ANALYSIS & PDE

Volume 10 No. 4 2017

PAATA IVANISVILI AND ALEXANDER VOLBERG

IMPROVING BECKNER'S BOUND VIA HERMITE FUNCTIONS

msp

ANALYSIS AND PDE
Vol. 10, No. 4, 2017
dx.doi.org/10.2140/apde.2017.10.929

msp

IMPROVING BECKNER'S BOUND VIA HERMITE FUNCTIONS

PAATA IVANISVILI AND ALEXANDER VOLBERG

We obtain an improvement of the Beckner inequality $\|f\|_2^2 - \|f\|_p^2 \leq (2-p)\|\nabla f\|_2^2$ valid for $p \in [1, 2]$ and the Gaussian measure. Our improvement is essential for the intermediate case $p \in (1, 2)$, and moreover, we find the natural extension of the inequality for any real p.

1. Introduction

1.1. ***The history of the problem.*** The Poincaré inequality [Nash 1958] for the standard Gaussian measure

$$d\gamma_n = \frac{e^{-|x|^2/2}}{\sqrt{(2\pi)^n}}\,dx$$

states that

$$\int_{\mathbb{R}^n} f^2\,d\gamma_n - \left(\int_{\mathbb{R}^n} f\,d\gamma_n\right)^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2\,d\gamma_n \tag{1}$$

for any smooth bounded function $f : \mathbb{R}^n \to \mathbb{R}$. Later William Beckner [1989] generalized (1) for any real power p, $1 \leq p \leq 2$, as follows:

$$\int_{\mathbb{R}^n} f^p\,d\gamma_n - \left(\int_{\mathbb{R}^n} f\,d\gamma_n\right)^p \leq \frac{p(p-1)}{2}\int_{\mathbb{R}^n} f^{p-2}|\nabla f|^2\,d\gamma_n \tag{2}$$

for any smooth bounded $f : \mathbb{R}^n \to (0, \infty)$. We caution the reader that in [Beckner 1989], inequality (2) was formulated in a slightly different but equivalent form (see Theorem 1, inequality (3) in that paper). It should be also mentioned that in the case $p = 2$, inequality (2) does coincide with (1) for all $f \geq 0$ but it does not imply the Poincaré inequality for the functions taking the negative values, especially when $\int_{\mathbb{R}^n} f\,d\gamma_n = 0$. If $p \to 1+$ then (2) provides us with log-Sobolev inequality (see [Beckner 1989]). In general, the constant $p(p-1)/2$ is sharp in the right-hand side of (2), as can be seen for $n = 1$ on the test functions $f(x) = e^{\varepsilon x}$ by sending $\varepsilon \to 0$.

Later Beckner's inequality (2) was studied by many mathematicians for different measures, in different settings and for different spaces as well. We refer the reader to [Arnold et al. 2007; Da Pelo et al. 2016;

Volberg is partially supported by the NSF grant DMS-1600065 and by the Hausdorff Institute for Mathematics, Bonn, Germany. This paper is also based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2017 semester.
MSC2010: primary 42B37, 52A40, 35K55, 42C05, 60G15; secondary 33C15, 46G12.
Keywords: Poincaré inequality, log-Sobolev inequality, Sobolev inequality, Beckner inequality, Gaussian measure, log-concave measures, semigroups, Hermite polynomials, Hermite differential equation, confluent hypergeometric functions, Turán's inequality, error term in Jensen's inequality, phi-entropy, phi-Sobolev, F-Sobolev, phi-divergence, information theory, backwards heat, Monge–Amperè with drift, exterior differential systems.

Barthe et al. 2006; 2007; Barthe and Roberto 2003; Bobkov and Tetali 2003; Bobkov and Götze 1999; Boucheron et al. 2005; Chafaï 2004; Wang 2005; Latała and Oleszkiewicz 2000; Kolesnikov 2007].

An analysis done in [Ivanisvili and Volberg 2015c] indicates that the right-hand side of (2) can be improved. In the present paper we address this issue: what is the precise estimate of the difference given in the left-hand side of (2), and can the requirement $p \in [1, 2]$ be avoided by slightly changing the right-hand side of (2)?

We give complete answers to these questions. For example, if $p = \frac{3}{2}$ we will obtain an improvement in Beckner's inequality (2):

$$\int_{\mathbb{R}^n} f^{3/2}\, d\gamma_n - \left(\int_{\mathbb{R}^n} f\, d\gamma_n\right)^{3/2} \le \int_{\mathbb{R}^n} \left(f^{3/2} - \frac{1}{\sqrt{2}}(2f - \sqrt{f^2+|\nabla f|^2})\sqrt{f+\sqrt{f^2+|\nabla f|^2}}\right) d\gamma_n. \quad (3)$$

The left-hand side of (3) coincides with the left-hand side of (2) for $p = \frac{3}{2}$, but the right-hand side of (3) is strictly smaller than the right-hand side in (2). Indeed, notice that we have the *pointwise* inequality

$$x^{3/2} - \frac{1}{\sqrt{2}}(2x - \sqrt{x^2+y^2})\sqrt{x+\sqrt{x^2+y^2}} \le \tfrac{3}{8}x^{-1/2}y^2 \quad \text{for all } x, y \ge 0, \quad (4)$$

which follows from the homogeneity, i.e., take $x = 1$, and the rest is a direct computation which follows by introducing a new variable

$$u := \sqrt{1+\sqrt{1+y^2}}.$$

As one can see, the improvement of Beckner's inequality (2) is essential. Indeed, if $y \to \infty$ then the right-hand side of (4) increases as y^2 whereas the left-hand side of (4) increases as $y^{3/2}$. Also notice that if $x \to 0$ then the difference of both sides of (4) tends to infinity. The only place where the quantities in (4) are comparable is when $y/x \to 0$.

1.2. ***Main results.*** Let k be a real parameter. Let $H_k(x)$ be the Hermite function which satisfies the Hermite differential equation

$$H_k'' - xH_k' + kH_k = 0, \qquad x \in \mathbb{R}, \quad (5)$$

and which grows relatively slowly, that is, $H_k(x) = x^k + o(x^k)$ as $x \to +\infty$. If k is a nonnegative integer then H_k is the *probabilists'* Hermite polynomial of degree k with the leading coefficient 1; for example, $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, etc. In general, for arbitrary $k \in \mathbb{R}$ one should think that H_k is the analytic extension of the Hermite polynomials in k (existence and many other properties will be mentioned in Section 2).

