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A FOURIER RESTRICTION THEOREM FOR A TWO-DIMENSIONAL SURFACE OF FINITE TYPE

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The problem of $L^q(\mathbb{R}^3) \to L^2(S)$ Fourier restriction estimates for smooth hypersurfaces S of finite type in $\mathbb{R}^3$ is by now very well understood for a large class of hypersurfaces, including all analytic ones. In this article, we take up the study of more general $L^q(\mathbb{R}^3) \to L^r(S)$ Fourier restriction estimates, by studying a prototypical model class of two-dimensional surfaces for which the Gaussian curvature degenerates in one-dimensional subsets. We obtain sharp restriction theorems in the range given by Tao in 2003 in his work on paraboloids. For high-order degeneracies this covers the full range, closing the restriction problem in Lebesgue spaces for those surfaces. A surprising new feature appears, in contrast with the nonvanishing curvature case: there is an extra necessary condition. Our approach is based on an adaptation of the bilinear method. A careful study of the dependence of the bilinear estimates on the curvature and size of the support is required.

1. Introduction

Let S be a smooth hypersurface in $\mathbb{R}^n$ with surface measure $d\sigma_S$. The Fourier restriction problem for S, proposed by E. M. Stein in the seventies, asks for the range of exponents q and r for which the estimate

$$\left(\int_S |\hat{f}|^r \, d\sigma_S\right)^{\frac{1}{r}} \le C\|f\|_{L^q(\mathbb{R}^n)} \tag{1-1}$$

holds true for every $f \in \mathcal{S}(\mathbb{R}^n)$, with a constant C independent of f. There was a lot of activity on this problem in the seventies and early eighties. The sharp range in dimension $n = 2$ for curves

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with nonvanishing curvature was determined through work by C. Fefferman [1970], E. M. Stein and A. Zygmund [1974]. In higher dimensions, the sharp $L^q - L^2$ result for hypersurfaces with nonvanishing Gaussian curvature was obtained by Stein [1986] and P. A. Tomas [1975] (see also [Strichartz 1977]). Some more general classes of surfaces were treated by A. Greenleaf [1981].

The question about the general $L^q - L^r$ restriction estimates is nevertheless still wide open. Fundamental progress has been made since the nineties, with contributions by many. Major new ideas were introduced in particular by J. Bourgain [1991; 1995b] and T. Wolff [1995], which led to important further steps towards an understanding of the case of nonvanishing Gaussian curvature. These ideas and methods were further developed by A. Moyua, A. Vargas, L. Vega and T. Tao [Moyua et al. 1996; 1999; Tao et al. 1998], who established the so-called bilinear approach (which had been anticipated in the work of C. Fefferman [1970] and had implicitly been present in the work of Bourgain [1995a]) for hypersurfaces with nonvanishing Gaussian curvature for which all principal curvatures have the same sign. The same method was applied to the light cone by Tao and Vargas [2000a; 2000b]. The climax of the application of that bilinear method to these types of surfaces is due to Tao [2001b] (for principal curvatures of the same sign), and Wolff [2001] and Tao [2001a] (for the light cone). In particular, in these last two papers the sharp linear restriction estimates for the light cone in $\mathbb{R}^4$ were obtained.

For the case of nonvanishing curvature but principal curvatures of different signs, analogous results in $\mathbb{R}^3$ were proved by S. Lee [2006] and Vargas [2005]. Results for the light cone were previously obtained in $\mathbb{R}^3$ by B. Barceló [1985], who also considered more general cones [Barceló 1986]. These results were improved to sharp theorems by S. Buschenhenke [2015]. The bilinear approach also produced results for hypersurfaces with $k \le n-2$ nonvanishing principal curvatures [Lee and Vargas 2010].

More recently, J. Bourgain and L. Guth [2011] made further important progress on the case of nonvanishing curvature by making use also of multilinear restriction estimates due to J. Bennett, A. Carbery and T. Tao [Bennett et al. 2006].

On the other hand, general finite-type surfaces in $\mathbb{R}^3$ (without assumptions on the curvature) have been considered in work by I. Ikromov, M. Kempe and D. Müller [Ikromov et al. 2010; Ikromov and Müller 2011; 2012; 2014], and the sharp range of Stein–Tomas-type $L^q - L^2$ restriction estimates has been determined for a large class of smooth, finite-type hypersurfaces, including all analytic hypersurfaces.

It is our aim in this work to take up the latter branch of development by considering a certain model class of hypersurfaces in dimension three with varying curvature and study more general $L^q - L^r$ restriction estimates. Our approach will again be based on the bilinear method.[1] In our model class, the degeneracy of the curvature will take place along one-dimensional subvarieties. For analytic hypersurfaces whose Gaussian curvature does not vanish identically, this kind of behavior is typical, even though in our model class the zero varieties will still be linear (or the union of two linear subsets). Even though our model class would seem to be among the simplest possible surfaces of such behavior, we will see that they require a very intricate study. We hope that this work will give some insight also for future research on more general types of hypersurfaces.

[1] When preparing this article, the multilinear approach seemed still not sufficiently developed for our needs, since estimates with sharp dependence on the transversality were lacking. For recent progress on this issue, we refer to [Ramos 2016].

Independently of our work, a result for rotationally invariant surfaces with degeneracy of the curvature at a single point has been obtained recently by B. Stovall [2015].

1A. ***Outline of the problem: the adjoint setting.*** We start with a description of the surfaces that we want to study. We will consider surfaces that are graphs of smooth functions defined on $Q =]0,1[\times]0,1[$,

$$\Gamma = \operatorname{graph}(\phi) = \{(\xi, \phi(\xi)) : \xi \in Q\}.$$

The surface Γ is equipped with the surface measure, σ_Γ. It will be more convenient to use duality and work in the adjoint setting. The adjoint restriction operator is given by

$$\mathcal{R}^* f(x) = \widehat{f\, d\sigma_\Gamma}(x) = \int_\Gamma f(\xi) e^{-ix\cdot\xi}\, d\sigma_\Gamma(\xi), \tag{1-2}$$

where $f \in L^s(\Gamma, \sigma_\Gamma)$. The restriction problem is therefore equivalent to the question of finding the appropriate range of exponents for which the estimate

$$\|\mathcal{R}^* f\|_{L^p(\mathbb{R}^3)} \le C \|f\|_{L^s(\Gamma, d\sigma_\Gamma)}$$

holds with a constant C independent of the function $f \in L^s(\Gamma, d\sigma_s)$. We shall require the following properties of the functions ϕ:

Let $m_1, m_2 \in \mathbb{R}$, $m_1, m_2 \ge 2$. We say that a function ϕ is of *normalized type* (m_1, m_2) if there exist $\phi_{(1)}, \phi_{(2)} \in C^\infty(]0,1[, \mathbb{R})$ and $a, b > 0$ such that

$$\phi(\xi_1, \xi_2) = \phi_{(1)}(\xi_1) + \phi_{(2)}(\xi_2) \tag{1-3}$$

on $]0,1[\times]0,1[$, where the derivatives of the $\phi_{(i)}$ satisfy

$$\phi''_{(i)}(\xi_i) \sim \xi_i^{m_i-2}, \tag{1-4}$$

$$|\phi^{(k)}_{(i)}(\xi_i)| \lesssim \xi_i^{m_i-k} \quad \text{for } k \ge 3. \tag{1-5}$$

The constants hidden in these estimates are assumed to be admissible in the sense that they only depend on m_1, m_2 and the order of the derivative, but not explicitly on the $\phi_{(i)}$.

Note we have restricted ourselves to the open square Q which does not contain the origin in order to allow also for noninteger values of m_1 and m_2.

One would of course expect that small perturbations of such functions ϕ, depending on both ξ_1 and ξ_2, should lead to hypersurfaces sharing the same restriction estimates as our model class above. However, such perturbation terms are not covered by our proof. It seems that the treatment of these more general situations would require even more intricate arguments, which will have to take the underlying geometry of the surface into account. We plan to study these questions in the future.

The prototypical example of a normalized function of type (m_1, m_2) is of course $\phi(\xi) = \xi_1^{m_1} + \xi_2^{m_2}$. For m_1 and m_2 integer, others arise simply as follows:

Remarks 1.1. (i) Let $\varepsilon > 0$ and $\varphi \in C^\infty(]-\varepsilon, \varepsilon[, \mathbb{R})$ be of *finite type* $m \in \mathbb{N}$ in 0, i.e.,

$$\varphi(0) = \varphi'(0) = \cdots = \varphi^{(m-1)}(0) = 0 \ne \varphi^{(m)}(0).$$

Assume $\varphi^{(m)}(0) > 0$. Then there exist $\varepsilon' \in (0, \varepsilon)$ such that

$$\varphi^{(k)}(t) \sim t^{m-k}$$

for all $0 \le k \le m$, $|t| < \varepsilon'$.

(ii) Further let $\phi(\xi) = \phi_{(1)}(\xi_1) + \phi_{(2)}(\xi_2)$, $|\xi| \le \varepsilon$, where $\phi_{(i)} \in C^\infty(]-\varepsilon, \varepsilon[, \mathbb{R})$ is of finite type m_i in 0 with $\phi_{(i)}^{(m_i)}(0) > 0$. Then there exists an $\bar{\varepsilon} > 0$ such that $y \mapsto \phi(\bar{\varepsilon} y)$ is of normalized finite type (m_1, m_2).

Proof. (i) Since φ has a zero of order m at the origin, we can find some $\varepsilon_0 > 0$, a smooth function $\chi_0 :]-\varepsilon_0, \varepsilon_0[\to]0, \infty[$ and a sign $\sigma = \pm 1$ such that

$$\varphi(t) = \sigma t^m \chi_0(t)$$

for all $|t| < \varepsilon_0$. It is then easy to see that this implies $\varphi^{(k)}(t) \sim t^{m-k}$.

(ii) Choose $\bar{\varepsilon} > 0$ such that for both $i = 1, 2$, $0 \le k \le m_i$ and all $0 \le t \le \bar{\varepsilon}$,

$$\phi_{(i)}^{(k)}(t) \sim t^{(m_i - k)}.$$

Then for all $0 \le s \le 1$,

$$\frac{d^k}{ds^k} \phi_{(i)}(\bar{\varepsilon} s) \sim \bar{\varepsilon}^k (\bar{\varepsilon} s)^{(m_i - k)} = \bar{\varepsilon}^{m_i} s^{(m_i - k)}. \qquad \square$$

In order to formulate our main theorem, adapting Varchenko's notion of height to our setting, we introduce the height h of the surface by

$$\frac{1}{h} = \frac{1}{m_1} + \frac{1}{m_2}.$$

Let us also put $\bar{m} = m_1 \vee m_2 = \max\{m_1, m_2\}$ and $m = m_1 \wedge m_2 = \min\{m_1, m_2\}$.

Theorem 1.2. *Let $p > \max\{\frac{10}{3}, h+1\}$, $1/s' \ge (h+1)/p$ and $1/s + (2\bar{m}+1)/p < (\bar{m}+2)/2$. Then $\mathcal{R}^*$ is bounded from $L^{s,t}(\Gamma, d\sigma_\Gamma)$ to $L^{p,t}(\mathbb{R}^3)$ for every $1 \le t \le \infty$.*

If in addition $s \le p$ or $1/s' > (h+1)/p$, then $\mathcal{R}^*$ is even bounded from $L^s(\Gamma, d\sigma_\Gamma)$ to $L^p(\mathbb{R}^3)$.

Remarks 1.3. (i) Notice that the "critical line" $1/s' = (h+1)/p$ and the line $1/s + (2\bar{m}+1)/p = (\bar{m}+2)/2$ in the $(1/s, 1/p)$-plane intersect at the point $(1/s_0, 1/p_0)$ given by

$$\frac{1}{s_0} = \frac{3\bar{m} + m - m\bar{m}}{4\bar{m} + 2m}, \qquad \frac{1}{p_0} = \frac{\bar{m} + m}{4\bar{m} + 2m}. \tag{1-6}$$

This shows in particular that the point $(1/s_0, 1/p_0)$ lies strictly above (if $m > 2$) or on the bisectrix $1/s = 1/p$ (if $m = 2$).

The condition $1/s' \ge (h+1)/p$ in the theorem is necessary and in fact dictated by homogeneity (Knapp box examples).

(ii) By (i), the condition

$$\frac{\bar{m}+2}{2} > \frac{2\bar{m}+1}{p} + \frac{1}{s} \tag{1-7}$$

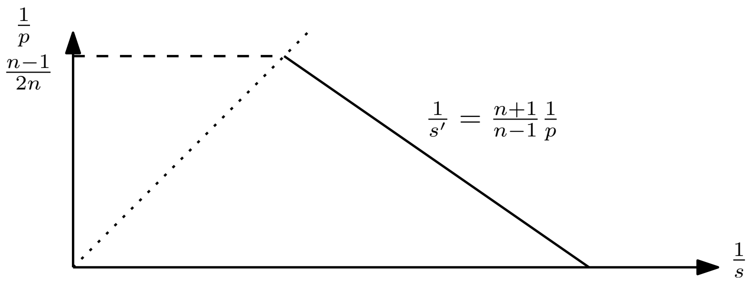

Figure 1. Conjectured range of p and s for nonvanishing Gaussian curvature.

only plays a role above the bisectrix. It is necessary too when $p < s$, hence, in view of (i), if $m > 2$. If $m = 2$, it is necessary with the possible exception of the case where

$$s_0 = p_0 = \frac{4(\bar{m}+1)}{\bar{m}+2},$$

for which we do not have an argument. Our proof in Section 1C will reflect the fact that for $m_j > 2$, the behavior of the operator must be worse than for the case $m_j = 2$.

(iii) From the first condition in the theorem, we see that $p \geq h+1$ is also necessary. Moreover, we shall show in Section 1C that strong-type estimates are not possible unless $s \leq p$ or $1/s' > h + 1/p$. The condition $p > \frac{10}{3}$ is due to the use of the bilinear method, as this exponent gives the sharp bilinear result for the paraboloid, and it is surely not sharp. Nevertheless, when $h > \frac{7}{3}$, we obtain the sharp result.

A new phenomenon appears in these surfaces. In the case of nonvanishing Gaussian curvature, it is conjectured that the sharp range is given by the homogeneity condition $1/s' \geq (h+1)/p$ (with $h = 2/(n-1)$, hence $h+1 = (n+1)/(n-1)$), and a second condition, $p > 2n/(n-1)$, due to the decay rate of the Fourier transform of the surface measure. A similar result is conjectured for the light cone (cf. Figure 1). In contrast to this, we show in our theorem that for the class of surfaces Γ under consideration a third condition appears, namely (1-7).

Let us briefly discuss the different situations that may arise in Theorem 1.2, depending on the choice of m_1 and m_2:

First observe that $1/p_0$ in (1-6) is above the critical threshold $1/p_c = \frac{3}{10}$, if $\bar{m} \leq 2m$. In this case, the new condition

$$\frac{1}{s} + \frac{2\bar{m}+1}{p} = \frac{\bar{m}+2}{2}$$

will not show up in our theorem. So for $\bar{m} \leq 2m$, we are in the situation of either Figure 2 $\big($if $h \leq \frac{7}{3}$, i.e., $h+1 \leq \frac{10}{3}\big)$ or of Figure 3 $\big($if $h > \frac{7}{3}\big)$. Notice that in the last case our theorem is sharp.

It might also be interesting to compare p_0 not only with the condition $p > \frac{10}{3}$, which is due to the bilinear method, but with the conjectured range $p > 3$. We always have $p_0 \geq 3$, while we have $p_0 = 3$ only if $m_1 = m_2$; i.e., a reasonable conjecture is that the new condition (1-7) should always appear for inhomogeneous surfaces with $m_1 \neq m_2$. In the case $\bar{m} > 2m$, our new condition might be visible.

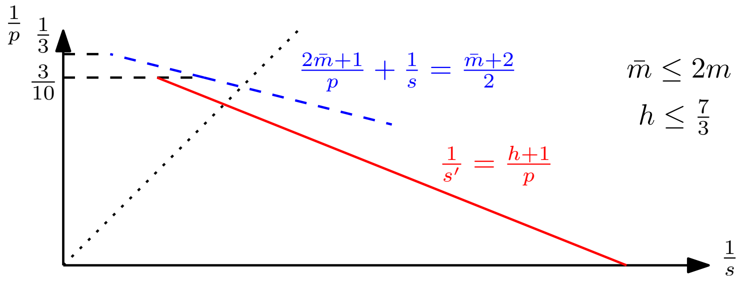

Figure 2. Range of p and s in Theorem 1.2.

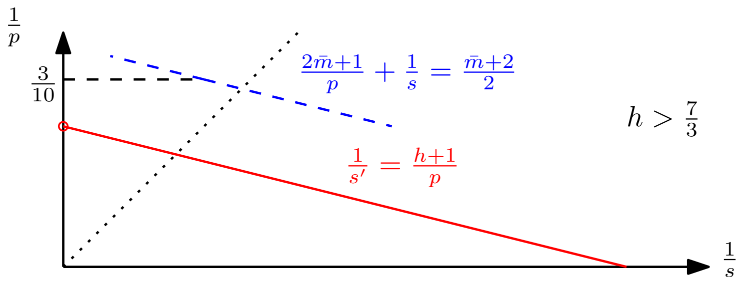

Figure 3. Range of p and s in Theorem 1.2.

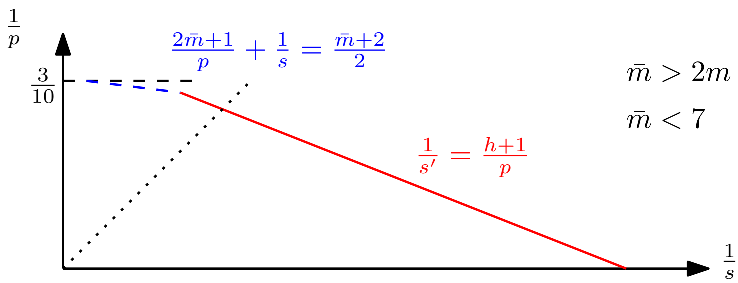

Figure 4. Range of p and s in Theorem 1.2.

Observe next that the line

$$\frac{1}{s}+\frac{2\bar{m}+1}{p}=\frac{\bar{m}+2}{2}$$

intersects the $(1/p)$-axis where

$$p=p_1=\frac{4\bar{m}+2}{\bar{m}+2}.$$

Thus there are two subcases:

For $\bar{m}<7$ we have $p_1<\frac{10}{3}$, corresponding to Figure 4, and our new condition appears.

For $\bar{m}\ge 7$ we may either have $p_0\ge p_1\ge h+1$ (which is equivalent to $\bar{m}m\ge 3\bar{m}+m$) and thus Figure 5 applies, or $p_0<p_1<h+1$ (which is equivalent to $\bar{m}m<3\bar{m}+m$), and we are in the situation of Figure 3; here again the new condition becomes relevant. Observe that in the two last mentioned cases, i.e., for $\bar{m}\ge 7$, our theorem is always sharp.

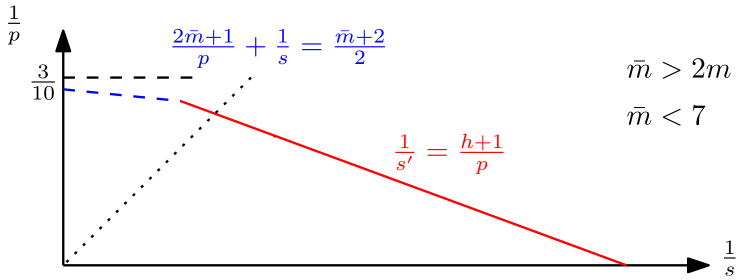

Figure 5. Range of p and s in Theorem 1.2.

Further observe that the appearance of a third condition, besides the classical ones, is natural: Fix $m_1 = 2$ and let $\bar{m} = m_2 \to \infty$. Then the contact order in the second coordinate direction degenerates. Hence, we would expect to find the same p-range as for a two-dimensional cylinder, which agrees with the range for a parabola in the plane, namely $p > 4$ (see [Fefferman 1970; Zygmund 1974]). Since $h \to 2$ as $\bar{m} = m_2 \to \infty$, the condition $p > \max\{\frac{10}{3}, h+1\}$ becomes $p > \frac{10}{3}$ in the limit, which would lead to a larger range than expected. However, the new extra condition

$$\frac{1}{s} + \frac{2\bar{m}+1}{p} < \frac{\bar{m}+2}{2}$$

becomes $p > 4$ for $\bar{m} \to \infty$, as is to be expected.

The restriction problem for the graph of functions $\phi(x) = \xi_1^{m_1} + \xi_2^{m_2}$ (and related surfaces) was studied by E. Ferreyra and M. Urciuolo [2009], however by simpler methods, which led to weaker results than ours. In their approach, they made use of the invariance of this surface under suitable nonisotropic dilations as well as of the one-dimensional results for curves. This allowed them to obtain some results for $p > 4$, in the region below the homogeneity line, i.e., for $1/s' > (h+1)/p$. Our results are stronger in two ways: they include the critical line and, more importantly, when $h < 3$, we obtain a larger range for p.

As for the points on the critical line in the range $p > 4$, let us indicate that these points can in fact also be obtained by means of a simple summation argument involving Lorentz spaces and real interpolation. This can be achieved by means of a summation trick going back to ideas by Bourgain [1985] (see for instance [Tao et al. 1998; Lee 2003]). Details are given in Section A1 of this article.

1B. ***Passage from surface to Lebesgue measure.*** We will always consider hypersurfaces $S = \{(\eta, \phi(\eta)) : \eta \in U\}$ which are the graphs of functions ϕ that are smooth on an open bounded subset $U \subset \mathbb{R}^d$ and continuous on the closure of U. The adjoint of the Fourier restriction operator associated to S is then given by

$$\mathcal{R}^* f(x,t) = \widehat{f\, d\sigma_S}(x,t) = \int_S f(\xi) e^{-i(x,t)\cdot\xi}\, d\sigma_S(\xi), \quad (x,t) \in \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^{d+1},$$

where $d\sigma_S = (1 + |\nabla\phi(\eta)|^2)^{\frac{1}{2}}\, d\eta$ denotes the Riemannian surface measure of S. Here, $f : S \to \mathbb{C}$ is a function on S, but we shall often identify it with the corresponding function $\tilde{f} : U \to \mathbb{C}$, given by

$\tilde{f}(\eta) = f(\eta, \phi(\eta))$. Correspondingly, we define

$$R^*_{\mathbb{R}^d} g(x,t) := \widehat{g\, d\nu}(x,t) := \int_U g(\eta) e^{-i(x\eta + t\phi(\eta))}\, d\eta, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^{d+1},$$

for every function $g \in L^1(U)$ on U. We shall occasionally address $d\nu = d\nu_S$ as the "Lebesgue measure" on S, in contrast with the surface measure $d\sigma = d\sigma_S$. Moreover, to emphasize which surface S is meant, we shall occasionally also write $R^*_{\mathbb{R}^d} = R^*_{S,\mathbb{R}^d}$. Observe that if there is a constant A such that

$$|\nabla\phi(\eta)| \le A, \quad \eta \in U \tag{1-8}$$

(this applies for instance to our class of hypersurfaces Γ, since we assume $m_i \ge 2$), then the Lebesgue measure $d\nu$ and the surface measure $d\sigma$ are comparable, up to some positive multiplicative constants depending only on A. Moreover, since

$$R^* f = R^*_{\mathbb{R}^d}\big(\tilde{f}(1 + |\nabla\phi(\eta)|^2)^{\frac{1}{2}}\big), \tag{1-9}$$

the L^q-norms $\|\tilde{f}\|_{L^q(d\eta)}$ and $\|f\|_{L^q(d\sigma_S)} = \|\tilde{f}(1+|\nabla\phi(\eta)|^2)^{\frac{1}{2}q}\|_{L^q(d\eta)}$ of $\tilde{f}$ and of f are comparable too. Throughout the article, we shall therefore apply the following.

Convention 1.4. Whenever $|\nabla\phi| \lesssim 1$, with some slight abuse of notation, we shall denote the function f on S and the corresponding function $\tilde{f}$ on U by the same symbol f, and write $R^*_{\mathbb{R}^d} f$ in place of $R^*_{\mathbb{R}^d} \tilde{f}$.

In view of these observations, we shall in the sequel mainly work with the operator $R^*_{\mathbb{R}^d}$ associated to the hypersurface Γ, in place of R^*.

1C. *Necessary conditions.* The condition $p > h+1$ is in some sense the weakest one. Indeed, the second condition already implies $p \ge h+1$, and even $p > h+1$ when $s < \infty$. Thus the condition $p > h+1$ only plays a role when the critical line $1/s' = (h+1)/p$ intersects the axis $1/s = 0$ at a point where $p > p_c = \frac{10}{3}$ (see Figure 3).

However, the condition $p > h+1$ is necessary as well (although some kind of weak-type estimate might hold true at the endpoint). This can be shown by analyzing the oscillatory integral defined by $R^*_{\mathbb{R}^d} 1$ (see [Sogge 1987] for similar arguments). For the sake of simplicity, we shall do this only for the model case $\phi(\xi) = \xi_1^{m_1} + \xi_2^{m_2}$ (the more general case can be treated by similar, but technically more involved arguments).

Lemma 1.5. *Assume $m \ge 2$.*

(i) *If $1 \ll \mu \ll \lambda \ll \mu^m$, then*

$$\left| \int_0^\delta e^{i(\mu\xi - \lambda(\xi^m + \mathcal{O}(\xi^{m+1})))}\, d\xi \right| \ge C_\delta \mu^{-\frac{m-2}{2m-2}} \lambda^{-\frac{1}{2m-2}},$$

provided $\delta > 0$ is sufficiently small.

(ii) *If $1 \ll \mu^m \ll \lambda$, $0 \le \alpha < 1$ and $0 \le \beta < 1$, then*

$$\left| \int_0^1 e^{i(\mu\xi - \lambda\xi^m)} \xi^{-\alpha} |\log(\xi/2)|^{-\beta}\, d\xi \right| \gtrsim \lambda^{\frac{\alpha-1}{m}} (\log\lambda)^{-\beta}.$$

Proof. (i) Apply the transformation $\xi \mapsto (\mu/\lambda)^{\frac{1}{m-1}}\xi$ to obtain

$$\int_0^\delta e^{i(\mu\xi-\lambda\xi^m+\mathcal{O}(\xi^{m+1}))}\,d\xi = \int_0^{\delta(\frac{\lambda}{\mu})^{1/(m-1)}} \Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{m-1}} e^{i(\frac{\mu^m}{\lambda})^{1/(m-1)}\phi(\xi)}\,d\xi = \int_0^1 + \int_1^{\delta(\frac{\lambda}{\mu})^{1/(m-1)}},$$

where $\phi(\xi)=\xi-\xi^m+\mathcal{O}((\mu/\lambda)^{\frac{1}{m-1}}\xi^{m+1})$, $(\mu/\lambda)^{\frac{1}{m-1}} \ll 1$ and $(\mu^m/\lambda)^{\frac{1}{m-1}} \gg 1$. The phase function $\phi(\xi)$ has a unique critical point at $\xi_0=\xi_0(\mu/\lambda)$ in [0, 1] lying very close to $m^{-\frac{1}{m-1}}$. Applying well-known asymptotic expansions for oscillatory integrals with nondegenerate critical points (see, e.g., [Stein 1993]), we find that

$$\Big|\int_0^1\Big| \gtrsim \Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{m-1}}\Big(\frac{\mu^m}{\lambda}\Big)^{-\frac{1}{2}\frac{1}{m-1}}.$$

Moreover, integrating by parts in the second integral leads to

$$\Big|\int_1^{\delta(\frac{\lambda}{\mu})^{1/(m-1)}}\Big| \lesssim C_1\Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{m-1}}\Big(\frac{\mu^m}{\lambda}\Big)^{-\frac{1}{m-1}},$$

provided δ is sufficiently small. These estimates imply

$$\Big|\int_0^1 \Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{m-1}} e^{i(\mu\xi-\lambda\xi^m+\mathcal{O}(\xi^{m+1}))}\,d\xi\Big| \gtrsim \Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{m-1}}\Big(\frac{\mu^m}{\lambda}\Big)^{-\frac{1}{2}\frac{1}{m-1}} = \mu^{-\frac{m-2}{2m-2}}\lambda^{-\frac{1}{2m-2}}.$$

(ii) Apply the change of variables $\xi \mapsto \lambda^{-\frac{1}{m}}\xi$ to obtain

$$\begin{aligned}\Big|\int_0^1 e^{i(\mu\xi-\lambda\xi^m)}\xi^{-\alpha}|\log(\xi/2)|^{-\beta}\,d\xi\Big| &\\ &= \lambda^{\frac{\alpha-1}{m}}|\log(\lambda^{-\frac{1}{m}})|^{-\beta}\Big|\int_0^{\lambda^{1/m}} e^{i(\mu\lambda^{-1/m}\xi-\xi^m)}\xi^{-\alpha}\Big(1+\frac{|\log(\xi/2)|}{\frac{1}{m}\log\lambda}\Big)^{-\beta}d\xi\Big|\\ &\gtrsim \lambda^{\frac{\alpha-1}{m}}(\log\lambda)^{-\beta}.\end{aligned}$$

Notice here that $\lambda^{\frac{1}{m}} \gg 1$ and $\mu\lambda^{-\frac{1}{m}} \ll 1$, and that, as $\lambda\to\infty$, the last oscillatory integral tends to $\int_0^\infty e^{-i\xi^m}\xi^{-\alpha}\,d\xi \neq 0$ (which is easily seen). □

Part (ii) of the lemma implies

$$\Big|\int_0^1 e^{-i(x_1\xi-x_3\xi^{m_1})}\,d\xi\Big| \gtrsim x_3^{-\frac{1}{m_1}}$$

for $1\le x_3<\infty$, $1\ll x_1^{m_1}\ll x_3$, and since $R^*_{\mathbb{R}^d}1=\widehat{d\nu}$, we find that

$$\begin{aligned}\|R^*_{\mathbb{R}^d}1\|_p^p &\ge \int_1^\infty\int_{1\ll x_2\ll x_3^{1/m_2}}\int_{1\ll x_1\ll x_3^{1/m_1}} |R^*_{\mathbb{R}^d}1(x_1,x_2,-x_3)|^p\,dx_1\,dx_2\,dx_3\\ &\gtrsim \int_1^\infty\int_{1\ll x_2\ll x_3^{1/m_2}}dx_2\int_{1\ll x_1\ll x_3^{1/m_1}}dx_1 x_3^{-p(\frac{1}{m_1}+\frac{1}{m_2})}\,dx_3\\ &\gtrsim \int_1^\infty x_3^{(1-p)(\frac{1}{m_1}+\frac{1}{m_2})}\,dx_3 = \int_1^\infty x_3^{-\frac{p-1}{h}}\,dx_3.\end{aligned}$$

If the adjoint Fourier restriction operator is bounded, the integral has to be finite; thus necessarily $(p-1)/h > 1$, i.e., $p > h+1$.

Next, to see that the condition (1-7), i.e.,

$$\frac{\bar{m}+2}{2} > \frac{2\bar{m}+1}{p} + \frac{1}{s},$$

is necessary in Theorem 1.2, we consider the subsurface

$$\Gamma_0 = \bigl\{(\xi, \phi(\xi)) : \xi \in [0,1] \times \bigl[\tfrac{1}{2}, \tfrac{1}{2}+\delta\bigr]\bigr\},$$

where $\delta > 0$ is assumed to be sufficiently small. On this subsurface, the principal curvature in the ξ_2-direction is bounded from below. This means that, after applying a suitable affine transformation of coordinates, the restriction problem for the surface Γ_0 is equivalent to the one for the surface

$$\Gamma_{m_1,2} = \bigl\{\bigl(\xi_1, \xi_2, \xi_1^{m_1} + c(\xi_2^2 + \mathcal{O}(\xi_2^3))\bigr) : (\xi_1, \xi_2) \in [0,1] \times [0,\delta]\bigr\},$$

where $c > 0$.

As stated in Remarks 1.3, the condition (1-7) only plays a role above the bisectrix $1/s = 1/p$. So, assume $p < s$ (as explained, this excludes only the case where $m = 2$ and $s = p = p_0$). Then we may choose $\beta < 1$ such that $\beta s > 1 > \beta p$. Assume $\mathcal{R}^*_\Gamma$ is bounded from $L^s(\Gamma)$ to $L^p(\mathbb{R}^3)$; i.e., $\mathcal{R}^*_{\Gamma_{m_1,2}}$ is bounded from $L^s(\Gamma_{m_1,2})$ to $L^p(\mathbb{R}^3)$. Passing again from the surface measure $d\sigma$ to the "Lebesgue measure" $d\nu$ on $\Gamma_{m_1,2}$, define $f(\xi_1,\xi_2) = \xi_1^{-1/s}\log(\xi_1/2)^{-\beta} \in L^s(\Gamma_{m_1,2}, d\nu)$. Then

$$\begin{aligned}\bigl|\widehat{f\,d\nu}(x_1,x_2,-t)\bigr| &= \left|\int_{[0,1]\times[0,\delta]} e^{i(x_1\xi_1 + x_2\xi_2 - t(\xi_1^{m_1} + c(\xi_2^2 + \mathcal{O}(\xi_2^3))))}\xi_1^{-\frac{1}{s}}\log(\xi_1/2)^{-\beta}\,d(\xi_1,\xi_2)\right| \\ &= \left|\int_0^1 e^{i(x_1\xi_1 - t\xi_1^{m_1})}\xi_1^{-\frac{1}{s}}\log(\xi_1/2)^{-\beta}\,d\xi_1\right|\left|\int_0^\delta e^{i(x_2\xi_2 - tc(\xi_2^2 + \mathcal{O}(\xi_2^3)))}\,d\xi_2\right|.\end{aligned}$$

We estimate the first integral by means of Lemma 1.5(ii), and for the second one we use Lemma 1.5(i) (with $m = 2$), which leads to

$$\begin{aligned}\infty > \|f\|_s^p \gtrsim \|\widehat{f\,d\nu}\|_p^p &\gtrsim \int_N^\infty \int_{t^{1/2}}^t \int_1^{t^{1/m_1}} t^{-\frac{p}{s'm_1}}t^{-\frac{p}{2}}(\log t)^{-\beta p}\,dx_1\,dx_2\,dt \\ &\approx \int_N^\infty t^{1-\frac{p}{2}}t^{\frac{1}{m_1}-\frac{p}{s'm_1}}(\log t)^{-\beta p}\,dt,\end{aligned}$$

provided N is chosen sufficiently large. This implies that necessarily

$$1 - \frac{p}{2} + \frac{1}{m_1} - \frac{p}{s'm_1} < -1,$$

which is equivalent to

$$\frac{m_1+2}{2} > \frac{2m_1+1}{p} + \frac{1}{s}.$$

Interchanging the roles of ξ_1 and ξ_2, we obtain the same inequality for m_2 and hence for $\bar{m} = m_1 \vee m_2$, and we arrive at (1-7).

Let us finally prove that on the critical line $1/s' = (h+1)/p$ one cannot have strong-type estimates above the bisectrix $1/s = 1/p$, i.e., for $s > p$. In this regime, we find some $1 > r > 0$ such that $1/s < r < 1/p$. Let

$$f(\xi) = \xi_2^{-\frac{m_2}{sh}} |\log(\xi_2/2)|^{-r} \chi_{\{\xi_1^{m_1} \le \xi_2^{m_2}\}}(\xi).$$

It is easy to check that $f \in L^s(\Gamma)$ since $1 < rs$. Now assume $1 \ll x_j^{m_j} \ll t$ for $j = 1, 2$; more precisely choose $N \gg 1$ and assume $N^2 \le N x_j^{m_j} \le t$ for $j = 1, 2$. Then

$$(R^*_{\mathbb{R}^d} f)(x_1, x_2, -t) = \int_0^1 e^{-i(x_2\xi_2 - t\xi_2^{m_2})} \xi_2^{-\frac{m_2}{sh}} |\log(\xi_2/2)|^{-r} \int_0^{\xi_2^{m_2/m_1}} e^{-i(x_1\xi_1 - t\xi_1^{m_1})}\, d\xi_1\, d\xi_2.$$

Since $x_1^{m_1} \ll t$ is equivalent to $(\xi_2^{m_2/m_1} x_1)^{m_1} \ll t\xi_2^{m_2}$, Lemma 1.5(ii) gives

$$\left| \int_0^{\xi_2^{m_2/m_1}} e^{-i(x_1\xi_1 - t\xi_1^{m_1})}\, d\xi_1 \right| = \xi_2^{\frac{m_2}{m_1}} \left| \int_0^1 e^{-i(\xi_2^{m_2/m_1} x_1 \eta - t\xi_2^{m_2}\eta^{m_1})}\, d\eta \right| \gtrsim t^{-\frac{1}{m_1}}.$$

Applying Lemma 1.5 once more, we obtain

$$\left|(R^*_{\mathbb{R}^d} f)(x_1, x_2, -t)\right| \gtrsim t^{-\frac{1}{m_1}} t^{\frac{1}{sh} - \frac{1}{m_2}} \log^{-r}(t/2) = t^{-\frac{1}{p}(1+\frac{1}{h})} \log^{-r}(t/2),$$

where we made use of $1/s' = (h+1)/p$. Thus we get

$$\begin{aligned} \|R^*_{\mathbb{R}^d} f\|_p^p &\gtrsim \int_{N^2}^{\infty} \int_{N^{1/m_2}}^{(t/N)^{1/m_2}} \int_{N^{1/m_1}}^{(t/N)^{1/m_1}} t^{-1-\frac{1}{h}} \log^{-rp}(t/2)\, dx_1\, dx_2\, dt \\ &\approx \int_{N^2}^{\infty} t^{-1} \log^{-rp}(t/2)\, dt = \infty, \end{aligned}$$

since $rp < 1$.

