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COMPLETE EMBEDDED COMPLEX CURVES IN THE BALL OF $\mathbb{C}^2$ CAN HAVE ANY TOPOLOGY

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In this paper we prove that the unit ball $\mathbb{B}$ of $\mathbb{C}^2$ admits complete properly embedded complex curves of any given topological type. Moreover, we provide examples containing any given closed discrete subset of $\mathbb{B}$.

1. Introduction

Yang [1977a; 1977b] raised the question of whether there exist complete bounded complex submanifolds of a complex Euclidean space $\mathbb{C}^N$ ($N > 1$). Recall that an immersed k-dimensional complex submanifold $\psi : M^k \to \mathbb{C}^N$ ($1 \le k < N$) is said to be *complete* if the Riemannian metric induced on M by the Euclidean metric in $\mathbb{C}^N$ via ψ is complete in the classical sense, or equivalently, if the image by ψ of any divergent path on M has infinite Euclidean length.

The case of main interest to us in this paper is when $k = 1$ and $N = 2$, i.e., complex curves in the complex Euclidean plane $\mathbb{C}^2$. There are many known examples of complete bounded *immersed* complex curves in $\mathbb{C}^2$, which have been obtained by different methods; see the works of Jones [1979] for discs, Martín, Umehara, and Yamada [Martín et al. 2009] for some finite topologies, Alarcón and López [2013a] for examples with arbitrary topology, and Alarcón and Forstnerič [2013] and Alarcón, Drinovec Drnovšek, Forstnerič, and López [Alarcón et al. 2015] for examples normalized by any given bordered Riemann surface. Furthermore, the curves in [Alarcón and López 2013a; Alarcón and Forstnerič 2013; Alarcón et al. 2015] can be chosen to be proper in any given convex domain of $\mathbb{C}^2$, in particular, in the open unit Euclidean ball, which throughout this paper will be denoted by $\mathbb{B}$.

On the other hand, using the techniques developed in the cited sources and taking into account that the general position of complex curves in $\mathbb{C}^N$ is embedded for all $N \ge 3$, it is not very hard to construct complete bounded *embedded* complex curves in $\mathbb{C}^N$ for any such N; see again [Alarcón and Forstnerič 2013; Alarcón et al. 2015]. For submanifolds of higher dimension, Alarcón and Forstnerič [2013] provided examples of complete bounded embedded k-dimensional complex submanifolds of $\mathbb{C}^{3k}$ for any $k \in \mathbb{N}$, whereas Drinovec Drnovšek [2015] proved that every bounded, strictly pseudoconvex, smoothly bounded domain of $\mathbb{C}^k$ admits a complete proper holomorphic embedding into the unit ball of $\mathbb{C}^N$ provided that the codimension $N - k$ is sufficiently large.

However, constructing complete bounded *embedded* complex curves in $\mathbb{C}^2$ (and, more generally, complete bounded embedded complex hypersurfaces of $\mathbb{C}^N$ for $N > 1$) is a much more arduous task; the

main reason why is that *self-intersections of complex curves in $\mathbb{C}^2$ are stable under small deformations*. It is therefore not surprising that the first known examples of such curves were only found almost four decades after Yang posed his question; they were given in [Alarcón and López 2016]. Their method, which in fact furnishes complete properly embedded complex curves in any convex domain of $\mathbb{C}^2$, is rather involved and relies, among other things, on a subtle self-intersection removal procedure that does not allow for the inference of any information on the topological type of the examples. A little later Globevnik [2015; 2016b], using a different technique, extended the results in [Alarcón and López 2016] by proving the existence of complete properly embedded complex hypersurfaces in any pseudoconvex domain D of $\mathbb{C}^N$ for any $N > 1$; this settles the *embedded Yang problem* in an optimal way in all dimensions. His examples are given as level sets of highly oscillating holomorphic functions $D \to \mathbb{C}$, and hence, again, no information on their topology is provided. In light of the above, the following questions naturally appear; see [Alarcón and López 2016, Question 1.5; Globevnik 2015, Questions 13.1 and 13.2]:

Problem 1.1. Is there any restriction on the topology of a complete bounded embedded complex hypersurface of $\mathbb{C}^N$? What if $N = 2$? For instance, do there exist complete proper holomorphic embeddings of the unit disk $\mathbb{D} \subset \mathbb{C}$ into the unit ball $\mathbb{B} \subset \mathbb{C}^2$?

The first approach to this problem was recently made in [Alarcón et al. 2016] by Alarcón, Globevnik, and López, who, with a conceptually new method based on the use of holomorphic automorphisms of $\mathbb{C}^N$, constructed complete closed complex hypersurfaces in the unit ball of $\mathbb{C}^N$ for any $N > 1$ with certain restrictions on their topology. In particular, for $N = 2$, they showed that the unit ball of $\mathbb{C}^2$ admits complete properly embedded complex curves with arbitrary *finite* topology, see Corollary 1.2 of that paper, thereby affirmatively answering the third question in Problem 1.1. Going further in this line, Globevnik [2016a] proved the existence of complete proper holomorphic embeddings $\mathbb{D} \hookrightarrow \mathbb{B}$ whose image contains any given closed discrete subset of $\mathbb{B}$. This is reminiscent of an old result by Forstnerič, Globevnik, and Stensønes [Forstnerič et al. 1996] asserting that, given a pseudoconvex Runge domain $D \subset \mathbb{C}^N$ ($N > 1$) and a closed discrete subset $\Lambda \subset D$, there is a proper holomorphic embedding $\mathbb{D} \hookrightarrow D$ whose image contains Λ. It is nevertheless true that these embeddings are not ensured to be complete in any case.

The aim of this paper is to settle Problem 1.1 for $N = 2$ by proving the existence of complete properly embedded complex curves in $\mathbb{B}$ with *arbitrary topology* (possibly infinite). This completely solves the problem and, in particular, generalizes the existence result [Alarcón et al. 2016, Corollary 1.2], which only deals with finite topological types. Moreover, we provide examples of such curves which contain any given closed discrete subset of $\mathbb{B}$, thereby extending the above-mentioned hitting result by Globevnik [2016a, Theorem 1.1].

