, \delta] \times [-\pi, -\delta])$, singular-second-coordinate $[\delta, \pi] \times [-\delta, \delta]$ and singular-center $[0, \delta] \times [-\delta, \delta]$.

The nonsingular region was integrated as before, using a 2D Gauss–Legendre quadrature of order 2. The singular-center region was integrated in the following way. Assuming $\operatorname{sign}(a) = \operatorname{sign}(b)$, $\operatorname{sign}(c) = \operatorname{sign}(d)$ and that we expand up to orders num_y, den_y, num_z, den_z:

$$\begin{aligned}\int_a^b \int_c^d \frac{\mathrm{Num}(y,z)}{\mathrm{Den}(y,z)}\,dy\,dz &\in \int_a^b \int_c^d \frac{\frac{1}{\mathrm{num\_}y!\mathrm{num\_}z!}\partial_y^{\mathrm{num\_}y}\partial_z^{\mathrm{num\_}z}\mathrm{Num}(A,B)\,y^{\mathrm{num\_}y}z^{\mathrm{num\_}z}}{\frac{1}{\mathrm{den\_}y!\mathrm{den\_}z!}\partial_y^{\mathrm{den\_}y}\partial_z^{\mathrm{den\_}z}\mathrm{Den}(A,B)\,y^{\mathrm{den\_}y}z^{\mathrm{den\_}z}}\,dy\,dz \\ &= \frac{1}{1+\mathrm{num\_}y-\mathrm{den\_}y}\,\frac{1}{1+\mathrm{num\_}z-\mathrm{den\_}z}\,\frac{\mathrm{den\_}y!\mathrm{den\_}z!}{\mathrm{num\_}y!\mathrm{num\_}z!} \\ &\qquad \times \frac{\partial_y^{\mathrm{num\_}y}\partial_z^{\mathrm{num\_}z}\mathrm{Num}(A,B)}{\partial_y^{\mathrm{den\_}y}\partial_z^{\mathrm{den\_}z}\mathrm{Den}(A,B)}\,y^{1+\mathrm{num\_}y-\mathrm{den\_}y}z^{1+\mathrm{num\_}z-\mathrm{den\_}z}\Big|_{z=c}^{z=d}\Big|_{y=a}^{y=b},\end{aligned}$$

where A is the convex hull of $\{0, a, b\}$ and B is the convex hull of $\{0, c, d\}$. For the singular-first-coordinate and singular-second-coordinate regions the same procedure was applied taking num_z = den_z = 0, $B = [c, d]$ and num_y = den_y = 0, $A = [a, b]$ respectively. A detailed list of the orders of each of the integrals can be found in Table 1 in Appendix C. Whenever the Taylor-based formulas failed because of 0 being enclosed in the denominator terms, we tried to integrate using 2D Gauss–Legendre of order 2.

In this two-dimensional setting, we used a geometric splitting (in both coordinates) in the nonsingular region, arithmetic in the singular-center and singular-first-coordinate and hybrid in the singular-second-coordinate (see below). The geometric splitting consists in splitting by the geometric mean as opposed to the arithmetic one (i.e., assuming a and b have the same sign and are nonzero, we split $[a, b]$ into $[a, \sqrt{ab}\,\operatorname{sign}(a)]$ and $[\sqrt{ab}\,\operatorname{sign}(a), b]$). While the arithmetic division minimizes the length of the longest piece after the division, the geometric one minimizes the piece with the biggest ratio between its endpoints.

This can be particularly useful in many cases: for example in order to avoid divisions by zero for integrands of the type $1/(y - A\sin(y))$, which is a simplified version of some of the denominators that appear in all the terms. Detailed results of the breakdown by region and by term can be found in Table 2 of Appendix C.

We chose $\delta = 2^{-5}$, and `AbsTol` and `RelTol` equal to 10^{-4}. We changed the number of maximum subdivision levels depending on the region and (possibly) depending on the terms. For the nonsingular region, the maximum number was 10 (2^{20} intervals). In the singular-first-coordinate, the maximum number of subdivisions was 8 (2^{16} intervals), and that number was also used in the singular-center region. The singular-second-coordinate region was treated differently: all terms other than B_{47} and B_{55} were further split into three subregions: $[\delta, 0.65]\times[-\delta, \delta]$, $[0.65, 0.95]\times[-\delta, \delta]$ and $[0.95, \pi]\times[-\delta, \delta]$ and setting the maximum number of subdivisions to 9 in each subregion. The first and second subregions were computed using arithmetic splitting, whereas the third one was split geometrically only in the first coordinate, and arithmetically in the second.

The singular-second-coordinate regions of the terms B_{47} and B_{55} are highly singular because of the cubic denominators and they required special precision. They were subdivided into six subregions: namely $[\delta, 0.325]\times[-\delta, \delta]$, $[0.325, 0.65]\times[-\delta, \delta]$, $[0.65, 0.775]\times[-\delta, \delta]$, $[0.775, 0.95]\times[-\delta, \delta]$, $[0.95, 1.5]\times[-\delta, \delta]$ and $[1.5, \pi]\times[-\delta, \delta]$. The maximum number of subdivisions was 10 in each subregion. The last two subregions were split geometrically in the first coordinate, arithmetically in the second. The other four subregions were split arithmetically in each of the coordinates.

The simulations were run on the NewComp cluster at Princeton University. Each of the programs was run on a core of 2 Xeon X5680 CPUs (6 cores each, 12 in total) at 3.33 GHz and 8 GB of RAM. The total runtime was about 3.5 min for the first part of Lemma 2.2. For the second part, the different runtimes are summarized in Table 3 of Appendix C.

Acknowledgements

Córdoba and Gómez-Serrano were partially supported by the grant MTM2014-59488-P (Spain) and ICMAT Severo Ochoa project SEV-2011-0087. Gómez-Serrano was partially supported by an AMS-Simons Travel Grant. Zlatoš acknowledges partial support by NSF grants DMS-1652284 and DMS-1656269. We thank Princeton University for computing facilities (NewComp cluster). We are also very grateful to the anonymous referees for their valuable comments and suggestions.

References

[Berselli et al. 2014] L. C. Berselli, D. Córdoba, and R. Granero-Belinchón, "Local solvability and turning for the inhomogeneous Muskat problem", *Interfaces Free Bound.* **16**:2 (2014), 175–213. MR Zbl

[Berz and Makino 1999] M. Berz and K. Makino, "New methods for high-dimensional verified quadrature", *Reliab. Comput.* **5**:1 (1999), 13–22. MR Zbl

[Castro et al. 2012] Á. Castro, D. Córdoba, C. Fefferman, F. Gancedo, and M. López-Fernández, "Rayleigh–Taylor breakdown for the Muskat problem with applications to water waves", *Ann. of Math.* (2) **175**:2 (2012), 909–948. MR Zbl

[Castro et al. 2013] Á. Castro, D. Córdoba, C. Fefferman, and F. Gancedo, "Breakdown of smoothness for the Muskat problem", *Arch. Ration. Mech. Anal.* **208**:3 (2013), 805–909. MR Zbl

[Cheng et al. 2016] C. H. A. Cheng, R. Granero-Belinchón, and S. Shkoller, "Well-posedness of the Muskat problem with H^2 initial data", *Adv. Math.* **286** (2016), 32–104. MR Zbl

[Constantin et al. 2013] P. Constantin, D. Córdoba, F. Gancedo, and R. M. Strain, "On the global existence for the Muskat problem", *J. Eur. Math. Soc.* (*JEMS*) **15**:1 (2013), 201–227. MR Zbl

[Constantin et al. 2016a] P. Constantin, D. Córdoba, F. Gancedo, L. Rodríguez-Piazza, and R. M. Strain, "On the Muskat problem: global in time results in 2D and 3D", *Amer. J. Math.* **138**:6 (2016), 1455–1494.

[Constantin et al. 2016b] P. Constantin, F. Gancedo, R. Shvydkoy, and V. Vicol, "Global regularity for 2D Muskat equations with finite slope", *Ann. Inst. H. Poincaré Anal. Non Linéaire* (online publication September 2016).

[Córdoba and Gancedo 2007] D. Córdoba and F. Gancedo, "Contour dynamics of incompressible 3-D fluids in a porous medium with different densities", *Commun. Math. Phys.* **273**:2 (2007), 445–471. MR Zbl

[Córdoba and Gancedo 2009] D. Córdoba and F. Gancedo, "A maximum principle for the Muskat problem for fluids with different densities", *Comm. Math. Phys.* **286**:2 (2009), 681–696. MR Zbl

[Córdoba et al. 2015] D. Córdoba, J. Gómez-Serrano, and A. Zlatoš, "A note on stability shifting for the Muskat problem", *Philos. Trans. Roy. Soc. A* **373**:2050 (2015), art. id. 20140278. MR Zbl

[Escher and Matioc 2011] J. Escher and B.-V. Matioc, "On the parabolicity of the Muskat problem: well-posedness, fingering, and stability results", *Z. Anal. Anwend.* **30**:2 (2011), 193–218. MR Zbl

[Friedman and Tao 2003] A. Friedman and Y. Tao, "Nonlinear stability of the Muskat problem with capillary pressure at the free boundary", *Nonlinear Anal.* **53**:1 (2003), 45–80. MR Zbl

[Gómez-Serrano and Granero-Belinchón 2014] J. Gómez-Serrano and R. Granero-Belinchón, "On turning waves for the inhomogeneous Muskat problem: a computer-assisted proof", *Nonlinearity* **27**:6 (2014), 1471–1498. MR Zbl

[Granero-Belinchón 2014] R. Granero-Belinchón, "Global existence for the confined Muskat problem", *SIAM J. Math. Anal.* **46**:2 (2014), 1651–1680. MR Zbl

[Hofschuster and Krämer 2004] W. Hofschuster and W. Krämer, "C-XSC 2.0 — a C++ library for extended scientific computing", pp. 15–35 in *Numerical software with result verification* (Dagstuhl, Germany, 2003), edited by R. Alt et al., Springer, Berlin, 2004.

[Krämer and Wedner 1996] W. Krämer and S. Wedner, "Two adaptive Gauss–Legendre type algorithms for the verified computation of definite integrals", *Reliab. Comput./Nadezhn. Vychisl.* **2**:3 (1996), 241–253. MR Zbl

[Lang 2001] B. Lang, "Derivative-based subdivision in multi-dimensional verified gaussian quadrature", pp. 145–152 in *Symbolic algebraic methods and verification methods* (Dagstuhl, Germany, 1999), edited by G. Alefeld et al., Springer, Vienna, 2001. MR Zbl

[Moore 1979] R. Moore, *Methods and applications of interval analysis*, Studies in applied and numerical mathematics **2**, SIAM, Philadelphia, PA, 1979. MR Zbl

[Muskat 1937] M. Muskat, "The flow of fluids through porous media", *J. Appl. Phys.* **8**:4 (1937), 274–282. Zbl

[Siegel et al. 2004] M. Siegel, R. E. Caflisch, and S. Howison, "Global existence, singular solutions, and ill-posedness for the Muskat problem", *Comm. Pure Appl. Math.* **57**:10 (2004), 1374–1411. MR Zbl

[Tucker 2011] W. Tucker, *Validated numerics: a short introduction to rigorous computations*, Princeton University Press, 2011. MR Zbl

Received 27 Jan 2016. Revised 26 Oct 2016. Accepted 26 Dec 2016.

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mathematical sciences publishers msp

ANALYSIS AND PDE
Vol. 10, No. 2, 2017
dx.doi.org/10.2140/apde.2017.10.379

DERIVATION OF AN EFFECTIVE EVOLUTION EQUATION FOR A STRONGLY COUPLED POLARON

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Fröhlich's polaron Hamiltonian describes an electron coupled to the quantized phonon field of an ionic crystal. We show that in the strong coupling limit the dynamics of the polaron are approximated by an effective nonlinear partial differential equation due to Landau and Pekar, in which the phonon field is treated as a classical field.

1. Introduction and main result

1A. ***Setting of the problem.*** In this paper we are interested in the dynamics of a strongly coupled polaron. A polaron is a model of an electron in an ionic lattice interacting with its surrounding polarization field. Fröhlich [1937] proposed a quantum-mechanical Hamiltonian, given in (1-1) below, in order to describe the dynamics of a polaron. In this model the phonon field is treated as a quantum field. The Fröhlich Hamiltonian depends on a single parameter $\alpha > 0$ which describes the strength of the coupling between the electron and the phonon field. Landau and Pekar [1948] proposed a system of nonlinear PDEs, see (1-8), (1-9) below, to describe the dynamics of a polaron and used this in their famous computation of the effective polaron mass (see [Spohn 1987] for an alternative approach). They treat the phonons as a classical field. The derivation of their equations is phenomenological and they do not comment on the relation between their equations and the dynamics generated by Fröhlich's Hamiltonian. Our purpose in this paper is to establish a connection between the two dynamics and to rigorously derive the Landau–Pekar equations from the Fröhlich dynamics in the strong coupling limit $\alpha \to \infty$ for a natural class of initial conditions and on certain time scales.

In order to describe this result in detail, we recall that the Fröhlich Hamiltonian acts in $\mathcal{L}^2(\mathbb{R}^3) \otimes \mathcal{F}$, where $\mathcal{L}^2(\mathbb{R}^3)$ corresponds to the electron and $\mathcal{F} = \mathcal{F}(\mathcal{L}^2(\mathbb{R}^3))$, the bosonic Fock space over $\mathcal{L}^2(\mathbb{R}^3)$, corresponds to the phonon field. The Hamiltonian is given by

$$p^2 + \sqrt{\alpha} \int_{\mathbb{R}^3} [e^{-ik\cdot x} a_k + e^{ik\cdot x} a_k^*] \frac{dk}{|k|} + \int_{\mathbb{R}^3} a_k^* a_k \, dk, \tag{1-1}$$

where $p := -i\nabla_x$ and x are momentum and position of the electron and a_k and a_k^* are annihilation and creation operators in $\mathcal{F}$ satisfying the commutation relations

$$[a_k, a_{k'}^*] = \delta(k - k'), \quad [a_k, a_{k'}] = 0, \quad \text{and} \quad [a_k^*, a_{k'}^*] = 0. \tag{1-2}$$

MSC2010: 35Q40, 46N50.
Keywords: polaron, dynamics, quantized field.

As mentioned before, the scalar $\alpha > 0$ describes the strength of the coupling between the electron and the phonon field and will be large in our study.

To facilitate later discussions we rescale the variables, as in [Frank and Schlein 2014],

$$x \mapsto \alpha^{-1} x, \quad k \mapsto \alpha k, \tag{1-3}$$

and find that the Hamiltonian in (1-1) is unitarily equivalent to $\alpha^2 \widetilde{H}_\alpha^F$, where the new Hamiltonian $\widetilde{H}_\alpha^F$, acting again in $\mathcal{L}^2(\mathbb{R}^3) \otimes \mathcal{F}$, is defined as

$$\widetilde{H}_\alpha^F := p^2 + \int_{\mathbb{R}^3} [e^{-ik\cdot x} b_k + e^{ik\cdot x} b_k^*] \frac{dk}{|k|} + \int_{\mathbb{R}^3} b_k^* b_k \, dk. \tag{1-4}$$

The new annihilation and creation operators $b_k := \alpha^{1/2} a_{\alpha k}$ and $b_k^* := \alpha^{1/2} a_{\alpha k}^*$ satisfy the commutation relations

$$[b_k, b_{k'}^*] = \alpha^{-2} \delta(k - k'), \quad [b_k, b_{k'}] = 0, \quad \text{and} \quad [b_k^*, b_{k'}^*] = 0. \tag{1-5}$$

We emphasize the α-dependence in (1-5).

We will discuss the dynamics generated by $\widetilde{H}_\alpha^F$ for initial conditions of the product form

$$\psi_0 \otimes W(\alpha^2 \varphi_0) \Omega. \tag{1-6}$$

Here, Ω denotes the vacuum in $\mathcal{F}$ and $W(f)$ denotes the Weyl operator,

$$W(f) := e^{b^*(f) - b(f)}, \tag{1-7}$$

so that $W(\alpha^2 \varphi)\Omega$ is a coherent state. This particular choice of initial conditions is motivated by Pekar's approximation [1946; 1951] to the ground state energy, which uses exactly states of this form. Pekar's approximation was made mathematically rigorous by Donsker and Varadhan [1983] (see [Lieb and Thomas 1997] for an alternative approach).

Clearly, the time-evolved state $e^{-i\widetilde{H}_\alpha^F t} \psi_0 \otimes W(\alpha^2 \varphi_0)\Omega$ with $t \neq 0$ will in general no longer have an exact product structure. However, we will see that for large α (and t of order one, or even larger) it can be approximated, in a certain sense, by a state of the product form $\psi_t \otimes W(\alpha^2 \varphi_t)\Omega$, where ψ_t and φ_t solve the Landau–Pekar equations

$$i \partial_t \psi_t(x) = \left[-\Delta + \int_{\mathbb{R}^3} [e^{-ik\cdot x} \varphi_t(k) + e^{ik\cdot x} \overline{\varphi_t(k)}] \frac{dk}{|k|} \right] \psi_t(x), \tag{1-8}$$

$$i \alpha^2 \partial_t \varphi_t(k) = \varphi_t(k) + |k|^{-1} \int_{\mathbb{R}^3} |\psi_t(x)|^2 e^{ik\cdot x} \, dx \tag{1-9}$$

with initial data ψ_0 and φ_0. Using standard methods one can show that for any $\psi_0 \in \mathcal{H}^1(\mathbb{R}^3)$, $\varphi_0 \in \mathcal{L}^2(\mathbb{R}^3)$ and $\alpha > 0$, the system (1-8), (1-9) has a global solution (ψ_t, φ_t), which satisfies

$$\|\psi_t\|_{\mathcal{L}^2(\mathbb{R}^3)} = \|\psi_0\|_{\mathcal{L}^2(\mathbb{R}^3)} \quad \text{and} \quad \mathcal{E}(\psi_t, \varphi_t) = \mathcal{E}(\psi_0, \varphi_0) \quad \text{for all } t \in \mathbb{R}$$

with the energy

$$\mathcal{E}(\psi, \varphi) := \int_{\mathbb{R}^3} |\nabla \psi|^2 \, dx + \int_{\mathbb{R}^3} |\psi(x)|^2 \int_{\mathbb{R}^3} \left(e^{-ik\cdot x} \varphi(k) + e^{ik\cdot x} \bar{\varphi}(k) \right) \frac{dk}{|k|} \, dx + \int_{\mathbb{R}^3} |\varphi(k)|^2 \, dk. \tag{1-10}$$

We refer to Lemma 2.1 and Proposition 2.2 for more details about the solution (ψ_t,φ_t). In the original work of Landau and Pekar the equations are given in a different, but equivalent form, and we explain this connection in Section 1D.

1B. *Main result.* In order to prove our main result we need the following regularity and decay assumptions on the initial data. We denote by $\mathcal{H}^m(\mathbb{R}^3)$ the Sobolev space of order m and by

$$\mathcal{L}^2_{(m)}(\mathbb{R}^3) := \mathcal{L}^2\big(\mathbb{R}^3,(1+k^2)^m\,dk\big) \tag{1-11}$$

the weighted $\mathcal{L}^2$ space with norm

$$\|\varphi\|_{\mathcal{L}^2_{(m)}} = \left(\int_{\mathbb{R}^3}(1+k^2)^m|\varphi(k)|^2\,dk\right)^{\frac{1}{2}}.$$

Our main result will be valid under the following:

Assumption 1.1. We assume $\psi_0\in\mathcal{H}^4(\mathbb{R}^3)$ and $\varphi_0\in\mathcal{L}^2_{(3)}(\mathbb{R}^3)$ with $\|\psi_0\|_{\mathcal{L}^2(\mathbb{R}^3)}=1$.

A first version of our main result concerns the approximation of the reduced density matrices of $e^{-i\widetilde{H}^F_\alpha t}\psi_0\otimes W(\alpha^2\varphi_0)\Omega$ in the trace norm.

Theorem 1.2. *Assume ψ_0 and φ_0 satisfy Assumption 1.1 and let (ψ_t,φ_t) be the solution of (1-8), (1-9) with initial condition (ψ_0,φ_0). Define*

$$\begin{aligned}\gamma_t^{\text{particle}} &:= \operatorname{Tr}_{\mathcal{F}}\big|e^{-i\widetilde{H}^F_\alpha t}\psi_0\otimes W(\alpha^2\varphi_0)\Omega\big\rangle\big\langle e^{-i\widetilde{H}^F_\alpha t}\psi_0\otimes W(\alpha^2\varphi_0)\Omega\big|,\\ \gamma_t^{\text{field}} &:= \operatorname{Tr}_{\mathcal{L}^2(\mathbb{R}^3)}\big|e^{-i\widetilde{H}^F_\alpha t}\psi_0\otimes W(\alpha^2\varphi_0)\Omega\big\rangle\big\langle e^{-i\widetilde{H}^F_\alpha t}\psi_0\otimes W(\alpha^2\varphi_0)\Omega\big|.\end{aligned}$$

Then, for all $\alpha\ge 1$ and all $t\in[-\alpha,\alpha]$,

$$\begin{aligned}\operatorname{Tr}_{\mathcal{L}^2(\mathbb{R}^3)}\big|\gamma_t^{\text{particle}}-|\psi_t\rangle\langle\psi_t|\big| &\le C\alpha^{-2}(1+t^2),\\ \operatorname{Tr}_{\mathcal{F}}\big|\gamma_t^{\text{field}}-|W(\alpha^2\varphi_t)\Omega\rangle\langle W(\alpha^2\varphi_t)\Omega|\big| &\le C\alpha^{-2}(1+t^2).\end{aligned}$$

Note that $\gamma_t^{\text{particle}}$, γ_t^{field}, $|\psi_t\rangle\langle\psi_t|$ and $|W(\alpha^2\varphi_t)\Omega\rangle\langle W(\alpha^2\varphi_t)\Omega|$ all have trace norm equal to one (in fact, they are nonnegative operators with trace one) and therefore Theorem 1.2 gives a nontrivial approximation up to times $t=o(\alpha)$. Already the approximation up to times of order one is significant since this is the time scale on which ψ_t changes. It is a bonus that the same approximation is in fact valid for much longer times.

We emphasize that the Landau–Pekar approximation to the Fröhlich dynamics depends on α (through (1-9)). As we will explain in Section 1C, without allowing for an α-dependence one cannot approximate $\gamma_t^{\text{particle}}$ with accuracy α^{-2} for times of order one.

We next present a more precise result which comes at the expense of a more complicated formulation. We approximate the state $e^{-i\widetilde{H}^F_\alpha t}\psi_0\otimes W(\alpha^2\varphi_0)\Omega$ itself in $\mathcal{L}^2(\mathbb{R}^3)\otimes\mathcal{F}$, and not only its reduced density matrices. However, it turns out that up to the desired order α^{-2} this is *not* possible in terms of simple product states. Instead, we need to include an explicit nonproduct state of order α^{-1} which takes correlations between the particle and the field into account. The key observation is that this term satisfies an almost orthogonality condition, so that it does not contribute to the reduced density matrices to order α^{-1}. For the statement we need the real scalar function ω defined as

$$\omega(t) := \alpha^2 \operatorname{Im}(\varphi_t, \partial_t \varphi_t) + \|\varphi_t\|^2_{\mathcal{L}^2(\mathbb{R}^3)}. \tag{1-12}$$

It will follow from Lemma 2.1 below that this function is uniformly bounded in $t \in \mathbb{R}$.

The following is our main result.

Theorem 1.3. *Assume ψ_0 and φ_0 satisfy Assumption 1.1 and let (ψ_t, φ_t) be the solution of (1-8), (1-9) with initial condition (ψ_0, φ_0). Then there is a decomposition*

$$e^{-i\widetilde{H}^F_\alpha t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega = e^{-i\int_0^t \omega(s)\,ds}\psi_t \otimes W(\alpha^2\varphi_t)\Omega + R(t) \tag{1-13}$$

and a constant $C > 0$ such that for all $\alpha \geq 1$ and all $t \in [-\alpha, \alpha]$,

$$\big\|\langle \Omega, W^*(\alpha^2\varphi_t)R(t)\rangle_{\mathcal{F}}\big\|_{\mathcal{L}^2(\mathbb{R}^3)} \leq C\alpha^{-2}|t|(1+|t|), \tag{1-14}$$

$$\big\|\langle \psi_t, W^*(\alpha^2\varphi_t)R(t)\rangle_{\mathcal{L}^2(\mathbb{R}^3)}\big\|_{\mathcal{F}} \leq C\alpha^{-2}|t|(1+|t|) \tag{1-15}$$

and

$$\|R(t)\|_{\mathcal{L}^2(\mathbb{R}^3)\otimes\mathcal{F}} \leq C\alpha^{-1}(1+|t|). \tag{1-16}$$

More precisely, (1-13) holds with $R(t) = R_1(t) + R_2(t)$ and with the bounds

$$\big\|\langle \Omega, W^*(\alpha^2\varphi_t)R_1(t)\rangle_{\mathcal{F}}\big\|_{\mathcal{L}^2(\mathbb{R}^3)} \leq C\alpha^{-2}t^2, \tag{1-17}$$

$$\big\|\langle \psi_t, W^*(\alpha^2\varphi_t)R_1(t)\rangle_{\mathcal{L}^2(\mathbb{R}^3)}\big\|_{\mathcal{F}} \leq C\alpha^{-2}t^2 \tag{1-18}$$

and

$$\|R_2(t)\|_{\mathcal{L}^2(\mathbb{R}^3)\otimes\mathcal{F}} \leq C\alpha^{-2}|t|(1+|t|), \quad \|R_1(t)\|_{\mathcal{L}^2(\mathbb{R}^3)\otimes\mathcal{F}} \leq C\alpha^{-1}(1+|t|). \tag{1-19}$$

Similarly as before, we note that for $t = o(\alpha)$ the term $R(t)$ is of lower order than the main term $e^{-i\int_0^t \omega(s)\,ds}\psi_t \otimes W(\alpha^2\varphi_t)\Omega$, which has constant norm equal to one.

The message of Theorem 1.3 is that, while $R(t)$ is in general not of order α^{-2} (for times of order one), it can be split into a piece which is, namely $R_2(t)$, and a piece which satisfies almost orthogonality conditions, so that it does not contribute to the reduced particle or field density matrices at order α^{-1} either. The term $R_1(t)$ is given explicitly in (2-16) below.

Theorem 1.3 implies Theorem 1.2 by a simple abstract argument, which we explain in Appendix D. In the following we concentrate on proving Theorem 1.3.

In Section 1C we compare Theorem 1.3 with a similar approximation in [Frank and Schlein 2014] where φ_t is independent of t. In Lemma 1.4 we show that this simpler approximation does not yield the same accuracy in terms of powers of α^{-1} as Theorem 1.3. In this sense Theorem 1.3 derives the Landau–Pekar dynamics from the Fröhlich dynamics and answers an open question in [Frank and Schlein 2014].

While it is necessary to take the time dependence of φ_t into account, this dependence is still weak for times of order α as considered in our theorems. The field φ_t changes by order one only on times of order α^2, and it would be desirable to extend Theorems 1.2 and 1.3 to this time scale, at least for a certain class of initial conditions. This remains an open problem.

The key point in Theorem 1.3 and novel aspect of this work are the almost orthogonality relations (1-14) and (1-15). As we will see in Section 1C, they will be crucial for deriving Theorem 1.2. Inequality (1-16) is not sufficient for this purpose. Let us discuss the motivation behind the almost orthogonality relations in more detail. We introduce the function

$$\tilde{\psi}_t := e^{-i\int_0^t \omega(s)\,ds}\psi_t \tag{1-20}$$

and consider the problem of approximating $e^{-i\widetilde{H}^F_\alpha t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega$ by a function of the form $\tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega$. (We do *not* assume at this point that $\tilde{\psi}_t$ and φ_t satisfy an equation.) Since $W(\alpha^2\varphi_t)$ is unitary, this is the same as the problem of choosing $\tilde{\psi}_t$ and φ_t so as to minimize the norm of the vector

$$W^*(\alpha^2\varphi_t)e^{-i\widetilde{H}^F_\alpha t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega - \tilde{\psi}_t \otimes \Omega. \tag{1-21}$$

Clearly, for given ψ_0, φ_0 and φ_t, the optimal choice for $\tilde{\psi}_t$ is

$$\tilde{\psi}_t = \langle \Omega, W^*(\alpha^2\varphi_t)e^{-i\widetilde{H}^F_\alpha t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega\rangle_{\mathcal{F}}. \tag{1-22}$$

In order to determine φ_t we only solve the simpler problem of minimizing the norm of the projection of (1-21) onto the subspace $\operatorname{span}\{\tilde{\psi}_t\} \otimes \mathcal{F}$. This norm could be made zero if we could achieve

$$\Omega = \langle \tilde{\psi}_t, W^*(\alpha^2\varphi_t)e^{-i\widetilde{H}^F_\alpha t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega\rangle_{\mathcal{L}^2}. \tag{1-23}$$

While it may not be possible to have exact equalities in (1-22) and (1-23), we will see that the Landau–Pekar equations yield almost equalities. In fact, the almost orthogonality relations (1-14) and (1-15) in our main theorem state exactly that:

$$\tilde{\psi}_t - \langle \Omega, W^*(\alpha^2\varphi_t)e^{-i\widetilde{H}^F_\alpha t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega\rangle_{\mathcal{F}} = O_{\mathcal{L}^2}\big(\alpha^{-2}|t|(1+|t|)\big), \tag{1-24}$$

$$\Omega - \langle \tilde{\psi}_t, W^*(\alpha^2\varphi_t)e^{-i\widetilde{H}^F_\alpha t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega\rangle_{\mathcal{L}^2} = O_{\mathcal{F}}\big(\alpha^{-2}|t|(1+|t|)\big). \tag{1-25}$$

1C. *Comparison with earlier results.* The problem of approximating the Fröhlich dynamics of a polaron was studied before in [Frank and Schlein 2014]. There a different and simpler effective equation is proposed in which only the particles move and the phonon field remains constant. In this subsection we show that Theorem 1.2 is *not* valid for these effective dynamics from [Frank and Schlein 2014], in the sense that the reduced phonon density matrix cannot be approximated to within an error α^{-2} for times of order one. The fact that our Theorem 1.2 achieves an approximation at this accuracy is because the phonon motion is taken into account in the Landau–Pekar equations. Technically this is reflected in the orthogonality conditions (1-14) and (1-15).

To be more specific we recall that in [Frank and Schlein 2014] it was shown that

$$\big\| e^{-i\widetilde{H}^F_\alpha t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega - e^{-i\|\varphi_0\|_2^2 t}\zeta_t \otimes W(\alpha^2\varphi_0)\Omega \big\|_{\mathcal{L}^2\otimes\mathcal{F}} \le C\alpha^{-1}(e^{C|t|}-1)^{\frac{1}{2}}, \tag{1-26}$$

where ζ_t denotes the solution of the *linear* equation

$$i\partial_t\zeta_t(x) = \left[-\Delta + \int_{\mathbb{R}^3} \big[e^{-ikx}\varphi_0(k) + e^{ik\cdot x}\overline{\varphi_0(k)}\big]\frac{dk}{|k|}\right]\zeta_t(x)$$

with initial condition ψ_0. We stress again that in this approximation, φ_0 does not evolve in time. An anonymous referee, to whom we are most grateful, has explained to us that the method of [Frank and Schlein 2014] actually leads to the bound

$$\left\| e^{-i\widetilde{H}_\alpha^F t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega - e^{-i\|\varphi_0\|_2^2 t}\zeta_t \otimes W(\alpha^2\varphi_0)\Omega \right\|_{\mathcal{L}^2\otimes\mathcal{F}} \le C(e^{C|t|/\alpha}-1)^{\frac{1}{2}}, \tag{1-27}$$

which provides an approximation even up to times of order $o(\alpha)$. With his/her kind permission we reproduce the argument in Appendix E.

As an aside we note that we recover a similar bound as a simple consequence of Theorem 1.3. (In fact, our new bound is better by a power of $\alpha^{-\frac{1}{2}}$ for times $t \gg 1$.) Namely, (1-16) says that

$$\left\| e^{-i\widetilde{H}_\alpha^F t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega - e^{-i\int_0^t \omega(s)\,ds}\psi_t \otimes W(\alpha^2\varphi_t)\Omega \right\|_{\mathcal{L}^2\otimes\mathcal{F}} \le C\alpha^{-1}(1+|t|). \tag{1-28}$$

(In [Frank and Schlein 2014] weaker regularity and decay assumptions are imposed on ψ_0 and φ_0, but we emphasize that (1-16) is also valid under weaker assumptions than those in Assumption 1.1. In fact, the latter assumption is needed to bound $R_2(t)$, whereas for (1-16) one can avoid the use of Duhamel's principle in Proposition 2.3.)

For the reduced density matrices, inequalities (1-26) and (1-27) give, using (D-1) and possibly changing the value of C,

$$\operatorname{Tr}_{\mathcal{L}^2}\left|\gamma_t^{\text{particle}} - |\zeta_t\rangle\langle\zeta_t|\right| \le C\min\left\{\alpha^{-1}(e^{C|t|}-1)^{\frac{1}{2}}, (e^{C|t|/\alpha}-1)^{\frac{1}{2}}\right\},$$
$$\operatorname{Tr}_{\mathcal{L}^2}\left|\gamma_t^{\text{field}} - |W(\alpha^2\varphi_0)\Omega\rangle\langle W(\alpha^2\varphi_0)\Omega|\right| \le C\min\left\{\alpha^{-1}(e^{C|t|}-1)^{\frac{1}{2}}, (e^{C|t|/\alpha}-1)^{\frac{1}{2}}\right\}.$$

These bounds behave like α^{-1} for times of order one.

The next result shows that in this approximation of γ_t^{field} by a time-independent φ_0 the order α^{-1} (for times of order one) cannot be improved in general.

Lemma 1.4. *In addition to Assumption 1.1 suppose that $\varphi_0 \not\equiv -\sigma_{\psi_0}$ in the notation (2-2). Then there are $\varepsilon > 0$, $C > 0$ and $c > 0$ such that for all $|t| \in [C\alpha^{-1}, \varepsilon]$ and all $\alpha \ge C/\varepsilon$,*

$$\operatorname{Tr}_{\mathcal{F}}\left|\gamma_t^{\text{field}} - |W(\alpha^2\varphi_0)\Omega\rangle\langle W(\alpha^2\varphi_0)\Omega|\right| \ge c\alpha^{-1}|t|.$$

This lemma should be contrasted with Theorem 1.2, which says that the time-dependent approximation $|W(\alpha^2\varphi_t)\Omega\rangle\langle W(\alpha^2\varphi_t)\Omega|$ is correct to order α^{-2} (for times of order one). This argument shows the importance of the orthogonality conditions (1-14) and (1-15). Indeed, if we would only use (1-16), we would arrive at (1-28) and this would again only give an approximation to order α^{-1} (for times of order one).

Since Theorem 1.2 is a consequence of Theorem 1.3 and since we showed that one cannot replace φ_t by φ_0 in Theorem 1.2, the same applies also to Theorem 1.3.

Let us consider our problem from a wider perspective. We have a composite quantum system $\mathcal{H}_1 \otimes \mathcal{H}_2$ and a Hamiltonian which couples the two subsystems. Each system has an effective "Planck constant" and the characteristic feature of the problem is that the Planck constant of one system goes to zero, whereas that of the other system remains fixed. Thus, one of the systems becomes classical, whereas the other one remains quantum-mechanical, and Ginibre, Nironi and Velo [Ginibre et al. 2006] used the term "partially classical limit" in a closely related context. (For us, the "Planck constant" of the phonons is α^{-2}, as can be seen from the commutation relations, whereas that of the electron is of order one.) A prime example of such a problem is the Born–Oppenheimer approximation, where the inverse square root of the nuclear mass plays the role of the small Planck constant.

Here, however, we consider the case where $\mathcal{H}_1 \otimes \mathcal{H}_2$ has infinitely many degrees of freedom. As is well known, our Hamiltonian is the Wick quantization of an energy functional on an infinite-dimensional phase space and the notion of "Planck constant" has a well-defined meaning through the commutation relations of the fields. (We emphasize that in our problem we can imagine that we have also a field Ψ for the electrons, but that we only consider the sector of a single electron.)

Although there is an enormous literature concerning the classical limit, starting with Hepp's work [1974], and although we believe that the question of a partially classical limit is a very natural one which appears in many models, we are only aware of the single work [Ginibre et al. 2006] prior to [Frank and Schlein 2014] on this question, and it studies fluctuation dynamics. Closer to our focus here are the works [Falconi 2013; Ammari and Falconi 2014] about the Nelson model with a cut-off where, however, a classical limit on *both* systems is taken. On the level of results, one obtains equations similar to the Landau–Pekar equations (without the factor α^2 in (1-9)), but the proofs are completely different, as [Ammari and Falconi 2014] relies on the Wigner measure approach from [Ammari and Nier 2008; 2009].

The polaron model, in contrast to the Nelson model, does not require a cut-off, although this is not obvious since the operator $\int e^{ik\cdot x} b_k |k|^{-1}\, dk$ and its adjoint are not bounded relative to the number operator. Lieb and Yamazaki [1958] devised a method to deal with this problem in the stationary case, but it is not clear to us how to apply their argument in a dynamical setting and we consider our solution of this problem as a technical novelty in this paper. Our methods apply equally well to a partially classical limit in the cut-off Nelson model and, in fact, the proofs in that case would be considerably shorter.

1D. ***An equivalent form of the Landau–Pekar equations.*** Often the Landau–Pekar equations are stated in the form

$$i\partial_t \psi_t = (-\Delta + |x|^{-1} * P_t)\psi_t, \tag{1-29}$$

$$\alpha^4 \partial_t^2 P_t = -P_t - (2\pi)^2 |\psi_t|^2 \tag{1-30}$$

for a real-valued polarization field P_t; see, e.g., [Landau and Pekar 1948; Devreese and Alexandrov 2009]. Let us show that this pair of equations is equivalent to the pair of equations that we discussed so far. In fact, assume that ψ_t and φ_t solve (1-8) and (1-9) and define

$$P_t(x) := (2\pi)^{-1} \operatorname{Re} \int_{\mathbb{R}^3} |k| \varphi_t(k) e^{-ik\cdot x}\, dk,$$

as well as the auxiliary function

$$Q_t(x) := (2\pi)^{-1} \operatorname{Im} \int_{\mathbb{R}^3} |k| \varphi_t(k) e^{-ik\cdot x}\, dk.$$

If we multiply (1-9) by $|k|$ and integrate with respect to $e^{-ik\cdot x}$, we obtain

$$i\alpha^2\partial_t(P_t+iQ_t)=P_t+iQ_t+(2\pi)^2|\psi_t|^2.$$

Since P_t and Q_t are real, this equation is equivalent to the pair of equations

$$\alpha^2\partial_t P_t=Q_t,\quad \alpha^2\partial_t Q_t=-P_t-(2\pi)^2|\psi_t|^2.$$

Here we can eliminate Q_t by differentiating the first equation and arrive at (1-30).

Moreover, the inversion formula

$$\varphi_t(k)=(2\pi)^{-2}|k|^{-1}\int_{\mathbb{R}^3}(P_t+iQ_t)e^{ik\cdot x}\,dx$$

implies

$$\int_{\mathbb{R}^3}\big(e^{-ik\cdot x}\varphi_t(k)+e^{ik\cdot x}\overline{\varphi_t(k)}\big)\frac{dk}{|k|}=|x|^{-1}*P_t,$$

which yields (1-29).

2. Outline of the proof

2A. *Well-posedness of the Landau–Pekar equations.* We begin by discussing the well-posedness of the equations for ψ_t and φ_t in (1-8) and (1-9). We use the following abbreviations for the coupling terms in these equations,

$$V_\varphi(x):=\int_{\mathbb{R}^3}\big[e^{-ik\cdot x}\varphi(k)+e^{ik\cdot x}\overline{\varphi(k)}\big]\frac{dk}{|k|}\tag{2-1}$$

and

$$\sigma_\psi(k):=|k|^{-1}\int_{\mathbb{R}^3}|\psi(x)|^2e^{ik\cdot x}\,dx.\tag{2-2}$$

The following lemma, which is proved in Appendix C, states global well-posedness in the energy space $\mathcal{H}^1(\mathbb{R}^3)\times\mathcal{L}^2(\mathbb{R}^3)$.

Lemma 2.1. *For any $(\psi_0,\varphi_0)\in\mathcal{H}^1(\mathbb{R}^3)\times\mathcal{L}^2(\mathbb{R}^3)$ there is a unique global solution (ψ_t,φ_t) of (1-8), (1-9). One has the conservation laws*

$$\|\psi_t\|_{\mathcal{L}^2}=\|\psi_0\|_{\mathcal{L}^2}\quad\textit{and}\quad\mathcal{E}(\psi_t,\varphi_t)=\mathcal{E}(\psi_0,\varphi_0)\quad\textit{for all } t\in\mathbb{R}.$$

Moreover, for all $\alpha>0$ and all $t\in\mathbb{R}$,

$$\|\psi_t\|_{\mathcal{H}^1}\lesssim 1,\quad \|\varphi_t\|_{\mathcal{L}^2}\lesssim 1\tag{2-3}$$

and

$$\|\partial_t\varphi_t\|_{\mathcal{L}^2}\lesssim\alpha^{-2},\quad \|\varphi_t-\varphi_s\|_{\mathcal{L}^2}\lesssim\alpha^{-2}|t-s|,\quad \|\sigma_{\psi_t}\|_{\mathcal{L}^2}\lesssim 1.\tag{2-4}$$

In the proof of our main result we need to go beyond the energy space $\mathcal{H}^1(\mathbb{R}^3)\times\mathcal{L}^2(\mathbb{R}^3)$. The following proposition states that if the initial conditions have more regularity and decay then, at least for a certain (long) time interval, we have bounds on the solution in the corresponding spaces. We will also need some bounds on the auxiliary functions $g_{s,t} : \mathbb{R}^3 \to \mathbb{C}$ defined by

$$g_{s,t}(x) := \int_{\mathbb{R}^3} [\overline{\varphi_t(k)} - \overline{\varphi_s(k)}] e^{ik\cdot x} \frac{dk}{|k|} \tag{2-5}$$

and $g_s : \mathbb{R}^3 \to \mathbb{C}$ defined by

$$g_s(x) := -\partial_s g_{s,t}(x) = \int_{\mathbb{R}^3} e^{ik\cdot x} \overline{\partial_s \varphi_s(k)} \frac{dk}{|k|}. \tag{2-6}$$

The following proposition will also be proved in Appendix C.

Proposition 2.2. *Let $\tau > 0$. If (ψ_0, φ_0) satisfies Assumption 1.1, then for all $\alpha > 0$ and for all $t, s \in [-\tau\alpha^2, \tau\alpha^2]$ we have*

$$\|\psi_t\|_{\mathcal{H}^4} \lesssim_\tau 1, \quad \|\varphi_t\|_{\mathcal{L}^2_{(3)}} \lesssim_\tau 1. \tag{2-7}$$

Moreover,

$$\|\partial_t \psi_t\|_{\mathcal{H}^2} \lesssim_\tau 1, \quad \|\partial_t \sigma_{\psi_t}\|_{\mathcal{L}^2} \lesssim_\tau 1 \tag{2-8}$$

and

$$\|g_{s,t}\|_\infty \lesssim_\tau \alpha^{-2}|t-s|, \quad \|g_s\|_\infty \lesssim \alpha^{-2}. \tag{2-9}$$

2B. ***Decomposition of the solution.*** We now decompose the solution $e^{-i\widetilde{H}^F_\alpha t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega$ as claimed in Theorem 1.3. In order to state this, we need to introduce some notations.

It will be convenient to work with the function $\tilde{\psi}_t$ from (1-20). Clearly, the bounds from Lemma 2.1 and Proposition 2.2 hold for $\tilde{\psi}_t$ as well. (For the bounds on $\partial_t \tilde{\psi}_t$ we use the fact that $|\omega(t)| \lesssim 1$ by Lemma 2.1.) Moreover, we note that $\tilde{\psi}_t$ and φ_t satisfy the modified equations

$$i\partial_t \tilde{\psi}_t(x) = \left[-\Delta + \int_{\mathbb{R}^3} [e^{-ik\cdot x}\varphi_t(k) + e^{ik\cdot x}\bar{\varphi}_t(k)] \frac{dk}{|k|} + \omega(t)\right] \tilde{\psi}_t(x), \tag{2-10}$$

$$i\alpha^2 \partial_t \varphi_t(k) = \varphi_t(k) + |k|^{-1} \int_{\mathbb{R}^3} |\tilde{\psi}_t(x)|^2 e^{ik\cdot x}\, dx. \tag{2-11}$$

Next, we define for $\psi \in \mathcal{L}^2(\mathbb{R}^3)$ with $\|\psi\| = 1$ the orthogonal projections in $\mathcal{L}^2(\mathbb{R}^3)$,

$$P_\psi := |\psi\rangle\langle\psi|, \quad P_\psi^\perp := 1 - P_\psi = 1 - |\psi\rangle\langle\psi|.$$

The effective Schrödinger operator H_φ in $\mathcal{L}^2(\mathbb{R}^3)$ is defined by

$$H_\varphi := -\Delta + V_\varphi + \int_{\mathbb{R}^3} |\varphi(k)|^2\, dk \tag{2-12}$$

with V_φ from (2-1). Moreover, let us introduce the operator

$$\widetilde{H}_\varphi := W^*(\alpha^2\varphi)\widetilde{H}^F_\alpha W(\alpha^2\varphi) \tag{2-13}$$

in $\mathcal{L}^2(\mathbb{R}^3) \otimes \mathcal{F}$. Using the commutation relations (see Lemma A.1) we find that

$$\widetilde{H}_\varphi = H_\varphi + \int_{\mathbb{R}^3} [e^{ik\cdot x} b_k^* + e^{-ik\cdot x} b_k] \frac{dk}{|k|} + \int_{\mathbb{R}^3} [\varphi(k) b_k^* + \bar{\varphi}(k) b_k]\, dk + \int_{\mathbb{R}^3} b_k^* b_k\, dk. \tag{2-14}$$

Finally, we introduce the vector

$$F_{t,s} := P^{\perp}_{\tilde\psi_s} \int_{\mathbb{R}^3} \big(e^{ik\cdot x} W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) b_k^* \tilde\psi_s \otimes \Omega\big) \frac{dk}{|k|} \tag{2-15}$$

and define

$$D_0 := \int_0^t e^{iH_{\varphi_t} s} F_{t,s}\, ds$$

and

$$D_1 := \int_0^t \int_0^{t-s} \int_{\mathbb{R}^3} \big(e^{i\widetilde H_{\varphi_t}(s+s_1)} e^{ik\cdot x} b_k^* e^{-iH_{\varphi_t}s_1} F_{t,s}\big) \frac{dk}{|k|}\, ds_1\, ds,$$
$$D_2 := \int_0^t \int_0^{t-s} \int_{\mathbb{R}^3} \big(e^{i\widetilde H_{\varphi_t}(s+s_1)} e^{-ik\cdot x} b_k e^{-iH_{\varphi_t}s_1} F_{t,s}\big) \frac{dk}{|k|}\, ds_1\, ds,$$
$$D_3 := \int_0^t \int_0^{t-s} \int_{\mathbb{R}^3} \big(e^{i\widetilde H_{\varphi_t}(s+s_1)} \varphi_t(k) b_k^* e^{-iH_{\varphi_t}s_1} F_{t,s}\big)\, dk\, ds_1\, ds,$$
$$D_4 := \int_0^t \int_0^{t-s} \int_{\mathbb{R}^3} \big(e^{i\widetilde H_{\varphi_t}(s+s_1)} \overline{\varphi_t(k)} b_k e^{-iH_{\varphi_t}s_1} F_{t,s}\big)\, dk\, ds_1\, ds,$$
$$D_5 := \int_0^t \int_0^{t-s} \int_{\mathbb{R}^3} \big(e^{i\widetilde H_{\varphi_t}(s+s_1)} b_k^* b_k e^{-iH_{\varphi_t}s_1} F_{t,s}\big)\, dk\, ds_1\, ds.$$

While these definitions might seem formal, we will show in Theorem 2.5 that each of $D_0, \dots, D_5$ belongs to $\mathcal{L}^2(\mathbb{R}^3) \otimes \mathcal{F}$.

With these notations, the promised representation formula for the solution looks as follows.

Proposition 2.3. *Assume that $(\tilde\psi_t, \varphi_t)$ satisfy (2-10), (2-11) with initial conditions (ψ_0, φ_0) where $\|\psi_0\|^2 = 1$. Then for any $t \in \mathbb{R}$ one has the decomposition*

$$e^{-i\widetilde H_\alpha^F t} \psi_0 \otimes W(\alpha^2\varphi_0)\Omega = \tilde\psi_t \otimes W(\alpha^2\varphi_t)\Omega + R_1(t) + R_2(t)$$

with

$$R_1(t) := -iW(\alpha^2\varphi_t) e^{-iH_{\varphi_t} t} D_0$$

and

$$R_2(t) := -W(\alpha^2\varphi_t) e^{-i\widetilde H_{\varphi_t} t} (D_1 + D_2 + D_3 + D_4 + D_5).$$

Clearly, in terms of the original function ψ_t, the term R_1 is explicitly given by

$$R_1(t) = -iW(\alpha^2\varphi_t) \int_0^t \Big[e^{-iH_{\varphi_t}(t-s) - i\int_0^s \omega(s_1)\, ds_1} P^{\perp}_{\psi_s} \int_{\mathbb{R}^3} \big(e^{ik\cdot x} W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) b_k^* \psi_s \otimes \Omega\big) \frac{dk}{|k|} \Big] ds. \tag{2-16}$$

The proof of Proposition 2.3 makes use of equations (2-10), (2-11) for $(\tilde\psi_t, \varphi_t)$ as well as the Duhamel formula. We single out the use of the equations in the following lemma.

Lemma 2.4. *Assume that $(\tilde\psi_t, \varphi_t)$ satisfy (2-10), (2-11) with initial conditions (ψ_0, φ_0) where $\|\psi_0\|^2 = 1$. Then for any $t \in \mathbb{R}$ one has*

$$e^{-i\widetilde H_\alpha^F t} \psi_0 \otimes W(\alpha^2\varphi_0)\Omega = \tilde\psi_t \otimes W(\alpha^2\varphi_t)\Omega - i\int_0^t e^{-i\widetilde H_\alpha^F (t-s)} W(\alpha^2\varphi_t) F_{t,s}\, ds. \tag{2-17}$$

Proof of Lemma 2.4. Applying the operator $e^{i\widetilde{H}_\alpha^F t}$ to both sides of (2-17) we see that we need to prove

$$\psi_0 \otimes W(\alpha^2\varphi_0)\Omega = e^{i\widetilde{H}_\alpha^F t}\tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega - i\int_0^t e^{i\widetilde{H}_\alpha^F s} W(\alpha^2\varphi_t) F_{t,s}\, ds.$$

This is clearly true at $t = 0$ and therefore we only need to show that the time derivatives of both sides coincide for all t; that is, in view of definition (2-15) of $F_{t,s}$,

$$0 = e^{i\widetilde{H}_\alpha^F t}\Big[i\widetilde{H}_\alpha^F \tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega + \partial_t\tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega + \tilde{\psi}_t \otimes \partial_t W(\alpha^2\varphi_t)\Omega \\ - iW(\alpha^2\varphi_t)P_{\tilde{\psi}_t}^{\perp}\int_{\mathbb{R}^3}(e^{ik\cdot x}b_k^*\tilde{\psi}_t\otimes\Omega)\frac{dk}{|k|}\Big].$$

This is, of course, the same as

$$i\widetilde{H}_\alpha^F \tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega + \partial_t\tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega + \tilde{\psi}_t \otimes \partial_t W(\alpha^2\varphi_t)\Omega \\ = iW(\alpha^2\varphi_t)P_{\tilde{\psi}_t}^{\perp}\int_{\mathbb{R}^3}(e^{ik\cdot x}b_k^*\tilde{\psi}_t\otimes\Omega)\frac{dk}{|k|}, \quad (2\text{-}18)$$

which is what we are going to show now.

We begin by rewriting the first term on the left side. Using (2-13) and (2-14) we obtain

$$i\widetilde{H}_\alpha^F \tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega = iH_{\varphi_t}\tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega + \tilde{\psi}_t W(\alpha^2\varphi_t)\Big[ib^*(\varphi_t) + i\int_{\mathbb{R}^3} e^{ik\cdot x}b_k^*\frac{dk}{|k|}\Big]\Omega.$$

In order to rewrite the third term on the left side of (2-18) we use the formula for $\partial_t W(\alpha^2\varphi_t)$ from (A-4) below and find

$$\tilde{\psi}_t \otimes \partial_t W(\alpha^2\varphi_t)\Omega = i\alpha^2(\operatorname{Im}(\varphi_t, \partial_t\varphi_t))\tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega + \alpha^2\tilde{\psi}_t \otimes W(\alpha^2\varphi_t)b^*(\partial_t\varphi_t)\Omega.$$

Thus, recalling the definition of ω in (1-12), we have shown that

$$i\widetilde{H}_\alpha^F \tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega + \partial_t\tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega + \tilde{\psi}_t \otimes \partial_t W(\alpha^2\varphi_t)\Omega \\ = \big[\partial_t + i(-\Delta + V_{\varphi_t} + \omega(t))\big]\tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega \quad (2\text{-}19) \\ + W(\alpha^2\varphi_t)\Big[\alpha^2 b^*(\partial_t\varphi_t) + ib^*(\varphi_t) + i\int_{\mathbb{R}^3} e^{ik\cdot x}b_k^*\frac{dk}{|k|}\Big](\tilde{\psi}_t\otimes\Omega). \quad (2\text{-}20)$$

At this point in the proof we use the equations for $\tilde{\psi}_t$ and φ_t. It follows from (2-10) that line (2-19) vanishes identically. For line (2-20) we use (2-11) to obtain

$$i\widetilde{H}_\alpha^F \tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega + \partial_t\tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega + \tilde{\psi}_t \otimes \partial_t W(\alpha^2\varphi_t)\Omega \\ = iW(\alpha^2\varphi_t)\Big[\int_{\mathbb{R}^3}\Big(-\int_{\mathbb{R}^3}|\tilde{\psi}_t(y)|^2 e^{ik\cdot y}\,dy + e^{ik\cdot x}\Big)b_k^*\frac{dk}{|k|}\Big](\tilde{\psi}_t\otimes\Omega) \\ = iW(\alpha^2\varphi_t)P_{\tilde{\psi}_t}^{\perp}\int_{\mathbb{R}^3}(e^{ik\cdot x}b_k^*\tilde{\psi}_t\otimes\Omega)\frac{dk}{|k|}. \quad (2\text{-}21)$$

Here we used the fact that $\|\tilde{\psi}_t\| = \|\psi_0\| = 1$ by assumption and Lemma 2.1, and therefore

$$P^{\perp}_{\tilde{\psi}_t} = 1 - |\tilde{\psi}_t\rangle\langle\tilde{\psi}_t|.$$

Equation (2-21) proves (2-18) and completes the proof. □

Having proved Lemma 2.4 we turn to the proof of Proposition 2.3.

Proof of Proposition 2.3. It follows from Lemma 2.4 and (2-13) that

$$e^{-i\tilde{H}_\alpha^F t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega = \tilde{\psi}_t \otimes W(\alpha^2\varphi_t)\Omega - iW(\alpha^2\varphi_t)\int_0^t e^{-i\tilde{H}_{\varphi_t}(t-s)}F_{t,s}\,ds.$$

In the time integral on the right side we use Duhamel's principle and (2-14),

$$e^{-i\tilde{H}_{\varphi_t}(t-s)} = e^{-iH_{\varphi_t}(t-s)} - i\int_0^{t-s} e^{-i\tilde{H}_{\varphi_t}(t-s-s_1)}\biggl(\int_{\mathbb{R}^3}[e^{ik\cdot x}b_k^* + e^{-ik\cdot x}b_k]\frac{dk}{|k|} + \int_{\mathbb{R}^3} b_k^* b_k\,dk + \int_{\mathbb{R}^3}[\varphi_t(k)b_k^* + \bar{\varphi}_t(k)b_k]\,dk\biggr)e^{-iH_{\varphi_t}s_1}\,ds_1.$$

Proposition 2.3 now follows easily from the definition of $D_0, \dots, D_5$. □

2C. *Reduction of the proof of the main result.* In the remainder of this paper we will prove the following.

Theorem 2.5. *Assume that ψ_0 and φ_0 satisfy Assumption 1.1, let $(\tilde{\psi}_t, \varphi_t)$ be the solution of (2-10), (2-11) with initial condition (ψ_0, φ_0) and let $D_0, \dots, D_5$ be as in Proposition 2.3. Then there is a constant $C > 0$ such that for all $\alpha \ge 1$ and $t \in [0, \alpha^2]$,*

$$\|D_0\|_{\mathcal{L}^2\otimes\mathcal{F}} \le C\alpha^{-1}(1+t), \tag{2-22}$$

$$\|D_1\|_{\mathcal{L}^2\otimes\mathcal{F}} \le C\alpha^{-2}t(1+t), \tag{2-23}$$

$$\|D_2\|_{\mathcal{L}^2\otimes\mathcal{F}} \le \alpha^{-2}t(1+t)(1+\alpha^{-1}t), \tag{2-24}$$

$$\|D_3\|_{\mathcal{L}^2\otimes\mathcal{F}} \le C\alpha^{-2}t(1+t)(1+\alpha^{-1}t), \tag{2-25}$$

$$\|D_4\|_{\mathcal{L}^2\otimes\mathcal{F}} \le C\alpha^{-2}t^2(1+\alpha^{-1}t), \tag{2-26}$$

$$\|D_5\|_{\mathcal{L}^2\otimes\mathcal{F}} \le C\alpha^{-3}t(1+t)(1+\alpha^{-2}t^2), \tag{2-27}$$

$$\bigl\|\langle\Omega, e^{-iH_{\varphi_t}t}D_0\rangle_{\mathcal{F}}\bigr\|_{\mathcal{L}^2(\mathbb{R}^3)} \le C\alpha^{-2}t^2, \tag{2-28}$$

$$\bigl\|\langle\tilde{\psi}_t, e^{-iH_{\varphi_t}t}D_0\rangle_{\mathcal{L}^2(\mathbb{R}^3)}\bigr\|_{\mathcal{F}} \le C\alpha^{-2}t^2(1+\alpha^{-2}t^2). \tag{2-29}$$

This theorem (and its analogue for $t \in [-\alpha^2, 0]$), together with the decomposition from Proposition 2.3 and the fact that the operators $W(\alpha^2\varphi_t)$, $e^{-iH_{\varphi_t}t}$ and $e^{-i\tilde{H}_{\varphi_t}t}$ are unitary, implies Theorem 1.3. In fact, (2-22) implies the second bound in (1-19), (2-23)–(2-27) imply the first bound in (1-19), (2-28) implies (1-17) and (2-29) implies (1-18).

We emphasize that Theorem 2.5 is valid up to times α^2. (In fact, since the proof only relies on Proposition 2.2, it is valid up to times $\tau\alpha^2$ for an arbitrary $\tau > 0$ with C depending on τ.) Consequently, the bounds in Theorem 1.3 are also valid up to times α^2. However, since the evolved state and the main term in the approximation both have norm one, the bounds are only meaningful for times up to $\varepsilon\alpha$ for some small $\varepsilon > 0$.

The basic intuition behind the bounds on D_k, $k = 0, \dots, 5$, is that each annihilation or creation operator is of order α^{-1} and therefore D_0, which contains only one creation operator, is of order α^{-1}, D_1, D_2, D_3, D_4, which contain two creation or annihilation operators, are of order α^{-2} and D_5, which contains three creation or annihilation operators, is of order α^{-3}. We illustrate this intuition in more detail in Section 2E with the simplest possible terms.

While this basic principle is true, it is oversimplifying the situation considerably as it does not take the slow-decaying terms $|k|^{-1}$ into account. The operator $\int e^{ik\cdot x} b_k^* |k|^{-1}\, dk$ and its adjoint are *not* bounded relative to the number operator $\int b_k^* b_k \, dx$. In fact, the treatment of these operators is the major difficulty that we have to overcome here.

At this point we have reduced the proof of Theorem 1.3 to the proof of Theorem 2.5, and the remainder of the paper is concerned with this. We bound D_0 in Section 3, D_1 in Section 4 and D_2 in Section 5. The terms D_3, D_4 and D_5, which are easier to bound than D_1 and D_2, are briefly discussed in Section 6. Finally, the bounds (2-28) and (2-29) will be proved in Subsections 7A and 7B, respectively.

2D. ***A further decomposition.*** Using the fact that $P^{\perp}_{\tilde{\psi}_t} = 1 - |\tilde{\psi}_t\rangle\langle\tilde{\psi}_t|$ (see the proof of Lemma 2.4), we have the decomposition

$$F_{t,s} = F^{(1)}_{t,s} - F^{(2)}_{t,s},$$

where

$$F^{(1)}_{t,s} := \int_{\mathbb{R}^3} \big(e^{ik\cdot x} W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) b_k^* \tilde{\psi}_s \otimes \Omega\big) \frac{dk}{|k|}$$

and, with the notation σ_ψ from (2-2),

$$F^{(2)}_{t,s} := \tilde{\psi}_s \otimes W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) b^*(\sigma_{\tilde{\psi}_s})\Omega.$$

Correspondingly, we define

$$D_i = D_{i1} - D_{i2} \quad \text{for } i = 0, 1, 2, 3, 4, 5.$$

In general, the terms D_{i2} are easier to deal with than the terms D_{i1}. The reason for this is that $e^{ik\cdot x}|k|^{-1} \notin L^2(\mathbb{R}^3)$, whereas $\sigma_{\tilde{\psi}_t} \in L^2(\mathbb{R}^3)$ by Lemma 2.1, so the operator $\int e^{ik\cdot x} b_k^* |k|^{-1}\, dk$ in $F^{(1)}_{t,s}$ is harder to control than the operator $b^*(\sigma_{\tilde{\psi}_s})$ in $F^{(2)}_{t,s}$.

For $k = 1, \dots, 5$, both operators D_{i1} and D_{i2} involve an operator b_k^*, b_k or $b_k^* b_k$ to the left of $F^{(1)}_{t,s}$ or $F^{(2)}_{t,s}$, which in turn involves an operator $W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)$. We now have the decomposition

$$D_{ij} = D_{ij1} + D_{ij2} \quad \text{for } i = 1, 2, 3, 4, 5 \text{ and } j = 1, 2,$$

where D_{ij1} denotes the expression with b_k, b_k^* or $b_k^* b_k$ commuted through the operator $W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)$ and D_{ij2} denotes the expression coming from the commutator. To be explicit, we display some exemplary cases,

$$D_{111}=\int_0^t\int_0^{t-s}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}e^{ik\cdot x}e^{-iH_{\varphi_t}s_1}e^{ik'\cdot x} \times W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b_k^*b_{k'}^*\tilde{\psi}_s\otimes\Omega\,\frac{dk'}{|k'|}\,\frac{dk}{|k|}\,ds_1\,ds, \tag{2-30}$$

$$D_{121}=\int_0^t\int_0^{t-s}\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}e^{ik\cdot x}e^{-iH_{\varphi_t}s_1}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b_k^*b^*(\sigma_{\tilde{\psi}_s})\tilde{\psi}_s\otimes\Omega\,\frac{dk}{|k|}\,ds_1\,ds, \tag{2-31}$$

$$D_{211}=\int_0^t\int_0^{t-s}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}e^{ik\cdot x}e^{-iH_{\varphi_t}s_1}e^{ik'\cdot x} \times W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b_k b_{k'}^*\tilde{\psi}_s\otimes\Omega\,\frac{dk'}{|k'|}\,\frac{dk}{|k|}\,ds_1\,ds, \tag{2-32}$$

$$D_{221}=\int_0^t\int_0^{t-s}\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}e^{ik\cdot x}e^{-iH_{\varphi_t}s_1}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b_k b^*(\sigma_{\tilde{\psi}_s})\tilde{\psi}_s\otimes\Omega\,\frac{dk}{|k|}\,ds_1\,ds. \tag{2-33}$$

The commutator terms can be computed with the help of Corollary A.2. Recalling the definition of the function $g_{s,t}$ in (2-5), we have, for instance,

$$D_{112}=-\int_0^t\int_0^{t-s}\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}g_{s,t}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)e^{-iH_{\varphi_t}s_1}e^{ik\cdot x}b_k^*\tilde{\psi}_s\otimes\Omega\,\frac{dk}{|k|}\,ds_1\,ds, \tag{2-34}$$

$$D_{122}=-\int_0^t\int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}g_{s,t}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)e^{-iH_{\varphi_t}s_1}b^*(\sigma_{\tilde{\psi}_s})\tilde{\psi}_s\otimes\Omega\,ds_1\,ds, \tag{2-35}$$

$$D_{212}=-\int_0^t\int_0^{t-s}\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}\overline{g_{s,t}}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)e^{-iH_{\varphi_t}s_1}e^{ik\cdot x}b_k^*\tilde{\psi}_s\otimes\Omega\,\frac{dk}{|k|}\,ds_1\,ds, \tag{2-36}$$

$$D_{222}=-\int_0^t\int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}\overline{g_{s,t}}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)e^{-iH_{\varphi_t}s_1}b^*(\sigma_{\tilde{\psi}_s})\tilde{\psi}_s\otimes\Omega\,ds_1\,ds. \tag{2-37}$$

2E. *Some warm-up bounds.* In order to prepare for the rather technical sections that follow, we will first focus on the terms that do not include a term of the form $|k|^{-1}$, that is, on the terms D_{02}, D_{32}, D_{42} and D_{52}. We hope that this explains the underlying mechanism of our proof and the intuition that each annihilation or creation operator is of size α^{-1}.

