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Alex Iosevich, Azita Mayeli and Jonathan Pakianathan

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THE FUGLEDE CONJECTURE HOLDS IN $\mathbb{Z}_p \times \mathbb{Z}_p$

ALEX IOSEVICH, AZITA MAYELI AND JONATHAN PAKIANATHAN

In this paper we study subsets E of $\mathbb{Z}_p^d$ such that any function $f : E \to \mathbb{C}$ can be written as a linear combination of characters orthogonal with respect to E. We shall refer to such sets as spectral. In this context, we prove the Fuglede conjecture in $\mathbb{Z}_p^2$, which says in this context that $E \subset \mathbb{Z}_p^2$ is spectral if and only if E tiles $\mathbb{Z}_p^2$ by translation. Arithmetic properties of the finite field Fourier transform, elementary Galois theory and combinatorial geometric properties of direction sets play the key role in the proof. The proof relies to a significant extent on the analysis of direction sets of Iosevich et al. (*Integers* **11** (2011), art. id. A39) and the tiling results of Haessig et al. (2011).

1. Introduction

Let $E \subset \mathbb{Z}_p^d$, where $\mathbb{Z}_p$, p prime, is the cyclic group of size p and $\mathbb{Z}_p^d$ is the d-dimensional vector space over $\mathbb{Z}_p$. We say that $L^2(E)$ has an orthogonal basis of exponentials (indexed by A) if the following conditions hold:

- (completeness) There exists $A \subset \mathbb{Z}_p^d$ such that for every function $f : E \to \mathbb{C}$ there exist complex numbers $\{c_a\}_{a\in A}$, $A \subset \mathbb{Z}_p^d$, such that
$$f(x) = \sum_{a\in A} \chi(x \cdot a)c_a$$
for all $x \in E$ where $\chi(u) = e^{2\pi iu/p}$. We shall refer to A as a *spectrum* of E. The expansion above can be applied to functions $f : \mathbb{Z}_p^d \to \mathbb{C}$ by restricting them to E but the equality holds only for $x \in E$.
- (orthogonality) The relation
$$\sum_{x\in E} \chi(x \cdot (a - a')) = 0$$
holds for every $a, a' \in A$, $a \neq a'$.

If these conditions hold, we refer to $E \subset \mathbb{Z}_p^d$ as a *spectral* set.

Definition 1.1 (spectral pair). A *spectral pair* (E, A) in $V = \mathbb{Z}_p^d$ is a spectral set E with an orthogonal basis of exponentials indexed by A.

Definition 1.2 (tiling pair). A *tiling pair* (E', A') consists of $E', A' \subset \mathbb{Z}_p^d$ such that every element $v \in V$ can be written uniquely as a sum $v = e' + a'$, $e' \in E'$, $a' \in A'$. Equivalently, (E', A') is a tiling pair if

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$\sum_{a' \in A'} E'(x-a') \equiv 1$ for every $x \in V$. We say that E' *tiles* V by translation if there exists $A' \subset V$ such that (E', V') is a tiling pair. Here and throughout $E(x)$ is the indicator function of E.

The study of the relationship between exponential bases and tiling has its roots in the celebrated Fuglede conjecture in $\mathbb{R}^d$, which says that if $E \subset \mathbb{R}^d$ is of positive Lebesgue measure, then $L^2(E)$ possesses an orthogonal basis of exponentials if and only if E tiles $\mathbb{R}^d$ by translation. Fuglede [1974] proved this conjecture in the case when either the tiling set or the spectrum is a lattice. Katz, Tao and the first author [Iosevich et al. 2003] proved that the Fuglede conjecture holds for convex planar domains.

Terry Tao [2004] disproved the Fuglede conjecture by exhibiting a spectral set in $\mathbb{R}^{12}$ which does not tile. The first step in his argument is the construction of a spectral subset of $\mathbb{Z}_3^5$ of size 6. It is easy to see that this set does not tile because 6 does not divide 3^5. As a by-product, this shows that spectral sets in $\mathbb{Z}_p^d$ do not necessarily tile. See [Kolountzakis and Matolcsi 2006], where the authors also disprove the reverse implication of the Fuglede conjecture. Tao's example raises the natural question of whether and when spectral sets in a variety of settings necessarily tile by translation and vice versa. In this paper we see that the Fuglede conjecture holds in two-dimensional vector spaces over prime fields.

