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ON THE GROWTH OF SOBOLEV NORMS FOR NLS ON 2- AND 3-DIMENSIONAL MANIFOLDS

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Using suitable modified energies, we study higher-order Sobolev norms' growth in time for the nonlinear Schrödinger equation (NLS) on a generic 2- or 3-dimensional compact manifold. In two dimensions, we extend earlier results that dealt only with cubic nonlinearities, and get polynomial-in-time bounds for any higher-order nonlinearities. In three dimensions, we prove that solutions to the cubic NLS grow at most exponentially, while for the subcubic NLS we get polynomial bounds on the growth of the H^2 norm.

1. Introduction

We are interested in long-time qualitative properties of solutions to the family of nonlinear Schrödinger equations

$$\begin{cases} i\,\partial_t u + \Delta_g u = |u|^{p-1}u, & (t,x) \in \mathbb{R} \times M^d, \\ u(0,x) = \varphi \in H^m(M^d), \end{cases} \tag{1}$$

where Δ_g is the Laplace–Beltrami operator associated with a d-dimensional compact Riemannian manifold (M^d, g) and $H^m(M^d)$, the standard Sobolev space associated to Δ_g, where $m \in \mathbb{N}$ with $m \geq 2$. More specifically we are interested in the analysis of the possible growth of higher-order Sobolev norms for large times, namely the behavior of the quantity $\|u(t,x)\|_{H^m(M^d)}$ for $m \geq 2$ and $t \gg 1$.

This issue of growth of higher-order Sobolev norms has garnered a lot of attention in recent years, mainly because of its connection with the so-called *weak wave turbulence*, e.g., a cascade of energy from low to high frequencies. In fact two main issues have been extensively studied in the literature: the first one concerns a priori bounds on how fast higher-order Sobolev norms can grow along the flow associated with Hamiltonian PDEs (see [Bourgain 1993; 1996; 1999a; 1999b; Colliander et al. 2012; Delort 2014; Sohinger 2011a; 2011b; 2012; Staffilani 1997; Thirouin 2017; Zhong 2008]); the second one concerns the existence of global solutions whose higher-order Sobolev norms are unbounded (see [Colliander et al. 2010; Gérard and Grellier 2016; 2015; Guardia 2014; Guardia et al. 2016; Guardia and Kaloshin 2015; Hani 2014; Hani et al. 2015; Haus and Procesi 2015; Xu 2015]).

Here, we aim at dealing with the first problem, namely to provide a priori bounds on the growth of higher-order Sobolev norms, or equivalently to understand how fast the dynamical system under consideration can move energy from the low frequencies to the high frequencies.

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First of all we point out that solutions to (1) enjoy so-called mass and energy conservation laws:

$$\int_{M^d} |u(t,x)|^2 \,\mathrm{dvol}_g = \int_{M^d} |\varphi(x)|^2 \,\mathrm{dvol}_g,$$

$$\int_{M^d} \Big(|\nabla_g u(t,x)|_g^2 + \frac{1}{p+1}|u(t,x)|^{p+1}\Big)\,\mathrm{dvol}_g = \int_{M^d} \Big(|\nabla_g \varphi(x)|_g^2 + \frac{1}{p+1}|\varphi(x)|^{p+1}\Big)\,\mathrm{dvol}_g,$$

where ∇_g and $|\cdot|_g$ are respectively the gradient and the norm associated with the metric g, and $|\cdot|$ denotes the modulus of any complex number. These conservation laws immediately imply that

$$\sup_{\mathbb{R}} \|u(t,x)\|_{H^1(M^d)} < \infty, \tag{2}$$

and therefore the growth in time of H^m norms is only of interest for $m \geq 2$.

In the sequel, with notation as above, we shall be interested in the following cases:

$(d, p) = (2, 2n+1)$ with $n \in \mathbb{N}$, $n \geq 1$ (2-dimensional manifold and odd integer nonlinearity),

$(d, p) = (3, 3)$ (3-dimensional manifold and cubic nonlinearity),

$(d, p) = (3, p)$ with $2 < p < 3$ (3-dimensional manifold and subcubic nonlinearity).

In those settings, existence of local solutions follows by classical arguments, provided one assumes the initial datum to be H^2. On the other hand, following [Burq et al. 2004], one can establish local (and hence global) Cauchy theory in H^1 for generic nonlinear potentials in the 2-dimensional case, as well as local (and global) Cauchy theory in $H^{1+\epsilon}$ for the cubic and subcubic NLS in the 3-dimensional case (see [Burq et al. 2003; 2004]). From now on and for the sake of simplicity, we shall assume existence and uniqueness of a global solution, and focus on estimating the growth of higher-order Sobolev norms. However, we point out that our argument not only provides polynomial bounds of such growth, but also yields an alternative proof of global existence in three dimensions.

We will use as a basic tool (in fact, as a black box) available Strichartz estimates on manifolds (see [Burq et al. 2004; Staffilani and Tataru 2002]) together with the introduction of suitable *modified energies*, which is the main new ingredient in this context. For this reason we will not discuss further the issue of global existence, which is indeed guaranteed by aforementioned previous results.

We first start with the 2-dimensional case. It is worth mentioning that, to the authors' knowledge, no results were available in the literature about growth of higher-order Sobolev norms for NLS with higher than cubic nonlinearities, although one may reasonably believe that this problem could be addressed, at least in two dimensions, by adapting the strategy pioneered by Bourgain (see for instance [Zhong 2008]). Nevertheless as a warm up we show how this problem can be handled by a completely different strategy, based on the introduction of suitable *modified energies*: its benefit relies on a clear decoupling between higher-order energy estimates relying on clever integration by parts and the (deep) input provided by dispersive estimates of Strichartz type. Moreover by using modified energies, one can deal as well with generic nonlinear potential $V(|u|^2)$ rather than $|u|^{p-1}$, where V may not necessarily be a pure power (see also Remark 1.7 below).

We emphasize that modified energies have proved useful in different contexts (see, for instance, [Chiron and Rousset 2009; Hunter et al. 2015; Koch and Tataru 2016; Kwon 2008; Ozawa and Visciglia 2016;

Raphaël and Szeftel 2009; Tsutsumi 1989]), but the present work seems to provide the first example where they are combined with dispersive bounds in order to get results on the growth of higher-order Sobolev norms.

We underline that our argument, being essentially based on integration by parts, relies on the time derivative of suitable higher-order energies $\mathcal{E}_m$, whose leading term is essentially the norm $\|u(t,x)\|^2_{H^m}$. In fact, for $m = 2k$ an even integer, one should think of $\|\partial_t^k u(t,x)\|^2_{L^2}$ as a good prototype of modified energy, up to lower-order terms. In other words, one should think of replacing Δ_g by ∂_t rather than the other way around when using the equation satisfied by u.

A direct consequence of this privileged use of ∂_t is that in our approach the geometry of the manifold is not directly involved in the computation, and integration by parts in the space variables, when required, is performed thanks to the following elementary identity, available on any generic manifold:

$$\Delta_g(fh) = h\Delta_g f + 2(\nabla_g f, \nabla_g h)_g + f\Delta_g h.$$

We also underline that the aforementioned energy $\mathcal{E}_m$ is not preserved along the flow; however, by computing its time derivative along solutions, we may estimate the resulting space-time integral taking advantage of dispersive bounds, namely Strichartz estimates with loss, which are available on a generic manifold (or better ones when available).

In order to state our result in two dimensions, we recall Strichartz estimates with loss:

$$\|e^{it\Delta_g}\varphi\|_{L^4((0,1)\times M^2)} \lesssim \|\varphi\|_{H^{s_0}(M^2)}. \tag{3}$$

It is well known that estimate (3) holds on $\mathbb{T}^2$ for any $s_0 > 0$ (see [Bourgain 1993; 1999a]) and on the sphere $\mathbb{S}^2$ for any $s_0 > \frac{1}{8}$ (see [Burq et al. 2004]). We can now state our first result, where we assume (3) to be satisfied for some s_0 in the range $\left[0, \frac{1}{4}\right]$. We recall that the existence of such an s_0 is guaranteed on every compact manifold M^2 by [Burq et al. 2004].

Theorem 1.1. *For every $\epsilon > 0$, $m \in \mathbb{N}$ with $m \geq 2$ and for every solution $u(t,x) \in \mathcal{C}_t(H^m(M^2))$ to* (1), *where $d = 2$ and $p = 2n+1$ for $n \geq 1$, we get*

$$\sup_{(0,T)} \|u(t,x)\|_{H^m(M^2)} \leq C(\max\{1,T\})^{\frac{m-1}{1-2s_0}+\epsilon}, \tag{4}$$

where $C = C(\epsilon, m, \|\varphi\|_{H^m}) > 0$ and $s_0 \in \left[0, \frac{1}{4}\right]$ is given in (3).

Notice that bounds from Theorem 1.1 also apply to solutions of NLS on $\mathbb{T}$. In fact the dynamics of NLS on $\mathbb{T}$ is a subset of the dynamics on $\mathbb{T}^2$, and this framework is covered by Theorem 1.1, where we can choose $s_0 = 0$. In particular, Theorem 1.1 recovers results from [Colliander et al. 2012] for solutions to NLS on $\mathbb{T}$ with $p > 5$. Notice that paper obtains a better $\mathbb{T}^{\frac{m-1}{2}+\epsilon}$ growth for $p = 5$ by implementing a normal-form method. We will address this better growth for all $p > 5$ with a suitable modification of our argument in a later work.

Remark 1.2. We underline that the main point in order to establish Theorem 1.1 is the following bound: for all $\tau \in (0,1)$, $\epsilon > 0$,

$$\|u(\tau)\|^2_{H^m(M^2)} - \|u(0)\|^2_{H^m(M^2)} \lesssim \sqrt{\tau}\,\|u\|^{\frac{2m-3+2s_0}{m-1}+\epsilon}_{L^\infty((0,\tau);H^m(M^2))} + \|u\|^{\frac{2m-4}{m-1}+\epsilon}_{L^\infty((0,\tau);H^m(M^2))}. \tag{5}$$

Once this bound is established, a classical argument (which in turn requires the local well-posedness of the Cauchy problem in the energy space H^1) leads to the polynomial growth. More specifically notice that the exponent $(m-1)/(1-2s_0)+\epsilon$ (which appears in the right-hand side of (4)) can be computed as the quantity $\frac{1}{2\gamma}$, where $2-2\gamma=(2m-3+2s_0)/(m-1)+\epsilon$ is the power of the first term in the right-hand side of (5). Next we choose $\tau=\tau(\|u(0)\|_{H^1})$ to be the time of existence provided by the H^1 local Cauchy theory. Then (5) gives

$$\|u(t+\tau)\|^2_{H^m(M^2)} \le \|u(t)\|^2_{H^m(M^2)} + C\big(\|u\|^{2-2\gamma}_{L^\infty((t,t+\tau);H^m(M^2))}+1\big).$$

As a byproduct of the local existence theory in H^1, and conservation of the energy, we get

$$\|u(t+\tau)\|^2_{H^m(M^2)} \le \|u(t)\|^2_{H^m(M^2)} + C\big(\|u(t)\|^{2-2\gamma}_{H^m(M^2)}+1\big).$$

Therefore the sequence $\alpha_n = 1+\|u(n\tau)\|^2_{H^m(M^2)}$ satisfies $\alpha_{n+1}\le\alpha_n+C\alpha_n^{1-\gamma}$, which in turn implies $\alpha_n \lesssim n^{1/\gamma}$, leading to (4) by induction on n.