Bound on D_{02}. We recall that

$$D_{02}=\int_0^t (e^{iH_{\varphi_t}s}\tilde{\psi}_s)\otimes\big(W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b^*(\sigma_{\tilde{\psi}_s})\Omega\big)\,ds$$

and, therefore, by Lemma 2.1,

$$\|D_{02}\|_{\mathcal{L}^2\otimes\mathcal{F}}\le\int_0^t\|\tilde{\psi}_s\|_2\|b^*(\sigma_{\tilde{\psi}_s})\Omega\|_{\mathcal{F}}\,ds=\alpha^{-1}\int_0^t\|\sigma_{\tilde{\psi}_s}\|_2\,ds\lesssim\alpha^{-1}t. \tag{2-38}$$

Bound on D_{32}. We have

$$D_{321}=\int_0^t\int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(e^{-iH_{\varphi_t}s_1}\tilde{\psi}_s)\otimes\big(W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b^*(\varphi_t)b^*(\sigma_{\tilde{\psi}_s})\Omega\big)\,ds_1\,ds$$

and, according to Corollary A.2,

$$D_{322} = -\int_0^t \int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(e^{-iH_{\varphi_t}s_1}\tilde{\psi}_s) \otimes \big((\varphi_t-\varphi_s,\varphi_t)W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b^*(\sigma_{\tilde{\psi}_s})\Omega\big)\,ds_1\,ds.$$

By the bounds from Lemma 2.1 we have

$$\big\|b^*(\varphi_t)b^*(\sigma_{\tilde{\psi}_s})\Omega\big\|_{\mathcal{F}} = \alpha^{-2}\big(\|\varphi_t\|_2^2\|\sigma_{\tilde{\psi}_s}\|_2^2 + |(\varphi_t,\sigma_{\tilde{\psi}_s})|^2\big)^{\frac{1}{2}} \lesssim \alpha^{-2},$$

and therefore, using also the conservation of the $\mathcal{L}^2$-norm of $\tilde{\psi}_s$,

$$\|D_{321}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2}t^2.$$

On the other hand, the bounds from Lemma 2.1 imply

$$\|b^*(\sigma_{\tilde{\psi}_s})\Omega\|_{\mathcal{F}} = \alpha^{-1}\|\sigma_{\tilde{\psi}_s}\|_2 \lesssim \alpha^{-1}, \quad |(\varphi_t-\varphi_s,\varphi_t)| \lesssim \alpha^{-2}|t-s|,$$

and therefore, using again the conservation of the $\mathcal{L}^2$-norm of $\tilde{\psi}_s$,

$$\|D_{322}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}t^3.$$

Thus, we have shown that

$$\|D_{32}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2}t^2(1+\alpha^{-1}t). \tag{2-39}$$

Bound on D_{42}. We have

$$D_{421} = \int_0^t \int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(e^{-iH_{\varphi_t}s_1}\tilde{\psi}_s) \otimes \big(W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b(\varphi_t)b^*(\sigma_{\tilde{\psi}_s})\Omega\big)\,ds_1\,ds$$

and, according to Corollary A.2,

$$D_{422} = -\int_0^t \int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(e^{-iH_{\varphi_t}s_1}\tilde{\psi}_s) \otimes \big((\varphi_t,\varphi_t-\varphi_s)W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b^*(\sigma_{\tilde{\psi}_s})\Omega\big)\,ds_1\,ds.$$

We commute once again and obtain

$$D_{421} = \int_0^t \int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(e^{-iH_{\varphi_t}s_1}\tilde{\psi}_s) \otimes \big(\alpha^{-2}(\varphi_t,\sigma_{\tilde{\psi}_s})W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)\Omega\big)\,ds_1\,ds.$$

According to Lemma 2.1 we have $|(\varphi_t,\sigma_{\tilde{\psi}_s})| \lesssim 1$. This and computations similar to those in the bound of D_{32} yield

$$\|D_{421}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2}t^2, \quad \|D_{422}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}t^3.$$

Thus, we have shown that

$$\|D_{42}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2}t^2(1+\alpha^{-1}t). \tag{2-40}$$

Bound on D_{52}. To simplify the notation, let us introduce

$$\mathcal{N} := \int_{\mathbb{R}^3} b_k^* b_k \, dk. \tag{2-41}$$

We have

$$D_{521} = \int_0^t \int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(e^{-iH_{\varphi_t}s_1}\tilde{\psi}_s) \otimes \big(W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)\mathcal{N}\, b^*(\sigma_{\tilde{\psi}_s})\Omega\big)\, ds_1\, ds.$$

Moreover, by Corollary A.2,

$$\begin{aligned}\big[\mathcal{N}, W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)\big] = {}&-W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)(b(\varphi_t)-b(\varphi_s))\\ &-W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)(b^*(\varphi_t)-b^*(\varphi_s))+W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)\|\varphi_t-\varphi_2\|_2^2,\end{aligned}$$

so

$$\begin{aligned}D_{522} = \int_0^t \int_0^{t-s} & e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(e^{-iH_{\varphi_t}s_1}\tilde{\psi}_s)\\ &\otimes \Big(W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)\big(-b(\varphi_t-\varphi_s)-b^*(\varphi_t-\varphi_s)+\|\varphi_t-\varphi_2\|_2^2\big)b^*(\sigma_{\tilde{\psi}_s})\Omega\Big)\, ds_1\, ds.\end{aligned}$$

We use $\mathcal{N} b^*(\sigma_{\tilde{\psi}_s}) = b^*(\sigma_{\tilde{\psi}_s})\mathcal{N} + \alpha^{-2} b^*(\sigma_{\tilde{\psi}_s})$ and obtain

$$D_{521} = \alpha^{-2}\int_0^t \int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(e^{-iH_{\varphi_t}s_1}\tilde{\psi}_s) \otimes \big(W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b^*(\sigma_{\tilde{\psi}_s})\Omega\big)\, ds_1\, ds.$$

Therefore, much as before,

$$\|D_{521}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}t^2.$$

For D_{522} we commute again to get

$$\begin{aligned}D_{522} = \int_0^t \int_0^{t-s} & e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(e^{-iH_{\varphi_t}s_1}\tilde{\psi}_s) \otimes \Big(W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)\\ &\times\big(-\alpha^{-2}(\varphi_t-\varphi_s,\sigma_{\tilde{\psi}_s})\Omega - b^*(\varphi_t-\varphi_s)b^*(\sigma_{\tilde{\psi}_s})\Omega + \|\varphi_t-\varphi_2\|_2^2\, b^*(\sigma_{\tilde{\psi}_s})\Omega\big)\Big)\, ds_1\, ds.\end{aligned}$$

For the second term on the right side we compute

$$\big\|b^*(\varphi_t-\varphi_s)b^*(\sigma_{\tilde{\psi}_s})\Omega\big\|_{\mathcal{F}} = \alpha^{-2}\big(\|\varphi_t-\varphi_s\|_2^2\|\sigma_{\tilde{\psi}_s}\|_2^2 + |(\varphi_t-\varphi_s,\sigma_{\tilde{\psi}_s})|^2\big)^{\frac{1}{2}}.$$

Using the bounds from Lemma 2.1 for $\|\varphi_t-\varphi_s\|_2$ we obtain that

$$\|D_{522}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-4}t^3(1+\alpha^{-1}t).$$

Thus, we have shown that

$$\|D_{52}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}t^2(1+\alpha^{-2}t^2). \tag{2-42}$$

3. Bound on D_0

We have already controlled D_{02} in (2-38), so it remains to consider D_{01}.

Bound on D_{01}. We recall that

$$D_{01} = \int_0^t e^{iH_{\varphi_t}s} \int_{\mathbb{R}^3} \big(e^{ik\cdot x} W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) b_k^* \tilde{\psi}_s \otimes \Omega\big) \frac{dk}{|k|}\, ds.$$

The main difficulty here, which we will encounter in various forms throughout this paper, is the unboundedness of the operator $\int e^{ik\cdot x} b_k^* |k|^{-1}\, dk$ (for any fixed $x \in \mathbb{R}^3$), since $e^{ik\cdot x}|k|^{-1} \notin \mathcal{L}^2(\mathbb{R}^3)$.

To overcome this difficulty we make use of the oscillatory behavior of $e^{ik\cdot x}$ via the formula

$$e^{ik\cdot x} = \frac{1 - ik\cdot\nabla_x}{1+|k|^2} e^{ik\cdot x} \tag{3-1}$$

and aim at integrating by parts with respect to x. However, this integration by parts creates a new difficulty: the resulting operator ∇_x is unbounded and has to be controlled.

To overcome this new difficulty, it will be desirable to have an operator $(-\Delta+1)^{-1}$ somewhere in the expression of D_{01} so that we can use it to control ∇_x, since obviously $\nabla_x(-\Delta+1)^{-1}$ is bounded. It is equivalent and technically more convenient to work with $(H_{\varphi_t}+M)^{-1}$, where $M>0$ is a large constant (independent of α and t), instead of $(-\Delta+1)^{-1}$. In order to create this term we first integrate by parts in s and make use of the identity

$$e^{iH_{\varphi_t}s} = -i(H_{\varphi_t}+M)^{-1} e^{-iMs} \partial_s[e^{i(H_{\varphi_t}+M)s}]. \tag{3-2}$$

We obtain, using the fact that H_{φ_t} commutes with $W(\alpha^2\varphi_s)$,

$$\begin{aligned}
D_{01} = &-ie^{iH_{\varphi_t}t}(H_{\varphi_t}+M)^{-1}\int_{\mathbb{R}^3} e^{ik\cdot x} b_k^* \tilde{\psi}_t \otimes \Omega \frac{dk}{|k|}\\
&+ iW^*(\alpha^2\varphi_t)W(\alpha^2\varphi_0)(H_{\varphi_t}+M)^{-1}\int_{\mathbb{R}^3} e^{ik\cdot x} b_k^* \tilde{\psi}_0 \otimes \Omega \frac{dk}{|k|}\\
&+ M\int_0^t e^{iH_{\varphi_t}s} W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)(H_{\varphi_t}+M)^{-1}\int_{\mathbb{R}^3} e^{ik\cdot x} b_k^* \tilde{\psi}_s \otimes \Omega \frac{dk}{|k|}\, ds\\
&+ i\int_0^t e^{iH_{\varphi_t}s} W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)(H_{\varphi_t}+M)^{-1}\int_{\mathbb{R}^3} e^{ik\cdot x} b_k^* \partial_s\tilde{\psi}_s \otimes \Omega \frac{dk}{|k|}\, ds\\
&+ i\int_0^t e^{iH_{\varphi_t}s} W^*(\alpha^2\varphi_t)(\partial_s W(\alpha^2\varphi_s))(H_{\varphi_t}+M)^{-1}\int_{\mathbb{R}^3} e^{ik\cdot x} b_k^* \tilde{\psi}_s \otimes \Omega \frac{dk}{|k|}\, ds\\
= {}& D_{011} + D_{012} + D_{013} + D_{014} + D_{015},
\end{aligned}$$

where the terms D_{01k} are defined in a natural way. We will prove the following lemma.

Lemma 3.1. *For $u \in \mathcal{H}^1(\mathbb{R}^3)$ and $f \in \mathcal{L}^2(\mathbb{R}^3)$,*

$$\left\| (-\Delta+1)^{-\frac{1}{2}} \int_{\mathbb{R}^3} e^{ik\cdot x} b_k^* u \otimes \Omega \frac{dk}{|k|} \right\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-1}\|u\|_{\mathcal{H}^1}$$

and

$$\left\| (-\Delta+1)^{-\frac{1}{2}} \int_{\mathbb{R}^3} e^{ik\cdot x} b^*(f) b_k^* u \otimes \Omega \frac{dk}{|k|} \right\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2}\|u\|_{\mathcal{H}^1}\|f\|_2.$$

We defer the proof of this lemma to the end of this section and first show how to use it to control D_{01}. By Corollary B.2 and Lemma 2.1, we can choose M large enough so that $(H_{\varphi_t}+M)^{-\frac{1}{2}}(-\Delta+1)^{\frac{1}{2}}$ is bounded uniformly in $t\in\mathbb{R}$. Moreover, by Proposition 2.2, $\tilde{\psi}_t$ and $\partial_t\tilde{\psi}_t$ belong to $H^1(\mathbb{R}^3)$ and have uniformly bounded norms for $t\in[0,\alpha^2]$; see also the remark at the beginning of Section 2B concerning the bounds on $\partial_t\tilde{\psi}_t$. These facts, together with the unitarity of $e^{iH_{\varphi_t}s}$, $W^*(\alpha^2\varphi_t)$ and $W(\alpha^2\varphi_s)$, imply that

$$\|D_{011}\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\alpha^{-1},\quad \|D_{012}\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\alpha^{-1}$$

and

$$\|D_{013}\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\alpha^{-1}t,\quad \|D_{014}\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\alpha^{-1}t.$$

In order to deal with the term D_{015} we make use of (A-4) and find

$$\begin{aligned}D_{015}=&-\int_0^t(\mathrm{Im}(\varphi_s,\alpha^2\partial_s\varphi_s))e^{iH_{\varphi_t}s}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)(H_{\varphi_t}+M)^{-1}\int_{\mathbb{R}^3}e^{ik\cdot x}b_k^*\tilde{\psi}_s\otimes\Omega\,\frac{dk}{|k|}\,ds\\&+i\int_0^t e^{iH_{\varphi_t}s}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)(H_{\varphi_t}+M)^{-1}\int_{\mathbb{R}^3}e^{ik\cdot x}b^*(\alpha^2\partial_s\varphi_s)b_k^*\tilde{\psi}_s\otimes\Omega\,\frac{dk}{|k|}\,ds\\&-i\int_0^t e^{iH_{\varphi_t}s}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)(H_{\varphi_t}+M)^{-1}\int_{\mathbb{R}^3}e^{ik\cdot x}b(\alpha^2\partial_s\varphi_s)b_k^*\tilde{\psi}_s\otimes\Omega\,\frac{dk}{|k|}\,ds\\=&\,D_{0151}+D_{0152}+D_{0153}.\end{aligned}$$

From Lemma 2.1 we know that $|(\varphi_s,\alpha^2\partial_s\varphi_s)|\lesssim 1$ and $\|\alpha^2\partial_s\varphi_s\|\lesssim 1$. Thus, the first and the second bounds in Lemma 3.1 imply, respectively,

$$\|D_{0151}\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\alpha^{-1}t,\quad \|D_{0152}\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\alpha^{-2}t.$$

For D_{0153} we use the commutation relations to rewrite it as

$$D_{0153}=-i\int_0^t e^{iH_{\varphi_t}s}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)(H_{\varphi_t}+M)^{-1}g_s\tilde{\psi}_s\otimes\Omega\,ds$$

with g_s from (2-6). Therefore, Proposition 2.2 yields

$$\|D_{0153}\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\alpha^{-2}t.$$

To summarize, we have shown that

$$\|D_{01}\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\alpha^{-1}(1+t).\tag{3-3}$$

Proof of Lemma 3.1. For any $\gamma\in\mathcal{L}^2(\mathbb{R}^3)\otimes\mathcal{F}$ and $(\Phi_k)_{k\in\mathbb{R}^3}\subset\mathcal{F}$, we use (3-1) to find

$$\begin{aligned}\left\langle\gamma,(-\Delta+1)^{-\frac{1}{2}}\int_{\mathbb{R}^3}e^{ik\cdot x}u\otimes\Phi_k\,\frac{dk}{|k|}\right\rangle_{\mathcal{L}^2\otimes\mathcal{F}}=&\left\langle\nabla(-\Delta+1)^{-\frac{1}{2}}\gamma,\int_{\mathbb{R}^3}\frac{ike^{ik\cdot x}}{|k|(1+|k|^2)}u\otimes\Phi_k\,dk\right\rangle_{\mathcal{L}^2\otimes\mathcal{F}}\\&+\left\langle(-\Delta+1)^{-\frac{1}{2}}\gamma,\int_{\mathbb{R}^3}\frac{ike^{ik\cdot x}}{|k|(1+|k|^2)}(\nabla u)\otimes\Phi_k\,dk\right\rangle_{\mathcal{L}^2\otimes\mathcal{F}}\\&+\left\langle(-\Delta+1)^{-\frac{1}{2}}\gamma,\int_{\mathbb{R}^3}\frac{e^{ik\cdot x}}{|k|(1+|k|^2)}u\otimes\Phi_k\,dk\right\rangle_{\mathcal{L}^2\otimes\mathcal{F}}.\end{aligned}$$

Clearly,

$$\|\nabla(-\Delta+1)^{-\frac{1}{2}}\gamma\|_{\mathcal{L}^2\otimes\mathcal{F}} \le \|\gamma\|_{\mathcal{L}^2\otimes\mathcal{F}} \quad \text{and} \quad \|(-\Delta+1)^{-\frac{1}{2}}\gamma\|_{\mathcal{L}^2\otimes\mathcal{F}} \le \|\gamma\|_{\mathcal{L}^2\otimes\mathcal{F}},$$

so

$$\begin{aligned}\Big\|(-\Delta+1)^{-\frac{1}{2}}\int_{\mathbb{R}^3} e^{ik\cdot x}u\otimes\Phi_k\,\frac{dk}{|k|}\Big\|_{\mathcal{L}^2\otimes\mathcal{F}}\\ \lesssim \|u\|_{\mathcal{H}^1}\sup_{x\in\mathbb{R}^d}\bigg(\Big\|\int_{\mathbb{R}^3}\frac{ike^{ik\cdot x}}{|k|(1+|k|^2)}\Phi_k\,dk\Big\|_{\mathcal{F}}+\Big\|\int_{\mathbb{R}^3}\frac{e^{ik\cdot x}}{|k|(1+|k|^2)}\Phi_k\,dk\Big\|_{\mathcal{F}}\bigg).\end{aligned}$$

If $\Phi_k = b_k^*\Omega$, we use the fact that

$$\frac{1}{|k|(1+|k|^2)},\ \frac{k}{|k|(1+|k|^2)} \in \mathcal{L}^2(\mathbb{R}^3)$$

to conclude that, uniformly in $x\in\mathbb{R}^3$,

$$\Big\|\int_{\mathbb{R}^3}\frac{ike^{ik\cdot x}}{|k|(1+|k|^2)}b_k^*\Omega\,dk\Big\|_{\mathcal{F}}\lesssim\alpha^{-1},\quad \Big\|\int_{\mathbb{R}^3}\frac{e^{ik\cdot x}}{|k|(1+|k|^2)}b_k^*\Omega\,dk\Big\|_{\mathcal{F}}\lesssim\alpha^{-1}.$$

This proves the first bound in the lemma. If $\Phi_k = b^*(f)b_k^*\Omega$, one can similarly show that

$$\Big\|\int_{\mathbb{R}^3}\frac{ike^{ik\cdot x}}{|k|(1+|k|^2)}b^*(f)b_k^*\Omega\,dk\Big\|_{\mathcal{F}}\lesssim\frac{\|f\|_2}{\alpha^2},\quad \Big\|\int_{\mathbb{R}^3}\frac{e^{ik\cdot x}}{|k|(1+|k|^2)}b^*(f)b_k^*\Omega\,dk\Big\|_{\mathcal{F}}\lesssim\frac{\|f\|_2}{\alpha^2}.$$

This proves the second bound in the lemma. □

4. Bound on D_1

Bound on D_{111}. We recall equation (2-30) for D_{111}. In this equation, we commute $e^{ik\cdot x}$ with $e^{-iH_{\varphi_t}s}$. Thus, if we introduce the operator

$$H_\varphi(k) := e^{ik\cdot x}H_\varphi e^{-ik\cdot x} = (i\nabla_x+k)^2+V_\varphi+\int_{\mathbb{R}^3}|\varphi(k)|^2\,dk, \tag{4-1}$$

we obtain

$$\begin{aligned}D_{111} = \int_0^t\int_0^{t-s}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}e^{-iH_{\varphi_t}(k)s_1}e^{i(k+k')\cdot x}\\ \times W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b_k^*b_{k'}^*\tilde{\psi}_s\otimes\Omega\,\frac{dk'}{|k'|}\frac{dk}{|k|}\,ds_1\,ds.\end{aligned}$$

Controlling D_{111} is harder than controlling D_{01} because there are two slowly decaying terms $|k|^{-1}$ and $|k'|^{-1}$. The beginning of the proof, however, is similar; namely, for a large constant $M>0$ to be specified, independent of t and α, we integrate by parts in s using

$$e^{i\widetilde{H}_{\varphi_t}s} = -i(\widetilde{H}_{\varphi_t}+M)^{-1}e^{-iMs}[\partial_s e^{i(\widetilde{H}_{\varphi_t}+M)s}].$$

In this way we obtain

$$
\begin{aligned}
D_{111} = -i\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} & e^{i\widetilde{H}_{\varphi_t} t}(\widetilde{H}_{\varphi_t}+M)^{-1} e^{-iH_{\varphi_t}(k)s_1} e^{i(k'+k)\cdot x} \\
& \times W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_{t-s_1}) b_k^* b_{k'}^* \tilde{\psi}_{t-s_1}\otimes\Omega \,\frac{dk'}{|k'|}\frac{dk}{|k|}\,ds_1 \\
+ i\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} & e^{i\widetilde{H}_{\varphi_t} s_1}(\widetilde{H}_{\varphi_t}+M)^{-1} e^{-iH_{\varphi_t}(k)s_1} e^{i(k'+k)\cdot x} \\
& \times W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_0) b_k^* b_{k'}^* \tilde{\psi}_0\otimes\Omega \,\frac{dk'}{|k'|}\frac{dk}{|k|}\,ds_1 \\
+ M\int_0^t\int_0^{t-s} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} & e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(\widetilde{H}_{\varphi_t}+M)^{-1} e^{-iH_{\varphi_t}(k)s_1} e^{i(k'+k)\cdot x} \\
& \times W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s) b_k^* b_{k'}^* \tilde{\psi}_s\otimes\Omega \,\frac{dk'}{|k'|}\frac{dk}{|k|}\,ds_1\,ds \\
+ i\int_0^t\int_0^{t-s} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} & e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(\widetilde{H}_{\varphi_t}+M)^{-1} e^{-iH_{\varphi_t}(k)s_1} e^{i(k'+k)\cdot x} \\
& \times W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s) b_k^* b_{k'}^* [\partial_s\tilde{\psi}_s]\otimes\Omega \,\frac{dk'}{|k'|}\frac{dk}{|k|}\,ds_1\,ds \\
+ i\int_0^t\int_0^{t-s} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} & e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(\widetilde{H}_{\varphi_t}+M)^{-1} e^{-iH_{\varphi_t}(k)s_1} e^{i(k'+k)\cdot x} \\
& \times W^*(\alpha^2\varphi_t)[\partial_s W(\alpha^2\varphi_s)] b_k^* b_{k'}^* \tilde{\psi}_s\otimes\Omega \,\frac{dk'}{|k'|}\frac{dk}{|k|}\,ds_1\,ds.
\end{aligned}
$$

We now use (2-13), which implies

$$
\begin{aligned}
(\widetilde{H}_{\varphi_t}+M)^{-1}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s) &= W^*(\alpha^2\varphi_t)(\widetilde{H}_\alpha^F+M)^{-1}W(\alpha^2\varphi_s) \\
&= W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)(\widetilde{H}_{\varphi_s}+M)^{-1},
\end{aligned}
$$

in order to commute $(\widetilde{H}_{\varphi_t}+M)^{-1}$ to the right through $W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)$. Moreover, we use Lemma A.3 to compute $\partial_s W(\alpha^2\varphi_s)$. In this way we obtain

$$
\begin{aligned}
D_{111} = -i\int_0^t & e^{i\widetilde{H}_{\varphi_t} t} W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s) Q_1\,ds \\
& + i\int_0^t e^{i\widetilde{H}_{\varphi_t} s_1} W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_0) Q_2\,ds_1 \\
& + M\int_0^t\int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)} W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s) Q_3\,ds_1\,ds \\
& + i\int_0^t\int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)} W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s) Q_4\,ds_1\,ds \\
& + i\int_0^t\int_0^{t-s} e^{i\widetilde{H}_{\varphi_t}(s+s_1)} W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s) Q_5\,ds_1\,ds
\end{aligned}
$$

with

$$
Q_1 := (\widetilde{H}_{\varphi_s}+M)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)(t-s)} e^{i(k'+k)\cdot x} b_k^* b_{k'}^* \tilde{\psi}_s\otimes\Omega\,\frac{dk'}{|k'|}\frac{dk}{|k|},
$$

$$
Q_2 := (\widetilde{H}_{\varphi_0}+M)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)s_1} e^{i(k'+k)\cdot x} b_k^* b_{k'}^* \tilde{\psi}_0\otimes\Omega\,\frac{dk'}{|k'|}\frac{dk}{|k|},
$$

$$Q_3 := (\widetilde{H}_{\varphi_s}+M)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)s_1} e^{i(k'+k)\cdot x} b_k^* b_{k'}^* \tilde{\psi}_s \otimes \Omega \frac{dk'}{|k'|}\frac{dk}{|k|},$$
$$Q_4 := (\widetilde{H}_{\varphi_s}+M)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)s_1} e^{i(k'+k)\cdot x} b_k^* b_{k'}^* [\partial_s \tilde{\psi}_s] \otimes \Omega \frac{dk'}{|k'|}\frac{dk}{|k|},$$
$$Q_5 := (\widetilde{H}_{\varphi_s}+M)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)s_1} e^{i(k'+k)\cdot x} \big(b^*(\alpha^2\partial_s\varphi_s) - b(\alpha^2\partial_s\varphi_s) + i\,\mathrm{Im}(\varphi_s, \alpha^2\partial_s\varphi_s)\big) b_k^* b_{k'}^* \tilde{\psi}_s \otimes \Omega \frac{dk'}{|k'|}\frac{dk}{|k|}.$$

(Here, we suppress the dependence on t, s and s_1 in the notation of the Q_j.)

In the remainder of this section we shall show that, uniformly for $0 \le s, s_1 \le t \le \alpha^2$,

$$\|Q_j\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2} \quad \text{if } j = 1,2,3,4,5. \tag{4-2}$$

This will imply that

$$\|D_{111}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2} t(1+t). \tag{4-3}$$

Since the operator $(\widetilde{H}_{\varphi_s}+M)^{-1}(-\Delta+\mathcal{N}+M)$ is *not* bounded, bounding the Q_j is rather involved. (Here $\mathcal{N}$ was introduced in (2-41).) With the notation

$$Z_\varphi := V_\varphi + \int_{\mathbb{R}^3} |\varphi(x)|^2\,dk + \int_{\mathbb{R}^3} (e^{-ik\cdot x} b_k + e^{ik\cdot x} b_k^*)\frac{dk}{|k|} + b(\varphi) + b^*(\varphi),$$

we abbreviate (2-14) as

$$\widetilde{H}_\varphi = -\Delta + \mathcal{N} + Z_\varphi.$$

Defining

$$\widetilde{Z}_\varphi := (-\Delta+\mathcal{N}+M)^{-\frac12} Z_\varphi (-\Delta+\mathcal{N}+M)^{-\frac12},$$

we have

$$\begin{aligned}(\widetilde{H}_\varphi+M)^{-1} &= (-\Delta+\mathcal{N}+M)^{-\frac12}(1+\widetilde{Z}_\varphi)^{-1}(-\Delta+\mathcal{N}+M)^{-\frac12}\\ &= (-\Delta+\mathcal{N}+M)^{-1} - (-\Delta+\mathcal{N}+M)^{-\frac12}(1+\widetilde{Z}_\varphi)^{-1}(-\Delta+\mathcal{N}+M)^{-\frac12} Z_\varphi (-\Delta+\mathcal{N}+M)^{-1}.\end{aligned}$$

It is not difficult to see that for every $\varepsilon > 0$ and $A > 0$ there is an M such that

$$\|\widetilde{Z}_\varphi\|_{\mathcal{L}^2\otimes\mathcal{F}\mapsto\mathcal{L}^2\otimes F} \le \varepsilon \tag{4-4}$$

for all φ with $\|\varphi\|_{\mathcal{L}^2} \le A$; for details of this argument we refer to [Frank and Schlein 2014]. Thus, using the bound on $\|\varphi_s\|_{\mathcal{L}^2}$ from Lemma 2.1, we can choose M in such a way that

$$\|\widetilde{Z}_{\varphi_s}\|_{\mathcal{L}^2\otimes\mathcal{F}\mapsto\mathcal{L}^2\otimes F} \le \tfrac12 \qquad \text{for all } s > 0.$$

Therefore, the operator $1+\widetilde{Z}_{\varphi_s}$ in the above formula for $(H_{\varphi_s}+M)^{-1}$ is invertible. We use this formula to decompose

$$\begin{aligned}Q_1 = \Big(1 - (-\Delta+\mathcal{N}+M)^{-\frac12}(1+\widetilde{Z}_{\varphi_s})^{-1}(-\Delta+\mathcal{N}+M)^{-\frac12}\Big(V_{\varphi_s} + \int_{\mathbb{R}^3}|\varphi_s(x)|^2\,dk + b(\varphi_s) + b^*(\varphi_s)\Big)\Big) Q_{10}\\ -(-\Delta+\mathcal{N}+M)^{-\frac12}(1+\widetilde{Z}_{\varphi_s})^{-1}(Q_{11}+Q_{12})\end{aligned} \tag{4-5}$$

with

$$Q_{10} := (-\Delta+\mathcal{N}+M)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)(t-s)}e^{i(k'+k)\cdot x}b_k^*b_{k'}^*\tilde{\psi}_s\otimes\Omega\,\frac{dk'}{|k'|}\frac{dk}{|k|},$$

$$Q_{11} := (-\Delta+\mathcal{N}+M)^{-\frac12}\Big(\int_{\mathbb{R}^3} e^{-ik''\cdot x}b_{k''}\frac{dk''}{|k''|}\Big)(-\Delta+\mathcal{N}+M)^{-1} \times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)(t-s)}e^{i(k'+k)\cdot x}b_k^*b_{k'}^*\tilde{\psi}_s\otimes\Omega\,\frac{dk'}{|k'|}\frac{dk}{|k|},$$

$$Q_{12} := (-\Delta+\mathcal{N}+M)^{-\frac12}\Big(\int_{\mathbb{R}^3} e^{ik''\cdot x}b_{k''}^*\frac{dk''}{|k''|}\Big)(-\Delta+\mathcal{N}+M)^{-1} \times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)(t-s)}e^{i(k'+k)\cdot x}b_k^*b_{k'}^*\tilde{\psi}_s\otimes\Omega\,\frac{dk'}{|k'|}\frac{dk}{|k|}.$$

Using (4-4), the fact that $(-\Delta+\mathcal{N}+M)^{-\frac12}(b(\varphi_s)+b^*(\varphi_s))$ is bounded uniformly in s, as well as the estimates $\|V_{\varphi_s}\|_\infty\lesssim 1$ (from (C-1) and Proposition 2.2) and $\|\varphi_s\|_2\lesssim 1$ (from Lemma 2.1), we conclude from (4-5) that

$$\|Q_1\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\|Q_{10}\|_{\mathcal{L}^2\otimes\mathcal{F}}+\|Q_{11}\|_{\mathcal{L}^2\otimes\mathcal{F}}+\|Q_{12}\|_{\mathcal{L}^2\otimes\mathcal{F}}.$$

We now bound the three terms on the right side separately.

Bound on Q_{10}. To control Q_{10} we prove an analogue of Lemma 3.1 for the case of two singularities.

Lemma 4.1. *For $u\in\mathcal{H}^2(\mathbb{R}^3)$, $f\in\mathcal{L}^2(\mathbb{R}^3)$ and $s\in\mathbb{R}$,*

$$\Big\|(-\Delta+1)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)s}e^{i(k+k')\cdot x}b_k^*b_{k'}^*u\otimes\Omega\,\frac{dk'\,dk}{|k'|\,|k|}\Big\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\alpha^{-2}\|u\|_{\mathcal{H}^2}.$$

Before proving this lemma we show how to use it to bound Q_{10}. Note that, since Q_{10} involves only $b_k^*b_{k'}^*\Omega$, the operator $(-\Delta+\mathcal{N}+M)^{-1}$ in its definition can be replaced by $(-\Delta+2\alpha^{-2}+M)^{-1}$. This observation, together with Lemma 4.1 and the uniform boundedness of $\tilde{\psi}_s$ in $\mathcal{H}^2$ for $s\in[0,\alpha^2]$ (see Proposition 2.2), proves that

$$\|Q_{10}\|_{\mathcal{L}^2\otimes\mathcal{F}}\lesssim\alpha^{-2}. \tag{4-6}$$

Proof of Lemma 4.1. We shall show that for any $\gamma\in\mathcal{L}^2(\mathbb{R}^3)\otimes\mathcal{F}$,

$$\Big|\Big\langle\gamma,(-\Delta+1)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)s}e^{i(k+k')\cdot x}b_k^*b_{k'}^*u\otimes\Omega\,\frac{dk'\,dk}{|k'|\,|k|}\Big\rangle\Big|\lesssim\alpha^{-2}\|\gamma\|_{\mathcal{L}^2\otimes\mathcal{F}}\|u\|_{\mathcal{H}^2}.$$

We integrate by parts twice in x and use (3-1) with k replaced by $k+k'$. A typical term that is obtained in this way in the inner product on the left side is

$$\Big\langle e^{iH_{\varphi_t}(k)s}\partial_{x_i}\partial_{x_j}(-\Delta+1)^{-1}\gamma,\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(k+k')\cdot x}b_k^*b_{k'}^*u\otimes\Omega\,\frac{(k_i+k_i')(k_j+k_j')\,dk'\,dk}{|k|\,|k'|\,(1+|k+k'|^2)^2}\Big\rangle.$$

Since $\partial_{x_i}\partial_{x_j}(-\Delta+1)^{-1}$ is bounded and $e^{iH_{\varphi_t}(k)s}$ is unitary, the vector on the left side of the inner product is bounded in norm by $\|\gamma\|_{\mathcal{L}^2\otimes\mathcal{F}}$. We now show that the vector on the right side of the inner product is bounded as well. We compute

$$\begin{aligned}\Big\| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(k+k')\cdot x} b_k^* b_{k'}^* u \otimes \Omega \frac{(k_i+k_i')(k_j+k_j')}{|k||k'|\,(1+|k+k'|^2)^2}\,dk'\,dk \Big\|^2_{\mathcal{L}^2\otimes\mathcal{F}} \\ = 2\alpha^{-4}\|u\|_2^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(k_i+k_i')^2(k_j+k_j')^2}{|k|^2|k'|^2(1+|k+k'|^2)^4}\,dk'\,dk.\end{aligned}$$

The desired bound now follows from the fact that the double integral on the right side is finite. Other terms that arise in the integration by parts are controlled similarly and we omit the details. ☐

Bound on Q_{11}. By considering the number of involved field particles, we can replace $\mathcal{N}$ in the definition of Q_{11} by numbers and obtain

$$\begin{aligned}Q_{11} = (-\Delta+\alpha^{-2}+M)^{-\frac12}\Big(\int_{\mathbb{R}^3} e^{-ik''\cdot x} b_{k''}\frac{dk''}{|k''|}\Big)(-\Delta+2\alpha^{-2}+M)^{-1} \\ \times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)(t-s)} e^{i(k'+k)\cdot x} b_k^* b_{k'}^* \tilde{\psi}_s \otimes \Omega \frac{dk'}{|k'|}\frac{dk}{|k|}.\end{aligned}$$

Next, by commuting $b_{k''}$ to the right,

$$\begin{aligned}Q_{11} = \alpha^{-2}(-\Delta+\alpha^{-2}+M)^{-\frac12}\int_{\mathbb{R}^3}\big((i\nabla-k')^2+2\alpha^{-2}+M\big)^{-1} \\ \times \int_{\mathbb{R}^3} e^{-ik'\cdot x} e^{-iH_{\varphi_t}(k)(t-s)} e^{i(k'+k)\cdot x} b_k^* \tilde{\psi}_s \otimes \Omega \frac{dk'}{|k'|^2}\frac{dk}{|k|} \\ + \alpha^{-2}(-\Delta+\alpha^{-2}+M)^{-\frac12}\int_{\mathbb{R}^3}\big((i\nabla-k)^2+2\alpha^{-2}+M\big)^{-1} \\ \times \int_{\mathbb{R}^3} e^{-ik\cdot x} e^{-iH_{\varphi_t}(k)(t-s)} e^{i(k'+k)\cdot x} b_{k'}^* \tilde{\psi}_s \otimes \Omega \frac{dk'}{|k'|}\frac{dk}{|k|^2}.\end{aligned}$$

It remains to compute the norm of this expression. Since this is considerably easier than for Q_{12}, we omit the details and only state the final result,

$$\|Q_{11}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}. \tag{4-7}$$

Bound on Q_{12}. In the same way as for Q_{11}, we can replace $\mathcal{N}$ by a number, so that

$$\begin{aligned}Q_{12} = (-\Delta+3\alpha^{-2}+M)^{-\frac12}\int_{\mathbb{R}^3} e^{ik''\cdot x} b_{k''}^*(-\Delta+2\alpha^{-2}+M)^{-1} \\ \times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-iH_{\varphi_t}(k)(t-s)} e^{i(k'+k)\cdot x} b_{k'}^* b_k^* \tilde{\psi}_s \otimes \Omega \frac{dk'}{|k'|}\frac{dk}{|k|}.\end{aligned}$$

Next, we commute $e^{ik''\cdot x}$ and $e^{i(k'+k)\cdot x}$ to the right and obtain

$$\begin{aligned}Q_{12} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} b_k^* b_{k'}^* b_{k''}^* e^{i(k+k'+k'')\cdot x}\big((i\nabla-k-k'-k'')^2+3\alpha^{-2}+M\big)^{-\frac12} \\ \times \big((i\nabla-k-k')^2+2\alpha^{-2}+M\big)^{-1} e^{-iH_{\varphi_t}(-k')(t-s)}\tilde{\psi}_s \otimes \Omega \frac{dk''}{|k''|}\frac{dk'}{|k'|}\frac{dk}{|k|}.\end{aligned}$$

We now compute the norm of this expression. For the part of the norm over $\mathcal{F}$, we use the fact that

$$\begin{aligned}\alpha^6\langle\Omega, b_{k_1}b_{k_2}b_{k_3}b_{k_4}^*b_{k_5}^*b_{k_6}^*\Omega\rangle &= \delta(k_1-k_4)\delta(k_2-k_5)\delta(k_3-k_6)+\delta(k_1-k_4)\delta(k_2-k_6)\delta(k_3-k_5)\\ &\quad+\delta(k_1-k_5)\delta(k_2-k_4)\delta(k_3-k_6)+\delta(k_1-k_5)\delta(k_2-k_6)\delta(k_3-k_4)\\ &\quad+\delta(k_1-k_6)\delta(k_2-k_4)\delta(k_3-k_5)+\delta(k_1-k_6)\delta(k_2-k_4)\delta(k_3-k_6)\end{aligned}$$

to write

$$\|Q_{12}\|^2_{\mathcal{L}^2\otimes\mathcal{F}} = \alpha^{-6}(X_1+\cdots+X_6), \tag{4-8}$$

where, for instance,

$$\begin{aligned}X_1 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\langle e^{-iH_{\varphi_t}(-k')(t-s)}\tilde{\psi}_s, \big((i\nabla-k-k'-k'')^2+3\alpha^{-2}+M\big)^{-1}\\ \times\big((i\nabla-k-k')^2+2\alpha^{-2}+M\big)^{-2}e^{-iH_{\varphi_t}(-k')(t-s)}\tilde{\psi}_s\Big\rangle\frac{dk''}{|k''|^2}\frac{dk'}{|k'|^2}\frac{dk}{|k|^2}\end{aligned}$$

and

$$\begin{aligned}X_2 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big\langle e^{-iH_{\varphi_t}(-k'')(t-s)}\tilde{\psi}_s, \big((i\nabla-k-k'-k'')^2+3\alpha^{-2}+M\big)^{-1}\\ \times\big((i\nabla-k-k'')^2+2\alpha^{-2}+M\big)^{-1}\big((i\nabla-k-k')^2+2\alpha^{-2}+M\big)^{-1}e^{-iH_{\varphi_t}(-k')(t-s)}\tilde{\psi}_s\Big\rangle\frac{dk''}{|k''|^2}\frac{dk'}{|k'|^2}\frac{dk}{|k|^2}.\end{aligned}$$

By the Schwarz inequality we have $|X_2|\le X_1$ and, similarly,

$$|X_j|\le X_1 \qquad \text{for all } j=1,\dots,6. \tag{4-9}$$

Thus it suffices to control X_1.

We first perform the k'' integral and then the k integral. We make use of the following bounds.

Lemma 4.2. *One has the operator inequalities*

$$\int_{\mathbb{R}^3}((i\nabla-k'')^2+1)^{-1}\frac{dk''}{|k''|^2}\lesssim 1, \tag{4-10}$$

$$\int_{\mathbb{R}^3}((i\nabla_x-k)^2+1)^{-2}\frac{dk}{|k|^2}\lesssim(-\Delta+1)^{-1}. \tag{4-11}$$

Before proving the lemma, let us see that they provide the desired bounds on X_1. First, conjugating (4-10) with $e^{i(k+k')\cdot x}$ and assuming that $M+3\alpha^2\ge 1$, we obtain, uniformly in $k,k'\in\mathbb{R}^3$,

$$\int_{\mathbb{R}^3}\big((i\nabla-k-k'-k'')^2+3\alpha^{-2}+M\big)^{-1}\frac{dk''}{|k''|^2}\lesssim 1. \tag{4-12}$$

Similarly, conjugating (4-11) with $e^{ik'\cdot x}$, we obtain, uniformly in $k'\in\mathbb{R}^3$,

$$\int_{\mathbb{R}^3}\big((i\nabla_x-k-k')^2+2\alpha^{-2}+M\big)^{-2}\frac{dk}{|k|^2}\lesssim((i\nabla-k')^2+1)^{-1}. \tag{4-13}$$

Inserting (4-12) and (4-13) into the definition of X_1, we obtain

$$X_1\lesssim\int_{\mathbb{R}^3}\langle e^{-iH_{\varphi_t}(-k')(t-s)}\tilde{\psi}_s, ((i\nabla-k')^2+1)^{-1}e^{-iH_{\varphi_t}(-k')(t-s)}\tilde{\psi}_s\rangle\frac{dk'}{|k'|^2}.$$

Since $(-\Delta+1)^{-\frac{1}{2}}(H_{\varphi_t}+M)^{\frac{1}{2}}$ is bounded, uniformly in t (by Corollary B.2 and Lemma 2.1), we also know that $((i\nabla-k')^2+1)^{-\frac{1}{2}}(H_{\varphi_t}(-k')+M)^{\frac{1}{2}}$ is bounded, uniformly in t. Thus,

$$
\begin{aligned}
X_1 &\lesssim \int_{\mathbb{R}^3} \langle e^{-iH_{\varphi_t}(-k')(t-s)}\tilde{\psi}_s, (H_{\varphi_t}(-k')+M)^{-1} e^{-iH_{\varphi_t}(-k')(t-s)}\tilde{\psi}_s\rangle \frac{dk'}{|k'|^2} \\
&= \int_{\mathbb{R}^3} \langle \tilde{\psi}_s, (H_{\varphi_t}(-k')+M)^{-1}\tilde{\psi}_s\rangle \frac{dk'}{|k'|^2} \\
&\lesssim \int_{\mathbb{R}^3} \langle \tilde{\psi}_s, ((i\nabla-k')^2+M)^{-1}\tilde{\psi}_s\rangle \frac{dk'}{|k'|^2}.
\end{aligned}
$$

Applying (4-10) again, we see that the latter expression is bounded by a constant times $\|\tilde{\psi}_s\|_{\mathcal{L}^2}^2 = 1$ by Lemma 2.1. This, together with (4-8) and (4-9), implies that

$$
\|Q_{12}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}. \tag{4-14}
$$

Proof of Lemma 4.2. We only prove (4-11), since the proof of (4-10) is similar and simpler. By applying a Fourier transform, we see that we need to prove

$$
\int_{\mathbb{R}^3} ((p+k)^2+1)^{-2} \frac{dk}{|k|^2} \lesssim (p^2+1)^{-1} \quad \text{for } p \in \mathbb{R}^3.
$$

We split the integral into the regions $4|k| > |p|+1$ and $4|k| \le |p|+1$. In the first region we bound $|k|^{-2} \le 16/(|p|+1)^2$ and note that

$$
\int_{\{4|k|>|p|+1\}} ((p+k)^2+1)^{-2}\,dk \le \int_{\mathbb{R}^3} ((p+k)^2+1)^{-2}\,dk = \int_{\mathbb{R}^3} (k^2+1)^{-2}\,dk < \infty.
$$

In the second region we distinguish the cases $|p|<1$ and $|p|\ge 1$. In the first case we bound

$$
\int_{\{4|k|\le|p|+1\}} ((p+k)^2+1)^{-2} \frac{dk}{|k|^2} \le \int_{\{4|k|\le|p|+1\}} \frac{dk}{|k|^2} \le \int_{\{|k|\le\frac{1}{2}\}} \frac{dk}{|k|^2} < \infty.
$$

For $|p|\ge 1$ we note that in the second region we have $2|k|\le|p|$ and therefore $(p+k)^2 \ge \frac{1}{4}p^2 \ge k^2$. Thus,

$$
((p+k)^2+1)^{-2} \le \left(\tfrac{1}{4}p^2+1\right)^{-1}(k^2+1)^{-1}.
$$

Since $(k^2+1)^{-1}|k|^{-2}$ is integrable, we obtain again a bound of the required form. □

Bounds on $Q_2,\dots,Q_5$. The terms $Q_2,\dots,Q_4$ are controlled in exactly the same way as Q_1. (For Q_4 we use the fact that $\|\partial_s\tilde{\psi}_s\|_{\mathcal{H}^2} \lesssim 1$ for $t \le \alpha^2$ by Proposition 2.2.) The argument for Q_5 is also similar. In fact, the term involving $\operatorname{Im}(\varphi_s, \alpha^2\partial_s\varphi_s)$ is controlled as before. For the term involving $b^*(\alpha^2\partial_s\varphi_s)$ we have to prove a simple extension of Lemma 4.1 where we have operators $b^*(f)b_k^*b_{k'}^*$ with $f \in \mathcal{L}^2$ (similarly as the second part in Lemma 3.1). Finally, the term involving $b(\alpha^2\partial_s\varphi_s)$ can be commuted to the right and therefore becomes a less singular term which can be controlled already with Lemma 3.1. These arguments prove (4-2) and complete the proof of (4-3).

Bound on D_{112}. The term D_{112} in (2-34) contains only one factor $|k'|^{-1}$ and can therefore be controlled essentially by the same method as D_{01}, based on Lemma 3.1. In order to create a factor of $(H_{\varphi_t}+M)^{-1}$, we integrate by parts in s_1. This, however, will create a factor of $\widetilde{H}_{\varphi_t}$ in one of the terms. When dealing with D_{211} we will explain how to remove this term by integrating by parts in s. Since $\|g_{s,t}\|_\infty \lesssim \alpha^{-2}|t-s|$ and $\|\partial_s g_{s,t}\|_\infty = \|g_s\|_\infty \lesssim \alpha^{-2}$ by Proposition 2.2, this factor behaves well in the bounds. When applying Lemma 3.1 we also use $\|\partial_s \tilde{\psi}_s\|_{\mathcal{H}^1} \lesssim 1$ from Proposition 2.2; see also the remark at the beginning of Section 2B concerning the bounds on $\partial_t \tilde{\psi}_t$. Without going into details we state the final result,

$$\|D_{112}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3} t^2(1+t). \tag{4-15}$$

Bound on D_{121}. Also the term D_{121} in (2-31) contains only one factor of $|k|^{-1}$ and can be controlled as just sketched for D_{112} and as explained in detail for D_{211}. In order to control the terms that appear when integrating by parts in s we make use of $\|\partial_s \sigma_{\tilde{\psi}_s}\|_{\mathcal{L}^2} \lesssim 1$ and $\|\partial_s \tilde{\psi}_s\|_{\mathcal{H}^1} \lesssim 1$ from Proposition 2.2 in addition to the bounds from Lemma 2.1. Moreover, we need an obvious extension of Lemma 3.1 to the case with $b^*(f_1)b^*(f_2)b_k^*$, which is proved in the same way. Combining all this, we end up with

$$\|D_{121}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2} t(1+t). \tag{4-16}$$

Bound on D_{122}. The term D_{122} contains no $|k|^{-1}$ term. Using $\|g_{s,t}\|_\infty \lesssim \alpha^{-2}|t-s|$ for $0 \le s \le t \le \alpha^2$ by Proposition 2.2 and $\|b(\sigma_{\tilde{\psi}_s})\Omega\|_{\mathcal{F}} = \alpha^{-1}\|\sigma_{\tilde{\psi}_s}\|_2 \lesssim \alpha^{-1}$ by Lemma 2.1, we obtain immediately

$$\|D_{122}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3} t^3. \tag{4-17}$$

5. Estimation on D_2

Bound on D_{211}. We recall equation (2-32) for D_{211}. In this equation we commute $e^{-ik\cdot x}$ through $e^{-iH_{\varphi_t}s_1}$, which introduces again the operator $H_{\varphi_t}(k)$ from (4-1), and we commute b_k with $b_{k'}^*$. In this way, we obtain

$$D_{211} = \alpha^{-2} \int_0^t \int_0^{t-s} \int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)} e^{-iH_{\varphi_t}(k)s_1} W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s)\tilde{\psi}_s \otimes \Omega \frac{dk}{|k|^2}\, ds_1\, ds.$$

The difficulty in controlling D_{211} comes again from the k-integral. It is not enough to bound the norm of the integrand as it stands, since $|k|^{-2}$ is not integrable. Thus, we need to gain some extra decay from $e^{-iH_{\varphi_t}(k)s_1}$. To get this decay, we integrate by parts in s_1 using

$$e^{-iH_{\varphi_t}(k)s_1} = i e^{iMs_1}(H_{\varphi_t}(k)+M)^{-1} \partial_{s_1} e^{-i[H_{\varphi_t}(k)+M]s_1} \tag{5-1}$$

with a large constant $M > 0$ independent of α and t. We obtain

$$\begin{aligned} D_{211} = i\alpha^{-2} \int_0^t \int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}t} (H_{\varphi_t}(k)+M)^{-1} e^{-iH_{\varphi_t}(k)(t-s)} W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s)\tilde{\psi}_s \otimes \Omega \frac{dk}{|k|^2}\, ds \\ - i\alpha^{-2} \int_0^t \int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}s} (H_{\varphi_t}(k)+M)^{-1} W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s)\tilde{\psi}_s \otimes \Omega \frac{dk}{|k|^2}\, ds \end{aligned}$$

$$
\begin{aligned}
&+\alpha^{-2} M \int_0^t \int_0^{t-s} \int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)} (H_{\varphi_t}(k)+M)^{-1} e^{-iH_{\varphi_t}(k)s_1} \\
&\qquad\qquad\qquad\qquad \times W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) \tilde{\psi}_s \otimes \Omega \frac{dk}{|k|^2}\, ds_1\, ds \\
&+\alpha^{-2} \int_0^t \int_0^{t-s} \int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)} \widetilde{H}_{\varphi_t} (H_{\varphi_t}(k)+M)^{-1} e^{-iH_{\varphi_t}(k)s_1} \\
&\qquad\qquad\qquad\qquad \times W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) \tilde{\psi}_s \otimes \Omega \frac{dk}{|k|^2}\, ds_1\, ds \\
&= D_{2111} + D_{2112} + D_{2113} + D_{2114},
\end{aligned}
$$

where D_{211k}, $k = 1, \dots, 4$, are naturally defined.

We first show how to deal with the terms D_{2111}, D_{2112} and D_{2113}. The term D_{2114} is harder because of the additional factor of $\widetilde{H}_{\varphi_t}$.

The following lemma quantifies in which sense the operator $(H_{\varphi_t} + M)^{-1}$ leads to additional decay in k.

Lemma 5.1. *For* $u \in \mathcal{H}^2(\mathbb{R}^3)$,

$$
\int_{\mathbb{R}^3} \| (|i\nabla + k|^2 + 1)^{-1} u \|_2 \frac{dk}{|k|^2} \lesssim \|u\|_{\mathcal{H}^2}. \tag{5-2}
$$

Proof. By Fourier transform, we have

$$
\| (|i\nabla + k|^2 + 1)^{-1} u \|_2^2 = \int_{\mathbb{R}^3} \frac{1}{(1+|p+k|^2)^2 (1+|p|^2)^2} (1+|p|^2)^2 |\hat{u}(p)|^2\, dp.
$$

We now observe that

$$
\frac{1}{(1+|p+k|^2)^2 (1+|p|^2)^2} \lesssim \frac{1}{(1+|k|^2)^2}.
$$

This can be proved by considering separately the regions where $|p| \le \frac{1}{2}|k|$ and $|p| \ge \frac{1}{2}|k|$. Thus,

$$
\| (|i\nabla + k|^2 + 1)^{-1} u \|_2^2 \lesssim \frac{1}{(1+|k|^2)^2} \|u\|_{\mathcal{H}^2}^2,
$$

and the claimed bound follows by integration over k. □

Let us return to the terms D_{2111}, D_{2112} and D_{2113}. It follows from Corollary B.2 by conjugating with the unitary $e^{ik\cdot x}$ that there is an $M > 0$ such that the operator $(H_{\varphi_t}(k) + M)^{-1}(|i\nabla + k|^2 + 1)$ is uniformly bounded in α and t. This, together with the boundedness of ψ_s in $\mathcal{H}^2$ for $s \in [0, \alpha^2]$ from Proposition 2.2, yields

$$
\int_{\mathbb{R}^3} \| (H_{\varphi_t}(k) + M)^{-1} \tilde{\psi}_s \|_2 \frac{dk}{|k|^2} \lesssim 1,
$$

and therefore

$$
\|D_{2111}\|_{\mathcal{L}^2 \otimes \mathcal{F}} \lesssim \alpha^{-2} t, \quad \|D_{2112}\|_{\mathcal{L}^2 \otimes \mathcal{F}} \lesssim \alpha^{-2} t, \quad \|D_{2113}\|_{\mathcal{L}^2 \otimes \mathcal{F}} \lesssim \alpha^{-2} t^2. \tag{5-3}
$$

We now turn to the term D_{2114}, which contains the operator $\widetilde{H}_{\varphi_t}$. The idea is to remove this operator by integrating by parts in s using

$$\widetilde{H}_{\varphi_t} e^{i\widetilde{H}_{\varphi_t}s} = -i\partial_s e^{i\widetilde{H}_{\varphi_t}s}. \tag{5-4}$$

This leads to

$$\begin{aligned}
D_{2114} = &-i\alpha^{-2}\int_0^t\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}t}(H_{\varphi_t}(k)+M)^{-1}e^{-iH_{\varphi_t}(k)(t-s_1)}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_{s_1})\tilde{\psi}_{s_1}\otimes\Omega\,\frac{dk}{|k|^2}\,ds_1\\
&+i\alpha^{-2}\int_0^t\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}s_1}(H_{\varphi_t}(k)+M)^{-1}e^{-iH_{\varphi_t}(k)s_1}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_0)\tilde{\psi}_0\otimes\Omega\,\frac{dk}{|k|^2}\,ds_1\\
&+i\alpha^{-2}\int_0^t\int_0^{t-s}\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(H_{\varphi_t}(k)+M)^{-1}e^{-iH_{\varphi_t}(k)s_1}\\
&\qquad\qquad\times W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)\partial_s\tilde{\psi}_s\otimes\Omega\,\frac{dk}{|k|^2}\,ds_1\,ds\\
&+i\alpha^{-2}\int_0^t\int_0^{t-s}\int_{\mathbb{R}^3} e^{i\widetilde{H}_{\varphi_t}(s+s_1)}(H_{\varphi_t}(k)+M)^{-1}e^{-iH_{\varphi_t}(k)s_1}\\
&\qquad\qquad\times W^*(\alpha^2\varphi_t)(\partial_s W(\alpha^2\varphi_s))\tilde{\psi}_s\otimes\Omega\,\frac{dk}{|k|^2}\,ds_1\,ds.
\end{aligned}$$

The first three terms on the right side can be bounded by Lemma 5.1 together with the uniform boundedness in $\mathcal{H}^2$ of $\tilde{\psi}_s$ and $\partial_s\tilde{\psi}_s$ in $[0,\alpha^2]$ from Proposition 2.2; see also the remark at the beginning of Section 2B concerning the bounds on $\partial_t\tilde{\psi}_t$. For the fourth term on the right side we use the formula (A-4) for $\partial_s W(\alpha^2\varphi_s)$. Then the term can be bounded by proceeding in the same way as for D_{015} and using Lemma 5.1 together with the fact that $\alpha^2\partial_s\varphi_s$ is uniformly bounded in $\mathcal{L}^2$ for all times by Lemma 2.1. To summarize, we obtain

$$\|D_{2114}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2}t(1+t), \tag{5-5}$$

and, because of (5-3),

$$\|D_{211}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2}t(1+t). \tag{5-6}$$

Bound on D_{212}. The term D_{212} involves a single difficult operator $\int b^*_{k'}e^{ik'\cdot x}|k'|^{-1}\,dk'$ and can be controlled using the technique from bounding D_{01}. We first integrate by parts with respect to s_1 using (5-1) (with $k=0$) to create a factor of $(H_{\varphi_t}+M)^{-1}$. Using this factor we can apply Lemma 3.1 as in the bound of D_{01}. In one of the terms, however, the integration by parts creates a factor $\widetilde{H}_{\varphi_t}$. We remove this operator via (5-4) by integrating by parts in s. The factor $g_{s,t}$ and its derivative $\partial_s g_{s,t} = -g_s$ are bounded by Proposition 2.2 and do not create any problems. Eventually, this shows that

$$\|D_{212}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}t^2(1+t). \tag{5-7}$$

Bound on D_{221}. The term D_{221} appears in (2-33). We use $b_k b^*(\sigma_{\tilde{\psi}_s})\Omega = \alpha^{-2}\sigma_{\tilde{\psi}_s}(k)\Omega$. By the Schwarz inequality, (C-2) and Lemma 2.1 we have $\big\||k|^{-1}\sigma_{\tilde{\psi}_s}(k)\Omega\big\|_1 \lesssim \|\sigma_{\tilde{\psi}_s}\|_{\mathcal{L}^2_{(1)}} \lesssim \|\psi_s\|^2_{H^1} \lesssim 1$. From this one easily concludes that

$$\|D_{221}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2}t^2.$$

Bound on D_{222}. The term D_{222} appears in (2-37). Using the bound on $g_{s,t}$ from Proposition 2.2 and the fact that $b(\sigma_{\tilde{\psi}_s})\Omega$ has norm of order α^{-1} by Lemma 2.1, one obtains

$$\|D_{222}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}t^3.$$

6. Bounds on D_3, D_4 and D_5

We recall that we have already controlled D_{32}, D_{42} and D_{52} in (2-39), (2-40) and (2-42). The remaining terms D_{31}, D_{41} and D_{51} have at most a single term $|k|^{-1}$ and can be bounded using the methods we have already developed. Therefore we will be rather brief.

For each of the terms D_{311}, D_{312}, D_{412}, D_{511} and D_{512} we first integrate by parts in s_1 to generate a factor of $(H_{\varphi_t}+M)^{-1}$, which allows us to apply Lemma 3.1. One of the terms, however, will involve $\widetilde{H}_{\varphi_t}$, which we have to remove by integrating by parts in s. Using the bounds from Lemma 2.1 and Proposition 2.2 we obtain

$$\|D_{311}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2}t(1+t), \quad \|D_{312}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}t^2(1+t), \quad \|D_{412}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}t^2(1+t),$$
$$\|D_{511}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-3}t(1+t), \quad \|D_{512}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-4}t^2(1+t+\alpha^{-1}t^2).$$

The remaining term D_{411} can be immediately bounded by

$$\|D_{411}\|_{\mathcal{L}^2\otimes\mathcal{F}} \lesssim \alpha^{-2}t^2.$$

7. Proof of the almost orthogonality relations

7A. *Proof of* (2-28). We recall that

$$\langle\Omega, e^{-iH_{\varphi_t}t}D_0\rangle_{\mathcal{F}} = \left\langle \Omega, \int_0^t e^{-iH_{\varphi_t}(t-s)}P^{\perp}_{\tilde{\psi}_s}\int_{\mathbb{R}^3}\big(e^{ik\cdot x}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)b_k^*\tilde{\psi}_s\otimes\Omega\big)\frac{dk}{|k|}\,ds\right\rangle_{\mathcal{F}}.$$

We commute the operator b_k^* to the left and use $b_k\Omega=0$. For the commutator we obtain from Corollary A.2 (with the definition (2-5) of $g_{s,t}$)

$$\begin{aligned}\langle\Omega, e^{-iH_{\varphi_t}t}D_0\rangle_{\mathcal{F}} &= \left\langle \Omega, \int_0^t e^{-iH_{\varphi_t}(t-s)}P^{\perp}_{\tilde{\psi}_s}g_{s,t}W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)\ \tilde{\psi}_s\otimes\Omega\,ds\right\rangle_{\mathcal{F}}\\ &= \int_0^t e^{-iH_{\varphi_t}(t-s)}P^{\perp}_{\tilde{\psi}_s}g_{s,t}\tilde{\psi}_s\big\langle\Omega, W^*(\alpha^2\varphi_t)W(\alpha^2\varphi_s)\ \Omega\big\rangle_{\mathcal{F}}\,ds.\end{aligned}$$

Thus,

$$\big\|\langle\Omega, e^{-iH_{\varphi_t}t}D_0\rangle_{\mathcal{F}}\big\|_{\mathcal{L}^2} \le t\sup_{0\le s\le t}\|g_{s,t}\|_\infty\|\tilde{\psi}_s\|_2.$$

Thus, by the bound on $g_{s,t}$ from Proposition 2.2 and the conservation of the $\mathcal{L}^2$ norm of $\tilde{\psi}_s$, we obtain the claimed bound (2-28).

7B. *Proof of* (2-29). For $\Phi \in \mathcal{F}$, let

$$\begin{aligned}\Theta_\Phi(t) &:= \langle \tilde\psi_t \otimes \Phi, e^{-iH_{\varphi_t} t} D_0 \rangle_{\mathcal{L}^2 \otimes \mathcal{F}} \\ &= \left\langle \tilde\psi_t \otimes \Phi, \int_0^t e^{-iH_{\varphi_t}(t-s)} P^\perp_{\tilde\psi_s} \int_{\mathbb{R}^3} \big(e^{ik\cdot x} W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) b_k^* \tilde\psi_s \otimes \Omega\big) \frac{dk}{|k|}\, ds \right\rangle_{\mathcal{L}^2 \otimes \mathcal{F}}.\end{aligned}$$

We shall show that

$$|\Theta_\Phi(t)| \lesssim \alpha^{-2} t^2 (1+\alpha^{-2}t^2) \|\Phi\|_{\mathcal{F}}, \tag{7-1}$$

which by duality implies (2-29).

Our goal will be to derive an ordinary differential equation for Θ_Φ. We use the presence of the operator $P^\perp_{\tilde\psi_s}$ to obtain (with inner products in $\mathcal{L}^2 \otimes \mathcal{F}$)

$$\begin{aligned}\partial_t \Theta_\Phi &= \left\langle \partial_t \tilde\psi_t \otimes \Phi, \int_0^t e^{-iH_{\varphi_t}(t-s)} P^\perp_{\tilde\psi_s} \int_{\mathbb{R}^3} \big(e^{ik\cdot x} W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) b_k^* \tilde\psi_s \otimes \Omega\big) \frac{dk}{|k|}\, ds \right\rangle \\ &\quad + \left\langle \tilde\psi_t \otimes \Phi, \int_0^t (\partial_t e^{-iH_{\varphi_t}(t-s)}) P^\perp_{\tilde\psi_s} \int_{\mathbb{R}^3} \big(e^{ik\cdot x} W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) b_k^* \tilde\psi_s \otimes \Omega\big) \frac{dk}{|k|}\, ds \right\rangle \\ &\quad + \left\langle \tilde\psi_t \otimes \Phi, \int_0^t e^{-iH_{\varphi_t}(t-s)} P^\perp_{\tilde\psi_s} \int_{\mathbb{R}^3} \big(e^{ik\cdot x} (\partial_t W^*(\alpha^2\varphi_t)) W(\alpha^2\varphi_s) b_k^* \tilde\psi_s \otimes \Omega\big) \frac{dk}{|k|}\, ds \right\rangle.\end{aligned}$$

For the first term we use equation (2-10) for $\partial_t \tilde\psi_t$. In the second term, we compute, using Duhamel's formula,

$$\begin{aligned}\partial_t e^{-iH_{\varphi_t}(t-s)} &= -iH_{\varphi_t} e^{-iH_{\varphi_t}(t-s)} - i \int_0^{t-s} e^{-iH_{\varphi_t}(t-s-s_1)} (\partial_t H_{\varphi_t}) e^{-iH_{\varphi_t} s_1}\, ds_1 \\ &= -i\big(H_{\varphi_t} + (t-s)\partial_t \|\varphi_t\|_2^2\big) e^{-iH_{\varphi_t}(t-s)} - i \int_0^{t-s} e^{-iH_{\varphi_t}(t-s-s_1)} (\partial_t V_{\varphi_t}) e^{-iH_{\varphi_t} s_1}\, ds_1.\end{aligned}$$

Note that the part involving H_{φ_t} will cancel the contribution from the first term, except for part of the constant $\omega(t)$. Finally, for the third term we use Lemma A.3 and Lemma A.1 to obtain

$$\begin{aligned}&\partial_t W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s) \\ &\quad = \alpha^2 W^*(\alpha^2\varphi_t)\big[b(\partial_t\varphi_t) - b^*(\partial_t\varphi_t) + i\,\mathrm{Im}(\varphi_t, \partial_t\varphi_t)\big] W(\alpha^2\varphi_s) \\ &\quad = \alpha^2 W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s)\big[b(\partial_t\varphi_t) - b^*(\partial_t\varphi_t) + 2i\,\mathrm{Im}(\partial_t\varphi_t, \varphi_s) + i\,\mathrm{Im}(\varphi_t, \partial_t\varphi_t)\big] \\ &\quad = \alpha^2 W^*(\alpha^2\varphi_t) W(\alpha^2\varphi_s)\big[b(\partial_t\varphi_t) - b^*(\partial_t\varphi_t) + 2i\,\mathrm{Im}(\partial_t\varphi_t, \varphi_s - \varphi_t) + i\,\mathrm{Im}(\partial_t\varphi_t, \varphi_t)\big].\end{aligned}$$

Putting all this into the above formula, we obtain

$$\partial_t \Theta_\Phi = M_1 + M_2 + M_3,$$

where the terms M_1, M_2 and M_3 are defined, using the notation

$$\Phi_{s,t} := W^*(\alpha^2\varphi_s) W(\alpha^2\varphi_t) \Phi,$$

by

$$M_1(t) := -i\int_0^t\int_0^{t-s}\Big\langle\tilde\psi_t\otimes\Phi_{s,t}, e^{-iH_{\varphi_t}(t-s-s_1)}(\partial_t V_{\varphi_t})e^{-iH_{\varphi_t}s_1}P^{\perp}_{\tilde\psi_s}\int_{\mathbb{R}^3}(e^{ik\cdot x}b_k^*\tilde\psi_s\otimes\Omega)\frac{dk}{|k|}\Big\rangle ds_1\,ds,$$

$$M_2(t) := \alpha^2\int_0^t\Big\langle\tilde\psi_t\otimes\Phi_{s,t}, e^{-iH_{\varphi_t}(t-s)}P^{\perp}_{\tilde\psi_s}\int_{\mathbb{R}^3}\big(e^{ik\cdot x}(b(\partial_t\varphi_t)-b^*(\partial_t\varphi_t))b_k^*\tilde\psi_s\otimes\Omega\big)\frac{dk}{|k|}\Big\rangle ds,$$

$$M_3(t) := \int_0^t m(s,t)\Big\langle\tilde\psi_t\otimes\Phi_{s,t},\ e^{-iH_{\varphi_t}(t-s)}P^{\perp}_{\tilde\psi_s}\int_{\mathbb{R}^3}(e^{ik\cdot x}b_k^*\tilde\psi_s\otimes\Omega)\frac{dk}{|k|}\Big\rangle ds$$

with

$$m(s,t) := -i(t-s)\partial_t\|\varphi_t\|_2^2 + 2i\alpha^2\,\mathrm{Im}(\partial_t\varphi_t, \varphi_s-\varphi_t).$$

Since $\Theta_\Phi(0)=0$, we conclude that

$$\Theta_\Phi(t) = \int_0^t (M_1(s)+M_2(s)+M_3(s))\,ds. \tag{7-2}$$

Below we shall show that

$$|M_1(t)| \lesssim \alpha^{-3}t^2\|\Phi\|_{\mathcal{F}},\quad |M_2(t)| \lesssim \alpha^{-2}t\|\Phi\|_{\mathcal{F}},\quad |M_3(t)| \lesssim \alpha^{-3}t^2\|\Phi\|_{\mathcal{F}}. \tag{7-3}$$

Together with (7-2) this will prove (7-1) and therefore (2-29).

Bound on M_1. Using the fact that $P^{\perp}_{\tilde\psi_s} = 1-|\tilde\psi_s\rangle\langle\tilde\psi_s|$ (see the proof of Lemma 2.4), we have the decomposition

$$M_1 = M_{11} - M_{12},$$

where

$$M_{11}(t) := -i\int_0^t\int_0^{t-s}\Big\langle\tilde\psi_t\otimes\Phi_{s,t}, e^{-iH_{\varphi_t}(t-s-s_1)}(\partial_t V_{\varphi_t})e^{-iH_{\varphi_t}s_1}\int_{\mathbb{R}^3}(e^{ik\cdot x}b_k^*\tilde\psi_s\otimes\Omega)\frac{dk}{|k|}\Big\rangle_{\mathcal{L}^2\otimes\mathcal{F}} ds_1\,ds$$

and, with $\sigma_{\tilde\psi_s}$ from (2-2),

$$M_{12}(t) := -i\int_0^t\int_0^{t-s}\langle\tilde\psi_t, e^{-iH_{\varphi_t}(t-s-s_1)}(\partial_t V_{\varphi_t})e^{-iH_{\varphi_t}s_1}\tilde\psi_s\rangle_{\mathcal{L}^2}\langle\Phi_{s,t}, b^*(\sigma_{\tilde\psi_s})\Omega\rangle_{\mathcal{F}}\,ds_1\,ds.$$

The second term is easy to control. In fact, the a priori bounds from Lemma 2.1 together with $\|\partial_t V_{\varphi_t}\|_\infty \lesssim \alpha^{-2}$ from (C-8) imply

$$\big|\langle\tilde\psi_t, e^{-iH_{\varphi_t}(t-s-s_1)}(\partial_t V_{\varphi_t})e^{-iH_{\varphi_t}s_1}\tilde\psi_s\rangle_{\mathcal{L}^2}\big| \lesssim \alpha^{-2}$$

and

$$\big|\langle\Phi_{s,t}, b^*(\sigma_{\tilde\psi_s})\Omega\rangle_{\mathcal{F}}\big| \lesssim \alpha^{-1}\|\Phi\|_{\mathcal{F}}.$$

This yields a bound of the form (7-3).

