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KOSZUL COMPLEXES, BIRKHOFF NORMAL FORM AND THE MAGNETIC DIRAC OPERATOR

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We consider the semiclassical Dirac operator coupled to a magnetic potential on a large class of manifolds, including all metric contact manifolds. We prove a sharp Weyl law and a bound on its eta invariant. In the absence of a Fourier integral parametrix, the method relies on the use of almost analytic continuations combined with the Birkhoff normal form and local index theory.

1. Introduction

Semiclassical analysis concerns the study of the spectrum of (h-)pseudodifferential operators $P_h : C^\infty(X) \to C^\infty(X)$, $h \in (0,1]$, in the limit $h \to 0$ and is now the subject of several texts [Dimassi and Sjöstrand 1999; Guillemin and Sternberg 2013; Ivrii 1998; 2017; Maslov and Fedoriuk 1981; Robert 1987; Zworski 2012]. Standard examples of such operators include the Schrödinger operator $P_h = -h^2\Delta_X + V$ on a compact n-dimensional Riemannian manifold X with potential $V \in C^\infty(X)$. The clearest asymptotic result is given by the celebrated Weyl law, see for example [Dimassi and Sjöstrand 1999, Chapter 10], on the asymptotic number of eigenvalues $N[a,b]$ in a fixed interval $[a,b]$. A related result is on the number of eigenvalues $N(-ch,ch)$ of P_h in the finer interval $(-ch,ch)$: assuming 0 is not a critical value of the symbol $\sigma(P) = p(x,\xi) \in C^\infty(T^*X)$, one has

$$N(-ch,ch) = O(h^{-n+1}) \tag{1-1}$$

as $h \to 0$, for all $c > 0$. Similar results also exist in the case where 0 is a Morse–Bott critical level for the symbol; see [Brummelhuis et al. 1995]. In the critical case, the exponent in the Weyl law may drop depending on the codimension of zero energy level $\Sigma_0^P := \{p(x,\xi) = 0\}$ and the signature of the normal Hessian. The Weyl laws thus obtained are sharp and are proved using a parametrix construction for the evolution operator $e^{\frac{it}{h}P_h}$ as a Fourier integral operator.

In the context of nonscalar operators $P_h : C^\infty(X;E) \to C^\infty(X;E)$ acting on sections of a vector bundle E, fewer result are known. The simplest case is when the nonscalar symbol $p(x,\xi) \in C^\infty(T^*X;E)$ is smoothly diagonalizable near the zero energy level $\Sigma_0^P = \{\det(p(x,\xi)) = 0\}$. In this case, similar Fourier integral methods apply; see [Emmrich and Weinstein 1996; Maslov and Fedoriuk 1981] or [Guillemin; Sandoval 1999] for an exposition in the microlocal/classical setting. For nonscalar operators

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another method is provided under the microhyperbolicity condition of Ivrii [1998, Chapters 2 and 3]; see also [Dimassi and Sjöstrand 1999, Chapter 12]. In this paper, we study the particular case of the magnetic Dirac operator where neither diagonalizability nor the microhyperbolicity condition is satisfied.

More precisely, let (X, g^{TX}) be an oriented Riemannian manifold of odd dimension $n = 2m+1$ equipped with a spin structure. Let S be the corresponding spin bundle and let L be an auxiliary Hermitian line bundle. Fix a unitary connection A_0 on L and let $a \in \Omega^1(X;\mathbb{R})$ be a one-form. This gives a family of unitary connections on L via $\nabla^h = A_0 + \frac{i}{h}a$ and a corresponding family of coupled magnetic Dirac operators

$$D_h := hD_{A_0} + ic(a) \tag{1-2}$$

for $h \in (0,1]$ and where c stands for the Clifford multiplication endomorphism (see Section 2B).

In order to derive sharp spectral asymptotics, we shall make a couple of restrictive assumptions on the one-form a and the metric g^{TX}. First, the one-form a will be assumed to be a contact one-form (i.e., one satisfying $a \wedge (da)^m > 0$). This gives rise to the contact hyperplane $H = \ker(a) \subset TX$ as well as the Reeb vector field R defined via $i_R da = 0,\ i_R a = 1$.

To state the assumption on the metric, consider the contracted endomorphism $\mathfrak{J} : T_xX \to T_xX$ defined at each point $x \in X$ via

$$da(v_1, v_2) = g^{TX}(v_1, \mathfrak{J}v_2) \quad \forall v_1, v_2 \in T_xX.$$

From the contact assumption, $\mathfrak{J}$ has a one-dimensional kernel spanned by the Reeb vector field R. The endomorphism $\mathfrak{J}$ is clearly antisymmetric with respect to the metric

$$g^{TX}(v_1, \mathfrak{J}v_2) = -g^{TX}(\mathfrak{J}v_1, v_2),$$

and hence its nonzero eigenvalues come in purely imaginary pairs $\pm i\mu,\ \mu > 0$. The assumption on the metric g^{TX} is then as follows.

Definition 1.1. We say that the metric g^{TX} is *suitable* to the contact form a if there exist positive constants $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_m$ (independent of $x \in X$) and a positive real function $\nu(x) > 0$ such that

$$\operatorname{Spec}(\mathfrak{J}_x) = \{0,\ \pm i\mu_1\nu(x),\ \pm i\mu_2\nu(x),\ \ldots,\ \pm i\mu_m\nu(x)\} \tag{1-3}$$

for all $x \in X$.

Before proceeding further, we give two examples of suitable metrics:

(1) In the case that the dimension of the manifold X is 3, any metric g^{TX} is suitable, as $\operatorname{Spec}(\mathfrak{J}_x) = \{0, \pm i|da|\}$ has only two nonzero eigenvalues.

(2) There is a smooth endomorphism $J : TX \to TX$ such that (X^{2m+1}, a, g^{TX}, J) is a metric contact manifold. That is, we have

$$\begin{aligned} J^2 v_1 &= -v_1 + a(v_1)R, \\ g^{TX}(v_1, Jv_2) &= da(v_1, v_2) \quad \forall v_1, v_2 \in T_xX. \end{aligned} \tag{1-4}$$

In this case the nonzero eigenvalues of $\mathfrak{J}_x = J_x$ are $\pm i$ (each with multiplicity m). For any given contact form a there exists an infinite-dimensional space of (g^{TX}, J) satisfying (1-4). This case in particular includes all strictly pseudoconvex CR manifolds.

In addition to the Weyl law we shall also be interested in the asymptotics of the eta invariant $\eta_h = \eta(D_h)$ of the Dirac operator, formally its signature (see Section 2A for a definition). The main result is now stated as follows.

Theorem 1.2. *Under the contact and suitability assumptions on a and g^{TX}, the Weyl counting function and eta invariant of D_h satisfy the sharp asymptotics*

$$N(-ch, ch) = O(h^{-m}), \tag{1-5}$$

$$\eta_h = O(h^{-m}) \tag{1-6}$$

as $h \to 0$.

We note that the exponents above are significantly lower than (1-1). This is again partly attributed to the high codimension of the zero energy level Σ_0^D. In this case $\Sigma_0^D = \{\xi = -a\} \subset T^*X$ is the graph of the contact form a, a submanifold of half-dimension $2m+1$ on which the canonical symplectic form is maximally nondegenerate of rank $2m$.

The proof of the asymptotic result Theorem 1.2 above will be based on a functional trace expansion. To state the trace expansion involved, set $\nu_0 := \mu_1[\min_{x\in X} \nu(x)]$ and choose $f \in C_c^\infty(-\sqrt{2\nu_0}, \sqrt{2\nu_0})$. Pick real numbers $0 < T' < T$ and let $\theta \in C_c^\infty((-T, T); [0, 1])$ such that $\theta(x) = 1$ on $(-T', T')$. Let

$$\mathcal{F}^{-1}\theta(x) := \check{\theta}(x) = \frac{1}{2\pi}\int e^{ix\xi}\theta(\xi)\,d\xi,$$

$$\mathcal{F}_h^{-1}\theta(x) := \frac{1}{h}\check{\theta}\left(\frac{x}{h}\right) = \frac{1}{2\pi h}\int e^{\frac{i}{h}x\xi}\theta(\xi)\,d\xi$$

be its classical and semiclassical inverse Fourier transforms respectively. We now have the following functional trace expansion for the magnetic Dirac operator $D = D_h$ given in (1-2).

Theorem 1.3. *Let a be a contact form, g^{TX} be a suitable metric and f be as above. There exist smooth functions $u_j \in C^\infty(\mathbb{R})$ such that there is a trace expansion*

$$\begin{aligned}\operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)(\mathcal{F}_h^{-1}\theta)(\lambda\sqrt{h}-D)\right] &= \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h}\check{\theta}\left(\frac{\lambda\sqrt{h}-D}{h}\right)\right]\\ &= h^{-m-1}\left(f(\lambda)\sum_{j=0}^{N-1} u_j(\lambda)h^{\frac{j}{2}} + O(h^{\frac{N}{2}})\right)\end{aligned} \tag{1-7}$$

for T sufficiently small and for each $N \in \mathbb{N}$, $\lambda \in \mathbb{R}$.

Again, the trace (1-7) should be compared with the wave trace expansions for scalar and microhyperbolic operators [Dimassi and Sjöstrand 1999, Chapters 10 and 12], although a different scale of size $\sqrt{h}$ is being used. In the absence of a Fourier integral parametrix or microhyperbolicity our strategy is to combine the use of almost analytic continuations with local index theory expansions. We first show that the trace is $O(h^\infty)$ in the region $\mathrm{spt}(\theta) \subset \{T > |x| \geq h^\varepsilon\}$, $\varepsilon \in \left(0, \frac{1}{2}\right)$ (see Lemma 3.1). Here the lack of microhyperbolicity for the symbol poses a difficulty in the use of almost analytic continuations [Dimassi and Sjöstrand 1999, Chapter 12]; see also [Dimassi and Sjöstrand 1996]. We however show that this can be overcome with a closer understanding of the total symbol of D via its Birkhoff normal form. It is in deriving the Birkhoff normal form that Koszul complexes are used and the assumptions on a, g^{TX} are required. The local index theory method [Bismut 1987; Ma and Marinescu 2007] finally provides the expansion in the region $\mathrm{spt}(\theta) \subset \{|x| < h^\varepsilon\}$ (see Lemma 3.2).

There is a large recent literature for semiclassical problems in the presence of magnetic fields. In particular the extensive book of Ivrii [2017] specifically considers the case of the magnetic Dirac operator in Chapter 17. The Birkhoff normal form here (5-13) generalizes Proposition 17.2.1 therein. Our use of normal forms should also be compared to their use in scalar cases from [Charles and Vũ Ngọc 2008; Helffer et al. 2016; Raymond and Vũ Ngọc 2015]. We note that some of the spectral literature on Dirac operators treats the massive case (e.g., mass $m = 1$ in [Helffer and Robert 1983]), where the mass term renders the symbol diagonalizable. The geometric Dirac operator considered here corresponds to the odd-dimensional purely massless case.

The asymptotic problem of the eta invariant (1-6) was earlier considered by the author in [Savale 2014], where a nonsharp estimate was proved, under no assumptions on a, g^{TX}, via the use of the heat trace. This asymptotic problem was first considered and applied in [Taubes 2007] in the proof of the three-dimensional Weinstein conjecture using Seiberg–Witten theory. The three-dimensional case has been further explored in [Tsai 2014].

The paper is organized as follows. In Section 2, we begin with preliminary notions used throughout the paper, including basic facts about Clifford representations, Dirac operators and the semiclassical calculus. In Section 2B1 we compute the spectrum of a model magnetic Dirac operator on $\mathbb{R}^m$ using Clifford representations and the harmonic oscillator. In Section 3 we perform certain reductions towards proving Theorem 1.3, including a time scale breakup of the trace into Lemmas 3.1 and 3.2. These reductions are then used in Section 4 to further reduce Lemma 3.1 to the case of a Euclidean magnetic Dirac operator on $\mathbb{R}^n$. In Section 5 we obtain the Birkhoff normal form for the Euclidean magnetic Dirac operator on $\mathbb{R}^n$ from Section 4. It is here in Section 5A that Koszul complexes are employed for the normal form. In Section 6 we show how the normal form is used in proving Lemma 3.1 via the use of almost analytic continuations. In Section 7 we prove Lemma 3.2 using the methods of local index theory. In Section 8 we show how to prove the spectral estimates of Theorem 1.2 via the trace expansion Theorem 1.3. Finally, in the Appendix we prove some spectral estimates useful in Sections 4 and 5.

2. Preliminaries

2A. ***Spectral invariants of the Dirac operator.*** Here we review the basic facts about Dirac operators used throughout the paper, with [Berline et al. 2004] providing a standard reference. Consider a compact, oriented, Riemannian manifold (X, g^{TX}) of odd dimension $n = 2m + 1$. Let X be equipped with spin structure, i.e., a principal $\mathrm{Spin}(n)$ bundle $\mathrm{Spin}(TX) \to \mathrm{SO}(TX)$ with an equivariant double covering of the principal $\mathrm{SO}(n)$-bundle of orthonormal frames $\mathrm{SO}(TX)$. The corresponding spin bundle $S = \mathrm{Spin}(TX) \times_{\mathrm{Spin}(n)} S_{2m}$ is associated to the unique irreducible representation of $\mathrm{Spin}(n)$. Let ∇^{TX} denote the Levi-Civita connection on TX. This lifts to the spin connection ∇^S on the spin bundle S. The Clifford multiplication endomorphism $c : T^*X \to S \otimes S^*$ may be defined (see Section 2B) satisfying

$$c(a)^2 = -|a|^2 \quad \forall a \in T^*X.$$

Let L be a Hermitian line bundle on X. Let A_0 be a fixed unitary connection on L and let $a \in \Omega^1(X; \mathbb{R})$ be a one-form on X. This gives a family $\nabla^h = A_0 + \frac{i}{h}a$ of unitary connections on L. We denote by $\nabla^{S\otimes L} = \nabla^S \otimes 1 + 1 \otimes \nabla^h$ the tensor product connection on $S \otimes L$. Each such connection defines a coupled Dirac operator

$$D_h := hD_{A_0} + ic(a) = hc \circ (\nabla^{S\otimes L}) : C^\infty(X; S \otimes L) \to C^\infty(X; S \otimes L)$$

for $h \in (0, 1]$. Each Dirac operator D_h is elliptic and self-adjoint. It hence possesses a discrete spectrum of eigenvalues.

We define the eta function of D_h by the formula

$$\eta(D_h, s) := \sum_{\substack{\lambda \neq 0 \\ \lambda \in \mathrm{Spec}(D_h)}} \mathrm{sign}(\lambda)|\lambda|^{-s} = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s-1}{2}} \mathrm{tr}(D_h e^{-tD_h^2})\, dt. \tag{2-1}$$

Here, and in the remainder of the paper, we use the convention that $\mathrm{Spec}(D_h)$ denotes a multiset with each eigenvalue of D_h being counted with its multiplicity. The above series converges for $\mathrm{Re}(s) > n$. It was shown in [Atiyah et al. 1975; 1976] that the eta function possesses a meromorphic continuation to the entire complex s-plane and has no pole at zero. Its value at zero is defined to be the eta invariant of the Dirac operator

$$\eta_h := \eta(D_h, 0).$$

By including the zero eigenvalue in (2-1), with an appropriate convention, we may define a variant, known as the reduced eta invariant, by

$$\bar{\eta}_h := \tfrac{1}{2}\{k_h + \eta_h\},$$

with $k_h = \dim\ker D_h$.

The eta invariant is unchanged under positive scaling:

$$\eta(D_h, 0) = \eta(cD_h, 0) \quad \forall c > 0. \tag{2-2}$$

Let $L_{t,h}$ denote the Schwartz kernel of the operator $D_h e^{-tD_h^2}$ on the product $X \times X$. Throughout the paper all Schwartz kernels will be defined with respect to the Riemannian volume density. Denote by $\mathrm{tr}(L_{t,h}(x, x))$ the pointwise trace of $L_{t,h}$ along the diagonal. We may now analogously define the function

$$\eta(D_h, s, x) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty t^{\frac{s-1}{2}} \mathrm{tr}(L_{t,h}(x, x))\, dt. \tag{2-3}$$

In [Bismut and Freed 1986, Theorem 2.6], it was shown that for $\mathrm{Re}(s) > -2$, the function $\eta(D_h, s, x)$ is holomorphic in s and smooth in x. From (2-3) it is clear that this is equivalent to

$$\mathrm{tr}(L_{t,h}) = O(t^{\frac{1}{2}}) \quad \text{as } t \to 0. \tag{2-4}$$

The eta invariant is then given by the convergent integral

$$\eta_h = \int_0^\infty \frac{1}{\sqrt{\pi t}} \,\mathrm{tr}(D_h e^{-tD_h^2})\, dt. \tag{2-5}$$

2B. *Clifford algebra and its representations.* Here we review the construction of the spin representation of the Clifford algebra. The following, being standard, is merely used to set up our conventions and subsequently compute the spectrum of the model magnetic Dirac operator on $\mathbb{R}^m$ in Section 2B1.

Consider a real vector space V of even dimension $2m$ with metric $\langle \cdot, \cdot \rangle$. Recall that its Clifford algebra $\mathrm{Cl}(V)$ is defined as the quotient of the tensor algebra $T(V) := \bigoplus_{j=0}^{\infty} V^{\otimes j}$ by the ideal generated from the relations $v \otimes v + |v|^2 = 0$. Fix a compatible almost complex structure J and split $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ into the $\pm i$ eigenspaces of J. The complexification $V \otimes \mathbb{C}$ carries an induced $\mathbb{C}$-bilinear inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$, as well as an induced Hermitian inner product $h^{\mathbb{C}}(\cdot, \cdot)$. Next, define $S_{2m} = \Lambda^* V^{1,0}$. Clearly S_{2m} is a complex vector space of dimension 2^m on which the unique irreducible (spin)-representation of the Clifford algebra $\mathrm{Cl}(V) \otimes \mathbb{C}$ is defined by the rule

$$c_{2m}(v)\omega = \sqrt{2}(v^{1,0} \wedge \omega - \iota_{v^{0,1}}\omega), \qquad v \in V,\ \omega \in S_{2m}.$$

The contraction above is taken with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. It is clear that $c_{2m}(v) : \Lambda^{\text{even/odd}} \to \Lambda^{\text{odd/even}}$ switches the odd and even factors. For the Clifford algebra $\mathrm{Cl}(W) \otimes \mathbb{C}$ of an odd-dimensional vector space $W = V \oplus \mathbb{R}[e_0]$ there are exactly two irreducible representations. These two (spin)-representations $S^+_{2m+1} = S^-_{2m+1} = \Lambda^* V^{1,0}$ are defined via

$$\begin{aligned} c^{\pm}_{2m+1}(v) &= c_{2m}(v), \qquad v \in V, \\ c^{+}_{2m+1}(e_0)\omega_{\text{even/odd}} &= -c^{-}_{2m+1}(e_0)\omega_{\text{even/odd}} = \pm i\,\omega_{\text{even/odd}}. \end{aligned} \tag{2-6}$$

Throughout the rest of the paper, we stick with the positive convention and use the shorthand $c = c_{2m}$, $c = c^+_{2m+1}$ when the indices $2m$, $2m+1$ are implicitly understood.

Pick an orthonormal basis $e_1, e_2, \ldots, e_{2m}$ for V in which the almost complex structure is given by $Je_{2j-1} = e_{2j}$, $1 \le j \le m$. An $h^{\mathbb{C}}$-orthonormal basis for $V^{1,0}$ is now given by $w_j = \frac{1}{\sqrt{2}}(e_{2j} + i e_{2j-1})$, $1 \le j \le m$. A basis for S_{2m} and $S^{\pm}_{2m+1}$ is given by $w_k = w_1^{k_1} \wedge \cdots \wedge w_m^{k_m}$ with $k = (k_1, k_2, \ldots, k_m) \in \{0,1\}^m$. Ordering the above chosen bases lexicographically in k, we may define the Clifford matrices, of rank 2^m, via

$$\gamma_j^m = c(e_j), \qquad 0 \le j \le 2m,$$

for each m. Again, we often write $\gamma_j^m = \gamma_j$ with the index m implicitly understood. Giving representations of the Clifford algebra, these matrices satisfy the relation

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij}. \tag{2-7}$$

Next, one may further define the Clifford quantization map on the exterior algebra

$$\begin{aligned} c &: \Lambda^* W \otimes \mathbb{C} \to \mathrm{End}(S_{2m}), \\ c(e_0^{k_0} \wedge \cdots \wedge e_{2m}^{k_{2m}}) &= c(e_0)^k \ldots c(e_{2m})^{k_{2m}}. \end{aligned} \tag{2-8}$$

An easy computation yields

$$c(e_0 \wedge \cdots \wedge e_{2m}) = i^{m+1}.$$

Furthermore, if $e_0 \wedge \cdots \wedge e_{2m}$ is designated to give a positive orientation for W then for $\omega \in \Lambda^k W$ we have

$$c(*\omega) = i^{m+1}(-1)^{\frac{k(k+1)}{2}} c(\omega), \tag{2-9}$$

$$c(\omega)^* = (-1)^{\frac{k(k+1)}{2}} c(\omega) \tag{2-10}$$

under the Hodge star and $h^{\mathbb{C}}$-adjoint. The Clifford quantization map (2-8) is a linear surjection with kernel spanned by elements of the form $*\omega - i^{m+1}(-1)^{\frac{k(k+1)}{2}}\omega$. Thus, in particular one has linear isomorphisms

$$c : \Lambda^{\text{even/odd}} W \otimes \mathbb{C} \to \mathrm{End}(S_{2m}). \tag{2-11}$$

Next, given $(r_1, \ldots, r_m) \in \mathbb{R}^m \setminus 0$, we define

$$I_r := \{ j \mid r_j \neq 0 \} \subset \{1, 2, \ldots, m\}, \tag{2-12}$$

$$Z_r := |I_r|, \tag{2-13}$$

$$V_r := \bigoplus_{j \in I_r} \mathbb{C}[w_j] \subset V^{1,0}, \tag{2-14}$$

$$w_r := \sum_{j=1}^{m} r_j w_j \in V_r. \tag{2-15}$$

Clearly, $\|w_r\| = |r|$. Denoting by $w_r^{\perp}$ the $h^{\mathbb{C}}$-orthogonal complement of $w_r \subset V_r$, one clearly has $V_r = \mathbb{C}[w_r] \oplus w_r^{\perp}$. Hence

$$\begin{aligned} \Lambda^{\text{even}} V_r &= (\Lambda^{\text{even}} w_r^{\perp}) \oplus \frac{w_r}{|r|} \wedge (\Lambda^{\text{odd}} w_r^{\perp}), \\ \Lambda^{\text{odd}} V_r &= (\Lambda^{\text{odd}} w_r^{\perp}) \oplus \frac{w_r}{|r|} \wedge (\Lambda^{\text{even}} w_r^{\perp}). \end{aligned} \tag{2-16}$$

Next, we define

$$\mathtt{i}_r : \Lambda^* V_r \to \Lambda^* V_r \qquad \text{via} \qquad \mathtt{i}_r(\omega) := \frac{w_r}{|r|} \wedge \omega, \quad \mathtt{i}_r\left(\frac{w_r}{|r|} \wedge \omega\right) := \omega \tag{2-17}$$

for $\omega \in \Lambda^* w_r^{\perp}$. Clearly, $\mathtt{i}_r^2 = 1$ with the decomposition (2-16) implying that

$$\begin{aligned} \mathtt{i}_r &: \Lambda^{\text{even}} V_r \to \Lambda^{\text{odd}} V_r, \\ \mathtt{i}_r &: \Lambda^{\text{odd}} V_r \to \Lambda^{\text{even}} V_r \end{aligned}$$

are linear isomorphisms. Next, the endomorphism

$$c\left(\frac{w_r - \bar{w}_r}{\sqrt{2}}\right) = (w_r \wedge + \iota_{\bar{w}_r}) : \Lambda^* V_r \to \Lambda^* V_r \tag{2-18}$$

has the form

$$c\left(\frac{w_r - \bar{w}_r}{\sqrt{2}}\right) = \begin{bmatrix} & |r|\mathtt{i}_r \\ |r|\mathtt{i}_r & \end{bmatrix} \tag{2-19}$$

with respect to the decomposition $\Lambda^* V_r = \Lambda^{\text{odd}} V_r \oplus \Lambda^{\text{even}} V_r$. This finally allows us to write the eigenspaces of (2-18) as

$$V_r^{\pm} = (1 \pm \mathtt{i}_r)(\Lambda^{\text{even}} V_r) \tag{2-20}$$

with eigenvalues $\pm |r|$ respectively.

