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# THE WEAK-$A_\infty$ PROPERTY OF HARMONIC AND $p$-HARMONIC MEASURES IMPLIES UNIFORM RECTIFIABILITY

STEVE HOFMANN, PHI LE, JOSÉ MARÍA MARTELL AND KAJ NYSTRÖM

Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an Ahlfors–David regular set of dimension $n$. We show that the weak-$A_\infty$ property of harmonic measure, for the open set $\Omega := \mathbb{R}^{n+1} \setminus E$, implies uniform rectifiability of $E$. More generally, we establish a similar result for the Riesz measure, $p$-harmonic measure, associated to the $p$-Laplace operator, $1 < p < \infty$.

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## 1. Introduction

In this paper we prove quantitative, scale invariant results of free boundary type, for harmonic measure and, more generally, for $p$-harmonic measure. More precisely, let $\Omega \subset \mathbb{R}^{n+1}$ be an open set (not necessarily connected nor bounded) satisfying an interior corkscrew condition, whose boundary is $n$-dimensional Ahlfors–David regular (ADR) (see Definition 2.1). Given these background hypotheses we prove that if $\omega$, the harmonic measure for $\Omega$, is absolutely continuous with respect to $\sigma$, and if the Poisson kernel $k = d\omega/d\sigma$ verifies an appropriate scale invariant higher integrability estimate (in particular, if $\omega$ belongs to weak-$A_\infty$ with respect to $\sigma$), then $\partial\Omega$ is uniformly rectifiable in the sense of [David and Semmes 1991; 1993]; see Theorem 1.1 and Corollary 1.5 below. In particular, our background hypotheses hold in the case that $\Omega := \mathbb{R}^{n+1} \setminus E$ is the complement of an ADR set of codimension 1, as in that case it is well known that the corkscrew condition is verified automatically in $\Omega$, i.e., in every ball $B = B(x, r)$ centered on $E$, there is some component of $\Omega \cap B$ that contains a point $Y$ with $\operatorname{dist}(Y, E) \approx r$. Furthermore, our argument is general enough to allow us to establish a nonlinear version of Theorem 1.1 (see Theorem 1.12 below) involving the $p$-Laplace operator, $p$-harmonic functions, and $p$-harmonic measure.

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To briefly outline previous work, in [Hofmann et al. 2014] the first and third authors, together with I. Uriarte-Tuero, proved the same result (cf. Theorem 1.1 and Corollary 1.5) under the additional strong hypothesis that $\Omega$ is a connected domain, satisfying an interior Harnack chain condition. In hindsight, under that extra assumption, one obtains the stronger conclusion that the exterior domain $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ in fact also satisfies a corkscrew condition, and hence that $\Omega$ is an NTA domain in the sense of [Jerison and Kenig 1982]; see [Azzam et al. 2014] for the details. Compared to [Hofmann et al. 2014] the main new advances in the present paper are two. First, we remove any connectivity hypothesis; in particular, we avoid the Harnack chain condition. Second, we are able to establish a version of our results also in the nonlinear case $1 < p < \infty$. Our main results — Theorem 1.1, Corollary 1.5, and Theorem 1.12 — are new even in the linear case $p = 2$.

Our approach is decidedly influenced by prior work of Lewis and Vogel [2006; 2007]. In particular, a version of Theorem 1.12 and Theorem 1.1 was proved in [Lewis and Vogel 2007], under the stronger hypothesis that $p$-harmonic measure $\mu$ itself is an Ahlfors–David regular measure, which in the linear case $p = 2$ implies that the Poisson kernel is a bounded, accretive function, i.e., $k \approx 1$. However, to weaken the hypotheses on $\omega$ and $\mu$, as we have done here, requires further considerations, which we discuss below in Section 1B.

To provide some additional context, we mention that out results here may be viewed as "large constant" analogues of results of Kenig and Toro [2003] in the linear case $p = 2$, and of J. Lewis and Nyström [2012], in the general $p$-harmonic case $1 < p < \infty$. These authors show that in the presence of a Reifenberg flatness condition and Ahlfors–David regularity, $\log k \in \mathrm{VMO}$ implies that the unit normal $\nu$ to the boundary belongs to VMO, where $k$ is either the Poisson kernel with pole at some fixed point or the density of $p$-harmonic Riesz measure associated to a particular ball $B(x, r)$. Moreover, under the same background hypotheses, the condition $\nu \in \mathrm{VMO}$ is equivalent to a uniform rectifiability (UR) condition with vanishing trace. Thus $\log k \in \mathrm{VMO} \Longrightarrow$ vanishing UR, given sufficient Reifenberg flatness. On the other hand, our large constant version "almost" says "$\log k \in \mathrm{BMO} \Longrightarrow$ UR". Indeed, it is well known that the $A_\infty$ condition, i.e., weak-$A_\infty$ plus the doubling property, implies that $\log k \in \mathrm{BMO}$, while if $\log k \in \mathrm{BMO}$ with small norm, then $k \in A_\infty$. We further note that, in turn, the results of [Kenig and Toro 2003] may be viewed as an "endpoint" version of the free boundary results of [Alt and Caffarelli 1981; Jerison 1990], which establish, again in the presence of Reifenberg flatness, that Hölder continuity of $\log k$ implies that of the unit normal $\nu$ (and indeed, that $\partial\Omega$ is of class $C^{1,\alpha}$ for some $\alpha > 0$).

**1A. *Statement of main results.*** Given an open set $\Omega \subset \mathbb{R}^{n+1}$, and a Euclidean ball $B = B(x, r) \subset \mathbb{R}^{n+1}$ centered on $\partial\Omega$, we let $\Delta = \Delta(x, r) := B \cap \partial\Omega$ denote the corresponding surface ball. For $X \in \Omega$, let $\omega^X$ be harmonic measure for $\Omega$, with pole at $X$. As mentioned above, all other terminology and notation will be defined below.

Concerning the Laplace operator and harmonic measure we prove the following results.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set whose boundary is Ahlfors–David regular of dimension $n$* (see Definition 2.1). *Suppose that there are positive constants $C_0$ and $c_0$, and an exponent $q > 1$, such* *that for every surface ball* $\Delta = \Delta(x, r)$, *with* $x \in \partial\Omega$ *and* $0 < r < \operatorname{diam}(\partial\Omega)$, *there exists* $X_\Delta \in B(x, r) \cap \Omega$, *with* $\operatorname{dist}(X_\Delta, \partial\Omega) \geq c_0 r$, *satisfying*

($\star$) ***scale-invariant higher integrability***: $\omega^{X_\Delta} \ll \sigma$ *in* $2\Delta$, *and* $k^{X_\Delta} := d\omega^{X_\Delta}/d\sigma$ *satisfies*

$$\int_{2\Delta} k^{X_\Delta}(y)^q \, d\sigma(y) \leq C_0 \sigma(\Delta)^{1-q}. \tag{1.2}$$

*Then* $\partial\Omega$ *is uniformly rectifiable and moreover the "UR character"* (see Definition 2.4) *depends only on* $n$, *the ADR constants*, $q$, $c_0$, *and* $C_0$.

The point $X_\Delta$ in Theorem 1.1 is a "corkscrew point" for $\Omega$, relative to $\Delta$. An open set $\Omega$ for which there is such a point relative to every surface ball $\Delta(x, r)$, $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, with a uniform constant $c_0$, is said to satisfy the "corkscrew condition" (see Definition 2.5 below).

**Remark 1.3.** We note that, in lieu of absolute continuity and ($\star$), only the following apparently weaker condition is actually used in the proof of Theorem 1.1:

($\star\star$) **local nondegeneracy**: there exist uniform constants $\eta$, $\beta > 0$ such that if $A \subset \Delta$ is Borel measurable, then

$$\sigma(A) \geq (1-\eta)\sigma(\Delta) \implies \omega^{X_\Delta}(A) \geq \beta\omega^{X_\Delta}(\Delta).[1] \tag{1.4}$$

Here $\Delta = \Delta(x, r)$ for $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and $X_\Delta \in B(x, r/2) \cap \Omega$ with $\operatorname{dist}(X_\Delta, \partial\Omega) \geq c_0 r/2$.[2] We observe that there turns out to be some flexibility in the choice of $X_\Delta$ (see the discussion at the beginning of Section 4), and consequently it is not hard to see that ($\star$) implies ($\star\star$); see Lemma 4.3.

We also have the following easy corollary of Theorem 1.1 (we shall give the short proof of the corollary in Section 5D).

**Corollary 1.5.** *Let* $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, *be an open set satisfying the corkscrew condition, whose boundary is Ahlfors–David regular of dimension* $n$. *Suppose further that for every ball* $B = B(x, r)$ *with* $x \in \partial\Omega$ *and* $0 < r < \operatorname{diam}(\partial\Omega)$, *and every* $Y \in \Omega \setminus B(x, 2r)$, *harmonic measure* $\omega^Y$ *belongs to weak-*$A_\infty(\Delta(x, r))$, *i.e., there is a constant* $C_0 \geq 1$ *and an exponent* $q > 1$, *each of which is uniform with respect to* $x$, $r$, *and* $Y$, *such that* $\omega^Y \ll \sigma$ *in* $\Delta(x, r)$, *and* $k^Y = d\omega^Y/d\sigma$ *satisfies*

$$\left( \fint_{\Delta'} k^Y(z)^q \, d\sigma(z) \right)^{1/q} \leq C_0 \fint_{2\Delta'} k^Y(z) \, d\sigma(z) \tag{1.6}$$

*for every surface ball centered on the boundary* $\Delta' = B' \cap \partial\Omega$ *with* $2B' \subset B(x, r)$. *Then* $\partial\Omega$ *is uniformly rectifiable, and moreover, the "UR character"* (see Definition 2.4) *depends only on* $n$, *the ADR constant of* $\partial\Omega$, $q$, $C_0$, *and the corkscrew constant.*

**Remark 1.7.** As mentioned above, the corkscrew condition is automatically satisfied in the case that $E$ is an $n$-dimensional ADR set (hence closed, see Definition 2.1 below), and $\Omega = \mathbb{R}^{n+1} \setminus E$ is its complement, with the corkscrew constant for $\Omega$ depending only on $n$ and the ADR constant of $E$. Thus, in particular,

[1] This formulation is adapted from [Mourgoglou and Tolsa 2015]; see the discussion in Section 1D.

[2] For aesthetic reasons, and for convenience in the sequel, in contrast to condition ($\star$), we prefer to state condition ($\star\star$) in terms of $\Delta$ rather than $2\Delta$, and with $X_\Delta \in B(x, r/2)$ rather than $B(x, r)$.

Corollary 1.5 applies in that setting, so in the presence of the weak reverse Hölder condition (1.6), we deduce that $E$ is uniformly rectifiable.

Combining Theorem 1.1 with the results in [Bortz and Hofmann 2015], we obtain as an immediate consequence a "big pieces" characterization of uniformly rectifiable sets of codimension 1, in terms of harmonic measure. Here and in the sequel, given an ADR set $E$, $Q$ denotes a "dyadic cube" on $E$ in the sense of [David and Semmes 1991; 1993; Christ 1990], and $\mathbb{D}(E)$ denotes the collection of all such cubes; see Lemma 2.6 below.

**Theorem 1.8.** *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an n-dimensional ADR set. Let $\Omega := \mathbb{R}^{n+1} \setminus E$. Then $E$ is uniformly rectifiable if and only if it has "big pieces of good harmonic measure estimates" in the following sense: for each $Q \in \mathbb{D}(E)$ there exists an open set $\widetilde{\Omega} = \widetilde{\Omega}_Q$ with the following properties, with uniform control of the various implicit constants:*

- *$\partial\widetilde{\Omega}$ is ADR;*
- *the interior corkscrew condition holds in $\widetilde{\Omega}$;*
- *$\partial\widetilde{\Omega}$ has a "big pieces" overlap with $E$, in the sense that $\sigma(Q \cap \partial\widetilde{\Omega}) \gtrsim \sigma(Q)$;*
- *for each surface ball $\Delta = \Delta(x,r) := B(x,r) \cap \partial\widetilde{\Omega}$ with $x \in \partial\widetilde{\Omega}$ and $r \in (0, \operatorname{diam}(\widetilde{\Omega}))$, there is an interior corkscrew point $X_\Delta \in \widetilde{\Omega}$ such that $\omega_{\widetilde{\Omega}}^{X_\Delta}$, the harmonic measure for $\widetilde{\Omega}$ with pole at $X_\Delta$, satisfies $\omega_{\widetilde{\Omega}}^{X_\Delta}(\Delta) \gtrsim 1$, and belongs to weak-$A_\infty(\Delta)$.*

The "only if" direction is proved in [Bortz and Hofmann 2015], and the open sets $\widetilde{\Omega}$ constructed in [Bortz and Hofmann 2015] even satisfy a 2-sided corkscrew condition, and moreover, $\widetilde{\Omega} \subset \Omega$ with $\operatorname{diam}(\widetilde{\Omega}) \approx \operatorname{diam}(Q)$. To obtain the converse direction, we simply observe that by Theorem 1.1, the subdomains $\widetilde{\Omega}$ have uniformly rectifiable boundaries, with uniform control of the "UR character" of each $\partial\widetilde{\Omega}$, and thus, by [David and Semmes 1993], $E$ is uniformly rectifiable.

To formulate our main result in the nonlinear setting we first need to introduce some notation. If $O \subset \mathbb{R}^{n+1}$ is an open set and $1 \leq p \leq \infty$, then by $W^{1,p}(O)$ we denote the space of equivalence classes of functions $f$ with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_{n+1}})$, both of which are $q$-th power integrable on $O$. Let $\|f\|_{1,p} = \|f\|_p + \||\nabla f|\|_p$ be the norm in $W^{1,p}(O)$, where $\|\cdot\|_q$ denotes the usual Lebesgue $p$ norm in $O$. Next, let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in $O$, and let $W_0^{1,p}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,p}(O)$. We let $W_{\mathrm{loc}}^{1,p}(O)$ be the set of all functions $u$ such that $u\Theta \in W_0^{1,p}(O)$ whenever $\Theta \in C_0^\infty(O)$.

Given an open set $O$ and $1 < p < \infty$, we say that $u$ is $p$-harmonic in $O$ provided $u \in W_{\mathrm{loc}}^{1,p}(O)$ and

$$\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2} \nabla u \cdot \nabla\Theta \, dX = 0, \quad \forall \Theta \in C_0^\infty(O). \tag{1.9}$$

Observe that if $u$ is smooth and $\nabla u \neq 0$ in $O$, then

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \quad \text{in } O, \tag{1.10}$$

and $u$ is a classical solution in $O$ to the $p$-Laplace partial differential equation. Here, as in the sequel, $\nabla\cdot$ is the divergence operator.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, not necessarily connected, with $n$-dimensional ADR boundary. Let $p \in (1, \infty)$. Given $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, let $u$ be a nonnegative $p$-harmonic function in $\Omega \cap B(x, r)$ which vanishes continuously on $\Delta(x, r) := B(x, r) \cap \partial\Omega$. Extend $u$ to all of $B(x, r)$ by putting $u \equiv 0$ on $B(x, r) \setminus \overline{\Omega}$. Then there exists a unique nonnegative finite Borel measure $\mu$ on $\mathbb{R}^{n+1}$, with support contained in $\Delta(x, r)$, such that

$$-\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dX = \int_{\partial\Omega} \phi \, d\mu, \quad \forall \phi \in C_0^\infty(B(x, r)); \tag{1.11}$$

see [Heinonen et al. 2006, Chapter 21] and Lemma 3.43 below. We refer to $\mu$ as the $p$-harmonic measure associated to $u$. In the case $p = 2$, and if $u$ is the Green function for $\Omega$ with pole at $X \in \Omega$, then the measure $\mu$ coincides with harmonic measure at $X$, $\omega = \omega^X$.

Concerning the $p$-Laplace operator, $p$-harmonic functions, and $p$-harmonic measure, we prove the following theorem.

**Theorem 1.12.** *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set whose boundary is Ahlfors–David regular of dimension $n$. Let $p$, $1 < p < \infty$, be given. Let $C$ be a sufficiently large constant* (*to be specified*), *depending only on $n$ and the ADR constant*, *and suppose that there exist $q > 1$ and a positive constant $C_0$ for which the following holds*: *for each $x \in \partial\Omega$ and each $0 < r < \operatorname{diam}(\partial\Omega)$, there is a nontrivial, nonnegative $p$-harmonic function $u = u_{x,r}$ in $\Omega \cap B(x, Cr)$, and corresponding $p$-harmonic measure $\mu = \mu_{x,r}$, such that $\mu \ll \sigma$ in $\Delta(x, Cr)$, and such that $k := d\mu/d\sigma$ satisfies*

$$\left( \fint_{\Delta(x,Cr)} k(y)^q \, d\sigma(y) \right)^{1/q} \leq C_0 \frac{\mu(\Delta(x, r))}{\sigma(\Delta(x, r))}. \tag{1.13}$$

*Then $\partial\Omega$ is uniformly rectifiable*, *and moreover the "UR character"* (see Definition 2.4) *depends only on $n$*, *the ADR constant*, *$p$*, *$q$*, *and $C_0$.*

Some remarks are in order concerning the hypotheses of Theorem 1.12. Let us observe that, in particular, Ahlfors–David regularity and (1.13) imply that

$$\mu(\Delta(x, Cr)) \leq C_1 \mu(\Delta(x, r)), \tag{1.14}$$

with $C_1 \approx C_0$. In the linear case, the latter estimate follows automatically, with $\mu = \omega^Y$ for some $Y \in B(x, r)$ such that $\operatorname{dist}(Y, E) \approx r$, and with $C_1$ depending only on $n$ and the ADR constant, by Bourgain's Lemma 3.1 below, even though $\omega^Y$ need not be a doubling measure (i.e., (1.14) says nothing about points other than $x$ nor about scales other than $r$). In the nonlinear case, it seems that we must impose condition (1.14) by hypothesis. We also observe that (1.13) holds in particular if $\mu \in$ weak-$A_\infty(\Delta(x, 2Cr))$ and satisfies (1.14) (with radius $2C$ in place of $C$). Of course, (1.14) holds trivially if $\mu$ is a doubling measure, but we do not assume doubling.

**Remark 1.15.** We note that, as in Remark 1.3, the proof of Theorem 1.12 will in fact use, in lieu of absolute continuity and (1.13), only the apparently weaker condition that there exist uniform constants $\eta, \beta \in (0, 1)$ such that for all $\Delta = \Delta(x, r)$, and for all Borel sets $A \subset \Delta$,

$$\sigma(A) \geq (1 - \eta)\sigma(\Delta) \implies \mu(A) \geq \beta\mu(\Delta). \tag{1.16}$$

**1B.** ***Brief outline of the proofs of the main results.*** As mentioned, the approach in the present paper is strongly influenced by prior work due to Lewis and Vogel [2006; 2007], who in the latter paper proved a version of Theorem 1.12, and Theorem 1.1, under the stronger hypothesis that $p$-harmonic measure $\mu$ itself is an Ahlfors–David regular measure. In the linear case $p = 2$, this implies that the Poisson kernel is a bounded, accretive function, i.e., $k \approx 1$. Assuming that $p$-harmonic measure $\mu$ is an Ahlfors–David regular measure, Lewis and Vogel were able to show that $E$ satisfies the so-called weak exterior convexity (WEC) condition, which characterizes uniform rectifiability [David and Semmes 1993]. To weaken the hypotheses on $\omega$ and $\mu$, as we have done here, requires two further considerations. The first is quite natural in this context: a stopping time argument, in the spirit of the proofs of the Kato square root conjecture [Hofmann and McIntosh 2002; Hofmann et al. 2002; Auscher et al. 2002a] (and of local $Tb$ theorems [Christ 1990; Auscher et al. 2002b; Hofmann 2006]), by means of which we extract ample dyadic sawtooth regimes on which averages of harmonic measure and $p$-harmonic measure are bounded and accretive; see Lemma 4.12 below. This allows us to use the arguments of [Lewis and Vogel 2007] within these good sawtooth regions. The second new consideration is necessitated by the fact that in our setting, the doubling property may fail for harmonic and $p$-harmonic measure. In the absence of doubling, we are unable to obtain the WEC condition directly. Nonetheless, we are able to follow the arguments of [Lewis and Vogel 2007] very closely up to a point, to obtain a condition on $\partial\Omega$ which we call the "weak half space approximation" (WHSA) property (see Definition 2.19). Indeed, extracting the essence of the argument of [Lewis and Vogel 2007], while dispensing with the doubling property, one realizes that the WHSA is precisely what one obtains. In the sequel, we present the argument of [Lewis and Vogel 2007] as Lemma 5.10. Finally, having obtained that $\partial\Omega$ satisfies the WHSA property, we are able to prove the following proposition stating that WHSA implies uniform rectifiability.

**Proposition 1.17.** *An $n$-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ is uniformly rectifiable if and only if it satisfies the WHSA property.*

While the WHSA condition, per se, is new, our proof of Proposition 1.17 is based on a modified version of part of the argument in [Lewis and Vogel 2007].

**1C.** ***Organization of the paper.*** The paper is organized as follows. In Section 2, we state several definitions, including definitions of ADR, UR, and dyadic grids, and introduce further notions and notation. In Section 3, we state, and either prove or give references for, the PDE estimates needed in the proofs of our main results. In Section 4, we begin the (simultaneous) proofs of Theorem 1.1 and Theorem 1.12 by giving some preliminary arguments. In Section 5, following [Lewis and Vogel 2006; 2007], we complete the proofs of Theorem 1.1 and Theorem 1.12, modulo Proposition 1.17. At the end of Section 5 we also give the (very short) proof of Corollary 1.5. In Section 6, we give the proof of Proposition 1.17, i.e., the proof of the fact that the WHSA condition implies uniform rectifiability.

**1D.** ***Discussion of recent related work.*** We note that some related work has recently appeared, or been carried out, while this manuscript was in preparation. In the setting of uniform domains with lower ADR boundary with locally finite $n$-dimensional Hausdorff measure, Mourgoglou [2015] has shown that rectifiability of the boundary implies absolute continuity of surface measure with respect to harmonic measure (for the Laplacian). Akman, Badger, Hofmann, and Martell [Akman et al. 2015], in the setting of uniform domains with ADR boundary, have characterized the rectifiability of the boundary in terms of the absolute continuity of harmonic measure and some elliptic measures and surface measure or in terms of some qualitative $A_\infty$ condition. Also, Azzam, Mourgoglou, and Tolsa [Azzam et al. 2015] have obtained that absolute continuity of harmonic measure with respect to surface measure on an $H^n$-finite piece of the boundary implies that harmonic measure is rectifiable in that piece. The setting is very general as they only assume a "porosity" (i.e., corkscrew) condition in the complement of $\partial\Omega$. In [Hofmann et al. 2015], Hofmann, Martell, Mayboroda, Tolsa, and Volberg prove the same result removing the porosity assumption. Both [Azzam et al. 2015] and the follow-up version [Hofmann et al. 2015] (which will be combined in the forthcoming paper [Azzam et al. 2016]) rely on recent deep results of [Nazarov et al. 2014a; 2014b], concerning connections between rectifiability and the behavior of Riesz transforms.

Finally, we discuss two closely related papers treating the case $p = 2$. First, we mention that a preliminary version of our results, treating only the linear harmonic case (i.e., Theorem 1.1 of the present paper) under hypothesis ($\star$), appeared earlier in the unpublished preprint [Hofmann and Martell 2015]. That result, again in the case $p = 2$, was then essentially reproved, by a different method, in [Mourgoglou and Tolsa 2015], but assuming condition ($\star\star$) in place of ($\star$). While the present paper was in preparation, we learned of the work in [Mourgoglou and Tolsa 2015], and we realized that our arguments (and those of [Hofmann and Martell 2015]), almost unchanged, also allow ($\star$) to be replaced by ($\star\star$) or its $p$-harmonic equivalent. The current version of this manuscript incorporates this observation.[3] Let us mention also that the approach in [Mourgoglou and Tolsa 2015] is based on showing that ($\star\star$) for harmonic measure implies $L^2$-boundedness of the Riesz transforms, and thus it is a quantitative version of the method of [Azzam et al. 2016]. An interesting feature of the proof in [Mourgoglou and Tolsa 2015] is that it works even without the lower bound in the Ahlfors–David condition; in that case, one may deduce rectifiability, as opposed to uniform rectifiability, of the underlying measure on $\partial\Omega$. On the other hand, it seems difficult to generalize the approach of [Mourgoglou and Tolsa 2015] to the $p$-Laplace setting, since it is based on Riesz transforms, which are tied to the linear harmonic case.

## 2. ADR, UR, and dyadic grids

**Definition 2.1** (Ahlfors–David regular (ADR)). We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension $n$, is ADR if it is closed and if there is some uniform constant $C$ such that

$$C^{-1} r^n \le \sigma(\Delta(x,r)) \le C r^n, \quad \forall r \in (0, \operatorname{diam}(E)),\ x \in E, \tag{2.2}$$

where $\operatorname{diam}(E)$ may be infinite. Here, $\Delta(x,r) := E \cap B(x,r)$ is the "surface ball" of radius $r$, and $\sigma := H^n|_E$ is the "surface measure" on $E$, where $H^n$ denotes $n$-dimensional Hausdorff measure.

**Definition 2.3** (uniformly rectifiable (UR)). An $n$-dimensional ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is UR if and only if it contains "big pieces of Lipschitz images" of $\mathbb{R}^n$ (BPLI). This means that there are positive

[3] We thank Mourgoglou and Tolsa for making their preprint available to us while our manuscript was in preparation.

constants $\theta$ and $M_0$, such that for each $x \in E$ and each $r \in (0, \operatorname{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz constant no larger than $M_0$, such that

$$H^n\big(E \cap B(x,r) \cap \rho(\{z \in \mathbb{R}^n : |z| < r\})\big) \geq \theta r^n.$$

We recall that $n$-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of $H^n$ measure 0, by a countable union of Lipschitz images of $\mathbb{R}^n$; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of ADR sets, the UR sets are precisely those for which all "sufficiently nice" singular integrals are $L^2$-bounded [David and Semmes 1991]. In fact, for $n$-dimensional ADR sets in $\mathbb{R}^{n+1}$, the $L^2$-boundedness of certain special singular integral operators (the "Riesz transforms") suffices to characterize uniform rectifiability (see [Mattila et al. 1996] for the case $n = 1$, and [Nazarov et al. 2014a] in general). We further remark that there exist sets that are ADR (and that even form the boundary of a domain satisfying interior corkscrew and Harnack chain conditions), but that are totally nonrectifiable (e.g., see the construction of Garnett's "4-corners Cantor set" in [David and Semmes 1993, Chapter1]). Finally, we mention that there are numerous other characterizations of UR sets (many of which remain valid in higher codimensions); see [David and Semmes 1991; 1993], and in particular Theorem 2.14 below. In this paper, we also present a new characterization of UR sets of codimension 1 (see Proposition 1.17 below), which will be very useful in the proof of Theorem 1.1.

**Definition 2.4** (UR character). Given a UR set $E \subset \mathbb{R}^{n+1}$, its "UR character" is just the pair of constants $(\theta, M_0)$ involved in the definition of uniform rectifiability, along with the ADR constant; or equivalently, the quantitative bounds involved in any particular characterization of uniform rectifiability.

**Definition 2.5** (corkscrew condition). Following [Jerison and Kenig 1982], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the "corkscrew condition" if for some uniform constant $c_0 > 0$ and for every surface ball $\Delta := \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a point $X_\Delta \in B(x, r) \cap \Omega$ such that $\operatorname{dist}(X_\Delta, \partial\Omega) \geq c_0 r$. The point $X_\Delta \subset \Omega$ is called a "corkscrew point" relative to $\Delta$.

**Lemma 2.6** (existence and properties of the "dyadic grid" [David and Semmes 1991; 1993; Christ 1990]). *Suppose that $E \subset \mathbb{R}^{n+1}$ is a closed $n$-dimensional ADR set. Then there exist constants $a_0 > 0$, $\gamma > 0$, and $C_* < \infty$, depending only on $n$ and the ADR constant, such that for each $k \in \mathbb{Z}$, there is a collection*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{J}_k\}$$

*of Borel sets ("cubes"), where $\mathfrak{J}_k$ denotes some (possibly finite) index set depending on $k$, satisfying*

(i) $E = \bigcup_j Q_j^k$ *for each* $k \in \mathbb{Z}$;

(ii) *if* $m \geq k$ *then either* $Q_i^m \subset Q_j^k$ *or* $Q_i^m \cap Q_j^k = \varnothing$;

(iii) *for each* $(j, k)$ *and each* $m < k$, *there is a unique* $i$ *such that* $Q_j^k \subset Q_i^m$;

(iv) $\operatorname{diam}(Q_j^k) \leq C_* 2^{-k}$;

(v) *each* $Q_j^k$ *contains some "surface ball"* $\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap E$;

(vi) $H^n\big(\{x \in Q_j^k : \operatorname{dist}(x, E \setminus Q_j^k) \leq \varrho\, 2^{-k}\}\big) \leq C_* \varrho^\gamma H^n(Q_j^k)$ *for all* $k$, $j$ *and for all* $\varrho \in (0, a_0)$.

Let us make a few remarks concerning this lemma, and discuss some related notation and terminology.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [1990], with the dyadic parameter $\frac{1}{2}$ replaced by some constant $\delta \in (0, 1)$. In fact, one may always take $\delta = \frac{1}{2}$ (cf. [Hofmann et al. 2017, Proof of Proposition 2.12]). In the presence of the Ahlfors–David property (2.2), the result already appears in [David and Semmes 1991; 1993].
- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$, in the case that the latter is finite.
- We denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant $Q_j^k$, i.e.,

$$\mathbb{D} := \bigcup_k \mathbb{D}_k,$$

  where, if $\operatorname{diam}(E)$ is finite, the union runs over those $k$ such that $2^{-k} \lesssim \operatorname{diam}(E)$.
- Properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r)$, and a surface ball $\Delta(x_Q, r) := B(x_Q, r) \cap E$ such that $r \approx 2^{-k} \approx \operatorname{diam}(Q)$ and

$$\Delta(x_Q, r) \subset Q \subset \Delta(x_Q, Cr) \tag{2.7}$$

  for some uniform constant $C$. We denote this ball and surface ball by

$$B_Q := B(x_Q, r), \qquad \Delta_Q := \Delta(x_Q, r), \tag{2.8}$$

  and we refer to the point $x_Q$ as the "center" of $Q$.
- Given a dyadic cube $Q \in \mathbb{D}$, we define its "$\kappa$-dilate" by

$$\kappa Q := E \cap B(x_Q, \kappa \operatorname{diam}(Q)). \tag{2.9}$$

- For a dyadic cube $Q \in \mathbb{D}_k$, we set $\ell(Q) = 2^{-k}$, and we refer to this quantity as the "length" of $Q$. Clearly, $\ell(Q) \approx \operatorname{diam}(Q)$.
- For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the "dyadic generation" to which $Q$ belongs, i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.
- For any $Q \in \mathbb{D}(E)$, we set $\mathbb{D}_Q := \{Q' \in \mathbb{D} : Q' \subset Q\}$.
- Given $Q_0 \in \mathbb{D}(E)$ and a family $\mathcal{F} = \{Q_j\} \subset \mathbb{D}$ of pairwise disjoint cubes, we set

$$\mathbb{D}_{\mathcal{F},Q_0} := \{Q \in \mathbb{D}_{Q_0} : Q \text{ is not contained in any } Q_j \in \mathcal{F}\} = \mathbb{D}_{Q_0} \setminus \Big( \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} \Big). \tag{2.10}$$

**Definition 2.11** ($\varepsilon$-local BAUP). Given $\varepsilon > 0$, we say that $Q \in \mathbb{D}(E)$ satisfies the *$\varepsilon$-local BAUP* condition if there is a family $\mathcal{P}$ of hyperplanes (depending on $Q$) such that every point in $10Q$ is at a distance at most $\varepsilon\ell(Q)$ from $\bigcup_{P \in \mathcal{P}} P$, and every point in $\big(\bigcup_{P \in \mathcal{P}} P\big) \cap B(x_Q, 10 \operatorname{diam}(Q))$ is at a distance at most $\varepsilon\ell(Q)$ from $E$.

**Definition 2.12** (BAUP). We say that an $n$-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ satisfies the condition of *bilateral approximation by unions of planes* (BAUP) if for some $\varepsilon_0 > 0$, and for every positive $\varepsilon < \varepsilon_0$, there is a constant $C_\varepsilon$ such that the set $\mathcal{B}$ of bad cubes in $\mathbb{D}(E)$, for which the $\varepsilon$-local BAUP condition fails, satisfies the packing condition

$$\sum_{Q'\subset Q,\, Q'\in\mathcal{B}} \sigma(Q') \le C_\varepsilon \sigma(Q), \quad \forall Q \in \mathbb{D}(E). \tag{2.13}$$

For future reference, we recall the following result of David and Semmes.

**Theorem 2.14** [David and Semmes 1993, Theorem I.2.18, p. 36]. *Let $E \subset \mathbb{R}^{n+1}$ be an $n$-dimensional ADR set. Then $E$ is uniformly rectifiable if and only if it satisfies BAUP.*

We remark that the definition of BAUP in [David and Semmes 1993] is slightly different in superficial appearance, but it is not hard to verify that the dyadic version stated here is equivalent to their condition. We note that we shall not need the full strength of this equivalence here, but only the fact that our version of BAUP implies the version in [David and Semmes 1993], and hence implies UR.

We also require a new characterization of UR sets of codimension 1, which is related to the BAUP and its variants. For a sufficiently large constant $K_0$ to be chosen (see Lemma 4.24 below), we set

$$B_Q^* := B(x_Q, K_0^2\ell(Q)), \qquad \Delta_Q^* := B_Q^* \cap E. \tag{2.15}$$

Given a small positive number $\varepsilon$, which we typically assume to be much smaller than $K_0^{-6}$, we also set

$$B_Q^{**} = B_Q^{**}(\varepsilon) := B(x_Q, \varepsilon^{-2}\ell(Q)), \qquad B_Q^{***} = B_Q^{***}(\varepsilon) := B(x_Q, \varepsilon^{-5}\ell(Q)). \tag{2.16}$$

**Definition 2.17** ($\varepsilon$-local WHSA). Given $\varepsilon > 0$, we say that $Q \in \mathbb{D}(E)$ satisfies the *$\varepsilon$-local WHSA* condition (or more precisely, the "$\varepsilon$-local WHSA with parameter $K_0$") if there is a half-space $H = H(Q)$, a hyperplane $P = P(Q) = \partial H$, and a fixed positive number $K_0$ satisfying

(1) $\operatorname{dist}(Z, E) \le \varepsilon\ell(Q)$ for every $Z \in P \cap B_Q^{**}(\varepsilon)$,

(2) $\operatorname{dist}(Q, P) \le K_0^{3/2}\ell(Q)$, and

(3) $H \cap B_Q^{**}(\varepsilon) \cap E = \varnothing$.

Note that part (2) of the previous definition says that the hyperplane $P$ has an "ample" intersection with the ball $B_Q^{**}(\varepsilon)$. Indeed,

$$\operatorname{dist}(x_Q, P) \lesssim K_0^{3/2}\ell(Q) \ll \varepsilon^{-2}\ell(Q). \tag{2.18}$$

**Definition 2.19** (WHSA). We say that an $n$-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ satisfies the *weak half-space approximation* property (WHSA) if for some pair of positive constants $\varepsilon_0$ and $K_0$, and for every positive $\varepsilon < \varepsilon_0$, there is a constant $C_\varepsilon$ such that the set $\mathcal{B}$ of bad cubes in $\mathbb{D}(E)$, for which the $\varepsilon$-local WHSA condition with parameter $K_0$ fails, satisfies the packing condition

$$\sum_{Q\subset Q_0,\, Q\in\mathcal{B}} \sigma(Q) \le C_\varepsilon \sigma(Q_0), \quad \forall Q_0 \in \mathbb{D}(E). \tag{2.20}$$

Next, we develop some further notation and terminology. Given a closed set $E$, set $\delta_E(Y) := \operatorname{dist}(Y, E)$, simply writing $\delta(Y)$ when the set has been fixed.

Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of $\Omega$, so that the cubes in $\mathcal{W}$ form a covering of $\Omega$ with nonoverlapping interiors, and which satisfy

$$4\,\mathrm{diam}(I) \le \mathrm{dist}(4I, \partial\Omega) \le \mathrm{dist}(I, \partial\Omega) \le 40\,\mathrm{diam}(I) \tag{2.21}$$

and

$$\mathrm{diam}(I_1) \approx \mathrm{diam}(I_2), \quad \text{whenever } I_1 \text{ and } I_2 \text{ touch.} \tag{2.22}$$

Assuming that $E = \partial\Omega$ is ADR and given $Q \in \mathbb{D}(E)$, for the same constant $K_0$ as in (2.15) we set

$$\mathcal{W}_Q := \{I \in \mathcal{W} : K_0^{-1}\ell(Q) \le \ell(I) \le K_0\ell(Q),\ \text{and}\ \mathrm{dist}(I, Q) \le K_0\,\ell(Q)\}\,. \tag{2.23}$$

Fix a small, positive parameter $\tau$, to be chosen momentarily, and given $I \in \mathcal{W}$, let

$$I^* = I^*(\tau) := (1+\tau)I \tag{2.24}$$

denote the corresponding "fattened" Whitney cube. We now choose $\tau$ sufficiently small that the cubes $I^*$ retain the usual properties of Whitney cubes, in particular that

$$\mathrm{diam}(I) \approx \mathrm{diam}(I^*) \approx \mathrm{dist}(I^*, E) \approx \mathrm{dist}(I, E).$$

We then define Whitney regions with respect to $Q$ by setting

$$U_Q := \bigcup_{I \in \mathcal{W}_Q} I^*. \tag{2.25}$$

We observe that these Whitney regions may have more than one connected component, but that the number of distinct components is uniformly bounded, depending only upon $K_0$ and dimension. We enumerate the components of $U_Q$ as $\{U_Q^i\}_i$. Moreover, we enlarge the Whitney regions as follows.

**Definition 2.26.** For $\varepsilon > 0$, and given $Q \in \mathbb{D}(E)$, we write $X \approx_{\varepsilon,Q} Y$ if $X$ may be connected to $Y$ by a chain of at most $\varepsilon^{-1}$ balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3\ell(Q) \le \delta(Y_k) \le \varepsilon^{-3}\ell(Q)$. Given a sufficiently small parameter $\varepsilon > 0$, we then set

$$\widetilde{U}_Q^i := \{X \in \mathbb{R}^{n+1} \setminus E : X \approx_{\varepsilon,Q} Y,\ \text{for some } Y \in U_Q^i\}. \tag{2.27}$$

**Remark 2.28.** Since $\widetilde{U}_Q^i$ is an enlarged version of $U_Q$, it may be that there are some $i \neq j$ for which $\widetilde{U}_Q^i$ meets $\widetilde{U}_Q^j$. This overlap will be harmless.

## 3. PDE estimates

In this section we recall several estimates for harmonic measure and harmonic functions, and also for $p$-harmonic measure and $p$-harmonic functions. Although some of the PDE results in the harmonic case $p = 2$ can be subsumed into the general $p$-harmonic theory, we choose to present some aspects of the harmonic theory separately, in part for the convenience of those readers who are more familiar with the case $p = 2$, and in part because the presence of the Green function is unique to that case.

**3A.** ***PDE estimates: the harmonic case.*** Next, we recall several facts concerning harmonic measure and Green's functions. Let $\Omega$ be an open set, not necessarily connected, and set $\delta(X) = \delta_{\partial\Omega}(X) = \operatorname{dist}(X, \partial\Omega)$.

**Lemma 3.1** [Bourgain 1987]. *Suppose that $\partial\Omega$ is $n$-dimensional ADR. Then there are uniform constants $c \in (0,1)$ and $C \in (1,\infty)$, depending only on $n$ and ADR, such that for every $x \in \partial\Omega$ and every $r \in (0, \operatorname{diam}(\partial\Omega))$, if $Y \in \Omega \cap B(x, cr)$ then*

$$\omega^Y(\Delta(x,r)) \geq \frac{1}{C} > 0. \tag{3.2}$$

We refer the reader to [Bourgain 1987, Lemma 1] for the proof. We note for future reference that in particular, given $X \in \Omega$, if $\hat{x} \in \partial\Omega$ satisfies $|X - \hat{x}| = \delta(X)$ and $\Delta_X := \partial\Omega \cap B(\hat{x}, 10\delta(X))$, then for a slightly different uniform constant $C > 0$,

$$\omega^X(\Delta_X) \geq \frac{1}{C}. \tag{3.3}$$

Indeed, the latter bound follows immediately from (3.2), and the fact that we can form a Harnack chain connecting $X$ to a point $Y$ that lies on the line segment from $X$ to $\hat{x}$ and satisfies $|Y - \hat{x}| = c\delta(X)$.

A proof of the next lemma may be found, e.g., in [Hofmann et al. $\geq$ 2017]. We note that, in particular, the ADR hypothesis implies that $\partial\Omega$ is Wiener regular at every point (see Lemma 3.27 below).

**Lemma 3.4.** *Let $\Omega$ be an open set with $n$-dimensional ADR boundary. There exist positive, finite constants $C$, depending only on dimension, and $c_\theta$, depending on dimension and $\theta \in (0,1)$, such that the Green function satisfies*

$$G(X,Y) \leq C|X-Y|^{1-n}; \tag{3.5}$$

$$c_\theta|X-Y|^{1-n} \leq G(X,Y), \quad \text{if } |X-Y| \leq \theta\delta(X),\ \theta \in (0,1); \tag{3.6}$$

$$G(X,\cdot) \in C(\overline{\Omega}\setminus\{X\}) \quad \text{and} \quad G(X,\cdot)|_{\partial\Omega} \equiv 0, \qquad \forall X \in \Omega; \tag{3.7}$$

$$G(X,Y) \geq 0, \qquad \forall X, Y \in \Omega,\ X \neq Y; \tag{3.8}$$

$$G(X,Y) = G(Y,X), \qquad \forall X, Y \in \Omega,\ X \neq Y; \tag{3.9}$$

*and for every $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$,*

$$\int_{\partial\Omega} \Phi\, d\omega^X - \Phi(X) = -\iint_\Omega \nabla_Y G(Y,X) \cdot \nabla\Phi(Y)\, dY, \qquad \forall X \in \Omega. \tag{3.10}$$

Next we present a version of one of the estimates obtained by Caffarelli, Fabes, Mortola, and Salsa in [Caffarelli et al. 1981], which remains true even in the absence of connectivity.

**Lemma 3.11** ("CFMS" estimates). *Suppose that $\partial\Omega$ is $n$-dimensional ADR. For every $Y \in \Omega$ and $X \in \Omega$ such that $|X - Y| \geq \delta(Y)/2$, we have*

$$\frac{G(Y,X)}{\delta(Y)} \leq C\frac{\omega^X(\Delta_Y)}{\sigma(\Delta_Y)}, \tag{3.12}$$

*where $\Delta_Y = B(\hat{y}, 10\delta(Y)) \cap E$, with $\hat{y} \in \partial\Omega$ such that $|Y - \hat{y}| = \delta(Y)$.*

For future use, we note that as a consequence of (3.12), it follows directly that for every $Q \in \mathbb{D}(\partial\Omega)$, if $Y \in B(x_Q, C\ell(Q))$ with $\delta(Y) \ge c\ell(Q)$, then there exists $\kappa = \kappa(C, c)$ such that

$$\frac{G(Y,X)}{\ell(Q)} \lesssim \frac{\omega^X(\kappa Q)}{\sigma(Q)} \lesssim \kappa^n \left( \fint_Q (\mathcal{M}\omega^X)^{1/2}\, d\sigma \right)^2, \quad \forall X \notin B(x_Q, \kappa\ell(Q)), \tag{3.13}$$

where $\kappa Q$ is defined in (2.9), and $\mathcal{M}$ is the usual Hardy–Littlewood maximal operator on $\partial\Omega$.

*Proof of Lemma 3.11.* We follow the well known argument of [Caffarelli et al. 1981] (see also [Kenig 1994, Lemma 1.3.3]). Fix $Y \in \Omega$ and write $B^Y = \overline{B(Y, \delta(Y)/2)}$. Consider the open set $\widehat{\Omega} = \Omega \setminus B^Y$ for which clearly $\partial\widehat{\Omega} = \partial\Omega \cup \partial B^Y$. Set

$$u(X) := G(Y,X)/\delta(Y), \qquad v(X) := \omega^X(\Delta_Y)/\sigma(\Delta_Y),$$

for every $X \in \widehat{\Omega}$. Note that both $u$ and $v$ are nonnegative harmonic functions in $\widehat{\Omega}$. If $X \in \partial\Omega$ then $u(X) = 0 \le v(X)$. Take now $X \in \partial B^Y$, so that $u(X) \lesssim \delta(Y)^{-n}$ by (3.5). On the other hand, if we fix $X_0 \in \partial B^Y$ with $X_0$ on the line segment that joints $Y$ and $\hat{y}$, then $2\Delta_{X_0} = \Delta_Y$, so that $v(X_0) \gtrsim \delta(Y)^{-n}$, by (3.3). By Harnack's inequality, we then obtain $v(X) \gtrsim \delta(Y)^{-n}$ for all $X \in \partial B^Y$. Thus, $u \lesssim v$ in $\partial\widehat{\Omega}$ and by the maximum principle this immediately extends to $\widehat{\Omega}$ as desired. □

**Lemma 3.14.** *Let $\partial\Omega$ be $n$-dimensional ADR. Let $B = B(x,r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and set $\Delta = B \cap \partial\Omega$. There exist constants $\kappa_0 > 2$, $C > 1$, and $M_1 > 1$, depending only on $n$ and the ADR constant of $\partial\Omega$, such that for $X \in \Omega \setminus B(x, \kappa_0 r)$, we have*

$$\sup_{\frac{1}{2}B} G(\cdot, X) \lesssim \frac{1}{|B|} \iint_B G(Y,X)\, dY \le Cr \frac{\omega^X(\Delta(x, M_1 r))}{\sigma(\Delta)}. \tag{3.15}$$

*Moreover, for each $\gamma \in (0,1]$,*

$$\frac{1}{|B|} \iint_{B \cap \{Y : \delta(Y) < \gamma r\}} G(Y,X)\, dY \le C\gamma^2 r \frac{\omega^X(\Delta(x, M_1 r))}{\sigma(\Delta)}, \tag{3.16}$$

*where $C$ depends on $n$ and the ADR constant of $\partial\Omega$.*

We note that in the previous estimates it is implicitly understood that $G(\cdot, X)$ is extended to be 0 outside of $\Omega$.

*Proof.* Extending $G(\cdot, X)$ to be 0 outside of $\Omega$, we obtain a subharmonic function in $B$. The first inequality in (3.15) follows immediately. The second inequality in (3.15) is just the special case $\gamma = 1$ of (3.16), so it suffices to prove the latter. Set $\Sigma_\gamma = \{I \in \mathcal{W} : I \cap B \ne \varnothing, \operatorname{dist}(I, \partial\Omega) < \gamma r\}$, and note that if $I \in \Sigma_\gamma$ then by (2.21),

$$40^{-1} \operatorname{dist}(I, \partial\Omega) \le \operatorname{diam}(I) \le \operatorname{dist}(I, \partial\Omega) < \gamma r \le r, \qquad \operatorname{dist}(I, x) \le r.$$

In particular, $I \subset B(x, 2r)$. Furthermore, we can find $\kappa_0$, depending only on dimension, such that $\operatorname{dist}(X, 4I) \ge 4r$ for every $I \in \Sigma_\gamma$ and $X \in \Omega \setminus B(x, \kappa_0 r)$. Let $Q_I \in \mathbb{D}$ be such that $\ell(Q_I) = \ell(I)$ and $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, Q_I)$. Then $\ell(Q_I) \le \gamma r$, and $Y(I)$, the center of $I$, satisfies $Y(I) \in B(x_{Q_I}, C\ell(Q_I))$

and $\delta(Y(I)) \approx \ell(I) = \ell(Q_I)$. Hence we can invoke (3.13) (taking $\kappa_0$ larger if needed) and obtain that for every $Y \in I$,

$$G(Y, X) \approx G(Y(I), X) \lesssim \ell(I) \frac{\omega^X(\kappa Q_I)}{\sigma(Q_I)},$$

where the first estimate uses Harnack's inequality in $2I \subset \Omega$. Hence,

$$\begin{aligned}\iint_{B\cap\{Y:\delta(Y)<\gamma r\}} G(Y, X)\, dY &\le \sum_{I\in\Sigma_\gamma} \iint_I G(Y, X)\, dY \lesssim \sum_{I\in\Sigma_\gamma} \ell(I)^2 \omega^X(\kappa Q_I) \\ &\le \sum_{k:2^{-k}\lesssim\gamma r} 2^{-2k} \sum_{I\in\Sigma_\gamma:\ell(I)=2^{-k}} \omega^X(\kappa Q_I) \lesssim (\gamma r)^2 \omega^X(\Delta(x, M_1 r)),\end{aligned}$$

where in the last step we have used that for each fixed $k$, the cubes $\kappa Q_I$ with $\ell(I) = 2^{-k}$ have uniformly bounded overlaps, and are all contained in $\Delta(x, M_1 r)$ for $M_1$ large enough. Dividing by $|B| \approx r^{n+1}$ and using the ADR property, we obtain the desired estimate. □

**3B. *PDE estimates: the p-harmonic case.*** We now recall several fundamental estimates for $p$-harmonic functions and $p$-harmonic measure, some of which generalize certain of the preceding estimates that we have stated in the harmonic case. We ask the reader to forgive a moderate amount of redundancy. Given a closed set $E$, as above we set $\delta(Y) := \operatorname{dist}(Y, E)$.

**Lemma 3.17.** *Let $p$, $1 < p < \infty$, be given. Let $u$ be a positive $p$-harmonic function in $B(X, 2r)$. Then*

$$\left(\frac{1}{|B(X, r/2)|} \iint_{B(X,r/2)} |\nabla u|^p\, dy\right)^{1/p} \le \frac{C}{r} \max_{B(X,r)} u, \tag{3.18}$$

$$\max_{B(X,r)} u \le C \min_{B(X,r)} u. \tag{3.19}$$

*Furthermore, there exists $\alpha = \alpha(p, n) \in (0, 1)$ such that if $Y, Y' \in B(X, r)$, then*

$$|u(Y) - u(Y')| \le C\left(\frac{|Y - Y'|}{r}\right)^\alpha \max_{B(X,2r)} u. \tag{3.20}$$

*Proof.* The inequality (3.18) is a standard energy estimate, (3.19) is the well known Harnack inequality for positive solutions to the $p$-Laplace operator, and (3.20) is a well known interior Hölder continuity estimate for solutions to equations of $p$-Laplace type. We refer to [Serrin 1964] for these results. □

**Definition 3.21.** Let $O \subset \mathbb{R}^{n+1}$ be open and let $K$ be a compact subset of $O$. Given $p$, $1 < p < \infty$, we let

$$\operatorname{Cap}_p(K, O) = \inf\left\{\iint_O |\nabla\phi|^p\, dY : \phi \in C_0^\infty(O),\ \phi \ge 1 \text{ in } K\right\}.$$

$\operatorname{Cap}_p(K, O)$ is referred to as the $p$-capacity of $K$ relative to $O$. The $p$-capacity of an arbitrary set $E \subset O$ is defined by

$$\operatorname{Cap}_p(E, O) = \inf_{\substack{E\subset G\subset O\\ G \text{ open}}} \sup_{\substack{K\subset G\\ K \text{ compact}}} \operatorname{Cap}_p(K, O). \tag{3.22}$$

**Definition 3.23.** Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let $x \in E$, $0 < r < \operatorname{diam}(E)$. Given $p$, $1 < p < \infty$, we say that $E \cap B(x, 4r)$ is $p$-thick if for every $x \in E \cap B(x, 4r)$ there exists $r_x > 0$ such that

$$\int_0^{r_x} \left[ \frac{\operatorname{Cap}_p(E \cap B(x, \rho), B(x, 2\rho))}{\operatorname{Cap}_p(B(x, \rho), B(x, 2\rho))} \right]^{1/(p-1)} \frac{d\rho}{\rho} = \infty.$$

We note that this definition is just the Wiener criterion in the $p$-harmonic case. As it can be seen in [Heinonen et al. 2006, Chapter 6], $p$-thickness implies that all points on $E \cap B(x, 4r)$ are regular for the continuous Dirichlet problem for $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$.

**Definition 3.24.** Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let $x \in E$, $0 < r < \operatorname{diam}(E)$. Given $p$, $1 < p < \infty$, and $\eta > 0$ we say that $E \cap B(x, 4r)$ is uniformly $p$-thick with constant $\eta$ if

$$\frac{\operatorname{Cap}_p(E \cap B(\hat{x}, \hat{r}), B(\hat{x}, 2\hat{r}))}{\operatorname{Cap}_p(B(\hat{x}, \hat{r}), B(\hat{x}, 2\hat{r}))} \geq \eta \tag{3.25}$$

whenever $\hat{x} \in E \cap B(x, 4r)$ and $B(\hat{x}, 2\hat{r}) \subset B(x, 4r)$.

**Remark 3.26.** In the case $p = 2$, the condition defined in Definition 3.24 is sometimes called the capacity density condition (CDC); see for instance [Aikawa 2004]. Note that uniform $p$-thickness is a strong quantitative version of the $p$-thickness defined above and hence of the Wiener regularity for the Laplace and the $p$-Laplace operator.

**Lemma 3.27.** *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be Ahlfors–David regular of dimension $n$. Let $p$, $1 < p < \infty$, be given. Then $E \cap B(x, 4r)$ is uniformly $p$-thick for some constant $\eta$, depending only on $p$, $n$, and the ADR constant, whenever $x \in E$, $0 < r < \frac{1}{4} \operatorname{diam} E$.*

*Proof.* We first observe that since the ADR condition is scale-invariant we may translate and rescale to prove (3.25) only for $\hat{x} = 0$ and $\hat{r} = 1$ (we would also need to rescale $E$, but abusing the notation we still call it $E$). Write $B = B(0, 1)$ and observe that, for every $1 < p < \infty$, [Heinonen et al. 2006, Example 2.12] gives

$$\operatorname{Cap}_p(B, 2B) = C(n, p). \tag{3.28}$$

The desired bound from below follows at once if $p > n + 1$ from the estimate in [Heinonen et al. 2006, Example 2.12]:

$$\operatorname{Cap}_p(E \cap B, 2B) \geq \operatorname{Cap}_p(\{0\}, 2B) = C(n, p)'.$$

Let us now consider the case $1 < p \leq n + 1$. Write $K = E \cap \overline{\tfrac{1}{2} B}$. Combining [Heinonen et al. 2006, Theorem 2.38; Adams and Hedberg 1999, Theorems 2.2.7 and 4.5.2] we have that

$$\operatorname{Cap}_p(E \cap B, 2B) \gtrsim \widetilde{\operatorname{Cap}}_p(K) \gtrsim \sup_\mu \left( \frac{\mu(K)}{\|W_p(\mu)\|_{L^1(\mu)}^{1/p'}} \right)^p. \tag{3.29}$$

In the previous expression the implicit constants depend only on $p$ and $n$; $\widetilde{\operatorname{Cap}}_p$ stands for the inhomogeneous $p$-capacity, that is,

$$\widetilde{\operatorname{Cap}}_p(K) = \inf \left\{ \iint_{\mathbb{R}^{n+1}} (|\phi|^p + |\nabla \phi|^p) \, dY : \phi \in C_0^\infty(\mathbb{R}),\ \phi \geq 1 \text{ in } K \right\};$$

the sup runs over all Radon positive measures supported on $K$; and

$$W_p(\mu)(y) := \int_0^1 \left( \frac{\mu(B(y,t))}{t^{n+1-p}} \right)^{p'-1} \frac{dt}{t}, \quad x \in \operatorname{supp} \mu.$$

We choose $\mu = H^n|_K$ and observe that, if $y \in \operatorname{supp} \mu \subset K \subset E$ and $0 < t < 1$, then, by ADR, $\mu(B(y,t)) = \sigma(B(y,t) \cap B(0, \frac{1}{2}) \lesssim t^n$. This easily gives $W_p(\mu)(y) \lesssim 1$ for every $y \in \operatorname{supp} \mu$ and, by ADR,

$$\int_K W_p(\mu)(y)\, d\mu(y) \le \mu(K) \le \sigma(B) \lesssim 1.$$

We can now use (3.29) and ADR again to conclude that

$$\operatorname{Cap}_p(E \cap B, 2B) \gtrsim \mu(K) \ge \sigma\big(B\big(0, \tfrac{1}{2}\big)\big)^p \gtrsim 1.$$

Combining this with (3.28) we readily obtain (3.25). □

**Lemma 3.30.** *Let $E \subset \mathbb{R}^{n+1}$, $n \ge 2$, be Ahlfors–David regular of dimension $n$. Let $p$, $1 < p < \infty$, be given. Let $x \in E$ and $0 < r < \operatorname{diam}(E)$. Then, given $f \in W^{1,p}(B(x,4r))$ there exists a unique $p$-harmonic function $u \in W^{1,p}(B(x,4r) \setminus E)$ such that $u - f \in W_0^{1,p}(B(x,4r) \setminus E)$. Furthermore, let $u, v \in W^{1,p}_{\mathrm{loc}}(B(x,4r) \setminus E)$ be a $p$-superharmonic function and a $p$-subharmonic function in $\Omega$, respectively. If $\inf\{u - v, 0\} \in W_0^{1,p}(B(x,4r) \setminus E)$, then $u \ge v$ a.e. in $B(x,4r) \setminus E$. Finally, every point $\hat{x} \in E \cap B(x,4r)$ is regular for the continuous Dirichlet problem for $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$.*

*Proof.* The first part of the lemma is a standard maximum principle. The fact that every $\hat{x} \in E \cap B(x,4r)$ is regular in the continuous Dirichlet problem for $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$ follows from the fact that Lemma 3.27 implies that $E \cap B(x,4r)$ is uniformly $p$-thick for every $1 < p < \infty$, and hence we can invoke [Heinonen et al. 2006, Chapter 6]. □

**Lemma 3.31.** *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 2$, be an open set whose boundary is Ahlfors–David regular of dimension $n$. Let $p$, $1 < p < \infty$, be given. Let $x \in \partial\Omega$ and consider $0 < r < \operatorname{diam}(\partial\Omega)$. Assume also that $u$ is nonnegative and $p$-harmonic in $B(x,4r) \cap \Omega$, continuous on $B(x,4r) \cap \overline{\Omega}$, and that $u = 0$ on $\partial\Omega \cap B(x,4r)$. Then, extending $u$ to be $0$ in $B(x,4r) \setminus \overline{\Omega}$, we have*

$$\left( \frac{1}{|B(x,r/2)|} \iint_{B(x,r/2)} |\nabla u|^p \, dy \right)^{1/p} \le \frac{C}{r} \left( \frac{1}{|B(x,r)|} \iint_{B(x,r)} u^{p-1} \right)^{1/(p-1)}. \tag{3.32}$$

*Furthermore, there exists $\alpha \in (0,1)$, depending only on $p$, $n$, and the ADR constant, such that if $Y, Y' \in B(x,r)$, then*

$$|u(Y) - u(Y')| \le C \left( \frac{|Y - Y'|}{r} \right)^{\alpha} \max_{B(x,2r)} u. \tag{3.33}$$

*Proof.* Since $u$, extended as above to all of $B(x,4r)$, is a nonnegative $p$-subsolution in $B(x,4r)$, (3.32) is just a standard energy or Caccioppoli estimate plus a standard interior estimate. Thus, we only prove (3.33). Since $E \cap B(x,4r)$ is uniformly $p$-thick as seen in Lemma 3.27, we can invoke [Heinonen et al. 2006, Theorem 6.38] to obtain that there exist $C \ge 1$ and $\alpha = \alpha \in (0,1)$, depending only on $n$, $p$, and the ADR constant, such that

$$\max_{B(x,\rho)} u \le C\Big(\frac{\rho}{r}\Big)^{\alpha} \max_{B(x,r)} u, \quad \text{whenever } 0<\rho\le r. \tag{3.34}$$

This, the triangle inequality, and elementary arguments give (3.33). □

**Lemma 3.35.** *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 2$, be an open set whose boundary is Ahlfors–David regular of dimension $n$. Let $p$, $1<p<\infty$, be given. Let $x \in \partial\Omega$ and consider $0<r<\operatorname{diam}(\partial\Omega)$. Assume also that $u$ is nonnegative and $p$-harmonic in $B(x,4r)\cap\Omega$, continuous on $B(x,4r)\cap\overline{\Omega}$, and that $u=0$ on $\partial\Omega\cap B(x,4r)$. Then, extending $u$ to be $0$ in $B(x,4r)\setminus\overline{\Omega}$, there exists $\alpha>0$ such that*

$$u(Y) \le C\Big(\frac{\delta(Y)}{r}\Big)^{\alpha}\bigg(\frac{1}{|B(x,2r)|}\iint_{B(x,2r)} u^{p-1}(Z)\,dZ\bigg)^{1/(p-1)} \tag{3.36}$$

*for all $Y \in B(x,r)$, where the constants $C$ and $\alpha$ depend only on $n$, $p$, and the ADR constant of $\partial\Omega$.*

*Proof.* This follows from Lemma 3.31 and standard estimates for $p$-subsolutions. Let us note that in the linear case (i.e, $p=2$) one can give an alternative proof based on Bourgain's Lemma 3.1 and an iteration argument (see [Hofmann et al. ≥ 2017] for details). □

**Lemma 3.37.** *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 2$, be an open set whose boundary is Ahlfors–David regular of dimension $n$. Let $p$, $1<p<\infty$, be given. Let $x\in\partial\Omega$ and consider $0<r<\operatorname{diam}(\partial\Omega)$. Assume also that $u$ is nonnegative and $p$-harmonic in $B(x,4r)\cap\Omega$, continuous on $B(x,4r)\cap\overline{\Omega}$, that $u=0$ on $\partial\Omega\cap B(x,4r)$, and that $u$ is extended to be $0$ in $B(x,4r)\setminus\overline{\Omega}$. Then $u$ has a representative in $W^{1,p}(B(x,4r))$ with Hölder continuous partial derivatives in $B(x,4r)\setminus\partial\Omega$. Furthermore, there exists $\beta\in(0,1]$ such that if $Y, Y' \in B(X,\hat{r}/2)$, with $B(X,4\hat{r})\subset B(x,4r)\setminus\partial\Omega$, then*

$$|\nabla u(Y)-\nabla u(Y')| \lesssim \Big(\frac{|Y-Y'|}{\hat{r}}\Big)^{\beta}\max_{B(X,\hat{r})}|\nabla u| \lesssim \frac{1}{\hat{r}}\Big(\frac{|Y-Y'|}{\hat{r}}\Big)^{\beta}\max_{B(X,2\hat{r})} u\,, \tag{3.38}$$

*where $\beta$ and the implicit constants depend only on $p$ and $n$. Furthermore, if*

$$\frac{u(Y)}{\delta(Y)} \approx |\nabla u(Y)|, \qquad Y\in B(X,3\hat{r}), \tag{3.39}$$

*then $u$ has continuous second derivatives in $B(X,3\hat{r})$, and there exists $C\ge 1$, depending only on $n$, $p$, and the implicit constants in (3.39), such that*

$$\max_{B(X,\hat{r}/2)}|\nabla^2 u| \le C\bigg(\frac{1}{|B(X,\hat{r})|}\iint_{B(X,\hat{r})}|\nabla^2 u(Y)|^2\,dY\bigg)^{1/2} \le C^2\frac{u(X)}{\delta(X)^2}. \tag{3.40}$$

*Proof.* For (3.38) we refer, for example, to [Tolksdorf 1984]; (3.40) is a consequence of (3.38), (3.39), and Schauder type estimates, see [Gilbarg and Trudinger 1983]. For a more detailed proof of (3.40), see [Lewis and Vogel 2006, Lemma 2.4(d)] for example. □

**Remark 3.41.** We note that the second inequality in (3.38) and (3.19) give

$$|\nabla u(Y)| \lesssim \frac{u(Y)}{\delta(Y)}, \qquad Y\in B(x,2r)\setminus\partial\Omega. \tag{3.42}$$

**Lemma 3.43.** *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set and assume that $\partial\Omega$ is Ahlfors–David regular of dimension $n$. Let $p$, $1 < p < \infty$, be given. Let $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and suppose that $u$ is nonnegative and $p$-harmonic in $B(x, 4r) \cap \Omega$, vanishing continuously on $B(x, 4r) \cap \Omega$ (hence $u$ is continuous in $B(x, 4r)$ after being extended by $0$ in $B(x, 4r) \setminus \overline{\Omega}$). There exists a unique finite positive Borel measure $\mu$ on $\mathbb{R}^{n+1}$, with support in $\partial\Omega \cap B(x, 4r)$, such that*

$$-\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dY = \int \phi \, d\mu \tag{3.44}$$

*whenever $\phi \in C_0^\infty(B(x, 4r))$. Furthermore, there exists $C < \infty$, depending only on $p$, $n$, and the ADR constant, such that*

$$\left(\frac{\max_{B(x,r)} u}{r}\right)^{p-1} \leq C \frac{\mu(\Delta(x, 2r))}{\sigma(\Delta(x, 2r))}. \tag{3.45}$$

Note that (3.45) is the $p$-harmonic analogue of Lemma 3.11.

*Proof.* For the proof of (3.44), see [Heinonen et al. 2006, Chapter 21]. Using Lemma 3.27 and Lemma 3.31, (3.45) follows directly from [Kilpeläinen and Zhong 2003, Lemma 3.1]; see also [Eremenko and Lewis 1991]. □

The following lemma generalizes Lemma 3.14 to the case $1 < p < \infty$.

**Lemma 3.46.** *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set and assume that $\partial\Omega$ is Ahlfors–David regular of dimension $n$. Let $p$, $1 < p < \infty$, be given. Let $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and suppose that $u$ and $\mu$ are as in Lemma 3.43. Then there exist constants $C$ and $M_1$, depending only on $n$ and the ADR constant, such that if $B(y, M_1 s) \subset B(x, 2r)$ with $y \in \partial\Omega$, then*

$$\max_{B(y,s/2)} u^{p-1} \lesssim \frac{1}{|B(y, s)|} \iint_{B(y,s)} u^{p-1}(Z) \, dZ \leq C s^{p-1} \frac{\mu(\Delta(y, M_1 s))}{\sigma(\Delta(y, s))}.$$

*Moreover, for all $\gamma \in (0, 1]$,*

$$\frac{1}{|B(y, s)|} \iint_{B(y,s)\cap\{Y:\delta(Y)\leq\gamma s\}} u^{p-1}(Z) \, dZ \leq C \gamma^p s^{p-1} \frac{\mu(\Delta(y, M_1 s))}{\sigma(\Delta(y, s))}.$$

We note that in the previous estimates it is implicitly understood that $u$ is extended to be $0$ on $B(x, 4r) \setminus \overline{\Omega}$.

*Proof.* Using (3.45), the proof of Lemma 3.46 is the same mutatis mutandi as that of Lemma 3.14. We omit further details. □

## 4. Proofs of Theorem 1.1 and Theorem 1.12: preliminary arguments

We start the proofs of Theorem 1.1 and Theorem 1.12 by giving some preliminary arguments. We first show that (1.2) implies (1.4). To this end, we claim that, without loss of generality, we may suppose that for a surface ball $\Delta = \Delta(x, r)$, the point $X_\Delta$ in the statement of Theorem 1.1 satisfies (3.2), i.e., there is some $c_1 = c_1(n, \text{ADR}) > 0$ such that

$$\omega^{X_\Delta}(\Delta) \geq c_1. \tag{4.1}$$

The only price to be paid is that the constants $c_0$, $C_0$ may now be slightly different (depending only on $n$ and ADR), and that (1.2) now holds with $\Delta$ in place of $2\Delta$, i.e., for the (possibly) new point $X_\Delta$, we have

$$\int_\Delta k^{X_\Delta}(y)^q \, d\sigma(y) \le C_0 \sigma(\Delta)^{1-q}. \tag{4.2}$$

Indeed, set $\Delta' := \Delta(x, r/2)$, and let $X' := X_{\Delta'} \in B(x, r/2) \cap \Omega$ be the point such that (1.2) holds for $\Delta'$. Fix $\hat{x} \in \partial\Omega$ such that $\delta(X') = |X' - \hat{x}|$. Suppose first that $\delta(X') \le r/4$, in which case $\Delta(\hat{x}, r/4) \subset \Delta$. Thus, if in addition $\delta(X') < cr/4$, where $c \in (0, 1)$ is the constant in Lemma 3.1, then we set $X_\Delta := X'$, and (4.1) holds by Lemma 3.1. On the other hand, if $cr/4 \le \delta(X_\Delta) \le r/4$, we select $X_\Delta$ along the line segment joining $X'$ to $\hat{x}$, such that $\delta(X_\Delta) = |X_\Delta - \hat{x}| = cr/8$, and (4.1) holds exactly as before. Moreover, (4.2) holds for this new $X_\Delta$, in the first case, immediately by (1.2) applied to $X' = X_{\Delta'}$, and in the second case, by moving from $X'$ to $X_\Delta$ via Harnack's inequality (which may be used within the touching ball $B(X', \delta(X'))$). Let us finally consider the case $\delta(X') > r/4$. Then we can use Harnack within the ball $B(X', r/4)$ to pass to a point $X''$ on the line segment joining $X'$ to $x$ such that $|X' - X''| = r/8$, and consequently $\delta(X'') \le |X'' - x| < 3r/8$ (since $X' \in B(x, r/2)$). Hence (1.2) holds (with different constant) for $\Delta'$ with $X''$ in place of $X_{\Delta'}$. Now take $\hat{x} \in \partial\Omega$ such that $\delta(X'') = |X'' - \hat{x}|$ and note that $\Delta(\hat{x}, r/4) \subset \Delta$. We can now repeat the previous argument with $X''$ in place of $X'$. Details are left to the interested reader.

Similarly, if (1.4) holds for $\Delta = \Delta(x, r)$, with $X_\Delta \in B(x, r/2) \cap \Omega$, then again without loss of generality we may suppose that (4.1) holds, possibly for a new $X_\Delta \in B(x, r) \cap \Omega$. Indeed if we let $X' \in B(x, r/2) \cap \Omega$ be the original point $X_\Delta$ for which (1.4) holds, we may then follow the argument in the previous paragraph, mutatis mutandi. We choose $\hat{x} \in \partial\Omega$ such that $\delta(X') = |X' - \hat{x}|$ and suppose first that $\delta(X') \le r/4$, so that $\Delta(\hat{x}, r/4) \subset \Delta$. Considering the same two cases as before we pick $X_\Delta$ and in either case (4.1) holds by Lemma 3.1 applied to the surface ball $\Delta(\hat{x}, r/4)$. Note that in the second case, (1.4) continues to hold for $X_\Delta$, with a different but still uniform $\beta$, using Harnack's inequality within the touching ball $B(X', \delta(X'))$ to move from $X'$ to $X_\Delta$. When $r/4 < \delta(X')$ we choose $X''$ as before, and by Harnack's inequality, (1.4) holds with $X''$ in place of $X'$, for a different but still uniform $\beta$. Again, if we let $\hat{x} \in \partial\Omega$ with $\delta(X'') = |X'' - \hat{x}|$, then $\Delta(\hat{x}, r/4) \subset \Delta$, and we may now repeat the previous argument with $X''$ in place of $X'$.

We are now ready to show that (1.2) implies (1.4).

**Lemma 4.3.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with $n$-dimensional ADR boundary, and let $\Delta = \Delta(x, r)$ be a surface ball on $\partial\Omega$. Let $\mu$ be a measure on $\partial\Omega$ such that $\mu|_\Delta \ll \sigma$, and such that for some $q > 1$ and $\Lambda < \infty$,*

$$⨍_\Delta k^q \, d\sigma \le \Lambda, \tag{4.4}$$

*where $k := d\mu/d\sigma$ on $\Delta$. Suppose also that*

$$\frac{\mu(\Delta)}{\sigma(\Delta)} \ge 1. \tag{4.5}$$

*Then there are constants $\eta, \beta \in (0, 1)$, depending only on $n$, $q$, $\Lambda$, and ADR, such that for any Borel set $A \subset \Delta$,*

$$\sigma(A) \ge (1 - \eta)\sigma(\Delta) \implies \mu(A) \ge \beta\mu(\Delta). \tag{4.6}$$

**Remark 4.7.** Let $k$ be a normalized version of harmonic measure: $k = c_1^{-1}\sigma(\Delta)k^{X_\Delta}$, with $X_\Delta$ a point for which (4.1) and (4.2) hold. Then clearly (4.4) and (4.5) hold for $k$, and the conclusion (4.6) is just a reformulation of (1.4). We note that in the sequel, we actually use only (4.6) or (1.4), rather than condition (4.4) or (4.2). Thus, Theorem 1.1 could just as well have been stated with condition ($\star\star$) (see Remark 1.3) in place of ($\star$).

*Proof of Lemma 4.3.* Set $F := \Delta \setminus A$, so $\sigma(F) \le \eta\sigma(\Delta)$. Then

$$\begin{aligned}\mu(F) = \int_F k\,d\sigma &\le \sigma(F)^{1/q'}\Big(\int_\Delta k^q\,d\sigma\Big)^{1/q}\\ &\le \Lambda^{1/q}\sigma(F)^{1/q'}\sigma(\Delta)^{1/q} \le \Lambda^{1/q}\eta^{1/q'}\sigma(\Delta) \le \Lambda^{1/q}\eta^{1/q'}\mu(\Delta),\end{aligned}$$

where in the last step we have used (4.5). Thus,

$$\mu(A) \ge \big(1 - \Lambda^{1/q}\eta^{1/q'}\big)\mu(\Delta) \ge \tfrac{1}{2}\mu(\Delta)$$

for $\eta$ small enough. This completes the proof. □

Fix $Q_0 \in \mathbb{D}(\partial\Omega)$. As in (2.8), we set $B_{Q_0} = B(x_{Q_0}, r_0)$, with $r_0 := r_{Q_0} \approx \ell(Q_0)$, so that $\Delta_{Q_0} = B_{Q_0} \cap \partial\Omega \subset Q_0$.

Proceeding first in the setting of Theorem 1.1, let $X_0 := X_{\Delta_{Q_0}}$ be the point relative to $\Delta = \Delta_{Q_0}$ such that (4.1) and (4.2) hold. Note that (4.1) trivially implies that

$$\omega^{X_0}(Q_0) \ge c_1.$$

With the pole $X_0$ fixed, we define the normalized harmonic measure and the normalized Green's function, respectively, by

$$\mu := \frac{1}{c_1}\sigma(Q_0)\omega^{X_0}, \qquad u(Y) := \frac{1}{c_1}\sigma(Q_0)G(X_0, Y). \tag{4.8}$$

Then under this normalization, setting $\|\mu\| = \mu(\partial\Omega)$, we have

$$1 \le \frac{\mu(Q_0)}{\sigma(Q_0)} \le \frac{\|\mu\|}{\sigma(Q_0)} \le C_1, \tag{4.9}$$

with $C_1 = 1/c_1$. Furthermore, we may apply Lemma 4.3 (using (4.1) and with $\Lambda \approx C_0/c_1$) to obtain (4.6) for $\mu$, with $\Delta = \Delta_{Q_0}$. In turn, the latter bound, in conjunction with (4.1) and ADR, clearly implies an analogous estimate for $Q_0$, namely that there are constants that we again call $\eta, \beta \in (0, 1)$ such that for any Borel set $A \subset Q_0$,

$$\sigma(A) \ge (1-\eta)\sigma(Q_0) \implies \mu(A) \ge \beta\mu(Q_0). \tag{4.10}$$

Here, of course, we may have different values of the parameters $\eta$ and $\beta$, but these have the same dependence as the original values, so for convenience we maintain the same notation.

In the $p$-harmonic case, proceeding under the setup of Theorem 1.12, we let $u$ and $\mu$ be the $p$-harmonic function and its associated $p$-harmonic measure, corresponding to the point $x = x_{Q_0}$ and the radius $r = Cr_0 := Cr_{Q_0}$, satisfying the hypotheses of Theorem 1.12, where we choose the constant $C$ depending only on $n$ and ADR, such that $Q_0 \subset \Delta(x_{Q_0}, Cr_0)$ (thus, in particular, $\mu$ is defined on $Q_0$). Since we assume that $u$ is nontrivial and nonnegative, we can apply Lemma 3.43 in $B(x_{Q_0}, Cr_0)$ and use (1.14) to conclude that $\mu(\Delta_{Q_0}) > 0$. We can therefore normalize $u$ and $\mu$ (abusing the notation we call the normalizations $u$ and $\mu$) so that $\mu(\Delta_{Q_0})/\sigma(Q_0) = 1$, and since $\Delta_{Q_0} \subset Q_0 \subset \Delta(x_{Q_0}, Cr_0)$ by (1.14), we also have $\mu(\Delta(x_{Q_0}, Cr_0))/\sigma(\Delta(x_{Q_0}, Cr_0)) \approx \mu(Q_0)/\sigma(Q_0) \approx 1$. Set $k := d\mu/d\sigma$. As above, by (1.13) and (1.14), we may then use Lemma 4.3 to see that again $\mu$ satisfies both (4.9), now with $\|\mu\| := \mu(\Delta(x_{Q_0}, Cr_0))$, and (4.10). The constants $C_1$, $\eta$, and $\beta$ depend on $C$, $n$, the ADR constant, $C_0$, and $q$.

**Remark 4.11.** Under the assumptions of Theorems 1.1 and 1.12 and throughout this section and Section 6, for $Q_0 \in \mathbb{D}(E)$ fixed, $u$ and $\mu$ will continue to denote the normalized Green function and harmonic measure or the normalized nonnegative $p$-harmonic solution and $p$-harmonic Riesz measure, as defined above. In particular, (4.9) and (4.10) hold for all $1 < p < \infty$.

As above, let $\mathcal{M}$ denote the usual Hardy–Littlewood maximal operator on $\partial\Omega$ and recall the definition of $\mathbb{D}_{\mathcal{F},Q_0}$ in (2.10).

**Lemma 4.12.** *Let $Q_0 \in \mathbb{D}$, and suppose that $\mu$ satisfies (4.9) and (4.10). Then there is a pairwise disjoint family $\mathcal{F} = \{Q_j\}_{j\ge1} \subset \mathbb{D}_{Q_0}$ such that*

$$\sigma\big(Q_0 \setminus \big(\bigcup_j Q_j\big)\big) \ge \frac{1}{C}\,\sigma(Q_0) \tag{4.13}$$

*and*

$$\frac{\beta}{2} \le \frac{\mu(Q)}{\sigma(Q)} \le \bigg( ⨍_Q (\mathcal{M}\mu)^{1/2}\, d\sigma \bigg)^2 \le C, \quad \forall\, Q \in \mathbb{D}_{\mathcal{F},Q_0}, \tag{4.14}$$

*where $C > 1$ depends only on $\eta$, $\beta$, $C_1$, $n$, and ADR.*

*Proof.* The proof is based on a stopping time argument similar to those used in the proof of the Kato square root conjecture [Hofmann and McIntosh 2002; Hofmann et al. 2002; Auscher et al. 2002a], and in local $Tb$ theorems. We begin by noting that

$$\|\mathcal{M}\mu\|_{L^{1,\infty}(\sigma)} := \sup_{\lambda>0} \lambda\sigma\{\mathcal{M}\mu > \lambda\} \lesssim \|\mu\| \lesssim \sigma(Q_0) \tag{4.15}$$

by the Hardy–Littlewood theorem and (4.9). Consequently, by Kolmogorov's criterion,

$$⨍_{Q_0} (\mathcal{M}\mu)^{1/2}\, d\sigma \le C = C(n, \mathrm{ADR}, C_1). \tag{4.16}$$

We now perform a stopping time argument to extract a family $\mathcal{F} = \{Q_j\}$ of dyadic subcubes of $Q_0$ that are maximal with respect to the property that either

$$\frac{\mu(Q_j)}{\sigma(Q_j)} < \frac{\beta}{2} \tag{4.17}$$

and/or

$$⨍_{Q_j} (\mathcal{M}\mu)^{1/2}\, d\sigma > K, \tag{4.18}$$

where $K \geq 1$ is a sufficiently large number to be chosen momentarily. Note that $Q_0 \notin \mathcal{F}$, by (4.9) and (4.16). We say that $Q_j$ is of "type I" if (4.17) holds, and of "type II" if (4.18) holds but (4.17) does not. Set $A := Q_0 \setminus \big(\bigcup_j Q_j\big)$, and $F := \bigcup_{Q_j \text{ type II}} Q_j$. Then by (4.9),

$$\sigma(Q_0) \leq \mu(Q_0) = \sum_{Q_j \text{ type I}} \mu(Q_j) + \mu(F) + \mu(A). \tag{4.19}$$

By definition of the type I cubes,

$$\sum_{Q_j \text{ type I}} \mu(Q_j) \leq \frac{\beta}{2} \sum_j \sigma(Q_j) \leq \frac{\beta}{2} \sigma(Q_0). \tag{4.20}$$

To handle the remaining terms, observe that

$$\begin{aligned} \sigma(F) = \sum_{Q_j \text{ type II}} \sigma(Q_j) &\leq \frac{1}{K} \sum_j \int_{Q_j} (\mathcal{M}\mu)^{1/2}\, d\sigma \\ &\leq \frac{1}{K} \int_{Q_0} (\mathcal{M}\mu)^{1/2}\, d\sigma \leq \eta\sigma(Q_0), \end{aligned} \tag{4.21}$$

by the definition of the type II cubes, (4.16), and the choice of $K = C\eta^{-1}$. By (4.10) and complementation, we therefore find that

$$\mu(F) \leq (1-\beta)\mu(Q_0). \tag{4.22}$$

Next, if $x \in A$, then every $Q \in \mathbb{D}_{Q_0}$ that contains $x$ must satisfy the opposite inequality to (4.18), and therefore, by Lebesgue's differentiation theorem,

$$\mathcal{M}\mu(x) \leq K^2, \quad \text{for } \sigma\text{-a.e. } x \in A.$$

Thus $\mu|_A \ll \sigma$, with $d\mu|_A/d\sigma \leq K^2$, and thus,

$$\mu(A) \leq K^2 \sigma(A).$$

Combining the latter estimate with (4.19), (4.20), and (4.22), we obtain

$$\beta\mu(Q_0) \leq \frac{\beta}{2}\sigma(Q_0) + K^2\sigma(A).$$

Using (4.9), we then find that

$$\beta\sigma(Q_0) \leq \beta\mu(Q_0) \leq \frac{\beta}{2}\sigma(Q_0) + K^2\sigma(A).$$

The conclusion of the lemma now follows readily. □

For future reference, let us note an easy consequence of the last inequality in (4.14) and the ADR property: for all $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, and for any constant $b > 1$, we have

$$\mu\big(\Delta(x_Q, b \operatorname{diam}(Q))\big) \lesssim b^n \sigma(Q) \bigg( \fint_Q (\mathcal{M}\mu)^{1/2}\, d\sigma \bigg)^2 \lesssim b^n \sigma(Q). \tag{4.23}$$

Recall that the ball $B_Q^*$ and surface ball $\Delta_Q^*$ are defined in (2.15).

**Lemma 4.24.** *Let $u$, $\mu$ be as in Remark 4.11. If the constant $K_0$ in (2.15) and (2.23) is chosen sufficiently large, then for each $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ with $\ell(Q) \le K_0^{-1}\ell(Q_0)$, there exists $Y_Q \in U_Q$ with*

$$\delta(Y_Q) \le |Y_Q - x_Q| \lesssim \ell(Q),$$

*where the implicit constant is independent of $K_0$, such that*

$$\frac{\mu(Q)}{\sigma(Q)} \le C|\nabla u(Y_Q)|^{p-1}, \tag{4.25}$$

*where $C$ depends on $K_0$ and the implicit constants in the hypotheses of Theorems 1.1 and 1.12.*

**Remark 4.26.** Recalling the construction at the beginning of Section 4, and the fact that we have defined $X_0 := X_{\Delta_{Q_0}}$, we see that $\ell(Q_0) \approx \delta(X_0) \ge K_0^{-1/2}\ell(Q_0)$, for $K_0$ chosen large enough. We note further that the point $Y_Q$ whose existence is guaranteed by Lemma 4.24 is essentially a corkscrew point relative to $Q$. Indeed, $\delta(Y_Q) \gtrsim K_0^{-1}\ell(Q)$ (since $Y \in U_Q$), and also $|Y_Q - x_Q| \lesssim \ell(Q)$ (with constant independent of $K_0$). With a slight abuse of terminology, we shall refer to $Y_Q$ as a corkscrew point relative to $Q$, with corkscrew constant depending on $K_0$.

*Proof of Lemma 4.24.* Fix $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) \le K_0^{-1}\ell(Q_0)$, where, as in Remark 4.26, we have chosen $K_0$ large enough that $\ell(Q_0) \approx \delta(X_0) \ge K_0^{-1/2}\ell(Q_0)$. Recall (2.7) and (2.8), and set $\hat{B}_Q = B(x_Q, \hat{r}_Q)$, $\hat{\Delta}_Q = \hat{B}_Q \cap \partial\Omega$, with $\hat{r}_Q \approx \ell(Q)$ and $Q \subset \hat{\Delta}_Q$. Let $0 \le \phi_Q \in C_0^\infty(2\hat{B}_Q)$, such that $\phi_Q \equiv 1$ in $\hat{B}_Q$ and $\|\nabla\phi_Q\| \lesssim \ell(Q)^{-1}$. Note that

$$K_0^{1/2}\ell(Q) \le K_0^{-1/2}\ell(Q_0) \le \delta(X_0) \le |X_0 - x_Q|,$$

which implies that $X_0 \notin 4\hat{B}_Q$ provided $K_0$ is large enough. Thus, by (3.10) in the linear case, or (3.44) in general,

$$\begin{aligned} \ell(Q)\mu(Q) \le \ell(Q)\int_{\partial\Omega}\phi_Q\,d\mu &\lesssim \iint_{\hat{B}_Q\cap\Omega}|\nabla u(Y)|^{p-1}\,dY \\ &\le \iint_{\hat{B}_Q\cap U_Q}|\nabla u(Y)|^{p-1}\,dY + \iint_{(\hat{B}_Q\cap\Omega)\setminus U_Q}|\nabla u(Y)|^{p-1}\,dY \\ &=: \mathcal{I} + \mathcal{II}. \end{aligned} \tag{4.27}$$

Notice that by construction,

$$(\hat{B}_Q \cap \Omega)\setminus U_Q \subset \{Y \in \hat{B}_Q : \delta(Y) \le CK_0^{-1}\ell(Q)\}.$$

We may therefore cover the latter region by a family of balls $\{B_k\}_k$, centered on $\partial\Omega$, of radius $CK_0^{-1}\ell(Q)$, such that their doubles $\{2B_k\}$ have bounded overlaps and satisfy

$$\bigcup_k 2B_k \subset \{Y \in 2\hat{B}_Q : \delta(Y) \le 2CK_0^{-1}\ell(Q)\} =: \Sigma(K_0).$$

By the boundary Cacciopoli estimate in Lemma 3.31, plus Hölder's inequality, we obtain

$$\begin{aligned}
\mathcal{II} &\le \sum_k \iint_{B_k} |\nabla u(Y)|^{p-1}\,dY \lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1} \sum_k \iint_{2B_k} |u(Y)|^{p-1}\,dY \\
&\lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1} \iint_{\Sigma(K_0)} |u(Y)|^{p-1}\,dY \\
&\lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1} K_0^{-p}\ell(Q)^p \mu(\Delta(x_Q, 2M_1\hat{r}_Q)) \\
&\lesssim K_0^{-1}\ell(Q)\sigma(Q) \le \frac{1}{2}\ell(Q)\mu(Q),
\end{aligned}$$

where in the last three steps we have used (3.16) (when $p = 2$) or Lemma 3.46 ($1 < p < \infty$), (4.23), and finally the choice of $K_0$ large enough. We can then hide this term on the left-hand side of (4.27), so that

$$\begin{aligned}
\ell(Q)\mu(Q) \lesssim \mathcal{I} = \iint_{\hat{B}_Q\cap U_Q} |\nabla u(Y)|^{p-1}\,dY &= \sum_i \iint_{\hat{B}_Q\cap U_Q^i} |\nabla u(Y)|^{p-1}\,dY \\
&\lesssim \ell(Q)^{n+1} \max_i \sup_{Y\in\hat{B}_Q\cap U_Q^i} |\nabla u(Y)|^{p-1} \\
&\approx \ell(Q)\sigma(Q) \max_i \sup_{Y\in\hat{B}_Q\cap U_Q^i} |\nabla u(Y)|^{p-1},
\end{aligned}$$

and we recall that $\{U_Q^i\}_i$ is an enumeration of the connected components of $U_Q$, and that the number of these components is uniformly bounded. Thus, for some $i$, there is a point $Y_Q \in \hat{B}_Q \cap U_Q^i$ such that $\mu(Q)/\sigma(Q) \lesssim |\nabla u(Y_Q)|^{p-1}$. To complete the proof, we simply observe that by construction, $\delta(Y_Q) \le |Y_Q - x_Q| \le \hat{r}_Q \lesssim \ell(Q)$. □

## 5. Proof of Theorem 1.1, Corollary 1.5, and Theorem 1.12

In this section we complete the proofs of Theorem 1.1 and Theorem 1.12 by proving that $E := \partial\Omega$ satisfies WHSA, and hence, by Proposition 1.17, $E$ is UR. The proof of Corollary 1.5 follows almost immediately from Theorem 1.1 and we supply the proof at the end of the section. Our approach to the proofs of Theorems 1.1 and 1.12 is a refinement and extension of the arguments in [Lewis and Vogel 2007], who, as mentioned in the introduction, treated the special case that $k \approx 1$.

We fix $Q_0 \in \mathbb{D}(E)$ and we let $u$ and $\mu$ be as in Remark 4.11. We recall that by (4.9),

$$\frac{\mu(Q_0)}{\sigma(Q_0)} \approx 1. \tag{5.1}$$

Let $\mathcal{F} = \{Q_j\}_j$ be the family of maximal stopping time cubes constructed in Lemma 4.12. Combining (4.25) and (4.14), we see that

$$|\nabla u(Y_Q)| \gtrsim 1, \quad \forall\, Q \in \mathbb{D}^*_{\mathcal{F},Q_0} := \{Q \in \mathbb{D}_{\mathcal{F},Q_0} : \ell(Q) \le K_0^{-1}\ell(Q_0)\}, \tag{5.2}$$

where $Y_Q \in U_Q$ is the point constructed in Lemma 4.24. We recall that the Whitney region $U_Q$ has a uniformly bounded number of connected components, which we have enumerated as $\{U_Q^i\}_i$. We now fix the particular $i$ such that $Y_Q \in U_Q^i \subset \widetilde{U}_Q^i$, where the latter is the enlarged Whitney region constructed in Definition 2.26.

For a suitably small $\varepsilon_0$, say $\varepsilon_0 \ll K_0^{-6}$, we fix an arbitrary positive $\varepsilon < \varepsilon_0$, and we fix also a large positive number $M$ to be chosen. For each point $Y \in \Omega$, we set

$$B_Y := \overline{B(Y, (1-\varepsilon^{2M/\alpha})\delta(Y))}, \qquad \widetilde{B}_Y := \overline{B(Y, \delta(Y))}, \tag{5.3}$$

where $0 < \alpha < 1$ is the exponent appearing in Lemma 3.35. For $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, we consider three cases.

**Case 0**: $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) > \varepsilon^{10}\ell(Q_0)$.

**Case 1**: $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) \le \varepsilon^{10}\ell(Q_0)$ and

$$\sup_{X\in\widetilde{U}_Q^i} \sup_{Z\in B_X} |\nabla u(Z) - \nabla u(Y_Q)| > \varepsilon^{2M}. \tag{5.4}$$

**Case 2**: $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) \le \varepsilon^{10}\ell(Q_0)$ and

$$\sup_{X\in\widetilde{U}_Q^i} \sup_{Z\in B_X} |\nabla u(Z) - \nabla u(Y_Q)| \le \varepsilon^{2M}. \tag{5.5}$$

We trivially see that the cubes in Case 0 satisfy a packing condition:

$$\sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \text{Case 0 holds}}} \sigma(Q) \le \sum_{\substack{Q\in\mathbb{D}_{Q_0}\\ \ell(Q)>\varepsilon^{10}\ell(Q_0)}} \sigma(Q) \lesssim (\log\varepsilon^{-1})\sigma(Q_0). \tag{5.6}$$

Note that in Case 1 and Case 2 we have $Q \in \mathbb{D}^*_{\mathcal{F},Q_0}$ (see (5.2)). Furthermore, if $\ell(Q) \le \varepsilon^{10}\ell(Q_0)$, then by (5.2), (3.42), and either (3.13) (which we apply in the case $p = 2$, with $X = X_0$, since $\ell(Q) \ll \ell(Q_0)$) or (3.45) (for general $p$, $1 < p < \infty$), and (4.14), we have

$$1 \lesssim |\nabla u(Y_Q)| \lesssim \frac{u(Y_Q)}{\delta(Y_Q)} \lesssim 1. \tag{5.7}$$

Regarding Case 1 we obtain the following packing condition.

**Lemma 5.8.** *Under the previous assumptions, the following packing condition holds*:

$$\frac{1}{\sigma(Q_0)} \sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \text{Case 1 holds}}} \sigma(Q) \le C(\varepsilon, K_0, M, \eta). \tag{5.9}$$

On the other hand, we show that the cubes in Case 2 satisfy the $\varepsilon$-local WHSA property. Given $\varepsilon > 0$, recall that $B_Q^{***}(\varepsilon) = B(x_Q, \varepsilon^{-5}\ell(Q))$ (see (2.16)). We also introduce

$$B_Q^{\text{big}} = B_Q^{\text{big}}(\varepsilon) := B(x_Q, \varepsilon^{-8}\ell(Q)), \qquad \Delta_Q^{\text{big}} := B_Q^{\text{big}} \cap E.$$

**Lemma 5.10.** *Fix $\varepsilon \in (0, K_0^{-6})$, and let $1 < p < \infty$. Suppose that $u$ is nonnegative and $p$-harmonic in $\Omega_Q := \Omega \cap B_Q^{\text{big}}$, $u \in C(\overline{\Omega_Q})$, $u \equiv 0$ on $\Delta_Q^{\text{big}}$. Suppose also that for some $i$, there exists a point $Y_Q \in U_Q^i$ such that*

$$|\nabla u(Y_Q)| \approx 1, \tag{5.11}$$

*and furthermore, that*

$$\sup_{B_Q^{***}} u \lesssim \varepsilon^{-5}\ell(Q) \tag{5.12}$$

*and*

$$\sup_{X,Y\in\widetilde{U}_Q^i}\ \sup_{Z_1\in B_Y,\, Z_2\in B_X} |\nabla u(Z_1)-\nabla u(Z_2)| \le 2\varepsilon^{2M}. \tag{5.13}$$

*Then $Q$ satisfies the $\varepsilon$-local WHSA, provided that $M$ is large enough, depending only on dimension and on the implicit constants in the stated hypotheses.*

Assuming these results momentarily, we can complete the proofs of Theorem 1.1 and Theorem 1.12 as follows. First we see that we can apply Lemma 5.10 to the cubes in Case 2. Indeed, let $Q$ be a cube such that $Q\in\mathbb{D}_{\mathcal{F},Q_0}$, $\ell(Q)\le\varepsilon^{10}\ell(Q_0)$, and (5.5) holds. Hence (5.11) follows by virtue of (5.7), while (5.12) holds by Lemma 3.14 applied with $B=2B_Q^{***}$ (or Lemma 3.46, with $B(y,s)=2B_Q^{***}$), and (4.23). Moreover, (5.13) follows trivially from (5.5). Thus, the hypotheses of Lemma 5.10 are all verified and hence $Q$ satisfies the $\varepsilon$-local WHSA condition. In particular, the cubes $Q\in\mathbb{D}_{\mathcal{F},Q_0}$, which belong to the bad collection $\mathcal{B}$ of cubes in $\mathbb{D}(E)$ for which the $\varepsilon$-local WHSA condition fails, must be as in Case 0 or Case 1. By (5.6) and (5.9) these cubes satisfy the packing estimate

$$\sum_{Q\in\mathcal{B}\cap\mathbb{D}_{\mathcal{F},Q_0}} \sigma(Q)\le C_\varepsilon\,\sigma(Q_0). \tag{5.14}$$

For each $Q_0\in\mathbb{D}(E)$, there is a family $\mathcal{F}\subset\mathbb{D}_{Q_0}$ for which (5.14), and also the "ampleness" condition (4.13), hold uniformly. We may therefore invoke a well known lemma of John–Nirenberg type to deduce that (2.20) holds for all $\varepsilon\in(0,\varepsilon_0)$, and therefore to conclude that $E$ satisfies the WHSA condition, Definition 2.19. Hence $E$ is UR by Proposition 1.17.

The rest of the section is devoted to the proof of Lemmas 5.8 and 5.10. We shall first prove Lemma 5.8 in the relatively simpler linear case $p=2$ (see Section 5A). The proof of Lemma 5.8 in the general case $1<p<\infty$ is a bit more delicate and given in Section 5B. Lemma 5.10 is proved in Section 5C. Finally, the proof of Corollary 1.5 is given in Section 5D.

Before passing to the subsections we first introduce some additional notation to be used in the sequel. We augment $\widetilde{U}_Q^i$ as follows. Set

$$\mathcal{W}_Q^{i,*} := \Big\{ I\in\mathcal{W} : I^* \text{ meets } B_Y \text{ for some } Y\in\Big(\bigcup_{X\in\widetilde{U}_Q^i} B_X\Big)\Big\} \tag{5.15}$$

(and define $\mathcal{W}_Q^{j,*}$ analogously for all other $\widetilde{U}_Q^j$), and set

$$U_Q^{i,*} := \bigcup_{I\in\mathcal{W}_Q^{i,*}} I^{**}, \qquad U_Q^* := \bigcup_j U_Q^{j,*}, \tag{5.16}$$

where $I^{**}=(1+2\tau)I$ is a suitably fattened Whitney cube, with $\tau$ fixed as above. By construction,

$$\widetilde{U}_Q^i \subset \bigcup_{X\in\widetilde{U}_Q^i} B_X \subset \bigcup_{Y\in\bigcup_{X\in\widetilde{U}_Q^i} B_X} B_Y \subset U_Q^{i,*},$$

and for all $Y \in U_Q^{i,*}$, we have that $\delta(Y) \approx \ell(Q)$ (depending of course on $\varepsilon$). Moreover, also by construction, there is a Harnack path connecting any pair of points in $U_Q^{i,*}$ (depending again on $\varepsilon$), and furthermore, for every $I \in \mathcal{W}_Q^{i,*}$ (or for that matter for every $I \in \mathcal{W}_Q^{j,*}$, $j \neq i$),

$$\varepsilon^s \ell(Q) \lesssim \ell(I) \lesssim \varepsilon^{-3}\ell(Q), \qquad \operatorname{dist}(I, Q) \lesssim \varepsilon^{-4}\ell(Q),$$

where $0 < s = s(M, \alpha)$. Thus, by Harnack's inequality and (5.7),

$$C^{-1}\delta(Y) \le u(Y) \le C\delta(Y), \quad \forall Y \in U_Q^{i,*}, \tag{5.17}$$

with $C = C(K_0, \varepsilon, M)$. Moreover, for future reference, we note that the upper bound for $u$ holds in all of $U_Q^*$, i.e.,

$$u(Y) \le C\delta(Y), \quad \forall Y \in U_Q^*, \tag{5.18}$$

by (3.12) or (3.45) and (4.14), where again $C = C(K_0, \varepsilon, M)$.

**5A. *Proof of Lemma 5.8 in the linear case* ($p = 2$).** Here we complete the proof of estimate (5.9) in the relatively simpler linear case $p = 2$. To start the proof of (5.9), we fix $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ so that Case 1 holds. We see that if we choose $Z$ as in (5.4), and use the mean value property of harmonic functions, then

$$\varepsilon^{2M} \le C_\varepsilon(\ell(Q))^{-(n+1)} \iint_{B_Z \cup B_{Y_Q}} |\nabla u(Y) - \vec{\beta}|\, dY,$$

where $\vec{\beta}$ is a constant vector at our disposal. By Poincaré's inequality (see, e.g., [Hofmann and Martell 2014, Section 4] in this context), we obtain that

$$\sigma(Q) \lesssim \iint_{U_Q^{i,*}} |\nabla^2 u(Y)|^2 \delta(Y)\, dY \lesssim \iint_{U_Q^{i,*}} |\nabla^2 u(Y)|^2 u(Y)\, dY,$$

where the implicit constants depend on $\varepsilon$, and in the last step we have used (5.17). Consequently,

$$\sum_{\substack{Q \in \mathbb{D}_{\mathcal{F},Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \lesssim \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F},Q_0} \\ \ell(Q) \le \varepsilon^{10}\ell(Q_0)}} \iint_{U_Q^*} |\nabla^2 u(Y)|^2 u(Y)\, dY \lesssim \iint_{\Omega^*_{\mathcal{F},Q_0}} |\nabla^2 u(Y)|^2 u(Y)\, dY, \tag{5.19}$$

where

$$\Omega^*_{\mathcal{F},Q_0} := \operatorname{int}\Bigg( \bigcup_{\substack{Q \in \mathbb{D}_{\mathcal{F},Q_0} \\ \ell(Q) \le \varepsilon^{10}\ell(Q_0)}} U_Q^* \Bigg), \tag{5.20}$$

and where we have used that the enlarged Whitney regions $U_Q^*$ have bounded overlaps.

Take an arbitrary $N > 1/\varepsilon$ (eventually $N \to \infty$), and augment $\mathcal{F}$ by adding to it all subcubes $Q \subset Q_0$ with $\ell(Q) \le 2^{-N}\ell(Q_0)$. Let $\mathcal{F}_N \subset \mathbb{D}_{Q_0}$ denote the collection of maximal cubes of this augmented family. Thus, $Q \in \mathbb{D}_{\mathcal{F}_N,Q_0}$ if and only if $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ and $\ell(Q) > 2^{-N}\ell(Q_0)$. Clearly, $\mathbb{D}_{\mathcal{F}_N,Q_0} \subset \mathbb{D}_{\mathcal{F}_{N'},Q_0}$ if $N \le N'$, and therefore $\Omega^*_{\mathcal{F}_N,Q_0} \subset \Omega^*_{\mathcal{F}_{N'},Q_0}$ (where $\Omega^*_{\mathcal{F}_N,Q_0}$ is defined as in (5.20) with $\mathcal{F}_N$ replacing $\mathcal{F}$).

By monotone convergence and (5.19), we have that

$$\sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \text{Case 1 holds}}} \sigma(Q) \lesssim \limsup_{N\to\infty} \iint_{\Omega^*_{\mathcal{F}_N,Q_0}} |\nabla^2 u(Y)|^2 u(Y)\,dY. \tag{5.21}$$

It therefore suffices to establish bounds for the latter integral that are uniform in $N$, with $N$ large.

Let us then fix $N > 1/\varepsilon$. Since $\Omega^*_{\mathcal{F}_N,Q_0}$ is a finite union of fattened Whitney boxes, we may now integrate by parts, using the identity $2|\nabla\partial_k u|^2 = \operatorname{div}\nabla(\partial_k u)^2$ for harmonic functions, to obtain that

$$\iint_{\Omega^*_{\mathcal{F}_N,Q_0}} |\nabla^2 u(Y)|^2 u(Y)\,dY \lesssim \int_{\partial\Omega^*_{\mathcal{F}_N,Q_0}} (|\nabla^2 u||\nabla u|u + |\nabla u|^3)\,dH^n \le C_\varepsilon H^n(\partial\Omega^*_{\mathcal{F}_N,Q_0}), \tag{5.22}$$

where in the second inequality we have used the standard estimates

$$\delta(Y)|\nabla^2 u(Y)|,\ |\nabla u(Y)| \lesssim \frac{u(Y)}{\delta(Y)},$$

along with (5.18). We observe that $\Omega^*_{\mathcal{F}_N,Q_0}$ is a sawtooth domain in the sense of [Hofmann et al. 2016], or to be more precise, it is a union of a bounded number, depending on $\varepsilon$, of such sawtooths, one for each maximal subcube of $Q_0$ with length on the order of $\varepsilon^{10}\ell(Q_0)$. By [Hofmann et al. 2016, Appendix A] each of the previous sawtooth domains is ADR uniformly in $N$. Hence, its union is upper ADR uniformly in $N$ with constant depending on the number of sawtooth domains in the union, which ultimately depends on $\varepsilon$. Therefore,

$$H^n(\partial\Omega^*_{\mathcal{F}_N,Q_0}) \le C_\varepsilon(\operatorname{diam}(\partial\Omega^*_{\mathcal{F}_N,Q_0}))^n \le C_\varepsilon\sigma(Q_0).$$

Combining the latter estimate with (5.21) and (5.22), we obtain (5.9), as desired, in the case $p = 2$.

**5B. *Proof of Lemma 5.8 in the general case* ($1 < p < \infty$).** Here we prove (5.9) for general $p$, $1 < p < \infty$, by proceeding along the lines of the proof of Lemma 2.5 in [Lewis and Vogel 2006]. We fix $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ so that Case 1, and hence (5.4), holds. Let us recall that we have verified estimates (5.7), (5.17), and (5.18) for all $p$, $1 < p < \infty$.

Recall that if $X \in \widetilde{U}^i_Q$, then by definition $X$ can be connected to some $\tilde{Y} \in U^i_Q$, and then to $Y_Q \in U^i_Q$, by a chain of at most $C\varepsilon^{-1}$ balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3\ell(Q) \le \delta(Y_k) \le \varepsilon^{-3}\ell(Q)$. Note that using the triangle inequality and the definition of $\widetilde{U}^i_Q$, we may suppose that $Y_{k+1} \in B(Y_k, 3\delta(Y_k)/4) \subset B_{Y_k}$; otherwise we increase the chain by introducing some intermediate points and the new chain will have essentially the same length. Fix now $Q$, a cube in Case 1, and by (5.4) we can pick $X \in \widetilde{U}^i_Q$ so that

$$\sup_{Y\in B_X} |\nabla u(Y) - \nabla u(Y_Q)| > \varepsilon^{2M}.$$

As observed previously, we can form a Harnack chain connecting $X$ and $Y_Q$ so that $Y_1 = Y_Q$ and $Y_l = X$ and $l \le C\varepsilon^{-1}$. Then the previous expression can be written as

$$\sup_{Y\in B_{Y_l}} |\nabla u(Y) - \nabla u(Y_1)| > \varepsilon^{2M}. \tag{5.23}$$

Obviously we may assume that

$$\sup_{Y\in B_{Y_j}} |\nabla u(Y)-\nabla u(Y_1)| \le \varepsilon^{2M} \tag{5.24}$$

whenever $1 < j \le l-1$, and $l > 1$, since otherwise we shorten the chain (and work with the first $Y_j$ for which (5.23) holds). This and the fact that $Y_{j+1}\in B_{Y_j}$ for every $1\le j\le l-1$ imply that

$$|\nabla u(Y_j)| \ge |\nabla u(Y_1)| - \varepsilon^{2M}, \quad \text{for } 1\le j\le l. \tag{5.25}$$

Furthermore, using the triangle inequality,

$$\varepsilon^{2M} \le \sup_{Y\in B_{Y_l}} |\nabla u(Y)-\nabla u(Y_l)| + \sum_{j=1}^{l-1} |\nabla u(Y_{j+1})-\nabla u(Y_j)|. \tag{5.26}$$

Hence, using this and the fact that $l\lesssim \varepsilon^{-1}$ we have that either

$$\begin{aligned} &\text{(i)} \quad \sup_{Y\in B_{Y_l}} |\nabla u(Y)-\nabla u(Y_l)| \ge \varepsilon^{2M+2}, \quad \text{or} \\ &\text{(ii)} \quad |\nabla u(Y_{j+1})-\nabla u(Y_j)| \ge \varepsilon^{2M+2}, \quad \text{for some } 1\le j\le l-1. \end{aligned} \tag{5.27}$$

By (5.18) and (3.42) we have

$$|\nabla u(Y)| \le C_\varepsilon, \quad \forall Y\in U_Q^*. \tag{5.28}$$

In scenario (i) of (5.27) we take $Y$, a point where the sup is attained. This choice, (5.28), and the first inequality in (3.38) imply that $|Y-Y_l|\approx_\varepsilon \ell(Q)$. We then construct $\Gamma_0(Q)$ a (possibly rotated) rectangle as follows. The base and the top are two $n$-dimensional cubes of side length $c_\varepsilon\ell(Q)$, with $c_\varepsilon$ chosen sufficiently small, centered respectively at the points $Y$ and $Y_l$, and lying in the two parallel hyperplanes passing through the points $Y$ and $Y_l$ and perpendicular to the vector joining these two points. Note that for this rectangle, all side lengths are of the order of $\ell(Q)$ with implicit constants possibly depending on $\varepsilon$. In scenario (ii) of (5.27) we do the same construction with $Y_{j+1}$ and $Y_j$ in place of $Y$ and $Y_l$ and define $\Gamma_0(Q)$ which verifies the same properties. Note that in either case, (5.28) and the first inequality in (3.38) give the property that

$$|\nabla u(Y)-\nabla u(W)| \ge \varepsilon^{2M+4} \tag{5.29}$$

whenever $W$ and $Y$ are in the base and top of the parallelepiped, respectively. By construction, at least the top, which we denote by $t(Q)$, is centered on $Y_j$, for some $1\le j\le l$. We observe that by (5.25) and (5.7), since $Y_1 := Y_Q$, and since $\varepsilon$ is very small, we have for each $Y_j$, $1\le j\le l$,

$$|\nabla u(Y_j)| \ge a, \tag{5.30}$$

for some uniform constant $a$ independent of $\varepsilon$. Therefore, by (3.38), we also have

$$|\nabla u(Y)| \ge \frac{a}{2}, \quad \forall Y\in t(Q), \tag{5.31}$$

provided that we take $c_\varepsilon$ small enough, since $\operatorname{diam}(t(Q))\approx c_\varepsilon\ell(Q)$. Moving downward, that is, from top to base, through $\Gamma_0(Q)$ along slices parallel to $t(Q)$, we stop the first time that we reach a slice $b(Q)$

which contains a point $Z$ with $|\nabla u(Z)| \le a/4$. If there is such a slice, we form a new rectangle $\Gamma(Q)$ with base $b(Q)$ and top $t(Q)$; otherwise, we set $\Gamma(Q) := \Gamma_0(Q)$, and let $b(Q)$ denote the base in this case as well. In either case, $\operatorname{dist}(b(Q), t(Q)) \approx \ell(Q)$, with implicit constants possibly depending on $\varepsilon$, by (3.38) and (5.31). Note that by construction and the continuity of $\nabla u$,

$$|\nabla u(Y)| \ge \frac{a}{4}, \quad \forall Y \in \Gamma(Q), \tag{5.32}$$

and that $|\Gamma(Q)| \approx \ell(Q)^{n+1}$, again with implicit constants which may depend on $\varepsilon$. Furthermore, if $\Gamma(Q) = \Gamma_0(Q)$, then (5.29) holds for all $W \in b(Q)$ and $Y \in t(Q)$. Otherwise, if $\Gamma(Q)$ is strictly contained in $\Gamma_0(Q)$, then, since $\operatorname{diam}(b(Q)) \approx c_\varepsilon \ell(Q)$ with $c_\varepsilon$ small, and since by construction $b(Q)$ contains a point $Z$ with $|\nabla u(Z)| = a/4$, it follows that $|\nabla u(W)| \le 3a/8$ for all $W \in b(Q)$, by (3.38). Hence, in either situation, since $a/8 \gg \varepsilon^{2M+4}$, we have

$$|\nabla u(Y) - \nabla u(W)| \ge \varepsilon^{2M+4}, \quad \forall W \in b(Q),\ Y \in t(Q). \tag{5.33}$$

We let $\gamma = a/8$ and set

$$F_\gamma(|\nabla u|) := \max(|\nabla u|^2 - \gamma^2, 0).$$

Then by (5.32) we see that

$$F_\gamma(|\nabla u|) \ge \frac{a^2}{64}, \quad \forall Y \in \Gamma(Q). \tag{5.34}$$

Furthermore, by (5.33), the fundamental theorem of calculus, (5.17), (5.32), and (5.34), we have

$$\ell(Q)^n \lesssim \iint_{\Gamma(Q)} u|\nabla^2 u|^2\, dX \lesssim \iint_{\Gamma(Q)} u F_\gamma(|\nabla u|)\, |\nabla u|^{p-2}\, |\nabla^2 u|^2\, dY,$$

where the implicit constants depend on $\varepsilon$. In particular, since $\Gamma(Q) \subset U_Q^{i,*} \subset U_Q^*$, by ADR we obtain

$$\sigma(Q) \lesssim \iint_{U_Q^*} u F_\gamma(|\nabla u|)\, |\nabla u|^{p-2}\, |\nabla^2 u|^2\, dY,$$

where the implicit constants still depend on $\varepsilon$, and this estimate holds for all cubes $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, so that Case 1 holds. Hence,

$$\sum_{\substack{Q \in \mathbb{D}_{\mathcal{F},Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \lesssim \iint_{\Omega^*_{\mathcal{F},Q_0}} u F_\gamma(|\nabla u|)\, |\nabla u|^{p-2}\, |\nabla^2 u|^2\, dY, \tag{5.35}$$

where $\Omega^*_{\mathcal{F},Q_0}$ was defined in (5.20) and where we have used that the enlarged Whitney regions $U_Q^*$ have bounded overlaps. To prove (5.9) in the general case $1 < p < \infty$, it therefore suffices to establish the local square function bound

$$\iint_{\Omega^*_{\mathcal{F},Q_0}} u F_\gamma(|\nabla u|)\, |\nabla u|^{p-2}\, |\nabla^2 u|^2\, dY \lesssim \sigma(Q_0), \tag{5.36}$$

where, as we recall, $u$ is a nonnegative $p$-harmonic function in the open set $\Omega_0 := \Omega \cap B(x_{Q_0}, Cr_{Q_0})$, vanishing on $\Delta(x_{Q_0}, Cr_{Q_0})$.