Next we present our result on the growth of higher-order Sobolev norms for the cubic NLS on a generic 3-dimensional compact manifold M^3. We recall that, following [Burq et al. 2004], the Cauchy problem is globally well-posed for every initial data $\varphi\in H^{1+\epsilon_0}(M^3)$, and that, following the crucial use of logarithmic Sobolev type inequalities, one can get the following double exponential bound,

$$\sup_{(0,T)}\|u(t,x)\|_{H^m(M^3)} \le C\exp(\exp(CT)).$$

Our main contribution is an improvement on the bound above; indeed, we will replace the double exponential with a single one. It should be emphasized that, in the 3-dimensional case, it is at best unclear to us how Bourgain's original argument and derivatives thereof could be used in order to get Theorem 1.3. More specifically, in three dimensions our use of modified energies appears to be a key tool in order to eliminate one of the two exponentials.

Theorem 1.3. *For every $m\in\mathbb{N}$ with $m\ge 2$ and for every solution $u(t,x)\in\mathcal{C}_t(H^m(M^3))$ to* (1), *where $(d,p)=(3,3)$, we have*

$$\sup_{(0,T)}\|u(t,x)\|_{H^m(M^3)} \le C\exp(CT),$$

where $C=C(m,\|\varphi\|_{H^m})>0$.

Remark 1.4. The proof of Theorem 1.3 follows by a straightforward iteration once the following bound is established: for all $\tau\in(0,1)$,

$$\|u(\tau)\|^2_{H^m(M^3)} - \|u(0)\|^2_{H^m(M^3)} \lesssim \tau\|u\|^2_{L^\infty((0,\tau);H^m(M^3))} + \|u\|^\gamma_{L^\infty((0,\tau);H^m(M^3))}, \tag{6}$$

where $\gamma\in(0,2)$ is a suitable number and the implicit constant depends only on the energy of u. Indeed, using (6) for τ small enough and the fact that $\gamma<2$, we get the bound

$$\|u\|^2_{L^\infty((0,\tau);H^m(M^3))} \le 2\|u(0)\|^2_{H^m(M^3)} + C.$$

Therefore the sequence $\alpha_n=\|u(n\tau)\|^2_{H^m(M^3)}$ satisfies $\alpha_{n+1}\le 2\alpha_n + C$, which implies the claimed exponential bound.

Remark 1.5. Notice that in Theorems 1.1 and 1.3 we provide bounds on the growth of the H^m Sobolev norm for initial data of regularity H^m, for a given, odd or even, integer m. We point out that most of the paper will be devoted to the case of even integers. In the last section we sketch how to adapt the argument to odd integers. Of course, if we assume the initial datum φ to be H^{m+1}, then the growth of H^m, with m odd, can be obtained by interpolation between the growth of the two norms H^{m+1} and H^{m-1}, with $m-1$ and $m+1$ even. Hence if the initial datum is smooth enough, it is not necessary to deal separately with the case m odd. However the situation is more delicate since we assume only the regularity H^m (with m odd) on the initial datum.

Finally, we end our presentation with a result dealing with NLS on a 3-dimensional compact manifold M^3 with subcubic nonlinearity, establishing polynomial growth for the H^2 Sobolev norm. It makes no sense to consider higher-order Sobolev norms, given that the nonlinearity is not smooth enough to guarantee that regularity H^m, with $m > 2$, is preserved along the evolution.

Nevertheless we emphasize that the next result appears to be the first one available in the literature about polynomial growth of any Sobolev norms above the energy, on a generic 3-dimensional compact manifold.

Theorem 1.6. *For every solution $u(t,x) \in \mathcal{C}_t(H^2(M^3))$ to* (1) *with $d = 3$ and $p \in (2,3)$ we have*

$$\sup_{(0,T)} \|u(t,x)\|_{H^2(M^3)} \le C(\max\{1,T\})^{\frac{4}{3-p}},$$

where $C = C(\|\varphi\|_{H^2}) > 0$.

Remark 1.7. The proof of Theorem 1.6 follows once the following local bound is established: for all $\tau \in (0,1)$,

$$\|u(\tau)\|^2_{H^2(M^3)} - \|u(0)\|^2_{H^2(M^3)} \lesssim \tau \|u\|^{\frac{p+5}{4}}_{L^\infty((0,\tau);H^2(M^3))} + \|u\|^{\gamma}_{L^\infty((0,\tau);H^2(M^3))} \tag{7}$$

for some $\gamma \in \left(0, \frac{p+5}{4}\right)$. In order to conclude the polynomial growth from (7), we can combine Remarks 1.2 and 1.4. In fact arguing as in Remark 1.4 we get

$$\|u\|^2_{L^\infty((0,\tau);H^m(M^3))} \le 2\|u(0)\|^2_{H^m(M^3)} + C.$$

Once this bound is established, the polynomial growth follows by using (7) and arguing exactly as in Remark 1.2.

Remark 1.8. Following our approach to proving (7), there is no need to restrict oneself to pure power nonlinearities. In particular, polynomial growth for solutions to NLS on generic 3-dimensional compact manifolds could be established for general higher-order Sobolev norms (namely H^m with $m \ge 2$), provided the subcubic nonlinearity is suitably regularized in order to guarantee that H^m regularity is preserved along the flow. Nevertheless, for the sake of simplicity we elected not to deal with the full generality in this work.

2. Linear Strichartz estimates

Strichartz estimates on M^2. In the sequel we shall make use without any further comment of the following Strichartz estimate, which was already recalled in the Introduction:

$$\|e^{it\Delta_g}\varphi\|_{L^4((0,1)\times M^2)} \lesssim \|\varphi\|_{H^{s_0}(M^2)}. \tag{8}$$

By using Duhamel formula we also have at our disposal an inhomogeneous estimate that we state as an independent proposition.

Proposition 2.1. *Let $v(t,x)$ be solution to*

$$\begin{cases} i\,\partial_t v + \Delta_g v = F, & (t,x) \in \mathbb{R} \times M^2, \\ u(0,x) = \varphi \in H^{s_0}(M^2). \end{cases}$$

Then we have, for $T \in (0,1)$,

$$\|v\|_{L^4((0,T)\times M^2)} \lesssim \|\varphi\|_{H^{s_0}(M^2)} + T\|F\|_{L^\infty((0,T);H^{s_0}(M^2))}. \tag{9}$$

Strichartz estimates on M^3. In the proofs of Theorems 1.3 and 1.6 we shall make use of the following suitable version of the endpoint Strichartz estimate:

Proposition 2.2. *Let $v(t,x)$ be solution to*

$$i\,\partial_t v + \Delta_g v = F, \quad (t,x) \in \mathbb{R} \times M^3.$$

Then we have, for $\tau \in (0,1)$,

$$\|v\|_{L^2((0,\tau);L^6(M^3))} \lesssim_\epsilon \|v\|_{L^\infty((0,\tau);H^\epsilon(M^3))} + \|v\|_{L^2((0,\tau);H^{1/2}(M^3))} + \|F\|_{L^2((0,\tau);L^{6/5}(M^3))}. \tag{10}$$

Notice that the above estimate may look somewhat unusual compared with the classical version of Strichartz estimates, where on the right-hand side one expects a norm involving the initial datum $v(0,x)$ and another norm involving the forcing term $F(t,x)$.

Nevertheless we underline that in the case $F = 0$, the estimate above reduces to the usual Strichartz estimate with loss of half of a derivative (see [Burq et al. 2004; Staffilani and Tataru 2002]). On the other hand, the main point of (10) is that no derivative losses occur on the forcing term $F(t,x)$ when this term is not identically zero, and the loss of derivative indeed occurs only for the solution $v(t,x)$ on the right-hand side. Estimates in this spirit are also of crucial importance in the low regularity well-posedness theory for quasilinear dispersive PDEs (see, e.g., [Koch and Tzvetkov 2003]). We emphasize that the estimate (10) comes from the following spectrally localized version (see [Burq et al. 2004; Staffilani and Tataru 2002] and for more details Proposition 5.4 in [Bouclet and Tzvetkov 2007]):

$$\|\pi_N v\|_{L^2((0,1);L^6(M^3))} \lesssim \|\pi_N v\|_{L^\infty((0,1);L^2(M^3))} + \|\pi_N v\|_{L^2((0,1);H^{1/2}(M^3))} + \|\pi_N F\|_{L^2((0,1);L^{6/5}(M^3))},$$

where (π_N) is the usual Littlewood–Paley spectral projector and N ranges over dyadic numbers. In fact by taking squares and summing over N we get (10), provided that we make use of the bound

$$\sum_N \|\pi_N v\|^2_{L^\infty((0,1);L^2(M^3))} \le \sum_N \frac{1}{N^\epsilon} \|v\|^2_{L^\infty((0,1);H^\epsilon(M^3))}$$

together with the equivalence of the L^r norm of v with the L^r $(1 < r < \infty)$ norm of its squared function $\big(\sum_N |\pi_N v|^2\big)^{\frac{1}{2}}$.

3. Modified energies associated with even Sobolev norms

Modified energies. In this subsection we consider the general Cauchy problem

$$\begin{cases} i\,\partial_t u + \Delta_g u = |u|^{p-1}u, & (t,x)\in\mathbb{R}\times M^d,\\ u(0,x)=\varphi\in H^{2k}(M^d), \end{cases} \tag{11}$$

where (M^d, g) is a compact d-dimensional Riemannian manifold.

In the sequel we shall extensively make use of the following bound without further notice:

$$\|u\|_{L^\infty(\mathbb{R};H^1(M^d))} \lesssim_{p,\|\varphi\|_{H^1}} 1. \tag{12}$$

For every solution $u(t,x)$ to the Cauchy problem (11) we introduce the following energy, to be used in connection with growth of the Sobolev norm H^{2k}:

$$\mathcal{E}_{2k}(u) = \|\partial_t^k u\|^2_{L^2(M^d)} - \frac{p-1}{4}\int_{M^d} \big|\partial_t^{k-1}\nabla_g(|u|^2)\big|_g^2\,|u|^{p-3}\,\mathrm{dvol}_g - \int_{M^d}\big|\partial_t^{k-1}(|u|^{p-1}u)\big|^2\,\mathrm{dvol}_g.$$

We have the following key identity.

Proposition 3.1. *Let $u(t,x)$ be a solution to* (11), *where $p=2n+1\geq 3$, with initial data $\varphi\in H^{2k}(M^d)$. Then we have*

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{2k}(u(t,x)) = &-\frac{p-1}{4}\int_{M^d}\big|\partial_t^{k-1}\nabla_g(|u|^2)\big|_g^2\,\partial_t(|u|^{p-3})\,\mathrm{dvol}_g + 2\int_{M^d}\partial_t^k(|u|^{p-1})\,\partial_t^{k-1}(|\nabla_g u|_g^2)\,\mathrm{dvol}_g\\ &+\sum_{j=0}^{k-1} c_j\int_{M^d}\partial_t^j\nabla_g(|u|^2)\,\partial_t^{k-1}\nabla_g(|u|^2)_g\,\partial_t^{k-j}(|u|^{p-3})\,\mathrm{dvol}_g\\ &+\mathrm{Re}\sum_{j=0}^{k-1} c_j\int_{M^d}\partial_t^j(|u|^{p-1})\,\partial_t^{k-j}u\,\partial_t^{k-1}(|u|^{p-1}\bar u)\,\mathrm{dvol}_g\\ &+\mathrm{Re}\sum_{j=0}^{k-2} c_j\int_{M^d}\partial_t^k(|u|^{p-1})\,\partial_t^j(\Delta_g\bar u)\,\partial_t^{k-1-j}u\,\mathrm{dvol}_g\\ &+\mathrm{Im}\sum_{j=1}^{k-1} c_j\int_{M^d}\partial_t^j(|u|^{p-1})\,\partial_t^{k-j}u\,\partial_t^k\bar u\,\mathrm{dvol}_g,\end{aligned} \tag{13}$$

where c_j denote explicit constants that may change from line to line.

