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CONCENTRATION ET RANDOMISATION UNIVERSELLE DE SOUS-ESPACES PROPRES

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Nous étudions des conditions nécessaires et suffisantes de convergence pour des séries aléatoires de fonctions propres dans L^p, avec p fini. De façon précise, nous montrons des résultats optimaux pour les harmoniques sphériques sur $\mathbb{S}^d$ et l'oscillateur harmonique sur $\mathbb{R}^d$ (cela améliore des résultats de Ayache–Tzvetkov, Grivaux et Imekraz–Robert–Thomann). Dans le cas multidimensionnel, nous utiliserons des séries aléatoires faisant intervenir des matrices aléatoires. Cela nous permettra de donner un éclairage sur la construction d'une famille de mesures construites par Burq–Lebeau sur l'espace de Hilbert d'une variété riemannienne compacte. En fait, nous montrons que c'est précisément parce que L^p est de cotype fini que cette construction est possible (il s'agit d'une version multidimensionnelle du théorème de Maurey–Pisier).

We study necessary and sufficient conditions of convergence for random series of eigenfunctions in L^p, for finite p. More precisely, we get optimal results for the spherical harmonics on $\mathbb{S}^d$ and for the harmonic oscillator on $\mathbb{R}^d$ (this improves results by Ayache–Tzvetkov, Grivaux and Imekraz–Robert–Thomann). In the multidimensional framework, our random series involve random matrices. This illuminates a construction, made by Burq–Lebeau, of a family of specific measures on the Hilbert space of a Riemannian boundaryless compact manifold. Actually, we show that the latter construction is possible because L^p has finite cotype (this is nothing but a multidimensional version of the Maurey–Pisier theorem).

1. Introduction

1A. ***Littérature existante.*** Cet article étudie de manière unifiée les questions suivantes :

MSC2010 : 15B52, 46B09, 60G50.
Mots-clefs : random matrix, eigenfunctions, Sobolev embedding.

(i) Fixons $p \in [1, +\infty[$. Peut-on trouver des conditions *déterministes* (nécessaires et suffisantes) qui assurent la convergence dans $L^p(\mathbb{S}^d)$ des séries *aléatoires* constituées par des harmoniques sphériques ?

(ii) Qu'en est-il pour l'oscillateur harmonique $-\Delta + |x|^2$ sur la variété non compacte $X = \mathbb{R}^d$?

(iii) Quelles séries aléatoires permettent de résoudre les questions précédentes ?

Dans le cas où les fonctions propres considérées ont des propriétés de concentration, nous nous proposons de démontrer des résultats optimaux qui complètent ceux obtenus séparément par Ayache et Tzvetkov [2008] et Grivaux [2010], ainsi qu'un résultat obtenu conjointement par Robert, Thomann et l'auteur dans [Imekraz et al. 2016]. En ce qui concerne la randomisation multidimensionnelle, nous obtenons une extension de deux résultats de Maurey et Maurey–Pisier. Cela donnera un éclairage sur la construction d'une famille de mesures sur $L^2(X)$ (où X est une variété riemannienne compacte sans bord) construites par Burq et Lebeau [2013].

Avant d'écrire des énoncés précis, il convient de citer les principaux travaux existants concernant cette problématique. La source de toute cette étude réside dans le théorème de Paley et Zygmund [1930; 1932] : si l'on considère une suite $(a_n)_{n\in\mathbb{Z}}$ appartenant à $\ell^2(\mathbb{Z})$ et une suite $(\varepsilon_n)_{n\in\mathbb{Z}}$ de variables aléatoires indépendantes suivant une loi $\frac{1}{2}$-Bernoulli à valeurs dans $\{-1, +1\}$ alors la série de Fourier aléatoire $\sum_{n\in\mathbb{Z}} \varepsilon_n a_n e^{inx}$ définit presque sûrement un élément de $L^p(\mathbb{T})$ pour tout réel $p \in [2, +\infty[$. La randomisation permet ainsi un gain d'intégrabilité alors qu'il n'y a évidemment aucun gain de régularité dans l'échelle des espaces de Sobolev $H^s(\mathbb{T})$. Le théorème de Paley–Zygmund peut être considéré comme une amélioration probabiliste de l'injection de Sobolev $H^{\frac{1}{2}-\frac{1}{p}}(\mathbb{T}) \subset L^p(\mathbb{T})$ valide pour tout réel $p \in [2, +\infty[$. Les démonstrations du théorème de Paley–Zygmund utilisent généralement l'inégalité de Khintchine. L'ouvrage de Kahane [1985] contient de nombreux résultats importants dans ce thème et introduit notamment des versions banachiques de l'inégalité de Khintchine, à savoir les inégalités de Kahane–Khintchine (voir plus loin (39)). Citons maintenant trois résultats connus concernant la randomisation dans les espaces de Lebesgue.

Le premier résultat est le théorème de Maurey et énonce ceci : pour toute suite $(u_n)_{n\in\mathbb{N}}$ de $L^p(X)$ où X est un espace mesuré σ-fini, on a l'équivalence

$$\sum \varepsilon_n u_n \quad \text{converge presque sûrement dans } L^p(X) \quad \Leftrightarrow \quad \sqrt{\sum_{n\in\mathbb{N}} |u_n|^2} \in L^p(X). \tag{1}$$

En fait, le théorème de Maurey permet de remplacer $L^p(X)$ par n'importe quel treillis de Banach qui dispose de la propriété de cotype fini [Maurey 1974, pages 21–22; Lindenstrauss et Tzafriri 1973, Theorem 1.d.6, Corollary 1.f.9].

Le deuxième résultat est le théorème de Maurey et Pisier [1976, corollaire 1.3] (voir le théorème 1.6 ci-dessous). Ce dernier assure, avec les mêmes notations, que les convergences des séries aléatoires $\sum \varepsilon_n u_n$ et $\sum g_n u_n$ dans $L^p(X)$ sont équivalentes où $(g_n)_{n\in\mathbb{N}}$ désigne une suite i.i.d. de gaussiennes suivant une loi $\mathcal{N}_{\mathbb{C}}(0, 1)$. En fait, le théorème de Maurey–Pisier permet de remplacer $L^p(X)$ par un espace de Banach complexe de cotype fini et les gaussiennes g_n par des variables aléatoires centrées plus générales.

Le troisième résultat a été obtenu par Figà-Talamanca et Rider [1966, Theorem 4 ; 1967, Corollary 4]. qui ont pu remplacer le tore $\mathbb{T}$ par un groupe compact quelconque G dans le théorème de Paley–Zygmund. On pourra aussi consulter [Figà-Talamanca 1971]. Marcus et Pisier [1981] ont résolu le problème de la convergence presque sûre des séries de Fourier dans $L^\infty(G)$ et leur analyse permet de retrouver les résultats dans l'échelle des espaces $L^p(G)$ avec $p \in [1, +\infty[$ à l'aide d'une version multidimensionnelle des inégalités de Kahane–Khintchine. Ces dernières inégalités, que nous choisissons de nommer les inégalités de Kahane–Khintchine–Marcus–Pisier (voir plus loin (40)), vont jouer un rôle important dans notre article.

Récemment, ces problèmes ont ressurgi en considérant les fonctions trigonométriques $x \mapsto e^{inx}$ non pas comme les caractères du groupe abélien compact $\mathbb{T}$ mais plutôt comme les modes propres de l'opérateur de Laplace–Beltrami $\frac{d^2}{dx^2}$. Deux types de résultat ont notamment motivé cette résurgence :

(i) Les bases hilbertiennes aléatoires de modes propres ont des propriétés non triviales comme l'ergodicité quantique [Zelditch 1992; Robert et Thomann 2015] ou des estimations de normes L^p bien meilleures que celles des bases hilbertiennes “canoniques” [Poiret et al. 2015; Burq et Lebeau 2013].

(ii) L'étude des équations non-linéaires de type Schrödinger ou ondes sur une variété riemannienne compacte X avec des conditions initiales à faible régularité dans les espaces de Sobolev $H^s(X)$ (on parle de régime sur-critique) est un problème difficile en toute dimension et tout spécialement en dimension $\dim(X) \geq 3$. La randomisation donne un gain d'intégrabilité qui permet de construire des solutions qui sont hors d'atteinte avec les méthodes déterministes actuelles. Ces travaux concernent les constructions de mesures de Gibbs (voir les articles [Bourgain 1994 ; 1996 ; Bourgain et Bulut 2014a ; 2014b ; 2014c ; Burq et al. 2013 ; Deng 2012 ; Tzvetkov 2008] et leurs références) ou des conditions initiales aléatoires plus générales [Burq et al. 2015 ; Burq et Tzvetkov 2008a ; 2008b ; 2014 ; Poiret et al. 2014]. Concernant ces derniers articles, on pourra consulter le séminaire Bourbaki [de Bouard 2015].

Les deux points précédents ont même été combinés par de Suzzoni [2014] qui a étudié l'équation cubique des ondes sur la sphère $\mathbb{S}^3$ à l'aide d'une base hilbertienne aléatoire de $L^2(\mathbb{S}^3)$.

Il est donc légitime d'étudier la randomisation non plus sur un groupe compact mais sur une variété riemannienne compacte X (que l'on supposera toujours lisse, sans bord et de dimension $d \geq 2$). Étant données une suite $(a_n) \in \ell^2(\mathbb{N})$, une famille *orthonormée* $(\phi_n)_{n\geq 0}$ de $L^2(X)$ constituée de fonctions propres de l'opérateur de Laplace–Beltrami Δ sur X et la fonction

$$\sum_{n\geq 0} a_n \phi_n \in L^2(X),$$

on s'intéresse à la convergence presque sûre dans $L^p(X)$ de la série aléatoire $\sum \varepsilon_n a_n \phi_n$. Bien entendu, le critère (1) répond à la question de façon théorique mais il ne paraît pas évident de le traduire en un comportement asymptotique sur la suite des coefficients $(a_n)_{n\in\mathbb{N}}$. Sans surprise, cette étude est intimement liée à la suite des normes $\|\phi_n\|_{L^p(X)}$ en vertu de l'inégalité triangulaire :

$$\forall p \in [2, +\infty[, \quad \left\| \sqrt{\sum_{n\in\mathbb{N}} |a_n|^2 |\phi_n|^2} \right\|_{L^p(X)} \leq \left(\sum_{n\in\mathbb{N}} |a_n|^2 \|\phi_n\|^2_{L^p(X)} \right)^{\frac{1}{2}}. \tag{2}$$

En fait, nous verrons que c'est l'éventuelle concentration des fonctions ϕ_n qui rentre en jeu. Rappelons les résultats connus. Tzvetkov montre ce que l'on appellera par la suite une *injection de Sobolev probabiliste* (selon la terminologie introduite par Burq et Lebeau [2013]) : pour tout réel $p \in]2, +\infty[$ il existe un réel explicite $\delta(d, p) < d\left(\frac{1}{2} - \frac{1}{p}\right)$ tel que

$$\sum_{n\geq 0} a_n \phi_n \in H^{\delta(d,p)}(X) \quad \Rightarrow \quad \sum_{n\geq 0} \varepsilon_n a_n \phi_n \quad \text{converge p.s. dans } L^p(X). \tag{3}$$

Avant de donner l'expression de $\delta(d, p)$, signalons que (3) améliore l'*injection de Sobolev déterministe*

$$\sum_{n\geq 0} a_n \phi_n \in H^{d\left(\frac{1}{2}-\frac{1}{p}\right)}(X) \quad \Rightarrow \quad \sum_{n\geq 0} a_n \phi_n \in L^p(X).$$

Par construction (voir [Tzvetkov 2010, Theorem 4]), le nombre $\delta(d, p)$ vient des inégalités de Sogge [1988] que nous rappelons : si l'on a $\Delta\phi_n = -\lambda_n^2 \phi_n$, avec $\lambda_n > 0$ alors on a

$$\|\phi_n\|_{L^p(X)} \leq C(X, p)\lambda_n^{\delta(d,p)}, \quad \delta(d, p) := \begin{cases} \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) & \text{si } 2 \leq p \leq \frac{2(d+1)}{d-1}, \\ \frac{d-1}{2} - \frac{d}{p} & \text{si } \frac{2(d+1)}{d-1} \leq p \leq +\infty, \end{cases} \tag{4}$$

où $C(X, p) \geq 1$ ne dépend que de X et p. Le cas $p = \infty$ est dû à Avakumovič, Levitan et Hörmander [Hörmander 1968] et les inégalités (4) sont optimales pour $X = \mathbb{S}^d$. Notons au passage que le critère de Maurey (1) et l'inégalité triangulaire (2) permettent de retrouver immédiatement l'implication (3).

De façon indépendante, des travaux font intervenir la notion de randomisation multidimensionnelle sur une variété riemannienne compacte X en tenant compte de la décomposition spectrale de l'opérateur de Laplace–Beltrami sur $L^2(X)$. Cette notion apparaît sous des formes en apparence différente, dans les travaux de Shiffman et Zelditch [2003; Zelditch 1992], dans celui de Burq et Lebeau [2013] ainsi que dans celui de Poiret, Robert et Thomann [Poiret et al. 2015] (pour l'oscillateur harmonique sur $\mathbb{R}^d$) avec des arguments de "concentration de la mesure" et de "grandes déviations". C'est dans l'article [Burq et Lebeau 2013] que le terme "injection de Sobolev probabiliste" apparaît pour la première fois pour exprimer rigoureusement le gain d'intégrabilité obtenu par la randomisation. Même si cela n'est pas explicitement écrit, il nous semble qu'un des intérêts du papier [Burq et Lebeau 2013] est précisément de s'émanciper de l'astuce de considérer des modes propres invariants par symétrie (ce qui permet usuellement de réduire un problème multidimensionnel à un problème unidimensionnel). C'est ainsi que Burq et Lebeau obtiennent un résultat d'existence locale pour l'équation semi-linéaire des ondes sur une variété riemannienne compacte en régime sur-critique et leurs solutions ne sont pas spectralement supportées par des sous-suites particulières de fonctions propres.

Dans notre article, nous étudions les propriétés de dualité et interpolation de nouveaux espaces de Banach associés à une suite de sous-espaces propres $(E_n)_{n\geq 1}$, dénommés plus loin *espaces de Lebesgue* *probabilistes* et notés $\mathbf{PL}^p\big(X, \bigoplus E_n\big)$. Afin de motiver cette étude dans le cadre de la randomisation des fonctions propres, nous écrivons dans les trois prochaines parties les applications que nous obtenons.

1B. ***Harmoniques sphériques de*** $\mathbb{S}^d$. Ayache et Tzvetkov [2008, Theorem 1] obtiennent un éclairage gaussien du théorème de Paley–Zygmund, à savoir l'équivalence pour tout $p \in [2, +\infty[$ des deux assertions suivantes :

(i) les fonctions propres ϕ_n sont uniformément bornées dans $L^p(X)$,

(ii) pour toute suite complexe $(a_n)_{n\in\mathbb{N}}$, la fonction gaussienne aléatoire $\sum_{n\geq 0} g_n a_n \phi_n$ appartient presque sûrement à $L^p(X)$ si et seulement si $(a_n)_{n\in\mathbb{N}}$ appartient à $\ell^2(\mathbb{N})$.

Alors que les preuves usuelles du théorème de Paley–Zygmund utilisent l'égalité $|e^{inx}| = 1$, l'équivalence (i) $\Leftrightarrow$ (ii) montre que c'est plutôt l'inégalité $\sup_{n\in\mathbb{N}} \|e^{inx}\|_{L^p_x(-\pi,\pi)} < +\infty$ qui est déterminante. En d'autres termes, seule l'explosion des normes $\|\phi_n\|_{L^p}$ peut contredire la conclusion du théorème de Paley–Zygmund. Dans ce cas, Ayache et Tzvetkov posent la question de calculer en fonction des coefficients a_n, la borne supérieure des réels $p \in [2, +\infty[$ tels que la série aléatoire $\sum g_n a_n \phi_n$ converge presque sûrement dans $L^p(X)$. Sans aucune information sur les fonctions ϕ_n, cette question est trop générale et Ayache et Tzvetkov examinent les modes propres radiaux ψ_n de l'opérateur Laplacien Δ avec condition de Dirichlet au bord sur la boule fermée unité $\overline{\mathbb{B}_d(0,1)}$ de $\mathbb{R}^d$. Il s'avère que les fonctions ψ_n ne sont pas uniformément bornées dans $L^p(\mathbb{B}_d(0,1))$ pour $p \gg 1$, et l'on en déduit l'existence de suites $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ telles que la série aléatoire gaussienne $\sum g_n a_n \psi_n$ diverge presque sûrement dans $L^p(\mathbb{B}_d(0,1))$. Il s'agit là d'une différence majeure avec le théorème de Paley–Zygmund sur le tore $\mathbb{T}^d$. Dans le cas des fonctions ψ_n, Ayache et Tzvetkov [2008, Theorem 4] obtiennent une réponse partielle. Dans [Grivaux 2010] apparaissent deux idées qui vont jouer un rôle important dans notre travail :

(i) D'une part, Grivaux remarque que le théorème de Maurey–Pisier assure l'équivalence des convergences des séries aléatoires $\sum \varepsilon_n a_n \psi_n$ et $\sum g_n a_n \psi_n$.

(ii) D'autre part, Grivaux répond à la question de Ayache et Tzvetkov en utilisant la concentration des fonctions ψ_n en l'origine $0 \in \mathbb{R}^d$.

Comme le remarquent Ayache et Tzvetkov [2008, Theorem 4, Remark (d)], l'analyse précédente se transfère sans problème aux fonctions propres zonales de l'opérateur de Laplace–Beltrami sur la sphère

$$\mathbb{S}^d := \big\{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} \mid x_1^2 + \cdots + x_{d+1}^2 = 1\big\}, \quad d \geq 2.$$

On notera μ_d la mesure volume de $\mathbb{S}^d$ et l'on rappelle que le spectre de l'opérateur de Laplace–Beltrami Δ est donné par la suite $(-n(n+d-1))_{n\in\mathbb{N}}$. Convenons aussi qu'une fonction sur $\mathbb{S}^d$ est zonale si elle ne dépend que de la première coordonnée x_1 d'un point $x \in \mathbb{S}^d$. D'après [Stein et Weiss 1971, Chapter IV.2, Theorem 2.14, page 149], on sait que l'opérateur de Laplace–Beltrami Δ admet une suite de fonctions propres zonales $(Z_n)_{n\geq 1}$ vérifiant :

$$Z_n(x) = n^{\frac{1}{2}} P_n^{\left(\frac{d-2}{2}, \frac{d-2}{2}\right)}(x_1), \quad \Delta Z_n = -n(n+d-1) Z_n, \quad \|Z_n\|_{L^2(\mathbb{S}^d)} \simeq_d 1, \tag{5}$$

où $P_n^{((d-2)/2,(d-2)/2)}$ est le n-ième polynôme de Jacobi, c'est-à-dire le n-ième polynôme orthogonal pour le poids $w \in [-1,1] \mapsto (1-w^2)^{\frac{d-2}{2}}$ et normalisé de sorte que

$$\forall n \in \mathbb{N}^\star, \quad P_n^{\left(\frac{d-2}{2},\frac{d-2}{2}\right)}(1) = \binom{n+\frac{d-2}{2}}{n} \simeq_d n^{\frac{d-2}{2}}.$$

On peut montrer que l'on a les estimations (voir partie 3D) :

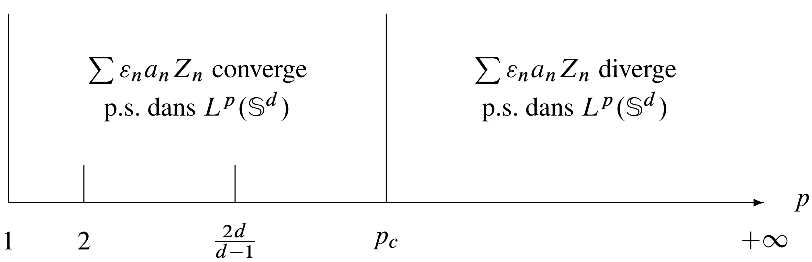
$$\begin{aligned} 1 \le p < \frac{2d}{d-1} \quad &\Rightarrow \quad \|Z_n\|_{L^p(\mathbb{S}^d)} \simeq_{d,p} 1, \\ p = \frac{2d}{d-1} \quad &\Rightarrow \quad \|Z_n\|_{L^p(\mathbb{S}^d)} \simeq_{d,p} \sqrt[p]{\ln(n+1)}, \\ \frac{2d}{d-1} < p < \infty \quad &\Rightarrow \quad \|Z_n\|_{L^p(\mathbb{S}^d)} \simeq_{d,p} n^{\frac{d-1}{2}-\frac{d}{p}}. \end{aligned} \tag{6}$$

D'après [Ayache et Tzvetkov 2008], on sait que pour toute suite $(a_n) \in \ell^2(\mathbb{N}^\star)$ et tout $p \in \left[2, \frac{2d}{d-1}\right[$, la série aléatoire $\sum g_n a_n Z_n$ converge presque sûrement dans $L^p(\mathbb{S}^d)$. Pour comprendre ce qu'il en est pour $p > \frac{2d}{d-1}$, on pourrait appliquer la méthode de Grivaux [2010]. Cette méthode nécessite de rappeler que les fonctions zonales Z_n se concentrent autour de deux boules centrées aux pôles $(\pm 1, 0, \dots, 0)$ de rayon d'ordre $\frac{1}{n}$ avec une amplitude d'ordre $n^{\frac{d-1}{2}}$. Remarquons au passage que les formules (6) expliquent que cette concentration polaire est significative dans $L^p(\mathbb{S}^d)$ pour $p > \frac{2d}{d-1}$. Notons maintenant

$$p_c := \sup\left\{ p > \frac{2d}{d-1} \;\middle|\; \frac{1}{n^{d+1}} \left(\sum_{k=1}^{n} k^{d-1} |a_k|^2 \right)^{\frac{p}{2}} = \mathcal{O}\left(\frac{1}{n}\right) \right\}. \tag{7}$$

Alors pour tout réel $p \in [2, +\infty[$, la méthode de Grivaux donne les implications suivantes :

$$\begin{aligned} p < p_c \Rightarrow \ &\text{la série aléatoire } \sum g_n a_n Z_n \text{ converge p.s. dans } L^p(\mathbb{S}^d), \\ p > p_c \Rightarrow \ &\text{la série aléatoire } \sum g_n a_n Z_n \text{ diverge p.s. dans } L^p(\mathbb{S}^d). \end{aligned}$$



Les arguments connus ne permettent pas de décider ce qu'il en est en $p = p_c$ (question soulevée dans [Imekraz et al. 2016, remark pages 2776–7] qui utilise notamment la méthode de Grivaux pour les fonctions propres radiales de l'opérateur $-\Delta + |x|^2$). Le premier résultat que nous énonçons élucide ce point.

Théorème 1.1. *On considère une suite complexe* $(a_n)_{n\geq 1}$ *à croissance polynomiale et un réel* $p \in \left]\frac{2d}{d-1}, +\infty\right[$. *La série aléatoire* $\sum g_n a_n Z_n$ *converge presque sûrement dans* $L^p(\mathbb{S}^d)$ *si et seulement si*

$$\sum_{n\geq 1} \frac{1}{n^{d+1}} \left(\sum_{k=1}^{n} k^{d-1} |a_k|^2 \right)^{\frac{p}{2}} < +\infty.$$

Le théorème précédent nous permet de préciser la formule (7) par comparaison logarithmique et rend légitime la mise en évidence de son terme $\mathcal{O}(\frac{1}{n})$. Examinons par exemple des coefficients de la forme

$$\alpha_n = \frac{1}{n^{d(\frac{1}{2}-\frac{1}{p})} \ln^{\frac{\beta}{p}}(n)}, \qquad \beta \geq 0, \quad p > \frac{2d}{d-1}, \quad n \geq 2.$$

La suite $(\alpha_n)_{n\geq 2}$ appartient à $\ell^2(\mathbb{N}\backslash\{0,1\})$ et l'on calcule facilement l'équivalent

$$\frac{1}{n^{d+1}} \left(\sum_{k=2}^{n} k^{d-1} |\alpha_k|^2 \right)^{\frac{p}{2}} \sim \left(\frac{p}{2d} \right)^{\frac{p}{2}} \frac{1}{n \ln^{\beta}(n)}, \quad n \to +\infty.$$

La borne supérieure (7) vaut donc p pour toute valeur de β, mais le théorème 1.1 permet d'affirmer que la série aléatoire $\sum_{n\geq 2} g_n \alpha_n Z_n$ converge presque sûrement dans $L^p(\mathbb{S}^d)$ si et seulement si $\beta > 1$.

Donnons une idée de la preuve du théorème 1.1. Nous justifierons rigoureusement que l'on peut assimiler $|Z_n|$ à sa restriction $\widetilde{Z}_n$ sur une boule de rayon d'ordre $\frac{1}{n}$ et centrée en l'un des pôles de symétrie. Cela nous permettra d'obtenir l'encadrement suivant pour tout $p > \frac{2d}{d-1}$:

$$\left\| \sqrt{\sum_{n\geq 1} |a_n \widetilde{Z}_n|^2} \right\|_{L^p(\mathbb{S}^d)} \leq \left\| \sqrt{\sum_{n\geq 1} |a_n Z_n|^2} \right\|_{L^p(\mathbb{S}^d)} \leq C(d,p) \left\| \sqrt{\sum_{n\geq 1} |a_n \widetilde{Z}_n|^2} \right\|_{L^p(\mathbb{S}^d)}, \tag{8}$$

où $C(d,p) \geq 1$ ne dépend que de d et p. Comme les normes dans $L^{\frac{p}{2}}(\mathbb{S}^d)$ des fonctions $\sum_{n\geq 1} |a_n \widetilde{Z}_n|^2$ sont calculables explicitement, on pourra finir en invoquant le critère (1). La majoration (8) est délicate et utilisera de nouveaux résultats abstraits d'interpolation développés dans la partie 2B. Signalons dès maintenant que l'interpolation (réelle ou complexe) des normes qui apparaissent dans (8) n'est pas gratuite et découlera des propriétés de concentration des fonctions zonales Z_n. Par exemple, il découlera de (6) et de la proposition 2.4 que l'interpolation n'est pas réalisée si l'on autorise p à parcourir un intervalle ouvert contenant $\frac{2d}{d-1}$.

Nos arguments permettent de traiter un autre cas important de fonctions propres, à savoir la suite $(Y_n)_{n\geq 1}$ des fonctions propres "de plus haut poids" qui se concentrent sur une géodésique. Ces dernières sont définies par les formules

$$\forall n \in \mathbb{N}^{\star}, \quad Y_n(x) := c_{d,n}(x_1 + i x_2)^n, \quad \|Y_n\|_{L^2(\mathbb{S}^d)} = 1, \tag{9}$$

où $c_{d,n} > 0$ est une constante de normalisation. On vérifie que l'on a $\Delta Y_n = -n(n+d-1)Y_n$ et $c_{d,n} \simeq_d n^{\frac{d-1}{4}}$. Les fonctions Y_n sont connues pour avoir des estimations de normes dans $L^p(\mathbb{S}^d)$ maximales pour $2 < p \leq \frac{2(d+1)}{d-1}$ et minimales pour $p \in [1, 2[$ parmi les modes propres $L^2(\mathbb{S}^d)$-normalisés associés à la valeur propre $-n(n+d-1)$ (et cela est même optimal d'après les inégalités de Sogge (4) et [Sogge et Zelditch 2011, Proposition 2]). De manière précise, il est connu et facile à vérifier que (9) implique les estimations suivantes

$$\forall p \in [1,+\infty[\cup\{+\infty\}, \quad \forall n \in \mathbb{N}^\star, \quad \|Y_n\|_{L^p(\mathbb{S}^d)} \simeq_{d,p} n^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})}. \tag{10}$$

On montrera le théorème suivant.

Théorème 1.2. *On considère une suite complexe $(a_n)_{n\geq 1}$ à croissance polynomiale. Pour tout réel $p \in]1,+\infty[$, la série aléatoire $\sum g_n a_n Y_n$ converge presque sûrement dans $L^p(\mathbb{S}^d)$ si et seulement si*

$$\sum_{n\geq 1} \frac{1}{n^{\frac{d+1}{2}}} \left(\sum_{k=1}^{n} k^{\frac{d-1}{2}} |a_k|^2 \right)^{\frac{p}{2}} < +\infty. \tag{11}$$

Comme application immédiate, on obtient le fait suivant qui contraste avec le théorème de Paley–Zygmund sur $\mathbb{T}$: la fonction $\sum_{n\geq 2}(\sqrt{n}\ln(n))^{-1} Y_n$ appartient à $L^2(\mathbb{S}^d)$ mais la série aléatoire

$$\sum \frac{g_n}{\sqrt{n}\ln(n)} Y_n$$

diverge presque sûrement dans $L^p(\mathbb{S}^d)$ pour tout réel $p > 2$. La preuve du théorème 1.2 consistera à estimer les normes

$$\left\| \sqrt{\sum_{n\geq 1} |a_n Y_n|^2} \right\|_{L^p(\mathbb{S}^d)}.$$

Cela sera très facile pour $p \in 2\mathbb{N}$ en utilisant des estimées optimales vérifiées par les intégrales

$$\int_{\mathbb{S}^d} |Y_{n_{(1)}}(x) \cdots Y_{n_{(p/2)}}(x)|^2 \, d\mu_d(x), \quad (n_{(1)},\ldots,n_{(p/2)}) \in (\mathbb{N}^\star)^{\frac{p}{2}}. \tag{12}$$

Pour traiter le cas général $p \in]1,+\infty[$, on raisonnera par interpolation et dualité. Mais comme précédemment, cela n'est pas gratuit. De façon précise, la concentration gaussienne de Y_n autour de la géodésique $\{x_1^2 + x_2^2 = 1\} \subset \mathbb{S}^d$ nous permettra de valider les hypothèses de nos nouveaux théorèmes d'interpolation et de dualité (théorèmes 2.5 et 2.6).

Remarque 1.3. La condition (11) est exactement celle que l'on obtient avec le critère (1) en remplaçant Y_n par la fonction $x \mapsto n^{\frac{d-1}{4}} \mathbf{1}_{V_n}(x)$ où $V_n \subset \mathbb{S}^d$ est une bande de largeur $\frac{1}{\sqrt{n}}$ autour de la géodésique d'équation $x_1^2 + x_2^2 = 1$. Par souci de simplification, nous n'avons traité que le cas des géodésiques sur $\mathbb{S}^d$ mais notre démarche est en fait plus générale. En effet, on devrait pouvoir estimer les intégrales (12) en utilisant seulement la concentration gaussienne de Y_n autour de la géodésique $\{x_1^2 + x_2^2 = 1\}$. Or ce type de concentration se réalise dans un autre cas important, à savoir celui d'une surface X qui admet une géodésique $\Gamma \subset X$ fermée elliptique et non-dégénérée. Il est alors connu, mais assez délicat à rédiger, que l'on peut construire des quasi-modes qui se concentrent autour de la géodésique Γ avec des estimations gaussiennes (ce sont les travaux de Ralston et Babich).

En résumé, les théorèmes 1.1 et 1.2 affirment que la concentration sur une zone déterminée (voisinage d'une géodésique ou d'un pôle) est la seule information pertinente pour étudier les séries aléatoires de fonctions propres. Malgré la ressemblance de leurs énoncés, les preuves des théorèmes 1.1 et 1.2 vont être différentes car la concentration des fonctions Y_n autour de la géodésique $\{x_1^2 + x_2^2 = 1\} \subset \mathbb{S}^d$ est bien meilleure que la concentration des fonctions zonales Z_n autour des pôles $(\pm 1, 0, \dots, 0)$ (voir la discussion concernant les estimées multilinéaires au début de la preuve de la proposition 3.1).

Les deux exemples de fonctions propres Y_n et Z_n nous permettront facilement de justifier l'optimalité de l'injection de Sobolev probabiliste (3) obtenue par Tzvetkov [2010]. Cela est parfaitement cohérent car les preuves de [Tzvetkov 2010] sont obtenues grâce aux inégalités de Sogge (4) qui sont précisément saturées pour les fonctions Y_n et Z_n.

1C. ***Oscillateur harmonique*** $-\Delta + |x|^2$ ***sur*** $\mathbb{R}^d$**.** Dans cette partie, nous expliquons notre principale application à l'oscillateur harmonique $-\Delta + |x|^2$ sur $L^2(\mathbb{R}^d)$ avec $d \geq 2$. Nous prolongeons l'étude entamée dans [Imekraz et al. 2016]. Pour tout entier $n \in \mathbb{N}$, on note $H_n \in \mathbb{R}[X]$ le n-ième polynôme de Hermite ainsi que h_n la n-ième fonction de Hermite :

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad \text{et} \quad h_n(x) := \frac{H_n(x)}{\sqrt{n! 2^n \sqrt{\pi}}} e^{-\frac{1}{2}x^2}.$$

On vérifie que l'on a $\|h_n\|_{L^2(\mathbb{R})} = 1$. On définit aussi le sous-espace suivant de $L^2(\mathbb{R}^d)$

$$E_{d,n} := \mathrm{Vect}\big\{h_{i_1} \otimes \cdots \otimes h_{i_d} \mid (i_1, \dots, i_d) \in \mathbb{N}^d,\ i_1 + \cdots + i_d = n\big\},$$

où l'on note classiquement $(h_{i_1} \otimes \cdots \otimes h_{i_d})(x) = h_{i_1}(x_1) \cdots h_{i_d}(x_d)$ pour tout $x \in \mathbb{R}^d$. Les fonctions $h_{i_1} \otimes \cdots \otimes h_{i_d}$ forment une base hilbertienne de $E_{d,n}$ et l'on vérifie aussi que l'on a

$$d_n := \dim(E_{d,n}) = \frac{(n+1) \cdots (n+d-1)}{(d-1)!} \sim \frac{n^{d-1}}{(d-1)!}, \quad n \to +\infty. \tag{13}$$

Il faut savoir de plus que $E_{d,n}$ est le sous-espace propre de $-\Delta + |x|^2$ associé à la valeur propre $d + 2n$ et que l'on a la somme directe orthogonale

$$L^2(\mathbb{R}^d) = \bigoplus_{n \in \mathbb{N}} E_{d,n}.$$

On veut associer à cette somme directe orthogonale des séries aléatoires et étudier leur convergence en probabilité dans $L^p(\mathbb{R}^d)$. Expliquons d'abord la démarche employée dans [Imekraz et al. 2016, (1.9)]. Les séries aléatoires précédemment utilisées sont de la forme

$$\omega \in \Omega \mapsto \sum_{n \in \mathbb{N}} \sum_{k=1}^{d_n} V_{n,k}(\omega) c_{n,k} \phi_{n,k},$$

où Ω est un univers probabilisé de référence, $(\phi_{n,k})_{1 \leq k \leq d_n}$ est une base hilbertienne de $E_{d,n}$, $(V_{n,k})_{n,k}$ est une famille de variables aléatoires, i.i.d., centrées, non nulles et dont tous les moments sont finis, et $(c_{n,k})_{n,k}$ est une suite de coefficients complexes. D'après le critère de Maurey (1) et le théorème de Maurey–Pisier, la convergence de ces séries aléatoires dans $L^p(\mathbb{R}^d)$ revient à étudier les normes

$$\Big\| \sum_{n \in \mathbb{N}} \sum_{k=1}^{d_n} |c_{n,k}|^2 |\phi_{n,k}(x)|^2 \Big\|_{L_x^{p/2}(\mathbb{R}^d)}. \tag{14}$$

L'expression précédente dépend a priori de chaque base hilbertienne $(\phi_{n,k})$ de $E_{d,n}$. Afin d'obtenir des résultats indépendants de ces bases, une condition de contrôle (nommée *squeezing condition*) sur les coefficients $(c_{n,k})$ a été imposée dans [Imekraz et al. 2016] (et même dans [Poiret et al. 2015 ; 2014 ; Robert et Thomann 2015]). Cette condition technique demande que les nombres $|c_{n,k}|^2$, d'un même paquet, soient du même ordre de grandeur, c'est-à-dire comparable à leur moyenne $\frac{1}{d_n}(|c_{n,1}|^2+\cdots+|c_{n,d_n}|^2)$. Ainsi, (14) se réduit à

$$\Big\| \sum_{n\in\mathbb{N}} \frac{1}{\dim(E_{d,n})} \Big(\sum_{k=1}^{d_n} |c_{n,k}|^2\Big)\Big(\sum_{k=1}^{d_n} |\phi_{n,k}(x)|^2\Big)\Big\|_{L_x^{p/2}(\mathbb{R}^d)}, \tag{15}$$

où la fonction $|\phi_{n,1}|^2+\cdots+|\phi_{n,d_n}|^2$ s'avère indépendante de la base hilbertienne choisie. Il est naturel de vouloir s'émanciper de la squeezing condition (cette dernière a été légèrement relaxée dans [Imekraz et al. 2016, Part 1.1.2] par comparaison aux travaux [Poiret et al. 2015 ; 2014 ; Robert et Thomann 2015]). Dans notre article, nous allons employer une autre méthode de randomisation qui ne dépend que de la suite $(E_{d,n})_{n\geq 0}$ et qui ne nécessite aucune condition de contrôle des coefficients. De façon précise, on va randomiser une fonction propre appartenant à $E_{d,n}$ en la faisant tourner uniformément et aléatoirement autour de l'origine. Nous noterons désormais $(W_n)_{n\geq 0}$ une suite de matrices aléatoires indépendantes et supposons que chaque matrice aléatoire W_n suit une loi uniforme dans le groupe unitaire $U_{d_n}(\mathbb{C})$. Notre résultat s'énonce comme suit :

Théorème 1.4. *Supposons $d\geq 2$ et considérons une suite $(u_n)_{\in\mathbb{N}}$ d'éléments de $L^2(\mathbb{R}^d)$ vérifiant $u_n\in E_{d,n}$ pour tout $n\in\mathbb{N}$. Pour tout réel $p\in[1,+\infty[$, les conditions suivantes sont équivalentes :*

(i) *la série aléatoire $\sum_n \big(\sum_{i,j=1}^{d_n} W_{n,i,j}\langle u_n,\phi_{n,j}\rangle \phi_{n,i}\big)$ converge presque sûrement dans $L^p(\mathbb{R}^d)$,*

(ii) *la série numérique $\sum_{n\geq 1} n^{\frac{d}{2}-1}\big(\sum_{k\geq n}\|u_k\|^2_{L^2(\mathbb{R}^d)}/k^{\frac{d}{2}}\big)^{\frac{p}{2}}$ est convergente.*

Pour tout $p\in[2,+\infty[$ on a l'injection de Sobolev probabiliste

$$\sum_{n\geq 1} n^{-d(\frac{1}{2}-\frac{1}{p})}\|u_n\|^2_{L^2(\mathbb{R}^d)}<+\infty \Rightarrow \sum_n \Big(\sum_{i,j=1}^{d_n} W_{n,i,j}\langle u_n,\phi_{n,j}\rangle\phi_{n,i}\Big) \text{ converge p.s. dans } L^p(\mathbb{R}^d). \tag{16}$$

Le théorème 1.4 donne donc une caractérisation explicite des séries aléatoires qui convergent presque sûrement dans $L^p(\mathbb{R}^d)$. Au passage, on obtient un résultat de type Paley–Zygmund :

$$\sum_{n\geq 1}\|u_n\|^2_{L^2(\mathbb{R}^d)}<+\infty \Rightarrow \forall p\in[2,+\infty[,\quad \sum_n \Big(\sum_{i,j=1}^{d_n} W_{n,i,j}\langle u_n,\phi_{n,j}\rangle\phi_{n,i}\Big) \text{ converge p.s. dans } L^p(\mathbb{R}^d).$$

Dans la partie 4, nous verrons que l'implication (16) est optimale par rapport à l'exposant $-d\big(\frac{1}{2}-\frac{1}{p}\big)$ et doit être vue comme une amélioration probabiliste de l'injection de Sobolev déterministe :

$$\sum_{n\geq 1} n^{d(\frac{1}{2}-\frac{1}{p})}\|u_n\|^2_{L^2(\mathbb{R}^d)}<+\infty \quad\Rightarrow\quad \sum_{n\geq 0} u_n\in L^p(\mathbb{R}^d). \tag{17}$$

Dans [Imekraz et al. 2016, Theorem 2.2], un cas particulier de (16) a été obtenu avec une *squeezing condition*. Nous écrirons aussi une implication duale à celle de (16), avec $p \in \,]1, 2]$, grâce à un nouveau théorème de dualité.

Au niveau probabiliste, la preuve du théorème 1.4 reposera sur une extension multidimensionnelle du critère de Maurey (1) qui fera apparaître l'expression (15). An niveau de l'analyse, nous utiliserons une nouvelle estimée de la fonction spectrale de l'oscillateur harmonique tronquée au n-ième niveau d'énergie (voir la proposition 4.1) :

$$\sum_{\substack{(i_1,\dots,i_d)\in\mathbb{N}^d \\ i_1+\cdots+i_d=n}} h_{i_1}(x_1)^2\cdots h_{i_d}(x_d)^2. \tag{18}$$

Didier Robert nous a signalé après la rédaction de notre article que des informations complémentaires sur la concentration de (18) ont été précédemment obtenues dans [Hanin et al. 2015].

1D. *Version multidimensionnelle du théorème de Maurey–Pisier.* Le théorème de Maurey–Pisier permet de remplacer les variables gaussiennes par des variables de Bernoulli dans les théorèmes 1.1 et 1.2. Il y a en réalité un phénomène d'universalité plus important : on peut remplacer les variables gaussiennes par toute une famille de variables aléatoires ayant des moments finis de tout ordre. Pour énoncer précisément ce résultat, rappelons la définition du cotype.

Définition 1.5. Considérons un réel $q \in [2, +\infty[$, un espace de Banach complexe B est de cotype q s'il existe $c > 0$ tel que pour tout $N \in \mathbb{N}^\star$ et tout $(u_n)_{1\le n\le N} \in B^N$ on a

$$\left(\sum_{n=1}^N \|u_n\|_B^q\right)^{1/q} \le c\mathbf{E}\left[\left\|\sum_{n=1}^N \varepsilon_n u_n\right\|_B\right]. \tag{19}$$

S'il existe un réel $q \in [2, +\infty[$ tel que l'inégalité précédente se réalise, alors on dit que B est de cotype fini.

Il faut considérer la propriété de cotype fini comme une propriété géométrique d'un espace de Banach. On vérifie que $L^p(X)$ est un espace de Banach de cotype $\max(2, p)$. La preuve du théorème de Maurey et Pisier [1976, corollaire 1.3] donne en fait le résultat suivant.

Théorème 1.6 (Maurey–Pisier). *Soit B un espace de Banach complexe. Alors les deux propriétés suivantes sont équivalentes :*

(a) *B est de cotype fini.*

(b) *Pour toute suite (u_n) de B et toute suite $(X_n)_{n\in\mathbb{N}}$ de variables aléatoires réelles, centrées, indépendantes et vérifiant*

$$\inf_{n\in\mathbb{N}} \mathbf{E}[|X_n|] > 0 \quad \textit{et} \quad \forall q \in [2, +\infty[,\ \sup_{n\in\mathbb{N}} \mathbf{E}[|X_n|^q] < +\infty,$$

les deux propriétés suivantes sont équivalentes :

(i) *la série aléatoire $\sum X_n u_n$ converge presque sûrement dans B,*

(ii) *la série aléatoire $\sum \varepsilon_n u_n$ converge presque sûrement dans B.*

En d'autres termes, l'universalité de la randomisation *unidimensionnelle* dans $L^p(X)$ se produit précisément parce que $L^p(X)$ dispose de la propriété de cotype fini. On veut naturellement généraliser ce phénomène pour les séries aléatoires qui apparaissent dans l'hypothèse (i) du théorème 1.4, c'est-à-dire trouver d'autres séries aléatoires multidimensionnelles pour lesquelles la conclusion du théorème 1.4 est encore valide.

Ce problème est lié à un fait remarqué par Burq et Lebeau [2013, appendice C] : si l'on considère X une variété riemannienne compacte sans bord alors on peut construire *beaucoup* de mesures ν sur $L^2(X)$ (même étrangères deux à deux) telles que $\nu(L^p(X)) = 1$ pour tout $p \in [2, +\infty[$. Rappelons simplement comment sont construites les mesures de Burq–Lebeau, disons dans le cas $X = \mathbb{S}^d$ (voir [Burq et Lebeau 2013, appendice C.1]). Notons $(E_n)_{n\geq 0}$ la suite des sous-espaces propres de l'opérateur de Laplace–Beltrami Δ sur $L^2(\mathbb{S}^d)$ et considérons une suite $(V_n)_{n\geq 1}$ de variables aléatoires indépendantes à valeurs dans $]0, +\infty[$ vérifiant les estimations de grandes déviations

$$\exists\gamma > 0,\ C > 0,\ c > 0, \quad \forall\rho > 0, \quad \sup_{n\in\mathbb{N}} \boldsymbol{P}(V_n \geq \rho) \leq Ce^{-c\rho^{\gamma}}. \tag{20}$$

On supposera que les variables aléatoires V_n et les matrices unitaires aléatoires W_n sont indépendantes dans leur ensemble. Considérons maintenant une suite $(u_n)_{n\geq 1}$ de $L^2(\mathbb{S}^d)$ vérifiant $u_n \in E_n$ pour tout $n \in \mathbb{N}$. Les mesures de Burq–Lebeau sont tout simplement les mesures images (c'est-à-dire les lois) des fonctions aléatoires

$$\begin{aligned} &\Omega \to L^2(\mathbb{S}^d),\\ &\omega \mapsto \sum_{n\in\mathbb{N}} \bigg(\sum_{i,j=1}^{d_n} V_n(\omega) W_{n,i,j}(\omega) \langle u_n, \phi_{n,j}\rangle \phi_{n,i} \bigg), \end{aligned} \tag{21}$$

où $(\phi_{n,1}, \dots, \phi_{n,d_n})$ est une base orthonormée du sous-espace propre E_n et Ω un univers probabilisé de référence. Par indépendance des variables aléatoires en jeu, cette construction équivaut à définir une mesure de probabilité sur $L^2(\mathbb{S}^d)$ comme un produit tensoriel infini de certaines mesures de probabilités respectivement supportées dans les sous-espaces E_n (voir [Burq et Lebeau 2013, pages 955]). En termes de séries aléatoires, le résultat de Burq–Lebeau se reformule donc ainsi : il existe *beaucoup* de séries aléatoires de la forme (21) qui convergent presque sûrement dans $L^p(\mathbb{S}^d)$. Ce résultat n'est pas purement théorique ; il fait partie d'un programme de recherche consistant à étudier le transfert de mesures par le flot d'équations aux dérivées partielles non linéaires (par exemple des mesures de Gibbs).

On va écrire une version multidimensionnelle du théorème 1.6 qui va expliquer pourquoi l'on peut bien construire des mesures de Burq–Lebeau (et au passage en construire des nouvelles). Afin de recouvrir les variables aléatoires X_n du théorème 1.6, nous allons considérer des matrices aléatoires plus générales. Introduisons la notion suivante qui généralise la notion de symétrie pour les variables aléatoires scalaires [Marcus et Pisier 1981, page 82, (2.3)].

Définition 1.7. Une matrice aléatoire $M : \Omega \to \mathcal{M}_d(\mathbb{R})$ est *orthogonalement invariante* si pour toute matrice orthogonale $P \in O_d(\mathbb{R})$ les matrices aléatoires M et PM ont la même loi. On définit de même l'invariance unitaire pour une matrice aléatoire $M : \Omega \to \mathcal{M}_d(\mathbb{C})$ en remplaçant le groupe orthogonal $O_d(\mathbb{R})$ par le groupe unitaire $U_d(\mathbb{C})$.

Fixons maintenant quelques notations usuelles d'algèbre linéaire :

$$\begin{aligned}
\forall y \in \mathcal{M}_{d,1}(\mathbb{C}) = \mathbb{C}^d, \quad |y| &:= \sqrt{|y_1|^2 + \cdots + |y_d|^2}, \\
\forall A \in \mathcal{M}_d(\mathbb{C}), \quad \|A\|_{\mathrm{op}} &:= \sup_{y \in \mathbb{C}^d \setminus \{0\}} \frac{|Ay|}{|y|}, \\
|A| &:= \sqrt{{}^t\bar{A}A} \in \mathcal{M}_d(\mathbb{C}), \\
\sigma(A) &:= \inf_{y \neq 0} \frac{|Ay|}{|y|}.
\end{aligned} \tag{22}$$

La matrice $|A|$ est une matrice hermitienne positive issue d'une décomposition polaire de $A = P|A|$ avec $P \in U_d(\mathbb{C})$ (une telle décomposition est unique dès lors que $\det(A) \neq 0$). Enfin, $\sigma(A)$ est la plus petite valeur singulière de A, il s'agit aussi de la plus petite valeur propre de $|A|$. Notre première version multidimensionnelle du théorème de Maurey–Pisier prend la forme suivante :

Théorème 1.8. *Fixons X un espace mesuré σ-fini et un réel $p \in [1, +\infty[$. Considérons une suite de sous-espaces non nuls $(E_n)_{n \geq 0}$ de $L^2(X) \cap L^p(X)$ de dimension finie. On notera $d_n = \dim(E_n)$ et $(\phi_{n,1}, \ldots, \phi_{n,d_n})$ une base hilbertienne de E_n. Considérons aussi une suite de matrices aléatoires $M_n : \Omega \to \mathcal{M}_{d_n}(\mathbb{R})$ indépendantes et orthogonalement invariantes telles que*

$$\inf_{n \in \mathbb{N}} \sigma(\mathbf{E}[|M_n|]) > 0 \quad et \quad \sup_{n \in \mathbb{N}} \mathbf{E}[\|M_n\|_{\mathrm{op}}^{\max(2,p)}] < +\infty. \tag{23}$$

Alors, pour toute suite $(u_n)_{\in \mathbb{N}}$, avec $u_n \in E_n$ pour tout $n \in \mathbb{N}$, les assertions suivantes sont équivalentes :

(i) *La série aléatoire $\sum_n \big(\sum_{i,j=1}^{d_n} M_{n,i,j} \langle u_n, \phi_{n,j} \rangle \phi_{n,i}\big)$ converge presque sûrement dans $L^p(\mathbb{R}^d)$.*

(ii) *La série aléatoire $\sum_n \big(\sum_{i,j=1}^{d_n} W_{n,i,j} \langle u_n, \phi_{n,j} \rangle \phi_{n,i}\big)$ converge presque sûrement dans $L^p(\mathbb{R}^d)$.*

Dans ce qui précède, on convient que $W_n : \Omega \to U_{d_n}(\mathbb{C})$ suit une loi uniforme dans le groupe unitaire et que les matrices aléatoires $(W_n)_{n \in \mathbb{N}}$ sont indépendantes.

La même conclusion est valide en remplaçant "orthogonalement invariante" par "unitairement invariante" au sens de la définition 1.7.

Faisons quelques remarques :

(a) L'assertion (20) est beaucoup plus restrictive que la condition $\sup_{n \geq 0} \mathbf{E}[|V_n|^{\max(2,p)}] < +\infty$. Le théorème 1.8 permet donc de construire des mesures de Burq–Lebeau sur $L^2(\mathbb{S}^d)$ qui ne sont pas accessibles par les arguments de grandes déviations utilisés dans [Burq et Lebeau 2013, appendice C].

(b) Le théorème 1.8 est indépendant d'une quelconque géométrie riemannienne sur X. En un certain sens, on peut dire que la philosophie globale est que c'est la géométrie de l'espace de Banach $L^p(X)$ qui est déterminante et non la géométrie de l'espace X. Cela explique au passage pourquoi les mesures de Burq et Lebeau [2013] peuvent être construites sur toute variété riemannienne compacte sans bord.

(c) La série aléatoire de l'assertion (i) du théorème 1.8 traduit l'idée que l'on applique un endomorphisme aléatoire à l'élément $u_n \in E_n$. Il s'agit d'une généralisation naturelle des séries aléatoires $\sum X_n u_n$ où $X_n : \Omega \to \mathbb{R}$ sont des variables aléatoires réelles.

(d) Les conditions (23) apparaissent sous une forme analogue dans l'étude des séries de Fourier aléatoires dans l'espace fonctionnel $L^\infty(G)$ sur un groupe compact G mais $\|M_n\|_{\mathrm{op}}^{\max(2,p)}$ est remplacé par $\|M_n\|_{\mathrm{op}}^2$ (voir [Marcus et Pisier 1981, page 97, Theorem 3.5]). Dans le cas où le groupe G est abélien et localement compact, Marcus et Pisier expliquent que l'exposant 2 est dû au fait que l'espace des fonctions "presque sûrement continues" sur G est curieusement de cotype 2 (alors que $L^\infty(G)$ n'est même pas de cotype fini dès que G est infini).

Notre preuve du théorème 1.8 n'est pas un prolongement de celle du théorème de Maurey et Pisier [1976, corollaire 1.3]. La preuve originelle utilise le théorème de factorisation de Pietsch tandis que notre preuve est plus simple et repose essentiellement sur le théorème de Fubini–Tonelli et les inégalités de Kahane–Khintchine–Marcus–Pisier. En fait, l'auteur ignore s'il est possible de démontrer le théorème 1.8 avec un théorème de factorisation. L'obstacle est le suivant : on veut démontrer un résultat de nature multidimensionnelle alors que la définition même d'un espace de Banach de cotype fini fait intervenir la randomisation unidimensionnelle (voir le membre droit de (19)). Pour autant, l'exposant $\max(2, p)$, qui apparaît dans (23), est le cotype de l'espace $L^p(X)$. Il est donc normal d'interpréter la preuve du théorème 1.8 en termes de cotype. Il se trouve que l'on peut contourner l'obstacle mentionné en considérant la structure de treillis de $L^p(X)$ (c'est-à-dire l'existence d'une relation d'ordre compatible avec la structure d'espace de Banach). Dans ce cas, il est connu que la propriété de "cotype fini" (19) peut être définie de manière déterministe sans variables de Bernoulli (qui sont par nature unidimensionnelle). Voici notre version multidimensionnelle du théorème 1.6 qui donne un éclairage sur les raisons géométriques de l'universalité de la randomisation *multidimensionnelle* dans les espaces de Banach $L^p(X)$, avec $p \in [1, +\infty[$.

Théorème 1.9. *Soit B un treillis de Banach complexe, alors se valent :*

(a) *L'espace de Banach B est de cotype fini.*

(b) *Pour toute suite d'entiers non nuls $(d_n)_{n\geq 0}$, pour toute suite $M_n : \Omega \to \mathcal{M}_{d_n}(\mathbb{R})$ de matrices aléatoires indépendantes, orthogonalement invariantes et vérifiant*

$$\inf_{n\in\mathbb{N}} \sigma(\mathbf{E}[|M_n|]) > 0 \quad \text{et} \quad \forall q \in [2, +\infty[, \ \sup_{n\in\mathbb{N}} \mathbf{E}[\|M_n\|_{\mathrm{op}}^q] < +\infty, \tag{24}$$

et pour toute suite de matrices $b_n \in \mathcal{M}_{d_n}(B)$, les deux propriétés suivantes sont équivalentes :

(i) *la série aléatoire $\sum \mathbf{tr}(M_n b_n)$ converge presque sûrement dans B,*

(ii) *la série aléatoire $\sum \mathbf{tr}(W_n b_n)$ converge presque sûrement dans B.*

La même conclusion est valide en remplaçant "orthogonalement invariante" par "unitairement invariante" au sens de la définition 1.7.

Le théorème 1.6 fournit évidemment le sens (b) $\Rightarrow$ (a) en choisissant $d_n = 1$ pour tout $n \in \mathbb{N}$. C'est donc le sens (a) $\Rightarrow$ (b) qui est nouveau (et c'est en fait le sens pratique). On retrouve les séries aléatoires du théorème 1.8 avec $b_n = [\langle u_n, \phi_{n,i}\rangle \phi_{n,j}] \in \mathcal{M}_{d_n}(L^p(X))$. La forme du théorème 1.9 nous a été inspirée par l'étude des séries aléatoires sur un groupe compact dans le livre de Marcus et Pisier [1981].

Signalons que certains espaces de Lorentz fournissent des exemples non triviaux de treillis de Banach de cotype fini [Creekmore 1981]. À l'instar du théorème 1.8, le théorème 2.21 donnera une version quantitative de l'hypothèse (24) afin de montrer que le bon exposant q à choisir est lié à la q-concavité du treillis B et que l'on peut effectivement choisir $q = \max(2, p)$ dans le cas de $B = L^p(X)$.

Le résultat suivant, prouvé dans les appendices B et C, donne des exemples de matrices aléatoires vérifiant les hypothèses (23) et (24).

Proposition 1.10. *Considérons des variables aléatoires* $X_{ij} : \Omega \to \mathbb{R}$ *centrées, i.i.d. et où* (i, j) *parcourt* $\mathbb{N}^{\star 2}$. *Pour tout* $n \in \mathbb{N}^\star$, *on note la matrice aléatoire* $M_n = \frac{1}{\sqrt{n}}[X_{ij}]$. *Il existe une constante universelle* $C \geq 1$ *telle que si l'on a* $0 < \mathbf{E}[|X_{1,1}|^4] < +\infty$ *alors*

$$\sigma(\mathbf{E}[|M_n|]) \geq \frac{\mathbf{E}[|X_{11}|]^2}{C\,\mathbf{E}[|X_{11}|^4]^{\frac{1}{4}}}. \tag{25}$$

En outre, si $X_{1,1}$ *admet un moment d'ordre* $p \geq 4$ *alors nous avons l'inégalité de Latała précisée*

$$\mathbf{E}[\|M_n\|_{\mathrm{op}}^p] \leq C(p)\mathbf{E}[|X_{1,1}|^p]. \tag{26}$$

Par conséquent, on a

$$\inf_{n\geq 1} \sigma(\mathbf{E}[|M_n|]) > 0 \quad \textit{et} \quad \sup_{n\geq 1} \mathbf{E}[\|M_n\|_{\mathrm{op}}^p] < +\infty.$$

Remarque 1.11. Si les variables aléatoires X_{ij} sont à valeurs complexes et d'espérance nulle, alors on peut aussi montrer (26) en considérant partie réelle et partie imaginaire des variables aléatoires X_{ij}. Quant à (25), une preuve similaire est valide mais l'on doit remplacer l'hypothèse "centrée" par "symétrique au sens complexe" : pour tout réel $\alpha \in \mathbb{R}$ les variables $e^{i\alpha} X_{ij}$ et X_{ij} ont la même loi.

Remarque 1.12. Dans le cadre des théorèmes 1.8 et 1.9, nous avons besoin de matrices aléatoires orthogonalement invariantes. Or les matrices $\frac{1}{\sqrt{n}}[X_{ij}]$ de la proposition 1.10 ne sont pas orthogonalement invariantes en général (le cas gaussien est une exception notable). Il suffit alors d'examiner des matrices aléatoires de la forme $\frac{1}{\sqrt{n}}\mathcal{E}_n[X_{ij}]$ où les matrices aléatoires $\mathcal{E}_n : \Omega \to O_{d_n}(\mathbb{R})$ et $[X_{ij}]$ sont indépendantes et où $\mathcal{E}_n$ suit une loi uniforme pour chaque $n \in \mathbb{N}^\star$. On remarque évidemment que $\frac{1}{\sqrt{n}}\mathcal{E}_n[X_{ij}]$ et $\frac{1}{\sqrt{n}}[X_{ij}]$ ont la même norme d'opérateur et que l'on a $|\mathcal{E}_n[X_{ij}]| = |[X_{ij}]|$ (au sens de (22)).

Nous n'avons pas trouvé les estimations de la proposition 1.10 dans des références déjà publiées mais elles doivent être connues des spécialistes. Par exemple, nous pensons qu'il doit être possible de prouver (26) avec la méthode des moments (par exemple expliquée dans [Tao 2012, Part 2.3]). Cependant, cette méthode est assez technique à mettre en place. Un cas particulier très bien compris dans la littérature des matrices aléatoires est le cas *sous-gaussien*. Parmi plusieurs définitions équivalentes, $X_{1,1}$ est sous-gaussienne si l'on a $\mathbf{E}[X_{1,1}] = 0$ et

$$\exists K > 0, \quad \forall p \in \mathbb{N}^\star, \quad \mathbf{E}[|X_{1,1}|^p]^{\frac{1}{p}} \leq K\sqrt{p}.$$

Cela équivaut par exemple à la condition

$$\exists K' > 0, \quad \forall t \in \mathbb{R}, \quad \mathbf{E}[\exp(tX_{11})] \leq \exp(K't^2).$$

D'après [Rudelson et Vershynin 2009, Proposition 2.3] (voir aussi la preuve de [Litvak et al. 2005, Fact 2.4]), on sait que l'on a

$$\forall p \in \mathbb{N}^\star, \quad \exists C(p,K) > 0, \quad \forall n \in \mathbb{N}^\star, \quad \mathbf{E}[\|M_n\|_{\mathrm{op}}^p] \leq C(p,K).$$

L'inégalité (26) est plus générale. Dans notre article, nous montrons que (26) est en réalité une conséquence de la preuve de l'inégalité de Latała

$$\mathbf{E}[\|M_n\|_{\mathrm{op}}] \leq C\mathbf{E}[|X_{1,1}|^4]^{\frac{1}{4}}. \tag{27}$$

et des inégalités de Kahane–Khintchine dans l'espace de Banach $(\mathcal{M}_n(\mathbb{R}), \|\cdot\|_{\mathrm{op}})$ (voir (39)). Cela explique pourquoi (26) implique (27) avec $p = 4$. La renormalisation de $[X_{ij}]$ par le facteur $\frac{1}{\sqrt{n}}$ et le fait que (27) nécessite au moins un moment d'ordre 4 peuvent surprendre a priori mais il s'agit de faits bien connus dans la théorie des matrices aléatoires (voir [Bai et al. 1988 ; Yin et al. 1988, Theorem 3.1]). Enfin, l'inégalité triviale $|X_{1,1}| \leq \sqrt{n}\|M_n\|_{\mathrm{op}}$ nous fait comprendre que l'inégalité (26) est optimale par rapport à la condition de moment d'ordre p.

Quant à l'inégalité (25), elle découlera d'un argument d'interpolation simplifiant celui du cas gaussien [Marcus et Pisier 1981, page 78, Proposition 1.5] et de l'inégalité de Latała (27).

1E. ***Organisation de l'article.*** La partie 2 contient les preuves des théorèmes 1.8 et 1.9. Mais elle est plus généralement dévolue à l'étude abstraite des espaces de Lebesgue probabilistes, notés $\mathbf{PL}^p(X, \bigoplus E_n)$ où X est un espace mesuré σ-fini et une suite $(E_n)_{n\in\mathbb{N}}$ de sous-espaces non nuls de *dimension finie* de $L^2(X)$. Expliquons brièvement la définition de ces nouveaux espaces de Banach. Notons $(\phi_{n,1}, \dots, \phi_{n,d_n})$ une base orthonormée de E_n pour tout $n \in \mathbb{N}$. Pour tout $p \in [1, +\infty[$, l'espace $\mathbf{PL}^p(X, \bigoplus E_n)$ sera défini comme l'espace vectoriel des suites $(u_n)_{n\in\mathbb{N}}$, avec $u_n \in E_n$ pour tout $n \in \mathbb{N}$, telles que la série aléatoire

$$\sum_n \Bigl(\sum_{i,j=1}^{d_n} W_{n,i,j} \langle u_n, \phi_{n,j} \rangle \phi_{n,i} \Bigr) \tag{28}$$

converge presque sûrement dans $L^p(X)$. Le cas $\dim(E_n) = 1$, pour tout $n \in \mathbb{N}$, est le plus simple à comprendre grâce au théorème de Maurey–Pisier et au critère de Maurey (1). Pour analyser les séries aléatoires (28) dans le cas $\dim(E_n) \neq 1$, nous allons utiliser une version multidimensionnelle des inégalités de Kahane–Khintchine, à savoir les inégalités de Kahane–Khintchine–Marcus–Pisier que nous avons mentionnées plus haut. L'idée maîtresse de notre article est que ces inégalités sont précisément celles qui permettent d'aborder de façon satisfaisante la randomisation multidimensionnelle avec des hypothèses optimales de moments. Le théorème 2.2 nous donnera notamment une extension multidimensionnelle du critère de Maurey avec la caractérisation suivante des espaces de Lebesgue probabilistes :

$$(u_n)_{n\in\mathbb{N}} \in \mathbf{PL}^p(X, \textstyle\bigoplus E_n) \quad \Leftrightarrow \quad \Bigl\| \sqrt{\sum_{n\in\mathbb{N}} \|u_n\|_{L^2(X)}^2 \frac{e_n(x)}{\dim(E_n)}} \Bigr\|_{L_x^p(X)} < +\infty,$$

où la fonction $e_n := |\phi_{n,1}|^2 + \cdots + |\phi_{n,d_n}|^2$ est indépendante de la base hilbertienne $(\phi_{n,1}, \dots, \phi_{n,d_n})$ (voir (33)).

A priori, on s'attend à ce que le dual de $\mathbf{PL}^p\big(X, \bigoplus E_n\big)$ soit $\mathbf{PL}^{\frac{p}{p-1}}\big(X, \bigoplus E_n\big)$ et l'on espère que les espaces $\mathbf{PL}^p\big(X, \bigoplus E_n\big)$ soient stables par interpolation complexe et réelle. En fait, cela s'avérera faux pour la randomisation des fonctions zonales sur la sphère $\mathbb{S}^d$ (voir la partie 3) : on n'a pas de dualité et l'interpolation se réalise seulement si p parcourt certains intervalles. Nous avons cependant des résultats positifs : les théorèmes 2.5 et 2.6 donnent des hypothèses pratiques sur les fonctions e_n qui assurent les propriétés attendues de dualité et d'interpolation. La stratégie consistera à montrer $\mathbf{PL}^p\big(X, \bigoplus E_n\big)$ est un rétracte convenable de l'espace de Bochner–Lebesgue $L^p(X, \ell^2(\mathbb{N}))$. Cela nous permettra de faire hériter les propriétés de dualité/interpolation des espaces $L^p(X, \ell^2(\mathbb{N}))$ aux espaces $\mathbf{PL}^p\big(X, \bigoplus E_n\big)$. Mais pour y arriver, nous serons obligés de justifier la continuité de certains projecteurs de $L^p(X, \ell^2(\mathbb{N}))$ à l'aide d'un nouveau critère de $\mathcal{R}$-bornitude sur $L^p(X)$. Les notions de rétracte et de $\mathcal{R}$-bornitude, que nous rappellerons plus loin, peuvent paraître abstraites mais elles sont incontournables sous une forme ou une autre (cela est formalisé par la proposition 2.31). C'est à ce moment qu'interviendra l'hypothèse abstraite

$$\sup_{n\in\mathbb{N}} \frac{\sqrt{e_n}}{\|\sqrt{e_n}\|_{L^p(X)}} \in L^{p,\infty}(X), \tag{29}$$

où $L_x^{p,\infty}(X)$ est l'usuel espace L^p faible. Curieusement, il se trouve que la propriété (29) est vérifiée dans des exemples importants issus de la physique mathématique où les fonctions e_n se concentrent sur une même région de X.

La partie 3 contient les preuves des théorèmes 1.1 et 1.2 grâce aux nouveaux arguments d'interpolation. Expliquons la principale idée dans le cas particulier $\dim(E_n) = 1$, c'est-à-dire si $E_n = \mathbb{C}\phi_n$. Supposons en outre que $|\phi_n|$ a tendance à se concentrer, en un sens à préciser, sur une partie $A_n \subset X$ avec une amplitude $c_n > 0$ quand n tend vers $+\infty$. Si l'on fixe une suite de coefficients $(a_n) \in \ell^2(\mathbb{N})$, alors il est légitime d'espérer l'équivalence suivante pour toute suite complexe $(a_n)_{n\in\mathbb{N}}$

$$\sqrt{\sum_{n\in\mathbb{N}} |a_n\phi_n|^2} \in L^p(X) \quad \Leftrightarrow \quad \sqrt{\sum_{n\in\mathbb{N}} |a_n c_n \mathbf{1}_{A_n}|^2} \in L^p(X).$$

En partitionnant les parties A_n en sous-parties disjointes, la condition du membre droit est explicite et résoudrait le problème de l'étude de la série aléatoire $\sum \varepsilon_n a_n \phi_n$. Si l'approximation de $|\phi_n|$ par $c_n\mathbf{1}_{A_n}$ n'est pas bonne, alors l'équivalence précédente n'est pas facile à prouver. L'hypothèse (29), avec $e_n = |\phi_n|^2$, a vocation à mesurer le reste de cette approximation et les arguments de dualité-interpolation permettent de simplifier les calculs.

La partie 4 est consacrée à la preuve du théorème 1.4 concernant l'oscillateur harmonique multidimensionnel.

Nous ajoutons trois appendices. Le premier est consacré à l'optimalité de l'exposant $\max(2, p)$ dans les différents théorèmes concernant $L^p(X)$ (ce fait est sans doute bien connu). Les deux derniers appendices contiennent la preuve des estimations matricielles de la proposition 1.10.

Conventions. Dans cet article, on fera les conventions suivantes :

— Sauf exception, la lettre d sera globalement réservée pour désigner la dimension des espaces en jeu ($\mathbb{R}^d$ ou $\mathbb{S}^d$) et sera supérieure ou égale à 2.

— Le triplet $(\Omega, \mathcal{F}, \boldsymbol{P})$ désignera un espace probabilisé de référence (par exemple $\Omega = [0,1]$ muni de la mesure de Lebesgue sur la tribu borélienne), on notera ω les éléments de Ω et $\mathbf{E}$ l'opérateur d'espérance.
— On notera C une constante universelle supérieure ou égale à 1 qui peut changer d'une ligne à l'autre.
— On notera $C(d,p)$ une constante supérieure ou égale à 1 qui ne dépend que des paramètres d et p et qui peut changer d'une ligne à l'autre.
— Si A et B sont deux nombres réels, la formule $A \lesssim B$ signifiera qu'il existe une constante universelle $C \geq 1$ telle que $A \leq CB$. De même, l'assertion $A \gtrsim B$ signifie $B \lesssim A$.
— On écrira $A \simeq B$ si l'on a $A \lesssim B$ et $B \lesssim A$.
— Si l'on fait intervenir des paramètres d et p, alors on notera $A \lesssim_{d,p} B$ pour signifier qu'il existe une constante $C(d,p) \geq 1$ qui ne dépend que de d et p telle que $A \leq C(d,p)B$. On définit de même les symboles $\gtrsim_{d,p}$ et $A \simeq_{d,p}$.

Par exemple, nous avons l'équivalence

$$\forall n \in \mathbb{N},\ \ A_n \simeq_{d,p} B_n \quad \Leftrightarrow \quad \exists C(d,p) \geq 1,\ \ \forall n \in \mathbb{N},\ \ \frac{A_n}{C(d,p)} \leq B_n \leq C(d,p) A_n.$$

2. Espaces de Lebesgue probabilistes

2A. ***Universalité de la randomisation multidimensionnelle dans L^p.*** On énonce les résultats qui vont impliquer le théorème 1.8 et nous seront utiles pour la suite. On utilisera la notation suivante pour la norme matricielle de Hilbert–Schmidt :

$$\forall A \in \mathcal{M}_N(\mathbb{C}), \quad \|A\|_{\mathrm{HS}} := \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}|^2}. \tag{30}$$

Désormais X désignera un espace mesuré σ-fini. Nous allons démontrer le théorème suivant.

Théorème 2.1. *Fixons un réel $p \in [1, +\infty[$, une suite d'entiers non nuls $(d_n)_{n \in \mathbb{N}}$, une suite de matrices $b_n \in \mathcal{M}_{d_n}(L^p(X))$ et une suite de matrices aléatoires $M_n : \Omega \to \mathcal{M}_{d_n}(\mathbb{R})$ indépendantes, orthogonalement invariantes et vérifiant*

$$\inf_{n \in \mathbb{N}} \sigma(\mathbf{E}[|M_n|]) > 0 \quad et \quad \sup_{n \in \mathbb{N}} \mathbf{E}[\|M_n\|_{\mathrm{op}}^{\max(2,p)}] < +\infty.$$

Alors les propriétés suivantes sont équivalentes :

(i) *la fonction $x \mapsto \sqrt{\sum_{n \geq 0} \|b_n(x)\|_{\mathrm{HS}}^2}$ appartient à $L^p(X)$,*

(ii) *la série aléatoire $\sum \sqrt{d_n}\, \mathbf{tr}(M_n b_n)$ converge presque sûrement dans $L^p(X)$,*

(iii) *la série aléatoire $\sum \sqrt{d_n}\, \mathbf{tr}(M_n b_n)$ converge dans $L^{\max(2,p)}(\Omega, L^p(X))$,*

(iv) *la série aléatoire $\sum \sqrt{d_n}\, \mathbf{tr}(M_n b_n)$ est bornée en probabilité dans $L^p(X)$ (voir définition (46) plus loin).*

La même conclusion est valide en supposant que chaque matrice aléatoire $M_n : \Omega \to \mathcal{M}_{d_n}(\mathbb{C})$ est unitairement invariante.

Le théorème 2.1 implique le suivant.

Théorème 2.2. *Fixons un réel $p \in [1, +\infty[$ et une suite de sous-espaces non nuls de dimension finie $(E_n)_{n\geq 0}$ de $L^2(X) \cap L^p(X)$. On notera $d_n = \dim(E_n)$ et $(\phi_{n,1}, \dots, \phi_{n,d_n})$ une base hilbertienne de E_n pour tout $n \in \mathbb{N}$. On définit de plus*

$$\forall x \in X, \quad e_n(x) = |\phi_{n,1}(x)|^2 + \cdots + |\phi_{n,d_n}(x)|^2. \tag{31}$$

Fixons de plus une suite de matrices aléatoires $M_n : \Omega \to \mathcal{M}_{d_n}(\mathbb{R})$ indépendantes, orthogonalement invariantes et vérifiant

$$\inf_{n\in\mathbb{N}} \sigma(\mathbf{E}[|M_n|]) > 0 \quad \textit{et} \quad \sup_{n\in\mathbb{N}} \mathbf{E}[\|M_n\|_{\mathrm{op}}^{\max(2,p)}] < +\infty.$$

Pour toute suite $(u_n)_{n\geq 0}$, avec $u_n \in E_n$, on définit la série aléatoire

$$\sum_n \bigg(\sum_{i,j=1}^{d_n} M_{n,i,j} \langle u_n, \phi_{n,j} \rangle \phi_{n,i} \bigg). \tag{32}$$

Les propriétés suivantes sont alors équivalentes :

(i) *la fonction $x \mapsto \sqrt{\sum_{n\geq 0} \|u_n\|_{L^2(X)}^2 e_n(x)/d_n}$ appartient à $L^p(X)$,*

(ii) *la série aléatoire* (32) *converge presque sûrement dans $L^p(X)$,*

(iii) *la série aléatoire* (32) *converge dans l'espace de Bochner–Lebesgue $L^{\max(2,p)}(\Omega, L^p(X))$,*

(iv) *la série aléatoire* (32) *est bornée en probabilité dans $L^p(X)$.*

Le même énoncé est valide si les matrices aléatoires $M_n : \Omega \to \mathcal{M}_{d_n}(\mathbb{C})$ sont supposées unitairement invariantes.

Démonstration. Il s'agit d'appliquer convenablement le théorème 2.1. Pour tout entier $n \in \mathbb{N}$, on écrit

$$\sum_{i=1}^{d_n} \sum_{j=1}^{d_n} M_{n,i,j} \langle u_n, \phi_{n,j} \rangle \phi_{n,i} = \sqrt{d_n}\, \mathbf{tr}(M_n b_n),$$

avec

$$\forall x \in X, \quad b_n(x) := \frac{1}{\sqrt{d_n}} [\langle u_n, \phi_{n,i} \rangle \phi_{n,j}(x)]_{i,j} \in \mathcal{M}_{d_n}(L_x^p(X)).$$

Cela nous donne

$$\|b_n(x)\|_{\mathrm{HS}}^2 = \sum_{i,j=1}^{d_n} \frac{1}{d_n} |\langle u_n, \phi_{n,i} \rangle \phi_{n,j}(x)|^2 = \frac{1}{d_n} \|u_n\|_{L^2(X)}^2 \sum_{j=1}^{d_n} |\phi_{n,j}(x)|^2 = \|u_n\|_{L^2(X)}^2 \frac{e_n(x)}{d_n}. \quad \square$$

La condition (i) du théorème 2.2 doit être vue comme une extension multidimensionnelle du critère de Maurey (1) et éclaire la discussion sur la squeezing condition (voir (14) et (15)). Remarquons aussi que la fonction (31) ne dépend que du sous-espace E_n en vertu des formules

$$\begin{aligned} e_n(x) &= \sup_{|a_1|^2+\cdots+|a_{d_n}|^2=1} |a_1\phi_{n,1}(x)+\cdots+a_{d_n}\phi_{n,d_n}(x)|^2 \\ &= \sup\bigl\{|u_n(x)|^2 \mid u_n \in E_n,\ \|u_n\|_{L^2(X)}=1\bigr\}. \end{aligned} \tag{33}$$

Par conséquent la condition (i) est *indépendante* de la suite des lois des matrices aléatoires M_n et des bases hilbertiennes $\phi_{n,1},\dots,\phi_{n,d_n}$ de E_n. Cela traduit précisément un phénomène d'universalité pour la randomisation dans $L^p(X)$ et implique le théorème 1.8. L'interprétation avec la notion de cotype sera faite dans la partie 2D.

2B. ***Interpolation et dualité des espaces de Lebesgue probabilistes*** $\mathbf{PL}^p$**.** Le théorème 2.2 amène à poser la définition suivante.

Définition 2.3. Fixons un réel $p\in[1,+\infty[$ et une suite de sous-espaces non nuls de dimension finie $(E_n)_{n\geq 0}$ de $L^2(X)\cap L^p(X)$. On notera $d_n=\dim(E_n)$ et $(\phi_{n,1},\dots,\phi_{n,d_n})$ une base hilbertienne de E_n pour tout $n\in\mathbb{N}$. On notera aussi

$$\forall x\in X,\quad e_n(x)=|\phi_{n,1}(x)|^2+\cdots+|\phi_{n,d_n}(x)|^2.$$

On note $\mathbf{PL}^p\bigl(X,\bigoplus E_n\bigr)$ l'espace vectoriel des suites $(u_n)_{n\geq 0}$, avec $u_n\in E_n$ pour tout $n\geq 0$, qui satisfont les assertions équivalentes (i), (ii), (iii), (iv) du théorème 2.2. On appelle $\mathbf{PL}^p\bigl(X,\bigoplus E_n\bigr)$ l'espace de Lebesgue probabiliste associé à la suite de sous-espaces $(E_n)_{n\geq 0}$ et on le munit de la norme

$$\|(u_n)\|_{\mathbf{PL}^p(X,\bigoplus E_n)} := \Biggl\| \sqrt{\sum_{n\geq 0}\|u_n\|^2_{L^2(X)}\frac{e_n(x)}{\dim(E_n)}}\Biggr\|_{L^p_x(X)}.$$

Faisons quelques remarques sur cette définition :

— Répétons que la fonction e_n est indépendante de la base $(\phi_{n,1},\dots,\phi_{n,d_n})$.

— On vérifie facilement que $\mathbf{PL}^p\bigl(X,\bigoplus E_n\bigr)$ est complet (voir (57)).

— Bien que défini abstraitement, nous verrons que dans les cas qui nous intéressent, on pourra considérer $\mathbf{PL}^p\bigl(X,\bigoplus E_n\bigr)$ comme un espace de distributions (par exemple si $X=\mathbb{S}^d$ ou $X=\mathbb{R}^d$).

— Si une suite (u_n), vérifiant $u_n\in E_n$ pour tout $n\in\mathbb{N}$, n'appartient pas à $\mathbf{PL}^p\bigl(X,\bigoplus E_n\bigr)$ alors les séries aléatoires (32) du théorème 2.2 divergent dans $L^p(X)$ avec une probabilité strictement positive et donc divergent presque sûrement d'après la loi du tout ou rien.

— Puisque les fonctions $\frac{1}{d_n}e_n$ sont des densités de probabilité sur X, on a l'égalité

$$\mathbf{PL}^2\bigl(X,\bigoplus E_n\bigr)=\Biggl\{(u_n)_{n\in\mathbb{N}}\ \Bigg|\ u_n\in E_n,\ \sum_{n\in\mathbb{N}}\|u_n\|^2_{L^2(X)}<+\infty\Biggr\}.$$

Si p est différent de 2, alors il semble nécessaire d'exploiter des propriétés des fonctions e_n afin de mieux comprendre l'espace $\mathbf{PL}^p\bigl(X,\bigoplus E_n\bigr)$. Nous proposons une approche par dualité et interpolation. Le prochain résultat montre que l'interpolation des espaces $\mathbf{PL}^p\bigl(X,\bigoplus E_n\bigr)$ n'est pas gratuite et implique des propriétés sur les fonctions e_n. Rappelons maintenant que les espaces interpolés des espaces $\mathbf{PL}^p\bigl(X,\bigoplus E_n\bigr)$ sont définis de façon abstraite mais peuvent être vus comme des sous-espaces vectoriels du même espace ambiant, à savoir $\prod_{n\in\mathbb{N}} E_n$. Dans la suite, on notera $[\cdot,\cdot]_\theta$ et $[\cdot,\cdot]_{\theta,p}$ les méthodes d'interpolation complexe et réelle.

Proposition 2.4. *Fixons des réels $p_1 < p < p_2$ appartenant à $[1,+\infty[$ et soit $\theta \in]0,1[$ le nombre défini par la relation $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Considérons une suite de sous-espaces non nuls $(E_n)_{n\geq 0}$ de $L^2(X)\cap L^{p_1}(X)\cap L^{p_2}(X)$ de dimension finie et supposons que l'on a l'égalité d'espaces vectoriels*

$$\mathbf{PL}^p\big(X,\bigoplus E_n\big) = \big[\mathbf{PL}^{p_1}\big(X,\bigoplus E_n\big),\mathbf{PL}^{p_2}\big(X,\bigoplus E_n\big)\big]_\theta$$

et que les normes des deux précédents espaces sont équivalentes. Alors

$$\sup_{n\in\mathbb{N}} \frac{\|\sqrt{e_n}\|_{L^{p_1}(X)}^{1-\theta}\|\sqrt{e_n}\|_{L^{p_2}(X)}^{\theta}}{\|\sqrt{e_n}\|_{L^p(X)}} < +\infty. \tag{34}$$

La même conclusion est valide en remplaçant la méthode d'interpolation complexe $[\cdot,\cdot]_\theta$ par la méthode d'interpolation réelle $[\cdot,\cdot]_{\theta,p}$.

Démonstration. On considère pour tout entier $k\in\mathbb{N}$ le "projecteur sur E_k" défini par

$$\Lambda_k : \prod_{n\in\mathbb{N}} E_n \to L^2(X), \quad (u_n)_{n\in\mathbb{N}} \mapsto u_k.$$

Le calcul de la norme d'opérateur de $\Lambda_k : \mathbf{PL}^p\big(X,\bigoplus E_n\big)\to L^2(X)$ est immédiat

$$\|\Lambda_k\|_{\mathbf{PL}^p(X,\bigoplus E_n)\to L^2(X)} = \sup_{(u_n)\neq 0} \frac{\|u_k\|_{L^2(X)}}{\Big\|\sqrt{\sum_{n\in\mathbb{N}}\|u_n\|_{L^2(X)}^2\frac{e_n}{d_n}}\Big\|_{L^p(X)}} = \frac{\sqrt{d_k}}{\|\sqrt{e(k,\cdot)}\|_{L^p(X)}}.$$

Par interpolation, il existe une constante $K>0$ telle que pour tout $k\in\mathbb{N}$ on a

$$\|\Lambda_k\|_{\mathbf{PL}^p(X,\bigoplus E_n)\to L^2(X)} \leq K\|\Lambda_k\|_{\mathbf{PL}^{p_1}(X,\bigoplus E_n)\to L^2(X)}^{1-\theta}\|\Lambda_k\|_{\mathbf{PL}^{p_2}(X,\bigoplus E_n)\to L^2(X)}^{\theta}. \qquad \square$$

L'inégalité (34) renverse l'inégalité usuelle $\|\sqrt{e_n}\|_{L^p(X)} \leq \|\sqrt{e_n}\|_{L^{p_1}(X)}^{1-\theta}\|\sqrt{e_n}\|_{L^{p_2}(X)}^{\theta}$. L'auteur ignore si (34) suffit pour interpoler les espaces $\mathbf{PL}^p\big(X,\bigoplus E_n\big)$. Afin d'obtenir des résultats positifs d'interpolation (et même de dualité), nous allons ajouter une hypothèse supplémentaire qui utilise l'espace de Lorentz $L^{1,\infty}(X)$. Il s'agit de l'espace vectoriel des fonctions mesurables $f : X\to\mathbb{C}\cup\{\infty\}$ telles que

$$\forall t>0, \quad \mu\{x\in X \mid |f(x)|>t\} = \mathcal{O}(t^{-1}),$$

où μ est la mesure de l'espace mesuré X. L'inclusion $L^1(X)\subset L^{1,\infty}(X)$ est toujours vraie et est généralement stricte. Le contre-exemple typique est $|x|^{-d}\in L_x^{1,\infty}(\mathbb{R}^d)\backslash L_x^1(\mathbb{R}^d)$.

Théorème 2.5. *Considérons $p_1 < p_2$ deux nombres appartenant à $]1,+\infty[$ et vérifiant la condition $\frac{1}{p_1}+\frac{1}{p_2}\leq 1$. Considérons une suite de sous-espaces non nuls $(E_n)_{n\geq 0}$ de $L^2(X)\cap L^{p_1}(X)\cap L^{p_2}(X)$*

de dimension finie. On suppose que les fonctions e_n satisfont les propriétés suivantes

$$\sup_{n\in\mathbb{N}}\left(\frac{\sqrt{e_n}}{\|\sqrt{e_n}\|_{L^{p_1}(X)}}\right)^{p_1}+\sup_{n\in\mathbb{N}}\left(\frac{\sqrt{e_n}}{\|\sqrt{e_n}\|_{L^{p_2}(X)}}\right)^{p_2}\in L^{1,\infty}(X), \tag{35}$$

$$\exists p_0\in\,]p_1,p_2[,\quad \sup_{n\in\mathbb{N}}\frac{\|\sqrt{e_n}\|_{L^{p_1}(X)}^{1-\theta_0}\|\sqrt{e_n}\|_{L^{p_2}(X)}^{\theta_0}}{\|\sqrt{e_n}\|_{L^{p_0}(X)}}<+\infty, \tag{36}$$

où $\theta_0\in\,]0,1[$ *est déterminé par la relation* $\frac{1}{p_0}=\frac{1-\theta_0}{p_1}+\frac{\theta_0}{p_2}$.

Alors les espaces $\mathbf{PL}^p\big(X,\bigoplus E_n\big)$, *pour* p *parcourant* $]p_1,p_2[$, *sont stables par interpolation* ***complexe*** : *en d'autres termes, si l'on a*

$$p_1<p_1'<p<p_2'<p_2,\quad \theta\in\,]0,1[,\quad \frac{1}{p}=\frac{1-\theta}{p_1'}+\frac{\theta}{p_2'},$$

alors on a l'égalité $\mathbf{PL}^p\big(X,\bigoplus E_n\big)=\big[\mathbf{PL}^{p_1'}\big(X,\bigoplus E_n\big),\mathbf{PL}^{p_2'}\big(X,\bigoplus E_n\big)\big]_\theta$ *avec équivalence des normes.*

De même, les espaces $\mathbf{PL}^p\big(X,\bigoplus E_n\big)$, *pour* p *parcourant* $]p_1,p_2[$, *sont stables par d'interpolation* ***réelle*** : *on a l'égalité* $\mathbf{PL}^p\big(X,\bigoplus E_n\big)=\big[\mathbf{PL}^{p_1'}\big(X,\bigoplus E_n\big),\mathbf{PL}^{p_2'}\big(X,\bigoplus E_n\big)\big]_{\theta,p}$ *avec équivalences des normes.*

La condition $\frac{1}{p_1}+\frac{1}{p_2}\leq 1$ est de nature technique et est peut-être inutile en toute généralité. Elle est toujours vérifiée si l'on a $2\leq p_1<p_2$. Une autre situation intéressante se produit si p_1 et p_2 sont deux exposants conjugués, i.e., vérifient $\frac{1}{p_1}+\frac{1}{p_2}=1$. Dans ce cas, il sera commode de considérer (36) avec $p_0=2$ et $\theta=\frac{1}{2}$ pour avoir l'assertion suivante qui est en apparence plus faible (mais équivalente comme le lemme 2.36 le montrera plus loin) :

$$\sup_{n\in\mathbb{N}}\frac{\|\sqrt{e_n}\|_{L^{p_1}(X)}\|\sqrt{e_n}\|_{L^{p_2}(X)}}{\dim(E_n)}<+\infty. \tag{37}$$

En prime de l'interpolation, le théorème suivant donne une propriété de dualité.

Théorème 2.6. *Fixons deux réels* $p_1<p_2$ *appartenant à* $]1,+\infty[$ *et vérifiant l'égalité* $\frac{1}{p_1}+\frac{1}{p_2}=1$. *Considérons une suite de sous-espaces non nuls* $(E_n)_{n\geq 0}$ *de* $L^{p_1}(X)\cap L^{p_2}(X)$ *de dimension finie. Pour tous* $p\in\,]p_1,p_2[$ *et* $(u,w)\in\mathbf{PL}^p\big(X,\bigoplus E_n\big)\times\mathbf{PL}^q\big(X,\bigoplus E_n\big)$, *avec* $q:=\frac{p}{p-1}$, *on a*

$$\sum_{n\geq 0}|\langle u_n,w_n\rangle_{L^2(X)}|\leq\|u\|_{\mathbf{PL}^p(X,\bigoplus E_n)}\|w\|_{\mathbf{PL}^q(X,\bigoplus E_n)}. \tag{38}$$

En d'autres termes, tout élément de $\mathbf{PL}^q\big(X,\bigoplus E_n\big)$ *induit canoniquement une forme linéaire bornée sur* $\mathbf{PL}^p\big(X,\bigoplus E_n\big)$.

En outre, si l'on suppose (35) *et* (37) *alors les espaces* $\mathbf{PL}^p\big(X,\bigoplus E_n\big)$, *pour* p *parcourant* $]p_1,p_2[$, *sont stables par dualité au sens suivant : l'injection canonique* $\Lambda_p:\mathbf{PL}^q\big(X,\bigoplus E_n\big)\to\mathbf{PL}^p\big(X,\bigoplus E_n\big)'$ *qui à un élément* $w\in\mathbf{PL}^q\big(X,\bigoplus E_n\big)$ *associe la forme linéaire*

$$\mathbf{PL}^p\big(X,\bigoplus E_n\big)\mapsto\mathbb{C},\quad u\mapsto\sum_{n\geq 0}\langle u_n,w_n\rangle_{L^2(X)},$$

est un isomorphisme d'espaces de Banach.

En d'autres termes, (35) et (37) impliquent que le dual de $\mathbf{PL}^p\big(X, \bigoplus E_n\big)$ est canoniquement isomorphe à $\mathbf{PL}^q\big(X, \bigoplus E_n\big)$ pour tout $p \in]p_1, p_2[$. Les deux théorèmes précédents appellent à quelques remarques :

(a) L'inégalité (38) implique l'estimation $\|\Lambda_p\| \leq 1$ mais nous verrons que l'injection canonique Λ_p n'est généralement pas une isométrie de $\mathbf{PL}^p\big(X, \bigoplus E_n\big)$ sur $\mathbf{PL}^q\big(X, \bigoplus E_n\big)$ (voir (62)). Cela contraste avec la dualité des espaces de Lebesgue.

(b) En pratique, les hypothèses (36) et (37) se réalisent si les densités $\frac{1}{d_n} e_n$ ont tendance à se concentrer uniformément sur un borélien $A_n \subset X$ de mesure finie et strictement positive, c'est-à-dire si l'on peut assimiler la densité de probabilité $(\dim(E_n))^{-1} e_n$ à une fonction de la forme $x \mapsto (\mu(A_n))^{-1} \mathbf{1}_{A_n}(x)$. Dans les exemples issus de la physique mathématique qui font intervenir des polynômes orthogonaux, on a très souvent des propriétés de concentration qui forcent $\|\sqrt{e_n}\|_{L^p(X)}$ à être équivalent à $n^{(a+b/p)}$, pour un certain couple $(a, b) \in \mathbb{R}^2$, si p parcourt un intervalle donné $]p_1, p_2[$. Ainsi, (36) sera vérifié. De même (37) sera vérifié si $\|\sqrt{e_n}\|_{L^p(X)}$ est équivalent à $n^{a\left(\frac{1}{2}-\frac{1}{p}\right)} \sqrt{\dim(E_n)}$ pour un certain réel a.

(c) L'hypothèse (35) est très importante dans nos preuves. Bien qu'il ne semble pas facile de l'interpréter, elle sera toujours réalisée dans les cas qui nous concernent. Il est sans doute possible de relaxer cette hypothèse (et peut-être même de la supprimer), mais permettons-nous d'expliquer l'intervention des espaces de Lorentz. D'une part, nos démonstrations utilisent l'interpolation réelle, théorie dans laquelle les espaces de Lorentz jouent un rôle clé. D'autre part, nous verrons des exemples bien concrets, à savoir les fonctions propres qui se concentrent sur une géodésique de la sphère $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, où l'hypothèse (35) ne se réalise pas si l'on remplace l'espace de Lorentz $L^{1,\infty}(\mathbb{S}^d)$ par l'espace de Lebesgue $L^1(\mathbb{S}^d)$ (voir la remarque 3.7).

2C. ***Preuve du théorème 2.1, randomisation avec des matrices aléatoires.*** Dans la théorie unidimensionnelle, les inégalités de Kahane–Khintchine, que nous rappelons, sont très importantes : pour tous réels $p > q \geq 1$, il existe une constante $K_{p,q} \geq 1$ telle que, pour tout espace de Banach B et tous éléments $u_0, \dots, u_N$ de B, les moments des variables aléatoires $\big\|\sum_{n=0}^N \varepsilon_n u_n\big\|_B$ sont du même ordre de grandeur au sens suivant

$$\mathbf{E}\left[\left\|\sum_{n=0}^N \varepsilon_n u_n\right\|_B^q\right]^{\frac{1}{q}} \leq \mathbf{E}\left[\left\|\sum_{n=0}^N \varepsilon_n u_n\right\|_B^p\right]^{\frac{1}{p}} \leq K_{p,q}\mathbf{E}\left[\left\|\sum_{n=0}^N \varepsilon_n u_n\right\|_B^q\right]^{\frac{1}{q}}. \tag{39}$$

D'après Kwapień, il existe une constante universelle $K > 0$ telle que l'on a $K_{p,q} \leq K_{p,1} \leq K\sqrt{p}$ [Li et Queffélec 2004, Part 3.III].

Fixons maintenant une suite d'entiers non nuls $(d_n)_{n\in\mathbb{N}}$ et notons $(\mathcal{E}_n)_n$ une suite de matrices aléatoires indépendantes. On supposera que la matrice aléatoire $\mathcal{E}_n$ suit une loi uniforme dans le groupe orthogonal $O_{d_n}(\mathbb{R})$ pour tout $n \in \mathbb{N}$. On définit de même W_n en remplaçant les groupes orthogonaux $O_{d_n}(\mathbb{R})$ par les groupes unitaires $U_{d_n}(\mathbb{C})$. Comme nous manipulerons des matrices de tailles différentes, il sera commode de faire le raccourci suivant : à la place de "*une suite* $(b_n)_{n\in\mathbb{N}}$ *de matrices carrées telles que pour tout* $n \in \mathbb{N}$ *la matrice* b_n *soit de taille* $d_n \times d_n$ *à coefficients dans* B", nous écrirons "*une suite de matrices carrées* $b_n \in \mathcal{M}_{d_n}(B)$". Dans ce contexte, les inégalités de Kahane–Khintchine démontrées par Marcus et Pisier [1981, page 81, (2.1) ; page 91, Corollary 2.12] s'énoncent comme suit :

Proposition 2.7. *Pour tous réels $p > q \geq 1$, il existe une constante $K_{p,q} \geq 1$ telle que, pour tout espace de Banach B, pour tout entier $N \in \mathbb{N}$ et toute suite de matrices carrées $b_n \in \mathcal{M}_{d_n}(B)$, nous avons*

$$\mathbf{E}\left[\left\|\sum_{n=0}^{N}\mathbf{tr}(\mathcal{E}_n b_n)\right\|_B^q\right]^{\frac{1}{q}} \leq \mathbf{E}\left[\left\|\sum_{n=0}^{N}\mathbf{tr}(\mathcal{E}_n b_n)\right\|_B^p\right]^{\frac{1}{p}} \leq K_{p,q}\mathbf{E}\left[\left\|\sum_{n=0}^{N}\mathbf{tr}(\mathcal{E}_n b_n)\right\|_B^q\right]^{\frac{1}{q}}. \tag{40}$$

De même que pour les inégalités de Kahane–Khintchine (39), *il existe une constante numérique $K \geq 1$ telle que l'on a $K_{p,q} \leq K_{p,1} \leq K\sqrt{p}$. Des inégalités similaires sont valides pour les matrices aléatoires W_n à la place de $\mathcal{E}_n$.*

La version originale des inégalités (40) utilise les séries aléatoires $\sum d_n\,\mathbf{tr}(\mathcal{E}_n b_n)$ qui sont plus adaptées à la théorie des séries de Fourier sur un groupe compact, mais l'on peut bien entendu englober l'entier d_n dans la matrice b_n (il est d'ailleurs remarquable que ces inégalités ne nécessitent aucune condition de croissance sur les dimensions d_n). Comme les inégalités (40) impliquent les inégalités (39), on se permet d'utiliser la même notation $K_{p,q}$. Les inégalités (40) jouent un rôle clé pour montrer le théorème suivant [Marcus et Pisier 1981, page 92, Corollary 2.14] où nous forçons l'apparition du terme multiplicatif $\sqrt{d_n}$ par cohérence avec la suite.

Théorème 2.8. *Considérons un espace de Banach complexe B et une suite $b_n \in \mathcal{M}_{d_n}(B)$ de matrices carrées. Les propriétés suivantes sont équivalentes :*

(i) *la série $\sum_n \sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n b_n)$ converge dans $L^p(\Omega, B)$ pour **un** réel $p \in [1, +\infty[$,*

(ii) *la série $\sum_n \sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n b_n)$ converge dans $L^p(\Omega, B)$ pour **tout** réel $p \in [1, +\infty[$,*

(iii) *la série $\sum_n \sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n b_n)$ converge presque sûrement dans B,*

(iv) *la série $\sum_n \sqrt{d_n}\,\mathbf{tr}(W_n b_n)$ converge dans $L^p(\Omega, B)$ pour **un** réel $p \in [1, +\infty[$,*

(v) *la série $\sum_n \sqrt{d_n}\,\mathbf{tr}(W_n b_n)$ converge dans $L^p(\Omega, B)$ pour **tout** réel $p \in [1, +\infty[$,*

(vi) *la série $\sum_n \sqrt{d_n}\,\mathbf{tr}(W_n b_n)$ converge presque sûrement dans B.*

D'après le théorème de Maurey–Pisier (théorème 1.6), il faut nécessairement faire une hypothèse supplémentaire si l'on veut ajouter d'autres lois matricielles en plus des matrices aléatoires $(\mathcal{E}_n)$ et (W_n). Le théorème 2.1 explique que l'on peut considérablement préciser le théorème 2.8 dans le cas particulier $B = L^p(X)$, avec $p \in [1, +\infty[$. Nous allons exploiter la preuve du théorème 2.8, c'est-à-dire l'utilisation systématique des inégalités (40) et d'un principe de contraction (théorème 2.16). La partie 2D donnera un éclairage sur les propriétés géométriques de l'espace de Banach $L^p(X)$ utilisées dans notre argumentation.

Dans le cas $B = \mathbb{C}$, les inégalités (40) donnent un résultat simple et bien connu dans le cas unidimensionnel. Le cas multidimensionnel fait intervenir les normes de Hilbert–Schmidt (30).

Lemme 2.9. *Pour toute suite de matrices $a_n \in \mathcal{M}_{d_n}(\mathbb{C})$ et pour tout $N \in \mathbb{N}$, on a*

$$\mathbf{E}\left[\left|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n a_n)\right|^2\right] = \sum_{n=0}^{N}\|a_n\|_{\mathrm{HS}}^2. \tag{41}$$

La même conclusion est valide en remplaçant $(\mathcal{E}_n)_{n\geq 0}$ par $(W_n)_{n\geq 0}$.

Démonstration. En notant dP la mesure de Haar normalisée du groupe compact $O_d(\mathbb{R})$ et en utilisant l'indépendance des matrices aléatoires $\mathcal{E}_n$, on a

$$\begin{aligned}\mathbf{E}\Big[\Big|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n a_n)\Big|^2\Big] &= \sum_{n_1=0}^{N}\sum_{n_2=0}^{N}\sqrt{d_{n_1}}\sqrt{d_{n_2}}\mathbf{E}\big[\mathbf{tr}(\mathcal{E}_n a_n)\overline{\mathbf{tr}(\mathcal{E}_n a_n)}\big] \\ &= \sum_{\substack{0\le n_1\le N\\ 0\le n_2\le N\\ n_1\neq n_2}}\sqrt{d_{n_1}d_{n_2}}\Big(\int_{O_{d_{n_1}}(\mathbb{R})}\mathbf{tr}(Pa_{n_1})\,dP\Big)\Big(\int_{O_{d_{n_2}}(\mathbb{R})}\overline{\mathbf{tr}(Pa_{n_2})}\,dP\Big) \\ &\qquad + \sum_{n=0}^{N} d_n\int_{O_{d_n}(\mathbb{R})}|\,\mathbf{tr}(Pa_n)|^2\,dP.\end{aligned}$$

Il nous suffit de prouver pour tout entier $d\in\mathbb{N}^\star$ et pour toute matrice $A\in\mathcal{M}_d(\mathbb{C})$:

$$\int_{O_d(\mathbb{R})}\mathbf{tr}(PA)\,dP = 0 \quad\text{et}\quad \int_{O_d(\mathbb{R})}|\,\mathbf{tr}(PA)|^2\,dP = \frac{\mathbf{tr}({}^t\bar{A}A)}{d}.$$

Le cas $d=1$ étant trivial, on suppose $d\ge 2$. La nullité de la première intégrale est claire par le changement de variables $P\mapsto -P$. Pour la seconde intégrale, il suffit de traiter le cas où tous les coefficients de A sont réels. En effet, le cas complexe découle du cas réel en écrivant $A=A_x+iA_y$ avec $A_x, A_y\in\mathcal{M}_d(\mathbb{R})$. En notant la matrice symétrique $|A|=\sqrt{{}^tAA}$, on peut factoriser $A=P|A|=PQDQ^{-1}$ avec $(P,Q)\in O_d(\mathbb{R})^2$ et D matrice diagonale dont les valeurs propres $\mu_1,\dots,\mu_d$ sont les valeurs singulières de A. Les propriétés d'invariance de la trace et de la mesure de Haar dP simplifient l'intégrale

$$\begin{aligned}\int_{O_d(\mathbb{R})}\mathbf{tr}(PA)^2\,dP &= \int_{O_d(\mathbb{R})}\mathbf{tr}(PD)^2\,dP \\ &= \int_{O_d(\mathbb{R})}(p_{11}\mu_1+\cdots+p_{dd}\mu_d)^2\,dP \\ &= \sum_{i,j=1}^{d}\mu_i\mu_j\int_{O_d(\mathbb{R})}p_{ii}p_{jj}\,dP.\end{aligned}$$

En effectuant les changements de coordonnées $P\mapsto E_{ij}P$ et $P\mapsto PE_{ij}$ où E_{ij} est la matrice orthogonale associée à la permutation qui transpose i et j ainsi que les d changements de coordonnées $P\mapsto\Delta_kP$ où Δ_k est la matrice diagonale

$$\mathrm{Diag}(1,\dots,1,\underbrace{-1}_{k},1,\dots,1),$$

on obtient

$$\begin{aligned}\forall i\neq j,&\quad \int_{O_d(\mathbb{R})}p_{ii}p_{jj}\,dP = 0,\\ \forall i,j,&\quad \int_{O_d(\mathbb{R})}p_{ij}^2\,dP = \int_{O_d(\mathbb{R})}p_{11}^2\,dP.\end{aligned}$$

Par moyenne, la dernière intégrale vaut

$$\frac{1}{d^2}\int_{O_d(\mathbb{R})}\sum_{i=1}^{d}\sum_{j=1}^{d}p_{ij}^2\,dP=\frac{1}{d^2}\int_{O_d(\mathbb{R})}\mathbf{tr}({}^tPP)dP=\frac{1}{d^2}\int_{O_d(\mathbb{R})}\mathbf{tr}(I_d)\,dP=\frac{1}{d}.$$

Cela prouve (41) car $\mathbf{tr}({}^tAA)=\mu_1^2+\cdots+\mu_d^2$. □

Nous avons à présent tous les moyens pour aborder la randomisation de matrices aléatoires plus générales.

