

# ANALYSIS & PDE

Volume 11      No. 1      2018

JESSICA ELISA MASSETTI

**A NORMAL FORM À LA MOSER FOR DIFFEOMORPHISMS AND  
A GENERALIZATION OF RÜSSMANN'S TRANSLATED CURVE  
THEOREM  
TO HIGHER DIMENSIONS**





# A NORMAL FORM À LA MOSER FOR DIFFEOMORPHISMS AND A GENERALIZATION OF RÜSSMANN'S TRANSLATED CURVE THEOREM TO HIGHER DIMENSIONS

JESSICA ELISA MASSETTI

We prove a discrete time analogue of Moser's normal form (1967) of real analytic perturbations of vector fields possessing an invariant, reducible, Diophantine torus; in the case of diffeomorphisms too, the persistence of such an invariant torus is a phenomenon of finite codimension. Under convenient nondegeneracy assumptions on the diffeomorphisms under study (a torsion property for example), this codimension can be reduced. As a by-product we obtain generalizations of Rüssmann's translated curve theorem in any dimension, by a technique of elimination of parameters.

1. Introduction and results	149
2. The normal form operator	153
3. Difference equations	155
4. Inversion of the operator $\phi$	159
5. A generalization of Rüssmann's theorem	163
Appendix A. The inverse function theorem and regularity of $\phi$	166
Appendix B. Inversion of a holomorphism of $\mathbb{T}_s^n$	167
Appendix C. Fourier norms	168
Acknowledgments	169
References	169

## 1. Introduction and results

Let  $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ ,  $a, b \in \mathbb{R}$ ,  $a < b$ , and consider the twist map

$$P : \mathbb{T} \times [a, b] \rightarrow \mathbb{T} \times \mathbb{R}, \quad (\theta, r) \mapsto (\theta + \alpha(r), r),$$

where  $\alpha'(r) > 0$ ; this map preserves circles  $r = r_0$ ,  $r_0 \in [a, b]$ , and rotates them by an angle which *increases* as  $r$  does (this is the twist property).

Moser [1962] proved that for any  $r_0 \in (a, b)$  such that  $\alpha(r_0)$  is Diophantine, if  $Q$  is an exact-area-preserving diffeomorphism sufficiently close to  $P$ , it has an invariant curve near  $r = r_0$  on which the dynamics is conjugated to the rotation  $\theta \mapsto \theta + \alpha(r_0)$ .

Rüssmann [1970] generalized this fundamental result to nonconservative twist diffeomorphisms of the annulus; see also [Bost 1986; Yoccoz 1992]. He showed that the persistence of a Diophantine invariant

---

MSC2010: 37-XX, 37C05, 37J40.

Keywords: normal forms, Diophantine tori, KAM, counter terms, translated tori.

circle is a phenomenon of codimension 1; in general the invariant curve does not persist but it is translated in the normal direction. It is the “theorem of the translated curve” (see below for a precise statement).

As in Kolmogorov’s theorem [1954], see also [Herman and Sergeraert 1971], the dynamics on the translated curve can be conjugated to the same initial Diophantine rotation because of the nondegeneracy (twist) of the map. Herman [1983] gave a proof of the translated curve theorem for diffeomorphisms with rotation number of constant type, then generalized Rüssmann’s result in higher dimensions to diffeomorphisms of  $\mathbb{T}^n \times \mathbb{R}$  ( $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ ) close enough to the rotation  $(\theta, r) \mapsto (\theta + \alpha, r)$ , where  $\alpha$  is a Diophantine vector, without assuming any twist hypothesis but introducing an external parameter in order to tune the frequency on the translated torus, yet breaking the dynamical conjugacy to the Diophantine rotation; see [Yoccoz 1992].

To our knowledge no further generalization in  $\mathbb{T}^n \times \mathbb{R}^m$  of Rüssmann’s theorem has been given so far.

The first purpose of this work is to prove a discrete-time analogue of Moser’s normal form [1967] of real analytic perturbations of vector fields on  $\mathbb{T}^n \times \mathbb{R}^m$  possessing a quasiperiodic Diophantine, reducible, invariant torus. The normal form will then be used to deduce “translated torus theorems” under convenient nondegeneracy assumptions. As a by-product, Rüssmann’s classical theorem will be a particular case of small dimension. While Rüssmann and Herman consider smooth or finite differentiable diffeomorphisms, we focus here on the analytic category. Let us state the main results.

**A normal form for diffeomorphisms.** Let  $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$  be the  $n$ -dimensional torus. Let  $V$  be the space of germs along  $\mathbb{T}^n \times \{0\}$  in  $\mathbb{T}^n \times \mathbb{R}^m = \{(\theta, r)\}$  of real analytic diffeomorphisms. Fix  $\alpha \in \mathbb{R}^n$  and  $A \in \text{GL}_m(\mathbb{R})$ , assuming that  $A$  is diagonalizable with (possibly complex) eigenvalues  $a_1, \dots, a_m \in \mathbb{C}$ .

Let  $U(\alpha, A)$  be the affine subspace of  $V$  of diffeomorphisms of the form

$$P(\theta, r) = (\theta + \alpha + O(r), A \cdot r + O(r^2)), \quad (1-1)$$

where  $O(r^k)$  are terms of order  $\geq k$  in  $r$  which may depend on  $\theta$ . For these diffeomorphisms,  $\mathbb{T}_0^n = \mathbb{T}^n \times \{0\}$  is an invariant, reducible,  $\alpha$ -quasiperiodic torus whose normal dynamics at the first order is characterized by  $a_1, \dots, a_m$ . We will collectively refer to  $\alpha_1, \dots, \alpha_n$  and  $a_1, \dots, a_m$  as the *characteristic frequencies* or *characteristic numbers* of  $\mathbb{T}_0^n$ .

Let now  $a_1, \dots, a_q \in \mathbb{C}$  be the pairwise distinct eigenvalues of  $A$ . We will impose the following Diophantine conditions for some  $\gamma > 0$  and  $\tau \geq 1$ :

$$\begin{aligned} \forall i = 1, \dots, q, \quad |a_i| = 1, \quad |k \cdot \alpha + \arg a_i - 2\pi l| &\geq \frac{\gamma}{|k|^\tau} \quad \forall (k, l) \in \mathbb{Z}^n \setminus \{0\} \times \mathbb{Z}, \\ \forall i, j = 1, \dots, q, \quad |a_i| = |a_j|, \quad |k \cdot \alpha + \arg a_i - \arg a_j - 2\pi l| &\geq \frac{\gamma}{|k|^\tau} \quad \forall (k, l) \in \mathbb{Z}^n \setminus \{0\} \times \mathbb{Z}, \end{aligned} \quad (1-2)$$

where  $\arg a_i \in [0, 2\pi[$  denotes the argument of the  $i$ -th eigenvalue  $a_i = |a_i|e^{i \arg a_i}$ .

**Remark 1.1.** Since  $A$  is in  $\text{GL}_m(\mathbb{R})$ , the possible complex eigenvalues come in couples, and conditions (1-2) imply the classical Diophantine condition on  $\alpha$  when  $i = j$ .

Let  $\mathcal{G}$  be the space of germs of real analytic isomorphisms of  $\mathbb{T}^n \times \mathbb{R}^m$  of the form

$$G(\theta, r) = (\varphi(\theta), R_0(\theta) + R_1(\theta) \cdot r), \quad (1-3)$$

where  $\varphi$  is a diffeomorphism of the torus fixing the origin and  $R_0, R_1$  are functions defined on the torus  $\mathbb{T}^n$  with values in  $\mathbb{R}^m$  and  $\text{GL}_m(\mathbb{R})$  respectively and such that  $\Pi_{\text{Ker}(A-I)} R_0(0) = 0$  and  $\Pi_{\text{Ker}[A, \cdot]} (R_1(0) - I) = 0$ , where  $I$  is the identity matrix in  $\text{Mat}_m(\mathbb{R})$  and  $\Pi$  is the projection on the indicated subspace.

Let us define the “correction map”

$$T_\lambda : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathbb{T}^n \times \mathbb{R}^m, \quad (\theta, r) \mapsto (\beta + \theta, b + (I + B) \cdot r),$$

where  $\beta \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$  and  $B \in \text{Mat}_m(\mathbb{R})$  are such that

$$(A - I) \cdot b = 0, \quad [A, B] = 0. \tag{1-4}$$

We will refer to translating parameters  $\lambda = (\beta, b + B \cdot r)$  as *corrections* or *counter terms*, and denote by  $\Lambda$  the space of such  $\lambda$ 's:

$$\Lambda = \{ \lambda = (\beta, b + B \cdot r) : (A - I) \cdot b = 0, [A, B] = 0 \}.$$

**Theorem A** (normal form). *Let  $(\alpha, A)$  satisfy the Diophantine condition (1-2). If  $Q$  is sufficiently close to  $P^0 \in U(\alpha, A)$ , there exists a unique triplet  $(G, P, \lambda) \in \mathcal{G} \times U(\alpha, A) \times \Lambda$  close to  $(\text{id}, P^0, 0)$ , such that*

$$Q = T_\lambda \circ G \circ P \circ G^{-1}.$$

In the neighborhood of  $(\text{id}, P^0, 0)$ , the  $\mathcal{G}$ -orbit of all  $P \in U(\alpha, A)$  has *finite* codimension. The proof is based on a relatively general inverse function theorem in analytic class (Theorem A.1 of the Appendix).

The idea of proving the finite codimension of a set of conjugacy classes of a diffeomorphism or of a vector field has been successfully exploited by many authors. Arnol'd [1961] first proved a normal form for diffeomorphisms of  $\mathbb{T}^n$ ; this was followed by Moser's normal forms for vector fields [Moser 1966; 1967; Wagener 2010; Massetti 2015a; 2015b]. Among others, we recall the work of Calleja, Celletti and de la Llave [Calleja et al. 2013] on conformally symplectic systems, Chenciner's study [1985a; 1985b; 1988] on the bifurcation of elliptic fixed points, Herman's twisted conjugacy for Hamiltonians [Féjóz 2004; 2010] (a generalization of [Arnol'd 1961]) and the work of Eliasson, Fayad and Krikorian [Eliasson et al. 2015] around the stability of KAM tori.

This technique allows us to study the persistence of an invariant torus in two steps: first, prove a normal form that does not depend on any nondegeneracy hypothesis (but that contains the hard analysis); second, reduce or eliminate the (finite-dimensional) corrections by the usual implicit function theorem, using convenient nondegeneracy assumptions on the system under consideration. This second step was probably not deeply understood before the 80s [Sevryuk 1999].

**A generalization of Rüssmann's theorem.** From the normal form of Theorem A, we see that when  $\lambda = 0$ ,  $Q = G \circ P \circ G^{-1}$ ; the torus  $G(\mathbb{T}_0^n)$  is invariant for  $Q$  and the first-order dynamics is given by  $P \in U(\alpha, A)$ . Conversely, whenever  $\lambda = (\beta, b)$ , the torus is *translated* and the  $\alpha$ -quasiperiodic tangential dynamics is *twisted* by the correction  $\beta$ :

$$Q(\varphi(\theta), R_0(\theta)) = (\beta + \varphi(\theta + \alpha), b + R_0(\theta + \alpha)).$$

We will loosely say that the torus  $\mathbb{T}_0^n$

- persists up to twist-translation when  $\lambda = (\beta, b)$ ,
- persists up to translation when  $\lambda = (0, b)$ .

We stress the fact that Theorem A not only gives the tangential dynamics to the torus, but also the normal one, of which Rüssmann's original statement is regardless:

**Theorem (Rüssmann).** *Let  $\alpha \in \mathbb{R}$  be Diophantine and  $P^0 : \mathbb{T} \times [-r_0, r_0] \rightarrow \mathbb{T} \times \mathbb{R}$  be of the form*

$$P^0(\theta, r) = (\theta + \alpha + t(r) + O(r^2), A^0 r + O(r^2)),$$

where  $A^0 \in \mathbb{R} \setminus \{0\}$ ,  $t(0) = 0$  and  $t'(r) > 0$ .