Proof. We start with the following computation:

$$\begin{aligned}\frac{d}{dt}\|\partial_t^k u\|^2_{L^2(M^d)} &= 2\,\mathrm{Re}(\partial_t^{k+1}u,\partial_t^k u) = 2\,\mathrm{Re}\big(\partial_t^k(-\Delta_g u + |u|^{p-1}u), i\,\partial_t^k u\big)\\ &= 2\,\mathrm{Im}\int_{M^d}(\partial_t^k\nabla_g u,\partial_t^k\nabla_g u)_g\,\mathrm{dvol}_g + 2\,\mathrm{Re}\big(\partial_t^k(|u|^{p-1}u), i\,\partial_t^k u\big),\end{aligned}$$

where (f, g) denotes the usual $L^2(M^d)$ scalar product $\int_{M^d} f \cdot \bar{g}\, \mathrm{dvol}_g$. Since the first term on the right-hand side vanishes identically we get

$$\begin{aligned}\frac{d}{dt}\|\partial_t^k u\|^2_{L^2(M^d)} &= 2\,\mathrm{Re}\big(\partial_t^k(|u|^{p-1}u), i\,\partial_t^k u\big)\\ &= 2\,\mathrm{Re}\big(\partial_t^k(|u|^{p-1})u, i\,\partial_t^k u\big) + 2\,\mathrm{Re}\big(|u|^{p-1}\,\partial_t^k u, i\,\partial_t^k u\big) + \mathrm{Re}\sum_{j=1}^{k-1} c_j\big(\partial_t^j(|u|^{p-1})\partial_t^{k-j}u, i\,\partial_t^k u\big),\end{aligned}$$

where c_j are suitable integers. Notice that the second term on the right-hand side vanishes identically and if we substitute for the equation again then we get

$$\begin{aligned}&\frac{d}{dt}\|\partial_t^k u\|^2_{L^2(M^d)}\\ &\quad= 2\,\mathrm{Re}\big(\partial_t^k(|u|^{p-1})u, -\Delta_g(\partial_t^{k-1}u)\big) + 2\,\mathrm{Re}\big(\partial_t^k(|u|^{p-1})u, \partial_t^{k-1}(|u|^{p-1}u)\big)\\ &\qquad+ \mathrm{Re}\sum_{j=1}^{k-1} c_j\big(\partial_t^j |u|^{p-1}\,\partial_t^{k-j}u, i\,\partial_t^k u\big)\\ &\quad= 2\,\mathrm{Re}\big(\partial_t^k(|u|^{p-1})u, -\Delta_g(\partial_t^{k-1}u)\big) + 2\,\mathrm{Re}\big(\partial_t^k(|u|^{p-1}u), \partial_t^{k-1}(|u|^{p-1}u)\big)\\ &\qquad+ \mathrm{Re}\sum_{j=0}^{k-1} c_j\big(\partial_t^j(|u|^{p-1})\,\partial_t^{k-j}u, \partial_t^{k-1}(|u|^{p-1}u)\big) + \mathrm{Re}\sum_{j=1}^{k-1} c_j\big(\partial_t^j |u|^{p-1}\,\partial_t^{k-j}u, i\,\partial_t^k u\big)\\ &\quad= 2\,\mathrm{Re}\big(\partial_t^k(|u|^{p-1})u, -\Delta_g(\partial_t^{k-1}u)\big) + \int_{M^d}\partial_t|\partial_t^{k-1}(|u|^{p-1}u)|^2\,\mathrm{dvol}_g\\ &\qquad+ \mathrm{Re}\sum_{j=0}^{k-1} c_j\big(\partial_t^j(|u|^{p-1})\,\partial_t^{k-j}u, \partial_t^{k-1}(|u|^{p-1}u)\big) + \mathrm{Re}\sum_{j=1}^{k-1} c_j\big(\partial_t^j(|u|^{p-1})\,\partial_t^{k-j}u, i\,\partial_t^k u\big). \quad (14)\end{aligned}$$

Next we focus on the first term on the right-hand side

$$2\,\mathrm{Re}\big(\partial_t^k(|u|^{p-1})u, -\Delta_g(\partial_t^{k-1}u)\big) = \int_{M^d}\partial_t^k(|u|^{p-1})\big(-\bar{u}\,\partial_t^{k-1}(\Delta_g u) - u\,\partial_t^{k-1}(\Delta_g\bar{u})\big)\,\mathrm{dvol}_g$$

and we notice

$$-\bar{u}\Delta_g(\partial_t^{k-1}u) - u\Delta_g(\partial_t^{k-1}\bar{u}) = \partial_t^{k-1}(-\bar{u}\Delta_g u - u\Delta_g\bar{u}) + \mathrm{Re}\sum_{j=0}^{k-2} c_j\,\partial_t^j(\Delta_g u)\,\partial_t^{k-1-j}\bar{u}.$$

Moreover we have the identity

$$\Delta_g(|u|^2) = u\Delta_g\bar{u} + \bar{u}\Delta_g u + 2|\nabla_g u|_g^2.$$

Hence,

$$\begin{aligned}&2\,\mathrm{Re}\big(\partial_t^k(|u|^{p-1})u, -\Delta_g\,\partial_t^{k-1}u\big)\\ &\quad= -\int_{M^d}\partial_t^k(|u|^{p-1})\,\partial_t^{k-1}\Delta_g(|u|^2)\,\mathrm{dvol}_g + 2\int_{M^d}\partial_t^k(|u|^{p-1})\,\partial_t^{k-1}(|\nabla_g u|_g^2)\,\mathrm{dvol}_g\\ &\qquad\qquad+ \mathrm{Re}\sum_{j=0}^{k-2} c_j\int_{M^d}\partial_t^k(|u|^{p-1})\,\partial_t^j(\Delta_g u)\,\partial_t^{k-1-j}\bar{u}\,\mathrm{dvol}_g\end{aligned}$$

$$= \int_{M^d} \big(\partial_t^k \nabla_g(|u|^{p-1}), \partial_t^{k-1} \nabla_g(|u|^2)\big)_g \,\mathrm{dvol}_g + 2\int_{M^d} \partial_t^k(|u|^{p-1})\, \partial_t^{k-1}(|\nabla_g u|_g^2)\,\mathrm{dvol}_g + \mathrm{Re}\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|u|^{p-1})\, \partial_t^j(\Delta_g u)\, \partial_t^{k-1-j}\bar{u}\,\mathrm{dvol}_g,$$

and by elementary computations we get

$$\cdots = \frac{p-1}{2}\int_{M^d} \big(\partial_t^k(\nabla_g(|u|^2)|u|^{p-3}), \partial_t^{k-1} \nabla_g(|u|^2)\big)_g \,\mathrm{dvol}_g + 2\int_{M^d} \partial_t^k(|u|^{p-1})\, \partial_t^{k-1}(|\nabla_g u|_g^2)\,\mathrm{dvol}_g + \mathrm{Re}\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|u|^{p-1})\, \partial_t^j(\Delta_g u)\, \partial_t^{k-1-j}\bar{u}\,\mathrm{dvol}_g.$$

Using the Leibniz rule to develop ∂_t^k we get

$$\begin{aligned}
\cdots &= \frac{p-1}{2}\int_{M^d} (\partial_t^k \nabla_g(|u|^2)|u|^{p-3}, \partial_t^{k-1} \nabla_g(|u|^2))_g \,\mathrm{dvol}_g \\
&\quad + \sum_{j=0}^{k-1} c_j \int_{M^d} (\partial_t^j \nabla_g(|u|^2), \partial_t^{k-1}\nabla_g(|u|^2))_g\, \partial_t^{k-j}(|u|^{p-3})\,\mathrm{dvol}_g \\
&\quad + 2\int_{M^d} \partial_t^k(|u|^{p-1})\, \partial_t^{k-1}(|\nabla_g u|_g^2)\,\mathrm{dvol}_g + \mathrm{Re}\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|u|^{p-1})\, \partial_t^j(\Delta_g u)\, \partial_t^{k-1-j}\bar{u}\,\mathrm{dvol}_g \\
&= \frac{p-1}{4}\int_{M^d} \partial_t |\partial_t^{k-1} \nabla_g(|u|^2)|_g^2 |u|^{p-3}\,\mathrm{dvol}_g \\
&\quad + \sum_{j=0}^{k-1} c_j \int_{M^d} (\partial_t^j \nabla_g(|u|^2), \partial_t^{k-1}\nabla_g(|u|^2))_g\, \partial_t^{k-j}(|u|^{p-3})\,\mathrm{dvol}_g \\
&\quad + 2\int_{M^d} \partial_t^k(|u|^{p-1})\, \partial_t^{k-1}(|\nabla_g u|_g^2)\,\mathrm{dvol}_g + \mathrm{Re}\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|u|^{p-1})\, \partial_t^j(\Delta_g u)\, \partial_t^{k-1-j}\bar{u}\,\mathrm{dvol}_g,
\end{aligned}$$

and we conclude by combining this identity with (14). □

Remark 3.2. In the specific case of the cubic NLS (i.e., (11) with $p = 3$) we have some simplifications; more precisely we get

$$\mathcal{E}_{2k}(u) = \|\partial_t^k u\|_{L^2(M^d)}^2 - \frac{1}{2}\int_{M^d} \big|\partial_t^{k-1}\nabla_g(|u|^2)\big|_g^2\,\mathrm{dvol}_g - \int_{M^d} \big|\partial_t^{k-1}(|u|^2 u)\big|^2\,\mathrm{dvol}_g$$

and also

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_{2k}(u(t,x)) \\
&= 2\int_{M^d} \partial_t^k(|u|^2)\, \partial_t^{k-1}(|\nabla_g u|_g^2)\,\mathrm{dvol}_g + \mathrm{Re}\sum_{j=0}^{k-2} c_j \int_{M^d} \partial_t^k(|u|^2)\, \partial_t^j(\Delta_g u)\, \partial_t^{k-1-j}\bar{u}\,\mathrm{dvol}_g \\
&\quad + \mathrm{Re}\sum_{j=0}^{k-1} c_j \int_{M^d} \partial_t^j(|u|^2)\, \partial_t^{k-j}u\, \partial_t^{k-1}(|u|^2\bar{u})\,\mathrm{dvol}_g + \mathrm{Im}\sum_{j=1}^{k-1} c_j \int_{M^d} \partial_t^j(|u|^2)\, \partial_t^{k-j}u\, \partial_t^k\bar{u}\,\mathrm{dvol}_g. \quad (15)
\end{aligned}$$

The norms $\|\partial_t^k u\|_{L^2}$ and $\|u\|_{H^{2k}}$ are comparable. The aim of this subsection is indeed to prove that the leading term in our modified energy $\mathcal{E}_{2k}(u)$ is equivalent to the Sobolev norm $\|u\|_{H^{2k}}$, provided that $u(t,x)$ is a solution to (11) with $d=2$ and $p\geq 3$ or $d=3$ and $p=3$.

Proposition 3.3. *Let $u(t,x)$ be solution to* (11), *where either $d=2$ and $p\geq 3$ is an integer*, *or $d=3$ and $p=3$. Then for every $k,s\in\mathbb{N}$ we have*

$$\|\partial_t^k u - i^k\Delta_g^k u\|_{H^s(M^d)} \lesssim_{\|\varphi\|_{H^1}} \|u\|_{H^{s+2k-1}(M^d)}. \tag{16}$$

Proof. We shall use the following identity (satisfied by every solution to (11) in any dimension d):

$$\partial_t^h u = i^h\Delta_g^h u + \sum_{j=0}^{h-1} c_j\,\partial_t^j\,\Delta_g^{h-j-1}(u|u|^{p-1}), \tag{17}$$

where $c_j\in\mathbb{C}$ are suitable coefficients. The elementary proof follows by induction on h and by using the equation solved by $u(t,x)$.