To start the proof of (5.36), for each $Q \in \mathbb{D}(E)$, we define a further fattening of $U_Q^*$ as follows. Set

$$U_Q^{i,**} := \bigcup_{I \in \mathcal{W}_Q^{i,*}} I^{***}, \qquad U_Q^{**} := \bigcup_i U_Q^{i,**},$$

$$U_Q^{i,***} := \bigcup_{I \in \mathcal{W}_Q^{i,*}} I^{****}, \qquad U_Q^{***} := \bigcup_i U_Q^{i,***},$$

where $I^{***} = (1+3\tau)I$ and $I^{****} = (1+4\tau)I$ are fattened Whitney regions, for some fixed small $\tau$ as above; see (5.15)–(5.16). Notice that $I^{**} \subset I^{***} \subset I^{****}$. We observe that the fattened Whitney regions $U_Q^{***}$ have bounded overlaps, say

$$\sum_{Q \in \mathbb{D}(E)} 1_{U_Q^{***}}(Y) \le M_0, \tag{5.37}$$

where $M_0 < \infty$ is a uniform constant depending on $K_0$, $\varepsilon$, $\tau$, and $n$. Next, let $\{\eta_Q\}_Q$ be a partition of unity adapted to $U_Q^{**}$. That is,

(1) $\sum_Q \eta_Q(Y) \equiv 1$ whenever $Y \in \Omega$,

(2) $\operatorname{supp} \eta_Q \subset U_Q^{**}$, and

(3) $\eta_Q \in C_0^\infty(\mathbb{R}^{n+1})$ with $0 \le \eta_Q \le 1$, $\eta_Q \ge c$ on $U_Q^*$, and $|\nabla \eta_Q| \le C\ell(Q)^{-1}$.

Set

$$\mathbb{D}_{\mathcal{F},Q_0,\varepsilon} := \{Q \in \mathbb{D}_{\mathcal{F},Q_0} : \ell(Q) \le \varepsilon^{10}\ell(Q_0)\},$$

and recall from (5.20) that

$$\Omega^*_{\mathcal{F},Q_0} := \operatorname{int}\Bigg(\bigcup_{Q \in \mathbb{D}_{\mathcal{F},Q_0,\varepsilon}} U_Q^*\Bigg).$$

Given a large number $N \gg \varepsilon^{-10}$, set

$$\Lambda = \Lambda(N) = \{Q \in \mathbb{D}(E) : U_Q^{**} \cap \Omega^*_{\mathcal{F},Q_0} \ne \varnothing \ \text{ and } \ \ell(Q) \ge N^{-1}\ell(Q_0)\}.$$

Eventually, we shall let $N \to \infty$. Let

$$I_1(N) := \sum_{Q \in \Lambda(N)} \iint u F_\gamma(|\nabla u|)\Bigg(\sum_{i,j=1}^{n+1} u^2_{y_i y_j}\Bigg)\eta_Q \, dY$$

and note, by positivity of $u$ and the properties of $\eta_Q$, that we then have

$$\iint_{\Omega^*_{\mathcal{F},Q_0}} u F_\gamma(|\nabla u|)\,|\nabla^2 u|^2 \, dY \lesssim \lim_{N \to \infty} I_1(N).$$

We now fix $N$. We intend to perform integration by parts and in this argument, we exploit that $|\nabla u|^2$ is a subsolution to a certain linear PDE defined based on $u$. To describe this in detail, let $Q \in \Lambda(N)$ be such that $F_\gamma(|\nabla u(Y)|) \ne 0$ for some $Y \in U_Q^{**}$. Then $|\nabla u(Y)| \ge \gamma$ and there exists $C = C(\gamma) \ge 1$ such that

$$C^{-1} \le |\nabla u(X)| \lesssim 1 \quad \text{whenever } X \in B(Y, \delta(Y)/C), \tag{5.38}$$

and where the upper bound follows from (5.18) and the lower bound uses also (3.38). Let $\zeta = \nabla u \cdot \xi$, for some $\xi \in \mathbb{R}^{n+1}$. Then $\zeta$ satisfies, at $X \in B(Y, \delta(Y)/C)$, the partial differential equation

$$L\zeta = \nabla \cdot \big[(p-2)|\nabla u|^{p-4}(\nabla u \cdot \nabla \zeta)\nabla u + |\nabla u|^{p-2}\,\nabla \zeta\big] = 0, \tag{5.39}$$

as is seen by a straightforward calculation from differentiating the $p$-Laplace partial differential equation for $u$ with respect to $\xi$. Note that (5.39) can be written in the form

$$L\zeta = \sum_{i,j=1}^{n+1} \frac{\partial}{\partial y_i}[b_{ij}(\,\cdot\,)\zeta_{y_j}(\,\cdot\,)] = 0, \tag{5.40}$$

where

$$b_{ij}(Y) = |\nabla u|^{p-4}[(p-2)u_{y_i}u_{y_j} + \delta_{ij}|\nabla u|^2](Y), \quad 1 \le i, j \le n+1, \tag{5.41}$$

and $\delta_{ij}$ is the Kronecker $\delta$. Clearly we also have

$$Lu(Y) = (p-1)\nabla \cdot [|\nabla u|^{p-2}\nabla u](Y) = 0. \tag{5.42}$$

In particular, $u$ and $(\nabla u \cdot \xi)$ for each $\xi \in \mathbb{R}^{n+1}$ all satisfy the divergence form partial differential equation (5.40).

It is easy to see that $(b_{ij})_{ij}$ satisfies the following degenerate ellipticity condition: for every $\xi \in \mathbb{R}^{n+1}$ one has

$$\begin{aligned}\sum_{i,j=1}^{n+1} b_{ij}\xi_i\xi_j &= (p-2)\,|\nabla u|^{p-4}\sum_{i,j=1}^{n+1} u_i u_j \xi_i \xi_j + |\nabla u|^{p-2}\sum_{i,j=1}^{n+1}\delta_{ij}\xi_i\xi_j \\ &= (p-2)\,|\nabla u|^{p-4}(\nabla u \cdot \xi)^2 + |\nabla u|^{p-2}\,|\xi|^2 \ge \min\{1, p-1\}\,|\nabla u|^{p-2}\,|\xi|^2,\end{aligned} \tag{5.43}$$

where the last inequality is immediate when $p \ge 2$ and uses the Cauchy–Schwarz inequality when $1 < p < 2$. Hence, $|\nabla u|^2$ is a subsolution to the PDE defined in (5.40), (5.41), as seen from the calculation

$$L(|\nabla u|^2) = 2\sum_{i,j,k=1}^{n+1} b_{ij}u_{y_iy_k}u_{y_jy_k} \gtrsim |\nabla u|^{p-2}\bigg(\sum_{i,j=1}^{n+1} u^2_{y_iy_j}\bigg). \tag{5.44}$$

Now, using (5.44) and the fact that (5.38) holds for every $Y$ such that $F_\gamma(|\nabla u(Y)|) \neq 0$, we see that $I_1(N) \lesssim J_1(N)$, where

$$J_1(N) := \sum_{Q\in\Lambda(N)} \iint u F_\gamma(|\nabla u|) L(|\nabla u|^2)\eta_Q\, dY.$$

Hence it suffices to establish bounds for the integral $J_1 := J_1(N)$ that are uniform in $N$, with $N$ large. In the following we let $v = F_\gamma(|\nabla u|)$ and note that $\nabla v = \nabla(|\nabla u|^2)$ whenever $v > 0$. Using this and integration by parts we see that

$$J_1 = -J_2 - J_3 - J_4,$$

where

$$J_2 = \sum_{Q\in\Lambda(N)} \iint v \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} v_{y_j} \eta_Q \, dY,$$

$$J_3 = \sum_{Q\in\Lambda(N)} \iint u \sum_{i,j=1}^{n+1} b_{ij} v_{y_i} v_{y_j} \eta_Q \, dY,$$

$$J_4 = \sum_{Q\in\Lambda(N)} \iint uv \sum_{i,j=1}^{n+1} b_{ij} v_{y_j} (\eta_Q)_{y_i} \, dY.$$

We estimate $J_4$ first. Set $\Lambda_1 = \Lambda_{11} \cup \Lambda_{12}$, where

$$\Lambda_{11} := \{Q \in \Lambda : U_Q^{**} \text{ meets } \Omega \setminus \Omega_{\mathcal{F},Q_0}\},$$

and

$$\Lambda_{12} := \{Q \in \Lambda : U_Q^{**} \text{ meets } U_{Q'}^{**} \text{ such that } \ell(Q') < N^{-1}\ell(Q_0)\}.$$

From the definition of $\eta_Q$, we obtain

$$|J_4| \lesssim \sum_{Q\in\Lambda_{11}} \iint uv \sum_{i,j=1}^{n+1} |u_{ij}||u_i||(\eta_Q)_j| \, dY + \sum_{Q\in\Lambda_{11}} \iint uv \sum_{i,j=1}^{n+1} |u_{ij}||u_i||(\eta_Q)_j| \, dY =: J_{51} + J_{52}.$$

Notice that, equivalently, $\Lambda_{11}$ is the subcollection of $Q \in \Lambda_1$ such that $U_Q^{**}$ meets $\partial\Omega^*_{\mathcal{F},Q_0}$. We start with $J_{51}$. Note that by (3.38), (5.18), and Harnack's inequality,

$$\delta(Y)|\nabla u(Y)| \lesssim u(Y) \lesssim \delta(Y) \approx \ell(Q) \tag{5.45}$$

whenever $Y \in U_Q^{***}$. Furthermore, if $v \neq 0$ for some $Y \in U_Q^{***}$, then using (5.38) and (3.40), we also have

$$(\delta(Y))^2 |\nabla^2 u(Y)| \lesssim u(Y) \lesssim \delta(Y) \approx \ell(Q). \tag{5.46}$$

In particular, $u|\nabla\eta_Q| \lesssim 1$ by construction of $\eta_Q$, $|\nabla u(Y)| \lesssim 1$ whenever $Y \in U_Q^{***}$, and $\delta(Y)|\nabla^2 u(Y)| \lesssim 1$ whenever $Y \in U_Q^{***}$ and $v \neq 0$. Thus,

$$J_{51} \lesssim \sum_{Q\in\Lambda_{11}} \ell(Q)^n \lesssim \sum_{Q\in\Lambda_{11}} H^n(U_Q^{***} \cap \partial\Omega^*_{\mathcal{F},Q_0}) \lesssim \sum_{Q\in\Lambda_{11}} H^n(\partial\Omega^*_{\mathcal{F},Q_0}) \lesssim \sigma(Q_0),$$

where we have used that $\partial\Omega^*_{\mathcal{F},Q_0}$ is ADR (see [Hofmann et al. 2016]), and the bounded overlap property (5.37). To estimate $J_{52}$, observe that for each $Q \in \Lambda_{12}$, we have $\ell(Q) \approx N^{-1}\ell(Q_0)$ by properties of Whitney regions. Hence, by a slightly simpler version of the argument used for $J_{51}$, we obtain

$$J_{52} \lesssim \sum_{Q\in\Lambda_{12}} \sigma(Q) \lesssim \sigma(Q_0).$$

Therefore, $|J_4| \lesssim J_{51} + J_{52} \lesssim \sigma(Q_0)$.

To handle $J_2$ we use the fact that $u$ is a solution to (5.40). Indeed, by integration by parts, using the identity $2vv_{y_j} = (v^2)_{y_j}$ we see that

$$2J_2 = \sum_{Q\in\Lambda(N)} \iint \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} (v^2)_{y_j} \eta_Q \, dY = -\sum_{Q\in\Lambda(N)} \iint \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} v^2 (\eta_Q)_{y_j} \, dY,$$

and by the same argument as in the estimate of $J_4$ we obtain $|J_2| \lesssim \sigma(Q_0)$.

To conclude, we collect the estimates for $J_2$ and $J_4$, and use the fact that $J_3$ is nonnegative by (5.43) to obtain $J_1(N) \lesssim \sigma(Q_0)$, with constants independent of $N$. The proof of (5.9) in the general case $1 < p < \infty$ is then complete.

**5C. *Proof of Lemma 5.10.*** To prove Lemma 5.10, we follow the corresponding argument in [Lewis and Vogel 2007] closely, but with some modifications due to the fact that in contrast to the situation in that paper, our solution $u$ need not be Lipschitz up to the boundary, and our harmonic/$p$-harmonic measures need not be doubling. It is the latter obstacle that has forced us to introduce the WHSA condition, rather than to work with the weak exterior convexity condition used by Lewis and Vogel. Lemma 5.10 is essentially a distillation of the main argument of the corresponding part of [Lewis and Vogel 2007], but with the doubling hypothesis removed.

In the remainder of this section, for convenience we use the notational convention that implicit and generic constants are allowed to depend upon $K_0$, but not on $\varepsilon$ or $M$. Dependence on the latter is stated explicitly. We first prove the following lemma. Recall that the balls $B_Y$ and $\widetilde{B}_Y$ are defined in (5.3).

**Lemma 5.47.** *Let $Y \in U_Q^i$, $X \in \widetilde{U}_Q^i$. Suppose first that $w \in \partial\widetilde{B}_Y \cap E$, and let $W$ be the radial projection of $w$ onto $\partial B_Y$. Then*

$$u(W) \lesssim \varepsilon^{2M-5}\delta(Y). \tag{5.48}$$

*If $w \in \partial\widetilde{B}_X \cap E$, and $W$ now is the radial projection of $w$ onto $\partial B_X$, then*

$$u(W) \lesssim \varepsilon^{2M-5}\ell(Q). \tag{5.49}$$

*Proof.* Since $K_0^{-1}\ell(Q) \lesssim \delta(Y) \lesssim K_0\ell(Q)$ for $Y \in U_Q^i$, it is enough to prove (5.49). To prove (5.49), we first note that

$$|W - w| = \varepsilon^{2M/\alpha}\delta(X) \lesssim \varepsilon^{2M/\alpha}\varepsilon^{-3}\ell(Q),$$

by definition of $B_X$, $\widetilde{B}_X$ and the fact that by construction of $\widetilde{U}_Q^i$,

$$\varepsilon^3\ell(Q) \lesssim \delta(X) \lesssim \varepsilon^{-3}\ell(Q), \quad \forall X \in \widetilde{U}_Q^i. \tag{5.50}$$

In addition, again by construction of $\widetilde{U}_Q^i$,

$$\operatorname{diam}(\widetilde{U}_Q^i) \lesssim \varepsilon^{-4}\ell(Q). \tag{5.51}$$

Consequently, $W \in \frac{1}{2}B_Q^{***} = B\big(x_Q, \frac{1}{2}\varepsilon^{-5}\ell(Q)\big)$, so by Lemma 3.35 and (5.12),

$$u(W) \lesssim \left(\frac{\varepsilon^{2M/\alpha}\varepsilon^{-3}\ell(Q)}{\varepsilon^{-5}\ell(Q)}\right)^{\alpha} \frac{1}{|B_Q^{***}|} \iint_{B_Q^{***}} u \lesssim \varepsilon^{2M+2\alpha-5}\ell(Q) \le \varepsilon^{2M-5}\ell(Q). \qquad \square$$

**Claim 5.52.** *Let $Y \in U_Q^i$. For all $W \in B_Y$,*

$$|u(W) - u(Y) - \nabla u(Y) \cdot (W - Y)| \lesssim \varepsilon^{2M} \delta(Y). \tag{5.53}$$

*Proof of Claim 5.52.* Let $W \in B_Y$. Then for some $\widetilde{W} \in B_Y$,

$$u(W) - u(Y) = \nabla u(\widetilde{W}) \cdot (W - Y).$$

We may then invoke (5.13), with $X = Y$, $Z_1 = \widetilde{W}$, and $Z_2 = Y$, to obtain (5.53). □

**Claim 5.54.** *Let $Y \in U_Q^i$. Suppose that $w \in \partial \widetilde{B}_Y \cap E$. Then*

$$|u(Y) - \nabla u(Y) \cdot (Y - w)| = |u(w) - u(Y) - \nabla u(Y) \cdot (w - Y)| \lesssim \varepsilon^{2M-5} \delta(Y). \tag{5.55}$$

*Proof of Claim 5.54.* Given $w \in \partial \widetilde{B}_Y \cap E$, let $W$ be the radial projection of $w$ onto $\partial B_Y$, so that $|W - w| = \varepsilon^{2M/\alpha} \delta(Y)$. Since $u(w) = 0$, by (5.48) we have

$$|u(W) - u(w)| = u(W) \lesssim \varepsilon^{2M-5} \delta(Y).$$

Since (5.53) holds for $W$, we obtain (5.55) by (5.11) and (5.13). □

To simplify notation, we now set $Y := Y_Q$, the point in $U_Q^i$ satisfying (5.11). By (5.11) and (5.13), for $\varepsilon < \frac{1}{2}$, and $M$ chosen large enough, we have that

$$|\nabla u(Z)| \approx 1, \quad \forall Z \in \widetilde{U}_Q^i. \tag{5.56}$$

By translation and rotation, we assume that $0 \in \partial \widetilde{B}_Y \cap E$ and that $Y = \delta(Y) e_{n+1}$, where as usual $e_{n+1} := (0, \dots, 0, 1)$.

**Claim 5.57.** *We claim that*

$$\Big| \nabla u(Y) \cdot e_{n+1} - |\nabla u(Y)| \Big| \lesssim \varepsilon^{2M-5}. \tag{5.58}$$

*Proof of Claim 5.57.* We apply (5.55), with $w = 0$, to obtain

$$|u(Y) - \nabla u(Y) \cdot Y| \lesssim \varepsilon^{2M-5} \delta(Y).$$

Combining the latter bound with (5.53), we find that

$$|u(W) - \nabla u(Y) \cdot W| = |u(W) - \nabla u(Y) \cdot Y - \nabla u(Y) \cdot (W - Y)| \lesssim \varepsilon^{2M-5} \delta(Y), \quad \forall W \in B_Y. \tag{5.59}$$

Fix $W \in \partial B_Y$ so that $\nabla u(Y) \cdot \dfrac{W - Y}{|W - Y|} = -|\nabla u(Y)|$. Since $|W - Y| = (1 - \varepsilon^{2M/\alpha}) \delta(Y)$, and since $u \geq 0$, we have

$$\begin{aligned} 0 &\leq |\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1} \leq |\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1} + \frac{u(W)}{\delta(Y)} \\ &\leq \frac{1}{\delta(Y)} \left( -\nabla u(Y) \cdot \frac{(W - Y)}{1 - \varepsilon^{2M/\alpha}} - \nabla u(Y) \cdot Y + u(W) \right) \\ &\lesssim (\varepsilon^{2M-5} + \varepsilon^{2M/\alpha}) \approx \varepsilon^{2M-5}, \end{aligned} \tag{5.60}$$

by (5.59) and (5.11). □

**Claim 5.61.** *Suppose that $M > 5$. Then*

$$\big||\nabla u(Y)|e_{n+1} - \nabla u(Y)\big| \lesssim \varepsilon^{M-3}. \tag{5.62}$$

*Proof of Claim 5.61.* By Claim 5.57,

$$\big||\nabla u(Y)|e_{n+1} - (\nabla u(Y)\cdot e_{n+1})e_{n+1}\big| \lesssim \varepsilon^{2M-5}.$$

Therefore, it is enough to consider $\nabla_{\|}u := \nabla u - (\nabla u(Y)\cdot e_{n+1})e_{n+1}$. Observe that

$$\begin{aligned}|\nabla_{\|}u(Y)|^2 &= |\nabla u(Y)|^2 - (\nabla u(Y)\cdot e_{n+1})^2\\ &= (|\nabla u(Y)| - \nabla u(Y)\cdot e_{n+1})(|\nabla u(Y)| + \nabla u(Y)\cdot e_{n+1}) \lesssim \varepsilon^{2M-5},\end{aligned}$$

by (5.58) and (5.11). □

Now for $Y = \delta(Y)e_{n+1} \in U_Q^i$ fixed as above, we consider another point $X \in \widetilde{U}_Q^i$. By definition of $\widetilde{U}_Q^i$, there is a polygonal path in $\widetilde{U}_Q^i$, joining $Y$ to $X$, with vertices

$$Y_0 := Y, Y_1, Y_2, \dots, Y_N := X, \quad N \lesssim \varepsilon^{-4},$$

such that $Y_{k+1} \in B_{Y_k} \cap B(Y_k, \ell(Q))$, $0 \le k \le N-1$, and such that the distance between consecutive vertices is at most $C\ell(Q)$. Indeed, by definition of $\widetilde{U}_Q^i$, we may connect $Y$ to $X$ by a polygonal path connecting the centers of at most $\varepsilon^{-1}$ balls, such that the distance between consecutive vertices is between $\varepsilon^3\ell(Q)/2$ and $\varepsilon^{-3}\ell(Q)/2$. If any such distance is greater than $\ell(Q)$, we take at most $C\varepsilon^{-3}$ intermediate vertices with distances on the order of $\ell(Q)$. The total length of the path is thus on the order of $N\ell(Q)$ with $N \lesssim \varepsilon^{-4}$. Furthermore, by (5.13) and (5.62),

$$\begin{aligned}\big|\nabla u(W) - |\nabla u(Y)|e_{n+1}\big| &\le |\nabla u(W) - \nabla u(Y)| + \big|\nabla u(Y) - |\nabla u(Y)|e_{n+1}\big|\\ &\lesssim \varepsilon^{2M} + \varepsilon^{M-3} \lesssim \varepsilon^{M-3}, \quad \forall W \in B_Z,\ \forall Z \in \widetilde{U}_Q^i.\end{aligned} \tag{5.63}$$

**Claim 5.64.** *Assume $M > 7$. Then for each $k = 1, 2, \dots, N$,*

$$\big|u(Y_k) - |\nabla u(Y)|Y_k\cdot e_{n+1}\big| \lesssim k\varepsilon^{M-3}\ell(Q). \tag{5.65}$$

*Moreover,*

$$\big|u(W) - |\nabla u(Y)|W_{n+1}\big| \lesssim \varepsilon^{M-7}\ell(Q), \quad \forall W \in B_X,\ \forall X \in \widetilde{U}_Q^i. \tag{5.66}$$

*Proof of Claim 5.64.* By (5.59) and (5.62), we have

$$\begin{aligned}\big|u(W) - |\nabla u(Y)|W_{n+1}\big| &\lesssim |u(W) - \nabla u(Y)\cdot W| + \big|(\nabla u(Y) - |\nabla u(Y)|e_{n+1})\cdot W\big|\\ &\lesssim \varepsilon^{2M-5}\delta(Y) + \varepsilon^{M-3}|W| \lesssim \varepsilon^{M-3}\ell(Q), \quad \forall W \in B_Y,\end{aligned} \tag{5.67}$$

since $\delta(Z) \approx \ell(Q)$, for all $Z \in U_Q^i$ (so in particular, for $Z = Y$), and since $|W| \le 2\delta(Y) \lesssim \ell(Q)$, for all $W \in B_Y$. Thus, (5.65) holds with $k = 1$, since $Y_1 \in B_Y$, by construction. Now suppose that (5.65) holds for all $1 \le i \le k$, with $k \le N$. Let $W \in B_{Y_k}$, so that $W$ may be joined to $Y_k$ by a line segment of length less than $\delta(Y_k) \lesssim \varepsilon^{-3}\ell(Q)$ (the latter bound holds by (5.50)). We note also that if $k \le N-1$, and if $W = Y_{k+1}$, then this line segment has length at most $\ell(Q)$, by construction. Then

$$\begin{aligned}\big|u(W) - |\nabla u(Y)|W_{n+1}\big| &\le \big|u(W) - u(Y_k) + |\nabla u(Y)|(Y_k - W)\cdot e_{n+1}\big| + \big|u(Y_k) - |\nabla u(Y)|Y_k \cdot e_{n+1}\big| \\ &= \big|(W - Y_k)\cdot \nabla u(W_1) + |\nabla u(Y)|(Y_k - W)\cdot e_{n+1}\big| + O(k\varepsilon^{M-3}\ell(Q)),\end{aligned}$$

where $W_1$ is an appropriate point on the line segment joining $W$ and $Y_k$, and where we have used that $Y_k$ satisfies (5.65). By (5.63), applied to $W_1$, we find in turn that

$$\big|u(W) - |\nabla u(Y)|W_{n+1}\big| \lesssim \varepsilon^{M-3}|W - Y_k| + k\varepsilon^{M-3}\ell(Q), \tag{5.68}$$

which, by our previous observations, is bounded by $C(k+1)\varepsilon^{M-3}\ell(Q)$ if $W = Y_{k+1}$, or by $(\varepsilon^{M-6} + k\varepsilon^{M-3})\ell(Q)$ in general. In the former case, we find that (5.65) holds for all $k = 1, 2, \dots, N$, and in the latter case, taking $k = N \lesssim \varepsilon^{-4}$, we obtain (5.66). □

**Claim 5.69.** *Let $X \in \widetilde{U}^i_Q$, and let $w \in E \cap \partial\widetilde{B}_X$. Then*

$$|\nabla u(Y)|\,|w_{n+1}| \lesssim \varepsilon^{M/2}\ell(Q). \tag{5.70}$$

*Proof of Claim 5.69.* Let $W$ be the radial projection of $w$ onto $\partial B_X$, so that

$$|W - w| = \varepsilon^{2M/\alpha}\delta(X) \lesssim \varepsilon^{(2M/\alpha)-3}\ell(Q), \tag{5.71}$$

by (5.50). We write

$$|\nabla u(Y)|\,|w_{n+1}| \le |\nabla u(Y)|\,|W - w| + \big|u(W) - |\nabla u(Y)|W_{n+1}\big| + u(W) =: I + II + u(W).$$

Note that $I \lesssim \varepsilon^{(2M/\alpha)-3}\ell(Q)$ by (5.71) and (5.11) (recall that $Y = Y_Q$), and that $II \lesssim \varepsilon^{M-7}\ell(Q)$ by (5.66). Furthermore, $u(W) \lesssim \varepsilon^{2M-5}\ell(Q)$, by (5.49). For $M$ chosen large enough, we obtain (5.70). □

We note that since we have fixed $Y = Y_Q$, it then follows from (5.70) and (5.11) that

$$|w_{n+1}| \lesssim \varepsilon^{M/2}\ell(Q), \quad \forall w \in E \cap \partial\widetilde{B}_X,\ \forall X \in \widetilde{U}^i_Q. \tag{5.72}$$

Recall that $x_Q$ denotes the "center" of $Q$ (see (2.7)–(2.8)). Set

$$\mathcal{O} := B(x_Q, 2\varepsilon^{-2}\ell(Q)) \cap \{W : W_{n+1} > \varepsilon^2\ell(Q)\}. \tag{5.73}$$

**Claim 5.74.** *For every point $X \in \mathcal{O}$, we have $X \approx_{\varepsilon,Q} Y$ (see Definition 2.26). Thus, in particular, $\mathcal{O} \subset \widetilde{U}^i_Q$.*

*Proof of Claim 5.74.* Let $X \in \mathcal{O}$. We need to show that $X$ may be connected to $Y$ by a chain of at most $\varepsilon^{-1}$ balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3\ell(Q) \le \delta(Y_k) \le \varepsilon^{-3}\ell(Q)$ (for convenience, we shall refer to such balls as "admissible"). We first observe that if $X = te_{n+1}$, with $\varepsilon^3\ell(Q) \le t \le \varepsilon^{-3}\ell(Q)$, then by an iteration argument using (5.72) (with $M$ chosen large enough), we may join $X$ to $Y$ by at most $C\log(1/\varepsilon)$ admissible balls. The point $(2\varepsilon)^{-3}\ell(Q)e_{n+1}$ may then be joined to any point of the form $(X', (2\varepsilon)^{-3}\ell(Q))$ by a chain of at most $C$ admissible balls, whenever $X' \in \mathbb{R}^n$ with $|X'| \le \varepsilon^{-3}\ell(Q)$. In turn, the latter point may then be joined to $(X', \varepsilon^3\ell(Q))$ by at most $C\log(1/\varepsilon)$ admissible balls. □

We note that Claim 5.74 implies that

$$E \cap O = \varnothing. \tag{5.75}$$

Indeed, $O \subset \widetilde{U}_Q^i \subset \Omega$. Let $P_0$ denote the hyperplane

$$P_0 := \{Z : Z_{n+1} = 0\}.$$

**Claim 5.76.** *If $Z \in P_0$, with $|Z - x_Q| \le \frac{3}{2}\varepsilon^{-2}\ell(Q)$, then*

$$\delta(Z) = \operatorname{dist}(Z, E) \le 16\varepsilon^2\ell(Q). \tag{5.77}$$

*Proof of Claim 5.76.* Observe that $B(Z, 2\varepsilon^2\ell(Q))$ meets $O$. Then by Claim 5.74, there is a point $X \in \widetilde{U}_Q^i \cap B(Z, 2\varepsilon^2\ell(Q))$. Suppose that (5.77) is false, which in particular implies that $\delta(X) \ge 14\varepsilon^2\ell(Q)$. Then $B(Z, 4\varepsilon^2\ell(Q)) \subset B_X$, so by (5.66), we have

$$\big|u(W) - |\nabla u(Y)|W_{n+1}\big| \le C\varepsilon^{M-7}\ell(Q), \quad \forall W \in B(Z, 4\varepsilon^2\ell(Q)). \tag{5.78}$$

In particular, since $Z_{n+1} = 0$, we may choose $W$ such that $W_{n+1} = -\varepsilon^2\ell(Q)$, to obtain that

$$|\nabla u(Y)|\varepsilon^2\ell(Q) \le C\varepsilon^{M-7}\ell(Q),$$

since $u \ge 0$. But for $\varepsilon < \frac{1}{2}$, and $M$ large enough, this is a contradiction, by (5.11) (recall that we have fixed $Y = Y_Q$). □

It now follows by Definition 2.17 that $Q$ satisfies the $\varepsilon$-local WHSA condition, with

$$P = P(Q) := \{Z : Z_{n+1} = \varepsilon^2\ell(Q)\}, \qquad H = H(Q) := \{Z : Z_{n+1} > \varepsilon^2\ell(Q)\}.$$

This concludes the proof of Lemma 5.10.

**5D. *Proof of Corollary 1.5.*** Now Corollary 1.5 follows almost immediately from Theorem 1.1. Let $B = B(x, r)$ and $\Delta = B \cap \partial\Omega$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. Let $c$ be the constant in Lemma 3.1. By hypothesis, there is a point $X_\Delta \in B \cap \Omega$ which is a corkscrew point relative to $\Delta$, that is, there is a uniform constant $c_0 > 0$ such that $\delta(X_\Delta) \ge c_0 r$. Thus, to apply Theorem 1.1, it remains only to verify hypothesis ($\star$). For a sufficiently large constant $C_1$, set $\Delta^{\text{fat}} = \Delta(x, C_1 r)$. Cover $\Delta^{\text{fat}}$ by a collection of surface balls $\{\Delta_i\}_{i=1}^N$ with $\Delta_i = B_i \cap \partial\Omega$ and $B_i := B(x_i, c_0 r/4)$, where $x_i \in \Delta^{\text{fat}}$ and where $N$ is uniformly bounded, depending only on $n$, $c_0$, $C_1$, and ADR. By construction, $X_\Delta \in \Omega \setminus 4B_i$, so by hypothesis, $\omega^{X_\Delta} \in \text{weak-}A_\infty(2\Delta_i)$. Hence, $\omega^{X_\Delta} \ll \sigma$ in $2\Delta_i$, and (1.6) holds with $Y = X_\Delta$, and with $\Delta' = \Delta_i$. Consequently, $\omega^{X_\Delta} \ll \sigma$ in $\Delta^{\text{fat}}$, and if we write $k^{X_\Delta} = d\omega^{X_\Delta}/d\sigma$, we obtain

$$\begin{aligned}
\int_{\Delta^{\text{fat}}} k^{X_\Delta}(z)^q \, d\sigma(z) &\le \sum_{i=1}^N \int_{\Delta_i} k^{X_\Delta}(z)^q \, d\sigma(z) \lesssim \sum_{i=1}^N \sigma(\Delta_i)\bigg(\fint_{2\Delta_i} k^{X_\Delta}(z)\, d\sigma(z)\bigg)^q \\
&\lesssim \sum_{i=1}^N \sigma(2\Delta_i)^{1-q}\omega^{X_\Delta}(2\Delta_i) \lesssim \sigma(\Delta^{\text{fat}})^{1-q},
\end{aligned}$$

where in the last estimate we have used the ADR property, the uniform boundedness of $N$, and the fact that $\omega^{X_\Delta}(2\Delta_i) \le 1$. By Theorem 1.1, it then follows that $\partial\Omega$ is UR as desired. □

## 6. Proof of Proposition 1.17

Here we prove Proposition 1.17. We first observe that if $E$ is UR then it satisfies the so-called "bilateral weak geometric lemma" (BWGL); see [David and Semmes 1991, Theorem I.2.4, p. 32]. In turn, in [David and Semmes 1991, Section II.2.1, p. 97], one can find a dyadic formulation of the BWGL as follows. Given $\varepsilon$ small enough and $k > 1$ large to be chosen, $\mathbb{D}(E)$ can be split in two collections, one of "bad cubes" and another of "good cubes", so that the "bad cubes" satisfy a packing condition and each "good cube" $Q$ verifies the following: there is a hyperplane $P = P(Q)$ such that $\operatorname{dist}(Z, E) \le \varepsilon\ell(Q)$ for every $Z \in P \cap B(x_Q, k\ell(Q))$, and $\operatorname{dist}(Z, P) \le \varepsilon\ell(Q)$ for every $Z \in B(x_Q, k\ell(Q)) \cap E$. In turn, this implies that $B(x_Q, k\ell(Q)) \cap E$ is sandwiched between two planes parallel to $P$ at distance $\varepsilon\ell(Q)$. Hence, at that scale, we have a half-space (indeed we have two) free of $E$, and clearly the $2\varepsilon$-local WHSA holds provided $K$ is taken of the order of $\varepsilon^{-2}$ or larger. Further details are left to the interested reader. Thus we obtain the easy implication UR $\Longrightarrow$ WHSA.

The main part of the proof is to establish the opposite implication. To this end, we assume that $E$ satisfies the WHSA property and show that $E$ is UR. Given a positive $\varepsilon < \varepsilon_0 \ll K_0^{-6}$, we let $\mathcal{B}_0$ denote the collection of bad cubes for which $\varepsilon$-local WHSA fails. By Definition 2.19, $\mathcal{B}_0$ satisfies the Carleson packing condition (2.20). We now introduce a variant of the packing measure for $\mathcal{B}_0$. We recall that $B_Q^* = B(x_Q, K_0^2\ell(Q))$, and given $Q \in \mathbb{D}(E)$, we set

$$\mathbb{D}_\varepsilon(Q) := \{Q' \in \mathbb{D}(E) : \varepsilon^{3/2}\ell(Q) \le \ell(Q') \le \ell(Q),\ \ Q' \text{ meets } B_Q^*\}. \tag{6.1}$$

Set

$$\alpha_Q := \begin{cases} \sigma(Q) & \text{if } \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases} \tag{6.2}$$

and define

$$\mathfrak{m}(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \alpha_Q, \qquad \mathbb{D}' \subset \mathbb{D}(E). \tag{6.3}$$

Then $\mathfrak{m}$ is a discrete Carleson measure, with

$$\mathfrak{m}(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0} \alpha_Q \le C_\varepsilon \sigma(Q_0), \qquad Q_0 \in \mathbb{D}(E). \tag{6.4}$$

Indeed, note that for any $Q'$, the cardinality of $\{Q : Q' \in \mathbb{D}_\varepsilon(Q)\}$ is uniformly bounded, depending on $n$, $\varepsilon$, and ADR, and that $\sigma(Q) \le C_\varepsilon \sigma(Q')$ if $Q' \in \mathbb{D}_\varepsilon(Q)$. Then given any $Q_0 \in \mathbb{D}(E)$,

$$\begin{aligned} \mathfrak{m}(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0 : \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) \neq \varnothing} \sigma(Q) &\le \sum_{Q' \in \mathcal{B}_0} \ \sum_{Q \subset Q_0 : Q' \in \mathbb{D}_\varepsilon(Q)} \sigma(Q) \\ &\le C_\varepsilon \sum_{Q' \in \mathcal{B}_0 :\, Q' \subset 2B_{Q_0}^*} \sigma(Q') \le C_\varepsilon \sigma(Q_0), \end{aligned}$$

by (2.20) and ADR.

To prove Proposition 1.17, we are required to show that the collection $\mathcal{B}$ of bad cubes for which the $\sqrt{\varepsilon}$-local BAUP condition fails satisfies a packing condition. That is, we establish the discrete Carleson measure estimate

$$\widetilde{\mathfrak{m}}(\mathbb{D}_{Q_0}) = \sum_{Q\subset Q_0: Q\in\mathcal{B}} \sigma(Q) \le C_\varepsilon \sigma(Q_0), \qquad Q_0\in\mathbb{D}(E). \tag{6.5}$$

To this end, by (6.4), it suffices to show that if $Q\in\mathcal{B}$, then $\alpha_Q\neq 0$ (and thus $\alpha_Q=\sigma(Q)$, by definition). In fact, we prove the contrapositive statement.

**Claim 6.6.** *Suppose that $\alpha_Q=0$. Then the $\sqrt{\varepsilon}$-local BAUP condition holds for $Q$.*

*Proof of Claim 6.6.* We first note that since $\alpha_Q=0$, then by definition of $\alpha_Q$,

$$\mathcal{B}_0\cap\mathbb{D}_\varepsilon(Q)=\varnothing. \tag{6.7}$$

Thus, the $\varepsilon$-local WHSA condition (Definition 2.17) holds for every $Q'\in\mathbb{D}_\varepsilon(Q)$ (in particular, for $Q$ itself). By rotation and translation, we may suppose that the hyperplane $P=P(Q)$ in Definition 2.17 is

$$P=\{Z\in\mathbb{R}^{n+1}: Z_{n+1}=0\},$$

and that the half-space $H=H(Q)$ is the upper half-space $\mathbb{R}^{n+1}_+=\{Z:Z_{n+1}>0\}$. We recall that by Definition 2.17, $P$ and $H$ satisfy

$$\operatorname{dist}(Z,E)\le\varepsilon\ell(Q), \qquad \forall Z\in P\cap B_Q^{**}(\varepsilon), \tag{6.8}$$

$$\operatorname{dist}(P,Q)\le K_0^{3/2}\ell(Q), \tag{6.9}$$

and

$$H\cap B_Q^{**}(\varepsilon)\cap E=\varnothing. \tag{6.10}$$

The proof now follows by a construction similar to that in [Lewis and Vogel 2007], used to establish the weak exterior convexity condition. By (6.10), there are two cases.

**Case 1**: $10Q\subset\{Z:-\sqrt{\varepsilon}\ell(Q)\le Z_{n+1}\le 0\}$. In this case, the $\sqrt{\varepsilon}$-local BAUP condition holds trivially for $Q$, with $\mathcal{P}=\{P\}$.

**Case 2**: There is a point $x\in 10Q$ such that $x_{n+1}<-\sqrt{\varepsilon}\ell(Q)$. In this case, we choose $Q'\ni x$ with $\varepsilon^{3/4}\ell(Q)\le\ell(Q')<2\varepsilon^{3/4}\ell(Q)$. Thus,

$$Q'\subset\left\{Z:Z_{n+1}\le-\tfrac{1}{2}\sqrt{\varepsilon}\ell(Q)\right\}. \tag{6.11}$$

Moreover, $Q'\in\mathbb{D}_\varepsilon(Q)$, so by (6.7), $Q'\notin\mathcal{B}_0$, i.e., $Q'$ satisfies the $\varepsilon$-local WHSA. Let $P'=P(Q')$ and $H'=H(Q')$ denote the hyperplane and half-space corresponding to $Q'$ in Definition 2.17, so that

$$\operatorname{dist}(Z,E)\le\varepsilon\ell(Q')\le 2\varepsilon^{7/4}\ell(Q), \qquad \forall Z\in P'\cap B_{Q'}^{**}(\varepsilon), \tag{6.12}$$

$$\operatorname{dist}(P',Q')\le K_0^{3/2}\ell(Q')\approx K_0^{3/2}\varepsilon^{3/4}\ell(Q)\ll\varepsilon^{1/2}\ell(Q) \tag{6.13}$$

(where the last inequality holds since $\varepsilon\ll K_0^{-6}$), and

$$H'\cap B_{Q'}^{**}(\varepsilon)\cap E=\varnothing, \tag{6.14}$$

where we recall that $B^{**}_{Q'}(\varepsilon) := B(x_{Q'}, \varepsilon^{-2}\ell(Q'))$ (see (2.16)). We note that

$$B^*_Q \subset \widetilde{B}_Q(\varepsilon) := B(x_Q, \varepsilon^{-1}\ell(Q)) \subset B^{**}_{Q'}(\varepsilon) \cap B^{**}_Q(\varepsilon), \tag{6.15}$$

by construction, since $\varepsilon \ll K_0^{-6}$. Let $\nu'$ denote the unit normal vector to $P'$, pointing into $H'$. Note that by (6.10), (6.12), and the definition of $H$,

$$P' \cap \widetilde{B}_Q(\varepsilon) \cap \{Z : Z_{n+1} > 2\varepsilon^{7/4}\ell(Q)\} = \varnothing. \tag{6.16}$$

Moreover, $\nu'$ points "downward", i.e., $\nu' \cdot e_{n+1} < 0$, as otherwise, $H' \cap \widetilde{B}_Q(\varepsilon)$ would meet $E$ by (6.8), (6.11), and (6.13). More precisely, we have the following.

**Claim 6.17.** *The angle $\theta$ between $\nu'$ and $-e_{n+1}$ satisfies $0 \le \theta \approx \sin\theta \lesssim \varepsilon$.*

Indeed, since $Q'$ meets $10Q$, (6.9) and (6.13) imply that $\operatorname{dist}(P, P') \lesssim K_0^{3/2}\ell(Q)$, and that the latter estimate is attained near $Q$. By (6.16) and a trigonometric argument, one then obtains Claim 6.17 (more precisely, one obtains $\theta \lesssim K_0^{3/2}\varepsilon$, but in this section, we continue to use the notational convention that implicit constants may depend upon $K_0$, but $K_0$ is fixed, and $\varepsilon \ll K_0^{-6}$). The interested reader could probably supply the remaining details of the argument that we have just sketched, but for the sake of completeness, we give the full proof at the end of this section.

We therefore take Claim 6.17 for granted, and proceed with the argument. We note first that every point in $(P \cup P') \cap B^*_Q$ is at a distance at most $\varepsilon\ell(Q)$ from $E$ by (6.8), (6.12), and (6.15). To complete the proof of Claim 6.6, it therefore remains only to verify the following. As with the previous claim, we provide a condensed proof immediately, and present a more detailed argument at the end of the section.

**Claim 6.18.** *Every point in $10Q$ lies within $\sqrt{\varepsilon}\ell(Q)$ of a point in $P \cup P'$.*

Suppose not. We could then repeat the previous argument, to construct a cube $Q''$, a hyperplane $P''$, a unit vector $\nu''$ forming a small angle with $-e_{n+1}$, and a half-space $H''$ with boundary $P''$, with the same properties as $Q'$, $P'$, $\nu'$, and $H'$. In particular, we have the respective analogues of (6.13) and (6.14), namely

$$\operatorname{dist}(P'', Q'') \le K_0^{3/2}\ell(Q') \approx K_0^{3/2}\varepsilon^{3/4}\ell(Q) \ll \varepsilon^{1/2}\ell(Q) \tag{6.19}$$

and

$$H'' \cap B^{**}_{Q''}(\varepsilon) \cap E = \varnothing, \tag{6.20}$$

Also, we have the analogue of (6.11), with $Q''$, $P'$ in place of $Q'$, $P$. Thus

$$\operatorname{dist}(Q'', P') \ge \tfrac{1}{2}\sqrt{\varepsilon}\ell(Q) \quad \text{and} \quad Q'' \cap H' = \varnothing. \tag{6.21}$$

In addition, as in (6.15), we also have $B^*_Q \subset B^{**}_{Q''}(\varepsilon)$. On the other hand, the angle between $\nu'$ and $\nu''$ is very small. Thus, combining (6.12), (6.19), and (6.21), we see that $H'' \cap B^*_Q$ captures points in $E$, which contradicts (6.20).

Claim 6.6 therefore holds (in fact, with a union of at most 2 planes), and thus we obtain the conclusion of Proposition 1.17. □

We now provide detailed proofs of Claims 6.17 and 6.18.

*Proof of Claim 6.17.* By (6.13) we can pick $x' \in Q'$, $y' \in P'$ such that $|y' - x'| \ll \varepsilon^{1/2}\ell(Q)$, and therefore $y' \in 11Q$. Also, from (6.9) and (6.10) we can find $\bar{x} \in Q$ such that $-K_0^{3/2}\ell(Q) < \bar{x}_{n+1} \le 0$. This and (6.11) yield

$$-2K_0^{3/2}\ell(Q) < y'_{n+1} < -\tfrac{1}{4}\sqrt{\varepsilon}\ell(Q). \tag{6.22}$$

Let $\pi$ be the orthogonal projection onto $P$. Let $Z \in P$ (i.e., $Z_{n+1} = 0$) be such that $|Z - \pi(y')| \le K_0^{3/2}\ell(Q)$. Then $Z \in B(x_Q, 4K_0^{3/2}\ell(Q)) \subset B_Q^*$. Hence $Z \in P \cap B_Q^{**}(\varepsilon)$ and by (6.8), $\operatorname{dist}(Z, E) \le \varepsilon\ell(Q)$. Then there exists $x_Z \in E$ with $|Z - x_Z| \le \varepsilon\ell(Q)$, which in turn implies that $|(x_Z)_{n+1}| \le \varepsilon\ell(Q)$. Note that $x_Z \in B(x_Q, 5K_0^{3/2}\ell(Q)) \subset B_Q^*$ and by (6.15), $x_Z \in E \cap B_Q^{**}(\varepsilon) \cap B_{Q'}^{**}(\varepsilon)$. This, (6.10), and (6.14) imply that $x_Z \notin H \cup H'$. Hence, $(x_Z)_{n+1} \le 0$ and $(x_Z - y') \cdot \nu' \le 0$, since $y' \in P'$ and $\nu'$ denote the unit normal vector to $P'$ pointing into $H'$. Using (6.22) we observe that

$$\tfrac{1}{8}\sqrt{\varepsilon}\ell(Q) < -\varepsilon\ell(Q) + \tfrac{1}{4}\sqrt{\varepsilon}\ell(Q) < (x_Z - y')_{n+1} < 2K_0^{3/2}\ell(Q), \tag{6.23}$$

and that

$$\begin{aligned}(x_Z - y')_{n+1}\nu'_{n+1} &\le -\pi(x_Z - y') \cdot \pi(\nu') \\ &\le |x_Z - z| - \pi(Z - y') \cdot \pi(\nu') \le \varepsilon\ell(Q) - \pi(Z - y') \cdot \pi(\nu').\end{aligned} \tag{6.24}$$

We prove that $\nu'_{n+1} < -\frac{1}{8} < 0$ by considering two cases.

**Case 1**: $|\pi(\nu')| \ge \frac{1}{2}$. We pick

$$Z_1 = \pi(y') + K_0^{3/2}\ell(Q)\frac{\pi(\nu')}{|\pi(\nu')|}.$$

By construction, $Z_1 \in P$ and $|Z_1 - \pi(y')| \le K_0^{3/2}\ell(Q)$. Hence, we can use (6.24) with $Z_1$:

$$\begin{aligned}(x_{Z_1} - y')_{n+1}\,\nu'_{n+1} &\le \varepsilon\ell(Q) - \pi(Z_1 - y') \cdot \pi(\nu') \\ &= \varepsilon\ell(Q) - K_0^{3/2}\ell(Q)\,|\pi(\nu')| \le -\tfrac{1}{4}K_0^{3/2}\ell(Q).\end{aligned}$$

This together with (6.23) give that $\nu'_{n+1} < -\frac{1}{8} < 0$.

**Case 2**: $|\pi(\nu')| < \frac{1}{2}$. This case is much simpler. Note first that $|\nu'_{n+1}|^2 = 1 - |\pi(\nu')|^2 > \frac{3}{4}$, and thus either $\nu'_{n+1} < -\frac{1}{2}\sqrt{3}$ or $\nu'_{n+1} > \frac{1}{2}\sqrt{3}$. We see that the second scenario leads to a contradiction. Assume then that $\nu'_{n+1} > \frac{1}{2}\sqrt{3}$. We take $Z_2 = \pi(y') \in P$, which clearly satisfies $|Z_2 - \pi(y')| \le K_0^{3/2}\ell(Q)$. Again (6.24) and (6.23) are applicable with $Z_2$:

$$\frac{1}{8}\sqrt{\varepsilon}\ell(Q)\,\frac{\sqrt{3}}{2} < (x_{Z_2} - y')_{n+1}\nu'_{n+1} \le \varepsilon\ell(Q) \ll \sqrt{\varepsilon}\,\ell(Q),$$

and we get a contradiction. Hence necessarily $\nu'_{n+1} \le -\frac{1}{2}\sqrt{3} < -\frac{1}{8} < 0$.

Having proved that $\nu'_{n+1} < -\frac{1}{8} < 0$, we estimate $\theta$, the angle between $\nu'$ and $-e_{n+1}$. Note first $\cos\theta = -\nu'_{n+1} > \frac{1}{8}$. If $\cos\theta = 1$ (which occurs if $\nu' = -e_{n+1}$), then $\theta = \sin\theta = 0$ and the proof is complete. Assume then that $\cos\theta \ne 1$, in which case $\frac{1}{8} < -\nu'_{n+1} < 1$ and hence $|\pi(\nu')| \ne 0$. Pick

$$Z_3 = y' + \frac{\ell(Q)}{2\varepsilon}\,\hat{\nu}', \qquad \hat{\nu}' = \frac{e_{n+1} - \nu'_{n+1}\nu'}{|\pi(\nu')|}.$$

Then $\hat{\nu}' \cdot \nu' = 0$ and hence $Z_3 \in P'$ as $y' \in P'$. Also, $|\hat{\nu}'| = 1$ and therefore $|Z_3 - y'| = \ell(Q)/(2\varepsilon)$. This in turn gives that $Z_3 \in \widetilde{B}_Q(\varepsilon)$. We have obtained that $Z_3 \in P' \cap \widetilde{B}_Q(\varepsilon)$, and hence $(Z_3)_{n+1} \le 2\varepsilon^{7/4}\ell(Q)$ by (6.16). This and (6.23) applied to $Z_3$ easily give

$$\begin{aligned} 4K_0^{3/2}\ell(Q) \ge 2\varepsilon^{7/4}\ell(Q) \ge (Z_3)_{n+1} &= y'_{n+1} + \frac{\ell(Q)}{2\varepsilon}\frac{1-(\nu'_{n+1})^2}{|\pi(\nu')|} \\ &= y'_{n+1} + \frac{\ell(Q)}{2\varepsilon}|\pi(\nu')| \ge -2\,K_0^{3/2}\ell(Q) + \frac{\ell(Q)}{2\varepsilon}|\pi(\nu')|. \end{aligned}$$

This readily yields $|\sin\theta| = |\pi(\nu')| \le 8K_0^{3/2}\varepsilon$, and the proof is complete. □

*Proof of Claim 6.18.* We want to prove that every point in $10Q$ lies within $\sqrt{\varepsilon}\ell(Q)$ of a point in $P \cup P'$. We argue by contradiction and hence we assume that there exists $x' \in 10Q$ with $\operatorname{dist}(x', P \cup P') > \sqrt{\varepsilon}\ell(Q)$. In particular, $x'_{n+1} < -\sqrt{\varepsilon}\ell(Q)$, and as observed above, we may repeat the previous argument to construct a cube $Q''$, a hyperplane $P''$, a unit vector $\nu''$ forming a small angle with $-e_{n+1}$, and a half-space $H''$ with boundary $P''$, with the same properties as $Q'$, $P'$, $\nu'$, and $H'$, namely (6.19), (6.21), and (6.20). Also,

$$\sqrt{\varepsilon}\ell(Q) \le \operatorname{dist}(x', P') \le \operatorname{diam}(Q'') + \operatorname{dist}(Q'', P') \le \tfrac{1}{2}\sqrt{\varepsilon}\,\ell(Q) + \operatorname{dist}(Q'', P'),$$

and, in addition, as in (6.15), we have $B_Q^* \subset B_{Q''}^{**}(\varepsilon)$.

By (6.19) there is $y'' \in Q''$ and $z'' \in P''$ such that $|y'' - z''| \ll \varepsilon^{1/2}\ell(Q)$. By (6.20) $y'' \notin H'$. Write $\pi'$ to denote the orthogonal projection onto $P'$ and note that (6.21) gives $\operatorname{dist}(y'', P') = |y'' - \pi'(y'')| \ge \frac{1}{2}\sqrt{\varepsilon}\,\ell(Q)$. Note also that

$$\begin{aligned} |y'' - \pi'(y'')| &= \operatorname{dist}(y'', P') \\ &\le |y'' - x'| + |x' - x| + \operatorname{diam}(Q') + \operatorname{dist}(Q', P') \le 11\operatorname{diam}(Q) \end{aligned}$$

and that

$$|\pi'(y'') - x_Q| \le |\pi'(y'') - y''| + |y'' - x'| + |x' - x_Q| < 22\operatorname{diam}(Q) < K_0^2\ell(Q).$$

Hence $\pi'(y'') \in B_Q^* \subset \widetilde{B}_Q(\varepsilon)$, and since $\pi'(y'') \in P'$, (6.12) gives $\tilde{y} \in E$ with $|\pi'(y'') - \tilde{y}| \le 2\varepsilon^{7/4}\ell(Q)$. Then $\tilde{y} \in 23Q \subset B_Q^* \cap E$ and $|\tilde{y} - z''| < 12\operatorname{diam}(Q)$. To complete our proof we just need to show that $\tilde{y} \in H''$, which contradicts (6.20).

Write $\nu''$ to denote the unit normal vector to $P''$ pointing into $H''$, and let us momentarily assume that

$$|\nu' - \nu''| \le 16\sqrt{2}\,K_0^{2/3}\varepsilon. \tag{6.25}$$

Recalling that $y'' \notin H'$, we then obtain that

$$\begin{aligned} \tfrac{1}{2}\sqrt{\varepsilon}\,\ell(Q) \le |y'' - \pi'(y'')| &= (\pi'(y'') - y'')\cdot\nu' \\ &\le |\pi'(y'') - \tilde{y}| + |\tilde{y} - z''|\,|\nu' - \nu''| + (\tilde{y} - z'')\cdot\nu'' + |z'' - y''| \\ &< \tfrac{1}{4}\sqrt{\varepsilon}\,\ell(Q) + (\tilde{y} - z'')\cdot\nu''. \end{aligned}$$

This immediately gives that $(\tilde{y}-z'')\cdot\nu'' > \frac{1}{4}\sqrt{\varepsilon}\,\ell(Q) > 0$, and hence $\tilde{y}\in H''$ as desired. Thus, to complete the proof we have to prove (6.25). We first note that if $|\alpha| < \frac{\pi}{4}$, then

$$1-\cos\alpha = 1-\sqrt{1-\sin^2\alpha} \le \sin^2\alpha.$$

In particular, we can apply this to $\theta$ (resp. $\theta'$), which is the angle between $\nu'$ (resp. $\nu''$) and $-e_{n+1}$, and as we showed that $|\sin\theta|$, $|\sin\theta'| \le 8K_0^{3/2}\varepsilon$, we see that

$$\sqrt{1-\cos\theta}+\sqrt{1-\cos\theta'} \le 16K_0^{3/2}\varepsilon.$$

Using the trivial formula

$$|a-b|^2 = 2(1-a\dot{b}), \qquad \forall a,b\in\mathbb{R}^{n+1},\ |a|=|b|=1,$$

we conclude that

$$\begin{aligned}|\nu'-\nu''| &\le |\nu'-(-e_{n+1})|+|(-e_{n+1})-\nu''|\\ &= \sqrt{2(1+\nu' e_{n+1})}+\sqrt{2(1+\nu'' e_{n+1})}\\ &= \sqrt{2(1-\cos\theta)}+\sqrt{2(1-\cos\theta')} \le 16\sqrt{2}K_0^{3/2}\varepsilon.\end{aligned}$$

This proves (6.25), and hence the proof of Claim 6.18 is complete. □

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# THE ONE-PHASE PROBLEM FOR HARMONIC MEASURE IN TWO-SIDED NTA DOMAINS

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We show that if $\Omega \subset \mathbb{R}^3$ is a two-sided NTA domain with AD-regular boundary such that the logarithm of the Poisson kernel belongs to $\mathrm{VMO}(\sigma)$, where $\sigma$ is the surface measure of $\Omega$, then the outer unit normal to $\partial\Omega$ belongs to $\mathrm{VMO}(\sigma)$ too. The analogous result fails for dimensions larger than 3. This answers a question posed by Kenig and Toro and also by Bortz and Hofmann.

## 1. Introduction

In this paper we study a one-phase free boundary problem in connection with the Poisson kernel. The study of this type of problems was initiated in the pioneering work [Alt and Caffarelli 1981], where they showed that for a Reifenberg flat domain with $n$-AD-regular boundary in $\mathbb{R}^{n+1}$, if the logarithm of the Poisson kernel is in $C^\alpha$ for some $\alpha > 0$, then the domain is of class $C^{1,\beta}$ for some $\beta > 0$. Later on, Jerison [1990] showed that, in fact, one can take $\beta = \alpha$. Kenig and Toro [1997; 1999; 2003] considered the endpoint case of the logarithm of the Poisson kernel being in VMO, and they obtained the following remarkable result:

**Theorem A** [Kenig and Toro 2003]. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a $\delta$-Reifenberg flat chord-arc domain for some $\delta > 0$ small enough. Denote by $\sigma$ the surface measure of $\Omega$ and by $h$ the Poisson kernel with a pole in $\Omega$ if $\Omega$ is bounded or with the pole at infinity if $\Omega$ is unbounded. Then* $\log h \in \mathrm{VMO}(\sigma)$ *if and only if the outer unit normal $\vec{n}$ to $\partial\Omega$ is in* $\mathrm{VMO}(\sigma)$.

A domain $\Omega \subset \mathbb{R}^{n+1}$ is called *chord-arc* if it is an NTA domain with $n$-AD-regular boundary. Its Poisson kernel with pole at $p \in \Omega$ equals $h = d\omega^p/d\sigma$, where $\omega^p$ stands for the harmonic measure of $\Omega$ with pole at $p$. For the definitions of Reifenberg flatness, NTA, and VMO, we refer the reader to Section 2.

We also remark that, in fact, Kenig and Toro [2003] proved a slightly weaker statement than the one in Theorem A. Indeed, instead of showing that when $\log h \in \mathrm{VMO}(\sigma)$, the outer unit normal $\vec{n}$ to $\partial\Omega$ is in $\mathrm{VMO}(\sigma)$, they proved that $\vec{n}$ belongs to $\mathrm{VMO}_{\mathrm{loc}}(\sigma)$ (which coincides with $\mathrm{VMO}(\sigma)$ when $\Omega$ is bounded). However, as we explain in Remark 9.1, a minor modification of their arguments in [Kenig and Toro 2003] proves the full statement above in Theorem A.

Without the Reifenberg flatness assumption and just assuming the NTA condition, the conclusion of the theorem above need not hold: Kenig and Toro [1999, Proposition 3.1] showed that for the Kowalski–Preiss

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cone $\Omega = \{(x, y, z, w) : x^2 + y^2 + z^2 > w^2\} \subset \mathbb{R}^4$, the harmonic measure with pole at infinity coincides with the surface measure modulo a constant factor, and thus has $\log h \in \mathrm{VMO}(\sigma)$, even though the outer unit normal is not in $\mathrm{VMO}(\sigma)$. In fact, a similar conical example in $\mathbb{R}^3$ was shown previously by Alt and Caffarelli [1981, Section 2.7].

It was conjectured by Kenig and Toro [2006] and Bortz and Hofmann [2016] that, instead of the Reifenberg flatness assumption, being a two-sided chord-arc domain should be enough for the implication $\log h \in \mathrm{VMO}(\sigma) \implies \vec{n} \in \mathrm{VMO}(\sigma)$. By a two-sided chord-arc domain we mean a chord-arc domain such that its exterior is also connected and chord-arc. Kenig and Toro showed that this holds under the additional assumption that the logarithm of the Poisson kernel of the exterior domain is also in $\mathrm{VMO}(\sigma)$. Their precise result reads as follows:

**Theorem B** [Kenig and Toro 2006, Corollary 5.2]. *Let $\Omega$ be a two-sided chord-arc domain in $\mathbb{R}^{n+1}$. Assume further that* $\log(d\omega/d\sigma), \log(d\omega_{\mathrm{ext}}/d\sigma) \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$. *Then* $\vec{n} \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$.

Bortz and Hofmann [2016] showed that this same result holds under the assumption that $\partial\Omega$ is uniformly $n$-rectifiable, so that the measure theoretic boundary has full surface measure, instead of the two-sided chord-arc condition above. The boundary of any two-sided chord-arc domain is always uniformly $n$-rectifiable by results due to David and Jerison [1990], and thus this is more general than Theorem B. We also note that, by Proposition 4.10 in [Hofmann et al. 2010], such domains with $\vec{n} \in \mathrm{VMO}_{\mathrm{loc}}(\sigma)$ are also vanishing Reifenberg flat. It is also worth mentioning that the arguments in [Bortz and Hofmann 2016] are very different from the ones in [Kenig and Toro 2006]: while the latter uses blow-up techniques, the former relies on the relationship between the Riesz transform and harmonic measure and exploit the jump relations for the gradient of the single layer potential.

In this paper we resolve the conjecture mentioned above:

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^3$ be a two-sided chord-arc domain. Denote by $\sigma$ the surface measure of $\Omega$ and by $h$ the Poisson kernel with a pole in $\Omega$ if $\Omega$ is bounded or with the pole at infinity if $\Omega$ is unbounded. If* $\log h \in \mathrm{VMO}(\sigma)$, *then the outer unit normal of $\Omega$ also belongs to* $\mathrm{VMO}(\sigma)$.

*On the other hand, for $d \geq 4$, there are two-sided chord-arc domains $\Omega \subset \mathbb{R}^d$ satisfying $h \equiv 1$ and such that the outer unit normal of $\Omega$ does not belong to* $\mathrm{VMO}(\sigma)$.

Most of the paper is devoted to proving the positive result stated in the theorem for $\mathbb{R}^3$. Our arguments use the powerful blow-up techniques developed by Kenig and Toro [2003]. Indeed, by arguments analogous to the ones of Kenig and Toro, we reduce our problem to the study of the case when $\Omega_\infty$ is an unbounded two-sided chord-arc domain such that its Poisson kernel with pole at infinity is constantly equal to 1. By combining a monotonicity formula due to Weiss [1998] and some topological arguments inspired by a work from Caffarelli, Jerison and Kenig [Caffarelli et al. 2004], we then show that for such domains all blow-downs are flat. This is probably one of the main novelties in our paper. Then an application of a variant of a well-known theorem of Alt and Caffarelli [1981] shows that $\Omega_\infty$ must be a half-space.

The aforementioned reduction of the problem to the case when the Poisson kernel with pole at infinity is constantly equal to 1 requires estimating from above the gradient of the Green function. This estimate is obtained in [Kenig and Toro 2003] under the assumption that the domain $\Omega$ is Reifenberg flat, and this is one of the main technical difficulties of that paper. In [Kenig and Toro 2006] it is shown how these estimates can be extended to the case when $\Omega$ is not Reifenberg flat. In our present paper we provide some alternative arguments to estimate the gradient of the Green function. The main difference with respect to the ones in [Kenig and Toro 2003; 2006] is that in the present paper we use the jump relations for the gradient of single layer potentials, instead of the (perhaps) less standard approach in the aforementioned works. We think that our approach has some independent interest (especially because of the connection between harmonic measure with pole at infinity and the Riesz transform that we describe in Section 3).

Concerning the negative result for dimensions $d \geq 4$ in Theorem 1.1, basically we recall in the last section of the paper an example of a conical domain in $\mathbb{R}^4$ by Guanghao Hong[1] [2015] such that the harmonic measure with pole at infinity coincides with surface measure, and so that the outer unit normal does not belong to VMO($\sigma$). One can check easily that such domain is two-sided NTA. Probably, this example was unnoticed in some recent works in this area.

## 2. Preliminaries

For $a, b \geq 0$, we will write $a \lesssim b$ if there is $C > 0$ so that $a \leq Cb$ and $a \lesssim_t b$ if the constant $C$ depends on the parameter $t$. We write $a \approx b$ to mean $a \lesssim b \lesssim a$ and define $a \approx_t b$ similarly.

**Definition 2.1.** Given a closed set $E$, $x \in \mathbb{R}^d$, $r > 0$, and $P$ a $d$-plane, we set

$$\Theta_E(x, r, P) = r^{-1} \max\big\{\sup_{y \in E \cap B(x,r)} \operatorname{dist}(y, P),\ \sup_{y \in P \cap B(x,r)} \operatorname{dist}(y, E)\big\}.$$

Also define

$$\Theta_E(x, r) = \inf_P \Theta_E(x, r, P),$$

where the infimum is over all $d$-planes $P$. A set $E$ is *$\delta$-Reifenberg flat* if $\Theta_E(x, r) < \delta$ for all $x \in E$ and $r > 0$, and is *vanishing Reifenberg flat* if

$$\lim_{r \to 0} \sup_{x \in E} \Theta_E(x, r) = 0.$$

**Definition 2.2.** Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and let $0 < \delta < \frac{1}{2}$. We say that $\Omega$ is a $\delta$-Reifenberg flat domain if it satisfies the following conditions:

(a) $\partial\Omega$ is $\delta$-Reifenberg flat.

(b) For every $x \in \partial\Omega$ and $r > 0$, denote by $\mathcal{P}(x, r)$ an $n$-plane that minimizes $\Theta_E(x, r)$. Then one of the connected components of

$$B(x, r) \cap \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, \mathcal{P}(x, r)) \geq 2\delta r\}$$

is contained in $\Omega$ and the other is contained in $\mathbb{R}^{n+1} \setminus \Omega$.

If, additionally, $\partial\Omega$ is vanishing Reifenberg flat, then $\Omega$ is said to be vanishing Reifenberg flat, too.

**Definition 2.3.** Let $\Omega \subset \mathbb{R}^{n+1}$. We say that $\Omega$ satisfies the *Harnack chain condition* if there is a uniform constant $C$ such that for every $\rho > 0$, $\Lambda \geq 1$, and every pair of points $x, y \in \Omega$ with $\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \partial\Omega) \geq \rho$ and $|x - y| < \Lambda\rho$, there is a chain of open balls $B_1, \dots, B_N \subset \Omega$, $N \leq C(\Lambda)$, with $x \in B_1$, $y \in B_N$,

[1] So the statement in the theorem referring to the case $d \geq 4$ should not be attributed to us.

$B_k \cap B_{k+1} \neq \varnothing$ and $C^{-1}\operatorname{diam}(B_k) \le \operatorname{dist}(B_k, \partial\Omega) \le C \operatorname{diam}(B_k)$. The chain of balls is called a *Harnack chain*. Note that if such a chain exists, then

$$u(x) \approx_N u(y).$$

For $C \ge 2$, the set $\Omega$ is a *$C$-corkscrew domain* if for all $\xi \in \partial\Omega$ and $r > 0$ there are two balls of radius $r/C$ contained in $B(\xi, r) \cap \Omega$ and $B(\xi, r)\backslash\Omega$ respectively. If

$$B(x, r/C) \subseteq B(\xi, r) \cap \Omega,$$

we call $x$ a *corkscrew point* for the ball $B(\xi, r)$. Finally, we say $\Omega$ is *$C$-nontangentially accessible (or $C$-NTA, or just NTA)* if it satisfies the Harnack chain condition and is a $C$-corkscrew domain. We say $\Omega$ is *two-sided $C$-NTA* if both $\Omega$ and $\Omega_{\text{ext}} := (\overline{\Omega})^c$ are $C$-NTA. Finally, it is *chord-arc* if, additionally, $\partial\Omega$ is $n$-AD-regular, meaning there is $C > 0$ so that, if $\sigma$ denotes surface measure, then

$$C^{-1} r^n < \sigma(B(x, r)) < C r^n \quad \text{for all } x \in \partial\Omega,\ 0 < r \le \operatorname{diam}(\Omega).$$

Any measure $\sigma$ that satisfies the preceding estimate for all $x \in \operatorname{supp}\sigma$ and $0 < r \le \operatorname{diam}(\operatorname{supp}\sigma)$ is called $n$-AD-regular.

**Definition 2.4.** Let $\sigma$ be an n-AD-regular measure in $\mathbb{R}^n$ and $f$ a locally integrable function with respect to $\sigma$. We say $f \in \mathrm{VMO}(\sigma)$ if

$$\lim_{r\to 0} \sup_{x\in\operatorname{supp}\sigma} ⨍_{B(x,r)} \left| f - ⨍_{B(x,r)} f\, d\sigma \right|^2 d\sigma = 0. \tag{2-1}$$

We say $f \in \mathrm{VMO}_{\text{loc}}(\sigma)$ if, for any compact set $K$,

$$\lim_{r\to 0} \sup_{x\in\operatorname{supp}\sigma\cap K} ⨍_{B(x,r)} \left| f - ⨍_{B(x,r)} f\, d\sigma \right|^2 d\sigma = 0.$$

It is well known that the space VMO coincides with the closure of the set of bounded uniformly continuous functions on $\operatorname{supp}\sigma$ in the BMO norm.

We also remark that one can find slightly different definitions of VMO in the literature. For example, in some references besides (2-1) the additional condition that

$$\lim_{r\to \infty} \sup_{x\in\operatorname{supp}\sigma} ⨍_{B(x,r)} \left| f - ⨍_{B(x,r)} f\, d\sigma \right|^2 d\sigma = 0$$

is required. In this case, it turns out that VMO coincides with the closure of the set of *compactly supported* continuous functions on $\operatorname{supp}\sigma$ in the BMO norm. However, the definition we will use in our paper is Definition 2.4 (as in other works like [Kenig and Toro 1999; 2003]).

## 3. The Riesz transform of the harmonic measure with pole at infinity

Readers that are familiar with the results in [Kenig and Toro 2003; 2006] may skip this section, as well as Sections 4 and 5, and go directly to Section 6 without much harm. In fact, in Sections 3–5 we provide the alternative arguments to estimate the gradient of the Green function that we mentioned in the Introduction. Our approach uses the jump relations for the gradient of the single layer potential (derived by Hofmann, Mitrea, and Taylor [Hofmann et al. 2010] in the case of chord-arc domains and somewhat more general settings). Modulo these standard relations, our arguments are reasonably self-contained and shorter than the ones in [Kenig and Toro 2003; 2006].

Recall from [Kenig and Toro 1999, Lemma 3.7] that if $\Omega \subset \mathbb{R}^{n+1}$ is an unbounded NTA domain, then there exist a function $u \in C(\overline{\Omega})$ and a measure $\omega$ in $\partial\Omega$ such that

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \partial\Omega, \qquad u > 0 \quad \text{in } \Omega, \tag{3-1}$$

and

$$\int_\Omega u\, \Delta\phi\, dm = \int_{\partial\Omega} \phi\, d\omega \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^{n+1}). \tag{3-2}$$

The function $u$ and the measure $\omega$ are unique modulo constant factors, and $u$ is the so-called Green function with pole at infinity and $\omega$ the harmonic measure with pole at infinity.

From now on, we will assume that $u$ is also defined in $\mathbb{R}^{n+1} \setminus \Omega$ and vanishes identically here, so that $u \in C(\mathbb{R}^{n+1})$.

Given a Radon measure $\mu$ in $\mathbb{R}^{n+1}$, its $n$-dimensional Riesz transform is defined by

$$\mathcal{R}\mu(x) = c_n \int \frac{x-y}{|x-y|^{n+1}}\, d\mu(y),$$

whenever the integral makes sense. We assume that the constant $c_n$ is chosen so that

$$K(x) := c_n \frac{x}{|x|^{n+1}}$$

coincides with the gradient of the fundamental solution of the Laplacian.

The main result of this section is the following.

**Proposition 3.1.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded NTA domain, and let $u$ and $\omega$ be the Green function and the associated harmonic measure with pole at infinity, respectively. Suppose that for all $x \in \partial\Omega$ there exist some constants $0 < \delta < 1$ and $C > 0$ (both possibly depending on $x$) such that*

$$\omega(B(x,r)) \le C\, r^{n+\delta} \quad \text{for all } r \ge 1. \tag{3-3}$$

*Then we have*

$$\mathcal{R}\omega(x) - \mathcal{R}\omega(y) = \nabla u(y) - \nabla u(x) \quad \text{for all } x, y \in \mathbb{R}^{n+1} \setminus \partial\Omega. \tag{3-4}$$

Some remarks are in order. First, it is easy to check that if the condition (3-3) holds for all $x \in \partial\Omega$, then it also holds for all $x \in \mathbb{R}^{n+1}$ (with some constants $C$, $\delta$ depending also on $x$). For the identity (3-4) to be true, it is important to define the Riesz transform so that its kernel is the gradient of the fundamental solution of the Laplacian, as we did above. On the other hand, the function $\mathcal{R}\omega$ is defined modulo a constant term (i.e., in a BMO sense). So for all $x, y \in \mathbb{R}^{n+1} \setminus \partial\Omega$, by definition we write

$$\mathcal{R}\omega(x) - \mathcal{R}\omega(y) = \int \big(K(x-z) - K(y-z)\big)\, d\omega(z).$$

Then it turns out that the integral on the right-hand side above is absolutely convergent. Indeed, defining $d = \max(2|x-y|, 1)$, we have

$$\int_{|x-z|\geq d} \big|K(x-z)-K(y-z)\big|\,d\omega(z) \lesssim \int_{|x-z|\geq d} \frac{|x-y|}{|x-z|^{n+1}}\,d\omega(z) \lesssim \sum_{k\geq 0} \frac{|x-y|}{(2^k d)^{n+1}}\,\omega(B(x,2^k d)) \lesssim_x \sum_{k\geq 0} \frac{|x-y|}{(2^k d)^{n+1}}\,(2^k d)^{k(n+\delta)} < \infty, \tag{3-5}$$

which implies

$$\int \big|K(x-z)-K(y-z)\big|\,d\omega(z) < \infty,$$

since $x, y \notin \operatorname{supp}\omega = \partial\Omega$.

Before proceeding with the proof of Proposition 3.1, we recall a few lemmas about NTA domains. These lemmas were originally shown in [Jerison and Kenig 1982] for bounded NTA domains, but as the arguments for these results are purely local, they also hold for unbounded NTA domains.

**Lemma 3.2** [Jerison and Kenig 1982, Lemma 4.4]. *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be NTA and $B$ a ball centered on $\partial\Omega$ with $0 < r(B) < \operatorname{diam}\partial\Omega$. Let $x_B$ be a corkscrew point for $B$ in $\Omega$ and let $g$ be the Green function for $\Omega$. Then*

$$\omega^z(B) \approx g(x_B, z)r^{1-n} \quad \text{for all } z \in \Omega\backslash 2B. \tag{3-6}$$

**Lemma 3.3** [Jerison and Kenig 1982, Lemma 4.10]. *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be an NTA domain and $B$ a ball centered on $\partial\Omega$ with $0 < Mr(B) < \operatorname{diam}\partial\Omega$, where $M$ depends on the NTA character of $\Omega$. Suppose $u, v$ are two positive harmonic functions in $\Omega$ vanishing continuously on $MB\cap\partial\Omega$ and let $x_B$ be a corkscrew point for $B$ in $\Omega$. Then*

$$\frac{u(z)}{v(z)} \approx \frac{u(x_B)}{v(x_B)} \quad \text{for all } z \in B\cap\Omega. \tag{3-7}$$

*Proof of Proposition 3.1.* As shown in [Kenig and Toro 1999, Section 3], the Green function $u$ and the harmonic measure $\omega$ with pole at infinity can be constructed as follows. Given a fixed point $a \in \Omega$ and a sequence of points $p_j \in \Omega$ such that $p_j \to \infty$, we consider the function

$$u_j(x) = \begin{cases} g(x, p_j)/g(a, p_j) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \overline{\Omega}, \end{cases}$$

and the measure

$$\omega_j = \frac{1}{g(a, p_j)}\,\omega^{p_j}.$$

Passing to a subsequence and relabeling if necessary, we may assume that $u_j$ is locally uniformly convergent and that $\omega_j$ is weakly convergent. Then it turns out that $u$ is the weak limit of the sequence $u_j$ and $\omega$ is the weak limit of $\omega_j$. For simplicity, we choose points $p_j$ such that $|p_j - a| \approx \operatorname{dist}(p_j, \partial\Omega) \to \infty$. Observe that by (3-6) and our definitions of $u$ and $\omega$, it follows that for all balls $B$ centered on $\partial\Omega$, if $x_B$ is a corkscrew point for $B$ in $\Omega$, then

$$\omega(B)r^{1-n} \approx u(x_B). \tag{3-8}$$

It is well known that the Green function $g(\cdot,\cdot)$ equals

$$g(x,p)=\mathcal{E}(x-p)-\int \mathcal{E}(x-z)\,d\omega^p(z) \quad \text{for } x,p\in\Omega,$$

where $\mathcal{E}$ stands for the fundamental solution of the Laplacian. On the other hand, the right-hand side above vanishes if $x\in\mathbb{R}^{n+1}\setminus\overline{\Omega}$, $p\in\Omega$. So we deduce that for all $x\notin\partial\Omega$,

$$\nabla u_j(x)=\frac{1}{g(a,p_j)}K(x-p_j)-\mathcal{R}\omega_j(x).$$

Thus, for all $x,y\notin\partial\Omega$,

$$\nabla u_j(y)-\nabla u_j(x)=\frac{1}{g(a,p_j)}\big(K(y-p_j)-K(x-p_j)\big)+\mathcal{R}\omega_j(x)-\mathcal{R}\omega_j(y).$$

Since $u_j$ is harmonic outside of $\partial\Omega$ and $u_j$ converges locally uniformly to $u$, it turns out that $\nabla u_j$ converges also locally uniformly to $\nabla u$ outside of $\partial\Omega$. Hence, to prove the proposition, it suffices to show that

$$\lim_{j\to\infty}\frac{1}{g(a,p_j)}\big(K(y-p_j)-K(x-p_j)\big)=0, \tag{3-9}$$

$$\lim_{j\to\infty}\big(\mathcal{R}\omega_j(x)-\mathcal{R}\omega_j(y)\big)=\mathcal{R}\omega(x)-\mathcal{R}\omega(y). \tag{3-10}$$

To prove the above identities, first we will estimate $g(a,p_j)$ in terms of $u$ and $\omega$. To this end, we will apply the boundary Harnack principle.

Let $\xi_j\in\partial\Omega$ be such that $|\xi_j-p_j|=\operatorname{dist}(p_j,\partial\Omega)$, and consider the ball $B(p_j)=B(\xi_j,|\xi_j-p_j|)$. Suppose that $|\xi_j-p_j|\gg\operatorname{dist}(a,\partial\Omega)$. Consider a corkscrew point $\tilde p_j\in\frac{1}{2}B(p_j)\cap\Omega$, so that $\operatorname{dist}(\tilde p_j,\partial\Omega)\approx r(B(p_j))$. Since $u$ and $g(\cdot,p_j)$ are harmonic in $\Omega\cap B(p_j)$ and vanish identically in $\partial\Omega$, we deduce from (3-7) that

$$\frac{g(\tilde p_j,p_j)}{g(a,p_j)}\approx\frac{u(\tilde p_j)}{u(a)},$$

since $a$ belongs to $CB(p_j)$ for some fixed constant $C$, and $\operatorname{dist}(a,\partial\Omega)\ll r(B(p_j))$ by assumption. Taking into account that by (3-8)

$$u(\tilde p_j)\approx u(p_j)\approx\omega(B(p_j))\,|p_j-\xi_j|^{1-n}\approx\omega(B(p_j))\,|p_j-a|^{1-n}$$

and

$$g(\tilde p_j,p_j)\approx\frac{1}{|\tilde p_j-p_j|^{n-1}}\approx\frac{1}{|p_j-a|^{n-1}},$$

we infer that

$$g(a,p_j)\approx\frac{u(a)}{\omega(B(p_j))}. \tag{3-11}$$

With (3-11) at hand, we are ready to prove (3-9):

$$\frac{1}{g(a,p_j)}\big|K(y-p_j)-K(x-p_j)\big|\lesssim\frac{\omega(B(p_j))}{u(a)}\,\frac{|x-y|}{|x-p_j|^{n+1}}.$$

For $j$ big enough, we have $r(B(p_j))\approx|x-p_j|$, and then we derive

$$\frac{\omega(B(p_j))}{|x-p_j|^{n+1}}\lesssim_x\frac{|x-p_j|^{n+\delta}}{|x-p_j|^{n+1}}=\frac{1}{|x-p_j|^{1-\delta}},$$

and thus

$$\frac{1}{g(a,\,p_j)}\,\big|K(y-p_j)-K(x-p_j)\big| \lesssim_x \frac{|x-y|}{u(a)}\,\frac{1}{|x-p_j|^{1-\delta}} \to 0 \quad \text{as } j\to\infty.$$

We turn our attention to the identity (3-10) now. Take an auxiliary radial $C^\infty$ function $\phi:\mathbb{R}^{n+1}\to\mathbb{R}$ such that $\chi_{B(0,1)}\le\phi\le\chi_{B(0,2)}$ and define $\phi_\varepsilon(z)=\phi(z/\varepsilon)$. For $\varepsilon>0$, we define

$$K_\varepsilon=(1-\phi_\varepsilon)\,K \quad\text{and}\quad \widetilde{K}_\varepsilon=\phi_\varepsilon\,K.$$

Notice that $K_\varepsilon$ and $\widetilde{K}_\varepsilon$ are standard Calderón–Zygmund kernels. We denote by $\mathcal{R}_\varepsilon$ and $\widetilde{\mathcal{R}}_\varepsilon$ the respective associated operators, so that, at least formally, $\widetilde{\mathcal{R}}_\varepsilon$ tends to $\mathcal{R}$ as $\varepsilon\to\infty$. Then we write

$$\begin{aligned}&\big|\big(\mathcal{R}\omega_j(x)-\mathcal{R}\omega_j(y)\big)-\big(\mathcal{R}\omega(x)-\mathcal{R}\omega(y)\big)\big|\\ &\quad\le\big|\big(\widetilde{\mathcal{R}}_\varepsilon\omega_j(x)-\widetilde{\mathcal{R}}_\varepsilon\omega_j(y)\big)-\big(\widetilde{\mathcal{R}}_\varepsilon\omega(x)-\widetilde{\mathcal{R}}_\varepsilon\omega(y)\big)\big|+\big|\mathcal{R}_\varepsilon\omega_j(x)-\mathcal{R}_\varepsilon\omega_j(y)\big|+\big|\mathcal{R}_\varepsilon\omega(x)-\mathcal{R}_\varepsilon\omega(y)\big|.\end{aligned}\tag{3-12}$$

Since the function $\widetilde{K}_\varepsilon(x-\cdot\,)-\widetilde{K}_\varepsilon(y-\cdot\,)$ is continuous on $\partial\Omega$ (recall that $x,y\in\mathbb{R}^{n+1}\setminus\partial\Omega$) and has compact support, we infer that

$$\big|\big(\widetilde{\mathcal{R}}_\varepsilon\omega_j(x)-\widetilde{\mathcal{R}}_\varepsilon\omega_j(y)\big)-\big(\widetilde{\mathcal{R}}_\varepsilon\omega(x)-\widetilde{\mathcal{R}}_\varepsilon\omega(y)\big)\big|\to 0 \quad\text{as } j\to\infty,\tag{3-13}$$

by the weak convergence of $\omega_j$ to $\omega$.

Concerning the second term on the right-hand side of (3-12), we will show below that

$$\big|\mathcal{R}_\varepsilon\omega_j(x)-\mathcal{R}_\varepsilon\omega_j(y)\big|\lesssim_x\frac{|x-y|}{u(a)}\bigg(\frac{1}{\varepsilon^{1-\delta}}+\frac{1}{|x-p_j|^{1-\delta}}\bigg).\tag{3-14}$$

The last term in (3-12) is estimated as in (3-5). Indeed, for $\varepsilon\gg|x-y|$,

$$\begin{aligned}\big|\mathcal{R}_\varepsilon\omega(x)-\mathcal{R}_\varepsilon\omega(y)\big|&\lesssim\int\big|K_\varepsilon(x-z)-K_\varepsilon(y-z)\big|\,d\omega(z)\lesssim\int_{|x-z|\ge\varepsilon/2}\frac{|x-y|}{|x-z|^{n+1}}\,d\omega(z)\\ &\lesssim\sum_{k\ge0}\frac{|x-y|}{(2^k\varepsilon)^{n+1}}\,\omega(B(x,2^k\varepsilon))\lesssim_x\sum_{k\ge0}\frac{|x-y|}{(2^k\varepsilon)^{n+1}}\,(2^k\varepsilon)^{n+\delta}\approx\frac{|x-y|}{\varepsilon^{1-\delta}}.\end{aligned}\tag{3-15}$$

From (3-12), (3-13), (3-14) and (3-15) we deduce that

$$\limsup_{j\to\infty}\big|\big(\mathcal{R}\omega_j(x)-\mathcal{R}\omega_j(y)\big)-\big(\mathcal{R}\omega(x)-\mathcal{R}\omega(y)\big)\big|\lesssim_x\frac{|x-y|}{u(a)\,\varepsilon^{1-\delta}}+\frac{|x-y|}{\varepsilon^{1-\delta}}.$$

Since this holds for any arbitrarily big $\varepsilon>0$, the limit vanishes and this concludes the proof of (3-4).

Finally we deal with the estimate (3-14). Arguing as in (3-15), with $\omega$ replaced by $\omega_j$, we obtain

$$\big|\mathcal{R}_\varepsilon\omega_j(x)-\mathcal{R}_\varepsilon\omega_j(y)\big|\lesssim\sum_{k\ge0}\frac{|x-y|}{(2^k\varepsilon)^{n+1}}\,\omega_j(B(x,2^k\varepsilon)).$$

We split the last sum according to whether $2^k\varepsilon\le|p_j-x|$ or not, so that

$$\big|\mathcal{R}_\varepsilon\omega_j(x)-\mathcal{R}_\varepsilon\omega_j(y)\big|\le S_1+S_2,$$

where

$$S_1 = \sum_{\substack{k\geq 0\\ 2^k\varepsilon\leq |p_j-x|}} \frac{|x-y|}{(2^k\varepsilon)^{n+1}}\,\omega_j(B(x,2^k\varepsilon)) \quad\text{and}\quad S_2 = \sum_{\substack{k\geq 0\\ 2^k\varepsilon> |p_j-x|}} \frac{|x-y|}{(2^k\varepsilon)^{n+1}}\,\omega_j(B(x,2^k\varepsilon)).$$

To estimate $S_1$ we use the fact that, for $2^k\varepsilon \leq |p_j - x|$,

$$\omega_j(B(x,2^k\varepsilon)) = \frac{1}{g(a,p_j)}\,\omega^{p_j}(B(x,2^k\varepsilon)) \approx \frac{1}{g(a,p_j)}\,\frac{\omega(B(x,2^k\varepsilon))}{\omega(B(p_j))}.$$

Hence, by (3-11),

$$\omega_j(B(x,2^k\varepsilon)) \approx \frac{\omega(B(x,2^k\varepsilon))}{u(a)},$$

and so

$$S_1 \lesssim \sum_{k\geq 0} \frac{|x-y|}{(2^k\varepsilon)^{n+1}}\,\frac{\omega(B(x,2^k\varepsilon))}{u(a)} \lesssim_x \sum_{k\geq 0} \frac{|x-y|}{(2^k\varepsilon)^{n+1}}\,\frac{(2^k\varepsilon)^{n+\delta}}{u(a)} \lesssim \frac{|x-y|}{u(a)\,\varepsilon^{1-\delta}}.$$

To estimate $S_2$ we use the trivial estimate

$$\omega_j(B(x,2^k\varepsilon)) = \frac{1}{g(a,p_j)}\,\omega^{p_j}(B(x,2^k\varepsilon)) \leq \frac{1}{g(a,p_j)} \approx \frac{\omega(B(p_j))}{u(a)}.$$

Therefore,

$$S_2 \approx \sum_{\substack{k\geq 0\\ 2^k\varepsilon> |p_j-x|}} \frac{|x-y|}{(2^k\varepsilon)^{n+1}}\,\frac{\omega(B(p_j))}{u(a)} \lesssim \frac{|x-y|}{|p_j-x|^{n+1}}\,\frac{\omega(B(p_j))}{u(a)}.$$

Assuming that $|p_j - x| \geq 1$, we have

$$\omega(B(p_j)) \lesssim_x r(B(p_j))^{n+\delta} \approx |p_j-x|^{n+\delta},$$

and thus

$$S_2 \lesssim_x \frac{|x-y|}{|p_j-x|^{1-\delta}}\,\frac{\omega(B(p_j))}{u(a)}.$$

From this estimate and the one for $S_1$, we obtain (3-14), as wished. □

We recall now the following version of the jump equations for the gradient of the single layer potential due to Hofmann, Mitrea and Taylor [Hofmann et al. 2010]:

**Proposition 3.4** [Hofmann et al. 2010, Proposition 3.30]. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain in $\mathbb{R}^{n+1}$ with uniformly rectifiable boundary such that $\sigma(\partial\Omega\setminus\partial_*\Omega)=0$, where $\partial_*\Omega$ stands for the measure theoretic boundary and $\sigma$ for the surface measure of $\Omega$. Let $f\in L^p(\sigma|_{\partial\Omega})$ for $1\leq p<\infty$. Then, for $\sigma$-a.e. $x\in\partial\Omega$,*

$$\lim_{\Gamma^-(x)\ni z\to x} \mathcal{R}(f\sigma)(z) = -\tfrac{1}{2}\,\vec{n}(x)\,f(x) + \operatorname{pv}\mathcal{R}(f\sigma)(x), \tag{3-16}$$

*and*

$$\lim_{\Gamma^+(x)\ni z\to x} \mathcal{R}(f\sigma)(z) = \tfrac{1}{2}\,\vec{n}(x)\,f(x) + \operatorname{pv}\mathcal{R}(f\sigma)(x), \tag{3-17}$$

*where $\Gamma^+(x)$ is a nontangential cone at $x$ relative to $\Omega$, (that is,*

$$\Gamma^+(x) = \{y \in \Omega : \operatorname{dist}(y, \Omega^c) > t|y-x|\}$$

*for some $t > 0$), $\Gamma^-(x)$ is a nontangential cone at $x$ relative to $\mathbb{R}^{n+1} \setminus \overline{\Omega}$, and $\vec{n}(x)$ is the outer normal to $\Omega$ at $x$.*

In particular, if $\Omega$ is a chord-arc domain in $\mathbb{R}^{n+1}$, then $\partial\Omega$ is uniformly rectifiable (see [David and Jerison 1990]) and $\sigma(\partial\Omega \setminus \partial_*\Omega) = 0$; thus the preceding proposition can be applied.

**Proposition 3.5.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain in $\mathbb{R}^{n+1}$. Let $\omega$ and $u$ be the harmonic measure and the Green function with a pole either at infinity or at some point $p \in \Omega$. Suppose that for each $x \in \partial\Omega$ there exist some constants $0 < \delta < 1$ and $C > 0$ such that*

$$\omega(B(x,r)) \le C\, r^{n+\delta} \quad \textit{for all } r \ge 1. \tag{3-18}$$

*Suppose $h := d\omega/d\sigma \in L^p_{\mathrm{loc}}(\sigma)$ for some $p \ge 1$. Then $\lim_{\Gamma^+(x)\ni z\to x} \nabla u(z)$ exists for $\sigma$-a.e. $x \in \partial\Omega$ and*

$$\lim_{\Gamma^+(x)\ni z\to x} \nabla u(z) = -h(x)\,\vec{n}(x). \tag{3-19}$$

*Proof.* Assume that the pole for $\omega$ and $u$ is at infinity (the arguments for the case when the pole is finite are analogous). Let $B$ be a ball centered at $\partial\Omega$. By Proposition 3.4, for $\sigma$-a.e. $x \in B$,

$$\lim_{\Gamma^-(x)\ni z\to x} \mathcal{R}(\chi_{2B}\omega)(z) = -\tfrac{1}{2}\,\vec{n}(x)\,h(x) + \operatorname{pv}\mathcal{R}(\chi_{2B}\omega),$$

and

$$\lim_{\Gamma^+(x)\ni z\to x} \mathcal{R}(\chi_{2B}\omega)(z) = \tfrac{1}{2}\,\vec{n}(x)\,h(x) + \operatorname{pv}\mathcal{R}(\chi_{2B}\omega)$$

In particular,

$$\lim_{\Gamma^+(x)\ni z\to x} \mathcal{R}(\chi_{2B}\omega)(z) - \lim_{\Gamma^-(x)\ni z\to x} \mathcal{R}(\chi_{2B}\omega)(z) = \vec{n}(x)h(x).$$

Using the condition (3-18), by estimates analogous to the ones in (3-5), it is immediate to check that

$$\lim_{\Gamma^+(x)\ni z\to x} \mathcal{R}(\chi_{2B}\omega)(z) - \lim_{\Gamma^-(x)\ni z\to x} \mathcal{R}(\chi_{2B}\omega)(z) = \lim_{\Gamma^+(x)\ni z\to x} \mathcal{R}\omega(z) - \lim_{\Gamma^-(x)\ni z\to x} \mathcal{R}\omega(z).$$

Then, by Proposition 3.1 we infer that

$$\lim_{\Gamma^-(x)\ni z\to x} \nabla u(z) - \lim_{\Gamma^+(x)\ni z\to x} \nabla u(z) = \vec{n}(x)h(x).$$

Since $u \equiv 0$ in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$, we have $\lim_{\Gamma^-(x)\ni z\to x} \nabla u(z) = 0$ and so

$$-\lim_{\Gamma^+(x)\ni z\to x} \nabla u(z) = \vec{n}(x)h(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega \cap B. \qquad \square$$

## 4. Some technical lemmas

From now on, given a domain $\Omega \subset \mathbb{R}^{n+1}$ and $x \in \mathbb{R}^{n+1}$, we define

$$d_\Omega(x) = \operatorname{dist}(x, \Omega^c).$$

The following is a well-known result. See, for example, [Jerison and Kenig 1982, Section 4].

**Lemma 4.1.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain and let $B$ be a ball centered at $\partial\Omega$. There exist some constants $C, \alpha > 0$ depending on the NTA character of $\Omega$ such that the following holds. If $u$ is a nonnegative harmonic function on $\Omega \cap 2B$ which vanishes continuously on $\partial\Omega \cap 2B$, then*

$$u(x) \leq C \left(\frac{d_\Omega(x)}{r(B)}\right)^\alpha \sup_{y \in \partial(2B)\cap\Omega} u(y) \quad \textit{for all } x \in B \cap \Omega.$$

*If $x_B$ is a corkscrew point for $B$, then*

$$\sup_{y \in B\cap\Omega} u(y) \leq C u(x_B).$$

We will also need the next auxiliary result.

**Lemma 4.2.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain. There exist some constants $C, \alpha > 0$ depending on the NTA character of $\Omega$ such that the Green function of $\Omega$ satisfies*

$$g(x, y) \leq C \frac{1}{|x-y|^{n-1}} \left(\frac{\min\big(d_\Omega(x), d_\Omega(y)\big)}{|x-y|}\right)^\alpha \quad \textit{for all } x, y \in \Omega. \tag{4-1}$$

*Proof.* It is enough to show that, for some $C, \alpha' > 0$,

$$g(x, y) \leq C \frac{1}{|x-y|^{n-1}} \left(\frac{d_\Omega(x)}{|x-y|}\right)^\alpha \quad \text{for all } x, y \in \Omega, \tag{4-2}$$

because then the analogous inequality interchanging $x$ by $y$ also holds, by symmetry.

Because of the trivial estimate $g(x, y) \lesssim 1/|x-y|^{n-1}$, to prove (4-2) we may assume $|x-y| > 10\, d_\Omega(x)$. Let $\xi_x \in \partial\Omega$ be such that $|\xi_x - x| = d_\Omega(x)$ and consider the ball $B := B(\xi_x, |x-y|/8)$. Clearly $x \in B$, as

$$|x - \xi_x| = d_\Omega(x) \leq \tfrac{1}{10}|x-y| = \tfrac{8}{10} r(B).$$

Note also that, for all $z \in \partial(2B)$,

$$|y - z| \geq |x - y| - |x - z| \geq 8\, r(B) - 4\, r(B) = 4\, r(B) \approx |x - y|.$$

Hence $g(z, y) \lesssim 1/|y-z|^{n-1} \lesssim 1/|x-y|^{n-1}$ for all $z \in \partial(2B)$. Thus, (4-2) follows from Lemma 4.1 applied to the function $g(\cdot\,, y)$. □

The following rather standard result is shown in [Kenig and Toro 2003, Theorem 2.1].

**Lemma 4.3.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain, $f \in \mathrm{VMO}(\sigma)$, and $h = e^f$. Then, for all $x \in \partial\Omega$, $0 < r \leq \mathrm{diam}(\Omega)$ and $1 < p < \infty$,*

$$\left(\fint_{B(x,r)} h^p \, d\sigma\right)^{1/p} \leq C_p \fint_{B(x,r)} h \, d\sigma \quad \text{and} \quad \left(\fint_{B(x,r)} h^{-p} \, d\sigma\right)^{1/p} \leq C_p \fint_{B(x,r)} h^{-1} \, d\sigma.$$

The next lemma is proven in [Jerison and Kenig 1982, Lemma 4.11]:

**Lemma 4.4.** *Let $\Omega$ be an NTA domain, $B$ a ball centered on $\partial\Omega$ with $0 < r(B) < \mathrm{diam}\,\partial\Omega$, and let $E \subseteq B \cap \partial\Omega$ be Borel. If $x_B$ is a corkscrew point for $B$ in $\Omega$, then*

$$\frac{\omega^z(E)}{\omega^z(B)} \approx \omega^{x_B}(E) \quad \text{for } z \in \Omega \backslash 2B. \tag{4-3}$$

Note that this implies that if $\omega$ is the harmonic measure with pole at infinity, we also have

$$\frac{\omega(E)}{\omega(B)} \approx \omega^{x_B}(E). \tag{4-4}$$

The next corollary is an easy consequence of the preceding lemma, as shown in [Kenig and Toro 2003, Corollary 2.4].

**Corollary 4.5.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. If the harmonic measure $\omega$ in $\Omega$ is such that $d\omega/d\sigma \in \mathrm{VMO}(\sigma)$, then, for all $\varepsilon > 0$, $x \in \partial\Omega$, $0 < r \leq \mathrm{diam}(\Omega)$ and $E \subset B(x,r) \cap \partial\Omega$,*

$$C(\varepsilon)^{-1}\left(\frac{\sigma(E)}{\sigma(B(x,r))}\right)^{1+\varepsilon} \leq \frac{\omega(E)}{\omega(B(x,r))} \leq C(\varepsilon)\left(\frac{\sigma(E)}{\sigma(B(x,r))}\right)^{1-\varepsilon}.$$

Let us remark that the pole of harmonic measure above can be either a point $p \in \Omega$ (in which case the constants also depend on $p$) or infinity in the case $\Omega$ is unbounded.

Another easy consequence of Lemma 4.3 is the following.

**Corollary 4.6.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. Suppose that the harmonic measure $\omega$ in $\Omega$ with pole at infinity is such that $\log(d\omega/d\sigma) \in \mathrm{VMO}(\sigma)$. For $z \in \Omega$, let $K_z = d\omega^z/d\sigma$ (i.e., $K_z$ is the Poisson kernel). For $1 < p < \infty$, if $x \in \partial\Omega$, $0 < r \leq \mathrm{diam}(\Omega)$, and $z \in \Omega \setminus B(x, 2r)$, then*

$$\left(\fint_{B(x,r)} (K_z)^p \, d\sigma\right)^{1/p} \leq C(p) \fint_{B(x,r)} K_z \, d\sigma.$$

For this corollary to hold we assume either the pole of $\omega$ is at $\infty$ if $\Omega$ is unbounded, or it is in $\Omega$.

*Proof.* Since $z \in \Omega \setminus B(x, 2r)$, if $z_0$ is a corkscrew point for $B(x,r)$, then whenever $B(y,s) \subset B(x,r)$ and all $0 < s < r/10$, by (4-3) and (4-4),

$$\frac{\omega(B(y,s))}{\omega(B(x,r))} \approx \omega^{z_0}(B(y,s)) \approx \frac{\omega^z(B(y,s))}{\omega^z(B(x,r))}.$$

Hence, by the Lebesgue differentiation theorem, if we define $h = d\omega/d\sigma$ for $\sigma$-a.e. $y \in B(x,r) \cap \partial\Omega$,

$$K_z(y) = \frac{d\omega^z}{d\sigma}(y) = \lim_{s\to 0} \frac{\omega^z(B(y,s))}{\sigma(B(y,s))} \approx \frac{\omega^z(B(x,r))}{\omega(B(x,r))} \lim_{s\to 0} \frac{\omega(B(y,s))}{\sigma(B(y,s))} = \frac{\omega^z(B(x,r))}{\omega(B(x,r))} h(y).$$

Therefore, by Lemma 4.3, since $\log h \in \mathrm{VMO}(\sigma)$,

$$\left(\fint_{B(x,r)} K_z(y)^p\, d\sigma(y)\right)^{1/p} \approx \frac{\omega^z(B(x,r))}{\omega(B(x,r))}\left(\fint_{B(x,r)} h(y)^p\, d\sigma(y)\right)^{1/p}$$
$$\lesssim \frac{\omega^z(B(x,r))}{\omega(B(x,r))}\fint_{B(x,r)} h(y)\, d\sigma(y) \approx \fint_{B(x,r)} K_z(y)\, d\sigma(y). \qquad \square$$

**Lemma 4.7.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. Suppose that the harmonic measure $\omega$ in $\Omega$ with pole either at infinity or at some fixed point $p \in \Omega$ is such that $\log(d\omega/d\sigma) \in \mathrm{VMO}(\sigma)$. Denote by $u$ the associated Green function. Then, for $\sigma$-a.e. $x \in \partial\Omega$, we have $\nabla u(z)$ converges to $-\vec{n}(x)(d\omega/d\sigma)(x)$ as $\Omega \ni z \to x$ nontangentially, where $\vec{n}$ is the outer unit normal of $\Omega$.*

This lemma is proved in [Kenig and Toro 2003] under the additional assumption that $\Omega$ is Reifenberg flat. In [Kenig and Toro 2006] it is shown how to prove this without the Reifenberg flatness assumption. The delicate arguments involved in [Kenig and Toro 2003; 2006] do not use the connection between harmonic measure and the Riesz transform and instead are of a more geometric nature.

*Proof.* This is an immediate consequence of Proposition 3.5 and Corollary 4.5. Indeed, this corollary, implies that for all $x \in \partial\Omega$ and all $0 < r_0 \le r \le \mathrm{diam}(\Omega)$,

$$\left(\frac{\sigma(B(x,r_0))}{\sigma(B(x,r))}\right)^{1+\varepsilon} \le C(\varepsilon)\,\frac{\omega(B(x,r_0))}{\omega(B(x,r))}.$$

Hence, using also the AD-regularity of $\sigma$ we get

$$\omega(B(x,r)) \le C(\varepsilon)\,\omega(B(x,r_0))\left(\frac{\sigma(B(x,r))}{\sigma(B(x,r_0))}\right)^{1+\varepsilon} \approx \frac{\omega(B(x,r_0))}{\sigma(B(x,r_0))^{1+\varepsilon}}\, r^{n(1+\varepsilon)}.$$

Therefore, choosing $\varepsilon = 1/(2n)$,

$$\omega(B(x,r)) \le C(x)\, r^{n+1/2} \quad \text{for } r \ge r_0.$$

So the assumption (3-18) in Proposition 3.5 holds and thus

$$\lim_{\Gamma^+(x)\ni z\to x} \nabla u(z) = -\frac{d\omega}{d\sigma}(x)\,\vec{n}(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega. \qquad \square$$

The next result is an auxiliary calculation which will be used several times in the next section. The arguments for the proof are quite standard. Similar calculations appear, for example, in the proofs of Lemma 5.2 of [Kenig and Toro 2006], Lemma 3.3 of [Kenig and Toro 2003] or Lemma 3.30 of [Hofmann and Martell 2014].

**Lemma 4.8.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain, and let $\omega$ be the harmonic measure in $\Omega$ with pole either at infinity or at some fixed point $p \in \Omega$. Let $B \subset \mathbb{R}^{n+1}$ be a ball centered at $\partial\Omega$ such that $p \notin 10B$. Then for any constant $\varepsilon > 0$,*

$$\int_{B\cap\Omega}\left(\frac{d_\Omega(y)}{r(B)}\right)^{\varepsilon}\frac{\omega(B(y,2d_\Omega(y)))}{d_\Omega(y)^{n+1}}\, dy \le C(\varepsilon)\,\omega(B).$$

*Proof.* We write

$$\int_{B\cap\Omega}\left(\frac{d_\Omega(y)}{r(B)}\right)^{\varepsilon}\frac{\omega(B(y,2d_\Omega(y)))}{d_\Omega(y)^{n+1}}\,dy \lesssim \sum_{j\geq 0}2^{-j\varepsilon}\int_{\substack{y\in B\cap\Omega\\ 2^{-j-1}r(B)<d_\Omega(y)\leq 2^{-j}r(B)}}\frac{\omega(B(y,2^{-j+1}r(B)))}{(2^{-j}r(B))^{n+1}}\,dy. \quad (4\text{-}5)$$

We define $A_j := \{y\in B\cap\Omega : 2^{-j-1}r(B) < d_\Omega(y)\leq 2^{-j}r(B)\}$. For each $y\in A_j$ consider a ball $B_y^j$ with radius $2^{-j+1}r(B)$ centered at a point $\xi_y\in\partial\Omega$ such that $|y-\xi_y| = d_\Omega(y)$. Clearly $y\in B_y^j$ for each $y\in A_j$, and thus we can extract a subfamily of pairwise disjoint balls $\{B_k^j\}_k\subset\{B_y^j\}_{y\in A_j}$ so that

$$A_j\subset\bigcup_k 3B_k^j.$$

Notice that for each $y\in B_k^j$, since $\omega$ is doubling,

$$\omega(B(y,2^{-j+1}r(B)))\leq\omega(6B_k^j)\lesssim\omega(B_k^j).$$

Therefore, taking also into account that the balls $B_k^j$ are contained in $6B$,

$$\int_{\substack{y\in B\cap\Omega:\\ 2^{-j-1}r(B)<d_\Omega(y)\leq 2^{-j}r(B)}}\frac{\omega(B(y,2^{-j+1}r(B)))}{(2^{-j}r(B))^{n+1}}\,dy \lesssim \sum_k\int_{B_k^j}\frac{\omega(B_k^j)}{(2^{-j}r(B))^{n+1}}\,dy$$
$$= C\sum_k\omega(B_k^j)\lesssim\omega(6B)\lesssim\omega(B).$$

Plugging this estimate into (4-5), the lemma follows. □

## 5. Estimates for the gradient of Green's function

The reader should compare the arguments in this section to the ones in Section 3 of [Kenig and Toro 2003] and Section 2 of [Kenig and Toro 2006], which in turn rely on the results in the Appendices A1 and A2 of [Kenig and Toro 2003].

**Lemma 5.1.** *Let $\Omega\subset\mathbb{R}^{n+1}$ be an unbounded chord-arc domain. Suppose that the harmonic measure $\omega$ in $\Omega$ with pole at infinity satisfies $\log(d\omega/d\sigma)\in\mathrm{VMO}(\sigma)$. Denote by $u$ the associated Green function. Then*

$$|\nabla u(x)|\leq\int_{\partial\Omega}\frac{d\omega}{d\sigma}(y)\,d\omega^x(y)\quad\textit{for all } x\in\Omega. \quad (5\text{-}1)$$

The proof of this lemma would be quite immediate if the function $d\omega/d\sigma$ inside the integral in (5-1) were compactly supported, taking into account that $\nabla u$ is harmonic. However, this is not the case and so the arguments are more delicate. The next auxiliary lemma will be used to take care of this question by a localization of singularities technique.

**Lemma 5.2.** *Under the assumptions of Lemma 5.1, suppose that $0\in\partial\Omega$. Fix $R>1$ large and let $\phi_R\in C_c^\infty(\mathbb{R}^{n+1})$ such that $\chi_{B(0,R)}\leq\phi_R\leq\chi_{B(0,2R)}$, $|\nabla^j\phi_R|\lesssim 1/R^j$ for $j=1,2$. For $x\in\Omega$, define*

$$w_R(x)=\int_\Omega g(x,y)\,\Delta[\phi_R\nabla u](y)\,dy.$$

*Then* $w_R \in C^{\alpha/2}(\overline{\Omega})$ *for some* $\alpha > 0$, $w_R|_{\partial\Omega} \equiv 0$, *and the following estimates hold for* $x \in \Omega$:

(a) $|w_R(x)| \lesssim \dfrac{\omega(B(0,R))}{R^n}\left(\dfrac{d_\Omega(x)}{R}\right)^{\alpha/2}$ *if* $|x| \le 4R$.

(b) $|w_R(x)| \lesssim \dfrac{\omega(B(0,R))}{|x|^{n-1+\alpha/2}\,R^{1-\alpha/2}}\left(\dfrac{d_\Omega(x)}{|x|}\right)^{\alpha/2}$ *if* $|x| > 4R$.