Let us finish this subsection by adding a few more observations and remarks.

(a) First, observe that Γ_0 is a subset of

$$\Gamma_1 = \{(\xi, \phi(\xi)) \in \Gamma : |\xi| \sim 1\}.$$

(b) One can use the dilations $(\xi_1, \xi_2) \mapsto (r^{1/m_1}\xi_1, r^{1/m_2}\xi_2)$, $r > 0$, in order to decompose $Q = [0, 1] \times [0, 1]$ into "dyadic annuli" which, after rescaling, reduces the restriction problem in many situations to the one for Γ_1 (this kind of approach is used extensively in [Ikromov et al. 2010; Ikromov and Müller 2011], as well as in [Ferreyra and Urciuolo 2009]).

Indeed, on the one hand, any restriction estimate on Γ clearly implies the same estimate also for the subsurface Γ_1. On the other hand, the estimates for the dyadic pieces sum up below the sharp critical line (this is the approach in [Ferreyra and Urciuolo 2009]), i.e., when $1/s' > (h+1)/p$. Moreover, in many situations one may apply Bourgain's summation trick in a similar way to that described in Section A1 in order to establish weak-type estimates also when $(1/s, 1/p)$ lies on the critical line, i.e., when $1/s' = (h+1)/p$. However, we shall not pursue this approach here, since it would not give too much of a simplification for us and since our approach (outlined in the next subsection) seems to lead to an even somewhat sharper

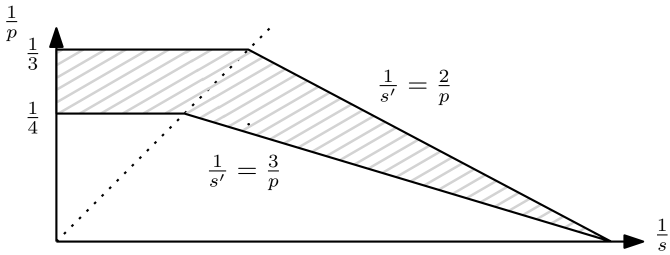

Figure 6. Region on which (1-10) is valid.

result. Moreover, it seems useful and more systematic to understand bilinear restriction estimates for quite general pairs of pieces of our surfaces Γ, and not only the ones which would arise from Γ_1.

(c) On Γ_1, one of the two principal curvatures may vanish, but not both. Notice also that by dividing Γ_1 into a finite number of pieces lying in sufficiently small angular sectors and applying a suitable affine transformation to each of them, we may reduce to surfaces of the form

$$\Gamma_{m,2} = \{(\xi_1, \xi_2, \psi_m(\xi_1) + \xi_2^2 + \mathcal{O}(\xi_2^3)) : \xi_1, \xi_2 \in [0, 1]\},$$

where $\psi_m(\xi_1) \sim \xi_1^m$ as before, with $m = m_1$ or $m = m_2$ (see also our previous discussion of necessary conditions). Applying then a further dyadic decomposition in ξ_1, we see that we may essentially reduce to subsurfaces on which $\xi_1 \sim \varepsilon$, with $\varepsilon > 0$ a small dyadic number. Note that on these we have nonvanishing Gaussian curvature, but the lower bounds of the curvature depend on $\varepsilon > 0$. A rescaling then leads to surfaces of the form

$$P_T = \bigl\{\bigl(\xi_1, \xi_2, \xi_1^2 + \xi_2^2 + \mathcal{O}(\xi_1^3 + T^{-1}\xi_2^3)\bigr) : \xi_1 \in [0, 1], \xi_2 \in [0, T]\bigr\},$$

with $T = \varepsilon^{-m/2} \gg 1$. A prototype of such a situation would be the part of the standard paraboloid lying above a very long-stretched rectangle. Although Fourier restriction estimates for the paraboloid have been studied extensively, the authors are not aware of any results that would give the right control on the dependence on the parameter $T \gg 1$. Indeed, one can prove that the lower bound

$$\|R_T^*\|_{L^s(P_T) \to L^p(\mathbb{R}^3)} \gtrsim T^{\left(\frac{1}{p} - \frac{1}{s}\right)_+} \tag{1-10}$$

for the adjoint restriction operator $R_T^* = R^*_{P_T,\mathbb{R}^2}$ associated to Lebesgue measure on P_T holds true for all s and p for which $(1/s, 1/p)$ lies within the shaded region in Figure 6, and a reasonable conjecture is that also the reverse inequality essentially holds true, maybe up to an extra factor T^δ, i.e., that

$$\|R_T^*\|_{L^s(P_T) \to L^p(\mathbb{R}^3)} \le C_\delta T^{\delta + \left(\frac{1}{p} - \frac{1}{s}\right)_+} \tag{1-11}$$

for every $\delta > 0$.

We give some hints as to why (1-10) holds true and why the inverse inequality (with δ-loss) seems a reasonable conjecture. Let $d\nu_T$ denote the "Lebesgue measure" on P_T. Then by Lemma 1.5,

$$|\widehat{d\nu_T}(x_1, x_2, t)| \gtrsim t^{-\frac{1}{2}} \int_0^T e^{i(x_2\xi_2 + t[\xi_2^2 + \mathcal{O}(T^{-1}\xi_2^3)])}\, d\xi_2 = T t^{-\frac{1}{2}} \int_0^1 e^{i(x_2 T\eta + tT^2[\eta^2 + \mathcal{O}(\eta^3)])}\, d\eta \gtrsim t^{-1},$$

provided $x_1 \ll t$ and $x_2 \ll Tt$ (we may arrange matters in the preceding reductions so that the error term $\mathcal{O}(\eta^3)$ is small compared to η^2). Hence, since we assume $p > 3$,

$$\|\widehat{d\nu_T}\|_{L^p(\mathbb{R}^3)} \gtrsim T^{\frac{1}{p}}.$$

Obviously $\|1\|_{L^s(P_T,d\nu_T)} = T^{1/s}$, so we see that

$$\|R_T^*\|_{L^s(P_T)\to L^p(\mathbb{R}^3)} \gtrsim T^{\left(\frac{1}{p}-\frac{1}{s}\right)}.$$

Restricting P_T to the region where $\xi_2 \le 1$, we see that also $\|R_T^*\|_{L^s(P_T)\to L^p(\mathbb{R}^3)} \gtrsim 1$, and combining these two lower bounds gives (1-10).

On the other hand, from Remark 4, (2.4) in [Ferreyra and Urciuolo 2009] we easily obtain by an obvious rescaling argument that for $1/s' = 3/p$ and $p > 4$ (hence $1/p < 1/s$), we have

$$\|R_T^*\|_{L^s(P_T,d\nu_T)\to L^p(\mathbb{R}^3)} \le C,$$

uniformly in T. It is conjectured that for the entire paraboloid $\mathcal{P} = \{(\xi_1,\xi_2,\xi_1^2+\xi_2^2) : (\xi_1,\xi_2) \in \mathbb{R}^2\}$, the adjoint restriction operator $R^*_{\mathcal{P},\mathbb{R}^2}$ is bounded for $1/s' = 2/p$ and $p > 3$ (hence $1/p < 1/s$). It would be reasonable to expect the same kind of behavior for suitable perturbations of the paraboloid, and subsets of those, such as P_T (maybe with an extra factor T^δ for any $\delta > 0$). By complex interpolation, the previous estimate in combination with the latter conjectural estimate would lead to

$$\|R_T^*\|_{L^s(P_T,d\nu_T)\to L^p(\mathbb{R}^3)} \le C_\delta T^\delta$$

for every $\delta > 0$, provided that $1/p < 1/s$ and $2/p < 1/s' < 3/p$. In combination with a trivial application of Hölder's inequality this leads to the conjecture (1-11),

$$\|R_T^*\|_{L^s(P_T)\to L^p(\mathbb{R}^3)} \le C_\delta T^{\delta+\left(\frac{1}{p}-\frac{1}{s}\right)_+}$$

for every $\delta > 0$, provided $(1/s, 1/p)$ lies within the shaded region in Figure 6.

1D. ***The strategy of the approach.*** We will study certain bilinear operators. For a suitable pair of subsurfaces $S_1, S_2 \subset S$ (we will be more specific on this point later), we seek to establish bilinear estimates

$$\|R^*_{\mathbb{R}^2} f_1 R^*_{\mathbb{R}^2} f_2\|_{L^p(\mathbb{R}^3)} \le C_p C(S_1,S_2)\, \|f_1\|_{L^2(S_1)} \|f_2\|_{L^2(S_2)},$$

for functions f_1, f_2 supported in S_1 and S_2, respectively.

For hypersurfaces with nonvanishing Gaussian curvature and principal curvatures of the same sign, the sharp estimates of this type, under the appropriate transversality assumption, appeared in [Tao 2003b] (after previous partial results in [Tao et al. 1998; Tao and Vargas 2000a]). For the light cone in any dimension, the analogous results were established in [Wolff 2001; Tao 2001a] (improving on earlier results in [Bourgain 1995a; Tao and Vargas 2000a]). For the case of principal curvatures of different sign, or with a smaller number of nonvanishing principal curvatures, sharp bilinear results are also known [Lee 2006; Vargas 2005; Lee and Vargas 2010].

What is crucial for us is to know how the constant $C(S_1, S_2)$ explicitly depends on the pair of surfaces S_1 and S_2, in order to be able to sum all the bilinear estimates that we obtain for pairs of pieces of our given surface, to pass to a linear estimate. Classically, this is done by proving a bilinear estimate for one "generic" class of subsurfaces. For instance, if S is the paraboloid, then other pairs of subsurfaces can be reduced to it by means suitable affine transformations and homogeneous rescalings. However, general surfaces do not come with such a kind of self-similarity under these transformations, and it is one of the features of this article that we establish new, very precise bilinear estimates.

The bounds on the constant $C(S_1, S_2)$ that we establish will depend on the size of the domains and local principal curvatures of the subsurfaces, and we shall have to keep track of these during the whole proof. In this sense, many of the lemmas are generalized, quantitative versions of well-known results from classical bilinear theory.

The pairs of subsurfaces that we would like to discuss are pieces of the surface sitting over two dyadic rectangles and satisfying certain separation or "transversality" assumptions. However, such a rectangle might touch one of the axes, where some principle curvature is vanishing. In this case we will decompose dyadically a second time. But even on these smaller sets, we do not have the correct "transversality" conditions; we first have to find a proper rescaling such that the scaled subsurfaces allow us to run the bilinear machinery.

The following section will begin with the bilinear argument to provide us with a very general bilinear result for sufficiently "good" pairs of surfaces. In the subsequent section, we construct a suitable scaling in order to apply this general result to our situation. After rescaling and several additional arguments, we pass to a global bilinear estimate and finally proceed to the linear estimate.

A few more remarks on the notion will be useful: as mentioned before, it is very important to know precisely how the constants depend on the specific choice of subsurfaces. Moreover, there will appear other constants, depending possibly on m_1, m_2, p, q, or other quantities, but not explicitly on the choice of subsurfaces. We will not keep track of such types of constants, since it would even set a false focus and distract the reader. Instead we will simply use the symbol $\lesssim$ for an inequality involving one of these constants of minor importance. To be more precise on this, later we introduce a family of pairs of subsurfaces $\mathcal{S}_0$. Then for quantities $A, B : \mathcal{S}_0 \to \mathbb{R}$ the inequality $A \lesssim B$ means there exists a constant $C > 0$ such that $A(S_1, S_2) \le CB(S_1, S_2)$ uniformly for all $(S_1, S_2) \in \mathcal{S}_0$.

Moreover, we will also use the notation $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We will even use this notation for vectors, meaning their entries are comparable in each coordinate. Similarly, we write $A \ll B$ if there exists a constant $c > 0$ such that $A(S_1, S_2) \le cB(S_1, S_2)$ for all $(S_1, S_2) \in \mathcal{S}_0$ and c is "small enough" for our purposes. This notion of being "sufficiently small" will in general depend on the situation and further constants, but the choice will be uniform in the sense that it will work for all pairs of subsurfaces in the class $\mathcal{S}_0$.

The inner product of two vectors x, y will usually be denoted by xy or $x \cdot y$, and occasionally also by $\langle x, y \rangle$.

2. General bilinear theory

2A. ***Wave packet decomposition.*** We begin with what is basically a well-known result, although we need a more quantitative version (cf. [Tao 2003a; Lee 2006]).

Lemma 2.1. *Let $U \subset \mathbb{R}^d$ be an open and bounded subset, and let $\phi \in C^\infty(U, \mathbb{R})$. We assume there exist constants $\kappa > 0$ and $D \le 1/\kappa$ such that $\|\partial^\alpha \phi\|_\infty \le A_\alpha \kappa D^{2-|\alpha|}$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \ge 2$. Then for every $R \ge 1$ there exists a wave packet decomposition adapted to ϕ with tubes of radius $R/D = R'$ and length $R^2/(D^2\kappa) = (R')^2/\kappa$, where we have put $R = R'D$.*

More precisely, consider the index sets $\mathcal{Y} = R'\mathbb{Z}^d$ and $\mathcal{V} = (R')^{-1}\mathbb{Z}^d \cap U$, and define for $w = (y, v) \in \mathcal{Y} \times \mathcal{V} = \mathcal{W}$ the tube

$$T_w = \left\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |t| \le \frac{(R')^2}{\kappa},\ |x - y + t\nabla\phi(v)| \le R'\right\}. \tag{2-1}$$

Then, given any function $f \in L^2(U)$, there exist functions (wave packets) $\{p_w\}_{w\in\mathcal{W}}$ and coefficients $c_w \in \mathbb{C}$ such that $R^*_{\mathbb{R}^d} f$ can be decomposed as

$$R^*_{\mathbb{R}^d} f(x,t) = \sum_{w\in\mathcal{W}} c_w\, p_w(x,t)$$

for every $t \in \mathbb{R}$ with $|t| \le (R')^2/\kappa$, in such a way that the following hold true:

(P1) $p_w = R^*_{\mathbb{R}^d}(\mathfrak{F}^{-1}_{\mathbb{R}^d}(p_w(\,\cdot\,, 0)))$.

(P2) $\operatorname{supp} \mathfrak{F}_{\mathbb{R}^{d+1}} p_w \subset B((v, \phi(v)), 2/R')$.

(P3) *p_w is essentially supported in T_w; i.e.,*

$$|p_w(x,t)| \le C_N (R')^{-\frac{d}{2}} \left(1 + \frac{|x - y + t\nabla\phi(v)|}{R'}\right)^{-N}$$

for every $N \in \mathbb{N}$. In particular, $\|p_w(\,\cdot\,, t)\|_2 \lesssim 1$.

(P4) *For all $W \subset \mathcal{W}$, we have $\left\|\sum_{w\in W} p_w(\,\cdot\,, t)\right\|_2 \lesssim |W|^{\frac{1}{2}}$.*

(P5) $\|c\|_{\ell^2} \lesssim \|f\|_{L^2}$.

Moreover, the constants arising explicitly (such as the C_N) or implicitly in these estimates can be chosen to depend only on the constants A_α but no further on the function ϕ, and also not on the other quantities R, D and κ (such constants will be called admissible).

Remarks 2.2. (i) Notice that no bound is required on $\nabla\phi$ at this stage; however, such bounds will become important later (for instance in (iii)).

(ii) Denote by $N(v)$ the normal vector at $(v, \phi(v))$ to the graph of ϕ which is given by $N(v) = (-\nabla\phi(v), 1)$. Since $(R')^2/\kappa \ge R'$, we may thus rewrite

$$T_w = (y, 0) + \left\{tN(v) : |t| \le \frac{(R')^2}{\kappa}\right\} + \mathcal{O}(R').$$

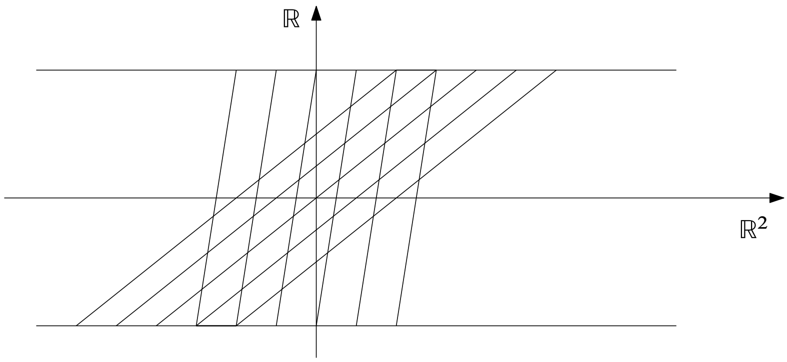

Figure 7. The tubes T_w fill a horizontal strip.

Moreover,

$$|x-y+t\nabla\phi(v)|=|(x,t)-(y,0)-tN(v)|\geq \operatorname{dist}((x,t),T_w).$$

It is then easily seen that (P3) can be rewritten as

$$|p_w(z)|\leq C_N(R')^{-\frac{d}{2}}\left(1+\frac{\operatorname{dist}(z,T_w)}{R'}\right)^{-N}$$

for all $z\in\mathbb{R}^{d+1}$ with $|\langle z,e_{d+1}\rangle|\leq (R')^2/\kappa$, where e_{d+1} denotes the last vector of the canonical basis of $\mathbb{R}^{d+1}$. This justifies the statement that "p_w is essentially supported in T_w".

(iii) Notice further that we can reparametrize the wave packets by lifting $\mathcal{V}$ to $\widetilde{\mathcal{V}}=\{(v,\phi(v)):v\in\mathcal{V}\}\subset S$. If we now assume $\|\nabla\phi\|\lesssim 1$, then we have $|(v,\phi(v))-(v',\phi(v'))|\sim|v-v'|$, and thus $\widetilde{\mathcal{V}}$ becomes an $(R')^{-1}$-net in S. Finally, we shall identify a parameter $y\in\mathbb{R}^d$ with the point $(y,0)$ in the hyperplane $\mathbb{R}^d\times\{0\}$.

Proof of Lemma 2.1. We will basically follow the proof by Lee [2006]; the only new feature consists in elaborating the precise role of the constant κ.

Let $\psi,\hat{\eta}\in C_0^\infty(B(0,1))$ be chosen in such a way that for $\eta_y(x)=\eta((x-y)/R')$, $\psi_v(\xi)=\psi(R'(\xi-v))$ we have $\sum_{v\in\mathcal{V}}\psi_v=1$ on U and $\sum_{y\in\mathcal{Y}}\eta_y=1$. We also choose a slightly bigger function $\tilde{\psi}\in C_0^\infty(B(0,3))$ such that $\tilde{\psi}=1$ on $B(0,2)\supset\operatorname{supp}\psi+\operatorname{supp}\hat{\eta}$, and put $\tilde{\psi}_v(\xi)=\tilde{\psi}(R'(\xi-v))$. Then the functions

$$F_{(y,v)}=\mathfrak{F}^{-1}_{\mathbb{R}^d}(\widehat{\psi_v f\,\eta_y})=(\psi_v f)*\check{\eta}_y,\quad y\in\mathcal{Y},\ v\in\mathcal{V},$$

are essentially well localized in both position and momentum/frequency space. Define $q_w=R^*_{\mathbb{R}^d}(F_w)$, $w=(y,v)\in\mathcal{W}$; up to a certain factor c_w, which will be determined later, these are already the announced wave packets, i.e., $q_w=c_w p_w$.

Since $f=\sum_{w\in W}F_w$, we then have the decomposition $R^*_{\mathbb{R}^d}f=\sum_{w\in W}q_w$. Let us concentrate on property (P3) — the other properties are then rather easy to establish. Since $\operatorname{supp}F_{(y,v)}\subset B(v,2/R')$, we have, for every $w=(y,v)\in\mathcal{W}$,

$$\begin{aligned}
q_w(x,t) &= \int e^{-i(x\xi+t\phi(\xi))}F_w(\xi)\,d\xi = \int e^{-i(x\xi+t\phi(\xi))}F_w(\xi)\tilde{\psi}_v(\xi)\,d\xi \\
&= (2\pi)^{-d}\iint e^{-i\left((x-z)\xi+t\phi(\xi)\right)}\tilde{\psi}_v(\xi)\,d\xi\,\widehat{F}_w(z)\,dz \\
&= (2\pi)^{-d}(R')^{-d}\iint e^{-i\left((x-z)(\frac{\xi}{R'}+v)+t\phi(\frac{\xi}{R'}+v)\right)}\tilde{\psi}(\xi)\,d\xi\,\widehat{F}_w(z)\,dz \\
&= (2\pi)^{-d}(R')^{-d}\int K(x-z,t)\widehat{F}_w(z)\,dz,
\end{aligned}$$

with the kernel

$$K(x,t)=\int e^{i\left(x(\frac{\xi}{R'}+v)+t\phi(\frac{\xi}{R'}+v)\right)}\tilde{\psi}(\xi)\,d\xi.$$

We claim that

$$|K(x,t)|\lesssim\left(1+\frac{|x+t\nabla\phi(v)|}{R'}\right)^{-N} \tag{2-2}$$

for every $N\in\mathbb{N}$. To this end, we shall estimate the oscillatory integral

$$K_\lambda=\int e^{i\lambda\Phi(\xi)}\tilde{\psi}(\xi)\,d\xi,$$

with phase

$$\Phi(\xi)=\frac{x\left(\frac{\xi}{R'}+v\right)+t\phi\left(\frac{\xi}{R'}+v\right)}{1+(R')^{-1}|x+t\nabla\phi(v)|},$$

where we put $\lambda=1+(R')^{-1}|x+t\nabla\phi(v)|$. In order to prove (2-2), we may assume $|x+t\nabla\phi(v)|\gg R'$. Then integrations by parts will lead to $|K_\lambda|\lesssim\lambda^{-N}$ for all $N\in\mathbb{N}$, hence to (2-2), provided we can show that

$$|\nabla\Phi(\xi)|\sim 1\quad\text{for all }\xi, \tag{2-3}$$

$$\|\partial^\alpha\Phi\|_\infty\lesssim 1\quad\text{for all }\alpha\geq 2, \tag{2-4}$$

and that the constants in these estimates are admissible. But,

$$\frac{|t|\left|\nabla\phi\left(\frac{\xi}{R'}+v\right)-\nabla\phi(v)\right|}{|x+t\nabla\phi(v)|}\ll\frac{|t|\left|\nabla\phi\left(\frac{\xi}{R'}+v\right)-\nabla\phi(v)\right|}{R'}\lesssim\frac{|t|}{(R')^2}\|\phi''\|_\infty\leq\frac{1}{\kappa}\|\phi''\|_\infty\leq 1$$

for every $\xi\in\operatorname{supp}\tilde{\psi}$, hence

$$|t|\left|\nabla\phi\left(\frac{\xi}{R'}+v\right)-\nabla\phi(v)\right|\ll|x+t\nabla\phi(v)|.$$

Thus

$$|\nabla\Phi(\xi)|=\frac{\left|x+t\nabla\phi\left(\frac{\xi}{R'}+v\right)\right|}{R'+|x+t\nabla\phi(v)|}=\frac{\left|x+t\nabla\phi(v)-t\left[\nabla\phi(v)-\nabla\phi\left(\frac{\xi}{R'}+v\right)\right]\right|}{R'+|x+t\nabla\phi(v)|}\sim\frac{|x+t\nabla\phi(v)|}{R'+|x+t\nabla\phi(v)|}\sim 1,$$

which verifies (2-3). And, for $|\alpha| \geq 2$ we have

$$|\partial^\alpha \Phi(\xi)| \leq \left| t(R')^{-|\alpha|}(\partial^\alpha \phi)\left(\frac{\xi}{R'} + v\right)\right| \lesssim \frac{(R')^2}{\kappa}(R')^{-|\alpha|}\kappa D^{2-|\alpha|} \leq (DR')^{2-|\alpha|} = R^{2-|\alpha|} \leq 1,$$

which gives (2-4). It is easily checked that the constants in these estimates can be chosen to be admissible. Following the proof in [Lee 2006], we conclude that

$$\begin{aligned}|q_w(x,t)| &\lesssim (R')^{-d} \int \left|K(x-z-y,t)\hat{F}_w(z+y)\right| dz \\ &= (R')^{-d} \int \left| K(x-z-y,t)\eta\left(\frac{z}{R'}\right)\widehat{\psi_v f}(z+y)\right| dz \\ &\lesssim \left(1 + \frac{|x-y+t\nabla\phi(v)|}{R'}\right)^{-N} M(\widehat{\psi_v f})(y),\end{aligned}$$

where M denotes the Hardy–Littlewood maximal operator. Thus, we obtain (P3) by choosing $c_w = c_{(y,v)} = (R')^{d/2} M(\widehat{\psi_v f})(y)$.

Properties (P1) and (P2) follow from the definition of the wave packets. From (P2) and (P3) we can deduce (P4). For (P5), we refer to [Lee 2006]. □

In view of our previous remarks, it is easy to restate Lemma 2.1 in a more coordinate-free way. For any given hyperplane $H = n^\perp \subset \mathbb{R}^{d+1}$, with n a unit vector (so that $\mathbb{R}^{d+1} = H + \mathbb{R}n$), define the partial Fourier (co)transform

$$\mathfrak{F}_H^{-1} f(\xi + tn) = \int_H f(x+tn)e^{ix\cdot\xi}\, dx, \quad \xi \in H,\ t \in \mathbb{R}.$$

Moreover, if $U \subset H$ is open and bounded, and if $\phi_H \in C^\infty(U, \mathbb{R})$ is given, then consider the smooth hypersurface $S = \{\eta + \phi_H(\eta)n : \eta \in U\} \subset \mathbb{R}^{d+1}$, and define the corresponding Fourier extension operator

$$R_H^* f(x+tn) = \int_U f(\eta)e^{-i(x\eta + t\phi_H(\eta))}\, d\eta = \int_U f(\eta)e^{-i\langle x+tn, \eta+\phi_H(\eta)n\rangle}\, d\eta$$

for $(x,t) \in H \times \mathbb{R}$ and $f \in L^2(U)$. Notice that $R^*_{\mathbb{R}^d}$ corresponds to the special case $H = \mathbb{R}^d \times \{0\}$, and thus by means of a suitable rotation, mapping e_{d+1} to n, we immediately obtain the following.

Corollary 2.3 (wave packet decomposition). *Let $U \subset H$ be an open and bounded subset, and let $\phi_H \in C^\infty(U, \mathbb{R})$. We assume that there are constants $\kappa > 0$ and $D \leq 1/\kappa$ such that $\|\phi_H^{(l)}\|_\infty \leq A_l \kappa D^{2-l}$ for every $l \in \mathbb{N}$ with $l \geq 2$, where $\phi_H^{(l)}$ denotes the total derivative of ϕ_H of order l, and in addition that $\|\phi'\|_\infty \leq A$. Then for every $R \geq 1$ there exists a wave packet decomposition adapted to S and the decomposition of $\mathbb{R}^{d+1}$ into $\mathbb{R}^{d+1} = H + \mathbb{R}n$, with tubes of radius $R/D = R'$ and length $R^2/(D^2\kappa) = (R')^2/\kappa$, where $R = R'D$.*

More precisely, there exists an R'-lattice $\mathcal{Y}$ in H and an $(R')^{-1}$-net $\mathcal{V}$ in S such that the following hold true: if we denote by $\mathcal{W}$ the index set $\mathcal{W} = \mathcal{Y} \times \mathcal{V}$ and associate to $w = (y,v) \in \mathcal{Y} \times \mathcal{V} = \mathcal{W}$ the tube-like set

$$T_w = y + \left\{tN(v) : |t| \leq \frac{(R')^2}{\kappa}\right\} + B(0, R'), \tag{2-5}$$

then for every given function $f \in L^2(U)$ there exist functions (wave packets) $\{p_w\}_{w\in\mathcal{W}}$ and coefficients $c_w \in \mathbb{C}$ such that for every $x = x' + tn \in \mathbb{R}^{d+1}$ with $|t| \le (R')^2/\kappa$ and $x' \in H$, we may decompose $R_H^* f(x)$ as

$$R_H^* f(x) = \sum_{w\in\mathcal{W}} c_w p_w(x),$$

in such a way that the following hold true:

(P1′) $p_w = R_H^*(\mathfrak{F}_H^{-1}(p_w|_H))$.

(P2′) $\operatorname{supp}\mathfrak{F}_{\mathbb{R}^{d+1}} p_w \subset B(v,(R')^{-1})$ *and* $\operatorname{supp}\mathfrak{F}_H(p_w(\cdot + tn)) \subset B(v', \mathcal{O}((R')^{-1}))$,*where v' denotes the orthogonal projection of $v \in S$ to H.*

(P3′) *p_w is essentially supported in T_w; i.e.,*

$$|p_w(x)| \le C_N (R')^{-1}\left(1 + \frac{\operatorname{dist}(x, T_w)}{R'}\right)^{-N}.$$

(P4′) *For all $W \subset \mathcal{W}$, we have* $\|\sum_{w\in W} p_w(\cdot + tn)\|_{L^2(H)} \lesssim |W|^{\frac{1}{2}}$.

(P5′) $\|c\|_{\ell^2} \lesssim \|f\|_{L^2}$.

Moreover, the constants arising in these estimates can be chosen to depend only on the constants A_l and A, but no further on the function ϕ_H, and also not on the other quantities R, D and κ (such constants will be called admissible).

Notice that, unlike as in Lemma 2.1, we may here choose an $(R')^{-1}$-net in S in place of an R' -lattice in H for the parameter set $\mathcal{V}$, because of our assumed bound on ϕ_H'.

It will become important that under suitable additional assumptions on the position of a given hyperplane H, we may reparametrize a given smooth hypersurface $S = \{(\xi, \phi(\xi)) : \xi \in U\}$ (where U is an open subset of $\mathbb{R}^d$) also of the form

$$S = \{\eta + \phi_H(\eta)n : \eta \in U_H\},$$

where U_H is an open subset of H and $\phi_H \in C^\infty(U_H, \mathbb{R})$.

Lemma 2.4 (reparametrization). *Let $H_1 = n_1^\perp$ and $H_2 = n_2^\perp$ be two hyperplanes in $\mathbb{R}^{d+1}$, where n_1 and n_2 are given unit vectors. Let $K = H_1 \cap H_2$, and choose unit vectors h_1, h_2 orthogonal to K such that $H_1 = K + \mathbb{R}h_1$ and $H_2 = K + \mathbb{R}h_2$. Let $U_1 \subset H_1$ be an open bounded subset such that for every $x' \in K$, the section $U_1^{x'} = \{u \in \mathbb{R} : x' + uh_1 \subset U_1\}$ is an (open) interval, and let $\phi_1 \in C^\infty(U_1, \mathbb{R})$ satisfy the assumptions of Corollary 2.3. Setting $B = \kappa D^2$ and $r = D^{-1}$, an equivalent way to state this is that there are constants $B, r > 0$ such that $Br \le 1$, $\|\phi_1'\|_\infty \le A$ and $\|\phi_1^{(l)}\|_\infty \le A_l B r^l$ for every $l \in \mathbb{N}$ with $l \ge 2$. Denote by S the hypersurface*

$$S = \{\eta + \phi_1(\eta)n_1 : \eta \in U_1\} \subset \mathbb{R}^{d+1},$$

and again by $v \mapsto N(v)$ the corresponding unit normal field on S.

Assume furthermore that the vector n_2 *is transversal to* S; *i.e.*, $|\langle n_2, N(v)\rangle| \geq a > 0$ *for all* $v \in S$. *Then there exist an open bounded subset* $U_2 \subset H_2$ *such that for every* $x' \in K$, *the section* $U_2^{x'} = \{s \in \mathbb{R} : x' + sh_2 \in U_2\}$ *is an interval, and a function* $\phi_2 \in C^\infty(U_2, \mathbb{R})$ *so that we may rewrite*

$$S = \{\xi + \phi_2(\xi)n_2 : \xi \in U_2\}. \tag{2-6}$$

Moreover, the derivatives of ϕ_2 *satisfy estimates of the same form as those of* ϕ_1, *up to multiplicative constants which are admissible*, *i.e.*, *which depend only on the constants* A_l, A *and* a.

Finally, given any $f_1 \in L^2(U_1)$, *there exists a unique function* $f_2 \in L^2(U_2)$ *such that*

$$R^*_{H_1} f_1 = R^*_{H_2} f_2, \tag{2-7}$$

and $\|f_1\|_2 \sim \|f_2\|_2$, *where the constants in these estimates are admissible.*

Proof. Assume that (2-6) holds true. Then, given any point $\eta + \phi_H(\eta)n_1 \in S$, with $\eta = x' + uh_1 \in U_1$, $x' \in K$, we find some $\xi = x' + sh_2 \in U_2$ such that

$$x' + uh_1 + \phi_1(x' + uh_1)n_1 = x' + sh_2 + \phi_2(x' + sh_2)n_2, \tag{2-8}$$

which shows that necessarily

$$s = \langle uh_1 + x' + \phi_1(x' + uh_1)n_1, h_2\rangle. \tag{2-9}$$

Let us therefore define the mapping $G : U_1 \to H_2$ by

$$G(x' + uh_1) = x' + \langle uh_1 + x' + \phi_1(x' + uh_1)n_1, h_2\rangle h_2.$$

Moreover, fixing an orthonormal basis $E_1, \dots, E_{d-1}$ of K and extending this by the vector h_1 or h_2 in order to obtain bases of H_1 and H_2 respectively and working in the corresponding coordinates, we may assume without loss of generality that U_1 is an open subset of $\mathbb{R}^{d-1} \times \mathbb{R}$, since $\dim K = d - 1$, and that G is a mapping $G : U_1 \to \mathbb{R}^{d-1} \times \mathbb{R}$, given by

$$G(x', u) = (x', g(x', u)),$$

where

$$g(x', u) = \langle x' + uh_1 + \phi_1(x', u)n_1, h_2\rangle.$$

To show that G is a diffeomorphism onto its image $U_2 = G(U_1)$, observe that

$$\partial_u G(x', u) = (0, \partial_u g(x', u)) = \big(0, \langle h_1 + \partial_u \phi_1(x', u)n_1, h_2\rangle\big).$$

On the other hand, the vector

$$N_0 = -\partial_u \phi_1(x', u)h_1 - \sum_{j=1}^{k} \partial_{x_j} \phi(x', u)E_j + n_1$$

is normal to S at the point $x' + uh_1 + \phi_1(x' + uh_1)n_1$ $\big($here $x' = \sum_{j=1}^{d-1} x_j E_j\big)$, and $|N_0| \sim 1$. Thus, our transversality assumption implies

$$|\langle -\partial_u \phi_1(x', u)h_1 + n_1, n_2\rangle| \gtrsim a > 0. \tag{2-10}$$

But, $\{h_j, n_j\}$ forms an orthonormal basis of $K^\perp$ for $j = 1, 2$, and thus, rotating all these vectors by an angle of $\pi/2$, we see that (2-10) is equivalent to $|\langle \partial_u \phi_1(x', u) n_1 + h_1, h_2 \rangle| \gtrsim a > 0$, so that

$$|\partial_u g(x', u)| \gtrsim a > 0.$$

Given the special form of G, this also implies

$$|\det G'(x', u)| = |\partial_u g(x', u)| \gtrsim a > 0.$$

Consequently, for x' fixed, the mapping $u \mapsto g(x', u)$ is a diffeomorphism from the interval $U_1^{x'}$ onto an open interval $U_2^{x'}$, and thus G is bijective onto its image U_2, in fact even a diffeomorphism, and U_2 fibers into the intervals $U_2^{x'}$. Indeed, the inverse mapping $F = G^{-1} : U_2 \to U_1$ of G is also of the form

$$F(x', s) = (x', f(x', s)),$$

where

$$g(x', f(x', s)) = s. \tag{2-11}$$

In combination with (2-8) this shows that (2-6) holds indeed true, with

$$\phi_2(x', s) = f(x', s)\langle h_1, n_2 \rangle + \phi_1(F(x', s))\langle n_1, n_2 \rangle. \tag{2-12}$$

Moreover, if $f_1 \in L^2(U_1)$, then, by (2-8) and a change of coordinates,

$$\begin{aligned} R^*_{H_1} f_1(y) &= \iint_{U_1} f_1(x', u) e^{-i\langle y, x' + uh_1 + \phi_1(x', u) n_1 \rangle} \, dx' du \\ &= \iint_{U_2} f_1(F(x', s)) |\det F'(x', s)| \, e^{-i\langle y, x' + sh_2 + \phi_2(x', s) n_2 \rangle} \, dx' ds, \end{aligned}$$

so that (2-7) holds true, with

$$f_2(x' + sh_2) = f_1(x' + f(x', s)h_1) \, |\det F'(x', s)|. \tag{2-13}$$

Our estimates for derivatives of F show that $|\det F'(x', s)| \sim 1$, with admissible constants, so that in particular $\|f_1\|_2 \sim \|f_2\|_2$.