First case: $d=2$, $p\geq 3$. We argue by induction on k, and hence we shall prove $k\Rightarrow k+1$. By (17) we aim at proving

$$\|\partial_t^j(u|u|^{p-1})\|_{H^{2k-2j+s}(M^2)} \lesssim \|u\|_{H^{s+2k+1}(M^2)}, \quad j=0,\dots,k, \tag{18}$$

by assuming the property (16) is true for k. By expanding the time and space derivatives on the left-hand side above, we deduce (18) by the chain of inequalities

$$\begin{aligned}\prod_{\substack{j_1+\cdots+j_p=j\\ s_1+\cdots+s_p=2k-2j+s}} \|\partial_t^{j_l}u\|_{W^{s_l,2p}(M^2)} &\lesssim \prod_{\substack{j_1+\cdots+j_p=j\\ s_1+\cdots+s_p=2k-2j+s}} \|\partial_t^{j_l}u\|_{H^{s_l+1}(M^2)}\\ &\lesssim \prod_{\substack{j_1+\cdots+j_p=j\\ s_1+\cdots+s_p=2k-2j+s}} \|u\|_{H^{2j_l+s_l+1}(M^2)},\end{aligned}$$

where we used the Sobolev embedding $H^1(M^2)\subset L^{2p}(M^2)$ and we have used the induction hypothesis at the last step. We can continue the estimate by a trivial interpolation argument as follows:

$$\cdots\lesssim\Bigg(\prod_{l=1,\dots,p}\|u\|^{\theta_l}_{H^{s+2k+1}(M^2)}\,\|u\|^{(1-\theta_l)}_{H^1(M^2)}\Bigg),$$

where

$$\theta_l(s+2k+1)+(1-\theta_l)=2j_l+s_l+1.$$

We conclude using (12), since $\sum_{l=1}^p\theta_l=1$ for $j=0,\dots,k$.

Second case: $d=3$, $p=3$. Arguing as above, and by assuming the result true for k, we are reduced to proving

$$\|\partial_t^j(u|u|^2)\|_{H^{2k-2j+s}(M^3)} \lesssim \|u\|_{H^{s+2k+1}(M^3)}, \quad j=0,\dots,k. \tag{19}$$

Expanding again the time and space derivatives on the left-hand side, we are reduced to the estimate

$$\begin{aligned}\|\partial_t^{j_1} u\|_{W^{k_1,6}(M^3)} \times \|\partial_t^{j_2} u\|_{W^{k_2,6}(M^3)} \times \|\partial_t^{j_3} u\|_{W^{k_3,6}(M^3)} \\ \lesssim \|\partial_t^{j_1} u\|_{H^{k_1+1}(M^3)} \times \|\partial_t^{j_2} u\|_{H^{k_2+1}(M^3)} \times \|\partial_t^{j_3} u\|_{H^{k_3+1}(M^3)} \\ \lesssim \|u\|_{H^{2j_1+k_1+1}(M^3)} \|u\|_{H^{2j_2+k_2+1}(M^3)} \|u\|_{H^{2j_3+k_3+1}(M^3)},\end{aligned}$$

where

$$j_1 + j_2 + j_3 = j, \quad k_1 + k_2 + k_3 = 2k - 2j + s.$$

Notice that we have used the Sobolev embedding $H^1(M^3) \subset L^6(M^3)$ and the induction hypothesis at the last step. By interpolation we have

$$\|u\|_{H^{2j_l+k_l+1}(M^3)} \lesssim \|u\|^{\theta_l}_{H^{s+2k+1}(M^3)} \|u\|^{1-\theta_l}_{H^1(M^3)}, \quad l = 1, 2, 3,$$

where

$$\theta_l(s + 2k + 1) + (1 - \theta_l) = 2j_l + k_l + 1,$$

and we conclude as above since $\sum_{l=1}^3 \theta_l = 1$ for $j = 0, \dots, k$. □

Strichartz estimates for nonlinear solutions. In this subsection we get a priori bounds for the Strichartz norms of solutions to (11) in dimension $d = 2$, with a general nonlinearity, and in dimension $d = 3$, with cubic nonlinearity. In the sequel we denote by $L^p_\tau X$ the space $L^p((0, \tau); X)$, where X is a Banach space and $p \in [1, \infty]$.

Proposition 3.4. *We have the following estimate for every solution $u(t, x)$ to* (11) *for $d = 2$ and $p = 2n + 1 \geq 3$ is an integer*: *for any $\epsilon > 0$ and $\tau \in (0, 1)$,*

$$\|\partial_t^j u\|_{L^4_\tau W^{s,4}(M^2)} \lesssim_{\epsilon, \|\varphi\|_{H^1}} \|u\|^{1-s_0}_{L^\infty_\tau H^{2j+s}(M^2)} \|u\|^{s_0}_{L^\infty_\tau H^{2j+s+1}(M^2)} \|u\|^{\epsilon}_{L^\infty_\tau H^{2j+2}(M^2)}. \tag{20}$$

Proof. We use (9), together with the equation solved by $\partial_t^j u$, and we get

$$\begin{aligned}&\|\partial_t^j u\|_{L^4_\tau W^{s,4}(M^2)} \\ &\quad\lesssim \|\partial_t^j u(0)\|_{H^{s+s_0}(M^2)} + \tau \|\partial_t^j (u|u|^{p-1})\|_{L^\infty_\tau H^{s+s_0}(M^2)} \\ &\quad\lesssim \|\partial_t^j u(0)\|^{1-s_0}_{H^s} \|\partial_t^j u(0)\|^{s_0}_{H^{s+1}(M^2)} + T \|\partial_t^j (u|u|^{p-1})\|^{1-s_0}_{L^\infty_\tau H^s(M^2)} \|\partial_t^j (u|u|^{p-1})\|^{s_0}_{L^\infty_\tau H^{s+1}(M^2)}.\end{aligned}$$

Notice that the first term on the right-hand side can be estimated by Proposition 3.3. Hence we shall complete the proof provided that for every $\epsilon > 0$,

$$\|\partial_t^j (u|u|^{p-1})\|_{H^s(M^2)} \lesssim_{\epsilon, \|\varphi\|_{H^1}} \|u\|_{H^{2j+s}(M^2)} \|u\|^{\epsilon}_{H^{2j+2}(M^2)} \quad \text{for all } j, s = 1, 2, \dots.$$

Expanding the time derivative ∂_t^j and using

$$\|fg\|_{H^r(M^2)} \lesssim \|f\|_{H^r(M^2)} \|g\|_{L^\infty(M^2)} + \|g\|_{H^r(M^2)} \|f\|_{L^\infty(M^2)},$$

we are reduced to estimating

$$\|\partial_t^{j_1} u\|_{H^s(M^2)} \times \|\partial_t^{j_2} u\|_{L^\infty(M^2)} \times \cdots \times \|\partial_t^{j_p} u\|_{L^\infty(M^2)},$$

where $j_1 + \cdots + j_p = j$. Notice that from

$$\|v\|_{L^\infty(M^2)} \lesssim_\epsilon \|v\|^{1-\epsilon}_{H^1(M^2)} \|v\|^{\epsilon}_{H^2(M^2)} \tag{21}$$

we get

$$\begin{aligned}
&\|\partial_t^{j_1} u\|_{H^s(M^2)} \times \|\partial_t^{j_2} u\|_{L^\infty(M^2)} \times \cdots \times \|\partial_t^{j_p} u\|_{L^\infty(M^2)} \\
&\quad \lesssim_\epsilon \|\partial_t^{j_1} u\|_{H^s(M^2)} \times \|\partial_t^{j_2} u\|^{1-\epsilon}_{H^1(M^2)} \times \|\partial_t^{j_2} u\|^{\epsilon}_{H^2} \times \cdots \times \|\partial_t^{j_p} u\|^{1-\epsilon}_{H^1(M^2)} \times \|\partial_t^{j_p} u\|^{\epsilon}_{H^2(M^2)},
\end{aligned}$$

and hence by (16)

$$\begin{aligned}
\cdots &\lesssim \|u\|_{H^{2j_1+s}(M^2)} \times \|u\|^{1-\epsilon}_{H^{2j_2+1}(M^2)} \times \|u\|^{\epsilon}_{H^{2j_2+2}(M^2)} \times \cdots \times \|u\|^{1-\epsilon}_{H^{2j_p+1}(M^2)} \times \|u\|^{\epsilon}_{H^{2j_p+2}(M^2)} \\
&\lesssim \|u\|^{\theta_1}_{H^{2j+s}(M^2)} \|u\|^{1-\theta_1}_{H^1(M^2)} \times \|u\|^{\theta_2(1-\epsilon)}_{H^{2j+s}} \|u\|^{(1-\theta_2)(1-\epsilon)}_{H^1(M^2)} \\
&\qquad \times \cdots \times \|u\|^{\theta_p(1-\epsilon)}_{H^{2j+s}(M^2)} \|u\|^{(1-\theta_p)(1-\epsilon)}_{H^1(M^2)} \times \|u\|^{\epsilon(p-1)}_{H^{2j+2}(M^2)},
\end{aligned}$$

where at the last step we have used an interpolation argument with

$$\theta_1(2j+s) + (1-\theta_1) = 2j_1 + s, \qquad \theta_l(2j+s) + (1-\theta_l) = 2j_l + 1, \quad l = 2, \dots, p.$$

Notice that we get $\sum_{l=1}^p \theta_l = 1$ and we conclude by (12). □

Proposition 3.5. *We have the following estimate for every solution $u(t,x)$ to* (11) *for $(p,l) = (3,3)$ and for every $\epsilon > 0$, $\tau \in (0,1)$:*

$$\begin{aligned}
\|\partial_t^j u\|_{L^2_\tau L^6(M^3)} &\lesssim_{\epsilon, \|\varphi\|_{H^1}} \|\partial_t^j u\|^{1-\epsilon}_{L^\infty_\tau L^2(M^3)} \|\partial_t^j u\|^{\epsilon}_{L^\infty_\tau H^1(M^3)} + \sqrt{\tau} \|u\|^{1/2}_{L^\infty_\tau H^{2j}(M^3)} \|u\|^{1/2}_{L^\infty_\tau H^{2j+1}(M^3)} \\
&\quad + \sqrt{\tau} \sum_{\substack{j_1+j_2+j_3=j \\ j_1=\max\{j_1,j_2,j_3\}}} \|u\|_{L^\infty_\tau H^{2j_1}(M^3)} \|u\|_{L^\infty_\tau H^{2j_2+1}(M^3)} \|u\|_{L^\infty_\tau H^{2j_3+1}(M^3)},
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
\|\partial_t^j u\|_{L^2_\tau W^{1,6}(M^3)} &\lesssim_{\epsilon, \|\varphi\|_{H^1}} \|\partial_t^j u\|^{1-\epsilon}_{L^\infty_\tau H^1(M^3)} \|\partial_t^j u\|^{\epsilon}_{L^\infty_\tau H^2(M^3)} + \sqrt{\tau} \|u\|^{1/2}_{L^\infty_\tau H^{2j+1}(M^3)} \|u\|^{1/2}_{L^\infty_\tau H^{2j+2}(M^3)} \\
&\quad + \sqrt{\tau} \sum_{j_1+j_2+j_3=j} \|u\|_{L^\infty_\tau H^{2j_1+1}(M^3)} \|u\|_{L^\infty_\tau H^{2j_2+1}(M^3)} \|u\|_{L^\infty_\tau H^{2j_3+1}(M^3)}.
\end{aligned} \tag{23}$$

Proof. We prove (23), the proof of (22) being similar. By using Strichartz estimates and the equation solved by $\partial_t^j u$ we get

$$\begin{aligned}
\|\partial_t^j u\|_{L^2_\tau W^{1,6}(M^3)} &\lesssim \|\partial_t^j u\|_{L^\infty_\tau H^{1+\epsilon}(M^3)} + \sqrt{T} \|\partial_t^j u\|_{L^\infty_\tau H^{3/2}(M^3)} + \|\partial_t^j (u|u|^2)\|_{L^2_\tau W^{1,6/5}(M^3)} \\
&\lesssim \|\partial_t^j u\|^{1-\epsilon}_{L^\infty_\tau H^1(M^3)} \|\partial_t^j u\|^{\epsilon}_{L^\infty_\tau H^2(M^3)} \\
&\qquad + \|\partial_t^j u\|^{1/2}_{L^\infty_\tau H^1(M^3)} \|\partial_t^j u\|^{1/2}_{L^\infty_\tau H^2(M^3)} + \|\partial_t^j (u|u|^2)\|_{L^2_\tau W^{1,6/5}(M^3)}.
\end{aligned}$$

Notice that by expanding the time derivative, and by using Hölder we get

$$\begin{aligned}\|\partial_t^j(u|u|^2)\|_{W^{1,6/5}(M^3)} &\lesssim \sum_{\substack{j_1+j_2+j_3=j\\ j_1=\max\{j_1,j_2,j_3\}}} \|\partial_t^{j_1}u\|_{H^1(M^3)}\,\|\partial_t^{j_2}u\|_{L^6(M^3)}\,\|\partial_t^{j_3}u\|_{L^6(M^3)}\\ &\lesssim \sum_{\substack{j_1+j_2+j_3=j\\ j_1=\max\{j_1,j_2,j_3\}}} \|\partial_t^{j_1}u\|_{H^1(M^3)}\,\|\partial_t^{j_2}u\|_{H^1(M^3)}\,\|\partial_t^{j_3}u\|_{H^1(M^3)}.\end{aligned}$$

We then conclude by using Proposition 3.3 in the special case of the cubic NLS on M^3. ☐

4. Polynomial growth of H^{2k} for pure power NLS on M^2

This section is devoted to the proof of Theorem 1.1 in the case $m = 2k$. We shall need the following estimate.