The main theorem of this paper can be stated as follows.

Theorem 1.2. *Let Λ be a closed discrete subset of the unit ball $\mathbb{B} \subset \mathbb{C}^2$. On each open connected orientable smooth surface M there exists a complex structure such that the open Riemann surface M admits a complete proper holomorphic embedding $M \hookrightarrow \mathbb{B}$ whose image contains Λ.*

It is perhaps worth mentioning that, choosing any closed discrete subset $\Lambda \subset \mathbb{B}$ such that $\bar{\Lambda} \setminus \Lambda = \mathrm{b}\mathbb{B} = \{\zeta \in \mathbb{C}^2 : |\zeta| = 1\}$, Theorem 1.2 trivially implies the following:

Corollary 1.3. *The unit ball $\mathbb{B} \subset \mathbb{C}^2$ contains complete properly embedded complex curves with any given topology and whose limit set equals $b\mathbb{B}$.*

Although it is not explicitly stated there, Corollary 1.3 in the simply connected case straightforwardly follows from the results by Globevnik [2016a].

Our method of proof exploits some ideas from both [Alarcón et al. 2016] and [Globevnik 2016a] (in particular, our construction technique is based on the use of holomorphic automorphisms of $\mathbb{C}^2$), but also from [Alarcón and López 2013b], where the authors constructed properly embedded complex curves in $\mathbb{C}^2$ with arbitrary topology. The latter contributes to the so-called embedding problem for open Riemann surfaces in $\mathbb{C}^2$, a long-standing open question in Riemann surface theory asking whether every open Riemann surface properly embeds in $\mathbb{C}^2$ as a complex curve [Bell and Narasimhan 1990, Conjecture 3.7, page 20]; for recent advances and a history of this classical problem we refer to [Forstnerič and Wold 2009; 2013]. It is shown in [Forstnerič and Wold 2009] that given a compact bordered Riemann surface $\overline{M} = M \cup bM$ admitting a smooth embedding $f : \overline{M} \hookrightarrow \mathbb{C}^2$ which is holomorphic in M, there is a proper holomorphic embedding $\tilde{f} : M \hookrightarrow \mathbb{C}^2$ which is as close as desired to f uniformly on a given compact subset of M. (A *compact bordered Riemann surface* $\overline{M}$ is a compact Riemann surface with boundary $\varnothing \neq bM \subset \overline{M}$ consisting of finitely many pairwise disjoint smooth Jordan curves; its interior $M = \overline{M} \setminus bM$ is called a *bordered Riemann surface*.) This fact and the arguments in its proof were key in the construction method in [Alarcón and López 2013b], which will be used in the proof of Theorem 1.2.

We strongly expect that the new construction techniques developed in this paper may be adapted to prove the statement of Theorem 1.2 but replacing the ball $\mathbb{B}$ by any convex domain of $\mathbb{C}^2$. The following questions, concerning pseudoconvex domains, remain open and seem to be much more challenging.

Problem 1.4. Let $D \subset \mathbb{C}^2$ be a pseudoconvex Runge domain. Does there exist a complete proper holomorphic embedding $\mathbb{D} \hookrightarrow D$? Given a closed discrete subset $\Lambda \subset D$, do there exist complete properly embedded complex curves in D containing Λ?

As we have already mentioned, every bordered Riemann surface M admits a complete proper holomorphic *immersion* $M \to \mathbb{B}$ [Alarcón and Forstnerič 2013], and if in addition there is a smooth embedding $\overline{M} \to \mathbb{C}^2$ that is holomorphic in M, then M properly holomorphically embeds into $\mathbb{C}^2$ [Forstnerič and Wold 2009]. It is however an open question, likely very difficult, whether every bordered Riemann surface admits a holomorphic embedding in $\mathbb{C}^2$ (even without requiring the embedding to have any global condition such as completeness or properness); see, e.g., the introduction of [Forstnerič and Wold 2009] or Section 8.9 in the monograph [Forstnerič 2011] for more information. Thus, one is also led to ask:

Problem 1.5. Let $\overline{M} = M \cup bM$ be a compact bordered Riemann surface and assume that there is a smooth embedding $\overline{M} \hookrightarrow \mathbb{C}^2$ which is holomorphic in M. Does M admit complete holomorphic embeddings $M \hookrightarrow \mathbb{C}^2$ with bounded image?

We hope to return to these interesting questions in a future work.

Organization of the paper. In Section 2 we set the notation that will be used throughout the paper and, with the aim of making it self-contained, state some already known results which will be used in the

proof of Theorem 1.2. In Section 3 we prove an approximation result by properly embedded complex curves in the unit ball $\mathbb{B} \subset \mathbb{C}^2$ (see Theorem 3.1) from which Theorem 1.2 will be easily derived.

2. Preliminaries

We denote by $|\cdot|$ and $\mathbb{B} = \{\zeta \in \mathbb{C}^2 : |\zeta| < 1\}$ the Euclidean norm and the unit Euclidean ball in $\mathbb{C}^2$. For a subset $C \subset \mathbb{C}^2$ we denote by $\bar{C}$, $\mathring{C}$, and $bC = \bar{C} \setminus \mathring{C}$ the topological closure, interior, and frontier of C in $\mathbb{C}^2$, respectively. Also, given a point $\xi \in \mathbb{C}^2$ and a number $r \in \mathbb{R}_+ = [0, +\infty[$ we write $\xi + rC = \{\xi + r\zeta : \zeta \in C\}$.

