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BILINEAR RESTRICTION ESTIMATES FOR SURFACES OF CODIMENSION BIGGER THAN 1

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In connection with the restriction problem in $\mathbb{R}^n$ for hypersurfaces including the sphere and paraboloid, the bilinear (adjoint) restriction estimates have been extensively studied. However, not much is known about such estimates for surfaces with codimension (and dimension) larger than 1. In this paper we show sharp bilinear $L^2 \times L^2 \to L^q$ restriction estimates for general surfaces of higher codimension. In some special cases, we can apply these results to obtain the corresponding linear estimates.

1. Introduction and statement of results

For a smooth hypersurface S such as the sphere or paraboloid in $\mathbb{R}^n$, $n \geq 3$, the L^p-L^q boundedness of the (adjoint) restriction operator (or the extension operator) $\widehat{f\,d\sigma}$ has been extensively studied since the late 1960s. Here $d\sigma$ denotes the induced Lebesgue measure on S. Specifically, when S is the sphere, it was conjectured by E. M. Stein [1993] that $\widehat{f\,d\sigma}$ should map $L^p(S)$ boundedly to $L^q(\mathbb{R}^n)$, precisely when $q \geq p'(n+1)/(n-1)$ and $q > 2n/(n-1)$. Since then, a large amount of literature has been devoted to this problem. Over the last couple of decades, the bilinear and multilinear approaches have proven to be quite effective, and through them substantial progress has been made. We refer the reader to [Bennett et al. 2006; Bourgain and Guth 2011; Guth 2016] for the most recent developments.

On the other hand, when the dimension of the manifold is 1, namely, when the associated surface is a curve, the restriction estimate is by now fairly well understood [Bak et al. 2002; 2009; 2013; Stovall 2016].

However, not much is known about the intermediate cases, namely, when the codimension k of the manifold is between 1 and $n-1$. The restriction problem for quadratic surfaces of codimension $k \geq 2$ was first studied by Christ [1982] and Mockenhaupt [1996]. They also considered the problem in a more general setting and found some necessary conditions on the curvature and codimension of the surface. For some surfaces they also established the optimal $L^2 \to L^q$ linear estimates, which may be regarded as generalizations of the Stein–Tomas restriction theorem; see also [Banner 2002]. Although there are some known cases in which the L^p-L^q boundedness is completely characterized, see for example [Bak and Ham 2014; Bak and Lee 2004; Oberlin 2005], for most surfaces with codimension bigger than 1, the current state of the restriction problem is hardly beyond that of the Stein–Tomas theorem.

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In this paper, we are concerned with restriction estimates for surfaces of codimension $k \geq 2$. To be more specific, let us set $k \geq 1$ and $I = [-1, 1]$. Let $\Phi : I^d \to \mathbb{R}^k$ be a smooth function given by

$$\Phi(\xi) = (\varphi_1(\xi), \varphi_2(\xi), \ldots, \varphi_k(\xi)).$$

The adjoint restriction operator (the extension operator) $E = E_\Phi$ for the surface $(\xi, \Phi(\xi)) \in \mathbb{R}^d \times \mathbb{R}^k$ is defined by

$$Ef(x,t) = \int_{I^d} e^{2\pi i (x\cdot\xi + t\cdot\Phi(\xi))} f(\xi)\, d\xi, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^k.$$

Specific examples of such operators with $2 \leq k \leq d-2$ can be found in [Bak and Ham 2014; Bak and Lee 2004; Christ 1982; Mockenhaupt 1996; Oberlin 2005]. (Also, see Section 5.)

There are some classes of surfaces for which the optimal L^2-L^q boundedness of E is well understood. In fact, using a Knapp-type example it is easy to see that E may be bounded from L^p to L^q only if $(d+2k)/q \leq d(1-1/p)$. Hence, the best possible L^2-L^q bound is that for $q = 2(d+2k)/d$. Christ [1982] and Mockenhaupt [1996] showed that this is true for a class of surfaces satisfying a suitable curvature condition. In particular, let M be a linear map from $\mathbb{R}^k$ to the space of $d \times d$ symmetric matrices and suppose that $\int_{S^{k-1}} |\det M(t)|^{-\gamma}\, d\sigma(t) < \infty$ for $\gamma = k/d$. Then it was proven in [Mockenhaupt 1996] that the extension operator E defined by $\Phi = \xi^t M(t)\xi$ is bounded from L^2 to $L^{2(d+2k)/d}$.

In order to obtain estimates for some $q < 2(d+2k)/d$ and $p > 2$, it seems necessary to consider methods other than the TT^* argument, which solely relies on the decay estimate for the Fourier transform of the surface measure. For this reason we wish to consider the bilinear restriction estimates for surfaces of codimension greater than 1 and try to obtain the best possible estimates.

Let S_1, S_2 be closed cubes contained in I^d and define

$$E_i f(x,t) = \int_{S_i} e^{2\pi i (x\cdot\xi + t\cdot\Phi(\xi))} f(\xi)\, d\xi, \quad i = 1, 2.$$

Let us consider the estimate

$$\|E_1 f\, E_2 g\|_{L^q(\mathbb{R}^{d+k})} \leq C \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}. \tag{1-1}$$

For the elliptic surfaces, bilinear estimates can be thought of as a generalization of linear estimates, since a linear restriction estimate follows from the corresponding bilinear one by an argument involving a Whitney decomposition; see, e.g., [Tao et al. 1998]. The advantage of the bilinear estimates is that a wider rage of boundedness is possible than for the linear estimate, provided that a separation condition holds between the supports of the functions f, g. For surfaces with codimension 1, the sharp bilinear (adjoint) restriction estimate for the cone was obtained by Wolff [2001], and for the paraboloid the corresponding estimate was proved by Tao [2003]. The bilinear approach has also been applied to the restriction problem for hyperbolic surfaces: Vargas [2005] used it for the saddle surface in $\mathbb{R}^3$ and, independently, Lee [2006] proved the bilinear estimate by extending Tao's method.[1] From these bilinear restriction estimates the corresponding linear ones have been obtained as well.

[1] For more general negatively curved surfaces in $\mathbb{R}^3$ and higher dimensions, Lee [2006] showed the bilinear restriction estimates. However, in higher dimensions the linear estimate could not be deduced from the bilinear one, because the separation condition needed to prove the bilinear estimate for hyperbolic surfaces was more complex than that for the elliptic surfaces.

In order to state our results, we first introduce some notation. For $\nu_1,\ \nu_2 \in I^d$, we define the $k \times d$ matrix $\boldsymbol{D}(\nu_1, \nu_2)$ by

$$\boldsymbol{D}(\nu_1,\nu_2) = \begin{pmatrix} \nabla\varphi_1(\nu_2) - \nabla\varphi_1(\nu_1) \\ \vdots \\ \nabla\varphi_k(\nu_2) - \nabla\varphi_k(\nu_1) \end{pmatrix}.$$

Here $\nabla\varphi_j$ is a row vector. Let $H\varphi$ denote the Hessian of φ and $\boldsymbol{D}^t(\nu_1,\nu_2)$ be the transpose of $\boldsymbol{D}(\nu_1,\nu_2)$. The following is our main theorem.

Theorem 1.1. *Let* $t = (t_1, \dots, t_k)$, $k \geq 1$. *Suppose that, for* $\nu \in S_1 \cup S_2$ *and* $|t| = 1$,

$$\det\Bigl(\sum_{i=1}^{k} t_i H\varphi_i(\nu)\Bigr) \neq 0 \tag{1-2}$$

and, for $\nu_1 \in S_1,\ \nu_2 \in S_2,\ \ |t| = 1$ *and for* $\nu = \nu_1, \nu_2$,

$$\det\Bigl[\boldsymbol{D}(\nu_1,\nu_2)\Bigl(\sum_{j=1}^{k} t_j H\varphi_j(\nu)\Bigr)^{-1}\boldsymbol{D}^t(\nu_1,\nu_2)\Bigr] \neq 0. \tag{1-3}$$

Then, for

$$q > \frac{d+3k}{d+k} \quad \textit{and} \quad \frac{1}{p} + \frac{d+3k}{d+k}\frac{1}{2q} < 1,$$

the estimate (1-1) holds.

As special cases of Theorem 1.1, one can deduce the known bilinear restriction theorems for the elliptic surfaces in [Tao 2003] and the negatively curved ones in [Vargas 2005; Lee 2006].

Let us set

$$\boldsymbol{M}(t,\nu_1,\nu_2,\nu) := \begin{pmatrix} 0 & \boldsymbol{D}(\nu_1,\nu_2) \\ \boldsymbol{D}^t(\nu_1,\nu_2) & \sum_{i=1}^{k} t_i H\varphi_i(\nu) \end{pmatrix}.$$

Assuming the condition (1-2), it is easy to see that (1-3) is equivalent to

$$\det \boldsymbol{M}(t,\nu_1,\nu_2,\nu) \neq 0 \tag{1-4}$$

for $\nu_1 \in S_1,\ \nu_2 \in S_2,\ |t| = 1$ and for $\nu = \nu_1, \nu_2$. (One can use the block matrix formula $\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \det(D)\det(A - BD^{-1}C)$.) The condition (1-4) may seem rather complicated, but such a condition appears naturally when one considers the bilinear $L^2 \times L^2 \to L^2$ estimate. When $k = 1$, it is closely related to the "rotational curvature"; see [Lee 2006] for more details. The necessity of the condition (1-4) will become clear in the course of the proof of Proposition 1.3 below.

From the condition (1-3) it follows that $\boldsymbol{D}(\nu_1,\nu_2)$ has rank k. So, the vectors

$$\{\nabla\varphi_i(\nu_2) - \nabla\varphi_i(\nu_1) : i = 1, \dots, k\}$$

are linearly independent. This means $d \geq k$. If $d = k$, then (1-4) implies (1-3), but otherwise (1-4) may hold without (1-3) being satisfied.

In fact, it is possible to obtain a local version (Theorem 1.2 below) of Theorem 1.1, which holds under a weaker assumption. Let $\boldsymbol{n}_1, \dots \boldsymbol{n}_{d-k}$ be orthonormal vectors (seen as row vectors), which are perpendicular to the span of $\{\nabla\varphi_i(\nu_2) - \nabla\varphi_i(\nu_1) : i = 1, \dots, k\}$ and set

$$\boldsymbol{N}(\nu_2, \nu_1) = \begin{pmatrix} \boldsymbol{n}_1 \\ \vdots \\ \boldsymbol{n}_{d-k} \end{pmatrix}.$$

Then we can replace the condition (1-4) with

$$\det\left[\boldsymbol{N}(\nu_2, \nu_1)\left(\sum_{i=1}^{k} t_i H\varphi_i(\nu)\right)\boldsymbol{N}^t(\nu_2, \nu_1)\right] \neq 0 \tag{1-5}$$

whenever $\nu_1 \in S_1,\ \nu_2 \in S_2,\ |t| = 1$ and $\nu = \nu_1, \nu_2$. It is easy to see that the value of this determinant is independent of the particular choice of orthonormal vectors $\boldsymbol{n}_1, \dots, \boldsymbol{n}_{d-k}$, and that the condition (1-5) is equivalent to (1-4) under the assumption (1-2).[2] If we have (1-5) instead of (1-3), then we don't need (1-2) to get (1-6) for any $\alpha > 0$. More precisely, we have

Theorem 1.2. *Suppose that, for any $\nu_1 \in S_1,\ \nu_2 \in S_2$, the vectors $\nabla\varphi_i(\nu_2) - \nabla\varphi_i(\nu_1),\ i = 1, \dots, k$, are linearly independent and that (1-5) holds for $\nu_1 \in S_1,\ \nu_2 \in S_2,\ |t| = 1$ and for $\nu = \nu_1, \nu_2$. Then, for any $\alpha > 0$, there is a constant C_α such that*

$$\|E_1 f\, E_2 g\|_{L^{(d+3k)/(d+k)}(Q_R)} \le C_\alpha R^\alpha \|f\|_2 \|g\|_2, \tag{1-6}$$

where Q_R is a cube of side length $R \gg 1$.

However, to obtain the global estimates $L^2 \times L^2 \to L^q$, for $q > (d+3k)/(d+k)$, we need to impose a decay condition on the Fourier transform of the surface measure, since it is needed to apply the epsilon removal lemma [Bourgain and Guth 2011]. Under the condition (1-2) such a decay estimate follows from the stationary phase method.