*Proof.* By the relationship between Green's function and harmonic measure, for all $y \in \Omega$ we have

$$u(y) \approx \frac{1}{d_\Omega(y)^{n-1}}\,\omega(B(y, 2d_\Omega(y))),$$

and by standard estimates for positive harmonic functions we derive

$$|\nabla u(y)| \lesssim \frac{u(y)}{d_\Omega(y)} \approx \frac{\omega(B(y,2d_\Omega(y)))}{d_\Omega(y)^n} \quad \text{and} \quad |\nabla^2 u(y)| \lesssim \frac{u(y)}{d_\Omega(y)^2} \approx \frac{\omega(B(y,2d_\Omega(y)))}{d_\Omega(y)^{n+1}}.$$

Thus,

$$\begin{aligned} |w_R(x)| &= \left|\int_\Omega g(x,y)\left(\Delta\phi_R(y)\,\nabla u(y) + 2\nabla\phi_R(y)\cdot\nabla^2 u(y)\right)dy\right| \\ &\lesssim \int_{A(0,R,2R)\cap\Omega} g(x,y)\left(\frac{\omega(B(y,2d_\Omega(y)))}{R^2\,d_\Omega(y)^n} + \frac{\omega(B(y,2d_\Omega(y)))}{R\,d_\Omega(y)^{n+1}}\right)dy \\ &\lesssim \int_{B(0,2R)\cap\Omega} g(x,y)\,\frac{\omega(B(y,2d_\Omega(y)))}{R\,d_\Omega(y)^{n+1}}\,dy. \end{aligned} \tag{5-2}$$

**Case 1:** $|x| \le 4R$.

We split the integral on the right-hand side of (5-2) as follows:

$$\begin{aligned} |w_R(x)| &\lesssim \int_{|y-x|\le d_\Omega(x)/2} g(x,y)\,\frac{\omega(B(y,2d_\Omega(y)))}{R\,d_\Omega(y)^{n+1}}\,dy + \int_{\substack{y\in B(0,2R)\cap\Omega \\ |y-x|>d_\Omega(x)/2}} g(x,y)\,\frac{\omega(B(y,2d_\Omega(y)))}{R\,d_\Omega(y)^{n+1}}\,dy \\ &=: I_1 + I_2. \end{aligned} \tag{5-3}$$

First we will deal with $I_1$. In the domain of integration of $I_1$ we have $d_\Omega(y) \approx d_\Omega(x)$. Taking into account that $\omega$ is doubling, in this case we derive $\omega(B(y,2d_\Omega(y))) \approx \omega(B(x,2d_\Omega(x)))$. Then using also the trivial estimate $g(x,y) \lesssim 1/|x-y|^{n-1}$, we get

$$I_1 \lesssim \int_{|y-x|\le d_\Omega(x)/2} \frac{1}{|x-y|^{n-1}}\,\frac{\omega(B(x,2d_\Omega(x))}{R\,d_\Omega(x)^{n+1}}\,dy \approx \frac{\omega(B(x,2d_\Omega(x))}{R\,d_\Omega(x)^{n-1}}.$$

Notice that, by Lemma 4.1,

$$u(x) \lesssim \left(\frac{d_\Omega(x)}{R}\right)^\alpha \sup_{y\in\partial B(0,8R)\cap\Omega} u(y) \lesssim \left(\frac{d_\Omega(x)}{R}\right)^\alpha u(x_R), \tag{5-4}$$

where $x_R$ is a corkscrew point for $B(0,R)$. That is, $x_R \in B(0,R)\cap\Omega$ and $d_\Omega(x_R) \approx R$. Hence using that $\omega(B(z,2d_\Omega(z))) \approx u(z)\,d_\Omega(z)^{n-1}$ both for $z = x$ and $z = x_R$, we deduce that

$$I_1 \lesssim \frac{\omega(B(x,2d_\Omega(x)))}{R\,d_\Omega(x)^{n-1}} \lesssim \left(\frac{d_\Omega(x)}{R}\right)^\alpha \frac{\omega(B(x_R,2d_\Omega(x_R)))}{R\,d_\Omega(x_R)^{n-1}} \approx \left(\frac{d_\Omega(x)}{R}\right)^\alpha \frac{\omega(B(0,R))}{R^n}. \tag{5-5}$$

We consider now the integral $I_2$ in (5-3). To estimate this we use the inequality

$$g(x,y) \lesssim \frac{1}{|x-y|^{n-1}} \left(\frac{d_\Omega(x)}{|x-y|}\right)^{\alpha/2} \left(\frac{d_\Omega(y)}{|x-y|}\right)^{\alpha/2}, \tag{5-6}$$

which is an immediate consequence of (4-1). To shorten notation, for each integer $j \geq 0$ we write $r_j := 2^j\, d_\Omega(x)$. Denote by $j_{\max}$ the least integer such that $B(0,2R) \subset B(x, r_{j_{\max}})$, so that $r_{j_{\max}} \approx R$. Then plugging the estimate (5-6) into $I_2$ and splitting, we obtain

$$I_2 \lesssim \sum_{0 \leq j \leq j_{\max}} \frac{1}{R\, r_j^{n-1}} \left(\frac{d_\Omega(x)}{r_j}\right)^{\alpha/2} \int_{\substack{y\in\Omega \\ r_{j-1}<|y-x|\leq r_j}} \left(\frac{d_\Omega(y)}{r_j}\right)^{\alpha/2} \frac{\omega(B(y, 2d_\Omega(y)))}{d_\Omega(y)^{n+1}}\, dy.$$

Let $\xi_x \in \partial\Omega$ be such that $|x-\xi_x| = d_\Omega(x)$. It is immediate to check that if $|y-x| \leq r_j = 2^j d_\Omega(x)$, then $y \in \bar{B}(\xi_x, 2r_j)$. So the last integral is bounded above by

$$\int_{\Omega\cap \bar{B}(\xi_x, 2r_j)} \left(\frac{d_\Omega(y)}{r_j}\right)^{\alpha/2} \frac{\omega(B(y, 2d_\Omega(y)))}{d_\Omega(y)^{n+1}}\, dy,$$

and then, by Lemma 4.8, this does not exceed $C\,\omega(B(\xi_x, r_j))$. Hence,

$$I_2 \lesssim \sum_{0 \leq j \leq j_{\max}} \frac{1}{R\, r_j^{n-1}} \left(\frac{d_\Omega(x)}{r_j}\right)^{\alpha/2} \omega(B(\xi_x, r_j)). \tag{5-7}$$

To estimate the right-hand side in the inequality above, we argue as in (5-4). We consider a corkscrew point $x_j$ in each ball $B(\xi_x, r_j)$, and then since $\operatorname{dist}(x_j, \partial\Omega) \approx r_j$, we deduce

$$u(x_j) \lesssim \left(\frac{r_j}{R}\right)^{\alpha} u(x_R)$$

(recall that $x_R$ is a corkscrew point for $B(0,R)$). Thus,

$$\frac{\omega(B(\xi_x, r_j))}{r_j^{n-1}} \lesssim \left(\frac{r_j}{R}\right)^{\alpha} \frac{\omega(B(0,R))}{R^{n-1}}.$$

Plugging this estimate into (5-7) we obtain

$$I_2 \lesssim \sum_{0 \leq j \leq j_{\max}} \left(\frac{d_\Omega(x)}{r_j}\right)^{\alpha/2} \left(\frac{r_j}{R}\right)^{\alpha} \frac{\omega(B(0,R))}{R^n} = \frac{d_\Omega(x)^{\alpha/2}}{R^\alpha} \frac{\omega(B(0,R))}{R^n} \sum_{0 \leq j \leq j_{\max}} r_j^{\alpha/2}.$$

Since the last sum is geometric, it turns out that

$$\sum_{0 \leq j \leq j_{\max}} r_j^{\alpha/2} \approx r_{j_{\max}}^{\alpha/2} \approx R^{\alpha/2}.$$

Therefore,

$$I_2 \lesssim \frac{d_\Omega(x)^{\alpha/2}}{R^{\alpha/2}} \frac{\omega(B(0,R))}{R^n}.$$

Together with the estimate for $I_1$ in (5-5), this yields the inequality (a) in the lemma.

**Case 2:** $|x| > 4R$.

To estimate the integral on the right-hand side of (5-2) we use the fact that, for $y \in B(0, 2R)$, by (4-1),

$$g(x, y) \lesssim \frac{1}{|x|^{n-1}} \left(\frac{d_\Omega(x)}{|x|}\right)^{\alpha/2} \left(\frac{d_\Omega(y)}{|x|}\right)^{\alpha/2},$$

taking into account that $|x - y| \approx |x|$. Then we get

$$\begin{aligned} |w_R(x)| &\lesssim \frac{1}{|x|^{n-1}} \left(\frac{d_\Omega(x)}{|x|}\right)^{\alpha/2} \int_{B(0,2R)\cap\Omega} \left(\frac{d_\Omega(y)}{|x|}\right)^{\alpha/2} \frac{\omega(B(y, 2d_\Omega(y)))}{R\, d_\Omega(y)^{n+1}}\, dy \\ &= \frac{1}{R|x|^{n-1}} \left(\frac{d_\Omega(x)}{|x|}\right)^{\alpha/2} \left(\frac{R}{|x|}\right)^{\alpha/2} \int_{B(0,2R)\cap\Omega} \left(\frac{d_\Omega(y)}{R}\right)^{\alpha/2} \frac{\omega(B(y, 2d_\Omega(y)))}{d_\Omega(y)^{n+1}}\, dy. \end{aligned}$$

By Lemma 4.8, the last integral above does not exceed $C\omega(B(0, R))$, and so we deduce that

$$|w_R(x)| \lesssim \frac{1}{R|x|^{n-1}} \left(\frac{d_\Omega(x)}{|x|}\right)^{\alpha/2} \left(\frac{R}{|x|}\right)^{\alpha/2} \omega(B(0, R)),$$

which gives the inequality (b) in the lemma. □

*Proof of Lemma 5.1.* The arguments are similar to the ones for [Kenig and Toro 2003, Theorem 3.1]. For the reader's convenience, we show the details below.

Suppose that $0 \in \partial\Omega$ and, for $R \geq 1$, let $\phi_R$ and $w_R$ be the functions introduced in Lemma 5.2. For $x \in \Omega$, we define

$$h_R(x) = \phi_R(x)\nabla u(x) - w_R(x).$$

Since $\Delta w_R = \Delta[\phi_R \nabla u]$ in $\Omega$, it turns out that $h_R$ is harmonic in $\Omega$. Further, the estimates (a) and (b) in Lemma 5.2, in particular, ensure that $w_R$ vanishes continuously at $\partial\Omega$. Thus $h_R$ vanishes on $\partial\Omega \setminus B(0, 2R)$.

By Lemma 4.7 it follows that $\nabla u(z)$ converges nontangentially to $-(d\omega/d\sigma)(y)\vec{n}(y)$ as $\Omega \ni z \to y$ for $\sigma$-a.e. $y \in \partial\Omega$. Also, as mentioned above, $w_R(z) \to 0$ as $z \to y$. Therefore, if we define

$$h(y) = \frac{d\omega}{d\sigma}(y),$$

we have

$$\lim_{\Gamma^+(y)\ni z\to y} h_R(z) = -\phi_R(y)\, h(y)\, \vec{n}(y) \quad \text{for } \sigma\text{-a.e. } y \in \partial\Omega.$$

We claim that for all $x \in \Omega$,

$$h_R(x) = -\int \phi_R(y)\, h(y)\, \vec{n}(y)\, d\omega^x(y). \tag{5-8}$$

To prove this, recalling that $h_R$ vanishes at $\infty$, by Theorem 5.8 and Lemma 8.3 in [Jerison and Kenig 1982] it suffices to show that $\mathcal{N}_1 h_R \in L^1(\omega^x)$ for all $x \in \Omega$, where $\mathcal{N}_1$ stands for the operator defined by

$$\mathcal{N}_1 h_R(y) = \sup_{z\in\Gamma_1^+(y)} h_R(z),$$

with $\Gamma_1^+(y) = \Gamma^+(y) \cap \bar{B}(y,1)$. By Lemma 5.2, $w_R$ is bounded, and thus $\mathcal{N}_1 w_R \in L^1(\omega^x)$. Hence it is enough to prove that $\mathcal{N}_1(\phi_R \nabla u) \in L^1(\omega^x)$. To this end, notice that if $z \in \Gamma_1^+(y)$, then

$$|\nabla u(z)| \lesssim \frac{u(z)}{d_\Omega(z)} \approx \frac{\omega(B(y, d_\Omega(z)))}{d_\Omega(z)^n}.$$

Thus,

$$\mathcal{N}_1(\phi_R \nabla u)(y) \lesssim \sup_{0<r\leq 1} \frac{\omega(B(y,r))}{r^n} = \sup_{0<r\leq 1} \frac{1}{r^n} \int_{B(y,r)} |h|\, d\sigma =: \mathcal{M}_1 h(y).$$

Also, $\mathcal{N}_1(\phi_R \nabla u)(y)$ vanishes outside of $B' := \bar{B}(0, 2R+1)$ because in this case $\phi_R(z) = 0$ whenever $z \in \Gamma_1^+(y)$. Therefore,

$$\int \mathcal{N}_1(\phi_R \nabla u)\, d\omega^x = \int_{B'} \mathcal{N}_1(\phi_R \nabla u)\, K_x\, d\sigma \lesssim \left( \int_{B'} |\mathcal{M}_1 h|^2\, d\sigma \right)^{1/2} \left( \int_{B'} (K_x)^2\, d\sigma \right)^{1/2}.$$

By the $L^2(\sigma)$-boundedness of $\mathcal{M}_1$, it follows that

$$\int_{B'} |\mathcal{M}_1 h|^2\, d\sigma = \int_{B'} |\mathcal{M}_1(\chi_{B''} h)|^2\, d\sigma < \infty,$$

where $B'' = \bar{B}(0, 2R+2)$. Also, by Corollary 4.6,

$$\int_{B'} (K_x)^2\, d\sigma < \infty,$$

and so $\mathcal{N}_1(\phi_R \nabla u) \in L^1(\omega^x)$ and (5-8) holds.

From the definition of $h_R$ and (5-8) we deduce that

$$\phi_R(x)\, \nabla u(x) = -\int \phi_R(y)\, h(y)\, \vec{n}(y)\, d\omega^x(y) + w_R(x). \tag{5-9}$$

Hence, letting $R \to \infty$,

$$|\nabla u(x)| \leq \int |h(y)|\, d\omega^x(y) + \liminf_{R\to\infty} |w_R(x)|.$$

By Lemma 5.2(a) and Corollary 4.5 (with $\varepsilon$ small enough), we deduce easily that $w_R(x) \to 0$ as $R \to \infty$, for any fixed $x \in \Omega$, and then the lemma follows. □

Now we wish to obtain a variant of Lemma 5.1 suitable for the case when the pole for harmonic measure is finite. This is what we do in the next lemma.

**Lemma 5.3.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. Suppose that the harmonic measure $\omega^p$ in $\Omega$ with pole at $p \in \Omega$ satisfies $\log(d\omega^p/d\sigma) \in \mathrm{VMO}(\sigma)$. Then, for all $x \in \Omega$ such that $d_\Omega(x) \leq d_\Omega(p)/8$ and all $q_x \in \partial\Omega$ such that $|x - q_x| \leq d_\Omega(p)/8$,*

$$|\nabla g(x,p)| \leq \int_{\partial\Omega} K_p(y)\, d\omega^x(y) + C\, \frac{\omega^p(B(q_x, d_\Omega(p)))}{d_\Omega(p)^n} \left( \frac{d_\Omega(x)}{d_\Omega(p)} \right)^{\alpha/2}. \tag{5-10}$$

*Proof.* Let $\xi \in \partial\Omega$ and take a $C^\infty$ function $\phi$ compactly supported in $B(\xi, d_\Omega(p)/4)$ which is identically 1 on $B(\xi, d_\Omega(p)/8)$, so that $|\nabla^j \phi| \lesssim 1/d_\Omega(p)^j$ for $j = 1, 2$. Note that, in particular, $\phi$ vanishes on $B(p, d_\Omega(p)/4)$. We consider the function

$$w_0(x) = \int_\Omega g(x, y)\, \Delta[\phi\, \nabla g(\cdot\,, p)](y)\, dy \quad \text{for } x \in \Omega.$$

We claim that

$$|w_0(x)| \lesssim \frac{\omega(B(\xi, d_\Omega(p)/8))}{d_\Omega(p)^n} \left(\frac{d_\Omega(x)}{d_\Omega(p)}\right)^{\alpha/2} \quad \text{if } |x-\xi| \le \frac{d_\Omega(p)}{4}. \tag{5-11}$$

The arguments to prove (5-11) are quite similar to the ones in Lemma 5.2. By the relationship between Green's function and harmonic measure and by standard estimates for positive harmonic functions, for all $y \in B(\xi, d_\Omega(p)/4) \cap \Omega$ we have

$$|\nabla g(y, p)| \lesssim \frac{g(y, p)}{d_\Omega(y)} \approx \frac{\omega^p(B(\xi, d_\Omega(p)/4))}{d_\Omega(y)^n} \quad \text{and} \quad |\nabla^2 g(y, p)| \lesssim \frac{g(y, p)}{d_\Omega(y)^2} \approx \frac{\omega^p(B(\xi, d_\Omega(p)/4))}{d_\Omega(y)^{n+1}}.$$

Thus,

$$\begin{aligned} |w_0(x)| &= \left| \int_\Omega g(x, y) \big(\Delta\phi(y)\, \nabla g(y, p) + 2\nabla\phi(y) \cdot \nabla^2 g(y, p)\big)\, dy \right| \\ &\lesssim \int_{A(\xi, d_\Omega(p)/8, d_\Omega(p)/4) \cap \Omega} g(x, y) \left( \frac{\omega^p(B(\xi, d_\Omega(p)/4))}{d_\Omega(p)^2\, d_\Omega(y)^n} + \frac{\omega^p(B(\xi, d_\Omega(p)/4))}{d_\Omega(p)\, d_\Omega(y)^{n+1}} \right) dy \\ &\lesssim \int_{B(\xi, d_\Omega(p)/4) \cap \Omega} g(x, y)\, \frac{\omega^p(B(\xi, d_\Omega(p)/4))}{d_\Omega(p)\, d_\Omega(y)^{n+1}}\, dy. \end{aligned}$$

Notice that the integral on the right-hand side above is very similar to the one on the right-hand side of (5-2). The reader can check that exactly the same arguments and estimates used to prove Lemma 5.2(a) yield (5-11), with $\xi$ instead of 0, $d_\Omega(p)/8$ instead of $R$, $\omega^p$ instead of $\omega$, and $g(y, p)$ instead of $u(y)$. We leave the details for the reader.

From (5-11) it follows that $w_0 \in C^{\alpha/2}(\overline{\Omega})$ and it vanishes at $\partial\Omega$. Further, the function defined by

$$h_0(x) = \phi(x) \nabla g(x, p) - w_0(x), \quad x \in \Omega,$$

is harmonic in $\Omega$, because $\Delta w_0 = \phi\, \nabla g(\cdot\,, p)$. Hence, arguing as in (5-9), we derive

$$\phi(x)\, \nabla g(x, p) = -\int \phi(y)\, K_p(y)\, \vec{n}(y)\, d\omega^x(y) + w_0(x).$$

If $|x - \xi| \le d_\Omega(p)/8$, then $\phi(x) = 1$ and from the last identity and the inequality (5-11) with $\xi = q_x$, we deduce (5-10). □

## 6. The pseudo-blow-up of harmonic measure is surface measure

Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. We recall that harmonic measure with either a finite pole $p \in \Omega$ or pole at infinity is in the $A_\infty(\sigma)$ class of weights by [David and Jerison 1990] or [Semmes 1990] and thus, the Poisson kernel $d\omega/d\sigma$ exists and is positive and finite. We denote by $u$ either the Green's function with pole at $p \in \Omega$ or with pole at infinity and by $h$ the corresponding Poisson kernel (see (3-2) for pole at infinity).

**6A.** ***Pseudo-blow-ups of chord-arc domains.*** Here we introduce the notion of *pseudo-blow-ups* from [Kenig and Toro 2003], but with a slight modification. Let $x_i \in \partial\Omega$ and let $\{r_i\}_{i\geq 1}$ be a sequence of positive numbers so that $\lim_{i\to\infty} r_i = 0$. Consider now the domains

$$\Omega_i = \frac{1}{r_i}(\Omega - x_i),$$

so that $\partial\Omega_i = (1/r_i)(\partial\Omega - x_i)$, and the functions $u_i$ in $\Omega_i$ defined by

$$u_i(x) = \frac{g(r_i x + x_i, p_i)}{r_i\, \omega^{p_i}(B(x_i, r_i))}\, \sigma(B(x_i, r_i)),$$

where either $p_i = \infty$ or $p_i \in \Omega\setminus\{x_i\}$ satisfies

$$\frac{p_i - x_i}{r_i} \to \infty \quad \text{as } i \to \infty.$$

Note that $u_i$ vanishes at $\partial\Omega_i$ and is harmonic in $\Omega_i \setminus \{(p_i - x_i)/r_i\}$. We denote by $d\omega_i = h_i\, d\sigma_i$ the harmonic measure of $\Omega_i$ with pole at infinity or $(p_i - x_i)/r_i$ depending on the pole of $u$, where $\sigma_i = \mathcal{H}^n|_{\partial\Omega_i}$. Moreover, the corresponding Poisson kernel[2] $h_i$ satisfies

$$h_i(x) = \frac{h(r_i x + x_i)}{\omega^{p_i}(B(x_i, r_i))}\, \sigma(B(x_i, r_i)).$$

**Theorem 6.1** [Kenig and Toro 2003, Theorem 4.1]. *If $\Omega \subset \mathbb{R}^{n+1}$ is a chord-arc domain, then there exists a subsequence satisfying*

$$\begin{aligned} \Omega_i &\to \Omega_\infty && \text{in the Hausdorff metric, uniformly on compact sets,} \\ \partial\Omega_i &\to \partial\Omega_\infty && \text{in the Hausdorff metric, uniformly on compact sets,} \end{aligned}$$

*where $\Omega_\infty$ is a chord-arc domain. Moreover, there exists $u_\infty \in C(\overline{\Omega}_\infty)$ such that $u_i \to u_\infty$ uniformly on compact sets which satisfies (3-1) for $\Omega = \Omega_\infty$. Furthermore, $\omega_i \to \omega_\infty$ weakly as Radon measures and $\omega_\infty$ is the harmonic measure of $\Omega_\infty$ with pole at infinity (corresponding to $u_\infty$).*

This was originally shown in [Kenig and Toro 2003] under the assumption that $p_i$ is a fixed point and $x_i$ converges to some point in $\partial\Omega$. However, the same proof gives the result above.

**Theorem 6.2.** *If $\Omega_\infty \subset \mathbb{R}^{n+1}$ and $u_\infty$ are as in Theorem 6.1, then*

$$\sup_{z\in\Omega_\infty} |\nabla u_\infty(z)| \leq 1. \tag{6-1}$$

**Theorem 6.3.** *If $\Omega_\infty \subset \mathbb{R}^{n+1}$ and $u_\infty$ and $\omega_\infty$ are as in Theorem 6.1, then*

$$\frac{d\omega_\infty}{d\sigma_\infty} \geq 1, \quad \mathcal{H}^n\text{-a.e. on } \partial\Omega_\infty, \tag{6-2}$$

*where $\sigma_\infty = \mathcal{H}^n|_{\partial\Omega_\infty}$.*

[2]In fact, this is the Poisson kernel of $\Omega_i$ with pole at $p_i$ modulo a constant factor.

Both theorems were proved in [Kenig and Toro 2003, Theorems 4.2 and 4.3] for Reifenberg flat domains with $n$-AD regular boundary, although, an inspection of the proofs shows that the same arguments, with very minor changes, work also for NTA domains with $n$-AD regular boundary, i.e., for chord-arc domains.

**Corollary 6.4.** *If $\Omega_\infty \subset \mathbb{R}^{n+1}$ and $u_\infty$ and $\omega_\infty$ are as in Theorem 6.1, then*

$$|\nabla u_\infty| = \frac{d\omega_\infty}{d\sigma_\infty} = 1 \quad \mathcal{H}^n\text{-a.e. on } \partial\Omega_\infty. \tag{6-3}$$

*Proof.* Combining (3-19) and (6-1) we get that $d\omega_\infty/d\sigma_\infty \le 1$ for $\mathcal{H}^n$-a.e. on $\partial\Omega_\infty$. Then (6-3) follows from (6-2). □

**Lemma 6.5.** *The subsequence introduced in Theorem 6.1 satisfies $\sigma_i \rightharpoonup \sigma_\infty$ weakly as Radon measures.*

*Proof.* This was essentially proved in Theorem 4.4 in [Kenig and Toro 2003]. The only difference is that instead of invoking [Kenig and Toro 2003, Theorem 2] in the proof, which is particular to the Reifenberg flat case, we just use Corollary 6.4. □

**6B. *Blow-downs of unbounded chord-arc domains.*** In the course of proving our main result we will need to construct the *blow-down* domain with respect to a fixed point $x_0 \in \partial\Omega$ of an unbounded chord-arc domain $\Omega$ such that $d\omega/d\sigma = 1$ $\sigma$-a.e. on $\partial\Omega$ (i.e., $\omega = \sigma$). To do so, we let $x_i = x_0$ for all $i \ge 1$ and a sequence of positive numbers $r_i$ such that $\lim_{i\to\infty} r_i = \infty$. Now we take $\Omega_i$ and $u_i$ as in the construction of pseudo-blow-ups in Section 6A and $p = p_i = \infty$. Then similar (but easier) arguments show that there exists a chord-arc domain $\widetilde{\Omega}$ such that

$$\Omega_i \to \widetilde{\Omega} \quad \text{in the Hausdorff metric, uniformly on compact sets,}$$
$$\partial\Omega_i \to \partial\widetilde{\Omega} \quad \text{in the Hausdorff metric, uniformly on compact sets.}$$

Moreover, there exists $\tilde{u} \in C(\overline{\widetilde{\Omega}})$ such that $u_i \to u_0$ uniformly on compact sets which satisfies

$$\Delta\tilde{u} = 0 \quad \text{in } \widetilde{\Omega}, \qquad \tilde{u} > 0 \quad \text{in } \widetilde{\Omega}, \qquad \tilde{u} = 0 \quad \text{in } \partial\widetilde{\Omega}.$$

## 7. Application of the monotonicity formula of Weiss: blow-downs are planes in $\mathbb{R}^3$

We first introduce the notion of a variational solution of the one-phase free boundary problem in an open ball $B \subset \mathbb{R}^{n+1}$,

$$\begin{cases} u \ge 0 & \text{in } B, \\ \Delta u = 0 & \text{in } B^+(u) := B \cap \{u > 0\}, \\ |\nabla u| = 1 & \text{on } F(u) := \partial B^+(u) \cap B. \end{cases} \tag{7-1}$$

**Definition 7.1.** We define $u \in W^{1,2}_{\text{loc}}(B)$ to be a *variational solution* of (7-1) if

(1) $u \in C(B) \cap C^2(B^+(u))$,

(2) $\chi_{\{u>0\}} \in L^1_{\text{loc}}(B)$ and

(3) the first variation with respect to the functional

$$F(v) := \int_B (|\nabla v|^2 + \chi_{\{v>0\}})\, dm \tag{7-2}$$

vanishes at $v = u$; i.e.,

$$0 = -\frac{d}{d\varepsilon} F(u(x+\varepsilon\phi(x)))|_{\varepsilon=0} = \int_B \big[(|\nabla u|^2 + \chi_{\{u>0\}}) \operatorname{div}\phi - 2\nabla u\, D\phi\, (\nabla u)^T\big]\, dm \tag{7-3}$$

for any $\phi \in C_c^\infty(B; \mathbb{R}^{n+1})$.

**Definition 7.2.** We say that $u$ is a *weak solution* of $\Delta u = \mathcal{H}^n(\partial\{u>0\} \cap \cdot)$ in $B$ if the following are satisfied:

(1) $u \in W^{1,2}_{\text{loc}}(B) \cap C(B^+(u))$, $u \geq 0$ in $B$, and $u$ is harmonic in the open set $\{u > 0\}$.

(2) *Nondegeneracy and regularity*: for any open $D \Subset B$ there exist $0 < c_D \leq C_D < \infty$ such that for any $B(x,r) \subset D$ satisfying $x \in \partial\{u > 0\}$ we have

$$c_D \leq r^{-n-1} \int_{\partial B(x,r)} u\, d\mathcal{H}^n \leq C_D. \tag{7-4}$$

(3) $\{u > 0\}$ is locally in $B$ a set of finite perimeter and

$$-\int \nabla u \cdot \nabla\zeta\, dm = \int_{\partial^*\{u>0\}} \zeta\, d\mathcal{H}^n \tag{7-5}$$

for any $\zeta \in C_c^\infty(B)$, where $\partial^*\{u > 0\}$ stands for the reduced boundary of $\{u > 0\}$.

Let us now record a useful lemma whose proof is contained in the one of [Weiss 1998, Theorem 5.1].

**Lemma 7.3.** *If $u$ is a weak solution of $\Delta u = \mathcal{H}^n(\partial\{u > 0\} \cap \cdot)$ in a ball $B$ in the sense of Definition 7.2, then it is also a variational solution in the ball $B$ in the sense of Definition 7.1.*

**Lemma 7.4.** *Assume that $\Omega_\infty$ is the blow-up domain and $u_\infty$ is the blow-up Green's function constructed in Theorem 6.1. If $B$ is a ball centered on $\partial\{u_\infty > 0\} = \partial\Omega_\infty$, then the extension by zero of $u_\infty$ outside $\{u_\infty > 0\}$ is a weak solution of $\Delta u = \mathcal{H}^n(\partial\{u > 0\} \cap \cdot)$ in $B$.*

*Proof.* By construction, $\Omega_\infty = \{u_\infty > 0\}$, $u_\infty > 0$ in $\Omega_\infty$, $u_\infty = 0$ in $\partial\Omega_\infty$, $u_\infty$ is harmonic in $\Omega_\infty$, $u_\infty \in C(\overline{\Omega}_\infty)$, and $|\nabla u_\infty| \leq 1$ in $\Omega_\infty$. Therefore, it is trivial to see that its extension by zero in the complement of $\Omega_\infty$ satisfies the condition (1) in Definition 7.2 for the ball $B$. Notice also that by Harnack's inequality at the boundary, if $x_r$ is a corkscrew point in $B(x,r) \cap \Omega_\infty$, it holds that

$$\max_{z \in \partial B(x,r) \cap \Omega_\infty} u_\infty(z) = \max_{z \in B(x,r) \cap \Omega_\infty} u_\infty(z) \approx u_\infty(x_r).$$

Therefore, we have that by (3-8) and Corollary 6.4,

$$r^{-n-1} \int_{\partial B(x,r)} u_\infty\, d\mathcal{H}^n \approx \frac{\mathcal{H}^n(\partial B(x,r))}{r^{n+1}}\, u_\infty(x_r) \approx \frac{\omega_\infty(B(x,r))}{\sigma_\infty(B(x,r))} = 1.$$

Since $\partial\Omega_\infty$ is $n$-AD regular, we have that $\mathcal{H}^n|_{\partial\Omega_\infty}$ is locally finite, and thus $\Omega_\infty$ is of locally finite perimeter in $\mathbb{R}^{n+1}$. By the generalized Gauss–Green formula for sets of locally finite perimeter, we infer that

$$\begin{aligned}\int_{\partial\Omega_\infty} \zeta\, d\mathcal{H}^n &= \int_{\partial\Omega_\infty} \zeta\, d\omega_\infty = \int_{\Omega_\infty} u_\infty\, \Delta\zeta\, dm \\ &= \int_{\Omega_\infty} \operatorname{div}(u_\infty \nabla\zeta)\, dm - \int_{\Omega_\infty} \nabla u_\infty \cdot \nabla\zeta\, dm = 0 - \int_{\Omega_\infty} \nabla u_\infty \cdot \nabla\zeta\, dm\end{aligned}$$

for any $\zeta \in C_c^\infty(\mathbb{R}^n)$. Note that $\mathcal{H}^n(\partial\Omega_\infty \setminus \partial^*\Omega_\infty) = 0$ in any NTA domain and thus, condition (3) in Definition 7.2 is satisfied. ☐

We state without proof a lemma from [Jerison and Kamburov 2016] which allows us to conclude that any blow-down domain of $\Omega_\infty$ is in fact a cone.

**Lemma 7.5** [Jerison and Kamburov 2016, Lemma 5.2]. *Let $u$ be a variational solution of (7-1) in $\mathbb{R}^{n+1}$ which is globally Lipschitz. Assume that $0 \in F(u)$ and consider a sequence $R_j \to \infty$. If the sequence*

$$v_j(x) = R_j^{-1} u(R_j x)$$

*converges uniformly on compact sets as $j \to \infty$, its limit is Lipschitz continuous and homogeneous of degree* 1.

**Lemma 7.6.** *Assume that $\Omega_\infty \subset \mathbb{R}^{n+1}$ is the blow-up domain and $u_\infty$ is the blow-up Green's function constructed in Theorem 6.1. If $x \in \partial\Omega_\infty$, then any blow-down domain of $\Omega_\infty$ at $x$ is a cone.*

By a cone we mean a set $F \subset \mathbb{R}^{n+1}$ such that if $x \in F$, then $\lambda x \in F$ for all $\lambda > 0$. A conical domain is a domain which is a cone.

*Proof.* It follows from Lemmas 7.3, 7.4 and 7.5 in view of Section 6B. ☐

**Lemma 7.7.** *If $\Omega_0 \subset \mathbb{R}^3$ is a conical two-sided NTA domain in $\mathbb{R}^3$ with 2-AD-regular boundary such that $d\omega_0/d\sigma_0 = 1$ $\sigma_0$-a.e. in $\partial\Omega_0$, then $\Omega_0$ is a half-space.*

*Proof.* Since $\Omega_0$ is a conical two-sided NTA domain, the intersection of $\Omega_0$ with the sphere $S^2$ is an open connected subset of $S^2$, and the interior of its complement should be another open connected set of $S^2$. Further, as shown in [Caffarelli et al. 2004, Remark 2 and p. 92] by studying the mean curvature of $\partial\Omega_0 \cap S^2$, one deduces that $\partial\Omega_0 \cap S^2$ is a convex curve and $\Omega_0^c$ is a convex cone. One can check that a convex cone in $\mathbb{R}^3$ is a Lipschitz domain, and also its exterior domain. Hence, by the results of Farina and Valdinoci [2010] (or by arguments analogous to the ones in [Caffarelli et al. 2004, p. 92]), $\Omega_0$ is a half-space. ☐

**Corollary 7.8.** *Suppose that $\Omega_0$ is a two-sided NTA domain in $\mathbb{R}^3$ with 2-AD-regular boundary such that $d\omega_0/d\sigma_0 = 1$ $\sigma_0$-a.e. in $\partial\Omega_0$. Then, for any $x \in \partial\Omega_0$,*

$$\lim_{r\to\infty} \Theta_{\partial\Omega_0}(x, r) = 0.$$

*Proof.* This is an immediate consequence of Lemmas 7.6 and 7.7. ☐

## 8. The Alt–Caffarelli theorem

The objective of this section is to explain how to prove the following lemma.

**Lemma 8.1.** *Let $\Omega_0$ be an NTA domain in $\mathbb{R}^{n+1}$ with $n$-AD-regular boundary with constant $C_0$. Suppose $0 \in \partial\Omega_0$ and*

$$\frac{d\omega_0}{d\sigma_0} \equiv 1 \quad \sigma_0\text{-a.e. in } \partial\Omega_0. \tag{8-1}$$

*There exists $\delta_0 > 0$ small enough depending on $n$, the NTA character of $\Omega_0$, and $C_0$ such that if $\mathbb{B} = B(0, 1)$ satisfies*

$$\Theta_{\partial\Omega_0}(\lambda\mathbb{B}) \leq \delta_0 \quad \text{for all } \lambda > 1, \tag{8-2}$$

*then $\Omega_0$ is a half-space.*

Before turning to the proof of this lemma, notice that an immediate consequence of this and Corollary 7.8 is the following.

**Corollary 8.2.** *Suppose that $\Omega_0$ is a two-sided chord-arc in $\mathbb{R}^3$ such that $d\omega_0/d\sigma_0 = 1$ $\sigma_0$-a.e. in $\partial\Omega_0$. Then, $\Omega_0$ is a half-space.*

Lemma 8.1 is essentially proven in [Kenig and Toro 2004], which assumes that the domain is Reifenberg flat. This is a variant of some of the results by Alt and Caffarelli [1981]. In [Kenig and Toro 2004] the authors also assume in the statement of their theorem that

$$|\nabla u_0| \leq \chi_\Omega, \tag{8-3}$$

where $u_0$ is its Green function with pole at infinity. However, this estimate is an immediate consequence of the assumptions of Lemma 8.1, especially (8-1), and Lemma 5.1. Thus, we will only explain how to read and adjust the proof in [Kenig and Toro 2004] in order to obtain the lemma, adding details where necessary.

**Lemma 8.3.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a two-sided $C$-corkscrew domain so that $\Omega_{\text{ext}}$ is also connected. Then whenever $\xi \in \partial\Omega$, $r > 0$, and $\beta_{\partial\Omega}(\xi, r, P) < 1/(2C)$ for some $n$-plane $P$,*

$$\Theta_{\partial\Omega}(\xi, r/2, P) \leq 2\,\beta_{\partial\Omega}(\xi, r, P) \tag{8-4}$$

*and there are half-spaces $H^\pm$ such that*

$$H^+ \cup H^- = \{y : \operatorname{dist}(y, P) > \beta_{\partial\Omega}(\xi, r, P)\},$$
$$H^+ \cap B(\xi, r) \subset \Omega \quad \text{and} \quad H^- \cap B(\xi, r) \subset \Omega_{\text{ext}}.$$

*In particular, if $\pi_P$ is the projection onto $P$, then $\pi_P(\partial\Omega \cap B(\xi, r)) \supseteq \pi_P(B(\xi, r/2))$.*

*Proof.* Without loss of generality, we assume $\xi = 0$, $r = 1$, so $B(\xi, r) = \mathbb{B} = B(0, 1)$. Let $\varepsilon = \beta_{\partial\Omega}(\xi, r, P)$. If $(H^+ \cup H^-) \cap \mathbb{B} \subset \Omega$, then

$$\Omega_{\text{ext}} \cap \mathbb{B} \subset \{y : \operatorname{dist}(y, P) \leq \varepsilon\},$$

but since $\Omega$ has exterior corkscrews, there must be

$$B(y, 1/C) \subset \mathbb{B} \cap \Omega_{\text{ext}} \subset \{y : \operatorname{dist}(y, P) \leq \varepsilon\},$$

which is a contradiction for $\varepsilon < 1/(2C)$. We also get a contradiction if $(H^+ \cup H^-) \cap \mathbb{B} \subset \Omega_{\text{ext}}$, and so $H^\pm \cap \mathbb{B}$ must be in two different components. Assume $H^+ \cap \mathbb{B} \subset \Omega$ and $H^- \subset \Omega_{\text{ext}}$. The last part of the lemma now follows from this, since for any $y \in \pi_P(B(\xi, r))$, the line $\pi_P^{-1}(y)$ must pass through both $H^+$ and $H^-$, and thus it must intersect $\partial\Omega$.

To prove (8-4) it suffices to show that if $x \in \frac{1}{2}\mathbb{B} \cap P$, then $\operatorname{dist}(x, \partial\Omega) \le 2\varepsilon$. Suppose there is $x \in \frac{1}{2}\mathbb{B} \cap P$ so that $B(x, 2\varepsilon) \subset (\partial\Omega)^c$. Then the set

$$U = \mathbb{B} \cap B(x, 2\varepsilon)^c \cap \{y : \operatorname{dist}(y, P) > \varepsilon\}$$

is a connected open subset of $(\partial\Omega)^c$, and hence $U \subset \Omega$ or $U \subset \Omega_{\text{ext}}$. Without loss of generality, we can assume the former case. Then

$$\Omega_{\text{ext}} \cap \mathbb{B} \subset \{y : \operatorname{dist}(y, P) \le \varepsilon\} \cup B(x, 2\varepsilon).$$

But by the exterior corkscrew condition, $B(y, 1/C) \subset \mathbb{B} \cap \Omega_{\text{ext}}$, which is impossible if $\varepsilon < 1/(2C)$. □

The following definition comes from [Kenig and Toro 2004], and it is a variant of one that appears in [Alt and Caffarelli 1981].

**Definition 8.4.** Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain. Let $x_0 \in \partial\Omega$, $\rho > 0$, $\sigma_+, \sigma \in (0, 1)$, $\nu \in \mathbb{S}^n$, and $v$ be the Green function with pole at infinity. We say $v \in F(\sigma_+, \sigma)$ in $B(x_0, \rho)$ in the direction $\nu \in \mathbb{S}^n$ if, for all $x \in B(x_0, \rho)$,

$$v(x) = 0 \quad \text{if } (x - x_0) \cdot \nu \ge \sigma_+ \rho \tag{8-5}$$

and

$$v(x) \ge -(x - x_0) \cdot \nu - \sigma\rho \quad \text{if } (x - x_0) \cdot \nu \le -\sigma\rho. \tag{8-6}$$

Observe that $v \equiv 0$ exactly on $\Omega^c$ and $v > 0$ exactly on $\Omega$, and so

$$v \in F(\sigma, \sigma) \text{ in direction } \nu \text{ in } B(x_0, \rho) \text{ implies } \beta_{\partial\Omega}(x_0, \rho) \le \sigma. \tag{8-7}$$

Indeed, assume $x_0 = 0$, $\rho = 1$, and note that by (8-5), since $v = 0$ only when $\Omega^c$, we have that for $x \in B(x_0, \rho)$,

$$\{x \in \mathbb{B} : x \cdot \nu \ge \sigma\} \subseteq \Omega^c.$$

By (8-6), if $x \cdot \nu < -\sigma\rho$, then

$$v(x) \ge -x \cdot \nu - \sigma > 0$$

and since $v(x) > 0$ only when $x \in \Omega$,

$$\{x \in \mathbb{B} : x \cdot \nu < -\sigma\} \subseteq \Omega.$$

Since $v$ is continuous, we thus have

$$\beta_{\partial\Omega}(0, 1) < \sigma.$$

**Lemma 8.5.** *Let $\Omega$ be a two-sided NTA domain and $v$ the Green function with pole at infinity. Let $x_0 \in \partial\Omega$, $\rho, \sigma > 0$, and $\nu \in \mathbb{S}^n$. If $v \in F(\sigma, 1)$ in $B(x_0, \rho)$ in the direction $\nu$, then $v \in F(2\sigma, C\sigma)$ in $B(x_0, \rho/2)$ in the same direction, where $C = C(n)$.*

*Proof.* The proof is exactly the same as in Lemma 0.4 in [Kenig and Toro 2004]. Its proof and that of Lemma 0.3 in the same paper, upon which it depends, do not require the Reifenberg flat assumption and the proofs are identical. □

**Lemma 8.6.** *Let $\Omega$ be a two-sided NTA domain and $v$ the Green function with pole at infinity. There is some $\varepsilon_0$ small enough so that the following holds. Let $x_0 \in \partial\Omega$, $\rho > 0$, and $\nu \in \mathbb{S}^n$. Given $\theta \in (0,1)$, there is $\sigma_\theta > 0$ and $\eta \in (0,1)$ so that if $0 < \sigma < \sigma_\theta$ and $v \in F(\sigma,\sigma)$ in $B(x_0,\rho)$ in the direction $\nu$ and $\beta_{\partial\Omega}(x_0, 2\rho) < \varepsilon_0$, then $v \in F(\theta\sigma, 1)$ in $B(x_0, \eta\rho)$ in some direction $\nu'$ such that $|\nu - \nu'| < C\sigma$.*

*Proof.* Again, the proof is exactly the same as that of Lemma 0.5 in [Kenig and Toro 2004]. The only time Kenig and Toro use the Reifenberg flatness assumption is to show that the intersection of a cylinder $C$ with the boundary (with axis passing through $Q_0$) has projection in the direction of the cylinder equal to the base of the cylinder (i.e., a ball); see right below equation (0.69) in [Kenig and Toro 2004]. However, we can just replace this with the assumption that $\beta_{\partial\Omega}(x_0, 2\rho) < \varepsilon_0$ is small and then apply Lemma 8.3. □

*Proof of Lemma 8.1.* Let $\theta' \in (0, 1/2)$ and $\delta_0 \in (0, \sigma_{n,\theta'}/(8+2C))$. Note that (8-2) implies that for $r > 1$, there is a plane $P_r$ so that

$$\beta_{\partial\Omega}(0, r, P_r) \le \Theta_{\partial\Omega}(0, r, P_r) \le \delta_0. \tag{8-8}$$

Let $L_r = P_{2r} - \pi_{P_{2r}}(0)$ and let $\nu_r \in \mathbb{S}^n$ be a unit vector orthogonal to $L_r$ so that $r\nu_r/2 \in \Omega^c$. Then

$$\{x \in B(0,r) : x\cdot\nu_r > \delta_0 r\} \subseteq \{x \in B(0,r) : \operatorname{dist}(x, L_r) > \delta_0 r\} \subseteq (\partial\Omega)^c.$$

Since $\{x \in B(0,r) : x\cdot\nu_r > \delta_0 r\}$ and $\Omega^c$ are connected and $r\nu_r/2$ is in their intersection, we actually have

$$\{x \in B(0,r) : x\cdot\nu_r > \delta_0 r\} \subseteq \Omega^c.$$

Hence, $v(x) = 0$ for $x \in B(0,r)$ such that $x\cdot\nu > \delta_0 r$. Furthermore, we trivially have

$$\{x \in B(0,r) : x\cdot\nu_r > r\} = \varnothing$$

and thus $v \in F(\delta_0, 1)$. Lemma 8.5 implies $v \in F(2\delta_0, C\delta_0)$ in $\frac{1}{2}r\mathbb{B}$ in the same direction, and so $v \in F(\delta,\delta)$ in $\frac{1}{2}r\mathbb{B}$, where $\delta = \max\{2, C\}$. Let $\theta' \in (0,1)$. By Lemma 8.6 and (8-8), there is $\eta' \in (0,1)$ (depending only on $\theta'$) so that $v \in F(\theta'\delta, 1)$ in $\frac{1}{2}(\eta' r)\mathbb{B}$. Again, by Lemma 8.5, we have $v \in F(2\theta'\delta, C\theta'\delta)$ in $\frac{1}{4}(\eta' r)\mathbb{B}$, and hence $v \in F(\theta\delta, \theta\delta)$ in $\eta r\mathbb{B}$, where $\theta = \max\{2\theta', C\theta'\}$ and $\eta = \frac{1}{4}\eta'$ in the direction of some vector $\nu \in \mathbb{S}^n$. By (8-7), we have

$$\beta_{\partial\Omega}(0, \eta r) < \theta\delta.$$

Iterating, we get that for all $m \in \mathbb{N}$,

$$v \in F(\theta^m\delta, \theta^m\delta) \quad \text{in } \eta^m r\mathbb{B} \tag{8-9}$$

and

$$\Theta_{\partial\Omega}(0, \eta^m r/2) \le 2\beta_{\partial\Omega}(0, \eta^m r) \le \theta^m\delta.$$

Let $1 < s \ll r$ and pick $m$ so that $\eta^{m+1} r \le s < \eta^m r$. Then this implies

$$\Theta_{\partial\Omega}(0, s/2) \le 2\Theta_{\partial\Omega}(0, \eta^m r/2) \le 2\theta^m\delta = 2\eta^{\log\theta/\log\eta m}\delta \le 2(\eta^{-1} s r^{-1})^{\log\theta/\log\eta}\delta.$$

Thus, by sending $r \to \infty$, we get $\Theta_{\partial\Omega}(0, s/2) = 0$. Since this holds for every $s > 1$, we have that $\partial\Omega$ is equal to an $n$-plane, and since $\Omega$ is connected, it must be a half-space. □

## 9. The proof of Theorem 1.1

Our arguments are very similar to the ones in [Kenig and Toro 2003]. The only difference is that in our pseudo-blow-ups we allow the points $x_i$ to escape to $\infty$. In this way, we are able to show that the outer unit normal $\vec{n}$ belongs to $\mathrm{VMO}(\sigma)$, not only to $\mathrm{VMO}_{\mathrm{loc}}(\sigma)$. For the reader's convenience, we replicate the arguments of [Kenig and Toro 2003] here.

Let

$$\ell = \lim_{r\to 0} \sup_{x\in\partial\Omega} \|\vec{n}\|_*(B(x,r)).$$

We will show $\ell = 0$. Let $x_i \in \partial\Omega$ and $r_i \downarrow 0$ be such that

$$\lim_{i\to\infty}\left(\fint_{B(x_i,r_i)} |\vec{n}-\vec{n}_{B(x_i,r_i)}|^2\, d\sigma\right)^{1/2} = \ell.$$

Let $\Omega_i = (1/r_i)(\Omega - x_i)$ and $u_i^{p_i}$, $\omega_i^{p_i}$ be as in Theorem 6.1. By this theorem, we can pass to a subsequence so that all these quantities converge to some $\Omega_\infty$, $u_\infty$, and $\omega_\infty$. By Lemma 6.5, $\sigma_i$ also converges to $\sigma_\infty = \mathcal{H}^n|_{\partial\Omega_\infty}$. By Lemma 7.7, $\Omega_\infty$ is a half-space (suppose it is $\mathbb{R}^{n+1}_+$) and $\omega_\infty = \mathcal{H}^n|_{\mathbb{R}^n}$. For $\phi$ a smooth, nonnegative, and compactly supported function with $\phi \ge \chi_{\mathbb{B}}$, and $\vec{n}_i$ the outer unit normal to $\partial\Omega_i$, we thus have

$$\begin{aligned}
\lim_{i\to\infty}\int_{\partial\Omega_i\cap\mathbb{B}} |\vec{n}_i+e_{n+1}|^2\, d\sigma_i &\le \lim_{i\to\infty}\int_{\partial\Omega_i} \phi\, |\vec{n}_i+e_{n+1}|^2\, d\sigma_i\\
&= \lim_{i\to\infty}\left(2\int_{\partial\Omega_i}\phi\, d\sigma_i + 2\int_{\partial\Omega_i}\phi\, \vec{n}_i\cdot e_{n+1}\, d\sigma_i\right)\\
&= 2\int_{\mathbb{R}^n}\phi\, d\sigma_\infty + 2\lim_{i\to\infty}\int_{\Omega_i}\operatorname{div}(\phi\, e_{n+1})\, dm\\
&= 2\int_{\mathbb{R}^n}\phi\, d\sigma_\infty + 2\int_{\mathbb{R}^{n+1}_+}\operatorname{div}(\phi\, e_{n+1})\, dm\\
&= 2\int_{\mathbb{R}^n}\phi\, d\sigma_\infty - 2\int_{\mathbb{R}^n}\phi\, e_{n+1}\cdot e_{n+1}\, d\sigma_\infty = 0
\end{aligned}$$

and hence

$$\ell = \lim_{i\to\infty}\left(\fint_{B(x_i,r_i)} |\vec{n}-\vec{n}_{B(x_i,r_i)}|^2 d\sigma\right)^{1/2} \le 2\lim_{i\to\infty}\left(\fint_{B(x_i,r_i)} |\vec{n}+e_{n+1}|^2 d\sigma\right)^{1/2} = 0. \qquad \square$$

**Remark 9.1.** The same arguments as above show that Theorem A by Kenig and Toro is valid as stated in the Introduction. That is, under the assumptions of Theorem A, one deduces that $\vec{n}\in\mathrm{VMO}(\sigma)$, instead of the weaker statement $\vec{n}\in\mathrm{VMO}_{\mathrm{loc}}(\sigma)$ proven in [Kenig and Toro 2003].

## 10. Counterexample for $\mathbb{R}^d$, $d \ge 4$

In this section we show that, for all $d \ge 4$, there exists a two-sided chord-arc unbounded domain $\Omega \subset \mathbb{R}^d$ for which the Poisson kernel with pole at infinity is constant and such that the outer unit normal is not in $\mathrm{VMO}(\sigma)$. Indeed, Hong [2015, Example 1] constructed $u \in C(\mathbb{R}^4)$ such that $u \ge 0$; $u(rx) = ru(x)$,

$r>0$; $\Delta u=0$ in $\Gamma=\{u>0\}$; $\partial\Gamma\setminus\{0\}$ is smooth; $\partial u/\partial\vec{n}=-1$, where $\vec{n}$ is the outward unit normal on $\Gamma$ and $u$ is singular, i.e., $u\neq x_1^+$ (modulo rotations). We describe his example in some detail below.

Since $u$ is homogeneous of degree 1, it is determined by its values on the unit sphere $\mathbb{S}^3\subset\mathbb{R}^4$. Further, $u$ solves the following overdetermined first eigenvalue problem on $\mathbb{S}^{d-1}$ for $d=4$:

$$\begin{cases}\Delta_{\mathbb{S}^{d-1}}u+(d-1)u=0 \quad\text{and}\quad u>0 & \text{in } \Omega:=\Gamma\cap\mathbb{S}^{d-1},\\ \dfrac{\partial u}{\partial\vec{n}}=-1 \quad\text{and}\quad u=0 & \text{in } \partial\Omega:=\partial\Gamma\cap\mathbb{S}^{d-1},\\ u\equiv 0 & \text{in } \Omega^c.\end{cases} \tag{10-1}$$

To be more precise, let us consider in $\mathbb{S}^3\subset\mathbb{R}^4$ the coordinates

$$\begin{aligned} x_1&=\cos\theta\cos\phi, & x_2&=\cos\theta\sin\phi,\\ x_3&=\sin\theta\cos\psi, & x_4&=\sin\theta\sin\psi,\end{aligned} \tag{10-2}$$

where $\theta\in[0,\pi/2]$ and $\phi,\psi\in[0,2\pi]$. Let $u(\theta,\phi,\psi)=\tau f(\theta)$, where $\tau>0$ and $f$ is a sufficiently nice function. To find $u$ that satisfies (10-1), it is enough to solve the ODE

$$\begin{cases}(\sin\theta\cos\theta f')'+\sin\theta\cos\theta f=0, & \theta\in(0,\pi/2),\\ f(0)=1,\ f'(0)=0.\end{cases}$$

Then it is shown in [Hong 2015] that there exists $\theta_0\in(0,\pi/2)$ such that $f(\theta_0)=0$, $f'(\theta_0)<0$ and $f'(\theta)>0$ for all $\theta\in(0,\theta_0)$. If $u$ is defined on $\mathbb{S}^3$ by $u(\theta,\phi,\psi)=(-1/f'(\theta_0))f(\theta)$ for all $\theta\in[0,\theta_0)$ and $u\equiv 0$ in $[\theta_0,\pi/2]$, then $v(x)=v(r\xi)=ru(\xi)$, for $r>0$ and $\xi\in\mathbb{S}^3$, is the solution to the one-phase free boundary problem we are after.

The above-mentioned construction provides us with a domain for which Theorem 1.1 does not hold. Indeed, let

$$\Omega:=\big\{x\in\mathbb{R}^4 : x=r\xi \text{ for some } \xi\in\mathbb{S}^3 \text{ satisfying (10-2) for } \theta\in[0,\theta_0)\big\}=\{v>0\},$$

whose boundary is given by all points $x\in\mathbb{R}^4$ so that $x=r\,\xi$ for some $r>0$ and $\xi\in\mathbb{S}^3$ that satisfies (10-2) for $\theta=\theta_0$. Remark here that as $v$ is a homogeneous, degree-1 function and $v\not\equiv x_1^+$ (under rotation), $\Omega$ is a cone in $\mathbb{R}^4$ but not a half-space. Thus, $\Omega$ is not a Reifenberg flat domain with vanishing constant, which infers that the outward unit normal $\vec{n}$ is not in $\mathrm{VMO}(\partial\Omega)$. Moreover, as the Poisson kernel $h=-\partial u/\partial\vec{n}=1$, it is clear that $\log h\in\mathrm{VMO}$. Therefore, it is enough to show that $\Omega$ is a two-sided chord-arc domain.

To this end, notice that every $x\in\partial\Omega$ satisfies the equation $x_1^2+x_2^2=\cos^2\theta_0$ $x_3^2+x_4^2=\sin^2\theta_0$ while, for $x\in\Omega$,

$$x_1^2+x_2^2=\cos^2\theta>\cos^2\theta_0 \quad\text{and}\quad x_3^2+x_4^2=\sin^2\theta<\sin^2\theta_0.$$

So $\Omega$ coincides with the set of those points $x\in\mathbb{R}^4$ such that

$$x_1^2+x_2^2>(x_3^2+x_4^2)\cot^2\theta_0. \tag{10-3}$$

Therefore, $\Omega$ is bi-Lipschitz equivalent to the domain $\{x\in\mathbb{R}^4 : x_1^2+x_2^2>x_3^2+x_4^2\}$, which is a well-known two-sided chord-arc domain. The AD-regularity is easier to see as the boundary is locally a Lipschitz graph away from the origin by the implicit function theorem, so it is locally AD-regular, and the fact that it is a cone easily gives that it is globally Ahlfors regular. Hence, $\Omega$ is also a two-sided chord-arc domain, which finishes our proof in $\mathbb{R}^4$.

If we set $D := \Omega \otimes \mathbb{R}^{d-4} \subset \mathbb{R}^d$, where $\Omega \subset \mathbb{R}^4$ is the domain just constructed, then $D$ is a two-sided chord-arc domain in $\mathbb{R}^d$ for which the Poisson kernel is constant and such that the outer unit normal is not in VMO($\sigma$).

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# FOCUSING QUANTUM MANY-BODY DYNAMICS, II: THE RIGOROUS DERIVATION OF THE 1D FOCUSING CUBIC NONLINEAR SCHRÖDINGER EQUATION FROM 3D

XUWEN CHEN AND JUSTIN HOLMER

We consider the focusing 3D quantum many-body dynamic which models a dilute Bose gas strongly confined in two spatial directions. We assume that the microscopic pair interaction is attractive and given by $a^{3\beta-1}V(a^{\beta}\,\cdot)$, where $\int V \leqslant 0$ and $a$ matches the Gross–Pitaevskii scaling condition. We carefully examine the effects of the fine interplay between the strength of the confining potential and the number of particles on the 3D $N$-body dynamic. We overcome the difficulties generated by the attractive interaction in 3D and establish new focusing energy estimates. We study the corresponding BBGKY hierarchy, which contains a diverging coefficient as the strength of the confining potential tends to $\infty$. We prove that the limiting structure of the density matrices counterbalances this diverging coefficient. We establish the convergence of the BBGKY sequence and hence the propagation of chaos for the focusing quantum many-body system. We derive rigorously the 1D focusing cubic NLS as the mean-field limit of this 3D focusing quantum many-body dynamic and obtain the exact 3D-to-1D coupling constant.

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## 1. Introduction

Since the experimental achievement of Bose–Einstein condensates (BEC) was reported in [Anderson et al. 1995; Davis et al. 1995] — a feat for which Cornell, Wieman and Ketterle won the 2001 Nobel Prize in Physics — the investigation of this new state of matter has become one of the most active areas of contemporary research. A BEC, first predicted theoretically by Einstein for noninteracting particles in 1925, is a peculiar gaseous state at which particles of integer spin (bosons) occupy a macroscopic quantum state.

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Let $t \in \mathbb{R}$ be the time variable and $\boldsymbol{r}_N = (r_1, r_2, \ldots, r_N) \in \mathbb{R}^{nN}$ be the position vector of $N$ particles in $\mathbb{R}^n$. Then, naively, BEC means that, up to a phase factor solely depending on $t$, the $N$-body wave function $\psi_N(t, \boldsymbol{r}_N)$ satisfies

$$\psi_N(t, \boldsymbol{r}_N) \sim \prod_{j=1}^{N} \varphi(t, r_j)$$

for some one-particle state $\varphi$. That is, every particle takes the same quantum state. Equivalently, there is the Penrose–Onsager formulation of BEC: if we let $\gamma_N^{(k)}$ be the $k$-particle marginal densities associated with $\psi_N$ by

$$\gamma_N^{(k)}(t, \boldsymbol{r}_k; \boldsymbol{r}'_k) = \int \psi_N(t, \boldsymbol{r}_k, \boldsymbol{r}_{N-k}) \overline{\psi}_N(t, \boldsymbol{r}'_k, \boldsymbol{r}_{N-k})\, d\boldsymbol{r}_{N-k}, \quad \boldsymbol{r}_k, \boldsymbol{r}'_k \in \mathbb{R}^{nk}, \tag{1}$$

then BEC equivalently means

$$\gamma_N^{(k)}(t, \boldsymbol{r}_k; \boldsymbol{r}'_k) \sim \prod_{j=1}^{k} \varphi(t, r_j)\bar{\varphi}(t, r'_j). \tag{2}$$

It is widely believed that the cubic nonlinear Schrödinger equation (NLS)

$$i\partial_t \phi = L\phi + \mu|\phi|^2\phi,$$

where $L$ is the Laplacian $-\triangle$ or the Hermite operator $-\triangle + \omega^2|x|^2$, fully describes the one-particle state $\varphi$ in (2), also called the condensate wave function since it characterizes the whole condensate. Such a belief is one of the main motivations for studying the cubic NLS. Here, the nonlinear term $\mu|\phi|^2\phi$ represents a strong on-site interaction taken as a mean-field approximation of the pair interactions between the particles: a repelling interaction gives a positive $\mu$, while an attractive interaction yields a $\mu < 0$. Gross and Pitaevskii proposed such a description of the many-body effect. Thus the cubic NLS is also called the Gross–Pitaevskii equation. Because the cubic NLS is a phenomenological equation of mean-field type, naturally, its validity has to be established rigorously from the many-body system which it is supposed to characterize.

In a series of works [Lieb et al. 2005; Adami et al. 2007; Elgart et al. 2006; Erdős et al. 2006; 2007; 2009; 2010; T. Chen and Pavlović 2011; 2014; X. Chen 2012a; 2013; Benedikter et al. 2015; X. Chen and Holmer 2013; Grillakis and Machedon 2013; Sohinger 2015], it has been proven rigorously that, for a repelling interaction potential with suitable assumptions, relation (2) holds; moreover, the one-particle state $\varphi$ solves the defocusing cubic NLS ($\mu > 0$).

It is then natural to ask if BEC happens (whether relation (2) holds) when we have attractive interparticle interactions and if the condensate wave function $\varphi$ satisfies a focusing cubic NLS ($\mu < 0$) if relation (2) does hold. In contemporary experiments, both positive [Khaykovich et al. 2002; Strecker et al. 2002] and negative [Cornish et al. 2000; Donley et al. 2001] results exist. To present the mathematical interpretations of the experiments, we adopt the notation

$$r_i = (x_i, z_i) \in \mathbb{R}^{2+1}$$

and investigate the procedure of laboratory experiments of BEC subject to attractive interactions according to [Cornish et al. 2000; Donley et al. 2001; Khaykovich et al. 2002; Strecker et al. 2002].

Step A. Confine a large number of bosons, whose interactions are originally *repelling*, inside a trap. Reduce the temperature of the system so that the many-body system reaches its ground state. It is expected that this ground state is a BEC state/factorized state. This step then corresponds to the following mathematical problem:

**Problem 1.** *Show that if $\psi_{N,0}$ is the ground state of the N-body Hamiltonian $H_{N,0}$ defined by*

$$H_{N,0} = \sum_{j=1}^{N} \left(-\triangle_{r_j} + \omega_{0,x}^2 |x_j|^2 + \omega_{0,z}^2 z_j^2\right) + \sum_{1 \leqslant i < j \leqslant N} \frac{1}{a^{3\beta-1}} V_0\left(\frac{r_i - r_j}{a^\beta}\right), \tag{3}$$

*where $V_0 \geqslant 0$, then the marginal densities $\{\gamma_{N,0}^{(k)}\}$ associated with $\psi_{N,0}$, defined in (1), satisfy relation (2).*

Here, the quadratic potential $\omega^2 |\cdot|^2$ stands for the trapping since [Cornish et al. 2000; Donley et al. 2001; Khaykovich et al. 2002; Strecker et al. 2002] and many other experiments of BEC use the harmonic trap and measure the strength of the trap with $\omega$. We use $\omega_{0,x}$ to denote the trapping strength in the $x$-direction and $\omega_{0,z}$ to denote the trapping strength in the $z$-direction, as we will explain later that in order to have a BEC with attractive interaction, either experimentally or mathematically, it is important to have $\omega_{0,x} \neq \omega_{0,z}$. Moreover, we define

$$\frac{1}{a} V_{0,a}(r) = \frac{1}{a^{3\beta-1}} V_0\left(\frac{r}{a^\beta}\right), \quad \beta > 0,$$

to be the interaction potential.[1] On the one hand, $V_{0,a}$ is an approximation of the identity as $a \to 0$ and hence matches the Gross–Pitaevskii description that the many-body effect should be modeled by an on-site strong self-interaction. On the other hand, the extra $1/a$ is to make sure that the Gross–Pitaevskii scaling condition is satisfied. This step is exactly the same as the preparation of the experiments with repelling interactions, and satisfactory answers to Problem 1 have been given in [Lieb et al. 2004].

Step B. Use the property of Feshbach resonance, strengthen the trap (increase $\omega_{0,x}$ or $\omega_{0,z}$) to make the interaction attractive and observe the evolution of the many-body system. This technique continuously controls the sign and the size of the interaction in a certain range.[2] The system is then time-dependent. In order to observe BEC, the factorized structure obtained in Step A must be preserved in time. Assuming this to be the case, we then reset the time so that $t = 0$ represents the point at which this Feshbach-resonance phase is complete. The subsequent evolution should then be governed by a focusing time-dependent $N$-body Schrödinger equation with an attractive-pair interaction $V$ subject to an asymptotically factorized initial datum. The confining strengths are different from Step A as well and we denote them by $\omega_x$ and $\omega_z$. A mathematically precise statement is the following:

---

[1] From here on, we consider the $\beta > 0$ case solely. For $\beta = 0$ (the Hartree dynamic), see [Fröhlich et al. 2009; Erdős and Yau 2001; Knowles and Pickl 2010; Rodnianski and Schlein 2009; Michelangeli and Schlein 2012; Grillakis et al. 2010; 2011; X. Chen 2012b; Ammari and Nier 2011; 2008; L. Chen et al. 2011].

[2] See [Cornish et al. 2000, Figure 1; Khaykovich et al. 2002, Figure 2; Strecker et al. 2002, Figure 1] for graphs of the relationship between $\omega$ and $V$.

**Problem 2.** *Let $\psi_N(t, \boldsymbol{x}_N)$ be the solution to the $N$-body Schrödinger equation*

$$i\partial_t\psi_N = \sum_{j=1}^{N}(-\triangle_{r_j} + \omega_x^2|x_j|^2 + \omega_z^2 z_j^2)\psi_N + \sum_{1\leqslant i<j\leqslant N} \frac{1}{a^{3\beta-1}} V\left(\frac{r_i - r_j}{a^\beta}\right)\psi_N, \tag{4}$$

*where $V \leqslant 0$, with $\psi_{N,0}$ from Step A as initial datum. Prove that the marginal densities $\{\gamma_N^{(k)}(t)\}$ associated with $\psi_N(t, \boldsymbol{x}_N)$ satisfy relation (2).*[3]

In the experiment [Cornish et al. 2000] by Cornell and Wieman's group (the JILA group), once the interaction is turned attractive, the condensate suddenly shrinks to below the resolution limit; then after $\sim 5$ ms, the many-body system blows up. That is, there is no BEC once the interaction becomes attractive. Moreover, there is no condensate wave function due to the absence of the condensate. Hence, the current NLS theory, which is about the condensate wave function when there is a condensate, cannot explain this 5 ms of time or the blow up. This is currently an open problem in the study of quantum many-body systems. The JILA group later conducted finer experiments and remarked in [Donley et al. 2001, p. 299] that these are simple systems with dramatic behavior, and this behavior provides puzzling results when mean-field theory is tested against them.

In [Khaykovich et al. 2002; Strecker et al. 2002], the particles are confined in a strongly anisotropic cigar-shape trap to simulate a 1D system. That is, $\omega_x \gg \omega_z$. In this case, the experiment is a success in the sense that one obtains a persistent BEC after the interaction is switched to attractive. Moreover, a soliton is observed in [Khaykovich et al. 2002] and a soliton train is observed in [Strecker et al. 2002]. The solitons in these two works have different motion patterns.

In [X. Chen and Holmer 2016b], we have studied the simplified 1D version of (4) as a model case and derived the 1D focusing cubic NLS from it. In the present paper, we consider the full 3D problem of (4), as in the experiments [Khaykovich et al. 2002; Strecker et al. 2002]: We take $\omega_z = 0$ and let $\omega_x \to \infty$ in (4). We derive rigorously the 1D cubic focusing NLS directly from a real 3D quantum many-body system. Here, "directly" means that we are not passing through any 3D cubic NLS. On the one hand, one infers from the experiment [Cornish et al. 2000] that not only it is very difficult to prove the 3D focusing NLS as the mean-field limit of a 3D focusing quantum many-body dynamic, but such a limit also may not be true. On the other hand, the route which first derives

$$i\partial_t\varphi = -\triangle_x + \omega^2|x|^2\varphi - \partial_z^2\varphi - |\varphi|^2\varphi \tag{5}$$

as an $N \to \infty$ limit, from the 3D $N$-body dynamic, and then considers the $\omega \to \infty$ limit of (5), corresponds to the iterated limit ($\lim_{\omega\to\infty}\lim_{N\to\infty}$) of the $N$-body dynamic; i.e., the 1D focusing cubic NLS coming from such a path approximates the 3D focusing $N$-body dynamic when $\omega$ is large and $N$ is infinity (if not substantially larger than $\omega$). In experiments, it is fully possible to have $N$ and $\omega$ comparable to each other. In fact, $N$ is about $10^4$ and $\omega$ is about $10^3$ in [Görlitz et al. 2001; Stock et al. 2005; Hadzibabic et al. 2006; Desbuquois et al. 2012]. Moreover, as seen in the experiment [Donley et al. 2001], even if $\omega_x$ is one digit larger than $\omega_z$, negative result persists if $N$ is three digits larger than $\omega_x$. Thus, in this

[3] Since $\omega \neq \omega_0$, $V \neq V_0$, one could not expect that $\psi_{N,0}$, the ground state of (3), is close to the ground state of (4).

paper, we derive rigorously the 1D focusing cubic NLS as the double limit ($\lim_{N,\omega\to\infty}$) of a real focusing 3D quantum $N$-body dynamic directly, without passing through any 3D cubic NLS. Furthermore, the interaction between the two parameters $N$ and $\omega$ plays a central role. To be specific, we establish the following theorem.

**Theorem 1.1** (main theorem). *Assume that the pair interaction $V$ is an even Schwartz class function which has a nonpositive integral, i.e., $\int_{\mathbb{R}^3} V(r)\,dr \leqslant 0$, but may not be negative everywhere. Let $\psi_{N,\omega}(t,\boldsymbol{r}_N)$ be the $N$-body Hamiltonian evolution $e^{itH_{N,\omega}}\psi_{N,\omega}(0)$ with the focusing $N$-body Hamiltonian $H_{N,\omega}$ given by*

$$H_{N,\omega}=\sum_{j=1}^{N}(-\triangle_{r_j}+\omega^2|x_j|^2)+\sum_{1\leqslant i<j\leqslant N}(N\omega)^{3\beta-1}V((N\omega)^{\beta}(r_i-r_j)) \tag{6}$$

*for some $\beta\in\left(0,\frac{3}{7}\right)$. Let $\{\gamma_{N,\omega}^{(k)}\}$ be the family of marginal densities associated with $\psi_{N,\omega}$. Suppose that the initial datum $\psi_{N,\omega}(0)$ verifies the following conditions*:

(a) *$\psi_{N,\omega}(0)$ is normalized; that is, $\|\psi_{N,\omega}(0)\|_{L^2}=1$,*

(b) *$\psi_{N,\omega}(0)$ is asymptotically factorized in the sense that*

$$\lim_{N,\omega\to\infty}\operatorname{Tr}\left|\frac{1}{\omega}\gamma_{N,\omega}^{(1)}\left(0,\frac{x_1}{\sqrt{\omega}},z_1;\frac{x_1'}{\sqrt{\omega}},z_1'\right)-h(x_1)h(x_1')\phi_0(z_1)\overline{\phi}_0(z_1')\right|=0 \tag{7}$$

*for some one-particle state $\phi_0\in H^1(\mathbb{R})$ and $h$ is the normalized ground state for the 2D Hermite operator $-\triangle_x+|x|^2$, i.e., $h(x)=\pi^{-\frac{1}{2}}e^{-\frac{1}{2}|x|^2}$.*

(c) *Away from the $x$-directional ground-state energy, $\psi_{N,\omega}(0)$ has finite energy per particle*:

$$\sup_{\omega,N}\frac{1}{N}\left\langle\psi_{N,\omega}(0),(H_{N,\omega}-2N\omega)\psi_{N,\omega}(0)\right\rangle\leqslant C.$$

*Then there exist $C_1$ and $C_2$ which depend solely on $V$ such that $\forall k\geqslant 1$, $t\geqslant 0$, and $\varepsilon>0$, we have the convergence in trace norm* (*propagation of chaos*)

$$\lim_{\substack{N,\omega\to\infty\\ C_1N^{v_1(\beta)}\leqslant\omega\leqslant C_2N^{v_2(\beta)}}}\operatorname{Tr}\left|\frac{1}{\omega^k}\gamma_{N,\omega}^{(k)}\left(t,\frac{\boldsymbol{x}_k}{\sqrt{\omega}},\boldsymbol{z}_k;\frac{\boldsymbol{x}_k'}{\sqrt{\omega}},\boldsymbol{z}_k'\right)-\prod_{j=1}^{k}h(x_j)h(x_j')\phi(t,z_j)\overline{\phi}(t,z_j')\right|=0, \tag{8}$$

*where $v_1(\beta)$ and $v_2(\beta)$ are defined by*

$$v_1(\beta)=\frac{\beta}{1-\beta}, \tag{9}$$

$$v_2(\beta)=\min\left(\frac{1-\beta}{\beta},\ \frac{\frac{3}{5}-\beta}{\beta-\frac{1}{5}}\mathbf{1}_{\beta\geqslant\frac{1}{5}}+\infty\cdot\mathbf{1}_{\beta<\frac{1}{5}},\ \frac{2\beta}{1-2\beta}-,\ \frac{\frac{7}{8}-\beta}{\beta}\right) \tag{10}$$

(see Figure 1) *and $\phi(t,z)$ solves the 1D focusing cubic NLS with the 3D-to-1D coupling constant $b_0\left(\int|h(x)|^4\,dx\right)$, that is,*

$$i\partial_t\phi=-\partial_z^2\phi-b_0\left(\int|h(x)|^4\,dx\right)|\phi|^2\phi\quad\text{in }\mathbb{R} \tag{11}$$

*with initial condition $\phi(0,z)=\phi_0(z)$ and $b_0=\left|\int V(r)\,dr\right|$.*

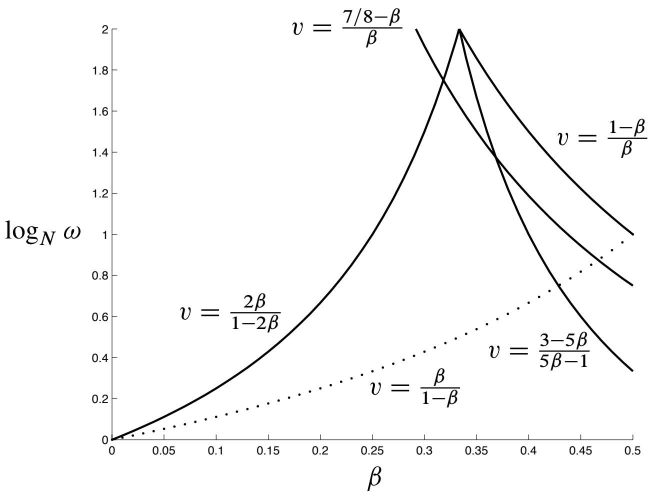


**Figure 1.** A graph of the various rational functions of $\beta$ appearing in (9) and (10). In Theorems 1.1 and 1.2, the limit $(N, \omega) \to \infty$ is taken with $v_1(\beta) \leqslant \log_N \omega \leqslant v_2(\beta)$. The region of validity is above the dashed curve and below the solid curves. It is a nonempty region for $0 < \beta \leqslant \frac{3}{7}$. As shown here, there are values of $\beta$ for which $v_1(\beta) \leqslant 1 \leqslant v_2(\beta)$, which allows $N \sim \omega$, as in [Cornish et al. 2000; Donley et al. 2001; Khaykovich et al. 2002; Strecker et al. 2002; Görlitz et al. 2001; Stock et al. 2005; Hadzibabic et al. 2006; Desbuquois et al. 2012]. Moreover, our result includes part of the $\beta > \frac{1}{3}$ self-interaction region. We will explain why we call the $\beta > \frac{1}{3}$ case self-interaction later in Introduction. We remark that it is not a coincidence that three restrictions intersect at $\beta = \frac{1}{3}$.

Theorem 1.1 is equivalent to the following theorem.

**Theorem 1.2** (main theorem). *Assume that the pair interaction $V$ is an even Schwartz class function which has a nonpositive integral, i.e., $\int_{\mathbb{R}^3} V(r)\,dr \leqslant 0$, but may not be negative everywhere. Let $\psi_{N,\omega}(t, \boldsymbol{r}_N)$ be the $N$-body Hamiltonian evolution $e^{itH_{N,\omega}}\psi_{N,\omega}(0)$, where the focusing $N$-body Hamiltonian $H_{N,\omega}$ is given by (6) for some $\beta \in \left(0, \frac{3}{7}\right)$. Let $\{\gamma^{(k)}_{N,\omega}\}$ be the family of marginal densities associated with $\psi_{N,\omega}$. Suppose that the initial datum $\psi_{N,\omega}(0)$ is normalized, asymptotically factorized in the sense of* (a) *and* (b) *of Theorem 1.1 and satisfies the energy condition that*

(c′) *there is a $C > 0$ such that*

$$\left\langle \psi_{N,\omega}(0), (H_{N,\omega} - 2N\omega)^k \psi_{N,\omega}(0) \right\rangle \leqslant C^k N^k, \quad \forall k \geqslant 1. \tag{12}$$

*Then there exist $C_1$, $C_2$ which depend solely on $V$ such that $\forall k \geqslant 1$, $\forall t \geqslant 0$, we have the convergence in trace norm (propagation of chaos)*

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_2(\beta)}}} \operatorname{Tr}\left| \frac{1}{\omega^k} \gamma^{(k)}_{N,\omega}\left(t, \frac{\boldsymbol{x}_k}{\sqrt{\omega}}, \boldsymbol{z}_k; \frac{\boldsymbol{x}'_k}{\sqrt{\omega}}, \boldsymbol{z}'_k\right) - \prod_{j=1}^{k} h(x_j)h(x'_j)\phi(t, z_j)\bar{\phi}(t, z'_j) \right| = 0,$$

*where $v_1(\beta)$ and $v_2(\beta)$ are given by* (9) *and* (10) *and $\phi(t, z)$ solves the* 1*D focusing cubic NLS* (11).

We remark that the assumptions in Theorem 1.1 are reasonable assumptions on the initial datum coming from Step A. In [Lieb et al. 2004, (1.10)], a satisfying answer has been found by Lieb, Seiringer, and Yngvason for Step A (Problem 1) in the $\omega_{0,x} \gg \omega_{0,z}$ case. For convenience, set $\omega_{0,z} = 1$ in the defocusing $N$-body Hamiltonian (3) in Step A. Let $\operatorname{scat}(W)$ denote the 3D scattering length of the potential $W$. By [Erdős et al. 2007, Lemma A.1], for $0 < \beta \leqslant 1$ and $a \ll 1$, we have

$$\operatorname{scat}\left(a \cdot \frac{1}{a^{3\beta}} V\left(\frac{r}{a^{\beta}}\right)\right) \sim \begin{cases} a/(8\pi) \int_{\mathbb{R}^3} V & \text{if } 0 < \beta < 1, \\ a \operatorname{scat}(V) & \text{if } \beta = 1. \end{cases}$$

Lieb et al. [2004, (1.10)] define the quantity $g = g(\omega_{0,x}, N, a)$ by

$$g := 8\pi a \omega_{0,x}\left(\int |h(x)|^4 \, dx\right).$$

Then if $Ng \sim 1$, they proved in Theorem 5.1 of the same work that BEC happens in Step A and the Gross–Pitaevskii limit holds.[4] To be specific, they proved that

$$\lim_{N,\omega_{0,x}\to\infty} \operatorname{Tr}\left| \frac{1}{\omega_{0,x}} \gamma^{(1)}_{N,\omega_{0,x}}\left(0, \frac{x_1}{\sqrt{\omega_{0,x}}}, z_1; \frac{x_1'}{\sqrt{\omega_{0,x}}}, z_1'\right) - h(x_1)h(x_1')\phi_0(z_1)\bar{\phi}_0(z_1') \right| = 0$$

provided that $\phi_0$ is the minimizer to the 1D defocusing NLS energy functional

$$E_{Ng} = \int_{\mathbb{R}} \left(|\partial_z \phi(z)|^2 + z^2|\phi(z)|^2 + 4\pi Ng|\phi(z)|^4\right) dz \tag{13}$$

subject to the constraint $\|\phi\|_{L^2(\mathbb{R})} = 1$. Hence, the assumptions in Theorem 1.1 are reasonable assumptions on the initial datum drawn from Step A. To be specific, we have chosen $a = (N\omega)^{-1}$ in the interaction so that $Ng \sim 1$ and assumptions (a), (b) and (c) are the conclusions of [Lieb et al. 2004, Theorem 5.1].[5]

The equivalence of Theorems 1.1 and 1.2 for asymptotically factorized initial data is well known. In the main part of this paper, we prove Theorem 1.2 in full detail. For completeness, we discuss briefly how to deduce Theorem 1.1 from Theorem 1.2 in Appendix B.

To our knowledge, Theorems 1.1 and 1.2 offer the first rigorous derivation of the 1D focusing cubic NLS (11) from the 3D focusing quantum $N$-body dynamic (6). Moreover, our result covers part of the $\beta > \frac{1}{3}$ self-interaction region in 3D. As pointed out in [Elgart et al. 2006], the study of Step B is of particular interest when $\beta \in \left(\frac{1}{3}, 1\right]$ in 3D. The reason is the following. The initial datum coming from Step A is the ground state of (3) with $\omega_{0,x}, \omega_{0,z} \neq 0$ and hence is localized in space. We can assume all $N$ particles are in a box of length 1. Let the effective radius of the pair interaction $V$ be $R_0$, then the effective radius of $V((N\omega)^{\beta}(r_i - r_j))$ is about $R_0/(N\omega)^{\beta}$. Thus every particle in the box interacts with $(R_0/(N\omega)^{\beta})^3 \times N$ other particles. Thus, for $\beta > \frac{1}{3}$ and large $N$, every particle interacts with only itself. This exactly matches the Gross–Pitaevskii theory that the many-body effect should be modeled

[4] This corresponds to Region 2 of [Lieb et al. 2004]. The other four regions are the ideal gas case, the 1D Thomas–Fermi case, the Lieb–Liniger case, and the Girardeau–Tonks case. As mentioned on page 388 of that work, BEC is not expected in the Lieb–Liniger and the Girardeau–Tonks cases, and is an open problem in the Thomas–Fermi case; we deal only with Region 2 in this paper.

[5] We remark that the interaction potential $N^{3\beta-1}\omega^{3\beta} V((N\omega)^{\beta}(r_i - r_j))$, which looks like a "direct" extension of the interaction potential from the $n$D-to-$n$D work, does not satisfy $Ng \sim 1$ in the $N, \omega \to \infty$ process.

by a strong on-site self-interaction. Therefore, for the mathematical justification of the Gross–Pitaevskii theory, it is of particular interest to prove Theorems 1.1 and 1.2 for self-interaction $\left(\beta > \frac{1}{3}\right)$.

A main tool used to prove Theorem 1.2 is the analysis of the BBGKY hierarchy of

$$\left\{ \tilde{\gamma}_{N,\omega}^{(k)}(t) = \frac{1}{\omega^k} \gamma_{N,\omega}^{(k)}\left(t, \frac{\boldsymbol{x}_k}{\sqrt{\omega}}, z_k; \frac{\boldsymbol{x}'_k}{\sqrt{\omega}}, z'_k\right) \right\}_{k=1}^{N}$$

as $N, \omega \to \infty$. In the classical setting, deriving equations of mean-field type by studying the limit of the BBGKY hierarchy was proposed by Kac and demonstrated by Lanford's work on the Boltzmann equation. In the quantum setting, the usage of the BBGKY hierarchy was suggested by Spohn [1980] and was proven successful by Elgart, Erdős, Schlein, and Yau in their fundamental papers [Elgart et al. 2006; Erdős et al. 2006; 2007; 2009; 2010],[6] which rigorously derive the 3D cubic defocusing NLS from a 3D quantum many-body dynamic with repulsive-pair interactions and no trapping. The Elgart–Erdös–Schlein–Yau program[7] consists of two principal parts: in one part, they consider the sequence of the marginal densities $\{\gamma_N^{(k)}\}$ associated with the Hamiltonian evolution $e^{itH_N}\psi_N(0)$, where

$$H_N = \sum_{j=1}^{N} -\triangle_{r_j} + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} N^{3\beta} V(N^{\beta}(r_i - r_j)),$$

and prove that an appropriate limit, as $N \to \infty$, solves the 3D Gross–Pitaevskii hierarchy

$$i\partial_t \gamma^{(k)} + \sum_{j=1}^{k} [\triangle_{r_k}, \gamma^{(k)}] = b_0 \sum_{j=1}^{k} \mathrm{Tr}_{r_{k+1}}[\delta(r_j - r_{k+1}), \gamma^{(k+1)}] \quad \text{for all } k \geqslant 1. \tag{14}$$

In another part, they show that hierarchy (14) has a unique solution which is therefore a completely factorized state. However, the uniqueness theory for hierarchy (14) is surprisingly delicate due to the fact that it is a system of infinitely many coupled equations over an unbounded number of variables. By assuming a space-time bound on the limit of $\{\gamma_N^{(k)}\}$, Klainerman and Machedon [2008] gave another uniqueness theorem regarding (14) through a collapsing estimate originating from the multilinear Strichartz estimates and a board game argument inspired by the Feynman graph argument in [Erdős et al. 2007].

The method by Klainerman and Machedon [2008] was taken up by Kirkpatrick, Schlein, and Staffilani [Kirkpatrick et al. 2011], who derived the 2D cubic defocusing NLS from the 2D quantum many-body dynamic; by T. Chen and Pavlović [2011], who considered the 1D and 2D three-body repelling interaction problem; by X. Chen [2012a; 2013], who investigated the defocusing problem with trapping in 2D and 3D; and by X. Chen and J. Homer [2013], who proved the effectiveness of the defocusing 3D to 2D reduction problem. Such a method has also inspired the study of the general existence theory of hierarchy (14); see [T. Chen et al. 2010; 2012; T. Chen and Pavlović 2010; Gressman et al. 2014; Sohinger and Staffilani 2015].

One main open problem in the uniqueness theory of Klainerman–Machedon type is the verification of the uniqueness condition in 3D, though it is fully solved in 1D and 2D using trace theorems in [Kirkpatrick et al. 2011]. For the 3D defocusing problem without traps, T. Chen and Pavlović [2014] showed that,

[6] Around the same time, there was the 1D defocusing work [Adami et al. 2007].

[7] See [Benedikter et al. 2015; Grillakis and Machedon 2013; Pickl 2011] for different approaches.

for $\beta \in \left(0, \frac{1}{4}\right)$, the limit of the BBGKY sequence satisfies the uniqueness condition.[8] X. Chen [2013] extended and simplified their method to study the 3D trapping problem for $\beta \in \left(0, \frac{2}{7}\right]$. The $\beta \in \left(0, \frac{2}{7}\right]$ result by X. Chen was then extended to $\beta \in \left(0, \frac{2}{3}\right)$ using $X_b$ spaces and Littlewood–Paley theory in [X. Chen and Holmer 2016c] and further to $\beta < 1$ in [X. Chen and Holmer 2016a] via correlation structures and many-body scattering process. The $\beta = 1$ case is still open.

Using a version of the quantum de Finetti theorem from [Lewin et al. 2014], T. Chen, Hainzl, Pavlović, and Seiringer provided an alternative proof to the uniqueness theorem in [Erdős et al. 2007] and showed that it is an unconditional uniqueness result in the sense of NLS theory. With this method, Sohinger [2015] derived the 3D defocusing cubic NLS in the periodic case. See also [X. Chen and Smith 2014; Hong et al. 2015].

Recently, the first step in the mass critical focusing case has been taken in [X. Chen and Holmer 2016d].

***Organization of the paper.*** We first outline the proof of our main theorem, Theorem 1.2, in Section 2. The components of the proof are in Sections 3, 4, and 5.

The first main part is the proof of the needed focusing energy estimate, stated and proved as Theorem 3.1 in Section 3. The main difficulty in establishing the energy estimate is understanding the interplay between two parameters $N$ and $\omega$. On the one hand, as suggested by the experiments [Cornish et al. 2000; Donley et al. 2001; Khaykovich et al. 2002; Strecker et al. 2002], in order to have to a tensor product state (BEC) in this focusing setting, one has to explore "the 1D feature" of the 3D focusing $N$-body Hamiltonian (6), which comes from a large $\omega$. At the same time, an $N$ too large would allow the 3D effect to dominate, and one has to avoid this. This suggests that an inequality of the form $N^{v_1(\beta)} \leqslant \omega$ is a natural requirement. On the other hand, according to the uncertainty principle, in 3D, as the $x$-component of the particles' position becomes more and more determined to be 0, the $x$-component of the momentum and thus the energy must blow up. Hence the energy of the system is dominated by its $x$-directional part, which is in fact infinity as $\omega \to \infty$. Since the particles are interacting via 3D potential, to avoid the excessive $x$-directional energy being transferred to the $z$-direction, during the $N, \omega \to \infty$ process, $\omega$ cannot be too large either. Such a problem is totally new and does not exist in the 1D model [X. Chen and Holmer 2016b]. It suggests that an inequality of the form $\omega \leqslant N^{v_2(\beta)}$ is a natural requirement.

The second main part of the proof is the analysis of the focusing "$\infty - \infty$" BBGKY hierarchy of

$$\left\{ \tilde{\gamma}^{(k)}_{N,\omega}(t) = \frac{1}{\omega^k} \gamma^{(k)}_{N,\omega}\left(t, \frac{\boldsymbol{x}_k}{\sqrt{\omega}}, z_k; \frac{\boldsymbol{x}'_k}{\sqrt{\omega}}, z'_k\right) \right\}_{k=1}^{N}$$

as $N, \omega \to \infty$. With our definition, the sequence of the marginal densities $\{\tilde{\gamma}^{(k)}_{N,\omega}\}_{k=1}^{N}$ satisfies the BBGKY hierarchy

$$i\partial_t \tilde{\gamma}^{(k)}_{N,\omega} = \omega \sum_{j=1}^{k} [-\Delta_{x_j} + |x_j|^2, \tilde{\gamma}^{(k)}_{N,\omega}] + \sum_{j=1}^{k} [-\partial_{z_j}^2, \tilde{\gamma}^{(k)}_{N,\omega}] \\ + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant k} [V_{N,\omega}(r_i - r_j), \tilde{\gamma}^{(k)}_{N,\omega}] + \frac{N-k}{N} \sum_{j=1}^{k} \operatorname{Tr}_{r_{k+1}} [V_{N,\omega}(r_j - r_{k+1}), \tilde{\gamma}^{(k+1)}_{N,\omega}],$$

[8] See also [T. Chen and Taliaferro 2014].

where $V_{N,\omega}$ is defined in (17). We call it an "$\infty-\infty$" BBGKY hierarchy because it is not clear whether the term

$$\omega[-\Delta_{x_j}+|x_j|^2,\,\tilde{\gamma}^{(k)}_{N,\omega}]$$

tends to a limit as $N,\omega\to\infty$. Since $\tilde{\gamma}^{(k)}_{N,\omega}$ is not a factorized state for $t>0$, one cannot expect the commutator to be zero; though it is zero if $\tilde{\gamma}^{(k)}_{N,\omega}$ is exactly the limit in (8). This is in strong contrast with the $n$D-to-$n$D work[9] [Adami et al. 2007; Elgart et al. 2006; Erdős et al. 2006; 2007; 2009; 2010; T. Chen and Pavlović 2011; 2014; X. Chen 2012a; 2013; Sohinger 2015] in which the formal limit of the corresponding BBGKY hierarchy is clear. With the aforementioned focusing energy estimate, we find that this diverging coefficient is counterbalanced by the limiting structure of the density matrices and establish the weak* compactness and convergence of this focusing BBGKY hierarchy in Sections 4 and 5.

## 2. Proof of the main theorem

We start by setting up some notation for the rest of the paper. Recall $h(x)=\pi^{-\frac{1}{2}}e^{-\frac{1}{2}|x|^2}$, which is the ground state for the 2D Hermite operator $-\triangle_x+|x|^2$; i.e., it solves $(-2-\Delta_x+|x|^2)h=0$. Then the normalized ground-state eigenfunction $h_\omega(x)$ of $-\triangle_x+\omega^2|x|^2$ is given by $h_\omega(x)=\omega^{\frac{1}{2}}h(\omega^{\frac{1}{2}}x)$; i.e., it solves

$$(-2\omega-\triangle_x+\omega^2|x|^2)h_\omega=0.$$

In particular, $h_1=h$. Noticing that both of the convergences (7) and (8) involve scaling, we introduce the rescaled solution

$$\tilde{\psi}_{N,\omega}(t,\boldsymbol{r}_N):=\frac{1}{\omega^{\frac{1}{2}N}}\psi_{N,\omega}\left(t,\frac{\boldsymbol{x}_N}{\sqrt{\omega}},\boldsymbol{z}_N\right) \tag{15}$$

and the rescaled Hamiltonian

$$\widetilde{H}_{N,\omega}=\left[\sum_{j=1}^{N}-\partial_{z_j}^2+\omega(-\Delta_x+|x|^2)\right]+\frac{1}{N}\sum_{1\leqslant i<j\leqslant N}V_{N,\omega}(r_i-r_j), \tag{16}$$

where

$$V_{N,\omega}(r)=N^{3\beta}\omega^{3\beta-1}V\left(\frac{(N\omega)^\beta}{\sqrt{\omega}}x,(N\omega)^\beta z\right). \tag{17}$$

Then

$$(\widetilde{H}_{N,\omega}\tilde{\psi}_{N,\omega})(t,\boldsymbol{x}_N,\boldsymbol{z}_N)=\frac{1}{\omega^{\frac{1}{2}N}}(H_{N,\omega}\psi_{N,\omega})\left(t,\frac{\boldsymbol{x}_N}{\sqrt{\omega}},\boldsymbol{z}_N\right),$$

and hence when $\psi_{N,\omega}(t)$ is the Hamiltonian evolution given by (6) and $\tilde{\psi}_{N,\omega}$ is defined by (15), we have

$$\tilde{\psi}_{N,\omega}(t,\boldsymbol{r}_N)=e^{it\widetilde{H}_{N,\omega}}\tilde{\psi}(0,\boldsymbol{r}_N).$$

[9] Here, "$n$D-to-$n$D" means "deriving the $n$D NLS equation from the $n$D many-body evolution".

If we let $\{\tilde{\gamma}^{(k)}_{N,\omega}\}_{k=1}^{N}$ be the marginal densities associated with $\tilde{\psi}_{N,\omega}$, then $\{\tilde{\gamma}^{(k)}_{N,\omega}\}_{k=1}^{N}$ satisfies the "$\infty-\infty$" focusing BBGKY hierarchy

$$\begin{aligned} i\partial_t \tilde{\gamma}^{(k)}_{N,\omega} &= \omega \sum_{j=1}^{k} [-\Delta_{x_j} + |x_j|^2, \tilde{\gamma}^{(k)}_{N,\omega}] + \sum_{j=1}^{k} [-\partial_{z_j}^2, \tilde{\gamma}^{(k)}_{N,\omega}] \\ &\quad + \frac{1}{N} \sum_{1\leqslant i<j\leqslant k} [V_{N,\omega}(r_i - r_j), \tilde{\gamma}^{(k)}_{N,\omega}] + \frac{N-k}{N} \sum_{j=1}^{k} \operatorname{Tr}_{r_{k+1}} [V_{N,\omega}(r_j - r_{k+1}), \tilde{\gamma}^{(k+1)}_{N,\omega}]. \end{aligned} \tag{18}$$

We will always take $\omega \geqslant 1$. For the rescaled marginals $\{\tilde{\gamma}^{(k)}_{N,\omega}\}_{k=1}^{N}$, we define

$$\widetilde{S}_j := \left[1 - \partial_{z_j}^2 + \omega(-\Delta_{x_j} + |x_j|^2 - 2)\right]^{\frac{1}{2}}. \tag{19}$$

Two immediate properties of $\widetilde{S}_j$ are the following. On the one hand,

$$\widetilde{S}_j^2 \big(h_1(x_j)\phi(z_j)\big) = h_1(x_j)(1 - \partial_{z_j}^2)\phi(z_j)$$

and thus the diverging parameter $\omega$ has no consequence when $\widetilde{S}_j$ is applied to a tensor product function $h_1(x_j)\phi(z_j)$ for which the $x_j$-component rests in the ground state. On the other hand, $\widetilde{S}_j \geqslant 0$ as an operator because $-\Delta_{x_j} + |x_j|^2 - 2 \geqslant 0$.

Now, noticing that the eigenvalues of $-\triangle_x + \omega^2|x|^2$ in 2D are $\{2(l+1)\omega\}_{l=0}^{\infty}$, let $P_{l\omega}$ be the orthogonal projection onto the eigenspace associated with eigenvalue $2(l+1)\omega$. That is, $I = \sum_{l=0}^{\infty} P_{l\omega}$, where $I$ is the identity operator on $L^2(\mathbb{R}^3)$. As a matter of notation for our multicoordinate problem, $P^j_{l\omega}$ will refer to the projection in $x_j$-coordinate at energy $2(l+1)\omega$; i.e.,

$$I = \prod_{j=1}^{k} \left(\sum_{l=0}^{\infty} P^j_{l\omega}\right). \tag{20}$$

In (20), $I$ is the identity operator on $L^2(\mathbb{R}^{3k})$. In particular, when $\omega = 1$, we use simply $P_l$. That is, $P_0$ denotes the orthogonal projection onto the ground state of $-\Delta_x + |x|^2$ and $P_{\geqslant 1}$ means the orthogonal projection onto all higher-energy modes of $-\Delta_x + |x|^2$ so that $I = P_0 + P_{\geqslant 1}$, where $I : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$. Since we will only use $P_0$ and $P_{\geqslant 1}$ for the $\omega = 1$ case, we define $\mathcal{P}^j_0$ and $\mathcal{P}^j_1$ to be respectively $P_0$ and $P_{\geqslant 1}$ acting on the $x_j$-variable, and

$$\mathcal{P}_\alpha = \mathcal{P}^1_{\alpha_1} \cdots \mathcal{P}^k_{\alpha_k} \tag{21}$$

for a $k$-tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_j \in \{0, 1\}$ and adopt the notation $|\alpha| = \alpha_1 + \cdots + \alpha_k$. Then

$$I = \sum_{\alpha} \mathcal{P}_\alpha, \tag{22}$$

where $I : L^2(\mathbb{R}^{3k}) \to L^2(\mathbb{R}^{3k})$.

We next introduce an appropriate topology on the density matrices, as was previously done in [Elgart et al. 2006; Erdős and Yau 2001; Erdős et al. 2006; 2007; 2009; 2010; Kirkpatrick et al. 2011; T. Chen and Pavlović 2011; X. Chen 2012a; 2013; X. Chen and Holmer 2013; 2016b; 2016c; Sohinger 2015].

Denote the spaces of compact operators and trace class operators on $L^2(\mathbb{R}^{3k})$ as $\mathcal{K}_k$ and $\mathcal{L}_k^1$, respectively. Then $(\mathcal{K}_k)' = \mathcal{L}_k^1$. By the fact that $\mathcal{K}_k$ is separable, we pick a dense countable subset

$$\{J_i^{(k)}\}_{i\geqslant 1} \subset \mathcal{K}_k$$

in the unit ball of $\mathcal{K}_k$ (so $\|J_i^{(k)}\|_{\mathrm{op}} \leqslant 1$, where $\|\cdot\|_{\mathrm{op}}$ is the operator norm). For $\gamma_1^{(k)}, \gamma_2^{(k)} \in \mathcal{L}_k^1$, we then define a metric $d_k$ on $\mathcal{L}_k^1$ by

$$d_k(\gamma_1^{(k)}, \gamma_2^{(k)}) = \sum_{i=1}^{\infty} 2^{-i} \left|\operatorname{Tr} J_i^{(k)}(\gamma_1^{(k)} - \gamma_2^{(k)})\right|.$$

A uniformly bounded sequence $\tilde{\gamma}_{N,\omega}^{(k)} \in \mathcal{L}_k^1$ converges to $\tilde{\gamma}^{(k)} \in \mathcal{L}_k^1$ with respect to the weak* topology if and only if

$$\lim_{N,\omega\to\infty} d_k(\tilde{\gamma}_{N,\omega}^{(k)}, \tilde{\gamma}^{(k)}) = 0.$$

For fixed $T > 0$, let $C([0,T], \mathcal{L}_k^1)$ be the space of functions of $t \in [0,T]$ with values in $\mathcal{L}_k^1$ which are continuous with respect to the metric $d_k$. On $C([0,T], \mathcal{L}_k^1)$, we define the metric

$$\hat{d}_k(\gamma^{(k)}(\cdot), \tilde{\gamma}^{(k)}(\cdot)) = \sup_{t\in[0,T]} d_k(\gamma^{(k)}(t), \tilde{\gamma}^{(k)}(t)),$$

and denote by $\tau_{\mathrm{prod}}$ the topology on the space $\bigoplus_{k\geqslant 1} C([0,T], \mathcal{L}_k^1)$ given by the product of topologies generated by the metrics $\hat{d}_k$ on $C([0,T], \mathcal{L}_k^1)$.

With the above topology on the space of marginal densities, we prove Theorem 1.2. The proof is divided into five steps.

Step I (focusing energy estimate). We first establish, via an elaborate calculation in Theorem 3.1, that one can compensate for the negativity of the interaction in the focusing many-body Hamiltonian (6) by adding a product of $N$ and some constant $\alpha$ depending on $V$, provided that $C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_2(\beta)}$, where $C_1$ and $C_2$ depend solely on $V$. Henceforth, though $H_{N,\omega}$ is not positive-definite, we derive, from the energy condition (12), an $H^1$-type energy bound:

$$\left\langle \psi_{N,\omega}, \, (\alpha + N^{-1} H_{N,\omega} - 2\omega)^k \psi_{N,\omega} \right\rangle \geqslant C^k \left\| \prod_{j=1}^{k} S_j \psi_{N,\omega} \right\|_{L^2(\mathbb{R}^{3N})}^2,$$

where

$$S_j := \left(1 - \Delta_{x_j} + \omega^2 |x_j|^2 - 2\omega - \partial_{z_j}^2\right)^{\frac{1}{2}}.$$

Since the quantity $\langle \psi_{N,\omega}, (H_{N,\omega} - 2N\omega)^k \psi_{N,\omega} \rangle$ is conserved by the evolution, via Corollary 3.2, we deduce the a priori bounds, crucial to the analysis of the "$\infty - \infty$" BBGKY hierarchy (18), on the scaled marginal densities,

$$\sup_t \operatorname{Tr}\left(\prod_{j=1}^{k} \widetilde{S}_j\right) \tilde{\gamma}_{N,\omega}^{(k)} \left(\prod_{j=1}^{k} \widetilde{S}_j\right) \leqslant C^k, \quad \sup_t \operatorname{Tr} \prod_{j=1}^{k} (1 - \triangle_{r_j}) \tilde{\gamma}_{N,\omega}^{(k)} \leqslant C^k,$$

$$\sup_t \operatorname{Tr} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta \leqslant C^k \omega^{-\frac{1}{2}|\alpha| - \frac{1}{2}|\beta|},$$

where $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ are defined as in (21). We remark that the quantity

$$\operatorname{Tr}(1-\triangle_{r_1})\tilde{\gamma}^{(1)}_{N,\omega}$$

is not the one-particle kinetic energy of the system; the one-particle kinetic energy of the system is $\operatorname{Tr}(1-\omega\triangle_{x_1}-\partial^2_{z_1})\tilde{\gamma}^{(1)}_{N,\omega}$ and grows like $\omega$. This is also in contrast to the $n$D-to-$n$D work,

Step II (compactness of BBGKY). We fix $T>0$ and work in the time interval $t\in[0,T]$. In Theorem 4.1, we establish the compactness of the BBGKY sequence $\{\Gamma_{N,\omega}(t)=\{\tilde{\gamma}^{(k)}_{N,\omega}\}_{k=1}^N\}\subset\bigoplus_{k\geqslant 1}C([0,T],\mathcal{L}^1_k)$ with respect to the product topology $\tau_{\text{prod}}$ even though hierarchy (18) contains attractive interactions and an indefinite $\infty-\infty$. Moreover, in Corollary 4.2, we prove that, to be compatible with the energy bound obtained in Step I, every limit point $\Gamma(t)=\{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ must take the form

$$\tilde{\gamma}^{(k)}\big(t,(\boldsymbol{x}_k,\boldsymbol{z}_k);(\boldsymbol{x}'_k,\boldsymbol{z}'_k)\big)=\bigg(\prod_{j=1}^k h_1(x_j)h_1(x'_j)\bigg)\tilde{\gamma}^{(k)}_z(t,\boldsymbol{z}_k;\boldsymbol{z}'_k),$$

where $\tilde{\gamma}^{(k)}_z=\operatorname{Tr}_x\tilde{\gamma}^{(k)}$ is the $z$-component of $\tilde{\gamma}^{(k)}$.

Step III (limit points of BBGKY satisfy GP). In Theorem 5.1, we prove that if $\Gamma(t)=\{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ is a $C_1N^{v_1(\beta)}\leqslant\omega\leqslant C_2N^{v_2(\beta)}$ limit point of $\{\Gamma_{N,\omega}(t)=\{\tilde{\gamma}^{(k)}_{N,\omega}\}_{k=1}^N\}$ with respect to the product topology $\tau_{\text{prod}}$, then $\{\tilde{\gamma}^{(k)}_z=\operatorname{Tr}_x\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ is a solution to the focusing coupled Gross–Pitaevskii (GP) hierarchy subject to initial data $\tilde{\gamma}^{(k)}_z(0)=|\phi_0\rangle\langle\phi_0|^{\otimes k}$ with coupling constant $b_0=\left|\int V(r)\,dr\right|$, which, written in differential form, is

$$i\partial_t\tilde{\gamma}^{(k)}_z=\sum_{j=1}^k[-\partial^2_{z_j},\tilde{\gamma}^{(k)}_z]-b_0\sum_{j=1}^k\operatorname{Tr}_{z_{k+1}}\operatorname{Tr}_x[\delta(r_j-r_{k+1}),\tilde{\gamma}^{(k+1)}].\tag{23}$$

Together with the limiting structure concluded in Corollary 4.2, we can further deduce $\{\tilde{\gamma}^{(k)}_z=\operatorname{Tr}_x\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ is a solution to the 1D focusing GP hierarchy subject to initial data $\tilde{\gamma}^{(k)}_z(0)=|\phi_0\rangle\langle\phi_0|^{\otimes k}$ with coupling constant $b_0\big(\int|h_1(x)|^4\,dx\big)$, which, written in differential form, is

$$i\partial_t\tilde{\gamma}^{(k)}_z=\sum_{j=1}^k[-\partial^2_{z_j},\tilde{\gamma}^{(k)}_z]-b_0\bigg(\int|h_1(x)|^4\,dx\bigg)\sum_{j=1}^k\operatorname{Tr}_{z_{k+1}}[\delta(z_j-z_{k+1}),\tilde{\gamma}^{(k+1)}_z].\tag{24}$$

Step IV (GP has a unique solution). When $\tilde{\gamma}^{(k)}_z(0)=|\phi_0\rangle\langle\phi_0|^{\otimes k}$, we know one solution to the 1D focusing GP hierarchy (24), namely $|\phi\rangle\langle\phi|^{\otimes k}$ if $\phi$ solves the 1D focusing NLS (11). Since we have proven the a priori bound,

$$\sup_t\operatorname{Tr}\bigg(\prod_{j=1}^k\langle\partial_{z_j}\rangle\bigg)\tilde{\gamma}^{(k)}_z\bigg(\prod_{j=1}^k\langle\partial_{z_j}\rangle\bigg)\leqslant C^k.$$

A trace theorem then shows that $\{\tilde{\gamma}^{(k)}_z\}$ verifies the requirement of the following uniqueness theorem and hence we conclude that $\tilde{\gamma}^{(k)}_z=|\phi\rangle\langle\phi|^{\otimes k}$.

**Theorem 2.1** [X. Chen and Holmer 2016b, Theorem 1.3]. [10]*Let*

$$B_{j,k+1}\gamma_z^{(k+1)} = \mathrm{Tr}_{z_{k+1}}[\delta(z_j - z_{k+1}), \gamma_z^{(k+1)}].$$

*If* $\{\gamma_z^{(k)}\}_{k=1}^{\infty}$ *solves the* 1*D focusing GP hierarchy* (24) *subject to zero initial data and the space-time bound*[11]

$$\int_0^T \left\| \left( \prod_{j=1}^k \langle \partial_{z_j} \rangle^{\varepsilon} \langle \partial_{z_j'} \rangle^{\varepsilon} \right) B_{j,k+1} \gamma_z^{(k+1)}(t, \cdot ; \cdot) \right\|_{L^2_{z,z'}} dt \leqslant C^k \tag{25}$$

*for some* $\varepsilon, C > 0$ *and all* $1 \leqslant j \leqslant k$, *then* $\forall k, t \in [0, T]$, *we have* $\gamma_z^{(k+1)} = 0$.

Thus the compact sequence $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N\}$ has only one $C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_2(\beta)}$ limit point, namely

$$\tilde{\gamma}^{(k)} = \prod_{j=1}^k h_1(x_j) h_1(x_j') \phi(t, z_j) \bar{\phi}(t, z_j').$$

We then infer from the definition of the topology that as trace class operators

$$\tilde{\gamma}_{N,\omega}^{(k)} \to \prod_{j=1}^k h_1(x_j) h_1(x_j') \phi(t, z_j) \bar{\phi}(t, z_j') \quad \text{weak*}.$$

Step V (weak* convergence upgraded to strong). Since the limit concluded in Step IV is an orthogonal projection, the well-known argument in [Erdős et al. 2010] upgrades the weak* convergence to strong. In fact, testing the sequence against the compact observable

$$J^{(k)} = \prod_{j=1}^k h_1(x_j) h_1(x_j') \phi(t, z_j) \bar{\phi}(t, z_j'),$$

and noticing the fact that $(\tilde{\gamma}_{N,\omega}^{(k)})^2 \leqslant \tilde{\gamma}_{N,\omega}^{(k)}$ since the initial data is normalized, we see that as Hilbert–Schmidt operators,

$$\tilde{\gamma}_{N,\omega}^{(k)} \to \prod_{j=1}^k h_1(x_j) h_1(x_j') \phi(t, z_j) \bar{\phi}(t, z_j') \quad \text{strongly}.$$

Since $\mathrm{Tr}\, \tilde{\gamma}_{N,\omega}^{(k)} = \mathrm{Tr}\, \tilde{\gamma}^{(k)}$, we deduce the strong convergence

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_2(\beta)}}} \mathrm{Tr} \left| \tilde{\gamma}_{N,\omega}^{(k)}(t, \boldsymbol{x}_k, \boldsymbol{z}_k; \boldsymbol{x}_k', \boldsymbol{z}_k') - \prod_{j=1}^k h_1(x_j) h_1(x_j') \phi(t, z_j) \bar{\phi}(t, z_j') \right| = 0$$

via Grümm's convergence theorem [Simon 2005, Theorem 2.19].[12]

[10] For other uniqueness theorems or related estimates regarding the GP hierarchies, see [Erdős et al. 2007; Klainerman and Machedon 2008; Kirkpatrick et al. 2011; Grillakis and Margetis 2008; X. Chen 2011; 2012a; Beckner 2014; Gressman et al. 2014; T. Chen et al. 2015; Hong et al. 2015; Sohinger 2015]

[11] Though the space-time bound (25) follows from a simple trace theorem here, verifying such a condition in 3D is highly nontrivial and is merely partially solved so far. See [T. Chen and Pavlović 2014; X. Chen 2013; X. Chen and Holmer 2016c].

[12] One can also use the argument in [X. Chen 2013, Appendix A] to conclude the convergence with general datum.

## 3. Focusing energy estimate

We find it more convenient to prove the energy estimate for $\psi_{N,\omega}$ and then convert it by scaling to an estimate for $\tilde{\psi}_{N,\omega}$; see (15). Note that, as an operator, we have the positivity

$$-\Delta_{x_j} + \omega^2 |x_j|^2 - 2\omega \geqslant 0.$$

Define

$$S_j := \left(1 - \Delta_{x_j} + \omega^2 |x_j|^2 - 2\omega - \partial_{z_j}^2\right)^{\frac{1}{2}} = \left(1 - 2\omega - \Delta_{r_j} + \omega^2 |x_j|^2\right)^{\frac{1}{2}},$$

and write

$$S^{(k)} = \prod_{j=1}^{k} S_j.$$

**Theorem 3.1** (energy estimate). *For $\beta \in \left(0, \frac{3}{7}\right)$, let*[13]

$$v_E(\beta) = \min\left(\frac{1-\beta}{\beta}, \frac{\frac{3}{5}-\beta}{\beta-\frac{1}{5}}\mathbf{1}_{\beta \geqslant \frac{1}{5}} + \infty \cdot \mathbf{1}_{\beta<\frac{1}{5}}, \frac{\frac{7}{8}-\beta}{\beta}\right). \tag{26}$$

*There are constants*[14] $C_1 = C_1(\|V\|_{L^1}, \|V\|_{L^\infty})$, $C_2 = C_2(\|V\|_{L^1}, \|V\|_{L^\infty})$, *and absolute constant* $C_3$, *and for each* $k \in \mathbb{N}$, *there is an integer* $N_0(k)$, *such that for any* $k \in \mathbb{N}$, $N \geqslant N_0(k)$ *and* $\omega$ *which satisfy*

$$C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_E(\beta)}, \tag{27}$$

*there holds*

$$\left\langle (\alpha + N^{-1} H_{N,\omega} - 2\omega)^k \psi, \psi \right\rangle \geqslant \frac{1}{2^k}\left(\|S^{(k)}\psi\|_{L^2}^2 + N^{-1}\|S_1 S^{(k-1)}\psi\|_{L^2}^2\right), \tag{28}$$

*where*

$$\alpha = C_3 \|V\|_{L^1}^2 + 1.$$

*Proof.* For smoothness of presentation, we postpone the proof to Section 3. □

Recall the rescaled operator (19),

$$\widetilde{S}_j = \left[1 - \partial_{z_j}^2 + \omega(-\Delta_{x_j} + |x_j|^2 - 2)\right]^{\frac{1}{2}}.$$

We notice that

$$(S_j \psi)(t, \boldsymbol{x}_N, \boldsymbol{z}_N) = \omega^{N/2} (\widetilde{S}_j \tilde{\psi})(t, \sqrt{\omega}\boldsymbol{x}_N, \boldsymbol{z}_N)$$

if $\tilde{\psi}_{N,\omega}$ is defined via (15). Thus we can convert the conclusion of Theorem 3.1 into statements about $\tilde{\psi}_{N,\omega}$, $\widetilde{S}_j$, and $\tilde{\gamma}_{N,\omega}^{(k)}$, which we will utilize in the rest of the paper.