Let A be a smoothly bounded compact domain in an open Riemann surface and let $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$. We denote by $\mathscr{A}^k(A)$ the space of functions $A \to \mathbb{C}$ of class $\mathscr{C}^k(A)$ which are holomorphic on the interior $\mathring{A} = \bar{A} \setminus bA$. If $N \in \mathbb{N}$ we will simply write $\mathscr{A}^k(A)$ instead of

$$\mathscr{A}^k(A)^N = \mathscr{A}^k(A) \times \overset{N \text{ times}}{\cdots} \times \mathscr{A}^k(A)$$

when there is no place for ambiguity. Given an immersion $\psi : A \to \mathbb{C}^N$ ($N > 1$) of class $\mathscr{C}^1(A)$, we denote by $\mathrm{dist}_\psi : A \times A \to \mathbb{R}_+$ the Riemannian distance in A induced by the Euclidean metric of $\mathbb{C}^N$ via ψ; that is,

$$\mathrm{dist}_\psi(p, q) := \inf\{\mathrm{length}(\psi(\gamma)) : \gamma \subset A \text{ path connecting } p \text{ and } q\}, \quad p, q \in A,$$

where $\mathrm{length}(\cdot)$ denotes the Euclidean length in $\mathbb{C}^N$.

Tangent balls. Given a point $\zeta \in \mathbb{C}^2 \setminus \{0\}$ and a number $r > 0$, we denote by $\mathscr{T}(\zeta, r)$ the closed ball with center ζ and radius r in the real affine hyperplane H_ζ tangent to the sphere $b(|\zeta|\mathbb{B})$ at the point ζ; that is,

$$\mathscr{T}(\zeta, r) := \{\xi \in H_\zeta : |\xi - \zeta| \le r\} \subset \mathbb{C}^2.$$

According to [Alarcón et al. 2016, Definition 1.3], the set $\mathscr{T}(\zeta, r)$ above is called *the tangent ball of center ζ and radius r*. A collection $\mathfrak{F} = \{\mathscr{T}(\zeta_j, r_j)\}_{j \in J}$ of tangent balls in $\mathbb{B} \subset \mathbb{C}^2$ will be called *tidy*, see Definition 1.4 of the same paper, if it satisfies the following requirements:

- $\mathscr{T}(\zeta_j, r_j) \subset \mathbb{B}$ for all $j \in J$ and the tangent balls in $\mathfrak{F}$ are pairwise disjoint.
- $t\overline{\mathbb{B}}$ intersects finitely many tangent balls in the family $\mathfrak{F}$ for all $0 < t < 1$.
- If $\mathscr{T}(\zeta, r)$, $\mathscr{T}(\zeta', r') \in \mathfrak{F}$ and $|\zeta| = |\zeta'|$, then $r = r'$.
- If $\mathscr{T}(\zeta, r)$, $\mathscr{T}(\zeta', r') \in \mathfrak{F}$ and $|\zeta| < |\zeta'|$, then $\mathscr{T}(\zeta, r) \subset |\zeta'|\mathbb{B}$.

Notice that a tidy collection $\mathfrak{F} = \{\mathscr{T}(\zeta_j, r_j)\}_{j \in J}$ of tangent balls in $\mathbb{B}$ consists of countably many elements; and so we may assume that $J \subset \mathbb{N} = \{1, 2, 3, \dots\}$. We denote by

$$|\mathfrak{F}| := \bigcup_{j \in J} \mathscr{T}(\zeta_j, r_j)$$

the union of all the tangent balls in a tidy collection. Note that if J is finite then $|\mathfrak{F}|$ is compact, whereas if J is infinite then $|\mathfrak{F}|$ is a proper subset of $\mathbb{B}$.

The following two results, involving tidy collections of tangent balls, are proved in [Alarcón et al. 2016], and will be invoked in our argumentation.

Lemma 2.1 [Alarcón et al. 2016, Lemma 2.4]. *Given numbers $0 < r < r' < 1$ and $\ell > 0$ there is a finite tidy collection $\mathfrak{F}$ of tangent balls in $\mathbb{B}$ such that $|\mathfrak{F}| \subset r'\mathbb{B} \setminus r\overline{\mathbb{B}}$ and the length of any path $\gamma : [0, 1] \to r'\overline{\mathbb{B}} \setminus ((r\mathbb{B}) \cup |\mathfrak{F}|)$ with $|\gamma(0)| = r$ and $|\gamma(1)| = r'$ is at least ℓ.*

The next result is not explicitly stated in [Alarcón et al. 2016] but straightforwardly follows from an inspection of the proof of Theorem 1.6 of that paper as a standard finite recursive application of their Lemma 3.1.

Lemma 2.2. *Let $0 < r < r' < 1$ be numbers and let $\mathfrak{F}$ be a finite tidy collection of tangent balls in $\mathbb{B}$ with $|\mathfrak{F}| \subset r'\mathbb{B} \setminus r\overline{\mathbb{B}}$. Then, given a properly embedded complex curve $Z \subset \mathbb{C}^2$ and a number $\epsilon > 0$, there is a holomorphic automorphism $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying the following properties:*

  (i) $\Phi(Z) \cap |\mathfrak{F}| = \varnothing$.

  (ii) $|\Phi(\zeta) - \zeta| < \epsilon$ *for all $\zeta \in r\overline{\mathbb{B}}$.*

Hitting and approximation lemmas. In this subsection we state two results which will also be used in our construction. The first one, due to Globevnik, will be key in order to achieve the hitting condition in the statement of Theorem 1.2.

Lemma 2.3 [Globevnik 2016a, Lemma 7.2]. *Given a finite subset $\Lambda \subset \mathbb{B}$ there exist numbers $\eta > 0$ and $\mu > 0$ such that the following holds. Given a number $0 < \delta < \eta$ and a map $\varphi : \Lambda \to \mathbb{C}^2$ such that*

$$|\varphi(\zeta) - \zeta| < \delta \quad \text{for all } \zeta \in \Lambda,$$

there exists a holomorphic automorphism $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying the following conditions:

  (i) $\Psi(\varphi(\zeta)) = \zeta$ *for all $\zeta \in \Lambda$,*

  (ii) $|\Psi(\zeta) - \zeta| < \mu\delta$ *for all $\zeta \in \overline{\mathbb{B}}$.*

The second result, which will help us to increase the topology in the recursive construction, is a particular case of [Alarcón and López 2013b, Theorem 4.5]; it also easily follows from the results in [Forstnerič and Wold 2009].