What remains is the control of the derivatives of ϕ_2. This somewhat technical part of the proof will be based on Faà di Bruno's theorem and is deferred until the Appendix (see Section A2). $\square$

We shall from now on restrict ourselves to dimension $d = 2$. The following lemma will deal with the separation of tubes along certain types of curves, for a special class of 2-hypersurfaces. It will later be applied to intersection curves of two hypersurfaces.

Lemma 2.5 (tube-separation along the intersection curve). *Let $\mathcal{Y}$, $\mathcal{V}$, $\mathcal{W}$, R, T_w be as in Corollary 2.3. Moreover assume $\phi \in C^\infty(U, \mathbb{R})$, $U \subset \mathbb{R}^2$, such that $\partial_i^2 \phi(x) \sim \kappa_i$ for all $x \in U$, $i = 1, 2$, and $\partial_1 \partial_2 \phi = 0$. Define $\kappa = \kappa_1 \vee \kappa_2$. Let $\gamma = (\gamma_1, \gamma_2)$ be a curve in U with $|\dot\gamma_i| \sim 1$ for $i = 1, 2$. Then for all pairs of points $v_1, v_2 \in \mathrm{im}(\gamma) + \mathcal{O}((R')^{-1})$ such that $v_1 - v_2 = j/R'$, where $j \in \mathbb{Z}^2$ and $|j| \gg 1$, the following separation condition holds true (again with constants in these estimates which are admissible in the* *obvious sense*):

$$|\nabla\phi(v_1)-\nabla\phi(v_2)|\sim|j|\frac{R'}{(R')^2/\kappa}.$$

Proof. Choose t_1, t_2 such that $v_i=\gamma(t_i)+\mathcal{O}((R')^{-1})$. Then

$$|\nabla\phi(v_i)-\nabla\phi(\gamma(t_i))|\le\|\phi''|_U\|_\infty|v_i-\gamma(t_i)|\lesssim\frac{\kappa}{R'}. \tag{2-14}$$

Therefore

$$\begin{aligned}\frac{|j|}{R'}&=|v_1-v_2|=|\gamma(t_1)-\gamma(t_2)|+\mathcal{O}((R')^{-1})\\&\sim|\dot\gamma_1||t_1-t_2|+|\dot\gamma_2||t_1-t_2|+\mathcal{O}((R')^{-1})\sim|t_1-t_2|+\mathcal{O}((R')^{-1}),\end{aligned}$$

and since $|j|\gg 1$, we see that $|t_1-t_2|\sim|j|/R'$. By our assumptions on ϕ and (2-14), we thus see that there exist s_1 and s_2 lying between t_1 and t_2 such that

$$\begin{aligned}|\nabla\phi(v_1)-\nabla\phi(v_2)|&\ge|\nabla\phi(\gamma(t_1))-\nabla\phi(\gamma(t_2))|-\kappa\mathcal{O}((R')^{-1})\\&\sim\big(|\partial_1^2\phi(\gamma(s_1))\dot\gamma(s_1)|+|\partial_2^2\phi(\gamma(s_2))\dot\gamma(s_2)|\big)\,|t_1-t_2|-\kappa\mathcal{O}((R')^{-1})\\&\sim(\kappa_1+\kappa_2)\frac{|j|}{R'}+\kappa\mathcal{O}((R')^{-1})\sim|j|\frac{\kappa}{R'},\end{aligned}$$

where we used again that $|j|\gg 1$. □

2B. ***A bilinear estimate for normalized hypersurfaces.*** In this section, we shall work under the following:

General Assumptions. Let $\phi\in C^\infty(\mathbb{R}^2)$ such that $\partial_1\partial_2\phi\equiv 0$, and let

$$S_j=\{(\eta,\phi(\eta):\eta\in U_j\},\quad U_j=r^{(j)}+[0,d_1^{(j)}]\times[0,d_2^{(j)}],\quad j=1,2,$$

where $r^{(j)}\in\mathbb{R}^2$ and $d_1^{(j)},d_2^{(j)}>0$. We assume the principal curvature of S_j in the direction of η_1 is comparable to $\kappa_1^{(j)}>0$, and in the direction of η_2 to $\kappa_2^{(j)}>0$, up to some fixed multiplicative constants. We then put for $j=1,2$,

$$\begin{gathered}\kappa^{(j)}=\kappa_1^{(j)}\vee\kappa_2^{(j)},\quad\bar\kappa_i=\kappa_i^{(1)}\vee\kappa_i^{(2)},\quad\bar\kappa=\bar\kappa_1\vee\bar\kappa_2=\kappa^{(1)}\vee\kappa^{(2)},\\ \bar d_i=d_i^{(1)}\vee d_i^{(2)},\quad D=\min_{i,j}d_i^{(j)}.\end{gathered} \tag{2-15}$$

The vector field $N=(-\nabla\phi,1)$ is normal to S_1 and S_2, and thus $N_0=N/|N|$ is a unit normal field to these hypersurfaces. We make the following additional assumptions:

(i) For all $i,j=1,2$ and all $\eta\in U_j$, we have

$$|\partial_i\phi(\eta)-\partial_i\phi(r^{(j)})|\lesssim\kappa_i^{(j)}d_i^{(j)}\quad\text{and}\quad\bar\kappa_i\bar d_i\lesssim 1 \tag{2-16}$$

(notice that the first inequality follows already from our earlier assumptions).

(ii) For all $\eta\in U_1\cup U_2$ and for all $\alpha\in\mathbb{N}^2$, $|\alpha|\ge 2$, we have $|\partial^\alpha\phi(\eta)|\lesssim\bar\kappa D^{2-|\alpha|}$.

(iii) For $i=1,2$, i.e., with respect to both variables, the following separation condition holds true:

$$|\partial_i\phi(\eta^1)-\partial_i\phi(\eta^2)|\sim 1\quad\text{for all }\eta^j\in U_j,\ j=1,2. \tag{2-17}$$

The set of all pairs (S_1, S_2) of hypersurfaces satisfying these properties will be denoted by $\mathcal{S}_0$ (note that it does depend on the constants hidden by the symbols $\lesssim$ and $\sim$).

The main goal of this chapter will be to establish a local, bilinear Fourier extension estimate on suitable cuboids adapted to the wave packets.

Theorem 2.6. *Assume $\frac{5}{3} \le p \le 2$. Let us choose $r \in \mathbb{R}^2$ such that $r = r^{(j)}$ if $\kappa^{(j)} = \kappa^{(1)} \wedge \kappa^{(2)}$. Then for every $\alpha > 0$ there exist constants $C_\alpha, \gamma_\alpha > 0$ such that for every pair $S = (S_1, S_2) \in \mathcal{S}_0$, every parameter $R \ge 1$ and all functions $f_j \in L^2(S_j)$, $j = 1, 2$, we have*

$$\| R^*_{\mathbb{R}^2} f_1 \, R^*_{\mathbb{R}^2} f_2 \|_{L^p(Q^0_{S_1,S_2}(R))} \le C_\alpha R^\alpha (\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} D^{3-\frac{5}{p}} \log^{\gamma_\alpha}(C_0(S)) \, \|f_1\|_2 \, \|f_2\|_2, \tag{2-18}$$

where

$$Q^0_{S_1,S_2}(R) = \left\{ x \in \mathbb{R}^3 : |x_i + \partial_i \phi(r) x_3| \le \frac{R^2}{D^2 \bar{\kappa}}, i = 1, 2, \ |x_3| \le \frac{R^2}{D^2 (\kappa^{(1)} \wedge \kappa^{(2)})} \right\}, \tag{2-19}$$

with

$$C_0(S) = \frac{\bar{d}_1^2 \bar{d}_2^2}{D^4} (D[\kappa^{(1)} \wedge \kappa^{(2)}])^{-\frac{1}{p}} (D\kappa^{(1)} D\kappa^{(2)})^{-\frac{1}{2}}. \tag{2-20}$$

Notice that $C_0(S) \gtrsim 1$.

Remark 2.7. If $\kappa^{(1)} = \kappa^{(2)} = \bar{\kappa}$, then r is not well defined. But in this case the two sets

$$Q^0_{S_1,S_2}(R; j) = \left\{ x \in \mathbb{R}^3 : |x_i + \partial_i \phi(r^{(j)}) x_3| \le \frac{R^2}{D^2 \bar{\kappa}}, i = 1, 2, \ |x_3| \le \frac{R^2}{D^2 \bar{\kappa}} \right\}, \quad j = 1, 2,$$

essentially coincide. Indeed, since $|\nabla\phi(r^{(1)}) - \nabla\phi(r^{(2)})| \sim 1$ (due to the transversality assumption (iii)), an easy geometric consideration shows that

$$a Q^0_{S_1,S_2}(R; 1) \subset Q^0_{S_1,S_2}(R; 2) \subset b Q^0_{S_1,S_2}(R; 1)$$

for some constants a, b which do not depend on R and the class $\mathcal{S}_0$ from which $S = (S_1, S_2)$ is taken.

By applying a suitable affine transformation whose linear part fixes the points of $\mathbb{R}^2 \times \{0\}$, if necessary, we may assume without loss of generality that $r = 0$ and $\nabla\phi(r) = 0$. Notice that conditions (i)–(iii) and the conclusion of the theorem are invariant under such affine transformations.

In fact, we shall then prove estimate (2-18) in the theorem on the even larger cuboid

$$Q_{S_1,S_2}(R) = \left\{ x \in \mathbb{R}^3 : |x_{i_0}| \le \frac{R^2}{D^2 \bar{\kappa}}, \ \|x\|_\infty \le \frac{R^2}{D^2 (\kappa^{(1)} \wedge \kappa^{(2)})} \right\} \tag{2-21}$$

for an appropriate choice of the coordinate direction x_{i_0}, $i_0 \in \{1, 2\}$, in which the cuboid has smaller side length. Later we shall need to combine different cuboids which may possibly have their smaller side lengths in different directions. Then it will become necessary to restrict to their intersection, which leads to (2-19).

Indeed, we shall see that there will be two directions in which the side length of the cuboids are dictated by the length of the wave packets, and one remaining third direction for which we shall have more freedom in choosing the side length.

Observe also that $\bar\kappa_i \bar d_i \lesssim 1$, and thus we may even assume without loss of generality that

$$\bar\kappa_i \bar d_i \ll 1 \quad \text{for all } i = 1, 2 \tag{2-22}$$

simply by decomposing S_1 and S_2 into a finite number of subsets for which the side lengths of corresponding rectangles U_j are sufficiently small fractions of the given $d_i^{(j)}$.

For $\eta^j \in U_j$ define

$$\begin{aligned}\phi_1(\eta) &= \phi(\eta-\eta^2)+\phi(\eta^2), \quad \eta \in \eta^2 + U_1,\\ \phi_2(\eta) &= \phi(\eta-\eta^1)+\phi(\eta^1), \quad \eta \in \eta^1 + U_2.\end{aligned}$$

The set $((\eta^2, \phi(\eta^2)) + S_1) \cap ((\eta^1, \phi(\eta^1)) + S_2) = \operatorname{graph}(\phi_1) \cap \operatorname{graph}(\phi_2)$ will be called an *intersection curve* of S_1 and S_2. It agrees with the graph of ϕ_1 (or ϕ_2) restricted to the set where $\psi = \phi_1 - \phi_2 = 0$. On this set, the normal field $N_j(\eta) = (-\nabla\phi_j(\eta), 1)$ forms the conical set

$$\Gamma_j = \{sN_j(\eta) : s \in \mathbb{R}, \psi(\eta) = 0\}.$$

In the sequel, we shall use the abbreviation $j + 1 \bmod 2 = 2$, if $j = 1$, and $j + 1 \bmod 2 = 1$, if $j = 2$.

Lemma 2.8. *Let $(S_1, S_2) \in \mathcal{S}_0$. Assume $\nabla\phi(r) = 0$ for some $r \in S_1 \cup S_2$ and $\bar\kappa_i \bar d_i \ll 1$. Then the following hold true*:

(a) *$D\kappa_i^{(j)} \ll 1$ for all $i, j = 1, 2$.*

(b) *$|\nabla\phi(\eta)| \lesssim 1$ for all $(\eta, \phi(\eta)) \in S_1 \cup S_2$.*

(c) *The unit normal fields on S_1 and S_2 are transversal*; *i.e.*,

$$|N_0(\eta^1) - N_0(\eta^2)| \sim 1 \quad \textit{for all } (\eta^j, \phi(\eta^j)) \in S_j. \tag{2-23}$$

(d) *N_j and $\Gamma_{j+1 \bmod 2}$ are transversal for $j = 1, 2$ and for any choice of intersection curve of S_1 and S_2.*

(e) *If γ is a parametrization by the arclength t of the projection of an intersection curve of S_1 and S_2 to the first two coordinates $\eta \in \mathbb{R}^2$, then $|\dot\gamma_1| \sim 1 \sim |\dot\gamma_2|$.*

Proof. We shall denote by $\eta = \tilde x \in \mathbb{R}^2$ the projection of a point in $x \in \mathbb{R}^3$ to its first two coordinates. Part (a) is clear since $D = \min_{i,j=1,2} d_i^{(j)}$. To prove (b), notice that for any $x, x' \in S_1 \cup S_2$ we have $|\nabla\phi(\tilde x) - \nabla\phi(\tilde x')| \lesssim 1$: if x and x' belong to different hypersurface S_j, we apply condition (iii) on page 838, and if x and x' are in the same hypersurface S_j, we use condition (a). Thus we have $|\nabla\phi(\tilde x)| = |\nabla\phi(\tilde x) - \nabla\phi(r)| \lesssim 1$ for all $x \in S_1 \cup S_2$.

This gives $|N(\tilde x)| = \sqrt{1 + |\nabla\phi(\tilde x)|^2} \sim 1$ for all $x \in S_1 \cup S_2$, which already implies the transversality of the normal fields:

$$|N_0(\eta^1) - N_0(\eta^2)| \sim |N(\eta^1) - N(\eta^2)| = |\nabla\phi(\eta^1) - \nabla\phi(\eta^2)| \sim 1$$

for all $(\eta^j, \phi(\eta^j)) \in S_j$, $j = 1, 2$.

We shall prove (e) first, since (e) will be needed for the proof of (d). It suffices to prove that $|\partial_i\psi(\eta)| \sim 1$ for all η such that $\eta - \eta^j \in U_{j+1 \bmod 2}$, $\eta^j \in U_j$, since the tangent to the curve γ at any point $\gamma(t)$ is orthogonal to $\nabla\psi(\gamma(t))$. But, in view of (2-17),

$$|\partial_i\psi(\eta)| = |\partial_i\phi(\eta-\eta^2)-\partial_i\phi(\eta-\eta^1)| \sim 1.$$

For (d), since the claim is symmetric in $j\in\{1,2\}$, it suffices to show that N_1 and Γ_2 are transversal. Since we have

$$|N_1(\eta)-N_1(\eta')| = |\nabla\phi_1(\eta)-\nabla\phi_1(\eta')| \lesssim \kappa_1^{(1)}d_1^{(1)}+\kappa_2^{(1)}d_2^{(1)} \ll 1$$

for all $\eta,\eta'\in U_1+\eta^2$, whereas $|N_1(\eta)|\sim 1$ for all $\eta\in U_1+\eta^2$, it is even enough to show that $N_1(\eta)$ and the tangent space $T_{N_2(\eta)}\Gamma_2$ of Γ_2 at the point $N_2(\eta)$ are transversal. Since γ is a parametrization by arclength of the zero set of ψ, the tangent space of Γ_2 at the point $N_2(\eta)$ for $\eta=\gamma(t)$ is spanned by $N_2(\eta)$ and $(-D^2\phi_2(\eta)\,\dot\gamma(t),0)$, where $D^2\phi_2$ denotes the Hessian matrix of ϕ_2. But, recalling that we assume $\partial_1\partial_2\phi\equiv 0$, we see that the vectors $N_2(\eta)$ and $(1/\kappa^{(2)})(-D^2\phi_2(\eta)\,\dot\gamma(t),0)$ form an "almost" orthonormal frame for the tangent space $T_{N_2(\eta)}\Gamma_2$, and thus the transversality can be checked by estimating the volume V of the parallelepiped spanned by $N_1(\eta)$ and these two vectors, which is given by

$$V=\begin{vmatrix} -\partial_1\phi_1(\eta) & -\partial_2\phi_1(\eta) & 1 \\ -\partial_1\phi_2(\eta) & -\partial_2\phi_2(\eta) & 1 \\ \frac{1}{\kappa^{(2)}}\partial_1^2\phi_2(\eta)\dot\gamma_1(t) & \frac{1}{\kappa^{(2)}}\partial_1^2\phi_2(\eta)\dot\gamma_2(t) & 0\end{vmatrix} = \frac{1}{\kappa^{(2)}}\big|-\partial_1^2\phi_2(\eta)\dot\gamma_1(t)\partial_2\psi(\eta)+\partial_2^2\phi_2(\eta)\dot\gamma_2(t)\partial_1\psi(\eta)\big|.$$

Since $\psi\circ\gamma=0$ by definition, we have $\partial_1\psi(\eta)\dot\gamma_1(t)+\partial_2\psi(\eta)\dot\gamma_2(t)$; hence

$$\partial_2\psi(\eta)=-\partial_1\psi(\eta)\frac{\dot\gamma_1(t)}{\dot\gamma_2(t)}.$$

Thus

$$V=\frac{|\partial_1\psi(\eta)|}{\kappa^{(2)}|\dot\gamma_2(t)|}\big(\partial_1^2\phi_2(\eta)\dot\gamma_1^2(t)+\partial_2^2\phi_2(\eta)\dot\gamma_2^2(t)\big)\sim\big|\partial_1\phi(\eta-\eta^2)-\partial_1\phi(\eta-\eta^1)\big|\frac{\kappa_1^{(2)}+\kappa_2^{(2)}}{\kappa^{(2)}}\sim 1. \qquad \square$$

We now come to the introduction of the wave packets that we shall use in the proof of Theorem 2.6. Let us assume without loss of generality that

$$\kappa^{(1)}\le\kappa^{(2)}, \tag{2-24}$$

i.e., $r=r^{(1)}$ and $\nabla\phi(r^{(1)})=0$.

Next, since S_1 is horizontal at $(r^{(1)},\phi(r^{(1)}))$, we may use the wave packet decomposition from Corollary 2.3, with normal n_1 and hyperplane H_1 given by

$$n_1=(0,0,1) \quad\text{and}\quad H_1=\mathbb{R}^2\times\{0\}$$

in order to obtain the decomposition

$$R^*_{\mathbb{R}^2}f_1=R^*_{H_1}f_1=\sum_{w_1\in\mathcal{W}_1}c_{w_1}p_{w_1},\qquad w_1\in\mathcal{W}_1, \tag{2-25}$$

into wave packets p_{w_1}, $w_1\in\mathcal{W}_1$ of length $(R')^2/\kappa^{(1)}$, directly by means of Lemma 2.1. By T_{w_1}, $w_1\in\mathcal{W}_1$, we denote the associated set of tubes. Recall that this decomposition is valid on the set $P_1=\mathbb{R}^2\times[-(R')^2/\kappa^{(1)},(R')^2/\kappa^{(1)}]$.

Let us next turn to S_2 and $R^*_{\mathbb{R}^2} f_2$. If we would keep the same coordinate system for S_2, we would have to truncate even further in x_3-direction, since $(R')^2/\kappa^{(2)} \le (R')^2/\kappa^{(1)}$. However, by (2-17) we have for $\eta \in U_2$ and both $i = 1$ and $i = 2$ that

$$|\langle e_i, N(\eta)\rangle| = |\partial_i \phi(\eta)| = |\partial_i \phi(\eta) - \partial\phi(r^{(1)})| \sim 1.$$

This means that we may apply Lemma 2.4 to S_2 in order to reparametrize S_2 by an open subset (denoted again by U_2) of the hyperplane $H_2 = n_2^\perp$ given by

$$n_2 = e_{i_0} \quad \text{and} \quad H_2 = \{n_2\}^\perp = \{e_{i_0}\}^\perp.$$

We may thus replace the function f_2 by a function (also denoted by f_2) on U_2 of comparable L^2-norm, and replace $R^*_{\mathbb{R}^2} f_2$ by $R^*_{H_2} f_2$ in the subsequent arguments.

Next, applying Corollary 2.3, now with $H = H_2$, for $i_0 = 1$, as well as for $i_0 = 2$, we may decompose $R^*_{H_2} f_2$ as

$$R^*_{H_2} f_2 = \sum_{w_2 \in \mathcal{W}_2} c_{w_2} p_{w_2}, \quad w_2 \in \mathcal{W}_2, \tag{2-26}$$

on the set

$$P_2 = \left\{ x \in \mathbb{R}^3 : |\langle x, n_2\rangle| \le \frac{(R')^2}{\kappa^{(2)}} \right\} = \left\{ x \in \mathbb{R}^3 : |x_{i_0}| \le \frac{(R')^2}{\kappa^{(2)}} \right\}$$

by means of wave packets of length $(R')^2/\kappa^{(2)}$. The associated set of tubes is denoted by T_{w_2}, $w_2 \in \mathcal{W}_2$.

In order to decide how to choose i_0, we observe that for $\eta \in U_1$, our definitions (2-15) in combination with the estimates (2-16) and (2-22) show that

$$|\partial_i \phi(\eta) - \partial_i \phi(r^{(1)})| \lesssim \kappa_i^{(1)} d_i^{(1)} \le \frac{\kappa_i^{(1)}}{\bar\kappa_i} \bar\kappa_i \bar d_i \ll \frac{\kappa_i^{(1)}}{\bar\kappa_i}.$$

Notice that the wave packets associated to S_1 are roughly pointing in the direction of $N(r^{(1)}) = (0, 0, 1)$. More precisely, if we project a wave packet pointing in the direction of $N(\eta)$, $\eta \in U_1$, to the coordinate x_i, $i = 1, 2$, then by the previous estimates we see that we obtain an interval of length comparable to

$$\left|\left\langle e_i, \frac{(R')^2}{\kappa^{(1)}} N(\eta)\right\rangle\right| = \frac{(R')^2}{\kappa^{(1)}} |\partial_i \phi(\eta)| = \frac{(R')^2}{\kappa^{(1)}} |\partial_i \phi(\eta) - \partial_i \phi(r^{(1)})| \ll \frac{(R')^2}{\bar\kappa} \frac{\bar\kappa}{\kappa^{(1)}} \frac{\kappa_i^{(1)}}{\bar\kappa_i}. \tag{2-27}$$

Let us therefore choose i_0 so that

$$\frac{\kappa_{i_0}^{(1)}}{\bar\kappa_{i_0}} = \frac{\kappa_1^{(1)}}{\bar\kappa_1} \wedge \frac{\kappa_2^{(1)}}{\bar\kappa_2}.$$

Then

$$\bar\kappa \frac{\kappa_{i_0}^{(1)}}{\bar\kappa_{i_0}} = (\bar\kappa_1 \vee \bar\kappa_2) \left(\frac{\kappa_1^{(1)}}{\bar\kappa_1} \wedge \frac{\kappa_2^{(1)}}{\bar\kappa_2} \right) \le \kappa_1^{(1)} \vee \kappa_2^{(1)} = \kappa^{(1)},$$

and thus by (2-27) and (2-24)

$$\left|\left\langle e_{i_0}, \frac{(R')^2}{\kappa^{(1)}} N(\eta)\right\rangle\right| \ll \frac{(R')^2}{\bar\kappa} = \frac{(R')^2}{\kappa^{(2)}}.$$

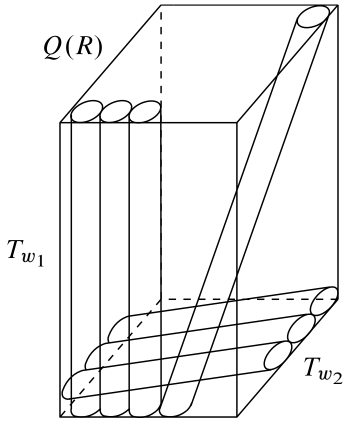

Figure 8. The wave packets filling the cuboid $Q_{S_1,S_2}(R)$.

This means that the geometry fits well: the wave packets associated to S_1 do not turn too much into the direction of x_{i_0}; projected to this coordinate, their length is smaller than the length of the wave packets associated to S_2, which are essentially pointing in the direction of the i_0-th coordinate axis (see Figure 8).

However, for the remaining coordinate direction x_i, $i \in \{1,2\} \setminus \{i_0\}$, we cannot guarantee such a behavior. But notice that by (2-24),

$$\begin{aligned} P_1 \cap P_2 &= \left(\mathbb{R}^2 \times \left[-\frac{(R')^2}{\kappa^{(1)}}, \frac{(R')^2}{\kappa^{(1)}}\right]\right) \cap \left\{x \in \mathbb{R}^3 : |x_{i_0}| \le \frac{(R')^2}{\kappa^{(2)}}\right\} \\ &= \left\{(x_1,x_2) \in \mathbb{R}^2 : |x_{i_0}| \le \frac{(R')^2}{\kappa^{(2)}}\right\} \times \left[-\frac{(R')^2}{\kappa^{(1)}}, \frac{(R')^2}{\kappa^{(1)}}\right] \\ &\supset \left\{x \in \mathbb{R}^3 : |x_{i_0}| \le \frac{(R')^2}{\bar{\kappa}},\ \|x\|_\infty \le \frac{(R')^2}{\kappa^{(1)} \wedge \kappa^{(2)}}\right\} = Q_{S_1,S_2}(R); \end{aligned}$$

i.e., on the cuboid $Q_{S_1,S_2}(R)$ we may apply our development into wave packets to the wave packets associated to the hypersurface S_1, as well as those associated to S_2.

For every $\alpha > 0$, let us denote by $E(\alpha)$ the following statement:

> There exist constants $C_\alpha > 0$ and $\gamma_\alpha > 0$ such that for all pairs $S = (S_1, S_2) \in \mathcal{S}_0$, all $R \ge 1$ and all $f_j \in L^2(U_j)$, $j = 1, 2$, (which we may also regard as functions on S_j) the following estimate holds true:
>
> $$\begin{aligned} &\|R^*_{H_1} f_1 R^*_{H_2} f_2\|_{L^p(Q_{S_1,S_2}(R))} \\ &\qquad \le C_\alpha R^\alpha \log^{\gamma_\alpha}(1+R)(\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} D^{3-\frac{5}{p}} \log^{\gamma_\alpha}(C_0(S)) \|f_1\|_2 \|f_2\|_2. \qquad (2\text{-}28) \end{aligned}$$
>
> Here, $C_0(S)$ denotes the constant defined in Theorem 2.6.

Our goal will be to show that $E(\alpha)$ holds true for every $\alpha > 0$, which would prove Theorem 2.6. To this end, we shall apply the method of induction on scales.

Observe that the intersection of two of the transversal tubes $T_{w_1}, w_1 \in \mathcal{W}_1$, and $T_{w_2}, w_2 \in \mathcal{W}_2$, will always be contained in a cube of side length $\mathcal{O}(R')$. Let us therefore decompose $\mathbb{R}^3$ by means of a grid of side length R' into cubes q of the same side length, and let $\{q\}_{q\in\mathcal{Q}}$ be a family of such cubes covering $Q_{S_1,S_2}(R)$. By c_q we shall denote the center of the cube q. Choose $\chi \in \mathcal{S}(\mathbb{R}^3)$ with $\operatorname{supp}\hat{\chi} \subset B(0,1)$ and $\hat{\chi}(0) = 1/(2\pi)^n$, and put $\chi_q(x) = \chi(x - c_q/(R'))$. Poisson's summation formula then implies $\sum \chi_q = 1$ on $\mathbb{R}^3$, so that in particular we may assume $\sum_{q\in\mathcal{Q}} \chi_q = 1$ on $Q_{S_1,S_2}(R)$.

Notice that our approach slightly differs from the standard usage of induction on scales, where χ_q is chosen to be the characteristic function of q, and not a smoothened version of it. The price we shall have to pay is that some arguments will become a bit more technical, but the compact Fourier support of the functions χ_q will become crucial later.

For a given index set $W_j \subset \mathcal{W}_j$, $j = 1, 2$, of wave packets (see (2-25), (2-26)), we denote by

$$W_j(q) = \{w_j \in W_j : T_{w_j} \cap R^\delta q \neq \varnothing\}$$

the collection of all the tubes of type j passing through (a slightly thickened) cube q. Here, $\delta > 0$ is a small parameter which will be fixed later, and $R^\delta q$ denotes the dilate of q by the factor R^δ having the same center c_q as q.

Let us denote by $\mathcal{N}$ the set $\mathcal{N} = \{2^n : n \in \mathbb{N}\} \cup \{0\}$. In order to count the magnitude of the number of wave packets W_j passing through a given cube q, we introduce the sets

$$Q^\mu = \{q : |W_j(q)| \sim \mu_j,\ j = 1, 2\}, \quad \mu = (\mu_1, \mu_2) \in \mathcal{N}^2.$$

Obviously the Q^μ form a partition of the family of all cubes $q \in \mathcal{Q}$. For $w_j \in W_j$, we further introduce the set of all cubes in Q^μ close to T_{w_j}:

$$Q^\mu(w_j) = \{q \in Q^\mu : T_{w_j} \cap R^\delta q \neq \varnothing\}.$$

Finally, we determine the number of such cubes by means of the sets

$$W_j^{\lambda_j,\mu} = \{w_j \in W_j : |Q^\mu(w_j)| \sim \lambda_j\}, \quad \lambda_j, \mu_1, \mu_2 \in \mathcal{N}.$$

For every fixed μ, the family $\{W_j^{\lambda_j,\mu}\}_{\lambda_j\in\mathcal{N}}$ forms a partition of W_j.

We are now in a position to reduce the statement $E(\alpha)$ to a formulation in terms of wave packets.

2C. *Reduction to a wave packet formulation.* Following basically a standard pigeonholing argument in combination with (P5), the estimate in $E(\alpha)$ can easily be reduced to a bilinear estimate for sums of wave packets (modulo an increase of the exponent γ_α by 5). It is in this reduction that some power of the logarithmic factor $\log(C_0(S))$ will appear, and we shall have to be a bit more precise than usual in order to identify $C_0(S)$ as the expression given by (2-20).

Lemma 2.9. *Let $\alpha > 0$. Assume there are constants $C_\alpha, \gamma_\alpha > 0$ such that for all $(S_1, S_2) \in \mathcal{S}_0$ (parametrized by the open subsets $U_j \subset H_j$) the following estimate is satisfied*:

Given any two families of wave packets $\{p_{w_1}\}_{w_1\in\mathcal{W}_1}$ and $\{p_{w_2}\}_{w_2\in\mathcal{W}_2}$ associated to S_1 and S_2 respectively, as in the wave packet decomposition Corollary 2.3, where the p_{w_j}, $j = 1, 2$, satisfy *uniformly the estimates in (P2)–(P5), for all $R \geq 1$, all $\lambda_j, \mu_j \in \mathcal{N}$ and all subsets $W_j \subset \mathcal{W}_j$, $j = 1, 2$, we have (with admissible constants)*

$$\Bigl\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j,\mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \Bigr\|_{L^p(Q_{S_1,S_2}(R))} \leq C_\alpha R^\alpha \log^{\gamma_\alpha}(1+R) (\kappa^{(1)}\kappa^{(2)})^{\frac12 - \frac1p} D^{3-\frac5p} \log^{\gamma_\alpha}(C_0(S)) |W_1|^{\frac12} |W_2|^{\frac12}. \tag{2-29}$$

Then $E(\alpha)$ holds true.

Proof. In order to show $E(\alpha)$, we may assume without loss of generality that $\|f_j\|_2 = 1$, $j = 1, 2$. Let us use the abbreviation $C_0(S) = C_0$.