For $q \ge 2$, the estimate (1-1) is relatively easier to prove under the conditions (1-2), (1-3). The following may be thought of as a generalization of Theorem 2.3 in [Tao et al. 1998], which is concerned with elliptic hypersurfaces; see also Theorem 4.2 in [Moyua et al. 1999]. A generalization to general hypersurfaces had already been observed in [Lee 2006]. As a byproduct this gives estimates for the endpoint cases of (p, q) satisfying

$$\frac{1}{p} + \frac{d+3k}{d+k}\frac{1}{2q} = 1, \quad q \ge 2.$$

Proposition 1.3. *Suppose the condition (1-4) holds for $\nu_1 \in S_1,\ \nu_2 \in S_2$ and $|t| = 1$. Then, for $q \ge 2$ and*

$$\frac{1}{p} + \frac{d+3k}{d+k}\frac{1}{2q} \le 1,$$

the estimate (1-1) holds.

[2]Indeed, if H, N and D are matrices of sizes $d \times d$, $(d-k) \times d$ and $k \times d$, respectively, such that $ND^t = 0$, $\det H \neq 0$ and $\operatorname{rank}(N^t\ D^t) = d$, then $\det(NHN^t) \neq 0$ if and only if $\det(DH^{-1}D^t) \neq 0$ because $\binom{NH}{D}(N^t\ D^t) = \begin{pmatrix} NHN^t & NHD^t \\ 0 & DD^t \end{pmatrix}$ and $\binom{N}{DH^{-1}}(N^t\ D^t) = \begin{pmatrix} NN^t & 0 \\ DH^{-1}N^t & DH^{-1}D^t \end{pmatrix}$.

Remark 1.4. In the proof of the above results we may assume that the aforementioned conditions hold uniformly, by breaking up the extension operator by decomposing S_1, S_2 into sufficiently small pieces. That is to say, there is a constant $c > 0$ such that for $\nu \in S_1 \cup S_2$ and $|t| = 1$,

$$\left|\det\left(\sum_{i=1}^{k} t_i H\varphi_i(\nu)\right)\right| \geq c \tag{1-7}$$

and, for $\nu_1, \nu_1' \in S_1$, $\nu_2, \nu_2' \in S_2$, $|t| \sim 1$ and for $\nu \in S_1 \cup S_2$,

$$\left|\det\left[\boldsymbol{D}(\nu_1, \nu_2)\left(\sum_{j=1}^{k} t_j H\varphi_j(\nu)\right)^{-1} \boldsymbol{D}^t(\nu_1', \nu_2')\right]\right| \geq c. \tag{1-8}$$

The same holds also for the conditions (1-4) and (1-5).

***Necessary conditions for* (1-1).** By modifying the examples in [Tao and Vargas 2000] with some specific surfaces we see that (1-1) cannot hold in general, unless

$$q \geq \frac{d+k}{d}, \tag{1-9}$$

$$\frac{1}{p} + \frac{d+3k}{d+k}\frac{1}{2q} \leq 1, \tag{1-10}$$

$$\frac{2(d-k)}{p} + \frac{d+3k}{q} \leq 2d. \tag{1-11}$$

In fact,

(i) (1-9) is necessary for (1-1) to hold under (1-2), and

(ii) so is (1-10) under the assumption that the matrix $\boldsymbol{D}(\nu_1, \nu_2)$ has rank k for $\nu_j \in S_j$, $j = 1, 2$.

However, in general, (1-11) is not necessarily required for (1-1), but as is well known there are various Φ satisfying (1-2) and (1-3) for which (1-1) fails if

$$\frac{2(d-k)}{p} + \frac{d+3k}{q} > 2d.$$

We show (i) and (ii) in the following paragraphs.

(i) By making use of the stationary phase method together with the condition (1-2) it is not difficult to see that, with suitable choice of x_0, there is a cube Q of side length $R \gg 1$ such that $|E_1(e^{-2\pi i x_0 \cdot \xi}\psi)| \sim |E_2\psi(x)| \sim R^{-d/2}$ on Q provided that supports of ψ_1, ψ_2 are small enough. We insert these into (1-1) to see $R^{-d/2}R^{-d/2}R^{(d+k)/q} \lesssim 1$, from which we get (1-9) by letting $R \to \infty$. (This can also be shown by making use of a wave packet decomposition, see Lemma 4.2, and randomization.)

(ii) For $j = 1, 2$, let Σ_j be the surface $\{(\xi, \Phi(\xi)) : \xi \in S_j\}$, and denote by $d\sigma_j$ the induced Lebesgue measure on Σ_j. To see (1-9) it is more convenient to consider $f \to \widehat{f\,d\sigma_j}$, instead of dealing with the operator E_j. Also, let ν_j be the center of cube S_j and let $\zeta_j = (\nu_j, \Phi(\nu_j)) \in \Sigma_j$, $j = 1, 2$. The normal space $\boldsymbol{N}_j$ to Σ_j at ζ_j is spanned by

$$\boldsymbol{n}_{j,i} = (-\nabla\varphi_i(\nu_j), e_i), \quad i = 1, 2, \ldots, k,$$

where $e_i \in \mathbb{R}^k$ is the usual unit vector with its i-th entry being equal to 1. Clearly, these vectors are linearly independent because $\boldsymbol{D}(\nu_1, \nu_2)$ has rank k. Let $\boldsymbol{p}_n$, $n = 1, \ldots, d-k$, be an orthonormal basis of the orthogonal complement of $\mathrm{span}\{\boldsymbol{n}_{j,i} : i = 1, 2, \ldots, k,\ j = 1, 2\}$. Let us set, for $j = 1, 2$,

$$\Lambda_j = \big\{\zeta \in \Sigma_j : |(\zeta - \zeta_j)\cdot \boldsymbol{n}_{3-j,i}| \le \delta,\ |(\zeta-\zeta_j)\cdot \boldsymbol{p}_n| \le \delta^{\frac{1}{2}},\ i = 1, \ldots, k,\ n = 1, \ldots, d-k\big\}.$$

Now, we set $f_j = \chi_{\Lambda_j}$, $j = 1, 2$. Then it is easy to see $|\widehat{f_j\, d\sigma_j}(x,t)| \gtrsim \delta^{(d+k)/2}$, $j = 1, 2$, provided that

$$|(x,t)\cdot \boldsymbol{n}_{\ell,i}| \le c\delta^{-1}, \quad |(x,t)\cdot \boldsymbol{p}_n| \le c\delta^{-\frac{1}{2}}, \quad i = 1, \ldots, k,\ \ \ell = 1, 2,\ \ n = 1, \ldots, d-k$$

with sufficiently small $c > 0$. (For example, see the proof Lemma 4.2.) Since (1-1) implies

$$\|\widehat{f_1\, d\sigma_1}\ \widehat{f_2\, d\sigma_2}\|_q \lesssim \|f_1\|_p\, \|f_2\|_p,$$

we get $\delta^{d+k-(d+3k)/(2q)} \le C\delta^{(d+k)/p}$ and (1-10) by letting $\delta \to 0$.

Restriction to complex surfaces. Using the above theorem we can obtain a bilinear restriction estimate for complex quadratic surfaces. To define the (Fourier) extension operator for a complex surface we first distinguish the dot product and the inner product for complex variables, and define an auxiliary product $\odot$. For $z, w \in \mathbb{C}^m$, we define $z\cdot w$, $\langle z, w\rangle$, $z \odot w$ by

$$z\cdot w = \sum_{j=1}^{m} z_j w_j, \quad \langle z, w\rangle = \sum_{j=1}^{m} z_j \bar{w}_j, \quad z\odot w = \mathrm{Re}\langle z, w\rangle.$$

Hence, if $z = x + iy$ and $w = u + iv$ for $x, y, u, v \in \mathbb{R}^m$, then $z\odot w = x\cdot u + y\cdot v$. If we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ in the usual way, then $z \odot w$ is just the inner product on $\mathbb{R}^{2m}$.

Let $n \ge 1$ be an integer and let D be a real symmetric invertible matrix. Then we define the complex quadratic surface $\gamma \subset \mathbb{C}^{n+1}$ by

$$\gamma(z) = \big(z, \tfrac{1}{2} z^t D z\big), \quad z \in \mathbb{C}^n. \tag{1-12}$$

Now we define the extension operator $E_\gamma f$ by

$$E_\gamma f(w) = \int_{\mathbb{C}^n} e^{2\pi i[w\odot\gamma(z)]} f(z)\, dz, \quad w \in \mathbb{C}^{n+1},$$

where we have written dz for $dx\, dy$, $z = x + iy$. The operator $E_\gamma f$ is an extension operator for surfaces of codimension 2 in $\mathbb{R}^{2n}$, which is given by $\big(x, y, \frac{1}{2}\mathrm{Re}(x+iy)^t D(x+iy), \frac{1}{2}\Im(x+iy)^t D(x+iy)\big)$, $x, y \in \mathbb{R}^n$. From Theorem 1.1 we can establish the following.

Corollary 1.5. *Let S_1, S_2 be closed cubes in $\mathbb{C}^n$. Suppose that, for any $z_1 \in S_1$ and $z_2 \in S_2$,*

$$|(z_2 - z_1)^t D(z_2 - z_1)| \ne 0. \tag{1-13}$$

Then, whenever f, g *are supported on* S_1, S_2, *respectively, for*

$$q > \frac{n+3}{n+1} \quad \text{and} \quad \frac{1}{p} + \frac{n+3}{n+1}\frac{1}{2q} < 1,$$

there is a constant C *such that*

$$\|E_\gamma f\, E_\gamma g\|_{L^q(\mathbb{C}^{n+1})} \le C\|f\|_{L^p(\mathbb{C}^n)}\|g\|_{L^p(\mathbb{C}^n)}.$$

This theorem can also be stated without using the complex number notation, but its use makes it easier to derive the linear estimates from the bilinear one. The condition (1-13) in $\mathbb{C}^2$ can be contrasted with that in $\mathbb{R}^2$. If $S_1, S_2 \subset \mathbb{R}^2$ and the eigenvalues of D have the same sign, then the condition (1-13) is always valid if $\operatorname{dist}(S_1, S_2) \neq 0$. But, when $S_1, S_2 \subset \mathbb{C}^2$, the condition (1-13) may fail even if the separation condition is satisfied. For instance, if D is the 2×2 identity matrix, the condition (1-13) becomes $|(v_1 - w_1)^2 + (v_2 - w_2)^2| \gtrsim 1$ with $z_1 = (v_1, v_2)$ and $z_2 = (w_1, w_2)$. Since we may factor $(v_1 - w_1)^2 + (v_2 - w_2)^2$ as $[(v_1 - w_1) + i(v_2 - w_2)][(v_1 - w_1) - i(v_2 - w_2)]$, the expression $|(v_1 - w_1)^2 + (v_2 - w_2)^2|$ may vanish even if $\operatorname{dist}(S_1, S_2) \gtrsim 1$. When D has eigenvalues with different signs, this phenomenon may occur even when $S_1, S_2 \subset \mathbb{R}^2$; for instance, if D is the 2×2 diagonal matrix with diagonal entries 1 and -1, then we have

$$x \cdot Dx = x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2).$$

This real-variable case was studied by Lee [2006] and Vargas [2005]. In the special case that the surface is two-dimensional they could deduce a linear estimate from the bilinear one.

By adapting their argument, we can obtain the following linear estimate.

Theorem 1.6. *Let* $n = 2$ *and* γ *be given by* (1-12) *with a nonsingular real symmetric matrix* D. *Then, for* $q > \frac{10}{3}$ *and* $\frac{1}{p} + \frac{2}{q} < 1$,

$$\|E_\gamma f\|_{L^q(\mathbb{C}^3)} \le C\|f\|_{L^p(\mathbb{C}^2)} \tag{1-14}$$

whenever f *is supported in a bounded set.*

By analogy with the corresponding problem for the paraboloid (elliptic or hyperbolic) in $\mathbb{R}^3$, it may be conjectured that (1-14) holds if and only if $q > 3$ and $\frac{1}{p} + \frac{2}{q} \le 1$. Theorem 1.6 extends the known (p, q) range for the operator $E_\gamma f$ when D is a nonsingular real symmetric matrix. This result is an analog of the adjoint Fourier restriction estimates for the hyperbolic paraboloid in $\mathbb{R}^3$, which is known to hold for the same range of p, q. As a special case of the results by Christ [1982, Lemma 4.3] and Mockenhaupt [1996, Theorem 2.11], it was previously known that Ef maps $L^2(\mathbb{R}^4)$ boundedly to $L^4(\mathbb{R}^6)$. Also, the slightly stronger Lorentz space estimate $\|Ef\|_{L^{4,2}(\mathbb{R}^6)} \le C\|f\|_{L^2(\mathbb{R}^4)}$ can be deduced by applying Theorem 1.1 in [Bak and Seeger 2011]. It is quite likely that the multilinear approach will yield further progress on these problems. We hope to return to this problem in the near future.

