ANALYSIS & PDE

Volume 10 No. 4 2017

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msp

ANALYSIS AND PDE
Vol. 10, No. 4, 2017
dx.doi.org/10.2140/apde.2017.10.893

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ON THE 3-DIMENSIONAL WATER WAVES SYSTEM ABOVE A FLAT BOTTOM

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As a starting point for studying the long-time behavior of the 3-dimensional water waves system in the flat bottom setting, we try to improve the understanding of the Dirichlet–Neumann operator in this set-up. As an application, we study the 3-dimensional gravity waves system and derive a new energy estimate of L^2–L^∞ type, which has good structure in the L^∞-type space. This has been used in our Ph.D. thesis (2016) to prove the global regularity of the 3-dimensional gravity waves system for suitably small initial data.

1. Introduction

1A. ***The full water waves system above a flat bottom.*** We are interested in the long-time behavior of the 3-dimensional water waves system for suitably small initial data in the flat-bottom setting.

The water waves system describes the evolution of an inviscid incompressible fluid with constant density (e.g., water) inside a time-dependent region $\Omega(t)$, which has a free interface $\Gamma(t)$ and a fixed flat bottom Σ. Above the domain $\Omega(t)$, there is a vacuum.

Without loss of generality, we normalize the depth of $\Omega(t)$ to be 1. In the Eulerian coordinate system, we can represent the domain $\Omega(t)$, the interface $\Gamma(t)$ and the bottom Σ as follows:

$$\Omega(t) := \{(x, y) : x \in \mathbb{R}^2,\ -1 \le y \le h(t, x)\},$$
$$\Gamma(t) := \{(x, y) : x \in \mathbb{R}^2,\ y = h(t, x)\}, \quad \Sigma := \{(x, y) : x \in \mathbb{R}^2,\ y = -1\}.$$

We remark that, for the case we are considering, the size of $h(t,\cdot)$ will be small for all time.

We assume that the velocity field is irrotational. The evolution of fluid is subject to the gravity effect or the surface tension effect. We can describe the evolution of fluid by the Euler equation as

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p - g(0,0,1), \\ \nabla\cdot u = 0, \quad \nabla\times u = 0, \quad u(0) = u_0, \end{cases} \tag{1-1}$$

MSC2010: 35Q35, 76B15.
Keywords: 3-dimensional water waves, finite depth, flat bottom, new energy estimate.

where g denotes the constant of the gravity effect.

Moreover, we have the boundary conditions

$$\begin{cases} u\cdot\vec{\boldsymbol{n}}=0 & \text{on } \Sigma,\\ P=\sigma H(h) & \text{on } \Gamma(t),\\ \partial_t+u\cdot\nabla \text{tangents to } \bigcup_t \Gamma(t) & \text{on } \Gamma(t),\end{cases} \tag{1-2}$$

where σ denotes the surface tension coefficient and $H(h)$ denotes the mean curvature of the interface, which is given by

$$H(h)=\nabla\cdot\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right).$$

The first boundary condition in (1-2) means that the fluid cannot go through the fixed bottom. The second boundary condition in (1-2) comes from the Young–Laplace equation for the pressure. The third boundary condition in (1-2) represents the kinematic boundary condition, which says that the free interface moves with the normal component of the velocity.

Recall that the velocity field is irrotational. Hence, we can represent it in terms of a velocity potential ϕ. We use ψ to denote the restriction of the velocity potential to the boundary $\Gamma(t)$, i.e., $\psi(t,x):=\phi(t,x,h(t,x))$. From the incompressible condition and the boundary conditions, we can derive the following Laplace equation with two boundary conditions, Neumann-type on the bottom and Dirichlet-type on the interface:

$$(\Delta_x+\partial_y^2)\phi=0,\qquad \frac{\partial\phi}{\partial\vec{\boldsymbol{n}}}\bigg|_{\Sigma}=0,\qquad \phi|_{\Gamma(t)}=\psi. \tag{1-3}$$

Hence, we can reduce (e.g., see [Zakharov 1968]) the motion of fluid inside the water region $\Omega(t)$ to the evolution of the height h and the restricted velocity potential ψ on the interface $\Gamma(t)$:

$$\begin{cases} \partial_t h=G(h)\psi,\\ \partial_t\psi=-gh+\sigma H(h)-\frac{1}{2}|\nabla\psi|^2+\dfrac{(G(h)\psi+\nabla h\cdot\nabla\psi)^2}{2(1+|\nabla h|^2)},\end{cases} \tag{1-4}$$

where $G(h)\psi=\sqrt{1+|\nabla h|^2}\mathcal{N}(h)\psi$ and $\mathcal{N}(h)\psi$ is the Dirichlet–Neumann operator on the interface.

The system (1-4) has the conservation law

$$\mathcal{H}\big(h(t),\psi(t)\big):=\left[\int \tfrac{1}{2}\psi(t)G(h(t))\psi(t)+\tfrac{1}{2}g|h(t)|^2+\frac{\sigma|\nabla h(t)|^2}{1+\sqrt{1+|\nabla h(t)|^2}}\right]=\mathcal{H}\big(h(0),\psi(0)\big).$$

Intuitively speaking, after diagonalizing the system (1-4), we find ourselves dealing with the following type of quasilinear dispersive equation:

$$(\partial_t+i\tilde{\Lambda})u=\mathcal{N}(u,\nabla u),\quad \tilde{\Lambda}=\sqrt{|\nabla|\tanh(|\nabla|)(g+\sigma|\nabla|^2)},\quad u=h+i\tilde{\Lambda}^{-1}|\nabla|\tanh|\nabla|\psi, \tag{1-5}$$

$$u:\mathbb{R}_t\times\mathbb{R}_x^2\to\mathbb{C}. \tag{1-6}$$

Readers can temporarily take (1-5) for granted. It will be much clearer after we obtain the linear term of the Dirichlet–Neumann operator, which is $|\nabla|\tanh|\nabla|\psi$, in Section 3.

1B. ***Motivation and the main result of this paper.*** Note that the best decay rate that one can expect for a 2-dimensional dispersive equation is $1/t$, which is critical in establishing the global regularity for small initial data.

For a 2-dimensional nonlinear dispersive equation, generally speaking, it is crucial to know what the quadratic terms are when studying the long-time behavior of the solution. Unfortunately, to the best of our knowledge, there is no previous work that addresses this issue for the water waves system in the flat-bottom setting. It motivated us to study the problem in this paper.

Identifying the quadratic terms requires a more careful analysis of the Dirichlet–Neumann operator in the flat-bottom setting. Note that the water waves system in the Eulerian coordinate formulation (1-4) is dimensionless. Since we don't want to limit our scope to the 3-dimensional setting, in this paper, we will identify structures inside the Dirichlet–Neumann operator as much as we can.

We summarize and explain several important properties of the Dirichlet–Neumann operator here to help readers understand the discussion of it in this paper. These properties will play important roles in the study of the long-time behavior of the water waves system.

(i) Unlike the infinite-depth setting, in the flat-bottom setting, we do not have the null structure in the low-frequency part. More precisely, if the frequencies of two inputs are 1 and 0 respectively, then the size of the symbol is 1 (flat-bottom setting) instead of 0 (infinite-depth setting).

We remark that the principal symbol of the Dirichlet–Neumann operator in the flat-bottom setting is still the same as in the infinite-depth setting. Intuitively speaking, the high-frequency parts of the Dirichlet–Neumann operator in the two settings are almost the same.

(ii) We give the explicit formula for the quadratic terms of the Dirichlet–Neumann operator, which provides the first step in studying the long-time behavior of (1-5).

(iii) We formulate the cubic and higher-order terms of the Dirichlet–Neumann operator in a fixed-point-type formulation, which provides a good way to control the cubic and higher-order terms over time.

As a starting point and also as an example, we study a specific setting of the water waves system (1-4), which is the gravity water waves system. More precisely, we consider the gravity effect and neglect the surface tension effect. After normalizing the gravity effect constant g to be 1, the system (1-4) is reduced to

$$\begin{cases} \partial_t h = G(h)\psi, \\ \partial_t \psi = -h - \frac{1}{2}|\nabla\psi|^2 + \dfrac{(G(h)\psi + \nabla h\cdot\nabla\psi)^2}{2(1+|\nabla h|^2)}. \end{cases} \tag{1-7}$$

Correspondingly, the diagonalized equation (1-5) is reduced to the quasilinear dispersive equation

$$(\partial_t + i\Lambda)u = \mathcal{N}(u,\nabla u), \quad \Lambda = \sqrt{|\nabla|\tanh|\nabla|}, \quad u = h + i\Lambda\psi. \tag{1-8}$$

For the water waves system in the flat-bottom setting, a typical issue is that the phases are highly degenerate at the low-frequency part. For example, we consider a phase associated with a quadratic term of (1-8),

$$\Lambda(|\xi|) - \Lambda(|\xi-\eta|) + \Lambda(|\eta|) \approx (|\xi| - |\xi-\eta| + |\eta|) - \tfrac{1}{6}(|\xi|^3 - |\xi-\eta|^3 + |\eta|^3), \quad |\eta| \le |\xi| \sim |\xi-\eta| \ll 1.$$

When ξ and $-\eta$ are in the same direction, the above phase is of size $|\xi|^2|\eta|$, which is highly degenerate. Because of this issue, generally speaking, there is no hope to prove the sharp $1/t$ decay rate of the nonlinear solution over time. As a result, a rough energy estimate is not sufficient to control the growth of energy in the long run. However, it turns out that there is a relatively simple way to control the growth of energy. It relies on two observations about the system (1-7):

(i) We can derive a new energy estimate of L^2-L^∞ type after carefully analyzing the structures inside the quadratic terms in (1-7). The input inside the quadratic terms is, roughly speaking, not put in L^∞ but rather in a weaker L^∞-type space, which has derivatives in front. See (1-12).

(ii) The low-frequency parts of the derivatives compensate for the decay rate of the solution of (1-7). We can prove that the solution with some derivatives in front decays sharply, despite the fact that the solution itself may not have the sharp decay rate. The proof of this fact involves a very delicate Fourier analysis. Interested readers are referred to [Wang 2016] for more details.

Before stating our main result, we define the function spaces

$$\|f\|_{\widetilde{W}^\gamma} := \sum_{k\geq 0, k\in\mathbb{Z}} 2^{\gamma k}\|P_k f\|_{L^\infty} + \|P_{\leq 0} f\|_{L^\infty}, \tag{1-9}$$

$$\|f\|_{\widehat{W}^{\gamma,\alpha}} := \sum_{k\in\mathbb{Z}} (2^{\alpha k} + 2^{\gamma k})\|P_k f\|_{L^\infty}, \quad 0\leq\alpha\leq\gamma, \qquad \|f\|_{\widehat{W}^\gamma} := \|f\|_{\widehat{W}^{\gamma,0}}. \tag{1-10}$$

Theorem 1.1. *Let $0<\delta<c$, $\alpha\in(0,1]$, and $N_0\geq 6$, where c is some sufficiently small constant. If the initial data $(h_0, \Lambda\psi_0)\in H^{N_0+1/2}(\mathbb{R}^2)\times H^{N_0}(\mathbb{R}^2)$ satisfies the smallness condition*

$$\|(h_0, \Lambda\psi_0)\|_{\widetilde{W}^4} \leq \delta, \tag{1-11}$$

then there exists $T>0$ such that the system (1-7) has the unique solution

$$(h, \Lambda\psi)\in C^0\big([0,T]; H^{N_0}(\mathbb{R}^2)\times H^{N_0}(\mathbb{R}^2)\big).$$

Moreover, we have a new type of energy estimate in the time interval of existence:

$$\frac{d}{dt}E_{N_0}(t) \lesssim_{N_0} \big[\|(h,\Lambda\psi)(t)\|_{\widehat{W}^{4,\alpha}} + \|(h,\Lambda\psi)(t)\|^2_{\widehat{W}^4}\big]E_{N_0}(t), \tag{1-12}$$

where the energy $E_{N_0}(t)$ is defined in (5-3). The size of energy is comparable to $\|(h,\Lambda\psi)(t)\|^2_{H^{N_0}}$.

Remark 1.2. Note that smallness condition is not assumed in [Alazard, Burq and Zuily 2011; 2014a; 2014b; Lannes 2005] to derive the local wellposedness. For the purpose of obtaining a global solution, we impose the smallness condition (1-11) to derive our desired estimate (1-12), which is the first step to obtaining global existence for small initial data.

In [Wang 2016], based on the results we obtained in this paper, we show that the solution of the system (1-7) exists globally and scatters to a linear solution. We will study the long-time behavior of the water waves system (1-4) in other settings in the future. For example, do we still have global solutions if only the surface tension is effective or both the gravity and the surface tension are effective? We expect that the results we obtained in this paper will be very helpful to the future study of the water waves system in the flat-bottom setting.

1C. ***Previous results.*** To be concise, we mainly discuss work on the local behavior of the water waves system in this subsection. For a more detailed discussion on the long-time behavior, please refer to the introduction of [Wang 2016].

Starting with [Nalimov 1974] and [Yosihara 1982], there has been a considerable amount of work on the local theory of the water waves system. In the framework of Sobolev spaces and without smallness assumptions on the initial data, the local wellposedness was first obtained by Wu [1997; 1999] for the gravity waves system. The local wellposedness was also obtained when the surface tension is effective by Beyer and Günther [1998]. Later, different methods were developed and many important results were obtained to improve our understanding of the local behavior of the water waves system. Among them, we mention [Christodoulou and Lindblad 2000; Ambrose and Masmoudi 2005; Lannes 2005; Shatah and Zeng 2008; Coutand and Shkoller 2007; Alazard, Burq and Zuily 2011; 2014a; 2014b].

Roughly speaking, the local existence for the water waves system (1-4) holds even when the initial interface has an unbounded curvature and the bottom is very rough. A fixed-length separation between the interface and the bottom is sufficient. See [Alazard, Burq and Zuily 2011; 2014a; 2014b; Lannes 2005] for more details and more precise descriptions.

1D. ***Main ideas and the outline of this paper.*** To prove our main theorem, we have to pay attention to both the low- and high-frequency parts.

For the high-frequency part, due to the quasilinear nature of the gravity waves system (1-7), we have to get around the difficulty of losing one derivative. Thanks to [Lannes 2005; Alazard and Métivier 2009; Alazard, Burq and Zuily 2011; 2014a; 2014b], we can utilize the paralinearization method to get around the potential loss of one derivative. However, for their purposes, only the high-frequency part has been carefully studied in their works. In this paper, we will do the paralinearization process and pay special attention to the low-frequency part at the same time.

For the low-frequency part, more careful estimates of the Dirichlet–Neumann operator are essential since it is not straightforward to see the fact that we can gain α derivatives for input in $\widehat{W}^{4,\alpha}$. For example, for the quadratic term $\nabla h \cdot \nabla \psi$ of the Dirichlet–Neumann operator, it is problematic to gain α derivatives when ψ has smaller frequency because the total number of derivatives of ψ in (1-12) is $1+\alpha$ in the low-frequency part when the input ψ of the quadratic terms is in L^∞.

To conclude the argument, we will use the hidden structure inside the system (1-7) for different scenarios. Without describing too many details, we give two examples as follows to explain the main ideas:

(i) When ψ has a smaller frequency inside $\nabla h \cdot \nabla \psi$, we can use the hidden symmetry to move one derivative from ∇h to $\nabla \psi$ during the energy estimate; hence we have two derivatives in total for ψ.

(ii) For some terms, e.g., the good remainder term of the paralinearization process, we can lower their regularities to L^2. Hence, we can put $\nabla \psi$ in L^2 and put ∇h in L^∞; as a result the desired estimate (1-12) also holds for this case.

Outline: In Section 2, we introduce notation and give a quick summary of paradifferential calculus. In Section 3, we study various properties of the Dirichlet–Neumann operator. In Section 4, we use the paralinearization method to show the good structures inside the system (1-7), which help us find good substitution variables. In Section 5, we prove the new energy estimate (1-12) by using the symmetries inside the equations satisfied by the good substitution variables. In the Appendix, we calculate explicitly the quadratic terms of good remainder terms. This is intended to help readers understand the fact that we can gain α derivatives in (1-12) for an input of quadratic terms, which lies in the L^∞-type space.

2. Preliminaries

2A. ***Notation.*** For any two numbers A and B, we use $A \lesssim B$ and $B \gtrsim A$ to denote $A \le CB$, where C is an absolute constant. We use $A \lesssim_\epsilon B$ to denote $A \le C_\epsilon B$, where the constant C_ϵ depends on ϵ. For an integer $k \in \mathbb{Z}$, we use k_+ to denote $\max\{k, 0\}$ and use k_- to denote $\min\{k, 0\}$.

Throughout this paper, we will abuse the notation of Λ. When there is no lower script associated with Λ, we let $\Lambda := \sqrt{\tanh(|\nabla|)|\nabla|}$, which is the linear operator associated with the system (1-8). For $p \in \mathbb{N}_+$, we use $\Lambda_p(\mathcal{N})$ to denote the p-th order terms of a nonlinearity $\mathcal{N}$ when a Taylor expansion for the nonlinearity $\mathcal{N}$ is available. For example, $\Lambda_2[\mathcal{N}]$ denotes the quadratic term of $\mathcal{N}$. We also use $\Lambda_{\ge p}[\mathcal{N}]$ to denote the p-th and higher-order terms. More precisely, $\Lambda_{\ge p}[\mathcal{N}] := \sum_{q \ge p} \Lambda_q[\mathcal{N}]$. In this paper, the Taylor expansion and $\Lambda_p[\cdot]$ are in terms of h and ψ when there is no special annotation.

We fix an even smooth function $\tilde{\psi} : \mathbb{R} \to [0, 1]$, which is supported in $\left[-\frac{3}{2}, \frac{3}{2}\right]$ and is equal to 1 in $\left[-\frac{5}{4}, \frac{5}{4}\right]$. For any $k \in \mathbb{Z}$, define

$$\psi_k(x) := \tilde{\psi}(x/2^k) - \tilde{\psi}(x/2^{k-1}), \quad \psi_{\le k}(x) := \tilde{\psi}(x/2^k), \quad \psi_{\ge k}(x) := 1 - \psi_{\le k-1}(x).$$

Denote the projection operators P_k, $P_{\le k}$ and $P_{\ge k}$ by the Fourier multipliers ψ_k, $\psi_{\le k}$ and $\psi_{\ge k}$ respectively. For a well-defined function f, we will also use the notation f_k to abbreviate $P_k f$.