*If  $Q$  is close enough to  $P^0$ , there exists a unique analytic curve  $\gamma : \mathbb{T} \rightarrow \mathbb{R}$  close to  $r = 0$ , an analytic diffeomorphism  $\varphi$  of  $\mathbb{T}$  fixing the origin close to the identity, and  $b \in \mathbb{R}$  close to 0, such that*

$$Q(\theta, \gamma(\theta)) = (\varphi \circ R_\alpha \circ \varphi^{-1}(\theta), b + \gamma(\varphi \circ R_\alpha \circ \varphi^{-1}(\theta))).$$

Note that  $t(r)$  may depend on the angles as well. In the original statement,  $A^0 = 1$ ; to consider this case with general  $A^0$  does not add any difficulty to the proof.

We will generalize Rüssmann's theorem on  $\mathbb{T}^n \times \mathbb{R}^n$ . At the expense of losing control on the final normal dynamics and conjugating  $T_\lambda^{-1} \circ Q$  to a diffeomorphism  $P$  whose invariant torus has a normal dynamics given by a different  $A$ , under convenient nondegeneracy conditions we can prove the existence of a twisted-translated or translated  $\alpha$ -quasiperiodic Diophantine torus by application of the classical implicit function theorem in finite dimension. The following results will be proved in Section 5, where a more functional statement will be given (Theorems 5.1 and 5.4).

On  $\mathbb{T}^n \times \mathbb{R}^n$ , let  $P \in U(\alpha, A)$ , defined in expression (1-1), be such that  $A$  is invertible and has simple, real eigenvalues  $a_1, \dots, a_n$ . This hypothesis clearly implies that the only frequencies that can cause small divisors are the tangential ones  $\alpha_1, \dots, \alpha_n$ , so that we only need to require the standard Diophantine hypothesis on  $\alpha$ .

**Theorem B.** *Let  $\alpha$  be Diophantine and let  $A \in \text{GL}_n(\mathbb{R})$  have simple, real eigenvalues. If  $Q$  is sufficiently close to  $P^0 \in U(\alpha, A)$ , there exists  $A'$  close to  $A$  such that the torus  $\mathbb{T}_0^n$  persists up to twist-translation and its final normal dynamics is given by  $A'$ .*

If, in addition,  $Q$  has a torsion property, we can prove the following theorem.

**Theorem C.** *Let  $\alpha$  be Diophantine and let  $A$  be invertible with simple, real eigenvalues. Let also*

$$P^0(\theta, r) = (\theta + \alpha + p_1(\theta) \cdot r + O(r^2), A \cdot r + O(r^2))$$

be such that

$$\det \left( \int_{\mathbb{T}^n} p_1(\theta) d\theta \right) \neq 0.$$

*If  $Q$  is sufficiently close to  $P^0$ , there exists  $A'$  close to  $A$  such that the torus  $\mathbb{T}_0^n$  persists up to translation and the final normal dynamics is given by  $A'$ .*

The paper is organized as follows: in Sections 2–3 we introduce the normal form operator, define conjugacy spaces and present the difference equations that will be solved to linearize the dynamics on the perturbed torus; in Section 4 we will prove Theorem A, while in Section 5 we will prove Theorems B and C.

## 2. The normal form operator

We will show that the operator

$$\phi : \mathcal{G} \times U(\alpha, A) \times \Lambda \rightarrow V, \quad (G, P, \lambda) \mapsto T_\lambda \circ G \circ P \circ G^{-1},$$

is a local diffeomorphism (in the sense of scales of Banach spaces) in a neighborhood of  $(\text{id}, P^0, 0)$ . Note that  $\phi$  is formally defined on the whole space but  $\phi(G, P, \lambda)$  is analytic in the neighborhood of  $T_0^n$  only if  $G$  is close enough to the identity with respect to the width of analyticity of  $P$ . See the subsection on page 155.

Although the difficulty to overcome in the proof is rather standard for conjugacy problems of this kind (proving the fast convergence of a Newton-like scheme), the procedure relies on a relatively general inverse function theorem (Theorem A.1 of the Appendix), following a strategy different from Zehnder's [1975]. Both Zehnder's approach and ours rely on the fact that the fast convergence of the Newton scheme is somewhat independent of the internal structure of the variables.

**Complex extensions.** Let us extend the tori

$$\mathbb{T}^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n \quad \text{and} \quad \mathbb{T}_0^n = \mathbb{T}^n \times \{0\} \subset \mathbb{T}^n \times \mathbb{R}^m,$$

as

$$\mathbb{T}_{\mathbb{C}}^n = \mathbb{C}^n / 2\pi \mathbb{Z}^n \quad \text{and} \quad \mathbb{T}_{\mathbb{C}}^n = \mathbb{T}_{\mathbb{C}}^n \times \mathbb{C}^m$$

respectively, and consider the corresponding  $s$ -neighborhoods defined using  $\ell^\infty$ -balls (in the real normal bundle of the torus):

$$\mathbb{T}_s^n = \left\{ \theta \in \mathbb{T}_{\mathbb{C}}^n : \max_{1 \leq j \leq n} |\text{Im } \theta_j| \leq s \right\} \quad \text{and} \quad \mathbb{T}_s^n = \left\{ (\theta, r) \in \mathbb{T}_{\mathbb{C}}^n : |(\text{Im } \theta, r)| \leq s \right\},$$

where  $|(\text{Im } \theta, r)| := \max(\max_{1 \leq j \leq n} |\text{Im } \theta_j|, \max_{1 \leq j \leq m} |r_j|)$ .

Let now  $f : \mathbb{T}_s^n \rightarrow \mathbb{C}$  be holomorphic on the interior of  $\mathbb{T}_s^n$ , continuous on  $\mathbb{T}_s^n$ , and consider its Fourier expansion  $f(\theta, r) = \sum_{k \in \mathbb{Z}^n} f_k(r) e^{i k \cdot \theta}$ , where  $k \cdot \theta = k_1 \theta_1 + \dots + k_n \theta_n$ . In this context we introduce the so-called weighted norm:

$$|f|_s := \sum_{k \in \mathbb{Z}^n} |f_k| e^{|k|s}, \quad |k| = |k_1| + \dots + |k_n|,$$

where  $|f_k| = \sup_{|r| < s} |f_k(r)|$ . Whenever  $f : \mathbb{T}_s^n \rightarrow \mathbb{C}^n$ , we have  $|f|_s = \max_{1 \leq j \leq n} (|f_j|_s)$ , where  $f_j$  is the  $j$ -th component of  $f(\theta, r)$ .

It is a trivial fact that the classical sup-norm is bounded from above by the weighted norm:

$$\sup_{z \in \mathbb{T}_s^n} |f(z)| \leq |f|_s$$

and that  $|f|_s < +\infty$  whenever  $f$  is analytic on its domain, which necessarily contains some  $\mathbb{T}_{s'}^n$  with  $s' > s$ . In addition, the following useful inequalities hold if  $f, g$  are analytic on  $\mathbb{T}_{s'}^n$ :

$$|f|_s \leq |f|_{s'} \quad \text{for } 0 < s < s',$$

and

$$|fg|_{s'} \leq |f|_{s'} |g|_{s'}.$$

Moreover, one can show that if  $f$  is analytic on  $\mathbb{T}_{s+\sigma}^n$  and  $G$  is a diffeomorphism of the form (1-3) sufficiently close to the identity, then  $|f \circ G|_s \leq C_G |f|_{s+\sigma}$ , where  $C_G$  is a positive constant depending on  $|G - \text{id}|_s$  (see Appendix C). For more details about the weighted norm, see for example [Meyer 1975; Chierchia 2003].

In general for complex extensions  $U_s$  and  $V_{s'}$ , we will denote by  $\mathcal{A}(U_s, V_{s'})$  the set of holomorphic functions from  $U_s$  to  $V_{s'}$  and by  $\mathcal{A}(U_s)$ , endowed with the  $s$ -weighted norm, the Banach space  $\mathcal{A}(U_s, \mathbb{C})$ .

Finally, let  $E$  and  $F$  be two Banach spaces:

- We indicate contractions with a dot “ $\cdot$ ”, with the convention that if  $l_1, \dots, l_{k+p} \in E^*$  and  $x_1, \dots, x_p \in E$ ,

$$(l_1 \otimes \dots \otimes l_{k+p}) \cdot (x_1 \otimes \dots \otimes x_p) = l_1 \otimes \dots \otimes l_k \langle l_{k+1}, x_1 \rangle \dots \langle l_{k+p}, x_p \rangle.$$

In particular, if  $l \in E^*$ , we simply write  $l^n = l \otimes \dots \otimes l$ .

- If  $f$  is a differentiable map between two open sets of  $E$  and  $F$ , then  $f'(x)$  is considered as a linear map belonging to  $F \otimes E^*$ ,  $f'(x) : \zeta \mapsto f'(x) \cdot \zeta$ ; the corresponding norm will be the standard operator norm

$$|f'(x)| = \sup_{\zeta \in E, |\zeta|_E=1} |f'(x) \cdot \zeta|_F.$$

**Spaces of conjugacies.** – We consider the neighborhood of the identity  $\mathcal{G}_s^\sigma$  in the space of germs of real holomorphic diffeomorphisms on  $\mathbb{T}_s^n$ , defined by

$$|\varphi - \text{id}|_s \leq \sigma$$

and

$$|R_0 + (R_1 - I) \cdot r|_s \leq \sigma,$$

where  $\varphi(0) = 0$ , and  $R_0$  and  $R_1$  satisfy  $\Pi_{\ker(A-I)} R_0(0) = 0$  and  $\Pi_{\ker([A, \cdot])} (R_1(0) - I) = 0$ .

The tangent space at the identity  $T_{\text{id}} \mathcal{G}_s^\sigma$  consists of maps  $\dot{G} \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{C}^{n+m})$ ,

$$\dot{G}(\theta, r) = (\dot{\varphi}(\theta), \dot{R}_0(\theta) + \dot{R}_1(\theta) \cdot r),$$

where  $\dot{\varphi} \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{C}^n)$ ,  $\dot{R}_0 \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{C}^m)$  and  $\dot{R}_1 \in \mathcal{A}(\mathbb{T}_s^n, \text{Mat}_m(\mathbb{C}))$ . We endow it with the norm

$$|\dot{G}|_s = \max_{1 \leq j \leq n+m} (|\dot{G}_j|_s).$$

- Let  $V_s$  be the subspace of  $\mathcal{A}(\mathbb{T}_s^n, \mathbb{T}_{\mathbb{C}}^n \times \mathbb{C}^m)$  of diffeomorphisms

$$Q : (\theta, r) \mapsto (f(\theta, r), g(\theta, r)),$$

where  $f \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{C}^n)$ ,  $g \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{C}^m)$ , endowed with the norm

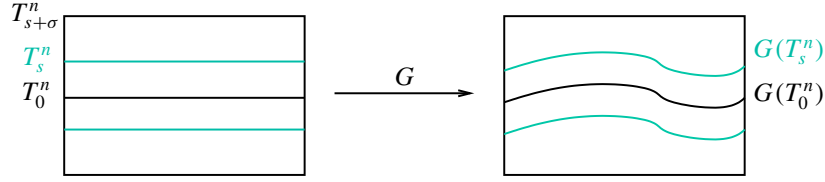
$$|Q|_s = \max (|f|_s, |g|_s).$$

- Let  $U_s(\alpha, A)$  be the affine subspace of  $V_s$  of those diffeomorphisms  $P$  of the form

$$P(\theta, r) = (\theta + \alpha + O(r), A \cdot r + O(r^2)).$$

We will indicate by  $p_i$  and  $P_i$  the coefficients of the order- $i$  term in  $r$ , in the  $\theta$ - and  $r$ -directions respectively.





