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FINITE-TIME DEGENERATION OF HYPERBOLICITY WITHOUT BLOWUP FOR QUASILINEAR WAVE EQUATIONS

JARED SPECK

In three spatial dimensions, we study the Cauchy problem for the wave equation $-\partial_t^2\Psi+(1+\Psi)^P\Delta\Psi=0$ for $P\in\{1,2\}$. We exhibit a form of stable Tricomi-type degeneracy formation that has not previously been studied in more than one spatial dimension. Specifically, using only energy methods and ODE techniques, we exhibit an open set of data such that Ψ is initially near 0, while $1+\Psi$ vanishes in finite time. In fact, generic data, when appropriately rescaled, lead to this phenomenon. The solution remains regular in the following sense: there is a high-order L^2-type energy, featuring degenerate weights only at the top-order, that remains bounded. When $P=1$, we show that any C^1 extension of Ψ to the future of a point where $1+\Psi=0$ must exit the regime of hyperbolicity. Moreover, the Kretschmann scalar of the Lorentzian metric corresponding to the wave equation blows up at those points. Thus, our results show that curvature blowup does not always coincide with singularity formation in the solution variable. Similar phenomena occur when $P=2$, where the vanishing of $1+\Psi$ corresponds to the failure of strict hyperbolicity, although the equation is hyperbolic at all values of Ψ.

The data are compactly supported and are allowed to be large or small as measured by an unweighted Sobolev norm. However, we assume that initially the spatial derivatives of Ψ are nonlinearly small relative to $|\partial_t\Psi|$, which allows us to treat the equation as a perturbation of the ODE $(d^2/dt^2)\Psi=0$. We show that for appropriate data, $\partial_t\Psi$ remains quantitatively negative, which simultaneously drives the degeneracy formation and yields a favorable spacetime integral in the energy estimates that is crucial for controlling some top-order error terms. Our result complements those of Alinhac and Lindblad, who showed that if the data are small as measured by a Sobolev norm with radial weights, then the solution is global.

1. Introduction

Many authors have studied model nonlinear wave equations as a means to gain insight into more challenging wave-like quasilinear equations, such as Einstein's equations of general relativity, the compressible Euler equations without vorticity, and the equations of elasticity. Motivated by the same considerations, in this paper, we study model quasilinear wave equations in three spatial dimensions. To simplify the presentation, we have chosen to restrict our attention to the 3-dimensional case only; with only modest additional effort, our results could be generalized to apply in any number of spatial dimensions. In a broad sense, we are interested in finding initial conditions without symmetry assumptions that lead to some kind of stable breakdown. In our main results, which we summarize just below (1A-2), we exhibit a type of stable degen-

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erate solution behavior, *distinct from blowup*, that, to the best of our knowledge, has not previously been studied in the context of quasilinear equations in more than one spatial dimension. Roughly, we show that there exists an open set of data such that certain principal coefficients in the equation vanish in finite time *without a singularity forming in the solution*. More precisely, the vanishing of the coefficients corresponds to the vanishing of the wave speed, which in turn is tied to other kinds of degeneracies described below. We note that Wong [2016] has obtained similar constructive results for axially symmetric timelike minimal submanifolds of Minkowski spacetime, a setting in which the equations of motion are a system of (effectively one-space-dimensional) quasilinear wave equations with principal coefficients that depend on the solution (but not its derivatives). Specifically, he showed that *all* axially symmetric solutions (without any smallness assumption) lead to a finite-time degeneracy caused by the vanishing of a principal coefficient in the evolution equations. We also note that in the case of one spatial dimension, results similar to ours are obtained in [Kato and Sugiyama 2013; Sugiyama 2013; 2016a; 2016b] using proofs by contradiction that rely on the method of Riemann invariants. However, since the method of Riemann invariants is not applicable in more than one spatial dimension and since we are interested in direct proofs, our approach here is quite different.

Through our study of model problems, we are aiming to develop approaches that might be useful for studying the kinds of degeneracies that might develop in solutions to more physically relevant quasilinear equations. One consideration behind this aim is that there are relatively few breakdown results for quasilinear equations compared to the semilinear case. A second consideration is that many of the techniques that have been used to study semilinear wave equations do not apply in the quasilinear case; see Section 1D for further discussion. A third consideration concerns fundamental limitations of semilinear model equations: they are simply incapable of exhibiting some of the most important degeneracies that can occur in solutions to quasilinear equations. In particular, the degeneracy exhibited by the solutions from our main results cannot occur in solutions to semilinear wave equations with principal part equal to the linear wave operator[1] $\square_m$. As a second example of breakdown that is unique to the quasilinear case, we note that the phenomenon of shock formation, described in more detail at the end of Section 1F, cannot occur in solutions to semilinear equations since, in the semilinear case, the evolution of characteristics is not influenced in any way[2] by the solution.

In view of the above discussion, it is significant that our analysis has robust features and could be extended to apply to a large class[3] of quasilinear equations. The robustness stems from the fact that our proofs are based only on energy estimates, ODE-type estimates, and the availability of an important monotonic spacetime integral (which we describe below) that arises in the energy estimates. However, rather than formulating a theorem about a general class of equations, we prefer to keep the paper short and to exhibit the main ideas by studying only the model equation (1A-1a) below in the cases $P = 1, 2$.

[1]Here, $\square_m := -\partial_t^2 + \Delta$ denotes the standard linear wave operator corresponding to the Minkowski metric $m := \mathrm{diag}(-1, 1, 1, 1)$ on $\mathbb{R}^{1+3}$.

[2]In the works on shock formation for quasilinear equations described in Section 1F, the intersection of the characteristics is tied to the blowup of some derivative of the solution.

[3]Of course, we can only hope to treat wave equations whose principal spatial coefficients vanish when evaluated at some finite values of the solution variable; equations such as $-\partial_t^2 \Psi + (1 + \Psi^2)\, \Delta \Psi = 0$ are manifestly immune to the kind of degeneracies under study here.

1A. ***Statement of the equations and summary of the main results.*** Specifically, in the cases $P = 1, 2$, we study the following model Cauchy problem on $\mathbb{R}^{1+3}$:

$$-\partial_t^2\Psi + (1+\Psi)^P \Delta\Psi = 0, \tag{1A-1a}$$

$$(\Psi|_{\Sigma_0}, \partial_t\Psi|_{\Sigma_0}) = (\mathring{\Psi}, \mathring{\Psi}_0), \tag{1A-1b}$$

where $(x^0 := t, x^1, x^2, x^3)$ is a fixed set of standard rectangular coordinates on $\mathbb{R}^{1+3}$, $\Delta := \sum_{a=1}^3 \partial_a^2$ is the standard Euclidean Laplacian on $\mathbb{R}^3$, and throughout,[4] $\Sigma_t := \{t\} \times \mathbb{R}^3 \simeq \mathbb{R}^3$. We sometimes denote the spatial coordinates by $\underline{x} := (x^1, x^2, x^3)$. Note that we can rewrite (1A-1a) as[5] $(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha\partial_\beta\Psi = 0$, where g is the Lorentzian (for $\Psi > -1$) metric

$$g := -dt^2 + (1+\Psi)^{-P}\sum_{a=1}^3 (dx^a)^2. \tag{1A-2}$$

This geometric perspective will be useful at various points in our discussion.

We now summarize our results; see Theorem 4.1 and Proposition 4.2 for precise statements.

Summary of the main results: In the case $P = 1$, there exists an open subset of $H^6(\mathbb{R}^3) \times H^5(\mathbb{R}^3)$ comprising compactly supported initial data $(\mathring{\Psi}, \mathring{\Psi}_0)$ such that the solution Ψ, its *spatial* derivatives, and its *mixed space-time* derivatives initially satisfy a nonlinear smallness condition compared to[6] $\max_{\Sigma_0}[\mathring{\Psi}_0]_-$ and $1/\|\mathring{\Psi}_0\|_{L^\infty(\Sigma_0)}$, and such that the solution has the following property: the coefficient $1+\Psi$ in (1A-1a) vanishes at some time $T_\star \in (0, \infty)$. In fact, the finite-time vanishing of $1+\Psi$ *always* occurs if $\mathring{\Psi}_0$ is nontrivial and the data are appropriately rescaled; see Remark 2.4. Moreover,

$$\Psi \in C\big([0, T_\star), H^6(\mathbb{R}^3)\big) \cap L^2\big([0, T_\star], H^6(\mathbb{R}^3)\big) \cap C\big([0, T_\star], H^5(\mathbb{R}^3)\big),$$

while for any $N < 5$,

$$\partial_t\Psi \in C\big([0, T_\star), H^5(\mathbb{R}^3)\big) \cap L^\infty\big([0, T_\star], H^5(\mathbb{R}^3)\big) C\big([0, T_\star], H^N(\mathbb{R}^3)\big).$$

In addition, the Kretschmann scalar $\mathrm{Riem}(g)^{\alpha\beta\gamma\delta}\,\mathrm{Riem}(g)_{\alpha\beta\gamma\delta}$ blows up precisely at points where $1+\Psi$ vanishes, where $\mathrm{Riem}(g)$ denotes the Riemann curvature of g. Finally, the solution exits the regime of hyperbolicity at time $T_\star$ and thus it cannot be continued beyond $T_\star$ as a classical solution to a hyperbolic equation. In the case $P = 2$, similar results hold, the main differences being that Ψ is not necessarily an element of $L^2\big([0, T_\star], H^6(\mathbb{R}^3)\big)$ and that the strict hyperbolicity[7] breaks down when $1+\Psi$ vanishes but hyperbolicity[8] does not.[9] This leaves open, in the case $P = 2$, the possibility of classically extending the solution past time $T_\star$; see Section 1C2.

[4]Here we use the notation "$\simeq$" to mean "diffeomorphic to".

[5]Throughout we use Einstein's summation convention.

[6]Throughout, $[p]- := |\min\{p, 0\}|$.

[7]Equation (1A-1a) is said to be strictly hyperbolic in the direction ω if the symbol $p(\xi) := -\xi_0^2 + (1+\Psi)^P \sum_{a=1}^3 \xi_a^2$ has the following property: for any one-form $\xi \neq 0$, the polynomial $s \to p(\xi + s\omega)$ has two distinct real roots. It is straightforward to see that (1A-1a) is strictly hyperbolic in the direction $\omega := (1, 0, 0, 0)$ if $1+\Psi > 0$, and that it is *not* strictly hyperbolic in any direction if $1+\Psi = 0$.

[8]Here, by hyperbolic (in the direction ω), we mean that for all one-forms $\xi \neq 0$, the polynomial $s \to p(\xi + s\omega)$ from Footnote 7 has only real roots. Such polynomials are known as *hyperbolic polynomials*.

[9]In the literature, equations exhibiting this kind of degeneracy are often referred to as *weakly hyperbolic*.

1B. ***Paper outline.*** The remainder of the paper is organized as follows.

- In Section 1C, we provide some initial remarks expanding upon various aspects of our results.
- In Section 1D, we mention some techniques that have been used in studying the breakdown of solutions to semilinear equations. As motivation for the present work, we point out some limitations of the semilinear techniques for the study of quasilinear equations.
- In Section 1E we provide a brief overview of the proof of our main results.
- In Section 1F, we describe some connections between our results and prior work on degenerate hyperbolic PDEs.
- In Section 1G we summarize our notation.
- In Section 2, we state our assumptions on the initial data and introduce bootstrap assumptions.
- In Section 3, we use the bootstrap and data-size assumptions of Section 2 to derive a priori pointwise estimates and energy estimates. From the energy estimates, we deduce improvements of the bootstrap assumptions.
- In Section 4, we use the estimates of Section 3 to prove our main results.

1C. ***Initial remarks on the main results.*** As far as we know, there are no prior results in the spirit of our main results in more than one spatial dimension. There are, however, examples in which the Cauchy problem for a quasilinear wave equation has been solved (for suitable data without symmetry assumptions) and such that it was shown that some derivative of the solution blows up in finite time while the solution itself remains bounded. One class of such examples comprises shock formation results, which we describe in more detail at the end of Section 1F. A second example is [Luk 2013] on the formation of weak null singularities in a family of solutions to the Einstein-vacuum equations. Specifically, Luk exhibited a stable family of solutions such that the Christoffel symbols (which are, roughly speaking, the first derivatives of the solution) blow up along a null boundary, while the metric (that is, the solution itself) extends continuously past the null boundary. We stress that the degeneracy we have exhibited in our main results is much less severe than in the above results; there is no blowup in our solutions, except possibly at the top derivative level, due to the degeneracy of the weights in the energy (1E-1).