Proposition 4.1. *Let us assume that $u(t,x)$ solves* (11) *with $d = 2$ and $p = 2n+1 \geq 3$. Then we have the following bound for every $\tau \in (0,1)$:*

$$\int_0^\tau |\textit{right-hand side of }(13)|\,ds \lesssim \sqrt{\tau}\,\|u\|_{L^\infty_\tau H^{2k}(M^2)}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + \|u\|_{L^\infty_\tau H^{2k}(M^2)}^{\frac{4k-4}{2k-1}+\epsilon}.$$

Proof. Since we work on a 2-dimensional compact manifold we simplify notation as follows: L^q, $W^{s,q}$, H^s denote the spaces $L^q(M^2)$, $W^{s,q}(M^2)$, $H^s(M^2)$. Moreover in the sequel we shall denote by $\epsilon > 0$ any arbitrary small constant whose value can change from line to line. We shall also make use of the inequality

$$\|u\|_{L^\infty_T H^s} \lesssim_{\|\varphi\|_{H^1}} \|u\|_{L^\infty_\tau H^{2k}}^{\frac{s-1}{2k-1}}, \quad s \in [1, 2k], \tag{24}$$

which in turn follows by combining an elementary interpolation inequality with (12).

Let I, II, III, IV, V, VI be the successive terms on each line of the right-hand side in (13). Estimating I can be reduced to controlling the terms

$$\int_0^T \|\partial_t^{k_1}u\|_{W^{1,4}}^2\,\|\partial_t^{k_2}u\|_{L^\infty}^2\,\|\partial_t u\|_{L^2}\,\|u\|_{L^\infty}^{p-4}\,ds, \quad k_1+k_2 = k-1, \tag{25}$$

and we have, by combining (21), Proposition 3.3, Proposition 3.4 and the Hölder inequality,

$$\begin{aligned}(25) &\lesssim \sqrt{T}\,\|u\|_{L^\infty_\tau H^{2k_2+1}}^{2(1-\epsilon)}\,\|u\|_{L^\infty H^{2k_2+2}}^{2\epsilon}\,\|u\|_{L^\infty_\tau H^2}\,\|\partial_t^{k_1}u\|_{L^4_T W^{1,4}}^2\\ &\lesssim \sqrt{T}\,\|u\|_{L^\infty_\tau H^{2k_2+1}}^{2(1-\epsilon)}\,\|u\|_{L^\infty H^2}\,\|u\|_{L^\infty_\tau H^{2k_1+1}}^{2(1-s_0)}\,\|u\|_{L^\infty_\tau H^{2k_1+2}}^{2s_0}\,\|u\|_{L^\infty_\tau H^{2k}}^{\epsilon} \lesssim \sqrt{\tau}\,\|u\|_{L^\infty_\tau H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon},\end{aligned}$$

where at the last step we have used (24). Notice that the value of $\epsilon > 0$ changes at each line, but can be chosen arbitrarily small. Concerning II, we are reduced to controlling

$$\int_0^T \|\partial_t^{j_1}u\|_{L^2}\Big(\prod_{h=2,\dots,p-1}\|\partial_t^{j_h}u\|_{L^\infty}\Big)\|\partial_t^{k_1}u\|_{W^{1,4}}\,\|\partial_t^{k_2}u\|_{W^{1,4}}, \tag{26}$$

where we assume $j_1 = \max\{j_1, j_2, \dots, j_{p-1}\}$ and

$$j_1 + \cdots + j_{p-1} = k, \quad k_1 + k_2 = k - 1.$$

By using the interpolation estimate (21) together with Proposition 3.3, Proposition 3.4 and the Hölder inequality, we get

$$\begin{aligned}(26) \lesssim \sqrt{T}\, \|u\|^{\epsilon}_{L^\infty_T H^{2k}} \|u\|_{L^\infty_T H^{2j_1}} \Big(\prod_{h=2,\dots,p-1} \|u\|^{1-\epsilon}_{L^\infty_T H^{2j_h+1}}\Big) \\ \times \|u\|^{1-s_0}_{L^\infty_T H^{2k_1+1}} \|u\|^{s_0}_{L^\infty_T H^{2k_1+2}} \|u\|^{1-s_0}_{L^\infty_T H^{2k_2+1}} \|u\|^{s_0}_{L^\infty_T H^{2k_2+2}} \\ \lesssim \sqrt{\tau}\, \|u\|^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}},\end{aligned}$$

where we used (24) at the last step. Next we deal with III, and it is sufficient to control

$$\int_0^\tau \|\partial_t^{h_1} u\|_{L^\infty} \|\partial_t^{h_2} u\|_{W^{1,4}} \|\partial_t^{m_1} u\|_{L^2} \Big(\prod_{i=2,\dots,p-3} \|\partial_t^{m_i} u\|_{L^\infty}\Big) \|\partial_t^{l_1} u\|_{L^\infty} \|\partial_t^{l_2} u\|_{W^{1,4}}, \tag{27}$$

where we assume $m_1 = \max\{m_1, m_2, \dots, m_{p-3}\}$ and

$$h_1 + h_2 = j \in [0, k-1], \quad m_1 + \cdots + m_{p-3} = k - j, \quad l_1 + l_2 = k - 1.$$

Arguing as above, it can be estimated by

$$\begin{aligned}(27) \lesssim \sqrt{\tau} \|u\|^{\epsilon}_{L^\infty_\tau H^{2k}} \|u\|_{L^\infty_\tau H^{2h_1+1}} \|u\|^{1-s_0}_{L^\infty_\tau H^{2h_2+1}} \|u\|^{s_0}_{L^\infty_\tau H^{2h_2+2}} \|u\|_{L^\infty_\tau H^{2m_1}} \\ \times \Big(\prod_{i=2,\dots,p-3} \|u\|^{1-\epsilon}_{L^\infty H^{2m_i+1}}\Big) \|u\|^{1-\epsilon}_{L^\infty_\tau H^{2l_1+1}} \|u\|^{1-s_0}_{L^\infty_\tau H^{2l_2+1}} \|u\|^{s_0}_{L^\infty_\tau H^{2l_2+2}} \\ \lesssim \sqrt{\tau} \|u\|^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}}.\end{aligned}$$

In order to treat IV we are reduced to controlling

$$\int_0^T \|\partial_t^{j_1} u\|_{L^2} \Big(\prod_{h=2,\dots,p-1} \|\partial_t^{j_h} u\|_{L^\infty}\Big) \|\partial_t^{j} (\Delta_g u)\|_{L^4} \|\partial_t^{k-1-j} \bar{u}\|_{L^4}, \tag{28}$$

where we assume $j_1 = \max\{j_1, j_2, \dots, j_{p-1}\}$ and

$$j_1 + \cdots + j_{p-1} = k,$$

and by an argument similar to those above we have

$$\begin{aligned}(28) \lesssim \sqrt{\tau} \|u\|^{\epsilon}_{L^\infty_\tau H^{2k}} \|u\|_{L^\infty_T H^{2j_1}} \Big(\prod_{h=2,\dots,p-1} \|u\|^{1-\epsilon}_{L^\infty_\tau H^{2j_h+1}}\Big) \\ \times \|u\|^{1-s_0}_{L^\infty_\tau H^{2j+2}} \|u\|^{s_0}_{L^\infty_\tau H^{2j+3}} \|u\|^{1-s_0}_{L^\infty_\tau H^{2k-2-2j}} \|u\|^{s_0}_{L^\infty_\tau H^{2k-2j-1}} \\ \lesssim \sqrt{\tau} \|u\|^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}}.\end{aligned}$$

In order to estimate V it is sufficient to control the terms

$$\int_0^T \|\partial_t^{m_1} u\|_{L^4}\Big(\prod_{i=2,\dots,p-1} \|\partial_t^{m_i} u\|_{L^\infty}\Big)\|\partial_t^{k-j} u\|_{L^4}\,\|\partial_t^k u\|_{L^2}, \tag{29}$$

where we assume $m_1 = \max\{m_1, m_2, \dots, m_{p-1}\}$ and

$$m_1 + \cdots + m_{p-1} = j,$$

and as usual we get

$$\begin{aligned}(29) &\lesssim \sqrt{\tau}\|u\|^{1+\epsilon}_{L^\infty_\tau H^{2k}}\,\|u\|^{1-s_0}_{L^\infty_\tau H^{2m_1}}\,\|u\|^{s_0}_{L^\infty_\tau H^{2m_1+1}}\Big(\prod_{i=2,\dots,p-1}\|u\|^{1-\epsilon}_{L^\infty_\tau H^{2m_i+1}}\Big)\\ &\qquad\qquad \times \|u\|^{1-s_0}_{L^\infty_\tau H^{2k-2j}}\,\|u\|^{s_0}_{L^\infty_\tau H^{2k-2j+1}}\\ &\lesssim \sqrt{\tau}\|u\|^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}}.\end{aligned}$$

We conclude with the estimate of VI, which in turn can be reduced to controlling

$$\int_0^T \|\partial_t^{m_1} u\|_{L^2}\Big(\prod_{i=2,\dots,p-1} \|\partial_t^{m_i} u\|_{L^\infty}\Big)\|\partial_t^{k-j} u\|_{L^\infty}\;\|\partial_t^{l_1} u\|_{L^2}\Big(\prod_{i=2,\dots,p} \|\partial_t^{l_i} u\|_{L^\infty}\Big), \tag{30}$$

where we assume $m_1 = \max\{m_1, m_2, \dots, m_{p-1}\}$ and

$$m_1 + \cdots + m_{p-1} = j, \quad l_1 + \cdots + l_p = k-1,$$

and we get

$$\begin{aligned}(30) &\lesssim \tau\|u\|^{\epsilon}_{L^\infty_\tau H^{2k}}\,\|u\|_{L^\infty_\tau H^{2m_1}}\Big(\prod_{i=2,\dots,p-1}\|u\|^{1-\epsilon}_{L^\infty_\tau H^{2m_i+1}}\Big)\\ &\qquad \times \|u\|^{1-\epsilon}_{L^\infty_\tau H^{2k-2j+1}}\,\|u\|_{L^\infty_\tau H^{2l_1}}\Big(\prod_{i=2,\dots,p-1}\|u\|^{1-\epsilon}_{L^\infty_\tau H^{2l_i+1}}\Big)\\ &\lesssim \tau\|u\|^{\frac{4k-4}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}}.\end{aligned}$$

□

The key estimate to deduce Theorem 1.1 is the following one (see Remark 1.2).