We now bound the integrand in M_{11}. We have

$$\left|\left\langle \tilde{\psi}_t\otimes\Phi_{s,t}, e^{-iH_{\varphi_t}(t-s-s_1)}(\partial_t V_{\varphi_t})e^{-iH_{\varphi_t}s_1}\int_{\mathbb{R}^3}(e^{ik\cdot x}b_k^*\tilde{\psi}_s\otimes\Omega)\frac{dk}{|k|}\right\rangle_{\mathcal{L}^2\otimes\mathcal{F}}\right|$$
$$\leq \left\|(H_{\varphi_t}+M)^{\frac12}\tilde{\psi}_t\otimes\Phi_{s,t}\right\|\left\|(H_{\varphi_t}+M)^{-\frac12}(\partial_t V_{\varphi_t})(H_{\varphi_t}+M)^{\frac12}\right\|\left\|(H_{\varphi_t}+M)^{-\frac12}\int_{\mathbb{R}^3}(e^{ik\cdot x}b_k^*\tilde{\psi}_s\otimes\Omega)\frac{dk}{|k|}\right\|.$$

By Corollary B.2 and an easy modification of its proof, for M sufficiently large (but independent of t and α), the operators $(H_{\varphi_t}+M)^{\pm\frac12}(-\Delta+1)^{\mp\frac12}$ are both bounded uniformly in t. Therefore Lemma 3.1 and the a priori bounds from Lemma 2.1 yield

$$\left|\left\langle \tilde{\psi}_t\otimes\Phi_{s,t}, e^{-iH_{\varphi_t}(t-s-s_1)}(\partial_t V_{\varphi_t})e^{-iH_{\varphi_t}s_1}\int_{\mathbb{R}^3}(e^{ik\cdot x}b_k^*\tilde{\psi}_s\otimes\Omega)\frac{dk}{|k|}\right\rangle_{\mathcal{L}^2\otimes\mathcal{F}}\right|$$
$$\lesssim \alpha^{-1}\|\tilde{\psi}_t\|_{\mathcal{H}^1}\|\Phi\|_{\mathcal{F}}\left\|(-\Delta+1)^{-\frac12}(\partial_t V_{\varphi_t})(-\Delta+1)^{\frac12}\right\|\|\psi_s\|_{\mathcal{H}^1}$$
$$\lesssim \alpha^{-1}\|\Phi\|_{\mathcal{F}}\left\|(-\Delta+1)^{-\frac12}(\partial_t V_{\varphi_t})(-\Delta+1)^{\frac12}\right\|.$$

Finally, using the fact that $\|\nabla\partial_t V_{\varphi_t}\|_\infty\lesssim\alpha^{-2}$ (see (C-8)), we obtain that the operator appearing in this bound has norm $\lesssim\alpha^{-2}$. Thus, we finally obtain

$$\left|\left\langle \tilde{\psi}_t\otimes\Phi_{s,t}, e^{-iH_{\varphi_t}(t-s-s_1)}(\partial_t V_{\varphi_t})e^{-iH_{\varphi_t}s_1}\int_{\mathbb{R}^3}(e^{ik\cdot x}b_k^*\tilde{\psi}_s\otimes\Omega)\frac{dk}{|k|}\right\rangle_{\mathcal{L}^2\otimes\mathcal{F}}\right|\lesssim\alpha^{-3},$$

which, when integrated over s_1 and s, leads to the bound in (7-3).

Bound on M_2. As for M_1, we use $P^{\perp}_{\tilde{\psi}_s}=1-|\tilde{\psi}_s\rangle\langle\tilde{\psi}_s|$ to get the decomposition

$$M_2=M_{21}-M_{22}$$

with

$$M_{21}(t):=\alpha^2\int_0^t\left\langle\tilde{\psi}_t\otimes\Phi_{s,t}, e^{-iH_{\varphi_t}(t-s)}\int_{\mathbb{R}^3}\left(e^{ik\cdot x}(b(\partial_t\varphi_t)-b^*(\partial_t\varphi_t))b_k^*\ \tilde{\psi}_s\otimes\Omega\right)\frac{dk}{|k|}\right\rangle ds$$

and, with $\sigma_{\tilde{\psi}_s}$ from (2-2),

$$M_{22}(t):=\alpha^2\int_0^t\langle\tilde{\psi}_t,\ e^{-iH_{\varphi_t}(t-s)}\tilde{\psi}_s\rangle_{\mathcal{L}^2}\left\langle\Phi_{s,t},\big(b(\partial_t\varphi_t)-b^*(\partial_t\varphi_t)\big)b^*(\sigma_{\tilde{\psi}_s})\Omega\right\rangle_{\mathcal{F}}ds.$$

Once again the bound on M_{22} is straightforward. Namely, we commute $b^*(\sigma_{\tilde{\psi}_s})$ to the left through $b(\partial_t\varphi_t)-b^*(\partial_t\varphi_t)$ and obtain

$$\left\langle\Phi_{s,t},\big(b(\partial_t\varphi_t)-b^*(\partial_t\varphi_t)\big)b^*(\sigma_{\tilde{\psi}_s})\Omega\right\rangle_{\mathcal{F}}=-\left\langle\Phi_{s,t},b^*(\sigma_{\tilde{\psi}_s})b^*(\partial_t\varphi_t)\Omega\right\rangle_{\mathcal{F}}+\alpha^{-2}(\partial_t\varphi_t,\sigma_{\tilde{\psi}_s})\langle\Phi_{s,t},\Omega\rangle_{\mathcal{F}}.$$

By similar computations as, for instance, in the bound on D_{32} and by the a priori bounds from Lemma 2.1, we obtain

$$\left|\left\langle\Phi_{s,t},\big(b(\partial_t\varphi_t)-b^*(\partial_t\varphi_t)\big)b^*(\sigma_{\tilde{\psi}_s})\Omega\right\rangle_{\mathcal{F}}\right|\lesssim\alpha^{-2}\|\Phi\|_{\mathcal{F}}\|\sigma_{\tilde{\psi}_s}\|\|\partial_t\varphi_t\|\lesssim\alpha^{-4}\|\Phi\|_{\mathcal{F}}.$$

By the conservation of the $\mathcal{L}^2$ norm of $\tilde{\psi}_t$ we conclude

$$|M_{22}(t)| \lesssim \alpha^{-2} t \|\Phi\|_{\mathcal{F}},$$

which is of the form claimed in (7-3).

We now discuss M_{21}. Again we commute b_k^* to the left through $b(\partial_t \varphi_t) - b^*(\partial_t \varphi_t)$ and obtain

$$M_{21} = M_{211} + M_{212},$$

where

$$M_{211}(t) := -\alpha^2 \int_0^t \left\langle \tilde{\psi}_t \otimes \Phi_{s,t}, e^{-iH_{\varphi_t}(t-s)} \int_{\mathbb{R}^3} \left(e^{ik\cdot x} b_k^* b^*(\partial_t \varphi_t) \tilde{\psi}_s \otimes \Omega \right) \frac{dk}{|k|} \right\rangle_{\mathcal{L}^2 \otimes \mathcal{F}} ds$$

and, with g_s from (2-6),

$$M_{212}(t) := \int_0^t \langle \tilde{\psi}_t, e^{-iH_{\varphi_t}(t-s)} g_s \tilde{\psi}_s \rangle_{\mathcal{L}^2} \langle \Phi_{s,t}, \Omega \rangle_{\mathcal{F}} \, ds.$$

Since $\|g_s\|_\infty \lesssim \alpha^{-2}$ by Proposition 2.2, we obtain immediately

$$|M_{212}(t)| \lesssim \alpha^{-2} t \|\Phi\|_{\mathcal{F}}.$$

To control M_{211} we bound

$$\begin{aligned} &\left| \left\langle \tilde{\psi}_t \otimes \Phi_{s,t}, e^{-iH_{\varphi_t}(t-s)} \int_{\mathbb{R}^3} \left(e^{ik\cdot x} b_k^* b^*(\partial_t \varphi_t) \tilde{\psi}_s \otimes \Omega \right) \frac{dk}{|k|} \right\rangle_{\mathcal{L}^2 \otimes \mathcal{F}} \right| \\ &\qquad \le \left\| (H_{\varphi_t} + M)^{\frac12} \tilde{\psi}_t \otimes \Phi_{s,t} \right\| \left\| (H_{\varphi_t} + M)^{-\frac12} \int_{\mathbb{R}^3} \left(e^{ik\cdot x} b_k^* b^*(\partial_t \varphi_t) \tilde{\psi}_s \otimes \Omega \right) \frac{dk}{|k|} \right\|. \end{aligned}$$

As for M_{11}, we use Lemma 2.1 and Corollary B.2 (and a simple extension of its proof) to choose M large enough, but independent of t and α, so that $(H_{\varphi_t} + M)^{\pm\frac12}(-\Delta+1)^{\mp\frac12}$ are both bounded uniformly in t. Therefore Lemma 3.1 and the a priori bounds from Lemma 2.1 yield

$$\begin{aligned} \left| \left\langle \tilde{\psi}_t \otimes \Phi_{s,t}, e^{-iH_{\varphi_t}(t-s)} \int_{\mathbb{R}^3} \left(e^{ik\cdot x} b_k^* b^*(\partial_t \varphi_t) \tilde{\psi}_s \otimes \Omega \right) \frac{dk}{|k|} \right\rangle_{\mathcal{L}^2 \otimes \mathcal{F}} \right| &\lesssim \alpha^{-2} \|\tilde{\psi}_t\|_{\mathcal{H}^1} \|\Phi\|_{\mathcal{F}} \|\partial_t \varphi_t\|_{\mathcal{L}^2} \|\tilde{\psi}_s\|_{\mathcal{H}^1} \\ &\lesssim \alpha^{-4} \|\Phi\|_{\mathcal{F}}. \end{aligned}$$

This, when integrated over s and multiplied by α^2, leads to the bound in (7-3).

Bound on M_3. The a priori bounds from Lemma 2.1 yield

$$|m(s,t)| \lesssim \alpha^{-2} |t - s|.$$

Moreover, applying Lemma 3.1 as in the bound on M_{21} we find that the absolute value of the inner product in the integral defining M_3 is bounded by a constant times $\alpha^{-1} \|\Phi\|_{\mathcal{F}}$. This yields the bound in (7-3).

This concludes the proof of (2-29).

Appendix A: Some properties of the Weyl operators

In this appendix we collect some standard properties of the Weyl operators $W(f)$ defined in (1-7) in terms of $b(f)$ and $b^*(f)$. They are well known, but we provide proofs for the sake of completeness. We recall that the commutation relations for b_k and b_k^* involve a factor α^{-2}.

Lemma A.1. *The operators b_k, b_k^* and $W(f)$ satisfy the following relations*,

$$b_k W(f) = W(f)(b_k + \alpha^{-2} f(k)) \quad \textit{and} \quad b_k^* W(f) = W(f)(b_k^* + \alpha^{-2} \bar{f}(k)). \tag{A-1}$$

Proof. For $t > 0$ we consider the operators

$$F_t := W(tf) = e^{t(b^*(f)-b(f))}, \tag{A-2}$$

which satisfy

$$\partial_t F_t = (b^*(f) - b(f))F_t, \quad F_0 = \mathrm{Id}.$$

Multiplying by b_k and using the commutation relations, we obtain the following equation for $b_k F_t$:

$$\partial_t b_k F_t = (b^*(f) - b(f)) b_k F_t + \alpha^{-2} f(k) F_t, \quad b_k F_0 = b_k.$$

Therefore, by Duhamel's principle applied to the latter equation,

$$b_k F_t = e^{t(b^*(f)-b(f))} b_k + \alpha^{-2} f(k) \int_0^t e^{(t-s)(b^*(f)-b(f))} F_s \, ds.$$

Recalling the definition of F_t in (A-2), we can rewrite this as

$$b_k F_t = F_t b_k + t\alpha^{-2} f(k) F_t. \tag{A-3}$$

At $t = 1$ we obtain the first identity in the lemma. The second one is proved similarly. □

By applying Lemma A.1 twice, we obtain:

Corollary A.2.

$$[b_k^*, W^*(f)W(g)] = -\alpha^{-2}(\bar{f}(k) - \bar{g}(k)) W^*(f)W(g),$$
$$[b_k, W^*(f)W(g)] = -\alpha^{-2}(f(k) - g(k))\, W^*(f)W(g).$$

Next, we'll consider the case where f depends (differentiably) on a parameter.

Lemma A.3.

$$\partial_t W(f_t) = \tfrac{1}{2}\alpha^{-2}\big((f_t, \partial_t f_t) - (\partial_t f_t, f_t)\big) W(f_t) + W(f_t)\big(b^*(\partial_t f_t) - b(\partial_t f_t)\big), \tag{A-4}$$

$$\partial_t W(f_t) = -\tfrac{1}{2}\alpha^{-2}\big((f_t, \partial_t f_t) - (\partial_t f_t, f_t)\big) W(f_t) + \big(b^*(\partial_t f_t) - b(\partial_t f_t)\big) W(f_t). \tag{A-5}$$

Proof. For $s > 0$ we consider the operators

$$F(s,t) := W(s f_t), \tag{A-6}$$

which satisfy

$$\partial_s F(s,t) = (b^*(f_t) - b(f_t)) F(s,t), \quad F(0,t) = \mathrm{Id}.$$

We differentiate this equation with respect to t and obtain

$$\begin{aligned}\partial_s\partial_t F(s,t) &= (b^*(f_t)-b(f_t))\partial_t F(s,t)+(b^*(\partial_t f_t)-b(\partial_t f_t))F(s,t),\\ \partial_t F(0,t) &= 0.\end{aligned}$$

Therefore, by Duhamel's principle,

$$\begin{aligned}\partial_t F(s,t) &= \int_0^s e^{(b^*(f_t)-b(f_t))(s-s_1)}(b^*(\partial_t f_t)-b(\partial_t f_t))F(s_1,t)\,ds_1\\ &= \int_0^s W((s-s_1)f_t)(b^*(\partial_t f_t)-b(\partial_t f_t))W(s_1 f_t)\,ds_1.\end{aligned}$$

In order to simplify the integrand we now use Lemma A.1 and obtain

$$(b^*(\partial_t f_t)-b(\partial_t f_t))W(s_1 f_t)=\alpha^{-2}W(s_1 f_t)s_1((f_t,\partial_t f_t)-(\partial_t f_t,f_t))+W(s_1 f_t)(b^*(\partial_t f_t)-b(\partial_t f_t)).$$

If we insert this into the above formula for $\partial_t F(s,t)$, we obtain

$$\partial_t F(s,t)=\alpha^{-2}\tfrac{1}{2}s^2 W(sf_t)((f_t,\partial_t f_t)-(\partial_t f_t,f_t))+sW(sf_t)(b^*(\partial_t f_t)-b(\partial_t f_t)).$$

At $s=1$, we obtain the first identity in the lemma. The second one is proved similarly. □

Lemma A.4. *For any* $f,g\in\mathcal{L}^2$,

$$\langle\Omega,W^*(g)W(f)\Omega\rangle=e^{i\alpha^{-2}\mathrm{Im}(g,f)-\alpha^{-2}\|f-g\|^2/2}.$$

Proof. Let $f_t:=tf+(1-t)g$ and $F(t):=\langle\Omega,W^*(g)W(f_t)\Omega\rangle$. By Lemma A.3, using that $\mathrm{Im}(f_t,\partial_t f_t)=\mathrm{Im}(f_t,f-g)=\mathrm{Im}(g,f)$,

$$\partial_t F(t)=\big\langle\Omega,W^*(g)W(f_t)\big(b^*(f-g)+i\alpha^{-2}\,\mathrm{Im}(g,f)\big)\Omega\big\rangle.$$

Next, by Corollary A.2, since $(g-f_t,f-g)=-t\|f-g\|^2$,

$$W^*(g)W(f_t)b^*(f-g)=b^*(f-g)W^*(g)W(f_t)+\alpha^{-2}(g-f_t,f-g)W^*(g)W(f_t),$$

so

$$\partial_t F(t)=\big(-\alpha^{-2}t\|f-g\|^2+i\alpha^{-2}\,\mathrm{Im}(g,f)\big)F(t).$$

Since $F(0)=1$, we conclude that

$$F(t)=e^{-\alpha^{-2}t^2\|f-g\|^2/2+i\alpha^{-2}t\,\mathrm{Im}(g,f)},$$

which, at $t=1$, gives the assertion. □

Appendix B: The effective Schrödinger operator

In this appendix we investigate the operator and form domains of the effective Schrödinger operator H_φ from (2-12) with potential V_φ from (2-1).

Lemma B.1. *For every $A > 0$ and $\varepsilon > 0$ there is an $M > 0$ such that if $\|\varphi\| \le A$, then for all $\psi \in \mathcal{H}^1(\mathbb{R}^3)$,*

$$\big\| |V_\varphi|^{\frac12} \psi \big\| \le \varepsilon \big\| (-\Delta + M)^{\frac12} \psi \big\|$$

and for all $\psi \in \mathcal{H}^2(\mathbb{R}^3)$,

$$\| V_\varphi \psi \| \le \varepsilon \| (-\Delta + M) \psi \|.$$

Proof. As in [Frank and Schlein 2014, Section 2.1], the Hardy–Littlewood–Sobolev inequality implies

$$\| V_\varphi \|_6 \lesssim \| \varphi \|_2. \tag{B-1}$$

This implies, by the Hölder and Sobolev inequalities,

$$\int_{\mathbb{R}^3} |V_\varphi| \, |\psi|^2 \, dx \le \| V_\varphi \|_6 \, \| \psi \|_{\frac{12}{5}}^2 \lesssim \| \varphi \|_2 \, \| \nabla \psi \|_2^{\frac12} \, \| \psi \|_2^{\frac32}$$

and

$$\int_{\mathbb{R}^3} |V_\varphi|^2 \, |\psi|^2 \, dx \le \| V_\varphi \|_6^2 \, \| \psi \|_3^2 \lesssim \| \varphi \|_2^2 \, \| \Delta \psi \|_2^{\frac12} \, \| \psi \|_2^{\frac32}.$$

These bounds easily imply the assertions of the lemma. □

Corollary B.2. *For every $A > 0$ there are $M > 0$ and $C > 0$ such that if $\|\varphi\|_2 \le A$ then for all $f \in \mathcal{L}^2(\mathbb{R}^3)$*

$$\| (H_\varphi + M)^{-\frac12} f \|_2 \le C \, \| (-\Delta + 1)^{-\frac12} f \|_2$$

and

$$\| (H_\varphi + M)^{-1} f \|_2 \le C \, \| (-\Delta + 1)^{-1} f \|_2.$$

Proof. To prove the first assertion, we write

$$(H_\varphi + M)^{-1} = (-\Delta + M)^{-\frac12} \big(1 + (-\Delta + M)^{-\frac12} V_\varphi (-\Delta + M)^{-\frac12} \big)^{-1} (-\Delta + M)^{-\frac12}$$

and note that according to Lemma B.1 we can choose M such that $\|\varphi\| \le A$ implies

$$\| (-\Delta + M)^{-\frac12} V_\varphi (-\Delta + M)^{-\frac12} \| \le \varepsilon^2.$$

Similarly, for the second assertion we write

$$(H_\varphi + M)^{-1} = \big(1 + (-\Delta + M)^{-1} V_\varphi \big)^{-1} (-\Delta + M)^{-1}$$

and choose M such that $\|\varphi\| \le A$ implies $\| (-\Delta + M)^{-1} V_\varphi \| \le \varepsilon$. □

Appendix C: Well-posedness of the Landau–Pekar equations

In this appendix we prove Lemma 2.1 and Proposition 2.2. Recall that the weighted spaces $\mathcal{L}^2_{(m)} = \mathcal{L}^2(\mathbb{R}^3; (1 + k^2)^m \, dk)$ were introduced in (1-11). We begin with some bounds on the coupling terms V_φ and σ_ψ introduced in (2-1) and (2-2).

Lemma C.1. *We have*

$$\|\partial^\beta V_\varphi\|_\infty \lesssim \|\varphi\|_{\mathcal{L}^2_{|\beta|+1}} \quad \textit{for all } \beta \in \mathbb{N}_0^3, \tag{C-1}$$

$$\|\sigma_\psi\|_{\mathcal{L}^2_{(1)}} \lesssim \|\psi\|^2_{\mathcal{H}^1}, \quad \|\sigma_\psi\|_{\mathcal{L}^2_{(3)}} \lesssim \|\psi\|^2_{\mathcal{H}^2}. \tag{C-2}$$

Proof. By the Schwarz inequality,

$$|\partial^\beta V_\varphi(x)| \le 2\int_{\mathbb{R}^3} |k|^{|\beta|-1}|\varphi(k)|\,dk \le 2\|\varphi\|_{\mathcal{L}^2_{|\beta|+1}} \left(\int_{\mathbb{R}^3} \frac{|k|^{2(|\beta|-1)}\,dk}{(1+k^2)^{2(|\beta|+1)}}\right)^{\frac{1}{2}}$$

and the last integral is finite.

We have

$$\|\sigma_\psi\|_2^2 = \left\| \frac{1}{|k|}\int_{\mathbb{R}^3} |\psi(x)|^2 e^{ik\cdot x}\,dx\right\|_2^2 = 2\pi^2 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\psi(x)|^2\,|\psi(y)|^2}{|x-y|}\,dx\,dy.$$

By the Hardy–Littlewood–Sobolev inequality, we know this is bounded by a constant times $\big\||\psi|^2\big\|^2_{\frac{6}{5}} = \|\psi\|^4_{\frac{12}{5}}$, which, by the Sobolev embedding theorem, is bounded by a constant times $\|\psi\|^4_{\mathcal{H}^1}$. Moreover, by Plancherel,

$$\|\sigma_\psi\|^2_{\mathcal{L}^2(|k|^{2m})} = \int_{\mathbb{R}^3} |k|^{2(m-1)} \left| \int_{\mathbb{R}^3} |\psi|^2 e^{ik\cdot x}\,dx\right|^2 dk = (2\pi)^3\big(|\psi|^2, (-\Delta)^{m-1}|\psi|^2\big).$$

In particular, for $m=1$ we get $\|\psi\|_4^4$, which by Sobolev is controlled by $\|\psi\|^2_{\mathcal{H}^1}$. For $m=3$, the claimed bound follows easily using $\|\psi\|_\infty \lesssim \|\psi\|_{\mathcal{H}^2}$ and again Sobolev. □

Proof of Lemma 2.1. Local well-posedness in $\mathcal{H}^1\times\mathcal{L}^2$ follows by a standard fixed-point argument and one sees that $\|\psi_t\|_2$ and $\mathcal{E}(\psi_t,\varphi_t)$ are conserved. One can use (B-1) and the Sobolev inequality to show that [Frank and Schlein 2014, Section 2.1],

$$\mathcal{E}(\psi,\varphi) \ge \|\nabla\psi\|_2^2 + \|\varphi\|_2^2 - C\|\varphi\|_2\|\nabla\psi\|_2^{\frac{1}{2}}\|\psi\|_2^{\frac{3}{2}} \tag{C-3}$$

for some universal constant $C>0$. This, together with conservation of $\mathcal{E}(\psi_t,\varphi_t)$, yields global well-posedness as well as the uniform bounds (2-3).

According to (C-2) and the first bound in (2-3), we have $\|\sigma_{\psi_t}\| \lesssim \|\psi_t\|^2_{\mathcal{H}^1} \lesssim 1$, which is the third bound in (2-4).

By equation (1-9) for φ_t we have

$$\|\alpha^2\partial_t\varphi_t\|_2 \le \|\varphi_t\|_2 + \|\sigma_{\psi_t}\|_2$$

and therefore, by the second bound in (2-3) and the third bound in (2-4), we obtain the first bound in (2-4).

Finally, $\varphi_t - \varphi_s = \int_s^t \partial_{s_1}\varphi_{s_1}\,ds_1$, so for $t>s$, by the first bound in (2-4),

$$\|\varphi_t-\varphi_s\|_2 \le \int_s^t \|\partial_{s_1}\varphi_{s_1}\|_2\,ds_1 \lesssim \alpha^{-2}|t-s|.$$

This proves the second bound in (2-4) and completes the proof of the lemma. □

Before dealing with $\mathcal{H}^4\times\mathcal{L}^2_{(3)}$-regularity in Proposition 2.2, we need to establish $\mathcal{H}^2\times\mathcal{L}^2_{(1)}$-regularity.

Lemma C.2. *If* $(\psi_0, \varphi_0) \in \mathcal{H}^2(\mathbb{R}^3) \times \mathcal{L}^2_{(1)}(\mathbb{R}^3)$, *then* $(\psi_t, \varphi_t) \in \mathcal{H}^2(\mathbb{R}^3) \times \mathcal{L}^2_{(1)}(\mathbb{R}^3)$ *for all* $t \in \mathbb{R}$ *and*

$$\|\psi_t\|_{\mathcal{H}^2} \lesssim 1 + \alpha^{-2}|t|, \quad \|\varphi_t\|_{\mathcal{L}^2_{(1)}(\mathbb{R}^3)} \lesssim 1 + \alpha^{-2}|t|$$

with implicit constants depending only on the initial data. Moreover,

$$\|\partial_t \psi_t\|_{\mathcal{L}^2} \lesssim 1 + \alpha^{-2}|t|, \quad \|\partial_t \sigma_{\psi_t}\|_{\mathcal{L}^2} \lesssim 1 + \alpha^{-2}|t|. \tag{C-4}$$

If, in addition, $\varphi_0 \in \mathcal{L}^2_{(m)}(\mathbb{R}^3)$, $m = 2, 3$, *then* $\varphi_t \in \mathcal{L}^2_{(m)}(\mathbb{R}^3)$ *for all* $t \in \mathbb{R}$ *and*

$$\|\varphi_t\|_{\mathcal{L}^2_{(m)}(\mathbb{R}^3)} \lesssim 1 + \alpha^{-6}|t|^3.$$

Proof. By a standard fixed-point argument one can show local existence of solutions in $\mathcal{H}^2 \times \mathcal{L}^2_{(1)}$. In the following we will construct a functional, which is equivalent to the $\mathcal{H}^2$ norm of ψ and which grows in a controlled way as time increases. This will prove, in particular, that ψ_t belongs to $\mathcal{H}^2$ for all times.

We claim that for every $A > 0$ there is a constant $M > 0$ such that

$$\mathcal{E}^{(2)}(\psi, \varphi) := \|(-\Delta + V_\varphi + M)\psi\|_2^2$$

satisfies

$$\tfrac{1}{2}\|\psi\|_{\mathcal{H}^2} \le (\mathcal{E}^{(2)}(\psi, \varphi))^{\frac{1}{2}} \le \tfrac{3}{2}\|\psi\|_{\mathcal{H}^2} \tag{C-5}$$

for all $\psi \in \mathcal{H}^2$ and all φ satisfying $\|\varphi\|_2 \le A$. In fact, much as in the proof of Corollary B.2, we have

$$\Big|\|(-\Delta + V_\varphi + M)\psi\|_2 - \|(-\Delta + M)\psi\|_2\Big| \le \|V_\varphi(-\Delta + M)^{-1}\| \, \|(-\Delta + M)\psi\|_2$$

and according to Lemma B.1 we can choose M such that the first factor on the right side is less than ε for $\|\varphi\|_2 \le A$.

According to Lemma 2.1 there is an $A > 0$ (depending only on $\|\psi_0\|_{\mathcal{H}^1}$ and $\|\varphi_0\|_{\mathcal{L}^2}$) such that $\|\varphi_t\|_{\mathcal{L}^2} \le A$ for all t. We choose M corresponding to this value of A and compute, using the equation for ψ_t,

$$\begin{aligned}
\partial_t &\mathcal{E}^{(2)}(\psi_t, \varphi_t) \\
&= 2\,\mathrm{Re}\big((-\Delta + V_{\varphi_t} + M)\psi_t, (-\Delta + V_{\varphi_t} + M)\partial_t\psi_t\big) + 2\,\mathrm{Re}\big((-\Delta + V_{\varphi_t} + M)\psi_t, (\partial_t V_{\varphi_t})\psi_t\big) \\
&= 2\,\mathrm{Re}\big((-\Delta + V_{\varphi_t} + M)\psi_t, (\partial_t V_{\varphi_t})\psi_t\big).
\end{aligned}$$

By the Schwarz and the Hölder inequalities,

$$\partial_t \mathcal{E}^{(2)}(\psi_t, \varphi_t) \le 2(\mathcal{E}^{(2)}(\psi_t, \varphi_t))^{\frac{1}{2}} \|\partial_t V_{\varphi_t}\|_6 \|\psi_t\|_3.$$

By (B-1) and Lemma 2.1, $\|\partial_t V_{\varphi_t}\|_6 \lesssim \|\partial_t \varphi_t\|_2 \lesssim \alpha^{-2}$, and by the Sobolev inequality and Lemma 2.1, $\|\psi_t\|_3 \lesssim \|\psi_t\|_{\mathcal{H}^1} \lesssim 1$. Thus,

$$\partial_t \mathcal{E}^{(2)}(\psi_t, \varphi_t) \lesssim \alpha^{-2}(\mathcal{E}^{(2)}(\psi_t, \varphi_t))^{\frac{1}{2}},$$

which implies $(\mathcal{E}^{(2)}(\psi_t, \varphi_t))^{\frac{1}{2}} \lesssim 1 + \alpha^{-2}|t|$. According to (C-5), this implies the claimed bound on $\|\psi_t\|_{\mathcal{H}^2}$.

The remaining bounds are proved in a straightforward way. We have

$$\|\partial_t \psi_t\|_2 \le \|-\Delta\psi_t\|_2 + \|V_{\varphi_t}\psi_t\|_2 \le \|\psi_t\|_{\mathcal{H}^2} + \|V_{\varphi_t}\|_6 \|\psi_t\|_3.$$

By the bound on $\|\psi_t\|_{\mathcal{H}^2}$ together with (B-1) and the bounds from Lemma 2.1, we obtain the first bound in (C-4). Moreover,

$$\partial_t \sigma_{\psi_t} = 2|k|^{-1} \int_{\mathbb{R}^3} \mathrm{Re}(\overline{\psi_t} \partial_t \psi_t) e^{ik\cdot x}\, dx$$

and so, by the Hardy–Littlewood–Sobolev inequality as in (B-1),

$$\|\partial_t \sigma_{\psi_t}\|_2 \lesssim \|\psi_t \partial_t \psi_t\|_{\frac{6}{5}} \le \|\psi_t\|_3 \|\partial_t \psi_t\|_2.$$

By the first bound in (C-4) and Lemma 2.1, we obtain the second bound in (C-4).

In order to deduce the bounds on φ_t, we use Duhamel's formula:

$$\varphi_t(k) = e^{-it/\alpha^2} \varphi_0(k) - i\alpha^{-2} \int_0^t e^{-i(t-s)/\alpha^2} \sigma_{\psi_s}(k)\, ds. \tag{C-6}$$

If $\varphi_0 \in \mathcal{L}^2_{(m)}$, $m = 1, 2, 3$, we deduce that $\varphi_t \in \mathcal{L}^2_{(m)}$ provided we can bound $\|\sigma_{\psi_s}\|_{\mathcal{L}^2_{(m)}}$. This quantity can by controlled by Sobolev norms of ψ_s according to (C-2). □

Proof of Proposition 2.2. The basic strategy is the same as in the proof of Lemma C.2, except that verifying the properties of the functional is more complicated in this case. Again we do not give the details of the local existence via a fixed-point argument.

We claim that for every $A > 0$ there is a constant $M > 0$ such that

$$\mathcal{E}^{(4)}(\psi, \varphi) := \|(-\Delta + V_\varphi + M)^2 \psi\|_2^2$$

satisfies

$$\tfrac{1}{2} \|\psi\|_{\mathcal{H}^4} \le (\mathcal{E}^{(4)}(\psi, \varphi))^{\frac{1}{2}} \le \tfrac{3}{2} \|\psi\|_{\mathcal{H}^4} \tag{C-7}$$

for all $\psi \in \mathcal{H}^4$ and all φ satisfying $\|\varphi\|_{\mathcal{L}^2_{(3)}} \le A$. To show this, we first observe that, as in the proof of Lemma C.2,

$$\begin{aligned} &\big|\|(-\Delta + V_\varphi + M)^2 \psi\|_2 - \|(-\Delta + M)(-\Delta + V_\varphi + M)\psi\|_2\big| \\ &\qquad\qquad \le \|V_\varphi(-\Delta + M)^{-1}\| \, \|(-\Delta + M)(-\Delta + V_\varphi + M)\psi\|_2 \end{aligned}$$

and that $\|V_\varphi(-\Delta + M)^{-1}\|$ can be made arbitrarily small for $\|\varphi\|_{\mathcal{L}^2}$ bounded by choosing M large. Thus, it suffices to show that $\|(-\Delta + M)(-\Delta + V_\varphi + M)\psi\|_2$ is equivalent to $\|(-\Delta + M)^2 \psi\|_2$. We compute

$$\begin{aligned} &\big|\|(-\Delta + M)(-\Delta + V_\varphi + M)\psi\|_2 - \|(-\Delta + V_\varphi + M)(-\Delta + M)\psi\|_2\big| \\ &\qquad\qquad \le \|(2\nabla V_\varphi \cdot \nabla + \Delta V_\varphi)(-\Delta + M)^{-1}\| \, \|(-\Delta + M)\psi\|_2. \end{aligned}$$

According to (C-1), the first factor on the right side can be made arbitrarily small for $\|\varphi\|_{\mathcal{L}^2_{(3)}}$ bounded by choosing M large. We conclude by applying the argument in Lemma C.2 again to compare $\|(-\Delta + V_\varphi + M)(-\Delta + M)\psi\|_2$ to $\|(-\Delta + M)^2 \psi\|_2$. This proves the claim.

According to Lemma C.2, for every $\tau > 0$ there is an $A > 0$ (depending only on $\|\psi_0\|_{\mathcal{H}^2}$, $\|\varphi_0\|_{\mathcal{L}^2_{(3)}}$ and τ) such that $\|\varphi_t\|_{\mathcal{L}^2_{(3)}} \le A$ for all $|t| \le \tau\alpha^2$. We choose M corresponding to this value of A and compute, using the equation for ψ_t,

$$\begin{aligned}\partial_t \mathcal{E}^{(4)}(\psi_t,\varphi_t) &= 2\operatorname{Re}\big((-\Delta+V_{\varphi_t}+M)^2\psi_t,(-\Delta+V_{\varphi_t}+M)^2\partial_t\psi_t\big)\\ &\qquad + 2\operatorname{Re}\big((-\Delta+V_{\varphi_t}+M)^2\psi_t,(\partial_t V_{\varphi_t})(-\Delta+V_{\varphi_t}+M)\psi_t\big)\\ &\qquad\quad + 2\operatorname{Re}\big((-\Delta+V_{\varphi_t}+M)^2\psi_t,(-\Delta+V_{\varphi_t}+M)(\partial_t V_{\varphi_t})\psi_t\big)\\ &= 4\operatorname{Re}\big((-\Delta+V_{\varphi_t}+M)^2\psi_t,(\partial_t V_{\varphi_t})(-\Delta+V_{\varphi_t}+M)\psi_t\big)\\ &\qquad - 2\operatorname{Re}\big((-\Delta+V_{\varphi_t}+M)^2\psi_t,(2\nabla\partial_t V_{\varphi_t}\cdot\nabla+\Delta\partial_t V_{\varphi_t})\psi_t\big).\end{aligned}$$

Therefore, by the Schwarz inequality,

$$\begin{aligned}&\partial_t \mathcal{E}^{(4)}(\psi_t,\varphi_t)\\ &\quad\le 2(\mathcal{E}^{(4)}(\psi_t,\varphi_t))^{\frac12}\big(2\|\partial_t V_{\varphi_t}\|_\infty\|(-\Delta+V_{\varphi_t}+M)\psi_t\|_2+2\|\nabla\partial_t V_{\varphi_t}\|_\infty\|\nabla\psi_t\|_2+\|\Delta\partial_t V_{\varphi_t}\|_\infty\|\psi_t\|_2\big).\end{aligned}$$

According to Lemma C.2 and (C-5), all terms involving ψ_t here are bounded by a constant for $|t|\le\tau\alpha^2$. Assume that we can prove that all terms involving φ_t here are bounded by a constant times α^{-2} for $|t|\le\tau\alpha^2$. Then we will have shown that

$$\partial_t \mathcal{E}^{(4)}(\psi_t,\varphi_t)\lesssim \alpha^{-2}(\mathcal{E}^{(4)}(\psi_t,\varphi_t))^{\frac12}$$

for $|t|\le\tau\alpha^2$, which implies that $(\mathcal{E}^{(4)}(\psi_t,\varphi_t))^{\frac12}\lesssim 1+\alpha^{-2}|t|\lesssim 1$ for $|t|\le\tau\alpha^2$. According to (C-7), this proves that $\|\psi_t\|_{\mathcal{H}^4}\lesssim 1$ for $|t|\le\tau\alpha^2$.

Thus, it remains to prove that for all multi-indices β with $|\beta|\le 2$,

$$\|\partial_x^\beta\partial_t V_{\varphi_t}\|_\infty\lesssim\alpha^{-2}\quad\text{for } |t|\le\tau\alpha^{-2}. \tag{C-8}$$

If we insert the equation of φ_t into the definition of V_{φ_t}, we find

$$\partial_t V_{\varphi_t}(x) = -i\alpha^{-2}\int_{\mathbb{R}^3}\big(e^{-ik\cdot x}\varphi_t(k)-e^{ik\cdot x}\overline{\varphi_t(k)}\big)\,\frac{dk}{|k|}. \tag{C-9}$$

(Note that the contribution from σ_{ψ_t} cancels.) Using this formula, we obtain

$$\|\partial_x^\beta\partial_t V_{\varphi_t}\|_\infty\lesssim\alpha^{-2}\|\varphi_t\|_{\mathcal{L}^2_{|\beta|+1}}$$

in the same way as we obtained (C-1). This implies (C-8) in view of the bounds on φ_t from Lemma C.2.

It is straightforward to deduce the remaining bounds claimed in the proposition. The bound on $\|\varphi_t\|_{\mathcal{L}^2_{(3)}}$ follows from Lemma C.2. Because of the equation for ψ_t, we have

$$\|\partial_t\psi_t\|_{\mathcal{H}^2}\le\|-\Delta\psi_t\|_{\mathcal{H}^2}+\|V_{\varphi_t}\psi_t\|_{\mathcal{H}^2}\lesssim\|\psi_t\|_{\mathcal{H}^4}+\sum_{|\beta|\le 2}\|\partial^\beta V_{\varphi_t}\|_\infty\|\psi_t\|_{\mathcal{H}^2}.$$

Using the fact that $\|\psi_t\|_{\mathcal{H}^4}\lesssim 1$ and $\|\varphi_t\|_{\mathcal{L}^2_{(3)}}\lesssim 1$, which by (C-1) controls $\|\partial^\beta V_{\varphi_t}\|_\infty$ for $|\beta|\le 2$, we conclude that $\|\partial_t\psi_t\|_{\mathcal{H}^2}\lesssim 1$. The second bound in (2-8) follows from Lemma C.2.

Finally, we need to prove the bounds on g_s and $g_{s,t}$. By the Schwarz inequality as in the proof of (C-1) together with the equation for φ_s we find

$$\|g_s\|_\infty\lesssim\|\partial_s\varphi_s\|_{\mathcal{L}^2_{(1)}}\le\alpha^{-2}(\|\varphi_s\|_{\mathcal{L}^2_{(1)}}+\|\sigma_{\psi_s}\|_{\mathcal{L}^2_{(1)}}).$$

According to (C-2) and Lemma 2.1 we have $\|\sigma_{\psi_s}\|_{\mathcal{L}^2_{(1)}} \lesssim \|\psi_s\|^2_{H^1} \lesssim 1$. Moreover, if $|t|, |s| \le \tau\alpha^2$, then Lemma C.2 implies $\|\varphi_s\|_{\mathcal{L}^2_{(1)}} \lesssim 1$. Thus,

$$\|g_s\|_\infty \lesssim \alpha^{-2},$$

as claimed. Moreover, $g_{s,t} = \int_s^t g_{s_1}\, ds_1$, so for $t > s$

$$\|g_{s,t}\|_\infty \le \int_s^t \|g_{s_1}\|_\infty\, ds_1 \lesssim \alpha^{-2}(t-s).$$

This proves (2-9). □

Appendix D: Reduced density matrices

Here we show how the approximation of $e^{-i\tilde{H}_\alpha^F t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega$ in Theorem 1.3 yields approximations to its reduced density matrices in Theorem 1.2. The argument relies on the following abstract lemma.

Lemma D.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces; let $\Psi, \Phi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$ such that*

$$\Psi = f \otimes g + \Phi,$$
$$\|f\|_{\mathcal{H}_1} \le C, \quad \|g\|_{\mathcal{H}_2} \le C, \quad \|\Phi\|_{\mathcal{H}_1\otimes\mathcal{H}_2} \le C\varepsilon,$$
$$\|\langle g, \Phi\rangle_{\mathcal{H}_2}\|_{\mathcal{H}_1} \le C\varepsilon^2, \quad \|\langle f, \Phi\rangle_{\mathcal{H}_1}\|_{\mathcal{H}_2} \le C\varepsilon^2$$

for some $C > 0$ and $\varepsilon > 0$. Define

$$\gamma_1 = \mathrm{Tr}_{\mathcal{H}_2} |\Psi\rangle\langle\Psi|, \quad \gamma_2 = \mathrm{Tr}_{\mathcal{H}_1} |\Psi\rangle\langle\Psi|.$$

Then

$$\mathrm{Tr}_{\mathcal{H}_1}\big|\gamma_1 - \|g\|^2_{\mathcal{H}_2} |f\rangle\langle f|\big| \le 3C^2\varepsilon^2, \quad \mathrm{Tr}_{\mathcal{H}_2}\big|\gamma_2 - \|f\|^2_{\mathcal{H}_1} |g\rangle\langle g|\big| \le 3C^2\varepsilon^2.$$

Before proving this lemma, let us use it to derive Theorem 1.2 from Theorem 1.3. We apply the lemma with $\mathcal{H}_1 = \mathcal{L}^2(\mathbb{R}^3),\ \mathcal{H}_2 = \mathcal{F},\ f = e^{-i\int_0^t \omega(s)\,ds}\psi_t,\ g = \Omega,$

$$\Psi = W^*(\alpha^2\varphi_t)e^{-i\tilde{H}_\alpha^F t}\psi_0 \otimes W(\alpha^2\varphi_0)\Omega, \quad \Phi = W^*(\alpha^2\varphi_t)R(t).$$

Then Theorem 1.3 implies that the assumptions of the lemma are satisfied with $\varepsilon = \alpha^{-1}(1+|t|)$. We have $\|f\|^2 = \|\psi_t\|^2 = \|\psi_0\|^2 = 1,\ \|g\|^2 = \|\Omega\|^2 = 1$ and $|f\rangle\langle f| = |\psi_t\rangle\langle\psi_t|$. Moreover,

$$\mathrm{Tr}_{\mathcal{H}_2} |\Psi\rangle\langle\Psi| = \gamma_t^{\text{particle}}, \quad \mathrm{Tr}_{\mathcal{H}_1} |\Psi\rangle\langle\Psi| = W^*(\alpha^2\varphi_t)\gamma_t^{\text{field}} W(\alpha^2\varphi_t).$$

Thus, the conclusion of Theorem 1.2 follows from the lemma.

We now turn to the proof of the lemma. It relies on the bound

$$\mathrm{Tr}_{\mathcal{H}_1}\big|\mathrm{Tr}_{\mathcal{H}_2} |\Psi_1\rangle\langle\Psi_2|\big| \le \|\Psi_1\|_{\mathcal{H}_1\otimes\mathcal{H}_2}\|\Psi_2\|_{\mathcal{H}_1\otimes\mathcal{H}_2} \tag{D-1}$$

valid for any vectors $\Psi_1, \Psi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$. For the proof of (D-1) recall the variational characterization of the trace norm,

$$\mathrm{Tr}_{\mathcal{H}_1} |K| = \sup_{(e_j),(e'_j)} \mathrm{Re} \sum_j \langle e_j, K e'_j\rangle_{\mathcal{H}_1},$$

where the supremum is over all orthonormal systems (e_j) and (e'_j) in $\mathcal{H}_1$. Thus, if (b_k) is an orthonormal basis in $\mathcal{H}_2$, then

$$\begin{aligned}\operatorname{Re}\sum_j \langle e_j, (\operatorname{Tr}_{\mathcal{H}_2}|\Psi_1\rangle\langle\Psi_2|)e'_j\rangle_{\mathcal{H}_1} &= \operatorname{Re}\sum_{j,k}\langle e_j\otimes b_k,\Psi_1\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}\langle\Psi_2,e'_j\otimes b_k\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}\\ &\le \Big(\sum_{j,k}\big|\langle e_j\otimes b_k,\Psi_1\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}\big|^2\Big)^{\frac12}\Big(\sum_{j,k}\big|\langle\Psi_2,e'_j\otimes b_k\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2}\big|^2\Big)^{\frac12}\\ &\le \|\Psi_1\|_{\mathcal{H}_1\otimes\mathcal{H}_2}\|\Psi_2\|_{\mathcal{H}_1\otimes\mathcal{H}_2},\end{aligned}$$

where the last inequality comes from the orthonormality of $(e_j\otimes b_k)$ and $(e'_j\otimes b_k)$. Therefore the variational characterization of the trace norm yields (D-1).

Proof. Since $\operatorname{Tr}_{\mathcal{H}_2}|f\otimes g\rangle\langle\Phi| = |f\rangle\langle\langle g,\Phi\rangle_{\mathcal{H}_2}|$, we have

$$\gamma_1 - \|g\|^2_{\mathcal{H}_2}|f\rangle\langle f| = |f\rangle\langle\langle g,\Phi\rangle_{\mathcal{H}_2}| + |\langle\Phi,g\rangle_{\mathcal{H}_2}\rangle\langle f| + \operatorname{Tr}_2|\Phi\rangle\langle\Phi|.$$

By (D-1) and the assumptions the trace norm, each one of the three operators on the right side is bounded by $C^2\varepsilon^2$. This proves the first inequality in the lemma. The second one is proved similarly. □

Finally, we show that the α^{-2} error bound in Theorem 1.2 (for times of order one) is due to the fact that φ_t is time-dependent. The proof makes use of the fact that for arbitrary normalized vectors a and b in a Hilbert space $\mathcal{H}$ one has

$$\operatorname{Tr}_{\mathcal{H}}\big||a\rangle\langle a| - |b\rangle\langle b|\big| = 2(1-|\langle a,b\rangle|^2)^{\frac12}, \tag{D-2}$$

as is easily verified.

Proof of Lemma 1.4. Because of Theorem 1.2, it suffices to prove that there are $\varepsilon>0$ and $c>0$ such that for all $|t|\le\varepsilon$ and all $\alpha\ge 1$,

$$\operatorname{Tr}_{\mathcal{F}}\big||W(\alpha^2\varphi_t)\Omega\rangle\langle W(\alpha^2\varphi_t)\Omega| - |W(\alpha^2\varphi_0)\Omega\rangle\langle W(\alpha^2\varphi_0)\Omega|\big| \ge c\alpha^{-1}|t|.$$

According to Lemma A.4 and (D-2), this is equivalent to

$$1-e^{-\alpha^2\|\varphi_t-\varphi_0\|_2^2} = 1-\big|\langle\Omega, W^*(\alpha^2\varphi_0)W(\alpha^2\varphi_t)\Omega\rangle\big|^2 \ge \tfrac14 c^2\alpha^{-2}t^2.$$

Since $\|\varphi_t-\varphi_0\|_2\lesssim\alpha^{-2}|t|$ by Lemma 2.1, it suffices to prove that there are $\varepsilon>0$ and $c'>0$ such that for all $|t|\le\varepsilon$ and all $\alpha\ge1$,

$$\|\varphi_t-\varphi_0\|_2\ge c'\alpha^{-2}|t|.$$

Since $\varphi_0+\sigma_{\psi_0}\not\equiv 0$, this will clearly follow if we can prove that for all $|t|\le\alpha^2$ and $\alpha\ge1$,

$$\big\|\varphi_t-\varphi_0+i\alpha^{-2}t(\varphi_0+\sigma_{\psi_0})\big\|_2\le C\alpha^{-2}t^2. \tag{D-3}$$

To prove this, we use equation (1-8) for φ_t to write

$$\varphi_t-\varphi_0=\int_0^t\partial_s\varphi_s\,ds = -i\alpha^{-2}\int_0^t(\varphi_s+\sigma_{\psi_s})\,ds = -i\alpha^{-2}t(\varphi_0+\sigma_{\psi_0})+r_t$$

with

$$r_t := -i\alpha^{-2}\int_0^t\int_0^s (\partial_{s_1}\varphi_{s_1} + \partial_{s_1}\sigma_{\psi_{s_1}})\, ds_1\, ds.$$

By Lemma 2.1 and Proposition 2.2, the $\mathcal{L}^2$-norm of the integrand of r_t is bounded by a constant uniformly in $|s_1| \le \alpha^2$ and $\alpha \ge 1$. This yields (D-3) and completes the proof. □

Appendix E: Improving the result of [Frank and Schlein 2014]

We now show how the techniques from [Frank and Schlein 2014] can be extended to times $|t| = o(\alpha)$. This argument is due to an anonymous referee, whom we thank for kind permission to include it in our paper.

Proposition E.1. *Let $\varphi \in L^2(\mathbb{R}^3)$ and $\alpha_0 > 0$. Assume that $\Psi \in L^2(\mathbb{R}^3) \otimes \mathcal{F}$ satisfies*

$$\|(p^2 + \mathcal{N} + 1)^{\frac{1}{2}}\Psi\| \le M, \quad \|(p^2+1)^{\frac{1}{2}}\mathcal{N}\Psi\| \le M\alpha^{-2}.$$

Then for all $\alpha \ge \alpha_0$ and all $t \in \mathbb{R}$,

$$\left\|e^{-i\widetilde{H}_\alpha^{\mathrm{F}} t}W(\alpha^2\varphi)\Psi - e^{-iH_\varphi t}W(\alpha^2\varphi)\Psi\right\|^2 \le M^2(1+2\alpha^{-1})(e^{C|t|/(2\alpha)} - 1),$$

where C depends only on α_0 and an upper bound on $\|\varphi\|_{\mathcal{L}^2}$.

Note that this result can be applied, in particular, to $\Psi = \psi \otimes \Omega$ with $\|\psi\|_{\mathcal{H}^1} \le M$. We also recall that the effective Schrödinger operator H_φ was defined in (2-12).

Proof. Let $A(t) := \left\|e^{-i\widetilde{H}_\alpha^{\mathrm{F}} t}W(\alpha^2\varphi)\Psi - e^{-iH_\varphi t}W(\alpha^2\varphi)\Psi\right\|^2$. It is shown in [Frank and Schlein 2014, Proposition 9] that $A'(t) = f(t) + g(t)$ with

$$f(t) \le CM\alpha^{-1}A(t)^{\frac{1}{2}}, \quad \int_0^T g(t)\, dt \le CM^2\alpha^{-2}T,$$

where C depends only on α_0 and an upper bound on $\|\varphi\|_{\mathcal{L}^2}$. We bound $f(t) \le \frac{1}{2}C\alpha^{-1}(A(t) + M^2)$ and therefore

$$A(T) \le \int_0^T f(t)\, dt + \int_0^T g(t)\, dt \le \tfrac{1}{2}C\alpha^{-1}\int_0^T A(t)\, dt + \tfrac{1}{2}CM^2\alpha^{-1}(1+2\alpha^{-1})T.$$

Thus,

$$A(T) + M^2(1+2\alpha^{-1}) \le M^2(1+2\alpha^{-1}) + \tfrac{1}{2}C\alpha^{-1}\int_0^T \big(A(t) + M^2(1+2\alpha^{-1})\big)\, dt$$

and, by Gronwall's inequality, for all $t \ge 0$

$$A(t) + M^2(1+2\alpha^{-1}) \le M^2(1+2\alpha^{-1})e^{Ct/(2\alpha)}.$$ □

Acknowledgements

The authors are grateful to J. Fröhlich, M. Lewin, B. Schlein and R. Seiringer for their helpful remarks at various stages of this project, as well as to the anonymous referees who helped improve this paper. Support through NSF grants PHY–1347399 and DMS–1363432 (R.L.F.) and DMS–1308985 and DMS–1443225 (Z.G.) is acknowledged.

Note added in proof

After this work was accepted for publication, the preprint by M. Griesemer [2016] appeared on the arXiv. This preprint studies the dynamics generated by the initial conditions given by the minimizing pair (ψ_*, φ_*) of the energy functional $\mathcal{E}(\psi, \varphi)$ under the constraint $\|\psi\| = 1$ up to times of order $o(\alpha^2)$.

References

[Ammari and Falconi 2014] Z. Ammari and M. Falconi, "Wigner measures approach to the classical limit of the Nelson model: convergence of dynamics and ground state energy", *J. Stat. Phys.* **157**:2 (2014), 330–362. MR Zbl

[Ammari and Nier 2008] Z. Ammari and F. Nier, "Mean field limit for bosons and infinite dimensional phase-space analysis", *Ann. Henri Poincaré* **9**:8 (2008), 1503–1574. MR Zbl

[Ammari and Nier 2009] Z. Ammari and F. Nier, "Mean field limit for bosons and propagation of Wigner measures", *J. Math. Phys.* **50**:4 (2009), art. id. 042107. MR Zbl

[Devreese and Alexandrov 2009] J. T. Devreese and A. S. Alexandrov, "Fröhlich polaron and bipolaron: recent developments", *Rep. Prog. Phys.* **72**:6 (2009), art. id. 066501.

[Donsker and Varadhan 1983] M. D. Donsker and S. R. S. Varadhan, "Asymptotics for the polaron", *Comm. Pure Appl. Math.* **36**:4 (1983), 505–528. MR Zbl

[Falconi 2013] M. Falconi, "Classical limit of the Nelson model with cutoff", *J. Math. Phys.* **54**:1 (2013), art. id. 012303. MR Zbl

[Frank and Schlein 2014] R. L. Frank and B. Schlein, "Dynamics of a strongly coupled polaron", *Lett. Math. Phys.* **104**:8 (2014), 911–929. MR Zbl

[Fröhlich 1937] H. Fröhlich, "Theory of electrical breakdown in ionic crystals", *Proc. R. Soc. Lond. A* **160**:901 (1937), 230–241.

[Ginibre et al. 2006] J. Ginibre, F. Nironi, and G. Velo, "Partially classical limit of the Nelson model", *Ann. Henri Poincaré* **7**:1 (2006), 21–43. MR Zbl

[Griesemer 2016] M. Griesemer, "On the dynamics of polarons in the strong-coupling limit", preprint, 2016. arXiv

[Hepp 1974] K. Hepp, "The classical limit for quantum mechanical correlation functions", *Comm. Math. Phys.* **35**:4 (1974), 265–277. MR

[Landau and Pekar 1948] L. D. Landau and S. I. Pekar, "Effective mass of a polaron", *Zh. Eksp. Teor. Fiz.* **18**:5 (1948), 419–423. In Russian.

[Lieb and Thomas 1997] E. H. Lieb and L. E. Thomas, "Exact ground state energy of the strong-coupling polaron", *Comm. Math. Phys.* **183**:3 (1997), 511–519. MR Zbl

[Lieb and Yamazaki 1958] E. H. Lieb and K. Yamazaki, "Ground-state energy and effective mass of the polaron", *Phys. Rev.* **111**:3 (1958), 728–733. Zbl

[Pekar 1946] S. Pekar, "Avto lokalizatsiya èlektrona v dièlektricheskoĭ polyarizuyushcheĭsya srede", *Zh. Eksp. Teor. Fiz.* **16**:4 (1946), 335–340.

[Pekar 1951] S. Pekar, *Issledovaniya po èlektronnoĭ teorii kristallov*, Gostekhizdat, Moscow, 1951. Translated as *Research in electron theory of crystals*, US Atomic Energy Com., Oak Ridge, TN, 1963.

[Spohn 1987] H. Spohn, "Effective mass of the polaron: a functional integral approach", *Ann. Physics* **175**:2 (1987), 278–318. MR

Received 30 Jun 2016. Accepted 28 Nov 2016.

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mathematical sciences publishers msp

ANALYSIS AND PDE
Vol. 10, No. 2, 2017
dx.doi.org/10.2140/apde.2017.10.423

TIME-WEIGHTED ESTIMATES IN LORENTZ SPACES AND SELF-SIMILARITY FOR WAVE EQUATIONS WITH SINGULAR POTENTIALS

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We show time-weighted estimates in Lorentz spaces for the linear wave equation with singular potential. As a consequence, assuming radial symmetry on initial data and potentials, we obtain well-posedness of global solutions in critical weak-L^p spaces for semilinear wave equations. In particular, we can consider the Hardy potential $V(x) = c|x|^{-2}$ for small $|c|$. Self-similar solutions are obtained for potentials and initial data with the right homogeneity. Our approach relies on performing estimates in the predual of weak-L^p, i.e., the Lorentz space $L^{(p',1)}$.

1. Introduction

We are concerned with the linear wave equation with potential

$$\begin{cases} \Box u + Vu = f(x,t), & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ \vec{u}(0) = (u(0,x), \partial_t u(0,x)) = (0,0), & x \in \mathbb{R}^n, \end{cases} \tag{1-1}$$

and the semilinear wave equation

$$\begin{cases} \Box u + Vu = \mu |u|^{p-1} u, & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ \vec{u}(0) = (u_0, u_1), & x \in \mathbb{R}^n, \end{cases} \tag{1-2}$$

where $\Box = \partial_t^2 - \Delta_x$, $n \geq 5$ odd, $\mu \in \{+1, -1\}$ (focusing or defocusing case) and $p > (n^2+n-4)/(n(n-3))$. The problems (1-1) and (1-2) are addressed in the radial setting.

The semilinear wave equation (1-2) with $V = 0$ has three notions of critical nonlinearity, namely the Strauss critical power $p = p_{\mathrm{str}}$, conformal critical power $p = p_{\mathrm{conf}}$ and energy critical power $p = p_{\mathrm{e}}$. The former p_{str} is the positive root of

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Strauss [1981] conjectured about the existence for $p > p_{\mathrm{str}}$ or nonexistence for $1 < p \leq p_{\mathrm{str}}$ of global solutions for (1-2) with small compact support initial data. The conjecture of Strauss has a nice history (see, e.g., [Wang and Yu 2012]) and was completed by [Yordanov and Zhang 2006; Zhou 2007] (see also [Lai and Zhou 2014]). The conformal power p_{conf} is linked to the conformal symmetry map

$$u(x,t) \mapsto u_{\mathrm{conf}}(x,t) = (t^2 - |x|^2)^{-\frac{n-1}{2}} u\left(\frac{x}{t^2-|x|^2}, \frac{t}{t^2-|x|^2}\right) \quad \text{for } |x| < |t|.$$

Ferreira was supported by FAPESP-SP and CNPQ, Brazil .
MSC2010: primary 35L05, 35L71, 35L15, 35A01, 35B06; secondary 35C06, 42B35.
Keywords: wave equations, singular potentials, self-similarity, radial symmetry, Lorentz spaces.

More precisely, u_{conf} solves (1-2) with $V = 0$ if u does and $p = p_{\text{conf}} = (n+3)/(n-1)$ for $n \geq 2$. The power $p_{\text{e}} = (n+2)/(n-2)$ ($p_{\text{e}} = \infty$ if $n = 2$) is connected to the scaling invariance of the conserved energy. In fact, for $p = p_{\text{e}}$ and $V = 0$, the conserved energy

$$E(u, \partial_t u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla_x u|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^n} |\partial_t u|^2 \, dx + \frac{\mu}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} \, dx$$

is invariant by the scaling map

$$u(x,t) \mapsto u_\gamma(x,t) := \gamma^{\frac{2}{p-1}} u(\gamma x, \gamma t), \quad \gamma > 0, \tag{1-3}$$

namely

$$E(u_\gamma, \partial_t u_\gamma) = \gamma^{\frac{4}{p-1}+2-n} E(u, \partial_t u) = E(u, \partial_t u).$$

We refer the reader to the classical papers [Grillakis 1990; 1992; Shatah and Struwe 1993; Struwe 1999] for results about solutions with finite energy.

A solution is called self-similar when it is invariant by (1-3), that is, $u(x,t) = u_\gamma(x,t)$. For a homogeneous function V of degree -2, equation (1-2) presents the same scaling as in the case $V = 0$. Taking $t = 0$, the map (1-3) induces the scaling for the initial data:

$$(u_0(x), u_1(x)) \mapsto \big(\gamma^{\frac{2}{p-1}} u_0(\gamma x), \gamma^{\frac{p+1}{p-1}} u_1(\gamma x)\big). \tag{1-4}$$

In other words, self-similar solutions of (1-2) are associated to initial data u_0 and u_1 homogeneous of degrees $-\frac{2}{p-1}$ and $-\frac{p+1}{p-1}$, respectively, that is, homogeneous functions of the form

$$u_0(x) = \varepsilon c_1 |x|^{-\frac{2}{p-1}} \quad \text{and} \quad u_1(x) = \varepsilon c_2 |x|^{-\frac{p+1}{p-1}}, \tag{1-5}$$

where $c_1, c_2 \in \mathbb{R}$ and $\varepsilon > 0$.

For $V = 0$, there are a number of results about self-similar solutions in different frameworks. The first work is due to Kavian and Weissler [1990], where the authors proved the nonexistence of radially symmetric self-similar solutions with finite energy $E(u, \partial_t u)$ for $n \geq 3$ and $p_{\text{e}} \leq p < \infty$. Working in the infinite energy space of all Bochner-measurable functions $u : (0, \infty) \to L^r(\mathbb{R}^n)$ such that

$$\sup_{t>0} t^\beta \|u(\cdot, t)\|_{L^r(\mathbb{R}^n)} < \infty, \tag{1-6}$$

Pecher [2000a] showed the existence of self-similar solutions for $\varepsilon > 0$ in (1-5) sufficiently small by considering $n = 3$ and $p_1 < p \leq p_{\text{conf}}$, where p_1 is the larger positive root of

$$(n^2 - n)p^2 - (n^2 + 3n - 2)p + 2 = 0.$$

The parameters $\beta > 0$ and $r > 2$ are taken in such a way that the norm (1-6) is scaling invariant. The approach in [Pecher 2000a] is based on $L^q - L^r$ dispersive estimates for the wave group

$$\omega(t) = (-\Delta)^{-\frac{1}{2}} \sin(t(-\Delta)^{\frac{1}{2}}). \tag{1-7}$$

In fact, the case of nonradial homogeneous data also was considered in [Pecher 2000a]. Moreover, replacing L^r by suitable homogeneous Sobolev spaces $\dot{H}^{k,l}$ with $k > 0$, the upper condition $p \leq p_{\text{conf}}$

was removed by him. Still for $n=3$, Pecher [2000b] obtained self-similar solutions (not necessarily radial) in the range $p_{\mathrm{str}} < p < p_{\mathrm{conf}}$ and showed that the lower bound p_{str} is sharp in the sense that in general no nontrivial self-similar solution exists even in the radial case when $p \le p_{\mathrm{str}}$. Unlike [Pecher 2000a], the paper [Pecher 2000b] developed pointwise estimates related to the weights $|x| \pm t$ and a norm due to F. John and did not employ L^p, Sobolev or Besov spaces. Hidano [2002] complemented these results by showing scattering and existence of self-similar solutions for (1-2) when $n=2,3$ and $p_{\mathrm{str}} < p < p_{\mathrm{conf}}$. The result of [Pecher 2000a] was proved to be true for $n=2,3,4,5$ by Ribaud and Youssfi [2002], recovering in particular $n=2,3$. Moreover, for $n \ge 6$ they considered $p \in (p_1, p_{\mathrm{conf}}] \cup [2,\infty)$ or $p \in (p_1, p_{\mathrm{conf}}] \cup (p_2, \infty)$, where p_2 is the larger positive root of

$$2(n+1)p^2 - (n^2+3n+4)p + (n^2+5n+2) = 0.$$

Note that $p_{\mathrm{str}} < p_1 < p_{\mathrm{conf}} < p_2$ for all dimensions in which these parameters are defined.

The weighted Strichartz estimate in $L^{(r,\infty)}(\mathbb{R}^{1+n}_+)$

$$\big\| |t^2-|x|^2|^a u \big\|_{L^{(r,\infty)}(\mathbb{R}^{1+n}_+)} \le C \big\| |t^2-|x|^2|^b f \big\|_{L^{(r',\infty)}(\mathbb{R}^{1+n}_+)} \tag{1-8}$$

was obtained by Kato and Ozawa [2003] for f radially symmetric in x-variables, $2<r<2(n+1)/(n-1)$ and suitable powers a,b. By using (1-8) and assuming $n \ge 3$ odd, they proved existence and uniqueness of radially symmetric self-similar solutions for (1-2) with initial data (1-5) provided that $p_{\mathrm{str}} < p < p_{\mathrm{conf}}$ and $\varepsilon > 0$ is small enough. In [Kato and Ozawa 2004], they extended their results to the case $n \ge 2$ even. By employing spherical harmonics and Sobolev spaces over the unit sphere, the condition of radial symmetry on u and f was removed in [Kato et al. 2007] for $2 \le n \le 5$. In the case $p \in \mathbb{N}$, $p > p_{\mathrm{conf}}$ and $V=0$, Planchon [2000] showed global well-posedness and existence of self-similar solutions for (1-2) in $L^\infty((0,\infty); \dot{B}^{s_p}_{2,\infty})$ for small data $(u_0,u_1) \in \dot{B}^{s_p}_{2,\infty} \times \dot{B}^{s_p-1}_{2,\infty}$ with $s_p = \frac{n}{2} - \frac{2}{p-1}$. Notice that the above results do not contradict the nonexistence result in [Kavian and Weissler 1990] because the obtained self-similar solutions have infinite energy.