For $k \in \mathbb{R}$, let R_k be the rightmost zero of $H_k(x)$ (see Lemma 7). If $k \le 0$ then we set $R_k = -\infty$. Define $F_k(x)$ as

$$F_k\left(\left|\frac{H_k'(q)}{H_k(q)}\right|\right) = \frac{H_{k+1}(q)}{H_k^{1+1/k}(q)} \quad \text{for } q \in (R_k, \infty). \quad (6)$$

We will see in the next section that $F_k \in C^2([0,\infty))$ is well defined and $F_k(0) = 1$. Moreover, if $k > -1$ then F_k will be a decreasing concave function, and if $k < -1$ then F_k will be an increasing convex function.

One may observe that

$$F_1(y) = 1 - y^2, \quad F_2(y) = \frac{1}{\sqrt{2}}(2 - \sqrt{1+y^2})\sqrt{1+\sqrt{1+y^2}}.$$

If $k = 0$ then definition (6) should be understood in the limiting sense as

$$F_{\exp}(H_{-1}(q)) = q \exp\left(\alpha - \int_1^q H_{-1}(s)\, ds\right) \quad \text{for all } q \in \mathbb{R}, \tag{7}$$

where

$$\alpha = \int_1^\infty \left(H_{-1}(s) - \frac{1}{s}\right) ds \approx -0.266\ldots. \tag{8}$$

Theorem 1. *For any $p \in \mathbb{R} \setminus [0, 1]$ and any smooth bounded $f \geq 0$ with $\int_{\mathbb{R}^n} f^p \, d\gamma_n < \infty$ we have*

$$\int_{\mathbb{R}^n} f^p F_{1/(p-1)}\left(\frac{|\nabla f|}{f}\right) d\gamma_n \leq \left(\int_{\mathbb{R}^n} f \, d\gamma_n\right)^p. \tag{9}$$

The inequality is reversed if $p \in (0, 1)$.

Proposition 2. *We have*

$$1 - \frac{p(p-1)}{2} t^2 \leq F_{1/(p-1)}(t) \quad \textit{for all } t \geq 0, \ p \in (1, 2]. \tag{10}$$

It remains to notice that estimate (10) applied to (9) immediately gives (2).

The improvement will be essential when $t \to \infty$. For example, it will become clear in the next section that as $t \to \infty$ we have

$$F_{1/(p-1)}(t) \sim -t^p \big(H'_{1/(p-1)}(R_{1/(p-1)})\big)^{1-p} \quad \text{for } p > 1. \tag{11}$$

Another immediate application of Theorem 1 is the following corollary.

Corollary 3. *For any $p \in (1, 2]$ and any smooth bounded $f \geq 0$ we have*

$$\int_{\mathbb{R}^n} f^p \, d\gamma_n - \left(\int_{\mathbb{R}^n} f \, d\gamma_n\right)^p \leq \big(H'_{1/(p-1)}(R_{1/(p-1)})\big)^{1-p} \int_{\mathbb{R}^n} |\nabla f|^p \, d\gamma_n. \tag{12}$$

Estimate (12) will follow by showing that, for any $y \geq 0$, the map

$$x \to x^p - x^p F_{1/(1-p)}\left(\frac{y}{x}\right) \tag{13}$$

is decreasing for $x > 0$, and the limit $x \to 0$ gives (12) by (11).

Appearance of the roots of Hermite functions in (12) seems quite unexpected, especially when these estimates are obtained on the Hamming cube. For example, in [Ivanisvili and Volberg 2016] we were able to extend (12) to the Hamming cube but for a particular power $p = \frac{3}{2}$:

$$\mathbb{E} f^{3/2} - (\mathbb{E} f)^{3/2} \leq \frac{1}{\sqrt{2}} \mathbb{E} |\nabla f|^{3/2}, \quad f : \{-1, 1\}^n \to \mathbb{R}_+. \tag{14}$$

We refer the reader to that paper for the notations, and we notice that the result announced there is a counterpart of (9) for $p = \frac{3}{2}$ on the Hamming cube, where the identity $x^{3/2} F_2(y/x) = \Re(x + iy)^{3/2}$

was used. Next, let $A \subset \{-1,1\}^n$, and let $w_A(x)$ denote the number of edges containing x between the set A and its complement. Clearly $w_A(x)$ lives on the *boundary* of the set A: $w_A(x) = 4|\nabla \mathbb{1}_A|^2$. If A has cardinality 2^{n-1} then the classical edge-isoperimetric inequality [Harper 1966] states that $\sum_{x\in\{-1,1\}^n} w_A(x) \geq 2^n$. On the other hand, taking $f = \mathbb{1}_A$ in (14) gives

$$\sum_{x\in\{-1,1\}^n} w_A(x)^{3/4} \geq (2-\sqrt{2})2^n,$$

which is a new edge-isoperimetric inequality and does not follow from the classical one.

Theorem 1 generates several inequalities. If $p \to 1+$ then (9) gives the log-Sobolev inequality. If $p = 2$ then (9) provides us with the Poincaré inequality. If $p \to \pm\infty$ then we obtain a new Sobolev inequality:

Corollary 4. *For any smooth bounded f we have*

$$\int_{\mathbb{R}^n} \exp(f)\, F_{\exp}(|\nabla f|)\, d\gamma_n \leq \exp\Bigl(\int_{\mathbb{R}^n} f\, d\gamma_n\Bigr),$$

where $F_{\exp}$ is defined in (7).