Lemma 2.4. *Let $\overline{M} = M \cup bM$ be a compact bordered Riemann surface and let $K \subset M$ be a connected, smoothly bounded, Runge compact domain which is a strong deformation retract of $\overline{M}$. Let $\phi : K \hookrightarrow \mathbb{C}^2$ be an embedding of class $\mathscr{A}^1(K)$ and assume that there exists a number $s > 0$ such that*

$$\phi(bK) \cap s\overline{\mathbb{B}} = \varnothing.$$

Then, given $\epsilon > 0$, there are an open domain $\Omega \subset M$ and a proper holomorphic embedding $\tilde{\phi} : \Omega \hookrightarrow \mathbb{C}^2$ such that:

  (i) $K \subset \Omega$ *and Ω is a deformation retract of M and homeomorphic to M.*

  (ii) $|\tilde{\phi}(p) - \phi(p)| < \epsilon$ *for all $p \in K$.*

  (iii) $\tilde{\phi}(\Omega \setminus \mathring{K}) \cap s\overline{\mathbb{B}} = \varnothing$.

Recall that a compact subset K of an open Riemann surface M is said to be *Runge* or *holomorphically convex* if every continuous function $K \to \mathbb{C}$, holomorphic in $\mathring{K}$, may be approximated uniformly on K by holomorphic functions $M \to \mathbb{C}$. The classical Mergelyan theorem ensures that $K \subset M$ is Runge if and only if $M \setminus K$ has no relatively compact connected components in M; see [Bishop 1958; Runge 1885; Mergelyan 1951].

3. Proof of the main theorem

In this section we prove the following more precise version of Theorem 1.2.

Theorem 3.1. *Let M be an open connected Riemann surface, let $K \subset M$ be a connected, smoothly bounded, Runge compact domain, let $0 < s < r < 1$ be numbers, and assume that there is an embedding $\psi : K \to \mathbb{C}^2$ of class $\mathscr{A}^1(K)$ such that*

$$\psi(\mathsf{b}K) \subset r\mathbb{B} \setminus s\overline{\mathbb{B}}. \tag{3-1}$$

Let $\Lambda \subset \mathbb{B}$ be a closed discrete subset such that

$$\Lambda \cap r\overline{\mathbb{B}} \subset \psi(\mathring{K}) \cap s\mathbb{B}. \tag{3-2}$$

Then, given $\epsilon > 0$, there are a domain $D \subset M$ and a complete proper holomorphic embedding $\tilde{\psi} : D \hookrightarrow \mathbb{B}$ satisfying the following properties:

(i) *$K \subset D$ and D is a deformation retract of M (and hence homeomorphic to M).*

(ii) *$|\tilde{\psi}(\zeta) - \psi(\zeta)| < \epsilon$ for all $\zeta \in K$.*

(iii) *$\Lambda \subset \tilde{\psi}(D)$.*

(iv) *$\tilde{\psi}(D \setminus K) \cap s\overline{\mathbb{B}} = \varnothing$.*

Proof. Pick a number $s < s_0' < r$ such that

$$\psi(\mathsf{b}K) \subset r\mathbb{B} \setminus s_0'\overline{\mathbb{B}}. \tag{3-3}$$

Such an s_0' exists in view of (3-1) by compactness of $\psi(\mathsf{b}K)$. Without loss of generality we assume that Λ is infinite, and hence, since it is closed in $\mathbb{B}$ and discrete, Λ is in bijection with $\mathbb{N}$ and for any ordering $\Lambda = \{p_i\}_{i \in \mathbb{N}}$ of Λ we have $\lim_{i \to \infty} |p_i| = 1$. Thus, there are sequences of numbers $\{r_j\}_{j \in \mathbb{N}}$, with $r_1 = r$, $\{r_j'\}_{j \in \mathbb{N}}$, $\{s_j\}_{j \in \mathbb{N}}$, and $\{s_j'\}_{j \in \mathbb{N}}$, satisfying the following properties:

(a) $s_{j-1}' < r_j < r_j' < s_j < s_j'$ for all $j \in \mathbb{N}$.

(b) $\lim_{j \to \infty} r_j = 1$.

(c) $\Lambda \subset s\mathbb{B} \cup \left(\bigcup_{j \in \mathbb{N}} (s_j'\mathbb{B} \setminus s_j\overline{\mathbb{B}}) \right)$, taking into account (3-2).

Write $\Lambda_0 := \Lambda \cap s\mathbb{B} = \Lambda \cap s_0'\mathbb{B}$ and $\Lambda_j = \Lambda \cap (s_j'\mathbb{B} \setminus s_j\overline{\mathbb{B}})$, $j \in \mathbb{N}$, and observe that Λ_j is finite for all $j \in \mathbb{Z}_+$. We also assume without loss of generality that $\Lambda_j \neq \varnothing$ for all $j \in \mathbb{Z}_+$. By (a) and (c) we have

$$\Lambda = \bigcup_{j \in \mathbb{Z}_+} \Lambda_j, \quad \Lambda_i \cap \Lambda_j = \varnothing \ \text{ for all } i, j \in \mathbb{Z}_+, \ i \neq j. \tag{3-4}$$

Let $\{\epsilon_j\}_{j\in\mathbb{N}} \searrow 0$ be a decreasing sequence of positive numbers such that

$$\sum_{j\in\mathbb{N}} \epsilon_j < \epsilon. \tag{3-5}$$

The precise values of the numbers ϵ_j, $j \in \mathbb{N}$, will be specified later.

Write $M_0 := K$ and let

$$M_0 \Subset M_1 \Subset M_2 \Subset \cdots \Subset M = \bigcup_{j\in\mathbb{Z}_+} M_j \tag{3-6}$$

be an exhaustion of M by connected, smoothly bounded, Runge compact domains such that the Euler characteristic satisfies $\chi(M_j \setminus \mathring{M}_{j-1}) \in \{0, -1\}$ for all $j \in \mathbb{N}$. Such an exhaustion exists by basic topological arguments; see, e.g., [Alarcón and López 2013b, Lemma 4.2] for a simple proof.

