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POSITIVITY FOR FOURTH-ORDER SEMILINEAR PROBLEMS
RELATED TO THE KIRCHHOFF–LOVE FUNCTIONAL

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We study the ground states of the following generalization of the Kirchhoff–Love functional,

$$J_\sigma(u) = \int_\Omega \frac{(\Delta u)^2}{2} - (1-\sigma) \int_\Omega \det(\nabla^2 u) - \int_\Omega F(x, u),$$

where Ω is a bounded convex domain in $\mathbb{R}^2$ with $C^{1,1}$ boundary and the nonlinearities involved are of sublinear type or superlinear with power growth. These critical points correspond to least-energy weak solutions to a fourth-order semilinear boundary value problem with Steklov boundary conditions depending on σ. Positivity of ground states is proved with different techniques according to the range of the parameter $\sigma \in \mathbb{R}$ and we also provide a convergence analysis for the ground states with respect to σ. Further results concerning positive radial solutions are established when the domain is a ball.

1. Introduction

The energy of a thin hinged plate under the action of a vertical external force of density f can be computed by the Kirchhoff–Love functional

$$I_\sigma(u) = \int_\Omega \frac{(\Delta u)^2}{2} - (1-\sigma) \int_\Omega \det(\nabla^2 u) - \int_\Omega f u,$$

where the bounded domain $\Omega \subset \mathbb{R}^2$ describes the shape of the plate and u its deflection from the original unloaded position. Since the plate is supposed to be hinged, the natural space in which to consider our problem is $H^2(\Omega) \cap H^1_0(\Omega)$. The coefficient σ, called the *Poisson ratio*, depends on the material and measures its transverse expansion (resp. contraction), according to its positive (resp. negative) sign, when subjected to an external compressing force. Due to some thermodynamic considerations in elasticity theory, the physical relevant interval for σ is $\left(-1, \frac{1}{2}\right)$. A detailed derivation of the model can be found in [Ventsel and Krauthammer 2001], while a mathematical analysis concerning the positivity-preserving property for I_σ has been carried out by Parini and Stylianou [2009]. Besides a further extension of their results, here we are interested in a direct generalization of the Kirchhoff–Love functional, namely when

the density f may depend also on the deflection of the plate itself:

$$J_\sigma(u) = \int_\Omega \frac{(\Delta u)^2}{2} - (1-\sigma)\int_\Omega \det(\nabla^2 u) - \int_\Omega F(x,u), \qquad (1\text{-}1)$$

where $F(x,s) = \int_0^s f(x,t)\,dt$, and furthermore we let $\sigma \in \mathbb{R}$. We are mainly interested in a power-type nonlinearity, namely

$$F(x,u) = \frac{g(x)|u|^{p+1}}{p+1}, \quad \text{where } g \in L^1(\Omega) \text{ and } g > 0 \text{ in } \Omega. \qquad (1\text{-}2)$$

In particular we look for existence and positivity of those critical points which have the lowest energy, referred to in the literature as *ground states*.

If the boundary is sufficiently smooth, searching for critical points of J_σ with the nonlinearity (1-2) is equivalent to finding weak solutions of the fourth-order semilinear boundary problem

$$\begin{cases} \Delta^2 u = g(x)|u|^{p-1}u & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa u_n = 0 & \text{on } \partial\Omega, \end{cases} \qquad (1\text{-}3)$$

where u_n stands for the normal derivative of u on $\partial\Omega$ and κ is the signed curvature of the boundary (positive on convex parts). These kinds of mixed boundary conditions are usually called *Steklov* from their first appearance in [Stekloff 1902] and they are an intermediate situation between Navier boundary conditions (when $\sigma = 1$) and Dirichlet boundary conditions ($u = u_n = 0$, seen as the limit case as $\sigma \to +\infty$).

Although fourth-order (or more generally, higher-order) problems have garnered attention even from the first decade of the 20th century, most of the literature deals with the Navier case, where the maximum principle still holds, or with Dirichlet boundary conditions, where Green's function arguments are available. Conversely, problems like (1-3) have been intensively studied only in the last decade, focusing on the associated boundary eigenvalue problems (see [Ferrero et al. 2005; Bucur et al. 2009]), the positivity-preserving property of the solution operator (see [Gazzola and Sweers 2008]) and some semilinear problems (see, for instance, [Berchio and Gazzola 2011; Berchio et al. 2006; 2007]).

This paper is a contribution to the study of semilinear subcritical biharmonic Steklov problems in low dimension. Here, we mainly focus on a nonlinearity of power-type as in [Berchio et al. 2007], where the critical exponent in high dimensions is considered and the domain is a ball. On the other hand, although some related subcritical problems have already appeared in [Berchio et al. 2006], we consider a slightly different kind of nonlinearity; we let σ be lying not only in the physical relevant interval, and the techniques involved are different.

Besides the existence of ground states for J_σ, we mainly investigate their positivity. The question is quite challenging since, like most fourth-order problems, one has to face the lack of a maximum principle. Moreover, we will show that positivity is strongly related to the parameter σ and different techniques are needed to cover different regions in which σ lies: the superharmonic method, some convergence arguments and the dual cones decomposition.

The main results contained in this paper may be summarized as follows:

Theorem 1.1 (existence, positivity). *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with $\partial\Omega$ of class $C^{1,1}$ and let $f(x, s) = g(x)|s|^{p-1}s$, with $p \in (0, 1) \cup (1, +\infty)$ and $g \in L^1(\Omega)$, $g > 0$ a.e. in Ω. Then there exist $\sigma^* \leq -1$ and $\sigma_1 > 1$ (depending on Ω and possibly infinite) such that the functional J_σ has no positive critical points if $\sigma \leq \sigma^*$, while it admits (at least) a positive ground state if $\sigma \in (\sigma^*, \sigma_1)$.*

Theorem 1.2 (convergence). *Under the previous assumptions for Ω and f, let $(u_k)_{k \in \mathbb{N}}$ be a sequence of ground states for the respective sequence of functionals $(J_{\sigma_k})_{k \in \mathbb{N}}$. Up to a subsequence,*

(i) *if $\sigma_k \searrow \sigma^*$, then $u_k \to 0$ in $H^2(\Omega)$ in the case $p > 1$, while $u_k \to +\infty$ in $L^\infty(\Omega)$ if $p \in (0, 1)$;*

(ii) *if $\sigma_k \to 1$, then $u_k \to \bar{u}$ in $W^{2,q}(\Omega)$ for every $q > 2$, where $\bar{u}$ is a ground state for the Navier problem;*

(iii) *if $\sigma_k \to +\infty$, then $u_k \to U$ in $H^2(\Omega)$, where U is a ground state for the Dirichlet problem.*

Notice that Theorem 1.1 might also be seen as an extension to the semilinear setting of the main positivity results established by Gazzola and Sweers [2008, Theorem 4.1] for the linear case.

Finally, we want to stress our attempt to impose only the strictly necessary assumptions on the domain in order to obtain our results and to have a well-defined second boundary condition in (1-3).

The paper is organized as follows: after a few preliminary results (Section 2), we establish existence (Section 3) and positivity (Section 4) of ground states of J_σ when σ belongs to the range $(-1, 1]$ (which contains the relevant physical interval) both for f sublinear and superlinear; the latter is due to an argument based on the Nehari manifold. Except for the last section, the rest of the paper is devoted to proving Theorem 1.1 in the cases $\sigma \leq -1$ (Section 5) and $\sigma > 1$ (Section 6). While the first situation is quite easy to handle, the positivity in the second is more delicate and requires different tools. In this context and also for this purpose, Theorem 1.2 will be established. Finally, Section 7 provides a further investigation in the case Ω is the unit ball, concerning generic positive radially symmetric solutions.

2. Notation and preliminary results

Throughout the paper, $\nabla^2 u$ stands for the Hessian matrix of u and the derivatives are denoted by subscripts $(u_x, u_{xy}, \dots)$. Moreover, n and τ will be the exterior normal and the tangent vector, and u_n and u_τ the normal and the tangential derivative of u. We say that u is *superharmonic* in Ω when $-\Delta u \geq 0$ in Ω and $u = 0$ on $\partial\Omega$; u is *strictly superharmonic* when we have in addition that $-\Delta u \not\equiv 0$.

Let $N \geq 2$. We say $\Omega \subset \mathbb{R}^N$ is a *domain* when it is open and connected; moreover, Ω has *a boundary of class $C^{k,1}$* if $\partial\Omega$ can be described in local coordinates by a C^k function with Lipschitz continuous k-th derivatives. Finally, Ω satisfies a *uniform external ball condition* if there exists $R > 0$ such that for all $x \in \partial\Omega$ there exists a ball B_R of radius R such that $x \in \partial B_R$ and $B_R \subset \mathbb{R}^N \setminus \bar{\Omega}$.

The topological dual of a normed space X is denoted by X^*; for $q \in [1, +\infty]$, the $L^q(\Omega)$ norm is denoted by $\|\cdot\|_q$ and we have

$$\big\||\nabla^k \cdot|\big\|_q := \left(\sum_{|\alpha|=k} \|D^\alpha \cdot\|_q^q\right)^{\frac{1}{q}},$$

where α is a multi-index.

Let us also recall that, if $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary, by Sobolev embeddings (see [Adams and Fournier 2003, Theorem 4.12, Part II]), $H^2(\Omega) \hookrightarrow C^{0,\lambda}(\overline{\Omega})$ for any $\lambda \in (0,1)$; thus we have continuous embedding in $L^q(\Omega)$ for every $q \in [1, \infty]$.

Finally, we present some very useful facts about equivalence of norms in $H^2(\Omega) \cap H_0^1(\Omega)$. The quoted results have been already obtained by Nazarov, Stylianou and Sweers in [Nazarov et al. 2012]; we will include the proof of the second equivalence in order to have a self-contained exposition.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ bounded with a Lipschitz boundary and $\sigma \in (-1, 1]$.*

(i) $\left\| |\nabla^2 \cdot| \right\|_2$ *and* $\| \cdot \|_{H^2(\Omega)}$ *are equivalent norms on* $H^2(\Omega) \cap H_0^1(\Omega)$.

(ii) *If $\sigma = 1$, assume additionally that Ω satisfies a uniform external ball condition. Then*

$$\|u\|_{H_\sigma(\Omega)} := \left(\int_\Omega (\Delta u)^2 - 2(1-\sigma) \int_\Omega \det(\nabla^2 u) \right)^{\frac{1}{2}} \tag{2-1}$$

defines a norm on $H^2(\Omega) \cap H_0^1(\Omega)$ equivalent to the standard norm.

Proof. We prove here only (ii) and we refer to [Nazarov et al. 2012, Corollary 5.4] for a proof of (i). Firstly

$$\|u\|_{H_\sigma(\Omega)}^2 = \int_\Omega u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 + 2\sigma(u_{xx}u_{yy} - u_{xy}^2)$$

$$\leq \left\| |\nabla^2 u| \right\|_2^2 + 2|\sigma| \left(\frac{u_{xx}^2 + u_{yy}^2}{2} + u_{xy}^2 \right) = (1 + |\sigma|) \left\| |\nabla^2 u| \right\|_2^2.$$

Moreover, if $\sigma \in (-1, 1)$, one has

$$\|u\|_{H_\sigma(\Omega)}^2 = \int_\Omega u_{xx}^2 + u_{yy}^2 + 2(1-\sigma)u_{xy}^2 + 2\sigma u_{xx}u_{yy}$$

$$\geq \int_\Omega u_{xx}^2 + u_{yy}^2 + 2(1-\sigma)u_{xy}^2 - |\sigma|(u_{xx}^2 + u_{yy}^2) \geq (1 - |\sigma|) \left\| |\nabla^2 u| \right\|_2^2. \tag{2-2}$$

The proof is completed by applying (i) and noticing that the map

$$(u, v)_{H_\sigma} \mapsto \int_\Omega \Delta u \, \Delta v - (1-\sigma) \int_\Omega u_{xx} v_{yy} + u_{yy} v_{xx} - 2u_{xy} v_{xy}$$

defines a scalar product on $H^2(\Omega) \cap H_0^1(\Omega)$ for every $\sigma \in (-1, 1)$ by the inequality (2-2). In the special case $\sigma = 1$, one has $\|u\|_{H_1(\Omega)} = \|\Delta u\|_2$, which is an equivalent norm on $H^2(\Omega) \cap H_0^1(\Omega)$ provided the external ball condition is satisfied (see [Adolfsson 1992]). □

In the following, $C_0 = C_0(\Omega)$ and $C_A = C_A(\Omega)$ indicate the smallest positive constants such that

$$\|u\|_{H^2(\Omega)}^2 := \|u\|_2^2 + \left\| |\nabla u| \right\|_2^2 + \left\| |\nabla^2 u| \right\|_2^2 \leq C_0 \left\| |\nabla^2 u| \right\|_2^2 \tag{2-3}$$

and

$$\|u\|_{H^2(\Omega)}^2 \leq C_A \|\Delta u\|_2^2 \tag{2-4}$$

for every $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

3. Existence of ground states

In this section we investigate the existence of critical points of the generalized Kirchhoff–Love functional $J_\sigma : H^2(\Omega) \cap H_0^1(\Omega) \to \mathbb{R}$ defined in (1-1) in the physical relevant interval $\sigma \in (-1, 1]$. Hereafter, we assume Ω to be a bounded domain in $\mathbb{R}^2$. Concerning the nonlinearity, the functional J_σ is well defined once we impose $F(\cdot, s) \in L^1(\Omega)$ and $F(x, \cdot) \in C^1(\mathbb{R})$ (and thus there exists $f(x, \cdot)$ continuous such that $F(x, s) = \int_0^s f(x, t) \, dt$) and a power-type growth control on F, namely the existence of $a, b \in L^1(\Omega)$ such that $|F(x, s)| \le a(x) + b(x)|s|^q$ for some $q > 0$. With these assumptions on F, it is a standard fact to prove that J_σ is a C^1 functional with Fréchet derivative

$$J_\sigma'(u)[v] = \int_\Omega \Delta u \, \Delta v - (1 - \sigma) \int_\Omega (u_{xx} v_{yy} + u_{yy} v_{xx} - 2 u_{xy} v_{xy}) - \int_\Omega f(x, u) v.$$

Notice that, if Ω satisfies the assumptions of Lemma 2.1, we can rewrite the functional as

$$J_\sigma(u) = \tfrac{1}{2} \|u\|_{H_\sigma(\Omega)}^2 - \int_\Omega F(x, u).$$

Our aim is to investigate the *ground states* of the functional J_σ, i.e., the critical points on which the functional assumes the lowest value. In fact, besides the interest from a physical point of view, we are able to characterize them variationally and thus to apply a larger number of analytical tools.

Since the geometry of the functional plays an important role, from now on we have to distinguish between the sublinear case, that is, when the density f has at most a slow linear growth in the real variable (as it will be specified in the following), and the superlinear case, the opposite one. In fact, we will see that in the first case J_σ behaves similarly to the linear Kirchhoff–Love functional studied in [Parini and Stylianou 2009] since it is coercive and ground states are global minima, while, in the second case, J_σ has a mountain pass geometry and the ground states are saddle points. Moreover, although in the sequel we will be mainly interested in the power-type nonlinearity as in (1-2), in the sublinear case we can easily generalize our analysis to a larger class of nonlinearities, as specified in Proposition 3.1.

We exclude from our analysis the case of general linear growths for the nonlinearity, for instance $f(x, u) = \lambda g(x) u$, since (1-3) becomes an eigenvalue problem and can be investigated with standard techniques (see also [Berchio et al. 2006, Theorem 4]).

Sublinear case.

Proposition 3.1. *With the assumptions for σ and Ω as in Lemma 2.1, let $p \in (0, 2)$ and suppose*

$$|F(x, s)| \le d(x) + c(x)|s|^p + \tfrac{1}{2}(1 - |\sigma|)C_0^{-1} s^2, \tag{H}$$

where $c, d \in L^1(\Omega)$. Then the functional J_σ is weakly lower semicontinuous and coercive; hence there exists a global minimizer of J_σ in $H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. Let $(u_k)_{k \in \mathbb{N}} \subset H^2(\Omega) \cap H_0^1(\Omega) \ni u$ be such that $u_k \rightharpoonup u$ weakly in $H^2(\Omega)$; since it is bounded in $H^2(\Omega)$ and consequently in $L^\infty(\Omega)$, one has

$$|F(x, u_k)| \le d(x) + c(x)M^p + \tfrac{1}{2}(1 - |\sigma|)C_0^{-1} M^2$$

for some $M > 0$, which is integrable over Ω. Moreover, by the compactness of the embedding $H^2(\Omega) \hookrightarrow L^p(\Omega)$, there exists a subsequence $(u_{k_j})_{j \in \mathbb{N}}$ such that $u_{k_j} \to u$ in $L^p(\Omega)$ for a suitable $p \geq 1$, so $F(x, u_{k_j}(x)) \to F(x, u(x))$ a.e. in Ω by continuity of $F(x, \cdot)$. Hence, by the dominated convergence theorem, we have $\int_\Omega F(x, u_{k_j}) \to \int_\Omega F(x, u)$. This, together with the weakly lower semicontinuity of the norm, implies the same property for J_σ. If $\sigma \in (-1, 1)$, by (2-3)

$$J_\sigma(u) \geq \tfrac{1}{2}(1 - |\sigma|) \big\| |\nabla^2 u| \big\|_2^2 - \|d\|_1 - C^p \|c\|_1 \|u\|_{H^2(\Omega)}^p - \tfrac{1}{2}(1 - |\sigma|)C_0^{-1} \|u\|_2^2$$

$$\geq \tfrac{1}{2}(1 - |\sigma|)C_0^{-1} \big\| |\nabla^2 u| \big\|_2^2 - \|c\|_1 C^p C_0^{\frac{p}{2}} \big\| |\nabla^2 u| \big\|_2^p - \|d\|_1;$$

by (i) of Lemma 2.1, we deduce that $J_\sigma(u) \to +\infty$ as $\|u\|_{H^2(\Omega)} \to +\infty$, since $p \in (0, 2)$. Easier computations provide a similar estimate to conclude the proof also if $\sigma = 1$. $\square$

Remark 3.2 (model case). As an application of Proposition 3.1, we may consider the following kind of sublinearity:

$$F(x, u) = g(x)|u|^{p+1} + d(x)u,$$

where $p \in (0, 1)$ and $d, g \in L^1(\Omega)$. In this case the functional is coercive and verifies (H). Notice also that if $g = 0$ we retrieve the linear Kirchhoff–Love functional considered in [Parini and Stylianou 2009].