Proposition 2.10. *Fixons $q\in[2,+\infty[$ et considérons une suite de matrices aléatoires $M_n:\Omega\to\mathcal{M}_{d_n}(\mathbb{R})$ indépendantes, orthogonalement invariantes et vérifiant l'hypothèse de bornitude des moments*

$$\sup_{n\in\mathbb{N}}\mathbf{E}[\|M_n\|_{\mathrm{op}}^q]<+\infty.$$

Il existe une constante universelle $K\geq 1$ telle que, pour toute suite $a_n\in\mathcal{M}_{d_n}(\mathbb{C})$, nous avons

$$\forall N\in\mathbb{N},\quad \mathbf{E}\left[\left|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(M_na_n)\right|^q\right]^{\frac{1}{q}}\leq K\sqrt{q}\left(\sup_{0\leq n\leq N}\mathbf{E}[\|M_n\|_{\mathrm{op}}^q]\right)^{\frac{1}{q}}\sqrt{\sum_{n=0}^{N}\|a_n\|_{\mathrm{HS}}^2}.\tag{42}$$

La même conclusion est valide en supposant que chaque matrice aléatoire M_n est à valeurs dans $\mathcal{M}_{d_n}(\mathbb{C})$ et est unitairement invariante.

Démonstration. Fixons $\mathcal{E}_0,\dots,\mathcal{E}_N$ des matrices aléatoires indépendantes (et indépendantes des matrices aléatoires M_n) qui suivent des lois uniformes dans les groupes orthogonaux de tailles respectives $d_0,\dots,d_N$. Utilisant la définition 1.7 et les inégalités de Kahane–Khintchine–Marcus–Pisier (40) et (41), on peut écrire

$$\begin{aligned}\mathbf{E}\left[\left|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(M_na_n)\right|^q\right]&=\mathbf{E}_{\omega'}\mathbf{E}_{\omega}\left[\left|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n(\omega')M_n(\omega)a_n)\right|^q\right]\\&=\mathbf{E}_{\omega}\mathbf{E}_{\omega'}\left[\left|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n(\omega')M_n(\omega)a_n)\right|^q\right]\\&\leq K^qq^{\frac{q}{2}}\mathbf{E}_{\omega}\left[\mathbf{E}_{\omega'}\left[\left|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n(\omega')M_n(\omega)a_n)\right|^2\right]^{\frac{q}{2}}\right]\\&\leq K^qq^{\frac{q}{2}}\mathbf{E}_{\omega}\left[\left(\sum_{n=0}^{N}\|M_n(\omega)a_n\|_{\mathrm{HS}}^2\right)^{\frac{q}{2}}\right]\\&\leq K^qq^{\frac{q}{2}}\mathbf{E}_{\omega}\left[\left(\sum_{n=0}^{N}\|M_n(\omega)\|_{\mathrm{op}}^2\|a_n\|_{\mathrm{HS}}^2\right)^{\frac{q}{2}}\right].\end{aligned}$$

L'inégalité triangulaire dans $L^{\frac{q}{2}}(\Omega)$ nous amène alors à la conclusion

$$\mathbf{E}\Big[\Big|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(M_n a_n)\Big|^q\Big] \le K^q q^{\frac{q}{2}} \Big(\sum_{n=0}^{N} \|a_n\|_{\mathrm{HS}}^2 \mathbf{E}[\|M_n\|_{\mathrm{op}}^q]^{\frac{2}{q}}\Big)^{\frac{q}{2}}$$
$$\le K^q q^{\frac{q}{2}} \Big(\sup_{0\le n\le N} \mathbf{E}[\|M_n\|_{\mathrm{op}}^q]\Big)\Big(\sum_{n=0}^{N} \|a_n\|_{\mathrm{HS}}^2\Big)^{\frac{q}{2}}. \qquad \square$$

Remarque 2.11. Dans le cas particulier $d_n = 1$, c'est-à-dire si les matrices aléatoires $M_n : \Omega \to \mathbb{R}$ sont en fait des *variables aléatoires symétriques*, l'inégalité (42) s'écrit pour tout réel $q \in [2, +\infty[$ et tous réels $a_0, \dots, a_N$:

$$\mathbf{E}\Big[\Big|\sum_{n=0}^{N} M_n a_n\Big|^q\Big]^{\frac{1}{q}} \le K\sqrt{q}\Big(\sup_{0\le n\le N} \mathbf{E}[|M_n|^q]\Big)^{\frac{1}{q}} \sqrt{\sum_{n=0}^{N} |a_n|^2}\,.$$

Du fait que la dimension considérée est 1, le principe de symétrisation montrerait que l'inégalité précédente est aussi vraie si les variables aléatoires M_n sont seulement centrées (voir la preuve du lemme B.1).

Voici deux lemmes spécifiques à $L^p(X)$.

Lemme 2.12. *Pour tous réels p et q appartenant à $[1, +\infty[$ et pour toute suite de matrices $b_n \in \mathcal{M}_{d_n}(L^p(X))$, on a*

$$\mathbf{E}\Big[\Big\|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n b_n(x))\Big\|_{L_x^p(X)}^q\Big]^{\frac{1}{q}} \simeq_{p,q} \Bigg\|\sqrt{\sum_{n=0}^{N} \|b_n(x)\|_{\mathrm{HS}}^2}\Bigg\|_{L_x^p(X)}. \tag{43}$$

La même conclusion est valide en remplaçant $(\mathcal{E}_n)_{n\ge 0}$ par $(W_n)_{n\ge 0}$.

Démonstration. D'après les inégalités de Kahane–Khintchine–Marcus–Pisier (40), il suffit de traiter le cas $q = p$. Le théorème de Fubini donne alors

$$\mathbf{E}\Big[\Big\|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n b_n(x))\Big\|_{L_x^p(X)}^p\Big] = \int_X \mathbf{E}\Big[\Big|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n b_n(x))\Big|^p\Big]\,d\mu(x).$$

De nouveau, les inégalités de Kahane–Khintchine–Marcus–Pisier sur $\mathbb{C}$ montrent que l'intégrale précédente est équivalente à la suivante

$$\int_X \mathbf{E}\Big[\Big|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n b_n(x))\Big|^2\Big]^{\frac{p}{2}}\,d\mu(x).$$

L'égalité (41) achève la preuve. $\square$

Lemme 2.13. *Considérons un réel $p \in [1, +\infty[$, un entier $N \in \mathbb{N}$, des matrices aléatoires*

$$M_0 : \Omega \to \mathcal{M}_{d_0}(\mathbb{C}), \dots, M_N : \Omega \to \mathcal{M}_{d_N}(\mathbb{C})$$

et des matrices

$$b_0 \in \mathcal{M}_{d_0}(L^p(X)), \dots, b_N \in \mathcal{M}_{d_N}(L^p(X)).$$

Alors on a l'inégalité :

$\forall q \in [p, +\infty[\cup \{+\infty\}$,

$$\mathbf{E}_\omega\left[\left\|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(M_n(\omega)b_n)\right\|_{L^p(X)}^q\right]^{\frac{1}{q}} \leq \left\|\mathbf{E}_\omega\left[\left|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(M_n(\omega)b_n)\right|^q\right]^{\frac{1}{q}}\right\|_{L^p(X)}.$$

Démonstration. Il s'agit de remarquer la continuité de l'injection canonique

$$L^p(X, L^q(\Omega)) \to L^q(\Omega, L^p(X))$$

par interpolation entre $q = p$ et $q = +\infty$. □

À partir de ce point, nous n'aurons besoin d'aucune autre propriété spécifique de l'espace de Banach $L^p(X)$. On obtient facilement le résultat suivant.

Corollaire 2.14. *Considérons $p \in [1, +\infty[$ et une suite de matrices aléatoires $M_n : \Omega \to \mathcal{M}_{d_n}(\mathbb{R})$ indépendantes, orthogonalement invariantes et vérifiant*

$$\sup_{n\in\mathbb{N}} \mathbf{E}\big[\|M_n\|_{\mathrm{op}}^{\max(2,p)}\big] < +\infty.$$

Il existe une constante universelle $K \geq 1$ telle que, pour toute suite de matrices $b_n \in \mathcal{M}_{d_n}(L^p(X))$ et tout entier $N \in \mathbb{N}$, on a

$$\left\|\sum_{n=0}^{N}\sqrt{d_n}\,\mathbf{tr}(M_n b_n)\right\|_{L^{\max(2,p)}(\Omega, L^p(X))} \leq K\sqrt{p}\left(\sup_{n\in\mathbb{N}} \mathbf{E}\big[\|M_n\|_{\mathrm{op}}^{\max(2,p)}\big]\right)^{\frac{1}{\max(2,p)}} \left[\int_X \left(\sum_{n=0}^{N}\|b_n(x)\|_{\mathrm{HS}}^2\right)^{\frac{p}{2}} d\mu(x)\right]^{\frac{1}{p}}. \quad (44)$$

Par conséquent, si

$$x \mapsto \sum_{n\geq 0}\|b_n(x)\|_{\mathrm{HS}}^2$$

appartient à $L^{p/2}(X)$, alors la série aléatoire $\sum\sqrt{d_n}\,\mathbf{tr}(M_n b_n)$ converge dans $L^{\max(2,p)}(\Omega, L^p(X))$ et presque sûrement dans $L^p(X)$.

La même conclusion est valide en supposant que chaque matrice aléatoire M_n est à valeurs dans $\mathcal{M}_{d_n}(\mathbb{C})$ et est unitairement invariante.

Démonstration. L'inégalité (44) découle d'une combinaison du lemme 2.13, l'inégalité (42) avec $q = \max(2, p)$ et de l'inégalité $\sqrt{\max(2, p)} \leq \sqrt{2p}$.

Pour obtenir la convergence dans $L^{\max(2,p)}(\Omega, L^p(X))$ de la série aléatoire $\sum\sqrt{d_n}\,\mathbf{tr}(M_n b_n)$, il suffit d'appliquer (44) à ses paquets de Cauchy. Ensuite, la convergence dans $L^{\max(2,p)}(\Omega, L^p(X))$ implique la convergence en probabilité. Puisque les termes de la série aléatoire $\sum\sqrt{d_n}\,\mathbf{tr}(M_n b_n)$ sont indépendants et sont à valeurs dans le sous-espace séparable de $L^p(X)$ engendré par les coefficients des matrices b_n, on en déduit sa convergence presque sûre (voir [Ledoux et Talagrand 1991, Theorem 6.1] ou [Li et Queffélec 2004, théorème II.3]). □

Remarque 2.15. On peut reformuler l'inégalité (44) à l'aide de (43) :

$$\Big\| \sum_{n=0}^{N} \sqrt{d_n}\, \mathbf{tr}(M_n b_n) \Big\|_{L^{\max(2,p)}(\Omega, L^p(X))} \leq C(p) \Big(\sup_{n\in\mathbb{N}} \mathbf{E}\big[\|M_n\|_{\mathrm{op}}^{\max(2,p)} \big] \Big)^{\frac{1}{\max(2,p)}} \Big\| \sum_{n=0}^{N} \sqrt{d_n}\, \mathbf{tr}(\mathcal{E}_n b_n) \Big\|_{L^{\max(2,p)}(\Omega, L^p(X))}.$$

Cette inégalité a la même forme que celle qui apparaît à la fin de la démonstration de [Maurey et Pisier 1976, page 69]. Il est donc légitime de croire qu'une démonstration avec un théorème de factorisation est possible.

Rappelons maintenant un principe de contraction pour les variables $\mathcal{E}_n$ et W_n obtenu par Marcus et Pisier.

Théorème 2.16. *Fixons un espace de Banach complexe B, un réel $q \in [1, +\infty[$, une suite de matrices $b_n \in \mathcal{M}_{d_n}(B)$ et une suite de matrices aléatoires $M_n : \Omega \to \mathcal{M}_{d_n}(\mathbb{R})$ indépendantes et orthogonalement invariantes. Alors on a*

$$\Big(\inf_{0\leq n\leq N} \sigma(\mathbf{E}[|M_n|]) \Big) \mathbf{E}\Big[\Big\| \sum_{n=0}^{N} \sqrt{d_n}\, \mathbf{tr}(\mathcal{E}_n b_n) \Big\|_B^q \Big]^{\frac{1}{q}} \leq \mathbf{E}\Big[\Big\| \sum_{n=0}^{N} \sqrt{d_n}\, \mathbf{tr}(M_n b_n) \Big\|_B^q \Big]^{\frac{1}{q}}. \tag{45}$$

Une inégalité similaire est valide en remplaçant $\mathcal{E}_n$ par W_n, $\mathcal{M}_{d_n}(\mathbb{R})$ par $\mathcal{M}_{d_n}(\mathbb{C})$ et l'invariance orthogonale par l'invariance unitaire.

Démonstration. On consultera [Marcus et Pisier 1981, page 82, Proposition 2.1, (2.7)] avec le terme $\|\mathbf{E}[|M_n|]^{-1}\|_{\mathrm{op}}^{-1} = \sigma(\mathbf{E}[|M_n|])$. □

Le résultat précédent semble suggérer qu'une minoration uniforme de la forme $\inf_{n\in\mathbb{N}} \sigma(\mathbf{E}[|M_n|]) > 0$ et la convergence presque sûre de la série aléatoire $\sum \sqrt{d_n}\, \mathbf{tr}(M_n b_n)$ impliquent celle de $\sum \sqrt{d_n}\, \mathbf{tr}(\mathcal{E}_n b_n)$. En fait, il est aisé de construire un contre-exemple dans le cas plus simple, à savoir $d_n = 1$ pour tout $n \in \mathbb{N}$ et $B = \mathbb{R}$. Pour cela, considérons une suite indépendante de variables aléatoires symétriques $X_n : \Omega \to \mathbb{R}$, avec $n \geq 1$, telle que la loi de X_n est

$$\frac{\delta_{-n^2}}{2n^2} + \Big(1 - \frac{1}{n^2}\Big)\delta_0 + \frac{\delta_{n^2}}{2n^2}.$$

On a une concentration en 0 au sens suivant : il existe une constante $C > 1$ telle que

$$\boldsymbol{P}[X_n = 0] = 1 - \frac{1}{n^2}, \quad \boldsymbol{P}[|X_n| \geq 1] = \frac{1}{n^2}, \quad \mathbf{E}[|X_n|] \in \Big[\frac{1}{C}, C\Big].$$

Le lemme de Borel–Cantelli assure que pour presque tout $\omega \in \Omega$, la suite $(X_n(\omega))$ stationne en 0, donc la série aléatoire $\sum X_n$ converge presque sûrement tandis que la série aléatoire $\sum \varepsilon_n$ diverge toujours. Cet exemple élémentaire montre qu'il faut nécessairement imposer des conditions supplémentaires sur les matrices aléatoires M_n pour obtenir la convergence presque sûre de la série aléatoire $\sum \mathbf{tr} \sqrt{d_n}\, \mathbf{tr}(\mathcal{E}_n b_n)$ (voir par exemple [Jain et Marcus 1975, Part 5 ; Imekraz et al. 2016, Theorem 5.2] dans le cas unidimensionnel). Un défaut du contre-exemple précédent est l'explosion des moments d'ordre strictement plus grand que 1. Cela nous mène à la proposition suivante dont l'inspiration vient de la preuve de [Marcus et Pisier 1981, page 55, Lemma 1.2]. Cet ouvrage examine la situation d'un espace de Banach de la forme $\mathcal{C}^0(K)$, où K est compact d'un groupe localement compact, à la place de $L^p(X)$. L'idée de la preuve se résume simplement : nous allons majorer les moments d'ordre 2 des sommes partielles de la série $\sum \sqrt{d_n}\,\mathbf{tr}(M_n b_n)$ par leurs moments d'ordre 1 (ce qui inverse l'ordre naturel), puis la bornitude presque sûre impliquera que les moments d'ordre 1 et 2 sont uniformément bornés. Par comparaison avec les variables aléatoires $\mathcal{E}_n$, nous obtiendrons la conclusion.

Proposition 2.17. *Fixons $p \in [1, +\infty[$ et une suite de matrices aléatoires $M_n : \Omega \to \mathcal{M}_{d_n}(\mathbb{R})$ indépendantes, orthogonalement invariantes et vérifiant*

$$\inf_{n\in\mathbb{N}} \sigma(\mathbf{E}[|M_n|]) > 0 \quad \textit{et} \quad \sup_{n\in\mathbb{N}} \mathbf{E}\big[\|M_n\|_{\mathrm{op}}^{\max(2,p)}\big] < +\infty.$$

Pour toute suite de matrices $b_n \in \mathcal{M}_{d_n}(L^p(X))$, si la série aléatoire $\sum \sqrt{d_n}\,\mathbf{tr}(M_n b_n)$ est bornée en probabilité dans $L^p(X)$, c'est-à-dire si l'on a

$$\lim_{t\to+\infty} \sup_{N\in\mathbb{N}} \boldsymbol{P}\left[\left\|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(M_n b_n)\right\|_{L^p(X)} > t\right] = 0, \tag{46}$$

alors

$$\sqrt{\sum_{n\in\mathbb{N}} \|b_n(x)\|_{\mathrm{HS}}^2} \in L^p_x(X).$$

De nouveau, la même conclusion est valide en supposant que chaque matrice aléatoire M_n est à valeurs dans $\mathcal{M}_{d_n}(\mathbb{C})$ et est unitairement invariante.

Démonstration. Posons pour tout entier $N \in \mathbb{N}$ la somme partielle

$$S_N = \sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(M_n b_n).$$

En outre, on considère $c > 1$ de sorte que l'on a pour tout $n \in \mathbb{N}$

$$\frac{1}{c} \le \sigma(\mathbf{E}[|M_n|]) \quad \text{et} \quad \mathbf{E}\big[\|M_n\|_{\mathrm{op}}^{\max(2,p)}\big] \le c.$$

Le principe de contraction (théorème 2.16) et les inégalités (43) montrent qu'il existe une constante $C(p,c) > 1$ telle que

$$\frac{1}{c}\mathbf{E}\left[\left\|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(\mathcal{E}_n b_n(x))\right\|_{L^p_x(X)}\right] \le \mathbf{E}[\|S_N\|_{L^p(X)}],$$

$$\frac{1}{C(p,c)}\left\|\sqrt{\sum_{n=0}^{N} \|b_n(x)\|_{\mathrm{HS}}^2}\right\|_{L^p_x(X)} \le \mathbf{E}[\|S_N\|_{L^p(X)}]. \tag{47}$$

Majorons maintenant les moments d'ordre 2. En invoquant (44) et quitte à augmenter $C(p,c)$, on a

$$\begin{aligned}\mathbf{E}[\|S_N\|^2_{L^p(X)}]^{\frac{1}{2}} &\leq \mathbf{E}\big[\|S_N\|^{\max(2,p)}_{L^p(X)}\big]^{\frac{1}{\max(2,p)}}\\ &\leq C(p,c)\Bigg\|\sqrt{\sum_{n=0}^{N}\|b_n(x)\|^2_{\mathrm{HS}}}\Bigg\|_{L^p_x(X)}\\ &\leq C(p,c)^2\mathbf{E}[\|S_N\|_{L^p(X)}].\end{aligned}$$

Il s'agit de l'inégalité d'inversion des moments que nous cherchions. Utilisons l'inégalité de Paley–Zygmund [Kahane 1985, page 8, Inequality II] pour obtenir

$$\boldsymbol{P}\big[\|S_N\|_{L^p(X)} \geq \tfrac{1}{2}\mathbf{E}[\|S_N\|_{L^p(X)}]\big] \geq \frac{\mathbf{E}[\|S_N\|_{L^p(X)}]^2}{4\mathbf{E}[\|S_N\|^2_{L^p(X)}]} \geq \frac{1}{4C(p,c)^4}. \tag{48}$$

D'après (46), il existe un réel $A > 0$ tel que

$$\forall N \in \mathbb{N}, \quad \frac{1}{4C(p,c)^4} > \boldsymbol{P}\big[\|S_N\|_{L^p(X)} \geq A\big].$$

Or (48) force chaque moment $\mathbf{E}[\|S_N\|_{L^p(X)}]$ à être majoré par $2A$. La conclusion découle en examinant (47). □

La preuve du théorème 2.1 est obtenue en combinant tous les arguments précédents. En effet, on vérifie les implications (i) ⇒ (iii) et (iv) ⇒ (i) grâce au corollaire 2.14 et à la proposition 2.17. L'implication (iii) ⇒ (ii) a été vue au cours de la preuve du corollaire 2.14. Quant à l'implication (ii) ⇒ (iv), elle est vraie en toute généralité.

2D. ***Randomisation dans un treillis de Banach de cotype fini et preuve du théorème 1.9.*** On explique comment les arguments développés dans la partie 2C s'étendent pour les treillis de Banach (pour cette notion, on se réfère au livre [Lindenstrauss et Tzafriri 1973]). Par souci de simplicité, on ne considérera que des espaces de Banach sur le corps des réels. Il est en fait possible de *complexifier* un treillis de Banach afin de déduire des résultats complexes à partir de résultats réels (voir [Lindenstrauss et Tzafriri 1973, page 43]). On notera $\leq$, $\vee$ et $|\cdot|$ respectivement la relation d'ordre, la borne supérieure et la valeur absolue. Pour tout $N \in \mathbb{N}^\star$, on notera aussi $\overline{\mathcal{H}_N}$ l'espace vectoriel réel des fonctions continues et 1-homogènes sur $\mathbb{R}^N$ ainsi que les projections sur les coordonnées :

$$\phi_i : (t_1, \ldots, t_N) \in \mathbb{R}^N \mapsto t_i \in \mathbb{R}.$$

On notera que $\overline{\mathcal{H}_N}$ est un treillis. Le théorème suivant assure l'existence d'un calcul fonctionnel dans un treillis de Banach basé sur $\overline{\mathcal{H}_N}$ (voir [Lindenstrauss et Tzafriri 1973, Theorem 1d1]).

Théorème 2.18. *Soit B un treillis de Banach et fixons $f_1, \ldots, f_N$ des éléments de B. Il existe une unique application linéaire*

$$\tau : \overline{\mathcal{H}_N} \to B, \quad F \mapsto F(f_1, \ldots, f_N)$$

qui vérifie les deux propriétés suivantes :

(i) $\phi_i(f_1, \dots, f_N) = f_i$ *pour tout entier* $i \in [1, N]$,

(ii) τ *est un morphisme de treillis* (*i.e.*, *conserve positivité*, *borne supérieure*, *borne inférieure et valeur absolue*).

Dans ce cas, *en notant* $f = |f_1| \vee \cdots \vee |f_N|$, *on a l'estimation de continuité*

$$\|F(f_1, \dots, f_N)\|_B \le \|f\|_B \sup_{\|t\|_\infty \le 1} |F(t_1, \dots, t_N)|. \tag{49}$$

Rappelons maintenant la définition suivante.

Définition 2.19. Un treillis de Banach B est q-concave, avec $q \in [1, +\infty[$, s'il existe un réel $M_{(q)}(B) > 0$ tel que l'on a l'inégalité suivante pour tout entier $N \in \mathbb{N}^\star$ et tout élément $(f_1, \dots, f_N) \in B^N$:

$$\left(\sum_{i=1}^N \|f_i\|_B^q\right)^{\frac{1}{q}} \le M_{(q)}(B) \left\| \left(\sum_{i=1}^N |f_i|^q\right)^{\frac{1}{q}} \right\|_B, \tag{50}$$

où le terme $\left(\sum_{i=1}^N |f_i|^q\right)^{\frac{1}{q}} \in B$ est défini par calcul fonctionnel.

La notion de q-concavité est reliée à celle de cotype comme le montre le résultat suivant :

Proposition 2.20 [Lindenstrauss et Tzafriri 1973, Proposition 1f3, Corollary 1f9]. *Soient* B *un treillis de Banach et un réel* $q \in [2, +\infty[$, *on a*

(i) *si* B *est* q*-concave*, *alors* B *est de cotype* q,

(ii) *si* B *est de cotype* $q \in [2, +\infty[$ *alors* B *est* $(q + \varepsilon)$*-concave pour tout* $\varepsilon \in]0, +\infty[$.

Par conséquent, *on a*

$$\inf\{q \ge 2 \mid B \textit{ est } q\textit{-concave}\} = \inf\{q \ge 2 \mid B \textit{ est de cotype } q\}. \tag{51}$$

On vérifie que $L^p(\mathbb{R})$ est q-concave si $1 \le p \le q$ (il s'agit d'interpoler l'injection canonique $L^p(\mathbb{R}, \ell^q(\mathbb{N})) \to \ell^q(\mathbb{N}, L^p(\mathbb{R}))$ entre $q = p$ et $q = \infty$) et la borne inférieure (51) vaut $\max(2, p)$. Un autre exemple intéressant est fourni par les espaces de Lorentz $L^{p,1}(\mathbb{R})$ (voir les calculs exacts dans [Creekmore 1981]).

Pour ce qui nous concerne, la proposition précédente est le point crucial qui permet de s'émanciper de la randomisation unidimensionnelle. En effet la définition de la q-concavité ne fait pas intervenir de variables aléatoires de Bernoulli (contrairement à la définition du cotype (19)). Nous disposons maintenant du vocabulaire adéquat pour énoncer une version multidimensionnelle et quantitative du théorème de Maurey–Pisier dans la catégorie des treillis de Banach. De façon précise, le théorème ci-dessous généralise le théorème 2.1 et implique le théorème 1.9.

Théorème 2.21. *Considérons un treillis de Banach* B q*-concave avec* $q \in [2, +\infty[$, *une suite d'entiers non nuls* $(d_n)_{n\in\mathbb{N}}$, *une suite de matrices* $b_n \in \mathcal{M}_{d_n}(B)$, *et une suite de matrices aléatoires* $M_n : \Omega \to \mathcal{M}_{d_n}(\mathbb{R})$ *indépendantes*, *orthogonalement invariantes et vérifiant*

$$\inf_{n\in\mathbb{N}} \sigma(\mathbf{E}[|M_n|]) > 0 \quad \textit{et} \quad \sup_{n\in\mathbb{N}} \mathbf{E}[\|M_n\|_{\mathrm{op}}^q] < +\infty.$$

Alors on a l'équivalence des propriétés suivantes :

(i) *les normes* $\left\| \sqrt{\sum_{n=0}^{N} \|b_n\|_{\mathrm{HS}}^2} \right\|_B$ *sont majorées indépendamment de* $N \in \mathbb{N}$,

(ii) *la série aléatoire* $\sum \sqrt{d_n}\, \mathbf{tr}(M_n b_n)$ *converge presque sûrement dans* B,

(iii) *la série aléatoire* $\sum \sqrt{d_n}\, \mathbf{tr}(M_n b_n)$ *converge dans* $L^q(\Omega, B)$,

(iv) *la série aléatoire* $\sum \sqrt{d_n}\, \mathbf{tr}(M_n b_n)$ *est bornée en probabilité dans* B.

Pour prouver le théorème 2.21, nous aurons besoin du lemme suivant. L'inégalité (53) ci-dessous est seulement une version généralisée de (50).

Lemme 2.22. *Considérons des variables aléatoires* $X_1, \ldots, X_N$ *appartenant à* $L^1(\Omega)$ *et des éléments* $f_1, \ldots, f_N$ *d'un treillis de Banach* B. *Alors on a "l'inégalité triangulaire"* :

$$\left\| \mathbf{E}_\omega \left[\left| \sum_{n=1}^{N} X_n(\omega) f_n \right| \right] \right\|_B \leq \mathbf{E}_\omega \left[\left\| \sum_{n=1}^{N} X_n(\omega) f_n \right\|_B \right]. \tag{52}$$

S'il existe un réel $q \in [1, +\infty[$ *tel que* B *soit* q*-concave et que les variables aléatoires* $X_1, \ldots, X_N$ *appartiennent à* $L^q(\Omega)$, *alors on a aussi*

$$\mathbf{E}_\omega \left[\left\| \sum_{n=1}^{N} X_n(\omega) f_n \right\|_B^q \right]^{\frac{1}{q}} \leq M_{(q)}(B) \left\| \mathbf{E}_\omega \left[\left| \sum_{n=1}^{N} X_n(\omega) f_n \right|^q \right]^{\frac{1}{q}} \right\|_B, \tag{53}$$

où les espérances dans le membre gauche de (52) *et dans le membre droit de* (53) *sont définies par calcul fonctionnel sur les* N *variables* $f_1, \ldots, f_N$.

Démonstration. On commence par (53). Pour tout entier $n \in [1, N]$, on note $(X_{n,k})_{k \in \mathbb{N}}$ une suite de variables aléatoires qui prend un nombre fini de valeurs et qui converge dans $L^q(\Omega)$ vers X_n. Pour tout $k \in \mathbb{N}$, il existe donc une partition finie

$$\Omega = \bigsqcup_{\ell=1}^{L} \Omega_{k,\ell}$$

en parties mesurables telle que chaque variable aléatoire $X_{n,k}$ prend une valeur fixe, disons $x_{n,k,\ell}$, sur $\Omega_{k,\ell}$. La q-concavité de B nous donne alors

$$\begin{aligned}\mathbf{E}_\omega \left[\left\| \sum_{n=1}^{N} X_{n,k}(\omega) f_n \right\|_B^q \right]^{\frac{1}{q}} &= \left(\sum_{\ell=1}^{L} \boldsymbol{P}(\Omega_{k,\ell}) \left\| \sum_{n=1}^{N} x_{n,k,\ell} f_n \right\|_B^q \right)^{\frac{1}{q}} \\ &\leq M_{(q)}(B) \left\| \left(\sum_{\ell=1}^{L} \boldsymbol{P}(\Omega_{k,\ell}) \left| \sum_{n=1}^{N} x_{n,k,\ell} f_n \right|^q \right)^{\frac{1}{q}} \right\|_B .\end{aligned}$$

Or on a évidemment pour tout $(t_1, \ldots, t_N) \in \mathbb{R}^N$

$$\left(\sum_{\ell=1}^{L} \boldsymbol{P}(\Omega_{k,\ell}) \left| \sum_{n=1}^{N} x_{n,k,\ell} t_n \right|^q \right)^{\frac{1}{q}} = \mathbf{E} \left[\left| \sum_{n=1}^{N} X_{n,k} t_n \right|^q \right]^{\frac{1}{q}}. \tag{54}$$

Si k tend vers $+\infty$, l'expression précédente converge uniformément, en tant que fonction de $(t_1, \ldots, t_N)$, sur chaque compact de $\mathbb{R}^N$ vers

$$\mathbf{E}\Big[\Big|\sum_{n=1}^{N} X_n t_n\Big|^q\Big]^{\frac{1}{q}}.$$

Le calcul fonctionnel du théorème 2.18 assure que l'égalité (54) est encore valide en substituant $(f_1, \ldots, f_N)$ à $(t_1, \ldots, t_N)$, ce qui nous donne

$$\mathbf{E}\Big[\Big\|\sum_{n=1}^{N} X_{n,k} f_n\Big\|_B^q\Big]^{\frac{1}{q}} \leq M_{(q)}(B)\Big\|\mathbf{E}\Big[\Big|\sum_{n=1}^{N} X_{n,k} f_n\Big|^q\Big]^{\frac{1}{q}}\Big\|_B. \tag{55}$$

L'estimation (49) de continuité du calcul fonctionnel assure aussi que le membre droit de (55) tend vers

$$M_{(q)}(B)\Big\|\mathbf{E}\Big[\Big|\sum_{n=1}^{N} X_n f_n\Big|^q\Big]^{\frac{1}{q}}\Big\|_B.$$

Enfin, il est clair que le membre gauche de (55) tend vers le membre gauche de (53). Pour démontrer (52), on refait la même démarche de densité dans $L^1(\Omega)$ à l'aide de l'inégalité triangulaire dans B :

$$\begin{aligned}\mathbf{E}_\omega\Big[\Big\|\sum_{n=1}^{N} X_{n,k}(\omega) f_n\Big\|_B\Big] &= \sum_{\ell=1}^{L} \boldsymbol{P}(\Omega_{k,\ell})\Big\|\sum_{n=1}^{N} x_{n,k,\ell} f_n\Big\|_B \\ &\geq \Big\|\sum_{\ell=1}^{L} \boldsymbol{P}(\Omega_{k,\ell})\Big|\sum_{n=1}^{N} x_{n,k,\ell} f_n\Big|\Big\|_B. \qquad \square\end{aligned}$$

Corollaire 2.23. *Considérons des matrices aléatoires* $M_0 : \Omega \to \mathcal{M}_{d_0}(\mathbb{R}), \ldots, M_N : \Omega \to \mathcal{M}_{d_N}(\mathbb{R})$ *dont les coefficients appartiennent à* $L^1(\Omega)$ *et des matrices* $b_0 \in \mathcal{M}_{d_0}(B), \ldots, b_N \in \mathcal{M}_{d_N}(B)$ *dont les coefficients appartiennent à un treillis de Banach* B*. Alors on a l'inégalité* :

$$\Big\|\mathbf{E}\Big[\Big|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(M_n b_n)\Big|\Big]\Big\|_B \leq \mathbf{E}\Big[\Big\|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(M_n b_n)\Big\|_B\Big].$$

S'il existe de plus un réel $q \in [1, +\infty[$ *tel que* B *soit* q*-concave et tel que les coefficients des matrices aléatoires* $M_0, \ldots, M_N$ *appartiennent à* $L^q(\Omega)$, *alors on a*

$$\mathbf{E}\Big[\Big\|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(M_n b_n)\Big\|_B^q\Big]^{\frac{1}{q}} \leq M_{(q)}(B)\Big\|\mathbf{E}\Big[\Big|\sum_{n=0}^{N} \sqrt{d_n}\,\mathbf{tr}(M_n b_n)\Big|^q\Big]^{\frac{1}{q}}\Big\|_B.$$

Il nous reste à remarquer que les inégalités de Kahane–Khintchine–Marcus–Pisier permettent d'étendre le théorème de Maurey [Lindenstrauss et Tzafriri 1973, Theorem 1.d.6 i)] au cas multidimensionnel.

Théorème 2.24. *Soit B un treillis de Banach q-concave, avec $q \in [2, +\infty[$ et considérons une suite de matrices $b_n \in \mathcal{M}_{d_n}(B)$. Alors, pour tout entier $N \in \mathbb{N}$, on a*

$$\frac{1}{K_{2,1}} \left\| \sqrt{\sum_{n=0}^{N} \|b_n\|_{\mathrm{HS}}^2} \right\|_B \leq \mathbf{E}\left[\left\| \sum_{n=0}^{N} \sqrt{d_n}\, \mathbf{tr}(\mathcal{E}_n b_n) \right\|_B \right] \leq M_{(q)}(B) K_{q,2} \left\| \sqrt{\sum_{n=0}^{N} \|b_n\|_{\mathrm{HS}}^2} \right\|_B, \tag{56}$$

où l'élément $\sqrt{\sum_{n=0}^{N} \|b_n\|_{\mathrm{HS}}^2} \in B$ est défini par calcul fonctionnel.

Démonstration. La minoration est vraie sans hypothèse de q-concavité. La version scalaire des inégalités de Kahane–Khintchine–Marcus–Pisier (voir (40) et (41)) rend triviale l'inégalité suivante si les matrices b_n sont à coefficients réels

$$\sqrt{\sum_{n=0}^{N} \|b_n\|_{\mathrm{HS}}^2} \leq K_{2,1} \mathbf{E}\left[\left| \sum_{n=0}^{N} \sqrt{d_n}\, \mathbf{tr}(\mathcal{E}_n b_n) \right| \right].$$

L'inégalité précédente s'étend par calcul fonctionnel si les coefficients des matrices b_n sont à coefficients dans B. La minoration de (56) découle alors du corollaire 2.23. On raisonne de même pour la majoration. □

Le corollaire 2.23 et le théorème 2.24 nous permettent de prouver le théorème 2.21. Il s'agit de reprendre mutatis mutandis les arguments de la partie 2C à partir du lemme 2.12.

2E. *Preuves des théorèmes 2.5 et 2.6, partie I : Rétracte d'un espace de Banach.* On conviendra que les éléments de $E_n \subset L^2(X)$ sont des fonctions de la variable $y \in X$ et l'on préférera les deux écritures

$$L^2(X) \to L^2_y(X) \quad \text{et} \quad \mathbf{PL}^p\big(X, \bigoplus E_n\big) \to \mathbf{PL}^p_y\big(X, \bigoplus E_n\big).$$

Il sera aussi commode de définir l'espace de Hilbert abstrait $\oplus E_n$, c'est-à-dire que l'on pose

$$\forall (u_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} E_n, \quad \|(u_n)_{n\in\mathbb{N}}\|_{\oplus E_n} := \left(\int_X \sum_{n\in\mathbb{N}} |u_n(y)|^2 \, d\mu(y) \right)^{\frac{1}{2}}.$$

Concernant l'espace de Bochner–Lebesgue, on conservera la lettre $x \in X$ pour écrire $L^p_x\big(X, \bigoplus E_n\big)$ au lieu de $L^p\big(X, \bigoplus E_n\big)$. La définition 2.3 dit exactement que l'opérateur linéaire suivant est isométrique

$$S_p : \mathbf{PL}^p_y\big(X, \bigoplus E_n\big) \to L^p_x\big(X, \bigoplus E_n\big), \quad (u_n(y))_{n\in\mathbb{N}} \mapsto \left(\frac{\sqrt{e_n(x)}}{\sqrt{d_n}} u_n(y) \right)_{n\in\mathbb{N}}. \tag{57}$$

Le résultat suivant est facile.

Proposition 2.25. *L'espace $\mathbf{PL}^p_y\big(X, \bigoplus E_n\big)$ est complet.*

Démonstration. Il s'agit de prouver que l'image de l'opérateur (57) est un sous-espace fermé de $L^p_x\big(X, \bigoplus E_n\big)$. Remarquons d'abord, pour tout $k \in \mathbb{N}$, la continuité du projecteur

$$L^p_x\big(X, \bigoplus E_n\big) \to L^p_x(X, L^2_y(X)), \quad (w_n(x,y))_{n\in\mathbb{N}} \mapsto w_k(x,y).$$

Il suffit donc de prouver que, pour tout $n \in \mathbb{N}$, $\{\sqrt{e_n(x)}u_n(y) \mid u_n \in E_n\}$ est un sous-espace vectoriel fermé de $L^p_x(X, L^2_y(X))$. L'hypothèse $E_n \subset L^p(X)$ de la définition 2.3 signifie précisément que $\sqrt{e_n(x)}$ appartient à $L^p_x(X)$. Soit $\big(\sqrt{e_n(x)}u_{n,\ell}(y)\big)_{\ell\in\mathbb{N}}$ une suite convergente dans $L^p_x(X, L^2_y(X))$ (avec $u_{n,\ell} \in E_n$ pour tout $\ell \in \mathbb{N}$). En particulier, $(u_{n,\ell}(y))_{\ell\in\mathbb{N}}$ est de Cauchy dans $E_n \subset L^2_y(X)$:

$$\|u_{n,\ell}(y) - u_{n,\ell'}(y)\|_{L^2_y(X)} = \frac{1}{\big\|\sqrt{e_n(x)}\big\|_{L^p_x}} \big\|\sqrt{e_n(x)}(u_{n,\ell}(y) - u_{n,\ell'}(y))\big\|_{L^p_x(X, L^2_y(X))}.$$

On déduit que $(u_{n,\ell})_{\ell\in\mathbb{N}}$ converge dans $L^2(X)$ vers une fonction $u_{n,\infty} \in L^2(X)$ et bien entendu que l'on a $u_{n,\infty} \in E_n$ (car E_n est de dimension finie). Il est alors immédiat que $\big(\sqrt{e_n(x)}u_{n,\ell}(y)\big)_{\ell\in\mathbb{N}}$ converge vers $\sqrt{e_n(x)}u_{n,\infty}(y)$ dans $L^p_x(X, L^2_y(X))$. □

Commençons par les points faciles ayant trait à la dualité et l'interpolation des espaces $\mathbf{PL}^p\big(X, \bigoplus E_n\big)$.

Dualité. Rappelons que l'on note $q = \frac{p}{p-1}$ l'exposant conjugué de p. On montre facilement l'inégalité (38) pour tout $(u, w) \in \mathbf{PL}^p_y\big(X, \bigoplus E_n\big) \times \mathbf{PL}^q_y\big(X, \bigoplus E_n\big)$:

$$\begin{aligned}\sum_{n\geq 0} |\langle u_n, w_n\rangle_{L^2(X)}| &= \int_X \sum_{n\in\mathbb{N}} \frac{e_n(x)}{d_n} |\langle u_n, w_n\rangle_{L^2(X)}|\, d\mu(x)\\ &\leq \int_X \sqrt{\sum_{n\geq 0} \frac{e_n(x)}{d_n}\|u_n(y)\|^2_{L^2_y(X)}}\sqrt{\sum_{n\geq 0} \frac{e_n(x)}{d_n}\|w_n(y)\|^2_{L^2_y(X)}}\, d\mu(x)\\ &\leq \|u\|_{\mathbf{PL}^p(X,\bigoplus E_n)}\|w\|_{\mathbf{PL}^q(X,\bigoplus E_n)}.\end{aligned}$$

L'injection canonique $\Lambda_p : \mathbf{PL}^q_y\big(X, \bigoplus E_n\big) \to \mathbf{PL}^p_y\big(X, \bigoplus E_n\big)'$, définie dans l'énoncé du théorème 2.6, est donc continue.

Interpolation. Rappelons que les espaces $\mathbf{PL}^p_y\big(X, \bigoplus E_n\big)$ et leurs interpolés complexes et réels peuvent être vus comme des sous-espaces de $\prod_{n\in\mathbb{N}} E_n$. En outre, on a le résultat suivant [Triebel 1978, Part 1.18.4].

Théorème 2.26. *Considérons deux espaces de Banach complexes B_1 et B_2 ainsi que des réels $p_1 < p < p_2$ appartenant à $[1, +\infty[$. En notant $\theta \in]0, 1[$ le réel qui vérifie $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, on a*

$$\big[L^{p_1}(X, B_1), L^{p_2}(X, B_2)\big]_\theta = L^p(X, [B_1, B_2]_\theta).$$

Si un opérateur linéaire T borné de $L^{p_1}(X, B_1)$ dans lui-même et de $L^{p_2}(X, B_2)$ dans lui-même alors il est aussi borné de $L^p(X, [B_1, B_2]_\theta)$ dans lui-même. Le même énoncé est valide en remplaçant la méthode d'interpolation complexe $[\cdot, \cdot]_\theta$ par la méthode d'interpolation réelle $[\cdot, \cdot]_{\theta,p}$.

Avec les notations du théorème précédent, on peut interpoler l'application (57) et assurer que l'opérateur

$$\big[\mathbf{PL}^{p_1}_y\big(X, \bigoplus E_n\big), \mathbf{PL}^{p_2}_y\big(X, \bigoplus E_n\big)\big]_\theta \to L^p_x\big(X, \bigoplus E_n\big), \quad (u_n(y))_{n\in\mathbb{N}} \mapsto \left(\frac{\sqrt{e_n(x)}}{\sqrt{d_n}} u_n(y)\right)_{n\in\mathbb{N}}$$

est borné. Cela implique l'inclusion continue

$$\big[\mathbf{PL}^{p_1}_y\big(X, \bigoplus E_n\big), \mathbf{PL}^{p_2}_y\big(X, \bigoplus E_n\big)\big]_\theta \subset \mathbf{PL}^p_y\big(X, \bigoplus E_n\big). \tag{58}$$

Vers la notion de rétracte d'un espace de Banach. Cependant, il ne paraît pas évident d'aller plus loin dans les deux analyses précédentes, c'est-à-dire de prouver que $\Lambda_p : \mathbf{PL}^q_y(X, \bigoplus E_n) \to \mathbf{PL}^p_y(X, \bigoplus E_n)'$ est surjective et que l'inclusion (58) est une égalité. La notion de rétracte d'un espace de Banach permet de reformuler la question.

Définition 2.27. Soient A et B deux $\mathbb{C}$-espaces vectoriels normés, on dit que A est un rétracte de B s'il existe deux applications linéaires bornées $S : A \to B$ et $R : B \to A$ telles que $RS = \mathrm{id}_A$.

Dans la définition précédente, il faut imaginer que B est un espace de référence qui est bien compris et A un sous-espace que l'on veut analyser. Le prototype de rétracte qu'il faut avoir à l'esprit est le cas où A est un sous-espace complémenté de B, c'est-à-dire image d'un projecteur borné, l'application S est alors l'application identité et R un projecteur de B sur A. En effet, on vérifie facilement le résultat suivant.

Proposition 2.28. *Avec les mêmes notations que dans la définition 2.27, on a :*

(i) *$SR : B \to B$ est un projecteur borné, son espace image est égal à $S(A)$ et est fermé.*

(ii) *S est un isomorphisme d'espaces vectoriels normés de A sur $S(A) \subset B$:*

$$\forall a \in A, \quad \frac{1}{\|R\|_{B\to A}} \|a\|_A \leq \|S(a)\|_B \leq \|S\|_{A\to B} \|a\|_A.$$

(iii) *Si B est complet alors A l'est aussi.*

En ce qui concerne la dualité, on a le corollaire facile.

Corollaire 2.29. *Avec les mêmes notations que dans la définition 2.27, si l'on note les applications duales (ou applications transposées) de S et R :*

$$\begin{aligned} {}^tS &: B' \to A', \quad \phi \mapsto \phi \circ S, \\ {}^tR &: A' \to B', \quad \psi \mapsto \psi \circ R, \end{aligned}$$

alors on a ${}^tS{}^tR = \mathrm{id}_{A'}$. Cela signifie que l'espace dual A' est un rétracte de B' et en particulier que ${}^tR$ est un isomorphisme de A' sur l'image du projecteur ${}^tR{}^tS = {}^t(SR) : B' \to B'$.

En ce qui concerne l'interpolation réelle ou complexe, la réponse est donnée par le résultat suivant.

Corollaire 2.30 [Triebel 1978, page 22, Theorem 1.2.4].. *On note $[\cdot,\cdot]$ une méthode d'interpolation. Soient (A_1, A_2) et (B_1, B_2) deux couples d'interpolation d'espaces de Banach, on suppose qu'il existe un opérateur linéaire borné $S : A_1 \to B_1$ et $S : A_2 \to B_2$ et un opérateur linéaire borné $R : B_1 \to A_1$ et $R : B_2 \to A_2$ qui satisfont :*

$$\forall a \in A_1 + A_2, \quad RS(a) = a.$$

Alors SR est un projecteur borné de l'espace interpolé $[B_1, B_2]$ et S réalise un isomorphisme d'espaces de Banach de l'espace interpolé $[A_1, A_2]$ sur $SR([B_1, B_2])$.

Démonstration. C'est immédiat puisque l'espace de Banach $[A_1, A_2]$ est un rétracte de $[B_1, B_2]$ par l'intermédiaire des opérateurs bornés $S : [A_1, A_2] \to [B_1, B_2]$ et $R : [B_1, B_2] \to [A_1, A_2]$. □

Pour comprendre les espaces duaux et interpolés des espaces $\mathbf{PL}_y^p\big(X,\bigoplus E_n\big)$, il suffit d'étudier si $\mathbf{PL}_y^p\big(X,\bigoplus E_n\big)$ est un rétracte de l'espace de Banach $L_x^p\big(X,\bigoplus E_n\big)$ par le biais de l'application S_p définie en (57) si p parcourt $]p_1,p_2[$ pour les théorèmes 2.5 et 2.6. En d'autres termes, on cherche un opérateur borné

$$R_p : L_x^p\big(X,\bigoplus E_n\big)\to \mathbf{PL}_y^p\big(X,\bigoplus E_n\big)$$

tel que $R_pS_p=\mathrm{id}_{\mathbf{PL}_y^p(X,\bigoplus E_n)}$.

Stratégie pour le théorème 2.6 de dualité. Puisque les fonctions $\frac{1}{d_n}e_n$ sont des densités de probabilité sur X, nous avons un candidat très naturel pour l'opérateur R_p. En effet, posons :

$$R_p : L_x^p\big(X,\bigoplus E_n\big)\to \mathbf{PL}_y^p\big(X,\bigoplus E_n\big),\quad (u_n(x,y))_{n\geq 0}\mapsto\left(\int_X u_n(x',y)\frac{\sqrt{e_n(x')}}{\sqrt{d_n}}\,d\mu(x')\right)_{n\geq 0}. \tag{59}$$

D'après la définition (57), on a bien formellement $R_pS_p=\mathrm{id}_{\mathbf{PL}_y^p(X,\bigoplus E_n)}$. Malheureusement, nous ne voyons aucune raison triviale assurant que R_p arrive bien dans $\mathbf{PL}_y^p\big(X,\bigoplus E_n\big)$! Puisque S_p est une isométrie, la bornitude de R_p équivaut à la bornitude du projecteur

$$\begin{aligned} S_pR_p : L_x^p\big(X,\bigoplus E_n\big)&\to L_x^p\big(X,\bigoplus E_n\big),\\ (u_n(x,y))_{n\in\mathbb{N}}&\mapsto\left(\frac{\sqrt{e_n(x)}}{\sqrt{d_n}}\int_X u_n(x',y)\frac{\sqrt{e_n(x')}}{\sqrt{d_n}}\,d\mu(x')\right)_{n\in\mathbb{N}}.\end{aligned} \tag{60}$$

Par souci pédagogique, permettons-nous de considérer comme exemple le cas unidimensionnel :

$$d_n=1,\quad e_n(x)=|\phi_n(x)|^2,\quad E_n=\mathbb{C}\phi_n,\quad \int_X|\phi_n(x)|^2d\mu(x)=1.$$

Dans ce cas, un élément de $L_x^p\big(X,\bigoplus E_n\big)$ est de la forme $(u_n(x)\phi_n(y))_{n\in\mathbb{N}}$ et il est donc évident que $L_x^p\big(X,\bigoplus E_n\big)$ s'identifie à $L_x^p(X,\ell^2(\mathbb{N}))$ par le biais de l'isométrie

$$L_x^p\big(X,\bigoplus E_n\big)\to L_x^p(X,\ell^2(\mathbb{N})),\quad (u_n(x)\phi_n(y))_{n\in\mathbb{N}}\mapsto(u_n(x))_{n\in\mathbb{N}}.$$

Par conséquent, la bornitude du projecteur S_pR_p équivaut à celle du projecteur

$$L_x^p(X,\ell^2(\mathbb{N}))\to L_x^p(X,\ell^2(\mathbb{N})),\quad (u_n(x))_{n\in\mathbb{N}}\mapsto\left(|\phi_n(x)|\int_X u_n(x')|\phi_n(x')|d\mu(x')\right)_{n\in\mathbb{N}}.$$

Revenons au cas général et rappelons que sous les hypothèses du théorème 2.6 de dualité, p_1 et p_2 sont supposés être deux exposants conjugués. La bornitude du projecteur S_pR_p sera prouvée dans la partie 2H pour tout $p\in\,]p_1,p_2[$ grâce à (35) et (37). La proposition suivante achèvera la preuve du théorème 2.6 de dualité et explique pourquoi la notion de rétracte est la bonne notion pour aborder la dualité des espaces $\mathbf{PL}^p\big(X,\bigoplus E_n\big)$.

Proposition 2.31. *Fixons $p\in\,]1,+\infty[$ et posons $q=\frac{p}{p-1}$ l'exposant conjugué. Pour tout $n\in\mathbb{N}$, on suppose que $\sqrt{e_n}$ appartient à $L^p(X)\cap L^q(X)$. Alors les propriétés suivantes sont équivalentes :*

(i) *l'injection canonique $\Lambda_p:\mathbf{PL}_y^q\big(X,\bigoplus E_n\big)\to\mathbf{PL}_y^p\big(X,\bigoplus E_n\big)'$ est surjective,*

(ii) *l'injection canonique $\Lambda_q:\mathbf{PL}_y^p\big(X,\bigoplus E_n\big)\to\mathbf{PL}_y^q\big(X,\bigoplus E_n\big)'$ est surjective,*

(iii) *l'injection canonique* $\Lambda_p : \mathbf{PL}^q_y\big(X, \bigoplus E_n\big) \to \mathbf{PL}^p_y\big(X, \bigoplus E_n\big)'$ *est un isomorphisme d'espaces de Banach,*

(iv) *l'injection canonique* $\Lambda_q : \mathbf{PL}^p_y\big(X, \bigoplus E_n\big) \to \mathbf{PL}^q_y\big(X, \bigoplus E_n\big)'$ *est un isomorphisme d'espaces de Banach,*

(v) *le projecteur* $S_p R_p$ *est borné sur* $L^p_x\big(X, \bigoplus E_n\big)$,

(vi) *le projecteur* $S_q R_q$ *est borné sur* $L^q_x\big(X, \bigoplus E_n\big)$.

Les assertions précédentes impliquent les assertions suivantes

$$\sup_{n\in\mathbb{N}} \left\| \frac{\sqrt{e_n}}{\sqrt{d_n}} \right\|_{L^p(X)} \left\| \frac{\sqrt{e_n}}{\sqrt{d_n}} \right\|_{L(^q(X)} < +\infty. \tag{61}$$

$$\|S_p R_p\| = \|S_q R_q\| = \|R_p\| = \|R_q\| = \|\Lambda_p^{-1}\| = \|\Lambda_q^{-1}\| \in [1, +\infty[. \tag{62}$$

Démonstration. Preuve de (61) : Pour tout $n \in \mathbb{N}$, on fixe $\phi \in E_n$ vérifiant $\|\phi\|_{L^2_y(X)} = 1$ et l'on note

$$u(x, y) = \Big(0, \dots, 0, \underbrace{\sqrt{e_n(x)}^{q-1} \phi(y)}_{n}, 0, \dots\Big) \in L^p_x\big(X, \textstyle\bigoplus E_n\big).$$

On peut alors écrire

$$\begin{aligned} \|S_p R_p u\|_{L^p(X,\bigoplus E_n)} &\le \|S_p R_p\| \times \|u\|_{L^p(X,\bigoplus E_n)} \\ \left\| \frac{\sqrt{e_n(x)}}{\sqrt{d_n}} \right\|_{L^p_x(X)} \times \left| \int_X \frac{\sqrt{e_n(x')}^q}{\sqrt{d_n}} \, d\mu(x') \right| &\le \|S_p R_p\| \times \left\| \sqrt{e_n(x)}^{q-1} \right\|_{L^p_x(X)} \\ \left\| \frac{\sqrt{e_n(x)}}{\sqrt{d_n}} \right\|_{L^p_x(X)} \times \frac{1}{\sqrt{d_n}} \left\| \sqrt{e_n(x)} \right\|^q_{L^q_x(X)} &\le \|S_p R_p\| \times \left\| \sqrt{e_n(x)} \right\|^{q-1}_{L^q_x(X)} \\ \left\| \frac{\sqrt{e_n(x)}}{\sqrt{d_n}} \right\|_{L^p_x(X)} \left\| \frac{\sqrt{e_n(x)}}{\sqrt{d_n}} \right\|_{L^q_x(X)} &\le \|S_p R_p\|. \end{aligned}$$

Équivalence des propriétés (i), (ii), (iii), (iv), (v) et (vi) : i) ⇔ (iii) et (ii) ⇔ (iv). Cela découle du théorème de l'application ouverte pour des bijections linéaires continues entre espaces de Banach.

v) ⇔ (vi). Puisque $\bigoplus E_n$ est un espace de Hilbert, les espaces de Bochner–Lebesgue $L^p_x\big(X, \bigoplus E_n\big)$ et $L^q_x\big(X, \bigoplus E_n\big)$ sont duaux l'un de l'autre et la dualité naturelle est donnée par

$$\forall (u, w) \in L^p_x\big(X, \textstyle\bigoplus E_n\big) \times L^q_x\big(X, \textstyle\bigoplus E_n\big), \quad \langle u, w\rangle := \displaystyle\int_X \left[\sum_{n\in\mathbb{N}} \int_X u_n(x, y) w_n(x, y) \, d\mu(y) \right] d\mu(x).$$

Or l'on a

$$\begin{aligned} \int_X \left[\sum_{n\in\mathbb{N}} \int_X |u_n(x, y)| \, |w_n(x, y)| \, d\mu(y) \right] d\mu(x) &\le \int_X \left[\sum_{n\in\mathbb{N}} \|u_n(x, y)\|_{L^2_y(X)} \|w_n(x, y)\|_{L^2_y(X)} \right] d\mu(x) \\ &\le \|u\|_{L^p_x(X,\bigoplus E_n)} \|w\|_{L^q_x(X,\bigoplus E_n)} \\ &< +\infty. \end{aligned}$$

Le théorème de Fubini assure que la dualité naturelle s'écrit aussi

$$\langle u, w\rangle = \sum_{n\in\mathbb{N}} \int_{X\times X} u_n(x,y) w_n(x,y)\, d\mu(x)\, d\mu(y).$$

La définition (60) nous permet alors d'écrire

$$\langle S_p R_p u, w\rangle = \sum_{n\in\mathbb{N}} \frac{1}{d_n} \int_{X\times X} \left[\sqrt{e_n(x)} \int_X u_n(x',y) \sqrt{e_n(x')}\, d\mu(x') \right] w_n(x,y)\, d\mu(x)\, d\mu(y).$$

Remarquons maintenant, pour tout $n \in \mathbb{N}$, la majoration triviale de

$$\iiint_{X^3} \sqrt{e_n(x)} |w_n(x,y)| \sqrt{e_n(x')} |u_n(x',y)|\, d\mu(x)\, d\mu(x')\, d\mu(y)$$

par

$$\begin{aligned}
&\left(\int_X \sqrt{e_n(x)} \|w_n(x,y)\|_{L^2_y(X)} d\mu(x) \right) \times \left(\int_X \sqrt{e_n(x')} \|u_n(x',y)\|_{L^2_y(X)}\, d\mu(x') \right) \\
&\qquad \le \left\| \sqrt{e_n(x)} \right\|_{L^p_x(X)} \|w_n(x,y)\|_{L^q_x(X, L^2_y(X))} \times \left\| \sqrt{e_n(x')} \right\|_{L^q_{x'}(X)} \|u_n(x',y)\|_{L^p_{x'}(X, L^2_y(X))} \\
&\qquad \le \left\| \sqrt{e_n(x)} \right\|_{L^p_x(X)} \|w\|_{L^q_x(X, \bigoplus E_n)} \left\| \sqrt{e_n(x')} \right\|_{L^q_{x'}(X)} \|u\|_{L^p_x(X, \bigoplus E_n)} \\
&\qquad < +\infty.
\end{aligned}$$

Le théorème de Fubini nous permet donc d'intervertir x et x' pour obtenir

$$\begin{aligned}
\langle S_p R_p u, w\rangle &= \sum_{n\in\mathbb{N}} \frac{1}{d_n} \int_{X\times X} \left[\sqrt{e_n(x)} \int_X w_n(x',y) \sqrt{e_n(x')} d\mu(x') \right] u_n(x,y)\, d\mu(x)\, d\mu(y) \\
&= \langle u, S_q R_q w\rangle.
\end{aligned}$$

Ainsi, $S_p R_p$ et $S_q R_q$ sont adjoints l'un de l'autre. La continuité de l'un implique la continuité de l'autre.

Pour la fin de la démonstration, on aura besoin de l'expression de ${}^t S_p$ (qui découle de (57)) :

$$\begin{aligned}
&{}^t S_p : L^q_x\big(X, \textstyle\bigoplus E_n\big) \to \mathbf{PL}^p_y\big(X, \textstyle\bigoplus E_n\big)', \\
&\qquad (w_n(x,y))_{n\in\mathbb{N}} \mapsto \left((u_n(y))_{n\ge 0} \mapsto \int_{X\times X} \sum_{n\in\mathbb{N}} w_n(x,y) u_n(y) \frac{\sqrt{e_n(x)}}{\sqrt{d_n}} d\mu(x) d\mu(y) \right).
\end{aligned}$$

iii) $\Rightarrow$ (vi). À l'aide de (59), on vérifie la formule $R_q = \Lambda_p^{-1} \circ {}^t S_p$. A fortiori, $S_q R_q$ est un opérateur borné.

iv) $\Rightarrow$ (v). On permute p et q dans l'argument précédent.

vi) $\Rightarrow$ (i). L'équivalence (v) $\Leftrightarrow$ (vi) assure que $S_p R_p$ est borné. Puisque S_p et S_q sont des isométries, R_p et R_q sont des opérateurs bornés et l'on a par construction

$$R_p S_p = \mathrm{id}_{\mathbf{PL}^p_y(X, \bigoplus E_n)} \quad \text{et} \quad R_q S_q = \mathrm{id}_{\mathbf{PL}^q_y(X, \bigoplus E_n)}.$$

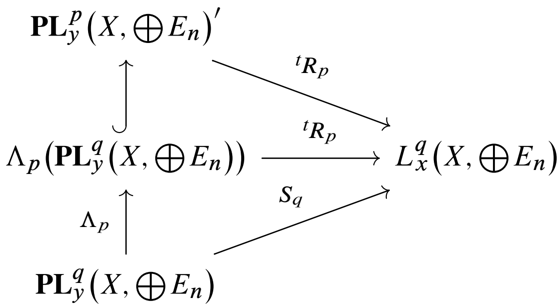
L'idée est d'exprimer Λ_p avec la formule suivante qui découle facilement de (57) :

$$\Lambda_p = {}^t S_p S_q.$$

Utilisant que $S_p R_p$ et $S_q R_q$ sont deux opérateurs duaux, on obtient

$${}^t R_p \Lambda_p = {}^t R_p {}^t S_p S_q = {}^t(S_p R_p) S_q = S_q R_q S_q = S_q. \tag{63}$$

Le diagramme suivant est donc commutatif :



On doit examiner le diagramme précédent en se rappelant que l'on a

$${}^t S_p {}^t R_p = \mathrm{id}_{\mathbf{PL}_y^p(X, \bigoplus E_n)'} \quad \text{et} \quad R_q S_q = \mathrm{id}_{\mathbf{PL}_y^q(X, \bigoplus E_n)}.$$

La proposition 2.28 et le corollaire 2.29 assurent que S_q réalise un isomorphisme d'espaces de Banach de $\mathbf{PL}_y^q(X, \bigoplus E_n)$ sur l'image du projecteur $S_q R_q$ de $L_x^q(X, \bigoplus E_n)$ et que ${}^t R_p$ réalise un isomorphisme d'espaces de Banach de $\mathbf{PL}_y^p(X, \bigoplus E_n)'$ sur l'image du même projecteur ${}^t R_p {}^t S_p = S_q R_q$ de $L_x^q(X, \bigoplus E_n)$. Grâce à (63), on vérifie que $\Lambda_p(\mathbf{PL}_y^q(X, \bigoplus E_n))$ et $\mathbf{PL}_y^p(X, \bigoplus E_n)'$ ont la même image par l'opérateur ${}^t R_p$:

$$\begin{aligned} {}^t R_p(\Lambda_p(\mathbf{PL}_y^q(X, \textstyle\bigoplus E_n))) &= S_q(\mathbf{PL}_y^q(X, \textstyle\bigoplus E_n)) = S_q R_q(L_x^q(X, \textstyle\bigoplus E_n)), \\ {}^t R_p(\mathbf{PL}_y^p(X, \textstyle\bigoplus E_n)') &= S_q R_q(L_x^q(X, \textstyle\bigoplus E_n)). \end{aligned}$$

Par application de ${}^t S_p$, on obtient (i) :

$$\Lambda_p(\mathbf{PL}_y^q(X, \textstyle\bigoplus E_n)) = \mathbf{PL}_y^p(X, \textstyle\bigoplus E_n)'.$$

v) $\Rightarrow$ (ii). On permute p et q dans la preuve précédente.

Preuve de (62) : Comme $S_p R_p$ est un projecteur, sa norme est supérieure ou égale à 1. On a déjà vu au cours de la démonstration précédente l'égalité $\|S_p R_p\| = \|S_q R_q\|$. Comme S_p et S_q sont des isométries, on déduit à la fois la formule $\|S_p R_p\| = \|R_p\|$ et $\|S_q R_q\| = \|R_q\|$ et la formule $\|R_p\| = \|{}^t R_p\| = \|\Lambda_p^{-1}\|$ (grâce à (63)). □

Stratégie pour le théorème 2.5. Il est naturel d'espérer utiliser une stratégie similaire pour prouver le théorème 2.5 d'interpolation en utilisant cette fois-ci le corollaire 2.30. On suppose donc seulement que l'on a $\frac{1}{p_1}+\frac{1}{p_2}\le 1$. La proposition 2.31 nous informe que la bornitude de R_p pour tout $p\in\,]p_1,p_2[$ implique l'assertion (61) :

$$\forall p\in\,]p_1,p_2[,\quad \sup_{n\in\mathbb{N}}\left\|\frac{\sqrt{e_n}}{\sqrt{d_n}}\right\|_{L^p(X)}\left\|\frac{\sqrt{e_n}}{\sqrt{d_n}}\right\|_{L^{\frac{p}{p-1}}(X)}<+\infty.$$

En raison de la symétrie entre p et $\frac{p}{p-1}$, les inégalités précédentes sont aussi valides si p parcourt le plus petit intervalle contenant $]p_1,p_2[$ et stable par la fonction $p\mapsto\frac{p}{p-1}$. Toutes ces inégalités sont trop violentes puisque l'on doit seulement se contenter de l'hypothèse (36). L'application R_p paraît donc inutilisable. Pour pallier ce problème, nous allons remplacer R_p par une application de la forme

$$R_{p,\psi}:L^p_x\big(X,\bigoplus E_n\big)\to\mathbf{PL}^p_y\big(X,\bigoplus E_n\big),\quad (u_n(x,y))_{n\in\mathbb{N}}\mapsto\left(\int_X u_n(x',y)\psi_n(x')\,d\mu(x')\right)_{n\ge 0},$$

où $(\psi_n)_{n\in\mathbb{N}}$ est une suite de fonctions de $L^{\frac{p_2}{p_2-1}}(X)\cap L^{\frac{p_1}{p_1-1}}(X)$ vérifiant

$$\int_X\frac{\sqrt{e_n(x)}}{\sqrt{d_n}}\psi_n(x)\,d\mu(x)=1.$$

De nouveau, on a bien $R_{p,\psi}S_p=\mathrm{id}_{\mathbf{PL}^p_y(X,\bigoplus E_n)}$ de manière formelle et l'on rencontre le même obstacle : il n'y a aucune raison pour que $R_{p,\psi}$ arrive bien dans $\mathbf{PL}^p_y\big(X,\bigoplus E_n\big)$. Une nouvelle fois, puisque S_p est une isométrie pour tout $p\in\,]p_1,p_2[$, la bornitude de $R_{p,\psi}$ équivaut à la bornitude de l'opérateur

$$\begin{aligned}S_pR_{p,\psi}:L^p_x\big(X,\bigoplus E_n\big)&\to L^p_x\big(X,\bigoplus E_n\big)\\(u_n(x,y))_{n\in\mathbb{N}}&\mapsto\left(\frac{\sqrt{e_n(x)}}{\sqrt{d_n}}\int_X u_n(x',y)\psi_n(x')\,d\mu(x')\right).\end{aligned}\tag{64}$$

Sous les hypothèses (35) et (36), l'existence d'une suite adéquate $(\psi_n)_{n\in\mathbb{N}}$ et la bornitude de $S_pR_{p,\psi}$, pour $p\in\,]p_1,p_2[$, seront établies dans la partie 2J.

On peut maintenant expliquer la preuve du théorème 2.5 d'interpolation. On remarque que les expressions de S_p et $R_{p,\psi}$ sont indépendantes de p. On fixe alors p_1' et p_2' deux réels appartenant à $]p_1,p_2[$. Pour tout $p\in\,]p_1',p_2'[$ on note $\theta'\in[0,1]$ l'unique réel vérifiant $\frac{1}{p}=\frac{1-\theta'}{p_1'}+\frac{\theta'}{p_2'}$. Le corollaire 2.30 assure que l'image de l'opérateur

$$S_p:\big[\mathbf{PL}^{p_1'}_y\big(X,\bigoplus E_n\big),\mathbf{PL}^{p_2'}_y\big(X,\bigoplus E_n\big)\big]_{\theta'}\to\big[L^{p_1'}_x\big(X,\bigoplus E_n\big),L^{p_2'}_x\big(X,\bigoplus E_n\big)\big]_{\theta'}$$

est $S_pR_{p,\psi}\big(\big[L^{p_1'}_x\big(X,\bigoplus E_n\big),L^{p_2'}_x\big(X,\bigoplus E_n\big)\big]_{\theta'}\big)$ et que S_p induit un isomorphisme sur son image. Le théorème 2.26 et le point (i) de la proposition 2.28 donnent alors

$$\begin{aligned}S_p\big(\big[\mathbf{PL}^{p_1'}_y\big(X,\bigoplus E_n\big),\mathbf{PL}^{p_2'}_y\big(X,\bigoplus E_n\big)\big]_{\theta'}\big)&=S_pR_{p,\psi}\big(\big[L^{p_1'}_x\big(X,\bigoplus E_n\big),L^{p_2'}_x\big(X,\bigoplus E_n\big)\big]_{\theta'}\big)\\&=S_pR_{p,\psi}\big(L^p_x\big(X,\bigoplus E_n\big)\big)\\&=S_p\big(\mathbf{PL}^p_y\big(X,\bigoplus E_n\big)\big).\end{aligned}$$

Se rappelant l'inclusion (58), on a donc le schéma

$$\begin{array}{ccc} \mathbf{PL}_y^p\big(X,\bigoplus E_n\big) & \xrightarrow{\quad S_p \quad} & L_x^p\big(X,\bigoplus E_n\big) \\ \big\uparrow & \nearrow_{S_p} & \\ \big[\mathbf{PL}_y^{p_1'}\big(X,\bigoplus E_n\big),\mathbf{PL}_y^{p_2'}\big(X,\bigoplus E_n\big)\big]_{\theta'} & & \end{array}$$

Or S_p est isométrique sur $\mathbf{PL}_y^p\big(X,\bigoplus E_n\big)$ et donc injectif. Cela nous amène à l'égalité

$$\big[\mathbf{PL}_y^{p_1'}\big(X,\bigoplus E_n\big),\mathbf{PL}_y^{p_2'}\big(X,\bigoplus E_n\big)\big]_{\theta'}=\mathbf{PL}_y^p\big(X,\bigoplus E_n\big).$$

Enfin, les normes des espaces $\big[\mathbf{PL}_y^{p_1'}\big(X,\bigoplus E_n\big),\mathbf{PL}_y^{p_2'}\big(X,\bigoplus E_n\big)\big]_{\theta'}$ et $\mathbf{PL}_y^p\big(X,\bigoplus E_n\big)$ sont équivalentes d'après le théorème du graphe fermé et l'inclusion continue (58). On aurait aussi pu invoquer le fait que S_p est un isomorphisme d'espaces de Banach de $\big[\mathbf{PL}_y^{p_1'}\big(X,\bigoplus E_n\big),\mathbf{PL}_y^{p_2'}\big(X,\bigoplus E_n\big)\big]_{\theta'}$ sur son image. La même argumentation est valide en remplaçant la méthode d'interpolation complexe $[\cdot,\cdot]_{\theta'}$ par la méthode d'interpolation réelle $[\cdot,\cdot]_{\theta',p}$.

2F. ***Preuves des théorèmes 2.5 et 2.6, partie II : Espaces de Lorentz.*** On effectue quelques rappels sur les espaces de Lorentz $L^{p,\infty}(X)$, avec $p\in\,]1,+\infty[$ (voir par exemple [Grafakos 2008, Chapter 1]). Afin d'exprimer l'inégalité de Hölder des espaces de Lorentz, il sera utile de remarquer la reformulation suivante de la quasi-norme $\|\cdot\|_{L^{p,\infty}(X)}$:

$$\begin{aligned}\forall f\in L^{p,\infty}(X)\quad \|f\|_{L^{p,\infty}(X)}&:=\inf\Big\{c>0\;\Big|\;\forall t>0\quad \mu\{x\in X\mid |f(x)|>t\}\le\frac{c^p}{t^p}\Big\}\\ &=\sup_{T>0}T^{\frac1p}f^\star(T),\end{aligned}$$

où $f^\star:[0,+\infty[\,\to[0,+\infty[$ est le réarrangement décroissant de f définie par la formule

$$f^\star(T):=\inf\{t>0\mid\mu\{|f|>t\}\le T\}.$$

En général, la quasi-norme $\|\cdot\|_{L^{p,\infty}(X)}$ ne vérifie pas l'inégalité triangulaire mais est toujours équivalente à la norme suivante dès lors que l'on a $p>1$ ([Grafakos 2008, page 13 and 64] ou [García-Cuerva et Rubio de Francia 1985, Part V, Lemma 2.8]) :

$$\forall f\in L^{p,\infty}(X),\quad |||f|||_{L^{p,\infty}(X)}:=\sup_{\substack{A\in\mathcal{B}(X)\\0<\mu(A)<+\infty}}\frac{\|f\mathbf{1}_A\|_{L^1(X)}}{\mu(A)^{1-\frac1p}},\tag{65}$$

où $\mathcal{B}(X)$ désigne l'ensemble des parties mesurables de X. Précisément, nous avons

$$\|f\|_{L^{p,\infty}(X)}\le|||f|||_{L^{p,\infty}(X)}\le\frac{p}{p-1}\|f\|_{L^{p,\infty}(X)}.$$

En d'autres termes, quitte à perdre une constante multiplicative, on pourra utiliser l'inégalité triangulaire dans $L^{p,\infty}(X)$. On aura aussi besoin de l'espace de Lorentz $L^{p,1}(X)$: il s'agit de l'espace vectoriel des fonctions mesurables $g : X \to \mathbb{C}$ qui vérifient

$$\|g\|_{L^{p,1}(X)} := \int_0^{+\infty} T^{\frac{1}{p}} g^\star(T)\, \frac{dT}{T} < +\infty.$$

Venons-en maintenant aux inégalités de Hardy–Littlewood et de Hölder, on a pour toutes fonctions $f \in L^{p,\infty}(X)$ et $g \in L^{\frac{p}{p-1},1}(X)$

$$\begin{aligned}\left|\int_X f(x)g(x)\,d\mu(x)\right| \le \int_0^{+\infty} f^\star(T)g^\star(T)\,dT &= \int_0^{+\infty} T^{\frac{1}{p}} f^\star(t) T^{\frac{p-1}{p}} g^\star(T)\,\frac{dT}{T} \\ &\le \|f\|_{L^{p,\infty}(X)}\, \|g\|_{L^{\frac{p}{p-1},1}(X)} \\ &\le |||f|||_{L^{p,\infty}(X)}\, \|g\|_{L^{\frac{p}{p-1},1}(X)}. \qquad (66)\end{aligned}$$

On peut aussi définir des espaces de Lorentz $L^{p,r}(X, B)$, avec $r \in \{1, \infty\}$ (et même tout réel $r \ge 1$), à valeurs dans un espace de Banach complexe B. Il s'agit des fonctions mesurables $f : X \to B$ telles que la fonction $x \mapsto \|f(x)\|_B$ appartient à $L^{p,r}(X)$. La théorie de l'interpolation réelle fait jouer un rôle important aux espaces de Lorentz, en particulier on a le résultat suivant [Triebel 1978, Part 1.18.6, Lemma, Theorem 2, (16), $q = p$; Part 1.18.7, Theorem 2].

Théorème 2.32. *Considérons un espace de Banach complexe B et trois réels $p_1 < p < p_2$ appartenant à $]1, +\infty[$. Soit $\theta \in]0, 1[$ l'unique réel tel que $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Pour tout $r \in \{1, +\infty\}$, on a pour la méthode d'interpolation réelle*

$$[L^{p_1,r}(X, B), L^{p_2,r}(X, B)]_{\theta,p} = L^p(X, B).$$

Si un opérateur linéaire T est borné de $L^{p_1,1}(X, B)$ à valeurs dans $L^{p_1,\infty}(X, B)$ et de $L^{p_2,1}(X, B)$ à valeurs dans $L^{p_2,\infty}(X, B)$ alors il est borné de $L^p(X, B)$ à valeurs dans $L^p(X, B)$ et il existe une constante $K > 0$ indépendante de T telle que

$$\|T\|_{L^p(X,B)\to L^p(X,B)} \le K \|T\|^{1-\theta}_{L^{p_1,1}(X,B)\to L^{p_1,\infty}(X,B)} \|T\|^{\theta}_{L^{p_2,1}(X,B)\to L^{p_2,\infty}(X,B)}.$$

2G. *Preuves des théorèmes 2.5 et 2.6, partie III : R-bornitude.* Énonçons deux faits qui vont justifier que la bornitude d'un opérateur linéaire sur $L^p_x(X, \ell^2(\mathbb{N}))$ n'est généralement pas simple et qui vont motiver l'approche qui va suivre.

Fait 1. *Soit $(P_n)_{n\ge0}$ une suite d'opérateurs uniformément bornés sur $L^2(X)$, c'est-à-dire que l'on a*

$$\sup_{n\ge0} \|P_n\|_{L^2(X)\to L^2(X)} < +\infty,$$

alors l'opérateur $\bigoplus P_n$ défini par l'expression suivante est borné sur $L^2(X, \ell^2(\mathbb{N}))$:

$$\forall (f_n)_{n\ge0} \in L^2(X, \ell^2(\mathbb{N})), \quad \big(\bigoplus P_n\big)(f_n) = (P_n f_n).$$

En effet, cela découle immédiatement des formules :

$$\|(f_n)\|_{L^2(X,\ell^2(\mathbb{N}))} = \sqrt{\int_X \sum_{n\ge0} |f_n(x)|^2\, d\mu(x)} = \sqrt{\sum_{n\ge0} \|f_n\|^2_{L^2(X)}}.$$

Fait 2. *Pour tout réel $p > 2$ il existe une suite $(P_n)_{n\geq 0}$ d'opérateurs uniformément bornés de $L^p(\mathbb{R})$ et tels que l'opérateur $\bigoplus P_n$ défini par*

$$\forall (f_n)_{n\geq 0} \in L^p(X, \ell^2(\mathbb{N})), \quad \Big(\bigoplus P_n\Big)(f_n) = (P_n f_n). \tag{67}$$

n'est pas borné sur $L^p(\mathbb{R}, \ell^2(\mathbb{N}))$.

L'exemple est élémentaire. On considère les isométries P_n définies par

$$\forall n \in \mathbb{N}, \quad \forall f \in L^p(\mathbb{R}), \quad \forall x \in \mathbb{R}, \quad (P_n f)(x) = f(x+n),$$

puis les fonctions $f_{N,n}$, paramétrées par $(n, N) \in \mathbb{N}^2$, définies par

$$\forall x \in \mathbb{R} \quad f_{N,n}(x) := \begin{cases} \mathbf{1}_{[n,n+1[}(x) & \text{si } n < N, \\ 0 & \text{si } n \geq N. \end{cases}$$

On a immédiatement $\|P_n\|_{L^p(X)\to L^p(X)} = 1$. De même pour tout $N \in \mathbb{N}^\star$, on a

$$\Big\|\Big(\bigoplus P_n\Big)(f_{N,n})\Big\|^p_{L^p(\mathbb{R},\ell^2(\mathbb{N}))} = \int_{\mathbb{R}} \bigg(\sum_{n=0}^{N-1} \mathbf{1}_{[0,1[}(x)\bigg)^{\frac{p}{2}} dx = N^{\frac{p}{2}},$$

$$\|(f_{N,n})\|^p_{L^p(\mathbb{R},\ell^2(\mathbb{N})} = \int_{\mathbb{R}} \bigg(\sum_{n=0}^{N-1} \mathbf{1}_{[n,n+1[}(x)\bigg)^{\frac{p}{2}} dx = N \ll N^{\frac{p}{2}}.$$

En faisant tendre N vers $+\infty$, on voit que l'opérateur $\bigoplus P_n$ n'est pas borné sur $L^p(\mathbb{R}, \ell^2(\mathbb{N}))$. L'exemple précédent utilise le fait que $\mathbb{R}$ n'est pas compact afin de s'échapper vers l'infini. En réalité, cela est un faux-semblant et l'on peut transférer la construction précédente sur $L^p(0,1)$ à l'aide d'une isométrie linéaire surjective adéquate de $L^p(0,1)$ sur $L^p(\mathbb{R})$. En outre, si l'on raisonne par dualité ou si l'on choisit $f_n = \mathbf{1}_{[0,1[}$ à la place de $\mathbf{1}_{[n,n+1[}$, la construction précédente s'adapte au cas $p < 2$. Ainsi, le fait 2 contraste fortement avec le fait 1 et montre que la seule condition

$$\sup_{n\in\mathbb{N}} \|P_n\|_{L^p(X)\to L^p(X)} < +\infty$$

ne suffit pas pour assurer la bornitude de l'opérateur $\bigoplus P_n$ sur $L^p(X, \ell^2(\mathbb{N}))$ pour $p \neq 2$, même si $\mu(X) < +\infty$. Un exemple qui illustre ce problème est la non continuité du multiplicateur de Fourier sur $L^p(\mathbb{R}^2)$ de l'indicatrice de la boule unité (voir la preuve de Fefferman [1971] dans laquelle les opérateurs P_n sont des projecteurs de $L^p(\mathbb{R}^2)$).

La notion mathématique qui s'est dégagée est la $\mathcal{R}$-bornitude. Étant donné un espace de Banach B, une suite $(P_n)_{n\geq 0}$ d'opérateurs linéaires bornés de B est dite $\mathcal{R}$-bornée s'il existe $K > 0$ telle que pour toute suite $(f_n)_{n\in\mathbb{N}}$ de B (nulle pour $n \gg 1$) on a un principe de contraction :

$$\mathbf{E}\bigg[\bigg\|\sum_{n\geq 0} \varepsilon_n P_n(f_n)\bigg\|_B\bigg] \leq K\mathbf{E}\bigg[\bigg\|\sum_{n\geq 0} \varepsilon_n f_n\bigg\|_B\bigg].$$

Dans le cas $B = L^p(X)$, il est classique que les deux espérances précédentes s'estiment avec le théorème de Fubini et les inégalités de Kahane–Khintchine dans $L^p(X)$ et dans $\mathbb{C}$ (voir (43) dans le cas unidimensionnel). L'inégalité précédente signifie alors que l'opérateur $\bigoplus P_n$, défini en (67), est borné sur $L^p(X, \ell^2(\mathbb{N}))$. Quitte à changer $K > 0$, cela équivaut à une estimation de la forme

$$\int_X \left(\sum_{n \geq 0} |(P_n f_n)(x)|^2 \right)^{\frac{p}{2}} d\mu(x) \leq K \int_X \left(\sum_{n \geq 0} |f_n(x)|^2 \right)^{\frac{p}{2}} d\mu(x).$$

Différents critères de $\mathcal{R}$-bornitude existent dans la littérature (voir par exemple [García-Cuerva et Rubio de Francia 1985, Chapter V ; Weis 2001, Section 2 ; Clément et al. 2000] et les références indiquées). Le lemme 2.33 donne un nouveau critère qui s'apparente au lemme de Schur et qui donne des conditions suffisantes pour obtenir la $\mathcal{R}$-bornitude. Ce critère suffira pour la théorie de l'interpolation et de la dualité des espaces $\mathbf{PL}^p\big(X, \bigoplus E_n\big)$ dans le cas unidimensionnel $d_n = \dim(E_n) = 1$. Pour comprendre le cas $d_n \neq 1$, nous aurons besoin d'une version légèrement plus sophistiquée du lemme 2.33, à savoir le lemme 2.34, mais le coeur de l'idée est dans la démonstration suivante.

Lemme 2.33. *Fixons des réels $p_1 < p < p_2$ appartenant à $]1, +\infty[$. Notons $q_1 := \frac{p_1}{p_1 - 1}$ et $q_2 := \frac{p_2}{p_2 - 1}$. Considérons $K_n : X^2 \to \mathbb{C}$ des fonctions mesurables, pour $n \in \mathbb{N}$, de sorte que l'on a*

$$\sup_{n \geq 0} \| K_n(x, x') \|_{L^{q_1, \infty}_{x'}(X)} \in L^{p_1, \infty}_x(X), \tag{68}$$

$$\sup_{n \geq 0} \| K_n(x, x') \|_{L^{q_2, \infty}_{x'}(X)} \in L^{p_2, \infty}_x(X), \tag{69}$$

$$\sup_{n \geq 0} \| K_n(x, x') \|_{L^{p_1, \infty}_{x}(X)} \in L^{q_1, \infty}_{x'}(X), \tag{70}$$

$$\sup_{n \geq 0} \| K_n(x, x') \|_{L^{p_2, \infty}_{x}(X)} \in L^{q_2, \infty}_{x'}(X). \tag{71}$$

Les deux assertions suivantes sont vraies :

(a) *Pour tout entier $n \in \mathbb{N}$ l'opérateur P_n de noyau K_n défini par*

$$\forall f \in L^p_x(X), \quad \forall x \in X, \quad (P_n f)(x) = \int_X K_n(x, x') f(x') \, d\mu(x'),$$

est borné sur $L^p(X)$ et la borne supérieure $\sup_{n \geq 0} \| P_n \|_{L^p(X) \to L^p(X)}$ est finie.

(b) *La famille d'opérateurs $(P_n)_{n \in \mathbb{N}}$ de $L^p(X)$ est $\mathcal{R}$-bornée.*

Démonstration. Nous aurons besoin des propriétés des espaces de Lorentz rappelées dans la partie 2F.

(a) On affirme que (68) implique la bornitude de l'opérateur

$$P_n : L^{p_1, 1}_x(X) \to L^{p_1, \infty}_x(X).$$

En effet, l'inégalité de Hölder (66) donne

$$|(P_n f)(x)| \leq \| K_n(x, x') \|_{L^{q_1, \infty}_{x'}(X)} \| f \|_{L^{p_1, 1}(X)}$$
$$\| P_n f \|_{L^{p_1, \infty}(X)} \leq \big\| \| K_n(x, x') \|_{L^{q_1, \infty}_{x'}(X)} \big\|_{L^{p_1, \infty}_x(X)} \| f \|_{L^{p_1, 1}(X)}.$$

Afin de faciliter la lecture de la preuve de l'assertion (b), nous donnons un autre argument qui utilise (70) et la norme $|||\cdot|||_{L^{p_1,\infty}(X)}$ de $L^{p_1,\infty}(X)$, définie en (65), car elle présente l'avantage de satisfaire l'inégalité triangulaire. En effet, on a

$$\begin{aligned}|||P_n f|||_{L^{p_1,\infty}(X)} &\leq \int_X |||K_n(x,x')|||_{L_x^{p_1,\infty}(X)} |f(x')|\, d\mu(x') \\ &\leq \big\| |||K_n(x,x')|||_{L_x^{p_1,\infty}(X)} \big\|_{L_{x'}^{q_1,\infty}(X)} \|f\|_{L^{p_1,1}(X)}.\end{aligned}$$

Les deux arguments précédents sont en fait duaux. En utilisant (69) ou (71), on obtient de même la bornitude de l'opérateur

$$P_n : L_x^{p_2,1}(X) \to L_x^{p_2,\infty}(X).$$

On conclut par interpolation réelle (à savoir le théorème 2.32 avec $B = \mathbb{C}$).

(b) On affirme que l'opérateur suivant est borné

$$\bigoplus P_n : L_x^{p_1,1}(X, \ell^\infty(\mathbb{N})) \to L_x^{p_1,\infty}(X, \ell^\infty(\mathbb{N})).$$

En effet, cela découle de l'inégalité de Hölder (66) et de (68) :

$$\begin{aligned}\sup_{n\in\mathbb{N}} |(P_n f_n)(x)| &= \sup_{n\in\mathbb{N}} \left| \int_X K_n(x,x') f_n(x')\, d\mu(x') \right| \\ &\leq \sup_{n\in\mathbb{N}} \int_X |K_n(x,x')| \sup_{m\in\mathbb{N}} |f_m(x')|\, d\mu(x') \\ &\leq \underbrace{\left(\sup_{n\in\mathbb{N}} \|K_n(x,x')\|_{L_{x'}^{q_1,\infty}(X)} \right)}_{\in L_x^{p_1,\infty}(X)} \times \left\| \sup_{m\in\mathbb{N}} |f_m(x')| \right\|_{L_{x'}^{p_1,1}(X)}.\end{aligned}$$

On s'attelle maintenant à prouver la bornitude de l'opérateur

$$\bigoplus P_n : L_x^{p_1,1}(X, \ell^1(\mathbb{N})) \to L_x^{p_1,\infty}(X, \ell^1(\mathbb{N})).$$

De nouveau, on utilise la norme $|||\cdot|||_{L^{p_1,\infty}(X)}$ de $L^{p_1,\infty}(X)$. L'hypothèse (70) et l'inégalité de Hölder (66) nous amènent aux estimations

$$\begin{aligned}\left|\left|\left| \sum_{n\in\mathbb{N}} |(P_n f_n)(x)| \right|\right|\right|_{L_x^{p_1,\infty}(X)} &= \left|\left|\left| \sum_{n\geq 0} \left| \int_X K_n(x,x') f_n(x')\, d\mu(x') \right| \right|\right|\right|_{L_x^{p_1,\infty}(X)} \\ &\leq \sum_{n\geq 0} \int_X |||K_n(x,x')|||_{L_x^{p_1,\infty}(X)} |f_n(x')|\, d\mu(x') \\ &\leq \sum_{n\geq 0} \int_X \left(\sup_{m\in\mathbb{N}} |||K_m(x,x')|||_{L_x^{p_1,\infty}(X)} \right) |f_n(x')|\, d\mu(x') \\ &\leq \int_X \left(\sup_{m\in\mathbb{N}} |||K_m(x,x')|||_{L_x^{p_1,\infty}(X)} \right) \left(\sum_{n\geq 0} |f_n(x')| \right) d\mu(x')\end{aligned}$$

$$\leq \left\| \sup_{m\in\mathbb{N}} |||K_m(x,x')|||_{L_x^{p_1,\infty}(X)} \right\|_{L_{x'}^{q_1,\infty}(X)} \left\| \sum_{n\geq 0} |f_n(x')| \right\|_{L_{x'}^{p_1,1}(X)}.$$

La démonstration est évidemment similaire en remplaçant p_1 par p_2 et en utilisant (71) et (69). On a ainsi obtenu la bornitude des quatre opérateurs

$$\begin{aligned}
\bigoplus P_n &: L_x^{p_1,1}(X,\ell^\infty(\mathbb{N})) \to L_x^{p_1,\infty}(X,\ell^\infty(\mathbb{N})),\\
\bigoplus P_n &: L_x^{p_2,1}(X,\ell^\infty(\mathbb{N})) \to L_x^{p_2,\infty}(X,\ell^\infty(\mathbb{N})),\\
\bigoplus P_n &: L_x^{p_1,1}(X,\ell^1(\mathbb{N})) \to L_x^{p_1,\infty}(X,\ell^1(\mathbb{N})),\\
\bigoplus P_n &: L_x^{p_2,1}(X,\ell^1(\mathbb{N})) \to L_x^{p_2,\infty}(X,\ell^1(\mathbb{N})).
\end{aligned}$$

Pour tout $p \in]p_1, p_2[$, le théorème 2.32 assure la bornitude des deux opérateurs suivants par interpolation réelle

$$\begin{aligned}
\bigoplus P_n &: L_x^{p}(X,\ell^\infty(\mathbb{N})) \to L_x^{p}(X,\ell^\infty(\mathbb{N})),\\
\bigoplus P_n &: L_x^{p}(X,\ell^1(\mathbb{N})) \to L_x^{p}(X,\ell^1(\mathbb{N})).
\end{aligned}$$

Par interpolation complexe (c'est-à-dire le théorème 2.26 avec $\theta = \frac{1}{2}$), on obtient la bornitude de l'opérateur

$$\bigoplus P_n : L_x^{p}(X,\ell^2(\mathbb{N})) \to L_x^{p}(X,\ell^2(\mathbb{N})). \qquad \square$$

Il s'agit maintenant d'écrire un résultat analogue au lemme 2.33 adaptée à la théorie générale avec $\dim(E_n) \neq 1$. Pour cela, on aura besoin d'espaces analogues à $\ell^1(\mathbb{N})$ et $\ell^\infty(\mathbb{N})$ adaptés aux sous-espaces E_n, ce sera le rôle joué par les espaces $(\bigoplus E_n)_{\ell^r}$ dans la preuve du résultat suivant.

Lemme 2.34. *Fixons des réels $p_1 < p < p_2$ appartenant à $]1, +\infty[$ et considérons $K_n : X^2 \to \mathbb{C}$ des fonctions mesurables, pour $n \in \mathbb{N}$, de sorte que*

$$\sup_{n\geq 0} \|K_n(x,x')\|_{L_{x'}^{q_1,\infty}(X)} \in L_x^{p_1,\infty}(X), \tag{72}$$

$$\sup_{n\geq 0} \|K_n(x,x')\|_{L_{x'}^{q_2,\infty}(X)} \in L_x^{p_2,\infty}(X), \tag{73}$$

$$\sup_{n\geq 0} \|K_n(x,x')\|_{L_{x}^{p_1,\infty}(X)} \in L_{x'}^{q_1,\infty}(X), \tag{74}$$

$$\sup_{n\geq 0} \|K_n(x,x')\|_{L_{x}^{p_2,\infty}(X)} \in L_{x'}^{q_2,\infty}(X). \tag{75}$$

Alors les deux assertions suivantes sont vraies

(a) *Pour tout entier $n \in \mathbb{N}$, l'opérateur P_n défini par*

$$\forall u \in L_x^p(X,E_n), \quad \forall (x,y) \in X^2, \quad (P_n u)(x,y) = \int_X K_n(x,x') u(x',y)\, d\mu(x'),$$

est borné sur $L_x^p(X,E_n)$ et la borne supérieure $\sup_{n\geq 0}\|P_n\|$ est finie.

(b) *L'opérateur $\bigoplus P_n$, défini par l'expression suivante, est borné*

$$\bigoplus P_n : L_x^p\big(X, \bigoplus E_n\big) \to L_x^p\big(X, \bigoplus E_n\big), \quad (u_n(x,y))_{n\in\mathbb{N}} \mapsto ((P_n u_n)(x,y))_{n\in\mathbb{N}}.$$

Démonstration. (a) On raisonne comme pour le lemme 2.33, c'est-à-dire que chaque opérateur P_n est borné de $L_x^{p_1,1}(X, E_n)$ à valeurs dans $L_x^{p_1,\infty}(X, E_n)$ et de $L_x^{p_2,1}(X, E_n)$ à valeurs dans $L_x^{p_2,\infty}(X, E_n)$ avec des normes majorées indépendamment de n. On conclut par interpolation réelle (c'est-à-dire le théorème 2.32 avec $B = E_n$).

(b) L'idée est d'interpoler l'espace de Hilbert $\bigoplus E_n$ entre les deux espaces

$$\big(\bigoplus E_n\big)_{\ell^1} := \big\{(u_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N}, u_n \in E_n,\ \textstyle\sum_{n\in\mathbb{N}} \|u_n\|_{L_y^2(X)} < +\infty\big\},$$
$$\big(\bigoplus E_n\big)_{\ell^\infty} := \big\{(u_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N}, u_n \in E_n,\ \sup_{n\in\mathbb{N}} \|u_n\|_{L_y^2(X)} < +\infty\big\},$$

qui sont munis de leurs normes naturelles. On vérifie aisément que les espaces $\big(\bigoplus E_n\big)_{\ell^1}$ et $\big(\bigoplus E_n\big)_{\ell^\infty}$ sont des rétractes, au sens de la définition 2.27, des espaces $\ell^1\big(\mathbb{N}, \bigoplus E_n\big)$ et $\ell^\infty\big(\mathbb{N}, \bigoplus E_n\big)$. Par suite, le corollaire 2.30 et la théorie de l'interpolation complexe des espaces $\ell^r\big(\mathbb{N}, \bigoplus E_n\big)$ (c'est-à-dire le théorème 2.26) montrent que l'on a l'égalité

$$\big[\big(\bigoplus E_n\big)_{\ell^1}, \big(\bigoplus E_n\big)_{\ell^\infty}\big]_{\frac{1}{2}} = \bigoplus E_n,$$

avec équivalence de normes. Nous pouvons donc reprendre l'argumentation du lemme 2.33 et il nous suffit manifestement de prouver la bornitude des quatre opérateurs

$$\begin{aligned}
\bigoplus P_n &: L_x^{p_1,1}\big(X, \big(\bigoplus E_n\big)_{\ell^1}\big) \to L_x^{p_1,\infty}\big(X, \big(\bigoplus E_n\big)_{\ell^1}\big),\\
\bigoplus P_n &: L_x^{p_2,1}\big(X, \big(\bigoplus E_n\big)_{\ell^1}\big) \to L_x^{p_2,\infty}\big(X, \big(\bigoplus E_n\big)_{\ell^1}\big),\\
\bigoplus P_n &: L_x^{p_1,1}\big(X, \big(\bigoplus E_n\big)_{\ell^\infty}\big) \to L_x^{p_1,\infty}\big(X, \big(\bigoplus E_n\big)_{\ell^\infty}\big),\\
\bigoplus P_n &: L_x^{p_2,1}\big(X, \big(\bigoplus E_n\big)_{\ell^\infty}\big) \to L_x^{p_2,\infty}\big(X, \big(\bigoplus E_n\big)_{\ell^\infty}\big).
\end{aligned}$$

Par symétrie évidente, il nous suffit de justifier la bornitude des deux opérateurs

$$\begin{aligned}
\bigoplus P_n &: L_x^{p_1,1}\big(X, \big(\bigoplus E_n\big)_{\ell^1}\big) \to L_x^{p_1,\infty}\big(X, \big(\bigoplus E_n\big)_{\ell^1}\big),\\
\bigoplus P_n &: L_x^{p_1,1}\big(X, \big(\bigoplus E_n\big)_{\ell^\infty}\big) \to L_x^{p_1,\infty}\big(X, \big(\bigoplus E_n\big)_{\ell^\infty}\big).
\end{aligned}$$

L'hypothèse (74) et l'inégalité de Hölder (66) nous donnent les majorations

$$\begin{aligned}
&\Big\|\Big\|\sum_{n\geq 0} \|(P_n u_n)(x,y)\|_{L_y^2(X)}\Big\|\Big\|_{L_x^{p_1,\infty}(X)}\\
&\qquad = \Big\|\Big\|\sum_{n\geq 0} \Big\| \int_X K_n(x,x') u_n(x',y)\, d\mu(x')\Big\|_{L_y^2(X)}\Big\|\Big\|_{L_x^{p_1,\infty}(X)}\\
&\qquad \leq \sum_{n\geq 0} \int_X \||K_n(x,x')\||_{L_x^{p_1,\infty}(X)} \|u_n(x',y)\|_{L_y^2(X)}\, d\mu(x')\\
&\qquad \leq \int_X \Big(\sup_{m\in\mathbb{N}} \||K_m(x,x')\||_{L_x^{p_1,\infty}(X)}\Big)\Big(\sum_{n\geq 0} \|u_n(x',y)\|_{L_y^2(X)}\Big)\, d\mu(x')\\
&\qquad \leq \Big\|\sup_{m\in\mathbb{N}} \||K_m(x,x')\||_{L_x^{p_1,\infty}(X)}\Big\|_{L_{x'}^{q_1,\infty}(X)} \Big\|\sum_{n\geq 0} \|u_n(x',y)\|_{L_y^2(X)}\Big\|_{L_{x'}^{p_1,1}(X)}.
\end{aligned}$$

Enfin, (72) et l'inégalité de Hölder (66) nous permettent d'écrire

$$\begin{aligned}\sup_{n\in\mathbb{N}}\|(P_nu_n)(x,y)\|_{L^2_y(X)} &= \sup_{n\in\mathbb{N}}\left\|\int_X K_n(x,x')u_n(x',y)\,d\mu(x')\right\|_{L^2_y(X)}\\ &\leq \sup_{n\in\mathbb{N}}\int_X |K_n(x,x')|\sup_{m\in\mathbb{N}}\|u_m(x',y)\|_{L^2_y(X)}\,d\mu(x')\\ &\leq \underbrace{\left(\sup_{n\in\mathbb{N}}\|K_n(x,x')\|_{L^{q_1,\infty}_{x'}(X)}\right)}_{\in L^{p_1,\infty}_x(X)}\times\left\|\sup_{m\in\mathbb{N}}\|u_m(x',y)\|_{L^2_y(X)}\right\|_{L^{p_1,1}_{x'}(X)}. \qquad \square\end{aligned}$$

2H. ***Preuve du théorème 2.6 : bornitude de S_pR_p.*** Sous les hypothèses du théorème 2.6 de dualité, les exposants p_1 et p_2 sont conjugués et l'on souhaite montrer que S_pR_p est un projecteur borné sur $L^p_x\big(X,\bigoplus E_n\big)$ pour tout $p\in]p_1,p_2[$. Avec les notations usuelles, on a $q_1=\frac{p_1}{p_1-1}=p_2$ et $q_2=\frac{p_2}{p_2-1}=p_1$. D'après la forme du projecteur (60), il s'agit d'appliquer le lemme 2.34 avec la suite de noyaux K_n définis par

$$\forall (x,x')\in X^2,\quad K_n(x,x')=\frac{1}{d_n}\sqrt{e_n(x)e_n(x')}.$$

Il est facile de constater que (35) et (37) impliquent l'assertion (72) (qui est identique à (75)). En effet, on a pour tout $(x,n)\in X\times\mathbb{N}$

$$\begin{aligned}\left\|\frac{1}{d_n}\sqrt{e_n(x)e_n(x')}\right\|_{L^{q_1,\infty}_{x'}(X)} &\leq \frac{\sqrt{e_n(x)}}{d_n}\|\sqrt{e_n}\|_{L^{q_1}(X)}\\ &\leq \frac{\sqrt{e_n(x)}}{\|\sqrt{e_n}\|_{L^{p_1}(X)}}\times\frac{\|\sqrt{e_n}\|_{L^{p_1}(X)}\|\sqrt{e_n}\|_{L^{q_1}(X)}}{d_n}.\end{aligned}$$

Les deux autres hypothèses (73) et (74) du lemme 2.34 se vérifient de la même façon. La preuve du théorème 2.6 de dualité est finie.