2B1. *Magnetic Dirac operator on* $\mathbb{R}^m$. We now define the magnetic Dirac operator on $\mathbb{R}^m$ via

$$D_{\mathbb{R}^m} = \sum_{j=1}^{m} \left(\frac{\mu_j}{2}\right)^{\frac{1}{2}} [\gamma_{2j}(h\partial_{x_j}) + i\gamma_{2j-1} x_j] \in \Psi^1_{\text{cl}}(\mathbb{R}^m; \mathbb{C}^{2^m}). \tag{2-21}$$

Its square is computed in terms of the harmonic oscillator

$$D^2_{\mathbb{R}^m} = \mathtt{H}_2 - ih\mathtt{R}_{2m+1}, \tag{2-22}$$

with

$$\mathtt{H}_2 = \frac{1}{2}\sum_{j=1}^{m} \mu_j[-(h\partial_{x_j})^2 + x_j^2], \quad \mathtt{R}_{2m+1} = \frac{1}{2}\sum_{j=1}^{m} \mu_j[\gamma_{2j-1}\gamma_{2j}]. \tag{2-23}$$

It is an easy exercise to show that

$$\mathtt{R}_{2m+1} w_k = \frac{i}{2}\left[\sum_{j=1}^{m} (-1)^{k_j - 1}\mu_j\right] w_k. \tag{2-24}$$

Next, define the lowering and raising operators $A_j = h\partial_{x_j} + x_j$ and $A_j^* = -h\partial_{x_j} + x_j$ for $1 \le j \le m$, and the Hermite functions

$$\psi_{\tau,k}(x) := \psi_\tau(x) \otimes w_k,$$
$$\psi_\tau(x) := \frac{1}{(\pi h)^{\frac{m}{4}} (2h)^{\frac{|\tau|}{2}} \sqrt{\tau!}} \left[\prod_{j=1}^{m} (A_j^*)^{\tau_j}\right] e^{-\frac{|x|^2}{2h}} \quad \text{for } \tau = (\tau_1, \tau_2, \ldots, \tau_m) \in \mathbb{N}_0^m. \tag{2-25}$$

It is well known that $\psi_{\tau,k}(x)$ form an orthonormal basis for $L^2(\mathbb{R}^m; \mathbb{C}^{2^m})$. Furthermore we have the standard relations

$$[A_j, A_j^*] = 2h, \quad \mathtt{H}_2 = \frac{1}{2}\sum_{j=1}^{m} \mu_j (A_j A_j^* - 1). \tag{2-26}$$

It is clear from (2-22), (2-24) and (2-26) that each $\psi_{\tau,k}(x)$ is an eigenvector of $D^2_{\mathbb{R}^m}$ with eigenvalue

$$\lambda_{\tau,k} = h\sum_{j=1}^{m}(2\tau_j + 1 + (-1)^{k_j-1})\frac{\mu_j}{2}.$$

Hence, clearly the kernel of $D_{\mathbb{R}^m}$ is one-dimensional and spanned by $\psi_{0,0} = e^{-\frac{|x|^2}{2h}}$. We now find a decomposition of $L^2(\mathbb{R}^m;\mathbb{C}^{2^m})$ into eigenspaces of $D_{\mathbb{R}^m}$. First, if we define

$$\bar\partial = \frac{1}{2}\sum_{j=1}^{m}\left(\frac{\mu_j}{2}\right)^{\frac{1}{2}} c(w_j)A_j, \tag{2-27}$$

then one quickly computes

$$\bar\partial^* = -\frac{1}{2}\sum_{j=1}^{m}\left(\frac{\mu_j}{2}\right)^{\frac{1}{2}} c(\bar w_j)A_j^* \tag{2-28}$$

and

$$D_{\mathbb{R}^m} = \sqrt{2}(\bar\partial + \bar\partial^*). \tag{2-29}$$

For each $\tau \in \mathbb{N}_0^m \setminus 0$, we define I_τ, V_τ as in (2-12), (2-14) and set

$$E_\tau := \bigoplus_{b\in\{0,1\}^{I_\tau}} \mathbb{C}\left[\prod_{j\in I_\tau}\left(\frac{c(w_j)A_j}{\sqrt{2\tau_j h}}\right)^{b_j}\psi_{\tau,0}\right].$$

It is clear that we have an orthogonal decomposition

$$L^2(\mathbb{R}^m;\mathbb{C}^{2^m}) = \mathbb{C}[\psi_{0,0}] \oplus \bigoplus_{\tau\in\mathbb{N}_0^m\setminus 0} E_\tau.$$

Furthermore, we have the isomorphism

$$\mathscr{I}_\tau : \Lambda^* V_\tau \to E_\tau,$$
$$\mathscr{I}_\tau\left(\bigwedge_{j\in I_\tau} w_j^{b_j}\right) := \prod_{j\in I_\tau}\left(\frac{c(w_j)A_j}{\sqrt{2\tau_j h}}\right)^{b_j}\psi_{\tau,0}.$$

Each E_τ hence has dimension 2^{Z_τ} and is closed under $c(w_j)A_j$ and $c(\bar w_j)A_j^*$ for $1 \le j \le m$. We again have

$$E_\tau = E_\tau^{\text{even}} \oplus E_\tau^{\text{odd}}, \quad \text{where } E_\tau^{\text{even/odd}} := \mathscr{I}_\tau(\Lambda^{\text{even/odd}} V_\tau), \tag{2-30}$$

thus giving the Landau decomposition

$$L^2(\mathbb{R}^m;\mathbb{C}^{2^m}) = \mathbb{C}[\psi_{0,0}] \oplus \bigoplus_{\tau\in\mathbb{N}_0^m\setminus 0}(E_\tau^{\text{even}} \oplus E_\tau^{\text{odd}}). \tag{2-31}$$

The Dirac operator $D_{\mathbb{R}^m}$ by virtue of (2-27)–(2-29) preserves and acts on E_τ via

$$c\left(\frac{w_{r_\tau} + \bar w_{r_\tau}}{\sqrt{2}}\right) = (w_{r_\tau} \wedge + \iota_{\bar w_{r_\tau}}),$$

under the isomorphism $\mathscr{I}_\tau$, where $r_\tau := (\sqrt{\tau_1\mu_1 h}, \dots, \sqrt{\tau_m\mu_m h})$ and w_{r_τ} is as in (2-15). Hence, if we define $\mathtt{i}_\tau := \mathscr{I}_\tau \mathtt{i}_{r_\tau} \mathscr{I}_\tau^{-1} : E_\tau^{\text{even/odd}} \to E_\tau^{\text{odd/even}}$, we have that the restriction of $D_{\mathbb{R}^m}$ to E_τ is of the form

$$D_{\mathbb{R}^m} = \begin{bmatrix} & |r_\tau|\mathtt{i}_\tau \\ |r_\tau|\mathtt{i}_\tau & \end{bmatrix} \tag{2-32}$$

via (2-19). Also note that since $E_\tau^{\text{even/odd}} \subset \mathscr{I}_\tau(C^\infty(\mathbb{R}^m) \otimes \Lambda^{\text{even/odd}} V^{1,0})$ respectively, one has

$$c(e_0)E_\tau^{\text{even/odd}} = \pm i E_\tau^{\text{even/odd}} \tag{2-33}$$

using (2-6). The eigenspaces for $D_{\mathbb{R}^m}$ are now given by

$$E_\tau^{\pm} = \mathscr{I}_\tau(V_\tau^{\pm}), \tag{2-34}$$

via (2-20) with eigenvalues $\pm|r_\tau| = \pm\sqrt{\mu.\tau h}$ respectively. We now summarize.

Proposition 2.1. *An orthogonal decomposition of $L^2(\mathbb{R}^m; \mathbb{C}^{2^m})$ consisting of eigenspaces of the magnetic Dirac operator $D_{\mathbb{R}^m}$ (2-21) is given by*

$$L^2(\mathbb{R}^m; \mathbb{C}^{2^m}) = \mathbb{C}[\psi_{0,0}] \oplus \bigoplus_{\tau \in \mathbb{N}_0^m \setminus 0} (E_\tau^+ \oplus E_\tau^-).$$

Here $E_\tau^{\pm}$, as in (2-34), have dimension $2^{Z_\tau - 1}$ and correspond to the eigenvalues $\pm\sqrt{\mu.\tau h}$ respectively.

2C. ***The semiclassical calculus.*** Finally, here we review the semiclassical pseudodifferential calculus used throughout the paper, with [Guillemin and Sternberg 2013; Zworski 2012] being the detailed references. Let $\mathfrak{gl}(l)$ denote the space of all $l \times l$ complex matrices. For $A = (a_{ij}) \in \mathfrak{gl}(l)$ we define $|A| = \max_{ij} |a_{ij}|$. Denote by $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^l)$ the space of Schwartz maps $f : \mathbb{R}^n \to \mathbb{C}^l$. We define the symbol space $S^m(\mathbb{R}^{2n}; \mathbb{C}^l)$ as the space of maps $a : (0,1]_h \to C^\infty(\mathbb{R}^{2n}_{x,\xi}; \mathfrak{gl}(l))$ such that each of the seminorms

$$\|a\|_{\alpha,\beta} := \sup\nolimits_{x,\xi,h} \langle\xi\rangle^{-m+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)|$$

is finite for all $\alpha, \beta \in \mathbb{N}_0^n$. Such a symbol is said to lie in the more refined class $a \in S_{\text{cl}}^m(\mathbb{R}^{2n}; \mathbb{C}^l)$ if there exists an h-independent sequence a_k, $k = 0, 1, \dots$ of symbols such that

$$a - \left(\sum_{k=0}^{N} h^k a_k\right) \in h^{N+1} S^m(\mathbb{R}^{2n}; \mathbb{C}^l) \quad \forall N. \tag{2-35}$$

Symbols as above can be Weyl quantized to define one-parameter families of operators $a^W : \mathcal{S}(\mathbb{R}^n; \mathbb{C}^l) \to \mathcal{S}(\mathbb{R}^n; \mathbb{C}^l)$ with Schwartz kernels given by

$$a^W := \frac{1}{(2\pi h)^n} \int e^{i(x-y).\frac{\xi}{h}} a\left(\frac{x+y}{2}, \xi; h\right) d\xi.$$

We denote by $\Psi_{\text{cl}}^m(\mathbb{R}^n; \mathbb{C}^l)$ the class of operators thus obtained by quantizing $S_{\text{cl}}^m(\mathbb{R}^{2n}; \mathbb{C}^l)$. This class of operators is closed under the standard operations of composition and formal-adjoint. Indeed, the Weyl symbols of the composition and adjoint satisfy

$$a^W \circ b^W = (a * b)^W := \big[e^{\frac{ih}{2}(\partial_{r_1}\partial_{s_2} - \partial_{r_2}\partial_{s_1})}(a(s_1, r_1; h) b(s_2, r_2; h))\big]^W_{x=s_1=s_2, \xi=r_1=r_2}, \\ (a^W)^* = (a^*)^W. \tag{2-36}$$

Furthermore the class is invariant under changes of coordinates and basis for $\mathbb{C}^l$. This allows one to define an invariant class of operators $\Psi^m_{\mathrm{cl}}(X; E)$ on $C^\infty(X; E)$ associated to any complex vector bundle on a smooth compact manifold X. These define uniformly in h bounded operators between the Sobolev spaces $H^s(X; E)$ and $H^{s-m}(X; E)$ with the h-dependent norm on each Sobolev space defined via

$$\|u\|_{H^s(X)} := \big\|(1 + h^2 \nabla^{E*}\nabla^E)^{\frac{s}{2}} u\big\|_{L^2}, \quad s \in \mathbb{R},$$

with respect to any metric g^{TX}, h^E on X, E and unitary connection ∇^E.

For $A \in \Psi^m_{\mathrm{cl}}(X; E)$, its principal symbol is well defined as an element in $\sigma(A) \in S^m(X; \mathrm{End}(E)) \subset C^\infty(X; \mathrm{End}(E))$. One has that $\sigma(A) = 0$ if and only if $A \in h\Psi^m_{\mathrm{cl}}(X; E)$. We remark that $\sigma(A)$ is the restriction of standard symbol in [Zworski 2012] to the refined class $\Psi^m_{\mathrm{cl}}(X; E)$ and is locally given by the first coefficient a_0 in the expansion of its Weyl symbol. The principal symbol satisfies the basic relations $\sigma(AB) = \sigma(A)\sigma(B)$ and $\sigma(A^*) = \sigma(A)^*$ with the formal adjoints being defined with respect to the same Hermitian metric h^E. The principal symbol map has an inverse given by the quantization map $\mathrm{Op}: S^m(X; \mathrm{End}(E)) \to \Psi^m_{\mathrm{cl}}(X; E)$ satisfying $\sigma(\mathrm{Op}(a)) = a \in S^m(X; \mathrm{End}(E))$. We often use the alternate notation $\mathrm{Op}(a) = a^W$. For a scalar function $b \in S^m(X)$, it is clear from the multiplicative property of the symbol that $[a^W, b^W] \in h\Psi^m_{\mathrm{cl}}(X; E)$ and we define $H_b(a) := \frac{i}{h}\sigma([a^W, b^W]) \in S^m(X; \mathrm{End}(E))$. If a is self adjoint and b real, then it is easy to see that $H_b(a)$ is self-adjoint. We then define $|H_b(a)| = \max_{\lambda \in \mathrm{Spec}\, H_b(a)} |\lambda|$.

The wavefront set of an operator $A \in \Psi^m_{\mathrm{cl}}(X; E)$ can be defined invariantly as a subset $\mathrm{WF}(A) \subset \overline{T^*X}$ of the fiberwise radial compactification of its cotangent bundle. If the local Weyl symbol of A is given by a then $(x_0, \xi_0) \notin \mathrm{WF}(A)$ if and only if there exists an open neighborhood $(x_0, \xi_0; 0) \in U \subset \overline{T^*X} \times (0, 1]_h$ such that $a \in h^\infty \langle \xi \rangle^{-\infty} C^k(U; \mathbb{C}^l)$ for all k. The wavefront set satisfies the basic properties

$$\mathrm{WF}(A + B) \subset \mathrm{WF}(A) \cup \mathrm{WF}(B), \quad \mathrm{WF}(AB) \subset \mathrm{WF}(A) \cap \mathrm{WF}(B) \quad \text{and} \quad \mathrm{WF}(A^*) = \mathrm{WF}(A).$$

The wavefront set $\mathrm{WF}(A) = \varnothing$ is empty if and only if $A \in h^\infty \Psi^{-\infty}(X; E)$. We say that two operators A and B are equal microlocally on $U \subset \overline{T^*X}$ if $\mathrm{WF}(A - B) \cap U = \varnothing$. We also define by $\Psi^c_{\mathrm{cl}}(X; E)$ the class of pseudodifferential operators A with wavefront set $\mathrm{WF}(A) \Subset T^*X$ compactly contained in the cotangent bundle. It is clear that $\Psi^c_{\mathrm{cl}}(X; E) \subset \Psi^{-\infty}_{\mathrm{cl}}(X; E)$.

An operator $A \in \Psi^m_{\mathrm{cl}}(X; E)$ is said to be elliptic if $\langle \xi \rangle^m \sigma(A)^{-1}$ exists and is uniformly bounded on T^*X. If $A \in \Psi^m_{\mathrm{cl}}(X; E)$, $m > 0$, is formally self-adjoint such that $A + i$ is elliptic then it is essentially self-adjoint (with domain $C^\infty_c(X; E)$) as an unbounded operator on $L^2(X; E)$. Its resolvent $(A - z)^{-1} \in \Psi^{-m}_{\mathrm{cl}}(X; E)$, $z \in \mathbb{C}$, $\mathrm{Im}\, z \neq 0$, now exists and is pseudodifferential by an application of Beals's lemma. The resolvent furthermore has an expansion $(A - z)^{-1} \sim \sum_{j=0}^\infty h^j \,\mathrm{Op}(a^z_j)$ in $\Psi^{-m}_{\mathrm{cl}}(X; E)$. Here each symbol appearing in the expansion has the form

$$a_j^z = (\sigma(A)-z)^{-1}a_{j,1}^z(\sigma(A)-z)^{-1}\cdots(\sigma(A)-z)^{-1}a_{j,2j}^z(\sigma(A)-z)^{-1} \in S^{-m}(X;\mathrm{End}(E)),$$

where $a_{j,k}^z$ is a polynomial in z symbols for $k=1,\dots,2j$. Given a Schwartz function $f\in\mathcal{S}(\mathbb{R})$, the Helffer–Sjöstrand formula now expresses the function $f(A)$ of such an operator in terms of its resolvent and an almost analytic continuation $\tilde{f}$ via

$$f(A)=\frac{1}{2\pi}\int_{\mathbb{C}}\bar{\partial}\tilde{f}(z)(A-z)^{-1}\,dz\,d\bar{z}.$$

Plugging the resolvent expansion into the above formula then shows that the above lies in and has an expansion $f(A)\sim\sum_{j=0}^{\infty}h^jA_j^f$ in $\Psi_{\mathrm{cl}}^{-\infty}(X;E)$. Finally, one defines the classical λ-energy level of A via

$$\Sigma_\lambda^A=\{(x,\xi)\in T^*X \mid \det(\sigma(A)(x,\xi)-\lambda I)=0\}.$$

Now, the form for the coefficients of the resolvent expansion also shows

$$\mathrm{WF}(f(A))\subset\Sigma_{\mathrm{spt}(f)}^A:=\bigcup_{\lambda\in\mathrm{spt}(f)}\Sigma_\lambda^A.$$

2C1. *The class $\Psi_\delta^m(X;E)$.* In Section 3 we shall need the more exotic class of symbols $S_\delta^m(\mathbb{R}^{2n};\mathbb{C})$ defined for each $0<\delta<\frac{1}{2}$. A function $a:(0,1]_h\to C^\infty(\mathbb{R}_{x,\xi}^{2n};\mathbb{C})$ is said to be in this class if and only if

$$\|a\|_{\alpha,\beta}:=\sup_{x,\xi,h}\langle\xi\rangle^{-m+|\beta|}h^{(|\alpha|+|\beta|)\delta}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi;h)| \tag{2-37}$$

is finite for all $\alpha,\beta\in\mathbb{N}_0^n$. This class of operators is closed under the standard operations of composition, adjoint and changes of coordinates allowing the definition of the exotic pseudodifferential algebra $\Psi_\delta^m(X)$ on a compact manifold. The class $S_\delta^m(X)$ is a family of functions $a:(0,1]_h\to C^\infty(T^*X;\mathbb{C})$ satisfying the estimates (2-37) in every coordinate chart and induced trivialization. Such a family can be quantized to $a^W\in\Psi_\delta^m(X)$ satisfying $a^Wb^W=(ab)^W+h^{1-2\delta}\Psi_\delta^{m+m'-1}(X)$ for another $b\in S_\delta^{m'}(X)$. The operators in $\Psi_\delta^0(X)$ are uniformly bounded on $L^2(X)$. Finally, the wavefront of an operator $A\in\Psi_\delta^m(X;E)$ is similarly defined and satisfies the same basic properties as before.

2C2. *Fourier integral operators.* We shall also need the local theory of Fourier integral operators. Let $\kappa:U\to V$ be an exact symplectomorphism between two open subsets $U\subset T^*X$ and $V\subset T^*Y$ inside cotangent spaces of manifolds of same dimension n. Assume that there exist local coordinates $(x_1,\dots,x_n)$,$(y_1,\dots y_n)$ on $\pi(U),\pi(V)$ respectively with induced canonical coordinates (x,ξ), (y,η) on U,V. A function $S(x,\eta)\in C^\infty(\Omega)$ on an open subset $\Omega\subset\mathbb{R}_{x,\eta}^{2n}$ is said to be a generating function for the graph of κ if the Lagrangian submanifolds

$$(T^*X)\times(T^*Y)^-\supset\Lambda_\kappa:=\{((x,\xi);\kappa(x,\xi))\mid(x,\xi)\in U\}\quad\text{and}\quad\{(x,\partial_xS;\partial_\eta S,\eta)\mid(x,\eta)\in\Omega\}$$

are equal. Here $(T^*Y)^-$ denotes the cotangent bundle with the negative canonical symplectic form. A generating function S always exists locally near any point on Λ_κ. Letting $a:(0,1]_h\to C_c^\infty(\Omega\times\pi(V);\mathbb{C})$, which admits an expansion $a(x,y,\eta;h) \sim \sum_{k=0}^{\infty} h^k a_k(x,y,\eta)$, one may now define a Fourier integral operator associated to κ via

$$A : L^2(Y) \to L^2(X),$$
$$(Af)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(S(x,\eta)-y.\eta)} a(x,y,\eta;h) f(y)\, dy\, d\eta.$$

The symbol of $\sigma(A) \in C_c^{\infty}(\Lambda_\kappa;\mathbb{C})$ is defined using the generating function via $\sigma(A)(x,\eta) = a_0(x,\partial_x S,\eta)$. The adjoint A^* is again a Fourier integral operator associated to the symplectomorphism κ^{-1}. The wavefront set of A maybe defined as a subset $\mathrm{WF}(A) \subset \overline{T^*X} \times \overline{T^*Y}$. A point $(x,\xi;y,\eta)$ is not in $\mathrm{WF}(A)$ if and only if there exist pseudodifferential operators $B \in \Psi_{\mathrm{cl}}^{m}(X)$, $C \in \Psi_{\mathrm{cl}}^{m'}(Y)$ with $(x,\xi;y,\eta) \in \mathrm{WF}(B) \times \mathrm{WF}(C)$ such that $\|BAC\|_{H^s(Y) \to H^{s'}(X)} = O(h^\infty)$ for each $s,s' \in \mathbb{R}$. It can be shown that the wavefront set is in fact a compact subset $\mathrm{WF}(A) \subset \Lambda_\kappa$. Given a pseudodifferential operator $B \in \Psi_{\mathrm{cl}}^{m}(X)$, Egorov's theorem says that the composite is a pseudodifferential operator $A^*BA \in \Psi_{\mathrm{cl}}^{m}(Y)$. Moreover its principal symbol is given via $\sigma(A^*BA) = (\kappa^{-1})^*|\sigma(A)|^2\sigma(B) \in C_c^\infty(V)$, where we have again used the identification of V with Λ_κ given by the generating function. Finally one has the wavefront relation $\mathrm{WF}(A^*BA) \subset \mathrm{WF}(A) \cap \mathrm{WF}(B)$, again using the identifications of U, V and Λ_κ.

An important special case arises when $\kappa = e^{tH_g}$ is the time t flow of a Hamiltonian $g \in S^m(T^*X)$. The operator $e^{\frac{it}{h}g^W}$, defined as a unitary operator via Stone's theorem, is now a Fourier integral operator associated to κ. Egorov's theorem now gives that the conjugation $e^{\frac{it}{h}g^W} A e^{-\frac{it}{h}g^W} \in \Psi_{\mathrm{cl}}^{m'}(X)$ is pseudodifferential for each $A \in \Psi_{\mathrm{cl}}^{m'}(X)$ with principal symbol $\sigma(e^{\frac{it}{h}g^W} A e^{-\frac{it}{h}g^W}) = (e^{tH_g})^*\sigma(A)$.

3. First reductions

The trace expansion theorem, Theorem 1.3, will be proved in two steps based on the following two lemmas. Below, τ, T, T', f, θ and D are the same as in Section 1.

Lemma 3.1. *Let $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $\vartheta \in C_c^\infty((T'h^\varepsilon, T); [-1,1])$. Then*

$$\operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)(\mathcal{F}_h^{-1}\vartheta)(\lambda\sqrt{h}-D)\right] = \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h}\check{\vartheta}\left(\frac{\lambda\sqrt{h}-D}{h}\right)\right] = O(h^\infty)$$

for all $\lambda \in \mathbb{R}$.

We note that in the above lemma the function ϑ is allowed to depend on h, while its support and range are contained in h-independent intervals.

Lemma 3.2. *There exist smooth functions $u_j \in C^\infty(\mathbb{R})$ such that for each $\lambda \in \mathbb{R}$ and $\varepsilon \in \left(0, \frac{1}{2}\right)$ one has a trace expansion*

$$\operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)(\mathcal{F}_h^{-1}\theta_\varepsilon)(\lambda\sqrt{h}-D)\right] = \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h^{1-\varepsilon}}\check{\theta}\left(\frac{\lambda\sqrt{h}-D}{h^{1-\varepsilon}}\right)\right] = h^{-m-1}\left(\sum_{j=0}^{N-1} c_j h^{\frac{j}{2}} + O(h^{\frac{N}{2}})\right),$$

where $\theta_\varepsilon(x) := \theta(x/h^\varepsilon)$.

We note that the trace expansion theorem, Theorem 1.3, follows from the above two lemmas by simply splitting

$$\theta(x) = \theta_\varepsilon(x) + \underbrace{[\theta(x) - \theta_\varepsilon(x)]}_{\vartheta(x)}$$

and applying Lemmas 3.2 and 3.1 to the first and second summands respectively. Lemma 3.2 is a relatively classical expansion proved via local index theory and will be deferred to Section 7. Our main occupation until then is in proving Lemma 3.1.

As a first step, for $\tau > 0$ fixed one chooses a microlocal partition of unity $A_\alpha \in \Psi^0_{\mathrm{cl}}(X)$, $0 \le \alpha \le N$, satisfying

$$\sum_{\alpha=0}^{N} A_\alpha = 1, \qquad \mathrm{WF}(A_0) \subset U_0 \subset \overline{T^*X} \setminus \Sigma^D_{(-\tau,\tau)}, \qquad \mathrm{WF}(A_\alpha) \Subset U_\alpha \subset \Sigma^D_{(-2\tau,2\tau)}, \quad 1 \le \alpha \le N, \tag{3-1}$$

subordinate to an open cover $\{U_\alpha\}_{\alpha=0}^N$ of T^*X. Clearly, it suffices to prove

$$\mathrm{tr}\left[A_\alpha f\left(\frac{D}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h} - D}{h}\right)A_\beta\right] = O(h^\infty) \tag{3-2}$$

for $1 \le \alpha, \beta \le N$ with $\mathrm{WF}(A_\alpha) \cap \mathrm{WF}(A_\beta) \ne \varnothing$.

By the Helffer–Sjöstrand formula we have the trace above is given by

$$\mathcal{T}^\vartheta_{\alpha\beta}(D) := \frac{1}{2\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)\check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right)\mathrm{tr}\left[A_\alpha\left(\frac{1}{\sqrt{h}}D - z\right)^{-1}A_\beta\right]dz\,d\bar{z} \tag{3-3}$$

for $\tilde{f}$ an almost analytic extension of f. We note that the resolvent, the above trace, and the left-hand side of (3-2) are well defined for any essentially self-adjoint pseudodifferential operator in place of D. The next reduction step attempts to modify D without affecting the asymptotics of $\mathcal{T}^\vartheta_{\alpha\beta}(D)$. To this end, choose open subsets $U_{\alpha\beta}$, $V_{\alpha\beta}$ such that

$$\begin{aligned} \mathrm{WF}(A_\alpha) \cap \mathrm{WF}(A_\beta) \subset\, & U_{\alpha\beta} \\ & \cap \\ \mathrm{WF}(A_\alpha) \cup \mathrm{WF}(A_\beta) \subset\, & V_{\alpha\beta} \Subset T^*X \end{aligned} \tag{3-4}$$

for each such pair α, β with $\mathrm{WF}(A_\alpha) \cap \mathrm{WF}(A_\beta) \ne \varnothing$. With $d = \sigma(D) \in C^\infty(X; i\mathfrak{u}(S))$, define the required exit time

$$T_{\alpha\beta} := \frac{1}{\inf_{g \in \mathcal{G}_{\alpha\beta}} |H_g d|}, \qquad \text{where } \mathcal{G}_{\alpha\beta} := \left\{g \in C^\infty(T^*X; [0,1]) \;\middle|\; g|_{U_{\alpha\beta}} = 1,\ g|_{V^c_{\alpha\beta}} = 0\right\}. \tag{3-5}$$

If one were to use a scalar symbol $d \in C^\infty(X)$ instead in (3-5), the required exit time $T_{\alpha\beta}$ would have the following significance: any Hamiltonian trajectory $\gamma(t) = e^{tH_d}$ with $\gamma(0) \in U_{\alpha\beta}$ and $\gamma(T) \in V^c_{\alpha\beta}$ would have length $T \ge T_{\alpha\beta}$ at least the required exit time. We now have the following.