Our main result is the following.

Theorem 1.3. *Let E be a subset of $\mathbb{Z}_p^d$, p an odd prime.*

(i) (*density*) *The space $L^2(E)$ has an orthogonal basis of exponentials indexed by A if and only if $|E| = |A|$ and $\widehat{E}(a-a') = 0$ for all distinct $a, a' \in A$.*

(ii) *If $E \subset \mathbb{Z}_p^d$ is spectral and $|E| > p^{d-1}$ then $E = A = \mathbb{Z}_p^d$.*

(iii) (*divisibility*) *If $E \subset \mathbb{Z}_p^d$ is spectral, then $|E|$ is 1 or a multiple of p.*

(iv) (*Fuglede conjecture in $\mathbb{Z}_p^2$*) *A set $E \subset \mathbb{Z}_p^2$ is a spectral set if and only if E tiles $\mathbb{Z}_p^2$ by translation.*

We note that the Fuglede conjecture holds trivially also in $\mathbb{Z}_p^1$, as a tiling set E must have $|E|$ divide p and thus must be a point or the whole space, and hence is also a spectral set. Conversely, a spectral set E must have size 1 or a multiple of p by the divisibility condition of the theorem above, and so also is either a point or the whole space, and hence is a tiling set. We also note the results of the theorem above also hold for $p = 2$ but we choose to focus on the odd prime case in the rest of the paper. Parts (i)–(iii) extend with no difficulty and indeed imply $|E| \in \{1, 2, 4\}$ if E is either a spectral set or a tiling set. As sets of size 2 are lines, which are both tiling sets and spectral sets, (iv) follows also.

2. Basic properties of spectra

Lemma 2.1. *Suppose that $L^2(E)$ has an orthogonal basis of exponentials and*

$$f : \mathbb{Z}_p^d \to \mathbb{C}.$$

Then the coefficients are given by

$$c_a(f) = |E|^{-1} \sum_{x \in E} \chi(-x \cdot a) f(x).$$

To prove this, observe that if $f(x) = \sum_{a\in A} \chi(x \cdot a)c_a$ for $x \in E$, then

$$|E|^{-1} \sum_{x\in E} \chi(-x \cdot a) f(x) = |E|^{-1} \sum_{x\in E} \sum_{b\in A} \chi(-x \cdot (a-b))c_b(f)$$
$$= |E|^{-1} \sum_{b\in A} \sum_{x\in E} \chi(-x \cdot (a-b))c_b(f) = c_a(f)$$

and the proof is complete.

Lemma 2.2 (delta function). *Suppose that $L^2(E)$ has an orthogonal basis of exponentials with the spectrum A. Let $\delta_0(x) = 1$ if $x = \vec{0}$ and 0 otherwise and suppose $\vec{0} \in E$. Then*

$$\delta_0(x) = |E|^{-1} \sum_{a\in A} \chi(x \cdot a).$$

To prove the lemma, observe that if $f(x) = \delta_0(x)$, then

$$c_a(f) = |E|^{-1} \sum_{x\in E} \chi(-x \cdot a)\delta_0(x) = |E|^{-1}.$$

The conclusion follows from Lemma 2.1.

Lemma 2.3 (Parseval). *Suppose that $L^2(E)$ has an orthogonal basis of exponentials and f is any function on $\mathbb{Z}_p^d$ with values in $\mathbb{C}$. Then*

$$\sum_{a\in A} |c_a(f)|^2 = |E|^{-1} \sum_{x\in E} |f(x)|^2.$$

Lemma 2.4 (density). *Suppose that $L^2(E)$ has an orthogonal basis of exponentials with the spectrum A. Then $|E| = |A|$.*

The set of functions $\{\chi(x \cdot a) : a \in A\}$ is, by completeness, a spanning set for $L^2(E)$ and, by orthogonality, a linearly independent set for $L^2(E)$ and hence is a basis for $L^2(E)$. Thus the cardinality of this set, which is $|A|$, is equal to the dimension of $L^2(E)$, which is $|E|$.