Finally if $p \to 0$ we obtain a new "negative log-Sobolev" inequality:

Corollary 5. *For any smooth bounded $f \geq 0$ with $\int_{\mathbb{R}^n} \ln f\, d\gamma_n > -\infty$ we have*

$$\int_{\mathbb{R}^n} -\ln f\, d\gamma_n + \ln\Bigl(\int_{\mathbb{R}^n} f\, d\gamma_n\Bigr) \leq \int_{\mathbb{R}^n} -F_{-\ln}\Bigl(\frac{|\nabla f|}{f}\Bigr)\, d\gamma_n,$$

where $F_{-\ln}(t)$ is defined as

$$F_{-\ln}\Bigl(\frac{H_{-2}(x)}{H_{-1}(x)}\Bigr) = \int_1^x H_{-1}(s)\, ds - c + \ln H_{-1}(x), \qquad x \in \mathbb{R}.$$

All these estimates extend to uniformly log-concave probability measures in the following sense (for the proof see Section 3).

Corollary 6. *Let $d\mu = e^{-U}\, dx$ be a probability measure, where* $\operatorname{Hess} U \geq R \cdot \mathrm{Id}$ *for some $R > 0$. For any $p \in \mathbb{R} \setminus [0,1]$ and any smooth bounded $f \geq 0$ with $\int_{\mathbb{R}^n} f^p\, d\mu < \infty$ we have*

$$\int_{\mathbb{R}^n} f^p F_{1/(p-1)}\Bigl(\frac{|\nabla f|}{f\sqrt{R}}\Bigr)\, d\mu \leq \Bigl(\int_{\mathbb{R}^n} f\, d\mu\Bigr)^p. \tag{15}$$

The inequality is reversed if $p \in (0,1)$.

The limiting cases of (15) when $p \to \pm\infty$ and $p \to 0$ should be understood in the sense of functions $F_{\exp}$ and $F_{-\ln}$ as in Corollary 4 and Corollary 5.

To summarize, the current paper provides us with estimates of Φ-entropy (see [Chafaï 2004])

$$\mathbf{Ent}^{\Phi}_{\gamma_n}(f) := \int_{\mathbb{R}^n} \Phi(f)\, d\gamma_n - \Phi\Bigl(\int_{\mathbb{R}^n} f\, d\gamma_n\Bigr)$$

for the following examples:

- $\Phi(x) = x^p$ for $p \in \mathbb{R} \setminus [0, 1]$ using Theorem 1.
- $\Phi(x) = -x^p$ for $p \in (0, 1)$ using Theorem 1.
- $\Phi(x) = e^x$ using Corollary 4, or by taking $p \to \pm\infty$ in Theorem 1.
- $\Phi(x) = -\ln x$ using Corollary 5, or by taking $p \to 0$ in Theorem 1.
- $\Phi(x) = x \ln x$ by taking $p \to 1$ in Theorem 1.

2. The proof of the theorem

The proof of the theorem amounts to checking that the real-valued function

$$M(x, y) = x^p F_k\left(\frac{y}{x}\right), \quad k = \frac{1}{1-p}, \tag{16}$$

defined on $[\varepsilon, \infty) \times [0, \infty)$ for any $\varepsilon > 0$, obeys a necessary smoothness condition, has a boundary condition $M(x, 0) = x^p$ and satisfies the partial differential inequality

$$\begin{pmatrix} M_{xx} + M_y/y & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} \leq 0, \tag{17}$$

with reversed inequality in (17) if $p \in (0, 1)$. Then by Theorem 1 in [Ivanisvili and Volberg 2015c] we obtain that

$$\int_{\mathbb{R}^n} f^p F_k\left(\frac{|\nabla f|}{f}\right) d\gamma_n = \int_{\mathbb{R}^n} M(f, |\nabla f|)\, d\gamma_n \leq M\left(\int_{\mathbb{R}^n} f\, d\gamma_n, 0\right) = \left(\int_{\mathbb{R}^n} f\, d\gamma_n\right)^p$$

for any smooth bounded $f \geq \varepsilon$, which is the statement of the theorem we want to prove (except we need to justify the passage to the limit $\varepsilon \to 0$ and this will be done later). Notice that the inequality is reversed if $p \in (0, 1)$; indeed, in this case we should work with $-M(x, y)$ instead of $M(x, y)$.

Next we will need some tools regarding the Hermite functions H_k.

2.1. *Properties of Hermite functions.*

H_k can be defined (see [Hayman and Ortiz 1975]) by

$$H_k(x) = -\frac{2^{-k/2} \sin(\pi k)\, \Gamma(k+1)}{2\pi} \sum_{n=0}^{\infty} \frac{\Gamma((n-k)/2)}{n!} (-x\sqrt{2})^n, \tag{18}$$

or in terms of the confluent hypergeometric functions (see [Durand 1975]) by

$$H_k(x) = \sqrt{\frac{2^k}{\pi}} \left[\cos\left(\frac{\pi k}{2}\right) \Gamma\left(\frac{k+1}{2}\right) {}_1F_1\left(-\frac{k}{2}, \frac{1}{2}; \frac{x^2}{2}\right) + t\sqrt{2} \sin\left(\frac{\pi k}{2}\right) \Gamma\left(\frac{k}{2}+1\right) {}_1F_1\left(\frac{1-k}{2}, \frac{3}{2}; \frac{x^2}{2}\right)\right]. \tag{19}$$

If k is a nonnegative integer then one should understand (18) and (19) in the limiting sense. Notice the recurrence properties

$$H_k'(x) = k H_{k-1}(x), \tag{20}$$

$$H_{k+1}(x) = x H_k(x) - H_k'(x). \tag{21}$$

These properties follow from (18) and the fact that $\Gamma(z+1) = z\Gamma(z)$.