Lemma 3.3. *Let $D' \in \Psi^1_{\mathrm{cl}}(X; E)$ be essentially self-adjoint such that $D = D'$ microlocally on $V_{\alpha\beta}$. Then for $\vartheta \in C_c^\infty((T'_{\alpha\beta} h^\varepsilon, T_{\alpha\beta}); [0,1])$, $0 < T'_{\alpha\beta} < T_{\alpha\beta}$, one has*

$$\mathcal{T}^{\vartheta}_{\alpha\beta}(D) = \mathcal{T}^{\vartheta}_{\alpha\beta}(D') \bmod h^\infty.$$

Proof. Let $B \in \Psi^0_{\mathrm{cl}}(X)$ be a microlocal cutoff such that $B = 0$ on $\mathrm{WF}(D - D')$ and $B = 1$ on $V_{\alpha\beta}$. Then $(1 - B)A_\beta = 0$ microlocally implies

$$\begin{aligned}\left(z - \frac{1}{\sqrt{h}}D\right) B \left(z - \frac{1}{\sqrt{h}}D'\right)^{-1} A_\beta \\ = A_\beta - \left[\frac{1}{\sqrt{h}}D, B\right](z - D')^{-1} A_\beta + B\left(\frac{1}{\sqrt{h}}D' - \frac{1}{\sqrt{h}}D\right)\left(z - \frac{1}{\sqrt{h}}D'\right)^{-1} A_\beta \pmod{h^\infty}\end{aligned} \tag{3-6}$$

in trace norm. Next, multiplying through by $A_\alpha\bigl(z - \frac{1}{\sqrt{h}}D\bigr)^{-1}$and using $A_\alpha B = A_\alpha$ microlocally gives

$$\begin{aligned}A_\alpha\left(z-\frac{1}{\sqrt{h}}D'\right)^{-1} A_\beta - A_\alpha\left(z-\frac{1}{\sqrt{h}}D\right)^{-1} A_\beta = A_\alpha\left(z-\frac{1}{\sqrt{h}}D\right)^{-1} B\left(\frac{1}{\sqrt{h}}D'-\frac{1}{\sqrt{h}}D\right)\left(z-\frac{1}{\sqrt{h}}D'\right)^{-1} A_\beta \\ -A_\alpha\left(z-\frac{1}{\sqrt{h}}D\right)^{-1}\left[\frac{1}{\sqrt{h}}D, B\right]\left(z-\frac{1}{\sqrt{h}}D'\right)^{-1} A_\beta \\ +O(|\mathrm{Im}\, z|^{-1}h^\infty)\end{aligned} \tag{3-7}$$

in trace norm. Now $B = 0$ on $\mathrm{WF}(D - D')$ gives that the first term on the right-hand side above is $O(|\mathrm{Im}\, z|^{-2}h^\infty)$.

We now estimate the second term. Let $S_{\alpha\beta} < S''_{\alpha\beta} < S'''_{\alpha\beta} < T_{\alpha\beta}$ and $S'_{\alpha\beta} > T'_{\alpha\beta}$ be such that $\vartheta \in C_c^\infty([S'_{\alpha\beta}h^\varepsilon, S_{\alpha\beta}]; [0,1])$. Let $g_0 \in \mathcal{G}_{\alpha\beta}$ with $|H_{g_0}(d)| \le 1/S'''_{\alpha\beta}$. Set $g = \alpha_z g_0$, where

$$\alpha_z = \min\left(\frac{S''_{\alpha\beta}\,\mathrm{Im}\, z}{\sqrt{h}\log(1/h)}, N\right),$$

with the constant $N > 0$ to be specified later. We note that

$$G = (e^{g \log \frac{1}{h}})^W \in h^{-N}\Psi^0_\delta(X)$$

for each $0 < \delta < \frac{1}{2}$. Since it has an elliptic symbol, we may construct its inverse by symbolic calculus $G^{-1} \in h^N \Psi^0_\delta(X)$. Moreover

$$G\left(z - \frac{1}{\sqrt{h}}D_h\right)G^{-1} = \left(z - \frac{1}{\sqrt{h}}D_h\right) + i\left(\alpha_z\sqrt{h}\log\frac{1}{h}\right)(H_{g_0}(d))^W, \tag{3-8}$$

with

$$R = O\left(h^{\frac{3}{2}}\alpha_z \log\frac{1}{h}\right) \quad \text{in } S^0_\delta(X). \tag{3-9}$$

Now, since

$$\left|\left(\alpha_z\sqrt{h}\log\frac{1}{h}\right)H_{g_0}(d)\right| \le \frac{S''_{\alpha\beta}}{S'''_{\alpha\beta}}|\mathrm{Im}\, z| < |\mathrm{Im}\, z|,$$

the inverse $G\big(z-\frac{1}{\sqrt{h}}D_h\big)^{-1}G^{-1}$ of the above exists and is $O(|\operatorname{Im} z|^{-1})$ in operator norm for $\operatorname{Im} z \neq 0$ and h sufficiently small.

Next, pick $C \in \Psi^0_{\mathrm{cl}}(X)$ such that $\mathrm{WF}(C) \subset U_{\alpha\beta}$ and $C=1$ on $\mathrm{WF}(A_\alpha) \cap \mathrm{WF}(A_\beta)$. Now $G = e^{\alpha_z \log \frac{1}{h}}$ on $\mathrm{WF}(CA_\alpha)$, $G = G^{-1} = I$ on $\mathrm{WF}(B) \setminus V_{\alpha\beta}$ and $[D_h, B] = 0$ on $V_{\alpha\beta}$ imply

$$e^{\alpha_z \log \frac{1}{h}} CA_\alpha \left(z-\frac{1}{\sqrt{h}}D_h\right)^{-1}\left[\frac{1}{\sqrt{h}}D_h, B\right] = CA_\alpha G\left(z-\frac{1}{\sqrt{h}}D_h\right)^{-1} G^{-1}\left[\frac{1}{\sqrt{h}}D_h, B\right] + O(|\operatorname{Im} z|^{-1}h^\infty)$$

in trace norm. The above is now $O(|\operatorname{Im} z|^{-1}h^{-n})$ in trace norm. Hence

$$CA_\alpha\left(z-\frac{1}{\sqrt{h}}D_h\right)^{-1}\left[\frac{1}{\sqrt{h}}D_h, B\right] = O\big(|\operatorname{Im} z|^{-1}h^{-n}\max(h^N, e^{-\frac{S''_{\alpha\beta}\operatorname{Im} z}{\sqrt{h}}})\big)$$

in trace norm. This and $CA_\alpha A_\beta = A_\alpha A_\beta$ now estimate the second term of (3-7) to give

$$A_\alpha\left(z-\frac{1}{\sqrt{h}}D'_h\right)^{-1}A_\beta - A_\alpha\left(z-\frac{1}{\sqrt{h}}D_h\right)^{-1}A_\beta = O\big(|\operatorname{Im} z|^{-2}h^{-n}\max(h^N, e^{-\frac{S''_{\alpha\beta}\operatorname{Im} z}{\sqrt{h}}})\big) \tag{3-10}$$

in trace norm.

Next, we have the Paley–Wiener estimate

$$\check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right) = \begin{cases} O(e^{\frac{S_{\alpha\beta}(\operatorname{Im} z)}{\sqrt{h}}}), & \operatorname{Im} z > 0, \\ O(e^{\frac{S'_{\alpha\beta}(\operatorname{Im} z)}{h^{1/2-\varepsilon}}}), & \operatorname{Im} z < 0. \end{cases} \tag{3-11}$$

Introduce $\psi \in C^\infty(\mathbb{R}; [0,1])$ such that

$$\psi(x) = \begin{cases} 1, & x \le 1, \\ 0, & x \ge 2. \end{cases}$$

Setting

$$\psi_M(z) = \psi\left(\frac{\operatorname{Im} z}{M\sqrt{h}\log(1/h)}\right)$$

for another constant $M > 1$ yet to be chosen, we have the estimate

$$\bar{\partial}(\psi_M \tilde{f}) = \begin{cases} O\left(\psi_M|\operatorname{Im} z|^N + \dfrac{1}{M\sqrt{h}\log(1/h)}1_{[1,2]}\left(\dfrac{\operatorname{Im} z}{M\sqrt{h}\log(1/h)}\right)\right), & \operatorname{Im} z > 0, \\ O(|\operatorname{Im} z|^N), & \operatorname{Im} z < 0. \end{cases} \tag{3-12}$$

Finally, (3-10)–(3-12), along with the observation

$$\psi_M|\operatorname{Im} z|^N = O\left(\left(M\sqrt{h}\log\frac{1}{h}\right)^N\right),$$

give

$$
\begin{aligned}
&\mathcal{T}^{\vartheta}_{\alpha\beta}(D') - \mathcal{T}^{\vartheta}_{\alpha\beta}(D) \\
&\quad = \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}(\psi_M \tilde{f})\check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right)\left[A_\alpha\left(z - \frac{1}{\sqrt{h}}D'_h\right)^{-1}A_\beta - A_\alpha\left(z - \frac{1}{\sqrt{h}}D_h\right)^{-1}A_\beta\right]dz\,d\bar{z} \\
&\quad = O(h^\infty) + O\left[\int_{\{M\sqrt{h}\log\frac{1}{h} \le \operatorname{Im} z \le 2M\sqrt{h}\log\frac{1}{h}\}} \frac{h^{-n}}{\sqrt{h}\log\frac{1}{h}} \max\left(h^N e^{\frac{S_{\alpha\beta}(\operatorname{Im} z)}{\sqrt{h}}}, e^{-\frac{(S''_{\alpha\beta} - S_{\alpha\beta})\operatorname{Im} z}{\sqrt{h}}}\right)\right] \\
&\quad = O\left[\max\left(h^{N - 2MS_{\alpha\beta} - n}, h^{M(S''_{\alpha\beta} - S_{\alpha\beta}) - n}\right)\right].
\end{aligned}
$$

Choosing $M \gg n/(S''_{\alpha\beta} - S_{\alpha\beta})$ and furthermore $N \gg 2MS_{\alpha\beta} + n$ gives the result. □

In the proof above we have closely followed [Dimassi and Sjöstrand 1999, Lemma 12.7]. Again, the proof above avoids the use of an unknown parametrix for $e^{\frac{it}{h}D}$ which, following the significance of the required exit time $T_{\alpha\beta}$ noted before, maybe used to give an alternate proof in the case when d is scalar.

4. Reduction to $\mathbb{R}^n$

In this section we shall further reduce to the case of a Dirac operator on $\mathbb{R}^n$. First we cover X by a finite set of Darboux charts $\{\varphi_s : \Omega_s \to \Omega_s^0 \subset \mathbb{R}^n\}_{s\in S}$ for the contact form a, centered at points $\{x_s\}_{s\in S} \in X$. By shrinking the partition of unity (3-1) we may assume that for each pair α, β, with $\mathrm{WF}(A_\alpha) \cap \mathrm{WF}(A_\beta) \neq \varnothing$, the open sets $V_{\alpha\beta} \subset T^*\Omega_s$ in (3-4) are contained in some Darboux chart. Now consider such a chart Ω_s with coordinates$(x_0, \ldots, x_{2m})$ centered at $x_s \in X$ and an orthonormal frame $\{e_j = w_j^k \partial_{x_k}\}$, $0 \le j \le 2m$, for the tangent bundle on Ω_s. We hence have

$$w_j^k g_{kl} w_r^l = \delta_{jr}, \tag{4-1}$$

where g_{kl} is the metric in these coordinates and the Einstein summation convention is being used. Let Γ^l_{jk} be the Christoffel symbols for the Levi-Civita connection in the orthonormal frame e_i satisfying $\nabla_{e_j} e_k = \Gamma^l_{jk} e_l$. This orthonormal frame induces an orthonormal frame u_q, $1 \le q \le 2^m$, for the spin bundle S. We further choose a local orthonormal section $\mathtt{1}(x)$ for the Hermitian line bundle L and define via $\nabla^{A_0}_{e_j}\mathtt{1} = \Upsilon_j(x)\mathtt{1}$, $0 \le j \le 2m$, the Christoffel symbols of the unitary connection A_0 on L. In terms of the induced frame $u_q \otimes \mathtt{1}$, $1 \le q \le 2^m$, for $S \otimes L$ the Dirac operator (1-2) has the form [Berline et al. 2004, Section 3.3]

$$D = \gamma^j w_j^k P_k + h\left(\tfrac{1}{4}\Gamma^l_{jk}\gamma^j\gamma^k\gamma_l + \Upsilon_j\gamma^j\right), \tag{4-2}$$

where

$$P_k = h\partial_{x_k} + i a_k, \tag{4-3}$$

and

$$a(x) = a_k\,dx^k = dx_0 + \sum_{j=1}^{m}(x_j\,dx_{j+m} - x_{j+m}\,dx_j) \tag{4-4}$$

is the standard contact one-form in these coordinates.

The expression in (4-2) is formally self-adjoint with respect to the Riemannian density $e^0\wedge\cdots\wedge e^{2m} = \sqrt{g}\,dx := \sqrt{g}\,dx^0\wedge\cdots\wedge dx^{2m}$ with $g=\det(g_{ij})$. To get an operator self-adjoint with respect to the Euclidean density dx, one expresses the Dirac operator in the framing $g^{\frac14}u_q\otimes 1$, $1\le q\le 2^m$. In this new frame the expression (4-2) for the Dirac operator needs to be conjugated by $g^{\frac14}$ and hence the term $h\gamma^j w_j^k g^{-\frac14}(\partial_{x_k}g^{\frac14})$ needs to be added. Hence, the Dirac operator in the new frame has the form

$$D=[\sigma^j w_j^k(\xi_k+a_k)]^W+hE\in\Psi^1_{\mathrm{cl}}(\Omega_s^0;\mathbb{C}^{2^m}),$$

with $\sigma^j=i\gamma^j$, for some self-adjoint endomorphism $E(x)\in C^\infty(\Omega_s^0;i\mathfrak{u}(\mathbb{C}^{2^m}))$.

The one-form a is extended to all of $\mathbb{R}^n$ by the same formula (4-4). The functions w_j^k are extended such that

$$(w_j^k\partial_{x_k}\otimes dx^j)|_{(K_s^0)^c}=\partial_{x_0}\otimes dx^0+\sum_{j=1}^m\mu_j^{\frac12}(\partial_{x_j}\otimes dx^j+\partial_{x_{j+m}}\otimes dx^{j+m})$$

$\big($and hence $g|_{(K_s^0)^c}=dx_0^2+\sum_{j=1}^m\mu_j(dx_j^2+dx_{j+m}^2)\big)$ outside a compact neighborhood $\Omega_s^0\Subset K_s^0$. These extensions may further be chosen such that the suitability assumption Definition 1.1 holds globally on $\mathbb{R}^n$ and for an extended positive function $\nu\in C_c^\infty(\mathbb{R}^n)$ satisfying

$$\nu_0\le\mu_1(\inf_{\mathbb{R}^n}\nu).\tag{4-5}$$

The endomorphism $E(x)\in C_c^\infty(\mathbb{R}^n;i\mathfrak{u}(\mathbb{C}^{2^m}))$ is extended to an arbitrary self-adjoint endomorphism of compact support. This now gives

$$D_0=[\sigma^j w_j^k(\xi_k+a_k)]^W+hE\in\Psi^1_{\mathrm{cl}}(\mathbb{R}^n;\mathbb{C}^{2^m}),\tag{4-6}$$

as a well defined formally self adjoint operator on $\mathbb{R}^n$. Furthermore, the symbol of D_0+i is elliptic in the class $S^0(m)$ for the order function

$$m=\Bigg(1+\sum_{k=0}^{2m}(\xi_k+a_k)^2\Bigg)^{\frac12},$$

and hence D_0 is essentially self adjoint; see [Dimassi and Sjöstrand 1999, Chapter 8]. Below $\vartheta\in C_c^\infty((T'_{\alpha\beta}h^\varepsilon,T_{\alpha\beta});[0,1])$, $0<T'_{\alpha\beta}<T_{\alpha\beta}$, as before and we set $V^0_{\alpha\beta}:=(d\varphi_s)^*V_{\alpha\beta}\subset T^*\Omega_s^0$.

Proposition 4.1. *There exist $A^0_\alpha, A^0_\beta\in\Psi^0_{\mathrm{cl}}(\mathbb{R}^n)$, with $\mathrm{WF}(A^0_\alpha)\cup\mathrm{WF}(A^0_\beta)\Subset V^0_{\alpha\beta}\subset T^*\tilde{\Omega}_s$, such that*

$$\mathcal{T}^\vartheta_{\alpha\beta}(D)=\underbrace{\mathrm{tr}\bigg[A^0_\alpha f\bigg(\frac{D_0}{\sqrt{h}}\bigg)\check{\vartheta}\bigg(\frac{\lambda\sqrt{h}-D_0}{h}\bigg)A^0_\beta\bigg]}_{:=\mathcal{T}^\vartheta_{\alpha\beta}(D_0)}\mod h^\infty.$$

Proof. Let $K'_{\alpha\beta}, K''_{\alpha\beta}$ and $V'_{\alpha\beta}, V''_{\alpha\beta}$ be compact and open subsets respectively satisfying $V_{\alpha\beta}\subset K'_{\alpha\beta}\subset V'_{\alpha\beta}\subset K''_{\alpha\beta}\subset V''_{\alpha\beta}\subset T^*\Omega_s$. Choose $D'\in\Psi^0_{\mathrm{cl}}(X;S)$ self-adjoint such that $D=D'$ microlocally on $K'_{\alpha\beta}$ and

$$\Sigma^{D'}_{(-\infty,2\tau]}\subset V'_{\alpha\beta}\tag{4-7}$$

and set $E = D' - 3\tau \in \Psi^0_{\mathrm{cl}}(X; S)$. Pick a cutoff function $\chi(x; y, \eta) \in C^\infty_c\big(\pi(V''_{\alpha\beta}) \times (d\varphi_s)^* V''_{\alpha\beta}; [0, 1]\big)$ such that $\chi = 1$ on $\pi(K''_{\alpha\beta}) \times (d\varphi_s)^* K''_{\alpha\beta}$. Now define the operator

$$U : L^2(\mathbb{R}^n; \mathbb{C}^{2^m}) \to L^2(X; S),$$
$$(Uf)(x) = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(\varphi_s(x)-y).\eta} \chi(x; y, \eta) f(y)\, dy\, d\eta, \quad x \in X.$$

The above is a semiclassical Fourier integral operator associated to symplectomorphism $\kappa = (d\varphi_s^{-1})^*$ given by the canonical coordinates. Its adjoint $U^* : L^2(X; S) \to L^2(\mathbb{R}^n; \mathbb{C}^{2^m})$ is again a semiclassical Fourier integral operator associated to the symplectomorphism $\kappa^{-1} = (d\varphi_s)^*$. A simple computation gives the following compositions are pseudodifferential with

$$UU^* = I \quad \text{microlocally on } K''_{\alpha\beta}, \tag{4-8}$$
$$U^*U = I \quad \text{microlocally on } \kappa(K''_{\alpha\beta}). \tag{4-9}$$

The composition

$$E' = E_0 := U^* E U \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^n; \mathbb{C}^{2^m})$$

is now a pseudodifferential operator by Egorov's theorem with symbol

$$\sigma(E_0) = (d\varphi_s)^* \chi^2.\sigma(E). \tag{4-10}$$

Similarly, $E'_0 := U E_0 U^* \in \Psi^0_{\mathrm{cl}}(X; S)$ and

$$\sigma(E'_0) = (d\varphi_s)^* \chi^4.\sigma(E_0). \tag{4-11}$$

By (4-7), (4-10) and (4-11) we have $\Sigma^{E_0}_{(-\infty,-\tau]} \subset \kappa(V'_{\alpha\beta})$ and $\Sigma^{E'_0}_{(-\infty,-\tau]} \subset V'_{\alpha\beta}$. Hence by Proposition A.6, E, E', E_0 and E'_0 all have discrete spectrum in $(-\infty, -\tau]$. We now select $g \in C^\infty_c(-5\tau, -\tau)$ such that $g = 1$ on $[-4\tau, -2\tau]$. We have

$$\mathrm{WF}(g(E)) \subset \Sigma^E_{\mathrm{spt}(g)} \subset \Sigma^E_{(-\infty,-\tau]} \subset V'_{\alpha\beta}.$$

Combined with (4-9) this gives $(U^*U - I)g(E) \in h^\infty \Psi^{-\infty}_{\mathrm{cl}}(X; S)$ and hence $\|(U^*U - I)g(E)\| = O(h^\infty)$ as an operator on $L^2(X; S)$. This in turn now gives

$$\big\|(U^*U - I)\Pi^E\big\|(\|E\|\,\|U\| + 1) = O(h^\infty), \tag{4-12}$$

with $\Pi^E = \Pi^E_{[-4\tau,-2\tau]}$ denoting the spectral projector of E onto the interval $[-4\tau, -2\tau]$. Similarly, we get

$$\big\|(UU^* - I)\Pi^{E_0}\big\|(\|E_0\|\,\|U^*\| + 1) = O(h^\infty). \tag{4-13}$$

Another easy computation gives $E = E'_0$ microlocally on $K''_{\alpha\beta}$ and we may similarly estimate

$$\big\|(E - E'_0)\Pi^{E'_0}\big\| = O(h^\infty). \tag{4-14}$$

Next we define $A^0_\alpha := U^* A_\alpha U,\ A^0_\beta := U^* A_\beta U \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^n)$ and again note

$$\begin{aligned} UA^0_\alpha A^0_\beta U^* &= A_\alpha A_\beta \quad \text{microlocally on } K''_{\alpha\beta}, \\ U^* A_\alpha A_\beta U &= A^0_\alpha A^0_\beta \quad \text{microlocally on } \kappa(K''_{\alpha\beta}). \end{aligned} \tag{4-15}$$

This again gives

$$\big\| [UA^0_\alpha A^0_\beta U^* - A_\alpha A_\beta]\Pi^E \big\| = O(h^\infty), \tag{4-16}$$

$$\big\| [U^* A_\alpha A_\beta U - A^0_\alpha A^0_\beta]\Pi^{E_0} \big\| = O(h^\infty). \tag{4-17}$$

Now using (4-12), (4-13), (4-14), (4-16), (4-17) and using the cyclicity of the trace we may apply Proposition A.5 of the Appendix with

$$\rho(x) = f\left(\frac{x+3\tau}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-3\tau-x}{h}\right)$$

to get

$$\operatorname{tr}\left[A_\alpha f\left(\frac{D'}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-D'}{h}\right)A_\beta\right] - \operatorname{tr}\left[A^0_\alpha f\left(\frac{D'_0}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-D'_0}{h}\right)A^0_\beta\right] = O(h^\infty)$$

for $D'_0 := E_0 + 3\tau$. Finally observing $D = D'$ on $V_{\alpha\beta}$, $D_0 = D'_0$ on $V^0_{\alpha\beta}$ and using Lemma 3.3 completes the proof. □

5. Birkhoff normal form for the Dirac operator

In this section we derive a Birkhoff normal form for the Dirac operator (4-6) on $\mathbb{R}^n$. First consider the function

$$f_0 := \left(x_0\xi_0 - \frac{x_0}{(\sqrt{2}-1)}\right)\frac{\ln 4}{\pi} + \sum_{j=1}^{m}(x_j x_{j+m} + \xi_j \xi_{j+m}).$$

If H_{f_0} and $e^{tH_{f_0}}$ denote the Hamilton vector field and time t flow of f_0 respectively then it is easy to compute

$$\begin{aligned} e^{\frac{\pi}{4}H_{f_0}}(x_0,\xi_0) &= \left(\sqrt{2}x_0, \frac{\xi_0+1}{\sqrt{2}}\right), \\ e^{\frac{\pi}{4}H_{f_0}}(x_j,\xi_j;x_{j+m},\xi_{j+m}) &= \left(\frac{x_j+\xi_{j+m}}{\sqrt{2}}, \frac{-x_{j+m}+\xi_j}{\sqrt{2}}; \frac{x_{j+m}+\xi_j}{\sqrt{2}}, \frac{-x_j+\xi_{j+m}}{\sqrt{2}}\right). \end{aligned}$$

We abbreviate $(x',\xi') = (x_1,\dots,x_m;\xi_1,\dots,\xi_m)$, $(x'',\xi'') = (x_{m+1},\dots,x_{2m};\xi_{m+1},\dots,\xi_{2m})$ and $(x,\xi) = (x_0,x',x'';\xi_0,\xi',\xi'')$. Further, let $o_N \subset S^1_{\mathrm{cl}}(\mathbb{R}^{2n};\mathbb{C}^l)$ denote the subspace of self-adjoint symbols $a : (0,1]_h \to C^\infty(\mathbb{R}^{2n}_{x,\xi}; i\mathfrak{u}(2^m))$ such that each of the coefficients a_k, $k = 0,1,2,\dots,$ in its symbolic expansion (2-35) vanishes to order N in (ξ_0,x',ξ') at 0. We also denote by o_N the space of Weyl quantizations of such symbols.