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[13] One notices that $v_E(\beta)$ is different from $v_2(\beta)$ in the sense that the term $2\beta/(1-2\beta)-$ is missing. That restriction comes from Theorem 5.1.

[14] By *absolute* constant we mean a constant independent of $V$, $N$, $\omega$, etc. Formulas for $C_1$, $C_2$ in terms of $\|V\|_{L^1}$, $\|V\|_{L^\infty}$ can, in principle, be extracted from the proof.

**Corollary 3.2.** *Define*

$$\widetilde{S}^{(k)} = \prod_{j=1}^{k} \widetilde{S}_j, \quad L^{(k)} = \prod_{j=1}^{k} \langle \nabla_{r_j} \rangle.$$

*Assume $C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_E(\beta)}$. Let $\tilde{\psi}_{N,\omega}(t) = e^{it\widetilde{H}_{N,\omega}} \tilde{\psi}_{N,\omega}(0)$ and $\{\tilde{\gamma}_{N,\omega}^{(k)}(t)\}$ be the associated marginal densities. Then for all $\omega \geqslant 1$, $k \geqslant 0$, $N$ large enough, we have the uniform-in-time bound*

$$\operatorname{Tr} \widetilde{S}^{(k)} \tilde{\gamma}_{N,\omega}^{(k)} \widetilde{S}^{(k)} = \| \widetilde{S}^{(k)} \tilde{\psi}_{N,\omega}(t) \|_{L^2(\mathbb{R}^{3N})}^2 \leqslant C^k. \tag{29}$$

*Consequently,*

$$\operatorname{Tr} L^{(k)} \tilde{\gamma}_{N,\omega}^{(k)} L^{(k)} = \| L^{(k)} \tilde{\psi}_{N,\omega}(t) \|_{L^2(\mathbb{R}^{3N})}^2 \leqslant C^k, \tag{30}$$

*and*

$$\| \mathcal{P}_\alpha \tilde{\psi}_{N,\omega} \|_{L^2(\mathbb{R}^{3N})} \leqslant C^k \omega^{-\frac{1}{2}|\alpha|}, \quad |\operatorname{Tr} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta | \leqslant C^k \omega^{-\frac{1}{2}|\alpha| - \frac{1}{2}|\beta|}, \tag{31}$$

*where $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ are defined as in (21).*

*Proof.* Substituting (15) into estimate (28) and rescaling, we obtain

$$\| \widetilde{S}^{(k)} \tilde{\psi}_{N,\omega}(t) \|_{L^2(\mathbb{R}^{3N})}^2 \leqslant C^k \big\langle \tilde{\psi}_{N,\omega}(t), \, (\alpha + N^{-1} \widetilde{H}_{N,\omega} - 2\omega)^k \tilde{\psi}_{N,\omega}(t) \big\rangle.$$

The quantity on the right-hand side is conserved; therefore

$$\| \widetilde{S}^{(k)} \tilde{\psi}_{N,\omega}(t) \|_{L^2(\mathbb{R}^{3N})}^2 = C^k \big\langle \tilde{\psi}_{N,\omega}(0), \, (\alpha + N^{-1} \widetilde{H}_{N,\omega} - 2\omega)^k \tilde{\psi}_{N,\omega}(0) \big\rangle.$$

Applying the binomial theorem twice,

$$\begin{aligned} \| \widetilde{S}^{(k)} \tilde{\psi}_{N,\omega}(t) \|_{L^2(\mathbb{R}^{3N})}^2 &\leqslant C^k \sum_{j=0}^{k} \binom{k}{j} \alpha^j \big\langle \tilde{\psi}_{N,\omega}(0), \, (N^{-1} \widetilde{H}_{N,\omega} - 2\omega)^{k-j} \tilde{\psi}_{N,\omega}(0) \big\rangle \\ &\leqslant C^k \sum_{j=0}^{k} \binom{k}{j} \alpha^j (C)^{k-j} \\ &= C^k (\alpha + C)^k \leqslant \widetilde{C}^k, \end{aligned}$$

where we used condition (12) in the second-to-last line. So we have proved (29). Putting (29) and (70) together, estimate (30) then follows.[15] The first inequality of (31) follows from (29) and (72). By Lemma A.5,

$$\operatorname{Tr} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta = \langle \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega} \rangle,$$

so the second inequality of (31) follows by Cauchy–Schwarz. □

[15] We remark that, though $L^{(k)} \leqslant 3^k \widetilde{S}^{(k)}$, it is not true that $L^{(k)} \leqslant C^k S^{(k)}$ for any $C$ independent of $\omega$ because of the ground-state case.

***Proof of the focusing energy estimate.*** Note that

$$N^{-1}H_{N,\omega}-2\omega=N^{-1}\sum_{i=1}^{N}(-\Delta_{r_i}+\omega^2|x_i|^2-2\omega)+N^{-2}\omega^{-1}\sum_{1\leqslant i<j\leqslant N}V_{N\omega}(r_i-r_j),$$

where we have used the notation[16]

$$V_{N\omega}(r)=(N\omega)^{3\beta}V((N\omega)^{\beta}r).$$

Define

$$H_{Kij}=(\alpha-\Delta_{r_i}+\omega^2|x_i|^2-2\omega)+(\alpha-\Delta_{r_j}+\omega^2|x_j|^2-2\omega),$$

where the $K$ stands for "kinetic" and

$$H_{Iij}=\omega^{-1}V_{N\omega ij}=\omega^{-1}V_{N\omega}(r_i-r_j),$$

where the $I$ is for "interaction". If we write

$$H_{ij}=H_{Kij}+H_{Iij},$$

then

$$\alpha+N^{-1}H_{N,\omega}-2\omega=\tfrac{1}{2}N^{-2}\sum_{1\leqslant i\neq j\leqslant N}H_{ij}=N^{-2}\sum_{1\leqslant i<j\leqslant N}H_{ij}. \tag{32}$$

We will first prove Theorem 3.1 for $k=1$ and $k=2$. Then, by a two-step induction (result known for $k$ implies result for $k+2$), we establish the general case. Before we proceed, we prove some estimates regarding the Hermite operator.

*Estimates needed to prove Theorem 3.1.*

**Lemma 3.3.** *Recall that $P_{l\omega}$ is the orthogonal projection onto the eigenspace of $-\Delta_x+\omega^2|x|^2$ associated with eigenvalue $2(l+1)\omega$. There is a constant independent of $l$ and $\omega$ such that*

$$\|P_{l\omega}f\|_{L_x^\infty}\leqslant C\omega^{\frac{1}{2}}\|f\|_{L_x^2}. \tag{33}$$

*Proof.* This estimate has more than one proof. It is a special result in 2D. It does not follow from the Strichartz estimates. For a modern argument which proves the estimate for general, at most quadratic potentials, see [Koch and Tataru 2005, Corollary 2.2]. In the special case of the quantum harmonic oscillator, one can also use a special property of 2D Hermite projection kernels to yield a direct proof without using Littlewood–Paley theory — see [Thangavelu 1993, Lemma 3.2.2; X. Chen 2011, Remark 8]. □

**Lemma 3.4.** *There is an absolute constant $C_3>0$ and a constant $C_1=C(\|V\|_{L^1},\|V\|_{L^\infty})$ such that if*

$$\omega\geqslant C_1N^{\frac{\beta}{1-\beta}}$$

[16] We remind the reader that this $V_{N\omega}$ is different from $V_{N,\omega}$ defined in (17).

*then*

$$\frac{1}{\omega}\int |V_{N\omega}(r_1-r_2)||\psi(r_1,r_2)|^2\,dr_1 \leqslant \tfrac{1}{100}\big\langle \psi(r_1,r_2),\,(-\Delta_{r_1}+\omega^2|x_1|^2-2\omega)\psi(r_1,r_2)\big\rangle_{r_1} + C_3\|V\|_{L^1}^2\,\|\psi(r_1,r_2)\|_{L^2_{r_1}}^2. \quad (34)$$

*The above estimate is performed in one coordinate only (taken to be $r_1$), and the other coordinate $r_2$ is effectively "frozen". In particular, let*

$$f(r_2,\dots,r_N)=\int |V_{N\omega}(r_1-r_2)||\psi_1(r_1,\dots,r_N)||\psi_2(r_1,\dots,r_N)|\,dr_1.$$

*Then*

$$f(r_2,\dots,r_N)\lesssim \omega\|S_1\psi_1(r_1,\dots,r_N)\|_{L^2_{r_1}}\|S_1\psi_2(r_1,\dots,r_N)\|_{L^2_{r_1}}. \quad (35)$$

*The implicit constant in the $\lesssim$ is an absolute constant times $\|V\|_{L^1}+\|V\|_{L^\infty}$.*

*Proof.* By Cauchy–Schwarz,

$$\int |V_{N\omega12}||\psi_1||\psi_2|\,dr_1 \leqslant \left(\int |V_{N\omega12}||\psi_1|^2\,dr_1\right)^{\frac12}\left(\int |V_{N\omega12}||\psi_2|^2\,dr_1\right)^{\frac12}.$$

Thus, assuming (34) and using the facts

$$S_1^2\geqslant 1,\quad S_1^2\geqslant(-\Delta_{r_1}+\omega^2|x_1|^2-2\omega),$$

we obtain (35). So we only need to prove (34).

Taking $P_{l\omega}$ to be the projection onto the $x_1$-component at the moment, we decompose $\psi$ into ground state, middle energies, and high energies as follows:

$$\psi=P_{0\omega}\psi+\sum_{\ell=1}^{e-1}P_{l\omega}\psi+P_{\geqslant e\omega}\psi,$$

where $e$ is an integer, and the optimal choice of $e$ is determined below. It then suffices to bound

$$A_{\text{low}}:=\frac{1}{\omega}\int |V_{N\omega}(r_1-r_2)||P_{0\omega}\psi(r_1,r_2)|^2\,dr_1, \quad (36)$$

$$A_{\text{mid}}:=\frac{1}{\omega}\int |V_{N\omega}(r_1-r_2)|\left|\sum_{l=2}^{e-1}P_{l\omega}\psi(r_1,r_2)\right|^2dr_1, \quad (37)$$

$$A_{\text{high}}:=\frac{1}{\omega}\int |V_{N\omega}(r_1-r_2)||P_{\geqslant e\omega}\psi(r_1,r_2)|^2\,dr_1. \quad (38)$$

For each estimate, we will only work in the $r_1=(x_1,z_1)$ component, and thus will not even write the $r_2$-variable. First we consider (36):

$$A_{\text{low}}\leqslant\frac{1}{\omega}\|V_{N\omega}\|_{L^1}\|P_{0\omega}\psi\|_{L_x^\infty L_z^\infty}^2.$$

By the standard 1D Sobolev-type estimate,

$$A_{\text{low}} \lesssim \frac{1}{\omega} \|V\|_{L^1} \|P_{0\omega} \partial_z \psi\|_{L_x^\infty L_z^2} \|P_{0\omega} \psi\|_{L_x^\infty L_z^2}.$$

Then using the estimate (33), we get

$$\begin{aligned} A_{\text{low}} &\lesssim \|V\|_{L^1} \|P_{0\omega} \partial_z \psi\|_{L_r^2} \|P_{0\omega} \psi\|_{L_r^2} \\ &\lesssim \|V\|_{L^1} \|\partial_z \psi\|_{L^2} \|\psi\|_{L^2} \\ &\lesssim \epsilon \|\partial_z \psi\|_{L^2}^2 + \frac{1}{\epsilon} \|V\|_{L^1}^2 \|\psi\|_{L^2}^2. \end{aligned}$$

Since $(-\Delta_r + \omega^2|x|^2 - 2\omega)$ is a sum of two positive operators, namely $-\Delta_x + \omega^2|x|^2 - 2\omega$ and $-\partial_z^2$, we conclude the estimate for $A_{\text{low}}$.

Now consider the middle harmonic energies given by (37). We aim to estimate $A_{\text{mid}}$. For any $l \geqslant 1$, we have

$$\|P_{l\omega}\psi\|_{L_z^\infty L_x^\infty} \leqslant \|P_{l\omega}\partial_z\psi\|_{L_z^2 L_x^\infty}^{\frac{1}{2}} \|P_{l\omega}\psi\|_{L_z^2 L_x^\infty}^{\frac{1}{2}}.$$

By (33),

$$\begin{aligned} \|P_{l\omega}\psi\|_{L_z^\infty L_x^\infty} &\lesssim \omega^{\frac{1}{2}} \|P_{l\omega}\partial_z\psi\|_{L_z^2 L_x^2}^{\frac{1}{2}} \|P_{l\omega}\psi\|_{L_z^2 L_x^2}^{\frac{1}{2}} \\ &= \omega^{\frac{1}{4}} \|P_{l\omega}\partial_z\psi\|_{L^2}^{\frac{1}{2}} \big(\|P_{l\omega}\psi\|_{L^2} l^{\frac{1}{2}} \omega^{\frac{1}{2}}\big)^{\frac{1}{2}} l^{-\frac{1}{4}} \\ &= \omega^{\frac{1}{4}} \|P_{l\omega}\partial_z\psi\|_{L_r^2}^{\frac{1}{2}} \big\|P_{l\omega}(-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}}\psi\big\|_{L^2}^{\frac{1}{2}} l^{-\frac{1}{4}}. \end{aligned}$$

Summing over $1 \leqslant l \leqslant e-1$, and using the Hölder inequality with exponents 4, 4, and 2, we get

$$\begin{aligned} \sum_{l=1}^{e-1} \|P_{l\omega}\psi\|_{L_z^\infty L_x^\infty} &\lesssim \omega^{\frac{1}{4}} \Big(\sum_{\ell=1}^{e-1} \|P_{l\omega}\partial_z\psi\|_{L^2}^2\Big)^{\frac{1}{4}} \Big(\sum_{l=1}^{e-1} \big\|P_{l\omega}(-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}}\psi\big\|_{L^2}^2\Big)^{\frac{1}{4}} \Big(\sum_{l=1}^{e} l^{-\frac{1}{2}}\Big)^{\frac{1}{2}} \\ &\lesssim \omega^{\frac{1}{4}} e^{\frac{1}{4}} \|\partial_z\psi\|_{L^2}^{\frac{1}{2}} \big\|(-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}}\psi\big\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

Applying this to estimate (37),

$$A_{\text{mid}} \lesssim \omega^{-\frac{1}{2}} e^{\frac{1}{2}} \|V\|_{L^1} \|\partial_z\psi\|_{L^2} \big\|(-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}}\psi\big\|_{L^2}.$$

Take $e$ to be the largest integer so that $\omega^{-\frac{1}{2}} e^{\frac{1}{2}} \|V\|_{L^1} \leqslant \epsilon$, i.e.,

$$e = \left\lfloor \frac{\epsilon^2}{\|V\|_{L^1}^2} \omega \right\rfloor, \tag{39}$$

and then we have

$$A_{\text{mid}} \lesssim \epsilon \|\partial_z\psi\|_{L^2}^2 + \epsilon \big\|(-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac{1}{2}}\psi\big\|_{L^2}^2.$$

For (38),

$$\begin{aligned} A_{\text{high}} &\lesssim \omega^{-1} \|V_{N\omega}\|_{L^\infty} \|P_{\geqslant e\omega}\psi\|_{L^2}^2 \\ &\lesssim \omega^{-2} e^{-1} \|V_{N\omega}\|_{L^\infty} \big\| e^{\frac12}\omega^{\frac12} P_{\geqslant e\omega}\psi\big\|_{L^2}^2 \\ &\lesssim \omega^{-2} e^{-1} (N\omega)^{3\beta} \|V\|_{L^\infty} \big\|(-\Delta_x + \omega^2|x|^2 - 2\omega)^{\frac12}\psi\big\|_{L^2}^2. \end{aligned}$$

We need

$$\omega^{-2} e^{-1} (N\omega)^{3\beta} \|V\|_{L^\infty} \leqslant \epsilon.$$

Substituting the specification of $e$ given by (39), we obtain

$$\omega^{-2}(N\omega)^{3\beta} \leqslant \frac{e\epsilon}{\|V\|_{L^\infty}} \leqslant \frac{\epsilon^3}{\|V\|_{L^\infty}\|V\|_{L^1}^2}\omega,$$

which is

$$N^{3\beta}\omega^{3\beta-3} \leqslant \frac{\epsilon^3}{\|V\|_{L^1}^2\|V\|_{L^\infty}}.$$

That is, $\omega \geqslant C_1 N^{\frac{\beta}{1-\beta}}$ as required in the statement of Lemma 3.4. □

In the following lemma, we have excited-state estimates and ground-state estimates, and the ground-state estimates are weaker (they involve a loss of $\omega^{\frac12}$).

**Lemma 3.5.** *Taking $\psi = \psi(r)$, we have the "excited-state" estimate*

$$\|\omega^{\frac12} P_{\geqslant 1\omega}\psi\|_{L^2} + \big\|\omega|x| P_{\geqslant 1\omega}\psi\big\|_{L^2} + \|\nabla_r P_{\geqslant 1\omega}\psi\|_{L^2} \lesssim \|S\psi\|_{L^2}, \tag{40}$$

*and the "ground-state" estimate*

$$\|\omega^{\frac12} P_{0\omega}\psi\|_{L^2} + \big\|\omega|x| P_{0\omega}\psi\big\|_{L^2} + \|\nabla_x P_{0\omega}\psi\|_{L^2} \lesssim \omega^{\frac12}\|\psi\|_{L^2}. \tag{41}$$

*We are, however, spared from the $\omega^{\frac12}$ loss when working only with the $z$-derivative:*

$$\|\partial_z P_{0\omega}\psi\|_{L^2} \lesssim \|S\psi\|_{L^2}. \tag{42}$$

*Putting the excited-state and ground-state estimates together gives*

$$\|\omega^{\frac12}\psi\|_{L^2} + \big\|\omega|x|\psi\big\|_{L^2} + \|\nabla_r\psi\|_{L^2} \lesssim \omega^{\frac12}\|S\psi\|_{L^2}. \tag{43}$$

*Proof.* For the excited-state estimates, we note

$$0 \leqslant \big\langle P_{\geqslant 1\omega}\psi,\ (-\Delta_x + \omega^2|x|^2 - 4\omega) P_{\geqslant 1\omega}\psi\big\rangle.$$

Adding $\frac32\|\partial_z P_{\geqslant 1\omega}\psi\|_{L^2}^2 + \frac12\|\nabla_x P_{\geqslant 1\omega}\psi\|_{L^2}^2 + \frac12\big\|\omega|x| P_{\geqslant 1\omega}\psi\big\|_{L^2}^2 + \|\omega^{\frac12} P_{\geqslant 1\omega}\psi\|_{L^2}^2$ to both sides, we get

$$\begin{aligned} &\tfrac32\|\partial_z P_{\geqslant 1\omega}\psi\|_{L^2}^2 + \tfrac12\|\nabla_x P_{\geqslant 1\omega}\psi\|_{L^2}^2 + \tfrac12\big\|\omega|x| P_{\geqslant 1\omega}\psi\big\|_{L^2}^2 + \|\omega^{\frac12} P_{\geqslant 1\omega}\psi\|_{L^2}^2 \\ &\qquad\leqslant \tfrac32\big\langle P_{\geqslant 1\omega}\psi, (-\Delta_r + \omega^2|x|^2 - 2\omega) P_{\geqslant 1\omega}\psi\big\rangle. \end{aligned}$$

This proves (40).

For the ground-state estimate (41), it suffices to prove

$$\left\|\omega|x|P_{0\omega}\psi\right\|_{L^2}+\|\nabla_x P_{0\omega}\psi\|_{L^2}\lesssim C\omega^{\frac{1}{2}}\|\psi\|_{L^2},$$

because

$$\|\omega^{\frac{1}{2}}P_{0\omega}\psi\|_{L^2}=\omega^{\frac{1}{2}}\|P_{0\omega}\psi\|_{L^2}\leqslant\omega^{\frac{1}{2}}\|\psi\|_{L^2}.$$

We notice that

$$\left\|\omega|x|f\right\|_{L^2}+\|\nabla_x f\|_{L^2}\sim\left\|(-\triangle_x+\omega^2|x|^2)^{\frac{1}{2}}f\right\|_{L^2}.$$

This estimate has been proved by many authors (see, for example, [Thangavelu 1993]), but is usually known as a scattering space $\Sigma$ estimate for PDE analysts. Then, since the eigenvalue for the ground-state Gaussian is exactly $2\omega$ in 2D, we have

$$\left\|(-\triangle_x+\omega^2|x|^2)^{\frac{1}{2}}P_{0\omega}\psi\right\|_{L^2}=\sqrt{2}\omega^{\frac{1}{2}}\|P_{0\omega}\psi\|_{L^2}\leqslant\sqrt{2}\omega^{\frac{1}{2}}\|\psi\|_{L^2}.$$

So we have proved (41).

For (42), we notice that

$$\|\partial_z P_{0\omega}\psi\|_{L^2}=\|P_{0\omega}(\partial_z\psi)\|_{L^2}\leqslant\|\partial_z\psi\|_{L^2}\lesssim\|S\psi\|_{L^2}. \qquad\square$$

**Lemma 3.6.** *We have the estimates*

$$\left\||V_{N\omega12}|^{\frac{1}{2}}S_1P^1_{0\omega}\psi_2\right\|_{L^2_{r_1}}\lesssim\omega^{\frac{1}{2}}N^{\frac{1}{4}}\|S_1\psi_2\|^{\frac{1}{2}}_{L^2}\big(N^{-\frac{1}{4}}\|S_1^2\psi_2\|^{\frac{1}{2}}_{L^2}\big),\tag{44}$$

$$\left\||V_{N\omega12}|^{\frac{1}{2}}S_1P^1_{\geqslant1\omega}\psi_2\right\|_{L^2_{r_1}}\lesssim N^{\frac{1}{2}\beta+\frac{1}{2}}\omega^{\frac{1}{2}\beta}\big(N^{-\frac{1}{2}}\|S_1^2\psi_2\|_{L^2_{r_1}}\big).\tag{45}$$

*In particular, if $\omega\geqslant C_1N^{\frac{\beta}{1-\beta}}$ then*

$$\int_{r_1}|V_{N\omega12}||\psi_1||S_1\psi_2|\,dr_1$$
$$\lesssim\omega N^{\frac{1}{4}}\|S_1\psi_1\|_{L^2}\|S_1\psi_2\|^{\frac{1}{2}}_{L^2}N^{-\frac{1}{4}}\|S_1^2\psi_2\|^{\frac{1}{2}}_{L^2}+(N\omega)^{\frac{1}{2}\beta+\frac{1}{2}}\|S_1\psi_1\|_{L^2}N^{-\frac{1}{2}}\|S_1^2\psi_2\|_{L^2}.\tag{46}$$

*Proof.* To prove (46), substituting $\psi_2=P^1_{0\omega}\psi_2+P^1_{\geqslant1\omega}\psi_2$, we obtain

$$\int_{r_1}|V_{N\omega12}||\psi_1||S_1\psi_2|\,dr_1\lesssim F_1+F_2,$$

where

$$\begin{aligned}F_1&=\int_{r_1}|V_{N\omega12}||\psi_1||P^1_{0\omega}S_1\psi_2|\,dr_1\\&\leqslant\left\||V_{N\omega12}|^{\frac{1}{2}}\psi_1\right\|_{L^2_{r_1}}\left\||V_{N\omega12}|^{\frac{1}{2}}P^1_{0\omega}S_1\psi_2\right\|_{L^2_{r_1}}\\&\leqslant\omega^{\frac{1}{2}}\|S_1\psi_1\|_{L^2_{r_1}}\left\||V_{N\omega12}|^{\frac{1}{2}}P^1_{0\omega}S_1\psi_2\right\|_{L^2_{r_1}},\end{aligned}$$

$$\begin{aligned}F_2&=\int_{r_1}|V_{N\omega12}||\psi_1||P^1_{\geqslant1\omega}S_1\psi_2|\,dr_1\\&\leqslant\omega^{\frac{1}{2}}\|S_1\psi_1\|_{L^2_{r_1}}\left\||V_{N\omega12}|^{\frac{1}{2}}P^1_{\geqslant1\omega}S_1\psi_2\right\|_{L^2_{r_1}}\end{aligned}$$

by Cauchy–Schwarz and estimate (35). Hence we only need to prove (44) and (45).

On the one hand, using the fact that $P^1_{0\omega} S_1 = (1-\partial^2_{z_1})^{\frac{1}{2}} P^1_{0\omega}$,

$$\big\| |V_{N\omega 12}|^{\frac{1}{2}} S_1 P^1_{0\omega} \psi_2 \big\|_{L^2_{r_1}} = \big\| |V_{N\omega 12}|^{\frac{1}{2}} (1-\partial^2_{z_1})^{\frac{1}{2}} P^1_{0\omega} \psi_2 \big\|_{L^2_{r_1}} \leqslant \|V_{N\omega 12}\|^{\frac{1}{2}}_{L^1_{r_1}} \|(1-\partial^2_{z_1})^{\frac{1}{2}} P^1_{0\omega} \psi_2\|_{L^\infty_{r_1}}.$$

By Sobolev in $z_1$ and the estimate (33) in $x_1$,

$$\big\| |V_{N\omega 12}|^{\frac{1}{2}} S_1 P^1_{0\omega} \psi_2 \big\|_{L^2_{r_1}} \lesssim \omega^{\frac{1}{2}} \|(1-\partial^2_{z_1})^{\frac{1}{2}} \psi_2\|^{\frac{1}{2}}_{L^2_{r_1}} \|(1-\partial^2_{z_1}) \psi_2\|^{\frac{1}{2}}_{L^2_{r_1}}.$$

That is, we get (44):

$$\big\| |V_{N\omega 12}|^{\frac{1}{2}} S_1 P^1_{0\omega} \psi_2 \big\|_{L^2_{r_1}} \lesssim \omega^{\frac{1}{2}} N^{\frac{1}{4}} \|S_1 \psi_2\|^{\frac{1}{2}}_{L^2} \big(N^{-\frac{1}{4}} \|S_1^2 \psi_2\|^{\frac{1}{2}}_{L^2}\big).$$

On the other hand,

$$\begin{aligned}\big\| |V_{N\omega 12}|^{\frac{1}{2}} S_1 P^1_{\geqslant 1\omega} \psi_2 \big\|_{L^2_{r_1}} &\lesssim \big\| |V_{N\omega 12}|^{\frac{1}{2}} \big\|_{L^3} \|P^1_{\geqslant 1\omega} S_1 \psi_2\|_{L^6_{r_1}} \\ &\lesssim (N\omega)^{\frac{1}{2}\beta} \|S_1^2 \psi_2\|_{L^2_{r_1}} \\ &= N^{\frac{1}{2}\beta + \frac{1}{2}} \omega^{\frac{1}{2}\beta} \big(N^{-\frac{1}{2}} \|S_1^2 \psi_2\|_{L^2_{r_1}}\big),\end{aligned}$$

which is (45). □

*The $k = 1$ case.* Recalling (32),

$$\big\langle \psi, (\alpha + N^{-1} H_{N,\omega} - 2\omega)\psi \big\rangle = \tfrac{1}{2} N^{-2} \sum_{1 \leqslant i \neq j \leqslant N} \langle H_{ij} \psi, \psi \rangle = \tfrac{1}{2} \langle H_{12} \psi, \psi \rangle,$$

where the second equality follows by symmetry. Hence we need to prove

$$\langle H_{12} \psi, \psi \rangle \geqslant \|S_1 \psi\|^2_{L^2}. \tag{47}$$

We prove (47) with the following lemma.

**Lemma 3.7.** *Recall $\alpha = C_3 \|V\|^2_{L^2} + 1$. If $\omega \geqslant C_1 N^{\frac{\beta}{1-\beta}}$ and $\psi_j(r_1, r_2) = \psi_j(r_2, r_1)$ for $j = 1, 2$, then*

$$\big| \langle H_{12} \psi_1, \psi_2 \rangle_{r_1 r_2} \big| \lesssim \|S_1 \psi_1\|_{L^2_{r_1 r_2}} \|S_1 \psi_2\|_{L^2_{r_1 r_2}}. \tag{48}$$

*Moreover,*

$$\|S_1 \psi\|^2_{L^2} \leqslant \langle H_{12} \psi, \psi \rangle \leqslant C \|S_1 \psi\|^2_{L^2}. \tag{49}$$

*Proof.* By Cauchy–Schwarz and (34),

$$\begin{aligned}\big| \langle \psi_1, H_{I12} \psi_2 \rangle_{r_1 r_2} \big| &= \omega^{-1} \big| \langle V_{N\omega 12} \psi_1, \psi_2 \rangle \big| \\ &\lesssim \Big( \omega^{-1} \int |V_{N\omega 12}| |\psi_1|^2 \Big)^{\frac{1}{2}} \Big( \omega^{-1} \int |V_{N\omega 12}| |\psi_2|^2 \Big)^{\frac{1}{2}} \\ &\lesssim \|S_1 \psi_1\|_{L^2} \|S_1 \psi_2\|_{L^2}.\end{aligned}$$

Thus

$$\left|\langle H_{12}\psi_1, \psi_2\rangle_{r_1r_2}\right| \leqslant \left|\langle H_{K12}\psi_1, \psi_2\rangle_{r_1r_2}\right| + \left|\langle H_{I12}\psi_1, \psi_2\rangle_{r_1r_2}\right|$$
$$\lesssim \|S_1\psi_1\|_{L^2_{r_1r_2}} \|S_1\psi_2\|_{L^2_{r_1r_2}},$$

which is (48). It remains to prove the first inequality in (49).

On the one hand, by (34), we have the lower bound for the potential term,

$$-\tfrac{1}{100}\left\langle\psi, (-\Delta_{r_1} + \omega^2|x_1|^2 - 2\omega)\psi\right\rangle_{r_1r_2} - C_3\|V\|^2_{L^1}\|\psi\|^2_{L^2_{r_1r_2}} \leqslant \omega^{-1}\langle V_{N\omega 12}\psi, \psi\rangle_{r_1r_2}.$$

Adding $\left\langle\psi, (\alpha-\Delta_{r_1}+\omega^2|x_1|^2-2\omega)\psi\right\rangle_{r_1r_2}$ to both sides and noticing the trivial inequalities $\alpha - C_3\|V\|^2_{L^2} = 1 \geqslant \frac{1}{2}$ and $\frac{99}{100} \geqslant \frac{1}{2}$, we have

$$\tfrac{1}{2}\left\langle\psi, (1-\Delta_{r_1} + \omega^2|x_1|^2 - 2\omega)\psi\right\rangle_{r_1r_2} \leqslant \left\langle\psi, \left(\alpha - \Delta_{r_1} + \omega^2|x_1|^2 - 2\omega + \omega^{-1}V_{N\omega 12}\right)\psi\right\rangle_{r_1r_2}. \tag{50}$$

On the other hand, we trivially have

$$\tfrac{1}{2}\left\langle\psi, (1-\Delta_{r_2} + \omega^2|x_2|^2 - 2\omega)\psi\right\rangle_{r_1r_2} \leqslant \left\langle\psi, (\alpha-\Delta_{r_2} + \omega^2|x_2|^2 - 2\omega)\psi\right\rangle_{r_1r_2} \tag{51}$$

because $\alpha > \frac{1}{2}$.

Adding estimates (50) and (51) together, we have

$$\tfrac{1}{2}\langle\psi, S_1^2\psi\rangle + \tfrac{1}{2}\langle\psi, S_2^2\psi\rangle \leqslant \langle H_{12}\psi, \psi\rangle.$$

By symmetry in $r_1$ and $r_2$, this is precisely (49). □

*The $k = 2$ case.* The $k = 2$ energy estimate is the lower bound

$$\tfrac{1}{4}\left(\langle S_1^2S_2^2\psi, \psi\rangle + N^{-1}\langle S_1^4\psi, \psi\rangle\right) \leqslant \left\langle(\alpha + N^{-1}H - 2\omega)^2\psi,\ \psi\right\rangle.$$

We will prove it under the hypothesis

$$N^{\frac{\beta}{1-\beta}} \leqslant \omega \leqslant N^{\min\left(\frac{1-\beta}{\beta}, 2\right)}.$$

We substitute into (32) to obtain

$$\left\langle(\alpha + N^{-1}H - 2\omega)^2\psi,\ \psi\right\rangle = \tfrac{1}{4}N^{-4}\sum_{\substack{1\leqslant i_1\neq j_1\leqslant N\\ 1\leqslant i_2\neq j_2\leqslant N}}\langle H_{i_1j_1}H_{i_2j_2}\psi,\ \psi\rangle = A_1 + A_2 + A_3,$$

where

- $A_1$ consists of those terms with $\{i_1, j_1\} \cap \{i_2, j_2\} = \varnothing$,
- $A_2$ consists of those terms with $\left|\{i_1, j_1\} \cap \{i_2, j_2\}\right| = 1$,
- $A_3$ consists of those terms with $\left|\{i_1, j_1\} \cap \{i_2, j_2\}\right| = 2$.

By symmetry, we have

$$A_1 = \tfrac{1}{4}\langle H_{12}H_{34}\psi, \psi\rangle,$$
$$A_2 = \tfrac{1}{2}N^{-1}\langle H_{12}H_{23}\psi, \psi\rangle,$$
$$A_3 = \tfrac{1}{2}N^{-2}\langle H_{12}H_{12}\psi, \psi\rangle.$$

We discard $A_3$ since $A_3 \geqslant 0$. By the analysis used in the $k=1$ case,

$$A_1 \geqslant \tfrac{1}{4}\|S_1 S_3 \psi\|_{L^2}^2.$$

The main piece of work in the $k=2$ case is to estimate $A_2$. Substituting $H_{12} = H_{K12} + H_{I12}$ and $H_{23} = H_{K23} + H_{I23}$, we obtain the expansion

$$A_2 = B_0 + B_1 + B_2,$$

where

$$\begin{aligned}
B_0 &= \tfrac{1}{2}N^{-1}\langle H_{K12}H_{K23}\psi, \psi\rangle,\\
B_1 &= \tfrac{1}{2}N^{-1}\langle H_{K12}H_{I23}\psi, \psi\rangle + \tfrac{1}{2}N^{-1}\langle H_{I12}H_{K23}\psi, \psi\rangle,\\
B_2 &= \tfrac{1}{2}N^{-1}\langle H_{I12}H_{I23}\psi, \psi\rangle.
\end{aligned}$$

Let $\sigma = \alpha - 1 \geqslant 0$. First note that

$$B_0 = \tfrac{1}{2}N^{-1}\big\langle (S_1^2 + S_2^2 + 2\sigma)(S_2^2 + S_3^2 + 2\sigma)\psi,\ \psi\big\rangle.$$

Since $S_1^2$, $S_2^2$, $S_3^2$ all commute,

$$B_0 \geqslant \tfrac{1}{2}N^{-1}\langle S_2^4\psi, \psi\rangle,$$

which is a component of the claimed lower bound.

Next, we consider $B_1$. By symmetry

$$B_1 = N^{-1}\operatorname{Re}\langle H_{K12}H_{I23}\psi, \psi\rangle.$$

Since every term in $B_1$ is estimated, we do not drop the imaginary part. Decompose $I = P_{0\omega}^2 + P_{\gtrsim 1\omega}^2$ in the right factor of $\psi$ as

$$B_1 = B_{10} + B_{11} + B_{12},$$

where

$$\begin{aligned}
B_{10} &= (N\omega)^{-1}\big\langle [(2\alpha - 1) + S_1^2]V_{N\omega 23}\psi,\ \psi\big\rangle,\\
B_{11} &= (N\omega)^{-1}\big\langle (-\Delta_{r_2} + \omega^2|x_2|^2 - 2\omega)V_{N\omega 23}\psi,\ P_{0\omega}^2\psi\big\rangle,\\
B_{12} &= (N\omega)^{-1}\big\langle (-\Delta_{r_2} + \omega^2|x_2|^2 - 2\omega)V_{N\omega 23}\psi,\ P_{\gtrsim 1\omega}^2\psi\big\rangle.
\end{aligned}$$

The term $B_{10}$ is the simplest. In fact, by estimate (35) at the $r_2$-coordinate, we have

$$\begin{aligned}
|B_{10}| &= \big|(N\omega)^{-1}\big\langle [(2\alpha - 1) + S_1^2]V_{N\omega 23}\psi,\ \psi\big\rangle\big|\\
&\lesssim N^{-1}\big(\|S_2\psi\|_{L^2}^2 + \|S_1 S_2\psi\|_{L^2}^2\big).
\end{aligned}$$

For $B_{12}$, we consider the four terms separately:

$$B_{12} = B_{121} + B_{122} + B_{123} + B_{124},$$

where

$$B_{121} = (N\omega)^{\beta-1}\langle(\nabla V)_{N\omega 23}\psi, \nabla_{r_2} P^2_{\geqslant 1\omega}\psi\rangle,$$
$$B_{122} = (N\omega)^{-1}\langle V_{N\omega 23}\nabla_{r_2}\psi, \nabla_{r_2} P^2_{\geqslant 1\omega}\psi\rangle,$$
$$B_{123} = (N\omega)^{-1}\langle V_{N\omega 23}\omega|x_2|\psi, \omega|x_2| P^2_{\geqslant 1\omega}\psi\rangle,$$
$$B_{124} = -2(N\omega)^{-1}\langle V_{N\omega 23}\omega^{\frac{1}{2}}\psi, \omega^{\frac{1}{2}} P^2_{\geqslant 1\omega}\psi\rangle.$$

By (35) applied with $r_1$ replaced by $r_3$, we obtain

$$|B_{121}| \lesssim (N\omega)^{\beta-1}\omega\|S_3\psi\|_{L^2}\left\|\nabla_{r_2} P^2_{\geqslant 1\omega} S_3\psi\right\|_{L^2}.$$

By (40),

$$|B_{121}| \lesssim (N\omega)^{\beta-1}\omega\|S_3\psi\|_{L^2}\|S_2 S_3\psi\|_{L^2},$$

which yields the requirement $\omega \leqslant N^{\frac{1-\beta}{\beta}}$. By (35) applied with $r_1$ replaced by $r_3$, we obtain

$$|B_{122}| \lesssim (N\omega)^{-1}\omega\|\nabla_{r_2} S_3\psi\|_{L^2}\left\|\nabla_{r_2} P_{\geqslant 1\omega} S_3\psi\right\|_{L^2}.$$

Utilizing (43) for the $\|\nabla_{r_2} S_3\psi\|_{L^2}$ term and (40) for the $\|\nabla_{r_2} P_{\geqslant 1\omega} S_3\psi\|_{L^2}$ term,

$$|B_{122}| \lesssim (N\omega)^{-1}\omega^{\frac{3}{2}}\|S_2 S_3\|^2_{L^2}.$$

This requires $\omega \leqslant N^2$. The terms $B_{123}$ and $B_{124}$ are estimated in the same way as $B_{122}$, yielding the requirement $\omega \leqslant N^2$. This completes the treatment of $B_{12}$.

For $B_{11}$, we move the operator $(-\Delta_{r_2} + \omega^2|x_2|^2 - 2\omega)$ over to the right, and use the fact that $(-\Delta_{r_2} + \omega^2|x_2|^2 - 2\omega)P^2_{0\omega}\psi = -\partial^2_{z_2} P^2_{0\omega}\psi$ to obtain

$$B_{11} = B_{111} + B_{112},$$

where

$$B_{111} = (N\omega)^{\beta-1}\langle(\partial_z V)_{N\omega 23}\psi, \partial_{z_2} P^2_{0\omega}\psi\rangle,$$
$$B_{112} = (N\omega)^{-1}\langle V_{N\omega 23}\partial_{z_2}\psi, \partial_{z_2} P^2_{0\omega}\psi\rangle.$$

By (35) applied with $r_1$ replaced by $r_3$, we obtain

$$|B_{111}| \lesssim (N\omega)^{\beta-1}\omega\|S_3\psi\|_{L^2}\|\partial_{z_2} P^2_{0\omega} S_3\psi\|_{L^2}.$$

Using (42) for the $\|\partial_{z_2} P^2_{0\omega} S_3\psi\|_{L^2}$ term (which saves us from the $\omega^{\frac{1}{2}}$ loss),

$$|B_{111}| \lesssim (N\omega)^{\beta-1}\omega\|S_3\psi\|_{L^2}\|S_2 S_3\psi\|_{L^2},$$

which again requires that $\omega \leqslant N^{\frac{1-\beta}{\beta}}$. By (35) applied with $r_1$ replaced by $r_3$, we obtain

$$|B_{112}| \lesssim (N\omega)^{-1}\omega\|\partial_{z_2} S_3\psi\|_{L^2}\|\partial_{z_2} P^2_{0\omega} S_3\psi\|_{L^2}.$$

Using (42),

$$|B_{112}| \lesssim (N\omega)^{-1}\omega\|S_2 S_3\psi\|^2_{L^2},$$

which has no requirement on $\omega$. This completes the treatment of $B_{11}$, and hence also $B_1$. Now let us consider $B_2$:

$$B_2 = N^{-1}\omega^{-2}\langle V_{N\omega 12}V_{N\omega 23}\psi,\ \psi\rangle,$$
$$|B_2| \leqslant N^{-1}\omega^{-2}\int |V_{N\omega 23}|\left(\int_{r_1}|V_{N\omega 12}||\psi(r_1,\dots,r_N)|^2\,dr_1\right)dr_2\cdots dr_N.$$

In the parentheses, apply estimate (35) in the $r_1$-coordinate to obtain

$$|B_2| \lesssim N^{-1}\omega^{-2}\omega\int_{r_2,\dots,r_N}|V_{N\omega 23}|\,\|S_1\psi\|^2_{L^2_{r_1}}\,dr_2\cdots dr_N.$$

By Fubini, the right-hand side is equal to

$$N^{-1}\omega^{-2}\omega\int_{r_1}\left(\int_{r_2,\dots,r_N}|V_{N\omega 23}||S_1\psi(r_1,\dots,r_N)|^2\,dr_2\cdots dr_N\right)dr_1.$$

In the parentheses, apply estimate (35) in the $r_2$-coordinate to obtain

$$|B_2| \lesssim N^{-1}\omega^{-2}\omega^2\|S_1S_2\psi\|^2_{L^2}.$$

Hence $B_2$ is bounded without additional restriction on $\omega$. Therefore we end the proof for the $k=2$ case.

*The $k$ case implies the $k+2$ case.* We assume that (28) holds for $k$. Applying it with $\psi$ replaced by $(\alpha+N^{-1}H_{N,\omega}-2\omega)\psi$,

$$\frac{1}{2^k}\left\|S^{(k)}(\alpha+N^{-1}H_{N,\omega}-2\omega)\psi\right\|_{L^2} \leqslant \left\langle(\alpha+N^{-1}H_{N,\omega}-2\omega)^{k+2}\psi,\ \psi\right\rangle.$$

Hence, to prove (28) in the case $k+2$, it suffices to prove

$$\tfrac{1}{4}\left(\|S^{(k+2)}\psi\|^2_{L^2}+N^{-1}\|S_1S^{(k+1)}\psi\|^2_{L^2}\right) \leqslant \left\|S^{(k)}(\alpha+N^{-1}H_{N,\omega}-2\omega)\psi\right\|^2_{L^2}. \tag{52}$$

To prove (52), we substitute (32) into

$$\left\langle S^{(k)}(\alpha+N^{-1}H_{N,\omega}-2\omega)\psi,\ S^{(k)}(\alpha+N^{-1}H_{N,\omega}-2\omega)\psi\right\rangle,$$

which gives

$$N^{-4}\sum_{\substack{1\leqslant i_1<j_1\leqslant N\\ 1\leqslant i_2<j_2\leqslant N}}\langle S^{(k)}H_{i_1j_1}\psi,\ S^{(k)}H_{i_2j_2}\psi\rangle.$$

We decompose into three terms

$$E_1+E_2+E_3$$

according to the location of $i_1$ and $i_2$ relative to $k$. We place no restriction on $j_1$, $j_2$ (other than $i_1<j_1$, $i_2<j_2$):

- $E_1$ consists of those terms for which $i_1\leqslant k$ and $i_2\leqslant k$.
- $E_2$ consists of those terms for which both $i_1>k$ and $i_2>k$.
- $E_3$ consists of those terms for which either ($i_1\leqslant k$ and $i_2>k$) or ($i_1>k$ and $i_2<k$).

We have $E_1 \geqslant 0$, and we discard this term. We extract the key lower bound from $E_2$ exactly as in the $k = 2$ case. In fact, inside $E_2$, we know $H_{i_1 j_1}$ and $H_{i_2 j_2}$ commute with $S^{(k)}$ because $j_1 > i_1 > k$ and $j_2 > i_2 > k$; hence we indeed face the $k = 2$ case again. This leaves us with $E_3$:

$$E_3 = 2N^{-4} \sum_{\substack{1\leqslant i_1<j_1\leqslant N\\ 1\leqslant i_2<j_2\leqslant N\\ i_1\leqslant k, i_2>k}} \operatorname{Re}\langle S^{(k)} H_{i_1 j_1}\psi,\, S^{(k)} H_{i_2 j_2}\psi\rangle.$$

We decompose $E_3$ as

$$E_3 = D_1 + D_2 + D_3,$$

where, in each case we require $i_1 \leqslant k$ and $i_2 > k$, but make the additional distinctions as follows:

- $D_1$ consists of those terms where $j_1 \leqslant k$.
- $D_2$ consists of those terms where $j_1 > k$ and $j_1 \in \{i_2, j_2\}$.
- $D_3$ consists of those terms where $j_1 > k$ and $j_1 \notin \{i_2, j_2\}$.

By symmetry,

$$\begin{aligned}
D_1 &= k^2 N^{-2}\big\langle S_1 \cdots S_k H_{12}\psi,\, S_1 \cdots S_k H_{(k+1)(k+2)}\psi\big\rangle,\\
D_2 &= k N^{-2}\big\langle S_1 \cdots S_k H_{1(k+1)}\psi,\, S_1 \cdots S_k H_{(k+1)(k+2)}\psi\big\rangle,\\
D_3 &= N^{-1}\big\langle S_1 \cdots S_k H_{1(k+1)}\psi,\, S_1 \cdots S_k H_{(k+2)(k+3)}\psi\big\rangle.
\end{aligned}$$

We begin with estimates for the term $D_1$. We decompose it as

$$D_1 = D_{11} + D_{12},$$

where

$$\begin{aligned}
D_{11} &= N^{-2}\big\langle H_{(k+1)(k+2)}[S_1 S_2, H_{12}] S_3 \cdots S_k\psi,\, S_1 \cdots S_k\psi\big\rangle,\\
D_{12} &= N^{-2}\big\langle H_{(k+1)(k+2)} H_{12} S_1 \cdots S_k\psi,\, S_1 \cdots S_k\psi\big\rangle.
\end{aligned}$$

By Lemmas 3.7 and A.3, $D_{12}$ is positive because $H_{(k+1)(k+2)}$ and $H_{12}$ commutes. Therefore we discard $D_{12}$. For $D_{11}$, we take $[V_{N\omega 12}, S_1 S_2] \sim (N\omega)^{2\beta}(\Delta V)_{N\omega 12}$. This gives

$$|D_{11}| \lesssim N^{2\beta-2}\omega^{2\beta-1}\big\langle H_{(k+1)(k+2)}(\Delta V)_{N\omega 12} S_3 \cdots S_k\psi,\, S_1 \cdots S_k\psi\big\rangle.$$

By using Lemma 3.7 in the $r_{k+1}$-coordinate to handle $H_{(k+1)(k+2)}$, we have

$$|D_{11}| \lesssim N^{2\beta-2}\omega^{2\beta-1}\big\||(\Delta V)_{N\omega 12}|^{\frac12} S_3 \cdots S_{k+1}\psi\big\|_{L^2}\big\||(\Delta V)_{N\omega 12}|^{\frac12} S_1 \cdots S_{k+1}\psi\big\|_{L^2}.$$

Using (35) in the first factor,

$$|D_{11}| \lesssim N^{2\beta-2}\omega^{2\beta-\frac12}\|S_1 S_3 \cdots S_{k+1}\psi\|_{L^2}\big\||(\Delta V)_{N\omega 12}|^{\frac12} S_1 \cdots S_{k+1}\psi\big\|_{L^2}.$$

Decomposing $\psi$ in the second factor into $P^1_{0\omega}\psi + P^1_{\geqslant 1\omega}\psi$ gives

$$\begin{aligned}
|D_{11}| \lesssim{}& N^{2\beta-2}\omega^{2\beta-\frac12}\|S_1 S_3 \cdots S_{k+1}\psi\|_{L^2}\\
&\times\Big(\big\||(\Delta V)_{N\omega 12}|^{\frac12} S_1 \cdots S_{k+1} P^1_{0\omega}\psi\big\|_{L^2} + \big\||(\Delta V)_{N\omega 12}|^{\frac12} S_1 \cdots S_{k+1} P^1_{\geqslant 1\omega}\psi\big\|_{L^2}\Big).
\end{aligned}$$

Applying Lemma 3.6,

$$|D_{11}| \lesssim N^{2\beta-2}\omega^{2\beta-\frac{1}{2}}\|S_1S_3\cdots S_{k+1}\psi\|_{L^2}\omega^{\frac{1}{2}}N^{\frac{1}{4}}\|S_1\cdots S_{k+1}\psi\|_{L^2}^{\frac{1}{2}}\big(N^{-\frac{1}{4}}\|S_1^2\cdots S_{k+1}\psi\|_{L^2}^{\frac{1}{2}}\big) + N^{2\beta-2}\omega^{2\beta-\frac{1}{2}}\|S_1S_3\cdots S_{k+1}\psi\|_{L^2}N^{\frac{\beta}{2}+\frac{1}{2}}\omega^{\frac{\beta}{2}}\big(N^{-\frac{1}{2}}\|S_1^2\cdots S_{k+1}\psi\|_{L^2}\big).$$

The coefficients simplify to $N^{2\beta-\frac{7}{4}}\omega^{2\beta}$ and $N^{\frac{5}{2}\beta-\frac{3}{2}}\omega^{\frac{5}{2}\beta-\frac{1}{2}}$. This gives the constraints

$$\omega \leqslant N^{\frac{7/4-2\beta}{2\beta}} \quad \text{and} \quad \omega \leqslant N^{\frac{3/5-\beta}{\beta-1/5}}.$$

The second one is the worst one. When combined with the lower bound $N^{\frac{\beta}{1-\beta}} \leqslant \omega$, it restricts us to $\beta \leqslant \frac{3}{7}$. Moreover, at $\beta = \frac{2}{5}$, the relation $\omega = N$ is within the allowable range.

We now find estimates for the term $D_2$. We write

$$D_2 = D_{21} + D_{22},$$

where

$$D_{21} = N^{-2}\langle H_{(k+1)(k+2)}[S_1, H_{1(k+1)}]S_2\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle,$$
$$D_{22} = N^{-2}\langle H_{(k+1)(k+2)}H_{1(k+1)}S_1\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle.$$

Let us begin with $D_{21}$. We use

$$[S_1, H_{1(k+1)}] \sim (N\omega)^{\beta}\omega^{-1}(\nabla V)_{N\omega 1(k+1)}$$

and

$$H_{(k+1)(k+2)} = 2\sigma + S_{k+1}^2 + S_{k+2}^2 + \omega^{-1}V_{N\omega(k+1)(k+2)}$$

to get

$$D_{21} = D_{210} + D_{211} + D_{212} + D_{213},$$

where

$$D_{210} = 2\sigma N^{-1}(N\omega)^{\beta-1}\big\langle(\nabla V)_{N\omega 1(k+1)}S_2\cdots S_k\psi,\ S_1\cdots S_k\psi\big\rangle,$$
$$D_{211} = N^{-1}(N\omega)^{\beta-1}\big\langle S_{k+1}^2(\nabla V)_{N\omega 1(k+1)}S_2\cdots S_k\psi,\ S_1\cdots S_k\psi\big\rangle,$$
$$D_{212} = N^{-1}(N\omega)^{\beta-1}\big\langle S_{k+2}^2(\nabla V)_{N\omega 1(k+1)}S_2\cdots S_k\psi,\ S_1\cdots S_k\psi\big\rangle,$$
$$D_{213} = N^{-2}(N\omega)^{\beta}\omega^{-2}\big\langle V_{N\omega(k+1)(k+2)}(\nabla V)_{N\omega 1(k+1)}S_2\cdots S_k\psi,\ S_1\cdots S_k\psi\big\rangle.$$

For $D_{211}$,

$$D_{211} = N^{-1}(N\omega)^{\beta-1}\big\langle[S_{k+1}, (\nabla V)_{Nw1(k+1)}]S_2\cdots S_k\psi,\ S_1\cdots S_k\psi\big\rangle + N^{-1}(N\omega)^{\beta-1}\big\langle(\nabla V)_{Nw1(k+1)}S_2\cdots S_kS_{k+1}\psi,\ S_1\cdots S_k\psi\big\rangle.$$

The first piece is estimated the same way as $D_{11}$. For the second term, using Lemma 3.6 in the $r_1$-coordinate,

$$|\cdot| \lesssim N^{-1}(N\omega)^{\beta-1}\omega N^{\frac{1}{4}}\|S_1\cdots S_{k+1}\psi\|_{L^2}\|S_1\cdots S_k\psi\|_{L^2}^{\frac{1}{2}}\big(N^{-\frac{1}{4}}\|S_1S_1\cdots S_k\psi\|_{L^2}\big) + N^{-1}(N\omega)^{\beta-1}(N\omega)^{\frac{1}{2}\beta+\frac{1}{2}}\|S_1\cdots S_{k+1}\psi\|_{L^2}\big(N^{-\frac{1}{2}}\|S_1S_1\cdots S_k\psi\|_{L^2}\big),$$

which gives the conditions $\omega \leqslant N^{\frac{7/4-\beta}{\beta}}$ and $\omega \leqslant N^{\frac{3-3\beta}{3\beta-1}}$. Since this results in conditions better than those produced for $D_{11}$, we neglect them.

For $D_{213}$, we apply estimate (35) in the $r_{k+2}$-coordinate and again in the $r_{k+1}$-coordinate to obtain

$$|D_{213}| \lesssim N^{-2}(N\omega)^{\beta}\omega^{-2}\omega^{2}\|S_2\cdots S_{k+2}\psi\|_{L^2}\|S_1\cdots S_{k+2}\psi\|_{L^2}.$$

This gives the requirement $\omega \leqslant N^{\frac{2-\beta}{\beta}}$, which is clearly weaker than $\omega \leqslant N^{\frac{1-\beta}{\beta}}$, so we drop it. The terms $D_{210}$ and $D_{212}$ are estimated in the same way. In fact, utilizing estimate (35) in the $r_{k+1}$-coordinate yields

$$|D_{210}| \lesssim N^{-1}(N\omega)^{\beta-1}\omega\|S_2\cdots S_k\psi\|_{L^2}\|S_1\cdots S_k\psi\|_{L^2},$$
$$|D_{212}| \lesssim N^{-1}(N\omega)^{\beta-1}\omega\|S_2\cdots S_{k+2}\psi\|_{L^2}\|S_1\cdots S_{k+2}\psi\|_{L^2}.$$

They give the same weaker condition $\omega \leqslant N^{\frac{2-\beta}{\beta}}$.

We now turn to $D_{22}$. Since $H_{(k+1)(k+2)}$ and $H_{1(k+1)}$ do not commute, we cannot directly quote Lemma 3.7 and conclude it is positive. We estimate it. By the definition of $H_{ij}$, we only need to look at the terms

$$\begin{aligned}
D_{220} &= N^{-2}\omega^{-1}\langle \sigma V_{N\omega 1(k+1)}S_1\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle,\\
D_{221} &= N^{-2}\omega^{-1}\langle S_{k+1}^2 V_{N\omega 1(k+1)}S_1\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle,\\
D_{222} &= N^{-2}\omega^{-1}\langle S_{k+2}^2 V_{N\omega 1(k+1)}S_1\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle,\\
D_{223} &= N^{-2}\omega^{-2}\langle V_{N\omega(k+1)(k+2)}V_{N\omega 1(k+1)}S_1\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle,\\
D_{224} &= N^{-2}\omega^{-1}\langle \sigma V_{N\omega(k+1)(k+2)}S_1\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle,\\
D_{225} &= N^{-2}\omega^{-1}\langle V_{N\omega(k+1)(k+2)}S_1^2 S_1\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle,\\
D_{226} &= N^{-2}\omega^{-1}\langle V_{N\omega(k+1)(k+2)}S_{k+1}^2 S_1\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle
\end{aligned}$$

because all the other terms inside the expansion of $D_{22}$ are positive. It is easy to tell the following: the terms $D_{220}$ and $D_{224}$ can be estimated in the same way as $D_{210}$, the terms $D_{221}$ and $D_{226}$ can be estimated in the same way as $D_{211}$, the terms $D_{222}$ and $D_{225}$ can be estimated in the same way as $D_{212}$, and the term $D_{223}$ can be estimated in the same way as $D_{213}$. Moreover, all the $D_{22}$ terms are better than the corresponding $D_{21}$ terms since they do not have a $(N\omega)^{\beta}$ in front of them. Hence, we get no new restrictions from $D_{22}$ and we conclude the estimate for $D_{22}$.

We now find estimates for the term $D_3$. Commuting terms as usual,

$$D_3 = D_{31} + D_{32},$$

where

$$\begin{aligned}
D_{31} &= N^{-1}\langle H_{(k+2)(k+3)}[S_1, H_{1(k+1)}]S_2\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle,\\
D_{32} &= N^{-1}\langle H_{(k+2)(k+3)}H_{1(k+1)}S_1\cdots S_k\psi,\ S_1\cdots S_k\psi\rangle.
\end{aligned}$$

Since $H_{(k+2)(k+3)}$ and $H_{1(k+1)}$ commute, $D_{32}$ is positive due to Lemmas 3.7 and A.3. Thus we discard $D_{32}$. For $D_{31}$, we use that

$$[S_1, H_{1(k+1)}] \sim (N\omega)^{\beta}\omega^{-1}(\nabla V)_{N\omega 1(k+1)}$$

together with estimate (35) in the $r_{k+1}$-coordinate (to handle $[S_1, H_{1(k+1)}]$) and Lemma 3.7 in the $r_{k+2}$-coordinate (to handle $H_{(k+2)(k+3)}$):

$$|D_{31}| \lesssim N^{-1}(N\omega)^{\beta} \|S_2 \cdots S_{k+2}\psi\|_{L^2} \|S_1 \cdots S_{k+2}\psi\|_{L^2}.$$

This term again yields to the restriction

$$\omega \leqslant N^{\frac{1-\beta}{\beta}}.$$

So far, we have proved that all the terms in $E_3$ can be absorbed into the key lower bound exacted from $E_2$ for all $N$ large enough as long as $C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_E(\beta)}$. Hence we have finished the two-step induction argument and established Theorem 3.1.

## 4. Compactness of the BBGKY sequence

**Theorem 4.1.** *Assume $C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_2(\beta)}$. Then the sequence*

$$\left\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^{N}\right\} \subset \bigoplus_{k\geqslant 1} C([0,T], \mathcal{L}_k^1),$$

*which satisfies the focusing "$\infty - \infty$" BBGKY hierarchy (18), is compact with respect to the product topology $\tau_{\text{prod}}$. For any limit point $\Gamma(t) = \{\tilde{\gamma}^{(k)}\}_{k=1}^{N}$, we have $\tilde{\gamma}^{(k)}$ is a symmetric nonnegative trace class operator with trace bounded by* 1.

*Proof.* By the standard diagonalization argument, it suffices to show the compactness of $\tilde{\gamma}_{N,\omega}^{(k)}$ for fixed $k$ with respect to the metric $\hat{d}_k$. By the Arzelà–Ascoli theorem, this is equivalent to the equicontinuity of $\tilde{\gamma}_{N,\omega}^{(k)}$. By [Erdős et al. 2010, Lemma 6.2], it suffices to prove that for every test function $J^{(k)}$ from a dense subset of $\mathcal{K}(L^2(\mathbb{R}^{3k}))$ and for every $\varepsilon > 0$, there exists $\delta(J^{(k)}, \varepsilon)$ such that for all $t_1, t_2 \in [0,T]$ with $|t_1 - t_2| \leqslant \delta$, we can write

$$\sup_{N,\omega} \left|\operatorname{Tr} J^{(k)} \tilde{\gamma}_{N,\omega}^{(k)}(t_1) - \operatorname{Tr} J^{(k)} \tilde{\gamma}_{N,\omega}^{(k)}(t_2)\right| \leqslant \varepsilon. \tag{53}$$

Here, we assume that our compact operators $J^{(k)}$ have been cut off in frequency as in Lemma A.6. Assume $t_1 \leqslant t_2$. Inserting the decomposition (22) on the left and right sides of $\gamma_{N,\omega}^{(k)}$, we obtain

$$\tilde{\gamma}_{N,\omega}^{(k)} = \sum_{\alpha,\beta} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta,$$

where the sum is taken over all $k$-tuples $\alpha$ and $\beta$ of the type described in (22).

To establish (53) it suffices to prove that, for each $\alpha$ and $\beta$, we have

$$\sup_{N,\omega} \left|\operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_1) - \operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_2)\right| \leqslant \varepsilon. \tag{54}$$

To this end, we establish the estimate

$$\begin{aligned} &\left|\operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_1) - \operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}_{N,\omega}^{(k)} \mathcal{P}_\beta(t_2)\right| \\ &\qquad \lesssim C|t_2 - t_1| \big(\mathbf{1}_{\alpha=0 \text{ and } \beta=0} + \max(1, \omega^{1-\frac{1}{2}|a|-\frac{1}{2}|\beta|}) \mathbf{1}_{\alpha\neq 0 \text{ or } \beta\neq 0}\big). \end{aligned} \tag{55}$$

At a glance, (55) seems not quite enough in the $|\alpha| = 0$ and $|\beta| = 1$ case (or vice versa) because it grows in $\omega$. However, we can also prove the (comparatively simpler) bound

$$\left|\operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta(t_2) - \operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta(t_1)\right| \lesssim \omega^{-\frac{1}{2}|\alpha| - \frac{1}{2}|\beta|}, \tag{56}$$

which provides a better power of $\omega$ but no gain as $t_2 \to t_1$. Interpolating between (55) and (56) in the $|\alpha| = 0$ and $|\beta| = 1$ case (or vice versa), we acquire

$$\left|\operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta(t_2) - \operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta(t_1)\right| \lesssim |t_2 - t_1|^{\frac{1}{2}},$$

which suffices to establish (54).

Below, we prove (55) and (56). We first prove (55). The BBGKY hierarchy (18) yields

$$\partial_t \operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \tag{57}$$

where

$$\mathrm{I} = -i\omega \sum_{j=1}^{k} \operatorname{Tr} J^{(k)} \big[-\Delta_{x_j} + |x_j|^2,\, \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta\big],$$

$$\mathrm{II} = -i \sum_{j=1}^{k} \operatorname{Tr} J^{(k)} \big[-\partial_{z_j}^2,\, \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta\big],$$

$$\mathrm{III} = \frac{-i}{N} \sum_{1 \leqslant i < j \leqslant k} \operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \big[V_{N,\omega}(r_i - r_j),\, \tilde{\gamma}^{(k)}_{N,\omega}\big] \mathcal{P}_\beta,$$

$$\mathrm{IV} = -i \frac{N-k}{N} \sum_{j=1}^{k} \operatorname{Tr} J^{(k)} \mathcal{P}_\alpha \big[V_{N,\omega}(r_j - r_{k+1}),\, \tilde{\gamma}^{(k+1)}_{N,\omega}\big] \mathcal{P}_\beta.$$

We first consider I. When $\alpha = \beta = 0$,

$$\begin{aligned} \mathrm{I} &= -i\omega \sum_{j=1}^{k} \operatorname{Tr} J^{(k)} \big[-\Delta_{x_j} + |x_j|^2,\, \mathcal{P}_{\mathbf{0}} \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_{\mathbf{0}}\big] \\ &= -i\omega \sum_{j=1}^{k} \operatorname{Tr} J^{(k)} \big[-2 - \Delta_{x_j} + |x_j|^2,\, \mathcal{P}_{\mathbf{0}} \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_{\mathbf{0}}\big] = 0, \end{aligned}$$

since constants commute with everything. When $\alpha \neq 0$ or $\beta \neq 0$, we apply Lemma A.5 and integrate by parts to obtain

$$\begin{aligned} |\mathrm{I}| &\leqslant \omega \sum_{j=1}^{k} \big|\langle J^{(k)} H_j \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega}\rangle - \langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega},\, H_j \mathcal{P}_\beta \tilde{\psi}_{N,\omega}\rangle\big| \\ &\leqslant \omega \sum_{j=1}^{k} \big(\big|\langle J^{(k)} H_j \mathcal{P}_\alpha \tilde{\psi}_{N,\omega},\, \mathcal{P}_\beta \tilde{\psi}_{N,\omega}\rangle\big| + \big|\langle H_j J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega},\, \mathcal{P}_\beta \tilde{\psi}_{N,\omega}\rangle\big|\big), \end{aligned}$$

where $H_j = -\Delta_{x_j} + |x_j|^2$. Hence

$$|\mathrm{I}| \lesssim \omega \sum_{j=1}^{k} \big(\|J^{(k)} H_j\|_{\mathrm{op}} + \|H_j J^{(k)}\|_{\mathrm{op}}\big) \|\mathcal{P}_\alpha \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|\mathcal{P}_\beta \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})}.$$

By the energy estimate (31),

$$\begin{aligned} &|\mathrm{I}| = 0, && \text{if } \alpha = 0 \text{ and } \beta = 0, \\ &|\mathrm{I}| \lesssim C_{k,J^{(k)}} \omega^{1-\frac{1}{2}|\alpha| - \frac{1}{2}|\beta|}, && \text{otherwise.} \end{aligned} \tag{58}$$

Next, consider II. Proceeding as in I, we have

$$|\mathrm{II}| \leqslant \sum_{j=1}^{k} \Big( \big|\langle J^{(k)} \partial_{z_j}^2 \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega}\rangle\big| + \big|\langle \partial_{z_j}^2 J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega}\rangle\big| \Big).$$

That is,

$$|\mathrm{II}| \leqslant \sum_{j=1}^{k} \big(\|J^{(k)} \partial_{z_j}^2\|_{\mathrm{op}} + \|\partial_{z_j}^2 J^{(k)}\|_{\mathrm{op}}\big) \|\mathcal{P}_\alpha \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|\mathcal{P}_\beta \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \leqslant C_{k,J^{(k)}}. \tag{59}$$

Now, consider III:

$$\begin{aligned} |\mathrm{III}| \leqslant N^{-1} \sum_{1 \leqslant i < j \leqslant k} &\big|\langle J^{(k)} \mathcal{P}_\alpha V_{N,\omega}(r_i - r_j) \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega}\rangle\big| \\ &+ N^{-1} \sum_{1 \leqslant i < j \leqslant k} \big|\langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta V_{N,\omega}(r_i - r_j) \tilde{\psi}_{N,\omega}\rangle\big|. \end{aligned}$$

That is,

$$\begin{aligned} |\mathrm{III}| \leqslant N^{-1} \sum_{1 \leqslant i < j \leqslant k} &\big|\langle J^{(k)} \mathcal{P}_\alpha L_i L_j W_{ij} L_i L_j \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega}\rangle\big| \\ &+ N^{-1} \sum_{1 \leqslant i < j \leqslant k} \big|\langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta L_i L_j W_{ij} L_i L_j \tilde{\psi}_{N,\omega}\rangle\big| \end{aligned}$$

if we write $L_i = (1 - \Delta_{r_i})^{\frac{1}{2}}$ and

$$W_{ij} = L_i^{-1} L_j^{-1} V_{N,\omega}(r_i - r_j) L_i^{-1} L_j^{-1}.$$

Hence

$$\begin{aligned} |\mathrm{III}| \leqslant N^{-1} \sum_{1 \leqslant i < j \leqslant k} &\|J^{(k)} L_i L_j\|_{\mathrm{op}} \|W_{ij}\|_{\mathrm{op}} \|L_i L_j \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|\mathcal{P}_\beta \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \\ &+ N^{-1} \sum_{1 \leqslant i < j \leqslant k} \|L_i L_j J^{(k)}\|_{\mathrm{op}} \|W_{ij}\|_{\mathrm{op}} \|L_i L_j \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|\mathcal{P}_\alpha \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})}. \end{aligned}$$

Since $\|W_{ij}\|_{\mathrm{op}} \lesssim \|V_{N,\omega}\|_{L^1} = \|V\|_{L^1}$ (independent of $N$, $\omega$) by Lemma A.1, the energy estimates (Corollary 3.2) imply that

$$|\mathrm{III}| \lesssim \frac{C_{k,J^{(k)}}}{N}. \tag{60}$$

Apply the same ideas to IV:

$$|\mathrm{IV}| \leqslant \sum_{j=1}^{k} \big|\langle J^{(k)} \mathcal{P}_\alpha L_j L_{k+1} W_{j(k+1)} L_j L_{k+1} \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta \tilde{\psi}_{N,\omega}\rangle\big| + \sum_{j=1}^{k} \big|\langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}, \mathcal{P}_\beta L_j L_{k+1} W_{j(k+1)} L_j L_{k+1} \tilde{\psi}_{N,\omega}\rangle\big|.$$

Then, since $J^{(k)} L_{k+1} = L_{k+1} J^{(k)}$,

$$\begin{aligned} |\mathrm{IV}| &\leqslant \sum_{j=1}^{k} \big(\|J^{(k)} L_j\|_{\mathrm{op}} + \|L_j J^{(k)}\|_{\mathrm{op}}\big) \|W_{j(k+1)}\|_{\mathrm{op}} \|L_j L_{k+1} \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|L_j \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \\ &\lesssim C_{k,J^{(k)}}. \end{aligned} \tag{61}$$

Integrating (57) from $t_1$ to $t_2$ and applying the bounds obtained in (58)–(61), we obtain (55).

Finally, we prove (56). By Lemma A.5,

$$\begin{aligned} \big|\mathrm{Tr}\, J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta(t_2) - \mathrm{Tr}\, J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta(t_1)\big| &\leqslant 2 \sup_t \big|\langle J^{(k)} \mathcal{P}_\alpha \tilde{\psi}_{N,\omega}(t), \mathcal{P}_\beta \tilde{\psi}_{N,\omega}(t)\rangle\big| \\ &\lesssim \|J^{(k)}\|_{\mathrm{op}} \|\mathcal{P}_\alpha \tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})} \|\mathcal{P}_\beta \tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})}; \end{aligned}$$

that is,

$$\big|\mathrm{Tr}\, J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta(t_2) - \mathrm{Tr}\, J^{(k)} \mathcal{P}_\alpha \tilde{\gamma}^{(k)}_{N,\omega} \mathcal{P}_\beta(t_1)\big| \lesssim \omega^{-\frac{1}{2}|\alpha| - \frac{1}{2}|\beta|}$$

once we apply (31). □

With Theorem 4.1, we can start talking about the limit points of $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}^{(k)}_{N,\omega}\}_{k=1}^N\}$. With the proofs of [X. Chen and Holmer 2013, Theorem 5 and Corollary 2], we arrive at the following corollary and theorem.

**Corollary 4.2.** *Let $\Gamma(t) = \{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ be a limit point of $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}^{(k)}_{N,\omega}\}_{k=1}^N\}$, with respect to the product topology $\tau_{\mathrm{prod}}$. Then $\tilde{\gamma}^{(k)}$ satisfies the a priori bound*

$$\mathrm{Tr}\, L^{(k)} \tilde{\gamma}^{(k)} L^{(k)} \leqslant C^k \tag{62}$$

*and takes the structure*

$$\tilde{\gamma}^{(k)}\big(t, (\boldsymbol{x}_k, \boldsymbol{z}_k); (\boldsymbol{x}'_k, \boldsymbol{z}'_k)\big) = \bigg(\prod_{j=1}^{k} h_1(x_j) h_1(x'_j)\bigg) \tilde{\gamma}^{(k)}_z(t, \boldsymbol{z}_k; \boldsymbol{z}'_k), \tag{63}$$

*where $\tilde{\gamma}^{(k)}_z = \mathrm{Tr}_x\, \tilde{\gamma}^{(k)}$.*

**Theorem 4.3.** *Assume $C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_2(\beta)}$. Then the sequence*

$$\{\Gamma_{z,N,\omega}(t) = \{\tilde{\gamma}^{(k)}_{z,N,\omega} = \mathrm{Tr}_x\, \tilde{\gamma}^{(k)}_{N,\omega}\}_{k=1}^N\} \subset \bigoplus_{k \geqslant 1} C\big([0,T], \mathcal{L}^1_k(\mathbb{R}^k)\big)$$

*is compact with respect to the one-dimensional version of the product topology $\tau_{\mathrm{prod}}$ used in Theorem 4.1.*

## 5. Limit points satisfy GP hierarchy

**Theorem 5.1.** *Let $\Gamma(t) = \{\tilde{\gamma}^{(k)}\}_{k=1}^{\infty}$ be a $C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_2(\beta)}$ limit point of $\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^{N}\}$ with respect to the product topology $\tau_{\mathrm{prod}}$. Then $\{\tilde{\gamma}_z^{(k)} = \mathrm{Tr}_x\, \tilde{\gamma}^{(k)}\}_{k=1}^{\infty}$ is a solution to the coupled focusing Gross–Pitaevskii hierarchy (23) subject to initial data $\tilde{\gamma}_z^{(k)}(0) = |\phi_0\rangle\langle\phi_0|^{\otimes k}$ with coupling constant $b_0 = |\int V(r)\,dr|$, which, rewritten in integral form, is*

$$\tilde{\gamma}_z^{(k)} = U^{(k)}(t)\tilde{\gamma}_z^{(k)}(0) + i b_0 \sum_{j=1}^{k} \int_0^t U^{(k)}(t-s)\,\mathrm{Tr}_{z_{k+1}}\,\mathrm{Tr}_x\big[\delta(r_j - r_{k+1}),\, \tilde{\gamma}^{(k+1)}(s)\big]\,ds, \tag{64}$$

*where $U^{(k)}(t) = \prod_{j=1}^{k} e^{it\partial_{z_j}^2} e^{-it\partial_{z_j'}^2}$.*

**Remark.** The proof of Theorem 5.1 is a bit special for the focusing case and is dimension- and scaling-dependent. So it does not follow from the 3D to 2D defocusing case [X. Chen and Holmer 2013, Theorem 4].

*Proof.* Passing to subsequences if necessary, we have

$$\begin{aligned} &\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)}\leqslant\omega\leqslant C_2 N^{v_2(\beta)}}} \sup_t \mathrm{Tr}\, J^{(k)}\big(\tilde{\gamma}_{N,\omega}^{(k)}(t) - \tilde{\gamma}^{(k)}(t)\big) = 0 \quad \forall J^{(k)} \in \mathcal{K}(L^2(\mathbb{R}^{3k})), \\ &\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)}\leqslant\omega\leqslant C_2 N^{v_2(\beta)}}} \sup_t \mathrm{Tr}\, J_z^{(k)}\big(\tilde{\gamma}_{z,N,\omega}^{(k)}(t) - \tilde{\gamma}_z^{(k)}(t)\big) = 0 \quad \forall J_z^{(k)} \in \mathcal{K}(L^2(\mathbb{R}^{k})) \end{aligned} \tag{65}$$

via Theorems 4.1 and 4.3.

To establish (64), it suffices to test the limit point against the test functions $J_z^{(k)} \in \mathcal{K}(L^2(\mathbb{R}^k))$, as in the proof of Theorem 4.3. We will prove that the limit point satisfies

$$\mathrm{Tr}\, J_z^{(k)}\tilde{\gamma}_z^{(k)}(0) = \mathrm{Tr}\, J_z^{(k)}|\phi_0\rangle\langle\phi_0|^{\otimes k} \tag{66}$$

and

$$\mathrm{Tr}\, J_z^{(k)}\tilde{\gamma}_z^{(k)}(t) = \mathrm{Tr}\, J_z^{(k)}U^{(k)}(t)\tilde{\gamma}_z^{(k)}(0) + i b_0 \sum_{j=1}^{k}\int_0^t \mathrm{Tr}\, J_z^{(k)}U^{(k)}(t-s)\big[\delta(r_j - r_{k+1}), \tilde{\gamma}^{(k+1)}(s)\big]\,ds. \tag{67}$$

To this end, we use the coupled focusing BBGKY hierarchy satisfied by $\tilde{\gamma}_{z,N,\omega}^{(k)}$, which, written in the form needed here, is

$$\mathrm{Tr}\, J_z^{(k)}\tilde{\gamma}_{z,N,\omega}^{(k)}(t) = A + \frac{i}{N}\sum_{i<j} B + i\left(1 - \frac{k}{N}\right)\sum_{j=1}^{k} D,$$

where

$$\begin{aligned} A &= \mathrm{Tr}\, J_z^{(k)}U^{(k)}(t)\tilde{\gamma}_{z,N,\omega}^{(k)}(0), \\ B &= \int_0^t \mathrm{Tr}\, J_z^{(k)}U^{(k)}(t-s)\big[-V_{N,\omega}(r_i - r_j),\, \tilde{\gamma}_{N,\omega}^{(k)}(s)\big]\,ds, \\ D &= \int_0^t \mathrm{Tr}\, J_z^{(k)}U^{(k)}(t-s)\big[-V_{N,\omega}(r_j - r_{k+1}),\, \tilde{\gamma}_{N,\omega}^{(k+1)}(s)\big]\,ds. \end{aligned}$$

By (65), we know

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)}\leqslant\omega\leqslant C_2 N^{v_2(\beta)}}} \operatorname{Tr} J_z^{(k)} \tilde{\gamma}_{z,N,\omega}^{(k)}(t) = \operatorname{Tr} J_z^{(k)} \tilde{\gamma}_z^{(k)}(t),$$

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)}\leqslant\omega\leqslant C_2 N^{v_2(\beta)}}} \operatorname{Tr} J_z^{(k)} U^{(k)}(t) \tilde{\gamma}_{z,N,\omega}^{(k)}(0) = \operatorname{Tr} J_z^{(k)} U^{(k)}(t) \tilde{\gamma}_z^{(k)}(0).$$

With the argument in [Lieb et al. 2005, p. 64], we infer, from assumption (b) of Theorem 1.1,

$$\tilde{\gamma}_{N,\omega}^{(1)}(0) \to |h_1 \otimes \phi_0\rangle\langle h_1 \otimes \phi_0| \quad \text{strongly in trace norm;}$$

that is,

$$\tilde{\gamma}_{N,\omega}^{(k)}(0) \to |h_1 \otimes \phi_0\rangle\langle h_1 \otimes \phi_0|^{\otimes k} \quad \text{strongly in trace norm.}$$

Thus we have checked (66), the left-hand side of (67), and the first term on the right-hand side of (67) for the limit point. We are left to prove that

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)}\leqslant\omega\leqslant C_2 N^{v_2(\beta)}}} \frac{B}{N} = 0,$$

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)}\leqslant\omega\leqslant C_2 N^{v_2(\beta)}}} \left(1 - \frac{k}{N}\right) D = b_0 \int_0^t J_x^{(k)} U^{(k)}(t-s) \big[\delta(r_j - r_{k+1}),\, \tilde{\gamma}^{(k+1)}(s)\big]\, ds.$$

We first use an argument similar to the estimates of II and III in the proof of Theorem 4.3 to prove that $|B|$ and $|D|$ are bounded for every finite time $t$. In fact, since $U^{(k)}$ is a unitary operator which commutes with Fourier multipliers, we have

$$\begin{aligned}
|B| &\leqslant \int_0^t \big|\operatorname{Tr} J_z^{(k)} U^{(k)}(t-s) \big[V_{N,\omega}(r_i - r_j),\, \tilde{\gamma}_{N,\omega}^{(k)}(s)\big]\big|\, ds \\
&= \int_0^t ds \big| \operatorname{Tr} L_i^{-1} L_j^{-1} J_z^{(k)} L_i L_j U^{(k)}(t-s) W_{ij} L_i L_j \tilde{\gamma}_{N,\omega}^{(k)}(s) L_i L_j \\
&\qquad\qquad - \operatorname{Tr} L_i L_j J_z^{(k)} L_i^{-1} L_j^{-1} U^{(k)}(t-s) L_i L_j \tilde{\gamma}_{N,\omega}^{(k)}(s) L_i L_j W_{ij} \big| \\
&\leqslant \int_0^t ds \|L_i^{-1} L_j^{-1} J_z^{(k)} L_i L_j\|_{\mathrm{op}} \|U^{(k)}\|_{\mathrm{op}} \|W_{ij}\| \operatorname{Tr} L_i L_j \tilde{\gamma}_{N,\omega}^{(k)}(s) L_i L_j \\
&\qquad\qquad + \int_0^t ds \|L_i L_j J_z^{(k)} L_i^{-1} L_j^{-1}\|_{\mathrm{op}} \|U^{(k)}\|_{\mathrm{op}} \|W_{ij}\| \operatorname{Tr} L_i L_j \tilde{\gamma}_{N,\omega}^{(k)}(s) L_i L_j \\
&\leqslant C_J t.
\end{aligned}$$

That is,

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)}\leqslant\omega\leqslant C_2 N^{v_2(\beta)}}} \frac{B}{N} = \lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)}\leqslant\omega\leqslant C_2 N^{v_2(\beta)}}} \frac{kD}{N} = 0.$$

We now use Lemma A.2 (stated and proved in Appendix A), which compares the $\delta$-function and its approximation, to prove

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)}\leqslant\omega\leqslant C_2 N^{v_2(\beta)}}} D = b_0 \int_0^t \operatorname{Tr} J_z^{(k)} U^{(k)}(t-s)\big[\delta(r_j - r_{k+1}),\, \tilde{\gamma}^{(k+1)}(s)\big]\, ds. \tag{68}$$

Pick a probability measure $\rho \in L^1(\mathbb{R}^3)$ and define $\rho_\alpha(r) = \alpha^{-3}\rho(r/\alpha)$. Letting $M_{s-t}^{(k)} = J_z^{(k)} U^{(k)}(t-s)$, we have

$$\big|\operatorname{Tr} J_z^{(k)} U^{(k)}(t-s)\big(-V_{N,\omega}(r_j - r_{k+1})\tilde{\gamma}_{N,\omega}^{(k+1)}(s) - b_0\delta(r_j - r_{k+1})\tilde{\gamma}^{(k+1)}(s)\big)\big| = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

where

$$\begin{aligned}
\mathrm{I} &= \big|\operatorname{Tr} M_{s-t}^{(k)}\big(-V_{N,\omega}(r_j - r_{k+1}) - b_0\delta(r_j - r_{k+1})\big)\tilde{\gamma}_{N,\omega}^{(k+1)}(s)\big|,\\
\mathrm{II} &= b_0\big|\operatorname{Tr} M_{s-t}^{(k)}\big(\delta(r_j - r_{k+1}) - \rho_\alpha(r_j - r_{k+1})\big)\tilde{\gamma}_{N,\omega}^{(k+1)}(s)\big|,\\
\mathrm{III} &= b_0\big|\operatorname{Tr} M_{s-t}^{(k)}\rho_\alpha(r_j - r_{k+1})\big(\tilde{\gamma}_{N,\omega}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s)\big)\big|,\\
\mathrm{IV} &= b_0\big|\operatorname{Tr} M_{s-t}^{(k)}\big(\rho_\alpha(r_j - r_{k+1}) - \delta(r_j - r_{k+1})\big)\tilde{\gamma}^{(k+1)}(s)\big|.
\end{aligned}$$

Consider I. Writing $V_\omega(r) = (1/\omega)V(x/\sqrt{\omega}, z)$, we have $V_{N,\omega} = (N\omega)^{3\beta} V_\omega((N\omega)^\beta r)$. Lemma A.2 then yields

$$\begin{aligned}
\mathrm{I} &\leqslant \frac{Cb_0}{(N\omega)^{\beta\kappa}}\left(\int |V_\omega(r)||r|^\kappa\, dr\right)\big(\|L_j J_z^{(k)} L_j^{-1}\|_{\mathrm{op}} + \|L_j^{-1} J_z^{(k)} L_j\|_{\mathrm{op}}\big) L_j L_{k+1}\tilde{\gamma}_{N,\omega}^{(k+1)}(s) L_j L_{k+1}\\
&= C_J \frac{\left(\int |V_\omega(r)||r|^\kappa\, dr\right)}{(N\omega)^{\beta\kappa}}.
\end{aligned}$$

Notice that $\left(\int |V_\omega(r)||r|^\kappa\, dr\right)$ grows like $(\sqrt{\omega})^\kappa$, so

$$I \leqslant C_J\left(\frac{\sqrt{\omega}}{(N\omega)^\beta}\right)^\kappa,$$

which converges to zero as $N, \omega \to \infty$ in the way in which $N \geqslant \omega^{\frac{1}{2\beta}-1+}$. So we have proved

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)}\leqslant\omega\leqslant C_2 N^{v_2(\beta)}}} I = 0.$$

Similarly, for II and IV, via Lemma A.2, we have

$$\mathrm{II} \leqslant Cb_0\alpha^\kappa\big(\|L_j J_z^{(k)} L_j^{-1}\|_{\mathrm{op}} + \|L_j^{-1} J_z^{(k)} L_j\|_{\mathrm{op}}\big)\operatorname{Tr} L_j L_{k+1}\tilde{\gamma}_{N,\omega}^{(k+1)}(s) L_j L_{k+1} \leqslant Cb_0\alpha^\kappa C_{J_z^{(k)}} C^2,$$

where the second inequality follows from Corollary 3.2, and

$$\mathrm{IV} \leqslant Cb_0\alpha^\kappa\big(\|L_j J_z^{(k)} L_j^{-1}\|_{\mathrm{op}} + \|L_j^{-1} J_z^{(k)} L_j\|_{\mathrm{op}}\big)\operatorname{Tr} L_j L_{k+1}\tilde{\gamma}^{(k+1)}(s) L_j L_{k+1} \leqslant Cb_0\alpha^\kappa C_{J_z^{(k)}} C^2,$$

where the second inequality follows from Corollary 4.2; that is,

$$\mathrm{II} \leqslant C_J \alpha^\kappa \quad \text{and} \quad \mathrm{IV} \leqslant C_J \alpha^\kappa,$$

due to the energy estimate (Corollary 4.2). Hence II and IV converge to 0 as $\alpha \to 0$, uniformly in $N$, $\omega$.

For III,

$$\mathrm{III} \leqslant b_0 \left| \mathrm{Tr}\, J_{s-t}^{(k)} \rho_\alpha(r_j - r_{k+1}) \frac{1}{1+\varepsilon L_{k+1}} \big(\tilde{\gamma}_{N,\omega}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s)\big) \right| \\ + b_0 \left| \mathrm{Tr}\, J_{s-t}^{(k)} \rho_\alpha(r_j - r_{k+1}) \frac{\varepsilon L_{k+1}}{1+\varepsilon L_{k+1}} \big(\tilde{\gamma}_{N,\omega}^{(k+1)}(s) - \tilde{\gamma}^{(k+1)}(s)\big) \right|.$$

The first term in the above estimate goes to zero as $N, \omega \to \infty$ for every $\varepsilon > 0$, since we have assumed condition (65) and $J_{s-t}^{(k)} \rho_\alpha(r_j - r_{k+1})(1+\varepsilon L_{k+1})^{-1}$ is a compact operator. Due to the energy bounds on $\tilde{\gamma}_{N,\omega}^{(k+1)}$ and $\tilde{\gamma}^{(k+1)}$, the second term tends to zero as $\varepsilon \to 0$, uniformly in $N$ and $\omega$.

Putting together the estimates for I–IV, we have justified limit (68). Hence, we have obtained Theorem 5.1. □

Combining Corollary 4.2 and Theorem 5.1, we see that $\tilde{\gamma}_z^{(k)}$ in fact solves the 1D focusing Gross–Pitaevskii hierarchy with the desired coupling constant $b_0\big(\int |h_1(x)|^4\, dx\big)$.

**Corollary 5.2.** *Let $\Gamma(t) = \{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ be a $N \geqslant \omega^{v(\beta)+\varepsilon}$ limit point of $\big\{\Gamma_{N,\omega}(t) = \{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^N\big\}$ with respect to the product topology $\tau_{\mathrm{prod}}$. Then $\{\tilde{\gamma}_z^{(k)} = \mathrm{Tr}_x\, \tilde{\gamma}^{(k)}\}_{k=1}^\infty$ is a solution to the 1D Gross–Pitaevskii hierarchy (24) subject to initial data $\tilde{\gamma}_z^{(k)}(0) = |\phi_0\rangle\langle\phi_0|^{\otimes k}$ with coupling constant $b_0\big(\int |h_1(x)|^4\, dx\big)$, which, rewritten in integral form, is*

$$\tilde{\gamma}_z^{(k)} = U^{(k)}(t)\tilde{\gamma}_z^{(k)}(0) \\ + i b_0 \bigg( \int |h_1(x)|^4\, dx \bigg) \sum_{j=1}^k \int_0^t U^{(k)}(t-s)\, \mathrm{Tr}_{z_{k+1}} \big[\delta(z_j - z_{k+1}),\, \tilde{\gamma}_z^{(k+1)}(s)\big]\, ds. \quad (69)$$

*Proof.* This is a direct computation by plugging (63) into (64). □

## Appendix A: Basic operator facts and Sobolev-type lemmas

**Lemma A.1** [Erdős et al. 2007, Lemma A.3]. *Let $L_j = (1 - \triangle_{r_j})^{\frac{1}{2}}$. Then we have*

$$\big\| L_i^{-1} L_j^{-1} V(r_i - r_j) L_i^{-1} L_j^{-1} \big\|_{\mathrm{op}} \leqslant C \|V\|_{L^1}.$$

**Lemma A.2.** *Let $f \in L^1(\mathbb{R}^3)$ be such that $\int_{\mathbb{R}^3} \langle r\rangle^{\frac{1}{2}} |f(r)|\, dr < \infty$ and $\int_{\mathbb{R}^3} f(r)\, dr = 1$ but we allow that $f$ not be nonnegative everywhere. Define $f_\alpha(r) = \alpha^{-3} f(r/\alpha)$. Then, for every $\kappa \in \big(0, \frac{1}{2}\big)$, there exists $C_\kappa > 0$ such that*

$$\big|\mathrm{Tr}\, J^{(k)} \big(f_\alpha(r_j - r_{k+1}) - \delta(r_j - r_{k+1})\big) \gamma^{(k+1)}\big| \\ \leqslant C_\kappa \bigg( \int |f(r)|\, |r|^\kappa\, dr \bigg) \alpha^\kappa \big( \|L_j J^{(k)} L_j^{-1}\|_{\mathrm{op}} + \|L_j^{-1} J^{(k)} L_j\|_{\mathrm{op}} \big) \mathrm{Tr}\, L_j L_{k+1} \gamma^{(k+1)} L_j L_{k+1}$$

*for all nonnegative $\gamma^{(k+1)} \in \mathcal{L}^1(L^2(\mathbb{R}^{3k+3}))$.*

*Proof.* This is the same as [X. Chen and Holmer 2016b, Lemma A.3; 2013, Lemma 2]. See [Kirkpatrick et al. 2011; T. Chen and Pavlović 2011; Erdős et al. 2007] for similar lemmas. ◻

**Lemma A.3** (some standard operator inequalities).

(1) *Suppose that* $A \geqslant 0$, $P_j = P_j^*$, *and* $I = P_0 + P_1$. *Then* $A \leqslant 2P_0AP_0 + 2P_1AP_1$.

(2) *If* $A \geqslant B \geqslant 0$, *and* $AB = BA$, *then* $A^\alpha \geqslant B^\alpha$ *for any* $\alpha \geqslant 0$.

(3) *If* $A_1 \geqslant A_2 \geqslant 0$, $B_1 \geqslant B_2 \geqslant 0$ *and* $A_iB_j = B_jA_i$ *for all* $1 \leqslant i, j \leqslant 2$, *then* $A_1B_1 \geqslant A_2B_2$.

(4) *If* $A \geqslant 0$ *and* $AB = BA$, *then* $A^{\frac{1}{2}}B = BA^{\frac{1}{2}}$.

*Proof.* For (1), $\|A^{\frac{1}{2}}f\|^2 = \|A^{\frac{1}{2}}(P_0+P_1)f\|^2 \leqslant 2\|A^{\frac{1}{2}}P_0f\|^2 + 2\|A^{\frac{1}{2}}P_1f\|^2$. For (3), $A_1B_1 \geqslant A_2B_1 = B_1A_2 \geqslant B_2A_2 = A_2B_2$. The rest, (2) and (4), are standard facts in operator theory. See, for example, [Reed and Simon 1978; Stein and Shakarchi 2005, Proposition 6.3]. ◻

**Lemma A.4.** *Recall*

$$\widetilde{S} = (1 - \partial_z^2 + \omega(-2 - \triangle_x + |x|^2))^{\frac{1}{2}}.$$

*We have*

$$\widetilde{S}^2 \gtrsim 1 - \Delta_r, \tag{70}$$

$$\widetilde{S}^2 P_{\geqslant 1} \gtrsim P_{\geqslant 1}(1 - \partial_z^2 - \omega\triangle_x + \omega|x|^2)P_{\geqslant 1}, \tag{71}$$

$$\widetilde{S}^2 P_{\geqslant 1} \gtrsim \omega P_{\geqslant 1}. \tag{72}$$

*Proof.* Directly from the definition of $\widetilde{S}$, we have

$$\underbrace{P_{\geqslant 1}(1 - \partial_z^2 - \omega\triangle_x + \omega|x|^2)P_{\geqslant 1}}_{\text{all terms positive}} = 2\omega P_{\geqslant 1} + \widetilde{S}^2 P_{\geqslant 1}. \tag{73}$$

The eigenvalues of the 2D Hermite operator $-\Delta_x + |x|^2$ are $\{2k+2\}_{k=0}^{\infty}$. So

$$2\omega P_{\geqslant 1} \leqslant \omega(-2 - \triangle_x + |x|^2)P_{\geqslant 1} \leqslant \widetilde{S}^2 P_{\geqslant 1}. \tag{74}$$

Inequalities (71) and (72) immediately follow from (73) and (74).

We now establish (70) using (71). On the one hand, we have

$$\widetilde{S}^2 \geqslant (1 - \partial_z^2). \tag{75}$$

On the other hand,

$$P_0(-\triangle_x)P_0 \lesssim 1 \leqslant \widetilde{S}^2 \tag{76}$$

since $P_0$ is merely the projection onto the smooth function $Ce^{-\frac{1}{2}|x|^2}$. Moreover, by (71),

$$P_{\geqslant 1}(-\triangle_x)P_{\geqslant 1} \leqslant \widetilde{S}^2 P_{\geqslant 1} \leqslant \widetilde{S}^2. \tag{77}$$

Thus Lemma A.3(1), (76) and (77) together imply,

$$-\triangle_x \lesssim \widetilde{S}^2. \tag{78}$$

The claimed inequality (70) then follows from (75) and (78). ◻

**Lemma A.5.** *Suppose $\sigma : L^2(\mathbb{R}^{3k}) \to L^2(\mathbb{R}^{3k})$ has kernel*

$$\sigma(\boldsymbol{r}_k, \boldsymbol{r}_k') = \int \psi(\boldsymbol{r}_k, \boldsymbol{r}_{N-k})\overline{\psi}(\boldsymbol{r}_k', \boldsymbol{r}_{N-k})\, d\boldsymbol{r}_{N-k}$$

*for some $\psi \in L^2(\mathbb{R}^{3N})$, and let $A, B : L^2(\mathbb{R}^{3k}) \to L^2(\mathbb{R}^{3k})$. Then the composition $A\sigma B$ has kernel*

$$(A\sigma B)(\boldsymbol{r}_k, \boldsymbol{r}_k') = \int (A\psi)(\boldsymbol{r}_k, \boldsymbol{r}_{N-k})(\overline{B^*\psi})(\boldsymbol{r}_k', \boldsymbol{r}_{N-k})\, d\boldsymbol{r}_{N-k}.$$

*It follows that*

$$\operatorname{Tr} A\sigma B = \langle A\psi, B^*\psi\rangle.$$

Let $\mathcal{K}_k$ denote the class of compact operators on $L^2(\mathbb{R}^{3k})$, let $\mathcal{L}_k^1$ denote the trace class operators on $L^2(\mathbb{R}^{3k})$, and let $\mathcal{L}_k^2$ denote the Hilbert–Schmidt operators on $L^2(\mathbb{R}^{3k})$. We have

$$\mathcal{L}_k^1 \subset \mathcal{L}_k^2 \subset \mathcal{K}_k.$$

For an operator $J$ on $L^2(\mathbb{R}^{3k})$, let $|J| = (J^*J)^{\frac{1}{2}}$ and denote by $J(\boldsymbol{r}_k, \boldsymbol{r}_k')$ the kernel of $J$ and by $|J|(\boldsymbol{r}_k, \boldsymbol{r}_k')$ the kernel of $|J|$, which satisfies $|J|(\boldsymbol{r}_k, \boldsymbol{r}_k') \geqslant 0$. Let

$$\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant 0$$

be the eigenvalues of $|J|$ repeated according to multiplicity (the *singular values* of $J$). Then

$$\begin{aligned}
\|J\|_{\mathcal{K}_k} &= \|\mu_n\|_{\ell_n^\infty} = \mu_1 = \|\,|J|\,\|_{\text{op}} = \|J\|_{\text{op}},\\
\|J\|_{\mathcal{L}_k^2} &= \|\mu_n\|_{\ell_n^2} = \|J(\boldsymbol{r}_k, \boldsymbol{r}_k')\|_{L^2(\boldsymbol{r}_k, \boldsymbol{r}_k')} = (\operatorname{Tr} J^*J)^{\frac{1}{2}},\\
\|J\|_{\mathcal{L}_k^1} &= \|\mu_n\|_{\ell_n^1} = \||J|(\boldsymbol{r}_k, \boldsymbol{r}_k)\|_{L^1(\boldsymbol{r}_k)} = \operatorname{Tr}|J|.
\end{aligned}$$

The topology on $\mathcal{K}_k$ coincides with the operator topology, and $\mathcal{K}_k$ is a closed subspace of the space of bounded operators on $L^2(\mathbb{R}^{3k})$.

**Lemma A.6.** *On the one hand, let $\chi$ be a smooth function on $\mathbb{R}^3$ such that $\chi(\xi) = 1$ for $|\xi| \leqslant 1$ and $\chi(\xi) = 0$ for $|\xi| \geqslant 2$. Let*

$$(Q_M f)(\boldsymbol{r}_k) = \int e^{i\boldsymbol{r}_k\cdot\boldsymbol{\xi}_k} \prod_{j=1}^k \chi(M^{-1}\xi_j)\hat{f}(\boldsymbol{\xi}_k)\, d\boldsymbol{\xi}_k.$$

*On the other hand, with respect to the spectral decomposition of $L^2(\mathbb{R}^2)$ corresponding to the operator $H_j = -\triangle_{x_j}^2 + |x_j|^2$, let $X_M^j$ be the orthogonal projection onto the sum of the first $M$ eigenspaces (in the $x_j$-variable only) and let*

$$R_M = \prod_{j=1}^k X_M^j.$$

*We then have the following:*

(1) *Suppose that $J$ is a compact operator. Then $J_M := R_M Q_M J Q_M R_M \to J$ in the operator norm.*

(2) *$H_j J_M$, $J_M H_j$, $\Delta_{r_j} J_M$ and $J_M \Delta_{r_j}$ are all bounded.*

(3) *There exists a countable dense subset $\{T_i\}$ of the closed unit ball in the space of bounded operators on $L^2(\mathbb{R}^{3k})$ such that each $T_i$ is compact and in fact for each $i$ there exists $M$ (depending on $i$) and $Y_i \in \mathcal{K}_k$ with $\|Y_i\|_{\text{op}} \leqslant 1$ such that $T_i = R_M Q_M Y_i Q_M R_M$.*

*Proof.* (1) If $S_n \to S$ strongly and $J \in \mathcal{K}_k$, then $S_n J \to SJ$ in the operator norm and $JS_n \to JS$ in the operator norm.

(2) This is straightforward.

(3) Start with a subset $\{Y_n\}$ of the closed unit ball in the space of bounded operators on $L^2(\mathbb{R}^{3k})$ such that each $Y_n$ is compact. Then let $\{T_i\}$ be an enumeration of the set $R_M Q_M Y_n Q_M R_M$, where $M$ ranges over the dyadic integers. By (1) this collection will still be dense. The $\{Y_i\}$ in the statement of (3) is just a reindexing of $\{Y_n\}$. □

## Appendix B: Deducing Theorem 1.1 from Theorem 1.2

We first give the following lemma.

**Lemma B.1.** *Assume $\tilde{\psi}_{N,\omega}(0)$ satisfies* (a), (b) *and* (c) *in Theorem 1.1. Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off such that $0 \leqslant \chi \leqslant 1$, $\chi(s) = 1$ for $0 \leqslant s \leqslant 1$ and $\chi(s) = 0$ for $s \geqslant 2$. For $\kappa > 0$, we define an approximation of $\tilde{\psi}_{N,\omega}(0)$ by*

$$\tilde{\psi}^{\kappa}_{N,\omega}(0) = \frac{\chi\big(\kappa(\widetilde{H}_{N,\omega} - 2N\omega)/N\big)\tilde{\psi}_{N,\omega}(0)}{\big\|\chi\big(\kappa(\widetilde{H}_{N,\omega} - 2N\omega)/N\big)\tilde{\psi}_{N,\omega}(0)\big\|}.$$

*This approximation has the following properties:*

(i) *$\tilde{\psi}^{\kappa}_{N,\omega}(0)$ verifies the energy condition*

$$\big\langle \tilde{\psi}^{\kappa}_{N,\omega}(0),\ (\widetilde{H}_{N,\omega} - 2N\omega)^k \tilde{\psi}^{\kappa}_{N,\omega}(0)\big\rangle \leqslant \frac{2^k N^k}{\kappa^k}.$$

(ii) $\sup_{N,\omega} \big\| \tilde{\psi}_{N,\omega}(0) - \tilde{\psi}^{\kappa}_{N,\omega}(0)\big\|_{L^2} \leqslant C\kappa^{\frac{1}{2}}$.

(iii) *For small enough $\kappa > 0$, we have $\tilde{\psi}^{\kappa}_{N,\omega}(0)$ is asymptotically factorized as well:*

$$\lim_{N,\omega\to\infty} \operatorname{Tr}\big|\tilde{\gamma}^{\kappa,(1)}_{N,\omega}(0, x_1, z_1; x_1', z_1') - h(x_1)h(x_1')\phi_0(z_1)\bar{\phi}_0(z_1')\big| = 0,$$

*where $\tilde{\gamma}^{\kappa,(1)}_{N,\omega}(0)$ is the one-particle marginal density associated with $\tilde{\psi}^{\kappa}_{N,\omega}(0)$, and $\phi_0$ is the same as in assumption* (b) *in Theorem 1.1.*

*Proof.* Let us write $\chi(\kappa(\widetilde{H}_{N,\omega} - 2N\omega))$ as $\chi$ and $\tilde{\psi}_{N,\omega}(0)$ as $\tilde{\psi}_{N,\omega}$. This proof closely follows [Erdős et al. 2010, Proposition 8.1(i)–(ii); 2007, Proposition 5.1(iii)].

Property (i) follows by definition. In fact, denote the characteristic function of $[0, \lambda]$ by $\mathbf{1}(s \leqslant \lambda)$. We see that

$$\chi\big(\kappa(\widetilde{H}_{N,\omega} - 2N\omega)/N\big) = \mathbf{1}\big(\widetilde{H}_{N,\omega} - 2N\omega \leqslant 2N/\kappa\big)\chi\big(\kappa(\widetilde{H}_{N,\omega} - 2N\omega)/N\big).$$

Thus

$$\langle\tilde{\psi}^{\kappa}_{N,\omega}(0),\,(\widetilde{H}_{N,\omega}-2N\omega)^k\tilde{\psi}^{\kappa}_{N,\omega}(0)\rangle=\left\langle\frac{\chi\tilde{\psi}_{N,\omega}}{\|\chi\tilde{\psi}_{N,\omega}\|},\,\mathbf{1}\big(\widetilde{H}_{N,\omega}-2N\omega\leqslant 2N/\kappa\big)(\widetilde{H}_{N,\omega}-2N\omega)^k\frac{\chi\tilde{\psi}_{N,\omega}}{\|\chi\tilde{\psi}_{N,\omega}\|}\right\rangle$$
$$\leqslant\big\|\mathbf{1}\big(\widetilde{H}_{N,\omega}-2N\omega\leqslant 2N/\kappa\big)(\widetilde{H}_{N,\omega}-2N\omega)^k\big\|_{\mathrm{op}}$$
$$\leqslant\frac{2^kN^k}{\kappa^k}.$$

We prove (ii) with a slightly modified proof of [Erdős et al. 2010, Proposition 8.1(ii)]. We still have

$$\|\tilde{\psi}^{\kappa}_{N,\omega}-\tilde{\psi}_{N,\omega}\|_{L^2}\leqslant\|\chi\tilde{\psi}_{N,\omega}-\tilde{\psi}_{N,\omega}\|_{L^2}+\left\|\frac{\chi\tilde{\psi}_{N,\omega}}{\|\chi\tilde{\psi}_{N,\omega}\|}-\chi\tilde{\psi}_{N,\omega}\right\|_{L^2}$$
$$\leqslant\|\chi\tilde{\psi}_{N,\omega}-\tilde{\psi}_{N,\omega}\|_{L^2}+\big|1-\|\chi\tilde{\psi}_{N,\omega}\|\big|$$
$$\leqslant 2\|\chi\tilde{\psi}_{N,\omega}-\tilde{\psi}_{N,\omega}\|_{L^2},$$

where

$$\|\chi\tilde{\psi}_{N,\omega}-\tilde{\psi}_{N,\omega}\|^2_{L^2}=\left\langle\psi_N,\,\left(1-\chi\left(\frac{\kappa(\widetilde{H}_{N,\omega}-2N\omega)}{N}\right)\right)^2\psi_N\right\rangle$$
$$\leqslant\left\langle\psi_N,\,\mathbf{1}\big(\frac{\kappa(\widetilde{H}_{N,\omega}-2N\omega)}{N}\geqslant 1\big)\psi_N\right\rangle.$$

To continue estimating, we notice that if $C\geqslant 0$, then $\mathbf{1}(s\geqslant 1)\leqslant\mathbf{1}(s+C\geqslant 1)$ for all $s$. So

$$\|\chi\tilde{\psi}_{N,\omega}-\tilde{\psi}_{N,\omega}\|^2_{L^2}\leqslant\left\langle\tilde{\psi}_{N,\omega},\,\mathbf{1}\left(\frac{\kappa(\widetilde{H}_{N,\omega}-2N\omega)}{N}\geqslant 1\right)\tilde{\psi}_{N,\omega}\right\rangle$$
$$\leqslant\left\langle\tilde{\psi}_{N,\omega},\,\mathbf{1}\left(\frac{\kappa(\widetilde{H}_{N,\omega}-2N\omega+N\alpha)}{N}\geqslant 1\right)\tilde{\psi}_{N,\omega}\right\rangle.$$

With the inequality $\mathbf{1}(s\geqslant 1)\leqslant s$ for all $s\geqslant 0$ and the fact that

$$\widetilde{H}_{N,\omega}-2N\omega+N\alpha\geqslant 0,$$

proved in Theorem 3.1, we arrive at

$$\|\chi\tilde{\psi}_{N,\omega}-\tilde{\psi}_{N,\omega}\|^2_{L^2}\leqslant\frac{\kappa}{N}\langle\tilde{\psi}_{N,\omega},\,(\widetilde{H}_{N,\omega}-2N\omega+N\alpha)\tilde{\psi}_{N,\omega}\rangle$$
$$\leqslant\frac{\kappa}{N}\langle\tilde{\psi}_{N,\omega},\,(\widetilde{H}_{N,\omega}-2N\omega)\tilde{\psi}_{N,\omega}\rangle+\alpha\kappa\langle\tilde{\psi}_{N,\omega},\,\tilde{\psi}_{N,\omega}\rangle.$$

Using (a) and (c) in the assumptions of Theorem 1.1, we deduce that

$$\|\chi\tilde{\psi}_{N,\omega}-\tilde{\psi}_{N,\omega}\|^2_{L^2}\leqslant C\kappa,$$

which implies

$$\|\tilde{\psi}^{\kappa}_{N,\omega}-\tilde{\psi}_{N,\omega}\|_{L^2}\leqslant C\kappa^{\frac{1}{2}}.$$

Property (iii) does not follow from the proof of [Erdős et al. 2010, Proposition 8.1(iii)] in which the positivity of $V$ is used. Instead (iii) follows from the proof of [Erdős et al. 2007, Proposition 5.1(iii)], which does not require $V$ to hold a definite sign. Lemma B.1 follows the same proof as [Erdős et al. 2007, Proposition 5.1(iii)] if one replaces $H_N$ by $(\widetilde{H}_{N,\omega} - 2N\omega)$ and $\widehat{H}_N$ by

$$\sum_{j\geqslant k+1}^{N} \left(-\partial_{z_j} + \omega(-2 - \Delta_{x_j} + |x_j|^2)\right) + \frac{1}{N} \sum_{k+1<i<j\leqslant N} V_{N,\omega}(r_i - r_j).$$

Notice that we are working with $V_{N,\omega} = (N\omega)^{3\beta} V_\omega((N\omega)^\beta r)$, where $V_\omega(r) = (1/\omega)V(x/\sqrt{\omega}, z)$; thus we get

$$(N\omega)^{\frac{3}{2}\beta} \|V_\omega\|_{L^2}^2 \sim \frac{(N\omega)^{\frac{3}{2}\beta}}{\omega}$$

instead of $N^{\frac{3}{2}\beta}$ in [Erdős et al. 2007, (5.20)] and hence we get $(N\omega)^{\frac{3}{2}\beta-1}$ in the estimate (5.18) of the same work, which tends to zero as $N, \omega \to \infty$ for $\beta \in \left(0, \frac{2}{3}\right)$. □

Via (i) and (iii) of Lemma B.1, $\tilde{\psi}^\kappa_{N,\omega}(0)$ verifies the hypothesis of Theorem 1.2 for small enough $\kappa > 0$. Therefore, for $\tilde{\gamma}^{\kappa,(1)}_{N,\omega}(t)$, the marginal density associated with $e^{it\widetilde{H}_{N,\omega}} \tilde{\psi}^\kappa_{N,\omega}(0)$, Theorem 1.2 gives the convergence

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_2(\beta)}}} \operatorname{Tr}\left|\tilde{\gamma}^{\kappa,(k)}_{N,\omega}(t, \boldsymbol{x}_k, \boldsymbol{z}_k; \boldsymbol{x}'_k, \boldsymbol{z}'_k) - \prod_{j=1}^{k} h_1(x_j)h_1(x'_j)\phi(t, z_j)\bar{\phi}(t, z'_j)\right| = 0 \tag{79}$$

for all small enough $\kappa > 0$, all $k \geqslant 1$, and all $t \in \mathbb{R}$.