2I. ***Preuve du théorème 2.5 : défaut d'interpolation.*** Nous allons définir la notion de *défaut d'interpolation* afin d'aborder l'interpolation des espaces $\mathbf{PL}^p\big(X,\bigoplus E_n\big)$. Commençons par introduire quelques notations. Pour tous réels p_1, p et p_2 appartenant à $[1,+\infty[$ et vérifiant $p_1\leq p\leq p_2$, on définit les nombres $\theta_1(p_1,p,p_2)$ et $\theta_2(p_1,p,p_2)$ tels que

$$\frac{\theta_1(p_1,p,p_2)}{p_1}+\frac{\theta_2(p_1,p,p_2)}{p_2}=\frac{1}{p}\quad\text{et}\quad \theta_1(p_1,p,p_2)+\theta_2(p_1,p,p_2)=1.$$

De façon précise, on a les formules

$$\theta_1(p_1,p,p_2)=\frac{\frac{1}{p}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{p_2}}\quad\text{et}\quad \theta_2(p_1,p,p_2)=\frac{\frac{1}{p_1}-\frac{1}{p}}{\frac{1}{p_1}-\frac{1}{p_2}}.$$

On peut alors poser la définition suivante.

Définition 2.35. Considérons p_1 et p_2 appartenant à $[1,+\infty[$ et vérifiant $p_1 < p_2$ ainsi qu'une fonction non nulle $\phi \in L^{p_1}(X) \cap L^{p_2}(X)$. Nous définissons $Q(\phi,[p_1,p_2]) \in [1,+\infty[$ le *défaut d'interpolation* de ϕ sur $[p_1,p_2]$ par la formule

$$Q(\phi,[p_1,p_2]) := \sup_{p\in[p_1,p_2]} \frac{\|\phi\|_{L^{p_1}(X)}^{\theta_1(p_1,p,p_2)} \|\phi\|_{L^{p_2}(X)}^{\theta_2(p_1,p,p_2)}}{\|\phi\|_{L^p(X)}}.$$

L'inégalité $Q(\phi,[p_1,p_2]) < +\infty$ est facile et nous allons la vérifier par convexité (voir le lemme 2.36 ci-après). Quant à l'inégalité $Q(\phi,[p_1,p_2]) \geq 1$, elle découle de l'inégalité de Hölder avec les exposants conjugués $\frac{p_1}{p\theta_1(p_1,p,p_2)}$ et $\frac{p_2}{p\theta_2(p_1,p,p_2)}$:

$$\begin{aligned}\left(\int_X |\phi(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} &= \left(\int_X |\phi(x)|^{p\theta_1(p_1,p,p_2)} |\phi(x)|^{p\theta_2(p_2,p,p_2)} \, d\mu(x)\right)^{\frac{1}{p}} \\ &\leq \left(\int_X |\phi(x)|^{p_1} \, d\mu(x)\right)^{\frac{\theta_1(p_1,p,p_2)}{p_1}} \left(\int_X |\phi(x)|^{p_2} \, d\mu(x)\right)^{\frac{\theta_2(p_1,p,p_2)}{p_2}}. \qquad (76)\end{aligned}$$

La condition d'égalité de l'inégalité de Hölder montre alors l'équivalence

$$Q(\phi,[p_1,p_2]) = 1 \quad \Leftrightarrow \quad \exists a > 0,\ \exists A \in \mathcal{B}(X),\ \ 0 < \mu(A) < +\infty,\ \ |\phi| = a\mathbf{1}_A,$$

où $\mathcal{B}(X)$ est l'ensemble des parties mesurables de X. Ainsi, le défaut d'interpolation de la fonction ϕ permet de tester si elle se concentre complètement sur une même partie de l'espace mesuré X. Par comparaison avec (76), le défaut d'interpolation permet de *minorer* $\|\phi\|_{L^p(X)}$ si l'on connaît $\|\phi\|_{L^{p_1}(X)}$ et $\|\phi\|_{L^{p_2}(X)}$:

$$\forall p \in [p_1,p_2], \quad \frac{\|\phi\|_{L^{p_1}(X)}^{\theta_1(p_1,p,p_2)} \|\phi\|_{L_2^{p}(X)}^{\theta_2(p_1,p,p_2)}}{Q(\phi,[p_1,p_2])} \leq \|\phi\|_{L^p(X)}. \qquad (77)$$

Le lemme suivant montre qu'il suffit d'examiner un seul point de l'intervalle $]p_1,p_2[$ pour contrôler $Q(\phi,[p_1,p_2])$.

Lemme 2.36. *Fixons des réels $p_1 < p < p_2$ appartenant à $[1,+\infty[$ et une suite de fonctions non nulles $(\phi_n)_{n\in\mathbb{N}}$ de $L^{p_1}(X) \cap L^{p_2}(X)$, alors on a l'équivalence :*

$$\sup_{n\in\mathbb{N}} \frac{\|\phi_n\|_{L^{p_1}(X)}^{\theta_1(p_1,p,p_2)} \|\phi_n\|_{L^{p_2}(X)}^{\theta_2(p_1,p,p_2)}}{\|\phi_n\|_{L^p(X)}} < +\infty \quad \Leftrightarrow \quad \sup_{n\in\mathbb{N}} Q(\phi_n,[p_1,p_2]) < +\infty.$$

Démonstration. Pour toute fonction non nulle $\phi \in L^{p_1}(X) \cap L^{p_2}(X)$, il est bien connu que la fonction

$$\Phi : \wp \in [p_1,p_2] \mapsto \ln(\|\phi\|_{L^{\wp}(X)})$$

est convexe par rapport à $\frac{1}{\wp}$ (voir les inégalités (76)). Introduisons la fonction $\widetilde{\Phi} : [p_1,p_2] \to \mathbb{R}$ définie par

$$\widetilde{\Phi}(\wp) = \ln\left(\frac{\|\phi\|_{L^{p_1}(X)}^{\theta_1(p_1,\wp,p_2)} \|\phi\|_{L^{p_2}(X)}^{\theta_2(p_1,\wp,p_2)}}{\|\phi\|_{L^{\wp}(X)}}\right) = \theta_1(p_1,\wp,p_2)\Phi(p_1) + \theta_2(p_1,\wp,p_2)\Phi(p_2) - \Phi(\wp).$$

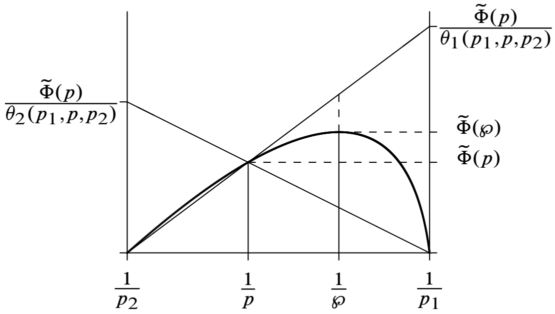


Figure 1. La fonction $\widetilde{\Phi}$.

La fonction $\widetilde{\Phi}$ s'annule en p_1 et p_2 et est concave par rapport à $\frac{1}{\wp}$; voir figure 1. Par concavité et application du théorème de Thalès, nous avons les estimations

$$\forall \wp \in [p_1, p_2], \quad \widetilde{\Phi}(\wp) \leq \widetilde{\Phi}(p) \max\left(\frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{p} - \frac{1}{p_2}}, \frac{\frac{1}{p_1} - \frac{1}{p_2}}{\frac{1}{p_1} - \frac{1}{p}} \right).$$

En passant à l'exponentielle, on obtient les estimations suivantes qui donnent la conclusion :

$$\frac{\|\phi\|_{L^{p_1}(X)}^{\theta_1(p_1,p,p_2)} \|\phi\|_{L^{p_2}(X)}^{\theta_2(p_1,p,p_2)}}{\|\phi\|_{L^p(X)}} \leq Q(\phi, [p_1, p_2]) \leq \left(\frac{\|\phi\|_{L^{p_1}(X)}^{\theta_1(p_1,p,p_2)} \|\phi\|_{L^{p_2}(X)}^{\theta_2(p_1,p,p_2)}}{\|\phi\|_{L^p(X)}} \right)^{C(p_1,p,p_2)}. \quad \square$$

On sait que pour tout réel $p \in]1, +\infty[$ et toute fonction $\phi \in L^p(X)$, il existe une fonction $\psi \in L^{\frac{p}{p-1}}(X)$ telle que

$$\int_X \phi(x)\psi(x)\, d\mu(x) = \|\phi\|_{L^p(X)} \|\psi\|_{L^{\frac{p}{p-1}}(X)}.$$

Par exemple, si ϕ est positive alors on peut choisir $\psi(x) = \phi(x)^{p-1}$. Le défaut d'interpolation permet de formuler des propriétés analogues si ϕ appartient à deux espaces de Lebesgue $L^{p_1}(X)$ et $L^{p_2}(X)$.

Proposition 2.37. *Fixons p_1 et p_2 appartenant à $]1, +\infty[$ et vérifiant $p_1 < p_2$ et $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$. Il existe un réel $r = r(p_1, p_2) > 1$ tel que pour toute fonction non nulle $\phi \in L^{p_1}(X) \cap L^{p_2}(X)$ on peut trouver une fonction $\psi \in L^{q_2}(X) \cap L^{q_1}(X)$, avec $q_2 = \frac{p_2}{p_2-1}$ et $q_1 = \frac{p_1}{p_1-1}$, de sorte que l'on a*

$$\int_X \phi(x)\psi(x)\, d\mu(x) = 1,$$

$$\|\phi\|_{L^{p_1}(X)} \|\psi\|_{L^{q_1}(X)} \leq Q(\phi, [p_1, p_2])^r, \quad \|\phi\|_{L^{p_2}(X)} \|\psi\|_{L^{q_2}(X)} \leq Q(\phi, [p_1, p_2])^r,$$

$$\frac{|\psi|^{q_1}}{\int_X |\psi(x)|^{q_1}\, d\mu(x)} \leq Q(\phi, [p_1, p_2])^{(r-1)q_1} \left[\frac{|\phi|^{p_1}}{\int_X |\phi(x)|^{p_1}\, d\mu(x)} + \frac{|\phi|^{p_2}}{\int_X |\phi(x)|^{p_2}\, d\mu(x)} \right],$$

$$\frac{|\psi|^{q_2}}{\int_X |\psi(x)|^{q_2}\, d\mu(x)} \leq Q(\phi, [p_1, p_2])^{(r-1)q_2} \left[\frac{|\phi|^{p_1}}{\int_X |\phi(x)|^{p_1}\, d\mu(x)} + \frac{|\phi|^{p_2}}{\int_X |\phi(x)|^{p_2}\, d\mu(x)} \right].$$

Démonstration. Commençons par le calcul élémentaire suivant :

$$p_2q_2 - p_1q_1 = \frac{p_2^2}{p_2-1} - \frac{p_1^2}{p_1-1} = \frac{p_2^2(p_1-1) - p_1^2(p_2-1)}{(p_2-1)(p_1-1)} = \frac{(p_2-p_1)(p_1p_2-p_1-p_2)}{(p_2-1)(p_1-1)}.$$

Par conséquent on a $p_2q_2 \geq p_1q_1$ et l'on peut choisir un réel $r \in \left[1+\frac{p_1}{q_2}, 1+\frac{p_2}{q_1}\right]$. En particulier, on a $r \in [p_1, p_2]$ et

$$p_1 \leq (r-1)q_2 \leq (r-1)q_1 \leq p_2. \tag{78}$$

On peut maintenant considérer la fonction ψ définie par

$$\forall x \in X, \quad \psi(x) = \begin{cases} \dfrac{1}{\int_X |\phi(x')|^r \, d\mu(x')} \times \dfrac{|\phi(x)|^r}{\phi(x)} & \text{si } \phi(x) \neq 0, \\ 0 & \text{si } \phi(x) = 0. \end{cases}$$

La définition précédente est licite car, d'après (76), la fonction ϕ appartient à $L^p(X)$ pour tout $p \in [p_1, p_2]$. Ensuite, il est clair que la fonction $\phi\psi$ est positive et d'intégrale égale à 1. Les inégalités (78) impliquent que ψ appartient à $L^{q_2}(X) \cap L^{q_1}(X)$. Grâce à (76) et (77), il vient

$$\|\phi\|_{L^{p_1}(X)} \|\psi\|_{L^{q_1}(X)} = \frac{\|\phi\|_{L^{p_1}(X)} \|\phi\|^{r-1}_{L^{(r-1)q_1}(X)}}{\|\phi\|^r_{L^r(X)}} \leq Q(\phi, [p_1, p_2])^r \|\phi\|^{\alpha}_{L^{p_1}(X)} \|\phi\|^{\beta}_{L^{p_2}(X)}, \tag{79}$$

où l'on a noté

$$\alpha := 1 + (r-1)\theta_1(p_1, (r-1)q_1, p_2) - r\theta_1(p_1, r, p_2),$$
$$\beta := (r-1)\theta_2(p_1, (r-1)q_1, p_2) - r\theta_2(p_1, r, p_2).$$

L'égalité $\theta_1 + \theta_2 = 1$ implique que l'on a $\alpha + \beta = 0$. En fait, il s'avère que $\alpha = \beta = 0$. Cela peut se voir par calcul, mais puisque α ne dépend que de (p_1, p_2, r), il suffit de traiter le cas particulier $X = \mathbb{R}$ muni de la mesure de Lebesgue. Pour tout $t > 0$, si l'on pose $\phi_t = \mathbf{1}_{[0,t]}$ alors on a $\psi_t = \frac{1}{t}\mathbf{1}_{[0,t]}$ et $\|\phi_t\|_{L^p} = t^{\frac{1}{p}}$ pour tout $p \in [1, +\infty[$ et donc

$$\|\phi_t\|_{L^{p_1}(X)} \|\psi_t\|_{L^{q_1}(X)} = t^{\frac{1}{p_1}+\frac{1}{q_1}-1} = 1 \quad \text{et} \quad Q(\phi_t, [p_1, p_2]) = 1.$$

L'inégalité (79) devient

$$\forall t > 0, \quad 1 \leq t^{\alpha\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}.$$

Cela force les égalités $\alpha = \beta = 0$. Un argument similaire permet d'estimer $\|\phi\|_{L^{p_2}(X)} \|\psi\|_{L^{q_2}(X)}$.

Passons aux estimations de $|\psi|^q$ avec $q \in \{q_1, q_2\}$. On se permet de noter $\theta_1 = \theta_1(p_1, (r-1)q, p_2)$ et $\theta_2 = \theta_2(p_1, (r-1)q, p_2)$. On obtient alors pour tout $x \in X$

$$\frac{|\psi(x)|^q}{\|\psi\|^q_{L^q(X)}} = \frac{|\phi(x)|^{(r-1)q}}{\|\phi\|^{(r-1)q}_{L^{(r-1)q}(X)}} \leq Q(\phi, [p_1, p_2])^{(r-1)q} \frac{|\phi(x)|^{(r-1)q}}{\|\phi\|^{(r-1)q\theta_1}_{L^{p_1}(X)} \|\phi\|^{(r-1)q\theta_2}_{L^{p_2}(X)}}.$$

Les égalités $(r-1)q\theta_1 + (r-1)q\theta_2 = (r-1)q$ et $\frac{(r-1)q\theta_1}{p_1} + \frac{(r-1)q\theta_2}{p_2} = 1$ nous donnent la conclusion

$$\frac{|\psi(x)|^q}{\|\psi\|^q_{L^q(X)}} \leq Q(\phi, [p_1, p_2])^{(r-1)q} \left(\frac{|\phi(x)|^{p_1}}{\|\phi\|^{p_1}_{L^{p_1}(X)}}\right)^{\frac{(r-1)q\theta_1}{p_1}} \left(\frac{|\phi(x)|^{p_2}}{\|\phi\|^{p_2}_{L^{p_2}(X)}}\right)^{\frac{(r-1)q\theta_2}{p_2}}$$

$$\le Q(\phi,[p_1,p_2])^{(r-1)q}\bigg(\frac{(r-1)q\theta_1}{p_1}\frac{|\phi(x)|^{p_1}}{\|\phi\|^{p_1}_{L^{p_1}(X)}}+\frac{(r-1)q\theta_2}{p_2}\frac{|\phi(x)|^{p_2}}{\|\phi\|^{p_2}_{L^{p_2}(X)}}\bigg)$$

$$\le Q(\phi,[p_1,p_2])^{(r-1)q}\bigg(\frac{|\phi(x)|^{p_1}}{\|\phi\|^{p_1}_{L^{p_1}(X)}}+\frac{|\phi(x)|^{p_2}}{\|\phi\|^{p_2}_{L^{p_2}(X)}}\bigg). \qquad \square$$

2J. ***Preuve du théorème 2.5 : bornitude de $S_p R_{p,\psi}$.*** Nous avons maintenant les moyens d'achever la preuve, expliquée dans la partie 2E, du théorème 2.5. Sous les hypothèses (35) et (36), il s'agit de justifier l'existence d'une suite $(\psi_n)_{n\in\mathbb{N}}$ de $L^{p_2/(p_2-1)}(X)\cap L^{p_1/(p_1-1)}(X)$ telle que

$$\forall n\in\mathbb{N},\quad \int_X \frac{\sqrt{e_n(x)}}{\sqrt{d_n}}\psi_n(x)\,d\mu(x)=1, \tag{80}$$

et que l'opérateur $S_p R_{p,\psi}$ soit borné sur $L^p_x\big(X,\bigoplus E_n\big)$ (voir (64)). L'hypothèse (36) et le lemme 2.36 nous apprennent que la suite des défauts d'interpolation $(Q(\sqrt{e_n},[p_1,p_2]))_{n\in\mathbb{N}}$ est bornée. Par homogénéité, on a aussi

$$\sup_{n\in\mathbb{N}} Q\bigg(\frac{\sqrt{e_n}}{\sqrt{d_n}},[p_1,p_2]\bigg)<+\infty.$$

Par suite, la proposition 2.37 nous assure l'existence d'une constante $K>0$ et d'une suite de fonctions $(\psi_n)_{n\in\mathbb{N}}$ de $L^{p_2/(p_2-1)}(X)\cap L^{p_1/(p_1-1)}(X)$ qui vérifient (80) et les estimations suivantes uniformément en n :

$$\bigg\|\frac{\sqrt{e_n}}{\sqrt{d_n}}\bigg\|_{L^{p_1}(X)}\|\psi_n\|_{L^{q_1}(X)}\le K,\qquad \bigg\|\frac{\sqrt{e_n}}{\sqrt{d_n}}\bigg\|_{L^{p_2}(X)}\|\psi_n\|_{L^{q_2}(X)}\le K,$$

ainsi que les inégalités suivantes pour tout $x\in X$

$$\bigg(\frac{|\psi_n(x)|}{\|\psi_n\|_{L^{q_1}(X)}}\bigg)^{q_1}+\bigg(\frac{|\psi_n(x)|}{\|\psi_n\|_{L^{q_2}(X)}}\bigg)^{q_2}\le K\bigg[\bigg(\frac{\sqrt{e_n(x)}}{\|\sqrt{e_n}\|_{L^{p_1}(X)}}\bigg)^{p_1}+\bigg(\frac{\sqrt{e_n(x)}}{\|\sqrt{e_n}\|_{L^{p_2}(X)}}\bigg)^{p_2}\bigg]. \tag{81}$$

On va appliquer le lemme 2.34 avec les noyaux K_n définis par

$$\forall n\in\mathbb{N},\quad \forall (x,x')\in X^2,\quad K_n(x,x'):=\frac{\sqrt{e_n(x)}}{\sqrt{d_n}}\psi_n(x').$$

Les propriétés (72) et (73) se traitent comme pour la bornitude de $S_p R_p$. Par exemple, l'inclusion continue $L^{q_1}(X)\subset L^{q_1,\infty}(X)$ nous permet d'écrire pour tout $x\in X$

$$\|K_n(x,x')\|_{L^{q_1,\infty}_{x'}(X)}\le\frac{\sqrt{e_n(x)}}{\sqrt{d_n}}\|\psi_n\|_{L^{q_1}(X)}\le K\frac{\sqrt{e_n(x)}}{\|\sqrt{e_n}\|_{L^{p_1}(X)}}.$$

L'hypothèse (35) implique alors (72). On montre de même (73). L'intérêt de la proposition 2.37 apparaît pour démontrer (74) et (75). En effet, on a

$$\forall x'\in X,\quad \|K_n(x,x')\|_{L^{p_1,\infty}_x(X)}\le|\psi_n(x')|\bigg\|\frac{\sqrt{e_n}}{\sqrt{d_n}}\bigg\|_{L^{p_1}(X)}\le K\frac{|\psi_n(x')|}{\|\psi_n\|_{L^{q_1}(X)}}.$$

D'après (35) et (81), on obtient

$$\left(\sup_{n\in\mathbb{N}} \|K_n(x,x')\|_{L_x^{p_1,\infty}(X)}\right)^{q_1} \in L_{x'}^{1,\infty}(X),$$

c'est-à-dire

$$\sup_{n\in\mathbb{N}} \|K_n(x,x')\|_{L_x^{p_1,\infty}(X)} \in L_{x'}^{q_1,\infty}(X).$$

On a ainsi obtenu (74). Et (75) se traite de même. On a validé toutes les hypothèses du lemme 2.34 et l'on peut donc conclure que $S_p R_{p,\psi}$ est borné sur $L_x^p\big(X, \bigoplus E_n\big)$. Cela achève la preuve du théorème 2.5 d'interpolation.

3. Espaces $\mathbf{PL}^p$ pour les harmoniques sphériques

3A. ***Reformulation des énoncés.*** On reprend les notations de la partie 1B. Commençons par remarquer que l'on peut naturellement identifier une somme d'une série $\sum a_n Z_n$ comme une distribution sur $\mathbb{S}^d$ dès lors que $(a_n)_{n\geq 1}$ est à croissance polynomiale. En effet, pour toute fonction test $\psi \in \mathcal{C}^\infty(\mathbb{S}^d)$ on a pour un paramètre $\varsigma \gg 1$:

$$\sum_{n\geq 1} |a_n \langle Z_n, \psi\rangle| \leq \sqrt{\sum_{n\geq 1} |a_n|^2 n^{-2\varsigma}} \sqrt{\sum_{n\geq 1} n^{2\varsigma} |\langle Z_n, \psi\rangle|^2}.$$

La série $\sum_{n\geq 1} |a_n|^2 n^{-2\varsigma}$ converge trivialement. Il en est de même de la seconde en utilisant la relation $(I-\Delta)^{\varsigma/2} Z_n = (1+n(n+d-1))^{\varsigma/2} Z_n$ et en faisant intervenir les espaces de Sobolev :

$$\begin{aligned}\sum_{n\geq 1} n^{2\varsigma} |\langle Z_n, \psi\rangle|^2 &\lesssim_{\varsigma,d} \sum_{n\geq 1} |\langle (I-\Delta)^{\varsigma/2} Z_n, \psi\rangle|^2 \\ &\lesssim_{\varsigma,d} \sum_{n\geq 1} |\langle Z_n, (I-\Delta)^{\varsigma/2}\psi\rangle|^2 \\ &\lesssim_{\varsigma,d} \|(I-\Delta)^{\varsigma/2}\psi\|_{L^2(\mathbb{S}^d)} := \|\psi\|_{H^\varsigma(\mathbb{S}^d)} < +\infty.\end{aligned}$$

On peut donc naturellement identifier la suite $(a_n Z_n)$ à la distribution $\sum_{n\geq 1} a_n Z_n$. Ainsi, l'espace $\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C} Z_n\big)$ sera vu comme un espace de distributions sur $\mathbb{S}^d$. Une injection de Sobolev est une inclusion de la forme $H^s \subset L^p$, il est donc légitime de définir une injection de Sobolev probabiliste comme une inclusion de la forme $H^s \subset \mathbf{PL}^p$. Nous pouvons maintenant énoncer le résultat suivant (qui implique le théorème 1.1).

Proposition 3.1. *On considère une suite complexe $(a_n)_{n\geq 1}$ à croissance polynomiale. Pour tout réel $p \in \left]\frac{2d}{d-1}, +\infty\right[$, la distribution $\sum_{n\geq 1} a_n Z_n$ appartient à $\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C} Z_n\big)$ si et seulement si l'on a*

$$\sum_{n\geq 1} \frac{1}{n^{d+1}} \left(\sum_{k=1}^{n} k^{d-1} |a_k|^2\right)^{\frac{p}{2}} < +\infty.$$

Les espaces $\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C} Z_n\big)$ sont stables par interpolation réelle et complexe pour p parcourant $\left]\frac{2d}{d-1}, +\infty\right[$ au sens du théorème 2.5. Enfin, les injections de Sobolev probabilistes des fonctions Z_n sont *données par les inclusions*

$$H_{\mathrm{zon}}^{\frac{d-1}{2}-\frac{d}{p}}(\mathbb{S}^d) \subset \mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C} Z_n\big) \subset \bigcap_{\varepsilon>0} H_{\mathrm{zon}}^{\frac{d-1}{2}-\frac{d}{p}-\varepsilon}(\mathbb{S}^d), \tag{82}$$

où l'on note

$$\forall s \in \mathbb{R}, \quad H_{\mathrm{zon}}^{s}(\mathbb{S}^d) := \left\{ \sum_{n\geq 1} a_n Z_n \;\middle|\; \sum_{n\geq 1} n^{2s} |a_n|^2 < +\infty \right\} \subset H^s(\mathbb{S}^d).$$

De même, le théorème 1.2 découle du résultat suivant.

Proposition 3.2. *On considère une suite complexe $(a_n)_{n\geq 1}$ à croissance polynomiale. Pour tout réel $p \in \,]1, +\infty[$, la distribution $\sum_{n\geq 1} a_n Y_n$ appartient à $\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C} Y_n\big)$ si et seulement si l'on a la condition*

$$\sum_{n\geq 1} \frac{1}{n^{\frac{d+1}{2}}} \left(\sum_{k=1}^{n} k^{\frac{d-1}{2}} |a_k|^2 \right)^{\frac{p}{2}} < +\infty. \tag{83}$$

En outre, les espaces $\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C} Y_n\big)$ sont stables par dualité et interpolation réelle et complexe pour p parcourant $]1, +\infty[$ au sens des théorèmes 2.5 et 2.6.

Par dualité, on va obtenir gratuitement la moitié des injections de Sobolev probabilistes des fonctions Y_n. Par commodité, on note

$$\forall s \in \mathbb{R}, \quad \widetilde{H}^s(\mathbb{S}^d) := \left\{ \sum_{n\geq 0} a_n Y_n \;\middle|\; \sum_{n\geq 1} n^{2s} |a_n|^2 < +\infty \right\} \subset H^s(\mathbb{S}^d).$$

Corollaire 3.3. *Considérons $p \in \,]2, +\infty[$ et $q = \frac{p}{p-1} \in \,]1, 2[$. Nous avons les inclusions*

$$\widetilde{H}^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})}(\mathbb{S}^d) \subset \mathbf{PL}^p(\mathbb{S}^d, \oplus \mathbb{C} Y_n) \subset \bigcap_{\varepsilon>0} \widetilde{H}^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})-\varepsilon}(\mathbb{S}^d),$$

$$\bigcup_{\varepsilon>0} \widetilde{H}^{\frac{-(d-1)}{2}(\frac{1}{q}-\frac{1}{2})+\varepsilon}(\mathbb{S}^d) \subset \mathbf{PL}^q(\mathbb{S}^d, \oplus \mathbb{C} Y_n) \subset \widetilde{H}^{\frac{-(d-1)}{2}(\frac{1}{q}-\frac{1}{2})}(\mathbb{S}^d).$$

Pour conclure cette partie, remarquons que (82) et le corollaire 3.3 montrent l'optimalité de l'exposant $\delta(d, p)$ dans l'injection de Sobolev probabiliste de Tzvetkov (3). Comme expliqué dans l'introduction, cela est lié au fait que les fonctions Y_n et Z_n optimisent par leur concentration les normes dans $L^p(\mathbb{S}^d)$.

3B. ***Preuve de la proposition 3.2, randomisation des fonctions Y_n.*** Nous commençons par traiter les fonctions Y_n car les idées d'interpolation sont plus simples. Rappelons en quel sens la fonction Y_n se concentre de façon gaussienne sur une bande de largeur $\frac{1}{\sqrt{n}}$ autour de la géodésique $\{x_1^2 + x_2^2 = 1\}$. On part des inégalités

$$\forall \delta \in \left[0, \frac{\pi}{2}\right], \quad 1 - \frac{\delta^2}{2} \leq \cos(\delta) \leq e^{-\frac{1}{2}\delta^2}.$$

Le nombre

$$\delta = \mathcal{A}(x_1, x_2) := \arccos\Big(\sqrt{x_1^2 + x_2^2}\Big)$$

désigne la distance géodésique d'un point $x \in \mathbb{S}^d$ au cercle $\{x_1^2+x_2^2=1\} \subset \mathbb{S}^d$, puis la définition $Y_n(x)=c_{d,n}(x_1+ix_2)^n$ et l'équivalent $c_{d,n} \simeq_d n^{\frac{d-1}{4}}$ assurent qu'il existe $C(d)>1$ de sorte que

$$\forall x \in \mathbb{S}^d, \quad \forall n \in \mathbb{N}^\star, \quad \frac{1}{C(d)} \mathbf{1}_{\{\mathcal{A}(x_1,x_2)\le 1/\sqrt{n}\}} \le \frac{|Y_n(x)|}{n^{\frac{d-1}{4}}} \le C(d) e^{-\frac{1}{2}n\mathcal{A}(x_1,x_2)^2}. \tag{84}$$

L'idée que l'on doit garder à l'esprit est que l'on peut sous certaines conditions assimiler $|Y_n|$ à la fonction $\widetilde{Y}_n$ définie comme suit

$$\widetilde{Y}_n(x) := n^{\frac{d-1}{4}} \mathbf{1}_{\{\mathcal{A}(x_1,x_2)\le 1/\sqrt{n}\}}.$$

Pour tout $p>1$, on peut alors vérifier les équivalences

$$\|\widetilde{Y}_n\|_{L^p(\mathbb{S}^d)} \simeq_{d,p} \|Y_n\|_{L^p(\mathbb{S}^d)} \simeq_{d,p} n^{\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})}, \tag{85}$$

ce qui signifie que la concentration autour de la géodésique $\{x_1^2+x_2^2=1\}$ est significative dans $L^p(\mathbb{S}^d)$.

Venons-en maintenant aux estimées multilinéaires des fonctions Y_n qui font *disparaître* la plus grande fréquence. Le résultat suivant énonce que les estimées multilinéaires des fonctions Y_n et $\widetilde{Y}_n$ sont équivalentes.

Lemme 3.4. *Pour tout entier $\alpha \ge 2$ et pour tous entiers naturels $n_1 \ge \cdots \ge n_\alpha \ge 1$, on a*

$$\int_{\mathbb{S}^d} |Y_{n_1}(x)\cdots Y_{n_\alpha}(x)|^2 \, d\mu_d(x) \simeq_{d,\alpha} (n_2\cdots n_\alpha)^{\frac{d-1}{2}},$$
$$\int_{\mathbb{S}^d} |\widetilde{Y}_{n_1}(x)\cdots \widetilde{Y}_{n_\alpha}(x)|^2 \, d\mu_d(x) \simeq_{d,\alpha} (n_2\cdots n_\alpha)^{\frac{d-1}{2}}.$$

Démonstration. On va invoquer l'argument algébrique de [Burq et al. 2005, pages 5–8], on a pour tout $x \in \mathbb{S}^d$:

$$Y_{n_1}(x)\cdots Y_{n_\alpha}(x) = c_{d,n_1}\cdots c_{d,n_\alpha}(x_1+ix_2)^{n_1+\cdots+n_\alpha} = \frac{c_{d,n_1}\cdots c_{d,n_\alpha}}{c_{d,n_1+\cdots+n_\alpha}} Y_{n_1+\cdots+n_\alpha}(x).$$

Et donc

$$\int_{\mathbb{S}^d} |Y_{n_1}(x)\dots Y_{n_\alpha}(x)|^2 \, d\mu_d(x) = \left(\frac{c_{d,n_1}\dots c_{d,n_\alpha}}{c_{d,n_1+\cdots+n_\alpha}}\right)^2 \simeq_{d,\alpha} \left(\frac{n_1\dots n_\alpha}{n_1+\cdots+n_\alpha}\right)^{\frac{d-1}{2}}.$$

On conclut en invoquant les inégalités $n_1 \le n_1+\cdots+n_\alpha \le \alpha n_1$.

Les intégrales multilinéaires des fonctions $\widetilde{Y}_n$ sont faciles à calculer l'aide d'une formule de changement de variables (voir plus loin (89)) en tenant compte que n_1 est le plus grand entier parmi $n_2,\dots,n_\alpha$:

$$\begin{aligned}
\frac{\int_{\mathbb{S}^d} |\widetilde{Y}_{n_1}(x)\dots \widetilde{Y}_{n_\alpha}(x)|^2 \, d\mu_d(x)}{(n_1\cdots n_\alpha)^{\frac{d-1}{2}}} &= \int_{\mathbb{S}^d} \mathbf{1}_{\{\mathcal{A}(x_1,x_2)\le 1/\sqrt{n_1}\}} \, d\mu_d(x) \\
&= \mu_{d-2}(\mathbb{S}^{d-2}) \int_{x_1^2+x_2^2<1} (1-x_1^2-x_2^2)^{\frac{d-3}{2}} \mathbf{1}_{\{\mathcal{A}(x_1,x_2)\le 1/\sqrt{n_1}\}} \, dx_1\, dx_2 \\
&= \mu_{d-2}(\mathbb{S}^{d-2}) \int_{\cos(1/\sqrt{n_1})}^1 (1-r^2)^{\frac{d-3}{2}} r\, dr
\end{aligned}$$

$$= \mu_{d-2}(\mathbb{S}^{d-2})\int_0^{1/\sqrt{n_1}} \sin^{d-2}(u)\cos(u)\,du$$
$$\simeq_d 1/n_1^{\frac{d-1}{2}}, \tag{86}$$

qui donne le résultat. □

Proposition 3.5. *Pour tout réel $p \in [1, +\infty[$ et pour toute suite complexe $(a_n)_{n\geq 1}$, on a*

$$\Big\| \sqrt{\sum_{n\geq 1} |a_n \widetilde{Y}_n|^2} \Big\|_{L^p(\mathbb{S}^d)} \simeq_{d,p} \Big[\sum_{n\geq 1} \frac{1}{n^{\frac{d+1}{2}}} \Big(\sum_{k=1}^n k^{\frac{d-1}{2}} |a_k|^2 \Big)^{\frac{p}{2}} \Big]^{\frac{1}{p}}.$$

Démonstration. Pour tout $x \in \mathbb{S}^d$, il s'agit de décomposer en supports disjoints

$$\sum_{n\geq 1} |a_n \widetilde{Y}_n(x)|^2 = \sum_{n\geq 1} n^{\frac{d-1}{2}} |a_n|^2 \mathbf{1}_{\{\mathcal{A}(x_1,x_2)\leq \frac{1}{\sqrt{n}}\}} = \sum_{n\geq 1} \Big(\sum_{k=1}^n k^{\frac{d-1}{2}} |a_k|^2 \Big) \mathbf{1}_{\{\frac{1}{\sqrt{n+1}} < \mathcal{A}(x_1,x_2) \leq \frac{1}{\sqrt{n}}\}}.$$

En utilisant une expression de la forme $\sin^{d-2}(\xi)\cos(\xi) = \xi^{d-2} G(\xi^2)$ avec G fonction holomorphe, on peut assurer l'existence d'un réel $c \in \mathbb{R}$ qui précise la formule (86) :

$$\int_{\mathbb{S}^d} \mathbf{1}_{\{\mathcal{A}(x_1,x_2)\leq 1/\sqrt{n}\}}\, d\mu_d(x) = \frac{\mu_{d-2}(\mathbb{S}^{d-2})}{n^{\frac{d-1}{2}}} \Big(1 + \frac{c}{n} + \mathcal{O}\Big(\frac{1}{n^2}\Big) \Big).$$

On conclut alors aisément. □

Le résultat suivant dit précisément que l'approximation de $|Y_n|$ par $\widetilde{Y}_n$ est légitime dans la théorie L^p probabiliste des modes propres Y_n.

Proposition 3.6. *Pour tout réel $p \in]1, +\infty[$ et pour toute suite complexe $(a_n)_{n\geq 1}$, on a*

$$\frac{1}{C(p,d)} \Big\| \sqrt{\sum_{n\geq 1} |a_n \widetilde{Y}_n|^2} \Big\|_{L^p(\mathbb{S}^d)} \leq \Big\| \sqrt{\sum_{n\geq 1} |a_n Y_n|^2} \Big\|_{L^p(\mathbb{S}^d)} \leq C(p,d) \Big\| \sqrt{\sum_{n\geq 1} |a_n \widetilde{Y}_n|^2} \Big\|_{L^p(\mathbb{S}^d)}. \tag{87}$$

Démonstration. Si $p = 2\alpha$ est un entier pair non nul, (87) découle du lemme 3.4 et des deux formules

$$\Big\| \sqrt{\sum_{n\geq 1} |a_n Y_n|^2} \Big\|_{L^p(\mathbb{S}^d)}^p = \sum_{n_1,\ldots,n_\alpha} |a_{n_1} \ldots a_{n_\alpha}|^2 \int_{\mathbb{S}^d} |Y_{n_1}(x) \ldots Y_{n_\alpha}(x)|^2\, d\mu_d(x),$$
$$\Big\| \sqrt{\sum_{n\geq 1} |a_n \widetilde{Y}_n|^2} \Big\|_{L^p(\mathbb{S}^d)}^p = \sum_{n_1,\ldots,n_\alpha} |a_{n_1} \ldots a_{n_\alpha}|^2 \int_{\mathbb{S}^d} |\widetilde{Y}_{n_1}(x) \ldots \widetilde{Y}_{n_\alpha}(x)|^2\, d\mu_d(x).$$

Pour assurer que l'équivalence (87) est encore valide pour tout $p \geq 2$, il nous suffit de prouver que les espaces $\mathbf{PL}^p(\mathbb{S}^d, \oplus\mathbb{C}Y_n)$ et $\mathbf{PL}^p(\mathbb{S}^d, \oplus\mathbb{C}\widetilde{Y}_n)$ sont stables par interpolation au sens du théorème 2.5. Pour conclure, il suffira d'invoquer un argument de dualité (grâce au théorème 2.6). En effet, pour la dualité on aurait pour $q = \frac{p}{p-1}$:

$$\Big\| \sqrt{\sum_{n\geq 1} |a_n \widetilde{Y}_n|^2} \Big\|_{L^q(\mathbb{S}^d)} \simeq_{d,p} \sup_{(b_n)_{n\geq 1}\neq 0} \frac{\big| \sum_{n\geq 1} \langle a_n \widetilde{Y}_n, b_n \widetilde{Y}_n \rangle \big|}{\Big\| \sqrt{\sum_{n\geq 1} |b_n \widetilde{Y}_n|^2} \Big\|_{L^p(\mathbb{S}^d)}}.$$

Quitte à remplacer b_n par $b_n/\|\widetilde{Y}_n\|^2_{L^2(\mathbb{S}^d)}$ et en remarquant l'équivalence $\|\widetilde{Y}_n\|_{L^2(\mathbb{S}^d)} \simeq_d 1$ (voir (85)), l'expression précédente devient

$$\begin{aligned}\Big\|\sqrt{\sum_{n\geq 1}|a_n\widetilde{Y}_n|^2}\Big\|_{L^q(\mathbb{S}^d)} &\simeq_{d,p} \sup_{(b_n)_{n\geq 1}\neq 0}\frac{\big|\sum_{n\geq 1}a_nb_n\big|}{\Big\|\sqrt{\sum_{n\geq 1}|b_n\widetilde{Y}_n|^2}\Big\|_{L^p(\mathbb{S}^d)}}\\ &\simeq_{d,p} \sup_{(b_n)_{n\geq 1}\neq 0}\frac{\big|\sum_{n\geq 1}a_nb_n\big|}{\Big\|\sqrt{\sum_{n\geq 1}|b_nY_n|^2}\Big\|_{L^p(\mathbb{S}^d)}}\\ &\simeq_{d,p} \Big\|\sqrt{\sum_{n\geq 1}|a_nY_n|^2}\Big\|_{L^q(\mathbb{S}^d)}.\end{aligned}$$

Validons maintenant les hypothèses des théorèmes d'interpolation et dualité. En examinant (85) et en se rappelant que $\frac{1}{2}-\frac{1}{p}$ change de signe en remplaçant p par son exposant conjugué $\frac{p}{p-1}$, on comprend que l'hypothèse (37) est vérifiée avec $\sqrt{e_n(x)} = |Y_n(x)|$ et $\sqrt{e_n(x)} = \|\widetilde{Y}_n\|^{-1}_{L^2(\mathbb{S}^d)}\widetilde{Y}_n(x)$.

Quant à l'hypothèse (35), elle va s'avérer être une conséquence de la concentration gaussienne des fonctions Y_n. En effet, grâce à (84) et (85), nous avons pour tout $n \in \mathbb{N}^\star$ et $x \in \mathbb{S}^d$,

$$\frac{|Y_n(x)|}{\|Y_n\|_{L^p(\mathbb{S}^d)}} \leq C(d,p)n^{\frac{d-1}{2p}}e^{-\frac{n}{2}\mathcal{A}(x_1,x_2)^2} \leq \frac{C(d,p)}{\mathcal{A}(x_1,x_2)^{\frac{d-1}{p}}}.$$

Et il se trouve que $x \mapsto \mathcal{A}(x_1,x_2)^{-\frac{(d-1)}{p}}$ appartient à l'espace de Lorentz $L^{p,\infty}(\mathbb{S}^d)$, ou ce qui revient au même que $x \mapsto \mathcal{A}(x_1,x_2)^{-(d-1)}$ appartient à $L^{1,\infty}(\mathbb{S}^d)$: la formule de changement de variables (89) donne pour tout $t > 1$

$$\begin{aligned}\mu_d\big\{x \in \mathbb{S}^d \mid \mathcal{A}(x_1,x_2)^{-(d-1)} > t\big\} &\simeq_d \int_{x_1^2+x_2^2<1}(1-x_1^2-x_2^2)^{\frac{d-3}{2}}\mathbf{1}_{\{\mathcal{A}(x_1,x_2)^{-(d-1)}>t\}}\,dx_1\,dx_2\\ &\simeq_d \int_{\cos(t^{-1/(d-1)})}^1(1-r^2)^{\frac{d-3}{2}}r\,dr\\ &\simeq_d \int_{\cos(t^{-1/(d-1)})}^1(1-r)^{\frac{d-3}{2}}\,dr\\ &\simeq_d [1-\cos(t^{-1/(d-1)})]^{\frac{d-1}{2}}\\ &\lesssim_d \frac{1}{t}.\end{aligned}\tag{88}$$

Comme $\mu_d(\mathbb{S}^d)$ est fini, l'estimation précédente est aussi valide si t appartient à $]0,1]$. □

Remarque 3.7. Concernant l'hypothèse (35), l'intérêt des espaces de Lorentz est désormais flagrant. En effet, en utilisant la minoration de (84), nous arrivons à

$$\forall x \in \mathbb{S}^d,\quad \sup_{n\geq 1}\frac{|Y_n(x)|^p}{\|Y_n\|^p_{L^p(\mathbb{S}^d)}} \geq C(d,p)\sup_{n\geq 1}n^{\frac{d-1}{2}}\mathbf{1}_{\{\mathcal{A}(x_1,x_2)\leq 1/\sqrt{n}\}} \geq C(d,p)\mathcal{A}(x_1,x_2)^{-(d-1)}.$$

Or (88) est une équivalence si t tend vers $+\infty$. Cela implique que $x \mapsto \mathcal{A}(x_1, x_2)^{-(d-1)}$ n'est pas intégrable sur $\mathbb{S}^d$. Il s'ensuit que $\sup_{n\geq 1} |Y_n|/\|Y_n\|_{L^p(\mathbb{S}^d)}$ n'appartient pas à $L^p(\mathbb{S}^d)$.

La proposition 3.2 est alors une conséquence des propositions 3.5 et 3.6 et des théorèmes 2.5 et 2.6. On conclut avec une formule standard de changement de variables.

Proposition 3.8. *Considérons un entier naturel $k \in [1, d]$ et une fonction intégrable $f : \mathbb{S}^d \to \mathbb{C}$ qui ne dépend que des k premières coordonnées $y = (x_1, x_2, \ldots, x_k)$. En notant $\widetilde{f}(y)$ la valeur commune des nombres $f(y, z)$ pour $(y, z) \in \mathbb{S}^d$, on a*

$$\int_{\mathbb{S}^d} f(x)\, d\mu_d(x) = \mu_{d-k}(\mathbb{S}^{d-k}) \int_{\mathbb{B}_k(0,1)} \widetilde{f}(y)(1-|y|^2)^{\frac{d-k-1}{2}}\, dy, \tag{89}$$

où l'on a noté $\mathbb{B}_k(0,1) := \{y \in \mathbb{R}^k, |y| < 1\}$. Dans le cas $k = d$, on convient que $\mu_0(\mathbb{S}^0) = 2$.

Démonstration. On donne les principales lignes la preuve du cas $k \leq d-1$. Le cas $k = d$ se traite de même. On introduit le changement de variables

$$\Psi : \mathbb{B}_k(0,1)\backslash\{0\} \times \mathbb{S}^{d-k} \to \mathbb{S}^d, \quad (y, u) \mapsto (y, (1-|y|^2)^{\frac{1}{2}} u)$$

dont la différentielle au point (y, u) est l'application linéaire

$$D_{(y,u)}\Psi : \mathbb{R}^k \times T_u\mathbb{S}^{d-k} \to T_{\Psi(y,u)}\mathbb{S}^d, \quad (\eta, w) \mapsto \left(\eta, \frac{-\langle y, \eta\rangle}{\sqrt{1-|y|^2}} u + (1-|y|^2)^{\frac{1}{2}} w\right).$$

L'injectivité de Ψ et l'inversibilité de $D_{(y,u)}\Psi$ pour tout (y, u) sont immédiates de sorte que le théorème d'inversion globale sur les variétés montre que Ψ est $\mathcal{C}^1$-difféomorphisme sur son image (qui est de mesure pleine dans $\mathbb{S}^d$). Il nous reste à étudier le transport des formes volumes. Considérons d'abord dans $\mathbb{R}^k$ une base orthonormée de la forme $\frac{y}{|y|}, \overrightarrow{\xi_2}, \ldots, \overrightarrow{\xi_k}$, puis dans $T_u\mathbb{S}^{d-k}$ une base orthonormée quelconque $\overrightarrow{\vartheta_1}, \ldots, \overrightarrow{\vartheta_{d-k}}$. Ainsi,

$$\left(\frac{y}{|y|}, 0\right), (\overrightarrow{\xi_2}, 0), \ldots, (\overrightarrow{\xi_k}, 0), (0, \overrightarrow{\vartheta_1}), \ldots, (0, \overrightarrow{\vartheta_{d-k}})$$

est une base orthonormée de $\mathbb{R}^k \times T_u\mathbb{S}^{d-k}$. Il s'avère que cette base orthonormée est envoyée sur une famille orthogonale de l'espace tangent $T_{\Psi(y,u)}\mathbb{S}^d$:

$$D_{(y,u)}\Psi\left(\frac{y}{|y|}, 0\right) = \frac{1}{\sqrt{1-|y|^2}}\left(\frac{y}{|y|}\sqrt{1-|y|^2}, -|y|u\right),$$

$$2 \leq i \leq k, \qquad D_{(y,u)}\Psi(\overrightarrow{\xi_i}, 0) = (\overrightarrow{\xi_i}, 0),$$

$$1 \leq j \leq d-k, \qquad D_{(y,u)}\Psi(0, \overrightarrow{\vartheta_j}) = \sqrt{1-|y|^2}(0, \overrightarrow{\vartheta_j}).$$

Par conséquent, $D_{(y,u)}\Psi$ multiplie localement les volumes par $(1-|y|^2)^{\frac{d-k-1}{2}}$. Le changement de variables $x=\Psi(y,u)$ nous amène à la formule

$$\begin{aligned}\int_{\mathbb{S}^d} f(x)\,d\mu_d(x) &= \int_{\mathbb{B}_k(0,1)\times\mathbb{S}^{d-k}} f\Big(y,u\sqrt{1-|y|^2}\Big)(1-|y|^2)^{\frac{d-k-1}{2}}\,dy\,d\mu_{d-k}(u)\\ &= \mu_{d-k}(\mathbb{S}^{d-k})\int_{\mathbb{B}_k(0,1)} \widetilde{f}(y)(1-|y|^2)^{\frac{d-k-1}{2}}\,dy. \qquad \square\end{aligned}$$

3C. ***Preuve du corollaire 3.3, injections de Sobolev probabilistes des fonctions Y_n.*** Nous avons vu dans la preuve de la proposition 3.6 que les espaces $\mathbf{PL}^p\big(\mathbb{S}^d,\bigoplus\mathbb{C}Y_n\big)$ sont stables par dualité. Comme il en est de même des espaces de Sobolev, on peut se restreindre au cas $p>2$.

L'inégalité triangulaire (2) et les estimations (10) donnent immédiatement l'inclusion

$$\widetilde{H}^{\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}(\mathbb{S}^d)\subset \mathbf{PL}^p\big(\mathbb{S}^d,\textstyle\bigoplus\mathbb{C}Y_n\big).$$

Montrons maintenant l'inclusion

$$\mathbf{PL}^p\big(\mathbb{S}^d,\textstyle\bigoplus\mathbb{C}Y_n\big)\subset \displaystyle\bigcap_{\varepsilon>0}\widetilde{H}^{\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)-\varepsilon}(\mathbb{S}^d).$$

Autrement dit, pour toute distribution $u=\sum_{n\geq 1}a_nY_n$, il s'agit de vérifier l'inégalité suivante

$$\|u\|_{H^{\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{\varepsilon}{2}}(\mathbb{S}^d)}\lesssim_{d,p,\varepsilon}\|u\|_{\mathbf{PL}^p(\mathbb{S}^d,\bigoplus\mathbb{C}Y_n)}.$$

On pose $S_0=0$ et $S_n=\sum_{k=1}^n k^{\frac{d-1}{2}}|a_k|^2$ pour tout $n\geq 1$. En utilisant une transformation d'Abel et une inégalité de Hölder, on a

$$\begin{aligned}\|u\|^2_{H^{\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{\varepsilon}{2}}(\mathbb{S}^d)} &= \sum_{n\geq 1} n^{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)-\varepsilon}|a_n|^2\\ &= \sum_{n\geq 1} n^{(d-1)\left(\frac{1}{2}-\frac{1}{p}\right)-\varepsilon-\left(\frac{d-1}{2}\right)}[S_n-S_{n-1}]\\ &= \sum_{n\geq 1} n^{-\frac{(d-1)}{p}-\varepsilon}[S_n-S_{n-1}]\\ &\lesssim_{d,p,\varepsilon} \lim_{N\to+\infty}\sum_{n=1}^N \frac{S_n}{n^{\frac{(d-1)}{p}+1+\varepsilon}}+\sup_{N\geq 1}\frac{S_N}{N^{\frac{(d-1)}{p}+\varepsilon}}\\ &\lesssim_{d,p,\varepsilon} \lim_{N\to+\infty}\sum_{n=1}^N \frac{S_n}{n^{\frac{(d-1)}{p}+1+\varepsilon}}+\sup_{N\geq 1}\sum_{n>N}\frac{S_n}{n^{\frac{(d-1)}{p}+1+\varepsilon}}\\ &\lesssim_{d,p,\varepsilon} \sum_{n\geq 1}\frac{S_n}{n^{\frac{(d-1)}{p}+1+\varepsilon}}\\ &\lesssim_{d,p,\varepsilon} \sum_{n\geq 1}\frac{1}{n^{1-\frac{2}{p}+\varepsilon}}\times\frac{S_n}{n^{\frac{d+1}{p}}}\lesssim_{d,p,\varepsilon}\Bigg(\sum_{n\geq 1}\frac{S_n^{\frac{p}{2}}}{n^{\frac{d+1}{2}}}\Bigg)^{\frac{2}{p}}.\end{aligned}$$

3D. ***Normes L^p des fonctions zonales.*** Parmi les modes propres de Δ, les fonctions Z_n sont connues pour maximiser la croissance des quotients $\|Z_n\|_{L^p(\mathbb{S}^d)}/\|Z_n\|_{L^2(\mathbb{S}^d)}$ si n tend vers $+\infty$ avec $p \geq 2(d+1)/(d-1)$ (et cela est même optimal d'après les inégalités de Sogge (4)). En l'occurrence, la formule de changement de variables (89) donne

$$\|Z_n\|^p_{L^p(\mathbb{S}^d)} = \int_{\mathbb{S}^d} |Z_n(x)|^p \, d\mu_d(x) = \mu_{d-1}(\mathbb{S}^{d-1}) \times n^{\frac{p}{2}} \int_{-1}^{1} \left|P_n^{\left(\frac{d-2}{2},\frac{d-2}{2}\right)}(x_1)\right|^p (1-x_1^2)^{\frac{d-2}{2}} \, dx_1,$$

puis [Szegő 1975, page 391] nous fournit les estimations des normes (6).

3E. ***Preuve de la proposition 3.1, randomisation des fonctions zonales Z_n.*** La description des espaces $\mathbf{PL}^p(\mathbb{S}^d, \bigoplus \mathbb{C} Z_n)$ est plus délicate que celle des espaces $\mathbf{PL}^p(\mathbb{S}^d, \bigoplus \mathbb{C} Y_n)$ car on ne peut pas raisonner par interpolation complexe en faisant parcourir p dans $2\mathbb{N}$. Cela dit, une fois cette description obtenue, les injections de Sobolev probabilistes (82) des fonctions Z_n se démontrent de façon rigoureusement semblable à celle des fonctions Y_n du corollaire 3.3 et l'on se permet d'en omettre la preuve. Avant d'expliquer plus en détail la difficulté rencontrée dans cette preuve par rapport à celle de la proposition 3.2, commençons par rappeler les estimations précises des polynômes de Jacobi.

Lemme 3.9. *Pour tout $\alpha > -1$, il existe des constantes $c = c(\alpha) \in \left]0, \frac{\pi}{2}\right]$ et $C(\alpha) \geq 1$ de sorte que pour tous $n \in \mathbb{N}^\star$ l'on a*

$$\Theta \in \left[0, \frac{c}{n}\right] \cup \left[\pi - \frac{c}{n}, \pi\right] \quad \Rightarrow \quad \frac{n^\alpha}{C(\alpha)} \leq |P_n^{(\alpha,\alpha)}(\cos(\Theta))| \leq C(\alpha) n^\alpha. \tag{90}$$

On note ensuite $N = n + \alpha + \frac{1}{2}$ et $\varrho = \frac{\pi}{2}\left(\alpha + \frac{1}{2}\right)$. Si Θ appartient à $[\frac{c}{n}, \pi - \frac{c}{n}]$ alors

$$P_n^{(\alpha,\alpha)}(\cos(\Theta)) = \frac{2^{\alpha+\frac{1}{2}}}{\sqrt{\pi n}(\sin\Theta)^{\alpha+\frac{1}{2}}} \left[\cos(N\Theta - \varrho) + \frac{\mathcal{O}_\alpha(1)}{n \sin(\Theta)}\right], \tag{91}$$

où le terme $\mathcal{O}_\alpha(1)$ vérifie $|\mathcal{O}_\alpha(1)| \leq C(\alpha)$.

Démonstration. Ces estimées découlent des formules (4.1.3), (4.21.7), (7.32.5) et (8.21.18) du livre [Szegő 1975]. La formule (4.1.3) nous donne $P_n^{(\alpha,\alpha)}(-x_1) = (-1)^n P_n^{(\alpha,\alpha)}(x_1)$, ce qui nous ramène au cas $\Theta \in \left[0, \frac{\pi}{2}\right]$. D'une part, on a toujours

$$P_n^{(\alpha,\alpha)}(1) = \binom{n+\alpha}{n} \geq \frac{n^\alpha}{C(\alpha)}.$$

Pour tout $x_1 \in [1 - 1/n^2, 1]$, on peut estimer grâce aux formules (4.21.7) page 63 et (7.32.5) page 169 :

$$\left|\frac{d}{dx_1} P_n^{(\alpha,\alpha)}(x_1)\right| = \tfrac{1}{2}\left|(n+2\alpha+1) P_{n-1}^{(\alpha+1,\alpha+1)}(x_1)\right| \leq C(\alpha) n^{\alpha+2}.$$

Choisissons $c(\alpha) = 1/(2C(\alpha)^2)$ de sorte que $1/C(\alpha) - c(\alpha)C(\alpha) = 1/(2C(\alpha))$. On a

$$1 - \frac{c(\alpha)}{n^2} \leq x_1 \leq 1 \quad \Rightarrow \quad P_n^{(\alpha,\alpha)}(x_1) \geq \frac{n^\alpha}{2C(\alpha)}.$$

De nouveau, d'après la formule (7.32.5) de la page 169 et quitte à augmenter $C(\alpha) > 1$, on a aussi $P_n^{(\alpha,\alpha)}(x_1) \leq C(\alpha) n^\alpha$. Cela nous donne (90). Quant à (91), c'est la formule (8.21.18) de la page 198. □

Dans la suite, on notera

$$\Theta := \arccos(x_1) \in [0, \pi]$$

la distance géodésique d'un point $x \in \mathbb{S}^d$ au pôle $(1, 0, \ldots, 0)$. D'après (5), (90), (91) avec $\alpha = \frac{d-2}{2}$, on a

$$\Theta \in \Big[0, \frac{c}{n}\Big] \cup \Big[\pi - \frac{c}{n}, \pi\Big] \quad \Rightarrow \quad \frac{n^{\frac{d-1}{2}}}{C(d)} \leq |Z_n(x)| \leq C(d) n^{\frac{d-1}{2}}, \tag{92}$$

$$\Theta \in \Big]\frac{c}{n}, \pi - \frac{c}{n}\Big[\quad \Rightarrow \quad |Z_n(x)| \leq \frac{C(d)}{\sin(\Theta)^{\frac{d-1}{2}}}. \tag{93}$$

C'est d'ailleurs avec ces estimations que l'on peut obtenir les estimations (6) des normes dans $L^p(\mathbb{S}^d)$ des fonctions Z_n. Ces dernières disent que seule la concentration au voisinage des pôles est significative dans l'échelle des espaces $L^p(\mathbb{S}^d)$, avec $p > \frac{2d}{d-1}$. Il est donc naturel de comparer Z_n à sa restriction $\widetilde{Z}_n$ au voisinage du pôle $(1, 0, \ldots, 0)$:

$$\widetilde{Z}_n(x) := \mathbf{1}_{[0, \frac{c}{n}]}(\Theta) \times Z_n(x).$$

La fonction $\widetilde{Z}_n$ se concentre sur une boule de centre $(1, 0, \ldots, 0)$ de rayon $\frac{c}{n}$ et avec une amplitude d'ordre $n^{(d-1)/2}$. La preuve de la proposition 3.2 consistait à comparer $|Y_n|$ à sa restriction $\widetilde{Y}_n$ autour d'une géodésique. Le lemme 3.4 assurait alors que les fonctions $|Y_n|$ ont les mêmes estimations multilinéaires que les fonctions $\widetilde{Y}_n$. Malheureusement, il est illusoire de refaire le même argument en approchant les fonctions Z_n par les fonctions $\widetilde{Z}_n$ par exemple pour étudier l'espace $\mathbf{PL}^6\big(\mathbb{S}^d, \bigoplus \mathbb{C} Z_n\big)$. Utilisant l'équivalent $\|Z_n\|_{L^2(\mathbb{S}^d)} \simeq_d 1$, (92) et [Burq et al. 2005, page 8], nous avons en effet pour tout entier $n \in \mathbb{N}^\star$,

$$\int_{\mathbb{S}^d} |\widetilde{Z}_1(x)|^4 |\widetilde{Z}_n(x)|^2 \, d\mu_d(x) \leq C(d) \int_{\mathbb{S}^d} |\widetilde{Z}_n(x)|^2 \, d\mu_d(x) \leq \frac{C(d)}{n} \ll \int_{\mathbb{S}^d} |Z_1(x)|^4 |Z_n(x)|^2 \, d\mu_d(x).$$

L'estimation précédente est une manifestation de la mauvaise qualité de l'approximation de Z_n par $\widetilde{Z}_n$ dans $L^2(\mathbb{S}^d)$, phénomène qui ne se produit pas pour les fonctions Y_n. On ne dispose donc pas d'un résultat analogue au lemme 3.4. Cela nous oblige à raisonner différemment.

Proposition 3.10. *Pour tout réel $p > \frac{2d}{d-1}$, il existe une constante $C(p, d) \geq 1$ telle que pour toute suite complexe $(a_n)_{n \geq 1}$ et tout $x \in \mathbb{S}^d$ vérifiant $\Theta(x) \in \big[0, \frac{\pi}{2}\big]$, on a*

$$\sqrt{\sum_{n \geq 1} |a_n Z_n(x)|^2} \leq C(p, d) \sqrt{\sum_{n \geq 1} |a_n \widetilde{Z}_n(x)|^2} + \frac{C(p, d)}{\sin^{\frac{d}{p}}(\Theta)} \bigg\| \sqrt{\sum_{n \geq 1} |a_n \widetilde{Z}_n|^2} \bigg\|_{L^p(\mathbb{S}^d)}. \tag{94}$$

Par conséquent, on a

$$\bigg\| \sqrt{\sum_{n \geq 1} |a_n Z_n|^2} \bigg\|_{L^{p,\infty}(\mathbb{S}^d)} \leq C(p, d) \bigg\| \sqrt{\sum_{n \geq 1} |a_n \widetilde{Z}_n|^2} \bigg\|_{L^p(\mathbb{S}^d)}. \tag{95}$$

Démonstration. Sans perte de généralité, on suppose que la suite (a_n) n'a qu'un nombre fini de termes non nuls. L'idée consiste essentiellement à décomposer les différentes fonctions en jeu en somme de fonctions à supports disjoints deux à deux. La formule de changement de variables (89) donne

$$\begin{aligned}\int_{\mathbb{S}^d} \mathbf{1}_{]\frac{c}{n+1},\frac{c}{n}]}(\Theta)\, d\mu_d(x) &= \mu_{d-1}(\mathbb{S}^{d-1}) \int_{-1}^{1} \mathbf{1}_{]\frac{c}{n+1},\frac{c}{n}]}(\Theta)(1-x_1^2)^{\frac{d-2}{2}}\, dx_1 \\ &= \mu_{d-1}(\mathbb{S}^{d-1}) \int_0^{\pi} \mathbf{1}_{]\frac{c}{n+1},\frac{c}{n}]}(\Theta) \sin(\Theta)^{d-1}\, d\Theta \\ &\simeq_d \frac{1}{n^{d+1}}.\end{aligned}$$

En posant $S_n = \sum_{k=1}^n k^{d-1}|a_k|^2$ pour tout $n \in \mathbb{N}^\star$, nous avons grâce à (92),

$$\sqrt{\sum_{n\geq 1} |a_n \widetilde{Z}_n(x)|^2} \simeq_d \sqrt{\sum_{n\geq 1} n^{d-1}|a_n|^2 \mathbf{1}_{[0,\frac{c}{n}]}(\Theta)} \simeq_d \sum_{n\geq 1} \sqrt{S_n}\mathbf{1}_{]\frac{c}{n+1},\frac{c}{n}]}(\Theta), \tag{96}$$

et donc

$$\left\| \sqrt{\sum_{n\geq 1} |a_n \widetilde{Z}_n|^2} \right\|_{L^p(\mathbb{S}^d)} \simeq_{d,p} \left(\sum_{n\geq 1} S_n^{\frac{p}{2}} \int_{\mathbb{S}^d} \mathbf{1}_{]\frac{c}{n+1},\frac{c}{n}]}(\Theta)\, d\mu_d(x) \right)^{\frac{1}{p}} \simeq_{d,p} \left(\sum_{n\geq 1} \frac{S_n^{\frac{p}{2}}}{n^{d+1}} \right)^{\frac{1}{p}}. \tag{97}$$

On peut maintenant estimer $Z_n(x) - \widetilde{Z}_n(x)$. En tenant compte de (93) et du fait que $\Theta(x)$ appartient à $\left[0, \frac{\pi}{2}\right]$, nous avons

$$\begin{aligned}\sqrt{\sum_{n\geq 1} |a_n|^2 |Z_n(x) - \widetilde{Z}_n(x)|^2} &\lesssim_d \left(\sum_{k\geq 1} |a_k|^2 \right)^{\frac{1}{2}} \frac{\mathbf{1}_{]c,\frac{\pi}{2}]}(\Theta)}{\sin(\Theta)^{\frac{d-1}{2}}} + \sum_{n\geq 1} \left(\sum_{k>n} |a_k|^2 \right)^{\frac{1}{2}} \frac{\mathbf{1}_{]\frac{c}{n+1},\frac{c}{n}]}(\Theta)}{\sin(\Theta)^{\frac{d-1}{2}}} \\ &\lesssim_d \underbrace{\left(\sum_{k\geq 1} |a_k|^2 \right)^{\frac{1}{2}} \mathbf{1}_{]c,\frac{\pi}{2}]}(\Theta)}_{:=A_1(\Theta)} + \underbrace{\sum_{n\geq 1} \left(n^{d-1} \sum_{k>n} |a_k|^2 \right)^{\frac{1}{2}} \mathbf{1}_{]\frac{c}{n+1},\frac{c}{n}]}(\Theta)}_{:=A_2(\Theta)}.\end{aligned}$$

On va faire quelques calculs avant d'attaquer l'estimation des termes $A_1(\Theta)$ et $A_2(\Theta)$. En convenant que $S_0 = 0$, nous pouvons effectuer une transformation d'Abel pour tout $n \in \mathbb{N}$

$$\sum_{k>n} |a_k|^2 = \sum_{k>n} \frac{k^{d-1}|a_k|^2}{k^{d-1}} = \frac{-S_n}{(n+1)^{d-1}} + \sum_{k>n} \left(\frac{1}{k^{d-1}} - \frac{1}{(k+1)^{d-1}} \right) S_k. \tag{98}$$

Puisque la suite $(a_n)_{n\geq 1}$ n'a qu'un nombre fini de termes non nuls, la suite $(S_n)_{n\geq 0}$ est bornée et la dernière série converge bien. On va exploiter l'inégalité $p > \frac{2d}{d-1}$, ou encore

$$\left(d - \frac{2(d+1)}{p} \right) \frac{p}{p-2} = d - \frac{2}{p-2} > 1.$$

L'inégalité de Hölder avec les exposants conjugués $\frac{p}{p-2}$, $\frac{p}{2}$:

$$(n+1)^{d-1} \sum_{k>n} \frac{S_k}{k^d} \leq (n+1)^{d-1} \sum_{k>n} \overbrace{\frac{1}{k^{d-\frac{2(d+1)}{p}}}}^{\in \ell^{\frac{p}{p-2}}(\mathbb{N})} \times \frac{S_k}{k^{\frac{2(d+1)}{p}}}$$

$$\begin{aligned}
&\leq (n+1)^{d-1}\Big(\sum_{k>n}\frac{1}{k^{d-\frac{2}{p-2}}}\Big)^{\frac{p-2}{p}}\Big(\sum_{k>n}\frac{S_k^{\frac{p}{2}}}{k^{d+1}}\Big)^{\frac{2}{p}}\\
&\lesssim_{d,p}\frac{(n+1)^{d-1}}{(n+1)^{\frac{p-2}{p}(d-\frac{2}{p-2}-1)}}\Big(\sum_{k>n}\frac{S_k^{\frac{p}{2}}}{k^{d+1}}\Big)^{\frac{2}{p}}\\
&\lesssim_{d,p}(n+1)^{\frac{2d}{p}}\Big(\sum_{k>n}\frac{S_k^{\frac{p}{2}}}{k^{d+1}}\Big)^{\frac{2}{p}} \qquad (99)\\
&\lesssim_{d,p}(n+1)^{\frac{2d}{p}}\Big\|\sqrt{\sum_{n\geq 1}|a_n\widetilde{Z}_n|^2}\Big\|^2_{L^p(\mathbb{S}^d)}, \qquad (100)
\end{aligned}$$

où l'on a utilisé (97). On peut alors contrôler $A_1(\Theta)$ avec (98) et (100) pour $n=0$:

$$A_1(\Theta)=\sqrt{\sum_{k\geq 1}|a_k|^2}\times \mathbf{1}_{]c,\frac{\pi}{2}[}(\Theta)\lesssim_{d,p}\Big\|\sqrt{\sum_{n\geq 1}|a_n\widetilde{Z}_n|^2}\Big\|_{L^p(\mathbb{S}^d)}\frac{1}{\sin^{d/p}(\Theta)}.$$

Pour contrôler $A_2(\Theta)$, on utilise (96), (98) et (100) :

$$\begin{aligned}
A_2(\Theta)&\lesssim_{d,p}\sum_{n\geq 1}\sqrt{S_n}\mathbf{1}_{]\frac{c}{n+1},\frac{c}{n}]}(\Theta)+\sum_{n\geq 1}n^{\frac{d}{p}}\Big\|\sqrt{\sum_{n\geq 1}|a_n\widetilde{Z}_n|^2}\Big\|_{L^p(\mathbb{S}^d)}\mathbf{1}_{]\frac{c}{n+1},\frac{c}{n}]}(\Theta)\\
&\lesssim_{d,p}\sqrt{\sum_{n\geq 1}|a_n\widetilde{Z}_n(x)|^2}+\Big\|\sqrt{\sum_{n\geq 1}|a_n\widetilde{Z}_n|^2}\Big\|_{L^p(\mathbb{S}^d)}\frac{1}{\sin^{d/p}(\Theta)}.
\end{aligned}$$

On a donc obtenu (94). Passons à l'inégalité (95). Comme les polynômes de Jacobi sont pairs ou impairs, les fonctions $x\mapsto |Z_n(x)|$ sont invariantes par la transformation $\Theta(x)\to\pi-\Theta(x)$. Se rappelant que $\|\cdot\|_{L^{p,\infty}(\mathbb{S}^d)}$ n'est pas une norme, on sait qu'il existe néanmoins une constante universelle $C\geq 1$ telle que

$$\Big\|\sqrt{\sum_{n\geq 1}|a_nZ_n|^2}\Big\|_{L^{p,\infty}(\mathbb{S}^d)}\leq C\Big\|\sqrt{\sum_{n\geq 1}|a_nZ_n|^2}\times\mathbf{1}_{[0,\frac{\pi}{2}]}(\Theta)\Big\|_{L^{p,\infty}(\mathbb{S}^d)}.$$

Pour obtenir l'estimation (94) à partir de (95), on remarque d'une part l'inégalité triviale

$$\Big\|\sqrt{\sum_{n\geq 1}|a_n\widetilde{Z}_n|^2}\Big\|_{L^{p,\infty}(\mathbb{S}^d)}\leq\Big\|\sqrt{\sum_{n\geq 1}|a_n\widetilde{Z}_n|^2}\Big\|_{L^p(\mathbb{S}^d)},$$

d'autre part que la fonction $x\mapsto\sin(\Theta(x))^{-d/p}$ appartient à $L^{p,\infty}(\mathbb{S}^d)$. Cela peut se vérifier avec la formule de changement de variable (89) mais il est plus simple de remarquer que sur un voisinage $\mathcal{V}$ du pôle $(1,0,\dots 0)\in\mathbb{S}^d$ nous avons l'équivalent $\sin(\Theta)^{-d/p}\sim\Theta^{-d/p}$ et que la mesure de $\mathbb{S}^d$ sur $\mathcal{V}$ est comparable à la mesure de Lebesgue d'un voisinage de l'origine de $\mathbb{R}^d$. □

Si l'on essaie d'estimer directement la norme dans $L^p(\mathbb{S}^d)$ de $\sqrt{\sum_{n\geq 1}|a_nZ_n|^2}$ en comparant Z_n avec $\widetilde{Z}_n$ à l'aide de (92) et (93), alors la preuve précédente montre que l'on commet une perte avec l'inégalité de Hölder. De façon précise, après application de l'inégalité de Hölder, la formule (99) et le contrôle de $\|A_2(\Theta)\|_{L^p(\mathbb{S}^d)}$ conduisent aux inégalités

$$\Big\|\sum_{n\geq 1} n^{\frac{d}{p}}\Big(\sum_{k>n}\frac{S_k^{\frac{p}{2}}}{k^{d+1}}\Big)^{\frac{1}{p}}\mathbf{1}_{]\frac{c}{n+1},\frac{c}{n}]}(\Theta)\Big\|^p_{L^p(\mathbb{S}^d)} \simeq_{d,p} \sum_{n\geq 1}\frac{n^{\frac{d}{p}p}}{n^{d+1}}\sum_{k>n}\frac{S_k^{\frac{p}{2}}}{k^{d+1}} \simeq_{d,p} \sum_{n\geq 1}\frac{1}{n}\sum_{k>n}\frac{S_k^{\frac{p}{2}}}{k^{d+1}} \simeq_{d,p} \sum_{k\geq 1}\frac{\ln(k+1)}{k^{d+1}}S_k^{\frac{p}{2}}.$$

Par comparaison avec (97), l'estimation précédente est mauvaise en raison du terme logarithmique. Un argument d'interpolation réelle bien connu va nous permettre de corriger le facteur logarithmique. C'est maintenant que le théorème 2.5 intervient via le lemme suivant.

Lemme 3.11. *La famille d'espaces de Banach $\big(\mathbf{PL}^p\big(\mathbb{S}^d,\bigoplus\mathbb{C}\widetilde{Z}_n\big)\big)_{p\in]1,+\infty[}$ est stable par interpolation réelle au sens du théorème 2.5.*

Démonstration. Il s'agit d'appliquer le théorème 2.5 avec

$$E_n=\mathbb{C}\widetilde{Z}_n \quad \text{et} \quad \sqrt{e_n(x)}=\frac{|\widetilde{Z}_n(x)|}{\|\widetilde{Z}_n\|_{L^2(\mathbb{S}^d)}}.$$

D'une part, on a le comportement asymptotique

$$\forall p>1, \quad \|\widetilde{Z}_n\|_{L^p(\mathbb{S}^d)}\simeq_{d,p} n^{\frac{d-1}{2}-\frac{d}{p}},$$

ce qui nous assure la validité de (36). Quant à l'hypothèse (35), on peut écrire d'après (92) :

$$\forall x\in\mathbb{S}^d, \quad \sup_{n\geq 1}\frac{|\widetilde{Z}_n(x)|}{\|\widetilde{Z}_n\|_{L^p(\mathbb{S}^d)}}\leq C(d,p)\sup_{n\geq 1}\frac{|\widetilde{Z}_n(x)|}{n^{\frac{d-1}{2}-\frac{d}{p}}}=C(d,p)\sup_{n\geq 1}n^{d/p}\mathbf{1}_{[0,\frac{c}{n}]}(\Theta)\leq\frac{C(d,p)}{\sin(\Theta)^{d/p}}.$$

Or l'on a remarqué à la fin de la proposition 3.10 que la fonction $x\mapsto\sin(\Theta)^{-d/p}$ appartient à $L^{p,\infty}(\mathbb{S}^d)$. □

Nous sommes en mesure d'obtenir la partie de la proposition 3.1 qui décrit $\mathbf{PL}^p\big(\mathbb{S}^d,\bigoplus\mathbb{C}Z_n\big)$.

Proposition 3.12. *Pour tout réel $p>\frac{2d}{d-1}$, et pour toute suite complexe $(a_n)_{n\geq 1}$ on a*

$$\Big\|\sqrt{\sum_{n\geq 1}|a_n\widetilde{Z}_n|^2}\Big\|_{L^p(\mathbb{S}^d)}\simeq_{d,p}\Big\|\sqrt{\sum_{n\geq 1}|a_nZ_n|^2}\Big\|_{L^p(\mathbb{S}^d)}. \tag{101}$$

En outre, on a

$$\Big\|\sqrt{\sum_{n\geq 1}|a_nZ_n|^2}\Big\|_{L^p(\mathbb{S}^d)}\simeq_{d,p}\Big[\sum_{n\geq 1}\frac{1}{n^{d+1}}\Big(\sum_{k=1}^{n}k^{d-1}|a_k|^2\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}}. \tag{102}$$

Démonstration. L'équivalence (102) découlera de (97) et (101). Montrons donc (101). L'inégalité $|\widetilde{Z}_n|\leq|Z_n|$ donne un sens de l'équivalence (101). L'autre sens équivaut à la bornitude de l'opérateur :

$$\mathbf{PL}^p\big(\mathbb{S}^d,\bigoplus\mathbb{C}\widetilde{Z}_n\big)\to L^p(\mathbb{S}^d,\ell^2(\mathbb{N}^\star)), \quad (a_n\widetilde{Z}_n)_{n\geq 1}\mapsto(a_nZ_n)_{n\geq 1}.$$

L'inégalité (95) de la proposition 3.10 montre, pour tout $p > \frac{2d}{d-1}$, la continuité de l'opérateur

$$\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C}\widetilde{Z}_n\big) \to L^{p,\infty}(\mathbb{S}^d, \ell^2(\mathbb{N}^\star)), \quad (a_n\widetilde{Z}_n)_{n\geq 1} \mapsto (a_n Z_n)_{n\geq 1}.$$

À présent, il s'agit de raisonner par interpolation réelle. Fixons deux réels p_1' et p_2' tels que

$$\frac{2d}{d-1} < p_1' < p < p_2'.$$

Considérons de plus l'unique réel $\theta' \in [0,1]$ tel que

$$\frac{1}{p} = \frac{1-\theta'}{p_1'} + \frac{\theta'}{p_2'}.$$

Par application de la méthode d'interpolation réelle $[\cdot,\cdot]_{\theta',p}$, l'opérateur suivant est borné :

$$\big[\mathbf{PL}^{p_1'}\big(\mathbb{S}^d, \bigoplus \mathbb{C}\widetilde{Z}_n\big), \mathbf{PL}^{p_2'}\big(\mathbb{S}^d, \bigoplus \mathbb{C}\widetilde{Z}_n\big)\big]_{\theta',p} \to \big[L_x^{p_1',\infty}(\mathbb{S}^d, \ell^2(\mathbb{N}^\star)), L^{p_2',\infty}(\mathbb{S}^d, \ell^2(\mathbb{N}^\star))\big]_{\theta',p}$$
$$(a_n\widetilde{Z}_n)_{n\geq 1} \mapsto (a_n Z_n)_{n\geq 1}.$$

L'espace de départ s'identifie à $\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C}\widetilde{Z}_n\big)$ d'après le lemme 3.11. Quant à l'espace d'arrivée, il s'identifie à $L_x^p(\mathbb{S}^d, \ell^2(\mathbb{N}^\star))$ d'après le théorème 2.32. Cela prouve (101). □

En apparence, on n'a pas montré l'interpolation des espaces $\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C}Z_n\big)$. La preuve de la proposition suivante montre en fait que cela est inclus dans l'inégalité (94).

Proposition 3.13. *Les espaces* $\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C}Z_n\big)$ *sont stables par interpolation réelle et complexe pour* p *parcourant* $\big]\frac{2d}{d-1}, +\infty\big[$ *au sens du théorème 2.5.*

Démonstration. On peut donner deux preuves. La proposition 3.12 montre que les espaces

$$\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C}Z_n\big) \quad \text{et} \quad \mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C}\widetilde{Z}_n\big)$$

sont isomorphes pour tout $p > \frac{2d}{d-1}$. Le lemme 3.11, qui contient aussi dans sa preuve l'interpolation complexe, donne alors la conclusion.

La seconde preuve utilise aussi la démonstration du lemme 3.11 avec l'argument additionnel suivant. D'après (94), on a pour tout $x \in \mathbb{S}^d$ vérifiant $\Theta(x) \in \big[0, \frac{\pi}{2}\big]$,

$$|Z_n(x)| \leq C(d,p)|\widetilde{Z}_n(x)| + \frac{C(d,p)}{\sin^{d/p}(\Theta)}\|\widetilde{Z}_n\|_{L^p(\mathbb{S}^d)}.$$

La symétrie $\Theta(x) \leftrightarrow \pi - \Theta(x)$ et le fait que $\sin^{-d}(\Theta)$ appartient à $L^{1,\infty}(\mathbb{S}^d)$ nous donnent

$$\sup_{n\in\mathbb{N}^\star} \frac{|Z_n(x)|}{\|Z_n\|_{L^p(\mathbb{S}^d)}} \in L_x^{p,\infty}(\mathbb{S}^d).$$

On conclut par application du théorème 2.5 pour les espaces $\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C}Z_n\big)$. □

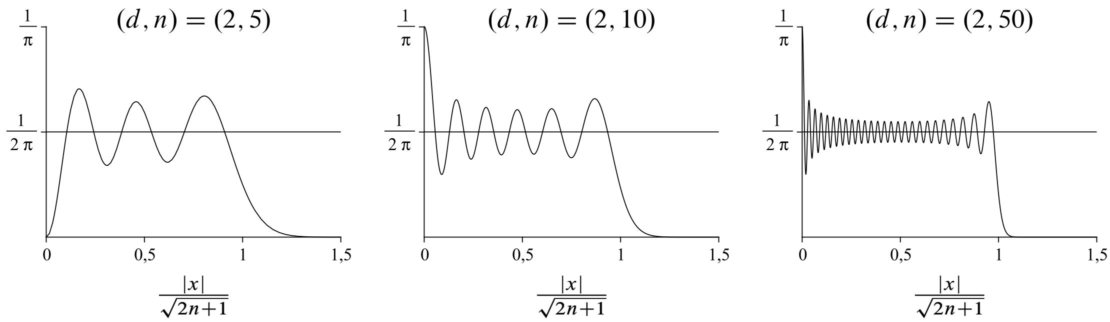


Figure 2. Graphes de la fonction $x \mapsto e_{d,n}(x)$.

4. Espaces $\mathbf{PL}^p$ pour l'oscillateur harmonique sur $\mathbb{R}^d$

4A. ***Reformulation des énoncés.*** On reprend les notations de la partie 1C. En particulier les résultats énoncés sont valides uniquement en dimension $d \geq 2$. Pour tout $p \in [1, +\infty[$, l'étude de l'espace $\mathbf{PL}^p\big(\mathbb{R}^d, \bigoplus E_{d,n}\big)$ passe par la compréhension de la localisation des fonctions

$$\begin{aligned} e_{d,n}(x) &:= \sup\big\{|u_n(x)|^2 \mid u_n \in E_{d,n},\ \|u_n\|_{L^2(\mathbb{R}^d)} = 1\big\} \\ &= \sum_{\substack{(i_1,\dots,i_d)\in\mathbb{N}^d \\ i_1+\cdots+i_d=n}} h_{i_1}(x_1)^2 \cdots h_{i_d}(x_d)^2. \end{aligned} \tag{103}$$

Suivant l'idée selon laquelle seule la concentration de $e_{d,n}$ devrait être significative, on démontre le résultat suivant.

Proposition 4.1. *Pour tout entier $n \in \mathbb{N}^\star$ et tout vecteur $x \in \mathbb{R}^d$ on a*

$$\begin{aligned} |x| \leq \sqrt{2(2n+1)} \quad &\Rightarrow \quad e_{d,n}(x) \leq C(d) n^{\frac{d}{2}-1}, \\ |x| \geq \sqrt{2(2n+1)} \quad &\Rightarrow \quad e_{d,n}(x) \leq C(d) e^{-\frac{|x|^2}{C(d)}}. \end{aligned} \tag{104}$$

Il existe aussi une constante universelle $\alpha \in]0, 1[$ et un entier $n(d) \in \mathbb{N}^\star$ tels que pour tout entier $n \geq n(d)$ on a, quitte à augmenter $C(d) \geq 1$, l'implication

$$\frac{C(d)}{\sqrt{2n+1}} \leq |x| \leq \alpha\sqrt{2n+1} \quad \Rightarrow \quad \frac{n^{\frac{d}{2}-1}}{C(d)} \leq e_{d,n}(x) \leq C(d) n^{\frac{d}{2}-1}. \tag{105}$$

On justifiera plus loin que la fonction $x \mapsto e_{d,n}(x)$ est invariante par rotation autour de 0, si bien que l'on a $e_{d,n}(x) = e_{d,n}((|x|, 0, \dots, 0))$. À titre d'exemple, on examine à la figure 2 les graphes pour $d = 2$ et $n \in \{5, 10, 50\}$ de $e_{d,n}(x)$ en fonction de $|x| \in \big[0, \frac{3}{2}\sqrt{2n+1}\big]$.

La majoration (104) est grossière et est obtenue grâce à des majorations classiques des fonctions de Hermite. Quant à la minoration de (105), sa démonstration est plus subtile et utilise des approximations essentiellement optimales des fonctions de Hermite dues à Muckhenhoupt. On notera que nous sommes obligés d'éviter, en toute rigueur, un voisinage de l'origine pour effectuer une minoration uniforme de $e_{d,n}$. En effet, pour tout entier n impair la fonction h_n est impaire et donc $e_{d,n}(0)$ est nul.

Après la rédaction de cet article, Didier Robert nous a signalé le lemme 10 de l'article [Hanin et al. 2015] dans lequel se trouvent des estimations plus précises si $|x|$ appartient à un compact de la forme $[a\sqrt{2n+d}, b\sqrt{2n+d}]$ avec $0 < a < b < 1$. Par souci de comparaison, on se permet d'écrire cette approximation :

$$e_{d,n}(x) = \frac{\mu_{d-1}(\mathbb{S}^{d-1})}{(2\pi)^d}(2n+d-|x|^2)^{\frac{d}{2}-1}\Big(1+\mathcal{O}_{a,b}\Big(\frac{1}{2n+d}\Big)\Big),$$

où $\mathcal{O}_{a,b}$ est uniforme en la condition $a \leq |x|/\sqrt{2n+d} \leq b$. Il s'agit d'une loi de Weyl locale pour l'oscillateur harmonique. Dans la théorie des espaces de Lebesgue probabilistes, il apparaîtra que l'on peut en fait considérer, en première approximation, que la fonction $e_{d,n}$ se localise uniformément sur le compact $\overline{\mathbb{B}_d(0, \sqrt{2n+1})}$ avec une amplitude d'ordre $n^{\frac{d}{2}-1}$.

Une application immédiate de la proposition 4.1 est donnée par les estimations

$$\forall p \in [1, +\infty[\cup \{+\infty\}, \quad \forall n \in \mathbb{N}^\star, \quad \|\sqrt{e_{d,n}}\|_{L^p(\mathbb{R}^d)} \simeq_{d,p} n^{\frac{d-1}{2}-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})} = n^{\frac{1}{2}[\frac{d}{2}-1+\frac{d}{p}]}. \quad (106)$$

La majoration $\|e_{d,n}\|_{L^{\frac{p}{2}}(\mathbb{R}^d)} \lesssim_{d,p} n^{\frac{d}{2}-1+\frac{d}{p}}$ est connue pour $p \geq 2$ et est généralement traitée par interpolation entre $p = 2$ et $p = +\infty$, mais nous ne connaissons pas de référence où l'optimalité est prouvée (voir [Poiret et al. 2015, Lemma 3.5] et les références indiquées). Les hypothèses du théorème 2.6 de dualité sont alors très faciles à vérifier. D'une part, (13) nous donne

$$\forall p \in]1, +\infty[, \quad \|\sqrt{e_{d,n}}\|_{L^p(\mathbb{R}^d)}\|\sqrt{e_{d,n}}\|_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} \lesssim_{d,p} \dim(E_{d,n}). \quad (107)$$

D'autre part, en utilisant (104) et (106), on a

$$\begin{aligned}
\forall n \in \mathbb{N}^\star, \quad \frac{\sqrt{e_{d,n}(x)}}{\|\sqrt{e_{d,n}}\|_{L^p(\mathbb{R}^d)}} &\lesssim_{d,p} \frac{1}{n^{\frac{d}{2p}}}\mathbf{1}_{\mathbb{B}_d(0,\sqrt{2(2n+1)})}(x) + \frac{e^{-\frac{|x|^2}{C(d)}}}{n^{\frac{1}{2}(\frac{d}{2}-1+\frac{d}{p})}}, \\
\sup_{n\geq 1} \frac{\sqrt{e_{d,n}(x)}}{\|\sqrt{e_{d,n}}\|_{L^p(\mathbb{R}^d)}} &\lesssim_{d,p} \frac{1}{|x|^{\frac{d}{p}}} + e^{-\frac{|x|^2}{C(d)}}, \\
\sup_{n\geq 0} \frac{\sqrt{e_{d,n}}}{\|\sqrt{e_{d,n}}\|_{L^p(\mathbb{R}^d)}} &\in L^{p,\infty}(\mathbb{R}^d). \qquad (108)
\end{aligned}$$

Ainsi, on sait par avance que les espaces $\mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_{d,n})$ sont stables par dualité. Tout comme pour les harmoniques sphériques, ces espaces vont s'identifier à des sous-espaces de distributions sur $\mathbb{R}^d$. Pour le voir, commençons par rappeler la définition des espaces de Sobolev naturellement associés à l'oscillateur harmonique. En notant $\Pi_n : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ le projecteur orthogonal sur

$$E_{d,n} = \ker(-\Delta + |x|^2 - 2n - d),$$

on a

$$\begin{aligned}\forall s \geq 0, \quad \mathcal{H}^s(\mathbb{R}^d) &:= \mathrm{Dom}((-\Delta + |x|^2)^{\frac{s}{2}}) \\ &= \{u \in L^2(\mathbb{R}^d) \mid (-\Delta + |x|^2)^{\frac{s}{2}} u \in L^2(\mathbb{R}^d)\} \\ &= \Big\{u \in L^2(\mathbb{R}^d) \,\Big|\, \sum_{n\in\mathbb{N}} (1+n)^s \|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)} < +\infty\Big\}.\end{aligned}$$

Rappelons aussi que cet espace abstrait admet, si $s \in \mathbb{N}$, la caractérisation fonctionnelle suivante (voir la preuve de [Yajima et Zhang 2004, Lemma 2.4 ; Imekraz et al. 2016, page 2765]) :

$$\mathcal{H}^s(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) \mid \forall (m_0, m_1, \dots, m_d) \in \mathbb{N}^d,\ m_0 + \cdots + m_d \leq s \Rightarrow |x|^{m_0} \partial_{x_1}^{m_1} \cdots \partial_{x_d}^{m_d} u \in L^2(\mathbb{R}^d)\}. \quad (109)$$

Toute fonction φ de l'espace de Schwartz $\mathcal{S}(\mathbb{R}^d) = \bigcap_{s\in\mathbb{N}} \mathcal{H}^s(\mathbb{R}^d)$ admet alors une décomposition

$$\varphi = \sum_{n\geq 0} \Pi_n(\varphi), \quad \forall \alpha > 0, \quad \forall n \in \mathbb{N}^\star, \quad \|\Pi_n(\phi)\|_{L^2(\mathbb{R}^d)} \leq \frac{C(\alpha)}{n^\alpha}.$$

On en déduit par dualité que toute distribution tempérée $u \in \mathcal{S}'(\mathbb{R}^d)$ admet une décomposition en série faiblement convergente pour la dualité $(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$:

$$u = \sum_{n\in\mathbb{N}} \Pi_n(u), \quad \exists \alpha > 0, \quad \forall n \in \mathbb{N}^\star, \quad \|\Pi_n(u)\|_{L^2(\mathbb{R}^d)} \leq C(\alpha) n^\alpha.$$

Cela nous amène à définir des espaces de Sobolev pour tout $s \in \mathbb{R}$:

$$\forall s \in \mathbb{R}, \quad \mathcal{H}^s(\mathbb{R}) = \Big\{u \in \mathcal{S}'(\mathbb{R}^d) \,\Big|\, \sum_{n\in\mathbb{N}} (1+n)^s \|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)} < +\infty\Big\}.$$

Nous pouvons maintenant énoncer un lemme d'identification.

Lemme 4.2. *Pour tous* $p \in [1, +\infty[$ *et* $(u_n)_{n\in\mathbb{N}} \in \mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_{d,n})$, *la série* $\sum u_n$ *converge pour la dualité* $(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ *vers une distribution tempérée. Par suite, on peut identifier* $\mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_{d,n})$ *au sous-espace des distributions tempérées* $u \in \mathcal{S}'(\mathbb{R}^d)$ *vérifiant*

$$\|u\|_{\mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_n)} = \left\| \sqrt{\sum_{n\in\mathbb{N}} \|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)} \frac{e_{d,n}(x)}{\dim(E_{d,n})}} \right\|_{L^p_x(\mathbb{R}^d)} < +\infty.$$

Démonstration. On invoque l'inégalité triviale :

$$\|u\|_{\mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_{d,n})} \geq \sup_{n\in\mathbb{N}} \frac{\|u_n\|_{L^2(\mathbb{R}^d)}}{\sqrt{\dim(E_{d,n})}} \|\sqrt{e_{d,n}}\|_{L^p(\mathbb{R}^d)}.$$

Les équivalents (13) et (106) assurent que $(\|u_n\|_{L^2(\mathbb{R}^d)})_{n\in\mathbb{N}}$ est à croissance polynomiale. □

La proposition 4.1, le théorème 2.6 de dualité et le théorème 2.5 d'interpolation vont nous permettre de décrire complètement les sous-espaces $\mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_{d,n})$ et leurs propriétés de dualité et d'interpolation pour $p \in]1, +\infty[$. Cela nous donne l'équivalence (i) $\Leftrightarrow$ (ii) du théorème 1.4.

Théorème 4.3. *Pour tout réel $p \in [1,+\infty[$ et toute distribution tempérée $u \in \mathcal{S}'(\mathbb{R}^d)$, on a*

$$\|u\|_{\mathbf{PL}^p(\mathbb{R}^d,\bigoplus E_{d,n})} \simeq_{d,p} \|\Pi_0(u)\|_{L^2(\mathbb{R}^d)} + \Bigg[\sum_{n\geq 1} n^{\frac{d}{2}-1}\Bigg(\sum_{k\geq n} \frac{\|\Pi_k(u)\|^2_{L^2(\mathbb{R}^d)}}{k^{\frac{d}{2}}}\Bigg)^{\frac{p}{2}}\Bigg]^{\frac{1}{p}}. \tag{110}$$

En outre, nous avons les propriétés de dualité et d'interpolation :

(i) *pour tout $p \in]1,+\infty[$, on pose $q = \frac{p}{p-1}$ l'exposant conjugué. L'injection canonique*

$$\Lambda_p : \mathbf{PL}^q\big(\mathbb{R}^d, \textstyle\bigoplus E_{d,n}\big) \to \mathbf{PL}^p\big(\mathbb{R}^d, \textstyle\bigoplus E_{d,n}\big)'$$

qui à un élément $w \in \mathbf{PL}^q\big(\mathbb{R}^d, \bigoplus E_{d,n}\big)$ associe la forme linéaire

$$u \in \mathbf{PL}^p\big(\mathbb{R}^d, \textstyle\bigoplus E_{d,n}\big) \mapsto \sum_{n\geq 0} \langle \Pi_n(u), \Pi_n(w)\rangle_{L^2(\mathbb{R}^d)}$$

est bien définie et est un isomorphisme d'espaces de Banach.

(ii) *les espaces $\big(\mathbf{PL}^p\big(\mathbb{R}^d, \bigoplus E_{d,n}\big)\big)_{p\in]1,+\infty[}$ sont stables par interpolation complexe et réelle au sens du théorème 2.5.*

Le théorème précédent ressemble aux propositions 3.1 et 3.2 mais sa preuve est bien plus simple car la concentration des fonctions $e_{d,n}$ est bien meilleure que celle des harmoniques sphériques Y_n et Z_n étudiées dans la partie 3. En effet, la proposition 4.1 assure que l'on peut brutalement contrôler $e_{d,n}(x)$ par le terme gaussien $e^{-|x|^2/C(d)} \ll 1$ à l'extérieur de la boule $\mathbb{B}_d(0, \sqrt{2(2n+1)})$.

L'équivalence de normes (110) implique déjà quelques faits non triviaux :

— pour tout $p \in [1,+\infty[$ on a l'inclusion $\mathbf{PL}^p\big(\mathbb{R}^d, \bigoplus E_{d,n}\big) \subset \mathcal{H}^{-\frac{d}{2}}(\mathbb{R}^d)$. Cela signifie qu'il faut un minimum de régularité pour espérer arriver presque sûrement dans $L^p(\mathbb{R}^d)$.

— pour tous réels $p_1 < p_2$ on a l'inclusion stricte $\mathbf{PL}^{p_1}\big(\mathbb{R}^d, \bigoplus E_{d,n}\big) \subset \mathbf{PL}^{p_2}\big(\mathbb{R}^d, \bigoplus E_{d,n}\big)$. D'une part, cela contraste fortement avec le fait que $L^{p_1}(\mathbb{R}^d)$ n'est pas inclus dans $L^{p_2}(\mathbb{R}^d)$. D'autre part, la conclusion du théorème de Paley–Zygmund est vérifiée pour l'oscillateur harmonique (choisir $p_1 = 2$).

Il est temps à présent d'énoncer les injections de Sobolev probabilistes de l'oscillateur harmonique multidimensionnel (ce qui donne la fin du théorème 1.4).

Théorème 4.4. *Pour tout réel $p \in]2,+\infty[$, on note $q = \frac{p}{p-1} \in]1,2[$. Nous avons les inclusions*

$$\mathcal{H}^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}(\mathbb{R}^d) \subset \mathbf{PL}^p\big(\mathbb{R}^d, \textstyle\bigoplus E_{d,n}\big) \subset \displaystyle\bigcap_{\varepsilon>0} \mathcal{H}^{-d\left(\frac{1}{2}-\frac{1}{p}\right)-\varepsilon}(\mathbb{R}^d),$$

$$\bigcup_{\varepsilon>0} \mathcal{H}^{d\left(\frac{1}{q}-\frac{1}{2}\right)+\varepsilon}(\mathbb{R}^d) \subset \mathbf{PL}^q\big(\mathbb{R}^d, \textstyle\bigoplus E_{d,n}\big) \subset \mathcal{H}^{d\left(\frac{1}{q}-\frac{1}{2}\right)}(\mathbb{R}^d).$$

Remarquons que pour tout $q \in [1,2]$, l'espace $\mathbf{PL}^q\big(\mathbb{R}^d, \bigoplus E_{d,n}\big) \subset L^2(\mathbb{R}^d)$ est un espace de fonctions (alors que pour les harmoniques sphériques Y_n, l'espace $\mathbf{PL}^p\big(\mathbb{S}^d, \bigoplus \mathbb{C}Z_n\big)$ est en général constitué de distributions). En choisissant $m_0 = 0$ dans (109), on voit que l'espace de Sobolev $\mathcal{H}^s(\mathbb{R}^d)$ est inclus dans l'usuel espace de Sobolev $H^s(\mathbb{R}^d)$. Dans le cas $p > 2$, les injections de Sobolev déterministes (17) s'écrivent alors

$$\mathcal{H}^{d(\frac{1}{2}-\frac{1}{p})}(\mathbb{R}^d) \subset H^{d(\frac{1}{2}-\frac{1}{p})}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d).$$

En autorisant un aléa pour arriver dans $L^p(\mathbb{R}^d)$, l'injection de Sobolev probabiliste

$$\mathcal{H}^{-d(\frac{1}{2}-\frac{1}{p})}(\mathbb{R}^d) \subset \mathbf{PL}^p\big(\mathbb{R}^d, \bigoplus E_{d,n}\big)$$

assure donc presque sûrement un gain de $2d\big(\frac{1}{2}-\frac{1}{p}\big)$ dérivées.

4B. ***Rappels sur les fonctions de Hermite.*** Par récurrence, on peut simplifier (103) en

$$e_{d,n}(x) = \sum_{\substack{(i_1,\dots,i_d)\in\mathbb{N}^d \\ i_1+\cdots+i_d=n}} h_{i_1}(x_1)^2 \cdots h_{i_d}(x_d)^2 = \sum_{k=0}^{n} e_{d-1,k}(x_1,\dots,x_{d-1}) h_{n-k}(x_d)^2.$$

Cela dit, nous allons utiliser d'autres formules plus maniables. Remarquons que toute rotation linéaire de $\mathbb{R}^d$ commute avec l'oscillateur harmonique $-\Delta + |x|^2$ et laisse donc invariant son sous-espace propre $E_{d,n}$ associé à la valeur propre $d + 2n$. On en déduit que la fonction $e_{d,n}$ de $E_{d,n}$ ne dépend que de la norme euclidienne

$$|x| = \sqrt{x_1^2 + \cdots + x_d^2}$$

et l'on peut écrire

$$e_{d,n}(x) = \sum_{k=0}^{n} e_{d-1,k}(\overbrace{|x|, 0, \dots, 0}^{\in\mathbb{R}^{d-1}}) h_{n-k}(0)^2 \tag{111}$$

$$= \sum_{k=0}^{n} e_{d-1,k}(\overbrace{0, \dots, 0}^{\in\mathbb{R}^{d-1}}) h_{n-k}(|x|)^2. \tag{112}$$

Le résultat suivant nous donne des estimations faciles en 0.

Proposition 4.5. *Pour tout entier $k \in \mathbb{N}$, on a $h_{2k+1}(0) = 0$ et l'équivalence $(-1)^k h_{2k}(0) \simeq \frac{1}{(k+1)^{1/4}}$. Pour tout entier $n \in \mathbb{N}$, on a*

$$n \in 2\mathbb{N}+1 \Rightarrow e_{d,n}(0) = 0, \quad n \in 2\mathbb{N} \Rightarrow e_{d,n}(0) \simeq_d n^{\frac{d}{2}-1}.$$

Démonstration. La première équivalence découle de la formule (5.5.5) de [Szegő 1975] :

$$(-1)^k h_{2k}(0) = (-1)^k \frac{H_{2k}(0)}{\sqrt{(2k)!4^k\sqrt{\pi}}} = \frac{\sqrt{(2k)!}}{k!2^k\pi^{1/4}} \sim \frac{1}{\sqrt{\pi}\sqrt[4]{k}}.$$

Pour le cas n impair, on a déjà remarqué l'égalité $e_{d,n}(0) = 0$ après l'énoncé de la proposition 4.1. Pour le cas n pair, on peut écrire :

$$\begin{aligned} e_{d,n}(0) &= \sum_{2i_1+\dots+2i_d=n} h_{2i_1}(0)^2 \cdots h_{2i_d}(0)^2 \\ &\simeq_d \sum_{2i_1+\dots+2i_d=n} \frac{1}{\sqrt{(1+i_1)\cdots(1+i_d)}} \\ &\gtrsim_d \sum_{2i_1+\dots+2i_d=n} n^{-\frac{d}{2}} \\ &\gtrsim_d n^{\frac{d}{2}-1}. \end{aligned}$$

La majoration $e_{d,n}(0) \lesssim_d n^{\frac{d}{2}-1}$ peut se démontrer par récurrence sur d en séparant les sommes suivantes selon que ℓ est plus petit ou plus grand que $\frac{n}{4}$:

$$e_{d,n}(0) = \sum_{\ell=0}^{n/2} e_{d-1,n-2\ell}(0) h_{2\ell}(0)^2 \simeq \sum_{\ell=0}^{n/2} \frac{e_{d-1,n-2\ell}(0)}{\sqrt{1+\ell}}. \qquad \square$$

Nous aurons besoin d'estimations précises concernant le comportement des fonctions de Hermite. Pour le résultat suivant, on fait référence à [Thangavelu 1993, Lemma 1.5.1] ou [Muckenhoupt 1970b, (2.3)].

Proposition 4.6. *Il existe deux constantes universelles $C \geq 1$ et $\gamma > 0$ telles que pour tous $x \in \mathbb{R}$ et $n \in \mathbb{N}$ on a*

$$|x| \leq \sqrt{2(2n+1)} \quad \Rightarrow \quad |h_n(x)| \leq \frac{C}{\sqrt[4]{|2n+1-x^2| + \sqrt[3]{2n+1}}},$$
$$|x| > \sqrt{2(2n+1)} \quad \Rightarrow \quad |h_n(x)| \leq C e^{-\gamma x^2}.$$

En particulier, on a

$$|x| \leq \sqrt{2n+1} \Rightarrow |h_n(x)| \leq \frac{C}{\sqrt[4]{2n+2-x^2}}. \tag{113}$$

Comme les fonctions h_n oscillent et s'annulent plusieurs fois, les estimations de la proposition 4.6 sont inutilisables pour minorer les fonctions $|h_n|$. Pour remédier à cela, on fait appel à un résultat d'approximation des fonctions de Hermite dû à Muckenhoupt [1970a, (2.4), page 421] et prouvé à partir de [Erdélyi 1960, 6.12, page 23].