Proposition 4.2. *Let us assume that $u(t,x)$ solves (11) with $d = 2$ and $p \ge 3$. Then we have the following bound for every $\tau \in (0,1)$ and for every $\epsilon > 0$:*

$$\|u(\tau)\|^2_{H^{2k}} - \|u(0)\|^2_{H^{2k}} \lesssim \sqrt{\tau}\,\|u\|^{\frac{4k-3+2s_0}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}} + \|u\|^{\frac{4k-4}{2k-1}+\epsilon}_{L^\infty_\tau H^{2k}}.$$

Proof. We write $\mathcal{E}_{2k}(u) = \|\partial_t^k u\|^2_{L^2} + \mathcal{R}_{2k}(u)$, where

$$\mathcal{R}_{2k}(u) = -\frac{p-1}{4}\int \big|\partial_t^{k-1}\nabla_g(|u|^2)\big|^2_g\,|u|^{p-3}\,\mathrm{dvol}_g - \int \big|\partial_t^{k-1}(|u|^{p-1}u)\big|^2\,\mathrm{dvol}_g.$$

We claim that

$$|\mathcal{R}_{2k}(u)| \lesssim_\epsilon \|u\|^{\frac{4k-4}{2k-1}+\epsilon}_{H^{2k}} + \|u\|^{\frac{4k-6}{2k-1}+\epsilon}_{H^{2k}}. \tag{31}$$

In fact notice that arguing as in the proof of Proposition 4.1 we get

$$\int \big|\partial_t^{k-1}\nabla_g(|u|^2)\big|_g^2\,|u|^{p-3}\,\mathrm{dvol}_g \lesssim \sum_{k_1+k_2=k-1} \|\partial_t^{k_1}u\|_{W^{1,2}}^2\,\|\partial_t^{k_2}u\|_{L^\infty}^2\,\|u\|_{L^\infty}^{p-3}$$
$$\lesssim \sum_{k_1+k_2=k-1}\|u\|_{H^{2k_1+1}}^2\,\|u\|_{H^{2k_2+1}}^2\,\|u\|_{H^{2k}}^{\epsilon} \lesssim \|u\|_{H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon}$$

and also

$$\int \big|\partial_t^{k-1}(|u|^{p-1}u)\big|^2\,\mathrm{dvol}_g \lesssim \sum_{j_1+\dots+j_p=k-1}\|\partial_t^{j_1}u\|_{L^2}^2\Big(\prod_{h=1,\dots,p}\|\partial_t^{j_h}u\|_{L^\infty}^2\Big)$$
$$\lesssim \sum_{j_1+\dots+j_p=k-1}\|u\|_{H^{2j_1}}^2\Big(\prod_{h=1,\dots,p}\|u\|_{H^{2j_h+1}(M^2)}^2\Big)\|u\|_{L^\infty H^{2k}}^{\epsilon} \lesssim \|u\|_{L^\infty H^{2k}}^{\frac{4k-6}{2k-1}+\epsilon}.$$

Next notice that if we integrate the identity (13) and we use Proposition 4.1 then

$$\|\partial_t^k u(\tau)\|_{L^2}^2 - \|\partial_t^k u(0)\|_{L^2}^2 \lesssim \sup_{(0,\tau)}|\mathcal{R}_{2k}(u)| + \sqrt{\tau}\|u\|_{L_\tau^\infty H^{2k}}^{\frac{4k-3+2s_0}{2k-1}+\epsilon} + \|u\|_{L_\tau^\infty H^{2k}}^{\frac{4k-4}{2k-1}+\epsilon}.$$

We conclude by using (31) and Proposition 3.3. □

5. Exponential growth for H^{2k} norms of solutions to the cubic NLS on M^3

The aim of this section is the proof of Theorem 1.3 in the case $m = 2k$.

The following is the 3-dimensional version of Proposition 4.1 for the cubic NLS.

Proposition 5.1. *Let us assume that $u(t,x)$ solves* (11) *with $d = 3$ and $p = 3$. Then we have the following bound for every $\tau \in (0,1)$*

$$\int_0^\tau |\textit{right-hand side of } (13)|\,ds \lesssim \tau\|u\|_{L_\tau^\infty H^{2k}}^2 + \|u\|_{L_\tau^\infty H^{2k}}^{\gamma}$$

for some $\gamma \in (0,2)$.

Proof. Since we work on a 3-dimensional compact manifold we simplify the notation as follows: L^q, $W^{s,q}$, H^s denote the spaces $L^q(M^3)$, $W^{s,q}(M^3)$, $H^s(M^3)$. In the sequel we shall also make use of the following inequalities, which in turn follow by combining an elementary interpolation inequality with (12). We also notice that by combining Proposition 3.3 and Proposition 3.5 with (24) we get

$$\|\partial_t^j u\|_{L_\tau^2 L^6} \lesssim_\epsilon \|u\|_{L_\tau^\infty H^{2j}}^{1-\epsilon}\,\|u\|_{L_\tau^\infty H^{2j+1}}^{\epsilon} + \sqrt{\tau}\|u\|_{L_\tau^\infty H^{2j}}^{1/2}\,\|u\|_{L_\tau^\infty H^{2j+1}}^{1/2}$$
$$+\sqrt{\tau}\sum_{\substack{j_1+j_2+j_3=j\\ j_1=\max\{j_1,j_2,j_3\}}}\|u\|_{L_\tau^\infty H^{2j_1}}\,\|u\|_{L_\tau^\infty H^{2j_2+1}}\,\|u\|_{L_\tau^\infty H^{2j_3+1}}$$
$$\lesssim \|u\|_{L_\tau^\infty H^{2k}}^{\frac{2j-1+\epsilon}{2k-1}} + \sqrt{\tau}\|u\|_{L_\tau^\infty H^{2k}}^{\frac{4j-1}{4k-2}} + \sqrt{\tau}\|u\|_{L_\tau^\infty H^{2k}}^{\frac{2j-1}{2k-1}}, \tag{32}$$

provided that $j \geq 1$, and

$$\begin{aligned}\|\partial_t^j u\|_{L^2_\tau W^{1,6}} &\lesssim_\epsilon \|u\|^{1-\epsilon}_{L^\infty_\tau H^{2j+1}} \|u\|^{\epsilon}_{L^\infty_\tau H^{2j+2}} + \sqrt{\tau}\|u\|^{1/2}_{L^\infty_\tau H^{2j+1}} \|u\|^{1/2}_{L^\infty_\tau H^{2j+2}} \\ &\quad + \sqrt{\tau} \sum_{j_1+j_2+j_3=j} \|u\|_{L^\infty_\tau H^{2j_1+1}} \|u\|_{L^\infty_\tau H^{2j_2+1}} \|u\|_{L^\infty_\tau H^{2j_3+1}} \\ &\lesssim \|u\|^{\frac{2j+\epsilon}{2k-1}}_{L^\infty_\tau H^{2k}} + \sqrt{\tau}\|u\|^{\frac{4j+1}{4k-2}}_{L^\infty_\tau H^{2k}} + \sqrt{\tau}\|u\|^{\frac{2j}{2k-1}}_{L^\infty_\tau H^{2k}}. \end{aligned} \tag{33}$$

We denote by I, II, III, IV the four terms on each line of the right-hand side in (15). We first estimate the term I. By developing the time derivatives ∂_t^k and ∂_t^{k-1}, and by using the Hölder inequality, we are reduced to estimating

$$\int_0^T \|\partial_t^{k_1} u\|_{L^2} \|\partial_t^{k_2} u\|_{L^6} \|\partial_t^{j_1} u\|_{W^{1,6}} \|\partial_t^{j_2} u\|_{W^{1,6}}\, ds, \tag{34}$$

where we can assume $k_1 \geq k_2$ and

$$j_1 + j_2 = k-1, \quad k_1 + k_2 = k.$$

Notice that by combining the Sobolev embedding $H^1(M^3) \subset L^6(M^3)$ with Proposition 3.3 for $d = 3$ and $p = 3$, and (24) we have

$$\begin{aligned}(34) &\lesssim \|u\|_{L^\infty_\tau H^{2k_1}} \|u\|_{L^\infty_\tau H^{2k_2+1}} \|\partial_t^{j_1} u\|_{L^2_\tau W^{1,6}} \|\partial_t^{j_2} u\|_{L^2_\tau W^{1,6}} \\ &\lesssim \|u\|_{L^\infty_\tau H^{2k}} \|\partial_t^{j_1} u\|_{L^2_\tau W^{1,6}} \|\partial_t^{j_2} u\|_{L^2_\tau W^{1,6}},\end{aligned}$$

and we can continue the estimate by using (33). Indeed we should estimate $\|\partial_t^j u\|_{L^2_\tau W^{1,6}}$ by three terms on the right-hand side in (33). However, we can consider only the term that gives the worst growth with respect to the power of $\|u\|_{L^\infty_\tau H^{2k}}$ (i.e., only the second term on the right-hand side of (33), as all the other terms give a smaller power of $\|u\|_{L^\infty_\tau H^{2k}}$). Summarizing we get

$$(34) \lesssim \tau \|u\|^2_{L^\infty_\tau H^{2k}} + \|u\|^\gamma_{L^\infty_\tau H^{2k}}$$

for a suitable $\gamma \in (0, 2)$. Next we estimate the term II, which can be reduced to estimating the terms

$$\int_0^T \|\partial_t^{k_1} u\|_{L^2} \|\partial_t^{k_2} u\|_{L^6} \|\partial_t^j \Delta_g u\|_{L^6} \|\partial_t^{k-1-j} u\|_{L^6}, \tag{35}$$

where we can assume $k_1 \leq k_2$ and

$$j = 0, \dots, k-2, \quad k_1 + k_2 = k.$$

By using the Sobolev embedding $H^1(M^3) \subset L^6(M^3)$ in conjunction with Proposition 3.3 we get

$$(35) \lesssim \|u\|_{L^\infty_\tau H^{2k_1}} \|\partial_t^{k_2} u\|_{L^2_\tau L^6} \|u\|_{L^\infty_\tau H^{2j+3}} \|\partial_t^{k-1-j} u\|_{L^2_\tau L^6}.$$

By using (32) and (24) we get

$$(35) \lesssim \|u\|_{L^\infty_\tau H^{2k_1}} \|u\|^{\frac{4k_2-1}{4k-2}}_{L^\infty_\tau H^{2k}} \|u\|_{L^\infty_\tau H^{2j+3}} \|u\|^{\frac{4(k-1-j)-1}{4k-2}}_{L^\infty_\tau H^{2k}} \lesssim \tau \|u\|^2_{L^\infty_\tau H^{2k}} + \|u\|^\gamma_{L^\infty_\tau H^{2k}},$$

where $\gamma \in (0,2)$. Concerning the term III we are reduced to

$$\int_0^T \|\partial_t^{j_1} u\|_{L^\infty} \|\partial_t^{j_2} u\|_{L^\infty} \|\partial_t^{k-j} u\|_{L^2} \|\partial_t^{k_1} u\|_{L^6} \|\partial_t^{k_2} u\|_{L^6} \|\partial_t^{k_3} u\|_{L^6},$$
$$j_1 + j_2 = j, \quad 0 \le j \le k-1, \quad k_1 + k_2 + k_3 = k-1. \tag{36}$$

By the Sobolev embeddings $H^1(M^3) \subset L^6(M^3)$ and $H^2(M^3) \subset L^\infty(M^3)$ and Proposition 3.3 we get

$$(36) \lesssim \|u\|_{L^\infty_\tau H^{2j_1+2}} \|u\|_{L^\infty_\tau H^{2j_2+2}} \|u\|_{L^\infty_\tau H^{2k-2j}} \|\partial_t^{k_1} u\|_{L^2_\tau L^6} \|\partial_t^{k_2} u\|_{L^2_\tau L^6} \|u\|_{L^\infty_\tau H^{2k_3+1}}.$$