First observe that for fixed q and v_j, the number of y_j such that the tube $T_{(y_j,v_j)}$ passes through $R^\delta q$ is bounded by $R^{c\delta}$, whereas the total number of $v_j \in V_j$ is bounded by

$$|V_j| \sim (R')^2 |U_j| \leq R^2 \frac{\bar d_1 \bar d_2}{D^2}. \tag{2-30}$$

Thus we have

$$|W_j(q)| \leq R^{2+c\delta} \frac{\bar d_1 \bar d_2}{D^2} \leq R^{2+c} \frac{\bar d_1^2 \bar d_2^2}{D^4} (D[\kappa^{(1)} \wedge \kappa^{(2)}])^{-\frac1p} (D\kappa^{(1)} D\kappa^{(2)})^{-\frac12} = R^{c'} C_0,$$

where we have used property (a) of Lemma 2.8. Consequently $Q^\mu = \varnothing$, if $\mu_j \gg R^{c'} C_0$ for some j. Similarly, the number of cubes q of side length R' such that $R^\delta q$ intersects with a tube T_{w_j} of length $(R')^2/\kappa^{(j)}$ is bounded by $R^{c\delta} R'/\kappa^{(j)} = R^{1+c\delta}/D\kappa^{(j)}$. Since $D \leq \bar d_1, \bar d_2$, this implies

$$|Q^\mu(w_j)| \leq \frac{R^{1+c\delta}}{D\kappa^{(j)}} \leq R^{c'} \frac{\bar d_1^4 \bar d_2^4}{D^8} (D[\kappa^{(1)} \wedge \kappa^{(2)}])^{-\frac2p} (D\kappa^{(1)} D\kappa^{(2)})^{-1} = R^{c'} C_0^2,$$

and thus $W_j^{\lambda_j,\mu} = \varnothing$, if $\lambda_j \gg R^{c'} C_0^2$. For $C \geq 0$ let us put $\mathcal{N}(C) = \{\nu \in \mathcal{N} : \nu \lesssim C\}$. Since $C_0 \gtrsim 1$, we then see that

$$\mathcal{Q} = \bigcup_{\mu_1,\mu_2 \in \mathcal{N}(R^{c'} C_0^2)} Q^\mu,$$

and for every fixed μ,

$$W_j = \bigcup_{\lambda_j \in \mathcal{N}(R^{c'} C_0^2)} W_j^{\lambda_j,\mu}.$$

These decompositions in combination with our assumed estimate (2-29) imply

$$\begin{aligned} \Bigl\| \prod_{j=1,2} \sum_{w_j \in W_j} p_{w_j} \Bigr\|_{L^p(Q_{S_1,S_2}(R))} &\leq \sum_{\lambda_1,\lambda_2,\mu_1,\mu_2 \in \mathcal{N}(R^{c'} C_0^2)} \Bigl\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j,\mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \Bigr\|_{L^p(Q_{S_1,S_2}(R))} \\ &\leq C_\alpha R^\alpha \log^4(R^{c'} C_0^2) \log^{\gamma_\alpha}(1+R) (\kappa^{(1)}\kappa^{(2)})^{\frac12 - \frac1p} D^{3-\frac5p} \log^{\gamma_\alpha}(C_0) |W_1|^{\frac12} |W_2|^{\frac12} \end{aligned}$$

for every $W_j \subset \mathcal{W}_j$, $j = 1, 2$; hence

$$\Big\| \prod_{j=1,2} \sum_{w_j \in W_j} p_{w_j} \Big\|_{L^p(Q_{S_1,S_2}(R))} \le C_\alpha R^\alpha \log^{\gamma_\alpha+4}(1+R)(\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} D^{3-\frac{5}{p}} \log^{\gamma_\alpha+4}(C_0)|W_1|^{\frac{1}{2}}|W_2|^{\frac{1}{2}}. \tag{2-31}$$

Recall next that $R^* f_j = \sum_{w_j \in \mathcal{W}_j} c_{w_j} p_{w_j}$. We introduce the subsets $W_j^k = \{w_j \in \mathcal{W}_j : |c_{w_j}| \sim 2^{-k}\}$, which allow us to partition $\mathcal{W}_j$ into $\bigcup_{k\in\mathbb{N}} W_j^k$. We fix some k_0, whose precise value will be determined later. Then

$$\Big\| \sum_{k>k_0} \sum_{w_1\in W_1^k} \sum_{w_2\in\mathcal{W}_2} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \Big\|_{L^p(Q_{S_1,S_2}(R))} \le |Q_{S_1,S_2}(R)|^{\frac{1}{p}} \sum_{k>k_0} \Big\| \sum_{w_1\in W_1^k} \sum_{w_2\in\mathcal{W}_2} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \Big\|_\infty.$$

The wave packets p_{w_j} are well separated with respect to the parameter y_j, and by (P4), their L^∞- norm is of order $\mathcal{O}((R')^{-1})$. Moreover, by (2-30) the number of v_j's is bounded by $R^2\bar{d}_1\bar{d}_2/D^2$. Furthermore, $|c_{w_1}| \lesssim 2^{-k}$ for every $w_1 \in W_1^k$, and by (P5) we have $|c_{w_2}| \le \|\{c_{w_2}\}_{w_2\in\mathcal{W}_2}\|_{\ell^2} \lesssim \|f_2\|_2 = 1$. Combining all this information, we may estimate

$$\begin{aligned}\Big\| \sum_{k>k_0} \sum_{w_1\in W_1^k} \sum_{w_2\in\mathcal{W}_2} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \Big\|_{L^p(Q_{S_1,S_2}(R))} &\lesssim \Big(\frac{(R')^6}{[\kappa^{(1)}\wedge\kappa^{(2)}]\kappa^{(1)}\kappa^{(2)}} \Big)^{\frac{1}{p}} \frac{\bar{d}_1^2\bar{d}_2^2}{D^4} R^4 (R')^{-2} \sum_{k>k_0} 2^{-k} \\ &\sim R^{\frac{6}{p}+2} C_0 D^{3-\frac{5}{p}} (\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} 2^{-k_0}.\end{aligned}$$

If we now choose $k_0 = \log_2 C_0 + \log R^{\frac{6}{p}+2}$, then we obtain

$$\Big\| \sum_{k>k_0} \sum_{w_1\in W_1^k} \sum_{w_2\in\mathcal{W}_2} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \Big\|_{L^p(Q_{S_1,S_2}(R))} \lesssim D^{3-\frac{5}{p}} (\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}}. \tag{2-32}$$

In a similar way we also get

$$\Big\| \sum_{k_1\le k_0} \sum_{w_1\in W_1^{k_1}} \sum_{k_2>k_0} \sum_{w_2\in W_2^{k_2}} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \Big\|_{L^p(Q_{S_1,S_2}(R))} \lesssim D^{3-\frac{5}{p}} (\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}}. \tag{2-33}$$

The remaining terms can simply be estimated by

$$\Big\| \sum_{k_1,k_2=1}^{k_0} \sum_{w_1\in W_1^{k_1}} \sum_{w_2\in\mathcal{W}_2^{k_2}} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \Big\|_{L^p(Q_{S_1,S_2}(R))} \lesssim \sum_{k_1,k_2=1}^{k_0} 2^{-k_1-k_2} \Big\| \sum_{w_1\in W_1^{k_1}} \sum_{w_2\in\mathcal{W}_2^{k_2}} c_{w_1} 2^{k_1} p_{w_1} c_{w_2} 2^{k_2} p_{w_2} \Big\|_{L^p(Q_{S_1,S_2}(R))}.$$

Since $|c_{w_j} 2^{k_j}| \sim 1$ for $w_j \in W_j^{k_j}$, it is appropriate to apply (2-31) to the modified wave packets $\tilde{p}_{w_j} = c_{w_j} 2^{k_j} p_{w_j}$:

$$\Big\| \sum_{k_1,k_2=1}^{k_0} \sum_{w_1 \in W_1^{k_1}} \sum_{w_2 \in \mathcal{W}_2^{k_2}} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \Big\|_{L^p(Q_{S_1,S_2}(R))}$$
$$\leq C_\alpha R^\alpha \log^{\gamma_\alpha+4}(1+R)(\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} D^{3-\frac{5}{p}} \log^{\gamma_\alpha+4}(C_0) \sum_{k_1,k_2=1}^{k_0} 2^{-k_1-k_2} |W_1^{k_1}|^{\frac{1}{2}} |W_2^{k_2}|^{\frac{1}{2}}.$$

But observe that by (P5),

$$\sum_{k_1,k_2=1}^{k_0} 2^{-k_1-k_2} |W_1^{k_1}|^{\frac{1}{2}} |W_2^{k_2}|^{\frac{1}{2}} \leq k_0 \Big(\sum_{k_1=1}^{k_0} |W_1^{k_1}| 2^{-2k_1} \sum_{k_2=1}^{k_0} |W_2^{k_2}| 2^{-2k_2} \Big)^{\frac{1}{2}}$$
$$\lesssim k_0 \Big(\sum_{k_1=1}^{k_0} \sum_{w_1 \in W_1^{k_1}} |c_{w_1}|^2 \sum_{k_2=1}^{k_0} \sum_{w_2 \in W_2^{k_2}} |c_{w_2}|^2 \Big)^{\frac{1}{2}} \lesssim k_0 \|f_1\|_2 \|f_2\|_2 = k_0,$$

and thus

$$\Big\| \sum_{k_1,k_2=1}^{k_0} \sum_{w_1 \in W_1^{k_1}} \sum_{w_2 \in W_1^{k_2}} c_{w_1} p_{w_1} c_{w_2} p_{w_2} \Big\|_{L^p(Q_{S_1,S_2}(R))}$$
$$\lesssim C_\alpha R^\alpha \log^{\gamma_\alpha+5}(1+R)(\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} D^{3-\frac{5}{p}} \log^{\gamma_\alpha+5}(C_0). \quad (2\text{-}34)$$

Combining (2-32)–(2-34), we find that

$$\|R_{H_1}^* f_1 R_{H_2}^* f_2\|_{L^p(Q_{S_1,S_2}(R))} = \Big\| \prod_{j=1,2} \sum_{w_j \in \mathcal{W}_j} c_{w_j} p_{w_j} \Big\|_{L^p(Q_{S_1,S_2}(R))}$$
$$\lesssim C_\alpha R^\alpha \log^{\gamma_\alpha+5}(1+R)(\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} D^{3-\frac{5}{p}} \log^{\gamma_\alpha+5}(C_0),$$

which verifies $E(\alpha)$. □

2D. *Bilinear estimates for sums of wave packets.* Let $v_j \in \mathcal{V}_j$, $j = 1, 2$, and define the ($\mathcal{O}(1/R')$ thickened) “intersection” of the transversal hypersurfaces S_1 and S_2 by

$$\Pi_{v_1,v_2} = (v_1 + S_2) \cap (v_2 + S_1) + \mathcal{O}((R')^{-1}).$$

For any subset $W_j \subset \mathcal{W}_j$, let

$$W_j^{\Pi_{v_1,v_2}} = \{w_j' \in W_j : v_j' + v_{j+1} \in \Pi_{v_1,v_2}\}$$

(where $j+1$ is to be interpreted mod 2 as before, i.e., we will use the shorthand notation $j+1 = j+1 \bmod 2$ in the sequel whenever $j+1$ appears as an index), and denote by

$$V_j = \{v_j' \in \mathcal{V}_j : (y_j', v_j') \in W_j \ \text{ for some } y_j' \in \mathcal{Y}\}$$

the $\mathcal{V}$-projection of W_j. Further let

$$\begin{aligned} V_j^{\Pi_{v_1,v_2}} &= \{v_j' \in \mathcal{V}_j : (y_j', v_j') \in W_j^{\Pi_{v_1,v_2}} \text{ for some } y_j' \in \mathcal{Y}_j\} \\ &= \{v_j' \in \mathcal{V}_j : \text{ there is some } y_j' \in \mathcal{Y}_j \text{ s.t. } (y_j', v_j') \in W_j \text{ and } v_j' + v_{j+1} \in \Pi_{v_1,v_2}\}. \end{aligned}$$

Lemma 2.10. *Let $W_j \subset \mathcal{W}_j$, $j = 1, 2$. Then*

$$\Big\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \Big\|_{L^1(Q_{S_1,S_2}(R))} \le \frac{(R')^2}{\sqrt{\kappa^{(1)}\kappa^{(2)}}} |W_1|^{\frac12} |W_2|^{\frac12}, \tag{2-35}$$

$$\Big\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \Big\|_{L^2(Q_{S_1,S_2}(R))} \lesssim (R')^{-\frac12} \min_j \sup_{v_1,v_2} |V_j^{\Pi_{v_1,v_2}}|^{\frac12} |W_1|^{\frac12} |W_2|^{\frac12}. \tag{2-36}$$

Proof. We shall closely follow the arguments in [Lee and Vargas 2010], in particular the proof of Lemma 2.2, with only slight modifications.

The first estimate is easy. Using Hölder's inequality, we see that

$$\begin{aligned} \Big\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \Big\|_{L^1(Q_{S_1,S_2}(R))} &\le \prod_{j=1,2} \Big\| \sum_{w_j \in W_j} p_{w_j} \Big\|_{L^2(Q_{S_1,S_2}(R))} \\ &\le \prod_{j=1,2} \bigg(\int_{-(R')^2/\kappa^{(j)}}^{(R')^2/\kappa^{(j)}} \Big\| \sum_{w_j \in W_j} p_{w_j}(\cdot + t n_j) \Big\|_{L^2(H_j)}^2 dt \bigg)^{\frac12} \\ &\lesssim \prod_{j=1,2} \frac{R'}{\sqrt{\kappa^{(j)}}} |W_j|^{\frac12}, \end{aligned}$$

where we have used (P4) in the last estimate. The second one is more involved. We write

$$\Big\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \Big\|_{L^2(Q_{S_1,S_2}(R))}^2 = \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} \sum_{v_1' \in V_1} \sum_{v_2' \in V_2} \Big\langle p_{w_1} \sum_{y_2' \in Y_2(v_2')} p_{w_2'},\, p_{w_2} \sum_{y_1' \in Y_1(v_1')} p_{w_1'} \Big\rangle,$$

where $Y_j(v_j') = \{y \in \mathcal{Y}_j : (y, v_j') \in W_j\}$ (recall that V_j is $\mathcal{V}$-projection of W_j). Since for $j = 1, 2$ the Fourier transform of $\sum_{y_{j+1}' \in Y_{j+1}(v_{j+1}')} p_{w_{j+1}'} p_{w_j}$ is supported in a ball of radius $\mathcal{O}((R')^{-1})$ centered at $v_{j+1}' + v_j$, we may assume that the intersection of these two balls is nonempty, and thus

$$v_1' + v_2 = v_2' + v_1 + \mathcal{O}((R')^{-1}).$$

Especially

$$v_{j+1}' + v_j \in \Pi_{v_1,v_2} \quad \text{and} \quad v_j' \in V_j^{\Pi_{v_1,v_2}}, \quad j = 1, 2.$$

This implies

$$\begin{aligned} &\Big\| \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} p_{w_1} p_{w_2} \Big\|_{L^2(Q_{S_1,S_2}(R))}^2 \\ &\qquad \le \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} \sum_{v_1' \in V_1^{\Pi_{v_1,v_2}}} \sum_{v_2'} \int_{\mathbb{R}^3} |p_{w_1} p_{w_2}|\, dx \Big\| \sum_{y_2' \in Y(v_2')} p_{w_2'} \Big\|_\infty \Big\| \sum_{y_1' \in Y(v_1')} p_{w_1'} \Big\|_\infty, \end{aligned}$$

where $v_2' = v_1' + v_2 - v_1 + \mathcal{O}((R')^{-1})$ in the rightmost sum. Observe that there are at most $\mathcal{O}(1)$ possible choices for v_2' such that

$$v_2' = v_1' + v_2 - v_1 + \mathcal{O}((R')^{-1}).$$

Since the wave packets p_{w_j} are essentially supported in the tubes T_{w_j}, which are well separated with respect to the parameter y, the sum in y_j' can be replaced by the supremum, up to some multiplicative constant. Since T_{w_1} and T_{w_2} satisfy the transversality condition (2-23), $p_{w_1}p_{w_2}$ decays rapidly away from the intersection $T_{w_1} \cap T_{w_2}$; i.e.,

$$\int_{\mathbb{R}^3} |p_{w_1}p_{w_2}|\,dx \lesssim \int_{\mathbb{R}^3} (R')^{-2}\Big(1+\frac{|x|}{R'}\Big)^{-N} dx = R' \int_{\mathbb{R}^3} (1+|x|)^{-N}\,dx \sim R'.$$

We thus obtain

$$\begin{aligned}\Big\| \sum_{\substack{w_1\in W_1\\ w_2\in W_2}} p_{w_1}p_{w_2} \Big\|^2_{L^2(Q_{S_1,S_2}(R))} &\lesssim R'|W_1||W_2| \sup_{v_1,v_2} |V_1^{\Pi_{v_1,v_2}}| \prod_{j=1,2} \sup_{w_j'\in W_j} \|p_{w_j'}\|_\infty \\ &\lesssim (R')^{-1}|W_1||W_2| \sup_{v_1,v_2} |V_1^{\Pi_{v_1,v_2}}|. \end{aligned} \tag{2-37}$$

Repeating the same computation with the roles of v_1' and v_2' interchanged gives (2-36). □

2E. *Basis of the induction-on-scales argument.* In order to start our induction on scales, we need to establish a base case estimate which will respect the form of our estimate (2-29). This will require a somewhat more sophisticated approach than what is done usually, based on the following.

Lemma 2.11. *Let $V_j \subset \mathcal{V}_j$. Then $\min_j \sup_{v_1\in\mathcal{V}_1, v_2\in\mathcal{V}_2} |V_j^{\Pi_{v_1,v_2}}| \lesssim R$.*

Proof. Define the graph mapping $\Phi : U_1 \cup U_2 \to S_1 \cup S_2,\ \Phi(x) = (x, \phi(x))$. If $v_j' = \Phi(x_j') \in V_j^{\Pi_{v_1,v_2}}$, then $v_j' + v_{j+1} \in \Pi_{v_1,v_2}$, and for $x_{j+1} = \Phi^{-1}(v_{j+1})$ we have $x_j' + x_{j+1} \in \gamma(I) + \mathcal{O}((R')^{-1})$, where $\gamma : I \to [U_1+x_2]\cap[U_2+x_1] \subset \mathbb{R}^2$ is a parametrization by arclength of the projection to the (x_1,x_2)-space of the intersection curve Π_{v_1,v_2}. Recall from Lemma 2.8(e) that our assumptions imply that then γ will be close to a diagonal, i.e., $|\dot{\gamma}_i| \sim 1,\ i = 1,2$.

For all $t, t' \in I$, we have $\gamma(t), \gamma(t') \in [U_1 + x_2] \cap [U_2 + x_1]$; hence

$$\min_j d_i^{(j)} \ge |\gamma_i(t) - \gamma_i(t')| \ge \min_{t''\in I} |\dot{\gamma}_i(t'')||t-t'| \sim |t-t'|.$$

This implies $|I| = \sup_{t,t'\in I} |t-t'| \lesssim \min_{i,j} d_i^{(j)} = D$; hence $L(\gamma) \lesssim D$, and thus

$$\begin{aligned}|V_j^{\Pi_{v_1,v_2}}| &\sim |\Phi^{-1}(V_j^{\Pi_{v_1,v_2}})| \\ &\le \big|\{x_j' \in \Phi^{-1}(\mathcal{V}_j) : x_j' \in \gamma(I) - x_{j+1} + \mathcal{O}((R')^{-1})\}\big| \\ &\lesssim L(\gamma)/((R')^{-1}) \\ &\lesssim DR' = R,\end{aligned}$$

since $\Phi^{-1}(\mathcal{V}_j)$ is an $(R')^{-1}$-grid in U_j. □

Corollary 2.12. *$E(1)$ holds true, provided $\frac{4}{3} \le p \le 2$.*

Proof. Due to Lemma 2.9, it is enough to show the corresponding estimate for wave packets (2-29) with $\alpha = 1$. But, estimating $|V_j^{\Pi_{v_1,v_2}}|$ on the right-hand side of (2-36) in Lemma 2.10 by means of Lemma 2.11, we obtain

$$\Bigl\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j,\mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \Bigr\|_{L^2(Q_{S_1,S_2}(R))} \lesssim (R')^{-\frac{1}{2}} R^{\frac{1}{2}} |W_1|^{\frac{1}{2}} |W_1|^{\frac{1}{2}}.$$

Interpolating this with the corresponding L^1-estimate that we obtain from (2-35), we arrive at

$$\begin{aligned} \Bigl\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j,\mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \Bigr\|_{L^p(Q_{S_1,S_2}(R))} &\lesssim (\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} (R')^{\frac{5}{p}-3} R^{1-\frac{1}{p}} |W_1|^{\frac{1}{2}} |W_1|^{\frac{1}{2}} \\ &\le (\kappa^{(1)}\kappa^{(2)})^{\frac{1}{2}-\frac{1}{p}} D^{3-\frac{5}{p}} R |W_1|^{\frac{1}{2}} |W_1|^{\frac{1}{2}}, \end{aligned}$$

provided $\frac{4}{3} \le p \le 2$. ☐

2F. ***Further decompositions.*** In a next step, by some slight modification of the usual approach, we introduce a further decomposition of the cuboid $Q_{S_1,S_2}(R)$ defined in (2-21) into smaller cuboids b whose dimensions are those of $Q_{S_1,S_2}(R)$ shrunk by a factor $R^{-2\delta}$; i.e., all of the b's will be translates of $Q_{S_1,S_2}(R^{1-\delta})$. Here, $\delta > 0$ is a sufficiently small parameter to be chosen later. Since

$$\frac{(R')^2}{\bar{\kappa}} R^{-2\delta} = \frac{R^{1-2\delta} R'}{D\bar{\kappa}} \ge R^{1-2\delta} R',$$

the smallest side length of b is still much larger than the side length $R^\delta R'$ of the thickened cubes $R^\delta q$ introduced at the end of Section 2B. Observe further that the number of cuboids b into which $Q_{S_1,S_2}(R)$ will be decomposed is of the order $R^{c\delta}$.[2]

If $\mu \in \mathcal{N}^2$ is a fixed pair of dyadic numbers, and if $w_j \in W_j$, then we assign to w_j a cuboid $b(w_j)$ in such a way that $b(w_j)$ contains a maximal number of q's from $Q^\mu(w_j)$ among all the cuboids b. We say that $b \sim w_j$ if b is contained in $10b(w_j)$ (the cuboid having the same center as $b(w_j)$ but scaled by a factor of 10). Notice that if $b \not\sim w_j$, then this does not necessarily mean that there are only few cubes $q \in Q^\mu(w_j)$ contained in b (since the cuboid $b(w_j)$ may not be unique), but it does imply that there are many cubes q lying "away" from b. To be more precise, if $b \not\sim w_j$, then

$$\bigl|\{q \in Q^\mu(w_j) : q \cap 5b = \varnothing\}\bigr| \ge \bigl|\{q \in Q^\mu(w_j) : q \subset b(w_j)\}\bigr| \gtrsim R^{-c\delta} |Q^\mu(w_j)|, \tag{2-38}$$

since only $\mathcal{O}(R^{2\delta})$ cuboids b meet T_{w_j}.

For a fixed b, we can decompose any given set $W_j \subset \mathcal{W}_j$ into $W_j^{\not\sim b} = \{w_j \in W_j : b \not\sim w_j\}$ and $W_j^{\sim b} = \{w_j \in W_j : b \sim w_j\}$. Thus we have

$$\Bigl\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j,\mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \Bigr\|_{L^p(Q_{S_1,S_2}(R))} \le \sum_b \Bigl\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j,\mu}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \Bigr\|_{L^p(b)} = \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{2-39}$$

[2] Here and in the subsequent considerations, c will denote some constant which is independent of R and S_1, S_2, but whose precise value may vary from line to line.

where

$$\mathrm{I} = \sum_b \Big\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j,\mu,\sim b}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \Big\|_{L^p(b)},$$

$$\mathrm{II} = \sum_b \Big\| \sum_{w_1 \in W_1^{\lambda_1,\mu,\not\sim b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2,\mu}} p_{w_2} \sum_{q \in Q^\mu} \chi_q \Big\|_{L^p(b)},$$

$$\mathrm{III} = \sum_b \Big\| \sum_{w_1 \in W_1^{\lambda_1,\mu,\sim b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2,\mu,\not\sim b}} p_{w_2} \sum_{q \in Q^\mu} \chi_q \Big\|_{L^p(b)}.$$

As usual in the bilinear approach, part I, which comprises the terms of highest density of wave packets over the cuboids b, will be handled by means of an inductive argument. The treatment of part II (and analogously of part III) will be based on a combination of geometric and combinatorial arguments. It is only here that the very choice of the $b(w_j)$ will become crucial.

Lemma 2.13. *Let $\alpha > 0$, and assume that $E(\alpha)$ holds true. Then*

$$I \le C_\alpha R^{\alpha(1-\delta)} \log^{\gamma_\alpha}(1+R)(\kappa^1\kappa^2)^{\frac12-\frac1p} D^{3-\frac5p} \log^{\gamma_\alpha}(C_0(S))|W_1|^{\frac12}|W_2|^{\frac12}.$$

Proof. To shorten notation, write $C_1 = C_\alpha(\kappa^1\kappa^2)^{\frac12-\frac1p} D^{3-\frac5p} \log^{\gamma_\alpha}(C_0(S))$. Recall the reproducing formula (P1) in Corollary 2.3: $p_{w_j} = R^*_{H_j}(\mathfrak{F}_{H_j}(p_{w_j}|_{H_j}))$. Since every cuboid b is a translate of $Q_{S_1,S_2}(R^{1-\delta})$, and since a translation of $R^*_{H_j} g$ corresponds to a modulation of the function g, we see that $E(\alpha)$ implies

$$\begin{aligned}
I &= \sum_b \Big\| \prod_{j=1,2} \sum_{w_j \in W_j^{\lambda_j,\mu,\sim b}} p_{w_j} \sum_{q \in Q^\mu} \chi_q \Big\|_{L^p(b)} \\
&\le \sum_b \Big\| \prod_{j=1,2} R^*_{H_j}\Big(\sum_{w_j \in W_j^{\lambda_j,\mu,\sim b}} \mathfrak{F}_{H_j}(p_{w_j}|_{H_j})\Big) \Big\|_{L^p(b)} \\
&\le C_1 (R^{1-\delta})^\alpha \log^{\gamma_\alpha}(1+R^{1-\delta}) \sum_b \prod_{j=1,2} \Big\| \sum_{w_j \in W_j^{\lambda_j,\mu,\sim b}} \mathfrak{F}_{H_j}(p_{w_j}|_{H_j}) \Big\|_{L^2(H_j)} \\
&\le C_1 R^{\alpha(1-\delta)} \log^{\gamma_\alpha}(1+R) \sum_b \prod_{j=1,2} |W_j^{\lambda_j,\mu,\sim b}|^{\frac12}.
\end{aligned}$$

In the last estimate, we have made use of property (P4). Moreover, using Hölder's inequality, we obtain

$$\sum_b \prod_{j=1,2} |W_j^{\lambda_j,\mu,\sim b}|^{\frac12} \le \prod_{j=1,2} \Big(\sum_b |W_j^{\lambda_j,\mu,\sim b}| \Big)^{\frac12},$$

where, due to Fubini's theorem (for sums),

$$\sum_b |W_j^{\lambda_j,\mu,\sim b}| = \sum_b |\{w_j \in W_j^{\lambda_j,\mu} : w_j \sim b\}| = \sum_{w_j \in W_j^{\lambda_j,\mu}} |\{b :\ b \sim w_j\}| \lesssim |W_j|.$$

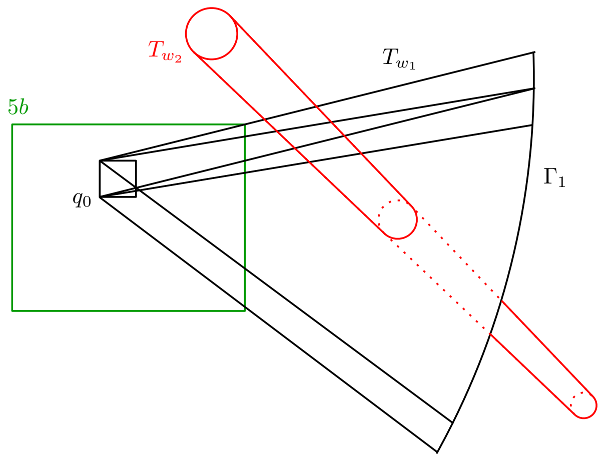

Figure 9. The geometry in Lemma 2.14.

In combination, these estimates yield

$$I \leq C_1 R^{\alpha(1-\delta)} \log^{\gamma_\alpha}(1+R) \prod_{j=1,2} |W_j|^{\frac{1}{2}}. \qquad \square$$

2G. ***The geometric argument.*** We next turn to the estimation of II and III. A crucial tool will be the following lemma, which is a variation of Lemma 2.3 in [Lee and Vargas 2010].

Lemma 2.14. *Let $\lambda_j, \mu_j \in \mathcal{N}$, $W_j \subset \mathcal{W}_j$, $v_j \in \mathcal{V}_j$, $j = 1, 2$, and let b and q_0 be cuboids from our collections such that $q_0 \cap 2b \neq \varnothing$. If we define $W_j^{\lambda_j,\mu,\not\sim b}(q_0) = W_j^{\lambda_j,\mu,\not\sim b} \cap W_j(q_0)$, then*

(i) $\lambda_1\mu_2 \left|[W_1^{\lambda_1,\mu,\not\sim b}(q_0)]^{\Pi_{v_1,v_2}}\right| \lesssim R^{c\delta}|W_2|$,

(ii) $\lambda_2\mu_1 \left|[W_2^{\lambda_2,\mu,\not\sim b}(q_0)]^{\Pi_{v_1,v_2}}\right| \lesssim R^{c\delta}|W_1|$.

Proof. We only show (i), since the proof of (ii) is analogous. Set

$$\Gamma_1 = \bigcup \{T_{w_1} : w_1 \in [W_1^{\lambda_1,\mu,\not\sim b}(q_0)]^{\Pi_{v_1,v_2}}\} \setminus 5b, \quad Q^\mu_{\Gamma_1} = \{q \in Q^\mu : R^\delta q \cap \Gamma_1 \neq \varnothing\}.$$

Since we have seen in Lemma 2.8(d) that T_{w_2} is transversal to Γ_1, we have

$$|Q^\mu_{\Gamma_1} \cap Q^\mu(w_2)| \lesssim R^{c\delta}. \tag{2-40}$$

Due to the separation of the tube directions, the sets $T_{w_1} \setminus 5b$ do not overlap too much. To be more precise, we claim that for all cubes $q \in Q^\mu_{\Gamma_1}$,

$$\left|\{w_1 \in [W_1^{\lambda_1,\mu,\not\sim b}(q_0)]^{\Pi_{v_1,v_2}} : R^\delta q \cap T_{w_1} \setminus 5b \neq \varnothing\}\right| \lesssim R^{c\delta}. \tag{2-41}$$

Indeed, let $w_1, w_1' \in [W_1^{\lambda_1,\mu,\not\sim b}(q_0)]^{\Pi_{v_1,v_2}}$ and $x \in R^\delta q \cap T_{w_1} \setminus 5b$, $x' \in R^\delta q \cap T_{w_1'} \setminus 5b$. The definition of $W_1(q_0)$ means that we can find $x_0 \in R^\delta q_0 \cap T_{w_1}$ and $x_0' \in R^\delta q_0 \cap T_{w_1'}$; then we may write

$$x = x_0 + |x - x_0| N(v_1) + \mathcal{O}(R') \quad \text{and} \quad x' = x_0' + |x' - x_0'| N_0(v_1') + \mathcal{O}(R'). \tag{2-42}$$

Furthermore we have

$$\big||x-x_0|-|x'-x_0'|\big| \le |x-x'|+|x_0-x_0'| = \mathcal{O}(R^{c\delta}R'). \tag{2-43}$$

Since T_{w_1} has length $(R')^2/\kappa^{(1)}$, so that the length of b in the direction of T_{w_1} is at least $R^{-2\delta}(R')^2/\kappa^{(1)}$, and since $x_0 \in R^\delta q_0 \subset 4b$ but $x \notin 5b$, we conclude that

$$R^{-2\delta}\frac{(R')^2}{\kappa^{(1)}} \le |x-x_0|. \tag{2-44}$$

Applying Lemma 2.5, and consecutively making use of the estimates (2-44), (2-43), (2-42) and again (2-43), we obtain

$$\begin{aligned}
R'|v_1-v_1'| &\lesssim \frac{R'}{\kappa^{(1)}}|N(v_1)-N(v_1')|\\
&\lesssim R^{2\delta}(R')^{-1}|x-x_0|\,|N_0(v_1)-N_0(v_1')|\\
&\lesssim R^{2\delta}(R')^{-1}\big||x-x_0|N_0(v_1)-|x'-x_0'|N_0(v_1')\big| + \mathcal{O}(R^{c\delta})\\
&\lesssim R^{2\delta}(R')^{-1}\big(|x-x'|+|x_0'-x_0|\big) + \mathcal{O}(R^{c\delta}) = \mathcal{O}(R^{c\delta}).
\end{aligned}$$

Recall also that the direction of a tube T_{w_1} with $w_1=(y_1,v_1)$ depends only on v_1, and thus the set of all these directions corresponding to the set

$$\big\{w_1 \in [W_1^{\lambda_1,\mu,\not\sim b}(q_0)]^{\Pi_{v_1,v_2}} : R^\delta q \cap T_{w_1}\setminus 5b\big\}$$

has cardinality $\mathcal{O}(R^{c\delta})$. But, for a fixed direction v_1, the number of parameters y_1 such that the tube $T_{(y_1,v_1)}$ passes through $R^{c\delta}q_0$ is bounded by $\mathcal{O}(R^{c\delta})$ anyway, and thus (2-41) holds true.

Recall next from (2-38) that for $w_1 \not\sim b$ we have $R^{-c\delta}|Q^\mu(w_1)| \lesssim |\{q \in Q^\mu(w_1) : q \cap 5b = \varnothing\}|$. Since for $w_1 \in W_1^{\lambda_1,\mu}$ we have $|Q^\mu(w_1)| \sim \lambda_1$, we may thus estimate

$$\begin{aligned}
R^{-c\delta}\lambda_1\big|[W_1^{\lambda_1,\mu,\not\sim b}(q_0)]^{\Pi_{v_1,v_2}}\big| &\lesssim R^{-c\delta}\sum\big|Q^\mu(w_1)\big|\\
&\lesssim \sum\big|\{q \in Q^\mu(w_1) : q\cap 5b=\varnothing\}\big|\\
&\le \sum\big|\{q \in Q^\mu : R^\delta q\cap T_{w_1}\ne\varnothing,\ R^\delta q\cap 5b=\varnothing\}\big|\\
&\le \sum\big|\{q \in Q^\mu : R^\delta q\cap (T_{w_1}\setminus 5b)\ne\varnothing\}\big|\\
&= \sum_{q\in Q^\mu}\big|\big\{w_1 \in [W_1^{\lambda_1,\mu,\not\sim b}(q_0)]^{\Pi_{v_1,v_2}} : R^\delta q\cap (T_{w_1}\setminus 5b)\ne\varnothing\big\}\big|\\
&= R^{c\delta}|Q^\mu_{\Gamma_1}|,
\end{aligned}$$

where sums are taken over $w_1 \in [W_1^{\lambda_1,\mu,\not\sim b}(q_0)]^{\Pi_{v_1,v_2}}$ unless otherwise indicated, and where we have used (2-41) in the last estimate. But, by (2-40), we also have

$$\mu_2|Q^\mu_{\Gamma_1}| = \sum_{q\in Q^\mu_{\Gamma_1}}|W_2(q)| \le \sum_{w_2\in W_2}|Q^\mu_{\Gamma_1}\cap Q^\mu(w_2)| \lesssim R^{c\delta}|W_2|,$$

and combining this with the previous estimate we arrive at the desired estimate in (i). □

Lemma 2.15. *Let $0<\delta<\frac{1}{4}$. Then*

$$\mathrm{II}=\sum_b\Big\|\sum_{w_1\in W_1^{\lambda_1,\mu,\not\sim b}}p_{w_1}\sum_{w_2\in W_2^{\lambda_2,\mu}}p_{w_2}\sum_{q\in Q^\mu}\chi_q\Big\|_{L^p(b)}\le C_\alpha R^{c\delta}(\kappa^1\kappa^2)^{\frac{1}{2}-\frac{1}{p}}D^{3-\frac{5}{p}}|W_1|^{\frac{1}{2}}|W_2|^{\frac{1}{2}} \tag{2-45}$$

and

$$\mathrm{III}=\sum_b\Big\|\sum_{w_1\in W_1^{\lambda_1,\mu,\sim b}}p_{w_1}\sum_{w_2\in W_2^{\lambda_2,\mu,\not\sim b}}p_{w_2}\sum_{q\in Q^\mu}\chi_q\Big\|_{L^p(b)}\le C_\alpha R^{c\delta}(\kappa^1\kappa^2)^{\frac{1}{2}-\frac{1}{p}}D^{3-\frac{5}{p}}|W_1|^{\frac{1}{2}}|W_2|^{\frac{1}{2}}. \tag{2-46}$$

Proof. We will only prove the first inequality; the proof of second one works in a similar way. Since the number of b's over which we sum in (2-45) is of the order $R^{c\delta}$, it is enough to show that for every fixed b

$$\Big\|\sum_{w_1\in W_1^{\lambda_1,\mu,\not\sim b}}p_{w_1}\sum_{w_2\in W_2^{\lambda_2,\mu}}p_{w_2}\sum_{q\in Q^\mu}\chi_q\Big\|_{L^p(b)}\le C_\alpha R^{c\delta}(\kappa^1\kappa^2)^{\frac{1}{2}-\frac{1}{p}}D^{3-\frac{5}{p}}|W_1|^{\frac{1}{2}}|W_2|^{\frac{1}{2}}. \tag{2-47}$$

For $p=1$, we apply (2-35) from Lemma 2.10:

$$\begin{aligned}\Big\|\sum_{w_1\in W_1^{\lambda_1,\mu,\not\sim b}}p_{w_1}\sum_{w_2\in W_2^{\lambda_2,\mu}}p_{w_2}\sum_{q\in Q^\mu}\chi_q\Big\|_{L^1(b)}&\lesssim\Big\|\sum_{w_1\in W_1^{\lambda_1,\mu,\not\sim b}}p_{w_1}\sum_{w_2\in W_2^{\lambda_2,\mu}}p_{w_2}\Big\|_{L^1(Q_{S_1,S_2}(R))}\\&\le\frac{(R')^2}{\sqrt{\kappa^{(1)}\kappa^{(2)}}}|W_1|^{\frac{1}{2}}|W_1|^{\frac{1}{2}}.\end{aligned}$$

For $p=2$, we claim that

$$\Big\|\sum_{w_1\in W_1^{\lambda_1,\mu,\not\sim b}}p_{w_1}\sum_{w_2\in W_2^{\lambda_2,\mu}}p_{w_2}\sum_{q\in Q^\mu}\chi_q\Big\|^2_{L^2(b)}\lesssim C_\alpha R^{c\delta}(R')^{-1}|W_1||W_2|. \tag{2-48}$$

The desired inequality (2-45) will then follow by means of interpolation with the previous L^1-estimate — notice here that $R^{5/p-3}\le 1$ since $\frac{5}{3}\le p$.

To prove (2-48), recall that the side lengths of b are of the form

$$\left(\frac{(R')^2}{\kappa^{(j)}}\right)R^{-2\delta}=\frac{R'}{D\kappa^{(j)}}R^{1-2\delta}\ge R'R^{1-2\delta},\quad j\in\{1,2\}.$$

If $q\cap 2b=\varnothing$, then for $x\in b$ we have $|x-c_q|\ge\inf_{y\notin 2b}|x-y|=d(x,(2b)^c)\ge R'R^{1-2\delta}$. Therefore for every $x\in b$,

$$\begin{aligned}\Big|\sum_{\substack{q\in Q^\mu\\ q\cap 2b=\varnothing}}\chi_q(x)\Big|&\le C_N\sum_{\substack{l\in\mathbb{N}\\ 2^l\ge R^{1-2\delta}}}\sum_{\substack{q\in Q^\mu\\ |x-c_q|\sim R'2^l}}\left(1+\frac{|x-c_q|}{R'}\right)^{-N-2}\lesssim C_N\sum_{\substack{l\in\mathbb{N}\\ 2^l\ge R^{1-2\delta}}}\big|\{q:|x-c_q|\sim R'2^l\}\big|2^{-(N+2)l}\\&\sim C_N\sum_{\substack{l\in\mathbb{N}\\ 2^l\ge R^{1-2\delta}}}2^{-Nl}\sim C_NR^{-(1-2\delta)N}=C_{\delta,N'}R^{-N'}.\end{aligned} \tag{2-49}$$

The last step requires that $\delta < \frac{1}{2}$. Choosing N sufficiently large, we see that by Lemma 2.10 and Lemma 2.11,

$$\begin{aligned}\Big\| \sum_{w_1 \in W_1^{\lambda_1,\mu,\not\sim b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2,\mu}} p_{w_2} \sum_{\substack{q \in Q^\mu \\ q \cap 2b = \varnothing}} \chi_q \Big\|_{L^2(b)}^2 &\lesssim \Big\| \sum_{w_1 \in W_1^{\lambda_1,\mu,\not\sim b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2,\mu}} p_{w_2} \Big\|_{L^2}^2 \Big\| \sum_{\substack{q \in Q^\mu \\ q \cap 2b = \varnothing}} \chi_q \Big\|_{L^\infty(b)}^2 \\ &\lesssim C_{\delta,N'} (R')^{-1} |W_1||W_2| \min_j \sup_{v_1,v_2} |V_j^{\Pi_{v_1,v_2}}| R^{-2N'} \\ &\lesssim C_{\delta,N'} (R')^{-1} |W_1||W_2| R^{1-2N'} \\ &\lesssim C'_{\delta,N''} (R')^{-1} |W_1||W_2| R^{-N''}.\end{aligned}$$

Thus it is enough to consider the sum over the set $Q_b^\mu = \{q \in Q^\mu : q \cap 2b \neq \varnothing\}$. For fixed w_1, w_2, we split this set into the subsets $Q_b^\mu(w_1, w_2) = Q_b^\mu \cap Q^\mu(w_1) \cap Q^\mu(w_2)$ and $Q_b^\mu \cap Q^\mu(w_1) \setminus Q^\mu(w_2)$ and

$$Q_b^\mu \setminus Q^\mu(w_1) = \big(Q_b^\mu \cap Q^\mu(w_2) \setminus Q^\mu(w_1)\big) \cup \big(Q_b^\mu \setminus (Q^\mu(w_2) \cap Q^\mu(w_1))\big).$$

Except for the first set, the contributions by the other subsets can be treated in the same way, since they are all special cases of the following situation:

Let $Q_0 = Q_0(w_1, w_2) \subset Q_b^\mu$ such that there exists an $j \in \{1, 2\}$ with $R^\delta q \cap T_{w_j} = \varnothing$ for all $q \in Q_0$. Then

$$\Big\| \sum_{w_1 \in W_1^{\lambda_1,\mu,\not\sim b}} p_{w_1} \sum_{w_2 \in W_2^{\lambda_2,\mu}} p_{w_2} \sum_{q \in Q_0} \chi_q \Big\|_{L^2(b)}^2 \lesssim C_\alpha R^{c\delta} (R')^{-1} |W_1||W_2|. \tag{2-50}$$

Notice that the right-hand side is just what we need for (2-48).