Superlinear case. This case is more involved and we have to restrict to the nonlinearity (1-2) with $p > 1$:

$$J_\sigma(u) := \int_\Omega \frac{(\Delta u)^2}{2} - (1 - \sigma) \int_\Omega \det(\nabla^2 u) - \int_\Omega \frac{g(x)|u|^{p+1}}{p+1}. \tag{3-1}$$

Here the functional is not coercive anymore: in fact, fixing any $u \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$, we have $J_\sigma(tu) \to -\infty$ as $t \to +\infty$. Following some arguments of [Castro et al. 1997; Grumiau and Parini 2008], we will make use of the method of the Nehari manifold to infer the existence of a (nontrivial) critical point. After some preliminary results, we will show that in our manifold the infimum of J_σ is attained and then, using a deformation lemma, we will prove it is a critical point for J_σ in $H^2(\Omega) \cap H_0^1(\Omega)$.

Let us define the *Nehari manifold* of J_σ as the set

$$\mathcal{N}_\sigma := \big\{ u \in (H^2(\Omega) \cap H_0^1(\Omega)) \setminus \{0\} \ \big| \ J_\sigma'(u)[u] = 0 \big\},$$

which clearly contains all nontrivial critical points of J_σ. First of all, notice that $u \in \mathcal{N}_\sigma$ if and only if

$$\int_\Omega (\Delta u)^2 - 2(1 - \sigma) \int_\Omega \det(\nabla^2 u) = \int_\Omega g(x)|u|^{p+1},$$

so one has two equivalent formulations for J_σ restricted on $\mathcal{N}_\sigma$,

$$J_{\sigma|_{\mathcal{N}_\sigma}}(u) = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_\Omega g(x)|u|^{p+1} = \left(\frac{1}{2} - \frac{1}{p+1} \right) \left(\int_\Omega (\Delta u)^2 - 2(1 - \sigma) \int_\Omega \det(\nabla^2 u) \right), \tag{3-2}$$

which implies $J_{\sigma|_{\mathcal{N}_\sigma}}(u) > 0$ for every $u \in \mathcal{N}_\sigma$.

A crucial step will be to study what happens on the half-lines of $H^2(\Omega) \cap H_0^1(\Omega)$:

Lemma 3.3. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$ and the half-line r_u be defined as $r_u := \{tu \mid t > 0\}$. The intersection between r_u and $\mathcal{N}_\sigma$ consists of a unique point $t^*(u)u$, where*

$$t^*(u) := \left(\frac{\int_\Omega (\Delta u)^2 - 2(1-\sigma)\int_\Omega \det(\nabla^2 u)}{\int_\Omega g(x)|u|^{p+1}} \right)^{\frac{1}{p-1}}. \tag{3-3}$$

Moreover $J_\sigma(t^*(u)u) = \max_{t>0} J_\sigma(tu)$.

Proof. For $t > 0$ and a fixed $u \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$, we have $tu \in \mathcal{N}_\sigma$ if and only if

$$t^2 \left[\int_\Omega (\Delta u)^2 - 2(1-\sigma)\int_\Omega \det(\nabla^2 u) \right] = t^{p+1} \int_\Omega g(x)|u|^{p+1},$$

from which we deduce $t = t^*(u)$. Moreover, define

$$\eta(t) := J_\sigma(tu) = \frac{t^2}{2} \left[\int_\Omega (\Delta u)^2 - 2(1-\sigma)\int_\Omega \det(\nabla^2 u) \right] - \frac{t^{p+1}}{p+1} \int_\Omega g(x)|u|^{p+1}.$$

If we look for $\bar{t} > 0$ such that $\eta'(\bar{t}) = 0$, we find again that $\bar{t} = t^*(u)$ and, since $\eta'(t)(t - t^*(u)) < 0$ for $t \neq t^*(u)$, we have that $t^*(u)u$ is the unique global maximum in the half-line r_u. $\square$

Lemma 3.4. *The Nehari manifold is bounded away from 0; i.e., $0 \notin \overline{\mathcal{N}}_\sigma$.*

Proof. Suppose first that $\sigma \in (-1, 1)$ and let $u \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$. By Lemmas 2.1 and 3.3 and the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, the following chain of inequalities holds:

$$(1 + |\sigma|)\|t^*(u)u\|_{H^2(\Omega)}^2 \geq \|t^*(u)u\|_{H_\sigma(\Omega)}^2 = (t^*(u))^{p+1} \int_\Omega g(x)|u|^{p+1}$$

$$\geq (C_0^{-1}(1-|\sigma|))^{\frac{p+1}{p-1}} \frac{\|u\|_{H^2(\Omega)}^{\frac{2(p+1)}{p-1}}}{\left(\int_\Omega g(x)|u|^{p+1}\right)^{\frac{2}{p-1}}}$$

$$\geq C(\Omega, p, \sigma) \frac{\|u\|_{H^2(\Omega)}^{\frac{2(p+1)}{p-1}}}{\left(\|g\|_1 \|u\|_{H^2(\Omega)}^{p+1}\right)^{\frac{2}{p-1}}} = \frac{C(\Omega, p, \sigma)}{\|g\|_1^{\frac{2}{p-1}}}.$$

If $\sigma = 1$, one can deduce the same result using the equivalent norm on $H^2(\Omega) \cap H_0^1(\Omega)$ given by $\|\Delta \cdot\|_2$. In both cases, there exists a uniform bound from below for the $H^2(\Omega)$ norm of the elements in the Nehari manifold and thus 0 cannot be a cluster point for $\mathcal{N}_\sigma$. $\square$

Proposition 3.5. *There exists $u \in \mathcal{N}_\sigma$ such that $J_\sigma(u) = \inf_{v \in \mathcal{N}_\sigma} J_\sigma(v) =: c$.*

Proof. As already noticed, $c \geq 0$, since it attains positive values on $\mathcal{N}_\sigma$. Let now $(u_k)_{k \in \mathbb{N}} \subset \mathcal{N}_\sigma$ be a minimizing sequence for J_σ: we claim that $(u_k)_{k \in \mathbb{N}}$ is bounded in $H^2(\Omega)$ norm. In fact, if $\sigma \in (-1, 1)$, there exists a constant $C > 0$ such that, for every $k \in \mathbb{N}$,

$$C \geq J_\sigma(u_k) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_k\|_{H_\sigma(\Omega)}^2 \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)(1 - |\sigma|)C_0^{-1}\|u_k\|_{H^2(\Omega)}^2,$$

while (2-4) provides the right estimate in the case $\sigma = 1$. Hence, there exists a subsequence $(u_{k_j})_{j \in \mathbb{N}} \subset \mathcal{N}_\sigma$ and $u \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$ such that $u_{k_j} \rightharpoonup u$ weakly in $H^2(\Omega)$ (and so weakly in $(H^2(\Omega) \cap H_0^1(\Omega),$ $\|\cdot\|_{H\sigma})$ by Lemma 2.1) and strongly in $L^\infty(\Omega)$, by compact embedding. Consider now $t^* = t^*(u)$ such that $t^* u \in \mathcal{N}_\sigma$: by weak semicontinuity of the norm,

$$
\begin{aligned}
c = \inf_{v \in \mathcal{N}_\sigma} J_\sigma(v) &\leq J(t^* u) = (t^*)^2 \left[\int_\Omega \frac{(\Delta u)^2}{2} - (1 - \sigma) \int_\Omega \det(\nabla^2 u) \right] - (t^*)^{p+1} \int_\Omega \frac{g(x)|u|^{p+1}}{p+1} \\
&\leq \liminf_{j \to +\infty} \left((t^*)^2 \left[\int_\Omega \frac{(\Delta u_{k_j})^2}{2} - (1 - \sigma) \int_\Omega \det(\nabla^2 u_{k_j}) \right] - (t^*)^{p+1} \int_\Omega \frac{g(x)|u_{k_j}|^{p+1}}{p+1} \right) \\
&= \liminf_{j \to +\infty} J_\sigma(t^* u_{k_j}) \leq \liminf_{j \to +\infty} J_\sigma(u_{k_j}) = c, \qquad\qquad\qquad\qquad (3\text{-}4)
\end{aligned}
$$

where the last inequality holds because the supremum of J_σ in each half-line $\{t u_{k_j} \mid t > 0\}$ is achieved exactly in u_{k_j} by Lemma 3.3. Hence, the infimum of J_σ on $\mathcal{N}_\sigma$ is attained on $t^* u$. $\square$

In the proof of Proposition 3.5 something weird happened: we took a minimizing sequence, which converges to an element u and we proved that there exists $\alpha = t^*(u) \in \mathbb{R}$ such that αu is the minimum point of our functional J_σ. One expects that the minimum is u itself and not a dilation of it. Indeed, one may prove that $t^* = 1$. In fact, with the same notation as in that proof, from (3-4) we deduce $J_\sigma(u_{k_j}) \to c = J_\sigma(t^* u)$ by construction and $t^* u \in \mathcal{N}_\sigma$, so

$$
J_\sigma(u_{k_j}) \to \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_\Omega g(x)|t^* u|^{p+1}.
$$

Moreover, we took the sequence to be in the Nehari manifold itself, so

$$
J_\sigma(u_{k_j}) = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_\Omega g(x)|u_{k_j}|^{p+1},
$$

and we have that $u_{k_j} \to u$ strongly in $L^\infty(\Omega)$; thus

$$
J_\sigma(u_{k_j}) \to \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_\Omega g(x)|u|^{p+1}.
$$

By the uniqueness of the limit, we must have $t^* = 1$, so $u \in \mathcal{N}_\sigma$.

Theorem 3.6. *The minimum u of J_σ in $\mathcal{N}_\sigma$ is a critical point for J_σ in $H^2(\Omega) \cap H_0^1(\Omega)$.*

Proof. Suppose by contradiction that u is not a critical point. Since the functional is C^1, there exists a ball centered in u and $\varepsilon > 0$ such that, for all $v \in B$,

$$
c - \varepsilon \leq J_\sigma(v) \leq c + \varepsilon, \quad \|J_\sigma'(v)\|_{(H^2(\Omega) \cap H_0^1(\Omega))^*} \geq \tfrac{1}{2}\varepsilon,
$$

where $c = J_\sigma(u) = \inf_{v \in \mathcal{N}_\sigma} J_\sigma(v)$. Notice that on the half-line r_u, the point u is the global maximum, so $J_\sigma(v) < c$ for each $v \in B \cap r_u$, $v \neq u$.

If we define $a = c - \varepsilon$, $b = c + \varepsilon$, $\delta = 8$, $S = \overline{B_r(u)}$ and $S_0 = \overline{H^2(\Omega) \cap H_0^1(\Omega) \setminus B'}$, where $r > 0$ such that $B_r(u) \Subset B' \Subset B$, applying [Gasiński and Papageorgiou 2006, Proposition 5.1.25], there exists a locally Lipschitz homotopy of homeomorphisms Γ_t on $H^2(\Omega) \cap H_0^1(\Omega)$ such that

(i) $t \mapsto J_\sigma(\Gamma(t, v))$ is decreasing in $B_r(u)$ and, in general, nonincreasing;

(ii) $J_\sigma(\Gamma(t, v)) = v$ for $v \in S_0$ and $t \in [0, 1]$, and so also for all $v \in \partial B$.

From (i) we deduce that $J_\sigma(\Gamma(t, v)) < c$ for every $v \in B \cap r_u$ and $t \neq 0$. Moreover, define the map $\psi : B \cap r_u \to \mathbb{R}$ such that

$$\psi(v) := J'_\sigma(\Gamma(1, v))[\Gamma(1, v)]$$

and consider $v \in \partial B \cap r_u$, so there exists $\alpha \neq 1$ such that $v = \alpha u$: we know from (ii) that $\Gamma(1, v) = v$ and, by Lemma 3.3, $J'_\sigma(\alpha u)[\alpha u] > 0$ if $\alpha \in (0, 1)$ and $J'_\sigma(\alpha u)[\alpha u] < 0$ if $\alpha \in (1, +\infty)$, so $\psi(v)(v - u) < 0$ on $\partial B \cap r_u$. As a result, since one can think of ψ as a continuous map from $[x_1, x_2] \to \mathbb{R}$, where x_1 and x_2 correspond to the intersections between the half-line r_u and the ball B, and since $\psi(x_1) > 0$ and $\psi(x_2) < 0$, there exists a zero of ψ in (x_1, x_2); i.e., there exists $\bar{v} \in B \cap r_u$ such that $J'_\sigma(\Gamma(1, \bar{v}))[\Gamma(1, \bar{v})] = 0$.

Setting $w := \Gamma(1, \bar{v})$, we have $w \in \mathcal{N}_\sigma$ and $J_\sigma(w) = J_\sigma(\Gamma(1, \bar{v})) < c = \inf_{v \in \mathcal{N}_\sigma} J_\sigma(v)$, a contradiction. $\square$

So far, we proved the existence of a ground state for J_σ. Actually, one can say more about the existence of general critical points by means of the Krasnoselskiĭ genus theory (see [Ambrosetti and Malchiodi 2007, Section 10.2]). In fact, since our framework is subcritical, it is quite standard to prove the Palais–Smale condition for J_σ by a compact embedding of $H^2(\Omega)$ in every Lebesgue space. Moreover, our functional is C^1, even and bounded from below on the unit sphere of $H^2(\Omega) \cap H^1_0(\Omega)$: indeed, if $\|u\|_{H_\sigma(\Omega)} = 1$, then $\|u\|_\infty < C$ for some $C > 0$, so

$$J_\sigma(u) = \frac{1}{2} - \int_\Omega \frac{g(x)|u|^{p+1}}{p+1} \geq \frac{1}{2} - \frac{C^{p+1}\|g\|_1}{p+1} > -\infty.$$

Hence, by [Ambrosetti and Malchiodi 2007, Proposition 10.8], one can ensure the existence of an infinite number of couples of critical points. The same argument may also be applied for the general sublinear case, provided $F(x, s) = F(x, -s)$ for every $s \in \mathbb{R}$.

4. An identity and the positivity of ground states in convex domains

The aim of this section is to prove positivity for the ground states found in the previous section. Notice that the problematic term in J_σ is the one involving the determinant of the Hessian matrix. In order to overcome this difficulty, we need to rewrite it in an equivalent way, transforming it into a boundary term which can be handled in order to prove the desired positivity. Nevertheless, since the signed curvature of the boundary will be involved, we need to impose some regularity on $\partial\Omega$. We will basically deduce the same statement as Lemma 2.5(ii) of [Parini and Stylianou 2009], but we will extend it to a larger class of domains.

A crucial identity. Our goal is to generalize the following result:

Theorem 4.1 [Parini and Stylianou 2009, Lemma 2.5]. *Let Ω be a bounded domain in $\mathbb{R}^2$ with $C^{2,1}$ boundary, and let κ be its signed curvature. Then for all $u \in H^2(\Omega)$ and for every $\varphi \in H^3(\Omega)$, defining $K(u) := \int_\Omega \det(\nabla^2 u)\,dx$, we have*

$$\langle K'(u), \varphi \rangle = \int_{\partial\Omega} (\kappa\varphi_n u_n + \varphi_{\tau\tau} u_n - \varphi_{\tau n} u_\tau). \tag{F$_{\text{PS}}$}$$

Hence, for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$K(u) = \frac{1}{2} \int_{\partial\Omega} \kappa u_n^2. \tag{F}$$

Going into the details of its proof, one can actually realize that the strong regularity assumption on the boundary was needed only to derive (F) from (F_{PS}) because the authors used the density of $H^3(\Omega) \cap H_0^1(\Omega)$ in $H^2(\Omega) \cap H_0^1(\Omega)$, which strongly relied on the fact that $\partial\Omega \in C^{2,1}$ (see [Parini and Stylianou 2009, Lemma 2.3]). Nevertheless, (F_{PS}) requires only that all the elements therein are well defined. Hence, our starting point is the following:

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,1}$. Then for every $v \in C^\infty(\overline{\Omega})$,*

$$K(v) = \frac{1}{2}\langle K'(v), v\rangle = \frac{1}{2} \int_{\partial\Omega} (\kappa v_n^2 - (v_{n\tau} + v_{\tau n})v_\tau). \tag{$F_{PS}2$}$$

Proof. One only has to notice that if $\partial\Omega \in C^{1,1}$, then κ is in $L^\infty(\partial\Omega)$ and

$$\int_{\partial\Omega} (v_{n\tau}v_\tau + v_n v_{\tau\tau}) = \int_{\partial\Omega} (v_n v_\tau)_\tau = 0,$$

as $\partial\Omega$ is a closed curve and by the definition of the tangential derivative (i.e., as $(d/ds)u(\gamma(s))$, where γ is the parametrization of the curve $\partial\Omega$ in the arch parameter s). $\qquad \square$

Our strategy consists of two steps: using ($F_{PS}2$), we will firstly prove that (F) holds also for every $v \in C_0^{1,1}(\overline{\Omega}) := \{u \in C^{1,1}(\overline{\Omega}) \mid u_{|\partial\Omega} = 0\}$; then, by a density result, we will transfer (F) from $C_0^{1,1}(\overline{\Omega})$ to $H^2(\Omega) \cap H_0^1(\Omega)$. We will make use of the following lemma, which makes a well-known result more precise:

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^1 and $u \in C^{1,1}(\overline{\Omega})$. Then there exists a sequence $(u_k)_{k\in\mathbb{N}} \in C^\infty(\overline{\Omega})$ such that $u_k \to u$ in $H^2(\Omega)$ and $\|u_k\|_{W^{2,\infty}(\Omega)} \le C \|u\|_{W^{2,\infty}(\Omega)}$ for some positive constant C.*

Proof. First of all notice that $C^{1,1}(\overline{\Omega})$ can be equivalently seen as $W^{2,\infty}(\Omega)$, which is a subset of $H^2(\Omega)$ since Ω is a bounded domain; moreover the fact that $C^\infty(\overline{\Omega})$ is dense in $H^2(\Omega)$ in $H^2(\Omega)$ norm if $\partial\Omega$ is of class C^1 is a standard fact (see [Evans 2010, Section 5.3.3, Theorem 3]), so the only statement to be verified is the $W^{2,\infty}(\Omega)$ estimate. Since the main tool in the proof of the $H^2(\Omega)$ convergence is the local approximation, which is achieved by mollification, we only have to prove that the same inequality holds there. So, let $v \in L^\infty(\Omega)$, $\varepsilon > 0$ and consider

$$v_\varepsilon(x) := (\eta_\varepsilon * v)(x) = \int_{B_\varepsilon(0)} \eta_\varepsilon(y)v(x-y)\, dy,$$

where η_ε is the standard mollifier in $\mathbb{R}^N$, that is $\eta_\varepsilon := \varepsilon^{-n}\eta(x/\varepsilon)$ and

$$\eta(x) = \widetilde{C} e^{\frac{1}{|x|^2-1}} \chi_{B_1(0)}(x),$$

where $\widetilde{C} > 0$ such that $\int_{B_1(0)} \eta(z)\, dz = 1$. So v_ε is in $\Omega_\varepsilon := \{x \in \Omega \mid d(x, \partial\Omega) > \varepsilon\}$ and we know that $v_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and η_ε is such that $\int_{B_\varepsilon(0)} \eta_\varepsilon(z)\, dz = 1$.