3. Proof of Theorem 1.3

Part (i) of Theorem 1.3 follows easily, as we have seen that if (E, A) is a spectral pair then $|E| = |A|$ and since the orthogonality property can be easily rewritten as $\widehat{E}(a - a') = 0$ for all $a \neq a'$, with $a, a' \in A$. Conversely if (E, A) has the last two properties, it is a spectral pair, as orthogonality implies $\{\chi(x \cdot a) : a \in A\}$ is linearly independent in $L^2(E)$ and $|A| = |E|$ ensures it is a basis and hence that completeness is satisfied.

Definition 3.1 [Iosevich et al. 2011]. We say that two vectors x and x' in $\mathbb{Z}_p^d$ point in the same direction if there exists $t \in \mathbb{F}_q^*$ such that $x' = tx$. Here $\mathbb{F}_q^*$ denotes the multiplicative group of $\mathbb{Z}_p$. Writing this equivalence as $x \sim x'$, we define the set of directions as the quotient

$$\mathcal{D}(\mathbb{Z}_p^d) = \mathbb{Z}_p^d / \sim . \tag{3-1}$$

Similarly, we can define the set of directions determined by $E \subset \mathbb{Z}_p^d$ by

$$\mathcal{D}(E) = E - E/\sim, \tag{3-2}$$

where

$$E - E = \{x - y : x, y \in E\},$$

with the same equivalence relation $\sim$ as in (3-1) above.

The following result, which is one of the two key tools in the proof of our main result, was previously established in [Iosevich et al. 2011].

Theorem 3.2. *A set E does not determine all directions if and only if there is a hyperplane H and $S \subseteq H$ such that E is the graph of a function $f : S \to \mathbb{Z}_p$ over H, which means that relative to some decomposition $\mathbb{Z}_p^d = H \oplus \mathbb{Z}_p$, we have $E = \{(x, f(x)) : x \in S\}$. In particular, if $|E| > p^{d-1}$, every possible direction is determined by E.*

The second main tool in our proof is the following result.

Theorem 3.3 [Haessig et al. 2015, Proposition 3.2]. *Let $E \subset \mathbb{Z}_p^d$. Then $\widehat{E}(m) = 0$ implies that $\widehat{E}(rm) = 0$ for all $r \in \mathbb{Z}_p^*$. Furthermore $\widehat{E}(m) = 0$ for $m \neq 0$ if and only if E is equidistributed on the p hyperplanes $H_t = \{x : x \cdot m = t\}$ for $t \in \mathbb{Z}_p$ in the sense that*

$$\sum_{x \cdot m = t} E(x) = |E \cap H_t|,$$

viewed as a function of t, is constant.

Note this last theorem is a fact about rational-valued functions over prime fields that is *not true* for complex-valued functions in general or over other fields. We give the proof of Theorem 3.3 at the end of the paper for the sake of completeness.

The proof of part (ii) of Theorem 1.3 follows fairly easily from combining Theorems 3.2 and 3.3. Indeed, suppose that $L^2(E)$ has an orthogonal basis of exponentials and $|E| > p^{d-1}$. By Lemma 2.4, $|E| = |A| > p^{d-1}$. By Theorem 3.2, $\mathcal{D}(A) = \mathcal{D}(\mathbb{Z}_p^d)$. Combining this with Theorem 3.3 implies that $\widehat{E}$ vanishes on $\mathbb{Z}_p^d \backslash \vec{0}$. It follows that $E = \mathbb{Z}_p^d$, as claimed.

Part (iii) of Theorem 1.3 is contained in the following result. A spectral pair is called trivial if $(E, A) =$ (point, another point) or $(E, A) = (\mathbb{Z}_p^d, \mathbb{Z}_p^d)$ or $(E, A) = (\varnothing, \varnothing)$. All other spectral pairs are called nontrivial.

Proposition 3.4. *Let p be an odd prime and (E, A) be a nontrivial spectral pair in $\mathbb{Z}_p^d$. Then $|E| = |A| = mp$, where $m \in \{1, 2, 3, \dots, p^{d-2}\}$.*

To prove Proposition 3.4, let (E, A) be a nontrivial spectral pair in $\mathbb{Z}_p^d$. Then part (i) of Theorem 1.3 shows that $|E| = |A|$ and $\widehat{E}(a - a') = 0$ for distinct $a, a' \in A$. Since the spectral pair (E, A) is nontrivial, $2 \leq |E| = |A| \leq p^{d-1}$ also. Thus taking two distinct elements $a, a' \in A$ shows that $\widehat{E}(\alpha) = 0$ for $\alpha = a - a' \neq 0$. Thus E is equidistributed on the p parallel hyperplanes

$$H_t = \{x : x \cdot \alpha = t\},$$

$t \in \mathbb{Z}_p$, by Theorem 3.3. Thus if E has $m \geq 1$ elements per hyperplane we have $|E| = |A| = mp$. Then $1 \leq m \leq p^{d-2}$ since $0 < |E| \leq p^{d-1}$. This proves part (iii) of Theorem 1.3.