For the proof of (2-50), assume without loss of generality that $j = 1$. Let $q \in Q_0$. Then $T_{w_1} \cap R^\delta q = \varnothing$, and for all $x \in (R^\delta/2)q$ we have

$$(R^\delta/2)R' \le \operatorname{dist}(x, (R^\delta q)^c) \le \operatorname{dist}(x, T_{w_1}).$$

Thus for every $x \in Q_{S_1,S_2}(R)$, we have $\operatorname{dist}(x, T_{w_1}) \ge (R^\delta/2)R'$ or $x \notin (R^\delta/2)q$. In the first case, we have

$$|p_{w_1}(x)| \le C_N (R')^{-1} \Big(1 + \frac{\operatorname{dist}(x, T_{w_1})}{R'}\Big)^{-2N} \le C'_N (R')^{-1} R^{-\delta N} \Big(1 + \frac{\operatorname{dist}(x, T_{w_1})}{R'}\Big)^{-N}. \tag{2-51}$$

One the other hand, in the second case, where $x \notin (R^\delta/2)q$, we have $(R^\delta/2)R' \le |x - c_q|$. Using the rapid decay of the Schwartz function ϕ we then see that

$$|\chi_q(x)| = \Big|\chi\Big(\frac{x - c_q}{R'}\Big)\Big| \le C_N \Big(\frac{|x - c_q|}{R'}\Big)^{-N} \le C'_N R^{-\delta N}. \tag{2-52}$$

Applying an argument similar to the one used in (2-49), we even obtain

$$\Big| \sum_{q \in Q_0} \chi_q(x) \Big| \le C''_N R^{-\delta N}$$

for all $x \notin (R^\delta/2)q$. To summarize, we obtain that for every $x \in Q_{S_1,S_2}(R)$,

$$\Big| p_{w_1} \sum_{q \in Q_0(w_1,w_2)} \chi_q \Big|(x) \le C(N,\delta)(R')^{-1} R^{-\delta N} \Big(1 + \frac{\operatorname{dist}(x, T_{w_1})}{R'}\Big)^{-N}. \tag{2-53}$$

This means that the expression $p_{w_1} \sum_{q \in Q_0(w_1,w_2)} \chi_q$ cannot only be estimated in the same way as the original wave packet p_{w_1}, but we even obtain an improved estimate because of an additional factor $R^{-\delta N}$. If we replace p_{w_1} by p_{w_2} on the left-hand side, we obtain in a similar way just the standard wave packet estimate

$$\Big| p_{w_2} \sum_{q \in Q_0(w_1,w_2)} \chi_q \Big|(x) \lesssim \|p_{w_2}\|_\infty \lesssim (R')^{-1} \Big(1 + \frac{\operatorname{dist}(x, T_{w_2})}{R'}\Big)^{-N}, \tag{2-54}$$

without an additional factor.

We can now finish the proof of (2-50), basically by following the ideas of the proof of the estimate (2-36) in Lemma 2.10. The crucial argument was the fact that the Fourier transform of $p_{w'_{j+1}} p_{w_j}$ is supported in $v'_{j+1} + v_j + \mathcal{O}((R')^{-1})$. Since $\operatorname{supp} \hat{\chi}_q = \operatorname{supp} \hat{\chi}(R' \cdot) \subset B(0, (R')^{-1})$, the Fourier support of $p_{w_1} p_{w_2} \sum_{q \in Q_0(w_1,w_2)} \chi_q$ remains essentially the same. It is at this point that we need that the functions χ_q have compact Fourier support. The modified wave packets $p_{w_i} \sum_{q \in Q_0(w_1,w_2)} \chi_q$ are still well separated with respect to the parameter y_i, for fixed direction v_i, thanks to (2-53) and (2-54). Thus the argument from Lemma 2.10 applies, and by the analogue of (2-37) we obtain

$$\begin{aligned}
&\Big\| \sum_{\substack{w_1 \in W_1^{\lambda_1,\mu,\not\sim b} \\ w_2 \in W_2^{\lambda_2,\mu}}} p_{w_1} p_{w_2} \sum_{q \in Q_0(w_1,w_2)} \chi_q \Big\|_{L^2(b)}^2 \\
&\quad \lesssim R' |W_1||W_2| \min_j \sup_{v_1,v_2} |V_j^{\Pi_{v_1,v_2}}| \sup_{\substack{w_1 \in W_1 \\ w'_2 \in W_2}} \Big\| p_{w'_2} \sum_{q \in Q_0(w_1,w'_2)} \chi_q \Big\|_\infty \sup_{\substack{w'_1 \in W_1 \\ w_2 \in W_2}} \Big\| p_{w'_1} \sum_{q \in Q_0(w'_1,w_2)} \chi_q \Big\|_\infty \\
&\quad \lesssim C_{\delta,N'} (R')^{-1} |W_1||W_2| \min_j \sup_{v_1,v_2} |V_j^{\Pi_{v_1,v_2}}| R^{-N'} \lesssim C_{\delta,N'} (R')^{-1} R^{1-N'} |W_1||W_2|.
\end{aligned}$$

In the second inequality, we have made use of (2-53) and (2-54), and the last one is based on Lemma 2.11. This concludes the proof of (2-50).

What remains to be controlled are the contributions by the cubes q from $Q_b^\mu(w_1,w_2)$. Notice that the kernel $K(q,q') = \chi_q(x)\chi_{q'}(x)$ satisfies Schur's test condition

$$\sup_q \sum_{q'} \chi_q(x)\chi_{q'}(x) \lesssim \sum_{q'} \chi_{q'}(x) \lesssim 1,$$

with a constant not depending on x. Let us put

$$f_q = \sum p_{w_1} p_{w_2},$$

where the sum is taken over $w_1 \in W_1^{\lambda_1,\mu,\not\sim b}(q)$ and $w_2 \in W_2^{\lambda_2,\mu}(q)$. Observe that for $w_1 \in W_1^{\lambda_1,\mu,\not\sim b}$ and $w_2 \in W_2^{\lambda_2,\mu}$, we have $q \in Q_b^\mu(w_1,w_2)$ if and only if $q \in Q_b^\mu$ and $w_1 \in W_1^{\lambda_1,\mu,\not\sim b}(q)$ and $w_2 \in W_2^{\lambda_2,\mu}(q)$.

Then we see that we may estimate

$$
\begin{aligned}
\Big\| \sum_{\substack{w_1\in W_1^{\lambda_1,\mu,\not\sim b}\\ w_2\in W_2^{\lambda_2,\mu}}} p_{w_1}p_{w_2}\sum_{q\in Q_b^{\mu}(w_1,w_2)}\chi_q\Big\|_{L^2(b)}^2 &= \Big\|\sum_{q\in Q_b^{\mu}}\chi_q f_q\Big\|_{L^2(b)}^2\\
&= \int_b \Big|\sum_{q,q'\in Q_b^{\mu}}\chi_q\chi_{q'}f_q f_{q'}\Big|\,dx = \int_b\Big|\sum_{q,q'\in Q_b^{\mu}} K(q,q')f_q f_{q'}\Big|\,dx\\
&\lesssim \int_b \sum_{q\in Q_b^{\mu}}|f_q|^2\,dx = \sum_{\substack{q\in Q^{\mu}\\ q\cap 2b\neq\varnothing}}\|f_q\|_{L^2(b)}^2.
\end{aligned}
$$

Invoking also Lemma 2.10 and Lemma 2.14(i), we thus obtain

$$
\begin{aligned}
&\Big\| \sum_{\substack{w_1\in W_1^{\lambda_1,\mu,\not\sim b}\\ w_2\in W_2^{\lambda_2,\mu}}} p_{w_1}p_{w_2}\sum_{q\in Q_b^{\mu}(w_1,w_2)}\chi_q\Big\|_{L^2(b)}^2\\
&\qquad\lesssim \sum_{\substack{q\in Q^{\mu}\\ q\cap 2b\neq\varnothing}} (R')^{-1}|W_1^{\lambda_1,\mu,\not\sim b}(q)||W_2^{\lambda_2,\mu}(q)|\sup_{v_1,v_2}\big|[W_1^{\lambda_1,\mu,\not\sim b}(q)]^{\Pi_{v_1,v_2}}\big|\\
&\qquad\lesssim R^{c\delta}(R')^{-1}\sum_{\substack{q\in Q^{\mu}\\ q\cap 2b\neq\varnothing}}|W_1^{\lambda_1,\mu}(q)||W_2(q)|\frac{|W_2|}{\lambda_1\mu_2}\\
&\qquad\lesssim R^{c\delta}(R')^{-1}\sum_{w_1\in W_1^{\lambda_1,\mu}}|Q^{\mu}(w_1)|\frac{|W_2|}{\lambda_1}\lesssim R^{c\delta}(R')^{-1}|W_1||W_2|.
\end{aligned}
$$

This completes the proof of estimate (2-45), and hence of Lemma 2.15. □

2H. *Induction on scales.* We can now easily complete the proof of Theorem 2.6 by following standard arguments.

Corollary 2.16. *There exist constants $c,\delta_0>0$ such that $c\delta_0>1$ and such that the following holds true*:

Whenever $\alpha>0$ is such that $E(\alpha)$ holds true, then $E(\max\{\alpha(1-\delta),c\delta\})$ holds true for every δ such that $0<\delta<\delta_0$.

Proof. Let us put $\delta_0=\frac14$. Then the previous Lemmas 2.13 and 2.15 imply

$$
\begin{aligned}
&\Big\|\prod_{j=1,2}\sum_{w_j\in W_j^{\lambda_j,\mu}}p_{w_j}\sum_{q\in Q^{\mu}}\chi_q\Big\|_{L^p(Q_{S_1,S_2}(R))}\\
&\qquad\le \mathrm{I}+\mathrm{II}+\mathrm{III}\\
&\qquad\lesssim \big(C_\alpha R^{\alpha(1-\delta)}\log^{\gamma_\alpha}(1+R)+C_\delta R^{c\delta}\big)(\kappa^1\kappa^2)^{\frac12-\frac1p}D^{3-\frac5p}\log^{\gamma_\alpha}(C_0)|W_1|^{\frac12}|W_2|^{\frac12}\\
&\qquad\lesssim C_{\alpha,\delta}R^{\alpha(1-\delta)\vee c\delta}\log^{\gamma_\alpha}(1+R)(\kappa^1\kappa^2)^{\frac12-\frac1p}D^{3-\frac5p}\log^{\gamma_\alpha}(C_0)|W_1|^{\frac12}|W_2|^{\frac12}
\end{aligned}
$$

whenever $\delta < \delta_0$, where $C_0 = C_0(S) \gtrsim 1$ is defined in (2-20). By Lemma 2.9, this estimate implies $E(\alpha(1-\delta) \vee c\delta)$. Finally, by simply increasing the constant c, if necessary, we may also ensure $c\delta_0 > 1$. ☐

Corollary 2.17. *$E(\alpha)$ holds true for every $\alpha > 0$.*

This completes also the proof of Theorem 2.6.

Proof. Define inductively the sequence $\alpha_0 = 1$, $\alpha_{j+1} = c\alpha_j/(c+\alpha_j)$, which is decreasing and converges to 0. It therefore suffices to prove that $E(\alpha_j)$ is valid for every $j \in \mathbb{N}$. But, by Corollary 2.12, $E(\alpha_0) = E(1)$ does hold true. Moreover, Corollary 2.16 shows that $E(\alpha_j)$ implies $E(\alpha_{j+1})$, for if we choose $\delta = \alpha_j/(c+\alpha_j)$, then $\delta < 1/c < \delta_0$ and $\alpha_j(1-\delta) = c\delta = \alpha_j c/(c+\alpha_j) = \alpha_{j+1}$, and thus we may conclude by induction. ☐

3. Scaling

For the proof of our main theorem, we shall have to perform a kind of Whitney-type decomposition of $S \times S$ into pairs of patches of hypersurfaces (S_1, S_2) and prove very precise bilinear restriction estimates for those. In order to reduce these estimates to Section 2B, we shall need to rescale simultaneously the hypersurfaces S_1, S_2 for each such pair (S_1, S_2) in a suitable way. To this end, we shall denote here and in the sequel by $R^*_{S_1,S_2}$ the bilinear Fourier extension operator

$$R^*_{S_1,S_2}(f_1, f_2) = R^*_{\mathbb{R}^2} f_1 \cdot R^*_{\mathbb{R}^2} f_2, \quad f_1 \in L^2(U_1),\ f_2 \in L^2(U_2)$$

associated to any pair of hypersurfaces (S_1, S_2) given as the graphs $S_j = \{(\xi, \phi_j(\xi)) : \xi \in U_j\}$, $j = 1, 2$.

The following trivial lemma comprises the effect of the type of rescaling that we shall need.

Lemma 3.1. *Let $S_j = \{(\xi, \phi(\xi) : \xi \in U_j\}$, where again $U_j \subset \mathbb{R}^d$ is open and bounded for $j = 1, 2$. Let $A \in \mathrm{GL}(d, \mathbb{R})$, $a > 0$, put $\phi^s(\eta) = (1/a)\phi(A\eta)$, and let*

$$S_j^s = \{(\eta, \phi^s(\eta)) : \eta \in U_j^s\}, \quad U_j^s = A^{-1}(U_j), \quad j = 1, 2.$$

For any measurable subset $Q^s \subset \mathbb{R}^{d+1}$, we set $Q = \{x : ({}^tAx', ax_{d+1}) \in Q^s\}$. Assume the following estimate holds true:

$$\|R^*_{S_1^s,S_2^s}(g_1, g_2)\|_{L^p(Q^s)} \le C_s \|g_1\|_2 \|g_2\|_2 \quad \text{for all } g_j \in L^2(U_j^s).$$

Then

$$\|R^*_{S_1,S_2}(f_1, f_2)\|_{L^p(Q)} \le C_s |\det A|^{\frac{1}{p'}} a^{-\frac{1}{p}} \|f_1\|_2 \|f_2\|_2 \quad \text{for all } f_j \in L^2(U_j).$$

We now return to our model hypersurface (see (1-3), (1-4) and (1-5)), which is the graph of

$$\phi(\xi_1, \xi_2) = \phi_{(1)}(\xi_1) + \phi_{(2)}(\xi_2)$$

on $]0, 1[\times]0, 1[$, where the derivatives of the $\phi_{(i)}$ satisfy

$$\phi''_{(i)}(\xi_i) \sim \xi_i^{m_i-2},$$
$$|\phi^{(k)}_{(i)}(\xi_i)| \lesssim \xi_i^{m_i-k} \quad \text{for } k \ge 3,$$

and where $m_1, m_2 \in \mathbb{R}$ are such that $m_i \ge 2$.

We shall apply the preceding lemma to pairs $S_1 = S$ and $S_2 = \widetilde{S}$ of patches of this hypersurface on which the following assumptions are met:

General Assumptions. Let $S = \{(\xi, \phi(\xi)) : \xi \in U\}$ and $\widetilde{S} = \{(\xi, \phi(\xi)) : \xi \in \widetilde{U}\}$, where $U = r + [0, d_1] \times [0, d_2]$ and $\widetilde{U} = \tilde{r} + [0, \tilde{d}_1] \times [0, \tilde{d}_2]$, with $r = (r_1, r_2)$ and $\tilde{r} = (\tilde{r}_1, \tilde{r}_2)$.

We assume that for $i = 1, 2$ we have $r_i \geq d_i$ and $\tilde{r}_i \geq \tilde{d}_i$, so that the principal curvature $\phi''_{(i)}$ of S with respect to ξ_i is comparable to $\kappa_i = r_i^{m_i - 2}$, and that of $\widetilde{S}$ is comparable to $\tilde{\kappa}_i = \tilde{r}_i^{m_i - 2}$. We put

$$\begin{gathered} \bar{d}_i = d_i \vee \tilde{d}_i, \quad \bar{r}_i = r_i \vee \tilde{r}_i, \quad \Delta r_i = r_i - \tilde{r}_i, \\ \kappa = \kappa_1 \vee \kappa_2, \quad \tilde{\kappa} = \tilde{\kappa}_1 \vee \tilde{\kappa}_2, \bar{\kappa}_i = \kappa_i \vee \tilde{\kappa}_i, \quad \bar{\kappa} = \kappa \vee \tilde{\kappa} = \bar{\kappa}_1 \vee \bar{\kappa}_2. \end{gathered} \tag{3-1}$$

In addition, *we assume that for each direction ξ_1 and ξ_2 the rectangle U or $\widetilde{U}$ respectively on which the corresponding principal curvature is bigger (which means that its projection to the ξ_i-axis is the one further to the right) has also bigger length in this direction.* This is easily seen to be equivalent to

$$(\kappa_i d_i) \vee (\tilde{\kappa}_i \tilde{d}_i) = \bar{\kappa}_i \bar{d}_i. \tag{3-2}$$

Last, but not least, we assume the rectangles U and $\widetilde{U}$ are separated with respect to both variables ξ_i, $i = 1, 2$, in the following sense:

$$\operatorname{dist}_{\xi_i}(U, \widetilde{U}) = \inf\bigl\{|\xi_i - \tilde{\xi}_i| : \xi \in U,\ \tilde{\xi} \in \widetilde{U}\bigr\} \sim |\Delta r_i| \sim \bar{d}_i. \tag{3-3}$$

Given these assumptions, we shall introduce a rescaling as follows: we put

$$a_1 = \bar{\kappa}_2 \bar{d}_2, \quad a_2 = \bar{\kappa}_1 \bar{d}_1, \tag{3-4}$$

and

$$\phi^s(\eta) = \frac{1}{a} \phi(A\eta) = \frac{1}{a_1 a_2} \phi(a_1 \eta_1, a_2 \eta_2). \tag{3-5}$$

The quantities that arise after this scaling will be denoted by a superscript s; i.e.,

$$\begin{gathered} r_i^s = \frac{r_i}{a_i}, \quad d_i^s = \frac{d_i}{a_i}, \quad \kappa_i^s = \frac{1}{a_1 a_2} a_i^2 \kappa_i = \frac{a_i}{a_{i+1 \bmod 2}} \kappa_i, \\ D^s = \min\{d_1^s, d_2^s, \tilde{d}_1^s, \tilde{d}_2^s\}, \quad U^s = r^s + [0, d_1^s] \times [0, d_2^s], \end{gathered}$$

with corresponding expressions for $\tilde{r}^s$, $\tilde{d}_i^s$, $\tilde{\kappa}_i^s$ and $\widetilde{U}^s$. For later use, recall also the normal field N on $S \cup \widetilde{S}$ defined by $N(\xi, \phi(\xi)) = (-\nabla\phi(\xi), 1)$ and the corresponding unit normal field $N_0 = N/|N|$. After scaling, the corresponding normal fields on $S^s \cup \widetilde{S}^s$ will be denoted by N^s and N_0^s. With our choice of scaling, the following lemma holds true:

Lemma 3.2 (scaling). (i) *For $i = 1, 2$ and all $\eta \in U^s$ and $\tilde{\eta} \in \widetilde{U}^s$ we have*

$$|\partial_i \phi^s(\eta) - \partial_i \phi^s(r^s)| \lesssim \kappa_i^s d_i^s \lesssim 1 \quad \textit{and} \quad |\partial_i \phi^s(\tilde{\eta}) - \partial_i \phi^s(\tilde{r}^s)| \lesssim \tilde{\kappa}_i^s \tilde{d}_i^s \lesssim 1.$$

Moreover, $\bar{\kappa}_i^s \bar{d}_i^s = 1$.

(ii) *For every $|\alpha| \geq 2$ and all $\eta \in U^s$ and $\tilde{\eta} \in \widetilde{U}^s$,*

$$|\partial^\alpha \phi^s(\eta)| \lesssim \kappa^s |d_1^s \wedge d_2^s|^{2-|\alpha|} \quad \text{and} \quad |\partial^\alpha \phi^s(\tilde{\eta})| \lesssim \tilde{\kappa}^s |\tilde{d}_1^s \wedge \tilde{d}_2^s|^{2-|\alpha|}.$$

(iii) *For $i = 1, 2$, i.e., with respect to both variables, the separation condition*

$$|\partial_i \phi^s(\eta) - \partial_i \phi^s(\tilde{\eta})| \sim 1 \quad \text{for all } \eta \in S,\ \tilde{\eta} \in \widetilde{S}$$

holds true.

In particular, the rescaled pair of hypersurfaces $(S^s, \widetilde{S}^s)$ satisfies the general assumptions (i)–(iii) introduced before Theorem 2.6.

Proof. Observe first that

$$\bar{d}_i^s = \frac{\bar{d}_i}{a_i}, \quad \bar{\kappa}_i^s = \frac{1}{a_1 a_2} a_i^2 \bar{\kappa}_i,$$

and thus, by the definition of a_i, we see that $\bar{\kappa}_i^s \bar{d}_i^s = 1$.

Next, in order to prove (i), observe that for $\eta \in U^s$,

$$|\partial_i \phi^s(\eta) - \partial_i \phi^s(r^s)| \leq \sup_{\eta' \in U} |\partial_i^2 \phi^s(\eta')| |\eta_i - r_i^s| \lesssim \kappa_i^s d_i^s,$$

with $\kappa_i^s d_i^s \leq \bar{\kappa}_i^s \bar{d}_i^s = 1$.

As for (ii), notice that also $\partial_1 \partial_2 \phi^s \equiv 0$. In the unscaled situation, we have for $k \geq 2$ and every $\xi \in U$,

$$|\partial_i^k \phi(\xi)| \lesssim \xi_i^{m_i - k} \sim \partial_i^2 \phi(\xi) \xi_i^{2-k} \sim \kappa_i \xi_i^{2-k}.$$

Thus, for $\eta \in U^s$, we find that

$$|\partial_i^k \phi^s(\eta)| = \frac{1}{a_1 a_2} a_i^k |\partial_i^k \phi(A\eta)| \lesssim \frac{1}{a_1 a_2} a_i^k \kappa_i (a_i \eta_i)^{2-k} = \frac{a_i^2}{a_1 a_2} \kappa_i \eta_i^{2-k} = \kappa_i^s \eta_i^{2-k}.$$

On the other hand, for $\eta \in U^s$ we have

$$\eta_i \geq r_i^s = \frac{r_i}{a_i} \geq \frac{d_i}{a_i} = d_i^s \geq d_1^s \wedge d_2^s,$$

and thus we conclude that

$$|\partial_i^k \phi^s(\eta)| \lesssim \kappa^s (d_1^s \wedge d_2^s)^{2-k}, \quad k \geq 2.$$

In the same way, we obtain the corresponding result for $\eta \in \widetilde{U}^s$. These estimates imply (ii).

Finally, in order to prove (iii), let $\xi = (\xi_1, \xi_2) \in U$ and $\tilde{\xi} = (\tilde{\xi} - 1, \tilde{\xi}_2) \in \widetilde{U}$. Then, by (3-3), we see that $|\xi_i - \tilde{\xi}_i| \sim \bar{d}_i$. Moreover, if for instance $r_i < \tilde{r}_i$ (the other case can be treated analogously), then by (3-3) we even have $r_i + d_i + c\bar{d}_i \leq \tilde{r}_i$ for some admissible constant $c > 0$ such that $c < 1$. But then $\kappa_i \lesssim |\phi''_{(i)}(t)| \leq \tilde{\kappa}_i$ for every t in between ξ_i and $\tilde{\xi}_i$, and moreover $\phi''_{(i)}(t) \sim \tilde{\kappa}_i = \bar{\kappa}_1$ on the subinterval $[\tilde{r}_i - c\bar{d}_i/4, \tilde{r}_i]$, and thus

$$|\partial_i \phi(\xi) - \partial_i \phi(\tilde{\xi})| = \left| \int_{\xi_i}^{\tilde{\xi}_i} \phi''_{(i)}(t)\, dt \right| \sim \bar{\kappa}_i \bar{d}_i = a_{i+1 \bmod 2};$$

hence

$$|\partial_i\phi^s(\eta)-\partial_i\phi^s(\tilde\eta)|=\frac{|\partial_i\phi(A\eta)-\partial_i\phi(A\tilde\eta)|}{a_{i+1 \bmod 2}}\sim 1. \tag{3-6}$$

This completes the proof. □

In view of Lemma 3.2, we may now apply Theorem 2.6 to the rescaled phase function ϕ^s. According to (2-19), the scaled cuboids are given by

$$Q^0_{S^s,\tilde S^s}(R)=\left\{x\in\mathbb{R}^3:\ |x_i+\partial_i\phi^s(r_0^s)x_3|\le\frac{R^2}{(D^s)^2\bar\kappa^s}, i=1,2,\ |x_3|\le\frac{R^2}{(D^s)^2(\kappa^s\wedge\tilde\kappa^s)}\right\},$$

with $r_0^s=r^s$ if $\kappa^s=\kappa^s\wedge\tilde\kappa^s$, and $r_0^s=\tilde r^s$ if $\tilde\kappa^s=\kappa^s\wedge\tilde\kappa^s$. Thus, if $\frac{5}{3}\le p\le 2$, then for every $\alpha>0$ we obtain the following estimate, valid for every $R\ge 1$:

$$\|R^*_{S^s,\tilde S^s}\|_{L^2\times L^2\to L^p(Q^s_{S^s,\tilde S^s}(R))}\le(\kappa^s\tilde\kappa^s)^{\frac12-\frac1p}(D^s)^{3-\frac5p}\log^{\gamma_\alpha}(C_0^s)C_\alpha R^\alpha,$$

with (compare to (2-20))

$$C_0^s=\frac{\bar d_1^{s\,2}\bar d_2^{s\,2}}{(D^s)^4}(D^s[\kappa^s\wedge\tilde\kappa^s])^{-\frac1p}(D^s\kappa^sD^s\tilde\kappa^s)^{-\frac12}.$$

Recall here that $R^*_{S,\tilde S}(f_1,f_2)=R^*_{\mathbb{R}^2}f_1\cdot R^*_{\mathbb{R}^2}f_2$, if $f_1\in L^2(U)$, $f_2\in L^2(\widetilde U)$. Scaling back by means of Lemma 3.1, we obtain

$$\begin{aligned}\|R^*_{S,\tilde S}\|_{L^2\times L^2\to L^p(Q_{S,\tilde S}(R))}&\le(a_1a_2)^{1-\frac2p}(\kappa^s\tilde\kappa^s)^{\frac12-\frac1p}(D^s)^{3-\frac5p}\log^{\gamma_\alpha}(C_0^s)C_\alpha R^\alpha\\&=(a_1a_2\kappa^s\cdot a_1a_2\tilde\kappa^s)^{\frac12-\frac1p}(D^s)^{3-\frac5p}\log^{\gamma_\alpha}(C_0^s)C_\alpha R^\alpha,\end{aligned} \tag{3-7}$$

where

$$\begin{aligned}Q_{S,\tilde S}(R)&=\left\{x\in\mathbb{R}^3:\ |a_ix_i+\partial_i\phi^s(r_0^s)a_1a_2x_3|\le\frac{R^2}{(D^s)^2\bar\kappa^s},\ i=1,2,\ |a_1a_2x_3|\le\frac{R^2}{(D^s)^2\kappa^s\wedge\tilde\kappa^s}\right\}\\&=\left\{x\in\mathbb{R}^3:\ |x_i+\partial_i\phi(r_0)x_3|\le\frac{R^2}{a_i(D^s)^2\bar\kappa^s},\ i=1,2,\ |x_3|\le\frac{R^2}{a_1a_2(D^s)^2\kappa^s\wedge\tilde\kappa^s}\right\}.\end{aligned}$$

But, by (3-4), we have

$$\bar\kappa^s=\bar\kappa_1^s\vee\bar\kappa_2^s=\frac{a_1}{a_2}\bar\kappa_1\vee\frac{a_2}{a_1}\bar\kappa_2=\frac{a_1}{\bar d_1}\vee\frac{a_2}{\bar d_2}=\frac{\bar\kappa_2\bar d_2^2\vee\bar\kappa_1\bar d_1^2}{\bar d_1\bar d_2} \tag{3-8}$$

and

$$D^s=\min\{d_1^s,d_2^s,\tilde d_1^s,\tilde d_2^s\}\le\min\left\{\frac{\bar d_1}{a_1},\frac{\bar d_2}{a_2}\right\}=(\bar\kappa^s)^{-1};$$

hence

$$a_i(D^s)^2\bar\kappa^s\le a_iD^s\le\bar d_i,\quad i=1,2,$$

and also

$$a_1a_2(D^s)^2(\kappa^s\wedge\tilde\kappa^s)\le D^sa_1a_2\le a_2\bar d_1\wedge a_1\bar d_2=\bar\kappa_1\bar d_1^2\wedge\bar\kappa_2\bar d_2^2.$$

These estimates imply that

$$Q_{S,\tilde S}(R) \supset Q^1_{S,\tilde S}(R), \tag{3-9}$$

if we put

$$Q^1_{S,\tilde S}(R) = \left\{ x \in \mathbb{R}^3 : \, |x_i + \partial_i \phi(r_0) x_3| \le \frac{R^2}{\bar d_i}, \, i = 1, 2, \, |x_3| \le \frac{R^2}{\bar\kappa_1 \bar d_1^2 \wedge \bar\kappa_2 \bar d_2^2} \right\}.$$

Moreover, by (3-4) we have

$$d_i^s = \frac{d_i}{a_i} = \frac{\bar\kappa_i \bar d_i}{a_1 a_2} d_i,$$

and

$$\min\{d_i^s, \tilde d_i^s\} = \frac{\bar\kappa_i}{a_1 a_2} \bar d_i \min\{d_i, \tilde d_i\} = \frac{\bar\kappa_i d_i \tilde d_i}{a_1 a_2}.$$

Furthermore,

$$a_1 a_2 \kappa^s \sim a_1 a_2 \left(\frac{a_2}{a_1} \kappa_2 + \frac{a_1}{a_2} \kappa_1 \right) = (\bar\kappa_1^2 \bar d_1^2 \kappa_2 + \bar\kappa_2^2 \bar d_2^2 \kappa_1) = \bar\kappa_1 \bar\kappa_2 \left(\bar\kappa_1 \bar d_1^2 \frac{\kappa_2}{\bar\kappa_2} + \bar\kappa_2 \bar d_2^2 \frac{\kappa_1}{\bar\kappa_1} \right). \tag{3-10}$$

Thus the product of the first two factors on the right-hand side of (3-7) can be rewritten as

$$\begin{aligned}
&(a_1 a_2 \kappa^s \cdot a_1 a_2 \tilde\kappa^s)^{\frac12 - \frac1p} (D^s)^{3-\frac5p} \\
&= (a_1 a_2)^{\frac5p - 3} (\bar\kappa_1 \bar\kappa_2)^{1-\frac2p} \left(\bar\kappa_1 \bar d_1^2 \frac{\kappa_2}{\bar\kappa_2} + \bar\kappa_2 \bar d_2^2 \frac{\kappa_1}{\bar\kappa_1} \right)^{\frac12 - \frac1p} \left(\bar\kappa_1 \bar d_1^2 \frac{\tilde\kappa_2}{\bar\kappa_2} + \bar\kappa_2 \bar d_2^2 \frac{\tilde\kappa_1}{\bar\kappa_1} \right)^{\frac12 - \frac1p} \min_i (\bar\kappa_i d_i \tilde d_i)^{3-\frac5p} \\
&= (\bar\kappa_1 \bar d_1 \bar\kappa_2 \bar d_2)^{\frac5p - 3} (\bar\kappa_1 \bar\kappa_2)^{1-\frac2p} \left(\bar\kappa_1 \bar d_1^2 \frac{\kappa_2}{\bar\kappa_2} + \bar\kappa_2 \bar d_2^2 \frac{\kappa_1}{\bar\kappa_1} \right)^{\frac12 - \frac1p} \left(\bar\kappa_1 \bar d_1^2 \frac{\tilde\kappa_2}{\bar\kappa_2} + \bar\kappa_2 \bar d_2^2 \frac{\tilde\kappa_1}{\bar\kappa_1} \right)^{\frac12 - \frac1p} \min_i (\bar\kappa_i d_i \tilde d_i)^{3-\frac5p} \\
&= (\bar\kappa_1 \bar\kappa_2)^{\frac3p - 2} (\bar d_1 \bar d_2)^{\frac5p - 3} \left(\bar\kappa_1 \bar d_1^2 \frac{\kappa_2}{\bar\kappa_2} + \bar\kappa_2 \bar d_2^2 \frac{\kappa_1}{\bar\kappa_1} \right)^{\frac12 - \frac1p} \left(\bar\kappa_1 \bar d_1^2 \frac{\tilde\kappa_2}{\bar\kappa_2} + \bar\kappa_2 \bar d_2^2 \frac{\tilde\kappa_1}{\bar\kappa_1} \right)^{\frac12 - \frac1p} \min_i (\bar\kappa_i d_i \tilde d_i)^{3-\frac5p}.
\end{aligned}$$

For $a, b \in (0, \infty)$ write

$$q(a,b) = \frac{a \vee b}{a \wedge b} = \frac{a}{b} \vee \frac{b}{a} \ge 1.$$

A lower bound for D^s is

$$D^s = \frac{d_1 \wedge \tilde d_1}{a_1} \wedge \frac{d_2 \wedge \tilde d_2}{a_2} \ge \left(\frac{d_1 \wedge \tilde d_1}{\bar d_1} \wedge \frac{d_2 \wedge \tilde d_2}{\bar d_2} \right) \left(\frac{\bar d_1}{a_1} \wedge \frac{\bar d_2}{a_2} \right) \ge \frac{1}{q(d_1, \tilde d_1) q(d_2, \tilde d_2)} \frac{1}{\bar\kappa^s}, \tag{3-11}$$

where we have used (3-8) in the last inequality. And, from formula (3-10) we can deduce

$$\kappa^s \gtrsim \frac{\bar\kappa_1 \bar\kappa_2}{a_1 a_2} (\bar\kappa_1 \bar d_1^2 \vee \bar\kappa_2 \bar d_2^2) \frac{\kappa_1}{\bar\kappa_1} \frac{\kappa_2}{\bar\kappa_2} \ge \frac{\bar\kappa_1 \bar d_1^2 \vee \bar\kappa_2 \bar d_2^2}{\bar d_1 \bar d_2} \frac{1}{q(\kappa_1, \bar\kappa_1) q(\kappa_2, \bar\kappa_2)} = \frac{\bar\kappa^s}{q(\kappa_1, \bar\kappa_1) q(\kappa_2, \bar\kappa_2)}, \tag{3-12}$$

where we have again applied (3-8) in the last step. Combining (3-11) and (3-12), we obtain

$$(D^s \kappa^s)^{-1} \lesssim \prod_{i=1,2} q(\kappa_i, \tilde\kappa_i) q(d_i, \tilde d_i), \tag{3-13}$$

and then by symmetry also

$$(D^s\tilde{\kappa}^s)^{-1} \lesssim \prod_{i=1,2} q(\kappa_i,\tilde{\kappa}_i)q(d_i,\tilde{d}_i).$$

We may now estimate the constant C_0^s in the following way, using (3-13) in the first inequality, (3-11) in the second one and (3-8) in the third one (being generous in the exponents, since C_0^s appears only logarithmically):

$$\begin{aligned}
C_0^s &= \frac{\bar{d_1^s}^2\bar{d_2^s}^2}{(D^s)^4}(D^s[\kappa^s\wedge\tilde{\kappa}^s])^{-\frac{1}{p}}(D^s\kappa^s D^s\tilde{\kappa}^s)^{-\frac{1}{2}} \le \frac{\bar{d_1^s}^2\bar{d_2^s}^2}{(D^s)^4}\Big(\prod_{i=1,2} q(\kappa_i,\tilde{\kappa}_i)q(d_i,\tilde{d}_i)\Big)^{\frac{1}{p}+1}\\
&\le \Big(\prod_{i=1,2} q(\kappa_i,\tilde{\kappa}_i)q(d_i,\tilde{d}_i)\Big)^{\frac{1}{p}+5}(\bar{d}_1^s\bar{d}_2^s)^2(\bar{\kappa^s})^4\\
&\le \Big(\prod_{i=1,2} q(\kappa_i,\tilde{\kappa}_i)q(d_i,\tilde{d}_i)\Big)^{\frac{1}{p}+5}\Big(\frac{\bar{d}_1\bar{d}_2}{a_1a_2}\Big)^2\Big(\frac{\bar{\kappa}_1\bar{d}_1^2\vee\bar{\kappa}_2\bar{d}_2^2}{\bar{d}_1\bar{d}_2}\Big)^4\\
&= \Big(\prod_{i=1,2} q(\kappa_i,\tilde{\kappa}_i)q(d_i,\tilde{d}_i)\Big)^{\frac{1}{p}+5}\Big(\frac{(\bar{\kappa}_1\bar{d}_1^2\vee\bar{\kappa}_2\bar{d}_2^2)^2}{\bar{\kappa}_1\bar{d}_1^2\bar{\kappa}_2\bar{d}_2^2}\Big)^2\\
&= \Big(\prod_{i=1,2} q(\kappa_i,\tilde{\kappa}_i)q(d_i,\tilde{d}_i)\Big)^{\frac{1}{p}+5}q(\bar{\kappa}_1\bar{d}_1^2,\bar{\kappa}_2\bar{d}_2^2)^2.
\end{aligned}$$

Combining all these estimates, we finally arrive at the following.