We also point out a connection between our work here and our joint works [Rodnianski and Speck 2014a; 2014b], in which we proved stable blowup results (without symmetry assumptions) for solutions to the linearized and nonlinear Einstein-scalar field and Einstein-stiff fluid systems. In the nonlinear problem, the wave speed became, relative to a geometrically defined coordinate system,[10] *infinite* at the singularity. Although the infinite wave speed is in the opposite direction of the degeneracy exhibited by our main results (in which the wave speed vanishes[11]), the analysis in [Rodnianski and Speck 2014a; 2014b] shares a key feature with that of the present work: the solution regime studied is such that the time derivatives dominate the evolution. That is, the spatial derivatives remain negligible, all the way up to the degeneracy; see

[10] Specifically, the Σ_t have constant mean curvature and the spatial coordinates are transported along the unit normal to Σ_t.
[11] Note that the effective wave speed for (1A-1a) is $(1+\Psi)^{P/2}$.

Section 1E for further discussion regarding this issue for the solutions under study here. Hence, those works and the present work all exhibit the stability of ODE-type behavior in some solutions to wave equations.

1C1. *Remarks on small data.* The methods of Alinhac [2003] and Lindblad [2008] yield that small-data solutions to (1A-1a) exist globally,[12] where the size of the data is measured by Sobolev norms with *radial weights*. Consequently, if $(\mathring{\Psi}, \mathring{\Psi}_0)$ are compactly supported data to which our main results apply, then for λ sufficiently large, the solution corresponding to the data $(\lambda^{-1}\mathring{\Psi}, \lambda^{-1}\mathring{\Psi}_0)$ is global. On the other hand, our main results apply to data that are allowed to be small in certain unweighted norms, as long as the spatial derivatives are "very small". How can we reconcile these two competing statements? The answer is that our data assumptions are nonlinear in nature and are *not* invariant under the rescaling $(\mathring{\Psi}, \mathring{\Psi}_0) \to (\lambda^{-1}\mathring{\Psi}, \lambda^{-1}\mathring{\Psi}_0)$ if λ is too large. We can sketch the situation as follows (see Section 2C for the precise nonlinear smallness assumptions that we use to close our proof): if ϵ is the size of $\nabla\Psi$ at time 0 (where ∇ denotes the spatial coordinate gradient) and δ is the size of $\partial_t\Psi$ at time 0, then, roughly speaking, some parts of our proof rely on[13] the assumption that $\epsilon \exp(C\delta^{-1}) \lesssim 1$. The point is that if λ is too large, then the assumption is not satisfied, the reason being that ϵ and δ both scale like λ^{-1}. One can contrast this against the discussion in Section 2D, where we note that a different scaling of the data always leads to our nonlinear smallness assumptions being satisfied.

1C2. *Remarks on extending the solution past the degeneracy.* It is of interest to know if and when the solutions provided by our main results can be extended, as solutions with some kind of Sobolev regularity,[14] past the time of first vanishing of $1+\Psi$. Although we do not address this question in this article, in this subsubsection, we describe what is known and some of the difficulties that one would encounter in attempting to answer it. The cases $P=1$ and $P=2$ in (1A-1a) correspond to different phenomena and hence we will discuss them separately, starting with the case $P=1$.

Interesting results have recently been obtained in [Lerner et al. 2015] for equations related to (1A-1a). They *suggest* that in the case $P=1$, it might not be possible to continue the solutions from our main results as Sobolev-class solutions in a spacetime neighborhood of a point at which $1+\Psi$ vanishes. Perhaps this is not surprising since, for the solutions under study, the case $P=1$ corresponds to (1A-1a) changing from hyperbolic to elliptic type past the degeneracy (at least for C^1 solutions). Specifically, those authors proved a type of Hadamard ill-posedness for certain initial data for a class of quasilinear first-order systems in n spatial dimensions of the form

$$\partial_t u + \sum_{a=1}^{n} A^a(t,x,u)\,\partial_a u = F(t,x,u), \tag{1C-1}$$

[12] Since the equations do not satisfy the null condition, the asymptotics of the solution can be distorted compared to the case of solutions to the linear wave equation.

[13] For example, a careful analysis of the proof of inequality (3C-4) yields that the constant C in front of the $\mathring{\epsilon}^2$ term on the right-hand side depends on $\exp(\mathring{\delta}_*^{-1})$, where $\mathring{\delta}_*$ is defined in (2A-2). See Section 3A for our conventions regarding the dependence of constants on various parameters.

[14] The Cauchy–Kovalevskaya theorem could be used to prove an (admittedly unsatisfying) result showing that in the cases $P=1,2$, one can extend *analytic* solutions to (1A-1a) to exist in a spacetime neighborhood of a point at which $1+\Psi$ vanishes. Note that this shows that the blowup of the curvature of the metric of (1A-1a) that occurs when $1+\Psi=0$ is not always an obstacle to continuing the solution classically.

where $(t,x)\in\mathbb{R}^{1+n}$, u is a map from $\mathbb{R}^{1+n}$ to $\mathbb{R}^N$ with n and N arbitrary, and the A^a are real $N\times N$ matrices. The authors proved several types of results in [Lerner et al. 2015], but here we describe only the ones that are most relevant for (1A-1a). Roughly, in Theorem 1.3 of that paper, for systems of type (1C-1) that satisfy some technical conditions, the authors studied perturbations of a background solution, denoted by $\phi=\phi(t,x)$, with the following property: the system (1C-1) is hyperbolic when evaluated at $(t,x,u)=(0,x,\mathring{\phi}(x))$, where $\mathring{\phi}(x):=\phi(0,x)$, but necessarily becomes elliptic at $(t,x,u)=(t,x,\phi(t,x))$ at any $t>0$ due to the branching[15] of the eigenvalues of the principal symbol. The assumptions of their Theorem 1.3 guarantee that the branching is stable under small perturbations. Roughly, for the solutions to (1A-1a) from our main results, a similar transition to ellipticity would occur in the case $P=1$ if one were able to classically extend the solution[16] past the time of first vanishing of $1+\Psi$. We now summarize the main aspects of [Lerner et al. 2015, Theorem 1.3]. We will use the notation $\mathring{u}$ to denote initial data for the system (1C-1) and u to denote the corresponding solution (if it exists). The theorem is, roughly, as follows: for any $m\in\mathbb{R}$ and $\alpha\in\left(\frac{1}{2},1\right]$, and any sufficiently small $T>0$, there is no H^m-neighborhood $\mathcal{U}$ of $\mathring{\phi}$ whose elements launch corresponding solutions obeying a bound roughly[17] of the type[18]

$$\sup_{\mathring{u}\in\mathcal{U}}\frac{\|u-\phi\|_{W_x^{1,\infty}L_t^\infty([0,T])}}{\|\mathring{u}-\mathring{\phi}\|_{H^m}^\alpha}<\infty.$$

Put differently, there exist data arbitrarily close to $\mathring{\phi}$ (as measured by a Sobolev norm of arbitrarily high order) such that either the solution does not exist or such that its deviation from ϕ becomes arbitrarily large in the low-order norm $\|\cdot\|_{W_x^{1,\infty}}$ in an arbitrarily short amount of time. It would be interesting to determine whether or not a similar result holds for initial data close to that of the data induced by the solutions to (1A-1a) from our main results at the time of first vanishing of $1+\Psi$.

We now discuss the case $P=2$. We are not aware of any results for Sobolev-class solutions to quasilinear equations that are relevant for extending solutions to (1A-1a) to exist in a spacetime neighborhood of a point at which $1+\Psi$ vanishes. As we will explain, the main technical difficulty that one encounters is that the solution might lose regularity past the degeneracy. In the case $P=2$, even though the strict hyperbolicity (see Footnote 7) of (1A-1a) breaks down when $1+\Psi$ vanishes (corresponding to a wave of zero speed), the hyperbolicity (see Footnote 8) of the equation nonetheless persists for *all* values of Ψ. The degeneracy is therefore less severe compared to the case $P=1$ and thus in principle, when $P=2$, the Sobolev-class solutions from our main results might be extendable, as a Sobolev-class solution, to a neighborhood of the points where $1+\Psi$ first vanishes. As we alluded to above, the lack of results in this direction might be tied to the following key difficulty: the best energy estimates available for degenerate[19] linear hyperbolic wave

[15] In [Lerner et al. 2015], the definition of hyperbolicity is that the polynomial (in λ) $p:=\det\left(\lambda I-\sum_{a=1}^n\xi_aA^a(t,x,u)\right)$ should have only real roots, which are eigenvalues of $\sum_{a=1}^n\xi_aA^a(t,x,u)$. Moreover, branching roughly means that the eigenvalues are real at $t=0$ but can have nonzero imaginary parts at arbitrarily small values of $t>0$.

[16] As we will explain, the solutions from our main results are such that Ψ is strictly decreasing in time at points where $1+\Psi$ vanishes.

[17] The precise results of [Lerner et al. 2015, Theorem 1.3] are localized in space, but here we omit those details for brevity.

[18] If $f=f(t,x)$, then $\|f\|_{W_x^{1,\infty}L_t^\infty([0,T])}:=\operatorname{ess\,sup}_{t\in[0,T]}\|f(t,\cdot)\|_{W^{1,\infty}}$.

[19] By degenerate, we mean that the wave equation is allowed to violate strict hyperbolicity at one or more points.

equations exhibit a loss of derivatives. By this, we roughly mean that the estimates for solutions Ψ to the linear equation are of the form $\|\Psi\|_{H^N(\Sigma_t)} \lesssim \|\mathring{\Psi}\|_{H^{N+d}(\Sigma_0)} + \|\mathring{\Psi}_0\|_{H^{N+d}(\Sigma_0)}$, where the loss of derivatives d (relative to the data) depends in a complicated way on the details of the degeneration of the coefficients in the equation; see Section 1F for further discussion. As is described in [Dreher 1999], in some cases, the loss of derivatives in the estimates is known to be saturated. Since proofs of well-posedness for nonlinear equations typically rely on estimates for linearized equations, any derivative loss would pose a serious obstacle to extending (in the case $P = 2$) the solution of (1A-1a) as a Sobolev-class solution in a spacetime neighborhood of points at which $1 + \Psi$ vanishes. At the very least, one would need to rely on a method capable of handling a finite loss of derivatives in solutions to quasilinear equations. As is well-known [Hamilton 1982], in some cases, it is sometimes possible to handle a finite loss of derivatives using the Nash–Moser framework.

Despite the lack of results concerning extending the solution to (1A-1a) as a Sobolev-class solution past points at which $1 + \Psi$ vanishes, there are constructive results in the class C^∞. Specifically, in one spatial dimension, Manfrin [1996] obtained well-posedness results that, for C^∞ initial data, allow one to locally continue the solution to (1A-1a) in the case $P = 2$ to a C^∞ solution that exists in a spacetime neighborhood of a point at which $1 + \Psi$ vanishes; see Section 1F for further discussion. Manfrin [1999] also derived similar results in more than one spatial dimension, again treating the case of C^∞ data/solutions. We are also aware of a few results [Dreher 1999; Han et al. 2003] for quasilinear equations in more than one spatial dimension in which the authors proved local well-posedness in Sobolev spaces for equations featuring a degeneracy related to — but distinct from — the one under study here. However, the degeneracy in those works was created by a "prescribed semilinear factor" rather than a quasilinear-type solution-dependent factor. For this reason, it is not clear that the techniques used in those works are of relevance for trying to extend solutions to (1A-1a) beyond points where $1 + \Psi$ vanishes; see the paragraph below (1F-5) for further discussion.

To close this subsubsection, we note that there are various well-posedness results [D'Ancona and Spagnolo 1992; Ebihara 1985; Ebihara et al. 1986] for degenerate wave equations of Kirchhoff type. An example of an equation of this type is

$$-\partial_t^2 \Psi + F\left(\int_\Omega |\nabla\Psi|^2 \, dx\right) \Delta\Psi = 0, \tag{1C-2}$$

where Ω is a bounded open set in $\mathbb{R}^n$ and $F = F(s) \geq 0$ satisfies various technical conditions (with $F = 0$ corresponding to the degeneracy). However, it remains open whether or not the techniques used in studying Kirchhoff-type equations are of relevance for proving local well-posedness for (1A-1a) (in the case $P = 2$) in regions where $1 + \Psi$ is allowed to vanish.

1D. ***Remarks on methods used for studying blowup in solutions to semilinear wave equations.*** Although we are not aware of any other results in the spirit of the present work, there are many results exhibiting the most well-known type of degeneracy that can occur in solutions to wave equations in three spatial dimensions: the finite-time *blowup* of initially smooth solutions. Our main goal in this subsection is to recall some of the most important of these results but, at the same time, to describe some limitations of the proof techniques for the study of more general equations. We will focus only on constructive[20] results, by which we mean that the proofs provide a detailed description of the degeneracy formation and the mechanisms driving it, as in the present work. Constructive results, especially those proved via robust techniques, are clearly desirable if one aims to understand the mechanisms of breakdown in solutions to realistic physical and geometric systems. They are also important if one aims to continue the solution past the breakdown, as is sometimes possible if it is not too severe; see, for example, [Christodoulou and Lisibach 2016] for a recent result in spherical symmetry concerning weakly locally extending solutions to the relativistic Euler equations past the first shock singularity. Importantly, we will confine our discussion to prior results for semilinear equations since, as we mentioned earlier, aside from the shock formation results described at the end of Section 1F, most constructive breakdown results for wave equations in three spatial dimensions are blowup results for semilinear equations.