Observe that if $d = 2$ and (E, A) is a nontrivial spectral pair, then $|E| = |A| = mp$ implies $|E| \geq p$, while $|E| \leq p$ by part (ii) of Theorem 1.3 and so $|E| = |A| = p$. Furthermore, by Theorem 3.2 above, A is a graph of a function $\mathbb{Z}_p \to \mathbb{Z}_p$ since $|A| = p$ and it does not determine all directions. Finally, since E is equidistributed on a family of p parallel lines and $|E| = p$, we see that E is also a graph of a function $\mathbb{Z}_p \to \mathbb{Z}_p$ with respect to some system of axes. The following is an immediate corollary of Proposition 3.4.

Corollary 3.5. *If E is a spectral set in $\mathbb{Z}_p^2$, p an odd prime, then E is either a point, a graph set of order p or the whole space and hence tiles $\mathbb{Z}_p^2$ in all cases.*

This corollary follows from Proposition 3.4 immediately once one notes that any graph set

$$E = \{(x, f(x)) : x \in \mathbb{Z}_p\}$$

for a function f, with respect to some coordinate systems, tiles $\mathbb{Z}_p^2$ using the tiling partner

$$A = \{(0, t) : t \in \mathbb{Z}_p\}.$$

To complete the proof of the Fuglede conjecture in two dimensions over prime fields, which is the content of part (iv) of Theorem 1.3, it remains to show that any tiling set is spectral since we have just shown that any spectral set tiles.

Proposition 3.6 (sets which tile by translation are spectral). *Let p be an odd prime, and let $E \subseteq \mathbb{Z}_p^2$. Suppose that E tiles $\mathbb{Z}_p^2$ by translation. Then E is a spectral set.*

We shall need the following result. We shall prove it at the end of the paper for the sake of completeness.

Theorem 3.7 [Haessig et al. 2015, Theorem 1.7]. *Let E be a set that tiles $\mathbb{Z}_p^2$. Then $|E| = 1$, p or p^2 and E is a graph if $|E| = p$.*

We include a proof of Theorem 3.7 at the end of this paper for completeness.

The cases $|E| = 1$, p^2 are trivially spectral sets so we may reduce to the case that E is a graph, i.e.,

$$E = \{xe_1 + f(x)e_2 : x \in \mathbb{Z}_p\},$$

where e_1, e_2 is a basis for $\mathbb{Z}_p^2$ and $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a function. By changing the function if necessary we can assume e_2 is orthogonal to e_1 as long as $e_1 \cdot e_1 \neq 0$, i.e., e_1 does not generate an isotropic line. This is always the case if $p \equiv 3 \bmod 4$. In the case when $p \equiv 1 \bmod 4$, it is possible that e_1 generates one of the two isotropic lines

$$\{(t, it) : t \in \mathbb{Z}_p\},$$

where i is one of the two distinct solutions of the equation $x^2+1=0$. The reason this case needs to be treated separately is that $(t_1, it_1)\cdot(t_2, it_2)=0$ for all $t_1, t_2 \in \mathbb{Z}_p$. To deal with this, we note that the other solution of the equation $x^2+1=0$ is given by $-i$ and we take e_2 to be on the other isotropic line in the plane, given by

$$\{(t, -it) : t \in \mathbb{Z}_p\},$$

with e_2 normalized so that $e_1 \cdot e_2 = 1$.

There are two situations to consider.