The Fourier transform is defined as

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\, dx.$$

For two well-defined functions f and g and a bilinear form $Q(f, g)$, we will use the convention that the symbol $q(\cdot\,,\cdot)$ of $Q(\cdot\,,\cdot)$ is defined in the following sense throughout this paper:

$$\mathcal{F}[Q(f, g)](\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \hat{f}(\xi - \eta)\hat{g}(\eta) q(\xi - \eta, \eta)\, d\eta. \tag{2-1}$$

Meanwhile, for a trilinear form $C(f, g, h)$, its symbol $c(\cdot\,,\cdot\,,\cdot)$ is defined in the following sense:

$$\mathcal{F}[C(f, g, h)](\xi) = \frac{1}{16\pi^4} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \hat{f}(\xi - \eta)\hat{g}(\eta - \sigma)\hat{h}(\sigma) c(\xi - \eta, \eta - \sigma, \sigma)\, d\eta\, d\sigma.$$

2B. ***Multilinear estimate.*** We define a class of symbols with an associated norm as

$$\mathcal{S}^\infty := \{m : \mathbb{R}^4 \text{ or } \mathbb{R}^6 \to \mathbb{C},\ m \text{ is continuous and } \|\mathcal{F}^{-1}(m)\|_{L^1} < \infty\},$$

$$\|m\|_{\mathcal{S}^\infty} := \|\mathcal{F}^{-1}(m)\|_{L^1}, \quad \|m(\xi, \eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} := \big\|m(\xi, \eta)\psi_k(\xi)\psi_{k_1}(\xi - \eta)\psi_{k_2}(\eta)\big\|_{\mathcal{S}^\infty},$$

$$\|m(\xi, \eta, \sigma)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} := \big\|m(\xi, \eta, \sigma)\psi_k(\xi)\psi_{k_1}(\xi - \eta)\psi_{k_2}(\eta - \sigma)\psi_{k_3}(\sigma)\big\|_{\mathcal{S}^\infty}.$$

Lemma 2.1. *Assume that $m, m' \in \mathcal{S}^\infty$ and $p, q, r, s \in [1, \infty]$. Then the estimates*

$$\|m \cdot m'\|_{\mathcal{S}^\infty} \lesssim \|m\|_{\mathcal{S}^\infty} \|m'\|_{\mathcal{S}^\infty}, \tag{2-2}$$

$$\left\| \mathcal{F}^{-1}\left[\int_{\mathbb{R}^2} m(\xi,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\eta \right] \right\|_{L^p} \lesssim \|m\|_{\mathcal{S}^\infty} \|f\|_{L^q} \|g\|_{L^r} \quad \text{if } \tfrac{1}{p} = \tfrac{1}{q} + \tfrac{1}{r}, \tag{2-3}$$

$$\left\| \mathcal{F}^{-1}\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} m'(\xi,\eta,\sigma) \hat{f}(\xi-\eta)\hat{h}(\sigma)\hat{g}(\eta-\sigma)\, d\eta\, d\sigma \right] \right\|_{L^p} \lesssim \|m'\|_{\mathcal{S}^\infty} \|f\|_{L^q} \|g\|_{L^r} \|h\|_{L^s} \tag{2-4}$$

hold for well-defined functions $f(x)$, $g(x)$, and $h(x)$, where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$.

To estimate the $\mathcal{S}^\infty_{k,k_1,k_2}$ norm and the $\mathcal{S}^\infty_{k,k_1,k_2,k_3}$ norm of symbols, we repeatedly use the following:

Lemma 2.2. *For $i \in \{1,2,3\}$, if $f : \mathbb{R}^{2i} \to \mathbb{C}$ is a smooth function and $k_1, \dots, k_i \in \mathbb{Z}$, then we have the estimate*

$$\left\| \int_{\mathbb{R}^{2i}} f(\xi_1,\dots,\xi_i) \prod_{j=1}^{i} e^{i x_j\cdot\xi_j} \psi_{k_j}(\xi_j)\, d\xi_1 \cdots d\xi_i \right\|_{L^1_{x_1,\dots,x_i}} \lesssim \sum_{m=0}^{i+1}\sum_{j=1}^{i} 2^{mk_j} \|\partial^m_{\xi_j} f\|_{L^\infty}. \tag{2-5}$$

Proof. The cases when $i = 1, 3$ can be estimated in the same way as the case when $i = 2$. We only do the case $i = 2$ in detail here. Through scaling, it is sufficient to prove the above estimate for the case when $k_1 = k_2 = 0$. From Plancherel's theorem, we have the two estimates

$$\left\| \int_{\mathbb{R}^{2i}} f(\xi_1,\xi_2) e^{i(x_1\cdot\xi_1 + x_2\cdot\xi_2)} \psi_0(\xi_1)\psi_0(\xi_2)\, d\xi_1\, d\xi_2 \right\|_{L^2_{x_1,x_2}} \lesssim \|f(\xi_1,\xi_2)\|_{L^\infty_{\xi_1,\xi_2}},$$

$$\left\| (|x_1|+|x_2|)^3 \int_{\mathbb{R}^{2i}} f(\xi_1,\xi_2) e^{i(x_1\cdot\xi_1 + x_2\cdot\xi_2)} \psi_0(\xi_1)\psi_0(\xi_2)\, d\xi_1\, d\xi_2 \right\|_{L^2_{x_1,x_2}} \lesssim \sum_{m=0}^{3} \left[\|\partial^m_{\xi_1} f\|_{L^\infty} + \|\partial^m_{\xi_2} f\|_{L^\infty} \right],$$

which are sufficient to finish the proof of (2-5). □

2C. *Paradifferential calculus.* In this subsection, we discuss some necessary background material from paradifferential calculus. For more details and related topics, please refer to [Métivier 2008].

Definition 2.3. Given $\rho \in \mathbb{N}_+$, $\rho \geq 0$ and $m \in \mathbb{R}$, we use $\Gamma^m_\rho(\mathbb{R}^2)$ to denote the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^2 \times (\mathbb{R}^2/\{0\})$, which are C^∞ with respect to ξ for $\xi \neq 0$. Moreover, they satisfy the estimate

$$\forall |\xi| \geq \tfrac{1}{2}, \quad \|\partial^\alpha_\xi a(\cdot,\xi)\|_{W^{\rho,\infty}} \lesssim_\alpha (1+|\xi|)^{m-|\alpha|}, \quad \alpha \in \mathbb{N}^2,$$

where $W^{\rho,\infty}$ is the usual Sobolev space. Note that $W^{\rho,\infty}$ contains the spaces $\widetilde{W}^\rho$ and $\widehat{W}^{\rho,\alpha}$, which are defined in (1-9) and (1-10), as subspaces.

Remark 2.4. In the above definitions, ρ is not necessarily an integer, but the integer case is sufficient for our purposes.

Definition 2.5. (i) We use $\dot{\Gamma}^m_\rho(\mathbb{R}^2)$ to denote the subspace of $\Gamma^m_\rho(\mathbb{R}^2)$ which consists of symbols that are homogeneous of degree m in ξ.

(ii) If $a = \sum_{0\leq j<\rho} a^{(m-j)}$, where $a^{(m-j)} \in \dot{\Gamma}^{m-j}_{\rho-j}(\mathbb{R}^2)$, then we say $a^{(m)}$ is the principal symbol of a.

(iii) An operator T is said to be of order m, where $m\in\mathbb{R}$, if for all $\mu\in\mathbb{R}$, it is bounded from $H^{\mu}(\mathbb{R}^2)$ to $H^{\mu-m}(\mathbb{R}^2)$. We use S^m to denote the set of all operators of order m.

For a symbol $a\in\Gamma^m_\rho$, we can define its norm as

$$M^m_\rho(a) := \sup_{|\alpha|\le 2+\rho}\ \sup_{|\xi|\ge\frac12} \big\|(1+|\xi|)^{|\alpha|-m}\,\partial_\xi^\alpha a(\cdot,\xi)\big\|_{W^{\rho,\infty}}.$$

For $a, f\in L^2$ and a pseudodifferential operator $\tilde a(x,\xi)$, we define the operators $T_a f$ and $T_{\tilde a} f$ as

$$T_a f=\mathcal{F}^{-1}\bigg[\int_{\mathbb{R}} \hat a(\xi-\eta)\theta(\xi-\eta,\eta)\hat f(\eta)\,d\eta\bigg],\quad T_{\tilde a} f=\mathcal{F}^{-1}\bigg[\int_{\mathbb{R}} \mathcal{F}_x(\tilde a)(\xi-\eta,\eta)\theta(\xi-\eta)\hat f(\eta)\,d\eta\bigg], \tag{2-6}$$

where the cut-off function $\theta(\xi-\eta,\eta)$ is given by

$$\theta(\xi-\eta,\eta)=\begin{cases}1 & \text{when } |\xi-\eta|\le 2^{-10}|\eta| \text{ and } |\eta|\ge 1,\\ 0 & \text{when } |\xi-\eta|\ge 2^{10}|\eta| \text{ or } |\eta|\le 1.\end{cases}$$

For two well-defined functions a and b, we have the paraproduct decomposition

$$ab = T_a b + T_b a + \mathcal{R}(a,b), \tag{2-7}$$

where $\mathcal{R}(a,b)$ contains those terms in which a and b have comparable size of frequencies or the frequency of the output is less than 1.

We have the following composition lemma for paradifferential operators. It can be found, for example, in [Alazard, Burq and Zuily 2011; Métivier 2008].

Lemma 2.6. *Let $m\in\mathbb{R}$ and $\rho>0$. If given symbols $a\in\Gamma^m_\rho(\mathbb{R}^d)$ and $b\in\Gamma^{m'}_\rho(\mathbb{R}^d)$ we define*

$$a\sharp b=\sum_{|\alpha|<\rho}\frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a\,\partial_x^\alpha b,$$

then for all $\mu\in\mathbb{R}$, there exists a constant K such that

$$\|T_aT_b - T_{a\sharp b}\|_{H^\mu\to H^{\mu-m-m'+\rho}}\le K M^m_\rho(a) M^{m'}_\rho(b). \tag{2-8}$$

Remark 2.7. It may be too early to give this remark here. However, we think that it is a good idea to keep the following simple observation in mind, which will be very helpful to see the equivalence relations later on. The simple observation is that if the symbols a and b all depend on ∇h instead of h, then the rough estimate (2-8) is sufficient to gain one derivative in the low-frequency part.

Lemma 2.8. *Let $m\in\mathbb{R}$, $\rho>0$ and $a\in\Gamma^m_\rho(\mathbb{R}^d)$. If we use $(T_a)^*$ to denote the adjoint operator of T_a and use $\bar a$ to denote the complex conjugate of a, then $(T_a)^*-T_{a^*}$ is of order $m-\rho$, where*

$$a^*=\sum_{|\alpha|<\rho}\frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha\partial_x^\alpha\bar a.$$

Moreover, the norm of the operator $(T_a)^*-T_{a^*}$ is bounded by $M^m_\rho(a)$.

Proof. See [Alazard, Burq and Zuily 2011, Theorem 3.10]. ☐

Remark 2.9. In most applications of Lemma 2.8, we have $m\le 1$. If we let $\rho=1$ in the above lemma, then it is easy to see $a^*=\bar a$. If, moreover, a is real, then $a^*=\bar a=a$.

3. Dirichlet–Neumann operator

The main goal of this section is to study various properties of the Dirichlet–Neumann operator, which provide a foundation for carrying out the processes of paralinearization and symmetrization in Section 4 and obtaining the new energy estimate (1-12) in Section 5. The study of the Dirichlet–Neumann operator is mainly reduced to a study of the velocity potential inside the water region $\Omega(t)$.

Recall the smallness condition (1-11) of the initial data. From the local wellposedness result of the gravity waves system (1-7), we know that there exists a positive time T such that the estimate

$$\sup_{t\in[0,T]} \|(h, \Lambda\psi)(t)\|_{\widetilde{W}^4} \le 2\delta \tag{3-1}$$

holds, which means that the L^∞ norm of solution remains small in the time interval $[0, T]$. Throughout the rest of this paper, we restrict ourselves to the time interval $[0, T]$.

3A. ***Type I formulation of the Laplace equation (1-3).*** In this subsection, we reduce the Laplace equation (1-3) to a favorable formulation so that we can solve it and identify the fixed-point-type structure inside the Laplace equation, which further enables us to estimate the Dirichlet–Neumann operator.

We do a change of variables and map the water region $\Omega(t)$ to the strip $\mathcal{S} := \mathbb{R}^2 \times [-1, 0]$ using

$$(x, y) \to (x, z), \quad z := \frac{y - h(t, x)}{h(t, x) + 1}.$$

Very naturally, the inverse transformation is given by

$$y = h + (h + 1)z.$$

Define the velocity potential in the (x, z)-coordinate system as $\varphi(x, z) := \phi(x, h + (h + 1)z)$. From direct computations, we have the identities

$$\phi(x,y)=\varphi\Big(x,\frac{y-h}{1+h}\Big),\quad \partial_y\phi=\frac{\partial_z\varphi}{1+h},\quad \partial_y^2\phi=\frac{\partial_z^2\varphi}{(1+h)^2}, \tag{3-2}$$

$$\partial_{x_i}\phi=\partial_{x_i}\varphi+\partial_z\varphi\Big[\frac{-\partial_{x_i}h}{1+h}-\frac{(y-h)\partial_{x_i}h}{(1+h)^2}\Big]=\partial_{x_i}\varphi-\frac{(y+1)\partial_{x_i}h}{(1+h)^2}\partial_z\varphi, \tag{3-3}$$

$$\partial_{x_i}^2\phi=\partial_{x_i}^2\varphi-2\frac{(y+1)\partial_{x_i}h}{(1+h)^2}\partial_z\partial_{x_i}\varphi+\Big[\frac{-(y+1)\partial_{x_i}^2h}{(1+h)^2}+2\frac{(y+1)(\partial_{x_i}h)^2}{(1+h)^3}\Big]\partial_z\varphi+\frac{(y+1)^2(\partial_{x_i}h)^2}{(1+h)^4}\partial_z^2\varphi.$$

From the above identities and (1-3), it is easy to derive the equation

$$(\Delta_x + \partial_y^2)\phi = 0 \quad\Longrightarrow\quad P_{x,z}\varphi := \big[\Delta_x + \tilde{a}\partial_z^2 + \tilde{b}\cdot\nabla\partial_z + \tilde{c}\partial_z\big]\varphi = 0, \tag{3-4}$$

where

$$\tilde{a} = \frac{(y+1)^2|\nabla h|^2}{(1+h)^4} + \frac{1}{(1+h)^2} = \frac{1+(z+1)^2|\nabla h|^2}{(1+h)^2}, \tag{3-5}$$

$$\tilde{b} = -2\frac{(y+1)\nabla h}{(1+h)^2} = \frac{-2(z+1)\nabla h}{1+h},\quad \tilde{c} = \frac{-(z+1)\Delta_x h}{(1+h)} + 2\frac{(z+1)|\nabla h|^2}{(1+h)^2}. \tag{3-6}$$

To sum up, we can reduce the Laplace equation (1-3) with two boundary conditions in terms of φ as follows:

$$P_{x,z}\varphi = 0, \quad \varphi|_{z=0} = \psi, \quad \partial_z \varphi|_{z=-1} = 0, \quad (x,z) \in \mathbb{R}^2 \times [-1,0]. \tag{3-7}$$

3B. *Type II formulation of the Laplace equation* (1-3). In this subsection, we reduce the Laplace equation (1-3) into another favorable formulation, which will be used to do the paralinearization of the Dirichlet–Neumann operator in Section 4A.

We remark that we don't use the type I formulation (3-7) to do the paralinearization process because the coefficients $\tilde{a}, \tilde{b}, \tilde{c}$ in (3-5) and (3-6) are very complicated, which complicates the paralinearization process and prevents us from seeing clearly the principal symbol of the Dirichlet–Neumann operator.

Recall the smallness condition (3-1). Since the height of interface is very small, we know that there exists a curve parallel to the interface $\Gamma(t)$ with depth $\frac{1}{2}$ inside $\Omega(t)$. More precisely, we have

$$\Omega_1(t) := \big\{(x,y) : x \in \mathbb{R}^2,\ h(t,x) - \tfrac{1}{2} \le y \le h(t,x)\big\}, \quad \Omega_1(t) \subset \Omega(t).$$

Define

$$\Omega_2(t) := \big\{(x,y) : x \in \mathbb{R}^2, h(t,x) - \tfrac{1}{4} \le y \le h(t,x)\big\}, \quad \Omega_2(t) \subset \Omega_1(t) \subset \Omega(t), \tag{3-8}$$

$$\tilde{\phi}(x,y) := \chi(y - h(t,x))\phi(x,y), \quad (x,y) \in \Omega_1(t), \quad \chi(z) = 1 \text{ if } z \ge -\tfrac{1}{4}, \quad \operatorname{supp}(\chi) \subset \big[-\tfrac{1}{2}, 0\big], \tag{3-9}$$

where $\chi(x)$ is a fixed Schwartz function.

Recall the Laplace equation (1-3). From (3-9), it is easy to derive the identities

$$\begin{aligned} \Delta_{x,y}\tilde{\phi} = \tilde{g} := \Delta_{x,y}[\chi\phi] - \chi\Delta_{x,y}\phi, \quad (x,y) \in \Omega_1(t), \\ \tilde{\phi}(x,y) = \phi(x,y), \quad \tilde{g}(x,y) = 0, \quad (x,y) \in \Omega_2(t). \end{aligned} \tag{3-10}$$

We can map the water region $\Omega_1(t)$ to the strip $\mathcal{S}' := \mathbb{R}^2 \times \big[-\frac{1}{2}, 0\big]$ by changing the coordinate system using

$$(x,y) \to (x,w), \quad w := y - h(t,x).$$

Define the velocity potential in the (x,w)-coordinate system as $\Phi(x,w) := \tilde{\phi}(x, \omega + h(t,x))$. Hence $\tilde{\phi}(x,y) = \Phi(x, y - h(t,x))$. From (3-10), it is easy to verify that the equality

$$\mathcal{P}_{x,w}\Phi := \big[\Delta_x + a'\partial_w^2 + b' \cdot \nabla\partial_w + c'\partial_w\big]\Phi = g'(x,w) := \tilde{g}(x, \omega + h(t,x)) \tag{3-11}$$

holds, where

$$a' = 1 + |\nabla h|^2, \quad b' = -2\nabla h, \quad c' = -\Delta h. \tag{3-12}$$

Remark 3.1. From (3-5), (3-6), and (3-12), it is easy to see that the coefficients in (3-11) satisfied by Φ are much easier and more favorable than the coefficients in (3-4) satisfied by φ. However, the formulation satisfied by Φ in (3-11) cannot be used as the starting point because we don't know the estimates of Φ in the first place.