**Figure 1.** Deformed complex domain.

– If  $G \in \mathcal{G}_s^\sigma$  and  $P$  is a diffeomorphism over  $G(T_s^n)$  we define the deformed norm

$$|P|_{G,s} := |P \circ G|_s,$$

depending on  $G$ ; this is in order not to shrink artificially the domains of analyticity. See Figure 1. The problem, in a smooth context, may be solved without changing the domain, by using plateau functions.

**The normal form operator.** By Theorem B.1 and Corollary B.2 the operator

$$\begin{aligned} \phi : \mathcal{G}_{s+\sigma}^{\sigma/n} \times U_{s+\sigma}(\alpha, A) \times \Lambda &\rightarrow V_s, \\ (G, P, \lambda) &\mapsto T_\lambda \circ G \circ P \circ G^{-1}, \end{aligned} \quad (2-1)$$

is now well defined. It would be more appropriate to write  $\phi_{s,\sigma}$  but, since these operators commute with source and target spaces, we will refer to them simply as  $\phi$ . We will always assume that  $0 < s < s + \sigma < 1$  and  $\sigma < s$ .

### 3. Difference equations

We present here three lemmata that we will use in the following in order to linearize the tangent and the normal dynamics of the torus (see Section 4).

Let  $\alpha \in \mathbb{R}^n$  and let  $M \in \text{GL}_m(\mathbb{R})$  have pairwise distinct eigenvalues  $\mu_1, \dots, \mu_m$ . We assume the following Diophantine conditions on  $\alpha$  and  $M$ :

$$|k \cdot \alpha - 2\pi l| \geq \frac{\gamma}{|k|^\tau} \quad \forall k \in \mathbb{Z}^n \setminus \{0\}, \quad \forall l \in \mathbb{Z}, \quad (3-1)$$

$$|k \cdot \alpha - \arg \mu_j - 2\pi l| \geq \frac{\gamma}{|k|^\tau} \quad \forall (k, l) \in \mathbb{Z}^n \setminus \{0\} \times \mathbb{Z}, \quad \forall j = 1, \dots, m : |\mu_j| = 1, \quad (3-2)$$

$$|k \cdot \alpha + \arg \mu_i - \arg \mu_j - 2\pi l| \geq \frac{\gamma}{|k|^\tau} \quad \forall (k, l) \in \mathbb{Z}^n \setminus \{0\} \times \mathbb{Z}, \quad \forall i, j = 1, \dots, m : |\mu_i| = |\mu_j|, \quad (3-3)$$

$$\begin{cases} |\mu_i| - |\mu_j| \geq \gamma & \forall i, j = 1, \dots, m, \quad i \neq j : |\mu_i| \neq |\mu_j|, \\ |1 - |\mu_j|| \geq \gamma & \text{if } |\mu_j| \neq 1, \end{cases} \quad (3-4)$$

$$\begin{cases} |\mu_i - \mu_j| \geq \gamma & \forall i, j = 1, \dots, m, \quad i \neq j : |\mu_i| = |\mu_j|, \\ |1 - \mu_j| \geq \gamma & \text{if } |\mu_j| = 1 \text{ and } \mu_j \neq 1, \end{cases} \quad (3-5)$$

$$\min_{1 \leq j \leq m} (|\mu_j|) \geq \gamma. \quad (3-6)$$

We first prove the following fundamental lemma, which is the heart of the proof of Theorem A and, more generally, of many stability results related to Diophantine rotations on the torus.

**Lemma 3.1.** *Let  $\alpha \in \mathbb{R}^n$  be Diophantine in the sense of (3-1) and let  $a, b \in \mathbb{C} \setminus \{0\}$ .*

(1) *If  $a = b$  and  $|a| \geq \gamma$ , for any  $g \in \mathcal{A}(\mathbb{T}_{s+\sigma}^n)$ , there exists a unique  $f$  of zero average which is complex analytic on  $\mathbb{T}_s^n$  and a unique  $\lambda \in \mathbb{R}$  such that*

$$\lambda + af(\theta + \alpha) - af(\theta) = g(\theta), \quad \lambda = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} g \, d\theta,$$

satisfying

$$|f|_s \leq \frac{C}{\gamma^2 \sigma^{\tau+n}} |g|_{s+\sigma},$$

where  $C$  is a constant depending only on  $n$  and  $\tau$ .

(2) *Let  $a \neq b$ .*

(i) *If  $|a| = |b|$  and*

$$\begin{cases} |a - b| \geq \gamma, \\ |a| \geq \gamma, \\ |k \cdot \alpha + \arg a - \arg b - 2\pi l| \geq \gamma/|k|^\tau \quad \forall (k, l) \in \mathbb{Z}^n \setminus \{0\} \times \mathbb{Z}, \end{cases} \quad (3-7)$$

for any  $g \in \mathcal{A}(\mathbb{T}_{s+\sigma}^n)$ , there exists a unique  $f$  which is complex analytic on  $\mathbb{T}_s^n$  such that

$$af(\theta + \alpha) - bf(\theta) = g(\theta), \quad (3-8)$$

satisfying

$$|f|_s \leq \frac{C}{\gamma^2 \sigma^{\tau+n}} |g|_{s+\sigma},$$

where  $C$  is a constant depending only on  $n, \tau$ .

(ii) *If  $|a| \neq |b|$  and  $||a| - |b|| \geq \gamma$ , for any  $g \in \mathcal{A}(\mathbb{T}_{s+\sigma}^n)$ , there exists a unique  $f$  which is complex analytic on  $\mathbb{T}_{s+\sigma}^n$  such that*

$$af(\theta + \alpha) - bf(\theta) = g(\theta),$$

satisfying

$$|f|_{s+\sigma} \leq \gamma^{-1} |g|_{s+\sigma}.$$

*Proof.* (1) Developing in Fourier series the equation yields

$$\lambda + a \sum_k (e^{ik \cdot \alpha} - 1) f_k e^{ik \cdot \theta} = \sum_k g_k e^{ik \cdot \theta};$$

letting  $\lambda = g_0$  we formally have

$$f(\theta) = \frac{1}{a} \sum_{k \neq 0} \frac{g_k}{e^{ik \cdot \alpha} - 1} e^{ik \cdot \theta}.$$

First note that the coefficients  $g_k$  decay exponentially, that is,

$$|g_k| = \left| \int_{\mathbb{T}^n} g(\theta) e^{-ik \cdot \theta} \frac{d\theta}{2\pi} \right| \leq |g|_{s+\sigma} e^{-|k|(s+\sigma)},$$

by deforming the path of integration to  $\text{Im } \theta_j = -\text{sgn}(k_j)(s + \sigma)$ .

Second, remark that for any  $x, y \in \mathbb{R}^+$ ,  $\varphi \in [0, 2\pi[$ ,

$$\begin{aligned} |x e^{i\varphi} - y|^2 &= (x - y)^2 \cos^2 \frac{\varphi}{2} + (x + y)^2 \sin^2 \frac{\varphi}{2} \\ &\geq (x + y)^2 \sin^2 \frac{\varphi}{2} = (x + y)^2 \sin^2 \frac{\varphi - 2\pi l}{2}, \end{aligned} \tag{3-9}$$

with  $l \in \mathbb{Z}$ . By choosing  $l \in \mathbb{Z}$  such that  $-\frac{\pi}{2} \leq \frac{1}{2}(\varphi - 2\pi l) \leq \frac{\pi}{2}$ , we get

$$|x e^{i\varphi} - y| \geq \frac{2}{\pi}(x + y) \frac{|\varphi - 2\pi l|}{2}, \tag{3-10}$$

by the classical inequality  $|\sin \delta| \geq \frac{2}{\pi}|\delta|$ , whenever  $-\frac{\pi}{2} \leq \delta \leq \frac{\pi}{2}$ .

In our case  $x = y = 1$ ,  $\varphi = k \cdot \alpha$  and for all  $k$ , by choosing  $l \in \mathbb{Z}$  such that  $-\frac{\pi}{2} \leq \frac{1}{2}(k \cdot \alpha - 2\pi l) \leq \frac{\pi}{2}$ , we get

$$|e^{i k \cdot \alpha} - 1| \geq \frac{4}{\pi} \frac{|k \cdot \alpha - 2\pi l|}{2} \geq \frac{2}{\pi} \frac{\gamma}{|k|^\tau},$$

by inequality (3-10) and the Diophantine condition (3-1).

We thus have

$$\begin{aligned} |f|_s &\leq \frac{\pi |g|_{s+\sigma}}{|a|^\gamma} \sum_k |k|^\tau e^{-|k|\sigma} \leq \frac{\pi 2^n |g|_{s+\sigma}}{|a|^\gamma} \sum_{\ell \geq 1} \binom{\ell+n+1}{\ell} e^{-\ell\sigma} \ell^\tau \\ &\leq \frac{\pi 4^n |g|_{s+\sigma}}{|a|^\gamma (n-1)!} \sum_{\ell \geq 1} (n+\ell-1)^{n-1+\tau} e^{-\ell\sigma} \\ &\leq \frac{\pi 4^n |g|_{s+\sigma}}{|a|^\gamma (n-1)!} \int_1^\infty (\ell+n-1)^{n+\tau-1} e^{-(\ell-1)\sigma} d\ell. \end{aligned}$$

The integral is equal to

$$\sigma^{-\tau-n} e^{n\sigma} \int_{n\sigma}^\infty \ell^{\tau+n-1} e^{-\ell} d\ell < \sigma^{-\tau-n} e^{n\sigma} \int_0^\infty \ell^{\tau+n-1} e^{-\ell} d\ell = \sigma^{-\tau-n} e^{n\sigma} \Gamma(\tau+n).$$

Hence  $f$ , of zero average, is complex analytic on  $\mathbb{T}_s^n$  and, since  $|a| \geq \gamma$ , it satisfies the claimed estimate.

(2i) Let  $a = |a|e^{i \arg a}$  and  $b = |b|e^{i \arg b}$  with the convention that  $\arg z = \pi$  ( $\arg z = 0$ ) if  $z \in \mathbb{R}^-$  (if  $z \in \mathbb{R}^+$ ).

The Fourier expansion gives

$$f_0 = \frac{g_0}{a - b}$$

and for all  $k \neq 0$

$$f_k = \frac{g_k}{e^{i \arg b} (|a| e^{i(k \alpha + \arg a - \arg b)} - |b|)} e^{i k \cdot \theta}.$$

In order to bound the divisors we apply the same inequalities as in (3-9)–(3-10), with  $\varphi = k \cdot \alpha + \arg a - \arg b$ . Since  $|a| = |b|$ , by conditions (3-7) we proceed as in the proof of point (1) to get the stated estimate. In the case where  $a$  (or  $b$ ) is real and  $\arg a$  (or  $\arg b$ ) is equal to  $\pi$ , we shall choose  $\hat{l} = 2l - 1$  (or  $\hat{l} = 2l + 1$ ) such that  $-\frac{\pi}{2} \leq \frac{1}{2}(k \cdot \alpha + \arg a - \pi \hat{l}) \leq \frac{\pi}{2}$  to conclude the estimate as in (3-10).

(2ii) This follows directly from the triangular inequality.