Corollary 3.3. *Let $\frac{5}{3}\le p\le 2$. For every $\alpha>0$ there exist $C_\alpha,\gamma_\alpha>0$ such that, for every pair of patches of hypersurfaces S and $\widetilde{S}$ as described in our general assumptions at the beginning of this section and every $R>0$, we have*

$$\begin{aligned}
\|R^*_{S,\widetilde{S}}\|_{L^2(S)\times L^2(\widetilde{S})\to L^p(Q^1_{S,\widetilde{S}}(R))} &\le C_\alpha R^\alpha(\bar{\kappa}_1\bar{\kappa}_2)^{\frac{3}{p}-2}(\bar{d}_1\bar{d}_2)^{\frac{5}{p}-3}\min_i(\bar{\kappa}_i d_i\tilde{d}_i)^{3-\frac{5}{p}}\\
&\quad\times\Big(\bar{\kappa}_1\bar{d}_1^2\frac{\kappa_2}{\bar{\kappa}_2}\vee\bar{\kappa}_2\bar{d}_2^2\frac{\kappa_1}{\bar{\kappa}_1}\Big)^{\frac{1}{2}-\frac{1}{p}}\Big(\bar{\kappa}_1\bar{d}_1^2\frac{\tilde{\kappa}_2}{\bar{\kappa}_2}\vee\bar{\kappa}_2\bar{d}_2^2\frac{\tilde{\kappa}_1}{\bar{\kappa}_1}\Big)^{\frac{1}{2}-\frac{1}{p}}\\
&\quad\times\Big[1+\log^{\gamma_\alpha}\Big(q(\bar{\kappa}_1\bar{d}_1^2,\bar{\kappa}_2\bar{d}_2^2)\prod_{i=1,2}q(d_i,\tilde{d}_i)q(\kappa_i,\tilde{\kappa}_i)\Big)\Big],
\end{aligned}\tag{3-14}$$

where, in correspondence with our Convention 1.4, we have put $R^*_{S,\widetilde{S}}(f_1,f_2)=R^*_{\mathbb{R}^2}f_1\cdot R^*_{\mathbb{R}^2}f_2$, $f_1\in L^2(S)$, $f_2\in L^2(\widetilde{S})$.

4. Globalization and ε-removal

4A. *General results.* The next task will be to extend our inequalities (3-14) from the cuboids $Q^1_{S,\widetilde{S}}(R)$ to the whole space, and to get rid of the factor R^α. There is a certain amount of “globalization” or “ε-removal” technique available for this purpose, in particular Lemma 2.4 by Tao and Vargas [2000a], which in return follows ideas from [Bourgain 1995b]. We shall need to adapt those techniques to our setting, in which it will be important to understand more precisely how the corresponding estimates will depend on the parameters κ_j and d_j, $j = 1, 2$.

To this end, let us consider two hypersurfaces S_1 and S_2 in $\mathbb{R}^{d+1}$, defined as graphs $S_j = \{(x, \phi_j(x)) : x \in U_j\}$, and assume there is a constant A such that

$$|\nabla \phi_j(x)| \le A \tag{4-1}$$

for all $x \in U_j$, $j = 1, 2$. We will consider the measures ν_j defined on S_j by

$$\int_{S_j} g \, d\nu_j = \int_{U_j} f(x, \phi_j(x)) \, dx.$$

Note that, under the assumption (4-1), these measures are equivalent to the surface measures on S_1 and S_2. We write again

$$R^*_{S_1,S_2}(f_1, f_2) = R^*_{\mathbb{R}^d} f_1 \, R^*_{\mathbb{R}^d} f_2.$$

Denote by $B(0, R) = \{x \in \mathbb{R}^{d+1} : |x| \le R\}$ the ball of radius R. Our main result in this section is the following.

Lemma 4.1. *Let $C_1, C_2, \alpha, s > 0$, $R_0 \ge 1$, $1 \le p_0 < p \le \infty$, and let S_1, S_2 be hypersurfaces with ν_1, ν_2, respectively, satisfying (4-1), and let μ be a positive Borel measure on $\mathbb{R}^{d+1}$. Assume that for all $R \ge R_0$ and all $f_j \in L^2(S_j, \nu_j)$, $j = 1, 2$,*

(i) $\|R^*_{S_1,S_2}(f_1, f_2)\|_{L^{p_0}(B(0,R),\mu)} \le C_1 R^\alpha \|f_1\|_{L^2(S_1,\nu_1)} \|f_2\|_{L^2(S_2,\nu_2)}$,

(ii) $|\widehat{d\nu_j}(x)| \le C_2(1 + |x|)^{-s}$ *for all* $x \in \mathbb{R}^{d+1}$,

and that $(1 + 2\alpha/s)/p < 1/p_0$. *Then*

$$\|R^*_{S_1,S_2}(f_1, f_2)\|_{L^p(\mathbb{R}^{d+1},\mu)} \le C' \|f_1\|_{L^2(S_1,\nu_1)} \|f_2\|_{L^2(S_2,\nu_2)} \tag{4-2}$$

for all $f_j \in L^2(S_j, \nu_j)$, $j = 1, 2$, where C' only depends on $C_1, C_2, R_0, \alpha, s, p, p_0$.

Proof. We shall follow the proof of Lemma 2.4 in [Tao and Vargas 2000a] and only briefly sketch the main arguments, indicating those changes in the proof that will be needed in our setting. The main difference with [Tao and Vargas 2000a] is that instead of a Stein–Tomas-type estimate, we will use the trivial bound

$$\|R^*_{\mathbb{R}^d} f_j\|_{L^\infty(\mathbb{R}^{d+1},\mu)} \le \|f_j\|_{L^1(\nu_j)} \le \|f_j\|^{\frac{1}{2}}_{L^2(S_j,\nu_j)} |\widehat{d\nu_j}(0)|^{\frac{1}{2}} \le C_2^{\frac{1}{2}} \|f_j\|^{\frac{1}{2}}_{L^2(S_j,\nu_j)}, \tag{4-3}$$

where we have used our hypothesis (ii).

By (4-3) and interpolation, it then suffices to prove a weak-type estimate of the form

$$\mu(E_\lambda) \lesssim \lambda^{-p}, \quad \lambda > 0, \tag{4-4}$$

assuming $\|f_j\|_{L^2(\nu_j)} = 1$, $j = 1, 2$. Here, $E_\lambda = \{\mathrm{Re}(R^*_{\mathbb{R}^d} f_1 \, R^*_{\mathbb{R}^d} f_2) > \lambda\}$. Given $\lambda > 0$, let us abbreviate $E = E_\lambda$. We may also assume $\mu(E) \gtrsim 1$. Chebyshev's inequality implies

$$\lambda \mu(E) \lesssim \|\chi_E R^*_{\mathbb{R}^d} f_1 \, R^*_{\mathbb{R}^d} f_2\|_{L^1(\mu)},$$

and thus it suffices to show

$$\|\chi_E R^*_{\mathbb{R}^d} g_1 \, R^*_{\mathbb{R}^d} g_2\|_{L^1(\mu)} \lesssim \mu(E)^{\frac{1}{p'}} \|g_1\|_{L^2(\nu_1)} \|g_2\|_{L^2(\nu_1)} \tag{4-5}$$

for arbitrary L^2-functions g_1 and g_2 (which are completely independent of f_1 and f_2).

To this end, fix g_2 with $\|g_2\|_{L^2(\nu_2)} \sim 1$, and define $T = T_{E,g_2}$ as the linear operator

$$Tg_1 = \chi_E R^*_{\mathbb{R}^d} g_1 \, R^*_{\mathbb{R}^d} g_2.$$

Then, (4-5) is equivalent to the inequality

$$\|Tg_1\|_{L^1(\mu)} \lesssim \mu(E)^{\frac{1}{p'}} \|g_1\|_{L^2(\nu_1)}.$$

By duality, it suffices to show

$$\|T^* F\|_{L^2(d\nu_1)} \lesssim \mu(E)^{\frac{1}{p'}} \|F\|_{L^\infty(\mu)},$$

where T^* is (essentially) the adjoint operator

$$T^* F = \mathcal{F}^{-1}(\chi_E R^*_{\mathbb{R}^d} g_2 F \mu),$$

and $\mathcal{F}^{-1}$ is the inverse Fourier transform. We may assume $\|F\|_{L^\infty}(\mu) \lesssim 1$.

By squaring this and applying Plancherel's theorem, we reduce ourselves to showing

$$|\langle \widetilde{F}\, d\mu * \widehat{d\nu_1}, \widetilde{F}\, d\mu\rangle| \lesssim \mu(E)^{\frac{2}{p'}}, \tag{4-6}$$

where $\widetilde{F} = \chi_E (R^*_{\mathbb{R}^d} g_2) F$. Note that the hypotheses on F and g_2 and inequality (4-3) imply

$$\|\widetilde{F}\|_{L^1(\mu)} = \|\chi_E (R^*_{\mathbb{R}^d} g_2) F\|_{L^1(\mu)} \le \|\chi_E\|_{L^1(\mu)} \|R^*_{\mathbb{R}^d} g_2\|_{L^\infty(\mu)} \|F\|_{L^\infty(\mu)} \lesssim \mu(E). \tag{4-7}$$

From this point on, we follow the proof of [Tao and Vargas 2000a] with the obvious changes. Let $R > 1$ be a quantity to be chosen later. Let ϕ be a bump function which equals 1 for $|x| \lesssim 1$ and vanishes for $|x| \gg 1$, and write $d\nu_1 = d\nu_1^R + d\nu_{1\,R}$, where

$$\widehat{d\nu_{1\,R}}(x) = \phi\left(\frac{x}{R}\right)\widehat{d\nu_1}(x). \tag{4-8}$$

From hypothesis (ii) we have

$$\|\widehat{d\nu_1^R}\|_\infty \lesssim R^{-s},$$

and so by (4-7) we have

$$|\langle \widetilde{F}\, d\mu * \widehat{d\nu_1^R}, \widetilde{F}\, d\mu\rangle| \lesssim R^{-s}\mu(E)^2.$$

We now choose R to be

$$R = \mu(E)^{\frac{2}{sp}}, \tag{4-9}$$

so that the contribution of $d\nu_1^R$ to (4-6) is acceptable. Thus (4-6) reduces to

$$|\langle \widetilde{F}\, d\mu * \widehat{d\nu_{1\,R}}, \widetilde{F}\, d\mu\rangle| \lesssim \mu(E)^{\frac{2}{p'}}.$$

Following the arguments in [Tao and Vargas 2000a] and skipping details, we may then reduce the problem to proving

$$\|\chi_E \hat{\tilde{g}}_1 \hat{\tilde{g}}_2\|_{L^1(\mu)} \lesssim R^{-\frac{1}{2}} R^{-\frac{1}{2}} \mu(E)^{\frac{1}{p'}} \|\tilde{g}_1\|_2 \|\tilde{g}_2\|_2,$$

where $\tilde{g}_i$ is an arbitrary function on the $1/R$ neighborhood of $S_{i,R}$ for $i = 1, 2$. By Hölder's inequality it suffices to show

$$\|\hat{\tilde{g}}_1 \hat{\tilde{g}}_2\|_{L^{p_0}(\mu)} \lesssim \mu(E)^{-\frac{1}{p_0'}} R^{-\frac{1}{2}} R^{-\frac{1}{2}} \mu(E)^{\frac{1}{p'}} \|\tilde{g}_1\|_2 \|\tilde{g}_2\|_2. \tag{4-10}$$

Moreover, using the first hypothesis of the lemma, we obtain

$$\|\hat{\tilde{g}}_1 \hat{\tilde{g}}_2\|_{L^{p_0}(\mu)} \lesssim R^{\alpha-1} \|\hat{\tilde{g}}_1\|_2 \|\hat{\tilde{g}}_2\|_2.$$

Comparing this with (4-10), we see that we will be done if

$$R^{\alpha} \lesssim \mu(E)^{-\frac{1}{p_0'}} \mu(E)^{\frac{1}{p'}} = \mu(E)^{\frac{1}{p_0} - \frac{1}{p}}.$$

But this follows from (4-9) and the assumption $(1 + 2\alpha/s)/p < 1/p_0$. $\square$

4B. ***Application to the setting of Section 3.*** Let us now come back to the situation described by our General Assumptions in Lemma 3.2; i.e., we are interested in pairs of surfaces $S = \operatorname{graph}(\phi|_U)$, $U = r + [0, d_1] \times [0, d_2]$, with principal curvatures on S comparable to $\kappa_i = r_i^{m_i - 2}$, $r_i \geq d_i$, and $\widetilde{S} = \operatorname{graph}(\phi|_{\widetilde{U}})$, with corresponding quantities $\tilde{r}_i$, $\tilde{d}_i$, $\tilde{\kappa}_i$, $\tilde{\kappa}$.

Recall also the notation defined in (1-3), (3-1), and assume the conditions (3-2) and (3-3) are satisfied.

We consider the measure ν_S supported on S given by

$$\int_S f \, d\nu_S := \int_U f(x_1, x_2, \phi(x_1, x_2)) \, dx_1 \, dx_2,$$

and define $\nu_{\widetilde{S}}$ on $\widetilde{S}$ analogously.

4B1. *Decay of the Fourier transform.*

Lemma 4.2. *Let $s = 1/(m_1 \vee m_2)$. For any $r^0 \in U \cup \widetilde{U}$ we then have the uniform estimate for $x \in \mathbb{R}^3$*

$$\begin{aligned} &|\widehat{d\nu_S}(x)| + |\widehat{d\nu_{\widetilde{S}}}(x)| \\ &\quad \leq C_s \bar{d}_1 \bar{d}_2 \big(1 + |\bar{d}_1(x_1 + \partial_1\phi(r^0)x_3)| + |\bar{d}_2(x_2 + \partial_2\phi(r^0)x_3)| + |(\bar{\kappa}_1 \bar{d}_1^2 \vee \bar{\kappa}_2 \bar{d}_2^2)x_3|\big)^{-s}. \end{aligned} \tag{4-11}$$

Proof. We only consider $\nu = \nu_S$, since the proof for $\nu_{\widetilde{S}}$ is analogous. Recall that ϕ splits into $\phi(x) = \phi_{(1)}(x_1) + \phi_{(2)}(x_2)$, so that

$$|\widehat{d\nu}(x)| = \left| \int_{r_1}^{r_1 + d_1} e^{-i(x_1\xi_1 + x_3\phi_{(1)}(\xi_1))} \, d\xi_1 \int_{r_2}^{r_2 + d_2} e^{-i(x_2\xi_2 + x_3\phi_{(2)}(\xi_2))} \, d\xi_2 \right|.$$

Next, for $i \in \{1, 2\}$, we have

$$\begin{aligned} I_i &= \left| \int_{r_i}^{r_i+d_i} e^{-i(x_i\xi_i + x_3\phi_{(i)}(\xi_i))}\, d\xi_i \right| = \left| \int_0^{d_i} e^{-i(x_i(r_i+y_i)+x_3\phi_{(i)}(r_i+y_i))}\, dy_i \right| \\ &= \left| \int_0^{d_i} e^{-i\left((x_i+\phi'_{(i)}(r_i)x_3)y_i + x_3(\phi_{(i)}(r_i+y_i)-\phi_{(i)}(r_i)-\phi'_{(i)}(r_i)y_i)\right)}\, dy_i \right| \\ &= d_i \left| \int_0^1 e^{-i((x_i+\phi'_{(i)}(r_i)x_3)d_i y_i + x_3\kappa_i d_i^2 \Psi_i(d_i y_i))}\, dy_i \right|, \end{aligned}$$

where $\Psi_i(y_i) = \big(\phi_{(i)}(r_i + d_1 y_i) - \phi_{(i)}(r_i) - \phi'_{(i)}(r_i) d_i y_i\big)/(\kappa_i d_i^2)$, so that in particular

$$\left| \frac{d}{dy_i}\Psi_i(y_i) \right| = \left| \frac{\phi'_{(i)}(r_i + d_i y_i) - \phi'_{(i)}(r_i)}{\kappa_i d_i^2} d_i \right| \lesssim \frac{\kappa_i d_i}{\kappa_i d_i^2} d_i \sim 1, \qquad \frac{d^2}{dy_i^2}\Psi_i(y_i) = \frac{\phi''_{(i)}(r_i + d_i y_i)}{\kappa_i d_i^2} d_i^2 \sim 1.$$

Therefore, by either applying van der Corput's lemma of order 2, or by integrating by parts (if $|d_i(x_i + \phi'_{(i)}(r_i)x_3)| \gg |\kappa_i d_i^2 x_3|$,) we obtain

$$I_i \lesssim d_i (1 + |d_i(x_i + \phi'_{(i)}(r_i)x_3)| + |\kappa_i d_i^2 x_3|)^{-\frac{1}{2}}. \tag{4-12}$$

We next claim that the distortion $d_i/\bar{d}_i$ in the side lengths is bounded by the distortion in the size of the space variable r_i, i.e.,

$$\frac{d_i}{\bar{d}_i} \lesssim \frac{r_i}{\bar{r}_i}. \tag{4-13}$$

If $r_i \sim \bar{r}_i$, the statement is obvious, so assume $r_i \ll \bar{r}_i$. Then $\tilde{r}_i = \bar{r}_i$, and furthermore by our assumptions we have $d_i \le r_i$ and $\bar{r}_i \sim |r_i - \tilde{r}_i| \lesssim \bar{d}_i$ (compare to the separation condition (3-3)). Thus (4-13) follows also in this case. As $\kappa_i = r_i^{m_i - 2}$, we conclude from (4-13) that

$$\frac{\kappa_i d_i^2}{\bar{\kappa}_i \bar{d}_i^2} \gtrsim \left(\frac{d_i}{\bar{d}_i} \right)^{m_i}. \tag{4-14}$$

In combination, the estimates (4-13) and (4-14) imply

$$1 + |d_i(x_i + \phi'_{(i)}(r_i)x_3)| + |\kappa_i d_i^2 x_3| \gtrsim \left(\frac{d_i}{\bar{d}_i} \right)^{m_i} \big(1 + |\bar{d}_i(x_i + \phi'_{(i)}(r_i)x_3)| + |\bar{\kappa}_i \bar{d}_i^2 x_3|\big).$$

Since we may replace the exponent $-\frac{1}{2}$ in the right-hand side of (4-12) by $-1/m_i$, we now see that we may estimate

$$I_i \lesssim \bar{d}_i (1 + |\bar{d}_i(x_i + \phi'_{(i)}(r_i)x_3)| + |\bar{\kappa}_i \bar{d}_i^2 x_3|)^{-\frac{1}{m_i}}. \tag{4-15}$$

Finally, in order to pass from the point r to an arbitrary point $r^0 \in U \cup \widetilde{U}$ in these estimates, observe that by (3-3) we have $|r_i - r_i^0| \le |r_i - \tilde{r}_i| + \bar{d}_i \sim \bar{d}_i$, and hence

$$\bar{d}_i |\phi'_{(i)}(r_i) - \phi'_{(i)}(r_i^0)| \le \bar{\kappa}_i |r_i - r_i^0| \bar{d}_i \lesssim \bar{\kappa}_i \bar{d}_i^2,$$

since $|\phi''_{(i)}| \lesssim \bar{\kappa}_i$ on $[r_i, r_i + d_i] \cup [\tilde{r}_i, \tilde{r}_i + \tilde{d}_i]$. Therefore (4-15) implies that also

$$I_i \lesssim \bar{d}_i (1 + |\bar{d}_i (x_i + \partial_i \phi(r^0) x_3)| + |\bar{\kappa}_i \bar{d}_i^2 x_3|)^{-\frac{1}{m_i}}.$$

The estimate (4-11) is now immediate. □

4B2. *Linear change of variables and verification of the assumptions of Lemma 4.1.* In view of Lemma 4.2, let us fix $r^0 \in U \cup \widetilde{U}$, and define the linear transformation $T = T_{S,\widetilde{S}}$ of $\mathbb{R}^3$ by

$$T(x) = \big(\bar{d}_1(x_1 + \partial_1 \phi(r^0) x_3), \bar{d}_2(x_2 + \partial_2 \phi(r^0) x_3), (\bar{\kappa}_1 \bar{d}_1^2 \vee \bar{\kappa}_2 \bar{d}_2^2) x_3\big).$$

Then estimate (4-11) reads

$$|\widehat{d\nu_S}(x)| + |\widehat{d\nu_{\widetilde{S}}}(x)| \le C_s \bar{d}_1 \bar{d}_2 (1 + |T(x)|)^{-s}.$$

Therefore, in order to apply Lemma 4.1, we will consider the rescaled surfaces

$$S_1 = (T^t)^{-1} S \quad \text{and} \quad S_2 = (T^t)^{-1} \widetilde{S}. \tag{4-16}$$

Then we find that

$$\begin{aligned} S_1 &= \{(T^t)^{-1}(x_1, x_2, \phi_{(1)}(x_1) + \phi_{(2)}(x_2)) : (x_1, x_2) \in U\} \\ &= \left\{\left(\frac{x_1}{\bar{d}_1}, \frac{x_2}{\bar{d}_2}, \frac{1}{\bar{\kappa}_1 \bar{d}_1^2 \vee \bar{\kappa}_2 \bar{d}_2^2}\big(-\partial_1 \phi(r_0) x_1 - \partial_2 \phi(r_0) x_2 + \phi_{(1)}(x_1) + \phi_{(2)}(x_2)\big)\right) : (x_1, x_2) \in U\right\} \\ &= \{(y_1, y_2, \psi(y_1, y_2)) : (y_1, y_2) \in U_1\}, \end{aligned}$$

where $U_1 = \{(y_1, y_2) = (x_1/\bar{d}_1, x_2/\bar{d}_2) : (x_1, x_2) \in U\}$ is a square of side length ≤ 1 and

$$\psi(y_1, y_2) = \frac{1}{\bar{\kappa}_1 \bar{d}_1^2 \vee \bar{\kappa}_2 \bar{d}_2^2}\big(-\bar{d}_1 \partial_1 \phi(r_0) y_1 - \bar{d}_2 \partial_2 \phi(r_0) y_2 + \phi_{(1)}(\bar{d}_1 y_1) + \phi_{(2)}(\bar{d}_2 y_2)\big).$$

We have a similar expression for S_2.

In S_1 we consider the measure $d\nu_1$ defined by

$$\int_{S_1} g \, d\nu_1 = \frac{1}{\bar{d}_1 \bar{d}_2} \int_S g((T^t)^{-1} x) \, d\nu_S(x).$$

By our definition of dν and ψ, this may be rewritten as

$$\begin{aligned} \int_{S_1} g \, d\nu_1 &= \frac{1}{\bar{d}_1 \bar{d}_2} \int_U g\big((T^t)^{-1}(x_1, x_2, \phi(x_1, x_2))\big) \, dx_1 \, dx_2 \\ &= \frac{1}{\bar{d}_1 \bar{d}_2} \int_U g\left(\frac{x_1}{\bar{d}_1}, \frac{x_2}{\bar{d}_2}, \psi\left(\frac{x_1}{\bar{d}_1}, \frac{x_2}{\bar{d}_2}\right)\right) dx_1 \, dx_2 = \int_{U_1} g(y_1, y_2, \psi(y_1, y_2)) \, dy_1 \, dy_2. \end{aligned}$$

Moreover, we have

$$\widehat{g \, d\nu_1}(\xi) = \frac{1}{\bar{d}_1 \bar{d}_2} \, \widehat{\big(g \circ (T^t)^{-1} d\nu_S\big)} \, (T^{-1} \xi), \tag{4-17}$$

and therefore

$$|\widehat{d\nu_1}(x)| \le C_s(1+|x|)^{-s}.$$

We have a similar estimate for $\widehat{d\nu_2}$. Thus, the hypothesis (ii) in Lemma 4.1 is satisfied. To check that condition (4-1) is satisfied for S_1 and S_2 too, we compute

$$\left|\frac{\partial\psi}{\partial y_1}\right| = \frac{1}{\bar\kappa_1\bar d_1^2 \vee \bar\kappa_2\bar d_2^2}\left|-\bar d_1\partial_1\phi(r_0)+\bar d_1\phi'_{(1)}(\bar d_1 y_1)\right|.$$

Writing $r_0 = (\bar d_1 y_{1,0}, \bar d_2 y_{2,0})$, we see that

$$\begin{aligned}\left|\frac{\partial\psi}{\partial y_1}\right| &= \frac{1}{\bar\kappa_1\bar d_1^2 \vee \bar\kappa_2\bar d_2^2}\left|-\bar d_1\phi'_{(1)}(\bar d_1 y_{1,0})+\bar d_1\phi'_{(1)}(\bar d_1 y_1)\right| \\ &\sim \frac{1}{\bar\kappa_1\bar d_1^2 \vee \bar\kappa_2\bar d_2^2}|\bar d_1^2(y_1-y_{1,0})\phi''_{(1)}| \le \frac{\bar\kappa_1\bar d_1^2}{\bar\kappa_1\bar d_1^2 \vee \bar\kappa_2\bar d_2^2}|y_1-y_{1,0}| \le C_{m_1,m_2},\end{aligned}$$

and in a similar way we find that the derivative with respect to y_2 is bounded. Hence, hypothesis (4-1) is satisfied for ψ in place of ϕ.

What remains to be checked is condition (i) in Lemma 4.1. Observe first that our local bilinear estimate for S and $\tilde S$ in Corollary 3.3 is restricted to cuboids (see (3-9))

$$Q^1(R) = Q^1_{S,\tilde S}(R) = \left\{x\in\mathbb{R}^3 : |x_i+\partial_i\phi(r^0)x_3| \le \frac{R}{\bar d_i}, i=1,2,\ |x_3| \le \frac{R}{\bar\kappa_1\bar d_1^2 \wedge \bar\kappa_2\bar d_2^2}\right\}, \tag{4-18}$$

where r^0 is either[3] r or $\tilde r$. Obviously $T^{-1}(B(0,R)) = \{x\in\mathbb{R}^3 : |Tx| \le R\} \subset Q^1(R)$.

Define

$$\begin{aligned} A &= (\bar\kappa_1\bar\kappa_2)^{-2}(\bar d_1\bar d_2)^{-3}\min_i(\bar\kappa_i d_i\tilde d_i)^3\left(\bar\kappa_1\bar d_1^2\frac{\kappa_2}{\bar\kappa_2}\vee\bar\kappa_2\bar d_2^2\frac{\kappa_1}{\bar\kappa_1}\right)^{\frac12}\left(\bar\kappa_1\bar d_1^2\frac{\tilde\kappa_2}{\bar\kappa_2}\vee\bar\kappa_2\bar d_2^2\frac{\tilde\kappa_1}{\bar\kappa_1}\right)^{\frac12}(1+\log^{\gamma_\alpha} Q), \\ B &= (\bar\kappa_1\bar\kappa_2)^{3}(\bar d_1\bar d_2)^{5}\min_i(\bar\kappa_i d_i\tilde d_i)^{-5}\left(\bar\kappa_1\bar d_1^2\frac{\kappa_2}{\bar\kappa_2}\vee\bar\kappa_2\bar d_2^2\frac{\kappa_1}{\bar\kappa_1}\right)^{-1}\left(\bar\kappa_1\bar d_1^2\frac{\tilde\kappa_2}{\bar\kappa_2}\vee\bar\kappa_2\bar d_2^2\frac{\tilde\kappa_1}{\bar\kappa_1}\right)^{-1}, \end{aligned} \tag{4-19}$$

where

$$Q = Q(S,\tilde S) = q(\bar\kappa_1\bar d_1^2, \bar\kappa_2\bar d_2^2)\prod_{i=1,2} q(d_i,\tilde d_i)q(\kappa_i,\tilde\kappa_i)$$

and $q(a,b) = (a\vee b)/a\wedge b \ge 1$ are defined to be the maximal quotient of a and b. In some sense Q is a "degeneracy quotient" that measures how much (for instance) quantities $d_i, \tilde d_i$ differ from their maximum $\bar d_i$.

Then the estimate (3-14) in Corollary 3.3, valid for $\frac53 \le p \le 2$, can be rewritten in terms of these quantities as

$$Q^1(R)\|R^*_{S,\tilde S}\|_{L^2(S)\times L^2(\tilde S)\to L^p(Q^1(R))} \le C_\alpha R^\alpha A B^{\frac1p}. \tag{4-20}$$

[3]Recall that we have some algorithm for how to choose r^0, but this will not be relevant here.

Now, in order to check hypothesis (i) in Lemma 4.1, let us choose for μ the measure on $\mathbb{R}^3$ given by

$$d\mu = \widetilde{B}^{-1} d\xi, \quad \text{where } \widetilde{B} = |\det T| \left(\frac{A}{\bar{d}_1 \bar{d}_2}\right)^{p_0} B,$$

and where $d\xi$ denotes the Lebesgue measure. Notice also that (4-17) implies that, for any measurable set $E \subset \mathbb{R}^3$ and any exponent p, we have

$$\|R^*_{S_1,S_2}(f_1, f_2)\|_{L^p(E,\mu)} = \frac{A^{-\frac{p_0}{p}} B^{-\frac{1}{p}}}{(\bar{d}_1\bar{d}_2)^{2-\frac{p_0}{p}}} \big\|R^*_{S,\widetilde{S}}\big(f_1 \circ (T^t)^{-1}, f_2 \circ (T^t)^{-1}\big)\big\|_{L^p(T^{-1}(E),d\xi)}. \tag{4-21}$$

In particular, we obtain

$$\begin{aligned}\|R^*_{S_1,S_2}(f_1, f_2)\|_{L^{p_0}(B(0,R),\mu)} &= \frac{A^{-1} B^{-\frac{1}{p_0}}}{\bar{d}_1\bar{d}_2} \big\|R^*_{S,\widetilde{S}}\big(f_1 \circ (T^t)^{-1}, f_2 \circ (T^t)^{-1}\big)\big\|_{L^{p_0}(T^{-1}(B(0,R)),d\xi)} \\ &\le \frac{A^{-1} B^{-\frac{1}{p_0}}}{\bar{d}_1\bar{d}_2} \big\|R^*_{S,\widetilde{S}}\big(f_1 \circ (T^t)^{-1}, f_2 \circ (T^t)^{-1}\big)\big\|_{L^{p_0}(Q^1(R),d\xi)}.\end{aligned}$$

Invoking (4-20), we thus see that for $\frac{5}{3} \le p_0 \le 2$ and every $\alpha > 0$,

$$\begin{aligned}\|R^*_{S_1,S_2}(f_1, f_2)\|_{L^{p_0}(B(0,R),\mu)} &\le \frac{1}{\bar{d}_1\bar{d}_2} C_\alpha R^\alpha \|f_1 \circ (T^t)^{-1}\|_{L^2(d\nu_S)} \|f_2 \circ (T^t)^{-1}\|_{L^2(d\nu_{\widetilde{S}})} \\ &= C_\alpha R^\alpha \|f_1\|_{L^2(d\nu_1)} \|f_2\|_{L^2(d\nu_2)},\end{aligned}$$

which shows that hypothesis (i) in the Lemma 4.1 is satisfied. Applying this lemma and using again identity (4-21) and the definitions of μ, ν_1 and ν_2, we find that for any g_1 and g_2 supported in S and $\widetilde{S}$, respectively, and any p satisfying the assumptions of Lemma 4.1, we have

$$\|R^*_{S,\widetilde{S}}(g_1, g_2)\|_{L^p(d\xi)} \le C(\bar{d}_1\bar{d}_2)^{1-\frac{p_0}{p}} A^{\frac{p_0}{p}} B^{\frac{1}{p}} \|g_1\|_{L^2(d\nu_1)} \|g_2\|_{L^2(d\nu_2)}. \tag{4-22}$$

Finally, putting $\varepsilon = 1 - p_0/p$, and recalling that we may choose α in Lemma 4.1 as small as we wish, then by applying Hölder's inequality in order to replace the L^2-norms on the right-hand side of (4-22) by the L^q-norms, we arrive at the following global estimate:

Theorem 4.3. *Let $\frac{5}{3} < p \le 2$, $q \ge 2$, $\varepsilon > 0$. Then there exist constants $C = C_{p,\varepsilon}$ and $\gamma = \gamma_{p,\varepsilon} > 0$ such that*

$$\begin{aligned}\|R^*_{S,\widetilde{S}}\|_{L^q(S)\times L^q(\widetilde{S}) \to L^p(\mathbb{R}^n)} \le\, & C(\bar{\kappa}_1\bar{\kappa}_2)^{\frac{3}{p}-2+2\varepsilon} (\bar{d}_1\bar{d}_2)^{\frac{5}{p}-3+4\varepsilon} \\ & \times (d_1 d_2 \tilde{d}_1 \tilde{d}_2)^{\frac{1}{2}-\frac{1}{q}} (1 + \log^\gamma Q) \min_i (\bar{\kappa}_i d_i \tilde{d}_i)^{3-3\varepsilon-\frac{5}{p}} \\ & \times \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\kappa_2}{\tilde{\kappa}_2} \vee \bar{\kappa}_2 \bar{d}_2^2 \frac{\kappa_1}{\bar{\kappa}_1}\right)^{\frac{1-\varepsilon}{2}-\frac{1}{p}} \left(\bar{\kappa}_1 \bar{d}_1^2 \frac{\tilde{\kappa}_2}{\bar{\kappa}_2} \vee \bar{\kappa}_2 \bar{d}_2^2 \frac{\tilde{\kappa}_1}{\bar{\kappa}_1}\right)^{\frac{1-\varepsilon}{2}-\frac{1}{p}}\end{aligned} \tag{4-23}$$

uniformly in S and $\widetilde{S}$, where $Q = q(\bar{\kappa}_1 \bar{d}_1^2, \bar{\kappa}_2 \bar{d}_2^2) \prod_{i=1,2} q\big(d_i, \tilde{d}_i\big) q(\kappa_i, \tilde{\kappa}_i)$ and $q(a,b) = (a \vee b)/a \wedge b$.

5. Dyadic summation

Recall that our hypersurface of interest is the graph of a smooth function $\phi(x_1,x_2)=\phi_{(1)}(x_1)+\phi_{(2)}(x_2)$ defined over the square $]0,1[\times]0,1[$. We assume ϕ to be extended continuously to the closed square $Q=[0,1]\times[0,1]$ (this extension will in the end not really play any role, but it will be more convenient to work with a closed square). By means of a kind of Whitney decomposition of the direct product $Q\times Q$ near the "diagonal", following some standard procedure in the bilinear approach, we can decompose $Q\times Q$ into products of congruent rectangles U and $\widetilde{U}$ of dyadic side lengths, which are "well-separated neighbors" in some sense. The next step will therefore consist in establishing bilinear estimates for pairs of subhypersurfaces supported over such pairs of neighboring rectangles. Notice that if one of these rectangles meets one of the coordinate axes, then the principal curvature in at least one coordinate direction will no longer be of a certain size, but will indeed go down to zero within this rectangle. We then perform an additional dyadic decomposition of this rectangle in order to achieve that both principal curvatures will be of a certain size on each of the dyadic subrectangles (see Figure 10). To these we can then apply our estimates from Theorem 4.3. Thus, in this section we shall work under the following:

General Assumptions. For $k_i,\tilde{k}_i,j_i\in\mathbb{N}$,

$$U=[k_1 2^{-j_1},(k_1+1)2^{-j_1}]\times[k_2 2^{-j_2},(k_2+1)2^{-j_2}],$$
$$\widetilde{U}=[\tilde{k}_1 2^{-j_1},(\tilde{k}_1+1)2^{-j_1}]\times[\tilde{k}_2 2^{-\tilde{j}_2},(\tilde{k}_2+1)2^{-\tilde{j}_2}],$$

are two congruent closed bidyadic rectangles in $[0,1]\times[0,1]$ whose side length and distance between them in the x_i-direction is equal to $\rho_i=2^{-j_i}$, both for $i=1$ and $i=2$.