Proposition 4.7. *Introduisons la fonction croissante $\Phi : [0, 1] \to [0, \frac{\pi}{4}]$ définie par*

$$\Phi(u) = \int_0^u \sqrt{1-s^2}\, ds = \tfrac{1}{2}\arcsin(u) + \tfrac{1}{2}u\sqrt{1-u^2}.$$

Il existe une constante universelle $C \geq 1$ telle que pour tous $n \in \mathbb{N}$ et $x \in [0, \sqrt{2n+1} - (2n+1)^{-\frac{1}{6}}]$ on a

$$\left| h_n(x) - \frac{\sqrt{2}}{\sqrt{\pi}(2n+1-x^2)^{\frac{1}{4}}} \cos\left[(2n+1)\Phi\left(\frac{x}{\sqrt{2n+1}}\right) - \frac{n\pi}{2}\right] \right| \leq \frac{C\sqrt{2n+1}}{(2n+1-x^2)^{\frac{7}{4}}}.$$

Démonstration. En fait, le terme principal est exprimé dans [Muckenhoupt 1970a, (2.4), page 421] sous la forme différente mais équivalente

$$\frac{\sqrt{2}}{\sqrt{\pi}(2n+1-x^2)^{\frac{1}{4}}}\cos\left[\tfrac{1}{4}(2n+1)[2\theta-\sin(2\theta)]-\tfrac{\pi}{4}\right],$$

avec $\theta=\frac{\pi}{2}-\arcsin\left(\frac{x}{\sqrt{2n+1}}\right)$ et donc $\sin(2\theta)=2\sin(\theta)\cos(\theta)=2\frac{x}{\sqrt{2n+1}}\sqrt{1-\frac{x^2}{2n+1}}$. □

Pour tout $\beta\in\,]0,1[$ et $x\in[0,\beta\sqrt{2n+1}]$, l'inégalité (113) et la proposition 4.7 nous amènent donc à

$$\left|h_n(x)^2-\frac{2}{\pi\sqrt{2n+1-x^2}}\cos^2\left[(2n+1)\Phi\left(\frac{x}{\sqrt{2n+1}}\right)-\frac{n\pi}{2}\right]\right|\le\frac{C(\beta)}{(2n+1)^{\frac{3}{2}}}. \tag{114}$$

C'est la formule précédente et la proposition 4.6 que nous allons utiliser pour comprendre la localisation spatiale de la fonction $e_{d,n}$ de l'oscillateur harmonique multidimensionnel.

4C. ***Preuve de la proposition 4.1, majoration* (104) *de la fonction*** $e_{d,n}$**.** À l'aide de [Koch et Tataru 2005, Corollary 3.2, case $n\ge 2$ and $p=\infty$, $\lambda=\sqrt{2n+d}$], on a pour tout $x\in\mathbb{R}^d$

$$\begin{aligned}e_{d,n}(x)&=\sup\left\{|u_n(x)|^2\mid u_n\in E_{d,n},\ \|u_n\|_{L^2(\mathbb{R}^d)}=1\right\}\\&\le\sup\left\{\|u_n\|^2_{L^\infty(\mathbb{R}^d)}\mid u_n\in E_{d,n},\ \|u_n\|_{L^2(\mathbb{R}^d)}=1\right\}\\&\le C(d)n^{\frac{d}{2}-1}.\end{aligned}$$

Il nous reste à analyser le cas $|x|>\sqrt{2(2n+1)}$. Nous ne traitons que le sous-cas $n\in 2\mathbb{N}$ car le sous-cas $n\in 2\mathbb{N}+1$ se traite de la même façon. La formule (112) et la proposition 4.5 donnent

$$e_{d,n}(x)=\sum_{k=0}^{n/2}h_{2k}(|x|)^2e_{d-1,n-2k}(0)\simeq_d\sum_{k=0}^{n/2}h_{2k}(|x|)^2(1+n-2k)^{\frac{d}{2}-\frac{3}{2}}.$$

Par suite, la proposition 4.6 nous donne

$$e_{d,n}(x)\le C(d)e^{-2\gamma|x|^2}\sum_{k=0}^{n/2}(1+n-2k)^{\frac{d}{2}-\frac{3}{2}}\le C(d)n^{\frac{d}{2}-\frac{1}{2}}e^{-2\gamma|x|^2}\le C(d)|x|^{d-1}e^{-2\gamma|x|^2}.$$

Quitte à augmenter $C(d)\ge 1$, le majorant précédent est majoré par $C(d)e^{\frac{-|x|^2}{C(d)}}$.

4D. ***Preuve de la proposition 4.1, minoration* (105) *de la fonction*** $e_{d,n}$**.** Pour tout entier $d\ge 2$, on va démontrer par récurrence sur d l'assertion

$$H(d):\qquad \forall\alpha\in\left]0,\sin\left(\tfrac{1}{4}\right)\right[,\quad \exists C(d,\alpha)>1,\quad \liminf_{n\to+\infty}\left(\inf_{\frac{C(d,\alpha)}{\sqrt{2n+1}}\le|x|\le\alpha\sqrt{2n+1}}\frac{e_{d,n}(x)}{n^{\frac{d}{2}-1}}\right)>0.$$

Dans toute cette preuve, on aura besoin d'un réel $\beta\in\left]\alpha,\sin\left(\frac{1}{4}\right)\right[$ et l'on choisit par simplicité

$$\beta:=\tfrac{1}{2}\left(\alpha+\sin\left(\tfrac{1}{4}\right)\right). \tag{115}$$

Premier cas : $H(2)$ avec $n \in 2\mathbb{N}$. Il s'agit du cas le plus technique. Tout d'abord, l'invariance radiale de $e_{2,n}$ et la proposition 4.5 donnent pour tout $x \in \mathbb{R}^2$,

$$e_{2,n}(x) = \sum_{k=0}^{n/2} h_{2k}(|x|)^2 h_{n-2k}(0)^2 \simeq \sum_{k=0}^{n/2} \frac{h_{2k}(|x|)^2}{\sqrt{1+n-2k}}.$$

Afin de pouvoir exploiter la formule (114), nous avons besoin de considérer des indices k du même ordre de grandeur que n. De façon précise, nous allons sélectionner les indices k tels que $\frac{\alpha^2}{2\beta^2}n \leq k \leq \frac{n}{2}$. Puisque l'on a $\beta > \alpha$, on déduit que l'on a

$$n \leq \frac{2\beta^2}{\alpha^2}k + \frac{1}{2}\left(\frac{\beta^2}{\alpha^2} - 1\right), \quad 2n+1 \leq \frac{\beta^2}{\alpha^2}(4k+1), \quad \alpha\sqrt{2n+1} \leq \beta\sqrt{4k+1}.$$

En utilisant (114) et en imposant $|x| \leq \alpha\sqrt{2n+1}$, il existe une constante $C(\alpha) > 1$ telle que

$$e_{2,n}(x) \gtrsim \sum_{\frac{\alpha^2}{2\beta^2}n \leq k \leq \frac{n}{2}} \frac{1}{\sqrt{(1+n-2k)k}} \cos^2\left[(4k+1)\Phi\left(\frac{|x|}{\sqrt{4k+1}}\right)\right] - \sum_{\frac{\alpha^2}{2\beta^2}n \leq k \leq \frac{n}{2}} \frac{C(\alpha)}{(4k+1)^{\frac{3}{2}}(1+n-2k)^{\frac{1}{2}}},$$

ce qui se minore grossièrement par

$$\frac{1}{C(\alpha)n} \sum_{\frac{\alpha^2}{2\beta^2}n \leq k \leq \frac{n}{2}} \cos^2\left[(4k+1)\Phi\left(\frac{|x|}{\sqrt{4k+1}}\right)\right] - \sum_{\frac{\alpha^2}{2\beta^2}n \leq k \leq \frac{n}{2}} \frac{C(\alpha)}{(4k+1)^{\frac{3}{2}}(1+n-2k)^{\frac{1}{2}}}.$$

Or on a immédiatement

$$\sum_{\frac{\alpha^2}{2\beta^2}n \leq k \leq \frac{n}{2}} \frac{C(\alpha)}{(4k+1)^{\frac{3}{2}}(1+n-2k)^{\frac{1}{2}}} \leq C(\alpha)\frac{\sqrt{n}}{n^{3/2}} = \frac{C(\alpha)}{n}.$$

Notons à présent $\lceil \frac{\alpha^2}{2\beta^2}n \rceil$ le plus petit entier supérieur ou égal à $\frac{\alpha^2}{2\beta^2}n$ et $\lfloor \frac{n}{2} \rfloor$ le plus grand entier inférieur ou égal à $\frac{n}{2}$. Si l'on définit

$$S(n,x,\alpha,\beta) := \sum_{\lceil \frac{\alpha^2}{2\beta^2}n \rceil \leq k \leq \lfloor \frac{n}{2} \rfloor} \cos\left[2(4k+1)\Phi\left(\frac{|x|}{\sqrt{4k+1}}\right)\right],$$

alors notre minoration se reformule en

$$e_{2,n}(x) \gtrsim_\alpha \frac{1}{4}\left(1 - \frac{\alpha^2}{\beta^2}\right) + \frac{1}{2n}S(n,x,\alpha,\beta) - \frac{C(\alpha)}{n}. \tag{116}$$

L'estimation grossière $|S(n,x,\alpha,\beta)| \leq \frac{n}{2}\left(1 - \frac{\alpha^2}{\beta^2}\right)$ ne suffit pas pour minorer $e_{2,n}(x)$. Nous allons raffiner cette estimation grâce à l'oscillation des termes. On va faire appel à la formule d'Euler–Maclaurin au rang 0 en posant

$$\forall t\in\left[\frac{\alpha^2}{2\beta^2}n,\tfrac12 n\right],\quad a_x(t):=2(4t+1)\Phi\left(\frac{|x|}{\sqrt{4t+1}}\right)=2(4t+1)\int_0^{\frac{|x|}{\sqrt{4t+1}}}\sqrt{1-s^2}\,ds,$$
$$a_x'(t)\ =4\arcsin\left(\frac{|x|}{\sqrt{4t+1}}\right)\in[0,4\arcsin(\beta)].$$

Cela nous permet de reformuler $S(n,x,\alpha,\beta)$ en

$$\frac{\cos\big[a_x\big(\lceil\frac{\alpha^2}{2\beta^2}n\rceil\big)\big]+\cos\big[a_x\big(\lfloor\frac n2\rfloor\big)\big]}{2}+\int_{\lceil\frac{\alpha^2}{2\beta^2}n\rceil}^{\lfloor\frac n2\rfloor}\cos(a_x(t))\,dt-\int_{\lceil\frac{\alpha^2}{2\beta^2}n\rceil}^{\lfloor\frac n2\rfloor}a_x'(t)\sin(a_x(t))\big[t-\lfloor t\rfloor-\tfrac12\big]\,dt.$$

La deuxième intégrale est contrôlée grossièrement par

$$\left|\int_{\lceil\frac{\alpha^2}{2\beta^2}n\rceil}^{\lfloor\frac n2\rfloor}a_x'(t)\sin(a_x(t))\big[t-\lfloor t\rfloor-\tfrac12\big]\,dt\right|\le\frac{4\arcsin(\beta)}{2}\left(\left\lfloor\frac n2\right\rfloor-\left\lceil\frac{\alpha^2}{2\beta^2}n\right\rceil\right)\le n\arcsin(\beta)\left(1-\frac{\alpha^2}{\beta^2}\right).$$

Passons au contrôle de la première intégrale. La forme de la minoration (105) nous autorise à supposer $x\neq0$. Remarquons maintenant que $t\mapsto a_x'(t)$ est strictement positive et décroissante. En utilisant une intégration par parties, on obtient

$$\int_{\lceil\frac{\alpha^2}{2\beta^2}n\rceil}^{\lfloor\frac n2\rfloor}\cos(a_x(t))\,dt=\left[\frac{\sin(a_x(t))}{a_x'(t)}\right]_{\lceil\frac{\alpha^2}{2\beta^2}n\rceil}^{\lfloor\frac n2\rfloor}-\int_{\lceil\frac{\alpha^2}{2\beta^2}n\rceil}^{\lfloor\frac n2\rfloor}\left(\frac{1}{a_x'(t)}\right)'\sin(a_x(t))\,dt$$
$$\left|\int_{\lceil\frac{\alpha^2}{2\beta^2}n\rceil}^{\lfloor\frac n2\rfloor}\cos(a_x(t))\,dt\right|\le\frac{1}{a_x'\big(\lfloor\frac n2\rfloor\big)}+\frac{1}{a_x'\big(\lceil\frac{\alpha^2n}{2\beta^2}\rceil\big)}+\frac{1}{a_x'\big(\lfloor\frac n2\rfloor\big)}-\frac{1}{a_x'\big(\lceil\frac{\alpha^2n}{2\beta^2}\rceil\big)}$$
$$\le\frac{2}{a_x'\big(\lfloor\frac n2\rfloor\big)}\le\frac{2}{a_x'\big(\frac n2\big)}\le\frac{\sqrt{2n+1}}{2|x|}.$$

La définition (115) de β nous autorise à majorer

$$|S(n,x,\alpha,\beta)|\le1+n\arcsin(\beta)\left(1-\frac{\alpha^2}{\beta^2}\right)+\frac{\sqrt{2n+1}}{2|x|}\le1+\frac n4\left(1-\frac{\alpha^2}{\beta^2}\right)+\frac{2n}{|x|\sqrt{2n+1}}.$$

On peut donc choisir $C(2,\alpha)>1$ de sorte que l'inégalité $C(2,\alpha)/\sqrt{2n+1}\le|x|$ implique

$$|S(n,x,\alpha,\beta)|\le1+\frac n3\left(1-\frac{\alpha^2}{\beta^2}\right).$$

On conclut à l'aide de (116).

Deuxième cas : $H(2)$ et $n\in2\mathbb N+1$. Par invariance radiale et par imparité des fonctions de Hermite h_{n-2k}, on obtient

$$e_{2,n}(x)=\sum_{k=1}^{(n+1)/2}h_{2k-1}(|x|)^2h_{n-2k+1}(0)^2\simeq\sum_{k=1}^{(n+1)/2}h_{2k-1}(|x|)^2\frac{1}{\sqrt{n-2k+2}}.$$

En notant K l'ensemble des indices k vérifiant

$$\frac{\alpha^2}{2\beta^2}n + \frac{\alpha^2+\beta^2}{4\beta^2} \le k \le \frac{n+1}{2},$$

on a

$$\forall k \in K, \qquad \frac{\alpha^2}{\beta^2}\Big(\frac{n}{2}+\frac{1}{4}\Big) \le k - \frac{1}{4}, \quad \alpha\sqrt{2n+1} < \beta\sqrt{4k-1}.$$

En utilisant (114) et la même argumentation que celle du premier cas, nous arrivons à

$$\begin{aligned} e_{2,n}(x) &\ge \frac{1}{C(\alpha)\sqrt{n}} \sum_{k\in K} \frac{1}{\sqrt{n-2k+2}} \sin^2\Big[(4k-1)\Phi\Big(\frac{x}{\sqrt{4k-1}}\Big)\Big] - \frac{C(\alpha)}{n} \\ &\ge \frac{1}{C(\alpha)n} \sum_{k\in K} \sin^2\Big[(4k-1)\Phi\Big(\frac{x}{\sqrt{4k-1}}\Big)\Big] - \frac{C(\alpha)}{n} \\ &\gtrsim_\alpha \frac{1}{4}\Big(1-\frac{\alpha^2}{\beta^2}\Big) - \frac{1}{2n}\sum_{k\in K} \cos\Big[2(4k-1)\Phi\Big(\frac{x}{\sqrt{4k-1}}\Big)\Big] - \frac{C(\alpha)}{n}. \end{aligned}$$

On finit exactement comme dans le premier cas.

Troisième cas : $H(d)$ avec $d > 2$ et $n \in 2\mathbb{N}$. On utilise la proposition 4.5 et l'invariance radiale de $e_{d,n}$ exprimée par la formule (111) pour obtenir

$$\begin{aligned} e_{d,n}(x) &= \sum_{k=0}^{n/2} e_{d-1,2k}(\overbrace{|x|,0,\dots,0}^{\in\mathbb{R}^{d-1}}) h_{n-2k}(0)^2 \\ &\simeq \sum_{k=0}^{n/2} \frac{e_{d-1,2k}(|x|,0,\dots,0)}{\sqrt{1+n-2k}} \\ &\gtrsim \sum_{\frac{\alpha^2}{2\beta^2}n \le k \le \frac{n}{2}} \frac{e_{d-1,2k}(|x|,0,\dots,0)}{\sqrt{1+n-2k}}. \end{aligned}$$

Comme dans le premier cas, les indices k sélectionnés vérifient l'inégalité $\alpha\sqrt{2n+1} \le \beta\sqrt{2(2k)+1}$. Puisque l'on a $\beta < \sin\big(\frac{1}{4}\big)$, l'hypothèse de récurrence $H(d-1)$ nous fournit un nombre $C(d-1,\beta) > 1$. En imposant la condition

$$\frac{\beta C(d-1,\beta)}{\alpha\sqrt{2n+1}} \le |x| \le \alpha\sqrt{2n+1},$$

on a

$$\frac{C(d-1,\beta)}{\sqrt{2(2k)+1}} \le |x| \le \beta\sqrt{2(2k)+1}.$$

Cela nous mène à

$$\begin{aligned} e_{d,n}(x) &\gtrsim_{d,\beta} \sum_{\frac{\alpha^2}{2\beta^2} n \le k \le \frac{n}{2}} \frac{k^{\frac{d-1}{2}-1}}{\sqrt{1+n-2k}} \\ &\gtrsim_{d,\beta} n^{\frac{d-1}{2}-1} \sum_{\frac{\alpha^2}{2\beta^2} n \le k \le \frac{n}{2}} \frac{1}{\sqrt{1+n-2k}} \\ &\gtrsim_{d,\beta} n^{\frac{d-1}{2}-1} \times \sqrt{n} = n^{\frac{d}{2}-1}. \end{aligned}$$

Dernier cas : $H(d)$ avec $d > 2$ et $n \in 2\mathbb{N}+1$. On se ramène à $H(d-1)$ comme dans le troisième cas.

4E. ***Preuve du théorème 4.3, randomisation des fonctions de Hermite.*** Nous avons déjà vérifié les hypothèses des théorème 2.6 de dualité et théorème 2.5 d'interpolation (voir (107) et (108)). On a donc les points (i) et (ii). Décomposant $u = \sum_{n \in \mathbb{N}} \Pi_n(u) \in \mathbf{PL}^p\big(\mathbb{R}^d, \bigoplus E_{d,n}\big)$, on souhaite maintenant montrer que la norme $\|\cdot\|_{\mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_{d,n})}$ est équivalente à la suivante

$$N(u) := \|\Pi_0(u)\|_{L^2(\mathbb{R}^d)} + \left[\sum_{n \ge 1} n^{\frac{d}{2}-1} \left(\sum_{k \ge n} \frac{\|\Pi_k(u)\|^2_{L^2(\mathbb{R}^d)}}{k^{\frac{d}{2}}}\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}.$$

On va appliquer la proposition 4.1. Quitte à augmenter l'entier $n(d)$, on peut supposer que l'on a

$$\frac{C(d)}{\sqrt{2n+1}} \le 1 \le \alpha\sqrt{2n+1} \quad \text{pour tout } n \ge n(d).$$

On vérifie facilement que la norme $\|\cdot\|_{\mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_{d,n})}$ domine la norme N :

$$\begin{aligned} \|u\|^p_{\mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_{d,n})} &\gtrsim_{d,p} \int_{\mathbb{R}^d} \left(\sum_{n \ge n(d)} (1+n)^{-(d-1)} \|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)} e_{d,n}(x)\right)^{\frac{p}{2}} dx \\ &\gtrsim_{d,p} \int_{\mathbb{R}^d} \left(\sum_{n \ge n(d)} (1+n)^{-\frac{d}{2}} \|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)} \mathbf{1}_{1 \le |x| \le \alpha\sqrt{2n+1}}\right)^{\frac{p}{2}} dx \\ &\gtrsim_{d,p} \int_{\mathbb{R}^d} \sum_{n \ge n(d)} \left(\sum_{k > n} (1+k)^{-\frac{d}{2}} \|\Pi_k(u)\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{p}{2}} \mathbf{1}_{\sqrt{2n+1} < \frac{|x|}{\alpha} \le \sqrt{2n+3}}\, dx \\ &\gtrsim_{d,p} \sum_{n > n(d)} n^{\frac{d}{2}-1} \left(\sum_{k \ge n} (1+k)^{-\frac{d}{2}} \|\Pi_k(u)\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{p}{2}}. \end{aligned}$$

Pour récupérer les premiers termes d'indice $n \le n(d)$, il s'agit de remarquer les inégalités triviales

$$\forall n \in \mathbb{N} \cap [0, n(d)], \quad \|u\|_{\mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_{d,n})} \gtrsim_{d,p} (1+n)^{-\frac{(d-1)}{2}} \|\Pi_n(u)\|_{L^2(\mathbb{R}^d)} \|\sqrt{e_{d,n}}\|_{L^p(\mathbb{R}^d)}.$$

On obtient alors facilement l'estimation $\|u\|_{\mathbf{PL}^p(\mathbb{R}^d, \bigoplus E_{d,n})} \gtrsim_{d,p} N(u)$.

Montrons maintenant l'estimation réciproque $\|u\|_{\mathbf{PL}^p(\mathbb{R}^d,\bigoplus E_{d,n})} \lesssim_{d,p} N(u)$ à l'aide des deux fonctions

$$A(u,x) := \sum_{n\geq 0}(1+n)^{-\frac{d}{2}}\|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)}\mathbf{1}_{0\leq|x|\leq\sqrt{2(2n+1)}},$$
$$B(u,x) := e^{-\frac{|x|^2}{C(d)}}\sum_{n\geq 0}(1+n)^{-(d-1)}\|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)}\mathbf{1}_{\sqrt{2(2n+1)}\leq|x|},$$

où le terme gaussien provient de la proposition 4.1. D'après la proposition 4.1, nous avons

$$\|u\|^p_{\mathbf{PL}^p(\mathbb{R}^d,\bigoplus E_{d,n})} \lesssim_{d,p} \int_{\mathbb{R}^d}[A(u,x)+B(u,x)]^{\frac{p}{2}}\,dx \lesssim_{d,p} \int_{\mathbb{R}^d}A(u,x)^{\frac{p}{2}}+B(u,x)^{\frac{p}{2}}\,dx.$$

Par une argumentation similaire à celle que nous venons d'employer, on vérifie

$$\int_{\mathbb{R}^d}A(u,x)^{\frac{p}{2}}dx \lesssim_{d,p} N(u)^p.$$

Par ailleurs, nous avons trivialement

$$\begin{aligned}B(u,x) &\leq e^{-\frac{|x|^2}{C(d)}}\sum_{n\geq 0}(1+n)^{-\frac{d}{2}}\|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)}\\ &\leq e^{-\frac{|x|^2}{C(d)}}\|u\|^2_{\mathcal{H}^{-\frac{d}{2}}(\mathbb{R}^d)},\end{aligned}$$
$$\int_{\mathbb{R}^d}B(u,x)^{\frac{p}{2}}\,dx \lesssim_{d,p} N(u)^p.$$

Nous pouvons conclure que l'on a $\|u\|^p_{\mathbf{PL}^p(\mathbb{R}^d,\bigoplus E_{d,n})} \lesssim_{d,p} N(u)^p$.

4F. ***Preuve du théorème 4.4, injections de Sobolev probabilistes hermitiennes.*** Le théorème 4.3 permet par dualité de se ramener au cas $p\in[2,+\infty[$. Commençons par l'inclusion

$$\mathcal{H}^{-d(\frac{1}{2}-\frac{1}{p})}(\mathbb{R}^d)\subset\mathbf{PL}^p\big(\mathbb{R}^d,\bigoplus E_{d,n}\big).$$

On invoque l'inégalité triangulaire dans $L^{\frac{p}{2}}(\mathbb{R}^d)$ et (106) pour obtenir pour tout $u\in\mathcal{H}^{-d(\frac{1}{2}-\frac{1}{p})}(\mathbb{R}^d)$:

$$\begin{aligned}\Big\|\sum_{n\geq 0}\|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)}\frac{e_{d,n}}{(1+n)^{d-1}}\Big\|_{L^{\frac{p}{2}}(\mathbb{R}^d)} &\leq \sum_{n\geq 0}\|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)}\frac{\|e_{d,n}\|_{L^{\frac{p}{2}}(\mathbb{R}^d)}}{(1+n)^{d-1}}\\ &\leq \sum_{n\geq 0}\frac{\|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)}}{(1+n)^{d(\frac{1}{2}-\frac{1}{p})}} = \|u\|^2_{\mathcal{H}^{-d(\frac{1}{2}-\frac{1}{p})}(\mathbb{R}^d)}.\end{aligned}$$

Montrons maintenant l'inclusion $\mathbf{PL}^p\big(\mathbb{R}^d,\bigoplus E_{d,n}\big)\subset\mathcal{H}^{-d(\frac{1}{2}-\frac{1}{p})-\varepsilon}(\mathbb{R}^d)$ pour tout $\varepsilon>0$. Pour tout $u\in\mathbf{PL}^p\big(\mathbb{R}^d,\bigoplus E_{d,n}\big)$, on doit montrer

$$\|u\|_{\mathcal{H}^{-d(\frac{1}{2}-\frac{1}{p})-\varepsilon}(\mathbb{R}^d)} \leq C(p,d,\varepsilon)\|u\|_{\mathbf{PL}^p(\mathbb{R}^d,\bigoplus E_{d,n})}. \tag{117}$$

Sans perte de généralité, on peut supposer que l'on a $\Pi_n(u) = 0$ pour $n \gg 1$. Posons à cet effet

$$\forall n \in \mathbb{N}, \quad R_n := \sum_{k \geq n} (1+k)^{-\frac{d}{2}} \|\Pi_k(u)\|^2_{L^2(\mathbb{R}^d)}.$$

La décroissance et la positivité de (R_n) permettent d'écrire

$$\begin{aligned}
\|u\|^2_{\mathcal{H}^{-d(\frac{1}{2}-\frac{1}{p})-\varepsilon}(\mathbb{R}^d)} &= \sum_{n \in \mathbb{N}} (1+n)^{-d(\frac{1}{2}-\frac{1}{p})-\varepsilon} \|\Pi_n(u)\|^2_{L^2(\mathbb{R}^d)} \\
&= \sum_{n \in \mathbb{N}} (1+n)^{\frac{d}{p}-\varepsilon} (R_n - R_{n+1}) \\
&\leq C(d,p,\varepsilon) \sum_{n \in \mathbb{N}} (1+n)^{\frac{d}{p}-\varepsilon-1} R_n \\
&\leq C(d,p,\varepsilon) \sum_{n \in \mathbb{N}} \frac{1}{(1+n)^{\varepsilon + \frac{p-2}{p}}} \times (1+n)^{\frac{d-2}{p}} R_n \\
&\leq C(d,p,\varepsilon) \Big(\sum_{n \in \mathbb{N}} (1+n)^{\frac{d}{2}-1} R_n^{\frac{p}{2}} \Big)^{\frac{2}{p}}.
\end{aligned}$$

On conclut avec (110).

Appendice A: Optimalité de l'exposant max(2, p) dans le théorème 2.1

Il suffit de comprendre le cas unidimensionnel $d_n = 1$. Examinons l'espace mesuré $X = \mathbb{N}\backslash\{0,1\}$ muni de la mesure de comptage si bien que l'on a $L^p(X) = \ell^p(\mathbb{N}\backslash\{0,1\})$. Pour tout réel $p \in [2,\infty[$, on note $(X_{n,p})_{n\geq 2}$ une suite i.i.d. de variables aléatoires symétriques, réelles et vérifiant

$$\forall t \gg 1, \quad \boldsymbol{P}[|X_{n,p}| \geq t] = \frac{\ln(t)}{t^p}.$$

Il est clair que $X_{n,p}$ n'a pas de moment d'ordre p et a des moments d'ordre $q \in [1,p[$. On a aussi

$$\text{p.s.} \quad \sup_{n\geq 2} \frac{|X_{n,p}|}{n^{\frac{1}{p}} \ln^{\frac{2}{p}}(n)} = +\infty. \tag{118}$$

En effet, il s'agit de remarquer que pour tout entier $K \in \mathbb{N}$, on a

$$\sum_{n\geq 2} \boldsymbol{P}\big[|X_{n,p}| \geq K n^{\frac{1}{p}} \ln^{\frac{2}{p}}(n)\big] = +\infty.$$

Par indépendance des variables $X_{n,p}$, il existe presque sûrement une infinité d'entiers $n \geq 2$ tels que $|X_{n,p}| \geq K n^{\frac{1}{p}} \ln^{\frac{2}{p}}(n)$. On en déduit facilement (118). Revenons à l'optimalité de l'exposant $\max(2,p)$.

Cas $p \in [1,2]$: On fixe u une suite non nulle appartenant à $\ell^p(\mathbb{N}\backslash\{0,1\})$ et l'on examine les deux séries aléatoires dans $\ell^p(\mathbb{N}\backslash\{0,1\})$

$$\sum \frac{\varepsilon_n}{\sqrt{n}\ln(n)} u \quad \text{et} \quad \sum \frac{X_{n,2}}{\sqrt{n}\ln(n)} u, \tag{119}$$

La série aléatoire $\sum \varepsilon_n/(\sqrt{n}\ln(n))$ converge presque sûrement dans $\mathbb{R}$. Il s'ensuit que la première série aléatoire dans (119) converge presque sûrement dans $\ell^p(\mathbb{N}\backslash\{0,1\})$. La divergence presque sûre de la seconde série aléatoire dans (119) découle de (118).

Cas $p \in [2,+\infty[$: Une démarche similaire est valide en examinant les deux séries aléatoires

$$\sum \frac{\varepsilon_n}{n^{\frac{1}{p}}\ln^{\frac{2}{p}}(n)} w_n \quad \text{et} \quad \sum \frac{X_{n,p}}{n^{\frac{1}{p}}\ln^{\frac{2}{p}}(n)} w_n,$$

où $w_n = (0,\dots,0,1,0,\dots) \in \ell^p(\mathbb{N}\backslash\{0,1\})$ est la suite qui admet 1 à la position n et 0 ailleurs. Il est clair que la première série converge de manière déterministe dans $\ell^p(\mathbb{N}\backslash\{0,1\})$. De nouveau, (118) implique la divergence presque sûre de la seconde série aléatoire.

Appendice B: Preuve de la proposition 1.10, inégalité de Latała précisée (26)

Notons $B_n = [\varepsilon_{ij}]$ la matrice aléatoire de taille $n \times n$ et dont les coefficients ε_{ij} sont des variables aléatoires i.i.d. qui suivent une loi $\frac{1}{2}$-Bernoulli à valeurs dans $\{-1,+1\}$. Les inégalités de Kahane–Khintchine (39) dans l'espace de Banach $(\mathcal{M}_n(\mathbb{R}), \|\cdot\|_{\mathrm{op}})$ nous donnent l'encadrement :

$$\forall q \in [1,+\infty[, \quad \forall n \in \mathbb{N}^\star, \quad \mathbf{E}[\|B_n\|_{\mathrm{op}}] \le \mathbf{E}[\|B_n\|_{\mathrm{op}}^q]^{\frac{1}{q}} \le K_{q,1}\mathbf{E}[\|B_n\|_{\mathrm{op}}].$$

Cela signifie que tous les moments $\mathbf{E}[\|B_n\|_{\mathrm{op}}^q]^{\frac{1}{q}}$ ont le même ordre de grandeur si n tend vers $+\infty$. Par ailleurs, la théorie des matrices aléatoires explique que le moment $\mathbf{E}[\|B_n\|_{\mathrm{op}}]$ est asymptotiquement de l'ordre de $\sqrt{n}$ (voir [Tao 2012, Part 2.3]). Nous allons exploiter cette idée pour démontrer l'inégalité (26). Commençons par le lemme élémentaire suivant qui s'apparente à une version commutative de (26).

Lemme B.1. *Considérons un réel $p \in [2,+\infty[$ ainsi que N variables aléatoires $U_1,\dots,U_N$ réelles, centrées, i.i.d. et ayant un moment d'ordre p. Nous avons l'inégalité*

$$\mathbf{E}\left[\left|\frac{U_1+\cdots+U_N}{\sqrt{N}}\right|^p\right] \le (C\sqrt{p})^p\mathbf{E}[|U_1|^p].$$

Démonstration. L'idée se résume en deux points : on se ramène au cas où les variables U_i sont symétriques et l'on invoque les inégalités de Kahane–Khintchine à l'aide du théorème de Fubini. Si nous notons $\widetilde{U}_1,\dots,\widetilde{U}_N$ des copies indépendantes des variables $U_1,\dots,U_N$ alors l'inégalité de Jensen pour l'espérance en les variables $\widetilde{U}_1,\dots,\widetilde{U}_N$ donne

$$\mathbf{E}[|U_1+\cdots+U_N|^p] \le \mathbf{E}[|U_1-\widetilde{U}_1+\cdots+U_N-\widetilde{U}_N|^p].$$

Rappelons alors l'égalité triviale

$$\mathbf{E}[|U_1-\widetilde{U}_1+\cdots+U_N-\widetilde{U}_N|^p] = \mathbf{E}_{\omega'}\mathbf{E}_\omega\big[\big|\varepsilon_1(\omega')(U_1(\omega)-\widetilde{U}_1(\omega))+\cdots+\varepsilon_N(\omega')(U_N(\omega)-\widetilde{U}_N(\omega))\big|^p\big].$$

Il s'agit maintenant d'utiliser le théorème de Fubini, les inégalités de Kahane–Khintchine (39) et (41) (avec $d_n = 1$) pour obtenir

$$\begin{aligned}
&\mathbf{E}[|U_1 + \cdots + U_N|^p] \\
&\quad \le \mathbf{E}_\omega \mathbf{E}_{\omega'}\big[\big|\varepsilon_1(\omega')|U_1(\omega) - \widetilde{U}_1(\omega)| + \cdots + \varepsilon_N(\omega')|U_N(\omega) - \widetilde{U}_N(\omega)|\big|^p\big] \\
&\quad \le K_{p,2}^p \mathbf{E}_\omega\big[\mathbf{E}_{\omega'}\big[\big|\varepsilon_1(\omega')|U_1(\omega) - \widetilde{U}_1(\omega)| + \cdots + \varepsilon_N(\omega')|U_N(\omega) - \widetilde{U}_N(\omega)|\big|^2\big]^{\frac{p}{2}}\big] \qquad (120) \\
&\quad \le (C\sqrt{p})^p \mathbf{E}_\omega\bigg[\bigg(\sum_{i=1}^N |U_i(\omega) - \widetilde{U}_i(\omega)|^2\bigg)^{\frac{p}{2}}\bigg] \qquad (121) \\
&\quad \le (C\sqrt{p})^p \bigg(\sum_{i=1}^N \|(U_i - \widetilde{U}_i)^2\|_{L^{\frac{p}{2}}(\Omega)}\bigg)^{\frac{p}{2}} = (C\sqrt{pN})^p \|(U_1 - \widetilde{U}_1)^2\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \\
&\quad \le (C\sqrt{pN})^p \|U_1\|_{L^p(\Omega)}^p. \qquad \square
\end{aligned}$$

La preuve précédente est similaire à celle de la proposition 2.10 avec $(d_n, a_n, M_n) = (1, 1, U_n)$ à ceci près que nous pouvons utiliser en plus le principe de symétrisation. Nous avons écrit la preuve précédente d'abord parce que nous aurons besoin plus loin de considérer des variables seulement centrées au lieu de symétriques, mais aussi pour des raisons pédagogiques. En effet, la disparition élémentaire de l'espérance $\mathbf{E}_{\omega'}$ de la ligne (120) à la ligne (121) peut être interprétée comme suit : l'espace de Banach $\mathbb{R}$ est de type 2. Nous allons utiliser un substitut non-commutatif de cette propriété pour démontrer (26). C'est l'objet de la proposition suivante dont la preuve est très technique.

Proposition B.2 [Latała 2005, Theorem 1]. *Il existe une constante universelle $C \ge 1$ telle que pour tout entier $n \in \mathbb{N}^\star$, si l'on considère n^2 variables aléatoires i.i.d. $(g_{ij})_{1 \le i,j \le n}$ qui suivent une loi normale $\mathcal{N}_\mathbb{R}(0,1)$ et une matrice $[a_{ij}] \in \mathcal{M}_n(\mathbb{R})$ alors*

$$\mathbf{E}[\|a_{ij} g_{ij}\|_{\text{op}}] \le C\Bigg(\sqrt[4]{\sum_{i,j=1}^n a_{ij}^4} + \max_{1\le i\le n}\sqrt{\sum_{j=1}^n a_{ij}^2} + \max_{1\le j\le n}\sqrt{\sum_{i=1}^n a_{ij}^2}\Bigg). \qquad (122)$$

Latała démontre la proposition précédente pour en déduire l'estimation (27) qui se reformule

$$\mathbf{E}[\|X_{ij}\|_{\text{op}}] \le C\sqrt{n}\,\mathbf{E}[|X_{1,1}|^4]^{\frac{1}{4}}.$$

On peut attaquer la preuve de (26) et l'on commence comme dans [Latała 2005]. On note $\widetilde{X}_{ij}$ des copies indépendantes des variables aléatoires X_{ij}. En particulier, on a $\mathbf{E}[\widetilde{X}_{ij}] = 0$. Pour tout réel $p \in [4, +\infty[$, l'argument classique de l'inégalité de Jensen en les variables $\widetilde{X}_{ij}$ donne

$$\mathbf{E}[\|X_{ij}\|_{\text{op}}^p] \le \mathbf{E}[\|X_{ij} - \widetilde{X}_{ij}\|_{\text{op}}^p] = \mathbf{E}[\|\varepsilon_{ij}(X_{ij} - \widetilde{X}_{ij})\|_{\text{op}}^p] \le 2^p \mathbf{E}[\|\varepsilon_{ij} X_{ij}\|_{\text{op}}^p], \qquad (123)$$

où les n^2 variables aléatoires ε_{ij} sont indépendantes entre elles et vis-à-vis des variables X_{ij} et $\widetilde{X}_{ij}$. On diffère maintenant de [Latała 2005] en faisant appel aux inégalités de Kahane–Khintchine (39) dans l'espace de Banach $(\mathcal{M}_n(\mathbb{R}), \|\cdot\|_{\text{op}})$ afin de récupérer le moment d'ordre p :

$$\mathbf{E}[\|\varepsilon_{ij} X_{ij}\|_{\text{op}}^p] = \mathbf{E}_\omega \mathbf{E}_{\omega'}[\|\varepsilon_{ij}(\omega') X_{ij}(\omega)\|_{\text{op}}^p] \le K_{p,1}^p \mathbf{E}_\omega[\mathbf{E}_{\omega'}[\|\varepsilon_{ij}(\omega') X_{ij}(\omega)\|_{\text{op}}]^p]. \qquad (124)$$

Et l'on reprend de nouveau l'argumentation de Latała. En supposant que toutes les variables aléatoires sont indépendantes, (122) et le principe de contraction (voir (45)) donnent pour tout $\omega \in \Omega$,

$$\mathbf{E}_{\omega'}[\|\varepsilon_{ij}(\omega')X_{ij}(\omega)\|_{\text{op}}] \leq \frac{1}{\mathbf{E}[|g_{11}|]}\mathbf{E}_{\omega'}[\|g_{ij}(\omega')X_{ij}(\omega)\|_{\text{op}}] = \frac{\sqrt{\pi}}{\sqrt{2}}\mathbf{E}_{\omega'}[\|g_{ij}(\omega')X_{ij}(\omega)\|_{\text{op}}]$$
$$\leq C\left(\sqrt[4]{\sum_{i,j=1}^{n} X_{ij}(\omega)^4} + \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n} X_{ij}(\omega)^2} + \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n} X_{ij}(\omega)^2}\right).$$

Reprenant (123), (124) et tenant compte que les variables X_{ij} sont i.i.d., on obtient

$$\mathbf{E}[\|X_{ij}\|_{\text{op}}^p] \leq C(p)\mathbf{E}\left[\left(\sum_{i,j=1}^{n} X_{ij}^4\right)^{\frac{p}{4}} + \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} X_{ij}^2\right)^{\frac{p}{2}} + \max_{1\leq j\leq n}\left(\sum_{i=1}^{n} X_{ij}^2\right)^{\frac{p}{2}}\right]$$
$$\leq C(p)\mathbf{E}\left[\left(\sum_{i,j=1}^{n} X_{ij}^4\right)^{\frac{p}{4}} + \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} X_{ij}^2\right)^{\frac{p}{2}}\right].$$

La fin de cette preuve est différente de [Latała 2005] car on doit tenir compte de l'inégalité $p \geq 4$. Le premier terme ne posera aucun problème tandis que le dernier est plus délicat, c'est pour cela que nous le forçons à être centré en majorant

$$\sum_{j=1}^{n} X_{ij}^2 \leq \sum_{j=1}^{n}\mathbf{E}[X_{ij}^2] + \left|\sum_{j=1}^{n} X_{ij}^2 - \mathbf{E}[X_{ij}^2]\right|.$$

Cela nous donne

$$\mathbf{E}[\|X_{ij}\|_{\text{op}}^p] \leq C(p)[A(1) + A(2) + A(3)],$$
$$A(1) := \mathbf{E}\left[\left(\sum_{i,j=1}^{n} X_{ij}^4\right)^{\frac{p}{4}}\right],$$
$$A(2) := \max_{1\leq i\leq n}\left(\sum_{j=1}^{n}\mathbf{E}[X_{ij}^2]\right)^{\frac{p}{2}} = n^{\frac{p}{2}}\mathbf{E}[X_{11}^2]^{\frac{p}{2}},$$
$$A(3) := \mathbf{E}\left[\max_{1\leq i\leq n}\left|\sum_{j=1}^{n} X_{ij}^2 - \mathbf{E}[X_{ij}^2]\right|^{\frac{p}{2}}\right].$$

L'inégalité de Hölder donne d'abord $A(2) \leq n^{\frac{p}{2}}\mathbf{E}[|X_{11}|^p]$, puis

$$A(1) \leq \mathbf{E}\left[(n^2)^{(1-\frac{4}{p})\frac{p}{4}}\sum_{i,j=1}^{n}|X_{ij}|^p\right] = n^{\frac{p}{2}}\mathbf{E}[|X_{11}|^p].$$

Concernant le terme $A(3)$, nous majorons grossièrement à l'aide du lemme B.1 et de l'inégalité $p \geq 4$

$$A(3) \leq \sum_{i=1}^{n}\mathbf{E}\left[\left|\sum_{j=1}^{n} X_{ij}^2 - \mathbf{E}[X_{ij}^2]\right|^{\frac{p}{2}}\right] \leq C(p)n^{1+\frac{p}{4}}\mathbf{E}\big[|X_{11}^2 - \mathbf{E}[X_{11}^2]|^{\frac{p}{2}}\big] \leq C(p)n^{1+\frac{p}{4}}\mathbf{E}[|X_{11}|^p]$$
$$\leq C(p)n^{\frac{p}{2}}\mathbf{E}[|X_{11}|^p].$$

Appendice C: Preuve de la proposition 1.10, minoration de la plus petite valeur singulière

On commence par un lemme dual au lemme B.1.

Lemme C.1. *Considérons N variables aléatoires $U_1, \dots, U_N$ réelles, centrées, i.i.d. et ayant un moment d'ordre* 1. *Pour tout $(y_1, \dots, y_N) \in \mathbb{R}^N$, nous avons l'inégalité*

$$\mathbf{E}[|U_1|]\sqrt{y_1^2 + \cdots + y_N^2} \le C\,\mathbf{E}[|y_1 U_1 + \cdots + y_N U_N|].$$

Démonstration. Soient $\widetilde{U}_1, \dots, \widetilde{U}_N$ des copies indépendantes de $U_1, \dots, U_N$. On a

$$\mathbf{E}\big[|y_1(U_1 - \widetilde{U}_1) + \cdots + y_N(U_N - \widetilde{U}_N)|\big] \le 2\mathbf{E}[|y_1 U_1 + \cdots + y_N U_N|].$$

Puisque les variables $U_i - \widetilde{U}_i$ sont symétriques, le principe de contraction (théorème 2.16), l'inégalité de Khintchine et l'inégalité de Jensen donnent

$$\begin{aligned}
\mathbf{E}[|U_1|]\sqrt{y_1^2 + \cdots + y_N^2} &= \mathbf{E}[|U_1|]\mathbf{E}[|\varepsilon_1 y_1 + \cdots + \varepsilon_N y_N|^2]^{\frac{1}{2}} \\
&\le \mathbf{E}[|U_1 - \widetilde{U}_1|] \times K_{2,1}\mathbf{E}[|\varepsilon_1 y_1 + \cdots + \varepsilon_N y_N|] \\
&\le K_{2,1}\mathbf{E}\big[|y_1(U_1 - \widetilde{U}_1) + \cdots + y_N(U_N - \widetilde{U}_N)|\big] \\
&\le 2K_{2,1}\mathbf{E}[|y_1 U_1 + \cdots + y_N U_N|]. \qquad \square
\end{aligned}$$

Passons à la preuve de (25). Fixons $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. Pour tout $\omega \in \Omega$, la diagonalisation de la matrice symétrique positive $|M_n(\omega)| = \sqrt{{}^t M_n(\omega) M_n(\omega)}$ dans une base orthonormée fournit l'inégalité

$$|M_n(\omega)y|^2 := \langle M_n(\omega)y, M_n(\omega)y\rangle = {}^t y |M_n(\omega)|^2 y \le \|M_n(\omega)\|_{\mathrm{op}} {}^t y |M_n(\omega)| y.$$

L'inégalité de Cauchy–Schwarz donne alors

$$\begin{gathered}
\mathbf{E}_\omega[|M_n(\omega)y|] \le \mathbf{E}_\omega\big[\sqrt{\|M_n(\omega)\|_{\mathrm{op}}}\sqrt{{}^t y|M_n(\omega)|y}\big] \\
\mathbf{E}_\omega[|M_n(\omega)y|]^2 \le \mathbf{E}_\omega[\|M_n(\omega)\|_{\mathrm{op}}]\mathbf{E}_\omega[{}^t y|M_n(\omega)|y].
\end{gathered}$$

On invoque alors l'inégalité de Latała (27) pour contrôler le moment d'ordre 1 de $\|M_n\|_{\mathrm{op}}$:

$$\mathbf{E}_\omega[|M_n(\omega)y|]^2 \le C\,\mathbf{E}[|X_{11}|^4]^{\frac{1}{4}} \times {}^t y \mathbf{E}_\omega[|M_n(\omega)|]y.$$

On va maintenant utiliser l'égalité

$$|M_n(\omega)y| = \sqrt{\frac{1}{n}\sum_{i=1}^n \left|\sum_{j=1}^n X_{ij}(\omega)y_j\right|^2} = \left|\left(\frac{1}{\sqrt{n}}\left|\sum_{j=1}^n X_{1j}(\omega)y_j\right|, \dots, \frac{1}{\sqrt{n}}\left|\sum_{j=1}^n X_{nj}(\omega)y_j\right|\right)\right|$$

à l'aide de l'inégalité triangulaire entre $\mathbf{E}_\omega$ et la norme euclidienne $|\cdot|$ de $\mathbb{R}^n$:

$$\left(\frac{1}{n}\sum_{i=1}^n \mathbf{E}_\omega\left[\left|\sum_{j=1}^n X_{ij}(\omega)y_j\right|\right]^2\right)^{\frac{1}{2}} \le \mathbf{E}_\omega[|M_n(\omega)y|].$$

Le lemme C.1 nous permet d'obtenir (25) :

$$\mathbf{E}[|X_{11}|]^2(y_1^2+\cdots+y_n^2) \leq C\mathbf{E}_\omega[|M_n(\omega)y|]^2,$$
$$\frac{\mathbf{E}[|X_{11}|]^2}{C\mathbf{E}[|X_{11}|^4]^{\frac{1}{4}}}(y_1^2+\cdots+y_n^2) \leq {}^t y\mathbf{E}[|M_n|]y.$$

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ASYMPTOTIC LIMITS AND STABILIZATION FOR THE 2D NONLINEAR MINDLIN–TIMOSHENKO SYSTEM

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Dedicated to Enrique Fernández-Cara on the occasion of his 60th birthday

We show how the so-called von Kármán model can be obtained as a singular limit of a Mindlin–Timoshenko system when the modulus of elasticity in shear k tends to infinity. This result gives a positive answer to a conjecture by Lagnese and Lions in 1988. Introducing damping mechanisms, we also show that the energy of solutions for this modified Mindlin–Timoshenko system decays exponentially, uniformly with respect to the parameter k. As $k \to \infty$, we obtain the damped von Kármán model with associated energy exponentially decaying to zero as well.

1. Introduction

The Mindlin–Timoshenko system of equations is a widely used and physically fairly complete mathematical model to describe the dynamics of a plate, taking into account transverse shear effects; see, e.g., [Lagnese and Lions 1988]. This model is used, for example, to model aircraft wings; see, for instance, [Doyle 1997]. To describe this model, let $\Omega \subset \mathbb{R}^2$ be an open bounded set whose boundary Γ is regular enough. Consider $\{\Gamma_0, \Gamma_1\}$ to be a partition of Γ. Let $T > 0$ be given and consider the cylinder $Q = \Omega \times (0, T)$, with lateral boundary $\Sigma = \Sigma_0 \cup \Sigma_1$, where $\Sigma_i = \Gamma_i \times (0, T)$, $i = 0, 1$. The two-dimensional Mindlin–Timoshenko system is

$$\begin{cases} \frac{1}{12}\rho h^3 \phi_{1tt} - L_1(\phi_1, \phi_2, \psi) = 0 & \text{in } Q, \\ \frac{1}{12}\rho h^3 \phi_{2tt} - L_2(\phi_1, \phi_2, \psi) = 0 & \text{in } Q, \\ \rho h \psi_{tt} - L_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2) = 0 & \text{in } Q, \\ \rho h \eta_{1tt} - L_4(\psi, \eta_1, \eta_2) = 0 & \text{in } Q, \\ \rho h \eta_{2tt} - L_5(\psi, \eta_1, \eta_2) = 0 & \text{in } Q. \end{cases} \tag{1-1}$$

We complete the system with the boundary conditions

$$\begin{gathered} \phi_1 = \phi_2 = \psi = \eta_1 = \eta_2 = 0 \quad \text{on } \Sigma_0, \\ \big\{\mathcal{B}_1(\phi_1, \phi_2),\, \mathcal{B}_2(\phi_1, \phi_2),\, \mathcal{B}_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2),\, \mathcal{B}_4(\eta_1, \eta_2),\, \mathcal{B}_5(\eta_1, \eta_2)\big\} = \{0, 0, 0, 0, 0\} \quad \text{on } \Sigma_1, \end{gathered} \tag{1-2}$$

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and initial data

$$\{\phi_1(\cdot\,,0),\phi_2(\cdot\,,0),\psi(\cdot\,,0),\eta_1(\cdot,0),\eta_2(\cdot\,,0)\}=\{\phi_{10},\phi_{20},\psi_0,\eta_{10},\eta_{20}\}\quad\text{in }\Omega, \tag{1-3a}$$

$$\{\phi_{1t}(\cdot\,,0),\phi_{2t}(\cdot\,,0),\psi_t(\cdot\,,0),\eta_{1t}(\cdot\,,0),\eta_{2t}(\cdot\,,0)\}=\{\phi_{11},\phi_{21},\psi_1,\eta_{11},\eta_{21}\}\quad\text{in }\Omega, \tag{1-3b}$$

where

$$\begin{aligned}
L_1(\phi_1,\phi_2,\psi)&=D\big(\phi_{1xx}+\tfrac{1}{2}(1-\mu)\phi_{1yy}+\tfrac{1}{2}(1+\mu)\phi_{2xy}\big)-k(\phi_1+\psi_x),\\
L_2(\phi_1,\phi_2,\psi)&=D\big(\phi_{2yy}+\tfrac{1}{2}(1-\mu)\phi_{2xx}+\tfrac{1}{2}(1+\mu)\phi_{1xy}\big)-k(\phi_2+\psi_y),\\
L_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2)&=k[(\psi_x+\phi_1)_x+(\psi_y+\phi_2)_y]+(N_1\psi_x+N_{12}\psi_y)_x+(N_2\psi_y+N_{12}\psi_x)_y,\\
L_4(\psi,\eta_1,\eta_2)Y&=N_{1x}+N_{12y},\\
L_5(\psi,\eta_1,\eta_2)&=N_{2y}+N_{12x},\\
\mathcal{B}_1(\phi_1,\phi_2)&=D\big[\nu_1\phi_{1x}+\mu\nu_1\phi_{2y}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})\nu_2\big],\\
\mathcal{B}_2(\phi_1,\phi_2)&=D\big[\nu_2\phi_{2y}+\mu\nu_2\phi_{1x}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})\nu_1\big],\\
\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2)&=k\left(\frac{\partial\psi}{\partial\nu}+\nu_1\phi_1+\nu_2\phi_2\right)+(\nu_1N_1+\nu_2N_{12})\psi_x+(\nu_2N_2+\nu_1N_{12})\psi_y,\\
\mathcal{B}_4(\psi,\eta_1,\eta_2)&=\nu_1N_1+\nu_2N_{12},\\
\mathcal{B}_5(\psi,\eta_1,\eta_2)&=\nu_2N_2+\nu_1N_{12},\\
N_1&=\frac{Eh}{1-\mu^2}\big(\eta_{1x}+\mu\eta_{2y}+\tfrac{1}{2}\psi_x^2+\tfrac{1}{2}\mu\psi_y^2\big),\\
N_2&=\frac{Eh}{1-\mu^2}\big(\eta_{2y}+\mu\eta_{1x}+\tfrac{1}{2}\psi_y^2+\tfrac{1}{2}\mu\psi_x^2\big),\\
N_{12}&=\frac{Eh}{2(1+\mu)}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y).
\end{aligned}$$

In system (1-1), subscripts mean partial derivatives. The vector $\nu=(\nu_1,\nu_2)$ represents the outward unit normal to Ω and $\frac{\partial}{\partial\nu}$ stands for the normal derivative. The unknowns are $\phi_1=\phi_1(x,y,t)$, $\phi_2=\phi_2(x,y,t)$, $\psi=\psi(x,y,t)$, $\eta_1=\eta_1(x,y,t)$, and $\eta_2=\eta_2(x,y,t)$. Physically, the functions ϕ_1 and ϕ_2 represent, respectively, the angles of rotation of the cross sections $x=\text{const.}$, $y=\text{const.}$ containing the filament which, when the plate is in equilibrium, is orthogonal to the middle surface at the point $(x,y,0)$. The function ψ is the vertical displacement, and η_1, η_2 are the in-plane displacement of the plate at time t of the cross section located at (x,y) units from the endpoint $(x,y)=(0,0)$. The positive constant h represents the thickness of the plate which, in this model, is considered to be small and uniform with respect to x. The constant ρ is the mass density per unit volume of the plate and the parameter k is the so-called modulus of elasticity in shear. The constant E is the Young's modulus and the constant μ, $0<\mu<\frac{1}{2}$, is the Poisson's ratio. The constant D is the modulus of flexural rigidity and is given by $D=Eh^3/(12(1-\mu^2))$. The constant k is given by the expression $k=\hat{k}Eh/(2(1+\mu))$, where $\hat{k}$ is a shear correction coefficient. For more details concerning the Mindlin–Timoshenko hypotheses and the governing equations see, for instance, [Lagnese and Lions 1988].

For the nonlinear system (1-1)–(1-3), Rahmani [2014] considered a plate reinforced by a thin stiffener on a portion of its boundary and modeled this junction through an approximate model where the stiffener has a role on its boundary conditions.

The linear version of system (1-1)–(1-3) is

$$\begin{cases} \frac{1}{12}\rho h^3\phi_{1tt} - L_1(\phi_1,\phi_2,\psi) = 0 & \text{in } Q, \\ \frac{1}{12}\rho h^3\phi_{2tt} - L_2(\phi_1,\phi_2,\psi) = 0 & \text{in } Q, \\ \rho h\psi_{tt} - \widetilde{L}_3(\phi_1,\phi_2,\psi) = 0 & \text{in } Q, \end{cases} \tag{1-4}$$

where L_1, L_2 are defined above and

$$\widetilde{L}_3(\phi_1,\phi_2,\psi) = k[(\psi_x+\phi_1)_x + (\psi_y+\phi_2)_y].$$

There are quite a few works on this system: Lagnese and Lions [1988] studied its well-posedness and analyzed its asymptotic limit when the parameter k tends to infinity. Lagnese [1989] studied problems of existence, uniqueness and some other important properties as the asymptotic behavior in time when some damping effects are considered. Chueshov and Lasiecka [2006] studied the dynamics for a class of Mindlin–Timoshenko plate models with nonlinear feedback forces and showed the existence of a compact global attractor for the system. Furthermore they studied its limiting properties when the shear modulus tends to infinity. Fernández Sare [2009] investigated system (1-4) with frictional dissipations acting on the equations for the rotation angles and proved that this system is not exponentially stable independent of any relations between the constants of the system. Moreover, he showed that the solution decays polynomially to zero, with rates that can be improved depending on the regularity of the initial data. Rahmani [2015] studied system (1-4) and obtained results similar to those in [Rahmani 2014] for the system (1-1)–(1-3).

If one assumes the filament of the plate to remain orthogonal to the deformed middle surface, the transverse shear effects are neglected, and the resulting model is the so-called von Kármán system; see [Lagnese and Lions 1988]:

$$\begin{cases} \rho h\psi_{tt} - \frac{1}{12}\rho h^3\Delta\psi_{tt} + D\Delta^2\psi - (N_1\psi_x + N_{12}\psi_y)_x - (N_2\psi_y + N_{12}\psi_x)_y = 0 & \text{in } Q, \\ \rho h\eta_{1tt} - (N_{1x} + N_{12y}) = 0 & \text{in } Q, \\ \rho h\eta_{2tt} - (N_{2y} + N_{12x}) = 0 & \text{in } Q, \end{cases} \tag{1-5}$$

with boundary conditions

$$\begin{gathered} \psi = \frac{\partial\psi}{\partial\nu} = \eta_1 = \eta_2 = 0 \quad \text{on } \Sigma_0, \\ D[\Delta\psi + (1-\mu)(2\nu_1\nu_2\psi_{xy} - \nu_1^2\psi_{yy} - \nu_2^2\psi_{xx})] = 0 \quad \text{on } \Sigma_1, \\ D\left[\frac{\partial(\Delta\psi)}{\partial\nu} + (1-\mu)\frac{\partial}{\partial\tau}[(\nu_1^2-\nu_2^2)\psi_{xy} + \nu_1\nu_2(\psi_{yy}-\psi_{xx})]\right] - \frac{1}{12}\rho h^3\frac{\partial\psi_{tt}}{\partial\nu} \\ -(\nu_1N_1+\nu_2N_{12})\psi_x - (\nu_2N_2+\nu_1N_{12})\psi_y = 0 \quad \text{on } \Sigma_1, \\ \nu_1N_1 + \nu_2N_{12} = 0 \quad \text{on } \Sigma_1, \\ \nu_2N_2 + \nu_1N_{12} = 0 \quad \text{on } \Sigma_1, \end{gathered} \tag{1-6}$$

and initial data

$$\{\psi(\cdot,0),\eta_1(\cdot,0),\eta_2(\cdot,0)\} = \{\psi_0,\eta_{10},\eta_{20}\} \quad \text{in } \Omega, \tag{1-7a}$$

$$\{\psi_t(\cdot,0),\eta_{1t}(\cdot,0),\eta_{2t}(\cdot,0)\} = \{\psi_1,\eta_{11},\eta_{21}\} \quad \text{in } \Omega. \tag{1-7b}$$

In (1-6), $\tau = (-\nu_2, \nu_1)$ is the tangent vector to Ω and $\partial/\partial\tau$ represents the tangential derivative. System (1-5)–(1-7) has been an object of study for many years. Let us mention some known results about this type of system. Lasiecka [1998] and Favini et al. [1996] studied well-posedness for this problem, as well as the regularity of its solution. Perla Menzala and Zuazua [1997] proved exponential decay rates for the energy of the system for a bounded smooth thermoelastic plate clamped on its boundary. A similar result was obtained by Kang [2013] for von Kármán equations with a memory term. Finally, for monotonic functions with certain growth properties at the origin and at infinity, Lagnese and Leuring [1991] showed that the one-dimensional von Kármán is uniformly asymptotically stable.

Neglecting the shear effects of the plate, obtaining system (1-5) is formally equivalent to considering the modulus of elasticity k tending to infinity in system (1-1), since k is inversely proportional to the shear angle. The present paper is devoted to analyzing the asymptotic limit of the nonlinear Mindlin–Timoshenko system (1-1) as $k \to \infty$. This problem was mentioned in [Lagnese and Lions 1988, p. 24], where it was conjectured that system (1-1) approaches, in some sense, the von Kármán system (1-5), as $k \to \infty$:

> One expects that, as $k \to \infty$, solutions of the system (1-1) will converge (in some sense) to solution of the von Kármán system (1-5). However, a rigorous proof of convergence is lacking and seems to be a difficult question.

In this direction, Lagnese and Lions [1988] proved (see also [Araruna and Zuazua 2008] for the one-dimensional case) that, in the linear case, the solution of the Mindlin–Timoshenko model (1-4) converges, as $k \to \infty$, towards to the solution of the Kirchhoff model (subject to appropriate boundary conditions)

$$\rho h\psi_{tt} - \tfrac{1}{12}\rho h^3\Delta\psi_{tt} + D\Delta^2\psi = 0. \tag{1-8}$$

Later on, in [Araruna et al. 2010], the authors studied the one-dimensional nonlinear Mindlin–Timoshenko system with an extra fourth-order regularizing term

$$\begin{cases} \tfrac{1}{12}\rho h^3\phi_{tt} - D\phi_{xx} + k(\phi+\psi_x) = 0 & \text{in } Q,\\ \rho h\psi_{tt} - k(\phi+\psi_x)_x - Eh\big[\psi_x\big(\eta_x + \tfrac{1}{2}\psi_x^2\big)\big]_x + \tfrac{1}{k}\psi_{xxxx} = 0 & \text{in } Q,\\ \rho h\eta_{tt} - Eh\big(\eta_x + \tfrac{1}{2}\psi_x^2\big)_x = 0 & \text{in } Q, \end{cases} \tag{1-9}$$

and showed that, as $k \to \infty$, the system (1-9) converges toward the one-dimensional von Kármám system

$$\begin{cases} \rho h\psi_{tt} - \tfrac{1}{12}\rho h^3\psi_{xxtt} + D\psi_{xxxx} - Eh\big[\psi_x\big(\eta_x + \tfrac{1}{2}\psi_x^2\big)\big]_x = 0 & \text{in } Q,\\ \rho h\eta_{tt} - Eh\big(\eta_x + \tfrac{1}{2}\psi_x^2\big)_x = 0 & \text{in } Q. \end{cases} \tag{1-10}$$

In the argument used in [Araruna et al. 2010], the use of the extra fourth-order regularizing term was indispensable, since it ensures the compactness of a family of solutions, as $k \to \infty$, allowing one to pass to the limit in the nonlinear term. Here, we study the nonlinear two-dimensional problem without any regularizing term. We prove that the Mindlin–Timoshenko system converges to the von Kármán one, therefore giving a positive answer for the 1988 Lagnese–Lions conjecture. We note that our argument here can be used for the one-dimensional case as well, ensuring the conjecture holds also in the one-dimensional case (as would be expected).

In the context of asymptotic limits, with respect to singular coefficients, Perla Menzala and Zuazua [Perla Menzala and Zuazua 1999] proved that the one-dimensional von Kármán system of equations approaches (weakly) to a nonlocal beam equation of Timoshenko type as a suitable parameter tends to zero. In [Perla Menzala and Zuazua 2000a], the authors considered a dynamical one-dimensional nonlinear von Kármán model depending on one parameter $\varepsilon > 0$ and studied its weak limit as $\varepsilon \to 0$. Furthermore, they proved that, depending on the type of boundary condition, the nonlinearity of the Timoshenko model may either vanish or may become a nonlinearity concentrated on the extremes of the beam. In [Perla Menzala and Zuazua 2000b], the full nonlinear dynamic von Kármán system of equations was considered and the authors showed how the so-called Timoshenko and Berger models for thin plates may be obtained as singular limits of the von Kármán system when a suitable parameter tends to zero. We also mention the work [Perla Menzala et al. 2002], where the authors obtained the stabilization of Berger–Timoshenko's equation as a limit of the uniform stabilization of the von Kármán system of beams and plates with respect to a singular parameter.

The second part of this work concerns stabilization. To our knowledge, exponential stability has not been investigated for the two-dimension nonlinear Mindlin–Timoshenko system, so we study decay properties of its solutions with both internal and boundary damping. More precisely, we show the following: adding appropriate damping terms, there is a uniform (with respect to k) rate of decay for the total energy of the solutions for (1-1) as $t \to \infty$. As a consequence of this analysis, we obtain a decay rate for the total energy of the solutions for the von Kármán system (as $t \to \infty$) as a singular limit of the uniform (with respect to k) decay rate of the energy of the Mindlin–Timoshenko system.

Let us mention some known results related to the stabilization. In the one-dimensional case, Araruna et al. [2010] showed the exponential stability of the nonlinear Mindlin–Timoshenko beam under internal damping. Stabilization results for the linear model were obtained in [Lagnese 1989; Kim and Renardy 1987] considering damping in both equations, and in [Alabau-Boussouira 2007] with a single nonlinear feedback control. In [Ammar-Khodja et al. 2003], the system is damped by a memory-type term. In the two-dimensional case, the uniform stabilization for linear Mindlin–Timoshenko model was studied in [Fernández Sare 2009] considering frictional dissipations acting on the equations for the rotations angle. Grobbelaar-Van Dalsen [2015] studied the polynomial decay rate of the Mindlin–Timoshenko plate model with thermal dissipation. Stabilization results were obtained in [Nicaise 2011] for the multidimensional case with nonconstant and nonsmooth coefficients, when the interior dissipation acts either on both equations or only on the elasticity equation. The stabilization of the von Kármán system, in the two-dimensional case, was studied in [Perla Menzala and Zuazua 1997], where the energy decreases along trajectories. Bradley and Lasiecka [1992] studied the local exponential stabilization for an unstructured perturbation and feedback controls. Kang [2013] proved the exponential decay for the nonlinear von Kármán system with memory.

This work is organized as follows. In Section 2, we rigorously study the behavior of the Mindlin–Timoshenko system towards the von Kármán system as $k \to \infty$. More precisely, we prove that solutions

$\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\}$ of (1-1)–(1-3) converge to $\{-\psi_x, -\psi_y, \psi, \eta_1, \eta_2\}$ as $k \to \infty$, where $\{\psi, \eta_1, \eta_2\}$ solves system (1-5)–(1-7). In Sections 3 and 4 we prove that, adding appropriate damping terms (internal and boundary, respectively), one can prove a uniform (in k) exponential decay property for the solutions of (1-1)–(1-3). Finally, in Section 5, we briefly discuss some related issues and open problems.

2. Asymptotic limit

In this section, we study the asymptotic limit of the solutions for the nonlinear Mindlin–Timoshenko system (1-1)–(1-3) as $k \to \infty$. To study this problem, we consider the Hilbert space

$$\mathcal{X} = [H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)]^2 \times [W^{1,4}(\Omega) \cap H^1_{\Gamma_0}(\Omega)] \times L^2(\Omega) \times [H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)]^2, \tag{2-1}$$

where $H^1_{\Gamma_0}(\Omega) = \{\varphi : \varphi \in H^1(\Omega),\ \varphi = 0 \text{ on } \Gamma_0\}$.

The energy $E_k(t)$ of solutions is given by

$$E_k(t) = \tfrac{1}{2}\Big\{ \tfrac{1}{12}\rho h^3\big[|\phi_{1t}|^2+|\phi_{2t}|^2\big]+\rho h\big[|\psi_t|^2+|\eta_{1t}|^2+|\eta_{2t}|^2\big]+k\big[|\phi_1+\psi_x|^2+|\phi_2+\psi_y|^2\big] \\ +F([b_{ij}],[b_{ij}])+D\Big[|\phi_{1x}|^2+|\phi_{2y}|^2+\tfrac{1}{2}(1-\mu)|\phi_{1y}+\phi_{2x}|^2+2\mu\int_\Omega(\phi_{1y}\phi_{2x})\,dx\,dy\Big]\Big\}, \tag{2-2}$$

where

$$b_{11} = \eta_{1x} + \tfrac{1}{2}\psi_x^2, \quad b_{22} = \eta_{2y} + \tfrac{1}{2}\psi_y^2, \quad b_{12} = b_{21} = \eta_{1y} + \eta_{2x} + \psi_x\psi_y,$$

and

$$F([b_{ij}]) = \frac{Eh}{1-\mu^2}\left\{\mu\begin{bmatrix} b_{11}+b_{22} & 0 \\ 0 & b_{11}+b_{22}\end{bmatrix} + c(1-\mu)\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right\}.$$

Note that

$$(F([b_{ij}]),[b_{ij}])_{(L^2(\Omega))^4} \\ = \left(\frac{Eh}{1-\mu^2}\left\{\mu\begin{bmatrix} b_{11}+b_{22} & 0 \\ 0 & b_{11}+b_{22}\end{bmatrix} + c(1-\mu)\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right\}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right) \\ = \frac{Eh}{1-\mu^2}\big\{\mu\big|\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\big|^2+(1-\mu)|b_{11}|^2+(1-\mu)|b_{22}|^2+\tfrac{1}{2}(1-\mu)|\eta_{1y}+\eta_{2x}+\psi_x\psi_y|^2\big\} > 0$$

since $Eh/(1-\mu^2) > 0$ and $0 < \mu < 1$, which shows that F is positive definite. Moreover, we have by [Lagnese 1989, Lemma 2.1] that

$$D\Big[|\phi_{1x}|^2 + |\phi_{2y}|^2 + \tfrac{1}{2}(1-\mu)|\phi_{1y}+\phi_{2x}|^2 + 2\mu\int_\Omega(\phi_{1y}\phi_{2x}\,dx\,dy)\Big] \ge C\|\phi_1\|^2_{H^1(\Omega)} + \|\phi_2\|^2_{H^1(\Omega)}.$$

So, the energy is positive. Furthermore,

$$E_k(t) = E_k(0) \quad \forall t \ge 0. \tag{2-3}$$

The main result of this paper is to give a positive response to a conjecture from [Lagnese and Lions 1988]. Our result is as follows.

Theorem 2.1. *Let* $\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\}$ *be a solution of the system* (1-1)–(1-3) *with initial data* $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$ *satisfying*

$$\phi_{10} + \psi_{0x} = 0 \quad \text{and} \quad \phi_{20} + \psi_{0y} = 0 \quad \text{in } \Omega. \tag{2-4}$$

Then, letting $k \to \infty$, *one gets*

$$\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\} \to \{-\psi_x, -\psi_y, \psi, \eta_1, \eta_2\} \quad \text{weak} * \text{ in } L^\infty(0, T, [H^1_{\Gamma_0}(\Omega)]^3 \times [L^2(\Omega)]^2),$$

where $\{\psi, \eta_1, \eta_2\}$ *solves* (1-5)–(1-7).

Remark 2.2. The variational formulation of system (1-5)–(1-7) is given by

$$\rho h \frac{d}{dt}(\psi_t, c) + \tfrac{1}{12}\rho h^3 \frac{d}{dt}(\nabla\psi_t, \nabla c) + \rho h \frac{d}{dt}(\eta_{1t}, d) + \rho h \frac{d}{dt}(\eta_{2t}, e) + (N_1\psi_x + N_{12}\psi_y, c_x) \\ + (N_2\psi_y + N_{12}\psi_x, c_y) + (N_1^k, d_x) + (N_{12}^k, d_y) + (N_2^k, e_y) + (N_{12}^k, e_x) + D(\Delta\psi, \Delta c) = 0, \tag{2-5}$$

for all $\{c, d, e\} \in [H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)] \times [H^1_{\Gamma_0}(\Omega)]^2$ and the initial conditions (1-6). In equation (2-5), $(\cdot\,,\cdot)$ represents the inner product in $L^2(\Omega)$. Furthermore, the system (1-5)–(1-7) is conservative; that is, its energy

$$E(t) = \tfrac{1}{2}\Big\{\rho h\big[|\psi_t|^2 + |\eta_{1t}|^2 + |\eta_{2t}|^2\big] + \tfrac{1}{12}\rho h^3|\nabla\psi_t|^2 + D|\Delta\psi|^2 \\ + \frac{Eh}{1-\mu}\int_\Omega \big[\eta_{1x} + \tfrac{1}{2}\psi_x^2\big]^2 + \big[\eta_{2y} + \tfrac{1}{2}\psi_y^2\big]^2 \\ + \big[\eta_{1x} + \eta_{2y} + \tfrac{1}{2}|\nabla\psi|^2\big]^2 + \tfrac{1}{2}(1-\mu)\big[\eta_{1y} + \eta_{2x} + \psi_x\psi_y\big]^2\,dx\,dy\Big\} \tag{2-6}$$

satisfies $E(t) = E(0)$ for all $t \in [0, T]$.

Proof of Theorem 2.1. For each $k > 0$ fixed, let $\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\}$ be the solution of system (1-1)–(1-3) with data $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$. Since the initial data $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\}$ satisfy the condition (2-4), one has, due to the conservation of energy (2-3),

$$E_k(t) \le C \quad \forall\, k > 0,\ \forall\, t > 0. \tag{2-7}$$

From now on, the letter C stands for a generic positive constant which may vary from line to line (unless otherwise stated). The estimate (2-7) implies that the sequences (in k)

$$(\phi_{1t}^k), \quad (\phi_{2t}^k), \quad (\psi_t^k), \quad (\eta_{1t}^k), \quad (\eta_{2t}^k), \quad \sqrt{k}(\phi_1^k + \psi_x^k), \quad \sqrt{k}(\phi_2^k + \psi_y^k), \quad (\phi_{1x}^k), \quad (\phi_{2y}^k), \\ (\phi_{1y}^k + \phi_{2x}^k), \quad \big(\eta_{1x}^k + \tfrac{1}{2}[\psi_x^k]^2\big), \quad \big(\eta_{2y}^k + \tfrac{1}{2}[\psi_y^k]^2\big), \quad \big(\eta_{1x}^k + \eta_{2y}^k + \tfrac{1}{2}[\nabla\psi^k]^2\big), \quad (\eta_{1y}^k + \eta_{2x}^k + \psi_x^k\psi_y^k)$$

are bounded in $L^\infty(0, T, L^2(\Omega))$. Furthermore,

$$[\phi_{1y}^k]_x = [\phi_{1x}^k]_y \in H^{-1}(\Omega) \quad \text{and} \quad [\phi_{2x}^k]_y = [\phi_{2y}^k]_x \in H^{-1}(\Omega),$$

since (ϕ_{1x}^k) and (ϕ_{2y}^k) are bounded in $L^\infty(0, T, L^2(\Omega))$. On the other hand,

$$[\phi_{1y}^k]_y = [\phi_{1y}^k + \phi_{2x}^k]_y - [\phi_{2x}^k]_y = [\phi_{1y}^k + \phi_{2x}^k]_y - [\phi_{2y}^k]_x \in H^{-1}(\Omega),$$

which implies that (ϕ^k_{1y}) is bounded in $L^\infty(0, T, L^2(\Omega))$. Similarly, one can show that (ϕ^k_{2x}) is bounded in $L^\infty(0, T, L^2(\Omega))$. Thus, the sequences (in k) (ϕ^k_1), (ϕ^k_2) and (ψ^k) are bounded in $L^\infty(0, T, H^1_{\Gamma_0}(0, L))$.

Since, for each k, we have η^k_{1t} and η^k_{2t} belong to $C^0([0, T], L^2(\Omega))$, we can write

$$\eta^k_1(t) = \eta_{10} + \int_0^t \eta^k_{1t}(s)\, ds \quad \text{and} \quad \eta^k_2(t) = \eta_{20} + \int_0^t \eta^k_{2t}(s)\, ds.$$

Therefore, since (η^k_{1t}) is bounded in $L^\infty(0, T, L^2(\Omega))$, the sequence (η^k_1) is bounded $L^\infty(0, T, L^2(\Omega))$. Indeed,

$$|\eta^k_1| = \left| \eta_{10} + \int_0^t \eta^k_{1t}\, ds \right| \le C + \int_0^t |\eta^k_{1t}|\, ds \le C.$$

Analogously, it follows that (η^k_2) is bounded in $L^\infty(0, T, L^2(\Omega))$. Therefore, the sequences (η^k_1), (η^k_2) are bounded in $L^\infty(0, T, L^2(\Omega))$. Extracting subsequences, without changing notation, one gets

$$\{\phi^k_1, \phi^k_2, \psi^k, \eta^k_1, \eta^k_2\} \to \{\phi_1, \phi_2, \psi, \eta_1, \eta_2\} \quad \text{weak} * \text{ in } L^\infty(0, T; [H^1_{\Gamma_0}(\Omega)]^3 \times [L^2(\Omega)]^2), \tag{2-8}$$

with

$$\phi_1 + \psi_x = 0 \quad \text{and} \quad \phi_2 + \psi_y = 0, \tag{2-9}$$

$$\{\phi^k_{1t}, \phi^k_{2t}, \psi^k_t, \eta^k_{1t}, \eta^k_{2t}\} \to \{\phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}\} \quad \text{weak} * \quad \text{in} \quad L^\infty(0, T; [L^2(\Omega)]^5), \tag{2-10}$$

$$\eta^k_{1x} + \tfrac{1}{2}[\psi^k_x]^2 \to \alpha \quad \text{weak} * \text{ in } L^\infty(0, T, L^2(\Omega)), \tag{2-11}$$

$$\eta^k_{2y} + \tfrac{1}{2}[\psi^k_y]^2 \to \beta \quad \text{weak} * \text{ in } L^\infty(0, T, L^2(\Omega)), \tag{2-12}$$

$$\eta^k_{1y} + \eta^k_{2x} + \psi^k_x \psi^k_y \to \gamma \quad \text{weak} * \text{ in } L^\infty(0, T, L^2(\Omega)). \tag{2-13}$$

Now, using a compactness theorem due to Aubin and Lions see [Simon 1987, Corollary 4], we obtain

$$\phi^k_1 \to \phi_1 \quad \text{strongly in } L^2(Q), \tag{2-14}$$

$$\phi^k_2 \to \phi_2 \quad \text{strongly in } L^2(Q). \tag{2-15}$$

Therefore, given $\varepsilon > 0$, for large enough k one has

$$|\psi^k_x + \phi_1| \le |\psi^k_x + \phi^k_1| + |\phi^k_1 - \phi_1| \le \frac{C}{\sqrt{k}} + \varepsilon.$$

Consequently,

$$\psi^k_x \to -\phi_1 \quad \text{in } L^2(Q). \tag{2-16}$$

On the other hand, we also have by the convergence (2-8) that

$$\psi^k_x \to \psi_x \quad \text{in } \mathcal{D}'(Q). \tag{2-17}$$

Combining (2-16) and (2-17), we obtain

$$\psi_x = -\phi_1.$$

In a similar way, we get

$$\psi_y = -\phi_2.$$

Therefore,

$$\psi^k \to \psi \quad \text{strongly in } L^\infty(0, T, H^1_{\Gamma_0}(\Omega)). \tag{2-18}$$

By the previous convergence we conclude that

$$[\psi^k_x]^2 \to [\psi_x]^2 \quad \text{strongly in } L^\infty(0, T, L^1(\Omega)), \tag{2-19}$$

$$[\psi^k_y]^2 \to [\psi_y]^2 \quad \text{strongly in } L^\infty(0, T, L^1(\Omega)). \tag{2-20}$$

On other hand, the sequences (η^k_1), (η^k_2) are bounded in $L^\infty(0, T, L^2(\Omega))$ and so

$$\eta^k_{1x} \to \eta_{1x} \quad \text{weak} * \text{ in } L^\infty(0, T, H^{-1}(\Omega)), \tag{2-21}$$

$$\eta^k_{2y} \to \eta_{2y} \quad \text{weak} * \text{ in } L^\infty(0, T, H^{-1}(\Omega)). \tag{2-22}$$

The same holds for (η^k_{1y}) and (η^k_{2x}). Combining the convergences (2-18)–(2-22), it follows that

$$\alpha = \eta_{1x} + \tfrac{1}{2}\psi_x^2, \quad \beta = \eta_{2y} + \tfrac{1}{2}\psi_y^2, \quad \gamma = \eta_{1y} + \eta_{2x} + \psi_x\psi_y,$$

$$N^k_1\psi^k_x + N^k_{12}\psi^k_y \to N_1\psi_x + N_{12}\psi_y \quad \text{weak} * \text{ in } L^\infty(0, T, L^2(\Omega)), \tag{2-23}$$

$$N^k_2\psi^k_y + N^k_{12}\psi^k_x \to N_2\psi_y + N_{12}\psi_x \quad \text{weak} * \text{ in } L^\infty(0, T, L^2(\Omega)). \tag{2-24}$$

For $\{a, b, c, d, e\} \in [H^1_{\Gamma_0}(\Omega)]^5$ satisfying

$$a + c_x = 0 \quad \text{and} \quad b + c_y = 0, \tag{2-25}$$

the variational formulation of problem (1-1)–(1-3) is

$$\begin{aligned}\tfrac{1}{12}\rho h^3 \frac{d}{dt}(\phi^k_{1t}, a) + \tfrac{1}{12}\rho h^3 \frac{d}{dt}(\phi^k_{2t}, b) + \rho h \frac{d}{dt}(\psi^k_t, c) + \rho h \frac{d}{dt}(\eta^k_{1t}, d) + \rho h \frac{d}{dt}(\eta^k_{2t}, e)\\ + D\big[(\phi^k_{1x}, a_x) + \tfrac{1}{2}(1-\mu)(\phi^k_{1y}, a_y) + \tfrac{1}{2}(1+\mu)(\phi^k_{2x}, a_y) + (\phi^k_{2y}, b_y) + \tfrac{1}{2}(1-\mu)(\phi^k_{2x}, b_x) + \tfrac{1}{2}(1+\mu)(\phi^k_{1y}, b_x)\big]\\ + (N^k_1\psi^k_x + N^k_{12}\psi^k_y, c_x) + (N^k_1, d_x) + (N^k_{12}, d_y) + (N^k_2\psi^k_y + N^k_{12}\psi^k_x, c_y) + (N^k_2, e_y) + (N^k_{12}, e_x) = 0.\end{aligned} \tag{2-26}$$

Using convergences (2-8), (2-10)–(2-13), (2-23) and (2-24) in equation (2-26), and applying identities (2-9) and (2-25), one obtains the weak formulation of the system (1-5)–(1-7) given in (2-5). To finish the proof, it remains to identify the initial data of the limit system. In view of the convergences (2-8), (2-10), and classical compactness arguments, one has $\{\psi^k, \eta^k_1, \eta^k_2\} \to \{\psi, \eta_1, \eta_2\}$ in $C^0([0, T]; [L^2(\Omega)]^3)$. Then,

$$\{\psi^k(\cdot\,, 0), \eta^k_1(\cdot\,, 0), \eta^k_2(\cdot\,, 0)\} \to \{\psi(\cdot\,, 0), \eta_1(\cdot\,, 0), \eta_2(\cdot\,, 0)\} \quad \text{in } [L^2(\Omega)]^3,$$

which combined with (1-3a) guarantees that $\{\psi(\cdot, 0), \eta_1(\cdot\,, 0), \eta_2(\cdot\,, 0)\} = \{\psi_0, \eta_{10}, \eta_{20}\}$. In order to identify $\{\psi_t(\cdot, 0), \eta_{1t}(\cdot\,, 0), \eta_{2t}(\cdot\,, 0)\}$, multiply both sides of (2-26) by the function $\theta_\delta \in H^1(0, T)$ defined by

$$\theta_\delta(t) = \begin{cases} -t/\delta + 1 & \text{if } t \in [0, \delta],\\ 0 & \text{if } t \in (\delta, T],\end{cases}$$

and integrate by parts to obtain

$$
\begin{aligned}
&-\frac{\rho h^3}{12}(\phi_{11},a)+\frac{\rho h^3}{12\delta}\int_0^\delta(\phi_{1t}^k,a)\,dt-\frac{\rho h^3}{12}(\phi_{21},b)+\frac{\rho h^3}{12\delta}\int_0^\delta(\phi_{2t}^k,b)\,dt-\rho h(\psi_1,c)\\
&\quad+\frac{\rho h}{\delta}\int_0^\delta(\psi_t^k,c)\,dt-\rho h(\eta_{11},d)+\frac{\rho h}{\delta}\int_0^\delta(\eta_{1t}^k,d)\,dt-\rho h(\eta_{21},e)+\frac{\rho h}{\delta}\int_0^\delta(\eta_{2t}^k,e)\,dt\\
&\quad+\int_0^T D\Big[(\phi_{1x}^k,a_x)+\frac{1-\mu}{2}(\phi_{1y}^k,a_y)+\frac{1+\mu}{2}(\phi_{2x}^k,a_y)+(\phi_{2y}^k,b_y)+\frac{1-\mu}{2}(\phi_{2x}^k,b_x)+\frac{1+\mu}{2}(\phi_{1y}^k,b_x)\Big]\theta_\delta\,dt\\
&\quad+\int_0^t(N_1^k\psi_x^k+N_{12}^k\psi_y^k,c_x)\theta_\delta\,dt+\int_0^T(N_2^k\psi_y^k+N_{12}^k\psi_x^k,c_y)\theta_\delta\,dt-\int_0^T(N_{1x}^k+N_{12y}^k,d)\theta_\delta\,dt\\
&\quad-\int_0^T(N_{2y}^k+N_{12x}^k,e)\theta_\delta\,dt=0.
\end{aligned}
\tag{2-27}
$$

Passing to the limit as $k\to\infty$ in the last equation, and using (2-8), (2-10)–(2-23), one obtains

$$
\begin{aligned}
&-\frac{\rho h^3}{12}(\phi_{11},c_x)+\frac{\rho h^3}{12\delta}\int_0^\delta(\psi_{xt},c_x)\,dt+\frac{\rho h^3}{12}(\phi_{21},c_y)+\frac{\rho h^3}{12\delta}\int_0^\delta(\psi_{yt},c_y)\,dt\\
&\quad-\rho h(\psi_1,c)+\frac{\rho h}{\delta}\int_0^\delta(\psi_t,c)\,dt-\rho h(\eta_{11},d)+\frac{\rho h}{\delta}\int_0^\delta(\eta_{1t},d)\,dt\\
&\quad-\rho h(\eta_{21},e)+\frac{\rho h}{\delta}\int_0^\delta(\eta_{2t},e)\,dt+D\int_0^T(\Delta\psi,\Delta c)\theta_\delta\,dt+\int_0^t(N_1\psi_x+N_{12}\psi_y,c_x)\theta_\delta\,dt\\
&\quad+\int_0^T(N_2\psi_y+N_{12}\psi_x,c_y)\theta_\delta\,dt-\int_0^T(N_{1x}+N_{12y},d)\theta_\delta\,dt-\int_0^T(N_{2y}+N_{12x},e)\theta_\delta\,dt=0.
\end{aligned}
$$

On the other hand, multiplying (2-5) by θ_δ and integrating in time, we get the identity

$$
\begin{aligned}
&-\tfrac{1}{12}\rho h^3(\Delta\psi_t(\,\cdot\,,0),c)-\rho h(\psi_t(\,\cdot\,,0),c)-\rho h(\eta_{1t}(\,\cdot\,,0),d)-\rho h(\eta_{2t},e)\\
&\qquad=-\tfrac{1}{12}\rho h^3(\phi_{11x}+\phi_{21y},c)-\rho h(\psi_1,c)-\rho h(\eta_{11},d)-\rho h(\eta_{21},e).
\end{aligned}
\tag{2-28}
$$

Therefore, $\big(-\tfrac{1}{12}h^2\Delta\psi+\psi\big)_t(\,\cdot\,,0)=\psi_1+\tfrac{1}{12}h^2(\phi_{11x}+\phi_{21y})$, $\eta_{1t}(\,\cdot\,,0)=\eta_{11}$, and $\eta_{2t}(\,\cdot\,,0)=\eta_{21}$. □

Remark 2.3. In order to fully identify the initial data of the solutions of the limit system (1-5)–(1-7) and, more precisely, to determine the initial data of ψ_t, an elliptic equation has to be solved. Namely, the initial datum for the velocity ψ_t in (1-7b) is determined by solving the elliptic equation

$$
\psi_t(\,\cdot\,,0)\in H^1_{\Gamma_0}(\Omega):\quad \big(-\tfrac{1}{12}h^2\Delta\psi+\psi\big)_t(\,\cdot\,,0)=\psi_1+\tfrac{1}{12}h^2(\phi_{11x}+\phi_{21y})\quad\text{in }\Omega,
$$

as the proof of the theorem showed. More precisely, this elliptic equation can be written in the variational form

$$
\tfrac{1}{12}h^2(\nabla\psi_t(\,\cdot\,,0),\nabla c)+(\psi_t(\,\cdot\,,0),c)=(\psi_1,c)-\tfrac{1}{12}h^2(\phi_{11},c_x)-\tfrac{1}{12}h^2(\phi_{21},c_y),
$$

where the terms ϕ_{11x} and ϕ_{21y} are not derived from ϕ_1 and ϕ_2, respectively, in the sense of transposition, but they are rather the linear mappings which, when acting on any element c of $H^1_{\Gamma_0}(\Omega)$, produce $-(\phi_{11},c_x)$ and $-(\phi_{21},c_y)$. The same can be said about $\Delta\psi_t(\,\cdot\,,0)$, yielding $-(\nabla\psi_t(\,\cdot\,,0),\nabla c)$.

3. Stability: internal feedback

In this section we analyze the plate model with hinged boundary conditions and in the presence of internal damping distributed all along the plate. Consider the system

$$\begin{cases} \frac{1}{12}\rho h^3\phi_{1tt} - L_1(\phi_1, \phi_2, \psi) + \phi_{1t} = 0 & \text{in } Q, \\ \frac{1}{12}\rho h^3\phi_{2tt} - L_2(\phi_1, \phi_2, \psi) + \phi_{2t} = 0 & \text{in } Q, \\ \rho h\psi_{tt} - L_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2) + \psi_t = 0 & \text{in } Q, \\ \rho h\eta_{1tt} - L_4(\psi, \eta_1, \eta_2) + \eta_{1t} = 0 & \text{in } Q, \\ \rho h\eta_{2tt} - L_5(\psi, \eta_1, \eta_2) + \eta_{2t} = 0 & \text{in } Q, \end{cases} \tag{3-1}$$

under boundary conditions (1-2) and initial data (1-3). The energy of solutions for (3-1), (1-2), (1-3) decreases in time. Indeed, the energy given by (2-2) obeys the energy dissipation law

$$\frac{d}{dt}E_k(t) = -\big(|\phi_{1t}(t)|^2 + |\phi_{2t}(t)|^2 + |\psi_t(t)|^2 + |\eta_{1t}(t)|^2 + |\eta_{2t}(t)|^2\big). \tag{3-2}$$

The aim of this section is to obtain exponential decay for the energy (2-6) associated to the solution of the von Kármán system

$$\begin{cases} \rho h\psi_{tt} - \frac{1}{12}\rho h^3\Delta\psi_{tt} + D\Delta^2\psi - [N_1\psi_x + N_{12}\psi_y]_x - [N_2\psi_y + N_{12}\psi_x]_y + \psi_t - \Delta\psi_t = 0 & \text{in } Q, \\ \rho h\eta_{1tt} - [N_{1x} + N_{12y}] + \eta_{1t} = 0 & \text{in } Q, \\ \rho h\eta_{2tt} - [N_{2y} + N_{12x}] + \eta_{2t} = 0 & \text{in } Q, \end{cases} \tag{3-3}$$

with boundary conditions (1-6) and initial data (1-7), as a limit (as $k \to \infty$) of the uniform stabilization of the dissipative Mindlin–Timoshenko system (3-1), (1-2), (1-3).

Analogously to the proof of Theorem 2.1, considering the initial data $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$ satisfying (2-4), system (3-3) can be obtained as a limit, as $k \to \infty$, of system (3-1), (1-2), (1-3).

Since the energy $E_k(t)$ in (2-2) is a nonincreasing function, we will show that this energy decays exponentially (as $t \to \infty$) uniformly with respect to k. More precisely, the following result holds:

Theorem 3.1. *Let* $\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\}$ *be the solution of system* (3-1), (1-2), (1-3) *for data* $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$. *There exists a constant* $\omega > 0$ *such that*

$$E_k(t) \le 4E_k(0)e^{-\omega t/2} \quad \forall\, t \ge 0. \tag{3-4}$$

Remark 3.2. As a consequence of inequality (3-4), if the initial data satisfy (2-4), letting $k \to \infty$ one recovers the exponential decay of the energy $E(t)$ associated to system (3-3), which is given by (2-6). This is in agreement with the results from [Perla Menzala et al. 2002] in the sense that the same decay rate for the solutions of the von Kármám system was obtained.

Proof of Theorem 3.1. For each $k \ge 1$ fixed, let $\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\}$ be the solution of system (3-1), (1-2), (1-3) with data $\{\phi_{10}, \phi_{20}, \psi_0, \eta_{10}, \eta_{20}\} \in \mathcal{X}$. From now on in this proof, we will omit the index k of the solution to simplify the notation. For an arbitrary $\lambda > 0$, define the perturbed energy

$$G_\lambda(t) := E_k(t) + \lambda F(t), \tag{3-5}$$

where F is the functional

$$F(t)=\theta\big(\tfrac{1}{12}\rho h^3\phi_{1t},\phi_1\big)+\theta\big(\tfrac{1}{12}\rho h^3\phi_{2t},\phi_2\big)+\theta(\rho h\psi_t,\psi)+2\theta(\rho h\eta_{1t},\eta_1)+2\theta(\rho h\eta_{2t},\eta_2), \tag{3-6}$$

where $\theta>0$ is a constant to be chosen later on. Let us bound each term on the right-hand side of identity (3-6) by an expression involving the energy (2-2).

- Analysis of $\theta\big(\tfrac{1}{12}\rho h^3\phi_{1t},\phi_1\big)+\theta\big(\tfrac{1}{12}\rho h^3\phi_{2t},\phi_2\big)$: using the Poincaré inequality, one obtains

$$\begin{aligned}\theta\big(\tfrac{1}{12}\rho h^3\phi_{1t},\phi_1\big)&+\theta\big(\tfrac{1}{12}\rho h^3\phi_{2t},\phi_2\big)\\ &\le C\theta\bigg(\tfrac{1}{12}\rho h^3|\phi_{1t}|^2+\tfrac{1}{12}\rho h^3|\phi_{2t}|^2+|\phi_{1x}|^2+|\phi_{2y}|^2+|\phi_{1y}+\phi_{2x}|^2-2\int_\Omega\phi_{1x}\phi_{2y}\,dx\,dy\bigg)\\ &\le C\theta E_k(t).\end{aligned}\tag{3-7}$$

- Analysis of $\theta(\rho h\psi_t(t),\psi(t))$: using the Poincaré inequality again, one gets

$$\begin{aligned}\theta(\rho h\psi_t,\psi)&\le C\theta\big(\rho h|\psi_t|^2+|\psi_x|^2+|\psi_y|^2\big)\\ &\le C\theta\big(\rho h|\psi_t|^2+|\phi_1+\psi_x|^2+|\phi_2+\psi_y|^2+|\phi_1|^2+|\phi_2|^2\big)\\ &\le C\theta\big(\rho h|\psi_t|^2+|\phi_1+\psi_x|^2+|\phi_2+\psi_y|^2+|\phi_{1x}|^2+|\phi_{1y}|^2+|\phi_{2x}|^2+|\phi_{2y}|^2\big)\\ &\le C\theta E_k(t).\end{aligned}\tag{3-8}$$

- Analysis of $2\theta(\rho h\eta_{1t},\eta_1)+2\theta(\rho h\eta_{2t},\eta_2)$: one has

$$\begin{aligned}&2\theta(\rho h\eta_{1t},\eta_1)+2\theta(\rho h\eta_{2t},\eta_2)\\ &\le C\theta\big(\rho h|\eta_{1t}|^2+|\eta_{1x}|^2+|\eta_{1y}|^2+\rho h|\eta_{2t}|^2+|\eta_{2x}|^2+|\eta_{2y}|^2\big)\\ &\le C\theta\big(\rho h|\eta_{1t}|^2+\rho h|\eta_{2t}|^2+\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2+\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2+|\eta_{1y}|^2+|\eta_{2x}|^2+\tfrac12|\psi_x^2|^2+\tfrac12|\psi_y^2|^2\big)\\ &\le C\theta\bigg(\rho h|\eta_{1t}|^2+\rho h|\eta_{2t}|^2+\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2+\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2+|\eta_{1y}+\eta_{2x}|^2-2\int_\Omega\eta_{1y}\eta_{2x}\,dx\,dy+|\nabla\psi|^2\bigg)\\ &\le C\theta E_k(t).\end{aligned}\tag{3-9}$$

According to the bounds (3-7)–(3-9), we conclude that

$$|F(t)|\le CE_k(t). \tag{3-10}$$

Now, using (3-5) and (3-10), one obtains

$$|G_\lambda(t)-E_k(t)|\le\lambda|F(t)|\le\lambda CE_k(t),$$

which is equivalent to

$$(1-\lambda C)E_k(t)\le G_\lambda(t)\le(1+\lambda C)E_k(t).$$

Taking $0<\lambda\le 1/(2C)$, one gets

$$\tfrac12E_k(t)\le G_\lambda(t)\le 2E_k(t). \tag{3-11}$$

Differentiating the functional F and using the equations in (3-1), one obtains

$$\begin{aligned}\frac{d}{dt}F(t) = &-\theta D|\phi_{1x}|^2-\tfrac{1}{2}(1-\mu)\theta D|\phi_{1y}|^2-\theta D\tfrac{1}{2}(1+\mu)\int_\Omega \phi_{2x}\phi_{1y}\,dx\,dy-\theta k|\phi_1|^2-\theta k\int_\Omega \psi_x\phi_1\,dx\,dy\\
&-\theta\int_\Omega \phi_{1t}\phi_1\,dx\,dy+\theta\tfrac{1}{12}\rho h^3|\phi_{1t}|^2-\theta D|\phi_{2y}|^2-\theta D\tfrac{1}{2}(1-\mu)|\phi_{2x}|^2-\theta D\tfrac{1}{2}(1+\mu)\int_\Omega \phi_{1y}\phi_{2x}\,dx\,dy\\
&-\theta k|\phi_2|^2-\theta k\int_\Omega \psi_y\phi_2\,dx\,dy+\theta\tfrac{1}{12}\rho h^3|\phi_{2t}|^2-\theta\int_\Omega \phi_{2t}\phi_2\,dx\,dy-\theta k|\psi_x|^2\\
&-\theta k\int_\Omega \phi_1\psi_x\,dx\,dy-\theta k|\psi_y|^2-\theta k\int_\Omega \phi_2\psi_y\,dx\,dy-\theta\int_\Omega [N_1\psi_x+N_{12}\psi_y]\psi_x\,dx\,dy\\
&-\theta\int_\Omega [N_2\psi_y+N_{12}\psi_x]\psi_y\,dx\,dy+\theta\rho h|\psi_t|^2-\theta\int_\Omega \psi_t\psi\,dx\,dy-2\theta\int_\Omega N_1\eta_{1x}\,dx\,dy\\
&-2\theta\int_\Omega N_{12}\eta_{1y}\,dx\,dy+2\theta\rho h|\eta_{1t}|^2-2\theta\int_\Omega N_2\eta_{2y}\,dx\,dy-2\theta\int_\Omega N_{12}\eta_{2x}\,dx\,dy\\
&+2\theta\rho h|\eta_{2t}|^2-2\theta\int_\Omega \eta_{1t}\eta_1\,dx\,dy-2\theta\int_\Omega \eta_{2t}\eta_2\,dx\,dy. \qquad (3\text{-}12)\end{aligned}$$

We bound each term on the right-hand side of identity (3-12) separately.

- Analysis of $-\theta(\phi_{1t},\phi_1)-\theta(\phi_{2t},\phi_2)$:

$$\begin{aligned}-\theta(\phi_{1t},\phi_1)-\theta(\phi_{2t},\phi_2) &\le \frac{\theta^2}{2\xi}|\phi_{1t}|^2+\frac{\xi}{2}|\phi_1|^2+\frac{\theta^2}{2\xi}|\phi_{2t}|^2+\frac{\xi}{2}|\phi_2|^2\\
&\le \frac{\theta^2}{2\xi}[|\phi_{1t}|^2+|\phi_{2t}|^2]+\frac{\xi C}{2}(|\phi_{1x}|^2+|\phi_{1y}|^2+|\phi_{2x}|^2+|\phi_{2y}|^2)\\
&= \frac{\theta^2}{2\xi}[|\phi_{1t}|^2+|\phi_{2t}|^2]+\frac{\xi C}{2}\left(|\phi_{1x}|^2+|\phi_{2y}|^2+|\phi_{1y}+\phi_{2x}|^2-2\int_\Omega \phi_{1y}\phi_{2x}\,dx\,dy\right)\\
&\le \frac{\theta^2}{2\xi}[|\phi_{1t}|^2+|\phi_{2t}|^2]+\xi C E_k(t), \qquad (3\text{-}13)\end{aligned}$$

where $\xi > 0$ is a real number to be appropriately chosen.