For $\tilde{\gamma}^{(k)}_{N,\omega}(t)$ in Theorem 1.1, we notice that, $\forall J^{(k)} \in \mathcal{K}_k$, $\forall t \in \mathbb{R}$, we have

$$\begin{aligned}
&\left|\operatorname{Tr} J^{(k)}\left(\tilde{\gamma}^{(k)}_{N,\omega}(t) - \left|h_1 \otimes \phi(t)\right\rangle\left\langle h_1 \otimes \phi(t)\right|^{\otimes k}\right)\right| \\
&\qquad \leqslant \left|\operatorname{Tr} J^{(k)}\left(\tilde{\gamma}^{(k)}_{N,\omega}(t) - \tilde{\gamma}^{\kappa,(k)}_{N,\omega}(t)\right)\right| + \left|\operatorname{Tr} J^{(k)}\left(\tilde{\gamma}^{\kappa,(k)}_{N,\omega}(t) - |h_1 \otimes \phi(t)\rangle\langle h_1 \otimes \phi(t)|^{\otimes k}\right)\right| \\
&\qquad = \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Convergence (79) then takes care of II. To handle I, part (ii) of Lemma B.1 yields

$$\left\|e^{it\widetilde{H}_{N,\omega}} \tilde{\psi}_{N,\omega}(0) - e^{it\widetilde{H}_{N,\omega}} \tilde{\psi}^\kappa_{N,\omega}(0)\right\|_{L^2} = \left\|\tilde{\psi}_{N,\omega}(0) - \tilde{\psi}^\kappa_{N,\omega}(0)\right\|_{L^2} \leqslant C\kappa^{\frac{1}{2}},$$

which implies

$$I = \left|\operatorname{Tr} J^{(k)}\left(\tilde{\gamma}^{(k)}_{N,\omega}(t) - \tilde{\gamma}^{\kappa,(k)}_{N,\omega}(t)\right)\right| \leqslant C\|J^{(k)}\|_{\mathrm{op}}\kappa^{\frac{1}{2}}.$$

Since $\kappa > 0$ is arbitrary, we deduce that

$$\lim_{\substack{N,\omega\to\infty \\ C_1 N^{v_1(\beta)} \leqslant \omega \leqslant C_2 N^{v_2(\beta)}}} \left|\operatorname{Tr} J^{(k)}\left(\tilde{\gamma}^{(k)}_{N,\omega}(t) - |h_1 \otimes \phi(t)\rangle\langle h_1 \otimes \phi(t)|^{\otimes k}\right)\right| = 0;$$

i.e., as trace class operators

$$\tilde{\gamma}^{(k)}_{N,\omega}(t) \to |h_1 \otimes \phi(t)\rangle\langle h_1 \otimes \phi(t)|^{\otimes k} \quad \text{weak*}.$$

Then again, Grümm's convergence theorem upgrades the above weak* convergence to strong. Hence, we have concluded Theorem 1.1 via Theorem 1.2 and Lemma B.1.

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# CONFORMALLY EUCLIDEAN METRICS ON $\mathbb{R}^n$ WITH ARBITRARY TOTAL $Q$-CURVATURE

ALI HYDER

We study the existence of solution to the problem

$$(-\Delta)^{n/2}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \qquad \kappa := \int_{\mathbb{R}^n} Qe^{nu}\,dx < \infty,$$

where $Q \geq 0$, $\kappa \in (0, \infty)$ and $n \geq 3$. Using ODE techniques, Martinazzi (for $n = 6$) and Huang and Ye (for $n = 4m + 2$) proved the existence of a solution to the above problem with $Q \equiv \text{constant} > 0$ and for every $\kappa \in (0, \infty)$. We extend these results in every dimension $n \geq 5$, thus completely answering the problem opened by Martinazzi. Our approach also extends to the case in which $Q$ is nonconstant, and under some decay assumptions on $Q$ we can also treat the cases $n = 3$ and $n = 4$.

## 1. Introduction

For a function $Q \in C^0(\mathbb{R}^n)$ we consider the problem

$$(-\Delta)^{n/2}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \qquad \kappa := \int_{\mathbb{R}^n} Qe^{nu}\,dx < \infty, \tag{1}$$

where for $n$ odd the nonlocal operator $(-\Delta)^{n/2}$ is defined on page 639.

Geometrically if $u$ is a smooth solution of (1) then the conformal metric $g_u := e^{2u}|dx|^2$ (here $|dx|^2$ is the Euclidean metric on $\mathbb{R}^n$) has the $Q$-curvature $Q$, at least when $n \geq 2$. Moreover, the total $Q$-curvature of the metric $g_u$ is $\kappa$.

Solutions to (1) have been classified in terms of their asymptotic behavior at infinity. More precisely we have the following:

**Theorem A** [Chen and Li 1991; Da Lio et al. 2015; Lin 1998; Martinazzi 2009a; Jin et al. 2015; Hyder 2015; Xu 2005]. *Let $n \geq 1$. Let $u$ be a solution of*

$$(-\Delta)^{n/2}u = (n-1)!\,e^{nu} \quad \textit{in } \mathbb{R}^n, \qquad \kappa := (n-1)!\int_{\mathbb{R}^n} e^{nu}\,dx < \infty. \tag{2}$$

*Then*

$$u(x) = \frac{(n-1)!}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{|y|}{|x-y|}\right)e^{nu(y)}\,dy + P(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + P(x) + o(\log|x|) \quad \textit{as } |x| \to \infty, \tag{3}$$

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*where* $\gamma_n := \frac{1}{2}(n-1)!|S^n|$, $\Lambda_1 := 2\gamma_n$, $o(\log|x|)/\log|x| \to 0$ *as* $|x| \to \infty$, $P$ *is a polynomial of degree at most* $n-1$ *and* $P$ *is bounded from above. If* $n \in \{3, 4\}$ *then* $\kappa \in (0, \Lambda_1]$ *and* $\kappa = \Lambda_1$ *if and only if* $u$ *is a spherical solution, that is,*

$$u(x) = u_{\lambda,x_0}(x) := \log \frac{2\lambda}{1+\lambda^2|x-x_0|^2} \tag{4}$$

*for some* $x_0 \in \mathbb{R}^n$ *and* $\lambda > 0$. *Moreover* $u$ *is spherical if and only if* $P$ *is constant* (*which is always the case when* $n \in \{1, 2\}$).

Chang and Chen [2001] showed the existence of nonspherical solutions to (2) in even dimension $n \geq 4$ for every $\kappa \in (0, \Lambda_1)$.

A partial converse to Theorem A has been proven in dimension 4 by Wei and Ye [2008] and extended by Hyder and Martinazzi [2015] for $n \geq 4$ even and Hyder [2016] for $n \geq 3$.

**Theorem B** [Wei and Ye 2008; Hyder and Martinazzi 2015; Hyder 2016]. *Let* $n \geq 3$. *Then for every* $\kappa \in (0, \Lambda_1)$ *and for every polynomial* $P$ *with*

$$\deg(P) \leq n-1 \quad \textit{and} \quad P(x) \xrightarrow{|x|\to\infty} -\infty,$$

*there exists a solution* $u$ *to* (2) *having the asymptotic behavior given by* (3).

Although the assumption $\kappa \in (0, \Lambda_1]$ is a necessary condition for the existence of a solution to (2) for $n = 3, 4$, it is possible to have a solution for $\kappa > \Lambda_1$ arbitrarily large in higher dimension, as shown by Martinazzi [2013] for $n = 6$. Huang and Ye [2015] extended Martinazzi's result in arbitrary even dimension $n$ of the form $n = 4m + 2$ for some $m \geq 1$, proving that for every $\kappa \in (0, \infty)$ there exists a solution to (2). The case $n = 4m$ remained open.

The ideas in [Martinazzi 2013; Huang and Ye 2015] are based upon ODE theory. One considers only radial solutions so that the equation in (2) becomes an ODE, and the result is obtained by choosing suitable initial conditions and letting one of the parameters go to $+\infty$ (or $-\infty$). However, this technique does not work if the dimension $n$ is a multiple of 4, and things get even worse in odd dimension since $(-\Delta)^{n/2}$ is nonlocal and ODE techniques cannot be used.

In this paper we extend the works of [Martinazzi 2013; Huang and Ye 2015] and completely solve the cases left open; namely we prove that when $n \geq 5$, problem (2) has a solution for every $\kappa \in (0, \infty)$. In fact we do not need to assume that $Q$ is constant, but only that it is radially symmetric with growth at infinity suitably controlled, or not even radially symmetric. Moreover, we are able to prescribe the asymptotic behavior of the solution $u$, as in (3), up to a polynomial of degree 4 which cannot be prescribed and in particular cannot be required to vanish when $\kappa \geq \Lambda_1$. This in turn, together with Theorem A, is consistent with the requirement $n \geq 5$, because only when $n \geq 5$ does the asymptotic expansion of $u$ at infinity admit polynomials of degree 4.

We prove the following two theorems.

**Theorem 1.1.** *Let* $n \geq 5$ *be an integer. Let* $P$ *be a polynomial on* $\mathbb{R}^n$ *with degree at most* $n-1$. *Let* $Q \in C^0(\mathbb{R}^n)$ *be such that* $Q(0) > 0$, $Q \geq 0$, $Qe^{nP}$ *is radially symmetric and*

$$\sup_{x\in\mathbb{R}^n} Q(x)e^{nP(x)} < \infty.$$

*Then for every $\kappa > 0$ there exists a solution $u$ to (1) such that*

$$u(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + P(x) + c_1|x|^2 - c_2|x|^4 + C + o(1) \quad \text{as } |x| \to \infty$$

*for some $c_1, c_2 > 0$ and $C \in \mathbb{R}$. In fact, there exists a radially symmetric function $v$ on $\mathbb{R}^n$ and a constant $c_v$ such that*

$$v(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + \frac{1}{2n}\Delta v(0)(|x|^4 - |x|^2) + o(1) \quad \text{as } |x| \to \infty,$$

*and*

$$u = P + v + c_v - |x|^4, \qquad x \in \mathbb{R}^n.$$

Taking $Q = (n-1)!$ and $P = 0$ in Theorem 1.1 one has the following corollary.

**Corollary 1.2.** *Let $n \geq 5$ and $\kappa \in (0, \infty)$. Then there exists a radially symmetric solution $u$ to (2) such that*

$$u(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + c_1|x|^2 - c_2|x|^4 + C + o(1) \quad \text{as } |x| \to \infty$$

*for some $c_1, c_2 > 0$ and $C \in \mathbb{R}$.*

Notice the polynomial part of the solution $u$ in Theorem 1.1 is not exactly the prescribed polynomial $P$ (compare to [Wei and Ye 2008; Hyder and Martinazzi 2015; Hyder 2016]). In general, without perturbing the polynomial part, it is not possible to find a solution for $\kappa \geq \Lambda_1$. For example, if $P$ is nonincreasing and nonconstant then there is no solution $u$ to (2) with $\kappa \geq \Lambda_1$ such that $u$ has the asymptotic behavior (3) (see Lemma 3.6 below). This justifies the term $c_1|x|^2$ in Theorem 1.1. Then the additional term $-c_2|x|^4$ is also necessary to avoid that $u(x) \geq \frac{1}{2}c_1|x|^2$ for $x$ large, which would contrast with the condition $\kappa < \infty$, at least if $Q$ does not decay fast enough at infinity. In the latter case, the term $-c_2|x|^4$ can be avoided, and one obtains an existence result also in dimensions 3 and 4.

**Theorem 1.3.** *Let $n \geq 3$. Let $Q \in C^0_{\mathrm{rad}}(\mathbb{R}^n)$ be such that $Q \geq 0$, $Q(0) > 0$ and*

$$\int_{\mathbb{R}^n} Q(x)e^{\lambda|x|^2}\,dx < \infty \quad \text{for every } \lambda > 0, \qquad \int_{B_1(x)} \frac{Q(y)}{|x-y|^{n-1}}\,dy \xrightarrow{|x|\to\infty} 0.$$

*Then for every $\kappa > 0$ there exists a radially symmetric solution $u$ to (1).*

The decay assumption on $Q$ in Theorem 1.3 is sharp in the sense that if $Qe^{\lambda|x|^2} \notin L^1(\mathbb{R}^n)$ for some $\lambda > 0$, then problem (1) might not have a solution for every $\kappa > 0$. For instance, if $Q = e^{-\lambda|x|^2}$ for some $\lambda > 0$, then (1) with $n = 3, 4$ and $\kappa > \Lambda_1$ has no solution (see Lemma 3.5 below).

The proof of Theorem 1.1 is based on the Schauder fixed point theorem, and the main difficulty is to show that the "approximate solutions" are precompact (see in particular Lemma 2.2). We will do that using blow up analysis (see for instance [Adimurthi et al. 2006; Martinazzi 2009b; Robert 2006]). In general, if $\kappa \geq \Lambda_1$ one can expect blow up, but we will construct our approximate solutions carefully in a way that this does not happen. For instance in [Wei and Ye 2008; Hyder and Martinazzi 2015] one looks for solutions of the form $u = P + v + c_v$, where $v$ satisfies the integral equation

$$v(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{nP(y)}e^{n(v(y)+c_v)}\,dy,$$

and $c_v$ is a constant such that

$$\int_{\mathbb{R}^n} Q e^{n(P+v+c_v)}\,dx = \kappa.$$

With such a choice we would not be able to rule out blow up. Instead, by looking for solutions of the form

$$u = P + v + P_v + c_v,$$

where a posteriori $P_v = -|x|^4$, $v$ satisfies

$$v(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y) e^{n(P(y)+P_v(y)+v(y)+c_v)}\,dy + \frac{1}{2n}(|x|^2 - |x|^4)|\Delta v(0)|, \tag{5}$$

and $c_v$ is again a normalization constant, one can prove that the integral equation (5) enjoys sufficient compactness, essentially due to the term $\frac{1}{2n}|x|^2|\Delta v(0)|$ on the right-hand side. Indeed a sequence of (approximate) solutions $v_k$ blowing up (for simplicity) at the origin, up to rescaling, leads to a sequence $(\eta_k)$ of functions satisfying, for every $R > 0$,

$$\int_{B_R} |\Delta \eta_k - c_k|\,dx \le C R^{n-2} + o(1) R^{n+2}, \quad o(1) \xrightarrow{k\to\infty} 0, \; c_k > 0,$$

and converging to $\eta_\infty$, solving (for simplicity here we ignore some cases)

$$(-\Delta)^{n/2}\eta_\infty = e^{n\eta_\infty} \quad \text{in } \mathbb{R}^n, \qquad \int_{\mathbb{R}^n} e^{n\eta_\infty}\,dx < \infty,$$

and

$$\int_{B_R} |\Delta \eta_\infty - c_\infty|\,dx \le C R^{n-2}, \quad c_\infty \ge 0, \tag{6}$$

where $c_\infty = 0$ corresponds to $\Delta\eta_\infty(0) = 0$ (see Subcase 1.1 in Lemma 2.2 with $x_k = 0$).

The estimate on $\|\Delta\eta_\infty\|_{L^1(B_R)}$ in (6) shows that the polynomial part $P_\infty$ of $\eta_\infty$, as in (3), has degree at most 2, and hence $\Delta P_\infty \le 0$ as $P_\infty$ is bounded from above. Therefore, $c_\infty = 0 = \Delta P_\infty$, $P_\infty$ is constant, and in particular $\eta_\infty$ is a spherical solution by Theorem A, that is, $\eta_\infty = u_{\lambda,x_0}$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, where $u_{\lambda,x_0}$ is given by (4). This leads to a contradiction as $\Delta\eta_\infty(0) = 0$ and $\Delta u_{\lambda,x_0} < 0$ in $\mathbb{R}^n$.

In this work we focus only on the case $Q \ge 0$ because the negative case is relatively well understood. For instance by a simple application of maximum principle, one can show that problem (1) has no solution with $Q \equiv \text{constant} < 0$, $n = 2$ and $\kappa > -\infty$, but when $Q$ is nonconstant, solutions do exist, as shown by Chanillo and Kiessling [2000] under suitable assumptions. Martinazzi [2008] proved that in higher even dimension $n = 2m \ge 4$, problem (1) with $Q \equiv \text{constant} < 0$ has solutions for some $\kappa$, and it has been shown in [Hyder and Martinazzi 2015] that actually for every $\kappa \in (-\infty, 0)$ and $Q$ a negative constant, (1) has a solution. The same result has been recently extended to odd dimension $n \ge 3$ in [Hyder 2016].

## 2. Proof of Theorem 1.1

We consider the space

$$X := \{v \in C^{n-1}(\mathbb{R}^n) : v \text{ is radially symmetric}, \|v\|_X < \infty\},$$

where

$$\|v\|_X := \sup_{x\in\mathbb{R}^n}\Bigg(\sum_{|\alpha|\le 3}(1+|x|)^{|\alpha|-4}|D^\alpha v(x)| + \sum_{3<|\alpha|\le n-1}|D^\alpha v(x)|\Bigg).$$

For $v \in X$ we set

$$A_v := \max\left\{0, \sup_{|x|\ge 10}\frac{v(x)-v(0)}{|x|^4}\right\}, \quad P_v(x) := -|x|^4 - A_v|x|^4.$$

Then

$$v(x) + P_v(x) \le v(0) - |x|^4 \quad \text{for } |x| \ge 10.$$

Let $c_v$ be the constant determined by

$$\int_{\mathbb{R}^n} K e^{n(v+c_v)}\,dx = \kappa, \quad K := Qe^{nP}e^{nP_v},$$

where the functions $Q$ and $P$ satisfy the hypotheses in Theorem 1.1. Since $Q > 0$ in a neighborhood of the origin, by a dilation argument we can assume that $Q > 0$ on $B_3$. More precisely, if $u$ is a solution to (1) then for any $\lambda > 0$, we know $u_\lambda(x) := u(\lambda x) + \log\lambda$ is also a solution to (1) with $Q$ replaced by $Q_\lambda$, where $Q_\lambda(x) := Q(\lambda x)$. Now for a suitable choice of $\lambda > 0$, one has $Q_\lambda > 0$ on $B_3$.

The function $u = P + P_v + v + c_v$ satisfies

$$(-\Delta)^{n/2}u = Qe^{nu}, \quad \kappa = \int_{\mathbb{R}^n} Qe^{nu}\,dx$$

if and only if $v$ satisfies

$$(-\Delta)^{n/2}v = Ke^{n(v+c_v)}.$$

For odd integer $n$, the operator $(-\Delta)^{n/2}$ is defined as follows:

**Definition.** Let $n$ be an odd integer. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We say that $u$ is a solution of

$$(-\Delta)^{n/2}u = f \quad \text{in } \mathbb{R}^n$$

if $u \in W^{n-1,1}_{\text{loc}}(\mathbb{R}^n)$ and $\Delta^{(n-1)/2}u \in L_{1/2}(\mathbb{R}^n)$ and for every test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}(-\Delta)^{(n-1)/2}u(-\Delta)^{1/2}\varphi\,dx = \langle f, \varphi\rangle.$$

Here $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space and the space $L_s(\mathbb{R}^n)$ is defined by

$$L_s(\mathbb{R}^n) := \left\{u \in L^1_{\text{loc}}(\mathbb{R}^n) : \|u\|_{L_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+2s}}\,dx < \infty\right\}, \quad s > 0.$$

For more details on the fractional Laplacian we refer the reader to [Di Nezza et al. 2012].

We define an operator $T : X \to X$ given by $T(v) = \bar{v}$, where

$$\bar{v}(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n}\log\left(\frac{1}{|x-y|}\right)K(y)e^{n(v(y)+c_v)}\,dy + \frac{1}{2n}(|x|^2 - |x|^4)|\Delta v(0)|.$$

**Lemma 2.1.** *Let $v$ solve $tT(v) = v$ for some $0 < t \le 1$. Then*

$$v(x) = \frac{t}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) K(y) e^{n(v(y)+c_v)}\, dy + \frac{t}{2n}(|x|^2 - |x|^4)|\Delta v(0)|, \tag{7}$$

$\Delta v(0) < 0$, *and* $v(x) \to -\infty$ *as* $|x| \to \infty$. *Moreover,*

$$\sup_{x \in B_1^c} v(x) = v(1) = \inf_{x \in B_1} v(x),$$

*and in particular* $A_v = 0$.

*Proof.* Since $v$ satisfies $tT(v) = v$, equation (7) follows from the definition of $T$. Differentiating under the integral sign and observing that $\Delta \log(1/|\cdot - y|) < 0$, from (7) one gets

$$\Delta v(x) < \frac{t}{2n} |\Delta v(0)| \Delta(|x|^2 - |x|^4), \quad x \in \mathbb{R}^n. \tag{8}$$

Taking $x = 0$ in (8) we obtain $\Delta v(0) < t|\Delta v(0)|$, which implies that $\Delta v(0) < 0$. Notice that the function

$$w(x) := v(x) + \frac{t}{2n} |\Delta v(0)| (|x|^4 - |x|^2)$$

is monotone decreasing as $\Delta w < 0$. This follows from (8) and the integral representation of radially symmetric functions given by

$$f(\xi) - f(\bar{\xi}) = \int_{\bar{\xi}}^{\xi} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r} \Delta f(x)\, dx\, dr, \quad 0 \le \bar{\xi} < \xi, \ \ \omega_{n-1} := |S^{n-1}|. \tag{9}$$

The monotonicity of $w$ implies that $\sup_{x \in B_1^c} v(x) = v(1) = \inf_{x \in B_1} v(x)$, and hence $A_v = 0$. Finally, together with $|\Delta v(0)| > 0$, we conclude that $\lim_{|x| \to \infty} v(x) = -\infty$ as $\lim_{|x| \to \infty} w(x) \le w(1)$. □

**Lemma 2.2.** *Let $(v, t) \in X \times (0, 1]$ satisfy $v = tT(v)$. Then there exists $C > 0$ (independent of $v$ and $t$) such that*

$$\sup_{B_{1/8}} w \le C, \quad w := v + c_v + \frac{1}{n} \log t.$$

*Proof.* Let us assume by contradiction that the conclusion of the lemma is false. Then there exists a sequence $w_k = v_k + c_{v_k} + \frac{1}{n} \log t_k$ such that $\max_{\bar{B}_{1/8}} w_k =: w_k(\theta_k) \to \infty$.

If $\theta_k$ is a point of local maxima of $w_k$, we set $x_k = \theta_k$. Otherwise, we can choose $x_k \in B_{1/4} \setminus B_{1/8}$ such that $x_k$ is a point of local maxima of $w_k$ and $w_k(x_k) \ge w_k(x)$ for every $x \in B_{|x_k|}$. This follows from the fact that

$$\inf_{B_{1/4} \setminus B_{1/8}} w_k \not\to \infty,$$

which is a consequence of

$$\int_{\mathbb{R}^n} K e^{n w_k}\, dx = t_k \kappa \le \kappa, \quad K > 0 \text{ on } B_3.$$

We set $\mu_k := e^{-w_k(x_k)}$. We distinguish the following cases.

**Case 1:** Up to a subsequence, $t_k \mu_k^2 |\Delta v_k(0)| \to c_0 \in [0, \infty)$.

We set

$$\eta_k(x) := v_k(x_k + \mu_k x) - v_k(x_k) = w_k(x_k + \mu_k x) - w_k(x_k).$$

Notice that by (7) we have, for some dimensional constant $C_1$,

$$\Delta\eta_k(x)=\mu_k^2\Delta v_k(x_k+\mu_k x)=C_1\frac{\mu_k^2}{\gamma_n}\int_{\mathbb{R}^n}\frac{K(y)e^{nw_k(y)}}{|x_k+\mu_k x-y|^2}\,dy+t_k\mu_k^2\left(1-\frac{4(n+2)}{2n}|x_k+\mu_k x|^2\right)|\Delta v_k(0)|,$$

so that

$$\begin{aligned}\int_{B_R}&\left|\Delta\eta_k(x)-t_k\mu_k^2|\Delta v_k(0)|\left(1-\frac{2(n+2)}{n}|x_k|^2\right)\right|dx\\
&\le\frac{C_1}{\gamma_n}\int_{\mathbb{R}^n}K(y)e^{nw_k(y)}\int_{B_R}\frac{\mu_k^2\,dx}{|x_k+\mu_k x-y|^2}\,dy+Ct_k\mu_k^2|\Delta v_k(0)|\int_{B_R}(\mu_k|x_k\cdot x|+\mu_k^2|x|^2)\,dx\\
&\le\frac{C_1}{\gamma_n}t_k\kappa\int_{B_R}\frac{1}{|x|^2}\,dx+Ct_k\mu_k^2|\Delta v_k(0)|\int_{B_R}(\mu_k|x|+\mu_k^2|x|^2)\,dx\\
&\le C\kappa t_kR^{n-2}+Ct_k\mu_k^2|\Delta v_k(0)|(\mu_kR^{n+1}+\mu_k^2R^{n+2}).\end{aligned}\tag{10}$$

The function $\eta_k$ satisfies

$$(-\Delta)^{n/2}\eta_k(x)=K(x_k+\mu_k x)e^{n\eta_k(x)}\quad\text{in }\mathbb{R}^n,\qquad\eta_k(0)=0.$$

Moreover, $\eta_k\le C(R)$ on $B_R$. This follows easily if $|x_k|\le\frac{1}{9}$, as in that case $\eta_k\le 0$ on $B_R$ for $k\ge k_0(R)$. On the other hand, for $\frac{1}{9}<|x_k|\le\frac{1}{4}$ one can use Lemma 2.4 (below). Therefore, by Lemma A.3 (and Lemmas 2.6, 2.7 if $n$ is odd), up to a subsequence, $\eta_k\to\eta$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$, where $\eta$ satisfies

$$(-\Delta)^{n/2}\eta=K(x_\infty)e^{n\eta}\quad\text{in }\mathbb{R}^n,\qquad K(x_\infty)\int_{\mathbb{R}^n}e^{n\eta}\,dx\le t_\infty\kappa<\infty,\qquad K(x_\infty)>0,$$

where (up to a subsequence) $t_k\to t_\infty$ and $x_k\to x_\infty$. Notice that $t_\infty\in(0,1]$, $x_\infty\in\bar{B}_{1/4}$ and for every $R>0$, by (10)

$$\int_{B_R}|\Delta\eta-c_0c_1|\,dx\le CR^{n-2},\qquad c_1=:1-\frac{2(n+2)}{n}|x_\infty|^2>0.\tag{11}$$

Hence by Theorem A we have

$$\eta(x)=P_0(x)-\alpha\log|x|+o(\log|x|)\quad\text{as }|x|\to\infty,$$

where $P_0$ is a polynomial of degree at most $n-1$, $P_0$ is bounded from above and $\alpha$ is a positive constant. In fact, by (11)

$$\int_{B_R}|\Delta P_0(x)-c_0c_1|\,dx\le CR^{n-2}\quad\text{for every }R>0.$$

Since $c_0$, $c_1\ge 0$, it follows that $P_0$ is a constant. This implies that $\eta$ is a spherical solution and in particular $\Delta\eta<0$ on $\mathbb{R}^n$, and therefore, again by (11), we have $c_0=0$.

We consider the following subcases.

**Subcase 1.1:** There exists $M>0$ such that $|x_k|/\mu_k\le M$.

We set $y_k:=-x_k/\mu_k$. Then (up to a subsequence) $y_k\to y_\infty\in B_{M+1}$. Therefore,

$$\Delta\eta(y_\infty)=\lim_{k\to\infty}\Delta\eta_k(y_k)=\lim_{k\to\infty}\mu_k^2\Delta v_k(0)=\frac{c_0}{t_\infty}=0,$$

a contradiction as $\Delta\eta<0$ on $\mathbb{R}^n$.

**Subcase 1.2:** Up to a subsequence, $|x_k|/\mu_k \to \infty$.

For any $N \in \mathbb{N}$ we can choose $\xi_{1,k}, \dots, \xi_{N,k} \in \mathbb{R}^n$ such that $|\xi_{i,k}| = |x_k|$ for all $i = 1, \dots, N$ and the balls $B_{2\mu_k}(\xi_{i,k})$ are disjoint for $k$ large enough. Since the $v_k$ are radially symmetric, the functions $\eta_{i,k} := v_k(\xi_{i,k} + \mu_k x) - v_k(\xi_{i,k}) \to \eta_i = \eta$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$. Therefore,

$$\lim_{k\to\infty} \int_{B_1} e^{n(v_k + c_{v_k})}\,dx \ge N \lim_{k\to\infty} \int_{B_{\mu_k}(\xi_{1,k})} e^{n(v_k + c_{v_k})}\,dx = N \frac{1}{t_\infty} \int_{B_1} e^{n\eta}\,dx.$$

This contradicts the fact that

$$\int_{B_1} K e^{n(v_k + c_{v_k})}\,dx \le \kappa, \qquad K > 0 \text{ on } B_3.$$

**Case 2:** Up to a subsequence, $t_k \mu_k^2 |\Delta v_k(0)| \to \infty$.

We choose $\rho_k > 0$ such that $t_k \rho_k^2 \mu_k^2 |\Delta v_k(0)| = 1$. We set

$$\psi_k(x) = v_k(x_k + \rho_k \mu_k x) - v_k(x_k).$$

Then one can get (similar to (10))

$$\int_{B_R} \left| \Delta \psi_k(x) - \left(1 - \frac{2(n+2)}{n} |x_k|^2\right) \right| dx$$
$$\le C_1 \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{B_R} \frac{\rho_k^2 \mu_k^2}{|x_k + \mu_k \rho_k x - y|^2}\,dx\,dy + C_2 \mu_k \rho_k \int_{B_R} (|x| + \mu_k \rho_k |x|^2)\,dx \xrightarrow{k\to\infty} 0,$$

thanks to Lemma 2.5 (below). Moreover, together with Lemma 2.4, $\psi_k$ satisfies

$$(-\Delta)^{n/2} \psi_k = o(1) \quad \text{in } B_R, \qquad \psi_k(0) = 0, \qquad \psi_k \le C(R) \text{ on } B_R.$$

Hence, by Lemma A.3 (and Lemma 2.6 if $n$ is odd), up to a subsequence, $\psi_k \to \psi$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$. Then $\psi$ must satisfy

$$\int_{B_1} |\Delta \psi - c_0|\,dx = 0, \qquad c_0 := 1 - \frac{2(n+2)}{n} |x_\infty|^2 > 0,$$

where (up to a subsequence) $x_k \to x_\infty$. This shows that $\Delta \psi(0) = c_0 > 0$, which is a contradiction as

$$\Delta \psi(0) = \lim_{k\to\infty} \Delta \psi_k(0) = \lim_{k\to\infty} \rho_k^2 \mu_k^2 \Delta v_k(x_k) \le 0.$$

Here, $\Delta v_k(x_k) \le 0$ follows from the fact that $x_k$ is a point of local maxima of $v_k$. □

A consequence of the local uniform upper bounds of $w$ are the following global uniform upper bounds:

**Lemma 2.3.** *There exists a constant $C > 0$ such that for all $(v, t) \in X \times (0, 1]$ with $v = tT(v)$ we have $|\Delta v(0)| \le C$ and*

$$v(x) + c_v + \frac{1}{n} \log t \le C \quad \text{on } \mathbb{R}^n.$$

*Proof.* By Lemma 2.2 we have

$$\sup_{B_{1/8}} w := \sup_{B_{1/8}} \left( v + c_v + \frac{1}{n} \log t \right) \le C.$$

Differentiating under the integral sign from (7), and recalling that $\Delta v(0) < 0$, we obtain

$$|\Delta v(0)| \le C \int_{B_{1/8}} \frac{1}{|y|^2} K(y) e^{nw(y)}\, dy + C \int_{B^c_{1/8}} \frac{1}{|y|^2} K(y) e^{nw(y)}\, dy$$
$$\le C \sup_{B_{1/8}} K \int_{B_{1/8}} \frac{1}{|y|^2}\, dy + C \int_{B^c_{1/8}} K e^{nw}\, dy \le C(\kappa, K).$$

By (8) we get

$$\Delta v(x) \le t|\Delta v(0)| \le C, \quad x \in \mathbb{R}^n,$$

and hence, together with (9)

$$v(x) = v(0) + \int_0^{|x|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r} \Delta v(y)\, dy\, dr \le v(0) + C|x|^2 \le C + v(0), \quad x \in B_2.$$

The lemma follows from Lemmas 2.1 and 2.2. □

*Proof of Theorem 1.1.* Let $v \in X$ be a solution of $v = tT(v)$ for some $0 < t \le 1$. Then $A_v = 0$ and $|\Delta v(0)| \le C$, thanks to Lemmas 2.1 and 2.3. Hence, for $0 \le |\beta| \le n-1$,

$$|D^\beta v(x)| \le C \int_{\mathbb{R}^n} \left| D^\beta \log\left(\frac{1}{|x-y|}\right)\right| K(y) e^{n(v(y)+c_v+(1/n)\log t)}\, dy + C|D^\beta(|x|^2 - |x|^4)|$$
$$\le C \int_{\mathbb{R}^n} \left| D^\beta \log\left(\frac{1}{|x-y|}\right)\right| e^{-|y|^4}\, dy + C|D^\beta(|x|^2 - |x|^4)|,$$

where in the second inequality we have used that

$$v(x) + c_v + \frac{1}{n}\log t \le C, \quad C \text{ is independent of } v \text{ and } t,$$

which follows from Lemma 2.3. Now as in Lemma 2.8 one can show that

$$\|v\|_X \le M,$$

and therefore, by Lemma A.1, the operator $T$ has a fixed point (say) $v$. Then

$$u = P + v + c_v - |x|^4$$

is a solution to the problem (1) and $u$ has the asymptotic behavior given by

$$u(x) = P(x) - \frac{2\kappa}{\Lambda_1}\log|x| + \frac{1}{2n}\Delta v(0)(|x|^4 - |x|^2) - |x|^4 + c_v + o(1) \quad \text{as } |x| \to \infty. \qquad □$$

Now we give a proof of the technical lemmas used in the proof of Lemma 2.2.

**Lemma 2.4.** *Let $\varepsilon > 0$. Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (7) or (14) for all $k \in \mathbb{N}$. Let $x_k \in B_1 \setminus B_\varepsilon$ be a point of maxima of $v_k$ on $\bar{B}_{|x_k|}$ and $v'_k(x_k) = 0$. Then*

$$v_k(x_k + x) - v_k(x_k) \le C(n, \varepsilon)|x|^2 t_k |\Delta v_k(0)|, \quad x \in B_1.$$

*Proof.* If $|x_k + x| \le |x_k|$ then $v_k(x_k + x) - v_k(x_k) \le 0$ as $v_k(x_k) \ge v_k(y)$ for every $y \in B_{|x_k|}$. For $|x_k| < |x_k + x|$, setting $a = a(k, x) := x_k + x$, and together with (9) we obtain

$$\begin{aligned} v_k(x_k + x) - v_k(x_k) &= \int_{|x_k|}^{|a|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r \setminus B_{|x_k|}} \Delta v_k(x)\, dx\, dr \\ &\le \int_{|x_k|}^{|a|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_{|a|} \setminus B_{|x_k|}} t_k |\Delta v_k(0)|\, dx\, d\rho \\ &\le C(n) t_k |\Delta v_k(0)| (|B_{|a|}| - |B_{|x_k|}|) \left( \frac{1}{|x_k|^{n-2}} - \frac{1}{|a|^{n-2}} \right) \\ &\le C(n, \varepsilon) t_k |x|^2 |\Delta v_k(0)|, \end{aligned}$$

where in the first equality we have used that

$$0 = v_k'(x_k) = \frac{1}{\omega_{n-1} |x_k|^{n-1}} \int_{B_{|x_k|}} \Delta v_k\, dx.$$

Hence we have the lemma. □

**Lemma 2.5.** *Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (7) for all $k \in \mathbb{N}$. Let $x_k \in B_1$ be a point of maxima of $v_k$ on $\bar{B}_{|x_k|}$ and $v_k'(x_k) = 0$. We set $w_k = v_k + c_{v_k} + \frac{1}{n} \log t_k$ and $\mu_k = e^{-w_k(x_k)}$. Let $\rho_k > 0$ be such that $t_k \rho_k^2 \mu_k^2 |\Delta v_k(0)| \le C$ and $\rho_k \mu_k \to 0$. Then for any $R_0 > 0$,*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{B_{R_0}} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2}\, dx\, dy =: \lim_{k\to\infty} I_k = 0.$$

*Proof.* In order to prove the lemma we fix $R > 0$ (large). We split $B_{R_0}$ into

$$A_1(R, y) := \left\{ x \in B_{R_0} : |x_k + \rho_k \mu_k x - y| > R \rho_k \mu_k \right\}, \quad A_2(R, y) := B_{R_0} \setminus A_1(R, y).$$

Then we can write $I_k = I_{1,k} + I_{2,k}$, where

$$I_{i,k} := \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{A_i(R,y)} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2}\, dx\, dy, \quad i = 1, 2.$$

Changing the variable $y \mapsto x_k + \rho_k \mu_k y$ and by Fubini's theorem, one gets

$$\begin{aligned} I_{2,k} &= \rho_k^n \int_{B_{R_0}} \int_{\mathbb{R}^n} K(x_k + \rho_k \mu_k y) e^{n \eta_k(y)} \frac{1}{|x - y|^2} \chi_{|x-y| \le R}\, dy\, dx \\ &\le \rho_k^n \int_{B_{R_0}} \int_{B_{R+R_0}} K(x_k + \rho_k \mu_k y) e^{n \eta_k(y)} \frac{1}{|x - y|^2}\, dy\, dx \\ &\le C(n, \varepsilon) (\sup\nolimits_{B_{R+R_0+1}} K e^{n \eta_k}) (R + R_0)^n R_0^{n-2} \rho_k^n, \end{aligned}$$

where $\eta_k(y) := w_k(x_k + \rho_k \mu_k y) - w_k(x_k)$. If $x_k \to 0$ then $\eta_k \le 0$ on $B_{R+R_0+1}$ for $k$ large. Otherwise, for $k$ large, $\rho_k \mu_k y \in B_1$ for every $y \in B_{R+R_0+1}$ and hence, by Lemma 2.4

$$\eta_k(y) = v_k(x_k + \rho_k \mu_k y) - v_k(x_k) \le C |\rho_k \mu_k y|^2 t_k |\Delta v_k(0)| \le C(R, R_0).$$

Therefore,

$$\lim_{k\to\infty} I_{2,k} = 0.$$

Using the definition of $c_v$ we bound

$$I_{1,k} \le \frac{|B_{R_0}|}{R^2}\int_{\mathbb{R}^n} K(y)e^{nw_k(y)}\,dy \le C(n,\kappa,R_0)\frac{1}{R^2}.$$

Since $R > 0$ is arbitrary, we conclude the lemma. □

We need the following two lemmas only for $n$ odd.

**Lemma 2.6.** *Let $n \ge 5$. Let $v$ be given by (7). For any $r > 0$ and $\xi \in \mathbb{R}^n$ we set*

$$w(x) = v(rx+\xi), \quad x \in \mathbb{R}^n.$$

*Then there exists $C > 0$ (independent of $v, t, r, \xi$) such that for every multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = n-1$ we have $\|D^\alpha w\|_{L_{1/2}(\mathbb{R}^n)} \le Ct(1 + r^4|\Delta v(0)|)$. Moreover, for any $\varepsilon > 0$ there exists $R > 0$ (independent of $r, \xi$ and $t$) such that*

$$\int_{B_R^c} \frac{|D^\alpha w(x)|}{1+|x|^{n+1}}\,dx < \varepsilon t(1 + r^4|\Delta v(0)|), \quad |\alpha| = n-1.$$

*Proof.* Differentiating under the integral sign we obtain

$$|D^\alpha w(x)| \le Ct\int_{\mathbb{R}^n} \frac{r^{n-1}}{|rx+\xi-y|^{n-1}} f(y)\,dy + Ctr^4|\Delta v(0)|, \quad f(y) := K(y)e^{n(v(y)+c_v)}.$$

If $n > 5$ then the above inequality is true without the term $Ctr^4|\Delta v(0)|$. Using a change of variable $y \mapsto \xi + ry$, we get

$$\int_\Omega \frac{|D^\alpha w(x)|}{1+|x|^{n+1}}\,dx \le Ctr^n\int_{\mathbb{R}^n} f(\xi+ry)\int_\Omega \frac{1}{|x-y|^{n-1}}\frac{1}{1+|x|^{n+1}}\,dx\,dy + Ctr^4|\Delta v(0)|\int_\Omega \frac{dx}{1+|x|^{n+1}}.$$

The lemma follows by taking $\Omega = \mathbb{R}^n$ or $B_R^c$. □

**Lemma 2.7.** *Let $\eta_k \to \eta$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$. We assume that for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{B_R^c} \frac{|\Delta^{(n-1)/2}\eta_k(x)|}{1+|x|^{n+1}}\,dx < \varepsilon \quad \text{for } k = 1, 2, \ldots. \tag{12}$$

*We further assume that*

$$(-\Delta)^{n/2}\eta_k = K(x_k+\mu_k x)e^{n\eta_k} \quad \text{in } \mathbb{R}^n, \qquad \int_{\mathbb{R}^n} |K(x_k+\mu_k x)|e^{n\eta_k(x)}\,dx \le C,$$

*where $x_k \to x_\infty$, $\mu_k \to 0$, $K$ is a continuous function and $K(x_\infty) > 0$. Then $e^{n\eta} \in L^1(\mathbb{R}^n)$ and $\eta$ satisfies*

$$(-\Delta)^{n/2}\eta = K(x_\infty)e^{n\eta} \quad \text{in } \mathbb{R}^n.$$

*Proof.* First notice that $\Delta^{(n-1)/2}\eta_k \to \Delta^{(n-1)/2}\eta$ in $L_{1/2}(\mathbb{R}^n)$, thanks to (12) and the convergence $\eta_k \to \eta$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$.

We claim that $\eta$ satisfies $(-\Delta)^{n/2}\eta = K(x_\infty)e^{n\eta}$ in $\mathbb{R}^n$ in the sense of distribution.

In order to prove the claim we let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} K(x_k+\mu_k x)e^{n\eta_k(x)}\varphi(x)\,dx = \int_{\mathbb{R}^n} K(x_\infty)e^{n\eta(x)}\varphi(x)\,dx,$$

and

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}(-\Delta)^{(n-1)/2}\eta_k(-\Delta)^{1/2}\varphi\,dx = \int_{\mathbb{R}^n}(-\Delta)^{(n-1)/2}\eta(-\Delta)^{1/2}\varphi\,dx.$$

We conclude the claim.

To complete the lemma first notice that $e^{n\eta} \in L^1(\mathbb{R}^n)$, which follows from the fact that for any $R > 0$

$$\int_{B_R} e^{n\eta}\,dx = \lim_{k\to\infty}\int_{B_R} e^{n\eta_k}\,dx = \lim_{k\to\infty}\int_{B_R}\frac{K(x_k+\mu_k x)}{K(x_\infty)}e^{n\eta_k(x)}\,dx \le \frac{C}{K(x_\infty)}.$$

We fix a function $\psi \in C_c^\infty(B_2)$ such that $\psi = 1$ on $B_1$. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we set $\varphi_k(x) = \varphi(x)\psi(x/k)$. The lemma follows by taking $k \to \infty$, thanks to the previous claim. □

**Lemma 2.8.** *The operator $T : X \to X$ is compact.*

*Proof.* Let $v_k$ be a bounded sequence in $X$. Then (up to a subsequence) $\{v_k(0)\}$, $\{\Delta v_k(0)\}$, $\{A_{v_k}\}$ and $\{c_{v_k}\}$ are convergent sequences. Therefore, $|\Delta v_k(0)|(|x|^2 - |x|^4)$ converges to some function in $X$. To conclude the lemma, it is sufficient to show that up to a subsequence $\{f_k\}$ converges in $X$, where $f_k$ is defined by

$$f_k(x) = \int_{\mathbb{R}^n}\log\left(\frac{1}{|x-y|}\right)Q(y)e^{nP(y)}e^{nP_{v_k}(y)}e^{n(v_k(y)+c_{v_k})}\,dy.$$

Differentiating under the integral sign, for $0 < |\beta| \le n-1$, one gets

$$|D^\beta f_k(x)| \le C\int_{\mathbb{R}^n}\frac{1}{|x-y|^{|\beta|}}Q(y)e^{nP(y)}e^{nP_{v_k}(y)}e^{n(v_k(y)+c_{v_k})}\,dy \le C\int_{\mathbb{R}^n}\frac{1}{|x-y|^{|\beta|}}e^{-|y|^4}\,dy \le C,$$

where the second inequality follows from the uniform bounds

$$|v_k(0)| \le C, \quad |c_{v_k}| \le C, \quad Qe^{nP} \le C, \quad \text{and} \quad v_k(x) + P_{v_k}(x) \le v_k(0) - |x|^4. \tag{13}$$

Indeed, for $0 < |\beta| \le n-1$

$$\lim_{R\to\infty}\sup_k\sup_{x\in B_R^c}|D^\beta f_k(x)| = 0,$$

and for every $0 < s < 1$ we have $\|D^{n-1}f_k\|_{C^{0,s}(B_R)} \le C(R,s)$. Finally, using (13) we have the bound

$$|f_k(x)| \le C\int_{\mathbb{R}^n}\big|\log|x-y|\big|e^{-|y|^4}\,dy \le C\log(2+|x|).$$

Thus, by Ascoli's theorem, up to a subsequence, $f_k \to f$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $f \in C^{n-1}(\mathbb{R}^n)$, and the global uniform estimates of $f_k$ and $D^\beta f_k$ would imply that $f_k \to f$ in $X$. □

## 3. Proof of Theorem 1.3

We consider the space

$$X := \{v \in C^{n-1}(\mathbb{R}^n) : v \text{ is radially symmetric}, \|v\|_X < \infty\},$$

where

$$\|v\|_X := \sup_{x\in\mathbb{R}^n}\Bigg(\sum_{|\alpha|\le 1}(1+|x|)^{|\alpha|-2}|D^\alpha v(x)| + \sum_{1<|\alpha|\le n-1}|D^\alpha v(x)|\Bigg).$$

For $v \in X$, let $c_v$ be the constant determined by

$$\int_{\mathbb{R}^n} Qe^{n(v+c_v)}\,dy = \kappa,$$

where $Q$ satisfies the hypothesis in Theorem 1.3. Again by a dilation argument we can assume $Q > 0$ on $B_3$.

We define an operator $T : X \to X$ given by $T(v) = \bar{v}$, where

$$\bar{v}(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n}\log\Big(\frac{1}{|x-y|}\Big)Q(y)e^{n(v(y)+c_v)}\,dy + \frac{1}{2n}|\Delta v(0)|\,|x|^2.$$

As in Lemma 2.8 one can show that the operator $T$ is compact.

The proofs of the following two lemmas are similar to those of Lemmas 2.1 and 2.5 respectively.

**Lemma 3.1.** *Let $v$ solve $tT(v) = v$ for some $0 < t \le 1$. Then $\Delta v(0) < 0$, and*

$$v(x) = \frac{t}{\gamma_n}\int_{\mathbb{R}^n}\log\Big(\frac{1}{|x-y|}\Big)Q(y)e^{n(v(y)+c_v)}\,dy + \frac{t}{2n}|\Delta v(0)|\,|x|^2. \tag{14}$$

**Lemma 3.2.** *Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (14) for all $k \in \mathbb{N}$. Let $x_k \in B_1$ be a point of maxima of $v_k$ on $\bar{B}_{|x_k|}$ and $v_k'(x_k) = 0$. We set $w_k = v_k + c_{v_k} + \frac{1}{n}\log t_k$ and $\mu_k = e^{-w_k(x_k)}$. Let $\rho_k > 0$ be such that $\rho_k^2 t_k\mu_k^2|\Delta v_k(0)| \le C$ and $\rho_k\mu_k \to 0$. Then for any $R_0 > 0$*

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} Q(y)e^{nw_k(y)}\int_{B_{R_0}}\frac{\rho_k^2\mu_k^2}{|x_k+\rho_k\mu_k x - y|^2}\,dx\,dy = 0.$$

Now we prove similar local uniform upper bounds to those in Lemma 2.2.

**Lemma 3.3.** *Let $(v, t) \in X \times (0, 1]$ satisfy (14). Then there exists $C > 0$ (independent of $v$ and $t$) such that*

$$\sup_{B_{1/8}} w \le C, \qquad w := v + c_v + \frac{1}{n}\log t.$$

*Proof.* The proof is very similar to that of Lemma 2.2. Here we briefly sketch it.

We assume by contradiction that the conclusion of the lemma is false. Then there exists a sequence of $(v_k, t_k)$ and a sequence of points $x_k$ in $B_{1/4}$ such that

$$w_k(x_k) \to \infty, \qquad w_k \le w_k(x_k) \text{ on } B_{|x_k|}, \qquad x_k \text{ is a point of local maxima of } v_k.$$

We set $\mu_k := e^{-w_k(x_k)}$ and we distinguish following cases.

**Case 1:** Up to a subsequence, $t_k\mu_k^2|\Delta v_k(0)| \to c_0 \in [0, \infty)$.

We set $\eta_k(x) := v_k(x_k + \mu_k x) - v_k(x_k)$. Then we have

$$\int_{B_R} \big|\Delta\eta_k - t_k\mu_k^2|\Delta v_k(0)|\big|\,dx \le C t_k R^{n-2}.$$

Now one can proceed exactly as in Case 1 in Lemma 2.2.

**Case 2:** Up to a subsequence, $t_k\mu_k^2|\Delta v_k(0)| \to \infty$.

We set $\psi_k(x) = v_k(x_k + \rho_k\mu_k x) - v_k(x_k)$, where $\rho_k$ is determined by $t_k\rho_k^2\mu_k^2|\Delta v_k(0)| = 1$. Then by Lemma 3.2

$$\int_{B_R} |\Delta\psi_k - 1|\,dx = o(1) \quad \text{as } k \to \infty.$$

Similar to Case 2 in Lemma 2.2 one can get a contradiction. □

With the help of Lemma 3.3 we prove:

**Lemma 3.4.** *There exists a constant $M > 0$ such that for all $(v, t) \in X \times (0, 1]$ satisfying (14) we have $\|v\| \le M$.*

*Proof.* Let $(v, t) \in X \times (0, 1]$ satisfy (14). We set $w := v + c_v + \frac{1}{n}\log t$.

First we show that $|\Delta v(0)| \le C$ for some $C > 0$ independent of $v$ and $t$. Indeed, differentiating under the integral sign, from (14), and together with Lemma 3.3, we get

$$\begin{aligned} |\Delta v(0)|(1+t) &\le C\int_{\mathbb{R}^n} \frac{1}{|y|^2} Q(y)e^{nw(y)}\,dy \\ &= C\int_{B_{1/8}} \frac{1}{|y|^2} Q(y)e^{nw(y)}\,dy + C\int_{B_{1/8}^c} \frac{1}{|y|^2} Q(y)e^{nw(y)}\,dy \\ &\le C\int_{B_{1/8}} \frac{1}{|y|^2} Q(y)\,dy + C\kappa \le C. \end{aligned}$$

Hence $|\Delta v(0)| \le C$.

We define a function $\xi(x) := v(x) - (t/2n)|\Delta v(0)||x|^2$. Then $\xi$ is monotone decreasing on $(0, \infty)$, which follows from the fact that $\Delta\xi \le 0$. Therefore,

$$\begin{aligned} w(x) &= \xi(x) + c_v + \frac{1}{n}\log t + \frac{t}{2n}|\Delta v(0)||x|^2 \\ &\le \xi\big(\tfrac{1}{8}\big) + c_v + \frac{1}{n}\log t + \frac{t}{2n}|\Delta v(0)||x|^2 \\ &\le w\big(\tfrac{1}{8}\big) + \frac{t}{2n}|\Delta v(0)||x|^2. \end{aligned}$$

Hence, $w(x) \le \lambda(1 + |x|^2)$ on $\mathbb{R}^n$ for some $\lambda > 0$ independent of $v$ and $t$. Using this in (14) one can show

$$|v(x)| \le C\log(2 + |x|) + C|x|^2,$$

and differentiating under the integral sign, from (14)

$$|D^\beta v(x)| \le C\int_{\mathbb{R}^n} \frac{1}{|x-y|^{|\beta|}} Q(y)e^{\lambda(1+|y|^2)}\,dy + C\big|D^\beta|x|^2\big|, \quad 0 < |\beta| \le n-1.$$

The lemma follows easily. □

*Proof of Theorem 1.3.* By the Schauder fixed point theorem (see Lemma A.1), the operator $T$ has a fixed point, thanks to Lemma 3.4. Let $v$ be a fixed point of $T$. Then $u = v + c_v$ is a solution of (1). □

Now we prove the nonexistence results stated in the Introduction.

**Lemma 3.5.** *Let $n \in \{3, 4\}$. If $Q(x) = e^{-\lambda|x|^2}$ for some $\lambda > 0$ then there is no solution to (1) with $\kappa > \Lambda_1$. If $Q \in C^1_{\mathrm{rad}}(\mathbb{R}^n)$ is of the form $Q = e^{\xi}$ and it satisfies*

$$Q' \le 0, \quad |x \cdot \nabla Q(x)| \le C, \quad \frac{\xi(x)}{|x|^2} \xrightarrow{|x|\to\infty} 0,$$

*then there is no radially symmetric solution to (1) with $\kappa > \Lambda_1$.*

*Proof.* First we consider the case when $Q = e^{-\lambda|x|^2}$. Let $u$ be a solution to (1) with $Q = e^{-\lambda|x|^2}$. Then the function $w(x) := u - (\lambda/n)|x|^2$ satisfies

$$(-\Delta)^{n/2} w = e^{nw}, \quad \kappa = \int_{\mathbb{R}^n} Q e^{nu}\, dx = \int_{\mathbb{R}^n} e^{nw}\, dx < \infty.$$

It follows from [Lin 1998; Jin et al. 2015] that $\kappa \le \Lambda_1$.

In order to prove the lemma for $Q = e^{\xi}$, we assume by contradiction that there is a solution $u$ to (1) with $\kappa > \Lambda_1$. We set

$$v(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{|y|}{|x-y|}\right) Q(y) e^{nu(y)}\, dy, \quad h := u - v.$$

Then $v(x) = -(2\kappa/\Lambda_1)\log|x| + o(\log|x|)$ as $|x| \to \infty$. Notice that $h$ is radially symmetric and $(-\Delta)^{n/2} h = 0$ on $\mathbb{R}^n$. Therefore, $h(x) = c_1 + c_2|x|^2$ for some $c_1, c_2 \in \mathbb{R}$. This follows easily if $n = 4$. For $n = 3$, first notice that $\Delta h \in L_{1/2}(\mathbb{R}^3)$. Hence, by [Jin et al. 2015, Lemma 15] $\Delta h \equiv$ constant. Now radial symmetry of $h$ implies that $h(x) = c_1 + c_2|x|^2$.

From a Pohozaev-type identity in [Xu 2005, Theorem 2.1], we get

$$\frac{\kappa}{\gamma_n}\left(\frac{\kappa}{\gamma_n} - 2\right) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} (x \cdot \nabla K(x)) e^{nv(x)}\, dx, \quad K := Q e^{nh}. \tag{15}$$

Since $\kappa > \Lambda_1 = 2\gamma_n$, from (15) we deduce that $x \cdot \nabla K(x) > 0$ for some $x \in \mathbb{R}^n$. Using that $Qe^{nu} \in L^1(\mathbb{R}^n)$ and that $\xi(x) = o(|x|^2)$ at infinity, one has $c_2 \le 0$. Therefore, $x \cdot \nabla K(x) \le 0$ in $\mathbb{R}^n$, a contradiction. □

The proof of the following lemma is similar to that of Lemma 3.5.

**Lemma 3.6.** *Let $\kappa \ge \Lambda_1$. Let $P$ be a nonconstant and nonincreasing radially symmetric polynomial of degree at most $n - 1$. Then there is no solution $u$ to (2) (with $n \ge 3$) such that $u$ has the asymptotic behavior given by*

$$u(x) = -\frac{2\kappa}{\Lambda_1} \log|x| + P(x) + o(\log|x|) \quad \text{as } |x| \to \infty.$$

## Appendix

**Lemma A.1** [Gilbarg and Trudinger 1998, Theorem 11.3]. *Let $T$ be a compact mapping of a Banach space $X$ into itself, and suppose that there exists a constant $M$ such that*

$$\|x\|_X < M$$

*for all $x \in X$ and $t \in (0, 1]$ satisfying $tTx = x$. Then $T$ has a fixed point.*

The following identity (16) is due to Pizzetti [1909]. Simple proofs of (16) and (17) can be found in Lemma 3 and Proposition 4, respectively, of [Martinazzi 2009a].

**Lemma A.2** [Pizzetti 1909; Martinazzi 2009a]. *Let $\Delta^m h = 0$ in $B_{4R} \subset \mathbb{R}^n$. For any $x \in B_R$ and $0 < r < R - |x|$ we have*

$$\frac{1}{|B_r|} \int_{B_r(x)} h(z)\,dz = \sum_{i=0}^{m-1} c_i r^{2i} \Delta^i h(x), \tag{16}$$

*where*

$$c_0 = 1, \qquad c_i = c(i, n) > 0 \quad \text{for } i \geq 1.$$

*Moreover, for every $k \geq 0$ there exists $C = C(k, R) > 0$ such that*

$$\|h\|_{C^k(B_R)} \leq C\|h\|_{L^1(B_{4R})}. \tag{17}$$

**Lemma A.3.** *Let $R > 0$ and $B_R \subset \mathbb{R}^n$. Let $u_k \in C^{n-1,\alpha}(\mathbb{R}^n)$ for some $\alpha \in \left(\frac{1}{2}, 1\right)$ be such that*

$$u_k(0) = 0, \quad \|u_k^+\|_{L^\infty(B_R)} \leq C, \quad \|(-\Delta)^{n/2} u_k\|_{L^\infty(B_R)} \leq C, \quad \int_{B_R} |\Delta u_k|\,dx \leq C.$$

*If $n$ is an odd integer, we also assume that $\|\Delta^{(n-1)/2} u_k\|_{L_{1/2}(\mathbb{R}^n)} \leq C$. Then (up to a subsequence) $u_k \to u$ in $C^{n-1}(B_{R/8})$.*

*Proof.* First we prove the lemma for $n$ even.

We write $u_k = w_k + h_k$, where

$$\begin{cases} (-\Delta)^{n/2} w_k = (-\Delta)^{n/2} u_k & \text{in } B_R, \\ \Delta^j w_k = 0 & \text{on } \partial B_R,\ j = 0, 1, \ldots, \frac{1}{2}(n-2). \end{cases}$$

Then by standard elliptic estimates, the $w_k$ are uniformly bounded in $C^{n-1,\beta}(B_R)$. Therefore,

$$|h_k(0)| \leq C, \quad \|h_k^+\|_{L^\infty(B_R)} \leq C, \quad \int_{B_R} |\Delta h_k|\,dx \leq C.$$

Since the $h_k$ are $\frac{n}{2}$-harmonic, the $\Delta h_k$ are $\left(\frac{n}{2}-1\right)$-harmonic in $B_R$, and by (17) we obtain

$$\|\Delta h_k\|_{C^n(B_{R/4})} \leq C\|\Delta h_k\|_{L^1(B_R)} \leq C.$$

Using the identity (16) we have the bound

$$\begin{aligned}\frac{1}{|B_R|}\int_{B_R(0)} h_k^-(z)\,dz &= \frac{1}{|B_R|}\int_{B_R(0)} h_k^+(z)\,dz - \frac{1}{|B_R|}\int_{B_R(0)} h_k(z)\,dz \\ &= \frac{1}{|B_R|}\int_{B_R(0)} h_k^+(z)\,dz - h_k(0) - \sum_{i=1}^{n/2-1} c_i R^{2i}\Delta^i h_k(0) \le C,\end{aligned}$$

and hence

$$\int_{B_R} |h_k(z)|\,dz = \int_{B_R} h_k^+(z)\,dz + \int_{B_R} h_k^-(z)\,dz \le C.$$

Again by (17) we obtain

$$\|h_k\|_{C^n(B_{R/4})} \le C\|h_k\|_{L^1(B_R)} \le C.$$

Thus, the $u_k$ are uniformly bounded in $C^{n-1,\beta}(B_{R/4})$ and (up to a subsequence) $u_k \to u$ in $C^{n-1}(B_{R/4})$ for some $u \in C^{n-1}(B_{R/4})$.

It remains to prove the lemma for $n$ odd.

If $n$ is odd then $\frac{1}{2}(n-1)$ is an integer. We split $\Delta^{(n-1)/2}u_k = w_k + h_k$, where

$$\begin{cases}(-\Delta)^{1/2}w_k = (-\Delta)^{1/2}\Delta^{(n-1)/2}u_k & \text{in } B_R,\\ w_k = 0 & \text{in } B_R^c.\end{cases}$$

Then by Lemmas A.4 and A.5 one has $\|\Delta^{(n-1)/2}u_k\|_{C^{1/2}(B_{R/2})} \le C$. Now one can proceed as in the case of even integer. ☐

**Lemma A.4** [Jin et al. 2015, Proposition 22]. *Let $u \in L_\sigma(\mathbb{R}^n)$ for some $\sigma \in (0,1)$ and $(-\Delta)^\sigma u = 0$ in $B_{2R}$. Then for every $k \in \mathbb{N}$,*

$$\|\nabla^k u\|_{C^0(B_R)} \le C(n,\sigma,k)\frac{1}{R^k}\left(R^{2\sigma}\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|u(x)|}{|x|^{n+2\sigma}}\,dx + \frac{\|u\|_{L^1(B_{2R})}}{R^n}\right),$$

*where $\alpha \in (0,1)$ and $k$ is a nonnegative integer.*

**Lemma A.5** [Ros-Oton and Serra 2014, Proposition 1.1]. *Let $\sigma \in (0,1)$. Let $u$ be a solution of*

$$\begin{cases}(-\Delta)^\sigma u = f & \text{in } B_R,\\ u = 0 & \text{in } B_R^c.\end{cases}$$

*Then*

$$\|u\|_{C^\sigma(\mathbb{R}^n)} \le C(R,\sigma)\|f\|_{L^\infty(B_R)}.$$

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# BOUNDARY ESTIMATES IN ELLIPTIC HOMOGENIZATION

ZHONGWEI SHEN

For a family of systems of linear elasticity with rapidly oscillating periodic coefficients, we establish sharp boundary estimates with either Dirichlet or Neumann conditions, uniform down to the microscopic scale, without smoothness assumptions on the coefficients. Under additional smoothness conditions, these estimates, combined with the corresponding local estimates, lead to the full Rellich-type estimates in Lipschitz domains and Lipschitz estimates in $C^{1,\alpha}$ domains. The $C^\alpha$, $W^{1,p}$, and $L^p$ estimates in $C^1$ domains for systems with VMO coefficients are also studied. The approach is based on certain estimates on convergence rates. As a biproduct, we obtain sharp $O(\varepsilon)$ error estimates in $L^q(\Omega)$ for $q = 2d/(d-1)$ and a Lipschitz domain $\Omega$, with no smoothness assumption on the coefficients.

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## 1. Introduction

The purpose of this paper is to establish sharp boundary estimates with either Dirichlet or Neumann conditions, uniform down to the microscopic scale, for a family of second-order elliptic systems in divergence form with rapidly oscillating coefficients, without any smoothness assumption on the coefficients. Under additional smoothness conditions, these estimates, combined with the corresponding local estimates, lead to the full Rellich-type estimates in Lipschitz domains and Lipschitz estimates in $C^{1,\alpha}$ domains. The $C^\alpha$, $W^{1,p}$, and $L^p$ estimates in $C^1$ domains for systems with VMO coefficients are also investigated. To fix the idea we shall consider the systems of linear elasticity with periodic coefficients in this paper. However, the same results, without the complications introduced by rigid displacements, hold for general second-order elliptic systems with periodic coefficients satisfying the stronger ellipticity

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condition (1-11) (the symmetry condition is also needed for Rellich estimates in Lipschitz domains). We further point out that although we restrict ourselves to the periodic case, our approach, which is based on certain estimates on convergence rates in $H^1$ and $L^2$, extends to nonperiodic settings, provided that the interior correctors or approximate correctors satisfy certain $L^2$ conditions. The compactness methods, which were introduced to the study of homogenization in [Avellaneda and Lin 1987] and have played an important role in establishing regularity results in the periodic setting (see, e.g., [Avellaneda and Lin 1987; 1989; Kenig et al. 2013; Kenig and Prange 2015]), are not used in this paper. As a biproduct of our new approach, we also obtain sharp $O(\varepsilon)$ error estimates in $L^q(\Omega)$ for $q = 2d/(d-1)$ and a Lipschitz domain $\Omega$, with no smoothness assumption on the coefficients.

More precisely, consider the systems of linear elasticity,

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial}{\partial x_j}\right], \quad \varepsilon > 0. \tag{1-1}$$

We will assume that $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \le i, j, \alpha, \beta \le d$ is real, bounded measurable, and satisfies the elasticity condition

$$\begin{gathered} a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y) = a_{\alpha j}^{i\beta}(y), \\ \kappa_1|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \le \kappa_2|\xi|^2 \end{gathered} \tag{1-2}$$

for a.e. $y \in \mathbb{R}^d$ and for any symmetric matrix $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times d}$, where $\kappa_1, \kappa_2 > 0$ (the summation convention is used throughout the paper). We will also assume that $A(y)$ is 1-periodic; i.e.,

$$A(y+z) = A(y) \quad \text{for a.e. } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \tag{1-3}$$

**Theorem 1.1.** *Suppose that $A$ satisfies conditions (1-2)–(1-3). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be the weak solution to the Dirichlet problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \qquad \text{and} \qquad u_\varepsilon = f \quad \text{on } \partial\Omega, \tag{1-4}$$

*where $F \in L^p(\Omega; \mathbb{R}^d)$ for $p = 2d/(d+1)$ and $f \in H^1(\partial\Omega; \mathbb{R}^d)$. Then, for $\varepsilon \le r < \operatorname{diam}(\Omega)$,*

$$\left\{\frac{1}{r}\int_{\Omega_r} |\nabla u_\varepsilon|^2\right\}^{1/2} \le C\big\{\|F\|_{L^p(\Omega)} + \|f\|_{H^1(\partial\Omega)}\big\}, \tag{1-5}$$

*where $\Omega_r = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < r\}$. The constant $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and the Lipschitz character of $\Omega$.*

Let $\mathcal{R}$ denote the space of rigid displacements,

$$\mathcal{R} = \big\{Mx + q : M^T = -M \in \mathbb{R}^{d\times d} \text{ and } q \in \mathbb{R}^d\big\}, \tag{1-6}$$

where $(Mx)^\alpha = M_i^\alpha x_i$ and $M^T$ denotes the transpose of matrix $M$. By $u \perp \mathcal{R}$ we mean $u \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$, i.e., $\int_\Omega u \cdot \phi = 0$ for any $\phi \in \mathcal{R}$. We will use $\partial u_\varepsilon/\partial \nu_\varepsilon$ to denote the conormal derivative of $u_\varepsilon$ associated with $\mathcal{L}_\varepsilon$.

**Theorem 1.2.** *Suppose that $A$ and $\Omega$ satisfy the same conditions as in Theorem 1.1. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be a weak solution to the Neumann problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \partial\Omega, \tag{1-7}$$

*where $F \in L^p(\Omega; \mathbb{R}^d)$ for $p = 2d/(d+1)$, $g \in L^2(\partial\Omega; \mathbb{R}^d)$ and $\int_\Omega F \cdot \phi + \int_{\partial\Omega} g \cdot \phi = 0$ for any $\phi \in \mathcal{R}$. Also assume that $u_\varepsilon \perp \mathcal{R}$. Then, for $\varepsilon \le r < \operatorname{diam}(\Omega)$,*

$$\left\{ \frac{1}{r} \int_{\Omega_r} |\nabla u_\varepsilon|^2 \right\}^{1/2} \le C \big\{ \|F\|_{L^p(\Omega)} + \|g\|_{L^2(\partial\Omega)} \big\}, \tag{1-8}$$

*where $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and the Lipschitz character of $\Omega$.*

Estimates (1-5) and (1-8), which are scaling-invariant, may be regarded as the Rellich estimates, uniform down to the scale $\varepsilon$, in Lipschitz domains for the elasticity operators $\mathcal{L}_\varepsilon$. Indeed, if the coefficient matrix $A$ is constant, then (1-5) and (1-8) hold for any $0 < r < \operatorname{diam}(\Omega)$. Suppose that $F = 0$ and $u_\varepsilon \in C^1(\overline{\Omega}; \mathbb{R}^d)$. By letting $r \to 0$, one recovers the full Rellich estimates in Lipschitz domains,

$$\|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} \le C \|u_\varepsilon\|_{H^1(\partial\Omega)} \quad \text{and} \quad \|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} \le C \left\| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right\|_{L^2(\partial\Omega)}, \tag{1-9}$$

which were proved in [Fabes et al. 1988; Dahlberg et al. 1988] for second-order elliptic systems with constant coefficients, using integration by parts (see [Kenig 1994] for references on related work on boundary value problems in Lipschitz domains). We should note that our proof of Theorems 1.1 and 1.2 uses the nontangential maximal function estimates in [Dahlberg et al. 1988]. On the other hand, under certain smoothness conditions on $A$, the Rellich estimates hold for the operator $\mathcal{L}_1$ on Lipschitz domains with $\operatorname{diam}(\Omega) \le 1$. By a blow-up argument as well as some localization procedures, this implies

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} &\le C \big\{ \|\nabla_{\tan} u_\varepsilon\|_{L^2(\partial\Omega)} + \varepsilon^{-1/2} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \big\}, \\ \|\nabla u_\varepsilon\|_{L^2(\partial\Omega)} &\le C \left\{ \left\| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right\|_{L^2(\partial\Omega)} + \varepsilon^{-1/2} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \right\}, \end{aligned} \tag{1-10}$$

where $\nabla_{\tan} u_\varepsilon$ denotes the tangential derivative of $u_\varepsilon$ on $\partial\Omega$. We emphasize that the estimates (1-10) are local and structure conditions such as periodicity are not needed. However, with the additional periodicity condition, one may combine the local estimates (1-10) with the estimates in Theorems 1.1 and 1.2 to obtain the full Rellich estimate (1-9), uniform in $\varepsilon$, for operators $\mathcal{L}_\varepsilon$ (see Remark 3.1). Thus we have been able to completely separate the large-scale regularity due to homogenization from the small-scale regularity due to smoothness of the coefficients.

Under the periodicity condition and the Hölder continuity condition on $A$, the uniform Rellich estimates (1-9) were proved in [Kenig and Shen 2011a; 2011b] for a family of elliptic operators $\{\mathcal{L}_\varepsilon\}$, where $\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla)$ and $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \le i, j \le d$ and $1 \le \alpha, \beta \le m$ satisfies the ellipticity condition

$$\mu|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha \xi_j^\beta \le \frac{1}{\mu}|\xi|^2 \tag{1-11}$$

for $y \in \mathbb{R}^d$ and $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times m}$ as well as the symmetry condition $A^* = A$, i.e., $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$. The results were used to establish the uniform solvability of the $L^2$ Dirichlet, regularity, and Neumann problems for the system $\mathcal{L}_\varepsilon(u_\varepsilon)=0$ in Lipschitz domains. It is worth pointing out that the Rellich estimates (1-9) are not accessible by compactness methods. One of the key steps in [Kenig and Shen 2011a; 2011b] uses integration by parts and relies on the observation that $\mathcal{L}_1(Q)=Q(\mathcal{L}_1)$, where

$$Q(u)(x',x_d)=u(x',x_d+1)-u(x',x_d).$$

As a result, the approach does not seem to apply if the coefficients are not periodic. We mention that even with periodic coefficients, the direct extension of the methods used in [Kenig and Shen 2011a; 2011b] is problematic for the system of elasticity, due to the weaker ellipticity condition and the lack of (uniform) Korn inequalities on boundary layers.

In this paper we develop a new approach to uniform boundary regularity in quantitative homogenization of elliptic equations and systems. Let $u_0$ denote the solution of the boundary value problem for the homogenized system with the same data. The basic idea is to consider the function

$$w_\varepsilon=u_\varepsilon-u_0-\varepsilon\chi_j^\beta(x/\varepsilon)K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right) \tag{1-12}$$

in $\Omega$, where $\chi=(\chi_j^\beta)$ denotes the matrix of correctors, $K_\varepsilon^2=K_\varepsilon\circ K_\varepsilon$ with $K_\varepsilon$ being a smoothing operator at scale $\varepsilon$, and $\eta_\varepsilon\in C_0^\infty(\Omega)$ is a cut-off function with support in $\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)\ge 3\varepsilon\}$. Using energy estimates for the operator $\mathcal{L}_\varepsilon$ as well as sharp boundary regularity estimates for $u_0$, we are able to bound

$$\varepsilon^{-1/2}\|w_\varepsilon\|_{H^1(\Omega)}$$

by the right-hand sides of estimates (1-5) and (1-8), respectively. This, together with sharp estimates for $u_0$, yields the desired estimates for

$$r^{-1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega_r)}$$

for $\varepsilon\le r<\operatorname{diam}(\Omega)$. We mention that since $\mathcal{L}_0$ has constant coefficients, the sharp boundary estimates in Lipschitz domains in terms of nontangential maximal functions are known [Fabes et al. 1988; Dahlberg et al. 1988]. Also, because of the use of the smoothing operator $K_\varepsilon$, which is motivated by [Pastukhova 2006; Suslina 2013a] (also see [Griso 2004; Onofrei and Vernescu 2007; Kenig et al. 2012; Suslina 2013b]), we only need to assume that

$$\sup_{x\in\mathbb{R}^d}\int_{B(x,1)}\bigl(|\chi(y)|^2+|\nabla\chi(y)|^2\bigr)\,dy<\infty,$$

and that a similar estimate holds for a dual corrector $\phi=(\phi_{kij}^{\alpha\beta})$ (see (2-5) for its definition). As such, it is possible to extend the approach to the almost-periodic or other nonperiodic settings. We plan to carry out this study in a separate work.

As we mentioned before, the estimates in Theorems 1.1 and 1.2 may be used to establish uniform solvability of $L^2$ boundary value problems for $\mathcal{L}_\varepsilon$ in Lipschitz domains [Kenig and Shen 2011a; 2011b]. They can also be used to obtain sharp $O(\varepsilon)$ error estimates in $L^q(\Omega)$ for $q=2d/(d-1)$ and a Lipschitz domain $\Omega$, with no smoothness assumption on the coefficients.

**Theorem 1.3.** *Suppose that $A$ and $\Omega$ satisfy the same conditions as in Theorem 1.1. Let $u_\varepsilon$ be a weak solution to (1-4) or (1-7), and $u_0$ the weak solution of the homogenized system with the same data. Suppose that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. In the case of the Neumann problem (1-7) we further assume that $u_\varepsilon, u_0 \perp \mathcal{R}$. Then*

$$\|u_\varepsilon - u_0\|_{L^q(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{1-13}$$

*where $q = p' = 2d/(d-1)$ and $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and $\Omega$.*

We remark that if $\Omega$ is $C^2$ and $u_\varepsilon = 0$ or $\partial u_\varepsilon/\partial \nu_\varepsilon = 0$ on $\partial\Omega$, the $O(\varepsilon)$ estimate

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|F\|_{L^2(\Omega)} \tag{1-14}$$

was proved in [Suslina 2013a; 2013b] for a broader class of elliptic operators with measurable periodic coefficients, which contains the systems of elasticity considered here (also see [Griso 2004; Onofrei and Vernescu 2007; Kenig et al. 2012; 2014] and their references for related work on convergence rates). Note that $q = 2d/(d-1) > 2$ and $\|u_0\|_{H^2(\Omega)} \leq C\|F\|_{L^2(\Omega)}$ if $\Omega$ is $C^2$ and $\mathcal{L}_0(u_0) = F$ in $\Omega$ with $u_0 = 0$ or $\partial u_0/\partial \nu_0 = 0$ on $\partial\Omega$. Thus our estimate (1-13) is stronger than (1-14). In the case of scalar elliptic equations with Dirichlet condition $u_\varepsilon = 0$ on $\partial\Omega$, it is known that $\|u_\varepsilon - u_0\|_{L^q(\Omega)} \leq C\varepsilon\|F\|_{L^p(\Omega)}$, where $1 < p < d$ and $1/q = 1/p - 1/d$ (see [Kenig et al. 2014, p. 1234]). Although the exponent $q = 2d/(d-1)$ may not be sharp, Theorem 1.3 seems to be the first result on the sharp $O(\varepsilon)$ estimate of $u_\varepsilon - u_0$ in $L^q(\Omega)$ with $q > 2$ for elliptic systems with bounded measurable periodic coefficients.

As we indicated above, the proof of Theorems 1.1 and 1.2 only uses the energy estimates in $L^2$ for $\mathcal{L}_\varepsilon$ and thus requires no smoothness assumptions on the coefficients. In the second part of this paper we apply the similar ideas in the $L^p$ setting for $1 < p < \infty$. To do this we first establish the $W^{1,p}$ estimates for the systems

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(h) \quad \text{in } \Omega, \tag{1-15}$$

where $h = (h_i^\alpha) \in L^p(\Omega; \mathbb{R}^{d\times d})$, with either the Dirichlet or Neumann boundary conditions, under the additional assumptions that $\Omega$ is $C^1$ and $A = A(y)$ belongs to $\mathrm{VMO}(\mathbb{R}^d)$. As a result, the $L^p$ analogues of estimates (1-5) and (1-8) are proved under these additional conditions, which are more or less sharp. Consequently, by combining the $L^p$ estimates on the boundary layer $\Omega_\varepsilon$ with local estimates for $\mathcal{L}_1$, which hold for Hölder continuous coefficients, we may obtain the uniform Rellich estimates in $L^p$ for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $C^1$ domains under the assumptions that $A$ is Hölder continuous and satisfies (1-2)–(1-3). By the method of layer potentials, this will lead to the uniform solvability of the $L^p$ Dirichlet, regularity, and Neumann problems in $C^1$ domains (details will be provided in a separate work). Previously, these results in $L^p$ are known only in $C^{1,\alpha}$ domains for operators $\mathcal{L}_\varepsilon$ with Hölder continuous coefficients satisfying (1-11) and $A^* = A$ [Kenig et al. 2013]. We remark that the $W^{1,p}$ estimates (local or global) for operators with nonsmooth coefficients in nonsmooth domains are of interest in their own rights and have been studied extensively in recent years (see [Caffarelli and Peral 1998; Auscher and Qafsaoui 2002; Wang 2003; Byun and Wang 2004; 2005; Shen 2005; 2008; Krylov 2007; Dong and Kim 2010; Kenig et al. 2013; Geng 2012; Geng et al. 2012] and their references). Our approach to the $W^{1,p}$ estimates is based on a real-variable argument, which originated in [Caffarelli and Peral 1998] and further developed in [Wang 2003; Shen 2005; 2007]. The required (weak) reverse Hölder estimates at the boundary are proved by combining the interior Lipschitz estimates down to the scale $\varepsilon$ with boundary $C^\alpha$ estimates.

Theorems 1.1 and 1.2 as well as their $L^p$ analogues, given in Section 7, are the main contributions of this paper. For a comprehensive study in the boundary regularity for $\mathcal{L}_\varepsilon$, in Sections 8 and 9, we investigate the boundary Lipschitz estimates, uniform down to the scale $\varepsilon$, for solutions in $C^{1,\alpha}$ domains with the Dirichlet or Neumann conditions. Let

$$\begin{aligned} D_r &= \big\{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \psi(x') < x_d < \psi(x') + r\big\}, \\ \Delta_r &= \big\{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } x_d = \psi(x')\big\}, \end{aligned} \tag{1-16}$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a $C^{1,\alpha}$ function for some $\alpha > 0$ with $\psi(0) = 0$ and $\|\nabla\psi\|_{C^\alpha(\mathbb{R}^{d-1})} \le M$.

**Theorem 1.4.** *Suppose that $A$ satisfies conditions (1-2)–(1-3). Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution to*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } D_1 \qquad \text{and} \qquad u_\varepsilon = f \quad \text{on } \Delta_1. \tag{1-17}$$

*Then, for $\varepsilon \le r < 1$,*

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left\{\left(\fint_{D_1} |\nabla u_\varepsilon|^2\right)^{1/2} + \|f\|_{C^{1,\sigma}(\Delta_1)} + \|F\|_{L^p(D_1)}\right\}, \tag{1-18}$$

*where $p > d$ and $\sigma \in (0, \alpha)$. The constant $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, $p$, $\sigma$, and $(\alpha, M)$.*

**Theorem 1.5.** *Suppose that $A$ satisfies (1-2)–(1-3). Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution to*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } D_1 \qquad \text{and} \qquad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on } \Delta_1. \tag{1-19}$$

*Then, for $\varepsilon \le r < 1$,*

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left\{\left(\fint_{D_1} |\nabla u_\varepsilon|^2\right)^{1/2} + \|g\|_{C^{\sigma}(\Delta_1)} + \|F\|_{L^p(D_1)}\right\}, \tag{1-20}$$

*where $p > d$ and $\sigma \in (0, \alpha)$. The constant $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, $p$, $\sigma$, and $(\alpha, M)$.*

As in the case of Rellich estimates, under additional smoothness conditions on $A$, using local Lipschitz estimates for $\mathcal{L}_1$ and a blow-up argument, one may derive from Theorems 1.4 and 1.5 the full boundary Lipschitz estimates

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \le C\left\{\left(\fint_{D_1} |u_\varepsilon|^2\right)^{1/2} + \|f\|_{C^{1,\sigma}(\Delta_1)} + \|F\|_{L^p(D_1)}\right\} \tag{1-21}$$

for solutions of (1-17), and

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \le C\left\{\left(\fint_{D_1} |u_\varepsilon|^2\right)^{1/2} + \|g\|_{C^{\sigma}(\Delta_1)} + \|F\|_{L^p(D_1)}\right\} \tag{1-22}$$

for solutions of (1-19). We remark that for elliptic systems satisfying the ellipticity condition (1-11), the periodicity condition (1-3) and the Hölder continuity condition, the estimate (1-21) was proved in [Avellaneda and Lin 1987], while (1-22) was established in [Kenig et al. 2013] under the additional symmetry condition $A^* = A$. This symmetry condition was removed recently in [Armstrong and Shen 2016]. However, our estimates in Theorems 1.4 and 1.5 are new for the system of elasticity.

Our proof of Theorems 1.4 and 1.5 also uses the function $w_\varepsilon$, given by (1-12). As a consequence of its estimates in $L^2$, for each $r \in \left(\varepsilon, \frac{1}{4}\right)$, we are able to construct a function $v$ such that $\mathcal{L}_0(v) = F$ in $D_r$ with the same (Dirichlet or Neumann) data on $\Delta_r$ as $u_\varepsilon$, and

$$\left(\fint_{D_r} |u_\varepsilon - v|^2\right)^{1/2} \le C(\varepsilon/r)^{1/2}\left\{\left(\fint_{D_{2r}} |u_\varepsilon|^2\right)^{1/2} + \text{terms involving given data}\right\}.$$

This allows us to use a general scheme for establishing Lipschitz estimates down to the scale $\varepsilon$, which was formulated recently in [Armstrong and Smart 2016] and used for interior estimates in stochastic homogenization with random coefficients (also see [Armstrong and Mourrat 2016] as well as related work in [Gloria and Otto 2011; 2012; Gloria et al. 2014; 2015]). Our argument is similar to (and somewhat simpler and more transparent than) that in [Armstrong and Shen 2016], where the scheme was adapted to prove the full boundary Lipschitz estimates for second-order elliptic systems with almost-periodic and Hölder continuous coefficients. As indicated earlier, we have been able to completely avoid the use of compactness methods (even in the case of $C^\alpha$ estimates). Although it is possible to prove the interior Lipschitz estimates as well as the boundary $C^\alpha$ estimates, down to the scale $\varepsilon$ without smoothness, by the compactness methods, as demonstrated in [Avellaneda and Lin 1987; Gu and Shen 2015], the compactness methods for boundary Lipschitz estimates require the same estimates for boundary correctors, which are not easy to establish [Avellaneda and Lin 1987; Kenig et al. 2013].

The paper is organized as follows. In Section 2 we establish some key convergence results in $H^1$. These results are used in Section 3 to prove Theorems 1.1 and 1.2. In Section 4 we study the convergence rates in $L^q$ for $q = 2d/(d-1)$ and give the proof of Theorem 1.3, which uses the estimates in Theorems 1.1 and 1.2 as well as a duality argument. In Sections 5 and 6 we obtain the boundary $C^\alpha$ and $W^{1,p}$ estimates, respectively, in $C^1$ domains for operators with VMO coefficients. These estimates are used in Section 7 to establish the $L^p$ analogues of (1-5) and (1-8) in $C^1$ domains. Finally, Theorem 1.4 is proved in Section 8, and Section 9 contains the proof of Theorem 1.5.

Throughout the paper we use $\fint_E u = (1/|E|) \int_E u$ to denote the average of $u$ over the set $E$. We will use $C$ and $c$ to denote constants that may depend on $d$, $\kappa_1$, $\kappa_2$, $A$ and $\Omega$, but never on $\varepsilon$.

## 2. Convergence rates in $H^1$

In this section we establish certain results on convergence rates in $H^1$, which will play a crucial role in the proof of our main results. Throughout the section we assume that $A = A(y)$ satisfies (1-2)–(1-3) and $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$.

Let $\chi = (\chi_j^\beta(y)) = (\chi_j^{\alpha\beta}(y))$ denote the matrix of correctors for $\mathcal{L}_\varepsilon$, where $1 \le j, \alpha, \beta \le d$. This means that $\chi_j^\beta \in H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ is 1-periodic, $\int_Y \chi_j^\beta = 0$, and

$$\mathcal{L}_1(\chi_j^\beta) = -\mathcal{L}_1(P_j^\beta) \quad \text{in } \mathbb{R}^d, \tag{2-1}$$

where $Y = [0, 1)^d$ and $P_j^\beta = y_j(0, \dots, 1, \dots, 0)$ with 1 in the $\beta$-th position. The homogenized operator is given by $\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla)$, where $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$ is the matrix of effective coefficients with

$$\hat{a}_{ij}^{\alpha\beta} = \fint_Y \left\{ a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k}(\chi_j^{\gamma\beta}) \right\}. \tag{2-2}$$

It is known that the constant matrix $\hat{A}$ satisfies the elasticity condition (1-2) [Oleĭnik et al. 1992; Jikov et al. 1994]. Define

$$b_{ij}^{\alpha\beta}(y) = a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k}(\chi_j^{\gamma\beta}) - \hat{a}_{ij}^{\alpha\beta}. \tag{2-3}$$

By the definition of $\hat{A}$ and (2-1),

$$\int_Y b_{ij}^{\alpha\beta} = 0 \quad \text{and} \quad \frac{\partial}{\partial y_i}(b_{ij}^{\alpha\beta}) = 0. \tag{2-4}$$

It follows that there exist $\phi_{kij}^{\alpha\beta} \in H^1_{\text{loc}}(\mathbb{R}^d)$ such that $\phi_{kij}^{\alpha\beta}$ is 1-periodic,

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k}(\phi_{kij}^{\alpha\beta}) \quad \text{and} \quad \phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta} \tag{2-5}$$

(see, e.g., [Jikov et al. 1994; Kenig et al. 2012]).

Fix $\varphi \in C_0^\infty\big(B\big(0, \tfrac{1}{4}\big)\big)$ such that $\varphi \ge 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. Define

$$K_\varepsilon(f)(x) = f * \varphi_\varepsilon(x) = \int_{\mathbb{R}^d} f(x-y)\varphi_\varepsilon(y)\,dy, \tag{2-6}$$

where $\varphi_\varepsilon(y) = \varepsilon^{-d}\varphi(y/\varepsilon)$.

**Lemma 2.1.** *Let $f \in L^p(\mathbb{R}^d)$ for some $1 \le p < \infty$. Then for any $g \in L^p_{\text{loc}}(\mathbb{R}^d)$,*

$$\|g(x/\varepsilon)K_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \le C \sup_{x\in\mathbb{R}^d} \left( ⨍_{B(x,1)} |g|^p \right)^{1/p} \|f\|_{L^p(\mathbb{R}^d)}, \tag{2-7}$$

*where $C$ depends only on $d$.*

*Proof.* By Hölder's inequality,

$$|K_\varepsilon(f)(x)|^p \le \frac{C}{|B(0,\varepsilon)|} \int_{\mathbb{R}^d} |f(y)|^p \chi_{B(x,\varepsilon)}(y)\,dy,$$

from which the estimate (2-7) follows readily by Fubini's theorem. □

It follows from (2-7) that if $g \in L^p_{\text{loc}}(\mathbb{R}^d)$ and is 1-periodic, then

$$\|g(x/\varepsilon)K_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \le C\|g\|_{L^p(Y)}\|f\|_{L^p(\mathbb{R}^d)}. \tag{2-8}$$

**Lemma 2.2.** *Let $f \in W^{1,q}(\mathbb{R}^d)$ for some $1 < q < \infty$. Then*

$$\|K_\varepsilon(f) - f\|_{L^q(\mathbb{R}^d)} \le C\varepsilon\|\nabla f\|_{L^q(\mathbb{R}^d)}. \tag{2-9}$$

*Moreover, if $p = 2d/(d+1)$,*

$$\begin{aligned} \|K_\varepsilon(f)\|_{L^2(\mathbb{R}^d)} &\le C\varepsilon^{-1/2}\|f\|_{L^p(\mathbb{R}^d)}, \\ \|f - K_\varepsilon(f)\|_{L^2(\mathbb{R}^d)} &\le C\varepsilon^{1/2}\|\nabla f\|_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{2-10}$$

*The constant $C$ depends only on $d$.*

*Proof.* To see (2-9), we note that

$$\|f(\cdot - y) - f(\cdot)\|_{L^q(\mathbb{R}^d)} \le |y|\, \|\nabla f\|_{L^q(\mathbb{R}^d)}$$

for any $y \in \mathbb{R}^d$. Thus, by Minkowski's inequality,

$$\begin{aligned}\|K_\varepsilon(f) - f\|_{L^q(\mathbb{R}^d)} &\le \int_{\mathbb{R}^d} \varphi_\varepsilon(y)\|f(\cdot - y) - f(\cdot)\|_{L^q(\mathbb{R}^d)}\, dy \\ &\le \int_{\mathbb{R}^d} \varphi_\varepsilon(y)|y|\, dy\, \|\nabla f\|_{L^q(\mathbb{R}^d)} \\ &= C\varepsilon\|\nabla f\|_{L^q(\mathbb{R}^d)}.\end{aligned}$$

Next, by Parseval's theorem and Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^d} |K_\varepsilon(f)|^2\, dx &= \int_{\mathbb{R}^d} |\hat{\varphi}(\varepsilon\xi)|^2 |\hat{f}(\xi)|^2\, d\xi \\ &\le \left(\int_{\mathbb{R}^d} |\hat{\varphi}(\varepsilon\xi)|^{2d}\, d\xi\right)^{1/d} \|\hat{f}\|^2_{L^{p'}(\mathbb{R}^d)} \\ &\le C\varepsilon^{-1}\|f\|^2_{L^p(\mathbb{R}^d)},\end{aligned}$$

where $\hat{f}$ denotes the Fourier transform of $f$, and we have used the Hausdorff–Young inequality $\|\hat{f}\|_{L^{p'}(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)}$. This gives the first inequality in (2-10). To see the second inequality, we note that $\hat{\varphi}(0) = \int_{\mathbb{R}^d} \varphi = 1$. It follows that

$$\begin{aligned}\|f - K_\varepsilon(f)\|_{L^2(\mathbb{R}^d)} &\le C\left\{\int_{\mathbb{R}^d} |\hat{\varphi}(\varepsilon\xi) - \hat{\varphi}(0)|^{2d}|\xi|^{-2d}\, d\xi\right\}^{1/(2d)} \|\widehat{\nabla f}\|_{L^{p'}(\mathbb{R}^d)} \\ &\le C\varepsilon^{1/2}\|\nabla f\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

where we have used $|\hat{\varphi}(\xi) - \hat{\varphi}(0)| \le C|\xi|$ for the last step. □

**Lemma 2.3.** *Let $u_\varepsilon, u_0 \in H^1(\Omega; \mathbb{R}^d)$. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$ in $\Omega$ and either $u_\varepsilon = u_0$ or $\partial u_\varepsilon/\partial \nu_\varepsilon = \partial u_0/\partial \nu_0$ on $\partial\Omega$. Let*

$$w_\varepsilon^\alpha = u_\varepsilon^\alpha - u_0^\alpha - \varepsilon\chi_j^{\alpha\beta}(x/\varepsilon)K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right),$$

*where $K_\varepsilon^2 = K_\varepsilon \circ K_\varepsilon$, $\eta_\varepsilon \in C_0^\infty(\Omega)$ and* $\operatorname{supp}(\eta_\varepsilon) \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge 3\varepsilon\}$. *Then*

$$\begin{aligned}\int_\Omega A(x/\varepsilon)\nabla w_\varepsilon \cdot \nabla w_\varepsilon\, dx &= \int_\Omega [\hat{A} - A(x/\varepsilon)][\nabla u_0 - K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)] \cdot \nabla w_\varepsilon\, dx \\ &\quad - \int_\Omega B(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon\, dx \\ &\quad - \varepsilon\int_\Omega A(x/\varepsilon)\chi(x/\varepsilon)\nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon\, dx, \end{aligned} \tag{2-11}$$

*where $B(y) = (b_{ij}^{\alpha\beta}(y))$ is defined in (2-3).*

*Proof.* We first note that if $u_\varepsilon = u_0$ on $\partial\Omega$, then $w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^d)$, as $K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \in C_0^\infty(\Omega)$. Since $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$ in $\Omega$, it follows that

$$\int_\Omega A(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla w_\varepsilon \, dx = \int_\Omega \hat{A}\nabla u_0 \cdot \nabla w_\varepsilon \, dx. \tag{2-12}$$

In the case of the Neumann condition $\partial u_\varepsilon/\partial\varepsilon = \partial u_0/\partial\nu_0$ on $\partial\Omega$, equation (2-12) continues to hold. This is because $w_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ and both sides of (2-12) are equal to

$$\langle \mathcal{L}_0(u_0), w_\varepsilon \rangle_{(H^1(\Omega))' \times H^1(\Omega)} + \left\langle \frac{\partial u_0}{\partial \nu_0}, w_\varepsilon \right\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}.$$

Using (2-12), we obtain

$$\begin{aligned}\int_\Omega A(x/\varepsilon)\nabla w_\varepsilon \cdot \nabla w_\varepsilon \, dx = \int_\Omega [\hat{A} - A(x/\varepsilon)]\nabla u_0 \cdot \nabla w_\varepsilon \, dx \\ - \int_\Omega A(x/\varepsilon)\nabla \chi(x/\varepsilon) K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon \, dx \\ - \varepsilon \int_\Omega A(x/\varepsilon)\chi(x/\varepsilon)\nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon \, dx,\end{aligned}$$

from which the formal (2-11) follows by the definition of $B(y)$. □

**Lemma 2.4.** *Let $\phi(y) = (\phi_{kij}^{\alpha\beta}(y))$ be defined by (2-5). Then*

$$\int_\Omega B(x/\varepsilon) K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon \, dx = -\varepsilon \int_\Omega \phi_{kij}^{\alpha\beta}(x/\varepsilon) \frac{\partial w_\varepsilon^\alpha}{\partial x_i} \cdot \frac{\partial}{\partial x_k} K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right) dx. \tag{2-13}$$

*Proof.* Using (2-5), we see that

$$\begin{aligned}B(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon) \cdot \nabla w_\varepsilon &= b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right) \cdot \frac{\partial w_\varepsilon^\alpha}{\partial x_i} \\ &= \varepsilon \frac{\partial}{\partial x_k}\left(\phi_{kij}^{\alpha\beta}(x/\varepsilon)\right) K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right) \cdot \frac{\partial w_\varepsilon^\alpha}{\partial x_i} \\ &= \varepsilon \frac{\partial}{\partial x_k}\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial w_\varepsilon^\alpha}{\partial x_i}\right\} K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right),\end{aligned}$$

from which equation (2-13) follows readily. □

**Lemma 2.5.** *Let $u_\varepsilon$ ($\varepsilon \geq 0$) be a solution to the Dirichlet problem (1-4) or the Neumann problem (1-7). Let $w_\varepsilon$ be defined as in Lemma 2.3 with $\eta_\varepsilon$ satisfying*

$$\begin{cases}\eta_\varepsilon \in C_0^\infty(\Omega), \quad 0 \leq \eta \leq 1, \\ \operatorname{supp}(\eta_\varepsilon) \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq 3\varepsilon\}, \\ \eta_\varepsilon = 1 \quad \text{on } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq 4\varepsilon\}, \\ |\nabla \eta_\varepsilon| \leq C\varepsilon^{-1}.\end{cases} \tag{2-14}$$

*Then*

$$\left| \int_\Omega A(x/\varepsilon)\nabla w_\varepsilon \cdot \nabla w_\varepsilon \, dx \right| \le C\|\nabla w_\varepsilon\|_{L^2(\Omega)}\big\{\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} + \|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\big\}. \quad (2\text{-}15)$$

*Proof.* It follows from Lemmas 2.3 and 2.4 by the Cauchy inequality that

$$\begin{aligned}\left| \int_\Omega A(x/\varepsilon)\nabla w_\varepsilon \cdot \nabla w_\varepsilon \, dx \right| &\le C\|\nabla w_\varepsilon\|_{L^2(\Omega)}\big\{\|\nabla u_0 - K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon\|\chi(x/\varepsilon)\nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\qquad + \varepsilon\|\phi(x/\varepsilon)\nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\big\} \\ &\le C\|\nabla w_\varepsilon\|_{L^2(\Omega)}\big\{\|\nabla u_0 - K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon\|\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\big\},\end{aligned}$$

where we have used Lemma 2.1 as well as the fact that $\chi, \phi \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and are 1-periodic for the last inequality. Observe that

$$\begin{aligned}\|\nabla u_0 - K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} &\le \|(\nabla u_0)(1-\eta_\varepsilon)\|_{L^2(\Omega)} + \|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\qquad + \left\| K_\varepsilon\big((u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\big)\right\|_{L^2(\Omega)} \\ &\le \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} + C\|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)}.\end{aligned}$$

Also,

$$\begin{aligned}\varepsilon\|\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} &\le \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon\|K_\varepsilon((\nabla u_0)(\nabla\eta_\varepsilon))\|_{L^2(\Omega)} \\ &\le \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} + C\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}.\end{aligned}$$

□

Finally, we are in a position to state and prove the main result of this section.

**Theorem 2.6.** *Suppose that $A(y)$ satisfies (1-2)–(1-3). Let $\Omega$ be a bounded Lipschitz domain. Let $u_\varepsilon$ ($\varepsilon \ge 0$) be the solutions to the Dirichlet problem (1-4) in $\Omega$ with $f \in H^1(\partial\Omega; \mathbb{R}^d)$ and $F \in L^p(\Omega; \mathbb{R}^d)$, where $p = 2d/(d+1)$. Then*

$$\left\| u_\varepsilon - u_0 - \varepsilon\chi_j^\beta(x/\varepsilon)K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)\right\|_{H_0^1(\Omega)} \le C\varepsilon^{1/2}\big\{\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}, \quad (2\text{-}16)$$

*where $\eta_\varepsilon \in C_0^\infty(\Omega)$ satisfies (2-14). The constant $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and the Lipschitz character of $\Omega$.*

*Proof.* Let $w_\varepsilon$ denote the function on the left-hand side of (2-16). Since $w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^d)$, it follows from (2-15) by the first Korn inequality [Oleĭnik et al. 1992] that

$$\|w_\varepsilon\|_{H_0^1(\Omega)} \le C\big\{\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} + \|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\big\}. \quad (2\text{-}17)$$

To bound the right-hand side of (2-17), we write $u_0 = v + h$, where

$$v(x) = \int_\Omega \Gamma_0(x-y)F(y)\,dy$$

and $\Gamma_0(x)$ denotes the matrix of fundamental solutions for the homogenized operator $\mathcal{L}_0$ in $\mathbb{R}^d$, with pole at the origin. Note that $\mathcal{L}_0(v) = F$ in $\Omega$, and by the well known singular integral and fractional integral estimates,

$$\|\nabla^2 v\|_{L^p(\mathbb{R}^d)} + \|\nabla v\|_{L^{p'}(\mathbb{R}^d)} \le C_p \|F\|_{L^p(\Omega)}, \tag{2-18}$$

where we have used the observation $1/p' = 1/p - 1/d$. Let $\boldsymbol{e} = (e_1, \dots, e_d) \in C_0^1(\mathbb{R}^d; \mathbb{R}^d)$ be a vector field such that $\langle \boldsymbol{e}, n\rangle \ge c_0 > 0$ on $\partial\Omega$ and $|\nabla \boldsymbol{e}| \le C r_0^{-1}$, where $r_0 = \operatorname{diam}(\Omega)$ and $n$ denotes the outward unit normal to $\partial\Omega$. It follows from the divergence theorem that

$$\begin{aligned} c_0 \int_{\partial\Omega} |\nabla v|^2 \, d\sigma &\le \int_{\partial\Omega} |\nabla v|^2 \langle \boldsymbol{e}, n\rangle \, d\sigma \\ &= \int_\Omega |\nabla v|^2 \operatorname{div}(\boldsymbol{e}) \, dx + \int_\Omega e_i \frac{\partial}{\partial x_i} \nabla v \cdot \nabla v \, dx \\ &\le C \biggl\{ r_0^{-1} \int_\Omega |\nabla v|^2 \, dx + \int_\Omega |\nabla v| \, |\nabla^2 v| \, dx \biggr\} \\ &\le C \bigl\{ r_0^{-1} \|\nabla v\|^2_{L^2(\Omega)} + \|\nabla v\|_{L^{p'}(\Omega)} \|\nabla^2 v\|_{L^p(\Omega)} \bigr\} \le C \|F\|^2_{L^p(\Omega)}, \end{aligned} \tag{2-19}$$

where we have used (2-18) for the last step. Note that the same argument also gives $\|\nabla v\|_{L^2(S_t)} \le C\|F\|_{L^p(\Omega)}$, where $S_t = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) = t\}$ for $0 < t < c r_0$. Consequently, by the coarea formula, we obtain

$$\biggl\{ \frac{1}{r} \int_{\widetilde{\Omega}_r} |\nabla v|^2 \, dx \biggr\}^{1/2} \le C \|F\|_{L^p(\Omega)}, \tag{2-20}$$

where $0 < r < \operatorname{diam}(\Omega)$ and $\widetilde{\Omega}_r = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) < r\}$.

Next, we observe that $\mathcal{L}_0(h) = 0$ in $\Omega$ and

$$\begin{aligned} \|h\|_{H^1(\partial\Omega)} &\le \|f\|_{H^1(\partial\Omega)} + \|v\|_{H^1(\partial\Omega)} \\ &\le \|f\|_{H^1(\partial\Omega)} + C\|F\|_{L^p(\Omega)}, \end{aligned}$$

where we have used (2-19) for the last inequality. It follows from the estimates for solutions of the $L^2$ regularity problem in Lipschitz domains for the operator $\mathcal{L}_0$ in [Dahlberg et al. 1988; Verchota 1986] that

$$\|(\nabla h)^*\|_{L^2(\partial\Omega)} \le C \bigl\{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \bigr\}, \tag{2-21}$$

where $(\nabla h)^*$ denotes the nontangential maximal function of $\nabla h$. This, together with (2-20), gives

$$\|\nabla u_0\|_{L^2(\Omega_r)} \le C r^{1/2} \bigl\{ \|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} \bigr\} \tag{2-22}$$

for any $0 < r < \operatorname{diam}(\Omega)$. As a result, the first term on the right-hand side of (2-17) is bounded by $C\varepsilon^{1/2}\{\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)}\}$.

To handle the third term on the right-hand side of (2-17), we use Lemma 2.2 to obtain

$$\begin{aligned} \varepsilon \|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} &\le \varepsilon \|K_\varepsilon((\nabla^2 v)\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon \|K_\varepsilon((\nabla^2 h)\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\le C\varepsilon^{1/2} \|(\nabla^2 v)\eta_\varepsilon\|_{L^p(\Omega)} + C\varepsilon \|(\nabla^2 h)\eta_\varepsilon\|_{L^2(\Omega)} \\ &\le C\varepsilon^{1/2} \|F\|_{L^p(\Omega)} + C\varepsilon \|\nabla^2 h\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})}. \end{aligned} \tag{2-23}$$

Since $\mathcal{L}_0(\nabla h) = 0$ in $\Omega$, we may use the interior estimate for $\mathcal{L}_0$,

$$|\nabla^2 h(x)| \le \frac{C}{\delta(x)} \left( \fint_{B(x,\delta(x)/8)} |\nabla h|^2 \right)^{1/2},$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, to show that

$$\begin{aligned} \|\nabla^2 h\|_{L^2(\Omega\setminus\Omega_{3\varepsilon})} &\le C\|(\nabla h)[\delta(x)]^{-1}\|_{L^2(\Omega\setminus\Omega_\varepsilon)} \\ &\le C\varepsilon^{-1/2}\big\{\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}, \end{aligned} \tag{2-24}$$

where the last inequality follows from (2-21). This, together with (2-23), gives

$$\varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\big\{\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}. \tag{2-25}$$

Finally, to bound the second term on the right-hand side of (2-17), we again write $u_0 = v + h$ as before. Note that by Lemma 2.2,

$$\begin{aligned} \|(\nabla v)\eta_\varepsilon - K_\varepsilon((\nabla v)\eta_\varepsilon)\|_{L^2(\Omega)} &\le \|\nabla v - K_\varepsilon(\nabla v)\|_{L^2(\mathbb{R}^d)} + \|(\nabla v)(1-\eta_\varepsilon)\|_{L^2(\Omega)} + \|K_\varepsilon((\nabla v)(1-\eta_\varepsilon))\|_{L^2(\Omega)} \\ &\le C\varepsilon^{1/2}\|\nabla^2 v\|_{L^p(\mathbb{R}^d)} + C\|\nabla v\|_{L^2(\widetilde{\Omega}_{8\varepsilon})} \\ &\le C\varepsilon^{1/2}\|F\|_{L^p(\Omega)}, \end{aligned}$$

where we have used (2-18) and (2-20) for the last inequality. Also, by Lemma 2.2,

$$\begin{aligned} \|(\nabla h)\eta_\varepsilon - K_\varepsilon((\nabla h)\eta_\varepsilon)\|_{L^2(\Omega)} &\le C\varepsilon\|\nabla((\nabla h)\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\le C\big\{\varepsilon\|\nabla^2 h\|_{L^2(\Omega\setminus\Omega_{3\varepsilon})} + \|\nabla h\|_{L^2(\Omega_{4\varepsilon})}\big\} \\ &\le C\varepsilon^{1/2}\big\{\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}. \end{aligned}$$

Consequently, the second term on the right-hand side of (2-17) is dominated by the right-hand side of (2-16). This completes the proof of Theorem 2.6. □

The next theorem is an analogue of Theorem 2.6 for the Neumann boundary conditions.

**Theorem 2.7.** *Suppose that $A = A(y)$ satisfies (1-2)–(1-3). Let $\Omega$ be a bounded Lipschitz domain. Let $u_\varepsilon$ ($\varepsilon \ge 0$) be the solutions to the Neumann problem (1-7) in $\Omega$ with $g \in L^2(\partial\Omega; \mathbb{R}^d)$ and $F \in L^p(\Omega; \mathbb{R}^d)$, where $p = 2d/(d+1)$. Also assume that $u_\varepsilon, u_0 \perp \mathcal{R}$. Then*

$$\left\| u_\varepsilon - u_0 - \varepsilon\chi_j^\beta(x/\varepsilon)K_\varepsilon^2\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)\right\|_{H^1(\Omega)} \le C\varepsilon^{1/2}\big\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}, \tag{2-26}$$

*where $\eta_\varepsilon \in C_0^\infty(\Omega)$ satisfies (2-14). The constant $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and the Lipschitz character of $\Omega$.*

*Proof.* The proof, which uses the estimate in Lemma 2.5, is similar to that of Theorem 2.6. We will only point out the differences and leave the details to the reader.

Let $w_\varepsilon$ denote the function on the left-hand side of (2-26). Let

$$\big\{\varphi_j : j = 1, \dots, J = \tfrac{1}{2}d(d+1)\big\}$$

be an orthonormal basis of $\mathcal{R}$, as a subspace of $L^2(\Omega; \mathbb{R}^d)$. By the second Korn inequality [Oleĭnik et al. 1992],

$$\|w_\varepsilon\|_{H^1(\Omega)} \le C\left|\int_\Omega A(x/\varepsilon)\nabla w_\varepsilon \cdot \nabla w_\varepsilon \, dx\right| + C\sum_{j=1}^{J}\left|\int_\Omega w_\varepsilon \cdot \varphi_j \, dx\right|. \tag{2-27}$$

Since $u_\varepsilon, u_0 \perp \mathcal{R}$, it follows that

$$\begin{aligned}\left|\int_\Omega w_\varepsilon \cdot \varphi_j \, dx\right| &\le C\varepsilon\|\chi(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\le C\varepsilon\|\nabla u_0\|_{L^2(\Omega)}.\end{aligned}$$

This, together with (2-27) and Lemma 2.5, shows that

$$\begin{aligned}&\|w_\varepsilon\|_{H^1(\Omega)} \\ &\le C\big\{\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} + \varepsilon\|\nabla u_0\|_{L^2(\Omega)} + \|(\nabla u_0)\eta_\varepsilon - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega)}\big\}.\end{aligned} \tag{2-28}$$

To bound the right-hand side of (2-28), we write $u_0 = v + h$, where $v$ is the same as in the proof of Theorem 2.6. Since $\mathcal{L}_0(h) = 0$ in $\Omega$ and

$$\begin{aligned}\left\|\frac{\partial h}{\partial \nu_0}\right\|_{L^2(\partial\Omega)} &\le \left\|\frac{\partial u_0}{\partial \nu_0}\right\|_{L^2(\partial\Omega)} + \left\|\frac{\partial v}{\partial \nu_0}\right\|_{L^2(\partial\Omega)} \\ &\le C\big\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\},\end{aligned}$$

we may use the estimates in [Dahlberg et al. 1988; Verchota 1986] for solutions of the $L^2$ Neumann problem for $\mathcal{L}_0$ in Lipschitz domains to obtain

$$\begin{aligned}\|(\nabla h)^*\|_{L^2(\partial\Omega)} &\le C\left\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)} + \sum_{j=1}^{J}\left|\int_\Omega h\cdot\varphi_j\right|\right\} \\ &\le C\big\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\},\end{aligned} \tag{2-29}$$

where we have used the assumption $u_0 \perp \mathcal{R}$. With the nontangential maximal function estimate (2-29) at our disposal, the rest of the proof is exactly the same as that of Theorem 2.6. □

**Remark 2.8.** Since

$$\|\chi(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega)} \le C\|\nabla u_0\|_{L^2(\Omega)},$$

it follows from the estimate (2-16) that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\big\{\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^2(\Omega)}\big\}, \tag{2-30}$$

where $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0) = F$ in $\Omega$ and $u_\varepsilon = u_0 = f$ on $\partial\Omega$. Similarly, the estimate (2-26) implies

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\big\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^2(\Omega)}\big\}, \tag{2-31}$$

where $u_\varepsilon, u_0$ are given in Theorem 2.7. These $O(\varepsilon^{1/2})$ estimates in $L^2$ are not sharp (see Section 4), but they will be sufficient for us to establish the boundary $C^\alpha$ and Lipschitz estimates.

## 3. Proof of Theorems 1.1 and 1.2

Theorems 1.1 and 1.2 are consequences of Theorems 2.6 and 2.7, respectively. We give the proof of Theorem 1.1. Theorem 1.2 follows from Theorem 2.7 in the same manner.

Without loss of generality we may assume that

$$\|f\|_{H^1(\partial\Omega)} + \|F\|_{L^p(\Omega)} = 1.$$

Let $w_\varepsilon$ denote the function on the left-hand side of (2-16). By Theorem 2.6, for $\varepsilon \le r < \operatorname{diam}(\Omega)$,

$$\begin{aligned}
\|\nabla u_\varepsilon\|_{L^2(\Omega_r)} &\le \|\nabla u_0\|_{L^2(\Omega_r)} + \|\nabla w_\varepsilon\|_{L^2(\Omega)} + \varepsilon \big\|\nabla\{\chi(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\big\|_{L^2(\Omega_r)} \\
&\le C r^{1/2} + \big\|\nabla\chi(x/\varepsilon)K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\big\|_{L^2(\Omega_r)} + \varepsilon\big\|\chi(x/\varepsilon)\nabla K_\varepsilon^2((\nabla u_0)\eta_\varepsilon)\big\|_{L^2(\Omega_r)} \\
&\le C r^{1/2} + C\|K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} + C\varepsilon\|\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})},
\end{aligned}$$

where we have used (2-22) and Lemma 2.1 as well as the fact that the operator $K_\varepsilon$ is a convolution with a kernel supported in $B(0, \varepsilon/4)$. Note that by (2-22) and (2-25),

$$\|K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} \le C\|\nabla u_0\|_{L^2(\Omega_{3r})} \le C r^{1/2},$$

and

$$\begin{aligned}
\varepsilon\|\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} &\le \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} + \varepsilon\|K_\varepsilon((\nabla u_0)(\nabla\eta_\varepsilon))\|_{L^2(\Omega_{2r})} \\
&\le \varepsilon\|K_\varepsilon((\nabla^2 u_0)\eta_\varepsilon)\|_{L^2(\Omega_{2r})} + C\|\nabla u_0\|_{L^2(\Omega_{3r})} \\
&\le C r^{1/2}.
\end{aligned}$$

The proof of Theorem 1.1 is complete.