We direct the reader interested to optimal estimates (with  $\sigma^\tau$  instead of  $\sigma^{\tau+n}$ ) to [Rüssmann 1976].  $\square$

Let now  $\alpha \in \mathbb{R}^n$  and  $M \in \mathrm{GL}_m(\mathbb{R})$  have simple eigenvalues such that<sup>1</sup>  $\mu_i \neq 1$  for all  $i = 1, \dots, m$ , and consider the operator

$$L_{1,M} : \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \mathbb{C}^m) \rightarrow \mathcal{A}(\mathbb{T}_s^n, \mathbb{C}^m), \quad f \mapsto f(\theta + \alpha) - M \cdot f(\theta).$$

**Lemma 3.2** (relocating the torus). *Let  $\alpha \in \mathbb{R}^n$  and  $M \in \mathrm{GL}_m(\mathbb{R})$ , a diagonalizable matrix with simple eigenvalues distinct from 1, satisfy the Diophantine conditions (3-1)–(3-2) and (3-4)–(3-6). For every  $g \in \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \mathbb{C}^m)$ , there exists a unique preimage  $f \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{C}^m)$  by  $L_{1,A}$ . Moreover, the following estimate holds:*

$$|f|_s \leq \frac{C_2}{\gamma^2} \frac{1}{\sigma^{n+\tau}} |g|_{s+\sigma},$$

where  $C_2$  is a constant depending only on the dimension  $n$  and the exponent  $\tau$ .

*Proof.* In the scalar case,  $m = 1$  and  $M = \mu \in \mathbb{R}$ . By expanding both sides of  $L_{1,M} f = g$ , the formal preimage is given by

$$f_k = \frac{g_k}{e^{ik\alpha} - \mu}$$

and the proof is recovered from Lemma 3.1(2ii). The diagonal case follows readily by working componentwise and taking into account condition (3-4).

Finally, if  $M$  is diagonalizable, let  $P \in \mathrm{GL}_m(\mathbb{C})$  be the diagonalizing matrix such that  $PMP^{-1} = \mathrm{diag}(\mu_1, \dots, \mu_m)$ ,  $\mu_i \in \mathbb{C}$ . By left multiplying both sides of  $f(\theta + \alpha) - M \cdot f(\theta) = g$  by  $P$ , we get

$$\tilde{f}(\theta + \alpha) - PMP^{-1} \tilde{f}(\theta) = \tilde{g},$$

where we have set  $\tilde{g} = Pg$  and  $\tilde{f} = Pf$ . By Lemma 3.1(2) and the Diophantine conditions (3-1)–(3-2) and (3-5)–(3-6),  $\tilde{f}$  satisfies the wanted estimates, and  $f = P^{-1} \tilde{f}$ .  $\square$

Finally, consider a holomorphic function  $F$  on  $\mathbb{T}_{s+\sigma}^n$  with values in  $\mathrm{Mat}_m(\mathbb{C})$  and define the operator

$$L_{2,M} : \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \mathrm{Mat}_m(\mathbb{C})) \rightarrow \mathcal{A}(\mathbb{T}_s^n, \mathrm{Mat}_m(\mathbb{C})), \\ F \mapsto F(\theta + \alpha) \cdot M - M \cdot F(\theta).$$

**Lemma 3.3** (straighten the first order dynamics). *Let  $\alpha \in \mathbb{R}^n$  and  $M \in \mathrm{GL}_m(\mathbb{R})$ , a diagonalizable matrix with simple eigenvalues distinct from 1, satisfy the Diophantine conditions (3-1) and (3-3)–(3-6). For every  $G \in \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \mathrm{Mat}_m(\mathbb{C}))$  such that  $\int_{\mathbb{T}^n} G_i^i / (2\pi)^n d\theta = 0$  there exists a unique  $F \in \mathcal{A}(\mathbb{T}_s^n, \mathrm{Mat}_m(\mathbb{C}))$ , having zero-average diagonal elements, such that the matrix equation*

$$F(\theta + \alpha) \cdot M - M \cdot F(\theta) = G(\theta)$$

is satisfied; moreover, the following estimate holds:

$$|F|_s \leq \frac{C_3}{\gamma^2} \frac{1}{\sigma^{n+\tau}} |G|_{s+\sigma},$$

where  $C_3$  is a constant depending only on the dimension  $n$  and the exponent  $\tau$ .

<sup>1</sup>In order not to burden the following statements, we suppose that  $M$  has simple spectrum and 1 does not belong to it. Just note that in the general case, one should introduce the correction  $\lambda$  meant to absorb the average of the given term in the homological equations when it is the case, as in Lemma 3.1(1); cf. conditions (1-4).



*Proof.* Let  $M = \text{diag}(\mu_1, \dots, \mu_m) \in \mathbb{R}^m$  and  $F \in \text{Mat}_m(\mathbb{C})$  be given; expanding  $L_{2,M}F = G$  we get  $m$  equations of the form

$$\mu_j(F_j^j(\theta + \alpha) - F_j^j(\theta)) = G_j^j, \quad j = 1, \dots, m,$$

and  $m^2 - m$  equations of the form

$$\mu_j F_j^i(\theta + \alpha) - \mu_i F_j^i(\theta) = G_j^i(\theta), \quad \forall i \neq j, \quad i, j = 1, \dots, m,$$

where we denoted by  $F_j^i$  the element corresponding to the  $i$ -th line and  $j$ -th column of the matrix  $F(\theta)$ . Taking into account the Diophantine conditions (3-1)–(3-4), the thesis follows from the same computations as Lemma 3.1(1) for the  $m$ -diagonal equations and point (2ii) for the  $(m^2 - m)$ -out diagonal ones.

Finally, to recover the general case, we consider the transition matrix  $P \in \text{GL}_m(\mathbb{C})$  such that  $PM P^{-1} = \text{diag}(\mu_1, \dots, \mu_m)$ ,  $\mu_i \in \mathbb{C}$ , and the equation

$$(PF(\theta + \alpha)P^{-1}PMP^{-1}) - PMP^{-1}PF(\theta)P^{-1} = PGP^{-1};$$

letting  $\tilde{F} = PFP^{-1}$  and  $\tilde{G} = PGP^{-1}$ , the equation is of the previous kind and by the Diophantine conditions (3-1) and (3-3)–(3-6),  $\tilde{F}$  satisfies the wanted estimates, and  $F = P^{-1}\tilde{F}P$ .  $\square$

**Remark 3.4.** The real analytic character of the solutions in Lemmata 3.2 and 3.3 follows from their uniqueness and the fact that the matrix  $M$  has real entries.

#### 4. Inversion of the operator $\phi$

The following theorem represents the main result of this first part, from which the normal form theorem, Theorem A, follows.

Let us fix  $P^0 \in U_s(\alpha, A)$  and note  $V_s^\sigma = \{Q \in V_s : |Q - P^0|_s < \sigma\}$ , the ball of radius  $\sigma$  centered at  $P^0$ .

**Theorem 4.1.** *The operator  $\phi$  is a local diffeomorphism in the sense that for any  $0 < \eta < s < s + \sigma < 1$  there exists  $\varepsilon > 0$  and a unique  $C^\infty$ -map  $\psi$ ,*

$$\psi : V_{s+\sigma}^\varepsilon \rightarrow \mathcal{G}_s^\eta \times U_s(\alpha, A) \times \Lambda,$$

such that  $\phi \circ \psi = \text{id}$ . Moreover,  $\psi$  is Whitney-smooth with respect to  $(\alpha, A)$ .

This result will follow from the inverse function theorem, Theorem A.1, and regularity propositions, Propositions A.2–A.4.

In order to solve locally  $\phi(x) = y$ , we use the remarkable idea of Kolmogorov and find the solution by composing infinitely many times the operator

$$x = (g, u, \lambda) \mapsto x + \phi'^{-1}(x) \cdot (y - \phi(x))$$

on extensions  $T_{s+\sigma}^n$  of shrinking width.

At each step of the induction, it is necessary that  $\phi'^{-1}(x)$  exists at an unknown  $x$  (not only at  $x_0$ ) in a whole neighborhood of  $x_0$  and that  $\phi'^{-1}$  and  $\phi''$  satisfy a suitable estimate, in order to control the convergence of the iterates.

The main step is to check the existence of a right inverse for

$$\phi'(G, P, \lambda) : T_G \mathcal{G}_{s+\sigma}^{\sigma/n} \times \vec{U}_{s+\sigma} \times \Lambda \rightarrow V_{G,s}$$

if  $G$  is close to the identity. We denote by  $\vec{U}$  the vector space directing  $U(\alpha, A)$ .

**Proposition 4.2.** *If  $(G, P, \lambda)$  is close enough to  $(\text{id}, P^0, 0)$  for all  $\delta Q \in V_{G,s+\sigma} = G^* \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \mathbb{C}^{n+m})$ , there exists a unique triplet  $(\delta G, \delta P, \delta \lambda) \in T_G \mathcal{G}_s \times \vec{U}_s \times \Lambda$  such that*

$$\phi'(G, P, \lambda) \cdot (\delta G, \delta P, \delta \lambda) = \delta Q. \quad (4-1)$$

Moreover, we have the estimate

$$\max(|\delta G|_s, |\delta P|_s, |\delta \lambda|) \leq \frac{C'}{\sigma^{\tau'}} |\delta Q|_{G,s+\sigma}, \quad (4-2)$$

where  $C'$  is a constant possibly depending on  $|((G - \text{id}), P - (\theta + \alpha, A \cdot r))|_{s+\sigma}$ .

*Proof.* Let a vector field  $\delta Q \in V_{G,s+\sigma}$  be given. Differentiating with respect to  $x = (G, P, \lambda)$ , we have

$$\delta(T_\lambda \circ G \circ P \circ G^{-1}) = T_{\delta\lambda} \circ (G \circ P \circ G^{-1}) + T'_\lambda \circ (G \circ P \circ G^{-1}) \cdot \delta(G \circ P \circ G^{-1});$$

hence

$$M \cdot (\delta G \circ P + G' \circ P \cdot \delta P - G' \circ P \cdot P' \cdot G'^{-1} \cdot \delta G) \circ G^{-1} = \delta Q - T_{\delta\lambda} \circ (G \circ P \circ G^{-1}),$$

where  $M = \begin{pmatrix} I & 0 \\ 0 & I+B \end{pmatrix}$ .

The data is  $\delta Q$ , while the unknowns are the ‘‘tangent vectors’’  $\delta P \in O(r) \times O(r^2)$ ,  $\delta G$  (geometrically, a vector field along  $G$ ) and  $\delta \lambda \in \Lambda$ .

Precomposing by  $G$ , we get the equivalent equation between germs along the standard torus  $\mathbb{T}_0^n$  (as opposed to  $G(\mathbb{T}_0^n)$ ):

$$M \cdot (\delta G \circ P + G' \circ P \cdot \delta P - G' \circ P \cdot P' \cdot G'^{-1} \cdot \delta G) = \delta Q \circ G - T_{\delta\lambda} \circ G \circ P;$$

multiplying both sides by  $(G'^{-1} \circ P)M^{-1}$ , we finally obtain

$$\dot{G} \circ P - P' \cdot \dot{G} + \delta P = G'^{-1} \circ P \cdot M^{-1} \delta Q \circ G + G'^{-1} \circ P \cdot M^{-1} T_{\delta\lambda} \circ G \circ P, \quad (4-3)$$

where  $\dot{G} = G'^{-1} \cdot \delta G$ .

Note that the term containing  $T_{\delta\lambda}$  is not constant; expanding along  $r = 0$ , it reads as

$$T_\lambda = G'^{-1} \circ P \cdot M^{-1} \cdot T_{\delta\lambda} \circ G \circ P = (\dot{\beta} + O(r), \dot{b} + \dot{B} \cdot r + O(r^2)).$$

The vector field  $\dot{G}$  (geometrically, a germ along  $\mathbb{T}_0^n$  of tangent vector fields) reads as

$$\dot{G}(\theta, r) = (\dot{\varphi}(\theta), \dot{R}_0(\theta) + \dot{R}_1(\theta) \cdot r).$$

The problem is now:  $G, \lambda, P, Q$  being given, find  $\dot{G}, \delta P$  and  $\dot{\lambda}$ , and hence  $\delta \lambda$  and  $\delta G$ .