From the above definitions, the following identities hold inside the water region $\Omega_2(t)$, see (3-8), and the corresponding regions in the new coordinate systems:

$$\Phi(x,w)=\varphi\Big(x,\frac{w}{1+h}\Big),\quad \varphi(x,z)=\Phi(x,(1+h)z),\quad (x,w)\in\mathbb{R}^2\times\big[-\tfrac14,0\big],$$
$$\partial_{x_i}\Phi=\partial_{x_i}\varphi-\frac{w\partial_z\varphi\partial_{x_i}h}{(1+h)^2},\quad \partial_w\Phi=\frac{\partial_z\varphi}{1+h}. \tag{3-13}$$

From (3-2) and (3-13), the Dirichlet–Neumann operator $G(h)\psi$ in terms of φ and Φ and the quadratic terms of $G(h)\psi$ are given by

$$G(h)\psi=[-\nabla h\cdot\nabla\phi+\partial_y\phi]|_{y=h}=\frac{1+|\nabla h|^2}{1+h}\partial_z\varphi|_{z=0}-\nabla\psi\cdot\nabla h, \tag{3-14}$$

$$G(h)\psi=(1+|\nabla h|^2)\partial_w\Phi|_{w=0}-\nabla h\cdot\nabla\psi, \tag{3-15}$$

$$\Lambda_2[G(h)\psi]=\Lambda_2[\partial_z\varphi|_{z=0}]-\Lambda_1[\partial_z\varphi|_{z=0}]h-\nabla\psi\cdot\nabla h. \tag{3-16}$$

3C. ***A fixed-point-type formulation for the Dirichlet–Neumann operator.*** In this subsection, our main goal is to obtain basic estimates for the Dirichlet–Neumann operator with special attention to the low-frequency part, which will further help us to obtain a new energy estimate.

To this end, we study the reduced Laplace equation (3-7) and formulate $\nabla_{x,z}\varphi$ into a fixed-point-type formulation, which enables us to use a fixed-point-type argument.

After moving all nonlinear terms to the right-hand, we can rewrite (3-7) as

$$\partial_z^2\varphi+\Delta_x\varphi=(\partial_z-|\nabla|)(\partial_z+|\nabla|)\varphi=g(z):=(1-\tilde a)\partial_z^2\varphi-\tilde b\cdot\nabla\partial_z\varphi-\tilde c\partial_z\varphi. \tag{3-17}$$

Now, we will solve $\varphi(z)$ from (3-17) by treating $g(z)$ in (3-17) as a given nonlinearity. Define $\tilde h(x,z):=(\partial_z-|\nabla|)\varphi$. Very naturally, we have

$$\begin{cases}(\partial_z+|\nabla|)\tilde h=g,\\ (\partial_z-|\nabla|)\varphi=\tilde h,\quad \varphi|_{z=0}=\psi,\ \partial_z\varphi|_{z=-1}=0.\end{cases} \tag{3-18}$$

We can solve the above system of equations with $\tilde h(-1)$ to be determined:

$$\tilde h(z)=e^{-z|\nabla|}\tilde h(-1)+\int_{-1}^z e^{-(z-z')|\nabla|}g(z')\,dz', \tag{3-19}$$

$$\begin{aligned}\varphi(z)&=e^{z|\nabla|}\varphi(0)+\int_0^z e^{(z-z')|\nabla|}\tilde h(z')\,dz'\\ &=e^{z|\nabla|}\psi-\int_z^0 e^{(z-z')|\nabla|}\Big[e^{-z'|\nabla|}\tilde h(-1)+\int_{-1}^{z'}e^{-(z'-s)|\nabla|}g(s)\,ds\Big]\,dz'\\ &=e^{z|\nabla|}\psi-\tfrac12|\nabla|^{-1}[e^{-z|\nabla|}-e^{z|\nabla|}]\tilde h(-1)-\int_{-1}^z\int_z^0 e^{(z+s-2z')|\nabla|}g(s)\,dz'\,ds\\ &\qquad\qquad-\int_z^0\int_s^0 e^{(z+s-2z')|\nabla|}g(s)\,dz'\,ds\\ &=e^{z|\nabla|}\psi-\tfrac12|\nabla|^{-1}[e^{-z|\nabla|}-e^{z|\nabla|}]\tilde h(-1)+\frac12\int_{-1}^0|\nabla|^{-1}e^{(z+s)|\nabla|}g(s)\,ds\\ &\qquad\qquad-\frac12\int_{-1}^0|\nabla|^{-1}e^{-|z-s||\nabla|}g(s)\,ds.\end{aligned} \tag{3-20}$$

The unknown $\tilde{h}(-1)$ is determined by the Neumann-type boundary condition $\partial_z\varphi|_{z=-1}=0$. We calculate $\partial_z\varphi$ from the formula (3-20) and have the equality

$$\partial_z\varphi=|\nabla|e^{z|\nabla|}\psi+\tfrac{1}{2}[e^{z|\nabla|}+e^{-z|\nabla|}]\tilde{h}(-1)+\frac{1}{2}\int_{-1}^{0}e^{(z+s)|\nabla|}g(s)\,ds-\frac{1}{2}\int_{-1}^{0}e^{-|z-s||\nabla|}\operatorname{sign}(s-z)g(s)\,ds.$$

After evaluating the above equality at the point $z=-1$, we have

$$\tilde{h}(-1)=-\frac{2|\nabla|e^{-|\nabla|}\psi}{e^{-|\nabla|}+e^{|\nabla|}}-\int_{-1}^{0}\frac{e^{(s-1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}g(s)\,ds, \tag{3-21}$$

which further gives us

$$\partial_z\varphi=\frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\psi-\frac{1}{2}\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\int_{-1}^{0}[e^{(s-1)|\nabla|}-e^{-(s+1)|\nabla|}]g(s)\,ds$$
$$+\frac{1}{2}\int_{-1}^{0}e^{(z+s)|\nabla|}g(s)ds-\frac{1}{2}\int_{-1}^{0}e^{-|z-s||\nabla|}\operatorname{sign}(s-z)g(s)\,ds. \tag{3-22}$$

Moreover, we can reduce (3-20):

$$\varphi(z)=\left[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\right]\psi+\frac{1}{2}|\nabla|^{-1}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\int_{-1}^{0}[e^{(s-1)|\nabla|}-e^{-(s+1)|\nabla|}]g(s)\,ds$$
$$+\frac{1}{2}\int_{-1}^{0}|\nabla|^{-1}e^{(z+s)|\nabla|}g(s)\,ds-\frac{1}{2}\int_{-1}^{0}|\nabla|^{-1}e^{-|z-s||\nabla|}g(s)\,ds. \tag{3-23}$$

However, we cannot use the formulation (3-23) to estimate the velocity potential and the Dirichlet–Neumann operator because $g(z)$ actually depends on the velocity potential $\varphi(z)$; see (3-17).

To get around this issue, we observe that there exists a fixed-point-type structure inside $g(z)$. Recall (3-17), (3-5), and (3-6). Note that

$$g=\partial_z\left[\frac{2h+h^2-(z+1)^2|\nabla h|^2}{(1+h)^2}\partial_z\varphi+\frac{2(z+1)\nabla h\cdot\nabla\varphi}{1+h}\right]-\frac{2\nabla h\cdot\nabla\varphi}{1+h}+\frac{(z+1)\Delta h}{1+h}\partial_z\varphi,$$

and

$$\frac{(z+1)\Delta h}{1+h}\partial_z\varphi=\nabla\cdot\left[\frac{(z+1)\nabla h\partial_z\varphi}{1+h}\right]+\frac{(z+1)|\nabla h|^2\partial_z\varphi}{(1+h)^2}-\partial_z\left[\frac{(z+1)\nabla h\cdot\nabla\varphi}{1+h}\right]+\frac{\nabla h\cdot\nabla\varphi}{1+h}.$$

Hence, we can decompose the nonlinearity $g(z)$ into three parts:

$$g(z)=\partial_z g_1(z)+g_2(z)+\nabla\cdot g_3(z), \tag{3-24}$$

where

$$g_1(z)=\frac{2h+h^2-(z+1)^2|\nabla h|^2}{(1+h)^2}\partial_z\varphi+\frac{(z+1)\nabla h\cdot\nabla\varphi}{1+h},\quad g_1(-1)=0, \tag{3-25}$$

$$g_2(z)=\frac{(z+1)|\nabla h|^2\partial_z\varphi}{(1+h)^2}-\frac{\nabla h\cdot\nabla\varphi}{1+h},\quad g_3(z)=\frac{(z+1)\nabla h\partial_z\varphi}{1+h}. \tag{3-26}$$

To simplify the notation, we define

$$\tilde{h}_1 := \frac{2h+h^2}{(1+h)^2}, \quad \tilde{h}_2 := \frac{|\nabla h|^2}{(1+h)^2}, \quad \tilde{h}_3 := \frac{\nabla h}{1+h}. \tag{3-27}$$

As a result, we have

$$g_1(z) = \tilde{h}_1 \partial_z \varphi - (z+1)^2 \tilde{h}_2 \partial_z \varphi + (z+1)\tilde{h}_3 \cdot \nabla\varphi, \tag{3-28}$$

$$g_2(z) = (z+1)\tilde{h}_2 \partial_z \varphi - \tilde{h}_3 \cdot \nabla\varphi, \quad g_3(z) = (z+1)\tilde{h}_3 \partial_z \varphi. \tag{3-29}$$

Note that $g_1(z)$, $g_2(z)$, and $g_3(z)$ are all linear with respect to $\nabla_{x,z}\varphi(z)$.

After decomposing $g(s)$ in (3-23) into three parts, $\partial_s g_1$, g_2 and $\nabla \cdot g_3$, we integrate by parts in s to move the derivative ∂_s in front of $\partial_s g_1$. As a result, we have

$$\begin{aligned}
\varphi(z) = &\left[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\right]\psi \\
&+\frac{1}{2}|\nabla|^{-1}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\int_{-1}^{0}\left[e^{(s-1)|\nabla|}(g_2+\nabla\cdot g_3-|\nabla|g_1)-e^{-(s+1)|\nabla|}(g_2+\nabla\cdot g_3+|\nabla|g_1)\right]ds \\
&+\frac{1}{2}\int_{-1}^{0}|\nabla|^{-1}e^{(z+s)|\nabla|}[g_2+\nabla\cdot g_3-|\nabla|g_1]\,ds \\
&-\frac{1}{2}\int_{-1}^{0}|\nabla|^{-1}e^{-|z-s||\nabla|}[g_2+\nabla\cdot g_3-\operatorname{sign}(z-s)|\nabla|g_1]\,ds.
\end{aligned} \tag{3-30}$$

Now, we know that the nonlinearity in (3-30) is linear with respect to $\nabla_{x,z}\varphi$.

To see the fixed-point-type structure of $\nabla_{x,z}\varphi$, we take the derivative $\nabla_{x,z}$ on both sides of (3-30). As a result, we derive a fixed-point-type formulation for $\nabla_{x,z}\varphi$:

$$\begin{aligned}
\nabla_{x,z}\varphi = &\left[\left[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\right]\nabla\psi, \frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\psi\right] \\
&+\frac{1}{2}\left[\frac{\nabla}{|\nabla|}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\int_{-1}^{0}\left[e^{(s-1)|\nabla|}(g_2+\nabla\cdot g_3-|\nabla|g_1)-e^{-(s+1)|\nabla|}(g_2+\nabla\cdot g_3+|\nabla|g_1)\right]ds,\right. \\
&\qquad\left.-\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\int_{-1}^{0}\left[e^{(s-1)|\nabla|}(g_2+\nabla\cdot g_3-|\nabla|g_1)-e^{-(s+1)|\nabla|}(g_2+\nabla\cdot g_3+|\nabla|g_1)\right]ds\right] \\
&+\frac{1}{2}\left[\int_{-1}^{0}\frac{\nabla}{|\nabla|}e^{(z+s)|\nabla|}[g_2+\nabla\cdot g_3-|\nabla|g_1]\,ds-\int_{-1}^{0}\frac{\nabla}{|\nabla|}e^{-|z-s||\nabla|}[g_2+\nabla\cdot g_3-\operatorname{sign}(z-s)|\nabla|g_1]\,ds,\right. \\
&\qquad\left.\int_{-1}^{0}e^{(z+s)|\nabla|}[g_2+\nabla\cdot g_3-|\nabla|g_1]\,ds-\int_{-1}^{0}e^{-|z-s||\nabla|}[\operatorname{sign}(s-z)(g_2+\nabla\cdot g_3)+|\nabla|g_1]\,ds\right] \\
&+[\mathbf{0}, g_1(z)].
\end{aligned} \tag{3-31}$$

To simplify the notation, we define operators

$$K_1(z,s) := \frac{1}{2}\left[\frac{\nabla}{|\nabla|}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{(s-1)|\nabla|}+\frac{\nabla}{|\nabla|}e^{(z+s)|\nabla|}, -\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{(s-1)|\nabla|}+e^{(z+s)|\nabla|}\right], \tag{3-32}$$

$$K_2(z,s) := \frac{1}{2}\left[\frac{\nabla}{|\nabla|}\frac{e^{-z|\nabla|}-e^{z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{-(s+1)|\nabla|}, -\frac{e^{z|\nabla|}+e^{-z|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}e^{-(s+1)|\nabla|}\right], \tag{3-33}$$

$$K_3(z,s) := \frac{1}{2}\left[\frac{\nabla}{|\nabla|}e^{-|z-s||\nabla|}, e^{-|z-s||\nabla|}\,\mathrm{sign}(s-z)\right]. \tag{3-34}$$

With the above operators, we can rewrite (3-31) as

$$\begin{aligned}\nabla_{x,z}\varphi = &\left[\left[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\right]\nabla\psi, \frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\psi\right] + [\mathbf{0}, g_1(z)] \\ &+ \int_{-1}^{0}[K_1(z,s)-K_2(z,s)-K_3(z,s)](g_2(s)+\nabla\cdot g_3(s))\,ds \\ &+ \int_{-1}^{0} K_3(z,s)|\nabla|\,\mathrm{sign}(z-s)g_1(s) - |\nabla|[K_1(z,s)+K_2(z,s)]g_1(s)\,ds. \end{aligned}\tag{3-35}$$

To make sure that we can conclude the fixed-point-type argument, we need to estimate the operators $K_i(z,s)$ so that the issue of losing derivatives does not exist. More precisely, the following lemma holds.

Lemma 3.2. *For $k,\gamma \geq 0$, we have the estimates*

$$\sum_{i=1}^{3}\left\|\int_{-1}^{0}K_i(z,s)\nabla g(s)\,ds\right\|_{L_z^\infty H^k} + \left\|\int_{-1}^{0}K_i(z,s)g(s)\,ds\right\|_{L_z^\infty H^k} \lesssim \|g(z)\|_{L_z^\infty H^k}, \tag{3-36}$$

$$\sum_{i=1}^{3}\left\|\int_{-1}^{0}K_i(z,s)\nabla g(s)\,ds\right\|_{L_z^\infty \widetilde{W}^\gamma} + \left\|\int_{-1}^{0}[K_1(z,s)-K_2(z,s)-K_3(z,s)]g(s)\,ds\right\|_{L_z^\infty \widetilde{W}^\gamma} \lesssim \|g(z)\|_{L_z^\infty \widetilde{W}^\gamma}. \tag{3-37}$$

Proof. We first prove the desired estimate (3-36). Recall (3-32), (3-33), and (3-34). From Lemma 2.2, we have

$$\sup_{z,s\in[-1,0]}\left\|\mathcal{F}^{-1}\left[\mathcal{F}\big([K_1(z,s)-K_2(z,s)-K_3(z,s)-[0,(1-\mathrm{sign}(s-z))/2]]\big)(\xi)\psi_{k_1}(\xi)\right]\right\|_{L^1} \lesssim 2^{k_{1,-}}, \tag{3-38}$$

$$\sup_{z,s\in[-1,0]}\sum_{i=1}^{3}\left\|\mathcal{F}^{-1}\left[\mathcal{F}\big(K_i(z,s)\big)(\xi)\psi_{k_1}(\xi)\right]\right\|_{L^1} \lesssim 1. \tag{3-39}$$

We will use above estimates for the case when $k_1 < 0$. However, when $k_1 \geq 0$, we cannot use the estimate (3-39) directly to estimate the left-hand side of (3-36); otherwise we lose one derivative. An important observation is that the integration with respect to s actually compensates for the loss.

For any fixed $k \geq 0$, $k \in \mathbb{Z}$, we have the following formulation in terms of the kernel:

$$\int_{-1}^{0} K_i(z,s)\nabla P_k[g(s)]\,ds = \int_{-1}^{0}\int_{\mathbb{R}^2} K_{i;k}(z,s,y)g(s,x-y)\,dy\,ds, \tag{3-40}$$

where

$$K_{i;k}(z,s,y)=\int_{\mathbb{R}^2} e^{iy\cdot\xi}\mathcal{F}(K_1(z,s))(\xi)\psi_k(\xi)\xi\,d\xi. \tag{3-41}$$

After integration by parts in ξ many times, we have the pointwise estimate

$$|K_{i;k}(z,s,y)|\lesssim 2^{3k}(1+2^k|y|+2^k|z-s|)^{-10} \tag{3-42}$$

for $i\in\{1,2,3\}$, which further implies that the kernel $K_{i;k}(z,s,y)$ belongs to $L^1_{s,y}$ for fixed z. Therefore, from (3-39) and (3-42), we have the estimate

$$\begin{aligned}&\big|\text{the left-hand side of (3-36)}\big|^2\\ &\quad\lesssim \sum_{k_1\le 0}\|P_{k_1}[g(z)]\|^2_{L^\infty_z L^2}+\sum_{i=1}^{3}\sum_{k_1\ge 0}2^{2kk_1}\|K_{i;k_1}(z,s)\|^2_{L^1_{s,y}}\|P_{k_1}[g(z)]\|^2_{L^\infty_z L^2}\lesssim \|g(z)\|^2_{L^\infty_z H^k},\end{aligned}$$

hence finishing the proof of (3-36). Very similarly, from (3-38), (3-39), and (3-42), our desired estimate (3-37) follows in the same way. □

From (3-35) and estimates in Lemma 3.2, now it is clear that we can estimate $\nabla_{x,z}\varphi$ by using a fixed-point-type argument.