By combining (32) with (24) we get

$$(36) \lesssim \tau \|u\|^2_{L^\infty_\tau H^{2k}} + \|u\|^\gamma_{L^\infty_\tau H^{2k}}$$

for $\gamma \in (0,2)$. Concerning IV, it is sufficient to estimate

$$\int_0^T \|\partial_t^{k} u\|_{L^2} \|\partial_t^{k-j} u\|_{L^6} \|\partial_t^{j_1} u\|_{L^6} \|\partial_t^{j_2} u\|_{L^6},$$
$$j_1 + j_2 = j, \quad 1 \le j \le k-1. \tag{37}$$

We can control it by using $H^1(M^3) \subset L^6(M^3)$ and Proposition 3.3:

$$(37) \lesssim \|u\|_{L^\infty_\tau H^{2k}} \|u\|_{L^\infty_\tau H^{2k-2j+1}} \|\partial_t^{j_1} u\|_{L^2_\tau L^6} \|\partial_t^{j_1} u\|_{L^2_\tau L^6}.$$

Again by (32) and (24) we get

$$(37) \lesssim \tau \|u\|^2_{L^\infty_\tau H^{2k}} + \|u\|^\gamma_{L^\infty_\tau H^{2k}}$$

for some $\gamma \in (0,2)$. □

In order to conclude the proof of Theorem 1.3, following the same argument as in the proof of Theorem 1.1, we have to split $\mathcal{E}_{2k}(u)$ as $\mathcal{E}_{2k}(u) = \|\partial_t^k u\|^2_{L^2} + \mathcal{R}_{2k}(u)$, where

$$\mathcal{R}_{2k}(u) = -\frac{1}{2} \int \left|\partial_t^{k-1} \nabla_g (|u|^2)\right|_g^2 \, \mathrm{dvol}_g - \int \left|\partial_t^{k-1} (|u|^2 u)\right|^2 \mathrm{dvol}_g,$$

and we need to estimate the term $\mathcal{R}_{2k}(u)$, namely

$$|\mathcal{R}_{2k}(u)| \lesssim \|u\|_{H^{2k}}^{\frac{4k-3}{2k-1}+\epsilon} + \|u\|_{H^{2k}}^{\frac{4k-5}{2k-1}+\epsilon},$$

which is a version of (31) in three dimensions. Once we prove this estimate, the conclusion is similar to Theorem 1.1. Notice that

$$\int \left|\partial_t^{k-1} \nabla_g (|u|^2)\right|_g^2 \mathrm{dvol}_g \lesssim \sum_{k_1+k_2=k-1} \|\partial_t^{k_1} u\|^2_{W^{1,2}} \|\partial_t^{k_2} u\|^2_{L^\infty}$$
$$\lesssim \sum_{k_1+k_2=k-1} \|u\|^2_{H^{2k_1+1}} \|u\|^{1-\epsilon}_{H^{2k_2+1}} \|u\|^{1+\epsilon}_{H^{2k_2+2}} \lesssim \|u\|_{H^{2k}}^{\frac{4k-3}{2k-1}+\epsilon},$$

where we used the estimate

$$\|v\|_{L^\infty} \lesssim \|v\|_{H^1}^{\frac{1-\epsilon}{2}} \|v\|_{H^2}^{\frac{1+\epsilon}{2}}, \tag{38}$$

which in turn follows by combining interpolation with Sobolev embedding, Proposition 3.3 and (24). Moreover we have

$$\begin{aligned}
&\int \left|\partial_t^{k-1}(|u|^2 u)\right|^2 \mathrm{dvol}_g \\
&\qquad \lesssim \sum_{j_1+j_2+j_3=k-1} \|\partial_t^{j_1} u\|_{L^2}^2 \, \|\partial_t^{j_2} u\|_{L^\infty}^2 \, \|\partial_t^{j_3} u\|_{L^\infty}^2 \\
&\qquad \lesssim \sum_{j_1+j_2+j_3=k-1} \|u\|_{H^{2j_1}}^2 \, \|u\|_{H^{2j_2+1}}^{1-\epsilon} \, \|u\|_{H^{2j_3+1}}^{1-\epsilon} \, \|u\|_{H^{2j_2+2}}^{1+\epsilon} \, \|u\|_{H^{2j_3+2}}^{1+\epsilon} \lesssim \|u\|_{H^{2k}}^{\frac{4k-5}{2k-1}+\epsilon},
\end{aligned}$$

where we used (38), Proposition 3.3 and (24).

6. Polynomial growth of H^2 for the subcubic NLS on M^3

Next we prove Theorem 1.6. We introduce the energy

$$\mathcal{F}_2(v(t,x)) = \int_{M^3} |\partial_t v|^2 \, \mathrm{dvol}_g - (p-1)\int_{M^3} |v|^{p-1} \left|\nabla_g |v|\right|^2 \mathrm{dvol}_g - \frac{p-1}{p}\int_{M^3} |v|^{2p} \, \mathrm{dvol}_g.$$

Proposition 6.1. *Let $u(t,x)$ be solution to* (11) *for $d=3$ and $2<p<3$. Then we have*

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F}_2 u(t,x) \\
&\quad = (p-1)(p-3)\int_{M^3} |u|^{p-2}\, \partial_t |u| \left|\nabla_g |u|\right|^2 \mathrm{dvol}_g + 2(p-1)\int_{M^3} |u|^{p-2}\, \partial_t |u| \, |\nabla_g u|_g^2 \, \mathrm{dvol}_g.
\end{aligned} \tag{39}$$

Proof. We start with the following computation:

$$\begin{aligned}
\frac{d}{dt}\|\partial_t u\|_{L^2}^2 &= 2\,\mathrm{Re}(\partial_t^2 u, \partial_t u) \\
&= 2\,\mathrm{Re}\big(\partial_t(-\Delta_g u + |u|^{p-1}u), i\,\partial_t u\big) \\
&= 2\,\mathrm{Im}\int_{M^3} (\partial_t \nabla_g u, \partial_t \nabla_g u)_g \, \mathrm{dvol}_g + 2\,\mathrm{Re}\big(\partial_t(|u|^{p-1}u), i\,\partial_t u\big),
\end{aligned}$$

where $(f,g) = \int_{M^3} f\bar{g}\, \mathrm{dvol}_g$. Since the first term vanishes, we get

$$\begin{aligned}
\frac{d}{dt}\|\partial_t u\|_{L^2}^2 &= 2\,\mathrm{Re}\big(\partial_t(|u|^{p-1})u, i\,\partial_t u\big) + 2\,\mathrm{Re}\big(|u|^{p-1}\,\partial_t u, i\,\partial_t u\big) \\
&= 2\,\mathrm{Re}\big(\partial_t(|u|^{p-1})u, -\Delta_g u\big) + 2\,\mathrm{Re}\big(\partial_t(|u|^{p-1})u, |u|^{p-1}u\big) \\
&= 2\,\mathrm{Re}\big(\partial_t(|u|^{p-1})u, -\Delta_g u\big) + \frac{p-1}{p}\frac{d}{dt}\int_{M^3} |u|^{2p} \, \mathrm{dvol}_g.
\end{aligned}$$

By using the identity

$$\Delta_g(|u|^2) = u\Delta_g \bar{u} + \bar{u}\Delta_g u + 2|\nabla_g u|_g^2,$$

we get

$$
\begin{aligned}
2\,\mathrm{Re}\big(\partial_t(|u|^{p-1})u, -\Delta_g u\big)
&= -(\partial_t|u|^{p-1}, \Delta_g|u|^2) + 2(\partial_t|u|^{p-1}, |\nabla_g u|_g^2)\\
&= (\partial_t\nabla_g|u|^{p-1}, \nabla_g|u|^2) + 2(\partial_t|u|^{p-1}, |\nabla_g u|_g^2)\\
&= 2(p-1)\big(\partial_t(|u|^{p-2}\nabla_g|u|), |u|\nabla_g|u|\big) + 2(\partial_t|u|^{p-1}, |\nabla_g u|_g^2)\\
&= 2(p-1)\frac{d}{dt}(|u|^{p-2}\nabla_g|u|, |u|\nabla_g|u|) - 2(p-1)(|u|^{p-2}\nabla_g|u|, \partial_t|u|\nabla_g|u|)\\
&\qquad - 2(p-1)(|u|^{p-2}\nabla_g|u|, |u|\nabla_g\,\partial_t|u|) + 2(\partial_t|u|^{p-1}, |\nabla_g u|_g^2)\\
&= 2(p-1)\frac{d}{dt}(|u|^{p-2}\nabla_g|u|, |u|\nabla_g|u|) - 2(p-1)(|u|^{p-2}\nabla_g|u|, \partial_t|u|\nabla_g|u|)\\
&\quad - (p-1)\frac{d}{dt}\big(|u|^{p-1}, \big|\nabla_g|u|\big|^2\big) + (p-1)\big(\partial_t|u|^{p-1}, \big|\nabla_g|u|\big|^2\big) + 2(\partial_t|u|^{p-1}, |\nabla_g u|_g^2). \quad \square
\end{aligned}
$$

The following proposition is a substitute for Proposition 5.1 in the subcubic case.

Proposition 6.2. *We have for every* $\tau \in (0,1)$

$$
\int_0^\tau |\textit{right-hand side of } (39)|\, ds \lesssim \tau\|u\|_{L_\tau^\infty H^2}^{\frac{p+5}{4}} + \|u\|_{L_\tau^\infty H^2}^{\gamma}
$$

for some $\gamma \in \big(0, \frac{p+5}{4}\big)$.

Proof. We can write the terms on the right-hand side of (39) as I and II. We estimate I and the estimate of II is similar. We estimate I as follows (we shall use the diamagnetic inequality in order to remove $|\cdot|$ inside the derivatives ∇_g and ∂_t) by the Hölder inequality:

$$
|I| \lesssim \|\partial_t u\|_{L_\tau^\infty L^2}\,\|u\|^2_{L_\tau^2 W^{1,\frac{12}{5-p}}}\,\|u\|_{L^6}^{p-2} \lesssim \tau^{\frac{6-2p}{8}}\|\partial_t\, u\|_{L_\tau^\infty L^2}\,\|u\|^2_{L_\tau^{\frac{8}{p+1}} W^{1,\frac{12}{5-p}}},
$$

where the pair $\big(\frac{8}{p+1}, \frac{12}{5-p}\big)$ is Strichartz admissible. Notice that by using the equation solved by $u(t,x)$, we are allowed to replace $\|\partial_t u\|_{L_\tau^\infty L^2}$ with $\|u\|_{L_\tau^\infty H^2}$ and hence

$$
|I| \lesssim \tau^{\frac{6-2p}{8}}\|u\|_{L_\tau^\infty H^2}\,\|u\|^2_{L_\tau^{\frac{8}{p+1}} W^{1,\frac{12}{5-p}}}.
$$

Next notice that we have the bound

$$
\|u\|_{L_\tau^{\frac{8}{p+1}} W^{1,\frac{12}{5-p}}} \lesssim \|u\|_{L_\tau^\infty H^1}^{\frac{3-p}{4}}\,\|u\|_{L_\tau^2 W^{1,6}}^{\frac{p+1}{4}},
$$

and hence due to the conservation of the energy, we can continue the estimate above as

$$
|I| \lesssim \tau^{\frac{6-2p}{8}}\|u\|_{L_\tau^\infty H^2}\,\|u\|_{L_\tau^2 W^{1,6}}^{\frac{p+1}{2}}.
$$

We can continue the estimate by using the Strichartz estimates (33) for $j = 0$ (which are still available for solutions to the subcubic NLS):

$$
|I| \lesssim \tau\|u\|_{L_\tau^\infty H^2}\,\|u\|_{L_\tau^\infty H^2}^{\frac{p+1}{4}} + \|u\|_{L_\tau^\infty H^2}^{\gamma}
$$

for some $\gamma \in \big(0, \frac{p+5}{4}\big)$ (indeed we have estimated the term $\|u\|_{L^2_\tau W^{1,6}}$ with the middle term on the right-hand side in (33) since it is the one that involves the larger power of $\|u\|_{L^\infty_\tau H^2}$, and the lower powers are absorbed in the term $\|u\|^\gamma_{L^\infty_\tau H^2}$). □

Integrating (39) on $[0, \tau]$ and arguing exactly as in the proofs of Theorems 1.1 and 1.3, we get the bound

$$\|u(\tau)\|^2_{H^2(M^3)} - \|u(0)\|^2_{H^2(M^3)} \lesssim \tau \|u\|^{\frac{p+5}{4}}_{L^\infty((0,\tau);H^2(M^3))} + \|u\|^\gamma_{L^\infty((0,\tau);H^2(M^3))}$$

for some $\gamma \in \big(0, \frac{p+5}{4}\big)$. This is sufficient to conclude Remark 1.7.