Specifically, most constructive breakdown results for wave equations in three spatial dimensions are blowup results for semilinear equations (or systems) of the form $\square_m \Psi = f(\Psi, \partial\Psi)$, where f is a smooth nonlinear term. Many important[21] approaches have been developed to prove constructive blowup for such equations, especially for scalar equations with $f = f(\Psi)$ given by a power law and for systems of wave-map type; see, for example, [Kenig and Merle 2008; Donninger 2010; Donninger and Schörkhuber 2012; 2014; Krieger and Schlag 2014; Krieger et al. 2008; 2009; Donninger et al. 2014; Rodnianski and Sterbenz 2010; Raphaël and Rodnianski 2012; Duyckaerts et al. 2012; Martel et al. 2014; Donninger and Krieger 2013]. There are also related results that are conditional in the sense that they do not guarantee that the solution will blow up. Instead they characterize the possible behaviors of the solution by providing information such as (i) how the singularity would form if the solution is not global and (ii) the structures of the data sets that lead to the various outcomes; see, for example, [Payne and Sattinger 1975; Struwe 2003; Nakanishi and Schlag 2011a; 2011b; 2012a; 2012b; Krieger et al. 2013a; 2013b; 2014; 2015; Killip et al. 2014].

Although the above results and others like them have yielded major advancements in our understanding of the blowup of solutions to semilinear equations, their proofs fundamentally rely on tools that are not typically applicable to quasilinear equations. Here are some important examples, where for brevity, we are not specific about exactly which semilinear equations have been treated with the stated technique:

- The existence of a conserved energy (which is not available for some important quasilinear equations, such as Einstein's equations[22]). This allows, among other things, for the application of techniques from Hamiltonian mechanics.

[20] Constructive proofs of blowup stand, of course, in contrast to proofs of breakdown by contradiction. There are many examples in the literature of proofs of blowup by contradiction for wave or wave-like equations. Two of the most important ones are Sideris' blowup result [1985] (proved by virial identity arguments) for the compressible Euler equations under a polytropic equation of state and John's proof [1981] of breakdown for several classes of semilinear and quasilinear wave equations in three spatial dimensions. See also the influential work [Levine 1974], in which he proved a nonconstructive blowup result for semilinear wave equations on an abstract Hilbert space.

[21] Arguably, the most sophisticated blowup results of this type have been proved for nonlinearities that correspond to energy critical equations.

[22] For asymptotically flat solutions to Einstein's equations, the ADM mass is conserved. However, in three spatial dimensions without symmetry assumptions, this quantity has thus far proven to be too weak to be of any use in controlling solutions.

• The invariance of the solutions under appropriate rescalings (which is not a feature of some important quasilinear equations, such as the compressible Euler equations[23]).

• The availability of well-posedness results in low regularity spaces such as the energy space (Lindblad [1998] showed that low regularity well-posedness fails for a large class of quasilinear equations in three spatial dimensions).

• The existence of a nontrivial ground state solution (corresponding to the existence of a soliton solution) and sharp classification results for the possible behaviors of the solution for initial data with energy less than the ground state: either there is finite-time blowup in both time directions or global existence, according to the sign of a functional (for quasilinear equations, there is no known analog of this kind of dichotomy). Moreover, in some cases, there are more complicated classification results available for solutions with energy just above the ground state.

• A characterization of a certain norm of the ground state as a size threshold separating global scattering solutions from ones that can blow up or exhibit other degenerate behavior (again, for quasilinear equations, there is no known analog of this kind of dichotomy).

• A characterization of the ground state as the universal blowup-profile under various assumptions.

• The availability of profile decompositions for sequences bounded in the natural energy space, which allows one to view the sequence as a superposition of linear solutions plus a small error (for quasilinear equations, there is no known analog of this).

• Channel-of-energy-type arguments showing that a portion of the solution propagates precisely at speed one (again, for quasilinear equations, there is no known[24] analog of this phenomenon).

• The possibility of sharply characterizing the spectrum, see for example [Costin et al. 2012], of linear operators tied to the dynamics (which, for quasilinear equations in many solution regimes, is exceedingly difficult).

Although the above methods are impressively powerful within their domain of applicability, since they do not seem to apply to quasilinear equations, we believe that it is important to develop new methods for studying the kinds of breakdown that can occur in the quasilinear case. It is for this reason that we have chosen to study the model wave equations (1A-1a).

1E. *Brief overview of the analysis.* As we mentioned earlier, the solutions that we study are such that $\mathring{\Psi}$, $\nabla\mathring{\Psi}_0$ (where ∇ denotes the spatial coordinate gradient), and sufficiently many of their spatial derivatives are "nonlinearly small" (in appropriate norms) compared to $[\mathring{\Psi}_0]_- := |\min\{\mathring{\Psi}_0, 0\}|$ and $1/\|\mathring{\Psi}_0\|_{L^\infty(\Sigma_0)}$. A key aspect of our work is that we are able to propagate the smallness, long enough for the coefficient $1+\Psi$ in (1A-1a) to vanish. Put differently, our main results show that under the smallness assumptions, the solution to (1A-1a) behaves in many ways like a solution to the second-order ODE $(d^2/dt^2)\Psi = 0$. The reason that Ψ vanishes for the first time is that $\partial_t\Psi$ is sufficiently negative at one or more spatial

[23] In particular, the fluid equation of state does not generally enjoy any useful scaling transformation properties.

[24] It is conceivable that channel-of-energy-type results might hold for certain quasilinear wave equations in various solution regimes, since channel-of-energy-type arguments seem to be somewhat stable under perturbations.

points, a condition that persists by the previous remarks. To control solutions, we use the (nonconserved) energy[25]

$$\mathcal{E}_{[2,5]}(t) := \sum_{k'=2}^{5} \int_{\Sigma_t} |\partial_t \nabla^{k'} \Psi|^2 + (1+\Psi)^P |\nabla \nabla^{k'} \Psi|^2 + |\nabla^{k'} \Psi|^2 \, d\underline{x}. \tag{1E-1}$$

We avoid using low-order energies corresponding to $k' = 0, 1$ in (1E-1) because for the solution regime under consideration such energies would contain terms that are allowed to be large, and we prefer to work only with small energies. Hence, to control the low-order derivatives of Ψ, we derive ODE-type estimates that rely in part on the energy estimates for its higher derivatives and Sobolev embedding. Analytically, the main challenge is that the vanishing of $1+\Psi$ leads to the degeneracy of the top-order spatial derivative terms in (1E-1), which makes it difficult to control some top-order error integrals in the energy estimates.

To close the energy estimates, we exploit the following monotonicity, which is available due to our assumptions on the data:

$\partial_t \Psi$ is *quantitatively strictly negative* in a neighborhood of points where $1+\Psi$ is close to 0.

This quantitative negativity yields, in our energy identities, the spacetime error integral

$$\int_{s=0}^{t} \int_{\Sigma_s} (\partial_t \Psi)(1+\Psi)^{P-1} |\nabla \nabla^{k'} \Psi|^2 \, d\underline{x} \, ds, \tag{1E-2}$$

which has a "friction-type" sign in regions where $1+\Psi$ is close to 0 but positive; see the spacetime integral on the left-hand side of (3C-3). It turns out that the availability of this spacetime integral compensates for the degeneracy of the energy (1E-1) and yields integrated control over the spatial derivatives up to top-order; *this is the key to closing the proof.*

1F. ***Comparing with and contrasting against other results for degenerate hyperbolic equations.*** For solutions such that $1+\Psi$ is near 0, (1A-1a) can be viewed as a "nearly degenerate" quasilinear hyperbolic PDE. For this reason, the proofs of our main results have ties to some prior results on degenerate hyperbolic PDEs, which we now discuss. In one spatial dimension, various aspects of degenerate hyperbolic PDEs have been explored in the literature, such as the branching of singularities [Alinhac 1978; Amano and Nakamura 1981; 1982; 1983; 1984], uniqueness of solutions for equations that are hyperbolic in one region but that can change type [Ruziev and Reissig 2016], and conditions that are *necessary* for well-posedness [Yagdzhyan 1989]. However, in the rest of this subsection, we will discuss only well-posedness results since they are most relevant in the context of our main results.

In one spatial dimension, there are many results on well-posedness, in various function spaces, for degenerate linear wave equations for the form

$$-\partial_t^2 \Psi + a(t,x)\, \partial_x^2 \Psi + b(t,x)\, \partial_x \Psi + c(t,x)\, \partial_t \Psi = f(t,x), \tag{1F-1}$$

where $a(t,x) \geq 0$ and $a(t,x) = 0$ corresponds to degeneracy via a breakdown of strict hyperbolicity. For example, if the coefficients $a(t,x)$, $b(t,x)$, and $c(t,x)$ are *analytic* and satisfy certain technical assumptions, then it is known [Nishitani 1984] that (1F-1) is well-posed for C^∞ data; see also [Nishitani

[25] Throughout, $d\underline{x} := dx^1 dx^2 dx^3$ denotes the standard Euclidean volume form on Σ_t.

1980] for similar results. There are also results on well-posedness for degenerate semilinear equations. For example, in [D'Ancona and Trebeschi 2001], the authors used a Nash–Moser argument to prove C^∞ local well-posedness for semilinear equations of the form

$$-\partial_t^2\Psi + a(t,x)\,\partial_x^2\Psi = f(t,x,\Psi,\partial_t\Psi,\partial_x\Psi), \tag{1F-2}$$

where $a(t,x)\geq 0$ is analytic and a and f satisfy appropriate technical assumptions. We clarify that in contrast to our work here, in the above works, the authors solved the equation in a spacetime neighborhood of points at which the degeneracy occurs.

A serious limitation of the above results is that techniques relying on analyticity assumptions are of little use for studying quasilinear Cauchy problems with Sobolev-class data, such as the problems we consider here. Fortunately, well-posedness results for degenerate linear equations that do not rely on analyticity assumptions are also known; see, for example, [Oleĭnik 1970; Taniguchi and Tozaki 1980; D'Ancona 1994; Han et al. 2006; Herrmann 2012; Herrmann et al. 2013; Han and Liu 2015]. We note in particular that the results of [Oleĭnik 1970; Taniguchi and Tozaki 1980; Han et al. 2006; Herrmann 2012; Herrmann et al. 2013; Han and Liu 2015] provide Sobolev estimates for the solution in terms of a Sobolev norm of the data, with a finite loss of derivatives. We also mention the related works [Ascanelli 2006; 2007], in which the author proves well-posedness results (in C^∞ and Gevrey spaces) for linear wave equations with two kinds of degeneracies: (i) the breakdown of strict hyperbolicity (corresponding to the vanishing of certain coefficients) and (ii) the blowup of the time derivatives of certain coefficients in the wave equation. We also mention the works [Ivriĭ 1975; Ishida and Yagdjian 2002], in which the authors obtain necessary and sufficient conditions for the Gevrey space well-posedness of degenerate linear hyperbolic equations.

Most relevant for our work here is Manfrin's aforementioned proof [1996] of C^∞ well-posedness for various degenerate *quasilinear* wave equations in one spatial dimension (see also [Manfrin 1999] for a similar result in more than one spatial dimension and the related work [Boiti and Manfrin 2000]), including those of the form

$$-\partial_t^2\Psi + \Psi^{2k}a(t,x,\Psi)\,\partial_x^2\Psi = f(t,x,\Psi), \tag{1F-3}$$

where $k\geq 1$ is an integer and $a(t,x,\Psi)$ is uniformly bounded from above and from below, strictly away from 0 (and $\Psi=0$ corresponds to the degeneracy). More precisely, for C^∞ initial data, Manfrin used weighted energy estimates and Nash–Moser estimates to prove local well-posedness for solutions to (1F-3). The energy estimate weights are complicated to construct and are based on dividing spacetime into various regions with the help of "separating functions" adapted to the degeneracy. Note that Manfrin's results apply to our model equation[26] (1A-1a) in the case $P=2$. However, it is an open problem whether or not his results can be extended to yield a local well-posedness result for (1F-3) with data in Sobolev spaces.

To further explain these results and their connection to our work here, we consider the simple Tricomi-type equation

$$-\partial_t^2\Psi + a(t)\,\Delta\Psi = 0, \tag{1F-4}$$

[26]More precisely, the role of "$1+\Psi$" in (1A-1a) is played by "Ψ" in (1F-3).

where $a(t) \geq 0$. It is known [Colombini and Spagnolo 1982] that in one spatial dimension, the linear (1F-4) can be *ill-posed*,[27] even if $a = a(t)$ is C^∞. Hence, it should not be taken for granted that we can (for suitable data) solve (1A-1a) in Sobolev spaces all the way up to the time of first vanishing of $1 + \Psi$. Roughly, what can go wrong in an attempt to solve the linear equation (1F-4) is that $a(t)$ can be highly oscillatory near a point t_0 with $a(t_0) = 0$. In fact, in the example from [Colombini and Spagnolo 1982], $a(t)$ oscillates *infinitely many* times near t_0. This generates, in the basic energy identity, an uncontrollable term involving the ratio $((d/dt)a(t))/a(t)$ and leads to ill-posedness in domains $[A, B) \times \mathbb{R}$ when $t_0 \in [A, B)$.