We are interested in solving the equation up to the 0-order in  $r$  in the  $\theta$ -direction, and up to the first order in  $r$  in the action direction; hence we consider the Taylor expansions along  $\mathbb{T}_0^n$  up to the needed order.

We remark that since  $\delta P = (O(r), O(r^2))$ , it will not intervene in the cohomological equations given out by (4-3), but will be uniquely determined by identification of the reminders.

Let us proceed to solve (4-3); taking its jet at the wanted order, it splits into the following three equations:

$$\begin{aligned} \dot{\varphi}(\theta + \alpha) - \dot{\varphi}(\theta) + p_1 \cdot \dot{R}_0 &= \dot{q}_0 + \dot{\beta}, \\ \dot{R}_0(\theta + \alpha) - A \cdot \dot{R}_0(\theta) &= \dot{Q}_0 + \dot{b}, \\ \dot{R}_1(\theta + \alpha) \cdot A - A \cdot \dot{R}_1(\theta) &= \dot{Q}_1 - (2P_2 \cdot \dot{R}_0 + \dot{R}'_0(\theta + \alpha) \cdot p_1) + \dot{B}. \end{aligned} \quad (4-4)$$

The first equation is the one straightening the tangential dynamics, while the second and the third ones are meant to relocate the torus and straighten the normal dynamics.

For the moment we solve the equations “modulo  $\dot{\lambda}$ ”; eventually  $\delta\lambda$  will be uniquely chosen to kill the average of the equation determining  $\dot{\varphi}$  and the constant component of the given terms in the second and third equation that belong to the kernels of  $A - I$  and  $[A, \cdot]$  respectively, and solve the cohomological equations.

In the following we will repeatedly apply Lemmata 3.1–3.3 and Cauchy’s inequality. Furthermore, we do not keep track of constants — just note that they may only depend on  $n$  and  $\tau$  (from the Diophantine condition) and on  $|G - \text{id}|_{s+\sigma}$  and  $|P - ((\theta + \alpha), A \cdot r)|_{s+\sigma}$ , and refer to them as  $C$ .

– First, the second equation has a solution

$$\dot{R}_0 = L_{1,A}^{-1}(\dot{Q}_0 + \dot{b} - \bar{b}),$$

where  $\bar{b} = \prod_{\text{Ker}(A-I)} \int_{\mathbb{T}^n} \dot{Q}_0 + \dot{b} / (2\pi)^n d\theta$ , and

$$|\dot{R}_0|_s \leq \frac{C}{\gamma^2 \sigma^{\tau+n}} |\dot{Q}_0 + \dot{b}|_{s+\sigma}.$$

– Second, we have

$$\dot{\varphi}(\theta + \alpha) - \dot{\varphi}(\theta) + p_1 \cdot \dot{R}_0 = \dot{q}_0 + \dot{\beta} - \bar{\beta},$$

where  $\bar{\beta} = \int_{\mathbb{T}^n} \dot{q}_0 - p_1 \cdot R_0 + \dot{\beta} / (2\pi)^n d\theta$ ; hence

$$\dot{\varphi} = L_{\alpha}^{-1}(\dot{q}_0 + \dot{\beta} - \bar{\beta}),$$

satisfying

$$|\dot{\varphi}|_{s-\sigma} \leq \frac{C}{\gamma^3 \sigma^{2(\tau+n)}} |\dot{q}_0 + \dot{\beta}|_{s+\sigma}.$$

– Third, the solution of the equation in  $\dot{R}_1$  is

$$\dot{R}_1 = L_{2,A}^{-1}(\tilde{Q}_1 + \dot{B} - \bar{B}),$$

where  $\tilde{Q}_1 = \dot{Q}_1 - (2P_2 \cdot \dot{R}_0 + \dot{R}'_0(\theta + \alpha) \cdot p_1)$ , and  $\bar{B} = \prod_{\text{Ker}[A, \cdot]} \int_{\mathbb{T}^n} \tilde{Q}_1 + \dot{B} / (2\pi)^n d\theta$ . It satisfies

$$|\dot{R}_1|_{s-2\sigma} \leq \frac{C}{\gamma^2 \sigma^{n+\tau}} |\tilde{Q}_1 + \dot{B}|_{s+\sigma}.$$

We now handle the unique choice of the correction  $\delta\lambda = (\delta\beta, \delta b + \delta B \cdot r)$  given by  $T_{\delta\lambda}$ . Letting  $\bar{\lambda} = (\bar{\beta}, \bar{b} + \bar{B} \cdot r)$ , the map  $f : \Lambda \rightarrow \Lambda$ ,  $\delta\lambda \mapsto -\bar{\lambda}$ , is well defined in the neighborhood of  $\delta\lambda = 0$ .

In particular  $f' = -\text{id}$  when  $G = \text{id}$ , and it will remain bounded away from 0 if  $G$  stays sufficiently close to the identity. In particular,  $\delta\lambda \mapsto -\bar{\lambda}$  is affine: the system in  $\Lambda$  to solve  $\lambda = 0$  is linear of the form  $\int_{\mathbb{T}^n} a(G, \dot{Q}) + A(G) \cdot \delta\lambda = 0$ , with diagonal close to 1 when  $G$  is close to the identity; hence  $f$  is invertible. Thus, there exists a unique  $\delta\lambda$  such that  $f(\delta\lambda) = 0$ , satisfying

$$|\delta\lambda| \leq \frac{C}{\gamma^2 \sigma^{\tau+n+1}} |\delta Q|_{G, s+\sigma}.$$

We finally have

$$|\dot{G}|_{s-2\sigma} \leq \frac{C}{\gamma^3} \frac{1}{\sigma^{2(\tau+n)+1}} |\delta Q|_{G, s+\sigma}.$$

Now, from the definition of  $\dot{G} = G'^{-1} \cdot \delta G$ , we get  $\delta G = G' \cdot \dot{G}$ . The unique solutions such that  $\delta\varphi(0) = 0$ ,  $\delta R_0(0) = 0$  and  $\delta R_1(0) = 0$  are easily determined, since  $G$  is close to the identity and similar estimates hold for  $\delta G$ :

$$|\delta G|_{s-2\sigma} \leq \sigma^{-1} (1 + |G - \text{id}|_s) \frac{C}{\sigma^{2(\tau+n)+1}} |\delta Q|_{G, s+\sigma}.$$

Finally, (4-3) uniquely determines  $\delta P$ .

Letting  $\tau' = 2(\tau + n) + 2$ , up to redefining  $\sigma' = \sigma/3$  and  $s' = s + \sigma$ , we have the stated estimates for all  $s', \sigma'$ , where  $s' < s' + \sigma'$ .  $\square$

**Proposition 4.3** (boundedness of  $\phi''$ ). *The bilinear map  $\phi''(x)$ ,*

$$\phi''(x) : (T_G \mathcal{G}_{s+\sigma}^{\sigma/n} \times \vec{U}_{s+\sigma} \times \Lambda)^{\otimes 2} \rightarrow \mathcal{A}(\mathbb{T}_s^n, \mathbb{T}_{\mathbb{C}}^n),$$

*satisfies the estimates*

$$|\phi''(x) \cdot \delta x^{\otimes 2}|_{G, s} \leq \frac{C''}{\sigma^{\tau''}} |\delta x|_{s+\sigma}^2,$$

where  $\tau'' \geq 1$  and  $C''$  is a constant depending on  $|x|_{s+\sigma}$ .

*Proof.* Differentiating  $\phi(x)$  twice yields

$$\begin{aligned} & -M \left\{ \left[ \delta G' \circ P \cdot \delta P + \delta G' \circ P \cdot \delta P + G'' \circ P \cdot \delta P^2 - (\delta G' \circ P + G'' \circ P \cdot \delta P) \cdot P' \cdot G'^{-1} \cdot \delta G \right. \right. \\ & \quad \left. \left. - G' \circ P \cdot (\delta P' \cdot (-G'^{-1} \cdot \delta G' \cdot G'^{-1}) \cdot \delta G) \right] \circ G^{-1} \right. \\ & \quad \left. + \left[ \delta G' \circ P \cdot \delta P + \delta G' \circ P \cdot \delta P + G'' \circ P \cdot \delta P^2 \right. \right. \\ & \quad \left. \left. - (\delta G' \circ P + G'' \circ P \cdot \delta P) \cdot P' \cdot G'^{-1} \cdot \delta G \right. \right. \\ & \quad \left. \left. - G' \circ P \cdot (\delta P' \cdot (-G'^{-1} \cdot \delta G' \cdot G'^{-1}) \cdot \delta G) \right] \circ G^{-1} \cdot (-G'^{-1} \cdot \delta G) \circ G^{-1} \right\}. \end{aligned}$$

Once we precompose with  $G$ , the estimate follows.  $\square$

The hypotheses of Theorem A.1 are satisfied; hence the existence of  $(G, P, \lambda)$  with  $Q = T_\lambda \circ G \circ P \circ G^{-1}$  is proved. Uniqueness and smoothness of the normal form follows from Propositions A.2–A.4. Theorem 4.1 follows, and hence Theorem A.



### 5. A generalization of Rüssmann's theorem

Theorem A provides a normal form that does not rely on any nondegeneracy assumption; thus, the existence of a translated Diophantine, reducible torus will be subordinated to eliminating the “parameters in excess”  $(\beta, B)$  using a nondegeneracy hypothesis. We will implicitly solve  $B = 0$  and  $\beta = 0$  by using the normal frequencies as *free* parameters and a torsion hypothesis respectively. Rüssmann's classical result will be the immediate small-dimensional case.

**Elimination of B.** Let  $\Delta_m^s(\mathbb{R}) \subset \text{GL}_m(\mathbb{R})$  be the open set of invertible matrices with simple, real eigenvalues. On  $\mathbb{T}^n \times \mathbb{R}^m$ , let us define

$$\widehat{U} = \bigcup_{A \in \Delta_m^s(\mathbb{R})} U(\alpha, A).$$

We recall that those  $P \in U(\alpha, A)$  are diffeomorphisms of the form

$$P(\theta, r) = (\theta + \alpha + O(r), A \cdot r + O(r^2)),$$

on a neighborhood of  $\mathbb{T}^n \times \{0\}$ .

The following theorem is an intermediate, yet fundamental result to prove the translated torus theorem, Theorem C, and holds without requiring any torsion assumption on the class of diffeomorphisms.

**Theorem 5.1** (twisted torus of codimension 1). *For every  $P^0 \in U_{s+\sigma}(\alpha, A^0)$  with  $\alpha$  Diophantine and  $A^0 \in \Delta_m^s(\mathbb{R})$ , there is a germ of  $C^\infty$ -maps*

$$\psi : V_{s+\sigma} \rightarrow \mathcal{G}_s \times \widehat{U}_s \times \Lambda(\beta, b), \quad Q \mapsto (G, P, \lambda),$$

at  $P^0 \mapsto (\text{id}, P^0, 0)$  such that  $Q = T_\lambda \circ G \circ P \circ G^{-1}$ , where  $\lambda = (\beta, b) \in \mathbb{R}^{n+1}$ .