We claim that $\|v_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \le \|v\|_{L^\infty(\Omega)}$. In fact,

$$\|v_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \le \sup_{x \in \Omega_\varepsilon} \int_{B_\varepsilon(0)} |\eta_\varepsilon(z)| \, |v(x-z)| \, dz \le \|v\|_{L^\infty(\Omega)} \int_{B_\varepsilon(0)} |\eta_\varepsilon(z)| \, dz = \|v\|_{L^\infty(\Omega)}.$$

Also for derivatives of v the same inequality holds, because for any admissible multi-index α we know that $D^\alpha(v_\varepsilon) = (D^\alpha(v))_\varepsilon$ (see [Gilbarg and Trudinger 1998, Lemma 7.3]).

At this point, following the aforementioned proof of [Evans 2010], it is easy to derive the desired result. $\qquad\square$

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,1}$. Then, for all $u \in C_0^{1,1}(\overline{\Omega})$,*

$$\int_\Omega \det(\nabla^2 u) = \frac{1}{2} \int_{\partial\Omega} \kappa u_n^2.$$

Proof. Applying Lemma 4.3, let $(u_k)_{k \in \mathbb{N}} \subset C^\infty(\overline{\Omega})$ be a sequence converging to u in $H^2(\Omega)$, whose norms in $W^{2,\infty}$ are controlled by the $W^{2,\infty}$ norm of u. By Corollary 4.2, the following identity holds:

$$K(u_k) = \frac{1}{2} \int_{\partial\Omega} \left[\kappa(u_k)_n^2 - ((u_k)_{n\tau} + (u_k)_{\tau n})(u_k)_\tau \right]. \tag{4-1}$$

By the convergence in $H^2(\Omega)$, one clearly has $K(u_k) \to K(u)$; moreover, since $\kappa \in L^\infty(\partial\Omega)$ and using the trace theorem, one can deduce also that

$$\int_{\partial\Omega} \kappa(u_k)_n^2 \to \int_{\partial\Omega} \kappa u_n^2.$$

Finally we have to consider the terms in which tangential derivatives are involved. Similarly to the normal derivative, one has $(u_k)_\tau \to u_\tau$ in $L^2(\partial\Omega)$, so $(u_k)_\tau \to 0$ in $L^2(\partial\Omega)$, since $u_{|\partial\Omega} = 0$. Furthermore,

$$(u_k)_{n\tau} = \nabla(u_k)_n \cdot \tau = \nabla(\nabla u_k \cdot n) \cdot \tau = (\nabla^2 u_k \cdot n + \nabla u_k \cdot \nabla n) \cdot \tau$$

and (see [Sperb 1981, Chapter 4])

$$(u_k)_{\tau n} = \sum_{i,j=1}^2 \frac{\partial^2 u_k}{\partial x_i \, \partial x_j} \tau_i n_j$$

and one can infer that $(u_k)_{n\tau}$ and $(u_k)_{\tau n}$ are uniformly bounded in $L^2(\partial\Omega)$. In fact, since the u_k are C^∞ functions and using Lemma 4.3,

$$\|(u_k)_{n\tau}\|_{L^2(\partial\Omega)} \le |\partial\Omega|^{\frac{1}{2}} \|(u_k)_{n\tau}\|_{L^\infty(\partial\Omega)} \le |\partial\Omega|^{\frac{1}{2}} \left(\left\| \nabla^2 u_k \cdot n \right\|_{L^\infty(\partial\Omega)} + \left\| \nabla u_k \cdot \nabla n \right\|_{L^\infty(\partial\Omega)} \right)$$

$$\le 2|\partial\Omega|^{\frac{1}{2}} \|n\|_{W^{1,\infty}(\partial\Omega)} \|u_k\|_{W^{2,\infty}(\Omega)} \le C(\Omega) \|u\|_{W^{2,\infty}(\Omega)}$$

and similarly for $(u_k)_{\tau n}$. Consequently,

$$\int_{\partial\Omega} \left((u_k)_{n\tau} + (u_k)_{\tau n} \right)(u_k)_\tau \to 0. \qquad\square$$

In order to extend (F) to the space $H^2(\Omega) \cap H_0^1(\Omega)$, we need a density result (Lemma 4.6 below) which is taken from [Stylianou 2010, Theorem 2.2.4] and which can be adapted to our context: in fact, it

concerns C^2 functions and diffeomorphisms but, with a little care, one can obtain the same result also in the class $C^{1,1}$.

Definition 4.5 [Adams and Fournier 2003, 3.40, p. 77]. Let Φ be a one-to-one transformation of a domain $\Omega \subset \mathbb{R}^N$ onto a domain $G \subset \mathbb{R}^N$ having inverse $\Psi := \Phi^{-1}$. We say that Φ is a $C^{1,1}$ diffeomorphism if, writing $\Phi = (\Phi_1, \ldots, \Phi_N)$ and $\Psi = (\Psi_1, \ldots, \Psi_N)$, then $\Phi_i \in C^{1,1}(\overline{\Omega})$ and $\Psi_i \in C^{1,1}(\overline{G})$ for every $i \in \{1, \ldots, N\}$.

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^N$ be bounded and open such that for every $x \in \partial\Omega$ there exists a $j \in \{0, \ldots, N-1\}$, $\varepsilon > 0$ and a $C^{1,1}$-diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ such that the following hold:*

- $\Phi(x) = 0$.
- $\Phi(B_\varepsilon(x) \cap \Omega) \subset S_j := \{x = (x_1, \ldots, x_N) \in \Omega \mid x_i > 0 , \forall i > j\}$.
- $\Phi(B_\varepsilon(x) \cap \partial\Omega) \subset \partial S_j$.

Then

$$\overline{C_0^{1,1}(\overline{\Omega})}^{\|\cdot\|_{H^2(\Omega)}} = H^2(\Omega) \cap H_0^1(\Omega).$$

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,1}$. Then, for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$,*

$$\int_\Omega \det(\nabla^2 u) = \frac{1}{2} \int_{\partial\Omega} \kappa u_n^2. \tag{F}$$

Proof. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$; since the assumptions on the boundary are clearly fulfilled if $\partial\Omega$ is of class $C^{1,1}$, applying Lemma 4.6 we get an approximating sequence $(u_k)_{k\in\mathbb{N}} \subset C_0^{1,1}(\overline{\Omega})$ converging in $H^2(\Omega)$ to u. With the same steps as in the proof of Proposition 4.4, by the $H^2(\Omega)$ convergence, we have both $K(u_k) \to K(u)$ and $\int_{\partial\Omega} \kappa (u_k)_n^2 \to \int_{\partial\Omega} \kappa u_n^2$ and one concludes by the uniqueness of the limit. $\square$

From the functional to the PDE. As already briefly mentioned in the Introduction, if the boundary is smooth enough ($\partial\Omega$ of class $C^{4,\alpha}$ for $\alpha > 0$), standard elliptic regularity results apply and one can integrate by parts the Euler–Lagrange equation from J_σ to see that critical points satisfy (1-3). On the other hand, assuming only that the boundary is of class $C^{1,1}$, the signed curvature is well defined in $L^\infty(\Omega)$ and we can have a weak formulation of problem (1-3). More precisely, in this case, by a *weak solution* of (1-3) here we mean a function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ which satisfies

$$\int_\Omega \Delta u \, \Delta\varphi - (1-\sigma) \int_{\partial\Omega} \kappa u_n \varphi_n = \int_\Omega g(x)|u|^{p-1}u\varphi \quad \text{for all } \varphi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{4-2}$$

Consequently, we can equivalently say "ground states of J_σ" or "ground state solutions for (1-3)". For a proof of the equivalence of the two problems, we refer to [Gazzola and Sweers 2008].

Positivity of ground states in convex domains. Assuming $\partial\Omega$ is of class $C^{1,1}$, Theorem 4.7 enables us to rewrite the functional J_σ in a more convenient way: in fact, we deduce that for every $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$J_\sigma(u) = \int_\Omega \frac{(\Delta u)^2}{2} - \frac{1-\sigma}{2} \int_{\partial\Omega} \kappa u_n^2 - \int_\Omega F(x, u), \tag{4-3}$$

where we recall that $F(x, s) = \int_0^s f(x, t)\, dt$.

With this formulation, now we are able to establish the positivity of ground states of the functional J_σ in convex domains with boundary of class $C^{1,1}$ if the density function $f(x, u)$ is nonnegative, both in the sublinear and superlinear case. We will make use of the method of the superharmonic function, which is quite a standard tool when dealing with fourth-order problems and which has already been successfully used, for instance, in [Berchio et al. 2006; Gazzola and Sweers 2008; Nazarov et al. 2012], and whose core is contained in the following lemma:

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded convex domain; fix $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and define $\tilde{u}$ as the unique solution in $H_0^1(\Omega)$ of the Poisson problem*

$$\begin{cases} -\Delta \tilde{u} = |\Delta u| & in \ \Omega, \\ \tilde{u} = 0 & on \ \partial\Omega. \end{cases} \tag{4-4}$$

Then $\tilde{u} \in H^2(\Omega) \cap H_0^1(\Omega)$ and either $\tilde{u} > |u|$ in Ω and $\tilde{u}_n^2 \geq u_n^2$ on $\partial\Omega$ or $\tilde{u} = u$ in Ω.

Proof. Since Ω is convex by assumption, it satisfies in particular a uniform external ball condition and thus, by [Adolfsson 1992], we infer $\tilde{u} \in H^2(\Omega)$. Suppose $\tilde{u} \not\equiv u$. Since in particular $-\Delta\tilde{u} \geq \Delta u$ holds, by the maximum principle for strong solutions (see [Gilbarg and Trudinger 1998, Theorem 9.6]), one has $\tilde{u} > -u$ in Ω and so $\tilde{u}_n \leq -u_n$. Similarly, $-\Delta\tilde{u} \geq -\Delta u$ implies also $\tilde{u} > u$ and $\tilde{u}_n \leq u_n$ and so, combining them, we have the result. $\qquad\square$

Proposition 4.9 (sublinear case). *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with $\partial\Omega$ of class $C^{1,1}$ and $\sigma \in (-1, 1]$. In addition to the assumption* (H), *suppose also that $f \geq 0$ and is positive for a subset of positive measure. If $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a nontrivial minimizer of J_σ, then u is strictly superharmonic in Ω, and thus positive.*

Proof. Firstly notice that $\kappa \geq 0$ a.e. on $\partial\Omega$ by the convexity of Ω. From u, define its superharmonic function $\tilde{u}$ as in Lemma 4.8. Supposing $\tilde{u} \not\equiv u$, by that result we can infer

$$J_\sigma(\tilde{u}) = \int_\Omega \frac{(\Delta\tilde{u})^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa\tilde{u}_n^2 - \int_\Omega F(x, \tilde{u}) \leq \int_\Omega \frac{(\Delta u)^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa u_n^2 - \int_\Omega F(x, \tilde{u}). \tag{4-5}$$

Nevertheless, since $(\partial F/\partial s) = f \geq 0$, we have also that $F(x, u) < F(x, \tilde{u})$, and thus $J_\sigma(\tilde{u}) < J_\sigma(u)$, which leads to a contradiction. Hence necessarily $\tilde{u}$ coincides with u, so $-\Delta u = -\Delta\tilde{u} = |\Delta u| \geq 0$. As $u = 0$ on $\partial\Omega$ and $u \not\equiv 0$, we deduce $u > 0$ in Ω. $\qquad\square$

It is clear that, when $f(x, 0) \neq 0$, by Proposition 3.1, we always find a *nontrivial* global minimizer, which is positive by Proposition 4.9. For homogeneous nonlinearities this is not true in general. Anyway, for our model $f(x, s) = g(x)|s|^{p-1}s$, if we restrict our attention to the Nehari set, we easily see

$$J_\sigma(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_{H_\sigma} < 0$$

for every $u \neq 0$. So it is clear that in the minimization process we do not fall on 0. The same argument holds for more general nonlinearities $f(x, u)$, provided

$$\tfrac{1}{2}f(x, u)u - F(x, u) < 0 \quad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega).$$

For instance this holds when $f(x, s) = g(x)|s|^{p-1}s + h(x)|s|^{q-1}s$ for $g, h > 0$, $p, q \in (0, 1)$.

Remark 4.10. We stress here that, as a direct consequence of Proposition 4.9, we have obtained the positivity-preserving property also in the case of f not depending on u, i.e., for the linear Kirchhoff–Love functional I_σ (see also Remark 3.2). This generalizes the corresponding result by Parini and Stylianou [2009, Theorem 3.1] for bounded convex domains assuming only $C^{1,1}$ regularity on the boundary.

In our sublinear model case $f(x, s) = g(x)|s|^{p-1}s$, $p \in (0, 1)$, something more may be deduced: in fact, Lemma 3.3 still applies and, with the same steps as in the proof of Lemma 3.4, (reversing the inequalities since now $p - 1 < 0$), one ends up with

$$\|u\|_{H^2(\Omega)} \leq \left(\frac{\|g\|_1 C(\Omega)}{(1 - |\sigma|)C_0^{-1}} \right)^{\frac{1}{1-p}} \quad \text{for all } u \in \mathcal{N}_\sigma.$$

As a result, we can state the following:

Proposition 4.11. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and let $g \in L^1(\Omega)$ be positive a.e. in Ω. For every $\sigma \in (-1, 1)$ fixed, all critical points of J_σ with $f(x, s) = g(x)|s|^{p-1}s$ and $p \in (0, 1)$ are uniformly bounded in $H^2(\Omega)$.*

Notice that by continuous embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, one may also infer an a priori L^∞ bound for all critical points of J_σ. The estimate becomes also uniform with respect to σ if we restrict $\sigma \in I \Subset (-1, 1)$.

Concerning the superlinear case with the nonlinearity (1-2), we obtain the same positivity result with the same assumptions on Ω and σ:

Proposition 4.12 (superlinear case). *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with $\partial\Omega$ of class $C^{1,1}$ and $\sigma \in (-1, 1]$. Moreover suppose $f(x, u) = g(x)|u|^{p-1}u$, where $p > 1$ and $g \in L^1(\Omega)$ positive a.e. in Ω. Then the ground states of the functional J_σ are positive in Ω.*

Proof. Suppose, by contradiction, that there exists $u \in \mathcal{N}_\sigma$ such that $J_\sigma(u) = \inf\{J_\sigma(v) \mid v \in \mathcal{N}_\sigma\}$ and u is not positive. With the same spirit of the proof of Proposition 4.9, consider the superharmonic function $\tilde{u}$ associated to u and suppose they are not the same. This time the inequality (4-5) is not sufficient to have a contradiction since we do not know whether $\tilde{u} \in \mathcal{N}_\sigma$. Nevertheless, by Lemma 3.3, there exists $t^* := t^*(\tilde{u}) \in \mathbb{R}^+$ such that $t^*\tilde{u} \in \mathcal{N}_\sigma$. Then,

$$
\begin{aligned}
J_\sigma(t^*\tilde{u}) &= (t^*)^2\left[\int_\Omega \frac{(\Delta\tilde{u})^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa\tilde{u}_n^2\right] - (t^*)^{p+1}\int_\Omega \frac{g(x)|\tilde{u}|^{p+1}}{p+1} \\
&< (t^*)^2\left[\int_\Omega \frac{(\Delta u)^2}{2} - \frac{1-\sigma}{2}\int_{\partial\Omega} \kappa u_n^2\right] - (t^*)^{p+1}\int_\Omega \frac{g(x)|u|^{p+1}}{p+1} = J_\sigma(t^*u) \leq J_\sigma(u), \quad (4\text{-}6)
\end{aligned}
$$

which is again a contradiction. Notice that the last inequality holds since, by Lemma 3.3, J_σ restricted to every half-line attains its maximum on the Nehari manifold. Thus necessarily $\tilde{u}$ coincides with u, which implies that u is strictly superharmonic and thus positive. $\square$

Remark 4.13. Notice that in the proofs of Propositions 4.9 and 4.12, if σ lies in the interval $(-1, 1]$, the assumption Ω convex was necessary to have the good inequality for the second term of J_σ; on the other hand, if $\sigma > 1$ we do not have anymore the right sign and we cannot conclude the argument.

5. Beyond the physical bounds: $\sigma \leq -1$

So far, we studied the existence of critical points of the functional J_σ with the assumption $\sigma \in (-1, 1]$, we described in a variational way the geometry of the ground states and we finally established their positivity. The aim of this section is to study what happens to the ground states of J_σ if we let the parameter be in the whole $\mathbb{R}$. Again, we are especially interested in studying their positivity.

Since the study is rather different if $\sigma \leq -1$ or $\sigma > 1$, we divide the subject into two sections. In both, we will always assume that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain of class $C^{1,1}$ so that Theorem 4.7 holds. Moreover, as it seems more interesting from a mathematical point of view, we mainly focus on the superlinear case $f(x, u) = g(x)|u|^{p-1}u$ with $p > 1$, pointing out, if needed, the necessary adaptation for the sublinear power $p \in (0, 1)$.