In all of the aforementioned well-posedness results, the technical conditions imposed on the coefficients rule out the infinite oscillatory behavior from [Colombini and Spagnolo 1982] that led to ill-posedness. To provide a more concrete example, we note that in one spatial dimension, Han [2010] derived degenerate energy estimates for linear wave equations of the form

$$-\partial_t^2 \Psi + a(t, x)\, \partial_x^2 \Psi + b_0(t, x)\, \partial_t \Psi + b(t, x)\, \partial_x \Psi + c(t, x)\Psi = f(t, x), \tag{1F-5}$$

where the coefficients satisfy certain technical conditions, including, roughly speaking, that $a(t, x) \geq 0$ behaves like $t^m + c_{m-1}(x)t^{m-1} + \cdots + c_1(x)t + c_0(x)$. In particular, even though a is allowed to vanish at some points, it does not exhibit highly oscillatory behavior in the t-direction. In [Han et al. 2006], similar results were derived in $n \geq 1$ spatial dimensions.

We now describe Dreher's Ph.D. thesis [1999], which involves the study of equations that share some common features with (1A-1a) near the degeneracy $1 + \Psi = 0$. Specifically, in his thesis, Dreher proved local well-posedness results in Sobolev spaces for several classes of quasilinear hyperbolic PDEs in any number of dimensions with various kinds of space and time degeneracies. However, a key difference between the equations studied by Dreher in his thesis and our work is that the degeneracies there were "prescribed" in the sense that they were caused only by degenerate semilinear factors that explicitly depend on the time and space variables. That is, if one deletes the degenerate semilinear factors, then one obtains a strictly hyperbolic PDE for which local well-posedness follows from standard techniques. Dreher made technical assumptions on the degenerate semilinear factors that were sufficient for proving well-posedness. In contrast, the degeneracy caused by $1 + \Psi = 0$ in (1A-1a) is "purely quasilinear" in nature. The following model equation in one spatial dimension gives a sense of the kinds of prescribed degeneracy treated by Dreher:

$$-\partial_t^2 \Psi + t^2 f(\Psi)\, \partial_x^2 \Psi = 0, \tag{1F-6}$$

where f is smooth and satisfies $f(\Psi) > 0$. We stress that the absence of strict hyperbolicity in a neighborhood of Σ_0 is *not* caused by the quasilinear factor $f(\Psi)$, but rather by the semilinear factor t^2. A related example, coming from geometry, is the aforementioned work [Han et al. 2003], in which the authors proved the existence of local C^k embeddings of surfaces of nonnegative Gaussian curvature into $\mathbb{R}^3$. The quasilinear system of PDEs studied there degenerated at points where the Gauss curvature of the surface vanishes. As in [Dreher 1999], the degeneracy was "prescribed" in the sense that it was caused

[27] In [Colombini and Spagnolo 1982], which addressed solutions in one spatial dimension, the authors exhibited a smooth function $a(t) \geq 0$ with $a(t_0) = 0$ for some $t_0 > 0$ and data such that there is no distributional solution to (1F-4) with the given data that extends past time t_0.

by the Gauss curvature (which is "known"). Hence, the authors were free to make technical assumptions on the Gauss curvature to ensure the local well-posedness of the PDE system.

We now give another example of prior work that is closely connected to our main results. In [Ruan et al. 2016], the authors proved local well-posedness in homogeneous Sobolev spaces on domains of the form $[0, T) \times \mathbb{R}^n$ for semilinear Tricomi equations of the form

$$-\partial_t^2 \Psi + t^P \Delta \Psi = f(\Psi), \tag{1F-7}$$

where $P \in \mathbb{N}$ and f is a nonlinearity such that f and f' obey certain $P, n-$dependent power-law growth bounds at ∞. See [Ruan et al. 2015a; 2015b] for related results. Note that the coefficient t^P in (1F-7) does not oscillate; once again, this is the key difference compared to the ill-posedness result for (1F-4) mentioned above. As we described in Section 1E, (1F-7) is a good model for the solutions to (1A-1a) provided by our main results in the sense that the degenerating coefficient $(1 + \Psi)^P$ in (1A-1a) behaves in some ways, when $1 + \Psi$ is small, like[28] the coefficient t^P (near $t = 0$) in (1F-7).

In view of the above discussion, we believe that one should not expect to be able to solve (1A-1a) in Sobolev spaces all the way up to points with $1 + \Psi = 0$ unless one makes assumptions on the data that preclude highly oscillatory behavior in regions where $1 + \Psi$ is small. In this article, we avoid the oscillatory behavior by exploiting the *relative* largeness of $[\partial_t \Psi]_-$ and the *relative* smallness of $\partial_t^2 \Psi$ in regions where $1 + \Psi$ is small, which are present at time 0 and which we propagate; see the estimates (3B-2) and (3C-5c). As we have mentioned, the relative largeness of $[\partial_t \Psi]_-$ can be viewed as a kind of monotonicity in the problem. One might say that this monotonicity makes up for the lack of remarkable structure in (1A-1a), including that it is not an Euler–Lagrange equation, its solutions admit no known coercive conserved quantities, and the nonlinearities are not signed. As we described in Section 1E, this monotonicity yields an important signed spacetime integral that we use to close the energy estimates; see the spacetime integral on the left-hand side of (3C-4). The largeness of $[\partial_t \Psi]_-$ is connected to so-called *Levi-type* conditions that have appeared in the literature. Roughly, a Levi condition is a quantitative relationship between the sizes of various coefficients in the equation and their derivatives. As an example, we note that in their study [D'Ancona and Trebeschi 2001] of well-posedness for (1F-2) with analytic coefficients, the authors studied linearized equations of the form (1F-1) under the Levi condition $|b(t, x)| \lesssim |a(t, x)| + |\partial_t \sqrt{a}(t, x)|$; the Levi condition allowed them, for the linearized equation, to construct suitable weights for the energy estimates (even at points where a vanishes), which were sufficient for proving well-posedness. In the problems under study here, the largeness of $[\partial_t \Psi]_-$ at points with $1 + \Psi = 0$ can be viewed as a Levi-type condition for the coefficient $(1 + \Psi)^P$ in (1A-1a), which allows us to control various error terms that arise when we derive energy estimates for the solution's higher derivatives.

The degenerate energy estimates featured in our proofs have some features in common with the foundational works [Alinhac 1999a; 1999b; 2001; 2002; Christodoulou 2007] on the formation of shock singularities in solutions to quasilinear wave equations in two or three spatial dimensions; see also the

[28] The key point is that since our solutions are such that $\partial_t \Psi < 0$ when $1 + \Psi = 0$, it follows that $1 + \Psi$ behaves, to first order, linearly in t near points where it vanishes.

follow-up works [Christodoulou and Miao 2014; Speck et al. 2016; Speck 2016; Ding et al. 2015a; 2015b; 2017] and the survey article [Holzegel et al. 2016]. In those works, the authors constructed a dynamic geometric coordinate system that degenerated in a precise fashion[29] as the shock formed. Consequently, relative to the geometric coordinates, the solution remains rather smooth,[30] which was a key fact used to control error terms. A crucial feature of the proofs is that the energy estimates[31] contained weights that vanished at the shock, which is in analogy with the vanishing of the weight $(1+\Psi)^P$ in (1E-1) at the degeneracy. A second crucial feature of the proofs of shock formation is that they relied on the fact that the weight has a *quantitatively strictly negative time derivative* in a neighborhood of points where it vanishes. This yields a critically important monotonic spacetime integral that is in analogy with the one, (1E-2), that we use to control various error terms in the present work.

We close this subsection by noting that the degeneracy that we encounter in our study of (1A-1a) is related to — but distinct from — a particular kind of absence of strict hyperbolicity that has been studied in the context of the compressible Euler equations for initial data satisfying the physical vacuum condition; see, for example, [Coutand et al. 2010; Coutand and Shkoller 2011; 2012; Jang and Masmoudi 2009; 2011]. The key difference between those works and ours is that in those works, the degeneracies occurred along the fluid-vacuum boundary in spacelike directions rather than a timelike one. In particular, the degeneracy was already present at time 0. More precisely, at time 0, the fluid density vanished at a certain rate, meaning that the density's derivative in the (spacelike) normal direction to the vacuum boundary satisfied a quantitative signed condition. It turns out that this condition yields a signed integral in the energy identities that is essential for closing the energy estimates. The signed integral exploited in those works is analogous to the integral (1E-2), but the integrals in the above papers were available because the solution's (spacelike) normal derivative had a sign, which is in contrast to the sign of the timelike derivative $\partial_t \Psi$ exploited in the present work. With the help of the signed integral, the authors of the above papers were able to prove degenerate energy estimates with weights that vanished at a certain rate in the normal direction to the vacuum boundary. Ultimately, these degenerate estimates allowed them to prove local well-posedness in Sobolev spaces with weights that degenerate at the fluid-vacuum boundary.

1G. *Notation.* In this subsection, we summarize some notation that we use throughout.

- $\{x^\alpha\}_{\alpha=0,1,2,3}$ denotes the standard rectangular coordinates on $\mathbb{R}^{1+3} = \mathbb{R}\times\mathbb{R}^3$, and $\partial_\alpha := \partial/\partial x^\alpha$ denotes the corresponding coordinate partial derivative vector fields; $x^0 \in \mathbb{R}$ is the time coordinate and $\underline{x} := (x^1, x^2, x^3) \in \mathbb{R}^3$ are the spatial coordinates.
- We often use the alternate notation $x^0 = t$ and $\partial_0 = \partial_t$.

[29] In essence, the authors straightened out the characteristics via a solution-dependent change of coordinates.

[30] The high-order geometric energies were allowed to blow up at the shock, which led to enormous technical complications in the proofs. Note that this possible high-order energy blowup is distinct from the formation of the shock singularity, which corresponds to the blowup of a low-order Cartesian coordinate partial derivative of the solution.

[31] There are many shock-formation results for solutions to quasilinear equations in one spatial dimension, with important contributions coming from Riemann [1860], Oleĭnik [1957], Lax [1957], Klainerman and Majda [1980], John [1974; 1981; 1984], and many others. However, those results are based exclusively on the method of characteristics and hence, unlike in the case of two or more spatial dimensions, the proofs do not rely on energy estimates.

- Greek "spacetime" indices such as α vary over 0, 1, 2, 3 and Latin "spatial" indices such as a vary over 1, 2, 3. We use primed indices, such as a', in the same way that we use their nonprimed counterparts. We use Einstein's summation convention in that repeated indices are summed over their respective ranges.
- We raise and lower indices with g^{-1} and g respectively (*not* with the Minkowski metric!).
- We sometimes omit the arguments of functions appearing in pointwise inequalities. For example, we sometimes write $|f| \leq C\mathring{\epsilon}$ instead of $|f(t,\underline{x})| \leq C\mathring{\epsilon}$.
- $\nabla^k\Psi$ denotes the array comprising all derivatives of order k of Ψ with respect to the rectangular spatial coordinate vector fields. We often use the alternate notation $\nabla\Psi$ in place of $\nabla^1\Psi$. For example, $\nabla\Psi := (\partial_1\Psi, \partial_2\Psi, \partial_3\Psi)$.
- $|\nabla^{\leq k}\Psi| := \sum_{k'=0}^{k} |\nabla^{k'}\Psi|$.
- $|\nabla^{[a,b]}\Psi| := \sum_{k'=a}^{b} |\nabla^{k'}\Psi|$.
- $H^N(\Sigma_t)$ denotes the standard Sobolev space of functions on Σ_t with corresponding norm

$$\|f\|_{H^N(\Sigma_t)} := \left\{ \sum_{a_1+a_2+a_3\leq N} \int_{\underline{x}\in\mathbb{R}^3} |\partial_1^{a_1}\partial_2^{a_2}\partial_3^{a_3} f(t,\underline{x})|^2\, d\underline{x} \right\}^{1/2}.$$

In the case $N = 0$, we use the notation "L^2" in place of "H^0".

- $L^\infty(\Sigma_t)$ denotes the standard Lebesgue space of functions on Σ_t with corresponding norm $\|f\|_{L^\infty(\Sigma_t)} := \operatorname{ess\,sup}_{\underline{x}\in\mathbb{R}^3} |f(t,\underline{x})|$.
- If A and B are two quantities, then we often write $A \lesssim B$ to indicate that "there exists a constant $C > 0$ such that $A \leq CB$".
- We sometimes write $\mathcal{O}(B)$ to denote a quantity A with the following property: there exists a constant $C > 0$ such that $|A| \leq C|B|$.