Using Egorov's theorem, the operator (4-6) is conjugated to

$$e^{\frac{i\pi}{4h}f_0^W} D_0 e^{-\frac{i\pi}{4h}f_0^W} = d_0^W, \tag{5-1}$$

with

$$d_0 = \sqrt{2}(\sigma^j w^0_{j,f_0}\xi_0 + \sigma^j w^k_{j,f_0}\xi_k + \sigma^j w^{k+m}_{j,f_0} x_k) + ho_0, \tag{5-2}$$

where

$$w^k_{j,f_0} = (e^{-\frac{\pi}{4}H_{f_0}})^* w^k_j. \tag{5-3}$$

Note that the index k ranges from 1 to m in the Einstein summation above. A Taylor expansion of d_0, given in (5-2), in (ξ_0, x', ξ') now gives $r^0_j \in o_2,\ 0 \le j \le 2m$, such that

$$d_0 = \sqrt{2}\sigma^j(\bar{w}^0_j\xi_0 + \bar{w}^k_j\xi_k + \bar{w}^{k+m}_j x_k) + \sigma^j r^0_j + ho_0$$

and where $\bar{w}^k_j(x_0, x'', \xi'') = w^k_j\big(x_0, -\frac{\xi''}{\sqrt{2}}, \frac{x''}{\sqrt{2}}\big)$. On squaring using (4-1) we obtain

$$(d_0^W)^2 = Q_0^W + ho_1 + o_3 + h^2 o_0,$$

with

$$Q_0 = \begin{bmatrix} x' & \xi_0 & \xi' \end{bmatrix} \begin{bmatrix} \bar{g}^{(k+m)(l+m)}(x_0, x'', \xi'') & \bar{g}^{(k+m)0}(x_0, x'', \xi'') & \bar{g}^{(k+m)l}(x_0, x'', \xi'') \\ \bar{g}^{0(l+m)}(x_0, x'', \xi'') & \bar{g}^{00}(x_0, x'', \xi'') & \bar{g}^{0l}(x_0, x'', \xi'') \\ \bar{g}^{k(l+m)}(x_0, x'', \xi'') & \bar{g}^{k0}(x_0, x'', \xi'') & \bar{g}^{kl}(x_0, x'', \xi'') \end{bmatrix} \begin{bmatrix} x' \\ \xi_0 \\ \xi' \end{bmatrix}.$$

Here $\bar{g}^{kl}(x_0, x'', \xi'') = 2g^{kl}\big(x_0, -\frac{\xi''}{\sqrt{2}}, \frac{x''}{\sqrt{2}}\big)$ and the g^{kl} are the components of the inverse metric on $T^*\mathbb{R}^n$.

Next we consider another function f_1 of the form

$$f_1 = \tfrac{1}{2}\begin{bmatrix} x' & \xi_0 & \xi' \end{bmatrix} \begin{bmatrix} \alpha_{m\times m}(x_0, x'', \xi'') & \gamma_{m\times m+1}(x_0, x'', \xi'') \\ \gamma^t_{m+1\times m}(x_0, x'', \xi'') & \beta_{m+1\times m+1}(x_0, x'', \xi'') \end{bmatrix} \begin{bmatrix} x' \\ \xi_0 \\ \xi' \end{bmatrix},$$

where α, β and γ are matrix-valued functions of the given orders, with α, β symmetric. An easy computation now shows

$$(e^{H_{f_1}})^* \begin{bmatrix} x' \\ \xi_0 \\ \xi' \end{bmatrix} = e^{\Lambda} \begin{bmatrix} x' \\ \xi_0 \\ \xi' \end{bmatrix} + o_2,$$

with

$$\Lambda(x_0, x'', \xi'') = \begin{bmatrix} 0 & -I_{m+1\times m+1} \\ I_{m\times m} & 0 \end{bmatrix} \begin{bmatrix} \alpha_{m\times m}(x_0, x'', \xi'') & \gamma_{m\times m+1}(x_0, x'', \xi'') \\ \gamma^t_{m+1\times m}(x_0, x'', \xi'') & \beta_{m+1\times m+1}(x_0, x'', \xi'') \end{bmatrix}.$$

From the suitability assumption (1-3), we have that there exist smooth matrix-valued functions α, β and γ such that

$$\begin{aligned} &\begin{bmatrix} x' & \xi_0 & \xi' \end{bmatrix} e^{\Lambda^t} \begin{bmatrix} \bar{g}^{(k+m)(l+m)}(x_0, x'', \xi'') & \bar{g}^{(k+m)0}(x_0, x'', \xi'') & \bar{g}^{(k+m)l}(x_0, x'', \xi'') \\ \bar{g}^{0(l+m)}(x_0, x'', \xi'') & \bar{g}^{00}(x_0, x'', \xi'') & \bar{g}^{0l}(x_0, x'', \xi'') \\ \bar{g}^{k(l+m)}(x_0, x'', \xi'') & \bar{g}^{k0}(x_0, x'', \xi'') & \bar{g}^{kl}(x_0, x'', \xi'') \end{bmatrix} e^{\Lambda} \begin{bmatrix} x' \\ \xi_0 \\ \xi' \end{bmatrix} \\ &\qquad = \xi_0^2 + \bar{\nu}\Bigg[\sum_{j=1}^m \mu_j(x_j^2 + \xi_j^2)\Bigg] + o_3, \end{aligned}$$

where

$$\bar{\nu}(x_0, x'', \xi'') = \nu\left(x_0, -\frac{\xi''}{\sqrt{2}}, \frac{x''}{\sqrt{2}}\right). \tag{5-4}$$

Letting

$$H_2 = \frac{1}{2}\sum_{j=1}^{m} \mu_j (x_j^2 + \xi_j^2),$$

Egorov's theorem now gives

$$e^{\frac{i}{h} f_1^W} d_0^W e^{-\frac{i}{h} f_1^W} = \left(\sum_{j=0}^{2m} \sigma_j b_j\right)^W + h o_0, \tag{5-5}$$

with

$$\sum_{j=0}^{2m} b_j^2 = (\xi_0^2 + 2\bar{\nu} H_2)^W + o_3.$$

Another Taylor expansion in the variables (x', ξ_0, ξ') gives $A = (a_{jk}(x_0, x'', \xi'')) \in C^\infty(\mathbb{R}^n_{(x_0, x'', \xi'')}; \mathfrak{so}(n))$ and $r_j \in o_2$, $j = 0, \dots, 2m$, such that

$$e^{-A}\begin{bmatrix} b_0 \\ \vdots \\ b_{2m} \end{bmatrix} = \begin{bmatrix} \xi_0 \\ (2\bar{\nu}\mu_1)^{\frac{1}{2}} x_1 \\ (2\bar{\nu}\mu_1)^{\frac{1}{2}} \xi_1 \\ \vdots \\ (2\bar{\nu}\mu_m)^{\frac{1}{2}} x_m \\ (2\bar{\nu}\mu_m)^{\frac{1}{2}} \xi_m \end{bmatrix} + \begin{bmatrix} r_0 \\ \vdots \\ r_{2m} \end{bmatrix}.$$

We may now set $c_A = \frac{1}{i} a_{jk} \sigma^j \sigma^k \in C^\infty(\mathbb{R}^n_{(x_0, x'', \xi'')}; i\,\mathfrak{u}(2^m))$ and compute

$$e^{i c_A^W} e^{\frac{i}{h} f_1^W} d_0^W e^{-\frac{i}{h} f_1^W} e^{-i c_A^W} = d_1^W, \tag{5-6}$$

where

$$d_1 = H_1 + \sigma^j r_j + h o_0, \tag{5-7}$$

$$H_1 := \xi_0 \sigma_0 + (2\bar{\nu})^{\frac{1}{2}} \sum_{j=1}^{m} \mu_j^{\frac{1}{2}} (x_j \sigma_{2j-1} + \xi_j \sigma_{2j}). \tag{5-8}$$

5A. ***Weyl product and Koszul complexes.*** We now derive a formal Birkhoff normal form for the symbol d_1 in (5-7). First denote by $R = C^\infty(x_0, x'', \xi'')$ the ring of real-valued functions in the given $2m+1$ variables. Further define

$$S := R[\![x', \xi_0, \xi'; h]\!],$$

the ring of formal power series in the further given $2m+2$ variables with coefficients in R. The ring $S \otimes \mathbb{C}$ is now equipped with the Weyl product

$$a * b := \left[e^{\frac{ih}{2}(\partial_{r_1}\partial_{s_2} - \partial_{r_2}\partial_{s_1})}(a(s_1, r_1; h) b(s_2, r_2; h))\right]_{x = s_1 = s_2, \xi = r_1 = r_2},$$

corresponding to the composition formula (2-36) for pseudodifferential operators, with

$$[a,b] := a * b - b * a$$

being the corresponding Weyl bracket. It is an easy exercise to show that for $a, b \in S$ real-valued, the commutator $i[a,b] \in S$ is real-valued.

Next, we define a filtration on S. Each monomial $h^k \xi_0^a (x')^\alpha (\xi')^\beta$ in S is given the weight $2k + a + |\alpha| + |\beta|$. The ring S is equipped with a decreasing filtration

$$S = O_0 \supset O_1 \supset \cdots \supset O_N \supset \ldots, \qquad \bigcap_N O_N = \{0\},$$

where O_N consists of those power series with monomials of weight N or more. It is an exercise to show that

$$O_N * O_M \subset O_{N+M},$$
$$[O_N, O_M] \subset ihO_{N+M-2}.$$

The associated grading is given by

$$S = \bigoplus_{N=0}^{\infty} S_N,$$

where S_N consists of those power series with monomials of weight exactly N. We also define the quotient ring $D_N := S/O_{N+1}$ whose elements may be identified with the set of homogeneous polynomials with monomials of weight at most N. The ring D_N is also similarly graded and filtered. In a similar vein, we may also define the ring

$$S(m) = S \otimes \mathfrak{gl}_{\mathbb{C}}(2^m)$$

of $R \otimes \mathfrak{gl}_{\mathbb{C}}(2^m)$-valued formal power series in $(x', \xi_0, \xi'; h)$. The ring $S(m)$ is equipped with an induced product $*$ and decreasing filtration

$$O_0(m) \supset O_1(m) \supset \cdots \supset O_N(m) \supset \cdots, \qquad \bigcap_N O_N(m) = \{0\},$$

where $O_N(m) = O_N \otimes \mathfrak{gl}_{\mathbb{C}}(2^m)$. It is again a straightforward exercise to show that for $a, b \in S \otimes i\mathfrak{u}_{\mathbb{C}}(2^m)$ self-adjoint, the commutator $i[a,b] \in S \otimes i\mathfrak{u}_{\mathbb{C}}(2^m)$ is self-adjoint.

5A1. *Koszul complexes.* Let us now again consider the $2m$ and $(2m+1)$-dimensional real inner product spaces $V = \mathbb{R}[e_1, \ldots, e_{2m}]$ and $W = \mathbb{R}[e_0] \oplus V$ from Section 2B. Considering the chain groups $D_N \otimes \Lambda^k V$, $k = 0, 1, \ldots, n$, one may define four differentials

$$w_x^0 = \sum_{j=1}^m \mu_j^{\frac{1}{2}} (x_j e_{2j-1} \wedge + \xi_j e_{2j} \wedge), \qquad i_x^0 = \sum_{j=1}^m \mu_j^{\frac{1}{2}} (x_j i_{e_{2j-1}} + \xi_j i_{e_{2j}}),$$
$$w_\partial^0 = \sum_{j=1}^m \mu_j^{\frac{1}{2}} (\partial_{x_j} e_{2j-1} \wedge + \partial_{\xi_j} e_{2j} \wedge), \quad i_\partial^0 = \sum_{j=1}^m \mu_j^{\frac{1}{2}} (\partial_{x_j} i_{e_{2j-1}} + \partial_{\xi_j} i_{e_{2j}}).$$

We equip D_N with the $R[\![h]\!]$-valued inner products, where the distinct monomials

$$\frac{1}{\sqrt{a!\alpha!\beta!}}\xi_0^a (x')^\alpha (\xi')^\beta$$

are orthonormal. With these inner products w_x^0, i_∂^0 and w_∂^0, i_x^0 are respectively adjoints. The combinatorial Laplacians $\Delta^0 = w_x^0 i_\partial^0 + i_\partial^0 w_x^0 = w_\partial^0 i_x^0 + i_x^0 w_\partial^0$ are computed to be equal and act on basis elements $\xi_0^a (x')^\alpha (\xi')^\beta \left(\bigwedge e_j^{\gamma_j}\right)$ via multiplication by $\mu.(2(\alpha+\beta)+\gamma)$. It now follows that these have (co-)homology only in degree zero given by $R[\![h]\!]$.

Similarly, we may consider the chain groups $D_N \otimes \Lambda^k W$, $k = 0, 1, \dots, n$; one may define four differentials

$$\begin{aligned} w_x &= \xi_0 e_0 \wedge + (2\bar{\nu})^{\frac{1}{2}} w_x^0, & i_x &= \xi_0 i_{e_0} + (2\bar{\nu})^{\frac{1}{2}} i_x^0, \\ w_\partial &= \partial_{\xi_0} e_0 \wedge + (2\bar{\nu})^{\frac{1}{2}} w_\partial^0, & i_\partial &= \partial_{\xi_0} i_{e_0} + (2\bar{\nu})^{\frac{1}{2}} i_\partial^0. \end{aligned}$$

Again these complexes have cohomology only in degree zero given by $R[\![h]\!]$.

Next, we define twisted Koszul differentials on $D_N \otimes \Lambda^k V$ via

$$\begin{aligned} \tilde{w}_\partial^0 &= \frac{i}{h}\sum_{j=1}^m \mu_j^{\frac{1}{2}} (\mathrm{ad}_{x_j} e_{2j-1} \wedge + \mathrm{ad}_{\xi_j} e_{2j} \wedge) = \sum_{j=1}^m \mu_j^{\frac{1}{2}} (\partial_{x_j} e_{2j} \wedge - \partial_{\xi_j} e_{2j-1} \wedge), \\ \tilde{i}_\partial^0 &= \frac{i}{h}\sum_{j=1}^m \mu_j^{\frac{1}{2}} (\mathrm{ad}_{x_j} i_{e_{2j-1}} + \mathrm{ad}_{\xi_j} i_{e_{2j}}) = \sum_{j=1}^m \mu_j^{\frac{1}{2}} (\partial_{x_j} i_{e_{2j}} - \partial_{\xi_j} i_{e_{2j-1}}). \end{aligned}$$

We note that the above are symplectic adjoints to their untwisted counterparts with respect to the symplectic pairing $\sum_{j=1}^m e_{2j-1} \wedge e_{2j}$ on V.

Similar twisted Koszul differentials on $D_N \otimes \Lambda^k W$ are defined via

$$\begin{aligned} \tilde{w}_\partial &= \frac{i}{h}\mathrm{ad}_{\xi_0} e_0 \wedge + (2\bar{\nu})^{\frac{1}{2}} \tilde{w}_\partial^0 = -\partial_{x_0} e_0 \wedge + (2\bar{\nu})^{\frac{1}{2}} \tilde{w}_\partial^0, \\ \tilde{i}_\partial &= \frac{i}{h} i_{e_0} \mathrm{ad}_{\xi_0} + (2\bar{\nu})^{\frac{1}{2}} \tilde{i}_\partial^0 = -\partial_{x_0} i_{e_0} + (2\bar{\nu})^{\frac{1}{2}} \tilde{i}_\partial^0. \end{aligned}$$

These twisted differentials correspond to the untwisted ones by a mere change of basis in V, W and hence also have (co-)homology only in degree zero given by $R[\![h]\!]$.

We now compute the twisted combinatorial Laplacian to be

$$\tilde{\Delta}^0 = \tilde{w}_\partial^0 i_x^0 + i_x^0 \tilde{w}_\partial^0 = -(w_x^0 \tilde{i}_\partial^0 + \tilde{i}_\partial^0 w_x^0) = \sum_{j=1}^m \mu_j [\xi_j \partial_{x_j} - x_j \partial_{\xi_j} + e_{2j} i_{e_{2j-1}} - e_{2j-1} i_{e_{2j}}].$$

One may similarly define $\tilde{\Delta} = \tilde{w}_\partial i_x + i_x \tilde{w}_\partial$. Next, we define the spaces of twisted $\tilde{\Delta}^0$-harmonic, ξ_0-independent elements

$$\begin{aligned} \mathcal{H}_N^k &= \left\{\omega \in D_N \otimes \Lambda^k W \;\middle|\; \tilde{\Delta}^0 \omega = 0,\ \partial_{\xi_0}\omega = 0\right\}, \\ \mathcal{H}^k &= \left\{\omega \in S \otimes \Lambda^k W \;\middle|\; \tilde{\Delta}^0 \omega = 0,\ \partial_{\xi_0}\omega = 0\right\}. \end{aligned}$$

We now prove a twisted version of the Hodge decomposition theorem.

Lemma 5.1. *The k-th chain group is spanned by three subspaces*:

$$D_N \otimes \Lambda^k W = \mathbb{R}[\operatorname{Im}(i_x \tilde{w}_\partial), \operatorname{Im}(\tilde{w}_\partial i_x), \mathcal{H}_N^k].$$

Proof. We first compute $\tilde{\Delta}$ in terms of $\tilde{\Delta}^0$ to be

$$\tilde{\Delta} = -\xi_0 \partial_{x_0} + 2\bar{\nu}\tilde{\Delta}^0 - 2(\partial_{x_0}\bar{\nu}^{\frac{1}{2}}) e_0 i_x^0.$$

Next, since $\tilde{\Delta}^0$ is skew-adjoint, we may decompose

$$D_N \otimes \Lambda^k W = E_0 \oplus \bigoplus_{\lambda > 0} [E_{i\lambda} \oplus E_{-i\lambda}]$$

into its eigenspaces. Following $[\tilde{\Delta}^0, \bar{\nu}] = 0$ we may now invert $\tilde{\Delta}$ on the nonzero eigenspaces of $\tilde{\Delta}^0$ above using the Volterra series:

$$\tilde{\Delta}^{-1} = (2\bar{\nu}\tilde{\Delta}^0)^{-1} \sum_{j=0}^{\infty} \big[(2\bar{\nu}\tilde{\Delta}^0)^{-1}(\xi_0 \partial_{x_0} + 2(\partial_{x_0}\bar{\nu}^{\frac{1}{2}}) e_0 i_x^0)\big]^j.$$

The sum above is finite since $\xi_0 \partial_{x_0} + 2(\partial_{x_0}\bar{\nu}^{\frac{1}{2}}) e_0 i_x^0$ is nilpotent on $D_N \otimes \Lambda^k W$. Thus we have

$$\bigoplus_{\lambda > 0} [E_{i\lambda} \oplus E_{-i\lambda}] \subset \operatorname{Im}(\tilde{\Delta}) \subset \mathbb{R}\big[\operatorname{Im}(i_x \tilde{w}_\partial), \operatorname{Im}(\tilde{w}_\partial i_x)\big].$$

Finally, we have the decomposition

$$E_0 = \bigoplus_{j=0}^{N} \xi_0^j \mathcal{H}_N^k$$

and we write each $\omega \in \xi_0^j \mathcal{H}_N^k$, $j \geq 1$, as

$$\omega = \omega_0 + \tilde{\Delta}\omega_1,$$

where

$$\omega_0 = \left[-2(\partial_{x_0}\bar{\nu}^{\frac{1}{2}}) e_0 i_x^0 \xi_0^{-1} \int_0^{x_0}\right]^j \omega \in \mathcal{H}_N^k,$$

$$\omega_1 = -\left(\xi_0^{-1} \int_0^{x_0}\right) \sum_{l=0}^{j-1} \left[-2(\partial_{x_0}\bar{\nu}^{\frac{1}{2}}) e_0 i_x^0 \xi_0^{-1} \int_0^{x_0}\right]^l \omega,$$

to complete the proof. □

5B. ***Formal Birkhoff normal form.*** The importance of the Koszul complexes introduced in the previous subsection is in continuing the Birkhoff normal form procedure for the symbol d_1 in (5-7). The remaining steps in the procedure are formal.

First let us define the Clifford quantization of an element in $a \in S \otimes \Lambda^k W$ using (2-8) as an element in

$$c_0(a) := i^{\frac{k(k+1)}{2}} c(a) \in S(m).$$

It is clear from (2-10) and (2-11) this gives an isomorphism

$$c_0 : S \otimes \Lambda^{\text{odd/even}} W \to S \otimes i\mathfrak{u}_{\mathbb{C}}(2^m) \tag{5-9}$$

of real elements of the even or odd exterior algebra with self-adjoint elements in $S(m)$. It is clear from (5-7) that

$$d_1 = H_1 + c_0(r) + hS \otimes i\mathfrak{u}_{\mathbb{C}}(2^m) \tag{5-10}$$

for $r := \sum_{j=1}^{n} r_j e_j \in O_2 \otimes W$.

For $a \in \Lambda^k W$, we define $[a] := \left[\frac{k}{2}\right]$. Now for $f \in O_N$, $N \geq 3$, and $a \in O_N \otimes \Lambda^{\text{even}} W$, $N \geq 1$, we may compute the conjugations

$$e^{\frac{i}{h} f} H_1 e^{-\frac{i}{h} f} = H_1 + c_0(\tilde{w}_\partial f) + O_N \otimes i\mathfrak{u}_{\mathbb{C}}(2^m), \tag{5-11}$$

$$e^{ic_0(a)} H_1 e^{-ic_0(a)} = H_1 + (-1)^{[a]+1} 2c_0(i_x a) + hc_0(\tilde{w}_\partial a) + O_{N+2} \otimes i\mathfrak{u}_{\mathbb{C}}(2^m) \tag{5-12}$$

in terms of the Koszul differentials.

We now come to the formal Birkhoff normal form for the symbol d_1.

Proposition 5.2. *There exist $f \in O_3$, $a \in O_2 \otimes \Lambda^{\text{even}} W$ and $\omega \in \mathcal{H}^{\text{odd}} \cap O_2$ such that*

$$e^{ic_0(a)} e^{\frac{i}{h} f} d_1 e^{-\frac{i}{h} f} e^{-ic_0(a)} = H_1 + c_0(\omega). \tag{5-13}$$

Proof. We first prove that for each $N \geq 1$, there exist $f_N \in O_3$, $a_N^0 \in O_1 \otimes \Lambda^2 W$, $\omega_N^0 \in \mathcal{H}^1 \cap O_2$ and $r_N^0 \in O_{N+1} \otimes W$ such that

$$\begin{gathered} e^{ic_0(a_N^0)} e^{\frac{i}{h} f_N} d_1 e^{-\frac{i}{h} f_N} e^{-ic_0(a_N^0)} = H_1 + c_0(\omega_N^0) + c_0(r_N^0) + hS \otimes i\mathfrak{u}_{\mathbb{C}}(2^m), \\ f_{N+1} - f_N \in O_{N+2}, \quad a_{N+1}^0 - a_N^0 \in O_N, \quad \omega_{N+1}^0 - \omega_N^0 \in O_{N+1}. \end{gathered} \tag{5-14}$$

The base case $N = 1$ is given by (5-10) with $a_1^0 = f_1 = \omega_1^0 = 0$ and $r_1^0 = r$. To complete the induction step we decompose r_N^0 as

$$r_N^0 = \underbrace{u_N^0}_{\in S_{N+1} \otimes W} + \underbrace{r_{N+1}^0}_{\in O_{N+2} \otimes W}. \tag{5-15}$$

Next we use Lemma 5.1 to find $b_N, g_N \in O_{N+1} \otimes W$ and $\upsilon_N^0 \in \mathcal{H}^1 \cap S_{N+1}$ such that

$$u_N^0 = \upsilon_N^0 - i_x \tilde{w}_\partial b_N^0 - \tilde{w}_\partial i_x g_N^0 + O_{N+2}. \tag{5-16}$$

Next, define $f_{N+1} = f_N + i_x g_N^0 \in O_3$, $a_{N+1}^0 = a_N^0 + \frac{1}{2} \tilde{w}_\partial b_N^0 \in O_1 \otimes \Lambda^2 W$ and $\omega_{N+1}^0 = \omega_N^0 + \upsilon_N^0$. We now use (5-11), (5-12), (5-15) and (5-16) to compute

$$\begin{aligned} &e^{ic_0(a_{N+1}^0)} e^{\frac{i}{h} f_{N+1}} d_1 e^{-\frac{i}{h} f_{N+1}} e^{-ic_0(a_{N+1}^0)} \\ &\qquad = e^{ic_0(\frac{1}{2}\tilde{w}_\partial b_N^0)} e^{\frac{i}{h} i_x g_N^0} H_1 e^{-\frac{i}{h} i_x g_N^0} e^{-ic_0(\frac{1}{2}\tilde{w}_\partial b_N^0)} + c_0(\omega_N^0) + c_0(r_N^0) + hS \otimes i\mathfrak{u}_{\mathbb{C}}(2^m) \\ &\qquad = H_1 + c_0(\omega_{N+1}^0) + c_0(r_{N+1}^0) + hS \otimes i\mathfrak{u}_{\mathbb{C}}(2^m), \end{aligned}$$

completing the induction step. Now setting $f = \lim_{N\to\infty} f_N$, $a_0 = \lim_{N\to\infty} a_N^0$ and $\omega_0 = \lim_{N\to\infty} \omega_N^0$ and letting $N \to \infty$ in (5-14) gives the relation

$$e^{ic_0(a_0)} e^{\frac{i}{h} f} d_1 e^{-\frac{i}{h} f} e^{-ic_0(a_0)} = H_1 + c_0(\omega_0) + hS \otimes i\mathfrak{u}_{\mathbb{C}}(2^m). \tag{5-17}$$