A Steklov eigenvalue problem. Let us begin by recalling some known facts about the eigenvalue problem associated to equation (1-3) (see [Gazzola and Sweers 2008] or, for the case $\kappa = 1$, [Bucur et al. 2009; Berchio et al. 2006]):

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u = d\kappa u_n & \text{on } \partial\Omega. \end{cases} \tag{5-1}$$

We define a *Steklov eigenvalue* to be a real value d such that (1-3) admits a nontrivial weak solution, named a *Steklov eigenfunction*, i.e., $u \in H^2(\Omega) \cap H_0^1(\Omega)$, $u \neq 0$, such that for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\int_\Omega \Delta u \, \Delta\varphi - d \int_{\partial\Omega} \kappa u_n \varphi_n = 0. \tag{5-2}$$

First of all, d must be positive. In fact, if u is a Steklov eigenfunction, taking $u = \varphi$ in (5-2),

$$d \int_{\partial\Omega} \kappa(u_n)^2 = \int_\Omega (\Delta u)^2 > 0,$$

since $\|\Delta \cdot \|_2$ is a norm in $H^2(\Omega) \cap H_0^1(\Omega)$. As $\kappa \geq 0$, we have both $d > 0$ and $\int_{\partial\Omega} \kappa u_n^2 > 0$. As a complementary result, in order to show nontrivial solutions of (5-1), without loss of generality, we can restrict to the subset

$$\mathcal{H} = \left\{ u \in H^2(\Omega) \cap H_0^1(\Omega) \,\bigg|\, \int_{\partial\Omega} \kappa(u_n)^2 \neq 0 \right\}.$$

Definition 5.1. We denote by $\tilde{\delta}_1(\Omega)$ the first Steklov eigenvalue for problem (5-1):

$$\tilde{\delta}_1(\Omega) := \inf_{\mathcal{H}\setminus\{0\}} \frac{\|\Delta u\|_2^2}{\int_{\partial\Omega} \kappa u_n^2}.$$

Proposition 5.2. *The first Steklov eigenvalue is attained, is positive and there exists a unique (up to a multiplicative constant) corresponding Steklov eigenfunction, which is positive in Ω.*

Proof. We refer to [Gazzola and Sweers 2008, Lemma 4.4], just noticing that the continuity of the curvature assumed therein was not necessary to obtain this result. $\qquad\square$

A nonexistence and an existence result. From Proposition 5.2, it is easy to deduce a nonexistence result for positive solutions if σ is negative enough:

Proposition 5.3. *If $\sigma \leq \sigma^* := 1 - \tilde{\delta}_1(\Omega)$, there is no nonnegative nontrivial solution for the Steklov boundary problem* (1-3).

Proof. Let u be a nonnegative solution for (1-3) and $\Phi_1 > 0$ be the first Steklov eigenfunction; we use Φ_1 as a test function in (4-2):

$$\int_\Omega \Delta u \, \Delta \Phi_1 - (1-\sigma) \int_{\partial\Omega} \kappa u_n (\Phi_1)_n = \int_\Omega g(x) u^p \Phi_1.$$

Then, interpreting u this time as a test function in (5-2), we have

$$\int_\Omega \Delta u \, \Delta \Phi_1 = \tilde{\delta}_1(\Omega) \int_{\partial\Omega} \kappa (\Phi_1)_n u_n.$$

Combining the two equalities,

$$(\tilde{\delta}_1(\Omega) - (1-\sigma)) \int_{\partial\Omega} \kappa (\Phi_1)_n u_n = \int_\Omega g(x) u^p \Phi_1 > 0.$$

Again by positivity of u and Φ_1, we have $u_n \leq 0$ and $(\Phi_1)_n \leq 0$ so, as $\kappa \geq 0$, we finally end up with $\tilde{\delta}_1(\Omega) - 1 + \sigma > 0$, which is exactly what we wanted. $\square$

Remark 5.4. We already proved that our problem (1-3) admits positive solutions whenever $\sigma \in (-1, 1]$ with the same assumptions on Ω. Hence, we infer that, $\tilde{\delta}_1(\Omega) \geq 2$ and we have equality if $\Omega = B_1(0)$ (see [Berchio et al. 2006, Proposition 12]). This result was already proved for C^2 bounded convex domains of $\mathbb{R}^2$ by Parini and Stylianou [2009, Remark 3.3], using Fichera's duality principle.

The next step is to investigate what happens if $\sigma \in (\sigma^*, -1]$ in case this interval is nonempty. We will show that the existence and the positivity results found for $\sigma \in (-1, 1]$ can be extended for this case. In fact, the only restriction we have to overcome is the fact that here Lemma 2.1 is not the right way to prove that the first two terms in the functional J_σ define indeed a norm on $H^2(\Omega) \cap H_0^1(\Omega)$.

Lemma 5.5. *For every $\sigma > \sigma^*$, the map*

$$u \mapsto \left[\int_\Omega (\Delta u)^2 - (1-\sigma) \int_{\partial\Omega} \kappa (u_n)^2 \right]^{\frac{1}{2}} := \|u\|_{H_\sigma}$$

is a norm in $H^2(\Omega) \cap H_0^1(\Omega)$ equivalent to the standard norm.

Proof. By the definition of $\tilde{\delta}_1(\Omega)$ as an inf, we have $\|\Delta u\|_2^2 \geq \tilde{\delta}_1(\Omega) \int_{\partial\Omega} \kappa u_n^2$ for each $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and so, if $d > 0$ (which corresponds to $\sigma < 1$),

$$\int_\Omega (\Delta u)^2 \geq \int_\Omega (\Delta u)^2 - d \int_{\partial\Omega} \kappa u_n^2 \geq \int_\Omega (\Delta u)^2 \left(1 - \frac{d}{\tilde{\delta}_1(\Omega)} \right). \tag{5-3}$$

On the other hand, if $d < 0$ (so that $\sigma > 1$),

$$\int_\Omega (\Delta u)^2 \leq \int_\Omega (\Delta u)^2 + |d| \int_{\partial\Omega} \kappa u_n^2 \leq \int_\Omega (\Delta u)^2 \left(1 + \frac{|d|}{\tilde{\delta}_1(\Omega)} \right).$$

As a result, we have to impose that $d < \tilde{\delta}_1(\Omega)$ to have the positivity of the constant in the first estimate, while no restriction occurs in the second. The proof is completed noticing that the map

$$(u, v)_{H_\sigma} \mapsto \int_\Omega \Delta u \, \Delta v - d \int_{\partial\Omega} \kappa u_n v_n$$

defines a scalar product on $H^2(\Omega) \cap H^1_0(\Omega)$ by inequality (5-3) for all $d < \tilde{\delta}_1(\Omega)$. □

Proposition 5.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with boundary $C^{1,1}$ and suppose $\sigma \in (\sigma^*, -1]$; then the functional J_σ admits a positive ground state.*

Proof. It is sufficient to notice that Lemma 3.4 holds for these values of σ if we replace Lemma 2.1 by Lemma 5.5, while all the other propositions that led to the existence and the positivity of ground states are not affected by this change. □

Remark 5.7 (sublinear case). Both Propositions 5.3 and 5.6 hold in the case of a function $f(x, u)$ which verifies the assumption (H) (modifying in a suitable way the constant in front of the quadratic term) and $f \geq 0$, $f \not\equiv 0$.

Approaching σ^*. As we know now the existence of positive ground state solutions for $\sigma \in (\sigma^*, 1]$ and that there are no positive solutions if $\sigma \leq \sigma^*$, a natural question that arises is determining the behaviour of a sequence $(u_k)_{k \in \mathbb{N}}$, each of them being a ground state for the respective functional J_{σ_k}, as $\sigma_k \searrow \sigma^*$. We will find an antipodal result for $f(x, u) = g(x)|u|^{p-1}u$ as $p \in (1, +\infty)$ or $p \in (0, 1)$.

The following proof is an adaptation of [Berchio et al. 2007, Theorem 1], which covers the critical case $f(x, u) = |u|^{2^*-2}u$ when the dimension is $N \geq 5$. Moreover, the authors considered a slightly different notion of solution, that is, the minimizers of the Rayleigh quotient associated to the boundary value problem:

$$R_\sigma(u) := \frac{\|\Delta u\|_2^2 - (1-\sigma)\int_{\partial\Omega} \kappa u_n^2}{\left(\int_\Omega g(x)|u|^{p+1}\right)^{\frac{2}{p+1}}} \tag{5-4}$$

Anyway, it is a standard fact to prove that every ground state of J_σ is also a minimizer of R_σ, while the converse is also true, up to a multiplication by a constant.

Theorem 5.8. *Let Ω be as in Proposition 5.6 and $\sigma_k \searrow \sigma^*$ as $k \to +\infty$. If $p \in (0, 1)$, then $\|u_k\|_\infty \to +\infty$, while, if $p > 1$, then $\|u_k\|_{H^2(\Omega)} \to 0$.*

Proof. Let $p > 0$, $p \neq 1$; by the remark above, each ground state u_k is such that

$$R_{\sigma_k}(u_k) = \inf_{0 \neq u \in H^2(\Omega) \cap H^1_0(\Omega)} R_{\sigma_k}(u) := \Sigma_{\sigma_k} \geq 0.$$

By Proposition 5.2, there exists a positive first Steklov eigenfunction Φ_1; since we have $\|\Delta\Phi_1\|_2^2 = (1-\sigma^*)\int_{\partial\Omega} \kappa (\Phi_1)_n^2$, we have

$$0 \leq \Sigma_{\sigma_k} \leq R_{\sigma_k}(\Phi_1) = (\sigma_k - \sigma^*)\frac{\int_{\partial\Omega} \kappa (\Phi_1)_n^2}{\left(\int_\Omega g(x)|\Phi_1|^{p+1}\right)^{\frac{2}{p+1}}} \to 0$$

as $k \to +\infty$. Moreover, since u_k is a ground state for J_{σ_k},

$$\|\Delta u_k\|_2^2 - (1-\sigma_k)\int_{\partial\Omega} \kappa (u_k)_n^2 = \int_\Omega g(x)|u_k|^{p+1}$$

and, since $R_{\sigma_k}(u_k) = \Sigma_{\sigma_k}$, we deduce

$$\left(\int_\Omega g(x)|u_k|^{p+1}\right)^{\frac{p-1}{p+1}} = \Sigma_{\sigma_k} \to 0.$$

Hence, if $p > 1$, then $\int_\Omega g(x)|u_k|^{p+1} \to 0$; otherwise, if $p \in (0,1)$, then $\int_\Omega g(x)|u_k|^{p+1} \to +\infty$, which implies, by the Hölder inequality as $g \in L^1(\Omega)$, that $\|u_k\|_\infty \to +\infty$.

We have now to prove that, if $p > 1$, this convergence to 0 is actually in the natural norm $H^2(\Omega)$. By Lemma 5.5, $\|\cdot\|_{H_{\sigma_k}}$ is a norm in $H^2(\Omega) \cap H_0^1(\Omega)$ for every k, so we are able to decompose in that norm the Hilbert space as $H^2(\Omega) \cap H_0^1(\Omega) = \text{span}(\Phi_1) \oplus [\text{span}(\Phi_1)]^\perp$. Thus, for every k there exist a unique $\alpha_k \in \mathbb{R}$ and $\psi_k \in [\text{span}(\Phi_1)]^\perp$ such that $u_k = \alpha_k \Phi_1 + \psi_k$.

Hence, for k large enough,

$$o(1) \geq \int_\Omega g(x)|u_k|^{p+1} = \|\Delta u_k\|_2^2 - (1-\sigma_k)\int_{\partial\Omega} \kappa (u_k)_n^2 = (u_k, u_k)_{H_{\sigma_k}}$$
$$= \alpha_k^2 (\Phi_1, \Phi_1)_{H_{\sigma_k}} + (\psi_k, \psi_k)_{H_{\sigma_k}}. \tag{5-5}$$

First of all,

$$(\Phi_1, \Phi_1)_{H_{\sigma_k}} = \|\Delta\Phi_1\|_2^2 - (1-\sigma_k)\int_{\partial\Omega} \kappa (\Phi_1)_n^2 = (\sigma_k - \sigma^*)\int_{\partial\Omega} \kappa (\Phi_1)_n^2. \tag{5-6}$$

Moreover, denoting by $\tilde\delta_2(\Omega)$ the second eigenvalue of the Steklov problem, i.e.,

$$\tilde\delta_2(\Omega) = \inf_{\text{span}(\Phi_1)^\perp \setminus \{0\}} \frac{\|\Delta v\|_2^2}{\int_{\partial\Omega} \kappa v_n^2},$$

and defining $\sigma^{**} := 1 - \tilde\delta_2(\Omega)$, we get

$$\|\Delta\psi_k\|_2^2 \geq (1-\sigma^{**})\int_{\partial\Omega} \kappa (\psi_k)_n^2,$$

from which

$$(\psi_k, \psi_k)_{H_{\sigma_k}} = \|\Delta\psi_k\|_2^2 - (1-\sigma_k)\int_{\partial\Omega} \kappa (\psi_k)_n^2 \geq \|\Delta\psi_k\|_2^2 - \frac{1-\sigma_k}{1-\sigma^{**}}\|\Delta\psi_k\|_2^2 = \frac{\sigma_k-\sigma^{**}}{1-\sigma^{**}}\|\Delta\psi_k\|_2^2. \tag{5-7}$$

As a result, combining (5-5) with (5-6) and (5-7), we get

$$o(1) \geq \int_\Omega g(x)|u_k|^{p+1} = \alpha_k^2(\sigma_k - \sigma^*)\int_{\partial\Omega} \kappa (\Phi_1)_n^2 + \frac{\sigma_k - \sigma^{**}}{1-\sigma^{**}}\|\Delta\psi_k\|_2^2.$$

Since we proved in Proposition 5.2 that the first Steklov eigenfunction is simple, we have $\sigma^{**} < \sigma^*$ and, recalling that $\sigma_k > \sigma^*$ by assumption, necessarily $\|\Delta\psi_k\|_2 \to 0$. Hence,

$$\int_\Omega g(x)|\alpha_k \Phi_1|^{p+1} \leq \int_\Omega g(x)\big[|u_k| + |\psi_k|\big]^{p+1} \leq 2^p \int_\Omega g(x)\big[|u_k|^{p+1} + |\psi_k|^{p+1}\big]$$
$$\leq 2^p \int_\Omega g(x)|u_k|^{p+1} + C^{p+1}(\Omega)\|g\|_1 \|\psi_k\|_{H^2(\Omega)} \to 0.$$

As a result, $\alpha_k \to 0$ and we finally obtain

$$\|u_k\|_{H^2(\Omega)} \le |\alpha_k| \|\Phi_1\|_{H^2(\Omega)} + \|\psi_k\|_{H^2(\Omega)} \to 0. \qquad \square$$

If we read carefully the proof of Theorem 5.8, we notice that the fact that each u_k is a ground state for J_σ was necessary only to deduce that $\int_\Omega g(x)|u_k|^{p+1} \to 0$, while to prove the convergence to 0 in $H^2(\Omega)$ norm it was only sufficient that each u_k is a critical point (actually, an element of the Nehari manifold $\mathcal{N}_{\sigma_k}$, since the only step of the proof involved is (5-5)). Consequently, we can directly state the following lemma, which will be useful when we will look at the radial case in Section 7:

Lemma 5.9. *Let $(u_k)_{k \in \mathbb{N}}$ be a sequence of critical points of J_{σ_k} in the superlinear case such that $\int_\Omega g(x)|u_k|^{p+1} \to 0$ as $\sigma_k \searrow \sigma^*$. Then $\|u_k\|_{H^2(\Omega)} \to 0$.*

6. Beyond the physical bounds: $\sigma > 1$

As briefly announced at the beginning of the previous section, here we want to investigate the behaviour of the ground states of J_σ when $\sigma > 1$. We assume again hereafter that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with $C^{1,1}$ boundary and (1-2) concerning the nonlinearity. As a consequence, the extension of the existence result is straightforward: in fact, in this case, by Lemma 5.5, $\| \cdot \|_{H_\sigma(\Omega)}$ is still a norm on $H^2(\Omega) \cap H^1_0(\Omega)$ and we can repeat the usual steps. Notice also that it is equivalent by these assumptions on Ω to consider critical points of J_σ as far as weak solutions of the semilinear problem (1-3).

The extension of positivity in this case seems not to be obvious, as already noticed in Remark 4.13. We will provide here two different proofs (which will produce two slightly different results); the first one relies on the study of the convergence of ground states as $\sigma \to 1$, which in the limit yields the Navier case, while the second is based on the method of dual cones by Moreau, connecting our semilinear problem with the linear one. We point out that the convergence result might be also of independent interest.

In the following, we will always consider the exponent of the nonlinearity (1-2) to be $p > 1$, but similar results can be proved also in the sublinear framework (see Remarks 6.6 and 6.22).