Notation. We adopt the usual convention to let C or c represent strictly positive constants, whose value may vary from line to line. But these constants will always be independent of f, for instance. We write $A \lesssim B$ or $B \gtrsim A$ to mean $A \le CB$, and $A \sim B$ means both $A \lesssim B$ and $B \lesssim A$.

2. $L^{\frac{4(d+k)}{3d+k}} \times L^{\frac{4(d+k)}{3d+k}} \to L^2$ estimates and proof of Proposition 1.3

In this section we show Proposition 1.3. Our proof here is different from that in [Tao et al. 1998]. Instead of making use of the boundedness of the averaging operator, we directly exploit the oscillatory decay estimate which is concealed in the averaging operator. For this we need the following lemma.

Lemma 2.1 [Greenleaf and Seeger 2002, Section 1.1]. *Let $a \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^N)$ and set*

$$T_\lambda f(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^N} e^{i\lambda\phi(x,y,\theta)} a(x,y,\theta)\, d\theta\, f(y)\, dy,$$

where ϕ is a smooth function on the support of a. Suppose

$$\det \begin{pmatrix} \phi''_{\theta\theta} & \phi''_{x\theta} \\ \phi''_{y\theta} & \phi''_{xy} \end{pmatrix} \neq 0$$

on the support of a whenever $\phi'_\theta = 0$. Then, $\|T_\lambda f\|_2 \lesssim \lambda^{-(d+N)/2} \|f\|_2$.

Proof of Proposition 1.3. By interpolation with the trivial $L^1 \times L^1 \to L^\infty$ estimate, it suffices to show

$$\|E_1 f_1\, E_2 f_2\|_2 \lesssim \|f_1\|_{\frac{4(d+k)}{3d+k}} \|f_2\|_{\frac{4(d+k)}{3d+k}}.$$

For fixed ξ_2, set

$$\Phi^{\xi_2}(\xi_1, \eta_1) = \Phi(\xi_1) + \Phi(\xi_2) - \Phi(\eta_1) - \Phi(\xi_1 + \xi_2 - \eta_1)$$

and

$$I^{\xi_2}(f_1, \bar{f}_1) = \iint \delta(\Phi^{\xi_2}(\xi_1, \eta_1))\, f_1(\xi_1)\, \bar{f}_1(\eta_1)\, d\xi_1\, d\eta_1,$$

where δ is the delta function. Its composition is well defined, since the vectors $\nabla\varphi_i(\nu_2) - \nabla\varphi_i(\nu_1)$, $i = 1, \dots, k$, are linearly independent.

By Plancherel's theorem

$$\begin{aligned} \|E_1 f_1\, E_2 f_2\|_2^2 &= \iiiint \delta\big(\xi_1 + \xi_2 - \eta_1 - \eta_2, \Phi(\xi_1) + \Phi(\xi_2) - \Phi(\eta_1) - \Phi(\eta_2)\big) \\ &\qquad\qquad \times f_1(\xi_1)\, f_2(\xi_2)\, \bar{f}_1(\eta_1)\, \bar{f}_2(\eta_2)\, d\xi_1\, d\xi_2\, d\eta_1\, d\eta_2 \\ &= \iiint \delta(\Phi^{\xi_2}(\xi_1, \eta_1))\, f_1(\xi_1)\, \bar{f}_1(\eta_1)\, f_2(\xi_2)\, \bar{f}_2(\xi_1 + \xi_2 - \eta_1)\, d\xi_1\, d\xi_2\, d\eta_1, \end{aligned}$$

where f_1, f_2 are assumed to be supported in S_1, S_2, respectively. We claim that

$$\|E_1 f_1\, E_2 f_2\|_2^2 \lesssim \|f_1\|_{p,1}\, \|f_2\|_1\, \|\bar{f}_1\|_{p,1}\, \|\bar{f}_2\|_\infty, \tag{2-1}$$

where $p = (d+k)/d$. Here $\|f\|_{r,s}$ denotes the norm of Lorentz space $L^{r,s}$. For this we may obviously assume that the functions $f_1, \bar{f}_1, f_2, \bar{f}_2$ are nonnegative. In order to show (2-1) it suffices to prove

$$|I^{\xi_2}(f, g)| \lesssim \|f\|_{p,1}\, \|\bar{g}\|_{p,1}. \tag{2-2}$$

Let ψ be a smooth function with compact Fourier support contained in $B(0,1)$ such that $\hat{\psi} = 1$ on $B\big(0, \frac{1}{2}\big)$. Since $h(0) = \lim_{j\to\infty} 2^{jk} \int_{\mathbb{R}^k} \psi(2^j x)\, h(x)\, dx$ for any Schwartz function h, we have

$\delta = \lim_{j\to\infty} 2^{jk}\psi(2^j x)$. So, we may write

$$\delta = \sum_{j=-\infty}^{\infty} [2^{(j+1)k}\psi(2^{j+1}x) - 2^{jk}\psi(2^j x)] = \sum_{j=-\infty}^{\infty} 2^{jk}\eta(2^j x),$$

where $\eta(x) := 2^k\psi(2x) - \psi(x)$. By the choice of ψ we see that the Fourier support of η is contained in $\{\xi : \frac{1}{2} < |\xi| \le 2\}$. We decompose $I^{\xi_2}(f, g)$ by making use of the above decomposition of δ to get

$$I^{\xi_2}(f, g) = \sum_{j=-\infty}^{\infty} I_j(f, g),$$

where

$$I_j(f, g) := 2^{kj} \iint \eta(2^j \Phi^{\xi_2}(\xi_1, \eta_1))\, f(\xi_1)\, g(\eta_1)\, d\xi_1\, d\eta_1.$$

It should be noted that we are assuming that f, g are supported on S_1 and $\xi_1 + \xi_2 - \eta_1 \in S_2$. Using the Fourier transform we write $I_j(f_1, \bar{f}_2)$ as

$$I_j(f, g) = 2^{kj} \int \biggl(\iint \hat{\eta}(\tau)\, e^{2^j \tau\cdot\Phi^{\xi_2}(\xi_1,\eta_1)}\, d\tau\, f(\xi_1)\, d\xi_1\biggr) g(\eta_1)\, d\eta_1.$$

Now, we will apply Lemma 2.1 to the double integral inside the parentheses. If we set $\phi(\xi_1, \eta_1, \tau) = \tau \cdot \Phi^{\xi_2}(\xi_1, \eta_1)$, then

$$\left|\det\begin{pmatrix} \phi''_{\tau\tau} & \phi''_{\tau\xi_1} \\ \phi''_{\eta_1\tau} & \phi''_{\xi_1\eta_1}\end{pmatrix}\right| = \left|\det\begin{pmatrix} 0 & \boldsymbol{D}(\xi_1, \xi_1+\xi_2-\eta_1) \\ \boldsymbol{D}(\eta_1, \xi_1+\xi_2-\eta_1)^t & \sum_{j=1}^{k} \tau_j\, H\varphi_j(\xi_1, \xi_1+\xi_2-\eta_1)\end{pmatrix}\right|.$$

So, by the condition (1-4) the last expression does not vanish since $|\tau| \sim 1$. Hence, by Lemma 2.1 it follows that

$$|I_j(f, g)| \lesssim 2^{-j\frac{d-k}{2}}\, \|f\|_2\, \|g\|_2.$$

On the other hand, we have the trivial bound $|I_j(f, g)| \lesssim 2^{kj}\, \|f\|_1\, \|g\|_1$. Now we may use a summation method (usually called Bourgain's summation trick) to obtain (2-2).

Considering $(f_1, \bar{f}_1, f_2, \bar{f}_2) \to \|Ef_1\, Ef_2\|_2^2$ as a quadrilinear mapping (replacing $\bar{f}_1$, $\bar{f}_2$ on the left-hand side by $\bar{f}_3$ and $\bar{f}_4$, respectively), we apply Christ's multilinear trick [1985]. By symmetry and interpolation we get the estimates

$$\left|\iint Ef_1\, Ef_2\, \overline{Ef_3\, Ef_4}\, dx\, dt\right| \lesssim \prod_{j=1}^{4} \|f_j\|_{p_j,1}$$

for $\left(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}, \frac{1}{p_4}\right)$ contained in the convex hull of the four points

$$v_1 = \Bigl(\frac{1}{p}, \frac{1}{p}, 1, 0\Bigr), \quad v_2 = \Bigl(\frac{1}{p}, \frac{1}{p}, 0, 1\Bigr), \quad v_3 = \Bigl(1, 0, \frac{1}{p}, \frac{1}{p}\Bigr), \quad v_4 = \Bigl(0, 1, \frac{1}{p}, \frac{1}{p}\Bigr)$$

which is contained in the 3-plane $\Pi = \{u_1 + u_2 + u_3 + u_4 = 1 + \frac{2}{p}\}$. The convex hull has a nonempty interior in Π, because $\det(v_1, v_2, v_3, v_4) \neq 0$ as long as $\frac{1}{p} \neq \frac{1}{2}$. Hence we may apply the multilinear trick to get

$$\|Ef_1\, Ef_2\|_2^2 \lesssim \|f_1\|_{\frac{4(d+k)}{3d+k},4} \|f_2\|_{\frac{4(d+k)}{3d+k},4} \|\bar{f}_1\|_{\frac{4(d+k)}{3d+k},4} \|\bar{f}_2\|_{\frac{4(d+k)}{3d+k},4}. \qquad \square$$

3. Transversality and the curvature conditions

In this section we prove several lemmas that will play crucial roles in proving Theorem 1.1. These lemmas are related to the curvature conditions.

For $R \gg 1$ and $\nu \in S_1 \cup S_2$, we set

$$\pi_\nu = \left\{(x,t) : \left| x + \left(\sum_{j=1}^{k} t_j \nabla\varphi_j(\nu)\right)\right| \le R^{\frac{1}{2}}\right\}, \quad R^\delta \pi_\nu = \pi_\nu + O(R^{\frac{1}{2}+\delta}).$$

Here, for any set $A \subset \mathbb{R}^{d+k}$ and $\rho > 0$, we have $A + O(\rho) = \{u \in R^{d+k} : \mathrm{dist}(u, A) \le C\rho\}$.

Lemma 3.1. *Suppose that the vectors $\nabla\varphi_j(\nu_2) - \nabla\varphi_j(\nu_1)$, $1 \le j \le k$, are linearly independent for all $\nu_1 \in S_1$ and $\nu_2 \in S_2$. Then, there is a constant C such that*

$$\pi_{\nu_1} \cap \pi_{\nu_2} \subset B(0, CR^{\frac{1}{2}}).$$

Proof. Since the set $\{\nabla\varphi_j(\nu_2) - \nabla\varphi_j(\nu_1)\}_{j=1}^{k}$ is linearly independent for all $\nu_1 \in S_1$ and $\nu_2 \in S_2$, the map $(t_1, \dots, t_k) \to (t_1, \dots, t_k)^t \boldsymbol{D}(\nu_1, \nu_2)$ is injective. So, by continuity and compactness it follows that there is a constant C such that, for all $\nu_1 \in S_1$ and $\nu_2 \in S_2$,

$$|(t_1, \dots, t_k)^t \boldsymbol{D}(\nu_1, \nu_2)| \ge C|(t_1, \dots, t_k)|.$$

If $(x,t) \in \pi_{\nu_1} \cap \pi_{\nu_2}$, then $\left|x + \left(\sum_{j=1}^{k} t_j \nabla\varphi_j(\nu_i)\right)\right| \le R^{\frac{1}{2}}$ for $i = 1, 2$. This gives

$$|(t_1, \dots, t_k)^t \boldsymbol{D}(\nu_1, \nu_2)| \le 2R^{\frac{1}{2}}.$$

Hence, the above inequality yields $|(t_1, \dots, t_k)| \le CR^{\frac{1}{2}}$. So, we also get $|x| \le CR^{\frac{1}{2}}$. □

As was already shown in [Lee 2006; Vargas 2005], a simple transversality condition between the two wave packets is not enough to obtain a bilinear estimate beyond the range of the linear $L^2 \to L^q$ estimate. So, we need to consider the Fourier supports of the wave packets to put a restriction on the permissible wave packets. This makes the geometry of the associated wave packets more favorable.