By $\varkappa_i$ we denote the maximum value of the principal curvature in the x_i-direction of both $S=\operatorname{graph}(\phi|_U)$ and $\widetilde{S}=\operatorname{graph}(\phi|_{\widetilde{U}})$.

Theorem 5.1. *Let $\frac{5}{3}<p<2$, $q\geq 2$, $\varepsilon>0$, and assume $(m_1\vee m_2+3)\bigl(\frac{1}{p}-\frac{1}{2}\bigr)<\frac{1}{q'}$. Then we have*

$$\|R^*_{S,\widetilde{S}}\|_{L^q(S)\times L^q(\widetilde{S})\to L^p(\mathbb{R}^3)}\leq C_{p,q,\varepsilon}(\rho_1\rho_2)^{\frac{2}{q'}-\frac{1}{p}}(\varkappa_1\rho_1^2\vee\varkappa_2\rho_2^2)^{\frac{1}{p}-1+\varepsilon}(\varkappa_1\rho_1^2\wedge\varkappa_2\rho_2^2)^{1-\frac{2}{p}-\varepsilon}. \tag{5-1}$$

Proof. If U does not intersect with the x_i-axis, then the principal curvature in x_i-direction on U is indeed comparable to $\varkappa_i$. Otherwise we decompose U further into sets with (roughly) constant principal curvatures in order to apply the previous results. More precisely, to each dyadic interval $I=[k2^{-j},(k+1)2^{-j}]$, $k,j\in\mathbb{N}$, we associate a family of subsets $\{I(l)\}_{l\in\mathcal{N}_0}$ with $\bigcup_{l\in\mathcal{N}_0}I(l)=I$, according to the following two alternatives:

(i) If $k>0$, then choose $\mathcal{N}_0=\{0\}$ and $I(0)=I$.

(ii) If $k=0$, then choose $\mathcal{N}_0=\mathbb{N}=\{1,2,3,\dots\}$ and $I(l)=[2^{-l}(k+1)2^{-j},2^{1-l}(k+1)2^{-j}]$.

If we write $U=I_1\times I_2$, then denote by $\{I_i(l_i)\}_{l_i\in\mathcal{N}_i}$ their associated family, and let $U(l)=I(l_1)\times I(l_2)$, $l=(l_1,l_2)\in\mathcal{N}=\mathcal{N}_1\times\mathcal{N}_2$ and $S(l)=\operatorname{graph}(\phi|_{U(l)})$. Define $\widetilde{\mathcal{N}}$, $\widetilde{U}(l)$ and $\widetilde{S}(l)$, $l\in\widetilde{\mathcal{N}}$, in an analogous manner. Other relevant quantities are the principal curvatures on $U(l)$, i.e.,

$$\kappa_i(l_i):=2^{-l_i(m_i-2)}\varkappa_i, \tag{5-2}$$

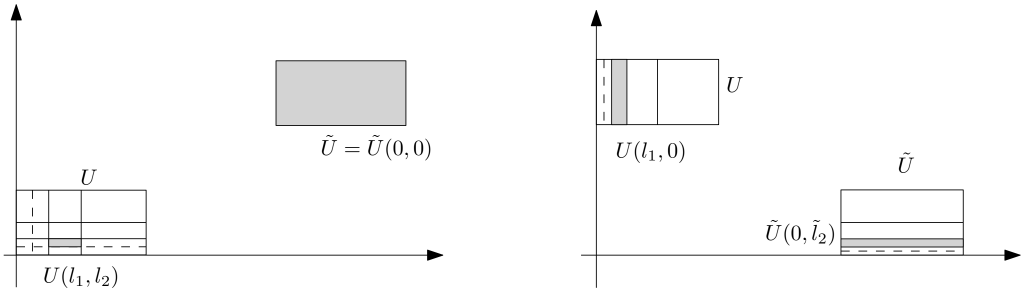

Figure 10. Two possibilities for the decomposition into subboxes.

and the side lengths of $U(l)$

$$d_i(l_i) = 2^{-l_i}\rho_i. \tag{5-3}$$

A simple but crucial observation is that since I_i and $\widetilde{I}_i$ are separated for both $i = 1$ and $i = 2$, we have $\mathcal{N}_i = \{0\}$ or $\widetilde{\mathcal{N}}_i = \{0\}$ (see Figure 10). Hence $l_i = 0$ or $\tilde{l}_i = 0$ for each pair $(l_i, \tilde{l}_i) \in \mathcal{N}_i \times \widetilde{\mathcal{N}}_i$, and thus

$$\bar{\kappa}_i(l_i, \tilde{l}_i) := \max\{\kappa_i(l_i), \tilde{\kappa}_i(\tilde{l}_i)\} = \max\{2^{-l_i(m_i-2)}, 2^{-\tilde{l}_i(m_i-2)}\}\varkappa_i = \varkappa_i, \tag{5-4}$$

$$\bar{d}_i(l_i, \tilde{l}_i) := \max\{d_i(l_i), \tilde{d}_i(\tilde{l}_i)\} = \max\{2^{-l_i}, 2^{-\tilde{l}_i}\}\rho_i = \rho_i. \tag{5-5}$$

We conclude that

$$\frac{\kappa_i(l_i)}{\bar{\kappa}_i(l_i, \tilde{l}_i)} = 2^{-l_i(m_i-2)}, \tag{5-6}$$

$$\frac{\tilde{\kappa}_i(l_i)}{\bar{\kappa}_i(l_i, \tilde{l}_i)} = 2^{-\tilde{l}_i(m_i-2)}\frac{d_i(l_i)}{\bar{d}_i} = 2^{-l_i}, \tag{5-7}$$

$$d_i(l_i)\tilde{d}_i(\tilde{l}_i) = 2^{-l_i-\tilde{l}_i}\rho_i^2 = 2^{-l_i\vee\tilde{l}_i}\rho_i^2. \tag{5-8}$$

Hence

$$Q = q(\bar{\kappa}_1\bar{d}_1^2, \bar{\kappa}_2\bar{d}_2^2)\prod_{i=1,2} q\big(d_i(l_i), \tilde{d}_i(\tilde{l}_i)\big)\, q\big(\kappa_i(l_i), \tilde{\kappa}_i(\tilde{l}_i)\big) \le \frac{\bar{\kappa}_1\bar{d}_1^2 \vee \bar{\kappa}_2\bar{d}_2^2}{\bar{\kappa}_1\bar{d}_1^2 \wedge \bar{\kappa}_2\bar{d}_2^2}\, 2^{m_1(l_1+\tilde{l}_1)+m_2(l_2+\tilde{l}_2)}. \tag{5-9}$$

Thus, if we apply inequality (4-23) from Theorem 4.3 to the pairs of hypersurfaces $S(l), \widetilde{S}(\tilde{l})$ and estimate by means of (5-4)–(5-9), then we get

$$\begin{aligned}
&\|R^*_{S,\widetilde{S}}\|_{L^q\times L^q\to L^p} \\
&\quad\le \sum_{l\in\mathcal{N}, \tilde{l}\in\widetilde{\mathcal{N}}} \|R^*_{S(l),\widetilde{S}(\tilde{l})}\|_{L^q\times L^q\to L^p} \\
&\quad\lesssim (\varkappa_1\rho_1^2\varkappa_2\rho_2^2)^{\frac{3}{p}-2+2\varepsilon}(\rho_1\rho_2)^{\frac{2}{q'}-\frac{1}{p}}\log^\gamma\left(\frac{\varkappa_1\rho_1^2}{\varkappa_2\rho_2^2}+\frac{\varkappa_2\rho_2^2}{\varkappa_1\rho_1^2}\right) \\
&\qquad\times\Bigg(\sum_{l\in\mathcal{N}, \tilde{l}\in\widetilde{\mathcal{N}}} [1+l_1+\tilde{l}_1+l_2+\tilde{l}_2]^\gamma \big(\varkappa_1\rho_1^2 2^{-l_1-\tilde{l}_1}\wedge\varkappa_2\rho_2^2 2^{-l_2-\tilde{l}_2}\big)^{3-3\varepsilon-\frac{5}{p}}\, 2^{-(l_1+\tilde{l}_1+l_2+\tilde{l}_2)(\frac{1}{2}-\frac{1}{q})}\Bigg) \\
&\qquad\times\big(\varkappa_1\rho_1^2 2^{-l_2(m_2-2)}\vee\varkappa_2\rho_2^2 2^{-l_1(m_1-2)}\big)^{\frac{1-\varepsilon}{2}-\frac{1}{p}}\big(\varkappa_1\rho_1^2 2^{-\tilde{l}_2(m_2-2)}\vee\varkappa_2\rho_2^2 2^{-\tilde{l}_1(m_1-2)}\big)^{\frac{1-\varepsilon}{2}-\frac{1}{p}}.
\end{aligned}$$

We claim

$$\begin{aligned}\sum_{l\in\mathcal{N},\tilde{l}\in\widetilde{\mathcal{N}}} &[1+l_1+\tilde{l}_1+l_2+\tilde{l}_2]^\gamma \big(\varkappa_1\rho_1^2 2^{-l_1-\tilde{l}_1}\wedge\varkappa_2\rho_2^2 2^{-l_2-\tilde{l}_2}\big)^{3-3\varepsilon-\frac{5}{p}} 2^{-(l_1+\tilde{l}_1+l_2+\tilde{l}_2)(\frac{1}{2}-\frac{1}{q})}\\ &\times\big(\varkappa_1\rho_1^2 2^{-l_2(m_2-2)}\vee\varkappa_2\rho_2^2 2^{-l_1(m_1-2)}\big)^{\frac{1-\varepsilon}{2}-\frac{1}{p}}\big(\varkappa_1\rho_1^2 2^{-\tilde{l}_2(m_2-2)}\vee\varkappa_2\rho_2^2 2^{-\tilde{l}_1(m_1-2)}\big)^{\frac{1-\varepsilon}{2}-\frac{1}{p}}\\ &\lesssim(\varkappa_1\rho_1^2\wedge\varkappa_2\rho_2^2)^{3-3\varepsilon-\frac{5}{p}}(\varkappa_1\rho_1^2\vee\varkappa_2\rho_2^2)^{1-\varepsilon-\frac{2}{p}}.\end{aligned}\tag{5-10}$$

Taking this for granted, we would arrive at estimate (5-1):

$$\begin{aligned}&\|R^*_{S,\tilde{S}}\|_{L^2\times L^2\to L^p(Q_{S,\tilde{S}}(R))}\\ &\quad\lesssim(\varkappa_1\rho_1^2\varkappa_2\rho_2^2)^{\frac{3}{p}-2+2\varepsilon}(\rho_1\rho_2)^{\frac{2}{q'}-\frac{1}{p}}(\varkappa_1\rho_1^2\vee\varkappa_2\rho_2^2)^{1-\varepsilon-\frac{2}{p}}(\varkappa_1\rho_1^2\wedge\varkappa_2\rho_2^2)^{3-3\varepsilon-\frac{5}{p}}\log^\gamma\left(\frac{\varkappa_1\rho_1^2}{\varkappa_2\rho_2^2}+\frac{\varkappa_2\rho_2^2}{\varkappa_1\rho_1^2}\right)\\ &\quad=(\rho_1\rho_2)^{\frac{2}{q'}-\frac{1}{p}}(\varkappa_1\rho_1^2\vee\varkappa_2\rho_2^2)^{\frac{1}{p}-1+\varepsilon}(\varkappa_1\rho_1^2\wedge\varkappa_2\rho_2^2)^{1-\frac{2}{p}-\varepsilon}\log^\gamma\left(\frac{\varkappa_1\rho_1^2}{\varkappa_2\rho_2^2}+\frac{\varkappa_2\rho_2^2}{\varkappa_1\rho_1^2}\right).\end{aligned}$$

We are thus left with the estimation of the dyadic sum in (5-10). Let

$$\mu=\frac{1}{p}-\frac{1-\varepsilon}{2}>0,\quad \nu=3-3\varepsilon-\frac{5}{p}>0,\quad \omega=\frac{1}{2}-\frac{1}{q}>0,\quad c_i=m_i-2.$$

Then $c_i\mu<\nu+\omega$ is equivalent to $m_i\big(\frac{1}{p}-\frac{1}{2}\big)+\mathcal{O}(\varepsilon)<\frac{1}{q'}$. This is satisfied since by our assumptions in the theorem we have $m_i\big(\frac{1}{p}-\frac{1}{2}\big)<\frac{1}{q'}$, and we can choose ε arbitrarily small.

Estimate (5-10) will then be an easy consequence of the next lemma. Indeed, recalling our earlier observation that for each pair $(l_i,\tilde{l}_i)\in\mathcal{N}_i\times\widetilde{\mathcal{N}}_i$ one of the entries l_i or $\tilde{l}_i$ must be zero, we see that we have to sum over at most two of the parameters l_1, l_2, $\tilde{l}_1$, $\tilde{l}_2$.

Thus, there are four possibilities: if exactly two of the parameters are nonzero, then there are two distinct cases: either these parameters belong to the same surface (i.e., $l_1=l_2=0$ or $\tilde{l}_1=\tilde{l}_2=0$), which correspond to the left picture in Figure 10, or the nonzero parameters belong to two different surfaces, as in the "over cross" situation shown in the picture on the right hand side of Figure 10. The remaining two possibilities are firstly that only one parameter l_1, l_2, $\tilde{l}_1$, $\tilde{l}_2$ is nonzero, which happens if only one of the rectangles $U,\widetilde{U}$ touches only one of the axes, and secondly the situation where both rectangles are located away from the axes. In this last situation, we have indeed no further decomposition and only one term to sum.

The first two of the aforementioned possibilities can be dealt with directly by the next lemma. But, notice that the corresponding sums of course dominate the sums over fewer parameters (or even none), which allows to also handle the remaining two possibilities. □

Lemma 5.2. *Let $\mu,\omega\geq 0$, $\nu>0$, $n,c_1,c_2\geq 0$ such that $(c_1\vee c_2)\mu<\nu+\omega$, and let $a,b\in\mathbb{R}_+$. Then*

$$\begin{aligned}\sum_{l_1,l_2\in\mathbb{N}}&(1+l_1+l_2)^n 2^{-(l_1+l_2)\omega}(a2^{-l_2c_2}\vee b)^{-\mu}(a\vee b2^{-l_1c_1})^{-\mu}(a2^{-l_1}\wedge b2^{-l_2})^\nu\\ &\leq\sum_{l_1,l_2\in\mathbb{N}}(1+l_1+l_2)^n 2^{-(l_1+l_2)\omega}(a\vee b)^{-\mu}(a2^{-l_2c_2}\vee b2^{-l_1c_1})^{-\mu}(a2^{-l_1}\wedge b2^{-l_2})^\nu\\ &\lesssim(a\vee b)^{-2\mu}(a\wedge b)^\nu.\end{aligned}$$

In the last estimate, the constant hidden by the symbol $\lesssim$ will depend only on the exponent n.

We remark that the bound in this lemma is essentially sharp, as one can immediately see by looking at the term with $l_1 = 0 = l_2$. Notice that the proof is easier when $\omega > 0$.

Proof. To prove the first inequality, observe that $a2^{-l_2c_2} \vee b2^{-l_1c_1}$ is bounded by $a2^{-l_2c_2} \vee b$ as well as by $a \vee b2^{-l_1c_1}$, and hence by the minimum of these expressions. Therefore we have

$$(a2^{-l_2c_2} \vee b) \wedge (a \vee b2^{-l_1c_1}) \geq a2^{-l_2c_2} \vee b2^{-l_1c_1},$$
$$(a2^{-l_2c_2} \vee b) \vee (a \vee b2^{-l_1c_1}) = a \vee b;$$

hence

$$(a2^{-l_2c_2} \vee b)(a \vee b2^{-l_1c_1}) \geq (a \vee b)(a2^{-l_2c_2} \vee b2^{-l_1c_1}).$$

Using the symmetry in this estimate, it suffices to estimate

$$S = a^\nu \sum_{\substack{l_1,l_2\in\mathbb{N} \\ a2^{-l_1}\leq b2^{-l_2}}} l_1^n l_2^n 2^{-(l_1+l_2)\omega} (a2^{-l_2c_2} \vee b2^{-l_1c_1})^{-\mu} 2^{-l_1\nu}.$$

On the one hand, we have

$$\begin{aligned} S &\leq a^\nu b^{-\mu} \sum_{l_1} l_1^n 2^{l_1(c_1\mu-\nu-\omega)} \sum_{l_2:a2^{-l_1}\leq b2^{-l_2}} l_2^n 2^{-l_2\omega} \\ &\leq a^\nu b^{-\mu} \log^{n+1}\Big(\frac{a}{b}+\frac{b}{a}\Big) \sum_{l_1} l_1^{2n+1} 2^{l_1(c_1\mu-\nu-\omega)} \lesssim a^\nu b^{-\mu} \log^{n+1}\Big(\frac{a}{b}+\frac{b}{a}\Big). \end{aligned}$$

In the case $\omega > 0$, we might get along even without the log-term. On the other hand,

$$\begin{aligned} S &\leq a^{\nu-\mu} \sum_{l_2} l_2^n 2^{l_2(c_2\mu-\omega)} \sum_{l_1:a2^{-l_1}\leq b2^{-l_2}} l_1^n 2^{-l_1(\nu+\omega)} \\ &\leq a^{\nu-\mu} \sum_{l_2} l_2^n 2^{l_2(c_2\mu-\omega)} \sum_{l_1:a2^{-l_1}\leq b2^{-l_2}} l_1^n 2^{-l_1\nu} \\ &\sim a^{\nu-\mu} \log^n\Big(\frac{a}{b}+\frac{b}{a}\Big)\Big(\frac{b}{a}\Big)^\nu \sum_{l_2} l_2^{2n} 2^{l_2(c_2\mu-\nu-\omega)} \sim a^{-\mu} b^\nu \log^n\Big(\frac{a}{b}+\frac{b}{a}\Big). \end{aligned}$$

Combining these two estimates, we obtain

$$\frac{S}{\log^{n+1}(\frac{a}{b}+\frac{b}{a})} \lesssim a^{-\mu}b^\nu \wedge a^\nu b^{-\mu} = (a \vee b)^{-\mu}(a \wedge b)^\nu. \qquad \square$$

6. Passage from bilinear to linear estimates

Recall that $\bar{m} = m_1 \vee m_2$, $m = m_1 \wedge m_2$ and $1/h = 1/m_1 + 1/m_2$. The first step to prove our main theorem, Theorem 1.2, is the following Lorentz space estimate for the adjoint restriction operator R^* associated to $\Gamma = \mathrm{graph}(\phi)$.

Theorem 6.1. *Let $p_0 = 1 + \bar{m}/(\bar{m}+m)$, $2p > \max\{\frac{10}{3}, 2p_0, h+1\}$ and $1/s' \geq (h+1)/(2p)$. Then R^* is bounded from $L^{s,t}(\Gamma, d\nu)$ to $L^{2p,t}(\mathbb{R}^3)$ for any $1 \leq t \leq \infty$.*

Proof. We begin by observing that we may assume

$$\frac{h+1}{p} > 1. \tag{6-1}$$

Indeed, if $2p \geq 2(h+1)$, then we have the Stein–Tomas-type result that R^* is bounded from $L^2(\Gamma, d\nu)$ to $L^{2p}(\mathbb{R}^3)$ (see [Ikromov et al. 2010; Ikromov and Müller 2011]). Interpolating this with the trivial estimate from $L^1(\Gamma, d\nu)$ to $L^\infty(\mathbb{R}^3)$ and applying Hölder's inequality on Γ, we see that the situation where $(h+1)/p \leq 1$ is settled in Theorem 6.1.

In the remaining cases, interpolation theory for Lorentz spaces (see, e.g., [Grafakos 2008]) shows that it suffices to prove the restricted weak-type estimate

$$\|\widehat{\chi_\Omega \, d\nu}\|_{2p} \lesssim |\Omega|^{\frac{1}{s}} \tag{6-2}$$

for any measurable set $\Omega \subset Q = [0,1] \times [0,1]$.

To this end we perform the kind of Whitney decomposition mentioned in Section 5 of $Q \times Q = \bigcup_j \bigcup_{k \approx \tilde{k}} \tau_{jk} \times \tau_{j\tilde{k}}$ into "well-separated neighboring rectangles" τ_{jk} and $\tau_{j\tilde{k}}$, where

$$\tau_{jk} = [(k_1-1)2^{-j_1}, k_1 2^{-j_1}] \times [(k_2-1)2^{-j_2}, k_2 2^{-j_2}],$$

and where $k \approx \tilde{k}$ means that $2 \leq |k_i - \tilde{k}_i| \leq C$, $i = 1, 2$ (see [Lee 2006; Vargas 2005]). Then we may estimate

$$\|\widehat{\chi_\Omega \, d\nu}\|_{2p}^2 = \|\widehat{\chi_\Omega \, d\nu}\,\widehat{\chi_\Omega \, d\nu}\|_p \leq \sum_j \Big(\sum_{k \sim \tilde{k}} \|\mathfrak{F}(\chi_{\Omega \cap \tau_{jk}} d\nu) \mathfrak{F}(\chi_{\Omega \cap \tau_{j\tilde{k}}} d\nu)\|_p^{p^*} \Big)^{\frac{1}{p^*}},$$

where

$$p^* = \min\{p, p'\}, \tag{6-3}$$

with $1/p + 1/p' = 1$. The last step can be obtained by interpolation between the case $p = 2$, where one may apply Plancherel's theorem, and the cases $p = 1$ and $p = \infty$, which are simply treated by means of the triangle inequality (see Lemma 6.3 in [Tao and Vargas 2000a]). We claim

$$(\bar{m}+3)\Big(\frac{1}{p} - \frac{1}{2}\Big) < \frac{h+1}{2p}. \tag{6-4}$$

Case 1: $\bar{m} \leq 2m$. Then $\bar{m} \leq 3h$ and

$$(\bar{m}+3)\Big(\frac{1}{p} - \frac{1}{2}\Big) - \frac{h+1}{2p} \leq 3(h+1)\Big(\frac{1}{p} - \frac{1}{2}\Big) - \frac{h+1}{2p} = (h+1)\Big(\frac{5}{2p} - \frac{3}{2}\Big) < 0$$

according to our assumptions.

Case 2: $\bar{m} > 2m$. Here,

$$h+1 = \frac{\bar{m}m + \bar{m} + m}{\bar{m}+m} > \frac{\bar{m}m + 3m}{\bar{m}+m} = (\bar{m}+3)\frac{m}{\bar{m}+m},$$

and thus

$$(\bar m+3)\Big(\frac1p-\frac12\Big)-\frac{h+1}{2p}<(\bar m+3)\Big(\frac1p-\frac12-\frac{m}{\bar m+m}\frac{1}{2p}\Big)$$
$$=(\bar m+3)\Big(\frac{1}{2p}\Big(1+\frac{\bar m}{\bar m+m}\Big)-\frac12\Big)<0,$$

because of our assumption $2p>2p_0$.

In both cases, these estimates show that we may choose $q\ge 2$ such that

$$(\bar m+3)\Big(\frac1p-\frac12\Big)<\frac{1}{q'}<\frac{h+1}{2p} \tag{6-5}$$

(recall here (6-1), which allows to choose $q\ge 2$).

The first inequality allows us to apply Theorem 5.1 to the pair of hypersurfaces

$$S_{jk}=\{(\xi,\phi(\xi)):\xi\in\tau_{jk}\}\quad\text{and}\quad S_{j\tilde k}=\{(\xi,\phi(\xi)):\xi\in\tau_{j\tilde k}\},$$

with

$$\rho_i=2^{-j_i},\quad \varkappa_i\sim(k_i2^{-j_i})^{m_i-2}\sim(\tilde k_i2^{-j_i})^{m_i-2},\quad \varkappa_i\rho_i^2\sim k_i^{m_i-2}2^{-j_im_i}. \tag{6-6}$$

Without loss of generality, we may assume

$$k\in I:=\{k:k_1^{m_1-2}2^{-j_1m_1}\ge k_2^{m_2-2}2^{-j_2m_2}\}, \tag{6-7}$$

i.e., $\varkappa_1\rho_1^2\ge\varkappa_2\rho_2^2$. Thus, by Theorem 5.1,

$$\begin{aligned}\|R^*_{S_{jk},S_{j\tilde k}}\|_{L^q(S_{jk})\times L^q(S_{j\tilde k})\to L^p(\mathbb{R}^3)}&\\ &\lesssim(\rho_1\rho_2)^{\frac{2}{q'}-\frac1p}(\varkappa_1\rho_1^2\vee\varkappa_2\rho_2^2)^{-\frac1p}\Big(\frac{\varkappa_1\rho_1^2\vee\varkappa_2\rho_2^2}{\varkappa_1\rho_1^2\wedge\varkappa_2\rho_2^2}\Big)^{\frac2p-1+\varepsilon}\\ &=2^{-(j_1+j_2)(\frac{2}{q'}-\frac1p)}k_1^{-(m_1-2)\frac1p}2^{j_1m_1\frac1p}\Big(\frac{k_1^{m_1-2}}{k_2^{m_2-2}}2^{-(j_1m_1-j_2m_2)}\Big)^{\frac2p-1+\varepsilon}=A_j\cdot B_{k,j}^{\frac1p},\end{aligned}$$

if we define

$$A_j=2^{-(j_1+j_2)(\frac{2}{q'}-\frac1p)}2^{j_1m_1\frac1p}2^{-(j_1m_1-j_2m_2)\frac2p-1+\varepsilon},$$
$$B_{j,\tilde k}\sim B_{k,j}=k_1^{-(m_1-2)}\Big(\frac{k_1^{m_1-2}}{k_2^{m_2-2}}\Big)^{2-p+\varepsilon p}.$$

Since $|\{\tilde k:k\sim\tilde k\}|\lesssim 1$ for fixed k, we conclude that

$$\begin{aligned}\|\widehat{\chi_\Omega\,d\nu}\|_{2p}^2&\lesssim\sum_j A_j\Big(\sum_{k\sim\tilde k}\big(B_{k,j}^{\frac1p}|\Omega\cap\tau_{jk}|^{\frac1q}|\Omega\cap\tau_{j\tilde k}|^{\frac1q}\big)^{p^*}\Big)^{\frac{1}{p^*}}\\ &\lesssim\sum_j A_j\Big(\sum_k B_{k,j}^{\frac{p^*}{p}}|\Omega\cap\tau_{jk}|^{\frac{2p^*}{q}}\Big)^{\frac{1}{p^*}}.\end{aligned}$$

Therefore we are reduced to showing

$$\sum_j A_j \Big(\sum_k B_{k,j}^{\frac{p^*}{p}} |\Omega \cap \tau_{jk}|^{\frac{2p^*}{q}} \Big)^{\frac{1}{p^*}} \lesssim |\Omega|^{\frac{2}{s}}. \tag{6-8}$$

6A. ***Further reduction.*** We have the decomposition

$$\frac{2p^*}{q} = \frac{\alpha}{r^*} + \frac{1}{r^{*\prime}},$$

where $r* \in [1, \infty]$ will be determined later, and introduce $r = r^* p^*/p$. Applying Hölder's inequality to the summation in k, with Hölder exponent $r^* \geq 1$, we get

$$\begin{aligned}
\Big(\sum_{k\in I} B_{k,j}^{\frac{p^*}{p}} |\Omega \cap \tau_{jk}|^{\frac{2p^*}{q}} \Big)^{\frac{1}{p^*}} &\leq \Big(\sum_{k\in I} B_{k,j}^{\frac{p^* r^*}{p}} |\Omega \cap \tau_{jk}|^{\alpha} \Big)^{\frac{1}{p^* r^*}} \Big(\sum_{k\in I} |\Omega \cap \tau_{jk}| \Big)^{\frac{1}{p^* r^{*\prime}}} \\
&\leq \Big(\sum_{k\in I} B_{k,j}^{r} \Big)^{\frac{1}{pr}} \min\{|\Omega|, 2^{-j_1-j_2}\}^{\frac{\alpha}{p^* r^*}} |\Omega|^{\frac{1}{p^*}(1-\frac{1}{r^*})} \\
&= \Big(\sum_{k\in I} B_{k,j}^{r} \Big)^{\frac{1}{pr}} \min\{|\Omega|, 2^{-j_1-j_2}\}^{\frac{2}{q}-\frac{1}{p^*}+\frac{1}{pr}} |\Omega|^{\frac{1}{p^*}-\frac{1}{pr}}.
\end{aligned}$$

Moreover we have $|\Omega| \leq |Q| = 1$, as well as $1/s' \geq h + 1/(2p)$, i.e., $2 - (h+1)/p \geq 2/s$. Therefore $|\Omega|^{2-(h+1)/p} \leq |\Omega|^{2/s}$, and thus in order to prove (6-8), it suffices to show that

$$|\Omega|^{2-\frac{h+1}{p}-\frac{1}{p^*}+\frac{1}{pr}} \gtrsim \sum_j A_j \min\{|\Omega|, 2^{-j_1-j_2}\}^{\frac{2}{q}-\frac{1}{p^*}+\frac{1}{pr}} \Big(\sum_{k\in I} B_{k,j}^{r} \Big)^{\frac{1}{rp}},$$

i.e., that

$$\begin{aligned}
|\Omega|^{2-\frac{h+1}{p}-\frac{1}{p^*}+\frac{1}{pr}} \gtrsim \sum_j 2^{-(j_1 m_1 - j_2 m_2)\frac{2}{p}-1+\varepsilon} 2^{-(j_1+j_2)(\frac{2}{q'}-\frac{1}{p})} 2^{j_1 m_1 \frac{1}{p}} \min\{|\Omega|, 2^{-j_1-j_2}\}^{\frac{2}{q}-\frac{1}{p^*}+\frac{1}{pr}} \\
\times \Big(\sum_{k\in I} k_1^{(m_1-2)(1-p+\varepsilon p)r} k_2^{(m_2-2)(p-2-\varepsilon p)r} \Big)^{\frac{1}{pr}}.
\end{aligned}$$

We apply the change of variables $l = j_1 + j_2 \in \mathbb{N}$, $l' = j_1 m_1 - j_2 m_2 \in \mathbb{Z}$, such that

$$j_1 = \frac{m_2 l + l'}{m_1 + m_2}.$$

Then the exponent in j_1, j_2 becomes

$$\begin{aligned}
(j_1 m_1 - j_2 m_2)\Big(1 - \frac{2}{p} - \varepsilon\Big) + (j_1+j_2)\Big(\frac{1}{p} - \frac{2}{q'}\Big) + j_1 m_1 \frac{1}{p} &= l'\Big(1 - \frac{2}{p} - \varepsilon\Big) + l\Big(\frac{1}{p} - \frac{2}{q'}\Big) + \frac{m_1 m_2 l + m_1 l'}{m_1 + m_2} \frac{1}{p} \\
&= \frac{l'}{p}\Big(p - \varepsilon p - \frac{m_1 + 2m_2}{m_1 + m_2}\Big) + l\Big(\frac{h+1}{p} - \frac{2}{q'}\Big).
\end{aligned}$$

The summation over $k \in I_{l'} = \{k_1^{m_1-2} \geq k_2^{m_2-2} 2^{l'}\}$ is independent of l, and thus we have finally reduced the proof of (6-8) to proving the following two decoupled estimates:[4]

$$\sum_{l'=-\infty}^{\infty} 2^{\frac{l'}{p}(p-\varepsilon p-\frac{m_1+2m_2}{m_1+m_2})} \Bigl(\sum_{k\in I_{l'}} k_1^{(m_1-2)(1-p+\varepsilon p)r} k_2^{(m_2-2)(p-2-\varepsilon p)r} \Bigr)^{\frac{1}{pr}} < \infty \tag{6-9}$$

and

$$\sum_{l=0}^{\infty} 2^{l(\frac{h+1}{p}-\frac{2}{q'})} \min\{|\Omega|, 2^{-l}\}^{\frac{2}{q}-\frac{1}{p^*}+\frac{1}{pr}} \lesssim |\Omega|^{2-\frac{h+1}{p}-\frac{1}{p^*}+\frac{1}{pr}}. \tag{6-10}$$

6B. *The case $m > 2$.* In this case we have both $m_1 > 2$ and $m_2 > 2$.

6B1. *Summation in k.* We compare the sum over k in (6-9) with an integral. We claim

$$\int_{\substack{k_1,k_2\geq 1 \\ k_1^{m_1-2}\geq k_2^{m_2-2}2^{l'}}} k_1^{(m_1-2)(1-p+\varepsilon p)r} k_2^{(m_2-2)(p-\varepsilon p-2)r}\, dk \lesssim \begin{cases} 2^{|l'|(\frac{1}{m_1-2}+(1-p+\varepsilon p)r)}, & l' \geq 0, \\ |l'| 2^{|l'|(\frac{1}{m_2-2}+(p-\varepsilon p-2)r)_+}, & l' < 0, \end{cases} \tag{6-11}$$

provided that

$$a := \frac{1}{m_1-2} + (1-p+\varepsilon p)r < 0 \tag{6-12}$$

and

$$a+b = \frac{1}{m_1-2} + \frac{1}{m_2-2} - r < 0, \tag{6-13}$$

where

$$b := \frac{1}{m_2-2} + (p-\varepsilon p-2)r \in \mathbb{R}. \tag{6-14}$$

For the moment, we will simply assume these conditions hold true. We shall collect several further conditions on the exponent r and verify at the end of this section that we can indeed find an r such that all these conditions are satisfied.

By means of the coordinate transformation $s = k_1^{m_1-2}$, $t = k_2^{m_2-2}$ (i.e., $dk \sim s^{\frac{1}{m_1-2}-1} t^{\frac{1}{m_2-2}-1} d(s,t)$), (6-11) simplifies to showing

$$J(a,b) = \iint_{\substack{s,t\geq 1 \\ s\geq t2^{l'}}} s^a t^b \frac{ds}{s}\frac{dt}{t} \lesssim \begin{cases} 2^{|l'|a}, & l' \geq 0, \\ 2^{|l'|b_+}, & l' < 0, \end{cases} \tag{6-15}$$

provided that $a < 0$, $a+b < 0$. Here we have set $b_+ = b \vee 0$. Changing t' to $s2^{-l'}/t$, the set of integration for the t-variable $\{t : t \geq 1, s2^{-l'}/t \geq 1\}$ transforms into $\{t' : s2^{-l'}/t' \geq 1, t' \geq 1\}$, and thus, since we

[4] Technically, we only have to sum over the smaller set $l' \in m_1\mathbb{N} - m_2\mathbb{N}$.

assume $a+b<0$,

$$\begin{aligned} J(a,b) &= \iint\limits_{\substack{s,t'\geq 1\\ s\geq t'2^{l'}}} s^a\Big(\frac{s}{t'}2^{-l'}\Big)^b \frac{ds}{s}\frac{dt'}{t'} = 2^{-l'b}\int_{t'=1}^{\infty} t'^{-b}\int_{s=1\vee t'2^{l'}}^{\infty} s^{a+b}\frac{ds}{s}\frac{dt'}{t'} \\ &= 2^{-l'b}\int_1^\infty (1\vee t'2^{l'})^{a+b}t'^{-b}\frac{dt'}{t'}. \end{aligned}$$

If $l'\geq 0$, then clearly $1\vee t'2^{l'}=t'2^{l'}$, and since $a<0$, we get

$$J(a,b)=2^{l'a}\int_1^\infty t'^a\frac{dt'}{t'}\sim 2^{l'a}.$$

And, if $l'<0$, then we can split it into

$$\begin{aligned} J(a,b) &= 2^{-l'b}\int_1^{2^{|l'|}} t'^{-b}\frac{dt'}{t'} + 2^{l'a}\int_{2^{|l'|}}^{\infty} t'^a\frac{dt'}{t'} = 2^{-l'b}\frac{1-2^{l'b}}{b}+\int_1^\infty u^a\frac{du}{u} \\ &\lesssim |l'|(2^{|l'|b}+1)\sim |l'|2^{|l'|b_+} \end{aligned}$$

(notice that the additional factor $|l'|$ arises in fact only when $b=0$). This proves (6-15).

6B2. *Summation in l'.* In order to apply (6-11) to (6-9), we split the sum in (6-9) into summation over $l'\geq 0$ and summation over $l'<0$. In the first case $l'\geq 0$, we obtain

$$\begin{aligned} &\sum_{l'\geq 0} 2^{\frac{l'}{p}(p-\varepsilon p-\frac{m_1+2m_2}{m_1+m_2})}\Big(\sum_{k\in I} k_1^{(m_1-2)(1-p+\varepsilon p)r}k_2^{(m_2-2)(p-\varepsilon p-2)r}\Big)^{\frac{1}{rp}} \\ &\qquad\lesssim \sum_{l'\geq 0} 2^{\frac{l'}{p}(p-\varepsilon p-1-\frac{m_2}{m_1+m_2})}2^{l'(\frac{1}{pr}\frac{1}{m_1-2}+\frac{1-p+\varepsilon p}{p})} \\ &\qquad= \sum_{l'\geq 0} 2^{\frac{l'}{p}(\frac{1}{r}\frac{1}{m_1-2}-\frac{m_2}{m_1+m_2})}. \end{aligned}\tag{6-16}$$

The sum is finite provided

$$\frac{1}{r}<\frac{m_2(m_1-2)}{m_1+m_2},\tag{6-17}$$

which gives yet another condition for our collection.