Case 1: e_1 and e_2 are orthogonal. Then we will take $A = \{xe_1 : x \in \mathbb{Z}_p\}$. To show that (E, A) is a spectral pair, we need only show that the set $\{\chi(ae_1 \cdot x) : a \in \mathbb{Z}_p\}$ is orthogonal in $L^2(E)$. By Theorem 3.3 this happens if and only if $\widehat{E}((a - a')e_1) = 0$ for all distinct $a, a' \in \mathbb{Z}_p$, which happens if and only if E equidistributes on the p parallel lines normal to e_1, i.e., on the p parallel lines of constant e_1-coordinate in the (e_1, e_2)-grid. This is clearly the case as E is a graph over the e_1-coordinate and hence has exactly one element on each of these parallel lines, so this case is proven.

Case 2: e_1 and e_2 generate the two isotropic lines in $\mathbb{Z}_p^2$, $p = 1 \bmod 4$. In this case $e_1 \cdot e_2 \neq 0$ but $e_1 \cdot e_1 = e_2 \cdot e_2 = 0$. Since E is equidistributed along the p parallel lines of constant e_1-coordinate, it is easy to see that these are the same family of lines as $H_t = \{x : x \cdot e_2 = t\}$, $t \in \mathbb{Z}_p$. Thus in this case using $A = \{ae_2 : a \in \mathbb{Z}_p\}$ we find that $\widehat{E}((a - a')e_2) = 0$ for distinct $a, a' \in \mathbb{Z}_p$ and so (E, A) is a spectral pair. Thus E is still spectral in this case and the theorem is proven in all cases.

4. Proof of Theorem 3.3

We include the proof of Theorem 3.3 for the sake of completeness. We have

$$\widehat{E}(m) = p^{-d} \sum_{x \in \mathbb{Z}_p^d} \chi(-x \cdot m) E(x) = 0$$

for some $m \neq (0, \ldots, 0)$. Let $\xi = \chi(-1) = e^{-2\pi i/p}$. Note that ξ is a primitive p-th root of unity. It follows that

$$0 = \sum_{x \in \mathbb{Z}_p^d} \xi^{x \cdot m} E(x) = \sum_{t \in \mathbb{Z}_p} \xi^t \sum_{x \cdot m = t} E(x).$$

Let

$$n(t) = \sum_{x \cdot m = t} E(x) \in \mathbb{Q},$$

so

$$\sum_{t \in \mathbb{Z}_p} \xi^t n(t) = 0.$$

This means that ξ is a root of the rational polynomial

$$P(u) = \sum_{t=0}^{p-1} n(t) u^t.$$

The minimal polynomial of ξ, over $\mathbb{Q}$, is

$$Q(u) = 1 + u + \cdots + u^{p-1},$$

so by elementary Galois theory, $P(u)$ is a constant multiple of $Q(u)$ since ξ is a root of the rational polynomial P and Q is the minimal polynomial of ξ. It follows that the coefficients of $n(t)$ are independent of t. This proves the second assertion of Theorem 3.3, namely that E is equidistributed on the hyperplanes $H_t = \{x \in \mathbb{Z}_p^d : x \cdot m = t\}$.

Let us now prove that if $\widehat{E}(m) = 0$ for some $m \neq (0, \ldots, 0)$, then $\widehat{E}(rm) = 0$ for all $r \neq 0$. We have

$$\sum_{x \in \mathbb{Z}_p^d} \chi(-x \cdot rm) E(x) = \sum_{t \in \mathbb{Z}_p} \xi^t \sum_{x \cdot rm = t} E(x) = \sum_{t \in \mathbb{Z}_p} \xi^t \sum_{x \cdot m = tr^{-1}} E(x) = \sum_{t \in \mathbb{Z}_p} \xi^t n(r^{-1}t).$$

For a fixed r, it follows from above that $n(r^{-1}t)$ is independent of t. Therefore

$$\sum_{t \in \mathbb{Z}_p} \xi^t n(r^{-1}t) = \sum_{t \in \mathbb{Z}_p} \xi^t n(t) = 0$$

and the proof of the claim follows. This completes the proof of Theorem 3.3.

Note the proof above generalizes to rational-valued functions but not to real- or complex-valued functions. The reason is that over $\mathbb{R}$ or $\mathbb{C}$, a polynomial that ξ is a root of need not be a multiple of $1 + x + x^2 + \cdots + x^{p-1}$; for example, $P(x) = x - \xi$ or $P(x) = (x - \xi)(x - \bar{\xi}) = x^2 - 2\cos(2\pi/p) + 1$.