Next we claim that for each $N \geq 0$, there exist $a_N \in O_1 \otimes \Lambda^{\text{even}} W$, $\omega_N \in \mathcal{H}^* \cap O_2$ such that

$$\begin{aligned} e^{ic_0(a_N)} e^{\frac{i}{h} f} d_1 e^{-\frac{i}{h} f} e^{-ic_0(a_N)} &= H_1 + c_0(\omega_N) + hO_N \otimes i\mathfrak{u}_{\mathbb{C}}(2^m), \\ a_{N+1} - a_N \in O_{N+1} \otimes \Lambda^{\text{even}} W, &\quad \omega_{N+1} - \omega_N \in \mathcal{H}^{\text{odd}} \cap O_N. \end{aligned} \tag{5-18}$$

The base case $N = 0$ is now provided by (5-17). To complete the induction step, we use the isomorphism (5-9) to decompose the remainder term in (5-18) above as

$$c_0(u_N) + ihO_{N+1} \otimes \mathfrak{u}_{\mathbb{C}}(2^m)$$

for $u_N \in S_N \otimes \Lambda^{\text{odd}} W$. Next we use Lemma 5.1 to find $b_N, g_N \in O_N \otimes \Lambda^{\text{odd}} W$ and $\upsilon_N \in \mathcal{H}^{\text{odd}} \cap S_N$ such that

$$u_N = \upsilon_N - i_x \tilde{w}_\partial b_N - \tilde{w}_\partial i_x g_N + O_{N+1}. \tag{5-19}$$

Now define $a_{N+1} = a_N + i_x g_N + \frac{1}{2} h(-1)^{[b_N]} \tilde{w}_\partial b_N \in O_1$ and $\omega_{N+1} = \omega_N + \upsilon_N$. We now use (5-11), (5-12), (5-15) and (5-19) to compute

$$e^{ic_0(a_{N+1})} e^{\frac{i}{h} f} d_1 e^{-\frac{i}{h} f} e^{-ic_0(a_{N+1})} = H_1 + c_0(\omega_{N+1}) + ihO_{N+1} \otimes \mathfrak{u}_{\mathbb{C}}(2^m),$$

completing the induction step. Now setting $a = \lim_{N\to\infty} a_N$ and $\omega = \lim_{N\to\infty} \omega_N$ and letting $N \to \infty$ in (5-18) gives the proposition. □

Finally, we show how the Birkhoff normal form maybe used to perform a further reduction on the trace. First note that we may similarly use (2-8) to define a self-adjoint Clifford–Weyl quantization map

$$c_0^W := \mathrm{Op} \otimes c_0 : S^0_{\text{cl}}(\mathbb{R}^{2n}; \mathbb{C}) \otimes \Lambda^{\text{odd/even}} W \to \Psi^0_{\text{cl}}(\mathbb{R}^n; \mathbb{C}^{2^m}),$$

which maps real-valued symbols $S^0_{\text{cl}}(\mathbb{R}^{2n}; \mathbb{R}) \otimes \Lambda^{\text{odd/even}} W$ to self-adjoint operators in $\Psi^0_{\text{cl}}(\mathbb{R}^n; \mathbb{C}^{2^m})$. Similarly we define a space of real-valued, twisted $\tilde{\Delta}^0$-harmonic, ξ_0- independent symbols

$$\mathcal{H}^k S^0_{\text{cl}} := \left\{ \omega \in S^0_{\text{cl}}(\mathbb{R}^{2n}; \mathbb{R}) \otimes \Lambda^k W \;\middle|\; \tilde{\Delta}^0 \omega = 0,\ \partial_{\xi_0} \omega = 0 \right\}.$$

Next, an application of Borel's lemma by virtue of (5-1), (5-6) and (5-13) gives the existence of

$$\begin{aligned} \bar{a} &\sim \sum_{j=0}^{\infty} h^j \bar{a}_j \in S^0_{\text{cl}}(\mathbb{R}^{2n}; \mathbb{R}) \otimes \Lambda^{\text{odd}} W, & \bar{f} &\sim \sum_{j=0}^{\infty} h^j \bar{f}_j \in S^0_{\text{cl}}(\mathbb{R}^{2n}; \mathbb{R}), \\ \bar{r} &\sim \sum_{j=0}^{\infty} h^j \bar{r}_j \in S^0_{\text{cl}}(\mathbb{R}^{2n}; \mathbb{R}) \otimes \Lambda^{\text{odd}} W, & \bar{\omega} &\sim \sum_{j=0}^{\infty} h^j \bar{\omega}_j \in \mathcal{H}^{\text{odd}} S^0_{\text{cl}} \end{aligned}$$

such that

$$e^{ic_0^W(\bar{a})} e^{\frac{i}{h} \bar{f}^W} d_0^W e^{-\frac{i}{h} \bar{f}^W} e^{-ic_0^W(\bar{a})} = \underbrace{H_1^W + c_0^W(\bar{\omega})}_{:=\bar{D}} + c_0^W(\bar{r}) \tag{5-20}$$

on $\overline{V}_{\alpha\beta} := e^{X_{\bar{f}_0}}(V^0_{\alpha\beta})$. Here $\{\bar{r}_j\}_{j\in\mathbb{N}_0}$, $\bar{f}_0$, $\bar{\omega}_0$ vanish to infinite, second and second order respectively along

$$\Sigma_0^{D_0} = \Sigma_0^{\bar{D}} = \Sigma_0^{\bar{D}+c_0^W(\bar{r})} = \{\xi_0 = x' = \xi' = 0\}.$$

Note that on account of (4-5) and (5-4) one again has

$$\nu_0 = \mu_1 \min_{x\in X} \nu(x) \le \mu_1 \inf_{\mathbb{R}^n_{x_0,x'',\xi''}} \bar{\nu}.$$

Furthermore, since $\bar{\omega}_0$ vanishes to second order we may choose $\bar{\omega}_0$ arbitrarily small satisfying the estimate

$$\|\bar{\omega}_0\|_{C^1} < \varepsilon \tag{5-21}$$

for any $\varepsilon > 0$, while still satisfying (5-20).

We note that $\bar{D} \in \Psi^1_{\mathrm{cl}}(\mathbb{R}^n; \mathbb{C}^{2^m})$, with $\bar{D} + i$ having an elliptic symbol in the class $S^0(\langle \xi_0, \xi' \rangle)$, and is hence essentially self-adjoint as an unbounded operator on $L^2(\mathbb{R}^n; \mathbb{C}^{2^m})$. The domain of its unique self-adjoint extension is $H^1(\mathbb{R}_{x_0}) \otimes L^2(\mathbb{R}^{n-1}_{x',x''}; \mathbb{C}^{2^m})$; see [Dimassi and Sjöstrand 1999, Chapter 8]. We now set

$$\bar{A}_\alpha := e^{ic_0^W(\bar{a})} e^{\frac{i}{h}\bar{f}^W} A^0_\alpha e^{-\frac{i}{h}\bar{f}^W} e^{-ic_0^W(\bar{a})}, \tag{5-22}$$

$$\begin{aligned} \mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D}) &:= \mathrm{tr}\left[\bar{A}_\alpha f\left(\frac{\bar{D}}{\sqrt{h}}\right) \check{\vartheta}\left(\frac{\lambda\sqrt{h} - \bar{D}}{h}\right) \bar{A}_\beta\right] \\ &= \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) \check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right) \mathrm{tr}\left[\bar{A}_\alpha \left(\frac{1}{\sqrt{h}}\bar{D} - z\right)^{-1} \bar{A}_\beta\right] dz\, d\bar{z}. \end{aligned} \tag{5-23}$$

Proposition 5.3. *We have*

$$\mathcal{T}^{\vartheta}_{\alpha\beta}(D_0) = \mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D}) \bmod h^\infty.$$

Proof. Since the conjugations in (5-1) and (5-20) are unitary and $\mathrm{WF}(\bar{A}_\alpha), \mathrm{WF}(\bar{A}_\beta) \subset \overline{V}_{\alpha\beta}$, we have

$$\mathcal{T}^{\vartheta}_{\alpha\beta}(D_0) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) \check{\vartheta}\left(\frac{\lambda - z}{\sqrt{h}}\right) \mathrm{tr}\left[\bar{A}_\alpha \left(\frac{1}{\sqrt{h}}(\bar{D} + c_0^W(\bar{r})) - z\right)^{-1} \bar{A}_\beta\right] dz\, d\bar{z}.$$

It now remains to do away with the $c_0^W(\bar{r})$ above. Since this term vanishes to infinite order along $\Sigma_0^{\bar{D}} = \Sigma_0^{\bar{D}+c_0^W(\bar{r})}$, we may use symbolic calculus to find $P_N, Q_N \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^n; \mathbb{C}^{2^m})$, for all $N \ge 1$, such that

$$c_0^W(\bar{r}) = P_N(\bar{D} + c_0^W(\bar{r}))^N, \tag{5-24}$$

$$c_0^W(\bar{r}) = Q_N(\bar{D})^N. \tag{5-25}$$

Modifying $\bar{D}$ outside a neighborhood of $\overline{V}_{\alpha\beta}$ using Lemma 3.3 and Proposition A.6 we may assume that $\bar{D}, \bar{D}+c_0^W(\bar{r})$ have discrete spectrum in $(-\sqrt{2\nu_0}, \sqrt{2\nu_0})$ and hence

$$\mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D}) = \operatorname{tr}\left[\bar{A}_\alpha f\left(\frac{\bar{D}}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-\bar{D}}{h}\right)\bar{A}_\beta\right],$$

$$\mathcal{T}^{\vartheta}_{\alpha\beta}(D_0) = \operatorname{tr}\left[\bar{A}_\alpha f\left(\frac{\bar{D}+c_0^W(\bar{r})}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-\bar{D}-c_0^W(\bar{r})}{h}\right)\bar{A}_\beta\right].$$

Next, with $\Pi^{\bar{D}} = \Pi^{\bar{D}}_{[-\sqrt{2\nu_0 h},\sqrt{2\nu_0 h}]}$ and $\Pi^{\bar{D}+c_0^W(\bar{r})} = \Pi^{\bar{D}+c_0^W(\bar{r})}_{[-\sqrt{2\nu_0 h},\sqrt{2\nu_0 h}]}$ denoting the spectral projections, (5-24) and (5-25) give

$$\|c_0^W(\bar{r})\Pi^{\bar{D}}\| = O(h^{\frac{N}{2}}), \quad \|c_0^W(\bar{r})\Pi^{\bar{D}+c_0^W(\bar{r})}\| = O(h^{\frac{N}{2}})$$

for each $N \geq 1$. Finally applying Proposition A.5 with

$$\rho(x) = f\left(\frac{x}{\sqrt{h}}\right)\check{\vartheta}\left(\frac{\lambda\sqrt{h}-x}{h}\right)$$

and using the cyclicity of the trace gives $\mathcal{T}^{\vartheta}_{\alpha\beta}(D_0) - \mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D}) = O(h^{-1}h^{\frac{N}{4096}})$, for all $N \geq 1$, completing the proof. □

6. Extension of a resolvent

In this section we complete the proof of Lemma 3.1. On account of the reductions in Propositions 4.1 and 5.3 in the previous sections, it suffices to now consider the trace $\mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D})$. First let $\bar{A}_\alpha = a_\alpha^W$, $\bar{A}_\beta = a_\beta^W$ for $a_\alpha, a_\beta \in S^0_{\mathrm{cl}}(\mathbb{R}^{2n})$. The conjugations

$$e^{\frac{it}{h}x_0}\bar{A}_\alpha e^{-\frac{it}{h}x_0} = a_{\alpha,t}^W \quad \text{and} \quad e^{\frac{it}{h}x_0}\bar{A}_\beta e^{-\frac{it}{h}x_0} = a_{\beta,t}^W$$

are easily computed in terms of the one-parameter family of symbols $a_{\alpha,t}(\xi_0,\dots) = a_\alpha(\xi_0+t,\dots)$, $a_{\beta,t} = a_\beta(\xi_0+t,\dots) \in S^0_{\mathrm{cl}}(\mathbb{R}^{2n})$, $t \in \mathbb{R}$, obtained by translating in the ξ_0-direction. One now introduces almost analytic continuations of the symbols $a_{\alpha,t}, a_{\beta,t} \in S^0_{\mathrm{cl}}(\mathbb{R}^{2n})$, defined for $t \in \mathbb{C}$, such that all the Fréchet seminorms of $\bar{\partial}a_{\alpha,t}, \bar{\partial}a_{\beta,t}$ are $O(|\operatorname{Im} t|^\infty)$. These may be further chosen to have the property that the wavefront sets of their quantizations have uniform compact support when t is restricted to compact subsets of $\mathbb{C}$. Again one clearly has

$$a_{\alpha,t}^W = e^{\frac{i\operatorname{Re}t}{h}x_0}(a_{\alpha,i\operatorname{Im}t})^W e^{-\frac{i\operatorname{Re}t}{h}x_0}, \tag{6-1}$$

$$a_{\beta,t}^W = e^{\frac{i\operatorname{Re}t}{h}x_0}(a_{\beta,i\operatorname{Im}t})^W e^{-\frac{i\operatorname{Re}t}{h}x_0}. \tag{6-2}$$

In similar vein we may define

$$\bar{D}_t := e^{-\frac{it}{h}x_0}\bar{D}e^{\frac{it}{h}x_0} = H_{1,t}^W + c_0^W(\bar{\omega}), \tag{6-3}$$

$$H_{1,t} = (\xi_0+t)\sigma_0 + (2\bar{\nu})^{\frac{1}{2}}\sum_{j=1}^m \mu_j^{\frac{1}{2}}(x_j\sigma_{2j-1}+\xi_j\sigma_{2j}) \in S^1_{\mathrm{cl}}(\mathbb{R}^{2n}) \tag{6-4}$$

for $t \in \mathbb{R}$, on account of the ξ_0-independence of $\bar{\omega}$. An almost analytic continuation of $\bar{D}_t$ is easily introduced by simply allowing $t \in \mathbb{C}$ to be complex in (6-4) above. The resolvent $(\bar{D}_t - z)^{-1} : L^2(\mathbb{R}^n; \mathbb{C}^{2^m}) \to L^2(\mathbb{R}^n; \mathbb{C}^{2^m})$ is well defined and holomorphic in the region $\operatorname{Im} z > |\operatorname{Im} t|$.

In the lemma below we set $t = i\gamma(M, \delta) := i2Mh^{\delta}\log\frac{1}{h}$, for $\delta = 1-\varepsilon \in \left(\frac{1}{2}, 1\right)$ with ε as in Lemma 3.1 and $M > 1$. We now have the following.

Lemma 6.1. *For h sufficiently small and for all $\varepsilon_0 > 0$, the resolvent*

$$\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma} - z\right)^{-1} : L^2(\mathbb{R}^n; \mathbb{C}^{2^m}) \to L^2(\mathbb{R}^n; \mathbb{C}^{2^m})$$

extends holomorphically, and is uniformly $O(h^{-\frac{1}{2}})$, in the region $\operatorname{Im} z > -Mh^{\delta-\frac{1}{2}}\log\frac{1}{h}$, $|\operatorname{Re} z| \leq \sqrt{2\nu_0} - \varepsilon_0$.

Proof. We begin with the orthogonal Landau decomposition (2-31)

$$L^2(\mathbb{R}^n; \mathbb{C}^{2^m}) = L^2(\mathbb{R}^{m+1}_{x_0,x''}) \otimes \underbrace{\left(\mathbb{C}[\psi_{0,0}] \oplus \bigoplus_{\Lambda \in \mu.(\mathbb{N}_0^m \setminus 0)} [E_\Lambda^{\text{even}} \oplus E_\Lambda^{\text{odd}}]\right)}_{=L^2(\mathbb{R}^m_{x'};\mathbb{C}^{2^m})}, \tag{6-5}$$

where

$$E_\Lambda^{\text{even}} := \bigoplus_{\substack{\tau \in \mathbb{N}_0^m \setminus 0 \\ \Lambda = \mu.\tau}} E_\tau^{\text{even}}, \quad E_\Lambda^{\text{odd}} := \bigoplus_{\substack{\tau \in \mathbb{N}_0^m \setminus 0 \\ \Lambda = \mu.\tau}} E_\tau^{\text{odd}} \tag{6-6}$$

according to the eigenspaces of the squared magnetic Dirac operator $D^2_{\mathbb{R}^m}$ (2-21) on $\mathbb{R}^m$. It is clear from (6-4) that

$$H^W_{1,t} = (\xi_0 + t)^W \sigma_0 + [(2\bar{\nu})^{\frac{1}{2}}]^W \otimes D_{\mathbb{R}^m}$$

in terms of the above decomposition. Furthermore one has the commutation relations

$$\begin{aligned} [\sigma_0, D^2_{\mathbb{R}^m}] &= 0, \\ [c_0^W(\bar{\omega}), D^2_{\mathbb{R}^m}] &= ihc_0^W(\tilde{\Delta}^0\bar{\omega}) = 0, \end{aligned}$$

since $\bar{\omega}$ is $\tilde{\Delta}^0$-harmonic. The above and (6-3) show that the $\left(\frac{1}{\sqrt{h}}\bar{D}_t - z\right)$ preserves the eigenspaces in the decomposition (6-5) for all $t \in \mathbb{C}$. It hence suffices to consider the restriction of $\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma} - z\right)$ to each eigenspace.

Let $E_0 := \mathbb{C}[\psi_{0,0}]$, $E_\Lambda := E_\Lambda^{\text{even}} \oplus E_\Lambda^{\text{odd}}$ and $\mathrm{P}_0, \mathrm{P}_\Lambda$ denote the projection onto the corresponding summands of (6-5). Define the restrictions

$$\begin{aligned} \Omega_0 &:= \mathrm{P}_0 c_0^W(\bar{\omega}) \mathrm{P}_0 \; : L^2(\mathbb{R}^{m+1}_{x_0,x''}) \to L^2(\mathbb{R}^{m+1}_{x_0,x''}), \\ \Omega_\Lambda &:= \mathrm{P}_\Lambda c_0^W(\bar{\omega}) \mathrm{P}_\Lambda \colon L^2(\mathbb{R}^{m+1}_{x_0,x''}; E_\Lambda^{\text{even}} \oplus E_\Lambda^{\text{odd}}) \to L^2(\mathbb{R}^{m+1}_{x_0,x''}; E_\Lambda^{\text{even}} \oplus E_\Lambda^{\text{odd}}), \quad \Lambda > 0. \end{aligned}$$

Now $\bar{\omega} \sim \sum_{j=0}^{\infty} h^j \bar{\omega}_j \in \mathcal{H}^{\text{odd}} S^0_{\text{cl}}$ with ξ_0-independent $\bar{\omega}_0$ vanishing to second order along $\Sigma_0^{D_0} = \Sigma_0^{\bar{D}} = \{\xi_0 = x' = \xi' = 0\}$. Hence we may decompose

$$\bar{\omega}_0 = \sum_{i \le j} [a_{ij} z_i z_j + \bar{a}_{ij} \bar{z}_i \bar{z}_j + b_{ij} \bar{z}_i z_j + \bar{b}_{ij} z_i \bar{z}_j]$$

in terms of the complex coordinates $z_j = x_j + i\xi_j$, $\bar{z}_j = x_j - i\xi_j$, $1 \le j \le m$, with $a_{ij}, b_{ij} \in S^0_{\text{cl}}(\mathbb{R}^{2n}; \mathbb{R}) \otimes \Lambda^{\text{odd}} W$. The self-adjoint Clifford–Weyl quantization now yields

$$c_0^W(\bar{\omega}_0) = \sum_{i \le j} \big[c_0^W(a_{ij}) A_i A_j + A_j^* A_i^* c_0^W(\bar{a}_{ij}) + c_0^W(b_{ij}) A_i^* A_j + A_j^* A_i c_0^W(\bar{b}_{ij})\big] + h\Psi^0_{\text{cl}}(\mathbb{R}^n; \mathbb{C}^{2^m})$$

in terms of the raising and lowering operators in (2-26). Since each lowering operator A_j annihilates $\psi_{0,0}$, this leads to the estimate

$$\|\Omega_0\| = O(h). \tag{6-7}$$

Next, on account of (5-21) one may also expand $\bar{\omega}_0 = \sum_{j=1}^{m} [a_j z_j + \bar{a}_j \bar{z}_j]$, with $a_j \in S^0_{\text{cl}}(\mathbb{R}^{2n}; \mathbb{R}) \otimes \Lambda^{\text{odd}} W$, satisfying $\|a_j\|_{C^0} \le \varepsilon < 1$. On self-adjoint quantization this now gives

$$c_0^W(\bar{\omega}_0) = \sum_{j=1}^{m} \big[c_0^W(a_j) A_j + A_j^* c_0^W(\bar{a}_j)\big] + h\Psi^0_{\text{cl}}(\mathbb{R}^n; \mathbb{C}^{2^m}),$$

where

$$\|c_0^W(a_j)\|_{L^2 \to L^2}, \ \|c_0^W(\bar{a}_j)\|_{L^2 \to L^2} = \|a_j\|_{C^0} + O(h) \le \varepsilon + O(h).$$

Knowing the action of the lowering and raising operators A_j, A_j^* on each eigenstate (2-25) of $D^2_{\mathbb{R}^m}$ gives the estimate

$$\|\Omega_\Lambda\| \le \varepsilon \sqrt{\Lambda h} + O(h), \tag{6-8}$$

with the $O(h)$ term above being uniform in Λ.

Next we compute the restriction of $\big(\frac{1}{\sqrt{h}} \bar{D}_{i\gamma} - z\big)$ to the E_0 eigenspace in (6-5) using (2-6) to be

$$D_{i\gamma,0}(z) := \mathrm{P}_0 \bigg(\frac{1}{\sqrt{h}} \bar{D}_{i\gamma} - z \bigg) \mathrm{P}_0 = \frac{1}{\sqrt{h}} [-\xi_0 - i\gamma - z\sqrt{h} + \Omega_0]. \tag{6-9}$$

The above is again understood as a closed unbounded operator on $L^2(\mathbb{R}^{m+1}_{x_0, x''})$ with domain $H^1(\mathbb{R}_{x_0}) \otimes L^2(\mathbb{R}^m_{x''})$. Set $R_{i\gamma,0}(z) = [r_{i\gamma,0}(z)]^W$, with

$$r_{i\gamma,0}(z) = \frac{\sqrt{h}}{-\xi_0 - i\gamma - z\sqrt{h}},$$

which is well defined for $\operatorname{Im} z > -\gamma/(2\sqrt{h}) = -M h^{\delta - \frac{1}{2}} \log \frac{1}{h}$, and compute

$$\begin{aligned} R_{i\gamma,0}(z) D_{i\gamma,0}(z) &= I + O(h^{1-\delta}), \\ D_{i\gamma,0}(z) R_{i\gamma,0}(z) &= I + O(h^{1-\delta}) \end{aligned}$$

using (6-7). This shows that the inverse $D_{i\gamma,0}(z)^{-1}$ exists and is $O(R_{i\gamma,0}(z)) = O(h^{\frac{1}{2}-\delta})$.

Next, we compute the restriction of $\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma}-z\right)$ to the E_Λ, $\Lambda>0$, eigenspace in (6-5). Using (2-32), (2-33) this has the form

$$D_{i\gamma,\Lambda}(z):=\mathrm{P}_\Lambda\left(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma}-z\right)\mathrm{P}_\Lambda=\frac{1}{\sqrt{h}}\begin{bmatrix}-\xi_0-i\gamma-z\sqrt{h} & (\sqrt{2\bar{v}\Lambda h})^W\\ (\sqrt{2\bar{v}\Lambda h})^W & \xi_0+i\gamma-z\sqrt{h}\end{bmatrix}+\frac{1}{\sqrt{h}}\Omega_\Lambda$$

with respect to the $\mathbb{Z}_2$-grading $E_\Lambda=E_\Lambda^{\text{even}}\oplus E_\Lambda^{\text{odd}}$. Here we leave the identification $\mathtt{i}_\tau$ in (2-32) between the odd and even parts as being understood. Set $R_{i\gamma,\Lambda}(z)=[r_{i\gamma,\Lambda}(z)]^W$, where

$$r_{i\gamma,\Lambda}(z):=\frac{\sqrt{h}}{z^2h-(\xi_0+i\gamma)^2-2\bar{v}\Lambda h}\begin{bmatrix}-\xi_0-i\gamma-z\sqrt{h} & (\sqrt{2\bar{v}\Lambda h})\\ (\sqrt{2\bar{v}\Lambda h}) & \xi_0+i\gamma-z\sqrt{h}\end{bmatrix},$$

which is well defined for $|\mathrm{Re}\,z|\le\sqrt{2v_0}-\varepsilon_0<\inf_{\mathbb{R}^n}\sqrt{2\bar{v}\Lambda}$, and h sufficiently small. We now compute

$$\begin{aligned}\|R_{i\gamma,\Lambda}(z)D_{i\gamma,\Lambda}(z)-I\|&\le C\varepsilon+O(h),\\ \|D_{i\gamma,\Lambda}(z)R_{i\gamma,\Lambda}(z)-I\|&\le C\varepsilon+O(h)\end{aligned}$$

using (6-8) with the constants above being uniform in Λ. Choosing ε sufficiently small in (5-21) shows that the inverse $D_{i\gamma,\Lambda}(z)^{-1}$ exists and is $O(R_{i\gamma,\Lambda}(z))=O(h^{-\frac{1}{2}})$ uniformly. ☐

We now finally finish the proof of Lemma 3.1.