For given $\nu_1 \in S_1$ and $\nu_2' \in S_2$ we define $\Pi_1^{\nu_1,\nu_2'}$ by

$$\Pi_1^{\nu_1,\nu_2'} = \{\nu_1' \in S_1 : \nu_1' + \nu_2' - \nu_1 \in S_2,\ \Phi(\nu_1) + \Phi(\nu_1' + \nu_2' - \nu_1) = \Phi(\nu_1') + \Phi(\nu_2')\}. \tag{3-1}$$

Since $\{\nabla\varphi_j(\nu_2) - \nabla\varphi_j(\nu_1)\}_{j=1}^{k}$ are linearly independent, by the implicit function theorem we may assume that $\Pi_1^{\nu_1,\nu_2'}$ is a smooth $(d-k)$-dimensional surface.[3] We now set

$$\Gamma_1^{\nu_1,\nu_2'}(R) = \bigcup_{\nu_1' \in \Pi_1^{\nu_1,\nu_2'}} R^\delta \pi_{\nu_1'},$$

which is an $O(R^{\frac{1}{2}+\delta})$ neighborhood of the conical set with k null directions. The transversality between $\Gamma_1^{\nu_1,\nu_2'}$ and the opposite plates π_{ν_2} is important. Such a transversality is made precise in the following (see Figure 1).

[3] We may need to assume that S_1 and S_2 are small enough.

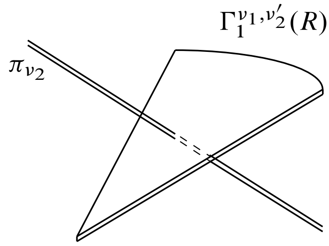

Figure 1. Transversality when $k = 1$ and $d = 2$.

Lemma 3.2. *Let* $0 < \delta \ll 1$, $u \in \mathbb{R}^{d+k}$ *and set*

$$\widetilde{\Gamma}_1^{\nu_1,\nu_2'}(R, R^\delta) = \{(x,t) \in \Gamma_1^{\nu_1,\nu_2'}(R) : R^{1-\delta} \le |(x,t)| \le CR\}.$$

Suppose that the conditions (1-2) and (1-3) hold. Then, if S_1 and S_2 are sufficiently small, there exist a constant C, independent of ν_1, ν_2', R, and a vector $u \in \mathbb{R}^{d+k}$ such that for some $u' \in \mathbb{R}^{d+k}$,

$$\widetilde{\Gamma}_1^{\nu_1,\nu_2'}(R, R^\delta) \cap (R^\delta \pi_{\nu_2} + u) \subset B(u', CR^{\frac{1}{2}+C\delta}).$$

Note that the set $\widetilde{\Gamma}_1^{\nu_1,\nu_2'}(R, R^\delta)$ can be represented as an $O(R^{\frac{1}{2}+\delta})$ neighborhood of a surface. Let us define the map $\Phi_1^{\nu_1,\nu_2'} : \Pi_1^{\nu_1,\nu_2'} \times \mathbb{R}^k \to \mathbb{R}^{d+k}$ by

$$\Phi_1^{\nu_1,\nu_2'}(\nu, t) = \Bigl(-\sum_{j=1}^{k} t_j \nabla\varphi_j(\nu), t\Bigr).$$

Then it is easy to see that

$$\widetilde{\Gamma}_1^{\nu_1,\nu_2'}(R, R^\delta) \subset \{\Phi_1^{\nu_1,\nu_2'}(\nu,t) : \nu \in \Pi_1^{\nu_1,\nu_2'},\ cR^{1-\delta} \le |t| \le CR\} + O(R^{\frac{1}{2}+\delta}).$$

Proof. After scaling it is sufficient to show that the intersection of the two sets

$$\Gamma_1 = \{\Phi_1^{\nu_1,\nu_2'}(\nu,t) : \nu \in \Pi_1^{\nu_1,\nu_2'},\ R^{-\delta} \le |t| \le C\} + O(R^{-\frac{1}{2}+\delta})$$

and

$$\mathfrak{C}_2(R^{-\frac{1}{2}+\delta}) = \Bigl\{\Bigl(-\sum_{j=1}^{k} t_j \nabla\varphi_j(\nu_2), t\Bigr) : |t| \le C\Bigr\} + \tilde{u} + O(R^{-\frac{1}{2}+\delta})$$

is contained in a ball of radius $CR^{-\frac{1}{2}+C\delta}$. For $j \ge -C$, let us set

$$\Gamma_1^j(R^{-\frac{1}{2}+\delta}) = \{\Phi_1^{\nu_1,\nu_2'}(\nu,t) : \nu \in \Pi_1^{\nu_1,\nu_2'},\ 2^{-j-1} \le |t| \le 2^{-j}\} + O(R^{-\frac{1}{2}+\delta}).$$

Using homogeneity and a dyadic decomposition in t for Γ_1, the matter can be reduced to the case $2^{-1} \le |t| \le 1$. That is to say,

$$\Gamma_1^0(R^{-\frac{1}{2}+\delta}) \cap \mathfrak{C}_2(R^{-\frac{1}{2}+\delta}) \subset B(u, C_0 R^{-\frac{1}{2}+\delta}) \tag{3-2}$$

for some u and $C_0 > 0$. In fact, applying the scaling change of variables $(x,t) \to 2^{-j}(x,t)$, followed by (3-2) and the reverse change of variables, we see that $\Gamma_1^j(R^{-\frac{1}{2}+\delta}) \cap \mathfrak{C}_2(R^{-\frac{1}{2}+\delta})$ is contained in a ball of radius $C_0 R^{-\frac{1}{2}+\delta}$. Since $\Gamma_1 \subset \bigcup_{2^{-1}R^{-\delta}\le 2^j \le C} \Gamma_1^j$, we know $\Gamma_1 \cap \mathfrak{C}_2(R^{-\frac{1}{2}+\delta})$ is contained in the union of as many as $\sim \log R$ such balls of radius $C_0 R^{-\frac{1}{2}+\delta}$. This union of balls is obviously contained in a ball of radius $CR^{-\frac{1}{2}+C\delta}$ since the set $\Gamma_1 \cap \mathfrak{C}_2(R^{-\frac{1}{2}+\delta})$ is connected.

Since we may assume that S_1 and S_2 are sufficiently small, in order to show (3-2) it is enough to show that the tangent spaces of the surfaces $\Phi_1^{\nu_1,\nu_2'} : \Pi_1^{\nu_1,\nu_2'} \times \{2^{-1} \le |t| \le 1\} \to \mathbb{R}^{d+k}$ and $\{(\sum_{j=1}^k t_j \nabla\varphi_j(\nu_2), t) : |t| \le C\}$ are uniformly transversal to each other. In fact, since all the underlying sets are compact, by continuity it is enough to check this at each point.

Let $u_0 = \Phi_1^{\nu_1,\nu_2'}(\nu_0, t_0)$ for $\nu_0 \in \Pi_1^{\nu_1,\nu_2'}$ and $2^{-1} \le |t_0| \le 1$. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d-k}$ be orthonormal vectors spanning the tangent space $T_{\nu_0}\Pi_1^{\nu_1,\nu_2'}$. Then the tangent space of the parametrized surface $\Phi_1^{\nu_1,\nu_2'} : \Pi_1^{\nu_1,\nu_2'} \times \{2^{-1} \le |t| \le 1\} \to \mathbb{R}^{d+k}$ at u_0 is spanned by the vectors

$$(\nabla\varphi_1(\nu_0), -1, 0, \dots, 0), \quad (\nabla\varphi_2(\nu_0), 0, -1, 0, \dots, 0), \quad \dots, \quad (\nabla\varphi_k(\nu_0), 0, \dots, 0, -1) \tag{3-3}$$

and

$$\left(\boldsymbol{v}_i\left(\sum_{j=1}^k t_{0,j} H\varphi_j(\nu_0)\right), 0, \dots, 0\right), \quad i = 1, \dots, d-k. \tag{3-4}$$

On the other hand, the k-dimensional plane $\{(-\sum_{j=1}^k t_j \nabla\varphi_j(\nu_2), t) : |t| \le C\}$ is spanned by

$$(\nabla\varphi_1(\nu_2), -1, 0, \dots, 0), \quad (\nabla\varphi_2(\nu_2), 0, -1, 0, \dots, 0), \quad \dots, \quad (\nabla\varphi_k(\nu_2), 0, \dots, 0, -1). \tag{3-5}$$

Hence it suffices to show that these $d+k$ vectors are linearly independent, or equivalently that the determinant of the matrix with these vectors as row vectors is nonzero. After Gaussian elimination it is enough to show

$$\det\begin{pmatrix} \boldsymbol{V}\left(\sum_{j=1}^k t_{0,j} H\varphi_j(\nu_0)\right) \\ \boldsymbol{D}(\nu_0, \nu_2) \end{pmatrix} \ne 0, \tag{3-6}$$

where $\boldsymbol{V}$ is the $(d-k)\times d$ matrix having $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d-k}$ as its row vectors. Now by (3-1) we note that the vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d-k}$ are orthogonal to the span of the vectors

$$\nabla\varphi_j(\nu_0 + \nu_2' - \nu_1) - \nabla\varphi_j(\nu_0), \quad j = 1, \dots, k.$$

Assuming S_2 is small enough, we may replace $\boldsymbol{D}(\nu_0, \nu_2)$ by $\boldsymbol{D}(\nu_0, \nu_0+\nu_2'-\nu_1)$. For simplicity we set $\tilde{\nu}_2 = \nu_0+\nu_2'-\nu_1$. (We may assume there is a $c > 0$ such that $\left|\det\left[\boldsymbol{N}(\nu_2,\nu_1)\left(\sum_{i=1}^k t_i H\varphi_i(\nu)\right)\boldsymbol{N}^t(\nu_2,\nu_1)\right]\right| > c$ for $\nu_1 \in S_2$ and $\nu_2 \in S_2$; see Remark 1.4.) Since $\left(\sum_{j=1}^k t_{0,j} H\varphi_j(\nu_0)\right)$ is invertible, we need only show

$$\det \boldsymbol{A} \ne 0, \quad \text{where } \boldsymbol{A} = \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{D}(\nu_0, \tilde{\nu}_2)\left(\sum_{j=1}^k t_{0,j} H\varphi_j(\nu_0)\right)^{-1} \end{pmatrix}.$$

Since $\boldsymbol{V}\boldsymbol{D}^t(\nu_0, \tilde{\nu}_2) = 0$, we note that the matrix $\boldsymbol{A}\,\left(\boldsymbol{V}^t\ \boldsymbol{D}^t(\nu_0, \tilde{\nu}_2)\right)$ equals

$$\begin{pmatrix} I_{d-k} & 0 \\ \boldsymbol{D}(\nu_0, \tilde{\nu}_2)\left(\sum_{j=1}^k t_{0,j} H\varphi_j(\nu_0)\right)^{-1} V^t & \boldsymbol{D}(\nu_0, \tilde{\nu}_2)\left(\sum_{j=1}^k t_{0,j} H\varphi_j(\nu_0)\right)^{-1} \boldsymbol{D}^t(\nu_0, \tilde{\nu}_2) \end{pmatrix}.$$

This matrix is clearly invertible thanks to (1-3). Hence, so is the matrix $\boldsymbol{A}$. ☐

Below we show that the following version of Lemma 3.2 holds, where we assume (1-5) instead of (1-3), dropping the condition (1-2).