Wave equations with singular potential arise in the study of stability of stationary solutions for a number of systems of PDEs, for example, wave-Schrödinger and Maxwell–Schrödinger ones (see, e.g., [D'ancona and Pierfelice 2005]). Unlike the case $p > p_{\mathrm{str}}$ and $V=0$, where no blow-up occurs for (1-2), Strauss and Tsutaya [1997] proved blow-up of solutions when $n=3$, $p>1$ and $V \in C^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ decays like $c/|x|^{(2-\varepsilon)}$ as $|x| \to \infty$ for $0<\varepsilon<2$. Also, they showed global existence for $\varepsilon<0$, $p>p_{\mathrm{str}}$ and

$$\Big\| (1+|x|)^{2-\varepsilon} \sum_{|\alpha|\le 2} |\partial_x^\alpha V(x)| \Big\|_{L^\infty}$$

small enough. Still considering small, smooth and rapidly decaying potentials, Yajima [1995] obtained $L^p - L^q$ dispersive estimates for the linear wave equation (1-1). The borderline case $\varepsilon=0$ corresponds to V homogeneous of degree $\sigma=-2$. In this case, the perturbation Vu has the same scaling of Δu and cannot be dealt with as a simple perturbation of lower order because it does not belong to the Kato class

$$\mathcal{K} = \Big\{ V \in L^1_{\mathrm{loc}} : \|V\|_{\mathcal{K}} = \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{2-n} |V(y)|\,dy < \infty,\ n \ge 3 \Big\}, \tag{1-9}$$

where $\|\cdot\|_{\mathcal{K}}$ is called the global Kato norm. Taking $n=3$, $\vec{u}(0)=(0,u_1)$ and $f=0$ in (1-1), Georgiev and Visciglia [2003] proved the dispersive estimate

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{n-1}{2}}\|u_1\|_{\dot{B}^1_{1,1}(\mathbb{R}^n)} \tag{1-10}$$

for potentials V that are Hölder continuous in $\mathbb{R}^3\backslash\{0\}$ satisfying

$$0 \le V(x) \le \frac{C}{|x|^{2-\varepsilon}+|x|^{2+\varepsilon}} \quad \text{for all } x\in\mathbb{R}^3.$$

D'ancona and Pierfelice [2005] improved the class of potentials to nonresonant $V\in\mathcal{K}$ and obtained, in particular, the estimate (1-10) for $V\in L^{\frac{n}{2}-\delta}\cap L^{\frac{n}{2}+\delta}\subset L^{(\frac{n}{2},1)}\subset\mathcal{K}$ with small $\delta>0$. Planchon et al. [2003a] proved (1-10) in the radial case for the critical potential $V(x)=c/|x|^2$ with $c\ge 0$. In the same work, they also proved a modified version of (1-10) for negative potentials $-((n-2)/2)^2<c<0$. Moreover, they showed that the classical $L^\infty - L^1$ estimate

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{n-1}{2}}\|(-\Delta)^{\frac{n-1}{4}}u_1\|_{L^1(\mathbb{R}^n)}$$

does not hold when $c<0$ and $V(x)=c/|x|^2$. In particular, this estimate is false for general $V\in L^{(\frac{n}{2},\infty)}$.

Burq et al. [2003] considered Strichartz estimates for (1-1) and showed

$$\|(-\Delta)^\sigma u\|_{L^p_t L^q_x} \le C(\|u_0\|_{\dot{H}^\gamma}+\|u_1\|_{\dot{H}^{\gamma-1}}) \tag{1-11}$$

for σ, p, q, γ satisfying suitable conditions. See also [Planchon et al. 2003b] for the radial case and [Burq et al. 2004] for a more general class of potentials satisfying $V\in C^1(\mathbb{R}^n\backslash\{0\})$ and $\sup_{x\in\mathbb{R}^n}|x|^2|V(x)|<\infty$, among some other conditions. Using (1-11), for $n\ge 2$, $p\ge p_{\text{conf}}$, $s_p=\frac{n}{2}-\frac{2}{p-1}$ and

$$\sqrt{c+\frac{(n-2)^2}{4}} > \frac{n-2}{2}-\frac{2}{p-1}+\max\left\{\frac{1}{2p},\ \frac{2}{(n+1)(p-1)}\right\},$$

the authors of [Burq et al. 2003] also showed global well-posedness of (1-2) provided that $(u_0,u_1)\in H^{s_p}\times H^{s_p-1}$ is small enough. This result has been extended to the range

$$1+\frac{4n}{(n-1)(n+1)} < p < p_{\text{conf}}$$

in [Miao et al. 2013] for $n\ge 3$ and small radial initial data.

In this paper we obtain estimates for solutions of (1-1) in weak-L^r ($L^{(r,\infty)}$) spaces for the case of small radial singular potentials $V\in L^{(\frac{n}{2},\infty)}$. Examples of those are Hardy potentials $V=c|x|^{-2}$ with $|c|$ small enough (see Remark 3.2(II)). More precisely, for certain conditions on r, s, we prove the estimate

$$\|u\|_{L^\infty(\mathbb{R};L^{(r,\infty)})} \le \frac{K}{1-KC_0\|V\|_{L^{(n/2,\infty)}}}\|f\|_{L^\infty(\mathbb{R};L^{(s,\infty)})}, \tag{1-12}$$

where C_0 and K are positive constants and V, f and u are radially symmetric in the x-variable. In our results, the potential V can have indefinite sign. Notice that taking $V=0$ in (1-12), one also obtains, in particular, an estimate for the linear wave equation $\Box u=f$. The estimate (1-12) can be regarded as an endpoint-inhomogeneous Strichartz-type estimate in weak-L^p spaces, specifically, from $L^{l_1}_t L^{(l_2,\infty)}_x$ to

$L_t^{m_1} L_x^{(m_2,\infty)}$ in the case $(l_1, m_1) = (\infty, \infty)$, which is important because it corresponds to the natural persistence space in existence results. Even when $V = 0$, notice that (1-12) cannot be obtained as a consequence of the inhomogeneous Strichartz estimates by Keel and Tao [1998] and Taggart [2010].

In order to obtain (1-12), we need to show a time-weighted estimate for the wave group (1-7) in the predual of $L^{(r,\infty)}$, i.e., the Lorentz space $L^{(r',1)}$, which is of its own interest (see Lemma 4.1). As will be seen below, this estimate will lead us to global well-posedness results for (1-2) in critical spaces. We denote the solution of the Cauchy problem for the linear homogeneous wave equation by

$$L_{\vec{u}(0)}(t) = \omega(t)u_1 + \dot{\omega}(t)u_0, \quad \text{where } \dot{\omega}(t) = \partial_t \omega(t), \tag{1-13}$$

and consider the space of initial data

$$\mathcal{I}_{\text{rad}} = \{(u_0, u_1) \in \mathcal{S}'_{\text{rad}} \times \mathcal{S}'_{\text{rad}} : L_{\vec{u}(0)}(t) \in L^\infty(\mathbb{R}; L_{\text{rad}}^{(r_0,\infty)}(\mathbb{R}^n))\}, \tag{1-14}$$

where $r_0 = \frac{n(p-1)}{2}$ and the subindex "rad" means space of radial distributions. The norm $\|\cdot\|_{\mathcal{I}_{\text{rad}}}$ is defined as

$$\|(u_0, u_1)\|_{\mathcal{I}_{\text{rad}}} = \sup_{t\in\mathbb{R}} \|L_{\vec{u}(0)}(t)\|_{L_{\text{rad}}^{(r_0,\infty)}}. \tag{1-15}$$

Applying the estimate (1-12), we obtain global well-posedness for (1-2) in the scaling-invariant space $E = L^\infty(\mathbb{R}; L_{\text{rad}}^{(r_0,\infty)}(\mathbb{R}^n))$ provided that $n \geq 5$ odd, $p > (n^2+n-4)/(n(n-3))$ and $\|(u_0, u_1)\|_{\mathcal{I}_{\text{rad}}}$ is small enough (see Theorem 3.3(I)). The continuous inclusion $(\dot{B}^{s_p}_{2,\infty} \times \dot{B}^{s_p-1}_{2,\infty})_{\text{rad}} \subset \mathcal{I}_{\text{rad}}$ holds true and so, in the radial case, our result extends the initial data class in [Planchon 2000]. In fact, we have $\dot{B}^{s_p}_{2,\infty} \subset L^{(\frac{n(p-1)}{2},\infty)}$ (see Remark 3.4(I)) and

$$\sup_{t\in\mathbb{R}} \|L_{\vec{u}(0)}(t)\|_{L^{(\frac{n(p-1)}{2},\infty)}} \leq C \sup_{t\in\mathbb{R}} \|L_{\vec{u}(0)}(t)\|_{\dot{B}^{s_p}_{2,\infty}} \leq C \|(u_0, u_1)\|_{\dot{B}^{s_p}_{2,\infty} \times \dot{B}^{s_p-1}_{2,\infty}}, \tag{1-16}$$

where the second inequality in (1-16) can be found in [Planchon 2000, estimate (29), p. 815]. Also, we have $\mathcal{K} \subsetneq L^{(\frac{n}{2},\infty)}$ and then our class of potentials is larger than the Kato one in the radial setting (see Remark 3.4(II)). Note that

$$p_{\text{str}} < p_{\text{conf}} < p_{\text{e}} < \frac{n^2+n-4}{n(n-3)}$$

and our range of admissible powers p differs from those of [Kato and Ozawa 2003; 2004; Planchon 2000; Ribaud and Youssfi 2002]. Finally, as a byproduct, we obtain the existence of radially symmetric self-similar solutions when u_0, u_1 and V are homogeneous of degrees $-\frac{2}{p-1}$, $-\frac{p+1}{p-1}$ and -2, respectively (see Theorem 3.3(II)).

This paper is organized as follows. In Section 2, we recall the definition of Lorentz spaces and some of their properties. Our results are stated in Section 3 and proved in Section 4.

2. Lorentz spaces

We start by recalling the decreasing rearrangement of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$,

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\} \quad \text{for } t > 0,$$

where $d_f(s) = \left|\{x \in \mathbb{R}^n : |f(x)| > s\}\right|$ is the distribution function of f. The Lorentz space $L^{(p,z)} = L^{(p,z)}(\mathbb{R}^n)$ is the vector space of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|f\|^*_{(p,z)} = \begin{cases} \left[\int_0^\infty \left(t^{\frac{1}{p}}[f^*(t)]\right)^z \frac{dt}{t}\right]^{\frac{1}{z}} < \infty & \text{for } 0 < p \le \infty,\ 1 \le z < \infty, \\ \sup_{t>0} t^{\frac{1}{p}}[f^*(t)] < \infty & \text{for } 0 < p \le \infty,\ z = \infty. \end{cases} \tag{2-1}$$

The space $L^{(\infty,z)}$ is trivial for $1 \le z < \infty$. Also, $L^{(p,p)}$ is the Lebesgue space L^p with $\|f\|^*_{(p,p)} = \|\cdot\|_{L^p}$ and $L^{(p,\infty)}$ is the so-called weak-L^p. The quantity $\|f\|^*_{(p,z)}$ defines a complete quasinorm on $L^{(p,z)}$ that in general is not a norm. Considering the double rearrangement

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds,$$

one can define the norm $\|f\|_{(p,z)}$ on $L^{(p,z)}$ by replacing f^* by f^{**} in (2-1). For $1 < p \le \infty$, we have the inequality

$$\|\cdot\|^*_{(p,z)} \le \|\cdot\|_{(p,z)} \le \frac{p}{p-1}\|\cdot\|^*_{(p,z)},$$

and then $\|\cdot\|_{(p,z)}$ and $\|\cdot\|^*_{(p,z)}$ are topologically equivalent. The pair $(L^{(p,z)}, \|\cdot\|_{(p,z)})$ is a Banach space. From now on, we consider $L^{(p,z)}$ endowed with $\|\cdot\|_{(p,z)}$ and $\|\cdot\|^*_{(p,z)}$ when $1 < p \le \infty$ and $0 < p \le 1$, respectively. The continuous inclusions

$$L^{(p,1)} \subset L^{(p,z_1)} \subset L^p \subset L^{(p,z_2)} \subset L^{(p,\infty)} \tag{2-2}$$

hold true for $1 \le z_1 \le p \le z_2 \le \infty$ and $1 \le p \le \infty$. Lorentz spaces have the same scaling as L^p-spaces, namely

$$\|\delta_c(f)\|_{(p,z)} = c^{-\frac{n}{p}}\|f\|_{(p,z)},$$

where δ_λ stands for the operator $\delta_c(f)(x) = f(cx)$.

Let $0 < \theta < 1$ and $1 \le z \le \infty$. Consider the interpolation functor $(\cdot,\cdot)_{\theta,z}$ constructed via the $K_{\theta,z}$-method and defined on the categories of quasinormed and normed spaces. For $0 < p_1 < p_2 \le \infty$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $1 \le z_1, z_2 \le \infty$, we have (see [Bergh and Löfström 1976, Theorems 5.3.1 and 5.3.2])

$$(L^{(p_1,z_1)}, L^{(p_2,z_2)})_{\theta,z} = L^{(p,z)}. \tag{2-3}$$

Moreover, $(\cdot,\cdot)_{\theta,z}$ is exact of exponent θ.

The pointwise product operator works well in Lorentz spaces; i.e., Hölder inequality is verified in this setting (see [Hunt 1966; O'Neil 1963]). Let $1 < p_1, p_2, p_3 \le \infty$ and $1 \le z_1, z_2, z_3 \le \infty$ be such that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{z_1} + \frac{1}{z_2} \ge \frac{1}{z_3}$. Then

$$\|fg\|_{(p_3,z_3)} \le C\|f\|_{(p_1,z_1)}\|g\|_{(p_2,z_2)}, \tag{2-4}$$

where the constant $C > 0$ is independent of f and g.

Finally, we recall that the dual space of $L^{(p,z)}$ is $L^{(p',z')}$ for $1 \le p, z < \infty$ (see [Grafakos 2004, p. 52]). Taking $z = 1$, we have $(L^{(p,1)})' = L^{(p',\infty)}$ for $1 \le p < \infty$.

3. Main results

Throughout the paper, the subindex "rad" means space of radial functions or distributions. For instance,

$$L^{(r,z)}_{\mathrm{rad}} = \{u \in L^{(r,z)} : u \text{ is radially symmetric}\}. \tag{3-1}$$

We define the open triangles $\Delta_{P_1P_2P_3}$ and $\Delta_{P_2P_4P_5}$ whose vertices P_i are

$$\begin{aligned} P_1 &= \Big(\frac{1}{2}+\frac{1}{n+1}, \frac{1}{2}-\frac{1}{n+1}\Big), \quad P_2 = \Big(\frac{1}{2}-\frac{1}{n-1}, \frac{1}{2}-\frac{1}{n-1}\Big), \\ P_3 &= \Big(\frac{1}{2}+\frac{1}{n-1}, \frac{1}{2}+\frac{1}{n-1}\Big), \quad P_4 = \Big(1, \frac{n-1}{2n}\Big) \quad \text{and} \quad P_5 = (1,1) \end{aligned} \tag{3-2}$$

(see Figure 1). The vertices P_2 and P_3 are defined as $(0,0)$ and $(1,1)$, respectively, when $n = 1, 2$.

Our first result consists in linear estimates in weak-L^p for the linear wave equation with singular potential.

Theorem 3.1. *Let $n \geq 5$ be odd and $\Delta_{P_2P_4P_5}$ be the open triangle defined by the points P_2, P_4 and P_5 in (3-2). If*

$$1 < r', s' < \frac{2(n-1)}{n-3} \quad \text{with} \ \Big(1-\frac{1}{r}, 1-\frac{1}{s}\Big) \in \Delta_{P_2P_4P_5} \ \text{and} \ \frac{1}{s}-\frac{1}{r}=\frac{2}{n}, \tag{3-3}$$

then there are $K, C_0 > 0$ such that the solution u of (1-1) satisfies

$$\sup_{t\in\mathbb{R}} \|u(\cdot,t)\|_{(r,\infty)} \leq \frac{K}{1-C_0K\|V\|_{(\frac{n}{2},\infty)}} \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{(s,\infty)} \tag{3-4}$$

for all $f \in L^\infty(\mathbb{R}; L^{(s,\infty)}_{\mathrm{rad}}(\mathbb{R}^n))$, provided that $V \in L^{(\frac{n}{2},\infty)}_{\mathrm{rad}}$ and $C_0K\|V\|_{(\frac{n}{2},\infty)} < 1$. The supremum in (3-4) is taken in the essential sense.

Some comments on Theorem 3.1 are in order.

Remark 3.2. (I) Let us point out that the range in Theorem 3.1 is not empty. In order to see this, set $w = 1-\frac{1}{r}$ and $h = \frac{1}{s}-\frac{1}{r}$. Now notice that $\big(1-\frac{1}{r}, 1-\frac{1}{s}\big) \in \Delta_{P_2P_4P_5}$ is equivalent to

$$\Big(1-\frac{1}{r}, 1-\frac{1}{s}\Big) = (w, w-h) \in \Delta_{P_2P_4P_5}. \tag{3-5}$$

In turn, for (3-5) we need only that $0 < h < \frac{n+1}{2n}$ holds true when $h = \frac{2}{n}$ and $n \geq 5$.

(II) The critical Hardy potential $V(x) = c_0|x|^{-2} \in L^{(\frac{n}{2},\infty)}_{\mathrm{rad}}(\mathbb{R}^n)$ is covered by Theorem 3.1 with $|c_0| < \big(C_0K\big\||x|^{-2}\big\|_{(\frac{n}{2},\infty)}\big)^{-1}$. The constant C_0 in (3-4) is that of the Hölder inequality

$$\|Vu\|_{(s,\infty)} \leq C_0\|V\|_{(\frac{n}{2},\infty)}\|u\|_{(r,\infty)} \quad \text{with} \ \frac{1}{s}=\frac{2}{n}+\frac{1}{r}.$$

(III) Taking $V = 0$, Theorem 3.1 also provides an estimate for the linear wave equation $\Box u = f$.

Let $\{\omega(t)\}_{t\in\mathbb{R}}$ be the wave group $\omega(t) = (-\Delta)^{-\frac{1}{2}}\sin(t(-\Delta)^{\frac{1}{2}})$ and define $\zeta(f)$ by

$$\zeta(f)(x,t) = \int_0^t \omega(t-s)f(s)\,ds. \tag{3-6}$$

Formally, the IVP (1-2) is equivalent to the integral equation

$$u = L_{\vec{u}(0)}(t) + \mathcal{N}(u) + \mathcal{T}(u), \tag{3-7}$$

where

$$\mathcal{N}(u) = \mu\zeta(|u|^{p-1}u) \quad \text{and} \quad \mathcal{T}(u) = -\zeta(Vu).$$

Solutions of (3-7) are called mild solutions for the Cauchy problem (1-2).

We will look for solutions of (3-7) in the Banach space $E = L^\infty(\mathbb{R}; L^{(r_0,\infty)}_{\mathrm{rad}})$ whose norm is

$$\|u\|_E = \sup_{t\in\mathbb{R}} \|u(\cdot,t)\|_{(r_0,\infty)}. \tag{3-8}$$

The supremum in (3-8) is taken in the essential sense. This space is invariant by the scaling (1-3) and allows the existence of self-similar solutions (i.e., $u = u_\gamma$).

Consider

$$A_1 = \left(\frac{n+1}{2(n-1)}, \frac{n+1}{2(n-1)} - \frac{2}{n}\right) \quad \text{and} \quad A_2 = \left(1, \frac{n-2}{n}\right).$$

Let $]A_1, A_2[$ be the open segment line. Notice that $]A_1, A_2[\in \Delta_{P_2P_4P_5} \backslash \Delta_{P_1P_2P_3}$ for all $n \geq 4$.

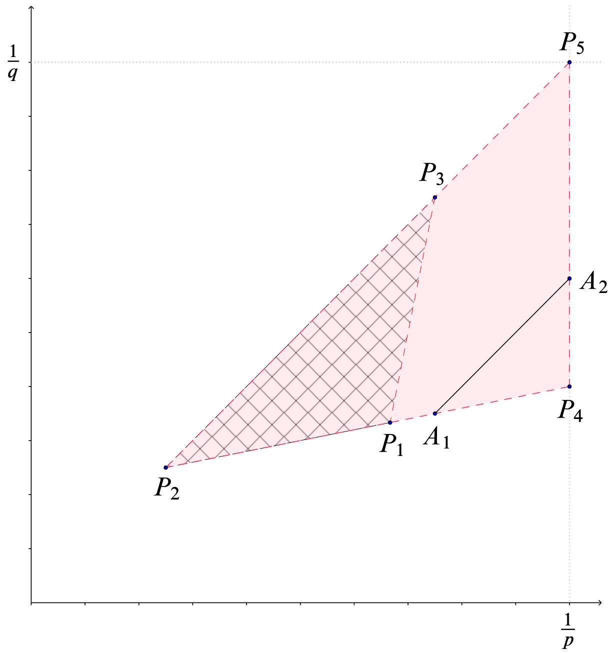

Figure 1. $]A_1, A_2[\in \Delta_{P_2P_4P_5} \backslash \Delta_{P_1P_2P_3}$.

Observe that $p > (n^2+n-4)/(n(n-3))$ is equivalent to

$$\left(1 - \frac{2}{n(p-1)}, 1 - \frac{2p}{n(p-1)}\right) \in\,]A_1, A_2[. \tag{3-9}$$

Our well-posedness and self-similarity results for (1-2) are stated below.

Theorem 3.3. *Let $n \geq 5$ be odd, $p > (n^2+n-4)/(n(n-3))$ and $r_0 = n(p-1)/2$. Suppose $(u_0, u_1) \in \mathcal{I}_{\text{rad}}$ and $V \in L_{\text{rad}}^{(\frac{n}{2},\infty)}$.*

(I) (***Global well-posedness***) *There are $\varepsilon, C_1 > 0$ such that if $\|(u_0, u_1)\|_{\mathcal{I}_{\text{rad}}} \leq \varepsilon$, then the IVP (1-2) has a unique mild solution $u \in L^\infty(\mathbb{R}; L_{\text{rad}}^{(r_0,\infty)})$ satisfying*

$$\sup_{t\in\mathbb{R}} \|u(\cdot\,, t)\|_{(r_0,\infty)} \leq \frac{2\varepsilon}{1-\eta}$$

provided that $\eta = C_1\|V\|_{(\frac{n}{2},\infty)} < 1$. Moreover, the solution u depends continuously on data (u_0, u_1) and potential V.

(II) (***Self-similarity***) *Under the hypotheses of item* (I), *the solution u is self-similar provided that u_0, u_1, V are homogeneous of degrees $-\frac{2}{p-1}, -\frac{p+1}{p-1}$ and -2, respectively.*

In what follows, we make some comments on Theorem 3.3.

Remark 3.4. (I) Taking $V = 0$, Theorem 3.3 provides a well-posedness result for semilinear wave equations in odd dimensions $n \geq 5$. Moreover, we have the continuous inclusions (see [Bergh and Löfström 1976, p. 154])

$$\dot{H}^s_{r_1} \hookrightarrow \dot{B}^s_{r_1,\infty} \hookrightarrow L^{(r_2,\infty)}, \tag{3-10}$$

where $\frac{1}{r_1} - \frac{s}{n} = \frac{1}{r_2}$ and $r_2 \geq r_1$. In particular, for $s_p = \frac{n}{2} - \frac{2}{p-1}$ we obtain $\dot{H}^{s_p} \hookrightarrow \dot{B}^{s_p}_{2,\infty} \hookrightarrow L^{(\frac{n(p-1)}{2},\infty)}$. In fact, the inclusions in (3-10) are strict and then the space $L^{(\frac{n(p-1)}{2},\infty)}$ is larger than $\dot{B}^{s_p}_{2,\infty}$, i.e., the one considered by Planchon [2000]. So, Theorem 3.3 extends the existence result of [Planchon 2000] in the case of radial solutions and $n \geq 5$ odd.

(II) Let $\mathcal{K}$ be the Kato class of potentials defined in (1-9). In view of the continuous strict inclusions

$$L^{\frac{n}{2}-\delta} \cap L^{\frac{n}{2}+\delta} \hookrightarrow L^{(\frac{n}{2},1)} \hookrightarrow \mathcal{K} \hookrightarrow L^{(\frac{n}{2},\infty)}, \quad \delta > 0, \tag{3-11}$$

our class for V is larger than $\mathcal{K}$ in the radial setting. For $L^{(\frac{n}{2},1)} \hookrightarrow \mathcal{K}$, we can use Hölder inequality (2-4) to obtain

$$\|V\|_{\mathcal{K}} \leq C \left\| \frac{1}{|x-y|^{(n-2)}} \right\|_{(\frac{n}{n-2},\infty)} \|V\|_{(\frac{n}{2},1)} \leq L\|V\|_{(\frac{n}{2},1)},$$

where

$$L = C\big\||x-y|^{-(n-2)}\big\|_{(\frac{n}{n-2},\infty)}$$

is a positive constant. Next recall that $f \in L^{(p,\infty)}$ if and only if there is a constant $C > 0$ such that

$$|E|^{\frac{1}{p}-1} \int_E |f(y)|\,dy \leq C \tag{3-12}$$

for every Borel set E. The supremum of the left-hand side of (3-12) over all Borel sets gives an equivalent norm in $L^{(p,\infty)}$. It follows from (3-12) that $\mathcal{K} \hookrightarrow L^{(\frac{n}{2},\infty)}$. In fact, it is sufficient to check (3-12) for every open ball $E = \mathcal{B}_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$. For that, we estimate

$$\int_{\mathcal{B}_r(x)} |V(y)|\,dy \leq \int_{\mathcal{B}_r(x)} \frac{r^{n-2}}{|x-y|^{n-2}} |V(y)|\,dy \leq C\|V\|_{\mathcal{K}} |\mathcal{B}_r(x)|^{1-\frac{2}{n}},$$

where $|\mathcal{B}_r(x)| = \big(\pi^{\frac{n}{2}}/\,\Gamma\big(\frac{n}{2}+1\big)\big) r^n$ is the volume of $\mathcal{B}_r(x)$.

4. Proofs

Collecting estimates in [Brenner 1975; Peral 1980; Strichartz 1970], we have that the wave group $\{\omega(t)\}_{t\in\mathbb{R}}$ is bounded from L^{l_1} to L^{l_2} at $t=1$, i.e.,

$$\|\omega(1)h\|_{l_2} \le M_1\|h\|_{l_1}, \tag{4-1}$$

provided that $(\frac{1}{l_1},\frac{1}{l_2}) \in \overline{\Delta_{P_1P_2P_3}}$, where $\overline{X}$ stands for the closure of X. It follows from scaling properties of $\omega(t)$ and L^p-spaces that

$$\|\omega(t)h\|_{l_2} \le M_1|t|^{-n(\frac{1}{l_1}-\frac{1}{l_2})+1}\|h\|_{l_1}. \tag{4-2}$$

Interpolating the estimate (4-2) (see, e.g., [Bergh and Löfström 1976]), we get

$$\|\omega(t)h\|_{(l_2,z)} \le M_2|t|^{-n(\frac{1}{l_1}-\frac{1}{l_2})+1}\|h\|_{(l_1,z)}, \tag{4-3}$$

where $1\le z\le\infty$. Assuming radial symmetry for h, the authors of [Ebert et al. 2016] extended the range of (4-2) to the closed triangle $\overline{\Delta_{P_2P_4P_5}}$ except for the semiopen segment line $]P_1,P_4]$ (see Figure 1). Thus, again by interpolation, for $(\frac{1}{l_1},\frac{1}{l_2})$ belonging to the open triangle $\Delta_{P_2P_4P_5}$ and $1\le z\le\infty$, we obtain the estimate

$$\|\omega(t)h\|_{(l_2,z)} \le M_3|t|^{-n(\frac{1}{l_1}-\frac{1}{l_2})+1}\|h\|_{(l_1,z)} \tag{4-4}$$

for all $h\in L^{(l_1,z)}_{\mathrm{rad}}(\mathbb{R}^n)$.

Yamazaki [2000] dealt with Navier–Stokes equations and Stokes and heat semigroups in weak-L^p spaces. The next estimate could be seen as a version of the Yamazaki estimate [2000, Corollary 2.3] for the wave group $\{\omega(t)\}_{t\in\mathbb{R}}$. Notice that it consists in a time-weighted estimate in preduals of weak-L^p spaces.

Lemma 4.1. *Let f be radially symmetric, $n\ge 3$ odd, and let $\Delta_{P_2P_4P_5}$ be the open triangle defined by the points P_2,P_4,P_5 in (3-2). If $1<d_1,d_2<2(n-1)/(n-3)$ (∞ if $n=3$) with $(\frac{1}{d_1},\frac{1}{d_2})\in\Delta_{P_2P_4P_5}$ then $|t|^{n(\frac{1}{d_1}-\frac{1}{d_2})-2}\omega(t)f\in L^1(\mathbb{R};L^{(d_2,1)}(\mathbb{R}^n))$ and there is $C>0$ such that*

$$\int_{\mathbb{R}} |t|^{\frac{n}{d_1}-\frac{n}{d_2}-2}\|\omega(t)f\|_{(d_2,1)}\,dt \le C\|f\|_{(d_1,1)} \tag{4-5}$$

for all $f\in L^{(d_1,1)}_{\mathrm{rad}}(\mathbb{R}^n)$.

Proof. Let p_1 and p_2 be such that $p_1<d_1<p_2$, $\frac{1}{d_1}-\frac{1}{p_2}<\frac{1}{n}$ and

$$\left(\frac{1}{p_j},\frac{1}{d_2}\right)\in\Delta_{P_2P_4P_5}\quad\text{for } j=1,2.$$

Using the estimate (4-4) with $(l_1,l_2,z)=(p_1,d_2,1)$ and $(l_1,l_2,z)=(p_2,d_2,1)$, we obtain

$$\|\omega(t)f\|_{(d_2,1)} \le C_k|t|^{1+\frac{n}{d_2}-\frac{n}{p_k}}\|f\|_{(p_k,1)}\quad\text{for } k=1,2. \tag{4-6}$$

Next consider the sublinear operator Ξ as a map from $L^{(p_1,1)}_{\mathrm{rad}}\cap L^{(p_2,1)}_{\mathrm{rad}}$ to a function $\Xi(f)(t)$ in $\mathbb{R}$ defined by

$$\Xi(f)(t)=|t|^{\frac{n}{d_1}-\frac{n}{d_2}-2}\|\omega(t)f\|_{(d_2,1)}.$$

In view of (4-6), we can estimate

$$\Xi(f)(t) \le C_k |t|^{\frac{n}{d_1}-\frac{n}{d_2}-2} |t|^{1+\frac{n}{d_2}-\frac{n}{p_k}} \|f\|_{(p_k,1)} = C_k |t|^{\frac{n}{d_1}-\frac{n}{p_k}-1} \|f\|_{(p_k,1)} \quad \text{for } k = 1, 2. \tag{4-7}$$

Hence, the operator Ξ is bounded from $L^{(p_k,1)}_{\mathrm{rad}}(\mathbb{R}^n)$ to $L^{(s_k,\infty)}(\mathbb{R})$, where $\frac{1}{s_k} = 1 - \left(\frac{n}{d_1} - \frac{n}{p_k}\right)$. Indeed, it follows from (4-7) that

$$\|\Xi(f)(t)\|_{L^{(s_k,\infty)}(\mathbb{R})} \le C_k \left\| |t|^{-\frac{1}{s_k}} \right\|_{L^{(s_k,\infty)}(\mathbb{R})} \|f\|_{(p_k,1)} \le L_k \|f\|_{(p_k,1)}, \tag{4-8}$$

where $L_k = C_k \left\| |t|^{-1/s_k} \right\|_{L^{(s_k,\infty)}(\mathbb{R})}$, $k = 1, 2$.

Take now $\theta \in (0, 1)$ such that $\frac{1}{d_1} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. So, we have $\frac{1-\theta}{s_1} + \frac{\theta}{s_2} = 1$ and $0 < s_1 < 1 < s_2$. By interpolation, it follows that

$$L^{(d_1,1)}_{\mathrm{rad}} = (L^{(p_1,1)}_{\mathrm{rad}}, L^{(p_2,1)}_{\mathrm{rad}})_{\theta,1} \quad \text{and} \quad L^1(\mathbb{R}) = \left(L^{(s_1,\infty)}(\mathbb{R}), L^{(s_2,\infty)}(\mathbb{R})\right)_{\theta,1}$$

and then

$$\begin{aligned} \|\Xi(f)(t)\|_{L^1(\mathbb{R})} &\le m_1^{1-\theta} m_2^{\theta} \|f\|_{(d_1,1)} \\ &\le L_1^{1-\theta} L_2^{\theta} \|f\|_{(d_1,1)}, \end{aligned}$$

where $m_k = \|\Xi(f)\|_{L^{(p_k,1)}_{\mathrm{rad}} \to L^{(s_k,\infty)}} \le L_k$. This gives us the estimate (4-5). □

Proof of Theorem 3.1. Let us rewrite $\zeta(f)$ in (3-6) as

$$\zeta(f)(x,t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} W(x-y, t-s) f(y,s)\, dy\, ds,$$

where the kernel W is given by

$$\widehat{W}(\xi, t-s) = \begin{cases} \sin((t-s)|\xi|)/|\xi|, & 0 < s < t, \\ 0, & \text{otherwise.} \end{cases}$$

Given a suitable function $\phi \in C^{\infty}(\mathbb{R}^n)$, we set

$$\langle \zeta(f), \phi \rangle = \int_{\mathbb{R}^n} \zeta(f)(x,t) \phi(x)\, dx.$$

Here all functions are considered to be radially symmetric. Using Tonelli's theorem and the Hölder inequality (2-4), we obtain

$$\begin{aligned} |\langle \zeta(f), \phi \rangle| &\le \int_{-\infty}^{\infty} \left\langle |f(\cdot,\tau)|, |\omega(t-\tau)\phi| \right\rangle d\tau \\ &\le C \int_{-\infty}^{\infty} \|f(\cdot,\tau)\|_{(s,\infty)} \|\omega(t-\tau)\phi\|_{(s',1)}\, d\tau. \end{aligned} \tag{4-9}$$

In (4-9) we have proceeded somewhat formally but we are going to see that its right-hand side is indeed finite, which justifies the above computations. Take $(d_1, d_2) = (r', s')$ and note that

$$\frac{n}{d_1} - \frac{n}{d_2} - 2 = \frac{n}{r'} - \frac{n}{s'} - 2 = 0.$$

Using duality in $L^{(p,z)}$, the inequality (4-9) and Lemma 4.1 with $(d_1,d_2)=(s',r')$, it follows that

$$
\begin{aligned}
\|\zeta(f)(\cdot,t)\|_{(r,\infty)} &= \sup_{\|\phi\|_{(r',1)}=1} |\langle \zeta(f),\phi\rangle| \\
&\le C \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{(s,\infty)} \sup_{\|\phi\|_{(r',1)}=1} \int_{-\infty}^{\infty} \|\omega(t-\tau)\phi\|_{(s',1)}\,d\tau \\
&\le K \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{(s,\infty)} \sup_{\|\phi\|_{(r',1)}=1} \{\|\phi\|_{(r',1)}\} \\
&\le K \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{(s,\infty)}
\end{aligned} \tag{4-10}
$$

for a.e. $t\in\mathbb{R}$. Next let $\tilde f = f + Vu$ and $u=\zeta(\tilde f)$ be the mild solution of (1-1). Since $\frac{1}{s}=\frac{1}{n/2}+\frac{1}{r}$, the Hölder inequality (2-4) gives

$$
\|Vu\|_{(s,\infty)} \le C_0\|V\|_{(\frac{n}{2},\infty)}\|u(\cdot,t)\|_{(r,\infty)}. \tag{4-11}
$$

Thus, in view of (4-10), we get

$$
\begin{aligned}
\sup_{t\in\mathbb{R}}\|u(\cdot,t)\|_{(r,\infty)} &\le K \sup_{t\in\mathbb{R}} \|\tilde f(\cdot,t)\|_{(s,\infty)} \\
&\le K \sup_{t\in\mathbb{R}} \|f(\cdot,t)\|_{(s,\infty)} + KC_0\|V\|_{(\frac{n}{2},\infty)} \sup_{t\in\mathbb{R}}\|u(\cdot,t)\|_{(r,\infty)},
\end{aligned}
$$

which implies the desired estimate because $KC_0\|V\|_{(\frac{n}{2},\infty)}<1$. □

Proof of Theorem 3.3. **Part (I).** (Well-posedness) Take $r=r_0=\frac{n(p-1)}{2}$ and $s=\frac{r_0}{p}$, and note that

$$
\frac{1}{s}-\frac{1}{r}=(p-1)\frac{1}{r_0}=\frac{2}{n}.
$$

In view of (3-9), we have $\left(1-\frac{1}{r},1-\frac{1}{s}\right)\in\Delta_{P_2P_4P_5}$ and then we can employ Theorem 3.1 with $V=0$ in order to obtain

$$
\|\zeta(f)\|_E \le K\|f\|_{L^\infty(\mathbb{R};L^{(r_0/p,\infty)}_{\mathrm{rad}})}. \tag{4-12}
$$

Since $\frac{2}{n}+\frac{1}{r_0}=\frac{p}{r_0}$, it follows from (4-12) and the Hölder inequality (2-4) that

$$
\begin{aligned}
\|\mathcal{T}(u)-\mathcal{T}(v)\|_E = \|\mathcal{T}(u-v)\|_E &= \sup_{t\in\mathbb{R}}\|\zeta(V(u-v))(\cdot,t)\|_{(r_0,\infty)} \\
&\le K\sup_{t\in\mathbb{R}}\|V(u-v)(\cdot,t)\|_{(\frac{r_0}{p},\infty)} \\
&\le KC_0\|V\|_{(\frac{n}{2},\infty)}\sup_{t\in\mathbb{R}}\|u(\cdot,t)-v(\cdot,t)\|_{(r_0,\infty)} \\
&\le \eta\|u-v\|_E,
\end{aligned} \tag{4-13}
$$

where $\eta=C_1\|V\|_{(\frac{n}{2},\infty)}$, $C_1=KC_0$, and C_0 is the constant in the Hölder inequality $\|Vh\|_{(\frac{r_0}{p},\infty)}\le C_0\|V\|_{(\frac{n}{2},\infty)}\|h\|_{(r_0,\infty)}$. Next recall the inequality

$$
\left||u|^{p-1}u-|v|^{p-1}v\right| \le C|u-v|(|u|^{p-1}+|v|^{p-1})
$$

and let $\frac{p}{r_0} = \frac{1}{r_0} + \frac{p-1}{r_0}$. Using the Hölder inequality (2-4), we can estimate

$$\begin{aligned}\big\| |u|^{p-1}u - |v|^{p-1}v \big\|_{(\frac{r_0}{p},\infty)} &\le C \big\| |u-v|(|u|^{p-1} + |v|^{p-1}) \big\|_{(\frac{r_0}{p},\infty)} \\ &\le C \|u-v\|_{(r_0,\infty)} \big\| (|u|^{p-1} + |v|^{p-1}) \big\|_{(\frac{r_0}{p-1},\infty)} \\ &\le C \|u-v\|_{(r_0,\infty)} \big(\|u\|^{p-1}_{(r_0,\infty)} + \|v\|^{p-1}_{(r_0,\infty)} \big). \end{aligned} \tag{4-14}$$

Estimates (4-12) and (4-14) yield

$$\begin{aligned}\|\mathcal{N}(u) - \mathcal{N}(v)\|_E &= \big\| \zeta(|u|^{p-1}u - |v|^{p-1}v)(\cdot,t) \big\|_E \\ &\le K \big\| (|u|^{p-1}u - |v|^{p-1}v)(\cdot,t) \big\|_{L^\infty(\mathbb{R}; L^{(r_0/p,\infty)}_{\text{rad}})} \\ &\le C_2 \|u-v\|_E (\|u\|^{p-1}_E + \|v\|^{p-1}_E). \end{aligned} \tag{4-15}$$

Let $\Psi(u) = L_{\vec{u}(0)}(t) + \mathcal{N}(u) + \mathcal{T}(u)$ be defined in the closed ball $B_\varepsilon = \{u \in E : \|u\|_E \le 2\varepsilon/(1-\eta)\}$, where $\varepsilon > 0$ and $\eta = C_1 \|V\|_{(n/2,\infty)} < 1$. We are going to show that Ψ is a contraction in B_ε for ε small enough. For $u, v \in B_\varepsilon$, we have

$$\begin{aligned}\|\Psi(u) - \Psi(v)\|_E &\le \|\mathcal{N}(u) - \mathcal{N}(v)\|_E + \|\mathcal{T}u - \mathcal{T}v\|_E \\ &\le C_2 \|u-v\|_E (\|u\|^{p-1}_E + \|v\|^{p-1}_E) + \eta \|u-v\|_E \\ &\le \|u-v\|_E \left(C_2 \left(\frac{2\varepsilon}{1-\eta} \right)^{p-1} + C_2 \left(\frac{2\varepsilon}{1-\eta} \right)^{p-1} + \eta \right) \\ &\le \left(C_2 \frac{2^p \varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta \right) \|u-v\|_E. \end{aligned} \tag{4-16}$$

Choose $\varepsilon > 0$ in such a way that

$$\left(C_2 \frac{2^p \varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta \right) < 1. \tag{4-17}$$

Moreover, taking $v = 0$ in (4-16), we arrive at

$$\|\Psi(u)\|_E \le \|L_{\vec{u}(0)}\|_E + \|\Psi(u) - \Psi(0)\|_E \le \varepsilon + \left(C_2 \frac{2^{p-1} \varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta \right) \frac{2\varepsilon}{1-\eta} < \frac{2\varepsilon}{1-\eta}$$

for all $u \in B_\varepsilon$. Hence, the map $\Psi : B_\varepsilon \to B_\varepsilon$ is a contraction in E. It follows that its fixed point in B_ε is the unique solution for (3-7) such that

$$\|u\|_E \le \frac{2\varepsilon}{1-\eta}.$$

The continuous dependence follows naturally from the above estimates and fixed point argument. We include its proof for the sake of completeness. Let $u, v \in B_\varepsilon$ be the unique mild solutions associated to data (u_0, u_1, V) and (v_0, v_1, U), respectively. Then, defining $L_{\vec{u}(0)}(t) = \omega(t)u_1 + \dot{\omega}(t)u_0$ and

$L_{\vec{v}(0)}(t) = \omega(t)v_1 + \dot{\omega}(t)v_0$, we have

$$\begin{aligned}\|u-v\|_E &\le \|L_{\vec{u}(0)} - L_{\vec{v}(0)}\|_E + \|\mathcal{N}(u)-\mathcal{N}(v)\|_E + \|\mathcal{T}(u)-\mathcal{T}(v)\|_E \\ &= \|L_{\vec{u}(0)} - L_{\vec{v}(0)}\|_E + \|\mathcal{N}(u)-\mathcal{N}(v)\|_E + \big\|\zeta\big((V-U)v + V(u-v)\big)\big\|_E \\ &\le \|L_{\vec{u}(0)} - L_{\vec{v}(0)}\|_E + C_2\frac{2^p\varepsilon^{p-1}}{(1-\eta)^{p-1}}\|u-v\|_E + KC_0\|V-U\|_{(\frac{n}{2},\infty)}\|v\|_E + \eta\|u-v\|_E \\ &\le \|(u_0-v_0,u_1-v_1)\|_{\mathcal{I}_{\mathrm{rad}}} + \left(C_2\frac{2^p\varepsilon^{p-1}}{(1-\eta)^{p-1}} + \eta\right)\|u-v\|_E + \frac{2KC_0\varepsilon}{1-\eta}\|V-U\|_{(\frac{n}{2},\infty)},\end{aligned}$$

which yields the desired continuity because of (4-17).

Part (II). (Self-similarity) First note that the homogeneous pair (u_0, u_1) is in $\mathcal{I}_{\mathrm{rad}}$. Due to the fixed point argument, the solution u in item (I) is the limit in $E = L^\infty(\mathbb{R}; L^{(r_0,\infty)}_{\mathrm{rad}})$ of the sequence

$$u^{(1)} = L_{\vec{u}(0)}(t) \quad\text{and}\quad u^{(k+1)} = L_{\vec{u}(0)}(t) + \mathcal{N}(u^{(k)}) + \mathcal{T}(u^{(k)}) \quad \text{for } k \in \mathbb{N}. \tag{4-18}$$

Using the homogeneity properties of u_0, u_1 and V, one can show that $u^{(k)}$ is invariant by (1-3), that is,

$$u^{(k)} = (u^{(k)})_\gamma := \gamma^{\frac{2}{p-1}} u^{(k)}(\gamma x, \gamma t).$$

Now, since $(E, \|\cdot\|_E)$ is invariant by (1-3), a change of variable gives

$$\|(u^{(k)})_\gamma - (u)_\gamma\|_E = \|(u^{(k)} - u)_\gamma\|_E = \|u^{(k)} - u\|_E \to 0, \quad \text{as } k \to \infty. \tag{4-19}$$

Since $(u^{(k)})_\gamma = u^{(k)}$, it follows that $u^{(k)}$ also converges to $(u)_\gamma$. Then, $u \equiv (u)_\gamma$ for each $\gamma > 0$, as required. □

References

[Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundlehren der Mathematischen Wissenschaften **223**, Springer, Berlin, 1976. MR Zbl

[Brenner 1975] P. Brenner, "On $L_p - L_{p'}$ estimates for the wave-equation", *Math. Z.* **145**:3 (1975), 251–254. MR Zbl

[Burq et al. 2003] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential", *J. Funct. Anal.* **203**:2 (2003), 519–549. MR Zbl

[Burq et al. 2004] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay", *Indiana Univ. Math. J.* **53**:6 (2004), 1665–1680. MR Zbl

[D'ancona and Pierfelice 2005] P. D'ancona and V. Pierfelice, "On the wave equation with a large rough potential", *J. Funct. Anal.* **227**:1 (2005), 30–77. MR Zbl

[Ebert et al. 2016] M. R. Ebert, R. A. Kapp, and T. Picon, "$L^1 - L^p$ estimates for radial solutions of the wave equation and application", *Ann. Mat. Pura Appl.* (4) **195**:4 (2016), 1081–1091. MR Zbl

[Georgiev and Visciglia 2003] V. Georgiev and N. Visciglia, "Decay estimates for the wave equation with potential", *Comm. Partial Differential Equations* **28**:7–8 (2003), 1325–1369. MR Zbl

[Grafakos 2004] L. Grafakos, *Classical and modern Fourier analysis*, Pearson Education, Upper Saddle River, NJ, 2004. MR Zbl

[Grillakis 1990] M. G. Grillakis, "Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity", *Ann. of Math.* (2) **132**:3 (1990), 485–509. MR Zbl

[Grillakis 1992] M. G. Grillakis, "Regularity for the wave equation with a critical nonlinearity", *Comm. Pure Appl. Math.* **45**:6 (1992), 749–774. MR Zbl

[Hidano 2002] K. Hidano, "Scattering and self-similar solutions for the nonlinear wave equation", *Differential Integral Equations* **15**:4 (2002), 405–462. MR Zbl

[Hunt 1966] R. A. Hunt, "On $L(p,q)$ spaces", *Enseignement Math.* (2) **12** (1966), 249–276. MR Zbl

[Kato and Ozawa 2003] J. Kato and T. Ozawa, "Weighted Strichartz estimates and existence of self-similar solutions for semilinear wave equations", *Indiana Univ. Math. J.* **52**:6 (2003), 1615–1630. MR Zbl

[Kato and Ozawa 2004] J. Kato and T. Ozawa, "Weighted Strichartz estimates for the wave equation in even space dimensions", *Math. Z.* **247**:4 (2004), 747–764. MR Zbl

[Kato et al. 2007] J. Kato, M. Nakamura, and T. Ozawa, "A generalization of the weighted Strichartz estimates for wave equations and an application to self-similar solutions", *Comm. Pure Appl. Math.* **60**:2 (2007), 164–186. MR Zbl

[Kavian and Weissler 1990] O. Kavian and F. B. Weissler, "Finite energy self-similar solutions of a nonlinear wave equation", *Comm. Partial Differential Equations* **15**:10 (1990), 1381–1420. MR Zbl

[Keel and Tao 1998] M. Keel and T. Tao, "Endpoint Strichartz estimates", *Amer. J. Math.* **120**:5 (1998), 955–980. MR Zbl

[Lai and Zhou 2014] N.-A. Lai and Y. Zhou, "An elementary proof of Strauss conjecture", *J. Funct. Anal.* **267**:5 (2014), 1364–1381. MR Zbl

[Miao et al. 2013] C. Miao, J. Zhang, and J. Zheng, "Strichartz estimates for wave equation with inverse square potential", *Commun. Contemp. Math.* **15**:6 (2013), art. id. 1350026. MR Zbl

[O'Neil 1963] R. O'Neil, "Convolution operators and $L(p,q)$ spaces", *Duke Math. J.* **30** (1963), 129–142. MR Zbl

[Pecher 2000a] H. Pecher, "Self-similar and asymptotically self-similar solutions of nonlinear wave equations", *Math. Ann.* **316**:2 (2000), 259–281. MR Zbl

[Pecher 2000b] H. Pecher, "Sharp existence results for self-similar solutions of semilinear wave equations", *NoDEA Nonlinear Differential Equations Appl.* **7**:3 (2000), 323–341. MR Zbl

[Peral 1980] J. C. Peral, "L^p estimates for the wave equation", *J. Funct. Anal.* **36**:1 (1980), 114–145. MR Zbl

[Planchon 2000] F. Planchon, "Self-similar solutions and semi-linear wave equations in Besov spaces", *J. Math. Pures Appl.* (9) **79**:8 (2000), 809–820. MR Zbl

[Planchon et al. 2003a] F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "Dispersive estimate for the wave equation with the inverse-square potential", *Discrete Contin. Dyn. Syst.* **9**:6 (2003), 1387–1400. MR Zbl

[Planchon et al. 2003b] F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh, "L^p estimates for the wave equation with the inverse-square potential", *Discrete Contin. Dyn. Syst.* **9**:2 (2003), 427–442. MR Zbl

[Ribaud and Youssfi 2002] F. Ribaud and A. Youssfi, "Global solutions and self-similar solutions of semilinear wave equation", *Math. Z.* **239**:2 (2002), 231–262. MR Zbl

[Shatah and Struwe 1993] J. Shatah and M. Struwe, "Regularity results for nonlinear wave equations", *Ann. of Math.* (2) **138**:3 (1993), 503–518. MR Zbl

[Strauss 1981] W. A. Strauss, "Nonlinear scattering theory at low energy", *J. Funct. Anal.* **41**:1 (1981), 110–133. MR Zbl

[Strauss and Tsutaya 1997] W. A. Strauss and K. Tsutaya, "Existence and blow up of small amplitude nonlinear waves with a negative potential", *Discrete Contin. Dynam. Systems* **3**:2 (1997), 175–188. MR Zbl

[Strichartz 1970] R. S. Strichartz, "Convolutions with kernels having singularities on a sphere", *Trans. Amer. Math. Soc.* **148** (1970), 461–471. MR Zbl

[Struwe 1999] M. Struwe, "Uniqueness for critical nonlinear wave equations and wave maps via the energy inequality", *Comm. Pure Appl. Math.* **52**:9 (1999), 1179–1188. MR Zbl

[Taggart 2010] R. J. Taggart, "Inhomogeneous Strichartz estimates", *Forum Math.* **22**:5 (2010), 825–853. MR Zbl

[Wang and Yu 2012] C. Wang and X. Yu, "Recent works on the Strauss conjecture", pp. 235–256 in *Recent advances in harmonic analysis and partial differential equations*, edited by A. R. Nahmod et al., Contemp. Math. **581**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Yajima 1995] K. Yajima, "The $W^{k,p}$-continuity of wave operators for Schrödinger operators", *J. Math. Soc. Japan* **47**:3 (1995), 551–581. MR Zbl

[Yamazaki 2000] M. Yamazaki, "The Navier–Stokes equations in the weak-L^n space with time-dependent external force", *Math. Ann.* **317**:4 (2000), 635–675. MR Zbl

[Yordanov and Zhang 2006] B. T. Yordanov and Q. S. Zhang, "Finite time blow up for critical wave equations in high dimensions", *J. Funct. Anal.* **231**:2 (2006), 361–374. MR Zbl

[Zhou 2007] Y. Zhou, "Blow up of solutions to semilinear wave equations with critical exponent in high dimensions", *Chin. Ann. Math. Ser. B* **28**:2 (2007), 205–212. MR Zbl

Received 2 Jul 2016. Accepted 12 Nov 2016.

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ANALYSIS AND PDE
Vol. 10, No. 2, 2017
dx.doi.org/10.2140/apde.2017.10.439

OPTIMAL WELL-POSEDNESS FOR THE INHOMOGENEOUS INCOMPRESSIBLE NAVIER–STOKES SYSTEM WITH GENERAL VISCOSITY

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In this paper we obtain new well-posedness results concerning a linear inhomogeneous Stokes-like system. These results are used to establish local well-posedness in the critical spaces for initial density ρ_0 and velocity u_0 such that $\rho_0 - \rho \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$, $u_0 \in \dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)$, $p \in \left(\frac{6}{5}, 4\right)$ for the inhomogeneous incompressible Navier–Stokes system with variable viscosity. To the best of our knowledge, regarding the 3-dimensional case, this is the first result in a truly critical framework for which one does not assume any smallness condition on the density.

1. Introduction

In this paper we deal with the well-posedness of the inhomogeneous, incompressible Navier–Stokes system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) D(u)) + \nabla P = 0, \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{1-1}$$

In the above, $\rho > 0$ stands for the density of the fluid, $u \in \mathbb{R}^n$ is the fluid's velocity field, while P is the pressure. The viscosity coefficient μ is assumed to be a smooth, strictly positive function of the density, while

$$D(u) = \nabla u + Du$$

is the deformation tensor. This system is used to study fluids obtained as a mixture of two (or more) incompressible fluids that have different densities: fluids containing a melted substance, polluted air/water etc.

There is a very rich literature devoted to the study of the well-posedness of (1-1). Briefly, the question of existence of weak solutions with finite energy was first considered by Kažihov [1974] (see also [Antontsev et al. 1990]) in the case of constant viscosity. The case with a general viscosity law was treated in [Lions 1996]. Weak solutions for more regular data were considered in [Desjardins 1997]. Recently, weak solutions were investigated by Huang, Paicu and Zhang in [Huang et al. 2013c].

This work was partially supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS - UEFISCDI, project number PN-II-RU-TE-2014-4-0320.
MSC2010: 35Q30, 76D05.
Keywords: inhomogeneous Navier–Stokes system, critical regularity, Lagrangian coordinates.

The unique solvability of (1-1) was first addressed in the seminal work of Ladyzhenskaya and Solonnikov [1975]. More precisely, considering $u_0 \in W^{2-2/p,p}(\Omega)$, with $p > 2$, a divergence-free vector field that vanishes on $\partial\Omega$ and $\rho_0 \in C^1(\Omega)$ bounded away from zero, they construct a global strong solution in the 2-dimensional case and a local solution in the 3-dimensional case. Moreover, if u_0 is small in $W^{2-2/p,p}(\Omega)$ then global well-posedness holds true.

The question of weak-strong uniqueness was addressed in [Choe and Kim 2003] for the case of sufficiently smooth data with vanishing viscosity.

Over the last thirteen years, efforts were made to obtain well-posedness results in the so-called critical spaces, i.e., the spaces which have the same invariance with respect to time and space dilation as the system itself, namely

$$\begin{cases} (\rho_0(x), u_0(x)) \to (\rho_0(lx), lu_0(lx)), \\ (\rho(t,x), u(t,x)) \to \big(\rho(l^2t, lx), lu(l^2t, lx), l^2P(l^2t, lx)\big). \end{cases}$$

For more details and explanations for this classical approach we refer to [Danchin 2003] or [Danchin and Mucha 2015]. In the Besov space context, which includes in particular the more classical Sobolev spaces, these are

$$\rho_0 - \bar\rho \in \dot B^{n/p_1}_{p_1,r_1} \quad \text{and} \quad u_0 \in \dot B^{n/p_2-1}_{p_2,r_2}, \tag{1-2}$$

where $\bar\rho$ is some constant density state and n is the space dimension. Working with densities close (in some appropriate norm) to a constant has led to a rich literature. In [Danchin 2003] local and global existence results are obtained for the case of constant viscosity and by taking the initial data

$$\rho_0 - \bar\rho \in L^\infty \cap \dot B^{n/2}_{2,\infty} \quad \text{and} \quad u_0 \in \dot B^{n/2-1}_{2,1}$$

and under the assumption that $\|\rho_0 - \bar\rho\|_{L^\infty \cap \dot B^{n/2}_{2,\infty}}$ is sufficiently small. The case with variable viscosity and for initial data

$$\rho_0 - \bar\rho \in \dot B^{n/p}_{p,1} \quad \text{and} \quad u_0 \in \dot B^{n/p-1}_{p,1},$$

$p \in [1, 2n)$, is treated in [Abidi 2007]. However, uniqueness is guaranteed once $p \in [1, n)$. These results were further extended by H. Abidi and M. Paicu [2007] by noticing that $\rho_0 - \bar\rho$ can be taken in a larger Besov space. B. Haspot [2012] established results in the same spirit as those mentioned above (however, the results are obtained in the nonhomogeneous framework and thus do not fall into the critical framework) in the case where the velocity field is not Lipschitz. Using the Lagrangian formulation, R. Danchin and P. B. Mucha [2012], established local and global results for (1-1) with constant viscosity when $\rho_0 - \bar\rho \in \mathcal{M}(\dot B^{n/p-1}_{p,1})$, $u_0 \in \dot B^{n/p-1}_{p,1}$ and under the smallness condition

$$\|\rho_0 - \bar\rho\|_{\mathcal{M}(\dot B^{n/p-1}_{p,1})} \ll 1,$$

where $\mathcal{M}(\dot B^{n/p-1}_{p,1})$ stands for the multiplier space of $\dot B^{n/p-1}_{p,1}$. In particular, functions with small jumps enter this framework. Moreover, as a consequence of their approach, the range of Lebesgue exponents for which uniqueness of solutions holds is extended to $p \in [1, 2n)$. In [Paicu and Zhang 2012; Huang et al. 2013a; 2013b; 2013c] the authors improve the smallness assumptions used in order to obtain global existence. To summarize, all the previous well-posedness results in critical spaces were established assuming the density is close in some sense to a constant state.

When the latter assumption is removed, one must impose more regularity on the data. For the case of constant viscosity, R. Danchin [2004] obtained local well-posedness and global well-posedness in dimension $n = 2$ for data drawn from the nonhomogeneous Sobolev spaces:

$$(\rho_0 - \bar{\rho}, u_0) \in H^{n/2+\alpha} \times H^{n/2-1+\beta}$$

with $\alpha, \beta > 0$. The same result for the case of the general viscosity law is established in [Abidi 2007]. For data with non-Lipschitz velocity results were established in [Haspot 2012]. Concerning rougher densities, considering $\rho_0 \in L^\infty(\mathbb{R}^d)$ bounded from below and $u_0 \in H^2(\mathbb{R}^d)$, Danchin and Mucha [2013] constructed a unique local solution. Again, supposing the density is close to some constant state, they proved global well-posedness. These results are generalized in [Paicu et al. 2013]. Taking the density as above, the authors construct a global unique solution provided that $u_0 \in H^s(\mathbb{R}^2)$ for any $s > 0$ in the 2-dimensional case and a local unique solution in the 3-dimensional case considering $u_0 \in H^1(\mathbb{R}^3)$. Moreover, assuming u_0 is suitably small, the solution constructed is global even in the 3-dimensional case.

In critical spaces of the Navier–Stokes system, i.e., (1-2) there are few well-posedness results. Very recently, in the 2-dimensional case and allowing variable viscosity, H. Xu, Y. Li and X. Zhai [2016] constructed a unique local solution to (1-1) provided that the initial data satisfy $\rho_0 - \bar{\rho} \in \dot{B}^{2/p}_{p,1}(\mathbb{R}^2)$ and $u_0 \in \dot{B}^{2/p-1}_{p,1}(\mathbb{R}^2)$. Moreover, if $\rho_0 - \bar{\rho} \in L^p \cap \dot{B}^{2/p}_{p,1}(\mathbb{R}^2)$ and the viscosity is supposed constant, their solution becomes global. In the 3-dimensional situation, to the best of our knowledge, the results that are closest to the critical regularity are those presented in [Abidi et al. 2012; 2013] (for a similar result in the periodic case one can consult [Poulon 2015]). More precisely, in three dimensions, assuming

$$\rho_0 - \bar{\rho} \in L^2 \cap \dot{B}^{3/2}_{2,1} \quad \text{and} \quad u_0 \in \dot{B}^{1/2}_{2,1}$$

and taking constant viscosity, H. Abidi, G. Gui and P. Zhang [Abidi et al. 2012] show the local well-posedness of system (1-1). Moreover, if the initial velocity is small then global well-posedness holds true. In [Abidi et al. 2013] they establish the same kind of result for initial data

$$\rho_0 - \bar{\rho} \in L^\lambda \cap \dot{B}^{3/\lambda}_{\lambda,1} \quad \text{and} \quad u_0 \in \dot{B}^{3/p-1}_{p,1},$$

where $\lambda \in [1, 2]$, $p \in [3, 4]$ are such that $\frac{1}{\lambda} + \frac{1}{p} > \frac{5}{6}$ and $\frac{1}{\lambda} - \frac{1}{p} \leq \frac{1}{3}$.

One of the goals of the present paper is to establish local well-posedness in the critical spaces

$$\rho_0 - \bar{\rho} \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3), \quad u_0 \in \dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3), \quad p \in \left(\tfrac{6}{5}, 4\right)$$

for system (1-1)

- with general smooth variable viscosity law,
- without any smallness assumption on the density,
- without any extra low frequencies assumption. In particular, we generalize the local existence and uniqueness result of [Abidi et al. 2012], thus achieving the critical regularity.

As in [Danchin and Mucha 2012], we will not work directly with system (1-1); instead we will use its Lagrangian formulation. By proceeding so, we are naturally led to consider the following Stokes problem with time-independent, nonconstant coefficients:

$$\begin{cases} \partial_t u - a\,\mathrm{div}(bD(u)) + a\nabla P = f, \\ \mathrm{div}\, u = \mathrm{div}\, R, \\ u_{|t=0} = u_0. \end{cases} \tag{1-3}$$

We establish global well-posedness results for system (1-3). This can be viewed as a first step towards generalizing the results of Danchin and Mucha [2015, Chapter 4] for the case of general viscosity and without assuming the density is close to a constant state. Let us mention that the estimates we obtain for system (1-3) have a wider range of applications: in a forthcoming paper we will investigate the well-posedness issue of the Navier–Stokes–Korteweg system under optimal regularity assumptions.

To summarize all the above, our main result reads:

Theorem 1.1. *Consider $p \in \left(\frac{6}{5}, 4\right)$. Assume that there exist positive constants $(\bar{\rho}, \rho_\star, \rho^\star)$ such that $\rho_0 - \bar{\rho} \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$ and $0 < \rho_\star < \rho_0 < \rho^\star$. Furthermore, consider u_0 a divergence-free vector field with coefficients in $\dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)$. Then, there exists a time $T > 0$ and a unique solution $(\rho, u, \nabla P)$ of system (1-1) with*

$$\rho-\bar{\rho}\in\mathcal{C}_T(\dot{B}^{3/p}_{p,1}(\mathbb{R}^3))\cap L^\infty_T(\dot{B}^{3/p}_{p,1}(\mathbb{R}^3)), \quad u\in\mathcal{C}_T(\dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)), \quad (\partial_t u, \nabla^2 u, \nabla P)\in L^1_T(\dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)).$$

One salutary feature of the Lagrangian formulation is that the density becomes independent of time. More precisely, considering $(\rho, u, \nabla P)$ a solution of (1-1) and denoting by X the flow associated to the vector field u,

$$X(t,y) = y + \int_0^t u(\tau, X(\tau, y))\,dy.$$

We introduce the new Lagrangian variables

$$\bar{\rho}(t,y) = \rho(t, X(t,y)), \quad \bar{u}(t,y) = u(t, X(t,y)) \quad \text{and} \quad \bar{P}(t,y) = P(t, X(t,y)).$$

Then, using the chain rule and Proposition 3.23 we gather that $\bar{\rho}(t,\cdot) = \rho_0$ and

$$\begin{cases} \rho_0\partial_t\bar{u} - \mathrm{div}(\mu(\rho_0)A_{\bar{u}}D_{A_{\bar{u}}}(\bar{u})) + A^T_{\bar{u}}\nabla\bar{P} = 0, \\ \mathrm{div}(A_{\bar{u}}\bar{u}) = 0, \\ \bar{u}_{|t=0} = u_0, \end{cases} \tag{1-4}$$

where $A_{\bar{u}}$ is the inverse of the differential of X, and

$$D_A(\bar{u}) = D\bar{u}A_{\bar{u}} + A^T_{\bar{u}}\nabla\bar{u}.$$

Note that we can give a meaning to (1-4) independently of the Eulerian formulation by stating

$$X(t,y) = y + \int_0^t \bar{u}(\tau, y)\,d\tau.$$

Theorem 1.1 will be a consequence of the following result:

Theorem 1.2. *Consider* $p \in \left(\frac{6}{5}, 4\right)$. *Assume there exists positive* $(\bar{\rho}, \rho_\star, \rho^\star)$ *such that* $\rho_0 - \bar{\rho} \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)$ *and* $0 < \rho_\star < \rho_0 < \rho^\star$. *Furthermore, consider* u_0 *a divergence-free vector field with coefficients in* $\dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)$. *Then, there exists a time* $T > 0$ *and a unique solution* $(\bar{u}, \nabla \bar{P})$ *of system* (1-4) *with*

$$\bar{u} \in \mathcal{C}_T(\dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)) \quad \text{and} \quad (\partial_t \bar{u}, \nabla^2 \bar{u}, \nabla \bar{P}) \in L^1_T(\dot{B}^{3/p-1}_{p,1}(\mathbb{R}^3)).$$

Moreover, there exists a positive constant $C = C(\rho_0)$ *such that*

$$\|u\|_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})} + \|(\nabla^2 u, \nabla P)\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \le \|u_0\|_{\dot{B}^{3/p-1}_{p,1}} \exp(C(T+1)).$$

The study of system (1-4) naturally leads to the Stokes-like system (1-3). In Section 2 we establish the global well-posedness of system (1-3). More precisely, we prove:

Theorem 1.3. *Consider* $n \in \{2, 3\}$ *and* $p \in (1, 4)$ *if* $n = 2$ *or* $p \in \left(\frac{6}{5}, 4\right)$ *if* $n = 3$. *Assume there exist positive constants* $(a_\star, b_\star, a^\star, b^\star, \bar{a}, \bar{b})$ *such that* $a - \bar{a} \in \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$, $b - \bar{b} \in \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ *and*

$$0 < a_\star \le a \le a^\star, \quad 0 < b_\star \le b \le b^\star.$$

Furthermore, consider the vector fields u_0 *and* f *with coefficients in* $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$ *and* $L^1_{\mathrm{loc}}(\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n))$ *respectively. Also, consider the vector field* $R \in (\mathcal{S}'(\mathbb{R}^n))^n$ *with*[1]

$$\mathcal{Q}R \in \mathcal{C}\big([0,\infty); \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)\big) \quad \text{and} \quad (\partial_t R, \nabla \operatorname{div} R) \in L^1_{\mathrm{loc}}(\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n))$$

such that

$$\operatorname{div} u_0 = \operatorname{div} R(0, \cdot).$$

Then, system (1-3) *has a unique global solution* $(u, \nabla P)$ *with*

$$u \in C\big([0,\infty), \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)\big) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L^1_{\mathrm{loc}}(\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)).$$

Moreover, there exists a constant $C = C(a, b)$ *such that*

$$\begin{aligned} &\|u\|_{L^\infty_t(\dot{B}^{n/p-1}_{p,1})} + \|(\partial_t u, \nabla^2 u, \nabla P)\|_{L^1_t(\dot{B}^{n/p-1}_{p,1})} \\ &\qquad \le \big(\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L^1_t(\dot{B}^{n/p-1}_{p,1})}\big) \exp(C(t+1)) \end{aligned} \tag{1-5}$$

for all $t \in [0, \infty)$.

The difficulty in establishing such a result comes from the fact that the pressure and velocity are "strongly" coupled as opposed to the case where ρ is close to a constant; see Remark 2.11 below. The key idea is to use the high-low frequency splitting technique first introduced in [Danchin 2007] combined with the particular structure of the divergence-free part of $a\nabla P$, i.e.,

$$\begin{aligned} \mathcal{P}(a\nabla P) = \mathcal{P}((a-\bar{a})\nabla P) &= \mathcal{P}((a-\bar{a})\nabla P) - (a-\bar{a})\mathcal{P}(\nabla P) \\ &:= [\mathcal{P}, a-\bar{a}]\nabla P, \end{aligned}$$

[1] $\mathcal{P}$ is the Leray projector over divergence-free vector fields, $\mathcal{Q} = \mathrm{Id} - \mathcal{P}$.

which is, loosely speaking, more regular than ∇P. Let us mention that a similar principle holds for u, which is divergence free:[2] whenever we estimate some term of the form $\mathcal{Q}(bM(D)u)$, where b lies in an appropriate Besov space and $M(D)$ is some pseudodifferential operator, we may write it as

$$\mathcal{Q}(bM(D)u) = [\mathcal{Q}, b]M(D)u$$

and use the fact that the latter expression is more regular than $M(D)u$; see Proposition 3.21.