**Remark 3.1.** Under certain smoothness conditions on $A$, it is possible to extend the Rellich estimates in [Dahlberg et al. 1988] for the Lamé systems with constant coefficients to the operator $\mathcal{L}_1$ with variable coefficients satisfying the condition (1-2). We refer the reader to [Kenig and Shen 2011b], where this is done in the case that the coefficients satisfy the ellipticity condition (1-11). It follows that if $\mathcal{L}_1(u) = 0$ in $D_2$, where $D_r$ is defined by (1-16) with $\psi(0) = 0$ and $\|\nabla\psi\|_\infty \le M$, then

$$\begin{cases}
\displaystyle\int_{\partial D_r} |\nabla u|^2\, d\sigma \le C\int_{\partial D_r}\left|\frac{\partial u}{\partial\nu}\right|^2 d\sigma + C\int_{D_r}|\nabla u|^2\, dx, \\
\displaystyle\int_{\partial D_r} |\nabla u|^2\, d\sigma \le C\int_{\partial D_r}|\nabla_{\tan} u|^2\, d\sigma + C\int_{D_r}|\nabla u|^2\, dx
\end{cases} \tag{3-1}$$

for any $r \in \left(1, \frac{3}{2}\right)$, where $C$ depends only on $d$, $A$, and $M$. By integrating both sides of the inequalities in (3-1) with respect to $r$ over $\left(1, \frac{3}{2}\right)$, we obtain

$$\begin{cases}
\displaystyle\int_{\Delta_1} |\nabla u|^2\, d\sigma \le C\int_{\Delta_2}\left|\frac{\partial u}{\partial\nu}\right|^2 d\sigma + C\int_{D_2}|\nabla u|^2\, dx, \\
\displaystyle\int_{\Delta_1} |\nabla u|^2\, d\sigma \le C\int_{\Delta_2}|\nabla_{\tan} u|^2\, d\sigma + C\int_{D_2}|\nabla u|^2\, dx,
\end{cases} \tag{3-2}$$

where $\Delta_r = \{(x', \psi(x')) \in \mathbb{R}^d : |x'| < r \text{ and } x_d = \psi(x')\}$. We now take advantage of the fact that the dependence of $C$ on $\psi$ is only through $M$. Since $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ implies $\mathcal{L}_1\{u_\varepsilon(\varepsilon x)\} = 0$, one may deduce from (3-2) that if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $D_{2\varepsilon}$, then

$$\begin{cases} \displaystyle\int_{\Delta_\varepsilon} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\Delta_{2\varepsilon}} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^2 d\sigma + \frac{C}{\varepsilon} \int_{D_{2\varepsilon}} |\nabla u_\varepsilon|^2 \, dx, \\ \displaystyle\int_{\Delta_\varepsilon} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\Delta_{2\varepsilon}} |\nabla_{\tan} u_\varepsilon|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{D_{2\varepsilon}} |\nabla u_\varepsilon|^2 \, dx. \end{cases} \tag{3-3}$$

Now, suppose that $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ and $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $\Omega$, where $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$. By covering $\partial\Omega$ with a finite number of suitable balls of size $c\varepsilon$, it follows from (3-3) that

$$\begin{cases} \displaystyle\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^2 d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^2 \, dx, \\ \displaystyle\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^2 \, dx. \end{cases} \tag{3-4}$$

Notice that up to this point, we have only used the smoothness condition of $A$, not the periodicity of $A$. With the additional periodicity condition we may invoke the estimates in Theorems 1.1 and 1.2 to bound the volume integrals of $|\nabla u_\varepsilon|^2$ over the boundary layer $\Omega_{c\varepsilon}$. This yields the full Rellich estimates,

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^2 d\sigma \tag{3-5}$$

if $u_\varepsilon \perp \mathcal{R}$, and

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \le C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^2 \, d\sigma + C r_0^{-2} \int_{\partial\Omega} |u_\varepsilon|^2 \, d\sigma. \tag{3-6}$$

It is well known that estimates (3-5)–(3-6) may be used to solve the $L^2$ boundary value problems in Lipschitz domains by the method of layer potentials. We refer the reader to [Kenig and Shen 2011b] for the case where $A(y)$ satisfies (1-11). The details for the system of linear elasticity have been carried out in a separate work [Geng et al. 2017].

## 4. Convergence rates in $L^q$ for $q = 2d/(d-1)$

We now establish sharp $O(\varepsilon)$ estimates for $\|u_\varepsilon - u_0\|_{L^q(\Omega)}$ with $q = 2d/(d-1)$, using Theorems 1.1 and 1.2 and a duality argument. Throughout this section we will assume that $\Omega$ is a bounded Lipschitz domain and $A = A(y)$ satisfies (1-2)–(1-3).

We start with the Dirichlet boundary condition.

**Lemma 4.1.** *Let $u_\varepsilon$ ($\varepsilon \ge 0$) be the solution of (1-4). Suppose that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. Then*

$$\left\| u_\varepsilon - u_0 - \varepsilon \chi_k(x/\varepsilon) K_\varepsilon\left( \frac{\partial \tilde{u}_0}{\partial x_k} \right) - v_\varepsilon \right\|_{H_0^1(\Omega)} \le C\varepsilon \|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^d)}, \tag{4-1}$$

*where* $\tilde{u}_0 \in H^2(\mathbb{R}^d; \mathbb{R}^d)$ *is an extension of* $u_0$ *and* $v_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ *is the weak solution to*

$$\mathcal{L}_\varepsilon(v_\varepsilon) = 0 \quad \text{in } \Omega \qquad \text{and} \qquad v_\varepsilon = -\varepsilon \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right) \quad \text{on } \partial\Omega. \tag{4-2}$$

*Proof.* Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right) - v_\varepsilon.$$

Using $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$ and $\mathcal{L}_\varepsilon(v_\varepsilon) = 0$ in $\Omega$, a direct computation shows that

$$\begin{aligned}
\mathcal{L}_\varepsilon(w_\varepsilon) &= -\frac{\partial}{\partial x_i}\left\{[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\frac{\partial u_0^\beta}{\partial x_j}\right\} - \mathcal{L}_\varepsilon\left\{\varepsilon \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right)\right\} \\
&= -\frac{\partial}{\partial x_i}\left\{[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right)\right]\right\} + \frac{\partial}{\partial x_i}\left\{b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right)\right\} \\
&\qquad + \varepsilon \frac{\partial}{\partial x_i}\left\{a_{ij}^{\alpha\beta}(x/\varepsilon)\chi_k^{\beta\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j \partial x_k}\right)\right\},
\end{aligned} \tag{4-3}$$

where $b_{ij}^{\alpha\beta}$ is defined by (2-3). Using (2-5), we see that

$$\frac{\partial}{\partial x_i}\left\{b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right)\right\} = \varepsilon \frac{\partial}{\partial x_i}\left\{\phi_{ikj}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j}\right)\right\}. \tag{4-4}$$

Indeed, the left-hand side of (4-4) equals

$$b_{ij}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_i \partial x_j}\right),$$

while the right-hand side equals

$$b_{kj}^{\alpha\beta}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k \partial x_j}\right) + \phi_{ikj}^{\alpha\beta}(x/\varepsilon)\frac{\partial^2}{\partial x_i \partial x_k} K_\varepsilon\left(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\right)$$

and the second term is zero due to the skew-symmetry $\phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}$.

It follows from (4-3) and (4-4) by Lemmas 2.1 and 2.2 that

$$\|\mathcal{L}_\varepsilon(w_\varepsilon)\|_{H^{-1}(\Omega)} \le C\varepsilon \|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^d)},$$

where $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and $\Omega$. Since $w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^d)$, this gives the estimate (4-1) by the energy estimate. □

The following theorem establishes the sharp $O(\varepsilon)$ estimate in $L^q$ with $q = 2d/(d-1)$ for the Dirichlet boundary condition.

**Theorem 4.2.** *Suppose that* $A$ *satisfies* (1-2)–(1-3). *Let* $\Omega$ *be a bounded Lipschitz domain in* $\mathbb{R}^d$. *Let* $u_\varepsilon$ $(\varepsilon \ge 0)$ *be the weak solution to Dirichlet problem* (1-4). *Assume that* $u_0 \in H^2(\Omega; \mathbb{R}^d)$. *Then*

$$\|u_\varepsilon - u_0\|_{L^q(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{4-5}$$

*where* $q = 2d/(d-1)$ *and* $C$ *depends only on* $d$, $\kappa_1$, $\kappa_2$, *and* $\Omega$.

*Proof.* We begin by choosing $\tilde{u}_0 \in H^2(\mathbb{R}^d; \mathbb{R}^d)$ such that $\tilde{u}_0 = u_0$ in $\Omega$ and $\|\tilde{u}_0\|_{H^2(\mathbb{R}^d)} \le C\|u_0\|_{H^2(\Omega)}$, where $C$ depends only on $\Omega$. Since $\Omega$ is Lipschitz, this is possible by an extension theorem due to A. Calderón [Stein 1970, Theorem 5, p. 181]. Next, since $H_0^1(\Omega) \subset L^q(\Omega)$ and

$$\left\| \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right)\right\|_{L^q(\Omega)} \le C\|\nabla \tilde{u}_0\|_{L^q(\mathbb{R}^d)} \le C\|u_0\|_{H^2(\Omega)},$$

in view of Lemma 4.1, it suffices to show that

$$\|v_\varepsilon\|_{L^q(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{4-6}$$

where $v_\varepsilon$ is given by (4-2).

To this end we fix $G \in L^p(\Omega; \mathbb{R}^d)$, where $p = q' = 2d/(d+1)$, and let $h_\varepsilon \in H_0^1(\Omega; \mathbb{R}^d)$ be the weak solution to

$$\mathcal{L}_\varepsilon(h_\varepsilon) = G \quad \text{in } \Omega \qquad \text{and} \qquad h_\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{4-7}$$

It follows from (4-2), (4-7), and the divergence theorem that

$$\begin{aligned}
\int_\Omega v_\varepsilon \cdot G\, dx &= -\int_{\partial\Omega} v_\varepsilon \cdot \frac{\partial h_\varepsilon}{\partial \nu_\varepsilon}\, d\sigma \\
&= \varepsilon \int_{\partial\Omega} \chi_k(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0}{\partial x_k}\right) \cdot \frac{\partial h_\varepsilon}{\partial \nu_\varepsilon}(\eta_\varepsilon - 1)\, d\sigma \\
&= \int_\Omega \frac{\partial \chi_k^{\alpha\gamma}}{\partial x_i}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\gamma}{\partial x_k}\right) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial h_\varepsilon^\beta}{\partial x_j}(\eta_\varepsilon - 1)\, dx \\
&\quad + \varepsilon \int_\Omega \chi_k^{\alpha\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_i \partial x_k}\right) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial h_\varepsilon^\beta}{\partial x_j}(\eta_\varepsilon - 1)\, dx \\
&\quad - \varepsilon \int_\Omega \chi_k^{\alpha\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\gamma}{\partial x_k}\right) G^\alpha (\eta_\varepsilon - 1)\, dx \\
&\quad + \varepsilon \int_\Omega \chi_k^{\alpha\gamma}(x/\varepsilon) K_\varepsilon\left(\frac{\partial \tilde{u}_0^\gamma}{\partial x_k}\right) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial h_\varepsilon^\beta}{\partial x_j} \frac{\partial \eta_\varepsilon}{\partial x_i}\, dx,
\end{aligned}$$

where $\eta_\varepsilon \in C_0^\infty(\Omega)$ satisfies (2-14). This implies

$$\begin{aligned}
\left|\int_\Omega v_\varepsilon \cdot G\, dx\right| &\le C \int_\Omega |\nabla \chi(x/\varepsilon)| |K_\varepsilon(\nabla \tilde{u}_0)| |\nabla h_\varepsilon| |\eta_\varepsilon - 1|\, dx \\
&\quad + C\varepsilon \int_\Omega |\chi(x/\varepsilon)| |K_\varepsilon(\nabla^2 \tilde{u}_0)| |\nabla h_\varepsilon| |\eta_\varepsilon - 1|\, dx \\
&\quad + C\varepsilon \int_\Omega |\chi(x/\varepsilon)| |K_\varepsilon(\nabla \tilde{u}_0)| |G| |\eta_\varepsilon - 1|\, dx \\
&\quad + C\varepsilon \int_\Omega |\chi(x/\varepsilon)| |K_\varepsilon(\nabla \tilde{u}_0)| |\nabla h_\varepsilon| |\nabla \eta_\varepsilon|\, dx.
\end{aligned} \tag{4-8}$$

Note that by Cauchy's inequality and (2-14), the first and fourth terms on the right-hand side of (4-8) are bounded by

$$C\biggl(\int_{\Omega_{4\varepsilon}} \bigl|\bigl(|\nabla\chi(x/\varepsilon)|+|\chi(x/\varepsilon)|\bigr)K_\varepsilon(\nabla\tilde{u}_0)\bigr|^2\,dx\biggr)^{1/2}\biggl(\int_{\Omega_{4\varepsilon}} |\nabla h_\varepsilon|^2\,dx\biggr)^{1/2}$$
$$\le C\biggl(\int_{\widetilde{\Omega}_{5\varepsilon}} |\nabla\tilde{u}_0|^2\,dx\biggr)^{1/2}\biggl(\int_{\Omega_{4\varepsilon}} |\nabla h_\varepsilon|^2\,dx\biggr)^{1/2},$$

where $\Omega_r=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)<r\}$, $\widetilde{\Omega}_r=\{x\in\mathbb{R}^d:\operatorname{dist}(x,\partial\Omega)<r\}$, and we have used Lemma 2.1 for the last inequality. Using the divergence theorem, as in (2-19), one may prove that

$$\|\nabla\tilde{u}_0\|_{L^2(S_r)}\le C\|\tilde{u}_0\|_{H^1(\mathbb{R}^d)}^{1/2}\|\tilde{u}_0\|_{H^2(\mathbb{R}^d)}^{1/2},$$

where $S_r=\{x\in\mathbb{R}^d:\operatorname{dist}(x,\partial\Omega)=r\}$. It follows by the coarea formula that

$$\|\nabla\tilde{u}_0\|_{L^2(\widetilde{\Omega}_r)}\le Cr^{1/2}\|\tilde{u}_0\|_{H^1(\mathbb{R}^d)}^{1/2}\|\tilde{u}_0\|_{H^2(\mathbb{R}^d)}^{1/2}. \tag{4-9}$$

This, together with the estimate in Theorem 1.1 for $h_\varepsilon$, shows that the first and fourth terms on the right-hand side of (4-8) are bounded by

$$C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)},$$

where $p=q'=2d/(d+1)$. Finally, we note that the second and third terms on the right-hand side of (4-8) are bounded by

$$C\varepsilon\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)}\|\nabla h_\varepsilon\|_{L^2(\Omega)}+C\varepsilon\|\nabla\tilde{u}_0\|_{L^q(\mathbb{R}^d)}\|G\|_{L^p(\Omega)}\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)}.$$

As a result, we have proved that

$$\biggl|\int_\Omega v_\varepsilon\cdot G\,dx\biggr|\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)},$$

which, by duality, gives the estimate (4-6) and completes the proof. □

Next we consider the solutions with the Neumann boundary conditions.

**Lemma 4.3.** *Let $u_\varepsilon$ ($\varepsilon\ge 0$) be the solutions of (1-7) such that $u_\varepsilon\perp\mathcal{R}$. Suppose that $u_0\in H^2(\Omega;\mathbb{R}^d)$. Then*

$$\biggl\|u_\varepsilon-u_0-\varepsilon\chi_k(x/\varepsilon)K_\varepsilon\Bigl(\frac{\partial\tilde{u}_0}{\partial x_k}\Bigr)-v_\varepsilon\biggr\|_{H^1(\Omega)}\le C\varepsilon\bigl\{\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)}+\|\nabla\tilde{u}_0\|_{L^2(\mathbb{R}^d)}\bigr\}, \tag{4-10}$$

*where $\tilde{u}_0$ is an extension of $u_0$ and $v_\varepsilon\in H^1(\Omega;\mathbb{R}^d)$ is the weak solution to*

$$\begin{cases}\mathcal{L}_\varepsilon(v_\varepsilon)=0 \quad \text{in } \Omega,\\[4pt] \dfrac{\partial v_\varepsilon}{\partial\nu_\varepsilon}=\dfrac{\varepsilon}{2}\Bigl(n_k\dfrac{\partial}{\partial x_i}-n_i\dfrac{\partial}{\partial x_k}\Bigr)\Bigl\{\phi_{kij}(x/\varepsilon)K_\varepsilon\Bigl(\dfrac{\partial\tilde{u}_0}{\partial x_j}\Bigr)\Bigr\} \quad \text{on } \partial\Omega,\\[4pt] v_\varepsilon\perp\mathcal{R}.\end{cases} \tag{4-11}$$

*Proof.* Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi_k(x/\varepsilon)K_\varepsilon\Big(\frac{\partial \tilde{u}_0}{\partial x_k}\Big) - v_\varepsilon.$$

Using $\partial u_\varepsilon/\partial\nu_\varepsilon = \partial u_0/\partial \nu_0$ on $\partial\Omega$, a direct computation shows that

$$\begin{aligned}
\frac{\partial w_\varepsilon}{\partial \nu_\varepsilon} &= \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} - \frac{\partial}{\partial \nu_\varepsilon}\Big\{\varepsilon\chi_k(x/\varepsilon)K_\varepsilon\Big(\frac{\partial \tilde{u}_0}{\partial x_k}\Big)\Big\} - \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}\\
&= n_i[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\Big[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\Big(\frac{\partial u_0^\beta}{\partial x_j}\Big)\Big] - n_i b_{ij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\Big(\frac{\partial u_0^\beta}{\partial x_j}\Big)\\
&\qquad - n_i a_{ij}^{\alpha\beta}(x/\varepsilon)\cdot\varepsilon\chi_k^{\beta\gamma}(x/\varepsilon)K_\varepsilon\Big(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j\partial x_k}\Big) - \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}. \qquad \text{(4-12)}
\end{aligned}$$

Using (2-5), we also see that

$$\begin{aligned}
n_i b_{ij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\Big(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\Big) + \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} &= \varepsilon n_i \frac{\partial}{\partial x_k}[\phi_{kij}^{\alpha\beta}(x/\varepsilon)]K_\varepsilon\Big(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\Big) + \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}\\
&= \frac{\varepsilon}{2}\Big(n_i\frac{\partial}{\partial x_k} - n_k\frac{\partial}{\partial x_i}\Big)[\phi_{kij}^{\alpha\beta}(x/\varepsilon)]K_\varepsilon\Big(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\Big) + \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}\\
&= -\varepsilon n_i \phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\Big(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k\partial x_j}\Big). \qquad \text{(4-13)}
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
\frac{\partial w_\varepsilon}{\partial \nu_\varepsilon} &= n_i[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\Big[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\Big(\frac{\partial u_0^\beta}{\partial x_j}\Big)\Big]\\
&\qquad + \varepsilon n_i \phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\Big(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k\partial x_j}\Big) - n_i a_{ij}^{\alpha\beta}(x/\varepsilon)\cdot\varepsilon\chi_k^{\beta\gamma}(x/\varepsilon)K_\varepsilon\Big(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j\partial x_k}\Big). \qquad \text{(4-14)}
\end{aligned}$$

Next, we note that as in the proof of Lemma 4.1,

$$\begin{aligned}
\mathcal{L}_\varepsilon(w_\varepsilon) &= -\frac{\partial}{\partial x_i}\Big\{[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\Big[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\Big(\frac{\partial \tilde{u}_0^\beta}{\partial x_j}\Big)\Big]\Big\} - \varepsilon\frac{\partial}{\partial x_i}\Big\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\Big(\frac{\partial^2 \tilde{u}_0^\beta}{\partial x_k\partial x_j}\Big)\Big\}\\
&\qquad + \varepsilon\frac{\partial}{\partial x_i}\Big\{a_{ij}^{\alpha\beta}(x/\varepsilon)\chi_k^{\beta\gamma}(x/\varepsilon)K_\varepsilon\Big(\frac{\partial^2 \tilde{u}_0^\gamma}{\partial x_j\partial x_k}\Big)\Big\}. \qquad \text{(4-15)}
\end{aligned}$$

Thus, by (1-2) and the energy estimate,

$$\begin{aligned}
&\|\nabla w_\varepsilon + (\nabla w_\varepsilon)^T\|_{L^2(\Omega)}\\
&\quad\le C\|\nabla w_\varepsilon\|_{L^2(\Omega)}\big\{\|\nabla u_0 - K_\varepsilon(\nabla\tilde{u}_0)\|_{L^2(\Omega)} + \varepsilon\|\phi(x/\varepsilon)K_\varepsilon(\nabla^2\tilde{u}_0)\|_{L^2(\Omega)} + \varepsilon\|\chi(x/\varepsilon)K_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega)}\big\}\\
&\quad\le C\varepsilon\|\nabla w_\varepsilon\|_{L^2(\Omega)}\|\nabla^2\tilde{u}_0\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where we have used Lemmas 2.1 and 2.2 for the last step. By the second Korn inequality, this implies

$$\begin{aligned}\|w_\varepsilon\|_{H^1(\Omega)} &\le C\varepsilon\|\nabla^2\tilde u_0\|_{L^2(\mathbb{R}^d)} + C\sum_{j=1}^J\left|\int_\Omega w_\varepsilon\cdot\varphi_j\,dx\right|\\ &\le C\varepsilon\|\nabla^2\tilde u_0\|_{L^2(\mathbb{R}^d)} + C\varepsilon\|\chi(x/\varepsilon)K_\varepsilon(\nabla\tilde u_0)\|_{L^2(\Omega)} \le C\varepsilon\big\{\|\nabla^2\tilde u_0\|_{L^2(\mathbb{R}^d)}+\|\nabla\tilde u_0\|_{L^2(\mathbb{R}^d)}\big\},\end{aligned}$$

where $\{\varphi_j : j = 1, \dots, J\}$ forms an orthonormal basis of $\mathcal{R}$, as a subspace of $L^2(\Omega; \mathbb{R}^d)$. □

The next theorem is an analogue of Theorem 4.2 for the Neumann boundary conditions.

**Theorem 4.4.** *Suppose that $A$ satisfies (1-2)–(1-3). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $u_\varepsilon$ ($\varepsilon \ge 0$) be the weak solutions to the Neumann problem (1-7) with the property $u_\varepsilon \perp \mathcal{R}$. Assume that $u_0 \in H^2(\Omega; \mathbb{R}^d)$. Then*

$$\|u_\varepsilon - u_0\|_{L^q(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{4-16}$$

*where $q = 2d/(d-1)$ and $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and $\Omega$.*

*Proof.* As in the proof of Theorem 4.2, it suffices to show that

$$\|v_\varepsilon\|_{L^q(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{4-17}$$

where $v_\varepsilon$ is given by (4-11). To this end we fix $G \in L^p(\Omega; \mathbb{R}^d)$ with $G \perp \mathcal{R}$ and let $h_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be the weak solution to

$$\mathcal{L}_\varepsilon(h_\varepsilon) = G \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial h_\varepsilon}{\partial\nu_\varepsilon} = 0 \quad \text{on } \partial\Omega, \tag{4-18}$$

with the property $h_\varepsilon \perp \mathcal{R}$. It follows from (4-18), (4-11), and Green's formula that

$$\begin{aligned}\int_\Omega v_\varepsilon\cdot G\,dx &= \int_\Omega A(x/\varepsilon)\nabla v_\varepsilon\cdot\nabla h_\varepsilon\,dx = \int_{\partial\Omega}\frac{\partial v_\varepsilon}{\partial\nu_\varepsilon}\cdot h_\varepsilon\,d\sigma\\ &= \frac{\varepsilon}{2}\int_{\partial\Omega}\left(n_k\frac{\partial}{\partial x_i} - n_i\frac{\partial}{\partial x_k}\right)\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial\tilde u_0^\beta}{\partial x_j}\right)\right\}\cdot h_\varepsilon^\alpha\,d\sigma\\ &= -\frac{\varepsilon}{2}\int_{\partial\Omega}\phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial\tilde u_0^\beta}{\partial x_j}\right)\cdot\left(n_k\frac{\partial}{\partial x_i} - n_i\frac{\partial}{\partial x_k}\right)h_\varepsilon^\alpha\cdot(1-\eta_\varepsilon)\,d\sigma\\ &= -\varepsilon\int_\Omega\frac{\partial}{\partial x_k}\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial\tilde u_0^\beta}{\partial x_j}\right)(1-\eta_\varepsilon)\right\}\cdot\frac{\partial h_\varepsilon^\alpha}{\partial x_i}\,dx,\end{aligned}$$

where $\eta_\varepsilon \in C_0^\infty(\Omega)$ satisfies (2-14) and we have used the divergence theorem as well as (2-5) for the last inequality. This leads to

$$\begin{aligned}\left|\int_\Omega v_\varepsilon\cdot G\,dx\right| \le C&\int_{\Omega_{4\varepsilon}}|\nabla\phi(x/\varepsilon)||K_\varepsilon(\nabla\tilde u_0)||\nabla h_\varepsilon|\,dx\\ &+ C\varepsilon\int_{\Omega_{4\varepsilon}}|\phi(x/\varepsilon)||K_\varepsilon(\nabla^2\tilde u_0)||\nabla h_\varepsilon|\,dx\\ &\quad+ C\varepsilon\int_{\Omega_{4\varepsilon}}|\phi(x/\varepsilon)||K_\varepsilon(\nabla\tilde u_0)||\nabla\eta_\varepsilon||\nabla h_\varepsilon|\,dx.\end{aligned} \tag{4-19}$$

Note that by the Cauchy inequality, the first and third term on the right-hand side of (4-19) are bounded by

$$C\big\|\big(|\nabla\phi(x/\varepsilon)|+|\phi(x/\varepsilon)|\big)K_\varepsilon(\nabla\tilde{u}_0)\big\|_{L^2(\Omega_{4\varepsilon})}\|\nabla h_\varepsilon\|_{L^2(\Omega_{4\varepsilon})} \le C\|\nabla\tilde{u}_0\|_{L^2(\tilde{\Omega}_{5\varepsilon})}\|\nabla h_\varepsilon\|_{L^2(\Omega_{4\varepsilon})} \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)},$$

where we have used Lemma 2.2 for the first inequality and Theorem 1.2 as well as estimate (4-9) for the second. Also, the second term on the right-hand side of (4-19) is bounded by

$$C\varepsilon\|\phi(x/\varepsilon)K_\varepsilon(\nabla^2\tilde{u}_0)\|_{L^2(\Omega)}\|\nabla h_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)}.$$

Hence we have proved that for any $G \in L^p(\Omega;\mathbb{R}^d)$ with the property $G \perp \mathcal{A}$,

$$\left|\int_\Omega v_\varepsilon\cdot G\,dx\right| \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|G\|_{L^p(\Omega)}.$$

Since $v_\varepsilon \perp \mathcal{A}$, this gives the estimate (4-17) by duality and completes the proof. □

Note that by combining Theorems 4.2 and 4.4, one obtains Theorem 1.3.

## 5. $C^\alpha$ estimates in $C^1$ domains

In this section we investigate uniform boundary $C^\alpha$ estimates in $C^1$ domains. The results will be used in the next section to establish uniform boundary $W^{1,p}$ estimates in $C^1$ domains. Throughout the section we will assume that the defining function $\psi$ in $D_r$ and $\Delta_r$ is $C^1$ and $\psi(0)=0$. To quantify the $C^1$ condition we further assume that

$$\sup\big\{|\nabla\psi(x')-\nabla\psi(y')| : x', y' \in \mathbb{R}^{d-1} \text{ and } |x'-y'| \le t\big\} \le \tau(t), \tag{5-1}$$

where $\tau(t) \to 0$ as $t \to 0^+$.

The rescaling argument is used frequently in this paper. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon)=F$ in $D_{2r}$ and $u_\varepsilon=f$ on $\Delta_{2r}$. Let $w(x)=u_\varepsilon(rx)$. Then

$$\mathcal{L}_{\varepsilon/r}(w)=G \quad \text{in } \widetilde{D}_2 \qquad \text{and} \qquad w=g \quad \text{on } \tilde{\Delta}_2,$$

where $G(x)=r^2F(rx)$, $g(x)=f(rx)$, and

$$\widetilde{D}_2=\big\{(x',x_d)\in\mathbb{R}^d : |x'|<2 \text{ and } \psi_r(x')<x_d<\psi_r(x')+2\big\},$$
$$\tilde{\Delta}_2=\big\{(x',x_d)\in\mathbb{R}^d : |x'|<2 \text{ and } x_d=\psi_r(x')\big\},$$

with $\psi_r(x')=r^{-1}\psi(rx')$. Note that $\psi_r(0)=0$ and $\|\nabla\psi_r\|_\infty=\|\nabla\psi\|_\infty$. Moreover, if $\psi$ is $C^1$ and satisfies (5-1), then $\psi_r$ satisfies (5-1) uniformly in $r$ for $0<r\le 1$.

**Lemma 5.1.** *Let $0<\varepsilon\le r\le 1$. Let $u_\varepsilon \in H^1(D_{2r};\mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon)=0$ in $D_{2r}$ with $u_\varepsilon=0$ on $\Delta_{2r}$. Then there exists $v\in H^1(D_r;\mathbb{R}^d)$ such that $\mathcal{L}_0(v)=0$ in $D_r$, $v=0$ on $\Delta_r$, and*

$$\left(\fint_{D_r}|u_\varepsilon-v|^2\right)^{1/2} \le C(\varepsilon/r)^{1/2}\left(\fint_{D_{2r}}|u_\varepsilon|^2\right)^{1/2}, \tag{5-2}$$

*where $\|\nabla\psi\|_\infty \le M$, and $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and $M$.*

*Proof.* By rescaling we may assume $r = 1$. By Caccioppoli's inequality,

$$\left(\fint_{D_{3/2}} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left(\fint_{D_2} |u_\varepsilon|^2\right)^{1/2}. \tag{5-3}$$

It follows from (5-3) and the coarea formula that there exists $t \in \left[\frac{5}{4}, \frac{3}{2}\right]$ such that

$$\|\nabla u_\varepsilon\|_{L^2(\partial D_t \setminus \Delta_2)} + \|u_\varepsilon\|_{L^2(\partial D_t \setminus D_2)} \le C\|u_\varepsilon\|_{L^2(D_2)}. \tag{5-4}$$

Let $v$ be the solution to the Dirichlet problem: $\mathcal{L}_0(v) = 0$ in $D_t$ and $v = u_\varepsilon$ on $\partial D_t$. Note that $v = 0$ on $\Delta_1$, and by Remark 2.8,

$$\|u_\varepsilon - v\|_{L^2(D_t)} \le C\varepsilon^{1/2}\|u_\varepsilon\|_{H^1(\partial D_t)}. \tag{5-5}$$

This, together with (5-4), gives

$$\|u_\varepsilon - v\|_{L^2(D_1)} \le \|u_\varepsilon - v\|_{L^2(D_t)} \le C\varepsilon^{1/2}\|u_\varepsilon\|_{L^2(D_2)}.$$ □

**Theorem 5.2.** *Suppose that $A = A(y)$ satisfies (1-2)–(1-3). Let $u_\varepsilon$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $D_1$ with $u_\varepsilon = 0$ on $\Delta_1$, where the defining function $\psi$ in $D_1$ and $\Delta_1$ is $C^1$. Then, for any $\alpha \in (0, 1)$ and $\varepsilon \le r \le \frac{1}{2}$,*

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C_\alpha r^{\alpha-1}\left(\fint_{D_1} |u_\varepsilon|^2\right)^{1/2}, \tag{5-6}$$

*where $C_\alpha$ depends only on $d$, $\alpha$, $\kappa_1$, $\kappa_2$, and the function $\tau(t)$ in (5-1).*

*Proof.* Fix $\beta \in (\alpha, 1)$. For each $r \in \left[\varepsilon, \frac{1}{2}\right]$, let $v = v_r$ be the function given by Lemma 5.1. By the boundary $C^\beta$ estimates in $C^1$ domains for the operator $\mathcal{L}_0$ (see, e.g., [Auscher and Qafsaoui 2002; Byun and Wang 2004]),

$$\left(\fint_{D_{\theta r}} |v|^2\right)^{1/2} \le C_0\theta^\beta\left(\fint_{D_r} |v|^2\right)^{1/2}$$

for any $\theta \in (0, 1)$, where $C_0$ depends only on $d$, $\kappa_1$, $\kappa_2$, $\beta$ and $\tau(t)$. It follows that

$$\begin{aligned}
\left(\fint_{D_{\theta r}} |u_\varepsilon|^2\right)^{1/2} &\le \left(\fint_{D_{\theta r}} |v|^2\right)^{1/2} + C\left(\fint_{D_{\theta r}} |u_\varepsilon - v|^2\right)^{1/2} \\
&\le C\theta^\beta\left(\fint_{D_r} |v|^2\right)^{1/2} + C\theta^{-d/2}\left(\fint_{D_r} |u_\varepsilon - v|^2\right)^{1/2} \\
&\le C_1\theta^\beta\left(\fint_{D_r} |u_\varepsilon|^2\right)^{1/2} + C_1\theta^{-d/2}(\varepsilon/r)^{1/2}\left(\fint_{D_{2r}} |u_\varepsilon|^2\right)^{1/2}
\end{aligned}$$

for any $\varepsilon \le r \le \frac{1}{2}$. We now choose $\theta \in \left(0, \frac{1}{4}\right)$ so small that $C_1\theta^{\beta-\alpha} < \frac{1}{4}$. With $\theta$ fixed, choose $N > 1$ large so that

$$C_1 2^\alpha \theta^{-d/2-\alpha} N^{-1/2} \le \tfrac{1}{4}.$$

It follows that if $r \ge N\varepsilon$,

$$\phi(\theta r) \le \tfrac{1}{4}\{\phi(r) + \phi(2r)\}, \tag{5-7}$$

where

$$\phi(r) = r^{-\alpha}\left(\fint_{D_r} |u_\varepsilon|^2\right)^{1/2}.$$

By integration we may deduce from (5-7) that

$$\int_{\theta a}^{\theta/2} \phi(r)\frac{dr}{r} \le \frac{1}{4}\int_a^{1/2} \phi(r)\frac{dr}{r} + \frac{1}{4}\int_{2a}^{1} \phi(r)\frac{dr}{r},$$

where $N\varepsilon \le a < \frac{1}{2}$. This implies

$$\int_{\theta a}^{1} \phi(r)\frac{dr}{r} \le C\int_{\theta/2}^{1} \phi(r)\frac{dr}{r} \le C\phi(1).$$

Hence, $\phi(r) \le C\phi(1)$ for any $r \in [\varepsilon, 1]$, and the estimate (5-6) now follows by Caccioppoli's inequality. □

**Remark 5.3.** Under the stronger assumption that the defining function $\phi$ for $D_1$ is $C^{1,\sigma}$ for some $\sigma > 0$, we will show in Section 8 that the estimate (5-6) holds for $\alpha = 1$. In particular, it follows from the argument in Section 7 that if $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(0, 1)$, then

$$\left(\fint_{B(0,r)} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left(\fint_{B(0,1)} |\nabla u_\varepsilon|^2\right)^{1/2} \tag{5-8}$$

for any $\varepsilon \le r < 1$. This is the interior Lipschitz estimate down to the scale $\varepsilon$.

A function $A$ is said to belong to VMO($\mathbb{R}^d$) if the left-hand side of (5-9) goes to zero as $t \to 0^+$. To quantify this assumption we assume that

$$\sup_{\substack{x\in\mathbb{R}^d\\ 0<r<t}} \fint_{B(x,r)} \left| A(y) - \fint_{B(x,r)} A \right| dy \le \rho(t), \tag{5-9}$$

where $\rho(t) \to 0$ as $t \to 0^+$.

The following corollary was essentially proved in [Avellaneda and Lin 1987] by a compactness method.

**Corollary 5.4.** *Suppose that $A$ satisfies (1-2)–(1-3). Also assume that $A \in$ VMO($\mathbb{R}^d$). Let $u_\varepsilon \in H^1(D_1; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $D_1$ with $u_\varepsilon = 0$ on $\Delta_1$. Then, for any $\alpha \in (0, 1)$,*

$$\|u_\varepsilon\|_{C^\alpha(D_{1/2})} \le C_\alpha\left(\fint_{D_1} |u_\varepsilon|^2\right)^{1/2}, \tag{5-10}$$

*where $C_\alpha$ depends only on $d, \kappa_1, \kappa_2, \alpha$, and the functions $\tau(t), \rho(t)$.*

*Proof.* We may assume $0 < \varepsilon < \frac{1}{2}$, as the case of $\varepsilon \ge \frac{1}{2}$ is local. Since $\mathcal{L}_1(u_\varepsilon(\varepsilon x)) = 0$, it follows from the boundary $C^\alpha$ estimates in $C^1$ domains (see, e.g., [Auscher and Qafsaoui 2002; Byun and Wang 2004]) for the operator $\mathcal{L}_1$ by rescaling that if $\alpha \in (0, 1)$ and $0 < r < \varepsilon$,

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C(r/\varepsilon)^{\alpha-1}\left(\fint_{D_\varepsilon} |\nabla u_\varepsilon|^2\right)^{1/2},$$

where $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, $\alpha$, $\tau(t)$ and $\rho(t)$. This, together with Theorem 5.2, shows that the estimate (5-6) holds for any $0 < r < \frac{1}{2}$. By combining (5-6) with a similar interior estimate, we obtain

$$r^{\alpha-1}\left(\fint_{B(x,r)\cap D_{1/2}} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\|u_\varepsilon\|_{L^2(D_1)} \tag{5-11}$$

for any $0 < r < c$ and $x \in D_{1/2}$. The estimate (5-10) follows from (5-11) by Campanato's characterization of Hölder spaces. □

The rest of this section is devoted to the boundary $C^\alpha$ estimates for solutions with the Neumann boundary conditions.

**Lemma 5.5.** *Let $0 < \varepsilon \le r \le 1$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $D_{2r}$ with $\partial u_\varepsilon/\partial \nu_\varepsilon = 0$ on $\Delta_{2r}$. Then there exists a function $w \in H^1(D_r; \mathbb{R}^d)$ such that $\mathcal{L}_0(w) = 0$, $\partial w/\partial \nu_0 = 0$ in $\Delta_r$, and*

$$\left(\fint_{D_r} |u_\varepsilon - w|^2\right)^{1/2} \le C(\varepsilon/r)^{1/2}\left(\fint_{D_{2r}} |u_\varepsilon|^2\right)^{1/2}, \tag{5-12}$$

*where $\|\nabla\psi\|_\infty \le M$, and $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and $M$.*

*Proof.* By rescaling we may assume $r = 1$. As in the proof of Lemma 5.1, there exists $t \in \left[\frac{5}{4}, \frac{3}{2}\right]$ such that

$$\|u_\varepsilon\|_{L^2(\partial D_t\setminus\Delta_2)} + \|\nabla u_\varepsilon\|_{L^2(\partial D_t\setminus\Delta_2)} \le C\|u_\varepsilon\|_{L^2(D_2)}.$$

Let $\phi_\varepsilon$ be a function in $\mathcal{R}$ such that $u_\varepsilon - \phi_\varepsilon \perp \mathcal{R}$ in $L^2(D_t; \mathbb{R}^d)$. Let $v$ be the solution to the Neumann problem: $\mathcal{L}_0(v) = 0$ in $D_t$ and $\partial v/\partial \nu_0 = \partial u_\varepsilon/\partial \nu_\varepsilon$ on $\partial D_t$, with $v \perp \mathcal{R}$. It follows from Remark 2.8 that

$$\begin{aligned}\|u_\varepsilon - \phi_\varepsilon - v\|_{L^2(D_1)} &\le \|u_\varepsilon - \phi_\varepsilon - v\|_{L^2(D_t)}\\ &\le C\varepsilon^{1/2}\left\|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right\|_{L^2(\partial D_t)}\\ &\le C\varepsilon^{1/2}\|u_\varepsilon\|_{L^2(D_2)}.\end{aligned}$$

It is easy to see that the function $w = v + \phi_\varepsilon$ satisfies the desired conditions. □

**Theorem 5.6.** *Suppose that $A = A(y)$ satisfies (1-2)–(1-3). Let $u_\varepsilon$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $D_1$ with $\partial u_\varepsilon/\partial \nu_\varepsilon = 0$ on $\Delta_1$, where the defining function $\psi$ in $D_1$ and $\Delta_1$ is $C^1$. Then, for any $\alpha \in (0, 1)$ and $\varepsilon \le r \le 1$,*

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \le C_\alpha r^{\alpha-1}\left(\fint_{D_1} |\nabla u_\varepsilon|^2\right)^{1/2}, \tag{5-13}$$

*where $C$ depends only on $d$, $\alpha$, $\kappa_1$, $\kappa_2$, and the function $\tau(t)$.*

*Proof.* Fix $\beta \in (\alpha, 1)$. For each $r \in \left[\varepsilon, \frac{1}{2}\right]$, let $w = w_r$ be the function given by Lemma 5.5. By the boundary $C^\beta$ estimates in $C^1$ domains for the operator $\mathcal{L}_0$,

$$\inf_{q\in\mathbb{R}^d}\left(\fint_{D_{\theta r}} |w - q|^2\right)^{1/2} \le C_0\theta^\beta \inf_{q\in\mathbb{R}^d}\left(\fint_{D_r} |w - q|^2\right)^{1/2},$$

where $C_0$ depends only on $d$, $\beta$, $\kappa_1$, $\kappa_2$, and $\tau(t)$. This, together with Lemma 5.5, gives

$$\begin{aligned}\inf_{q\in\mathbb{R}^d}\biggl(\fint_{D_{\theta r}}|u_\varepsilon-q|^2\biggr)^{1/2} &\le C\inf_{q\in\mathbb{R}^d}\biggl(\fint_{D_{\theta r}}|w-q|^2\biggr)^{1/2}+\biggl(\fint_{D_{\theta r}}|u_\varepsilon-w|^2\biggr)^{1/2}\\ &\le C\theta^\beta\inf_{q\in\mathbb{R}^d}\biggl(\fint_{D_r}|w-q|^2\biggr)^{1/2}+C_0\theta^{-d/2}\biggl(\fint_{D_r}|u_\varepsilon-w|^2\biggr)^{1/2}\\ &\le C\theta^\beta\inf_{q\in\mathbb{R}^d}\biggl(\fint_{D_r}|u_\varepsilon-q|^2\biggr)^{1/2}+C\theta^{-d/2}(\varepsilon/r)^{1/2}\biggl(\fint_{D_{2r}}|u_\varepsilon|^2\biggr)^{1/2}.\end{aligned}$$

By replacing $u_\varepsilon$ with $u_\varepsilon-q$, we obtain

$$\phi(\theta r)\le C\theta^{\beta-\alpha}\phi(r)+C\theta^{-\alpha-d/2}(\varepsilon/r)^{1/2}\phi(2r)$$

for any $r\in\bigl[\varepsilon,\frac{1}{2}\bigr]$, where

$$\phi(r)=r^{-\alpha}\inf_{q\in\mathbb{R}^d}\biggl(\fint_{D_r}|u_\varepsilon-q|^2\biggr)^{1/2}.$$

By the integration argument used in the proof of Theorem 5.2, we may conclude that $\phi(r)\le C\phi(1)$ for $r\in\bigl[\varepsilon,\frac{1}{2}\bigr]$, which yields (5-13) by Caccioppoli's inequality. □

**Remark 5.7.** Under the stronger condition that the defining function for $D_1$ and $\Delta_1$ is $C^{1,\sigma}$ for some $\sigma>0$, we will show in Section 9 that the estimate (5-13) holds for $\alpha=1$.

The following corollary was essentially proved in [Kenig et al. 2013] by a compactness method.

**Corollary 5.8.** *Suppose that $A$ satisfies (1-2)–(1-3). Also assume that $A\in\mathrm{VMO}(\mathbb{R}^d)$. Let $u_\varepsilon\in H^1(D_1;\mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon)=0$ in $D_1$ with $\partial u_\varepsilon/\partial\nu_\varepsilon=0$ on $\Delta_1$. Then, for any $\alpha\in(0,1)$,*

$$\|u_\varepsilon\|_{C^\alpha(D_{1/2})}\le C_\alpha\biggl(\fint_{D_1}|u_\varepsilon|^2\biggr)^{1/2},\tag{5-14}$$

*where $C_\alpha$ depends only on $d$, $\kappa_1$, $\kappa_2$, $\alpha$, and the functions $\tau(t)$, $\rho(t)$.*

*Proof.* As in the case of the Dirichlet boundary condition, the additional smoothness assumption $A\in\mathrm{VMO}(\mathbb{R}^d)$ ensures that the estimate (5-13) holds for any $r\in\bigl(0,\frac{1}{2}\bigr)$ (see, e.g., [Byun and Wang 2005] for estimates for $\mathcal{L}_1$). This, together with the interior estimates, gives the estimate (5-14) by the use of Campanato's characterization of Hölder spaces. □

## 6. $W^{1,p}$ estimates in $C^1$ domains

In this section we study the uniform $W^{1,p}$ estimates in $C^1$ domains. Throughout the section we will assume that $A=A(y)$ satisfies (1-2)–(1-3), $A\in\mathrm{VMO}(\mathbb{R}^d)$, and $\Omega$ is $C^1$. Our goal is to prove the following two theorems.

**Theorem 6.1.** *Suppose that $A$ satisfies (1-2)–(1-3). Also assume that $A\in\mathrm{VMO}(\mathbb{R}^d)$. Let $1<p<\infty$ and $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^d$. Let $u_\varepsilon\in W^{1,p}(\Omega;\mathbb{R}^d)$ be a weak solution to the Dirichlet problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon)=\mathrm{div}(f)\quad\textit{in }\Omega\qquad\textit{and}\qquad u_\varepsilon=0\quad\textit{on }\partial\Omega,\tag{6-1}$$

*where* $f = (f_i^\alpha) \in L^p(\Omega; \mathbb{R}^{d\times d})$*. Then*

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p \|f\|_{L^p(\Omega)}, \tag{6-2}$$

*where* $C_p$ *depends only on* $d$, $p$, $A$, *and* $\Omega$.

**Theorem 6.2.** *Suppose that* $A$ *satisfies the same conditions as in Theorem 6.1. Let* $1 < p < \infty$ *and* $\Omega$ *be a bounded* $C^1$ *domain in* $\mathbb{R}^d$*. Let* $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ *be a weak solution to the Neumann problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(f) \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = -n\cdot f \quad \text{on } \partial\Omega, \tag{6-3}$$

*where* $f = (f_i^\alpha) \in L^p(\Omega; \mathbb{R}^{d\times d})$*. Assume that* $u_\varepsilon \perp \mathcal{R}$*. Then*

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p \|f\|_{L^p(\Omega)}, \tag{6-4}$$

*where* $C_p$ *depends only on* $d$, $p$, $A$, *and* $\Omega$.

Recall that a function $u_\varepsilon$ is called a weak solution of (6-1) if $u_\varepsilon \in W_0^{1,p}(\Omega; \mathbb{R}^d)$ and

$$\int_\Omega a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial u_\varepsilon^\beta}{\partial x_j} \cdot \frac{\partial \varphi^\alpha}{\partial x_i}\, dx = -\int_\Omega f_i^\alpha \cdot \frac{\partial \varphi^\alpha}{\partial x_i}\, dx \tag{6-5}$$

for any $\varphi = (\varphi^\alpha) \in C_0^\infty(\Omega; \mathbb{R}^d)$. Similarly, $u_\varepsilon$ is called a weak solution of (6-3) if $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ and (6-5) holds for any $\varphi = (\varphi^\alpha) \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$. Under the assumptions that $A \in \mathrm{VMO}(\mathbb{R}^d)$ and $\Omega$ is $C^1$, the existence and uniqueness of solutions of (6-1) and (6-3) are more or less well known (see [Auscher and Qafsaoui 2002; Byun and Wang 2004; 2005] for references). The main interest here is that the constants $C$ in the $W^{1,p}$ estimates (6-2) and (6-4) are independent of $\varepsilon$. We mention that for $\mathcal{L}_\varepsilon$ with coefficients satisfying (1-3), (1-11) and the Hölder continuity condition, estimates (6-2) and (6-4) were established in [Avellaneda and Lin 1987; 1991; Shen 2008; Kenig et al. 2013]. The results were extended to the case of almost-periodic coefficients in [Armstrong and Shen 2016]. Also, for $\mathcal{L}_\varepsilon$ with coefficients satisfying (1-2)–(1-3) in Lipschitz domains, some partial results may be found in [Geng et al. 2012].

Theorems 6.1 and 6.2 are proved by a real-variable argument. The required weak reverse Hölder inequalities (6-6) and (6-2) for $p > 2$ are established by combining local estimates for $\mathcal{L}_1$ and boundary Hölder estimates in Section 4 with the interior Lipschitz estimates, up to the scale $\varepsilon$.

**Lemma 6.3.** *Let* $u_\varepsilon \in H^1(B(x_0, 2r); \mathbb{R}^d)$ *be a weak solution to* $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ *in* $B(x_0, 2r)$ *for some* $x_0 \in \mathbb{R}^d$ *and* $r > 0$*. Then, for any* $2 < p < \infty$,

$$\left( \fint_{B(x_0,r)} |\nabla u_\varepsilon|^p \right)^{1/p} \le C_p \left( \fint_{B(x_0,2r)} |\nabla u_\varepsilon|^2 \right)^{1/2}, \tag{6-6}$$

*where* $C_p$ *depends only on* $d$, $p$, $\kappa_1$, $\kappa_2$, *and the function* $\rho(t)$ *in* (5-9).

*Proof.* By translation and dilation we may assume that $x_0 = 0$ and $r = 1$. We may also assume that $0 < \varepsilon < \frac{1}{4}$. The case $\varepsilon \ge \frac{1}{4}$ for $B(0, 1)$ is local, since $A(x/\varepsilon)$ satisfies the smoothness condition (5-9) uniformly in $\varepsilon$. For each $y \in B(0,1)$, we use the local $W^{1,p}$ estimates for the operator $\mathcal{L}_1$ (see, e.g., [Auscher and Qafsaoui 2002; Byun and Wang 2004]) and a simple blow-up argument to show that

$$\left(\fint_{B(y,\varepsilon/2)} |\nabla u_\varepsilon|^p\right)^{1/p} \le C\left(\fint_{B(y,\varepsilon)} |\nabla u_\varepsilon|^2\right)^{1/2}. \tag{6-7}$$

By the interior Lipschitz estimate, up to the scale $\varepsilon$, we have

$$\left(\fint_{B(y,\varepsilon)} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left(\fint_{B(y,1)} |\nabla u_\varepsilon|^2\right)^{1/2}. \tag{6-8}$$

We point out that the estimate (6-8) will be proved in Section 8 with no smoothness assumption on $A$ (see Theorem 8.6). Hence, for any $y \in B(0,1)$,

$$\begin{aligned}\left(\fint_{B(y,\varepsilon/2)} |\nabla u_\varepsilon|^p\right)^{1/p} &\le C\left(\fint_{B(y,1)} |\nabla u_\varepsilon|^2\right)^{1/2}\\ &\le C\|\nabla u_\varepsilon\|_{L^2(B(0,2))}.\end{aligned} \tag{6-9}$$

By covering $B(0,1)$ with balls of radius $\varepsilon/2$, we may deduce (6-6) readily from (6-9). □

**Lemma 6.4.** *Let $u_\varepsilon \in H^1(D_{2r};\mathbb{R}^d)$ be a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon)=0$ in $D_{2r}$ with either $u_\varepsilon=0$ or $\partial u_\varepsilon/\partial\nu_\varepsilon=0$ in $\Delta_{2r}$, where $0<r\le 1$. Then, for any $2<p<\infty$,*

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^p\right)^{1/p} \le C_p\left(\fint_{D_{2r}} |\nabla u_\varepsilon|^2\right)^{1/2}, \tag{6-10}$$

*where $C$ depends only on $d$, $p$, $\kappa_1$, $\kappa_2$, $\tau(t)$ in (5-1), and $\rho(t)$ in (5-9).*

*Proof.* Note that the function $r^{-1}\psi(rx')$ satisfies the condition (5-1) uniformly for $0<r\le 1$. Thus, by rescaling, it suffices to prove the lemma for $r=1$. Using Lemma 6.3, Theorem 5.2 and Theorem 5.6, we obtain

$$\begin{aligned}\left(\fint_{B(y,\delta(y)/8)} |\nabla u_\varepsilon|^p\right)^{1/p} &\le C\left(\fint_{B(y,\delta(y)/4)} |\nabla u_\varepsilon|^2\right)^{1/2}\\ &\le C_\alpha[\delta(y)]^{\alpha-1}\|\nabla u_\varepsilon\|_{L^2(D_2)}\end{aligned} \tag{6-11}$$

for any $\alpha \in (0,1)$, where $y \in D_1$ and $\delta(y)=\mathrm{dist}(y,\partial D_2)$. We now fix $\alpha \in \left(1-\frac{1}{p},1\right)$. It follows from (6-11) that

$$\int_{D_1}\left(\fint_{B(y,\delta(y)/8)} |\nabla u_\varepsilon|^p\,dx\right)dy \le C\|\nabla u_\varepsilon\|^p_{L^2(D_2)}. \tag{6-12}$$

Using the fact that $\delta(x)\approx\delta(y)$ if $y\in D_1$ and $|y-x|<\frac{1}{8}\delta(y)$, it is not hard to verify that (6-12) implies (6-10). □

*Proof of Theorems 6.1 and 6.2.* By duality and a density argument it suffices to consider the case where $p>2$ and $f=(f_i^\alpha)\in C_0^1(\Omega;\mathbb{R}^{d\times d})$. Furthermore, by a real-variable argument, which originated in [Caffarelli and Peral 1998] and further developed in [Shen 2005; 2007], one only needs to establish weak reverse Hölder inequalities for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon)=0$ in $B(x_0,r)\cap\Omega$ with either $u_\varepsilon=0$ or $\partial u_\varepsilon/\partial\nu_\varepsilon=0$

on $B(x_0, r) \cap \partial\Omega$, where $x_0 \in \overline{\Omega}$ and $0 < r < c_0 \operatorname{diam}(\Omega)$. These inequalities are exactly those given by Lemmas 6.3 and 6.4. We omit the details and refer the reader to [Shen 2005; 2008; Geng 2012] for details in the case of scalar elliptic equations. □

**Remark 6.5.** Suppose that $A$ and $\Omega$ satisfy the same conditions as in Theorem 6.1. By some fairly standard extension and duality arguments (see, e.g., [Kenig et al. 2013]), one may deduce from Theorem 6.1 that the solution of the Dirichlet problem

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(h) + F \quad \text{in } \Omega \qquad \text{and} \qquad u_\varepsilon = f \quad \text{on } \partial\Omega$$

satisfies

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p \big\{ \|h\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} + \|f\|_{W^{1/p,p}(\partial\Omega)} \big\}$$

for any $1 < p < \infty$, where $W^{\alpha,p}(\partial\Omega)$ denotes the Sobolev space on $\partial\Omega$ of order $\alpha$ with exponent $p$. Similarly, the solutions of the Neumann problem

$$\mathcal{L}_\varepsilon(u_\varepsilon) = \operatorname{div}(h) + F \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = -n \cdot h + g \quad \text{on } \partial\Omega$$

with $u_\varepsilon \perp \mathcal{R}$ satisfies

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p \big\{ \|h\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\partial\Omega)} \big\},$$

where $W^{-1/p,p}(\partial\Omega)$ is the dual of $W^{1/p,p'}(\partial\Omega)$.

## 7. $L^p$ estimates in $C^1$ domains

The $W^{1,p}$ estimates in the last section allow us to establish the Rellich-type estimates in $L^p$, down to the scale $\varepsilon$, in $C^1$ domains under the additional assumption that $A$ belongs to $\mathrm{VMO}(\mathbb{R}^d)$.

**Theorem 7.1.** *Suppose that $A = A(y)$ satisfies (1-2)–(1-3). Also assume that $A \in \mathrm{VMO}(\mathbb{R}^d)$. Let $1 < p < \infty$ and $\Omega$ be a bounded $C^1$ domain in $\mathbb{R}^d$. Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ be a weak solution to the Dirichlet problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \qquad \text{and} \qquad u_\varepsilon = f \quad \text{in } \partial\Omega, \tag{7-1}$$

*where $F \in L^p(\Omega; \mathbb{R}^d)$ and $f \in W^{1,p}(\partial\Omega; \mathbb{R}^d)$. Then, for any $\varepsilon \le r < \operatorname{diam}(\Omega)$,*

$$\left\{ \frac{1}{r} \int_{\Omega_r} |\nabla u_\varepsilon|^p \right\}^{1/p} \le C_p \big\{ \|F\|_{L^p(\Omega)} + \|f\|_{W^{1,p}(\partial\Omega)} \big\}, \tag{7-2}$$

*where $\Omega_r = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) < r\}$. The constant $C_p$ depends only on $d$, $p$, $A$ and $\Omega$.*

**Theorem 7.2.** *Suppose that $A$ and $\Omega$ satisfy the same conditions as in Theorem 7.1. Let $1 < p < \infty$. Let $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^d)$ be a weak solution to the Neumann problem*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{in } \partial\Omega, \tag{7-3}$$

*where* $F \in L^p(\Omega;\mathbb{R}^d)$, $g \in L^p(\partial\Omega;\mathbb{R}^d)$ *and* $\int_\Omega F + \int_{\partial\Omega} g = 0$. *Also assume that* $u_\varepsilon \perp \mathcal{R}$. *Then, for any* $\varepsilon \le r < \operatorname{diam}(\Omega)$,

$$\left\{\frac{1}{r}\int_{\Omega_r} |\nabla u_\varepsilon|^p\right\}^{1/p} \le C_p\big\{\|F\|_{L^p(\Omega)} + \|g\|_{L^p(\partial\Omega)}\big\}, \tag{7-4}$$

*where* $C_p$ *depends only on* $d$, $p$, $A$ *and* $\Omega$.

The proof of Theorems 7.1 and 7.2 follows a similar line of argument as for Theorems 1.1 and 1.2 by considering

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi_j^\beta(x/\varepsilon)K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right), \tag{7-5}$$

where $u_0$ is the solution of the homogenized problem, $K_\varepsilon$ is a smoothing operator defined by (2-6), and $\eta_\varepsilon \in C_0^\infty(\Omega)$ is a cut-off function satisfying (2-14).

Throughout this section we will assume that $\Omega$ is $C^1$ and $A$ satisfies (1-2)–(1-3) and (5-9).

**Lemma 7.3.** *Let* $u_\varepsilon$ $(\varepsilon \ge 0)$ *be the solutions of the Dirichlet problems* (7-1). *Let* $w_\varepsilon$ *be defined by* (7-5). *Then*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p\varepsilon^{1/p}\big\{\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}, \tag{7-6}$$

*where* $C_p$ *depends only on* $d$, $p$, $A$ *and* $\Omega$.

*Proof.* A direct computation shows that

$$\begin{aligned}\mathcal{L}_\varepsilon(w_\varepsilon) = &-\frac{\partial}{\partial x_i}\left\{[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)\right]\right\}\\ &+\frac{\partial}{\partial x_i}\left\{b_{ij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)\right\}\\ &+\varepsilon\frac{\partial}{\partial x_i}\left\{a_{ij}^{\alpha\beta}(x/\varepsilon)\chi_k^{\beta\gamma}(x/\varepsilon)\frac{\partial}{\partial x_j}\left(K_\varepsilon\left(\frac{\partial u_0^\gamma}{\partial x_k}\eta_\varepsilon\right)\right)\right\},\end{aligned}$$

where $b_{ij}^{\alpha\beta}(y)$ is defined by (2-3). Using (2-5), we obtain

$$\frac{\partial}{\partial x_i}\left\{b_{ij}^{\alpha\beta}(x/\varepsilon)K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)\right\} = -\varepsilon\frac{\partial}{\partial x_i}\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial}{\partial x_k}\left(K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)\right)\right\}.$$

It follows that

$$\begin{aligned}\mathcal{L}_\varepsilon(w_\varepsilon) = &-\frac{\partial}{\partial x_i}\left\{[\hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)]\left[\frac{\partial u_0^\beta}{\partial x_j} - K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)\right]\right\}\\ &-\varepsilon\frac{\partial}{\partial x_i}\left\{\phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial}{\partial x_k}\left(K_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\eta_\varepsilon\right)\right)\right\}\\ &+\varepsilon\frac{\partial}{\partial x_i}\left\{a_{ij}^{\alpha\beta}(x/\varepsilon)\chi_k^{\beta\gamma}(x/\varepsilon)\frac{\partial}{\partial x_j}\left(K_\varepsilon\left(\frac{\partial u_0^\gamma}{\partial x_k}\eta_\varepsilon\right)\right)\right\}.\end{aligned} \tag{7-7}$$

Since $w_\varepsilon = 0$ on $\partial\Omega$, we may apply the $W^{1,p}$ estimate in Theorem 6.1 to obtain

$$\begin{aligned}\|w_\varepsilon\|_{W^{1,p}(\Omega)} &\le C\big\{\|\nabla u_0 - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} + \varepsilon\|\phi(x/\varepsilon)\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} \\ &\qquad\qquad + \varepsilon\|\chi(x/\varepsilon)\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)}\big\} \\ &\le C\big\{\|\nabla u_0 - K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} + \varepsilon\|\nabla((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)}\big\} \\ &\le C\big\{\|\nabla u_0\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|(\nabla^2 u_0)\eta_\varepsilon\|_{L^2(\Omega)}\big\},\end{aligned} \tag{7-8}$$

where we have used Lemmas 2.1 and 2.2 for the second and third inequalities.

We now write $u_0 = v + w$, where

$$v(x) = \int_\Omega \Gamma_0(x-y)F(y)\,dy \tag{7-9}$$

and $\Gamma_0(x-y)$ denotes the matrix of fundamental solutions for the operator $\mathcal{L}_0$ in $\mathbb{R}^d$, with pole at the origin. Note that $\|v\|_{W^{2,p}(\mathbb{R}^d)} \le C_p\|F\|_{L^p(\Omega)}$ and

$$\|\nabla v\|_{L^p(S_t)} \le C_p\|F\|_{L^p(\Omega)},$$

where $S_t = \{x \in \mathbb{R}^d : \operatorname{dist}(x,\partial\Omega) = t\}$ for $t$ small (see the proof of Theorem 2.6). It follows that

$$\|\nabla v\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|\nabla^2 v\|_{L^p(\Omega)} \le C\varepsilon^{1/p}\|F\|_{L^p(\Omega)}. \tag{7-10}$$

Finally, we observe that $\mathcal{L}_0(w) = 0$ in $\Omega$ and

$$\|w\|_{W^{1,p}(\partial\Omega)} \le \|f\|_{W^{1,p}(\partial\Omega)} + \|v\|_{W^{1,p}(\partial\Omega)} \le C\big\{\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}.$$

It follows from the solvability of the $L^p$ regularity problem for the operator $\mathcal{L}_0$ in $C^1$ domain $\Omega$, which follows from [Fabes et al. 1978; Lewis et al. 1993; Hofmann et al. 2015], that

$$\|(\nabla w)^*\|_{L^p(\partial\Omega)} \le C\big\{\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}.$$

Also, using the interior estimate

$$|\nabla^2 w(x)| \le \frac{C}{\delta(x)}\bigg(\fint_{B(x,\delta(x)/8)} |\nabla w|^p\bigg)^{1/p},$$

where $\delta(x) = \operatorname{dist}(x,\partial\Omega)$, we may show that

$$\begin{aligned}\int_{\Omega\setminus\Omega_{3\varepsilon}} |\nabla^2 w|^p\,dx &\le C\int_{\Omega\setminus\Omega_{2\varepsilon}} |\nabla w(x)|^p[\delta(x)]^{-p}\,dx \\ &\le C\varepsilon^{1-p}\|(\nabla w)^*\|^p_{L^p(\partial\Omega)} \le C\varepsilon^{1-p}\big\{\|f\|^p_{W^{1,p}(\partial\Omega)} + \|F\|^p_{L^p(\Omega)}\big\}.\end{aligned}$$

As a result, we obtain

$$\|\nabla w\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|(\nabla^2 w)\eta_\varepsilon\|_{L^p(\Omega)} \le C\varepsilon^{1/p}\big\{\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}.$$

This, together with the estimate (7-10) for $v$, gives

$$\|\nabla u_0\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|(\nabla^2 u_0)\eta_\varepsilon\|_{L^p(\Omega)} \le C\varepsilon^{1/p}\big\{\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}, \tag{7-11}$$

which, in view of (7-8), completes the proof. □

*Proof of Theorem 7.1.* Without loss of generality we may assume that

$$\|f\|_{W^{1,p}(\partial\Omega)} + \|F\|_{L^p(\Omega)} = 1.$$

Let $\varepsilon \le r < \operatorname{diam}(\Omega)$. It follows from Lemma 7.3 that

$$\begin{aligned}
&\|\nabla u_\varepsilon\|_{L^p(\Omega_r)} \\
&\quad \le \|\nabla u_0\|_{L^p(\Omega_r)} + C\|\nabla\chi(x/\varepsilon)K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega_r)} + C\varepsilon\|\chi(x/\varepsilon)\nabla K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega_r)} + C\varepsilon^{1/p} \\
&\quad \le C\|\nabla u_0\|_{L^p(\Omega_{2r})} + C\varepsilon\|\nabla((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} + C\varepsilon^{1/p} \\
&\quad \le C\|\nabla u_0\|_{L^p(\Omega_{2r})} + C\varepsilon^{1/p},
\end{aligned} \tag{7-12}$$

where we have used Lemma 2.1 for the second inequality and (7-11) for the third. An inspection of the proof of Lemma 7.3 shows that

$$\|\nabla u_0\|_{L^p(\Omega_{2r})} \le C r^{1/p},$$

which, in view of (7-12), gives

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_r)} \le C r^{1/p}.$$

□

To prove Theorem 7.2, we need the following lemma.

**Lemma 7.4.** *Let $u_\varepsilon$ ($\varepsilon \ge 0$) be solutions of the Neumann problem (7-3). Also assume that $u_\varepsilon, u_0 \perp \mathcal{R}$. Let $w_\varepsilon$ be defined by (7-5). Then*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p \varepsilon^{1/p}\big\{\|g\|_{L^p(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}, \tag{7-13}$$

*where $C_p$ depends only on $d$, $p$, $A$ and $\Omega$.*

*Proof.* The proof is similar to that of Lemma 7.3. Let $\phi_\varepsilon$ be a function in $\mathcal{R}$ such that $w_\varepsilon - \phi_\varepsilon \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. It follows from the formula (7-7) and the $W^{1,p}$ estimates in Theorem 6.2 that

$$\|w_\varepsilon - \phi_\varepsilon\|_{W^{1,p}(\Omega)} \le C\big\{\|\nabla u_0\|_{L^p(\Omega_{4\varepsilon})} + \varepsilon\|(\nabla^2 u_0)\eta_\varepsilon\|_{L^2(\Omega)}\big\}. \tag{7-14}$$

To estimate the right-hand side of (7-14), we proceed as in the proof of Lemma 7.3, but use the nontangential maximal function estimate [Fabes et al. 1978; Lewis et al. 1993; Hofmann et al. 2015]

$$\|(\nabla w)^*\|_{L^p(\partial\Omega)} \le C\left\|\frac{\partial w}{\partial\nu_0}\right\|_{L^p(\partial\Omega)},$$

where $\mathcal{L}_0(w) = 0$ in $\Omega$ and $w \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. As a result, we obtain

$$\|w_\varepsilon - \phi_\varepsilon\|_{W^{1,p}(\Omega)} \le C\varepsilon^{1/p}\big\{\|g\|_{L^p(\partial\Omega)} + \|F\|_{L^p(\Omega)}\big\}. \tag{7-15}$$

Finally, note that since $u_\varepsilon - u_0 \perp \mathcal{R}$,

$$\begin{aligned}
\|\phi_\varepsilon\|_{W^{1,p}(\Omega)} &\le C\varepsilon\|\chi(x/\varepsilon)K_\varepsilon((\nabla u_0)\eta_\varepsilon)\|_{L^p(\Omega)} \\
&\le C\varepsilon\|\nabla u_0\|_{L^p(\Omega)}.
\end{aligned}$$

This, together with (7-15), yields the estimate (7-13). □

*Proof of Theorem 7.2.* The estimate (7-4) follows from (7-13), as in the case of the Dirichlet conditions. We omit the details. □

**Remark 7.5.** Under certain smoothness condition on $A$, such as Hölder continuity, it is possible to solve the $L^p$ Dirichlet, regularity, and Neumann problems for $\mathcal{L}_1(u) = 0$ in $C^1$ domains for any $1 < p < \infty$. By the same localization procedure and blow-up argument as in Remark 3.1, this implies

$$\begin{cases} \displaystyle\int_{\partial\Omega} |\nabla u_\varepsilon|^p \, d\sigma \le C \int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^p d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^p \, dx, \\ \displaystyle\int_{\partial\Omega} |\nabla u_\varepsilon|^p \, d\sigma \le C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^p \, d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{c\varepsilon}} |\nabla u_\varepsilon|^p \, dx, \end{cases} \tag{7-16}$$

where $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $\Omega$. It then follows from Theorems 7.1 and 7.2 that

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^p \, d\sigma \le C \int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^p d\sigma \tag{7-17}$$

if $u_\varepsilon \perp \mathcal{R}$, and

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^p \, d\sigma \le C \int_{\partial\Omega} |\nabla_{\tan} u_\varepsilon|^p \, d\sigma + C \int_{\partial\Omega} |u_\varepsilon|^p \, d\sigma. \tag{7-18}$$

As in the case $p = 2$, by the method of layer potentials, estimates (7-17)–(7-18) lead to the uniform solvability of the $L^p$ Dirichlet, regularity, and Neumann problems in $C^1$ domains. The details will be given elsewhere.

## 8. Lipschitz estimates in $C^{1,\alpha}$ domains, part I

In this section we investigate the Lipschitz estimates, down to the scale $\varepsilon$, in $C^{1,\alpha}$ domains with Dirichlet boundary conditions and give the proof of Theorem 1.4. The Neumann boundary conditions will be treated in the next section. The proof of Theorems 1.4 and 1.5 is based on a general scheme for establishing Lipschitz estimates at large scales in homogenization, recently formulated in [Armstrong and Smart 2016] for interior estimates. Our approach to the boundary Lipschitz estimates in $C^{1,\alpha}$ domains is similar to that used in [Armstrong and Shen 2016] for elliptic systems with almost-periodic coefficients. We remark that Lemma 8.5, which is a continuous version of Lemma 3.1 in [Armstrong and Shen 2016] and whose proof is simpler, makes the argument more transparent.

Let $D_r$ and $\Delta_r$ be defined by (1-16) with $\psi(0) = 0$ and $\|\nabla\psi\|_\infty \le M$.

**Lemma 8.1.** *Let $u_\varepsilon \in H^1(D_2; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $D_2$ with $u_\varepsilon = f$ on $\Delta_2$. Then there exists $v \in H^1(D_1; \mathbb{R}^d)$ such that $\mathcal{L}_0(v) = F$ in $D_1$, $v = f$ on $\Delta_1$, and*

$$\|u_\varepsilon - v\|_{L^2(D_1)} \le C\varepsilon^{1/2}\big\{\|u_\varepsilon\|_{L^2(D_2)} + \|F\|_{L^2(D_2)} + \|f\|_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|_{L^\infty(\Delta_2)}\big\}, \tag{8-1}$$

*where $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and $M$.*

*Proof.* By Caccioppoli's inequality,

$$\int_{D_{3/2}} |\nabla u_\varepsilon|^2 \le C\left\{\int_{D_2} |u_\varepsilon|^2 + \int_{D_2} |F|^2 + \|f\|^2_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|^2_{L^\infty(\Delta_2)}\right\}.$$

By the coarea formula this implies that there exists some $t \in \left[\frac{5}{4}, \frac{3}{2}\right]$ such that

$$\int_{\partial D_t \setminus \Delta_2} (|\nabla u_\varepsilon|^2 + |u_\varepsilon|^2) \le C\left\{\int_{D_2} |u_\varepsilon|^2 + \int_{D_2} |F|^2 + \|f\|^2_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|^2_{L^\infty(\Delta_2)}\right\}.$$

Let $v$ be the weak solution to the Dirichlet problem,

$$\mathcal{L}_0(v) = F \quad \text{in } D_t \qquad \text{and} \qquad v = u_\varepsilon \quad \text{on } \partial D_t.$$

It follows from Remark 2.8 that

$$\begin{aligned} \|u_\varepsilon - v\|_{L^2(D_1)} &\le \|u_\varepsilon - v\|_{L^2(D_t)} \\ &\le C\varepsilon^{1/2}\left\{\|u_\varepsilon\|_{H^1(\partial D_t)} + \|F\|_{L^2(D_t)}\right\} \\ &\le C\varepsilon^{1/2}\left\{\|u_\varepsilon\|_{L^2(D_2)} + \|F\|_{L^2(D_2)} + \|f\|_{L^\infty(\Delta_2)} + \|\nabla_{\tan} f\|_{L^\infty(\Delta_2)}\right\}, \end{aligned}$$

where $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and $M$. □

**Lemma 8.2.** *Let $\varepsilon \le r < 1$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $D_{2r}$ with $u_\varepsilon = f$ on $\Delta_{2r}$. Then there exists $v \in H^1(D_r; \mathbb{R}^d)$ such that $\mathcal{L}_0(v) = F$ in $D_r$, $v = f$ on $\Delta_r$, and*

$$\begin{aligned} &\left(\fint_{D_r} |u_\varepsilon - v|^2\right)^{1/2} \\ &\quad \le C(\varepsilon/r)^{1/2}\left\{\left(\fint_{D_{2r}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{D_{2r}} |F|^2\right)^{1/2} + \|f\|_{L^\infty(\Delta_{2r})} + r\|\nabla_{\tan} f\|_{L^\infty(\Delta_{2r})}\right\}, \end{aligned} \tag{8-2}$$

*where $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and $M$.*

*Proof.* This follows from Lemma 8.1 by rescaling. □

In the rest of this section we will assume that the defining function $\psi$ in the definition of $D_r$ and $\Delta_r$ is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$ with $\psi(0) = 0$ and $\|\nabla\psi\|_{C^\alpha(\mathbb{R}^{d-1})} \le M$.

**Lemma 8.3.** *Let $v$ be a solution of $\mathcal{L}_0(v) = F$ in $D_r$ with $v = f$ on $\Delta_r$. For $0 < t \le r$, define*

$$\begin{aligned} G(t; v) = \frac{1}{t} \inf_{\substack{M \in \mathbb{R}^{d\times d} \\ q \in \mathbb{R}^d}} \Bigg\{&\left(\fint_{D_t} |v - Mx - q|^2\right)^{1/2} + t^2\left(\fint_{D_t} |F|^p\right)^{1/p} + \|f - Mx - q\|_{L^\infty(\Delta_t)} \\ &+ t\|\nabla_{\tan}(f - Mx - q)\|_{L^\infty(\Delta_t)} + t^{1+\sigma}\|\nabla_{\tan}(f - Mx - q)\|_{C^{0,\sigma}(\Delta_t)}\Bigg\}, \end{aligned} \tag{8-3}$$

*where $p > d$ and $\sigma \in (0, \alpha)$. Then there exists $\theta \in \left(0, \frac{1}{4}\right)$, depending only on $d$, $p$, $\kappa_1$, $\kappa_2$, $\sigma$, $\alpha$ and $M$, such that*

$$G(\theta r; v) \le \tfrac{1}{2} G(r; v). \tag{8-4}$$

*Proof.* The lemma follows from the boundary $C^{1,\alpha}$ estimates for elasticity systems with constant coefficients. We refer the reader to [Armstrong and Shen 2016, Lemma 7.1] for the case $\mathcal{L}_0(v) = 0$. The argument for the general case $F \in L^p$ with $p > d$ is the same. □

**Lemma 8.4.** *Let* $0 < \varepsilon < \frac{1}{2}$. *Let* $u_\varepsilon$ *be a solution of* $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ *in* $D_1$ *with* $u_\varepsilon = f$ *on* $\Delta_1$. *Define*

$$H(r) = \frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{d\times d}\\ q\in\mathbb{R}^d}} \left\{ \left( \fint_{D_r} |u_\varepsilon - Mx - q|^2 \right)^{1/2} + r^2 \left( \fint_{D_r} |F|^p \right)^{1/p} + \|f - Mx - q\|_{L^\infty(\Delta_r)} + r\|\nabla_{\tan}(f - Mx - q)\|_{L^\infty(\Delta_r)} + r^{1+\sigma} \|\nabla_{\tan}(f - Mx - q)\|_{C^{0,\sigma}(\Delta_r)} \right\} \tag{8-5}$$

*and*

$$\Phi(r) = \frac{1}{r} \inf_{q\in\mathbb{R}^d} \left\{ \left( \fint_{D_{2r}} |u_\varepsilon - q|^2 \right)^{1/2} + r^2 \left( \fint_{D_{2r}} |F|^p \right)^{1/p} + \|f - q\|_{L^\infty(\Delta_{2r})} + r\|\nabla_{\tan} f\|_{L^\infty(\Delta_{2r})} \right\}, \tag{8-6}$$

*where* $p > d$ *and* $\sigma \in (0, \alpha)$. *Then*

$$H(\theta r) \le \tfrac{1}{2} H(r) + C\left(\frac{\varepsilon}{r}\right)^{1/2} \Phi(2r) \tag{8-7}$$

*for any* $r \in \left[\varepsilon, \frac{1}{2}\right]$, *where* $\theta \in \left(0, \frac{1}{4}\right)$ *is given by Lemma 8.3.*

*Proof.* Fix $r \in \left[\varepsilon, \frac{1}{2}\right]$. Let $v$ be a solution of $\mathcal{L}_0(v) = F$ in $D_r$ with $v = f$ on $\Delta_r$. Observe that

$$\begin{aligned} H(\theta r) &\le \frac{1}{\theta r} \left( \fint_{D_{\theta r}} |u_\varepsilon - v|^2 \right)^{1/2} + G(\theta r; v) \\ &\le \frac{1}{\theta r} \left( \fint_{D_{\theta r}} |u_\varepsilon - v|^2 \right)^{1/2} + \tfrac{1}{2} G(r; v) \\ &\le \frac{C}{r} \left( \fint_{D_r} |u_\varepsilon - v|^2 \right)^{1/2} + \tfrac{1}{2} H(r), \end{aligned}$$

where we have used Lemma 8.3 for the second inequality. This, together with Lemma 8.2, gives

$$H(\theta r) \le \tfrac{1}{2} H(r) + C\left(\frac{\varepsilon}{r}\right)^{1/2} \frac{1}{r} \left\{ \left( \fint_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left( \fint_{D_{2r}} |F|^2 \right)^{1/2} + \|f\|_{L^\infty(\Delta_{2r})} + r\|\nabla_{\tan} f\|_{L^\infty(\Delta_{2r})} \right\}.$$

Since $H(r)$ remains invariant if we subtract a constant from $u_\varepsilon$, the inequality (8-7) follows. □

**Lemma 8.5.** *Let* $H(r)$ *and* $h(r)$ *be two nonnegative continuous functions on the interval* $(0, 1]$. *Let* $0 < \varepsilon < \frac{1}{4}$. *Suppose that there exists a constant* $C_0$ *such that*

$$\begin{cases} \max\limits_{r \le t \le 2r} H(t) \le C_0 H(2r), \\ \max\limits_{r \le t, s \le 2r} |h(t) - h(s)| \le C_0 H(2r) \end{cases} \tag{8-8}$$

*for any* $r \in \left[\varepsilon, \frac{1}{2}\right]$. *We further assume that*

$$H(\theta r) \le \tfrac{1}{2} H(r) + C_0 \omega(\varepsilon/r) \{H(2r) + h(2r)\} \tag{8-9}$$

*for any* $r \in \left[\varepsilon, \frac{1}{2}\right]$, *where* $\theta \in \left(0, \frac{1}{4}\right)$ *and* $\omega$ *is a nonnegative increasing function* $[0, 1]$ *such that* $\omega(0) = 0$ *and*

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty. \tag{8-10}$$

*Then*

$$\max_{\varepsilon\le r\le 1}\{H(r)+h(r)\}\le C\{H(1)+h(1)\}, \tag{8-11}$$

*where $C$ depends only on $C_0$, $\theta$, and $\omega$.*

*Proof.* It follows from (8-8) that

$$h(r)\le h(2r)+C_0H(2r)$$

for any $\varepsilon\le r\le \frac{1}{2}$. Hence,

$$\begin{aligned}\int_a^{1/2}\frac{h(r)}{r}\,dr&\le\int_a^{1/2}\frac{h(2r)}{r}\,dr+C_0\int_a^{1/2}\frac{H(2r)}{r}\,dr\\&=\int_{2a}^{1}\frac{h(r)}{r}\,dr+C_0\int_{2a}^{1}\frac{H(r)}{r}\,dr,\end{aligned}$$

where $\varepsilon\le a\le\frac{1}{4}$. This implies

$$\begin{aligned}\int_a^{2a}\frac{h(r)}{r}\,dr&\le\int_{1/2}^{1}\frac{h(r)}{r}\,dr+C\int_{2a}^{1}\frac{H(r)}{r}\,dr\\&\le C\{h(1)+H(1)\}+C\int_{2a}^{1}\frac{H(r)}{r}\,dr,\end{aligned}$$

which, by (8-8), gives

$$\begin{aligned}h(a)&\le C\left\{H(2a)+h(1)+H(1)+\int_{2a}^{1}\frac{H(r)}{r}\,dr\right\}\\&\le C\left\{h(1)+H(1)+\int_{a}^{1}\frac{H(r)}{r}\,dr\right\}\end{aligned} \tag{8-12}$$

for any $a\in\left[\varepsilon,\frac{1}{4}\right]$.

Next, we use (8-9) and (8-12) to obtain

$$H(\theta r)\le\tfrac{1}{2}H(r)+C\omega(\varepsilon/r)\{h(1)+H(1)\}+C\omega(\varepsilon/r)\int_r^1\frac{H(r)}{r}\,dr.$$

It follows that

$$\int_{\alpha\theta\varepsilon}^{\theta}\frac{H(r)}{r}\,dr\le\frac{1}{2}\int_{\alpha\varepsilon}^{1}\frac{H(r)}{r}\,dr+C_\alpha\{h(1)+H(1)\}+C\int_{\alpha\varepsilon}^{1}\omega(\varepsilon/r)\left\{\int_r^1\frac{H(t)}{t}\,dt\right\}\frac{dr}{r},$$

where $\alpha>1$ and we have used the condition (8-10). Using (8-10) and the observation that

$$\int_{\alpha\varepsilon}^{1}\omega(\varepsilon/r)\left\{\int_r^1\frac{H(t)}{t}\,dt\right\}\frac{dr}{r}=\int_{\alpha\varepsilon}^{1}H(t)\left\{\int_{\varepsilon/t}^{1/\alpha}\frac{\omega(s)}{s}\,ds\right\}\frac{dt}{t}\le(4C)^{-1}\int_{\alpha\varepsilon}^{1}H(t)\frac{dt}{t}$$

if $\alpha>\alpha_0(\omega)$, we see that

$$\int_{\alpha\theta\varepsilon}^{\theta}\frac{H(r)}{r}\,dr\le\frac{1}{2}\int_{\alpha\varepsilon}^{1}\frac{H(r)}{r}\,dr+C_\alpha\{h(1)+H(1)\}+\frac{1}{4}\int_{\alpha\varepsilon}^{1}\frac{H(r)}{r}\,dr.$$

It follows that

$$\int_{\varepsilon}^{1} \frac{H(r)}{r}\, dr \le C\{h(1)+H(1)\}, \tag{8-13}$$

which, together with (8-8) and (8-12), yields the estimate (8-11). □

*Proof of Theorem 1.4.* We may assume that $0<\varepsilon<\frac{1}{4}$. Let $u_\varepsilon$ be a solution of $\mathcal{L}_\varepsilon(u_\varepsilon)=F$ in $D_1$ with $u_\varepsilon = f$ on $\Delta_1$, where $F\in L^p(D_1)$ for some $p>d$ and $f\in C^{1,\sigma}(\Delta_1)$ for some $\sigma\in(0,\alpha)$. For $r\in(0,1)$, we define the function $H(r)$ by (8-5). It is easy to see that $H(t)\le CH(2r)$ if $t\in(r,2r)$.

Next, we let $h(r)=|M_r|$, where $M_r$ is the $d\times d$ matrix such that

$$\begin{aligned} H(r)=\frac{1}{r}\inf_{q\in\mathbb{R}^d}\biggl\{\biggl(\fint_{D_r}|u_\varepsilon - M_r x - q|^2\biggr)^{1/2} + r^2\biggl(\fint_{D_r}|F|^p\biggr)^{1/p} + \|f-M_r x - q\|_{L^\infty(\Delta_r)} \\ + r\|\nabla_{\tan}(f-M_r x-q)\|_{L^\infty(\Delta_r)} + r^{1+\sigma}\|\nabla_{\tan}(f-M_r x - q)\|_{C^{0,\sigma}(\Delta_r)}\biggr\}. \end{aligned}$$

Let $t, s\in[r,2r]$. Using

$$\begin{aligned} |M_t-M_s| &\le \frac{C}{r}\inf_{q\in\mathbb{R}^d}\biggl(\fint_{D_r}|(M_t-M_s)x-q|^2\biggr)^{1/2} \\ &\le \frac{C}{t}\inf_{q\in\mathbb{R}^d}\biggl(\fint_{D_t}|u_\varepsilon - M_t x-q|^2\biggr)^{1/2} + \frac{C}{s}\inf_{q\in\mathbb{R}^d}\biggl(\fint_{D_s}|u_\varepsilon - M_s x - q|^2\biggr)^{1/2} \\ &\le C\{H(t)+H(s)\} \\ &\le CH(2r), \end{aligned}$$

we obtain

$$\max_{r\le t,s\le 2r}|h(t)-h(s)|\le CH(2r).$$

Furthermore, if $\Phi$ is defined by (8-6), then

$$\Phi(r)\le H(2r)+h(2r).$$

In view of Lemma 8.4 this gives

$$H(\theta r)\le \tfrac{1}{2}H(r)+C\omega(\varepsilon/r)\{H(2r)+h(2r)\}$$

for $r\in\bigl[\varepsilon,\frac{1}{2}\bigr]$, where $\omega(t)=t^{1/2}$. Thus the functions $H(r)$ and $h(r)$ satisfy the conditions (8-8), (8-9) and (8-10) in Lemma 8.5. Consequently, we obtain that for $r\in\bigl[\varepsilon,\frac{1}{2}\bigr]$,

$$\begin{aligned} \inf_{q\in\mathbb{R}^d}\frac{1}{r}\biggl(\fint_{D_r}|u_\varepsilon - q|^2\biggr)^{1/2} &\le C\{H(r)+h(r)\} \\ &\le C\{H(1)+h(1)\} \\ &\le C\biggl\{\biggl(\fint_{D_1}|u_\varepsilon|^2\biggr)^{1/2} + \|F\|_{L^p(D_1)} + \|f\|_{C^{1,\sigma}(\Delta_1)}\biggr\}, \end{aligned}$$

which, together with Caccioppoli's inequality, gives the estimate (1-18). □

The argument used in this section may be used to prove the interior Lipschitz estimates, down to the scale $\varepsilon$.

**Theorem 8.6.** *Suppose that $A$ satisfies (1-2)–(1-3). Let $u_\varepsilon \in H^1(B(x_0, R); \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > 0$, where $F \in L^p(B(x_0, R); \mathbb{R}^d)$ for some $p > d$. Then, for $\varepsilon \le r < R$,*

$$\left(\fint_{B(x_0,r)} |\nabla u_\varepsilon|^2\right)^{1/2} \le C\left\{\left(\fint_{B(x_0,R)} |\nabla u_\varepsilon|^2\right)^{1/2} + R\left(\fint_{B(x_0,R)} |F|^p\right)^{1/p}\right\}, \tag{8-14}$$

*where $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and $p$.*

## 9. Lipschitz estimates in $C^{1,\alpha}$ domains, part II

In this section we study the Lipschitz estimate, down to the scale $\varepsilon$, with Neumann boundary conditions, and give the proof of Theorem 1.5. Throughout this section we will assume that the defining function $\psi$ in $D_r$ and $\Delta_r$ is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$ and $\|\nabla\psi\|_{C^\alpha(\mathbb{R}^{d-1})} \le M$.

**Lemma 9.1.** *Let $\Omega$ be a bounded Lipschitz domain. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be a weak solution to the Neumann problem: $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $\Omega$ and $\partial u_\varepsilon/\partial\nu_\varepsilon = g$ on $\partial\Omega$. Then there exists $w \in H^1(\Omega; \mathbb{R}^d)$ such that $\mathcal{L}_0(w) = F$ in $\Omega$, $\partial w/\partial\nu_0 = g$ on $\partial\Omega$, and*

$$\|u_\varepsilon - w\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\left\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^2(\Omega)}\right\}. \tag{9-1}$$

*Proof.* Choose $\phi_\varepsilon \in \mathcal{R}$ such that $u_\varepsilon - \phi_\varepsilon \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. Let $u_0$ be the weak solution to the Neumann problem: $\mathcal{L}_0(u_0) = F$ in $\Omega$ and $\partial u_0/\partial\nu_0 = g$ on $\partial\Omega$ with the property $u_0 \perp \mathcal{R}$. It follows from Remark 2.8 that

$$\|u_\varepsilon - \phi_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\left\{\|g\|_{L^2(\partial\Omega)} + \|F\|_{L^2(\Omega)}\right\}.$$

By letting $w = u_0 + \phi_\varepsilon$ this gives (9-1). □

**Lemma 9.2.** *Let $\varepsilon \le r < 1$. Let $u_\varepsilon \in H^1(D_{2r}; \mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $D_{2r}$ with $\partial u_\varepsilon/\partial\nu_\varepsilon = g$ on $\Delta_{2r}$. Then there exists $w \in H^1(D_r; \mathbb{R}^d)$ such that $\mathcal{L}_0(w) = F$ in $D_r$, $\partial w/\partial\nu_0 = g$ on $\Delta_r$, and*

$$\left(\fint_{D_r} |u_\varepsilon - w|^2\right)^{1/2} \le C(\varepsilon/r)^{1/2}\left\{\left(\fint_{D_{2r}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{D_{2r}} |F|^2\right)^{1/2} + r\|g\|_{L^\infty(\Delta_{2r})}\right\}, \tag{9-2}$$

*where $C$ depends only on $d$, $\kappa_1$, $\kappa_2$, and $M$.*

*Proof.* By rescaling we may assume $r = 1$. As in the case of Dirichlet conditions in Lemma 8.2, the desired estimate follows from Lemma 9.1 by using the coarea formula and the Caccioppoli inequality

$$\int_{D_{3/2}} |\nabla u_\varepsilon|^2 \le C\left\{\int_{D_2} |u_\varepsilon|^2 + \int_{D_2} |F|^2 + \|g\|^2_{L^\infty(\Delta_2)}\right\}, \tag{9-3}$$

where $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $D_2$ and $\partial u_\varepsilon/\partial\nu_\varepsilon = g$ on $\Delta_2$. □

**Lemma 9.3.** *Let $w$ be a solution of $\mathcal{L}_0(w) = F$ in $D_r$ with $\partial w/\partial \nu_0 = g$ on $\Delta_r$. For $0 < t \le r$, define*

$$I(t; w) = \frac{1}{t} \inf_{\substack{M\in\mathbb{R}^{d\times d}\\ q\in\mathbb{R}^d}} \left\{ \left( \fint_{D_t} |w - Mx - q|^2 \right)^{1/2} + t^2 \left( \fint_{D_t} |F|^p \right)^{1/p} + t \left\| \frac{\partial}{\partial \nu_0}(w - Mx) \right\|_{L^\infty(\Delta_t)} + t^{1+\sigma} \left\| \frac{\partial}{\partial \nu_0}(w - Mx) \right\|_{C^{0,\sigma}(\Delta_t)} \right\}, \quad (9\text{-}4)$$

*where $p > d$ and $\sigma \in (0, \alpha)$. Then there exists $\theta \in \left(0, \frac{1}{4}\right)$, depending only on $d$, $p$, $\kappa_1$, $\kappa_2$, $\sigma$, $\alpha$ and $M$, such that*

$$I(\theta r; w) \le \tfrac{1}{2} I(r; w). \quad (9\text{-}5)$$

*Proof.* By rescaling we may assume $r = 1$. The lemma then follows from the boundary $C^{1,\sigma}$ estimates with Neumann boundary conditions in $C^{1,\alpha}$ domains for elasticity systems with constant coefficients. □

**Lemma 9.4.** *Let $0 < \varepsilon < \frac{1}{2}$. Let $u_\varepsilon$ be a solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $D_1$ with $\partial u_\varepsilon/\partial \nu_\varepsilon = g$ on $\Delta_1$, where $F \in L^p(D_1; \mathbb{R}^d)$ for some $p > d$ and $g \in C^\sigma(\Delta_1; \mathbb{R}^d)$ for some $\sigma \in (0, \alpha)$. Define*

$$J(r) = \frac{1}{r} \inf_{\substack{M\in\mathbb{R}^{d\times d}\\ q\in\mathbb{R}^d}} \left\{ \left( \fint_{D_r} |u_\varepsilon - Mx - q|^2 \right)^{1/2} + r^2 \left( \fint_{D_r} |F|^p \right)^{1/p} + r \left\| g - \frac{\partial}{\partial \nu_0}(Mx) \right\|_{L^\infty(\Delta_r)} + r^{1+\sigma} \left\| g - \frac{\partial}{\partial \nu_0}(Mx) \right\|_{C^{0,\sigma}(\Delta_r)} \right\} \quad (9\text{-}6)$$

*and*

$$\Psi(r) = \frac{1}{r} \inf_{q\in\mathbb{R}^d} \left\{ \left( \fint_{D_{2r}} |u_\varepsilon - q|^2 \right)^{1/2} + r^2 \left( \fint_{D_{2r}} |F|^p \right)^{1/p} + r \|g\|_{L^\infty(\Delta_{2r})} \right\}. \quad (9\text{-}7)$$

*Then*

$$J(\theta r) \le \tfrac{1}{2} J(r) + C(\varepsilon/r)^{1/2} \Psi(2r) \quad (9\text{-}8)$$

*for any $r \in \left[\varepsilon, \frac{1}{2}\right]$, where $\theta \in \left(0, \frac{1}{4}\right)$ is given by Lemma 9.3.*

*Proof.* Fix $r \in \left[\varepsilon, \frac{1}{2}\right]$. Let $w$ be the function in $H^1(D_r; \mathbb{R}^d)$ given by Lemma 9.2. Then

$$\begin{aligned} J(\theta r) &\le I(\theta r; w) + \frac{1}{\theta r} \left( \fint_{D_{\theta r}} |u_\varepsilon - w|^2 \right)^{1/2} \\ &\le \tfrac{1}{2} I(r; w) + \frac{1}{\theta r} \left( \fint_{D_{\theta r}} |u_\varepsilon - w|^2 \right)^{1/2} \\ &\le \tfrac{1}{2} J(r) + \frac{C}{r} \left( \fint_{D_r} |u_\varepsilon - w|^2 \right)^{1/2}, \end{aligned}$$

where we have used Lemma 9.3 for the second inequality. In view of Lemma 9.2, this gives

$$J(\theta r) \le \tfrac{1}{2} J(r) + \frac{C}{r} \left\{ \left( \fint_{D_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left( \fint_{D_{2r}} |F|^p \right)^{1/p} + r \|g\|_{L^\infty(\Delta_{2r})} \right\},$$

from which the estimate (9-8) follows, as the function $J(r)$ is invariant if we replace $u_\varepsilon$ by $u_\varepsilon - q$ for any $q \in \mathbb{R}^d$. □

*Proof of Theorem 1.5.* With Lemma 9.4 at our disposal, Theorem 1.5 follows from Lemma 8.5, as in the case of Dirichlet boundary conditions. We omit the details. □

As we indicate in the Introduction, under additional smoothness conditions, the full Lipschitz estimates, uniform in $\varepsilon$, follow from Theorem 1.4, Theorem 1.5, and local Lipschitz estimates by a blow-up argument.

**Corollary 9.5.** *Suppose that $A$ satisfies (1-2)–(1-3). Also assume that $A$ is Hölder continuous. Let $u_\varepsilon \in H^1(B(0,1);\mathbb{R}^d)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(0,1)$, where $F \in L^p(B(0,1);\mathbb{R}^d)$ for some $p > d$. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(B(0,1/2))} \le C_p\big\{\|u_\varepsilon\|_{L^2(B(0,1))} + \|F\|_{L^p(B(0,1))}\big\}, \tag{9-9}$$

*where $C_p$ depends only on $d$, $p$ and $A$.*

**Corollary 9.6.** *Suppose that $A$ satisfies (1-2)–(1-3). Also assume that $A$ is Hölder continuous. Let $u_\varepsilon \in H^1(D_1;\mathbb{R}^d)$ be a weak solution of $\mathcal{L}(u_\varepsilon) = F$ in $D_1$ with $u_\varepsilon = f$ on $\Delta_1$, where the defining function $\psi$ in $D_1$ and $\Delta_1$ is $C^{1,\alpha}$ with $\|\nabla\psi\|_{C^\alpha(\mathbb{R}^{d-1})} \le M$ for some $\alpha > 0$. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \le C\big\{\|u_\varepsilon\|_{L^2(D_1)} + \|F\|_{L^p(D_1)} + \|f\|_{C^{1,\sigma}(\Delta_1)}\big\}, \tag{9-10}$$

*where $p > d$, $\sigma \in (0,\alpha)$, and $C$ depends only on $d$, $p$, $\sigma$, $A$, $\alpha$ and $M$.*

**Corollary 9.7.** *Suppose that $A$, $D_1$ and $\Delta_1$ satisfy the same conditions as in Corollary 9.6. Let $u_\varepsilon \in H^1(D_1;\mathbb{R}^d)$ be a weak solution of $\mathcal{L}(u_\varepsilon) = F$ in $D_1$ with $\partial u_\varepsilon/\partial\nu_\varepsilon = g$ on $\Delta_1$. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(D_{1/2})} \le C\big\{\|u_\varepsilon\|_{L^2(D_1)} + \|F\|_{L^p(D_1)} + \|g\|_{C^\sigma(\Delta_1)}\big\}, \tag{9-11}$$

*where $p > d$, $\sigma \in (0,\alpha)$, and $C$ depends only on $d$, $p$, $\sigma$, $A$, $\alpha$ and $M$.*

As we mentioned in Introduction, for $\mathcal{L}_\varepsilon$ with coefficients satisfying (1-11), (1-3) and the Hölder continuity condition, estimates (9-9) and (9-10) were proved in [Avellaneda and Lin 1987], while (9-11) was established in [Kenig et al. 2013; Armstrong and Shen 2016].

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# CONVEX INTEGRATION FOR THE MONGE–AMPÈRE EQUATION IN TWO DIMENSIONS

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This paper concerns the questions of flexibility and rigidity of solutions to the Monge–Ampère equation, which arises as a natural geometrical constraint in prestrained nonlinear elasticity. In particular, we focus on degenerate, i.e., "flexible", weak solutions that can be constructed through methods of convex integration à la Nash and Kuiper and establish the related $h$-principle for the Monge–Ampère equation in two dimensions.

***An errata was submitted on 23 Aug 2023 and posted online on 30 Jan 2024.*** (Theorem 1.4 is unproved.)

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## 1. Introduction

In this paper we study the $\mathcal{C}^{1,\alpha}$ solutions to the Monge–Ampère equation in two dimensions,

$$\mathcal{D}et\,\nabla^2 v := -\tfrac{1}{2}\,\mathrm{curl}\,\mathrm{curl}(\nabla v \otimes \nabla v) = f \quad \text{in } \Omega \subset \mathbb{R}^2. \tag{1-1}$$

Our results concern the dichotomy of "rigidity vs. flexibility", in the spirit of the analogous results and techniques appearing in the contexts of the low codimension isometric immersion problem [Nash 1954; Kuiper 1955a; 1955b; Borisov 1959; 2004; Conti et al. 2012] and Onsager's conjecture for Euler equations [Székelyhidi 2013; De Lellis and Székelyhidi 2009; 2013; Constantin et al. 1994; Eyink 1994].

In the first, main part of the paper we show that below the regularity threshold $\alpha < \frac{1}{7}$, the very weak $\mathcal{C}^{1,\alpha}(\bar{\Omega})$ solutions to (1-1), as defined below, are dense in the set of all continuous functions (see Theorems 1.1 and 1.2). These flexibility statements are a consequence of the convex integration $h$-principle, which is a method proposed in [Gromov 1986] for solving certain partial differential relations

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and which turns out to be applicable to our setting of the Monge–Ampère equation as well. Here, we directly adapt the iteration method of Nash [1954] and Kuiper [1955a; 1955b] in order to construct the oscillatory solutions to (1-1).[1]

In the second part of the paper we prove that the same class of very weak solutions fails the above flexibility in the regularity regime $\alpha > \frac{2}{3}$. Our results are parallel with those concerning isometric immersions [Borisov 1959; Conti et al. 2012; Pakzad 2004], Euler equations [Constantin et al. 1994; Eyink 1994], the Perona–Malik equation [Kim and Yan 2015a; 2015b], the active scalar equation [Isett and Vicol 2015], and should also be compared with results on the regularity of Sobolev solutions to the Monge–Ampère equation [Pakzad 2004; Šverák 1991; Lewicka et al. 2017; Jerrard and Pakzad 2017], whose study is important in the context of nonlinear elasticity, and with the rigidity results for the Monge–Ampère functions [Jerrard 2008; 2010].

***The weak determinant Hessian.*** Let $\Omega \subset \mathbb{R}^2$ be an open set. Given a function $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we define its very weak Hessian (denoted by $\mathcal{H}_2^*$ in [Iwaniec 2001; Fonseca and Malý 2005]) as

$$\mathcal{D}et\, \nabla^2 v = -\tfrac{1}{2}\, \mathrm{curl}\, \mathrm{curl}(\nabla v \otimes \nabla v),$$

understood in the sense of distributions. A straightforward approximation argument shows that if $v \in W^{2,2}_{\mathrm{loc}}$ then $L^1_{\mathrm{loc}}(\Omega) \ni \mathcal{D}et\, \nabla^2 v = \det \nabla^2 v$ a.e. in $\Omega$, where $\nabla^2 v$ stands for the Hessian matrix field of $v$. We also remark that this notion of the very weak Hessian is distinct from the distributional Hessian $\mathrm{Det}\, \nabla^2 v = \mathrm{Det}\, \nabla(\nabla v)$ (denoted by $\mathcal{H}u$ in [Iwaniec 2001; Fonseca and Malý 2005]), which is defined through the distributional determinant Det,

$$\mathrm{Det}\, \nabla \psi = -\,\mathrm{div}(\psi_2\, \nabla^{\perp} \psi_1) = \partial_2(\psi_2\, \partial_1 \psi_1) - \partial_1(\psi_2\, \partial_2 \psi_1) \quad \text{for } \psi = (\psi_1, \psi_2) \in W^{1,4/3}(\Omega, \mathbb{R}^2).$$

Contrary to the distributional Hessian, the very weak Hessian is not continuous with respect to the weak topology. Indeed, an example of a sequence $v_n \in W^{1,2}(\Omega)$ is constructed in [Iwaniec 2001], where $\mathcal{D}et\, \nabla^2 v = -1$ while $v_n$ converges weakly to 0. One consequence of the proof of our Theorem 1.1 below is that $\mathcal{D}et\, \nabla^2$ is actually weakly discontinuous everywhere in $W^{1,2}(\Omega)$ (see Corollary 6.2).

Here is our first main result:

**Theorem 1.1.** *Let $f \in L^{7/6}(\Omega)$ on an open, bounded, simply connected $\Omega \subset \mathbb{R}^2$. Fix an exponent*

$$\alpha < \tfrac{1}{7}.$$

*Then the set of $\mathcal{C}^{1,\alpha}(\bar{\Omega})$ solutions to (1-1) is dense in the space $\mathcal{C}^0(\bar{\Omega})$. More precisely, for every $v_0 \in \mathcal{C}^0(\bar{\Omega})$ there exists a sequence $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$, converging uniformly to $v_0$ and satisfying*

$$\mathcal{D}et\, \nabla^2 v_n = f \quad \text{in } \Omega. \tag{1-2}$$

*When $f \in L^p(\Omega)$ and $p \in \left(1, \frac{7}{6}\right)$, the same result is true for any $\alpha < 1 - \frac{1}{p}$.*

[1] We remark that the recent work of De Lellis, Inauen and Székelyhidi [De Lellis et al. 2015] showed that the flexibility exponent $\frac{1}{7}$ can be improved to $\frac{1}{5}$ in the case of the isometric immersion problem in two dimensions. We expect similar improvement to be possible also in the present case of equation (1-1); this will be investigated in our future work.

In order to better understand Theorem 1.1, we point out a connection between the solutions to (1-1) and the isometric immersions of Riemannian metrics, motivated by a study of nonlinear elastic plates. Since on a simply connected domain $\Omega$, the kernel of the differential operator curl curl consists of the fields of the form $\operatorname{sym}\nabla w$, a solution to (1-1) with the vanishing right-hand side $f \equiv 0$ can be characterized by the criterion

$$\exists w : \Omega \to \mathbb{R}^2 \quad \text{such that} \quad \tfrac{1}{2}\nabla v \otimes \nabla v + \operatorname{sym}\nabla w = 0 \quad \text{in } \Omega. \tag{1-3}$$

The equation in (1-3) can be seen as an equivalent condition for the one-parameter family of deformations

$$\phi_\varepsilon = \mathrm{id} + \varepsilon v e_3 + \varepsilon^2 w : \Omega \to \mathbb{R}^3,$$

given through the out-of-plane displacement $v$ and the in-plane displacement $w$ (albeit with different orders of magnitude $\varepsilon$ and $\varepsilon^2$), to form a second-order infinitesimal isometry (bending), i.e., to induce the change of metric on the plate $\Omega$ whose second-order terms in $\varepsilon$ disappear:

$$(\nabla\phi_\varepsilon)^T\nabla\phi_\varepsilon - \mathrm{Id}_2 = o(\varepsilon^2).$$

In this context, we take the cue about Theorem 1.1 from the celebrated work of Nash [1954] and Kuiper [1955a; 1955b], where they show the density of codimension-one $\mathcal{C}^1$ isometric immersions of Riemannian manifolds in the set of short mappings. Since we are now dealing with the second-order infinitesimal isometries rather than the exact isometries, the classical metric pull-back equation

$$y^* g_e = h$$

for a mapping $y$ from $(\Omega, h)$ into $\mathbb{R}^3$ equipped with the standard Euclidean metric $g_e$ is replaced by the compatibility equation of the tensor $T(v, w) = \frac{1}{2}\nabla v \otimes \nabla v + \operatorname{sym}\nabla w$ with a matrix field $A_0$ that satisfies $-\operatorname{curl}\operatorname{curl} A_0 = f$:

$$T(v, w) = A_0. \tag{1-4}$$

Note that there are many potential choices for $A_0$; for example, one may take $A_0(x) = \lambda(x)\mathrm{Id}_2$ with $\Delta\lambda = -f$ in $\Omega$. Again, equation (1-4) states precisely that the metric $(\nabla\phi_\varepsilon)^T\nabla\phi_\varepsilon$ agrees with the given metric $h = \mathrm{Id}_2 + 2\varepsilon^2 A_0$ on $\Omega$, up to terms of order $\varepsilon^2$. The Gauss curvature $\kappa$ of the metric $h$ satisfies

$$\kappa(h) = \kappa(\mathrm{Id}_2 + 2\varepsilon^2 A_0) = -\varepsilon^2 \operatorname{curl}\operatorname{curl} A_0 + o(\varepsilon^2),$$

while $\kappa((\nabla\phi_\varepsilon)^T\nabla\phi_\varepsilon) = -\varepsilon^2 \operatorname{curl}\operatorname{curl}\big(\frac{1}{2}\nabla v \otimes \nabla v + \operatorname{sym} w\big) + o(\varepsilon^2)$, so the problem (1-1) can also be interpreted as seeking all appropriately regular out-of-plane displacements $v$ that can be matched, by a higher order in-plane displacement perturbation $w$, to achieve the prescribed Gauss curvature $f$ of $\Omega$, at its highest-order term.

In this paper, similar to the isometric immersion case, we show that solutions to (1-4) are ample. We design a scheme inspired by the work of Nash and Kuiper, which pushes a "short infinitesimal isometry", i.e., a couple $(v_0, w_0)$ such that $T(v_0, w_0) < A_0$, towards an exact solution to (1-4) in successive small steps. Note that both $y^* g_e = (\nabla y)^T\nabla y$ and the term $\nabla v \otimes \nabla v$ in $T(v, w)$ have a quadratic structure, which is crucial in the analysis of [Nash 1954; Kuiper 1955a; 1955b] and also of this paper. Here, not only does the presence of the linear term $\operatorname{sym}\nabla w$ in $T(u, w)$ not destroy the adaptation of the Nash–Kuiper scheme, but it actually allows for this construction to work.

***Convex integration for the Monge–Ampère equation in two dimensions.*** As we will see in Section 4, Theorem 1.1 follows easily from the statement of our next main result:

**Theorem 1.2.** *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $v_0 \in \mathcal{C}^1(\bar{\Omega})$, $w_0 \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ and $A_0 \in \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, for some $\beta \in (0, 1)$, be such that*

$$\exists c_0 > 0 \quad \textit{such that} \quad A_0 - \left(\tfrac{1}{2}\nabla v_0 \otimes \nabla v_0 + \operatorname{sym}\nabla w_0\right) > c_0 \mathrm{Id}_2 \quad \textit{in } \bar{\Omega}. \tag{1-5}$$

*Then, for every exponent $\alpha$ in the range*

$$0 < \alpha < \min\left\{\tfrac{1}{7}, \tfrac{1}{2}\beta\right\},$$

*there exist sequences $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $w_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ which converge uniformly to $v_0$ and $w_0$, respectively, and which satisfy*

$$A_0 = \tfrac{1}{2}\nabla v_n \otimes \nabla v_n + \operatorname{sym}\nabla w_n \quad \textit{in } \bar{\Omega}. \tag{1-6}$$

The above result is the Monge–Ampère analogue of [Conti et al. 2012, Theorem 1], where the authors improved on the Nash–Kuiper method to obtain higher regularity within the flexibility regime. In our paper, we adapt similar methods to the system (1-6).

The term *convex integration* usually refers to a collection of approaches that allow for constructing anomalous solutions to nonlinear PDEs; in particular, flexibility-type results for the isometric immersion problem were obtained via the above-mentioned iteration scheme of Nash and Kuiper. From a geometric perspective, they are special cases of *h-principle*, a notion which was developed by Gromov [1986] for studying partial differential relations; see also [Eliashberg and Mishachev 2002]. From another perspective, one seeks weak solutions of a differential inclusion $Lu(x) \in K$ in $\Omega$ by investigating certain classes of subsolutions, e.g., functions $u$ that satisfy $Lu(x) \in \operatorname{conv} K$, where the original constraint set $K$ is replaced by its convex hull $\operatorname{conv} K$ [Tartar 1979; Dacorogna and Marcellini 1997; Müller and Šverák 2003]. This approach leads to the density of very weak solutions, satisfying $Lu \in L^\infty(\Omega)$, in the set of subsolutions. When $K$ is a continuum, the regularity may be improved to $Lu \in \mathcal{C}^0(\Omega)$ by applying the correcting iterations.

Recently, similar techniques were advanced in the context of fluid dynamics and yielded many interesting results for the Euler equations. De Lellis and Székelyhidi [2009] proved the existence of weak solutions with bounded velocity and pressure, their nonuniqueness and the existence of energy-decreasing solutions. In [De Lellis and Székelyhidi 2013], using iteration methods à la Nash and Kuiper, they proved the existence of continuous periodic solutions of the three-dimensional incompressible Euler equations, which dissipate the total kinetic energy. These results are to be contrasted with [Constantin et al. 1994; Eyink 1994], where it was shown that $\mathcal{C}^{0,\alpha}$ solutions of the Euler equations are energy conservative if $\alpha > \frac{1}{3}$. There have been several improvements of [De Lellis and Székelyhidi 2009; 2013] recently, towards a proof of Onsager's conjecture, which puts the Hölder regularity threshold for the energy conservation of the weak solutions to the Euler equations at $\mathcal{C}^{0,1/3}$ [Isett 2012; 2013; 2016; Buckmaster et al. 2013; 2015; 2016; Choffrut and Székelyhidi 2014]. The stationary incompressible Euler equation has been studied in [Choffrut and Székelyhidi 2014], where the existence of bounded anomalous solutions has been proved. The authors indicate that in two dimensions, the relaxation set corresponding to the appropriate subsolutions is smaller than in the case of the evolutionary equations. In this context, we noticed a connection between our reformulation of the Monge–Ampère equation and the steady-state Euler equation, which lead to our modest Corollary 4.1.