2. Assumptions on the initial data and bootstrap assumptions

In this short section, we state our assumptions on the data $(\Psi|_{\Sigma_0}, \partial_t\Psi|_{\Sigma_0}) := (\mathring{\Psi}, \mathring{\Psi}_0)$ for the model equation (1A-1a) and formulate bootstrap assumptions that are convenient for studying the solution. We also show that there exist data satisfying the assumptions.

2A. ***Assumptions on the data.*** We assume that the initial data are compactly supported and satisfy the size assumptions

$$\|\nabla^{\leq 4}\mathring{\Psi}\|_{L^\infty(\Sigma_0)} + \|\nabla^{[1,3]}\mathring{\Psi}_0\|_{L^\infty(\Sigma_0)} + \|\nabla^2\mathring{\Psi}\|_{H^4(\Sigma_0)} + \|\nabla^2\mathring{\Psi}_0\|_{H^3(\Sigma_0)} \leq \mathring{\epsilon}, \quad \|\mathring{\Psi}_0\|_{L^\infty(\Sigma_0)} \leq \mathring{\delta}, \quad \text{(2A-1)}$$

where $\mathring{\epsilon}$ and $\mathring{\delta}$ are two data-size parameters that we will discuss below (roughly, $\mathring{\epsilon}$ will have to be small for our proofs to close). Roughly speaking, in our analysis, we will approximately propagate the above size assumptions all the way up until the time of breakdown in hyperbolicity, except at the top derivative level. More precisely, we are not able to uniformly control the top-order spatial derivatives of Ψ in the norm $\|\cdot\|_{L^2(\Sigma_t)}$ up to the time of breakdown due to the presence of degenerate weights in our energy (see Definition 3.4).

Before we can proceed, we must first introduce the crucial parameter $\mathring{\delta}_*$ that controls the time of first breakdown in hyperbolicity; our analysis shows that for $\mathring{\epsilon}$ sufficiently small, the time of first breakdown is $\{1+\mathcal{O}(\mathring{\epsilon})\}\mathring{\delta}_*^{-1}$; see also Remark 2.2.

Definition 2.1 (the parameter that controls the time of breakdown in hyperbolicity). We define the data-dependent parameter $\mathring{\delta}_*$ as

$$\mathring{\delta}_* := \max_{\Sigma_0}[\mathring{\Psi}_0]_-. \tag{2A-2}$$

Remark 2.2 (the relevance of $\mathring{\delta}_*$). The solutions that we study are such that[32] $\mathring{\Psi} \sim 0$ and $\partial_t^2\Psi \sim 0$ (throughout the evolution). Hence, by the fundamental theorem of calculus, we have $\partial_t\Psi(t,\underline{x}) \sim \mathring{\Psi}_0(\underline{x})$ and $1+\Psi(t,\underline{x}) \sim 1+t\mathring{\Psi}_0(\underline{x})$. From this last expression, we see that $1+\Psi$ is expected to vanish for the first time at approximately $t=\mathring{\delta}_*^{-1}$. See Lemma 3.1 for the precise statements.

2B. *Bootstrap assumptions.* In proving our main results, we find it convenient to rely on a set of bootstrap assumptions, which we provide in this subsection.

The size of $T_{(\mathrm{Boot})}$: We assume that $T_{(\mathrm{Boot})}$ is a bootstrap time with

$$0 < T_{(\mathrm{Boot})} \le 2\mathring{\delta}_*^{-1}. \tag{2B-1}$$

The assumption (2B-1) gives us a sufficient margin of error to prove that finite-time degeneration of hyperbolicity occurs, as we explained in Remark 2.2.

Degeneracy has not yet occurred: We assume that for $(t,\underline{x}) \in [0,T_{(\mathrm{Boot})}) \times \mathbb{R}^3$ we have

$$0 < 1+\Psi(t,\underline{x}). \tag{2B-2}$$

L^∞ *bootstrap assumptions*: We assume that for $t \in [0,T_{(\mathrm{Boot})})$, we have

$$\|\Psi\|_{L^\infty(\Sigma_t)} \le 2\mathring{\delta}_*^{-1}\mathring{\delta} + \epsilon^{1/2}, \tag{2B-3a}$$

$$\|\partial_t\Psi\|_{L^\infty(\Sigma_t)} \le \mathring{\delta} + \epsilon^{1/2}, \tag{2B-3b}$$

$$\|\nabla^{[1,3]}\Psi\|_{L^\infty(\Sigma_t)} \le \epsilon, \quad \|\partial_t\nabla^{[1,3]}\Psi\|_{L^\infty(\Sigma_t)} \le \epsilon, \quad \|\partial_t^2\nabla^{\le 1}\Psi\|_{L^\infty(\Sigma_t)} \le \epsilon, \tag{2B-3c}$$

where $\epsilon > 0$ is a small bootstrap parameter; we describe our smallness assumptions in the next subsection.

Remark 2.3 (the solution remains compactly supported in space). From (2B-3a), we deduce that the wave speed $(1+\Psi)^{P/2}$ associated to (1A-1a) remains uniformly bounded from above for $(t,\underline{x}) \in [0,T_{(\mathrm{Boot})}) \times \mathbb{R}^3$. Hence, there exists a large, data-dependent ball $B \subset \mathbb{R}^3$ such that $\Psi(t,\underline{x})$ vanishes for $(t,\underline{x}) \in [0,T_{(\mathrm{Boot})}) \times B^c$, where B^c denotes the complement of B in $\mathbb{R}^3$.

2C. *Smallness assumptions.* For the rest of the article, when we say that "A is small relative to B," we mean that $B > 0$ and that there exists a continuous increasing function $f : (0,\infty) \to (0,\infty)$ such that $A \le f(B)$. In principle, the functions f could always be chosen to be polynomials with positive

[32] Here "$A \sim B$" imprecisely indicates that A is well-approximated by B.

coefficients or exponential functions. However, to avoid lengthening the paper, we typically do not specify the form of f.

Throughout the rest of the paper, we make the following relative smallness assumptions. We continually adjust the required smallness in order to close our estimates.

- The bootstrap parameter ϵ from Section 2B is small relative to $\mathring{\delta}^{-1}$, where $\mathring{\delta}$ is the data-size parameter from (2A-1).
- ϵ is small relative to the data-size parameter $\mathring{\delta}_*$ from (2A-2).

The first assumption will allow us to control error terms that, roughly speaking, are of size $\epsilon\mathring{\delta}^k$ for some integer $k \geq 0$. The second assumption is relevant because the expected degeneracy-formation time is approximately $\mathring{\delta}_*^{-1}$ (see Remark 2.2); the assumption will allow us to show that various error products featuring a small factor ϵ remain small for $t \leq 2\mathring{\delta}_*^{-1}$, which is plenty of time for us to show that $1+\Psi$ vanishes.

Finally, we assume that

$$\epsilon^{3/2} \leq \mathring{\epsilon} \leq \epsilon, \tag{2C-1}$$

where $\mathring{\epsilon}$ is the data smallness parameter from (2A-1).

2D. *Existence of data.* It is easy to construct data such that the parameters $\mathring{\epsilon}$, $\mathring{\delta}$, and $\mathring{\delta}_*$ satisfy the relative size assumptions stated in Section 2C. For example, we can start with *any* smooth compactly supported data $(\mathring{\Psi}, \mathring{\Psi}_0)$ such that $\min_{\mathbb{R}^3} \mathring{\Psi}_0 < 0$. We then consider the one-parameter family

$$\big({}^{(\lambda)}\mathring{\Psi}(\underline{x}), {}^{(\lambda)}\mathring{\Psi}_0(\underline{x})\big) := \big(\lambda^{-1}\mathring{\Psi}(\underline{x}), \mathring{\Psi}_0(\lambda^{-1}\underline{x})\big).$$

One can check that for $\lambda > 0$ sufficiently large, all of the size assumptions of Section 2C are satisfied. The proof relies on the simple scaling identities

$$\nabla^k\, {}^{(\lambda)}\mathring{\Psi}(\underline{x}) = \lambda^{-1}(\nabla^k\, \mathring{\Psi})(\underline{x}), \tag{2D-1a}$$

$$\nabla^k\, {}^{(\lambda)}\mathring{\Psi}_0(\underline{x}) = \lambda^{-k}(\nabla^k\, \mathring{\Psi}_0)(\lambda^{-1}\underline{x}) \tag{2D-1b}$$

and

$$\|\nabla^k\, {}^{(\lambda)}\mathring{\Psi}\|_{L^2(\Sigma_0)} = \lambda^{-1}\|\mathring{\Psi}\|_{L^2(\Sigma_0)}, \tag{2D-2a}$$

$$\|\nabla^k\, {}^{(\lambda)}\mathring{\Psi}_0\|_{L^2(\Sigma_0)} = \lambda^{3/2-k}\|\mathring{\Psi}_0\|_{L^2(\Sigma_0)}. \tag{2D-2b}$$

Remark 2.4 (degeneracy occurs for solutions launched by any appropriately rescaled nontrivial data). The discussion in Section 2D can easily be extended to show that if $\mathring{\Psi}_0$ is nontrivial, then one *always* generates data to which our results apply by considering the rescaled data $({}^{(\lambda)}\mathring{\Psi}, {}^{(\lambda)}\mathring{\Psi}_0)$ with λ sufficiently large. More precisely, if $\min_{\mathbb{R}^3} \mathring{\Psi}_0 = 0$, then we must have $\max_{\mathbb{R}^3} \mathring{\Psi}_0 > 0$; in this case, the degeneracy in the solution generated by the rescaled data occurs in the past rather than the future.

3. A priori estimates

In this section, we use the data-size assumptions and the bootstrap assumptions of Section 2 to derive a priori estimates for the solution. This is the main step in the proof our results.

3A. ***Conventions for constants.*** In our estimates, the explicit constants $C > 0$ and $c > 0$ are free to vary from line to line. *These constants are allowed to depend on the data-size parameters* $\mathring{\delta}$ *and* $\mathring{\delta}_*^{-1}$ *from Section 2A*. However, the constants can be chosen to be independent of the parameters $\mathring{\epsilon}$ and ϵ whenever $\mathring{\epsilon}$ and ϵ are sufficiently small relative to $\mathring{\delta}^{-1}$ and $\mathring{\delta}_*$ in the sense described in Section 2C. For example, under our conventions, we have that $\mathring{\delta}_*^{-2}\epsilon = \mathcal{O}(\epsilon)$.

3B. ***Pointwise estimates.*** In this subsection, we derive pointwise estimates for Ψ and the inhomogeneous terms in the commuted wave equation.

We start with a simple lemma that provides sharp pointwise estimates for Ψ and $\partial_t \Psi$.

Lemma 3.1 (pointwise estimates for Ψ and $\partial_t \Psi$). *Under the data-size assumptions of Section 2A, the bootstrap assumptions of Section 2B, and the smallness assumptions of Section 2C, the following pointwise estimates hold for* $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$:

$$\Psi(t, \underline{x}) = t\mathring{\Psi}_0(\underline{x}) + \mathcal{O}(\epsilon), \tag{3B-1a}$$

$$\partial_t \Psi(t, \underline{x}) = \mathring{\Psi}_0(\underline{x}) + \mathcal{O}(\epsilon). \tag{3B-1b}$$

Proof. To derive (3B-1b), we first use the bootstrap assumptions to deduce $\|(1+\Psi)^P \Delta\Psi\|_{L^\infty(\Sigma_t)} \leq C\epsilon$. Hence, from (1A-1a), we deduce the pointwise bound $|\partial_t^2\Psi| \leq C\epsilon$. From this estimate and the fundamental theorem of calculus, we conclude the desired bound (3B-1b). The bound (3B-1a) then follows from the fundamental theorem of calculus, (3B-1b), and the small-data bound $\|\mathring{\Psi}\|_{L^\infty(\Sigma_0)} \leq \mathring{\epsilon} \leq \epsilon$. □

The next proposition captures the monotonicity that is present at points where $1 + \Psi$ is small. It is of critical importance for the energy estimates.

Proposition 3.2 (monotonicity near the degeneracy). *Under the data-size assumptions of Section 2A, the bootstrap assumptions of Section 2B, and the smallness assumptions of Section 2C, the following statement holds for* $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$:

$$\Psi(t, \underline{x}) \leq -\tfrac{1}{2} \quad \Longrightarrow \quad \partial_t \Psi(t, \underline{x}) \leq -\tfrac{1}{8}\mathring{\delta}_*, \tag{3B-2}$$

where $\mathring{\delta}_*$ *is the data-dependent parameter from Definition 2.1.*

Proof. To prove (3B-2), we first use the estimates (3B-1a) and (3B-1b) to deduce that $\Psi(t, \underline{x}) = t\,\partial_t\Psi(t, \underline{x}) + \mathcal{O}(\epsilon)$. Hence, if $\Psi(t, \underline{x}) \leq -\frac{1}{2}$, then $t\,\partial_t\Psi(t, \underline{x}) \leq -\frac{1}{4}$. Recalling that $0 \leq t < 2\mathring{\delta}_*^{-1}$, see (2B-1), we conclude (3B-2). □

We now derive pointwise estimates for the inhomogeneous terms in the commuted wave equation.