The proof of Theorem 1.3 in the 3-dimensional case is more subtle. Loosely speaking, in order to close the estimates for system (1-3) one should work in a space on which the solution operator corresponding to the elliptic equation $\operatorname{div}(a\nabla P) = \operatorname{div} f$ is continuous. It is for this reason that we first prove a more restrictive result by demanding extra low-frequency information on the initial data. Then, using a perturbative version of Danchin and Mucha's results [2015] we arrive at constructing a solution with the optimal regularity. Uniqueness is obtained by a duality method.

Once the estimates of Theorem 1.3 are established, we proceed with the proof of Theorem 1.2, which is the object of Section 3. Finally, we show the equivalence between system (1-4) and system (1-1) thus achieving the proof of Theorem 1.1. We end this paper with an Appendix where results of Littlewood–Paley theory used through the text are gathered.

We end this section with some observations regarding the global existence issue. As opposed to the case when ρ is supposed to be a small perturbation of a constant state, when considering the linearized system of the Lagrangian formulation, i.e., system (1-3), we obtain the estimates (1-5), which have a time-dependent right-hand side term. This in particular prevents us from adapting the arguments from [Danchin and Mucha 2012] to our situation and obtaining a global solution for system (1-4) and consequently for the system (1-1). In fact, even if we were able to construct such a solution for system (1-4), it is not clear how we could go back into the original formulation as passing from the Eulerian formulation to the Lagrangian one needs some smallness condition on the $\|\cdot\|_{L^1_t(\dot{B}^{n/p}_{p,1})}$-norm of the velocity.

2. The Stokes system with nonconstant coefficients

Pressure estimates. Before handling system (1-3) we shall study the elliptic equation

$$\operatorname{div}(a\nabla P) = \operatorname{div} f. \tag{2-1}$$

For the reader's convenience let us cite the following classical result, a proof of which can be found, for instance, in [Danchin 2010]:

Proposition 2.1. *Consider $a \in L^\infty(\mathbb{R}^n)$ and a constant $a_\star$ such that*

$$a \geq a_\star > 0.$$

For all vector fields f with coefficients in $L^2(\mathbb{R}^n)$, there exists a tempered distribution P unique up to constant functions such that $\nabla P \in L^2(\mathbb{R}^n)$ and equation (2-1) is satisfied. In addition, we have

$$a_\star \|\nabla P\|_{L^2} \leq \|\mathcal{Q} f\|_{L^2}.$$

[2] and thus $\mathcal{Q}u = 0$.

Recently, regarding the 2-dimensional case, Xu et al. [2016], studied the elliptic equation (2-1) with the data $(a-\bar{a}, f)$ in Besov spaces. Using a different approach, we obtain estimates in both 2-dimensional and 3-dimensional situations. Let us also mention that our method allows to obtain a wider range of indices than the one of [Xu et al. 2016, Proposition 3.1(i)]. We choose to focus on the 3-dimensional case. We aim at establishing the following result:

Proposition 2.2. *Consider $p \in \left(\frac{6}{5}, 2\right)$ and $q \in [1,\infty)$ such that $\frac{1}{p}-\frac{1}{q} \le \frac{1}{2}$. Assume there exist positive constants $(\bar{a}, a_\star, a^\star)$ such that $a-\bar{a} \in \dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$ and $0 < a_\star \le a \le a^\star$. Furthermore, consider $f \in \dot{B}^{3/p-3/2}_{p,2}(\mathbb{R}^3)$. Then there exists a tempered distribution P unique up to constant functions such that $\nabla P \in \dot{B}^{3/p-3/2}_{p,2}(\mathbb{R}^3)$ and equation (2-1) is satisfied. Moreover, the following estimate holds true*:

$$\|\nabla P\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \left(\frac{1}{\bar{a}} + \left\|\frac{1}{a}-\frac{1}{\bar{a}}\right\|_{\dot{B}^{3/q}_{q,1}}\right)\left(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}\right)\|\mathcal{Q}f\|_{\dot{B}^{3/p-3/2}_{p,2}}. \tag{2-2}$$

Remark 2.3. Working in Besov spaces with third index $r=2$ is enough in view of the applications that we have in mind. However, similar estimates do hold true when the third index is chosen in the interval $[1,2]$.

Proof. Because $p<2$, Proposition 3.7 ensures that $\dot{B}^{3/p-3/2}_{p,2} \hookrightarrow L^2 = \dot{B}^0_{2,2}$ and owing to Proposition 2.1, we get the existence of $P \in \mathcal{S}'(\mathbb{R}^3)$ with $\nabla P \in L^2$ and

$$a_\star\|\nabla P\|_{L^2} \le \|\mathcal{Q}f\|_{L^2}. \tag{2-3}$$

Moreover, as $\mathcal{Q}$ is a continuous operator on L^2, we deduce from (2-1) that

$$\mathcal{Q}(a\nabla P) = \mathcal{Q}f. \tag{2-4}$$

Using the Bony decomposition (see Definition 3.14 and the remark that follows) and the fact that $\mathcal{P}(\nabla P)=0$, we write

$$\mathcal{P}(a\nabla P) = \mathcal{P}(\dot{T}'_{\nabla P}(a-\bar{a})) + [\mathcal{P}, \dot{T}_{a-\bar{a}}]\nabla P.$$

Using Proposition 3.16 along with Proposition 3.7 and relation (2-3), we get

$$\|\mathcal{P}(\dot{T}'_{\nabla P}(a-\bar{a}))\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \|\nabla P\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{3/p-3/2}_{p^\star,2}} \lesssim \frac{1}{a_\star}\|\mathcal{Q}f\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}, \tag{2-5}$$

where

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{p^\star}.$$

Next, proceeding as in Proposition 3.20 we get

$$\left\|[\mathcal{P}, \dot{T}_{a-\bar{a}}]\nabla P\right\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \|\nabla a\|_{\dot{B}^{3/p-5/2}_{p^\star,2}}\|\nabla P\|_{L^2} \lesssim \frac{1}{a_\star}\|\mathcal{Q}f\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}. \tag{2-6}$$

Putting together relations (2-5) and (2-6) we get

$$\|\mathcal{P}(a\nabla P)\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \frac{1}{a_\star}\|\mathcal{Q}f\|_{L^2}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}.$$

Combining this with (2-4) and Proposition 3.7, we find

$$\|a\nabla P\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \Big(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}\Big)\|\mathcal{Q}f\|_{\dot{B}^{3/p-3/2}_{p,2}}.$$

Of course, writing

$$\nabla P = \frac{1}{a}a\nabla P,$$

using product rules one gets

$$\|\nabla P\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \Big(\frac{1}{\bar{a}}+\Big\|\frac{1}{a}-\frac{1}{\bar{a}}\Big\|_{\dot{B}^{3/q}_{q,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}\Big)\|\mathcal{Q}f\|_{\dot{B}^{3/p-3/2}_{p,2}} \tag{2-7}$$

This concludes the proof. □

Applying the same technique as above leads to the 2-dimensional estimate:

Proposition 2.4. *Consider $p\in(1,2)$ and $q\in[1,\infty)$ such that $\frac{1}{p}-\frac{1}{q}\le\frac{1}{2}$. Assume there exists positive constants $(\bar{a},a_\star,a^\star)$ such that $a-\bar{a}\in\dot{B}^{2/q}_{q,1}(\mathbb{R}^2)$ and $0<a_\star\le a\le a^\star$. Furthermore, consider $f\in\dot{B}^{2/p-1}_{p,2}(\mathbb{R}^2)$. Then there exists a tempered distribution P unique up to constant functions such that $\nabla P\in\dot{B}^{2/p-1}_{p,2}(\mathbb{R}^2)$ and equation (2-1) is satisfied. Moreover, the following estimate holds true:*

$$\|\nabla P\|_{\dot{B}^{2/p-1}_{p,2}} \lesssim \Big(\frac{1}{\bar{a}}+\Big\|\frac{1}{a}-\frac{1}{\bar{a}}\Big\|_{\dot{B}^{2/q}_{q,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{2/q}_{q,1}}\Big)\|\mathcal{Q}f\|_{\dot{B}^{2/p-1}_{p,2}}. \tag{2-8}$$

Let us point out that the restriction $p>\frac{6}{5}$ comes from the fact that we need $\frac{3}{p}-\frac{5}{2}<0$ in relation (2-6). In two dimensions, instead of $\frac{3}{p}-\frac{5}{2}$ we will have $\frac{2}{p}-2$, which is negative provided $p>1$.

The next result covers the range of integrability indices larger than 2:

Proposition 2.5. *Consider $p\in(2,6)$ and $q\in[1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}\ge\frac{1}{2}$. Assume there exist positive constants $(\bar{a},a_\star,a^\star)$ such that $a-\bar{a}\in\dot{B}^{3/q}_{q,1}(\mathbb{R}^3)$ and $0<a_\star\le a\le a^\star$. Furthermore, consider $f\in\dot{B}^{3/p-3/2}_{p,2}(\mathbb{R}^3)$ and a tempered distribution P with $\nabla P\in\dot{B}^{3/p-3/2}_{p,2}(\mathbb{R}^3)$ such that equation (2-1) is satisfied. Then, the following estimate holds true:*

$$\|\nabla P\|_{\dot{B}^{3/p-3/2}_{p,2}} \lesssim \Big(\frac{1}{\bar{a}}+\Big\|\frac{1}{a}-\frac{1}{\bar{a}}\Big\|_{\dot{B}^{3/q}_{q,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}\Big)\|\mathcal{Q}f\|_{\dot{B}^{3/p-3/2}_{p,2}}. \tag{2-9}$$

Proof. Notice that p', the conjugate Lebesgue exponent of p, satisfies $p'\in\big(\frac{6}{5},2\big)$ and $\frac{1}{p'}-\frac{1}{q}\le\frac{1}{2}$. Thus, by Proposition 2.2, for any g belonging to the unit ball of $\mathcal{S}\cap\dot{B}^{3/p'-3/2}_{p',2}$ there exists a $P_g\in\mathcal{S}'(\mathbb{R}^3)$ with $\nabla P_g\in\mathcal{S}\cap\ \dot{B}^{3/p'-3/2}_{p',2}$ such that

$$\operatorname{div}(a\nabla P_g)=\operatorname{div} g$$

and

$$\|\nabla P_g\|_{\dot{B}^{3/p'-3/2}_{p',2}} \lesssim \Big(\frac{1}{\bar{a}}+\Big\|\frac{1}{a}-\frac{1}{\bar{a}}\Big\|_{\dot{B}^{3/q}_{q,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar{a}\|_{\dot{B}^{3/q}_{q,1}}\Big).$$

We write

$$\begin{aligned}\langle\nabla P,g\rangle &= -\langle P,\operatorname{div} g\rangle = -\langle P,\operatorname{div}(a\nabla P_g)\rangle\\ &= -\langle\operatorname{div}\mathcal{Q}f,P_g\rangle = \langle\mathcal{Q}f,\nabla P_g\rangle,\end{aligned}$$

and consequently

$$
\begin{aligned}
|\langle \nabla P, g\rangle| &\lesssim \|\mathcal{Q} f\|_{\dot B^{3/p-3/2}_{p,2}} \|\nabla P_g\|_{\dot B^{3/p'-3/2}_{p',2}} \\
&\lesssim \Big(\frac{1}{\bar a} + \Big\|\frac{1}{a} - \frac{1}{\bar a}\Big\|_{\dot B^{3/q}_{q,1}}\Big)\Big(1 + \frac{1}{a_\star}\|a-\bar a\|_{\dot B^{3/q}_{q,1}}\Big)\|\mathcal{Q} f\|_{\dot B^{3/p-3/2}_{p,2}}.
\end{aligned}
$$

Using Proposition 3.8, we get that relation (2-9) holds true. □

As in the previous situation, by applying the same technique we get a similar result in two dimensions:

Proposition 2.6. *Consider $p \in (2,\infty)$ and $q \in [1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} \ge \frac{1}{2}$. Assume there exist positive constants $(\bar a, a_\star, a^\star)$ such that $a - \bar a \in \dot B^{2/q}_{q,1}(\mathbb{R}^2)$ and $0 < a_\star \le a \le a^\star$. Furthermore, consider $f \in \dot B^{2/p-1}_{p,2}(\mathbb{R}^2)$ and a tempered distribution P with $\nabla P \in \dot B^{2/p-1}_{p,2}(\mathbb{R}^2)$ such that equation (2-1) is satisfied. Then, following estimate holds true*:

$$
\|\nabla P\|_{\dot B^{2/p-1}_{p,2}} \lesssim \Big(\frac{1}{\bar a} + \Big\|\frac{1}{a} - \frac{1}{\bar a}\Big\|_{\dot B^{2/q}_{q,1}}\Big)\Big(1 + \frac{1}{a_\star}\|a-\bar a\|_{\dot B^{2/q}_{q,1}}\Big)\|\mathcal{Q} f\|_{\dot B^{2/p-1}_{p,2}}. \tag{2-10}
$$

Some preliminary results. In this section we derive estimates for a Stokes-like problem with time-independent, nonconstant coefficients. Before proceeding to the actual proof, for the reader's convenience, let us cite the following results which were established by Danchin and Mucha [2009; 2015]. These results correspond to the case where a and b are constants:

Proposition 2.7. *Consider $u_0 \in \dot B^{n/p-1}_{p,1}$ and $(f, \partial_t R, \nabla \operatorname{div} R) \in L^1_T(\dot B^{n/p-1}_{p,1})$ with $\mathcal{Q} R \in C_T(\dot B^{n/p-1}_{p,1})$ such that*

$$
\operatorname{div} u_0 = \operatorname{div} R(0,\cdot).
$$

Then, the system

$$
\begin{cases}
\partial_t u - \bar a \bar b \Delta u + \bar a \nabla P = f,\\
\operatorname{div} u = \operatorname{div} R,\\
u_{|t=0} = u_0
\end{cases}
$$

has a unique solution $(u, \nabla P)$ with

$$
u \in \mathcal{C}\big([0,T); \dot B^{n/p-1}_{p,1}\big) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L^1_T(\dot B^{n/p-1}_{p,1})
$$

and the following estimate is valid:

$$
\|u\|_{L^\infty_T(\dot B^{n/p-1}_{p,1})} + \|(\partial_t u, \bar a \bar b \nabla^2 u, \bar a \nabla P)\|_{L^1_T(\dot B^{n/p-1}_{p,1})} \lesssim \|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f, \partial_t R, \bar a \bar b \nabla \operatorname{div} R)\|_{L^1_T(\dot B^{n/p-1}_{p,1})}.
$$

As a consequence of the previous result, one can establish via a perturbation argument:

Proposition 2.8. *Consider $u_0 \in \dot B^{n/p-1}_{p,1}$ and $(f, \partial_t R, \nabla \operatorname{div} R) \in L^1_T(\dot B^{n/p-1}_{p,1})$ with $\mathcal{Q} R \in C_T(\dot B^{n/p-1}_{p,1})$ such that*

$$
\operatorname{div} u_0 = \operatorname{div} R(0,\cdot).
$$

Then, there exists an $\eta = \eta(\bar a)$ small enough such that for all $c \in \dot B^{n/p}_{p,1}$ with

$$
\|c\|_{\dot B^{n/p}_{p,1}} \le \eta,
$$

the system

$$\begin{cases}\partial_t u - \bar{a}\bar{b}\Delta u + (\bar{a}+c)\nabla P = f,\\ \operatorname{div} u = \operatorname{div} R,\\ u_{|t=0} = u_0\end{cases}$$

has a unique solution $(u, \nabla P)$ *with*

$$u \in \mathcal{C}\big([0,T); \dot{B}^{n/p-1}_{p,1}\big) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L^1_T(\dot{B}^{n/p-1}_{p,1})$$

and the following estimate is valid:

$$\|u\|_{L^\infty_T(\dot{B}^{n/p-1}_{p,1})} + \|(\partial_t u, \bar{a}\bar{b}\nabla^2 u, \bar{a}\nabla P)\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})} \lesssim \|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|(f, \partial_t R, \bar{a}\bar{b}\nabla \operatorname{div} R)\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})}.$$

In all that follows we denote by E_{loc} the space of $(u, \nabla P)$ such that

$$u \in \mathcal{C}\big([0,\infty); \dot{B}^{n/p-1}_{p,1}\big) \quad \text{and} \quad (\nabla^2 u, \nabla P) \in L^1_{\text{loc}}(\dot{B}^{n/p-1}_{p,1}) \times L^1_{\text{loc}}(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1}).$$

Additionally, we introduce the space E_T of $u \in \mathcal{C}_T(\dot{B}^{n/p-1}_{p,1})$ with $\nabla^2 u \in L^1_T(\dot{B}^{n/p-1}_{p,1})$ and $\nabla P \in L^1_T(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})$ such that

$$\|(u,\nabla P)\|_{E_T} = \|u\|_{L^\infty_T(\dot{B}^{n/p-1}_{p,1})} + \|\nabla^2 u\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})} + \|\nabla P\|_{L^1_T(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})} < \infty.$$

The first ingredient in proving Theorem 1.3 is the following:

Proposition 2.9. *Consider* $n \in \{2,3\}$ *and* $p \in (1,4)$ *if* $n = 2$ *or* $p \in \big(\frac{6}{5}, 4\big)$ *if* $n = 3$. *Assume there exist positive constants* $(a_\star, b_\star, a^\star, b^\star, \bar{a}, \bar{b})$ *such that* $a - \bar{a} \in \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$, $b - \bar{b} \in \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ *and*

$$0 < a_\star \le a \le a^\star, \quad 0 < b_\star \le b \le b^\star.$$

Furthermore, consider u_0, f *vector fields with coefficients in* $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$ *and* $L^1_{\text{loc}}(\dot{B}^{n/p-n/2}_{p,2}(\mathbb{R}^n) \cap \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n))$ *respectively and a vector field* $R \in (\mathcal{S}'(\mathbb{R}^n))^n$ *with*

$$(\partial_t R, \nabla \operatorname{div} R) \in L^1_{\text{loc}}\big(\dot{B}^{n/p-n/2}_{p,2}(\mathbb{R}^n) \cap \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)\big) \quad \text{and} \quad \mathcal{Q}R \in \mathcal{C}\big([0,\infty); \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)\big)$$

such that

$$\operatorname{div} u_0 = \operatorname{div} R(0,\cdot).$$

Then, there exists a constant C_{ab} *depending on* a *and* b *such that any solution* $(u, \nabla P) \in E_T$ *of the Stokes system* (1-3) *will satisfy*

$$\|u\|_{L^\infty_t(\dot{B}^{n/p-1}_{p,1})} + \|\nabla^2 u\|_{L^1_t(\dot{B}^{n/p-1}_{p,1})} + \|\nabla P\|_{L^1_t(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})} \le \big(\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L^1_t(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})}\big) \exp(C_{ab}(t+1)) \tag{2-11}$$

for all $t \in (0, T]$.

Before proceeding with the proof, a few remarks are in order:

Remark 2.10. Proposition 2.9 is different from Theorem 1.3 when $n=3$. Indeed, in the 3-dimensional case the theory is more subtle and thus, as a first step we construct a unique solution for the case of more regular initial data.

Remark 2.11. The difficulty when dealing with the Stokes system with nonconstant coefficients lies in the fact that the pressure and the velocity u are coupled. Indeed, in the constant coefficients case, in view of

$$\operatorname{div} u = \operatorname{div} R,$$

one can apply the divergence operator in the first equation of (1-3) in order to obtain the following elliptic equation verified by the pressure:

$$a\Delta P = \operatorname{div}(f - \partial_t R + 2ab\nabla \operatorname{div} R). \tag{2-12}$$

From (2-12) we can construct the pressure. Having built the pressure, the velocity satisfies a classical heat equation. In the nonconstant coefficient case, proceeding as above we find that

$$\operatorname{div}(a\nabla P) = \operatorname{div}\big(f - \partial_t R + a \operatorname{div}(bD(u))\big). \tag{2-13}$$

Therefore the strategy used in the previous case is not well-adapted. We will establish a priori estimates and use a continuity argument like in [Danchin 2014]. In order to be able to close the estimates on u, we have to bound $\|a\nabla P\|_{L^1_t(\dot B^{n/p-1}_{p,1})}$ in terms of

$$\|u\|^{\beta}_{L^\infty_t(\dot B^{n/p-1}_{p,1})}\|\nabla^2 u\|^{1-\beta}_{L^1_t(\dot B^{n/p-1}_{p,1})}$$

for some $\beta\in(0,1)$. Thus, the difficulty is to find estimates for the pressure which do not feature the time derivative of the velocity.

In view of Proposition 2.7, consider $(u_L,\nabla P_L)$, the unique solution of the system

$$\begin{cases} \partial_t u - \bar a \operatorname{div}(\bar b D(u)) + \bar a\nabla P = f,\\ \operatorname{div} u = \operatorname{div} R,\\ u_{|t=0} = u_0,\end{cases} \tag{2-14}$$

with

$$u_L \in \mathcal{C}\big([0,\infty); \dot B^{n/p-1}_{p,1}\big) \quad \text{and} \quad (\partial_t u_L, \nabla^2 u_L, \nabla P_L) \in L^1_{\mathrm{loc}}(\dot B^{n/p-1}_{p,1}).$$

Recall that for any $t\in[0,\infty)$ we have

$$\|u_L\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + \|(\partial_t u_L, \bar a\bar b\nabla^2 u_L, \bar a\nabla P_L)\|_{L^1_t(\dot B^{n/p-1}_{p,1})} \le C\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R, \bar a\bar b\nabla \operatorname{div} R)\|_{L^1_t(\dot B^{n/p-1}_{p,1})}\big). \tag{2-15}$$

In what follows, we will use the notation

$$\tilde u = u - u_L, \quad \nabla\widetilde{P} = \nabla P - \nabla P_L. \tag{2-16}$$

Obviously, we have

$$\operatorname{div}\tilde u = 0. \tag{2-17}$$

Thus, the system (1-3) is recast into

$$\begin{cases} \partial_t \tilde{u} - a\,\mathrm{div}(bD(\tilde{u})) + a\nabla \widetilde{P} = \tilde{f}, \\ \mathrm{div}\,\tilde{u} = 0, \\ \tilde{u}_{|t=0} = 0, \end{cases} \tag{2-18}$$

where

$$\tilde{f} = a\,\mathrm{div}(bD(u_L)) - \bar{a}\,\mathrm{div}(\bar{b}D(u_L)) - (a-\bar{a})\nabla P_L.$$

Using the last equality along with Proposition 3.17, we infer

$$\begin{aligned} \|\tilde{f}\|_{\dot{B}^{n/p-1}_{p,1}} &\le \big\|a\,\mathrm{div}(bD(u_L)) - \bar{a}\,\mathrm{div}(\bar{b}D(u_L))\big\|_{\dot{B}^{n/p-1}_{p,1}} + \|(a-\bar{a})\nabla P_L\|_{\dot{B}^{n/p-1}_{p,1}} \\ &\lesssim (\bar{a} + \|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}})(\bar{b} + \|b-\bar{b}\|_{\dot{B}^{n/p}_{p,1}})\|\nabla u_L\|_{\dot{B}^{n/p}_{p,1}} + \|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}}\|\nabla P_L\|_{\dot{B}^{n/p-1}_{p,1}}. \end{aligned} \tag{2-19}$$

Let us estimate the pressure $a\nabla\widetilde{P}$. First, we write

$$\|a\nabla\widetilde{P}\|_{\dot{B}^{n/p-1}_{p,1}} \le \|\mathcal{Q}(a\nabla\widetilde{P})\|_{\dot{B}^{n/p-1}_{p,1}} + \|\mathcal{P}(a\nabla\widetilde{P})\|_{\dot{B}^{n/p-1}_{p,1}}.$$

Applying the $\mathcal{Q}$ operator in the first equation of (2-18) we get

$$\mathcal{Q}(a\nabla\widetilde{P}) = \mathcal{Q}\tilde{f} + \mathcal{Q}(a\,\mathrm{div}(bD(\tilde{u}))).$$

Thus, we get

$$\|\mathcal{Q}(a\nabla\widetilde{P})\|_{\dot{B}^{n/p-1}_{p,1}} \le \|\mathcal{Q}\tilde{f}\|_{\dot{B}^{n/p-1}_{p,1}} + \|\mathcal{Q}(a\,\mathrm{div}(bD(\tilde{u})))\|_{\dot{B}^{n/p-1}_{p,1}}. \tag{2-20}$$

Let

$$\mathcal{Q}\big(a\,\mathrm{div}(bD(\tilde{u}))\big) = \mathcal{Q}\big(D(\tilde{u})\dot{S}_m(a\nabla b)\big) + \mathcal{Q}\big(\dot{S}_m(ab-\bar{a}\bar{b})\Delta\tilde{u}\big) \tag{2-21}$$

$$+\, \mathcal{Q}\big(D(\tilde{u})(\mathrm{Id}-\dot{S}_m)(a\nabla b)\big) \tag{2-22}$$

$$+\, \mathcal{Q}\big((\mathrm{Id}-\dot{S}_m)(ab-\bar{a}\bar{b})\Delta\tilde{u}\big), \tag{2-23}$$

where $m \in \mathbb{N}$ will be chosen later. According to Proposition 3.17 we have

$$\big\|\mathcal{Q}\big(D(\tilde{u})\dot{S}_m(a\nabla b)\big)\big\|_{\dot{B}^{n/p-1}_{p,1}} \lesssim \|\dot{S}_m(a\nabla b)\|_{\dot{B}^{n/p-1/2}_{p,1}}\|\nabla\tilde{u}\|_{\dot{B}^{n/p-1/2}_{p,1}}. \tag{2-24}$$

Owing to the fact that $\tilde{u}$ is divergence free we can write

$$\mathcal{Q}(\dot{S}_m(ab-\bar{a}\bar{b})\Delta\tilde{u}) = [\mathcal{Q}, \dot{S}_m(ab-\bar{a}\bar{b})]\Delta\tilde{u}, \tag{2-25}$$

such that applying Proposition 3.21 we get

$$\begin{aligned} \big\|\mathcal{Q}\big(\dot{S}_m(ab-\bar{a}\bar{b})\Delta\tilde{u}\big)\big\|_{\dot{B}^{n/p-1}_{p,1}} &\lesssim \big\|\big(\dot{S}_m(a\nabla b), \dot{S}_m(b\nabla a)\big)\big\|_{\dot{B}^{n/p-1/2}_{p,1}}\|\Delta\tilde{u}\|_{\dot{B}^{n/p-3/2}_{p,1}} \\ &\lesssim \big\|\big(\dot{S}_m(a\nabla b), \dot{S}_m(b\nabla a)\big)\big\|_{\dot{B}^{n/p-1/2}_{p,1}}\|\nabla\tilde{u}\|_{\dot{B}^{n/p-1/2}_{p,1}}. \end{aligned} \tag{2-26}$$

The last two terms of (2-21)–(2-23) are estimated as follows:

$$\big\|\mathcal{Q}\big((\mathrm{Id}-\dot S_m)(a\nabla b)D(\tilde u)\big)+\mathcal{Q}\big((\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\Delta\tilde u\big)\big\|_{\dot B^{n/p-1}_{p,1}} \tag{2-27}$$

$$\lesssim \big(\|(\mathrm{Id}-\dot S_m)(a\nabla b)\|_{\dot B^{n/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{n/p}_{p,1}}\big)\|\nabla\tilde u\|_{\dot B^{n/p}_{p,1}}. \tag{2-28}$$

Thus, putting together relations (2-20)–(2-28) we get

$$\|\mathcal{Q}(a\nabla\widetilde P)\|_{\dot B^{n/p-1}_{p,1}}\lesssim\|\mathcal{Q}\tilde f\|_{\dot B^{n/p-1}_{p,1}}+\big\|\big(\dot S_m(a\nabla b),\dot S_m(b\nabla a)\big)\big\|_{\dot B^{n/p-1/2}_{p,1}}\|\nabla\tilde u\|_{\dot B^{n/p-1/2}_{p,1}}$$
$$+\|\nabla\tilde u\|_{\dot B^{n/p}_{p,1}}\big(\|(\mathrm{Id}-\dot S_m)(a\nabla b,b\nabla a)\|_{\dot B^{n/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{n/p}_{p,1}}\big). \tag{2-29}$$

Next, we turn our attention towards $\mathcal{P}(a\nabla\widetilde P)$. The 2-dimensional case and the 3-dimensional case have to be treated differently.

The 3-dimensional case. Noticing that

$$\mathcal{P}(a\nabla\widetilde P)=\mathcal{P}\big((\mathrm{Id}-\dot S_m)(a-\bar a)\nabla\widetilde P\big)+[\mathcal{P},\dot S_m(a-\bar a)]\nabla\widetilde P,$$

and using again Proposition 3.21 combined with Propositions 2.2 and 2.5 we get

$$\|\nabla\widetilde P\|_{\dot B^{3/p-3/2}_{p,2}}+\|\mathcal{P}(a\nabla\widetilde P)\|_{\dot B^{3/p-1}_{p,1}}$$
$$\lesssim\|\nabla\widetilde P\|_{\dot B^{3/p-3/2}_{p,2}}+\|\mathcal{P}((\mathrm{Id}-\dot S_m)(a-\bar a)\nabla\widetilde P)\|_{\dot B^{3/p-1}_{p,1}}+\|[\mathcal{P},\dot S_m(a-\bar a)]\nabla\widetilde P\|_{\dot B^{3/p-1}_{p,1}} \tag{2-30}$$
$$\lesssim\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{3/p}_{p,1}}\|\nabla\widetilde P\|_{\dot B^{3/p-1}_{p,1}}+(1+\|\dot S_m\nabla a\|_{\dot B^{3/p-1/2}_{p,2}})\|\nabla\widetilde P\|_{\dot B^{3/p-3/2}_{p,2}} \tag{2-31}$$
$$\lesssim\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{3/p}_{p,1}}\Big(\frac{1}{\bar a}+\Big\|\frac1a-\frac1{\bar a}\Big\|_{\dot B^{3/p}_{p,1}}\Big)\|a\nabla\widetilde P\|_{\dot B^{3/p-1}_{p,1}} \tag{2-32}$$
$$+\widetilde C(a)\big(1+\|\dot S_m\nabla a\|_{\dot B^{3/p-1/2}_{p,2}}\big)\big(\|\tilde f\|_{\dot B^{3/p-3/2}_{p,2}}+\|a\,\mathrm{div}(bD(\tilde u))\|_{\dot B^{3/p-3/2}_{p,2}}\big), \tag{2-33}$$

where

$$\widetilde C(a)=\Big(\frac1{\bar a}+\Big\|\frac1a-\frac1{\bar a}\Big\|_{\dot B^{3/p}_{p,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar a\|_{\dot B^{3/p}_{p,1}}\Big).$$

We observe that

$$\|a\,\mathrm{div}(bD(\tilde u))\|_{\dot B^{3/p-3/2}_{p,2}}\lesssim(\bar a+\|a-\bar a\|_{\dot B^{3/p}_{p,1}})(\bar b+\|b-\bar b\|_{\dot B^{3/p}_{p,1}})\|\nabla\tilde u\|_{\dot B^{3/p-1/2}_{p,1}}. \tag{2-34}$$

Putting together (2-30)–(2-33) along with (2-34) we get

$$\|\nabla\widetilde P\|_{\dot B^{3/p-3/2}_{p,2}}+\|\mathcal{P}(a\nabla\widetilde P)\|_{\dot B^{3/p-1}_{p,1}}$$
$$\lesssim\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{3/p}_{p,1}}\Big(\frac1{\bar a}+\Big\|\frac1a-\frac1{\bar a}\Big\|_{\dot B^{3/p}_{p,1}}\Big)\|a\nabla\widetilde P\|_{\dot B^{3/p-1}_{p,1}}+\widetilde C(a)(1+\|\dot S_m\nabla a\|_{\dot B^{3/p-1/2}_{p,2}})$$
$$\times\big(\|\tilde f\|_{\dot B^{3/p-3/2}_{p,2}}+(\bar a+\|a-\bar a\|_{\dot B^{3/p}_{p,1}})(\bar b+\|b-\bar b\|_{\dot B^{3/p}_{p,1}})\|\nabla\tilde u\|_{\dot B^{3/p-1/2}_{p,1}}\big). \tag{2-35}$$

Combining (2-29) with (2-35) yields

$$\|\nabla\widetilde{P}\|_{\dot{B}^{3/p-3/2}_{p,2}}+\|a\nabla\widetilde{P}\|_{\dot{B}^{3/p-1}_{p,1}}\lesssim T^1_m(a,b)\|a\nabla\widetilde{P}\|_{\dot{B}^{3/p-1}_{p,1}}+T^2_m(a,b)\|\tilde{f}\|_{\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1}}\\ +T^3_m(a,b)\|\nabla\tilde{u}\|_{\dot{B}^{3/p-1/2}_{p,1}}+T^4_m(a,b)\|\nabla\tilde{u}\|_{\dot{B}^{3/p}_{p,1}},$$

where

$$T^1_m(a,b)=\|(\mathrm{Id}-\dot{S}_m)(a-\bar{a})\|_{\dot{B}^{3/p}_{p,1}}\Big(\frac{1}{\bar{a}}+\Big\|\frac{1}{a}-\frac{1}{\bar{a}}\Big\|_{\dot{B}^{3/p}_{p,1}}\Big),$$
$$T^2_m(a,b)=\widetilde{C}(a)(1+\|\dot{S}_m\nabla a\|_{\dot{B}^{3/p-1/2}_{p,2}}),$$
$$T^3_m(a,b)=\|(\dot{S}_m(a\nabla b),\dot{S}_m(b\nabla a))\|_{\dot{B}^{3/p-1/2}_{p,1}}\\ +\widetilde{C}(a)(1+\|\dot{S}_m\nabla a\|_{\dot{B}^{3/p-1/2}_{p,2}})(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}})(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{3/p}_{p,1}}),$$
$$T^4_m(a,b)=\|(\mathrm{Id}-\dot{S}_m)(a\nabla b,b\nabla a)\|_{\dot{B}^{3/p-1}_{p,1}}+\|(\mathrm{Id}-\dot{S}_m)(ab-\bar{a}\bar{b})\|_{\dot{B}^{3/p}_{p,1}}.$$

Observe that m could be chosen large enough such that $T^1_m(a,b)$ and $T^4_m(a,b)$ can be made arbitrarily small. Thus, there exists a constant C_{ab} depending on a and b such that

$$\|\nabla\widetilde{P}\|_{\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1}}\le C_{ab}\big(\|\tilde{f}\|_{\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1}}+\|\nabla\tilde{u}\|_{\dot{B}^{3/p-1/2}_{p,1}}\big)+\eta\|\nabla\tilde{u}\|_{\dot{B}^{3/p}_{p,1}},\tag{2-36}$$

where η can be made arbitrarily small (of course, with the price of increasing the constant C_{ab}). Let us take a look at the $\dot{B}^{3/p-3/2}_{p,2}$-norm of $\tilde{f}$; we get

$$\|\tilde{f}\|_{\dot{B}^{3/p-3/2}_{p,2}}\le\big\|a\,\mathrm{div}(bD(u_L))-\bar{a}\,\mathrm{div}(\bar{b}D(u_L))\big\|_{\dot{B}^{3/p-3/2}_{p,2}}+\|(a-\bar{a})\nabla P_L\|_{\dot{B}^{3/p-3/2}_{p,2}}\\ \lesssim(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}})(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{3/p}_{p,1}})\|\nabla u_L\|_{\dot{B}^{3/p-1/2}_{p,1}}+\|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}}\|\nabla P_L\|_{\dot{B}^{3/p-3/2}_{p,2}}.\tag{2-37}$$

As $u_L\in C\big([0,\infty),\dot{B}^{3/p-1}_{p,1}\big)\cap L^1\big([0,\infty),\dot{B}^{3/p+1}_{p,1}\big)$ and $\mathcal{Q}$ is a continuous operator on homogeneous Besov spaces from

$$\mathrm{div}(u_L-R)=0,$$

we deduce

$$\mathcal{P}(u_L-R)=u_L-R,$$

which implies

$$\mathcal{Q}u_L=\mathcal{Q}R.$$

By applying the operator $\mathcal{Q}$ in the first equation of system (2-14) we get

$$\bar{a}\nabla P_L=\mathcal{Q}f-\mathcal{Q}\partial_t u_L+\bar{a}\bar{b}\mathcal{Q}\Delta u_L+\bar{a}\bar{b}\nabla\,\mathrm{div}\,R\\ =\mathcal{Q}f-\mathcal{Q}\partial_t R+2\bar{a}\bar{b}\nabla\,\mathrm{div}\,R$$

and thus

$$\|\nabla P_L\|_{\dot{B}^{3/p-3/2}_{p,2}}\le\frac{1}{\bar{a}}\|\mathcal{Q}f\|_{\dot{B}^{3/p-3/2}_{p,2}}+\frac{1}{\bar{a}}\|\partial_t\mathcal{Q}R\|_{\dot{B}^{3/p-3/2}_{p,2}}+2\bar{b}\|\nabla\,\mathrm{div}\,R\|_{\dot{B}^{3/p-3/2}_{p,2}}.$$

In view of (2-36), (2-19), (2-37) and interpolation we gather that there exists a constant C_{ab} such that

$$
\begin{aligned}
&\|\nabla\widetilde{P}\|_{\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1}}\\
&\quad\le C_{ab}\big(\|\nabla u_L\|_{\dot B^{3/p-1/2}_{p,1}}+\|\nabla P_L\|_{\dot B^{3/p-3/2}_{p,2}}+\|(\nabla^2u_L,\nabla P_L)\|_{\dot B^{3/p-1}_{p,1}}+\|\nabla\tilde u\|_{\dot B^{3/p-1/2}_{p,1}}\big)+\eta\|\nabla\tilde u\|_{\dot B^{3/p}_{p,1}} &\text{(2-38)}\\
&\quad\le C_{ab}\|(\mathcal{Q}f,\partial_t\mathcal{Q}R,\nabla\operatorname{div}R)\|_{\dot B^{3/p-3/2}_{p,2}}+C_{ab}\|u_L\|_{\dot B^{3/p-1}_{p,1}} &\text{(2-39)}\\
&\qquad\qquad+C_{ab}\|(\nabla^2u_L,\nabla P_L)\|_{\dot B^{3/p-1}_{p,1}}+C_{ab}\|\tilde u\|_{\dot B^{3/p-1}_{p,1}}+2\eta\|\nabla\tilde u\|_{\dot B^{3/p}_{p,1}}, &\text{(2-40)}
\end{aligned}
$$

where, again, at the price of increasing C_{ab}, we can make η arbitrarily small.

The 2-dimensional case. In this case, using again Proposition 3.21 combined with Propositions 2.4 and 2.6 we get

$$
\begin{aligned}
&\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,2}}+\|\mathcal{P}(a\nabla\widetilde{P})\|_{\dot B^{2/p-1}_{p,1}}\\
&\qquad\lesssim\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,2}}+\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{2/p}_{p,1}}\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}+\|[\mathcal{P},\dot S_m(a-\bar a)]\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}\\
&\qquad\lesssim\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{2/p}_{p,1}}\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}+(1+\|\nabla\dot S_m a\|_{\dot B^{2/p}_{p,2}})\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,2}}\\
&\qquad\lesssim\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{2/p}_{p,1}}\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}\\
&\qquad\qquad+\widetilde{C}(a)(1+\|\nabla\dot S_m a\|_{\dot B^{2/p}_{p,2}})\big(\|\tilde f\|_{\dot B^{2/p-1}_{2,2}}+\|\mathcal{Q}(a\operatorname{div}(bD(\tilde u)))\|_{\dot B^{2/p-1}_{p,2}}\big)
\end{aligned}
$$

where, as before

$$
\widetilde{C}(a)=\Big(\frac{1}{\bar a}+\Big\|\frac{1}{a}-\frac{1}{\bar a}\Big\|_{\dot B^{2/p}_{p,1}}\Big)\Big(1+\frac{1}{a_\star}\|a-\bar a\|_{\dot B^{2/p}_{p,1}}\Big).
$$

As we have already estimated $\|\mathcal{Q}(a\operatorname{div}(bD(\tilde u)))\|_{\dot B^{2/p-1}_{p,2}}$ in (2-29), we gather

$$
\begin{aligned}
\|\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,2}}+\|a\nabla\widetilde{P}\|_{\dot B^{2/p}_{p,1}}&\lesssim T^1_m(a,b)\|a\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}+T^2_m(a,b)\|\tilde f\|_{\dot B^{2/p-1}_{p,1}}\\
&\qquad+T^3_{m,M}(a,b)\|\nabla\tilde u\|_{\dot B^{2/p-1/2}_{p,1}}+T^4_{m,M}(a,b)\|\nabla\tilde u\|_{\dot B^{2/p}_{p,1}}, \qquad\text{(2-41)}
\end{aligned}
$$

where

$$
\begin{aligned}
T^1_m(a,b)&=\|(\mathrm{Id}-\dot S_m)(a-\bar a)\|_{\dot B^{2/p}_{p,1}}\Big(\frac{1}{\bar a}+\Big\|\frac{1}{a}-\frac{1}{\bar a}\Big\|_{\dot B^{2/p}_{p,1}}\Big),\\
T^2_m(a,b)&=\widetilde{C}(a)(1+\|\nabla\dot S_m a\|_{\dot B^{2/p}_{p,2}}),\\
T^3_{m,M}(a,b)&=\|(\dot S_m(a\nabla b),\dot S_m(b\nabla a))\|_{\dot B^{2/p}_{p,1}}+\widetilde{C}(a)(1+\|\nabla\dot S_m a\|_{\dot B^{2/p}_{p,2}})\|(\dot S_M(a\nabla b),\dot S_M(b\nabla a))\|_{\dot B^{2/p}_{p,1}},\\
T^4_{m,M}(a,b)&=\|(\mathrm{Id}-\dot S_m)(a\nabla b)\|_{\dot B^{2/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{2/p}_{p,1}}\\
&\qquad+\widetilde{C}(a)(1+\|\nabla\dot S_m a\|_{\dot B^{2/p}_{p,2}})\big(\|(\mathrm{Id}-\dot S_M)(a\nabla b)\|_{\dot B^{2/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_M)(ab-\bar a\bar b)\|_{\dot B^{2/p-1}_{p,1}}\big).
\end{aligned}
$$

First, we fix an $\eta>0$. Let us fix an $m\in\mathbb{N}$ such that $T^1_m(a,b)\|a\nabla\widetilde{P}\|_{\dot B^{2/p-1}_{p,1}}$ can be "absorbed" by the left-hand side of (2-41) and such that

$$
\|(\mathrm{Id}-\dot S_m)(a\nabla b)\|_{\dot B^{2/p-1}_{p,1}}+\|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{2/p}_{p,1}}\le\tfrac12\eta.
$$

Next, we see that by choosing M large enough we have

$$T^4_{m,M}(a,b) \le \eta.$$

Thus, using interpolation we can write

$$\|\nabla \widetilde{P}\|_{\dot B^{2/p-1}_{p,2}} + \|a\nabla \widetilde{P}\|_{\dot B^{2/p-1}_{p,1}} \le C_{ab}\big(\|(\nabla^2 u_L, \nabla P_L)\|_{\dot B^{2/p-1}_{2,1}} + \|\tilde u\|_{\dot B^{2/p-1}_{p,1}}\big) + 2\eta\|\nabla^2 \tilde u\|_{\dot B^{2/p-1}_{p,1}}. \tag{2-42}$$

End of the proof of Proposition 2.9. Obviously, combining the two estimates (2-38)–(2-40) and (2-42) we can continue in a unified manner the rest of the proof of Proposition 2.9. First, choose $m \in \mathbb{N}$ large enough such that

$$\bar a\bar b + \dot S_m(ab - \bar a\bar b) \ge \tfrac{1}{2} a_\star b_\star.$$

We apply $\dot\Delta_j$ to (2-18) and we write

$$\begin{aligned}
&\partial_t \tilde u_j - \operatorname{div}\big((\bar a\bar b + \dot S_m(ab-\bar a\bar b))\nabla \tilde u_j\big)\\
&\quad = \tilde f_j - \dot\Delta_j(a\nabla\widetilde{P}) + \dot\Delta_j \operatorname{div}\big((\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\nabla\tilde u\big) + \operatorname{div}[\dot\Delta_j, \dot S_m(ab-\bar a\bar b)]\nabla\tilde u\\
&\qquad + \dot\Delta_j(D\tilde u\dot S_m(b\nabla a)) + \dot\Delta_j\big(D\tilde u(\mathrm{Id}-\dot S_m)(b\nabla a)\big) + \dot\Delta_j(\nabla\tilde u\dot S_m(a\nabla b)) + \dot\Delta_j\big(\nabla\tilde u(\mathrm{Id}-\dot S_m)(a\nabla b)\big).
\end{aligned}$$

Multiplying the last relation by $|\tilde u_j|^{p-1}\operatorname{sgn}\tilde u_j$, integrating and using Lemma 8 from Appendix B of [Danchin 2010], we get

$$\begin{aligned}
\|\tilde u_j\|_{L^p} + a_\star b_\star 2^{2j} C\int_0^t \|\tilde u_j\|_{L^p} \lesssim{}& \int_0^t \|\tilde f_j\|_{L^p} + \int_0^t \|\dot\Delta_j(a\nabla\widetilde{P})\|_{L^p} + \int_0^t \big\|\operatorname{div}[\dot\Delta_j, \dot S_m(ab-\bar a\bar b)]\nabla\tilde u\big\|_{L^p}\\
&+ \int_0^t \big\|\dot\Delta_j \operatorname{div}\big((\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\nabla\tilde u\big)\big\|_{L^p}\\
&+ \int_0^t \|\dot\Delta_j(D\tilde u\dot S_m(b\nabla a))\|_{L^p} + \int_0^t \big\|\dot\Delta_j\big(D\tilde u(\mathrm{Id}-\dot S_m)(b\nabla a)\big)\big\|_{L^p}\\
&+ \int_0^t \|\dot\Delta_j(\nabla\tilde u\dot S_m(a\nabla b))\|_{L^p} + \int_0^t \big\|\dot\Delta_j\big(\nabla\tilde u(\mathrm{Id}-\dot S_m)(a\nabla b)\big)\big\|_{L^p}.
\end{aligned}$$

Multiplying the last relation by $2^{j(n/p-1)}$, performing an $\ell^1(\mathbb{Z})$-summation and using Proposition 3.19 to deal with $\big\|\operatorname{div}[\dot\Delta_j, \dot S_m(ab-\bar a\bar b)]\nabla\tilde u\big\|_{\dot B^{n/p-1}_{p,1}}$ along with (2-38)–(2-40) and (2-41) to deal with the pressure, we get

$$\begin{aligned}
&\|\tilde u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + a_\star b_\star C\|\nabla^2\tilde u\|_{L^1_t(\dot B^{n/p-1}_{p,1})}\\
&\quad \lesssim \|\tilde f\|_{L^1_t(\dot B^{n/p-1}_{p,1})} + C\int_0^t \|a\nabla\widetilde{P}\|_{\dot B^{n/p-1}_{p,1}} + \int_0^t \|(\dot S_m(b\nabla a), \dot S_m(a\nabla b))\|_{\dot B^{n/p}_{p,1}}\|\nabla\tilde u\|_{\dot B^{n/p-1}_{p,1}}\\
&\qquad\qquad + T_m(a,b)\|\nabla^2\tilde u\|_{L^1_t(\dot B^{n/p-1}_{p,1})}\\
&\quad \le C_{ab}(1+t)\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap \dot B^{n/p-1}_{p,1})}\big) + C_{ab}\int_0^t \|\tilde u\|_{\dot B^{n/p-1}_{p,1}}\\
&\qquad\qquad + (T_m(a,b)+\eta)\|\nabla^2\tilde u\|_{L^1_t(\dot B^{n/p-1}_{p,1})}
\end{aligned} \tag{2-43}$$

where

$$T_m(a,b) = \|(\mathrm{Id}-\dot S_m)(b\nabla a)\|_{\dot B^{n/p-1}_{p,1}} + \|(\mathrm{Id}-\dot S_m)(a\nabla b)\|_{\dot B^{n/p-1}_{p,1}} + \|(\mathrm{Id}-\dot S_m)(ab-\bar a\bar b)\|_{\dot B^{n/p-1}_{p,1}}. \tag{2-44}$$

Assuming m is large enough and η is small enough, we can "absorb" $(T_m(a,b)+\eta)\|\nabla^2 u\|_{L^1_t(\dot B^{n/p-1}_{p,1})}$ into the left-hand side of (2-43). Thus, we end up with

$$\|\tilde u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + a_\star b_\star \tfrac12 C\|\nabla^2\tilde u\|_{L^1_t(\dot B^{n/p-1}_{p,1})} \le C_{ab}(1+t)\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big) + C_{ab}\int_0^t \|\tilde u\|_{\dot B^{n/p-1}_{p,1}}$$

such that using Grönwall's lemma, (2-15) and the classical inequality

$$1+t^\alpha \le C_\alpha \exp(t)$$

yields

$$\|\tilde u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + a_\star b_\star \tfrac12 C\|\nabla^2\tilde u\|_{L^1_t(\dot B^{n/p-1}_{p,1})} \le C_{ab}\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)\exp(C_{ab}t). \tag{2-45}$$

Using the fact that $u=u_L+\tilde u$ along with (2-15) and (2-45) gives us

$$\|u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + a_\star b_\star \tfrac12 C\|\nabla^2 u\|_{L^1_t(\dot B^{n/p-1}_{p,1})} \le C_{ab}\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)\exp(C_{ab}t). \tag{2-46}$$

Next, using (2-38)–(2-40) and (2-42) combined with (2-15), we infer

$$\begin{aligned}\|\nabla P\|_{L^1_t((\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})} &\le C_a\|a\nabla P_L\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})} + C_a\|a\nabla\widetilde P\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})} &&(2\text{-}47)\\ &\le C_{ab}\big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)\exp(C_{ab}t). &&(2\text{-}48)\end{aligned}$$

Combining (2-48) with (2-46) we finally get

$$\|u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + \|\nabla^2 u\|_{L^1_t(\dot B^{n/p-1}_{p,1})} + \|\nabla P\|_{L^1_t((\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})} \le \big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)\exp(C_{ab}(t+1)). \tag{2-49}$$

Obviously, by obtaining the last estimate we conclude the proof of Proposition 2.9.

Next, let us deal with the existence part of the Stokes problem with the coefficients having regularity as in Proposition 2.9. More precisely, we have:

Proposition 2.12. *Consider (a,b,u_0,f,R) as in the statement of Proposition 2.9. Then, there exists a unique solution $(u,\nabla P)\in E_{\mathrm{loc}}$ of the Stokes system (1-3). Furthermore, there exists a constant C_{ab} depending on a and b such that*

$$\|u\|_{L^\infty_t(\dot B^{n/p-1}_{p,1})} + \|\nabla^2 u\|_{L^1_t(\dot B^{n/p-1}_{p,1})} + \|\nabla P\|_{L^1_t((\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})} \le \big(\|u_0\|_{\dot B^{n/p-1}_{p,1}} + \|(f,\partial_t R,\nabla\operatorname{div} R)\|_{L^1_t(\dot B^{n/p-n/2}_{p,2}\cap\dot B^{n/p-1}_{p,1})}\big)\exp(C_{ab}(t+1)) \tag{2-50}$$

for all $t>0$.

Proof. The uniqueness property is a direct consequence of the estimates of Proposition 2.9. The proof of existence relies on Proposition 2.9 combined with a continuity argument as used in [Danchin 2014]; see also [Krylov 2008]. Let us introduce

$$(a_\theta, b_\theta) = (1-\theta)(\bar{a}, \bar{b}) + \theta(a, b)$$

and consider the Stokes system

$$\begin{cases} \partial_t u - a_\theta(\operatorname{div}(b_\theta D(u)) - \nabla P) = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u_{|t=0} = u_0. \end{cases} \tag{$\mathcal{S}_\theta$}$$

First of all, a more detailed analysis of the estimates established in Proposition 2.9 enables us to conclude that the constant $C_{a_\theta b_\theta}$ appearing in (2-49) is uniformly bounded with respect to $\theta \in [0, 1]$ by a constant $c = c_{ab}$. Indeed, repeating the estimation process carried out in Proposition 2.9 with (a_θ, b_θ) instead of (a, b) amounts to replacing $(a - \bar{a})$ and $(b - \bar{b})$ with $\theta(a - \bar{a})$ and $\theta(b - \bar{b})$. Taking into account Proposition 3.12 and the remark that follows we get that there exists

$$c := \sup_{\theta \in [0,1]} C_{a_\theta b_\theta} < \infty.$$

Let us take $T > 0$ and consider $\mathcal{E}_T$, the set of those $\theta \in [0, 1]$ such that for any (u_0, f, R) as in the statement of Proposition 2.9 problem $(\mathcal{S}_\theta)$ admits a unique solution $(u, \nabla P) \in E_T$ which satisfies

$$\begin{aligned} &\|u\|_{L_t^\infty(\dot{B}^{n/p-1}_{p,1})} + \|\nabla^2 u\|_{L_t^1(\dot{B}^{n/p-1}_{p,1})} + \|\nabla P\|_{L_t^1(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})} \\ &\qquad \le \big(\|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}^{n/p-n/2}_{p,2} \cap \dot{B}^{n/p-1}_{p,1})}\big) \exp(c(t+1)) \end{aligned} \tag{2-51}$$

for all $t \in [0, T]$. According to Proposition 2.7, $0 \in \mathcal{E}_T$.

Suppose $\theta \in \mathcal{E}_T$. First we denote by $(u_\theta, \nabla P_\theta) \in E_T$ the unique solution of $(\mathcal{S}_\theta)$. We consider the space

$$E_{T,\text{div}} = \{(\tilde{w}, \nabla \widetilde{Q}) \in E_T : \operatorname{div} \tilde{w} = 0\}$$

and let $S_{\theta\theta'}$ be the operator which associates to $(\tilde{w}, \nabla \widetilde{Q}) \in E_{T,\text{div}}$, the unique solution $(\tilde{u}, \nabla \widetilde{P})$ of

$$\begin{cases} \partial_t \tilde{u} - a_\theta(\operatorname{div}(b_\theta D(\tilde{u})) - \nabla \widetilde{P}) = g_{\theta\theta'}(u_\theta, \nabla P_\theta) + g_{\theta\theta'}(\tilde{w}, \nabla \widetilde{Q}), \\ \operatorname{div} u = 0, \\ u_{|t=0} = 0, \end{cases} \tag{2-52}$$

where

$$g_{\theta\theta'}(u, \nabla P) = (a_\theta - a_{\theta'})\nabla P + a_{\theta'} \operatorname{div}(b_{\theta'} D(u)) - a_\theta \operatorname{div}(b_\theta D(u)). \tag{2-53}$$

Obviously, $S_{\theta\theta'}$ maps $E_{T,\text{div}}$ into $E_{T,\text{div}}$. We claim that there exists a positive quantity $\varepsilon = \varepsilon(T) > 0$ such that if $|\theta - \theta'| \le \varepsilon(T)$ then $S_{\theta\theta'}$ has a fixed point $(\tilde{u}^\star, \nabla \widetilde{P}^\star)$ in a suitable ball centered at the origin of the space $E_{T,\text{div}}$. Obviously,

$$(\tilde{u}^\star + u_\theta, \nabla \widetilde{P}^\star + \nabla P_\theta)$$

will solve $(\mathcal{S}_{\theta'})$ in E_T.

First, we note that, as a consequence of Proposition 2.9, we have

$$\|(\tilde{u},\nabla\widetilde{P})\|_{E_T} \le \big(\|g_{\theta\theta'}(u_\theta,\nabla P_\theta)\|_{L^1_T(\dot{B}^{n/p-n/2}_{p,2}\cap\dot{B}^{n/p-1}_{p,1})}+\|g_{\theta\theta'}(\tilde{w},\nabla\widetilde{Q})\|_{L^1_T(\dot{B}^{n/p-n/2}_{p,2}\cap\dot{B}^{n/p-1}_{p,1})}\big)\exp(c(T+1)). \tag{2-54}$$

Observe that

$$\|(a_\theta-a_{\theta'})\nabla P\|_{L^1_T(\dot{B}^{n/p-n/2}_{p,2}\cap\dot{B}^{n/p-1}_{p,1})} \le |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}}\|\nabla P\|_{L^1_T(\dot{B}^{n/p-n/2}_{p,2}\cap\dot{B}^{n/p-1}_{p,1})}. \tag{2-55}$$

Next, we write

$$a_{\theta'}\operatorname{div}(b_{\theta'}D(u))-a_\theta\operatorname{div}(b_\theta D(u))=(a_{\theta'}-a_\theta)\operatorname{div}(b_{\theta'}D(u))+a_\theta\operatorname{div}((b_{\theta'}-b_\theta)D(u)).$$

The first term of the last identity is estimated as follows:

$$\|(a_{\theta'}-a_\theta)\operatorname{div}(b_{\theta'}D(u))\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})} \le |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}}(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{n/p}_{p,1}})\|D(u)\|_{L^1_T(\dot{B}^{n/p}_{p,1})}.$$

Regarding the second term, we have

$$\|a_\theta\operatorname{div}((b_{\theta'}-b_\theta)D(u))\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})} \le |\theta-\theta'|\,\|b-\bar{b}\|_{\dot{B}^{n/p}_{p,1}}(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}})\|D(u)\|_{L^1_T(\dot{B}^{n/p}_{p,1})}$$

and thus

$$\big\|a_{\theta'}\operatorname{div}(b_{\theta'}D(u))-a_\theta\operatorname{div}(b_\theta D(u))\big\|_{L^1_T(\dot{B}^{n/p-1}_{p,1})} \le |\theta-\theta'|(\bar{a}+\|a-\bar{a}\|_{\dot{B}^{n/p}_{p,1}})(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{n/p}_{p,1}})\|Du\|_{L^1_T(\dot{B}^{n/p}_{p,1})}. \tag{2-56}$$

The only thing left is to treat the $L^1_t(\dot{B}^{3/p-3/2}_{p,2})$-norm of $a_{\theta'}\operatorname{div}(b_{\theta'}D(u))-a_\theta\operatorname{div}(b_\theta D(u))$ in the case where $n=3$. Using the fact that $\nabla u\in L^{4/3}_T(\dot{B}^{3/p-1/2}_{p,1})$, we can write

$$\begin{aligned}
\|(a_{\theta'}-a_\theta)\operatorname{div}(b_{\theta'}D(u))\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2})}
&\le |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}}\|\operatorname{div}(b_{\theta'}D(u))\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,1})}\\
&\le |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}}(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{3/p}_{p,1}})\|Du\|_{L^1_T(\dot{B}^{3/p-1/2}_{p,1})} &(2\text{-}57)\\
&\le |\theta-\theta'|\,\|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}}(\bar{b}+\|b-\bar{b}\|_{\dot{B}^{3/p}_{p,1}})T^{1/4}\|u\|^{1/4}_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})}\|u\|^{3/4}_{L^1_T(\dot{B}^{3/p+1}_{p,1})} &(2\text{-}58)\\
&\le |\theta-\theta'|C(T,a,b)\big(\|u\|_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})}+\|\nabla^2u\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}\big) &(2\text{-}59)
\end{aligned}$$

and, proceeding in a similar manner, we can estimate $\big\|a_\theta\operatorname{div}\big((b_{\theta'}-b_\theta)D(u)\big)\big\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2})}$.

Combining (2-55), (2-56) along with (2-59) we get

$$\|g_{\theta\theta'}(u,\nabla P)\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1})} \le |\theta-\theta'|C(T,a,b)\big(\|u\|_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})}+\|\nabla^2u\|_{L^1_T(\dot{B}^{3/p+1}_{p,1})}+\|\nabla P\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1})}\big). \tag{2-60}$$

Substituting this into (2-54), we get

$$\|(\tilde{u},\nabla\widetilde{P})\|_{E_T}\le|\theta-\theta'|C(T,a,b)\big(\|(u_\theta,\nabla P_\theta)\|_{E_T}+\|(\tilde{w},\nabla\widetilde{Q})\|_{E_T}\big),$$

and by linearity

$$\left\|(\tilde{u}^1-\tilde{u}^2,\nabla\widetilde{P}^1-\nabla\widetilde{P}^2)\right\|_{E_T}\le|\theta-\theta'|C(T,a,b)\left\|(\tilde{w}^1-\tilde{w}^2,\nabla\widetilde{Q}^1-\nabla\widetilde{Q}^2)\right\|_{E_T},$$

where for $k=1,2$,

$$(\tilde{u}^i,\nabla\widetilde{P}^i)=S_{\theta\theta'}((\tilde{w}^i,\nabla\widetilde{Q}^i)).$$

Thus one can choose $\varepsilon(T)$ small enough such that $|\theta-\theta'|\le\varepsilon(T)$ gives us a fixed point of the solution operator $S_{\theta\theta'}$ in $B_{E_{T,\mathrm{div}}}(0,2\|(u_\theta,\nabla P_\theta)\|_{E_T})$.

Thus, for all $T>0$, we have $E_T=[0,1]$ and owing to the uniqueness property and to Proposition 2.9, we can construct a unique solution $(u,\nabla P)\in E_{\mathrm{loc}}$ to (1-3) such that for all $t>0$ the estimate (2-11) is valid. □

The proof of Theorem 1.3 in the case $n=3$. As discussed earlier, in dimension $n=3$, Proposition 2.9 is weaker than Theorem 1.3, as one requires additional low-frequency information on the data

$$(f,\partial_t R,\nabla\operatorname{div}R)\in L^1_t(\dot{B}^{3/p-3/2}_{p,2}).$$

Thus, we have to bring an extra argument in order to conclude the validity of Theorem 1.3. This is the object of interest of this section.

Existence. We begin by taking $m\in\mathbb{N}$ large enough and owing to Proposition 2.8 we can consider $(u^1,\nabla P^1)$, the unique solution with $u^1\in\mathcal{C}(\mathbb{R}^+;\dot{B}^{3/p-1}_{p,1})$ and $(\partial_t u^1,\nabla^2u^1,\nabla P^1)\in L^1_{\mathrm{loc}}(\dot{B}^{3/p-1}_{p,1})$ of the system

$$\begin{cases}\partial_t u-\bar{a}\bar{b}\operatorname{div}D(u)+(\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P=f,\\ \operatorname{div}u=\operatorname{div}R,\\ u_{|t=0}=u_0,\end{cases}$$

which also satisfies

$$\|u^1\|_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})}+\|(\partial_t u^1,\bar{a}\bar{b}\nabla^2u^1,\bar{a}\nabla P^1)\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}\\ \le C\big(\|u_0\|_{\dot{B}^{3/p-1}_{p,1}}+\|(f,\partial_t R,\bar{a}\bar{b}\nabla\operatorname{div}R)\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}\big)$$

for all $T>0$. Let us consider

$$G(u^1,\nabla P^1)=a\operatorname{div}(bD(u^1))-\bar{a}\operatorname{div}(\bar{b}D(u^1))-((\mathrm{Id}-\dot{S}_{-m})(a-\bar{a}))\nabla P^1.$$

We claim $G(u^1,\nabla P^1)\in L^1_{\mathrm{loc}}(\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1})$. Indeed

$$a\operatorname{div}(bD(u^1))-\bar{a}\operatorname{div}(\bar{b}D(u^1))=(a-\bar{a})\operatorname{div}(bD(u^1))+\bar{a}\operatorname{div}((b-\bar{b})D(u^1))$$

and proceeding as in (2-56) and (2-58) we get

$$\begin{aligned}\left\|a\operatorname{div}(bD(u^1))-\bar{a}\operatorname{div}(\bar{b}D(u^1))\right\|_{L^1_t(\dot{B}^{3/p-3/2}_{p,2}\cap\dot{B}^{3/p-1}_{p,1})}&\\ \le C_{ab}(1+t^{1/4})\big(\|u^1\|_{L^\infty_t(\dot{B}^{3/p-1}_{p,1})}+\|u^1\|_{L^1_t(\dot{B}^{3/p+1}_{p,1})}\big)&\\ \le\exp(C_{ab}(t+1))\big(\|u_0\|_{\dot{B}^{3/p-1}_{p,1}}+\|(f,\partial_t R,\nabla\operatorname{div}R)\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}\big).&\quad(2\text{-}61)\end{aligned}$$

Next, we obviously have

$$\big\|\big((\mathrm{Id}-\dot S_{-m})(a-\bar a)\big)\nabla P^1\big\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\le C\|(a-\bar a)\|_{\dot B^{3/p}_{p,1}}\|\nabla P^1\|_{L^1_t(\dot B^{3/p-1}_{p,1})}. \tag{2-62}$$

Using the fact that the product maps $\dot B^{3/p-1/2}_{p,1}\times\dot B^{3/p-1}_{p,1}\to\dot B^{3/p-3/2}_{p,2}$, we get

$$\big\|\big((\mathrm{Id}-\dot S_{-m})(a-\bar a)\big)\nabla P^1\big\|_{L^1_t(\dot B^{3/p-3/2}_{p,2})}\le C\|(\mathrm{Id}-\dot S_{-m})(a-\bar a)\|_{\dot B^{3/p-1/2}_{p,1}}\|\nabla P^1\|_{L^1_t(\dot B^{3/p-1}_{p,1})}. \tag{2-63}$$

Of course

$$\begin{aligned}\|(\mathrm{Id}-\dot S_{-m})(a-\bar a)\|_{\dot B^{3/p-1/2}_{p,1}}&\le C\sum_{j\ge -m}2^{j(3/p-1/2)}\|\dot\Delta_j(a-\bar a)\|_{L^2}\le C2^{m/2}\sum_{j\ge -m}2^{3/pj}\|\dot\Delta_j(a-\bar a)\|_{L^2}.\\&\le C2^{m/2}\|a-\bar a\|_{\dot B^{3/p}_{p,1}}\end{aligned}$$

so that the first term on the right-hand side of (2-63) is finite. We thus gather from (2-61), (2-62) and (2-63) that $G(u^1,\nabla P^1)\in L^1_{\mathrm{loc}}(\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1})$ and that for all $t>0$ there exists a constant C_{ab} such that

$$\|G(u^1,\nabla P^1)\|_{L^1_t(\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1})}\le(\|u_0\|_{\dot B^{3/p-1}_{p,1}}+\|(f,\partial_tR,\nabla\,\mathrm{div}\,R)\|_{L^1_t(\dot B^{3/p-1}_{p,1})})\exp(C_{ab}(t+1)).$$

According to Proposition 2.12, there exists a unique solution $(u^2,\nabla P^2)\in E_{\mathrm{loc}}$ of the system

$$\begin{cases}\partial_tu-a\,\mathrm{div}(bD(u))+a\nabla P=G(u^1,\nabla P^1),\\ \mathrm{div}\,u=0,\\ u_{|t=0}=0,\end{cases}$$

which satisfies the estimate

$$\begin{aligned}&\|u^2\|_{L^\infty_t(\dot B^{3/p-1}_{p,1})}+\|(\nabla^2u^2,\nabla P^2)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\\&\qquad\le\|G(u^1,\nabla P^1)\|_{L^1_t(\dot B^{3/p-3/2}_{p,2}\cap\dot B^{3/p-1}_{p,1})}\exp(C_{ab}(t+1))\\&\qquad\le\big(\|u_0\|_{\dot B^{3/p-1}_{p,1}}+\|(f,\partial_tR,\nabla\,\mathrm{div}\,R)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\big)\exp(C_{ab}(t+1)).\end{aligned}$$

We observe that

$$(u,\nabla P):=(u^1+u^2,\nabla P^1+\nabla P^2)$$

is a solution of (1-3) which satisfies

$$\begin{aligned}&\|u\|_{L^\infty_t(\dot B^{3/p-1}_{p,1})}+\|(\nabla^2u,\nabla P)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\\&\qquad\le\big(\|u_0\|_{\dot B^{3/p-1}_{p,1}}+\|(f,\partial_tR,\nabla\,\mathrm{div}\,R)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\big)\exp(C_{ab}(t+1)).\end{aligned} \tag{2-64}$$

Of course, using again the first equation of (1-3) we get

$$\|\partial_tu\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\le C_{ab}\|(f,\nabla^2u,\nabla P)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}$$

and thus, we get the estimate

$$\begin{aligned}&\|u\|_{L^\infty_t(\dot B^{3/p-1}_{p,1})}+\|(\partial_tu,\nabla^2u,\nabla P)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\\&\qquad\le\big(\|u_0\|_{\dot B^{3/p-1}_{p,1}}+\|(f,\partial_tR,\nabla\,\mathrm{div}\,R)\|_{L^1_t(\dot B^{3/p-1}_{p,1})}\big)\exp(C_{ab}(t+1)).\end{aligned} \tag{2-65}$$

Uniqueness. Next, let us prove the uniqueness property. Let us suppose there exists a $T > 0$ and a pair $(u, \nabla P)$ that solves

$$\begin{cases} \partial_t u - a\operatorname{div}(bD(u)) + a\nabla P = 0, \\ \operatorname{div} u = 0, \\ u_{|t=0} = 0, \end{cases} \tag{2-66}$$

with

$$u \in C_T(\dot{B}^{3/p-1}_{p,1}) \text{ and } (\partial_t u, \nabla^2 u, \nabla P) \in L^1_T(\dot{B}^{3/p-1}_{p,1}).$$

Observe that we cannot directly conclude to the uniqueness property by appealing to Proposition 2.12 because the pressure does not belong (a priori) to $L^1_T(\dot{B}^{3/p-3/2}_{p,2})$. Recovering this low-frequency information is done in the following lines. Suppose $3 < p < 4$. Applying the operator $\mathcal{Q}$ in the first equation of (2-66) we can write

$$\mathcal{Q}\big((\bar{a} + \dot{S}_{-m}(a-\bar{a}))\nabla P\big) = \mathcal{Q}\big(a\operatorname{div}(bD(u))\big) - \mathcal{Q}\big((\mathrm{Id} - \dot{S}_{-m})(a-\bar{a})\nabla P\big),$$

where $m \in \mathbb{N}$ will be fixed later. We observe that

$$\begin{aligned} &\big\|\mathcal{Q}\big((\bar{a} + \dot{S}_{-m}(a-\bar{a}))\nabla P\big)\big\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,1})} \\ &\quad\lesssim \big\|\mathcal{Q}\big(a\operatorname{div}(bD(u))\big)\big\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,1})} + \big\|\mathcal{Q}\big((\mathrm{Id} - \dot{S}_{-m})(a-\bar{a})\nabla P\big)\big\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,1})} \\ &\quad\lesssim T^{1/4}(\bar{a} + \|a-\bar{a}\|_{\dot{B}^{3/p}_{p,1}})(\bar{b} + \|b-\bar{b}\|_{\dot{B}^{3/p}_{p,1}})\|\nabla u\|_{L^{4/3}_T(\dot{B}^{3/p-1/2}_{p,1})} \\ &\qquad\qquad + \|(\mathrm{Id} - \dot{S}_{-m})(a-\bar{a})\|_{\dot{B}^{3/p-1/2}_{p,1}}\|\nabla P\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}. \end{aligned}$$

Consequently, we get

$$\mathcal{Q}\big((\bar{a} + \dot{S}_{-m}(a-\bar{a}))\nabla P\big) \in L^1_T(\dot{B}^{3/p-3/2}_{p,1}). \tag{2-67}$$

Let us observe that the condition $p \in (3,4)$ ensures that $\dot{B}^{3/p}_{p,1}$ is contained in the multiplier space of $\dot{B}^{-3/p+1}_{p',2} = \dot{B}^{3/p'-2}_{p',2}$. More precisely, we get:

Proposition 2.13. *Consider $p \in (3,4)$ and $(u,v) \in \dot{B}^{3/p}_{p,1} \times \dot{B}^{-3/p+1}_{p',2}$. Then $uv \in \dot{B}^{-3/p+1}_{p',2}$ and*

$$\|uv\|_{\dot{B}^{-3/p+1}_{p',2}} \lesssim \|u\|_{\dot{B}^{3/p}_{p,1}}\|v\|_{\dot{B}^{-3/p+1}_{p',2}}.$$

Proof. Indeed, considering $(u,v) \in \dot{B}^{3/p}_{p,1} \times \dot{B}^{-3/p+1}_{p',2}$ and using the Bony decomposition we get

$$\|\dot{T}_u v\|_{\dot{B}^{-3/p+1}_{p',2}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot{B}^{-3/p+1}_{p',2}}.$$

Next, considering

$$\frac{1}{p'} = \frac{1}{2} + \frac{1}{p^\star},$$

we see

$$\begin{aligned} 2^{j(-3/p+1)}\|\dot{\Delta}_j \dot{T}'_v u\|_{L^{p'}} &\lesssim \sum_{\ell \geq j-3} 2^{(-3/p+1)(j-\ell)} 2^{(-3/p+1)\ell}\|S_{\ell+1}v\|_{L^2}\|\dot{\Delta}_\ell u\|_{L^{p^\star}} \\ &= \sum_{\ell \geq j-3} 2^{(-3/p+1)(j-\ell)} 2^{-1/2\ell}\|S_{\ell+1}v\|_{L^2} 2^{3/p^\star \ell}\|\dot{\Delta}_\ell u\|_{L^{p^\star}} \end{aligned}$$

such that, with the help of Proposition 3.10, we get

$$\|\dot{T}'_v u\|_{\dot{B}^{-3/p+1}_{p',2}} \lesssim \|v\|_{\dot{H}^{-1/2}} \|u\|_{\dot{B}^{3/p^\star}_{p^\star,1}} \lesssim \|v\|_{\dot{B}^{-3/p+1}_{p',2}} \|u\|_{\dot{B}^{3/p}_{p,1}}. \qquad \square$$

Proposition 2.14. *Consider $p \in (3,4)$. Furthermore, consider a constant $\bar{c} > 0$ and $c \in \dot{B}^{3/p}_{p,1}$. Then there exists a universal constant $\eta > 0$ such that if*

$$\|c\|_{\dot{B}^{3/p}_{p,1}} \le \eta,$$

then for any $\psi \in \dot{B}^{3/p'-3/2}_{p',2} \cap \dot{B}^{3/p'-2}_{p',2}$ there exists a unique solution $\nabla P \in \dot{B}^{3/p'-3/2}_{p',2} \cap \dot{B}^{3/p'-2}_{p',2}$ of the elliptic equation

$$\operatorname{div}((\bar{c}+c)\nabla P) = \operatorname{div}\psi.$$

Moreover, the following estimate holds true:

$$\|\nabla P\|_{\dot{B}^{3/p'-\sigma}_{p',2}} \lesssim \|\mathcal{Q}\psi\|_{\dot{B}^{3/p'-\sigma}_{p',2}},$$

where $\sigma \in \{\frac{3}{2}, 2\}$.