Write $D_0 := M_0 = K$ and set $\psi_0 := \psi$. Fix a point $p_0 \in \mathring{K}$. We shall recursively construct a sequence $\Xi_j = \{D_j, \psi_j\}$, $j \in \mathbb{N}$, where D_j is a connected, smoothly bounded, Runge compact domain in M and $\psi_j : D_j \hookrightarrow \mathbb{C}^2$ is an embedding of class $\mathscr{A}^1(D_j)$ such that the following hold for all $j \in \mathbb{N}$:

(1_j) $D_{j-1} \Subset D_j \Subset M_j$ and D_j is a strong deformation retract of M_j.

(2_j) $|\psi_j(p) - \psi_{j-1}(p)| < \epsilon_j$ for all $p \in D_{j-1}$.

(3_j) $\mathrm{dist}_{\psi_j}(p_0, \mathrm{b}D_j) > j$.

(4_j) $\psi_j(\mathrm{b}D_j) \subset r_{j+1}\mathbb{B} \setminus s'_j\overline{\mathbb{B}}$.

(5_j) $\psi_j(D_j \setminus \mathring{D}_{j-1}) \cap s'_{j-1}\overline{\mathbb{B}} = \varnothing$.

(6_j) $\Lambda_i \subset \psi_j(\mathring{D}_i)$ for all $i \in \{0, \ldots, j\}$.

Assume for a moment that we have already constructed a sequence $\{\Xi_j\}_{j\in\mathbb{N}}$ enjoying the above conditions. By (1_j) and (3-6), we obtain that

$$D := \bigcup_{j\in\mathbb{Z}_+} D_j$$

is a domain in M satisfying property (i) in the statement of the theorem. We claim that, if the number $\epsilon_j > 0$ is chosen small enough at each step in the recursive construction, the sequence $\{\psi_j\}_{j\in\mathbb{Z}_+}$ converges uniformly on compacta in D to a limit map

$$\tilde{\psi} = \lim_{j\to\infty} \psi_j : D \to \mathbb{C}^2$$

satisfying the conclusion of the theorem. Indeed, by (2_j) and (3-5) the limit map $\tilde{\psi}$ exists and satisfies (ii); recall that $K = D_0$. Since each map ψ_j is an embedding of class $\mathscr{A}^1(D_j)$, choosing the number $\epsilon_j > 0$ sufficiently small at each step in the recursive construction, we have $\tilde{\psi} : D \to \mathbb{C}^2$ is an injective holomorphic immersion. Moreover, (3_j) ensures that $\tilde{\psi}$ is complete, whereas (5_j), (a), and (b) guarantee (iv) and the facts that $\tilde{\psi}(D) \subset \mathbb{B}$ and that $\tilde{\psi} : D \to \mathbb{B}$ is a proper map; recall that $s < s'_0$ and observe that $\lim_{j\to\infty} s'_j = 1$. Thus, $\tilde{\psi} : D \hookrightarrow \mathbb{B}$ is a proper embedding. Finally, (6_j) and (3-4) imply that $\Lambda \subset \tilde{\psi}(D)$, thereby proving (iii).

To conclude the proof, it therefore suffices to construct a sequence $\Xi_j = \{D_j, \psi_j\}$, $j \in \mathbb{N}$, satisfying properties (1_j)–(6_j) above. We proceed by induction. The basis is given by the compact domain D_0

and the map ψ_0; observe that (3_0) holds since $p_0 \in \mathring{D}_0$ and ψ_0 is an immersion, $(4_0) =$ (3-3) (recall that $r_1 = r$), (6_0) is implied by (c) and (3-2), and (2_0), (5_0), and the first part of (1_0) are vacuous conditions. Finally, the second part of (1_0) is obvious since $D_0 = M_0$.

For the inductive step, assume that we have $\Xi_{j-1} = \{D_{j-1}, \psi_{j-1}\}$ enjoying the desired properties for some $j \in \mathbb{N}$ and let us construct $\Xi_j = \{D_j, \psi_j\}$. We distinguish two cases.

Case 1: Assume that $\chi(M_j \setminus \mathring{M}_{j-1}) = 0$. In this case M_{j-1} is a strong deformation retract of M_j.

Write $\Lambda' = \bigcup_{i=0}^{j} \Lambda_i$ and let $\eta > 0$ and $\mu > 0$ be the numbers given by Lemma 2.3 applied to the finite subset $\Lambda' \subset \mathbb{B}$.

By (3_{j-1}) and (4_{j-1}) there is another number $\eta' > 0$ with the following property:

(A1) If $\phi : D_{j-1} \to \mathbb{C}^2$ is an immersion of class $\mathscr{A}^1(D_{j-1})$ such that $|\phi(p) - \psi_{j-1}(p)| < \eta'$ for all $p \in D_{j-1}$, then $\mathrm{dist}_\phi(p_0, bD_{j-1}) > j - 1$ and $\phi(bD_{j-1}) \subset r_j \mathbb{B} \setminus s'_{j-1}\overline{\mathbb{B}}$.

On the other hand, Lemma 2.1 gives a finite tidy collection $\mathfrak{F}$ of tangent balls in $\mathbb{B}$ satisfying the following conditions:

(A2) $|\mathfrak{F}| \subset r'_j \mathbb{B} \setminus r_j \overline{\mathbb{B}}$.

(A3) The length of any path $\gamma : [0, 1] \to r'_j \overline{\mathbb{B}} \setminus ((r_j \mathbb{B}) \cup |\mathfrak{F}|)$ with $|\gamma(0)| = r_j$ and $|\gamma(1)| = r'_j$ is greater than 2.