- Analysis of $-\theta(\psi_t(t),\psi(t))$:

$$\begin{aligned}-\theta(\psi_t,\psi) &\le \frac{\theta^2}{2\xi}|\psi_t|^2+\frac{\xi}{2}|\psi|^2\\
&\le \frac{\theta^2}{2\xi}|\psi_t|^2+\frac{\xi C}{2}(|\psi_x|^2+|\psi_y|^2)\\
&\le \frac{\theta^2}{2\xi}|\psi_t|^2+\frac{\xi C}{2}(|\phi_1+\psi_x|^2+|\phi_2+\psi_y|^2+|\phi_1|^2+|\phi_2|^2)\\
&\le \frac{\theta^2}{2\xi}|\psi_t|^2+\frac{\xi C}{2}(|\phi_1+\psi_x|^2+|\phi_2+\psi_y|^2+|\phi_{1x}|^2+|\phi_{1y}|^2+|\phi_{2x}|^2+|\phi_{2y}|^2)\\
&\le \frac{\theta^2}{2\xi}|\psi_t|^2+\xi C E_k(t). \qquad (3\text{-}14)\end{aligned}$$

• Analysis of $-2\theta(\eta_{1t},\eta_1)-2\theta(\eta_{1t},\eta_1)$:

$$
\begin{aligned}
&-2\theta(\eta_{1t},\eta_1)-2\theta(\eta_{1t},\eta_1)\\
&\quad\le \frac{\theta^2}{2\xi}|\eta_{1t}|^2+\frac{\xi}{2}|\eta_1|^2+\frac{\theta^2}{2\xi}|\eta_{2t}|^2+\frac{\xi}{2}|\eta_2|^2\\
&\quad\le \frac{\theta^2}{2\xi}[|\eta_{1t}|^2+|\eta_{2t}|^2]+\frac{\xi C}{2}\big[|\eta_{1x}|^2+|\eta_{1y}|^2+|\eta_{2x}|^2+|\eta_{2y}|^2\big]\\
&\quad\le \frac{\theta^2}{2\xi}[|\eta_{1t}|^2+|\eta_{2t}|^2]+\frac{\xi C}{2}\Big[\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2+\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2+|\eta_{1y}+\eta_{2x}|^2-2\int_\Omega\eta_{1y}\eta_{2x}\,dx\,dy+\tfrac12\psi_x^2+\tfrac12\psi_y^2\Big]\\
&\quad\le \frac{\theta^2}{2\xi}[|\eta_{1t}|^2+|\eta_{2t}|^2]+\frac{\xi C}{2}\Big[\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2+\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2+|\eta_{1y}+\eta_{2x}|^2-2\int_\Omega\eta_{1y}\eta_{2x}\,dx\,dy+|\nabla\psi|^2\Big]\\
&\quad\le \frac{\theta^2}{2\xi}[|\eta_{1t}|^2+|\eta_{2t}|^2]+\xi C E_k(t).
\end{aligned}\tag{3-15}
$$

Using bounds (3-13)–(3-15), one obtains, from (3-12),

$$
\begin{aligned}
\frac{d}{dt}F(t)\le{}&-\theta D|\phi_{1x}|^2-\theta D|\phi_{2y}|^2-\theta k|\phi_1+\psi_x|^2-\theta k|\phi_2+\psi_y|^2-\theta D\,\frac{1-\mu}{2}\,|\phi_{1y}+\phi_{2x}|^2\\
&\qquad-2\theta D\mu\int_\Omega\phi_{1y}\phi_{2x}\,dx\,dy-2\theta\frac{Eh}{1-\mu^2}\,\frac{1-\mu}{2}\,|\eta_{1y}+\eta_{2x}+\psi_x\psi_y|^2-2\theta\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2\\
&\qquad-2\theta\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2-2\mu\theta\big|\eta_{2y}+\tfrac12\psi_y^2\big|^2-2\mu\theta\big|\eta_{1x}+\tfrac12\psi_x^2\big|^2+3\xi C E_k(t)+\theta\frac{\rho h^3}{12}\big[|\phi_{1t}|^2+|\phi_{2t}|^2\big]\\
&\qquad+\theta\rho h\big[|\psi_t|^2+2|\eta_{1t}|^2+2|\eta_{2t}|^2\big]+\frac{\theta^2}{2\xi}\big[|\phi_{1t}|^2+|\phi_{2t}|^2\big]+\frac{\theta^2}{2\xi}|\psi_t|^2+\frac{\theta^2}{2\xi}\big[|\eta_{1t}|^2+|\eta_{2t}|^2\big]\\
\le{}&-(\theta-3\xi C)E_k(t)+\Big(\theta\frac{\rho h^3}{12}+\frac{\theta^2}{2\xi}\Big)\big[|\phi_{1t}|^2+|\phi_{2t}|^2\big]+\Big(\theta\rho h+\frac{\theta^2}{2\xi}\Big)|\psi_t|^2\\
&+\Big(2\theta\rho h+\frac{\theta^2}{2\xi}\Big)\big[|\eta_{1t}|^2+|\eta_{2t}|^2\big].
\end{aligned}\tag{3-16}
$$

Therefore,

$$
\frac{d}{dt}F(t)\le-(\theta-3\xi C)E_k(t)+C\big[|\phi_{1t}|^2+|\phi_{2t}|^2+|\psi_t|^2+|\eta_{1t}|^2+|\eta_{2t}|^2\big].\tag{3-17}
$$

Considering the derivative of the expression (3-5), and observing (3-2) and (3-17), one has

$$
\frac{d}{dt}G_\lambda(t)\le-\lambda(\theta-3\xi C)E_k(t)-(1-\lambda C)\big[|\phi_{1t}|^2+|\phi_{2t}|^2+|\psi_t|^2+|\eta_{1t}|^2+|\eta_{2t}|^2\big].
$$

Choosing $\lambda\le 1/(2C)$ and $\xi<\theta/3$, one obtains, according to (3-11),

$$
\frac{d}{dt}G_\lambda(t)\le-\lambda(\theta-3\xi C)E_k(t)\le-\tfrac12 wG_\lambda(t)\quad\forall t\ge0,
$$

where $\omega=\lambda(\theta-3\xi C)$. Therefore,

$$
G_\lambda(t)\le G_\lambda(0)e^{-wt/2}.\tag{3-18}
$$

Combining (3-11) and (3-18), one gets (3-4). □

4. Stability: boundary feedback

In this section we analyze the plate model in the case where the energy of the Mindlin–Timoshenko system is dissipated through boundary feedback mechanisms. Let us assume that $\Gamma_i \neq \varnothing$ $(i = 0, 1)$, and we consider the system (1-1) with boundary conditions

$$\begin{gathered}\phi_1 = \phi_2 = \psi = \eta_1 = \eta_2 = 0 \quad \text{on } \Sigma_0,\\ \{\mathcal{B}_1(\phi_1,\phi_2),\ \mathcal{B}_2(\phi_1,\phi_2),\ \mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2),\ \mathcal{B}_4(\eta_1,\eta_2),\ \mathcal{B}_5(\eta_1,\eta_2)\}\\ = -\{\phi_{1t},\phi_{2t},\psi_t,\eta_{1t},\eta_{2t}\} \quad \text{on } \Sigma_1,\end{gathered} \tag{4-1}$$

and initial data (1-3). The energy of this system obeys the dissipation law

$$\frac{d}{dt}E_k(t) = -\int_{\Gamma_1}\big[(\phi^k_{1t})^2 + (\phi^k_{2t})^2 + (\psi^k_t)^2 + (\eta^k_{1t})^2 + (\eta^k_{2t})^2\big]\,d\Gamma.$$

Consequently,

$$E_k(t) \le E_k(0) \quad \forall\, t \ge 0.$$

We are interested in studying the asymptotic behavior of $E_k(t)$ as $t \to \infty$.

The variational formulation of (1-1), (4-1), (1-3) is given by

$$\begin{aligned}&\tfrac{1}{12}\rho h^3\frac{d}{dt}(\phi^k_{1t},a)+\tfrac{1}{12}\rho h^3\frac{d}{dt}(\phi^k_{2t},b)+\rho h\frac{d}{dt}(\psi^k_t,c)+\rho h\frac{d}{dt}(\eta^k_{1t},d)+\rho h\frac{d}{dt}(\eta^k_{2t},e)\\ &\quad+k\big[(\phi^k_1+\psi^k_x,a+c_x)+(\phi^k_2+\psi^k_y,b+c_y)\big]\\ &\quad+D\big[(\phi^k_{1x},a_x)+\tfrac12(1-\mu)(\phi^k_{1y},a_y)+\tfrac12(1+\mu)(\phi^k_{2x},a_y)+(\phi^k_{2y},b_y)+\tfrac12(1-\mu)(\phi^k_{2x},b_x)+\tfrac12(1+\mu)(\phi^k_{1y},b_x)\big]\\ &\quad+(N^k_1\psi^k_x+N^k_{12}\psi^k_y,c_x)+(N^k_1,d_x)+(N^k_{12},d_y)+(N^k_2\psi^k_y+N^k_{12}\psi^k_x,c_y)\\ &\quad+(N^k_2,e_y)+(N^k_{12},e_x)+\int_{\Gamma_1}\big[\phi^k_{1t}a+\phi^k_{2t}b+\psi^k_tc+\eta^k_{1t}d+\eta^k_{2t}e\big]\,d\Gamma=0\end{aligned} \tag{4-2}$$

for all $\{a, b, c, d, e\} \in [H^1_{\Gamma_0}(\Omega)]^5$.

Remark 4.1. Using arguments similar to those in Section 2, considering initial data in a suitable class and satisfying (2-4), we can prove that the system (1-1), (4-1), (1-3) converges (as $k \to \infty$) toward the dissipative von Kármán system (1-5) with boundary conditions

$$\begin{gathered}\psi = \frac{\partial\psi}{\partial\nu} = \eta_1 = \eta_2 = 0 \quad \text{on } \Sigma_0,\\ D\big[\Delta\psi+(1-\mu)(2\nu_1\nu_2\psi_{xy}-\nu_1^2\psi_{yy}-\nu_2^2\psi_{xx})\big] = -(\nu_1\psi_{xt}+\nu_2\psi_{yt}) \quad \text{on } \Sigma_1,\\ D\Big[\frac{\partial(\Delta\psi)}{\partial\nu}+(1-\mu)\frac{\partial}{\partial\tau}\big[(\nu_1^2-\nu_2^2)\psi_{xy}+\nu_1\nu_2(\psi_{yy}-\psi_{xx})\big]\Big]\\ -\tfrac{1}{12}\rho h^3\frac{\partial\psi_{tt}}{\partial\nu}-(\nu_1N_1+\nu_2N_{12})\psi_x-(\nu_2N_2+\nu_1N_{12})\psi_y = \frac{\partial}{\partial\tau}(-\nu_1\psi_{yt}+\nu_2\psi_{xt})-\psi_t \quad \text{on } \Sigma_1,\\ \nu_1N_1+\nu_2N_{12} = -\eta_{1t} \quad \text{on } \Sigma_1,\\ \nu_2N_2+\nu_1N_{12} = -\eta_{2t} \quad \text{on } \Sigma_1,\end{gathered} \tag{4-3}$$

and initial data (1-7).

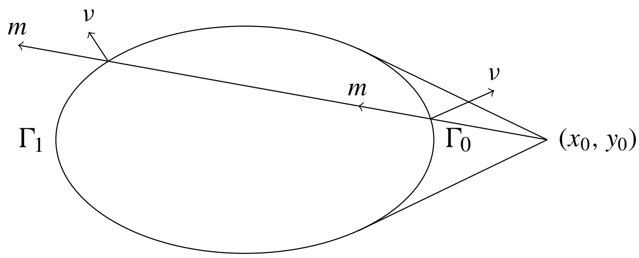

Figure 1. Example for which condition (4-4) is satisfied.

In order to establish the uniform asymptotic stability of system (1-1), (4-1), (1-3), some restrictions are needed on the geometry of $\Omega,\ \Gamma_0$ and Γ_1. Let us introduce a vector field $m = m(x, y)$ in $\mathbb{R}^2$ defined by

$$m(x, y) = (x, y) - (x_0, y_0),$$

where (x_0, y_0) is a fixed point of $\mathbb{R}^2$. We assume that Γ_0 and Γ_1 are such that

$$m \cdot \nu \leq 0 \quad \text{on } \Gamma_0, \qquad m \cdot \nu \geq 0 \quad \text{on } \Gamma_1. \tag{4-4}$$

Let us consider $G = [g_{ij}]$ the 5×5 matrix such that

$$g_{ij} = 0, \quad i \neq j, \qquad \text{and} \qquad (m \cdot \nu) g_{ii} = 1, \quad i = 1, \dots, 5.$$

Note that $g_{ij} \in C^1(\bar{\Gamma}_1)$. Moreover, there are positive constants g_0 and G_0 such that

$$g_0 |\varsigma|^2 \leq G\varsigma \cdot \varsigma \leq G_0 |\varsigma|^2 \quad \forall\, \varsigma \in \mathbb{R}^5, \text{ on } \Gamma_1. \tag{4-5}$$

Before establishing the main result of this section, we will state and prove the following two lemmas.

Lemma 4.2. *Let $\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\}$ and $\{\phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}\}$ be regular enough. Then*

$$\begin{aligned} &\int_\Omega \big[\phi_{1t} L_1(\phi_1, \phi_2, \psi) + \phi_{2t} L_2(\phi_1, \phi_2, \psi) \\ &\qquad + \psi_t L_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2) + \eta_{1t} L_4(\psi, \eta_1, \eta_2) + \eta_{2t} L_5(\psi, \eta_1, \eta_2)\big]\, dx\, dy \\ &\qquad\quad + a(\phi_1, \phi_2, \psi, \eta_1, \eta_2, \phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}) \\ &= \int_\Gamma \big[\phi_{1t} \mathcal{B}_1(\phi_1, \phi_2) + \phi_{2t} \mathcal{B}_2(\phi_1, \phi_2) + \psi_t \mathcal{B}_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2) + \eta_{1t} \mathcal{B}_4(\psi, \eta_1, \eta_2) + \eta_{2t} \mathcal{B}_5(\psi, \eta_1, \eta_2)\big]\, d\Gamma, \end{aligned} \tag{4-6}$$

with

$$\begin{aligned} &a(\phi_1, \phi_2, \psi, \eta_1, \eta_2, \phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}) \\ &\qquad := a_0(\phi_1, \phi_2, \phi_{1t}, \phi_{2t}) + k a_1(\phi_1, \phi_2, \psi, \phi_{1t}, \phi_{2t}, \psi_t) + a_2(\psi, \eta_1, \eta_2, \psi_t, \eta_{1t}, \eta_{2t}), \end{aligned}$$

where

$$\begin{aligned} &a_0(\phi_1, \phi_2, \phi_{1t}, \phi_{2t}) \\ &\qquad = D \int_\Omega \big[\phi_{1x}\phi_{1tx} + \phi_{2y}\phi_{2ty} + \mu \phi_{1x}\phi_{2ty} + \mu \phi_{1tx}\phi_{2y} + \tfrac{1}{2}(1-\mu)(\phi_{1y} + \phi_{2x})(\phi_{1ty} + \phi_{2tx})\big]\, dx\, dy, \end{aligned}$$

$$a_1(\phi_1,\phi_2,\psi,\phi_{1t},\phi_{2t},\psi_t)=\int_\Omega\big[(\phi_1+\psi_x)(\phi_{1t}+\psi_{tx})+(\phi_2+\psi_y)(\phi_{2t}+\psi_{ty})\big]\,dx\,dy,$$

and

$$\begin{aligned}
&a_2(\psi,\eta_1,\eta_2,\psi_t,\eta_{1t},\eta_{2t})\\
&\quad=\frac{Eh}{1-\mu^2}\int_\Omega\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)(\eta_{1tx}+\psi_x\psi_{tx})\\
&\qquad+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)(\eta_{2ty}+\psi_y\psi_{ty})\\
&\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)(\eta_{1tx}+\eta_{2ty}+\nabla\psi\cdot\nabla\psi_t)\\
&\qquad+\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)(\eta_{1ty}+\eta_{2tx}+\psi_x\psi_{ty}+\psi_y\psi_{tx})\Big]\,dx\,dy.
\end{aligned}$$

Remark 4.3. Here and elsewhere in this section we use the term "regular enough" to ensure that all integrals are well defined (see Section 5 for additional comments on this point).

Proof of Lemma 4.2. By definition of the operators $L_i(\phi_1,\phi_2,\psi,\eta_1,\eta_2)$ $(i=1,\dots,5)$, one has

$$\begin{aligned}
&\int_\Omega\big[\phi_{1t}L_1(\phi_1,\phi_2,\psi)+\phi_{2t}L_2(\phi_1,\phi_2,\psi)+\psi_tL_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\
&\qquad\qquad+\eta_{1t}L_4(\psi,\eta_1,\eta_2)+\eta_{2t}L_5(\psi,\eta_1,\eta_2)\big]\,dx\,dy\\
&=\int_\Omega\Big\{\phi_{1t}\big[D\big(\phi_{1xx}+\tfrac12(1-\mu)\phi_{1yy}+\tfrac12(1+\mu)\phi_{2xy}\big)-k(\phi_1+\psi_x)\big]\\
&\quad+\phi_{2t}\big[D\big(\phi_{2yy}+\tfrac12(1-\mu)\phi_{2xx}+\tfrac12(1+\mu)\phi_{1xy}\big)-k(\phi_2+\psi_y)\big]\\
&\quad+\psi_t\big\{k\big[(\psi_x+\phi_1)_x+(\psi_y+\phi_2)_y\big]+(N_1\psi_x+N_{12}\psi_y)_x+(N_2\psi_y+N_{12}\psi_x)_y\big\}\\
&\quad+\eta_{1t}[N_{1x}+N_{12y}]+\eta_{2t}[N_{2y}+N_{12x}]\Big\}\,dx\,dy.
\end{aligned}$$

Through integration by parts one obtains

$$\begin{aligned}
&\int_\Omega\big[\phi_{1t}L_1(\phi_1,\phi_2,\psi)+\phi_{2t}L_2(\phi_1,\phi_2,\psi)+\psi_tL_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\
&\qquad\qquad+\eta_{1t}L_4(\psi,\eta_1,\eta_2)+\eta_{2t}L_5(\psi,\eta_1,\eta_2)\big]\,dx\,dy\\
&=-a_0(\phi_1,\phi_2,\phi_{1t},\phi_{2t})-ka_1(\phi_1,\phi_2,\psi,\phi_{1t},\phi_{2t},\psi_t)\\
&\quad-\int_\Omega\big[(N_1\psi_x+N_{12}\psi_y)\psi_{tx}+(N_{2\psi_y+N_{12}\psi_x})\psi_{ty}+\eta_{1tx}N_1+\eta_{1tyN_{12}}+\eta_{2ty}N_2+\eta_{2tx}N_{12}\big]\,dx\,dy\\
&\quad+\int_\Gamma\Big\{\phi_{1t}D\big[\phi_{1x}\nu_1+\tfrac12(1-\mu)\phi_{1y}\nu_2+\tfrac12(1-\mu)\phi_{2x}\nu_2+\mu\phi_{2y\nu_1}\big]\\
&\qquad+\phi_{2t}D\big[\phi_{2y}\nu_2+\tfrac12(1-\mu)\phi_{2x}\nu_1+\tfrac12(1-\mu)\phi_{1y}\nu_1+\mu\phi_{1x\nu_2}\big]\\
&\qquad+\psi_tk\big[(\phi_1+\psi_x)\nu_1+(\phi_2+\psi_y)\nu_2\big]+(N_1\psi_x+N_{12}\psi_y)\nu_1\\
&\qquad+(N_2\psi_y+N_{12}\psi_x)\nu_2+\eta_{1t}(N_1\nu_1+N_{12}\nu_2)+\eta_{2t}(N_2\nu_2+N_{12}\nu_1)\Big\}\,d\Gamma.
\end{aligned}$$

Finally, using the definition of N_1, N_2 and N_{12}, one has

$$\begin{aligned}
&\int_\Omega\big[\phi_{1t}L_1(\phi_1,\phi_2,\psi)+\phi_{2t}L_2(\phi_1,\phi_2,\psi)+\psi_tL_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\
&\qquad\qquad+\eta_{1t}L_4(\psi,\eta_1,\eta_2)+\eta_{2t}L_5(\psi,\eta_1,\eta_2)\big]\,dx\,dy
\end{aligned}$$

$$= -a_0(\phi_1,\phi_2,\phi_{1t},\phi_{2t}) - ka_1(\phi_1,\phi_2,\psi,\phi_{1t},\phi_{2t},\psi_t) - a_2(\psi,\eta_1,\eta_2,\psi_t,\eta_{1t},\eta_{2t}) + \int_\Gamma \{\phi_{1t}\mathcal{B}_1(\phi_1,\phi_2) + \phi_{2t}\mathcal{B}_2(\phi_1,\phi_2) + \psi_t\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + \eta_{1t}\mathcal{B}_4(\psi,\eta_1,\eta_2) + \eta_{2t}\mathcal{B}_5(\psi,\eta_1,\eta_2)\}\,d\Gamma. \qquad \square$$

Lemma 4.4. *Consider* $\{\phi_1,\phi_2,\psi,\eta_1,\eta_2\}$ *to be regular enough. Then*

$$\int_\Omega \big[(m\cdot\nabla\phi_1)L_1(\phi_1,\phi_2,\psi)+(m\cdot\nabla\phi_2)L_2(\phi_1,\phi_2,\psi)+(m\cdot\nabla\psi)L_3(\phi_1,\phi_2,\psi) +(m\cdot\nabla\eta_1)L_4(\psi,\eta_1,\eta_2)+(m\cdot\nabla\eta_2)L_5(\psi,\eta_1,\eta_2)\big]\,dx\,dy$$
$$= k\int_\Omega \big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy$$
$$-\tfrac12\int_\Gamma m\cdot\nu\bigg\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big] + \frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12(1)\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2 +\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\bigg\}\,d\Gamma$$
$$+\int_\Gamma \big[(m\cdot\nabla\phi_1)\mathcal{B}_1(\phi_1,\phi_2)+(m\cdot\nabla\phi_2)\mathcal{B}_2(\phi_1,\phi_2)+(m\cdot\nabla\psi)\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) +(m\cdot\nabla\eta_1)\mathcal{B}_4(\psi,\eta_1,\eta_2)+(m\cdot\nabla\eta_2)\mathcal{B}_5(\psi,\eta_1,\eta_2)\big]\,d\Gamma. \tag{4-7}$$

Proof. Analogously to the proof of Lemma 4.2,

$$\int_\Omega \big[(m\cdot\nabla\phi_1)L_1(\phi_1,\phi_2,\psi)+(m\cdot\nabla\phi_2)L_2(\phi_1,\phi_2,\psi)+(m\cdot\nabla\psi)L_3(\phi_1,\phi_2,\psi) + (m\cdot\nabla\eta_1)L_4(\psi,\eta_1,\eta_2)+(m\cdot\nabla\eta_2)L_5(\psi,\eta_1,\eta_2)\big]\,dx\,dy$$
$$= -a\big(\phi_1,\phi_2,\psi,\eta_1,\eta_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2\big) + \int_\Gamma \big[(m\cdot\nabla\phi_1)\mathcal{B}_1(\phi_1,\phi_2)+(m\cdot\nabla\phi_2)\mathcal{B}_2(\phi_1,\phi_2)+(m\cdot\nabla\psi)\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + (m\cdot\nabla\eta_1)\mathcal{B}_4(\psi,\eta_1,\eta_2)+(m\cdot\nabla\eta_2)\mathcal{B}_5(\psi,\eta_1,\eta_2)\big]\,d\Gamma. \tag{4-8}$$

In this way, to prove (4-7) we have only to study the term

$$a\big(\phi_1,\phi_2,\psi,\eta_1,\eta_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2\big). \tag{4-9}$$

Note that

$$a_0(\phi_1,\phi_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2) = D\int_\Omega \big[\phi_{1x}(m\cdot\nabla\phi_1)_x+\phi_{2y}(m\cdot\nabla\phi_2)_y+\mu\phi_{1x}(m\cdot\nabla\phi_2)_y+\mu\phi_{2y}(m\cdot\nabla\phi_1)_x + \tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})\big((m\cdot\nabla\phi_1)_y+(m\cdot\nabla\phi_2)_x\big)\big]\,dx\,dy$$
$$= \frac{D}{2}\int_\Omega \operatorname{div}\big\{m\big[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]\big\}\,dx\,dy$$
$$= \frac{D}{2}\int_\Gamma m\cdot\nu\big[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]\,d\Gamma, \tag{4-10}$$

$$
\begin{aligned}
&a_1(\phi_1, \phi_2, \psi, m\cdot\nabla\phi_1, m\cdot\nabla\phi_2, m\cdot\nabla\psi)\\
&\quad= \int_\Omega \big[(\phi_1+\psi_x)((m\cdot\nabla\phi_1)+(m\cdot\nabla\psi)_x)+(\phi_2+\psi_y)((m\cdot\nabla\phi_2)+(m\cdot\nabla\psi)_y)\big]\,dx\,dy\\
&\quad= \frac{1}{2}\int_\Omega \operatorname{div}\big\{m[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2]\big\}\,dx\,dy - \int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy\\
&\quad= \frac{1}{2}\int_\Gamma \big\{m\cdot\nu[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2]\big\}\,d\Gamma - \int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy,
\end{aligned}
\tag{4-11}
$$

and

$$
\begin{aligned}
&a_2(\psi,\eta_1,\eta_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2)\\
&=\frac{Eh}{1-\mu^2}\int_\Omega\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)\big((m\cdot\nabla\eta_1)_x+\psi_x(m\cdot\nabla\psi)_x\big)+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)\big((m\cdot\nabla\eta_2)_y+\psi_y(m\cdot\nabla\psi)_y\big)\\
&\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)\big((m\cdot\nabla\eta_1)_x+(m\cdot\nabla\eta_2)_y+\nabla\psi\cdot\nabla(m\cdot\nabla\psi)\big)\\
&\qquad+\tfrac12(1-\mu)u(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)\big((m\cdot\nabla\eta_1)_y+(m\cdot\nabla\eta_2)_x+\psi_x(m\cdot\nabla\psi)_y+\psi_y(m\cdot\nabla\psi)_x\big)\Big]\,dx\,dy\\
&=\frac{Eh}{2(1-\mu^2)}\int_\Omega\operatorname{div}\Big\{m\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)\big(\phi_{2y}+\tfrac12\psi_y^2\big)^2\\
&\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2+\tfrac12(1-\mu)u(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,dx\,dy\\
&=\frac{Eh}{2(1-\mu^2)}\int_\Gamma m\cdot\nu\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)^2\\
&\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2+\tfrac12(1-\mu)u(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\,d\Gamma.
\end{aligned}
\tag{4-12}
$$

Plugging (4-10)–(4-12) in (4-9) we get

$$
\begin{aligned}
&a(\phi_1,\phi_2,\psi,\eta_1,\eta_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2)\\
&\quad=\tfrac12\int_\Gamma m\cdot\nu\Big\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\big]\\
&\qquad+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12(1)\psi_x^2\big)^2+(1-\mu)\big(\phi_{2y}+\tfrac12\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2\\
&\qquad+\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,d\Gamma\\
&\qquad-k\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy.
\end{aligned}
\tag{4-13}
$$

Equation (4-7) follows directly from (4-8) and (4-13). □

The main result in this section is the following.

Theorem 4.5. *Assume the geometric condition* (4-4) *holds. Let* $\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\}$ *be a regular enough solution of system* (1-1), (4-1), (1-3). *Then, there exist positive constants* C *and* ω *such that*

$$E_k(t) \le C E_k(0)e^{-\omega t} \quad \forall t \ge 0. \tag{4-14}$$

Remark 4.6. For regular enough initial data satisfying (2-4), one obtains, as a consequence of inequality (4-14), exponential decay for the energy $E(t)$ associated to system (1-5), (4-3), (1-7) as $k \to \infty$. This decay rate for the limit system is in agreement with the results from [Perla Menzala et al. 2002].

Remark 4.7. The case $\Gamma_0 = \varnothing$ is not considered in this paper. In this case, one cannot ensure that the energy decays to zero for every finite energy solution of (1-1), (4-1), (1-3) regardless of how the feedbacks are chosen. Indeed, defining

$$\{\phi_{11}, \phi_{21}, \psi_1, \eta_{11}, \eta_{21}\} = \big\{\alpha, \beta, -\alpha x - \beta y + \gamma, -\tfrac{1}{2}\alpha^2 x - \tfrac{1}{2}\alpha\beta y + c_1, -\tfrac{1}{2}\beta^2 y - \tfrac{1}{2}\alpha\beta x + c_2\big\},$$

where α, β, γ, c_1 and c_2 are nonzero constants, and $\{\phi_{10}, \phi_{20}, \psi_0, \eta_{10}, \eta_{20}\}$ such that

$$L_i(\phi_{10}, \phi_{20}, \psi_0, \eta_{10}, \eta_{20}) = 0, \quad i = 1, \dots, 5,$$

$$\big\{\mathcal{B}_1(\phi_{10}, \phi_{20}), \mathcal{B}_2(\phi_{10}, \phi_{20}), \mathcal{B}_3(\phi_{10}, \phi_{20}, \psi_0, \eta_{10}, \eta_{20}), \mathcal{B}_4(\psi_0, \eta_{10}, \eta_{20}), \mathcal{B}_5(\psi_0, \eta_{10}, \eta_{20})\big\} = -\{\phi_{11}, \phi_{21}, \psi_1, \eta_{11}, \eta_{21}\},$$

it is not difficult to check that

$$\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\} = t\{\phi_{11}, \phi_{21}, \psi_1, \eta_{11}, \eta_{21}\} + \{\phi_{10}, \phi_{20}, \psi_0, \eta_{10}, \eta_{20}\}$$

is a solution of (1-1), (4-1), (1-3). However, for this solution,

$$E(t) = \tfrac{1}{2}\big[\tfrac{1}{12}\rho h^3(|\phi_{11}|^2 + |\phi_{21}|^2) + \rho h(|\psi_1|^2 + |\eta_1|^2 + |\eta_2|^2)\big] = \text{const.} > 0.$$

Proof of Theorem 4.5. We divide the proof into three steps:

Step 1: We apply Lemma 4.4 to the solution of (1-1), (4-1), (1-3) and integrate the resulting identity with respect to t from 0 to T to obtain

$$\begin{aligned}
&\rho h\int_0^T \big[\tfrac{1}{12}h^2(\phi_{1tt}, m\cdot\nabla\phi_1)+\tfrac{1}{12}h^2(\phi_{2tt}, m\cdot\nabla\phi_2)+(\psi_{tt}, m\cdot\nabla\psi)+(\eta_{1tt}, m\cdot\nabla\eta_1)+(\eta_{2tt}, m\cdot\nabla\eta_2)\big]\,dt \\
&\qquad -k\int_0^T\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy \\
&= -\tfrac{1}{2}\int_0^T\int_\Gamma m\cdot\nu\bigg\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\big] \\
&\qquad +\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac{1}{2}\psi_x^2\big)^2+(1-\mu)(\eta_{2y}+\tfrac{1}{2}\psi_y^2)^2 \\
&\qquad +\mu(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2)^2+\tfrac{1}{2}(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\bigg\}\,d\Gamma \\
&\quad +\int_0^T\int_{\Gamma_0}\big[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5\big]\,d\Gamma \\
&\quad -\int_0^T\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2)+\psi_t(m\cdot\nabla\psi)+\eta_{1t}(m\cdot\nabla\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2)\big]\,d\Gamma. \qquad (4\text{-}15)
\end{aligned}$$

Both of the integrals on the left-hand side of (4-15) may be interpreted in the $L^2(Q)$ scalar product since $\{\phi_{1tt}, \phi_{2tt}, \psi_{tt}, \eta_{1tt}, \eta_{2tt}\} \in C([0,\infty), [L^2(\Omega)]^5)$. The first integral on the left-hand side may be written as

$$\begin{aligned}
&\rho h\int_0^T\int_\Omega\big\{\tfrac{1}{12}h^2\big[\phi_{1tt}(m\cdot\nabla\phi_1)+\phi_{2tt}(m\cdot\nabla\phi_2)\big]+\psi_{tt}(m\cdot\nabla\psi)+\eta_{1tt}(m\cdot\nabla\eta_1)+\eta_{2tt}(m\cdot\nabla\eta_2)\big\}\,dx\,dy\,dt \\
&= \Upsilon_1 - \rho h\int_0^T\int_\Omega\big[\tfrac{1}{12}h^2\big(\phi_{1t}(m\cdot\nabla\phi_{1t})+\phi_{2t}(m\cdot\nabla\phi_{2t})\big) \\
&\qquad +\psi_t(m\cdot\nabla\psi_t)+\eta_{1t}(m\cdot\nabla\eta_{1t})+\eta_{2t}(m\cdot\nabla\eta_{2t})\big]\,dx\,dy\,dt, \qquad (4\text{-}16)
\end{aligned}$$

where

$$\mathrm{Y}_1=\rho h\int_\Omega\big\{\tfrac{1}{12}h^2\big[\phi_{1t}(m\cdot\nabla\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2)\big]+\psi_t(m\cdot\nabla\psi)+\eta_{1t}(m\cdot\nabla\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2)\big\}\,dx\,dy\bigg|_0^T. \tag{4-17}$$

A typical term of the last integral in (4-16) is (except for a constant factor)

$$\begin{aligned}\int_0^T(\phi_{1t},m\cdot\nabla\phi_{1t})\,dt&=\tfrac{1}{2}\int_0^T\int_\Omega \operatorname{div}(m\phi_{1t}^2)\,dx\,dy\,dt-\int_0^T\int_\Omega\phi_{1t}^2\,dx\,dy\,dt\\&=\tfrac{1}{2}\int_0^T\int_{\Gamma_1}(m\cdot\nu)\phi_{1t}^2\,d\Gamma\,dt-\int_0^T\int_\Omega\phi_{1t}^2\,dx\,dy\,dt.\end{aligned}$$

The other terms of that integral are treated similarly. Thus, it follows that

$$\begin{aligned}&\rho h\int_0^T\int_\Omega\big\{\tfrac{1}{12}h^2\big[\phi_{1t}(m\cdot\nabla\phi_{1t})+\phi_{2t}(m\cdot\nabla\phi_{2t})\big]+\psi_t(m\cdot\nabla\psi_t)+\eta_{1t}(m\cdot\nabla\eta_{1t})+\eta_{2t}(m\cdot\nabla\eta_{2t})\big\}\,dx\,dy\,dt\\&\quad=\tfrac{1}{2}\rho h\int_0^T\int_{\Gamma_1}m\cdot\nu\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,d\Gamma\,dt\\&\qquad\qquad-\int_0^T\int_\Omega\rho h\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\,dt.\end{aligned}\tag{4-18}$$

Combining (4-15), (4-16) and (4-18), one has

$$\begin{aligned}&\mathrm{Y}_1+\int_0^T\int_\Omega\rho h\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\,dt-k\int_0^T\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy\,dt\\&=\mathrm{J}_1-\mathrm{J}_2+\int_0^T\int_{\Gamma_0}\big[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5\big]\,d\Gamma\,dt\\&\quad-\int_0^T\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2)+\psi_t(m\cdot\nabla\psi)+\eta_{1t}(m\cdot\nabla\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2)\big]\,d\Gamma\,dt,\end{aligned}\tag{4-19}$$

where

$$\mathrm{J}_1=\tfrac{1}{2}\rho h\int_0^T\int_{\Gamma_1}m\cdot\nu\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,d\Gamma\,dt,\tag{4-20}$$

and

$$\begin{aligned}\mathrm{J}_2=\tfrac{1}{2}\int_0^T\int_{\Gamma_1}m\cdot\nu\bigg\{&D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]\\&+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\big]\\&\quad+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac{1}{2}\psi_x^2\big)^2+(1-\mu)(\eta_{2y}+\tfrac{1}{2}\psi_y^2)^2\\&\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\big)^2+\tfrac{1}{2}(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\bigg\}\,d\Gamma\,dt.\end{aligned}\tag{4-21}$$

Let us examine the integrals on Γ_0 in the right-hand side of (4-19). Since $\phi_1=\phi_2=\psi=\eta_1=\eta_2=0$ on Γ_0, we have $\nabla\phi_1=\nu((\partial\phi_1)/(\partial\nu))$ on Γ_0 and similarly for the other functions. Therefore,

$$\int_{\Gamma_0} m\cdot\nu\Big\{D\big[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)\big]\Big\}\,d\Gamma$$
$$+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac{1}{2}\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac{1}{2}\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\big)^2$$
$$+\tfrac{1}{2}(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]$$
$$=\int_{\Gamma_0} m\cdot\nu\Bigg\{D\bigg[\Big(\nu_1\frac{\partial\phi_1}{\partial\nu}+\nu_2\frac{\partial\phi_2}{\partial\nu}\Big)^2-(1-\mu)\nu_1\nu_2\frac{\partial\phi_1}{\partial\nu}\frac{\partial\phi_2}{\partial\nu}\bigg]+k\Big(\frac{\partial\psi}{\partial\nu}\Big)^2$$
$$+\frac{Eh}{1-\mu^2}\bigg[(1-\mu)\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}\Big(\frac{\partial\psi}{\partial\nu}\Big)^2\Big)^2$$
$$-2(1-\mu)\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\frac{1}{2}\Big(\nu_1\frac{\partial\psi}{\partial\nu}\Big)^2\Big)\Big(\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}\Big(\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2\Big)$$
$$+\mu\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}\Big|\Big(\frac{\partial\psi}{\partial\nu}\Big)\Big|^2\Big)^2$$
$$+\frac{1-\mu}{2}\Big(\nu_2\frac{\partial\eta_1}{\partial\nu}+\nu_1\frac{\partial\eta_2}{\partial\nu}+\nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}\Big)\bigg]\Bigg\}\,d\Gamma. \quad (4\text{-}22)$$

Furthermore,

$$\int_{\Gamma_0}\big[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5\big]\,d\Gamma$$
$$=\int_{\Gamma_0} m\cdot\nu\Bigg\{\frac{D}{2}\bigg[(1-\mu)\Big(\Big(\frac{\partial\phi_1}{\partial\nu}\Big)^2+\Big(\frac{\partial\phi_2}{\partial\nu}\Big)^2\Big)+(1+\mu)\Big(\nu_1\frac{\partial\phi_1}{\partial\nu}+\nu_2\frac{\partial\phi_2}{\partial\nu}\Big)^2\bigg]+k\Big(\frac{\partial\psi}{\partial\nu}\Big)^2+N_1\Big(\nu_1\frac{\partial\psi}{\partial\nu}\Big)^2$$
$$+2N_{12}\nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}+N_2\Big(\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2+N_1\nu_1\frac{\partial\eta_1}{\partial\nu}+N_{12}\nu_2\frac{\partial\eta_1}{\partial\nu}+N_2\nu_2\frac{\partial\eta_2}{\partial\nu}+N_{12}\nu_1\frac{\partial\eta_2}{\partial\nu}\Bigg\}\,d\Gamma. \quad (4\text{-}23)$$

Since

$$-\tfrac{1}{2}\Big\{D\big[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)\big]$$
$$+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac{1}{2}\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac{1}{2}\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\big)^2$$
$$+\tfrac{1}{2}(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}$$
$$+\big[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5\big]$$
$$=\frac{1}{2}\Bigg\{D\bigg[\Big(\nu_1\frac{\partial\phi_1}{\partial\nu}+\nu_2\frac{\partial\phi_2}{\partial\nu}\Big)^2+\frac{1-\mu}{2}\Big(\nu_2\frac{\partial\phi_1}{\partial\nu}-\nu_1\frac{\partial\psi_2}{\partial_\nu}\Big)^2\bigg]+k\Big(\frac{\partial\psi}{\partial\nu}\Big)^2$$
$$+\frac{Eh}{1-\mu^2}\bigg[(1-\mu)\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\frac{1}{2}\Big(\nu_1\frac{\partial\psi}{\partial\nu}\Big)^2\Big)^2+(1-\mu)\Big(\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}\Big(\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2\Big)^2$$
$$+\mu\Big(\nu_1\frac{\partial\eta_1}{\partial\nu}+\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}|\nabla\psi|^2\Big)^2+\frac{1-\mu}{2}\Big(\nu_2\frac{\partial\eta_1}{\partial\nu}+\nu_1\frac{\partial\eta_2}{\partial\nu}+\nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2\bigg]\Bigg\}, \quad (4\text{-}24)$$

we conclude from (4-19) and (4-24) that

$$
\begin{aligned}
&\mathrm{Y}_1+\int_0^T\!\!\int_\Omega \rho h\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\,dt-k\int_0^T\!\!\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy\,dt\\
&\quad=\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2+\int_0^T\!\!\int_{\Gamma_0}\big[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5\big]\,d\Gamma\,dt\\
&\qquad-\int_0^T\!\!\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2)+\psi_t(m\cdot\nabla\psi)+\eta_{1t}(m\cdot\nabla\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2)\big]\,d\Gamma\,dt,
\end{aligned}
\tag{4-25}
$$

where

$$
\begin{aligned}
\mathrm{J}_0=\frac{1}{2}\int_0^T\!\!\int_{\Gamma_0}\Bigg\{&D\Bigg[\bigg(\nu_1\frac{\partial\phi_1}{\partial\nu}+\nu_2\frac{\partial\phi_2}{\partial\nu}\bigg)^2+\frac{1-\mu}{2}\bigg(\nu_2\frac{\partial\phi_1}{\partial\nu}-\nu_1\frac{\partial\psi_2}{\partial_\nu}\bigg)^2\Bigg]+k\bigg(\frac{\partial\psi}{\partial\nu}\bigg)^2\\
&+\frac{Eh}{1-\mu^2}\Bigg[(1-\mu)\bigg(\nu_1\frac{\partial\eta_1}{\partial\nu}+\frac{1}{2}\Big(\nu_1\frac{\partial\psi}{\partial\nu}\Big)^2\bigg)^2+(1-\mu)\bigg(\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}\Big(\nu_2\frac{\partial\psi}{\partial\nu}\Big)^2\bigg)^2\\
&+\mu\bigg(\nu_1\frac{\partial\eta_1}{\partial\nu}+\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}|\nabla\psi|^2\bigg)^2+\frac{1-\mu}{2}\bigg(\nu_2\frac{\partial\eta_1}{\partial\nu}+\nu_1\frac{\partial\eta_2}{\partial\nu}+\nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}\bigg)^2\Bigg]\Bigg\}\,d\Gamma.
\end{aligned}
$$

Now, use (4-6) with $\{\phi_1,\phi_2,0,\eta_1,\eta_2\}$ in the third term on the left-hand side of (4-25) to obtain

$$
\begin{aligned}
&\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1tt}\phi_1+\phi_{2tt}\phi_2)+\eta_{1tt}\eta_1+\eta_{2tt}\eta_2\big]\,dx\,dy\\
&\qquad-k\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy+a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2,0,\eta_1,\eta_2)\\
&\quad=-\int_{\Gamma_1}\big[\phi_{1t}\phi_1+\phi_{2t}\phi_2+\eta_{1t}\eta_1+\eta_{2t}\eta_2\big]\,d\Gamma.
\end{aligned}
\tag{4-26}
$$

Integrate identity (4-26) with respect to t from 0 to T. After an integration by parts in the first term, one obtains

$$
\begin{aligned}
&\mathrm{Y}_2-\rho h\int_0^T\!\!\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\,dt\\
&\qquad+k\int_0^T\!\!\int_\Omega\big[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2\big]\,dx\,dy\,dt+\int_0^T\big[a_0(\phi_1,\phi_2)+a_2(\eta_1,\eta_2)\big]\,dt\\
&\quad=-\int_0^T\!\!\int_{\Gamma_1}\big[\phi_{1t}\phi_1+\phi_{2t}\phi_2+\eta_{1t}\eta_1+\eta_{2t}\big]\,d\Gamma\,dt
\end{aligned}
\tag{4-27}
$$

where

$$
\mathrm{Y}_2=\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}\phi_1+\phi_{2t}\phi_2)+\eta_{1t}\eta_1+\eta_{2t}\eta_2\big]\,dx\,dy\bigg|_0^T.
\tag{4-28}
$$

Multiply (4-27) by $1-\varepsilon$, with $\varepsilon \in (0, 1)$, and add the product to (4-25) to get

$$\begin{aligned}(1-\varepsilon)\rho h\int_0^T\int_\Omega \psi_t^2\,dx\,dy\,dt+\varepsilon\int_0^T\int_\Omega \rho h\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\,dt\\ +(1-\varepsilon)\int_0^T a_0(\phi_1,\phi_2)\,dt+(1-\varepsilon)\int_0^T a_2(\psi,\eta_1,\eta_2)\,dt-\varepsilon k\int_0^T a_1(\phi_1,\phi_2,\psi,\phi_1,\phi_2,0)\,dt+\mathrm{Y}_1+(1-\varepsilon)\mathrm{Y}_2\\ =\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2-\int_0^T\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi)\\ +\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma\,dt.\end{aligned} \tag{4-29}$$

Now, use (4-6) with $\{0, 0, \psi, 0, 0\}$. After an integration by parts in t one obtains

$$\begin{aligned}\mathrm{Y}_3-\rho h\int_0^T\int_\Omega \psi_t^2\,dx\,dy\,dt+k\int_0^T a_1(\phi_1,\phi_2,\psi,0,0,\psi)\,dt+\int_0^T a_2(\psi,\eta_1,\eta_2,\psi,0,0)\,dt\\ =-\int_0^T\int_{\Gamma_1}\psi\psi_t\,d\Gamma\,dt,\end{aligned} \tag{4-30}$$

where

$$\mathrm{Y}_3=\rho h\int_\Omega \psi_t\psi\,dx\,dy\Big|_0^T. \tag{4-31}$$

Multiply identity (4-30) by ε and add the product to (4-29) to obtain

$$\begin{aligned}(1-2\varepsilon)\rho h\int_0^T\int_\Omega \psi_t^2\,dx\,dy\,dt+\varepsilon\int_0^T\int_\Omega \rho h\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\,dt\\ +(1-\varepsilon)\int_0^T\big[a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2,0,\eta_1\eta_2)\big]\,dt\\ +\varepsilon k\int_0^T a_1(\phi_1,\phi_2,\psi)\,dt-2\varepsilon k\int_0^T a_1(\phi_1,\phi_2,\psi,\phi_1,\phi_2,0)\,dt\\ +\varepsilon\int_0^T a_2(\psi,\eta_1,\eta_2,\psi,0,0)\,dt+\mathrm{Y}_1+(1-\varepsilon)\mathrm{Y}_2+\varepsilon\mathrm{Y}_3\\ =\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2-\int_0^T\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)\\ +\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma\,dt.\end{aligned} \tag{4-32}$$

<u>Step 2</u>: Define the functional

$$\begin{aligned}\rho_\varepsilon(t)=\rho h\Big[\tfrac{1}{12}h^2(\phi_{1t}(t),m\cdot\nabla\phi_1(t))+\tfrac{1}{12}h^2(\phi_{2t}(t),m\cdot\nabla\phi_2(t))+(\psi_t(t),m\cdot\nabla\psi(t))\\ +(\eta_{1t}(t),m\cdot\nabla\eta_1(t))+(\eta_{2t}(t),m\cdot\nabla\eta_2(t))\Big]\\ +(1-\varepsilon)\rho h\big\{\tfrac{1}{12}h^2\big[(\phi_{1t}(t),\phi_1(t))+(\phi_{2t}(t),\phi_2(t))\big]+(\eta_{1t}(t),\eta_1(t))+(\eta_{2t}(t),\eta_2(t))\big\}\\ +\varepsilon\rho h(\psi_t(t),\psi(t)).\end{aligned} \tag{4-33}$$

From identities (4-17), (4-28), and (4-31), one sees that

$$\mathrm{Y}_1 + (1-\varepsilon)\mathrm{Y}_2 + \varepsilon \mathrm{Y}_3 = \rho_\varepsilon(T) - \rho_\varepsilon(0). \tag{4-34}$$

Since (4-32) is valid for all $T > 0$, we differentiate in T and obtain, writing t in place of T,

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) = {}& \frac{d}{dt}(\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2)-(1-2\varepsilon)\rho h\int_\Omega \psi_t^2\,dx\,dy-\varepsilon\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\\ &-(1-\varepsilon)\big[a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2,0,\eta_1,\eta_2)\big]-\varepsilon k a_1(\phi_1,\phi_2,\psi)\,dt\\ &+2\varepsilon k a_1(\phi_1\phi_2,\psi,\phi_1,\phi_2,0)-\varepsilon a_2(\psi,\eta_1,\eta_2,\psi,0,0)\\ &-\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)\\ &\qquad+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)+\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma,\end{aligned} \tag{4-35}$$

where the right-hand side is evaluated at t. Now, let $\delta > 0$ and consider the perturbed energy $F_{\varepsilon,\delta}(t)$ given by

$$F_{\varepsilon,\delta}(t) = E_k(t) + \delta\rho_\varepsilon(t). \tag{4-36}$$

We are going to prove that for all ε, δ sufficiently small, one has

$$\frac{d}{dt}F_{\varepsilon,\delta}(t) \le -\tfrac{1}{2}\varepsilon\delta E_k(t) - \tfrac{1}{2}\delta E_\Gamma(t), \tag{4-37}$$

where

$$\begin{aligned}E_\Gamma(t) = {}& \tfrac{1}{2}\rho h\int_{\Gamma_1} m\cdot\nu\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,d\Gamma\\ &+\tfrac{1}{2}\int_\Gamma |m\cdot\nu|\Big\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac{1}{2}(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\big]\\ &\qquad+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac{1}{2}\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac{1}{2}\psi_y^2\big)^2\\ &\qquad\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\big)^2+\tfrac{1}{2}(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,d\Gamma.\end{aligned} \tag{4-38}$$

We begin the proof of inequality (4-37) estimating $(d/dt)\rho_\varepsilon(t)$. First of all, we bound the term $a_1(\phi_1, \phi_2, \psi, \phi_1\phi_2, 0)$ in (4-35). For any $\xi > 0$, we have

$$\big|a_1(\phi_1,\phi_2,\psi,\phi_1,\phi_2,0)\big| \le \frac{\xi}{2}a_1(\phi_1,\phi_2,\psi)+\frac{1}{2\xi}a_1(\phi_1,\phi_2,0).$$

Since $\Gamma_0 \ne \varnothing$, according to [Lagnese 1989, Lemma 2.1] there is a constant γ_0 (depending on the geometry of Ω and on the parameters μ and D) such that

$$a_1(\phi_1,\phi_2,0) = \|\phi_1\|^2+\|\phi_2\|^2 \le \gamma_0 a_0(\phi_1,\phi_2).$$

Therefore,

$$\big|a_1(\phi_1,\phi_2,\psi,\phi_1,\phi_2,0)\big| \le \frac{\xi}{2}a_1(\phi_1,\phi_2,\psi)+\frac{\gamma_0}{2\xi}a_0(\phi_1,\phi_2). \tag{4-39}$$

Use inequality (4-39) in identity (4-35) to get

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) \le \frac{d}{dt}(\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2)-(1-2\varepsilon)\rho h\int_\Omega \psi_t^2\,dx\,dy-\varepsilon\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\\ -\Big(1-\varepsilon-\frac{\varepsilon\gamma_0 k}{\xi}\Big)a_0(\phi_1,\phi_2)-(1-\varepsilon)a_2(\psi,\eta_1,\eta_2,0,\eta_1,\eta_2)\\ -\varepsilon k(1-\xi)a_1(\phi_1,\phi_2,\psi)\,dt-\varepsilon a_2(\psi,\eta_1,\eta_2,\psi,0,0)\\ -\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)\\ +\psi_t(m\cdot\nabla\psi+\varepsilon\psi)+\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma.\end{aligned}$$

Fix $\xi=\frac{1}{2}$, and then choose $\varepsilon>0$ so that $1-\varepsilon-2\varepsilon\gamma_0 k\ge\varepsilon$; that is,

$$0<\varepsilon\le\frac{1}{2(1+\gamma_0 k)}. \tag{4-40}$$

For such ε, one has

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) \le \frac{d}{dt}(\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2)-(1-2\varepsilon)\rho h\int_\Omega \psi_t^2\,dx\,dy-\varepsilon\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy\\ -\varepsilon a_0(\phi_1,\phi_2)-(1-\varepsilon)a_2(\psi,\eta_1,\eta_2,0,\eta_1,\eta_2)-\tfrac{1}{2}k\varepsilon a_1(\phi_1,\phi_2,\psi)-\varepsilon a_2(\psi,\eta_1,\eta_2,\psi,0,0)\\ -\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)\\ +\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma\,dt\\ \le \frac{d}{dt}(\mathrm{J}_0+\mathrm{J}_1-\mathrm{J}_2)-(1-2\varepsilon)\rho h\int_\Omega \psi_t^2\,dx\,dy-\varepsilon E_k(t)\\ -\tfrac{1}{2}\varepsilon\Big\{\rho h\int_\Omega\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,dx\,dy+a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\Big\}\\ -\int_{\Gamma_1}[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)\\ +\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)]\,d\Gamma\,dt.\end{aligned} \tag{4-41}$$

We estimate the last term on the right-hand side of (4-41) as follows:

$$\begin{aligned}\Big|\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)\\ +\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma\Big|\\ \le\frac{1}{2\xi}\int_{\Gamma_1}[\phi_{1t}^2+\phi_{2t}^2+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2]\,d\Gamma\\ +\frac{\xi}{2}\int_{\Gamma_1}\big[(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2+(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)^2+(m\cdot\nabla\psi+\varepsilon\psi)^2\\ +(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)^2+(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)^2\big]\,d\Gamma\\ =-\frac{1}{2\xi}\frac{d}{dt}E_k(t)+\frac{\xi}{2}\int_{\Gamma_1}\big[(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2+(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)^2+(m\cdot\nabla\psi+\varepsilon\psi)^2\\ +(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)^2+(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)^2\big]\,d\Gamma.\end{aligned} \tag{4-42}$$

Looking for the last integral in (4-42), it follows by (4-5) that

$$\begin{aligned}\int_{\Gamma_1} \big[(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2+(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)^2+(m\cdot\nabla\psi+\varepsilon\psi)^2 \\ +(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)^2+(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)^2\big]\,d\Gamma \\ \le G_0\int_{\Gamma_1} m\cdot\nu\big[(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2+(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)^2 \\ +(m\cdot\nabla\psi+\varepsilon\psi)^2+(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)^2+(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)^2\big]\,d\Gamma. \end{aligned}\tag{4-43}$$

We now bound the right-hand side of inequality (4-43). Its first term is bounded by

$$\begin{aligned}\int_{\Gamma_1} m\cdot\nu(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2\,d\Gamma &\le 2\int_{\Gamma_1} m\cdot\nu\big[(m\cdot\nabla\phi_1)^2+(1-\varepsilon)^2\phi_1^2\big]\,d\Gamma \\ &\le 2R^2\int_{\Gamma_1} m\cdot\nu|\nabla\phi_1|^2\,d\Gamma+2(1-\varepsilon)^2R\int_{\Gamma_1}\phi_1^2\,d\Gamma,\end{aligned}$$

where $R=\sup_{\Gamma_1} m(x,y)$. The other terms can be bounded analogously. Therefore, one gets

$$\begin{aligned}\int_{\Gamma_1} \big[(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)^2+(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)^2 \\ +(m\cdot\nabla\psi+\varepsilon\psi)^2+(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)^2+(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)^2\big]\,d\Gamma \\ \le 2G_0R^2\int_{\Gamma_1} m\cdot\nu\big[|\nabla\phi_1|^2+|\nabla\phi_2^2+|\nabla\psi|^2+|\nabla\eta_1|^2+|\nabla\eta_2|^2\big]\,d\Gamma \\ +2G_0(1-\varepsilon)^2R\int_{\Gamma_1}\big[\phi_1^2+\phi_2^2+\psi^2+\eta_1^2+\eta_2^2\big]\,d\Gamma.\end{aligned}\tag{4-44}$$

For $k\ge k_0>0$ we have, according to [Lagnese 1989, Lemma 2.1] and to trace theory,

$$\int_{\Gamma_1}\big[\phi_1^2+\phi_2^2+\psi^2+\eta_1^2+\eta_2^2\big]\,d\Gamma\le\gamma_1\big[a_0(\phi_1,\phi_2)+ka_1(\phi_1,\phi_2,\psi)+a_2(\psi,\eta_1,\eta_2)\big].\tag{4-45}$$

In addition,

$$\begin{aligned}\int_{\Gamma_1} m\cdot\nu\big[|\nabla\phi_1|^2+|\nabla\phi_2|^2+|\nabla\psi|^2+|\nabla\eta_1|^2+|\nabla\eta_2|^2\big]\,d\Gamma \\ \le\tilde\gamma_2\bigg[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+\int_{\Gamma_1}(\phi_1^2+\phi_2^2+\eta_1^2+\eta_2^2)\bigg] \\ \le\gamma_2\big[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_0(\phi_1,\phi_2)+a_2(\eta_1,\eta_2)\big],\end{aligned}\tag{4-46}$$

where the constants γ_1, γ_2 depend only on Ω, D, μ, and k_0, and

$$\begin{aligned}&a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\ &\quad=2\frac{d}{dt}\mathrm{J}_2=\int_\Gamma m\cdot\nu\bigg\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]\\ &\qquad+k[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2]\\ &\qquad+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)(\eta_{2y}+\tfrac12\psi_y^2)^2\\ &\qquad+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2+\tfrac12(1-\mu)(\eta_{1y}\eta_{2x}+\psi_x\psi_y)^2\Big]\bigg\}\,d\Gamma.\end{aligned}\tag{4-47}$$

From (4-42)–(4-47), we obtain the estimate

$$\begin{aligned}
&\left|\int_{\Gamma_1}\big[\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)\right.\\
&\qquad\left.+\eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)\big]\,d\Gamma\right|\\
&\le -\frac{1}{2\xi}\frac{d}{dt}E_k(t)+\xi G_0R^2\gamma_2\big[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\big]\\
&\qquad+\xi G_0\gamma_1(1-\varepsilon)^2R\big[a_0(\phi_1,\phi_2)+ka_1(\phi_1,\phi_2,\psi)+a_2(\psi,\eta_1,\eta_2)\big]\\
&\le -\frac{1}{2\xi}\frac{d}{dt}E_k(t)+\xi G_0R^2\gamma_2\big[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\big]\\
&\qquad+\xi G_0\gamma_1(1-\varepsilon)^2RE_k(t).
\end{aligned}\tag{4-48}$$

Using (4-48) in (4-41), it follows that

$$\begin{aligned}
\frac{d}{dt}\rho_\varepsilon(t)\le\frac{d}{dt}\mathrm{J}_0&+\tfrac12 c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)-\frac{1}{2\xi}\frac{d}{dt}E_k(t)-\big(\tfrac12-\xi\gamma_2G_0R^2\big)a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\
&-\big[\varepsilon-2\xi\gamma_1G_0(1-\varepsilon)^2R\big]E_k(t)-\big[\tfrac12\varepsilon-\xi\gamma_2G_0R^2\big]\big[a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\big],
\end{aligned}\tag{4-49}$$

where

$$c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)=2\frac{d}{dt}J_1=\rho h\int_{\Gamma_1}m\cdot\nu\big[\tfrac{1}{12}h^2(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\big]\,d\Gamma.$$

From the definition of J_0 and the first of the geometric assumptions in (4-4), we have

$$\begin{aligned}
\frac{d}{dt}\mathrm{J}_0=\tfrac12\int_0^T\!\!\int_{\Gamma_0}m\cdot\nu\Big\{&D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]\\
&+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\big]\\
&+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2\\
&\qquad+\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,d\Gamma\,dt\\
+\tfrac14(1-\mu)&\int_{\Gamma_0}m\cdot\nu\left(\nu_2\frac{\partial\phi_1}{\partial\nu}+\nu_1\frac{\partial\phi_2}{\partial\nu}\right)^2d\Gamma\\
\le -\tfrac12 a_{\Gamma_0}&(\phi_1,\phi_2,\psi,\eta_1,\eta_2),
\end{aligned}\tag{4-50}$$

where

$$\begin{aligned}
&a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\
&=\tfrac12\int_0^T\!\!\int_{\Gamma_0}|m\cdot\nu|\Big\{D\big[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac12(1-\mu)(\phi_{1y}+\phi_{2x})^2\big]+k\big[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\big]\\
&\qquad+\frac{Eh}{1-\mu^2}\Big[(1-\mu)\big(\eta_{1x}+\tfrac12\psi_x^2\big)^2+(1-\mu)\big(\eta_{2y}+\tfrac12\psi_y^2\big)^2+\mu\big(\eta_{1x}+\eta_{2y}+\tfrac12|\nabla\psi|^2\big)^2\\
&\qquad\qquad+\tfrac12(1-\mu)(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Big]\Big\}\,d\Gamma.
\end{aligned}$$

Substituting (4-50) in the right-hand side of (4-49), one gets the estimate

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) \le \tfrac{1}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) - \frac{1}{2\xi}\frac{d}{dt}E_k(t)\\ -\left(\tfrac{1}{2}-\xi\gamma_2G_0R^2\right)a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) - [\varepsilon-2\xi\gamma_1G_0(1-\varepsilon)^2R]E_k(t)\\ -\left[\tfrac{1}{2}\varepsilon-\xi\gamma_2G_0R^2\right]\left[a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\right] - \tfrac{1}{2}a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2).\end{aligned} \tag{4-51}$$

Now, taking $\varepsilon < \frac{1}{2}$ and choosing $\xi > 0$ small enough such that

$$2\xi\gamma_1G_0(1-\varepsilon)^2R \le \tfrac{1}{2}\varepsilon, \quad \xi\gamma_2G_0R^2 \le \tfrac{1}{4}\varepsilon < \tfrac{1}{4},$$

we can guarantee from inequality (4-51) that

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) \le -\frac{1}{2\xi}\frac{d}{dt}E_k(t) - \tfrac{1}{2}\varepsilon E_k(t) + \tfrac{1}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\ -\tfrac{1}{4}a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) - \tfrac{1}{4}a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2).\end{aligned}$$

Let us consider

$$F_{\varepsilon,\delta}(t) = E_k(t) + \delta\rho_\varepsilon(t)$$

with $\delta > 0$. Therefore,

$$\begin{aligned}\frac{d}{dt}F_{\varepsilon,\delta}(t) &= \frac{d}{dt}E_k(t) + \delta\frac{d}{dt}\rho_\varepsilon(t)\\ &= \left(1-\frac{\delta}{2\xi}\right)\frac{d}{dt}E_k(t) - \frac{\delta\varepsilon}{2}E_k(t) + \frac{\delta}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) - \frac{\delta}{4}a_\Gamma(\phi_1,\phi_2,\psi,\eta_1,\eta_2),\end{aligned}$$

where

$$a_\Gamma(\phi_1,\phi_2,\psi,\eta_1,\eta_2) = a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2).$$

From (4-5), we get

$$\begin{aligned}\frac{d}{dt}E_k(t) &= -\int_{\Gamma_1}[\phi_{1t}^2+\phi_{2t}^2+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2]\,d\Gamma\\ &\le -g_0\int_{\Gamma_1} m\cdot\nu[\phi_{1t}^2+\phi_{2t}^2+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2]\,d\Gamma\\ &\le -\frac{g_0}{\rho h}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2),\end{aligned} \tag{4-52}$$

provided $\frac{1}{12}h^2 \le 1$ (as we may assume). Therefore

$$\begin{aligned}\frac{d}{dt}F_{\varepsilon,\delta}(t) &= -\left[\frac{g_0}{\rho h}\left(1-\frac{\delta}{2\xi}\right)-\frac{\delta}{2}\right]c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) - \frac{\varepsilon\delta}{2}E_k(t) - \frac{\delta}{4}a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\ &\le -\frac{\varepsilon\delta}{2}E_k(t) - \frac{\delta}{4}\left[c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\right]\\ &= -\frac{\varepsilon\delta}{2}E_k(t) - \frac{\delta}{2}E_\Gamma(t),\end{aligned} \tag{4-53}$$

with $\delta > 0$ being chosen such that

$$\frac{g_0}{\rho h}\left(1-\frac{\delta}{2\xi}\right) - \frac{\delta}{2} \ge \frac{\delta}{4}.$$

Step 3: To get the exponential decay of $E_k(t)$ using inequality (4-53), we need to compare $E_k(t)$ and $F_{\varepsilon,\delta}(t)$. To carry this out, we use the definition (4-33) of $\rho_\varepsilon(t)$ and [Lagnese 1989, Lemma 2.1] to obtain

$$|\rho_\varepsilon(t)| \le C E_k(t),$$

where C depends on Ω, D, μ, and K_0 ($K \ge K_0 > 0$) but not on ε. Consequently

$$|F_{\varepsilon,\delta} - E(t)| = \delta\rho_\varepsilon(t) \le \delta C E_k(t).$$

Therefore,

$$(1-\delta C)E_k(t) \le F_{\varepsilon,\delta}(t) \le (1+\delta C)E_k(t).$$

Moreover, since

$$E_k(t) + \frac{1}{\varepsilon}E_\Gamma(t) \ge E_k(t),$$

one gets

$$\frac{d}{dt}F_{\varepsilon,\delta} \le -\omega F_{\varepsilon,\delta},$$

where $\omega = \delta\varepsilon/(2(1+\delta C))$. As a consequence of (4-33), (4-36) and of the choice of ε (see (4-40)), we conclude that there exist positive constants $C > 0$ and $\omega > 0$ such that

$$E_k(t) \le C E_k(0)\mathrm{e}^{-\omega t}$$

for every $t > 0$ and every solution of (1-1), (4-1), (1-3). □

5. Further comments and open problems

(1) Although we know the physical deduction for the nonlinear Mindlin–Timoshenko system (1-1)–(1-3), see for example [Lagnese and Lions 1988; Rahmani 2014], we are not aware of results concerning well-posedness and regularity for all $k > 0$. However, since our main goal was to give a positive response to the Lagnese–Lions conjecture, what we can say is that, for k large enough and for initial data in the space $\mathcal{X}$, the system (1-1)–(1-3) is very close to the known von Kármán system (1-5)–(1-7) (see Theorem 2.1). On the other hand, there is extensive literature dealing with well-posedness, regularity, stability, etc., for system (1-5)–(1-7); see [Favini et al. 1996; Lagnese 1989; Lagnese and Leugering 1991; Lasiecka 1998; Perla Menzala et al. 2002]. In Section 4 we analyzed the asymptotic behavior (as $t \to \infty$) for the solution of the nonlinear Mindlin–Timoshenko system with boundary feedback. To this end, we had to request an additional regularity for their solutions. For this reason, in all results of that section, we have used the expression "regular enough" to the solutions, in order to ensure that, under certain restrictions, the results hold. In our case, for instance, if we consider the solution $\{\phi_1(t), \phi_2(t), \psi(t), \eta_1(t), \eta_2(t)\} \in [H^2 \cap H^1_{\Gamma_0}]^2 \times [H^3 \cap H^1_{\Gamma_0}] \times [H^2 \cap H^1_{\Gamma_0}]^2$, the stability results hold. For the linear Mindlin–Timoshenko system, this issue was treated in [Lagnese 1989, Remark 3.1].

(2) In the proofs of Theorems 2.1, 3.1, and 4.5, we have considered the case where the initial data are fixed. The same results hold if we consider the case where they do depend on k, provided we assume the initial data $\{\phi^k_{10}, \phi^k_{11}, \phi^k_{20}, \phi^k_{21}, \psi^k_0, \psi^k_1, \eta^k_{10}, \eta^k_{11}, \eta^k_{20}, \eta^k_{21}\}$ to be such that the initial energy $E_k(0)$ remains bounded and such that they converge weakly to $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\}$ in the corresponding spaces.

(3) It would be interesting to analyze whether the same stabilization results (Theorems 3.1, 4.5) hold considering the systems (3-1), (1-2), (1-3) and (1-1), (4-1), (1-3) with less damping terms. To eliminate some of these dissipative terms is a difficult task due to the complex nonlinearities involved. In this context, we can mention the works [Alabau-Boussouira 2007; Alabau-Boussouira and Léautaud 2012; Alabau-Boussouira et al. 2011; Ammar-Khodja et al. 2007; Soufyane 1999], which have obtained stability for some hyperbolic systems without damping terms in some of their equations.

(4) Another interesting and difficult problem is to obtain the same result in Theorem 3.1 when the damping mechanisms act in an arbitrary small region of the plate. The difficulty for this case, of course, consists in getting a unique continuation result for the Mindlin–Timoshenko system. On this subject, we mention [Cavalcanti et al. 2014; Charles et al. 2013; Geredeli and Lasiecka 2013; Komornik and Zuazua 1990; Zuazua 1990], which have obtained decay rates for the energy of various hyperbolic systems considering both linear and nonlinear localized damping terms.

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FINITE TIME BLOWUP FOR A SUPERCRITICAL DEFOCUSING NONLINEAR SCHRÖDINGER SYSTEM

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We consider the global regularity problem for defocusing nonlinear Schrödinger systems

$$i\partial_t + \Delta u = (\nabla_{\mathbb{R}^m} F)(u) + G$$

on Galilean spacetime $\mathbb{R} \times \mathbb{R}^d$, where the field $u : \mathbb{R}^{1+d} \to \mathbb{C}^m$ is vector-valued, $F : \mathbb{C}^m \to \mathbb{R}$ is a smooth potential which is positive, phase-rotation-invariant, and homogeneous of order $p+1$ outside of the unit ball for some exponent $p > 1$, and $G : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^m$ is a smooth, compactly supported forcing term. This generalises the scalar defocusing nonlinear Schrödinger (NLS) equation, in which $m = 1$ and $F(v) = 1/(p+1)|v|^{p+1}$. It is well known that in the energy-subcritical and energy-critical cases when $d \leq 2$ or $d \geq 3$ and $p \leq 1 + 4/(d-2)$, one has global existence of smooth solutions from arbitrary smooth compactly supported initial data $u(0)$ and forcing term G, at least in low dimensions. In this paper we study the supercritical case where $d \geq 3$ and $p > 1 + 4/(d-2)$. We show that in this case, there exists a smooth potential F for some sufficiently large m, positive and homogeneous of order $p+1$ outside of the unit ball, and a smooth compactly supported choice of initial data $u(0)$ and forcing term G for which the solution develops a finite time singularity. In fact the solution is locally discretely self-similar with respect to parabolic rescaling of spacetime. This demonstrates that one cannot hope to establish a global regularity result for the scalar defocusing NLS unless one uses some special property of that equation that is not shared by these defocusing nonlinear Schrödinger systems.

As in a previous paper of the author (*Anal. PDE* **9**:8 (2016), 1999–2030) considering the analogous problem for the nonlinear wave equation, the basic strategy is to first select the mass, momentum, and energy densities of u, then u itself, and then finally design the potential F in order to solve the required equation.

1. Introduction

The (inhomogeneous) *nonlinear Schrödinger equation* (NLS) takes the form

$$i\partial_t u + \Delta u = \pm|u|^{p-1}u + G,$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is the unknown field of one time variable t and d spatial variables $x_1, \dots, x_d$, $p > 1$ is an exponent, $\Delta = \partial_{x_j}\partial_{x_j}$ is the spatial Laplacian (with the usual summation conventions), $\partial_t, \partial_{x_1}, \dots, \partial_{x_d}$ are the partial derivatives in time and space, $G : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is a forcing term, and $\pm$ is either the $+$ sign (defocusing case) or $-$ sign (focusing case). As is well known, in the homogeneous case $G = 0$, this equation has (formally, at least) a conserved Hamiltonian

$$H(u) := \int_{\mathbb{R}^d} \tfrac{1}{2}|\nabla u|^2 \pm \frac{1}{p+1}|u|^{p+1}\,dx$$

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which is nonnegative in the defocusing case; this Hamiltonian plays a crucial role in the large data global regularity theory for such equations.

In this paper, we will study the following generalisation of the defocusing NLS equation, in which the unknown field u is allowed to be vector-valued, but for which one continues to have a nonnegative conserved Hamiltonian in the homogeneous case. Let $\mathbb{C}^m$ be a standard finite-dimensional complex vector space, with the real inner product

$$\langle (z_1, \dots, z_m), (w_1, \dots, w_m) \rangle_{\mathbb{C}^m} := \operatorname{Re} \sum_{j=1}^{m} z_j \bar{w}_j$$

and norm $\|z\|_{\mathbb{C}^m} := \langle z, z \rangle_{\mathbb{C}^m}^{1/2}$.

A function $F : \mathbb{C}^m \to \mathbb{R}$ is said to be *phase-rotation-invariant and homogeneous of order* α for some real α if we have

$$F(\lambda v) = |\lambda|^\alpha F(v) \tag{1-1}$$

for all $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^m$; thus, for instance, $F(e^{i\theta} v) = F(v)$ for all $\theta \in \mathbb{R}$ and $v \in \mathbb{C}^m$. In particular, differentiating (1-1) at $\lambda = 1$ we obtain *Euler's identity*

$$\langle v, (\nabla_{\mathbb{C}^m} F)(v) \rangle_{\mathbb{C}^m} = \alpha F(v), \tag{1-2}$$

as well as the variant

$$\langle iv, (\nabla_{\mathbb{C}^m} F)(v) \rangle_{\mathbb{C}^m} = 0 \tag{1-3}$$

for all $v \in \mathbb{C}^m$ where a gradient $\nabla_{\mathbb{C}^m} F(v) \in \mathbb{C}^m$ exists. Here the gradient $\nabla_{\mathbb{C}^m} F(v)$ is defined via duality by the formula

$$\langle (\nabla_{\mathbb{C}^m} F)(v), w \rangle_{\mathbb{C}^m} = \frac{d}{dt} F(v + tw)|_{t=0} \tag{1-4}$$

for all test directions $w \in \mathbb{C}^m$. When α is not an integer, it is not possible for such homogeneous functions to be smooth at the origin unless they are identically zero (this can be seen by performing a Taylor expansion of F around the origin). To avoid this technical issue, we also introduce the notion of F being *phase-rotation-invariant and homogeneous of order* α *outside of the unit ball*, by which we mean that (1-1) holds for $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^m$ whenever $|\lambda|, \|v\|_{\mathbb{C}^m} \geq 1$, or whenever $|\lambda| = 1$.

Define a *potential* to be a function $F : \mathbb{C}^m \to \mathbb{R}$ that is smooth away from the origin; if F is also smooth at the origin, we call it a *smooth potential*. We say that the potential is *defocusing* if F is positive away from the origin, and *focusing* if F is negative away from the origin. In this paper we consider nonlinear Schrödinger systems of the form

$$i\partial_t u + \Delta u = (\nabla_{\mathbb{C}^m} F)(u) + G, \tag{1-5}$$

where the unknown field $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^m$ is assumed to be smooth, $F : \mathbb{C}^m \to \mathbb{R}$ is a smooth potential, and $G : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^m$ is a smooth compactly supported forcing term. In the homogeneous case $G = 0$, this is (formally, at least) a Hamiltonian evolution equation, with Hamiltonian

$$H(u) := \int_{\mathbb{R}^d} \tfrac{1}{2} \|\nabla u\|_{\mathbb{R}^d \otimes \mathbb{C}^m}^2 + F(u)\, dx$$

which is nonnegative when F is defocusing, where the quantity $\|\nabla u\|^2_{\mathbb{R}^d \otimes \mathbb{C}^m}$ is given by the formula

$$\|\nabla u\|^2_{\mathbb{R}^d \otimes \mathbb{C}^m} := \langle \partial_{x_j} u, \partial_{x_j} u \rangle_{\mathbb{C}^m}$$

with the usual summation conventions. By Noether's theorem, the phase rotation invariance of this Hamiltonian yields (formally, at least) the conservation of mass $\int_{\mathbb{R}^d} \|u\|^2_{\mathbb{C}^m}\, dx$, while the translation invariance of the Hamiltonian similarly yields conservation of the momentum $2 \int_{\mathbb{R}^d} \langle \partial_j u, iu \rangle_{\mathbb{C}^m}\, dx$. In fact one has a conserved (pseudo-)stress-energy tensor $T_{\alpha\beta}$, which we will take advantage of later in the paper.

Remark 1.1. One could of course consider other generalisations of the scalar NLS equation in which the nonlinear term $(\nabla_{C^m} F)(u)$ is replaced by other functions of u; for instance, in view of the form $\pm |u|^{p-1} u$ of the scalar nonlinearity, one might consider nonlinearities of the form $A(u)u$ for some scalar-valued or matrix-valued function $A(u)$ of u. However, such equations would in general fail to have a conserved Hamiltonian (or a conserved pseudo-stress-energy tensor) and would thus presumably have worse behaviour at long times starting from large initial data.

We will restrict attention to potentials F which are phase-rotation-invariant and homogeneous outside of the unit ball of order $p+1$ for some exponent $p > 1$. The scalar NLS equation introduced earlier then corresponds to the case when $m = 1$ and $F(v) = |v|^{p+1}/(p+1)$ (for defocusing NLS) or $F(v) = -|v|^{p+1}/(p+1)$ (for focusing NLS), with the caveat that one needs to restrict p to be an odd integer if one wants these potentials to be smooth at the origin.

The natural initial value problem to study here is the Cauchy initial value problem, in which one specifies a smooth initial position $u_0 : \mathbb{R}^d \to \mathbb{C}^m$ and forcing term $G : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^m$, as well as the potential F, and asks for a smooth solution u to (1-5) with $u(0, x) = u_0(x)$. To avoid ill-posedness issues relating to the infinite speed of propagation of the Schrödinger equation, we will require the data u_0 and G to be compactly supported in space, and restrict attention to solutions u that are in the Schwartz class.

Standard energy methods (see, e.g., [Cazenave 2003; Tao 2006]) show that for any choice of smooth compactly supported data $u_0 : \mathbb{R}^d \to \mathbb{C}^m$ and smooth compactly supported forcing term $G : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^m$, one can construct a unique smooth solution u to (1-5) in $(-T_-, T_+) \times \mathbb{R}^d$ for some $0 < T_-, T_+ \leq \infty$ which is Schwartz in space, with T_-, T_+ maximal amongst all such solutions. Furthermore, if $T_+ < \infty$, then $\|u(t)\|_{L^\infty}$ goes to infinity as $t \to T_+$, and similarly for T_-. In these latter situations we say that the initial value problem exhibits finite time blowup.

The *global regularity problem* for a given choice of potential F asks if the latter situation does not occur, that is to say, for every choice of smooth, compactly supported data u_0, G there is a smooth global solution.

The answer to this question depends in a somewhat complicated way on the dimension d, the exponent p, and whether the potential F is focusing or defocusing; the literature here is vast and the following discussion is not meant to be comprehensive. Readers may consult the texts [Cazenave 2003; Bourgain 1999; Tao 2006] for more complete references. See Table 1 for an oversimplified summary of the situation.

Consider first the *mass subcritical* case $p < 1 + \frac{4}{d}$. It is known in this case from Strichartz estimates and contraction mapping arguments (see, e.g., [Cazenave 2003]) that the initial value problem is globally well-posed in the Sobolev space $H^1(\mathbb{R}^d)$, regardless of whether the potential F is defocusing or not; in the low-dimensional case $d \leq 3$, Strichartz estimates then place the solution locally in the space $L^4_t L^\infty_x(\mathbb{R} \times \mathbb{R}^d)$, which is sufficient when combined with standard persistence of regularity arguments based on the energy method (see, e.g., [Tao 2006, Proposition 3.11]) to show that solutions remain smooth for all time. The case $d = 4$ can be handled by modifications of the arguments in [Ryckman and Visan 2007]. The global regularity question in higher dimensions $d > 4$ is still not fully resolved; note that for the analogous question for the nonlinear wave equation (NLW), it was shown recently in [Tao 2016b] that global regularity can in fact fail in extremely high dimensions $d \geq 11$, even in the "extremely subcritical" case when the potential F and all of its derivatives are bounded.

Now consider the case when p is *mass critical or supercritical* in the sense that $p \geq 1 + \frac{4}{d}$, but is also *energy critical or subcritical* in the sense that either $d < 3$, or $p \leq 1 + \frac{4}{d-2}$. In the case of the focusing NLS, the well-known virial argument of Glassey [1977] shows that finite time blowup can[1] occur (and in fact *must* occur if the initial data has negative Hamiltonian). If instead the potential is defocusing, then it is known that the initial value problem is globally well-posed in the energy space $H^1(\mathbb{R}^d)$. In energy-subcritical situations when $d < 3$ or $p < 1 + \frac{4}{d-2}$, this claim can again be established from Strichartz estimates and contraction mapping arguments; see, e.g., [Cazenave 2003; Bourgain 1999; Tao 2006]. The energy-critical case when $d \geq 3$ and $p = 1 + \frac{4}{d-2}$ is more delicate; in the case of scalar NLS (in which $m = 1$ and $F(u) = |u|^{p+1}/(p+1)$), the $d = 3$ case was established in [Colliander et al. 2008] (after several previous partial results), and the higher-dimensional cases $d = 4$ and $d > 4$ were treated[2] in [Ryckman and Visan 2007] and [Visan 2007] respectively. It is likely that these results can be extended to more general defocusing potentials, though we do not attempt this here. Again, in low-dimensional cases $d \leq 3$, this H^1 local well-posedness can be used in conjunction with Strichartz estimates to establish global regularity; see, e.g., [Bourgain 1999; Colliander et al. 2004; Tao 2006]; the $d = 4$ case was treated in [Ryckman and Visan 2007]. As before, the status of the global regularity question in higher dimensions $d > 4$ is not yet fully resolved.

Finally, we turn to the *energy-supercritical* case when $d \geq 3$ and $p > 1 + \frac{4}{d-2}$, which is the main focus of this paper. The Glassey virial argument [1977] continues to show that finite time blowup can occur here in the focusing case. In the defocusing case, the situation is less well understood. There are a number of results [Burq et al. 2005; 2007; Christ et al. 2003; Carles 2007a; 2007b; Alazard and Carles 2009] that demonstrate that the solution map, if it exists at all, is highly unstable, although one can at least construct global weak solutions, which are not known to be unique; see [Ginibre and Velo 1985; Alazard and Carles 2009; Tao 2009].

The main result of this paper is to show that, at least for certain choices of defocusing potential F and data u_0, G, one in fact has blowup in finite time.

[1]Global regularity can however be restored if one imposes a suitable smallness condition on the data u_0, G; see, e.g., [Cazenave 2003].

[2]These papers are primarily concerned with the homogeneous case $G = 0$, but one can use the stability properties of NLS (see, e.g., [Tao and Visan 2005]) to extend from the homogeneous case to the inhomogeneous case, at least in the context of H^1 global well-posedness.

mass	energy	defocusing	focusing
subcritical	subcritical	global well-posedness	global well-posedness
critical	subcritical	global well-posedness	finite time blowup
supercritical	subcritical	global well-posedness	finite time blowup
supercritical	critical	global well-posedness	finite time blowup
supercritical	supercritical	finite time blowup	finite time blowup

Table 1. A somewhat oversimplified summary of whether nonlinear Schrödinger systems are necessarily globally well-posed, or can admit finite time blowup solutions, for various criticality types of exponents and for both focusing and defocusing nonlinearities. Theorem 1.2 establishes the bottom entry on the third column.

Theorem 1.2 (finite time blowup). *Let $d \geq 3$, let $p > 1 + \frac{4}{d-2}$, and let m be a sufficiently large integer. Then there exists a defocusing smooth potential $F : \mathbb{C}^m \to \mathbb{R}$ that is phase-rotation-invariant and homogeneous of order $p+1$ outside of the unit ball, and a smooth compactly supported choice of initial data $u_0 : \mathbb{R}^d \to \mathbb{C}^m$ and forcing term $G : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}^m$, such that there is a smooth, compactly supported solution $u : [0,1) \times \mathbb{R}^d \to \mathbb{C}^m$ to the nonlinear Schrödinger system* (1-5) *with the property that $\|u(t)\|_{L^\infty(\mathbb{R}^d)}$ goes to infinity as $t \to 1^-$.*

When combined with the known uniqueness theory for (1-5) (see, e.g., [Cazenave 2003; Tao 2006]), we see that there cannot be any smooth global solution to (1-5) with this data that is Schwartz in space (one can relax the Schwartz requirement considerably, but we will not attempt to do so here). The presence of the forcing term G is an unfortunate artefact of our method, which (due to the absence of finite speed of propagation for Schrödinger equations) requires one to use the forcing term to truncate a solution to a homogeneous equation that decays too slowly at infinity. It is however reasonable to conjecture that the above theorem can be strengthened by making G vanish (with u now being Schwartz in space rather than compactly supported).

We have not attempted to optimise the value of m produced by the arguments in this paper, but it will grow quadratically in the dimension d: $m = O(d^2)$. It would of course be of great interest to set m equal to 1 in order to have the blowup result apply to the *scalar* defocusing NLS; however, our method requires a lot of "freeness" to the solution u (in particular invoking a version of the Nash embedding theorem [1956]), and it does not seem possible to adapt it for this purpose. Nevertheless, Theorem 1.2 does construct a "barrier" against any attempt to prove global regularity for the scalar supercritical NLS, in that any such attempt must crucially rely on some property of the scalar equation that is not enjoyed by the vector-valued equations considered here. For instance, this theorem rules out any approach to global regularity for scalar supercritical NLS that relies on somehow manipulating the conservation laws of mass, momentum and energy to generate new a priori bounds on the solution.

Remark 1.3. We have only stated blowup in the L^∞ norm in Theorem 1.2; however, the construction of blowup is locally discretely self-similar, so one can in fact establish blowup of any subcritical norm, as well as blowup of any critical spacetime norm that involves integration in the time variable (such as the Strichartz norm $L^{(p-1)(d+1)/2}_{t,x}([0,1) \times \mathbb{R}^d))$. It also blows up in the critical purely spatial norm $\dot{H}^{\frac{d}{2}-\frac{2}{p-1}}(\mathbb{R}^d)$; see Remark 3.2 below. In particular this rules out the solution being continuable as a strong solution in a subcritical or critical norm; not coincidentally, these are also the norms which arise in the well-posedness (and persistence of regularity) results for these equations. Meanwhile, due to conservation of mass and energy, the solution stays bounded in the supercritical norm $H^1(\mathbb{R}^d)$. It is likely that one can continue the solution beyond the blowup time as a weak solution without any guarantees of continuity or uniqueness, for instance, by adding a small dissipative term and taking a weak limit to create a viscosity solution. We do not pursue these matters here.

Theorem 1.2 is an analogue of the recent finite time blowup result by the author [Tao 2016a] for vector-valued defocusing NLW equations, and the argument follows broadly similar lines, in particular performing a sequence of "quantifier elimination" steps, each of which removes one or more of the unknown fields from the problem.

The first reduction is to reduce matters to constructing a discretely self-similar solution to a homogeneous NLS system (1-5), in which G is now zero, the potential F is homogeneous everywhere (not just outside the unit ball), and the solution u obeys the discrete self-similarity relationship $u(4t, 2x) = e^{i\alpha}2^{-\frac{2}{p-1}}u(t,x)$ (the phase rotation α is needed for technical reasons, but can be ignored for a first reading). In order to perform this reduction, it will be important that the self-similar solution u remains smooth all the way up to the initial time slice $t = 0$ (except at the spacetime origin $(t,x) = (0,0)$ where a singularity occurs). See Theorem 3.1 for a precise statement of the claim needed.

Now that the forcing term G is eliminated from the problem, the next step is to eliminate the potential F by first locating a self-similar field u, and then constructing a homogeneous defocusing potential F to solve (1-5) with that given u. In order for this to be possible, the field u (as well as the "potential energy" field $V = F(u)$) have to obey some differential equations (related to the conservation of mass, momentum, and energy and the Euler identity (1-2)), as well as some positivity and regularity hypotheses; see Theorem 4.2 for a precise statement. The derivation of Theorem 3.1 from Theorem 4.2 relies on a classical extension theorem of Seeley [1964] that allows one to extend a smooth function on a submanifold with boundary to a smooth function on the entire manifold.

The differential equations alluded to in the previous paragraph can be expressed in terms of the potential energy field V and the "Gram-type matrix" $G[u,u]$ of u, which is a $(2d+4)\times(2d+4)$ matrix consisting of inner products $\langle D_1 u, D_2 u\rangle_{\mathbb{C}^m}$ for various differential operators

$$D_1, D_2 \in \{1, i, \partial_{x_1}, \dots, \partial_{x_d}, \partial_t, i\partial_{x_1}, \dots, i\partial_{x_d}, i\partial_t\}.$$

The coefficients of the Gram-type matrix $G[u,u]$ necessarily obey a number of constraints; for instance, $G[u,u]$ will be symmetric and positive definite, and one has the Leibniz type identities

$$\partial_{x_j}\langle u, u\rangle_{\mathbb{C}^m} = 2\langle u, \partial_{x_j} u\rangle_{\mathbb{C}^m},$$
$$\partial_{x_j}\langle iu, \partial_{x_k} u\rangle_{\mathbb{C}^m} - \partial_{x_k}\langle iu, \partial_{x_j} u\rangle_{\mathbb{C}^m} = 2\langle i\partial_{x_j} u, \partial_{x_k} u\rangle_{\mathbb{C}^m}.$$

One can then eliminate the field u in favour of the Gram-type matrix by reducing Theorem 4.2 to a statement about the existence of a certain matrix G of fields (as well as a potential field V) obeying the above-mentioned constraints and differential equations; see Theorem 5.4 for a precise statement. Conveniently, the constraints and equations are now *linear* in the fields G, V, in contrast to the nonlinear nature of the original equation (1-5). In order to reconstruct the field u from the Gram-type matrix G, one needs a "partially complexified" version of the Nash embedding theorem [1956]; this is the main reason[3] why the target dimension m is required to be large. Unfortunately, the existing forms of the Nash embedding theorem in the literature are not quite suitable for this application, and we need to adapt the *proof* of that theorem to establish the embedding theorem required (which we formalise as Proposition 5.2). The proof of this embedding theorem is given in the Appendix.

The Gram-type matrix G contains a large number of fields, while simultaneously being required to obey a large number of constraints. One can cut down the degrees of freedom considerably, as well as the number of constraints, by requiring the Gram-matrix to be homogeneous with respect to parabolic scaling, and also to be rotation-invariant in a certain tensorial sense. This reduces the number of independent components of G and V to seven scalar fields $g_{1,1}$, $g_{\partial_r,\partial_r}$, $g_{\partial_\omega,\partial_\omega}$, $g_{\partial_r,\partial_t}$, $g_{1,i\partial_r}$, $g_{1,i\partial_t}$, v which obey a certain number of conservation laws, positivity hypotheses, and some additional constraints such as homogeneity; see Theorem 5.4 for a precise statement. The fields g_{D_1,D_2} for various differential operators D_1, D_2 are supposed to be proxies for the inner products $\langle D_1 u, D_2 u\rangle_{\mathbb{C}^m}$, while v is a proxy for the potential energy $V(u)$. (Strictly speaking, G contains another scalar field $g_{\partial_t,\partial_t}$, a proxy for $\|\partial_t u\|^2_{\mathbb{C}^m}$, which is independent of the other fields, but it is essentially unconstrained by any of the conservation laws, and can be set to be extremely large and then ignored.)

Amongst the various constraints between the remaining scalar fields is the energy conservation law, which in this notation becomes

$$\partial_t\big(\tfrac{1}{2}g_{\partial_r,\partial_r}+\tfrac{1}{2}(d-1)g_{\partial_\omega,\partial_\omega}+v\big)-\Big(\partial_r+\frac{d-1}{r}\Big)g_{\partial_r,\partial_t}=0. \tag{1-6}$$

This law can be used in the energy subcritical case to rule out the type of discretely self-similar solutions we are trying to construct here; with a bit more effort involving an additional Morawetz-type identity arising from momentum conservation, one can also rule out such solutions in the energy-critical case. However, in the energy-supercritical case it turns out that the conservation law (1-6) is easy to satisfy, basically because the scalar field $g_{\partial_r,\partial_t}$ (representing energy current) that appears in this law has no presence in any of the other conservation laws, allowing the energy to be transported spatially at an essentially arbitrary rate. In the energy-supercritical case, the total energy becomes infinite, and so it becomes possible to eliminate the field $g_{\partial_r,\partial_t}$ and the energy conservation equation (1-6), reducing one to a variant of Theorem 5.4 with one fewer scalar field and one fewer constraint equation. One can similarly use another constraint

$$g_{1,i\partial_t}+\tfrac{1}{2}\Big(\partial_r^2+\frac{d-1}{r}\partial_r\Big)g_{1,1}-g_{\partial_r,\partial_r}-(d-1)g_{\partial_\omega,\partial_\omega}=(p+1)v$$

[3]It may be possible to cut down the dimension m substantially by restricting attention to solutions that are spherically symmetric, thus effectively reducing the dimension d to 1. However, we were not able to achieve this, mainly because we could not construct a stress-energy tensor with the required properties for which the angular stress $T_{\omega\omega}$ vanished. On the other hand, we could not definitively rule out the existence of such a tensor either.

(which ultimately arises from the Euler identity (1-2)) to easily eliminate the field $g_{1,i\partial_t}$ (which makes no appearance in any of the other constraints), leaving one with just five remaining scalar fields $g_{1,1}$, $g_{\partial_r,\partial_r}$, $g_{\partial_\omega,\partial_\omega}$, $g_{1,i\partial_r}$, v; see Theorem 6.2 for a precise statement.

An inspection of the remaining constraints reveals that the potential field v and the angular stress $g_{\partial_\omega,\partial_\omega}$ play almost[4] the same roles, and some elementary manipulations allow one to effectively absorb the potential v into the angular stress $g_{\partial_\omega,\partial_\omega}$ (and also the radial stress $g_{\partial_r,\partial_r}$), allowing one to reduce to the case $v = 0$; see Theorem 7.1 for a precise statement. Now there are just four independent scalar fields $g_{1,1}$, $g_{\partial_r,\partial_r}$, $g_{\partial_\omega,\partial_\omega}$, $g_{1,i\partial_r}$ that one needs to locate.

One of the remaining constraints is the momentum conservation law, which can be rewritten as

$$\partial_r(r^{d-1}g_{\partial_r,\partial_r}) = (d-1)r^{d-2}g_{\partial_\omega,\partial_\omega} + S_1,$$

where S_1 is the field

$$S_1 := \tfrac{1}{4}r^{d-1}\Big(\partial_r\Big(\partial_r^2 + \frac{d-1}{r}\partial_r\Big)g_{1,1} + 2\partial_t g_{1,i\partial_r}\Big).$$

One can integrate this law to obtain a representation of the radial stress $g_{\partial_r,\partial_r}$ as a certain integral involving $g_{\partial_\omega,\partial_\omega}$ and S_1. The requirement that $g_{\partial_r,\partial_r}$ be smooth up to the initial time $t = 0$ enforces some asymptotic vanishing conditions on the integrand, while the positive definiteness of the Gram matrix enforces an additional inequality on the integral. Once these conditions are satisfied, one can then eliminate the field $g_{\partial_r,\partial_r}$ from the problem, leaving only three fields $g_{1,1}$, $g_{\partial_\omega,\partial_\omega}$, $g_{1,i\partial_r}$ to construct. See Theorem 8.1 for a precise statement.

The angular stress $g_{\partial_\omega,\partial_\omega}$ is now only constrained by a nonnegativity condition and by the constraints on the integral involving $g_{\partial_\omega,\partial_\omega}$ and S_1 mentioned above. It is then not difficult to eliminate $g_{\partial_\omega,\partial_\omega}$, and reduce matters to locating just two fields $g_{1,1}$, $g_{1,i\partial_r}$ that obey a mass conservation law

$$\partial_t g_{1,1} = 2\Big(\partial_r + \frac{d-1}{r}\Big)g_{1,i\partial r}$$

together with a number of technical additional conditions, mostly involving integrals of the quantity S_1 mentioned above. See Theorem 9.1 for a precise statement.

The mass conservation law can be solved explicitly by using the ansatz

$$g_{1,1} = 2r^{1-d}\partial_r(r^d W),$$
$$g_{1,i\partial_r} = r^{1-d}\partial_t(r^d W)$$

for a suitable scalar field W. Now that there is only one field W to choose, it becomes possible to write down an explicit choice of this field that obeys the few remaining constraints required of it; we do so in Section 11.

[4]This phenomenon is analogous to the well-known fact that when applying separation of variables in polar coordinates to the free Schrödinger equation $i\partial_t u + \Delta u = 0$ in which $u(t, r\omega) = v(t,r)Y_\ell(\omega)$ for some spherical harmonic Y_ℓ of degree ℓ, the effect of the spherical harmonic is identical to that of a (defocusing) Coulomb type potential $\ell(\ell+1)/r^2$.

2. Notation

Throughout this paper, the spatial dimension d, the target dimension m, and the exponent p will be fixed. Unless otherwise stated, we will always be assuming the energy supercritical hypotheses

$$d \geq 3, \quad p > 1 + \frac{4}{d-2}. \tag{2-1}$$

We will also assume that the target dimension m is sufficiently large depending on d. In particular, all the theorems in subsequent sections will implicitly have these hypotheses present (though from Theorem 5.4 onwards, the target dimension m plays no further role as the field u is eliminated at that point).

We use the asymptotic notation $X = O(Y)$ or $X \ll Y$ to denote the estimate $|X| \leq CY$ for some C depending on the above parameters p, d. In some cases we will explicitly allow the implied constant C to depend on additional parameters.

Most of our analysis will take place in the spacetime region

$$H_d := ([0, +\infty) \times \mathbb{R}^d) \backslash \{(0,0)\} \tag{2-2}$$

or the one-dimensional variant

$$H_1 := ([0, +\infty) \times \mathbb{R}) \backslash \{(0,0)\}, \tag{2-3}$$

that is to say, on the portion of spacetime consisting of the present $t = 0$ and future $t > 0$, but with the spacetime origin $(0,0)$ removed. On these regions we introduce the parabolic magnitude function $\rho : H_d \to \mathbb{R}$ or $\rho : H_1 \to \mathbb{R}$ defined by

$$\rho(t,x) := (t^2 + |x|^4)^{\frac{1}{4}} \tag{2-4}$$

for $(t,x) \in H_d$, or

$$\rho(t,r) := (t^2 + r^4)^{\frac{1}{4}} \tag{2-5}$$

for $(t,r) \in H_1$. We also introduce the discrete scaling operator $T : H_d \to H_d$ by the formula

$$T(t,x) := (4t, 2x) \tag{2-6}$$

(thus, for instance, $\rho \circ T = 2\rho$) and let $T^{\mathbb{Z}} := \{T^n : n \in \mathbb{Z}\}$ be the group of scalings generated by T. A key point is that the quotient space $H_d / T^{\mathbb{Z}}$ of spacetime by discrete scalings is compact; indeed one can view this space as the set $\{(t,x) \in H_d : 1 \leq \rho \leq 2\}$ with the boundaries $\rho = 1$ and $\rho = 2$ identified. We have chosen to use the scaling $(t,x) \mapsto (4t, 2x)$ in (2-6) to generate the discrete self-similarity, but this is an arbitrary choice, and one could just as well have used another scaling $(t,x) \mapsto (\lambda_0^2 t, \lambda_0 x)$ for some fixed $\lambda_0 > 1$.

3. Reduction to constructing a discretely self-similar solution

We begin the proof of Theorem 1.2. In analogy with the argument in [Tao 2016a], the first step is to reduce to locating a discretely self-similar solution to a homogeneous nonlinear Schrödinger equation, thus eliminating the role of the forcing term G. In the previous paper [Tao 2016a], one could use the finite speed of propagation of nonlinear wave equations to restrict spacetime to a light cone $\{(t,x) : t > 0, |x| \leq t\}$ for the purposes of locating this solution. In the current context of nonlinear Schrödinger equations, one has infinite speed of propagation, and so one can only restrict to the region H_d defined in (2-2). To get from here to Theorem 1.2, one must now apply a spatial cutoff, which is responsible for the forcing term G that is present in this paper but not in the previous work [Tao 2016a].

We turn to the details. We will derive Theorem 1.2 from:

Theorem 3.1 (first reduction). *There exists a defocusing potential $F : \mathbb{C}^m \to \mathbb{R}$ which is phase-rotation-invariant and homogeneous of order $p+1$ and a smooth function $u : H_d \to \mathbb{C}^m \backslash \{0\}$ that solves* (1-5) (*with* $G = 0$) *on its domain and is nowhere vanishing, and also discretely self-similar in the sense that*

$$u(T(t,x)) = e^{i\alpha} 2^{-\frac{2}{p-1}} u(t,x) \tag{3-1}$$

for all $(t,x) \in H_d$, and some $\alpha \in \mathbb{R}$, where T is the scaling (2-6).

A key point here is that u is smooth all the way up to the boundary of the region H_d (except at the spacetime origin $(0,0)$), rather than merely being smooth in the interior. The exponent $-\frac{2}{p-1}$ is mandated by dimensional analysis considerations; the phase shift α is needed for more technical reasons, representing a "total charge" coming from the nonzero momentum density. It would be natural to consider solutions that are continuously self-similar in the sense that

$$u(\lambda^2 t, \lambda x) = \lambda^{-\frac{2}{p-1} + i\frac{\alpha}{\log 2}} u(t,x)$$

for all $\lambda > 0$ (not just powers of 2) but we were unable to construct such a solution. In the analogous situation for the NLW, such continuously self-similar solutions can be ruled out by ad hoc methods for some ranges of d, p, as was shown in [Tao 2016a, Proposition 2.2].

Let us assume Theorem 3.1 for the moment, and show how it implies Theorem 1.2. Let F, u be as in Theorem 3.1. Since u is smooth and nonzero on the compact region $\{(t,x) \in H_d : 1 \le \rho \le 2\}$, it is bounded from below in this region. By replacing u with Cu and F with $v \mapsto C^2 F(v/C)$ for some large constant C, we may thus assume that

$$\|u(t,x)\|_{\mathbb{C}^m} \ge 1$$

whenever $(t,x) \in H_d$ with $1 \le \rho \le 2$. Using the discrete self-similarity property (3-1), we then have this bound whenever $\rho \le 2$; in fact we have a lower bound on $\|u(t,x)\|_{\mathbb{C}^m}$ that goes to infinity as $(t,x) \to 0$, ensuring in particular that $\|u(t)\|_{L^\infty(\mathbb{R}^d)}$ goes to infinity as $t \to 0$.

Using a smooth cutoff function, one can find a smooth defocusing potential $F_1 : \mathbb{R}^m \to \mathbb{R}$ that is phase-rotation-invariant and agrees with F in the region $\{v \in \mathbb{C}^m : \|v\|_{\mathbb{C}^m} \ge 1\}$. Then u solves (1-5) with this potential in the truncated region $\{(t,x) \in H_d : \rho \le 2\}$, and in particular in the region $\{(t,x) \in H_d : t, |x| \le 1\}$. Next, let $\varphi : \mathbb{R}^d \to [0,1]$ be a smooth function supported on the ball $\{x \in \mathbb{R}^d : |x| \le 1\}$ that equals 1 on $\{x \in \mathbb{R}^d : |x| \le \frac{1}{2}\}$, and define the functions $\tilde{u} : [0,1) \times \mathbb{R}^d \to \mathbb{C}^m$, $\widetilde{F} : \mathbb{C}^m \to \mathbb{R}$, $\widetilde{G} : [0,1) \times \mathbb{R}^d \to \mathbb{C}^m$ by the formulae

$$\begin{aligned}\tilde{u}(t,x) &:= \bar{u}(1-t,x)\varphi(x),\\ \widetilde{F}(v) &:= F_1(\bar{v}),\\ \widetilde{G}(t,x) &:= i\partial_t \tilde{u}(t,x) + \Delta\tilde{u}(t,x) - \widetilde{F}(\tilde{u}(t,x)).\end{aligned}$$

It is clear that $\widetilde{F}$ is a smooth defocusing potential that is phase-rotation-invariant and homogeneous of degree $p+1$ outside of the unit ball, while $\tilde{u}, \widetilde{G}$ are smooth functions supported on the regions $\{(t,x) \in [0,1) \times \mathbb{R}^d : |x| \leq 1\}$ and $\{(t,x) \in [0,1) \times \mathbb{R}^d : \frac{1}{2} \leq |x| \leq 1\}$, with $\|\tilde{u}(t)\|_{L^\infty}$ going to infinity as $t \to 1$. This gives Theorem 1.2 (with u, F, G replaced by $\tilde{u}, \widetilde{F}, \widetilde{G}$ respectively).

It remains to prove Theorem 3.1. This will be the focus of the remaining sections of the paper. We remark that with the reduction to Theorem 3.1, we have effectively "compactified" spacetime, as the discretely self-similar solution can be viewed as a solution (interpreted geometrically as a section of an appropriate vector bundle) on the smooth compact manifold[5] with boundary $H_d / T^{\mathbb{Z}}$.

Remark 3.2. From the above construction we see that the solution $\tilde{u}$ used to demonstrate Theorem 1.2 will stay smooth at the blowup time $t=1$ away from the spatial origin, though it will be discretely self-similar near the origin at that time; in particular the critical Sobolev norm $\dot{H}^{\frac{d}{2}-\frac{2}{p-1}}(\mathbb{R}^d)$ will be infinite at time $t=1$, so the blowup is of "type I" in nature. This is consistent with the results in [Killip and Visan 2010] which rule out "type II" blowup for energy supercritical nonlinear Schrödinger equations, at least in dimensions 5 and higher. Beyond the blowup time, one can still continue the solution as a weak solution, although we have nothing new to say about the uniqueness (or lack thereof) of such a solution, or of its regularity.