However, if we do it naively, then the resulting estimate will not tell the difference between $\nabla_x\varphi$ and $\partial_z\varphi$. To capture the fact that $\partial_z\varphi$ actually has two derivatives at the low-frequency part, while $\nabla_x\varphi$ only has one derivative, we decompose $\nabla_{x,z}\varphi$ as

$$\nabla_{x,z}\varphi=\Lambda_1[\nabla_{x,z}\varphi]+\Lambda_{\ge 2}[\nabla_{x,z}\varphi]. \tag{3-43}$$

From (3-35), it is easy to see that $\Lambda_1[\nabla_{x,z}\varphi]$ is given by

$$\Lambda_1[\nabla_{x,z}\varphi]=\bigg[\bigg[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\bigg]\nabla\psi,\ \frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\psi\bigg]. \tag{3-44}$$

From (3-44), it is easy to see that $\Lambda_1[\partial_z\varphi]$ has two derivatives at the low-frequency part. Now, the goal is reduced to estimating $\Lambda_{\ge 2}[\nabla_{x,z}\varphi]$, which is done again by a fixed-point-type argument.

Recall (3-35). To identify the fixed-point-type structure inside $\Lambda_{\ge 2}[\nabla_{x,z}\varphi]$, it is sufficient to reformulate $\Lambda_{\ge 2}[g_i(z)]$, $i\in\{1,2,3\}$.

Recall (3-28) and (3-29). After using the decomposition (3-43) for $\nabla_{x,z}\varphi$ in $g_i(z)$, $i\in\{1,2,3\}$, we have the decomposition of $\Lambda_{\ge 2}[g_i(z)]$, $i\in\{1,2,3\}$,

$$\begin{aligned}\Lambda_{\ge 2}[g_1(z)]&=\tilde h_1\Lambda_{\ge 2}[\partial_z\varphi]-(z+1)^2\tilde h_2\Lambda_{\ge 2}[\partial_z\varphi]+(z+1)\tilde h_3\cdot\Lambda_{\ge 2}[\nabla\varphi]\\ &\quad+\tilde h_1\Lambda_1[\partial_z\varphi]-(z+1)^2\tilde h_2\Lambda_1[\partial_z\varphi]+(z+1)\tilde h_3\cdot\Lambda_1[\nabla\varphi],\end{aligned} \tag{3-45}$$

$$\Lambda_{\ge 2}[g_2(z)]=(z+1)\tilde h_2\Lambda_{\ge 2}[\partial_z\varphi]-\tilde h_3\cdot\Lambda_{\ge 2}[\nabla\varphi]+(z+1)\tilde h_2\Lambda_1[\partial_z\varphi]-\tilde h_3\cdot\Lambda_1[\nabla\varphi], \tag{3-46}$$

$$\Lambda_{\ge 2}[g_3(z)]=(z+1)\tilde h_3\Lambda_{\ge 2}[\partial_z\varphi]+(z+1)\tilde h_3\Lambda_1[\partial_z\varphi]. \tag{3-47}$$

From (3-45), (3-46), and (3-47), now it is easy to see that there exists a fixed-point-type structure for $\Lambda_{\ge 2}[\nabla_{x,z}\varphi]$ in $\Lambda_{\ge 2}[g_i(z)]$, $i\in\{1,2,3\}$. From the standard fixed-point-type argument and the estimates in Lemma 3.2, we obtain basic estimates for $\Lambda_{\geq 2}[\nabla_{x,z}\varphi]$, which further give us more precise estimates for $\nabla_{x,z}\varphi$ from (3-43).

More precisely, our main results in this subsection are summarized as follows,

Lemma 3.3. *For $\gamma', k' \geq 1,\ 0 < \delta \ll 1,\ \alpha \in (0,1]$, if $h \in \widetilde{W}^{\gamma'} \cap H^{k'}$ satisfies the smallness assumption*

$$\|h\|_{\widetilde{W}^{\gamma'}} < \delta, \tag{3-48}$$

then the following L^2-type estimate and L^∞-type estimate for the velocity potential φ hold:

$$\|\nabla_{x,z}\varphi\|_{L^\infty_z H^k} \lesssim \|\nabla\psi\|_{H^k} + \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W}^0}, \tag{3-49}$$

$$\|\nabla_x\varphi\|_{L^\infty_z \widetilde{W}^\gamma} \lesssim \|\nabla\psi\|_{\widetilde{W}^\gamma}, \quad \|\partial_z\varphi\|_{L^\infty_z \widetilde{W}^\gamma} \lesssim \|\nabla\psi\|_{\widehat{W}^{\gamma,\alpha}} + \|h\|_{\widetilde{W}^{\gamma+1}}\|\nabla\psi\|_{\widetilde{W}^\gamma}, \tag{3-50}$$

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z \widetilde{W}^\gamma} \lesssim \|\nabla\psi\|_{\widetilde{W}^\gamma}\|h\|_{\widetilde{W}^{\gamma+1}}, \tag{3-51}$$

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} \lesssim \|h\|_{\widetilde{W}^1}\||\nabla|\psi\|_{H^k} + \|\nabla\psi\|_{\widetilde{W}^0}\|h\|_{H^{k+1}}, \tag{3-52}$$

where $k \leq k'-1$ and $1 \leq \gamma \leq \gamma'-1$. In the above estimates, the range of z for the L^∞_z norm is $[-1,0]$.

Proof. We first estimate $\Lambda_{\geq 2}[\nabla_{x,z}\varphi]$. Recall (3-35), (3-45), (3-46) and (3-47). From estimate (3-37) in Lemma 3.2, we have

$$\begin{aligned}\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z \widetilde{W}^\gamma} &\lesssim \|\Lambda_{\geq 2}[(g_1(z), g_2(z), g_3(z))]\|_{L^\infty_z \widetilde{W}^\gamma} \\ &\lesssim \|h\|_{\widetilde{W}^{\gamma+1}}\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z \widetilde{W}^\gamma} + \|h\|_{\widetilde{W}^{\gamma+1}}\|\nabla\psi\|_{\widetilde{W}^\gamma}.\end{aligned}$$

Hence, by the smallness condition (3-48),

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z \widetilde{W}^\gamma} \lesssim \|h\|_{\widetilde{W}^{\gamma+1}}\|\nabla\psi\|_{\widetilde{W}^\gamma}. \tag{3-53}$$

Very similarly, from estimate (3-36) in Lemma 3.2, we have

$$\begin{aligned}&\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} \\ &\lesssim \|\Lambda_{\geq 2}[(g_1(z), g_2(z), g_3(z))]\|_{L^\infty_z H^k} \\ &\lesssim \|h\|_{\widetilde{W}^1}\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} + \|h\|_{H^{k+1}}\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z \widetilde{W}^0} + \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W}^0} + \|\nabla\psi\|_{H^k}\|h\|_{\widetilde{W}^1} \\ &\lesssim \|h\|_{\widetilde{W}^1}\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} + \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W}^0}(1 + \|h\|_{\widetilde{W}^1}) + \|\nabla\psi\|_{H^k}\|h\|_{\widetilde{W}^1}.\end{aligned}$$

Again, by the smallness assumption (3-48), we conclude

$$\|\Lambda_{\geq 2}[\nabla_{x,z}\varphi]\|_{L^\infty_z H^k} \lesssim \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W}^0} + \|\nabla\psi\|_{H^k}\|h\|_{\widetilde{W}^1}. \tag{3-54}$$

From estimates (3-53) and (3-54) and the explicit formulas of $\Lambda_1[\nabla_{x,z}\varphi]$ in (3-44), we have

$$\begin{gathered}\|\nabla_{x,z}\varphi\|_{L^\infty H^k} \lesssim \|\nabla\psi\|_{H^k} + \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W}^0}, \quad \|\nabla_x\varphi\|_{L^\infty_z \widetilde{W}^\gamma} \lesssim \|\nabla\psi\|_{\widetilde{W}^\gamma}, \\ \|\partial_z\varphi\|_{L^\infty_z \widetilde{W}^\gamma} \lesssim \|\Lambda^2\psi\|_{\widetilde{W}^\gamma} + \|h\|_{\widetilde{W}^{\gamma+1}}\|\nabla\psi\|_{\widetilde{W}^\gamma} \lesssim \|\nabla\psi\|_{\widehat{W}^{\gamma,\alpha}} + \|h\|_{\widetilde{W}^{\gamma+1}}\|\nabla\psi\|_{\widetilde{W}^\gamma}.\end{gathered}$$

□

3D. ***The quadratic terms of the Dirichlet–Neumann operator.*** The content of this subsection is not related to the proof of our main theorem. However, it is crucial to the study of the long-time behavior of the water waves system in the flat-bottom setting.

Generally speaking, the main enemies of the global existence for a 2-dimensional dispersive equation are the quadratic terms. The first step is to know exactly what the enemies are. Surprisingly, as a byproduct of the fixed-point-type formulation (3-35), we can calculate explicitly the quadratic terms of the Dirichlet–Neumann operator.

More precisely, the main result of this subsection is stated as follows:

Lemma 3.4. *In terms of h and ψ, the quadratic terms of the Dirichlet–Neumann operator are*

$$\Lambda_2[G(h)\psi] = -\nabla\cdot(h\nabla\psi) - |\nabla|\tanh|\nabla|(h|\nabla|\tanh|\nabla|\psi). \tag{3-55}$$

Remark 3.5. Before we proceed to prove the above lemma, we compare the main difference between the flat-bottom setting, which is less studied, and the infinite depth setting, which is recently well-studied. In the infinite-depth setting, the quadratic terms of the Dirichlet–Neumann operator are

$$\text{(infinite-depth setting)}\quad \Lambda_2[G(h)\psi] = -\nabla\cdot(h\nabla\psi) - |\nabla|(h|\nabla|\psi). \tag{3-56}$$

If the frequency η of ψ is of size 1 and the frequency $\xi-\eta$ of h is of size 0, from (3-55) and (3-56), it is easy to check the size of the symbol of quadratic terms:

$$\text{(flat-bottom setting)}\quad \xi\cdot\eta - |\xi|\,|\eta|\tanh|\xi|\tanh|\eta| = \frac{4|\xi|^2}{(e^{|\xi|}+e^{-|\xi|})^2} \sim 1,$$

$$\text{(infinite-depth setting)}\quad -|\xi|\,|\eta| + \xi\cdot\eta = 0.$$

That is to say, unlike the infinite-depth setting, we do not have the null structure at the low-frequency part in the flat-bottom setting. As a result, we expect a much stronger nonlinear effect from the quadratic terms, which makes the global regularity problem in the flat-bottom setting more delicate and more difficult than the infinite-depth setting.

Proof of Lemma 3.4. Recall (3-14) and (3-44). We have

$$\Lambda_2[G(h)\psi] = \Lambda_2[\partial_z\varphi|_{z=0}] - h|\nabla|\tanh|\nabla|\psi - \nabla h\cdot\nabla\psi. \tag{3-57}$$

Hence, the problem is reduced to calculating explicitly the quadratic terms of $\partial_z\varphi|_{z=0}$. Recalling (3-35),

$$\begin{aligned}
\Lambda_2[\partial_z\varphi|_{z=0}] &= -\frac{1}{e^{|\nabla|}+e^{-|\nabla|}}\int_{-1}^{0}[e^{(s-1)|\nabla|}-e^{-(s+1)|\nabla|}][\Lambda_2[g_2+\nabla\cdot g_3]]\,ds\\
&\quad+\frac{1}{e^{|\nabla|}+e^{-|\nabla|}}\int_{-1}^{0}[e^{(s-1)|\nabla|}+e^{-(s+1)|\nabla|}]|\nabla|[\Lambda_2[g_1]]\,ds\\
&\quad+\int_{-1}^{0}e^{s|\nabla|}\Lambda_2[g_2+\nabla\cdot g_3-|\nabla|g_1]\,ds+\Lambda_2[g_1(0)]\\
&=\int_{-1}^{0}\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\Lambda_2[g_2+\nabla\cdot g_3]\,ds\\
&\quad-\int_{-1}^{0}\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\Lambda_2[g_1(s)]\,ds+\Lambda_2[g_1(0)].
\end{aligned}\tag{3-58}$$

From (3-25), (3-26), and (3-44), it is easy to derive the equalities

$$
\begin{aligned}
\Lambda_2[g_1(s)] &= 2h\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\psi+(s+1)\nabla h\cdot\nabla\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\psi,\\
\Lambda_2[g_1(0)] &= 2h|\nabla|\tanh|\nabla|\psi+\nabla h\cdot\nabla\psi,\\
\Lambda_2[g_2(s)] &= -\nabla h\cdot\nabla\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\psi,\\
\Lambda_2[g_3(s)] &= (s+1)\nabla h\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\psi.
\end{aligned}
\tag{3-59}
$$

After plugging in the above explicit formula of $\Lambda_2[g_i(z)]$, $i\in\{1,2,3\}$, the goal is to calculate explicitly the symbols of the two integrals in (3-58). Define

$$
Q_1(h,\psi):=\int_{-1}^{0}\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\Lambda_2[g_2+\nabla\cdot g_3]\,ds=Q_{1,1}(h,\psi)+Q_{1,2}(h,\psi),\tag{3-60}
$$

$$
Q_2(h,\psi):=-\int_{-1}^{0}\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\Lambda_2[g_1]\,ds=Q_{2,1}(h,\psi)+Q_{2,2}(h,\psi),\tag{3-61}
$$

where

$$
\begin{aligned}
Q_{1,1}(h,\psi) &= \int_{-1}^{0}\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\left[\nabla\cdot[(s+1)\nabla h\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\psi\right]ds,\\
Q_{1,2}(h,\psi) &= \int_{-1}^{0}\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\left[-\nabla h\cdot\nabla\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\psi\right]ds,\\
Q_{2,1}(h,\psi) &= -\int_{-1}^{0}\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\left[2h\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\psi\right]ds,\\
Q_{2,2}(h,\psi) &= -\int_{-1}^{0}\frac{e^{(s+1)|\nabla|}-e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}|\nabla|\left[(s+1)\nabla h\cdot\nabla\frac{e^{(s+1)|\nabla|}+e^{-(s+1)|\nabla|}}{e^{|\nabla|}+e^{-|\nabla|}}\psi\right]ds.
\end{aligned}
$$

The symbol $q_{i,j}(\xi-\eta,\eta)$ of the bilinear operator $Q_{i,j}(h,\psi)$, $i,j\in\{1,2\}$, is given by

$$
\begin{aligned}
q_{1,1}(\xi-\eta,\eta) &= \frac{-\xi\cdot(\xi-\eta)|\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\int_{-1}^{0}(s+1)[e^{(s+1)|\xi|}+e^{-(s+1)|\xi|}][e^{(s+1)|\eta|}-e^{-(s+1)|\eta|}]\,ds\\
&= \frac{-\xi\cdot(\xi-\eta)|\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\Bigg[\frac{(|\xi|+|\eta|-1)e^{|\xi|+|\eta|}-(-|\xi|-|\eta|-1)e^{-|\xi|-|\eta|}}{(|\xi|+|\eta|)^2}\\
&\qquad+\frac{(|\eta|-|\xi|-1)e^{|\eta|-|\xi|}-(|\xi|-|\eta|-1)e^{|\xi|-|\eta|}}{(|\xi|-|\eta|)^2}\Bigg],
\end{aligned}
\tag{3-62}
$$

$$
\begin{aligned}
q_{1,2}(\xi-\eta,\eta) &= \frac{(\xi-\eta)\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\int_{-1}^{0}[e^{(s+1)|\xi|}+e^{-(s+1)|\xi|}][e^{(s+1)|\eta|}+e^{-(s+1)|\eta|}]\,ds\\
&= \frac{(\xi-\eta)\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\left[\frac{e^{|\xi|+|\eta|}-e^{-|\xi|-|\eta|}}{|\xi|+|\eta|}+\frac{e^{|\xi|-|\eta|}-e^{|\eta|-|\xi|}}{|\xi|-|\eta|}\right],
\end{aligned}
\tag{3-63}
$$

$$\begin{aligned}
q_{2,1}(\xi-\eta,\eta) &= \frac{-2|\xi||\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\int_{-1}^{0}[e^{(s+1)|\xi|}-e^{-(s+1)|\xi|}][e^{(s+1)|\eta|}-e^{-(s+1)|\eta|}]\,ds \\
&= \frac{-2|\xi||\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\left[\frac{e^{|\xi|+|\eta|}-e^{-|\xi|-|\eta|}}{|\xi|+|\eta|}-\frac{e^{|\xi|-|\eta|}-e^{|\eta|-|\xi|}}{|\xi|-|\eta|}\right],
\end{aligned} \tag{3-64}$$

$$\begin{aligned}
q_{2,2}(\xi-\eta,\eta) &= \frac{|\xi|(\xi-\eta)\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\int_{-1}^{0}(s+1)[e^{(s+1)|\xi|}-e^{-(s+1)|\xi|}][e^{(s+1)|\eta|}+e^{-(s+1)|\eta|}]\,ds \\
&= \frac{|\xi|(\xi-\eta)\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\left[\frac{(|\xi|+|\eta|-1)e^{|\xi|+|\eta|}-(-|\xi|-|\eta|-1)e^{-|\xi|-|\eta|}}{(|\xi|+|\eta|)^2}\right. \\
&\qquad\left.-\frac{(|\eta|-|\xi|-1)e^{|\eta|-|\xi|}-(|\xi|-|\eta|-1)e^{|\xi|-|\eta|}}{(|\xi|-|\eta|)^2}\right].
\end{aligned} \tag{3-65}$$

In the above computations, we have used the simple fact

$$\int_{-1}^{0}(s+1)e^{(s+1)a}\,ds = \frac{1+(a-1)e^a}{a^2}.$$

From (3-57)–(3-61), we have

$$\Lambda_2[G(h)\psi] := \widetilde{Q}(h,\psi) = Q_1(h,\psi)+Q_2(h,\psi)+h|\nabla|\tanh|\nabla|\psi.$$

Therefore, the symbol $\tilde{q}(\xi-\eta,\eta)$ of $\widetilde{Q}(h,\psi)$ is given by

$$\tilde{q}(\xi-\eta,\eta) = \sum_{i,j=1,2} q_{i,j}(\xi-\eta,\eta) + \frac{e^{|\eta|}-e^{-|\eta|}}{e^{|\eta|}+e^{-|\eta|}}|\eta|.$$

Although the above formulae look complicated, actually there are cancellations inside. Note that

$$\begin{aligned}
&q_{1,2}(\xi-\eta,\eta)+q_{2,1}(\xi-\eta,\eta)+\frac{e^{|\eta|}-e^{-|\eta|}}{e^{|\eta|}+e^{-|\eta|}}|\eta| \\
&\quad= \frac{\xi\cdot\eta}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\left[\frac{e^{|\xi|+|\eta|}-e^{-|\xi|-|\eta|}}{|\xi|+|\eta|}+\frac{e^{|\xi|-|\eta|}-e^{|\eta|-|\xi|}}{|\xi|-|\eta|}\right] \\
&\qquad -\frac{|\xi||\eta|}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\left[\frac{[e^{|\xi|+|\eta|}-e^{-|\xi|-|\eta|}]}{|\xi|+|\eta|}-\frac{e^{|\xi|-|\eta|}-e^{|\eta|-|\xi|}}{|\xi|-|\eta|}\right],
\end{aligned} \tag{3-66}$$

$$\begin{aligned}
&q_{1,1}(\xi-\eta,\eta)+q_{2,2}(\xi-\eta,\eta) \\
&\quad= \frac{(-|\xi||\eta|+\xi\cdot\eta)}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\,\frac{(|\xi|+|\eta|-1)e^{|\xi|+|\eta|}-(-|\xi|-|\eta|-1)e^{-|\xi|-|\eta|}}{|\xi|+|\eta|} \\
&\qquad -\frac{(\xi||\eta|+\xi\cdot\eta)}{(e^{|\xi|}+e^{-|\xi|})(e^{|\eta|}+e^{-|\eta|})}\,\frac{(|\eta|-|\xi|-1)e^{|\eta|-|\xi|}-(|\xi|-|\eta|-1)e^{|\xi|-|\eta|}}{|\xi|-|\eta|}.
\end{aligned} \tag{3-67}$$

From (3-66) and (3-67), now it is easy to verify

$$\tilde{q}(\xi-\eta,\eta) = \xi\cdot\eta - |\xi||\eta|\tanh|\xi|\tanh|\eta|. \tag{3-68}$$

Hence our desired equality (3-55) holds. □

Lemma 3.6. *For $k_1, k_2, k \in \mathbb{Z}$, the following estimate holds for the symbol of the quadratic terms for the Dirichlet–Neumann operator*:

$$\|\tilde{q}(\xi-\eta,\eta)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{k+k_2}. \tag{3-69}$$

Proof. From (3-68) and the estimate in Lemma 2.2, it is straightforward to derive the above estimate. □

3E. *A fixed-point-type formulation for* $\Lambda_{\geq 3}[\nabla_{x,z}\varphi]$. As in the previous subsection, the content of this subsection is not related to the proof of the main theorem but is related to the future study of the long-time behavior of the water waves system in different settings.