In the second case $l'<0$, we have

$$\begin{aligned} &\sum_{l'<0} 2^{\frac{l'}{p}(p-\varepsilon p-\frac{m_1+2m_2}{m_1+m_2})}\Big(\sum_{k\in I} k_1^{(m_1-2)(1-p+\varepsilon p)r}k_2^{(m_2-2)(p-\varepsilon p-2)r}\Big)^{\frac{1}{rp}} \\ &\qquad\lesssim \sum_{l'<0} 2^{\frac{l'}{p}(p-\varepsilon p-\frac{m_1+2m_2}{m_1+m_2})}|l'|2^{|l'|(\frac{1}{pr}\frac{1}{m_2-2}+\frac{p-\varepsilon p-2}{p})_+} \\ &\qquad= \sum_{l'<0} |l'|2^{\frac{l'}{p}(p-\varepsilon p-\frac{m_1+2m_2}{m_1+m_2}-(\frac{1}{r}\frac{1}{m_2-2}+p-\varepsilon p-2)_+)}. \end{aligned}\tag{6-18}$$

Notice that for sufficiently small $\varepsilon > 0$ we have $p-\varepsilon p > p_0 = 1+\bar{m}/(\bar{m}+m) \geq 1+m_2/(m_1+m_2)$, and therefore

$$p-\varepsilon p-\frac{m_1+2m_2}{m_1+m_2} > 0. \tag{6-19}$$

Thus the last sum in (6-18) converges in the case where

$$\frac{b}{r} = \frac{1}{r}\frac{1}{m_2-2}+p-\varepsilon p-2 \leq 0.$$

This shows that we only need to discuss the case where $b > 0$, in which we need that

$$0 < p-\varepsilon p-\frac{m_1+2m_2}{m_1+m_2}-\frac{1}{r}\frac{1}{m_2-2}-p+\varepsilon p+2 = \frac{m_1}{m_1+m_2}-\frac{1}{r}\frac{1}{m_2-2},$$

which is equivalent to

$$\frac{1}{r} < \frac{m_1(m_2-2)}{m_1+m_2}. \tag{6-20}$$

Notice that this is of the same form as (6-17), only with the roles of m_1 and m_2 interchanged.

6B3. *Summation in l.* Recall that we want to show estimate (6-10), i.e.,

$$\sum_{l=0}^{\infty} 2^{l\left(\frac{h+1}{p}-\frac{2}{q'}\right)} \min\{2^{-l}, |\Omega|\}^{\frac{2}{q}-\frac{1}{p^*}+\frac{1}{pr}} \lesssim |\Omega|^{2-\frac{h+1}{p}-\frac{1}{p^*}+\frac{1}{pr}}.$$

We claim it is sufficient to show that for $\mu > 0$ and $\nu-\mu > 0$,

$$\int_0^\infty e^{x\mu} \min\{e^{-x}, A\}^\nu \, dx \lesssim A^{\nu-\mu}. \tag{6-21}$$

Indeed, given (6-21), we apply it with $A = |\Omega|$, $\mu = (h+1)/p-2/q'$ and $\nu = 2/q-1/p^*+1/(pr)$. Due to the choice of q in (6-5), we have $\mu > 0$. Moreover we want

$$0 < \nu-\mu = 2-\frac{1}{p^*}+\frac{1}{pr}-\frac{h+1}{p} = \frac{1}{p}\left(2p-h-1-\frac{p}{p*}+\frac{1}{r}\right).$$

Notice that if $p \leq 2$, then $p/p* = 1$, but if $p > 2$, then $p/p* = p(1-1/p) = p-1$. Thus $p/p* = 1+(p-2)_+$ for all $1 \leq p \leq \infty$, i.e., the condition which is required here is

$$\frac{1}{r} > h+2-2p+(p-2)_+. \tag{6-22}$$

In order to verify (6-21), observe that

$$\int_0^\infty e^{x\mu} \min\{e^{-x}, A\}^\nu \, dx = \int_{\ln A}^\infty e^{y\mu} A^{-\mu} \min\{e^{-y}A, A\}^\nu \, dy = A^{\nu-\mu} \int_{\ln A}^\infty e^{y\mu} \min\{e^{-y}, 1\}^\nu \, dy.$$

The last integral can be estimated by

$$\int_{\ln A}^\infty e^{y\mu} \min\{e^{-y}, 1\}^\nu \, dy \leq \int_{-\infty}^0 e^{y\mu} \, dy + \int_0^\infty e^{-y(\nu-\mu)} \, dy,$$

which is convergent since $\mu > 0$ and $\nu-\mu > 0$.

It still remains to be checked whether there exists some $1 \le r^* < \infty$ (for $m > 2$) for which r satisfies the conditions (6-12), (6-13), (6-17), (6-20) and (6-22).

This task will be accomplished in Lemma 6.2. First, we discuss the situation where $m = 2$.

6C. *The case m = 2.* We will just give some hints for how to modify the previous proof for this situation. In this case, $r = \infty$ turns out to be an appropriate choice, and the inequality that we need to start the argument with here reads

$$\Big(\sum_{k\in I} B_{k,j}^{\frac{p^*}{p}} |\Omega\cap\tau_{jk}|^{\frac{2p^*}{q}}\Big)^{\frac{1}{p^*}} \le (\sup_{k\in I} B_{k,j})^{\frac{1}{p}} \min\{|\Omega|, 2^{-j_1-j_2}\}^{\frac{2}{q}-\frac{1}{p^*}} |\Omega|^{\frac{1}{p^*}}.$$

This is very easy to prove, provided $2p^*/q \ge 1$ (notice that this condition corresponds to our previous decomposition of $2p^*/q$ when $r = \infty$). To see that indeed $2p^*/q \ge 1$, recall from (6-5) that $1/q' < (h+1)/(2p)$. Then, it is enough to check that $2p^*(1-(h+1)/(2p)) > 1$, i.e., $h+1-2p+p/p^* < 0$. The last condition is equivalent to $h+2-2p+(p-2)_+ < 0$. However, this is what we shall indeed verify in the proof of Lemma 6.2 (compare to estimate (6-30) when $m = 2$).

Observe next that we may rewrite the integral in (6-11) in terms of the L^r-norm as

$$\big\|(k_1^{(m_1-2)(1-p+\varepsilon p)} k_2^{(m_2-2)(p-\varepsilon p-2)})_{k\in I_{l'}}\big\|_r \lesssim \begin{cases} 2^{|l'|(\frac{1}{r}\frac{1}{m_1-2}+1-p+\varepsilon p)}, & l' \ge 0,\\ 2^{|l'|(\frac{1}{r}\frac{1}{m_2-2}+p-\varepsilon p-2)_+}, & l' < 0,\end{cases}$$

provided the conditions (6-12) and (6-13) hold true, i.e., that

$$\frac{1}{m_1-2} + (1-p+\varepsilon p)r < 0 \quad\text{and}\quad \frac{1}{m_1-2} + \frac{1}{m_2-2} - r < 0.$$

This gives rise to the conjecture that (for $r = \infty$) we should have

$$\sup_{k\in I} k_1^{(m_1-2)(1-p+\varepsilon p)} k_2^{(m_2-2)(p-\varepsilon p-2)} \le \sup_{s\ge t2^{l'}} s^{1-p+\varepsilon p} t^{p-\varepsilon p-2} \lesssim \begin{cases} 2^{|l'|(1-p+\varepsilon p)}, & l' \ge 0,\\ 2^{|l'|(p-\varepsilon p-2)_+}, & l' < 0,\end{cases} \tag{6-23}$$

which would suffice in this case. But notice that the conditions (6-12) and (6-13) are formally fulfilled for $r = \infty$, and it is then easy to check that (6-23) indeed holds true, even in the case $m = 2$.

6C1. *Summation in l'.* The summation in l' becomes simpler here. We split again into the sums over $l' \ge 0$ and $l' < 0$, and obtain for the first half of the sum in (6-16)

$$\sum_{l'\ge 0} 2^{\frac{l'}{p}(p-\varepsilon p-\frac{m_1+2m_2}{m_1+m_2})} 2^{l'\frac{1-p+\varepsilon p}{p}} = \sum_{l'\ge 0} 2^{-\frac{l'}{p}\frac{m_2}{m_1+m_2}} < \infty.$$

The second part of the sum becomes (compare to (6-18))

$$\sum_{l'<0} 2^{\frac{l'}{p}(p-\varepsilon p-\frac{m_1+2m_2}{m_1+m_2}-(p-\varepsilon p-2)_+)}.$$

We already know from (6-19) that $p-\varepsilon p-(m_1+2m_2)/(m_1+m_2)>0$. Thus the sum converges if $p-\varepsilon p\le 2$. For $p-\varepsilon p>2$, notice that

$$p-\varepsilon p-\frac{m_1+2m_2}{m_1+m_2}-(p-\varepsilon p-2)_+=\frac{m_1}{m_1+m_2}>0,$$

and thus the sum is finite.

6C2. *Summation in l.* It remains to show that

$$\sum_{l=0}^{\infty}2^{l\left(\frac{h+1}{p}-\frac{1}{q'}\right)}\min\{|\Omega|,2^{-l}\}^{\frac{1}{q}-\frac{1}{p^*}}\lesssim|\Omega|^{2-\frac{h+1}{p}-\frac{1}{p^*}},$$

which is the special case $r=\infty$ of (6-10). We saw that this holds true provided (6-22) is valid, i.e., if $1/r>h+2-2p+(p-2)_+$.

However, if $m=2$, then

$$2p>p_0=\frac{2\bar{m}}{\bar{m}+2}+2=h+2.$$

Thus for the case $p\le 2$ we have $h+2-2p+(p-2)_+=h+2-2p<0$. For the case $p>2$ notice that

$$h+2-2p+(p-2)_+=h-p=\frac{2\bar{m}}{\bar{m}+2}-p<2-p<0. \qquad \square$$

6D. ***Final considerations.*** We finally verify that there is indeed always some r for which all necessary conditions (6-12), (6-13), (6-17), (6-20) and (6-22) are satisfied in the case $m>2$. Recall that

$$\frac{2p^*}{q}=\frac{\alpha}{r^*}+\frac{1}{r^{*\prime}},$$

and notice that it will suffice to verify the following equivalent inequalities:

$$\frac{1}{r}<(m_1-2)(p-1), \tag{6-24}$$

$$\frac{1}{r}<\frac{(m_1-2)(m_2-2)}{m_1+m_2-4}, \tag{6-25}$$

$$\frac{1}{r}<\frac{m_2(m_1-2)}{m_1+m_2}, \tag{6-26}$$

$$\frac{1}{r}<\frac{m_1(m_2-2)}{m_1+m_2}, \tag{6-27}$$

$$\frac{1}{r}>h+2-2p+(p-2)_+. \tag{6-28}$$

Lemma 6.2. *Assume $m>2$ and $2p>\max\{2p_0,h+1\}$, where we recall that $p_0=1+\bar{m}/(\bar{m}+m)$. Define*

$$J=]0,1+(p-2)_+]\cap\left]h+2-2p+(p-2)_+,\frac{\bar{m}(m-2)}{\bar{m}+m}\right[.$$

Then $J \neq \varnothing$, and for every $1/r \in J$ we have

$$r^* = \frac{rp}{p^*} \geq 1, \quad \textit{and} \quad \alpha = r^*\left(\frac{2p^*}{q} - \frac{1}{r^{*\prime}}\right) > 0, \tag{6-29}$$

and moreover the inequalities (6-24), (6-25), (6-26), (6-27) and (6-28) are valid.

Proof. First of all, we will show that $J \neq \varnothing$. We need to see that

$$h + 2 - 2p + (p-2)_+ < \frac{\bar{m}(m-2)}{\bar{m}+m} = h - \frac{2\bar{m}}{\bar{m}+m}, \tag{6-30}$$

i.e., that $2p_0 = 2 + 2\bar{m}/(\bar{m}+m) < 2p - (p-2)_+$. For the case $p \leq 2$, this holds true since $2p > 2p_0$. If $p > 2$, observe that

$$h + 2 - 2p + (p-2)_+ = h - p < h - 2 \leq h - \frac{2\bar{m}}{\bar{m}+m} = \frac{\bar{m}(m-2)}{\bar{m}+m}.$$

Thus both intervals used for the definition of J are not empty, but we still have to check that their intersection is not trivial. Since we assume $2p > h+1$, we have

$$h + 2 - 2p + (p-2)_+ < 1 + (p-2)_+.$$

And, for $m > 2$, we also have $0 < \bar{m}(m-2)/(\bar{m}+m)$, which shows that $J \neq \varnothing$.

Next, if $1/r \in J$, then in particular $1/r \leq 1 + (p-2)_+ = p/p^*$, and thus $r^* = rp/p^* \geq 1$. To prove (6-29), observe that due to our choice of q in (6-5) we have $1/q > 1 - (h+1)/(2p)$, and thus it suffices to prove that

$$2p^*\left(1 - \frac{h+1}{2p}\right) > \frac{1}{r^{*\prime}} = 1 - \frac{p^*}{rp}.$$

This inequality is equivalent to

$$\frac{1}{r} > \frac{p}{p*} + h + 1 - 2p = h + 2 - 2p + (p-2)_+,$$

and thus is satisfied.

Considering the remaining conditions listed before the statement of the lemma, notice that (6-28) is immediate by the definition of J. Furthermore we have

$$\frac{1}{r} < \frac{\bar{m}(m-2)}{\bar{m}+m} = \frac{m_1 m_2 - 2\bar{m}}{m_1 + m_2} \leq \frac{m_1 m_2 - 2m_i}{m_1 + m_2}$$

for both $i = 1, 2$, which gives (6-26) and (6-27). To obtain (6-24), we estimate

$$\frac{1}{r} < \frac{\bar{m}(m-2)}{\bar{m}+m} \leq \frac{\bar{m}(m_1-2)}{\bar{m}+m} = (p_0 - 1)(m_1 - 2) < (p - \varepsilon p - 1)(m_1 - 2).$$

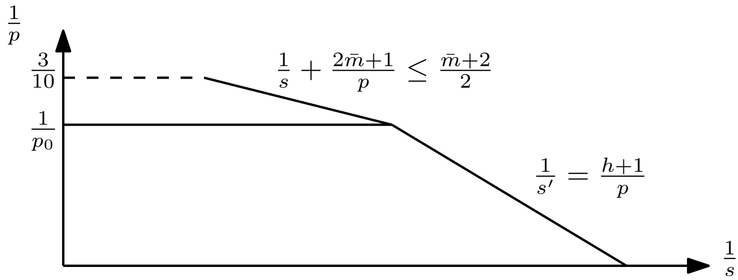

Figure 11. Range of p and q in Theorem 1.2.

Finally, observe that we have the following equivalences:

$$\begin{aligned}\frac{\bar{m}(m-2)}{\bar{m}+m} \le \frac{(m_1-2)(m_2-2)}{m_1+m_2-4} \quad &\Longleftrightarrow \quad \frac{\bar{m}}{\bar{m}+m} \le \frac{\bar{m}-2}{\bar{m}+m-4} \\ &\Longleftrightarrow \quad \bar{m}(\bar{m}+m)-4\bar{m} \le \bar{m}(\bar{m}+m)-2(\bar{m}+m) \\ &\Longleftrightarrow \quad m \le \bar{m}.\end{aligned}$$

Hence (6-25) holds true as well. □

6E. *Finishing the proof.* We can now conclude the proof of our main result, Theorem 1.2:

Corollary 6.3. *Let $2p > \max\{\frac{10}{3}, h+1\}$, $1/s' \ge (h+1)/(2p)$ and $1/s + (2\bar{m}+1)/(2p) < (\bar{m}+2)/2$. Then R^* is bounded from $L^{s,t}(\Gamma)$ to $L^{2p,t}(\mathbb{R}^3)$ for every $1 \le t \le \infty$. If moreover $s \le 2p$ or $1/s' > (h+1)/(2p)$, then R^* is bounded from $L^s(\Gamma)$ to $L^{2p}(\mathbb{R}^3)$.*

Proof. The crucial observation is that the intersection point of the two lines

$$\frac{1}{s'} = \frac{h+1}{2p} \quad \text{and} \quad \frac{2\bar{m}+1}{2p} + \frac{1}{s} = \frac{\bar{m}+2}{2}$$

has the p-coordinate $p = \hat{p}_0 = 1 + \bar{m}/(\bar{m}+m)$ (comparing with (1-6), notice that $\hat{p}_0 = p_0/2$). So, what remains is to establish estimates for R^* for the missing points $(1/s, 1/p)$ lying within the sectorial region defined by the conditions $(2\bar{m}+1)/(2p) + 1/s < (\bar{m}+2)/2$ and $1/p > 1/\hat{p}_0$ (the region above the horizontal threshold line $1/p = 1/p_0$ from Theorem 6.1 (see Figure 11).

Notice also that if $m \ge \bar{m}/2$, then $\hat{p}_0 \le \frac{5}{3}$, i.e., $p_0 \le \frac{10}{3}$, and hence the condition $1/s + (2\bar{m}+1)/(2p) < (\bar{m}+2)/2$ becomes redundant.

Moreover, the condition $1/s + (2\bar{m}+1)/(2p) < (\bar{m}+2)/2$ does only depend on $\bar{m}$, and not on m, whereas the condition $1/s' = (h+1)/(2p)$ depends on the height h, i.e., on both m_1 and m_2.

This leads to the following *heuristic idea*: Assume we fix $\bar{m}$ and consider a family of surfaces $\Gamma_{\bar{m},m^\sharp}$ corresponding to exponents $m_1 = \bar{m}$ and $m_2 = m^\sharp$ for different exponents $m < m^\sharp$ such that $\Gamma_{\bar{m},m} = \Gamma$ (think for instance of the graph of $x_1^{m^\sharp} + x_2^{\bar{m}}$ for $m^\sharp \ne m$). Let us then compare the restriction estimates that we have so far for the surface $\Gamma = \Gamma_{\bar{m},m}$ with the ones for the hypersurfaces $\Gamma_{\bar{m},m^\sharp}$. Denote by h and $h^\sharp$ the heights of these hypersurfaces. Then $h < h^\sharp$, so that the critical line $1/s' = (h^\sharp+1)/(2p)$ lies below the critical line $1/s' = (h+1)/(2p)$ for Γ, but its intersection point with the corresponding

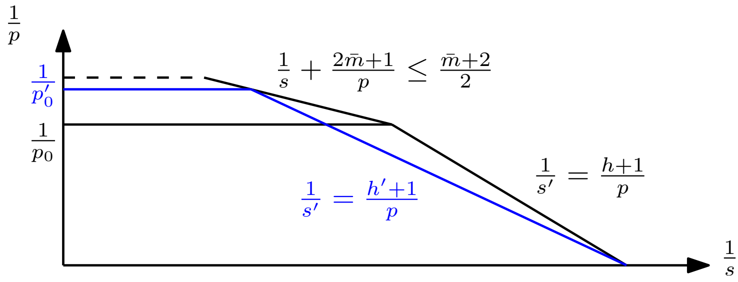

Figure 12. Variation of the minimal exponent.

horizontal threshold line $1/p = 1/p_0^\sharp$, where $p_0^\sharp = 2 + 2\bar{m}/(\bar{m}+m^\sharp) < p_0$, lies above the previous intersection point (see Figure 12).

This suggests that for our theorem, it should essentially be sufficient to "increase" $m^\sharp$ until $\bar{m} = 2m^\sharp$, because then we would have $p_0^\sharp = 2 + 2\bar{m}/(\bar{m}+m^\sharp) = \frac{10}{3}$. In other words, for any point $(1/s, 1/(2p))$ fulfilling the assumptions of Theorem 1.2, we would find an $m^\sharp \in \,]m, \bar{m}/2]$ such that $(1/s, 1/(2p))$ satisfies the requirements of Theorem 6.1 corresponding to the surface $\Gamma_{\bar{m},m'}$. Thus we would obtain the restriction estimate for the surface $\Gamma_{\bar{m},m^\sharp}$ at the point $(1/s, 1/(2p))$. However, since this surface has "less curvature" than $\Gamma_{\bar{m},m}$, as $m^\sharp > m$, the corresponding restriction inequality should hold true for $\Gamma_{\bar{m},m} = \Gamma$ as well.

To turn these heuristics into a solid proof, we just need to check that the bound for the bilinear operator that we obtained in Theorem 5.1 is increasing in m. Recall that for subsurfaces $S, \tilde{S} \subset S_{\bar{m},m}$ under the assumptions of the aforementioned theorem we obtained the bound

$$\|R^*_{S,\tilde{S}}\|_{L^s(S)\times L^s(\tilde{S})\to L^p(\mathbb{R}^3)} \lesssim C_{\bar{m},m} := (\rho_1\rho_2)^{\frac{2}{s'}-\frac{1}{p}}(\varkappa_1\rho_1^2 \vee \varkappa_2\rho_2^2)^{\frac{1}{p}-1+\varepsilon}(\varkappa_1\rho_1^2 \wedge \varkappa_2\rho_2^2)^{1-\frac{2}{p}-\varepsilon},$$

which we apply to $\rho_i = 2^{-j_i}$ and $\varkappa_i = (k_i 2^{-j_i})^{m_i-2}$ (see (6-6)). If we denote by $\rho_i^\sharp, \varkappa_i^\sharp$ the corresponding quantities associated to the exponents $\bar{m}$ and $m^\sharp$, then clearly $\rho_i^\sharp = \rho_i$ and $\varkappa_i^\sharp \leq \varkappa_i$. Since we seek to extend the range of validity of Theorem 6.1, we may assume that $2p \leq p_0 < 4$, and moreover that $2p \geq p_0^\sharp > 2$. Then we have $1/p - 1 + \varepsilon < 0$ and $1 - 2/p - \varepsilon < 0$ for sufficient small $\varepsilon > 0$, and hence

$$C_{\bar{m},m} \leq (\rho_1^\sharp\rho_2^\sharp)^{\frac{2}{s'}-\frac{1}{p}}\big(\varkappa_1^\sharp(\rho_1^\sharp)^2 \vee \varkappa_2^\sharp(\rho_2^\sharp)^2\big)^{\frac{1}{p}-1+\varepsilon}\big(\varkappa_1^\sharp(\rho_1^\sharp)^2 \wedge \varkappa_2^\sharp(\rho_2^\sharp)^2\big)^{1-\frac{2}{p}-\varepsilon} = C_{\bar{m},m^\sharp}.$$

Proceeding with the latter estimate from here on as before in our proof of Theorem 6.1, but working now with $m^\sharp$ in place of m, we arrive at the statement of Corollary 6.3. ☐

Appendix

A1. ***A short argument to improve [Ferreyra and Urciuolo 2009] to the critical line.*** We consider the set $A_0 = \{x \in \mathbb{R}^2 : \frac{1}{2} < |x| \leq 1\}$ and define $H = 2\bar{m}/(2+\bar{m})$. Note that $H < h$. Ferreyra and Urciuolo proved that for every p for which $p > 4$ and $1/s' > (H+1)/p$, there is a constant $C_{p,s} > 0$ such that, for every function f_0 with $\operatorname{supp} f_0 \subset A$, we have

$$\|R^*_{\mathbb{R}^2} f_0\|_p \le C_{p,s}\|f_0\|_s.$$

Rescaling this, we obtain

$$\|R^*_{\mathbb{R}^2} f_j\|_p \le C_{p,s} 2^{\frac{j}{h}(-\frac{1}{s'}+\frac{h+1}{p})}\|f_j\|_s \tag{A-1}$$

for every function f_j such that

$$\operatorname{supp} f_j \subset \big\{(x_1,x_2) : 2^{-\frac{j+1}{m_1}} \le x_1 \le 2^{-\frac{j}{m_1}},\ 2^{-\frac{j+1}{m_2}} \le x_2 \le 2^{-\frac{j}{m_2}}\big\}$$

and the same range of p, s.

Given a function f supported in the unit ball of $\mathbb{R}^2$, we decompose $f = \sum_{j=0}^{\infty} f_j$, where the functions f_j have supports as above. Then,

$$\big|\{x : |R^*_{\mathbb{R}^2} f(x)| > \lambda\}\big| \le \left|\left\{x : \left|\sum_{j=J}^{\infty} R^*_{\mathbb{R}^2} f_j(x)\right| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x : \left|\sum_{1}^{j=J} R^*_{\mathbb{R}^2} f_j(x)\right| > \frac{\lambda}{2}\right\}\right|,$$

for some J to be chosen appropriately. Using Chebyshev's inequality, the last expression can be bounded above by

$$\left(\frac{2}{\lambda}\right)^{p_1}\left\|\sum_{j=J}^{\infty} R^*_{\mathbb{R}^2} f_j\right\|^{p_1}_{L^{p_1}} + \left(\frac{2}{\lambda}\right)^{p_2}\left\|\sum_{j=1}^{J} R^*_{\mathbb{R}^2} f_j\right\|^{p_2}_{L^{p_2}}.$$

Let us choose exponents $p_1 > p > p_2$ such that $1/s' = (h+1)/p$ and $(h+1)/p_2 > 1/s' > (h+1)/p_1 > (H+1)/p$. We use the triangle inequality and (A-1) and sum the resulting geometric series, obtaining the inequality

$$\big|\{x : |R^*_{\mathbb{R}^2} f(x)| > \lambda\}\big| \lesssim \left(\frac{2}{\lambda}\right)^{p_1} 2^{\frac{J}{h}(-\frac{p_1}{s'}+h+1)}\|f\|^{p_1}_{L^s} + \left(\frac{2}{\lambda}\right)^{p_2} 2^{\frac{J}{h}(-\frac{p_2}{s'}+h+1)}\|f\|^{p_2}_{L^s}.$$

By choosing J such that $2^J = \big(\|f\|_{L^s}/\lambda\big)^{hs'}$, we then arrive at the weak-type estimate

$$\big|\{x\ :\ |R^*_{\mathbb{R}^2} f(x)| > \lambda\}\big| \lesssim \left(\frac{\|f\|_{L^s}}{\lambda}\right)^{(h+1)s'} = \left(\frac{\|f\|_{L^s}}{\lambda}\right)^{p}.$$

From this, by interpolation with the trivial bound $\|R^*_{\mathbb{R}^2}\|_{L^1\to L^\infty} \le 1$, we obtain the desired strong-type estimate.

A2. ***Faà di Bruno's theorem and completion of the proof of Lemma 2.4.*** The formula of Faà di Bruno is a chain rule for higher-order derivatives of the composition of two functions. This is well known for functions in one real variable. However, we need a version for several variables.

Lemma A.1 (formula of Faà di Bruno). *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, and let $g = (g^1, \dots, g^m) \in C^\infty(U, V)$ and $f \in C^\infty(V, \mathbb{R}^l)$. For $\alpha \in \mathbb{N}^n$, we put $A_\alpha = \{\gamma \in \mathbb{N}^n : 1 \le |\gamma| \le |\alpha|\}$. Then $f \circ g$ is smooth, and for* *every $\alpha \in \mathbb{N}^n$ we have*

$$\partial^\alpha (f \circ g) = \alpha! \sum_{1 \le |\beta| \le |\alpha|} (\partial^\beta f) \circ g \sum_k \prod_{j=1}^m \prod_{\gamma \in A_\alpha} \left(\frac{\partial^\gamma g^j}{\gamma!} \right)^{k_\gamma^j},$$

where the sum in k is over all mappings $k : \{1, \dots, m\} \times A_\alpha \to \mathbb{N}$, $(j, \gamma) \mapsto k_\gamma^j$, such that

$$\sum_{\gamma \in A_\alpha} k_\gamma^j = \beta_j \tag{A-1}$$

for all $j = 1, \dots, m$ and

$$\sum_{j=1}^m \sum_{\gamma \in A_\alpha} k_\gamma^j \gamma = \alpha. \tag{A-2}$$

Proof. The elegant short proof in [Spindler 2005] for the one-dimensional case can easily be adapted to the higher-dimensional situation. □

We now come back to the proof of Lemma 2.4 and establish the still-missing estimates for the derivatives of the function ϕ_2 (given explicitly by (2-12)). Notice that these estimates cannot simply be obtained by means of a scaling argument, since the first-order derivatives are assumed to exhibit a different behavior than the higher-order derivatives.

We shall not really make use of formula (2-12), but rather proceed as follows: denoting by $e_1, \dots, e_{d+1}$ the canonical basis of $\mathbb{R}^{d+1}$, after applying a suitable orthogonal transformation to $\mathbb{R}^{d+1}$ we may and shall assume $n_1 = (0, \dots, 0, 1) = e_{d+1}$, and $E_1 = e_1, \dots, E_{d-1} = e_{d-1}$ and $e_d = h_1$ (recall here from the first part of the proof of Lemma 2.4 that $E_1, \dots, E_{d-1}$ is an orthonormal basis of $K = H_1 \cap H_2$). Then we may regard U_1 as a subset of $\mathbb{R}^d$, and we consider the function

$$H(\eta, \tau) = \tau - \phi_1(\eta), \quad \eta \in U_1,\ \tau \in \mathbb{R},$$

whose set of zeros agrees exactly with S. Observe first that the derivatives of H satisfy almost the same kind of estimates as ϕ_1:

$$\|H'\|_\infty \le \sqrt{A^2 + 1}, \qquad \|H^{(l)}\|_\infty \le A_l B r^l \quad \text{for every } l \ge 2. \tag{A-3}$$

Let $\psi(\xi) = \xi + \phi_2(\xi) n_2$, $\xi \in U_2$, be the parametrization of S induced by ϕ_2. Moreover, we introduce coordinates on U_2 by writing $\xi = \xi_1 E_1 + \cdots + \xi_{d-1} E_{d-1} + \xi_d h_2$. Then obviously

$$H(\psi(\xi)) = 0 \quad \text{for all } \xi \in U_2. \tag{A-4}$$

Furthermore,

$$\frac{\partial \psi}{\partial \xi_j} = E_j + \frac{\partial \phi_2}{\partial \xi_j} n_2, \quad j = 1, \dots, d-1, \qquad \frac{\partial \psi}{\partial \xi_d} = h_2 + \frac{\partial \phi_2}{\partial \xi_d} n_2, \tag{A-5}$$

and

$$\partial_\xi^\alpha \psi = \partial_\xi^\alpha \phi_2 \, n_2 \quad \text{for all } \alpha \in \mathbb{N}^d,\ |\alpha| \ge 2. \tag{A-6}$$

From (A-4) and (A-5) we obtain that for $j = 1, \dots, d$,

$$\frac{\partial \phi_2}{\partial \xi_j}(\xi) = -\frac{\langle (\nabla H)(\psi(\xi)), \tilde{e}_j \rangle}{\langle (\nabla H)(\psi(\xi)), n_2 \rangle}, \tag{A-7}$$

if we put $\tilde{e}_j = E_j = e_j$, if $j = 1, \dots, d-1$ and $\tilde{e}_d = h_2$. Notice also that our transversality condition $|\langle n_2, N(v)\rangle| \geq a > 0$ for all $v \in S$ implies $|\langle (\nabla H)(\psi(\xi)), n_2\rangle| \geq a$. Thus (A-7) implies

$$\left| \frac{\partial \phi_2}{\partial \xi_j}(\xi) \right| \leq \frac{A+1}{a}. \tag{A-8}$$

It remains to show that

$$\|\partial^\alpha \phi_2\|_\infty = \|\partial^\alpha \psi\|_\infty \leq \widetilde{A}_l B r^{|\alpha|} \tag{A-9}$$

for every $|\alpha| \geq 2$, where we have used the abbreviation $\partial = \partial_\xi$. By induction, we may assume that for every $\gamma \in \mathbb{N}^d$ with $2 \leq |\gamma| < |\alpha|$ inequality (A-9) holds true.[5] Applying the partial derivative of order α to (A-4) yields

$$\partial^\alpha (H \circ \psi) = 0.$$

We apply the formula of Faà di Bruno (Lemma A.1). First, we discuss the summands in Faà di Bruno's formula with $|\beta| = 1$, say $\beta = e_{j_0}$ for some $j_0 = 1, \dots, m$. How many k's are there for which $\sum_{\gamma \in A_\alpha} k_\gamma^j = \beta_j = \delta_{j j_0}$ and $\sum_{j=1}^m \sum_{\gamma \in A_\alpha} k_\gamma^j \gamma = \alpha$? By the first condition, there exists a γ_0 such that $k_{\gamma_0}^{j_0} = 1$ and $k_\gamma^j = 0$ for $j \neq j_0$ or $\gamma \neq \gamma_0$. But then the second condition implies $\gamma_0 = \alpha$. Thus we obtain

$$\begin{aligned}\sum_{|\beta|=1} (\partial^\beta H) \circ \psi \sum_k \prod_{j=1}^m \prod_{\gamma \in A_\alpha} \left(\frac{\partial^\gamma \psi^j}{\gamma!} \right)^{k_\gamma^j} &= \sum_{j_0=1}^m (\partial_{j_0} H) \circ \psi \left(\frac{\partial^\alpha \psi^{j_0}}{\alpha!} \right)^{k_\alpha^{j_0}} \\ &= \frac{1}{\alpha!} \langle (\nabla H) \circ \psi, \partial^\alpha \psi \rangle = \frac{\partial^\alpha \phi_2}{\alpha!} \langle (\nabla H) \circ \psi, n_2 \rangle,\end{aligned}$$

where we have used (A-6) once more. This implies

$$|\partial^\alpha \phi_2| \leq \frac{\alpha!}{a} \left| \sum_{|\beta|=2}^{|\alpha|} (\partial^\beta H) \circ \psi \sum_k \prod_{j=1}^m \prod_{\gamma \in A_\alpha} \left(\frac{\partial^\gamma \psi^j}{\gamma!} \right)^{k_\gamma^j} \right|,$$

where the sum in k is over all mappings $k : \{1, \dots, m\} \times A_\alpha \to \mathbb{N}$, $(j, \gamma) \mapsto k_\gamma^j$ such that $\sum_{\gamma \in A_\alpha} k_\gamma^j = \beta_j$ for all $j = 1, \dots, m$ and $\sum_{j=1}^m \sum_{\gamma \in A_\alpha} k_\gamma^j \gamma = \alpha$. Observe that for all k appearing in the above sum, we have $k_\alpha^j = 0$ for all $j = 1, \dots, m$:

For, otherwise there would be some j_0 such that $k_\alpha^{j_0} = 1$ and $k_\gamma^j = 0$ if $\gamma \neq \alpha$ or $j \neq j_0$, a contradiction to $2 \leq |\beta| = \sum_{j,\gamma} k_\gamma^j$.

Thus, if $k_\gamma^j \neq 0$ for an exponent in the above sum, then we have $|\gamma| < |\alpha|$, and therefore our induction hypothesis implies the following:

[5] At the start of the induction with $|\alpha| = 2$, the range of such γ's is empty.

If $|\gamma| \geq 2$, then we may estimate $|\partial^\gamma \psi^j| \leq A_{|\gamma|} B r^{|\gamma|}$. And, if $|\gamma| = 1$, then in view of (A-5) and (A-8), we may estimate $|\partial^\gamma \psi^j| \leq 1 + (A+1)/a \lesssim 1$. Making also use of (A-3), we then arrive at the estimation

$$|\partial^\alpha \phi_2| \lesssim \sum_{|\beta|=2}^{|\alpha|} B r^{|\beta|} \sum_k \prod_{j=1}^{m} \prod_{|\gamma|=2}^{|\alpha|} [B r^{|\gamma|}]^{k_\gamma^j}$$
$$\leq \sum_{|\beta|=2}^{|\alpha|} \sum_k B^{1+\sum_j \sum_{|\gamma|=2}^{|\alpha|} k_\gamma^j} r^{|\beta|+\sum_j \sum_{|\gamma|=2}^{|\alpha|} k_\gamma^j |\gamma|}.$$

Notice that we have

$$|\beta| = \sum_j \beta_j = \sum_j \sum_{|\gamma|=1}^{|\alpha|} k_\gamma^j = \sum_j \sum_{|\gamma|=2}^{|\alpha|} k_\gamma^j + \sum_j \sum_{|\gamma|=1} k_\gamma^j |\gamma|,$$

and thus

$$B^{1+\sum_j \sum_{|\gamma|=2}^{|\alpha|} k_\gamma^j} r^{|\beta|+\sum_j \sum_{|\gamma|=2}^{|\alpha|} k_\gamma^j |\gamma|} = B r^{\sum_j \sum_{|\gamma|=1}^{|\alpha|} k_\gamma^j |\gamma|} (Br)^{\sum_j \sum_{|\gamma|=2}^{|\alpha|} k_\gamma^j} \leq B r^{|\alpha|},$$

where we have made use of our assumption $Br \leq 1$. This proves also (A-9).

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