Convergence of ground states of J_σ to ground states of J_{NAV} as $\sigma \to 1$. In this section, $(u_k)_{k \in \mathbb{N}}$ will always denote a sequence of ground states solutions of the Steklov problems

$$\begin{cases} \Delta^2 u = g(x)|u|^{p-1}u & \text{in } \Omega, \\ u = \Delta u - (1 - \sigma_k)\kappa u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{6-1}$$

for a sequence $(\sigma_k)_{k \in \mathbb{N}}$ converging to 1. Moreover, in order to underline the peculiarity of the problem when $\sigma = 1$, we set $J_{\mathrm{NAV}} := J_1$, whose critical points are the weak solution of the following Navier problem:

$$\begin{cases} \Delta^2 u = g(x)|u|^{p-1}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6-2}$$

Finally, $\bar{u}$ will always denote a ground state of J_{NAV}. Our main result is to prove the convergence $u_k \to \bar{u}$ in the natural norm, i.e., in $H^2(\Omega)$, as $\sigma_k \to 1$, no matter if σ_k is less or greater than 1. First of all, a weaker result is enough:

Lemma 6.1. *Let $(u_k)_{k \in \mathbb{N}}$ and $\bar{u}$ be as specified above. If $u_k \rightharpoonup \bar{u}$ weakly in $H^2(\Omega)$, then (up to a subsequence) $u_k \to \bar{u}$ strongly in $H^2(\Omega)$ as $\sigma_k \to 1$.*

Proof. As $u_k \rightharpoonup \bar{u}$ weakly in $H^2(\Omega)$, there exists $M > 0$ such that $\|u_k\|^2_{H^2(\Omega)} \le M$. Moreover, for each $k \in \mathbb{N}$, we know u_k is a solution of (6-1) and $\bar{u}$ of the Navier problem (6-2); thus, for every test function $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\int_\Omega \Delta u_k \, \Delta \varphi - (1 - \sigma_k) \int_{\partial\Omega} \kappa(u_k)_n \varphi_n = \int_\Omega g(x)|u_k|^{p-1}u_k\varphi,$$

$$\int_\Omega \Delta \bar{u} \, \Delta \varphi = \int_\Omega g(x)|\bar{u}|^{p-1}\bar{u}\varphi. \tag{6-3}$$

Hence

$$C_A^{-1}\|u_k - \bar{u}\|^2_{H^2(\Omega)}$$

$$\le \|\Delta u_k - \Delta \bar{u}\|^2_2 = \int_\Omega \Delta u_k \, \Delta(u_k - \bar{u}) - \int_\Omega \Delta \bar{u} \, \Delta(u_k - \bar{u})$$

$$= (1 - \sigma_k)\int_{\partial\Omega} \kappa(u_k)_n(u_k - \bar{u})_n + \left[\int_\Omega g(x)|u_k|^{p-1}u_k(u_k - \bar{u}) - \int_\Omega g(x)|\bar{u}|^{p-1}\bar{u}(u_k - \bar{u})\right].$$

For the first term,

$$\left|(1 - \sigma_k)\int_{\partial\Omega} \kappa(u_k)_n(u_k - \bar{u})_n\right| \le |1 - \sigma_k|C_T^2\|\kappa\|_{L^\infty(\partial\Omega)}\|u_k\|_{H^2(\Omega)}\|u_k - \bar{u}\|_{H^2(\Omega)}$$

$$\le |1 - \sigma_k|C_T^2\|\kappa\|_{L^\infty(\partial\Omega)}M(M + \|\bar{u}\|_{H^2(\Omega)}) \to 0,$$

where C_T is the constant in the trace theorem. Concerning the second, it is enough to invoke the dominated convergence theorem as we have pointwise convergence and since

$$\left|g(x)\big(|u_k|^{p-1}u_k - |\bar{u}|^{p-1}\bar{u}\big)(u_k - \bar{u})\right| \le |g(x)|\big[C(\Omega)^p M^p + |\bar{u}|^p\big]\big[C(\Omega)M + \bar{u}\big] \in L^1(\Omega),$$

where $C(\Omega)$ is the constant in the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. $\square$

Remark 6.2. This result holds not only for ground states, but for generic solutions; i.e., if $(u_k)_{k \in \mathbb{N}}$ is a sequence of weak solutions of the Steklov problem (6-1) and $\bar{u}$ a weak solution of the Navier problem (6-2) and we know that $u_k \rightharpoonup \bar{u}$ weakly in $H^2(\Omega)$, then, up to a subsequence, it converges strongly too.

A crucial observation is that the Nehari manifolds are nested with respect to the parameter σ:

Lemma 6.3. *Let $\sigma_1 < \sigma_2$ and fix $u \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$. Then*

$$t^*_{\sigma_1}(u) \le t^*_{\sigma_2}(u).$$

Proof. In fact, $-(1 - \sigma_1) < -(1 - \sigma_2)$ and so

$$t^*_{\sigma_1}(u) = \left(\frac{\int_\Omega(\Delta u)^2 - (1 - \sigma_1)\int_{\partial\Omega}\kappa u_n^2}{\int_\Omega g(x)|u|^{p+1}}\right)^{\frac{1}{p-1}} \le \left(\frac{\int_\Omega(\Delta u)^2 - (1 - \sigma_2)\int_{\partial\Omega}\kappa u_n^2}{\int_\Omega g(x)|u|^{p+1}}\right)^{\frac{1}{p-1}} = t^*_{\sigma_2}(u). \quad \square$$

Notice that if $u \in H_0^2(\Omega)$ then one has the equality; if we suppose moreover that $\kappa > 0$ a.e., we deduce also the converse.

Proposition 6.4. *The sequence of ground states* $(u_k)_{k \in \mathbb{N}}$ *is bounded in* $H^2(\Omega)$.

Proof. Set $k_{\max}$ such that $\sigma_{k_{\max}} = \max\{(\sigma_k)_{k \in \mathbb{N}}, 1\}$ and so $u_{k_{\max}}$ is a ground state for $J_{\sigma_{k_{\max}}}$ (with the convention that if $\sigma_{k_{\max}} = 1$, then $u_{k_{\max}}$ is a ground state for J_{NAV}).

Defining $w_k := t^*_{\sigma_k}(u_{k_{\max}}) u_{k_{\max}} \in \mathcal{N}_{\sigma_k}$, that is, the "projection" of $u_{k_{\max}}$ on the Nehari manifold $\mathcal{N}_{\sigma_k}$ along its half-line, one has

$$\int_{\Omega} g(x)|u_k|^{p+1} \leq \int_{\Omega} g(x)|w_k|^{p+1} \leq \int_{\Omega} g(x)|u_{k_{\max}}|^{p+1}. \tag{6-4}$$

Indeed, the first inequality comes from the fact that u_k is a ground state of J_{σ_k}, which has the equivalent formulation (3-2); the second is obtained by Lemma 6.3 since

$$\int_{\Omega} g(x)|w_k|^{p+1} = (t^*_{\sigma_k}(u_{k_{\max}}))^{p+1} \int_{\Omega} g(x)|u_{k_{\max}}|^{p+1}$$

$$\leq (t^*_{\sigma_{k_{\max}}}(u_{k_{\max}}))^{p+1} \int_{\Omega} g(x)|u_{k_{\max}}|^{p+1} = \int_{\Omega} g(x)|u_{k_{\max}}|^{p+1}.$$

Furthermore, for a generic $\sigma > 0$ (and here we can assume it without loss of generality),

$$\int_{\Omega} (\Delta u)^2 - (1-\sigma) \int_{\partial\Omega} \kappa u_n^2 \geq \min\{\sigma, 1\} C_A(\Omega) \|u\|^2_{H^2(\Omega)}. \tag{6-5}$$

In fact, if $\sigma \in [1, +\infty)$ the proof is straightforward since $-(1-\sigma) \geq 0$; otherwise, if $\sigma \in (0,1)$,

$$\int_{\Omega} (\Delta u)^2 - (1-\sigma) \int_{\partial\Omega} \kappa u_n^2 = \int_{\Omega} (\Delta u)^2 + 2(1-\sigma) \int_{\Omega} (-\det(\nabla^2 u))$$

$$= \int_{\Omega} [u_{xx}^2 + u_{yy}^2 + 2\sigma u_{xx} u_{yy} + 2(1-\sigma) u_{xy}^2] \geq \sigma \int_{\Omega} (\Delta u)^2 + 2(1-\sigma) \int_{\Omega} u_{xy}^2$$

$$\geq \sigma \int_{\Omega} (\Delta u)^2 \geq \sigma C_A^{-1}(\Omega) \|u\|^2_{H^2(\Omega)}.$$

As a result, combining (6-4) with (6-5), we get

$$\|u_k\|^2_{H^2(\Omega)} \leq \frac{C_A(\Omega)}{\min\{\sigma_k, 1\}} \int_{\Omega} g(x)|u_k|^{p+1} \leq \frac{C_A(\Omega)}{\min\{\sigma_k, 1\}} \int_{\Omega} g(x)|u_{k_{\max}}|^{p+1},$$

which is the estimate we needed. □

As a direct consequence of Proposition 6.4, the sequence $(u_k)_{k \in \mathbb{N}}$, up to a subsequence, is weakly convergent to some $u_\infty \in H^2(\Omega) \cap H_0^1(\Omega)$ with strong convergence in $L^\infty(\Omega)$. It is also easy to see that u_∞ is a weak solution of the Navier problem (6-2): it is enough to apply to (6-3) the weak convergence in $H^2(\Omega)$, the strong convergence in $L^2(\partial\Omega)$ of the normal derivatives and the dominated convergence theorem. As a consequence, by Lemma 6.1, the convergence $u_k \to u_\infty$ is strong in $H^2(\Omega)$.

Theorem 6.5. *Let* $\sigma_k \to 1$ *and* Ω *be a bounded convex domain in* $\mathbb{R}^2$ *with boundary of class* $C^{1,1}$. *Then the sequence* $(u_k)_{k \in \mathbb{N}}$ *of ground state solutions for the Steklov problems* (6-1) *admits a subsequence* $(u_{k_j})_{j \in \mathbb{N}}$ *which converges in* $H^2(\Omega)$ *to* u_∞, *which is a ground state for the Navier problem* (6-2), *and thus strictly superharmonic.*

Proof. Clearly, as u_∞ is weak solution of (6-2), we have $J_{\mathrm{NAV}}(u_\infty) \geq \inf_{\mathcal{N}_{\mathrm{NAV}}} J_{\mathrm{NAV}}$. Now we have to prove the converse inequality. Firstly, we have $J_{\mathrm{NAV}}(u_\infty) \leq \liminf_{k \to +\infty} J_{\sigma_k}(u_k)$. Indeed,

$$
\begin{aligned}
\liminf_{k \to +\infty} J_{\sigma_k}(u_k) &= \liminf_{k \to +\infty} \int_\Omega \frac{(\Delta u_k)^2}{2} - \lim_{k \to +\infty} \frac{1-\sigma_k}{2} \int_{\partial\Omega} \kappa (u_k)_n^2 - \lim_{k \to +\infty} \int_\Omega \frac{g(x)|u_k|^{p+1}}{p+1} \\
&\geq \int_\Omega \frac{(\Delta u_\infty)^2}{2} - \int_\Omega \frac{g(x)|u_\infty|^{p+1}}{p+1} = J_{\mathrm{NAV}}(u_\infty),
\end{aligned}
$$

having used the compactness of the map $\partial_n : H^1(\Omega)^2 \to L^2(\Omega)$ and the dominated convergence theorem. Moreover, if we suppose $\sigma_k < 1$ for k large enough, by Lemma 6.3 (with a similar argument to that in (6-4)), for all $k \in \mathbb{N}$ we have

$$
J_{\sigma_k}(u_k) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega g(x)|u_k|^{p+1} \leq \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega g(x)|u_\infty|^{p+1} = J_{\mathrm{NAV}}(u_\infty), \qquad (6\text{-}6)
$$

so in this case we are done. If otherwise $\sigma_k > 1$ for a infinite number of indices, (6-6) does not hold. In this case, without loss of generality, we can assume that $\sigma_k \searrow 1$. By the existence theorems in Section 3, we know that there exists a ground state $\bar{u} \in H^2(\Omega) \cap H_0^1(\Omega)$ for J_{NAV} and we define $\bar{u}_k := t_{\sigma_k}^*(\bar{u})\bar{u}$ to be the "projection" on the Nehari manifold $\mathcal{N}_{\sigma_k}$. Then $\|\bar{u}_k - \bar{u}\|_{H^2(\Omega)} = |1 - t_{\sigma_k}^*(\bar{u})| \|\bar{u}\|_{H^2(\Omega)}$ with

$$
1 - (t_{\sigma_k}^*(\bar{u}))^{p-1} \stackrel{[\bar{u} \in \mathcal{N}_{\mathrm{NAV}}]}{=} (t_{\mathrm{NAV}}^*(\bar{u}))^{p-1} - (t_{\sigma_k}^*(\bar{u}))^{p-1} = 2(1-\sigma_k) \frac{\int_\Omega \det(\nabla^2 \bar{u})}{\int_\Omega g(x)|\bar{u}|^{p+1}} \to 0,
$$

so $\bar{u}_k \to \bar{u}$ in $H^2(\Omega)$, which implies

$$
\int_\Omega g(x)|\bar{u}_k|^{p+1} \to \int_\Omega g(x)|\bar{u}|^{p+1}. \qquad (6\text{-}7)
$$

Nevertheless, since u_k is a ground state of J_{σ_k},

$$
\int_\Omega g(x)|\bar{u}_k|^{p+1} \stackrel{[\bar{u}_k \in \mathcal{N}_{\sigma_k}]}{=} \left(\frac{1}{2} - \frac{1}{p+1}\right) J_{\sigma_k}(\bar{u}_k) \geq \left(\frac{1}{2} - \frac{1}{p+1}\right) J_{\sigma_k}(u_k) \stackrel{[u_k \in \mathcal{N}_{\sigma_k}]}{=} \int_\Omega g(x)|u_k|^{p+1}; \qquad (6\text{-}8)
$$

furthermore, since we assumed $\sigma_k > 1$ and by Lemma 6.3,

$$
\begin{aligned}
\int_\Omega g(x)|u_k|^{p+1} &\geq \int_\Omega g(x)|t_{\mathrm{NAV}}^*(u_k)u_k|^{p+1} = \left(\frac{1}{2} - \frac{1}{p+1}\right) J_{\mathrm{NAV}}(t_{\mathrm{NAV}}^*(u_k)u_k) \\
&\geq \left(\frac{1}{2} - \frac{1}{p+1}\right) J_{\mathrm{NAV}}(\bar{u}) = \int_\Omega g(x)|\bar{u}|^{p+1}. \qquad (6\text{-}9)
\end{aligned}
$$

Combining (6-7), (6-8) and (6-9), we find that

$$
\int_\Omega g(x)|u_k|^{p+1} \to \int_\Omega g(x)|\bar{u}|^{p+1}, \qquad (6\text{-}10)
$$

from which $J_{\sigma_k}(u_k) \to J_{\mathrm{NAV}}(\bar{u})$, which completes our equality.

To conclude, notice that we have already obtained in the proof of Proposition 4.12 that ground states of the Navier equation (6-2) are strictly superharmonic. $\qquad \square$

Remark 6.6. The same analysis may be adapted also for the sublinear case $p \in (0, 1)$, paying attention to some minor changes: for instance, Lemma 6.3 holds with the reverse inequality, but this compensates for the fact that this time the coefficient $\frac{1}{2} - \frac{1}{p}$ in the equivalent formulation of J_σ is negative.

Regularity of solutions and $W^{2,q}$ convergence of ground states. The convergence result of the previous section will be used to derive positivity of ground states when σ lies in a right neighborhood of 1. Nevertheless, we will need a $C^{0,1}$ convergence to be able to control the normal derivatives on the boundary; thus we have to upgrade our convergence to a stronger norm. The first step will be to investigate, for a fixed $\sigma > \sigma^*$, the regularity of solutions of (1-3) and (6-2) with just a slightly more regular boundary (actually, we will have to impose that $\partial\Omega$ is of class C^2). This will be obtained by means of the following lemma by Gazzola, Grunau and Sweers, which follows from a result by Agmon, Douglis and Nirenberg [Agmon et al. 1959, Theorem 15.3′, p. 707]:

Lemma 6.7 [Gazzola et al. 2010, Corollary 2.23]. *Let $q > 1$ and take an integer $m \geq 4$. Assume that $\partial\Omega \in C^m$ and $a \in C^{m-2}$. Then there exists $C = C(m, q, a, \Omega) > 0$ such that*

$$\|u\|_{W^{m,q}(\Omega)} \leq C\big(\|u\|_q + \|\Delta^2 u\|_{W^{m-4,q}(\Omega)} + \|u\|_{W^{m-1/q,q}(\partial\Omega)} + \|\Delta u - a u_n\|_{W^{m-2-1/q,q}(\partial\Omega)}\big)$$

for every $u \in W^{m,q}(\Omega)$. The same statement holds for any $m \geq 2$ provided the norms on the right-hand side are suitably interpreted.

Hence we have to define $\Delta^2 u$ as a distribution in $W^{-2,q}(\Omega)$, i.e., acting on functions in $W_0^{2,q'}(\Omega)$. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be a weak solution of (1-3); we define the linear functional over $H^2(\Omega)$

$$\Delta^2 u : H^2(\Omega) \ni \varphi \mapsto \langle \Delta^2 u, \varphi \rangle := \int_\Omega \Delta u \, \Delta\varphi,$$

which is well defined and continuous. If we let

$$u_g^p : \varphi \mapsto \langle u_g^p, \varphi \rangle := \int_\Omega g(x)|u|^{p-1}u\varphi,$$

it is clearly well defined and continuous on $W_0^{2,q'}(\Omega)$ and, by the weak formulation of the PDE, on the subset $H_0^2(\Omega)$ it acts identically as $\Delta^2 u$. As a result, we define

$$\Delta^2 u : W_0^{2,q'}(\Omega) \ni \varphi \mapsto \langle \Delta^2 u, \varphi \rangle := \int_\Omega g(x)|u|^{p-1}u\varphi. \tag{6-11}$$

Proposition 6.8. *If $\partial\Omega \in C^2$, for every $\sigma > \sigma^*$ the weak solutions of Steklov and Navier problems (6-1) and (6-2) lie in $W^{2,q}(\Omega)$ for every $q > 2$.*

Proof. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be a weak solution of (1-3). Applying Lemma 6.7 with $m = 2$ and $a = (1-\sigma)\kappa \in C^0(\partial\Omega)$ ($a = 0$ for the Navier case), we find

$$\|u\|_{W^{2,q}(\Omega)} \leq C(q, \sigma, \Omega)\big(\|u\|_q + \|\Delta^2 u\|_{W^{-2,q}(\Omega)}\big),$$

which is well defined in view of (6-11). Since

$$\|\Delta^2 u\|_{W^{-2,q}(\Omega)} = \sup_{0 \neq \varphi \in W_0^{2,q'}(\Omega)} \frac{\left|\int_\Omega g(x)|u|^{p-1}u\varphi\right|}{\|\varphi\|_{W_0^{2,q'}(\Omega)}} \leq C(p, q, \Omega)\|g\|_1 \|u\|_{H^2(\Omega)}^p, \tag{6-12}$$

we finally deduce from (6-12) that

$$\|u\|_{W^{2,q}(\Omega)} \le C(q,\sigma,\Omega)\big(\|u\|_q + C(p,q,\Omega)\|g\|_1 \|u\|_{H^2(\Omega)}^p\big) < +\infty. \qquad \square$$

We stress that we did not use either the fact that u is a ground state solution, or its positivity: the above result holds true for every weak solution of Steklov and Navier problems.

Let us now recall the following interpolation result:

Lemma 6.9 (interpolation of fractional Sobolev spaces, [Brezis and Mironescu 2001, Corollary 2]). *For* $0 \le s_1 < s_2 < +\infty$, $1 < p_1, p_2 < +\infty$, *for every* s, p *such that* $s = \theta s_1 + (1-\theta)s_2$ *and* $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, *we have*

$$\|f\|_{W^{s,p}(\mathbb{R}^N)} \le C\|f\|_{W^{s_1,p_1}(\mathbb{R}^N)}^{\theta} \|f\|_{W^{s_2,p_2}(\mathbb{R}^N)}^{1-\theta}.$$

Proposition 6.10. *Let* Ω *be of class* C^2 *and* $(u_k)_{k\in\mathbb{N}}$ *be a sequence of weak solutions for the Steklov problems* (6-1) *converging in* $H^2(\Omega)$ *to* $\bar{u}$, *a weak solution for the Navier problem* (6-2). *Then the convergence is in* $W^{2,q}(\Omega)$ *for every* $q \ge 2$.