Lemma 3.3. *Suppose that, for any $\nu_1 \in S_1$, $\nu_2 \in S_2$, $\nabla\varphi_i(\nu_2) - \nabla\varphi_i(\nu_1)$, $i = 1, \dots, k$ are linearly independent and (1-5) holds for $\nu_1 \in S_1$, $\nu_2 \in S_2$, $|t| = 1$ and for $\nu = \nu_1, \nu_2$. If S_1 and S_2 are sufficiently small, there is a constant C, independent of ν_1, ν_2', R, and u such that, for some $u' \in \mathbb{R}^{d+1}$,*

$$\widetilde{\Gamma}_1^{\nu_1,\nu_2'}(R, R^\delta) \cap (R^\delta \pi_{\nu_2} + u) \subset B(u', CR^{\frac{1}{2}+C\delta}).$$

Proof. It is sufficient to show that (3-6) holds. As before, under the assumption that S_2 is small enough, we can replace $\boldsymbol{D}(\nu_0, \nu_2)$ with $\boldsymbol{D}(\nu_0, \tilde{\nu}_2)$, where $\tilde{\nu}_2 = \nu_0 + \nu_2' - \nu_1$. We need only show that

$$\det \begin{pmatrix} \boldsymbol{V}\big(\sum_{j=1}^k t_{0,j} H\varphi_j(\nu_0)\big) \\ \boldsymbol{D}(\nu_0, \tilde{\nu}_2) \end{pmatrix} \neq 0.$$

Since vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d-k}$ are orthogonal to the row vectors of $\boldsymbol{D}(\nu_0, \tilde{\nu}_2)$, by multiplying the nonsingular matrix $(\boldsymbol{V}^t\ \boldsymbol{D}^t(\nu_0, \tilde{\nu}_2))$ by the matrix inside the determinant from the right, we see that the above is equivalent to

$$\det \begin{pmatrix} \boldsymbol{V}\big(\sum_{j=1}^k t_{0,j} H\varphi_j(\nu_0)\big)\boldsymbol{V}^t & \boldsymbol{V}\big(\sum_{j=1}^k t_{0,j} H\varphi_j(\nu_0)\big)\boldsymbol{D}^t(\nu_0, \tilde{\nu}_2) \\ 0 & \boldsymbol{D}(\nu_0, \tilde{\nu}_2)\boldsymbol{D}^t(\nu_0, \tilde{\nu}_2) \end{pmatrix} \neq 0.$$

Since the matrix $\boldsymbol{D}(\nu_0, \tilde{\nu}_2)\, \boldsymbol{D}^t(\nu_0, \tilde{\nu}_2)$ is nonsingular, it is clear that the above is equivalent to

$$\det\left[\boldsymbol{V}\left(\sum_{j=1}^k t_{0,j} H\varphi_j(\nu_0)\right)\boldsymbol{V}^t\right] \neq 0,$$

which is the condition (1-5). □

4. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. Our proof is similar to that in [Lee 2006]; also see [Tao 2003]. To prove Theorem 1.1, we need only show that, for $p > (d+3k)/(d+k)$,

$$\|E_1 f\ E_2 g\|_p \le C\|f\|_2\,\|g\|_2$$

since we can obtain the desired conclusion by interpolating this estimate with the trivial estimate $\|E_1 f\ E_2 g\|_\infty \le \|f\|_1\,\|g\|_1$. By an ϵ-removal argument [Tao and Vargas 2000; Bourgain and Guth 2011], it is sufficient to show that (1-6) holds for any $\alpha > 0$. In fact, by the assumption that $\sum_{j=1}^k t_j H\varphi_j(\nu)$ is nonsingular for $\nu \in \operatorname{supp} f \cup \operatorname{supp} g$ as long as $|t| = 1$, it follows that

$$|E_\kappa(a_\kappa)(x,t)| \lesssim (|x| + |t|)^{-\frac{d}{2}}, \quad \kappa = 1, 2,$$

where a_1, a_2 are smooth bump functions which vanish on the supports of f and g, respectively. This can be shown by the stationary phase method. Hence, the arguments in the papers mentioned above work here without modification.

Proposition 4.1. *Let* $0 < \delta \ll 1$. *If* (1-6) *holds, then for any* $\epsilon > 0$

$$\|E_1 f\, E_2 g\|_{L^{(d+3k)/(d+k)}(Q_R)} \le C_\epsilon R^{\max(\alpha(1-\delta), C\delta)+\epsilon} \|f\|_2 \|g\|_2, \tag{4-1}$$

with C *independent of* δ.

By iterating finitely many times the implication in Proposition 4.1, we can easily obtain the estimate (1-6) for any $\alpha > 0$.

Wave packet decomposition. In this section we decompose the function Ef into wave packets. Let $R \gg 1$. We define

$$\mathcal{L} = \mathcal{L}(R) := R^{\frac{1}{2}}\mathbb{Z}^d, \quad \mathcal{V} = \mathcal{V}(R) := R^{-\frac{1}{2}}\mathbb{Z}^d.$$

Let ψ be a nonnegative Schwartz function such that $\hat{\psi}$ is supported on $B(0,1)$ and $\sum_{k\in\mathbb{Z}^d} \psi(\cdot - k) = 1$. Also, let ζ be a smooth function supported on $B(0,1)$ and $\sum_{k\in\mathbb{Z}^d} \zeta(\cdot - k) = 1$.

For $\ell \in \mathcal{L}$, $\nu \in \mathcal{V}$ we set $\psi_\ell(x) := \psi((x-\ell)/R^{\frac{1}{2}})$, $\zeta_\nu(\xi) =: \zeta(R^{\frac{1}{2}}(\xi - \nu))$, and for a given function f, we define $f_{\ell,\nu}$ by

$$f_{\ell,\nu} = \mathcal{F}(\psi_\ell \mathcal{F}^{-1}(\zeta_\nu f)),$$

where $\mathcal{F}$, $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively. Then, it follows that $f = \sum_{\nu\in\mathcal{V}} \sum_{\ell\in\mathcal{L}} f_{\ell,\nu}$. Hence we may write

$$Ef = \sum_{\nu\in\mathcal{V}} \sum_{\ell\in\mathcal{L}} Ef_{\ell,\nu}. \tag{4-2}$$

Lemma 4.2. *If* $|t| \lesssim R$, *then*

$$|Ef_{\ell,\nu}(x,t)| \le C_N \Big(1 + R^{-\frac{1}{2}} \Big| x - \ell + \sum_{j=1}^{k} t_j \nabla\varphi_j(\nu) \Big| \Big)^{-N} M(\mathcal{F}^{-1}(\zeta_\nu f))(\ell) \tag{4-3}$$

for all $N \ge 0$. *Here,* Mf *is the Hardy–Littlewood maximal function of* f.

Proof. Since $f_{\ell,\nu}$ is supported in $B(\nu, 3R^{-\frac{1}{2}})$, multiplying by a harmless smooth bump function $\tilde{\chi}$ supported in $B(0,5)$ and satisfying $\tilde{\chi} = 1$ on $B(0,3)$, we may write

$$Ef_{\ell,\nu}(x,t) = \int K(x-z,t)\psi_\ell(z)\mathcal{F}^{-1} f_\nu(z)\, dz,$$

where $K(x,t) = \int e^{2\pi i(x\cdot\xi + t\cdot\Phi(\xi))} \chi(R^{\frac{1}{2}}(\xi - \nu))\, d\xi$. Changing variables $\xi \to R^{-\frac{1}{2}}\xi + \nu$,

$$K(x,t) = R^{-\frac{d}{2}} e^{2\pi i x\cdot\nu} \int e^{2\pi i(R^{-1/2}x\cdot\xi + t\cdot\Phi(R^{-1/2}\xi+\nu))} \chi(\xi)\, d\xi.$$

Since $|t| \lesssim R$, we know $\nabla_\xi(R^{-\frac{1}{2}}x\cdot\xi + t\cdot\Phi(R^{-\frac{1}{2}}\xi + \nu)) = R^{-\frac{1}{2}}\big(x + \sum_{j=1}^{k} t_j \nabla\varphi_j(\nu)\big) + O(1)$. This follows by Taylor's expansion. Hence, by repeated integration by parts we get

$$|K(x,t)| \le C_N R^{-\frac{d}{2}} \Big(1 + R^{-\frac{1}{2}} \Big| x + \sum_{j=1}^{k} t_j \nabla\varphi_j(\nu) \Big| \Big)^{-N}.$$

Once this is established, (4-3) follows by a standard argument. See [Lee 2006] for the details. □

From the above lemma we see that $Ef_{\ell,\nu}$ is essentially supported on

$$\pi_{\ell,\nu} = \pi_\nu + (\ell, 0). \tag{4-4}$$

If $\pi = \pi_{\ell,\nu}$, we define $\nu(\pi) = \nu$, which may be considered as the (generalized) direction of π.

The following is the main lemma of this section.

Lemma 4.3. *Let $R \gg 1$. Then, Ef can be rewritten as*

$$Ef(x,t) = \sum_{(\ell,\nu)\in\mathcal{L}\times\mathcal{V}} c_{\ell,\nu} P_{\ell,\nu}(x,t) \tag{4-5}$$

and $c_{\ell,\nu}$, $P_{\ell,\nu}$ satisfy the following:

(i) *$\mathcal{F}(P_{\ell,\nu}(\cdot,t))$ is supported in the disc $D(\nu, CR^{-\frac{1}{2}})$.*

(ii) *If $|t| \lesssim R$, then for any $N \ge 0$*

$$|P_{\ell,\nu}(x,t)| \le C_N R^{-\frac{d}{4}} \left(1 + R^{-\frac{1}{2}} \left| x - \ell + \left(\sum_{j=1}^{k} t_j \nabla\varphi_j(\nu)\right)\right|\right)^{-N}.$$

(iii) $\left(\sum_{(\ell,\nu)\in\mathcal{L}\times\mathcal{V}} |c_{\ell,\nu}|^2\right)^{\frac{1}{2}} \lesssim \|f\|_2$.

(iv) *If $|t| \lesssim R$, then $\left\|\sum_{(\ell,\nu)\in\mathcal{W}} P_{\ell,\nu}(\cdot,t)\right\|_2^2 \lesssim \#\mathcal{W}$ for any $\mathcal{W} \subset \mathcal{L}\times\mathcal{V}$.*

Proof. We define $c_{\ell,\nu}$ and $P_{\ell,\nu}$ by

$$c_{\ell,\nu} = R^{\frac{d}{4}} M(\mathcal{F}^{-1} f_\nu)(\ell), \quad P_{\ell,\nu}(x,t) = c_{\ell,\nu}^{-1} Ef_{\ell,\nu}(x,t),$$

where M denotes the Hardy–Littlewood maximal function. Then we have (4-5) from (4-2). Since $Ef_{\ell,\nu}(\cdot, y) = \mathcal{F}^{-1}(e^{2\pi i y\Phi} f_{\ell,\nu})$, we know $Ef_{\ell,\nu}(\cdot, y)$ has Fourier support contained in supp $f_{\ell,\nu}$, which is in turn contained in $D(\nu, CR^{-\frac{1}{2}})$. Thus (i) follows and so does (ii) from Lemma 4.2.

In order to show (iii), note that

$$\sum_{(\ell,\nu)\in\mathcal{L}\times\mathcal{V}} |c_T|^2 = R^{\frac{d}{2}} \sum_{(\ell,\nu)\in\mathcal{L}\times\mathcal{V}} M(\mathcal{F}^{-1}(\zeta_\nu f))(\ell)^2. \tag{4-6}$$

Since $\zeta_\nu f$ is supported on $B(\nu, CR^{\frac{1}{2}})$, we have $M(\mathcal{F}^{-1}(\zeta_\nu f))(x) \sim M(\mathcal{F}^{-1}(\zeta_\nu f))(x')$ if $|x - x'| \lesssim R^{\frac{1}{2}}$. Hence, from the Hardy–Littlewood maximal theorem and Plancherel's theorem we have that, for each ν,

$$R^{\frac{d}{2}} \sum_{(\ell,\nu)\in\mathcal{L}\times\mathcal{V}} |M(\mathcal{F}^{-1}(\zeta_\nu f))(\ell)|^2 \lesssim \int |M(\mathcal{F}^{-1}(\zeta_\nu f))(x)|^2 \, dx \lesssim \|\zeta_\nu f\|_2^2.$$

Combining this and (4-6) we obtain $\sum_{(\ell,\nu)\in\mathcal{L}\times\mathcal{V}} |c_{\ell,\nu}|^2 \lesssim \sum_{\nu\in\mathcal{V}} \|\zeta_\nu f\|_2^2 \lesssim \|f\|_2^2$, and (iii).