Proof of Lemma 3.1. As noted in the beginning of the section, on account of (3-2), (3-3) and the reductions in Propositions 4.1 and 5.3, it suffices to show $\mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D})=O(h^\infty)$. We now define the trace

$$\tau_{\alpha\beta,t}(z):=\mathrm{tr}\left[a^W_{\alpha,t}\left(\frac{1}{\sqrt{h}}\bar{D}_t-z\right)^{-1}a^W_{\beta,t}\right],\quad \mathrm{Im}\,z>|\mathrm{Im}\,t|,\tag{6-10}$$

in terms of the almost analytic continuations. We clearly have

$$\begin{aligned}\tau_{\alpha\beta,t}(z)&=O(h^{-n}|\mathrm{Im}\,z|^{-1}),\\ \frac{\partial}{\partial\bar{t}}\tau_{\alpha\beta,t}(z)&=O(h^{-n}|\mathrm{Im}\,t|^\infty|\mathrm{Im}\,z|^{-2}).\end{aligned}$$

Furthermore, by (6-1)–(6-3) $\tau_{\alpha\beta,t}(z)$ only depends on $\mathrm{Im}\,t$ and we have

$$\tau_{\alpha\beta,i\,\mathrm{Im}\,t}(z)=\tau_{\alpha\beta,0}(z)+O(h^{-n}|\mathrm{Im}\,t|^\infty|\mathrm{Im}\,z|^{-2}).\tag{6-11}$$

As before, we again introduce $\psi\in C^\infty(\mathbb{R};[0,1])$ such that

$$\psi(x)=\begin{cases}1, & x\le 1,\\ 0, & x\ge 2,\end{cases}$$

and set

$$\psi_M(z)=\psi\left(\frac{\mathrm{Im}\,z}{M\sqrt{h}\log\frac{1}{h}}\right).$$

The estimates (3-11), (3-12) along with the observation $\psi_M |\mathrm{Im}\, z|^N = O\big(\big(M\sqrt{h}\log\frac{1}{h}\big)^N\big)$ now give

$$\begin{aligned}\mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D}) &= \frac{1}{\pi}\int_{\mathbb{C}} \bar\partial(\psi_M \tilde f)\check\vartheta\left(\frac{\lambda - z}{\sqrt{h}}\right)\tau_{\alpha\beta,0}(z)\,dz\,d\bar z\\ &= O(h^\infty) + \frac{1}{\pi}\int_{\{M\sqrt{h}\log\frac{1}{h}\le \mathrm{Im}\, z\le 2M\sqrt{h}\log\frac{1}{h}\}} \bar\partial(\psi_M \tilde f)\check\vartheta\left(\frac{\lambda - z}{\sqrt{h}}\right)\tau_{\alpha\beta,0}(z)\,dz\,d\bar z.\end{aligned}$$

Using (6-11) and $\gamma = 2Mh^\delta \log\frac{1}{h}$, $\delta\in\big(\frac12, 1\big)$, the above now equals

$$\mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D}) = O(h^\infty) + \frac{1}{\pi}\int_{\{M\sqrt{h}\log\frac{1}{h}\le \mathrm{Im}\, z\le 2M\sqrt{h}\log\frac{1}{h}\}} \bar\partial(\psi_M \tilde f)\check\vartheta\left(\frac{\lambda - z}{\sqrt{h}}\right)\tau_{\alpha\beta,i\gamma}(z)\,dz\,d\bar z.$$

Since the resolvent $\big(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma} - z\big)^{-1}$, and hence the trace $\tau_{\alpha\beta,i\gamma}(z)$, extends holomorphically to $\mathrm{Im}\, z > -Mh^{\delta-\frac12}\log\frac{1}{h}$, $|\mathrm{Re}\, z| \le \sqrt{2\nu_0} - \varepsilon_0$ by Lemma 6.1 we may replace the integral in the last line above:

$$\begin{aligned}\mathcal{T}^{\vartheta}_{\alpha\beta}(\bar{D}) &= O(h^\infty) + \frac{1}{\pi}\int_{\{-1/2Mh^{\delta-1/2}\log\frac{1}{h}\le \mathrm{Im}\, z\le -\frac14 Mh^{\delta-1/2}\log\frac{1}{h}\}} \bar\partial(\psi_M \tilde f)\check\vartheta\left(\frac{\lambda - z}{\sqrt{h}}\right)\tau_{\alpha\beta,i\gamma}(z)\,dz\,d\bar z\\ &= O(h^\infty) + O\left[\int_{\{-1/2Mh^{\delta-1/2}\log\frac{1}{h}\le \mathrm{Im}\, z\le -\frac14 Mh^{\delta-1/2}\log\frac{1}{h}\}} \frac{h^{-n-1/2}}{\sqrt{h}\log(1/h)} e^{\frac{S'_{\alpha\beta}(\mathrm{Im}\, z)}{h^{1/2-\varepsilon}}}\,dz\,d\bar z\right]\\ &= O\big[h^{\frac{M}{4}(S'_{\alpha\beta})-n-\frac12}\big]\end{aligned}$$

using (3-11) and $O(h^{-\frac12})$ estimate on the resolvent $\big(\frac{1}{\sqrt{h}}\bar{D}_{i\gamma} - z\big)^{-1}$. Choosing M sufficiently large now gives the result. □

7. Local trace expansion

In this section we prove Lemma 3.2. This is a relatively classical trace expansion. A parametrix construction for the operator $e^{\frac{it}{h}D_h^2}$ may potentially be employed in its proof since the principal symbol of D_h^2 is Morse–Bott critical, as in [Brummelhuis et al. 1995]. However Lemma 3.2 would require an understanding of the large time behavior of parametrix left open in that paper; see [Camus 2004; Khuat-Duy 1997]. Here we prove the expansion using the alternate methods of local index theory. The expansion is analogous to the heat trace expansions arising in the analysis of the Bergman kernel [Bismut 1987; Ma and Marinescu 2007]. Here we adopt a modification of the approach in [Ma and Marinescu 2007, Chapters 1 and 4].

First, fix a point $p\in X$. On account of Definition 1.1 there is an orthonormal basis $e_{0,p} = R_p$, $e_{j,p}$, $e_{j+m,p}$, $j = 1,\dots,m$, of T_pX consisting of eigenvectors of $\mathfrak{J}_p$ with eigenvalues $0, \pm\lambda_{j,p} (:= \pm i\mu_j\nu(p))$, $j = 1,\dots,m$, such that

$$da(p) = \sum_{j=1}^m \lambda_j(p) e^*_{j,p}\wedge e^*_{j+m,p}. \tag{7-1}$$

Using the parallel transport from this basis, fix a geodesic coordinate system $(x_0,\dots,x_{2m})$ on an open neighborhood of $p\in\Omega$. Let $e_j = w_j^k\partial_{x_k}$, $0\le j\le 2m$, be the local orthonormal frame of TX obtained by parallel transport of $e_{j,p} = \partial_{x_j}|_p$, $0\le j\le 2m$, along geodesics. Hence we again have $w_j^k g_{kl} w_r^l = \delta_{jr}$, $w_j^k|_p = \delta_j^k$, with the g_{kl} being the components of the metric in these coordinates. Choose an orthonormal basis $\{u_q(p)\}_{q=1}^{2^m}$for S_p in which Clifford multiplication

$$c(e_j)|_p = \gamma_j \tag{7-2}$$

is standard. Choose an orthonormal basis $\mathtt{l}_p$ for L_p. Parallel transport the bases $\{u_q(p)\}_{q=1}^{2^m}$, $\mathtt{l}_p$ along geodesics using the spin connection ∇^S and unitary family of connections $\nabla^h = A_0 + \frac{i}{h}a$ to obtain trivializations $\{u_q\}_{q=1}^{2^m}$, $\mathtt{l}$ of S, L on Ω. Since Clifford multiplication is parallel, the relation (7-2) now holds on Ω. The connection $\nabla^{S\otimes L} = \nabla^S \otimes 1 + 1 \otimes \nabla^h$ can be expressed in this frame and these coordinates as

$$\nabla^{S\otimes L} = d + A_j^h\, dx^j + \Gamma_j\, dx^j, \tag{7-3}$$

where each A_j^h is a Christoffel symbol of ∇^h and each Γ_j is a Christoffel symbol of the spin connection ∇^S. Since the section l is obtained via parallel transport along geodesics, the connection coefficient A_j^h may be written in terms of the curvature $F_{jk}^h\, dx^j \wedge dx^k$ of ∇^h via

$$A_j^h(x) = \int_0^1 d\rho(\rho x^k F_{jk}^h(\rho x)). \tag{7-4}$$

The dependence of the curvature coefficients F_{jk}^h on the parameter h is seen to be linear in $\frac{1}{h}$ via

$$F_{jk}^h = F_{jk}^0 + \frac{i}{h}(da)_{jk} \tag{7-5}$$

despite the fact that they are expressed in the h-dependent frame $\mathtt{l}$. This is because a gauge transformation from an h-independent frame into $\mathtt{l}$ changes the curvature coefficient by conjugation. Since L is a line bundle, this is conjugation by a function and hence does not change the coefficient. Furthermore, the coefficients in the Taylor expansion of (7-5) at 0 maybe expressed in terms of the covariant derivatives $(\nabla^{A_0})^l F_{jk}^0$, $(\nabla^{A_0})^l (da)_{jk}$ evaluated at p. Next, using the Taylor expansion

$$(da)_{jk} = (da)_{jk}(0) + x^l a_{jkl}, \tag{7-6}$$

we see that the connection $\nabla^{S\otimes L}$ has the form

$$\nabla^{S\otimes L} = d + \left[\frac{i}{h}\left(\frac{x^k}{2}(da)_{jk}(0) + x^k x^l A_{jkl}\right) + x^k A_{jk}^0 + \Gamma_j\right] dx^j, \tag{7-7}$$

where

$$A_{jk}^0 = \int_0^1 d\rho(\rho F_{jk}^0(\rho x)), \quad A_{jkl} = \int_0^1 d\rho(\rho a_{jkl}(\rho x))$$

and Γ_j are all independent of h. Finally from (7-2) and (7-7) may write down the expression for the Dirac operator (1-2) also given as $D = hc \circ (\nabla^{S\otimes L})$ in terms of the chosen frame and coordinates to be

$$D = \gamma^r w_r^j \left[h\partial_{x_j} + i\tfrac{1}{2}x^k (da)_{jk}(0) + i x^k x^l A_{jkl} + h(x^k A_{jk}^0 + \Gamma_j)\right] \tag{7-8}$$

$$\begin{aligned} &= \gamma^r \left[w_r^j h\partial_{x_j} + i w_r^j \tfrac{1}{2}x^k (da)_{jk}(0) + \tfrac{1}{2}hg^{-\frac{1}{2}}\partial_{x_j}(g^{\frac{1}{2}} w_r^j)\right] \\ &\quad + \gamma^r \left[i w_r^j x^k x^l A_{jkl} + h w_r^j (x^k A_{jk}^0 + \Gamma_j) - \tfrac{1}{2}hg^{-\frac{1}{2}}\partial_{x_j}(g^{\frac{1}{2}} w_r^j)\right] \in \Psi_{\mathrm{cl}}^1(\Omega_s^0; \mathbb{C}^{2^m}). \end{aligned} \tag{7-9}$$

In the second expression above, both square brackets are self-adjoint with respect to the Riemannian density $e^1 \wedge \cdots \wedge e^n = \sqrt{g}\, dx := \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n$ with $g = \det(g_{ij})$. Again one may obtain an expression self-adjoint with respect to the Euclidean density dx in the framing $g^{\frac{1}{4}}u_q \otimes \mathtt{1}$, $1 \le q \le 2^m$, with the result being an addition of the term $h\gamma^j w_j^k g^{-\frac{1}{4}}(\partial_{x_k} g^{\frac{1}{4}})$.

Let i_g be the injectivity radius of g^{TX}. Define the cutoff $\chi \in C_c^\infty(-1,1)$ such that $\chi = 1$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$. We now modify the functions w_j^k, outside the ball $B_{i_g/2}(p)$, such that $w_j^k = \delta_j^k$ (and hence $g_{jk} = \delta_{jk}$) are standard outside the ball $B_{i_g}(p)$ of radius i_g centered at p. This again gives

$$\begin{aligned}\mathbb{D} = \gamma^r\big[&w_r^j h\partial_{x_j} + i w_r^j \tfrac{1}{2} x^k (da)_{jk}(0) + \tfrac{1}{2} h g^{-\frac{1}{2}} \partial_{x_j}(g^{\frac{1}{2}} w_r^j)\big] \\ &+ \chi(|x|/i_g)\gamma^r\big[i w_r^j x^k x^l A_{jkl} + h w_r^j (x^k A_{jk}^0 + \Gamma_j) - \tfrac{1}{2} h g^{-\frac{1}{2}} \partial_{x_j}(g^{\frac{1}{2}} w_r^j)\big] \in \Psi_{\mathrm{cl}}^1(\mathbb{R}^n; \mathbb{C}^{2^m}) \quad (7\text{-}10)\end{aligned}$$

as a well defined operator on $\mathbb{R}^n$ formally self adjoint with respect to $\sqrt{g}\,dx$. Since $\mathbb{D} + i$ is elliptic in the class $S^0(m)$ for the order function

$$m = \sqrt{1 + g^{jl}\big(\xi_j + \tfrac{1}{2} x^k (da)_{jk}(0)\big)\big(\xi_l + \tfrac{1}{2} x^r (da)_{lr}(0)\big)},$$

the operator $\mathbb{D}$ is essentially self adjoint.

Proposition 7.1. *There exist tempered distributions $u_j \in \mathcal{S}'(\mathbb{R}_s)$, $j = 0, 1, 2, \dots$, such that one has a trace expansion*

$$\operatorname{tr} \phi\left(\frac{D}{\sqrt{h}}\right) = h^{-\frac{n}{2}} \left(\sum_{j=0}^N u_j(\phi) h^{\frac{j}{2}}\right) + h^{\frac{N+1-n}{2}} O\left(\sum_{k=0}^{n+1} \|\langle\xi\rangle^N \hat{\phi}^{(k)}\|_{L^1}\right) \tag{7-11}$$

for each $N \in \mathbb{N}$, $\phi \in \mathcal{S}(\mathbb{R}_s)$.

Proof. We begin by writing $\phi = \phi_0 + \phi_1$, with

$$\phi_0(s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi s} \hat{\phi}(\xi) \chi\left(\frac{2\xi\sqrt{h}}{i_g}\right) d\xi, \quad \phi_1(s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi s} \hat{\phi}(\xi) \left[1 - \chi\left(\frac{2\xi\sqrt{h}}{i_g}\right)\right] d\xi$$

via Fourier inversion.

First considering ϕ_1, integration by parts gives the estimate

$$|s^{n+1}\phi_1(s)| \le C_N h^{\frac{N-1}{2}} \left(\sum_{k=0}^{n+1} \|\xi^N \hat{\phi}^{(k)}\|_{L^1}\right)$$

for all $N \in \mathbb{N}$. Hence,

$$\left\| D^{n+1-a} \phi_1\left(\frac{D}{\sqrt{h}}\right) D^a \right\|_{L^2 \to L^2} = C_N h^{\frac{n+N}{2}} \left(\sum_{k=0}^{n+1} \|\xi^N \hat{\phi}^{(k)}\|_{L^1}\right)$$

for all $N \in \mathbb{N}$, for all $a = 0, \dots, n+1$. The semiclassical elliptic estimate and Sobolev's inequality now give the estimate

$$\left|\phi_1\left(\frac{D}{\sqrt{h}}\right)\right|_{C^0(X \times X)} \le C_N h^{\frac{n+N}{2}} \left(\sum_{k=0}^{n+1} \|\xi^N \hat{\phi}^{(k)}\|_{L^1}\right), \tag{7-12}$$

for all $N \in \mathbb{N}$, on the Schwartz kernel.

Next, considering ϕ_0, we first use the change of variables $\alpha = \xi\sqrt{h}$ to write

$$\phi_0\left(\frac{D}{\sqrt{h}}\right) = \frac{1}{2\pi\sqrt{h}}\int_{\mathbb{R}} e^{i\alpha(D_{A_0}+ih^{-1}c(a))}\hat{\phi}\left(\frac{\alpha}{\sqrt{h}}\right)\chi\left(\frac{2\alpha}{i_g}\right)d\alpha.$$

Now since $D = \mathbb{D}$ on $B_{i_g/2}(p)$, we may use the finite propagation speed of the wave operators $e^{i\alpha h^{-1}D}$, $e^{i\alpha h^{-1}\mathbb{D}}$ [Ma and Marinescu 2007, Theorem D.2.1] to conclude

$$\phi_0\left(\frac{D}{\sqrt{h}}\right)(p,\cdot) = \phi_0\left(\frac{\mathbb{D}}{\sqrt{h}}\right)(0,\cdot). \tag{7-13}$$

The right-hand side above is defined using functional calculus of self-adjoint operators, with standard local elliptic regularity arguments implying the smoothness of its Schwartz kernel. By virtue of (7-12), a similar estimate for $\phi_1\big(\frac{\mathbb{D}}{\sqrt{h}}\big)$, and (7-13) it now suffices to consider $\phi\big(\frac{\mathbb{D}}{\sqrt{h}}\big)$.

We now introduce the rescaling operator

$$\mathscr{R} : C^\infty(\mathbb{R}^n;\mathbb{C}^{2^m}) \to C^\infty(\mathbb{R}^n;\mathbb{C}^{2^m}),\quad (\mathscr{R}s)(x) := s\left(\frac{x}{\sqrt{h}}\right).$$

Conjugation by $\mathscr{R}$ amounts to the rescaling of coordinates $x \to x\sqrt{h}$. A Taylor expansion in (7-10) now gives the existence of classical (h-independent) self-adjoint, first-order differential operators $\mathtt{D}_j = a_j^k(x)\partial_{x_k} + b_j(x)$, $j = 0, 1, \dots$, with polynomial coefficients (of degree at most $j+1$) as well as h-dependent self-adjoint, first-order differential operators $\mathrm{E}_j = \sum_{|\alpha|=N+1} x^\alpha\big[c^k_{j,\alpha}(x;h)\partial_{x_k} + d_{j,\alpha}(x;h)\big]$, $j = 0, 1, \dots$, with uniformly C^∞ bounded coefficients $c^k_{j,\alpha}$, $d_{j,\alpha}$ such that

$$\mathscr{R}\mathbb{D}\mathscr{R}^{-1} = \sqrt{h}\mathtt{D}, \tag{7-14}$$

with

$$\mathtt{D} = \left(\sum_{j=0}^{N} h^{\frac{j}{2}}\mathtt{D}_j\right) + h^{\frac{N+1}{2}}\mathrm{E}_{N+1}\quad \forall N. \tag{7-15}$$

The coefficients of the polynomials $a_j^k(x)$, $b_j(x)$ again involve the covariant derivatives of the curvatures F^{TX}, F^{A_0} and da evaluated at p. Furthermore, the leading term in (7-15) is easily computed as

$$\mathtt{D}_0 = \gamma^j\big[\partial_{x_j} + i\tfrac{1}{2}x^k(da)_{jk}(0)\big] \tag{7-16}$$

$$= \gamma^0\partial_{x_0} + \underbrace{\gamma^j\big[\partial_{x_j} + \tfrac{1}{2}i\lambda_j(p)x_{j+m}\big] + \gamma^{j+m}\big[\partial_{x_{j+m}} - \tfrac{1}{2}i\lambda_j(p)x_j\big]}_{:=\mathtt{D}_{00}} \tag{7-17}$$

using (7-1), (7-6). It is now clear from (7-14) that

$$\phi\left(\frac{\mathbb{D}}{\sqrt{h}}\right)(x,x') = h^{-\frac{n}{2}}\phi(\mathtt{D})\left(\frac{x}{\sqrt{h}},\frac{x'}{\sqrt{h}}\right). \tag{7-18}$$

Next, let $I_j = \big\{k = (k_0, k_1, \dots)\,\big|\, k_\alpha \in \mathbb{N}, \sum k_\alpha = j\big\}$ denote the set of partitions of the integer j and set

$$\mathtt{C}_j^z = \sum_{k\in I_j}(z-\mathtt{D}_0)^{-1}\big[\Pi_\alpha[\mathtt{D}_{k_\alpha}(z-\mathtt{D}_0)^{-1}]\big]. \tag{7-19}$$

Local elliptic regularity estimates again give

$$(z-\mathtt{D})^{-1} = O_{L^2_{\mathrm{loc}}\to L^2_{\mathrm{loc}}}(|\mathrm{Im}\, z|^{-1}) \qquad \text{and} \qquad \mathtt{C}^z_j = O_{L^2_{\mathrm{loc}}\to L^2_{\mathrm{loc}}}(|\mathrm{Im}\, z|^{-j-1}), \quad j=0,1,\dots.$$

A straightforward computation using (7-15) then yields

$$(z-\mathtt{D})^{-1} - \left(\sum_{j=0}^{N} h^{\frac{j}{2}} \mathtt{C}^z_j\right) = O_{L^2_{\mathrm{loc}}\to L^2_{\mathrm{loc}}}((|\mathrm{Im}\, z|^{-1} h^{\frac{1}{2}})^{N+1}). \tag{7-20}$$

A similar expansion as (7-15) for the operator $(1+\mathtt{D}^2)^{\frac{n+1}{2}}(z-\mathtt{D})$ also gives the bounds

$$(1+\mathtt{D}^2)^{-\frac{n+1}{2}}(z-\mathtt{D})^{-1} - \left(\sum_{j=0}^{N} h^{\frac{j}{2}} \mathtt{C}^z_{j,n+1}\right) = O_{H^s_{\mathrm{loc}}\to H^{s+n+1}_{\mathrm{loc}}}((|\mathrm{Im}\, z|^{-1} h^{\frac{1}{2}})^{N+1}) \tag{7-21}$$

for all $s\in\mathbb{R}$, for classical (h-independent) Sobolev spaces H^s_{loc}. Here each $\mathtt{C}^z_{j,n+1}$ satisfies

$$\mathtt{C}^z_{j,n+1} = O_{H^s_{\mathrm{loc}}\to H^{s+n+1}_{\mathrm{loc}}}(|\mathrm{Im}\, z|^{-j-1})$$

with leading term

$$\mathtt{C}^z_{0,n+1} = (1+\mathtt{D}_0^2)^{-\frac{n+1}{2}}(z-\mathtt{D}_0)^{-1}.$$

Finally, plugging the expansion (7-21) into the Helffer–Sjöstrand formula

$$\phi(\mathtt{D}) = -\frac{1}{2\pi}\int_{\mathbb{C}} \bar\partial\tilde\rho(z)(1+\mathtt{D}^2)^{-\frac{n+1}{2}}(z-\mathtt{D})^{-1}\, dz\, d\bar z,$$

with $\rho(x) := \langle x\rangle^{n+1}\phi(x)$, gives

$$\phi(\mathtt{D})(0,0) = \left(\sum_{j=0}^{N} h^{\frac{j}{2}} U_{j,p}(\phi)\right) + h^{\frac{N+1}{2}} O\left(\sum_{k=0}^{n+1} \|\langle\xi\rangle^N \hat\phi^{(k)}\|_{L^1}\right) \tag{7-22}$$

using Sobolev's inequality. Here each

$$U_{j,p}(\phi) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar\partial\tilde\rho(z)\mathtt{C}^z_{j,n+1}(0,0)\, dz\, d\bar z \in \mathrm{End}\, S^{TX}_p \tag{7-23}$$

defines a smooth family (in $p\in X$) of distributions U_j and the remainder term in (7-22) comes from the estimate

$$\bar\partial\tilde\rho = O\left(|\mathrm{Im}\, z|^{N+1} \sum_{k=0}^{n+1} \|\langle\xi\rangle^N \hat\phi^{(k)}\|_{L^1}\right)$$

on the almost analytic continuation; see [Zworski 2012, Section 3.1]. Integrating the trace of (7-22) over X and using (7-18) gives (7-11). □

Next we would like to understand the structure of the distributions u_j appearing in (7-11). Clearly,

$$u_j = \int_X u_{j,p}, \quad \text{with } u_{j,p} := \mathrm{tr}\, U_{j,p} \in C^\infty(X; \mathcal{S}'(\mathbb{R}_s)), \tag{7-24}$$

is the smooth family of tempered distributions parametrized by X defined via the pointwise trace of (7-23). Letting $H(s) \in \mathcal{S}'(\mathbb{R}_s)$ denote the Heaviside distribution, we now define the following elementary tempered distributions:

$$v_{a;p}(s) := s^a, \quad a \in \mathbb{N}_0, \tag{7-25}$$

$$v_{a,b,c,\Lambda;p}(s) := \partial_s^a\big[|s|s^b(s^2-2\nu_p\Lambda)^{c-\frac{1}{2}}H(s^2-2\nu_p\Lambda)\big], \quad (a,b,c;\Lambda) \in \mathbb{N}_0\times\mathbb{Z}\times\mathbb{N}_0\times\mu.(\mathbb{N}_0^m\setminus 0). \tag{7-26}$$

Proposition 7.2. *For each j, the distribution (7-24) can be written in terms of (7-25), (7-26):*

$$u_{j,p}(s) = \sum_{a\le 2j+2} c_{j;a}(p)s^a + \sum_{\substack{\Lambda\in\mu.(\mathbb{N}_0^m\setminus 0)\\ a,|b|,c\le 4j+4}} c_{j;a,b,c,\Lambda}(p)v_{a,b,c,\Lambda;p}(s). \tag{7-27}$$

Moreover, the coefficient functions $c_{j;a}, c_{j;a,b,c,\Lambda} \in C^\infty(X)$ above are evaluations at p of polynomials in the covariant derivatives (with respect to $\nabla^{TX}\otimes 1 + 1\otimes\nabla^{A_0}$) of the curvatures F^{TX}, F^{A_0} of the Levi-Civita connection ∇^{TX}, ∇^{A_0} and da.