Proof. The proof is standard. Under some smallness condition on $c \in \dot{B}^{3/p}_{p,1}$, the operator

$$\nabla R \to \nabla P = \frac{1}{\bar{c}}\mathcal{Q}(\psi - c\nabla R)$$

has a fixed point in a suitable chosen ball of the space $\dot{B}^{3/p'-3/2}_{p',2} \cap \dot{B}^{3/p'-2}_{p',2}$. $\square$

Choose $m \in \mathbb{N}$ such that $\|\dot{S}_{-m}(a-\bar{a})\|_{\dot{B}^{3/p}_{p,1}}$ is small enough that we can apply Proposition 2.14 with $\bar{a}$ and $\dot{S}_{-m}(a-\bar{a})$ instead of $\bar{c}$ and c, and we consider ψ a vector field with coefficients in $\mathcal{S}$. As the Schwartz class is included in $\dot{B}^{3/p'-3/2}_{p',2} \cap \dot{B}^{3/p'-2}_{p',2}$, let us consider $\nabla P_\psi \in \dot{B}^{3/p'-3/2}_{p',2} \cap \dot{B}^{3/p'-2}_{p',2}$, the solution of the equation

$$\operatorname{div}\big((\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P_\psi\big) = \operatorname{div}\psi,$$

the existence of which is granted by Proposition 2.14. Then, using Propositions 3.8 and 3.9, we can write[3]

$$\begin{aligned}
&\langle \nabla P, \psi\rangle_{\mathcal{S}'\times\mathcal{S}} \\
&= \sum_j \langle \dot{\Delta}_j \nabla P, \tilde{\Delta}_j \psi\rangle = \sum_j -\langle \dot{\Delta}_j P, \tilde{\Delta}_j \operatorname{div}\psi\rangle && \text{(2-68)}\\
&= \sum_j -\big\langle \dot{\Delta}_j P, \tilde{\Delta}_j \operatorname{div}\big((\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P_\psi\big)\big\rangle = \sum_j \big\langle \dot{\Delta}_j \nabla P, \tilde{\Delta}_j\big((\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P_\psi\big)\big\rangle && \text{(2-69)}\\
&= \sum_j \big\langle \dot{\Delta}_j\big(\bar{a}+\dot{S}_{-m}(a-\bar{a})\big)\nabla P, \tilde{\Delta}_j \nabla P_\psi\big\rangle = \sum_j \big\langle \dot{\Delta}_j \mathcal{Q}\big((\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P\big), \tilde{\Delta}_j \nabla P_\psi\big\rangle && \text{(2-70)}\\
&\lesssim \big\|\mathcal{Q}\big((\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P\big)\big\|_{\dot{B}^{3/p-3/2}_{p,2}} \|\nabla P_\psi\|_{\dot{B}^{3/p'-3/2}_{p',1}} && \text{(2-71)}\\
&\lesssim \big\|\mathcal{Q}\big((\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P\big)\big\|_{\dot{B}^{3/p-3/2}_{p,2}} \|\psi\|_{\dot{B}^{3/p'-3/2}_{p',1}}. && \text{(2-72)}
\end{aligned}$$

[3] We define $\tilde{\Delta}_j := \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}$.

Taking the supremum over all $\psi \in \mathcal{S}$ with $\|\psi\|_{\dot{B}^{3/p'-3/2}_{p',2}} \le 1$, by (2-67) and Proposition 3.8, it follows that $\nabla P \in L^1_T(\dot{B}^{3/p-3/2}_{p,2})$ and that

$$\|\nabla P\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2})} \lesssim \left\|\mathcal{Q}\big((\bar{a}+\dot{S}_{-m}(a-\bar{a}))\nabla P\big)\right\|_{L^1_T(\dot{B}^{3/p-3/2}_{p,2})}.$$

According to the uniqueness property of Proposition 2.12 we conclude that $(u, \nabla P) = (0, 0)$.

Observe that in the case $p \in \left(\frac{6}{5}, 3\right]$, owing to the fact that $\dot{B}^{3/p-1}_{p,1} \hookrightarrow \dot{B}^{3/q-1}_{q,1}$ for any $q \in (3, 4)$ and $u \in C_T(\dot{B}^{3/p-1}_{p,1})$ along with $(\partial_t u, \nabla^2 u, \nabla P) \in L^1_T(\dot{B}^{3/p-1}_{p,1})$, we get $u \in C_T(\dot{B}^{3/q-1}_{q,1})$ along with $(\partial_t u, \nabla^2 u, \nabla P) \in L^1_T(\dot{B}^{3/q-1}_{q,1})$. Thus, by the uniqueness property for the case $q \in (3, 4)$, we conclude that $(u, \nabla P)$ is identically null for $p \in \left(\frac{6}{5}, 3\right]$.

3. Proof of Theorem 1.2

In the rest of the paper we aim to prove Theorem 1.2. Thus, from now on we will work in a 3-dimensional framework.

The linear theory. Let us fix some notation. The space $\widetilde{F}_T$ consists of $(\tilde{w}, \nabla\widetilde{Q})$ with $\tilde{w} \in \mathcal{C}_T(\dot{B}^{3/p-1}_{p,1})$ and $(\partial_t \tilde{w}, \nabla^2 \tilde{w}, \nabla\widetilde{Q}) \in L^1_T(\dot{B}^{3/p-1}_{p,1})$ with the norm

$$\|(\tilde{w}, \nabla\widetilde{Q})\|_{\widetilde{F}_T} = \|\tilde{w}\|_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})} + \|(\partial_t \tilde{w}, \nabla^2 \tilde{w}, \nabla\widetilde{Q})\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}. \tag{3-1}$$

For any time-dependent vector field $\bar{v}$ we define

$$X_{\bar{v}}(t, x) = x + \int_0^t \bar{v}(\tau, x)\, d\tau,$$

and $A_{\bar{v}} = (DX_{\bar{v}})^{-1}$. Also, let us denote by $\operatorname{adj}(DX_{\bar{v}})$ the adjugate matrix (i.e., the transpose of the cofactor matrix) of $DX_{\bar{v}}$ and $J_{\bar{v}} = \det(DX_{\bar{v}})$.

Before attacking the well-posedness of (1-4), we first have to solve the linear system

$$\begin{cases} \rho_0 \partial_t \bar{u} - \operatorname{div}\big(\mu(\rho_0) A_{\bar{v}} D_{A_{\bar{v}}}(\bar{u})\big) + A^T_{\bar{v}} \nabla \bar{P} = 0, \\ \operatorname{div}(\operatorname{adj}(DX_{\bar{v}})\bar{u}) = 0, \\ \bar{u}_{|t=0} = u_0, \end{cases} \tag{3-2}$$

where $\bar{v} \in \mathcal{C}_T(\dot{B}^{3/p-1}_{p,1})$ with $\nabla\bar{v} \in L^1_T(\dot{B}^{3/p}_{p,1}) \cap L^2_T(\dot{B}^{3/p-1}_{p,1})$ is such that

$$\|\nabla\bar{v}\|_{L^2_T(\dot{B}^{3/p-1}_{p,1})} + \|\nabla\bar{v}\|_{L^1_T(\dot{B}^{3/p}_{p,1})} \le 2\alpha \tag{3-3}$$

for a suitably small α. Obviously, this will be achieved using the estimates of the Stokes system established in the previous section; see Theorem 1.3. Let us write (3-2) in the form

$$\begin{cases} \partial_t \bar{u} - \frac{1}{\rho_0} \operatorname{div}\big(\mu(\rho_0) D(\bar{u})\big) + \frac{1}{\rho_0} \nabla \bar{P} = \frac{1}{\rho_0} F_{\bar{v}}(\bar{u}, \nabla \bar{P}), \\ \operatorname{div} \bar{u} = \operatorname{div}\big((\operatorname{Id} - \operatorname{adj}(DX_{\bar{v}}))\bar{u}\big), \\ \bar{u}_{t=0} = u_0, \end{cases}$$

with

$$F_{\bar{v}}(\bar{w},\nabla\bar{Q}) := \operatorname{div}\big(\mu(\rho_0)A_{\bar{v}}D_{A_{\bar{v}}}(\bar{w}) - \mu(\rho_0)D(\bar{w})\big) + (\mathrm{Id} - A_{\bar{v}}^T)\nabla\bar{Q}.$$

Consider $(u_L,\nabla P_L)$ with $u_L \in \mathcal{C}(\mathbb{R}^+,\dot{B}^{3/p-1}_{p,1})$ and $(\partial_t u_L,\nabla^2 u_L,\nabla P_L) \in L^1_{\mathrm{loc}}(\dot{B}^{3/p-1}_{p,1})$, the unique solution of

$$\begin{cases} \partial_t u_L - \frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0)D(u_L)\big) + \frac{1}{\rho_0}\nabla P_L = 0, \\ \operatorname{div} u_L = 0, \\ u_{L|t=0} = u_0, \end{cases} \tag{3-4}$$

for which we know that

$$\|(u_L,\nabla P_L)\|_{E_T} \le \|u_0\|_{\dot{B}^{3/p-1}_{p,1}} \exp(C_{\rho_0}(T+1)).$$

Moreover, T can be chosen small enough such that

$$\|\nabla u_L\|_{L^2_T(\dot{B}^{3/p-1}_{p,1})} + \|(\partial_t u_L,\nabla^2 u_L,\nabla P_L)\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \le \alpha. \tag{3-5}$$

Following the idea in [Danchin and Mucha 2012], and owing to Theorem 1.3, we consider the operator

$$\Phi(\tilde{w},\nabla\widetilde{Q}) = (\tilde{u},\nabla\widetilde{P}), \tag{3-6}$$

which associates to $(\tilde{w},\nabla\widetilde{Q}) \in \widetilde{F}_T$ the unique solution $(\tilde{u},\nabla\widetilde{P}) \in \widetilde{F}_T$ of

$$\begin{cases} \partial_t \tilde{u} - \frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0)D(\tilde{u})\big) + \frac{1}{\rho_0}\nabla\widetilde{P} = \frac{1}{\rho_0}F_{\bar{v}}(u_L+\tilde{w},\nabla P_L+\nabla\widetilde{Q}), \\ \operatorname{div}\tilde{u} = \operatorname{div}\big((\mathrm{Id} - \operatorname{adj}(DX_{\bar{v}}))(u_L+\tilde{w})\big), \\ \tilde{u}_{|t=0} = 0. \end{cases}$$

We will show in the following that for any $R>0$ there exists a sufficiently small $T>0$ such that there exists a fixed point for Φ in the ball of radius R centered at the origin of $\widetilde{F}_T$. More precisely, according to Theorem 1.3 we get

$$\begin{aligned} \|\Phi(\tilde{w},\nabla\widetilde{Q})\|_{\widetilde{F}_T} \le{}& \Big\|\frac{1}{\rho_0}F_{\bar{v}}(u_L+\tilde{w},\nabla P_L+\nabla\widetilde{Q})\Big\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \\ &+ \|\partial_t(\mathrm{Id}-\operatorname{adj}(DX_{\bar{v}}))(u_L+\tilde{w})\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \\ &\quad+ \big\|\nabla\operatorname{div}\big((\mathrm{Id}-\operatorname{adj}(DX_{\bar{v}}))(u_L+\tilde{w})\big)\big\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}. \end{aligned} \tag{3-7}$$

We begin by treating the first term:

$$\begin{aligned} &\Big\|\frac{1}{\rho_0}F_{\bar{v}}(u_L+\tilde{w},\nabla P_L+\nabla\widetilde{Q})\Big\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \\ &\qquad\lesssim \Big(\frac{1}{\bar{\rho}} + \Big\|\frac{1}{\rho_0}-\frac{1}{\bar{\rho}}\Big\|_{\dot{B}^{3/p}_{p,1}}\Big)\|F_{\bar{v}}(u_L+\tilde{w},\nabla P_L+\nabla\widetilde{Q})\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}. \end{aligned} \tag{3-8}$$

We write

$$
\begin{aligned}
T_1 &= \operatorname{div}\big(\mu(\rho_0)A_{\bar v}D_{A_{\bar v}}(u_L+\tilde w)\big)-\operatorname{div}\big(\mu(\rho_0)D(u_L+\tilde w)\big)\\
&= \operatorname{div}\big(\mu(\rho_0)(A_{\bar v}-\mathrm{Id})D_{A_{\bar v}}(u_L+\tilde w)\big)+\operatorname{div}\big(\mu(\rho_0)D_{A_{\bar v}-\mathrm{Id}}(u_L+\tilde w)\big)\\
&= \operatorname{div}\big(\mu(\rho_0)(A_{\bar v}-\mathrm{Id})D_{A_{\bar v}-\mathrm{Id}}(u_L+\tilde w)\big)+\operatorname{div}\big(\mu(\rho_0)(A_{\bar v}-\mathrm{Id})D(u_L+\tilde w)\big)+\operatorname{div}\big(\mu(\rho_0)D_{A_{\bar v}-\mathrm{Id}}(u_L+\tilde w)\big).
\end{aligned}
$$

Thus, using (3-22) of Proposition 3.27 along with product laws in Besov spaces (see Proposition 3.17) we get the following bound for T_1:

$$
\begin{aligned}
\|T_1\|_{L^1_T(\dot B^{3/p-1}_{p,1})} &\lesssim C_{\rho_0}\|A_{\bar v}-\mathrm{Id}\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\big(1+\|A_{\bar v}-\mathrm{Id}\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\big)\big(\|\nabla u_L\|_{L^1_T(\dot B^{3/p}_{p,1})}+\|\nabla\tilde w\|_{L^1_T(\dot B^{3/p}_{p,1})}\big)\\
&\lesssim C_{\rho_0}\|\nabla\bar v\|_{L^1_T(\dot B^{3/p}_{p,1})}\big(1+\|\nabla\bar v\|_{L^1_T(\dot B^{3/p}_{p,1})}\big)\big(\|\nabla u_L\|_{L^1_T(\dot B^{3/p}_{p,1})}+\|\nabla\tilde w\|_{L^1_T(\dot B^{3/p}_{p,1})}\big)\\
&\lesssim C_{\rho_0}\alpha\big(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}\big). \qquad (3\text{-}9)
\end{aligned}
$$

The second term is estimated as

$$
\begin{aligned}
\|(\mathrm{Id}-A^T_{\bar v})(\nabla P_L+\nabla\widetilde Q)\|_{L^1_T(\dot B^{3/p-1}_{p,1})} &\lesssim \|\nabla\bar v\|_{L^1_T(\dot B^{3/p}_{p,1})}\big(\|\nabla P_L\|_{L^1_T(\dot B^{3/p-1}_{p,1})}+\|\nabla\widetilde Q\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\big)\\
&\lesssim \alpha(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}) \qquad (3\text{-}10)
\end{aligned}
$$

so that combining (3-8), (3-9) and (3-10) we get

$$
\Big\|\frac{1}{\rho_0}F_{\bar v}(u_L+\tilde w,\nabla P_L+\nabla\widetilde Q)\Big\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\lesssim C_{\rho_0}\alpha(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}). \qquad (3\text{-}11)
$$

In order to treat the second term of (3-7) we use the estimates (3-23) and (3-24) of Proposition 3.27 along with Hölder's inequality in order to obtain

$$
\begin{aligned}
&\|\partial_t(\mathrm{Id}-\operatorname{adj}(DX_{\bar v}))(u_L+\tilde w)\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\
&\quad\lesssim \|(u_L+\tilde w)\partial_t\operatorname{adj}(DX_{\bar v})\|_{L^1_T(\dot B^{3/p-1}_{p,1})}+\|(\mathrm{Id}-\operatorname{adj}(DX_{\bar v}))(\partial_t u_L+\partial_t\tilde w)\|_{L^1_T(\dot B^{3/p-1}_{p,1})} \qquad (3\text{-}12)\\
&\quad\lesssim \|\partial_t\operatorname{adj}(DX_{\bar v})\|_{L^2_T(\dot B^{3/p-1}_{p,1})}\|u_L+\tilde w\|_{L^2_T(\dot B^{3/p}_{p,1})}+\|\mathrm{Id}-\operatorname{adj}(DX_{\bar v})\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\|\partial_t u_L+\partial_t\tilde w\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\
&\quad\lesssim \|\nabla\bar v\|_{L^2_T(\dot B^{3/p-1}_{p,1})}\big(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}\big)+\alpha(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T})\\
&\quad\lesssim \alpha(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}). \qquad (3\text{-}13)
\end{aligned}
$$

Treating the last term of (3-7) is done with the aid of Corollary 3.24:

$$
\begin{aligned}
\operatorname{div}((\mathrm{Id}-\operatorname{adj}(DX_{\bar v}))(u_L+\tilde w)) &= (Du_L+D\tilde w):(\mathrm{Id}-J_{\bar v}A_{\bar v})\\
&= J_{\bar v}(Du_L+D\tilde w):(\mathrm{Id}-A_{\bar v})+(1-J_{\bar v})(\operatorname{div}u_L+\operatorname{div}\tilde w).
\end{aligned}
$$

Thus, using the estimates (3-22) and (3-26) of Proposition 3.27, we may write

$$\begin{aligned}
&\big\|\nabla \operatorname{div}\big((\mathrm{Id}-\operatorname{adj}(DX_{\bar v}))(u_L+\tilde w)\big)\big\|_{L^1_T(\dot B^{3/p-1}_{p,1})}\\
&\qquad\lesssim \big\|J_{\bar v}(Du_L+D\tilde w):(\mathrm{Id}-A_{\bar v})\big\|_{L^1_T(\dot B^{3/p}_{p,1})}+\big\|(1-J_{\bar v})(\operatorname{div} u_L+\operatorname{div}\tilde w)\big\|_{L^1_T(\dot B^{3/p}_{p,1})}\\
&\qquad\lesssim \big(1+\|J_{\bar v}-1\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\big)\|\mathrm{Id}-A_{\bar v}\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\|Du_L+D\tilde w\|_{L^1_T(\dot B^{3/p}_{p,1})}\\
&\qquad+\|(J_{\bar v}-1)\|_{L^\infty_T(\dot B^{3/p}_{p,1})}\|\operatorname{div} u_L+\operatorname{div}\tilde w\|_{L^1_T(\dot B^{3/p}_{p,1})}\\
&\qquad\lesssim \alpha(1+\alpha)(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}).
\end{aligned}\tag{3-14}$$

Combining the estimates (3-11), (3-13) and (3-14) we get

$$\|\Phi(\tilde w,\nabla\widetilde Q)\|_{\widetilde F_T}\lesssim \alpha(\alpha+\|(\tilde w,\nabla\widetilde Q)\|_{F_T}).\tag{3-15}$$

Thus, for a suitably small α the operator Φ maps the ball of radius R centered at the origin of $\widetilde F_T$ into itself. Due to the linearity of Φ, one can repeat the above arguments in order to show that Φ is a contraction for small values of α. This concludes the existence of a fixed point of Φ, say $(\tilde u^\star,\nabla\widetilde P^\star)\in\widetilde F_T$. Of course,

$$(\bar u,\nabla\bar P)=(\tilde u^\star,\nabla\widetilde P^\star)+(u_L,\nabla P_L)$$

is a solution of (3-2).

Proof of Theorem 1.2. Consider T small enough such that $(u_L,\nabla P_L)$, the solution of (3-4), satisfies

$$\|\nabla u_L\|_{L^2_T(\dot B^{3/p-1}_{p,1})}+\|\nabla u_L\|_{L^1_T(\dot B^{3/p}_{p,1})}\le\alpha,$$

and consider the closed set

$$\widetilde F_T(\alpha)=\big\{(\tilde v,\nabla\widetilde Q)\in F_T:\tilde v_{|t=0}=0,\ \|(\tilde v,\nabla\widetilde Q)\|_{F_T}\le R\alpha\big\}$$

with R sufficiently small such that

$$\|\nabla\tilde v\|_{L^2_T(\dot B^{3/p-1}_{p,1})}+\|\nabla\tilde v\|_{L^1_T(\dot B^{3/p}_{p,1})}\le\alpha.\tag{3-16}$$

Let us consider the operator S which associates to $(\tilde v,\nabla\widetilde Q)\in\widetilde F_T(\alpha)$, the solution of

$$\begin{cases}\partial_t\tilde u-\frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0)D(\tilde u)\big)+\frac{1}{\rho_0}\nabla\widetilde P=\frac{1}{\rho_0}F_{(u_L+\tilde v)}(u_L+\tilde u,\nabla P_L+\nabla\widetilde P),\\ \operatorname{div}\big(\operatorname{adj}(DX_{u_L+\tilde v})(u_L+\tilde u)\big)=0,\\ \tilde u_{|t=0}=0\end{cases}$$

constructed in the previous section. We will show that for suitably small T and α, the operator S maps the closed set $\widetilde F_T(\alpha)$ into itself and that S is a contraction. First of all, recalling that $(\tilde u,\nabla\widetilde P)$ is in fact the fixed point of the operator Φ defined in (3-6) and using the estimates established in the last section, we conclude that

$$\|(\tilde u,\nabla\widetilde P)\|_{\widetilde F_T}=\|S(\tilde v,\nabla\widetilde Q)\|_{F_T}\le R\alpha\tag{3-17}$$

for some small enough T.

Next, we will deal with the stability estimates. For $i = 1, 2$, let us consider $(\tilde{v}_i, \nabla \widetilde{Q}_i) \in \widetilde{F}_T(\alpha)$ and $(\tilde{u}_i, \nabla \widetilde{P}_i) = S(\tilde{v}_i, \nabla \widetilde{Q}_i) \in \widetilde{F}_T(\alpha)$. Defining

$$\begin{aligned}(\delta\tilde{v}, \nabla\delta\widetilde{Q}) &= (\tilde{v}_1 - \tilde{v}_2, \nabla\widetilde{Q}_1 - \nabla\widetilde{Q}_2),\\ (\delta\tilde{u}, \nabla\delta\widetilde{P}) &= (\tilde{u}_1 - \tilde{u}_2, \nabla\widetilde{P}_1 - \nabla\widetilde{P}_2),\end{aligned}$$

we see

$$\begin{cases} \partial_t \delta\tilde{u} - \frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0) D(\delta\tilde{u})\big) + \frac{1}{\rho_0}\nabla\delta\widetilde{P} = \frac{1}{\rho_0}\widetilde{F},\\ \operatorname{div}\delta\tilde{u} = \operatorname{div}\widetilde{G},\\ \delta\tilde{u}_{|t=0} = 0,\end{cases}$$

where

$$\begin{aligned}\widetilde{F} &= F_1(\delta\tilde{v}, u_L + \tilde{u}_1) + F_1(u_L + \tilde{v}_2, \delta\tilde{u}) + F_2(\delta\tilde{v}, \nabla P_L + \nabla\widetilde{P}_1) + F_2(u_L + \tilde{v}_2, \nabla\delta\widetilde{P}),\\ \widetilde{G} &= -\big(\operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \mathrm{Id}\big)\delta\tilde{u} - \big(\operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \operatorname{adj}(DX_{(u_L+\tilde{v}_2)})\big)(u_L + \tilde{u}_2) := \widetilde{G}_1 + \widetilde{G}_2,\end{aligned}$$

and

$$\begin{aligned}F_1(\bar{v}, \bar{w}) &= \operatorname{div}\big(\mu(\rho_0) A_{\bar{v}} D_{A_{\bar{v}}}(\bar{w}) - \mu(\rho_0) D(\bar{w})\big),\\ F_2(\bar{v}, \nabla\overline{Q}) &= (\mathrm{Id} - A_{\bar{v}}^T)\nabla\overline{Q}.\end{aligned}$$

According to Theorem 1.3 we get

$$\|(\delta\tilde{u}, \nabla\delta\widetilde{P})\|_{\widetilde{F}_T} \lesssim C_{\rho_0}\big(\|\widetilde{F}\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} + \|\nabla\operatorname{div}\widetilde{G}\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} + \|\partial_t\widetilde{G}\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}\big). \tag{3-18}$$

Proceeding as in relations (3-8) and (3-9) we get

$$\begin{aligned}\|\widetilde{F}\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} &\lesssim \|\nabla\delta\tilde{v}\|_{L^1_T(\dot{B}^{3/p}_{p,1})}\|\nabla u_L + \nabla\tilde{u}_1\|_{L^1_T(\dot{B}^{3/p}_{p,1})} + \|\nabla u_L + \nabla\tilde{v}_2\|_{L^1_T(\dot{B}^{3/p}_{p,1})}\|\nabla\delta\tilde{u}\|_{L^1_T(\dot{B}^{3/p}_{p,1})}\\ &\quad + \|\nabla\delta\tilde{v}\|_{L^1_T(\dot{B}^{3/p}_{p,1})}\|\nabla P_L + \nabla\widetilde{P}_1\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}\\ &\quad + \|\nabla u_L + \nabla\tilde{v}_2\|_{L^1_T(\dot{B}^{3/p}_{p,1})}\|\nabla\delta\widetilde{P}\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}\\ &\lesssim \alpha\|(\nabla\delta\tilde{v}, \nabla\delta\widetilde{Q})\|_{L^1_T(\dot{B}^{3/p}_{p,1})} + \alpha\|(\delta\tilde{u}, \nabla\delta\widetilde{P})\|_{\widetilde{F}_T}.\end{aligned} \tag{3-19}$$

Of course, we will use the smallness of α to absorb $\alpha\|(\nabla\delta\tilde{u}, \nabla\delta\widetilde{P})\|_{L^1_T(\dot{B}^{3/p}_{p,1})}$ into the left-hand side of (3-18).

Next, we treat $\|\nabla\operatorname{div}\widetilde{G}\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})}$. Using Proposition 3.23, we can write

$$\begin{aligned}-\operatorname{div}\widetilde{G}_2 &= \operatorname{div}\big(\big(\operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \operatorname{adj}(DX_{(u_L+\tilde{v}_2)})\big)(u_L + \tilde{u}_2)\big)\\ &= \operatorname{div}\big(\operatorname{adj}(DX_{(u_L+\tilde{v}_1)})(u_L + \tilde{u}_2)\big) - \operatorname{div}\big(\operatorname{adj}(DX_{(u_L+\tilde{v}_2)})(u_L + \tilde{u}_2)\big)\\ &= J_{u_L+\tilde{v}_1} D(u_L + \tilde{u}_2) : A_{(u_L+\tilde{v}_1)} - J_{(u_L+\tilde{v}_2)} D(u_L + \tilde{u}_2) : A_{(u_L+\tilde{v}_2)}\\ &= (J_{u_L+\tilde{v}_1} - J_{(u_L+\tilde{v}_2)}) D(u_L + \tilde{u}_2) : A_{(u_L+\tilde{v}_1)} + J_{(u_L+\tilde{v}_2)} D(u_L + \tilde{u}_2) : (A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}),\end{aligned}$$

and thus, using Propositions 3.27 and 3.28 we get

$$
\begin{aligned}
&\|\nabla \operatorname{div} \widetilde{G}_2\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \\
&\quad\lesssim \|(J_{u_L+\tilde v_1} - J_{(u_L+\tilde v_2)})\|_{L^\infty_T(\dot{B}^{3/p}_{p,1})} \|(Du_L, D\tilde u_2)\|_{L^1_T(\dot{B}^{3/p}_{p,1})} (1 + \|\mathrm{Id} - A_{(u_L+\tilde v_1)}\|_{L^\infty_T(\dot{B}^{3/p}_{p,1})}) \\
&\qquad\quad + (1 + \|(J_{u_L+\tilde v_2} - 1)\|_{L^\infty_T(\dot{B}^{3/p}_{p,1})}) \|(Du_L, D\tilde u_2)\|_{L^1_T(\dot{B}^{3/p}_{p,1})} \|A_{(u_L+\tilde v_1)} - A_{(u_L+\tilde v_2)}\|_{L^\infty_T(\dot{B}^{3/p}_{p,1})} \\
&\quad\lesssim \alpha \|\nabla \delta \tilde v\|_{L^1_T(\dot{B}^{3/p}_{p,1})}.
\end{aligned}
\tag{3-20}
$$

Next, using again Proposition 3.23 we see

$$
\begin{aligned}
-\operatorname{div} \widetilde{G}_1 = \operatorname{div}\big((\operatorname{adj}(DX_{(u_L+\tilde v_1)}) - \mathrm{Id})\delta\tilde u\big) &= D\delta\tilde u : (J_{u_L+\tilde v_1} A_{(u_L+\tilde v_1)} - \mathrm{Id}) \\
&= J_{u_L+\tilde v_1} D\delta\tilde u : (A_{(u_L+\tilde v_1)} - \mathrm{Id}) + (J_{u_L+\tilde v_1} - 1) \operatorname{div} \delta\tilde u
\end{aligned}
$$

and consequently

$$
\begin{aligned}
&\|\nabla \operatorname{div} \widetilde{G}_1\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \\
&\quad\lesssim \big\| J_{u_L+\tilde v_1} D\delta\tilde u : (A_{(u_L+\tilde v_1)} - \mathrm{Id})\big\|_{L^1_T(\dot{B}^{3/p}_{p,1})} + \big\|(J_{u_L+\tilde v_1} - 1)\operatorname{div}\delta\tilde u\big\|_{L^1_T(\dot{B}^{3/p}_{p,1})} \\
&\quad\lesssim \big(1 + \|J_{u_L+\tilde v_1} - 1\|_{L^1_T(\dot{B}^{3/p}_{p,1})}\big) \|D\delta\tilde u : (A_{(u_L+\tilde v_1)} - \mathrm{Id})\|_{L^1_T(\dot{B}^{3/p}_{p,1})} + \|J_{u_L+\tilde v_1} - 1\|_{L^1_T(\dot{B}^{3/p}_{p,1})} \|\operatorname{div}\delta\tilde u\|_{L^1_T(\dot{B}^{3/p}_{p,1})} \\
&\quad\lesssim \alpha \|(\delta\tilde u, \nabla\delta P)\|_{\widetilde{F}_T}.
\end{aligned}
\tag{3-21}
$$

Combining (3-20) with (3-21) yields

$$
\|\nabla \operatorname{div} \widetilde{G}\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \lesssim \alpha \|\nabla\delta\tilde v\|_{L^1_T(\dot{B}^{3/p}_{p,1})} + \alpha \|(\delta\tilde u, \nabla\delta P)\|_{\widetilde{F}_T}. \tag{3-22}
$$

Again, we will use the smallness of α to absorb $\alpha\|(\nabla\delta\tilde u, \nabla\delta\widetilde{P})\|_{L^1_T(\dot{B}^{3/p}_{p,1})}$ into the left-hand side of (3-22).

Finally, we write

$$
\begin{aligned}
\partial_t\big[\big(\operatorname{adj}(DX_{(u_L+\tilde v_1)}) - \operatorname{adj}(DX_{(u_L+\tilde v_2)})\big)(u_L + \tilde u_2)\big] \\
= \big(\partial_t \operatorname{adj}(DX_{(u_L+\tilde v_1)}) - \partial_t \operatorname{adj}(DX_{(u_L+\tilde v_2)})\big)(u_L + \tilde u_2) \\
+ \operatorname{adj}(DX_{(u_L+\tilde v_1)}) - \operatorname{adj}(DX_{(u_L+\tilde v_2)})(\partial_t u_L + \partial_t \tilde u_2).
\end{aligned}
$$

Using Proposition 3.28 gives us

$$
\begin{aligned}
&\big\|\partial_t\big[(\operatorname{adj}(DX_{(u_L+\tilde v_1)}) - \operatorname{adj}(DX_{(u_L+\tilde v_2)}))(u_L + \tilde u_2)\big]\big\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \\
&\qquad\lesssim \big\|\partial_t \operatorname{adj}(DX_{(u_L+\tilde v_1)}) - \partial_t \operatorname{adj}(DX_{(u_L+\tilde v_2)})\big\|_{L^2_T(\dot{B}^{3/p-1}_{p,1})} \|u_L\|_{L^2_T(\dot{B}^{3/p}_{p,1})} \\
&\qquad\quad + \big\|\partial_t \operatorname{adj}(DX_{(u_L+\tilde v_1)}) - \partial_t \operatorname{adj}(DX_{(u_L+\tilde v_2)})\big\|_{L^1_T(\dot{B}^{3/p}_{p,1})} \|\tilde u_2\|_{L^\infty_T(\dot{B}^{3/p-1}_{p,1})} \\
&\qquad\quad + \big\|\operatorname{adj}(DX_{(u_L+\tilde v_1)}) - \operatorname{adj}(DX_{(u_L+\tilde v_2)})\big\|_{L^\infty_T(\dot{B}^{3/p}_{p,1})} \|\partial_t u_L + \partial_t \tilde u_2\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \\
&\qquad\lesssim \alpha \|\delta\tilde v\|_{L^2_T(\dot{B}^{3/p}_{p,1})}.
\end{aligned}
$$

The conclusion is

$$
\|\partial_t \widetilde{G}\|_{L^1_T(\dot{B}^{3/p-1}_{p,1})} \lesssim \alpha \|\delta\tilde v\|_{L^2_T(\dot{B}^{3/p}_{p,1})}. \tag{3-23}
$$

Combining (3-19), (3-22) and (3-23) we get if α is chosen sufficiently small then

$$\|((\delta\tilde{u}, \nabla\delta\widetilde{P}))\|_{\widetilde{F}_T} \leq \tfrac{1}{2}\|((\delta\tilde{v}, \nabla\delta\widetilde{Q}))\|_{F_T} \tag{3-24}$$

and the operator S is also a contraction over $\widetilde{F}_T(\alpha)$. Thus, according to Banach's theorem there exists a fixed point $(\bar{u}^\star, \nabla\bar{P}^\star)$ of S. Obviously,

$$(\bar{u}, \nabla\bar{P}) = (u_L, \nabla P_L) + (\bar{u}^\star, \nabla\bar{P}^\star)$$

is a solution of

$$\begin{cases} \rho_0\partial_t\bar{u} - \operatorname{div}\big(\mu(\rho_0)A_{\bar{u}}D_{A_{\bar{u}}}(\bar{u})\big) + A_{\bar{u}}^T\nabla\bar{P} = 0, \\ \operatorname{div}(\operatorname{adj}(DX_{\bar{u}})\bar{u}) = 0, \\ \bar{u}_{|t=0} = u_0. \end{cases} \tag{3-25}$$

In view of Proposition 3.26 we also get $J_{\bar{u}} = 1$. Thus, the second equation of (3-25) becomes

$$\operatorname{div}(A_{\bar{u}}\bar{u}) = 0.$$

The only thing left to prove is the uniqueness property. Consider $(\bar{u}^1, \nabla\bar{P}^1)$, $(\bar{u}^2, \nabla\bar{P}^2) \in F_T$, two solutions of (3-25) with the same initial data $u_0 \in \dot{B}^{3/p-1}_{p,1}$. With $(u_L, \nabla P_L)$ defined above, we let

$$(\tilde{u}^i, \nabla\widetilde{P}^i) = (\bar{u}^i, \nabla\bar{P}^i) - (u_L, \nabla P_L) \quad \text{for } i = 1, 2$$

such that the system verified by $(\tilde{u}^i, \nabla\widetilde{P}^i)$ is

$$\begin{cases} \partial_t\tilde{u}^i - \frac{1}{\rho_0}\operatorname{div}\big(\mu(\rho_0)D(\tilde{u}^i)\big) + \frac{1}{\rho_0}\nabla\widetilde{P}^i = \frac{1}{\rho_0}F_{(u_L+\tilde{u}^i)}(u_L + \tilde{u}^i, \nabla P_L + \nabla\widetilde{P}^i), \\ \operatorname{div}\big(A_{(u_L+\tilde{u}^i)}(u_L + \tilde{u}^i)\big) = 0, \\ \tilde{u}_{|t=0} = 0. \end{cases}$$

We are now in the position of performing exactly the same computations as above so that we obtain a time T' sufficiently small such that

$$(\bar{u}^1, \nabla\bar{P}^1) = (\bar{u}^2, \nabla\bar{P}^2) \quad \text{on } [0, T'].$$

It is classical that the above local uniqueness property extends to all of $[0, T]$. ☐

Proof of Theorem 1.1. Considering $(\rho_0, u_0) \in \dot{B}^{3/p}_{p,1} \times \dot{B}^{3/p-1}_{p,1}$ and applying Theorem 1.2, there exists a positive $T > 0$ such that we may construct a solution $(\bar{u}, \nabla\bar{P})$ to the system (1-4) in F_T. Then, working with a smaller T if needed and considering $X_{\bar{u}}$, the "flow" of $\bar{u}$ defined by (3-17), by using Proposition 3.26 from the Appendix, one obtains that $X_{\bar{u}}$ is a measure preserving C^1-diffeomorphism over $\mathbb{R}^n$ for all $t \in [0, T]$. Thus we may introduce the Eulerian variable:

$$\rho(t, x) = \rho_0(X_{\bar{u}}^{-1}(t, x)), \quad u(t, x) = \bar{u}(t, X_{\bar{u}}^{-1}(t, x)) \quad \text{and} \quad P(t, x) = \bar{P}(t, X_{\bar{u}}^{-1}(t, x)).$$

Then, Proposition 3.23 ensures that $(\rho, u, \nabla P)$ is a solution of (1-1). As $DX_{\bar{u}} - \mathrm{Id}$ belongs to $\dot{B}^{3/p}_{p,1}$, using Proposition 3.22, we may conclude that $(\rho, u, \nabla P)$ has the announced regularity.

The uniqueness property comes from the fact that considering two solutions $(\rho^i, u^i, \nabla P^i)$ of (1-1), $i = 1, 2$, and considering Y_{u^i}, the flow of u^i, we find that $\big(u^i(t, Y_{u^i}(t, y)), \nabla P^i(t, Y_{u^i}(t, y))\big)$ are solutions of the system (1-4) with the same data. Thus, they are equal according to the uniqueness property announced in Theorem 1.2. Thus, on some nontrivial interval $[0, T'] \subset [0, T]$ (chosen such that condition (3-19) holds), the solutions $(\rho^i, u^i, \nabla P^i)$ are equal. This local uniqueness property obviously entails uniqueness on all of $[0, T]$. □

Appendix

We present here a few results of Fourier analysis used through the text. The full proofs along with other complementary results can be found in [Bahouri et al. 2011, Chapter 2].

Let us introduce the dyadic partition of the space:

Proposition A.1. *Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. There exists a radial function $\varphi \in \mathcal{D}(\mathcal{C})$ valued in the interval* $[0, 1]$ *such that*

$$\textit{for all } \xi \in \mathbb{R}^n \backslash \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{A-1}$$

$$2 \le |j - j'| \quad \Rightarrow \quad \operatorname{Supp}(\varphi(2^{-j}\cdot)) \cap \operatorname{Supp}(\varphi(2^{-j'}\cdot)) = \varnothing. \tag{A-2}$$

Also, the following inequality holds:

$$\textit{for all } \xi \in \mathbb{R}^n \backslash \{0\}, \quad \frac{1}{2} \le \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \le 1. \tag{A-3}$$

From now on we fix functions χ and φ satisfying the assertions of the above proposition and denote by $\tilde{h}$ and h their Fourier inverses.

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\dot{S}_j$ are

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y)u(x - y)\, dy,$$
$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{jn} \int_{\mathbb{R}^n} \tilde{h}(2^j y)u(x - y)\, dy$$

for all $j \in \mathbb{Z}$.

Definition A.2. We denote by $\mathcal{S}'_h$ the space of tempered distributions such that

$$\lim_{j \to -\infty} \|\dot{S}_j u\|_{L^\infty} = 0.$$

Let us now define the homogeneous Besov spaces:

Definition A.3. Let s be a real number and $(p, r) \in [1, \infty]$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is the subset of tempered distributions $u \in S'_h$ such that

$$\|u\|_{\dot{B}^s_{p,r}} := \big\|(2^{js}\|\dot{\Delta}_j u\|_{L^2})_{j \in \mathbb{Z}}\big\|_{\ell^r(\mathbb{Z})} < \infty.$$

The next propositions gather some basic properties of Besov spaces.

Proposition A.4. *Let us consider $s \in \mathbb{R}$ and $p, r \in [1, \infty]$ such that*

$$s < \frac{n}{p} \quad \textit{or} \quad s = \frac{n}{p} \ \textit{ and } \ r = 1. \tag{A-4}$$

Then $(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}})$ is a Banach space.

Proposition 3.5. *A tempered distribution $u \in S'_h$ belongs to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ if and only if there exists a sequence $(c_j)_j$ such that $(2^{js}c_j)_j \in \ell^r(\mathbb{Z})$ with norm 1 and a constant $C = C(u) > 0$ such that for any $j \in \mathbb{Z}$ we have*

$$\|\dot{\Delta}_j u\|_{L^p} \leq C c_j.$$

Proposition 3.6. *Consider s_1 and s_2 two real numbers such that $s_1 < s_2$ and $\theta \in (0,1)$. Then, there exists a constant $C > 0$ such that for all $r \in [1,\infty]$ we have*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq \|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}},$$
$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_2-s_1}\left(\frac{1}{\theta}+\frac{1}{1-\theta}\right)\|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}.$$

Proposition 3.7. (1) *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then, for any real number s, the space $\dot{B}^s_{p_1,r_1}$ is continuously embedded in $\dot{B}^{s-n(1/p_1-1/p_2)}_{p_2,r_2}$.*

(2) *Let $1 \leq p < \infty$. Then, $\dot{B}^{n/p}_{p,1}$ is continuously embedded in $(C_0(\mathbb{R}^n), \|\cdot\|_{L^\infty})$, the space of continuous functions vanishing at infinity.*

Proposition 3.8. *For all $1 \leq p, r \leq \infty$ and $s \in \mathbb{R}$,*

$$\begin{cases} \dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'} \to \mathbb{R}, \\ (u,v) \to \sum_j \langle \dot{\Delta}_j u, \tilde{\Delta}_j v\rangle, \end{cases} \tag{3-5}$$

where $\tilde{\Delta}_j := \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}$, defines a continuous bilinear functional on $\dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'}$. Denote by $Q^{-s}_{p',r'}$ the set of functions $\varphi \in \mathcal{S} \cap \dot{B}^{-s}_{p',r'}$ such that $\|\varphi\|_{\dot{B}^{-s}_{p',r'}} \leq 1$. If $u \in \mathcal{S}'_h$, then we have

$$\|u\|_{\dot{B}^s_{p,r}} \lesssim \sup_{\varphi \in Q^{-s}_{p',r'}} \langle u, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}}.$$

Proposition 3.9. *Consider $1 < p, r < \infty$ and $s \in \mathbb{R}$. Furthermore, let $u \in \dot{B}^s_{p,r}$, $v \in \dot{B}^{-s}_{p',r'}$ and $\rho \in L^\infty \cap \mathcal{M}(\dot{B}^s_{p,r}) \cap \mathcal{M}(\dot{B}^{-s}_{p',r'})$. Then, we have*

$$(\rho u, v) = \sum_j \sum_j \langle \dot{\Delta}_j u, \tilde{\Delta}_j(\rho v)\rangle = (u, \rho v). \tag{3-6}$$

The proof of Proposition 3.9 follows from a density argument. Relation (3-6) clearly holds for functions from the Schwartz class: then we may write

$$\int_{\mathbb{R}^n} \rho u v = (\rho u, v) = (u, \rho v).$$

The conditions $1 < p, r < \infty$ and $s \in \mathbb{R}$ ensure that u and v may be approximated by Schwartz functions.

An important feature of Besov spaces with negative index of regularity is the following:

Proposition 3.10. *Let $s < 0$ and $1 \leq p, r \leq \infty$. Let u be a distribution in S'_h. Then, u belongs to $\dot{B}^s_{p,r}$ if and only if*

$$(2^{js}\|\dot{S}_j u\|_{L^p})_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z}).$$

Moreover, there exists a constant C depending only on the dimension n such that

$$C^{-|s|+1}\|u\|_{\dot{B}^s_{p,r}} \le \big\|(2^{js}\|\dot{S}_j u\|_{L^p})_{j\in\mathbb{Z}}\big\|_{\ell^r(\mathbb{Z})} \le C\Big(1+\frac{1}{|s|}\Big)\|u\|_{\dot{B}^s_{p,r}}.$$

The next proposition tells us how certain multipliers act on Besov spaces.

Proposition 3.11. *Consider A a smooth function on $\mathbb{R}^n\backslash\{0\}$ which is homogeneous of degree m. Then, for any $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ such that*

$$s-m<\frac{n}{p} \quad \text{or} \quad s-m=\frac{n}{p} \text{ and } r=1,$$

the operator[4] $A(D)$ maps $\dot{B}^s_{p,r}$ continuously into $\dot{B}^{s-m}_{p,r}$.

The next proposition describes how smooth functions act on homogeneous Besov spaces.

Proposition 3.12. *Let f be a smooth function on $\mathbb{R}$ which vanishes at 0. Consider $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ such that*

$$0<s<\frac{n}{p} \quad \text{or} \quad s=\frac{n}{p} \text{ and } r=1.$$

Then for any real-valued function $u\in\dot{B}^s_{p,r}\cap L^\infty$, the function $f\circ u$ is in $\dot{B}^s_{p,r}\cap L^\infty$ and we have

$$\|f\circ u\|_{\dot{B}^s_{p,r}} \le C(f',\|u\|_{L^\infty})\|u\|_{\dot{B}^s_{p,r}}.$$

Remark 3.13. The constant $C(f',\|u\|_{L^\infty})$ appearing above can be taken to be

$$\sup_{i\in\overline{1,[s]+1}} \|f^{(i)}\|_{L^\infty([-M\|u\|_{L^\infty},-M\|u\|_{L^\infty}])},$$

where M is a constant depending only on the dimension n.

Commutator and product estimates. Next, we want to see how the product acts in Besov spaces. The Bony decomposition, introduced in [Bony 1981], offers a mathematical framework to obtain estimates of the product of two distributions, when the latter is defined.

Definition 3.14. Given two tempered distributions $u,v\in S'_h$, the homogeneous paraproduct of v by u is defined as

$$\dot{T}_u v=\sum_{j\in\mathbb{Z}}\dot{S}_{j-1}u\dot{\Delta}_j v. \tag{3-7}$$

The homogeneous remainder of u and v is defined by

$$\dot{R}(u,v)=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j u\dot{\Delta}'_j v, \tag{3-8}$$

where

$$\dot{\Delta}'_j=\dot{\Delta}_{j-1}+\dot{\Delta}_j+\dot{\Delta}_{j+1}.$$

[4] $A(D)w=\mathcal{F}^{-1}(A\mathcal{F}w)$.

Remark 3.15. Notice that at a formal level, one has the following decomposition of the product of two (sufficiently well-behaved) distributions:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v) = \dot{T}_u v + \dot{T}'_v u.$$

The next result describes how the paraproduct and remainder behave.

Proposition 3.16. (1) *Assume* $(s, p, p_1, p_2, r) \in \mathbb{R}\times[1,\infty]^4$ *such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad s < \frac{n}{p} \qquad or \qquad s = \frac{n}{p} \;\; and \;\; r = 1.$$

Then, the paraproduct maps $L^{p_1} \times \dot{B}^s_{p_2,r}$ *into* $\dot{B}^s_{p,r}$ *and the following estimate holds*:

$$\|\dot{T}_f g\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^{p_1}} \|g\|_{\dot{B}^s_{p_2,r}}.$$

(2) *Assume* $(s, p, p_1, p_2, r, r_1, r_2) \in \mathbb{R}\times[1,\infty]^6$ *and* $\nu > 0$ *such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$s < \frac{n}{p} - \nu \qquad or \qquad s = \frac{n}{p} - \nu \;\; and \;\; r = 1.$$

Then, the paraproduct maps $\dot{B}^{-\nu}_{p_1,r_1} \times \dot{B}^{s+\nu}_{p_2,r_2}$ *into* $\dot{B}^s_{p,r}$ *and the following estimate holds*:

$$\|\dot{T}_f g\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{\dot{B}^{-\nu}_{p_1,r_1}} \|g\|_{\dot{B}^{s+\nu}_{p_2,r_2}}.$$

(3) *Consider* $(s_1, s_2, p, p_1, p_2, r, r_1, r_2) \in \mathbb{R}^2 \times [1,\infty]^6$ *such that*

$$0 < s_1 + s_2 < \frac{n}{p} \qquad or \qquad s_1 + s_2 = \frac{n}{p} \;\; and \;\; r = 1.$$

Then, the remainder maps $\dot{B}^{s_1}_{p_1,r_1} \times \dot{B}^{s_2}_{p_2,r_2}$ *into* $\dot{B}^{s_1+s_2}_{p,r}$ *and*

$$\|\dot{R}(f,g)\|_{\dot{B}^{s_1+s_2}_{p,r}} \leq \|f\|_{\dot{B}^{s_1}_{p_1,r_1}} \|g\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

As a consequence we obtain the following product rules in Besov space:

Proposition 3.17. *Consider* $p \in [1,\infty]$ *and the real numbers* $\nu_1 \geq 0$ *and* $\nu_2 \geq 0$ *with*

$$\nu_1 + \nu_2 < \frac{n}{p} + \min\left\{\frac{n}{p}, \frac{n}{p'}\right\}.$$

Then, the following estimate holds:

$$\|fg\|_{\dot{B}^{n/p-\nu_1-\nu_2}_{p,1}} \lesssim \|f\|_{\dot{B}^{n/p-\nu_1}_{p,1}} \|g\|_{\dot{B}^{n/p-\nu_2}_{p,1}}.$$

Proposition 3.18. *Consider* θ *a* $\mathcal{C}^1$ *function on* $\mathbb{R}^n$ *such that* $(1+|\cdot|)\hat{\theta} \in L^1$. *Let us also consider* $p, q \in [1,\infty]$ *such that*

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} \leq 1.$$

Then, there exists a constant C such that for any Lipschitz function a with gradient in L^p, any function $b \in L^q$ and any positive λ,

$$\big\|[\theta(\lambda^{-1}D), a]b\big\|_{L^r} \le C\lambda^{-1}\|\nabla a\|_{L^p}\|b\|_{L^q}.$$

In particular, when $\theta = \varphi$ and $\lambda = 2^j$ we get

$$\big\|[\dot\Delta_j, a]b\big\|_{L^r} \le C2^{-j}\|\nabla a\|_{L^p}\|b\|_{L^q}.$$

Proposition 3.19. *Assume s, ν and $p \in [1,\infty]$ are such that*

$$0 \le \nu \le \frac{n}{p} \quad \text{and} \quad -1 - \min\left\{\frac{n}{p}, \frac{n}{p'}\right\} < s \le \frac{n}{p} - \nu.$$

Then, there exists a constant C depending only on s, ν, p and n such that for all $l \in \overline{1,n}$ we have for some sequence $(c_j)_{j\in\mathbb{Z}}$ with $\|(c_j)_{j\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})} = 1$,

$$\big\|\partial_l[a, \dot\Delta_j]w\big\|_{L^p} \le Cc_j 2^{-js}\|\nabla a\|_{\dot B^{n/p-\nu}_{p,1}}\|w\|_{\dot B^{s+\nu}_{p,1}}$$

for all $j \in \mathbb{Z}$.

For a proof of the above results we refer the reader to the Appendix of [Danchin 2014, Lemmas A.5 and A.6].

Proposition 3.20. *Consider a homogeneous function $A : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ of degree 0. Let us consider $s \in \mathbb{R}$, $0 < \nu \le 1$ and $p, r, r_1, r_2 \in [1,\infty]$ such that*

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$s < \frac{n}{p} - \nu \quad \text{or} \quad s = \frac{n}{p} - \nu \ \text{ and } \ r_2 = 1. \tag{3-9}$$

Moreover, assume $w \in \dot B^{s+\nu}_{p,r_2}$ and $a \in L^\infty$ with $\nabla a \in \dot B^{-\nu}_{\infty,r_1}$. Then, the following estimate holds:

$$\big\|[A(D), \dot T_a]w\big\|_{\dot B^{s+1}_{p,r}} \lesssim \|\nabla a\|_{\dot B^{-\nu}_{\infty,r_1}}\|w\|_{\dot B^{s+\nu}_{p,r_2}}. \tag{3-10}$$

As this result is of great importance in the analysis of the pressure term, we present a sketched proof below (see also [Bahouri et al. 2011, Chapter 2, Lemma 2.99]).

Proof. The fact that $a \in L^\infty$, along with relation (3-9), guarantees that $A(D)w \in \dot B^{s+\nu}_{p,r}$ and that the paraproducts $\dot T_a w$ and $\dot T_a A(D)w$ are well-defined. We observe that there exists a function $\tilde\varphi$ supported in some annulus which equals 1 on the support of φ such that one may write (of course it is here that we use the homogeneity of A)

$$[A(D), \dot T_a]w = \sum_j \big[(A\tilde\varphi)(2^{-j}D), \dot S_{j-1}a\big]\dot\Delta_j w.$$

But according to Proposition 3.18 we have

$$2^{j(s+1)}\big\|\big[(A\tilde\varphi)(2^{-j}D), \dot S_{j-1}a\big]\dot\Delta_j w\big\|_{L^p} \lesssim 2^{-j\nu}\|\nabla\dot S_{j-1}a\|_{L^\infty}2^{j(s+\nu)}\|\dot\Delta_j w\|_{L^p}.$$

The last relation obviously implies (3-10). ∎

As a consequence of the above proposition and Proposition 3.16 we get the following:

Proposition 3.21. *Let us consider a homogeneous function* $A:\mathbb{R}^n\backslash\{0\}\to\mathbb{R}$ *of degree* 0, $s\in\mathbb{R}$, $0<\nu\le 1$ *and* $p,r,r_1,r_2\in[1,\infty]$ *such that*

$$\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$$

and

$$-1-\min\left\{\frac{n}{p},\frac{n}{p'}\right\}<s<\frac{n}{p}-\nu \quad \text{or} \quad s=\frac{n}{p}-\nu \ \text{ and } \ r=r_2=1. \tag{3-11}$$

assume $w\in\dot{B}^{s+\nu}_{p,r_2}$ *and* $a\in L^\infty$ *with* $\nabla a\in\dot{B}^{-\nu}_{\infty,r_1}$. *Then, the following estimate holds*:

$$\left\|[A(D),a]w\right\|_{\dot{B}^{s+1}_{p,r}}\lesssim\|\nabla a\|_{\dot{B}^{n/p-\nu}_{p,r_1}}\|w\|_{\dot{B}^{s+\nu}_{p,r_2}}.$$

Properties of Lagrangian coordinates. The following results are gathered from [Danchin 2014] and [Danchin and Mucha 2012]. More precisely, proofs of Propositions 3.22, 3.23, the estimate (3-26) of Proposition 3.27 and the estimate (3-31) of Proposition 3.28 can be found in [Danchin 2014, pp. 782–786]. Propositions 3.27 and 3.28 can be found in the Appendix of [Danchin and Mucha 2012]. Proposition 3.26 is inspired by [Danchin and Mucha 2012].

Proposition 3.22. *Let* X *be a globally defined bi-Lipschitz diffeomorphism of* $\mathbb{R}^3$ *and* $-\frac{3}{p'}<s\le\frac{3}{p}$. *Then* $a\to a\circ X$ *is a self-map over* $\dot{B}^s_{p,1}$ *whenever*

(1) $s\in(0,1)$;

(2) $s\ge 1$ *and* $(DX-\mathrm{Id})\in\dot{B}^{3/p}_{p,1}$.

The following result interferes in a crucial manner in the proof of the well-posedness result for the inhomogeneous incompressible Navier–Stokes system.

Proposition 3.23. *Let* m *be a* C^1 *scalar function over* $\mathbb{R}^3$ *and* $u\in\mathbb{R}^3$ *a* C^1 *vector field. Let* X *be a* C^1 *diffeomorphism and we define* $J:=\det(DX)$. *Suppose* $J>0$. *Then, the following relations hold*:

$$(\nabla m)\circ X=J^{-1}\operatorname{div}(\operatorname{adj}(DX)m\circ X), \tag{3-12}$$

$$(\operatorname{div}u)\circ X=J^{-1}\operatorname{div}(\operatorname{adj}(DX)u\circ X). \tag{3-13}$$

Corollary 3.24. *Let* m *be a* C^1 *scalar function over* $\mathbb{R}^3$ *and* $u\in\mathbb{R}^3$ *be a* C^1 *vector field. Let* X *be a* C^1 *diffeomorphism and* $J:=\det(DX)$. *Suppose* $J>0$. *Then, we have*

$$J^{-1}\operatorname{div}(\operatorname{adj}(DX)u)=Du:(DX)^{-1}, \tag{3-14}$$

$$J^{-1}\operatorname{div}(\operatorname{adj}(DX)m)=[(DX)^{-1}]^T\nabla m. \tag{3-15}$$

Proof. In order to ease reading, we define $F_{|x}:=F(x)$. Writing u as $u\circ X\circ X^{-1}$, using the chain rule and Einstein convention over repeated index, we write

$$\begin{aligned}(\operatorname{div}u)_{|x}&=\partial_k(u^i\circ X)_{|X^{-1}(x)}\partial_i(X^{-1})^k_{|x}\\&=D(u\circ X)_{|X^{-1}(x)}:D(X^{-1})_{|x}\\&=D(u\circ X)_{|X^{-1}(x)}:(DX)^{-1}_{|X^{-1}(x)},\end{aligned}$$

and thus, we get

$$(\operatorname{div} u)\circ X = D(u\circ X) : (DX)^{-1}. \tag{3-16}$$

Then, using (3-13) and (3-16) we get

$$\begin{aligned} J^{-1}\operatorname{div}(\operatorname{adj}(DX)u) &= J^{-1}\operatorname{div}\bigl(\operatorname{adj}(DX)u\circ X^{-1}\circ X\bigr) = (\operatorname{div} u\circ X^{-1})\circ X \\ &= D(u\circ X^{-1}\circ X) : (DX)^{-1} = Du : (DX)^{-1}. \end{aligned}$$

In a similar manner we prove (3-15). □

For any $\bar v$ a time-dependent vector field we set

$$X_{\bar v}(t,x) = x + \int_0^t \bar v(\tau,x)\,d\tau \tag{3-17}$$

and we define

$$A_{\bar v} = (DX_{\bar v})^{-1}. \tag{3-18}$$

It is crucial to know (in order to pass back in Eulerian coordinates, for instance) when $X_{\bar v}$ is a global diffeomorphism. In order to achieve this, we will use the following theorem due to Hadamard:

Theorem 3.25 (Hadamard). *Let $X:\mathbb{R}^n\to\mathbb{R}^n$ a function of class C^1. Then, the following are equivalent*:

(1) *X is a local diffeomorphism and $\lim_{|x|\to\infty}|X(x)| = \infty$.*

(2) *X is a global C^1-diffeomorphism over $\mathbb{R}^n$.*

For a proof of this result one can consult, for instance, [Katriel 1994].