Finally, we consider (iv). Since $\sum_{\ell:(\ell,\nu)\in\mathcal{W}} P_{\ell,\nu}(\cdot,t)$ is Fourier-supported in $D(\nu, CR^{-\frac{1}{2}})$, which has bounded overlap as ν varies over $\mathcal{V}$, by Plancherel's theorem,

$$\left\| \sum_{(\ell,\nu)\in\mathcal{W}} P_{\ell,\nu}(\cdot,t) \right\|_2^2 \lesssim \sum_{\nu\in\mathcal{V}} \left\| \sum_{\ell:(\ell,\nu)\in\mathcal{W}} P_{\ell,\nu}(\cdot,t) \right\|_2^2.$$

From (ii) it is easy to see that $\|\sum_{\ell:(\ell,\nu)\in\mathcal{W}} P_{\ell,\nu}(\cdot\,,t)\|_2^2 \lesssim \#\{\ell:(\ell,\nu)\in\mathcal{W}\}$. Hence, combining this with the above gives (iv). □

Dyadic pigeonholing and reduction. In the remainder of this section we will prove Proposition 4.1. For simplicity we set

$$p_0 = \frac{d+3k}{d+k}.$$

By translation invariance we may assume that Q_R is centered at the origin. Let

$$\mathcal{W}_i \subset \{(\ell,\nu)\in\mathcal{L}\times\mathcal{V} : \nu\in S_i + O(R^{-\frac{1}{2}})\},\quad i=1,2.$$

By Lemma 4.3 and the standard reduction with pigeonholing, which may only cause a loss of $(\log R)^C$, see [Lee 2006; Tao 2003], the matter is reduced to showing

$$\Big\| \sum_{\omega_1\in\mathcal{W}_1} P_{\omega_1} \sum_{\omega_2\in\mathcal{W}_2} P_{\omega_2}\Big\|_{L^{p_0}(Q_R)} \lessapprox (R^{(1-\delta)\alpha} + R^{C\delta})(\#\mathcal{W}_1\#\mathcal{W}_2)^{\frac{1}{2}}$$

whenever P_{ω_1}, P_{ω_2} satisfy (i), (ii), (iv) in Lemma 4.3. Here $A \lessapprox B$ means $A\le C_\epsilon R^\epsilon B$ for any $\epsilon>0$.

By a further pigeonholing argument we specify the associated quantities in dyadic scales. Let $\mathcal{Q}$ be a collection of almost disjoint cubes of the same side length $\sim R^{\frac{1}{2}}$ which cover Q_R. For each $q\in\mathcal{Q}$ we define

$$\mathcal{W}_j(q) = \{\omega_j\in\mathcal{W}_j : \pi_{\omega_j}\cap R^\delta q\neq\varnothing\}.$$

For dyadic numbers ρ_1, ρ_2 with $1\le\rho_1,\rho_2\le R^{100d}$, we define

$$\mathcal{Q}(\rho_1,\rho_2) = \{q\in\mathcal{Q} : \rho_j \le \#\mathcal{W}_j(q) < 2\rho_j,\ j=1,2\}. \tag{4-7}$$

For $\omega\in\mathcal{W}_1\cup\mathcal{W}_2$, we set

$$\lambda(\omega;\rho_1,\rho_2) = \#\{q\in\mathcal{Q}(\rho_1,\rho_2) : \pi_\omega\cap R^\delta q\neq\varnothing\}.$$

For a dyadic number $1\le\lambda\le R^{100d}$, we define

$$\mathcal{W}_j[\lambda;\rho_1,\rho_2] = \{\omega_j\in\mathcal{W}_j : \lambda\le\lambda(\omega_j;\rho_1,\rho_2)<2\lambda\},\quad j=1,2. \tag{4-8}$$

By a standard pigeonhole argument, it is sufficient to show

$$\Big(\sum_{q\in\mathcal{Q}(\rho_1,\rho_2)} \Big\| \sum_{\omega_1\in\mathcal{W}_1[\lambda_1;\rho_1,\rho_2]} P_{\omega_1} \sum_{\omega_2\in\mathcal{W}_2[\lambda_2;\rho_1,\rho_2]} P_{\omega_2}\Big\|_{L^{p_0}(q)}^{p_0}\Big)^{\frac{1}{p_0}} \lessapprox (R^{(1-\delta)\alpha}+R^{C\delta})\,(\#\mathcal{W}_1\,\#\mathcal{W}_2)^{\frac{1}{2}}. \tag{4-9}$$

For the rest of the proof we assume that $q\in\mathcal{Q}(\rho_1,\rho_2)$, $\omega_1\in\mathcal{W}_1[\lambda_1;\rho_1,\rho_2]$ and $\omega_2\in\mathcal{W}_2[\lambda_1;\rho_1,\rho_2]$ if it is not mentioned otherwise. So, the above sums are denoted simply by $\sum_q$, $\sum_{\omega_1}$, and $\sum_{\omega_2}$, respectively.

Induction argument. For brevity let us put

$$\Delta = \bigcup_{q \in \mathcal{Q}(\rho_1,\rho_2)} q.$$

Let $\{B\}$ be a collection of almost disjoint cubes of the same side length $R^{1-\delta}$, which cover Q_R. Then

$$\text{the left-hand side of (4-9)} \le \sum_B \Big\| \sum_{\omega_1} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \Big\|_{L^{p_0}(\Delta \cap B)}. \tag{4-10}$$

We define a relation $\sim$ between ω_1 (or ω_2) and the cubes in $\{B\}$. For each $\omega \in \mathcal{W}_1[\lambda_1; \rho_1, \rho_2] \cup \mathcal{W}_2[\lambda_2; \rho_1, \rho_2]$, we define $B^*(\omega) \in \{B\}$ to be the cube which maximizes the quantity

$$\#\big\{q \in \mathcal{Q}(\rho_1, \rho_2) : \pi_\omega \cap R^\delta q \neq \varnothing,\, q \cap B \neq \varnothing\big\}. \tag{4-11}$$

Then the relation $\sim$ is defined as

$$\omega \sim B \quad \text{if } B \cap 10B^*(\omega) \neq \varnothing.$$

Here $10B^*(\omega)$ is the cube which has the same center as $B^*(\omega)$ and side length 10 times as large as that of $B^*(\omega)$. Using this relation we divide the sum into three parts to get

$$\begin{aligned}\sum_B \Big\| \sum_{\omega_1} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \Big\|_{L^{p_0}(\Delta \cap B)} &\le \sum_B \Big\| \sum_{\substack{\omega_1: \\ \omega_1 \sim B}} P_{\omega_1} \sum_{\substack{\omega_2: \\ \omega_2 \sim B}} P_{\omega_2} \Big\|_{L^{p_0}(\Delta \cap B)} \\ &\quad + \sum_B \Big\| \sum_{\substack{\omega_1: \\ \omega_1 \sim B}} P_{\omega_1} \sum_{\substack{\omega_2: \\ \omega_2 \nsim B}} P_{\omega_2} \Big\|_{L^{p_0}(\Delta \cap B)} + \sum_B \Big\| \sum_{\substack{\omega_1: \\ \omega_1 \nsim B}} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \Big\|_{L^{p_0}(\Delta \cap B)}.\end{aligned}$$

We will first show that

$$\sum_B \Big\| \sum_{\substack{\omega_1: \\ \omega_1 \sim B}} P_{\omega_1} \sum_{\substack{\omega_2: \\ \omega_2 \sim B}} P_{\omega_2} \Big\|_{L^{p_0}(\Delta \cap B)} \lesssim R^{(1-\delta)\alpha} (\#\mathcal{W}_1 \#\mathcal{W}_2)^{\frac{1}{2}}. \tag{4-12}$$

By applying the hypothesis (1-6), (iv) in Lemma 4.3, and the Cauchy–Schwarz inequality,

$$\sum_B \Big\| \sum_{\substack{\omega_1: \\ \omega_1 \sim B}} P_{\omega_1} \sum_{\substack{\omega_2: \\ \omega_2 \sim B}} P_{\omega_2} \Big\|_{L^{p_0}(\Delta \cap B)} \le C R^{(1-\delta)\alpha} \prod_{j=1}^{2} \Big(\sum_B \#\{\omega_j : \omega_j \sim B\} \Big)^{\frac{1}{2}}.$$

From the definition of the relation $\sim$ it is clear that $\#\{B : \omega_j \sim B\} \le C$. Hence, for $j = 1, 2$

$$\sum_B \#\{\omega_j : \omega_j \sim B\} = \sum_{\omega_j} \#\{B : \omega_j \sim B\} \lesssim W_j.$$

By inserting this into the previous inequality, we get (4-12).

Now, to prove (4-9) it is enough to show

$$\Big\|\sum_{\substack{\omega_1:\\ \omega_1\sim B}} P_{\omega_1}\sum_{\substack{\omega_2:\\ \omega_2\nsim B}} P_{\omega_2}\Big\|_{L^{p_0}(\Delta\cap B)} \lessapprox R^{C\delta}(\#\mathcal{W}_1\,\#\mathcal{W}_2)^{\frac12}$$

and

$$\Big\|\sum_{\substack{\omega_1:\\ \omega_1\nsim B}} P_{\omega_1}\sum_{\omega_2} P_{\omega_2}\Big\|_{L^{p_0}(\Delta\cap B)} \lessapprox R^{C\delta}(\#\mathcal{W}_1\,\#\mathcal{W}_2)^{\frac12}. \tag{4-13}$$

The proofs of these two estimates are similar. So, we will only prove (4-13). By Plancherel's theorem, $\|Ef(\cdot,t)\|_2 \le \|f\|_2$ for all $t\in\mathbb{R}^k$. Integration in t gives $\|Ef\|_{L^2(Q_R)} \lesssim R^k/2\|f\|_2$. By the Schwarz inequality it follows that

$$\|E_1 f E_2 g\|_{L^1(Q_R)} \lesssim R^k\|f\|_2\,\|g\|_2.$$

Combining this with (iv) in Lemma 4.3 yields

$$\Big\|\sum_{\substack{\omega_1:\\ \omega_1\nsim B}} P_{\omega_1}\sum_{\omega_2} P_{\omega_2}\Big\|_{L^1(\Delta\cap B)} \lessapprox R^{k}(\#\mathcal{W}_1\#\mathcal{W}_2)^{\frac12}. \tag{4-14}$$

Hence, the (4-13) follows from interpolation between (4-14) and

$$\Big\|\sum_{\substack{\omega_1:\\ \omega_1\nsim B}} P_{\omega_1}\sum_{\omega_2} P_{\omega_2}\Big\|_{L^2(\Delta\cap B)} \lessapprox R^{C\delta}R^{-\frac{d-k}{4}}(\#\mathcal{W}_1\#\mathcal{W}_2)^{\frac12}. \tag{4-15}$$

Now it remains to show the L^2-estimate (4-15).

L^2 estimate. To prove (4-15) it suffices to show

$$\sum_{\substack{q\in\mathcal{Q}(\rho_1,\rho_2)\\ q\subset 2B}}\Big\|\sum_{\substack{\omega_1:\\ \omega_1\nsim B}} P_{\omega_1}\sum_{\omega_2} P_{\omega_2}\Big\|_{L^2(q)}^2 \lessapprox R^{C\delta}R^{-\frac{d-k}{2}}\#\mathcal{W}_1\#\mathcal{W}_2. \tag{4-16}$$

For $j=1,2$, let us set

$$\mathcal{W}_j(q)=\{\omega_j\in\mathcal{W}_i[\lambda_j;\rho_1,\rho_2]:\omega_j\cap R^\delta q\neq\varnothing\},\quad \mathcal{W}_j^{\nsim B}(q)=\{\omega_j\in\mathcal{W}_j(q):\omega_j\nsim B\}.$$

Then by (ii) in Lemma 4.3 we may discard some harmless terms, whose contributions are $O(R^{-C\delta})$. Hence, it suffices to show

$$\sum_{\substack{q\in\mathcal{Q}(\rho_1,\rho_2)\\ q\subset 2B}}\Big\|\sum_{\omega_1\in\mathcal{W}_1^{\nsim B}(q)} P_{\omega_1}\sum_{\omega_2(q)} P_{\omega_2}\Big\|_2^2 \lessapprox R^{C\delta}R^{-\frac{d-k}{2}}\#\mathcal{W}_1\,\#\mathcal{W}_2. \tag{4-17}$$

By using Plancherel's theorem we write

$$\Big\|\sum_{\omega_1\in\mathcal{W}_1^{\nsim B}(q)} P_{\omega_1}\sum_{\omega_2\in\mathcal{W}_2(q)} P_{\omega_2}\Big\|_2^2=\sum_{\omega_1\in\mathcal{W}_1^{\nsim B}(q)}\ \sum_{\omega_2'\in\mathcal{W}_2(q)}\ \sum_{\omega_1'\in\mathcal{W}_1^{\nsim B}(q)}\ \sum_{\omega_2\in\mathcal{W}_2(q)}\langle \widehat{P}_{\omega_1}*\widehat{P}_{\omega_2},\widehat{P}_{\omega_1'}*\widehat{P}_{\omega_2'}\rangle.$$

Let us write $\omega_j=(\ell_j,\nu_j)$, $\omega'_j=(\ell'_j,\nu'_j)$, $j=1,2$. For any $\nu_1\in S_1$, $\nu'_2\in S_2$, we define $\mathcal{W}_1^{\sim B}(q;\nu_1,\nu'_2)$ by

$$\mathcal{W}_1^{\sim B}(q;\nu_1,\nu'_2)=\{\omega'_1=(\ell'_1,\nu'_1)\in\mathcal{W}_1^{\sim B}(q):\nu'_1\in\Pi_1^{\nu_1,\nu'_2}+O(R^{-\frac12})\}.$$