Proof. Let $q \ge 2$ and apply the regularity estimate of Lemma 6.7 to $u_k - \bar{u}$ with $m = 2$, $a = 0$:

$$\|u_k - \bar{u}\|_{W^{2,q}(\Omega)}$$
$$\le C(q,\Omega)\big(\|u_k - \bar{u}\|_q + \|\Delta^2 u_k - \Delta^2 \bar{u}\|_{W^{-2,q}(\Omega)} + |1 - \sigma_k|\|\kappa(u_k)_n\|_{W^{-1/q,q}(\partial\Omega)}\big), \quad (6\text{-}13)$$

using that on $\partial\Omega$ we have $\Delta(u_k - \bar{u}) - a(u_k - \bar{u})_n = \Delta u_k - \Delta\bar{u} = (1 - \sigma_k)\kappa(u_k)_n$.

By (6-11) and the dominated convergence theorem,

$$\|\Delta^2 u_k - \Delta^2 \bar{u}\|_{W^{-2,q}(\Omega)} = \sup_{0 \ne \varphi \in W_0^{2,q'}(\Omega)} \frac{\left|\int_\Omega g(x)|u_k|^{p-1}u_k\varphi - \int_\Omega g(x)|\bar{u}|^{p-1}\bar{u}\varphi\right|}{\|\varphi\|_{W_0^{2,q'}(\Omega)}} \to 0,$$

similarly to (6-12). We need now to prove that $(\kappa(u_k)_n)_{k\in\mathbb{N}}$ is bounded in $W^{-\frac{1}{q},q}(\partial\Omega)$. Notice that if we provide a uniform bound in $L^q(\partial\Omega)$, then we are done. In fact $W^{-\frac{1}{q},q}(\partial\Omega) := W^{\frac{1}{q},q'}(\partial\Omega)^*$ and $W^{\frac{1}{q},q'}(\partial\Omega) \hookrightarrow L^{q'}(\partial\Omega)$, so we directly infer $W^{-\frac{1}{q},q}(\partial\Omega) \hookleftarrow L^q(\partial\Omega)$.

Moreover, it is known that, with our assumptions on $\partial\Omega$, the normal trace of functions in $W^{s,p}(\Omega)$ lies in $L^p(\partial\Omega)$, provided $s > 1 + \frac{1}{p}$ (for this and some further sharper results, see [Marschall 1987, Theorem 2]). Hence,

$$\|\kappa(u_k)_n\|_{W^{-1/q,q}(\partial\Omega)} \le C(q,\Omega)\|\kappa(u_k)_n\|_{L^q(\partial\Omega)} \le C(q,\Omega)\|\kappa\|_{L^\infty(\partial\Omega)}\|(u_k)_n\|_{L^q(\partial\Omega)}$$
$$\le C(q,\Omega,s)\|\kappa\|_{L^\infty(\partial\Omega)}\|u_k\|_{W^{s,q}(\Omega)} \qquad (6\text{-}14)$$

for some $s > 1 + \frac{1}{q}$. Thus, we need to find an appropriate fractional Sobolev space in which $H^2(\Omega)$ is embedded. We claim that $H^2(\Omega) \hookrightarrow W^{1+\frac{3}{2q},q}(\Omega)$. Actually, it is enough to prove that $H^1(\Omega) := W^{1,2}(\Omega) \hookrightarrow W^{\frac{3}{2q},q}(\Omega)$ by the definition of $W^{s,p}(\Omega)$ for $s > 1$. So, let $u \in W^{1,2}(\Omega)$; by the Stein total extension theorem [Adams and Fournier 2003, Theorem 5.24] there exists $U \in W^{1,2}(\mathbb{R}^2)$ such that $U_{|\Omega} = u$ a.e. and $\|U\|_{W^{1,2}(\mathbb{R}^2)} \le C\|u\|_{W^{1,2}(\Omega)}$ for some positive constant independent of u. Applying the interpolation result Lemma 6.9 to U with $\theta = \frac{3}{2q}$ and the Sobolev embedding $W^{1,2}(\mathbb{R}^2) \hookrightarrow L^{4q-6}(\mathbb{R}^2)$

since $4q - 6 \geq 2$,

$$\|U\|_{W^{3/2q,q}(\mathbb{R}^2)} \leq C \|U\|_{W^{1,2}(\mathbb{R}^2)}^{\frac{3}{2q}} \|U\|_{L^{4q-6}(\mathbb{R}^2)}^{1-\frac{3}{2q}} \leq C_1 \|U\|_{W^{1,2}(\mathbb{R}^2)}.$$

Hence,

$$\|u\|_{W^{3/(2q),q}(\Omega)} = \|U\|_{W^{3/(2q),q}(\Omega)} \leq \|U\|_{W^{3/(2q),q}(\mathbb{R}^2)} \leq C_1 \|U\|_{W^{1,2}(\mathbb{R}^2)} \leq C_2 \|u\|_{W^{1,2}(\Omega)}.$$

As a result, noticing that $s = 1 + \frac{3}{2q} > 1 + \frac{1}{q}$, we can continue (6-14), obtaining

$$\|\kappa(u_k)_n\|_{W^{-1/q,q}(\partial\Omega)} \leq C(q,\Omega)\|\kappa\|_{L^\infty(\partial\Omega)}\|u_k\|_{W^{1+3/(2q),q}(\Omega)} \leq \widetilde{C}(q,\Omega)\|\kappa\|_{L^\infty(\partial\Omega)}\|u_k\|_{H^2(\Omega)},$$

which is uniformly bounded in k. Combining estimate (6-13) with the ones above for the second and the third terms of (6-13), we finally end up with the strong convergence in $W^{2,q}(\Omega)$. $\qquad\square$

Extending positivity, part 1: A convergence argument. Let us start by noticing that, by Morrey's embeddings, the convergence in $W^{2,q}(\Omega)$ for every $q \geq 2$ of Proposition 6.10 implies the convergence in $C^{1,\alpha}(\overline{\Omega})$ for every $\alpha < 1$, thus in particular in $C^1(\overline{\Omega})$. This will be the main ingredient in the next proof.

Proposition 6.11. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain of class C^2 and $(\sigma_k)_{k \in \mathbb{N}}$ be a sequence of parameters such that $\sigma_k \searrow 1$ and $(u_k)_{k \in \mathbb{N}}$ be a sequence of ground states for the functional J_{σ_k}. Then there exists a subsequence $(u_{k_j})_{j \in \mathbb{N}}$ and $j_0 \in \mathbb{N}$ such that $u_{k_j} > 0$ in Ω for every $j \geq j_0$.*

Proof. By Propositions 6.8 and 6.10 and by the previous observation, we know that, up to a subsequence, $u_k \to \bar{u}$ in $C^1(\overline{\Omega})$ for some $\bar{u}$, a ground state for J_{NAV}.

Since Ω has a C^2 boundary, the interior sphere condition holds and one can extend the outer normal vector n in a small neighborhood $\omega_0 \subset \Omega$ of $\partial\Omega$ and thus define here $\bar{u}_n := \nabla\bar{u} \cdot n$ (see [Sperb 1981, Chapter 4]). Moreover, since $\bar{u}$ is strictly superharmonic, the normal derivative $\bar{u}_n$ is negative on $\partial\Omega$ and, by compactness of $\partial\Omega$ and continuity of $\bar{u}_n$, there exists $\alpha > 0$ such that

$$\bar{u}_{n|\partial\Omega} \leq -\alpha < 0.$$

Hence, again by continuity, there exists a second neighborhood $\omega \subset \omega_0$ of $\partial\Omega$ such that

$$\bar{u}_{n|\omega} \leq -\tfrac{2}{3}\alpha < 0.$$

Take now $\varepsilon_1 = \frac{1}{3}\alpha$: by the $C^1(\overline{\Omega})$ convergence, there exists $k_1 \in \mathbb{N}$ such that for every $k \geq k_1$ and $x \in \omega$,

$$|(u_k)_n(x)| \geq |\bar{u}_n(x)| - |(u_k)_n(x) - \bar{u}_n(x)| > \tfrac{2}{3}\alpha - \big\|n\big\|_{L^\infty(\omega)}\big\|\nabla u_k - \nabla\bar{u}\big\|_{L^\infty(\Omega)} > \tfrac{2}{3}\alpha - \varepsilon_1 > \tfrac{1}{3}\alpha.$$

By the interior sphere condition, the map $\omega \to \partial\Omega$, $x \mapsto x_0$, such that $d(x, x_0) = \inf\{d(x,y) | y \in \partial\Omega\}$ is well defined and the vector $x - x_0$ has the same direction as $n(x)$ and $n(x_0)$. Hence by the Lagrange theorem and recalling that $u_{k|\partial\Omega} = 0$, for $x \in \omega$,

$$|u_k(x)| = |u_k(x) - u_k(x_0)| \geq \min_{y \in [x_0, x]} |(u_k)_n(y)| |x - x_0| > \tfrac{1}{3}\alpha |x - x_0| > 0. \tag{6-15}$$

Moreover, notice that by compactness of $\Omega_0 := \Omega \setminus \omega$, the remaining part of Ω, we have

$$\bar{u}_{|\Omega_0} \geq \min_{\Omega_0} \bar{u} := m > 0$$

and so by the uniform convergence it is easy to deduce that, for k large enough, $u_k(x) > \frac{1}{2}m$ for every $x \in \Omega_0$. The result follows by combining this with (6-15). $\square$

Theorem 6.12. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain of class C^2; then there exists $\sigma_1 > 1$ such that for every $\sigma \in (1, \sigma_1)$ the ground states of J_σ are positive in Ω.*

Proof. By contradiction, suppose that such σ_1 does not exist. Hence we would be able to find a sequence $(\sigma_k) \searrow 1$ such that for each of them there exists a ground state u_k for J_{σ_k} which is not positive. This would contradict Proposition 6.11. $\square$

Remark 6.13. As we are dealing with continuous functions, since $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$, we are interested in the strict positivity *everywhere in Ω* and not only a.e. in Ω. Theorem 6.12 gives a positive answer for this question: in fact, as $\bar{u} \in H^2(\Omega) = W^{2,N}(\Omega)$ is strictly superharmonic, by the strong maximum principle for strong solutions [Gilbarg and Trudinger 1998, Theorem 9.6], we deduce that it cannot achieve its minimum on the interior of Ω; thus $\bar{u}(x) > 0$ *for every $x \in \Omega$*. By the C^1 convergence we deduce the same strict inequality for u_σ, with $\sigma \in (1, \sigma_1)$.

Extending positivity, part 2: Moreau dual cones decomposition. Our aim is to investigate a further extension of the positivity result found in Theorem 6.12, possibly for the whole range $\sigma \in (1, +\infty)$. It seems natural if we think of the following fact: similarly to what we already obtained for the Navier problem, one can prove the convergence in $H^2(\Omega)$, as $\sigma \to +\infty$, of a sequence of ground states of J_σ to a least-energy solution of the Dirichlet problem

$$\begin{cases} \Delta^2 u = g(x)|u|^{p-1}u & \text{in } \Omega, \\ u = u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{6-16}$$

at least when κ is positive a.e. on $\partial\Omega$. Since we already know that in some cases the ground states of (6-16) are positive (for instance if Ω is a ball, see [Ferrero et al. 2007]), we expect to be able to completely extend positivity for such domains.

After a brief explanation of the convergence just mentioned above, we will apply Moreau's method of dual cones to infer the intervals of positivity for the semilinear problem. At the end, one may also compare the resulting analysis with the respective one for the linear problem with the same boundary conditions, due to Gazzola and Sweers [2008].

The Dirichlet problem. The argument is similar to what we used in the subsection on page 961 for the convergence to the Navier problem, but now we have to pay attention to the fact that in this case the two functional spaces are different ($H^2(\Omega) \cap H^1_0(\Omega)$ for the Steklov problem and $H^2_0(\Omega)$ for the Dirichlet). We are not giving here the details of the proof of the existence of ground states of (6-16), as it can be obtained as for the Steklov framework by the Nehari method of Section 3. In the following, we assume Ω to be a bounded convex domain in $\mathbb{R}^2$ with boundary of class $C^{1,1}$ and $\sigma > 1$. We suppose also that the curvature κ is positive a.e, that is $\partial\Omega$ has parts that are not flat. Moreover, as usual, u_k will always denote a ground state for J_{σ_k} and $\bar{u}$ a ground state for $J_{\text{DIR}} : H^2_0(\Omega) \to \mathbb{R}$ defined as

$$J_{\text{DIR}}(u) = \frac{1}{2} \int_\Omega (\Delta u)^2 - \frac{1}{p+1} \int_\Omega g(x)|u|^{p+1},$$

whose critical points are weak solutions of (6-16). Moreover, as in the Steklov case, we define the Nehari manifold for J_{DIR}:

$$\mathcal{N}_{\mathrm{DIR}} := \left\{ u \in H_0^2(\Omega) \backslash \{0\} \mid J'_{\mathrm{DIR}}(u)[u] = 0 \right\}.$$

First of all, notice that, by the definition of J_σ, for each σ,

$$J_\sigma \big|_{H_0^2(\Omega)} = J_{\mathrm{DIR}}, \tag{6-17}$$

so $\mathcal{N}_\sigma$ restricted to the subspace $H_0^2(\Omega)$ coincides with $\mathcal{N}_{\mathrm{DIR}}$.

Theorem 6.14. *Let $\sigma_k \to +\infty$ and Ω be a bounded convex domain in $\mathbb{R}^2$ with boundary of class $C^{1,1}$. Assume also that the curvature k is positive a.e on $\partial\Omega$. Then the sequence $(u_k)_{k \in \mathbb{N}}$ of ground states of $(J_{\sigma_k})_{k \in \mathbb{N}}$ admits a subsequence $(u_{k_j})_{j \in \mathbb{N}}$ convergent in $H^2(\Omega)$ to $\bar{u}$, which is a ground state for the Dirichlet problem* (6-16).

Proof. We follow the same steps as in the subsection on page 961 to deduce Theorem 6.5. Firstly, we prove that $(u_k)_{k \in \mathbb{N}}$ is bounded in $H^2(\Omega)$. Indeed, fix $\bar{w} \in H_0^2(\Omega)$, a ground state for the Dirichlet problem (6-16). Then

$$\|\Delta u_k\|_2^2 \leq \int_\Omega (\Delta u_k)^2 - (1 - \sigma_k) \int_{\partial\Omega} \kappa(u_k)_n^2 = \int_\Omega g(x)|u_k|^{p+1} = \inf_{v \in \mathcal{N}_{\sigma_k}} \int_\Omega g(x)|v|^{p+1}$$

$$\leq \inf_{v \in \mathcal{N}_{\sigma_k} \cap H_0^2(\Omega)} \int_\Omega g(x)|v|^{p+1} = \int_\Omega g(x)|\bar{w}|^{p+1}. \tag{6-18}$$

Hence, there exists $\bar{u} \in H^2(\Omega) \cap H_0^1(\Omega)$ such that, up to a subsequence, $u_k \rightharpoonup \bar{u}$ weakly in $H^2(\Omega)$. Moreover, (6-18) implies that

$$0 \leq (\sigma_k - 1) \int_{\partial\Omega} \kappa(u_k)_n^2 \leq \int_\Omega g(x)|u_k|^{p+1} \leq C(\Omega, p)\|g\|_1 \|u_k\|_{H^2(\Omega)}^{p+1} \leq D(\Omega, p, g)$$

and, taking into account that $\sigma_k \to +\infty$, we deduce that

$$\int_{\partial\Omega} \kappa(u_k)_n^2 \to 0.$$

Furthermore, by the compactness of the map $\partial_n : H^2(\Omega) \to L^2(\partial\Omega)$, we have also that

$$\int_{\partial\Omega} \kappa(u_k)_n^2 \to \int_{\partial\Omega} \kappa\bar{u}_n^2.$$

Hence, combining the two and recalling that we assumed $\kappa > 0$ on $\partial\Omega$, we deduce that $\bar{u}_n \equiv 0$ on $\partial\Omega$ and thus $\bar{u} \in H_0^2(\Omega)$.

Finally, testing the weak formulation of problem (6-1) with $\varphi \in H_0^2(\Omega)$ and passing to the limit as $k \to +\infty$, we deduce that

$$\int_\Omega \Delta\bar{u}\,\Delta\varphi = \int_\Omega g(x)|\bar{u}|^{p-1}\bar{u}\varphi,$$

so $\bar{u}$ is a solution of the Dirichlet problem (6-16) and, similarly to Lemma 6.1, we can prove that the convergence is strong in $H^2(\Omega)$. It remains to prove that $\bar{u}$ is actually a ground state for J_{DIR}. Let $\bar{w} \in H_0^2(\Omega)$ be a ground state solution of J_{DIR}. Then, by (6-17),

$$m = J_{\mathrm{DIR}}(\bar{w}) = J_{\sigma_k}(t_{\sigma_k}^*(\bar{w})\bar{w}) \geq \inf_{\mathcal{N}_{\sigma_k} \cap H_0^2(\Omega)} J_{\sigma_k} \geq \inf_{\mathcal{N}_{\sigma_k}} J_{\sigma_k} = J_{\sigma_k}(u_k);$$

hence we deduce that $m \geq \liminf_{k \to +\infty} J_{\sigma_k}(u_k)$. Moreover, by strong convergence,

$$J_{\mathrm{DIR}}(\bar{u}) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega g(x)|\bar{u}|^{p+1} = \lim_{k \to +\infty} \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega g(x)|u_k|^{p+1} = \lim_{k \to +\infty} J_{\sigma_k}(u_k).$$

Finally, since $\bar{u}$ is a solution of the Dirichlet problem (6-16), we have $\bar{u} \in \mathcal{N}_{\mathrm{DIR}}$, so

$$m \leq J_{\mathrm{DIR}}(\bar{u}) \leq \liminf_{k \to +\infty} J_{\sigma_k}(u_k) \leq m. \qquad \square$$

Moreau dual cones decomposition. So far, we have proved the existence of ground states for the Dirichlet problem (6-16) and the convergence result as $\sigma \to +\infty$. Proving positivity of ground states of (6-16) is quite a hard subject, since it strongly relies on the geometry of the domain, even in the linear case, where $f(x,u) = f(x)$: we refer to [Sweers 2001] for a short survey. Anyway, there are some cases in which it holds: for instance, the Dirichlet problem in the ball has been studied in [Ferrero et al. 2007], which covers the case where $g \equiv 1$, but whose arguments hold also in the general situation.