Proposition 3.26. *Let us consider $\bar v \in C_b\bigl([0,T], \dot B^{3/p-1}_{p,1}\bigr)$ with $\partial_t \bar v$, $\nabla^2\bar v \in L^1_T(\dot B^{3/p-1}_{p,1})$. Then, there exists a positive α such that if*

$$\|\nabla \bar v\|_{L^1_T(\dot B^{3/p}_{p,1})} \le \alpha, \tag{3-19}$$

then, for any $t\in[0,T]$, we have $X_{\bar v}(t,\cdot)$ introduced in (3-17) is a global C^1-diffeomorphism over $\mathbb{R}^3$ and $\det(DX_{\bar v}) > 0$. *Moreover, if*

$$\operatorname{div}(\operatorname{adj}(DX_{\bar v})\bar v) = 0 \tag{3-20}$$

then, $X_{\bar v}$ is measure-preserving, i.e.,

$$\det DX_{\bar v} = 1. \tag{3-21}$$

Proof. Differentiating $X_{\bar v}$, we obtain

$$DX_{\bar v}(t,\cdot) = \mathrm{Id} + \int_0^t D\bar v(\tau,\cdot)\,d\tau$$

and because of the embedding of $\dot B^{3/p}_{p,1}$ into the space of continuous functions, see Proposition 3.7, we conclude $X_{\bar v}\in C^1([0,T]\times\mathbb{R}^3)$. We observe that

$$\begin{aligned} \|DX_{\bar v}(t,\cdot) - \mathrm{Id}\|_{L^\infty(\mathbb{R}^3)} &\le \int_0^t \|D\bar v(\tau,\cdot)\|_{L^\infty}\,d\tau \\ &\le C\|\nabla \bar v\|_{L^1_t(\dot B^{3/p}_{p,1})} \le \alpha C. \end{aligned}$$

Thus choosing α sufficiently small ensures that $X_{\bar{v}}(t,\cdot)$ is a local C^1-diffeomorphism over $\mathbb{R}^3$. The second condition of Hadamard's theorem is verified in the following lines. Using the triangle inequality we get

$$\begin{aligned}|X_{\bar{v}}(t,x)| &\geq |x| - \int_0^t |\bar{v}(\tau,x)|d\tau \geq |x| - \int_0^t \|\bar{v}(\tau,\cdot)\|_{L^\infty} d\tau \\ &\geq |x| - C\int_0^t \|\bar{v}(\tau,\cdot)\|_{\dot{B}^{3/p}_{p,1}} d\tau \\ &\geq |x| - C\sqrt{t}\|\bar{v}\|_{L^2_t(\dot{B}^{3/p}_{p,1})}.\end{aligned}$$

The conclusion is that $X_{\bar{v}}(t,\cdot)$ is a global C^1-diffeomorphism over $\mathbb{R}^3$. Let us define $J_{\bar{v}} := \det DX_{\bar{v}} \neq 0$. Using Jacobi's formula we get

$$J_{\bar{v}}(t,x) = 1 + \int_0^t \operatorname{tr}\bigl(D\bar{v}(\tau,x)\operatorname{adj}(DX_{\bar{v}})(\tau,x)\bigr)\,d\tau.$$

Recall that according to [Danchin and Mucha 2012, Lemma A.4] we may write

$$\mathrm{Id} - \operatorname{adj}(DX_{\bar{v}}) = \int_0^t (D\bar{v} - \operatorname{div}\bar{v}\,\mathrm{Id})\,d\tau + P_2\biggl(\int_0^t D\bar{v}\,d\tau\biggr),$$

where the coefficients of the matrix $P_2 : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ are at least quadratic polynomial functions of degree $n-1$. Using this identity combined with the embedding $L^\infty \hookrightarrow \dot{B}^{3/p}_{p,1}$ and the smallness condition (3-19) we get $J_{\bar{v}} > 0$. In order to prove the second part of the proposition, let us define

$$v(t,x) := \bar{v}(t, X_{\bar{v}}^{-1}(t,x)).$$

Using relation (3-20) combined with (3-13) we get

$$0 = J_{\bar{v}}^{-1}\operatorname{div}(\operatorname{adj}(DX_{\bar{v}})\bar{v}) = \operatorname{div}(\bar{v}\circ X_{\bar{v}}^{-1})\circ X_{\bar{v}},$$

which implies

$$\operatorname{div} v = \operatorname{div}(\bar{v}\circ X_{\bar{v}}^{-1}) = 0.$$

Since $X_{\bar{v}}$ can be viewed as being the flow of v, using Jacobi's formula we can conclude the validity of (3-21). Indeed, we have

$$\begin{aligned}X_{\bar{v}}(t,x) &= x + \int_0^t \bar{v}(\tau,x)d\tau \\ &= x + \int_0^t \bar{v}(\tau, X_{\bar{v}}^{-1}(\tau, X_{\bar{v}}(\tau,x)))\,d\tau \\ &= x + \int_0^t v(\tau, X_{\bar{v}}(\tau,x))\,d\tau.\end{aligned}$$

Then, Jacobi's formula implies

$$\det(DX_{\bar{v}})(t,x) = \exp\biggl(\int_0^t (\operatorname{div} v)(\tau, X_{\bar{v}}(\tau,x))\biggr) = 1. \qquad \square$$

Proposition 3.27. *Consider $\bar v \in C_b\big([0,T], \dot B^{3/p-1}_{p,1}\big)$ with $\partial_t \bar v$, $\nabla^2 \bar v \in L^1_T(\dot B^{3/p-1}_{p,1})$ satisfying the smallness condition (3-3). Let $X_{\bar v}$ be defined by (3-17) and $J_{\bar v} = \det DX_{\bar v}$. Then for all $t \in [0,T]$,*

$$\|\mathrm{Id} - A_{\bar v}(t)\|_{\dot B^{3/p}_{p,1}} \lesssim \|\nabla \bar v\|_{L^1_t(\dot B^{3/p}_{p,1})}, \tag{3-22}$$

$$\|\mathrm{Id} - \mathrm{adj}(DX_{\bar v})(t)\|_{\dot B^{3/p}_{p,1}} \lesssim \|\nabla \bar v\|_{L^1_t(\dot B^{3/p}_{p,1})}, \tag{3-23}$$

$$\|\partial_t\, \mathrm{adj}(DX_{\bar v})(t)\|_{\dot B^{3/p-1}_{p,1}} \lesssim \|\nabla \bar v(t)\|_{\dot B^{3/p-1}_{p,1}}, \quad \text{if } p < 6, \tag{3-24}$$

$$\|\partial_t\, \mathrm{adj}(DX_{\bar v})(t)\|_{\dot B^{3/p}_{p,1}} \lesssim \|\nabla \bar v(t)\|_{\dot B^{3/p}_{p,1}}, \tag{3-25}$$

$$\|J^{\pm}_{\bar v}(t) - 1\|_{\dot B^{3/p}_{p,1}} \lesssim \|\nabla \bar v\|_{L^1_t(\dot B^{3/p}_{p,1})}. \tag{3-26}$$

In order to establish stability estimates we use the following:

Proposition 3.28. *Let $\bar v_1, \bar v_2 \in C_b\big([0,T], \dot B^{3/p-1}_{p,1}\big)$ with $\partial_t \bar v_1, \partial_t \bar v_2$, $\nabla^2 \bar v_1, \nabla^2 \bar v_2 \in L^1_T(\dot B^{3/p-1}_{p,1})$, both satisfying the smallness condition (3-19) and $\delta v = \bar v_2 - \bar v_1$. Then we have*

$$\|A_{\bar v_1} - A_{\bar v_2}\|_{L^\infty_T(\dot B^{3/p}_{p,1})} \lesssim \|\nabla \delta v\|_{L^1_T(\dot B^{3/p}_{p,1})}, \tag{3-27}$$

$$\|\mathrm{adj}(DX_{\bar v_1}) - \mathrm{adj}(DX_{\bar v_2})\|_{L^\infty_T(\dot B^{3/p}_{p,1})} \lesssim \|\nabla \delta v\|_{L^1_T(\dot B^{3/p}_{p,1})}, \tag{3-28}$$

$$\|\partial_t\, \mathrm{adj}(DX_{\bar v_1}) - \partial_t\, \mathrm{adj}(DX_{\bar v_2})\|_{L^1_T(\dot B^{3/p}_{p,1})} \lesssim \|\nabla \delta v\|_{L^1_T(\dot B^{3/p}_{p,1})}, \tag{3-29}$$

$$\|\partial_t\, \mathrm{adj}(DX_{\bar v_1}) - \partial_t\, \mathrm{adj}(DX_{\bar v_2})\|_{L^2_T(\dot B^{3/p-1}_{p,1})} \lesssim \|\nabla \delta v\|_{L^2_T(\dot B^{3/p-1}_{p,1})}, \quad \text{if } p < 6, \tag{3-30}$$

$$\|J^{\pm}_{\bar v_1}(t) - J^{\pm}_{\bar v_2}(t)\|_{\dot B^{3/p}_{p,1}} \lesssim \|\nabla \delta v\|_{L^1_t(\dot B^{3/p}_{p,1})}. \tag{3-31}$$

Acknowledgements

After submitting this work, we were informed of the existence of [Zhai and Yin 2017], which contains similar results obtained using another method.

The author is grateful to the anonymous referee for his comments and suggestions that have considerably improved the quality of this work.

References

[Abidi 2007] H. Abidi, "Équation de Navier–Stokes avec densité et viscosité variables dans l'espace critique", *Rev. Mat. Iberoam.* **23**:2 (2007), 537–586. MR Zbl

[Abidi and Paicu 2007] H. Abidi and M. Paicu, "Existence globale pour un fluide inhomogène", *Ann. Inst. Fourier* (*Grenoble*) **57**:3 (2007), 883–917. MR Zbl

[Abidi et al. 2012] H. Abidi, G. Gui, and P. Zhang, "On the wellposedness of three-dimensional inhomogeneous Navier–Stokes equations in the critical spaces", *Arch. Ration. Mech. Anal.* **204**:1 (2012), 189–230. MR Zbl

[Abidi et al. 2013] H. Abidi, G. Gui, and P. Zhang, "Well-posedness of 3-D inhomogeneous Navier–Stokes equations with highly oscillatory initial velocity field", *J. Math. Pures Appl.* (9) **100**:2 (2013), 166–203. MR Zbl

[Antontsev et al. 1990] S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids*, Studies in Mathematics and its Applications **22**, North-Holland, Amsterdam, 1990. MR Zbl

[Bahouri et al. 2011] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften **343**, Springer, Heidelberg, 2011. MR Zbl

[Bony 1981] J.-M. Bony, "Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires", *Ann. Sci. École Norm. Sup.* (4) **14**:2 (1981), 209–246. MR Zbl

[Choe and Kim 2003] H. J. Choe and H. Kim, "Strong solutions of the Navier–Stokes equations for nonhomogeneous incompressible fluids", *Comm. Partial Differential Equations* **28**:5–6 (2003), 1183–1201. MR Zbl

[Danchin 2003] R. Danchin, "Density-dependent incompressible viscous fluids in critical spaces", *Proc. Roy. Soc. Edinburgh Sect. A* **133**:6 (2003), 1311–1334. MR Zbl

[Danchin 2004] R. Danchin, "Local and global well-posedness results for flows of inhomogeneous viscous fluids", *Adv. Differential Equations* **9**:3–4 (2004), 353–386. MR Zbl

[Danchin 2007] R. Danchin, "Well-posedness in critical spaces for barotropic viscous fluids with truly not constant density", *Comm. Partial Differential Equations* **32**:7–9 (2007), 1373–1397. MR Zbl

[Danchin 2010] R. Danchin, "On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework", *J. Differential Equations* **248**:8 (2010), 2130–2170. MR Zbl

[Danchin 2014] R. Danchin, "A Lagrangian approach for the compressible Navier–Stokes equations", *Ann. Inst. Fourier* (*Grenoble*) **64**:2 (2014), 753–791. MR Zbl

[Danchin and Mucha 2009] R. Danchin and P. B. Mucha, "A critical functional framework for the inhomogeneous Navier–Stokes equations in the half-space", *J. Funct. Anal.* **256**:3 (2009), 881–927. MR Zbl

[Danchin and Mucha 2012] R. Danchin and P. B. Mucha, "A Lagrangian approach for the incompressible Navier–Stokes equations with variable density", *Comm. Pure Appl. Math.* **65**:10 (2012), 1458–1480. MR Zbl

[Danchin and Mucha 2013] R. Danchin and P. B. Mucha, "Incompressible flows with piecewise constant density", *Arch. Ration. Mech. Anal.* **207**:3 (2013), 991–1023. MR Zbl

[Danchin and Mucha 2015] R. Danchin and P. B. Mucha, *Critical functional framework and maximal regularity in action on systems of incompressible flows*, Mém. Soc. Math. Fr. (N.S.) **143**, Société Mathématique de France, Paris, 2015. MR Zbl

[Desjardins 1997] B. t. Desjardins, "Regularity results for two-dimensional flows of multiphase viscous fluids", *Arch. Rational Mech. Anal.* **137**:2 (1997), 135–158. MR Zbl

[Haspot 2012] B. Haspot, "Well-posedness for density-dependent incompressible fluids with non-Lipschitz velocity", *Ann. Inst. Fourier* (*Grenoble*) **62**:5 (2012), 1717–1763. MR Zbl

[Huang et al. 2013a] J. Huang, M. Paicu, and P. Zhang, "Global solutions to 2-D inhomogeneous Navier–Stokes system with general velocity", *J. Math. Pures Appl.* (9) **100**:6 (2013), 806–831. MR Zbl

[Huang et al. 2013b] J. Huang, M. Paicu, and P. Zhang, "Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system with rough density", pp. 159–180 in *Studies in phase space analysis with applications to PDEs*, edited by M. Cicognani et al., Progr. Nonlinear Differential Equations Appl. **84**, Springer, 2013. MR Zbl

[Huang et al. 2013c] J. Huang, M. Paicu, and P. Zhang, "Global well-posedness of incompressible inhomogeneous fluid systems with bounded density or non-Lipschitz velocity", *Arch. Ration. Mech. Anal.* **209**:2 (2013), 631–682. MR Zbl

[Katriel 1994] G. Katriel, "Mountain pass theorems and global homeomorphism theorems", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **11**:2 (1994), 189–209. MR Zbl

[Kažihov 1974] A. V. Kažihov, "Solvability of the initial-boundary value problem for the equations of the motion of an inhomogeneous viscous incompressible fluid", *Dokl. Akad. Nauk SSSR* **216** (1974), 1008–1010. MR

[Krylov 2008] N. V. Krylov, *Lectures on elliptic and parabolic equations in Sobolev spaces*, Graduate Studies in Mathematics **96**, American Mathematical Society, Providence, RI, 2008. MR Zbl

[Ladyzhenskaya and Solonnikov 1975] O. A. Ladyzhenskaya and V. A. Solonnikov, "The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids", *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **52** (1975), 52–109. In Russian; translated in *J. Soviet Math.* **9**:5 (1978), 697–749. MR Zbl

[Lions 1996] P.-L. Lions, *Mathematical topics in fluid mechanics, I: Incompressible models*, Oxford Lecture Series in Mathematics and its Applications **3**, Oxford University Press, New York, 1996. MR Zbl

[Paicu and Zhang 2012] M. Paicu and P. Zhang, "Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system", *J. Funct. Anal.* **262**:8 (2012), 3556–3584. MR Zbl

[Paicu et al. 2013] M. Paicu, P. Zhang, and Z. Zhang, "Global unique solvability of inhomogeneous Navier–Stokes equations with bounded density", *Comm. Partial Differential Equations* **38**:7 (2013), 1208–1234. MR Zbl

[Poulon 2015] E. Poulon, "Well-posedness for density-dependent incompressible viscous fluids on the torus $\mathbb{T}^3$", preprint, 2015. arXiv

[Xu et al. 2016] H. Xu, Y. Li, and X. Zhai, "On the well-posedness of 2-D incompressible Navier–Stokes equations with variable viscosity in critical spaces", *J. Differential Equations* **260**:8 (2016), 6604–6637. MR Zbl

[Zhai and Yin 2017] X. Zhai and Z. Yin, "Global well-posedness for the 3D incompressible inhomogeneous Navier–Stokes equations and MHD equations", *J. Differential Equations* **262**:3 (2017), 1359–1412. MR Zbl

Received 26 Jul 2016. Revised 18 Oct 2016. Accepted 28 Nov 2016.

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ANALYSIS AND PDE
Vol. 10, No. 2, 2017
dx.doi.org/10.2140/apde.2017.10.481

msp

GLOBAL DYNAMICS BELOW THE STANDING WAVES FOR THE FOCUSING SEMILINEAR SCHRÖDINGER EQUATION WITH A REPULSIVE DIRAC DELTA POTENTIAL

MASAHIRO IKEDA AND TAKAHISA INUI

We consider the focusing mass-supercritical semilinear Schrödinger equation with a repulsive Dirac delta potential on the real line $\mathbb{R}$:

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u + \gamma\delta_0 u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}), \end{cases}$$

where $\gamma \le 0$, δ_0 denotes the Dirac delta with the mass at the origin, and $p > 5$. By a result of Fukuizumi, Ohta, and Ozawa (2008), it is known that the system above is locally well-posed in the energy space $H^1(\mathbb{R})$ and there exist standing wave solutions $e^{i\omega t}Q_{\omega,\gamma}(x)$ when $\omega > \frac{1}{2}\gamma^2$, where $Q_{\omega,\gamma}$ is a unique radial positive solution to $-\frac{1}{2}\partial_x^2 Q + \omega Q - \gamma\delta_0 Q = |Q|^{p-1}Q$. Our aim in the present paper is to find a necessary and sufficient condition on the data below the standing wave $e^{i\omega t}Q_{\omega,0}$ to determine the global behavior of the solution. The similar result for NLS without potential ($\gamma = 0$) was obtained by Akahori and Nawa (2013); the scattering result was also extended by Fang, Xie, and Cazenave (2011). Our proof of the scattering result is based on the argument of Banica and Visciglia (2016), who proved all solutions scatter in the defocusing and repulsive case ($\gamma < 0$) by the Kenig–Merle method (2006). However, the method of Banica and Visciglia cannot be applicable to our problem because the energy may be negative in the focusing case. To overcome this difficulty, we use the variational argument based on the work of Ibrahim, Masmoudi, and Nakanishi (2011). Our proof of the blow-up result is based on the method of Du, Wu, and Zhang (2016). Moreover, we determine the global dynamics of the radial solution whose mass-energy is larger than that of the standing wave $e^{i\omega t}Q_{\omega,0}$. The difference comes from the existence of the potential.

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1. Introduction

1A. ***Background.*** We consider the focusing mass-supercritical semilinear Schrödinger equation with a repulsive Dirac delta potential on the real line $\mathbb{R}$:

MSC2010: 35P25, 35Q55, 47J35.
Keywords: global dynamics, standing waves, nonlinear Schrödinger equation, Dirac delta potential.

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u + \gamma\delta_0 u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \tag{δNLS}$$

where $\gamma \leq 0$, δ_0 denotes the Dirac delta with the mass at the origin, and $p > 5$. The system (δNLS) appears in a wide variety of physical models with a point defect on the line; see [Goodman et al. 2004] and the references therein. We define the Schrödinger operator H_γ as the formulation of a formal expression $-\frac{1}{2}\partial_x^2 - \gamma\delta_0$:

$$H_\gamma \varphi := -\tfrac{1}{2}\partial_x^2 \varphi, \quad \varphi \in \mathcal{D}(H_\gamma),$$
$$\mathcal{D}(H_\gamma) := \{\varphi \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\setminus\{0\}) : \partial_x\varphi(0+) - \partial_x\varphi(0-) = -2\gamma\varphi(0)\}.$$

H_γ is a nonnegative self-adjoint operator on $L^2(\mathbb{R})$ (see [Albeverio et al. 2005] for more details), which implies that (δNLS) is locally well-posed in the energy space $H^1(\mathbb{R})$.

Proposition 1.1 [Fukuizumi et al. 2008, Section 2; Cazenave 2003, Theorem 3.7.1]. *For any $u_0 \in H^1(\mathbb{R})$, there exist $T_\pm = T_\pm(\|u_0\|_{H^1}) > 0$ and a unique solution*

$$u \in C\big((-T_-, T_+); H^1(\mathbb{R})\big) \cap C^1\big((-T_-, T_+); H^{-1}(\mathbb{R})\big)$$

of (δNLS). Moreover, the following statements hold:

- (*blow-up criterion*) *$T_\pm = \infty$, or $T_\pm < \infty$ and $\lim_{t\to\pm T_\pm} \|\partial_x u(t)\|_{L^2} = \infty$, where the double-sign corresponds.*
- (*conservation laws*) *The energy E and the mass M are conserved by the flow; i.e.,*

$$E(u(t)) = E(u_0), \quad M(u(t)) = M(u_0) \quad \text{for any } t \in (-T_-, T_+),$$

where for $\varphi \in H^1(\mathbb{R})$, we define E and M as

$$E(\varphi) = E_\gamma(\varphi) := \tfrac{1}{4}\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma|\varphi(0)|^2 - \frac{1}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}, \tag{1-1}$$
$$M(\varphi) := \tfrac{1}{2}\|\varphi\|_{L^2}^2. \tag{1-2}$$

We investigate the global behaviors of the solution. By the choice of the initial data, (δNLS) has various solutions, for example, scattering solutions, blow-up solutions, and so on. Let us recall the definitions of scattering and blow-up. Let u be a solution to (δNLS) on the maximal existence time interval $(-T_-, T_+)$.

Definition 1.1 (scattering). We say that the solution u to (δNLS) scatters if and only if $T_\pm = \infty$ and there exist $u_\pm \in H^1(\mathbb{R})$ such that

$$\|u(t) - e^{-itH_\gamma}u_\pm\|_{H^1} \to 0 \quad \text{as } t \to \pm\infty,$$

where $\{e^{-itH_\gamma}\}$ denotes the evolution group of $i\partial_t u - H_\gamma u = 0$.

Definition 1.2 (blow-up). We say that the solution u to (δNLS) blows up in positive time (resp. negative time) if and only if $T_+ < \infty$ (resp. $T_- < \infty$).

Since a pioneer work by Kenig and Merle [2006], the global dynamics without assuming smallness for focusing nonlinear Schrödinger equations have been studied. For the focusing cubic semilinear Schrödinger equation in three dimensions, Holmer and Roudenko [2008] proved that $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$ implies scattering and, on the other hand, $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} > \|Q\|_{L^2}\|\nabla Q\|_{L^2}$ implies finite-time blow-up if the initial data $u_0 \in H^1(\mathbb{R}^3)$ is radially symmetric and satisfies the mass-energy condition $M(u_0)E(u_0) < M(Q)E(Q)$, where Q is the ground state. For nonradial solutions, Duyckaerts, Holmer, and Roudenko [Duyckaerts et al. 2008] proved the scattering part and Holmer and Roudenko [2010] proved the solutions in the above blow-up region blow up in finite time or grow up in infinite time. Fang, Xie, and Cazenave [Fang et al. 2011] extended the scattering result and Akahori and Nawa [2013] extended both the scattering and the blow-up result to mass-supercritical and energy-subcritical Schrödinger equations in general dimensions.

Recently, Banica and Visciglia [2016] proved all solutions scatter in the defocusing case. On the other hand, in the focusing case, (δNLS) has blow-up solutions and nonscattering global solutions. Thus, their method cannot be applicable to our problem.

1B. ***Main results.*** To state our main result, we introduce several notations.

Let ω be a positive parameter that denotes the frequency. We define action S_ω and a functional P as

$$S_\omega(\varphi) = S_{\omega,\gamma}(\varphi) := E(\varphi) + \omega M(\varphi) = \tfrac{1}{4}\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma|\varphi(0)|^2 + \tfrac{1}{2}\omega\|\varphi\|_{L^2}^2 - \frac{1}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}, \quad (1\text{-}3)$$

$$P(\varphi) = P_\gamma(\varphi) := \tfrac{1}{2}\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma|\varphi(0)|^2 - \frac{p-1}{2(p+1)}\|\varphi\|_{L^{p+1}}^{p+1}, \quad (1\text{-}4)$$

where P appears in the virial identity (see [Le Coz et al. 2008]).

We often omit the index γ. We sometimes insert 0 into γ, such as $S_{\omega,0}$ and P_0.

We consider the three minimizing problems

$$n_\omega := \inf\{S_\omega(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ P(\varphi)=0\}, \quad (1\text{-}5)$$

$$r_\omega := \inf\{S_\omega(\varphi) : \varphi \in H^1_{\mathrm{rad}}(\mathbb{R})\setminus\{0\},\ P(\varphi)=0\}, \quad (1\text{-}6)$$

$$l_\omega := \inf\{S_{\omega,0}(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ P_0(\varphi)=0\}, \quad (1\text{-}7)$$

where $H^1_{\mathrm{rad}}(\mathbb{R}) := \{\varphi \in H^1(\mathbb{R}) : \varphi(x) = \varphi(-x)\}$.

Equation (1-7) is nothing but the minimizing problem for the nonlinear Schrödinger equation without a potential, and l_ω is positive and is attained by

$$Q_{\omega,0}(x) := \left\{\frac{(p+1)\omega}{2}\operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{\sqrt{2}}|x|\right)\right\}^{\frac{1}{p-1}},$$

which is a unique positive solution of

$$-\tfrac{1}{2}\partial_x^2 Q + \omega Q = |Q|^{p-1}Q. \quad (1\text{-}8)$$

For n_ω and r_ω, we prove the following statements, some of which were proved by Fukuizumi and Jeanjean [2008].

Proposition 1.2. *Let γ be strictly negative. Then the following statements are true*:

(1) *$n_\omega = l_\omega$ and n_ω is not attained.*

(2) *$n_\omega < r_\omega$ and*

$$\begin{cases} r_\omega = 2l_\omega & \text{if } 0 < \omega \le \frac{1}{2}\gamma^2, \\ r_\omega < 2l_\omega & \text{if } \omega > \frac{1}{2}\gamma^2. \end{cases}$$

(3) *If $\omega > \frac{1}{2}\gamma^2$, then r_ω is attained by*

$$Q_\omega(x) = Q_{\omega,\gamma}(x) := \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2\left(\frac{(p-1)\sqrt{\omega}}{\sqrt{2}} |x| + \tanh^{-1}\left(\frac{\gamma}{\sqrt{2\omega}} \right) \right) \right\}^{\frac{1}{p-1}},$$

which is a unique positive solution of $-\frac{1}{2}\partial_x^2 Q + \omega Q - \gamma\delta_0 Q = |Q|^{p-1}Q$. On the other hand, r_ω is not attained if $0 < \omega \le \frac{1}{2}\gamma^2$.

The function $e^{i\omega t}Q_\omega$ with $\omega > \frac{1}{2}\gamma^2$ is a global nonscattering solution to (δNLS), which is called the standing wave. The fact that $n_\omega \neq r_\omega$ comes from the existence of the potential, which means that the following main result in the radial case does not follow from that in the nonradial case.

By using the minimizing problems, we define subsets in $H^1(\mathbb{R})$ for $\omega > 0$ as follows:

$$\mathcal{N}_\omega^+ := \{\varphi \in H^1(\mathbb{R}) : S_\omega(\varphi) < n_\omega,\, P(\varphi) \ge 0\},$$
$$\mathcal{N}_\omega^- := \{\varphi \in H^1(\mathbb{R}) : S_\omega(\varphi) < n_\omega,\, P(\varphi) < 0\},$$

and

$$\mathcal{R}_\omega^+ := \{\varphi \in H^1_{\mathrm{rad}}(\mathbb{R}) : S_\omega(\varphi) < r_\omega,\, P(\varphi) \ge 0\},$$
$$\mathcal{R}_\omega^- := \{\varphi \in H^1_{\mathrm{rad}}(\mathbb{R}) : S_\omega(\varphi) < r_\omega,\, P(\varphi) < 0\}.$$

We state one of our main results, which treats the nonradial case. We classify the global behavior of the solution whose action is less than n_ω.

Theorem 1.3 (nonradial case). *Let $\omega > 0$. Let u be a solution to (δNLS) on $(-T_-, T_+)$ with the initial data $u_0 \in H^1(\mathbb{R})$.*

(1) *If the initial data u_0 belongs to $\mathcal{N}_\omega^+$, then the solution u scatters.*

(2) *If the initial data u_0 belongs to $\mathcal{N}_\omega^-$, then one of the following four cases holds*:

(a) *The solution u blows up in both time directions.*

(b) *The solution u blows up in a positive time, and u is global toward negative time and*

$$\limsup_{t\to-\infty} \|\partial_x u(t)\|_{L^2} = \infty.$$

(c) *The solution u blows up in a negative time, and u is global toward positive time and*

$$\limsup_{t\to\infty} \|\partial_x u(t)\|_{L^2} = \infty.$$

(d) *The solution u is global in both time directions and*

$$\limsup_{t\to\pm\infty} \|\partial_x u(t)\|_{L^2} = \infty.$$

Proposition 1.2 and a direct calculation give $n_\omega = l_\omega = \omega^{\frac{p+3}{2(p-1)}} S_{1,0}(Q_{1,0})$. By these relations, we can rewrite the main theorem in the nonradial case into a version independent of the frequency ω.

Corollary 1.4. *We define the subsets $\mathcal{N}^\pm$ in $H^1(\mathbb{R})$ as*

$$\mathcal{N}^+ := \{\varphi \in H^1(\mathbb{R}) : E(\varphi)M(\varphi)^\sigma < E_0(Q_{1,0})M(Q_{1,0})^\sigma,\ P(\varphi) \geq 0\},$$
$$\mathcal{N}^- := \{\varphi \in H^1(\mathbb{R}) : E(\varphi)M(\varphi)^\sigma < E_0(Q_{1,0})M(Q_{1,0})^\sigma,\ P(\varphi) < 0\},$$

where $\sigma := (p+3)/(p-5)$. Let u be a solution to (δNLS) on $(-T_-, T_+)$ with the initial data $u_0 \in H^1(\mathbb{R})$. Then, we can prove the same conclusion as in Theorem 1.3, where $\mathcal{N}_\omega^\pm$ is replaced by $\mathcal{N}^\pm$, respective of the sign.

The equivalency is proved in the Appendix.

Next, we state the other main result for radial solutions. If we restrict solutions to (δNLS) to radial solutions, then we can classify the global behavior of the radial solutions whose action is larger than n_ω and less than r_ω.

Theorem 1.5 (radial case). *Let $\omega > 0$ and u be a solution to (δNLS) with the initial data $u_0 \in H^1_{\mathrm{rad}}(\mathbb{R})$. Then, we can prove the same conclusion as in Theorem 1.3, where $\mathcal{N}_\omega^\pm$ is replaced by $\mathcal{R}_\omega^\pm$, respective of the sign.*

Remark 1.1. Even if solutions to (δNLS) are restricted to radial ones, the possibility that (b)–(d) (grow-up) occurs cannot be excluded since we consider one spatial dimension. In [Le Coz et al. 2008], it was proved that if the initial data satisfies $xu_0 \in L^2$ and $P(u_0) < 0$, then the solution blows up in a finite time in both time directions.

1C. *Difficulties and idea for the proofs.* Our proof of the scattering part is based on the argument of Banica and Visciglia [2016], where they proved all solutions scatter in the defocusing case. We also use a concentration compactness argument (see Sections 3C–3E) and a rigidity argument (see Section 3E). In the focusing case, it is not clear that each profile has positive energy when we use profile decomposition. To prove this with $\gamma = 0$, the orthogonality property of the functional P_0 was used in [Fang et al. 2011; Akahori and Nawa 2013]. However, it is not easy to prove the orthogonality of the functional P_γ because of the presence of the Dirac delta potential ($\gamma \neq 0$). To overcome this difficulty, we use the Nehari functional $I_{\omega,\gamma}$ (see (2-7) for the definition) instead of P_γ. Then we can prove that the subsets for the data defined by I_ω instead of P are the same as the subsets $\mathcal{N}_\omega^\pm$ (see Proposition 2.15) using an argument similar to that of [Ibrahim et al. 2011].

Theorem 1.5 (radial case) does not follow from Theorem 1.3 (nonradial case) since we treat solutions whose action is larger than or equal to n_ω in Theorem 1.5. Recently, Killip, Murphy, Visan, and Zheng [Killip et al. 2016] also considered a similar problem and extended the region to classify solutions under radial assumption for NLS with the inverse-square potential. They used the radial Sobolev inequality, which is only effective in higher dimensions, to prove a translation parameter in the linear profile decomposition is bounded. However, this method cannot be applied to our problem. In the one-dimensional case, it is not clear whether the translation parameter is bounded or not. To avoid this difficulty, we use the fact that the translation parameter $-x_n$ appears in the profile decomposition if x_n appears (see Theorem 3.5 for more detail).

Next, we explain the blow-up results. Holmer and Roudenko [2010] proved a blow-up result for the cubic Schrödinger equation without potentials in three dimensions by applying the Kenig–Merle method [2006]. Recently, Du, Wu, and Zhang [Du et al. 2016] gave a simpler proof for blow-up, in which they only used the localized virial identity. We apply their method to the equation with a potential.

1D. *Construction of the paper.* In Section 2, we consider the minimizing problems from the viewpoint of variational argument. We prove the existence and nonexistence of a minimizer for r_ω and n_ω, and that the subsets for the data defined by I_ω instead of P are the same as the subsets in $H^1(\mathbb{R})$ defined by P in this section. In Section 3, we prove the scattering results by a concentration compactness argument and a rigidity argument. We explain the necessity of the Nehari functional I_ω instead of P. In Section 4, we prove the blow-up results, based on the argument of Du et al. [2016].

2. Minimizing problems and variational structure

2A. *Minimizing problems.* Let (α,β) satisfy the conditions

$$\alpha>0,\quad 2\alpha-\beta\geq 0,\quad 2\alpha+\beta\geq 0,\quad (\alpha,\beta)\neq(0,0). \tag{2-1}$$

We set

$$\bar{\mu}:=\max\{2\alpha-\beta,2\alpha+\beta\},\quad \underline{\mu}:=\min\{2\alpha-\beta,2\alpha+\beta\}.$$

We define a scaling transformation and a derivative of functional as

$$\varphi_\lambda^{\alpha,\beta}(x):=e^{\alpha\lambda}\varphi(e^{-\beta\lambda}x), \tag{2-2}$$

$$\mathcal{L}_{\lambda_0}^{\alpha,\beta}S(\varphi):=\partial_\lambda S(\varphi_\lambda^{\alpha,\beta})|_{\lambda=\lambda_0}, \tag{2-3}$$

$$\mathcal{L}^{\alpha,\beta}S(\varphi):=\mathcal{L}_0^{\alpha,\beta}S(\varphi) \tag{2-4}$$

for any function φ and any functional $S:H^1(\mathbb{R})\to\mathbb{R}$. We define functionals $K_\omega^{\alpha,\beta}$ by

$$\begin{aligned}K_\omega^{\alpha,\beta}(\varphi)&=K_{\omega,\gamma}^{\alpha,\beta}(\varphi)\\&:=\mathcal{L}^{\alpha,\beta}S_\omega(\varphi)\\&=\partial_\lambda S_\omega(e^{\alpha\lambda}\varphi(e^{-\beta\lambda}\,\cdot))|_{\lambda=0}\\&=\tfrac14(2\alpha-\beta)\|\partial_x\varphi\|_{L^2}^2+\tfrac12\omega(2\alpha+\beta)\|\varphi\|_{L^2}^2-\gamma\alpha|\varphi(0)|^2-\frac{(p+1)\alpha+\beta}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}.\end{aligned} \tag{2-5}$$

We especially use the following functionals:

$$P(\varphi)=P_\gamma(\varphi):=K_\omega^{\frac12,-1}(\varphi)=\tfrac12\|\partial_x\varphi\|_{L^2}^2-\tfrac12\gamma|\varphi(0)|^2-\frac{p-1}{2(p+1)}\|\varphi\|_{L^{p+1}}^{p+1}, \tag{2-6}$$

$$I_\omega(\varphi)=I_{\omega,\gamma}(\varphi):=K_\omega^{1,0}(\varphi)=\tfrac12\|\partial_x\varphi\|_{L^2}^2-\gamma|\varphi(0)|^2+\omega\|\varphi\|_{L^2}^2-\|\varphi\|_{L^{p+1}}^{p+1}. \tag{2-7}$$

Remark 2.1. Both the functional P, which appears in the virial identity (3-2), and the Nehari functional I_ω are used to prove the scattering results. It is proved in Proposition 2.15 that P and I_ω have same sign under a condition for the action. To prove this, we introduce the parameter (α, β) based on [Ibrahim et al. 2011].

We also use $J_\omega^{\alpha,\beta}$ defined by

$$J_\omega^{\alpha,\beta}(\varphi) = J_{\omega,\gamma}^{\alpha,\beta}(\varphi) := S_\omega(\varphi) - \frac{K_\omega^{\alpha,\beta}(\varphi)}{\bar{\mu}}. \tag{2-8}$$

Lemma 2.1. *We have the relations*

$$(\mathcal{L}^{\alpha,\beta} - \bar{\mu})\|\partial_x\varphi\|_{L^2}^2 = \begin{cases} 0 & \text{if } \beta \le 0, \\ -2\beta\|\partial_x\varphi\|_{L^2}^2 & \text{if } \beta > 0, \end{cases}$$

$$(\mathcal{L}^{\alpha,\beta} - \bar{\mu})\|\varphi\|_{L^2}^2 = \begin{cases} 2\beta\|\varphi\|_{L^2}^2 & \text{if } \beta \le 0, \\ 0 & \text{if } \beta > 0, \end{cases}$$

$$(\mathcal{L}^{\alpha,\beta} - \bar{\mu})|\varphi(0)|^2 = \begin{cases} \beta|\varphi(0)|^2 & \text{if } \beta \le 0, \\ -\beta|\varphi(0)|^2 & \text{if } \beta > 0, \end{cases}$$

$$(\mathcal{L}^{\alpha,\beta} - \bar{\mu})\|\varphi\|_{L^{p+1}}^{p+1} = \begin{cases} ((p-1)\alpha + 2\beta)\|\varphi\|_{L^{p+1}}^{p+1} & \text{if } \beta \le 0, \\ (p-1)\alpha\|\varphi\|_{L^{p+1}}^{p+1} & \text{if } \beta > 0. \end{cases}$$

In particular,

$$\bar{\mu}J_\omega^{\alpha,\beta} = (\bar{\mu} - \mathcal{L}^{\alpha,\beta})S_\omega(\varphi) \ge |\beta| \min\{\tfrac{1}{2}\|\partial_x\varphi\|_{L^2}^2, \omega\|\varphi\|_{L^2}^2\} - \tfrac{1}{2}\gamma|\beta|\,|\varphi(0)|^2 + \frac{(p-5)\alpha}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}.$$

Moreover, *we have*

$$\begin{aligned} -(\mathcal{L}^{\alpha,\beta} - \bar{\mu})(\mathcal{L}^{\alpha,\beta} - \underline{\mu})S_\omega(\varphi) &= (\mathcal{L}^{\alpha,\beta} - \bar{\mu})(\mathcal{L}^{\alpha,\beta} - \underline{\mu})\left(\tfrac{1}{2}\gamma|\varphi(0)|^2 + \frac{\|\varphi\|_{L^{p+1}}^{p+1}}{p+1}\right) \\ &\ge -\tfrac{1}{2}\gamma|\beta|^2|\varphi(0)|^2 + \frac{(p-5)\alpha}{p+1}\mathcal{L}^{\alpha,\beta}\|\varphi\|_{L^{p+1}}^{p+1} \ge \frac{(p-5)\alpha\bar{\mu}}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Proof. These relations are obtained by simple calculations. We only note that

$$(p-1)\alpha + 2\beta = (p-5)\alpha + 2(2\alpha + \beta) \ge (p-5)\alpha. \qquad \square$$

By this lemma and $p > 5$, we find that $J_\omega^{\alpha,\beta}(\varphi) \ge 0$ for any $\varphi \in H^1(\mathbb{R})$. Next, we see that $K_\omega^{\alpha,\beta}$ is positive near the origin in $H^1(\mathbb{R})$.

Lemma 2.2. *Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})\setminus\{0\}$ be bounded in $L^2(\mathbb{R})$ such that $\|\partial_x\varphi_n\|_{L^2} \to 0$ as $n \to \infty$. Then $K_\omega^{\alpha,\beta}(\varphi_n) > 0$ for large $n \in \mathbb{N}$.*

Proof. By $\gamma < 0$, $p > 5$, and the Gagliardo–Nirenberg inequality, we have

$$K_\omega^{\alpha,\beta}(\varphi_n) \ge \tfrac{1}{4}(2\alpha - \beta)\|\partial_x\varphi_n\|_{L^2}^2 - \frac{(p+1)\alpha + \beta}{p+1}C\|\partial_x\varphi_n\|_{L^2}^{\frac{1}{2}(p-1)}\|\varphi_n\|_{L^2}^{\frac{1}{2}(p+3)} > 0$$

for sufficiently large $n \in \mathbb{N}$, where C is a positive constant. $\square$

We define the following minimizing problems for $\omega > 0$ and (α, β) satisfying (2-1):

$$n_\omega^{\alpha,\beta} := \inf\{S_\omega(\varphi) : \varphi \in H^1(\mathbb{R}) \setminus \{0\},\ K_\omega^{\alpha,\beta}(\varphi) = 0\}, \tag{2-9}$$

$$r_\omega^{\alpha,\beta} := \inf\{S_\omega(\varphi) : \varphi \in H^1_{\mathrm{rad}}(\mathbb{R}) \setminus \{0\},\ K_\omega^{\alpha,\beta}(\varphi) = 0\}, \tag{2-10}$$

$$l_\omega^{\alpha,\beta} := \inf\{S_{\omega,0}(\varphi) : \varphi \in H^1(\mathbb{R}) \setminus \{0\},\ K_{\omega,0}^{\alpha,\beta}(\varphi) = 0\}. \tag{2-11}$$

If $(\alpha, \beta) = \left(\frac{1}{2}, -1\right)$, these are nothing but n_ω, r_ω, and l_ω. We prove that these minimizing problems are independent of (α, β) and Proposition 1.2 holds in the following subsections.

2B. *Radial minimizing problem.* First, we consider the radial minimizing problem $r_\omega^{\alpha,\beta}$. For $\gamma \le 0$, $S_\omega : H^1_{\mathrm{rad}}(\mathbb{R}) \to \mathbb{R}$ satisfies the following mountain pass structure:

(1) $S_\omega(0) = 0$.

(2) There exist $\delta, \rho > 0$ such that $S_\omega(\varphi) > \delta$ for all φ with $\|\varphi\|_{H^1} = \rho$.

(3) There exists $\psi \in H^1_{\mathrm{rad}}(\mathbb{R})$ such that $S_\omega(\psi) < 0$ and $\|\psi\|_{H^1} > \rho$.

Indeed, (1) is trivial, (2) can be proved by the Gagliardo–Nirenberg inequality, and (3) is obtained by a scaling argument.

Let

$$\mathcal{C} := \left\{c \in C\left([0, 1] : H^1_{\mathrm{rad}}(\mathbb{R})\right) : c(0) = 0,\ S_\omega(c(1)) < 0\right\},$$
$$b := \inf_{c \in \mathcal{C}} \max_{t \in [0,1]} S_\omega(c(t)).$$

Lemma 2.3. *The identity $b = r_\omega^{\alpha,\beta}$ holds.*

Proof. First, we prove $b \le r_\omega^{\alpha,\beta}$. To see this, it is sufficient to prove the existence of $\{c_n\} \subset \mathcal{C}$ such that $\max_{t \in [0,1]} S_\omega(c_n(t)) \to r_\omega^{\alpha,\beta}$ as $n \to \infty$. We take a minimizing sequence $\{\varphi_n\}$ for $r_\omega^{\alpha,\beta}$, namely,

$$S_\omega(\varphi_n) \to r_\omega^{\alpha,\beta} \quad \text{as } n \to \infty \qquad \text{and} \qquad K_\omega^{\alpha,\beta}(\varphi_n) = 0 \quad \text{for all } n \in \mathbb{N}.$$

We set $\tilde{c}_n(\lambda) := \mathcal{L}_\lambda^{\alpha,\beta} \varphi_n$ for $\lambda \in \mathbb{R}$. Then, we see that $S_\omega(\tilde{c}_n(\lambda)) < 0$ for large λ. Moreover,

$$\max_{\lambda \in \mathbb{R}} S_\omega(\tilde{c}_n(\lambda)) = S_\omega(\tilde{c}_n(0)) = S_\omega(\varphi_n) \to r_\omega^{\alpha,\beta} \quad \text{as } n \to \infty$$

since $K_\omega^{\alpha,\beta}(\varphi_n) = 0$ for all $n \in \mathbb{N}$. We define $\mathscr{C}_n(t)$ for $t \in [-L, L]$ such that

$$\mathscr{C}_n(t) := \begin{cases} \tilde{c}_n(t) & \text{if } -\frac{1}{2}L \le t \le L, \\ \left(\frac{2}{L}(t + L)\right)^M \tilde{c}_n\left(-\frac{L}{2}\right) & \text{if } -L \le t < -\frac{1}{2}L. \end{cases}$$

$\mathscr{C}$ is continuous in $H^1(\mathbb{R})$ and we have $S_\omega(\mathscr{C}_n(L)) < 0$ and $\max_{t \in [-L,L]} S_\omega(\mathscr{C}_n(t)) = S_\omega(\varphi_n) \to r_\omega^{\alpha,\beta}$ when $L > 0$ and $M = M(n)$ are sufficiently large. By changing variables, we obtain a desired sequence $c_n \in \mathcal{C}$. Next, we prove $b \ge r_\omega^{\alpha,\beta}$. It is sufficient to prove

$$c([0, 1]) \cap \left\{\varphi \in H^1_{\mathrm{rad}}(\mathbb{R}) \setminus \{0\} : K_\omega^{\alpha,\beta}(\varphi) = 0\right\} \ne \varnothing \quad \text{for all } c \in \mathcal{C}.$$

We take an arbitrary $c \in \mathcal{C}$. Now, $c(0) = 0$ and $S_\omega(c(1)) < 0$. Therefore, $K_\omega^{\alpha,\beta}(c(t)) > 0$ for some $t \in (0, 1)$ by Lemma 2.2 and $K_\omega^{\alpha,\beta}(c(1)) \le ((p+1)\alpha + \beta)S_\omega(c(1)) < 0$. By continuity, there exists $t_0 \in (0, 1)$ such that $K_\omega^{\alpha,\beta}(c(t_0)) = 0$. Thus, we get $b = r_\omega^{\alpha,\beta}$. □

Next, we prove the existence and nonexistence of a minimizer for the minimizing problem $r_\omega^{\alpha,\beta}$. See [Fukuizumi and Jeanjean 2008, Lemmas 15, 19, 20, 21, and 25] for the proofs of the following Lemmas 2.4, 2.5, 2.6, 2.7, and 2.8, respectively.

The following lemma means that it is sufficient to find a nonnegative minimizer.

Lemma 2.4. *If $\varphi \in H^1(\mathbb{R})$ is a minimizer of $r_\omega^{\alpha,\beta}$, then $|\varphi| \in H^1(\mathbb{R})$ is also a minimizer.*

Definition 2.1 (Palais–Smale sequence). We say that $\{\varphi_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$ is a Palais–Smale sequence for S_ω at the level c if and only if the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfies

$$S_\omega(\varphi_n) \to c \qquad \text{and} \qquad S'_\omega(\varphi_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}) \qquad \text{as } n \to \infty.$$

By the mountain pass theorem, we obtain a Palais–Smale sequence at the level $b = r_\omega^{\alpha,\beta}$. We may assume that the sequence is bounded.

Lemma 2.5. *Any Palais–Smale sequence of S_ω considered on $H^1_{\mathrm{rad}}(\mathbb{R})$ is also a Palais–Smale sequence of S_ω considered on $H^1(\mathbb{R})$. In particular, a critical point of S_ω considered on $H^1_{\mathrm{rad}}(\mathbb{R})$ is also a critical point of S_ω considered on $H^1(\mathbb{R})$.*

Lemma 2.6. *Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$ be a bounded Palais–Smale sequence at the level c for S_ω. Then there exists a subsequence still denoted by $\{\varphi_n\}$ for which the following holds: there exist a critical point φ_0 of S_ω, an integer $k \ge 0$, for $j = 1, \dots, k$, a sequence of points $\{x_n^j\} \subset \mathbb{R}$, and nontrivial solutions $v^j(x)$ of the equation (1-8) satisfying*

$$\varphi_n \rightharpoonup \varphi_0 \quad \textit{weakly in } H^1(\mathbb{R}),$$
$$S_\omega(\varphi_n) \to c = S_\omega(\varphi_0) + \sum_{j=1}^{k} S_{\omega,0}(v^j),$$
$$\varphi_n - \Big(\varphi_0 + \sum_{j=1}^{k} v^j(x - x_n^j)\Big) \to 0 \quad \textit{strongly in } H^1(\mathbb{R}),$$
$$|x_n^j| \to \infty, \quad |x_n^j - x_n^i| \to \infty \quad \textit{for } 1 \le j \ne i \le k$$

as $n \to \infty$, where we agree that in the case $k = 0$, the above holds without v^j and x_n^j.

Lemma 2.7. *Assume that*

$$r_\omega^{\alpha,\beta} < 2 l_\omega^{\alpha,\beta}.$$

Then the bounded Palais–Smale sequence at the level $r_\omega^{\alpha,\beta}$ admits a strongly convergent subsequence.

Lemma 2.8. *If $\varphi \in H^1(\mathbb{R}) \setminus \{0\}$ is a critical point of S_ω, that is, φ satisfies*

$$-\tfrac{1}{2}\partial_x^2\varphi + \omega\varphi - \gamma\delta_0\varphi = |\varphi|^{p-1}\varphi \tag{2-12}$$

in the distribution sense, then it satisfies

$$\begin{aligned}
&\varphi \in C^j(\mathbb{R}\setminus\{0\}) \cap C(\mathbb{R}), \quad j=1,2,\\
&-\tfrac{1}{2}\partial_x^2\varphi + \omega\varphi = |\varphi|^{p-1}\varphi, \quad x \neq 0,\\
&\partial_x\varphi(0+) - \partial_x\varphi(0-) = -2\gamma\varphi(0),\\
&\partial_x\varphi(x), \varphi(x) \to 0 \quad \text{as } |x| \to \infty.
\end{aligned}$$

Lemma 2.9. *There exists a unique positive classical solution φ of (2-12) if and only if $\omega > \frac{1}{2}\gamma^2$. It is nothing but Q_ω. If $0 < \omega \le \frac{1}{2}\gamma^2$, then the classical solution does not exist.*

Proof. We have a unique positive classical solution $Q_{\omega,0}$ of (1-8). If $\omega > \frac{1}{2}\gamma^2$, then we get a classical solution φ of (2-12) by the translation of $Q_{\omega,0}$. See [Fukuizumi and Jeanjean 2008] for more detail. □

Lemma 2.10. *The inequality $r_\omega^{\alpha,\beta} < 2l_\omega^{\alpha,\beta}$ holds when $\omega > \frac{1}{2}\gamma^2$.*

Proof. When $\omega > \frac{1}{2}\gamma^2$, we know Q_ω is well defined. We find that Q_ω satisfies $K_\omega^{\alpha,\beta}(Q_\omega) = 0$ and $S_\omega(Q_\omega) < 2l_\omega^{\alpha,\beta}$ by direct calculations. □

By Lemmas 2.7 and 2.10, we find that when $\omega > \frac{1}{2}\gamma^2$, the function Q_ω attains $r_\omega^{\alpha,\beta}$.

Lemma 2.11. *If $0 < \omega \le \frac{1}{2}\gamma^2$, then $r_\omega^{\alpha,\beta} = 2l_\omega^{\alpha,\beta}$ holds.*

Proof. Suppose that $r_\omega^{\alpha,\beta} < 2l_\omega^{\alpha,\beta}$. By Lemmas 2.7 and 2.8, we have a unique positive classical solution of (2-12), which contradicts Lemma 2.9. Thus, it suffices to show $r_\omega^{\alpha,\beta} \le 2l_\omega^{\alpha,\beta}$ for all $\omega > 0$. Let

$$\varphi_n(x) := Q_{\omega,0}(x-n) + Q_{\omega,0}(x+n).$$

Then, $S_\omega(\varphi_n) \to 2l_\omega$ and $K_\omega^{\alpha,\beta}(\varphi_n) \to 0$ as $n \to \infty$. Thus, there exists a sequence $\{\lambda_n\}$ such that $K_\omega^{\alpha,\beta}(\lambda_n\varphi_n) = 0$ and $\lambda_n \to 1$ as $n \to \infty$. Therefore, we have $S_\omega(\lambda_n\varphi_n) \to 2l_\omega$ as $n \to \infty$ and $K_\omega^{\alpha,\beta}(\lambda_n\varphi_n) = 0$ for all $n \in \mathbb{N}$. This means that $r_\omega^{\alpha,\beta} \le 2l_\omega^{\alpha,\beta}$. □

Remark 2.2. The rearrangement argument implies

$$l_\omega^{\alpha,\beta} = \inf\{S_{\omega,0}(\varphi) : \varphi \in H^1_{\mathrm{rad}}(\mathbb{R})\setminus\{0\},\ K_{\omega,0}^{\alpha,\beta}(\varphi) = 0\}.$$

Therefore, the arguments in Section 2B do work for $l_\omega^{\alpha,\beta}$.

2C. *Nonradial minimizing problem.*

In this subsection, we prove $n_\omega^{\alpha,\beta} = l_\omega^{\alpha,\beta}$ and $n_\omega^{\alpha,\beta}$ is not attained.

Lemma 2.12. *We have*

$$l_\omega^{\alpha,\beta} = j_\omega^{\alpha,\beta} := \inf\{J_{\omega,0}^{\alpha,\beta}(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ K_{\omega,0}^{\alpha,\beta}(\varphi) \le 0\}.$$

Proof. First, we prove $j_\omega^{\alpha,\beta} \le l_\omega^{\alpha,\beta}$:

$$\begin{aligned}
j_\omega^{\alpha,\beta} &\le \inf\{J_{\omega,0}^{\alpha,\beta}(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ K_{\omega,0}^{\alpha,\beta}(\varphi) = 0\}\\
&= \inf\{S_{\omega,0}(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ K_{\omega,0}^{\alpha,\beta}(\varphi) = 0\}\\
&= l_\omega^{\alpha,\beta}.
\end{aligned}$$

Next, we prove $l_\omega^{\alpha,\beta} \le j_\omega^{\alpha,\beta}$. We take $\varphi \in H^1(\mathbb{R})\setminus\{0\}$ such that $K_{\omega,0}^{\alpha,\beta}(\varphi) \le 0$. If $K_{\omega,0}^{\alpha,\beta}(\varphi) = 0$, then

$$l_\omega^{\alpha,\beta} \le S_{\omega,0}(\varphi) = J_{\omega,0}^{\alpha,\beta}(\varphi).$$

If $K_{\omega,0}^{\alpha,\beta}(\varphi) < 0$, then there exists $\lambda_* \in (0,1)$ such that $K_{\omega,0}^{\alpha,\beta}(\lambda_*\varphi) = 0$. Indeed, this follows from continuity and the fact that $K_{\omega,0}^{\alpha,\beta}(\lambda\varphi) > 0$ holds for small $\lambda \in (0,1)$ by Lemma 2.2. By $\lambda_* < 1$,

$$l_\omega^{\alpha,\beta} \le S_{\omega,0}(\lambda_*\varphi) = J_{\omega,0}^{\alpha,\beta}(\lambda_*\varphi) \le J_{\omega,0}^{\alpha,\beta}(\varphi).$$

Therefore, we have $l_\omega^{\alpha,\beta} \le J_{\omega,0}^{\alpha,\beta}(\varphi)$ for any $\varphi \in H^1(\mathbb{R})\setminus\{0\}$ such that $K_{\omega,0}^{\alpha,\beta}(\varphi) \le 0$. This implies $l_\omega^{\alpha,\beta} \le j_\omega^{\alpha,\beta}$. Hence, we get $l_\omega^{\alpha,\beta} = j_\omega^{\alpha,\beta}$. □

Let $\tau_y\varphi(x) := \varphi(x-y)$ throughout this paper.

Proposition 2.13. *The identity $n_\omega^{\alpha,\beta} = l_\omega^{\alpha,\beta}$ holds.*

Proof. First, we prove $n_\omega^{\alpha,\beta} \ge l_\omega^{\alpha,\beta}$. We take an arbitrary $\varphi \in H^1(\mathbb{R})\setminus\{0\}$ such that $K_\omega^{\alpha,\beta}(\varphi) = 0$. Since $K_{\omega,0}^{\alpha,\beta}(\varphi) \le K_\omega^{\alpha,\beta}(\varphi) = 0$ due to $\gamma \le 0$, by Lemma 2.12, we then have

$$l_\omega^{\alpha,\beta} \le J_{\omega,0}^{\alpha,\beta}(\varphi) \le J_\omega^{\alpha,\beta}(\varphi),$$

which implies

$$\begin{aligned} l_\omega^{\alpha,\beta} &\le \inf\{J_\omega^{\alpha,\beta}(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ K_\omega^{\alpha,\beta}(\varphi)=0\} \\ &= \inf\{S_\omega(\varphi) : \varphi \in H^1(\mathbb{R})\setminus\{0\},\ K_\omega^{\alpha,\beta}(\varphi)=0\} = n_\omega^{\alpha,\beta}. \end{aligned}$$

Next, we prove $n_\omega^{\alpha,\beta} \le l_\omega^{\alpha,\beta}$. We note that $Q_{\omega,0}$ attains $l_\omega^{\alpha,\beta}$. Then, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ with $y_n \to \infty$ as $n \to \infty$ such that $S_\omega(\tau_{y_n}Q_{\omega,0}) \to S_{\omega,0}(Q_{\omega,0}) = l_\omega^{\alpha,\beta}$ as $n \to \infty$. For this $\{y_n\}$,

$$K_\omega^{\alpha,\beta}(\tau_{y_n}Q_{\omega,0}) \ge K_{\omega,0}^{\alpha,\beta}(\tau_{y_n}Q_{\omega,0}) = K_{\omega,0}^{\alpha,\beta}(Q_{\omega,0}) = 0$$

holds for all $n \in \mathbb{N}$. Since $K_\omega^{\alpha,\beta}(\lambda\tau_{y_n}Q_{\omega,0}) < 0$ for large $\lambda > 1$ and $K_\omega^{\alpha,\beta}(\tau_{y_n}Q_{\omega,0}) > 0$, there exists $\lambda_n > 1$ such that $K_\omega^{\alpha,\beta}(\lambda_n\tau_{y_n}Q_{\omega,0}) = 0$ by continuity. For this $\{\lambda_n\}$, we have $\lambda_n \to 1$ as $n \to \infty$. Indeed, since

$$\begin{aligned} 0 &= K_\omega^{\alpha,\beta}(\lambda_n\tau_{y_n}Q_{\omega,0}) \\ &= \lambda_n^2\big(\tfrac14(2\alpha-\beta)\|\partial_x\tau_{y_n}Q_{\omega,0}\|_{L^2}^2 + \tfrac12\omega(2\alpha+\beta)\|\tau_{y_n}Q_{\omega,0}\|_{L^2}^2 - \gamma\alpha|\tau_{y_n}Q_{\omega,0}(0)|^2\big) \\ &\qquad - \lambda_n^{p+1}\frac{(p+1)\alpha+\beta}{p+1}\|\tau_{y_n}Q_{\omega,0}\|_{L^{p+1}}^{p+1}, \end{aligned}$$

and $K_{\omega,0}^{\alpha,\beta}(\tau_{y_n}Q_{\omega,0}) = 0$, we have

$$\begin{aligned} 0 &= \tfrac14(2\alpha-\beta)\|\partial_x\tau_{y_n}Q_{\omega,0}\|_{L^2}^2 + \tfrac12\omega(2\alpha+\beta)\|\tau_{y_n}Q_{\omega,0}\|_{L^2}^2 - \gamma\alpha|\tau_{y_n}Q_{\omega,0}(0)|^2 \\ &\qquad - \lambda_n^{p-1}\frac{(p+1)\alpha+\beta}{p+1}\|\tau_{y_n}Q_{\omega,0}\|_{L^{p+1}}^{p+1} \\ &= (1-\lambda_n^{p-1})\frac{(p+1)\alpha+\beta}{p+1}\|\tau_{y_n}Q_{\omega,0}\|_{L^{p+1}}^{p+1} - \gamma\alpha|\tau_{y_n}Q_{\omega,0}(0)|^2 \\ &= (1-\lambda_n^{p-1})\frac{(p+1)\alpha+\beta}{p+1}\|Q_{\omega,0}\|_{L^{p+1}}^{p+1} - \gamma\alpha|\tau_{y_n}Q_{\omega,0}(0)|^2. \end{aligned}$$

Therefore, $\lambda_n \to 1$, since $|\tau_{y_n} Q_{\omega,0}(0)| \to 0$ as $n \to \infty$. Hence, $S_\omega(\lambda_n \tau_{y_n} Q_{\omega,0}) \to S_{\omega,0}(Q_{\omega,0}) = l_\omega^{\alpha,\beta}$ as $n \to \infty$ and $K_\omega^{\alpha,\beta}(\lambda_n \tau_{y_n} Q_{\omega,0}) = 0$ for all $n \in \mathbb{N}$. This implies $n_\omega^{\alpha,\beta} \le l_\omega^{\alpha,\beta}$. □

Proposition 2.14. *For any $\omega > 0$, the minimizing problem $n_\omega^{\alpha,\beta}$ is not attained; namely, there does not exist $\varphi \in H^1(\mathbb{R})$ such that $K_\omega^{\alpha,\beta}(\varphi) = 0$ and $S_\omega(\varphi) = n_\omega^{\alpha,\beta}$.*

Proof. We assume that φ attains $n_\omega^{\alpha,\beta}$. If $\varphi(0) = 0$, then $S_{\omega,0}(\varphi) = S_\omega(\varphi) = n_\omega^{\alpha,\beta} = l_\omega^{\alpha,\beta}$ and $K_{\omega,0}^{\alpha,\beta}(\varphi) = K_\omega^{\alpha,\beta}(\varphi) = 0$ holds; that is, φ also attains $l_\omega^{\alpha,\beta}$. By the uniqueness of the ground state for $l_\omega^{\alpha,\beta}$, we know $\varphi = Q_{\omega,0}$. However, $Q_{\omega,0}(0) \neq 0$. Therefore, $\varphi(0) \neq 0$. Now, $|\varphi(x)| \to 0$ as $x \to \infty$ since $\varphi \in H^1(\mathbb{R})$. Hence, $|\varphi(0)| > |\varphi(y)|$ for sufficiently large $|y|$. Thus,

$$K_\omega^{\alpha,\beta}(\tau_y \varphi) < K_\omega^{\alpha,\beta}(\varphi) = 0.$$

Since $K_\omega^{\alpha,\beta}(\lambda \tau_y \varphi) > 0$ for small $\lambda \in (0,1)$ by Lemma 2.2 and $K_\omega^{\alpha,\beta}(\tau_y \varphi) \le 0$, there exists $\lambda_* \in (0,1)$ such that $K_\omega^{\alpha,\beta}(\lambda_* \tau_y \varphi) = 0$ by continuity. By the definition of $n_\omega^{\alpha,\beta}$,

$$n_\omega^{\alpha,\beta} \le J_\omega^{\alpha,\beta}(\lambda_* \tau_y \varphi) < J_\omega^{\alpha,\beta}(\tau_y \varphi) < J_\omega^{\alpha,\beta}(\varphi) \le n_\omega^{\alpha,\beta}.$$

This is a contradiction. □

Since $S_{\omega,0}(Q_{\omega,0}) = l_\omega^{\alpha,\beta} = n_\omega^{\alpha,\beta}$ and $S_{\omega,\gamma}(Q_{\omega,\gamma}) = r_\omega^{\alpha,\beta}$ if $\omega > \frac{1}{2}\gamma^2$, and $2l_\omega^{\alpha,\beta} = r_\omega^{\alpha,\beta}$ if $\omega \le \frac{1}{2}\gamma^2$ hold, we find that $r_\omega^{\alpha,\beta}$, $l_\omega^{\alpha,\beta}$ and $n_\omega^{\alpha,\beta}$ are independent of (α,β) and so we denote $r_\omega^{\alpha,\beta}$, $l_\omega^{\alpha,\beta}$ and $n_\omega^{\alpha,\beta}$ by r_ω, l_ω and n_ω respectively and obtain Proposition 1.2.

2D. ***Variational structure.*** We define subsets $\mathcal{N}_\omega^{\alpha,\beta,\pm}$ and $\mathcal{R}_\omega^{\alpha,\beta,\pm}$ in $H^1(\mathbb{R})$ such that

$$\begin{aligned}
\mathcal{N}_\omega^{\alpha,\beta,+} &:= \{\varphi \in H^1(\mathbb{R}) : S_\omega(\varphi) < n_\omega,\ K_\omega^{\alpha,\beta}(\varphi) \ge 0\},\\
\mathcal{N}_\omega^{\alpha,\beta,-} &:= \{\varphi \in H^1(\mathbb{R}) : S_\omega(\varphi) < n_\omega,\ K_\omega^{\alpha,\beta}(\varphi) < 0\},\\
\mathcal{R}_\omega^{\alpha,\beta,+} &:= \{\varphi \in H^1_{\mathrm{rad}}(\mathbb{R}) : S_\omega(\varphi) < r_\omega,\ K_\omega^{\alpha,\beta}(\varphi) \ge 0\},\\
\mathcal{R}_\omega^{\alpha,\beta,-} &:= \{\varphi \in H^1_{\mathrm{rad}}(\mathbb{R}) : S_\omega(\varphi) < r_\omega,\ K_\omega^{\alpha,\beta}(\varphi) < 0\}.
\end{aligned}$$

We note that $\mathcal{N}_\omega^{\pm} = \mathcal{N}_\omega^{\frac{1}{2},-1,\pm}$ and $\mathcal{R}_\omega^{\pm} = \mathcal{R}_\omega^{\frac{1}{2},-1,\pm}$. From now on, let $(m_\omega, \mathcal{M}_\omega^{\alpha,\beta,\pm})$ denote either $(n_\omega, \mathcal{N}_\omega^{\alpha,\beta,\pm})$ or $(r_\omega, \mathcal{R}_\omega^{\alpha,\beta,\pm})$. The following proposition implies that P and I_ω have same sign if $S_\omega < m_\omega$.

Proposition 2.15. *For any (α,β) satisfying (2-1), $\mathcal{M}_\omega^{\pm} = \mathcal{M}_\omega^{\alpha,\beta,\pm}$.*

Proof. It is easy to check that $\mathcal{M}_\omega^{\alpha,\beta,\pm}$ are open subsets in $H^1(\mathbb{R})$ because of Lemma 2.2. Moreover, we have $0 \in \mathcal{M}_\omega^{\alpha,\beta,+}$ and $\mathcal{M}_\omega^{\alpha,\beta,+} \cup \mathcal{M}_\omega^{\alpha,\beta,-}$ is independent of (α,β). And $\mathcal{M}_\omega^{\alpha,\beta,+}$ are connected if $\mu > 0$ by the scaling contraction argument (see the proof of Lemma 2.9 in [Ibrahim et al. 2011]). Then $\mathcal{M}_\omega^{\alpha,\beta,+} = \mathcal{M}_\omega^{\alpha',\beta',+}$ for $(\alpha,\beta) \neq (\alpha',\beta')$ such that $2\alpha-\beta > 0,\ 2\alpha+\beta > 0$ and $2\alpha'-\beta' > 0,\ 2\alpha'+\beta' > 0$. Of course, then $\mathcal{M}_\omega^{\alpha,\beta,-} = \mathcal{M}_\omega^{\alpha',\beta',-}$.

We take $\{(\alpha_n, \beta_n)\}$ satisfying $2\alpha_n - \beta_n > 0$ and $2\alpha_n + \beta_n > 0$ for all $n \in \mathbb{N}$ and (α_n, β_n) converges to some (α, β) such that $\underline{\mu} = 0$. Then $K_\omega^{\alpha_n,\beta_n} \to K_\omega^{\alpha,\beta}$, and so

$$\mathcal{M}_\omega^{\alpha,\beta,\pm} \subset \bigcup_{n\in\mathbb{N}} \mathcal{M}_\omega^{\alpha_n,\beta_n,\pm}.$$

Since each set in the right-hand side is independent of (α, β), so is the left. □

Let $\|\varphi\|_{\mathcal{H}}^2 := \frac{1}{4}\|\partial_x\varphi\|_{L^2}^2 + \frac{1}{2}\omega\|\varphi\|_{L^2}^2 - \frac{1}{2}\gamma|\varphi(0)|^2$.

Lemma 2.16. *If $P(\varphi) \geq 0$, then*

$$S_\omega(\varphi) \leq \|\varphi\|_{\mathcal{H}}^2 \leq \frac{p-1}{p-5} S_\omega(\varphi),$$

which means that $S_\omega(\varphi)$ is equivalent to $\|\varphi\|_{H^1}^2$.

Proof. The left inequality is trivial. We consider the right inequality:

$$\begin{aligned}
0 &\leq 2P(\varphi) \\
&\leq \|\partial_x\varphi\|_{L^2}^2 - \gamma|\varphi(0)|^2 - \frac{p-1}{p+1}\|\varphi\|_{L^{p+1}}^{p+1} \\
&= -\tfrac{1}{4}(p-5)\|\partial_x\varphi\|_{L^2}^2 + \tfrac{1}{2}(\gamma(p-3))|\varphi(0)|^2 + (p-1)E(\varphi) \\
&\leq -\tfrac{1}{4}(p-5)\|\partial_x\varphi\|_{L^2}^2 + \tfrac{1}{2}(\gamma(p-5))|\varphi(0)|^2 + (p-1)E(\varphi).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\tfrac{1}{4}(p-5)\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{2}\gamma(p-5)|\varphi(0)|^2 + \tfrac{1}{2}(p-5)\omega\|\varphi\|_{L^2}^2 &\leq (p-1)E(\varphi) + (p-5)\omega M(\varphi) \\
&\leq (p-1)(E(\varphi) + \omega M(\varphi)).
\end{aligned}$$

Hence, we obtain

$$\|\varphi\|_{\mathcal{H}}^2 \leq \frac{p-1}{p-5} S_\omega(\varphi).$$ □

Lemma 2.17. *If $u_0 \in \mathcal{M}_\omega^+$, then the corresponding solution u stays in $\mathcal{M}_\omega^+$ for all $t \in (-T_-, T_+)$. Moreover, if $u_0 \in \mathcal{M}_\omega^-$, then the corresponding solution u stays in $\mathcal{M}_\omega^-$ for all $t \in (-T_-, T_+)$.*

Proof. Let $u_0 \in \mathcal{M}_\omega^+$. Since the energy and the mass are conserved, $u(t) \in \mathcal{M}_\omega^+ \cup \mathcal{M}_\omega^-$ for all $t \in (-T_-, T_+)$. We assume that there exists $t_{**} > 0$ such that $u(t_{**}) \in \mathcal{M}_\omega^-$. By continuity, there exists $t_* \in (0, t_{**})$ such that $P(u(t_*)) = 0$ and $P(u(t)) < 0$ for $t \in (t_*, t_{**}]$. By the definition of m_ω, if $u(t_*) \neq 0$, then

$$m_\omega > E(u_0) + \omega M(u_0) = E(u(t_*)) + \omega M(u(t_*)) \geq m_\omega.$$

This is a contradiction. Thus, $u(t_*) = 0$. By the uniqueness of solution, $u = 0$ for all time. This contradicts $u(t_{**}) \in \mathcal{M}_\omega^-$. By the same argument, the second statement can be proved. □

Lemmas 2.16 and 2.17 imply that all the solutions in $\mathcal{M}_\omega^+$ are global in both time directions.