Although, intuitively speaking, the quadratic terms are the leading terms for the dispersive equation (1-8) in 2 dimensions, we also have to control the cubic and higher-order remainder terms to see that their effects are indeed small over time. In this subsection, our goal is to formulate $\Lambda_{\geq 3}[\nabla_{x,z}\varphi]$ into a fixed-point-type formulation, which provides a good way to estimate the cubic and higher-order remainder terms.

Recall the fixed-point-type formulation of $\nabla_{x,z}\varphi$ in (3-35), we truncate it at the cubic-and-higher level and get

$$\begin{aligned}\Lambda_{\geq 3}[\nabla_{x,z}\varphi] = [\mathbf{0}, \Lambda_{\geq 3}[g_1(z)]] + \int_{-1}^{0} [K_1(z,s) - K_2(z,s) - K_3(z,s)]\big(\Lambda_{\geq 3}[g_2(s)] + \nabla\cdot\Lambda_{\geq 3}[g_3(s)]\big)\, ds \\ + \int_{-1}^{0} K_3(z,s)|\nabla|\operatorname{sign}(z-s)\Lambda_{\geq 3}[g_1(s)] - |\nabla|[K_1(z,s)+K_2(z,s)]\Lambda_{\geq 3}[g_1(s)]\, ds.\end{aligned} \tag{3-70}$$

Recall (3-28) and (3-29). Similar to the decomposition we did in (3-45)–(3-47), we can separate $\Lambda_{\geq 3}[g_i(z)]$, $i \in \{1,2,3\}$, into two parts: (i) one of them contains $\Lambda_{\geq 3}[\nabla_{x,z}\varphi]$, which involves the fixed-point structure; (ii) the other part does not depend on $\Lambda_{\geq 3}[\nabla_{x,z}\varphi]$, and hence can be estimated directly.

More precisely, we decompose $\Lambda_{\geq 3}[g_i(z)]$, $i \in \{1,2,3\}$, as follows:

$$\begin{aligned}\Lambda_{\geq 3}[g_1(z)] = \tilde{h}_1\Lambda_{\geq 3}[\partial_z\varphi] - (z+1)^2\tilde{h}_2\Lambda_{\geq 3}[\partial_z\varphi] + (z+1)\tilde{h}_3\cdot\Lambda_{\geq 3}[\nabla\varphi] \\ + \sum_{i=1,2} \Lambda_{\geq 3-i}[\tilde{h}_1]\Lambda_i[\partial_z\varphi] - (z+1)^2\Lambda_{\geq 3-i}[\tilde{h}_2]\Lambda_i[\partial_z\varphi] + (z+1)\Lambda_{\geq 3-i}[\tilde{h}_3]\cdot\Lambda_i[\nabla\varphi],\end{aligned} \tag{3-71}$$

$$\begin{aligned}\Lambda_{\geq 3}[g_2(z)] = (z+1)\tilde{h}_2\Lambda_{\geq 3}[\partial_z\varphi] - \tilde{h}_3\cdot\Lambda_{\geq 3}[\nabla\varphi] \\ + \sum_{i=1,2} (z+1)\Lambda_{\geq 3-i}[\tilde{h}_2]\Lambda_i[\partial_z\varphi] - \Lambda_{\geq 3-i}[\tilde{h}_3]\cdot\Lambda_i[\nabla\varphi],\end{aligned} \tag{3-72}$$

$$\Lambda_{\geq 3}[g_3(z)] = (z+1)\tilde{h}_3\Lambda_{\geq 3}[\partial_z\varphi] + \sum_{i=1,2} (z+1)\Lambda_{\geq 3-i}[\tilde{h}_3]\Lambda_i[\partial_z\varphi]. \tag{3-73}$$

From (3-27), it is easy to verify that

$$\Lambda_{\geq 2}[\tilde{h}_1] = h^2 - (2h+h^2)\tilde{h}_1, \quad \Lambda_{\geq 2}[\tilde{h}_2] = \tilde{h}_2, \quad \Lambda_{\geq 2}[\tilde{h}_3] = -h\tilde{h}_3. \tag{3-74}$$

We can summarize the above decomposition in the following lemma.

Lemma 3.7. *We have*

$$\Lambda_{\geq 3}[\nabla_{x,z}\varphi(z)] = \sum_{i=1}^{3} C_z^i(h,\psi,\tilde{h}_i) + h\widetilde{C}_z^i(h,\psi,\tilde{h}_i) + T_z^i(\tilde{h}_i,\Lambda_{\geq 3}[\nabla_{x,z}\varphi]),$$

where C_z^i and $\widetilde{C}_z^i$ are some trilinear operators and T_z^i is some bilinear operator. Assume that the corresponding symbols are $c_z^i(\cdot,\cdot,\cdot)$, $\tilde{c}_z^i(\cdot,\cdot,\cdot)$, and $t_z^i(\cdot,\cdot)$ respectively. Then we have the estimates

$$\sup_{z\in[-1,0]} \sum_{i=1}^{3} \|c_z^i(\xi_1,\xi_2,\xi_3)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} + \|\tilde{c}_z(\xi_1,\xi_2,\xi_3)\|_{\mathcal{S}^\infty_{k,k_1,k_2,k_3}} \lesssim 2^{3\max\{k_1,k_2,k_3\}_+}, \tag{3-75}$$

$$\sup_{z\in[-1,0]} \sum_{i=1}^{3} \|t_z^i(\xi_1,\xi_2)\|_{\mathcal{S}^\infty_{k,k_1,k_2}} \lesssim 2^{3\max\{k_1,k_2\}_+}. \tag{3-76}$$

Proof. The proof is straightforward. From Lemma 2.2, our desired estimates (3-75) and (3-76) can be derived by checking the symbol of each term inside the equations (3-70), (3-71), (3-72) and (3-73). Note that there are at most three derivatives in total. □

4. Paralinearization and symmetrization of the system

Since the gravity waves system (1-7) is quasilinear and lacks symmetric structures inside, we cannot use this system directly to do the energy estimate because of the difficulty of losing one derivative.

To identify the hidden symmetries inside the gravity waves system (1-7) and get around the issue of losing derivatives, we use the method of paralinearization and symmetrization which was introduced and studied in [Alazard and Métivier 2009; Alazard, Burq and Zuily 2011; 2014a; 2014b]. Interested readers may refer to those works for more details. Here, we only briefly discuss this method to help readers understand how this method works and get a sense of what they will read about in this section.

For a fully nonlinear term, it is very hard to tell which part actually loses derivatives and which part does not lose derivatives, which is clearly very important to get around the issue of losing derivatives. With the help of the paralinearization process, we can identify the part that actually loses derivatives, which is the real issue. In Section 4A, we will do the paralinearization process for the nonlinearity of the equation satisfied by the height h, which is the Dirichlet–Neumann operator. In Section 4B, we will do the paralinearization process for the nonlinearity of the equation satisfied by the velocity potential ψ.

Knowing which part loses derivatives is certainly very helpful, but it does not imply that we can get around the issue of losing derivatives because the original system lacks good symmetric structures. With the help of the symmetrization process, in Section 4C, we identify good substitution variables so that the system of equations satisfied by the good substitution variables has the requisite symmetries. Moreover, the good substitution variables have size of energy comparable to that of the original variables. Therefore, instead of doing the energy estimate for the original variables, we do an energy estimate for the good substitution variables.

4A. ***Paralinearization of the Dirichlet–Neumann operator.*** In this subsection, our main goal is to identify which part of the Dirichlet–Neumann operator actually loses derivatives by using the paralinearization method. In the meantime, we also pay attention to the low-frequency part for the purpose of proving our new energy estimate (1-12).

More precisely, the goal of this subsection is to prove the following proposition.

Proposition 4.1. *Let $k \geq 6$, $\alpha \in (0,1]$. Assume that $(h, \Lambda\psi) \in H^k$ and h satisfies the smallness condition (3-48). Then we have*

$$G(h)\psi = T_\lambda(\psi - T_B h) - T_V \cdot \nabla h + F(h)\psi = \Lambda^2(\psi - T_B h) + T_{\lambda - |\xi|}(\psi - T_B h) - T_V \cdot \nabla h + \widetilde{F}(h)\psi, \quad (4\text{-}1)$$

where

$$\lambda := \sqrt{(1+|\nabla h|^2)|\xi|^2 - (\nabla h \cdot \xi)^2}, \quad (4\text{-}2)$$

$$B \overset{\text{abbr}}{=} B(h)\psi = \frac{G(h)\psi + \nabla h \cdot \nabla \psi}{1+|\nabla h|^2}, \qquad V \overset{\text{abbr}}{=} V(h)\psi = \nabla\psi - B\nabla h. \quad (4\text{-}3)$$

The good remainder terms $F(h)\psi$ and $\widetilde{F}(h)\psi$ do not lose derivatives and satisfy the estimate

$$\|\Lambda_{\geq 2}[F(h)\psi]\|_{H^k} + \|\widetilde{F}(h)\psi\|_{H^k} \lesssim_k \big[\|(h,\Lambda\psi)\|_{\widehat{W}^{4,\alpha}} + \|(h,\Lambda\psi)\|^2_{\widehat{W}^4}\big]\big[\|h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big]. \quad (4\text{-}4)$$

Remark 4.2. We remark that, unlike the infinite-depth setting, the good remainder term $F(h)\psi$ in (4-1) actually contains a linear term, which is $[\tanh(|\nabla|) - 1]|\nabla|\psi \in H^\infty$.

For simplicity, we define the following equivalence relation. For two well-defined nonlinearities A and B, which are nonlinear with respect to h and ψ, we say

$$A \approx B \iff A - B \text{ is a good error term in the sense of (4-5)},$$

$$\|\text{good error term}\|_{H^k} \lesssim_k \big[\|(h,\Lambda\psi)\|_{\widehat{W}^{4,\alpha}} + \|(h,\Lambda\psi)\|^2_{\widehat{W}^4}\big]\big[\|h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big], \quad \alpha \in (0,1],\ k \geq 0. \quad (4\text{-}5)$$

Recall (3-15). Note that, essentially speaking, the only fully nonlinear term inside the Dirichlet–Neumann operator $G(h)\psi$ is $\partial_w \Phi|_{w=0}$. So the task is reduced to identifying which part of $\partial_w\Phi$ actually loses a derivative.

To this end, we will show that there exists a pseudodifferential operator $A(x,\xi)$ such that $\partial_w\Phi - T_A(\Phi - T_{\partial_w\Phi}h)$ actually does not lose derivatives, where $\Phi - T_{\partial_w\Phi}h$ is the so-called good unknown variable. This step is very nontrivial and technical. Unfortunately, to the best of our knowledge, there is no physical intuitive explanation available. It relies heavily on the study of good structures for the Laplace equation (3-11). We do this step in detail in the following subsubsection.

4A1. *Paralinearization of the Laplace equation* (3-11). Recall due to (3-11) and the fact that $g'(\cdot, w) = 0$ when $w \in [-\frac{1}{4}, 0]$, we have

$$\begin{gathered}[\Delta_x + a'\partial_w^2 + b' \cdot \nabla\partial_w + c'\partial_w]\Phi = 0, \\ a' = 1 + |\nabla h|^2 \approx 1 + 2T_{\nabla h} \cdot \nabla h, \quad b' = -2\nabla h, \quad c' = -\Delta h.\end{gathered} \quad (4\text{-}6)$$

We remark that w is also restricted inside $[-\frac{1}{4}, 0]$ in the rest of this paper.

Before proceeding to the paralinearization process for (4-6), we need some necessary estimates of Φ. Essentially speaking, under a certain smallness condition, the size of Φ is comparable to φ. Note that we already have necessary estimates of φ; see Lemma 3.3. More precisely, from the definition of Φ (see Section 3B) and estimates of φ in Lemma 3.3, the following lemma holds.

Lemma 4.3. *Under the smallness estimate (3-1), we have the following estimates for $k \geq 1, \gamma \leq 3$:*

$$\sup_{w\in[-1/4,0]} \|\nabla_{x,w}\Phi\|_{H^k} \lesssim \|\nabla\psi\|_{H^k} + \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W}^1}, \tag{4-7}$$

$$\sup_{w\in[-1/4,0]} \|\nabla_x\Phi\|_{\widetilde{W}^\gamma} \lesssim \|\nabla\psi\|_{\widetilde{W}^\gamma}, \qquad \sup_{w\in[-1/4,0]} \|\partial_w\Phi\|_{\widetilde{W}^\gamma} \lesssim \|\nabla\psi\|_{\widehat{W}^{\gamma,\alpha}} + \|h\|_{\widetilde{W}^{\gamma+1}}\|\nabla\psi\|_{\widetilde{W}^\gamma}, \tag{4-8}$$

$$\sup_{w\in[-1/4,0]} \|\Lambda_{\geq 2}[\nabla_{x,w}\Phi]\|_{L^2} \lesssim \big[\|(h,\Lambda\psi)\|_{\widehat{W}^{2,\alpha}} + \|(h,\Lambda\psi)\|^2_{\widehat{W}^2}\big]\big[\|h\|_{H^1} + \|\nabla\psi\|_{L^2}\big]. \tag{4-9}$$

Proof. This is postponed to the end of this subsection for the purpose of improving the presentation. □

After paralinearizing (4-6), we have

$$P\Phi + 2T_{\partial_w^2\Phi}T_{\nabla h}\cdot\nabla h - 2T_{\nabla\partial_w\Phi}\cdot\nabla h - T_{\partial_w\Phi}\Delta h \approx 0, \tag{4-10}$$

where

$$P := [\Delta + T_{a'}\partial_w^2 + T_{b'}\cdot\nabla\partial_w + T_{c'}\partial_w]. \tag{4-11}$$

To see why the equivalence relation (4-10) holds, we mention that we can always put ∇h in L^∞ and put $\partial_w\Phi$ and $\partial_w^2\Phi$ in L^2.

Define $W := \Phi - T_{\partial_w\Phi}h$. As in [Alazard, Burq and Zuily 2011], we claim that $PW \approx 0$ when $w \in \big[-\frac{1}{4}, 0\big]$. After using (2-7) and the composition in Lemma 2.6, the following equivalence relations hold:

$$\begin{aligned}
PW \approx 0 \iff & P[T_{\partial_w\Phi}h] + 2T_{\partial_w^2\Phi}T_{\nabla h}\cdot\nabla h - 2T_{\nabla\partial_w\Phi}\cdot\nabla h - T_{\partial_w\Phi}\Delta h \approx 0 && (4\text{-}12)\\
\iff & \big[T_{a'}T_{\partial_w^3\Phi}h + T_{b'}\cdot\nabla T_{\partial_w^2\Phi}h + T_{c'}T_{\partial_w^2\Phi}h + 2T_{\nabla\partial_w\Phi}\cdot\nabla h\big] \\
& \qquad + \big[2T_{\partial_w^2\Phi}T_{\nabla h}\cdot\nabla h - 2T_{\nabla\partial_w\Phi}\cdot\nabla h\big] \approx 0 && (4\text{-}13)\\
\iff & \big[T_{b'}\cdot T_{\partial_w^2\Phi}\nabla h + 2T_{\nabla\partial_w\Phi}\cdot\nabla h\big] + \big[2T_{\partial_w^2\Phi}T_{\nabla h}\cdot\nabla h - 2T_{\nabla\partial_w\Phi}\cdot\nabla h\big] \approx 0 && (4\text{-}14)\\
\iff & T_{[b'\partial_w^2\Phi + 2\partial_w^2\Phi\nabla h]}\cdot\nabla h \approx 0 && (4\text{-}15)\\
\iff & 0 \approx 0, \quad \text{as } b' = -2\nabla h. && (4\text{-}16)
\end{aligned}$$

Obviously, (4-16) holds. Hence, we can reverse the directions of all arrows back to conclude $PW \approx 0$.

Although tedious, it is not difficult to verify that all $\approx$ equivalence relations hold in all the above equations. As a typical example, we give a detailed proof of (4-13) here. To prove (4-13), it is sufficient to estimate $T_{\Delta\partial_w\Phi}h$. From the estimate (4-8) in Lemma 4.3, we have

$$\sup_{w\in[-1/4,0]} \|T_{\Delta\partial_w\Phi}h\|_{H^k} \lesssim \sup_{w\in[-1/4,0]} \|h\|_{H^k}\|\partial_w\Phi\|_{\widetilde{W}^2} \lesssim [\|\Lambda\psi\|_{\widehat{W}^{4,\alpha}} + \|(h,\Lambda\psi)\|^2_{\widehat{W}^4}]\|h\|_{H^k}.$$

Hence, the equivalence relation (4-13) holds. All other equivalence relations can be obtained very similarly.

The next step is to decompose the equation $PW \approx 0$ into a forward evolution equation and a backward evolution equation. As a result, from Lemma 4.6, we can show that $\partial_w W - T_A W$ actually does not lose derivatives. Note that $\partial_w W - T_A W \approx \partial_w \Phi - T_A(\Phi - T_{\partial_w \Phi} h)$. Hence, our desired result is obtained.