Then $\widehat{P}_{\omega_1}*\widehat{P}_{\omega_2}$ is supported on the $O(R^{-\frac12})$-neighborhood of the point $(\nu_1+\nu_2,\Phi(\nu_1)+\Phi(\nu_2))$. So the inner product $\langle\widehat{P}_{\omega_1}*\widehat{P}_{\omega_2},\widehat{P}_{\omega'_1}*\widehat{P}_{\omega'_2}\rangle$ vanishes unless

$$\nu_1+\nu_2=\nu'_1+\nu'_2+O(R^{-\frac12}),\quad \Phi(\nu_1)+\Phi(\nu_2)=\Phi(\nu'_1)+\Phi(\nu'_2)+O(R^{-\frac12}).$$

Thus, for given ν_1 and ν'_2, we see that ν'_1 is contained in an $O(R^{-\frac12})$-neighborhood of $\Pi_1^{\nu_1,\nu'_2}$, which is defined by (3-1). Once ν_1, ν'_1 and ν'_2 are given, then there are only $O(1)$ many ν_2, since ν_2 should be in an $O(R^{-\frac12})$-neighborhood of the point $\nu_1+\nu'_1-\nu'_2$. Therefore,

$$\Bigg\|\sum_{\omega_1\in\mathcal{W}_1^{\sim B}(q)}P_{\omega_1}\sum_{\omega_2\in\mathcal{W}_2(q)}P_{\omega_2}\Bigg\|_2^2\lesssim R^{-\frac{d-k}{2}}\sum_{\omega_1\in\mathcal{W}_1^{\sim B}(q)}\sum_{\omega'_2\in\mathcal{W}_2(q)}\#\mathcal{W}_1^{\sim B}(q;\nu_1,\nu'_2),$$

where we also used

$$|\langle P_{\omega_1}P_{\omega_2},P_{\omega'_1}P_{\omega'_2}\rangle|\lesssim R^{-\frac{d-k}{2}}.$$

This follows from (ii) in Lemma 4.3 and the transversality between π_{ω_1} ($\pi_{\omega'_1}$) and π_{ω_2} ($\pi_{\omega'_2}$), respectively. Hence, (4-17) follows if we show

$$\max_{q\subset 2B,\,\nu_1,\nu'_2}\#\mathcal{W}_1^{\sim B}(q;\nu_1,\nu'_2)\sum_{\substack{q\in\mathcal{Q}(\rho_1,\rho_2)\\ q\subset 2B}}\#\mathcal{W}_1^{\sim B}(q)\#\mathcal{W}_2(q)\lesssim R^{C\delta}\#\mathcal{W}_1\,\#\mathcal{W}_2. \tag{4-18}$$

We will prove (4-18), assuming for the moment that

$$\max_{q\subset 2B,\,\nu_1,\nu'_2}\#\mathcal{W}_1^{\sim B}(q;\nu_1,\nu'_2)\lesssim R^{C\delta}\frac{\#\mathcal{W}_2}{\lambda_1\rho_2}. \tag{4-19}$$

To this end it is enough to show

$$\sum_{\substack{q\in\mathcal{Q}(\rho_1,\rho_2)\\ q\subset 2B}}\#\mathcal{W}_1^{\sim B}(q)\#\mathcal{W}_2(q)\lesssim\lambda_1\rho_2\#\mathcal{W}_1.$$

Recalling $\#\mathcal{W}_2(q)\lesssim\rho_2$, we see that the left-hand side is bounded by

$$C\rho_2\sum_{q\in\mathcal{Q}(\rho_1,\rho_2)}\#\mathcal{W}_1(q).$$

Changing the order of summation, we see this in turn is bounded by $C\rho_2\sum_{\omega_1}\#\{q\in\mathcal{Q}(\rho_1,\rho_2):\pi_{w_1}\cap R^\delta q\}$. Since

$$\#\{q\in\mathcal{Q}(\rho_1,\rho_2):\pi_{w_1}\cap R^\delta q\}\lesssim\lambda_1,$$

the desired inequality (4-18) follows.

***Proof of* (4-19).** Fix $q \subset 2B$, $\nu_1 \in S_1$ and $\nu_2' \in S_2$. Let us consider the set

$$\boldsymbol{S} := \{(\tilde q, \omega_1, \omega_2) \in \mathcal{Q}(\rho_1, \rho_2) \times \mathcal{W}_1^{\nsim B}(q; \nu_1, \nu_2') \times \mathcal{W}_2 : \pi_{\omega_1} \cap R^\delta \tilde q \neq \varnothing,\ \pi_{\omega_2} \cap R^\delta \tilde q \neq \varnothing,\ \mathrm{dist}(\tilde q, q) \ge R^{1-\delta}\}.$$

To prove (4-19) it suffices to show

$$R^{-C\delta} \lambda_1 \rho_2 \# \mathcal{W}_1^{\nsim B}(q; \nu_1, \nu_2')) \lesssim \#\boldsymbol{S} \lesssim R^{C\delta} \# \mathcal{W}_2. \tag{4-20}$$

For the lower bound it is enough to show that, for each $\omega_1 \in \mathcal{W}_1^{\nsim B}(q; \nu_1, \nu_2')$,

$$\#\{(\tilde q, \omega_2) \in \mathcal{Q}(\rho_1, \rho_2) \times \mathcal{W}_2 : \pi_{\omega_1} \cap R^\delta \tilde q \neq \varnothing,\ \pi_{\omega_2} \cap R^\delta \tilde q \neq \varnothing,\ \mathrm{dist}(\tilde q, q) \ge R^{1-\delta}\} \ge R^{-C\delta} \lambda_1 \rho_2.$$

By (4-8), ω_1 contains as many as $O(\lambda_1)$ cubes $\tilde q$ in $\mathcal{Q}(\rho_1, \rho_2)$. (Recall that we are assume assuming $q \in \mathcal{Q}(\rho_1, \rho_2)$, $\omega_1 \in \mathcal{W}_1[\lambda_1; \rho_1, \rho_2]$ and $\omega_2 \in \mathcal{W}_2[\lambda_1; \rho_1, \rho_2]$.) Let $B^*(\omega_1) \in \mathcal{Q}$ be the cube which maximizes the quantity given by (4-11) with $\omega = \omega_1$. Since $\omega_1 \nsim B$, it follows from the definition of the relation $\sim$ that $\mathrm{dist}(B^*(\omega_1), B) \gtrsim R^{1-\delta}$. Since $\pi_{\omega_1} + O(R^{\frac12+\delta})$ can be covered by $R^{C\delta}$ cubes B, by a simple pigeonholing argument we get

$$\#\{\tilde q \in \mathcal{Q}(\rho_1, \rho_2) : \pi_{\omega_1} \cap R^\delta \tilde q \neq \varnothing,\ \mathrm{dist}(\tilde q, q) \ge R^{1-\delta}\} \gtrsim R^{-C\delta} \lambda_1.$$

Next, for the upper bound it suffices to show that, for any $\omega_2 \in \mathcal{W}_2$,

$$\#\{(\tilde q, \omega_1) \in \mathcal{Q}(\rho_1, \rho_2) \times \mathcal{W}_1^{\nsim B}(q; \nu_1, \nu_2') : \pi_{\omega_1} \cap R^\delta \tilde q \neq \varnothing,\ \pi_{\omega_2} \cap R^\delta \tilde q \neq \varnothing,\ \mathrm{dist}(\tilde q, q) \gtrsim R^{1-\delta}\} \lesssim R^{C\delta}. \tag{4-21}$$

Let z_0 be the center of q. Then, by the definition of $\mathcal{W}_1^{\nsim B}(q_0; \nu_1, \nu_2')$, it follows that

$$\bigcup_{\omega_1 \in \mathcal{W}_1^{\nsim B}(q; \nu_1, \nu_2')} \pi_{\omega_1} \subset \Gamma_1^{\nu_1, \nu_2'}(C R^{\frac12+\delta}) + z_0.$$

If $\omega_2 \in \mathcal{W}_2$, then it follows from Lemma 3.2 that the intersection

$$\pi_{\omega_2} \cap \Big(\bigcup_{\omega_1 \in \mathcal{W}_1^{\nsim B}(q; \nu_1, \nu_2')} \pi_{\omega_1} \Big)$$

is contained in a cube of side length $O(R^{\frac12+\delta})$. Thus, there are at most $O(R^{C\delta})$ choices of balls $\tilde q \in \mathcal{Q}(\rho_1, \rho_2)$ such that $(\tilde q, \omega_1)$ is contained in the set in (4-21). On the other hand, since $\mathrm{dist}(\tilde q, q) \gtrsim R^{1-C\delta}$, we have

$$\#\{w_1 \in \mathcal{W}_1^{\nsim B}(q; \nu_1, \nu_2') : \pi_{\omega_1} \cap R^\delta \tilde q \neq \varnothing,\ \pi_{\omega_1} \cap R^\delta q \neq \varnothing\} \lesssim R^{C\delta}. \tag{4-22}$$

To see this, by scaling it is enough to check that the map $S_1 \ni \nu \mapsto \sum_{i=1}^k t_j \nabla \varphi_i(\nu)$ is one-to-one whenever $|t| = 1$. But this follows from the condition (1-2) if we take S_1 to be small enough. Thus we obtain the claim (4-21). Hence, we also have (4-9), which finishes the proof of Proposition 4.1. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Thanks to Lemma 3.3, the line of argument in the proof of Theorem 1.1 works without modification except that we need to show (4-22). However, to prove (4-22) we don't need to show $S_1 \ni \nu \mapsto \sum_{i=1}^k t_j \nabla\varphi_i(\nu)$ is one-to-one. Instead, as is clear after rescaling, it is enough to show that $\Pi^{\nu_1,\nu_2'} \ni \nu \mapsto \sum_{i=1}^k t_j \nabla\varphi_i(\nu)$ is one-to-one. Let $\boldsymbol{t}_1, \dots, \boldsymbol{t}_{d-k}$ be a set of vectors spanning the tangent space of $\Pi^{\nu_1,\nu_2'}$ at ν_0. Then the above follows if we show that the matrix

$$(\boldsymbol{t}_1^t, \dots, \boldsymbol{t}_{d-k}^t)\biggl(\sum_{i=1}^k t_j H\varphi_i(\nu_0)\biggr)$$

has rank $d-k$ for $|t| = 1$. In fact, $\boldsymbol{t}_1, \dots, \boldsymbol{t}_{d-k}$ are almost normal to the span of $\{\nabla\varphi_i(\nu_2) - \nabla\varphi_i(\nu_1) : i = 1, \dots, k\}$. These vectors are close to $\boldsymbol{n}_1, \dots, \boldsymbol{n}_{d-k}$. Hence, assuming that S_1 and S_2 are small enough, the above follows if we show $\boldsymbol{N}(\nu_2, \nu_1)\sum_{i=1}^k t_j H\varphi_i(\nu_0)$ has rank $d-k$. This clearly follows from (1-5). ☐

5. Restriction estimates for complex surfaces

In this section we provide the proofs of Corollary 1.5 and Theorem 1.6. In what follows we set $k = 2$, $d = 2n$.

Proof of Corollary 1.5. Let φ_1, φ_2 be given by $\frac{1}{2} z^t D z = \varphi_1 + i\varphi_2$ so that

$$\varphi_1(x, y) = (x^t D x - y^t D y), \quad \varphi_2(x, y) = x^t D y, \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

In order to prove Corollary 1.5 we need only to show that the condition (1-13) implies the assumptions in Theorem 1.1.