We also notice that

$$H_k(x) := e^{x^2/4} D_k(x), \tag{22}$$

where $D_k(x)$ is the *parabolic cylinder function*; i.e., it is the solution of the equation

$$D_k'' + \left(k + \frac{1}{2} - \frac{x^2}{4}\right) D_k = 0.$$

Since $H_k(x)$ is an entire function in x and k (see [Temme 2015] for the parabolic cylinder function), sometimes it will be convenient to write $H(x, k)$ instead of $H_k(x)$. The precise asymptotic for $x \to +\infty$, $x > 0$ and any $k \in \mathbb{R}$ is given by

$$H_k(x) \sim x^k \cdot \sum_{n=0}^{\infty} (-1)^n \frac{(-k)_{2n}}{n!\,(2x^2)^n}. \tag{23}$$

Here $(a)_n = 1$ if $n = 0$, and $(a)_n = a(a+1)\cdots(a+n-1)$ if $n > 0$. When $x \to -\infty$ we have

$$H_k(x) \sim |x|^k \cos(k\pi) \sum_{n=0}^{\infty} (-1)^n \frac{(-k)_{2n}}{n!\,(2x^2)^n} + \frac{\sqrt{2\pi}}{\Gamma(-k)} |x|^{-k-1} e^{x^2/2} \sum_{n=0}^{\infty} \frac{(1+k)_{2n}}{n!\,(2x^2)^n}. \tag{24}$$

We refer the reader to [Temme 2015; Olver et al. 2010]. For instance, for (23) we can use the asymptotic formula (12.9.1) in [Olver et al. 2010] for the parabolic cylinder function. To verify (24) we can express $H_k(-x)$ as a linear combination of two parabolic cylinder functions but with argument x instead of $-x$, see (12.2.15) in [Olver et al. 2010], and then we can use (12.9.1) and (12.9.2) in the same paper.

Next we will need the result of Elbert and Muldoon [1999] which describes the behavior of the real zeros of $H_k(x)$ for any real k.

Lemma 7. *For $k \le 0$, the function $H_k(x)$ has no real zeros, and it is positive on the real axis. For $n < k \le n+1$, $n = 0, 1, \ldots,$ the function $H_k(x)$ has $n+1$ real zeros. Each zero is increasing function of k on its interval of definition.*

The proof of the lemma is Theorem 3.1 in [Elbert and Muldoon 1999]. It is explained in the paper that as k passes through each nonnegative integer n a new leftmost zero appears at $-\infty$, while the rightmost zero passes through the largest zero of $H_k(x)$. They also include more precise information about the asymptotic behavior of the zeros as $k \to \infty$.

Further we will need Turán's inequality for H_k for any real k.

Lemma 8. *We have Turán's inequality*:

$$H_k^2(x) - H_{k-1}(x) H_{k+1}(x) > 0 \quad \textit{for all } k \in \mathbb{R},\ x \ge L_k, \tag{25}$$

where L_k denotes the leftmost zero of H_k. If $k \le 0$ then $L_k = -\infty$.

The lemma is known as Turán's inequality when k is a nonnegative integer. Unfortunately we could not find the reference in the case when k is different from a positive integer; therefore we decided to include the proof of the lemma.

The following is borrowed from [Madhava Rao and Thiruvenkatachar 1949].

Proof. Take $f(x) = e^{-x^2/2}\big(H_k^2(x) - H_{k-1}(x)H_{k+1}(x)\big)$. Asymptotic formulas (23) and (24) imply that

$$\begin{aligned} \lim_{x\to+\infty} f(x) &= 0 \quad \text{for all } k\in\mathbb{R}, \\ f(x) &\sim \sqrt{2\pi}|x| > 0 \quad \text{for } x\to-\infty,\ k=0, \\ f(x) &\sim \frac{2\pi e^{x^2/2}}{\Gamma(-k)\Gamma(-k+1)}|x|^{-2k-2} \quad \text{for } x\to-\infty,\ k\notin\{0\}\cup\mathbb{N}. \end{aligned} \tag{26}$$

On the other hand, notice that by (20) and (21) we have

$$f'(x) = -e^{-x^2/2}H_kH_{k-1}. \tag{27}$$

If $k\le 0$ then by Lemma 7 we have $f'<0$, and because of the conditions $f(-\infty)=+\infty$ and $f(\infty)=0$ we obtain that $f>0$ on $\mathbb{R}$. To verify the statement for $k>0$ we notice that

$$f''(x) = e^{-x^2/2}(H_k^2 - kH_{k-1}^2). \tag{28}$$

Now we notice that if $H_k(c)=0$ then $H_{k-1}(c)\neq 0$. Indeed, assume to the contrary that $H_{k-1}(c)=0$. Then by (20) we have $H_k'(c)=0$ and by (5) we obtain $H_k''(c)=0$, and again taking derivatives in (20) we obtain that $H_{k-2}(c)=0$. Repeating this process we obtain that $H_{k-N}(c)=0$ for any large integer $N>0$. But this contradicts Lemma 7.

Thus by (27) and (28) we obtain that c is a local minimum of f if and only if $H_{k-1}(c)=0$. Then $f(c)=e^{-x^2/2}H_k^2(c)>0$. Finally we obtain that $f:[L_k,\infty)\to\mathbb{R}$ is positive on its local minimum points, $f(\infty)=0$ and $f(L_k)>0$ (because H_{k-1} and H_{k+1} have opposite signs at zeros of H_k by (21)). Therefore $f>0$ on $[L_k,\infty)\to\mathbb{R}$ and the lemma is proved. □

Remark 9. If $k\in\mathbb{N}$ then H_k is the probabilists' Hermite polynomial of degree k, so $f(x)$ will be even and inequality (25) will hold for all $x\in\mathbb{R}$, which confirms the classical Turán's inequality. However, if $k>0$ but $k\notin\mathbb{N}$ then (25) fails when $x\to-\infty$; see (26).

Finally the next corollary together with Lemma 7 implies that

$$\left|\frac{H_k'}{H_k}\right| = \operatorname{sign}(k)\frac{H_k'(q)}{H_k(q)}$$

is positive and decreasing for $q\in(R_k,\infty)$ and $k\in\mathbb{R}\setminus\{0\}$.

Corollary 10. *For any $x\ge L_k$ and any $k\in\mathbb{R}\setminus\{0\}$ we have*

$$\operatorname{sign}[(H_k')^2 - H_kH_k''] = \operatorname{sign}(k).$$

Proof. The proof follows from Lemma 8 and the identity

$$k(H_k^2 - H_{k-1}H_{k+1}) = (H_k')^2 - H_kH_k'' \tag{29}$$

by (5), (20) and (21). □

2.2. ***Checking the partial differential inequality.*** Let $p=1+1/k$. Further we assume $k\neq 0,-1$. Define $F=F_k$ as in the Introduction:

$$F(t)=\frac{H_{k+1}(q)}{H_k^{1+1/k}(q)},\quad \text{where } \left|\frac{H_k'(q)}{H_k(q)}\right|=t,\ q\in(R_k,\infty),\ t\in(0,\infty). \tag{30}$$

Notice that by Corollary 10, the function

$$\left|\frac{H_k'(q)}{H_k(q)}\right|=\operatorname{sign}(k)\frac{H_k'(q)}{H_k(q)}$$

is positive decreasing in q for $q\in(R_k,\infty)$; moreover, by (23) and (20) we have $H_k'(q)/H_k(q)\sim k/q$ when $q\to+\infty$. From the same asymptotic formulas it follows that when $t\to 0+$ we have

$$F(t)=1-\frac{p(p-1)}{2}t^2+O(t^4).$$

Therefore F is a well-defined function and $F\in C^2([0,\infty))$.