More precisely, we have the following lemma.

Lemma 4.4. *There exist two symbols $a = a(x,\xi)$ and $A(x,\xi)$ with*

$$a = a^{(1)} + a^{(0)}, \quad A = A^{(1)} + A^{(0)},$$

where

$$\begin{aligned}
a^{(1)}(x,\xi) &= \frac{1}{1+|\nabla h|^2}\left(i\nabla h\cdot\xi - \sqrt{(1+|\nabla h|^2)|\xi|^2 - (\nabla h\cdot\xi)^2}\right),\\
A^{(1)}(x,\xi) &= \frac{1}{1+|\nabla h|^2}\left(i\nabla h\cdot\xi + \sqrt{(1+|\nabla h|^2)|\xi|^2 - (\nabla h\cdot\xi)^2}\right),\\
a^{(0)}(x,\xi) &= \frac{1}{A^{(1)} - a^{(1)}}\left(i\partial_\xi a^{(1)}\cdot\partial_x A^{(1)} - \frac{\Delta h\ a^{(1)}}{1+|\nabla h|^2}\right),\\
A^{(0)}(x,\xi) &= \frac{1}{a^{(1)} - A^{(1)}}\left(i\partial_\xi a^{(1)}\cdot\partial_x A^{(1)} - \frac{\Delta h\ A^{(1)}}{1+|\nabla h|^2}\right),
\end{aligned} \tag{4-17}$$

such that

$$P = T_{a'}(\partial_w - T_a)(\partial_w - T_A) + R_0 + R_1\partial_w, \quad a'(a+A) = ib'\cdot\xi + c', \tag{4-18}$$

$$a'\Big[a^{(1)}A^{(1)} + \frac{1}{i}\partial_\xi a^{(1)}\cdot\partial_x A^{(1)} + a^{(1)}A^{(0)} + a^{(0)}A^{(1)}\Big] = a'(a\,\sharp\,A) = -|\xi|^2, \tag{4-19}$$

where

$$R_0 = T_{a'}T_aT_A - \Delta, \quad R_1 = -T_{a'}T_{a+A} + T_{b'}\cdot\nabla + T_{c'}. \tag{4-20}$$

Moreover, the following estimate holds for good error operators R_0 and R_1:

$$\|R_0 f\|_{H^k} + \|R_1 f\|_{H^{k+1}} \lesssim \|\nabla h\|_{\widetilde{W}^3}\|f\|_{H^k}. \tag{4-21}$$

Proof. Most parts of above lemma are cited directly from [Alazard, Burq and Zuily 2011, Lemma 3.18]. Given the a priori decomposition (4-18), from (4-11), we can calculate explicitly the formulae of R_0 and R_1, which are given in (4-20). Note that as a' doesn't depend on ξ, from (4-18)–(4-20), we have the identities

$$\begin{aligned}
R_1 &= -T_{a'}T_{a+A} + T_{a'(a+A)} = -T_{a'}T_{a+A} + T_{(a'\sharp(a+A))},\\
R_0 &= T_{a'}[T_aT_A - T_{a\sharp A}] + T_{a'}T_{a\sharp A} - T_{a'(a\sharp A)} = T_{a'}[T_aT_A - T_{a\sharp A}] + T_{(a'-1)}T_{a\sharp A} - T_{(a'-1)\sharp(a\sharp A)}.
\end{aligned}$$

From explicit formulations of a', a and A, we can see that $a', a'-1 \in \Gamma^0_2(\mathbb{R}^2)$ $a, A, a+A \in \Gamma^1_2(\mathbb{R}^2)$ and $a\,\sharp\,A \in \Gamma^2_2(\mathbb{R}^2)$. The following estimates on their symbolic bounds hold:

$$M^2_2(a\,\sharp\,A) + M^0_2(a') \lesssim 1, \quad M^1_2(a) + M^1_2(A) + M^1_2(a+A) \lesssim \|\nabla h\|_{\widetilde{W}^3}, \quad M^0_2(a'-1) \lesssim \|\nabla h\|^2_{\widetilde{W}^3}.$$

From estimate (2-8) in Lemma 2.6, we have

$$\|R_1 f\|_{H^{k+1}} \lesssim M_2^0(a')M_2^1(a+A)\|f\|_{H^k} \lesssim \|\nabla h\|_{\widetilde{W}^3}\|f\|_{H^k},$$
$$\|R_0 f\|_{H^k} \lesssim \big[M_2^0(a')M_2^1(a)M_2^1(A) + M_2^0(a'-1)M_2^2(a\sharp A)\big]\|f\|_{H^k} \lesssim \|\nabla h\|^2_{\widetilde{W}^3}\|f\|_{H^k}.$$

Hence finishing the proof of (4-21). □

In the following lemma, we prove that $\partial_w W - T_A W$ doesn't lose a derivative.

Lemma 4.5. *Let $A(x,\xi)$ be as defined in Lemma 4.4. For $k \geq 1$, we have the estimate*

$$\big\|\Lambda_{\geq 2}[(\partial_w W - T_A W)]|_{w=0}\big\|_{H^k} \lesssim_k \big[\|(h,\Lambda\psi)\|_{\widehat{W}^{4,\alpha}} + \|(h,\Lambda\psi)\|^2_{\widehat{W}^4}\big]\big[\|h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big]. \quad (4\text{-}22)$$

Proof. Recalling the decomposition of the operator P in (4-18) and the fact that $PW \approx 0$, we have

$$T_{a'}(\partial_w - T_a)(\partial_w - T_A)W \approx -R_0 W - R_1\partial_w W,$$

which further gives us

$$(\partial_w - T_a)(\partial_w - T_A)W \approx \tilde{g},$$

where

$$\tilde{g} = T_{a'^{-1}}[-R_0 W - R_1\partial_w W] + [I - T_{a'^{-1}}T_{a'}](\partial_w - T_a)(\partial_w - T_A)W,$$
$$= T_{a'^{-1}}[-R_0 W - R_1\partial_w W] + \big[I - T_1 + T_{(a'-1)(a'^{-1}-1)} - T_{(a'-1)}T_{(a'^{-1}-1)}\big](\partial_w - T_a)(\partial_w - T_A)W.$$

From the estimate (4-21) in Lemma 4.4, and the fact that $T_{(a'-1)(a'^{-1}-1)} - T_{(a'-1)}T_{(a'^{-1}-1)}$ is of order -2, we have

$$\sup_{w\in[-1/4,0]}\|\Lambda_{\geq 2}[\tilde{g}(w)]\|_{H^k} \lesssim \|\nabla h\|_{\widetilde{W}^3}\big[\|P_{\geq 1/2}[W]\|_{H^k} + \|\partial_w W\|_{H^{k-1}}\big]$$
$$\lesssim \big[\|(h,\Lambda\psi)\|_{\widehat{W}^{4,\alpha}} + \|(h,\Lambda\psi)\|^2_{\widehat{W}^4}\big]\big[\|h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big].$$

Note that $[\partial_w^2 + \Delta]\Lambda_1[W(w)] = [\partial_w^2 + \Delta]\Lambda_1[\Phi(w)] = 0$ when $w \in \big[-\frac{1}{4}, 0\big]$; see (4-6). It is easy to see that we have the equivalence relation

$$(\partial_w - T_a)\Lambda_{\geq 2}[(\partial_w - T_A)W] + \Lambda_{\geq 2}\big[(\partial_w - T_a)\Lambda_1[(\partial_w - T_A)W]\big] \approx \Lambda_{\geq 2}[\tilde{g}]. \quad (4\text{-}23)$$

Note that

$$\Lambda_1[\partial_w\Phi] = \Lambda_1[\partial_z\varphi(w/(1+h))] = \frac{e^{(w+1)|\nabla|} - e^{-(w+1)|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}}|\nabla|\psi,$$
$$\Lambda_1[\Phi] = \Lambda_1[\varphi(w/(1+h))] = \frac{e^{-(w+1)|\nabla|} + e^{(w+1)|\nabla|}}{e^{-|\nabla|} + e^{|\nabla|}}\psi.$$

It is easy to verify that

$$\Lambda_1[(\partial_w - T_A)W] = \Lambda_1[\partial_w\Phi - T_{|\xi|}\Phi] \in H^\infty,$$
$$\big\|\Lambda_{\geq 2}\big[(\partial_w - T_a)\Lambda_1[(\partial_w - T_A)W]\big]\big\|_{H^k} \lesssim \|\nabla h\|_{\widetilde{W}^3}\|\nabla\psi\|_{H^{k-1}}.$$

Therefore, from (4-23), we have

$$(\partial_w - T_a)\Lambda_{\geq 2}[(\partial_w - T_A)W] \approx \Lambda_{\geq 2}[\tilde{g}].$$

We reformulate the above equation as

$$(\partial_w + T_{-a})\Lambda_{\geq 2}[(\partial_w - T_A)W] = \Lambda_{\geq 2}[\tilde{g}] + \hat{g},$$

where

$$\hat{g} = \text{error term from} \approx \text{equivalence relation.}$$

Recalling the precise formula of a in Lemma 4.4, we know that $-a$ satisfies the assumption in Lemma 4.6. We can first choose a series of constants $\{\tau_i\}_{i=1}^{k}$ such that $\tau_{i+1} = 4\tau_i$ and $\tau_k \geq -\frac{1}{5}$ and then keep iterating the estimate (4-25). As a result, we have the estimate

$$\begin{aligned}\big\|\Lambda_{\geq 2}[(\partial_w - T_A)W|_{w=0}]\big\|_{H^k} &\lesssim_k \sup_{w\in[-1/5,0]} \big[\|\Lambda_{\geq 2}[(\partial_w - T_A)W(w,\cdot)]\|_{L^2} + \|\tilde{g}(w)\|_{H^{k-1+\epsilon}} + \|\hat{g}(w)\|_{H^{k-1+\epsilon}}\big] \\ &\lesssim_k \big[\|(h,\Lambda\psi)\|_{\widehat{W}^{4,\alpha}} + \|(h,\Lambda\psi)\|^2_{\widehat{W}^4}\big]\big[\|h\|_{H^k} + \|\nabla\psi\|_{H^{k-1}}\big], \end{aligned} \tag{4-24}$$

which concludes the proof. □

Lemma 4.6. *Let $a \in \Gamma_2^1(\mathbb{R}^2)$ and suppose it satisfies the assumption* $\mathrm{Re}[a(x,\xi)] \geq c|\xi|$ *for some positive constant c. If u solves the equation*

$$(\partial_w + T_a)u(w,\cdot) = g(w,\cdot),$$

then we know that the following estimate holds for any fixed and sufficiently small constant τ, and arbitrarily small constant $\epsilon > 0$:

$$\sup_{w\in[\tau,0]} \|u(w)\|_{H^k} \lesssim M_2^1(a)\frac{1+|\tau|}{|\tau|}\Big[\sup_{z\in[4\tau,0]} \|u(w)\|_{H^{k-2(1-\epsilon)}} + \sup_{z\in[4\tau,0]} \|g(z)\|_{H^{\mu-(1-\epsilon)}}\Big]. \tag{4-25}$$

Proof. A detailed proof can be found in [Alazard and Delort 2015] by combining Lemma 2.2.7 and the proof of Lemma 2.2.8. □

4A2. *Paralinearization of the Dirichlet–Neumann operator.* In this subsubsection, we use the result we obtained in the last subsubsection, which is the fact that $\partial_w W - T_A W$ doesn't lose derivatives, to identify which part of the Dirichlet–Neumann operator loses derivatives.

Recall (3-15). For the reader's convenience, we rewrite it as

$$G(h)\psi = \big((1+|\nabla h|^2)\partial_w\Phi - \nabla h\cdot\nabla\Phi\big)\big|_{w=0}.$$

Define

$$\underline{V} := \nabla\Phi - \partial_w\Phi\nabla h, \quad \underline{V}\big|_{w=0} = V.$$

Now we let w be inside the range $\big[-\frac{1}{4}, 0\big]$ instead of being restricted to the boundary. By using (2-7) and Lemma 2.6, we have the paralinearization result

$$\begin{aligned} +|\nabla h|^2)\partial_w\Phi - \nabla h\cdot\nabla\Phi &\approx T_{1+|\nabla h|^2}\partial_w\Phi + 2T_{\partial_w\Phi}T_{\nabla h}\cdot\nabla h - T_{\nabla h}\cdot\nabla\Phi - T_{\nabla\Phi}\cdot\nabla h \\ &\approx T_{1+|\nabla h|^2}\partial_w\Phi + T_{2\nabla h\partial_w\Phi - \nabla\Phi}\cdot\nabla h - T_{\nabla h}\cdot\nabla\Phi \end{aligned}$$

$$
\begin{aligned}
&= T_{1+|\nabla h|^2}\partial_w(W+T_{\partial_w\Phi}h)+T_{2\nabla h\partial_w\Phi-\nabla\Phi}\cdot\nabla h-T_{\nabla h}\cdot\nabla(W+T_{\partial_w\Phi}h)\\
&\approx T_{1+|\nabla h|^2}\partial_w W+T_{2\nabla h\partial_w\Phi-\nabla\Phi}\cdot\nabla h-T_{\nabla h}\cdot\nabla W-T_{\nabla h\partial_w\Phi}\cdot\nabla h\\
&= T_{1+|\nabla h|^2}\partial_w W-T_{\nabla h}\cdot\nabla W-T_{\underline{V}}\cdot\nabla h\\
&= T_{1+|\nabla h|^2}[\partial_w W-T_A W]+[T_{1+|\nabla h|^2}T_A-T_{\nabla h}\cdot\nabla]W-T_{\underline{V}}\cdot\nabla h\\
&= T_\lambda W-T_{\underline{V}}\cdot\nabla h+T_{1+|\nabla h|^2}[\partial_w W-T_A W]+R_2 W,
\end{aligned}
\tag{4-26}
$$

where

$$R_2 := [T_{1+|\nabla h|^2}T_A-T_{(1+|\nabla h|^2)A^{(1)}}],$$

and where λ is given in (4-2). In (4-26), we used the identity

$$\lambda=(1+|\nabla h|^2)A^{(1)}-i\xi\cdot\nabla h,$$

where $A^{(1)}$ is given in (4-17). Note that

$$R_2=T_{(1+|\nabla h|^2)}T_{A^{(0)}}+\big[T_{(1+|\nabla h|^2)}T_{A^{(1)}}-T_{(1+|\nabla h|^2)A^{(1)}}\big]=T_{a'}T_{A^0}+\big[T_{(a'-1)}T_{A^{(1)}}-T_{(a'-1)\sharp A^{(1)}}\big].$$

Now, it is easy to see that R_2 is an operator of order 0 with an upper bound given by $\|\nabla h\|_{\widetilde{W}^3}$. Hence, we have the estimate

$$
\begin{aligned}
\|R_2 W|_{w=0}\|_{H^k} &= \big\|R_2[P_{\geq 1/2}[W]|_{w=0}]\big\|_{H^k}\\
&\lesssim \|\nabla h\|_{\widetilde{W}^3}\big\|P_{\geq 1/2}[\psi-T_{B(h)\psi}h]\big\|_{H^k}\\
&\lesssim \big[\|(h,\Lambda\psi)\|_{\widehat{W}^{4,\alpha}}+\|(h,\Lambda\psi)\|^2_{\widehat{W}^4}\big]\big[\|h\|_{H^k}+\|\nabla\psi\|_{H^{k-1}}\big].
\end{aligned}
\tag{4-27}
$$

Combining (4-26), (4-27), and the estimate (4-22) in Lemma 4.5, it's easy to see that Proposition 4.1 holds.

Now, we give the postponed proof of Lemma 4.3.

Proof of Lemma 4.3. For fixed $w\in\big[-\frac{1}{4},0\big]$, it's easy to see $\Phi(w)=\varphi(w/(1+h(x)))$ and that we have the identity

$$\nabla_{x,w}\Phi=\nabla_{x,z}\varphi(w/(1+h))+\bigg[\frac{-w\nabla h\partial_z\varphi(w/(1+h))}{(1+h)^2},\frac{-h\partial_z\varphi(w/(1+h))}{1+h}\bigg]. \tag{4-28}$$

Therefore, we know that the leading term of $\nabla_{x,w}\Phi(w)$ is $\nabla_{x,z}\varphi(w/(1+h))$. Under the smallness estimate (3-1), to estimate $\nabla_{x,w}\Phi$, it is sufficient to estimate $\nabla_{x,z}\varphi(w/(1+h))$.