Thus, there is a third number $\eta'' > 0$ enjoying the following property:

(A4) If $\alpha : [0, 1] \to r'_j \overline{\mathbb{B}} \setminus r_j \mathbb{B}$ is a path satisfying that there is another path $\gamma : [0, 1] \to r'_j \overline{\mathbb{B}} \setminus ((r_j \mathbb{B}) \cup |\mathfrak{F}|)$ such that $|\gamma(0)| = r_j$, $|\gamma(1)| = r'_j$, and $|\gamma(x) - \alpha(x)| < \eta''$ for all $x \in [0, 1]$, then the length of α is greater than 1.

Next, pick a number t such that $s'_{j-1} < t < r_j$ and

$$\psi_{j-1}(bD_{j-1}) \subset r_j \mathbb{B} \setminus t\overline{\mathbb{B}}; \tag{3-7}$$

the existence of such number t is ensured by (a) and (4_{j-1}). Finally, choose a number

$$0 < \delta < \min\left\{\eta, \frac{\eta'}{\mu + 1}, \frac{\eta''}{\mu + 1}, \frac{\epsilon_j}{\mu + 1}, \frac{t - s'_{j-1}}{\mu}, \frac{r_{j+1} - s'_j}{2\mu}\right\}. \tag{3-8}$$

Also fix a number $\tau_1 > 0$ which will be specified later.

Taking into account (3-7), Lemma 2.4 furnishes an open domain $\Omega \Subset \mathring{M}_j$ and a proper holomorphic embedding $\phi_1 : \Omega \hookrightarrow \mathbb{C}^2$ such that the following hold:

(B1) $D_{j-1} \subset \Omega$ and Ω is a deformation retract of $\mathring{M}_j$ and is homeomorphic to $\mathring{M}_j$. In particular, the second part of (1_{j-1}) and the fact that M_{j-1} is a strong deformation retract of M_j ensure that $\Omega \setminus D_{j-1}$ consists of a finite collection of pairwise disjoint open annuli.

(B2) $|\phi_1(p) - \psi_{j-1}(p)| < \tau_1$ for all $p \in D_{j-1}$.

(B3) $\phi_1(\Omega \setminus \mathring{D}_{j-1}) \cap t\overline{\mathbb{B}} = \varnothing$.

In view of (3-7) and (B2), and choosing $\tau_1 > 0$ small enough, we also have:

(B4) $\phi_1(bD_{j-1}) \subset r_j \mathbb{B} \setminus t\overline{\mathbb{B}}$.

Now, given a number $\tau_2 > 0$ which will be specified later and taking into account (A2), Lemma 2.2 provides a holomorphic automorphism $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ such that

(C1) $\Phi(\phi_1(\Omega)) \cap |\mathfrak{F}| = \varnothing$.

(C2) $|\Phi(\zeta) - \zeta| < \tau_2$ for all $\zeta \in r_j \overline{\mathbb{B}}$.

Write

$$\phi_2 := \Phi \circ \phi_1 : \Omega \hookrightarrow \mathbb{C}^2. \tag{3-9}$$

By (B1), (B3), (B4), (C2), and (a), and choosing $\tau_2 > 0$ sufficiently small, we have:

(D1) $\phi_2(\mathrm{b}D_{j-1}) \subset r_j \mathbb{B} \setminus t\overline{\mathbb{B}}$.

(D2) $\phi_2(\Omega \setminus \mathring{D}_{j-1}) \cap t\overline{\mathbb{B}} = \varnothing$.

and, taking into account (B1), (D1), and the maximum principle for the function $|\phi_2|$, there exists a smoothly bounded compact domain $\Upsilon \subset \Omega$ such that:

(D3) $D_{j-1} \Subset \Upsilon$ and D_{j-1} is a strong deformation retract of Υ.

(D4) $\phi_2(\mathrm{b}\Upsilon) \subset \mathrm{b}(t'\mathbb{B})$ and meets transversely there for some number t' with $r'_j < t' < s_j$.

Next, choose a point $q_0 \in \mathrm{b}\Upsilon$ and a smooth embedded compact path $\lambda \subset \mathbb{C}^2 \setminus t'\mathbb{B}$ having $\phi_2(q_0)$ as an endpoint, meeting $\mathrm{b}(t'\mathbb{B})$ transversely there, and being otherwise disjoint from $t'\overline{\mathbb{B}}$. Assume also that

$$\Lambda_j \subset \lambda \subset s'_j \mathbb{B} \setminus t'\mathbb{B}. \tag{3-10}$$

Also take a smooth embedded compact path $\gamma \subset \Omega \setminus \mathring{\Upsilon}$ having q_0 as an endpoint, meeting $\mathrm{b}\Upsilon$ transversely there, and being otherwise disjoint from Υ. Extend ϕ_2, with the same name, to a smooth embedding $\Upsilon \cup \gamma \hookrightarrow \mathbb{C}^2$ such that

$$\phi_2(\gamma) = \lambda. \tag{3-11}$$

Observe that $\Upsilon \cup \gamma$ is a Runge compact subset of Ω; take into account (B1) and (D3). Thus, given $\tau_3 > 0$ to be specified later, Mergelyan's theorem applied to $\phi_2 : \Upsilon \cup \gamma \hookrightarrow \mathbb{C}^2$ ensures the existence of a smoothly bounded compact domain $\Upsilon' \subset \Omega$ and an embedding $\phi_3 : \Upsilon' \hookrightarrow \mathbb{C}^2$ of class $\mathscr{A}^1(\Upsilon')$ such that:

(E1) $\Upsilon \cup \gamma \subset \mathring{\Upsilon}'$ and Υ' is a strong deformation retract of M_j. In particular, D_{j-1} is a strong deformation retract of Υ'; see (D3).

(E2) $|\phi_3(p) - \phi_2(p)| < \tau_3$ for all $p \in \Upsilon \cup \gamma$.

Furthermore, if we take Υ' close enough to $\Upsilon \cup \gamma$ and if $\tau_3 > 0$ is chosen sufficiently small, we obtain in view of (E2) that:

(E3) $\phi_3(\Upsilon' \setminus \mathring{\Upsilon}) \subset s'_j \mathbb{B} \setminus r'_j \overline{\mathbb{B}}$. See (D4), (a), (3-10), and (3-11).