Lemma 3.3 (pointwise estimates for the inhomogeneous terms). *Let* Ψ *be a solution to the wave equation (1A-1a). For* $k = 2, 3, 4, 5$ *and* $P = 1, 2$, *consider following wave equation,*[33] *obtained by commuting*

[33]We do not bother to state the precise form of $F^{(k)}$ here.

(1A-1a) *with* ∇^k:

$$-\partial_t^2 \nabla^k \Psi + (1+\Psi)^P \Delta \nabla^k \Psi = F^{(k)}. \tag{3B-3}$$

Under the data-size assumptions of Section 2A, the bootstrap assumptions of Section 2B, and the smallness assumptions of Section 2C, the following pointwise estimates hold for $(t, \underline{x}) \in [0, T_{(\mathrm{Boot})}) \times \mathbb{R}^3$:

$$|F^{(k)}| \le C\epsilon |\nabla^{[2,k+1]}\Psi| \qquad (P=1), \tag{3B-4}$$

$$|F^{(k)}| \le C\epsilon (1+\Psi)|\nabla^{k+1}\Psi| + \epsilon|\nabla^{[2,k]}\Psi| \quad (P=2). \tag{3B-5}$$

Proof. We first consider the case $P = 1$. Commuting (1A-1a) with ∇^k, we compute that

$$|F^{(k)}| \le C \sum_{\substack{a+b\le k+2 \\ 1\le a,\, 2\le b\le k+1}} |\nabla^a \Psi||\nabla^b \Psi|.$$

The desired estimate (3B-4) then follows as a simple consequence of this bound and the bootstrap assumptions. The proof of (3B-5) is similar, the difference being that when $P = 2$, we have the bound

$$|F^{(k)}| \le C\epsilon(1+\Psi)|\nabla^{k+1}\Psi| + C \sum_{\substack{a+b\le k+2 \\ 1\le a\le k,\, 2\le b\le k}} |\nabla^a \Psi||\nabla^b \Psi|. \qquad \square$$

3C. *Energy estimates.* We will use the following energy, which corresponds to between two and five commutations of the wave equation with ∇, in order to control solutions.

Definition 3.4 (the energy). We define

$$\mathcal{E}_{[2,5]}(t) := \sum_{k'=2}^{5} \int_{\Sigma_t} |\partial_t \nabla^{k'}\Psi|^2 + (1+\Psi)^P |\nabla\nabla^{k'}\Psi|^2 + |\nabla^{k'}\Psi|^2 \, d\underline{x}. \tag{3C-1}$$

We now provide the basic energy identity satisfied by solutions.

Lemma 3.5 (basic energy identity). *Let* Ψ *be a solution to the wave equation* (1A-1a). *Let* $\mathcal{E}_{[2,5]}$ *be the energy defined in* (3C-1) *and let* $F^{(k)}$ *be the inhomogeneous term from* (3B-3). *Then for* $P = 1, 2$, *we have the energy identity*

$$\begin{aligned}
\mathcal{E}_{[2,5]}(t) = \mathcal{E}_{[2,5]}(0) &+ P\sum_{k'=2}^{5} \int_{s=0}^{t} \int_{\Sigma_s} (\partial_t \Psi)(1+\Psi)^{P-1} |\nabla\nabla^{k'}\Psi|^2 \, d\underline{x}\, ds \\
&- 2P \sum_{k'=2}^{5} \int_{s=0}^{t} \int_{\Sigma_s} (1+\Psi)^{P-1} (\nabla\Psi)\cdot(\nabla\nabla^{k'}\Psi)(\partial_t\nabla^{k'}\Psi)\, d\underline{x}\, ds \\
&- 2\sum_{k'=2}^{5} \int_{s=0}^{t} \int_{\Sigma_s} (\partial_t\nabla^{k'}\Psi) F^{(k')}\, d\underline{x}\, ds + 2\sum_{k'=2}^{5} \int_{s=0}^{t} \int_{\Sigma_s} (\partial_t\nabla^{k'}\Psi)(\nabla^{k'}\Psi)\, d\underline{x}\, ds.
\end{aligned} \tag{3C-2}$$

Proof. The identity (3C-2) is standard and can verified by taking the time derivative of both sides of (3C-1), using (3B-3) for substitution, integrating by parts over Σ_t, and then integrating the resulting identity in time. $\square$

With the help of Lemma 3.5, we now derive an inequality satisfied by the energy $\mathcal{E}_{[2,5]}$.

Proposition 3.6 (integral inequality for the energy). *Let $\mathcal{E}_{[2,5]}$ be the energy defined in (3C-1). Let $\mathbf{1}_{\{-1<\Psi\leq-1/2\}}$ be the characteristic function of the spacetime subset $\{(t,\underline{x}) \mid -1<\Psi(t,\underline{x})\leq-\frac{1}{2}\}$ and define $\mathbf{1}_{\{-1/2<\Psi\}}$ analogously. Let $\mathring{\delta}_*$ be the data-size parameter from Definition 2.1. Under the data-size assumptions of Section 2A, the bootstrap assumptions of Section 2B, and the smallness assumptions of Section 2C, the following integral inequality holds for $P=1,2$ and $t\in[0,T_{(\mathrm{Boot})})$:*

$$\begin{aligned}
\mathcal{E}_{[2,5]}(t) &+ \tfrac{1}{8}P\mathring{\delta}_*\sum_{k'=2}^{5}\int_{s=0}^{t}\int_{\Sigma_s}\mathbf{1}_{\{-1<\Psi\leq-1/2\}}(1+\Psi)^{P-1}|\nabla\nabla^{k'}\Psi|^2\,d\underline{x}\,ds \\
&\leq \mathcal{E}_{[2,5]}(0) + C\sum_{k'=2}^{5}\int_{s=0}^{t}\int_{\Sigma_s}|\partial_t\nabla^{k'}\Psi|^2\,d\underline{x}\,ds + C\sum_{k'=2}^{5}\int_{s=0}^{t}\int_{\Sigma_s}|\nabla^{k'}\Psi|^2\,d\underline{x}\,ds \\
&\quad + C\sum_{k'=2}^{5}\int_{s=0}^{t}\int_{\Sigma_s}\mathbf{1}_{\{-1/2<\Psi\}}(1+\Psi)^{P-1}|\nabla\nabla^{k'}\Psi|^2\,d\underline{x}\,ds \\
&\quad + C\epsilon\sum_{k'=2}^{5}\int_{s=0}^{t}\int_{\Sigma_s}\mathbf{1}_{\{-1<\Psi\leq-1/2\}}(1+\Psi)^{2(P-1)}|\nabla\nabla^{k'}\Psi|^2\,d\underline{x}\,ds. \qquad (3C\text{-}3)
\end{aligned}$$

Proof. We must bound the terms appearing in the energy identity (3C-2). We give the proof only for the case $P=1$ since the case $P=2$ can be handled using similar arguments. We start by bounding the first sum on the right-hand side of (3C-2); this is the only term that requires careful treatment. We split the integration domain $[0,t]\times\mathbb{R}^3$ into two pieces: a piece in which $-1<\Psi\leq-\frac{1}{2}$ and a piece in which $\Psi>-\frac{1}{2}$. To bound the first piece, we use the estimate (3B-2) to deduce that whenever $-1<\Psi\leq-\frac{1}{2}$, the integrand satisfies $(\partial_t\Psi)|\nabla\nabla^{k'}\Psi|^2\leq-\frac{1}{8}\mathring{\delta}_*|\nabla\nabla^{k'}\Psi|^2$. We can therefore bring all of the corresponding integrals over to the left-hand side of (3C-3) as *positive* integrals, as is indicated there. To bound the second piece, we use the estimate (3B-1b) to bound $\partial_t\Psi$ in L^∞ by $\leq C$, which allows us to bound the integrand by $C|\nabla\nabla^{k'}\Psi|^2$. It follows that since $\Psi>-\frac{1}{2}$ (by assumption), the integrals under consideration are bounded by the third sum on the right-hand side of (3C-3).

To bound the second sum on the right-hand side of (3C-2), we first use the bootstrap assumption (2B-3c) to bound the integrand factor $\nabla\Psi$ in L^∞ by $\leq\epsilon$. Thus, using Young's inequality, we bound the terms under consideration by the sum of the first, third, and fourth sums on the right-hand side of (3C-3).

To bound the third sum on the right-hand side of (3C-2), we use (3B-4) and Young's inequality to bound the integrand by $C\epsilon\sum_{k'=2}^{5}|\partial_t\nabla^{k'}\Psi|^2+C\epsilon\sum_{k'=2}^{6}|\nabla^{k'}\Psi|^2$. It is easy to see that the corresponding integrals are bounded by the right-hand side of (3C-3).

Finally, using Young's inequality, we bound last sum on the right-hand side of (3C-2) by the first two sums on the right-hand side of (3C-3). □

In the next proposition, we use Proposition 3.6 to derive our main a priori energy estimates. We also derive improvements of the bootstrap assumptions (2B-3).

Proposition 3.7 (a priori energy estimates and improvement of the bootstrap assumptions). *Let $\mathring{\delta}_*$ be the data-size parameter from (2A-2) and let $\mathbf{1}_{\{-1<\Psi\leq-1/2\}}$ be the characteristic function of the spacetime* *subset* $\left\{(t,\underline{x}) \mid -1 < \Psi(t,\underline{x}) \leq -\frac{1}{2}\right\}$. *There exists a constant* $C > 0$ *such that under the data-size assumptions of Section 2A, the bootstrap assumptions of Section 2B, and the smallness assumptions of Section 2C, the following a priori energy estimate holds for* $P = 1, 2$ *and* $t \in [0, T_{(\mathrm{Boot})})$:

$$\mathcal{E}_{[2,5]}(t) + \tfrac{1}{16} P \mathring{\delta}_* \sum_{k'=2}^{5} \int_{s=0}^{t} \int_{\Sigma_s} \mathbf{1}_{\{-1<\Psi\leq -1/2\}} (1+\Psi)^{P-1} |\nabla\nabla^{k'}\Psi|^2 \, d\underline{x}\, ds \leq C\mathring{\epsilon}^2. \tag{3C-4}$$

Moreover, we have the following estimates, which are a strict improvement of the bootstrap assumptions (2B-3) for $\mathring{\epsilon}$ *sufficiently small*:

$$\|\Psi\|_{L^\infty(\Sigma_t)} \leq 2\mathring{\delta}_*^{-1}\mathring{\delta} + C\mathring{\epsilon}, \tag{3C-5a}$$

$$\|\partial_t\Psi\|_{L^\infty(\Sigma_t)} \leq \mathring{\delta} + C\mathring{\epsilon}, \tag{3C-5b}$$

$$\|\nabla^{[1,3]}\Psi\|_{L^\infty(\Sigma_t)} \leq C\mathring{\epsilon}, \quad \|\partial_t\nabla^{[1,3]}\Psi\|_{L^\infty(\Sigma_t)} \leq C\mathring{\epsilon}, \quad \|\partial_t^2\nabla^{\leq 1}\Psi\|_{L^\infty(\Sigma_t)} \leq C\mathring{\epsilon}. \tag{3C-5c}$$

Proof. We give the proof only for the case $P = 1$ since the case $P = 2$ can be handled using similar arguments. To obtain (3C-4), we first note that for ϵ sufficiently small relative to $\mathring{\delta}_*$, we can absorb the last sum on the right-hand side of (3C-3) into the second term on the left-hand side, which at most reduces its coefficient from $\frac{1}{8}\mathring{\delta}_*$ to $\frac{1}{16}\mathring{\delta}_*$, as is stated on the left-hand side of (3C-4). Moreover, since $\mathbf{1}_{\{-1/2<\Psi\}} \leq C\mathbf{1}_{\{-1/2<\Psi\}}(1+\Psi)$, we have the bound $\int_{\Sigma_s}\mathbf{1}_{\{-1/2<\Psi\}}|\nabla\nabla^{k'}\Psi|^2\,d\underline{x} \leq C\mathcal{E}_{[2,5]}(s)$ for the terms in the next-to-last sum on the right-hand side of (3C-3). The remaining Σ_s integrals are easily seen to be bounded in magnitude by $\leq C\mathcal{E}_{[2,5]}(s)$. Also using the data bound $\mathcal{E}_{[2,5]}(0) \leq C\mathring{\epsilon}^2$, which follows from our data assumptions (2A-1), we obtain

$$\mathcal{E}_{[2,5]}(t) + \tfrac{1}{16}\mathring{\delta}_* \sum_{k'=2}^{5} \int_{s=0}^{t}\int_{\Sigma_s} \mathbf{1}_{\{-1<\Psi\leq -1/2\}}|\nabla\nabla^{k'}\Psi|^2\,d\underline{x}\,ds \leq C\mathring{\epsilon}^2 + c\int_{s=0}^{t}\mathcal{E}_{[2,5]}(s)\,ds. \tag{3C-6}$$

From (3C-6), Grönwall's inequality, and (2B-1), we conclude

$$\mathcal{E}_{[2,5]}(t) + \tfrac{1}{16}\mathring{\delta}_* \sum_{k'=2}^{5} \int_{s=0}^{t}\int_{\Sigma_s} \mathbf{1}_{\{-1<\Psi\leq -1/2\}}|\nabla\nabla^{k'}\Psi|^2\,d\underline{x}\,ds \leq C\exp(ct)\mathring{\epsilon}^2 \leq C\mathring{\epsilon}^2,$$

which is the desired bound (3C-4).