In this paper we use a direct iteration method to construct exact solutions of (1-1). The recasting of the statement and the proof in the language of convex integration might shed more light on the structure of the Monge–Ampère equation, but it would not improve the results and therefore we do not address this task. We note, however, that constructing Lipschitz continuous piecewise affine approximating solutions to (1-6) for $A_0 \equiv 0$ is quite straightforward and could be used to prove a convex integration density result via the Baire category method, as was done in [De Lellis and Székelyhidi 2009] for the Euler equations (see also Figure 1 and the corresponding explanation).

***Rigidity versus flexibility.*** The flexibility results obtained in view of the $h$-principle are usually coupled with the rigidity results for more regular solutions. Rigidity of isometric immersions of elliptic metrics for $\mathcal{C}^{1,\alpha}$ isometries [Borisov 1959; De Lellis and Székelyhidi 2009] with $\alpha > \frac{2}{3}$, or the energy conservation of weak solutions of the Euler equations for $\mathcal{C}^{0,\alpha}$ solutions with $\alpha > \frac{1}{3}$, are results of this type. For the Monge–Ampère equations, we recall two recent statements regarding solutions with Sobolev regularity: Following the well-known unpublished work by Šverák [1991], we proved in [Lewicka et al. 2017] that if $v \in W^{2,2}(\Omega)$ is a solution to (1-1) with $f \in L^1(\Omega)$ and $f \geq c > 0$ in $\Omega$, then in fact $v$ must be $\mathcal{C}^1$ and globally convex (or concave). On the other hand, if $f = 0$ then likewise $v \in \mathcal{C}^1(\Omega)$ and $v$ must be developable [Pakzad 2004] (see also [Jerrard 2008; 2010; Jerrard and Pakzad 2017]). A clear statement of rigidity is still lacking for the general $f$, as is the case for isometric immersions, where rigidity results are usually formulated only for elliptic [Conti et al. 2012] or Euclidean metrics [Pakzad 2004; Liu and Pakzad 2015; Jerrard and Pakzad 2017].

In this paper, we prove the rigidity properties of solutions to (1-1) in the Hölder regularity context when $f \equiv 0$. Namely, we prove:

**Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain and let*

$$\tfrac{2}{3} < \alpha < 1.$$

*If $v \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ is a solution to $\mathcal{D}et\, \nabla^2 v = 0$ in $\overline{\Omega}$, then $v$ must be developable. More precisely, for all $x \in \Omega$ either $v$ is affine in a neighbourhood of $x$, or there exists a segment $l_x$ joining $\partial\Omega$ on its both ends such that $\nabla v$ is constant on $l_x$.*

We also announce the following parallel rigidity result for $f \geq c > 0$, which will be the subject of the forthcoming paper [Lewicka and Pakzad $\geq$ 2017]:

**Theorem 1.4.** *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain and let*

$$\tfrac{2}{3} < \alpha < 1.$$

*If $v \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ is a solution to $\mathcal{D}et\, \nabla^2 v = f$ in $\overline{\Omega}$, where $f$ is a positive Dini continuous function, then $v$ is convex. In fact, it is also an Alexandrov solution to* $\det \nabla^2 v = f$ *in $\Omega$.*

In proving Theorem 1.3, we use a commutator estimate for deriving a degree formula in Proposition 7.1. Similar commutator estimates are used in [Constantin et al. 1994] for the Euler equations and in [Conti et al. 2012] for the isometric immersion problem; this is not surprising, since the presence of a quadratic term plays a major role in all three cases, allowing for the efficiency of the convex integration and iteration methods. Let us also mention that it is still unknown which value of $\alpha$ is the critical value for the rigidity-flexibility dichotomy, but it is conjectured to be $\frac{1}{3}$, $\frac{1}{2}$ or $\frac{2}{3}$.

***Notation.*** By $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ we denote the space of symmetric $2\times 2$ matrices, and by $\mathbb{R}^{2\times 2}_{\mathrm{sym},>}$ we denote the cone of symmetric, positive definite $2\times 2$ matrices. The space of Hölder continuous functions $\mathcal{C}^{k,\alpha}(\bar\Omega)$ consists of restrictions of functions $f\in\mathcal{C}^{k,\alpha}(\mathbb{R}^2)$ to $\Omega\subset\mathbb{R}^2$. Then, the $\mathcal{C}^k(\bar\Omega)$ norm of such a restriction is denoted by $\|f\|_k$, while its Hölder norm $\mathcal{C}^{k,\alpha}(\bar\Omega)$ is $\|f\|_{k,\alpha}$. By $C>0$ we denote a universal constant which is independent of all parameters, unless indicated otherwise.

## 2. The $\mathcal{C}^1$ approximations: preliminary results

In this and the next section we prove a weaker version of the result in Theorem 1.2. Namely:

**Theorem 2.1.** *Let $\Omega\subset\mathbb{R}^2$ be an open and bounded domain. Let $v_0\in\mathcal{C}^\infty(\bar\Omega)$, $w_0\in\mathcal{C}^\infty(\bar\Omega,\mathbb{R}^2)$ and $A_0\in\mathcal{C}^\infty(\bar\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})$ be such that*

$$\exists c_0>0 \quad \textit{such that} \quad A_0-\big(\tfrac12\nabla v_0\otimes\nabla v_0+\operatorname{sym}\nabla w_0\big)>c_0\,\mathrm{Id}_2 \quad \textit{in } \bar\Omega. \tag{2-1}$$

*Then there exist sequences $v_n\in\mathcal{C}^1(\bar\Omega)$ and $w_n\in\mathcal{C}^1(\bar\Omega,\mathbb{R}^2)$ which converge uniformly to $v_0$ and $w_0$ respectively, and which satisfy*

$$A_0=\tfrac12\nabla v_n\otimes\nabla v_n+\operatorname{sym}\nabla w_n \quad \textit{in } \bar\Omega. \tag{2-2}$$

We start with a series of preliminary lemmas whose details we provide for the sake of completeness. The first is an observation in convex integration, pertaining to solving an appropriate differential inclusion to be used for constructing the one-dimensional oscillatory perturbations in $v_n$ and $w_n$. As always, $C>0$ is a universal constant, independent of all parameters, in particular independent of the function $a$ below.

**Lemma 2.2.** *Let $a\in\mathcal{C}^\infty(\bar\Omega)$ be a nonnegative function on an open and bounded set $\Omega\subset\mathbb{R}^2$. There exists a smooth 1-periodic field $\Gamma=(\Gamma_1,\Gamma_2)\in\mathcal{C}^\infty(\bar\Omega\times\mathbb{R},\mathbb{R}^2)$ such that the following holds for all $(x,t)\in\bar\Omega\times\mathbb{R}$:*

$$\begin{aligned}\Gamma(x,t+1)&=\Gamma(x,t),\\ \tfrac12|\partial_t\Gamma_1(x,t)|^2+\partial_t\Gamma_2(x,t)&=a(x)^2,\end{aligned} \tag{2-3}$$

*together with the uniform bounds*

$$\begin{aligned}|\Gamma_1(x,t)|+|\partial_t\Gamma_1(x,t)|&\le Ca(x), & |\nabla_x\Gamma_1(x,t)|&\le C|\nabla a(x)|,\\ |\Gamma_2(x,t)|+|\partial_t\Gamma_2(x,t)|&\le Ca(x)^2, & |\nabla_x\Gamma_2(x,t)|&\le C|a(x)||\nabla a(x)|.\end{aligned} \tag{2-4}$$

*Proof.* Firstly, note that there exists a smooth 1-periodic function $\gamma\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^2)$ such that for all $t\in\mathbb{R}$,

$$\gamma(t+1)=\gamma(t),\qquad \int_0^1\gamma(t)\,\mathrm{d}t=(0,0),\qquad \gamma(t)\in P:=\big\{(s_1,s_2)\in\mathbb{R}^2:\tfrac12 s_1^2+s_2=1,\ |s_1|\le 2\big\}.$$

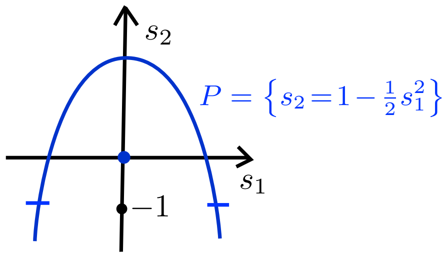


**Figure 1.** The parabola $P$ in the one-dimensional convex integration problem of Lemma 2.2.

The existence of $\gamma$ is a consequence of the fundamental lemma of convex integration, since the intended average $(0, 0)$ lies in the convex hull of the parabola $P$ (see Figure 1). Indeed, one can take

$$\gamma(t) = \big(2\cos(2\pi t), -\cos(4\pi t)\big) \in P.$$

It is now enough to ensure that $\partial_t \Gamma_1 = a(x)\gamma_1(x)$ and $\partial_t \Gamma_2 = a(x)^2\gamma_2(x)$ to obtain (2-3). Namely

$$\Gamma_1(x, t) = \frac{a(x)}{\pi}\sin(2\pi t), \qquad \Gamma_2(x, t) = -\frac{a(x)^2}{4\pi}\sin(4\pi t).$$

We see directly that the bounds in (2-4) hold. □

To compare with the problem of isometric immersions, note that in that context, a one-dimensional convex integration lemma is similarly proved in [Székelyhidi 2013, Figure 2, p. 11], where instead of a parabola, the constraint set consists of a full circle.

We will also need a special case of [Conti et al. 2012, Lemma 3] about decomposition of positive definite symmetric matrices into rank-one matrices.

**Lemma 2.3.** *There exists a sufficiently small constant $r_0 > 0$ such that the following holds. For every positive definite symmetric matrix $G_0 \in \mathbb{R}^{2\times 2}_{\mathrm{sym},>}$, there are three unit vectors $\{\xi_k \in \mathbb{R}^3\}_{k=1}^3$ and three linear functions $\{\Phi_k : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}\}_{k=1}^3$ such that for any $G \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ we have*

$$\forall G \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \qquad G = \sum_{k=1}^{3} \Phi_k(G)\xi_k \otimes \xi_k, \tag{2-5}$$

*and each $\Phi_k$ is strictly positive on the ball $B(G_0, r(G_0)) \subset \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ with radius $r(G_0) = r_0/|G_0{}^{-1/2}|^2$.*

*Proof.* (1) First, assume that $G_0 = \mathrm{Id}_2$. Set

$$\zeta_1 = \frac{1}{\sqrt{12}}(2+\sqrt{2}, -2+\sqrt{2}), \qquad \zeta_2 = \frac{1}{\sqrt{12}}(-2+\sqrt{2}, 2+\sqrt{2}), \qquad \zeta_3 = \frac{1}{\sqrt{2}}(1, 1).$$

In order to check that the matrices

$$\zeta_1 \otimes \zeta_1 = \frac{1}{12}\begin{bmatrix} 6+4\sqrt{2} & -2 \\ -2 & 6-4\sqrt{2} \end{bmatrix}, \quad \zeta_2 \otimes \zeta_2 = \frac{1}{12}\begin{bmatrix} 6-4\sqrt{2} & -2 \\ -2 & 6+4\sqrt{2} \end{bmatrix}, \quad \zeta_3 \otimes \zeta_3 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

form a basis of the three-dimensional space $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$, we validate that

$$\det\left(\frac{1}{12}\begin{bmatrix}6+4\sqrt{2} & 6-4\sqrt{2} & 6\\ -2 & -2 & 6\\ 6-4\sqrt{2} & 6+4\sqrt{2} & 6\end{bmatrix}\right)\neq 0.$$

Consequently, there exist linear mappings $\{\Psi_k : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}\}_{k=1}^3$ yielding the unique decomposition

$$\forall G \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \qquad G = \sum_{k=1}^{3} \Psi_k(G)\zeta_k \otimes \zeta_k. \tag{2-6}$$

Now, since $\mathrm{Id}_2 = \frac{3}{4}\zeta_1 \otimes \zeta_1 + \frac{3}{4}\zeta_2 \otimes \zeta_2 + \frac{1}{2}\zeta_3 \otimes \zeta_3$, the continuity of each function $\Psi_k$ implies its positivity in a neighbourhood of $\mathrm{Id}_2$ of some appropriate radius $r_0$.

(2) For an arbitrary $G_0 \in \mathbb{R}^{2\times 2}_{\mathrm{sym},>}$ we set

$$\forall k = 1, \dots, 3, \quad \xi_k = \frac{1}{|G_0^{1/2}\zeta_k|}G_0^{1/2}\zeta_k \quad \text{and} \quad \Phi_k(G) = |G_0^{1/2}\zeta_k|^2\Psi_k(G_0^{-1/2}GG_0^{-1/2}).$$

Then, in view of (2-6) we obtain (2-5):

$$\forall G \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \quad G = G_0^{-1/2}\left(\sum_{k=1}^{3} \Psi_k(G_0^{-1/2}GG_0^{-1/2})\zeta_k \otimes \zeta_k\right)G_0^{1/2} = \sum_{k=1}^{3} \Phi_k(G)\xi_k \otimes \xi_k.$$

Finally, if $|G-G_0| < r(G_0)$ then $\left|G_0^{-1/2}GG_0^{-1/2} - \mathrm{Id}_2\right| \le |G_0^{-1/2}|^2\,|G-G_0| < r_0$, and so indeed $\Phi_k(G) > 0$, since $\Psi_k(G_0^{-1/2}GG_0^{-1/2}) > 0$. □

The above result can be localized in the following manner, similar to [Székelyhidi 2013, Lemma 3.3]:

**Lemma 2.4.** *There exist sequences of unit vectors $\{\eta_k \in \mathbb{R}^2\}_{k=1}^\infty$ and nonnegative smooth functions $\{\phi_k \in \mathcal{C}_c^\infty(\mathbb{R}^{2\times 2}_{\mathrm{sym},>})\}_{k=1}^\infty$ such that*

$$\forall G \in \mathbb{R}^{2\times 2}_{\mathrm{sym},>}, \qquad G = \sum_{k=1}^{\infty} \phi_k(G)^2\eta_k \otimes \eta_k \tag{2-7}$$

*and such that:*

(i) *For all $G \in \mathbb{R}^{2\times 2}_{\mathrm{sym},>}$, at most $N_0$ terms of the sum in (2-7) are nonzero. The constant $N_0$ is independent of $G$.*

(ii) *For every compact $K \subset \mathbb{R}^{2\times 2}_{\mathrm{sym},>}$, there exists a finite set of indices $J(K) \subset \mathbb{N}$ such that $\phi_k(G) = 0$ for all $k \notin J(K)$ and $G \in K$.*

*Proof.* (1) Let $r_0$ be as in Lemma 2.3 and additionally ensure that

$$r_0 < \tfrac{1}{8}. \tag{2-8}$$

Recall that for each $G \in \mathbb{R}^{2\times 2}_{\mathrm{sym},>}$ we have defined $r(G) = r_0/|G^{-1/2}|^2$ and that $B(G, r(G)) \subset \mathbb{R}^{2\times 2}_{\mathrm{sym},>}$. We first construct a locally finite covering of $\mathbb{R}^{2\times 2}_{\mathrm{sym},>}$ with properties corresponding to (i) and (ii).

Since the set $\mathbb{R}^{2\times 2}_{\mathrm{sym},>}$ is a cone, we have

$$\mathbb{R}^{2\times 2}_{\mathrm{sym},>} = \bigcup_{k\in\mathbb{Z}} 2^k\mathcal{C}_0, \quad \text{where } \mathcal{C}_0 = \left\{G \in \mathbb{R}^{2\times 2}_{\mathrm{sym},>} : \tfrac{1}{2} \le |G| \le 1\right\}. \tag{2-9}$$

The collection $\{B(G,r(G))\}_{G\in\mathcal{C}_0}$ covers the sector $\mathcal{C}_0$ by balls that have uniformly bounded radii $r(G)\le r_0|G|/\sqrt{2}\le r_0$. Hence, by the Besicovitch covering theorem, it has a countable subcovering $\mathcal{G}_0=\bigcup_{\sigma=1}^{\sigma_0}\mathcal{G}_0^\sigma$, consisting of $\sigma_0\in\mathbb{N}$ countable families $\{\mathcal{G}_0^\sigma\}_{\sigma=1}^{\sigma_0}$ of pairwise disjoint balls.

Note that for all $c>0$ one has $r(cG)=cr(G)$ and so $B(cG,r(cG))=cB(G,r(G))$. Consequently, the collections $\mathcal{G}_k^\sigma=\{2^kB:B\in\mathcal{G}_0^\sigma\}$ each consist of countably many pairwise disjoint balls, and $\mathcal{G}_k=\bigcup_{\sigma=1}^{\sigma_0}\mathcal{G}_k^\sigma$ is a covering of the dilated sector $2^k\mathcal{C}_0$ for every $k\in\mathbb{Z}$. Define

$$\forall\sigma=1,\dots,\sigma_0,\quad \mathcal{G}_{\text{even}}^\sigma=\bigcup_{2|k}\mathcal{G}_k^\sigma\quad\text{and}\quad \mathcal{G}_{\text{odd}}^\sigma=\bigcup_{2|(k+1)}\mathcal{G}_k^\sigma. \tag{2-10}$$

Clearly, in view of (2-9), the $2\sigma_0$ families in (2-10) form a covering of $\mathbb{R}^{2\times2}_{\text{sym},>}$, namely

$$\mathcal{G}=\bigcup_{\sigma=1}^{\sigma_0}\mathcal{G}_{\text{even}}^\sigma\cup\bigcup_{\sigma=1}^{\sigma_0}\mathcal{G}_{\text{odd}}^\sigma.$$

We now prove that each of the families in $\mathcal{G}$ consists of pairwise disjoint balls. We argue by contradiction. Assume that

$$\exists G\in B(G_1,r(G_1))\cap B(G_2,r(G_2))\quad\text{for some } B(G_1,r(G_1))\in\mathcal{G}_{2k_1}^\sigma,\ B(G_2,r(G_2))\in\mathcal{G}_{2k_2}^\sigma.$$

Without loss of generality we may take $k_1=0$ and $k_2=k\ge1$, so that

$$\tfrac{1}{2}\le|G_1|\le1\quad\text{and}\quad 2^{2k-1}\le|G_2|\le2^{2k}.$$

This yields a contradiction with (2-8), in view of

$$\begin{aligned}2^{2k-1}-1&\le|G_2|-|G_1|\le|G_2-G_1|\le|G_2-G|+|G-G_1|\\&\le r(G_2)+r(G_1)=r_0\left(\frac{1}{|G_2^{-1/2}|^2}+\frac{1}{|G_1^{-1/2}|^2}\right)\le\frac{r_0}{\sqrt{2}}(|G_2|+|G_1|)\le r_0(2^{2k}+1).\end{aligned}$$

(2) Note that $\mathcal{G}$ can be assumed locally finite, by paracompactness. We write $\mathcal{G}=\{B_i=B(G_i,r(G_i))\}_{i=1}^\infty$ and let $\{\theta_i\in\mathcal{C}_c^\infty(B_i)\}_{i=1}^\infty$ be a partition of unity subordinated to $\mathcal{G}$. For each $i\in\mathbb{N}$, let $\{\xi_{k,G_i}\}_{k=1}^3$ and $\{\Phi_{k,G_i}\}_{k=1}^3$ be the unit vectors and the linear functions as in Lemma 2.3. Then

$$\forall G\in\mathbb{R}^{2\times2}_{\text{sym},>},\quad G=\sum_{i\in\mathbb{N}}\theta_i(G)G=\sum_{i\in\mathbb{N}}\sum_{k=1}^3\theta_i(G)\Phi_{k,G_i}(G)\xi_{k,G_i}\otimes\xi_{k,G_i},$$

and we see that (2-7) holds by taking

$$\eta_{i,k}=\xi_{k,G_i}\quad\text{and}\quad\phi_{i,k}=(\theta_i\Phi_{k,G_i}).$$

Since $\operatorname{supp}\phi_{i,k}\subset\mathcal{B}_i$ and since each $G$ belongs to at most $2\sigma_0$ balls $B_i$, we see that (i) holds with $N_0=6\sigma_0$. On the other hand, condition (ii) follows by the local finiteness of $\mathcal{G}$. □

## 3. The $\mathcal{C}^1$ approximations: a proof of Theorem 2.1

The first result in the approximating sequence construction is what corresponds to a "step" in the terminology of Nash and Kuiper.

**Proposition 3.1.** *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set. Given are functions $v \in \mathcal{C}^\infty(\bar\Omega)$ and $w \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$, a nonnegative function $a \in \mathcal{C}^\infty(\bar\Omega)$, and a unit vector $\eta \in \mathbb{R}^2$. Then, for every $\lambda > 1$ there exist approximations $\tilde v_\lambda \in \mathcal{C}^\infty(\bar\Omega)$ and $\tilde w_\lambda \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$ satisfying the bounds*

$$\Big\| \big(\tfrac{1}{2}\nabla\tilde v_\lambda \otimes \nabla\tilde v_\lambda + \operatorname{sym}\nabla\tilde w_\lambda\big) - \big(\tfrac{1}{2}\nabla v\otimes\nabla v + \operatorname{sym}\nabla w + a^2\eta\otimes\eta\big)\Big\|_0 \\ \leq \frac{C}{\lambda}\|a\|_0(\|\nabla a\|_0 + \|\nabla^2 v\|_0) + \frac{C}{\lambda^2}\|\nabla a\|_0^2, \tag{3-1}$$

$$\|\tilde v_\lambda - v\|_0 \leq \frac{C}{\lambda}\|a\|_0 \quad \textit{and} \quad \|\tilde w_\lambda - w\|_0 \leq \frac{C}{\lambda}\|a\|_0(\|a\|_0 + \|\nabla v\|_0), \tag{3-2}$$

*and for all $x \in \bar\Omega$,*

$$\begin{aligned} |\nabla\tilde v_\lambda(x) - \nabla v(x)| &\leq Ca(x) + \frac{C}{\lambda}\|\nabla a\|_0, \\ |\nabla\tilde w_\lambda(x) - \nabla w(x)| &\leq Ca(x)(\|a\|_0 + \|\nabla v\|_0) + \frac{C}{\lambda}\big(\|a\|_0(\|\nabla a\|_0 + \|\nabla^2 v\|_0) + \|\nabla a\|_0\|\nabla v\|_0\big). \end{aligned} \tag{3-3}$$

*Proof.* Using the 1-periodic functions $\Gamma_i$ from Lemma 2.2, we define $\tilde v_\lambda$ and $\tilde w_\lambda$ as $\lambda$-periodic perturbations of $v$, $w$ in the direction $\eta$:

$$\begin{aligned} \tilde v_\lambda(x) &= v(x) + \frac{1}{\lambda}\Gamma_1(x, \lambda x\cdot\eta), \\ \tilde w_\lambda(x) &= w(x) - \frac{1}{\lambda}\Gamma_1(x, \lambda x\cdot\eta)\nabla v(x) + \frac{1}{\lambda}\Gamma_2(x, \lambda x\cdot\eta)\eta. \end{aligned} \tag{3-4}$$

The error estimates in (3-2) follow immediately from (2-4). The pointwise error estimates (3-3) follow from (2-4) in view of

$$\nabla\tilde v_\lambda(x) = \nabla v(x) + \frac{1}{\lambda}\nabla_x\Gamma_1(x, \lambda x\cdot\eta) + \partial_t\Gamma_1(x, \lambda x\cdot\eta)\eta,$$

$$\nabla\tilde w_\lambda(x) = \nabla w(x) - \frac{1}{\lambda}\nabla v(x)\otimes\nabla_x\Gamma_1(x, \lambda x\cdot\eta) - \partial_t\Gamma_1(x, \lambda x\cdot\eta)\eta\otimes\nabla v(x) - \frac{1}{\lambda}\Gamma_1(x, \lambda x\cdot\eta)\nabla^2 v(x) \\ + \frac{1}{\lambda}\eta\otimes\nabla_x\Gamma_2(x, \lambda x\cdot\eta) + \partial_t\Gamma_2(x, \lambda x\cdot\eta)\eta\otimes\eta.$$

Finally, we compute

$$\tfrac{1}{2}\nabla\tilde v_\lambda(x)\otimes\nabla\tilde v_\lambda(x) - \tfrac{1}{2}\nabla v(x)\otimes\nabla v(x) \\ = \boxed{\frac{1}{\lambda}\operatorname{sym}\big(\nabla v(x)\otimes\nabla_x\Gamma_1(x, \lambda x\cdot\eta)\big) + \partial_t\Gamma_1(x, \lambda x\cdot\eta)\operatorname{sym}(\nabla v(x)\otimes\eta)} + \boxed{\boxed{\tfrac{1}{2}|\partial_t\Gamma_1(x, \lambda x\cdot\eta)|^2\eta\otimes\eta}} \\ + \frac{1}{\lambda}\partial_t\Gamma_1(x, \lambda x\cdot\eta)\operatorname{sym}\big(\eta\otimes\nabla_x\Gamma_1(x, \lambda x\cdot\eta)\big) + \frac{1}{2\lambda^2}\nabla_x\Gamma_1(x, \lambda x\cdot\eta)\otimes\nabla_x\Gamma_1(x, \lambda x\cdot\eta),$$

and

$$\operatorname{sym}\nabla\tilde w_\lambda(x) - \operatorname{sym}\nabla w(x) = \boxed{-\frac{1}{\lambda}\operatorname{sym}\big(\nabla v(x)\otimes\nabla_x\Gamma_1(x, \lambda x\cdot\eta)\big) - \partial_t\Gamma_1(x, \lambda x\cdot\eta)\operatorname{sym}(\nabla v(x)\otimes\eta)} \\ - \frac{1}{\lambda}\Gamma_1(x, \lambda x\cdot\eta)\nabla^2 v(x) + \frac{1}{\lambda}\operatorname{sym}\big(\eta\otimes\nabla_x\Gamma_2(x, \lambda x\cdot\eta)\big) + \boxed{\boxed{\partial_t\Gamma_2(x, \lambda x\cdot\eta)\eta\otimes\eta}}.$$

We see that the terms in boxes cancel out, while the terms in double boxes add up to $a(x)^2\eta\otimes\eta$ by (2-3). Consequently,

$$\begin{aligned}\big(\tfrac{1}{2}\nabla\tilde v_\lambda(x)\otimes\nabla\tilde v_\lambda(x)+\operatorname{sym}\nabla\tilde w_\lambda(x)\big)-\big(\tfrac{1}{2}\nabla v(x)\otimes\nabla v(x)+\operatorname{sym}\nabla w(x)+a(x)^2\eta\otimes\eta\big)\\ =\frac{1}{\lambda}\Big(\partial_t\Gamma_1(x,\lambda x\cdot\eta)\operatorname{sym}\big(\eta\otimes\nabla_x\Gamma_1(x,\lambda x\cdot\eta)\big)-\Gamma_1(x,\lambda x\cdot\eta)\nabla^2v(x)+\operatorname{sym}\big(\eta\otimes\nabla_x\Gamma_2(x,\lambda x\cdot\eta)\big)\Big)\\ +\frac{1}{2\lambda^2}\nabla_x\Gamma_1(x,\lambda x\cdot\eta)\otimes\nabla_x\Gamma_1(x,\lambda x\cdot\eta),\end{aligned}$$

which implies (3-1) in view of the bounds in (2-4). □

We now complete the "stage" in the approximating sequence construction.

**Proposition 3.2.** *Let $\Omega\subset\mathbb{R}^2$ be an open and bounded domain. Let $v\in\mathcal{C}^\infty(\bar\Omega)$, $w\in\mathcal{C}^\infty(\bar\Omega,\mathbb{R}^2)$ and $A\in\mathcal{C}^\infty(\bar\Omega,\mathbb{R}^{2\times2}_{\mathrm{sym}})$ be such that the deficit function $\mathcal{D}$ defined below is positive definite in $\bar\Omega$:*

$$\exists c>0\quad\text{such that}\quad\mathcal{D}=A-\big(\tfrac{1}{2}\nabla v\otimes\nabla v+\operatorname{sym}\nabla w\big)>c\,\mathrm{Id}_2\quad\text{in }\bar\Omega.\tag{3-5}$$

*Fix $\varepsilon>0$. Then there exist $\tilde v\in\mathcal{C}^\infty(\bar\Omega)$ and $\tilde w\in\mathcal{C}^\infty(\bar\Omega,\mathbb{R}^2)$ such that the new deficit $\widetilde{\mathcal{D}}$ is still positive definite, and bounded by $\varepsilon$ together with the error in the approximations $\tilde v$, $\tilde w$; namely,*

$$\exists\tilde c>0\quad\text{such that}\quad\widetilde{\mathcal{D}}=A-\big(\tfrac{1}{2}\nabla\tilde v\otimes\nabla\tilde v+\operatorname{sym}\nabla\tilde w\big)>\tilde c\,\mathrm{Id}_2\quad\text{in }\bar\Omega,\tag{3-6}$$

$$\|\widetilde{\mathcal{D}}\|_0<\varepsilon\quad\text{and}\quad\|\tilde v-v\|_0+\|\tilde w-w\|_0<\varepsilon.\tag{3-7}$$

*Moreover, we have the uniform gradient error bounds*

$$\begin{aligned}\|\nabla\tilde v-\nabla v\|_0&\le CN_0^{1/2}\|\mathcal{D}\|_0^{1/2}\\ \|\nabla\tilde w-\nabla w\|_0&\le CN_0(\|\nabla v\|_0+\|\mathcal{D}\|_0^{1/2})\|\mathcal{D}\|_0^{1/2},\end{aligned}\tag{3-8}$$

*where the constant $N_0\in\mathbb{N}$ is as in Lemma 2.4.*

*Proof.* (1) Note that the image $\mathcal{D}(\bar\Omega)$ is a compact subset of $\mathbb{R}^{2\times2}_{\mathrm{sym},>}$. By Lemma 2.4 and rearranging the indices, if needed, so that $J(\mathcal{D}(\bar\Omega))=\{1,\dots,N\}$ in (ii), we get

$$\forall x\in\bar\Omega,\quad\mathcal{D}(x)=\sum_{k=1}^N b_k(x)^2\eta_k\otimes\eta_k,\quad\text{where }b_k=\phi_k\circ\mathcal{D}\in\mathcal{C}^\infty(\bar\Omega).\tag{3-9}$$

Let now $a_k=(1-\delta)^{1/2}b_k$, with $\delta>0$ so small that

$$\mathcal{D}-\sum_{k=1}^N a_k^2\eta_k\otimes\eta_k=\delta\mathcal{D}\quad\text{and}\quad\delta\|\mathcal{D}\|_0<\tfrac{1}{2}\varepsilon.\tag{3-10}$$

We set $v_1=v$, $w_1=w$. For $k=1,\dots,N$ we inductively define $v_{k+1}\in\mathcal{C}^\infty(\bar\Omega)$ and $w_{k+1}\in\mathcal{C}^\infty(\bar\Omega,\mathbb{R}^2)$, by means of Proposition 3.1 applied to $v_k$, $w_k$, $a_k$, $\eta_k$ and with $\lambda_k>1$ sufficiently large, as indicated below. We then finally set $\tilde v=v_{N+1}$ and $\tilde w=w_{N+1}$.

(2) To prove the estimates (3-6)–(3-8), we start by observing that since by Lemma 2.4(i) at most $N_0$ terms in the expansion (3-9) are nonzero, we have

$$\sum_{k=1}^{N} a_k(x) \le \sum_{k=1}^{N} b_k(x) \le N_0^{1/2}\Big(\sum_{k=1}^{N} b_k(x)^2\Big)^{1/2} = N_0^{1/2}(\mathrm{Trace}\,\mathcal{D}(x))^{1/2}$$
$$\le N_0^{1/2}(\sqrt{2}\,|\mathcal{D}(x)|)^{1/2} \le C N_0^{1/2}\|\mathcal{D}\|_0^{1/2}. \tag{3-11}$$

Further, by (3-1) and (3-10),

$$\begin{aligned}\widetilde{\mathcal{D}} &= \mathcal{D} - \big(\big(\tfrac12\nabla\tilde v\otimes\nabla\tilde v + \mathrm{sym}\,\nabla\tilde w\big) - \big(\tfrac12\nabla v\otimes\nabla v + \mathrm{sym}\,\nabla w\big)\big)\\
&= \mathcal{D} - \sum_{k=1}^{N}\big(\big(\tfrac12\nabla v_{k+1}\otimes\nabla v_{k+1} + \mathrm{sym}\,\nabla w_{k+1}\big) - \big(\tfrac12\nabla v_k\otimes\nabla v_k + \mathrm{sym}\,\nabla w_k\big)\big)\\
&= \Big(\mathcal{D} - \sum_{k=1}^{N} a_k^2\eta_k\otimes\eta_k\Big) - \sum_{k=1}^{N}\big(\big(\tfrac12\nabla v_{k+1}\otimes\nabla v_{k+1} + \mathrm{sym}\,\nabla w_{k+1}\big) - \big(\tfrac12\nabla v_k\otimes\nabla v_k + \mathrm{sym}\,\nabla w_k + a_k^2\eta_k\otimes\eta_k\big)\big)\\
&= \delta\mathcal{D} + \sum_{k=1}^{N}\mathcal{O}\Big(\frac{1}{\lambda_k}\big(\|a_k\|_0\|\nabla a_k\|_0 + \|\nabla a_k\|_0^2 + \|a_k\|_0\|\nabla^2 v_k\|_0\big)\Big).\end{aligned}$$

Choosing at each step $\lambda_k$ sufficiently large with respect to the given $a_k$ and the already generated $v_k$, we may ensure the smallness of the error term in the right-hand side above and hence the positive definiteness of $\widetilde{\mathcal{D}}$ in (3-6), because of the uniform positive definiteness of $\delta\mathcal{D} > c\delta\,\mathrm{Id}_2$ in $\overline{\Omega}$. Likewise, the first inequality in (3-7) follows already when the error is smaller than $\frac12\varepsilon$.

The same reasoning proves the error bounds on $\tilde v - v$ and $\tilde w - w$ in (3-7), in view of (3-2):

$$\tilde v(x) - v(x) = \sum_{k=1}^{N}(v_{k+1}(x) - v_k(x)) = \sum_{k=1}^{N}\mathcal{O}\Big(\frac{1}{\lambda_k}\|a_k\|_0\Big),$$
$$\tilde w(x) - w(x) = \sum_{k=1}^{N}(w_{k+1}(x) - w_k(x)) = \sum_{k=1}^{N}\mathcal{O}\Big(\frac{1}{\lambda_k}\big(\|a_k\|_0^2 + \|\nabla a_k\|_0\|\nabla v_k\|_0\big)\Big).$$

(3) To obtain the first error bound in (3-8), use (3-3) and (3-11):

$$|\nabla\tilde v(x) - \nabla v(x)| \le \sum_{k=1}^{N}|\nabla v_{k+1}(x) - \nabla v_k(x)| \le C\sum_{k=1}^{N}a_k(x) + \sum_{k=1}^{N}\mathcal{O}\Big(\frac{1}{\lambda_k}\|a_k\|_0^2\Big) \le CN_0^{1/2}\|\mathcal{D}\|_0^{1/2},$$

where again, by adjusting $\lambda_k$ at each step, we ensure the controllability of the error term with respect to the nonnegative quantity $N_0^{1/2}\|\mathcal{D}\|_0^{1/2}$. Likewise,

$$\forall k = 1,\dots,N,\quad |\nabla v_k(x)| \le |\nabla v(x)| + \sum_{i=1}^{k-1}|\nabla v_{i+1}(x) - \nabla v_i(x)| \le \|\nabla v\|_0 + CN_0^{1/2}\|\mathcal{D}\|_0^{1/2},$$

and obviously by (3-11),

$$a_k(x) \le \sum_{i=1}^{k-1}a_i(x) \le CN_0^{1/2}\|\mathcal{D}\|_0^{1/2},$$

which by (3-11) yield

$$\sum_{k=1}^{N} a_k(x)(\|a_k\|_0 + \|\nabla v_k\|_0) \le C(\|\nabla v\|_0 + N_0^{1/2}\|\mathcal{D}\|_0^{1/2}) \sum_{k=1}^{N} a_k(x) \le CN_0(\|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2})\|\mathcal{D}\|_0^{1/2}.$$

Consequently and by (3-3), we get the last gradient error bound in (3-8):

$$\begin{aligned}|\nabla \tilde{w}(x) - \nabla w(x)|& \\ &\le \sum_{k=1}^{N} |\nabla w_{k+1}(x) - \nabla w_k(x)| \\ &\le C\sum_{k=1}^{N} a_k(x)(\|a_k\|_0 + \|\nabla v_k\|_0) + \sum_{k=1}^{N} \mathcal{O}\Big(\frac{1}{\lambda_k}\big(\|a_k\|_0\,\|\nabla a_k\|_0 + \|a_k\|_0\,\|\nabla^2 v_k\|_0 + \|\nabla a_k\|_0\,\|\nabla v_k\|_0\big)\Big) \\ &\le CN_0(\|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2})\|\mathcal{D}\|_0^{1/2}.\end{aligned}$$

This concludes the proof of the stage approximation construction. □

We now finally give:

*Proof of Theorem 2.1.* (1) Fix $\varepsilon > 0$. It suffices to construct $v \in \mathcal{C}^1(\bar{\Omega})$ and $w \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ such that

$$A_0 = \tfrac{1}{2}\nabla v \otimes \nabla v + \operatorname{sym}\nabla w \quad \text{in } \bar{\Omega} \tag{3-12}$$

and

$$\|v - v_0\|_0 + \|w - w_0\|_0 < \varepsilon. \tag{3-13}$$

The exact solution $(v, w)$ of (3-12) will be obtained as the $\mathcal{C}^1$ limit of sequences of successive approximations $\{v_k \in \mathcal{C}^\infty(\bar{\Omega}), w_k \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^2)\}_{k=0}^\infty$, where $v_0$ and $w_0$ are given in the statement of the theorem and satisfy (2-1), while $v_{k+1}$ and $w_{k+1}$ are defined inductively by means of Proposition 3.2 applied to $v_k$, $w_k$ and $\varepsilon_k > 0$, under the requirement

$$\sum_{k=1}^{\infty} \varepsilon_k < \varepsilon \quad \text{and} \quad \sum_{k=1}^{\infty} \varepsilon_k^{1/2} < 1. \tag{3-14}$$

In agreement with our notation convention, we introduce the $k$-th deficit $\mathcal{D}_k$, which is positive definite by (3-6):

$$\forall k \ge 0, \quad \mathcal{D}_k := A_0 - \big(\tfrac{1}{2}\nabla v_k \otimes \nabla v_k + \operatorname{sym}\nabla w_k\big) \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R}^{2\times 2}_{\mathrm{sym},>}).$$

By (3-7) it follows that

$$\|v_k - v\|_0 + \|w_k - w\|_0 \le \sum_{i=0}^{k-1} \|v_{i+1} - v_i\|_0 + \sum_{i=0}^{k-1} \|w_{i+1} - w_i\|_0 < \sum_{i=1}^{k-1} \varepsilon_i < \sum_{i=1}^{\infty} \varepsilon_i.$$

Thus, $\{v_k\}_{k=0}^\infty$ and $\{w_k\}_{k=0}^\infty$ converge uniformly in $\bar{\Omega}$, respectively, to $v$ and $w$ which satisfy (3-13) in view of (3-14).

(2) We now show that this convergence is in $\mathcal{C}^1$. Indeed, by (3-7) $\|\mathcal{D}_k\|_0 < \varepsilon_k$, so by (3-8)

$$\|\nabla v_{k+m} - \nabla v_k\|_0 \le \sum_{i=k}^{m-1} \|\nabla v_{i+1} - \nabla v_i\|_0 \le C N_0^{1/2} \sum_{i=k}^{m-1} \|\mathcal{D}_i\|_0^{1/2} \le C N_0^{1/2} \sum_{i=k}^{m-1} \varepsilon_i^{1/2}. \tag{3-15}$$

In particular, in view of (3-14) the sequence $\{\|\nabla v_k\|_0\}_{k=0}^{\infty}$ is bounded, so we further have

$$\|\nabla w_{k+m} - \nabla w_k\|_0 \le \sum_{i=k}^{m-1} \|\nabla w_{i+1} - \nabla w_i\|_0 \le C N_0 \sum_{i=k}^{m-1} (\|\nabla v_i\|_0 + \|\mathcal{D}_i\|_0^{1/2}) \|\mathcal{D}_i\|_0^{1/2} \le \widetilde{C} N_0 \sum_{i=k}^{m-1} \varepsilon_i^{1/2}, \tag{3-16}$$

where the constant $\widetilde{C}$ is independent of $k$ and $m$. Through the above assertions (3-15) and (3-16), in view of the second condition in (3-14), we conclude that $\{v_k\}_{k=1}^{\infty}$ and $\{w_k\}_{k=0}^{\infty}$ are Cauchy sequences that converge in $\mathcal{C}^1(\bar{\Omega})$ to $v \in \mathcal{C}^1(\bar{\Omega})$ and $w \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$, respectively. Finally,

$$\left\| A_0 - \left(\tfrac{1}{2} \nabla v \otimes \nabla v + \operatorname{sym} \nabla w\right) \right\|_0 = \lim_{k\to\infty} \|\mathcal{D}_k\|_0 \le \lim_{k\to\infty} \varepsilon_k = 0$$

implies (3-12) and completes the proof of Theorem 2.1. □

**Remark 3.3.** In addition to the uniform convergence postulated in Theorem 2.1, one also has

$$\forall n, \quad \|\nabla v_n\|_0 \le \|\nabla v_0\|_0 + C N_0^{1/2}.$$

Using notation as in the proof above and recalling (3-15) and (3-14), this bound follows by

$$\|\nabla v - \nabla v_0\|_0 = \lim_{k\to\infty} \|\nabla v_k - \nabla v_0\|_0 \le \lim_{k\to\infty} \left( C N_0^{1/2} \sum_{i=0}^{k-1} \varepsilon_i^{1/2} \right) \le C N_0^{1/2}.$$

## 4. The $\mathcal{C}^{1,\alpha}$ approximations: a proof of Theorem 1.1, preliminary results and some heuristics towards the proof of Theorem 1.2

Theorem 1.1 follows easily from Theorem 1.2, which will be proved in the next section.

*Proof of Theorem 1.1.* Since $\mathcal{C}^1(\bar{\Omega})$ is dense in $\mathcal{C}^0(\bar{\Omega})$, we may without loss of generality assume that $v_0 \in \mathcal{C}^1(\bar{\Omega})$. Set $w_0 = 0$ and $A_0 = (\lambda + c)\mathrm{Id} \in \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, where $c$ is a constant and $\lambda$ is constructed as follows.

Extend the function $f$ to $f \in L^p(\Omega_\varepsilon)$ defined on an open smooth set $\Omega_\varepsilon \supset \bar{\Omega}$ and solve

$$-\Delta\lambda = f \quad \text{in } \Omega_\varepsilon, \qquad \lambda = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Since $\lambda \in W^{2,p}(\Omega_\varepsilon)$, Morrey's theorem implies that $\lambda \in \mathcal{C}^{0,\beta}(\bar{\Omega})$ for every $\beta \in (0, 1)$ when $p \ge 2$, and for $\beta = 2 - \frac{2}{p}$ when $p \in (1, 2)$. Also, for $c$ large enough, condition (1-5) on the positive definiteness of the defect is satisfied. On the other hand,

$$-\operatorname{curl}\operatorname{curl} A_0 = -\Delta(\lambda + c) = f,$$

so the result follows directly from Theorem 1.2, since $\frac{1}{2}\left(2 - \frac{2}{p}\right) \ge \frac{1}{7}$ is equivalent to $p \ge \frac{7}{6}$. □

Our next simple corollary concerns the steady-state Euler equations with the exchanged roles of the given pressure $q$ and the unknown forcing term $\nabla^\perp g$.

**Corollary 4.1.** *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $q \in \mathcal{C}^{0,\beta}(\bar\Omega)$ for some $\beta \in (0,1)$ and fix $\varepsilon > 0$. Then for every exponent $\alpha$ in the range $0 < \alpha < \min\{\frac{1}{7}, \frac{1}{2}\beta\}$, there exist sequences $\{u_n \in \mathcal{C}^{0,\alpha}(\bar\Omega, \mathbb{R}^2)\}_{n=1}^\infty$ and $\{g_n \in \mathcal{C}^{0,\alpha}(\bar\Omega)\}_{n=1}^\infty$ solving in $\Omega$ the system*

$$\operatorname{div}(u_n \otimes u_n) - \nabla q = \nabla^\perp g_n, \quad \operatorname{div} u_n = 0, \tag{4-1}$$

*and such that $u_n = \nabla^\perp v_n$ and $g_n = \operatorname{curl} w_n$, where each $v_n \in \mathcal{C}^{1,\alpha}(\bar\Omega)$ and $w_n \in \mathcal{C}^1(\bar\Omega, \mathbb{R}^2)$, while the sequence $\{v_n\}_{n=1}^\infty$ is dense in $\mathcal{C}^0(\bar\Omega)$ and $\|w_n\|_0 < \varepsilon$ for every $n \geq 1$.*

*Proof.* As before, since $\mathcal{C}^1(\bar\Omega)$ is dense in $\mathcal{C}^0(\bar\Omega)$, it is enough to take $v_0 \in \mathcal{C}^1(\bar\Omega)$ and approximate it by a sequence $\{v_n \in \mathcal{C}^{1,\alpha}(\bar\Omega)\}_{n=1}^\infty$ with the properties as in the statement of the corollary. Let $w_0 = 0$ and let $c > 0$ be a sufficiently large constant, so that $(q + c)\mathrm{Id}_2 - \nabla v_0 \otimes \nabla v_0$ is strictly positive definite in $\bar\Omega$. By Theorem 1.2, there exist sequences $v_n \in \mathcal{C}^{1,\alpha}(\bar\Omega)$ and $w_n \in \mathcal{C}^{1,\alpha}(\bar\Omega, \mathbb{R}^2)$ which converge uniformly to $v_0$ and $w_0$ and which satisfy

$$(q + c)\mathrm{Id}_2 = \nabla v_n \otimes \nabla v_n + 2 \operatorname{sym} \nabla w_n \quad \text{in } \bar\Omega.$$

Taking the cofactor of both sides in the above matrix identity, we get

$$(q + c)\mathrm{Id}_2 = \nabla^\perp v_n \otimes \nabla^\perp v_n + 2 \operatorname{cof}(\operatorname{sym} \nabla w_n).$$

Taking the row-wise divergence, we obtain (4-1) with $u_n = \nabla^\perp v_n$ and $g_n = \operatorname{curl} w_n$, since $\operatorname{div} \operatorname{cof} \nabla w_n = 0$, while $(\operatorname{div} \operatorname{cof}(\nabla w_n)^T)^\perp = -\nabla(\operatorname{curl} w_n)$. □

Towards a proof of Theorem 1.2 we will derive a sequence of approximation results, and then combine them with Theorem 2.1 in Section 6. For completeness, we first prove a simple, useful result:

**Lemma 4.2.** *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Given are functions $f \in \mathcal{C}^N(\bar\Omega, \mathbb{R}^n)$ and $\psi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$. Then*

$$\forall k = 0, \ldots, N, \quad \|\psi \circ f\|_k \leq M \|f\|_k,$$

*where the constant $M > 0$ depends on the dimensions $n, m$, the differentiability order $N$, the domain $\Omega$, the norm $\|\psi\|_N$ on the compact set $f(\bar\Omega)$ and the norm $\|f\|_0$, but it does not depend on the higher norms of $f$.*

*Proof.* The statement is obvious for $k = 0$. Fix $k \in \{1, \ldots, N\}$ and let $m = (m_1, \ldots, m_k)$ be any $k$-tuple of nonnegative integers such that $\sum_{i=1}^k i m_i = k$. Defining $|m| = \sum_{i=1}^k m_i$ and using the interpolation inequality [Adams and Fournier 2003]

$$\forall i = 1, \ldots, k, \quad \|f\|_i \leq M_0 \|f\|_0^{1-i/k} \|f\|_k^{i/k},$$

valid with a constant $M_0 > 0$ depending on $n$, $N$ and $\Omega$, we get

$$\prod_{i=1}^k \|\nabla^i f\|_0^{m_i} \leq M_0^{|m|} \prod_{i=1}^k \|f\|_0^{m_i - i m_i/k} \|f\|_k^{i m_i/k} = M_0^{|m|} \|f\|_0^{|m|-1} \|f\|_k,$$

with $|m| := m_1 + \cdots + m_j$. Calculating the partial derivatives in $\nabla^k(\psi \circ f)$ by the Faà di Bruno formula gives hence the desired estimate

$$\|\nabla^k(\psi \circ f)\|_0 \leq M \sum_m \prod_{i=1}^k \|\nabla^i f\|_0^{m_i} \leq M\|f\|_k.$$

Above, the summation extends over all multiindices $m = (m_1, \dots, m_k)$ with the properties listed at the beginning of the proof. □

We recall the following estimates which have been proved in [Conti et al. 2012]:

**Lemma 4.3.** *Let $\varphi \in \mathcal{C}_c^\infty(B(0,1), \mathbb{R})$ be a standard mollifier supported on the ball $B(0,1) \subset \mathbb{R}^n$, that is, a nonnegative, smooth and radially symmetric function such that $\int_{\mathbb{R}^n} \varphi = 1$. Denote*

$$\forall l \in (0,1), \quad \varphi_l(x) = \frac{1}{l^n}\varphi\left(\frac{x}{l}\right).$$

*Then, for every $f, g \in \mathcal{C}^0(\mathbb{R}^n)$ we have*

$$\forall k, j \geq 0, \quad \|f * \varphi_l\|_{k+j} \leq \frac{C}{l^k}\|f\|_j, \tag{4-2}$$

$$\forall k \geq 0, \quad \|f * \varphi_l - f\|_k \leq \frac{C}{l^{k-2}}\|f\|_2, \tag{4-3}$$

$$\forall \alpha \in (0,1], \quad \|f * \varphi_l - f\|_0 \leq C l^\alpha \|f\|_{0,\alpha}, \tag{4-4}$$

$$\forall \alpha \in (0,1], \quad \|f * \varphi_l\|_1 \leq \frac{C}{l^{1-\alpha}}\|f\|_{0,\alpha}, \tag{4-5}$$

$$\forall k \geq 0,\ \forall \alpha \in (0,1], \quad \big\|(fg) * \varphi_l - (f * \varphi_l)(g * \varphi_l)\big\|_k \leq \frac{C}{l^{k-2\alpha}}\|f\|_{0,\alpha}\,\|g\|_{0,\alpha}, \tag{4-6}$$

*with the uniform constants $C > 0$ depending only on the smoothness exponents $k, j, \alpha$.*

*Proof.* The estimate (4-2) follows directly from the definition of convolution. To prove (4-3), note that for every $x \in \mathbb{R}^n$,

$$\begin{aligned}
\big|\nabla^k(f * \varphi_l - f)(x)\big| &= \left|\int_{\mathbb{R}^n} \varphi_l(y)\big(\nabla^k f(x-y) - \nabla^k f(x)\big)\,\mathrm{d}y\right| \\
&= \left|\int_{\mathbb{R}^n} \nabla^k \varphi_l(y)\big(f(x-y) - f(x)\big)\,\mathrm{d}y\right| = \frac{1}{l^k}\left|\int_{\mathbb{R}^n} \frac{1}{l^n}\nabla^k\varphi\left(\frac{y}{l}\right)\big(\nabla f(x)\cdot y + r_x(y)\big)\,\mathrm{d}y\right| \\
&= \frac{1}{l^k}\left|\int_{\mathbb{R}^n} \frac{1}{l^n}\nabla^k\varphi\left(\frac{y}{l}\right) r_x(y)\,\mathrm{d}y\right| \leq \frac{C}{l^k}\sup_{x \in \mathbb{R}^n, |y|<l}|r_x(y)| \leq \frac{C}{l^{k-2}}\|f\|_2,
\end{aligned}$$

where we integrated by parts, discarded the contribution with the symmetric term $\nabla f(x) \cdot y$, which integrates to 0, and estimated the Taylor's formula remainder term

$$r_x(y) = f(x-y) - f(x) - \nabla f(x)\cdot y = \|f\|_2 \mathcal{O}(|y|^2).$$

The proof of (4-4) follows similarly by

$$\left|\nabla^k(f*\varphi_l - f)(x)\right| = \left|\int_{\mathbb{R}^n} \varphi_l(y)|y|^\alpha \frac{f(x-y)-f(x)}{|y|^\alpha}\,\mathrm{d}y\right| \le Cl^\alpha \|f\|_{0,\alpha}\int_{\mathbb{R}^n}\varphi_l(y)\,\mathrm{d}y \le Cl^\alpha\|f\|_{0,\alpha},$$

while for (4-5) we write

$$\begin{aligned}\left|\nabla(f*\varphi_l)(x)\right| &= \left|\int_{\mathbb{R}^n} f(x-y)\frac{1}{l^{n+1}}\nabla\varphi_l\Big(\frac{y}{l}\Big)\,\mathrm{d}y\right| = \frac{1}{l}\left|\int_{\mathbb{R}^n}\frac{f(x-y)-f(x)}{|y|^\alpha}\frac{|y|^\alpha}{l}\frac{1}{l^n}\nabla\varphi_l\Big(\frac{y}{l}\Big)\,\mathrm{d}y\right| \\ &\le Cl^{\alpha-1}\|f\|_{0,\alpha}\int_{\mathbb{R}^n}\frac{1}{l^n}\left|\nabla\varphi_l\Big(\frac{y}{l}\Big)\right|\mathrm{d}y \le \frac{C}{l^{1-\alpha}}\|f\|_{0,\alpha}.\end{aligned}$$

Finally, for the crucial commutator estimate (4-6) we refer to [Conti et al. 2012, Lemma 1]. □

***A heuristic overview of the next two sections.*** Let us attempt to follow the construction in Sections 2 and 3, but with the goal of controlling the higher Hölder norms of the iterations, and hence also quantifying the growth of the $\mathcal{C}^2$ norms of $v$, $w$. Let $A \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ be the target matrix field and let $v_1 \in \mathcal{C}^\infty(\bar\Omega)$, $w_1 \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^2)$ be given at an input of a "stage". As in Proposition 3.2, we decompose the defect $\mathcal{D} = A - \big(\frac{1}{2}\nabla v_1\otimes\nabla v_1 + \mathrm{sym}\,\nabla w_1\big)$ into a linear combination $\sum_{k=1}^N a_k^2\eta_k\otimes\eta_k$ of rank-one symmetric matrices with smooth coefficients given by Lemma 2.4. We define

$$v_{k+1}(x) = v_k(x) + \frac{1}{\lambda}\Gamma_1(x, \lambda x\cdot\eta_k), \qquad w_{k+1}(x) = w_k(x) - \frac{1}{\lambda}\Gamma_1(x,\lambda x\cdot\eta_k)\nabla v_k(x) + \frac{1}{\lambda}\Gamma_2(x,\lambda x\cdot\eta_k)\eta_k.$$

This yields, by applying Lemma 4.2 to $\psi(x) = x^2$ and $f = a_k$,

$$\forall m = 0,\dots,3, \quad \|\nabla^m v_{k+1} - \nabla^m v_k\|_0 \le C\sum_{\substack{i+j=m\\ 0\le i,j\le m}}\|a_k\|_i\lambda^{j-1},$$

$$\forall m = 0,\dots,2, \quad \|\nabla^m w_{k+1} - \nabla^m w_k\|_0 \le C\sum_{\substack{i+j=m\\ 0\le i,j\le m}}\|a_k\|_i\lambda^{j-1} + C\sum_{\substack{i+j+s=m\\ 0\le i,j,s\le m}}\|a_k\|_i\lambda^{j-1}\|\nabla^{s+1}v_k\|_0,$$

On the other hand, applying Lemma 4.2 to $\psi = \phi_k$ defined in Lemma 2.4 and to $f = \mathcal{D}$, we get

$$\forall k = 1,\dots,N, \quad \|a_k\|_2 \le C(\|v_1\|_3^2 + \|w_1\|_3 + \|A\|_2).$$

Now, in order to control the $\mathcal{C}^{1,\alpha}$ norm of $v_{N+1}$ through interpolation, we need to control the norm $\|v_{N+1}\|_2$, which in turn depends on $\|a_k\|_2$. The above estimate shows that at the end of each stage, the $\mathcal{C}^2$ norm of $a_k$ is determined by the $\mathcal{C}^3$ norms of the given $v_1$ and $w_1$ of the previous stage. Further, the $\mathcal{C}^2$ norm of $w_{N+1}$ is only controlled by the $\mathcal{C}^3$ norm of $v_0$ and also of all the $a_k$. One might hope to control $\|a_k\|_3$ if the deficit $\mathcal{D}$ is small enough, but the dependence of $\|w_{N+1}\|_2$ on $\|v_0\|_3$ cannot be easily bypassed. Recalling that we need infinitely many stages in the construction, this implies that a direct estimate cannot be obtained in this manner, unless we deal with analytic data similarly to [Borisov 2004]. We thus need to modify the previous simplistic approach.

The appropriate modification is achieved by introducing a mollification before each stage. This technique was first introduced in [Conti et al. 2012] for the isometric immersion problem, in order to control the loss of regularity through the stages and to improve on results in [Borisov 2004]. Indeed, we note that the loss of derivatives in the above estimates is accompanied by a similar gain in the powers of $\lambda$, in a manner such that the total order of derivatives, plus the order of powers needed to control $\|v_{N+1}\|_2$ and $\|w_{N+1}\|_2$ is constant. If we replace $v_1$ and $w_1$ by their mollifications on the scale $l \sim \lambda^{-1}$, each derivative loss can be estimated by one power of $\lambda$, and $\|v_0\|_2$ and $\|w_0\|_2$ will control $\|v_{N+1}\|_2$ and $\|w_{N+1}\|_2$. One problem still remains to be taken care of: does the deficit $\mathcal{D}$ decrease at the end of each stage? As the calculation below will show, a mollification of order $\lambda^{-1}$ does not suffice to this end, and we need to mollify at a larger scale of $l > \lambda^{-1}$.

This is indeed how we want proceed. In practice, we let the mollification scale be $l = \delta/M$ and we treat $\nabla v$ "like $a$", controlling its $j$-th norm by $\delta l^{-j}$. We then "sacrifice" one $l$ in order to gain one $\delta$; instead of $\|\nabla(v * \varphi_l)\|_j \le C\|v\|_1 l^{-j}$, we use $\|\nabla(v * \varphi_l)\|_j \le C(\|v\|_2 l) l^{-j}$, choosing $l$ such that $l\|v\|_2 < \delta$ and obtaining the desired bound (5-2).

Finally, note that the loss of $N$ powers of $\lambda l > 1$ in the control of the $\mathcal{C}^2$ norms at the end of each stage is the main reason why the described scheme does not deliver better than $\mathcal{C}^{1,1/7}$ estimates, even for the optimal $N = 3$ from the decomposition in Lemma 2.3.

## 5. The $\mathcal{C}^{1,\alpha}$ approximations: a "step" and a "stage" in a proof of Theorem 1.2

In this section, we develop the approximation technique that will be used for a proof of Theorem 1.2 in the next section. The first result is a variant of Proposition 3.1 in which we accomplish the "step" of the Nash–Kuiper construction with extra estimates on the higher derivatives.

**Proposition 5.1.** *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set. Given are functions $v \in \mathcal{C}^3(\bar{\Omega})$, $w \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$, a nonnegative function $a \in \mathcal{C}^3(\bar{\Omega})$ and a unit vector $\eta \in \mathbb{R}^2$. Let $\delta, l \in (0, 1)$ be two parameter constants such that*

$$\|a\|_m \le \frac{\delta}{l^m} \quad \forall m = 0, \dots, 3, \qquad \textit{and} \qquad \|\nabla v\|_m \le \frac{\delta}{l^m} \quad \forall m = 1, 2. \tag{5-1}$$

*Then for every $\lambda > 1/l$ there exist approximating functions $\tilde{v}_\lambda \in \mathcal{C}^3(\bar{\Omega})$ and $\tilde{w}_\lambda \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ satisfying the following bounds, with a universal constant $C > 0$ independent of all parameters*:

$$\left\| \left(\tfrac{1}{2}\nabla\tilde{v}_\lambda \otimes \nabla\tilde{v}_\lambda + \operatorname{sym}\nabla\tilde{w}_\lambda\right) - \left(\tfrac{1}{2}\nabla v \otimes \nabla v + \operatorname{sym}\nabla w + a^2 \eta\otimes\eta\right)\right\|_0 \le C\frac{\delta^2}{\lambda l}, \tag{5-2}$$

$$\|\tilde{v}_\lambda - v\|_m \le C\delta\lambda^{m-1} \quad \forall m = 0, \dots, 3, \tag{5-3}$$

$$\|\tilde{w}_\lambda - w\|_m \le C\delta\lambda^{m-1}(1 + \|\nabla v\|_0) \quad \forall m = 0, \dots, 2. \tag{5-4}$$

*Proof.* We define $\tilde{v}_\lambda$, $\tilde{w}_\lambda$ as in the proof of Proposition 3.1:

$$\tilde{v}_\lambda(x) = v(x) + \frac{1}{\lambda}\Gamma_1(x, \lambda x\cdot\eta), \qquad \tilde{w}_\lambda(x) = w(x) - \frac{1}{\lambda}\Gamma_1(x, \lambda x\cdot\eta)\nabla v(x) + \frac{1}{\lambda}\Gamma_2(x, \lambda x\cdot\eta)\eta.$$

Firstly, (5-2) follows immediately from (3-1) in view of (5-1), because $\lambda l > 1$:

$$\frac{1}{\lambda}\|a\|_0(\|\nabla a\|_0 + \|\nabla^2 v\|_0) + \frac{1}{\lambda^2}\|\nabla a\|_0^2 \le 2\frac{\delta}{\lambda}\frac{\delta}{l} + \frac{1}{\lambda^2}\frac{\delta^2}{l^2} \le 3\frac{\delta^2}{\lambda l}.$$

To check (5-3), we compute directly as in Lemma 2.2:

$$\nabla^m(\tilde{v}_\lambda - v)\|_0 \le \frac{C}{\lambda}\|\nabla^m\Gamma_1(x,\lambda x\cdot\eta)\|_0 \le \frac{C}{\lambda}\sum_{\substack{i+j=m\\0\le i,j\le m}}\|a\|_j\lambda^j \le \frac{C}{\lambda}\sum_{i=0}^{m}\frac{\delta}{l^i}\lambda^{m-i} \le C\delta\lambda^{m-1}$$

by (5-1) and noting again $\lambda l > 1$. Similarly,

$$\begin{aligned}\|\nabla^m(\tilde{w}_\lambda - w)\|_0 &\le \frac{C}{\lambda}\big(\|\nabla^m\Gamma_2(x,\lambda x\cdot\eta)\|_0 + \|\nabla^m\Gamma_1(x,\lambda x\cdot\eta)\nabla v\|_0\big)\\ &\le \frac{C}{\lambda}\bigg(\sum_{\substack{i+j=m\\0\le i,j\le m}}\|a^2\|_i\lambda^j + \sum_{\substack{i+j+s=m\\0\le i,j,s\le m}}\|a\|_i\lambda^j\|\nabla v\|_s\bigg)\\ &\le \frac{C}{\lambda}\bigg(\sum_{i=1}^{m}\frac{\delta}{l^i}\lambda^{m-i} + \sum_{\substack{0\le i+s\le m\\0\le i,s\le m}}\frac{\delta}{l^i}\lambda^{m-(i+s)}\frac{\delta}{l^s} + \sum_{\substack{i+j=m\\0\le i,j\le m}}\frac{\delta}{l^i}\lambda^j\|\nabla v\|_0\bigg)\\ &\le \frac{C}{\lambda}\bigg(\sum_{i=1}^{m}\frac{\delta}{l^i}\lambda^{m-i}\bigg)(1+1+\|\nabla v\|_0) \le C\delta\lambda^{m-1}(1+\|\nabla v\|_0),\end{aligned}$$

where we applied Lemma 4.2 to $\psi(x) = x^2$ and $f = a$ in view of (5-1) yielding $\|a\|_0 \le 1$, so that $\|a^2\|_i \le C\|a\|_i \le C\delta/l^i$. This achieves (5-4) and completes the proof of the proposition. □

We now accomplish the "stage" in the Hölder regular approximation construction.

**Proposition 5.2.** *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain. Let $v \in \mathcal{C}^2(\bar{\Omega})$, $w \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ and $A \in \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ for some $\beta \in (0, 1)$ be such that the deficit $\mathcal{D}$ is appropriately small*:

$$\mathcal{D} = A - \big(\tfrac{1}{2}\nabla v\otimes\nabla v + \operatorname{sym}\nabla w\big), \quad 0 < \|\mathcal{D}\|_0 < \delta_0 \ll 1. \tag{5-5}$$

*Then, for every two parameter constants $M, \sigma$ satisfying*

$$M > \max\{\|v\|_2, \|w\|_2, 1\} \quad \textit{and} \quad \sigma > 1, \tag{5-6}$$

*there exist $\tilde{v} \in \mathcal{C}^2(\bar{\Omega})$ and $\tilde{w} \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ such that the following error bounds hold for $\tilde{v}$, $\tilde{w}$ and the new deficit $\widetilde{\mathcal{D}} = A - \big(\frac{1}{2}\nabla\tilde{v}\otimes\nabla\tilde{v} + \operatorname{sym}\nabla\tilde{w}\big)$*:

$$\|\widetilde{\mathcal{D}}\|_0 \le C\bigg(\frac{\|A\|_{0,\beta}}{M^\beta}\|\mathcal{D}\|_0^{\beta/2} + \frac{1}{\sigma}\|\mathcal{D}\|_0\bigg), \tag{5-7}$$

$$\|\tilde{v} - v\|_1 \le C\|\mathcal{D}\|_0^{1/2} \quad \textit{and} \quad \|\tilde{w} - w\|_1 \le C(1+\|\nabla v\|_0)\|\mathcal{D}\|_0^{1/2}, \tag{5-8}$$

$$\|\tilde{v}\|_2 \le CM\sigma^3 \quad \textit{and} \quad \|\tilde{w}\|_2 \le C(1+\|\nabla v\|_0)M\sigma^3. \tag{5-9}$$

*The constant $C > 0$ is universal and independent of all parameters.*

*Proof.* Analogously to [Conti et al. 2012, Proposition 4], the proof is split into three parts.

Part 1: mollification. Let $\varphi \in \mathcal{C}_c^\infty(B(0,1))$ be the standard mollifier in two dimensions, as in Lemma 4.3. Since $v$, $w$ and $A$ can be extended on the whole $\mathbb{R}^2$, with all their relevant norms increased at most $C$ times ($C$ depends here on the curvature of the boundary $\partial\Omega$), we may define

$$\mathfrak{v} = v * \varphi_l, \quad \mathfrak{w} := w * \varphi_l, \quad \mathfrak{A} := A * \varphi_l \quad \text{with } l = \frac{\|\mathcal{D}\|_0^{1/2}}{M} < 1.$$

Applying Lemma 4.3 and noting (5-6), we immediately get the following uniform error bounds for $\mathfrak{v}$, $\mathfrak{w}$, $\mathfrak{A}$ and for the induced deficit $\mathfrak{D} = \mathfrak{A} - \left(\frac{1}{2}\nabla\mathfrak{v}\otimes\nabla\mathfrak{v} + \operatorname{sym}\nabla\mathfrak{w}\right)$:

$$\begin{aligned} \|\mathfrak{v} - v\|_1 + \|\mathfrak{w} - w\|_1 &\le Cl(\|v\|_2 + \|w\|_2) \le C\|\mathcal{D}\|_0^{1/2}, \\ \|\mathfrak{A} - A\|_0 &\le Cl^\beta \|A\|_{0,\beta}, \\ \|\mathfrak{D}\|_m &\le \|\mathcal{D}*\varphi_l\|_m + \left\|(\nabla v * \varphi_l)\otimes(\nabla v*\varphi_l) - (\nabla v\otimes\nabla v)*\varphi_l\right\|_m \\ &\le \frac{C}{l^m}\|\mathcal{D}\| + \frac{C}{l^{m-2}}\|v\|_2^2 \le \frac{C}{l^m}\|\mathcal{D}\|_0 \quad \forall m = 0,\dots,3. \end{aligned} \tag{5-10}$$

In the proof of the last inequality above, we used (4-6) with the Hölder exponent $\alpha = 1$.

We note that so far we have simply exchanged the lower regularity fields $v$, $w$, $A$ with their smooth approximations, at the expense of the error that, as we shall see below, is compatible with the that postulated in (5-7)–(5-9). The following estimate, however, reflects the advantage of averaging through mollification that results in the control of the $\mathcal{C}^3$ norm of $\mathfrak{v}$ by the $\mathcal{C}^2$ norm:

$$\forall m = 1,2, \quad \|\nabla\mathfrak{v}\|_m \le \|\mathfrak{v}\|_{m+1} \le \frac{C}{l^{m-1}}\|v\|_2 \le \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2}, \tag{5-11}$$

where again we used Lemma 4.3 and (5-6). Note that the scaling bound (5-11) is consistent with the second requirement in (5-1) of Proposition 5.1. We also record the simple bound

$$\|\mathfrak{w}\|_2 \le C\|w\|_2 \le CM. \tag{5-12}$$

Part 2: modification and positive definiteness. Contrary to the "stage" construction in the proof of Proposition 3.2, we do not know whether the original defect $\mathcal{D}$ (and hence the induced defect $\mathfrak{D}$) is positive definite, so that Lemma 2.4 could be used. In any case, we need to keep the number of terms in the decomposition (3-9) into rank-one matrices as small as possible.

We now further modify $\mathfrak{w}$ in order to use the optimal decomposition in (2-5). Let $r_0$ be as in Lemma 2.3 and define

$$\mathfrak{w}' = \mathfrak{w} - 2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}\mathrm{id}_2, \quad \mathfrak{D}' = \mathfrak{A} - \left(\tfrac{1}{2}\nabla\mathfrak{v}\otimes\nabla\mathfrak{v} + \operatorname{sym}\nabla\mathfrak{w}'\right).$$

Clearly, by (5-10) we get

$$\|\mathfrak{w}' - \mathfrak{w}\|_2 \le C(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0) \le C\|\mathcal{D}\|_0. \tag{5-13}$$

Note now that

$$\mathfrak{D}'(x) = 2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}\mathrm{Id}_2 + \mathfrak{D}(x) = 2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}\left(\mathrm{Id}_2 + \frac{r_0}{2(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}\mathfrak{D}\right) \quad \forall x \in \bar{\Omega}.$$

By Lemma 2.3 we may apply (2-5) to the scaled defect

$$G = \mathrm{Id}_2 + \frac{r_0}{2(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}\mathfrak{D}$$

and arrive at

$$\mathfrak{D}'(x) = \sum_{k=1}^{3} 2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}\Phi_k(G(x))\xi_k \otimes \xi_k = \sum_{k=1}^{3} a_k^2(x)\xi_k \otimes \xi_k \quad \forall x \in \bar{\Omega}, \tag{5-14}$$

where

$$\left\{ a_k = \left( 2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0}\Phi_k \circ G \right)^{1/2} \right\}_{k=1}^{3}$$

are positive smooth functions on $\bar{\Omega}$. We claim that

$$\forall k = 1, \dots, 3,\ \forall m = 0, \dots, 3, \qquad \|a_k\|_m \le \frac{C}{l^m}\|\mathcal{D}\|_0^{1/2}. \tag{5-15}$$

Indeed, for $m = 0$ this inequality follows directly by $\|\mathfrak{D}\|_0 \le C\|\mathcal{D}\|_0$. For $m = 1, \dots, 3$ we use Lemma 4.2 on each $\psi = \Phi_k^{1/2}$ and $f = G$, where noting that $\|G\|_0 \le C$ and recalling (5-10) yields

$$\begin{aligned}
\|a_k\|_m &\le \left( 2\frac{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}{r_0} \right)^{1/2} C\|G\|_m \\
&\le C(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)^{1/2}\left( C + \frac{r_0}{2(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)}\|\mathfrak{D}\|_m \right) \\
&\le C\left( (\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)^{1/2} + \frac{1}{(\|\mathfrak{D}\|_0 + \|\mathcal{D}\|_0)^{1/2}}\frac{1}{l^m}\|\mathcal{D}\|_0 \right) \le C\left( \|\mathcal{D}\|_0^{1/2} + \frac{1}{l^m}\|\mathcal{D}\|_0^{1/2} \right)
\end{aligned} \tag{5-16}$$

and hence achieves (5-15). Note that the scaling bound (5-15) is consistent with the first requirement in (5-1) of Proposition 5.1.

Part 3: iterating the one-dimensional oscillations. We set $v_1 = \mathfrak{v}$, $w_1 = \mathfrak{w}$ and inductively define $v_{k+1} \in \mathcal{C}^3(\bar{\Omega})$ and $w_{k+1} \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ for $k = 1, 2, 3$ by means of Proposition 5.1 applied to $v_k$, $w_k$, the function $a_k$ and the unit vector $\xi_k$ appearing in (5-14), with the parameters

$$l_k = \frac{l}{\sigma^{k-1}} < 1, \quad \lambda_k = \frac{\sigma}{l_k} = \frac{1}{l_{k+1}} > \frac{1}{l_k},$$

and with the remaining three parameters

$$\delta_3 \ge \delta_2 \ge \delta_1 = \max_{m=1,2}\{l^m\|\nabla\mathfrak{v}\|_m\} + \max_{\substack{m=0,\dots,3\\ k=1,\dots,3}}\{l^m\|a_k\|_m\} \tag{5-17}$$

as indicated below. We then finally set $\tilde{v} = v_4$ and $\tilde{w} = w_4$.

We start by checking that the assumptions of Proposition 5.1 are satisfied. Namely, we claim that $\delta_k, l_k \in (0, 1)$, together with

$$\|a_k\|_m \le \frac{\delta_k}{l_k^m} \quad \forall m = 0, \dots, 3 \qquad \text{and} \qquad \|\nabla v_k\|_m \le \frac{\delta_k}{l_k^m} \quad \forall m = 1, 2, \tag{5-18}$$

at each iteration step $k = 1, 2, 3$, if only the constant $\delta_0$ in (5-5) is appropriately small.

Indeed, $\delta_1 \le C\|\mathcal{D}\|_0^{1/2}$ in view of (5-11) and (5-15), so $\delta_1 < 1$ if only $\delta_0 \ll 1$. Further, by the definition (5-17) it follows that

$$\|a_k\|_m = \frac{1}{l^m} l^m \|a_k\|_m \le \frac{\delta_1}{l^m} \le \frac{\delta_k}{l_k^m},$$

so the first assertion in (5-18) holds. For the second assertion, we see directly that it holds when $k = 1$, as

$$\|\nabla v_1\|_m = \frac{1}{l^m} l^m \|\nabla \mathfrak{v}\|_m \le \frac{\delta_1}{l^m}.$$

On the other hand, using induction on $k$ and exploiting (5-3), we get

$$\begin{aligned}\|\nabla v_{k+1}\|_m &\le \|\nabla v_k\|_m + \|\nabla v_{k+1} - \nabla v_k\|_m \le \frac{\delta_k}{l_k^m} + C\delta_k \lambda_k^m \\ &\le \delta_k \left( \frac{1}{l_{k+1}^m} + \frac{C}{l_{k+1}^m} \right) = C\frac{\delta_k}{l_{k+1}^m} \le \frac{\delta_{k+1}}{l_{k+1}^m} \quad \forall m = 1, 2, \ \forall k = 1, 2.\end{aligned}$$

The proof of (5-18) is now complete for the choice $\delta_{k+1} = C\delta_k$, where $C > 1$ is, as always, an appropriately large universal constant. Consequently, $\delta_2, \delta_3 \le C\|\mathcal{D}\|_0^{1/2} < 1$ if only $\delta_0 \ll 1$.

(4) We now directly verify the concluding estimates of Proposition 5.2. We have, in view of the definition of $\mathfrak{D}'$ and (5-14),

$$\begin{aligned}\widetilde{\mathcal{D}} &= A - \mathfrak{A} + \mathfrak{D}' + \left(\tfrac{1}{2}\nabla v_1 \otimes \nabla v_1 + \operatorname{sym}\nabla w_1\right) - \left(\tfrac{1}{2}\nabla v_4 \otimes \nabla v_4 + \operatorname{sym}\nabla w_4\right) \\ &= A - \mathfrak{A} - \sum_{k=1}^{3}\Big(\left(\tfrac{1}{2}\nabla v_{k+1} \otimes \nabla v_{k+1} + \operatorname{sym}\nabla w_{k+1}\right) - \left(\tfrac{1}{2}\nabla v_k \otimes \nabla v_k + \operatorname{sym}\nabla w_k + a_k \xi_k \otimes \xi_k\right)\Big),\end{aligned}$$

and thus by (5-10), (5-2) and the definition of $l$, (5-7) follows:

$$\begin{aligned}\|\widetilde{\mathcal{D}}\|_0 &\le \|A - \mathfrak{A}\|_0 + C\sum_{k=1}^{3} \frac{\delta_k^2}{\lambda_k l_k} \le C\left( l^\beta \|A\|_{0,\beta} + \delta_3^2 \sum_{k=1}^{3} \frac{1}{\lambda_k l_k} \right) \\ &\le C\left( \frac{\|\mathcal{D}\|_0^{\beta/2}}{M^\beta}\|A\|_{0,\beta} + 3\frac{\delta_3^2}{\sigma} \right) \le C\left( \frac{\|\mathcal{D}\|_0^{\beta/2}}{M^\beta}\|A\|_{0,\beta} + \frac{1}{\sigma}\|\mathcal{D}\|_0 \right).\end{aligned}$$

We now check (5-8), using (5-10), (5-13) and (5-4):

$$\|\tilde{v} - v\|_1 \le \|\mathfrak{v} - v\|_1 + \sum_{k=1}^{3} \|v_{k+1} - v_k\|_1 \le C\|\mathcal{D}\|_0^{1/2} + C\sum_{k=1}^{3}\delta_k \le C\|\mathcal{D}\|_0^{1/2},$$

$$\begin{aligned}\|\tilde{w} - w\|_1 &\le \|\mathfrak{w} - w\|_1 + \|\mathfrak{w}' - \mathfrak{w}\|_1 + \sum_{k=1}^{3}\|w_{k+1} - w_k\|_1 \\ &\le C\left( \|\mathcal{D}\|_0^{1/2} + \|\mathcal{D}\|_0 + \sum_{k=1}^{3}\delta_k(1 + \|\nabla v_k\|_0) \right) \le C\|\mathcal{D}\|_0^{1/2}\left(1 + \sum_{k=1}^{3}\|\nabla v_k\|_0\right) \\ &\le C\|\mathcal{D}\|_0^{1/2}\left(1 + \|\nabla v\|_0 + \|\mathfrak{v} - v\|_1 + \sum_{k=1}^{2}\|v_{k+1} - v_k\|_1\right) \\ &\le C\|\mathcal{D}\|_0^{1/2}(1 + \|\nabla v\|_0 + \|\mathcal{D}\|_0^{1/2}) \le C\|\mathcal{D}\|_0^{1/2}(1 + \|\nabla v\|_0).\end{aligned} \tag{5-19}$$

Finally, the first bound in (5-9) follows by (5-11) and (5-3),

$$\|\tilde{v}\|_2 \le \|\mathfrak{v}\|_2 + \sum_{k=1}^{3} \|v_{k+1} - v_k\|_2 \le \frac{C}{l}\|\mathcal{D}\|_0^{1/2} + C\sum_{k=1}^{3} \delta_k \lambda_k$$
$$\le \frac{C}{l}\|\mathcal{D}\|_0^{1/2} + C\delta_3 \sum_{k=1}^{3} \frac{\sigma^k}{l} \le \frac{C}{l}\|\mathcal{D}\|_0^{1/2}(1+\sigma^3) \le CM\sigma^3,$$

while the second bound is obtained by

$$\|\tilde{w}\|_2 \le \|\mathfrak{w}\|_2 + \|\mathfrak{w}' - \mathfrak{w}\|_2 + \sum_{k=1}^{3} \|w_{k+1} - w_k\|_2 \le C\Big(M + \|\mathcal{D}\|_0 + \sum_{k=1}^{3} \delta_k \lambda_k (1 + \|\nabla v_k\|_0)\Big)$$
$$\le C\Big(M + \delta_3 \sum_{k=1}^{3} \frac{\sigma^3}{l}(1 + \|\nabla v_k\|_0)\Big) \le CM\Big(1 + \sigma^3 + \sigma^3 \sum_{k=1}^{3} \|\nabla v_k\|_0\Big)$$
$$\le CM\sigma^3\Big(1 + \sum_{k=1}^{3} \|\nabla v_k\|_0\Big) \le CM\sigma^3(1 + \|\nabla v\|_0)$$

in view of (5-12), (5-13) and reasoning as in (5-19). □

## 6. The $\mathcal{C}^{1,\alpha}$ approximations: a proof of Theorem 1.2

We are now in a position to state the final intermediary approximation result, parallel to [Conti et al. 2012, Theorem 1].

**Theorem 6.1.** *Assume that $\Omega \subset \mathbb{R}^2$ is an open, bounded domain. Given are functions $v \in \mathcal{C}^2(\bar{\Omega})$, $w \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)$ and $A \in \mathcal{C}^{0,\beta}(\bar{\Omega}, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ for some $\beta \in (0, 1)$, such that the deficit $\mathcal{D}$ below is appropriately small*:

$$\mathcal{D} = A - \big(\tfrac{1}{2}\nabla v \otimes \nabla v + \operatorname{sym} \nabla w\big), \quad 0 < \|\mathcal{D}\|_0 < \delta_0 \ll 1. \tag{6-1}$$

*Fix the exponent*

$$0 < \alpha < \min\big\{\tfrac{1}{7}, \tfrac{1}{2}\beta\big\}. \tag{6-2}$$

*Then, there exist $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ such that*

$$\tfrac{1}{2}\nabla \bar{v} \otimes \nabla \bar{v} + \operatorname{sym} \nabla \bar{w} = A, \tag{6-3}$$
$$\|\bar{v} - v\|_1 \le C\|\mathcal{D}\|_0^{1/2} \quad \text{and} \quad \|\bar{w} - w\|_1 \le C(1 + \|\nabla \bar{v}\|_0)\|\mathcal{D}\|_0^{1/2}, \tag{6-4}$$

*where $C > 0$ is a constant depending on $\alpha$ but independent of all other parameters.*

*Proof.* The exact solution to (6-3) will be obtained as the $\mathcal{C}^{1,\alpha}$ limit of sequences of successive approximations $\{v_k \in \mathcal{C}^2(\bar{\Omega}), w_k \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^2)\}_{k=1}^{\infty}$.

Part 1: induction on stages. We set $v_0 = v$ and $w_0 = w$. Given $v_k$ and $w_k$, define $v_{k+1}$ and $w_{k+1}$ by applying Proposition 5.2 with parameters $\sigma$ and $M_k$ that will be appropriately chosen below and that satisfy

$$M_k > \max\{\|v_k\|_2, \|w_k\|_2, 1\} \quad \text{and} \quad \sigma > 1. \tag{6-5}$$

Following our notational convention, we define the $k$-th deficit $\mathcal{D}_k = A - \big(\frac{1}{2}\nabla v_k \otimes \nabla v_k + \operatorname{sym}\nabla w_k\big)$. In view of Proposition 5.2, we get

$$\|\mathcal{D}_{k+1}\|_0 \le C\bigg(\frac{\|A\|_{0,\beta}}{M_k^\beta}\|\mathcal{D}_k\|_0^{\beta/2} + \frac{1}{\sigma}\|\mathcal{D}_k\|_0\bigg), \tag{6-6}$$

$$\|v_{k+1} - v_k\|_1 \le C\|\mathcal{D}_k\|_0^{1/2} \quad \text{and} \quad \|w_{k+1} - w_k\|_1 \le C(1 + \|\nabla v_n\|_0)\|\mathcal{D}_k\|_0^{1/2}, \tag{6-7}$$

$$\|v_{k+1}\|_2 \le CM_k\sigma^3 \quad \text{and} \quad \|w_{k+1}\|_2 \le C(1 + \|\nabla v_k\|_0)M_k\sigma^3, \tag{6-8}$$

provided that (5-5) holds for each $\mathcal{D}_k$. We shall now validate this requirement, with the parameters

$$M_k = \big(\mathfrak{C}(1 + \|\nabla v_0\|_0)\sigma^3\big)^k M_0. \tag{6-9}$$

In fact, we will inductively prove that one can have

$$\|\mathcal{D}_k\|_0 \le \frac{1}{\sigma^{sk}}\|\mathcal{D}\|_0 \quad \text{with any } 0 < s < \min\bigg\{1, \frac{6\beta}{2-\beta}\bigg\}. \tag{6-10}$$

Fix $s$ as indicated in (6-10). Clearly, (6-10) and (6-5) hold for $k = 0$. By (6-6) and the induction assumption we obtain the bound

$$\sigma^{s(k+1)}\frac{\|\mathcal{D}_{k+1}\|_0}{\|\mathcal{D}\|_0} \le \frac{C\|A\|_{0,\beta}\,\|\mathcal{D}\|_0^{\beta/2-1}\sigma^s}{M_0^\beta}\frac{1}{\mathfrak{C}^{k\beta}}\bigg(\frac{\sigma^{(1-\beta/2)(s-6\beta/(2-\beta))}}{(1 + \|\nabla v_0\|_0)^\beta}\bigg)^k + C\sigma^{s-1}. \tag{6-11}$$

We see that in view of the condition on $s$ in (6-10), both $\sigma^{s-1}$ and $\sigma^{(1-\beta/2)(s-6\beta/(2-\beta))}$ are smaller than 1. Further, it is possible to choose $\sigma > 1$ so that the second term in (6-11) is smaller than $\frac{1}{2}$ and so that the quotient term in parentheses above is also smaller than 1. Then, choose $M_0$ so that (6-5) holds for $k = 0$ together with

$$\frac{C\|A\|_{0,\beta}\,\|\mathcal{D}\|_0^{\beta/2-1}\sigma^s}{M_0^\beta} < \frac{1}{2}.$$

This results in the first term in (6-11) being smaller than $\frac{1}{2}$ if $\mathfrak{C} \ge 1$. Consequently, we get that $\sigma^{s(k+1)}\|\mathcal{D}_{k+1}\|_0/\|\mathcal{D}\|_0 \le 1$ as needed in (6-10).

Observe now that by (6-7) and by the established (6-10),

$$\begin{aligned}\forall k \ge 0, \quad \|\nabla v_k\|_0 &\le \|\nabla v_0\|_0 + \sum_{i=0}^{k-1}\|v_{i+1} - v_i\|_1 \le \|\nabla v_0\|_0 + C\sum_{i=0}^{k-1}\|\mathcal{D}_i\|_0^{1/2} \\ &\le \|\nabla v_0\|_0 + C\bigg(\sum_{i=0}^{\infty}\frac{1}{\sigma^{si/2}}\bigg)\|\mathcal{D}\|_0^{1/2} = \|\nabla v_0\|_0 + \frac{C}{1 - \sigma^{-s/2}}\|\mathcal{D}\|_0^{1/2} \\ &\le \|\nabla v_0\|_0 + C\|\mathcal{D}\|_0^{1/2}\end{aligned} \tag{6-12}$$

if only, say, $\sigma^s > 4$, which can be easily achieved through the choice of $\sigma$. Now, by (6-8) and (6-12),

$$\frac{\|v_{k+1}\|_2}{M_{k+1}} \le \frac{1}{\mathfrak{C}} \frac{C}{(1+\|\nabla v_0\|_0)},$$
$$\frac{\|w_{k+1}\|_2}{M_{k+1}} \le \frac{1}{\mathfrak{C}} \frac{C(1+\|\nabla v_k\|_0)}{(1+\|\nabla v_0\|_0)} \le \frac{1}{\mathfrak{C}} \frac{C(1+\|\nabla v_0\|_0+\|\mathcal{D}\|_0^{1/2})}{(1+\|\nabla v_0\|_0)}.$$

Hence, taking the constant $\mathfrak{C} \gg 1$ large enough, we see that both quantities above can be made smaller than 1, proving therefore the required (6-5).

Part 2: $\mathcal{C}^{1,\alpha}$ control of the approximating sequences $v_n$ and $w_n$. Let now $\alpha$ be an exponent as in (6-2). Choose $s$ satisfying (6-10) and

$$\alpha(6+s)-s < 0. \tag{6-13}$$

It is an easy calculation that $s$ satisfying (6-10) and (6-13) exists if and only if the exponent $\alpha$ is in the range (6-2). Indeed, (6-13) is equivalent to $\alpha < s/(6+s)$, while (6-10) is equivalent to

$$0 < \frac{s}{6+s} < \min\left\{\tfrac{1}{7}, \tfrac{1}{2}\beta\right\}.$$

We will prove that the sequences $\{v_k, w_k\}_{k=0}^{\infty}$ are Cauchy in $\mathcal{C}^{1,\alpha}(\bar{\Omega})$. Firstly, by (6-7), (6-12), (6-10),

$$\begin{aligned} \|v_{k+1}-v_k\|_1 &\le C\|\mathcal{D}_k\|_0^{1/2} \le \frac{C}{\sigma^{sk/2}}\|\mathcal{D}\|_0^{1/2}, \\ \|w_{k+1}-w_k\|_1 &\le C(1+\|\nabla v_k\|_0)\|\mathcal{D}_k\|_0^{1/2} \le \frac{C}{\sigma^{sk/2}}(1+\|\nabla v_0\|_0+\|\mathcal{D}\|_0^{1/2})\|\mathcal{D}\|_0^{1/2}, \end{aligned} \tag{6-14}$$

so we see right away that they are Cauchy in $\mathcal{C}^1(\bar{\Omega})$. On the other hand, by (6-8), (6-12), (6-10),

$$\|v_{k+1}-v_k\|_2+\|w_{k+1}-w_k\|_2 \le C(1+\|\nabla v_k\|_0)M_k\sigma^3 \le C(1+\|\nabla v_0\|_0+\|\mathcal{D}\|_0^{1/2})\big(\mathfrak{C}(1+\|\nabla v_0\|_0)\sigma^3\big)^k M_0,$$

so the sequences have the tendency to diverge in $\mathcal{C}^2(\bar{\Omega})$. Interpolating now the $\mathcal{C}^{1,\alpha}$ norm by [Adams and Fournier 2003],

$$\|f\|_{0,\alpha} \le \|f\|_1^{\alpha}\,\|f\|_0^{1-\alpha},$$

we obtain

$$\begin{aligned} \|\nabla(v_{k+1}-v_k)\|_{0,\alpha}+\|\nabla(w_{k+1}-w_k)\|_{0,\alpha} &\le C_0^{\alpha}(C_0\sigma^3)^{k\alpha}M_0^{\alpha}\cdot C_0^{1-\alpha}\frac{1}{\sigma^{sk(1-\alpha)/2}} \\ &= C_0M_0^{\alpha}(C_0^{\alpha})^h(\sigma^{\frac{1}{2}(\alpha(6+s)-s)})^k, \end{aligned} \tag{6-15}$$

where by $C_0$ we denoted an upper bound of all quantities involving $C$, $v_0$, $\mathcal{D}$. It is clear that choosing $\sigma$ sufficiently large (so that $C_0\sigma^{3-s/2} < 1$), the resulting bound (6-15) implies that $\{\nabla v_k, \nabla w_k\}_{k=0}^{\infty}$ are Cauchy in $\mathcal{C}^{0,\alpha}(\bar{\Omega})$, provided that (6-13) holds. We see that the choice of exponent range in (6-2) so that the above construction technique works, is optimal.

Part 3: Concluding, we see that $\{v_k, w_k\}_{k=0}^{\infty}$ converge to some $\bar{v} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ and $\bar{w} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}, \mathbb{R}^2)$. Since the defects in the approximating sequence obeys $\lim_{k\to\infty}\|\mathcal{D}_k\|_0 = 0$ by (6-10), we immediately get (6-3).

Additionally, by (6-14),

$$\|\bar v - v\|_1 \le \sum_{k=0}^{\infty} \|v_{k+1}-v_k\|_1 \le C\Big(\sum_{k=0}^{\infty}\frac{1}{\sigma^{sk/2}}\Big)\|\mathcal{D}\|_0^{1/2} = \frac{C}{1-\sigma^{-s/2}}\|\mathcal{D}\|_0^{1/2} \le C\|\mathcal{D}\|_0^{1/2},$$

$$\|\bar w - w\|_1 \le \sum_{k=0}^{\infty} \|w_{k+1}-w_k\|_1 \le C\Big(\sum_{k=0}^{\infty}\frac{1}{\sigma^{sk/2}}\Big)(1+\|\nabla v\|_0)\|\mathcal{D}\|_0^{1/2} \le C(1+\|\nabla v\|_0)\|\mathcal{D}\|_0^{1/2},$$

completing the proof of (6-4). □

We are now ready to give:

*Proof of Theorem 1.2.* Fix a sufficiently small $\varepsilon > 0$. We will construct $\bar v \in \mathcal{C}^{1,\alpha}(\bar\Omega)$ and $\bar w \in \mathcal{C}^{1,\alpha}(\Omega, \mathbb{R}^2)$ such that

$$A_0 = \tfrac12 \nabla \bar v \otimes \nabla \bar v + \operatorname{sym}\nabla \bar w \quad \text{in } \bar\Omega \tag{6-16}$$

and

$$\|\bar v - v_0\|_0 + \|\bar w - w_0\|_0 < \varepsilon. \tag{6-17}$$

In order to apply Theorem 6.1, we need to decrease the deficit $A_0 - \big(\tfrac12 \nabla v_0 \otimes \nabla v_0 + \operatorname{sym}\nabla w_0\big)$ so that it obeys (6-1). This will be done in three steps.

First, let $\tilde v_0 \in \mathcal{C}^\infty(\bar\Omega)$, $\tilde w_0 \in \mathcal{C}^\infty(\Omega, \mathbb{R}^2)$ and $\tilde A_0 \in \mathcal{C}^\infty(\bar\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ be such that

$$\begin{gathered}\|\tilde v_0 - v_0\|_1 + \|\tilde w_0 - w_0\|_1 + \|\tilde A_0 - A_0\|_0 < \varepsilon^2,\\ \exists \tilde c_0 > 0 \quad \text{such that} \quad A_0 - \big(\tfrac12 \nabla \tilde v_0 \otimes \nabla \tilde v_0 + \operatorname{sym}\nabla \tilde w_0\big) > \tilde c_0 \mathrm{Id}_2 \quad \text{in } \bar\Omega.\end{gathered} \tag{6-18}$$

Second, by Theorem 2.1 and Remark 3.3, there exist $v \in \mathcal{C}^1(\bar\Omega)$ and $w \in \mathcal{C}^1(\Omega, \mathbb{R}^2)$ such that

$$\begin{gathered}\tilde A_0 = \tfrac12 \nabla v \otimes \nabla v + \operatorname{sym}\nabla w \quad \text{in } \bar\Omega,\\ \|v - \tilde v_0\|_0 + \|w - \tilde w_0\|_0 < \varepsilon^2 \quad \text{and} \quad \|\nabla v - \nabla \tilde v_0\|_0 \le C.\end{gathered} \tag{6-19}$$

Third, let $\tilde v \in \mathcal{C}^2(\bar\Omega)$ and $\tilde w \in \mathcal{C}^2(\Omega, \mathbb{R}^2)$ be such that

$$\|v - \tilde v\|_1 + \|w - \tilde w\|_1 < \varepsilon^2. \tag{6-20}$$

By (6-19), (6-20) and (6-18), we get

$$\begin{aligned}\big\|A_0 - \big(\tfrac12 \nabla \tilde v \otimes \nabla \tilde v + \operatorname{sym}\nabla \tilde w\big)\big\|_0 &\\ \le \|A_0 - \tilde A_0\|_0 &+ \big\|\big(\tfrac12 \nabla \tilde v \otimes \nabla \tilde v + \operatorname{sym}\nabla \tilde w\big) - \big(\tfrac12 \nabla v \otimes \nabla v + \operatorname{sym}\nabla w\big)\big\|_0\\ \le \|A_0 - \tilde A_0\|_0 &+ (\|\nabla v\|_0 + \|\nabla \tilde v\|_0)\|\nabla v - \nabla \tilde v\|_0 + \|\nabla w - \nabla \tilde w\|_0\\ \le \varepsilon^2 &+ (2\|\nabla v_0\|_0 + 2\varepsilon^2 + C)\varepsilon^2 + \varepsilon^2 < \delta_0,\end{aligned} \tag{6-21}$$

as required in Theorem 6.1, if only $\varepsilon$ is small enough. We now apply Theorem 6.1 to $\tilde v$, $\tilde w$ and the original field $A_0$, and get $\bar v \in \mathcal{C}^{1,\alpha}(\bar\Omega)$ and $\bar w \in \mathcal{C}^{1,\alpha}(\Omega, \mathbb{R}^2)$ satisfying (6-16) and such that

$$\begin{aligned}\|\bar v - v_0\|_0 + \|\bar w - w_0\|_0 &\le C(1 + \|\nabla \tilde v\|_0)\big\|A_0 - \big(\tfrac12 \nabla \tilde v \otimes \nabla \tilde v + \operatorname{sym}\nabla \tilde w\big)\big\|_0 + 3\varepsilon^2\\ &\le C(1 + \varepsilon^2 + \|\nabla v_0\|_0)^2\varepsilon^2 + 3\varepsilon^2\end{aligned}$$

by (6-4), (6-21), (6-20), (6-19) and (6-18). Clearly (6-17) follows, if $\varepsilon$ is small enough. □

The following corollary is of independent interest:

**Corollary 6.2.** *Let $\Omega$, $f$, $p$, $\alpha$ be as in the statement of Theorem 1.1. Let $q \geq 2$. Then, for all $v_0 \in W^{1,q}(\Omega)$, there exists a sequence $v_n \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ weakly converging to $v_0$ in $W^{1,q}(\Omega)$, and such that $\mathcal{D}et\,\nabla^2 v_n = f$ in $\Omega$.*

*Proof.* Let $\bar{v}_n \in \mathcal{C}^1(\bar{\Omega})$ converge to $v_0$ in $W^{1,q}(\Omega)$. For every $\bar{v}_n$, consider the approximating sequence $\{v_{n,k} \in \mathcal{C}^{1,\alpha}(\bar{\Omega})\}_{k=1}^{\infty}$ as in Theorem 1.1, converging uniformly to $\bar{v}_n$. Define now $\{v_n\}$ to be an appropriate diagonal sequence, so that it converges to $v_0$ in $L^q(\Omega)$. We will check that $\{v_n\}$ is bounded in $W^{1,q}$.

The boundedness of $\|v_n\|_{L^q}$ is clear from the convergence statement. On the other hand, the proof of Theorem 1.2 gives, by (6-4), (6-18), (6-19), (6-20) and (6-21),

$$|\nabla v_n(x)| \leq |\nabla \bar{v}_n(x)| + 2\varepsilon^2 + C + C\delta_0^{1/2} \leq |\nabla \bar{v}_n(x)| + C \qquad \forall x \in \Omega.$$

Consequently, $\|\nabla v_n\|_{L^q} \leq \|\nabla \bar{v}_n\|_{L^q} + C \leq C$, which concludes the proof. ☐

## 7. Rigidity results for $\alpha > \frac{2}{3}$: a proof of Theorem 1.3

The crucial element in the proof of the rigidity Theorems 1.3 and 1.4 is the following result, which is the "small slope analogue" of [Conti et al. 2012, Proposition 6]:

**Proposition 7.1.** *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, simply connected domain. Assume that for some $\alpha \in \left(\frac{2}{3}, 1\right)$, the function $v \in \mathcal{C}^{1,\alpha}(\bar{\Omega})$ is a solution to*

$$\mathcal{D}et\,\nabla^2 v = f \quad \text{in } \bar{\Omega},$$

*where $f \in L^p(\Omega)$ and $p > 1$. Then the following degree formula holds true for every open subset $U$ compactly contained in $\Omega$ and every $g \in L^\infty(\mathbb{R}^2)$ with $\operatorname{supp} g \subset \mathbb{R}^2 \setminus \nabla v(\partial U)$:*

$$\int_U (g \circ \nabla v) f = \int_{\mathbb{R}^2} g(y) \deg(\nabla v, U, y)\, dy. \tag{7-1}$$

*Above, $\deg(\psi, U, y)$ denotes the Brouwer degree of a continuous function $\psi : \bar{U} \to \mathbb{R}^2$ at a point $y \in \mathbb{R}^2 \setminus \psi(\partial U)$.*

*Proof.* (1) Fix $U$ and $g$ as in the statement of the proposition. We refer to [Lloyd 1978] for the definition and properties of the Brouwer degree; recall first that $\deg(\nabla v, U, \cdot)$ is well defined on the open set $\mathbb{R}^2 \setminus \nabla v(\partial U)$. In fact, this function is constant on each connected component $\{U_i\}_{i=0}^{\infty}$ of $\mathbb{R}^2 \setminus \nabla v(\partial U)$ and it equals 0 on the only unbounded component $U_0 \subset \mathbb{R}^2 \setminus \nabla v(\bar{U})$. Thus, without loss of generality, we may assume that $g$ is compactly supported and that $\operatorname{supp} g \subset \bigcup_{k=1}^{\infty} U_k$. By compactness, there must be $\operatorname{supp} g \subset \bigcup_{k=1}^{N} U_k$ for some $N$, and consequently the integral in the right-hand side of (7-1) is well defined.

Let now $\left\{g_i \in \mathcal{C}_c^\infty\left(\bigcup_{k=1}^{N} U_k\right)\right\}_{i=1}^{\infty}$ be a sequence pointwise converging to $g$ and such that $\|g_i\|_0 \leq \|g\|_{L^\infty}$ for all $i$. It is sufficient to prove the formula (7-1) for each $g_i$ and pass to the limit by the dominated convergence theorem. To simplify the notation, we drop the index $i$, and so in what follows we assume that $g \in \mathcal{C}_c^\infty(\mathbb{R}^2 \setminus \nabla v(\partial U))$.

As in the proof of Theorem 1.1, let $A \in W^{2,p}(\Omega) \cap \mathcal{C}^{0,\beta}(\bar{\Omega})$ be such that $\operatorname{curl}\operatorname{curl} A = -f$. Here, we take $\beta = \min\{2 - \frac{2}{p}, \alpha\} \in (0, 1)$. Consequently, in view of the simple connectedness of $\Omega$, there exists $w \in \mathcal{C}^{1,\beta}(\bar{\Omega}, \mathbb{R}^2)$ such that

$$A = \tfrac{1}{2}\nabla v \otimes \nabla v + \operatorname{sym} \nabla w.$$

For a standard 2-dimensional mollifier $\varphi \in \mathcal{C}_c^\infty(B(0, 1))$ as in Lemma 4.3, define

$$\forall l \in (0, 1), \qquad v_l = v * \varphi_l, \qquad w_l = w * \varphi_l, \qquad A_l = A * \varphi_l,$$

and apply the degree formula (change of variable formula [Evans and Gariepy 1992; Ambrosio et al. 2000]) to the smooth functions $g$ and $\nabla v_l$, noting that for sufficiently small $l$, we have $g \in \mathcal{C}_c^\infty(\mathbb{R}^2 \setminus \nabla v_l(\partial U))$:

$$\int_U (g \circ \nabla v_l) \det \nabla^2 v_l = \int_{\mathbb{R}^2} g(y) \deg(\nabla v_l, U, y)\, dy. \tag{7-2}$$

We see that $\nabla v_l$ converge uniformly to $\nabla v$, so by [Kavian 1993, Proposition 2.1] we obtain that for $l$ sufficiently small, and for all $y \in \operatorname{supp} g$, we have $\deg(\nabla v, U, y) = \deg(\nabla v_l, U, y)$. Thus

$$\lim_{l\to 0} \int_{\mathbb{R}^2} g(y) \deg(\nabla v_l, U, y)\, dy = \int_{\mathbb{R}^2} g(y) \deg(\nabla v, U, y)\, dy.$$

Another proof of integrability of the Brouwer degree, in a more general context, can be found in [Olbermann 2015]. Now, to conclude the proof in view of (7-2), it suffices to show that

$$\lim_{l\to 0} \int_U (g \circ \nabla v_l) \det \nabla^2 v_l = \int_U (g \circ \nabla v) f. \tag{7-3}$$

(2) Following [Conti et al. 2012; Constantin et al. 1994] we use a commutator estimate to get (7-3). As $f = -\operatorname{curl}\operatorname{curl} A$, we have

$$\begin{aligned}\left|\int_U (g\circ\nabla v_l) \det \nabla^2 v_l - (g\circ\nabla v) f\right| &\le \left|\int_U (g\circ\nabla v_l)(\det \nabla^2 v_l + \operatorname{curl}\operatorname{curl} A_l)\right| \\ &\quad + \left|\int_U (g\circ\nabla v_l) \operatorname{curl}\operatorname{curl}(A_l - A)\right| + \left|\int_U ((g\circ\nabla v_l) - (g\circ\nabla v)) f\right|. \end{aligned} \tag{7-4}$$

The second term above is bounded by $C \int_U |\nabla^2 A_l - \nabla^2 A| \le C \|A_l - A\|_{W^{2,p}(\Omega)}$, hence it converges to 0. The third term also converges to 0 by the dominated convergence theorem, since $g \circ \nabla v_l$ converges to $g \circ \nabla v$. In order to deal with the first term in (7-4), observe that $\det \nabla^2 v_l = -\operatorname{curl}\operatorname{curl}\big(\frac{1}{2}\nabla v_l \otimes \nabla v_l + \operatorname{sym} \nabla w_l\big)$ and integrate by parts, in view of $g \circ \nabla v_l = 0$ on $\partial U$:

$$\begin{aligned}\left|\int_U (g \circ \nabla v_l)(\det \nabla^2 v_l + \operatorname{curl}\operatorname{curl} A_l)\right| &= \left|\int_U \big\langle \nabla^\perp (g \circ \nabla v_l), \operatorname{curl}\big(\tfrac{1}{2}\nabla v_l \otimes \nabla v_l + \operatorname{sym} \nabla w_l - A_l\big)\big\rangle\right| \\ &\le C \|\nabla g\|_0 \, \|\nabla^2 v_l\|_0 \, \big\|\nabla v_l \otimes \nabla v_l - (\nabla v \otimes \nabla v) * \varphi_l\big\|_1 \\ &\le C \frac{1}{l^{1-\alpha}} \|\nabla v\|_{0,\alpha} \cdot \frac{1}{l^{1-2\alpha}} \|\nabla v\|_{0,\alpha}^2 = C \frac{1}{l^{2-3\alpha}} \|\nabla v\|_{0,\alpha}^3, \end{aligned} \tag{7-5}$$

where we used Lemma 4.3. Clearly, for $\alpha > \frac{2}{3}$ the right-hand side in (7-5) converges to 0 as $l \to 0$. By (7-4), this implies (7-3) and concludes the proof. ☐

Below, we present all the details of the proof of Theorem 1.3. The proof of Theorem 1.4 will be postponed to [Lewicka and Pakzad ≥ 2017].

*Proof of Theorem 1.3.* (1) By Proposition 7.1 it follows that for all open sets $U \subset \bar{U} \subset \Omega$,

$$\deg(\nabla v, U, y) = 0 \quad \forall y \in \mathbb{R}^2 \setminus \nabla v(\partial U). \tag{7-6}$$

We would like to conclude [Pogorelov 1956; 1973] that the image set $\nabla v(U)$ is of measure 0. This will result in the developability of $v$, by the main statement of [Korobkov 2007]. However, we note that (Malý, personal communication, 2016) for each $\alpha \in (0, 1)$, there exists a map in $C^{0,\alpha}(\Omega, \mathbb{R}^2)$ whose local degree vanishes everywhere, but whose image is onto the unit square. This example can be constructed through a similar approach to that in [Malý and Martio 1995, Section 5]. Therefore, we will additionally exploit the gradient structure of $\nabla v$, using ideas of [Kirchheim 2001, Chapter 2], in combination with the commutator estimate technique of the proof of Proposition 7.1.

Let $v_l = v * \varphi_l$ be as in the proof of Proposition 7.1 and for every $\delta > 0$ define

$$u_{l,\delta}(x_1, x_2) = \nabla v_l(x_1, x_2) + \delta(-x_2, x_1), \quad u_\delta(x_1, x_2) = \nabla v(x_1, x_2) + \delta(-x_2, x_1).$$

Fix an open set $U$ with smooth boundary and compactly contained in $\Omega$. Let $g \in \mathcal{C}_c^\infty(\mathbb{R}^2 \setminus \nabla v(\partial U))$, and use the change of variable formula to $g$ and $u_{l,\delta}$:

$$\int_U (g \circ u_{l,\delta})(\det \nabla^2 v_l + \delta^2) = \int_{\mathbb{R}^2} g(y) \deg(u_{l,\delta}, U, y)\, dy, \tag{7-7}$$

where we noted that $\det \nabla u_{l,\delta} = \det \nabla^2 v_l + \delta^2$. The integral in the right-hand side of (7-7) is well defined for sufficiently small $l$ and $\delta$, because then $y \in \operatorname{supp} g$ implies $y \notin u_{l,\delta}(\partial U)$.

Passing to the limit, we immediately obtain

$$\lim_{l\to 0} \int_{\mathbb{R}^2} g(y) \deg(u_{l,\delta}, U, y)\, dy = \int_{\mathbb{R}^2} g(y) \deg(u_\delta, U, y)\, dy, \tag{7-8}$$

while to the left hand side of (7-7) we apply the estimate

$$\left| \int_U (g \circ u_{l,\delta})(\det \nabla^2 v_l + \delta^2) - (g \circ u_\delta)\delta^2 \right| \le \left| \int_U (g \circ u_{l,\delta}) \det \nabla^2 v_l \right| + \left| \int_U (g \circ u_{l,\delta} - g \circ u_\delta)\delta^2 \right|.$$

The second term above clearly converges to 0 as $l \to 0$, because $u_{l,\delta}$ converge to $u_\delta$. The first term also converges to 0 as $\alpha > \frac{2}{3}$, where we reason exactly as in (7-4) and (7-5), keeping in mind that $f = 0$. We hence conclude

$$\lim_{l\to 0} \int_U (g \circ u_{l,\delta})(\det \nabla^2 v_l + \delta^2) = \int_U (g \circ u_\delta)\delta^2.$$

In view of (7-8) and (7-7) this implies

$$\forall 0 < \delta \ll 1, \quad \int_U (g \circ u_\delta)\delta^2 = \int_{\mathbb{R}^2} g(y) \deg(u_\delta, U, y)\, dy.$$

Consequently,

$$\forall 0 < \delta \ll 1,\ \forall y \in u_\delta(U) \setminus u_\delta(\partial U), \quad \deg(u_\delta, U, y) \ge 1. \tag{7-9}$$

(2) We now claim that

$$\nabla v(U) \subset \nabla v(\partial U). \tag{7-10}$$

To prove (7-10) we argue by contradiction, assuming that for some $x_0 \in U$ there is $y_0 = \nabla v(x_0) \in \nabla v(U) \setminus \nabla v(\partial U)$. Note that for $\delta$ small enough, we have $y_0 \notin u_\delta(\partial U)$, because $u_\delta$ converges uniformly to $\nabla v$ as $\delta \to 0$. We distinguish two cases:

(i) There exist sequences $\{x_k \in U\}_{k=1}^{\infty}$ and $\delta_k \to 0^+$ as $k \to \infty$ such that $y_0 = u_{\delta_k}(x_k)$ for all $k$. In view of (7-9) we get $\deg(u_{\delta_k}, U, y_0) \geq 1$, contradicting (7-6).

(ii) For all $\delta$ small enough, $y_0 \notin u_\delta(\bar{U})$. In this case, we must have $\deg(u_\delta, U, y_0) = 0$. But on the other hand, there exists a ball $B(y_0, 2r) \subset \mathbb{R}^2 \setminus \nabla v(\partial U)$, so also $B(y_0, r) \subset \mathbb{R}^2 \setminus u_\delta(\partial U)$ for all small $\delta$. Consequently, continuity of the degree yields that $\deg(u_\delta, U, z) = 0$ for every $z \in B(y_0, r)$. In particular, $\deg(u_\delta, U, u_\delta(x_0)) = 0$, because $\lim_{\delta\to 0} u_\delta(x_0) = \nabla v(x_0) = y_0$. This finally contradicts (7-9), as $u_\delta(x_0) \in u_\delta(U) \setminus u_\delta(\partial U)$.

Our claim (7-10) is now established. Since the set $\nabla v(\partial U)$ is the image of a Hausdorff one-dimensional set $\partial U$ under a $\mathcal{C}^{0,\alpha}$, $\alpha > \frac{1}{2}$, deformation $\nabla v$, it has Lebesgue measure 0 (see [Conti et al. 2012, Lemma 4]). Thus $\nabla v(U)$ must have measure 0 for every smooth $U$ compactly contained in $\Omega$. The same then must be true for the entire set $\Omega$, i.e., $|\nabla v(\Omega)| = 0$, and we consequently obtain

$$\operatorname{Int}(\nabla v(\Omega)) = \varnothing. \tag{7-11}$$

(3) By [Korobkov 2009, Corollary 1.1.2], condition (7-11) implies that every point $y \in \Omega$ has a convex open neighbourhood $\Omega_y$ such that for every point $x \in \Omega_y$ there is a line $L_x$ passing through $x$ so that $\nabla v$ is constant on $L_x \cap \Omega_y$. The same result in the present dimensionality has been first established in [Korobkov 2007]; see also the footnote on p. 875 in [Korobkov 2009] for an explanation.