(E4) $\phi_3(\mathrm{b}D_{j-1}) \subset r_j \mathbb{B} \setminus t\overline{\mathbb{B}}$. See (D1) and (D3).

(E5) $\phi_3(\Upsilon' \setminus \mathring{D}_{j-1}) \cap t\overline{\mathbb{B}} = \varnothing$. See (D2).

(E6) $\phi_3(\Upsilon') \cap |\mathfrak{F}| = \varnothing$. See (3-9), (C1), (E3), and (A2).

Given $\tau_4 > 0$ to be specified later, applying Lemma 2.4 once again, we obtain, in view of (E3), a domain $\Omega' \Subset M_j$ and a proper holomorphic embedding $\phi_4 : \Omega' \hookrightarrow \mathbb{C}^2$ such that:

(F1) $\Upsilon' \subset \Omega'$ and Ω' is a deformation retract of $\mathring{M}_j$ and homeomorphic to $\mathring{M}_j$. In particular, $\Omega' \setminus \Upsilon'$ consists of a finite collection of pairwise disjoint open annuli. See (B1).

(F2) $|\phi_4(p) - \phi_3(p)| < \tau_4$ for all $p \in \Upsilon'$.

(F3) $\phi_4(\Omega' \setminus \mathring{\Upsilon}') \cap r'_j \overline{\mathbb{B}} = \varnothing$.

If $\tau_4 > 0$ is chosen sufficiently small then, in view of (F2), we also have:

(F4) $\phi_4(bD_{j-1}) \subset r_j \mathbb{B} \setminus t \overline{\mathbb{B}}$. See (E4).

(F5) $\phi_4(\Omega' \setminus \mathring{D}_{j-1}) \cap t \overline{\mathbb{B}} = \varnothing$. See (F3) and (E5), and recall that $t < r_j < r'_j$.

(F6) $\phi_4(\Omega') \cap |\mathfrak{F}| = \varnothing$. See (E6), (F3), and (A2).

Assume now that the numbers τ_i, $i = 1, \ldots, 4$, are chosen small enough such that $\sum_{i=1}^4 \tau_i < \delta$, where $\delta > 0$ is the number in (3-8). Thus, by (B2), (C2), (3-9), (E2), and (F2), we have:

(F7) $|\phi_4(p) - \psi_{j-1}(p)| < \sum_{i=1}^4 \tau_i < \delta$ for all $p \in D_{j-1}$.

(F8) $|\phi_4(p) - \phi_2(p)| < \tau_3 + \tau_4 < \sum_{i=1}^4 \tau_i < \delta$ for all $p \in \Upsilon \cup \gamma$.

By (6_{j-1}), (c), (3-10), and (3-11), we infer that $\Lambda_i \subset \psi_{j-1}(\mathring{D}_i)$, $i = 0, \ldots, j-1$, and $\Lambda_j \subset \phi_2(\gamma)$. Thus, (F7) and (F8) guarantee the existence of an injective map $\varphi : \Lambda' \to \phi_4(\Omega') \subset \mathbb{C}^2$ such that:

(G1) $|\varphi(\zeta) - \zeta| < \delta$ for all $\zeta \in \Lambda'$.

(G2) $\varphi(\Lambda_i) \subset \phi_4(\mathring{D}_i)$ for all $i \in \{0, \ldots, j-1\}$.

(G3) $\varphi(\Lambda_j) \subset \phi_4(\gamma) \subset \phi_4(\mathring{\Upsilon}' \setminus \Upsilon)$.

Observe that, since $0 < \delta < \eta$ where η is given by Lemma 2.3 for the subset $\Lambda' \subset \mathbb{B}$, see (3-8), every map $\varphi : \Lambda' \to \mathbb{C}^2$ satisfying (G1) is injective. In view of (3-8) and (G1), Lemma 2.3 provides a holomorphic automorphism $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ such that:

(H1) $\Psi(\varphi(\zeta)) = \zeta$ for all $\zeta \in \Lambda'$.

(H2) $|\Psi(\zeta) - \zeta| < \mu\delta$ for all $\zeta \in \overline{\mathbb{B}}$, where μ is the number given by Lemma 2.3 for the set Λ' which appears in (3-8).

Consider the proper holomorphic embedding

$$\phi := \Psi \circ \phi_4 : \Omega' \hookrightarrow \mathbb{C}^2.$$

Properties (F4), (F7), (H2), and (3-8), together with the maximum principle, ensure that

$$|\phi(p) - \psi_{j-1}(p)| < (\mu + 1)\delta < \min\{\eta', \eta'', \epsilon_j\} \quad \text{for all } p \in D_{j-1}. \tag{3-12}$$

This inequality and (A1) guarantee that

$$\text{dist}_\phi(p_0, bD_{j-1}) > j - 1 \tag{3-13}$$

and

$$\phi(bD_{j-1}) \subset r_j \mathbb{B} \setminus s'_{j-1} \overline{\mathbb{B}}. \tag{3-14}$$

Furthermore, since $\phi_4 : \Omega' \hookrightarrow \mathbb{C}^2$ is a proper holomorphic embedding, (F4), (F1), and the maximum principle ensure the existence of a smoothly bounded compact domain $D_j \subset \mathring{M}_j$ satisfying condition (1_j) and such that

$$\phi_4(bD_j) \subset (r_{j+1} - \mu\delta)\mathbb{B} \setminus (s'_j + \mu\delta)\overline{\mathbb{B}} \subset \mathbb{B}; \tag{3-15}$$

observe that $r_{j+1} - \mu\delta > s'_j + \mu\delta > r_j$ by (3-8) and (a). Thus, (H2) gives that

$$\phi(bD_j) \subset r_{j+1}\mathbb{B} \setminus s'_j\overline{\mathbb{B}}. \tag{3-16}$$