Take a positive $\varepsilon>0$ and define $M(x,y)$ as in (16):

$$M(x,y):=x^pF\left(\frac{y}{x}\right)\quad \text{for } y\geq 0,\ x>\varepsilon>0. \tag{31}$$

Clearly $M(x,\sqrt{y})\in C^2([\varepsilon,\infty)\times[0,\infty))$. By Theorem 1 in [Ivanisvili and Volberg 2015c] we have the inequality

$$\int_{\mathbb{R}^n}M(f,|\nabla f|)\,d\gamma_n\leq M\left(\int_{\mathbb{R}^n}f\,d\gamma_n,0\right) \tag{32}$$

for all smooth bounded $f\geq\varepsilon$ if (17) holds. In terms of F (see (31)) condition (17) takes the form

$$tFF''p(p-1)+F'F''-t(p-1)^2(F')^2\geq 0,\quad \text{i.e., the determinant of (17) is nonnegative,} \tag{33}$$

$$F''(t+t^3)+F'(2t^2+1-2pt^2)+Fp(p-1)t\leq 0,\quad \text{i.e., the trace of (17) is nonpositive,} \tag{34}$$

where $t=y/x$ is the argument of F. In fact we will show that we have equality in (33) instead of inequality; therefore the sign of (17) will depend on the sign of the trace in (34). We will see that inequality (34) will be reversed for $p\in(0,1)$.

From (30), (29), (20) and (21) we obtain

$$F'(t)=-\frac{k+1}{|k|}\frac{1}{H_k^{1/k}}, \tag{35}$$

$$F''(t)=\frac{F'}{|k|}\frac{H_kH_{k-1}}{H_k^2-H_{k+1}H_{k-1}}, \tag{36}$$

$$F(t)=-\frac{|k|}{k+1}\frac{H_{k+1}}{H_k}F'. \tag{37}$$

If we plug (36) and (37) into (33) we obtain that the left-hand side of (33) is zero. If we plug (36) and (37) into (34) we obtain

$$\text{left-hand side of (34)} = \left[\frac{(kH_{k-1}^2 - H_k^2 + H_{k-1}H_{k+1})^2 + H_{k-1}^2 H_k^2}{H_k^2(H_k^2 - H_{k+1}H_{k-1})}\right] F'.$$

Thus the sign of left-hand side of (34) coincides with the sign of F', which coincides with $\operatorname{sign}(-(k+1))$. The condition $p \in \mathbb{R} \setminus [0, 1]$ implies that $k > -1$ and therefore (17) holds. The condition $p \in (0, 1)$ implies that $k < -1$ and therefore the inequality in (17) is reversed.

Thus we have obtained (32) for smooth bounded functions $f \geq \varepsilon$. Next we claim that for an arbitrary smooth bounded $f \geq 0$ with $\int_{\mathbb{R}^n} f^p \, d\gamma_n < \infty$, we can apply the inequality to $f_\varepsilon := f + \varepsilon$ and send ε to 0 in (9). Indeed, it follows from (6) and (23) that as $t \to \infty$ we have

$$F(t) \sim \begin{cases} -t^{1+1/k}(H_k'(R_k))^{-1/k} & \text{for } k > 0, \\ \operatorname{sign}(-1-k)\left(\dfrac{e^{t^2/2}\sqrt{2\pi}}{t|\Gamma(-1-k)|}\right)^{-1/k} |1+k|^{1+1/k} & \text{for } k < 0,\ k \neq -1. \end{cases} \tag{38}$$

Thus for $p > 1$, that is, $k > 0$, the claim about the limit follows from the estimate $|F(t)| \leq C_1 + C_2 t^p$ together with the Lebesgue dominated convergence theorem.

If $p < 0$, that is, $k \in (-1, 0)$, we rewrite (9) in a standard way as

$$\int_{\mathbb{R}^n} f_\varepsilon^p \, d\gamma_n - \left(\int_{\mathbb{R}^n} f_\varepsilon \, d\gamma_n\right)^p \leq \int_{\mathbb{R}^n} f_\varepsilon^p \left(1 - F\left(\frac{|\nabla f|}{f_\varepsilon}\right)\right) d\gamma_n. \tag{39}$$

Since f is bounded, $f \geq 0$ and $\int_{\mathbb{R}^n} f^p \, d\gamma_n < \infty$, there is no issue with the left-hand side of (39) when $\varepsilon \to 0$. For the right-hand side of (39) we notice that the function $x^p(1 - F(y/x))$ is nonnegative and decreasing in x. Then the claim follows from the monotone convergence theorem. The nonnegativity follows from the observation that $F(0) = 1$ and $F' < 0$ (see (35) where we have $k > -1$). The monotonicity follows from (6), (35), (20) and the straightforward computations

$$\frac{\partial}{\partial x}(x^p(1 - F(y/x))) = x^{p-1}(p - pF(t) + tF'(t)) = x^{p-1} p \left[1 - \frac{q}{H_k^{1/k}(q)}\right], \tag{40}$$

where

$$|k| \frac{H_{k-1}(q)}{H_k(q)} = t = \frac{y}{x}$$

and $q \in (R_k, \infty)$. The last expression in (40) is negative because

$$1 \geq F(t) = \frac{H_{k+1}}{H_k^{1+1/k}} = \frac{qH_k - kH_{k-1}}{H_k^{1+1/k}} > \frac{q}{H_k^{1/k}}.$$

Finally if $p \in (0, 1)$, that is, $k < -1$, we have the opposite inequality in (39). In this case the situation is absolutely the same as for $k \in (-1, 0)$ except now we should consider the function $x^p(F(y/x) - 1)$, which is nonnegative and decreasing in x; see (40). This finishes the proof of the theorem.