Proof. It suffices to consider the restriction of u_j to the interval $(-\sqrt{2\nu M}, \sqrt{2\nu M})$ for each $0 < M \notin \mu.(\mathbb{N}_0^m \setminus 0)$. We begin by finding the spectrum of the operator $\mathtt{D}_{00}$ in (7-17). To this end, define the unitary operator $\mathtt{U}_\lambda : C^\infty(\mathbb{R}^n;\mathbb{C}^{2^m}) \to C^\infty(\mathbb{R}^n;\mathbb{C}^{2^m})$,

$$(\mathtt{U}_\lambda s)(x_0,x_1,x_2,\dots) = \Big(\prod_{j=1}^m \lambda_j\Big)s(x_0,\lambda_1^{-\frac{1}{2}}x_1,\lambda_1^{-\frac{1}{2}}x_2,\lambda_2^{-\frac{1}{2}}x_3,\lambda_2^{-\frac{1}{2}}x_4,\dots)$$

and

$$f = \sum_{j=1}^m (x_jx_{j+m} + \xi_j\xi_{j+m}) \in C^\infty(\mathbb{R}^{2m}).$$

Next, as in (5-1) we compute the conjugate

$$e^{\frac{i\pi}{4}f_0^W}\mathtt{U}_\lambda\mathtt{D}_{00}\mathtt{U}_\lambda^* e^{-\frac{i\pi}{4}f_0^W} = [2\nu(p)]^{\frac{1}{2}}D_{\mathbb{R}^m}|_{h=1}$$

of the operator in (7-17) in terms of the magnetic Dirac operator on $\mathbb{R}^m$ (2-21) evaluated at $h=1$. Hence the eigenspaces of $\mathtt{D}_{00}$ are

$$\mathtt{U}_\lambda^* e^{-\frac{i\pi}{4}f_0^W}(E_0\otimes L^2(\mathbb{R}^{m+1}_{x_0,x''})), \quad \mathtt{U}_\lambda^* e^{-\frac{i\pi}{4}f_0^W}(E_\Lambda^\pm\otimes L^2(\mathbb{R}^{m+1}_{x_0,x''})); \Lambda\in\mu.(\mathbb{N}_0^m\setminus 0),$$

with eigenvalues 0, $\pm\sqrt{2\nu\Lambda}$ respectively, where

$$E_0 := \mathbb{C}[\psi_{0,0}|_{h=1}], \quad E_\Lambda^\pm = \bigoplus_{\substack{\tau\in\mathbb{N}_0^m\setminus 0\\ \Lambda=\mu.\tau}} E_\tau^\pm\Big|_{h=1}$$

are as in (6-5). We again let $\mathtt{P}_0$, $\mathtt{P}_\Lambda^\pm$ denote the respective projections onto the eigenspaces of $\mathtt{D}_{00}$ and $\mathtt{P}_\Lambda = \mathtt{P}_\Lambda^+ \oplus \mathtt{P}_\Lambda^-$. We also denote by $\mathtt{P}_{>M} = \oplus_{\Lambda>M}\mathtt{P}_\Lambda$ the projection onto eigenspaces with eigenvalue greater than $\sqrt{2\nu M}$ in absolute value.

Now, since expansions in L^2_{loc} are unique, it suffices to work with the resolvent expansion (7-20) in the computation of u_j. The j-th term in the expansion is of the form

$$\mathtt{C}^z_j = \sum_{k\in I_j} (z-\mathtt{D}_0)^{-1}[\Pi_\alpha \mathtt{D}_{k_\alpha}(z-\mathtt{D}_0)^{-1}], \tag{7-28}$$

where each $\mathtt{D}_{k_\alpha}$ is a differential operator with polynomial coefficients involving the covariant derivatives of the curvatures F^{TX}, F^{A_0} and da. Now using (7-17) we decompose each resolvent term above according to the eigenspaces of $\mathtt{D}_{00}$:

$$\begin{aligned}(z-\mathtt{D}_0)^{-1} = \mathtt{P}_0\Big(\frac{1}{z-\gamma^0\partial_{x_0}}\Big)\mathtt{P}_0 \\ \oplus \bigoplus_{\Lambda\in\mu.\mathbb{N}_0^m\cap(0,M)} \mathtt{P}_\Lambda\Big(\frac{z+\gamma^0\partial_{x_0}+\mathtt{D}_{00}}{z^2+\partial^2_{x_0}-2\nu\Lambda}\Big)\mathtt{P}_\Lambda \oplus \mathtt{P}_{>M}\Big(\frac{z+\gamma^0\partial_{x_0}+\mathtt{D}_{00}}{z^2+\partial^2_{x_0}-\mathtt{D}^2_{00}}\Big)\mathtt{P}_{>M}.\end{aligned} \tag{7-29}$$

Next, we plug (7-29) into (7-28). This gives an expansion for $\mathtt{C}^z_j$ with some of the terms given by

$$T^z[\Pi_\alpha \mathtt{D}_{k_\alpha} T^z], \quad \text{where } T^z = \mathtt{P}_{>M}\Big(\frac{z+\gamma^0\partial_{x_0}+\mathtt{D}_{00}}{z^2+\partial^2_{x_0}-\mathtt{D}^2_{00}}\Big)\mathtt{P}_{>M},$$

and which are holomorphic for $\operatorname{Re} z \in (-\sqrt{2\nu M}, \sqrt{2\nu M})$. For the rest of the terms in $\mathtt{C}^z_j$, we use the commutation relations

$$\begin{gathered}[\gamma^0,\mathtt{P}_0] = [\gamma^0,\mathtt{P}_\Lambda] = [\gamma^0,\mathtt{P}_{>M}] = 0,\\ [\partial_{x_0},\mathtt{P}_0] = [\partial_{x_0},\mathtt{P}_\Lambda] = [\partial_{x_0},\mathtt{P}_{>M}] = 0,\\ [\partial_{x_0},\mathtt{D}_{00}] = 0,\\ [(z^2+\partial^2_{x_0}-2\nu\Lambda)^{-1}, x_j] = \delta_{0j}(z^2+\partial^2_{x_0}-2\nu\Lambda)^{-2}\partial_{x_0},\\ [(z^2+\partial^2_{x_0}-2\nu\Lambda)^{-1}, \partial_{x_J}] = 0,\end{gathered}$$

as well as the Clifford relations (2-7). This now gives a finite sum of terms of the form

$$T^z_0\Big[\prod_{k=1}^K S_k T^z_k\Big] \times \Big[\prod_{\Lambda\in\mu.\mathbb{N}_0^m\cap(0,M)} \frac{1}{(z^2+\partial^2_{x_0}-2\nu\Lambda)^{a_\Lambda}}\Big](z-\gamma^0\partial_{x_0})^{-a_0} z^{b_1} x_0^{b_2}\partial_{x_0}^{b_3}, \tag{7-30}$$

$a_0+\Sigma a_\Lambda \le 2j+2$, $b_1, b_2, b_3 \le j+1$, where each S_k is a differential operator in $(x'x'')$ (i.e., independent of x_0) with polynomial coefficients and each T^z_k is equal to one of

$$\mathtt{P}_0,\quad \mathtt{P}_\Lambda,\quad \mathtt{P}_\Lambda\mathtt{D}_{00}\mathtt{P}_\Lambda,\quad \mathtt{P}_{>M}\Big(\frac{1}{z^2+\partial^2_{x_0}-\mathtt{D}^2_{00}}\Big)\mathtt{P}_{>M},\quad\text{or}\quad \mathtt{P}_{>M}\Big(\frac{\mathtt{D}_{00}}{z^2+\partial^2_{x_0}-\mathtt{D}^2_{00}}\Big)\mathtt{P}_{>M}, \tag{7-31}$$

with at least one occurrence of $\mathtt{P}_0, \mathtt{P}_\Lambda$ or $\mathtt{P}_\Lambda\mathtt{D}_{00}\mathtt{P}_\Lambda$ in (7-30). Now using partial fractions, (7-30) may be written as a sum of terms of the forms

$$T_0^z\left[\prod_{k=1}^K S_k T_k^z\right]\times(z-\gamma^0\partial_{x_0})^{-a_0}z^{b_1}x_0^{b_2}\partial_{x_0}^{b_3},$$

$$T_0^z\left[\prod_{k=1}^K S_k T_k^z\right]\times(z^2+\partial_{x_0}^2-2\nu\Lambda)^{-a_\Lambda}z^{b_1}x_0^{b_2}\partial_{x_0}^{b_3},\quad \Lambda\in\mu.\mathbb{N}_0^m\cap(0,M), \tag{7-32}$$

$a_0, a_\Lambda \le 2j+2$, $b_1, b_2, b_3 \le j+1$. Next, we plug (7-32) into the Helffer–Sjöstrand formula and use the analyticity of $\mathtt{P}_{>M}(1/(z^2+\partial_{x_0}^2-\mathtt{D}_{00}^2))\mathtt{P}_{>M}$ and $\mathtt{P}_{>M}(\mathtt{D}_{00}/(z^2+\partial_{x_0}^2-\mathtt{D}_{00}^2))\mathtt{P}_{>M}$ for $\operatorname{Re} z\in(-\sqrt{2\nu M},\sqrt{2\nu M})$. This gives

$$U_{j,p}(\phi)=-\frac{1}{2\pi}\int_{\mathbb{C}}\bar\partial\tilde\phi(z)\mathtt{C}_j^z(0,0)\,dz\,d\bar z$$

for $\phi\in C_c^\infty(-\sqrt{2\nu M},\sqrt{2\nu M})$, as a sum of terms of the form

$$\left(T_0^0\left[\prod_{k=1}^K S_k T_k^0\right]\times x_0^{b_2}\partial_{x_0}^{b_3}\phi_0(\gamma^0\partial_{x_0})\right)(0,0),$$

$$\left(T_0^0\left[\prod_{k=1}^K S_k T_k^0\right]\times x_0^{b_2}\partial_{x_0}^{b_3}\phi_\Lambda(-\partial_{x_0}^2+2\nu\Lambda)\right)(0,0),\quad \Lambda\in\mu.\mathbb{N}_0^m\cap(0,M), \tag{7-33}$$

where each T_k^0 is equal to one of

$$\mathtt{P}_0,\quad \mathtt{P}_\Lambda,\quad \mathtt{P}_\Lambda\mathtt{D}_{00}\mathtt{P}_\Lambda,\quad \mathtt{P}_{>M}\left(\frac{1}{2\nu\Lambda-\mathtt{D}_{00}^2}\right)\mathtt{P}_{>M},\quad\text{or}\quad \mathtt{P}_{>M}\left(\frac{\mathtt{D}_{00}}{2\nu\Lambda-\mathtt{D}_{00}^2}\right)\mathtt{P}_{>M},$$

and

$$\phi_0(s)=\frac{(-1)^{a_0-1}}{(a_0-1)!}x^{b_1}\phi(s),$$

$$\phi_\Lambda(s^2)=\frac{(-1)^{a_\Lambda-1}}{(a_\Lambda-1)!}\left\{\left[\partial_r^{a_\Lambda-1}\left(\frac{r^{b_1}\phi(r)}{(r-s)^{a_\Lambda}}\right)\right]\Big|_{r=-s}-\left[\partial_r^{a_\Lambda-1}\left(\frac{r^{b_1}\phi(r)}{(r+s)^{a_\Lambda}}\right)\right]\Big|_{r=s}\right\}.$$

At least one occurrence of $\mathtt{P}_0$,$\mathtt{P}_\Lambda$ and $\mathtt{P}_\Lambda\mathtt{D}_{00}\mathtt{P}_\Lambda$ in (7-33) gives the smoothness of the kernel.

Finally, an elementary computation involving Laplace transforms using the knowledge of the heat kernel

$$e^{t\partial_{x_0}^2}(x_0,y_0)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{|x_0-y_0|^2}{4t}}$$

gives

$$x_0^{b_2}\partial_{x_0}^{b_3}\phi_0(\gamma^0\partial_{x_0})(0,0)=\frac{(-\frac12)^{[\frac{b_3+1}{2}]}}{\sqrt{\pi}\,\Gamma\left(\left[\frac{b_3+1}{2}\right]+\frac12\right)}\delta_{0b_2}v_{b_3;p}(\phi_0),$$

$$x_0^{b_2}\partial_{x_0}^{b_3}\phi_\Lambda(-\partial_{x_0}^2+2\nu\Lambda)(0,0)=\begin{cases}\dfrac{(-\frac12)^{\frac{b_3}{2}}}{4\pi\Gamma\left(\frac{b_3}{2}-\frac12\right)}\delta_{0b_2}v_{0,0,\frac{b_3}{2},\Lambda;p}(\phi_\Lambda(s^2)), & b_3\text{ even},\\ 0, & b_3\text{ odd},\end{cases}$$

completing the proof. □

As an immediate corollary of Proposition 7.2, we have that the distributions u_j are smooth near 0.

Corollary 7.3. *For each j,*

$$\operatorname{sing\,spt}(u_j) \subset \mathbb{R} \setminus (-\sqrt{2\nu_0}, \sqrt{2\nu_0}).$$

Proof. This follows immediately from (7-24)–(7-27) on noting that the distributions $v_{a;p}$ are smooth, while $v_{a,b,c,\Lambda;p} = 0$ on $\mathbb{R} \setminus (-\sqrt{2\nu_0}, \sqrt{2\nu_0})$ for each $p \in X$. □

We next give the exact computation for the first coefficient u_0 of Proposition 7.1. In the computation below, recall that $Z_\tau = |I_\tau|$, as in (2-13), denotes the number of nonzero components of $\tau \in \mathbb{N}_0^m \setminus 0$.

Proposition 7.4. *The first coefficient u_0 of (7-11) is given by*

$$u_{0,p} = c_{0;0} + \sum_{\Lambda \in \mu.(\mathbb{N}_0^m \setminus 0)} c_{0;0,0,0,\Lambda}(p) v_{0,0,0,\Lambda;p}(s), \tag{7-34}$$

where

$$c_{0;0} = \frac{\nu_p^m \left(\prod_{j=1}^m \mu_j\right)}{(4\pi)^{\frac{n}{2}}},$$

$$c_{0;0,0,0,\Lambda}(p) = \frac{\nu_p^m \left(\prod_{j=1}^m \mu_j\right)}{(4\pi)^{\frac{n}{2}}} \dim(E_\Lambda) = \frac{\nu_p^m \left(\prod_{j=1}^m \mu_j\right)}{(4\pi)^{\frac{n}{2}}} \Bigg(\sum_{\substack{\tau \in \mathbb{N}_0^m \setminus 0 \\ \mu.\tau = \Lambda}} 2^{Z_\tau} \Bigg). \tag{7-35}$$

Proof. First note that the square of (7-16) gives the harmonic oscillator

$$\mathrm{D}_0^2 = -\delta^{jk} \partial_{x_j} \partial_{x_k} - i (da)^j_k(0) x^k \partial_{x_j} + \frac{1}{4} x^k x_l (da)^j_k(0)(da)^l_j(0) + \frac{i}{2} \gamma^j \gamma^k (da)_{jk}(0).$$

The heat kernel $e^{-t\mathrm{D}_0^2}$ of the above is given by Mehler's formula [Berline et al. 2004, Section 4.2]

$$e^{-t\mathrm{D}_0^2}(x,y) = \frac{1}{(4\pi t)^m} \frac{e^{-\frac{(x_0-y_0)^2}{4t}}}{\sqrt{4\pi t}} \det{}^{\frac{1}{2}} \left(\frac{itda(0)}{\sinh itda(0)} \right) e^{-tc(ida(0))} \tag{7-36}$$

$$\times \exp\left\{ -\frac{\lambda_j}{4\tanh \lambda_j t} \left((x_j - y_j)^2 + (x_{j+m} - y_{j+m})^2 \right) + \frac{\lambda_j}{2} \tanh\left(\frac{\lambda_j t}{2} \right) (x_j y_j + x_{j+m} y_{j+m}) \right\}. \tag{7-37}$$

Next, using (7-1) we compute

$$e^{-tc(ida(0))} = \prod_{j=1}^m \left[\cosh(t\lambda_j) - ic(e_j)c(e_{j+m}) \sinh(t\lambda_j) \right]. \tag{7-38}$$

For $I \subset \{2, \dots, m\}$ and $\omega_I = \bigwedge_{j \in I} (e_j \wedge e_{j+m})$, the commutation

$$c(e_1)c(e_{m+1})c(\omega_I) = \tfrac{1}{2} [c(e_1), c(e_{m+1})c(\omega_I)]$$

shows that the only traceless terms in (7-38) are the constants. Hence, Mehler's formula (7-36) gives

$$\begin{aligned}
\operatorname{tr} e^{-t\mathrm{D}_0^2}(0,0) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \det{}^{\frac{1}{2}}\left(\frac{itda(0)}{\tanh itda(0)}\right) = \frac{t^{-\frac{1}{2}}}{(4\pi)^{\frac{n}{2}}}\left(\prod_{j=1}^{m} \frac{\lambda_j}{\tanh t\lambda_j}\right) \\
&= \frac{t^{-\frac{1}{2}}}{(4\pi)^{\frac{n}{2}}}\left[\prod_{j=1}^{m} \lambda_j(1+2e^{-2t\lambda_j}+2e^{-4t\lambda_j}+\cdots)\right] \\
&= \frac{t^{-\frac{1}{2}}}{(4\pi)^{\frac{n}{2}}}\left(\prod_{j=1}^{m} \lambda_j\right)\left(\sum_{\tau\in\mathbb{N}_0^m} 2^{Z_\tau} e^{-2t\tau.\lambda}\right) \\
&= \frac{\nu_p^m\left(\prod_{j=1}^{m}\mu_j\right)}{(4\pi)^{\frac{n}{2}}}\left(t^{-\frac{1}{2}}\sum_{\tau\in\mathbb{N}_0^m} 2^{Z_\tau} e^{-2t\tau.\lambda}\right) = u_{0,p}(e^{-ts^2}),
\end{aligned} \tag{7-39}$$

with $u_{0,p}$ as in (7-34) and the last line above following from an easy computation of Laplace transforms; see [Savale 2014, Section 4]. Furthermore, differentiating Mehler's formula using (7-16) gives

$$\operatorname{tr} \mathrm{D}_0 e^{-t\mathrm{D}_0^2}(0,0) = 0 = u_{0,p}(se^{-ts^2}) \tag{7-40}$$

since the right-hand side of (7-34) is an even distribution. From (7-39) and (7-40) we have that the evaluations of both sides of (7-34) on e^{-ts^2}, se^{-ts^2} are equal. Differentiating with respect to t and setting $t=1$ gives that the two sides of (7-34) evaluate equally on $s^k e^{-s^2}$ for all $k\in\mathbb{N}_0$. The proposition now follows from the density of this collection in $\mathcal{S}(\mathbb{R}_s)$. □

We now complete the proof of Lemma 3.2.

Proof of Lemma 3.2. We begin by writing

$$\operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h^{1-\varepsilon}}\check{\theta}\left(\frac{\lambda\sqrt{h}-D}{h^{1-\varepsilon}}\right)\right] = \frac{h^{-\frac{1}{2}}}{2\pi}\int dt\, \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)e^{it(\lambda-\frac{D}{\sqrt{h}})}\right]\theta(th^{\frac{1}{2}-\varepsilon}). \tag{7-41}$$

Next, the expansion result, Proposition 7.1, with $\phi(x) = f(x)e^{it(\lambda-x)}$, combined with the smoothness of u_j on $\operatorname{spt}(f)\subset(-\sqrt{2\nu_0},\sqrt{2\nu_0})$ from Corollary 7.3 gives

$$\operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)e^{it(\lambda-\frac{D}{\sqrt{h}})}\right] = e^{it\lambda}h^{-\frac{n}{2}}\left(\sum_{j=0}^{N} h^{\frac{j}{2}}\widehat{fu_j}(t)\right) + h^{\frac{N+1-n}{2}}\underbrace{O\left(\sum_{k=0}^{n+1}\|\langle\xi\rangle^N\hat{\phi}^{(k)}(\xi-t)\|_{L^1}\right)}_{=O(\langle t\rangle^N)}. \tag{7-42}$$

Finally, plugging (7-42) into (7-41) and using $\theta(th^{\frac{1}{2}-\varepsilon}) = 1+O(h^\infty)$ gives via Fourier inversion

$$\frac{h^{-\frac{1}{2}}}{2\pi}\int dt\, \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)e^{it(\lambda-\frac{D}{\sqrt{h}})}\right]\theta(th^{\frac{1}{2}-\varepsilon}) = h^{-m-1}\left(\sum_{j=0}^{N} h^{\frac{j}{2}} f(\lambda)u_j(\lambda)\right) + O(h^{\varepsilon(N+1)-m-1})$$

as required. □

8. Asymptotics of spectral invariants

In this section we prove Theorem 1.2 on the asymptotics of the spectral invariants.

Proof of Theorem 1.2. To prove the Weyl law (1-5), we choose $\theta \in C_c^\infty((-T,T);[0,1])$ such that $\theta(x)=1$ on $(-T',T')$, $T'<T$, $\check{\theta}(\xi)\geq 0$ and $\check{\theta}(\xi)\geq 1$ for $|\xi|\leq c$ in Theorem 1.3. Choosing $f(x)\geq 0$ with $f(0)=1$, the trace expansion (1-7) with $\lambda=0$ now gives

$$\frac{1}{h}N(-ch,ch)(1+O(\sqrt{h}))\leq \operatorname{tr}\left[f\left(\frac{D}{\sqrt{h}}\right)\frac{1}{h}\check{\theta}\left(\frac{-D}{h}\right)\right]=O(h^{-m-1})$$

proving (1-5).

To prove the estimate (1-6) on the eta invariant, we first use its invariance under positive scaling (2-2) and the formula (2-5) to write

$$\begin{aligned}\eta_h=\eta\left(\frac{D}{\sqrt{h}}\right)&=\int_0^\infty dt\,\frac{1}{\sqrt{\pi t}}\operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]\\&=\int_0^1 dt\,\frac{1}{\sqrt{\pi t}}\operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]+\int_1^\infty dt\,\frac{1}{\sqrt{\pi t}}\operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right].\end{aligned}\tag{8-1}$$

Next, [Savale 2014, equation 4.5, p. 859] with $r=\frac{1}{h}$ translates to the estimate

$$\operatorname{tr}\left[\frac{D}{\sqrt{h}}e^{-\frac{t}{h}D^2}\right]=O(h^{-m}e^{ct}).\tag{8-2}$$

Plugging, (8-2) into the first integral of (8-1) gives

$$\eta_h=O(h^{-m})+\operatorname{tr}E\left(\frac{D}{\sqrt{h}}\right),\tag{8-3}$$

where

$$E(x)=\operatorname{sign}(x)\operatorname{erfc}(|x|)=\operatorname{sign}(x)\cdot\frac{2}{\sqrt{\pi}}\int_{|x|}^\infty e^{-s^2}\,ds$$

with the convention $\operatorname{sign}(0)=0$. The function $E(x)$ above is rapidly decaying with all derivatives, odd and smooth on $\mathbb{R}_x\setminus 0$. We may hence choose functions $f\in C_c^\infty(-\sqrt{2\nu_0},\sqrt{2\nu_0})$, $g\in C_c^\infty(\mathbb{R}_{<0})$ such that

$$f(x)+g(x)=E(x)\quad\text{for } x\leq 0.$$

Define the spectral measure

$$\mathfrak{M}_f(\lambda'):=\sum_{\lambda\in\operatorname{Spec}\left(\frac{D}{\sqrt{h}}\right)}f(\lambda)\delta(\lambda-\lambda').$$

It is clear that the expansion (1-7) to its first term may be written as

$$\mathfrak{M}_f*(\mathcal{F}_h^{-1}\theta_{\frac{1}{2}})(\lambda)=h^{-m-\frac{1}{2}}(f(\lambda)u_0(\lambda)+O(h^{\frac{1}{2}})),$$

where $\theta_{\frac{1}{2}}(x)=\theta(x/\sqrt{h})$ as before. Since both sides above involve Schwartz functions in λ, the remainder maybe replaced by $O(h^{\frac{1}{2}}/\langle\lambda\rangle^2)$. One may then integrate the equation to obtain

$$\int_{-\infty}^{0} d\lambda \int d\lambda' (\mathcal{F}_h^{-1}\theta_{\frac{1}{2}})(\lambda-\lambda')\mathfrak{M}_f(\lambda') = h^{-m-\frac{1}{2}}\left(\int_{-\infty}^{0} d\lambda f(\lambda)u_0(\lambda) + O(h^{\frac{1}{2}})\right). \tag{8-4}$$

Next we observe

$$\int_{-\infty}^{0} d\lambda (\mathcal{F}_h^{-1}\theta_{\frac{1}{2}})(\lambda-\lambda') = \int_{-\infty}^{0} dt\check{\theta}\left(t-\frac{\lambda'}{\sqrt{h}}\right) = 1_{(-\infty,0]}(\lambda') + O\left(\left\langle\frac{\lambda'}{\sqrt{h}}\right\rangle^{-\infty}\right). \tag{8-5}$$

While the Weyl law yields

$$\int d\lambda'\mathfrak{M}_f(\lambda')O\left(\left\langle\frac{\lambda'}{\sqrt{h}}\right\rangle^{-\infty}\right) = O(h^{-m}). \tag{8-6}$$

Substituting (8-5) and (8-6) into (8-4) gives

$$\sum_{\substack{\lambda\le 0\\ \lambda\in\operatorname{Spec}\left(\frac{D}{\sqrt{h}}\right)}} f(\lambda) = h^{-m-\frac{1}{2}}\left(\int_{-\infty}^{0} d\lambda f(\lambda)u_0(\lambda)\right) + O(h^{-m}).$$

This combined with

$$\operatorname{tr} g\left(\frac{D}{\sqrt{h}}\right) = h^{-m-\frac{1}{2}}u_0(g) + O(h^{-m})$$

then gives

$$\sum_{\substack{\lambda\le 0\\ \lambda\in\operatorname{Spec}\left(\frac{D}{\sqrt{h}}\right)}} E(\lambda) = h^{-m-\frac{1}{2}}\left(\int_{-\infty}^{0} d\lambda E(\lambda)u_0(\lambda)\right) + O(h^{-m}),$$

where the integral makes sense from the formula (7-34) for u_0. A similar formula for

$$\sum_{\substack{\lambda\ge 0\\ \lambda\in\operatorname{Spec}\left(\frac{D}{\sqrt{h}}\right)}} E(\lambda)$$

now gives

$$\operatorname{tr} E\left(\frac{D}{\sqrt{h}}\right) = h^{-m-\frac{1}{2}}\left(\int_{-\infty}^{\infty} d\lambda E(\lambda)u_0(\lambda)\right) + O(h^{-m}).$$

Since E is odd and u_0 is even from (7-34), the integral above is zero and hence $\eta_h = \operatorname{tr} E(D/\sqrt{h}) = O(h^{-m})$ from (8-3) as required. □

In the above proof we have used a Tauberian argument, as in [Dimassi and Sjöstrand 1999, Chapter 10]. A similar argument along with the trace expansion theorem, Theorem 1.3, also gives a true Weyl law in $O(\sqrt{h})$-sized intervals: the number of eigenvalues $N(-c\sqrt{h}, c\sqrt{h})$, $0<c<\sqrt{2\nu_0}$, in the given interval satisfies

$$N(-c\sqrt{h}, c\sqrt{h}) = h^{-m-\frac{1}{2}}\left[\frac{2c}{(4\pi)^m}\int_X \nu^m\left(\prod_{j=1}^{m}\mu_j\right)dx\right] + O(h^{-m}). \tag{8-7}$$

The leading term of the above may possibly be obtained by squaring the Dirac operator and using the spectral estimates on an $O(h)$-sized interval near the critical level for D_h^2, as in [Brummelhuis et al. 1995].