**Corollary 5.2** (twisted torus). *If 1 does not belong to the spectrum of  $A^0$ , the translation correction  $b$  is 0.*

*Proof.* Denote by  $\phi_A$  the operator  $\phi$ , as now we want  $A$  to vary. Let us define the map

$$\widehat{\psi} : \Delta_m^s(\mathbb{R}) \times V_{s+\sigma} \rightarrow \mathcal{G}_s \times \widehat{U}_s \times \Lambda, \quad (A, Q) \mapsto \widehat{\psi}_A(Q) := \phi_A^{-1}(Q) = (G, P, \lambda),$$

in the neighborhood of  $(A^0, P^0)$  such that  $Q = T_\lambda \circ G \circ P \circ G^{-1}$ , where  $\lambda = (\beta, b, B \cdot r)$ ,  $\beta \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ , such that  $(A - I) \cdot b = 0$  and  $B \in \text{Mat}_m(\mathbb{R})$  satisfies  $[B, A] = 0$ . Equivalently,  $B$  is simultaneously diagonalizable with  $A$ , since  $A$  has simple spectrum; we can thus restrict our analysis to a neighborhood of  $A^0$  in the subspace of those matrices commuting with  $A^0$ . Note that we can choose such a neighborhood so that it is contained in  $\Delta_m^s(\mathbb{R})$ . Then we study the dependence of  $B$  on  $A$  in their diagonal forms.

Without loss of generality, let  $A^0$  be in its canonical form, and let  $\Delta_{A^0}$  be the subspace of diagonal matrices, namely the matrices which commute with  $A^0$ . Consider the restriction of  $\widehat{\psi}$  to  $\Delta_{A^0}$ . Let  $A \in \Delta_{A^0}$  be close to  $A^0$ , let  $\delta A := A^0 - A$  and write  $P^0$  as

$$P^0(\theta, r) = (\theta + \alpha + O(r), (A^0 - \delta A) \cdot r + \delta A \cdot r + O(r^2));$$

we remark that  $P^0 = T_\lambda \circ P_A$ , where

$$\lambda = (0, B(A) = (A^0 - A) \cdot A^{-1}), \quad [B(A), A] = 0,$$

and  $P_A = (\theta + \alpha + O(r), A \cdot r + O(r^2))$ ,  $A = A^0 - \delta A$ .<sup>2</sup> Note that, since  $A \in \Delta_{A^0}$  has simple spectrum,  $B$  is indeed in  $\Delta_{A^0}$ .

According to Theorem A,  $\phi_A(\text{id}, P_A, \lambda) = P^0$ ; thus locally for all  $A \in \Delta_{A^0}$  close to  $A^0$  we have

$$\hat{\psi}(A, P^0) = (\text{id}, P_A, B \cdot r), \quad B(A, P^0) = (A^0 - A) \cdot A^{-1} = \delta A \cdot (A^0 - \delta A)^{-1},$$

and, in particular  $B(A^0, P^0) = 0$  and

$$\left. \frac{\partial B}{\partial A} \right|_{A=A^0} = -(A^0)^{-1},$$

which is invertible, since  $A^0$  is so. Hence  $A \mapsto B(A)$  is a local diffeomorphism on  $\Delta_{A^0}$  and by the implicit function theorem (in finite dimension) locally for all  $Q$  close to  $P^0$  there exists a unique  $\bar{A}$  such that  $B(\bar{A}, Q) = 0$ . It remains to define  $\psi(Q) = \hat{\psi}(\bar{A}, Q)$ .  $\square$

The proof of Corollary 5.2 is immediate, by conditions (1-4).

**Remark 5.3.** This twisted-torus theorem relies on the peculiarity of the normal dynamics of the torus  $\mathbb{T}_0^n$ . The direct applicability of the implicit function theorem is subordinated to the fact that no arithmetic condition is required on the characteristic (normal) frequencies so that the correction  $A^0 + \delta A$  is well defined; beyond that, since having simple, real eigenvalues is an open property, the needed counter term  $B$  is indeed a diagonal matrix, so that the number of free frequencies (parameters) is enough to solve, implicitly,  $B(A) = 0$ . The generic case of complex eigenvalues is more delicate since one should guarantee that corrections  $A^0 + \delta A$  at each step satisfy the Diophantine condition (1-2). It seems reasonable to think that one would need more parameters to control this issue, using the Whitney smoothness of  $\phi$  on  $A$ , and verify that the measure of such stay positive; see [Féjzo 2004].

**Elimination of  $\beta$ .** If  $Q$  satisfies a torsion hypothesis, the existence of a translated Diophantine torus can be proved.

**Theorem 5.4** (translated Diophantine torus). *Let  $\alpha$  be Diophantine. On a neighborhood of  $\mathbb{T}^n \times \{0\} \subset \mathbb{T}^n \times \mathbb{R}^n$ , let  $P^0 \in U(\alpha, A^0)$  be a diffeomorphism of the form*

$$P^0(\theta, r) = (\theta + \alpha + p_1(\theta) \cdot r + O(r^2), A^0 \cdot r + O(r^2)),$$

where  $A^0$  is invertible and has simple, real eigenvalues and such that

$$\det \left( \int_{\mathbb{T}^n} p_1(\theta) d\theta \right) \neq 0.$$

If  $Q$  is close enough to  $P^0$ , there exists a unique  $A'$ , close to  $A^0$ , and a unique  $(G, P, b) \in \mathcal{G} \times U(\alpha, A') \times \mathbb{R}^n$  such that  $Q = T_b \circ G \circ P \circ G^{-1}$ .

Phrasing the thesis, the graph of  $\gamma = R_0 \circ \varphi^{-1}$  is a translated torus on which the dynamics is conjugated to  $R_\alpha$  by  $\varphi$  (remember the form of  $G \in \mathcal{G}$  given in (1-3)). Before proceeding with the proof of Theorem 5.4, let us consider a parameter  $c \in B_1^n(0)$  (the unit ball in  $\mathbb{R}^n$ ) and the family of maps defined by  $Q_c(\theta, r) :=$

<sup>2</sup>The terms  $O(r^2)$  contain a factor  $(I + \delta A \cdot A^{-1})^{-1}$ .

$Q(\theta, c + r)$  obtained by translating the action coordinates. Considering the corresponding normal form operators  $\phi_c$ , the parametrized version of Theorem A follows readily.

Now, if  $Q_c$  is close enough to  $P_c^0$ , Theorem 5.1 asserts the existence of  $(G_c, P_c, \lambda_c) \in \mathcal{G} \times U(\alpha, A) \times \Lambda(\beta, b)$  such that

$$Q_c = T_\lambda \circ G_c \circ P_c \circ G_c^{-1}.$$

Hence we have a family of tori parametrized by  $\tilde{c} = c + \int_{\mathbb{T}^n} \gamma / (2\pi)^n d\theta$ ,

$$Q(\theta, \tilde{c} + \tilde{\gamma}(\theta)) = (\beta(c) + \varphi \circ R_\alpha \circ \varphi^{-1}(\theta), b(c) + \tilde{c} + \tilde{\gamma}(\varphi \circ R_\alpha \circ \varphi^{-1}(\theta))),$$

where  $\gamma := R_0 \circ \varphi^{-1}$  and  $\tilde{\gamma} = \gamma - \int_{\mathbb{T}^n} \gamma / (2\pi)^n d\theta$ .

*Proof.* Let  $\hat{\varphi}$  be the function defined on  $\mathbb{T}^n$  taking values in  $\text{Mat}_n(\mathbb{R})$  that solves the (matrix of) difference equation

$$\hat{\varphi}(\theta + \alpha) - \hat{\varphi}(\theta) + p_1(\theta) = \int_{\mathbb{T}^n} p_1(\theta) \frac{d\theta}{(2\pi)^n},$$

and let  $F : (\theta, r) \mapsto (\theta + \hat{\varphi}(\theta) \cdot r, r)$ . The diffeomorphism  $F$  restricts to the identity at  $\mathbb{T}_0^n$ . At the expense of substituting  $P^0$  and  $Q$  with  $F \circ P^0 \circ F^{-1}$  and  $F \circ Q \circ F^{-1}$  respectively, we can assume that

$$P^0(\theta, r) = (\theta + \alpha + p_1 \cdot r + O(r^2), A^0 \cdot r + O(r^2)), \quad p_1 = \int_{\mathbb{T}^n} p_1(\theta) \frac{d\theta}{(2\pi)^n}.$$

The germs so obtained from the initial  $P^0$  and  $Q$  are close to one another.

The proof will follow from Theorem 5.1 and the elimination of the parameter  $\beta \in \mathbb{R}^n$  obstructing the rotation conjugacy.

In line with the previous reasoning, we want to show that the map  $c \mapsto \beta(c)$  is a local diffeomorphism. It suffices to show this for the trivial perturbation  $P_c^0$ . The Taylor expansion of  $P_c^0$  directly gives the normal form. In particular  $b(c) = A^0 \cdot c + O(c^2)$ , while the map  $c \mapsto \beta(c) = p_1 \cdot c + O(c^2)$  is such that  $\beta(0) = 0$  and  $\beta'(0) = p_1$ , which is invertible by twist hypothesis, and thus a local diffeomorphism. Hence, the analogous map for  $Q_c$ , which is a small  $C^1$ -perturbation, is a local diffeomorphism too and, together with Theorem 5.1, there exists unique  $c \in \mathbb{R}^n$  and  $A \in \text{Mat}_n(\mathbb{R})$  such that  $(\beta, B) = (0, 0)$ .  $\square$

**Remark 5.5.** The theorem holds also on  $\mathbb{T}^n \times \mathbb{R}^m$ , with  $m \geq n$ , requiring that

$$\text{rank} \left( \int_{\mathbb{T}^n} p_1(\theta) d\theta \right) = n.$$

This guarantees that  $c \mapsto \beta(c)$  is submersive, but  $c$  solving  $\beta(c) = 0$  would no more be uniquely determined.

**Remark 5.6.** Theorem 5.4 generalizes the classical translated curve theorem of Rüssmann in higher dimension, in the case of normally hyperbolic systems such that  $A$  has simple, real, nonzero eigenvalues, for general perturbations.

We stress the fact that if  $P^0$  was of the form

$$P^0(\theta, r) = (\theta + \alpha + O(r), I \cdot r + O(r^2)),$$

like in the original frame studied by Rüssmann, we would need a whole matrix  $B \in \text{Mat}_n(\mathbb{R})$  in order to solve the homological equations, and, having just  $n$  characteristic frequencies at our disposal, it is hopeless to completely solve  $B = 0$  and eliminate the whole obstruction. The torus would not be just translated.

### Appendix A: The inverse function theorem and regularity of $\phi$

We state here the implicit function theorem we use to prove Theorem A as well as the regularity statements needed to guarantee uniqueness and smoothness of the normal form. These results follow from [Féjóz 2010; 2017]. Note that we endowed functional spaces with weighted norms, and bounds appearing in Propositions 4.2–4.3 may depend on  $|x|_s$  (as opposed to the analogous statements in [Féjóz 2010; 2017]); for the corresponding proofs taking into account these (slight) differences, we send the reader to [Massetti 2015a; 2015b] and the proof of Moser's theorem therein.

Let  $E = (E_s)_{0 < s < 1}$  and  $F = (F_s)_{0 < s < 1}$  be two decreasing families of Banach spaces with increasing norms  $|\cdot|_s$  and let  $B_s^E(\sigma) = \{x \in E : |x|_s < \sigma\}$  be the ball of radius  $\sigma$  centered at 0 in  $E_s$ .

On account of composition operators, we additionally endow  $F$  with some deformed norms which depend on  $x \in B_s^E(s)$  such that

$$|y|_{0,s} = |y|_s \quad \text{and} \quad |y|_{\hat{x},s} \leq |y|_{x,s+|x-\hat{x}|_s}.$$

Consider then operators commuting with inclusions  $\phi : B_{s+\sigma}^E(\sigma) \rightarrow F_s$ , with  $0 < s < s + \sigma < 1$ , such that  $\phi(0) = 0$ .

We then suppose that if  $x \in B_{s+\sigma}^E(\sigma)$  then  $\phi'(x) : E_{s+\sigma} \rightarrow F_s$  has a right inverse  $\phi'^{-1}(x) : F_{s+\sigma} \rightarrow E_s$  (for the particular operators  $\phi$  of this work,  $\phi'$  is both left- and right-invertible).