Recall that according to the fixed-point-type formulation of $\nabla_{x,z}\varphi$ in (3-35), we study the linear term on the right-hand side of (3-35) first. Define

$$
\begin{aligned}
p_\pm(w,x,\xi) &:= \pm\frac{e^{-(w+1)|\xi|}(e^{h(x)w|\xi|/(1+h(x))}-1)}{e^{-|\xi|}+e^{|\xi|}}+\frac{e^{(w+1)|\xi|}(e^{-h(x)w|\xi|/(1+h(x))}-1)}{e^{-|\xi|}+e^{|\xi|}}\\
&= \sum_{n\geq 1}\frac{1}{n!}\bigg[\pm\frac{e^{-(w+1)|\xi|}}{e^{|\xi|}+e^{-|\xi|}}(w|\xi|)^n\bigg(\frac{h(x)}{1+h(x)}\bigg)^n+\frac{e^{|\xi|}}{e^{|\xi|}+e^{-|\xi|}}e^{w|\xi|}(w|\xi|)^n\bigg(\frac{h(x)}{1+h(x)}\bigg)^n\bigg]\\
&= \sum_{n\geq 1}\frac{1}{n!}\big[\pm f_n^1(w,\xi)g_n(x)+f_n^2(w,\xi)g_n(x)\big],
\end{aligned}
$$

where

$$f_n^1(w,\xi) := \frac{e^{-(w+1)|\xi|}}{e^{|\xi|}+e^{-|\xi|}}(w|\xi|)^n, \quad f_n^2(w,\xi) := \frac{e^{|\xi|}}{e^{|\xi|}+e^{-|\xi|}}e^{w|\xi|}(w|\xi|)^n, \quad g_n(x) := \left(\frac{h(x)}{1+h(x)}\right)^n.$$

It is easy to verify that

$$\Lambda_1[\nabla_x\varphi](x, w/(1+h(x))) = \Lambda_1[\nabla_x\varphi](x,w) + P_1\psi(x,w),$$
$$\Lambda_1[\partial_z\varphi](x, w/(1+h(x))) = \Lambda_1[\partial_z\varphi](x,w) + P_2\psi(x,w),$$

where

$$P_1\psi(x,w) := \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\hat{\psi}(\xi) i\xi p_+(w,x,\xi)\,d\xi, \quad P_2\psi(x,w) := \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\hat{\psi}(\xi)|\xi| p_-(w,x,\xi)\,d\xi.$$

We will show that, under the smallness estimate (3-1), the size of $\Lambda_1[\nabla_{x,z}\varphi](x, w/(1+h(x)))$ is almost same as the size of $\Lambda_1[\nabla_{x,z}\varphi](x,w)$. For $k \in \mathbb{Z}$, we define

$$\begin{aligned} p_{\pm,k}(w,x,\xi) &:= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{ix\cdot\sigma}\mathcal{F}_x(p)(w,\sigma,\xi)\psi_k(\sigma)\,d\sigma = \frac{1}{4\pi^2}\sum_{n\geq 1}\frac{1}{n!}\big[\pm f_n^1(w,\xi) + f_n^2(w,\xi)\big] P_k[g_n](x), \\ P_{1,k}\psi(x,w) &:= \int_{\mathbb{R}^2} e^{ix\cdot\xi}\hat{\psi}(\xi) i\xi p_{+,k}(w,x,\xi)\,d\xi, \\ P_{2,k}\psi(x,w) &:= \int_{\mathbb{R}^2} e^{ix\cdot\xi}\hat{\psi}(\xi)|\xi| p_{-,k}(w,x,\xi)\,d\xi. \end{aligned} \tag{4-29}$$

Since $P_2\psi$ can be treated in the same way as $P_1\psi$, we only estimate $P_1\psi$ in detail here. We have the decomposition

$$P_1\psi = \sum_{k_1,k_2\in\mathbb{Z}} P_{1,k_1}\psi_{k_2} = \mathrm{I} + \mathrm{II}, \quad \mathrm{I} = \sum_{k_2\leq k_1} P_{1,k_2}\psi_{k_1}, \quad \mathrm{II} = \sum_{k_1\leq k_2} P_{1,k_2}\psi_{k_1}.$$

From the bilinear estimate of L^2-L^∞ type (2-3) in Lemma 2.1, it is easy to see that we have the following estimates

$$\|\mathrm{I}\|_{H^k} \lesssim \left(\sum_{k_1} 2^{2k_1+2kk_{1,+}}\|P_{k_1}\psi\|_{L^2}^2\left(\sum_{n\geq 1}\frac{1}{n!}\|P_{\leq k_1}g_n\|_{L^\infty}\right)^2\right)^{\frac{1}{2}} \lesssim \|\nabla\psi\|_{H^k}\|h\|_{\widetilde{W}^1},$$
$$\|\mathrm{II}\|_{H^s} \lesssim \sum_{n\geq 1}\frac{1}{n!}\sum_{k_2\leq k_1} 2^{k_2+kk_{1,+}}\|P_{k_1}g_n\|_{L^2}\|P_{k_2}\psi\|_{L^\infty} \lesssim \|h\|_{H^{s+1}}\|\nabla\psi\|_{\widetilde{W}^1},$$
$$\|\mathrm{I}\|_{\widetilde{W}^\gamma} \lesssim \|h\|_{\widetilde{W}^\gamma}\|\nabla\psi\|_{\widetilde{W}^\gamma}, \quad \|\mathrm{II}\|_{\widetilde{W}^\gamma} \lesssim \|h\|_{\widetilde{W}^\gamma}\|\nabla\psi\|_{\widetilde{W}^\gamma}.$$

Therefore

$$\big\|\Lambda_1[\nabla_{x,z}\varphi]\big(x, w/(1+h(x))\big)\big\|_{H^k} \lesssim \|\nabla\psi\|_{H^k} + \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W}^1},$$
$$\big\|\Lambda_1[\nabla_x\varphi]\big(x, w/(1+h(x))\big)\big\|_{\widetilde{W}^\gamma} \lesssim \|\nabla\psi\|_{\widetilde{W}^\gamma} + \|h\|_{\widetilde{W}^\gamma}\|\nabla\psi\|_{\widetilde{W}^\gamma} \lesssim \|\nabla\psi\|_{\widetilde{W}^\gamma},$$
$$\big\|\Lambda_1[\partial_z\varphi]\big(x, w/(1+h(x))\big)\big\|_{\widetilde{W}^\gamma} \lesssim \|\Lambda^2\psi\|_{\widetilde{W}^\gamma} + \|h\|_{\widetilde{W}^\gamma}\|\nabla\psi\|_{\widetilde{W}^\gamma}.$$

From the above estimates and (4-28), we conclude

$$\|\Lambda_1[\nabla_x\Phi]\|_{\widetilde{W}^\gamma} \lesssim \|\nabla\psi\|_{\widetilde{W}^\gamma}, \quad \|\Lambda_1[\partial_w\Phi]\|_{\widetilde{W}^\gamma} \lesssim \|\Lambda^2\psi\|_{\widetilde{W}^\gamma} + \|h\|_{\widetilde{W}^\gamma}\|\nabla\psi\|_{\widetilde{W}^\gamma}, \tag{4-30}$$

$$\begin{aligned}\|\Lambda_1[\nabla_{x,w}\Phi]\|_{H^k} &\lesssim \big\|\Lambda_1[\nabla_{x,z}\varphi]\big(x, w/(1+h)\big)\big\|_{H^k} + \big\|\Lambda_1[\nabla_{x,z}\varphi]\big(x, w/(1+h)\big)\big\|_{\widetilde{W}^0}\|h\|_{H^{k+1}} \\ &\lesssim \|\nabla\psi\|_{H^k} + \|h\|_{H^{k+1}}\|\nabla\psi\|_{\widetilde{W}^1}.\end{aligned} \tag{4-31}$$

Following a similar procedure, we can handle the integral part in (3-35) in the same way. Similar to what we did in the proof of Lemma 3.2, we use the size of the symbol directly when $|\xi| \le 1$ and estimate the associated kernel when $|\xi| \ge 1$. As a result, we have the estimates

$$\begin{aligned}&\big\|\text{all terms in the right-hand side of (3-35) except for the linear part } \big(x, w/(1+h(x))\big)\big\|_{H^k} \\ &\quad\lesssim \sum_{i=1}^{3}\|g_i(z,\cdot)\|_{L_z^\infty H^k} + \|h\|_{H^{k+1}}\|g_i(z,\cdot)\|_{L_z^\infty\widetilde{W}^0} \lesssim \|\nabla\psi\|_{\widetilde{W}^0}\|h\|_{H^{k+1}} + \|\nabla\psi\|_{H^k}\|h\|_{\widetilde{W}^0} \\ &\quad\lesssim \|\nabla\psi\|_{\widetilde{W}^0}\|h\|_{H^{k+1}} + \|\nabla\psi\|_{H^k}\|h\|_{\widetilde{W}^0}.\end{aligned} \tag{4-32}$$

$$\begin{aligned}&\big\|\text{all terms in the right-hand side of (3-35) except for the linear part } \big(x, w/(1+h(x))\big)\big\|_{\widetilde{W}^\gamma} \\ &\quad\lesssim \sum_{i=1,2,3}\|g_i(z,\cdot)\|_{L_z^\infty\widetilde{W}^\gamma} \lesssim \|h\|_{\widetilde{W}^{\gamma+1}}\|\nabla\psi\|_{\widetilde{W}^\gamma}.\end{aligned} \tag{4-33}$$

From (4-28), (4-30)–(4-33), now it's easy to see that estimates (4-7) and (4-8) hold.

Now, we proceed to prove (4-9). From (4-28) and the same procedure as above, we have the estimate

$$\begin{aligned}\|\Lambda_{\ge 2}[\nabla_{x,w}\Phi]\|_{L^2} &\lesssim \big\|\Lambda_{\ge 2}[\nabla_{x,z}\varphi]\big(x, w/(1+h(x))\big)\big\|_{L^2} + \|h\|_{H^1}\big\|\partial_z\varphi\big(x, w/(1+h(x))\big)\big\|_{\widetilde{W}^0} \\ &\lesssim \|h\|_{H^1}\big[\|\Lambda\psi\|_{\widehat{W}^{2,\alpha}} + \|h\|_{\widetilde{W}^1}\|\nabla\psi\|_{\widetilde{W}^1}\big] + \sum_{i=1}^{3}\|g_i\|_{L_z^\infty L^2}.\end{aligned} \tag{4-34}$$

Recall (3-25) and (3-26). Note that $\nabla\varphi$ appears together with ∇h inside the quadratic terms of $g_i(z)$, $i \in \{1,2,3\}$. When estimating the $L_z^\infty L^2$ norm of $g_i(z)$, $i \in \{1,2,3\}$, we always put $\nabla\varphi$ in L^2 and put $\partial_z\varphi$ in L^∞. As a result, the following estimate holds, i.e., our desired estimate (4-9) holds:

$$\begin{aligned}\text{(4-34)} &\lesssim \|h\|_{H^1}\|\Lambda\psi\|_{\widehat{W}^{2,\alpha}} + \big[\|(h,\Lambda\psi)\|_{\widehat{W}^{2,\alpha}} + \|(h,\Lambda\psi)\|_{\widehat{W}^2}\big](\|h\|_{H^1} + \|\nabla\psi\|_{L^2}) \\ &\lesssim \big[\|(h,\Lambda\psi)\|_{\widehat{W}^{2,\alpha}} + \|(h,\Lambda\psi)\|_{\widehat{W}^2}\big](\|h\|_{H^1} + \|\nabla\psi\|_{L^2}).\end{aligned}$$

□

4B. ***Paralinearization of the equation satisfied by the velocity potential.*** In this subsection, our main goal is to do the paralinearization process for the nonlinearity of the equation satisfied by ψ in (1-7), which shows which part of the nonlinearity actually loses derivatives.

More precisely, the main result of this subsection is stated in the following proposition,

Proposition 4.7. *We have the paralinearization*

$$\tfrac{1}{2}|\nabla\psi|^2 - \frac{(\nabla h\cdot\nabla\psi + G(h)\psi)^2}{2(1+|\nabla h|^2)} \approx T_V\cdot\nabla[\psi - T_{B(h)\psi}h] - T_B G(h)\psi \tag{4-35}$$

for the nonlinearity of the equation satisfied by ψ.

Proof. Recall that $V=\nabla\psi-\nabla hB$. From (2-7) and the composition Lemma 2.6, we have

$$\begin{aligned}
&\tfrac{1}{2}|\nabla\psi|^2-\frac{(\nabla h\cdot\nabla\psi+G(h)\psi)^2}{2(1+|\nabla h|^2)}\\
&\qquad=\tfrac{1}{2}|\nabla\psi|^2-\tfrac{1}{2}(1+|\nabla h|^2)B^2\\
&\qquad=\tfrac{1}{2}|V|^2+V\cdot\nabla hB+\tfrac{1}{2}|\nabla h|^2B^2-\tfrac{1}{2}(1+|\nabla h|^2)B^2=\tfrac{1}{2}|V|^2+V\cdot\nabla hB-\tfrac{1}{2}B^2\\
&\qquad\approx T_V\cdot V+T_B(V\cdot\nabla h)+T_{V\cdot\nabla h}B-T_BB=T_V\cdot V+T_{V\cdot\nabla h}B-T_BG(h)\psi\\
&\qquad=T_V\cdot\nabla\psi-T_V\cdot(\nabla hB)+T_{V\cdot\nabla h}B-T_BG(h)\psi\\
&\qquad\approx T_V\cdot\nabla\psi-T_VT_B\cdot\nabla h-[T_V\cdot T_{\nabla h}-T_{V\cdot\nabla h}]B-T_BG(h)\psi\\
&\qquad\approx T_V\cdot[\nabla\psi-T_B\cdot\nabla h]-T_BG(h)\psi\approx T_V\cdot\nabla[\psi-T_{B(h)\psi}h]-T_BG(h)\psi. \qquad\square
\end{aligned}$$

4C. ***Symmetrization of the full system.*** Based on the paralinearization results we obtained in previous subsections, in this subsection, we will find out the good substitution variables by doing the symmetrization process such that the resulting system has the requisite symmetric structures inside.

Define $\omega=\psi-T_{B(h)\psi}h$, which is the so-called good unknown variable. After combining the good decomposition (4-1) in Proposition 4.1 and the good decomposition (4-35) in Proposition 4.7, we reduce the system of equations satisfied by h and ψ to the system of equations satisfied by h and ω,

$$\begin{cases}\partial_t h=\Lambda^2\omega+T_{\lambda-|\xi|}\omega-T_V\cdot\nabla h+\widetilde{F}(h)\psi,\\ \partial_t\omega=-T_ah-T_V\cdot\nabla\omega+f',\end{cases}\tag{4-36}$$

where

$$a:=1+\partial_tB+V\cdot\nabla B,$$

which is the so-called Taylor coefficient, and f' is a good error term in the sense of estimate (4-5).

However, the system (4-36) cannot be used to do the energy estimate. When using the system (4-36) to do the energy estimate, one might find that the term

$$\int_{\mathbb{R}^2}\partial_x^Nh\,\partial_x^N[T_{\lambda-|\xi|}\omega]+\partial_x^N\Lambda\omega\partial_x^N\Lambda[-T_ah],\quad\text{where } N \text{ is the prescribed top derivative level,}\tag{4-37}$$

loses one derivative and cannot be simply treated.

To get around this difficulty, we will symmetrize the system (4-36) by following the same procedures in [Alazard, Burq and Zuily 2014a]. Define

$$U_1:=h+T_\alpha h,\quad U_2:=\Lambda[\omega+T_\beta\omega]\approx[T_{\sqrt{\lambda}-|\xi|^{1/2}}\omega]+\Lambda\omega,\quad U:=U_1+iU_2,\tag{4-38}$$

where

$$\alpha:=\sqrt{a}-1,\quad \omega:=\psi-T_{B(h)\psi}h,\quad \beta:=\sqrt{\lambda/|\xi|}-1=\sqrt[4]{(1+|\nabla h|^2)-(\nabla h\cdot\xi/|\xi|)^2}-1.\tag{4-39}$$

Note that

$$\Lambda_1[a]=\Lambda_1[\partial_t\Lambda^2\psi]=-\Lambda^2h,\quad \Lambda_1[\partial_ta]=-\Lambda^4\psi,\quad \Lambda_1[\partial_t\alpha]=-\tfrac{1}{2}\Lambda^4\psi,\quad \Lambda_1[\alpha]=-\tfrac{1}{2}\Lambda^2h.$$

We take the estimates of the Taylor coefficient in Lemma 4.8 for granted first. Then it is easy to see that the following estimate and equivalence relations hold:

$$\|(U_1,U_2)-(h,\Lambda\psi)\|_{H^k}\lesssim \|(h,\Lambda\psi)\|_{\widetilde{W}^3}\|(h,\Lambda\psi)\|_{H^k}, \tag{4-40}$$

$$\begin{aligned}\partial_t U_1 &\approx \Lambda^2\omega+[T_{\lambda-|\xi|}\omega]-T_V\cdot\nabla h+T_{\partial_t\alpha}h+T_\alpha[\partial_t h]\\ &\approx [\Lambda^2-T_{|\xi|}]\omega+T_\lambda\omega-T_V\cdot\nabla U_1+T_V\cdot T_\alpha\nabla h+T_\alpha\big[T_\lambda\omega-T_V\cdot\nabla h\big]\\ &\approx [\Lambda^2-T_{|\xi|}]\Lambda^{-1}U_2+T_{\lambda\sqrt{a}}\omega-T_V\cdot\nabla U_1\approx \Lambda U_2+[T_{\sqrt{\lambda}\alpha}T_{\sqrt{\lambda}}\omega]-T_V\cdot\nabla U_1\\ &\approx \Lambda U_2+[T_{\sqrt{\lambda}\alpha}U_2]-T_V\cdot\nabla U_1,\end{aligned} \tag{4-41}$$

$$\begin{aligned}\partial_t U_2 &\approx \Lambda(1+T_\beta)[-T_a h-T_V\cdot\nabla\omega]+\Lambda T_{\partial_t\beta}\omega\approx -\Lambda U_1-[T_{\sqrt{\lambda}}T_\alpha U_1]-T_V\cdot\nabla T_{\sqrt{\lambda}}\omega+T_{\partial_t\beta}\Lambda\omega\\ &\approx -\Lambda U_1-[T_{\sqrt{\lambda}\alpha}U_1]-T_V\cdot\nabla U_2+T_{\partial_t\beta}U_2.\end{aligned} \tag{4-42}$$

Hence, the problematic terms in (4-37) become the terms (modulo good error terms)

$$\int_{\mathbb{R}^2}\partial_x^N U_1\partial_x^N[T_{\sqrt{\lambda}\alpha}U_2]-\partial_x^N U_2\partial_x^N[T_{\sqrt{\lambda}\alpha}U_1],\quad \text{where } N \text{ is the prescribed top derivative level.} \tag{4-43}$$

Therefore, we can move derivatives in (4-43) around so that these cubic terms do not lose derivatives. See (5-5) for more details.

4D. ***Estimates of the Taylor coefficient.*** The main goal of this subsection is to obtain some basic estimates for the Taylor coefficient, which are necessary for the energy estimate.