Our strategy is mainly inspired by this last work and it was firstly applied to fourth-order problems by Gazzola and Grunau [2001]. Briefly, we use Moreau decomposition in dual cones (for the original paper, see [Moreau 1962]) to obtain from a supposed sign-changing ground state solution u, a function w of one sign and in the same space with a strictly lower energy level, leading to a contradiction. In our case, in order to apply this machinery, we have to impose that the associated linear problem is positivity preserving: this will be the connection between the two problems.

Definition 6.15. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{1,1}$ and fix $\sigma \in \mathbb{R}$. The linear Steklov boundary problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{6-19}$$

is *positivity preserving* in Ω if there exists a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \geq 0$ implies $u \geq 0$, and this holds for each $f \in L^2(\Omega)$. We shorten this by saying that "Ω is a [PPP$_\sigma$] domain for (6-19)".

Definition 6.16. Let H be a Hilbert space with scalar product $(\cdot,\cdot)_H$ and $K \subset H$ be a nonempty closed convex cone. Its *dual cone* K^* is defined as

$$K^* := \{w \in H \mid (w,v)_H \leq 0 \text{ for all } v \in K\}.$$

Theorem 6.17 (Moreau dual cone decomposition, [Gazzola et al. 2010, Theorem 3.4]). *Let H be a Hilbert space with scalar product $(\cdot,\cdot)_H$ and K and K^* as before. Then for every $u \in H$, there exists a unique couple $(u_1, u_2) \in K \times K^*$ such that $u = u_1 + u_2$ and $(u_1, u_2)_H = 0$.*

Our aim is to apply this result with $(H, \|\cdot\|_H) = (H^2(\Omega) \cap H_0^1(\Omega); \|\cdot\|_{H_\sigma})$, where $\|\cdot\|_{H_\sigma}$ is the norm (2-1), and $K := \{v \in H \mid v \geq 0\}$, the cone of nonnegative functions, looking for a decomposition of each element of the space in positive and negative "parts". Hence we need a characterization of the dual cone K^*:

Lemma 6.18. *If Ω is a $[PPP_\sigma]$ domain for (6-19) for a fixed $\sigma \in \mathbb{R}$, then $K^* \subseteq \{w \in H \mid w < 0 \text{ a.e.}\} \cup \{0\}$.*

Proof. We adapt here the proof of [Gazzola et al. 2010, Proposition 3.6]. Let $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$ and let $v_\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique weak solution of the linear problem

$$\begin{cases} \Delta^2 v_\varphi = \varphi & \text{in } \Omega, \\ v_\varphi = \Delta v_\varphi - (1-\sigma)\kappa(v_\varphi)_n = 0 & \text{on } \partial\Omega; \end{cases}$$

that is, for every test function $w \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$(v_\varphi, w)_{H_\sigma} := \int_\Omega \Delta v_\varphi \, \Delta w - (1-\sigma) \int_{\partial\Omega} \kappa(v_\varphi)_n w_n = \int_\Omega \varphi w.$$

Hence, suppose $w = u \in K^*$: as Ω is a $[PPP_\sigma]$ domain and $\varphi \geq 0$, we deduce that $v_\varphi \geq 0$, so $v_\varphi \in K$ and thus $(v_\varphi, u)_{H_\sigma} \leq 0$. As a result, we have obtained that for every $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$, we have $\int_\Omega \varphi u \leq 0$, which implies that $u \leq 0$ a.e. in Ω.

Moreover, let us suppose that the null set of u, namely $N := \{x \in \Omega \mid u(x) = 0\}$, has positive measure, consider $\psi := \chi_N \neq 0$ and let v_0 be the unique solution of the linear Navier problem

$$\begin{cases} \Delta^2 v_0 = \psi & \text{in } \Omega, \\ v_0 = \Delta v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{6-20}$$

Then v_0 is strictly superharmonic by the maximum principle; thus $v_0 > 0$ and, by the Hopf Lemma, $(v_0)_n < 0$. As a result, for any function $v \in H^2(\Omega) \cap H_0^1(\Omega)$ one can produce two positive constants α, β such that $v + \alpha v_0 \geq 0$ and $v - \beta v_0 \leq 0$. Moreover we claim that $(u, v_0)_{H_\sigma} \geq 0$. In fact, as v_0 is the weak solution of (6-20) and by the definition of ψ,

$$\int_\Omega \Delta u \, \Delta v_0 = \int_\Omega u\psi = \int_N u = 0.$$

Thus, since $\sigma > 1$, $\kappa \geq 0$, $u_n \leq 0$ as $u \geq 0$, and $(v_0)_n < 0$,

$$(u, v_0)_{H_\sigma} := \int_\Omega \Delta u \, \Delta v_0 - (1-\sigma) \int_{\partial\Omega} \kappa u_n (v_0)_n \geq 0.$$

As a result, recalling that $u \in K^*$, $v + \alpha v_0 \in K$ and $v - \beta v_0 \in (-K)$, we have the chain of inequalities

$$0 \geq (u, v+\alpha v_0)_{H_\sigma} = (u, v)_{H_\sigma} + \alpha(u, v_0)_{H_\sigma} \geq (u, v)_{H_\sigma} \geq (u, v)_{H_\sigma} - \beta(u, v_0)_{H_\sigma} = (u, v-\beta v_0)_{H_\sigma} \geq 0,$$

which implies that $(u, v)_{H_\sigma} = 0$, and this holds for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Hence this is true also for v defined as the unique solution of the Steklov problem

$$\begin{cases} \Delta^2 v = u & \text{in } \Omega, \\ v = \Delta v - (1-\sigma)\kappa v_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{6-21}$$

and, using u as a test function, we deduce that

$$0 = (u, v)_{H_\sigma} = \int_\Omega u^2 = \|u\|_2^2,$$

which implies $u = 0$ a.e. $\square$

Proposition 6.19. *Let $\sigma > 1$ and suppose Ω is a $[PPP_\sigma]$ domain for* (6-19). *Then the ground states of J_σ are a.e. strictly of only one sign.*

Proof. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be such a ground state and suppose by contradiction that u is sign-changing. Denoting as before the cone of nonnegative functions by K, by Moreau decomposition there exists a unique couple $(u_1, u_2) \in K \times K^*$ such that $u = u_1 + u_2$ and $(u_1, u_2)_{H_\sigma} = 0$. Hence we know that $u_1 \geq 0$ and, by Lemma 6.18, $u_2 < 0$. Moreover, u is supposed to change sign, so $u_1 \neq 0$.

Defining $w := u_1 - u_2 \in H^2(\Omega) \cap H_0^1(\Omega)$, we have $w > |u|$. Indeed,

$$w = u_1 - u_2 > u_1 + u_2 = u, \quad w = u_1 - u_2 > -u_1 - u_2 = -u.$$

Consequently, $\int_\Omega g(x)|w|^{p+1} > \int_\Omega g(x)|u|^{p+1}$ and, since the decomposition is orthogonal under that norm, $\|w\|_{H_\sigma}^2 = \|u_1\|_{H_\sigma}^2 + \|u_2\|_{H_\sigma}^2 = \|u\|_{H_\sigma}^2$. Moreover, by Lemma 3.3, there exists $t^* := t^*(w) \in (0, +\infty)$ such that $w^* := t^*(w)w \in \mathcal{N}_\sigma$. Hence we deduce

$$J_\sigma(w^*) = \frac{(t^*)^2}{2}\|w\|_{H_\sigma}^2 - \frac{(t^*)^{p+1}}{p+1}\int_\Omega g(x)|w|^{p+1}$$

$$< \frac{(t^*)^2}{2}\|u\|_{H_\sigma}^2 - \frac{(t^*)^{p+1}}{p+1}\int_\Omega g(x)|u|^{p+1} = J_\sigma(t^*(w)u) \leq J_\sigma(u),$$

since u is the maximum of J_σ on the half-line $\{tu \mid t \in (0, +\infty)\}$ by Lemma 3.3; thus we have a contradiction again, since u was the infimum of J_σ on the Nehari manifold $\mathcal{N}_\sigma$. Hence we infer that $u \geq 0$.

Finally, as u is a critical point of J_σ, we have for each a positive test function $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$(u, \varphi)_{H_\sigma} = \int_\Omega \Delta u\,\Delta\varphi - (1-\sigma)\int_{\partial\Omega}\kappa u_n\varphi_n = \int_\Omega g(x)u^p\varphi \geq 0,$$

which implies that $-u \in K^*$. Applying now Lemma 6.18, we get $-u < 0$, that is, $u > 0$. $\square$

As a consequence, the problem of proving positivity of ground state is led back to a problem of positivity preserving for the linear problem, which was already tackled and solved by Gazzola and Sweers [2008].

Theorem 6.20. *Let $\sigma > 1$ and $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with $\partial\Omega$ of class C^2. There exists $\tilde{\delta}_c(\Omega) \in (1, +\infty]$ such that if $\sigma \in (1, \tilde{\delta}_c(\Omega))$, the ground states of the functional J_σ are a.e. strictly of only one sign.*

Proof. We follow the notation of [Gazzola and Sweers 2008]. Choosing $\beta = \kappa$ in Theorem 4.1(iii) of that paper, we infer the existence of $\delta_{c,\kappa}(\Omega) \in [-\infty, 0)$ such that if $(1-\sigma)\kappa \geq \delta_{c,\kappa}(\Omega)\kappa$, then the positivity preserving for problem (6-19) holds in Ω. Hence, defining $\tilde{\delta}_c(\Omega) := 1 + |\delta_{c,\kappa}(\Omega)|$, we can apply Proposition 6.19, provided $\sigma < \tilde{\delta}_c(\Omega)$. $\square$

Comparing Theorems 6.20 and 6.12, one may argue that we have nothing more than what we already knew: in both we obtain the existence of $\sigma_1 = \sigma_1(\Omega) > 1$ such that for all $\sigma \in (1, \sigma_1)$ the ground state solutions of problem (1-3) are positive. Nevertheless, in Theorem 6.20 we get further precise information about how the interval of positivity depends on the domain, relating it strongly with the positivity-preserving property. This fact is striking in the case of the disk and allows us to finally answer the question which opened the section.

Corollary 6.21. *Let $B \subset \mathbb{R}^2$ be a disk and let $\sigma > 1$. Then the ground states of the functional J_σ are a.e. strictly of only one sign.*

Proof. It is enough to notice that here $\kappa = 1$ and applying [Gazzola and Sweers 2008, Theorem 2.9] one can deduce $\delta_{c,\kappa}(B) = -\infty$, which implies $\tilde{\delta}_c(B) = +\infty$. □

One should finally notice that here the positivity found by the dual cones method is up to a subset of the domain with zero Lebesgue measure, so *almost everywhere* in Ω. This is the price we have to pay to extend the positivity beyond the parameter σ_1 found in Theorem 6.12 (see also Remark 6.13).

Remark 6.22. Again, up to some easy modifications in the proofs, both the convergence in Theorem 6.14 and the positivity result in Theorem 6.20 hold also in the sublinear case $p \in (0, 1)$.

7. Radial case

This section is devoted to some further investigations when the domain is a disk in $\mathbb{R}^2$ and the function g is radial, regarding existence, positivity and some qualitative properties of radially symmetric solutions. Moreover, we establish the counterpart of the convergence results of Sections 5 and 6, but for general radial positive solutions.

For simplicity, we focus on the problem

$$\begin{cases} \Delta^2 u = g(x)|u|^{p-1}u & \text{in } B, \\ u = \Delta u - (1-\sigma)u_n = 0 & \text{on } \partial B, \end{cases} \tag{7-1}$$

where $B := B_1(0) \subset \mathbb{R}^2$, $g = g(|x|)$ lies in $L^1(B)$ and it is strictly positive inside B. Moreover, we let $\sigma \in \mathbb{R}$ and $p \in (0, 1) \cup (1, +\infty)$ to cover both the sublinear and the superlinear case. Notice that the curvature does not appear in the mixed boundary condition since $\kappa(B) \equiv 1$.

Positive radially decreasing solutions and global bounds. First of all, by Proposition 5.3, our analysis concerns only the range $\sigma > -1$: in fact, if $\Omega = B$, one has $\sigma^* = -1$, since the first Steklov eigenvalue $\tilde{\delta}_1(B)$ is 2 (see [Berchio et al. 2006, Proposition 12]).

Retracing exactly the same steps of Sections 3 and 4, it is quite easy to obtain the existence of a positive radial solution. In fact, confining ourselves to the closed subspace of radial functions

$$H_{\text{rad}}(B) := \left\{ u \in H^2(B) \cap H_0^1(B) \mid u(x) = u(|x|) \text{ for all } x \in B \right\} = \text{Fix}_{O(2)}(H^2(B) \cap H_0^1(B)),$$

we deduce the existence of a critical point of J_σ restricted to $H_{\text{rad}}(B)$. Then it is enough to notice that J_σ is invariant under the action of $O(2)$ and to apply the principle of symmetric criticality due to Palais

(see [Willem 1996, Theorem 1.28]), retrieving that these points are critical for J_σ also with respect to the whole space.

Finally, if we restrict to the interval $(-1, 1]$, the positivity of such critical points is proved as in Propositions 4.9 and 4.12, realizing that the superharmonic function of a radially symmetric function is radial too (see (4-4)). On the other hand, if $\sigma > 1$, one can apply the dual cone decomposition to the Hilbert space $H_{\text{rad}}(B)$ and argue as in Lemma 6.18 and Proposition 6.19, taking into account that B is a [PPP$_\sigma$] domain for every $\sigma > -1$. Summarizing, we have shown the following:

Proposition 7.1. *Let $p \in (0, 1) \cup (1, +\infty)$, $g = g(|x|) \in L^1(B)$, $g > 0$. If $\sigma \leq -1$, there is no positive nonnegative nontrivial solution for (7-1), while, if $\sigma > -1$, there exists at least a positive radial solution, which is strictly superharmonic whenever $\sigma \in (-1, 1]$.*

Now, we want to prove some qualitative properties of radial positive solutions of (7-1). The first result concerns the radial behaviour, while the second the uniform boundedness in $L^\infty(B)$. Before proving these results, one should notice that such solutions are strong solutions, namely in $W^{4,q}(B)$, provided $g \in L^q(B)$ for some $q > 2$ and also classical assuming in addition that $g \in W^{1,q}(B)$ for some $q > 2$. This is a straightforward application of Lemma 6.7 combined with Morrey's embeddings.

Lemma 7.2. *Let $B := B_R(0)$ be the ball of radius R in $\mathbb{R}^2$ centered in 0, $q > 2$ and $\tilde{h} \in W^{2,q}(B)$ be radial. Defining $h : [0, R] \to \mathbb{R}$ to be its restriction to the radial variable, for all $t \in [0, R]$ the following equality holds*:

$$th'(t) = \int_0^t s \, \Delta h(s) \, ds. \tag{7-2}$$

Proof. If h is of class C^2, it comes directly from integration by parts and from the radial representation of the laplacian as

$$\Delta \tilde{h}(x) = h''(|x|) + \frac{1}{|x|} h'(|x|).$$

Otherwise, let $(\tilde{f}_k)_{k \in \mathbb{N}} \subset C^\infty(\bar{B})$ be such that $\tilde{f}_k \to \tilde{h}$ in $W^{2,q}(B)$, so in $C^1(\bar{B})$. Since $\tilde{h}$ is radial, we claim that it is possible to choose each $\tilde{f}_k$ to be radial and we denote its restriction to the radial variable as f_k. If so, for every $k \in \mathbb{N}$, we have

$$tf_k'(t) = \int_0^t s \, \Delta f_k(s) \, ds.$$

As a result, as $k \to +\infty$,

$$\left| \int_0^t s(\Delta f_k(s) - \Delta h(s)) \, ds \right| = \frac{1}{2\pi} \| \Delta \tilde{f}_k - \Delta \tilde{h} \|_{L^1(B_t(0))} \leq C(q) \| \tilde{f}_k - \tilde{h} \|_{W^{2,q}(B)} \to 0.$$

The result is proved by the convergence in $C^1(\bar{B})$ and the uniqueness of the limit. Now we have to justify our previous claim. Since $\tilde{h} \in W^{2,q}(B)$, we have

$$\sum_{i,\alpha} \int_B \left| \frac{\partial^\alpha \tilde{h}}{\partial i^\alpha}(x, y) \right|^q dx \, dy < +\infty,$$

where $i \in \{x, y\}$ and α is a multi-index of length $0 \le |\alpha| \le 2$. Since each $(\partial^\alpha \tilde{h})/(\partial i^\alpha)$ is radial, this is equivalent to saying that $h \in W^{2,q}([0, R], r)$, that is, the weighted Sobolev space with weight r. Hence, by [Kufner 1985, Theorem 7.4] ($M = \{0\}$, $\varepsilon = 1$ in notation therein), there exists a sequence $(f_k)_{k \in \mathbb{N}} \subset C^\infty([0, R])$ such that $f_k \to h$ in $W^{2,q}([0, R], r)$, that is

$$\sum_{i,\alpha} \int_0^R r \left| \frac{\partial^\alpha h}{\partial i^\alpha}(r) - f_k(r) \right|^q dr \to 0.$$

Hence, defining $F_k(x) := f_k(|x|)$, each $F_k \in C^\infty(\bar{B})$ is radial and

$$\|\tilde{h} - F_k\|_{W^{2,q}(B)} = \sum_{i,\alpha} \int_B \left| \frac{\partial^\alpha \tilde{h}}{\partial i^\alpha}(x, y) - F_k(x, y) \right|^q dx\, dy = 2\pi \sum_{i,\alpha} \int_0^R r \left| \frac{\partial^\alpha h}{\partial i^\alpha}(r) - f_k(r) \right|^q dr \to 0,$$

and the claim is proved. $\qquad\square$

Proposition 7.3 (radial decay). *Assume $g \in L^q(B)$ for some $q > 2$, g is radial and $g > 0$, and let $u \not\equiv 0$ be a nonnegative radial solution of (7-1) with $\sigma \in (-1, 1]$ and $p \in (0, 1) \cup (1, +\infty)$. Then u is strictly radially decreasing; thus $u > 0$ in B.*

Proof. By the assumption on g, we infer that u is a strong solution; thus $w := \Delta u \in W^{2,q}(B)$. Since $\Delta w = \Delta^2 u = g(|x|)u^p \ge 0$ in $[0, 1]$, applying Lemma 7.2, we have $w' > 0$ in $(0, 1]$. Hence Δu is strictly increasing in $(0, 1]$. Moreover, since u is nonnegative and $u(1) = 0$, we have $u'(1) \le 0$; hence, using the second boundary condition, $\Delta u(1) = (1 - \sigma)u'(1) \le 0$. Since Δu is strictly increasing in $(0, 1]$, we deduce that $\Delta u < 0$ in $[0, 1)$, and finally, applying again Lemma 7.2, $u' < 0$ in $(0, 1]$. $\qquad\square$

In the next result we find a uniform upper bound for positive radial solutions of (7-1), which may be seen as a superlinear counterpart of Proposition 4.11. We will make use of a blow up method which goes back to Gidas and Spruck [1981], and which was adapted to the polyharmonic case by Reichel and Weth [2009; 2010]. Briefly, our argument will be the following: supposing the existence of a sequence of positive radial solutions with diverging L^∞ norm, we rescale each of them in order to have another sequence of functions with the same L^∞ norm, satisfying the same equation in nested domains which tend to occupy the whole $\mathbb{R}^2$. Then we show that, up to a subsequence, it converges uniformly on compact subsets to a continuous nonnegative but nontrivial function. This turns out to be a solution of the same equation on $\mathbb{R}^2$, which is a contradiction with the following Liouville-type result by Wei and Xu, with $N = 2$ and $m = 2$:

Lemma 7.4 [Wei and Xu 1999, Theorem 1.4]. *Let $m \in \mathbb{N}$ and assume that $p > 1$ if $N \le 2m$ and $1 < p \le (N + 2m)/(N - 2m)$ if $N > 2m$. If u is a classical nonnegative solution of*

$$(-\Delta)^m u = u^p \quad in \ \mathbb{R}^N,$$

then $u \equiv 0$.