The estimates (3C-5c) for $\nabla^{[2,3]}\Psi$ and $\partial_t\nabla^{[2,3]}\Psi$ then follow from (3C-4) and the Sobolev embedding result $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. Next, we take up to one ∇ derivative of (1A-1a) and use the already obtained L^∞ estimates and the bootstrap assumptions to obtain the bound $\|\partial_t^2\nabla^{\leq 1}\Psi\|_{L^\infty(\Sigma_t)} \leq C\mathring{\epsilon}$ stated in (3C-5c). Using this bound, the fundamental theorem of calculus, and the data assumptions $\|\nabla\mathring{\Psi}_0\|_{L^\infty(\Sigma_0)} \leq \mathring{\epsilon}$ and $\|\mathring{\Psi}_0\|_{L^\infty(\Sigma_0)} \leq \mathring{\delta}$, we obtain the bounds $\|\partial_t\nabla\Psi\|_{L^\infty(\Sigma_t)} \leq C\mathring{\epsilon}$ and $\|\partial_t\Psi\|_{L^\infty(\Sigma_t)} \leq \mathring{\delta} + C\mathring{\epsilon}$, which in particular yields (3C-5b). Using a similar argument based on the already obtained bound $\|\partial_t\nabla\Psi\|_{L^\infty(\Sigma_t)} \leq C\mathring{\epsilon}$, we deduce $\|\nabla\Psi\|_{L^\infty(\Sigma_t)} \leq C\mathring{\epsilon}$. Similarly, from the already obtained bound $\|\partial_t\Psi\|_{L^\infty(\Sigma_t)} \leq \mathring{\delta} + C\mathring{\epsilon}$, the fundamental theorem of calculus, the initial data bound $\|\mathring{\Psi}\|_{L^\infty(\Sigma_0)} \leq \mathring{\epsilon}$, and the fact that $0 \leq t < T_{(\mathrm{Boot})} \leq 2\mathring{\delta}_*^{-1}$, we deduce $\|\Psi\|_{L^\infty(\Sigma_t)} \leq 2\mathring{\delta}_*^{-1}\mathring{\delta} + C\mathring{\epsilon}$, that is, (3C-5a). □

4. The main results

We now derive our main results, namely Theorem 4.1 and Proposition 4.2.

Theorem 4.1 (stable finite-time degeneration of hyperbolicity). *Let* $(\mathring{\Psi}, \mathring{\Psi}_0) \in H^6(\mathbb{R}^3) \times H^5(\mathbb{R}^3)$ *be compactly supported initial data (1A-1b) for the wave equation (1A-1a) with* $P \in \{1, 2\}$ *and let* Ψ *denote the corresponding solution. Let*

$$\mathcal{M}(t) := \min_{(s,\underline{x}) \in [0,t]\times\mathbb{R}^3} \{1 + \Psi(s, \underline{x})\}. \tag{4-1}$$

Let $\mathring{\epsilon}$, $\mathring{\delta}$, *and* $\mathring{\delta}_*$ *be the data-size parameters from (2A-1)–(2A-2) and assume that* $\mathring{\delta} > 0$ *and* $\mathring{\delta}_* > 0$. *Note that if* $\mathring{\epsilon}$ *is sufficiently small, then* $\mathcal{M}(0) = 1 + \mathcal{O}(\mathring{\epsilon}) > 0$. *If* $\mathring{\epsilon}$ *is sufficiently small relative to* $\mathring{\delta}^{-1}$ *and* $\mathring{\delta}_*$ *in the sense described in Section 2C, then the following conclusions hold.*

Breakdown in hyperbolicity precisely at time $T_\star$: *There exists a* $T_\star > 0$ *satisfying*

$$T_\star = \{1 + \mathcal{O}(\mathring{\epsilon})\}\mathring{\delta}_*^{-1} \tag{4-2}$$

such that the solution exists classically on the slab $[0, T_\star) \times \mathbb{R}^3$ *and such that the following inequality holds for* $0 \le t < T_\star$:

$$\mathcal{M}(t) > 0. \tag{4-3}$$

Moreover,

$$\lim_{t \uparrow T_\star} \mathcal{M}(t) = 0. \tag{4-4}$$

Regularity properties on $[0, T_\star) \times \mathbb{R}^3$: *On the slab* $[0, T_\star) \times \mathbb{R}^3$, *the solution satisfies the energy bounds (3C-4), the* L^∞ *estimates (3C-5) and the pointwise estimates (3B-1) (with* $C\mathring{\epsilon}$ *on the right-hand side in place of* ϵ *in the latter two estimates). Moreover,*

$$\Psi \in C\big([0, T_\star), H^6(\mathbb{R}^3)\big) \cap L^\infty\big([0, T_\star), H^5(\mathbb{R}^3)\big), \tag{4-5a}$$

$$\partial_t \Psi \in C\big([0, T_\star), H^5(\mathbb{R}^3)\big) \cap L^\infty\big([0, T_\star), H^5(\mathbb{R}^3)\big). \tag{4-5b}$$

Regularity properties on $[0, T_\star] \times \mathbb{R}^3$: Ψ *extends to a classical solution on the closed slab* $[0, T_\star] \times \mathbb{R}^3$ *enjoying the following regularity properties: for any* $N < 5$, *we have*

$$\Psi \in C\big([0, T_\star], H^5(\mathbb{R}^3)\big), \tag{4-6a}$$

$$\partial_t \Psi \in C\big([0, T_\star], H^N(\mathbb{R}^3)\big) \cap L^\infty\big([0, T_\star], H^5(\mathbb{R}^3)\big). \tag{4-6b}$$

In particular, the L^∞ *estimates (3C-5) and the pointwise estimates (3B-1) (with* $C\mathring{\epsilon}$ *on the right-hand side in place of* ϵ *in these estimates) hold on* $[0, T_\star] \times \mathbb{R}^3$. *Moreover, in the case*[34] $P = 1$, *we have*

$$\Psi \in L^2\big([0, T_\star], H^6(\mathbb{R}^3)\big). \tag{4-7}$$

Description of the breakdown along $\Sigma_{T_\star}$: *The set*

$$\Sigma_{T_\star}^{\text{Degen}} := \{q \in \Sigma_{T_\star} \mid 1 + \Psi(q) = 0\} \tag{4-8}$$

[34] In the proof of the theorem, we clarify why our proof of (4-7) relies on the assumption $P = 1$.

is nonempty and we have the estimate

$$\sup_{\Sigma_{T_\star}^{\text{Degen}}} \partial_t \Psi \le -\tfrac{1}{8}\mathring{\delta}_*. \tag{4-9}$$

In particular, in the case $P = 1$, the hyperbolicity of the wave equation breaks down on $\Sigma_{T_\star}^{\text{Degen}}$ in the following sense: if $q \in \Sigma_{T_\star}^{\text{Degen}}$, then any C^1 extension of Ψ to any spacetime neighborhood Ω_q containing q necessarily contains points p such that (1A-1a) is elliptic at $\Psi(p)$. In contrast, in the case $P = 2$, only the ***strict*** *hyperbolicity (in the sense of Footnote 7) of (1A-1a) breaks down for the first time at $T_\star$.*

Proof. Let $T_\star$ be the supremum of the set of times $T_{(\text{Boot})}$ subject to inequality (2B-1) and such that the solution exists classically on the slab $[0, T_{(\text{Boot})}) \times \mathbb{R}^3$, has the same Sobolev regularity as the initial data, and satisfies the bootstrap assumptions of Section 2B with $\epsilon := C_*\mathring{\epsilon}$, where C_* is described just below. By standard local well-posedness, see for example [Hörmander 1997], if $\mathring{\epsilon}$ is sufficiently small and $C_* > 1$ is sufficiently large, note that this is consistent with the assumed inequalities (2C-1), then $T_\star > 0$. Next, we state the following standard continuation result, which can be proved, for example, by making straightforward modifications to the proof of [Hörmander 1997, Theorem 6.4.11]: if $T_\star < 2\mathring{\delta}_*^{-1}$, then the solution can be classically continued to a slab of the form $[0, T_\star + \triangle] \times \mathbb{R}^3$ (for some $\triangle > 0$ with $T_\star + \Delta < 2\mathring{\delta}_*^{-1}$) on which the solution has the same Sobolev regularity as the initial data and on which the bootstrap assumptions hold, as long as the bootstrap inequalities (2B-3) are strictly satisfied for $t \in [0, T_\star)$ and $\inf_{t\in[0,T_\star)} \mathcal{M}(t) > 0$. It follows that either (i) $T_\star = 2\mathring{\delta}_*^{-1}$, (ii) that the bootstrap inequalities (2B-3) are saturated at some time $t \in [0, T_\star)$, or (iii) that $\inf_{t\in[0,T_\star)} \mathcal{M}(t) = 0$. If C_* is chosen to be sufficiently large and $\mathring{\epsilon}$ is chosen to be sufficiently small, then the a priori estimates (3C-5) ensure that the bootstrap inequalities (2B-3) are in fact strictly satisfied for $t \in [0, T_\star)$. Moreover, from (2A-2) and the estimate (3B-1a) (which is now known to hold with ϵ replaced by $C\mathring{\epsilon}$), we see that $\min_{\Sigma_t}(1+\Psi) = 1 - \mathring{\delta}_* t + \mathcal{O}(\mathring{\epsilon})$ and thus, in fact, case (iii) occurs with $T_\star = \mathring{\delta}_*^{-1} + \mathcal{O}(\mathring{\epsilon}) = \{1 + \mathcal{O}(\mathring{\epsilon})\}\mathring{\delta}_*^{-1} < 2\mathring{\delta}_*^{-1}$ and $\lim_{t\uparrow T_\star} \mathcal{M}(t) = 0$. From the above reasoning, we easily deduce that the energy bound (3C-4) holds for $t \in [0, T_\star)$ and, since the a priori estimates (3C-5) show that the bootstrap assumptions hold with ϵ replaced by $C\mathring{\epsilon}$, that the pointwise estimates (3B-1) hold for $(t, \underline{x}) \in [0, T_\star) \times \mathbb{R}^3$ with ϵ replaced by $C\mathring{\epsilon}$.

In the rest of this proof, we sometimes silently use the simple facts that $\Psi, \partial_t\Psi \in L^\infty\big([0, T_\star), H^1(\mathbb{R}^3)\big)$. These facts do not follow from the energy estimates (3C-4) (since the energy does not directly control Ψ, $\partial_t\Psi$, $\nabla\Psi$ or $\nabla\partial_t\Psi$), but instead follow from (3C-5) and the compactly supported (in space) nature of the solution. To proceed, we easily conclude from the definition of $\mathcal{E}_{[2,5]}(t)$ and the fact that the estimates (3C-4) and (3C-5b)–(3C-5c) hold on $[0, T_\star)$ that $\partial_t\Psi \in L^\infty\big([0, T_\star], H^5(\mathbb{R}^3)\big)$, as is stated in (4-6b). Also, this fact trivially implies the corresponding statement in (4-5b), where the closed time interval is replaced with $[0, T_\star)$. The facts that $\Psi \in C\big([0, T_\star), H^6(\mathbb{R}^3)\big)$ and $\partial_t\Psi \in C\big([0, T_\star), H^5(\mathbb{R}^3)\big)$, as is stated in (4-5a) and (4-5b), are standard results that can be proved using energy estimates and simple facts from functional analysis. We omit the details and instead refer the reader to [Speck 2008, Section 2.7.5]. We note that in proving these "soft" facts, it is important that $\mathcal{M}(t) > 0$ for $t \in [0, T_\star)$, which implies that standard techniques for strictly hyperbolic equations can be used. To obtain the conclusion

$\Psi \in L^2([0,T_\star], H^6(\mathbb{R}^3))$ in the case $P=1$, as is stated in (4-7), we simply use the fact that the energy bounds (3C-4) hold on $[0,T_\star)$ (including the bound for the spacetime integral term on the left-hand side). Note that the same argument does not apply in the case $P=2$ since in this case, the spacetime integral on the left-hand side of (3C-4) features the degenerate weight $1+\Psi$. The fact that $\Psi \in C([0,T_\star], H^5(\mathbb{R}^3))$, as is stated in (4-6a), is a simple consequence of the fundamental theorem of calculus, the fact that $\mathring{\Psi} \in H^6(\Sigma_0)$, and the already proven fact that $\partial_t \Psi \in L^\infty([0,T_\star], H^5(\mathbb{R}^3))$. To obtain that for $N<5$ we have $\partial_t \Psi \in C([0,T_\star], H^N(\mathbb{R}^3))$, as is stated in (4-6b), we first use (1A-1a), the fact that $\Psi \in C([0,T_\star], H^5(\mathbb{R}^3))$, and the standard Sobolev calculus to obtain $\partial_t^2 \Psi \in C([0,T_\star], H^3(\mathbb{R}^3))$. Hence, from the fundamental theorem of calculus and the fact that $\mathring{\Psi}_0 \in H^5(\Sigma_0)$, we obtain $\partial_t \Psi \in C([0,T_\star], H^3(\mathbb{R}^3))$. From this fact and the fact $\partial_t \Psi \in L^\infty([0,T_\star], H^5(\mathbb{R}^3))$, we obtain, by interpolating[35] between L^2 and H^5, the desired conclusion $\partial_t \Psi \in C([0,T_\star], H^N(\mathbb{R}^3))$.