Let us set $z_j = x_j + i y_j \in \mathbb{C}^n$ for $j = 1, 2$, $\delta_x = x_2 - x_1$, and $\delta_y = y_2 - y_1$. Then a computation shows that the associated matrix $\boldsymbol{M}(t, z_1, z_2, z)$ is given by

$$\boldsymbol{M}(t, z_1, z_2, z) = \begin{pmatrix} 0 & 0 & \delta_x^t D & -\delta_y^t D \\ 0 & 0 & \delta_y^t D & \delta_x^t D \\ D\delta_x & D\delta_y & t_1 D & t_2 D \\ -D\delta_y & D\delta_x & t_2 D & -t_1 D \end{pmatrix}.$$

Note that

$$\sum_{j=1}^2 t_j H\varphi_j = \begin{pmatrix} t_1 D & t_2 D \\ t_2 D & -t_1 D \end{pmatrix}.$$

Then, it is easy to see that the inverse of $\sum_{j=1}^2 t_j H\varphi_j$ is

$$(t_1^2 + t_2^2)^{-1} \begin{pmatrix} t_1 D^{-1} & t_2 D^{-1} \\ t_2 D^{-1} & -t_1 D^{-1} \end{pmatrix}.$$

So, the assumption (1-2) holds. Hence, it suffices to show that (1-13) implies (1-3). By the block matrix formula we only need to check

$$\det\left[\begin{pmatrix} \delta_x^t D & -\delta_y^t D \\ \delta_y^t D & \delta_x^t D \end{pmatrix} \begin{pmatrix} t_1 D^{-1} & t_2 D^{-1} \\ t_2 D^{-1} & -t_1 D^{-1} \end{pmatrix} \begin{pmatrix} D\delta_x & D\delta_y \\ -D\delta_y & D\delta_x \end{pmatrix}\right] \neq 0.$$

By a direct computation it is not difficult to see that the left-hand side equals[4]

$$-(t_1^2+t_2^2)\big((\delta_x^t D\delta_x-\delta_y^t D\delta_y)^2+4(\delta_x^t D\delta_y)^2\big).$$

Since $(z_2-z_1)^t D(z_2-z_1)=\delta_x^t D\delta_x-\delta_y^t D\delta_y+2i\delta_x^t D\delta_y$, it is now clear that (1-13) implies (1-3). ☐

Proof of Theorem 1.6. From the bilinear estimate we can get the linear estimate by adapting the arguments in [Tao et al. 1998; Vargas 2005; Lee 2006]. Since D is nonsingular and symmetric, by making use of linear transforms we may assume that

$$D=\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix},$$

and so we have either $\Phi(z_1,z_2)=z_1^2+z_2^2=(z_1+iz_2)(z_1-iz_2)$ or $\Phi(z_1,z_2)=z_1^2-z_2^2=(z_1+z_2)(z_1-z_2)$. By a linear change of variables the problem can be further reduced to showing Theorem 1.6 when $\Phi(z_1,z_2)=z_1z_2$.

The following is an immediate consequence of Theorem 1.1 and the translation invariance of the bilinear estimate.

Lemma 5.1. *Let $\Phi(z_1,z_2)=z_1z_2$ and $Q_1, Q_2\subset\mathbb{C}^2$ be closed cubes. Assume that*

$$2^4\ge|z_1-w_1|\ge 2^{-1}\quad \textit{and}\quad 2^4\ge|z_2-w_2|\ge 2^{-1}$$

whenever $(z_1,z_2)\in Q_1$ and $(w_1,w_2)\in Q_2$. If $\operatorname{supp}(f)\subset Q_1$ and $\operatorname{supp}(g)\subset Q_2$, then for $q>\frac{10}{3}$ and $\frac{1}{p}+\frac{5}{3q}<1$,

$$\|Ef\,Eg\|_{\frac{q}{2}}\le C_{p,q}\,\|f\|_p\,\|g\|_p.$$

In the next lemma the hypothesis of "nonvanishing rotational curvature" is weakened to the usual separation condition. But then, for the conclusion to hold, the pair $\big(\frac{1}{p},\frac{1}{q}\big)$ needs to satisfy a more restrictive condition. This lemma is an analog of Proposition 4.1 in [Lee 2006].

Lemma 5.2. *Let Q_1, Q_2 be closed cubes in $\mathbb{C}^2$ such that $\operatorname{dist}(Q_1,Q_2)\ge 1$. If $\operatorname{supp}(f)\subset Q_1$ and $\operatorname{supp}(g)\subset Q_2$, then there is a constant $C_{p,q}$ such that*

$$\|Ef\,Eg\|_{\frac{q}{2}}\le C_{p,q}\,\|f\|_p\,\|g\|_p\quad \textit{if}\ \frac{1}{p}+\frac{2}{q}<1,\ q>\frac{10}{3},$$

or

$$\|E\chi_F\,E\chi_G\|_{\frac{q}{2}}\lesssim\|f\|_{p,1}\,\|g\|_{p,1}\quad \textit{if}\ \frac{1}{p}+\frac{2}{q}=1,\ q>\frac{10}{3}.$$

By translation it is clear that in Lemmas 5.1 and 5.2 the same estimate holds with Q_1, Q_2 replaced by Q_1+a, Q_2+a, respectively, for any $a\in\mathbb{C}^2$. It is possible to prove the strong-type estimate $\|E\chi_F\,E\chi_G\|_{q/2}\lesssim\|f\|_p\,\|g\|_p$ for $\frac{1}{p}+\frac{2}{q}=1$, $q>\frac{10}{3}$ by making use of the asymmetric estimates which are obtained in the course of proof of Proposition 1.3 and the bilinear interpolation; see, e.g., [Bergh and Löfström 1976, Section 3.13, 5(b)]. However, we have decided not to include the details here, because it does not seem to have any consequences for linear estimates.

[4] In fact, the product of the three matrices is equal to $\begin{pmatrix}t_1 & -t_2\\ t_2 & t_1\end{pmatrix}\begin{pmatrix}\delta_x^t D\delta_x-\delta_y^t D\delta_y & 2\delta_x^t D\delta_y\\ 2\delta_x^t D\delta_y & \delta_y^t D\delta_y-\delta_x^t D\delta_x\end{pmatrix}$.

Proof of Lemma 5.2. By interpolation it suffices to consider the case $\frac{10}{3} < q \le 4$ and $p \le q$. By decomposition of the domains, followed by translation and scaling, we may assume that $Q_1 = H_1 \times K$ and $Q_2 = H_2 \times K$, where $\operatorname{dist}(H_1, H_2) \ge 2^{-1}$ and K is the unit cube in $\mathbb{C}$, centered at the origin.

By a Whitney decomposition, we get

$$(K \times K) \setminus D = \bigcup_{j>1} \bigcup_{\substack{(k,k'):\\ I_k^j \sim I_{k'}^j}} I_k^j \times I_{k'}^j$$

where $D = \{(z_2, w_2) : z_2 = w_2\}$, and $\{I_k^j\}_k$ are the dyadic cubes in $\mathbb{C}$ of side length 2^{-j}, and as usual the notation $I_k^j \sim I_{k'}^j$ means that the parent cubes of I_k^j and $I_{k'}^j$ are adjacent, while I_k^j and $I_{k'}^j$ are not.

Let us set

$$f_k^j(z_1, z_2) = \chi_{I_k^j}(z_2) f(z_1, z_2), \quad g_k^j(w_1, w_2) = \chi_{I_k^j}(w_2) g(w_1, w_2).$$

Then, since the cubes $I_k^j \times I_{k'}^j$ are almost disjoint, we may write

$$Ef\ Eg = \sum_j \sum_{\substack{(k,k'):\\ I_k^j \sim I_{k'}^j}} E(f_k^j)\, E(g_{k'}^j).$$

Since $q > \frac{10}{3}$, we get

$$\|Ef\ Eg\|_{\frac{q}{2}} \le \sum_j \Big\| \sum_{I_k^j \sim I_{k'}^j} E(f_k^j)\, E(g_{k'}^j) \Big\|_{\frac{q}{2}} \lesssim \sum_j \Big(\sum_{I_k^j \sim I_{k'}^j} \|E(f_k^j)\, E(g_{k'}^j)\|_{\frac{q}{2}}^{\frac{q}{2}} \Big)^{\frac{2}{q}},$$

where the last inequality follows from Lemma 6.1 in [Tao et al. 1998]. Here, we used the fact that for each fixed j the supports of the Fourier transforms of $E(f_k^j)\, E(g_{k'}^j)$ have uniformly bounded overlap as (k, k') varies, provided that $I_k^j \sim I_{k'}^j$. This is a consequence of the Whitney decomposition. We now claim that if $I_k^j \sim I_{k'}^j$, then

$$\|E(f_k^j)\, E(g_{k'}^j)\|_{\frac{q}{2}} \lesssim 2^{4j(\frac{1}{p}+\frac{2}{q}-1)} \|f_k^j\|_p \|g_{k'}^j\|_p \tag{5-1}$$

when $\frac{1}{p} + \frac{5}{3q} < 1$, $q > \frac{10}{3}$. This is an easy consequence of a translated version of Lemma 5.1. Assuming this for the moment, we will finish the proof. Since $q \ge p$, for $\frac{1}{p} + \frac{5}{3q} < 1$, $4 > q > \frac{10}{3}$, we have

$$\|Ef\ Eg\|_{\frac{q}{2}} \le \sum_j 2^{4j(\frac{1}{p}+\frac{2}{q}-1)} \Big(\sum_{I_k^j \sim I_{k'}^j} \|f_k^j\|_p^{\frac{q}{2}} \|g_{k'}^j\|_p^{\frac{q}{2}} \Big)^{\frac{2}{q}}$$

$$\lesssim \sum_j 2^{4j(\frac{1}{p}+\frac{2}{q}-1)} \Big(\sum_{I^j} \|f_k^j\|_p^p \Big)^{\frac{1}{p}} \Big(\sum_{J^j} \|g_{k'}^j\|_p^p \Big)^{\frac{1}{p}} \lesssim \sum_j 2^{4j(\frac{1}{p}+\frac{2}{q}-1)} \|f\|_p \|g\|_p.$$

Now take $f = \chi_F$ and $g = \chi_G$ for measurable sets F, G contained in V_1, V_2, respectively. Fix p, q with $4 > q > \frac{10}{3}$, $\frac{1}{p} + \frac{2}{q} = 1$, and choose p_1 and p_2 such that $\frac{1}{p_j} + \frac{5}{3q} < 1$, $j = 1, 2$, and

$$\frac{1}{p_1} + \frac{2}{q} - 1 = \eta, \quad \frac{1}{p_2} + \frac{2}{q} - 1 = -\eta$$

for some small $\eta > 0$. Then by applying the last estimate for $p = p_1$ and $p = p_2$, we obtain

$$\|E\chi_F\, E\chi_G\|_{\frac{q}{2}} \lesssim \sum_j \min\{2^{4j\eta}|F|^{\eta+1-\frac{2}{q}}|G|^{\eta+1-\frac{2}{q}},\ 2^{-4j\eta}|F|^{-\eta+1-\frac{2}{q}}|G|^{-\eta+1-\frac{2}{q}}\}$$
$$\lesssim |F|^{1-\frac{2}{q}}|G|^{1-\frac{2}{q}} = |F|^{1/p}|G|^{\frac{1}{p}}.$$

This shows the estimate $\|E\chi_F\, E\chi_G\|_{q/2} \lesssim \|f\|_{p,1}\,\|g\|_{p,1}$ for $\frac{1}{p}+\frac{2}{q}=1,\ 4>q>\frac{10}{3}$.

Now it remains to show (5-1). Clearly, I_k^j and $I_{k'}^j$ are contained in a ball of radius 2^{2-j} and $\mathrm{dist}(I_k^j, I_{k'}^j) \geq 2^{-1-j}$. Hence, by a change of variables,

$$E(f_k^j)(w) = 2^{-2j}\, E(f_k^j(\cdot, 2^{-j}\,\cdot))(w_1, 2^{-j}w_2, 2^{-j}w_3),$$
$$E(g_{k'}^j)(w) = 2^{-2j}\, E(g_{k'}^j(\cdot, 2^{-j}\,\cdot))(w_1, 2^{-j}w_2, 2^{-j}w_3).$$

Then we see that $\operatorname{supp} f_k^j(\cdot, 2^{-j}\,\cdot) \subset H_1 \times \tilde{I}_1$ and $g_{k'}^j(\cdot, 2^{-j}\,\cdot) \subset H_2 \times \tilde{I}_2$ if $\mathrm{dist}(\tilde{I}_1, \tilde{I}_2) \geq 2^{-1}$ and $\tilde{I}_1, \tilde{I}_2$ are contained in a ball of radius $\leq 2^3$. The assumption of Lemma 5.1 is satisfied with $f = f_k^j(\cdot, 2^{-j}\,\cdot)$ and $g = g_{k'}^j(\cdot, 2^{-j}\,\cdot)$. Hence we may apply it to $E(f_k^j(\cdot, 2^{-j}\,\cdot))E(f_k^j(\cdot, 2^{-j}\,\cdot))$ and get (5-1). □

Once Lemma 5.2 is established, the usual argument in [Tao et al. 1998], used to deduce linear estimates from bilinear ones, works without modification. We omit the details. □

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