Lemma 4.8. *Under the smallness estimate (3-1), for $\gamma\le 3,\ \gamma_1\le 2$ we have the estimates*

$$\begin{aligned}\|a-1\|_{H^k}&\lesssim \|\nabla h\|_{H^k}+\|(h,\nabla\psi)\|_{H^k}\|(h,\nabla\psi)\|_{\widetilde{W}^2},\\ \|a-1\|_{\widetilde{W}^\gamma}&\lesssim \|\nabla h\|_{\widetilde{W}^\gamma}+\big[\|\nabla\psi\|_{\widetilde{W}^{\gamma+1}}+\|\nabla h\|_{\widetilde{W}^\gamma}\big]^2,\\ \|\partial_t a\|_{H^k}&\lesssim \|\nabla\psi\|_{H^{k+1}}+\|(h,\nabla\psi)\|_{\widetilde{W}^3}\|(h,\nabla\psi)\|_{H^{k+1}},\\ \|\partial_t a\|_{\widetilde{W}^{\gamma_1}}&\lesssim \|\Lambda^2\psi\|_{\widetilde{W}^{\gamma_1+1}}+\|(\nabla h,\nabla\psi)\|^2_{\widetilde{W}^{\gamma_1+1}}.\end{aligned}$$

Proof. Recall (4-3), (1-7), and

$$a=1+V\cdot\nabla B+\partial_t B,\quad B=\partial_z\varphi/(1+h)|_{z=0}.$$

To estimate a and $\partial_t a$, it is sufficient to estimate $\partial_z\partial_t\varphi$ and $\partial_z\partial_t^2\varphi$. From the fixed-point-type formulation of $\nabla_{x,z}\varphi$ in (3-35), we can derive the equality

$$\begin{aligned}\nabla_{x,z}\partial_t\varphi=&\bigg[\bigg[\frac{e^{-(z+1)|\nabla|}+e^{(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}\bigg]\nabla\partial_t\psi,\ \frac{e^{(z+1)|\nabla|}-e^{-(z+1)|\nabla|}}{e^{-|\nabla|}+e^{|\nabla|}}|\nabla|\partial_t\psi\bigg]+[\mathbf{0},\partial_t g_1(z)]+\\ &+\int_{-1}^0[K_1(z,s)-K_2(z,s)-K_3(z,s)](\partial_t g_2(s)+\nabla\cdot\partial_t g_3(s))\,ds\\ &+\int_{-1}^0 K_3(z,s)|\nabla|\operatorname{sign}(z-s)\partial_t g_1(s)-|\nabla|[K_1(z,s)+K_2(z,s)]\partial_t g_1(s)\,ds.\end{aligned} \tag{4-44}$$

Following the same fixed-point-type argument that we used in the proof of Lemma 3.3, we can derive the estimates

$$\begin{aligned}
&\|\nabla_{x,z}\partial_t\varphi\|_{L^\infty_z\widetilde{W}^\gamma}\\
&\quad\lesssim \|\nabla h\|_{\widetilde{W}^\gamma}+\|\partial_t h\|_{\widetilde{W}^{\gamma+1}}\|\nabla_{x,z}\varphi\|_{L^\infty_z\widetilde{W}^\gamma}\lesssim \|\nabla h\|_{\widetilde{W}^\gamma}+\big[\|\nabla\psi\|_{\widetilde{W}^{\gamma+1}}+\|\nabla h\|_{\widetilde{W}^\gamma}\big]^2, \qquad (4\text{-}45)\\
&\|\nabla_{x,z}\partial_t\varphi\|_{L^\infty_z H^k}\\
&\quad\lesssim \|\nabla\partial_t\psi\|_{H^k}+\|h\|_{H^{k+1}}\|\nabla_{x,z}\partial_t\varphi\|_{\widetilde{W}^1}+\|\partial_t h\|_{H^{k+1}}\|\nabla_{x,z}\varphi\|_{\widetilde{W}^1}+\|\partial_t h\|_{\widetilde{W}^1}\|\nabla_{x,z}\varphi\|_{L^\infty_z H^k}\\
&\quad\lesssim \|\nabla h\|_{H^k}+\|(h,\nabla\psi)\|_{H^k}\|(h,\nabla\psi)\|_{\widetilde{W}^2}.
\end{aligned}$$

We can take another time derivative at both sides of (4-44) to derive a fixed-point-type formulation for $\nabla_{x,z}\partial_t^2\varphi$. Following a similar argument, we can derive the estimate

$$\|\nabla_{x,z}\partial_t^2\varphi\|_{L^\infty_z\widetilde{W}^\gamma}\lesssim \|\partial_t^2\psi\|_{\widetilde{W}^{\gamma+1}}+\|\partial_t h\|_{\widetilde{W}^{\gamma+1}}\|\partial_t\nabla_{x,z}\varphi\|_{L^\infty_z\widetilde{W}^\gamma}+\|\partial_t^2 h\|_{\widetilde{W}^{\gamma+1}}\|\nabla_{x,z}\varphi\|_{L^\infty_z\widetilde{W}^\gamma}.$$

Recalling the system of equations satisfied by h and ψ in (1-7), we have

$$\begin{aligned}
\partial_t^2 h&=\partial_t G(h)\psi=\partial_t[(1+|\nabla h|^2)B-\nabla h\cdot\nabla\psi],\\
\partial_t^2\psi&=-\partial_t h+\nabla\psi\cdot\nabla\partial_t\psi+(1+|\nabla h|^2)B\partial_t B+\nabla h\cdot\nabla\partial_t h B^2.
\end{aligned}$$

Hence,

$$\|\partial_t^2\psi\|_{\widetilde{W}^{\gamma+1}}\lesssim \|\Lambda^2\psi\|_{\widetilde{W}^{\gamma+1}}+\big[\|\nabla h\|_{\widetilde{W}^{\gamma+1}}+\|\nabla\psi\|_{\widetilde{W}^{\gamma+1}}\big]^2.$$

Combining the above estimate, (4-45) and (3-50) in Lemma 3.3, we have

$$\|\nabla_{x,z}\partial_t^2\varphi\|_{L^\infty_z\widetilde{W}^\gamma}\lesssim \|\Lambda^2\psi\|_{\widetilde{W}^{\gamma+1}}+\big[\|\nabla h\|_{\widetilde{W}^{\gamma+1}}+\|\nabla\psi\|_{\widetilde{W}^{\gamma+1}}\big]^2.$$

Following the same argument, we derive the L^2-type estimate of $\partial_t^2\varphi$,

$$\begin{aligned}
\|\nabla_{x,z}\partial_t^2\varphi\|_{L^\infty_z H^k}\lesssim{}& \|\partial_t^2\psi\|_{H^{k+1}}+\|\partial_t h\|_{H^{k+1}}\|\nabla_{x,z}\partial_t\varphi\|_{L^\infty_z\widetilde{W}^1}+\|\partial_t h\|_{\widetilde{W}^1}\|\nabla_{x,z}\partial_t\varphi\|_{L^\infty_z H^k}\\
&+\|\partial_t^2 h\|_{H^k}\|\nabla_{x,z}\varphi\|_{L^\infty_z\widetilde{W}^1}+\|\partial_t^2 h\|_{\widetilde{W}^1}\|\nabla_{x,z}\varphi\|_{L^\infty_z H^k}\\
&+\|h\|_{H^{k+1}}\|\partial_t^2\nabla_{x,z}\varphi\|_{L^\infty_z\widetilde{W}^1}+\|h\|_{\widetilde{W}^1}\|\nabla_{x,z}\partial_t^2\varphi\|_{L^\infty_z H^k},
\end{aligned}$$

which further gives us the estimate

$$\begin{aligned}
\|\nabla_{x,z}\partial_t^2\varphi\|_{L^\infty_z H^k}&\lesssim \|\Lambda^2\psi\|_{H^{k+1}}+\|\nabla h\|_{\widetilde{W}^2}\|\Lambda^2\psi\|_{H^{k+1}}+\|\nabla\psi\|_{\widetilde{W}^2}\|h\|_{H^{k+1}}\\
&\lesssim \|\nabla\psi\|_{H^{k+1}}+\|(h,\nabla\psi)\|_{\widetilde{W}^3}\|(h,\nabla\psi)\|_{H^{k+1}}.
\end{aligned}$$

Therefore, our desired estimates of the Taylor coefficient hold. □

5. Energy estimate

The goal in this section is to prove our main result, Theorem 1.1. Since the energy of (U_1, U_2) is comparable with the energy of $(h, \Lambda\psi)$, see(4-40), it is sufficient to estimate the energy of (U_1, U_2). Let

N_0 be the prescribed top regularity level. From (4-41) and (4-42), we know that the system of equations satisfied by (U_1, U_2) is given by

$$\begin{cases} \partial_t U_1 - \Lambda U_2 = T_{\sqrt{\lambda}\alpha} U_2 - T_V . \nabla U_1 + \mathcal{R}_1, \\ \partial_t U_2 + \Lambda U_1 = -T_{\sqrt{\lambda}\alpha} U_1 - T_V \cdot \nabla U_2 + \mathcal{R}_2. \end{cases} \tag{5-1}$$

The precise formulations of good remainder terms $\mathcal{R}_1$ and $\mathcal{R}_2$ are not so important in the energy estimate. From (4-41) and (4-42), we know that they are good error terms, i.e.,

$$\|\mathcal{R}_1\|_{H^{N_0}} + \|\mathcal{R}_2\|_{H^{N_0}} \lesssim_{N_0} \big[\|(h, \Lambda\psi)\|_{\widehat{W}^{4,\alpha}} + \|(h, \Lambda\psi)\|^2_{\widehat{W}^4}\big]\|(\eta, \Lambda\psi)\|_{H^{N_0}}. \tag{5-2}$$

Define the energy of U_1 and U_2 as

$$E_{N_0}(t) := \frac{1}{2}\Big[\|U_1\|_{L^2} + \|U_2\|_{L^2} + \sum_{\substack{k+j=N_0\\ 0\le k,j\in\mathbb{Z}}} \|\partial_1^k \partial_2^j U_1\|^2_{L^2} + \|\partial_1^k \partial_2^j U_2\|^2_{L^2}\Big]. \tag{5-3}$$

From (5-1), we have

$$\begin{aligned} \Big|\frac{d}{dt} E_{N_0}(t)\Big| &\lesssim \|(U_1, U_2)\|_{H^{N_0}} \|(\mathcal{R}_1, \mathcal{R}_2)\|_{H^{N_0}} \\ &\quad + \sum_{\substack{k+j=N_0\\ 0\le k,j\in\mathbb{Z}}} \Big|\int_{\mathbb{R}^2} \big[\partial_1^k \partial_2^j U_1 \partial_1^k \partial_2^j [-T_V \cdot \nabla U_1] + \partial_1^k \partial_2^j U_2 \partial_1^k \partial_2^j [-T_V \cdot \nabla U_2]\big]\, dx\Big| \\ &\quad + \Big|\int_{\mathbb{R}^2} \partial_1^k \partial_2^j U_1 \partial_1^k \partial_2^j ([T_{\sqrt{\lambda}\alpha} U_2]) - \partial_1^k \partial_2^j U_2 \partial_1^k \partial_2^j ([T_{\sqrt{\lambda}\alpha} U_1])\Big| \\ &\lesssim_{N_0} \big[\|\nabla V\|_{\widetilde{W}^1} + \|(h, \Lambda\psi)\|_{\widehat{W}^{4,\alpha}} + \|(h, \Lambda\psi)\|^2_{\widehat{W}^4}\big] \|(U_1, U_2)\|^2_{H^{N_0}} + \mathcal{E}_{N_0} \\ &\lesssim_{N_0} \big[\|(h, \Lambda\psi)\|_{\widehat{W}^{4,\alpha}} + \|(h, \Lambda\psi)\|^2_{\widehat{W}^4}\big] \|(U_1, U_2)\|^2_{H^{N_0}} + \mathcal{E}_{N_0}, \end{aligned} \tag{5-4}$$

where

$$\begin{aligned} \mathcal{E}_{N_0} &= \sum_{\substack{k+j=N_0\\ 0\le k,j\in\mathbb{Z}}} \Big|\int_{\mathbb{R}^2} \big[\partial_1^k \partial_2^j U_1 T_{\sqrt{\lambda}\alpha} \partial_1^k \partial_2^j U_2 - \partial_1^k \partial_2^j U_2 T_{\sqrt{\lambda}\alpha} \partial_1^k \partial_2^j U_1\big]\Big| \\ &= \sum_{\substack{k+j=N_0\\ 0\le k,j\in\mathbb{Z}}} \Big|\int_{\mathbb{R}^2} \big[\partial_1^k \partial_2^j U_1 \big(T_{\sqrt{\lambda}\alpha} - (T_{\sqrt{\lambda}\alpha})^*\big) \partial_1^k \partial_2^j U_2\big]\Big|. \end{aligned} \tag{5-5}$$

Recall that $\sqrt{\lambda}\alpha \in M_1^{1/2}$ is a symbol of order $\frac{1}{2}$. Note that it is real. Hence, from Lemma 2.8, we know that $(\sqrt{\lambda}\alpha)^*$ is $\sqrt{\lambda}\alpha$, and that the operator $(T_{\sqrt{\lambda}\alpha})^* - T_{\sqrt{\lambda}\alpha}$ is of order $-\frac{1}{2}$. As a result, the estimate

$$\mathcal{E}_{N_0} \lesssim M_1^{1/2}(\sqrt{\lambda}\alpha)\|(U_1, U_2)\|^2_{H^{N_0}} \lesssim \|\nabla h\|_{\widetilde{W}^2} \|(U_1, U_2)\|^2_{H^{N_0}} \tag{5-6}$$

holds. Combining the above estimate with (5-4) and (4-40), we have

$$\Big|\frac{d}{dt} E_{N_0}(t)\Big| \lesssim_{N_0} \big[\|(h, \Lambda\psi)\|_{\widehat{W}^{4,\alpha}} + \|(h, \Lambda\psi)\|^2_{\widehat{W}^4}\big] \|(h, \Lambda\psi)\|^2_{H^{N_0}}.$$

Appendix: Quadratic terms of the good remainders

In this section, we calculate explicitly the quadratic terms of the good reminder terms $\mathcal{R}_1$ and $\mathcal{R}_2$ to help readers understand the fact that we can gain one derivative in the new energy estimate (1-12) for the inputs of quadratic terms, which are put in the L^∞-type space. Recall (4-38) and (4-39). We have

$$\Lambda_1[B] = \Lambda^2\psi, \quad \Lambda_1[a] = \Lambda_1[\partial_t B] = -\Lambda^2 h, \quad \Lambda_1[\alpha] = -\tfrac{1}{2}\Lambda^2 h, \quad \Lambda_1[\beta] = 0.$$

Recall (5-1). By using the above definitions, we can reduce the equations satisfied by U_1 and U_2 to the equations

$$\begin{cases} \partial_t h - \Lambda^2\psi = \widetilde{Q}_1(h,\psi) + \Lambda_2[\mathcal{R}_1](h,\psi) + \text{cubic and higher}_1, \\ \partial_t \Lambda\psi + \Lambda h = \widetilde{Q}_2(h,\psi) + \Lambda_2[\mathcal{R}_2](h,\psi) + \text{cubic and higher}_2, \end{cases} \tag{5-7}$$

satisfied by h and ψ, where

$$\widetilde{Q}_1(h,\psi) = -\Lambda^2(T_{\Lambda^2\psi}h) + \tfrac{1}{2}T_{\Lambda^2 h}\Lambda^2\psi + \tfrac{1}{2}T_{\Lambda^4\psi}h - \tfrac{1}{2}(T_{\Lambda^2 h}|\nabla|^{1/2}\Lambda\psi) - T_{\nabla\psi}\cdot\nabla h,$$
$$\widetilde{Q}_2(h,\psi) = \Lambda(T_{\Lambda^2\psi}\Lambda^2\psi - T_{\Lambda^2 h}h) + \Lambda\big(\tfrac{1}{2}T_{\Lambda^2 h}h\big) + \tfrac{1}{2}(T_{\Lambda^2 h}|\nabla|^{1/2}h) - T_{\nabla\psi}\cdot\nabla\Lambda\psi.$$

Recall (1-7) and (3-55) in Lemma 3.4. We have

$$\begin{aligned} \Lambda_2[\mathcal{R}_1](h,\psi) &= \Lambda_2[G(h)\psi] - \widetilde{Q}_1(h,\psi) \\ &= -\nabla\cdot(T_h\nabla\psi) - \nabla\cdot\mathcal{R}(h,\nabla\psi) - T_{\Delta^2\psi}h - \Lambda^2(T_h\Lambda^2\psi) \\ &\qquad - \Lambda^2\mathcal{R}(h,\Lambda^2\psi) + \tfrac{1}{2}T_{\Lambda^2 h}\Lambda(\Lambda - |\nabla|^{1/2})\psi - \tfrac{1}{2}T_{\Lambda^4\psi}h, \end{aligned} \tag{5-8}$$

$$\begin{aligned} \Lambda_2[\mathcal{R}_2](h,\psi) &= \Lambda\big[-\tfrac{1}{2}|\nabla\psi|^2 + \tfrac{1}{2}|\Lambda^2\psi|^2\big] - \widetilde{Q}_2(h,\psi) \\ &= \Lambda(-T_{\nabla\psi}\cdot\nabla\psi) + T_{\nabla\psi}\cdot\nabla\Lambda\psi \\ &\qquad + \tfrac{1}{2}\big(-\Lambda(T_{\Lambda^2 h}h) + T_{\Lambda^2 h}|\nabla|^{1/2}h\big) - \tfrac{1}{2}\Lambda\mathcal{R}(\nabla\psi,\nabla\psi) + \tfrac{1}{2}\Lambda\mathcal{R}(\Lambda^2\psi,\Lambda^2\psi). \end{aligned} \tag{5-9}$$

Note that

$$\Lambda - |\nabla|^{1/2} = |\nabla|^{1/2}(\sqrt{\tanh|\nabla|} - 1) = \frac{-2e^{-|\nabla|}|\nabla|^{1/2}}{(\sqrt{\tanh|\nabla|}+1)(e^{|\nabla|}+e^{-|\nabla|})}. \tag{5-10}$$

Now, it is easy to see that $\Lambda_2[\mathcal{R}_2](h,\psi)$ and $\Lambda_2[\mathcal{R}_2](h,\psi)$ do not lose derivatives. It remains to check that we can gain one derivative in the L^∞-type space. By (5-8) and (5-9), it is sufficient to check the term

$$-\nabla\cdot\big(T_h\nabla\psi\big) - \Lambda^2(T_h\Lambda^2\psi). \tag{5-11}$$

The corresponding symbol for the above quadratic terms is

$$\big(\xi\cdot\eta - |\xi||\eta|\tanh|\xi|\tanh|\eta|\big)\theta(\xi-\eta,\eta), \quad |\xi-\eta| \ll |\xi| \sim |\eta|.$$

We decompose this symbol into two parts:

$$p_1(\xi-\eta,\eta) = \xi\cdot\eta - |\xi||\eta| = -\tfrac{1}{2}|\xi-\eta|^2 + \tfrac{1}{2}|\xi|^2 + \tfrac{1}{2}|\eta|^2 - |\xi||\eta| = -\tfrac{1}{2}|\xi-\eta|^2 + \tfrac{1}{2}(|\xi|-|\eta|)^2,$$
$$p_2(\xi-\eta,\eta) = |\xi||\eta|(1 - \tanh|\xi|\tanh|\eta|).$$

Now, it is clear that the first part of (5-11), which is determined by $p_1(\xi-\eta,\eta)$, does not lose derivatives and gains two derivatives for h. For the second part of (5-11), which is determined by $p_2(\xi-\eta,\eta)$, we can lower its regularity to L^2. Hence, we can place ψ in L^∞ and h in L^2. As a result, we always gain one derivative for inputs of quadratic terms that are in L^∞.

Acknowledgements

I would like to thank my Ph.D. advisor Alexandru Ionescu for many helpful discussions and suggestions. The first version of this paper was done when I was visiting Fudan University and BICMR, Peking University. I would like to thank Zhen Lei and BICMR for their warm hospitality during the visits.

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Received 18 Feb 2016. Revised 16 Jan 2017. Accepted 25 Feb 2017.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 10 No. 4 2017