In our proof we will make also use of the following local regularity estimate, which is a particular case of a more general result by Reichel and Weth:

Lemma 7.5 [Reichel and Weth 2009, Corollary 6]. *Let* $\Omega = B_R(0) \subset \mathbb{R}^N$, $m \in \mathbb{N}$, $h \in L^p(\Omega)$ *for some* $p \in (1, +\infty)$ *and suppose* $u \in W^{2m,p}(\Omega)$ *satisfies*

$$(-\Delta)^m u = h \quad \text{in } \Omega.$$

Then there exists a constant $C = C(R, N, p, m)$ *such that for any* $\delta \in (0, 1)$,

$$\|u\|_{W^{2m,p}(B_{\delta R}(0))} \leq \frac{C}{(1-\delta)^{2m}} \big(\|h\|_{L^p(B_R(0))} + \|u\|_{L^p(B_R(0))} \big).$$

Proposition 7.6. *Let* $\sigma \in (-1, 1]$. *Let* $g \in L^q(B)$ *for some* $q > 2$, g *be radial and* $g > 0$. *Suppose also that* g *is continuous in* 0. *Then, there exists* $C > 0$ *independent of* σ *such that* $\|u\|_\infty \leq C$ *for every* u *radial positive solution of* (7-1).

Proof. By contradiction, suppose there exists a sequence $(v_k)_{k \in \mathbb{N}}$ of radial positive solutions such that $\|v_k\|_\infty \nearrow +\infty$. According to Proposition 7.3, each v_k is radially decreasing, so $v_k(0) = \|v_k\|_\infty \nearrow +\infty$. For each $k \geq 1$, define

$$u_k(x) = \lambda_k^{\frac{4}{p-1}} v_k(\lambda_k x),$$

where $\lambda_k \in \mathbb{R}^+$ are such that $\lambda_k^{\frac{4}{p-1}} = 1/v_k(0)$. With this choice, each u_k satisfies

$$\begin{cases} \Delta^2 u_k = g(|\lambda_k x|) u_k^p & \text{in } B_{\frac{1}{\lambda_k}}(0), \\ u_k = \Delta u_k - (1-\sigma)\lambda_k (u_k)_n = 0 & \text{on } \partial B_{\frac{1}{\lambda_k}}(0), \end{cases} \tag{7-3}$$

is in $W^{4,q}(B_{\frac{1}{\lambda_k}}(0))$, radially decreasing and

$$\|u_k\|_{L^\infty(B_{1/\lambda_k}(0))} = u_k(0) = \lambda_k^{\frac{4}{p-1}} v_k(0) = 1. \tag{7-4}$$

We claim that the sequence $(u_k)_{k \in \mathbb{N}}$ is uniformly bounded on compact sets of $\mathbb{R}^2$ in $W^{4,q}$ norm. In fact, let $K \subset \mathbb{R}^2$ be compact; then there exists $\rho > 0$ such that $B_\rho(0) \supset K$ and, for k large enough, each u_k is in K since $B_{\frac{1}{\lambda_k}}(0) \supset B_{2\rho}(0)$ definitively. For such k, by (7-4) and applying Lemma 7.5 with $\Omega = B_{2\rho}(0)$, $m = N = 2$ and $\delta = \frac{1}{2}$,

$$\|u_k\|_{W^{4,q}(K)} \leq \|u_k\|_{W^{4,q}(B_\rho(0))} \leq \frac{C(\rho, q)}{1/2^4} (\|\Delta^2 u_k\|_{L^q(B_{2\rho}(0))} + \|u_k\|_{L^q(B_{2\rho}(0))})$$

$$\leq 16 C(\rho, q)(\|g(|\lambda_k \cdot|)\|_{L^q(B_{2\rho}(0))} + |B_{2\rho}(0)|^{\frac{1}{q}}). \tag{7-5}$$

Moreover, fixing $\varepsilon > 0$ and supposing k large enough,

$$\|g(|\lambda_k \cdot|)\|_{L^q(B_{2\rho}(0))} = (4\pi\rho^2)^{\frac{1}{q}} \left(\frac{1}{|B_{2\rho\lambda_k}(0)|} \int_{B_{2\rho\lambda_k}(0)} |g(y)|^q \, dy \right)^{\frac{1}{q}} \leq (4\pi\rho^2)^{\frac{1}{q}} g(0) + \varepsilon, \tag{7-6}$$

where the last inequality follows from the Lebesgue differentiation theorem. Hence, combining (7-5) with (7-6), we infer $\|u_k\|_{W^{4,q}(K)} \leq C(p, q, K, g)$, which is uniform on k. Incidentally, notice that this

constant does not depend on σ. Hence we find $u \in W^{4,q}(K)$ such that, up to subsequences, $u_k \to u$ in $C^3(K)$, where $u \in C^3(\mathbb{R}^2)$, $u \geq 0$ and $u(0) = 1$ by (7-4) and satisfying

$$\Delta^2 u = g(0)u^p \quad \text{in } \mathbb{R}^2.$$

So, by a bootstrap method, we deduce that u is also a classical solution. Finally, setting for all $x \in \mathbb{R}^2$ $w(x) := u(bx)$ with $b := g(0)^{-\frac{1}{4}}$, one has w is a nonnegative solution of

$$\Delta^2 w = w^p \quad \text{in } \mathbb{R}^2,$$

with $w(0) = u(0) = 1$, which contradicts Lemma 7.4. $\qquad\square$

Convergence results. We want to investigate what happens at the endpoints of the interval $(-1, 1]$ in which σ lies, by means of the last results. More precisely, our aim is to examine if any result similar to Theorems 5.8 and 6.5 can be found assuming $(u_k)_{k \in \mathbb{N}}$ to be a sequence of positive radial solutions of (7-1) with $\sigma = \sigma_k$ but without imposing any "minimizing" requirement. Unless otherwise stated, we assume $g \equiv 1$ and $p > 1$.

Let us start with the behaviour for $\sigma \to 1$, where the main ideas are taken from the same result for ground states. Notice that we know everything for the Navier problem in the ball: in fact, Dalmasso [1995] proved that there exists a unique positive solution, which is radially symmetric and radially decreasing thanks to a result by Troy [1981].

Proposition 7.7. *Let $(u_k)_{k \in \mathbb{N}}$ be a sequence of positive radial solutions of (7-1) with $\sigma_k \nearrow 1$. Then $u_k \to \bar{u}$ in $H^2(B)$, where $\bar{u}$ is the unique positive solution of the Navier problem.*

Proof. We firstly claim that such a sequence is bounded in $H^2(B)$. Indeed, by Proposition 7.6,

$$\|u_k\|_{H^2(B)}^2 \leq C_0 \|\Delta u_k\|_2^2 \leq C_0 \left(1 - \frac{1 - \sigma_k}{2}\right)^{-1} \|u_k\|_{H_{\sigma_k}}^2 = \frac{2C_0}{1 + \sigma_k} \|u_k\|_{p+1}^{p+1} \leq 2\pi C_0 C^{p+1}.$$

Hence, we can extract a subsequence $(u_{k_j})_{j \in \mathbb{N}}$ such that there exists $\bar{v} \in H^2(B) \cap H_0^1(B)$ such that $u_{k_j} \rightharpoonup \bar{v}$ weakly in $H^2(B)$. By Lemma 6.1, together with Remark 6.2, one can infer that this subsequence is actually strongly convergent in $H^2(B)$ and then that $\bar{v}$ is a weak solution of the Navier problem (thus classical by regularity theory). Moreover, since the convergence is pointwise, we immediately deduce that $\bar{v}$ is nonnegative, radially symmetric and radially nonincreasing. Nevertheless, by Proposition 7.3, $\bar{v}$ is actually strictly decreasing and positive in B, so it coincides with the unique positive solution $\bar{u}$ of the Navier problem. By the uniqueness of the limit and applying Urysohn subsequence principle, we retrieve the convergence of the whole sequence $(u_k)_{k \in \mathbb{N}}$ from which we started. $\qquad\square$

Let us now investigate the case $\sigma \to -1$. As already noticed in Lemma 5.9, it is enough to understand the behaviour of the $L^{p+1}(B)$ norm of a sequence of solutions to infer the convergence in $H^2(B)$ norm. Since the proof of Theorem 5.8 strongly relies on the fact that it deals with ground states, we need a different technique. The first step is a Pohozaev-type identity by Mitidieri [1993]: it will allow us to prove an inequality involving $L^p(B)$ and $L^{p+1}(B)$ norms which, combined with the uniform bound of Proposition 7.6, will lead us to the convergence result.

Lemma 7.8 [Mitidieri 1993, Proposition 2.2]. *Let Ω be a smooth domain and $u \in C^4(\overline{\Omega})$. The following identity holds*:

$$\int_{\Omega} (\Delta^2 u) x \cdot \nabla u - \frac{N}{2} \int_{\Omega} (\Delta u)^2 - (N-2) \int_{\Omega} \nabla \Delta u \cdot \nabla u$$
$$= -\frac{1}{2} \int_{\partial\Omega} (\Delta u)^2 x \cdot n + \int_{\partial\Omega} \left((\Delta u)_n (x \cdot \nabla u) + u_n (x \cdot \nabla \Delta u) - \nabla \Delta u \cdot \nabla u (x \cdot n) \right).$$

Corollary 7.9. *Suppose u is a positive solution for problem* (7-1) *with $g \equiv 1$. Then the following identity holds*:

$$\int_{\partial B_R} \left((\Delta u)_n + (1-\sigma)\left(1 - \frac{1-\sigma}{2}\right) u_n \right) u_n = -\left(1 + \frac{2}{p+1}\right) \int_{B_R} u^{p+1}. \tag{7-7}$$

Proof. By similar computations as in the proof contained in [Berchio et al. 2007, Section 6], from Lemma 7.8 one infers

$$\left(\frac{N-4}{2} - \frac{N}{p+1} \right) \int_{\Omega} u^{p+1} = \int_{\partial\Omega} \left(x \cdot \nabla \Delta u + \tfrac{1}{2} N(1-\sigma)\kappa u_n - \tfrac{1}{2}(1-\sigma)^2 \kappa^2 u_n (x \cdot n) \right) u_n. \tag{7-8}$$

If $N = 2$ and $\Omega = B$, we have $x = n$ and $\kappa = 1$, so $x \cdot \nabla \Delta u = (\Delta u)_n$ and (7-7) follows. $\square$

The next result follows from some ideas of Berchio and Gazzola: we give here a sketch, while we refer to [Berchio and Gazzola 2011, Proposition 4], for a more detailed proof.

Lemma 7.10. *Let $\sigma \in (-1, 1)$ and u be a positive radial solution of problem* (7-1) *with $g \equiv 1$. Then the following estimate holds*:

$$\|u\|_{p+1}^{p+1} \geq \tfrac{3}{64}\left(1 - \tfrac{3}{64}(1-\sigma)\right) \frac{1}{\pi(1+\sigma)} \frac{p+1}{p+3} \|\Delta^2 u\|_1^2. \tag{7-9}$$

Proof. By radial symmetry, (7-7) reduces to

$$2(\Delta u)'(1) u'(1) + (1-\sigma)(1+\sigma)(u'(1))^2 = -\frac{p+3}{p+1}\frac{1}{\pi}\int_B u^{p+1}. \tag{7-10}$$

Moreover, by the divergence theorem we have

$$u'(1) = \frac{1}{2\pi}\int_B \Delta u \quad \text{and} \quad (\Delta u)'(1) = \frac{1}{2\pi}\int_B \Delta^2 u,$$

so, taking the first Steklov eigenfunction $w(x) = \frac{1}{4}(1 - |x|^2)$ and after some elementary computations, one gets

$$\left(\int_B \Delta^2 u - (1-\sigma)\int_B w \, \Delta^2 u \right) \int_B w \, \Delta^2 u = \frac{p+3}{p+1}(1+\sigma)\pi \int_B u^{p+1}. \tag{7-11}$$

Noticing that $0 \leq w \leq \frac{1}{4}$, we have

$$\frac{3}{64}\int_B \Delta^2 u \leq \int_B w \, \Delta^2 u \leq \frac{1}{4}\int_B \Delta^2 u.$$

Hence, defining now $d := (1-\sigma)$, $s := \int_B w \, \Delta^2 u$ and $A := \int_B \Delta^2 u$, the left-hand side of (7-11) becomes

$$As - ds^2, \quad \text{with } s \in \left[\tfrac{3}{64} A; \tfrac{1}{4} A \right].$$

Since $d > 0$, we know $\psi : s \mapsto As - ds^2$ is a concave function, so it attains its minimum on the extremal values of the interval: in this case, with $0 < d < 2$, one has

$$\psi(s) \geq \tfrac{3}{64}\big(1 - \tfrac{3}{64}d\big)A^2.$$

Combining this with (7-11), one finds the desired estimate (7-9). □

Theorem 7.11. *Let $\sigma_k \searrow -1$ and $(u_k)_{k \in \mathbb{N}}$ be a sequence of positive radial functions, each of them a solution of the problem* (7-1) *with $g \equiv 1$ and $\sigma = \sigma_k$. Then, $u_k \to 0$ in $H^2(B)$.*

Proof. By Lemma 5.9, it is enough to prove the convergence in $L^{p+1}(B)$ norm. Since every solution of (7-1) is smooth, we have $\|\Delta^2 u_k\|_1 = \|u_k\|_p^p$. Moreover, by the uniform L^∞ estimate found in Proposition 7.6, we know that there exists a constant $C > 0$ not depending on σ_k, such that

$$\|u_k\|_{p+1}^{p+1} \leq \|u_k\|_\infty^{p+1}|B| \leq \pi C^{p+1}.$$

As a result, using the estimate provided by Lemma 7.10, one has

$$\frac{1 + \sigma_k}{1 - \tfrac{3}{64}(1 - \sigma_k)} \geq \frac{p+1}{p+3}\frac{3}{64\pi^2 C^{p+1}}\|u_k\|_p^{2p}, \qquad (7\text{-}12)$$

so, letting $\sigma_k \to -1$ we deduce $\|u_k\|_p \to 0$. This, together with the $L^\infty(B)$ estimate of Proposition 7.6, gives us the convergence in $L^{p+1}(B)$ and so the desired result. □

8. Open problems

We end our paper with some unsolved questions that would complete the present investigation.

- *If Ω is a ball, are the ground states of J_σ radially symmetric?*

In fact, we deduced the existence of ground states and radial solutions which are indeed ground states among all possible radial solutions; both of them are positive and have the same behaviour when $\sigma \to -1$ and $\sigma \to 1$. But no standard techniques such as the Talenti symmetrization principle seem to apply (except for the Navier case) to prove that these classes of functions are indeed the same.

- *Are the radial positive solutions radially decreasing if $\sigma > 1$?*

Indeed, the radial decay property proved in Proposition 7.3 does not apply in this setting and, by now, we cannot extend Proposition 7.6 for these values of σ.

Moreover, in the spirit of [Dalmasso 1995] and [Ferrero et al. 2007]:

- *Can we say something about the uniqueness of (at least) the positive radially symmetric ground state of J_σ for some values of σ?*

Finally, all the techniques developed in Section 3 strongly relied on the assumptions we made on the boundary, that is, $\partial\Omega$ of class $C^{1,1}$, in order to have $\kappa \in L^\infty(\partial\Omega)$. In particular, Theorem 4.7 allowed us to rewrite in an appropriate way our functional. Also the convexity played a crucial role in proving the positivity: see in particular Propositions 4.9, 4.12 and 5.6 as well as Theorem 6.20.

- *May we deduce the positivity of ground states of J_σ when the domain Ω is not convex anymore or with less regularity on the boundary*?

Since in the Navier case their positivity is always assured simply by the maximum principle, we expect that, even without the convexity assumption, it continues to hold whenever σ belongs to a neighborhood of 1 which may depend on "how far" the domain is from being convex.

Concerning the regularity of the boundary, if we consider the particular case of a convex polygon $\mathcal{P}$, it is known that ground states of J_σ are positive for every σ: in fact, the superharmonic method applies easily once we have $\int_{\mathcal{P}} \det(\nabla^2 u) = 0$ thanks to a result by Grisvard [1992, Lemma 2.2.2]. We believe that positivity for ground states of J_σ still holds imposing, for instance, only Lipschitz regularity for $\partial\Omega$.

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