5. Proof of Theorem 3.7

Let A denote the set that tiles E. Note that $|E|\,|A| = p^2$, so $|E| = 1$, p or p^2. If $|E| = 1$ then E is a point and we are done. If $|E| = p^2$ then E is the whole plane and we are done, so without loss of generality let $|E| = p$.

If $\widehat{E}(m)$ never vanishes then E is a point and we are done. On the other hand if $\widehat{E}(m) = 0$ for some nonzero m, then it vanishes on L, the line passing through the origin and $m \neq \vec{0}$. Thus if we set $L^\perp$ to be the line through the origin, perpendicular to m, we see that

$$\widehat{L^\perp}(s)\widehat{E}(s) = 0$$

for all nonzero s. This is because by a straightforward calculation

$$\widehat{L^\perp}(s) = q^{-(d-1)} L(s).$$

Since $|L^\perp| = p = |E|$ we then see that E tiles $\mathbb{F}_p^2$ by $L^\perp$.

Since $\widehat{E}(m) = 0$ for some nonzero vector m, we see that E is equidistributed on the set of p lines $H_t = \{x : x \cdot m = t\}$, $t \in \mathbb{F}_p$. Since $|E| = p$ this means there is exactly one point of E on each of these lines.

We will now choose a coordinate system in which E will be represented as a graph of a function. The coordinate system will either be an orthogonal system or an isotropic system depending on the nature of the vector m. There are two cases to consider.

Case 1: $m \cdot m \neq 0$: We may set $e_1 = m$ and e_2 a vector orthogonal to m. Now $\{e_1, e_2\}$ is an orthogonal basis because e_2 does not lie on the line through m, as this line is not isotropic. If we take a general vector $hx = x_1 e_1 + x_2 e_2$ we see that $hx \cdot m = x_1(m \cdot m)$ and so the lines H_t, $t \in \mathbb{F}_p$, are the same as the lines of constant x_1-coordinate with respect to this orthogonal basis $\{e_1, e_2\}$. Thus there is a unique value of x_2 for any given value of x_1 so that $x_1 e_1 + x_2 e_2 \in E$. Thus $E = \{x_1 e_1 + f(x_1) e_2 : x_1 \in \mathbb{Z}_p\} = \text{Graph}(f)$ for some function $f : \mathbb{F}_p \to \mathbb{Z}_p$.

Case 2: $m \cdot m = 0$: We may set $e_1 = m$. In this case any vector orthogonal to e_1 lies on the line generated by e_1 and so cannot be part of a basis with e_1. Instead we select e_2 off the line generated by e_1 and scale it so that $e_1 \cdot e_2 = 1$. Then by subtracting a suitable multiple of e_1 from e_2 one can also ensure $e_2 \cdot e_2 = 0$.

Thus $\{e_1, e_2\}$ is a basis consisting of two linearly independent isotropic vectors. With respect to this basis, the dot product is represented by the matrix

$$\mathbb{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which exhibits the plane as the hyperbolic plane. This case can only occur when $p = 1 \mod 4$.

Note when we express a general vector $x = x_1 e_1 + x_2 e_2$ with respect to this basis we have $x \cdot m = x_2$; thus the lines $\{H_t : t \in \mathbb{Z}_p\}$ are the same as the lines of constant x_2-coordinate with respect to this basis and E has a unique point on each of these lines. Thus $E = \{f(x_2)e_1 + x_2 e_2 : x_2 \in \mathbb{Z}_p\} = \mathrm{Graph}(f)$ is a graph with respect to this isotropic coordinate system.

Finally we note any function $f : \mathbb{Z}_p \to \mathbb{F}_p$ is given by a polynomial of degree at most $p-1$, explicitly expressed in the form

$$f(x) = \sum_{k \in \mathbb{Z}_p} f(k) \frac{\Pi_{j \neq k}(x - j)}{\Pi_{j \neq k}(k - j)}.$$

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ALEX IOSEVICH: iosevich@math.rochester.edu
Department of Mathematics, University of Rochester, Rochester, NY 14627, United States

AZITA MAYELI: amayeli@qcc.cuny.edu
Department of Mathematics and Computer Science, Queensborough Community College, 222-05 56th Ave., Bayside, NY 11364, United States

JONATHAN PAKIANATHAN: jonpak@math.rochester.edu
Department of Mathematics, University of Rochester, Rochester, NY 14627, United States

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