Now let us show Proposition 2. Since $F(0) = 1$, it is enough to show a stronger inequality, namely $F' + p(p-1)t \geq 0$. From (35) and the fact that $k > 1$ since $p \in (1, 2)$, we obtain that it is enough to show the inequality

$$-\frac{p}{H_k^{1/k}} + p(p-1)\frac{H_k'}{H_k} \geq 0 \quad \text{for all } k \geq 1,\ q \in (R_k, \infty).$$

Using (20) and $p = 1 + 1/k$ we notice that the inequality can be rewritten as $1 \geq H_k(q)/H_{k-1}^{k/(k-1)}(q)$ for all $q \in (R_k, \infty)$. To verify the last inequality recall that $F(0) = 1$ and $F'(t) < 0$. Therefore $F(t) \leq 1$. We also recall the definition of $F(t)$; see (30). It follows that $1 \geq F = H_{k+1}/H_k^{1+1/k}$ for all $k > 0$. The last inequality is the same as

$$1 \geq \frac{H_k(q)}{H_{k-1}^{k/(k-1)}(q)} \quad \text{for all } q \in (R_k, \infty),\ k > 1. \tag{41}$$

This finishes the proof of Proposition 2.

To verify Corollary 3 we only need to prove the monotonicity of the map (13) for $p \in (1, 2]$, that is, $k \geq 1$, and the rest will follow from (38). If $k = 1$ there is nothing to prove; therefore we assume $k > 1$. By (40) it is enough to show that $L(q) := H_k^{1/k}(q) - q \leq 0$ for $q \in (R_k, \infty)$. The growth condition (24) on H_k implies that $\lim_{q\to\infty} L(q) = 0$. If $L'(q) \geq 0$ then we are done. Using (20) we notice that $L'(q) \geq 0$ is equivalent to (41), which was already proved.

2.3. ***Proof of Corollaries 4 and 5.*** Notice that as $y \to 0$ we have

$$F_{\exp}(y) = 1 - \frac{y^2}{2} + O(y^4) \quad \text{and} \quad F_{-\ln}(y) = -\frac{y^2}{2} + O(y^4).$$

One can check that

$$M_{\exp}(x, y) := e^x F_{\exp}(y), \quad M_{\exp}(x, 0) = e^x, \quad M_{\exp}(x, \sqrt{y}) \in C^2(\mathbb{R} \times \mathbb{R}_+),$$
$$M_{-\ln}(x, y) := -\ln(x) + F_{-\ln}\left(\frac{y}{x}\right), \quad M_{-\ln}(x, 0) = -\ln x, \quad x > 0,$$

and $M_{-\ln}(x, \sqrt{y}) \in C^2([\varepsilon, \infty) \times \mathbb{R}^+)$ for any $\varepsilon > 0$. By straightforward computations we notice that if we set $\psi(q) = \alpha - \int_1^q H_{-1}(s)\, ds$ then using the identity $1 = qH_{-1}(q) + H_{-2}(q)$ we obtain

$$F_{\exp}(H_{-1}) = qe^{\psi}, \quad F'_{\exp}(H_{-1}) = -e^{\psi} \quad \text{and} \quad F''_{\exp}(H_{-1}) = -\frac{H_{-1}}{H_{-2}}.$$

Similarly we compute that

$$F'_{-\ln}\left(\frac{H_{-2}}{H_{-1}}\right) = -H_{-1} \quad \text{and} \quad F''_{-\ln}\left(\frac{H_{-2}}{H_{-1}}\right) = -\frac{H_{-2}H_{-1}^2}{H_{-1}^2 - H_{-2}}.$$

Next one notices that $M_{\exp}$ and $M_{-\ln}$ satisfy (17) (in fact the determinant of (17) is zero). Then by Theorem 1 in [Ivanisvili and Volberg 2015c] we obtain the corollaries. The passage to the limit for $M_{-\ln}(x, y)$ when $\varepsilon \to 0$ follows from the monotone convergence theorem. Indeed, we notice that $-F_{-\ln}(y/x) \geq 0$ is decreasing in x. We apply Corollary 5 to $f_\varepsilon = f + \varepsilon$ and send $\varepsilon \to 0$.

2.3.1. *How we guessed the functions $M_{\exp}$ and $M_{-\ln}$.* One may ask how to find the functions $M_{\exp}$ and $M_{-\ln}$. To find $M_{\exp}$ we should apply (9) to functions $f = e^{g/p}$, where g is some fixed function. Then (9) takes the form

$$\int_{\mathbb{R}^n} e^g F_{1/(p-1)}\left(\frac{|\nabla g|}{p}\right) d\gamma_n \le \left(\int_{\mathbb{R}^n} e^{g/p}\, d\gamma_n\right)^p. \tag{42}$$

Now we take $p \to \infty$. The right-hand side of (42) tends to $\exp(\int_{\mathbb{R}^n} g\, d\gamma_n)$. For the left-hand side of (42) we should compute the limit

$$F_{\exp}(t) := \lim_{p\to\infty} F_{1/(p-1)}\left(\frac{t}{p}\right) = \lim_{p\to\infty} F_{1/(p-1)}\left(\frac{t}{p-1}\right) = \lim_{k\to 0+} F_k(tk).$$

In fact all equalities can be justified by direct calculations using the fact that $H_k(x) = H(x,k)$ is the entire function of x and k; see [Temme 2015] for the parabolic cylinder function and formula (22).