Moreover, (3-15) and the maximum principle imply that $\phi_4(D_j) \subset (r_{j+1} - \mu\delta)\mathbb{B} \subset \overline{\mathbb{B}}$, and hence (H2) gives that

$$|\phi(p) - \phi_4(p)| < \mu\delta \quad \text{for all } p \in D_j. \tag{3-17}$$

Set

$$\psi_j := \phi|_{D_j} : D_j \to \mathbb{C}^2.$$

We claim that the pair $\Xi_j = \{D_j, \psi_j\}$ satisfies conditions (1_j)–(6_j). Indeed, (1_j) has been already checked, (2_j) is implied by (3-12), and $(4_j) = $ (3-16). On the other hand, since $\phi_4(D_j) \subset \mathbb{B}$, properties (F5) and (H2) ensure that $\varnothing = \psi_j(D_j \setminus \mathring{D}_{j-1}) \cap (t - \mu\delta)\overline{\mathbb{B}} \supset \psi_j(D_j \setminus \mathring{D}_{j-1}) \cap s'_{j-1}\overline{\mathbb{B}}$, where the latter inclusion follows from the fact that $s'_{j-1} < t - \mu\delta$; see (3-8). This proves (5_j). Further, (G2) and (H1) guarantee that $\Lambda_i \subset \psi_j(\mathring{D}_i)$ for all $i \in \{0, \ldots, j-1\}$, whereas (G3), (H1), and (3-16) ensure that $\Lambda_j \subset \psi_j(\mathring{D}_j)$; thereby proving (6_j). Finally, (3-14), (3-16), (3-17), (F6), and (A4) guarantee that $\text{dist}_{\psi_j}(bD_{j-1}, bD_j) > 1$; take into account that $s'_j > r'_j$ in view of (a). This and (3-13) prove (3_j); recall that $p_0 \in \mathring{D}_{j-1} \Subset D_j$.

This concludes the construction of Ξ_j in Case 1.

Case 2: Assume that $\chi(M_j \setminus \mathring{M}_{j-1}) = -1$. In this case there is a smooth embedded compact path $\gamma \subset \mathring{M}_j \setminus \mathring{D}_{j-1}$ with its two endpoints lying in bD_{j-1}, meeting transversely there, and otherwise disjoint from D_{j-1}, such that $D_{j-1} \cup \gamma$ is a strong deformation retract of M_j.

By (4_{j-1}) we may extend ψ_{j-1}, with the same name, to a smooth embedding $D_{j-1} \cup \gamma \hookrightarrow \mathbb{C}^2$ such that $\psi_{j-1}(\gamma) \subset r_j\mathbb{B} \setminus s'_{j-1}\overline{\mathbb{B}}$. Thus, in view of properties (1_{j-1})–(6_{j-1}), Mergelyan's theorem with interpolation applied to $\psi_{j-1} : D_{j-1} \cup \gamma \hookrightarrow \mathbb{C}^2$ guarantees the existence of a connected, smoothly bounded, compact domain $D'_{j-1} \subset \mathring{M}_j$ and an embedding $\psi'_{j-1} : D'_{j-1} \hookrightarrow \mathbb{C}^2$ of class $\mathscr{A}^1(D'_{j-1})$ with the following properties:

$(1'_{j-1})$ $D_{j-1} \Subset D'_{j-1} \Subset M_j$ and D'_{j-1} is a strong deformation retract of M_j.

$(2'_{j-1})$ $|\psi'_{j-1}(p) - \psi_{j-1}(p)| < \epsilon_j/2$ for all $p \in D_{j-1}$.

$(3'_{j-1})$ $\text{dist}_{\psi'_{j-1}}(p_0, bD'_{j-1}) > j - 1$.

$(4'_{j-1})$ $\psi'_{j-1}(bD'_{j-1}) \subset r_j\mathbb{B} \setminus s'_{j-1}\overline{\mathbb{B}}$.

$(5'_{j-1})$ $\psi'_{j-1}(D'_{j-1} \setminus \mathring{D}_{j-1}) \cap s'_{j-1}\overline{\mathbb{B}} = \varnothing$.

$(6'_{j-1})$ $\Lambda_i \subset \psi'_{j-1}(\mathring{D}_i)$ for all $i \in \{0, \ldots, j-1\}$.

Since $\chi(M_j \setminus \mathring{D}'_{j-1}) = 0$, this reduces the proof to Case 1 and hence concludes the construction of the sequence $\Xi_j = \{D_j, \psi_j\}$ meeting properties (1_j)–(6_j), $j \in \mathbb{N}$.

This completes the proof of Theorem 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To finish the paper we show how Theorem 3.1 implies Theorem 1.2.

Proof of Theorem 1.2. Let Λ and M be as in Theorem 1.2 and assume without loss of generality that $0 \in \Lambda$. By [Alarcón and López 2013b, Main Theorem, page 1795], there exists a complex structure on M such that the open Riemann surface M admits a proper holomorphic embedding $\phi : M \hookrightarrow \mathbb{C}^2$; up to composing with a translation, we may assume that $0 \in \phi(M)$. Take a number $0 < r < 1$ such that $\Lambda \cap r\overline{\mathbb{B}} = \{0\}$; since $\Lambda \subset \mathbb{B}$ is closed and discrete, every small enough $r > 0$ meets this requirement. Thus, there are a small number $0 < s < r$ and a smoothly bounded compact disk $K \subset M$ such that $\phi(bK) \subset r\mathbb{B} \setminus s\overline{\mathbb{B}}$ and $0 \in \phi(\mathring{K})$.

Now, Theorem 3.1 applied to $\psi := \phi|_K$ furnishes a domain $D \subset M$ homeomorphic to M and a complete proper holomorphic embedding $\tilde{\psi} : D \hookrightarrow \mathbb{B}$ with $\Lambda \subset \tilde{\psi}(D)$. Since D is homeomorphic to the smooth surface M, there exists a complex structure on M (possibly different from the one used in the previous paragraph) such that the open Riemann surface M is biholomorphic to D. $\square$

Acknowledgements

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