4. Eliminating the potential

We now exploit the freedom to select the defocusing potential F from Theorem 3.1 by eliminating it from the equations of motion. To motivate this elimination, let us formally manipulate the equation

$$i\partial_t u + \Delta u = (\nabla_{\mathbb{C}^m} F)(u),$$

where F is assumed to be defocusing, phase-rotation-invariant, and homogeneous of order $p+1$, in order to derive equations that do not explicitly involve F.

From (1-2), (1-3) we have the identities

$$\langle i\partial_t u + \Delta u, u\rangle_{\mathbb{C}^m} = (p+1)V, \tag{4-1}$$

$$\langle i\partial_t u + \Delta u, iu\rangle_{\mathbb{C}^m} = 0, \tag{4-2}$$

where we define the *potential energy density* V by

$$V := F(u).$$

Note that the defocusing nature of F makes V nonnegative. From (1-4) and the chain rule we also have the additional identities

$$\langle i\partial_t u + \Delta u, \partial_{x_j} u\rangle_{\mathbb{C}^m} = \partial_{x_j} V, \tag{4-3}$$

$$\langle i\partial_t u + \Delta u, \partial_t u\rangle_{\mathbb{C}^m} = \partial_t V \tag{4-4}$$

for $j = 1, \ldots, d$. We have thus obtained $d+3$ equations involving the fields u, V that do not directly involve the nonlinearity F.

[5]This manifold is diffeomorphic to the solid torus $\overline{B(0,1)} \times (\mathbb{R}/\mathbb{Z})$, where $\overline{B(0,1)}$ is the closed unit ball in $\mathbb{R}^d$. However, we will not need to use the diffeomorphism type of the manifold $H_d / T^{\mathbb{Z}}$ in this paper.

Remark 4.1. The equations (4-1)–(4-4) are closely related to the usual conservation laws for the nonlinear Schrödinger equation. Indeed, if we define the pseudo-stress-energy-tensor

$$\begin{aligned} T_{00} &:= \|u\|^2_{\mathbb{C}^m}, \\ T_{0j} = T_{j0} &:= 2\langle \partial_{x_j} u, iu\rangle_{\mathbb{C}^m}, \\ T_{jk} &:= 4\langle \partial_{x_j} u, \partial_{x_k} u\rangle_{\mathbb{C}^m} + \delta_{jk} 2(p-1)V - \delta_{jk}\Delta(\|u\|^2_{\mathbb{C}^m}) \end{aligned}$$

for $j = 1, \dots, d$, where δ_{jk} is the Kronecker delta, and also define the energy density

$$E := \tfrac{1}{2}\langle \partial_{x_j} u, \partial_{x_j} u\rangle_{\mathbb{C}^m} + V$$

(with the usual summation conventions) and energy current

$$J_j := -\langle \partial_{x_j} u, \partial_t u\rangle_{\mathbb{C}^m}$$

for $j = 1, \dots, k$, then one can easily use (4-2) to deduce the mass conservation law

$$\partial_t T_{00} + \partial_{x_j} T_{j0} = 0$$

and similarly use (4-1), (4-3) to deduce the momentum conservation law

$$\partial_t T_{0k} + \partial_{x_j} T_{jk} = 0$$

for $k = 1, \dots, d$. From (4-1), (4-4) we can also obtain the energy conservation law

$$\partial_t E + \partial_{x_j} J_j = 0.$$

Finally, we can rewrite (4-1) in a way that does not explicitly involve second derivatives of u as

$$\langle iu_t, u\rangle_{\mathbb{C}^m} + \tfrac{1}{2}\Delta T_{00} - \langle \partial_{x_j} u, \partial_{x_j} u\rangle_{\mathbb{C}^m} = (p+1)V. \tag{4-5}$$

Conversely, if we take (4-5) as a definition of the potential energy density V, then the above conservation laws can be used to recover (4-2)–(4-4).

Now assume that u obeys the discrete self-similarity hypothesis (3-1) and is nowhere vanishing. We recall that the complex projective space $\mathbb{CP}^{m-1}$ is the quotient space

$$\mathbb{CP}^{m-1} := (\mathbb{C}^m \backslash \{0\})/\mathbb{C}^\times$$

of the manifold[6] $\mathbb{C}^m\backslash\{0\}$ by the action of the multiplicative complex group $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$ by scalar multiplication. Let $\pi : \mathbb{C}^m\backslash\{0\} \to \mathbb{CP}^{m-1}$ be the projection map; then $\pi \circ u : H_d \to \mathbb{CP}^{m-1}$ is a smooth map which is invariant under the action of $T^{\mathbb{Z}}$, and thus descends to a smooth map $\theta : H_d/T^{\mathbb{Z}} \to \mathbb{CP}^{m-1}$. We will derive Theorem 3.1 from:

Theorem 4.2 (second reduction). *There exists a smooth nowhere vanishing function $u : H_d \to \mathbb{C}^m\backslash\{0\}$ which is discretely self-similar in the sense of* (3-1) *for some $\alpha \in \mathbb{R}$, and a smooth function $V : H_d \to \mathbb{R}$ such that the defocusing property*

$$V > 0 \tag{4-6}$$

[6]For the purpose of defining tangent spaces, cotangent spaces, differentials, etc., we will view spaces such as $\mathbb{C}^m\backslash\{0\}$ as real manifolds (of dimension $2m$) rather than complex manifolds, although we will certainly also use the complex structure.

and the equations of motion (4-1)–(4-4) *hold on all of* H_d. *Furthermore, the map* $\theta : H_d/T^{\mathbb{Z}} \to \mathbb{CP}^{m-1}$ *defined above is a smooth embedding, that is to say, it is injective and immersed in the sense that the* $d+1$ *derivatives* $\partial_t\theta(t,x), \partial_{x_1}\theta(t,x), \ldots, \partial_{x_d}(t,x)$ *are linearly independent in the tangent space of* $\mathbb{CP}^{m-1}$ *at* $\theta(t,x)$ *for all* $(t,x) \in H_d$.

Let us assume Theorem 4.2 for now and see how it implies Theorem 3.1. Let d, p, m, u, V, θ be as in Theorem 4.2. To prove Theorem 3.1, it will suffice to produce a defocusing potential $F : \mathbb{C}^m \to \mathbb{R}$, phase-rotation-invariant and homogeneous of degree $p+1$, such that the identity

$$i\partial_t u + \Delta u = (\nabla_{\mathbb{C}^m} F)(u) \tag{4-7}$$

holds on all of H_d. Since u never vanishes, we can of course remove the origin 0 from the domain of F, working instead on the manifold $\mathbb{C}^m\backslash\{0\}$.

We now consider the subset Γ of $\mathbb{C}^m\backslash\{0\}$ defined by

$$\Gamma := \{zu(t,x) : (t,x) \in H_d,\ z \in \mathbb{C}^\times\}$$

or equivalently

$$\Gamma = \pi^{-1}(\theta(H_d/T^{\mathbb{Z}})).$$

This is a $(d+2)$-dimensional $\mathbb{C}^\times$-invariant smooth submanifold (with boundary) of $\mathbb{C}^m\backslash\{0\}$. The values of the potential F and its gradient $\nabla_{\mathbb{C}^m} F$ on Γ are determined by the data u, V. Indeed, if F was phase-rotation-invariant, homogeneous of degree $p+1$, and obeyed (4-7), then from (1-2), (4-1) and homogeneity we must have

$$F(zu(t,x)) = \frac{|z|^{p+1}}{p+1} V(t,x) \tag{4-8}$$

and

$$(\nabla_{\mathbb{C}^m} F)(zu(t,x)) = |z|^{p-1} z(i\partial_t u(t,x) + \Delta u(t,x)) \tag{4-9}$$

for all $(t,x) \in H_d$ and $z \in \mathbb{C}^\times$. Conversely, if we can locate a defocusing potential F that is phase-rotation-invariant, homogeneous of degree $p+1$, and obeys the identities (4-8), (4-9) on Γ, then we of course have (4-7) after specialising (4-9) to the case $z = 1$.

It remains to construct such an F. In view of the constraints (4-8), (4-9), it is natural to introduce the functions $F_0 : \Gamma \to \mathbb{R}$ and $F_1 : \Gamma \to \mathbb{C}^m$ by the formulae

$$F_0(zu(t,x)) := \frac{|z|^{p+1}}{p+1} V(t,x), \tag{4-10}$$

$$F_1(zu(t,x)) := |z|^{p-1} z(i\partial_t u(t,x) + \Delta u(t,x)) \tag{4-11}$$

for all $(t,x) \in H_d$ and $z \in \mathbb{C}^\times$. As we are assuming θ to be injective, we see that $zu(t,x) = z'u(t',x')$ occurs if and only if $(t',x') = T^n(t,x)$ and $z' = 2^{\frac{2}{p-1}n} z$ for some integer n. On the other hand, from (3-1), (4-1) we have

$$V(T^n(t,x)) = 2^{-\frac{2(p+1)}{p-1}n} V(t,x)$$

and similarly from (3-1) we have

$$i\partial_t u(T^n(t,x)) + \Delta u(T^n(t,x)) = 2^{-\frac{2p}{p-1}n}(i\partial_t u(t,x) + \Delta u(t,x))$$

and so we see that the functions F_0, F_1 are well defined. As θ is also a smooth embedding, the functions F_0, F_1 are also smooth on Γ; from (4-6) we know that F_0 is strictly positive. By construction we clearly have the homogeneity relations

$$\begin{aligned} F_0(zv) &= |z|^{p+1} F_0(v), \\ F_1(zv) &= |z|^{p-1} z F_1(v) \end{aligned} \tag{4-12}$$

for all $v \in \Gamma$ and $z \in \mathbb{C}^\times$. Our task is to extend $F_0 : \Gamma \to \mathbb{R}$ to a defocusing potential $F : \mathbb{C}^m \backslash \{0\} \to \mathbb{R}$ that continues to obey the relation (4-12), and such that $\nabla_{\mathbb{C}^m} F$ agrees with F_1 on Γ.

At any given point $zu(t,x)$ of Γ, the tangent space $T_{zu(t,x)}\Gamma$ is spanned (as a real vector space) by the vectors $zu(t,x)$, $izu(t,x)$, $z\partial_t u(t,x)$, and $z\partial_{x_j} u(t,x)$ for $j = 1, \dots, d$. From (4-1)–(4-4), (4-10), (4-11) and linearity, we conclude the identity

$$dF_0(v)(w) = \langle F_1(v), w\rangle_{\mathbb{C}^m} \tag{4-13}$$

for any $v \in \Gamma$ and $w \in T_v\Gamma$, where $dF_0(v) \in T_v^*\Gamma$ is the differential of F_0 at v, or equivalently $dF_0(v)(w)$ is the directional derivative of F_0 at v along the tangent vector w. To put it another way, if we use the inner product $\langle\ ,\ \rangle_{\mathbb{C}^m}$ to identify $\mathbb{C}^m$ with the dual space $(\mathbb{C}^m)^* = T_v^*\mathbb{C}^m$ (viewed as real vector spaces), then $dF_0(v)$ is the projection of $F_1(v)$ to $T_v^*\Gamma$ (using the dual of the inclusion map from $T_v\Gamma$ to $T_v\mathbb{C}^m$).

It will be convenient to normalise out the homogeneity on F, F_0, F_1. Define the normalised functions $F_0^{(1)} : \Gamma \to \mathbb{R}$ and $F_1^{(1)} : \Gamma \to \mathbb{C}^m$ by the formulae

$$\begin{aligned} F_0^{(1)}(v) &:= \|v\|_{\mathbb{C}^m}^{-p-1} F_0(v), \\ F_1^{(1)}(v) &:= \|v\|_{\mathbb{C}^m}^{-p-1} F_1(v) - (p+1)\|v\|_{\mathbb{C}^m}^{-p-3} F_0(v) v. \end{aligned}$$

Then $F_0^{(1)}$, $F_1^{(1)}$ are smooth, with the homogeneity relations

$$\begin{aligned} F_0^{(1)}(zv) &= F_0^{(1)}(v), \\ F_1^{(1)}(zv) &= |z|^{-2} z F_1^{(1)}(v) \end{aligned}$$

for all $v \in \Gamma$ and $z \in \mathbb{C}^\times$; also, from (4-13) and the product rule we see that

$$dF_0^{(1)}(v)(w) = \langle F_1^{(1)}(v), w\rangle_{\mathbb{C}^m} \tag{4-14}$$

for any $v \in \Gamma$ and $w \in T_v\Gamma$. Finally, $F_0^{(1)}$ is clearly everywhere positive.

Since $F_0^{(1)} : \Gamma \to \mathbb{R}$ is invariant under the action of $\mathbb{C}^\times$, it descends to a smooth positive function $F_0^{(2)} : \theta(H_d/T^{\mathbb{Z}}) \to \mathbb{R}$ on the quotient space $\Gamma/\mathbb{C}^\times = \theta(H_d/T^{\mathbb{Z}})$; thus

$$F_0^{(2)}(\pi(v)) = F_0^{(1)}(v)$$

for all $v \in \Gamma$. For any $v \in \Gamma$, we define the covector $F_1^{(2)}(\pi(v)) \in T^*_{\pi(v)}\mathbb{CP}^{m-1}$ by the formula

$$F_1^{(2)}(\pi(v))(\pi_{*,v}(w)) := \langle F_1^{(1)}(v), w\rangle_{\mathbb{C}^m}$$

for all $w \in T_v\mathbb{C}^m \equiv \mathbb{C}^m$, where $\pi_{*,v} : T_v\mathbb{C}^m \to T_{\pi(v)}\mathbb{CP}^{m-1}$ is the projection map. Note from (4-14) and the $\mathbb{C}^\times$-invariance of $F_0^{(1)}$ that $F_1^{(1)}(v)$ is orthogonal to the kernel of $\pi_{*,v}$; this and the homogeneity of $F_0^{(1)}$, $F_1^{(1)}$ ensure that $F_1^{(2)}$ is well defined and smooth on $\pi(\Gamma) = \theta(H_d/T^{\mathbb{Z}})$. From (4-14) we see

$$dF_0^{(2)}(\tilde{v})(\tilde{w}) = F_1^{(2)}(\tilde{v})(\tilde{w}) \tag{4-15}$$

for all $\tilde{v} \in \theta(H_d/T^{\mathbb{Z}})$ and $\tilde{w} \in T_{\tilde{v}}\theta(H_d/T^{\mathbb{Z}})$; in other words, $F_1^{(2)}$ agrees with $dF_0^{(2)}$ at any point $\tilde{v}$ on the compact manifold with boundary $\theta(H_d/T^{\mathbb{Z}})$, after restricting to the tangent space $T_{\tilde{v}}\theta(H_d/T^{\mathbb{Z}})$ of that manifold.

One can view $H_d/T^{\mathbb{Z}}$ as a smooth compact submanifold (with smooth boundary) of $(\mathbb{R}\times\mathbb{R}^d\backslash\{0,0\})/T^{\mathbb{Z}}$. The function $\theta : H_d/T^{\mathbb{Z}} \to \mathbb{CP}^{m-1}$ can be extended smoothly to an open neighbourhood of this submanifold using a classical theorem of Seeley [1964]; the embedded copy $\theta(H_d/T^{\mathbb{Z}})$ of $H_d/T^{\mathbb{Z}}$ in $\mathbb{CP}^{m-1}$ can then similarly be extended to a slightly larger open manifold of the same dimension $d+1$. A further application of Seeley's theorem allows one to smoothly extend $F_0^{(2)}$ to this enlargement of $\theta(H_d/T^{\mathbb{Z}})$. Using this extension as well as (4-15) and Fermi normal coordinates (using, for instance, the Fubini–Study metric on $\mathbb{CP}^{m-1}$), one can then obtain a smooth extension $F_0^{(3)}$ of $F_0^{(2)}$ to an open neighbourhood U of the embedded copy $\theta(H_d/T^{\mathbb{Z}})$ of $H_d/T^{\mathbb{Z}}$ in $\mathbb{CP}^{m-1}$ in such a fashion that $dF_0^{(3)} = F_1^{(2)}$ on $\theta(H_d/T^{\mathbb{Z}})$. By shrinking U if necessary one can ensure that $F_0^{(3)}$ is positive on all of U. If one then sets $F_0^{(4)} : \mathbb{CP}^{m-1} \to \mathbb{R}$ to be the function defined by

$$F_0^{(4)} := \varphi F_0^{(3)} + (1-\varphi)$$

for some smooth cutoff $\varphi : \mathbb{CP}^{m-1} \to [0,1]$ that is supported on U that equals 1 on a neighbourhood of $\theta(H_d/T^{\mathbb{Z}})$, we see that $F_0^{(4)} : \mathbb{CP}^{m-1} \to \mathbb{R}$ is a positive smooth extension of $F_0^{(2)}$ such that $dF_0^{(4)} = F_1^{(2)}$ on $\theta(H_d/T^{\mathbb{Z}})$.

If we now set $F : \mathbb{C}^m\backslash\{0\} \to \mathbb{R}$ to be the function

$$F(v) := \|v\|_{\mathbb{C}^m}^{p+1} F_0^{(4)}(\pi(v))$$

then F is a defocusing potential that is phase-rotation-invariant and homogeneous of degree $p+1$. By construction, F agrees with F_0 on Γ, and

$$d(\|v\|_{\mathbb{C}^m}^{-p-1} F)(v)(w) = \langle F_1^{(1)}(v), w\rangle_{\mathbb{C}^m}$$

for all $v \in \Gamma$ and $w \in T_v\mathbb{C}^m \equiv \mathbb{C}^m$. By the product rule and construction of $F_1^{(1)}$, this implies

$$dF(v)(w) = \langle F_1(v), w\rangle_{\mathbb{C}^m}$$

for all $v \in \Gamma$ and $w \in T_v\mathbb{C}^m$, and thus

$$\nabla_{\mathbb{C}^m} F = F_1$$

on Γ, as desired.

It remains to establish Theorem 4.2. This will be the focus of the remaining sections of the paper.

operator D	parabolic order $\operatorname{ord}(D)$
$1,\ i$	0
$\partial_{x_j},\ i\partial_{x_j},\ \partial_r,\ i\partial_r,\ \partial_\omega$	1
$\partial_t,\ i\partial_t$	2

Table 2. The parabolic order $\operatorname{ord}(D)$ of various differential operators D (or formal differential operators) used in this paper. Some of the operators in this table will only be defined in subsequent sections.

5. Eliminating the field

In view of Remark 4.1, the constraints (4-1)–(4-4) that need to be satisfied in Theorem 4.2 can be expressed in terms of the pseudo-stress-energy tensor T_{00}, T_{0j}, T_{jk}, as well as the energy density E and the energy current J_j. These quantities in turn depend linearly on the potential energy density V and the components of the $(2d+4)\times(2d+4)$ Gram-type matrix $G[u,u]$, where we define

$$G[u,v] := (\langle D_1 u, D_2 v\rangle_{\mathbb{C}^m})_{D_1,D_2\in\mathcal{D}} \tag{5-1}$$

for any smooth $u,v : H_d \to \mathbb{C}^m$, where $\mathcal{D}$ is the finite set of differential operators

$$\mathcal{D} := \{1, i, \partial_{x_1}, \dots, \partial_{x_d}, \partial_t, i\partial_{x_1}, \dots, i\partial_{x_d}, i\partial_t\}.$$

For our later arguments, it will be crucial to observe that the component $\langle \partial_t u, \partial_t u\rangle_{\mathbb{C}^m} = \langle i\partial_t u, i\partial_t u\rangle_{\mathbb{C}^m}$ of the Gram-type matrix $G[u,u]$ is *not* used to determine the quantities T_{00}, T_{0j}, T_{jk}, E, J_j, and in particular will be allowed to be extremely large compared to the other components of this matrix.

As in [Tao 2016a], the strategy of proof of Theorem 4.2 will be to eliminate the role of the field u by reformulating the problem in terms of V and the Gram-type matrix $G[u,u]$ (or on closely related quantities such as T_{00}, T_{0j}, T_{jk}, E, J_j). To do this, it is natural to ask what constraints a $(2d+4)\times(2d+4)$ matrix-valued function G on H_d has to obey in order to be expressible as a Gram-type matrix $G[u,u]$ of a smooth field $u : H_d \to \mathbb{C}^m$ obeying the homogeneity condition (3-1). Certainly we will have homogeneity relations of the form

$$\langle D_1 u(4t,2x), D_2 u(4t,2x)\rangle_{\mathbb{C}^m} = 2^{-\frac{4}{p-1}-\operatorname{ord}(D_1)-\operatorname{ord}(D_2)}\langle D_1 u(t,x), D_2 u(t,x)\rangle_{\mathbb{C}^m},$$

where the *parabolic order* $\operatorname{ord}(D)$ of a differential operator $D\in\mathcal{D}$ is defined by Table 2. Also, it is clear that the matrix $G[u,u]$ is real symmetric and positive semidefinite, with the additional constraint

$$\langle iD_1 u, iD_2 u\rangle_{\mathbb{C}^m} = \langle D_1 u, D_2 u\rangle_{\mathbb{C}^m} \tag{5-2}$$

for $D_1, D_2 = 1, \partial_{x_1}, \dots, \partial_{x_d}, \partial_t$. From the product rule we also have the additional constraints

$$\langle u, D_1 u\rangle_{\mathbb{C}^m} = \tfrac{1}{2} D_1 \langle u, u\rangle_{\mathbb{C}^m} \tag{5-3}$$

and

$$D_1\langle u, iD_2 u\rangle_{\mathbb{C}^m} - D_2\langle u, iD_1 u\rangle_{\mathbb{C}^m} = 2\langle D_1 u, iD_2 u\rangle_{\mathbb{C}^m} \tag{5-4}$$

for $D_1, D_2 = \partial_{x_1}, \dots, \partial_{x_d}, \partial_t$. Finally we have

$$\langle iD_1 u, D_2 u\rangle_{\mathbb{C}^m} = -\langle D_1 u, iD_2 u\rangle_{\mathbb{C}^m} \tag{5-5}$$

for $D_1, D_2 = 1, \partial_{x_1}, \dots, \partial_{x_d}, \partial_t$. One could then hope that these were essentially the complete list of constraints on the Gram-type matrix $G[u,u]$. In the real case, in which u takes values in the real Euclidean space $\mathbb{R}^m$ (with the usual inner product $\langle\ ,\ \rangle_{\mathbb{R}^m}$), and the set of operators $\mathcal{D}$ is reduced to the $d+2$ operators

$$D_{\mathbb{R}} := \{1, \partial_{x_1}, \dots, \partial_{x_d}, \partial_t\},$$

one can obtain such a claim using the Nash embedding theorem [1956]:

Proposition 5.1. *Let $(G_{D_1,D_2})_{D_1,D_2\in\mathcal{D}_{\mathbb{R}}}$ be a $(d+2)\times(d+2)$ matrix of smooth functions $G_{D_1,D_2} : H_d \to \mathbb{R}$ obeying the following hypotheses*:

(i) *For each $(t,x) \in H_d$, the matrix $(G_{D_1,D_2}(t,x))_{D_1,D_2\in\mathcal{D}_{\mathbb{R}}}$ is symmetric and strictly positive definite.*

(ii) *One has the scaling law*

$$G_{D_1,D_2}(4t,2x) = 2^{-\frac{4}{p_1}-\operatorname{ord}(D_1)-\operatorname{ord}(D_2)} G_{D_1,D_2}(t,x) \tag{5-6}$$

for all $D_1, D_2 \in \mathcal{D}_{\mathbb{R}}$ and $(t,x) \in H_d$.

(iii) *We have the identity*

$$G_{1,D_1}(t,x) = G_{D_1,1}(t,x) = \tfrac{1}{2} G_{1,1}(t,x) \tag{5-7}$$

for all $(t,x) \in H_d$ and $D_1 \in \mathcal{D}_{\mathbb{R}}\backslash\{1\}$.

Suppose also that m is an integer that is sufficiently large depending on d. Then there exists a smooth function $u : H_d \to \mathbb{R}^m$ that is nowhere vanishing and obeys the discrete self-similarity (3-1) *with $\alpha = 0$ such that*

$$G_{D_1,D_2}(t,x) = \langle D_1 u(t,x), D_2 u(t,x)\rangle_{\mathbb{R}^m} \tag{5-8}$$

for all $D_1, D_2 \in \mathcal{D}_{\mathbb{R}}$ and $(t,x) \in H_d$. Furthermore, the function $\theta : (t,x) \mapsto u(t,x)/\|u(t,x)\|_{\mathbb{R}^m}$, when descended to the quotient space $H_d/T^{\mathbb{Z}}$, is a smooth embedding.

Proof. Observe from the chain and quotient rules that if u is smooth and obeys (5-8), then u is nowhere vanishing (since $G_{1,1}$ is strictly positive) and the direction map $\theta : (t,x) \mapsto u(t,x)/\|u(t,x)\|_{\mathbb{R}^m}$ obeys the identity

$$g_{D_1,D_2}(t,x) = \langle D_1\theta(t,x), D_2\theta(t,x)\rangle_{\mathbb{R}^m} \tag{5-9}$$

for $(t,x) \in H_d$ and $D_1, D_2 \in \mathcal{D}_{\mathbb{R}}\backslash\{1\}$, where the functions $g_{D_1,D_2} : H_d \to \mathbb{R}$ are given by the formula

$$g_{D_1,D_2} := \frac{G_{D_1,D_2}}{G_{1,1}} - \frac{G_{1,D_1}G_{1,D_2}}{G_{1,1}^2}.$$

Motivated by this, our strategy will be to construct the direction map θ obeying (5-9) first, and use this to then reconstruct u.

Since $G_{1,1}$ is strictly positive, the $(d+1)\times(d+1)$-matrix $g = (g_{D_1,D_2})_{D_1,D_2\in\mathcal{D}_\mathbb{R}\backslash\{1\}}$ is smooth and symmetric; from the hypothesis (ii), the matrix g is $T^\mathbb{Z}$-invariant, and can thus (by slight abuse of notation) be viewed as a function on the quotient space $H_d/T^\mathbb{Z}$. From the identity

$$\sum_{D_1,D_2\in\mathcal{D}_\mathbb{R}\backslash\{1\}} g_{D_1,D_2}\, a_{D_1}\, a_{D_2} = \sum_{D_1,D_2\in\mathcal{D}} G_{D_1,D_2}\, b_{D_1}\, b_{D_2}$$

for all reals a_D, $D\in\mathcal{D}_\mathbb{R}\backslash\{1\}$, where

$$b_D := \frac{a_D}{G_{1,1}^{1/2}} \quad\text{and}\quad b_1 := -\frac{\sum_{D\in\mathcal{D}_\mathbb{R}\backslash\{1\}} a_D G_{1,D}}{G_{1,1}^{3/2}},$$

and the hypothesis (i), we see that the matrix g is strictly positive. Thus $(H_d/T^\mathbb{Z}, g)$ can be viewed as a smooth compact $(d+1)$-dimensional Riemannian manifold with smooth boundary. If m_0 is a large enough integer, we may then apply the Nash embedding theorem [1956] (see also [Günther 1991]) and find a smooth isometric embedding of $(H_d/T^\mathbb{Z}, g)$ into a Euclidean space $\mathbb{R}^{m_0}$. As observed in [Tao 2016a, §4], any compact region of $\mathbb{R}^{m_0}$ may be isometrically embedded into the unit sphere S^{m-1} of $\mathbb{R}^m$ if $m\ge 2m_0+2$. Thus, for m large enough, we may find an isometric embedding $\theta : H_d/T^\mathbb{Z}\to S^{m-1}$ of $(H_d/T^\mathbb{Z}, g)$ into unit sphere S^{m-1} of $\mathbb{R}^m$ (with the induced Euclidean metric); thus θ is a smooth embedding and obeys the identity (5-9) (after lifting up from $H_d/T^\mathbb{Z}$ to H_d). If one then defines the function $u : H_d\to\mathbb{R}^m$ by the formula

$$u(t,x) := \theta(t,x)G_{1,1}(t,x)^{\frac{1}{2}}$$

then u is smooth, nowhere vanishing, and obeys (3-1) with $\alpha = 0$, and from a routine calculation using the product and chain rules (as well as hypothesis (iii)) we have the required identity (5-8) for all $D_1, D_2\in\mathcal{D}_\mathbb{R}\backslash\{1\}$; it is also clear that (5-8) holds when $D_1 = D_2 = 1$. Differentiating the latter identity in space or time using (iii) and the product rule, we obtain the remaining cases of (5-8), and the claim follows. □

One could use the literature on the Nash embedding theorem to extract an explicit value of m as a function of d in the above proposition, but we will not seek to optimise this value here. For future reference, we observe that the above argument also gives the variant of Proposition 5.1 in which the half-space H_d is replaced by the punctured spacetime $\mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\}$.

In view of the above proposition, one could conjecture a complex analogue of the proposition, in which one uses $\mathcal{D}$ in place of $\mathcal{D}_\mathbb{R}$ and $\mathbb{C}^m$ in place of $\mathbb{R}^m$, with the additional constraints (5-4), (5-5) imposed. This conjecture may well be false in full generality (note that the complex version of the Nash embedding theorem is false; for instance, Liouville's theorem prevents compact complex manifolds without boundary from being holomorphically embedded into $\mathbb{C}^m$). Nevertheless we could adapt the *proof* of the Nash embedding theorem to obtain a partial complex analogue of Proposition 5.1, in which we do not seek to control the $\langle\partial_t u, \partial_t u\rangle_{\mathbb{C}^m}$ component of the Gram-like matrix (5-1), and in which we also have an additional curl-free property of a certain combination of components of this matrix. While this falls well short of a true complex version of Proposition 5.1, it will suffice for our purposes. Specifically, we have:

Proposition 5.2. *Let $G = (G_{D_1,D_2})_{D_1,D_2 \in \mathcal{D}}$ be a $(2d+4) \times (2d+4)$ matrix of smooth functions $G_{D_1,D_2} : H_d \to \mathbb{R}$ obeying the following hypotheses*:

(i) *For each $(t,x) \in H_d$, the matrix $(G_{D_1,D_2}(t,x))_{D_1,D_2 \in \mathcal{D}}$ is symmetric and strictly positive definite.*

(ii) *One has the scaling law* (5-6) *for all $D_1, D_2 \in \mathcal{D}$ and $(t,x) \in H_d$.*

(iii) *We have the identity* (5-7), *as well as the additional identities*

$$D_1 G_{1,iD_2}(t,x) - D_2 G_{1,iD_1}(t,x) = 2G_{D_1,iD_2}(t,x) \tag{5-10}$$

for all $(t,x) \in H_d$ and $D_1, D_2 \in \mathcal{D}_{\mathbb{R}} \backslash \{1\}$, and

$$G_{iD_1,iD_2}(t,x) = G_{D_1,D_2}(t,x), \tag{5-11}$$

$$G_{D_1,iD_2}(t,x) = -G_{D_2,iD_1}(t,x) \tag{5-12}$$

for all $(t,x) \in H_d$ and $D_1, D_2 \in \mathcal{D}_{\mathbb{R}}$ (in particular we have $G_{D_1,iD_1} = 0$).

(iv) *The vector field $(G_{1,i\partial_{x_j}}/G_{1,1})_{j=1}^d$ is curl-free*; *that is to say,*

$$\partial_{x_k} \frac{G_{1,i\partial_{x_j}}}{G_{1,1}}(t,x) - \partial_{x_j} \frac{G_{1,i\partial_{x_k}}}{G_{1,1}}(t,x) = 0$$

for all $j,k \in 1,\dots,d$ and $(t,x) \in H_d$.

Suppose also that m is an integer that is sufficiently large depending on d. Then there exists a smooth function $u : H_d \to \mathbb{C}^m$ that is nowhere vanishing and obeying the discrete self-similarity (3-1) *for some $\alpha \in \mathbb{R}$ such that*

$$G_{D_1,D_2}(t,x) = \langle D_1 u(t,x), D_2 u(t,x) \rangle_{\mathbb{C}^m} \tag{5-13}$$

for all $(t,x) \in H_d$ and all $D_1, D_2 \in \mathcal{D}$ other than $(D_1,D_2) = (\partial_t, \partial_t), (i\partial_t, i\partial_t)$. Furthermore, the function $\theta : H_d/T^{\mathbb{Z}} \to \mathbb{CP}^{m-1}$, formed by descending the map $\pi \circ u : H_d \to \mathbb{CP}^{m-1}$ to $H_d/T^{\mathbb{Z}}$, is a smooth embedding.

Remark 5.3. The condition (iv) differs from the other hypotheses in that it is not necessitated by the conclusions of this theorem. However, this condition turns out to be convenient in the proof of Proposition 5.2, as it will allow us to "gauge transform away" the $G_{1,i\partial_{x_j}}$ components; see Proposition A.1. However, this additional constraint (iv) will end up not being harmful to our argument, because we will eventually reduce to the case where the matrix G is spherically symmetric in the sense that $G_{1,1}(t,x) = g(t,|x|)$ and $G_{1,i\partial_{x_j}}(t,x) = (x_j/|x|)h(t,|x|)$ for some functions g, h, in which case the condition (iv) is automatically satisfied.

The proof of Proposition 5.2 is rather lengthy, and the methods of proof (based on the proof of the Nash embedding theorem) are not used elsewhere in the paper. We therefore defer this proof to the Appendix. Combining Proposition 5.2 with Remark 4.1, we thus see that Theorem 4.2 will now follow from the following theorem in which the field u has been eliminated.

Theorem 5.4 (third reduction). *There exists a smooth* $(2d+3)\times(2d+3)$ *matrix* $G = (G_{D_1,D_2})_{D_1,D_2\in\mathcal{D}}$ *of smooth functions* $G_{D_1,D_2} : H_d \to \mathbb{R}$ *and an additional smooth function* $V : H_d \to \mathbb{R}$ *obeying the following properties*:

(i) *For each* $(t,x) \in H_d$, *the matrix* $(G_{D_1,D_2}(t,x))_{D_1,D_2\in\mathcal{D}}$ *is symmetric and strictly positive definite.*

(ii) *One has the scaling law* (5-6) *for all* $D_1, D_2 \in \mathcal{D}$ *and* $(t,x) \in H_d$.

(iii) *We have the identities* (5-7), (5-10) *for all* $(t,x) \in H_d$ *and* $D_1, D_2 \in \mathcal{D}_{\mathbb{R}}\backslash\{1\}$, *and* (5-11), (5-12) *for all* $(t,x) \in H_d$ *and* $D_1, D_2 \in \mathcal{D}_{\mathbb{R}}$.

(iv) *The vector* $(G_{1,i\partial_{x_j}}/G_{1,1})_{j=1}^d$ *is curl-free on* H_d.

(v) *One has the defocusing property* (4-6).

(vi) *If one defines the pseudo-stress-energy tensor*

$$\begin{aligned} T_{00} &:= G_{1,1}, \\ T_{0j} = T_{j0} &:= -2G_{i\partial_{x_j},1}, \\ T_{jk} &:= 4G_{\partial_{x_j},\partial_{x_k}} + \delta_{jk}2(p-1)V - \delta_{jk}\Delta G_{1,1} \end{aligned}$$

for $j,k = 1,\dots,d$, *as well as the energy density*

$$E := \tfrac{1}{2}G_{\partial_{x_j},\partial_{x_j}} + V$$

(*with the usual summation conventions*) *and energy current*

$$J_j := -G_{\partial_{x_j},\partial_t}$$

for $j = 1,\dots,d$, *then one has the identity*

$$G_{i\partial_t,1} + \tfrac{1}{2}\Delta G_{1,1} - G_{\partial_{x_j},\partial_{x_j}} = (p+1)V \tag{5-14}$$

and the conservation laws

$$\partial_t T_{00} + \partial_{x_j} T_{j0} = 0, \tag{5-15}$$

$$\partial_t T_{0k} + \partial_{x_j} T_{jk} = 0, \tag{5-16}$$

$$\partial_t E + \partial_{x_j} J_j = 0 \tag{5-17}$$

for $k = 1,\dots,d$.

Note carefully that the components $G_{\partial_t,\partial_t}$, $G_{i\partial_t,i\partial_t}$ of G are not used in the hypotheses (ii)–(vi) above (and only influence (i) through the requirement of being positive definite). The scaling law (5-6) only applies directly to the components of G, but from the potential identity (5-14) we see that we also have a corresponding scaling law

$$V(T(t,x)) = 2^{-\frac{4}{p-1}-2}V(t,x)$$

for the potential V.

It remains to prove Theorem 5.4. This will be the objective of the remaining sections of the paper.

6. Spherical symmetry and scale invariance

At first glance, the hypotheses required in Theorem 5.4 of the unknown fields G, V appear to be more complicated than those in previous formulations of the problem, such as Theorem 3.1. However, there is one notable way in which the hypotheses of Theorem 5.4 are much better than those in previous formulations: as they only involve linear equalities and inequalities (as well as claims of positive definiteness), the constraints determine a *convex* set in the phase space of possible values for the fields G, V. This can be compared to previous formulations in which the conditions on the unknown field u were quadratic or otherwise nonlinear in nature.

One consequence of this convexity is that if there is at least one solution G, V to Theorem 5.4, then there is a solution G, V which is spherically symmetric in a tensorial sense, or more precisely that

$$\begin{aligned} G_{1,1}(t,x) &= g_{1,1}(t,|x|), \\ G_{\partial_{x_j},\partial_{x_k}}(t,x) &= \frac{x_j x_k}{|x|^2} g_{\partial_r,\partial_r}(t,|x|) + \left(\delta_{jk} - \frac{x_j x_k}{|x|^2}\right) g_{\partial_\omega \partial_\omega}(t,|x|), \\ G_{\partial_{x_j},\partial_t}(t,x) &= \frac{x_j}{|x|} g_{\partial_r,\partial_t}(t,|x|), \\ G_{1,i\partial_{x_j}}(t,x) &= \frac{x_j}{|x|} g_{1,i\partial_r}(t,|x|), \\ G_{1,i\partial_t}(t,x) &= g_{1,i\partial_t}(t,|x|), \\ V(t,x) &= v(t,|x|) \end{aligned}$$

for some functions g_{11}, $g_{\partial_r,\partial_r}$, $g_{\partial_\omega,\partial_\omega}$, $g_{\partial_r,\partial_t}$, $g_{1,i\partial_r}$, $g_{1,i\partial_t}$; we omit here for brevity some analogous constraints on the remaining components of G which are either constrained completely by the fields already listed, or (in the case of $G_{\partial_t,\partial_t}$ and $G_{i\partial_t,i\partial_t}$) are not relevant for the theorem. This is basically because we can average the original solution G, V over rotations (letting the orthogonal group $\mathrm{SO}(d)$ act on tensors in an appropriate fashion) and use convexity to obtain a spherically symmetric solution. For similar reasons (averaging now over dilations rather than rotations), one can assume without loss of generality that the solution G, V is not only *discretely* self-similar in the sense of (5-6), but is in fact *continuously* self-similar in the sense that the identity

$$G_{D_1,D_2}(\lambda^2 t, \lambda x) = \lambda^{-\frac{4}{p_1}-d_1-d_2} G_{D_1,D_2}(t,x)$$

holds for all $D_1, D_2 \in \mathcal{D}$, $(t,x) \in H_d$, and $\lambda > 0$, where d_1, d_2 denotes the degrees of D_1, D_2 respectively as before.

Remark 6.1. Note that the reduction to spherical symmetry of the fields G, V in Theorem 5.4 does *not* mean that we can reduce to spherically symmetric u in the original formulation (Theorem 1.2) of the results in this paper, because it is possible for a nonspherically symmetric field u to have a spherically symmetric Gram matrix (e.g., if $d = 2$ and u is equivariant rather than invariant with respect to rotations). Indeed, a spherically symmetric u would have a vanishing $g_{\partial_\omega,\partial_\omega}$ field, whereas in our construction we will insist instead that this field be positive. Similarly, we cannot necessarily reduce to solutions in Theorem 1.2 that are continuously self-similar.

We now perform these reductions by showing that Theorem 5.4 is a consequence of (and is in fact equivalent to) the following spherically symmetric, continuously self-similar version. Recall that the domain H_1 is given by (2-3). It will be convenient to make the following definition: we say that a function $F : H_1 \to \mathbb{R}$ *scales like* ρ^α for some $\alpha \in \mathbb{R}$ if one has

$$F(\lambda^2 t, \lambda r) = \lambda^\alpha F(t,r) \tag{6-1}$$

for all $(t,x) \in H_1$ and $\lambda > 0$. Here we recall $\rho : H_1 \to \mathbb{R}$ was defined in (2-5). We also note the following "factor theorem" on H_1: if $F : H_1 \to \mathbb{R}$ is a smooth function that vanishes on the time axis $r = 0$, then the quotient $F(t,r)/r$ has a removable singularity at $r = 0$, in the sense that there is a smooth function $G : H_1 \to \mathbb{R}$ such that $F(t,r) = rG(t,r)$ for all $(t,r) \in H_1$ (so that G can be viewed as the smooth completion of $F(t,r)/r$. Indeed, from the fundamental theorem of calculus, one can take

$$G(t,r) := \int_0^1 (\partial_r F)(t, sr)\, ds.$$

Iterating this, we see that if k is a positive integer, and $F : H_1 \to \mathbb{R}$ is smooth and vanishes to order k on the time axis $r = 0$ (in the sense that $F(t,r) = O(r^k)$ as $r \to 0$ for any fixed t), then F/r^k has a removable singularity on the time axis.

Theorem 6.2 (fourth reduction). *There exist smooth fields* $g_{1,1}$, $g_{\partial_r,\partial_r}$, $g_{\partial_\omega,\partial_\omega}$, $g_{\partial_r,\partial_t}$, $g_{1,i\partial_r}$, $g_{1,i\partial_t}$, $v : H_1 \to \mathbb{R}$ *obeying the following properties*:

(i) *One has the "positive definite" inequalities*

$$\left(\tfrac{1}{2}\partial_r g_{1,1}\right)^2 + g_{1,i\partial_r}^2 < g_{1,1} g_{\partial_r,\partial_r}, \tag{6-2}$$

$$g_{1,1}, g_{\partial_r,\partial_r}, g_{\partial_\omega,\partial_\omega} > 0 \tag{6-3}$$

pointwise on H_1.

(ii) *For each* $(D_1, D_2) = (1,1)$, (∂_r, ∂_r), $(\partial_\omega, \partial_\omega)$, (∂_r, ∂_t), $(1, i\partial_r)$, $(1, i\partial_t)$, *the field* g_{D_1,D_2} *scales like* $\rho^{-\frac{4}{p-1} - \mathrm{ord}(D_1) - \mathrm{ord}(D_2)}$, *where we recall the parabolic order* $\mathrm{ord}(D)$ *of a differential operator* $D \in \{1, \partial_r, i\partial_r, \partial_\omega, \partial_t, i\partial_t\}$ *is given by Table 2. Similarly, we require that* v *scales like* $\rho^{-\frac{4}{p-1}-2}$. *See Table 3 for a summary of these scaling requirements.*

(v) *One has the defocusing property* $v > 0$ *pointwise on* H_1.

(vi) *If one defines the mass density*

$$T_{00} := g_{1,1},$$

the radial momentum density

$$T_{0r} := -2 g_{1,i\partial_r},$$

the radial stress

$$T_{rr} := 4 g_{\partial_r,\partial_r} + 2(p-1)v - \left(\partial_r^2 + \frac{d-1}{r}\partial_r\right) g_{1,1}, \tag{6-4}$$

exponent	fields	parity
2	t	even
1	ρ	even
1	r	odd
$-\frac{4}{p-1}+d-4$	$S_1,\ S_2$	same as d
$-\frac{4}{p-1}$	$g_{1,1},\ T_{00},\ W$	even
$-\frac{4}{p-1}-1$	$g_{1,i\partial_r},\ T_{0r}$	odd
$-\frac{4}{p-1}-2$	$g_{\partial_r,\partial_r},\ g_{\partial_\omega,\partial_\omega},\ g_{1,i\partial_t},\ v,\ T_{rr},\ T_{\omega\omega},\ E,\ Z$	even
$-\frac{4}{p-1}-3$	$g_{\partial_r,\partial_t},\ J_r$	odd

Table 3. The scaling exponent of various fields on H_1 used in this paper, as well as their parity in r (even or odd). Some of the fields in this table will only be defined in subsequent sections.

the angular stress

$$T_{\omega\omega} := 4g_{\partial_\omega,\partial_\omega} + 2(p-1)v - \Big(\partial_r^2 + \frac{d-1}{r}\partial_r\Big)g_{1,1}, \tag{6-5}$$

the energy density

$$E := \tfrac{1}{2}g_{\partial_r,\partial_r} + \tfrac{1}{2}(d-1)g_{\partial_\omega,\partial_\omega} + v, \tag{6-6}$$

and radial energy current

$$J_r := -g_{\partial_r,\partial_t}, \tag{6-7}$$

then one has the potential identity

$$g_{1,i\partial_t} + \tfrac{1}{2}\Big(\partial_r^2 + \frac{d-1}{r}\partial_r\Big)g_{1,1} - g_{\partial_r,\partial_r} - (d-1)g_{\partial_\omega,\partial_\omega} = (p+1)v \tag{6-8}$$

and the conservation laws

$$\partial_t T_{00} + \Big(\partial_r + \frac{d-1}{r}\Big)T_{0r} = 0, \tag{6-9}$$

$$\partial_t T_{0r} + \Big(\partial_r + \frac{d-1}{r}\Big)T_{rr} - \frac{d-1}{r}T_{\omega\omega} = 0, \tag{6-10}$$

$$\partial_t E + \Big(\partial_r + \frac{d-1}{r}\Big)J_r = 0. \tag{6-11}$$

with a removable singularity at $r=0$ (see Remark 6.3 below).

(vii) *The functions $g_{1,1},\ g_{\partial_r,\partial_r},\ g_{\partial_\omega,\partial_\omega},\ g_{1,i\partial_t},\ v$ are even in r, while $g_{\partial_r,\partial_t},\ g_{1,i\partial_r}$ are odd in r (see Table 3). Furthermore, $g_{\partial_r,\partial_r} - g_{\partial_\omega,\partial_\omega}$ vanishes on the time axis $r=0$.*

Remark 6.3. At first glance, the quantities $T_{rr}, T_{\omega,\omega}$, as well as (6-8)–(6-11), appear to have singularities on the time axis $r=0$, due to the factors of $\frac{1}{r}$. However, these factors are removable due to the symmetry hypotheses in (vii). Indeed, for each fixed time t, one can Taylor expand the even function $g_{1,1}$ as

$g_{1,1} = a + br^2 + \cdots$, and then one sees that the quantity $\left(\partial_r^2 + \frac{d-1}{r}\partial_r\right)g_{1,1}$ extends smoothly across $r = 0$ (which is unsurprising given that this operator is nothing more than the Laplacian on spherically symmetric functions). Thus T_{rr} and $T_{\omega\omega}$ extend smoothly to $r = 0$. Also, the difference $T_{rr} - T_{\omega\omega}$ vanishes at $r = 0$, so the singularity for (6-10) is also removable. Finally, the functions T_{0r}, J_r are odd in r and so the singularity in (6-9), (6-11) is also removable.

Remark 6.4. It is not difficult to use Table 3 to perform a "dimensional analysis" to verify that the requirements in Theorem 6.2(vi) are consistent with the scaling and parity requirements in Theorem 6.2(ii), (vii). One can use the continuous self-similarity (ii) to eliminate the time variable, so that Theorem 6.2 becomes an ODE assertion about the existence of some scalar functions on $\mathbb{R}$. However, it will be convenient (and more physically natural) to continue to work with both the time variable t and the spatial variable r, rather than with just one of these variables. It is also worth noting that the components $g_{\partial_r,\partial_t}$ and $g_{1,i\partial_t}$ have only a small role to play in the above theorem, basically appearing only in the constraints (6-11) and (6-8) respectively; crucially, they do not appear at all in the positive definite constraints in (i), thanks to the previously observed absence of the fields $g_{\partial_t,\partial_t}$ or $g_{i\partial_t,i\partial_t}$. As such, we will be able to eliminate these fields from the problem in the next section.

Let us now see how Theorem 6.2 implies Theorem 5.4. Let the fields

$$g_{1,1},\ g_{\partial_r,\partial_r},\ g_{\partial_\omega,\partial_\omega},\ g_{\partial_r,\partial_t},\ g_{1,i\partial_r},\ g_{1,i\partial_t},\ v$$

be as in Theorem 6.2. Let $A > 0$ be a large quantity to be chosen later. We then define the functions G_{D_1,D_2} for $D_1, D_2 \in \mathcal{D} : H_d \to \mathbb{R}$ and $V : H_d \to \mathbb{R}$ by the formulae in Table 4.

It is a classical result of Whitney [1943] that a smooth function $g(t,r)$ that is even in r can be thought of as a smooth function of (t,r^2) (where the latter is viewed on the half-line $[0,+\infty)$), while an odd function of t,r that is odd in r can be thought of as r times a smooth function of (t,r^2); see, e.g., [Tao 2016b, Corollary 2.2]. In particular, we see that $g_{1,1}(t,|x|)$, $g_{1,i\partial_t}(t,|x|)$, $v(t,|x|)$, $g_{\partial_r,\partial_r}(t,|x|) - g_{\partial_\omega,\partial_\omega}(t,|x|)$ are smooth functions of t,x, while $g_{\partial_r,\partial_t}(t,|x|)$, $g_{1,i\partial_r}(t,|x|)$ are $|x|$ times a smooth function of t,x. Finally, $g_{\partial_r,\partial_r}(t,|x|) - g_{\partial_\omega,\partial_\omega}(t,|x|)$ is $|x|^2$ times a smooth function of t,x, due to the hypothesis that $g_{\partial_r,\partial_r} - g_{\partial_\omega,\partial_\omega}$ vanishes on the time axis. From this and Table 4, we can check that all of the functions G_{D_1,D_2}, V have removable singularities on the time axis and thus define smooth functions on H_d.

From tedious direct calculation using Table 4, we can verify the symmetry $G_{D_2,D_1} = G_{D_1,D_2}$ and the properties claimed in Theorem 5.4(ii), (iii). Since

$$\frac{G_{1,i\partial_{x_j}}}{G_{1,1}}(t,x) = \frac{x_j}{|x|}\frac{g_{1,i\partial_r}}{g_{1,1}}(t,|x|)$$

(away from the time axis at least), we have

$$\partial_{x_k}\frac{G_{1,i\partial_{x_j}}}{G_{1,1}}(t,x) = \left(\frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3} + \frac{x_j x_k}{|x|^2}\partial_r\right)\frac{g_{1,i\partial_r}}{g_{1,1}}(t,|x|);$$

as the right-hand side is symmetric in j and k, we have the curl-free property in Theorem 5.4(iv) (after removing the singularity at the time axis). The positivity property in Theorem 5.4(v) is clear. For in

fields	value at (t,x)
$G_{1,1},\ G_{i,i}$ $G_{1,i},\ G_{i,1}$	$g_{1,1}(t,\|x\|)$ 0
$G_{1,\partial_{x_j}},\ G_{\partial_{x_j},1},\ G_{i,i\partial_{x_j}},\ G_{i\partial_{x_j},i}$ $G_{1,i\partial_{x_j}},\ G_{i\partial_{x_j},1},\ -G_{i,\partial_{x_j}},\ -G_{\partial_{x_j},i}$	$\frac{1}{2}\partial_{x_j}G_{1,1}(t,x)$ $(x_j/r)g_{1,i\partial_r}(t,\|x\|)$
$G_{1,\partial_t},\ G_{\partial_t,1},\ G_{i,i\partial_t},\ G_{i\partial_t,i}$ $G_{1,i\partial_t},\ G_{i\partial_t,1},\ -G_{i,\partial_t},\ -G_{\partial_t,i}$	$\frac{1}{2}\partial_t G_{1,1}(t,x)$ $g_{1,i\partial_t}(t,\|x\|)$
$G_{\partial_{x_j},\partial_{x_k}},\ G_{i\partial_{x_j},i\partial_{x_k}}$ $G_{\partial_{x_j},i\partial_{x_k}},\ G_{i\partial_{x_k},\partial_{x_j}}$	$(x_jx_k/\|x\|^2)g_{\partial_r,\partial_r}(t,\|x\|)+(\delta_{jk}-x_jx_k/\|x\|^2)g_{\partial_\omega,\partial_\omega}(t,\|x\|)$ 0
$G_{\partial_{x_j},\partial_t},\ G_{\partial_t,\partial_{x_j}},\ G_{i\partial_{x_j},i\partial_t},\ G_{i\partial_t,i\partial_{x_j}}$ $G_{\partial_{x_j},i\partial_t},\ G_{i\partial_t,\partial_{x_j}},\ -G_{i\partial_{x_j},\partial_t},\ -G_{\partial_t,i\partial_{x_j}}$	$(x_j/\|x\|)g_{\partial_r,\partial_t}(t,\|x\|)$ $\frac{1}{2}(\partial_{x_j}G_{1,i\partial_t}(t,x)-\partial_t G_{1,i\partial_{x_j}}(t,x))$
$G_{\partial_t,\partial_t},\ G_{i\partial_t,i\partial_t}$ $G_{\partial_t,i\partial_t},\ G_{i\partial_t,\partial_t}$	$A\rho(t,x)^{-\frac{4}{p-1}-4}$ 0
V	$v(t,\|x\|)$

Table 4. Components of G and V, and their values at a given point (t,x) of H_d; thus, for instance, $G_{1,1}(t,x)$ and $G_{i,i}(t,x)$ are both set equal to $g_{1,1}(t,|x|)$. Here $j,k=1,\dots,d$ are arbitrary.

Theorem 5.4(vi), we observe from the constructions of the various fields that

$$\begin{aligned}
T_{00}(t,x) &= T_{00}(t,|x|),\\
T_{0j}(t,x) = T_{j0}(t,x) &= \frac{x_j}{|x|}T_{0r}(t,|x|),\\
T_{jk}(t,x) &= \frac{x_jx_k}{|x|^2}T_{rr}(t,|x|)+\left(\delta_{jk}-\frac{x_jx_k}{|x|^2}\right)T_{\omega\omega}(t,|x|),\\
E(t,x) &= E(t,|x|),\\
J_j(t,x) &= \frac{x_j}{|x|}J_r(t,|x|)
\end{aligned}$$

and then it is a routine matter to derive (5-14)–(5-17) from (6-8)–(6-11), again working away from the time axis and then using smoothness to remove the singularity.

The only remaining task is to check Theorem 5.4(i); that is to say, we need to verify that for $(t,x)\in H_d$, the matrix $(G_{D_1,D_2}(t,x))_{D_1,D_2\in\mathcal{D}}$ is strictly positive definite. In view of (5-6), it suffices to do so in a fundamental domain for $H_d/T^{\mathbb{Z}}$, such as $\{(t,x): 1\le\rho<2\}$. By continuity, we can also avoid the time axis $x=0$ as long as our positive definiteness is uniform in t,x. Henceforth we fix (t,x) in this region and suppress dependence on t,x. If we let $\vec{a}:=(a_D)_{D\in\mathcal{D}}$ be a tuple of real numbers, not all zero, our task is to show that

$$\sum_{D_1,D_2\in\mathcal{D}} a_{D_1}a_{D_2}G_{D_1,D_2} > \varepsilon|\vec{a}|^2$$

for some $\varepsilon > 0$ uniform in t, x. The left-hand side can be expanded out as

$$\begin{aligned}(a_1^2+a_i^2)G_{1,1} &+ 2(a_1a_{\partial_{x_j}}+a_ia_{i\partial_{x_j}})G_{1,\partial_{x_j}} + 2(a_1a_{i\partial_{x_j}}-a_ia_{\partial_{x_j}})G_{1,i\partial_{x_j}}\\ &+2(a_1a_{\partial_t}+a_ia_{i\partial_t})G_{1,\partial_t}+2(a_1a_{i\partial_t}-a_ia_{\partial_t})G_{1,i\partial_t}\\ &+2(a_{\partial_{x_j}}a_{\partial_{x_k}}+a_{i\partial_{x_j}}a_{i\partial_{x_k}})G_{\partial_{x_j},\partial_{x_k}}\\ &+2(a_{\partial_{x_j}}a_{\partial_t}+2a_{i\partial_{x_j}}a_{i\partial_t})G_{\partial_{x_j},\partial_t}+(a_{\partial_{x_j}}a_{i\partial_t}-a_{i\partial_{x_j}}a_{\partial_t})G_{\partial_{x_j},i\partial_t}\\ &+(a_{\partial_t}^2+a_{i\partial_t}^2)G_{\partial_t,\partial_t},\end{aligned}$$

where we use the usual summation conventions. If we define

$$\vec{b} = (a_1, a_i, a_{\partial_{x_1}}, \dots, a_{\partial_{x_d}}, a_{i\partial_{x_1}}, \dots, a_{i\partial_{x_d}})$$

to be the spatial components of $\vec{a}$, then all the cross-terms in the above expression involving one copy of a_{∂_t} or $a_{i\partial_t}$ and one term from $\vec{b}$ can be controlled via Cauchy–Schwarz as

$$O\big(|\vec{b}|(a_{\partial_t}^2+a_{i\partial_t}^2)^{\frac{1}{2}}\big),$$

where the implied constants can depend on G but are uniform in t, x in the fundamental domain. On the other hand, from construction of $G_{\partial_t,\partial_t}$, the term $(a_{\partial_t}^2+a_{i\partial_t}^2)G_{\partial_t,\partial_t}$ is bounded from below by $cA(a_{\partial_t}^2+a_{i\partial_t}^2)$ for some absolute constant $c > 0$. By the inequality of arithmetic and geometric means, it will thus suffice (for A large enough) to obtain the bound

$$\begin{aligned}(a_1^2+a_i^2)G_{1,1}+2(a_1a_{\partial_{x_j}}+a_ia_{i\partial_{x_j}})G_{1,\partial_{x_j}}&+2(a_1a_{i\partial_{x_j}}-a_ia_{\partial_{x_j}})G_{1,i\partial_{x_j}}\\ &+(a_{\partial_{x_j}}a_{\partial_{x_k}}+a_{i\partial_{x_j}}a_{i\partial_{x_k}})G_{\partial_{x_j},\partial_{x_k}} \geq 2\varepsilon|\vec{b}|^2 \quad (6\text{-}12)\end{aligned}$$

for some $\varepsilon > 0$ independent of A.

If we set $a_{\partial_r}, a_{i\partial_r} \in \mathbb{R}$ and $a_{\partial_\omega}, a_{i\partial_\omega} \in \mathbb{R}^d$ to be the quantities

$$\begin{aligned}a_{\partial_r} &:= \frac{x_j}{|x|}a_{\partial_{x_j}},\\ a_{i\partial_r} &:= \frac{x_j}{|x|}a_{i\partial_{x_j}},\\ a_{\partial_\omega} &:= \left(a_{\partial_{x_j}}-\frac{x_j}{|x|}a_{\partial_r}\right)_{j=1}^d,\\ a_{i\partial_\omega} &:= \left(a_{i\partial_{x_j}}-\frac{x_j}{|x|}a_{i\partial_r}\right)_{j=1}^d\end{aligned}$$

then the left-hand side of (6-12) can be written as

$$\begin{aligned}(a_1^2+a_i^2)g_{1,1}+2(a_1a_{\partial_r}+a_ia_{i\partial_r})g_{1,\partial_r}&+2(a_1a_{i\partial_r}-a_ia_{\partial_r})g_{1,i\partial_r}\\ &+(a_{\partial_r}^2+a_{i\partial_r}^2)g_{\partial_r,\partial_r}+(|a_{\partial_\omega}|^2+|a_{i\partial_\omega}|^2)g_{\partial_\omega,\partial_\omega},\end{aligned}$$

where we suppress the dependence on t and $|x|$ in the g-terms. The claim now follows from the Cauchy–Schwarz inequality, the Legendre identity

$$(a_1 a_{\partial_r} + a_i a_{i\partial_r})^2 + (a_1 a_{i\partial_r} - a_i a_{\partial_r})^2 = (a_1^2 + a_i^2)(a_{\partial_r}^2 + a_{i\partial_r}^2)$$

and the hypotheses (6-2), (6-3).

It remains to prove Theorem 6.2. This will be the objective of the remaining sections of the paper.

7. Eliminating the energy conservation law and the potential energy identity

To motivate the next reduction, assume for the moment that the fields

$$g_{1,1},\ g_{\partial_r,\partial_r},\ g_{\partial_\omega,\partial_\omega},\ g_{\partial_r,\partial_t},\ g_{1,i\partial_r},\ g_{1,i\partial_t},\ v$$

obey the hypotheses and conclusions of Theorem 6.2, and let $T_{00}, T_{0r}, T_{rr}, T_{\omega\omega}, E, J_r$ be as in that theorem. The pointwise conservation laws (6-9)–(6-11) can then be written in a familiar integral form. For instance, multiplying the pointwise mass conservation law (6-9) by r^{d-1} and then integrating on a fixed interval $[0, R]$, one obtains the integral mass conservation identity

$$\partial_t \int_0^R T_{00}(t,r)\, r^{d-1}\, dr = -R^{d-1} T_{0r}(t,R),$$

and similarly the pointwise energy conservation law (6-11) gives the integral energy conservation identity

$$\partial_t \int_0^R E(t,r)\, r^{d-1}\, dr = -R^{d-1} J_r(t,R). \tag{7-1}$$

Applying the same manipulations to (6-10) gives a more complicated identity

$$\partial_t \int_0^R T_{0r}(t,r)\, r^{d-1}\, dr = -R^{d-1} T_{rr}(t,R) + (d-1)\int_0^R T_{\omega\omega}(t,r)\, r^{d-2}\, dr;$$

if one sets $d = 3$ for sake of discussion, applies (6-4), (6-5), and integrates by parts, one obtains the local Morawetz identity

$$\begin{aligned}\partial_t \int_0^R T_{0r}(t,r)\, r^2\, dr = -R^2 T_{rr}(t,R) - 2R\partial_r g_{1,1}(t,R) - 2g_{1,1}(t,R) + 2g_{1,1}(t,0)\\ + \int_0^R (8g_{\partial\omega,\partial\omega}(t,r) + 4(p-1)V(t,r))r\, dr.\end{aligned}$$

These sorts of identities are often used in subcritical situations to help establish global regularity of solutions to NLS. For instance, suppose we are in the energy-subcritical situation where $d < 3$, or $d \geq 3$ and $p < 1 + \frac{4}{d-2}$, rather than in the energy supercritical situation (2-1) that is the focus of this paper. We apply (7-1) with $R = 1$ (say) to conclude that $\int_0^1 E(t,r)\, r^{d-1}\, dr$ stays bounded as $t \to 0^+$. But from the scaling hypothesis (ii) and (6-6), the energy density E scales like $\rho^{-\frac{4}{p-1}-2}$, and hence (on setting $\lambda = t^{-\frac{1}{2}}$ and integrating r from 0 to 1)

$$\int_0^{t^{-1/2}} E(1,r)\, r^{d-1}\, dr = t^{\frac{2}{p-1}-\frac{d-2}{2}} \int_0^1 E(t,r)\, r^{d-1}\, dr.$$

In the energy-subcritical case, the exponent $\frac{2}{p-1} - \frac{d-2}{2}$ is positive, and hence $\int_0^{t^{-1/2}} E(1,r)\, r^{d-1}\, dr$ goes to zero as $t \to 0^+$. In the defocusing setting $v > 0$, the energy density E is strictly positive, giving a contradiction.

Now we return to the energy-supercritical situation of Theorem 6.2. In this case, the local energy conservation law (7-1) does not lead to a contradiction, but still manages to impose a one-dimensional linear constraint on the energy density E. (Note that the energy current J_r is almost arbitrary, since there are almost no constraints on the field $g_{\partial_r,\partial_t}$ in Theorem 6.2 other than through the energy conservation law.) Namely, from (7-1), the smoothness of J_r on H_1, Taylor expansion, and the fundamental theorem of calculus we have the asymptotic

$$\int_0^1 E(t,r)\, r^{d-1}\, dr = P_k(t) + O(t^{k+1})$$

as $t \to 0$, where $k \geq 0$ is an integer to be chosen later, P_k is a polynomial of degree at most k, and the implied constant in the $O(\,)$ notation is allowed to depend on k and on the data in Theorem 6.2. As E scales like $\rho^{-\frac{4}{p-1}-2}$, we can then conclude the asymptotic

$$\int_0^R E(1,r)\, r^{d-1}\, dr = R^{d-2-\frac{4}{p-1}}(P_k(1/R^2) + O(R^{-2k-2})) \tag{7-2}$$

as $R \to \infty$. Again using the fact that E scales like $\rho^{-\frac{4}{p-1}-2}$, we also have the asymptotic

$$E(1,r)r^{d-1} = r^{d-3-\frac{4}{p-1}}(Q_k(1/r^2) + O(r^{-2k-2})) \tag{7-3}$$

as $r \to \infty$, for some polynomial Q_k of degree at most k.

Now take k to be the largest integer such that

$$d - 2 - \frac{4}{p-1} - 2k \geq 0; \tag{7-4}$$

note from the energy-supercriticality hypothesis (2-1) that k is nonnegative. If strict inequality holds in (7-4), then the error term $R^{d-2-\frac{4}{p-1}}O(R^{-2k-2})$ in (7-2) goes to zero at infinity, while the error term $r^{d-3-\frac{4}{p-1}}O(r^{-2k-2})$ in (7-3) is absolutely integrable in r (for r near zero this follows from the local integrability of $r^{d-3-\frac{4}{p-1}}Q_k(1/r^2)$ and the triangle inequality). Integrating (7-3) and comparing with (7-2), we see that $R^{d-2-\frac{4}{p-1}}P_k(1/R^2)$ must be a primitive of $r^{d-3-\frac{4}{p-1}}Q_k(1/r^2)$, and one has vanishing renormalised total energy in the sense that

$$\lim_{R\to\infty} \int_0^R \big(E(1,r)r^{d-1} - r^{d-3-\frac{4}{p-1}}Q_k(1/r^2)\big)\, dr = 0 \tag{7-5}$$

since otherwise there would have to be a constant term in $R^{d-2-\frac{4}{p-1}}P_k(1/R^2)$, which is not possible when strict inequality occurs in (7-4). If instead equality holds in (7-4), then the same analysis yields instead that the degree k coefficient of Q_k must vanish (that is to say, Q_k in fact has degree at most $k-1$), since otherwise there would have to be a $\log R$ term present in (7-2), which is not the case.

As it turns out, though, in the energy-supercritical case the linear constraint that we have just obtained is "dense" rather than "closed", in the sense that data that does not obey this constraint can be perturbed (in a natural topology) to obey the constraint. (In other words, the linear functional that defines the constraint is unbounded with respect to a certain natural norm.) Informally speaking, this will be because for self-similar solutions to an energy-supercritical problem there will be an infinite amount of energy near spatial infinity that is available to "spend" to perform such a perturbation. As such, the constraint can be eliminated entirely; we can also easily eliminate the potential identity (6-8) due to the fact that the field $g_{1,i\partial_t}$ appearing in that identity is almost completely unconstrained outside of that identity. More precisely, we can deduce Theorem 6.2 from:

Theorem 7.1 (fifth reduction). *There exist smooth fields* $g_{1,1},\ g_{\partial_r,\partial_r},\ g_{\partial_\omega,\partial_\omega},\ g_{1,i\partial_r},\ v : H_1 \to \mathbb{R}$ *obeying the following properties*:

(i) *One has the positive definite inequalities* (6-2), (6-3) *pointwise on* H_1.

(ii) *For* $(D_1, D_2) = (1, 1), (\partial_r, \partial_r), (\partial_\omega, \partial_\omega), (1, i\partial_r)$, *the field* g_{D_1,D_2} *scales like* $\rho^{-\frac{4}{p-1}-\mathrm{ord}(D_1)-\mathrm{ord}(D_2)}$. *Similarly, we require that* v *scales like* $\rho^{-\frac{4}{p-1}-2}$.

(v) *One has the defocusing property* $v > 0$ *pointwise on* H_1.

(vi) *If one defines the mass density*

$$T_{00} := g_{1,1},$$

the radial momentum density

$$T_{0r} := -2g_{1,i\partial_r},$$

the radial stress

$$T_{rr} := 4g_{\partial_r,\partial_r} + 2(p-1)v - \Big(\partial_r^2 + \frac{d-1}{r}\partial_r\Big)g_{1,1},$$

and the angular stress

$$T_{\omega\omega} := 4g_{\partial_\omega,\partial_\omega} + 2(p-1)v - \Big(\partial_r^2 + \frac{d-1}{r}\partial_r\Big)g_{1,1},$$

then one has the conservation laws (6-9), (6-10) *with removable singularity at* $r = 0$.

(vii) *The functions* $g_{1,1},\ g_{\partial_r,\partial_r},\ g_{\partial_\omega,\partial_\omega},\ v$ *are even in* r, *while* $g_{1,i\partial_r}$ *is odd in* r. *Furthermore, the function* $g_{\partial_r,\partial_r} - g_{\partial_\omega,\partial_\omega}$ *vanishes on the time axis* $r = 0$.

Let us now see how Theorem 7.1 implies Theorem 6.2. By Theorem 7.1, we may find fields $g_{1,1}$, $g_{\partial_r,\partial_r},\ g_{\partial_\omega,\partial_\omega},\ g_{1,i\partial_r},\ v$ obeying the conclusions of that theorem. Define the energy density $E : H_d \to \mathbb{R}$ by the formula (6-6). Clearly E is smooth and scales like $\rho^{-\frac{4}{p-1}-2}$. As in the previous discussion, we let k be the largest integer obeying (7-4), so that $k \geq 0$; then $E(1, r)$ has an asymptotic expansion of the form (7-3) as $r \to \infty$ for some polynomial Q_k of degree at most k. Let us call the energy density E *good* if one of the following conditions is satisfied:

- If strict inequality holds in (7-4), we call E good if we have the asymptotic vanishing property (7-5). (Note that the limit in (7-5) exists because the integrand will be absolutely integrable, thanks to (7-3).)
- If instead equality holds in (7-4), we call E good if the degree k component of Q_k vanishes, or equivalently that Q_k has degree at most $k - 1$.

Let us suppose first that E is good, and conclude the proof of Theorem 6.2. Using the data provided by Theorem 7.1, and comparing the conclusions of that theorem with that of Theorem 6.2, we see that it will suffice to produce smooth fields $g_{\partial_r,\partial_t}, g_{1,i\partial_t} : H_1 \to \mathbb{R}$ scaling like $\rho^{-\frac{4}{p-1}-3}$ and $\rho^{-\frac{4}{p-1}-2}$ respectively obeying the potential identity (6-8) and the energy conservation law (6-11), where J_r is defined by (6-7); also, we require $g_{\partial_r,\partial_t}$ to be odd in r, and $g_{1,i\partial_t}$ to be even in r.

It is clear from (6-8) how one should construct $g_{1,i\partial_t}$; namely one should set

$$g_{1,i\partial_t} := (p+1)v - \tfrac{1}{2}\Big(\partial_r^2 + \frac{d-1}{r}\partial_r\Big)g_{1,1} + g_{\partial_r,\partial_r} + (d-1)g_{\partial_\omega,\partial_\omega}.$$

Clearly $g_{1,i\partial_t}$ is smooth on H_1 and even in r, thanks to (vii). It is clear from the scaling laws for v, $g_{1,1}$, $g_{\partial_r,\partial_r}$, $g_{\partial_\omega,\partial_\omega}$ that the field $g_{1,i\partial_t}$ scales like $\rho^{-\frac{4}{p-1}-2}$ as required, and the identity (6-8) is clear from construction.

In a similar fashion, after using (6-7) to rewrite (6-11) as

$$\partial_t(r^{d-1}E) = \partial_r(r^{d-1}g_{\partial_r,\partial_t})$$

it is clear from the fundamental theorem of calculus that we should define $g_{\partial_r,\partial_t}$ by the formula

$$g_{\partial_r,\partial_t}(t,R) := \frac{1}{R^{d-1}}\int_0^R \partial_t E(t,r)\, r^{d-1}\,dr \tag{7-6}$$

in the interior $(0,+\infty)\times\mathbb{R}$ of H_1, where we adopt the convention $\int_0^R = -\int_R^0$ when R is negative. Note that the expression $(t,R)\mapsto \int_0^R \partial_t E(t,r)\, r^{d-1}\,dr$ is smooth and vanishes to order at least d on the time axis $R=0$ when $t>0$, so the above definition of $g_{\partial_r,\partial_t}(t,R)$ extends smoothly to the entire interior of H_1 (including the time axis). Since E is even in r and scales like $\rho^{-\frac{4}{p-1}-2}$, we know $g_{\partial_r,\partial_t}$ is odd in r and scales like $\rho^{-\frac{4}{p-1}-3}$ in the interior of H_1. After defining J_r by (6-7), we see from the fundamental theorem of calculus that (6-11) is obeyed in the interior of H_1. To complete the list of requirements stated in Theorem 6.2, it will suffice to show that $g_{\partial_r,\partial_t}$ extends smoothly to the boundary component $\{(0,r) : r\neq 0\}$ of H_1. As $g_{\partial_r,\partial_t}$ is odd in r and scales like $\rho^{-\frac{4}{p-1}-3}$, it suffices to show that $t\mapsto g_{\partial_r,\partial_t}(t,1)$ can be smoothly extended to $t=0$. From (7-6) we have

$$g_{\partial_r,\partial_t}(t,1) = \partial_t \int_0^1 E(t,r)\, r^{d-1}\,dr,$$

so it will suffice to show that the function $f : t\mapsto \int_0^1 E(t,r)\, r^{d-1}\,dr$ for $t>0$ can be smoothly extended to $t=0$.

From (6-1) one has

$$E(t,r) = t^{-\frac{2}{p-1}-1} E\Big(1, \frac{r}{\sqrt{t}}\Big), \tag{7-7}$$

so from a change of variables we have

$$f(t) = t^{\frac{d-2}{2}-\frac{2}{p-1}} \int_0^{t^{-1/2}} E(1,r)\, r^{d-1}\,dr.$$

Recalling the polynomial Q_k introduced previously, we thus have $f(t) = U_1(t) + U_2(t)$, where

$$U_1(t) := t^{\frac{d-2}{2}-\frac{2}{p-1}} \int_0^{t^{-1/2}} r^{d-3-\frac{4}{p-1}} Q_k(1/r^2)\, dr,$$
$$U_2(t) := t^{\frac{d-2}{2}-\frac{2}{p-1}} \int_0^{t^{-1/2}} \big(r^{d-1}E(1,r) - r^{d-3-\frac{4}{p-1}} Q_k(1/r^2)\big)\, dr.$$

Since Q_k has degree at most k, and at most $k-1$ when equality occurs in (7-4), the expression $r^{d-3-\frac{4}{p-1}} Q_k(1/r^2)$ is a linear combination of monomials $r^{d-3-\frac{4}{p-1}-2j}$ where $0 \le j \le k$, or $0 \le j \le k-1$ when equality occurs in (7-4). In particular, from (7-4) we see that the exponent in these monomials is strictly greater than -1, so the integral is absolutely convergent. Performing the integral, we see that $U_1(t)$ is a polynomial in t and thus clearly smoothly extendible to $t = 0$. It thus remains to show that U_2 is also smoothly extendible to $t = 0$.

First suppose that strict inequality occurs in (7-4). From (7-3) we know that the integrand $r^{d-1}E(1,r) - r^{d-3-\frac{4}{p-1}} Q_k(1/r^2)$ is absolutely integrable near $r = \infty$; from the smoothness of E and the absolute integrability of the U_1 integrand we also have absolute integrability near $r = 0$. From (7-5) we thus have

$$U_2(t) = -t^{\frac{d-2}{2}-\frac{2}{p-1}} \int_{t^{-1/2}}^{\infty} \big(r^{d-1}E(1,r) - r^{d-3-\frac{4}{p-1}} Q_k(1/r^2)\big)\, dr. \tag{7-8}$$

Making the change of variables $r = 1/\sqrt{st}$ and noting from (7-7) that

$$E\left(1, \frac{1}{\sqrt{st}}\right) = E(st, 1)(st)^{\frac{2}{p-1}+1},$$

this becomes

$$U_2(t) = -\frac{1}{2} \int_0^1 \frac{E(st,1) - Q_k(st)}{s^{k+1}}\, s^{\frac{2}{p-1}-\frac{d}{2}+k}\, ds.$$

The function $(s,t) \mapsto E(st,1) - Q_k(st)$ is smooth and vanishes to order $k+1$ at $s = 0$ thanks to (7-3) and rescaling, so the factor $(E(st,1) - Q_k(st))/s^{k+1}$ is smooth in $t \in [0,1]$, uniformly in $s \in [0,1]$. By definition of k, the weight $s^{\frac{2}{p-1}-\frac{d}{2}+k}$ is absolutely integrable on $[0,1]$. From repeated differentiation under the integral sign we conclude that U_2 extends smoothly to $[0,1]$ as desired.

Now suppose that equality occurs in (7-4). Now we do not necessarily have the vanishing property (7-5), so we need to adjust (7-8) to

$$U_2(t) = At^{\frac{d-2}{2}-\frac{2}{p-1}} - t^{\frac{d-2}{2}-\frac{2}{p-1}} \int_{t^{-1/2}}^{\infty} \big(r^{d-1}E(1,r) - r^{d-3-\frac{4}{p-1}} Q_k(1/r^2)\big)\, dr$$

for some quantity A depending on E, d, p but not on t. But in this case $\frac{d-2}{2} - \frac{2}{p-1}$ is an integer, so the monomial $At^{\frac{d-2}{2}-\frac{2}{p-1}}$ clearly extends smoothly to $t = 0$. Repeating the previous arguments we then obtain the smooth extension of U_2 to $t = 0$ as required.

We have completed the derivation of Theorem 6.2 from Theorem 7.1 under the hypothesis that E is good. It remains to handle the situation in which the energy density E produced by Theorem 7.1 is not good. In this case, we will perturb the data $g_{1,1}$, $g_{\partial_r,\partial_r}$, $g_{\partial_\omega,\partial_\omega}$, $g_{1,i\partial_r}$, v provided by Theorem 7.1 to make the energy density E good, without losing any of the properties listed in Theorem 7.1.

More precisely, we will consider perturbations of the form

$$\begin{aligned}
\tilde g_{1,1} &:= g_{1,1},\\
\tilde g_{\partial_r,\partial_r} &:= g_{\partial_r,\partial_r} - (p-1)Z,\\
\tilde g_{\partial_\omega,\partial_\omega} &:= g_{\partial_\omega,\partial_\omega} - (p-1)Z,\\
\tilde g_{1,i\partial r} &:= g_{1,i\partial_r},\\
\tilde v &:= v + 2Z,
\end{aligned}$$

where $Z : H_1 \to \mathbb{R}$ is a smooth function, even in r, which vanishes on the time axis $r = 0$, and which scales like $\rho^{-\frac{4}{p-1}-2}$ to be chosen later. It is clear that this perturbed data $\tilde g_{1,1},\ \tilde g_{\partial_r,\partial_r},\ \tilde g_{\partial_\omega,\partial_\omega},\ \tilde g_{1,i\partial_r},\ \tilde v$ continues to obey the scaling properties (ii) and symmetry properties (vii) of Theorem 7.1; the conservation laws (vi) are also maintained since the densities $T_{00}, T_{0r}, T_{rr}, T_{\omega\omega}$ are completely unchanged by this perturbation. The positive definite inequalities (i) and the defocusing property (v) might not be preserved in general, but will be maintained if the perturbation Z is sufficiently small in a suitable (scale-invariant) sense which we will make precise later. Finally, from (6-6) we see that the perturbed energy density $\widetilde E$ is related to the original energy density E by the formula

$$\widetilde E = E + \bigl(2 - \tfrac12 d(p-1)\bigr)Z. \tag{7-9}$$

In the energy-critical situation $p > 1 + \frac{4}{d-2}$, we avoid the mass-critical exponent $p = 1 + \frac4d$ and so the expression $2 - \frac{d(p-1)}{2}$ appearing in (7-9) is nonzero (in fact it is positive). This gives us substantial flexibility to modify the energy density E, and in particular to perturb it to be good.

We turn to the details. First suppose that strict inequality occurs in (7-4). We let B denote the quantity

$$B := \int_0^\infty \bigl(r^{d-1}E(1,r) - r^{d-3-\frac{4}{p-1}}Q_k(1/r^2)\bigr)\,dr, \tag{7-10}$$

which is well-defined since the integrand is absolutely integrable. We introduce a smooth nonnegative even function $\psi : \mathbb{R} \to \mathbb{R}$, supported in $[-2,-1] \cup [1,2]$, and normalised so that

$$\int_0^\infty \psi(r)\, r^{d-1}\,dr = 1.$$

We let $R > 1$ be a large quantity to be chosen later, and use the perturbation

$$Z(t,r) := -\frac{B}{2 - \frac12 d(p-1)} R^{-d} t^{-\frac{2}{p-1}-1}\psi\Bigl(\frac{r}{Rt^{\frac12}}\Bigr)$$

for $t > 0$, with $Z(t,r)$ vanishing at $t = 0$. By construction, $Z : H_1 \to \mathbb{R}$ is smooth, even, and scales like $\rho^{-\frac{4}{p-1}-2}$, with the function $r \mapsto Z(1,r)$ supported on $[-2R,-R] \cup [R,2R]$ and obeying the normalisation

$$\int_0^\infty Z(t,r)\, r^{d-1} = \frac{B}{2 - \frac{d(p-1)}{2}}.$$

Comparing this with (7-9) and (7-10) we see that

$$\int_0^\infty \bigl(r^{d-1}\widetilde E(1,r) - r^{d-3-\frac{4}{p-1}}Q_k(1/r^2)\bigr)\,dr = 0$$

so that $\widetilde{E}$ is good (note that $\widetilde{E}$ obeys the same asymptotics (7-3) as E with the same polynomial Q_k). It remains to choose the parameter R so that the perturbed fields $\tilde{g}_{1,1}, \tilde{g}_{\partial_r,\partial_r}, \tilde{g}_{\partial_\omega,\partial_\omega}, \tilde{g}_{1,i\partial_r}, \tilde{v}$ continue to obey the properties (i), (v).

We begin with (v) for $\tilde{v}$. By hypothesis, the original field v is continuous, scales like $\rho^{-\frac{4}{p-1}-2}$, and everywhere positive, which (by the compactness of $H_1/T^{\mathbb{Z}}$) implies a pointwise bound

$$v(t,r) > \varepsilon\rho^{-\frac{4}{p-1}-2}$$

on H_1 for some $\varepsilon > 0$. In order for $\widetilde{V}$ to also obey (v), it thus suffices to obtain the pointwise bound

$$\frac{B}{2-\frac{d(p-1)}{2}} R^{-d} t^{-\frac{2}{p-1}-1} \psi\left(\frac{r}{Rt^{\frac{1}{2}}}\right) \leq \varepsilon\rho^{-\frac{4}{p-1}-2}.$$

On the support of $\psi(r/(Rt^{\frac{1}{2}}))$, we know t is comparable to $(R^{-1}\rho)^2$; thus the left-hand side is $O(AR^{\frac{4}{p-1}-d+2}\rho^{-\frac{4}{p-1}-2})$. As we are in the energy-supercritical situation, the exponent $\frac{4}{p-1}-d+2$ is negative, and so we obtain the required bound if R is large enough.

Similarly, from (i), scaling and compactness, we obtain the pointwise bounds

$$g_{\partial_r,\partial_r},\ g_{\partial_\omega,\partial_\omega} > \varepsilon'\rho^{-\frac{4}{p-1}-2},$$

$$g_{\partial_r,\partial_r} - \frac{\left(\frac{1}{2}\partial_r g_{1,1}\right)^2 + g_{1,i\partial_r}^2}{g_{1,1}} > \varepsilon'\rho^{-\frac{4}{p-1}-2}$$

on H_1 for some $\varepsilon' > 0$, and by arguing as before we see that these properties will be preserved by the perturbation if R is large enough. The claim follows.

Now suppose instead that (7-4) holds with equality; from energy-supercriticality this implies that k is positive. We write

$$Q_k(s) = Q_{k-1}(s) + Cs^k \tag{7-11}$$

for all $s \in \mathbb{R}$ and some real number C, where Q_{k-1} is a polynomial of degree at most $k-1$. We let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth nonnegative even function, vanishing near the origin and equal to 1 near $\pm\infty$, let $R \geq 1$ be a large parameter to be chosen later, and set

$$Z(t,r) := -\frac{C}{2-\frac{1}{2}d(p-1)}|r|^{-d} t^k \eta\left(\frac{r}{Rt^{\frac{1}{2}}}\right) \tag{7-12}$$

on H_1, with the convention that $\eta(r/(Rt^{\frac{1}{2}})) = 1$ when $t = 0$. It is clear that $Z : H_1 \to \mathbb{R}$ is smooth, even in r, and vanishing near the time axis, and as (7-4) holds with equality we have Z scaling like $\rho^{-\frac{4}{p-1}-2}$ as required. From (7-9), (7-3), (7-11), and (7-12) we have

$$\widetilde{E}(1,r) = r^{-2-\frac{4}{p-1}}(Q_{k-1}(1/r^2) + O(r^{-2k-2}))$$

as $r \to \infty$, where the implied constant in the $O(\,)$ notation can depend on R. In particular, $\widetilde{E}$ is good. It remains to show that the properties in (i) and (v) are maintained by the perturbation. By repeating the previous arguments, it suffices to ensure that one has the pointwise bound

$$\frac{C}{2-\frac{1}{2}d(p-1)}|r|^{-d}t^k\eta\left(\frac{r}{Rt^{\frac{1}{2}}}\right) \le \varepsilon\rho^{-\frac{4}{p-1}-2},$$

where $\varepsilon > 0$ is a quantity not depending on R. But on the support of $\eta(r/(Rt^{\frac{1}{2}}))$, we know $|r|$ is comparable to ρ and t is $O(r^2/R^2)$, so the right-hand side is $O(CR^{-2k}\rho^{-\frac{4}{p-1}-2})$ (since (7-4) holds with equality), and the claim follows by taking R large enough. This completes the derivation of Theorem 6.2 from Theorem 7.1.

It remains to prove Theorem 7.1. This will be the objective of the remaining sections of the paper.

8. Eliminating the potential

We now make an easy reduction by eliminating the role of the potential energy density v.

Let $d \ge 1$ and $p > 1$, and suppose we have fields $g_{1,1}, g_{\partial_r,\partial_r}, g_{\partial_\omega,\partial_\omega}, g_{1,i\partial_r}, v$ obeying the properties claimed in Theorem 7.1. If we then define the modified fields $\tilde{g}_{1,1}$, $\tilde{g}_{\partial_r,\partial_r}$, $\tilde{g}_{\partial_\omega,\partial_\omega}$, $\tilde{g}_{1,i\partial_r}$, $\tilde{v}$ by

$$\begin{aligned} \tilde{g}_{1,1} &:= g_{1,1}, \\ \tilde{g}_{\partial_r,\partial_r} &:= g_{\partial_r,\partial_r} + \tfrac{1}{2}(p-1)v, \\ \tilde{g}_{\partial_\omega,\partial_\omega} &:= g_{\partial_\omega,\partial_\omega} + \tfrac{1}{2}(p-1)v, \\ \tilde{g}_{1,i\partial_r} &:= g_{1,i\partial_r}, \\ \tilde{v} &:= 0, \end{aligned}$$

then one easily verifies that these new fields also obey the claims of Theorem 7.1, except with the defocusing property $v > 0$ replaced by $v = 0$ (note that the new fields have exactly the same stresses $T_{rr}, T_{\omega\omega}$ as the original fields). In the converse direction, it turns out that we can replace the defocusing property $v > 0$ in Theorem 7.1(v) by $v = 0$. More precisely, we can deduce Theorem 7.1 from:

Theorem 8.1 (sixth reduction). *Then there exist smooth fields $g_{1,1}$, $g_{\partial_r,\partial_r}$, $g_{\partial_\omega,\partial_\omega}$, $g_{1,i\partial_r} : H_1 \to \mathbb{R}$ obeying the following properties*:

(i) *One has the positive definite inequalities*

$$g_{1,1}, g_{\partial_\omega,\partial_\omega} > 0, \tag{8-1}$$

$$g_{\partial_r,\partial_r} > \frac{\left(\frac{1}{2}\partial_r g_{1,1}\right)^2 + g_{1,i\partial_r}^2}{g_{1,1}} \tag{8-2}$$

pointwise on H_1.

(ii) *The fields $g_{1,1}, g_{\partial_r,\partial_r}, g_{\partial_\omega,\partial_\omega}, g_{1,i\partial_r}$ scale like $\rho^{-\frac{4}{p-1}}$, $\rho^{-\frac{4}{p-1}-2}$, $\rho^{-\frac{4}{p-1}-2}$, and $\rho^{-\frac{4}{p-1}-1}$ respectively.*

(vi) *One has the mass conservation law*

$$\partial_t g_{1,1} = 2\left(\partial_r + \frac{d-1}{r}\right)g_{1,i\partial_r} \tag{8-3}$$

and momentum conservation law

$$4\Big(\partial_r + \frac{d-1}{r}\Big)g_{\partial_r,\partial_r} = 4\frac{d-1}{r}g_{\partial_\omega,\partial_\omega} + \partial_r\Big(\partial_r^2 + \frac{d-1}{r}\partial_r\Big)g_{1,1} + 2\partial_t g_{1,i\partial_r} \tag{8-4}$$

with removable singularity at $r = 0$.

(vii) *The functions* $g_{1,1}, g_{\partial_r,\partial_r}, g_{\partial_\omega,\partial_\omega}$ *are even in* r, *while* $g_{1,i\partial_r}$ *is odd in* r. *Furthermore*, $g_{\partial_r,\partial_r} - g_{\partial_\omega,\partial_\omega}$ *vanishes on the time axis* $r = 0$.

Let us now see how Theorem 8.1 implies Theorem 7.1. Suppose that $g_{1,1}, g_{\partial_r,\partial_r}, g_{\partial_\omega,\partial_\omega}, g_{1,i\partial_r}, v$ obeys the properties claimed by Theorem 8.1. Let $\varepsilon > 0$ be a small quantity to be chosen later, and introduce the modified fields

$$\begin{aligned}
\tilde g_{1,1} &:= g_{1,1}\\
\tilde g_{\partial_r,\partial_r} &:= g_{\partial_r,\partial_r} - \frac{p-1}{2}\varepsilon\rho^{-\frac{4}{p-1}-2}\\
\tilde g_{\partial_\omega,\partial_\omega} &:= g_{\partial_\omega,\partial_\omega} - \frac{p-1}{2}\varepsilon\rho^{-\frac{4}{p-1}-2}\\
\tilde g_{1,i\partial_r} &:= g_{1,i\partial_r}\\
\tilde v &:= \varepsilon\rho^{-\frac{4}{p-1}-2}.
\end{aligned}$$

The properties (ii), (v), (vii) of Theorem 7.1 are easily verified to be obeyed by these new fields. Using (8-3), (8-4) and the definitions of $T_{00}, T_{0r}, T_{rr}, T_{r\omega}$ in Theorem 7.1(vi), we see that the conservation laws (6-9), (6-10) are obeyed by the original fields $g_{1,1}, g_{\partial_r,\partial_r}, g_{\partial_\omega,\partial_\omega}, g_{1,i\partial_r}$ (with $v = 0$), and hence by the new fields $\tilde g_{1,1}, \tilde g_{\partial_r,\partial_r}, \tilde g_{\partial_\omega,\partial_\omega}, \tilde g_{1,i\partial r}, \tilde v$ since the stress-energy densities $T_{00}, T_{0r}, T_{rr}, T_{\omega\omega}$ for these new fields are identical to those for the original fields. By using compactness as in the previous section, we also see that the positive definite inequalities (i) will also be obeyed if ε is small enough, and the claim follows.

It remains to prove Theorem 8.1. This will be the objective of the remaining sections of the paper.

Remark 8.2. The reduction to the case $v = 0$ does *not* mean that the finite time blowup in Theorem 1.2 is arising from a vanishing potential $F = 0$, and indeed such a vanishing is not possible since the linear Schrödinger equation will not create singularities in finite time from smooth, compactly supported data. Instead, the $v = 0$ case roughly speaking corresponds to the case where $F(x)$ is very close to zero when x lies in the range of the solution map $u : H_d \to \mathbb{C}^m$, but is allowed to be much larger than zero elsewhere; in particular, $\nabla F(x)$ does not need to vanish or be small on the range of u.

9. Eliminating the radial stress

Having eliminated the potential energy density v from the problem, we now turn our attention to eliminating the radial stress $g_{\partial_r,\partial_r}$. To motivate this reduction, assume for the moment that the hypotheses and conclusions of Theorem 8.1 hold. Multiplying the momentum conservation law (8-4) by $\frac{1}{4}r^{d-1}$, we arrive at the identity

$$\partial_r(r^{d-1}g_{\partial_r,\partial_r}) = S_2 \tag{9-1}$$

where $S_2 : H_1 \to \mathbb{R}$ is the function

$$S_2 := (d-1)r^{d-2} g_{\partial_\omega,\partial_\omega} + S_1 \tag{9-2}$$

and $S_1 : H_1 \to \mathbb{R}$ is the function

$$S_1 := \tfrac{1}{4} r^{d-1}\Big(\partial_r\Big(\partial_r^2 + \frac{d-1}{r}\partial_r\Big) g_{1,1} + 2\partial_t g_{1,i\partial_r}\Big). \tag{9-3}$$

As $r^{d-1} g_{\partial_r,\partial_r}$ vanishes on the time axis $r=0$, we can therefore solve for $g_{\partial_r,\partial_r}(t,R)$ for (t,R) in the interior of H_1 by the formula

$$g_{\partial_r,\partial_r}(t,R) := \frac{1}{R^{d-1}} \int_0^R S_2(t,r)\,dr,$$

noting that the right-hand side has a removable singularity at $R=0$ since the integral vanishes to order at least $d-1$ there; a Taylor expansion at $R=0$ then also reveals that $g_{\partial_r,\partial_r} - g_{\partial_\omega,\partial_\omega}$ vanishes at $R=0$. However, it is not immediately clear that the right-hand side will extend smoothly to the boundary $\{(0,R) : R \neq 0\}$ of H_1, due to the singularity of the integrand at the spacetime origin. As in Section 7, this requires an additional "good" hypothesis on the asymptotic expansion of the right-hand side of (9-1). More precisely, we can deduce Theorem 8.1 from

Theorem 9.1 (seventh reduction). *Then there exist smooth fields $g_{1,1}, g_{\partial_\omega,\partial_\omega}, g_{1,i\partial_r} : H_1 \to \mathbb{R}$ obeying the following properties*:

(i) *One has the positive definite inequalities* (8-1) *pointwise on H_1.*

(ii) *The fields $g_{1,1}$, $g_{\partial_\omega,\partial_\omega}$, $g_{1,i\partial_r}$ scale like $\rho^{-\frac{4}{p-1}}$, $\rho^{-\frac{4}{p-1}-2}$, and $\rho^{-\frac{4}{p-1}-1}$ respectively.*

(vi) *One has the conservation law* (8-3) *with removable singularity at $r=0$.*

(vii) *The functions $g_{1,1}$, $g_{\partial_\omega,\partial_\omega}$ are even in r, while $g_{1,i\partial_r}$ is odd in r.*

(viii) *There is an $\varepsilon > 0$ such that one has the pointwise inequality*

$$\frac{1}{R^{d-1}} \int_0^R S_2(1,r)\,dr \geq \frac{\big(\frac{1}{2}\partial_r g_{1,1}\big)^2 + g_{1,i\partial_r}^2}{g_{1,1}}(1,R) + \varepsilon\rho(1,R)^{-\frac{4}{p-1}-2}$$

for all $R>0$, where $S_2 : H_1 \to \mathbb{R}$ is the function defined by (9-2).

(ix) *Let $k \geq -1$ be the largest integer such that*

$$d - 3 - \frac{4}{p-1} - 2k \geq 0. \tag{9-4}$$

As S_2 is smooth and scales like $\rho^{-\frac{4}{p-1}+d-4}$, there is an asymptotic of the form

$$S_2(1,r) = r^{d-4-\frac{4}{p-1}}\big(R_k(1/r^2) + O(r^{-2k-2})\big) \tag{9-5}$$

as $r \to \infty$ for some polynomial R_k of degree at most k (this forces R to vanish in the case $k=-1$, where we adopt the convention that 0 has degree $-\infty$). If strict inequality holds in (9-4), *we require* *that*

$$\int_0^\infty \big(S_2(1,r) - r^{d-4-\frac{4}{p-1}} R_k(1/r^2)\big)\, dr = 0 \tag{9-6}$$

(*note that the integrand is absolutely integrable by* (9-4), (9-5), *and the smoothness of* S_2). *If instead equality holds in* (9-4) (*which can only occur if* $k \ge 0$), *we require that the degree* k *coefficient of* R_k *vanishes, so that* R_k *actually has degree at most* $k-1$.

Let us now see how Theorem 9.1 implies Theorem 8.1. Let $d \ge 3$ and $p > 1 + \frac{4}{d-2}$, and let $g_{1,1}, g_{\partial_\omega,\partial_\omega}, g_{1,i\partial_r} : H_1 \to \mathbb{R}$ be as in Theorem 9.1. The function S_2 defined in (9-2) scales like $\rho^{-\frac{4}{p-1}+d-4}$, vanishes to order at least $d-2$ at $r=0$, and has the same parity in r as r^{d-2}. We may then define

$$g_{\partial_r,\partial_r}(t,R) := \frac{1}{R^{d-1}} \int_0^R S_2(t,r)\, dr \tag{9-7}$$

for (t,R) in the interior of H_1. The integral $\int_0^R S_1(t,r)\,dr$ vanishes to order at least $d-1$ at the time axis $R=0$, so there is a removable singularity on that axis; by Taylor expansion we see that $g_{\partial_r,\partial_r} - g_{\partial_\omega,\partial_\omega}$ vanishes. It is also easy to see that $g_{\partial_r,\partial_r}$ is even in r and scales like $\rho^{-\frac{4}{p-1}-2}$, and from the fundamental theorem of calculus we see that $g_{\partial_r,\partial_r}$ obeys (9-1). If we could show that $g_{\partial_r,\partial_r}$ extends smoothly to the boundary $\{(0,R) : R \ne 0\}$ of H_1, then from Theorem 9.1(viii) we obtain (8-2) at $(1,R)$ for all $R>0$ (with a gap of at least $\varepsilon\rho^{-\frac{4}{p-1}-2}$); using scaling, symmetry and a limiting argument we would obtain (8-2) throughout H_1, and we would obtain all the requirements for Theorem 8.1.

Thus the only remaining difficulty is to ensure the smooth extension. We argue as in Section 7. By scaling and symmetry it suffices to show that the function $t \mapsto g_{\partial_r,\partial_r}(t,1)$ extends smoothly to $t=0$. If strict inequality occurs in (9-4), then from (9-6), (9-7) we can write

$$g_{\partial_r,\partial_r}(1,R) = \frac{1}{R^{d-1}} \int_0^R r^{d-4-\frac{4}{p-1}} R_k(1/r^2)\, dr - \frac{1}{R^{d-1}} \int_R^\infty (S_2(1,r) - r^{d-4-\frac{4}{p-1}} R_k(1/r^2))\, dr,$$

and hence by rescaling

$$g_{\partial_r,\partial_r}(t,1) = t^{-\frac{2}{p-1}-1} g_{\partial_r,\partial_r}(1, t^{-\frac{1}{2}}) = Y_1(t) + Y_2(t),$$

where the functions $Y_1, Y_2 : (0,+\infty) \to \mathbb{R}$ are defined by the formulae

$$Y_1(t) := t^{\frac{d-3}{2}-\frac{2}{p-1}} \int_0^{t^{-1/2}} r^{d-4-\frac{4}{p-1}} R_k(1/r^2)\, dr,$$

$$Y_2(t) := -t^{\frac{d-3}{2}-\frac{2}{p-1}} \int_{t^{-1/2}}^\infty (S_2(1,r) - r^{d-4-\frac{4}{p-1}} R_k(1/r^2))\, dr.$$

The function Y_1 is a polynomial and thus smoothly extends to $t=0$. As for Y_2, we make the change of variables $r = (st)^{-\frac{1}{2}}$ to write

$$Y_2(t) = -\frac{1}{2} \int_0^1 \frac{S_2(st,1) - R_k(st)}{s^{k+1}}\, s^{\frac{2}{p-1}-\frac{d-3}{2}+k}\, ds.$$

As in Section 7, $(S_2(st,1)-R_k(st))/s^{k+1}$ is smooth in $t\in[0,1]$ uniformly in $s\in[0,1]$, and the weight $s^{\frac{2}{p-1}-\frac{d-3}{2}+k}$ is absolutely integrable, so we obtain a smooth extension to $t=0$ as required. The case when equality occurs in (9-4) is treated by adding a monomial term $At^{\frac{d-3}{2}-\frac{2}{p-1}}$ to Y_2 precisely as in Section 7.

It remains to prove Theorem 9.1. This will be the objective of the remaining sections of the paper.

10. Eliminating the angular stress

Now we turn to eliminating the angular stress $g_{\partial_\omega,\partial_\omega}$ from the problem. It will be natural to divide into the *stress-subcritical* case $d-3-\frac{4}{p-1}<0$, the *stress-critical* case $d-3-\frac{4}{p-1}=0$, and the *stress-supercritical* case $d-3-\frac{4}{p-1}>0$ (note that all three of these cases can occur in the energy-supercritical regime (2-1)).

Assume that all the conclusions of Theorem 9.1 are satisfied. In the stress-subcritical case $d-3-\frac{4}{p-1}<0$, the exponent k in Theorem 9.1(ix) is equal to -1; thus R_k vanishes, S_2 is absolutely integrable, and the condition (9-6) becomes

$$\int_0^\infty S_2(1,r)\,dr=0.$$

From Theorem 9.1(viii) we thus have

$$\frac{1}{R^{d-1}}\int_R^\infty S_2(1,r)\,dr\le-\frac{\left(\frac{1}{2}\partial_r g_{1,1}\right)^2+g_{1,i\partial_r}^2}{g_{1,1}}(1,R)-\varepsilon\rho(1,R)^{-\frac{4}{p-1}-2}$$

for any $R>0$. Applying (9-2), (8-1), we obtain the constraint

$$\frac{1}{R^{d-1}}\int_R^\infty S_1(1,r)\,dr\le-\frac{\left(\frac{1}{2}\partial_r g_{1,1}\right)^2+g_{1,i\partial_r}^2}{g_{1,1}}(1,R)-\varepsilon\rho(1,R)^{-\frac{4}{p-1}-2}$$

on the fields $g_{1,1},\ g_{1,i\partial_r}$ for all $R>0$. By scale invariance, we then have

$$\frac{1}{R^{d-1}}\int_R^\infty S_1(t,r)\,dr\le-\frac{\left(\frac{1}{2}\partial_r g_{1,1}\right)^2+g_{1,i\partial_r}^2}{g_{1,1}}(t,R)-\varepsilon\rho(t,R)^{-\frac{4}{p-1}-2}\tag{10-1}$$

for all $t,R>0$.

Now suppose we are in the stress-critical case $d-3-\frac{4}{p-1}=0$. Then $k=0$, and from Theorem 9.1(ix) we have

$$\lim_{r\to\infty} rS_2(1,r)=0.$$

As S_1 scales like $\rho^{\frac{4}{p-1}+d-4}=\rho^{-1}$, the limit $\lim_{r\to\infty} rS_1(1,r)$ exists; from (9-2), (8-1), we conclude the constraint

$$\lim_{r\to\infty} rS_1(1,r)\le 0.$$

Finally, in the stress-supercritical case $d-3-\frac{4}{p-1}>0$, there is no obvious way to extract a constraint on $g_{1,1},\ g_{1,i\partial_r}$ from the properties in Theorem 9.1 that involve $g_{\partial_\omega,\partial_\omega}$.