It is clear that $F_{\exp}(0) = 1$. Next if we take $k \to 0+$ in (6) we obtain

$$\lim_{k\to 0+} F_k\left(\left|\frac{H_k'}{H_k}\right|\right) = \lim_{k\to 0+} F_k\left(k\frac{H_{k-1}}{H_k}\right) = \lim_{k\to 0+} F_k\left(k\frac{H_{-1}}{H_0}\right) = F_{\exp}(H_{-1}).$$

On the other hand, for the right-hand side of (6) we have

$$\lim_{k\to 0+} \frac{H_{k+1}(q)}{H_k^{1+1/k}} = q \lim_{k\to 0+} H_k^{-1/k}.$$

Here we have used $H_0(q) = 1$ and $H_1(q) = q$. Thus it remains to find $\lim_{k\to 0+} H_k^{-1/k}$. If we take the derivative in k of (20) we obtain $H_{xk}(x,k) = H(x,k-1) + kH_k(x,k)$ (here subindices denote partial derivatives). Now taking $k = 0$ we obtain $H_{xk}(x,0) = H(x,-1)$. Thus $H_k(x,0)$ is an antiderivative of $H(x,-1) = H_{-1}$. So

$$\lim_{k\to 0+} H_k^{-1/k} = \lim_{k\to 0+} \exp\left(-\frac{1}{k}\ln\bigl(1 + kH_k(x,0) + o(k)\bigr)\right) = \exp\left(-\int H_{-1}(s)\, ds\right).$$

Finally we obtain

$$F_{\exp}(H_{-1}(q)) = q \exp\left(C - \int_1^q H_{-1}\right). \tag{43}$$

In order to satisfy the condition $F_{\exp}(0) = 1$, the constant c must be chosen as $C = \int_1^\infty (H_{-1} - 1/s)\, ds$; indeed send $q \to \infty$ in (43). This gives Corollary 4. It is worth mentioning that we have also obtained

$$H_k(x,0) = \int_1^x H_{-1}(s)\, ds - \alpha;$$

see (8).

To find $M_{-\ln}$, let $F(x,k) := F_k(x)$. Let $F_k(x,k)$ denote the partial derivative in k of $F(x,k)$. If we send $p \to 0,\ p < 0$ in (9) and compare the terms of order p we obtain

$$\int_{\mathbb{R}^n} \left(\ln f - F_k\left(\frac{|\nabla f|}{f}, -1\right)\right) d\gamma_n \ge \ln\left(\int_{\mathbb{R}^n} f\, d\gamma_n\right).$$

It remains to find the function $F_k(x,-1)$. Let us equate terms of order $(k+1)$ as $k\to -1,\ k<-1$ in

$$F\left(\frac{|H_x(x,k)|}{H(x,k)},k\right)=\frac{H(x,k+1)}{H(x,k)^{1+1/k}}.$$

Straightforward computation shows that

$$F_k\left(\frac{H_{-2}(x)}{H_{-1}(x)},-1\right)=H_k(x,0)+\ln H_{-1}(x)=\int_1^x H_{-1}(s)\,ds-\alpha+\ln H_{-1}(x),$$

where

$$\alpha=\int_1^{\infty}\left(H_{-1}(s)-\frac{1}{s}\right)ds.$$

3. Concluding remarks

The reader may wonder how we guessed the choice (16). Of course it was not a random guess. Function (16) is the best possible in the sense that the determinant of (17) is identically zero:

$$\begin{aligned} M_{yy}\left(M_{xx}+\frac{M_y}{y}\right)-M_{xy}^2=0,\\ M(x,0)=x^p \quad \text{for } x\ge 0. \end{aligned} \tag{44}$$

Initially this was the way we started looking for $M(x,y)$ as the solution of the Monge–Ampère equation with a drift (44). By a proper change of variables, the equation reduces to the backwards heat equation (see [Ivanisvili and Volberg 2015c] for more details where the connection with the theory of exterior differential systems of R. Bryant et al. [1991] was exploited)

$$u_{xx}+u_t=0, \tag{45}$$

$$u(x,0)=Cx^{p/(p-1)} \quad \text{for } x\ge 0. \tag{46}$$

One can notice that the Hermite polynomials do satisfy (45) and (46) when $p/(p-1)$ is a positive integer. In general, one should invoke Hermite functions and this is the reason for the appearance of these functions in our theorem.

Another possibility is to assume that $M(x,y)$ should be homogeneous of degree p, which forces M to have the form (31) for some F. Next setting $h=F/F'$ and further by a subtle change of variables, one obtains Hermite differential equation (5).

Nevertheless, for the formal proof of Theorem 1 we do not need to go through the details. We have $M(x,y)$ defined by (16) and we just need to check that it satisfies the desired properties.

That $M(x,y)$ satisfies (17) makes it possible to extend Theorem 1 in a semigroup setting for uniformly log-concave probability measures. Indeed, let $d\mu=e^{-U}\,dx$, where $\operatorname{Hess} U\ge R\cdot\mathrm{Id}$, $R>0$. Let $L=\Delta-\nabla U\cdot\nabla$ and $P_t=e^{tL}$ be the semigroup with generator L; see [Ivanisvili and Volberg 2015c; Bakry et al. 2014].

Corollary 11. *For any $p\in\mathbb{R}\setminus[0,1]$ and any smooth bounded $f\ge 0$ with $\int_{\mathbb{R}^n} f^p\,d\mu<\infty$ we have*

$$P_t\left[f^pF_{1/(p-1)}\left(\frac{|\nabla f|}{f\sqrt{R}}\right)\right]\le (P_tf)^pF_{1/(p-1)}\left(\frac{|\nabla P_tf|}{P_tf\sqrt{R}}\right).$$

The inequality is reversed if $p\in(0,1)$.

Proof. Notice that $\widetilde{M}(x, y) = M(x, y/\sqrt{R})$ satisfies (17). Now it remains to use inequality (2.3) from [Ivanisvili and Volberg 2015c]. □

By taking $t \to \infty$ and using the fact that $|\nabla P_t f| \le e^{-tR} P_t |\nabla f|$, we obtain Corollary 6.

Finally we would like to mention that having characterization (17) of functional inequalities (32) makes our approach to problem (9) systematic. Very similar *local estimates* happen to rule some *global inequalities*. We refer the reader to our recent papers on this subject [Ivanisvili and Volberg 2015a;2015b; Ivanisvili 2016].

Acknowledgements

We are very grateful to Robert Bryant who suggested a change of variables in (33), and an anonymous referee for valuable suggestions.