Next, we note that the arguments given in the first paragraph of this proof imply that $\mathcal{M}$ extends as a continuous decreasing function defined for $t \in [0,T_\star]$ such that $\mathcal{M}(t) > 0$ for $t \in [0,T_\star)$ and such that $\mathcal{M}(T_\star) = 0$. Also using that $\Psi \in C([0,T_\star], H^5(\mathbb{R}^3)) \subset C([0,T_\star], C^3(\mathbb{R}^3))$, we deduce, in view of definitions (4-1) and (4-8), that $\Sigma_{T_\star}^{\text{Degen}}$ is nonempty. Moreover, from (3B-2) and the fact that $\partial_t \Psi \in C([0,T_\star], H^{4.9}(\mathbb{R}^3)) \subset C([0,T_\star], C^3(\mathbb{R}^3))$, we find that the estimate (4-9) holds. In addition, in view of (4-9), we see that in the case $P=1$, if $q \in \Sigma_{T_\star}^{\text{Degen}}$, then any C^1 extension of Ψ to a neighborhood of q contains points p such that $1+\Psi(p) < 0$, which renders (1A-1a) elliptic. This is in contrast to the case $P=2$ in the sense that (1A-1a) is hyperbolic for all values of Ψ. □

Theorem 4.1 yields that Ψ remains regular, all the way up to the time $T_\star$. However, as the next proposition shows, a type of invariant blowup does in fact occur at time $T_\star$ in both the cases $P=1,2$. The blowup is tied to the Riemann curvature of the metric g.

Proposition 4.2 (blowup of the Kretschmann scalar). *Let $g = g(\Psi)$ denote the spacetime metric defined in (1A-2) and let $\operatorname{Riem}(g)$ denote the Riemann curvature tensor[36] of g. Under the assumptions and conclusions of Theorem 4.1, we have the following estimate for the Kretschmann scalar $\operatorname{Riem}(g)^{\alpha\beta\gamma\delta}\operatorname{Riem}(g)_{\alpha\beta\gamma\delta}$ on $[0,T_\star]\times\mathbb{R}^3$:*

$$\operatorname{Riem}(g)^{\alpha\beta\gamma\delta}\operatorname{Riem}(g)_{\alpha\beta\gamma\delta} = \frac{15}{2}\frac{(\partial_t\Psi)^4}{(1+\Psi)^4} + \mathcal{O}\left(\frac{\mathring{\epsilon}}{(1+\Psi)^3}\right) \quad (P=1), \tag{4-10a}$$

$$\operatorname{Riem}(g)^{\alpha\beta\gamma\delta}\operatorname{Riem}(g)_{\alpha\beta\gamma\delta} = 60\frac{(\partial_t\Psi)^4}{(1+\Psi)^4} + \mathcal{O}\left(\frac{\mathring{\epsilon}}{(1+\Psi)^3}\right) \quad (P=2). \tag{4-10b}$$

In particular, $\operatorname{Riem}(g)^{\alpha\beta\gamma\delta}\operatorname{Riem}(g)_{\alpha\beta\gamma\delta}$ is bounded for $0 \le t < T_\star$, while by (3B-2) and (4-10) at time $T_\star$, $\operatorname{Riem}(g)^{\alpha\beta\gamma\delta}\operatorname{Riem}(g)_{\alpha\beta\gamma\delta}$ blows up precisely on the subset $\Sigma_{T_\star}^{\text{Degen}}$ defined in (4-8).

Proof. We prove only (4-10a) since the proof of (4-10b) is similar. The identities that we state in this proof rely on the form of the metric (1A-2). We first note the following simple decomposition formula,

[35] Here, we mean the following standard inequality: if $f \in H^5(\Sigma_t)$ and $0 \le N \le 5$, then there exists a constant $C_N > 0$ such that $\|f\|_{H^N(\Sigma_t)} \le C_N \|f\|_{L^2(\Sigma_t)}^{1-N/5} \|f\|_{H^5(\Sigma_t)}^{N/5}$.

[36] Our sign convention for curvature is $D_\alpha D_\beta X_\mu - D_\beta D_\alpha X_\mu = \operatorname{Riem}(g)_{\alpha\beta\mu\nu} X^\nu$, where D denotes the Levi-Civita connection of g and X is an arbitrary smooth vector field.

which relies on the standard symmetry and antisymmetry properties of the Riemann curvature tensor:

$$\mathrm{Riem}(g)^{\alpha\beta\gamma\delta}\,\mathrm{Riem}(g)_{\alpha\beta\gamma\delta} = \mathrm{Riem}(g)_{ab}^{\ \ cd}\,\mathrm{Riem}(g)_{cd}^{\ \ ab} + 4\,\mathrm{Riem}(g)_{a0}^{\ \ c0}\,\mathrm{Riem}(g)_{c0}^{\ \ a0} - 4 g_{cc'} g_{dd'} g^{bb'}\,\mathrm{Riem}(g)_{0b}^{\ \ cd}\,\mathrm{Riem}(g)_{0b'}^{\ \ c'd'}. \tag{4-11}$$

Next, we let $\underline{g}$ denote the first fundamental form of Σ_t relative to g; that is, $\underline{g}_{ij} = g_{ij} = (1+\Psi)^{-P}\delta_{ij}$ for $i, j = 1, 2, 3$, where δ_{ij} denotes the standard Kronecker delta. We also let (recalling that $P = 1$ in the present context)

$$k^i_{\ j} := -(\underline{g}^{-1})^{ia}\big(\tfrac{1}{2}\mathcal{L}_{\partial_t}\underline{g}_{aj}\big) = \tfrac{1}{2}\{\partial_t \ln(1+\Psi)\}\delta^i_{\ j} \tag{4-12}$$

denote the $\big(\text{type } \binom{1}{1}\big)$ second fundamental form of Σ_t relative to g, where $\mathcal{L}_{\partial_t}$ denotes Lie differentiation with respect to the vector field ∂_t and $\delta^i_{\ j}$ denotes the standard Kronecker delta. Standard calculations based on the Gauss and Codazzi equations for the Lorentzian manifold $(\mathbb{R}^{1+3}, g)$ yield, see for example [Rodnianski and Speck 2014b, Appendix A], the identities

$$\mathrm{Riem}(g)_{ab}^{\ \ cd} = k^c_{\ a} k^d_{\ b} - k^d_{\ a} k^c_{\ b} + \triangle_{ab}^{\ \ cd}, \tag{4-13}$$

$$\mathrm{Riem}(g)_{a0}^{\ \ c0} = (\partial_t \ln(1+\Psi)) k^c_{\ a} + k^c_{\ e} k^e_{\ a} + \triangle_{a0}^{\ \ c0}, \tag{4-14}$$

$$\mathrm{Riem}(g)_{0b}^{\ \ cd} = \triangle_{0b}^{\ \ cd}, \tag{4-15}$$

where the error terms are

$$\triangle_{ab}^{\ \ cd} := \mathrm{Riem}(\underline{g})_{ab}^{\ \ cd}, \tag{4-16}$$

$$\triangle_{a0}^{\ \ c0} := -\frac{1}{1+\Psi}\partial_t((1+\Psi)k^c_{\ a}), \tag{4-17}$$

$$\triangle_{0b}^{\ \ cd} := (\underline{g}^{-1})^{ce}\,\partial_e(k^d_{\ b}) - (\underline{g}^{-1})^{de}\,\partial_e(k^c_{\ b}) + (\underline{g}^{-1})^{ce}\,\Gamma_{e\ f}^{\ d} k^f_{\ b} - (\underline{g}^{-1})^{ce}\Gamma_{e\ b}^{\ f} k^d_{\ f} - (\underline{g}^{-1})^{de}\Gamma_{e\ f}^{\ c} k^f_{\ b} + (\underline{g}^{-1})^{de}\Gamma_{e\ b}^{\ f} k^c_{\ f}, \tag{4-18}$$

and

$$\Gamma_{j\ k}^{\ i} := \tfrac{1}{2}(\underline{g}^{-1})^{ai}\{\partial_j \underline{g}_{ak} + \partial_k \underline{g}_{ja} - \partial_a \underline{g}_{jk}\} \tag{4-19}$$

are the Christoffel symbols[37] of $\underline{g}$. In (4-16), $\mathrm{Riem}(\underline{g})$ denotes the Riemann curvature tensor of $\underline{g}$. We note that in deriving (4-14) and (4-17), we have used the simple identity

$$-\partial_t(k^c_{\ a}) = (\partial_t \ln(1+\Psi))k^c_{\ a} - \frac{1}{1+\Psi}\partial_t((1+\Psi)k^c_{\ a}).$$

We will use the estimates of Theorem 4.1 to show that

$$\triangle_{ab}^{\ \ cd} := \mathcal{O}(\mathring{\epsilon})\frac{1}{1+\Psi}, \tag{4-20}$$

$$\triangle_{a0}^{\ \ c0} := \mathcal{O}(\mathring{\epsilon}), \tag{4-21}$$

$$\triangle_{0b}^{\ \ cd} := \mathcal{O}(\mathring{\epsilon})\frac{1}{1+\Psi}. \tag{4-22}$$

[37] Our index conventions for the Christoffel symbols are different than the ones used in many works on differential geometry.

The desired bound (4-10a) then follows from (4-11), (4-12), (4-13)–(4-15), (4-20)–(4-22), the simple estimates $g^{ij} = \mathcal{O}(1)(1+\Psi)$ and $g_{ij} = \mathcal{O}(1)(1+\Psi)^{-1}$, and straightforward calculations.

It remains for us to prove (4-20)–(4-22). To prove (4-20), we first use (4-19) and (3C-5) to deduce

$$\Gamma_{j\ k}^{\ i} = \mathcal{O}(\mathring{\epsilon})\frac{1}{1+\Psi}, \quad \partial_l \Gamma_{j\ k}^{\ i} = \mathcal{O}(\mathring{\epsilon})\frac{1}{(1+\Psi)^2}. \tag{4-23}$$

Since $\mathrm{Riem}(\underline{g})_{ab}^{\quad cd}$ has the schematic structure $\mathrm{Riem}(\underline{g})_{ab}^{\quad cd} = \underline{g}^{-1}\underline{\partial}\Gamma + \underline{g}^{-1}\Gamma\cdot\Gamma$ (where $\underline{\partial}$ denotes the gradient with respect to the spatial coordinates), we deduce from (4-23) and the simple estimate $(\underline{g}^{-1})^{ij} = \mathcal{O}(1)(1+\Psi)$ that

$$\mathrm{Riem}(\underline{g})_{ab}^{\quad cd} = \mathcal{O}(\mathring{\epsilon})\frac{1}{1+\Psi},$$

which yields (4-20). To prove (4-21), we first use (4-12), (1A-1a), and the estimates (3C-5a) and (3C-5c) to deduce $\partial_t((1+\Psi)k^c_{\ a}) = \frac{1}{2}\partial_t^2\Psi\delta^c_{\ a} = \frac{1}{2}(1+\Psi)\,\Delta\Psi\delta^c_{\ a} = \mathcal{O}(\mathring{\epsilon})(1+\Psi)$. From this bound and (4-17), we conclude (4-21). To prove (4-22), we first use (4-12) and the estimates (3C-5) to deduce

$$k^i_{\ j} = \mathcal{O}(1)\frac{1}{1+\Psi}, \quad \partial_l k^i_{\ j} = \mathcal{O}(\mathring{\epsilon})\frac{1}{(1+\Psi)^2}. \tag{4-24}$$

From (4-23), (4-24), and the simple estimate $(\underline{g}^{-1})^{ij} = \mathcal{O}(1)(1+\Psi)$, we conclude (4-22). □

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