8A. *Sharpness of the result.* Here, we finally show that the result Theorem 1.2 is sharp. The worst case example was already noted in [Savale 2014, Section 5] for η_h. To recall, we let Y be a complex manifold of dimension $2m$ with complex structure J and a Riemannian metric g^{TY}. Fix a positive, holomorphic, Hermitian line bundle $\mathcal{L} \to Y$. The curvature $F^{\mathcal{L}}$ of the Chern connection is thus a positive $(1,1)$-form. Let X be the total space of the unit circle bundle $S^1 \to X \xrightarrow{\pi} Y$ of $\mathcal{L}$. The Chern connection gives a splitting of the tangent bundle

$$TX = TS^1 \oplus \pi^* TY, \tag{8-8}$$

where TS^1 is the vertical tangent space spanned by the generator e of the S^1 action. Define a metric g^{TS^1} on TS^1 via $\|e\|_{g^{TS^1}} = 1$. A metric on X can now be given using the splitting (8-8) via

$$g^{TX} = g^{TS^1} \oplus \varepsilon^{-1} \pi^* g^{TY}$$

for any $\varepsilon > 0$. A spin structure on Y corresponds to a holomorphic, Hermitian square root $\mathcal{K}$ of the canonical line bundle $K_Y = \mathcal{K}^{\otimes 2}$. Fixing such a spin structure as well as the trivial spin structure on TS^1 gives a spin structure on X. Finally $a = e^* \in \Omega^1(X)$, while the auxiliary line bundle is chosen to be trivial $L = \mathbb{C}$ with the family of connections $\nabla^h = d + \frac{i}{h} a$. We now have the required family of Dirac operators D_h (1-2). One may check that (X^{2m+1}, a, g^{TX}, J) here gives a metric contact structure (1-4) and hence the assumption Definition 1.1 is satisfied.

Denote by $\Delta^p_{\bar\partial_k} : \Omega^{0,p}(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k}) \longrightarrow \Omega^{0,p}(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k})$ the Hodge Laplacian acting on $(0, p)$ forms on X. Its null-space is given by the cohomology $H^p(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k})$ of the tensor product via Hodge theory. Let $e^{p,k}_\mu$ denote the dimension of a each positive eigenspace with eigenvalue $\frac{1}{2}\mu^2 \in \mathrm{Spec}^+(\Delta^p_{\bar\partial_k})$. The spectrum of D_h was now computed in Proposition 5.2 of [Savale 2014].

Proposition 8.1. *The spectrum of D_h is given by eigenvalues of the following types*:

- *Type* 1.

$$\lambda = (-1)^p h \left(k + \left(\varepsilon - \frac{m}{2} \right) - \frac{1}{h} \right), \tag{8-9}$$

$0 \le p \le m$, $k \in \mathbb{Z}$, *with multiplicity* $\dim H^p(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k})$.

- *Type* 2.

$$\lambda = h \left[\frac{1}{2} \left((-1)^{p+1} \varepsilon \pm \sqrt{ \left(2k + \varepsilon(2p - m) - \frac{2}{h} + 1 \right)^2 + 4\mu^2 \varepsilon } \right) \right], \tag{8-10}$$

$0 \le p \le m$, $k \in \mathbb{Z}$, $\frac{1}{2}\mu^2 \in \mathrm{Spec}^+(\Delta^p_{\bar\partial_k})$, *with multiplicity* $d^{p,k}_\mu := e^{p,k}_\mu - e^{p-1,k}_\mu + \cdots + (-1)^p e^{0,k}_\mu$.

As observed in [Savale 2014], by choosing

$$\varepsilon < \inf_{k,p} \left\{ \tfrac{1}{2}\mu^2 \in \mathrm{Spec}^+(\Delta^p_{\bar\partial_k}) \right\},$$

the eigenvalues of type 2 are either positive or negative depending on the sign appearing in (8-10). Hence the dimension of the kernel k_h of D_h is now given by the eigenvalues of type 1:

$$k_h = \begin{cases} \dim H^*(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k}), & \frac{1}{h} = k + \left(\varepsilon - \frac{m}{2}\right), \\ 0, & \text{otherwise}. \end{cases} \tag{8-11}$$

Now by a combination of Kodaira vanishing and Hirzebruch–Riemann–Roch,

$$\dim H^*(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k}) = \dim H^0(X; \mathcal{K} \otimes \mathcal{L}^{\otimes k}) = \chi(X, \mathcal{K} \otimes \mathcal{L}^{\otimes k}) = \int_X \mathrm{ch}(\mathcal{K} \otimes \mathcal{L}^{\otimes k})\, \mathrm{td}(X)$$

for $k \gg 0$, where $\chi(X, \mathcal{K} \otimes \mathcal{L}^{\otimes k})$, $\mathrm{ch}(\mathcal{K} \otimes \mathcal{L}^{\otimes k})$ and $\mathrm{td}(X)$ denote Euler characteristic, Chern character and Todd genus respectively. Hence (8-11), (8-12) show that the kernel and hence the counting function are discontinuous of order $O(h^{-m}) = k_h \le N(-ch, ch)$ in this example. A similar discontinuity of the eta invariant of $O(h^{-m})$ was proved in Theorem 5.3 of [Savale 2014].

Appendix: Some spectral estimates

In this appendix we prove some spectral estimates used in Sections 4 and 5; see [Helffer 1988, Section 4.1] for some related estimates.

Let H be a separable Hilbert space. Let $A : H \to H$ be a bounded self-adjoint operator. The resolvent set and the spectrum of A are defined to be

$$\begin{aligned} R(A) &= \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is invertible}\}, \\ \mathrm{Spec}(A) &= \mathbb{C} \setminus R(A). \end{aligned}$$

Since A is self-adjoint, $\mathrm{Spec}(A) \subset \mathbb{R}$. We may now define the following subsets of the spectrum:

$$\begin{aligned} \mathrm{EssSpec}(A) &= \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not Fredholm}\}, \\ \mathrm{DiscSpec}(A) &= \mathrm{Spec}(A) \setminus \mathrm{EssSpec}(A). \end{aligned}$$

We shall consider $\mathrm{DiscSpec}(A)$ above as a multiset with the multiplicity function $m^A : \mathrm{DiscSpec}(A) \to \mathbb{N}_0$ defined by $m^A(\lambda) = \dim \ker(A)$. We may then find a countable set of orthonormal eigenvectors $v_1^A, v_2^A, v_3^A, \ldots$, with eigenvalues $\lambda_1^A \le \lambda_2^A \le \lambda_3^A \le \cdots$ such that $\mathrm{DiscSpec}(A)$ and $\{\lambda_1^A, \lambda_2^A, \ldots\}$ are equal as multisets. Now let $[a, b] \subset \mathbb{R}$ be a finite closed interval such that $\mathrm{EssSpec}(A) \cap [a, b] = \varnothing$ (i.e., A has discrete spectrum in $[a, b]$). Then

$$H^A_{[a,b]} = \bigoplus_{\lambda \in \mathrm{Spec}(A) \cap [a,b]} \ker(A - \lambda)$$

is a finite-dimensional vector subspace of H. We denote by

$$\Pi^A_{[a,b]} : H \to H^A_{[a,b]} \subset H$$

the orthogonal projection onto $H^A_{[a,b]}$ and by $N^A_{[a,b]}$ the dimension of $H^A_{[a,b]}$. The operator $\rho(A) : H \to H$ may now be defined for any function $\rho \in C_c^0([a, b])$ by functional calculus.

Lemma A.1. *Let $v \in H$ and $\lambda \in [a,b]$. Assume there exists $\varepsilon > 0$ such that A has discrete spectrum in $[a-\sqrt{\varepsilon}, b+\sqrt{\varepsilon}]$ and $\|(A-\lambda)v\| \le \varepsilon\|v\|$. Then*

$$\|\Pi^A_{[a-\sqrt{\varepsilon},b+\sqrt{\varepsilon}]}v - v\| \le \sqrt{\varepsilon}\|v\|, \tag{A-1}$$

$$\|(\rho(A)-\rho(\lambda))v\| \le 3\sqrt{\varepsilon}\|\rho\|_{C^{0,1}}\|v\| \tag{A-2}$$

for any Holder continuous function $\rho \in C^{0,1}_c([a,b])$.

Proof. We abbreviate $\Pi = \Pi^A_{[a-\sqrt{\varepsilon},b+\sqrt{\varepsilon}]}$. Let $H_0 := H^A_{[a-\sqrt{\varepsilon},b+\sqrt{\varepsilon}]} = \Pi H$, which by assumption is a finite-dimensional vector space. Let $H_0^\perp$ be the orthogonal complement of H_0. By assumption, $\mathrm{Spec}((A-\lambda)^2|_{H_0^\perp}) \cap [-\varepsilon,\varepsilon] = \varnothing$. Hence by the mini-max principle for self-adjoint operators bounded from below [Dimassi and Sjöstrand 1999, Lemma 4.21], we have $\varepsilon \le (A-\lambda)^2|_{H_0^\perp}$. Hence

$$\begin{aligned}\|\Pi v - v\|^2\varepsilon &\le \|(A-\lambda)(\Pi v - v)\|^2 \\ &\le \|(A-\lambda)(\Pi v - v)\|^2 + \|(A-\lambda)\Pi v\|^2 = \|(A-\lambda)v\|^2 \le \varepsilon^2\|v\|^2\end{aligned}$$

since $(A-\lambda)(\Pi v - v)$ and $(A-\lambda)\Pi v$ are orthogonal. This gives

$$\|\Pi v - v\| < \sqrt{\varepsilon}\|v\|. \tag{A-3}$$

To prove (A-2) first note that $\|\Pi' v - v\| < \sqrt{\varepsilon}\|v\|$, for $\Pi' = \Pi^A_{[\lambda-\sqrt{\varepsilon},\lambda+\sqrt{\varepsilon}]}$, by the same argument. We now have

$$\begin{aligned}\|(\rho(A)-\rho(\lambda))v\| &\le \|(\rho(A)-\rho(\lambda))(\Pi' v - v)\| + \|(\rho(A)-\rho(\lambda))\Pi' v\| \\ &\le 2\sqrt{\varepsilon}\|\rho\|_{C^{0,1}}\|v\| + \sqrt{\varepsilon}\|\rho\|_{C^{0,1}}\|v\|. \qquad \square\end{aligned}$$

Before stating the next lemma we need the following definition.

Definition A.2. Given $0 < \varepsilon < 1$, a set of vectors $w_1, w_2, \dots, w_N \in H$ is called an ε-almost orthonormal set of eigenvectors (ε-AOSE for short) of A if

(1) $\big|\|w_j\|^2 - 1\big| < \varepsilon$ for all j,

(2) $|\langle w_j, w_k\rangle| < \varepsilon$ for all $j \ne k$,

(3) $\|(A-\mu_j)w_j\| < \varepsilon$ for some $\mu_j \in \mathbb{R}$, for all j.

Lemma A.3. *Assume that $H_0 \subset H$ has finite dimension M and is mapped onto itself by A. Let $w_1, w_2, \dots, w_N \in H_0$ be an ε-AOSE of A for some $\varepsilon < 1/(2(M+1))$. Then there exist orthonormal $w'_1, w'_2, \dots, w'_{M-N} \in H_0$ such that $\|(A-\mu'_j)w'_j\| < 4M\varepsilon$ for some $\mu'_j \in \mathbb{R}$, for all j. Furthermore $\langle w_j, w'_k\rangle = 0$ for each j,k.*

Proof. It follows from $\varepsilon < 1/(2(M+1))$ that $w_1, w_2, \dots, w_N$ are linearly independent. Let W denote their span and $W^\perp \subset H_0$ its orthogonal complement. Let $\Pi, \Pi^\perp$ be the orthogonal projections onto $W, W^\perp$ and consider the operator $A_0 := \Pi^\perp A \Pi^\perp : W^\perp \to W^\perp$. Let $w'_1, w'_2, \dots, w'_{M-N} \in W^\perp$ be an orthogonal basis of eigenvectors of A_0. Hence

$$\Pi^\perp A w'_j = \mu'_j w'_j$$

for some $\mu_j' \in \mathbb{R}$, for all j. Also

$$|\langle Aw_j', w_k\rangle| = |\langle w_j', (A-\mu_k)w_k\rangle| < \varepsilon.$$

It then follows that $\|\Pi A w_j'\| \leq 2M\varepsilon\sqrt{1+\varepsilon} < 4M\varepsilon$ giving the result. □

Lemma A.4. *Given $N \in \mathbb{N}$, let $0 < \varepsilon < (\|A\| + |a| + |b| + N + 1)^{-4}$. Let $w_1, w_2, \ldots, w_N \in H$ be an ε-AOSE for A. Assume that A has discrete spectrum in $[a-\varepsilon^{\frac{1}{8}}, b+\varepsilon^{\frac{1}{8}}]$. Then there exist orthonormal vectors $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_N \in H$ which span the same subspace of H as $w_1, w_2, \ldots, w_N$. Moreover $\|w_j - \bar{w}_j\| < \sqrt{\varepsilon}$ and $\|(\rho(A)-\rho(\mu_j))\bar{w}_j\| \leq 3\varepsilon^{\frac{1}{8}}\|\rho\|_{C^{0,1}}$ for $1 \leq j \leq N$ and any Holder continuous function $\rho \in C_c^{0,1}([a,b])$.*

Proof. Again it follows easily that the vectors w_j, $1 \leq j \leq N$, are linearly independent. Let $W \subset H$ be their span and choose an orthonormal basis e_i, $1 \leq j \leq N$, for W. We write

$$w_j = \sum_{k=1}^{N} m_{jk} e_k.$$

If we consider the matrix $M = [m_{jk}]$, then assumptions (1) and (2) of Definition A.2 are equivalent to $|M^*M - I| < \varepsilon$. Consider the polar decomposition $M = UP$, where U is unitary and P is a positive semidefinite Hermitian matrix. We have $|P^*P - I| < \varepsilon$ and hence $\|P^*P - I\| < N\varepsilon$. Thus any eigenvalue λ^P of P, being nonnegative, satisfies $|\lambda^P - 1| < \varepsilon$ and we have $\|P - I\| < N\varepsilon$. Thus $\|M - U\| = \|UP - U\| < N\varepsilon$. If we now let $U = [u_{jk}]$ and $\bar{w}_j = \sum_{k=1}^{N} u_{jk}e_k$, then the $\bar{w}_j$ are clearly orthonormal and satisfy $\|w_j - \bar{w}_j\| < \sqrt{\varepsilon}$. This last inequality along with assumption (3) of Definition A.2 easily gives

$$\|(A-\mu_j)\bar{w}_j\| < \varepsilon^{\frac{1}{4}}.$$

Now Lemma A.1 gives

$$\|\Pi\bar{w}_j - \bar{w}_j\| < \varepsilon^{\frac{1}{8}}, \tag{A-4}$$

$$\|(\rho(A)-\rho(\mu_j))\bar{w}_j\| < 3\varepsilon^{\frac{1}{8}}\|\rho\|_{C^{0,1}}, \tag{A-5}$$

completing the proof. □

Next, let H' be another separable Hilbert space. Let $U : H \to H'$ be a bounded operator. Let $B, D : H' \to H'$ and $C : H \to H$ be bounded self-adjoint operators. Define $A' = UAU^* : H' \to H'$, $B' = U^*BU : H \to H$, $C' = UCU^* : H' \to H'$ and $D' = U^*DU : H \to H$. In the next proposition we assume that there exists $\delta > 0$ such that A, A', B and B' have discrete spectrum in $[a-\delta, b+\delta]$. We also abbreviate $N^A = N^A_{[a-\delta,b+\delta]}$ and $\Pi^A = \Pi^A_{[a-\delta,b+\delta]}$ and similarly define $N^{A'}$, N^B, $N^{B'}$, $\Pi^{A'}$, Π^B, $\Pi^{B'}$.

Proposition A.5. *Suppose there exists $0 < \varepsilon < L^{-2048}$, with*

$$L = 25\big\{\|A\| + \|A'\| + \|B\| + \|B'\| + \|C\| + \|D\| + N^A + N^{A'} + N^B + N^{B'} + |a| + |b| + \delta^{-1} + 1\big\},$$

such that

(1) $\|(U^*U - I)\Pi^A\|\big(\|A\|\,\|U\| + 1\big) < \varepsilon$ *and* $\|(UU^* - I)\Pi^B\|\big(\|B\|\,\|U^*\| + 1\big) < \varepsilon$,

(2) $\|(A'-B)\Pi^{A'}\| < \varepsilon$ *and* $\|(A-B')\Pi^{B'}\| < \varepsilon$,

(3) $\|(C'-D)\Pi^{A}\| < \varepsilon$ *and* $\|(C-D')\Pi^{B}\| < \varepsilon$.

Then we have

$$\left|\mathrm{tr}[C\rho(A)] - \mathrm{tr}[D\rho(B)]\right| \le \varepsilon^{\frac{1}{2048}}\|\rho\|_{C^1}$$

for any $\rho \in C^1_c([a,b])$.

Proof. Let $(\mathrm{DiscSpec}(A), m^A) \cap [a,b] = \{\lambda^A_{a_1}, \lambda^A_{a_2}, \dots, \lambda^A_{a_N}\}$, with $N = N^A_{[a,b]}$, as multisets. Let $\rho^+(x) = \frac{1}{2}(\rho(x) + |\rho(x)|)$ and $\rho^-(x) = \frac{1}{2}(\rho(x) - |\rho(x)|)$. We then have $\rho^+, \rho^- \in C^{0,1}_c([a,b])$ with $\|\rho^+\|_{C^{0,1}} \le \|\rho\|_{C^1}$, $\|\rho^-\|_{C^{0,1}} \le \|\rho\|_{C^1}$. We further decompose $C = C^+ + C^-$, $D = D^+ + D^-$ into their positive and nonpositive parts. Clearly

$$\mathrm{tr}[C^+\rho^+(A)] = \sum_{j=1}^{N} \rho^+(\lambda_{a_j})\langle v_{a_j}, C^+ v_{a_j}\rangle.$$

Next we consider $w_j = U v_{a_j} \in H'$. From assumption (1) we have

$$\left\|(A'-\lambda_{a_j})w_j\right\| = \left\|(UAU^* - \lambda_{a_j})Uv_{a_j}\right\| \le \left\|(U^*U - I)\Pi^A_{[a,b]}\right\|\|A\|\|U\| < \varepsilon.$$

Similar estimates give $\left|\|w_j\|^2 - 1\right| < \varepsilon$ and $|\langle w_j, w_k\rangle| < \varepsilon$ for $j \ne k$. Now by Lemma A.1 we have $\|\Pi w_j - w_j\| < (2\varepsilon)^{\frac{1}{2}}$ with $\Pi = \Pi^{A'}_{[a-\sqrt{2\varepsilon}, b+\sqrt{2\varepsilon}]}$. Following this and using assumption (3) we have

$$\begin{aligned}\left\|(B-\lambda_{a_j})w_j\right\| &\le \left\|(A'-\lambda_{a_j})w_j\right\| + \left\|(B-A')\Pi w_j\right\| + \left\|(B-A')(\Pi w_j - w_j)\right\| \\ &\le \varepsilon + \varepsilon\sqrt{1+\varepsilon} + (2\varepsilon)^{\frac{1}{2}}(\|A'\| + \|B\|) \\ &< \varepsilon^{\frac{1}{4}} \le \varepsilon^{\frac{1}{8}}\|w_j\|.\end{aligned}$$

Next define $w^0_j := \Pi^B_{[a-\varepsilon^{1/16}, b+\varepsilon^{1/16}]} w_j$. By Lemma A.1,

$$\|w^0_j - w_j\| \le \varepsilon^{\frac{1}{16}}\|w_j\|. \tag{A-6}$$

From here it follows immediately that $w^0_1, w^0_2, \dots, w^0_N$ form an $\varepsilon^{\frac{1}{64}}$-AOSE of B. If we let $H_0 = H^B_{[a-\varepsilon^{1/16}, b+\varepsilon^{1/16}]}$, then by Lemma A.4 there exist orthonormal $\bar{w}_1, \bar{w}_2, \dots, \bar{w}_N \in H_0$ which span the same subspace of H_0 as the w^0_j. Furthermore

$$\|w^0_j - \bar{w}_j\| < \varepsilon^{\frac{1}{128}} \tag{A-7}$$

and $\|(\rho^+(B) - \rho^+(\lambda_{a_j}))\bar{w}_j\| \le 3\|\rho\|_{C^1}\varepsilon^{\frac{1}{512}}$. From (A-6) and (A-7) we also have $\|w_j - \bar{w}_j\| < \varepsilon^{\frac{1}{256}}$. From Lemma A.3 there exist orthonormal $w'_1, w'_2, \dots, w'_{M-N}$ with $M = N^B_{[a-\varepsilon^{1/16}, b+\varepsilon^{1/16}]}$ such that $\langle w'_i, \bar{w}_j\rangle = 0$ and $\|(B-\mu'_j)w'_j\| < 4M\varepsilon^{\frac{1}{64}} < \varepsilon^{\frac{1}{128}}$. Hence by Lemma A.1, $\|(\rho^+(B) - \rho^+(\mu'_j))w'_j\| \le$

$3\|\rho\|_{C^1}\varepsilon^{\frac{1}{256}}$. We now have

$$\begin{aligned}
\operatorname{tr}[D^+\rho^+(B)] &= \sum_{j=1}^{N}\langle \bar{w}_j, D^+\rho^+(B)\bar{w}_j\rangle + \sum_{j=1}^{M-N}\langle w'_j, D^+\rho^+(B)w'_j\rangle \\
&\geq \sum_{j=1}^{N}\rho^+(\lambda_{a_j})\langle \bar{w}_j, D^+\bar{w}_j\rangle + \sum_{j=1}^{M-N}\rho^+(\mu'_j)\langle w'_j, D^+w'_j\rangle - 3\varepsilon^{\frac{1}{512}}M\|D\|\|\rho\|_{C^1} \\
&\geq \sum_{j=1}^{N}\rho^+(\lambda_{a_j})\langle \bar{w}_j, D^+\bar{w}_j\rangle - 3\varepsilon^{\frac{1}{512}}M\|D\|\|\rho\|_{C^1} \\
&\geq \sum_{j=1}^{N}\rho^+(\lambda_{a_j})\langle w_j, D^+w_j\rangle - 6\varepsilon^{\frac{1}{512}}M\|D\|\|\rho\|_{C^1} \\
&\geq \sum_{j=1}^{N}\rho^+(\lambda_{a_j})\langle v_{a_j}, C^+v_{a_j}\rangle - 6\varepsilon^{\frac{1}{512}}M(\|D\|+1)\|\rho\|_{C^1} \\
&\geq \operatorname{tr}[C^+\rho^+(A)] - \varepsilon^{\frac{1}{1024}}\|\rho\|_{C^1}.
\end{aligned}$$

Reversing the roles of H and H' gives

$$\left|\operatorname{tr}[D^+\rho^+(B)] - \operatorname{tr}[C^+\rho^+(A)]\right| \leq \varepsilon^{\frac{1}{1024}}\|\rho\|_{C^1}.$$

Similar estimates with $C^+\rho^-(A)$, $C^-\rho^+(A)$ and $C^-\rho^-(A)$ give the result. $\square$

Finally, we now give a criterion implying the discreteness of spectrum for pseudodifferential operators required by the preceding propositions in this appendix.

Proposition A.6. *Let $A \in \Psi^m_{\mathrm{cl}}(\mathbb{R}^n;\mathbb{C}^l)$ and $I = [a,b] \subset \mathbb{R}$ be a closed interval such that the I-energy band*

$$\Sigma^A_I := \bigcup_{\lambda\in I}\Sigma^A_\lambda$$

is bounded. Then for $h < h_0$ sufficiently small

$$\operatorname{EssSpec}(A) \cap I = \varnothing.$$

Proof. Let $\sigma(A) = a(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ and $\Sigma_I(a) \subset B_R$ be some open ball of finite radius R around the origin. For $\lambda \in I$ and $(x,\xi) \notin B_R$, we hence have that $a_{-1} := (a(x,\xi)-\lambda)^{-1}$ exists. Let $\chi \in C^\infty_c(-4R,4R)$ such that $\chi(x) = 1$ for $x < 2R$. Set $\phi(x) = 1-\chi(x)$ and define

$$A_{-1} = \left[\phi(|(x,\xi)|)a_{-1}(x,\xi)\right]^W \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^n;\mathbb{C}^l).$$

Then since it has vanishing symbol, we have

$$(A-\lambda)A_{-1} - (I - \chi(|(x,\xi)|)^W) = hR \in h\Psi^0_{\mathrm{cl}}(\mathbb{R}^n;\mathbb{C}^l).$$

Next, we clearly have $I + hR$ is invertible for $h < h_0$ sufficiently small. Also, $\chi(|(x,\xi)|)^W$ is trace class by [Hörmander 1994, Lemma 19.3.2]. Hence if $S := A_{-1}(I + hR)^{-1}$, then $(A-\lambda)S - I$ is trace class. By a similar argument, $S(A-\lambda) - I$ is trace class. Hence by Proposition 19.1.14 of [Hörmander 1994], $A - \lambda$ is Fredholm. $\square$

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