References

[Arnold et al. 2007] A. Arnold, J.-P. Bartier, and J. Dolbeault, "Interpolation between logarithmic Sobolev and Poincaré inequalities", *Commun. Math. Sci.* **5**:4 (2007), 971–979. MR Zbl

[Bakry et al. 2014] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der Mathematischen Wissenschaften **348**, Springer, 2014. MR Zbl

[Barthe and Roberto 2003] F. Barthe and C. Roberto, "Sobolev inequalities for probability measures on the real line", *Studia Math.* **159**:3 (2003), 481–497. MR Zbl

[Barthe et al. 2006] F. Barthe, P. Cattiaux, and C. Roberto, "Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry", *Rev. Mat. Iberoam.* **22**:3 (2006), 993–1067. MR Zbl

[Barthe et al. 2007] F. Barthe, P. Cattiaux, and C. Roberto, "Isoperimetry between exponential and Gaussian", *Electron. J. Probab.* **12**:44 (2007), 1212–1237. MR Zbl

[Beckner 1989] W. Beckner, "A generalized Poincaré inequality for Gaussian measures", *Proc. Amer. Math. Soc.* **105**:2 (1989), 397–400. MR Zbl

[Bobkov and Götze 1999] S. G. Bobkov and F. Götze, "Exponential integrability and transportation cost related to logarithmic Sobolev inequalities", *J. Funct. Anal.* **163**:1 (1999), 1–28. MR Zbl

[Bobkov and Tetali 2003] S. Bobkov and P. Tetali, "Modified log-Sobolev inequalities, mixing and hypercontractivity", pp. 287–296 in *Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing* (San Diego, CA, 2003), ACM, New York, 2003. MR Zbl

[Boucheron et al. 2005] S. Boucheron, O. Bousquet, G. Lugosi, and P. Massart, "Moment inequalities for functions of independent random variables", *Ann. Probab.* **33**:2 (2005), 514–560. MR Zbl

[Bryant et al. 1991] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths, *Exterior differential systems*, Mathematical Sciences Research Institute Publications **18**, Springer, 1991. MR Zbl

[Chafaï 2004] D. Chafaï, "Entropies, convexity, and functional inequalities: on Φ-entropies and Φ-Sobolev inequalities", *J. Math. Kyoto Univ.* **44**:2 (2004), 325–363. MR Zbl

[Da Pelo et al. 2016] P. Da Pelo, A. Lanconelli, and A. I. Stan, "An extension of the Beckner's type Poincaré inequality to convolution measures on abstract Wiener spaces", *Stoch. Anal. Appl.* **34**:1 (2016), 47–64. MR Zbl

[Durand 1975] L. Durand, "Nicholson-type integrals for products of Gegenbauer functions and related topics", pp. 353–374 in *Theory and application of special functions* (Madison, WI, 1975), edited by R. A. Askey, Academic Press, New York, 1975. MR Zbl

[Elbert and Muldoon 1999] A. Elbert and M. E. Muldoon, "Inequalities and monotonicity properties for zeros of Hermite functions", *Proc. Roy. Soc. Edinburgh Sect. A* **129**:1 (1999), 57–75. MR Zbl

[Harper 1966] L. H. Harper, "Optimal numberings and isoperimetric problems on graphs", *J. Combinatorial Theory* **1** (1966), 385–393. MR Zbl

[Hayman and Ortiz 1975] W. K. Hayman and E. L. Ortiz, "An upper bound for the largest zero of Hermite's function with applications to subharmonic functions", *Proc. Roy. Soc. Edinburgh Sect. A* **75**:3 (1975), 182–197. MR Zbl

[Ivanisvili 2016] P. Ivanisvili, "Boundary value problem and the Ehrhard inequality", preprint, 2016. arXiv

[Ivanisvili and Volberg 2015a] P. Ivanisvili and A. Volberg, "Bellman partial differential equation and the hill property for classical isoperimetric problems", preprint, 2015. arXiv

[Ivanisvili and Volberg 2015b] P. Ivanisvili and A. Volberg, "Hessian of Bellman functions and uniqueness of the Brascamp–Lieb inequality", *J. Lond. Math. Soc.* (2) **92**:3 (2015), 657–674. MR Zbl

[Ivanisvili and Volberg 2015c] P. Ivanisvili and A. Volberg, "Isoperimetric functional inequalities via the maximum principle: the exterior differential systems approach", preprint, 2015. arXiv

[Ivanisvili and Volberg 2016] P. Ivanisvili and A. Volberg, "Poincaré inequality 3/2 on the Hamming cube", preprint, 2016. arXiv

[Kolesnikov 2007] A. V. Kolesnikov, "Modified log-Sobolev inequalities and isoperimetry", *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **18**:2 (2007), 179–208. MR Zbl

[Latała and Oleszkiewicz 2000] R. Latała and K. Oleszkiewicz, "Between Sobolev and Poincaré", pp. 147–168 in *Geometric aspects of functional analysis*, Lecture Notes in Math. **1745**, Springer, 2000. MR Zbl

[Madhava Rao and Thiruvenkatachar 1949] B. S. Madhava Rao and V. R. Thiruvenkatachar, "On an inequality concerning orthogonal polynomials", *Proc. Indian Acad. Sci., Sect. A.* **29** (1949), 391–393. MR Zbl

[Nash 1958] J. Nash, "Continuity of solutions of parabolic and elliptic equations", *Amer. J. Math.* **80** (1958), 931–954. MR Zbl

[Olver et al. 2010] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (editors), *NIST handbook of mathematical functions*, Cambridge University Press, 2010. MR Zbl

[Temme 2015] N. M. Temme, *Asymptotic methods for integrals*, Series in Analysis **6**, World Scientific Publishing, Hackensack, NJ, 2015. MR Zbl

[Wang 2005] F.-Y. Wang, "A generalization of Poincaré and log-Sobolev inequalities", *Potential Anal.* **22**:1 (2005), 1–15. MR Zbl

Received 28 Jun 2016. Revised 30 Jan 2017. Accepted 18 Mar 2017.

PAATA IVANISVILI: ivanishvili.paata@gmail.com
Department of Mathematics, Kent State University, Kent, OH 44240, United States

ALEXANDER VOLBERG: volberg@math.msu.edu
Department of Mathematics, Michigan State University, East Lansing, MI 48824, United States

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2017 is US $265/year for the electronic version, and $470/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 10 No. 4 2017