We now prove that $v$ is developable. Fix $x_0 \in \Omega$ and let $[y, z] \subset \bar{\Omega}$ be the maximal segment passing through $x_0$ on which $\nabla v = \nabla v(x_0)$ is constant. Assume that $[y, z]$ does not extend to the boundary $\partial\Omega$, i.e., $y \in \Omega$. We will prove that then $\nabla v$ must be constant in an open neighbourhood of $x_0$. In fact, we will show that

$$V = \operatorname{Int}\big((\nabla v)^{-1}(\nabla v(x_0))\big) \supset (y, z). \tag{7-12}$$

Let $(p, q) = L_y \cap \Omega_y$. By the maximality of $[y, z]$, the segment $(p, q)$ is not an extension of (is not parallel to) $[y, z]$. Also, $\nabla v = \nabla v(x_0)$ on $(p, q)$. Take any $y_1 \in (y, z) \cap \Omega_y$ and define the open triangle $T = \operatorname{Int}(\operatorname{span}\{p, q, y_1\})$. It is easy to notice that every line passing through any point $x \in T$ must intersect at least one of the segments $(p, q)$ or $(y, y_1)$. Since $T \subset \Omega_y$, it follows that $\nabla v(x) = \nabla v(x_0)$. Hence

$$(y, y_1) \subset T \subset V$$

and, in particular, the set $V$ in (7-12) is nonempty.

To prove (7-12) assume, by contradiction, that there exists $y_2 \in [y_1, z)$ so that

$$(y, y_2) \subset V \quad \text{but} \quad (y, y_3) \not\subset V \quad \forall y_3 \in (y_2, z). \tag{7-13}$$

Now, the intersection $\Omega_{y_2} \cap V$ contains an open arc $C$ crossing the segment $(y, y_2) \cap \Omega_{y_2}$. As above, we argue that every point in a sufficiently small open neighbourhood of the segment $I = (y, z) \cap \Omega_{y_2}$ must have the property that every line passing through it intersects $C$ or $I$, where $\nabla v = \nabla v(x_0)$. Consequently $I \subset V$, contradicting (7-13) and establishing (7-12). □

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# KINETIC FORMULATION OF VORTEX VECTOR FIELDS

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This article focuses on gradient vector fields of unit Euclidean norm in $\mathbb{R}^N$. The stream functions associated to such vector fields solve the eikonal equation and the prototype is given by the distance function to a closed set. We introduce a kinetic formulation that characterizes stream functions whose level sets are either spheres or hyperplanes in dimension $N \geq 3$. Our main result proves that the kinetic formulation is a selection principle for the vortex vector field whose stream function is the distance function to a point.

## 1. Introduction

In this article, we analyze the following type of vortex vector field:

$$u^\star : \mathbb{R}^N \to \mathbb{R}^N, \quad u^\star(x) = \frac{x}{|x|} \quad \text{for every } x \in \mathbb{R}^N \setminus \{0\}$$

in dimension $N \geq 2$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. This structure arises in many physical models such as micromagnetics, liquid crystals, superconductivity, elasticity. Clearly, $u^\star$ is smooth away from the origin: in fact, 0 is a topological singularity of degree 1 since the jacobian is $\det \nabla u^\star = V_N \delta_0$, where $\delta_0$ is the Dirac measure at the origin and $V_N$ is the volume of the unit ball in $\mathbb{R}^N$. Also, $u^\star$ is a curl-free unit-length vector field; i.e.,

$$|u^\star| = 1 \quad \text{and} \quad \nabla \times u^\star = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{1}$$

Moreover, there is a stream function $\psi^\star : \mathbb{R}^N \to \mathbb{R}$ associated to $u^\star$ by the equation

$$u^\star = \nabla \psi^\star;$$

indeed, one may consider $\psi^\star$ as the distance function at the origin, i.e., $\psi^\star(x) = |x|$ for $x \in \mathbb{R}^N$, and $\psi^\star$ represents the viscosity solution of the eikonal equation

$$|\nabla \psi^\star| = 1$$

under an appropriate boundary condition at infinity (e.g., $\lim_{|x|\to\infty}(\psi^\star(x) - |x|) = 0$).

Note that conversely, these properties characterize the vortex vector field: if $u : \mathbb{R}^N \to \mathbb{R}^N$ is a nonconstant vector field that is smooth away from the origin and satisfies (1) then $u = \pm u^\star$ in $\mathbb{R}^N$. Indeed, this classically follows by the method of characteristics: the flow associated to $u$ by

$$\partial_t X(t, x) = u(X(t, x)) \tag{2}$$

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with the initial condition $X(0,x)=x$ for $x\neq 0$ yields straight lines $\{X(t,x)\}_t$ given by $X(t,x)=x+tu(x)$ along which $u$ is constant, i.e., $u(X(t,x))=u(x)$. Since $u$ is nonconstant and two characteristics can intersect only at the origin (which is the prescribed point-singularity of $u$), every characteristic passes through the origin[1] and therefore, $u$ coincides with $u^\star$ or $-u^\star$. Caffarelli and Crandall [2010] proved this result under a weaker regularity hypothesis for the vector field $u=\nabla\psi$: if $\psi$ is assumed only pointwise differentiable away from a set $S$ of vanishing Hausdorff $\mathcal{H}^1$-measure (i.e., $\mathcal{H}^1(S)=0$) and $|\nabla\psi|=1$ in $\mathbb{R}^N\setminus S$, then $\psi=\pm\psi^\star$ (up to a translation and an additive constant). We also refer to [DiPerna and Lions 1989] for weaker regularity assumptions on $u$ in the framework of Sobolev spaces.

Our aim is to prove a kinetic characterization of the vortex vector field that does not assume any initial regularity on $u$. This kinetic formulation will characterize stream functions whose level sets are totally umbilical hypersurfaces in dimension $N\geq 3$, i.e., either pieces of spheres or hyperplanes. In order to introduce the kinetic formulation of the vortex vector field, we start by presenting the case of dimension $N=2$ and then we extend it to dimensions $N\geq 3$.

**1.1.** ***Kinetic formulation in dimension N = 2.*** Let $\Omega\subset\mathbb{R}^2$ be an open set and $u:\Omega\to\mathbb{R}^2$ be a Lebesgue-measurable vector field that satisfies

$$|u|=1 \text{ a.e. in } \Omega \quad\text{and}\quad \nabla\times u=0 \text{ distributionally in } \Omega. \tag{3}$$

The main feature of the kinetic formulation relies on the concept of weak characteristic for a nonsmooth vector field $u$. We start by noting that (2) has a proper meaning only if some notion of trace of $u$ can be defined on curves $\{X(t,x)\}_t$, which in general is a consequence of the regularity assumption on $u$ (see [DiPerna and Lions 1989]). To overcome this difficulty, the following notion of "weak characteristic" is introduced for measurable vector fields $u$ (see, e.g., [Lions, Perthame, and Tadmor 1994; Jabin and Perthame 2001]): for every direction $\xi\in\mathbb{S}^1$, one defines the function $\chi(\cdot,\xi):\Omega\to\{0,1\}$ by

$$\chi(x,\xi)=\begin{cases}1 & \text{for } u(x)\cdot\xi>0,\\ 0 & \text{for } u(x)\cdot\xi\leq 0.\end{cases} \tag{4}$$

In the case of a smooth vector field $u$ in a neighborhood of a point $x_0\in\Omega$, then $\chi(\cdot,\xi)$ mimics the characteristic of $u$ of normal direction $\xi=(\xi_1,\xi_2)$ (see Figure 1); formally, if $\xi^\perp=(-\xi_2,\xi_1)=\pm u(x_0)$, then either $\nabla\chi(\cdot,\xi)$ locally vanishes (if $u$ is constant in a neighborhood of $x_0$), or $\nabla\chi(\cdot,\xi)$ is a measure concentrated on the characteristic $\{X(t,x_0)\}_t$ given by (2) with constant measure density $\pm\xi$. In other words, we have the following "kinetic formulation" of the problem (see, e.g., [DeSimone, Müller, Kohn and Otto 2001; Jabin and Perthame 2001]):

**Proposition 1** (kinetic formulation in dimension $N=2$). *Let $\Omega\subset\mathbb{R}^2$ be an open set and $u:\Omega\to\mathbb{R}^2$ be a smooth vector field. If $u$ satisfies (3) then*

$$\xi^\perp\cdot\nabla_x\chi(\cdot,\xi)=0 \quad \textit{distributionally in } \Omega \textit{ for every } \xi\in\mathbb{S}^1. \tag{5}$$

[1]This argument is clear in dimension $N=2$; for dimensions $N\geq 3$, one needs an additional argument showing that two characteristics are coplanar, as we will see later in the proof of Theorem 8.

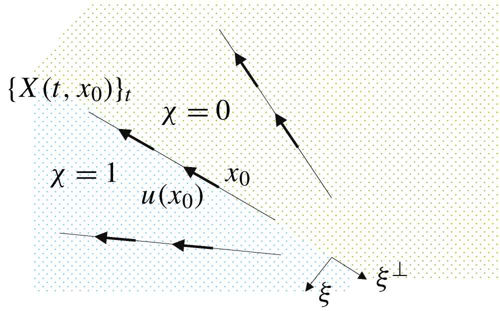


**Figure 1.** Characteristics of $u$.

We mention that the kinetic formulation (5) holds under the weaker Sobolev regularity $W^{1/p,p}$ for $p \in [1, 3]$ (see [Ignat 2011; 2012a; 2012b; De Lellis and Ignat 2015]). Note that the knowledge of $\chi(\cdot, \xi)$ in every direction $\xi \in \mathbb{S}^1$ determines completely a vector field $u$ with $|u| = 1$ due to the averaging formula

$$u(x) = \frac{1}{2} \int_{\mathbb{S}^1} \xi \chi(x, \xi) \, d\mathcal{H}^1(\xi) \qquad \text{for a.e. } x \in \Omega. \tag{6}$$

Thanks to (6), we deduce that the kinetic formulation (5) incorporates the fact that $\nabla \times u = 0$ (see Proposition 5 below). Therefore, the curl-free condition will be no longer mentioned in the following statements whenever (5) is assumed to hold true for unit-length vector fields $u$.

The main question is whether the kinetic formulation (5) characterizes the vortex vector field in $\mathbb{R}^2$. First of all, (5) induces a regularizing effect for Lebesgue-measurable unit-length vector fields $u$. Indeed, the classical "kinetic averaging lemma" (see, e.g., [Golse, Lions, Perthame, and Sentis 1988]) shows that a measurable vector field $u : \Omega \to \mathbb{S}^1$ satisfying (5) belongs to $H^{1/2}_{\text{loc}}(\Omega)$ due to the averaging formula (6).[2] Moreover, Jabin, Otto, and Perthame [2002] improved the regularizing effect by showing that $u$ is locally Lipschitz away from vortex point-singularities[3] and $u$ coincides with the vortex vector field around these singularities:

**Theorem 2** [Jabin, Otto, and Perthame 2002]. *Let $\Omega \subset \mathbb{R}^2$ be an open set and $u : \Omega \to \mathbb{R}^2$ be a Lebesgue-measurable vector field satisfying $|u| = 1$ a.e. in $\Omega$ together with the kinetic formulation (5). Then $u$ is locally Lipschitz continuous inside $\Omega$ except at a locally finite number of singular points. Moreover, every singular point $P$ of $u$ corresponds to a vortex singularity of topological degree 1 of $u$; i.e., there exists a sign $\gamma = \pm 1$ such that*

$$u(x) = \gamma u^\star(x - P) \qquad \textit{for every } x \neq P \textit{ in any convex neighborhood of } P \textit{ in } \Omega.$$

*In particular, if $\Omega = \mathbb{R}^2$ and $u$ is nonconstant, then $u$ coincides with $u^\star$ or $-u^\star$ (up to a translation).*

This result leads to the following interpretation of the kinetic formulation in dimension $N = 2$: equation (5) is a selection principle for the viscosity solutions of the eikonal equation $|\nabla \psi| = 1$ in the sense that the solutions $\psi$ are smooth (more precisely, they belong to the Sobolev space $W^{2,\infty}_{\text{loc}}$) away from

[2] For the improved regularizing effect for scalar conservation laws, see [Otto 2009; Golse and Perthame 2013].

[3] This regularity is optimal; see, e.g., Proposition 1 in [Ignat 2012b].

point-singularities. Clearly, these solutions are induced by the viscosity solutions of the eikonal equation under some appropriate boundary condition. Conversely, in the spirit of [Caffarelli and Crandall 2010], it was shown by Ignat [2012b] and De Lellis and Ignat [2015] that for any vector field $u$ satisfying (3) together with an initial Sobolev regularity $W^{1/p,p}$, $p \in [1, 3]$ (i.e., excluding jump line-singularities), the kinetic formulation (5) holds true and therefore, one obtains the regularizing effect in Theorem 2.

**Remark 3.** The result of Jabin, Otto, and Perthame [2002] was motivated by the study of zero-energy states in a line-energy Ginzburg–Landau model in dimension 2. More precisely, one considers the energy functional $E_\varepsilon : H^1(\Omega, \mathbb{R}^2) \to \mathbb{R}_+$ defined for $\varepsilon > 0$ as

$$E_\varepsilon(u_\varepsilon) = \varepsilon \int_\Omega |\nabla u_\varepsilon|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega (1 - |u_\varepsilon|^2)^2 \, dx + \frac{1}{\varepsilon} \|\nabla \times u_\varepsilon\|^2_{H^{-1}(\Omega)}, \quad u_\varepsilon \in H^1(\Omega, \mathbb{R}^2), \tag{7}$$

where $\Omega$ is a domain in $\mathbb{R}^2$ and $H^{-1}(\Omega)$ is the dual of the Sobolev space $H^1_0(\Omega)$. (We refer to [Ambrosio, De Lellis, and Mantegazza 1999; Aviles and Giga 1999; DeSimone, Müller, Kohn and Otto 2001; Jabin, Otto, and Perthame 2002; Jabin and Perthame 2001; Jin and Kohn 2000; Rivière and Serfaty 2001] for the analysis of this model.) A vector field $u : \Omega \to \mathbb{R}^2$ is called zero-energy state if there exists a family $\{u_\varepsilon \in H^1(\Omega, \mathbb{R}^2)\}_{\varepsilon \to 0}$ satisfying

$$u_\varepsilon \to u \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad E_\varepsilon(u_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

Obviously, a zero-energy state $u$ satisfies (3). The result of Jabin, Otto, and Perthame [2002] shows that every zero-energy state $u$ satisfies (5) and therefore, $u$ shares the structure stated in Theorem 2.

**1.2. *Kinetic formulation in dimension $N \geq 3$.*** Our main interest consists in defining a kinetic formulation for the vortex vector field in dimension $N \geq 3$. Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}^N$ be a Lebesgue-measurable vector field. For every direction $\xi \in \mathbb{S}^{N-1}$, we consider the characteristic function $\chi(\cdot, \xi)$ defined at (4) and we denote the orthogonal hyperplane to $\xi$ by

$$\xi^\perp := \{v \in \mathbb{R}^N : v \cdot \xi = 0\}.$$

**Definition 4** (kinetic formulation). We say that a measurable vector field $u$ satisfies the kinetic formulation if the following equation holds true:

$$v \cdot \nabla_x \chi(\cdot, \xi) = 0 \quad \text{distributionally in } \Omega \text{ for every } \xi \in \mathbb{S}^{N-1} \text{ and } v \in \xi^\perp. \tag{8}$$

Roughly speaking, (8) means that $\nabla_x \chi(\cdot, \xi)$ is a distribution pointing in direction $\pm\xi$. Note that the kinetic formulation (8) only carries out the information of the direction of the vector field $u$ (i.e., it gives no information about the Euclidean norm of $u$). Imposing the unit-length constraint, $u$ will satisfy a similar averaging formula (6) which justifies that the curl-free constraint $\nabla \times u = 0$ is incorporated in the kinetic formulation (8).

**Proposition 5.** *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}^N$ be Lebesgue measurable with $|u| = 1$ a.e. in $\Omega$. Then*

$$u(x) = \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \xi \chi(x, \xi) \, d\mathcal{H}^{N-1}(\xi) \quad \textit{for a.e. } x \in \Omega, \tag{9}$$

*where $V_{N-1}$ is the volume of the unit ball in $\mathbb{R}^{N-1}$. Moreover, if $u$ satisfies the kinetic formulation (8) then $\nabla \times u = 0$ distributionally in $\Omega$.*

**Remark 6.** We highlight that Proposition 1 is *false* in dimension $N \geq 3$; i.e., there are smooth curl-free vector fields with values into the unit sphere $\mathbb{S}^{N-1}$ that do not satisfy the kinetic formulation (8). For example, in dimension $N = 3$, considering the vortex-line vector field

$$u_0(x) = \frac{(x_1, x_2, 0)}{\sqrt{x_1^2 + x_2^2}} \quad \text{in } \Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 1\},$$

then $u_0$ is smooth in $\Omega$ and satisfies (3). However, (8) fails. Indeed, let $\xi = \frac{1}{\sqrt{2}}(1, 0, 1)$. Then $u_0(x) \cdot \xi = 0$ for $x \in \Omega$ is equivalent to $x_1 = 0$ and therefore,

$$\nabla_x \chi(\cdot, \xi) = e_1 \mathcal{H}^2 \llcorner \{x \in \Omega : x_1 = 0\},$$

where $e_1 = (1, 0, 0)$. Now, taking $v = \frac{1}{\sqrt{2}}(-1, 0, 1)$, we have $v \cdot \xi = 0$ (i.e., $v \in \xi^\perp$) and $v \cdot \nabla_x \chi(\cdot, \xi) \neq 0$ in $\mathcal{D}'(\Omega)$.

As Remark 6 has already revealed, the kinetic equation (8) in dimension $N \geq 3$ plays a different role than in dimension $N = 2$ because the gradient $\nabla \chi(\cdot, \xi)$ is expected to concentrate on hypersurfaces (not on the line characteristics of $u$). In fact, the geometric interpretation of (8) can be regarded in terms of the stream function $\psi$ of a nonconstant vector field $u = \nabla \psi$: the level sets of $\psi$ are expected to be pieces of spheres of codimension 1 where the characteristics of $u$ represent the normal directions to these spheres.

**Theorem 7.** *Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be an open set and $\psi : \Omega \to \mathbb{R}$ be a smooth stream function such that $u = \nabla \psi$ satisfies the kinetic formulation (8). Assume $|u|$ never vanishes on a level set $\{x \in \Omega : \psi(x) = \alpha\}$ for some $\alpha \in \mathbb{R}$ and let $\mathcal{S}$ be a connected component of $\{\psi = \alpha\}$. Then $\mathcal{S}$ is locally a totally umbilical hypersurface, that is, either a piece of an $(N-1)$-sphere or a piece of a hyperplane.*

Note that Theorem 7 fails in dimension $N = 2$: a level set of a smooth stream function $\psi$ of $u = \nabla \psi$ satisfying (3) (and therefore, $u$ satisfies the kinetic formulation (5) by Proposition 1) does not have, in general, constant curvature.[4]

## 2. Main results

Our main result shows that the kinetic formulation (8) is a characterization of the vortex vector field $u^\star$ in dimension $N \geq 3$.

**Theorem 8.** *Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be a connected open set and $u : \Omega \to \mathbb{R}^N$ be a nonconstant Lebesgue-measurable vector field satisfying $|u| = 1$ a.e. in $\Omega$ together with the kinetic equation (8). Then $u$ coincides with the vortex vector field $u^\star$ or $-u^\star$ up to a translation.*

Note that in dimension $N = 2$, this result is true for the domain $\Omega = \mathbb{R}^2$, but it is in general false for other domains $\Omega$ where there exist nonconstant smooth vector fields $u$ in $\Omega$ different than vortex

[4] If $\Gamma \subset \mathbb{R}^2$ is a smooth curve of nonconstant curvature, then one takes $\psi$ to be the distance function to $\Gamma$ in a small neighborhood $\Omega$ of $\Gamma$ (with the convention that $\Gamma$ is withdrawn from that neighborhood, i.e., $\Gamma \cap \Omega = \varnothing$, so that $\psi$ is smooth in $\Omega$).

vector fields that satisfy (3) and thus, (5) (by Proposition 1). The main difference in dimension $N \geq 3$ is the following: if $u$ is a smooth vector field with (3) that is neither constant nor a vortex vector field, then the kinetic formulation (8) doesn't hold for $u$ (see Remark 6). Hence, in dimension $N \geq 3$, the zero-energy states of $E_\varepsilon$ defined in (7) do not satisfy in general the kinetic equation (8). Therefore, the kinetic formulation (8) is more rigid in dimension $N \geq 3$ since it selects only the vortex vector fields, as they correspond to smooth solutions of the eikonal equation with level sets of constant sectional curvature (by Theorem 7).

Let us explain the strategy of the proof of Theorem 8. The key point relies on a relation of order of the level sets of the stream function associated to $u$: for every two Lebesgue points $x, y \in \Omega$ of $u$ such that the segment $[x, y]$ lies in $\Omega$ and for every direction $\xi \in \mathbb{S}^{N-1}$ orthogonal to $x - y$, one has

$$u(x) \cdot \xi > 0 \implies u(y) \cdot \xi \geq 0.$$

The next step consists in defining the trace of $u$ on each segment $\Sigma \subset \Omega$; more precisely, similar to the procedure of [Jabin, Otto, and Perthame 2002], there exists a trace $\tilde{u} \in L^\infty(\Sigma, \mathbb{S}^{N-1})$ of $u$ such that $u(P) = \tilde{u}(P)$ for each Lebesgue point $P \in \Sigma$ of $u$. Moreover, if the trace $\tilde{u}$ of $u$ is collinear with the segment $\Sigma$ at some Lebesgue point, then $\tilde{u}$ is $\mathcal{H}^1$-almost everywhere collinear with $\Sigma$ (which coincides with the classical principle of characteristics for smooth vector fields $u$). The final step consists in proving that every two characteristics are coplanar. Then one concludes by the following geometrical fact specific to dimension $N \geq 3$:

**Proposition 9.** *Let $N \geq 3$ and $\mathcal{D}$ be a set of lines in $\mathbb{R}^N$ such that every two lines of $\mathcal{D}$ are coplanar, but $\mathcal{D}$ is not planar (i.e., there is no 2-dimensional plane containing $\mathcal{D}$). Then either all lines of $\mathcal{D}$ are collinear, or all lines of $\mathcal{D}$ pass through a same point (that is a vortex point).*

In view of Theorem 8, it is natural to ask if one can characterize other types of unit-length curl-free vector fields $u$ by weakening the kinetic formulation (8), in particular, vector fields having a vortex-line singularity. In dimension $N \geq 3$, the prototype of a vortex-line vector field is given by

$$u_0(x', x_N) = \nabla |x'|,$$

where $x = (x', x_N)$ and $x' = (x_1, \ldots, x_{N-1})$; clearly, $u_0$ is smooth away from the vortex-line $\{x \in \mathbb{R}^N : x' = 0\}$ where (3) holds true. Defining

$$\mathcal{E} := \{\xi \in \mathbb{S}^{N-1} : \xi_N = 0\} = \mathbb{S}^{N-2} \times \{0\},$$

using the notation (4), we have that $u_0$ satisfies the following kinetic formulation in $\Omega = \mathbb{R}^N$:

$$\forall \xi \in \mathcal{E},\ \forall v \in \xi^\perp, \quad v \cdot \nabla_x \chi(\cdot, \xi) = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{10}$$

Note that (10) is a weakened form of (8): the quantity $v \cdot \nabla_x \chi(\cdot, \xi)$ vanishes for directions $\xi \in \mathcal{E}$ (and $v \in \xi^\perp$) and fails to vanish for $\mathcal{H}^{N-1}$-a.e. direction $\xi \in \mathbb{S}^{N-1}$. As opposed to (8) (in view of (9)), the kinetic formulation (10) does not force a unit-length vector field $u$ to be curl-free; it only implies that

$$\nabla' \times \frac{u'}{|u'|} = 0 \quad \text{in } \{|u'| \neq 0\} = \{u \neq \pm e_N\},$$

where $e_N = (0, \dots, 0, 1)$, $u' = (u_1, \dots, u_{N-1})$ and $\nabla' = (\partial_1, \dots, \partial_{N-1})$. Since we are looking for a characterization of vortex-line vector fields (that are in particular curl-free), we will impose that

$$\partial_k u_N = \partial_N u_k \quad \text{in } \Omega \text{ for } k = 1, \dots, N-1. \tag{11}$$

We will prove the following result:

**Theorem 10.** *Let $N \geq 4$, $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}^N$ be a Lebesgue-measurable vector field satisfying $|u| = 1$ a.e. on $\Omega$ together with (10) and (11). Then in every ball included in $\{x \in \Omega : u(x) \neq \pm e_N\}$, there exists a stream function $\psi = \psi(\alpha, \beta)$ solving the eikonal equation in dimension 2 such that*

$$u(x) = \nabla_x[\psi(\alpha, \beta)],$$

*where*

(1) *either $\alpha = |x' - P'|$ and $\beta = x_N$ for some point $P' \in \mathbb{R}^{N-1}$;*

(2) *or $\alpha = w' \cdot x'$ and $\beta = x_N$ for some vector $w' \in \mathbb{S}^{N-2}$.*

Therefore, the weakened kinetic formulation (10), together with (11), is not enough to select vortex-line vector fields which correspond to the stream function $\psi(\alpha, \beta) = \pm\alpha$ in case (1) of Theorem 10. Similar results to Theorem 10 hold for similar kinetic formulations corresponding to vector fields having vortex-sheet singularities of dimension $k$ in $\mathbb{R}^N$ with $N \geq k + 3$.

The outline of this paper is as follows: in Section 3, we characterize the level sets of smooth stream functions associated to vector fields that satisfy the kinetic formulation (8). In particular, we prove Proposition 1 and Theorem 7. Section 4 is devoted to proving fine properties of Lebesgue points of $u$ needed in Section 5, where the notion of the trace on lines for a vector field $u$ satisfying (8) is defined. Section 6 is the core of this paper: using this notion of trace and the geometric arguments of Proposition 9, we prove our main result in Theorem 8. Section 7 deals with the study of the weakened kinetic formulation (10).

## 3. Level sets of the stream function

This section is devoted to the study of the level sets of smooth stream functions $\psi$ associated to vector fields $u = \nabla\psi$ satisfying (8). We start by proving that $|\nabla\psi|$ is locally constant on each level set of $\psi$.

**Lemma 11.** *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open set and $\psi : \Omega \to \mathbb{R}$ be a smooth stream function such that $u = \nabla\psi$ satisfies the kinetic formulation (8). Assume $|u|$ never vanishes on a level set $\{x \in \Omega : \psi(x) = \alpha\}$ for some $\alpha \in \mathbb{R}$ and let $\mathcal{S}$ be a connected component of $\{\psi = \alpha\}$. Then $|u|$ is constant on $\mathcal{S}$. Moreover, there exists a neighborhood $\omega$ of $\mathcal{S}$, a smooth solution $\tilde{\psi} : \omega \to \mathbb{R}$ of the eikonal equation and a diffeomorphism $t \mapsto F(t)$ such that $\psi = F(\tilde{\psi})$ in $\omega$ (in particular, $\nabla\tilde{\psi}$ satisfies (8)).*

*Proof.* Since $|u| \neq 0$ on $\mathcal{S}$ and $u$ is smooth in $\Omega$, we can define

$$v = \frac{u}{|u|} \quad \text{in a neighborhood of } \mathcal{S}.$$

For simplicity of notation, we suppose that $\Omega$ is this neighborhood, i.e., $|u| \neq 0$ in $\Omega$. Then $v$ satisfies (8) because $u$ satisfies it, too; since $v$ is smooth in $\Omega$, Proposition 5 implies $\nabla \times v = 0$ in $\Omega$. (The proof of Proposition 5 is independent of Lemma 11; we will admit it here and prove it later in Section 4.) As a consequence, in any simply connected domain $\omega \subset \Omega$, the Poincaré lemma yields the existence of a smooth function $\tilde{\psi}$ such that $v = u/|u| = \nabla\tilde{\psi}$ in $\omega$, i.e.,

$$\nabla\psi = u = |u|v = |u|\nabla\tilde{\psi} \quad \text{in } \omega.$$

Therefore, $\psi$ and $\tilde{\psi}$ have the same level sets in $\omega$. Without loss of generality, we may assume that $\tilde{\psi} = 0$ on $\omega \cap \mathcal{S}$. Now, for every $P' \in \omega \cap \mathcal{S}$, we consider the flow associated to $v$,

$$\begin{cases} \dot{X}(P',t) = \nabla\tilde{\psi}(X(P',t)), \\ X(P',0) = P'. \end{cases} \tag{12}$$

Call $I_{P'}$ the maximal interval where the solution $X(P',\cdot)$ exists. Obviously, the flow is unique and smooth, satisfying

$$\ddot{X}(P',t) = \nabla^2\tilde{\psi}(X)\cdot\dot{X} = \nabla^2\tilde{\psi}(X)\cdot\nabla\tilde{\psi}(X) = 0 \quad \text{in } I_{P'}$$

because $\nabla^2\tilde{\psi}$ is a symmetric matrix and $|\nabla\tilde{\psi}| = 1$ in $\omega$. Consequently, $\dot{X}(P',\cdot)$ is constant in $I_{P'}$ so that

$$\nabla\tilde{\psi}(X(P',t)) = \nabla\tilde{\psi}(P'), \qquad \frac{d}{dt}[\tilde{\psi}(X(P',t))] = 1, \qquad X(P',t) = P' + t\nabla\tilde{\psi}(P').$$

Therefore, since $\tilde{\psi} = 0$ on $\omega \cap \mathcal{S}$, we have

$$\tilde{\psi}(X(P',t)) = t \quad \text{for all } P' \in \omega \cap \mathcal{S} \text{ and } t \in I_{P'}.$$

Identifying the level sets of $\tilde{\psi}$ (and of $\psi$, too) using the flow, i.e., $\{\tilde{\psi} = t\} = \{X(P',t) : P' \in \omega \cap \mathcal{S}\}$, we can define

$$F(t) := \psi(X(P',t)) \quad \text{for} \quad P' \in \omega \cap \mathcal{S},\ t \in I_{P'}.$$

The function $F$ is a diffeomorphism: $F$ is smooth (because $\psi$ and $X$ are smooth) and we have

$$\frac{d}{dt}F(t) = \nabla\psi(X(P',t))\cdot\dot{X}(P',t) \overset{(12)}{=} \nabla\psi(X(P',t))\cdot\frac{\nabla\psi}{|\nabla\psi|}(X(P',t)) = |u|(X(P',t)) \neq 0.$$

In particular, $|u|$ is constant on $\{\tilde{\psi} = 0\} = \{\psi = F(0)\} = \omega \cap \mathcal{S}$. Since $\omega$ was arbitrarily chosen, we deduce that $|u|$ is locally constant on $\mathcal{S}$; because $\mathcal{S}$ is connected, it follows that $|u|$ is constant on $\mathcal{S}$. Since the flow $\{X(P',t) : P' \in \mathcal{S}, t \in I_{P'}\}$ covers a neighborhood of $\mathcal{S}$, the last statement of the lemma follows. □

**3.1. *The case of dimension $N = 2$.*** In the special case of dimension $N = 2$, we start by proving that every smooth curl-free vector field of unit length satisfies the kinetic formulation (5). This result can be found already in [DeSimone, Müller, Kohn and Otto 2001; Jabin and Perthame 2001]. For completeness, we will present two easy and self-contained proofs. The first one is based on the geometry of the flow (2) (as heuristically described in Section 1), while the second proof is based on the concept of entropy introduced in [DeSimone, Müller, Kohn and Otto 2001].

*Proof of Proposition 1: first method.* We can assume that $\xi = e_1$ and $\xi^\perp = e_2$ (otherwise, one considers a rotation $R \in SO(2)$ such that $e_1 = R\xi$ and $\tilde{u}(x) := Ru(R^{-1}x)$ in a neighborhood of a point $x \in \Omega$).

Naturally, $\Omega$ can be written as a countable union of squares whose edges are parallel with $e_1$ and $e_2$. Therefore, using a partition of unity, it is enough to prove the statement for $\Omega = (-1, 1)^2$:

$$\forall \varphi \in C_c^\infty(\Omega), \quad 0 = \int_\Omega \varphi\, \xi^\perp \cdot \nabla_x \chi(x, \xi)\, dx \overset{\xi = e_1}{=} \int_\Omega \varphi\, \partial_2 \chi(x, e_1)\, dx = -\int_{\Omega \cap \{u_1 > 0\}} \partial_2 \varphi\, dx.$$

For that, we consider the flow (2) and by the proof of Lemma 11, we have, for every $x \in \Omega$, that $\{X(t, x)\}_t$ is a straight line given by $X(t, x) = x + tu(x)$ and $u(X(t, x)) = u(x)$ for all $t$. Since $u$ is smooth, there is no crossing between two characteristics in $\Omega$. We claim that

$$\Omega \cap \{u_1 > 0\} = \bigsqcup_{k \in K} A_k,$$

where $\{A_k\}_{k \in K}$ is a (at most) countable set of pairwise disjoint rectangles of type $(a_k, b_k) \times (-1, 1) \subset \Omega = (-1, 1)^2$. Note first that $\Omega \cap \{u_1 = 0\}$ is the intersection of $\Omega$ by vertical lines. Indeed, if $u_1(x) = 0$, then $u(x) \parallel e_2$. By the characteristic method, for all $t$, we have $u_1(x + tu(x)) = 0$ and $u_1$ vanishes on the vertical line passing through $x$. Now $\{x_1 \in (-1, 1) : u_1(x_1, 0) = 0\}^c$ is an open set in $(-1, 1)$ and therefore, we can write

$$\big\{x_1 \in (-1, 1) : u_1(x_1, 0) = 0\big\}^c = \bigsqcup_{k \in \widetilde{K}} (a_k, b_k),$$

where $\widetilde{K}$ is at most countable. For $k \in \widetilde{K}$, we define $A_k := (a_k, b_k) \times (-1, 1)$. By continuity, $u_1$ is either positive or negative on $A_k$. Defining $K := \{k \in \widetilde{K} : u_1 > 0 \text{ on } A_k\}$, the claim is proved. Now, for $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega \cap \{u_1 > 0\}} \partial_2 \varphi = \sum_k \int_{A_k} \partial_2 \varphi = \sum_k \int_{a_k}^{b_k} \int_{-1}^{1} \partial_2 \varphi = 0,$$

because $\partial_2 \varphi$ can be seen as a signed Radon measure for $\varphi \in C_c^\infty(\Omega)$ and the proposition is proved. □

*Proof of Proposition 1: second method.* The following proof links the kinetic formulation (5) with the theory of entropy solutions for scalar conservation laws (see, e.g., [DeSimone, Müller, Kohn and Otto 2001]). Indeed, if $u$ is a smooth vector field satisfying (3), then formally, $u_1 = -h(u_2) := \pm\sqrt{1 - u_2^2}$ so that $\nabla \times u = 0$ can be rewritten as

$$\partial_1 u_2 + \partial_2 [h(u_2)] = 0; \tag{13}$$

thus, $u_2$ can be formally interpreted as a solution of the above scalar conservation law in the variables (time, space) $= (x_1, x_2)$. Based on the concept of entropy solution of (13) introduced via the pairs (entropy, entropy-flux), the following applications (called "elementary entropies") were used in [DeSimone, Müller, Kohn and Otto 2001]. More precisely, for every $\xi \in \mathbb{S}^1$, the map $\Phi^\xi : \mathbb{S}^1 \to \mathbb{R}^2$ is defined as

$$\text{for } z \in \mathbb{S}^1, \quad \Phi^\xi(z) = \begin{cases} \xi^\perp & \text{for } z \cdot \xi > 0, \\ 0 & \text{for } z \cdot \xi \le 0. \end{cases}$$

Then the kinetic formulation (5) can be written as

$$\nabla \cdot \big[\Phi^\xi(u)\big] = 0 \quad \text{distributionally in } \Omega. \tag{14}$$

In order to prove (14), we will approximate $\Phi^\xi$ by a sequence of smooth maps $\{\Phi_k : \mathbb{S}^1 \to \mathbb{R}^2\}$ such that $\{\Phi_k\}$ is uniformly bounded, $\lim_k \Phi_k(z) = \Phi^\xi(z)$ for every $z \in \mathbb{S}^1$ and $\Phi_k$ satisfies (14) for every $k$. Following the ideas in [DeSimone, Müller, Kohn and Otto 2001] (see also [Ignat and Merlet 2012]), this smoothing result comes from the following observation: there exists a (unique) $2\pi$-periodic piecewise $C^1$ function $\varphi : \mathbb{R} \to \mathbb{R}$ associated to $\Phi^\xi$ via the equation

$$\Phi^\xi(z) = -\varphi'(\theta)z + \varphi(\theta)z^\perp \quad \text{for every } z = e^{i\theta} \in \mathbb{S}^1. \tag{15}$$

In fact, $\varphi$ is given by

$$\varphi(\theta) = \Phi^\xi(z)\cdot z^\perp = \xi \cdot z \mathbb{1}_{\{z\cdot\xi>0\}} = \cos(\theta-\theta_0)\mathbb{1}_{\{\theta-\theta_0\in(-\pi/2,\pi/2)\}} \quad \text{for } z = e^{i\theta},\ \theta \in (-\pi+\theta_0, \pi+\theta_0),$$

where $\xi = e^{i\theta_0} \in \mathbb{S}^1$ with $\theta_0 \in (-\pi, \pi]$. In (15), the distributional derivative $\varphi'$ is given by

$$\varphi'(\theta) = -\sin(\theta-\theta_0)\mathbb{1}_{\{\theta-\theta_0\in(-\pi/2,\pi/2)\}} \quad \text{for } \theta \in (-\pi+\theta_0, \pi+\theta_0).$$

Now, one regularizes $\varphi$ by $2\pi$-periodic functions $\varphi_k \in C^\infty(\mathbb{R})$ that are uniformly bounded in $W^{1,\infty}(\mathbb{R})$ with $\lim_k \varphi_k(\theta) = \varphi(\theta)$ and $\lim_k \varphi_k'(\theta) = \varphi'(\theta)$ for every $\theta \in \mathbb{R}$. Then we define $\Phi_k$ as in (15) for the functions $\varphi_k$:

$$\Phi_k(z) = -\varphi_k'(\theta)z + \varphi_k(\theta)z^\perp \quad \text{for } z = e^{i\theta} \in \mathbb{S}^1.$$

Let us now check that $\{\Phi_k\}_k$ are indeed the desired (smooth) approximating maps of $\Phi^\xi$. For that, first, note that differentiating the above equation defining $\Phi_k$, one obtains

$$\frac{\partial \Phi_k}{\partial\theta}(z)\cdot z^\perp = 0 \quad \text{for every } z = e^{i\theta} \in \mathbb{S}^1. \tag{16}$$

Next, we prove that $\Phi_k$ satisfies (14). Indeed, we can locally write $u = e^{i\Theta}$ in every ball $B \subset \Omega$ for some smooth lifting $\Theta : B \to \mathbb{R}$ that satisfies

$$\nabla\Theta \cdot u = \nabla \times u = 0 \quad \text{in } B.$$

This means that $\nabla\Theta = \lambda u^\perp$ in $B$ for some smooth function $\lambda : B \to \mathbb{R}$. Therefore, it follows that

$$\nabla\cdot[\Phi_k(u)] = \frac{\partial\Phi_k}{\partial\theta}(e^{i\Theta})\cdot\nabla\Theta = \lambda\frac{\partial\Phi_k}{\partial\theta}(u)\cdot u^\perp \overset{(16)}{=} 0 \quad \text{in } B.$$

Passing to limit $k \to \infty$, the dominated convergence theorem yields

$$\int_B \Phi^\xi(u)\cdot\nabla\zeta\,dx = 0 \quad \text{for every } \zeta \in C_c^\infty(B).$$

The conclusion is now straightforward. □

Note that another interest of this second method is that it can be adapted to vector fields $u \in W^{1/p,p}$ for $p \in [1,3]$. For such vector fields, there is a priori no trace of $u$ on a segment, so the flow (2) does not have a proper meaning anymore; see [Ignat 2012b; De Lellis and Ignat 2015] for more details.

**3.2.** ***The case of dimension $N \geq 3$.*** The aim of this subsection is to prove Theorem 7. We divide the proof in several steps, each being stated as a lemma.

**Lemma 12.** *Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}^N$ be a smooth vector field satisfying (8). We define*

$$\widetilde{\Omega} := \left\{ x \in \Omega : u(x) \neq 0,\ \nabla\left(\frac{u}{|u|}\right)(x) \neq 0 \right\}$$

*and for every $x \in \widetilde{\Omega}$,*

$$\mathbb{S}_x := u(x)^{\perp} \cap \mathbb{S}^{N-1} = \left\{ \xi \in \mathbb{S}^{N-1} : u(x) \cdot \xi = 0 \right\} \approx \mathbb{S}^{N-2}.$$

*Then we have for all $x \in \widetilde{\Omega}$ and for $\mathcal{H}^{N-2}$-a.e. $\xi \in \mathbb{S}_x$ that the set*

$$\{y \in \widetilde{\Omega} : u(y) \cdot \xi = 0\} = \widetilde{\Omega} \cap \partial\{u \cdot \xi > 0\}$$

*is a hyperplane around $x$ that is oriented by the normal vector $\xi$. Moreover,*

$$\nabla_x \chi(\,\cdot\,, \xi) = \pm \xi \mathcal{H}^{N-1} \llcorner \partial\{u \cdot \xi > 0\} \qquad \textit{locally around } x. \tag{17}$$

*Proof.* As in the proof of Lemma 11, we set $v = u/|u|$ on $\widetilde{\Omega}$. Then $v$ is a smooth unit-length vector field in $\widetilde{\Omega}$ that satisfies (8) (because $u$ satisfies it, too) and by Proposition 5, we have that $v$ is curl-free in $\widetilde{\Omega}$. Let $x \in \widetilde{\Omega}$; in particular, $\nabla v(x) \neq 0$. First, we show that $\{y \in \widetilde{\Omega} : u(y) \cdot \xi = 0\}$ is a smooth $(N-1)$-manifold around $x$. Since $v$ is curl-free, we know that $\nabla v(x) = (\partial_j v_i(x))_{i,j}$ is symmetric. By differentiating the relation $|v(x)| = 1$, it follows that

$$\nabla v(x)^T v(x) = \nabla v(x)\, v(x) = 0,$$

which means $v(x) \in \operatorname{Ker} \nabla v(x)$. We will prove that

$$\mathcal{H}^{N-2}(\mathbb{S}_x \cap \operatorname{Ker} \nabla v(x)) = 0.$$

Assume by contradiction that $\mathbb{S}_x \cap \operatorname{Ker} \nabla v(x)$ has positive $\mathcal{H}^{N-2}$-measure. Since $\operatorname{Ker} \nabla v(x)$ is a linear space, we have $\mathbb{S}_x \subset \operatorname{Ker} \nabla v(x)$, that is, $\nabla v(x)\xi = 0$ for all $\xi \in \mathbb{S}_x$. Moreover, since $v(x) \in \operatorname{Ker} \nabla v(x)$ and $\mathbb{S}_x \subset v(x)^{\perp}$, it follows that $\nabla v(x) = 0$, which is a contradiction with the assumption $\nabla v(x) \neq 0$. Therefore, $\nabla v(x)\xi \neq 0$ for $\mathcal{H}^{N-2}$-a.e. $\xi \in \mathbb{S}_x$ and $\{y \in \widetilde{\Omega} : v(y) \cdot \xi = 0\} = \{y \in \widetilde{\Omega} : u(y) \cdot \xi = 0\}$ is a smooth $(N-1)$-manifold around $x$.

It remains to prove that this manifold is a piece of hyperplane oriented by $\xi$ where (17) holds true. For that, let $\varphi \in C_c^{\infty}(\widetilde{\Omega}, \mathbb{R}^N)$ be supported in a ball $B \subset \widetilde{\Omega}$ centered at $x$. By the Gauss theorem, we have

$$-\big\langle \nabla_x \chi(\,\cdot\,, \xi), \varphi \big\rangle = \int_B \nabla \cdot \varphi(y) \chi(y, \xi)\, dy = \int_{\{y \in B : u(y) \cdot \xi > 0\}} \nabla \cdot \varphi\, dy = \int_{B \cap \partial\{u \cdot \xi > 0\}} \varphi \cdot \nu\, d\mathcal{H}^{N-1}(y),$$

where $\nu$ is the unit outer normal vector to the $(N-1)$-manifold $\partial\{u(y) \cdot \xi > 0\}$. This proves that locally around $x$, we have

$$\nabla_x \chi(x, \xi) = -\nu \mathcal{H}^{N-1} \llcorner \big(B \cap \partial\{u \cdot \xi > 0\}\big).$$

Because of (8), we know that $\nabla_x \chi(x, \xi)$ and $\xi$ are collinear. Since $\nu$ is smooth on $B \cap \partial\{u \cdot \xi > 0\}$, this implies $\nu = \xi$ or $\nu = -\xi$ on $B \cap \partial\{u \cdot \xi > 0\}$. The conclusion is now straightforward. □

We now state the following result, which is the key point in proving Theorem 7.

**Lemma 13.** *Under the hypotheses of Theorem 7, every point $x \in \mathcal{S}$ is an umbilical point*; *i.e., there exists $\lambda(x) \in \mathbb{R}$ such that*

$$Du(x) = \lambda(x)\mathrm{Id} : T_x\mathcal{S} \to \mathbb{R}^{N-1},$$

*where $u$ is proportional to the Gauss map on $\mathcal{S}$, $T_x\mathcal{S}$ is the tangent plane to the hypersurface $\mathcal{S}$ at $x$ and* Id *is the identity matrix.*

*Proof.* Recall that $|u|$ is constant on $\mathcal{S}$ by Lemma 11 so that $u/|u|$ is the normal vector (i.e., the Gauss map) at the hypersurface $\mathcal{S}$. Therefore,

$$D\left(\frac{u}{|u|}\bigg|_{\mathcal{S}}\right) = \frac{1}{|u|} D(u|_{\mathcal{S}}) \quad \text{in } \mathcal{S},$$

where $D(u|_{\mathcal{S}})$ is the differential of $u$ restricted to $\mathcal{S}$ as a map with values into the sphere $\mathbb{S}^{N-1}$ (up to the multiplicative constant $|u|$). As in the proofs of Lemmas 11 and 12, we may assume that $u$ never vanishes in $\Omega$ and set $v = u/|u|$ in $\Omega$. Then $v$ is a smooth unit-length vector field in $\Omega$ that satisfies (8) and by Proposition 5, $v$ is curl-free so that locally $v = \nabla\tilde{\psi}$ for a smooth stream function $\tilde{\psi}$. Since $\nabla\psi = u = |u|\nabla\tilde{\psi}$, we know that $\psi$ and $\tilde{\psi}$ have the same level sets; in particular, $\mathcal{S}$ is a level set of $\tilde{\psi}$. Therefore, replacing $u$ by $v$, we may assume in the following that

$$|u| = 1 \quad \text{in } \Omega.$$

Let $x \in \mathcal{S}$. We want to show that $x$ is an umbilical point of $\mathcal{S}$. This is clear if $\nabla u(x) = 0$. Therefore, we assume in the following that $x \in \tilde{\Omega} \cap \mathcal{S}$, as defined in Lemma 12; i.e.,

$$\nabla u(x) \neq 0.$$

Since (9) holds for the unit-length vector field $u$, by differentiating (9), we obtain

$$\nabla u(x) = \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \xi \otimes \nabla_x \chi(x, \xi),$$

where $V_{N-1}$ is the volume of the unit ball in $\mathbb{R}^{N-1}$. The above integrand is to be understood as an absolutely continuous measure with respect to the Hausdorff $\mathcal{H}^{N-2}$ measure concentrated on the set $\mathbb{S}_x$ (defined at Lemma 12). For that, we check first that the support of the integrand lies on $\mathbb{S}_x$. Indeed, if $\xi \in \mathbb{S}^{N-1}$ with $u(x) \cdot \xi \neq 0$, then $\nabla_x \chi(\cdot, \xi) = 0$ in the open set $\{u \cdot \xi \neq 0\}$ around $x$. Therefore, the integrand has support on the set $\xi \in \mathbb{S}_x$, where (17) holds true for $\mathcal{H}^{N-2}$-a.e. $\xi \in \mathbb{S}_x$ and the density of the measure is equal to $\pm\xi \otimes \xi \mathcal{H}^{N-2} \llcorner \mathbb{S}_x$. Since $\mathbb{S}_x \subset u(x)^\perp = T_x\mathcal{S}$, the density $\xi \otimes \xi$ with $\xi \in \mathbb{S}_x$ already identifies $\nabla u(x) \equiv Du(x)$. Next we compute this quantity by exploring the sign of the density $\pm\xi \otimes \xi$:

<u>Case $N = 3$</u>. We show that there are at most two nonzero vectors $\pm\xi_0 \in \mathbb{S}_x \approx \mathbb{S}^1$ such that $\nabla u(x)\xi_0 = 0$. Assume by contradiction that there are more than two vectors as above; i.e., there exists another nonzero vector $\tilde{\xi}_0 \neq \pm\xi_0$ in $\mathbb{S}_x$ such that $\nabla u(x)\xi_0 = \nabla u(x)\tilde{\xi}_0 = 0$. Because of $|u| = 1$, we know that $\nabla u(x)u(x) = 0$. Since the set $\{u(x), \xi_0, \tilde{\xi}_0\}$ spans $\mathbb{R}^3$, we have $\nabla u(x) = 0$, which contradicts the hypothesis $x \in \tilde{\Omega}$. Therefore, $\nabla u(x)\xi \neq 0$ for every $\xi \in \mathbb{S}_x \setminus \{\pm\xi_0\}$ (or for every $\xi \in \mathbb{S}_x$ if $\xi_0$ does not exist) and by Lemma 12,

$\partial\{u(y)\cdot\xi > 0\}$ is a smooth surface around $x$ oriented by $\xi$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the two connected components of $\mathbb{S}_x \backslash \{\pm\xi_0\}$ (with the convention that $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$ in the case $\nabla u(x)\xi \neq 0$ for every $\xi \in \mathbb{S}_x$). For $j = 1, 2$, we associate to a point $\xi \in \mathcal{C}_j$ the unit outer normal vector field $\nu(\xi) \in \{\pm\xi\}$ to the plane $\partial\{u\cdot\xi > 0\}$ around $x$. Since the map $\xi \in \mathcal{C}_j \to \nu(\xi)$ is smooth (by the implicit function theorem) and $\mathcal{C}_j$ is connected, we deduce that $\nu$ is constant on $\mathcal{C}_j$. Thus it follows that

$$\pi \nabla u(x) = \gamma_1 \int_{\mathcal{C}_1} \xi \otimes \xi \, d\xi + \gamma_2 \int_{\mathcal{C}_2} \xi \otimes \xi \, d\xi,$$

with $V_2 = \pi$ and $\gamma_{1,2} \in \{\pm 1\}$ (with the convention that $\gamma_1 = \gamma_2 = \pm 1/2$ if $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$). It remains to show that $\int_{\mathcal{C}_j} \xi \otimes \xi \, d\xi$ is proportional to the identity matrix Id, $j = 1, 2$. Up to a rotation, we can suppose that $u(x) = e_3$ and $\mathcal{C}_1 = \{\xi \in \mathbb{S}^1 \times \{0\} : \xi_2 > 0\} \approx \{(\cos\theta, \sin\theta) : \theta \in (0, \pi)\}$. We have

$$\int_{\mathcal{C}_1} \xi \otimes \xi \, d\xi \approx \int_0^\pi \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} d\theta = \frac{\pi}{2}\mathrm{Id}$$

(the conclusion follows similarly if $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$).

Case $N > 3$. Let $\mathcal{C} = \operatorname{Ker} \nabla u(x) \cap \mathbb{S}_x$. We know that $u(x) \in \operatorname{Ker} \nabla u(x)$ and $u(x)$ is orthogonal to $\mathbb{S}_x$, which is isomorphic to $\mathbb{S}^{N-2}$. Since $\nabla u(x) \neq 0$ (i.e., the dimension of $\operatorname{Ker} \nabla u(x)$ is at most $N-1$), we have two situations (as in the case $N = 3$):

• Situation 1: $\dim \operatorname{Ker} \nabla u(x) = N-1$, which leads to $\mathcal{C}$ isomorphic to $\mathbb{S}^{N-3}$. In this situation, $\mathbb{S}_x \backslash \mathcal{C}$ is the partition of two connected sets $\mathcal{C}_1$ and $\mathcal{C}_2$ that are isomorphic to the half-sphere

$$\mathbb{S}_+^{N-2} = \{\xi = (\xi_1, \ldots, \xi_{N-1}) \in \mathbb{S}^{N-2} : \xi_1 > 0\}.$$

The same argument as in the case $N = 3$ shows that the sign of the unit outer normal field $\nu(\xi) \in \{\pm\xi\}$ to the hyperplane $\partial\{u\cdot\xi > 0\}$ is constant when $\xi$ covers $\mathcal{C}_j$, $j = 1, 2$, so that

$$V_{N-1} \nabla u(x) = \gamma_1 \int_{\mathcal{C}_1} \xi \otimes \xi \, d\xi + \gamma_2 \int_{\mathcal{C}_2} \xi \otimes \xi \, d\xi,$$

with $\gamma_1, \gamma_2 \in \{\pm 1\}$.

• Situation 2: $\dim \operatorname{Ker} \nabla u(x) \leq N-2$, which leads to the manifold $\mathcal{C}$ of dimension $\leq N-4$. In other words, $\mathbb{S}_x \backslash \mathcal{C}$ is connected and covers a.e. point of $\mathbb{S}_x$. The above formula holds for $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{S}_x$ and $\gamma_1 = \gamma_2 = \pm 1/2$.

We now compute $\nabla u(x)$. For that, we may assume (up to a rotation) that $u(x) = e_N$ and $\mathcal{C}_1 = \mathbb{S}_+^{N-2}$. Since $\mathbb{S}_+^{N-2}$ is invariant under the change of coordinate $\xi_d \mapsto -\xi_d$ for some $2 \leq d \leq N-1$, we have for every $1 \leq j \leq N-1$ with $j \neq d$,

$$\int_{\mathbb{S}_+^{N-2}} \xi_j \xi_d \, d\xi = -\int_{\mathbb{S}_+^{N-2}} \xi_j \xi_d \, d\xi = 0,$$

leading to

$$\int_{\mathbb{S}_+^{N-2}} \xi \otimes \xi \, d\xi = \int_{\mathbb{S}_+^{N-2}} \begin{pmatrix} \xi_1^2 & & 0 \\ & \ddots & \\ 0 & & \xi_{N-1}^2 \end{pmatrix} d\xi = \frac{\mathcal{H}^{N-2}(\mathbb{S}^{N-2})}{2(N-1)}\mathrm{Id}. \qquad \square$$

*Proof of Theorem 7.* By Lemma 13, every point in $\mathcal{S}$ is an umbilical point. Such a hypersurface is called *totally umbilical*. A classical result in differential geometry states that a totally umbilical hypersurface is either a piece of an $(N-1)$-sphere or a piece of a hyperplane (see, e.g., [Hicks 1965, Chapter 2, page 36]). □

We have the following consequence of Lemma 11 and Theorem 8 (whose proof is independent of this section):

**Corollary 14.** *Under the hypotheses of Theorem 7, there exists a neighborhood $\omega$ of $\mathcal{S}$ and a diffeomorphism $t \to F(t)$ such that either $\psi = F(|x-P|)$ for every $x \in \omega$ for a point $P \in \mathbb{R}^N$, or $\psi = F(x \cdot \xi)$ for every $x \in \omega$ for a vector $\xi \in \mathbb{S}^{N-1}$.*

## 4. Several properties on the set of Lebesgue points

Let $\Omega \subset \mathbb{R}^N$ be an open set and $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Recall that $x_0 \in \Omega$ is a *Lebesgue point* of $u$ if there exists a vector $u_0 \in \mathbb{R}^N$ such that

$$\lim_{r\to 0} \fint_{B_r(x_0)} |u(x) - u_0|\, dx = 0. \tag{18}$$

In this case, we write $u(x_0) = u_0$, which is the limit of the average $\fint$ of $u$ on the ball $B_r(x_0)$ as $r \to 0$. We denote by $\mathrm{Leb} \subset \Omega$ the set of Lebesgue points of $u$. It is well known that $\mathcal{H}^N(\Omega \setminus \mathrm{Leb}) = 0$ and one can replace the ball $B_r(x_0)$ by the cube $x_0 + (-r, r)^N$ in the definition (18) to recover the same set of Lebesgue points.

*Proof of Proposition 5.* We start by proving (9) for a fixed vector $u(x) \in \mathbb{S}^{N-1}$. By rotating the axes if necessary, we may assume that $u(x) = e_N$. Then we compute

$$\int_{\mathbb{S}^{N-1}} \xi\chi(x,\xi)\, d\mathcal{H}^{N-1}(\xi) = \int_{\mathbb{S}^{N-1}\cap\{\xi_N>0\}} \xi\, d\mathcal{H}^{N-1}(\xi) = \Big(\int_{\mathbb{S}^{N-1}\cap\{\xi_N>0\}} \xi_N\, d\mathcal{H}^{N-1}(\xi)\Big) e_N$$

because the integrand is odd in the variables $\xi_j$ for $j = 1, \dots, N-1$. Defining $\xi' := (\xi_1, \dots, \xi_{N-1})$, the half-sphere $\mathbb{S}^{N-1} \cap \{\xi_N > 0\}$ is the graph of the map

$$\xi' \in B^{N-1} \mapsto \xi_N = \sqrt{1 - |\xi'|^2}$$

so that we have

$$\int_{\mathbb{S}^{N-1}\cap\{\xi_N>0\}} \xi_N\, d\mathcal{H}^{N-1}(\xi) = \int_{B^{N-1}} \sqrt{1-|\xi'|^2}\, \frac{d\xi'}{\sqrt{1-|\xi'|^2}} = \mathcal{H}^{N-1}(B^{N-1}) = V_{N-1}.$$

The second statement naturally reduces (by a slicing argument) to the case of dimension $N = 2$. In that case, for any $\varphi \in C_c^\infty(\Omega)$, we have $\nabla \times u = \partial_1 u_2 - \partial_2 u_1$ and

$$\begin{aligned}\int_\Omega \varphi \nabla \times u\, dx &= -\int_\Omega \nabla^\perp \varphi \cdot u\, dx \\ &\overset{(6)}{=} \frac{1}{2}\int_\Omega \int_{\mathbb{S}^1} \nabla\varphi \cdot \xi^\perp \chi(x,\xi)\, d\mathcal{H}^1(\xi)\, dx = \frac{1}{2}\int_{\mathbb{S}^1} d\mathcal{H}^1(\xi) \int_\Omega \nabla\varphi\cdot\xi^\perp\chi(x,\xi)\, dx \overset{(5)}{=} 0.\end{aligned}$$
□

The following lemma yields the relation between the Lebesgue points of $u$ and Lebesgue points of the functions $\{\chi(\,\cdot\,,\xi)\}_{\xi\in\mathbb{S}^{N-1}}$ defined in (4).

**Lemma 15.** *Let $\Omega\subset\mathbb{R}^N$ be an open set and $u\in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$.*

(i) *If $|u|=1$ a.e. in $\Omega$ and $x_0$ is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ for almost every $\xi\in\mathbb{S}^{N-1}$, then $x_0$ is a Lebesgue point of $u$ and (9) holds at $x_0$.*

(ii) *Let $x_0$ be a Lebesgue point of $u$ and $\xi\in\mathbb{S}^{N-1}$. If $u(x_0)\cdot\xi\neq 0$, then $x_0$ is a Lebesgue point of $\chi(\,\cdot\,,\xi)$. Conversely, if $x_0$ is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ with $\chi(x_0,\xi)=1$ (resp. $=0$), then $u(x_0)\cdot\xi\geq 0$ (resp. $\leq 0$).*

*Proof.* To prove (i), we apply Proposition 5. Indeed, if $x_0$ is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ for a.e. $\xi\in\mathbb{S}^{N-1}$, then Fubini's theorem implies

$$
\begin{aligned}
&\fint_{B_r(x_0)}\left|u(x)-\frac{1}{V_{N-1}}\int_{\mathbb{S}^{N-1}}\xi\,\chi(x_0,\xi)\,d\mathcal{H}^{N-1}(\xi)\right|dx\\
&\qquad\overset{(9)}{\leq}\frac{1}{V_{N-1}}\fint_{B_r(x_0)}\int_{\mathbb{S}^{N-1}}\bigl|\xi\bigl(\chi(x,\xi)-\chi(x_0,\xi)\bigr)\bigr|\,d\mathcal{H}^{N-1}(\xi)\,dx\\
&\qquad\leq\frac{1}{V_{N-1}}\int_{\mathbb{S}^{N-1}}\left(\fint_{B_r(x_0)}|\chi(x,\xi)-\chi(x_0,\xi)|\,dx\right)d\mathcal{H}^{N-1}(\xi)\xrightarrow{r\to 0}0,
\end{aligned}
$$

where we used the dominated convergence theorem.

Next we prove (ii). We treat the case $u(x_0)\cdot\xi>0$. For that, we have

$$
\begin{aligned}
\int_{B_r(x_0)}\bigl|\chi(x,\xi)-1\bigr|\,dx&=\frac{1}{u(x_0)\cdot\xi}\int_{B_r(x_0)\cap\{u\cdot\xi\leq 0\}}u(x_0)\cdot\xi\,dx\\
&\leq\frac{1}{u(x_0)\cdot\xi}\int_{B_r(x_0)\cap\{u\cdot\xi\leq 0\}}\underbrace{(u(x_0)\cdot\xi-u(x)\cdot\xi)}_{\geq u(x_0)\cdot\xi>0}\,dx\leq\frac{1}{u(x_0)\cdot\xi}\int_{B_r(x_0)}|u(x)-u(x_0)|\,dx.
\end{aligned}
$$

Since $x_0$ is a Lebesgue point of $u$, it follows that $x_0$ is a Lebesgue point for $\chi(\,\cdot\,,\xi)$ with $\chi(x_0,\xi)=1$. The case $u(x_0)\cdot\xi<0$ can be shown similarly and we obtain $\chi(x_0,\xi)=0$. The last statement is a direct consequence of the above lines (using a contradiction argument). □

**Remark 16.** (a) Note that the condition $u(x_0)\cdot\xi\neq 0$ is essential in Lemma 15(ii). Indeed, if one considers the vortex vector field $u(x)=x/|x|$ for $x\in\mathbb{R}^N\setminus\{0\}$, then for every $\xi\in\mathbb{S}^{N-1}$, any point $x_0\in\xi^{\perp}\setminus\{0\}$ is a Lebesgue point of $u$ (because $u$ is smooth around $x_0$) and satisfies

$$u(x_0)\cdot\xi=0,$$

but $x_0$ is not a Lebesgue point of $\chi(\,\cdot\,,\xi)$ because

$$\fint_{B_r(x_0)}\left|\chi(x,\xi)-\fint_{B_r(x_0)}\chi(\,\cdot\,,\xi)\right|dx=\fint_{B_r(x_0)}\tfrac{1}{2}\,dx\nrightarrow 0\qquad\text{as } r\to 0,$$

where we used that

$$\fint_{B_r(x_0)}\chi(x,\xi)\,dx=\frac{\mathcal{H}^N\bigl(\{x\in B_r(x_0):x\cdot\xi>0\}\bigr)}{\mathcal{H}^N(B_r(x_0))}\overset{x=y+x_0}{=}\frac{\mathcal{H}^N\bigl(\{y\in B_r(0):y\cdot\xi>0\}\bigr)}{\mathcal{H}^N(B_r(0))}=\frac{1}{2}.$$

(b) Note that in Lemma 15(ii), one cannot conclude in general that $u(x_0)\cdot\xi > 0$ provided that $\chi(x_0,\xi)=1$. Indeed, consider for example $\xi = e_N$, $u(x)\cdot\xi = u_N(x) := |x|$ for $x\in\mathbb{R}^N$ and set $x_0=0$; then $\chi(\,\cdot\,,\xi)=1$ in $\mathbb{R}^N\setminus\{x_0\}$, $x_0$ is a Lebesgue point of $u_N$ and $\chi(\,\cdot\,,\xi)$ with $\chi(x_0,\xi)=1$, but $u_N(x_0)=0$.

We now prove one of the key tools in the proof of Theorem 8, which mimics the relation of the ordering of level sets of a stream function when (8) holds true. It is a generalization of Proposition 3.1 in [Jabin, Otto, and Perthame 2002] to the case of dimension $N$:

**Proposition 17** (ordering). *Let $N\geq 2$, $\Omega\subset\mathbb{R}^N$ be an open set and $u\in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ satisfy the kinetic formulation (8). Assume that $y,z\in$ Leb are two different Lebesgue points of $u$ such that the closed segment $[yz]$ is included in $\Omega$. Then for every direction $\xi\in\mathbb{S}^{N-1}$ with $\xi\in(z-y)^\perp$, we have*

$$u(y)\cdot\xi > 0 \ (\textit{resp.} < 0) \quad\Longrightarrow\quad u(z)\cdot\xi\geq 0 \ (\textit{resp.} \leq 0); \tag{19}$$

*moreover, $y$ and $z$ are Lebesgue points of $\chi(\,\cdot\,,\xi)$ and $\chi(y,\xi)=\chi(z,\xi)$. As a consequence, if $u\neq 0$ a.e. in $\Omega$, then for a.e. $y\in\Omega$, $\mathcal{H}^{N-1}$-a.e. $\xi\in\mathbb{S}^{N-1}$ and $\mathcal{H}^{N-1}$-a.e. $v\in\xi^\perp$ with the segment $[y,y+v]$ included in $\Omega$, we have that $y$ and $y+v$ are Lebesgue points of $u$ and*

$$\chi(y,\xi)=\chi(y+v,\xi). \tag{20}$$

*Proof.* First, we consider the case $u(y)\cdot\xi>0$. By Lemma 15(ii), $y$ is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ and $\chi(y,\xi)=1$. Let

$$\left\{\rho_\varepsilon(\,\cdot\,)=\frac{1}{\varepsilon^N}\rho\left(\frac{\cdot}{\varepsilon}\right)\right\}_{\varepsilon>0}$$

be a standard family of mollifiers, where $\rho$ is a nonnegative radial smooth function having as support the unit ball $\operatorname{supp}\rho = B_1\subset\mathbb{R}^N$ and $\int_{B_1}\rho\,dx=1$. Set the convoluted function

$$\chi_\varepsilon := \rho_\varepsilon * \chi(\,\cdot\,,\xi)$$

in a neighborhood $\omega\subset\Omega$ of the segment $[yz]$ for $\varepsilon>0$ sufficiently small. Then $\chi_\varepsilon$ is smooth in $\omega$ and for every Lebesgue point $x\in\omega$ of $\chi(\,\cdot\,,\xi)$ we have $\chi_\varepsilon(x)\to\chi(x,\xi)$ as $\varepsilon\to 0$ because

$$\begin{aligned}|\chi_\varepsilon(x)-\chi(x,\xi)| &= \left|\int_{B_\varepsilon(0)}\bigl(\chi(x-\tilde{x},\xi)-\chi(x,\xi)\bigr)\rho_\varepsilon(\tilde{x})\,d\tilde{x}\right|\\ &\leq \frac{\sup\rho}{\varepsilon^N}\int_{B_\varepsilon(0)}\bigl|\chi(x-\tilde{x},\xi)-\chi(x,\xi)\bigr|\,d\tilde{x}\\ &\leq C\,⨍_{B_\varepsilon(x)}\bigl|\chi(\tilde{y},\xi)-\chi(x,\xi)\bigr|\,d\tilde{y}\xrightarrow{\varepsilon\to 0}0.\end{aligned}$$

In particular, $\lim_{\varepsilon\to 0}\chi_\varepsilon(y)=\chi(y,\xi)=1$. Let $v=z-y$. We will show that $\chi(y+v,\xi)=1$. For that, we have $v\in\xi^\perp$ and

$$v\cdot\nabla_x\chi_\varepsilon = v\cdot\nabla_x\chi(\,\cdot\,,\xi)*\rho_\varepsilon \overset{(8)}{=} 0 \quad\text{in } \omega.$$

Then

$$\chi_\varepsilon(y+v)-\chi_\varepsilon(y)=\int_0^1 v\cdot\nabla_x\chi_\varepsilon(y+tv)\,dt=0$$

so that

$$\lim_{\varepsilon\to 0}\chi_\varepsilon(z)=\lim_{\varepsilon\to 0}\chi_\varepsilon(y)=\chi(y,\xi)=1.$$

This implies that $u(z)\cdot\xi\geq 0$. Assume by contradiction that $u(z)\cdot\xi<0$. By Lemma 15(ii), $z$ is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ with $\chi(z,\xi)=0$ so that

$$\lim_{\varepsilon\to 0}\chi_\varepsilon(z)=\chi(z,\xi)=0,$$

which contradicts the above statement. We prove now the following:

**Claim.** If $\chi_\varepsilon(z)\to 1$ as $\varepsilon\to 0$, then $z$ is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ with $\chi(z,\xi)=1$.

*Proof of Claim.* Let $\{\varepsilon_k\}$ be a sequence converging to 0 as $k\to\infty$. For $k$ large enough, we define $f_k:B_1\to\{0,1\}$ by $f_k(x)=\chi(z-\varepsilon_k x,\xi)$ for every $x\in B_1$. Then the sequence $\{f_k\}$ is bounded in $L^2(B_1)$ and up to a subsequence, $f_k\rightharpoonup f$ weakly in $L^2(B_1)$, where the limit $f:B_1\to\mathbb{R}$ takes values in $[0,1]$. Therefore, we have for our smooth mollifier $\rho\in L^2(B_1)$ that

$$\int_{B_1}\rho f_k\,dx\to\int_{B_1}\rho f\,dx\quad\text{as } k\to\infty.$$

Note now that by the change of variable $\tilde{x}=z-\varepsilon_k x$ we obtain by our assumption:

$$\int_{B_1}\rho(x)f_k(x)\,dx=\int_{B_{\varepsilon_k}(z)}\rho_{\varepsilon_k}(z-\tilde{x})\chi(\tilde{x},\xi)\,d\tilde{x}=\chi_{\varepsilon_k}(z)\to 1\quad\text{as } k\to\infty;$$

therefore, $\int_{B_1}\rho f\,dx=1$. Since 1 is the maximal value of $f$ and $\rho$ is nonnegative with the integral on $B_1$ equal to 1, we deduce that $f=1$ in $\operatorname{supp}\rho=B_1$. It follows by the change of variable $\tilde{x}=z-\varepsilon_k x$ that

$$\fint_{B_{\varepsilon_k}(z)}|\chi(\tilde{x},\xi)-1|\,d\tilde{x}=1-\fint_{B_1(0)}f_k(x)\,dx\to 0\quad\text{as } k\to\infty,$$

because $f_k\rightharpoonup 1$ weakly in $L^2(B_1)$. Since the limit is unique for every subsequence $\varepsilon_k\to 0$, we conclude that $z$ is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ with $\chi(z,\xi)=1$, which proves the claim. □

For the case $u(y)\cdot\xi<0$, i.e., $\chi(y,\xi)=0$ by Lemma 15(ii), one applies the above argument by replacing $\xi$ with $-\xi$ and obtains that $z$ is a Lebesgue point of $\chi(\,\cdot\,,-\xi)$ with $\chi(z,-\xi)=1$. It follows that $z$ is a Lebesgue point of $\chi(\,\cdot\,,\xi)$ with $\chi(z,\xi)=0$ because

$$\fint_{B_r(z)}|\chi(x,\xi)|\,dx\leq\frac{\mathcal{H}^N\big(\{x\in B_r(z):u(x)\cdot\xi\geq 0\}\big)}{\mathcal{H}^N(B_r(z))}=1-\fint_{B_r(z)}\chi(x,-\xi)\,dx\to 0$$

as $r\to 0$. One also concludes that $u(z)\cdot\xi\leq 0$ by Lemma 15(ii).

For the last statement, we have for a.e. $y\in\Omega$ that $y$ is a Lebesgue point of $u$ with $u(y)\neq 0$. Then for $\mathcal{H}^{N-1}$-a.e. direction $\xi\in\mathbb{S}^{N-1}$, we have that $u(y)\cdot\xi\neq 0$ and $y+v$ is a Lebesgue point of $u$ for $\mathcal{H}^{N-1}$-a.e. $v\in\xi^\perp$ with the segment $[y,y+v]\subset\Omega$. By the above argument, we get (20). □

## 5. Notion of the trace on lines

The $H^{1/2}$-regularity for $N$-dimensional unit-length vector fields $u$ satisfying the kinetic formulation (8) (see [Golse, Lions, Perthame, and Sentis 1988]) is a priori not enough to define the notion of the trace of $u$ on 1-dimensional lines. However, using the ideas in [Jabin, Otto, and Perthame 2002] for dimension 2, we will define a notion of the trace of $u$ on segments (in the sense of Lebesgue points) in any dimension $N \geq 2$.

**Proposition 18** (trace). *Let $N \geq 2$, $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{S}^{N-1}$ be a Lebesgue-measurable vector field satisfying the kinetic formulation (8). Assume that the segment*

$$L := \{0\}^{N-1} \times [-1, 1] \text{ is included in } \Omega.$$

*Then there exists a Lebesgue-measurable function $\tilde{u} : (-1, 1) \to \mathbb{R}^N$ such that*

$$\lim_{r\to 0} \fint_{(-r,r)^{N-1}} \int_{-1}^{1} |u(x', x_N) - \tilde{u}(x_N)|\, dx_N\, dx' = 0, \tag{21}$$

*where $x = (x', x_N)$, $x' = (x_1, \dots, x_{N-1})$. Moreover, for $\mathcal{H}^1$-a.e. $x_N \in (-1, 1)$,*

$$\tilde{u}(x_N) = \lim_{r\to 0} \fint_{(-r,r)^{N-1}} u(x', x_N)\, dx' \quad \text{and} \quad |\tilde{u}(x_N)| = 1. \tag{22}$$

*Finally, every Lebesgue point $x \in \mathrm{Leb}$ of $u$ lying inside $L$ is a Lebesgue point of $\tilde{u}$ and $u(x) = \tilde{u}(x_N)$. The vector field $\tilde{u}$ is called the trace of $u$ on the segment $L$.*

*Proof.* To simplify the writing, we assume that $\Omega = \mathbb{R}^N$. We divide the proof into several steps:

*Step 1: defining the 1-dimensional function $\tilde{\chi}(\cdot, \xi)$ for suitable directions $\xi \in \mathbb{S}^{N-1}$.* Let $\mathcal{D}$ be the set of directions $\xi \in \mathbb{S}^{N-1}$ such that $\xi_N \neq 0$ and (20) holds true for the triple $(y, y + v, \xi)$ for a.e. $y \in \Omega$ and $\mathcal{H}^{N-1}$-a.e. $v \in \xi^\perp$ (with the segment $[y, y + v] \subset \Omega$, where $y$ and $y + v$ are Lebesgue points of $u$). By Proposition 17, we know that $\mathcal{D}$ covers $\mathbb{S}^{N-1}$ up to a set of zero $\mathcal{H}^{N-1}$-measure. For such a direction $\xi \in \mathcal{D}$, we can choose a point $y_\xi \in \Omega$ (in a neighborhood of $L$) such that the map $\xi \in \mathcal{D} \mapsto y_\xi \in \Omega$ is Lebesgue measurable, the point $y_\xi + t\xi \in \Omega$ is a Lebesgue point of $\chi(\cdot, \xi)$ for $\mathcal{H}^1$-a.e. $t \in \mathbb{R}$, the function $t \mapsto \chi(y_\xi + t\xi, \xi)$ is $\mathcal{H}^1$-measurable (by Fubini's theorem) and (20) holds true for the triple $(y_\xi + t\xi,\ y_\xi + t\xi + v,\ \xi)$ for $\mathcal{H}^{N-1}$-a.e. $v \in \xi^\perp$ and $\mathcal{H}^1$-a.e. $t$. Define the 1-dimensional function

$$s \mapsto \tilde{\chi}(s, \xi) := \chi\big(y_\xi + (s - y_\xi \cdot \xi)\xi,\ \xi\big) \in \{0, 1\}.$$

Then we have that for a.e. $x \in \Omega$ in a neighborhood of $L$,

$$\tilde{\chi}(x \cdot \xi, \xi) = \chi\big(y_\xi - y_\xi \cdot \xi\xi + x \cdot \xi\xi,\ \xi\big) \overset{(20)}{=} \chi(x, \xi), \tag{23}$$

because

$$v = y_\xi - y_\xi \cdot \xi\xi + x \cdot \xi\xi - x \in \xi^\perp.$$

*Step 2: for $\xi \in \mathcal{D}$ and for every Lebesgue point $P = (0, \dots, 0, P_N) \in L$ of $\chi(\cdot, \xi)$ with $P_N \in (-1, 1)$, the point $P \cdot \xi$ is a Lebesgue point of $\tilde\chi(\cdot, \xi)$ and $\tilde\chi(P_N\xi_N, \xi) = \chi(P, \xi)$.* Indeed, since $\xi_N \neq 0$, we have

$$\begin{aligned}
&\fint_{P_N\xi_N-r|\xi_N|}^{P_N\xi_N+r|\xi_N|} \big|\tilde\chi(t,\xi)-\chi(P,\xi)\big|\,dt \\
&\quad= \fint_{(-r,r)^{N-1}} dx' \fint_{P_N-r}^{P_N+r} \big|\tilde\chi(\tilde x_N\xi_N,\xi)-\chi(P,\xi)\big|\,d\tilde x_N && (\text{since } t=\tilde x_N\xi_N)\\
&\quad= \fint_{(-r,r)^{N-1}} dx' \fint_{P_N-x'\cdot\xi'/\xi_N-r}^{P_N-x'\cdot\xi'/\xi_N+r} \big|\underbrace{\tilde\chi(x'\cdot\xi'+x_N\xi_N,\xi)}_{\overset{(23)}{=}\chi(x,\xi)}-\chi(P,\xi)\big|\,dx_N && (\text{since } x'\cdot\xi'+x_N\xi_N=\tilde x_N\xi_N)\\
&\quad\le \fint_{(-r,r)^{N-1}} dx' \frac{1}{2r}\int_{P_N-\tilde r}^{P_N+\tilde r} \big|\chi(x,\xi)-\chi(P,\xi)\big|\,dx_N \\
&\quad\le C\fint_{P+(-\tilde r,\tilde r)^N} \big|\chi(x,\xi)-\chi(P,\xi)\big|\,dx \to 0 && \text{as } r\to 0,
\end{aligned}$$

where we used that $|x'\cdot\xi'| \le r\sqrt{N-1}$ for $x' \in (-r, r)^{N-1}$ and $\tilde r = (\sqrt{N-1}/|\xi_N| + 1)r$. Thus, $P_N\xi_N$ is a Lebesgue point of $\tilde\chi(\cdot, \xi)$. In particular, we have by Fubini's theorem, for every $\alpha > 0$,

$$\begin{aligned}
&\fint_{-\alpha r}^{\alpha r} d\tilde t \fint_{P_N\xi_N-r|\xi_N|+\tilde t}^{P_N\xi_N+r|\xi_N|+\tilde t} \big|\tilde\chi(t,\xi)-\tilde\chi(P_N\xi_N,\xi)\big|\,dt \\
&\quad= \frac{1}{4\alpha|\xi_N|r^2}\int_{-\alpha r}^{\alpha r}\int_{P_N\xi_N-r(|\xi_N|+\alpha)}^{P_N\xi_N+r(|\xi_N|+\alpha)} \big|\tilde\chi(t,\xi)-\tilde\chi(P_N\xi_N,\xi)\big|\,\mathbb{1}_{(P_N\xi_N-r|\xi_N|+\tilde t,\,P_N\xi_N+r|\xi_N|+\tilde t)}(t)\,dt\,d\tilde t \\
&\quad= \frac{1}{4\alpha|\xi_N|r^2}\int_{P_N\xi_N-r(|\xi_N|+\alpha)}^{P_N\xi_N+r(|\xi_N|+\alpha)} \big|\tilde\chi(t,\xi)-\tilde\chi(P_N\xi_N,\xi)\big|\,dt\int_{-\alpha r}^{\alpha r}\mathbb{1}_{(-P_N\xi_N-r|\xi_N|+t,\,-P_N\xi_N+r|\xi_N|+t)}(\tilde t)\,d\tilde t \\
&\quad\le \frac{1}{2|\xi_N|r}\int_{P_N\xi_N-r(|\xi_N|+\alpha)}^{P_N\xi_N+r(|\xi_N|+\alpha)} \big|\tilde\chi(t,\xi)-\tilde\chi(P_N\xi_N,\xi)\big|\,dt \to 0 \quad \text{as } r\to 0.
\end{aligned} \tag{24}$$

*Step 3: proof of (21).* For $\xi \in \mathcal{D}$, we have, for small $r > 0$,

$$\begin{aligned}
&\fint_{(-r,r)^{N-1}}\int_{-1}^{1}\big|\chi(x,\xi)-\tilde\chi(x_N\xi_N,\xi)\big|\,dx'\,dx_N \\
&\qquad\overset{(23)}{=} \fint_{(-r,r)^{N-1}}\int_{-1}^{1}\big|\tilde\chi(x'\cdot\xi'+x_N\xi_N,\xi)-\tilde\chi(x_N\xi_N,\xi)\big|\,dx'\,dx_N \\
&\qquad\le \frac{1}{|\xi_N|}\sup_{|\tilde t|\le r\sqrt{N-1}}\int_{-|\xi_N|}^{|\xi_N|}\big|\tilde\chi(t+\tilde t,\xi)-\tilde\chi(t,\xi)\big|\,dt \quad (\text{since } t=x_N\xi_N)
\end{aligned}$$

because $|x'\cdot\xi'| \le r\sqrt{N-1}$. Since the 1-dimensional function $t \mapsto \tilde\chi(t, \xi)$ belongs to $L^\infty$, its $L^1$-modulus of continuity present in the above right-hand side tends to 0 as $r \to 0$, which leads to

$$\lim_{r\to 0}\fint_{(-r,r)^{N-1}}\int_{-1}^{1}\big|\chi(x,\xi)-\tilde\chi(x_N\xi_N,\xi)\big|\,dx'\,dx_N = 0.$$

This formula can be interpreted as the notion of the trace of $\chi(\cdot,\xi)$ on the segment $L$ and yields (21). Indeed, due to (9), we define for a.e. $x_N \in (-1,1)$,

$$\tilde{u}(x_N) = \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \xi\, \tilde{\chi}(x_N\xi_N, \xi)\, d\mathcal{H}^{N-1}(\xi)$$

and we obtain, by Fubini's theorem,

$$\int_{(-r,r)^{N-1}} \int_{-1}^{1} \big|u(x', x_N) - \tilde{u}(x_N)\big|\, dx'\, dx_N \overset{(9)}{\leq} \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} \bigg( \int_{(-r,r)^{N-1}} \int_{-1}^{1} \big|\chi(x,\xi) - \tilde{\chi}(x_N\xi_N, \xi)\big|\, dx'\, dx_N \bigg)\, d\mathcal{H}^{N-1}(\xi) \xrightarrow{r\to 0} 0,$$

where we used the dominated convergence theorem.

*Step 4: proof of* (22). By Step 3, we deduce that

$$⨍_{(-r,r)^{N-1}} u(x', \cdot)\, dx' \xrightarrow{r\to 0} \tilde{u} \quad \text{in } L^1((-1,1));$$

therefore, the first statement in (22) follows immediately. Moreover,

$$\begin{aligned} ⨍_{-1}^{1} \big||\tilde{u}(x_N)| - 1\big|\, dx_N &= ⨍_{(-r,r)^{N-1}} \int_{-1}^{1} \big||\tilde{u}(x_N)| - |u(x', x_N)|\big|\, dx'\, dx_N \\ &\leq ⨍_{(-r,r)^{N-1}} \int_{-1}^{1} \big|\tilde{u}(x_N) - u(x', x_N)\big|\, dx'\, dx_N \overset{(21)}{\longrightarrow} 0 \quad \text{as } r \to 0; \end{aligned}$$

thus, $|\tilde{u}(x_N)| = 1$ for $\mathcal{H}^1$-a.e. $x_N \in (-1,1)$.

*Step 5: conclusion.* Let $P = (0, \dots, 0, P_N) \in \text{Leb}$ be a Lebesgue point of $u$ with $P_N \in (-1,1)$. We want to show that $P_N$ is a Lebesgue point of $\tilde{u}$ and $\tilde{u}(P_N) = u(P)$. For that, we know by Lemma 15 that $P$ is a Lebesgue point of $\chi(\cdot,\xi)$ for every direction $\xi \in \mathbb{S}^{N-1}$ with $u(P)\cdot\xi \neq 0$. If in addition $\xi \in \mathcal{D}$, we know by Step 2 that $P\cdot\xi$ is also a Lebesgue point of $\tilde{\chi}(\cdot,\xi)$. By the same argument as in Step 3, we have

$$\begin{aligned} &⨍_{P+(-r,r)^N} |u(x', x_N) - \tilde{u}(x_N)|\, dx'\, dx_N \\ &\quad\leq \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} ⨍_{P+(-r,r)^N} \big| \underbrace{\chi(x,\xi)}_{\overset{(23)}{=} \tilde{\chi}(x'\cdot\xi' + x_N\xi_N, \xi)} - \tilde{\chi}(x_N\xi_N, \xi)\big|\, dx'\, dx_N\, d\mathcal{H}^{N-1}(\xi) \\ &\quad\leq \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1}(\xi) \bigg[ ⨍_{P+(-r,r)^N} \big|\tilde{\chi}(x'\cdot\xi' + x_N\xi_N, \xi) - \tilde{\chi}(P_N\xi_N, \xi)\big|\, dx \\ &\qquad\qquad + ⨍_{P_N-r}^{P_N+r} \big|\tilde{\chi}(x_N\xi_N, \xi) - \tilde{\chi}(P_N\xi_N, \xi)\big|\, dx_N \bigg] \\ &\quad\leq \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1}(\xi) ⨍_{(-r,r)^{N-1}} dx' ⨍_{P_N\xi_N - r|\xi_N| + x'\cdot\xi'}^{P_N\xi_N + r|\xi_N| + x'\cdot\xi'} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N, \xi)\big|\, dt \\ &\qquad\qquad + \frac{1}{V_{N-1}} \int_{\mathbb{S}^{N-1}} d\mathcal{H}^{N-1}(\xi) ⨍_{P_N\xi_N - r|\xi_N|}^{P_N\xi_N + r|\xi_N|} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N, \xi)\big|\, dt. \end{aligned}$$

Using the dominated convergence theorem twice, we conclude that the above right-hand side vanishes as $r \to 0$. Indeed, the second integrand converges to 0 as $r \to 0$ by Step 2 for a.e. $\xi \in \mathbb{S}^{N-1}$. For the first integrand, we proceed as follows: for $\mathcal{H}^{N-1}$-a.e. direction $\xi$, we may assume that $|\xi'| > 0$ and $\xi_N \neq 0$ so that there exists a rotation $R' \in \mathrm{SO}(N-1)$ with $R'\xi' = |\xi'|e_1$ and we have by the change of variables $\tilde{x}' = R'x'$ and $\hat{r} = r\sqrt{N-1}$,

$$\fint_{(-r,r)^{N-1}} dx' \fint_{P_N\xi_N - r|\xi_N| + x'\cdot\xi'}^{P_N\xi_N + r|\xi_N| + x'\cdot\xi'} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N,\xi)\big|\,dt$$
$$\leq C \fint_{\{|\tilde{x}'|<\hat{r}\}} d\tilde{x}' \fint_{P_N\xi_N - r|\xi_N| + \tilde{x}_1|\xi'|}^{P_N\xi_N + r|\xi_N| + \tilde{x}_1|\xi'|} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N,\xi)\big|\,dt$$
$$\leq C \fint_{-|\xi'|\hat{r}}^{|\xi'|\hat{r}} \fint_{P_N\xi_N - r|\xi_N| + \tilde{t}}^{P_N\xi_N + r|\xi_N| + \tilde{t}} \big|\tilde{\chi}(t,\xi) - \tilde{\chi}(P_N\xi_N,\xi)\big|\,dt\,d\tilde{t} \xrightarrow{(24)} 0 \quad \text{as } r \to 0. \qquad \square$$

## 6. Proof of Theorem 8

We start by showing some preliminary results that reveal the geometric consequences of the kinetic formulation (8). The following lemma is the first step in proving that $u$ is constant along the characteristics and is reminiscent of the ideas presented in [Jabin, Otto, and Perthame 2002]:

**Lemma 19.** *Let $\Omega \subset \mathbb{R}^N$ be an open set such that $L = \{0\}^{N-1} \times [-1,1] \subset \Omega$ and $u : \Omega \to \mathbb{S}^{N-1}$ be a Lebesgue-measurable vector field satisfying the kinetic formulation (8). Assume that the origin $O \in \Omega \cap \mathrm{Leb}$ is a Lebesgue point of $u$ and $u(O) = e_N$. Then for every Lebesgue point $x_N \in (-1,1)$ of $\tilde{u}$, we have*

$$\tilde{u}(x_N) = \pm e_N,$$

*where $\tilde{u}$ is the trace of $u$ on $L$ defined at Proposition 18.*

*Proof.* Without loss of generality we assume that $\Omega$ is a convex open neighborhood of $L$. By Proposition 18, we know that $O$ is also a Lebesgue point of $\tilde{u}$ and $\tilde{u}(0) = e_N$; moreover, $|\tilde{u}| = 1$ a.e. in $(-1,1)$. Let $x_N \in (-1,1) \setminus \{0\}$ be a Lebesgue point of $\tilde{u}$ such that $\mathcal{H}^{N-1}$-a.e. $z \in \Omega \cap (x_N e_N + e_N^{\perp})$ is a Lebesgue point of $u$ and such that the following holds true (see Proposition 18):

$$\lim_{r\to 0} \fint_{(-r,r)^{N-1}} \big|u(x', x_N) - \tilde{u}(x_N)\big|\,dx' = 0. \tag{25}$$

Our goal is to prove that the component $\tilde{u}_i(x_N)$ of $\tilde{u}(x_N)$ in direction $e_i$ vanishes for every $i = 1, \dots, N-1$. For that, we follow the ideas in [Jabin, Otto, and Perthame 2002]. Let $\varepsilon > 0$ be small and define the following subsets $E_i^-$ and $E_i^+$ (depending on $\varepsilon$) of the hyperplane $(x_N e_N + e_N^{\perp})$ for $1 \leq i \leq N-1$:

$$E_i^{\pm} = \big\{z \in \Omega \cap \mathrm{Leb} : z_N = x_N,\ \varepsilon|x_N| \geq \pm z_i > 0\big\}.$$

By our assumption, the sets $E_i^{\pm}$ contain many points (e.g., for $i = 1$, the set $E_1^+$ covers the $(N-1)$-parallelepiped $(0,r) \times (-r,r)^{N-2} \times \{x_N\}$ up to a set of zero $\mathcal{H}^{N-1}$-measure for $r < \varepsilon$). For $z \in E_i^+$, we set $y = -z_i e_N + x_N e_i$ if $x_N > 0$ and $y = z_i e_N - x_N e_i$ if $x_N < 0$. Obviously, $z \cdot y = 0$; that is, $y \in z^{\perp}$.

By the convexity of Ω, the segment $[Oz]$ lies in Ω so that by Proposition 17 we have if $x_N > 0$ (resp. $x_N < 0$), then $u(O)\cdot y = -z_i < 0$ (resp. $u(O)\cdot y = z_i > 0$) so that $u(z)\cdot y \le 0$ (resp. $\ge 0$). It follows that

$$u_i(z) \le \frac{z_i}{x_N} u_N(z) \le \varepsilon \quad \left(\text{resp. } u_i(z) \ge \frac{-z_i}{|x_N|} u_N(z) \ge -\varepsilon\right),$$

because $|u_N(z)| \le 1$. Similarly, for $z \in E_i^-$, one uses $y = z_i e_N - x_N e_i$ if $x_N > 0$ and $y = -z_i e_N + x_N e_i$ if $x_N < 0$ and deduces that $u_i(z) \ge -\varepsilon$ if $x_N > 0$ and $u_i(z) \le \varepsilon$ if $x_N < 0$. We conclude that $\tilde{u}_i(x_N) \in [-\varepsilon, \varepsilon]$. Indeed, let us set $i = 1$ for simplicity of notation; by (25), we have

$$\tilde{u}_1(x_N) = \lim_{r\to 0} \fint_{(0,r)\times(-r,r)^{N-2}} u_1(x', x_N)\, dx' \le \varepsilon \quad \text{if } x_N > 0 \quad (\text{resp. } \ge -\varepsilon \text{ if } x_N < 0)$$

and also,

$$\tilde{u}_1(x_N) = \lim_{r\to 0} \fint_{(-r,0)\times(-r,r)^{N-2}} u_1(x', x_N)\, dx' \ge -\varepsilon \quad \text{if } x_N > 0 \quad (\text{resp. } \le \varepsilon \text{ if } x_N < 0).$$

Passing to the limit $\varepsilon \to 0$, we conclude that $\tilde{u}_i(x_N) = 0$ for $i = 1$ (similarly, for every $1 \le i \le N-1$). Obviously, $\mathcal{H}^1$-a.e. $x_N \in (-1, 1)$ satisfies this property. As a consequence, if $P_N \in (-1, 1)$ is a Lebesgue point of $\tilde{u}$ then for every $1 \le i \le N-1$,

$$\tilde{u}_i(P_N) = \lim_{r\to 0} \fint_{P_N - r}^{P_N + r} \tilde{u}_i(x_N)\, dx_N = 0.$$

Since $|\tilde{u}(P_N)| = 1$, we deduce that $\tilde{u}_N(P_N) = \pm 1$, that is, $\tilde{u}(P_N) = \pm e_N$. □

We now prove the main result:

*Proof of Theorem 8.* We first treat the case where Ω is a ball and then the general case of a connected open set.

Case I: Ω is a ball. Since $u$ is not a constant vector field, there exist two Lebesgue points $P_0, P_1 \in \Omega \cap \text{Leb}$ of $u$ such that

$$u(P_0) \ne u(P_1).$$

Let $D_0$ (resp. $D_1$) be the line directed by $u(P_0)$ (resp. $u(P_1)$) that passes through $P_0$ (resp. $P_1$).

*Step 1: we show that $D_0$ and $D_1$ are coplanar.* Assume by contradiction that $D_0$ and $D_1$ are not coplanar; in particular $|u(P_0)\cdot u(P_1)| < 1$. Set $A \in D_0$ and $B \in D_1$ such that

$$0 < |A - B| = \min_{x\in D_0, y \in D_1} |x - y|.$$

Obviously, the segment $[AB]$ is orthogonal to $D_0$ and $D_1$. Let $O$ be the middle point of the segment $[AB]$ (see Figure 2). Let

$$w_1 = u(P_0), \quad w_2 = \frac{\overrightarrow{OA}}{|\overrightarrow{OA}|} \quad \text{and} \quad w_3 = \alpha u(P_0) + \beta u(P_1),$$

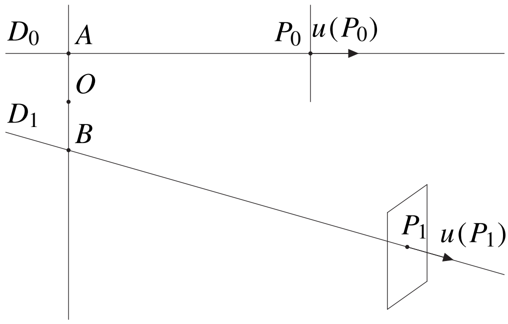


**Figure 2.** Two noncoplanar lines $D_0$ and $D_1$.

where

$$\alpha = \frac{-u(P_0)\cdot u(P_1)}{\sqrt{1-\big(u(P_0)\cdot u(P_1)\big)^2}} \quad \text{and} \quad \beta = \frac{1}{\sqrt{1-\big(u(P_0)\cdot u(P_1)\big)^2}} > 0. \tag{26}$$

The choice of $\alpha$ and $\beta$ is done in order to ensure that $w_1 \cdot w_3 = 0$ and $|w_3|^2 = 1$, which finally yields the orthonormal basis $w_1$, $w_2$ and $w_3$. Note now that the vectors $u(P_0)$ and $u(P_1)$ have the following components in the basis $(w_1, w_2, w_3)$:

$$u(P_0) = (1, 0, 0) \quad \text{and} \quad u(P_1) = \Big(-\frac{\alpha}{\beta}, 0, \frac{1}{\beta}\Big).$$

We want to find the expression of $\overrightarrow{P_0P_1}$ in that basis, too. For that, we have

$$\overrightarrow{P_0P_1} = \overrightarrow{P_0A} + \overrightarrow{AB} + \overrightarrow{BP_1},$$

which implies the existence of three real numbers $\lambda, \tilde{\lambda}, \hat{\lambda} \in \mathbb{R}$ with $\tilde{\lambda} \neq 0$ such that

$$\overrightarrow{P_0P_1} = \lambda w_1 + \tilde{\lambda} w_2 + \hat{\lambda} u(P_1) = \lambda w_1 + \tilde{\lambda} w_2 + \hat{\lambda}\Big(\frac{1}{\beta} w_3 - \frac{\alpha}{\beta} w_1\Big).$$

Thus, $\overrightarrow{P_0P_1}$ has the following components in the basis $(w_1, w_2, w_3)$:

$$\overrightarrow{P_0P_1} = \Big(\lambda - \frac{\alpha}{\beta}\hat{\lambda}, \tilde{\lambda}, \frac{\hat{\lambda}}{\beta}\Big).$$

Define the vector $\xi := (1, s, -\beta) \neq 0$, written in our basis where

$$s := \frac{\hat{\lambda}(\alpha+\beta)}{\beta\tilde{\lambda}} - \frac{\lambda}{\tilde{\lambda}}.$$

Then $[P_0P_1] \subset \Omega$ (since $\Omega$ is a ball) and

$$\overrightarrow{P_0P_1} \cdot \xi = 0, \quad \text{i.e., } \xi \in P_0P_1^{\perp},$$

$$u(P_0)\cdot \xi = 1 > 0, \quad u(P_1)\cdot\xi = u(P_0)u(P_1) - 1 < 0,$$

which contradicts Proposition 17. Thus, $D_0$ and $D_1$ are coplanar.

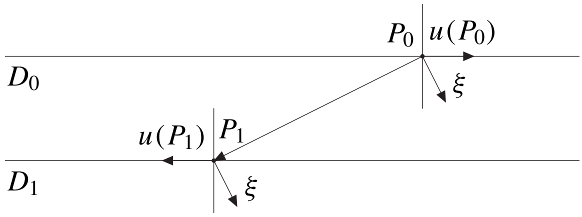


**Figure 3.** Two parallel lines $D_0$ and $D_1$.

*Step 2: we show that $D_0$ and $D_1$ must intersect* ($D_0$ *might coincide with* $D_1$). Assume by contradiction that $D_0$ and $D_1$ are parallel and $D_0 \neq D_1$. This means that $u(P_0) = -u(P_1)$ (because of our choice $u(P_0) \neq u(P_1)$). Set $(w_1, w_2)$ to be an orthonormal basis in the 2-dimensional plane $\Pi$ determined by $D_0$ and $D_1$ with $w_1 = u(P_0)$. In the basis $(w_1, w_2)$, we write $\overrightarrow{P_0P_1} = (\lambda, \tilde{\lambda})$, where $\tilde{\lambda} \neq 0$ (since $D_0 \neq D_1$), and set $\xi = (-\tilde{\lambda}, \lambda)$ to be an orthogonal vector to $\overrightarrow{P_0P_1}$ in $\Pi$ (see Figure 3). Then one checks that $u(P_0) \cdot \xi = -\tilde{\lambda}$ and $u(P_1) \cdot \xi = \tilde{\lambda}$ have different signs, which again contradicts Proposition 17.

*Step 3: there exists a point $O \in D_0$ with $O \neq P_0$, $P_1$ and a sign $\gamma \in \{\pm 1\}$ such that*

$$u(P_i) = \gamma \frac{\overrightarrow{OP_i}}{|\overrightarrow{OP_i}|}, \quad i = 0, 1.$$

If $D_0 = D_1$, then $u(P_0) = -u(P_1)$, so any point $O \in D_0$ located between $P_0$ and $P_1$ leads to the conclusion. Otherwise, $D_0 \neq D_1$ and we define $\{O\} = D_0 \cap D_1$. First, we prove that $O \neq P_0, P_1$. Assume by contradiction that $O = P_0 \in D_0 \cap D_1$. Then by Proposition 18 we know that $P_0$ and $P_1$ are Lebesgue points of the trace $\tilde{u}$ of $u$ on the segment $D_1 \cap \Omega$ (directed by $u(P_1)$) with $\tilde{u}(P_0) = u(P_0)$ and $\tilde{u}(P_1) = u(P_1)$ so that by Lemma 19, we should have $u(P_0)$ is parallel with $u(P_1)$, which is a contradiction with $D_0 \neq D_1$. So, $O \neq P_0, P_1$. Next, note that for any orthogonal vector $\xi$ to $\overrightarrow{P_0P_1}$ in the plane determined by $D_0$ and $D_1$, we have by Proposition 17 that $u(P_0) \cdot \xi$ and $u(P_1) \cdot \xi$ have the same sign, i.e.,

$$(u(P_0) \cdot \xi) \cdot (u(P_1) \cdot \xi) \geq 0. \tag{27}$$

Write now

$$\overrightarrow{OP_0} = \lambda u(P_0) \quad \text{and} \quad \overrightarrow{OP_1} = \tilde{\lambda} u(P_1)$$

with $\lambda, \tilde{\lambda}$ nonzero real numbers. The conclusion of Step 3 is equivalent to proving that $\lambda$ and $\tilde{\lambda}$ have the same sign. For that, as in Step 1, we choose the orthonormal basis $w_1 = u(P_0)$ and $w_2 = \alpha u(P_0) + \beta u(P_1)$ with $\alpha \in \mathbb{R}$ and $\beta > 0$ given in (26) (recall that $|u(P_0) \cdot u(P_1)| < 1$ because of the assumption $D_0 \neq D_1$). Since $\overrightarrow{P_0P_1} = \overrightarrow{OP_1} - \overrightarrow{OP_0} = \tilde{\lambda} u(P_1) - \lambda u(P_0)$, we write, in the basis $(w_1, w_2)$,

$$u(P_0) = (1, 0), \quad u(P_1) = \left(-\frac{\alpha}{\beta}, \frac{1}{\beta}\right), \quad \overrightarrow{P_0P_1} = \left(-\lambda - \frac{\alpha}{\beta}\tilde{\lambda}, \frac{\tilde{\lambda}}{\beta}\right).$$

Then for the orthogonal vector $\xi = (\tilde{\lambda}, \lambda\beta + \alpha\tilde{\lambda}) \neq 0$ to $\overrightarrow{P_0P_1}$, we have by (27) that

$$0 \leq (u(P_0) \cdot \xi) \cdot (u(P_1) \cdot \xi) = \tilde{\lambda} \cdot \lambda.$$

*Step 4: conclusion*. For every Lebesgue point $P \in \text{Leb} \cap \Omega$ of $u$, we consider the line $D$ passing through $P$ and directed by $u(P)$. Call $\mathcal{D}$ the set of these lines. Obviously, $\mathcal{D}$ covers $\mathcal{H}^N$-almost all of the ball $\Omega$ (since $\mathcal{H}^N(\Omega \setminus \text{Leb}) = 0$); in particular, $\mathcal{D}$ is not planar. By Step 1, we know that every two lines in $\mathcal{D}$ are coplanar. Then Proposition 9 (whose proof is presented below) implies that either all these lines are parallel, or they pass through the same point $O$. Since $u$ is nonconstant, we deduce by Step 2 that only the last situation holds true. By Step 3, we conclude that $u = \gamma u^\star(\cdot - O)$ a.e. in $\Omega$.

Case II: $\Omega$ is a connected open set. By Case I, we know that in every open ball $B \subset \Omega$ around a Lebesgue point of $u$, the vector field $u$ is either a vortex-type vector field in $B$, or $u$ is constant in $B$. Since $u$ is nonconstant in $\Omega$, there exists a Lebesgue point $P_0$ of $u$ and a ball $B_0 \subset \Omega$ around $P_0$ such that $u$ is a vortex-type vector field in $B_0$; say for simplicity $u = u^\star$. Let $P \neq P_0$ be any other Lebesgue point of $u$. Since $\Omega$ is path-connected, there exists a path $\Gamma \subset \Omega$ from $P_0$ to $P$. Then we can cover the path $\Gamma$ by a finite number of open balls $\{B_j\}_{0 \le j \le n}$ such that $P \in B_n$, $B_j \cap B_{j+1} \neq \varnothing$ for $0 \le j \le n-1$ and $u$ is either constant or a vortex-type vector field in any $B_j$. Since $u = u^\star$ in $B_0$ and $B_0 \cap B_1$ is a nonempty open set, the analysis in Case I yields $u = u^\star$ in $B_1$ and by induction, $u = u^\star$ in $B_n$, which is a neighborhood of $P$. □

Let us now present the proof of the geometric result in Proposition 9, which is independent of the previous results:

*Proof of Proposition 9*. Assume that there are two lines $D_0, D_1 \in \mathcal{D}$ that are not collinear. Since $D_0$ and $D_1$ are coplanar, they intersect at a point $P$. Call $\Pi$ the plane determined by $D_0$ and $D_1$. We show that all the lines in $\mathcal{D}$ pass through $P$. Let $D_2 \in \mathcal{D}$ be any line not included in $\Pi$ (such a line exists because $\mathcal{D}$ is not planar). We know that $D_2$ is coplanar with $D_0$ and $D_1$, respectively. Then $D_2$ cannot be parallel with $D_0$ (otherwise, $D_2 \parallel D_0$ and $D_2 \cap D_1 \neq \varnothing$ imply that $D_2 \subset \Pi$, which is a contradiction with our assumption). Similarly, $D_2$ cannot be parallel with $D_1$. Therefore, $D_2$ intersects both $D_0$ and $D_1$. Since $D_2$ is not included in $\Pi$, the intersection points coincide with $P$. Let now $D_3 \in \mathcal{D}$ be any line included in $\Pi$ (different than $D_0$ and $D_1$). Then $D_3$ is not included in the plane determined by $D_1$ and $D_2$. The previous argument leads again to $P \in D_3$, which concludes our proof. □

## 7. Vector fields of vortex-line type

We will prove the characterization of the weakened kinetic formulation (10) in Theorem 10. This result is in the spirit of Corollary 14 and leads to vector fields that have vortex-line singularities.

*Proof of Theorem 10*. For $x \in \mathbb{R}^N$, recall the notation $x = (x', x_N)$ with $x' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$. As the result is local in the set $\{u_N \neq \pm 1\}$, we will assume that $\omega = B' \times (-1, 1)$ is included in that set, where $B'$ is the unit ball in $\mathbb{R}^{N-1}$. Let $\xi' \in \mathbb{S}^{N-2}$ and $\xi = (\xi', 0) \in \mathcal{E}$. Since $e_N \in \xi^\perp$, we deduce by (10) that

$$e_N \cdot \nabla_x \chi(\cdot, \xi) = \partial_N \chi(\cdot, \xi) = 0 \quad \text{in } \mathcal{D}'(\omega). \tag{28}$$

We know that the point $(x', t)$ is a Lebesgue point of $\chi(\cdot, \xi)$ for $\mathcal{H}^{N-1}$-a.e. $x' \in B'$ and $\mathcal{H}^1$-a.e. $t \in (-1, 1)$ and the convolution argument in the proof of Proposition 17 yields

$$\chi(x, \xi) = \chi(x + te_N, \xi) \quad \text{for } \mathcal{H}^N\text{-a.e. } x \in \omega \text{ and } \mathcal{H}^1\text{-a.e. } t.$$

Then one can define the measurable function $\tilde{\chi}(\,\cdot\,,\xi') : B' \to \{0,1\}$ by

$$\tilde{\chi}(x',\xi') := \chi(x,\xi) = \mathbb{1}_{\{x\in\omega : u'(x)\cdot\xi'>0\}} \quad \text{for } \mathcal{H}^N\text{-a.e. } x=(x',t)\in\omega.$$

Set

$$\tilde{u}(x') = \frac{1}{V_{N-2}} \int_{\mathbb{S}^{N-2}} \xi' \tilde{\chi}(x',\xi')\, d\mathcal{H}^{N-2}(\xi'), \qquad x'\in B'.$$

Thanks to (9),

$$\tilde{u}(x') = \frac{u'(x)}{|u'(x)|} \quad \text{for } \mathcal{H}^N\text{-a.e. } x=(x',t)\in\omega\subset\{|u'|>0\}.$$

In particular, $\tilde{\chi}(x',\xi') = \mathbb{1}_{\{x'\in B' : \tilde{u}(x')\cdot\xi'>0\}}$ in $B'$ for every $\xi'\in\mathbb{S}^{N-2}$. Therefore, we deduce by (10) that $\tilde{u} : B' \to \mathbb{S}^{N-2}$ satisfies

$$\forall \xi'\in\mathbb{S}^{N-2},\ \forall v'\in(\xi')^{\perp}, \qquad v'\cdot\nabla'_{x'}\tilde{\chi}(x',\xi') = 0 \text{ in } B',$$

where $\nabla'_{x'} = (\partial_1,\ldots,\partial_{N-1})$. As $N-1\geq 3$, Theorem 8 yields either $\tilde{u}(x') = w'$ for almost every $x'\in B'$, where $w'\in\mathbb{S}^{N-2}$ is a constant vector, or $\tilde{u}(x') = \gamma(x'-P')/|x'-P'|$ for almost every $x'\in B'$, where $\gamma\in\{\pm1\}$ and $P'\in\mathbb{R}^{N-1}$ is some point. This means that for a.e. $x\in\omega$,

$$\text{either} \quad u'(x) = |u'(x)|w' \quad \text{or} \quad u'(x) = \gamma|u'(x)|\frac{x'-P'}{|x'-P'|}.$$

<u>Case 1</u>. Let $u'(x) = |u'(x)|w'$ for a.e. $x\in\omega$. By (11), we have for $k\in\{1,\ldots,N-1\}$,

$$\partial_k u_N = \partial_N u_k = w_k \partial_N(|u'|) \quad \text{in } \omega, \tag{29}$$

which yields, for all $k, j\in\{1,\ldots,N-1\}$,

$$w_j\partial_k u_N = w_k\partial_j u_N \quad \text{in } \omega.$$

Therefore, $u_N(x) = g(\alpha, x_N)$ in $\omega$ for some 2-dimensional function $g$ with the new variable $\alpha := \alpha(x) = x'\cdot w'$. Moreover, by (29), the function $g$ satisfies the following: since $w_k\neq 0$ for some $k\in\{1,\ldots,N-1\}$ (because $w\in\mathbb{S}^{N-1}$), the equation $|u'|^2 + u_N^2 = 1$ a.e. in $\omega$ implies

$$w_k\partial_\alpha g = \partial_k u_N \overset{(29)}{=} w_k\partial_N(|u'|) = w_k\partial_N(\sqrt{1-g^2}).$$

The Poincaré lemma yields the existence of a stream function $\psi(\alpha, x_N)$ such that $g = \partial_N\psi$ and $\sqrt{1-g^2} = \partial_\alpha\psi$ so that $u(x) = \nabla_x[\psi(\alpha, x_N)]$ and therefore, $\psi$ satisfies the 2-dimensional eikonal equation

$$(\partial_\alpha\psi)^2 + (\partial_N\psi)^2 = 1.$$

<u>Case 2</u>. Let $u'(x) = \gamma|u'(x)|(x'-P')/|x'-P'|$ for a.e. $x\in\omega$. As above, we have, for $k\in\{1,\ldots,N-1\}$,

$$\partial_k u_N = \partial_N u_k = \gamma\frac{x_k-P_k}{|x'-P'|}\partial_N(|u'|) \quad \text{in } \omega \tag{30}$$

and we deduce that, for all $k, j\in\{1,\ldots,N-1\}$,

$$(x_j-P_j)\partial_k u_N = (x_k-P_k)\partial_j u_N \quad \text{in } \omega.$$

Therefore, $u_N(x) = g(\alpha, x_N)$ in $\omega$ for some 2-dimensional function $g$ with the new variable $\alpha := \alpha(x) = |x'|$. By (30), we conclude as above that there exists a stream function $\psi$ solving the eikonal equation in the variables $(\alpha, x_N)$ such that

$$u(x) = \nabla_x[\psi(\alpha, x_N)]. \qquad \square$$

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