Suppose  $\phi$  is at least twice differentiable.

Let  $\tau := \tau' + \tau''$  and  $C := C' C''$ .

**Theorem A.1** (inverse function theorem). *Assume*

$$|\phi'^{-1}(x) \cdot \delta y|_s \leq \frac{C'}{\sigma^{\tau'}} |\delta y|_{x,s+\sigma}, \quad (\text{A-1})$$

$$|\phi''(x) \cdot \delta x^{\otimes 2}|_{x,s} \leq \frac{C''}{\sigma^{\tau''}} |\delta x|_{s+\sigma}^2 \quad \forall s, \sigma : 0 < s < s + \sigma < 1, \quad (\text{A-2})$$

where  $C'$  and  $C''$  depend on  $|x|_{s+\sigma}$ , and  $\tau', \tau'' \geq 1$ .

For any  $s, \sigma, \eta$  with  $\eta < s$  and  $\varepsilon \leq \eta \sigma^{2\tau} / (2^{8\tau} C^2)$  ( $C \geq 1$ ,  $\sigma < 3C$ ),  $\phi$  has a right inverse  $\psi : B_{s+\sigma}^F(\varepsilon) \rightarrow B_s^E(\eta)$ . In other words,  $\phi$  is locally surjective:

$$B_{s+\sigma}^F(\varepsilon) \subset \phi(B_s^E(\eta)).$$

**Proposition A.2** (Lipschitz continuity of  $\psi$ ). *Let  $\sigma < s$ . If  $y, \hat{y} \in B_{s+\sigma}^F(\varepsilon)$  with  $\varepsilon = 3^{-4\tau} 2^{-16\tau} \sigma^{6\tau} / (4C^3)$ , the following inequality holds:*

$$|\psi(y) - \psi(\hat{y})|_s \leq L |y - \hat{y}|_{x,s+\sigma},$$

with  $L = 2C' / \sigma^{\tau'}$ . In particular,  $\psi$  being the unique local right inverse of  $\phi$ , it is also its unique left inverse.



**Proposition A.3** (smooth differentiation of  $\psi$ ). *Let  $\sigma < s < s + \sigma$  and  $\varepsilon$  be as in Proposition A.2. There exists a constant  $K$  such that for every  $y, \hat{y} \in B_{s+\sigma}^F(\varepsilon)$  we have*

$$|\psi(\hat{y}) - \psi(y) - \phi'^{-1}(\psi(y))(\hat{y} - y)|_s \leq K(\sigma)|\hat{y} - y|_{x,s+\sigma}^2,$$

*and the map  $\psi' : B_{s+\sigma}^F(\varepsilon) \rightarrow L(F_{s+\sigma}, E_s)$  defined locally by  $\psi'(y) = \phi'^{-1}(\psi(y))$  is continuous. In particular  $\psi$  has the same degree of smoothness as  $\phi$ .*

It is sometimes convenient to extend  $\psi$  to non-Diophantine characteristic frequencies  $(\alpha, A)$ . Whitney smoothness guarantees that such an extension exists. Let suppose that  $\phi(x) = \phi_\nu(x)$  depends on some parameter  $\nu \in B^k$  (the unit ball of  $\mathbb{R}^k$ ) and that it is  $C^1$  with respect to  $\nu$  and that estimates on  $\phi_\nu'^{-1}$  and  $\phi_\nu''$  are uniform with respect to  $\nu$  over some closed subset  $D$  of  $\mathbb{R}^k$ .

**Proposition A.4** (Whitney differentiability). *Let us fix  $\varepsilon, \sigma, s$  as in Proposition A.2. The map  $\psi : D \times B_{s+\sigma}^F(\varepsilon) \rightarrow B_s^E(\eta)$  is  $C^1$ -Whitney differentiable and extends to a map  $\psi : \mathbb{R}^{2n} \times B_{s+\sigma}^F(\varepsilon) \rightarrow B_s^E(\eta)$  of class  $C^1$ . If  $\phi$  is  $C^k$ ,  $1 \leq k \leq \infty$ , with respect to  $\nu$ , this extension is  $C^k$ .*

### Appendix B: Inversion of a holomorphism of $\mathbb{T}_s^n$

We present here a classical result and a lemma that justify the well-definedness of the normal form operator  $\phi$  defined in Section 1.

Complex extensions of manifolds are defined with the help of the  $\ell^\infty$ -norm.

Let

$$\begin{aligned} \mathbb{T}_{\mathbb{C}}^n &= \mathbb{C}^n / 2\pi\mathbb{Z}^n \quad \text{and} \quad \mathbb{T}_{\mathbb{C}}^n = \mathbb{T}_{\mathbb{C}}^n \times \mathbb{C}^m, \\ \mathbb{T}_s^n &= \{\theta \in \mathbb{T}_{\mathbb{C}}^n : |\theta| := \max_{1 \leq j \leq n} |\operatorname{Im} \theta_j| \leq s\}, \quad \mathbb{T}_s^n = \{(\theta, r) \in \mathbb{T}_{\mathbb{C}}^n : |(\operatorname{Im} \theta, r)| \leq s\}, \end{aligned}$$

where  $|(\operatorname{Im} \theta, r)| := \max_{1 \leq j \leq n} \max(|\operatorname{Im} \theta_j|, |r_j|)$ .

Let also define  $\mathbb{R}_s^n := \mathbb{R}^n \times (-s, s)$  and consider the universal covering of  $\mathbb{T}_s^n$ ,  $p : \mathbb{R}_s^n \rightarrow \mathbb{T}_s^n$ .

**Theorem B.1.** *Let  $v : \mathbb{T}_s^n \rightarrow \mathbb{C}^n$  be a vector field such that  $|v|_s < \sigma/n$ . The map  $\operatorname{id} + v : \mathbb{T}_{s-\sigma}^n \rightarrow \mathbb{R}_s^n$  induces a map  $\varphi = \operatorname{id} + v : \mathbb{T}_{s-\sigma}^n \rightarrow \mathbb{T}_s^n$  which is a biholomorphism and there is a unique biholomorphism  $\psi : \mathbb{T}_{s-2\sigma}^n \rightarrow \mathbb{T}_{s-\sigma}^n$  such that  $\varphi \circ \psi = \operatorname{id}_{\mathbb{T}_{s-2\sigma}^n}$ .*

*In particular the following hold:*

$$|\psi - \operatorname{id}|_{s-2\sigma} \leq |v|_{s-\sigma}$$

*and, if  $|v|_s < \sigma/(2n)$ ,*

$$|\psi' - \operatorname{id}|_{s-2\sigma} \leq \frac{2}{\sigma}|v|_s.$$

For the proof we again direct readers to [Massetti 2015a; 2015b].

**Corollary B.2** (well-definedness of the normal form operator  $\phi$ ). *For all  $s, \sigma$  if  $G \in \mathcal{G}_{s+\sigma}^{\sigma/n}$ , then  $G^{-1} \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{T}_{s+\sigma}^n)$ .*

*Proof.* We recall the form of  $G \in \mathcal{G}_{s+\sigma}^{\sigma/n}$ :

$$G(\theta, r) = (\varphi(\theta), R_0(\theta) + R_1(\theta) \cdot r).$$

$G^{-1}$  reads as

$$G^{-1}(\theta, r) = (\varphi^{-1}(\theta), R_1^{-1} \circ \varphi^{-1}(\theta) \cdot (r - R_0 \circ \varphi^{-1}(\theta))).$$

Up to rescaling norms by a factor of  $\frac{1}{2}$ , like  $\|x\|_s := \frac{1}{2}|x|$ , the statement is straightforward and follows from Theorem B.1. By abuse of notations, we keep on denoting  $\|x\|_s$  by  $|x|_s$ .  $\square$

### Appendix C: Fourier norms

Let  $\mathcal{A}(\mathbb{T}_s^n, \mathbb{C})$  be the space of holomorphic functions on  $\mathbb{T}_s^n$  with values in  $\mathbb{C}$ , endowed with the norm

$$\|f\|_s = \sum_k \sup_{|r| < s} |f_k(r)| e^{|k|s}, \quad |k| = |k_1| + \dots + |k_n|.$$

If  $f \in \mathcal{A}(\mathbb{T}_s^n, \text{Mat}_m(\mathbb{C}))$ , the definition of the norm is adapted in the obvious way and the expression  $|f_k(r)|$  denotes the standard operator norm  $\sup_{|\xi|=1} |f_k(r)\xi|$ . If  $f : \mathbb{T}_s^n \rightarrow \mathbb{C}^n$ , then  $\|f\|_s = \max_{1 \leq j \leq n} (\|f^j\|_s)$ .

**Lemma C.1.** *Let  $f \in \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \mathbb{C})$  and let  $h \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{C}^n)$  be such that  $\|h\|_s < \sigma/e$ , then*

$$\|f(\theta, r + h(\theta, r))\|_s \leq \frac{1}{1 - e\|h\|_s/\sigma} \|f\|_{s+\sigma}.$$

*Proof.* Let  $f(\theta, r + h(\theta, r)) = \sum_n D^n f(\theta, r) h^n(\theta, r)/n!$  be the Taylor expansion of  $f$ . Then

$$\|f(\theta, r + h(\theta, r))\|_s \leq \sum_k \sup_{|r| < s} \left( \sum_n \frac{1}{n!} \sum_{\ell+k^1+\dots+k^n=k} |D^n f_\ell(r)| |h_{k^1}(r)| \dots |h_{k^n}(r)| \right) e^{|k|s},$$

where  $k^i \in \mathbb{Z}^n$ ,  $i = 1, \dots, n$ , are the Fourier indexes. Since  $|k| \leq |\ell| + |k^1| + \dots + |k^n|$ ,

$$\begin{aligned} \|f(\theta, r + h(\theta, r))\|_s &\leq \sum_k \sup_{|r| < s} \left( \sum_n \frac{1}{n!} \sum_{\ell+k^1+\dots+k^n=k} |D^n f_\ell(r)| e^{|\ell|s} |h_{k^1}(r)| e^{|k^1|s} \dots |h_{k^n}(r)| e^{|k^n|s} \right) \\ &\leq \sum_k \left( \sum_n \frac{1}{n!} \sum_{\ell+k^1+\dots+k^n=k} \sup_{|r| < s} |D^n f_\ell(r)| e^{|\ell|s} |h_{k^1}(r)| e^{|k^1|s} \dots |h_{k^n}(r)| e^{|k^n|s} \right) \\ &\leq \sum_k \left( \sum_n \frac{1}{n!} \sum_{\ell+k^1+\dots+k^n=k} \sup_{|r| < s} |D^n f_\ell(r)| e^{|\ell|s} |h_{k^1}(r)| e^{|k^1|s} \dots |h_{k^n}(r)| e^{|k^n|s} \right) \\ &\leq \sum_n \frac{1}{n!} \sum_\ell \sup_{|r| < s} |D^n f_\ell(r)| e^{|\ell|s} \sum_{k^1} \sup_{|r| < s} |h_{k^1}(r)| e^{|k^1|s} \dots \sum_{k^n} \sup_{|r| < s} |h_{k^n}(r)| e^{|k^n|s} \\ &\leq \sum_\ell \left( \sum_n \frac{n^n}{n!} \sup_{|r| < s+\sigma} |f_\ell(r)| \frac{\|h\|_s^n}{\sigma^n} \right) e^{|\ell|(s+\sigma)}, \end{aligned}$$

where the last estimate follows from the fact that  $(D^n f)_\ell = D^n(f_\ell)$  and the classical Cauchy's estimate by observing that for all  $|r| < s$  letting  $\mathbb{R}_{s+\sigma}^n \ni \xi \neq 0$ , the analytic function  $\varphi(t) = f(r + t\xi)$  on the complex disc  $|t| < \sigma/|\xi|$  satisfies  $d^n \varphi/dt^n|_{t=0} = D^n f(r)\xi^n$ . The factor  $n^n$  comes from the classical bound on the norm of a symmetric multilinear mapping by the associated homogeneous polynomial; see for example [Harris 1975].