As it turns out, the obstructions listed above to eliminating $g_{\partial_\omega,\partial_\omega}$ are essentially the only ones. More precisely, Theorem 9.1 is a consequence of:

Theorem 10.1 (eighth reduction). *Then there exist smooth fields $g_{1,1}$, $g_{1,i\partial_r} : H_1 \to \mathbb{R}$ obeying the following properties*:

(i) *One has the positive definite inequality $g_{1,1} > 0$ pointwise on H_1.*

(ii) *$g_{1,1}$ and $g_{1,i\partial_r}$ scale like $\rho^{-\frac{4}{p-1}}$ and $\rho^{-\frac{4}{p-1}-1}$ respectively.*

(vi) *One has the conservation law* (8-3) *on H_1 with removable singularity at $r = 0$.*

(vii) *The function $g_{1,1}$ is even in r, while $g_{1,i\partial_r}$ is odd in r.*

(x) *In the stress-subcritical case, we have the constraint* (10-1) *for all $R, t > 0$ and some $\varepsilon > 0$, where S_1 is defined by* (9-3). *In the stress-critical case, we have the constraint*

$$\lim_{r\to\infty} rS_1(1,r) < 0.$$

In the stress-supercritical case, we impose no constraint here.

In the remainder of this section we show how Theorem 9.1 implies Theorem 8.1. Let $g_{1,1}$, $g_{1,i\partial_r}$, be as in Theorem 9.1. It will suffice to locate a smooth field $g_{\partial_\omega,\partial_\omega} : H_1 \to \mathbb{R}$, scaling like $\rho^{-\frac{4}{p-1}-2}$, even in r, and vanishing at $r = 0$, which is strictly positive and such that the function $S_2 : H_1 \to \mathbb{R}$ defined by (9-2) obeys the properties claimed in Theorem 9.1(viii), (ix).

We begin with the stress-critical case $d - 3 - \frac{4}{p-1} = 0$, which is the simplest. From Theorem 9.1(x) we can write

$$\lim_{r\to\infty} rS_1(1,r) = -c \tag{10-2}$$

for some $c > 0$. Let $\psi : \mathbb{R} \to [0,1]$ be a smooth even function, supported on $[-2,2]$, that equals 1 on $[-1,1]$, and choose

$$g_{\partial_\omega,\partial_\omega}(t,r) := \frac{c}{d-1}\rho^{1-d} + At^{\frac{1-d}{2}}\psi(r/t^2)$$

for some large $A > 0$ to be chosen later. Clearly $g_{\partial_\omega,\partial_\omega}$ is strictly positive, smooth, even in r, and scales like $\rho^{-\frac{4}{p-1}-2} = \rho^{1-d}$. From (9-2), (10-2) we see that

$$\lim_{r\to\infty} rS_1(1,r) = 0.$$

It remains to establish the property in Theorem 9.1(viii) with (say) $\varepsilon = 1$. That is to say, we need to show

$$A\frac{1}{R^{d-1}}\int_0^R r^{d-2}\psi(r)\,dr \ge f(R) \tag{10-3}$$

for all $R > 0$, where

$$f(R) := \frac{\left(\frac{1}{2}\partial_r g_{1,1}\right)^2 + g_{1,i\partial_r}^2}{g_{1,1}}(1,R) + \rho(1,R)^{-\frac{4}{p-1}-2} - \frac{1}{R^{d-1}}\int_0^R (S_1(1,r) + cr^{d-2}\rho^{1-d})\,dr.$$

The function $S_1(1,r) + cr^{d-2}\rho^{1-d}$ scales like ρ^{-1}, and by (10-2) we have

$$\lim_{r\to\infty} r(S_1(1,r) + cr^{d-2}\rho^{1-d}) = 0,$$

so we have an asymptotic of the form

$$S_1(1,r) + cr^{d-2}\rho^{1-d} = O(1/r^3)$$

as $r \to \infty$. In particular, the integral $\int_0^R (S_1(1,r) + cr^{d-2}\rho^{1-d})\,dr$ is bounded in R. The first two terms in the definition of $f(R)$ come from evaluating smooth functions scaling like $\rho^{-\frac{4}{p-1}-2} = \rho^{1-d}$ at $(1, R)$. As such we conclude a bound of the form

$$f(R) = O((1+R)^{1-d})$$

for all $R > 0$, where the implied constant does not depend on A. On the other hand, from the construction of ψ, the expression $\frac{1}{R^{d-1}} \int_0^R r^{d-2}\psi(r)\,dr$ is bounded below by $\frac{1}{d-1}$ when $R \le 1$ and by $\frac{1}{(d-1)R^{d-1}}$ for $R \ge 1$, so we obtain the required bound (10-3) by choosing A large enough.

A similar argument lets us treat the stress-supercritical case in which $d - 3 - \frac{4}{p-1} = 2k$ for some positive integer k, as follows. The function S_1 is smooth and scales like $\rho^{d-4-\frac{4}{p-1}} = \rho^{2k-1}$, and thus we have an asymptotic of the form

$$S_1(1,r) = r^{2k-1}\big(R_{k-1}(1/r^2) + c_k/r^{2k} + O(r^{-2k-2})\big) \tag{10-4}$$

as $r \to +\infty$, for some real number c_k and some polynomial R_{k-1} of degree at most $k-1$. Let $\psi : \mathbb{R} \to [0, 1]$ be a smooth cutoff as before, let $A > 0$ be a large parameter to be chosen later, and set

$$g_{\partial_\omega,\partial_\omega}(t,r) := \left(-\frac{c_k}{d-1}|r|^{1-d}t^k + A|r|^{1-d+2k}\right)(1-\psi(r/t^2)) + +At^{\frac{1-d}{2}+k}\psi(r/t^2).$$

If A is large enough, it is easy to verify that $g_{\partial_\omega,\partial_\omega}$ is strictly positive, smooth, even in r, and scales like $\rho^{-\frac{4}{p-1}-2} = \rho^{1-d+2k}$. From (9-2), (10-4) we have the asymptotic

$$S_1(1,r) = r^{2k-1}\big(A + R_{k-1}(1/r^2) + O(r^{-2k-2})\big)$$

as $r \to +\infty$.

It remains to establish the property in Theorem 9.1(viii) with (say) $\varepsilon = 1$. As before, we rewrite this desired inequality as

$$A\frac{1}{R^{d-1}} \int_0^R \big(r^{d-2}\psi(r) + r^{2k-1}(1-\psi(r))\big)\,dr \ge f_k(R) \tag{10-5}$$

for all $R > 0$, where

$$f_k(R) := \frac{\big(\frac{1}{2}\partial_r g_{1,1}\big)^2 + g_{1,i\partial_r}^2}{g_{1,1}}(1,R) + \rho(1,R)^{-\frac{4}{p-1}-2} - \frac{1}{R^{d-1}}\int_0^R \left(S_1(1,r) - c_k\frac{1-\psi(r/t^2)}{r}\right)dr.$$

As before, the first two terms $f_k(R)$ come from evaluating a smooth function scaling like $\rho^{-\frac{4}{p-1}-2} = \rho^{1-d+2k}$ at $(1, R)$, while the integrand $S_1(1,r) - c_k(1-\psi(r/t^2))/r$ is of size $O((1+r)^{2k-1})$. We conclude that

$$f_k(R) = O((1+R)^{1-d+2k})$$

(with implied constant independent of A), while from direct computation we have

$$\frac{1}{R^{d-1}} \int_0^R \big(r^{d-2}\psi(r) + r^{2k-1}(1-\psi(r))\big)\, dr \geq c(1+R)^{1-d+2k}$$

for all $R > 0$ and some quantity $c > 0$ depending on d, k, ψ. The claim then follows by taking A large enough.

It remains to prove Theorem 10.1. This will be the objective of the final section of the paper.

11. Conclusion of the argument

The mass conservation law (8-3) can be rewritten as

$$\partial_t(r^{d-1} g_{1,1}) = 2\partial_r(r^{d-1} g_{1,i\partial_r}).$$

It is thus clear that this law will be satisfied for $r \neq 0$ (with removable singularity at $r = 0$) if one uses the ansatz

$$g_{1,1} = 2r^{1-d}\partial_r(r^d W) = 2r\partial_r W + 2dW, \tag{11-1}$$

$$g_{1,i\partial_r} = r^{1-d}\partial_t(r^d W) = r\partial_t W \tag{11-2}$$

for some smooth function $W : H_1 \to \mathbb{R}$. In order to obey the conditions (i), (ii), (vii) of Theorem 10.1, we should impose the following conditions on W:

(i) One has $\partial_r(r^d W(t,r)) > 0$ for all $r > 0$ and $t \geq 0$. Furthermore, $W(1,0) > 0$.

(ii) W scales like $\rho^{-\frac{4}{p-1}}$.

(vii) W is even in r.

It is clear that if W is smooth and obeys the above properties (i), (ii), (vii), and $g_{1,1}$, $g_{1,i\partial_r}$ are then defined by (11-1), (11-2), then the properties (i), (ii), (vi), (vii) of Theorem 10.1 are satisfied. Such a function W is easy to construct, indeed one can just take $W(t,r) := \rho^{-\frac{4}{p-1}}$ (noting from the energy supercriticality hypothesis (2-1) that $d - \frac{4}{p-1} > 2 > 0$; hence the derivative

$$\partial_r(r^d W) = \left(\frac{d}{r} - \frac{4}{p-1}\frac{r^3}{\rho^4}\right) r^d W$$

is positive for $r > 0$). This already establishes Theorem 10.1 in the stress-supercritical case $d-3-\frac{4}{p-1} > 0$.

It remains to handle the stress-critical case $d-3-\frac{4}{p-1} = 0$ and the stress-subcritical case $d-3-\frac{4}{p-1} < 0$. Here the difficulty is that there is an additional constraint in Theorem 10.1(x) that needs to be satisfied. If one sets $W^0 := \rho^{-\frac{4}{p-1}}$ and defines the initial fields $g^0_{1,1}$, $g^0_{1,i\partial_r}$ by the formulae (11-1), (11-2), that is to say,

$$g^0_{1,1} = 2r\partial_r W^0 + 2dW^0, \tag{11-3}$$

$$g^0_{1,i\partial_r} = r\partial_t W^0, \tag{11-4}$$

and then defines the initial field S_1^0 by the analogue of (9-3), namely

$$S_1^0 := \tfrac{1}{4} r^{d-1}\Big(\partial_r\Big(\partial_r^2 + \frac{d-1}{r}\partial_r\Big) g_{1,1}^0 + 2\partial_t g^0_{1,i\partial_r}\Big),$$

then there is no guarantee that the constraint in Theorem 10.1(x) will be obeyed for these choices of $g_{1,1}$, $g_{1,i\partial_r}$. Instead, we select a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ supported on $[-1, 1]$, such that $\psi''(t) \geq 0$ for all $t \geq 0$ and $\psi''(t) = 1$ for $0 \leq t \leq \frac{1}{2}$, let $\delta > 0$ be an even smaller parameter, and let $W : H_1 \to \mathbb{R}$ be the function defined for all $(t, r) \in H_1$ by the formula

$$W(t,r) := W^0(t,r) - \delta^{\frac{3}{2}} \rho^{-\frac{4}{p-1}} \psi\Big(\frac{t}{\delta\rho}\Big),$$

and then define $g_{1,1},\ g_{1,i\partial_r}$, S_1 by (11-1), (11-2), (9-3). Clearly W obeys the required properties (ii) and (vii). We now claim that the property (i) also holds if δ is small enough. Note that $W(t,r)$ is equal to $W^0(t,r)$ unless $t = O(\delta\rho)$; thus it suffices to verify (i) in the regime $t = O(\delta\rho)$. By rescaling we may normalise $r = 1$ and $t = O(\delta)$. In this regime we have

$$\partial_r(r^d W(t,r)) = \partial_r(r^d W^0(t,r)) + O(\delta^{\frac{1}{2}}),$$

and from the fact that W^0 obeys (i), the quantity $\partial_r(r^d W^0(t,r))$ is bounded away from zero uniformly in δ in the regime $r = 1,\ t = O(\delta)$, so the claim follows.

We now claim that Theorem 10.1(x) holds for δ small enough. In the stress-supercritical case there is nothing to prove. In the remaining cases, we need to study the quantity $S_1(t,r)$. By construction, this quantity is equal to $S_1^0(t,r)$ except in the regime $r = O(\delta\rho)$. Now we rescale and study $S_1(t,1)$ in the regime $r = O(\delta)$. From (11-1), (11-2) we have

$$g_{1,1} = g_{1,1}^0 - 2r\delta^{\frac{3}{2}} \partial_r\Big(\rho^{-\frac{4}{p-1}}\psi\Big(\frac{t}{\delta\rho}\Big)\Big) + 2d\delta^{\frac{3}{2}}\rho^{-\frac{4}{p-1}}\psi\Big(\frac{t}{\delta\rho}\Big)$$

and

$$g_{1,i\partial_r} = g^0_{1,i\partial_r} + r\delta^{\frac{3}{2}}\partial_t\Big(\rho^{-\frac{4}{p-1}}\psi\Big(\frac{t}{\delta\rho}\Big)\Big).$$

Using the identities $\partial_t\rho = t/(2\rho^3),\ \ \partial_r\rho = r^3/\rho^3$ we can obtain the bounds

$$\partial_r^j g_{1,1} = \partial_r^j g^0_{1,1} + O(\delta^{\frac{3}{2}}), \tag{11-5}$$

$$g_{1,i\partial_r} = g^0_{1,i\partial_r} + O(\delta^{\frac{1}{2}}), \tag{11-6}$$

$$\partial_t g_{1,i\partial_r} = -2\delta^{-\frac{1}{2}}\rho^{-\frac{4}{p-1}-2}\psi''\Big(\frac{t}{\delta\rho}\Big) + O(1) \tag{11-7}$$

for $j = 0, 1, 2, 3$ in the regime $r = 1,\ t = O(\delta)$. In particular, from (9-3) we have the bounds

$$S_1(1,t) = -\delta^{-\frac{1}{2}}\rho^{-\frac{4}{p-1}-2}\psi''\Big(\frac{t}{\delta\rho}\Big) + O(1) \tag{11-8}$$

in the region $r = 1,\ t = O(\delta)$; this bound is also true in the larger range $r = 1,\ t = O(1)$ since $S_1 = S_1^0$ and $\psi'' = 0$ when t is much larger than δ. In particular, for δ small enough we have

$$\lim_{t\to 0^+} S_1(t,1) < 0;$$

as S_1 scales like $\rho^{-\frac{4}{p-1}+d-2}$, this is equivalent to which by rescaling is equivalent to

$$\lim_{r\to\infty} \rho^{\frac{4}{p-1}-d+2} S_1(1,r) < 0.$$

In the stress-critical case $d - 3 - \frac{4}{p-1} = 0$, this gives Theorem 10.1(x). Now suppose we are in the stress-subcritical case $d - 3 - \frac{4}{p-1} < 0$. From (11-8), we have the bounds

$$-\frac{1}{t}\int_0^t S_1(t',1)\left(\frac{t'}{t}\right)^{\frac{2}{p-1}-\frac{d-1}{2}} dt' \gg \delta^{-\frac{1}{2}}\left(1+\frac{t}{\delta}\right)^{\frac{d-3}{2}-\frac{2}{p-1}} - O(1)$$

for all $0 < t \le 1$; note in the stress-subcritical case that the exponent $\frac{2}{p-1} - \frac{d-1}{2}$ is at least -1. We have

$$\delta^{-\frac{1}{2}}\left(1+\frac{t}{\delta}\right)^{\frac{d-3}{2}-\frac{2}{p-1}} \gg \delta^{\frac{2}{p-1}-\frac{d-2}{2}}.$$

By energy supercriticality (2-1), the exponent here is negative, and thus if δ is small enough we have

$$-\frac{1}{t}\int_0^t S_1(t',1)\left(\frac{t'}{t}\right)^{\frac{2}{p-1}-\frac{d-1}{2}} dt' \gg \delta^{\frac{2}{p-1}-\frac{d-2}{2}}$$

for all $0 < t \le 1$. As S_1 scales like $\rho^{-\frac{4}{p-1}+d-2}$, this bound is equivalent to

$$-\frac{1}{R^{d-1}}\int_R^\infty S_1(1,r)\,dr \gg \delta^{\frac{2}{p-1}-\frac{d-2}{2}}\rho^{-\frac{4}{p-1}-2}$$

for $1 \le R < \infty$; since $S_1(1,r) = S_1^0(1,r) = O(1)$ when $0 \le R \le 1$, we conclude that this bound also holds for $0 < R < 1$ if δ is small enough. Meanwhile, from (11-5), (11-6) we have

$$\frac{\left(\frac{1}{2}\partial_r g_{1,1}\right)^2 + g_{1,i\partial_r}^2}{g_{1,1}}(t,1) = O(1)$$

for $0 < t \le 1$, and hence by rescaling

$$\frac{\left(\frac{1}{2}\partial_r g_{1,1}\right)^2 + g_{1,i\partial_r}^2}{g_{1,1}}(1,R) = O(\rho^{-\frac{4}{p-1}-2})$$

for $1 \le R < \infty$; since

$$\begin{aligned} g_{1,1}(1,R) &= g_{1,1}^0(1,R) \gg 1,\\ g_{1,i\partial_r}(1,R) &= g_{1,i\partial_r}^0(1,R) = O(1),\\ \partial_r g_{1,1}(1,R) &= \partial_r g_{1,1}^0(1,R) = O(1) \end{aligned}$$

for $0 \leq R \leq 1$, this bound also holds for $0 < R \leq 1$. We conclude that for δ small enough, the conclusion of Theorem 10.1(x) holds (with $\varepsilon = 1$) in the stress subcritical case. This covers all the cases required for Theorem 10.1, and thus (finally!) completes the proof of Theorem 1.2.

Appendix: Proof of Nash-type embedding theorem

The purpose of this appendix is to prove Proposition 5.2.

We can use the hypothesis in Proposition 5.2(iv) to make a "gauge transformation" to reduce to the case when the components $G_{1,i\partial_{x_j}}$ vanish:

Proposition A.1. *In order to prove Proposition 5.2, it suffices to do so under the additional hypothesis that $G_{1,i\partial_{x_j}}$ vanishes identically for all $j = 1, \dots, d$, and in which we now require $\alpha = 0$ in* (3-1).

We remark from (5-10) that the vanishing of $G_{1,i\partial_{x_j}}$ also implies the vanishing of $G_{\partial_{x_j},i\partial_{x_k}}$.

Proof. Let the hypotheses be as in Proposition 5.2, and let $\vec{g} : H_d \to \mathbb{R}^d$ denote the vector field

$$\vec{g} := \left(\frac{G_{1,i\partial_{x_j}}}{G_{1,1}} \right)_{j=1}^{d}.$$

From hypothesis (iv) we know that $\vec{g}$ is curl-free, so in particular

$$\int_\gamma \vec{g}(t,x) \cdot ds = 0$$

for all $t > 0$ and all closed curves γ in $\mathbb{R}^d$, where ds is the length element. Taking limits as $t \to 0$, we conclude that

$$\int_\gamma \vec{g}(0,x) \cdot ds = 0$$

for all $t > 0$ and all closed curves γ in $\mathbb{R}^d \backslash \{0\}$. In particular, $\vec{g}(0,\cdot)$ is exact, and so we can find a smooth function $P_0 : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ such that

$$\vec{g}(0,x) = \nabla P_0(x) \tag{A-1}$$

for all $x \in \mathbb{R}^d \backslash \{0\}$. Observe from (5-6) that the vector field $\vec{g}$ has the homogeneity

$$\vec{g}(4t, 2x) = \tfrac{1}{2}\vec{g}(t,x) \tag{A-2}$$

for all $(t,x) \in H_d$. In particular, (A-1) continues to hold when P_0 is replaced by the rescaling $x \mapsto P_0(2x)$. Integrating, we conclude that

$$P_0(2x) = P_0(x) + \alpha \tag{A-3}$$

for all $x \in \mathbb{R}^d \backslash \{0\}$ and some $\alpha \in \mathbb{R}$.

From (A-2) and the smoothness of $\vec{g}$ up to the boundary of H_d, we see for fixed $t \geq 0$ that one has the asymptotic

$$\vec{g}(t,x) - \vec{g}(0,x) = O(1/|x|^2)$$

as $x \to \infty$, and similarly for all spacetime derivatives of $\vec{g}$ (in fact one gains additional powers of $|x|$ with each derivative). If we then define the function $P : H_d \to \mathbb{R}$ by

$$P(t,x) := P_0(x) - \int_\gamma (\vec{g}(t,x) - \vec{g}(0,x)) \cdot ds$$

where γ is an arbitrary curve from x to ∞ in $\mathbb{R}^d \backslash \{0\}$ that is eventually linear, then we see from Stokes' theorem that P is well-defined, and it is clear from construction that P is smooth and obeys the identity

$$\vec{g}(t,x) = \nabla P(t,x)$$

for all $(t,x) \in H_d$. Furthermore, from (5-6) and (A-3) we see that

$$P(4t, 2x) = P(t,x) + \alpha \tag{A-4}$$

for all $(t,x) \in \mathbb{R}^d$.

We now introduce the "gauge transformed" matrix $G' = (G'_{D_1,D_2})_{D_1,D_2 \in \mathcal{D}}$ by setting

$$\begin{aligned}
G'_{1,1} = G'_{i,i} &:= G_{1,1}, \\
G'_{1,i} = G'_{i,1} &:= 0, \\
G'_{1,D_1} = G'_{D_1,1} = G'_{i,iD_1} = G'_{iD_1,i} &:= G_{1,D_1}, \\
G'_{1,iD_1} = G'_{iD_1,1} = -G_{i,D_1} = -G_{D_1,i} &:= G_{1,D_1} - G_{1,1} D_1 P, \\
G'_{D_1,D_2} = G'_{iD_1,iD_2} &:= G_{D_1,D_2} - G_{1,iD_2} D_1 P - G_{1,iD_1} D_2 P + (D_1 P)(D_2 P) G_{1,1}, \\
G'_{D_1,iD_2} = G_{iD_2,D_1} &:= G_{D_1,iD_2} - (D_2 P) G_{1,D_1} + (D_1 P) G_{1,D_2}
\end{aligned}$$

for $D_1, D_2 \in \mathcal{D}_\mathbb{R} \backslash \{1\}$. The motivation for this matrix is that the requirement (5-13) can be seen to be equivalent to the requirement

$$G'_{D_1,D_2}(t,x) = \langle D_1(ue^{iP})(t,x), D_2(ue^{iP})(t,x) \rangle_{\mathbb{C}^m} \tag{A-5}$$

for $D_1, D_2 \in \mathcal{D}$, as can be seen from many applications of the product and Leibniz rules.

It is easy to see that G' is smooth and real symmetric and obeys the scaling relation (5-6). We observe the identity

$$\sum_{D_1,D_2 \in \mathcal{D}} G'_{D_1,D_2} a_{D_1} a_{D_2} = \sum_{D_1,D_2 \in \mathcal{D}} G_{D_1,D_2} b_{D_1} b_{D_2}$$

for all real numbers a_D, $D \in \mathcal{D}$, where

$$\begin{aligned}
b_1 &:= a_1 - \sum_{D \in \mathcal{D}_\mathbb{R}} a_{iD} DP, \\
b_i &:= a_1 + \sum_{D \in \mathcal{D}_\mathbb{R}} a_D DP, \\
b_{D_1} &:= a_{D_1}, \\
b_{iD_1} &:= a_{iD_1}.
\end{aligned}$$

From this we see that G' is strictly positive definite, and thus obeys the property (i). Routine calculation shows that it also obeys the conditions (ii), (iii), (iv), and that the components $G'_{1,i\partial_{x_j}}$ vanish for $j =$

$1, \dots, d$. By hypothesis, we may thus find a smooth function $u' : H_d \to \mathbb{C}^m$ that is nowhere vanishing and obeys the discrete self-similarity (3-1) with α replaced by 0, such that

$$G'_{D_1,D_2}(t,x) = \langle D_1 u'(t,x), D_2 u'(t,x)\rangle_{\mathbb{C}^m}$$

for all $(t,x) \in H_d$ and all $D_1, D_2 \in \mathcal{D}$ other than $(D_1, D_2) = (\partial_t, \partial_t), (i\partial_t, i\partial_t)$. Furthermore, the function $\theta : H_d / T^{\mathbb{Z}} \to \mathbb{CP}^{m-1}$, formed by descending the map $\pi \circ u' : H_d \to \mathbb{CP}^{m-1}$ to $H_d/T^{\mathbb{Z}}$, is a smooth embedding. If we then set $u := u' e^{iP}$, one checks from the equivalence of (5-13) and (A-5) that u obeys all the properties required for Proposition 5.2. □

It remains to prove Proposition 5.2 under the additional hypothesis that $G_{1,i\partial_{x_j}} = 0$ and with the requirement $\alpha = 0$. It will be convenient to work with a reduced "basis" of components of G, in order to eliminate the various constraints between the components of G. Let $\mathcal{P} \subset \mathcal{D}^2$ denote the following set of pairs in $\mathcal{D}$:

$$\mathcal{P} := \{(1, D) : D = 1, i\partial_{x_1}, \dots, i\partial_{x_d}, i\partial_t\} \cup \{(\partial_{x_j}, \partial_{x_k}) : 1 \le j \le k \le d\} \cup \{(\partial_{x_j}, \partial_t) : 1 \le j \le d\}$$

and then define the reduction $G_{\mathcal{P}} : H_d \to \mathbb{R}^{\mathcal{P}}$ of the matrix G as

$$G_{\mathcal{P}} := (G_{D_1,D_2})_{(D_1,D_2)\in\mathcal{P}} \tag{A-6}$$

and the Gram-type matrix $G_{\mathcal{P}}[u,v] : H_d \to \mathbb{R}^{\mathcal{P}}$ of two smooth functions $u, v : H_d \to \mathbb{C}^m$ for some $m \ge 1$ by the formula

$$G_{\mathcal{P}}[u,v] := (\langle D_1 u, D_2 v\rangle_{\mathbb{C}^m})_{(D_1,D_2)\in\mathcal{P}}.$$

Observe from the hypotheses (5-7), (5-11)–(5-12) (as well as the symmetry $G_{D_1,D_2} = G_{D_2,G_1}$) on the matrix G, as well as the analogous identities (5-2)–(5-5) (as well as the symmetry $\langle D_1 u, D_2 u\rangle_{\mathbb{C}^m} = \langle D_2 u, D_1\rangle_{\mathbb{C}^m}$) on the Gram-type matrix $G[u,u]$, that if u obeyed the equations

$$G_{\mathcal{P}}[u,u] = G_{\mathcal{P}} \tag{A-7}$$

(that is to say, (5-13) holds for all $(D_1, D_2) \in \mathcal{P}$) then in fact one has (5-13) for all pairs (D_1, D_2) in $\mathcal{D}^2$ other than (∂_t, ∂_t) and $(i\partial_t, i\partial_t)$. Thus, our task reduces to that of locating a smooth, nowhere vanishing map $u : H_d \to \mathbb{C}^m$ which obeys the discrete self-similarity (3-1) and the equation (A-7).

In order to avoid technicalities involving elliptic theory for manifolds with boundary, it will be convenient to replace the half-space H_d with the punctured spacetime $\mathbb{R} \times \mathbb{R}^d \backslash \{(0,0)\}$, so that the quotient

$$M := (\mathbb{R} \times \mathbb{R}^d \backslash \{(0,0)\}) / T^{\mathbb{Z}}$$

is now a smooth compact manifold without boundary. More precisely, we will show:

Proposition A.2. *Let $G_{\mathcal{P}} = (G_{D_1,D_2})_{(D_1,D_2)\in\mathcal{P}}$ be a tuple of smooth functions*

$$G_{D_1,D_2} : \mathbb{R} \times \mathbb{R}^d \backslash \{(0,0)\} \to \mathbb{R}$$

obeying the scaling law (5-6). *Suppose also that the fields* $G_{1,i\partial_{x_j}}$ *vanish for* $j = 1, \dots, d$, *and that the* $(d+1) \times (d+1)$ *matrix*

$$(G_{D_1,D_2})_{D_1,D_2 \in \{1,\partial_{x_1},\dots,\partial_{x_d}\}} \tag{A-8}$$

is strictly positive definite on all of $\mathbb{R} \times \mathbb{R}^d \backslash \{0\}$, *where we define*

$$G_{1,\partial_{x_j}} = G_{\partial_{x_j},1} := \tfrac{1}{2} \partial_{x_j} G_{1,1}$$

for $j = 1, \dots, d$ *and*

$$G_{\partial_{x_k},\partial_{x_j}} := G_{\partial_{x_j},\partial_{x_k}}$$

for $1 \leq j < k \leq d$. *Then, if* m *is an integer that is sufficiently large depending on* d, *there exists a smooth nowhere vanishing function* $u : \mathbb{R} \times \mathbb{R}^d \backslash \{(0,0)\} \to \mathbb{C}^m$ *obeying* (3-1) *with* $\alpha = 0$ *such that the map* $\pi \circ u$ *is a smooth embedding of* M *into* $\mathbb{CP}^{m-1}$, *and such that*

$$G_{\mathcal{P}}[u,u] = G_{\mathcal{P}}$$

on all of $\mathbb{R} \times \mathbb{R}^d \backslash \{(0,0)\}$.

We now explain why Proposition A.2 gives us Proposition 5.2. Let G_{D_1,D_2}, $D_1, D_2 \in \mathcal{D}$ be as in Proposition A.2, with $G_{1,i\partial_{x_j}} = 0$. For each $D_1, D_2 \in \mathcal{D}$, the function $\rho^{\frac{4}{p-1} + \operatorname{ord}(D_1) + \operatorname{ord}(D_2)} G_{D_1,D_2}$ is T-invariant and may thus be viewed as a smooth function on the quotient space $H_d / T^{\mathbb{Z}}$. Using the extension theorem[7] of Seeley [1964], we may smoothly extend this function to the larger space $(\mathbb{R} \times \mathbb{R}^d \backslash \{(0,0)\})/T^{\mathbb{Z}}$; lifting this extension back up to $\mathbb{R} \times \mathbb{R}^d \backslash \{(0,0)\}$ and dividing by $\rho^{\frac{4}{p-1} + \operatorname{ord}(D_1) + \operatorname{ord}(D_2)}$, we obtain a smooth extension of G_{D_1,D_2} for $(D_1, D_2) \in \mathcal{P}$ from H_d to $\mathbb{R} \times \mathbb{R}^d \to \{(0,0)\}$ that continues to obey the scaling properties (5-6). Of course we can arrange matters so that one retains the symmetry property $G_{D_1,D_2} = G_{D_2,D_1}$ with this extension, as well as the vanishing property $G_{1,i\partial_{x_j}} = 0$. By continuity, the matrix (A-8) will remain strictly positive definite in an open neighbourhood of H_d. By smoothly interpolating the G_{D_1,D_2} with another set of functions for which the matrix (A-8) is strictly positive definite everywhere (while also still obeying (5-6); this is easily achieved by keeping the diagonal terms $G_{1,1}$, $G_{\partial_{x_j},\partial_{x_j}}$ large and positive), one can assume without loss of generality that (A-8) is in fact positive definite on *all* of $\mathbb{R} \times \mathbb{R}^d \backslash \{(0,0)\}$. If one now applies Proposition A.2 and then restricts back to H_d, one obtains the claim.

It remains to establish Proposition A.2. If we knew that the component $G_{1,i\partial_t}$ of G vanished (in addition to the vanishing of $G_{1,i\partial_{x_j}}$ that is already assumed), one could obtain this claim immediately from Proposition 5.1, by embedding $\mathbb{R}^m$ into $\mathbb{C}^m$ and noting that the inner products $\langle u, i\partial_{x_j} u \rangle_{\mathbb{C}^m}$ and $\langle u, i\partial_t u \rangle_{\mathbb{C}^m}$ automatically vanish if u takes values in $\mathbb{R}^m$. (In this case, we could also recover the (∂_t, ∂_t) case of (5-13).) Thus the only obstacle to address is the nonvanishing of $G_{1,i\partial_t}$. Our strategy, inspired by the usual proofs of the Nash embedding theorem, will be to modify $G_{\mathcal{P}}$ by subtracting the contribution of a suitable "short map" that is designed to mostly eliminate the $G_{1,i\partial_t}$-component (while creating only small perturbations in the remaining components of $G_{\mathcal{P}}$), and then use the perturbative argument[8] [Günther 1991] to construct a solution u for this perturbative version of $G_{\mathcal{P}}$.

[7] One can also use the classical extension theorem of Whitney [1934].

[8] One could also use the Nash–Moser iteration scheme here, although this would be more complicated technically.

We turn to the details. The map $u \mapsto G_{\mathcal{P}}[u,u]$ defined by (A-6) is quadratic in u, rather than linear. Nevertheless, it does have the following very convenient additivity property: given two maps $u_1 : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^{m_1}$ and $u_2 : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^{m_2}$ into two finite-dimensional complex vector spaces, one has the identity

$$G_{\mathcal{P}}[(u_1,u_2),(u_1,u_2)] = G_{\mathcal{P}}[u_1,u_1] + G_{\mathcal{P}}[u_2,u_2], \tag{A-9}$$

where the *pairing* $(u_1,u_2) : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^{m_1+m_2}$ of u_1,u_2 is the map defined by the formula

$$(u_1,u_2)(t,x) := (u_1(t,x), u_2(t,x))$$

where we identify $\mathbb{C}^{m_1}\times\mathbb{C}^{m_2}$ with $\mathbb{C}^{m_1+m_2}$ in the obvious fashion. Note also that if u_1,u_2 are smooth and obey (3-1) with $\alpha = 0$, then the pairing (u_1,u_2) does also; and if one of u_1,u_2 is an embedding and nowhere vanishing and the other is merely a smooth map that is allowed to vanish, then the pairing (u_1,u_2) will be an embedding that is nowhere vanishing.

Next, we (again inspired by the usual proofs of the Nash embedding theorem) define a smooth map $u : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^m$ to be *free* if, for any $(t,x) \in \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\}$, the vectors $u(t,x)$, $\partial_{x_j}u(t,x)$ (for $1 \le j \le d$), $\partial_t u(t,x)$, $\partial_{x_j}\partial_{x_k}u(t,x)$ (for $1 \le j \le k \le d$), and $\partial_{x_j}\partial_t u(t,x)$ (for $1 \le j \le d$) are all linearly independent over the complex numbers $\mathbb{C}$ in $\mathbb{C}^m$. We observe that if m is sufficiently large (depending only on d), then there is at least one free map into $\mathbb{C}^m$ that obeys the discrete self-similarity (3-1). Indeed, from the Whitney embedding theorem there is a smooth embedding $v : M \to \mathbb{R}^{m_0}$ whenever m_0 is sufficiently large depending on d. If we then define the map $w : M \to \mathbb{R}^{1+m_0+\binom{m_0}{2}}$ by the formula

$$w := \big(1, (v_j)_{1\le j\le m_0}, (v_j v_k)_{1\le j<k\le m_0}\big),$$

where $v_1,\dots,v_{m_0} : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\}/T^{\mathbb{Z}} \to \mathbb{R}$ are the components of v, then one verifies from the chain rule and the immersed nature of v that w is free over $\mathbb{R}$, and hence free over $\mathbb{C}$ if one embeds $\mathbb{R}^{1+m_0+\binom{m_0}{2}}$ into $\mathbb{C}^{1+m_0+\binom{m_0}{2}}$. If one then defines the map $u_0 : H_d \to \mathbb{C}^{1+m_0+\binom{m_0}{2}}$ by the formula

$$u_0(t,x) := \rho^{-\frac{2}{p-1}} w(\pi(t,x)),$$

we see from a further application of the chain rule that u_0 is smooth, free, nowhere vanishing, and obeys the discrete self-similarity relation (3-1). By multiplying u_0 by a sufficiently small positive constant (which does not affect the properties of u stated above), and using the compactness of M and the positive definiteness of the $(d+2)\times(d+2)$ matrix-valued function $(G_{D_1,D_2})_{D_1,D_2\in\mathcal{D}_{\mathbb{R}}}$, we can also assume that u_0 is a *short map* in the sense that the $(d+2)\times(d+2)$ matrix-valued function

$$\big(G_{D_1,D_2} - \langle D_1u_0, D_2u_0\rangle_{\mathbb{C}^{1+m_0+\binom{m_0}{2}}}\big)_{D_1,D_2\in\mathcal{D}_{\mathbb{R}}}$$

is strictly positive definite on all of $\mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\}$. Applying Proposition 5.1, we see (for m_1 sufficiently large depending on d) we may find a smooth nowhere vanishing map $u_1 : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{R}^{m_1}$ obeying the discrete self-similarity property (3-1) with $\alpha = 0$, with $u_1/\|u_1\|_{\mathbb{R}^m}$ a smooth embedding of M into S^{m_1-1}, such that

$$G_{D_1,D_2} - \langle D_1u_0, D_2u_0\rangle_{\mathbb{C}^{1+m_0+\binom{m_0}{2}}} = \langle D_1u_1, D_2u_1\rangle_{\mathbb{C}^{m_1}} \tag{A-10}$$

on $\mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\}$ for all $D_1, D_2 \in \mathcal{D}_{\mathbb{R}}$. This identity also is obeyed when $(D_1, D_2) = (1, i\partial_{x_j})$ for some $j = 1, \dots, d$, since all three terms in the identity vanish in this case. On the other hand, (A-10) can fail when $(D_1, D_2) = (1, i\partial_t)$, since $G_{1,i\partial_t}$ is not assumed to vanish. In particular, the vector-valued function

$$G_{\mathcal{P}} - G_{\mathcal{P}}[u_0, u_0] - G_{\mathcal{P}}[u_1, u_1]$$

has all components vanishing except for the $(1, i\partial_t)$-component, which is equal to $G_{1,i\partial_t}$. To address this remaining component, we proceed by the following argument. Using a smooth partition of unity, we can find a finite number $a_1, \dots, a_k : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{R}$ of smooth functions, each of which is supported in a ball of radius $\frac{1}{1000}$ in the region $\left\{(t,x) \in H_d : \frac{1}{2} \le \rho \le 2\right\}$, such that

$$1 = \sum_{n\in\mathbb{Z}} \sum_{l=1}^{k} a_l^2(T^{-n}(t,x)) \tag{A-11}$$

for all $(t,x) \in H_d$, where k depends only on d. Meanwhile, the function $\rho^{\frac{4}{p-1}+2} G_{1,i\partial_t}(t,x)$ is T-invariant and thus descends to a smooth function of $H_d/T^{\mathbb{Z}}$. This function can be written as the difference of two squares $f_+^2 - f_-^2$ for some smooth $f_\pm : H_d/T^{\mathbb{Z}} \to \mathbb{R}$ (e.g., by setting f_- to be a large positive constant and then solving for f_+); thus

$$G_{1,i\partial_t}(t,x) = \rho^{-\frac{4}{p-1}-2} f_+(\pi(t,x))^2 - \rho^{-\frac{4}{p-1}-2} f_-(\pi(t,x))^2.$$

Multiplying this with (A-11), we obtain the decomposition

$$G_{1,i\partial_t}(t,x) = \sum_{n\in\mathbb{Z}} \sum_{l=1}^{k} 2^{-(\frac{4}{p-1}+2)n} b_{l,+}^2(T^{-n}(t,x)) - 2^{-(\frac{4}{p-1}+2)n} b_{l,-}^2(T^{-n}(t,x)),$$

where

$$b_{l,\pm}(t,x) := a_l(t,x)\rho^{-\frac{2}{p-1}-1} f_\pm(\pi(t,x)).$$

Note that for fixed l, the functions $b_{l,+}^2(T^{-n}(t,x))$ have disjoint supports as n varies, and similarly for $b_{l,-}^2(T^{-n}(t,x))$.

Next, let $\varepsilon > 0$ be a small parameter to be chosen later, and let $u_{2,\varepsilon} : H_d \to \mathbb{C}^{2k}$ be the map

$$u_{2,\varepsilon}(t,x) := \left(\left(\sum_{n\in\mathbb{Z}} \varepsilon 2^{-\frac{2n}{p-1}} b_{l,+}(T^{-n}(t,x)) e^{i\left(\frac{t}{\varepsilon 4^n}\right)^2}\right)_{l=1}^{k}, -\left(\sum_{n\in\mathbb{Z}} \varepsilon 2^{-\frac{2n}{p-1}} b_{l,-}(T^{-n}(t,x)) e^{i\left(\frac{t}{\varepsilon 4^n}\right)^2}\right)_{l=1}^{k}\right).$$

One can check that $u_{2,\varepsilon}$ is smooth and obeys the discrete self-similarity property (3-1). Direct computation using (A-9) and the chain and product rules gives the identity

$$G_{\mathcal{P}} - G_{\mathcal{P}}[u_0, u_0] - G_{\mathcal{P}}[u_1, u_1] - G_{\mathcal{P}}[u_{2,\varepsilon}, u_{2,\varepsilon}] = \varepsilon^2 H_{\mathcal{P}},$$

where $H_{\mathcal{P}} = (H_{D_1,D_2})_{(D_1,D_2)\in\mathcal{P}}$ is a smooth function from H_d to $\mathbb{C}^{\mathcal{P}}$ that is independent of ε and obeys the scaling property (5-6). The precise value of $H_{\mathcal{P}}$ is not important for our purposes, but for sake of explicitness we can evaluate the components of this matrix to be given by the formulae

$$
\begin{aligned}
H_{1,1}(t,x) &= -\sum_{\pm}\sum_{n\in\mathbb{Z}}\sum_{l=1}^{k} 2^{-\frac{4}{p-1}n} b_{l,\pm}^2(T^{-n}(t,x)),\\
H_{1,i\partial_{x_j}}(t,x) &= 0,\\
H_{1,i\partial_t}(t,x) &= 0,\\
H_{\partial_{x_j},\partial_{x_{j'}}}(t,x) &= -\sum_{\pm}\sum_{n\in\mathbb{Z}}\sum_{l=1}^{k} 2^{-(\frac{4}{p-1}+2)n}(\partial_{x_j} b_{l,\pm}\partial_{x_{j'}} b_{l,\pm})(T^{-n}(t,x)),\\
H_{\partial_{x_j},\partial_t}(t,x) = H_{\partial_t,\partial_{x_j}}(t,x) &= -\sum_{\pm}\sum_{n\in\mathbb{Z}}\sum_{l=1}^{k} 2^{-(\frac{4}{p-1}+3)n}(\partial_{x_j} b_{l,\pm}\partial_t b_{l,\pm})(T^{-n}(t,x))
\end{aligned}
$$

for $j, j' = 1, \dots, d$. It is important here that the pairs (∂_t, ∂_t), $(i\partial_t, i\partial_t)$ do not appear in $\mathcal{P}$, as these would introduce terms in $H_{\mathcal{P}}$ that are of order $1/\varepsilon^4$, which is unacceptably large for our purposes.

Proposition A.2 (and hence Proposition 5.2) may now be deduced from the following perturbative claim:

Proposition A.3. *Let the notation and hypotheses be as above. If $\varepsilon > 0$ is sufficiently small, then there exists a smooth map $u_{0,\varepsilon} : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^{1+m_0+\binom{m_0}{2}}$ obeying the discrete self-similarity property* (3-1) *with $\alpha = 0$, such that*

$$G_{\mathcal{P}}[u_{0,\varepsilon}, u_{0,\varepsilon}] = G_{\mathcal{P}}[u_0, u_0] + \varepsilon^2 H_{\mathcal{P}}.$$

Indeed, one can now take u to be the tuple $u := (u_{0,\varepsilon}, u_1, u_{2,\varepsilon})$ for a sufficiently small ε, giving the claim (for m large enough). Note that as u_1 was already a smooth nonvanishing embedding, u will be also, regardless of how badly $u_{0,\varepsilon}$ and $u_{2,\varepsilon}$ vanish or fail to be an embedding.

It remains to prove Proposition A.3. In order to be able to work on the compact manifold M rather than the noncompact space $\mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\}$, it will be convenient to normalise u_0 and the differential operators in $\mathcal{D}$ and $\mathcal{P}$ to be T-invariant. More precisely, let us introduce the T-invariant vector fields

$$X_j := \rho\partial_{x_j}, \quad X_t := \rho^2\partial_t$$

on $\mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\}$ (or the quotient space M) for $j = 1, \dots, d$, where we identify vector fields with first-order differential operators in the usual fashion. We also introduce the pairs of rescaled differential operators

$$\mathcal{P}' := \{(1,1)\}\cup\{(X_j, X_k) : 1 \le j \le k \le d\}\cup\{(X_j, X_t) : 1 \le j \le d\}\cup\{(1, iX_j) : 1 \le j \le d\}\cup\{(1, iX_t)\}$$

and then define

$$G_{\mathcal{P}'}[u,v] := (\langle D_1 u, D_2 v\rangle_{\mathbb{C}^m})_{(D_1,D_2)\in\mathcal{P}'}$$

for smooth $u, v : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^m$. Note that the operators in $\mathcal{P}'$ commute with the dilation operator T; in particular, if u is T-invariant, then so is $G_{\mathcal{P}'}[u,u]$.

Proposition A.3 is then a consequence of:

Proposition A.4. *Let m be a positive integer. Let $u : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^m$ be a smooth map which is T-invariant and free, and let $H_{\mathcal{P}'} : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{R}$ be smooth and T-invariant. Then, if $\varepsilon > 0$ is small enough, there exists a smooth map $u_\varepsilon : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^m$ that is smooth and T-invariant such that*

$$G_{\mathcal{P}'}[u_\varepsilon, u_\varepsilon] = G_{\mathcal{P}'}[u,u] + \varepsilon^2 H_{\mathcal{P}'}. \tag{A-12}$$

To see why Proposition A.4 implies Proposition A.3, we observe that if $u_0 : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^m$ is smooth and obeys (3-1) with $\alpha = 0$, and we set $u : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^m$ to be the map $u := \rho^{\frac{2}{p-1}} u_0$, then u is T-invariant, and we have the linear relation

$$G_{\mathcal{P}'}[u,u](t,x) = S_{t,x} G_{\mathcal{P}}[u_0,u_0](t,x)$$

for some invertible linear transformation $S_{t,x} : \mathbb{R}^{\mathcal{P}} \to \mathbb{R}^{\mathcal{P}'}$. The exact form of $S_{t,x}$ is not important, but for sake of explicitness we can compute $S_{t,x}(G_{D_1,D_2})_{(D_1,D_2)\in\mathcal{P}} := (G'_{D_1,D_2})_{(D_1,D_2)\in\mathcal{P}'}$, where

$$\begin{aligned}
G'_{1,1} &:= \rho^{\frac{4}{p-1}} G_{1,1},\\
G'_{X_j,X_k} &:= \rho^{\frac{4}{p-1}}\Big(\rho^2 G_{\partial_{x_j},\partial_{x_k}} + \frac{2}{p-1}\rho(\partial_{x_j}\rho)G_{1,\partial_{x_k}} + \frac{2}{p-1}\rho(\partial_{x_k}\rho)G_{\partial_{x_j},1} + \frac{4}{(p-1)^2}(\partial_{x_j}\rho)(\partial_{x_k}\rho)G_{1,1}\Big),\\
G'_{X_j,X_t} &:= \rho^{\frac{4}{p-1}}\Big(\rho^3 G_{\partial_{x_j},\partial_t} + \frac{2}{p-1}\rho^2(\partial_{x_j}\rho)G_{1,\partial_t} + \frac{2}{p-1}\rho^2(\partial_t\rho)G_{\partial_{x_j},1} + \frac{4}{(p-1)^2}\rho(\partial_{x_j}\rho)(\partial_t\rho)G_{1,1}\Big),\\
G'_{1,iX_j} &:= \rho^{\frac{4}{p-1}}\rho G_{1,i\partial_{x_j}},\\
G'_{1,iX_t} &:= \rho^{\frac{4}{p-1}}\rho^2 G_{1,i\partial_t}.
\end{aligned}$$

Also, from the product rule we see that u_0 is free if and only if u is free. If one then applies Proposition A.4 with

$$H_{\mathcal{P}'}(t,x) := S_{t,x} H_{\mathcal{P}}(t,x)$$

(which one verifies to be T-invariant), then for ε small enough, one can find a map $u_\varepsilon : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^m$ that is smooth and T-invariant, such that

$$G_{\mathcal{P}'}[u_\varepsilon,u_\varepsilon](t,x) = S_{t,x}G_{\mathcal{P}}[u,u](t,x) + \varepsilon^2 S_{t,x}H_{\mathcal{P}}(t,x) \tag{A-13}$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\}$. If we then define $u_{0,\varepsilon} : \mathbb{R}\times\mathbb{R}^d\backslash\{(0,0)\} \to \mathbb{C}^m$ to be the map $u_{0,\varepsilon} := \rho^{-\frac{2}{p-1}} u_\varepsilon$, then $G_{\mathcal{P}'}[u_\varepsilon,u_\varepsilon](t,x) = S_{t,x}G_{\mathcal{P}}[u_{0,\varepsilon},u_{0,\varepsilon}](t,x)$, so on applying $S_{t,x}^{-1}$ to (A-13) we obtain Proposition A.3 as claimed.

It remains to prove Proposition A.4. Henceforth the reference solution u will be held fixed, as well as the range dimension m. If we write $u_\varepsilon = u + v$, then we can rewrite (A-12) as

$$L_u v = \varepsilon^2 H_{\mathcal{P}'} - G_{\mathcal{P}'}[v,v], \tag{A-14}$$

where L_u is the linear operator defined on smooth functions $u : M \to \mathbb{C}^m$ by setting $L_u v : M \to \mathbb{R}^{\mathcal{P}'}$ to be the function

$$L_u v := G_{\mathcal{P}'}[u,v] + G_{\mathcal{P}'}[v,u].$$

Our task is now to find a smooth solution $v : M \to \mathbb{C}^m$ to (A-14). In coordinates, we can expand $L_u v = ((L_u v)_{D_1,D_2})_{(D_1,D_2)\in\mathcal{P}'}$ as

$$\begin{aligned}
(L_u v)_{1,1} &:= 2\langle v, u\rangle_{\mathbb{C}^m},\\
(L_u v)_{X_j,X_k} &:= \langle X_j v, X_k u\rangle_{\mathbb{C}^m} + \langle X_j u, X_k v\rangle_{\mathbb{C}^m}\\
&= X_j\langle v, X_k u\rangle_{\mathbb{C}^m} + X_k\langle v, X_j u\rangle_{\mathbb{C}^m} - \langle v, (X_j X_k + X_k X_j)u\rangle_{\mathbb{C}^m},\\
(L_u v)_{X_j,X_t} &:= \langle X_j v, X_t u\rangle_{\mathbb{C}^m} + \langle X_j u, X_t v\rangle_{\mathbb{C}^m}\\
&= X_j\langle v, X_t u\rangle_{\mathbb{C}^m} + X_t\langle v, X_j u\rangle_{\mathbb{C}^m} - \langle v, (X_j X_t + X_t X_j)u\rangle_{\mathbb{C}^m},\\
(L_u v)_{1,iX_j} &:= \langle v, iX_j u\rangle_{\mathbb{C}^m} + \langle u, iX_j v\rangle_{\mathbb{C}^m}\\
&= 2\langle v, iX_j u\rangle_{\mathbb{C}^m} - X_j\langle v, iu\rangle_{\mathbb{C}^m},\\
(L_u v)_{1,iX_t} &:= \langle v, iX_t u\rangle_{\mathbb{C}^m} + \langle u, iX_t v\rangle_{\mathbb{C}^m}\\
&= 2\langle v, iX_t u\rangle_{\mathbb{C}^m} - X_t\langle v, iu\rangle_{\mathbb{C}^m}.
\end{aligned}$$

Observe that the components of $L_u v$ are expressed in terms of the coefficients $\langle v, Du\rangle_{\mathbb{C}^m}$, where D ranges over the collection

$$\mathcal{F} := \{1, i, X_t, iX_t\} \cup \{X_j : 1 \le j \le d\} \cup \{iX_j : 1 \le k \le d\} \cup \{X_j X_k + X_k X_j : 1 \le j \le k \le d\}$$

of T-invariant differential operators (which may thus be viewed as differential operators on M). As u is free, we see at each point in M that the vectors Du, $D \in \mathcal{F}$ are linearly independent over $\mathbb{R}$. By Cramer's rule, we may thus find smooth dual fields $w_D : M \to \mathbb{C}^m$ (depending on u), which are pointwise real linear combinations of the Du, $D \in \mathcal{F}$, such that

$$\langle w_{D_1}, D_2 u\rangle_{\mathbb{C}^m} = \delta_{D_1,D_2} \tag{A-15}$$

pointwise on M, where δ_{D_1,D_2} is the Kronecker delta (equal to 1 when $D_1 = D_2$, and zero otherwise). This provides a zeroth-order right-inverse Z_u to L_u, defined on any smooth collection $F = (F_{D_1,D_2})_{(D_1,D_2)\in\mathcal{P}'}$ of functions $F_{D_1,D_2} : M \to \mathbb{R}$ by setting $Z_u F : M \to \mathbb{C}^m$ to be the function

$$Z_u F := \tfrac{1}{2} F_{1,1} w_1 - \sum_{1\le j\le k\le d} F_{X_j,X_k} w_{X_j X_k + X_k X_j} - \sum_{j=1}^{d} F_{X_j,X_t} w_{X_j X_t + X_t X_j} - \sum_{j=1}^{d} F_{1,iX_j} w_{iX_j} - F_{1,iX_t} w_{iX_t}.$$

One can easily check from (A-15) and the expansion of L_u in coordinates that Z_u is indeed a right-inverse for L_u; that is to say,

$$L_u Z_u F = F$$

for all smooth $F : M \to \mathbb{C}^m$.

One could now try to locate a solution to (A-14) using this left-inverse by solving the equation

$$v = Z_u \varepsilon^2 H_{\mathcal{P}'} - Z_u G_{\mathcal{P}'}[v, v],$$

which would imply (A-14). Here we face the familiar problem of *loss of derivatives*, since the Gram-type operator $G_{\mathcal{P}'}$ is first-order whereas Z_u is zeroth-order. It is possible to recover this loss of derivative problem for ε small enough using the technique of Nash–Moser iteration as in [Nash 1956]. However, we instead follow the simpler approach of [Günther 1991], by obtaining a decomposition of the form

$$G_{\mathcal{P}'}[v, v] = L_u Q_0[v, v] + Q_1[v, v], \tag{A-16}$$

where Q_0, Q_1 are “zeroth-order” operators. We will then be able to use a contraction mapping argument to obtain a solution to the equation

$$v = Z_u \varepsilon^2 H_{\mathcal{P}'} - Q_0[v, v] - Z_u Q_1[v, v] \tag{A-17}$$

for ε small enough; applying L_u to both sides, we obtain a solution to (A-14) as desired.

It remains to obtain the decomposition (A-16) and solve (A-17). We will need an elliptic second-order operator $-\Delta$ on M. The precise choice of $-\Delta$ is not important, but for the sake of concreteness we will take Δ to be the Laplace–Beltrami operator on M with the Riemannian metric

$$ds^2 := \sum_{j=1}^{d} \rho^2 dx_j^2 + \rho^4 dt^2$$

(noting that the right-hand side is T-invariant and thus descends to a metric on M), with the sign chosen so that $-\Delta$ is positive semidefinite; in particular, one can define the resolvent operator $(1-\Delta)^{-1}$ on smooth functions on M. We can then expand

$$G_{\mathcal{P}'}[v, v] = -(1-\Delta)^{-1} F + (1-\Delta)^{-1} Q_2[v, v], \tag{A-18}$$

where

$$F := G_{\mathcal{P}'}[\Delta v, v] + G_{\mathcal{P}'}[v, \Delta v],$$
$$Q_2[v, v] := G_{\mathcal{P}'}[v, v] - \Delta G_{\mathcal{P}'}[v, v] + G_{\mathcal{P}'}[\Delta v, v] + G_{\mathcal{P}'}[v, \Delta v].$$

Observe from the Leibniz rule that $Q_2[v, v]$ takes the schematic form

$$Q_2[v, v] = \sum_{0 \le a, b \le 2} O(\nabla^a v \nabla^b v),$$

where the gradient ∇ is with respect to the Riemannian metric ds^2 (and the implied coefficients in the $O(\,)$ notation are smooth on M); the point is that the “carré du champ”-type expression

$$-\Delta G_{\mathcal{P}'}[v, v] + G_{\mathcal{P}'}[\Delta v, v] + G_{\mathcal{P}'}[v, \Delta v]$$

does not have any terms involving third or higher derivatives after cancelling out the top-order terms. Thus, Q_2 is a “zeroth-order operator”; for instance, it is a bounded bilinear operator on the Hölder space $C^{2,\alpha}(M)$ for any $0 < \alpha < 1$, as can be seen by classical Schauder estimates.

The components of F can be expanded using the Leibniz rule as

$$F_{1,1} = 2\langle \Delta v, v\rangle_{\mathbb{C}^m},$$
$$F_{X_j, X_k} = X_j \langle \Delta v, X_k v\rangle_{\mathbb{C}^m} + X_k \langle \Delta v, X_j v\rangle_{\mathbb{C}^m} - \langle \Delta v, (X_j X_k + X_k X_j) v\rangle_{\mathbb{C}^m},$$

$$\begin{aligned}
F_{X_j,X_t} &= X_j\langle \Delta v, X_t v\rangle_{\mathbb{C}^m} + X_t\langle \Delta v, X_j v\rangle_{\mathbb{C}^m} - \langle \Delta v, (X_jX_t + X_tX_j)v\rangle_{\mathbb{C}^m},\\
F_{1,iX_j} &= -X_j\langle \Delta v, iv\rangle_{\mathbb{C}^m} + 2\langle \Delta v, iX_j v\rangle_{\mathbb{C}^m},\\
F_{1,iX_t} &= -X_t\langle \Delta v, iv\rangle_{\mathbb{C}^m} + 2\langle \Delta v, iX_t v\rangle_{\mathbb{C}^m}.
\end{aligned}$$

Comparing this with (A-15) and the components of L_u, we can then write

$$F = L_u Q_3[v,v] + Q_4[v,v], \tag{A-19}$$

where $Q_3[v,v] : M \to \mathbb{C}^m$ is the function

$$Q_3[v,v] := \langle \Delta v, v\rangle_{\mathbb{C}^m} w_1 + \sum_{k=1}^{d} \langle \Delta v, X_k v\rangle_{\mathbb{C}^m} w_{X_k} + \langle \Delta v, X_t v\rangle_{\mathbb{C}^m} w_{X_t} + \langle \Delta v, iv\rangle_{\mathbb{C}^m} w_i$$

and $Q_4[v,v] : M \to \mathbb{R}^{\mathcal{P}'}$ is given in components as

$$\begin{aligned}
Q_4[v,v]_{1,1} &:= 0,\\
Q_4[v,v]_{X_j,X_k} &:= -\langle \Delta v, (X_jX_k + X_kX_j)v\rangle_{\mathbb{C}^m},\\
Q_4[v,v]_{X_j,X_t} &:= -\langle \Delta v, (X_jX_t + X_tX_j)v\rangle_{\mathbb{C}^m},\\
Q_4[v,v]_{1,iX_j} &:= 2\langle \Delta v, iX_j v\rangle_{\mathbb{C}^m},\\
Q_4[v,v]_{1,iX_t} &:= 2\langle \Delta v, iX_t v\rangle_{\mathbb{C}^m}.
\end{aligned}$$

Observe that, as with $Q_2[v,v]$, the expressions $Q_3[v,v]$ and $Q_4[v,v]$ both take the schematic form $\sum_{0\le a,b\le 2} O(\nabla^a v \nabla^b v)$, as they does not contain any terms involving third or higher derivatives.

Using the identity

$$\begin{aligned}
(1-\Delta)^{-1}L_u &= L_u(1-\Delta)^{-1} + (1-\Delta)^{-1}[L_u, 1-\Delta](1-\Delta)^{-1}\\
&= L_u(1-\Delta)^{-1} - (1-\Delta)^{-1}[L_u, \Delta](1-\Delta)^{-1},
\end{aligned}$$

where $[A,B] = AB - BA$ denotes the commutator of A, B, as well as (A-18), (A-19), we obtain an expansion of the form (A-16) with

$$\begin{aligned}
Q_0[v,v] &:= -(1-\Delta)^{-1} Q_3[v,v],\\
Q_1[v,v] &:= (1-\Delta)^{-1}(Q_2[v,v] - Q_4[v,v]) + (1-\Delta)^{-1}[L_u,\Delta](1-\Delta)^{-1}Q_3[v,v].
\end{aligned}$$

Observe that the commutator $[L_u, \Delta]$ is a second-order differential operator on M with smooth coefficients. From Schauder theory we then conclude that (after depolarisation) Q_0, Q_1 are bounded bilinear operators on the Hölder space $C^{2,\alpha}(M)$ for any fixed $0 < \alpha < 1$. As such, the contraction mapping theorem then guarantees a solution v to (A-17) in the function space $C^{2,\alpha}(M)$ if ε is sufficiently small (depending on u and α). We are almost done, except that we have not established that v is smooth. However, from further application of Schauder theory one can establish estimates of the form

$$\|Q_i[v,v]\|_{C^{k,\alpha}(M)} \le C_{u,\alpha}\|v\|_{C^{k,\alpha}(M)}\|v\|_{C^{2,\alpha}(M)} + C_{k,u,\alpha}\|v\|^2_{C^{k-1,\alpha}(M)}$$

for any $k \geq 2$ and $i = 1, 2$, where the quantities $C_{u,\alpha}, C_{k,u,\alpha}$ depend only on the subscripted parameters. Crucially, the leading constant $C_{u,\alpha}$ is independent of k. As such, a routine induction argument shows that if ε is sufficiently small (depending on u and α, but not on k) that all the iterates used in the contraction mapping theorem to construct v, and hence v itself, are bounded in $C^{k,\alpha}(M)$ for any given $k \geq 2$, and so v is smooth as required. This (finally!) completes the proof of Proposition 5.2.

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A SUBLINEAR VERSION OF SCHUR'S LEMMA AND ELLIPTIC PDE

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We study the weighted norm inequality of $(1, q)$-type,

$$\|\boldsymbol{G}\nu\|_{L^q(\Omega, d\sigma)} \le C\|\nu\| \quad \text{for all } \nu \in \mathscr{M}^+(\Omega),$$

along with its weak-type analogue, for $0 < q < 1$, where $\boldsymbol{G}$ is an integral operator associated with the nonnegative kernel G on $\Omega \times \Omega$. Here $\mathscr{M}^+(\Omega)$ denotes the class of positive Radon measures in Ω; $\sigma, \nu \in \mathscr{M}^+(\Omega)$, and $\|\nu\| = \nu(\Omega)$.

For both weak-type and strong-type inequalities, we provide conditions which characterize the measures σ for which such an embedding holds. The strong-type $(1, q)$-inequality for $0 < q < 1$ is closely connected with existence of a positive function u such that $u \ge \boldsymbol{G}(u^q\sigma)$, i.e., a supersolution to the integral equation

$$u - \boldsymbol{G}(u^q\sigma) = 0, \quad u \in L^q_{\text{loc}}(\Omega, \sigma).$$

This study is motivated by solving sublinear equations involving the fractional Laplacian,

$$(-\Delta)^{\frac{\alpha}{2}}u - u^q\sigma = 0,$$

in domains $\Omega \subseteq \mathbb{R}^n$ which have a positive Green function G for $0 < \alpha < n$.

1. Introduction

Let Ω be a locally compact, Hausdorff space, and $\mathscr{M}^+(\Omega)$ denote the class of all positive Radon measures (locally finite) in Ω. For a nonnegative, lower semicontinuous kernel $G : \Omega \times \Omega \to [0, +\infty]$, we denote by

$$\boldsymbol{G}\nu(x) = \int_\Omega G(x, y)\, d\nu(y), \quad x \in \Omega,$$

the potential of $\nu \in \mathscr{M}^+(\Omega)$.

Let $\sigma \in \mathscr{M}^+(\Omega)$, and let $0 < q < 1$. We study the weighted norm inequality

$$\|\boldsymbol{G}\nu\|_{L^q(\Omega,\sigma)} \le \varkappa\|\nu\| \quad \text{for all } \nu \in \mathscr{M}^+(\Omega), \tag{1-1}$$

for some positive constant $\varkappa$, where we use the notation $\|\nu\| = \nu(\Omega)$ if $\nu \in \mathscr{M}^+(\Omega)$ is a finite measure.

The main goal of this paper is to show that (1-1) is connected to existence of a measurable function u such that

$$u \ge \boldsymbol{G}(u^q\sigma), \quad 0 < u < +\infty \;\; d\sigma\text{-a.e. in } \Omega, \tag{1-2}$$

under certain assumptions on G.

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The restrictions on the kernel G studied here include that it satisfies a *weak maximum principle*, and is *quasisymmetric* (see the definitions in Section 2 below). These restrictions are satisfied by the Green kernel associated with the Laplacian, the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, and kernels associated with more general elliptic operators, see [Ancona 2002], as well as radially decreasing convolution kernels $G(x, y) = k(|x-y|)$ on $\mathbb{R}^n$ [Adams and Hedberg 1996, Section 2.6].

For such kernels G, we show that (1-1) holds if and only if there exists $u \in L^q(\Omega, \sigma)$ which satisfies (1-2). The additional condition that $u \in L^q(\Omega, \sigma)$ can be dropped using a weighted modification of (1-1) discussed below.

This equivalence provides a sublinear version of Schur's lemma for linear integral operators; see [Gagliardo 1965]. Without the restriction that G satisfies the weak maximum principle, (1-2) with $u \in L^q(\Omega, \sigma)$ does not imply in general that (1-1) holds even for positive symmetric kernels G. A counterexample is discussed in Section 7 below.

Under further mild assumptions on G (the *nondegeneracy* of the kernel; see Section 2), we establish that there exists a solution $u \in L^q(\Omega, \sigma)$ to the integral equation

$$u - \boldsymbol{G}(u^q\sigma) = 0, \quad 0 < u < +\infty \;\; d\sigma\text{-a.e. in } \Omega. \tag{1-3}$$

Such integral equations arise from the study of the sublinear elliptic boundary value problem

$$\begin{cases} -\Delta u - u^q\sigma = 0, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-4}$$

where $0 < q < 1$, $\Omega \subseteq \mathbb{R}^n$ is an open domain, and $\sigma \in L^1_{\text{loc}}(\Omega)$, or more generally $\sigma \in \mathscr{M}^+(\Omega)$.

In the following, we will consider the application of our general results to solving the equation involving the fractional Laplacian

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u - u^q\sigma = 0, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1-5}$$

Note that $(-\Delta)^{\frac{\alpha}{2}}$ is a nonlocal operator for $\alpha \neq 2k$ ($k \in \mathbb{N}$), and consequently a condition that $u = 0$ on $\partial\Omega$ is ill-posed.

If $(-\Delta)^{\frac{\alpha}{2}}$ has a nonnegative Green's kernel, then applying the Green's operator $\boldsymbol{G}$ to both sides, we obtain the equivalent problem (1-3).

It is well known that G satisfies the maximum principle in Ω in the classical case $\alpha = 2$ [Maria 1934], and for $0 < \alpha \leq 2$ [Frostman 1950; Fuglede 1960]. For the case $2 < \alpha < n$, we can consider Green's kernels G for nice domains $\Omega \subseteq \mathbb{R}^n$, such as the balls or half-spaces, where the Green's kernel is known to be positive, quasimetrically modifiable, and consequently satisfies the weak maximum principle, which is enough for our purposes; see [Frazier et al. 2014].

In particular, for the entire space $\Omega = \mathbb{R}^n$, the Green's kernel is the Newtonian kernel if $\alpha = 2$, $n \geq 3$, and the Riesz kernel of order α if $0 < \alpha < n$. Sublinear equations of the type (1-5) in this case were treated earlier in [Cao and Verbitsky 2015; 2016; 2017].

For the weighted norm inequality (1-1), we show that it holds if and only if the associated integral equation has a nontrivial supersolution, and actually a solution in a slightly more specific setup.

Theorem 1.1. *Let $\sigma \in \mathscr{M}^+(\Omega)$ and $0 < q < 1$. Suppose G is a lower semicontinuous, quasisymmetric kernel which satisfies the weak maximum principle. Then the following statements are equivalent*:

(1) *There exists a positive constant $\varkappa = \varkappa(\sigma, G)$ such that*

$$\|\boldsymbol{G}\nu\|_{L^q(\Omega,\sigma)} \le \varkappa\,\|\nu\| \quad \text{for all } \nu \in \mathscr{M}^+(\Omega).$$

(2) *There exists a supersolution $u \in L^q(\Omega, d\sigma)$ such that* (1-2) *holds.*

(3) *There exists a solution $u \in L^q(\Omega, d\sigma)$ to* (1-3) *provided additionally that G is nondegenerate with respect to σ.*

To some degree, the class of measures σ for which (1-1) holds, and consequently those measures for which there is a positive supersolution u, can be understood in terms of energy norms of the type

$$\|\boldsymbol{G}\sigma\|^s_{L^s(\Omega,\sigma)} = \int_\Omega (\boldsymbol{G}\sigma)^s \, d\sigma < +\infty \tag{1-6}$$

for certain values of $s > 0$. This condition with $s = r/(1-q)$ characterizes the existence of supersolutions $u \in L^r(\Omega, \sigma)$ satisfying (1-2) in the case $r > q$, and is equivalent to the corresponding (p, r)-inequality

$$\|\boldsymbol{G}(f\, d\sigma)\|_{L^r(\Omega,\sigma)} \le C\,\|f\|_{L^p(\Omega,\sigma)} \quad \text{for all } f \in L^p(\Omega, \sigma), \tag{1-7}$$

if $0 < r < p$ and $p > 1$; see [Verbitsky 2017].

In the case of Riesz potentials on $\Omega = \mathbb{R}^n$, weighted norm inequalities (1-7) for $0 < r < p$ and $p > 1$ were studied earlier in [Cascante et al. 2006; Maz'ya 2011; Verbitsky 1999].

This study is concerned in a sense with the end-point case of (1-7) corresponding to $p = 1$ and $0 < r = q < 1$, where it is more natural to use $\mathscr{M}^+(\Omega)$ in place of $L^1(\Omega, \sigma)$ as in (1-1). We have the following result.

Theorem 1.2. *Let $\sigma \in \mathscr{M}^+(\Omega)$ and $0 < q < 1$. Suppose G is a quasisymmetric, nondegenerate kernel which satisfies the weak maximum principle*:

(1) *If* (1-1) *holds, then $\boldsymbol{G}\sigma \in L^{\frac{q}{1-q}}(\Omega, \sigma)$.*

(2) *If $\boldsymbol{G}\sigma \in L^{\frac{q}{1-q}, q}(\Omega, \sigma)$, then* (1-1) *holds.*

Here $L^{s,q}$ is the corresponding Lorentz space; see [Stein and Weiss 1971].

In Lemma 5.1 below, we will show that, without the assumption that G satisfies the weak maximum principle, condition (1-6) with $s = q/(1-q)$ is necessary for the existence of a (super)solution $u \in L^q(\Omega, \sigma)$ only if $q \in (0, q_0]$, where

$$q_0 = \tfrac{1}{2}(\sqrt{5} - 1) = 0.61\ldots$$

denotes the conjugate golden ratio. In the case $q \in (q_0, 1)$, the optimal value of s in (1-6) is $s = 1 + q$, provided σ is a finite measure. For general measures σ, the existence of a positive solution $u \in L^q(\Omega, \sigma)$ does not guarantee that (1-6) holds if $s = q/(1-q)$ and $q \in (q_0, 1)$, or $s \ne q/(1-q)$ for all $q \in (0, 1)$, even for symmetric nondegenerate kernels G (see Section 7).

Another characterization of (1-1) can be deduced from [Maurey 1974] (see also [Pisier 1986]): it is equivalent to the existence of a nonnegative function $F \in L^1(\Omega, \sigma)$ which satisfies

$$\sup_{y \in \Omega} \int_\Omega G(x, y)\, F(x)^{1-\frac{1}{q}}\, d\sigma(x) < +\infty.$$

This is a dual reformulation of (1-1), which does not require G to satisfy the weak maximum principle. In the discrete case where Ω consists of a finite number of points, it represents the duality of the two basic concave programming problems; see [Berge and Ghouila-Houri 1965, Section 5.7].

These characterizations have focused on the sublinear case $0 < q < 1$. Note that in the case $q \geq 1$, obviously (1-1) holds if and only if

$$\sup_{y \in \Omega} \int_\Omega G(x, y)^q\, d\sigma(x) < +\infty.$$

We also give characterizations of the weak-type $(1, q)$-inequality

$$\|\boldsymbol{G}\nu\|_{L^{q,\infty}(\Omega, d\sigma)} \leq C\,\|\nu\| \quad \text{for all } \nu \in \mathscr{M}^+(\Omega), \tag{1-8}$$

for any $q > 0$, in terms of energy estimates, as well as capacities (see Section 6 below). Some results of this type were discussed in [Quinn and Verbitsky 2017] under more restrictive assumptions on the kernel G, along with analogous characterizations of both strong-type and weak-type $(1, q)$-inequalities involving fractional maximal operators and Carleson measure inequalities for the Poisson kernel.

In Section 3, we demonstrate how to remove the extra assumption imposed in Theorem 1.1 that a (super)solution u is in $L^q(\Omega, \sigma)$ globally. We prove the following theorem where we only assume that $u \in L^q_{\mathrm{loc}}(\sigma)$, or equivalently, $0 < u < +\infty$ $d\sigma$-a.e., provided the kernel G satisfies a weak form of the complete maximum principle, or alternatively if G is a quasimetric kernel (see definitions in Sections 2 and 3).

With a special function m satisfying $0 < m < +\infty$ $d\sigma$-a.e., known as a *modifier*, see, e.g., [Frazier et al. 2014; Hansen and Netuka 2012], we can modify the kernel G, so that the modified kernel

$$K(x, y) = \frac{G(x, y)}{m(x)\, m(y)}, \quad x, y \in \Omega, \tag{1-9}$$

satisfies the weak maximum principle. This makes it possible to apply Theorem 1.1 with K in place of G, and consider $u \in L^q_{\mathrm{loc}}(\sigma)$. A typical modifier that works for general kernels G which satisfy the complete maximum principle is given by

$$g(x) = \min\{1, G(x, x_0)\}, \quad x \in \Omega, \tag{1-10}$$

where x_0 is a fixed pole in Ω [Hansen and Netuka 2012, Section 8].

Theorem 1.3. *Let $\sigma \in \mathscr{M}^+(\Omega)$ and $0 < q < 1$. Suppose G is a quasisymmetric nondegenerate kernel, continuous in the extended sense on $\Omega \times \Omega$, which either* (A) *satisfies the complete maximum principle, or* (B) *is quasimetrically modifiable with modifier given by* (1-10)*. Then the following statements are equivalent*:

(1) *There exists a positive constant* $\varkappa$ *such that the weighted norm inequality*

$$\|\boldsymbol{G}\nu\|_{L^q(g\,d\sigma)} \le \varkappa \int_\Omega g\,d\nu \quad \textit{for all } \nu \in \mathscr{M}^+(\Omega), \tag{1-11}$$

holds, *where the modifier* $g(x)$ *is given by* (1-10) *for some* $x_0 \in \Omega$.

(2) *There exists a positive* (*super*)*solution* u *to the equation* $u = \boldsymbol{G}(u^q\,d\sigma)$ *such that* $u \in L^q_{\mathrm{loc}}(\sigma)$ (*or equivalently* $0 < u < +\infty$ $d\sigma$*-a.e.*)

Theorem 1.3 yields a characterization of the existence of weak solutions $u \in L^q_{\mathrm{loc}}(\sigma)$ to the fractional Laplacian equation (1-5) in general domains Ω with positive Green's function G for $0 < \alpha \le 2$, or nice domains (the entire space $\mathbb{R}^n$, or balls or half-spaces in $\mathbb{R}^n$) for $0 < \alpha < n$ as discussed above. In the classical case $\alpha = 2$, such solutions are the so-called very weak solutions to the boundary value problem (1-4) for bounded C^2-domains Ω; see, e.g., [Frazier and Verbitsky 2017; Marcus and Véron 2014].

2. Background on integral kernels

Let $G : X \times Y \to [0, +\infty]$ be a lower semicontinuous nonnegative kernel, where following the framework of Fuglede [1960; 1965], we will assume that X, Y are locally compact Hausdorff spaces. Every kernel in this paper will be assumed to be of this type, even if not stated explicitly. For most of the following, in particular in the context of strong-type $(1, q)$-weighted norm inequalities, we will be working in the case $X = Y = \Omega$.

We denote by $\mathscr{M}^+(X)$ the collection of all nonnegative, locally finite, Borel measures on X, and we write S_ν for the support of $\nu \in \mathscr{M}^+(X)$ and $\|\nu\| := \nu(X)$ when ν is a finite measure.

For $\nu \in \mathscr{M}^+(Y)$, we define the potential of ν by

$$\boldsymbol{G}\nu(x) := \int_Y G(x, y)\,d\nu(y) \quad \text{for all } x \in X,$$

and for $\mu \in \mathscr{M}^+(X)$ we have the potential with the adjoint kernel

$$\boldsymbol{G}^*\mu(y) := \int_X G(x, y)\,d\mu(x) \quad \text{for all } y \in Y.$$

Let $X = Y = \Omega$, where Ω is a locally compact Hausdorff space with countable base. The operator $\boldsymbol{G}$ with kernel G on $\Omega \times \Omega$ is said to satisfy the *weak maximum principle* (with constant $h \ge 1$) provided that

$$\boldsymbol{G}\nu(x) \le M \quad \text{for all } x \in S_\nu,$$

implies

$$\boldsymbol{G}\nu(x) \le h\,M \quad \text{for all } x \in \Omega,$$

for any constant $M > 0$ and $\nu \in \mathscr{M}^+(\Omega)$.

When $h = 1$, we say that $\boldsymbol{G}$ satisfies the *strong maximum principle*.

We say that a kernel G satisfies the *complete maximum principle* with constant $h \ge 1$ if, for any $\mu, \nu \in \mathscr{M}^+(\Omega)$, and constant $c \ge 0$, the inequality

$$\boldsymbol{G}\mu(x) \le h\,[\boldsymbol{G}\nu(x) + c], \tag{2-1}$$

for all $x \in S_\mu$, implies that this inequality holds for all $x \in \Omega$, provided $\boldsymbol{G}\mu < \infty$ $d\mu$-a.e. This is a form of the *domination principle*, see [Doob 1984, Section 1.V.10], which holds for Green's kernel associated with $(-\Delta)^{\frac{\alpha}{2}}$ in the case $0 < \alpha \le 2$ with constant $h = 1$.

A kernel $G : \Omega \times \Omega \to (0, +\infty]$ is *quasisymmetric* provided there exists a positive constant a such that

$$a^{-1} G(y, x) \le G(x, y) \le a\, G(y, x) \quad \text{for all } x, y \in \Omega.$$

If G is a quasisymmetric kernel, note that we can construct a symmetric kernel G^s given by

$$G^s(x, y) := G(x, y) + G(y, x)$$

which is both symmetric and comparable to G. Indeed,

$$\Big(1 + \frac{1}{a}\Big) G(y, x) \le G^s(x, y) \le (1 + a) G(y, x), \quad x, y \in \Omega.$$

We denote the integral operator with kernel G^s by $\boldsymbol{G}^s$.

Remark 2.1. The inequality

$$\|\boldsymbol{G}\nu\|_{L^q(\Omega,\sigma)} \le \varkappa\, \|\nu\| \quad \text{for all } \nu \in \mathscr{M}^+(\Omega)$$

is equivalent to

$$\|\boldsymbol{G}^s\nu\|_{L^q(\Omega,\sigma)} \le \varkappa_a\, \|\nu\| \quad \text{for all } \nu \in \mathscr{M}^+(\Omega),$$

with only a change in the constant, so that $\varkappa_a$ depends only on $\varkappa$ and a.

Similarly, there is a supersolution u to the inequality

$$u \ge \boldsymbol{G}(u^q \sigma)$$

if and only if there is a supersolution u_s to the symmetrized inequality

$$u_s \ge \boldsymbol{G}^s(u_s^q \sigma).$$

Indeed, the first equivalence of the remark follows directly from the equivalence of G and G^s. The second equivalence can be shown by scaling u appropriately.

When $0 < q < 1$, G is a kernel on Ω, and $\sigma \in \mathscr{M}^+(\Omega)$, we are interested in *positive solutions* $u \in L^q(\sigma)$ to the integral equation

$$u = \boldsymbol{G}(u^q \sigma), \quad u > 0 \;\; d\sigma\text{-a.e. in } \Omega, \tag{2-2}$$

and *positive supersolutions* $u \in L^q(\sigma)$ to the integral inequality

$$u \ge \boldsymbol{G}(u^q \sigma), \quad u > 0 \;\; d\sigma\text{-a.e. in } \Omega. \tag{2-3}$$

In Section 3, we will discuss how to find solutions $u \in L^q_{\text{loc}}(\sigma)$ instead of $u \in L^q(\sigma)$ in the case that the kernels are quasimetrically modifiable, or satisfy the complete maximum principle. This corresponds to the so-called "very weak" solutions to the sublinear boundary value problem (1-4); see [Frazier and Verbitsky 2017; Marcus and Véron 2014].

Lemma 2.2. *Let G be a lower semicontinuous kernel on $\Omega \times \Omega$, which is nonzero along the diagonal. Let $\sigma \in \mathscr{M}^+(\Omega)$. Suppose that* (2-2) *or* (2-3) *holds, where $u < +\infty$ $d\sigma$-a.e. Then $u \in L^q_{\mathrm{loc}}(\Omega, \sigma)$.*

Proof. We consider the case where (2-3) holds. Let $K \subset \Omega$ be a compact set. Then $K_0 = K \cap \{u < +\infty\}$ is a compact set in $\Omega_0 = \Omega \cap \{u < +\infty\}$. For each $x \in K_0$, set $c_x := \min\{1, G(x,x)\}$. By lower semicontinuity, there exists an open neighborhood $U_x \subset \Omega_0$ such that $G(x,y) > \frac{1}{2} c_x > 0$ for $y \in U_x$. Since K_0 is compact, there exists a finite refinement of the collection $\{U_x\}$ which covers K_0, denoted by $\{U_{x_i}\}_{i=1}^N$. Then

$$\int_K u^q \, d\sigma = \int_{K_0} u^q \, d\sigma \le \sum_{i=1}^N \int_{U_{x_i}} u^q \, d\sigma \le \sum_{i=1}^N \frac{2}{c_i} \int_{U_{x_i}} G(x_i, y) \, u^q(y) \, d\sigma(y) \le \sum_{i=1}^N \frac{2}{c_i} u(x_i) < +\infty,$$

and thus $u \in L^q_{\mathrm{loc}}(\Omega, \sigma)$. □

For a measure $\lambda \in \mathscr{M}^+(\Omega)$, the *energy of μ* is given by

$$\mathscr{E}(\lambda) := \int_\Omega \boldsymbol{G} \lambda \, d\lambda.$$

The value of the energy of an extremal measure will be shown to be connected with the capacity. Following the convention of Fuglede [1960], we say that a kernel $G : \Omega \times \Omega \to (-\infty, +\infty]$ is *positive* if $G(x,y) \ge 0$ for every pair $(x,y) \in \Omega \times \Omega$. A kernel G is *strictly positive* if G is positive and additionally $G(x,x) > 0$ for every $x \in \Omega$. We say a kernel is *pseudopositive* if $\mathscr{E}(\mu) \ge 0$ for every measure $\mu \in \mathscr{M}^+(\Omega)$ with compact support. A kernel is *strictly pseudopositive* if $\mathscr{E}(\mu) > 0$ for every $\mu \ne 0$, $\mu \in \mathscr{M}^+(\Omega)$ with compact support. A positive kernel is obviously pseudopositive, and a kernel is strictly positive if and only if it is strictly pseudopositive [Fuglede 1960, p. 150].

The kernel G is said to be *degenerate* with respect to $\sigma \in \mathscr{M}^+(\Omega)$ provided there exists a set $A \subset \Omega$ with $\sigma(A) > 0$ and

$$G(\cdot, y) = 0 \quad d\sigma\text{-a.e. for } y \in A.$$

Otherwise, we will say that G is *nondegenerate* with respect to σ. (The notion of nondegeneracy appeared in special conditions in [Sinnamon 2002] in the context of (p,q)-inequalities for positive operators $T : L^p \to L^q$ in the case $1 < q \le p < +\infty$.) We will sometimes rule out degenerate kernels from study since the corresponding integral equations (1-3) cannot have positive solutions.

3. Modified kernels and $L^q_{\mathrm{loc}}(\sigma)$ solutions

In this section, we wish to describe how to find local solutions $u \in L^q_{\mathrm{loc}}(\sigma)$ to the equation

$$\begin{cases} u = \boldsymbol{G}(u^q \sigma) & d\sigma\text{-a.e. in } \Omega, \\ u \in L^q_{\mathrm{loc}}(\sigma) \end{cases} \tag{3-1}$$

from global solutions $v \in L^q(\omega) = L^q(\Omega, \omega)$ to the equation

$$\begin{cases} v = \boldsymbol{K}(v^q \omega) & d\omega\text{-a.e. in } \Omega, \\ v \in L^q(\omega). \end{cases} \tag{3-2}$$

Here, K is the modified kernel (1-9) with modifier (1-10) denoted by

$$g(x) = \min\{1, G(x, x_0)\},$$

where $x_0 \in \Omega$ is a fixed pole, $v := u/g$, and $d\omega := g(x)^{1+q}\, d\sigma$.

In this case, we introduce the relevant $(1, q)$-weighted norm inequalities for this section:

$$\|\boldsymbol{G}\nu\|_{L^q(g\,d\sigma)} \le \varkappa \int_\Omega g\, d\nu \quad \text{for all } \nu \in \mathscr{M}^+(\Omega), \tag{3-3}$$

$$\|\boldsymbol{K}\nu\|_{L^q(\omega)} \le \varkappa \|\nu\| \qquad \text{for all } \nu \in \mathscr{M}^+(\Omega). \tag{3-4}$$

Note that (3-4) is simply (3-3) restated with $\boldsymbol{K}$ and ω in place of $\boldsymbol{G}$ and σ.

In this section, we consider two classes of kernels — quasimetrically modifiable kernels and kernels satisfying the complete maximum principle — and show that if these kernels are modified, the modified kernels then satisfy the weak maximum principle and thus Theorem 1.1 applies when (1-1) holds with $\boldsymbol{K}$ and ω in place of $\boldsymbol{G}$ and σ. For domains $\Omega \subset \mathbb{R}^n$ satisfying the boundary Harnack principle, such as bounded Lipschitz domains and NTA domains, the Green's kernels G for the Laplacian and fractional Laplacian (in the case $0 < \alpha \le 2$) are quasimetrically modifiable. Examples of quasimetric kernels and quasimetrically modifiable kernels can be found in [Frazier et al. 2014].

We say that $d(x, y) : \Omega \times \Omega \to [0, +\infty)$ satisfies the quasimetric triangle inequality with quasimetric constant $\kappa > 0$ provided

$$d(x, y) \le \kappa[d(x, z) + d(z, y)] \tag{3-5}$$

for any $x, y, z \in \Omega$, and $d(x, y) \ne 0$ for some $x, y \in \Omega$. Without loss of generality we may assume $\kappa \ge \frac{1}{2}$. We say that G is a *quasimetric* kernel with quasimetric constant κ provided G is symmetric and $d(x, y) := 1/G(x, y)$ satisfies (3-5).

We say the kernel G is *quasimetrically modifiable* with constant κ if there exists a measurable function $m : \Omega \to (0, +\infty)$, called a modifier, such that

$$K(x, y) := \frac{G(x, y)}{m(x)\, m(y)} \tag{3-6}$$

defines a quasimetric kernel with quasimetric constant κ.

Remark 3.1. The two modifiers we will primarily work with are $G^{x_0}(x) := G(x, x_0)$ and $g(x) := \min\{1, G^{x_0}(x)\}$ for some fixed pole $x_0 \in \Omega$. Further development and discussion of quasimetric kernels can be found in [Frazier et al. 2014; Hansen 2005; Hansen and Netuka 2012; Kalton and Verbitsky 1999].

Remark 3.2. Since we wish to apply our existence theorems for supersolutions to the modified kernel K, we will sometimes require additionally that either $G(x, y)$ is continuous off the diagonal, or continuous on $\Omega \times \Omega$ in the extended sense, so that $K(x, y)$ will be lower semicontinuous.

We recall the so-called Ptolemy's inequality for quasimetric spaces [Frazier et al. 2014]: if d is a quasimetric with constant κ on Ω, then

$$d(x, z)d(y, w) \le 4\kappa^2\big[d(x, y)d(z, w) + d(y, z)d(x, w)\big] \tag{3-7}$$

for any $w, x, y, z \in \Omega$. The following lemma is immediate from (3-7); see also [Hansen and Netuka 2012, Proposition 8.1 and Corollary 8.2].

Lemma 3.3. *If G is a quasimetric kernel on Ω with quasimetric constant κ, then*

$$K(x, y) = \frac{G(x, y)}{G^{x_0}(x)G^{x_0}(y)}$$

is a quasimetric kernel on $\Omega \setminus \{x : G(x, x_0) = +\infty\}$ with quasimetric constant $4\kappa^2$.

We will need an analogous statement for modifiers g in place of G^{x_0}. (See [Hansen and Netuka 2012, Corollary 8.4], where a similar result is proved for Green's functions associated with a Brelot space.)

Lemma 3.4. *Let $x_0 \in \Omega$, and let $g(x) = \min\{1, G(x, x_0)\}$. If G is a quasimetric kernel on Ω with quasimetric constant κ, then*

$$K(x, y) = \frac{G(x, y)}{g(x)g(y)}$$

is a quasimetric kernel on $\Omega \setminus \{x : G(x, x_0) = +\infty\}$ with quasimetric constant $4\kappa^2$.

Proof. By (3-7), we have

$$\frac{1}{G(x, y)}\frac{1}{G(z, x_0)} \le 4\kappa^2\left[\frac{1}{G(x, z)}\frac{1}{G(y, x_0)} + \frac{1}{G(x, x_0)}\frac{1}{G(z, y)}\right],$$

from which it follows that

$$\frac{g(x)g(y)}{G(x, y)} \le 4\kappa^2\left[\frac{g(x)}{G(x, z)} + \frac{g(y)}{G(z, y)}\right]G(z, x_0).$$

Now we wish to consider several cases in order to replace $G(z, x_0)$ with $g(z)$. If $G(z, x_0) \le 1$, then we are done. We focus on the case where $G(z, x_0) > 1$, which implies $g(z) = 1$.

First, consider the subcase where $G(y, x_0) > 1$ and $G(x, x_0) > 1$. Then $g(x) = g(y) = 1$ and our desired result is precisely the quasimetric triangle inequality for G.

We now consider the case where $G(y, x_0) < 1$ and $G(y, x_0) \le G(x, x_0)$ (the case $G(x, x_0) < 1$ and $G(x, x_0) \le G(y, x_0)$ is similar). In this case, $g(y) = G(y, x_0)$ and $g(y) \le g(x)$. This reduces to showing

$$\frac{g(x)g(y)}{G(x, y)} \le 4\kappa^2\left[\frac{g(x)}{G(x, z)} + \frac{g(y)}{G(y, z)}\right].$$

Since $g(x) \le 1$, using the quasimetric triangle inequality for $d(x, y)$, we deduce

$$\frac{g(x)g(y)}{G(x, y)} \le \frac{g(y)}{G(x, y)} \le \kappa\left[\frac{g(y)}{G(x, z)} + \frac{g(y)}{G(y, z)}\right] \le 4\kappa^2\left[\frac{g(x)}{G(x, z)} + \frac{g(y)}{G(y, z)}\right],$$

which is the desired inequality. □

Note that, under the assumptions of Lemma 3.4, when G is finite off the diagonal, then K is a quasimetric kernel on the punctured domain $\Omega \setminus \{x_0\}$.

Lemma 3.5. *Let K be a quasimetric kernel with quasimetric constant κ. Then K satisfies the weak maximum principle with constant $h = 2\kappa$.*

Proof. For $x, y \in \Omega$, let $d(x, y) = 1/K(x, y)$. Suppose $\mu \in \mathscr{M}^+(\Omega)$ and $\boldsymbol{K}\mu(x) \le 1$ on S_μ, where we may assume without loss of generality that S_μ is a compact set in Ω. Suppose $x \in \Omega \setminus S_\mu$. Let $x' \in S_\mu$ be a point which "minimizes" (up to an $\epsilon > 0$) the quasidistance between x and S_μ. For all $y \in S_\mu$, note that

$$d(y, x') \le \kappa[d(y, x) + d(x', x)] \le (2\kappa + \epsilon)\, d(x, y).$$

This implies that $K(x, y) \le (2\kappa + \epsilon)\, K(x', y)$, and consequently

$$\boldsymbol{K}\mu(x) \le (2\kappa + \epsilon)\, \boldsymbol{K}\mu(x') \le 2\kappa + \epsilon.$$

Letting $\epsilon \to 0$, we deduce that K satisfies the weak maximum principle with constant $h = 2\kappa$. □

Lemma 3.6. *Let G be a positive kernel on Ω and let K be the modified kernel*

$$K(x, y) = \frac{G(x, y)}{g(x)\, g(y)}.$$

If G satisfies the complete maximum principle (2-1) *with constant $h \ge 1$, then K satisfies the weak maximum principle with the same constant.*

Proof. Let $\mu \in \mathscr{M}^+(\Omega)$. First, we claim that $d\nu := d\mu/g \in \mathscr{M}^+(\Omega)$. Let $F \subset \Omega$ be a compact set. By lower semicontinuity of g, it follows that $1 \ge g(x) \ge c > 0$ on F, and so $\nu(F) \le (1/c)\mu(F)$. This shows that ν is locally finite, and $S_\mu = S_\nu$.

Now suppose $\boldsymbol{K}\mu \le 1$ on S_μ. We wish to show that $\boldsymbol{K}\mu \le h$ on Ω. Notice that $\boldsymbol{G}\nu \le g(x)$ on S_ν, where $d\nu = d\mu/g$. Consequently, $\boldsymbol{G}\nu \le 1$ and $\boldsymbol{G}\nu \le \boldsymbol{G}\delta_{x_0}$ on S_ν. By the complete maximum principle with constant $h \ge 1$, it follows that $\boldsymbol{G}\nu \le h$ on Ω, and at the same time $\boldsymbol{G}\nu \le h\, \boldsymbol{G}\delta_{x_0}$ on Ω. Hence, $\boldsymbol{G}\nu \le h\, g(x)$ on Ω. Converting our expression back to terms of $\boldsymbol{K}$ and μ proves the claim. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $d\omega = g^{1+q}\, d\sigma$. It is easy to see by the definitions of $\boldsymbol{G}$, $\boldsymbol{K}$, u, v, and ω that (3-3) and (3-4) are equivalent. If (3-3) holds, then, by Theorem 1.1, there exists a solution $v \in L^q(\omega)$ to (3-2). Then by Lemma 2.2, we have $u := g v \in L^q_{\text{loc}}(\sigma)$, and u is a solution to (3-1).

Conversely, suppose $u \in L^q_{\text{loc}}(\sigma)$ is a supersolution to (3-1). Note that $v \in L^q(\omega)$ if and only if

$$\int_\Omega u(x)^q g(x)\, d\sigma(x) < +\infty \tag{3-8}$$

holds. Since $u \in L^q_{\text{loc}}(\sigma)$, we have $u(x) < +\infty$ $d\sigma$-a.e. Further,

$$\int_\Omega g(x)\, u(x)^q\, d\sigma(x) \le \int_\Omega G(x, x_0)\, u(x)^q\, d\sigma(x) \le u(x_0),$$

which establishes that $v \in L^q(\omega)$ provided $u(x_0) < +\infty$. Since (A) or (B) holds, by Lemma 3.5 and Lemma 3.6 it follows that K satisfies the weak maximum principle. Therefore, by Theorem 1.1, inequality (3-4) holds and so (3-3) holds as well. □

Remark 3.7. It follows from the proof of Theorem 1.3 that statement (1) holds for the weight $g(x) = \min\{G(x, x_0), 1\}$ with any $x_0 \in \Omega$ provided $u(x_0) < +\infty$, where u is the supersolution in statement (2).

Consequently, if statement (1) holds for at least one $x_0 \in \Omega$, then it holds for every $x_0 \in \Omega$, except possibly a set of σ-measure zero. In the case of Green's kernel associated with $(-\Delta)^{\frac{\alpha}{2}}$ in the case $0 < \alpha \leq 2$, it is easy to see that we can use any $x_0 \in \Omega$, since otherwise $u \equiv +\infty$ in Ω.

4. Summary of potential theory

A major tool in the proofs of both the strong-type and weak-type results will be the notions of capacity of a set and the associated equilibrium measure. We will start by describing potentials of kernels on $X \times Y$ used in the context of weak-type inequalities; then we will narrow our focus to kernels on $\Omega \times \Omega$ in the case $X = Y = \Omega$ having in mind applications to strong-type counterparts.

For a kernel $G : X \times Y \to [0, +\infty]$, we will be using several related notions of capacity. Let $K \subset X$ be a compact set. The initial two capacities we consider,

$$\mathrm{cap}_0(K) := \sup\{\mu(K) : \mu \in \mathscr{M}^+(K),\ \boldsymbol{G}^*\mu(y) \leq 1 \text{ for all } y \in Y\}, \tag{4-1}$$

$$\mathrm{cont}(K) := \inf\{\lambda(Y) : \lambda \in \mathscr{M}^+(Y),\ \boldsymbol{G}\lambda(x) \geq 1 \text{ for all } x \in K\}, \tag{4-2}$$

are discussed by Fuglede [1965] and Brelot [1960].

In fact, Fuglede [1965] showed that these two notions of capacity (content) coincide with the use of von Neumann's minimax theorem. The study of capacities provides characterizations of weak-type inequalities like (1-8), as we will see in Section 6.

In the case $G : \Omega \times \Omega \to [0, +\infty]$, we consider the *Wiener capacity*

$$\mathrm{cap}_1(K) := \sup\{\mu(K) : \mu \in \mathscr{M}^+(K),\ \boldsymbol{G}^*\mu(y) \leq 1 \text{ for all } y \in S_\mu\}$$

for compact sets $K \subset \Omega$.

The extremal measure μ which attains the capacity will be referred to as the *equilibrium measure*; it exists under certain assumptions on G (see Theorem 4.3 below).

Unless otherwise noted, we will work with this capacity. Note that $\mathrm{cap}_0(K) \leq \mathrm{cap}_1(K)$, and in the case where $\boldsymbol{G}$ satisfies the weak maximum principle we have $\mathrm{cap}_1(K) \leq h\,\mathrm{cap}_0(K)$. Capacity can also be computed via an extremal energy problem:

$$\mathrm{cap}_1(K) = (w[K])^{-1}$$

where

$$w[K] := \inf\{\mathscr{E}(\mu) : \mu \in \mathscr{M}^+(K),\ \mu(K) = 1\}.$$

We say that a property holds *nearly everywhere* (or n.e.) on K when the exceptional set $Z \subset K$ has capacity $\mathrm{cap}_1(Z) = 0$. The following lemmas will help us to work with sets of zero capacity.

Lemma 4.1. *If $\mu \in \mathscr{M}^+(K)$, $\mu \not\equiv 0$, and $\mathrm{cap}_1(K) = 0$, then $\boldsymbol{G}^*\mu = +\infty$ $d\mu$-a.e. in K.*

Proof. Set

$$E = \{x \in K : \boldsymbol{G}^*\mu(x) < +\infty\}.$$

Notice that $E = \bigcup_{n=1}^{\infty} F_n$, where $F_n = \{x \in K : \boldsymbol{G}^*\mu(x) \leq n\}$ is a closed set by the lower semicontinuity of G, and consequently is a compact subset of K. In particular, E is a Borel set.

Suppose that $\mathrm{cap}_1(K) = 0$. Then $\mathrm{cap}_1(F_n) = 0$, and hence $\mu(F_n) = 0$, for every $n = 1, 2, \dots,$ in view of the definition of $\mathrm{cap}_1(F_n)$. It follows that

$$\mu(E) \le \sum_{n=1}^{\infty} \mu(F_n) = 0.$$

This proves that $\boldsymbol{G}^*\mu = +\infty$ $d\mu$-a.e. on K. □

Lemma 4.2. *Let $q > 0$. Suppose $\sigma \in \mathscr{M}^+(K)$, and $\boldsymbol{G}^*(u^q\sigma) \le u$ $d\sigma$-a.e., where $\int_K u^q \, d\sigma < +\infty$ for every compact set $K \subset \Omega$. Then $d\omega := u^q \, d\sigma$ is absolutely continuous with respect to capacity; i.e.,* $\mathrm{cap}_1(K) = 0$ *yields $\omega(K) = 0$. If in addition $u > 0$ $d\sigma$-a.e. on K, where* $\mathrm{cap}_1(K) = 0$, *then $\sigma(K) = 0$.*

Proof. Suppose K is a compact set subset of Ω. Since

$$\boldsymbol{G}^*\omega \le u \quad d\sigma\text{-a.e.},$$

we deduce

$$\int_K (\boldsymbol{G}^*\omega)^q \, d\sigma \le \int_K u^q \, d\sigma = \omega(K) < \infty.$$

Hence $\sigma(\{x \in K : \boldsymbol{G}^*\omega = +\infty\}) = 0$. Since ω is absolutely continuous with respect to σ, it follows that $\omega(\{x \in K : \boldsymbol{G}^*\omega = +\infty\}) = 0$. If $\mathrm{cap}_1(K) = 0$, then by the previous lemma, $\omega(K) = 0$. This yields $\sigma(K) = 0$, unless $u = 0$ $d\sigma$-a.e. on K. □

The following result of [Fuglede 1960] will be important in deriving the inequality (1-1) from a known positive supersolution for (2-3).

Theorem 4.3. *Let G denote a symmetric, pseudopositive kernel, and K a compact set with* $\mathrm{cap}_1 K < +\infty$. *The two maxima problems*

$$\lambda(K) = \mathit{maximum} \quad (\mathit{where}\ \lambda \in \mathscr{M}^+(K),\ \boldsymbol{G}\lambda \le 1\ \mathit{on}\ S_\lambda),$$
$$2\lambda(K) - \mathscr{E}(\lambda) = \mathit{maximum} \quad (\mathit{where}\ \lambda \in \mathscr{M}^+(K)),$$

have precisely the same solutions, and the value of each of the two maxima is the Wiener capacity $\mathrm{cap}_1 K$. *The class of all solutions is compact in the vague topology on $\mathscr{M}^+$ and consists of all measures $\lambda \in \mathscr{M}^+(K)$ for which*

$$\mathscr{E}(\lambda) = \lambda(\Omega) = \mathrm{cap}_1 K.$$

The potential of any solution has the following properties:

(1) $\boldsymbol{G}\lambda(x) \ge 1$ *nearly everywhere in K.*

(2) $\boldsymbol{G}\lambda(x) \le 1$ *on S_λ.*

(3) $\boldsymbol{G}\lambda(x) = 1$ *$d\lambda$-a.e. in Ω.*

Note that the extremal measure λ in Theorem 4.3 is the equilibrium measure for the set K. We observe that the previous theorem requires the capacity of the compact set K to be finite. To deal with this requirement, we will make sure that the kernel is strictly pseudopositive.

Remark 4.4. Let G be a kernel on Ω. Then $\mathrm{cap}_1\, K < +\infty$ for every compact $K \subset \Omega$ if and only if G is strictly pseudopositive.