7. Growth of odd Sobolev norms H^{2k+1}

We point out that if we assume the initial datum φ to be H^{2k+2}, then the estimate

$$\sup_{(0,T)} \|u(t,x)\|_{H^{2k+1}(M^2)} \le C(\max\{1,T\})^{\frac{2k}{1-2s_0}+\epsilon},$$

stated in Theorem 1.1, follows by interpolation between the following bounds, which have been already proved by looking at growth of even Sobolev norms:

$$\sup_{(0,T)} \|u(t,x)\|_{H^{2k+2}(M^2)} \le C(\max\{1,T\})^{\frac{2k+1}{1-2s_0}+\epsilon},$$
$$\sup_{(0,T)} \|u(t,x)\|_{H^{2k}(M^2)} \le C(\max\{1,T\})^{\frac{2k-1}{1-2s_0}+\epsilon}.$$

A similar argument follows in order to prove Theorem 1.3 for $m = 2k+1$.

However, the main point in this section is that we assume the initial datum φ to be only in H^{2k+1}, and hence the argument above cannot be applied.

The proofs of Theorems 1.1 and 1.3 (which have been proved in the case $m = 2k$) can be adapted to the case $m = 2k+1$ by using the modified energies

$$\begin{aligned}\mathcal{E}_{2k+1}(u) = \frac{1}{2}\|\partial_t^k \nabla_g u\|^2_{L^2} + \frac{1}{2}\int |u|^{p-1}\,|\partial_t^k u|^2\,\mathrm{dvol}_g + \frac{p-1}{8}\int |u|^{p-3}\big|\partial_t^k(|u|^2)\big|^2\,\mathrm{dvol}_g \\ -\operatorname{Re}\sum_{j=1}^{k-1} c_j \int \partial_t^j u\,\partial_t^{k-j}(|u|^{p-1})\,\partial_t^k \bar{u}\,\mathrm{dvol}_g \\ -\sum_{j=1}^{k-1} c_j \int \partial_t^{k-j}(|u|^{p-3})\,\partial_t^j(|u|^2)\,\partial_t^k(|u|^2)\,\mathrm{dvol}_g.\end{aligned} \tag{40}$$

Indeed we have the following proposition, from which one may conclude the proof of Theorems 1.1 and 1.3 in the case $m = 2k+1$, exactly as we did in the case $m = 2k$. We leave details to the reader.

Proposition 7.1. *Let $u(t,x)$ be a solution to* (1) *with initial datum φ in H^{2k+1}. Then we have the identity*

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_{2k+1}(u(t,x)) = &\frac{1}{2}\int \partial_t(|u|^{p-1})|\partial_t^k u|^2 \,\mathrm{dvol}_g - \mathrm{Re}\sum_{j=1}^{k-1} c_j \int \partial_t^{j+1} u\, \partial_t^{k-j}(|u|^{p-1})\, \partial_t^k \bar{u}\,\mathrm{dvol}_g \\
&- \mathrm{Re}\sum_{j=1}^{k-1} c_j \int \partial_t^{j} u\, \partial_t^{k-j+1}(|u|^{p-1})\, \partial_t^k \bar{u}\,\mathrm{dvol}_g + \frac{p-1}{8}\int_{M^2} \partial_t(|u|^{p-3})\big|\partial_t^k(|u|^2)\big|^2 \,\mathrm{dvol}_g \\
&+ \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j+1}(|u|^{p-3})\, \partial_t^{j}(|u|^2)\, \partial_t^{k}(|u|^2)\,\mathrm{dvol}_g \\
&+ \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j}(|u|^{p-3})\, \partial_t^{j+1}(|u|^2)\, \partial_t^{k}(|u|^2)\,\mathrm{dvol}_g \\
&+ \sum_{j=1}^{k} c_j \int \partial_t^{k}(|u|^{p-1})\, \partial_t^{j} u\, \partial_t^{k+1-j}\bar{u}\,\mathrm{dvol}_g,
\end{aligned}
$$

where $c_j \in \mathbb{R}$ are explicit real numbers that can change in different lines.

Proof. First of all notice that we have

$$
\begin{aligned}
\mathrm{Re}(i\,\partial_t^{k+1}u, \partial_t^k u) &= \mathrm{Re}(\partial_t^k(-\Delta_g u), \partial_t^k u) + \mathrm{Re}(\partial_t^k(u|u|^{p-1}), \partial_t^k u) \\
&= \|\partial_t^k \nabla_g u\|_{L^2}^2 + \mathrm{Re}(\partial_t^k(u|u|^{p-1}), \partial_t^k u).
\end{aligned}
$$

Due to the identity above and by taking the time derivative, we get

$$
\begin{aligned}
\frac{d}{dt}\big(\|\partial_t^k \nabla_g u\|_{L^2}^2 + \mathrm{Re}(\partial_t^k(u|u|^{p-1}), \partial_t^k u)\big) &= \frac{d}{dt}\mathrm{Re}(i\,\partial_t^{k+1}u, \partial_t^k u) = \mathrm{Re}(i\,\partial_t^{k+2}u, \partial_t^k u) \\
&= \mathrm{Re}(\partial_t^{k+1}(-\Delta_g u), \partial_t^k u) + \mathrm{Re}(\partial_t^{k+1}(|u|^{p-1}u), \partial_t^k u) \\
&= \frac{1}{2}\frac{d}{dt}\|\partial_t^k \nabla_g u\|_{L^2}^2 + \mathrm{Re}(\partial_t^{k+1}(|u|^{p-1}u), \partial_t^k u).
\end{aligned}
$$

Next we focus on the second term on the right-hand side:

$$
\begin{aligned}
&\mathrm{Re}(\partial_t^{k+1}(|u|^{p-1}u), \partial_t^k u) \\
&\quad= \frac{d}{dt}\mathrm{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^k u) - \mathrm{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^{k+1} u) \\
&\quad= \frac{d}{dt}\mathrm{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^k u) - \mathrm{Re}(\partial_t^k(|u|^{p-1})u, \partial_t^{k+1} u) - \mathrm{Re}(|u|^{p-1}\partial_t^k u, \partial_t^{k+1} u) \\
&\qquad+ \mathrm{Re}\sum_{j=1}^{k-1} c_j (\partial_t^j u\, \partial_t^{k-j}(|u|^{p-1}), \partial_t^{k+1} u) \\
&\quad= \frac{d}{dt}\mathrm{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^k u) - \mathrm{Re}(\partial_t^k(|u|^{p-1})u, \partial_t^{k+1} u) - \frac{1}{2}\frac{d}{dt}\int |u|^{p-1}|\partial_t^k u|^2\,\mathrm{dvol}_g \\
&\qquad+ \frac{1}{2}\int \partial_t(|u|^{p-1})|\partial_t^k u|^2\,\mathrm{dvol}_g + \mathrm{Re}\sum_{j=1}^{k-1} c_j (\partial_t^j u\, \partial_t^{k-j}(|u|^{p-1}), \partial_t^{k+1} u)
\end{aligned}
$$

$$= \frac{d}{dt}\operatorname{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^k u) - \operatorname{Re}(\partial_t^k(|u|^{p-1})u, \partial_t^{k+1}u) - \frac{1}{2}\frac{d}{dt}\int |u|^{p-1}|\partial_t^k u|^2 \,\mathrm{dvol}_g + \frac{1}{2}\int \partial_t(|u|^{p-1})|\partial_t^k u|^2 \,\mathrm{dvol}_g + \frac{d}{dt}\operatorname{Re}\sum_{j=1}^{k-1} c_j(\partial_t^j u\, \partial_t^{k-j}(|u|^{p-1}), \partial_t^k u) - \operatorname{Re}\sum_{j=1}^{k-1} c_j(\partial_t^{j+1} u\, \partial_t^{k-j}(|u|^{p-1}), \partial_t^k u) - \operatorname{Re}\sum_{j=1}^{k-1} c_j(\partial_t^j u\, \partial_t^{k-j+1}(|u|^{p-1}), \partial_t^k u).$$

Next we deal with the third term on the right-hand side:

$$-\operatorname{Re}(\partial_t^k(|u|^{p-1})u, \partial_t^{k+1}u) = -\frac{1}{2}\int \partial_t^k(|u|^{p-1})\,\partial_t^{k+1}(|u|^2)\,\mathrm{dvol}_g + \sum_{j=1}^{k} c_j \int \partial_t^k(|u|^{p-1})\,\partial_t^j u\, \partial_t^{k+1-j}\bar{u}\,\mathrm{dvol}_g,$$

and we notice that $\partial_t^k(|u|^{p-1}) = \frac{1}{2}(p-1)\,\partial_t^{k-1}(\partial_t(|u|^2)|u|^{p-3})$. Hence we can continue the identity above as follows:

$$\cdots = -\frac{p-1}{4}\int |u|^{p-3}\,\partial_t^k(|u|^2)\,\partial_t^{k+1}(|u|^2)\,\mathrm{dvol}_g + \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j}(|u|^{p-3})\,\partial_t^j(|u|^2)\,\partial_t^{k+1}(|u|^2)\,\mathrm{dvol}_g + \sum_{j=1}^{k} c_j \int \partial_t^k(|u|^{p-1})\,\partial_t^j u\,\partial_t^{k+1-j}\bar{u}\,\mathrm{dvol}_g$$

$$= -\frac{p-1}{8}\frac{d}{dt}\int |u|^{p-3}\big|\partial_t^k(|u|^2)\big|^2\,\mathrm{dvol}_g + \frac{p-1}{8}\int \partial_t(|u|^{p-3})\big|\partial_t^k(|u|^2)\big|^2\,\mathrm{dvol}_g + \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j}(|u|^{p-3})\,\partial_t^j(|u|^2)\,\partial_t^{k+1}(|u|^2)\,\mathrm{dvol}_g + \sum_{j=1}^{k} c_j \int \partial_t^k(|u|^{p-1})\,\partial_t^j u\,\partial_t^{k+1-j}\bar{u}\,\mathrm{dvol}_g.$$

Then by elementary considerations

$$\cdots = -\frac{p-1}{8}\frac{d}{dt}\int |u|^{p-3}\big|\partial_t^k(|u|^2)\big|^2\,\mathrm{dvol}_g + \frac{p-1}{8}\int \partial_t(|u|^{p-3})\big|\partial_t^k(|u|^2)\big|^2\,\mathrm{dvol}_g + \frac{d}{dt}\sum_{j=1}^{k-1} c_j \int \partial_t^{k-j}(|u|^{p-3})\,\partial_t^j(|u|^2)\,\partial_t^k(|u|^2)\,\mathrm{dvol}_g + \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j+1}(|u|^{p-3})\,\partial_t^j(|u|^2)\,\partial_t^k(|u|^2)\,\mathrm{dvol}_g + \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j}(|u|^{p-3})\,\partial_t^{j+1}(|u|^2)\,\partial_t^k(|u|^2)\,\mathrm{dvol}_g + \sum_{j=1}^{k} c_j \int \partial_t^k(|u|^{p-1})\,\partial_t^j u\,\partial_t^{k+1-j}\bar{u}\,\mathrm{dvol}_g. \qquad \square$$

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