It thus follows that

$$\begin{aligned} \|f(\theta, r + h(\theta, r))\|_s &\leq 2 \sum_{\ell} \sup_{|r| < s + \sigma} |f_{\ell}(r)|_{s + \sigma} \sum_{n \geq 1} \frac{e^n}{\sqrt{2\pi n}} \left(\frac{\|h\|_s}{\sigma}\right)^n e^{|\ell|(s + \sigma)} \\ &\leq 2 \|f\|_{s + \sigma} \frac{1}{2(1 - e\|h\|_s/\sigma)}; \end{aligned}$$

hence the stated bound. □

**Lemma C.2.** *Let  $f \in \mathcal{A}(\mathbb{T}_{s+\sigma}^n, \mathbb{C})$  and  $h \in \mathcal{A}(\mathbb{T}_s^n, \mathbb{C})$  be such that  $\|h\|_s < \sigma$ ; then*

$$\|f(\theta + h(\theta), r)\|_s \leq \|f\|_{s + \sigma}.$$

For the proof, see [Chierchia 2003, Appendix B] for example.

### Acknowledgments

The author has been supported by the ERC starting grant HamPDEs. The author is grateful for the mathematical (and moral) support and advice of A. Chenciner and J. Féjoz all the way through her Ph.D, at the Astronomie et Systèmes Dynamiques group of the IMCCE, Observatoire de Paris. This work owes a lot to the meaningful talks of those times. Thank you also to L. Biasco, T. Castan, B. Fayad, J. Laskar, J.-P. Marco, P. Mastrolia, L. Niederman, P. Robutel, A. Pousse, and M. Procesi for the enlightening discussions we had and their constant interest in this (and future) work.

### References

- [Arnol'd 1961] V. I. Arnol'd, “Small denominators, I: Mapping the circle onto itself”, *Izv. Akad. Nauk SSSR Ser. Mat.* **25** (1961), 21–86. In Russian. MR Zbl
- [Bost 1986] J.-B. Bost, “Torsion invariants des systèmes dynamiques hamiltoniens (d’après Kolmogorov, Arnol'd, Moser, Rüssmann, Zehnder, Herman, Pöschel, . . .)”, exposé 639, pp. 113–157 in *Seminar Bourbaki*, 1984/85, Astérisque **133-134**, Société Mathématique de France, Paris, 1986. MR Zbl
- [Calleja et al. 2013] R. C. Calleja, A. Celletti, and R. de la Llave, “A KAM theory for conformally symplectic systems: efficient algorithms and their validation”, *J. Differential Equations* **255**:5 (2013), 978–1049. MR Zbl
- [Chenciner 1985a] A. Chenciner, “Bifurcations de points fixes elliptiques, I: Courbes invariantes”, *Inst. Hautes Études Sci. Publ. Math.* **61** (1985), 67–127. MR Zbl
- [Chenciner 1985b] A. Chenciner, “Bifurcations de points fixes elliptiques, II: Orbites périodiques et ensembles de Cantor invariants”, *Invent. Math.* **80**:1 (1985), 81–106. MR Zbl
- [Chenciner 1988] A. Chenciner, “Bifurcations de points fixes elliptiques, III: Orbites périodiques de “petites” périodes et élimination résonnante des couples de courbes invariantes”, *Inst. Hautes Études Sci. Publ. Math.* **66** (1988), 5–91. MR Zbl
- [Chierchia 2003] L. Chierchia, “KAM lectures”, pp. 1–55 in *Dynamical systems, I*, Scuola Norm. Sup., Pisa, 2003. MR Zbl
- [Eliasson et al. 2015] L. H. Eliasson, B. Fayad, and R. Krikorian, “Around the stability of KAM tori”, *Duke Math. J.* **164**:9 (2015), 1733–1775. MR Zbl
- [Féjoz 2004] J. Féjoz, “Démonstration du ‘théorème d’Arnold’ sur la stabilité du système planétaire (d’après Herman)”, *Ergodic Theory Dynam. Systems* **24**:5 (2004), 1521–1582. MR Zbl
- [Féjoz 2010] J. Féjoz, *Mouvements périodiques et quasi-périodiques dans le problème des  $n$  corps*, Mémoire d’habilitation, Université Pierre et Marie Curie, 2010, available at <https://tel.archives-ouvertes.fr/tel-00702650>.

- [Féjóz 2017] J. Féjóz, “Introduction to KAM theory, with a view to celestial mechanics”, pp. 387–433 in *Variational methods in imaging and geometric control*, edited by M. Bergounioux et al., Radon Series on Computational and Applied Mathematics **18**, de Gruyter, Berlin, 2017.
- [Harris 1975] L. A. Harris, “Bounds on the derivatives of holomorphic functions of vectors”, pp. 145–163 in *Analyse fonctionnelle et applications* (Rio de Janeiro, 1972), Actualités Aci. Indust. **1367**, Hermann, Paris, 1975. MR Zbl
- [Herman 1983] M.-R. Herman, *Sur les courbes invariantes par les difféomorphismes de l’anneau, Vol. 1*, Astérisque **103**, Société Mathématique de France, Paris, 1983. MR Zbl
- [Herman and Sergeraert 1971] M. Herman and F. Sergeraert, “Sur un théorème d’Arnold et Kolmogorov”, *C. R. Acad. Sci. Paris Sér. A-B* **273** (1971), A409–A411. MR Zbl
- [Kolmogorov 1954] A. N. Kolmogorov, “On conservation of conditionally periodic motions for a small change in Hamilton’s function”, *Dokl. Akad. Nauk SSSR (N.S.)* **98** (1954), 527–530. In Russian. MR Zbl
- [Massetti 2015a] J. E. Massetti, “Normal forms for perturbations of systems possessing a diophantine invariant torus”, preprint, 2015. arXiv
- [Massetti 2015b] J. E. Massetti, *Quasi-périodicité et quasi-conservativité*, Ph.D. thesis, Observatoire de Paris, 2015.
- [Meyer 1975] K. R. Meyer, “The implicit function theorem and analytic differential equations”, pp. 191–208 in *Dynamical systems* (Warwick, 1974), edited by A. Manning, Lecture Notes in Mathematics **468**, Springer, 1975. MR Zbl
- [Moser 1962] J. Moser, “On invariant curves of area-preserving mappings of an annulus”, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* **1962** (1962), 1–20. MR Zbl
- [Moser 1966] J. Moser, “A rapidly convergent iteration method and non-linear differential equations, II”, *Ann. Scuola Norm. Sup. Pisa* (3) **20** (1966), 499–535. MR Zbl
- [Moser 1967] J. Moser, “Convergent series expansions for quasi-periodic motions”, *Math. Ann.* **169** (1967), 136–176. MR Zbl
- [Rüssmann 1970] H. Rüssmann, “Kleine Nenner, I: Über invariante Kurven differenzierbarer Abbildungen eines Kreisringes”, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* **1970** (1970), 67–105. MR Zbl
- [Rüssmann 1976] H. Rüssmann, “On optimal estimates for the solutions of linear difference equations on the circle”, *Celestial Mech.* **14:1** (1976), 33–37. MR Zbl
- [Sevryuk 1999] M. B. Sevryuk, “The lack-of-parameters problem in the KAM theory revisited”, pp. 568–572 in *Hamiltonian systems with three or more degrees of freedom* (S’Agaró, 1995), edited by C. Simó, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **533**, Kluwer Acad. Publ., Dordrecht, 1999. MR Zbl
- [Wagener 2010] F. Wagener, “A parametrised version of Moser’s modifying terms theorem”, *Discrete Contin. Dyn. Syst. Ser. S* **3:4** (2010), 719–768. MR Zbl
- [Yoccoz 1992] J.-C. Yoccoz, “Travaux de Herman sur les tores invariants”, exposé 754, pp. 311–344 in *Séminaire Bourbaki*, 1991/92, Astérisque **206**, Société Mathématique de France, Paris, 1992. MR Zbl
- [Zehnder 1975] E. Zehnder, “Generalized implicit function theorems with applications to some small divisor problems, I”, *Comm. Pure Appl. Math.* **28** (1975), 91–140. MR Zbl

Received 31 Aug 2016. Revised 14 Jun 2017. Accepted 28 Jul 2017.

JESSICA ELISA MASSETTI: [jmassetti@mat.uniroma3.it](mailto:jmassetti@mat.uniroma3.it)

Università degli Studi Roma Tre, Dipartimento di Matematica e Fisica, Sezione Matematica, Rome, Italy



# Analysis & PDE

msp.org/apde

## EDITORS

EDITOR-IN-CHIEF

Patrick Gérard  
patrick.gerard@math.u-psud.fr  
Université Paris Sud XI  
Orsay, France

## BOARD OF EDITORS

Nicolas Burq	Université Paris-Sud 11, France nicolas.burq@math.u-psud.fr	Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Sun-Yung Alice Chang	Princeton University, USA chang@math.princeton.edu	Tristan Rivière	ETH, Switzerland riviere@math.ethz.ch
Michael Christ	University of California, Berkeley, USA mchrist@math.berkeley.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Wilhelm Schlag	University of Chicago, USA schlag@math.uchicago.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Sylvia Serfaty	New York University, USA serfaty@cims.nyu.edu
Vaughan Jones	U.C. Berkeley & Vanderbilt University vaughan.f.jones@vanderbilt.edu	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Herbert Koch	Universität Bonn, Germany koch@math.uni-bonn.de	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Gilles Lebeau	Université de Nice Sophia Antipolis, France lebeau@unice.fr	András Vasy	Stanford University, USA andras@math.stanford.edu
Richard B. Melrose	Massachusetts Inst. of Tech., USA rbm@math.mit.edu	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Frank Merle	Université de Cergy-Pontoise, France Frank.Merle@u-cergy.fr	Steven Zelditch	Northwestern University, USA zelditch@math.northwestern.edu
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu
Clément Mouhot	Cambridge University, UK c.mouhot@dpms.cam.ac.uk		

## PRODUCTION

production@msp.org  
Silvio Levy, Scientific Editor

---

See inside back cover or [msp.org/apde](http://msp.org/apde) for submission instructions.

---

The subscription price for 2018 is US \$275/year for the electronic version, and \$480/year (+\$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.


---

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

---

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2018 Mathematical Sciences Publishers

# ANALYSIS & PDE

Volume 11 No. 1 2018

---

Analytic torsion, dynamical zeta functions, and the Fried conjecture SHU SHEN	1
Existence theorems of the fractional Yamabe problem SEUNGHYEOK KIM, MONICA MUSSO and JUNCHENG WEI	75
On the Fourier analytic structure of the Brownian graph JONATHAN M. FRASER and TUOMAS SAHLSTEN	115
Nodal geometry, heat diffusion and Brownian motion BOGDAN GEORGIEV and MAYUKH MUKHERJEE	133
A normal form à la Moser for diffeomorphisms and a generalization of Rüssmann's translated curve theorem to higher dimensions JESSICA ELISA MASSETTI	149
Global results for eikonal Hamilton–Jacobi equations on networks ANTONIO SICONOLFI and ALFONSO SORRENTINO	171
High-frequency approximation of the interior Dirichlet-to-Neumann map and applications to the transmission eigenvalues GEORGI VODEV	213
Hardy–Littlewood inequalities on compact quantum groups of Kac type SANG-GYUN YOUN	237



2157-5045(2018)11:1;1-T