Proposition 2.18 (uniform bounds on P). *There exists $\delta > 0$ such that for any $\varphi \in H^1(\mathbb{R})$ with $S_\omega(\varphi) < m_\omega$, we have*

$$P(\varphi) \geq \min\{2(m_\omega - S_\omega(\varphi), \delta\|\varphi\|_{\mathcal{H}}^2\} \quad \textit{or} \quad P(\varphi) \leq -2(m_\omega - S_\omega(\varphi)).$$

Proof. We may assume $\varphi \neq 0$. Now $s(\lambda) := S_\omega(\varphi^\lambda)$ and $n(\lambda) := \|\varphi^\lambda\|_{L^{p+1}}^{p+1}$, where $\varphi^\lambda(x) := e^{\frac{1}{2}\lambda}\varphi(e^\lambda x)$ for $\lambda \in \mathbb{R}$. Then $s(0) = S_\omega(\varphi)$ and $s'(0) = P(\varphi)$, and we have

$$s(\lambda)=\tfrac{1}{4}e^{2\lambda}\|\partial_x\varphi\|_{L^2}^2+\tfrac{1}{2}\omega\|\varphi\|_{L^2}^2-\tfrac{1}{2}\gamma e^\lambda|\varphi(0)|^2-\frac{e^{\frac{p-1}{2}\lambda}}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}, \quad n(\lambda)=e^{\frac{p-1}{2}\lambda}\|\varphi\|_{L^{p+1}}^{p+1},$$

$$s'(\lambda)=\tfrac{1}{2}e^{2\lambda}\|\partial_x\varphi\|_{L^2}^2-\tfrac{1}{2}\gamma e^\lambda|\varphi(0)|^2-\frac{e^{\frac{p-1}{2}\lambda}(p-1)}{2(p+1)}\|\varphi\|_{L^{p+1}}^{p+1}, \qquad n'(\lambda)=\tfrac{1}{2}e^{\frac{p-1}{2}\lambda}(p-1)\|\varphi\|_{L^{p+1}}^{p+1},$$

$$s''(\lambda)=e^{2\lambda}\|\partial_x\varphi\|_{L^2}^2-\tfrac{1}{2}\gamma e^\lambda|\varphi(0)|^2-\frac{e^{\frac{p-1}{2}\lambda}(p-1)^2}{4(p+1)}\|\varphi\|_{L^{p+1}}^{p+1}, \qquad n''(\lambda)=\tfrac{1}{4}e^{\frac{p-1}{2}\lambda}(p-1)^2\|\varphi\|_{L^{p+1}}^{p+1}.$$

By an easy calculation, we have

$$s'' = 2s' + \tfrac{1}{2}\gamma|\varphi(0)|^2 - \frac{p-5}{2(p+1)}n' \le 2s' - \frac{p-5}{2(p+1)}n' \le 2s'.$$

First, we consider $P < 0$. We have $s'(\lambda) > 0$ for sufficiently small $\lambda < 0$. Therefore, by continuity, there exists $\lambda_0 < 0$ such that $s'(\lambda) < 0$ for $\lambda_0 < \lambda \le 0$ and $s'(\lambda_0) = 0$. Integrating the inequality on $[\lambda_0, 0]$, we have

$$s'(0) - s'(\lambda_0) \le 2(s(0) - s(\lambda_0)).$$

Therefore, we obtain

$$P(\varphi) \le -2(m_\omega - S_\omega(\varphi)).$$

Next, we consider $P \ge 0$. If

$$4P(\varphi) \ge \frac{p-5}{2(p+1)}\mathcal{L}^{\frac{1}{2},-1}\|\varphi\|_{L^{p+1}}^{p+1},$$

then, by adding

$$\frac{p-5}{2}P(\varphi) \ge \frac{p-5}{2}\ \|\varphi\|_{\mathcal{H}}^2 - \frac{p-5}{2(p+1)}\mathcal{L}^{\frac{1}{2},-1}\|\varphi\|_{L^{p+1}}^{p+1}$$

to both sides, we get

$$\left\{4 + \tfrac{1}{2}(p-5)\right\}P(\varphi) \ge \tfrac{1}{2}(p-5)\|\varphi\|_{\mathcal{H}}^2.$$

Thus, we get $P(\varphi) \ge \delta\|\varphi\|_{\mathcal{H}}^2$. If

$$4P(\varphi) < \frac{p-5}{2(p+1)}\mathcal{L}^{\frac{1}{2},-1}\|\varphi\|_{L^{p+1}}^{p+1},$$

then

$$0 < 4s' < \frac{p-5}{2(p+1)}n' \tag{2-13}$$

at $\lambda = 0$. Moreover,

$$s'' \le 4s' - 2s' - \frac{p-5}{2(p+1)}n' < -2s'$$

holds at $\lambda = 0$. Now let λ increase. As long as (2-13) holds and $s' > 0$, we have $s'' < 0$ and so s' decreases and s increases. Since $p > 5$, we also have

$$n'' \geq 2n' \geq 4n > 0$$

for all $\lambda \geq 0$ Hence, (2-13) is preserved until s' reaches 0, which it does at finite $\lambda_1 > 0$. Integrating $s'' < -2s'$ on $[0, \lambda_1]$, we obtain

$$s'(\lambda_1) - s'(0) < -2(s(\lambda_1) - s(0)).$$

Therefore, by the definition of m_ω,

$$P(\varphi) > 2(m_\omega - S_\omega(\varphi)). \qquad \square$$

3. Proof of the scattering part

3A. *Strichartz estimates and small data scattering.* We recall the Strichartz estimates and a small data scattering result in this subsection. See [Banica and Visciglia 2016, Sections 3.1 and 3.2] for the proofs. We define the exponents r, a, and b as

$$r = p+1, \quad a := \frac{2(p-1)(p+1)}{p+3}, \quad b := \frac{2(p-1)(p+1)}{(p-1)^2 - (p-1) - 4}.$$

Then we have the following estimates.

Lemma 3.1 (Strichartz estimates). *We have*

$$\begin{aligned}
\|e^{-itH_\gamma}\varphi\|_{L_t^a L_x^r} &\lesssim \|\varphi\|_{H^1}, \\
\|e^{-itH_\gamma}\varphi\|_{L_t^{p-1} L_x^\infty} &\lesssim \|\varphi\|_{H^1}, \\
\left\| \int_0^t e^{-i(t-s)H_\gamma} F(s)\, ds \right\|_{L_t^a L_x^r} &\lesssim \|F\|_{L_t^{b'} L_x^{r'}}, \\
\left\| \int_0^t e^{-i(t-s)H_\gamma} F(s)\, ds \right\|_{L_t^{p-1} L_x^\infty} &\lesssim \|F\|_{L_t^{b'} L_x^{r'}},
\end{aligned}$$

where b' denotes the Hölder conjugate of b, namely, $1/b' + 1/b = 1$.

Proposition 3.2. *Let the solution $u \in C(\mathbb{R} : H^1(\mathbb{R}))$ to (δNLS) satisfy $u \in L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. Then the solution u scatters.*

For the proof of Proposition 3.2, see [Banica and Visciglia 2016, Proposition 3.1].

The analogous statement to Proposition 3.2 for the following semilinear Schrödinger equation without potentials is well known:

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u + |u|^{p-1}u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}), \end{cases} \tag{NLS}$$

where $p > 5$.

Proposition 3.3 (small data scattering). *Let $\varphi \in H^1(\mathbb{R})$ and u, v denote the solutions to* (δNLS), (NLS), *respectively, with the initial data φ. Then, there exist $\varepsilon > 0$ and $C > 0$ independent of ε such that u and v are global and they satisfy $\|u\|_{L_t^q L_x^r(\mathbb{R})} < C\|\varphi\|_{H^1}$ and $\|v\|_{L_t^q L_x^r(\mathbb{R})} < C\|\varphi\|_{H^1}$, if $\|\varphi\|_{H^1} < \varepsilon$.*

For the proof of Proposition 3.3, see [Banica and Visciglia 2016, Proposition 3.2].

3B. ***Linear profile decomposition and its radial version.*** To prove the scattering results, we introduce the linear profile decomposition theorems. The linear profile decomposition for nonradial data, Proposition 3.4, is obtained in [Banica and Visciglia 2016].

Proposition 3.4 (linear profile decomposition). *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R})$. Then, up to subsequence, we can write*

$$\varphi_n = \sum_{j=1}^{J} e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j + W_n^J \quad \forall J \in \mathbb{N},$$

where $t_n^j \in \mathbb{R}$, $x_n^j \in \mathbb{R}$, $\psi^j \in H^1(\mathbb{R})$, and the following hold:

- *For any fixed j, we have*

$$\begin{array}{llll} \text{either} & t_n^j = 0 \ \text{ for any } n \in \mathbb{N}, & \text{or} & t_n^j \to \pm\infty \ \text{ as } n \to \infty, \\ \text{either} & x_n^j = 0 \ \text{ for any } n \in \mathbb{N}, & \text{or} & x_n^j \to \pm\infty \ \text{ as } n \to \infty. \end{array}$$

- *Orthogonality of the parameters:*

$$|t_n^j - t_n^k| + |x_n^j - x_n^k| \to \infty \quad \text{as } n \to \infty, \ \forall j \neq k.$$

- *Smallness of the reminder:*

$$\forall \varepsilon > 0, \exists J = J(\varepsilon) \in \mathbb{N} \quad \text{such that} \quad \limsup_{n\to\infty} \|e^{-itH_\gamma} W_n^J\|_{L_t^\infty L_x^\infty} < \varepsilon.$$

- *Orthogonality in norms: for any $J \in \mathbb{N}$,*

$$\|\varphi_n\|_{L^2}^2 = \sum_{j=1}^{J} \|\psi^j\|_{L^2}^2 + \|W_n^J\|_{L^2}^2 + o_n(1), \quad \|\varphi_n\|_H^2 = \sum_{j=1}^{J} \|\tau_{x_n^j}\psi^j\|_H^2 + \|W_n^J\|_H^2 + o_n(1),$$

where $\|v\|_H^2 := \frac{1}{2}\|\partial_x v\|_{L^2}^2 - \gamma|v(0)|^2$. Moreover, we have

$$\|\varphi_n\|_{L^q}^q = \sum_{j=1}^{J} \|e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j\|_{L^q}^q + \|W_n^J\|_{L^q}^q + o_n(1), \quad q \in (2,\infty), \ \forall J \in \mathbb{N},$$

and in particular, for any $J \in \mathbb{N}$,

$$S_\omega(\varphi_n) = \sum_{j=1}^{J} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + S_\omega(W_n^J) + o_n(1),$$

$$I_\omega(\varphi_n) = \sum_{j=1}^{J} I_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + I_\omega(W_n^J) + o_n(1).$$

Proof. See [Banica and Visciglia 2016, Theorem 2.1 and Section 2.2]. □

Remark 3.1. It is not clear whether

$$P(\varphi_n) = \sum_{j=1}^{J} P(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j) + P(W_n^J) + o_n(1) \quad \forall J \in \mathbb{N}$$

holds or not. That is why we use the Nehari functional I_ω to prove the scattering results.

We introduce the reflection operator $\mathcal{R}$ such that $\mathcal{R}\varphi(x) := \varphi(-x)$.

Proposition 3.4 is insufficient to prove the scattering result for radial data. We need the following linear profile decomposition for radial solutions, which is a key ingredient.

Theorem 3.5 (linear profile decomposition for radial data). *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1_{\mathrm{rad}}(\mathbb{R})$. Then, up to subsequence, we can write*

$$\varphi_n = \frac{1}{2}\sum_{j=1}^{J}\bigl(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + e^{it_n^j H_\gamma}\tau_{-x_n^j}\mathcal{R}\psi^j\bigr) + \frac{1}{2}(W_n^J + \mathcal{R}W_n^J) \quad \forall J \in \mathbb{N}, \tag{3-1}$$

where $t_n^j \in \mathbb{R}$, $x_n^j \in \mathbb{R}$, $\psi^j \in H^1(\mathbb{R})$, and the following hold:

- *For any fixed j, we have*

$$\begin{aligned} &\text{either} \quad t_n^j = 0 \quad \text{for any } n \in \mathbb{N}, \quad \text{or} \quad t_n^j \to \pm\infty \quad \text{as } n \to \infty,\\ &\text{either} \quad x_n^j = 0 \quad \text{for any } n \in \mathbb{N}, \quad \text{or} \quad x_n^j \to \pm\infty \quad \text{as } n \to \infty. \end{aligned}$$

- *Orthogonality of the parameters:*

$$|t_n^j - t_n^k| \to \infty, \quad \text{or} \quad |x_n^j - x_n^k| \to \infty \quad \text{and} \quad |x_n^j + x_n^k| \to \infty \quad \text{as } n \to \infty,\ \forall j \neq k.$$

- *Smallness of the reminder:*

$$\forall \varepsilon > 0, \quad \exists J = J(\varepsilon) \in \mathbb{N} \quad \text{such that} \quad \limsup_{n\to\infty} \|e^{-itH_\gamma} W_n^J\|_{L_t^\infty L_x^\infty} < \varepsilon.$$

- *Orthogonality in norms: for any $J \in \mathbb{N}$,*

$$\|\varphi_n\|_{L^2}^2 = \sum_{j=1}^{J}\bigl\|\tfrac{1}{2}(\tau_{x_n^j}\psi^j + \tau_{-x_n^j}\mathcal{R}\psi^j)\bigr\|_{L^2}^2 + \bigl\|\tfrac{1}{2}(W_n^J + \mathcal{R}W_n^J)\bigr\|_{L^2}^2 + o_n(1),$$

$$\|\varphi_n\|_{H}^2 = \sum_{j=1}^{J}\bigl\|\tfrac{1}{2}(\tau_{x_n^j}\psi^j + \tau_{-x_n^j}\mathcal{R}\psi^j)\bigr\|_{H}^2 + \bigl\|\tfrac{1}{2}(W_n^J + \mathcal{R}W_n^J)\bigr\|_{H}^2 + o_n(1).$$

Moreover, for any $q \in (2,\infty)$, we have

$$\|\varphi_n\|_{L^q}^q = \sum_{j=1}^{J}\bigl\|\tfrac{1}{2}e^{it_n^j H_\gamma}(\tau_{x_n^j}\psi^j + \tau_{-x_n^j}\mathcal{R}\psi^j)\bigr\|_{L^q}^q + \bigl\|\tfrac{1}{2}(W_n^J + \mathcal{R}W_n^J)\bigr\|_{L^q}^q + o_n(1) \quad \forall J \in \mathbb{N},$$

and in particular, for any $J \in \mathbb{N}$,

$$S_\omega(\varphi_n) = \sum_{j=1}^{J} S_\omega\big(\tfrac{1}{2}e^{it_n^j H_\gamma}(\tau_{x_n^j}\psi^j + \tau_{-x_n^j}\mathcal{R}\psi^j)\big) + S_\omega\big(\tfrac{1}{2}(W_n^J + \mathcal{R}W_n^J)\big) + o_n(1),$$

$$I_\omega(\varphi_n) = \sum_{j=1}^{J} I_\omega\big(\tfrac{1}{2}e^{it_n^j H_\gamma}(\tau_{x_n^j}\psi^j + \tau_{-x_n^j}\mathcal{R}\psi^j)\big) + I_\omega\big(\tfrac{1}{2}(W_n^J + \mathcal{R}W_n^J)\big) + o_n(1).$$

Proof. Since $\{\varphi_n\}$ is bounded in $H^1(\mathbb{R})$, we can apply the linear profile decomposition without the radial assumption, Proposition 3.4, and obtain the following: for any $J \in \mathbb{N}$ and $j \in \{1, 2, \dots, J\}$, up to subsequence, there exist $\{t_n^j\}_{n\in\mathbb{N}}$, $\{x_n^j\}_{n\in\mathbb{N}}$, and $\psi^j \in H^1(\mathbb{R})$ such that we can write

$$\varphi_n = \sum_{j=1}^{J} e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + W_n^J.$$

Since φ_n is radial,

$$2\varphi_n(x) = \varphi_n(x) + \varphi_n(x) = \varphi_n(x) + \varphi_n(-x) = \varphi_n(x) + \mathcal{R}\varphi_n(x).$$

By combining the identities, we get

$$\begin{aligned} 2\varphi_n(x) &= \sum_{j=1}^{J} e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + W_n^J + \mathcal{R}\Bigg(\sum_{j=1}^{J} e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + W_n^J\Bigg) \\ &= \sum_{j=1}^{J}\big(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + e^{it_n^j H_\gamma}\tau_{-x_n^j}\mathcal{R}\psi^j\big) + W_n^J + \mathcal{R}W_n^J, \end{aligned}$$

where we have used $\mathcal{R}e^{it_n^j H_\gamma} = e^{it_n^j H_\gamma}\mathcal{R}$ and $\mathcal{R}\tau_y = \tau_{-y}\mathcal{R}$, which gives (3-1).

We only prove the orthogonality of the parameters. If $x_n^j + x_n^k \to \bar{x} \in \mathbb{R}$ and $t_n^j = t_n^k$ for $j < k$, then we replace $\psi^j + \tau_{-\bar{x}}\mathcal{R}\psi^k$ by ψ^j and 0 by ψ^k and regard the remainder terms as W_n^J. By this replacement, we have $|x_n^j - x_n^k| \to \infty$ and $|x_n^j + x_n^k| \to \infty$ as $n \to \infty$ when $t_n^j = t_n^k$. The orthogonality in norms follows from the orthogonality of the parameters by a standard argument. □

Lemma 3.6. *Let k be a nonnegative integer and, for $l \in \{0, 1, 2, \dots, k\}$, we have $\varphi_l \in H^1(\mathbb{R})$ (or $\varphi_l \in H^1_{\mathrm{rad}}(\mathbb{R})$) satisfying*

$$S_\omega\Bigg(\sum_{l=0}^{k}\varphi_l\Bigg) \le m_\omega - \delta, \quad S_\omega\Bigg(\sum_{l=0}^{k}\varphi_l\Bigg) \ge \sum_{l=0}^{k} S_\omega(\varphi_l) - \varepsilon,$$

$$I_\omega\Bigg(\sum_{l=0}^{k}\varphi_l\Bigg) \ge -\varepsilon, \qquad I_\omega\Bigg(\sum_{l=0}^{k}\varphi_l\Bigg) \le \sum_{l=0}^{k} I_\omega(\varphi_l) + \varepsilon$$

for δ, ε satisfying $2\varepsilon < \delta$. Then $\varphi_l \in \mathcal{M}_\omega^+$ for all $l \in \{0, 1, 2, \dots, k\}$. Namely, we have $0 \le S_\omega(\varphi_l) < m_\omega$ and $I_\omega(\varphi_l) \ge 0$ for all $l \in \{0, 1, 2, \dots, k\}$.

Proof. We assume there exists an $l \in \{0, 1, 2, \dots, k\}$ such that $I_\omega(\varphi_l) < 0$. Then, we have $J_\omega^{1,0}(\varphi_l) \geq m_\omega$. Indeed, there exists $\lambda_* \in (0, 1)$ such that $I_\omega(\lambda_* \varphi_l) = 0$ since $I_\omega(\varphi_l) < 0$ and $I_\omega(\lambda \varphi_l) > 0$ for small $\lambda \in (0, 1)$ by Lemma 2.2. Thus, we obtain

$$m_\omega \leq S_\omega(\lambda_* \varphi_l) = J_\omega^{1,0}(\lambda_* \varphi_l) \leq J_\omega^{1,0}(\varphi_l).$$

By the positivity of $J_\omega = J_\omega^{1,0}$ and the assumptions, we obtain

$$\begin{aligned} m_\omega \leq J_\omega(\varphi_l) &\leq \sum_{l=0}^{k} J_\omega(\varphi_l) \\ &= \sum_{l=0}^{k} \Big(S_\omega(\varphi_l) - \frac{1}{2} I_\omega(\varphi_l) \Big) \\ &= \sum_{l=0}^{k} S_\omega(\varphi_l) - \frac{1}{2} \sum_{l=0}^{k} I_\omega(\varphi_l) \\ &\leq S_\omega\bigg(\sum_{l=0}^{k} \varphi_l \bigg) + \varepsilon - \frac{1}{2} \bigg(I_\omega\bigg(\sum_{l=0}^{k} \varphi_l \bigg) - \varepsilon \bigg) \leq m_\omega - \delta + \varepsilon + \varepsilon < m_\omega. \end{aligned}$$

This is a contradiction. So, $I_\omega(\varphi_l) \geq 0$ for all $l \in \{0, 1, 2, \dots, k\}$. Moreover, for any $l \in \{0, 1, 2, \dots, k\}$, we have

$$S_\omega(\varphi_l) = J_\omega(\varphi_l) + \tfrac{1}{2} I_\omega(\varphi_l) \geq 0,$$

and

$$S_\omega(\varphi_l) \leq \sum_{l=0}^{k} S_\omega(\varphi_l) \leq S_\omega\bigg(\sum_{l=0}^{k} \varphi_l \bigg) + \varepsilon \leq m_\omega - \delta + \varepsilon < m_\omega.$$

Therefore, we get $\varphi_l \in \mathcal{M}_\omega^+$ for all $l \in \{0, 1, 2, \dots, k\}$. □

3C. *Perturbation lemma and nonlinear profile decomposition.* We use a perturbation lemma and lemmas for nonlinear profiles. The proofs of these results are the same as in the defocusing case (see [Banica and Visciglia 2016]).

Lemma 3.7. *For any $M > 0$, there exist $\varepsilon = \varepsilon(M) > 0$ and $C = C(M) > 0$ such that the following occurs. Let $v \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$ be a solution of the integral equation with source term e:*

$$v(t) = e^{-itH_\gamma} \varphi + i \int_0^t e^{-i(t-s)H_\gamma} \big(|v(s)|^{p-1} v(s) \big)\, ds + e(t)$$

with $\|v\|_{L_t^a L_x^r} < M$ and $\|e\|_{L_t^a L_x^r} < \varepsilon$. Moreover assume $\varphi_0 \in H^1(\mathbb{R})$ is such that $\|e^{-itH_\gamma} \varphi_0\|_{L_t^a L_x^r} < \varepsilon$. Then the solution $u(t, x)$ to (δNLS) with initial condition $\varphi + \varphi_0$,

$$u(t) = e^{-itH_\gamma} (\varphi + \varphi_0) + i \int_0^t e^{-i(t-s)H_\gamma} \big(|u(s)|^{p-1} u(s) \big)\, ds,$$

satisfies $u \in L_t^a L_x^r$ and moreover $\|u - v\|_{L_t^a L_x^r} < C\varepsilon$.

See [Fang et al. 2011, Proposition 4.7] and [Banica and Visciglia 2016, Proposition 3.3] for the proof.

Following Lemmas 3.8, 3.9, and 3.10 can be proved in the same manner as [Banica and Visciglia 2016, Propositions 3.4, 3.5, and 3.6], respectively.

Lemma 3.8. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $|x_n| \to \infty$ as $n \to \infty$, $u_0 \in H^1(\mathbb{R})$ and $U \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$ be a solution of (NLS) with the initial data u_0. Then we have*

$$U_n(t) = e^{-itH_\gamma}\tau_{x_n}u_0 + i\int_0^t e^{-i(t-s)H_\gamma}\big(|U_n(s)|^{p-1}U_n(s)\big)\,ds + g_n(t),$$

where $U_n(t,x) = U(t,x-x_n)$ and $\|g_n\|_{L_t^aL_x^r} \to 0$ as $n \to \infty$.

Lemma 3.9. *Let $\varphi \in H^1(\mathbb{R})$. Then there exist solutions $W_\pm \in C(\mathbb{R}_\pm : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_\pm : L_x^r(\mathbb{R}))$ to (δNLS) such that*

$$\|W_\pm(t,\cdot) - e^{-itH_\gamma}\varphi\|_{H^1} \to 0 \quad \text{as } t \to \pm\infty.$$

Moreover, if $\{t_n\}_{n\in\mathbb{N}}$ is such that $t_n \to \mp\infty$ as $n \to \infty$ and $W_\pm$ is global, then

$$W_{\pm,n}(t) = e^{-itH_\gamma}\varphi_n + i\int_0^t e^{-i(t-s)H_\gamma}\big(|W_{\pm,n}(s)|^{p-1}W_{\pm,n}(s)\big)\,ds + f_{\pm,n}(t),$$

where $\varphi_n = e^{it_nH_\gamma}\varphi$, $W_{\pm,n}(t,x) = W_\pm(t-t_n,x)$, $\|f_{\pm,n}\|_{L_t^aL_x^r} \to 0$ as $n \to \infty$, and the double-sign corresponds.

Lemma 3.10. *Let $\{t_n\}_{n\in\mathbb{N}}$, $\{x_n\}_{n\in\mathbb{N}}$ be sequences of real numbers such that $t_n \to \mp\infty$ and $|x_n| \to \infty$ as $n \to \infty$, $\varphi \in H^1(\mathbb{R})$ and $V_\pm \in C(\mathbb{R}_\pm : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_\pm : L_x^r(\mathbb{R}))$ be solutions of (NLS) such that*

$$\|V_\pm(t,\cdot) - e^{-itH_0}\varphi\|_{H^1} \to 0 \quad \text{as } t \to \pm\infty.$$

Then we have

$$V_{\pm,n}(t,x) = e^{-itH_\gamma}\varphi_n + i\int_0^t e^{-i(t-s)H_\gamma}\big(|V_{\pm,n}(s)|^{p-1}V_{\pm,n}(s)\big)\,ds + e_{\pm,n}(t,x),$$

where $\varphi_n = e^{it_nH_\gamma}\tau_{x_n}\varphi$, $V_{\pm,n}(t,x) = V_\pm(t-t_n,x-x_n)$, $\|e_{\pm,n}\|_{L_t^aL_x^r} \to 0$ as $n \to \infty$, and the double-sign corresponds.

3D. *Construction of a critical element.*

We define the critical action level S_ω^c for fixed ω as

$$S_\omega^c := \sup\big\{S : S_\omega(\varphi) < S \text{ for any } \varphi \in \mathcal{M}_\omega^+ \text{ implies } u \in L_t^aL_x^r\big\}.$$

By the small data scattering result Proposition 3.3, we obtain $S_\omega^c > 0$. We prove $S_\omega^c = m_\omega$ by contradiction.

We assume $S_\omega^c < m_\omega$. By this assumption, we can take a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_\omega^+$ such that $S_\omega(\varphi_n) \to S_\omega^c$ as $n \to \infty$, and $\|u_n\|_{L_t^aL_x^r(\mathbb{R})} = \infty$ for all $n \in \mathbb{N}$, where u_n is a global solution to (δNLS) with the initial data φ_n. Then, we obtain the following lemma.

Lemma 3.11 (critical element). *We assume $S_\omega^c < m_\omega$. Then we find a global solution $u^c \in C(\mathbb{R} : H^1(\mathbb{R}))$ of (δNLS) which satisfies $u^c(t) \in \mathcal{M}_\omega^+$ for any $t \in \mathbb{R}$ and*

$$S_\omega(u^c) = S_\omega^c, \quad \|u^c\|_{L_t^aL_x^r(\mathbb{R})} = \infty.$$

This u^c is called a critical element.

Proof. First, we consider the nonradial case.

Case 1: nonradial data. By $\varphi_n \in \mathcal{N}_\omega^+$ and Lemma 2.16, we have

$$\|\varphi_n\|_{H^1}^2 \lesssim \|\varphi_n\|_{\mathcal{H}}^2 \lesssim E(\varphi_n) + \omega M(\varphi_n) < n_\omega$$

for all $n \in \mathbb{N}$. Since $\{\varphi_n\}$ is a bounded sequence in $H^1(\mathbb{R})$, we apply the linear profile decomposition, Proposition 3.4, and then obtain

$$\varphi_n = \sum_{j=1}^{J} e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j + W_n^J \quad \forall J \in \mathbb{N}.$$

By the orthogonality of the functionals in Proposition 3.4, we have

$$S_\omega(\varphi_n) = \sum_{j=1}^{J} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + S_\omega(W_n^J) + o_n(1),$$

$$I_\omega(\varphi_n) = \sum_{j=1}^{J} I_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + I_\omega(W_n^J) + o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$.

By these decompositions and $S_\omega(\varphi_n) < n_\omega$, we can find $\delta, \varepsilon > 0$ satisfying $2\varepsilon < \delta$ and

$$S_\omega(\varphi_n) \le n_\omega - \delta, \quad S_\omega(\varphi_n) \ge \sum_{j=0}^{J} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + S_\omega(W_n^J) - \varepsilon,$$

$$I_\omega(\varphi_n) \ge -\varepsilon, \qquad I_\omega(\varphi_n) \le \sum_{j=0}^{J} I_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + I_\omega(W_n^J) + \varepsilon$$

for large n. Therefore, by Lemma 3.6, we see that

$$e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j \in \mathcal{N}_\omega^+ \quad \text{and} \quad W_n^J \in \mathcal{N}_\omega^+ \quad \text{for large } n,$$

which means that, by Lemma 2.16,

$$S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) \ge 0 \quad \text{and} \quad S_\omega(W_n^J) \ge 0 \quad \text{for large } n.$$

So, we have

$$S_\omega^c = \limsup_{n\to\infty} S_\omega(\varphi_n) \ge \limsup_{n\to\infty} \sum_{j=1}^{J} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j)$$

for any J. We prove $S_\omega^c = \limsup_{n\to\infty} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j)$ for some j. We may assume $j = 1$ by reordering. If this is proved, then we find that $J = 1$ and $W_n^J \to 0$ in $L_t^\infty H_x^1$ as $n \to \infty$. Indeed, $\limsup_{n\to\infty} S_\omega(W_n^1) = 0$ holds and thus $\limsup_{n\to\infty} \|W_n^1\|_{H^1} = 0$ holds by $\|W_n^1\|_{H^1} \approx S_\omega(W_n^1)$ since W_n^1 belongs to $\mathcal{N}_\omega^+$ for large $n \in \mathbb{N}$. On the contrary, we assume $S_\omega^c = \limsup_{n\to\infty} S_\omega(e^{-it_n^j H_\gamma} \tau_{x_n^j} \psi^j)$

fails for all j. Then, for all j, there exists $\delta = \delta_j > 0$ such that

$$\limsup_{n\to\infty} S_\omega(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j) < S_\omega^c - \delta.$$

By reordering, we can choose $0 \le J_1 \le J_2 \le J_3 \le J_4 \le J_5 \le J$ such that

$$\begin{aligned}
1 \le j \le J_1 &: & t_n^j &= 0 \quad \forall n, & x_n^j &= 0 \quad \forall n,\\
J_1+1 \le j \le J_2 &: & t_n^j &= 0 \quad \forall n, & \lim_{n\to\infty}|x_n^j| &= \infty,\\
J_2+1 \le j \le J_3 &: & \lim_{n\to\infty} t_n^j &= +\infty, & x_n^j &= 0 \quad \forall n,\\
J_3+1 \le j \le J_4 &: & \lim_{n\to\infty} t_n^j &= -\infty, & x_n^j &= 0 \quad \forall n,\\
J_4+1 \le j \le J_5 &: & \lim_{n\to\infty} t_n^j &= +\infty, & \lim_{n\to\infty}|x_n^j| &= \infty,\\
J_5+1 \le j \le J &: & \lim_{n\to\infty} t_n^j &= -\infty, & \lim_{n\to\infty}|x_n^j| &= \infty.
\end{aligned}$$

Above we are assuming that if $a > b$, then there is no j such that $a \le j \le b$. Notice that $J_1 \in \{0, 1\}$ by the orthogonality of the parameters. We may treat only the case $J_1 = 1$ here. The case $J_1 = 0$ is easier. We have $0 < S_\omega(\psi^1) < S_\omega^c - \delta$ by $(t_n^j, x_n^j) = (0, 0)$ and the assumption. Hence, by the definition of S_ω^c, we can find $N \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$ such that

$$N(t, x) = e^{-itH_\gamma}\psi^1 + i\int_0^t e^{-i(t-s)H_\gamma}(|N(s)|^{p-1}N(s))\,ds.$$

For every j such that $J_1 + 1 \le j \le J_2$, let U^j be the solution of (NLS) with the initial data ψ^j. Since we have $\tau_{x_n^j}\psi^j \in \mathcal{N}_\omega^+$, we know ψ^j satisfies

$$S_{\omega,0}(\psi^j) \le S_\omega(\tau_{x_n^j}\psi^j) \le S_\omega^c < n_\omega = l_\omega$$

and $P_0(\psi^j) \ge 0$. (since $0 > P_0(\psi^j) = \lim_{n\to\infty} P(\tau_{x_n^j}\psi^j) \ge 0$ if we assume $P_0(\psi^j) < 0$.) Therefore, we see that the solution U^j scatters by [Fang et al. 2011; Akahori and Nawa 2013]; that is, $U^j(t, x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $U_n^j(t, x) := U^j(t, x - x_n^j)$.

For every j such that $J_2 + 1 \le j \le J_3$, we associate with profile ψ^j the function

$$W_-^j(t, x) \in C(\mathbb{R}_- : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_- : L_x^r(\mathbb{R}))$$

by Lemma 3.9. We claim that $W_-^j(t, x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. Indeed, by the assumption, we see that $S_\omega(W_-^j) = \lim_{n\to\infty} S_\omega(e^{it_n^j H_\gamma}\psi^j) < S_\omega^c$, since $e^{it_n^j H_\gamma}\psi^j \to W_-^j$ in $H^1(\mathbb{R})$ with $t_n^j \to \infty$ as $n \to \infty$. Therefore, by the definition of S_ω^c, we obtain $W_-^j(t, x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $W_{-,n}^j(t, x) := W_-^j(t - t_n^j, x)$.

For every j such that $J_3 + 1 \le j \le J_4$, we associate with profile ψ^j the function

$$W_+^j(t, x) \in C(\mathbb{R}_+ : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_+ : L_x^r(\mathbb{R}))$$

by Lemma 3.9. And the same argument as above gives us that $W_+^j(t, x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $W_{+,n}^j(t, x) := W_+^j(t - t_n^j, x)$.

For every j such that $J_4 + 1 \le j \le J_5$, we associate with profile ψ^j the function

$$V_-^j(t,x) \in C(\mathbb{R}_- : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_- : L_x^r(\mathbb{R}))$$

by Lemma 3.10. We will prove $V_-^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. Now,

$$\limsup_{n\to\infty} S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) < S_\omega^c - \delta$$

holds by the assumption. Here, since e^{-itH_γ} is unitary in $L^2(\mathbb{R})$ and conserves the linear energy, and $\gamma \le 0$, we have

$$\begin{aligned}
S_\omega(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) &= E(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) + \omega M(e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j) \\
&= \|\tau_{x_n^j} \psi^j\|_{\mathcal{H}}^2 - \frac{1}{p+1} \|e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j\|_{L^{p+1}}^{p+1} \\
&\ge \tfrac{1}{4} \|\partial_x (\tau_{x_n^j} \psi^j)\|_{L^2}^2 + \tfrac{1}{2}\omega \|\tau_{x_n^j} \psi^j\|_{L^2}^2 - \frac{1}{p+1} \|e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j\|_{L^{p+1}}^{p+1} \\
&= \tfrac{1}{4} \|\partial_x \psi^j\|_{L^2}^2 + \tfrac{1}{2}\omega \|\psi^j\|_{L^2}^2 - \frac{1}{p+1} \|e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j\|_{L^{p+1}}^{p+1}.
\end{aligned}$$

Since $t_n^j \to \infty$, we have $\|e^{it_n^j H_\gamma} \tau_{x_n^j} \psi^j\|_{L^{p+1}}^{p+1} \to 0$ as $n \to \infty$ by [Banica and Visciglia 2016, Section 2, (2.4)]. Therefore, we obtain

$$\tfrac{1}{4} \|\partial_x \psi^j\|_{L^2}^2 + \tfrac{1}{2}\omega \|\psi^j\|_{L^2}^2 \le S_\omega^c - \delta.$$

Since ψ^j is the final state of V_-^j, we have

$$S_{\omega,0}(V_-^j) = \tfrac{1}{4} \|\partial_x \psi^j\|_{L^2}^2 + \tfrac{1}{2}\omega \|\psi^j\|_{L^2}^2 \le S_\omega^c - \delta < n_\omega = l_\omega.$$

By [Fang et al. 2011; Akahori and Nawa 2013], we have $V_-^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $V_{-,n}^j(t,x) := V_-^j(t - t_n^j, x - x_n^j)$.

For every j such that $J_5 + 1 \le j \le J$, we associate with profile ψ^j the function

$$V_+^j(t,x) \in C(\mathbb{R}_+ : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_+ : L_x^r(\mathbb{R}))$$

by Lemma 3.10. And the same argument as above gives us that $V_+^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $V_{+,n}^j(t,x) := V_+^j(t - t_n^j, x - x_n^j)$.

We define the nonlinear profile as

$$Z_n^J := N + \sum_{j=J_1+1}^{J_2} U_n^j + \sum_{j=J_2+1}^{J_3} W_{-,n}^j + \sum_{j=J_3+1}^{J_4} W_{+,n}^j + \sum_{j=J_4+1}^{J_5} V_{-,n}^j + \sum_{j=J_5+1}^{J_6} V_{+,n}^j.$$

By Lemmas 3.8, 3.9, and 3.10, we have

$$Z_n^J = e^{-itH_\gamma}(\varphi_n - W_n^J) + i z_n^J + r_n^J,$$

where $\|r_n^J\|_{L_t^a L_x^r} \to 0$ as $n \to \infty$ and

$$
\begin{aligned}
z_n^J(t) := \int_0^t & e^{-i(t-s)H_\gamma}\big(|N(s)|^{p-1}N(s)\big)\,ds \\
&+ \sum_{j=J_1+1}^{J_2} \int_0^t e^{-i(t-s)H_\gamma}\big(|U_n^j(s)|^{p-1}U_n^j(s)\big)\,ds \\
&+ \sum_{j=J_2+1}^{J_3} \int_0^t e^{-i(t-s)H_\gamma}\big(|W_{-,n}^j(s)|^{p-1}W_{-,n}^j(s)\big)\,ds \\
&+ \sum_{j=J_3+1}^{J_4} \int_0^t e^{-i(t-s)H_\gamma}\big(|W_{+,n}^j(s)|^{p-1}W_{+,n}^j(s)\big)\,ds \\
&+ \sum_{j=J_4+1}^{J_5} \int_0^t e^{-i(t-s)H_\gamma}\big(|V_{-,n}^j(s)|^{p-1}V_{-,n}^j(s)\big)\,ds \\
&+ \sum_{j=J_5+1}^{J} \int_0^t e^{-i(t-s)H_\gamma}\big(|V_{+,n}^j(s)|^{p-1}V_{+,n}^j(s)\big)\,ds.
\end{aligned}
$$

We also have

$$\left\| z_n^J - \int_o^t e^{-i(t-s)H_\gamma}\big(|Z_n^J(s)|^{p-1}Z_n^J(s)\big)\,ds \right\|_{L_t^a L_x^r} \to 0 \quad \text{as } n \to \infty.$$

Therefore, we get

$$Z_n^J = e^{-itH_\gamma}(\varphi_n - W_n^J) + i\int_o^t e^{-i(t-s)H_\gamma}\big(|Z_n^J(s)|^{p-1}Z_n^J(s)\big)\,ds + s_n^J,$$

with $\|s_n^J\|_{L_t^a L_x^r} \to 0$ as $n \to \infty$. In order to apply the perturbation lemma, Lemma 3.7, we need a bound on $\sup_J(\limsup_{n\to\infty}\|Z_n^J\|_{L_t^a L_x^r})$. We have

$$
\begin{aligned}
\limsup_{n\to\infty}(\|Z_n^J\|_{L_t^a L_x^r})^p \le 2\|N\|_{L_t^a L_x^r}^p &+ 2\sum_{j=J_1+1}^{J_2}\|U^j\|_{L_t^a L_x^r}^p + 2\sum_{j=J_2+1}^{J_3}\|W_-^j\|_{L_t^a L_x^r}^p \\
&+2\sum_{j=J_3+1}^{J_4}\|W_+^j\|_{L_t^a L_x^r}^p + 2\sum_{j=J_4+1}^{J_5}\|V_-^j\|_{L_t^a L_x^r}^p + 2\sum_{j=J_5+1}^{J}\|V_+^j\|_{L_t^a L_x^r}^p,
\end{aligned}
$$

where we have used Corollary A.2 in [Banica and Visciglia 2016]. For simplicity, a_n^j denotes $2\|N\|_{L_t^a L_x^r}^p$ if $1 \le j \le J_1$, $2\|U_n^j\|_{L_t^a L_x^r}^p = 2\|U^j\|_{L_t^a L_x^r}^p$ if $J_1+1 \le j \le J_2$, and so on. Then, the above inequality means

$$\limsup_{n\to\infty}(\|Z_n^J\|_{L_t^a L_x^r})^p \le \sum_{j=1}^{J} a_n^j.$$

There exists a finite set $\mathcal{J}$ such that $\|\psi^j\|_{H^1} < \varepsilon_0$ for any $j \notin \mathcal{J}$, where ε_0 is the universal constant in the small data scattering result, Proposition 3.3. By Proposition 3.3 and the orthogonalities in H-norm and L^2-norm,

$$\begin{aligned}\limsup_{n\to\infty}(\|Z_n^J\|_{L_t^a L_x^r})^p &\le \limsup_{n\to\infty}\sum_{j=1}^J a_n^j = \limsup_{n\to\infty}\sum_{j\in\mathcal{J}} a_n^j + \limsup_{n\to\infty}\sum_{j\notin\mathcal{J}} a_n^j \\ &\lesssim \limsup_{n\to\infty}\sum_{j\in\mathcal{J}} a_n^j + \limsup_{n\to\infty}\sum_{j\notin\mathcal{J}} \|e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j\|_{\mathcal{H}} \\ &\lesssim \limsup_{n\to\infty}\sum_{j\in\mathcal{J}} a_n^j + \limsup_{n\to\infty}\|\varphi_n\|_{\mathcal{H}} \\ &\lesssim \limsup_{n\to\infty}\sum_{j\in\mathcal{J}} a_n^j + n_\omega \lesssim \sum_{j\in\mathcal{J}} a^j + n_\omega \le M,\end{aligned}$$

where M is independent of J.

By Lemma 3.7 and Proposition 3.4, we can choose J large enough in such a way that

$$\limsup_{n\to\infty}\|e^{-itH_\gamma}W_n^J\|_{L_t^a L_x^r} < \varepsilon,$$

where $\varepsilon = \varepsilon(M) > 0$. Then, we get the fact that u_n scatters for large n, and this contradicts $\|u_n\|_{L_t^a L_x^r} = \infty$.

Therefore, we obtain $J = 1$ and

$$\varphi_n = e^{it_n^1 H_\gamma}\tau_{x_n^1}\psi^1 + W_n^1, \qquad S_\omega^c = \limsup_{n\to\infty} S_\omega(e^{it_n^1 H_\gamma}\tau_{x_n^1}\psi^1), \qquad W_n^1 \to 0 \quad \text{in } L_t^\infty H_x^1.$$

By the same argument as [Banica and Visciglia 2016], we get $x_n^1 = 0$. Let u^c be the nonlinear profile associated with ψ^1. Then, $S_\omega^c = S_\omega(u^s)$ and the global solution u^c does not scatter by a contradiction argument and the perturbation lemma (see the proof of Proposition 6.1 in [Fang et al. 2011] for more detail).

Case 2: radial data. We only focus on the difference of the proof between the radial case and the nonradial case, which is in the profiles. By the linear profile decomposition for the radial data Theorem 3.5, we have

$$\varphi_n = \frac{1}{2}\sum_{j=1}^J \bigl(e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j + e^{it_n^j H_\gamma}\tau_{-x_n^j}\mathcal{R}\psi^j\bigr) + \frac{1}{2}(W_n^J + \mathcal{R}W_n^J) \quad \forall J \in \mathbb{N}.$$

For every j such that $J_1 + 1 \le j \le J_2$, let U^j be the solution to (NLS) with the initial data $\frac{1}{2}\psi^j$. Since we have

$$\tfrac{1}{2}\tau_{x_n^j}\psi^j + \tfrac{1}{2}\tau_{-x_n^j}\mathcal{R}\psi^j \in \mathcal{R}_\omega^+,$$

ψ^j satisfies $S_{\omega,0}\bigl(\frac{1}{2}\psi^j\bigr) < l_\omega$ and $P_0\bigl(\frac{1}{2}\psi^j\bigr) \ge 0$. Indeed, if we assume $S_{\omega,0}\bigl(\frac{1}{2}\psi^j\bigr) \ge l_\omega$, then by Theorem 3.5 and $\gamma \le 0$,

$$\begin{aligned} r_\omega > S_\omega^c &\ge \limsup_{n\to\infty} S_\omega(\varphi_n) \ge \limsup_{n\to\infty}\bigl(S_\omega\bigl(\tau_{x_n^j}\tfrac{1}{2}\psi^j\bigr) + S_\omega(\tau_{-x_n^j}\mathcal{R}\tfrac{1}{2}\psi^j)\bigr) \\ &\ge \limsup_{n\to\infty}\bigl(S_{\omega,0}\bigl(\tau_{x_n^j}\tfrac{1}{2}\psi^j\bigr) + S_{\omega,0}\bigl(\tau_{-x_n^j}\mathcal{R}\tfrac{1}{2}\psi^j\bigr)\bigr) \\ &= S_{\omega,0}\bigl(\tfrac{1}{2}\psi^j\bigr) + S_{\omega,0}\bigl(\tfrac{1}{2}\psi^j\bigr) \ge 2l_\omega.\end{aligned}$$

This contradicts $r_\omega \le 2l_\omega$. Moreover, we see that

$$2P_0\big(\tfrac{1}{2}\psi^j\big) = \limsup_{n\to\infty}\big(P_0\big(\tau_{x_n^j}\tfrac{1}{2}\psi^j\big) + P_0\big(\tau_{-x_n^j}\mathcal{R}\tfrac{1}{2}\psi^j\big)\big) = \limsup_{n\to\infty} P\big(\tau_{x_n^j}\tfrac{1}{2}\psi^j + \tau_{-x_n^j}\mathcal{R}\tfrac{1}{2}\psi^j\big) \ge 0.$$

Therefore, by [Fang et al. 2011; Akahori and Nawa 2013], we have

$$U^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R})).$$

We set $U_n^j(t,x) := U^j(t, x - x_n^j)$.

For every j such that $J_4 + 1 \le j \le J_5$, we associate with profile ψ^j the function

$$V_-^j(t,x) \in C(\mathbb{R}_- : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R}_- : L_x^r(\mathbb{R}))$$

by Lemma 3.10. We prove $V_-^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. Now, by the assumption, we have

$$\limsup_{n\to\infty} 2S_\omega\big(\tfrac{1}{2}e^{it_n^j H_\gamma}\tau_{x_n^j}\psi^j\big) = \limsup_{n\to\infty}\big\{S_\omega(e^{it_n^j H_\gamma}\tau_{x_n^j}\tfrac{1}{2}\psi^j) + S_\omega\big(\mathcal{R}e^{it_n^j H_\gamma}\tau_{x_n^j}\tfrac{1}{2}\psi^j\big)\big\} < S_\omega^c - \delta.$$

In the same argument as that for V_-^j in the nonradial case, we obtain

$$\tfrac{1}{4}\big\|\partial_x \tfrac{1}{2}1\psi^j\big\|_{L^2}^2 + \tfrac{1}{2}\omega\big\|\tfrac{1}{2}\psi^j\big\|_{L^2}^2 \le \tfrac{1}{2}(S_\omega^c - \delta).$$

Now, since ψ^j is the final state of V_-^j, we have

$$S_{\omega,0}(V_-^j) = \tfrac{1}{4}\big\|\partial_x \tfrac{1}{2}\psi^j\big\|_{L^2}^2 + \tfrac{1}{2}\omega\big\|\tfrac{1}{2}\psi^j\big\|_{L^2}^2 \le \tfrac{1}{2}(S_\omega^c - \delta) < \tfrac{1}{2}r_\omega \le l_\omega.$$

By [Fang et al. 2011; Akahori and Nawa 2013], we have $V_-^j(t,x) \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L_t^a(\mathbb{R} : L_x^r(\mathbb{R}))$. We set $V_{-,n}^j(t,x) := V_-^j(t - t_n^j, x - x_n^j)$.

Other statements are the same as in the nonradial case. □

3E. *Extinction of the critical element.* We assume that $\|u^c\|_{L_t^a((0,\infty):L_x^r)} = \infty$, where such u^c is called a forward critical element, and we prove $u^c = 0$. In the case of $\|u^c\|_{L_t^a((-\infty,0):L_x^r)} = \infty$, the same argument as below does work.

Lemma 3.12. *Let u be a forward critical element. Then the orbit of u, $\{u(t,x) : t > 0\}$, is precompact in $H^1(\mathbb{R})$. And then, for any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{|x|>R} |\partial_x u(t,x)|^2\,dx + \int_{|x|>R} |u(t,x)|^2\,dx + \int_{|x|>R} |u(t,x)|^{p+1}\,dx < \varepsilon \quad \text{for any } t \in \mathbb{R}_+.$$

This lemma is obtained in the same way as the defocusing case (see [Banica and Visciglia 2016]).

Now, we prove $u = 0$ by the localized virial identity and contradiction. Let $u \ne 0$. For $\varphi : \mathbb{R}_+ \to \mathbb{R}$, we define a function I by

$$I(t) := \int_{\mathbb{R}} \varphi(|x|)|u(t,x)|^2\,dx.$$

Then, by a direct calculation and using (δNLS), we have

$$
\begin{aligned}
I'(t) &= \operatorname{Im}\int_{\mathbb{R}} \partial_x(\varphi(|x|))\overline{u(t,x)}\partial_x u(t,x)\,dx,\\
I''(t) &= \operatorname{Re}\int_{\mathbb{R}} \partial_x^2(\varphi(|x|))|\partial_x u(t,x)|^2\,dx - \gamma\partial_x^2(\varphi(|x|))|_{x=0}|u(t,0)|^2\\
&\quad - \frac{p-1}{p+1}\operatorname{Re}\int_{\mathbb{R}} \partial_x^2(\varphi(|x|))|u(t,x)|^{p+1}\,dx - \frac{1}{4}\operatorname{Re}\int_{\mathbb{R}} \partial_x^4(\varphi(|x|))|u(t,x)|^2\,dx\\
&\quad - 2\gamma\operatorname{Re}\{\partial_x(\varphi(|x|))|_{x=0}u(t,0)\overline{\partial_x u(t,0)}\}.
\end{aligned}
$$

Taking $\varphi = \varphi(r)$ such that, for $R > 0$,

$$
0 \le \varphi \le r^2, \quad |\varphi'| \lesssim r, \quad |\varphi''| \le 2, \quad |\varphi^{(4)}| \le \frac{4}{R^2},
$$

and

$$
\varphi(r) = \begin{cases} r^2, & 0 \le r \le R,\\ 0, & r \ge 2R, \end{cases}
$$

we obtain

$$
\begin{aligned}
I''(t) &= 4P(u(t)) + \operatorname{Re}\int_{\mathbb{R}} (\partial_x^2(\varphi(|x|))-2)|\partial_x u(t,x)|^2\,dx - \frac{p-1}{p+1}\operatorname{Re}\int_{\mathbb{R}} (\partial_x^2(\varphi(|x|))-2)|u(t,x)|^{p+1}\,dx\\
&\quad -\tfrac{1}{4}\operatorname{Re}\int_{\mathbb{R}} \partial_x^4(\varphi(|x|))|u(t,x)|^2\,dx = 4P(u(t)) + R_1 + R_2 + R_3,
\end{aligned} \tag{3-2}
$$

where

$$
\begin{aligned}
R_1 &:= \operatorname{Re}\int_{\mathbb{R}} (\partial_x^2(\varphi(|x|))-2)|\partial_x u(t,x)|^2\,dx,\\
R_2 &:= -\frac{p-1}{p+1}\operatorname{Re}\int_{\mathbb{R}} (\partial_x^2(\varphi(|x|))-2)|u(t,x)|^{p+1}\,dx,\\
R_3 &:= -\tfrac{1}{4}\operatorname{Re}\int_{\mathbb{R}} \partial_x^4(\varphi(|x|))|u(t,x)|^2\,dx.
\end{aligned}
$$

By the property of φ, we have

$$
\begin{aligned}
|R_1| &= \left|\operatorname{Re}\int_{\mathbb{R}} \{\partial_x^2(\varphi(|x|))-2\}|\partial_x u(t,x)|^2\,dx\right| \le C\int_{|x|>R} |\partial_x u(t,x)|^2\,dx,\\
|R_2| &= \left|\frac{p-1}{p+1}\operatorname{Re}\int_{\mathbb{R}} \{\partial_x^2(\varphi(|x|))-2\}|u(t,x)|^{p+1}\,dx\right| \le C\int_{|x|>R} |u(t,x)|^{p+1}\,dx,\\
|R_3| &= \left|\tfrac{1}{4}\operatorname{Re}\int_{\mathbb{R}} \partial_x^4(\varphi(|x|))|u(t,x)|^2\,dx\right| \le C\int_{|x|>R} |u(t,x)|^2\,dx.
\end{aligned}
$$

Therefore, we obtain

$$
I''(t) = 4P(u(t)) - C\left(\int_{|x|>R} |\partial_x u(t,x)|^2\,dx + \int_{|x|>R} |u(t,x)|^2\,dx + \int_{|x|>R} |u(t,x)|^{p+1}\,dx\right).
$$

We note that there exists $\delta > 0$ independent of t such that $P(u(t)) > \delta$ by Proposition 2.18 since u belongs to $\mathcal{M}_\omega^+$. Therefore, by Lemma 3.12, if we take $\varepsilon \in (0, 3\delta)$, then there exists $R > 0$ such that $I''(t) \geq \delta$ for any $t \in \mathbb{R}_+$. On the other hand, the mass conservation law gives $I(t) \leq R^2 \|u(t)\|_{L^2}^2 < C$, where C is independent of t, for any $t \in \mathbb{R}_+$. Hence, we obtain a contradiction.

4. Proof of the blow-up part

To prove the blow-up results, we use the method of Du et al. [2016]. On the contrary, we assume that the solution u to (δNLS) with $u_0 \in \mathcal{M}_\omega^-$ is global in the positive time direction and $\sup_{t\in\mathbb{R}_+} \|\partial_x u(t)\|_{L^2}^2 < C_0 < \infty$. Then, we have $\sup_{t\in\mathbb{R}_+} \|u(t)\|_{L^q} < \infty$ for any $q > p+1$ by energy conservation and the Sobolev embedding.

For $R > 0$, we take φ such that

$$\varphi(r) = \begin{cases} 0, & 0 < r < \frac{1}{2}R, \\ 1, & r \geq R, \end{cases}$$

$$0 \leq \varphi \leq 1, \quad \varphi' \leq \frac{4}{R}.$$

By the fundamental formula and the Hölder inequality, we have

$$\begin{aligned} I(t) &= I(0) + \int_0^t I'(s)\,ds \leq I(0) + \int_0^t |I'(s)|\,ds \\ &\leq I(0) + t\|\varphi'\|_{L^\infty} \|u(t)\|_{L^2}^2 \|\partial_x u(t)\|_{L^2}^2 \\ &\leq I(0) + \frac{8M(u)C_0 t}{R}. \end{aligned}$$

Here, we note that $I(0) \leq \int_{|x|>R/2} |u(0,x)|^2\,dx = o_R(1)$ and $\int_{|x|>R} |u(t,x)|^2\,dx \leq I(t)$. Therefore, we obtain the following lemma.

Lemma 4.1. *Let $\eta_0 > 0$ be fixed. Then, for any $t \leq \eta_0 R/(8M(u)C_0)$, we have*

$$\int_{|x|>R} |u(t,x)|^2\,dx \leq o_R(1) + \eta_0.$$

We take another φ such that

$$0 \leq \varphi \leq r^2, \quad |\varphi'| \lesssim r, \quad |\varphi''| \leq 2, \quad |\varphi^{(4)}| \leq \frac{4}{R^2},$$

and

$$\varphi(r) = \begin{cases} r^2, & 0 \leq r \leq R, \\ 0, & r \geq 2R. \end{cases}$$

Then we have the following lemma.

Lemma 4.2. *There exist two constants $C = C(p, M(u), C_0) > 0$ and $\theta_q > 0$ such that*

$$I''(t) \leq 4P(u(t)) + C\|u\|_{L^2(|x|>R)}^{\theta_q} + CR^{-2}\|u\|_{L^2(|x|>R)}^2.$$

Proof. By (3-2), we have

$$I''(t) = 4P(u(t)) + R_1 + R_2 + R_3.$$

First, we prove $R_1 \le 0$. By the definition of φ, we see that

$$R_1 = \mathrm{Re}\int_{\mathbb{R}} (\partial_x^2(\varphi(|x|)) - 2)|\partial_x u(t,x)|^2\,dx = \mathrm{Re}\int_{\mathbb{R}} (\varphi''(|x|) - 2)|\partial_x u(t,x)|^2\,dx \le 0.$$

Next, we consider R_2. By the Hölder inequality, we have

$$\begin{aligned} R_2 &= -\frac{p-1}{p+1}\,\mathrm{Re}\int_{\mathbb{R}} (\partial_x^2(\varphi(|x|)) - 2)|u(t,x)|^{p+1}\,dx \\ &\le C\int_{|x|>R} |u(t,x)|^{p+1}\,dx \\ &\le C\|u\|_{L^q(|x|>R)}^{1-\theta_q}\|u\|_{L^2(|x|>R)}^{\theta_q} \\ &\le C\|u\|_{L^2(|x|>R)}^{\theta_q}, \end{aligned}$$

where $q > p+1$ and $0 < \theta_q \le 1$, since $\sup_{t\in\mathbb{R}_+}\|u(t)\|_{L^q} < \infty$. Finally, we consider R_3:

$$R_3 = -\tfrac{1}{4}\,\mathrm{Re}\int_{\mathbb{R}} \partial_x^4(\varphi(|x|))|u(t,x)|^2\,dx \le CR^{-2}\int_{|x|>R} |u(t,x)|^2\,dx = CR^{-2}\|u\|_{L^2(|x|>R)}^2. \qquad \square$$

Proof of Theorem 1.3(2) (and Theorem 1.5(2)). Since $u(t)$ belongs to $\mathcal{M}_\omega^-$, there exists $\delta > 0$ independent of t such that $P(u(t)) < -\delta$ for all $t \in \mathbb{R}_+$ by Proposition 2.18. Therefore, we obtain

$$I''(t) \le -4\delta + C\|u\|_{L^2(|x|>R)}^{\theta_q} + CR^{-2}\|u\|_{L^2(|x|>R)}^2.$$

We take $\eta_0 > 0$ such that $C\eta_0^{\theta_q} + C\eta_0^2 < \delta$. By Lemma 4.1, for $t \in [0, \eta_0 R/(8M(u)C_0)]$, we have

$$I''(t) \le -3\delta + o_R(1).$$

Let $T := \eta_0 R/(8M(u)C_0)$. Integrating the above inequality from 0 to T, we get

$$I(T) \le I(0) + I'(0)T + \tfrac{1}{2}(-3\delta + o_R(1))T^2.$$

For sufficiently large $R > 0$, we have $-3\delta + o_R(1) < -2\delta$. Thus, we get

$$I(T) \le I(0) + I'(0)\eta_0 R/(8M(u)C_0) - \alpha_0 R^2,$$

where $\alpha_0 := \delta\eta_0^2/(8M(u)C_0)^2 > 0$, and we can prove $I(0) = o_R(1)R^2$ and $I'(0) = o_R(1)R$. Indeed,

$$\begin{aligned} I(0) &\le \int_{|x|<\sqrt{R}} |x|^2|u_0(x)|^2\,dx + \int_{\sqrt{R}<|x|<2R} |x|^2|u_0(x)|^2\,dx \\ &\lesssim M(u)R + R^2\int_{\sqrt{R}<|x|} |u_0(x)|^2\,dx \\ &= o_R(1)R^2, \end{aligned}$$

and

$$\begin{aligned}
I'(0) &\le \int_{|x|<\sqrt{R}} |\varphi'(|x|)||u_0(x)||\partial_x u_0(x)|\,dx + \int_{\sqrt{R}<|x|<2R} |\varphi'(|x|)||u_0(x)||\partial_x u_0(x)|\,dx \\
&\le \int_{|x|<\sqrt{R}} |x||u_0(x)||\partial_x u_0(x)|\,dx + \int_{\sqrt{R}<|x|<2R} |x||u_0(x)||\partial_x u_0(x)|\,dx \\
&\lesssim \|u_0\|_{H^1}^2 \sqrt{R} + R\int_{\sqrt{R}<|x|} |u_0(x)||\partial_x u_0(x)|\,dx \\
&= o_R(1)R.
\end{aligned}$$

Therefore, we see that

$$I(T) \le o_R(1)R^2 - \alpha_0 R^2.$$

For sufficiently large $R > 0$, we have $o_R(1) - \alpha_0 < 0$. However, this contradicts

$$I(T) = \int_{\mathbb{R}} \varphi(|x|)|u(T,x)|^2\,dx > 0.$$

This argument can be applied in the negative time direction. □

Appendix: Rewriting the main theorem into a version independent of the frequency

We prove Corollary 1.4. To see this, it is sufficient to prove the following lemma.

Lemma A.1. *Let $\varphi \in H^1(\mathbb{R})$. The following statements are equivalent*:

(1) *There exists $\omega > 0$ such that $S_\omega(\varphi) < l_\omega = n_\omega$.*

(2) *φ satisfies $E(\varphi)M(\varphi)^\sigma < E_0(Q_{1,0})M(Q_{1,0})^\sigma$.*

Proof. If $\varphi = 0$, the statement holds. Let $\varphi \in H^1(\mathbb{R}) \setminus \{0\}$ be fixed. We define $f(\omega) := l_\omega - S_\omega(\varphi)$. Then, (1) is true if and only if $\sup_{\omega>0} f(\omega) > 0$. Noting that $l_\omega = \omega^{\frac{p+3}{2(p-1)}} S_{1,0}(Q_{1,0})$, we know f has a maximum at $\omega = \omega_0$, where

$$\omega_0 := \left(\frac{M(\varphi)}{\frac{p+3}{2(p-1)} S_{1,0}(Q_{1,0})} \right)^{-\frac{2(p-1)}{p-5}} > 0.$$

Therefore, (1) is equivalent to $f(\omega_0) > 0$. Now, since

$$\begin{aligned}
f(\omega_0) &= \left(\frac{M(\varphi)}{\frac{p+3}{2(p-1)} S_{1,0}(Q_{1,0})} \right)^{-\frac{p+3}{p-5}} S_{1,0}(Q_{1,0}) - \left(\frac{M(\varphi)}{\frac{p+3}{2(p-1)} S_{1,0}(Q_{1,0})} \right)^{-\frac{2(p-1)}{p-5}} M(\varphi) - E(\varphi) \\
&= \frac{\left(\frac{p+3}{2(p-1)} S_{1,0}(Q_{1,0})\right)^{\frac{2(p-1)}{p-5}}}{M(\varphi)^{\frac{p+3}{p-5}}} - E(\varphi) > 0,
\end{aligned}$$

we have

$$\left(\frac{p+3}{2(p-1)} S_{1,0}(Q_{1,0}) \right)^{\frac{2(p-1)}{p-5}} > E(\varphi)M(\varphi)^{\frac{p+3}{p-5}}.$$

Noting $Q_{1,0}$ satisfies

$$\|Q_{1,0}\|_{L^2}^2 = \frac{p+3}{2(p-1)}\|\partial_x Q_{1,0}\|_{L^2}^2 = \frac{p+3}{2(p+1)}\|Q_{1,0}\|_{L^{p+1}}^{p+1},$$

we have

$$\left(\frac{p+3}{2(p-1)}S_{1,0}(Q_{1,0})\right)^{\frac{2(p-1)}{p-5}} = E_0(Q_{1,0})M(Q_{1,0})^{\frac{p+3}{p-5}}. \qquad \square$$

Acknowledgments

The authors would like to express deep appreciation to Professor Kenji Nakanishi for many useful suggestions, comments and constant encouragement. Ikeda is supported by Grant-in-Aid for JSPS Fellows 26.1884 and Grant-in-Aid for Young Scientists (B) 15K17571. Inui is supported by Grant-in-Aid for JSPS Fellows 15J02570.

References

[Akahori and Nawa 2013] T. Akahori and H. Nawa, "Blowup and scattering problems for the nonlinear Schrödinger equations", *Kyoto J. Math.* **53**:3 (2013), 629–672. MR Zbl

[Albeverio et al. 2005] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, 2nd ed., AMS Chelsea, Providence, RI, 2005. MR Zbl

[Banica and Visciglia 2016] V. Banica and N. Visciglia, "Scattering for NLS with a delta potential", *J. Differential Equations* **260**:5 (2016), 4410–4439. MR Zbl

[Cazenave 2003] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics **10**, American Mathematical Society, Providence, RI, 2003. MR Zbl

[Du et al. 2016] D. Du, Y. Wu, and K. Zhang, "On blow-up criterion for the nonlinear Schrödinger equation", *Discrete Contin. Dyn. Syst.* **36**:7 (2016), 3639–3650. MR Zbl

[Duyckaerts et al. 2008] T. Duyckaerts, J. Holmer, and S. Roudenko, "Scattering for the non-radial 3D cubic nonlinear Schrödinger equation", *Math. Res. Lett.* **15**:6 (2008), 1233–1250. MR Zbl

[Fang et al. 2011] D. Fang, J. Xie, and T. Cazenave, "Scattering for the focusing energy-subcritical nonlinear Schrödinger equation", *Sci. China Math.* **54**:10 (2011), 2037–2062. MR Zbl

[Fukuizumi and Jeanjean 2008] R. Fukuizumi and L. Jeanjean, "Stability of standing waves for a nonlinear Schrödinger equation with a repulsive Dirac delta potential", *Discrete Contin. Dyn. Syst.* **21**:1 (2008), 121–136. MR Zbl

[Fukuizumi et al. 2008] R. Fukuizumi, M. Ohta, and T. Ozawa, "Nonlinear Schrödinger equation with a point defect", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25**:5 (2008), 837–845. MR Zbl

[Goodman et al. 2004] R. H. Goodman, P. J. Holmes, and M. I. Weinstein, "Strong NLS soliton-defect interactions", *Phys. D* **192**:3–4 (2004), 215–248. MR Zbl

[Holmer and Roudenko 2008] J. Holmer and S. Roudenko, "A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation", *Comm. Math. Phys.* **282**:2 (2008), 435–467. MR Zbl

[Holmer and Roudenko 2010] J. Holmer and S. Roudenko, "Divergence of infinite-variance nonradial solutions to the 3D NLS equation", *Comm. Partial Differential Equations* **35**:5 (2010), 878–905. MR Zbl

[Ibrahim et al. 2011] S. Ibrahim, N. Masmoudi, and K. Nakanishi, "Scattering threshold for the focusing nonlinear Klein–Gordon equation", *Anal. PDE* **4**:3 (2011), 405–460. MR Zbl

[Kenig and Merle 2006] C. E. Kenig and F. Merle, "Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case", *Invent. Math.* **166**:3 (2006), 645–675. MR Zbl

[Killip et al. 2016] R. Killip, J. Murphy, M. Visan, and J. Zheng, "The focusing cubic NLS with inverse-square potential in three space dimensions", preprint, 2016. arXiv

[Le Coz et al. 2008] S. Le Coz, R. Fukuizumi, G. Fibich, B. Ksherim, and Y. Sivan, "Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential", *Phys. D* **237**:8 (2008), 1103–1128. MR Zbl

Received 16 Sep 2016. Accepted 28 Nov 2016.

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