Indeed, see [Fuglede 1960, p. 162], since K is compact, the minimization problem

$$w(K) = \inf \mathscr{E}(\mu),$$

taken over all unit measures $\mu \in \mathscr{M}^+(K)$, attains its minimum. Therefore, $w(K) > 0$ by the strict pseudopositivity of the kernel, and thus $\mathrm{cap}_1(K) = 1/w(K) < +\infty$.

Conversely, if $\mathrm{cap}_1\, K < +\infty$ for every compact $K \subset \Omega$, then for each $x_0 \in \Omega$, we see that the point mass δ_{x_0} is the extremal measure for $w(K)$, with $G(x_0, x_0) = w(\{x_0\}) = 1/\mathrm{cap}_1\, K > 0$. This shows that the kernel is strictly positive, and therefore is strictly pseudopositive.

5. Proof of strong-type results

The proof of Theorem 1.1 is broken in parts contained within the following subsections. As shown in Section 3, we can find solutions $u \in L^q_{\mathrm{loc}}(\sigma)$ by passing to a modified kernel and determining solutions $v \in L^q(\omega)$. Going from the inequality (1-1) to supersolution (2-3) follows from a lemma due to Gagliardo [1965], see also [Szeptycki 1984], and does not require G to be quasisymmetric or to satisfy the weak maximum principle. However, the converse statement does not hold without the weak maximum principle. Indeed, we provide an example of such a kernel in Section 7.

Proof of Theorem 1.1. That (1) $\Longrightarrow$ (2) follows from Lemma 5.7 and Remark 5.5. That (2) $\Longrightarrow$ (3) follows from Lemma 5.8. The implication (3) $\Longrightarrow$ (2) is trivial, and (2) $\Longrightarrow$ (1) follows from Lemma 5.11. ☐

Energy estimates. Important to our study of the strong-type inequality (1-1) are energy conditions of the type

$$\int_\Omega (\boldsymbol{G}\sigma)^s \, d\sigma < \infty \tag{5-1}$$

for some $s > 0$. Note that when $s = 1$, we are computing the energy $\mathscr{E}(\sigma)$ introduced above. We first start with providing a proof of Theorem 1.2.

Proof of Theorem 1.2. (1) First, suppose that the strong-type inequality (1-1) holds, where G is a quasisymmetric kernel with quasisymmetry constant a. (Notice that the weak maximum principle is not used in the proof of this statement.) By Maurey's theorem [1974], (1-1) yields the existence of a nonnegative function $F \in L^1(\sigma)$, $F > 0$ $d\sigma$-a.e., so that $\|\boldsymbol{G}^*(F^{1-\frac{1}{q}}\, d\sigma)\|_{L^\infty(d\sigma)} \le 1$. Hence, by quasisymmetry of G it follows that $\|\boldsymbol{G}(F^{1-\frac{1}{q}}\, d\sigma)\|_{L^\infty(d\sigma)} \le a$, and by Hölder's inequality with exponents $1/q$ and $1/(1-q)$, we deduce

$$\begin{aligned}\boldsymbol{G}\sigma(x) &= \int_\Omega F(y)^{-1} G(x,y) F(y)\, d\sigma(y) \\ &\le [\boldsymbol{G}(F^{1-\frac{1}{q}} d\sigma)(x)]^q [\boldsymbol{G}(F\, d\sigma)(x)]^{1-q} \le a^q [\boldsymbol{G}(F\, d\sigma)(x)]^{1-q} \quad d\sigma\text{-a.e.}\end{aligned}$$

Using the preceding inequality, Hölder's inequality, and Fubini's theorem, we estimate

$$\begin{aligned}\int_\Omega (\boldsymbol{G}\sigma)^{\frac{q}{1-q}}\,d\sigma &\le a^{\frac{q^2}{1-q}}\int_\Omega [\boldsymbol{G}(F\,d\sigma)]^q F^{-1}F\,d\sigma\\ &\le a^{\frac{q^2}{1-q}}\left[\int_\Omega \boldsymbol{G}(F\,d\sigma)F^{1-\frac{1}{q}}\,d\sigma\right]^q \|F\|^{1-q}_{L^1(\sigma)}\\ &= a^{\frac{q^2}{1-q}}\left[\int_\Omega \boldsymbol{G}^*(F^{1-\frac{1}{q}}\,d\sigma)F\,d\sigma\right]^q \|F\|^{1-q}_{L^1(\sigma)} \le a^{\frac{q^2}{1-q}}\,\|F\|_{L^1(d\sigma)}<\infty.\end{aligned}$$

Thus we have established $\boldsymbol{G}\sigma \in L^{\frac{q}{1-q}}(\sigma)$ provided (1-1) holds for $q \in (0,1)$.

(2) Now, suppose that $\boldsymbol{G}\sigma \in L^{\frac{q}{1-q},q}(\sigma)$. We note that σ is absolutely continuous with respect to capacity. Indeed, suppose this were not the case; then by Lemma 4.1, $\boldsymbol{G}\sigma = +\infty$ on a set of positive σ measure. This contradicts $\|\boldsymbol{G}\sigma\|_{L^{q/(1-q),q}(\sigma)} < +\infty$. By nondegeneracy, we know $\boldsymbol{G}\sigma \not\equiv 0$ on a set of positive σ measure, and hence division by $\boldsymbol{G}\sigma$ is well defined. By duality we find

$$\begin{aligned}\|\boldsymbol{G}\nu\|^q_{L^q(\sigma)} &= \left\|\left(\frac{\boldsymbol{G}\nu}{\boldsymbol{G}\sigma}\right)^q(\boldsymbol{G}\sigma)^q\right\|_{L^1(\sigma)}\\ &\le \left\|\left(\frac{\boldsymbol{G}\nu}{\boldsymbol{G}\sigma}\right)^q\right\|_{L^{1/q,\infty}(\sigma)}\|(\boldsymbol{G}\sigma)^q\|_{L^{1/(1-q),1}(\sigma)}\\ &= \left\|\frac{\boldsymbol{G}\nu}{\boldsymbol{G}\sigma}\right\|^q_{L^{1,\infty}(\sigma)}\|\boldsymbol{G}\sigma\|^q_{L^{q/(1-q),q}(\sigma)} \le C\,\|\nu\|^q,\end{aligned}$$

where the last inequality holds by Lemma 5.10. Thus we have established the strong-type inequality (1-1). □

As the above proof shows, the energy condition is closely related to the existence of the strong-type inequality. The following lemma shows that knowing only that a supersolution exists allows us to obtain similar energy estimates. These estimates will allow us later to construct solutions to our integral equation from supersolutions.

In the next lemma, we deduce (5-1) for various values of s without assuming that (1-1) holds, and without using the weak maximum principle, for general quasisymmetric kernels G.

Let $q_0 = \frac{1}{2}(\sqrt{5}-1) = 0.61\ldots$ denote the conjugate golden ratio.

Lemma 5.1. *Suppose G is a quasisymmetric kernel on $\Omega \times \Omega$ with quasisymmetry constant a. Let $\sigma \in \mathscr{M}^+(\Omega)$. Suppose there is a positive supersolution $u \in L^q(\Omega,\sigma)$ to* (2-3).

(a) *Let $0 < q \le q_0$. Then* (5-1) *holds with $s = q/(1-q)$, and*

$$\int_\Omega (\boldsymbol{G}\sigma)^{\frac{q}{1-q}}\,d\sigma \le c\int_\Omega u^q\,d\sigma, \tag{5-2}$$

where $c = a^{\frac{q^2}{1-q}}$.

(b) *If $q_0 < q < 1$, and σ is a finite measure, then* (5-1) *holds for $0 < s \le 1+q$, and*

$$\int_\Omega (\boldsymbol{G}\sigma)^s\,d\sigma \le c\left[\int_\Omega u^q\,d\sigma\right]^{\frac{s(1-q)}{q}}[\sigma(\Omega)]^{1-\frac{s(1-q)}{q}}, \tag{5-3}$$

where $c = a^{\frac{s}{1+q}}$.

For symmetric kernels G, both (5-2) *and* (5-3) *hold with $c = 1$.*

Remark 5.2. For $q_0 < q < 1$, statement (a) generally fails. More precisely, there exists a strictly positive symmetric kernel G and measure σ such that there is a positive solution $u \in L^q(\Omega, \sigma)$ to (2-2), but $\int_\Omega (\boldsymbol{G}\sigma)^{\frac{q}{1-q}}\, d\sigma = +\infty$; see Section 7.

Remark 5.3. The exponents $s = q/(1-q)$ and $s = 1+q$ in statements (a) and (b) respectively are sharp; i.e., there exist symmetric kernels G for which (5-1) fails if $s \neq q/(1-q)$ in the case of general measures σ, and if $s > \min\{q/(1-q), 1+q\}$ in the case of finite measures σ; see Section 7 and [Verbitsky 2017].

Proof. Suppose u is a positive supersolution satisfying (2-3). Suppose

$$q \le s \le \min\left\{\frac{q}{1-q}, 1+q\right\}.$$

Let $r = s/q$. By Hölder's inequality with exponents r and $r' = r/(r-1)$,

$$\begin{aligned}\boldsymbol{G}\sigma(x) = \int_\Omega u^{\frac{q}{r}} u^{-\frac{q}{r}} G(x,y)\, d\sigma(y) &\le [\boldsymbol{G}(u^q\, d\sigma(x)]^{\frac{1}{r}} [\boldsymbol{G}(u^{-\frac{q}{r-1}}\, d\sigma)(x)]^{\frac{1}{r'}} \\ &\le [u(x)]^{\frac{1}{r}} [\boldsymbol{G}(u^{-\frac{q}{r-1}}\, d\sigma)(x)]^{\frac{1}{r'}}.\end{aligned}$$

Suppose $r' \ge s$. Using the preceding inequality, Hölder's inequality with exponents $r'/s = 1/(s-q)$ and $(r'/s)' = 1/(1+q-s)$, and Fubini's theorem, we estimate

$$\begin{aligned}\int_\Omega (\boldsymbol{G}\sigma)^s\, d\sigma &\le \int_\Omega u^q [\boldsymbol{G}(u^{-\frac{q}{r-1}}\, d\sigma)]^{s-q}\, d\sigma \\ &\le \left[\int_\Omega \boldsymbol{G}(u^{-\frac{q}{r-1}}\, d\sigma) u^q\, d\sigma\right]^{s-q} \left[\int_\Omega u^q\, d\sigma\right]^{1+q-s} \\ &= \left[\int_\Omega \boldsymbol{G}^*(u^q\, d\sigma) u^{-\frac{q}{r-1}}\, d\sigma\right]^{s-q} \left[\int_\Omega u^q\, d\sigma\right]^{1+q-s} \\ &\le a^{s-q} \left[\int_\Omega u^{1-\frac{q}{r-1}}\, d\sigma\right]^{s-q} \left[\int_\Omega u^q\, d\sigma\right]^{1+q-s}.\end{aligned}$$

Here

$$1 - \frac{q}{r-1} = \frac{s-(q+q^2)}{s-q}.$$

Setting $s = q/(1-q)$ where $q + q^2 \le 1$, so that $r = 1/(1-q)$, $r' = 1/q \ge s$ and $1 - q/(r-1) = q$, we obtain

$$\int_\Omega (\boldsymbol{G}\sigma)^{\frac{q}{1-q}}\, d\sigma \le a^{\frac{q^2}{1-q}} \int_\Omega u^q\, d\sigma$$

for all $0 < q \le q_0$.

If σ is a finite measure, $q_0 < q < 1$, $s = 1+q$, and $r = 1/q + 1$, using the preceding estimates we deduce

$$\int_\Omega (\boldsymbol{G}\sigma)^{1+q}\, d\sigma \le a \int_\Omega u^{1-q^2}\, d\sigma \le a \left[\int_\Omega u^q\, d\sigma\right]^{\frac{1-q^2}{q}} [\sigma(\Omega)]^{\frac{q+q^2-1}{q}}.$$

Hence, for $0 < s \le 1+q$, by Jensen's inequality,

$$\int_\Omega (\boldsymbol{G}\sigma)^s\, d\sigma \le \left[\int_\Omega (\boldsymbol{G}\sigma)^{1+q}\, d\sigma\right]^{\frac{s}{1+q}} [\sigma(\Omega)]^{1-\frac{s}{1+q}} \le a^{\frac{s}{1+q}} \left[\int_\Omega u^q\, d\sigma\right]^{\frac{s(1-q)}{q}} [\sigma(\Omega)]^{1-\frac{s(1-q)}{q}}. \qquad \square$$

Remark 5.4. Inequality (5-1) with $s = q/(1-q)$ is known for quasimetric kernels provided a supersolution u satisfying (2-3) exists.

Construction of supersolutions. In the following, we construct a supersolution $\phi \in L^1(\Omega, \sigma)$ to the problem

$$\phi \geq [\boldsymbol{G}(\phi \, d\sigma)]^q > 0 \quad d\sigma\text{-a.e. in } \Omega.$$

Remark 5.5. If ϕ solves the above inequality, then $u = \phi^{\frac{1}{q}}$ solves (2-3).

We then are able to use the energy estimates shown above to construct positive solutions to the integral equation (2-2) when the kernel G is nondegenerate.

The existence of supersolutions will follow from a lemma due to [Gagliardo 1965]; see also [Szeptycki 1984]. Let B be a Banach space. A convex cone $P \subset B$ is *strictly convex at the origin* if the convex combination of two elements of P equals zero only if both of those elements are zero; i.e., $\alpha\phi_1 + \beta\phi_2 = 0$ implies $\phi_1 = \phi_2 = 0$, whenever $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

Lemma 5.6 [Gagliardo 1965]. *Let B be a Banach space and let $P \subset B$ be a convex cone which is strictly convex at the origin. Let $S : P \to P$ be a continuous mapping. Assume the following conditions hold*:

(1) *If $(\phi_n) \subset P$, $\phi_{n+1} - \phi_n \in P$, and if $\|\phi_n\|_B \leq M$ for all $n = 1, 2, \ldots$, then there exists $\phi \in P$ such that $\|\phi_n - \phi\|_B \to 0$.*

(2) *For $\phi, \psi \in P$ such that $\phi - \psi \in P$, we have $S\phi - S\psi \in P$.*

(3) *If $\|\phi\|_B \leq 1$ and if $\phi \in P$, then $\|Su\|_B \leq 1$.*

Then for every $\lambda > 0$ there exists $\phi \in P$ such that $(1+\lambda)\phi - S\phi \in P$ and $0 < \|\phi\|_B \leq 1$. Moreover, for every $\psi \in P$ such that $0 < \|\psi\|_B \leq \lambda/(1+\lambda)$, we can pick ϕ so that $\phi = \psi + (1/(1+\lambda))S\phi$.

We will apply this lemma to $B = L^1(\sigma)$ and $P := \{\phi \in L^1(\sigma) : \phi \geq 0 \, d\sigma\text{-a.e.}\}$. In our case, it is easy to see that Lemma 5.6 gives not only that $\|\phi\|_B > 0$, but further that $\phi > 0$ $d\sigma$-a.e.

Lemma 5.7. *Let (Ω, σ) be a sigma-finite measure space. Suppose the strong-type inequality* (1-1) *holds. Then, for every $\lambda > 0$, there is a positive supersolution $\phi \in L^1(\sigma)$ such that*

$$\phi \geq [\boldsymbol{G}(\phi \, d\sigma)]^q$$

with $\|\phi\|_{L^1(\sigma)} \leq (1+\lambda)^{\frac{1}{1-q}} \varkappa^{\frac{q}{1-q}}$.

Proof. The supersolution ϕ_0 can be constructed using Lemma 5.6. Indeed, let $S : L^1(\sigma) \to L^1(\sigma)$ be given by

$$S\phi := \frac{1}{\varkappa^q}[\boldsymbol{G}(\phi \, d\sigma)]^q$$

for all $\phi \in L^1(\sigma)$. Inequality (1-1) gives that S is a continuous operator. Moreover, by (1-1) we can establish condition (3) of Lemma 5.6. Suppose that $\|\phi\|_{L^1(\sigma)} \leq 1$; then

$$\|S(\phi)\|_{L^1(\sigma)} = \frac{1}{\varkappa^q} \int_\Omega [\boldsymbol{G}(\phi \, d\sigma)]^q \, d\sigma \leq \frac{1}{\varkappa^q} \varkappa^q \left(\int_\Omega \phi \, d\sigma \right)^q \leq 1.$$

Therefore, by Lemma 5.6, there exists $\phi \in L^1(\sigma)$ such that

$$(1+\lambda)\phi \geq \frac{1}{\varkappa^q}[\boldsymbol{G}(\phi\, d\sigma)]^q,$$

$\|\phi\|_{L^1(\sigma)} \leq 1$, and $\phi > 0$ $d\sigma$-a.e. We can renormalize with $\phi_0 := a\phi$, with the choice

$$a := \left[\frac{1}{(1+\lambda)\varkappa^q}\right]^{\frac{1}{1-q}},$$

and see that ϕ_0 satisfies

$$\phi_0 \geq [\boldsymbol{G}(\phi_0\, d\sigma)]^q,$$

with

$$\|\phi_0\|_{L^1(\sigma)} \leq (1+\lambda)^{\frac{1}{1-q}} \varkappa^{\frac{q}{1-q}},$$

and $\phi_0 > 0$ $d\sigma$-a.e. □

Lemma 5.8. *If there exists a positive supersolution $u_0 \in L^q(\Omega, \sigma)$ satisfying* (2-3), *then there exists a positive solution $v \in L^q(\Omega, d\sigma)$ such that $v = \boldsymbol{G}(v^q\, d\sigma)$ $d\sigma$-a.e., unless G is degenerate. In the latter case of the degenerate kernel, the equation $v = \boldsymbol{G}(v^q\, d\sigma)$ does not have a positive solution $v \in L^q(\Omega, \sigma)$.*

Proof. Let $u_0 \in L^q(\Omega, \sigma)$ be the positive supersolution to (2-3). We can define by induction the nonincreasing sequence of supersolutions $\{u_n\}_{n=0}^{\infty}$ given by

$$u_{n+1} := \boldsymbol{G}[u_n^q\, d\sigma], \quad n = 0, 1, 2, \ldots,$$

where $u_n \downarrow v$, and $v \in L^q(\Omega, d\sigma)$ is a nonnegative solution by the dominated convergence theorem.

It remains to check that the solution v is positive $d\sigma$-a.e. provided the kernel is nondegenerate. This can be done by finding a lower bound on the supersolutions u_n by using Lemma 5.1 with u_n in place of u and σ_K in place of σ for an arbitrary compact set $K \subset \Omega$. Notice that by induction each $u_n > 0$ $d\sigma$-a.e. since G is nondegenerate. Consequently,

$$\int_K (\boldsymbol{G}\sigma_K)^s\, d\sigma \leq C_K \left[\int_K u_n^q\, d\sigma\right]^r,$$

where $s = \min\{q/(1-q), 1+q\}$, $r > 0$, and C_K does not depend on n. Letting $n \to +\infty$ in the preceding inequality, we deduce

$$\int_K (\boldsymbol{G}\sigma_K)^s\, d\sigma \leq C_K \left[\int_K v^q\, d\sigma\right]^r.$$

Thus, if $v = 0$ on K then $\boldsymbol{G}\sigma_K = 0$ $d\sigma$-a.e. on K, and hence $G(\cdot, y) = 0$ $d\sigma$-a.e. for $y \in K$. Hence, $\sigma(K) = 0$; that is, $v > 0$ $d\sigma$-a.e.

If the kernel is degenerate, then clearly a positive solution does not exist. Indeed, if G were degenerate, then there would exist a set K such that $\sigma(K) > 0$ and $G(x, \cdot) = 0$ $d\sigma$-a.e. for $x \in K$. This implies that, for every solution u, we have $u(x) = \int_\Omega G(x, y)\, u^q\, d\sigma(y) = 0$ for $x \in K$ $d\sigma$-a.e., which shows that a positive solution u does not exist. □

Corollary 5.9. *If inequality* (1-1) *holds, and there exists a solution $u \in L^q(\Omega, \sigma)$ to* (2-2), *it follows that*

$$\|u\|_{L^q(\Omega,\sigma)} \leq \varkappa^{\frac{1}{1-q}}.$$

Proof. By applying (1-1) to $\nu := u^q\sigma$, we get

$$\nu(\Omega) = \int_\Omega (\boldsymbol{G}\nu)^q \, d\sigma \le \varkappa^q \nu(\Omega)^q. \qquad \square$$

Derivation of inequality. In establishing a converse result, we appeal to potential theory, and in particular some results due to [Fuglede 1960]. The necessary definitions and results are summarized in Section 4.

We will need the following weak-type inequality.

Lemma 5.10. *Let G be a symmetric, nonnegative kernel satisfying a weak maximum principle. Suppose $\omega \in \mathscr{M}^+(\Omega)$ is absolutely continuous with respect to capacity. Then*

$$\left\| \frac{\boldsymbol{G}\nu}{\boldsymbol{G}\omega} \right\|_{L^{1,\infty}(\Omega,\omega)} \le h\|\nu\| \tag{5-4}$$

for any $\nu \in \mathscr{M}^+(\Omega)$.

Proof. Let $t > 0$. Define

$$E_t := \left\{ x \in \Omega : \frac{\boldsymbol{G}\nu}{\boldsymbol{G}\omega}(x) > t \right\}.$$

We claim that compact subsets $K \subset E_t$ have finite capacity. This requires that $G(x,x) > 0$ on E_t. Letting $A := \{x \in \Omega : G(x,x) = 0\}$, we claim $A \cap E_t = \varnothing$. Indeed, by the weak maximum principle, since $\boldsymbol{G}\delta_x(x) = 0$ for any $x \in A$, we have $\boldsymbol{G}\delta_x(y) = 0$ for every $y \in \Omega$. Thus, $G(x,y) = 0$ on $A \times \Omega$. Further, for any measure $\nu \in \mathscr{M}^+(\Omega)$, we have $\boldsymbol{G}\nu(x) = 0$ for $x \in A$. Adapting the convention $\frac{0}{0} = 0$, we see then that $E_t \cap A = \varnothing$ as claimed.

Let $K \subset \Omega$ be a compact set. We can find an equilibrium measure $\mu \in \mathscr{M}^+(K)$ such that $\boldsymbol{G}\mu \ge 1$ n.e. on K and $\boldsymbol{G}\mu \le 1$ on S_μ. Thus, if $N := \{x \in K : \boldsymbol{G}\mu(x) < 1\}$, then we have $\omega(N) = 0$, since ω is absolutely continuous with respect to capacity. By the weak maximum principle, $\boldsymbol{G}\mu \le 1$ on S_μ yields $\boldsymbol{G}\mu \le h$ on Ω.

We deduce the estimate

$$\omega(K) \le \int_K \boldsymbol{G}\mu \, d\omega = \int_K \boldsymbol{G}\omega_K \, d\mu \le \int_K \frac{\boldsymbol{G}\nu}{t} \, d\mu = \frac{1}{t}\int_\Omega \boldsymbol{G}\mu \, d\nu \le \frac{1}{t}\int_\Omega h \, d\nu = \frac{h}{t}\nu(\Omega).$$

Therefore we have $\omega(K) \le h\nu(\Omega)/t$ for any compact set $K \subset E_t$. Taking the supremum over all such K, we find

$$\omega(E_t) \le \frac{h}{t}\nu(\Omega)$$

for all $t > 0$. This establishes (5-4). $\square$

Lemma 5.11. *Let G be a quasisymmetric kernel which satisfies the weak maximum principle. Suppose there is a positive supersolution u to* (2-2) *such that $u \in L^q(\Omega,\sigma)$. Then* (1-1) *holds.*

Proof. Without loss of generality we may assume that G is symmetric (see Remark 2.1). Let $u \in L^q(\Omega,\sigma)$ be a positive supersolution; i.e., $\boldsymbol{G}(u^q\sigma) \le u$. Let the measure ω be given by $d\omega := u^q \, d\sigma$. By Lemma 4.2, we know that ω is absolutely continuous with respect to capacity. Suppose $\nu \in \mathscr{M}^+(\Omega)$. If $\nu(\Omega) = +\infty$, there is nothing to prove. In the case that $\nu(\Omega) < +\infty$, we can normalize the measure and work with the case $\nu(\Omega) = 1$.

Since u is a positive supersolution, we have $(\boldsymbol{G}\omega)^q\,d\sigma \le d\omega$. We estimate

$$\begin{aligned}\int_\Omega (\boldsymbol{G}\nu)^q\,d\sigma &= \int_\Omega \left(\frac{\boldsymbol{G}\nu}{u}\right)^q u^q\,d\sigma \le \int_\Omega \left(\frac{\boldsymbol{G}\nu}{\boldsymbol{G}\omega}\right)^q d\omega \\ &= q\int_0^\beta \omega\left(\left\{\frac{\boldsymbol{G}\nu}{\boldsymbol{G}\omega} > t\right\}\right) t^{q-1}\,dt + q\int_\beta^\infty \omega\left(\left\{\frac{\boldsymbol{G}\nu}{\boldsymbol{G}\omega} > t\right\}\right) t^{q-1}\,dt = I + II\end{aligned}$$

for any $\beta > 0$.

For integral I, we see that $I \le \beta^q \omega(\Omega) = \beta^q \int_\Omega u^q\,d\sigma$.

By Lemma 5.10, we have the weak-type bound

$$\omega\left(\left\{\frac{\boldsymbol{G}\nu}{\boldsymbol{G}\omega} > t\right\}\right) \le \frac{h\nu(\Omega)}{t} = \frac{h}{t}.$$

With this estimate, we find $II \le q/(1-q)h\beta^{q-1}$. Thus, with the choice of $\beta = h/(\omega(\Omega))$, we deduce

$$\int_\Omega (\boldsymbol{G}\nu)^q\,d\sigma \le \frac{h^q}{1-q}\left(\int_\Omega u^q\,d\sigma\right)^{1-q}.$$

Therefore, in the general case with $\nu \in \mathscr{M}^+(\Omega)$, we obtain the desired inequality

$$\int_\Omega (\boldsymbol{G}\nu)^q\,d\sigma \le \frac{h^q}{1-q}\left(\int_\Omega u^q\,d\sigma\right)^{1-q} \nu(\Omega)^q.$$

It is important to note that in the above inequality, we have the constant on the right-hand side in terms of the norm $\|u\|_{L^q(\Omega,\sigma)}$. This implies that (1-1) holds with

$$\varkappa \le \frac{h}{(1-q)^{\frac{1}{q}}}\,\|u\|^{1-q}_{L^q(\Omega,\sigma)},$$

where $\varkappa$ is the least constant in (1-1). ☐

6. Weak-type results

In addition to characterizing the strong-type inequality (1-1), we study in this section the analogous weak-type $(1,q)$-inequality

$$\|\boldsymbol{G}\nu\|_{L^{q,\infty}(X,\sigma)} \le C\,\|\nu\| \quad \text{for all } \nu \in \mathscr{M}^+(Y) \tag{6-1}$$

in a more general setting where G is a kernel on $X \times Y$ and $\sigma \in \mathscr{M}^+(X)$. We give various characterizations of (6-1) using capacities, as well as noncapacitary terms, for all $0 < q < \infty$.

A complete characterization of (6-1) in terms of the capacity $\mathrm{cap}_0(\cdot)$ (see Section 4 above) is given in the following proposition. Note that this result does not require G to satisfy the weak maximum principle on Ω, does not restrict to the case $X = Y$, and does not place any restriction on the range of $q > 0$.

Proposition 6.1. *Let G be a kernel on $X \times Y$. Suppose $0 < q < +\infty$ and $\sigma \in \mathscr{M}^+(X)$. Then there exists a positive constant C such that* (6-1) *holds if and only if*

$$\sigma(K) \le C^q\,(\mathrm{cap}_0(K))^q \quad \textit{for all compact sets } K \subset X, \tag{6-2}$$

where C is the same between both statements.

Proof. (⇒) Without loss of generality we may assume that $C = 1$. Let $K \subset X$ be a compact set. If $\mathrm{cap}_0\, K = +\infty$, there is nothing to show, so we assume $\mathrm{cap}_0\, K < +\infty$. Then for every $\epsilon > 0$, there exists a measure $\lambda \in \mathscr{M}^+(Y)$ so that $\boldsymbol{G}\lambda(x) \geq 1$ on K and $\lambda(Y) \leq \mathrm{cap}_0(K) + \epsilon$. Then by (6-1),

$$\sigma(K) \leq \|\boldsymbol{G}\lambda\|^q_{L^{q,\infty}(K,\sigma)} \leq \lambda(Y)^q \leq (\mathrm{cap}_0(K) + \epsilon)^q.$$

Letting $\epsilon \to 0$, we establish the capacity inequality (6-2).

(⇐) Suppose $\sigma(K) \leq (\mathrm{cap}_0(K))^q$ for any compact $K \subset X$. For $t > 0$, let $E_t := \{x \in X : \boldsymbol{G}\nu(x) > t\}$. Let $K \subset \Omega_t$ be a compact set. For $\epsilon > 0$, by the dual definition of capacity (4-2), we can find a measure $\mu \in \mathscr{M}^+(K)$ such that $\boldsymbol{G}^*\mu(y) \leq 1$ for all $y \in Y$ and $\mathrm{cap}_0(K) \leq \mu(K) + \epsilon$.

Then by Fubini's theorem,

$$\sigma(K)^{\frac{1}{q}} \leq \mu(K) + \epsilon \leq \frac{1}{t}\int_K \boldsymbol{G}\nu(x)\, d\mu(x) + \epsilon = \frac{1}{t}\int_Y \boldsymbol{G}^*\mu(y)\, d\nu(y) + \epsilon \leq \frac{\nu(Y)}{t} + \epsilon.$$

By exhausting over all compact sets $K \subset E_t$ and letting $\epsilon \to 0$, we establish the weak-type $(1, q)$-inequality

$$\sigma(E_t)^{\frac{1}{q}} \leq \frac{\nu(Y)}{t}$$

for all $t > 0$, which proves (6-1). □

In the case $q > 1$ we can use the duality $L^{q,\infty}(X, \sigma) = [L^{q',1}(X, \sigma)]^*$, $1/q + 1/q' = 1$, to show that it suffices to verify (6-1) on point masses $\nu = \delta_x$, $x \in X$. This leads to a simple noncapacitary characterization of (6-1).

Proposition 6.2. *Let G be a kernel on $X \times Y$. Suppose $1 < q < +\infty$, and $\sigma \in \mathscr{M}^+(X)$. Then the following statements are equivalent*:

(1) *There exists a positive constant C such that* (6-1) *holds.*

(2) *The following condition holds*:

$$\sup_{y \in Y} \|G(\cdot\,, y)\|_{L^{q,\infty}(X,\sigma)} < +\infty. \tag{6-3}$$

(3) *There exists a positive constant C such that*, *for all measurable sets $E \subset X$*,

$$\sup_{y \in Y} \boldsymbol{G}^*\sigma_E(y) \leq C\,\sigma(E)^{\frac{1}{q'}}. \tag{6-4}$$

Proof. By duality, statement (1) is equivalent to

$$\int_X (\boldsymbol{G}\nu)\phi\, d\sigma \leq c\|\nu\|\,\|\phi\|_{L^{q',1}(\sigma)} \quad \text{for all } \phi \in L^{q',1}(X, \sigma),\ \nu \in \mathscr{M}^+(Y).$$

Equivalently, by Fubini's theorem,

$$\int_Y \boldsymbol{G}^*(\phi\sigma)\, d\nu \leq c\|\nu\|\,\|\phi\|_{L^{q',1}(\sigma)} \quad \text{for all } \phi \in L^{q',1}(X, \sigma),\ \nu \in \mathscr{M}^+(Y).$$

Clearly, the preceding inequality holds if and only it holds for all $\nu = \delta_y$; that is,

$$\boldsymbol{G}^*(\phi\sigma)(y) = \int_X G(x, y)\,\phi(x)\, d\sigma(x) \leq c\|\phi\|_{L^{q',1}(\sigma)} \quad \text{for all } \phi \in L^{q',1}(X, \sigma),\ y \in Y.$$

Using duality again, we see that the preceding inequality is equivalent to

$$\|G(\,\cdot\,, y)\|_{L^{q,\infty}(\sigma)} \le c \quad \text{for all } y \in Y. \tag{6-5}$$

This establishes (1) $\Longleftrightarrow$ (2).

The equivalence (2) $\Longleftrightarrow$ (3) follows from the well-known fact that, for $q > 1$, we have $\|f\|_{L^{q,\infty}(X,\sigma)}$ is equivalent to the norm

$$\sup_{E\subset X} \frac{1}{\sigma(E)^{\frac{1}{q'}}} \int_E |f|\, d\sigma(x).$$

Applying this to $f(\cdot) = G(y,\cdot)$, for a fixed $y \in Y$, we see that

$$\boldsymbol{G}^*\sigma_E(y) = \int_E G(y,x)\, d\sigma(x) \le C\,\sigma(E)^{\frac{1}{q'}},$$

where C does not depend on $y \in Y$ and $E \subset X$, if and only if (6-5) holds. □

Remark 6.3. For Riesz kernels $I_\alpha(x) = |x|^{\alpha-n}$ $(0 < \alpha < n)$ on $\mathbb{R}^n$, condition (6-3) means that $\sigma(B(x,r)) \le C\, r^{(n-\alpha)q}$ for all balls $B(x,r)$ in $\mathbb{R}^n$. This condition was used by D. Adams in the context of (p,q)-inequalities for $q > p > 1$; the capacitary condition (6-2) was introduced by V. Maz'ya [2011]; see also [Adams and Hedberg 1996].

There are more direct characterizations of the weak-type $(1,q)$-inequality in the case $0 < q \le 1$ if $X = Y = \Omega$, and additionally if G is quasisymmetric and satisfies the weak maximum principle. Notice that in this case $\mathrm{cap}_0(\cdot)$ is equivalent to the Wiener capacity $\mathrm{cap}_1(\cdot)$.

Theorem 6.4. *Let $\sigma \in \mathscr{M}^+(\Omega)$, and $0 < q < \infty$. Suppose G is a quasisymmetric kernel on $\Omega\times\Omega$ which satisfies the weak maximum principle. Then the following statements are equivalent*:

(1) *There exists a positive constant c such that*

$$\|\boldsymbol{G}\nu\|_{L^{q,\infty}(\Omega,\sigma)} \le c\,\|\nu\| \quad \textit{for all } \nu \in \mathscr{M}^+(\Omega).$$

(2) *There exists a positive constant C such that*

$$\sigma(K) \le C\,(\mathrm{cap}_1(K))^q \quad \textit{for all compact sets } K \subset \Omega.$$

(3) $\boldsymbol{G}\sigma \in L^{\frac{q}{1-q},\infty}(\Omega,\sigma)$, *when* $0 < q < 1$.

The details of this theorem can be found in [Quinn and Verbitsky 2017].

We finally consider (6-1) in the case $q = 1$, i.e., the weak-type $(1,1)$-inequality, along with its (p,p)-analogues for $1 < p < +\infty$, under the same assumptions as in Theorem 6.4.

Theorem 6.5. *Let $\sigma \in \mathscr{M}^+(\Omega)$. Suppose G is a quasisymmetric kernel on $\Omega\times\Omega$ which satisfies the weak maximum principle. Then the following statements are equivalent*:

(1) *There exists a positive constant c such that*

$$\|\boldsymbol{G}\nu\|_{L^{1,\infty}(\sigma)} \le c\,\|\nu\| \quad \textit{for all } \nu \in \mathscr{M}^+(\Omega). \tag{6-6}$$

(2) *If* $1 < p < +\infty$, *then there exists a positive constant* c *such that*

$$\|\boldsymbol{G}(f\,d\sigma)\|_{L^p(\sigma)} \le c\,\|f\|_{L^p(\sigma)} \quad \text{for all } f \in L^p(\Omega,\sigma). \tag{6-7}$$

(3) *There exists a positive constant* c *such that*

$$\iint_{K\times K} G(x,y)\,d\sigma(x)\,d\sigma(y) \le c\,\sigma(K) \quad \text{for all compact sets } K \subset \Omega. \tag{6-8}$$

(4) *If* G *is a quasimetric kernel then*

$$\iint_{B\times B} G(x,y)\,d\sigma(x)\,d\sigma(y) \le c\,\sigma(B) \tag{6-9}$$

for all quasimetric balls $B = B(x,r)$, *where* $B(x,r) = \{y \in \Omega : d(x,y) < r\}$, $d(x,y) = 1/G(x,y)$ $(x, y \in \Omega, r > 0)$.

Remark 6.6. The equivalence of statements (2) and (4) of Theorem 6.5 in the case of quasimetric kernels G is due to F. Nazarov; see [Nikolski and Verbitsky 2017, Theorem 4.6]. It can be deduced from more general results on operators with nonpositive kernels in the framework of nonhomogeneous harmonic analysis; see [Hytönen 2010]. The weak-type $(1,1)$-inequality in Theorem 6.5 may be new.

Proof. As above in the case of strong-type $(1,q)$-inequalities, we may assume without loss of generality that G is a symmetric kernel such that $G(x,x) > 0$ for all $x \in \Omega$. The latter condition ensures that $\mathrm{cap}_1(K) < \infty$ for any compact set $K \in \Omega$. By Proposition 6.1, the weak-type $(1,1)$-inequality (6-6) is equivalent to the condition

$$\sigma(K) \le C\ \mathrm{cap}_1(K) \quad \text{for all compact sets } K \subset \Omega. \tag{6-10}$$

From the discussion in Section 4 it follows that, for any compact set $K \subset \Omega$,

$$\mathrm{cap}_1(K) = \sup\left\{\mu(K) : \frac{1}{\mu(K)}\iint_{K\times K} G(x,y)\,d\mu(x)\,d\mu(y) \le 1\right\},$$

where the supremum is taken over all $\mu \in \mathscr{M}^+(K)$ such that $\mu(K) > 0$. Taking $\mu = (1/C)\,\sigma$, where C is the constant in (6-10), we see that (6-8) implies (6-10), and consequently, (6-6). This proves (3) $\Longrightarrow$ (1).

Conversely, suppose that (6-10) holds. Let $1 < p < +\infty$. We first prove the corresponding weak-type (p,p)-inequality

$$\|\boldsymbol{G}(g\,d\sigma)\|_{L^{p,\infty}(\sigma)} \le c\,\|g\|_{L^p(\sigma)}, \tag{6-11}$$

where c is independent of g. Here without loss of generality we may assume that $g \in L^p(\Omega,\sigma)$, $g \ge 0$, is compactly supported. For a fixed $t > 0$, denote by E_t the set

$$E_t = \{x \in \Omega : \boldsymbol{G}(g\,d\sigma)(x) > t\}.$$

Notice that

$$\boldsymbol{G}(g\,d\sigma_{E_t^c}) \le \boldsymbol{G}(g\,d\sigma) \le t \quad \text{on } E_t^c.$$

Consequently, by the weak maximum principle

$$\boldsymbol{G}(g\, d\sigma_{E_t^c}) \le h\, t \quad \text{on } \Omega.$$

Denote by K an arbitrary compact subset of the set F_t defined by

$$F_t = \{x \in \Omega : \boldsymbol{G}(g\, d\sigma)(x) > (h+1)\, t\}.$$

We observe that by the preceding estimates,

$$F_t \subset \{x \in \Omega : \boldsymbol{G}(g\, d\sigma_{E_t}) > t\}.$$

We denote by μ the equilibrium measure μ associated with $\mathrm{cap}_1(K)$, which is supported on K, and has the property $\boldsymbol{G}\mu \le 1$ on K. Hence $\boldsymbol{G}\mu \le h$ on Ω by the weak maximum principle.

Since $K \subset F_t$, by (6-10) we estimate

$$\sigma(K) \le C\ \mathrm{cap}_1(K) = C\,\mu(K) \le \frac{C}{t}\int_K \boldsymbol{G}(g\, d\sigma_{E_t})\, d\mu = \frac{C}{t}\int_\Omega (\boldsymbol{G}\mu)\, g\, d\sigma_{E_t} \le \frac{C h}{t}\int_{E_t} g\, d\sigma.$$

From this, by Jensen's inequality we deduce

$$\sigma(K) \le \frac{C h}{t}\,\sigma(E_t)^{\frac{1}{p'}}\,\|g\|_{L^p(\sigma)}.$$

Taking the supremum over all $K \subset F_t$, we see that

$$\sigma(F_t) \le \frac{C h}{t}\,\sigma(E_t)^{\frac{1}{p'}}\,\|g\|_{L^p(\sigma)}.$$

Multiplying both sides of the preceding inequality by t^p and taking the supremum over all $t \in (0, t_0)$ we get

$$\sup_{0<t<t_0}\,[t^p\,\sigma(F_t)] \le C h \sup_{0<t<t_0}\,[t^p\,\sigma(E_t)]^{\frac{1}{p'}}\,\|g\|_{L^p(\sigma)}.$$

Here the right-hand side is finite for any $t_0 > 0$ since g is compactly supported, and consequently $g \in L^1(\Omega, \sigma)$, so that

$$\sup_{0<t<t_0}\,[t^p\,\sigma(E_t)] \le t_0^{p-1} \sup_{0<t<\infty}\,[t\,\sigma(E_t)] \le t_0^{p-1}\,\|g\|_{L^1(\sigma)} < \infty.$$

Notice that

$$\sup_{0<t<t_0}\,[t^p\,\sigma(F_t)] = \frac{1}{(h+1)^p} \sup_{0<\tau<(h+1)t_0}\,[\tau^p\,\sigma(E_\tau)] \ge \frac{1}{(h+1)^p} \sup_{0<\tau<t_0}\,[\tau^p\,\sigma(E_\tau)].$$

Combining the preceding estimates we deduce

$$\sup_{0<\tau<t_0}\,[\tau^p\,\sigma(E_\tau)]^{\frac{1}{p}} \le C h\,(h+1)^p\,\|g\|_{L^p(\sigma)}.$$

Letting $t_0 \to +\infty$, we obtain

$$\sup_{0<\tau<+\infty}\,[\tau^p\,\sigma(E_\tau)]^{\frac{1}{p}} \le C h (h+1)^p\,\|g\|_{L^p(\sigma)}.$$

This proves the weak-type (p, p)-inequality (6-11) for all $1 < p < +\infty$, which by the Marcinkiewicz interpolation theorem yields (6-7) for all $1 < p < +\infty$.

For any measurable set $E \subset \Omega$ and $1 < p < +\infty$, letting $g = \chi_E$ in (6-7), or (6-11), we deduce by Jensen's inequality

$$\iint_{E\times E} G(x, y)\, d\sigma(x)\, d\sigma(y) \le \|\boldsymbol{G}(\chi_E\, \sigma)\|_{L^p(\sigma)}\, \sigma(E)^{\frac{1}{p'}} \le C\, \sigma(E). \tag{6-12}$$

In particular, (6-8) and (6-9) hold. This proves (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1).

If G is a quasimetric kernel, then (4) $\Longrightarrow$ (2) for $p = 2$; see Remark 6.6. Conversely, (6-7) for $p = 2$ yields (6-12) for any measurable $E \subset \Omega$, so that (2) $\Longrightarrow$ (4). □

7. Breaking the inequality: a counterexample

In this section, we provide some examples which demonstrate that our main results may fail in the absence of the weak maximum principle, first for nonnegative symmetric kernels G, and then for strictly positive kernels. More specifically, we justify the following remarks.

Remark 7.1. Without the weak maximum principle, for a symmetric kernel G there can be a positive solution to $u = \boldsymbol{G}(u^q\, d\sigma)$ with $u \in L^q(\Omega, \sigma)$ but there is no constant $0 < \varkappa < +\infty$ such that the inequality $\int_\Omega (\boldsymbol{G}\,\nu)^q\, d\sigma \le \varkappa^q\, \nu(\Omega)^q$ holds for all $\nu \in \mathscr{M}^+(\Omega)$.

First, we present some minor computations for 2×2 matrices which we will employ extensively below. Suppose that we have a discrete kernel $G(x_i, x_j) = g_{ij}$ $(i = 1, 2)$ on $\Omega = \{x_1, x_2\}$, where x_1, x_2 are distinct points, and

$$G = [g_{ij}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that this kernel does not satisfy the weak maximum principle.

Suppose we have the measure $\sigma = (\sigma_1, \sigma_2)$ on Ω, and $u = (u_1, u_2)$, where $u_i, \sigma_i \ge 0$ $(i = 1, 2)$. Then, if u is a solution to the equation $u = \boldsymbol{G}(u^q\, d\sigma)$, we have the system of equations

$$u_1 = u_2^q \sigma_2, \quad u_2 = u_1^q \sigma_1,$$

which we can solve explicitly for u in terms of q and σ:

$$u_1 = (\sigma_1^q \sigma_2)^{\frac{1}{1-q^2}}, \quad u_2 = (\sigma_2^q \sigma_1)^{\frac{1}{1-q^2}}.$$

We compute the norm of u in $L^q(\sigma)$ to be

$$\|u\|_{L^q(\sigma)}^q = u_1^q \sigma_1 + u_2^q \sigma_2 = (\sigma_1^q \sigma_2)^{\frac{q}{1-q^2}} \sigma_1 + (\sigma_1 \sigma_2^q)^{\frac{q}{1-q^2}} \sigma_2.$$

Now suppose we have a kernel G on the discrete set of distinct points $\Omega = \{x_k\}_{k=1}^\infty$. This kernel will consist of the above blocks placed along the diagonal and zeros elsewhere:

$$G = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & 0 & 1 & \\ & & & & 1 & 0 & \\ & & & & & & \ddots \end{bmatrix}. \tag{7-1}$$

Then we find that for $\sigma = (\sigma_k)_{k=1}^{\infty}$, as above, the equation $u = \boldsymbol{G}(u^q\, d\sigma)$ has a solution

$$\begin{aligned} u &= (u_1, u_2, \dots, u_{2k-1}, u_{2k}, \dots) \\ &= \big((\sigma_1^q \sigma_2)^{\frac{1}{1-q^2}}, (\sigma_1 \sigma_2^q)^{\frac{1}{1-q^2}}, \dots, (\sigma_{2k-1}^q \sigma_{2k})^{\frac{1}{1-q^2}}, (\sigma_{2k-1}\sigma_{2k}^q)^{\frac{1}{1-q^2}}, \dots\big), \end{aligned} \tag{7-2}$$

with norm

$$\|u\|_{L^q(\sigma)}^q = \sum_{k=1}^{\infty} u_k^q \sigma_k = \sum_{k=1}^{\infty} \big((\sigma_{2k-1}^q \sigma_{2k})^{\frac{q}{1-q^2}} \sigma_{2k-1} + (\sigma_{2k-1}\sigma_{2k}^q)^{\frac{q}{1-q^2}} \sigma_{2k}\big). \tag{7-3}$$

We would now like to create a measure σ for which $\|u\|_{L^q(\sigma)} < +\infty$. Set $\sigma_{2k-1} = a^k$ and $\sigma_{2k} = b^{-k}$. Then the k-th pair of terms in the sum are

$$\begin{aligned} (\sigma_{2k-1}^q \sigma_{2k})^{\frac{q}{1-q^2}} \sigma_1 + (\sigma_{2k-1}\sigma_{2k}^q)^{\frac{q}{1-q^2}} \sigma_{2k} &= (a^{kq} b^{-k})^{\frac{q}{1-q^2}} a^k + (a^k b^{-kq})^{\frac{q}{1-q^2}} b^{-k} \\ &= \left[\left(\frac{a^q}{b}\right)^{\frac{q}{1-q^2}} a\right]^k + \left[\left(\frac{a}{b^q}\right)^{\frac{q}{1-q^2}\frac{1}{b}}\right]^k \\ &= \left[\left(\frac{a}{b^q}\right)^{\frac{1}{1-q^2}}\right]^k + \left[\left(\frac{a^q}{b}\right)^{\frac{1}{1-q^2}}\right]^k. \end{aligned}$$

We wish to choose $a, b > 0$ so that

$$a < b^q, \quad a^q < b.$$

Note that this reduces down to choosing $1 < a < b^q$. If this holds, then $a^q < a < b^q < b$, so $a^q < b$. Therefore, with appropriate choices of a, b, we have $\|u\|_{L^q(\sigma)} < +\infty$.

Now we wish to show that

$$\sup_{\nu \in \mathscr{M}^+(\Omega)} \frac{\int_\Omega (\boldsymbol{G}\nu)^q\, d\sigma}{\nu(\Omega)^q} = +\infty. \tag{7-4}$$

Note that the ratio on the left-hand side can be written as

$$\frac{\int_\Omega (\boldsymbol{G}\nu)^q\, d\sigma}{\nu(\Omega)^q} = \frac{\sum_{k=1}^{\infty} (\nu_{2k}^q \sigma_{2k-1} + \nu_{2k-1}^q \sigma_{2k})}{\big(\sum_{k=1}^{\infty} \nu_k\big)^q}.$$

Setting $\nu_{2k-1} = \sigma_{2k}^{\frac{1}{1-q}}$, $\nu_{2k} = \sigma_{2k-1}^{\frac{1}{1-q}}$ for $k = 1, 2, \dots, n$, and $\nu_k = 0$ for $k > 2n$, we obtain

$$\frac{\int_\Omega (\boldsymbol{G}\nu)^q\, d\sigma}{\nu(\Omega)^q} = \frac{\sum_{k=1}^{2n} \sigma_k^{\frac{1}{1-q}}}{\big(\sum_{k=1}^{2n} \sigma_k^{\frac{1}{1-q}}\big)^q} = \left(\sum_{k=1}^{2n} \sigma_k^{\frac{1}{1-q}}\right)^{1-q}.$$

Since $0 < q < 1$, and $\sigma_{2k-1} = a^k$ where $a > 1$, the partial sums on the right go to $+\infty$ as $n \to +\infty$, which yields (7-4). This justifies Remark 7.1.

Remark 7.2. The preceding example employs a block matrix kernel which fails to satisfy the weak maximum principle based on a construction with zeros along the diagonal. We have seen that such kernels allow for compact sets $K \Subset \Omega$ to have infinite capacity, i.e., $\text{cap}_1 K = +\infty$, which we would like to rule out. With this in mind, we can adapt the above construction so that (7-4) holds for a symmetric kernel G such that $G(x, x) > 0$ for all $x \in \Omega$, i.e., G is strictly positive, but nevertheless the equation $u = \boldsymbol{G}(u^q\, d\sigma)$ has a positive solution $u \in L^q(\Omega, \sigma)$.

Specifically, we adjust each block along the diagonal so that we have kernel $\widetilde{G}$ in place of G. Let

$$\widetilde{G} = \begin{bmatrix} a & 1 \\ 1 & 1/a \end{bmatrix},$$

where $a > 0$ is a constant to be specified. Note that $\boldsymbol{G}\nu \le \widetilde{\boldsymbol{G}}\nu$, so we can invoke the above computations to see that (7-4) holds for $\widetilde{\boldsymbol{G}}$ as well. We decompose $\widetilde{G}$ as

$$\widetilde{G} = G + G_a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}.$$

As shown above, there is a positive solution $u \in L^q(\Omega, \sigma)$ to $u = \boldsymbol{G}(u^q\, d\sigma)$. By scaling, for $\tilde{u} = 2^{\frac{1}{1-q}} u$, we have $\frac{1}{2}\tilde{u} = \boldsymbol{G}(\tilde{u}^q\, d\sigma)$. Following the appropriate choice of a, we can then ensure that $\frac{1}{2}\tilde{u} = \boldsymbol{G_a}(\tilde{u}^q\, d\sigma)$. This establishes that $\tilde{u}$ is a solution, since we have

$$\tilde{u} = \tfrac{1}{2}\tilde{u} + \tfrac{1}{2}\tilde{u} = \boldsymbol{G}(\tilde{u}^q\, d\sigma) + \boldsymbol{G_a}(\tilde{u}^q\, d\sigma) = \widetilde{\boldsymbol{G}}(\tilde{u}^q\, d\sigma).$$

The choice of a should be so that

$$a\tilde{u}_1^q \sigma_1 = \tfrac{1}{2}\tilde{u}_1, \quad \tfrac{1}{a}\,\tilde{u}_2^q \sigma_2 = \tfrac{1}{2}\tilde{u}_2,$$

where a is uniquely determined by $a = (\sigma_2/\sigma_1)^{\frac{1}{1+q}}$.

With this choice of a, let $a = a_k$ for each block, where a_k depends on q and the values of σ_{2k-1} and σ_{2k} defined for the k-th block, as specified above. Thus, we have a positive solution $\tilde{u} = \widetilde{\boldsymbol{G}}(\tilde{u}^q\, d\sigma)$, where $\tilde{u} \in L^q(\Omega, \sigma)$, but (7-4) holds with $\widetilde{\boldsymbol{G}}$ in place of $\boldsymbol{G}$, which justifies Remark 7.2.

The following example shows that the restriction on $q \in (0, q_0]$, where $q_0 = \frac{1}{2}(\sqrt{5}-1)$, in Lemma 5.1(a) is sharp.

Remark 7.3. Let $q \in (q_0, 1)$. Without the weak maximum principle, for a symmetric kernel G there can be a positive solution to $u = \boldsymbol{G}(u^q\, d\sigma)$ with $u \in L^q(\Omega, \sigma)$, but

$$\int_\Omega (\boldsymbol{G}\sigma)^{\frac{q}{1-q}}\, d\sigma = +\infty.$$

To construct such an example we employ the above construction of the block matrix kernel G given by (7-1). Then there exists a positive solution $u \in L^q(\Omega, \sigma)$ to $u = \boldsymbol{G}(u^q\, d\sigma)$ given by (7-2) with finite norm (7-3) provided

$$\|u\|^q_{L^q(\sigma)} = \sum_{k=1}^{\infty} \big((\sigma^q_{2k-1}\sigma_{2k})^{\frac{q}{1-q^2}} \sigma_{2k-1} + (\sigma_{2k-1}\sigma^q_{2k})^{\frac{q}{1-q^2}} \sigma_{2k} \big) < \infty.$$

At the same time we can pick σ_k so that, for $q \in (q_0, 1)$, we have

$$\int_\Omega (\boldsymbol{G}\sigma)^{\frac{q}{1-q}}\, d\sigma = \sum_{k=1}^{\infty} (\sigma_{2k-1}\sigma_{2k}^{\frac{q}{1-q}} + \sigma_{2k-1}^{\frac{q}{1-q}}\sigma_{2k}) = +\infty.$$

Indeed, setting $\sigma_{2k-1} = 1$ and $\sigma_{2k} = 1/k$, we see that

$$\|u\|^q_{L^q(\sigma)} = \sum_{k=1}^{\infty} (k^{-\frac{q}{1-q^2}} + k^{-\frac{1}{1-q^2}}) < \infty,$$

since both $q/(1-q^2) > 1$ and $1/(1-q^2) > 1$. On the other hand,

$$\int_\Omega (\boldsymbol{G}\sigma)^{\frac{q}{1-q}}\, d\sigma = \sum_{k=1}^{\infty} (k^{-\frac{q}{1-q}} + k^{-1}) = +\infty.$$

A slight modification of this example as in Remark 7.2 produces a strictly positive kernel G with the same properties.

Remark 7.4. There are analogous examples that show that the exponents $s = q/(1-q)$ (for general measures σ) and $s = 1 + q$ (for finite measures σ) in statements (a) and (b) of Lemma 5.1, respectively, are sharp as well. We omit the details.

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RADIAL FOURIER MULTIPLIERS IN $\mathbb{R}^3$ AND $\mathbb{R}^4$

LAURA CLADEK

We prove that for radial Fourier multipliers $m : \mathbb{R}^3 \to \mathbb{C}$ supported compactly away from the origin, T_m is restricted strong type (p, p) if $K = \hat{m}$ is in $L^p(\mathbb{R}^3)$, in the range $1 < p < \frac{13}{12}$. We also prove an L^p characterization for radial Fourier multipliers in four dimensions; namely, for radial Fourier multipliers $m : \mathbb{R}^4 \to \mathbb{C}$ supported compactly away from the origin, T_m is bounded on $L^p(\mathbb{R}^4)$ if and only if $K = \hat{m}$ is in $L^p(\mathbb{R}^4)$, in the range $1 < p < \frac{36}{29}$. Our method of proof relies on a geometric argument that exploits bounds on sizes of multiple intersections of 3-dimensional annuli to control numbers of tangencies between pairs of annuli in three and four dimensions.

1. Introduction and statement of results

In this paper we study radial multiplier transformations whose symbol is compactly supported away from the origin. These are operators T_m defined via the Fourier transform by

$$\mathcal{F}[T_m f](\xi) = m(\xi)\hat{f}(\xi),$$

where the function $m : \mathbb{R}^d \to \mathbb{C}$ is bounded, measurable, radial and supported in a compact subset of $\{\xi : \frac{1}{2} < |\xi| < 2\}$.

In the cases $p \neq 1, 2$, it is generally believed that it is impossible to give a reasonable characterization of all multiplier operators which are bounded on L^p. However, for *radial* Fourier multipliers, a characterization can be obtained for an appropriate range of p. Heo, Nazarov, and Seeger [Heo et al. 2011] proved a strikingly simple characterization of radial multipliers that are bounded on $L^p(\mathbb{R}^d)$ in dimensions $d \geq 4$ for $1 < p < (2d-2)/(d+1)$.

Theorem A. *Let $d \geq 2$. If $m : \mathbb{R}^d \to \mathbb{C}$ is radial and supported in a compact subset of $\{\xi : \frac{1}{2} < |\xi| < 2\}$, the multiplier operator T_m is bounded on $L^p(\mathbb{R}^d)$ if and only if the kernel $K = \hat{m}$ is in $L^p(\mathbb{R}^d)$, in the range $1 < p < (2d-2)/(d+1)$.*

The characterization in [Heo et al. 2011] was motivated by the earlier work [Garrigós and Seeger 2008], where the authors obtained a similar characterization of all convolution operators with radial kernels acting on the space L^p_{rad} of radial L^p functions, in the larger range $1 < p < 2d/(d+1)$.

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Theorem B. *Let $d \geq 2$. If $m : \mathbb{R}^d \to \mathbb{C}$ is radial and supported in a compact subset of $\{\xi : \frac{1}{2} < |\xi| < 2\}$, the multiplier operator T_m is bounded on $L^p_{\text{rad}}(\mathbb{R}^d)$ if and only if the kernel $K = \hat{m}$ is in $L^p(\mathbb{R}^d)$, in the range $1 < p < 2d/(d+1)$.*

This range $1 < p < 2d/(d+1)$ is the optimal range for their result to hold, since for $p \geq 2d/(d+1)$ one may construct radial kernels in L^p that have Fourier transforms which are supported compactly away from the origin but which are also unbounded. By the same reasoning, the range $1 < p < 2d/(d+1)$ is also the largest possible range in which one could hope for the characterization from Theorem A to hold. Thus one might propose the following conjecture, which we will refer to as the "radial Fourier multiplier conjecture".

Conjecture 1.1. *Let $d \geq 2$. If $m : \mathbb{R}^d \to \mathbb{C}$ is radial and supported in a compact subset of $\{\xi : \frac{1}{2} < |\xi| < 2\}$, the multiplier operator T_m is bounded on $L^p(\mathbb{R}^d)$ if and only if the kernel $K = \hat{m}$ is in $L^p(\mathbb{R}^d)$, in the range $1 < p < 2d/(d+1)$.*

One can appreciate the strength of this conjecture by noting that since $2d/(d+1)$ is the critical value for the Bochner–Riesz conjecture, the Bochner–Riesz conjecture (and hence also the restriction and Kakeya conjectures) would follow as a special case from Conjecture 1.1. However, the statement of Conjecture 1.1 is far more general than the Bochner–Riesz conjecture, since it makes no a priori assumptions whatsoever on the regularity of the multiplier.

The arguments of [Heo et al. 2011] did not yield any results about radial Fourier multipliers in $\mathbb{R}^3$. We will improve a key lemma of that paper in three dimensions to obtain a characterization of restricted strong type (p, p) boundedness of compactly supported radial Fourier multipliers $m : \mathbb{R}^3 \to \mathbb{C}$, in the range $1 < p < \frac{13}{12}$.

Theorem 1.2. *Let m be a radial Fourier multiplier in $\mathbb{R}^3$ supported in $\{\frac{1}{2} < |\xi| < 2\}$ and let $K = \mathcal{F}^{-1}[m]$. Then for $1 < p < \frac{13}{12}$, if $K \in L^p$ then the multiplier operator T_m is restricted strong type (p, p), and moreover,*

$$\|K * f\|_{L^p(\mathbb{R}^3)} \lesssim_p \|K\|_{L^p(\mathbb{R}^3)} \|f\|_{L^{p,1}(\mathbb{R}^3)}.$$

Remark 1.3. Our proof will also show that $\|K * f\|_{L^p} \lesssim_p \|K\|_{L^{p,1}} \|f\|_{L^p}$, and we expect that $\|K\|_{L^{p,1}}$ could be improved to $\|K\|_{L^p}$.

We will also prove a full L^p characterization for compactly supported radial Fourier multipliers in $\mathbb{R}^4$ in the range $1 < p < \frac{36}{29}$, which improves on Heo, Nazarov, and Seeger's result.

Theorem 1.4. *Let m be a radial Fourier multiplier in $\mathbb{R}^4$ supported in $\{\frac{1}{2} < |\xi| < 2\}$ and let $K = \mathcal{F}^{-1}[m]$. Then for $1 < p < \frac{36}{29}$, if $K \in L^p(\mathbb{R}^4)$, the multiplier operator T_m is bounded on $L^p(\mathbb{R}^4)$, and moreover,*

$$\|K * f\|_{L^p(\mathbb{R}^4)} \lesssim_p \|K\|_{L^p(\mathbb{R}^4)} \|f\|_{L^p(\mathbb{R}^4)}.$$

Our proofs of Theorems 1.2 and 1.4 refine the arguments of [Heo et al. 2011] while simultaneously incorporating new geometric input. A key divergence from their arguments is the exploitation of the underlying "tensor product structure" inherent in the problem, a notion which will become clearer later. This, combined with a geometric argument involving sizes of multiple intersections of 3-dimensional annuli, allows one to take advantage of improved scalar product estimates which were not used by Heo et al.

However, since we exploit the tensor product structure of the problem, we are currently not able to deduce any local smoothing results for the wave equation as corollaries, as was able to be done in [Heo et al. 2011].

The outline of the paper is as follows. The first portion of the paper will be devoted to the proof of Theorem 1.2, which is less technical than the proof of Theorem 1.4. The second portion will give the proof of Theorem 1.4. At the end, we provide as an Appendix the proof of the geometric lemma used in the proofs of both theorems.

2. Preliminaries and reductions

We will now collect some necessary preliminary results and reductions. Versions of these results can be found in [Heo et al. 2011], but we reproduce them here for completeness. In general, this section will very closely follow that paper, and for convenience we choose to adopt similar notation.

Discretization and density decomposition of sets. The first step will be to discretize our problem, and in preparation for this we will first need to introduce some notation. Let $\mathcal{Y}$ be a 1-separated set of points in $\mathbb{R}^3$ and let $\mathcal{R}$ be a 1-separated set of radii ≥ 1. Let $\mathcal{E} \subset \mathcal{Y} \times \mathcal{R}$ be a finite set that is also a *product*, i.e., $\mathcal{E} = \mathcal{E}_Y \times \mathcal{E}_R$, where $\mathcal{E}_Y \subset \mathcal{Y}$ and $\mathcal{E}_R \subset \mathcal{R}$. (The assumption that $\mathcal{E}$ is a product was not used in [Heo et al. 2011], but will be crucial for our argument.)

Let

$$u \in \mathcal{U} = \{2^\nu : \nu = 0, 1, 2, \ldots\}$$

be a collection of dyadic indices. For each k, let $\mathfrak{B}_k$ denote the collection of all 4-dimensional balls of radius $\leq 2^k$. For a ball B, let $\operatorname{rad} B$ denote the radius of B. Following [Heo et al. 2011], define

$$\begin{aligned}
\mathcal{R}_k &:= \mathcal{R} \cap [2^k, 2^{k+1}),\\
\mathcal{E}_k &:= \mathcal{E} \cap (\mathcal{Y} \times \mathcal{R}_k),\\
\widehat{\mathcal{E}}_k(u) &:= \{(y, r) \in \mathcal{E}_k : \exists B \in \mathfrak{B}_k \text{ such that } \#(\mathcal{E}_k \cap B) \geq u \operatorname{rad} B\},\\
\mathcal{E}_k(u) &= \widehat{\mathcal{E}}_k(u) \setminus \bigcup_{\substack{u' \in \mathcal{U}\\ u' > u}} \widehat{\mathcal{E}}_k(u').
\end{aligned}$$

We will refer to u as the *density* of the set $\mathcal{E}_k(u)$. Note that we have the decomposition

$$\mathcal{E}_k = \bigcup_{u \in \mathcal{U}} \mathcal{E}_k(u).$$

Let σ_r denote the surface measure on rS^2, the 2-sphere of radius r centered at the origin. Now fix a smooth, radial function ψ_0 which is supported in the ball centered at the origin of radius $\frac{1}{10}$ such that $\hat{\psi}_0$ vanishes to order 40 at the origin. Let $\psi = \psi_0 * \psi_0$. For $y \in \mathcal{Y}$ and $r \in \mathcal{R}$, define

$$F_{y,r} = \sigma_r * \psi(\,\cdot - y).$$

For a given function $c : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$, further define

$$G_{u,k} := \sum_{(y,r) \in \mathcal{E}_k(u)} c(y, r) F_{y,r}, \quad G_u := \sum_{k \geq 0} G_{u,k} \quad \text{and} \quad G_k := \sum_{u \in \mathcal{U}} G_{u,k}.$$

An interpolation lemma. As a preliminary tool, we will need the following dyadic interpolation lemma.

Lemma 2.1. *Let $0 < p_0 < p_1 < \infty$. Let $\{F_j\}_{j\in\mathbb{Z}}$ be a sequence of measurable functions on a measure space $\{\Omega, \mu\}$, and let $\{s_j\}$ be a sequence of nonnegative numbers. Assume that for all j, the inequality*

$$\|F_j\|_{p_\nu}^{p_\nu} \le 2^{jp_\nu} M^{p_\nu} s_j \tag{2-1}$$

holds for $\nu = 0$ and $\nu = 1$. Then for all $p \in (p_0, p_1)$, there is a constant $C = C(p_0, p_1, p)$ such that

$$\Big\|\sum_j F_j\Big\|_p^p \le C^p M^p \sum_j 2^{jp} s_j. \tag{2-2}$$

The discretized L^p inequality. Our goal is to prove the following proposition, which we will see implies our main result for compactly supported multipliers.

Proposition 2.2. *Let $\mathcal{E}$ and $\mathcal{E}_k$ be as above (recall that $\mathcal{E}$ has product structure). Let $c : \mathcal{E} \to \mathbb{C}$ be a function satisfying $|c(y,r)| \le 1$ for all $(y,r) \in \mathcal{E}$. Then for $1 < p < \frac{13}{12}$,*

$$\Big\|\sum_{(y,r)\in\mathcal{E}} c(y,r) F_{y,r}\Big\|_p^p \lesssim_p \sum_k 2^{2k} \#\mathcal{E}_k.$$

Using the dyadic interpolation lemma (Lemma 2.1), we obtain the following corollary.

Corollary 2.3. *Let E be any measurable set of finite measure, and χ_E its characteristic function. Suppose that f is a measurable function satisfying $|f| \le \chi_E$. Then for $1 < p < \frac{13}{12}$, we have*

$$\Big\|\sum_{(y,r)\in\mathcal{Y}\times\mathcal{R}} \gamma(r) f(y) F_{y,r}\Big\|_p \lesssim \Big(\sum_{(y,r)\in\mathcal{Y}\times\mathcal{R}} |\gamma(r)\chi_E(y)|^p r^2\Big)^{1/p}. \tag{2-3}$$

Also

$$\Big\|\int_{\mathbb{R}^d}\int_1^\infty h(r) f(y) F_{y,r}\,dr\,dy\Big\|_p \lesssim \Big(\int_{\mathbb{R}^d}\int_1^\infty |h(r)\chi_E(y)|^p r^2\,dr\,dy\Big)^{1/p}. \tag{2-4}$$

Proof that Proposition 2.2 implies Corollary 2.3. For $j \in \mathbb{Z}$, define the level sets

$$\mathcal{E}^j := \big\{(y,r) \in \mathcal{Y}\times\mathcal{R} : 2^{j-1} < |\gamma(r)\chi_E(y)| \le 2^j\big\}.$$

Notice that $\mathcal{E}^j$ has product structure, so Proposition 2.2 implies that for $1 < p < \frac{13}{12}$,

$$\Big\|\sum_{(y,r)\in\mathcal{E}^j} \gamma(r) f(y) F_{y,r}\Big\|_p^p \lesssim_p 2^{jp} \sum_{(y,r)\in\mathcal{E}^j} r^2.$$

Now apply Lemma 2.1 with $F_j = \sum_{(y,r)\in\mathcal{E}^j} \gamma(r) f(y) F_{y,r}$ and $M = 1$ and $s_j = \sum_{(y,r)\in\mathcal{E}^j} r^2$ to obtain (2-3).

Now we prove (2-4). Let $y = z + w$ for $z \in \mathbb{Z}^3$ and $w \in Q_0 := [0, 1)^3$ and $r = n + \tau$ for $n \in \mathbb{N}$ and $0 \le \tau < 1$. By Minkowski's inequality and (2-3),

$$\begin{aligned}\left\| \int_{\mathbb{R}^d} \int_1^\infty h(r) f(y) F_{y,r}\, dr\, dy \right\|_p &\lesssim \iint_{Q_0 \times [0,1)} \left\| \sum_{z \in \mathbb{Z}^d} \sum_{n=1}^\infty h(n+\tau) f(z+w) F_{z+w,n+\tau} \right\|_p dw\, d\tau \\ &\lesssim \iint_{Q_0 \times [0,1)} \left(\sum_{z \in \mathbb{Z}^d} \sum_{n=1}^\infty \left| h(n+\tau) \chi_E(z+w) \right|^p (n+\tau)^2 \right)^{1/p} dw\, d\tau \\ &\lesssim \left(\int_{\mathbb{R}^d} \int_1^\infty |h(r)\chi_E(y)|^p r^2\, dr\, dy \right)^{1/p},\end{aligned}$$

where in the last step we have used Hölder's inequality. □

Support size estimates vs. L^2 inequalities. As in [Heo et al. 2011], we will show that the functions $G_{u,k}$ either have relatively small support size or satisfy relatively good L^2 bounds. We begin with a support-size bound from that paper that improves as the density u increases.

Lemma C. *For all $u \in \mathcal{U}$, the Lebesgue measure of the support of $G_{u,k}$ is $\lesssim u^{-1} 2^{2k} \#\mathcal{E}_k$.*

We will prove the following L^2 inequality, which in some sense an improved version of Lemma 3.6 from [Heo et al. 2011], although the hypotheses are different since it is crucial that we assume that the underlying set $\mathcal{E}$ has product structure. This inequality improves as the density u decreases. In [Heo et al. 2011], the analogous L^2 inequality proved is

$$\|G_u\|_2^2 \lesssim u^{2/(d-1)} \log(2+u) \sum_k 2^{k(d-1)} \#\mathcal{E}_k, \tag{2-5}$$

and when $d = 3$ the term $u^{2/(d-1)}$ is equal to u. One may check that combining (2-5) with Lemma C as in the proof of Lemma 2.5 below yields no result in three dimensions. We use geometric methods to improve on (2-5) in three dimensions, and our argument will rely on Lemma A.1 proved later in the Appendix.

Lemma 2.4. *Let $\mathcal{E}$, $\mathcal{E}_k$, and G_u be as above (recall that $\mathcal{E}$ has product structure). Assume $|c(y, r)| \le 1$ for $(y, r) \in \mathcal{Y} \times \mathcal{R}$. Then for every $\epsilon > 0$,*

$$\|G_u\|_2^2 \lesssim_\epsilon u^{11/13+\epsilon} \sum_k 2^{2k} \#\mathcal{E}_k.$$

Combining Lemma C and Lemma 2.4, we obtain the following L^p bound.

Lemma 2.5. *For $p \le 2$, for every $\epsilon > 0$,*

$$\|G_u\|_p \lesssim_\epsilon u^{-(1/p-12/13-\epsilon)} \left(\sum_k 2^{2k} \#\mathcal{E}_k \right)^{1/p}.$$

Proof of Lemma 2.5 given Lemma C and Lemma 2.4. By Hölder's inequality,

$$\begin{aligned}\|G_u\|_p &\lesssim (\mathrm{meas}(\mathrm{supp}\, G_u))^{1/p-1/2} \|G_u\|_2 \\ &\lesssim_\epsilon u^{-1/p+1/2} u^{11/26+\epsilon} \left(\sum_k 2^{2k} \#\mathcal{E}_k \right)^{1/p} \lesssim_\epsilon u^{12/13-1/p+\epsilon} \left(\sum_k 2^{2k} \#\mathcal{E}_k \right)^{1/p}.\end{aligned}$$ □

Summing over $u \in \mathcal{U}$, we obtain Proposition 2.2. Thus to prove Proposition 2.2 it suffices to prove Lemma 2.4.

Compactly supported multipliers. Following [Heo et al. 2011], we now show how one may deduce Theorem 1.2 from Corollary 2.3. Suppose that $m : \mathbb{R}^3 \to \mathbb{C}$ is a bounded, measurable, radial function with compact support inside $\{\xi : \frac{1}{2} < |\xi| < 2\}$. Then $K = \mathcal{F}^{-1}[m]$ is radial, and so we may write $K(\cdot) = \kappa(|\cdot|)$ for some $\kappa : \mathbb{R} \to \mathbb{C}$. Fix a radial Schwartz function η_0 such that $\hat{\eta}_0(\xi) = 1$ on $\operatorname{supp} m$ and such that η_0 has Fourier support in $\{\frac{1}{4} < |\xi| < 4\}$. Set $\eta = \mathcal{F}^{-1}[(\hat{\psi})^{-1}\hat{\eta}_0]$. We have $K * f = \eta * \psi * K * f$. Let $K_0 = K\chi_{\{x:|x|\le 1\}}$ and write $K = K_0 + K_\infty$. Since $\|K_0\|_1 \lesssim \|K\|_p$, it suffices to show that the operator $f \mapsto \eta * \psi * K_\infty * f$ is restricted strong type (p, p) with operator norm $\lesssim_p \|K\|_p$. Let E be a measurable set of finite measure, and suppose that $|f| \le \chi_E$. We may write

$$\psi * K_\infty * f = \int_1^\infty \int \psi * \sigma_r(\cdot - y)\kappa(r) f(y)\, dy\, dr.$$

By Corollary 2.3, we have

$$\|\eta * \psi * K_\infty * f\|_p \lesssim_p \|\psi * K_\infty * f\|_p \lesssim_p \left(\int |\kappa(r)|^p r^2\, dr\right)^{1/p} \left(\int |\chi_E(y)|^p\, dy\right)^{1/p},$$

which implies the result of Theorem 1.2.

3. Proof of the L^2 inequality

We have shown in Section 2 that to prove our main result Theorem 1.2 it remains to prove Lemma 2.4, and this section is dedicated to the proof of that lemma. The proof will rely on a geometric lemma about sizes of multiple intersections of 3-dimensional annuli, which is stated and proved in the Appendix.

Estimates for scalar products. In order to obtain the desired L^2 estimate, we need to examine pairwise interactions of the form $\langle F_{y,r}, F_{y',r'}\rangle$. By applying Plancherel's theorem and writing $\widehat{F}_{y,r}$ and $\widehat{F}_{y',r'}$ as expressions involving Bessel functions, the authors of [Heo et al. 2011] obtained the following estimates for $|\langle F_{y,r}, F_{y',r'}\rangle|$.

Lemma 3.1. *For any choice of $r, r' > 1$ and $y, y' \in \mathbb{R}^3$,*

$$|\langle F_{y,r}, F_{y',r'}\rangle| \lesssim \frac{rr'}{1 + |y - y'| + |r - r'|}.$$

The proof of this lemma used only the decay and not the oscillation of the Bessel functions. By exploiting the oscillation of the Bessel functions, one may obtain the following improved bounds, which are crucial for our purposes. Since we will use this lemma in three and four dimensions, we state it in terms of dimension d, where the functions $F_{y,r}$ are defined analogously in d dimensions as they are defined previously in three dimensions.

Lemma 3.2. *For any choice of $r, r' > 1$ and $y, y' \in \mathbb{R}^d$ and any $N > 0$,*

$$|\langle F_{y,r}, F_{y',r'}\rangle| \le C_N (rr')^{(d-1)/2} (1 + |y - y'| + |r - r'|)^{-(d-1)/2} \sum_{\pm,\pm} \bigl(1 + \bigl|r \pm r' \pm |y - y'|\bigr|\bigr)^{-N}.$$

Proof of Lemma 3.2. We may write $\hat\sigma_1$ in terms of Bessel functions as $\hat\sigma_1(\xi) = B_d(|\xi|)$, where

$$B_d(s) = c_d s^{-(d-2)/2} J_{(d-2)/2}$$

and J denotes the standard Bessel function. This implies

$$\hat\sigma_r(\xi) = r^{d-1} B_d(r|\xi|).$$

Since $\hat\psi$ is radial, we may write $\hat\psi(\xi) = a(|\xi|)$ for some rapidly decaying function a that vanishes to high order (say $10d$) at the origin. By Plancherel, we have

$$\begin{aligned}\langle F_{y,r}, F_{y',r'}\rangle &= \int \hat\sigma_r(\xi)\hat\sigma_{r'}(\xi)|\hat\psi(\xi)|^2 e^{i\langle y'-y,\xi\rangle}\,d\xi\\ &= c_d(rr')^{d-1}\int B_d(r\rho)B_d(r'\rho)B_d(|y-y'|\rho)|a(\rho)|^2\rho^{d-1}\,d\rho.\end{aligned}$$

We will use the following well-known asymptotic expansion, which holds for $|x| \ge 1$ and any M:

$$B_d(x) = \sum_{\nu=0}^{M}(c^+_{\nu,k,d}e^{ix} + c^-_{\nu,k,d}e^{-ix})x^{-\nu-(d-1)/2} + x^{-M}E_{M,k,d}(x)$$

where for any $k_1 \ge 0$,

$$|E^{(k_1)}_{M,k,d}(x)| \le C(M,k,k_1,d).$$

Using this expansion together with the higher order of vanishing of a at the origin, one sees that there is a fixed Schwartz function η so that we obtain for any $N > 0$,

$$\langle F_{y,r}, F_{y',r'}\rangle \lesssim (rr')^{(d-1)/2}(1+|y-y'|)^{-(d-1)/2}\sum_{\pm,\pm}\eta(r\pm r'\pm|y-y'|) + (1+|r-r'|+|y-y'|)^{-N}.$$

In fact, we may take η to be the Fourier transform of $|a(\,\cdot\,)|^2\rho^{\alpha(d)}$ for some appropriate exponent $\alpha(d)$. □

Another preliminary reduction. Recall that our goal is to estimate the L^2 norm of $G_u = \sum_{k\ge 0} G_{u,k}$. Let $N(u)$ be a sufficiently large number to be chosen later (it will be some harmless constant depending on u that is essentially $O(\log(2+u))$). We split the sum in k as $\sum_{k\le N(u)} G_{u,k} + \sum_{k>N(u)} G_{u,k}$ and apply Cauchy–Schwarz to obtain

$$\Big\|\sum_k G_{u,k}\Big\|_2^2 \lesssim N(u)\Big[\sum_k \|G_{u,k}\|_2^2 + \sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle|\Big]. \tag{3-1}$$

We may thus separately estimate $\sum_k \|G_{u,k}\|_2^2$ and $\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle|$, which divides the proof of the L^2 estimate into two cases, the first being the case of "comparable radii" and the second being the case of "incomparable radii".

Comparable radii. We will first estimate $\sum_k \|G_{u,k}\|_2^2$. Our goal will be to prove the following lemma.

Lemma 3.3. *For every $\epsilon > 0$,*

$$\|G_{u,k}\|_2^2 \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)u^{11/13+\epsilon}. \tag{3-2}$$

Fix k and u. We first observe that for $(y, r), (y', r') \in \mathcal{E}_k(u)$, we have $\langle F_{y,r}, F_{y',r'}\rangle = 0$ unless $|(y, r) - (y', r')| \le 2^{k+5}$. To estimate $\|G_{u,k}\|_2^2$ for a fixed k, we would thus like to bound

$$\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|$$

for all $0 \le m \le k+4$.

Now fix $m \le k+4$. Let $\mathcal{Q}_{u,k,m}$ be a collection of almost disjoint cubes $Q \subset \mathbb{R}^4$ of side length 2^{m+5} such that $\mathcal{E}_k(u) \subset \bigcup_{Q\in\mathcal{Q}_{u,k,m}} Q$ and so that every Q has nonempty intersection with $\mathcal{E}_k(u)$. Let Q^* denote the 2^5-dilate of Q and $\mathcal{Q}^*_{u,k,m}$ the corresponding collection of dilated cubes. Observe that

$$\begin{aligned}\|G_{u,k}\|_2^2 &\lesssim \sum_{0\le m\le k+4}\Bigg(\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|\Bigg) + \sum_{(y,r)\in\mathcal{E}_k(u)} \|F_{y,r}\|_2^2\\ &\lesssim \sum_{0\le m\le k+4}\Bigg(\sum_{Q\in\mathcal{Q}_{u,k,m}}\Bigg(\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|\Bigg)\Bigg) + \sum_{(y,r)\in\mathcal{E}_k(u)} \|F_{y,r}\|_2^2. \qquad (3\text{-}3)\end{aligned}$$

Now we introduce some terminology that will be useful. For a subset $\mathcal{S} \subset \mathcal{Y}\times\mathcal{R}$, define its $\mathcal{Y}$- and $\mathcal{R}$-projections by

$$\mathcal{S}_Y = \{y \in \mathcal{Y} : \exists\, (y, r) \in \mathcal{S}\},$$
$$\mathcal{S}_R = \{r \in \mathcal{R} : \exists\, (y, r) \in \mathcal{S}\}.$$

Also define the *product-extension* $\mathcal{S}^\times$ of $\mathcal{S} \subset \mathcal{Y}\times\mathcal{R}$ to be the set $\mathcal{S}_Y \times \mathcal{S}_R$. We also define some parameters associated with a fixed $Q \in \mathcal{Q}_{u,k,m}$. Let $N_{R,Q}$ be the cardinality of the $\mathcal{R}$-projection of $\mathcal{E}_k \cap Q^*$, i.e.,

$$N_{R,Q} := \#((\mathcal{E}_k \cap Q^*)_R) = \#\{r : \exists (y, r) \in \mathcal{E}_k \cap Q^*\}.$$

Similarly define

$$N_{Y,Q} := \#((\mathcal{E}_k \cap Q^*)_Y) = \#\{y : \exists (y, r) \in \mathcal{E}_k \cap Q^*\}.$$

We also note the following important observation, which we will use repeatedly. Using the definition of the sets $\mathcal{E}_k(u)$ and the fact that $\mathcal{E}_k$ has product structure, one may see that if $Q \in \mathcal{Q}_{u,k,m}$ is such that $(\mathcal{E}_k(u) \cap Q^*)$ is nonempty, then

$$|N_{Y,Q}\cdot N_{R,Q}| \lesssim |\mathcal{E}_k \cap Q^*| \lesssim u2^m. \qquad (3\text{-}4)$$

We remark that the product structure of the sets $\mathcal{E}_k$ is related to the "tensor product structure" intrinsic to radial Fourier multipliers, mentioned in Section 1. Now with (3-3) in mind, we will prove the following lemma.

Lemma 3.4. *For each $Q \in \mathcal{Q}_{u,k,m}$, we have the estimates*

$$\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim N_{R,Q}(\#(\mathcal{E}_k\cap Q^*))2^{2(k-m/2)}(m\log(u))\max(u^{5/6}2^{5m/6}, u2^{m/2}) \quad (3\text{-}5)$$

and

$$\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim 2^{2(k-m/2)}(\#(\mathcal{E}_k\cap Q^*))u2^m(N_{R,Q})^{-1}. \tag{3-6}$$

We will then choose the better estimate from Lemma 3.4 depending on $N_{R,Q}$ and sum over all $Q\in\mathcal{Q}_{u,k,m}$ and then over all $m\ge u^a$, where a is a number to be chosen later. We will then use other methods to deal with the case $m\le u^a$, from which we will then obtain Lemma 3.3.

Proof of Lemma 3.4. We will first prove (3-5). By incurring a factor of $N_{R,Q}^2$, to estimate

$$\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|$$

it suffices to estimate, for a fixed pair r_1, r_2,

$$\sum_{\substack{(y,r_1),(y',r_2)\in(\mathcal{E}_k(u)^\times\cap Q^*)\\ 2^m\le|(y,r_1)-(y',r_2)|\le 2^{m+1}}} |\langle F_{y,r_1}, F_{y',r_2}\rangle|,$$

i.e., to restrict (y,r) and (y',r') to lie in fixed rows of the product-extension of $\mathcal{E}_k(u)\cap Q^*$. (Our estimates will not depend on the particular choice of r_1 and r_2.)

Now, referring to the estimate in Lemma 3.2, we see that for fixed y, r_1, r_2 we have that $|\langle F_{y,r_1}, F_{y',r_2}\rangle|$ decays rapidly as y' moves away from the set $\{y' : |y-y'|=|r_1-r_2| \text{ or } |y-y'|=r_1+r_2\}$, which is contained in a union of two annuli of thickness 2 and radii $|r_1-r_2|$ and r_1+r_2 centered at y.

Let $s\ge 0$, fix $t\le 2^{m+10}$, and define $K_k(Q,s,t)$ to be the number of points $y\in(\mathcal{E}_k(u)\cap Q^*)_Y$ such that there are at least 2^s many points $y'\in(\mathcal{E}_k\cap Q^*)_Y$ such that y' lies in the annulus of inner radius t and thickness 3 centered at y. That is, define

$$K_k(Q,s,t) := \#\big\{y\in(\mathcal{E}_k(u)\cap Q^*)_Y : \text{there exist at least } 2^s \text{ many points } y'\in(\mathcal{E}_k\cap Q^*)_Y \text{ such that } \big||y'-y|-\big(t+\tfrac{3}{2}\big)\big|\le\tfrac{3}{2}\big\}.$$

In view of the observation in the previous paragraph, for a given s and a fixed number $t\le 2^{m+10}$, we would like to prove a bound on $K_k(Q,s,t)$. Our bound will depend on s and m but be independent of the choice of $t\le 2^{m+10}$. For this reason, we define the quantity

$$K_k^*(Q,s) := \max_{0\le t\le 2^{m+10}} K_k(Q,s,t),$$

and we will see that $K_k^*(Q,s)$ satisfies the same bound we prove for $K_k(Q,s,t)$. Our bound for $K_k(Q,s,t)$ will decay as 2^s gets larger and closer to $N_{Y,Q}$; in other words, "most" of the points y in $(\mathcal{E}_k(u)\cap Q^*)_Y$ cannot have a large proportion of other points in $(\mathcal{E}_k\cap Q^*)_Y$ lie in the annulus of inner radius t and thickness 3 centered at y. If we take $t=|r_1-r_2|$ or $t=r_1+r_2$, we see that this implies that "most" of the $F_{y,r}$ with $(y,r)\in(\mathcal{E}_k(u)\cap Q^*)_Y\times\{r_1\}$ do not "interact badly" (where by badly we mean to the worst possible extent allowed by Lemma 3.2, i.e., internal tangencies of annuli) with most of the other $F_{y',r'}$ where $(y',r')\in(\mathcal{E}_k\cap Q^*)_Y\times\{r_2\}$. This will allow us to obtain (3-5), which is a good estimate in the case that $N_{R,Q}$ is small.

More precisely, we will prove

$$K_k^*(Q,s) \lesssim \max\bigl[u2^m N_{Y,Q}^{5/3} 2^{-2s},\ u2^{m/2} N_{Y,Q} 2^{-s}\bigr]. \tag{3-7}$$

Combining this with the trivial bound $K_k^*(Q,s) \lesssim N_{Y,Q}$ yields

$$K_k^*(Q,s) \lesssim \max\bigl[\min(u2^m N_{Y,Q}^{5/3} 2^{-2s}, N_{Y,Q}), \min(u2^{m/2} N_{Y,Q} 2^{-s}, N_{Y,Q})\bigr]. \tag{3-8}$$

Note that (3-7) gives decay in the number of points $K_k^*(Q,s)$ (i.e., $K_k^*(Q,s) \ll N_{Y,Q}$) if we have that

(1) $N_{Y,Q}^{5/3} 2^{-2s} u 2^m \ll N_{Y,Q}$, i.e., if $2^s \gg N_{Y,Q}^{1/3} u^{1/2} 2^{m/2}$, and also

(2) $N_{Y,Q} 2^{-s} u 2^{m/2} \ll N_{Y,Q}$, i.e., if $2^s \gg u 2^{m/2}$.

Using Lemma 3.2, we may bound

$$\begin{aligned}
&\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\
&\quad\lesssim \sum_{r_1,r_2\in(\mathcal{E}_k(u)\cap Q^*)_R} \Biggl(\sum_{\substack{y,y'\in(\mathcal{E}_k(u)\cap Q^*)_Y\\ 2^m\le|(y,r_1)-(y',r_2)|\le 2^{m+1}}} |\langle F_{y,r_1}, F_{y',r_2}\rangle| \Biggr) \\
&\quad\lesssim 2^{2(k-m/2)} \sum_{r_1,r_2\in(\mathcal{E}_k(u)\cap Q^*)_R} \Biggl(\sum_{0\le a\le m+10} \Biggl(\sum_{y\in(\mathcal{E}_k(u)\cap Q^*)_Y} \sum_{\substack{y'\in(\mathcal{E}_k(u)\cap Q^*)_Y\\ \min_{\pm,\pm}(1+|r_1\pm r_2\pm|y-y'||)\approx 2^a}} 2^{-aN} \Biggr)\Biggr) \\
&\quad\lesssim 2^{2(k-m/2)} \sum_{r_1,r_2\in(\mathcal{E}_k(u)\cap Q^*)_R} \Biggl(\sum_{0\le a\le m+10} 2^{-aN} \Biggl(\sum_{s\ge 0:2^s\le 2N_{Y,Q}} K_k^*(Q,s) 2^s \Biggr)\Biggr) \\
&\quad\lesssim 2^{2(k-m/2)} N_{Q,R}^2 \sum_{s\ge 0:2^s\le 2N_{Y,Q}} K_k^*(Q,s) 2^s.
\end{aligned} \tag{3-9}$$

Assuming (3-8) holds, we have

$$\begin{aligned}
&\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\
&\quad\lesssim N_{R,Q}^2 2^{2(k-m/2)} \sum_{s\ge 0:2^s\lesssim N_{Y,Q}} \max\bigl[\min(u2^m N_{Y,Q}^{5/3} 2^{-s}, N_{Y,Q}2^s), \min(u2^{m/2} N_{Y,Q}, N_{Y,Q}2^s)\bigr] \\
&\quad\lesssim N_{R,Q}^2 2^{2(k-m/2)} \max\Biggl[\sum_{s\ge 0:2^s\lesssim N_{Y,Q}} \min(u2^m N_{Y,Q}^{5/3} 2^{-s}, N_{Y,Q}2^s), \sum_{s\ge 0:2^s\lesssim N_{Y,Q}} \min(u2^{m/2} N_{Y,Q}, N_{Y,Q}2^s) \Biggr]
\end{aligned} \tag{3-10}$$

Now, note that $u2^m N_{Y,Q}^{5/3} 2^{-s} \ge N_{Y,Q} 2^s$ if and only if $2^s \le u^{1/2} 2^{m/2} N_{Y,Q}^{1/3}$. Thus choosing the better estimate in the term $\min(u2^m N_{Y,Q}^{5/3} 2^{-s}, N_{Y,Q} 2^s)$ depending on s yields that

$$\sum_{s\ge 0:2^s\lesssim N_{Y,Q}} \min(u2^m N_{Y,Q}^{5/3} 2^{-s}, N_{Y,Q}2^s) \lesssim u^{1/2} 2^{m/2} N_{Y,Q}^{4/3}.$$

Note that $u2^{m/2}N_{Y,Q} \ge N_{Y,Q}2^s$ if and only if $2^s \le u2^{m/2}$. Thus choosing the better estimate in the term $\min(u2^{m/2}N_{Y,Q},\, N_{Y,Q}2^s)$ depending on s yields that

$$\sum_{s\ge 0:2^s\lesssim N_{Y,Q}} \min(u2^{m/2}N_{Y,Q},\, N_{Y,Q}2^s) \lesssim \log(N_{Y,Q})N_{Y,Q}\, u2^{m/2}.$$

It follows that the left-hand side of (3-10) is bounded by

$$\begin{aligned} N_{R,Q}^2 2^{2(k-m/2)} N_{Y,Q} \log(N_{Y,Q}) \max(N_{Y,Q}^{1/3}u^{1/2}2^{m/2},\, u2^{m/2}) \\ \lesssim N_{R,Q}^2 2^{2(k-m/2)} N_{Y,Q}(m\log(u)) \max(u^{5/6}2^{5m/6},\, u2^{m/2}) \\ \lesssim N_{R,Q}(\#(\mathcal{E}_k\cap Q^*))2^{2(k-m/2)}(m\log(u)) \max(u^{5/6}2^{5m/6},\, u2^{m/2}), \end{aligned} \tag{3-11}$$

which proves (3-5). This will be a good estimate when $N_{R,Q}$ is small.

Thus to prove (3-5) it remains to prove (3-7). We will in fact prove (3-7) with $K_k^*(Q,s)$ replaced by $K_k(Q,s,t)$, uniformly in $t \le 2^{m+10}$. Fix $t \le 2^{m+10}$ and let $j = \lceil \log_2(t)\rceil$ and cover $(\mathcal{E}_k(u)\cap Q^*)_Y$ by $\lesssim 2^{3(m-j)}$ many 3-dimensional almost disjoint balls of radius 2^{j+5}; denote this collection of balls as $\mathfrak{B} = \{B_i\}$. For each i, we define a collection of "special" points $A_i(Q,s,t)$ to be the set of all points $y \in (\mathcal{E}_k(u)\cap Q^*)_Y \cap B_i$ such that there are $\ge 2^s$ many points $y' \in (\mathcal{E}_k\cap Q^*)_Y$ such that y' lies in the annulus of radius t and thickness 3 centered at y. That is, we define

$$A_{k,i}(Q,s,t) := \big\{ y \in (\mathcal{E}_k(u)\cap Q^*)_Y \cap B_i :\ \text{there exist at least } 2^s \text{ many points} \\ y' \in (\mathcal{E}_k\cap Q^*)_Y \text{ such that } \big||y'-y| - \big(t+\tfrac{3}{2}\big)\big| \le \tfrac{3}{2}\big\}.$$

Let $K_{k,i}(Q,s,t)$ denote the cardinality of $A_{k,i}(Q,s,t)$. Now cover each B_i with $\lesssim 2^{3(j-l)}$ many almost disjoint 3-dimensional balls $\{B_{i,\alpha}\}_\alpha$ of radius 2^l for some $l \le j$. Each such ball contains at most $u2^l$ many points of $A_{k,i}(Q,s,t)$, so for a fixed i there must be $\gtrsim K_{k,i}(Q,s,t)(u2^l)^{-1}$ many balls $B_{i,\alpha}$ that contain at least one point in $A_{k,i}(Q,s,t)$. Thus there must be at least $\gtrsim K_{k,i}(Q,s,t)(u2^l)^{-1}$ many such points in $B_i \cap A_{k,i}(Q,s,t)$ spaced apart by $\gtrsim 2^l$; call this set $D_{k,i}(Q,s,t)$. But by Lemma A.1, which we prove later in the Appendix, the size of 3-fold intersections of annuli of radius $t \approx 2^j$ and thickness 3 spaced apart by $\approx 2^l$ with centers lying in a ball of radius 2^{j-5} is bounded above by $2^{3(j-l)}$ provided that $l \ge j/2+20$.

It follows that if $l \ge j/2+20$, then for each of these $\approx K_{k,i}(Q,s,t)(u2^l)^{-1}$ many points $p \in D_{k,i}(Q,s,t)$, there can be

$$\lesssim K_{k,i}(Q,s,t)^2(u2^l)^{-2}2^{3(j-l)}$$

points lying inside the t-annulus centered at p that are simultaneously contained in at least two other different t-annuli centered at points in $D_{k,i}(Q,s,t)$. This implies that if $N_{Y,Q,i}$ denotes the cardinality of $(\mathcal{E}_k\cap Q^*)_Y \cap B_i^*$, where $B_i^* = 10B_i$, then we have

$$N_{Y,Q,i} \gtrsim K_{k,i}(Q,s,t)(u2^l)^{-1}2^s, \tag{3-12}$$

which is essentially 2^s times the number of points in $D_{k,i}(Q,s,t)$, provided that 2^s is much bigger than the total number of points lying inside a t-annulus centered at p that are simultaneously contained in at least two other different t-annuli centered at points in $D_{k,i}(Q,s,t)$, i.e., provided that

$$K_{k,i}(Q,s,t)^2(u2^l)^{-2}2^{3(j-l)} \ll 2^s \tag{3-13}$$

and

$$l \geq j/2+20.$$

Solving for 2^l in (3-13) yields

$$2^l \gg K_{k,i}(Q,s,t)^{2/5}2^{3j/5}u^{-2/5}2^{-s/5}. \tag{3-14}$$

Thus choosing a minimal l such that

$$2^l \gg \max\bigl[K_{k,i}(Q,s,t)^{2/5}2^{3j/5}u^{-2/5}2^{-s/5},\ 2^{j/2}\bigr]$$

for a sufficiently large implied constant and substituting into (3-12) yields

$$K_{k,i}(Q,s,t) \lesssim \max\bigl[u2^m N_{Y,Q,i}^{5/3}2^{-2s},\ u2^{m/2}N_{Y,Q,i}2^{-s}\bigr], \tag{3-15}$$

and summing over all i and using the almost-disjointness of the B_i^* gives

$$K_k(Q,s,t) \lesssim \max\bigl[u2^m N_{Y,Q}^{5/3}2^{-2s},\ u2^{m/2}N_{Y,Q}2^{-s}\bigr]. \tag{3-16}$$

Taking the maximum over all $0 \leq t \leq 2^{m+10}$ proves (3-7) and hence also (3-5).

It remains to prove (3-6), which will be a good estimate in the case that $N_{R,Q}$ is large. For a fixed $(y,r)\in Q^*$ and a fixed $y' \in (\mathcal{E}_k(u)\cap Q^*)_Y$, there are at most two values of r' away from which $\langle F_{y,r}, F_{y',r'}\rangle$ decays rapidly. Thus using Lemma 3.2 we may estimate

$$\begin{aligned}
&\sum_{\substack{(y,r),(y',r')\in(\mathcal{E}_k(u)\cap Q^*)\\ 2^m\leq|(y,r)-(y',r')|\leq 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\
&\quad\lesssim \sum_{0\leq a\leq m+10}\Biggl(\sum_{(y,r)\in(\mathcal{E}_k(u)\cap Q^*)}\Biggl(\sum_{y'\in(\mathcal{E}_k(u)\cap Q^*)_Y}\Biggl(\sum_{\substack{r'\in(\mathcal{E}_k(u)\cap Q^*)_R\\ 2^m\leq|(y,r)-(y',r')|\leq 2^{m+1}\\ \min_{\pm,\pm}(1+|r\pm r'\pm|y-y'||)\approx 2^a}} 2^{-Na}2^{2(k-m/2)}\Biggr)\Biggr)\Biggr) \\
&\quad\lesssim 2^{2(k-m/2)}(\#(\mathcal{E}_k(u)\cap Q^*))N_{Y,Q} \lesssim 2^{2(k-m/2)}(\#(\mathcal{E}_k(u)\cap Q^*))u2^m(N_{Q,R})^{-1},
\end{aligned} \tag{3-17}$$

and the proof of (3-6) is complete. □

We will now use Lemma 3.4 to prove Lemma 3.3.

Proof of Lemma 3.3. Fix an $a>0$ to be determined later. As in [Heo et al. 2011], we let $G_k=\sum_\mu G_{k,\mu}$, where for each positive integer μ we set

$$I_{k,\mu} = \bigl[2^k+(\mu-1)u^a,\ 2^k+\mu u^a\bigr),$$

$$\mathcal{E}_{k,\mu} = \mathcal{E}_k \cap (\mathcal{Y}\times I_{k,\mu}),$$

$$G_{k,\mu} = \sum_{(y,r)\in\mathcal{E}_{k,\mu}} c(y,r)F_{y,r} \quad\text{and}\quad G_{k,\mu,r} = \sum_{y:(y,r)\in\mathcal{E}_k} c(y,r)F_{y,r}.$$

We have

$$\|G_k\|_2^2 \lesssim \Big\|\sum_{\mu} G_{k,\mu}\Big\|_2^2 \lesssim \sum_{\mu} \|G_{k,\mu}\|_2^2 + \sum_{\mu'>\mu+10} |\langle G_{k,\mu'}, G_{k,\mu}\rangle|. \tag{3-18}$$

By Cauchy–Schwarz,

$$\|G_{k,\mu}\|_2^2 \lesssim u^a \sum_{r\in\mathcal{I}_{k,\mu}\cap\mathcal{R}} \|G_{k,\mu,r}\|_2^2.$$

Write

$$G_{k,\mu,r} = \Big(\sum_{y:(y,r)\in\mathcal{E}_{k,\mu}} c(y,r)\psi_0(\,\cdot - y)\Big) * (\sigma_r * \psi_0).$$

By the Fourier decay of σ_r and the order of vanishing of ψ_0 at the origin, we have

$$\|\hat{\sigma}_r \hat{\psi}_0\|_\infty \lesssim r.$$

Since the square of the L^2 norm of $\sum_{y:(y,r)\in\mathcal{E}_{k,\mu}} c(y,r)\psi_0(\,\cdot - y)$ is $\lesssim \#\{y\in\mathcal{Y} : (y,r)\in\mathcal{E}_{k,\mu}\}$, we have

$$\sum_{\mu} \|G_{k,\mu}\|_2^2 \lesssim u^a \sum_{\mu} \sum_{r\in\mathcal{I}_{k,\mu}\cap\mathcal{R}} \|G_{k,\mu,r}\|_2^2 \lesssim u^a 2^{2k} \#\mathcal{E}_k. \tag{3-19}$$

By (3-18), it remains to estimate $\sum_{\mu'>\mu+10} |\langle G_{k,\mu'}, G_{k,\mu}\rangle|$.

Fix $\epsilon > 0$. We will use (3-5) when $N_{R,Q} \le 2^{m\epsilon} \min(u^{1/12+a/12}, u^{a/4})$ and (3-6) when $N_{R,Q} \ge 2^{m\epsilon} \min(u^{1/12+a/12}, u^{a/4})$. We write

$$\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ |(y,r)-(y',r')|\ge u^a}} |\langle F_{y,r}, F_{y',r'}\rangle|$$

$$\lesssim \sum_{m:2^m\ge u^a} \Bigg(\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ |(y,r)-(y',r')|\approx 2^m}} \Bigg(\sum_{\substack{Q\in\mathcal{Q}_{u,k,m}\\ N_{R,Q}\le 2^{m\epsilon}\min(u^{1/12+a/12},u^{a/4})}} |\langle F_{y,r}, F_{y',r'}\rangle| + \sum_{\substack{Q\in\mathcal{Q}_{u,k,m}\\ N_{R,Q}\ge 2^{m\epsilon}\min(u^{1/12+a/12},u^{a/4})}} |\langle F_{y,r}, F_{y',r'}\rangle| \Bigg)\Bigg).$$

One sees that

$$\sum_{\substack{(y,r),(y',r')\in\mathcal{E}_k(u)\\ |(y,r)-(y',r')|\ge u^a}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim I + II, \tag{3-20}$$

where using (3-5) when $N_{R,Q} \le 2^{m\epsilon} \min(u^{1/12+a/12}, u^{a/4})$ and summing over all $Q \in \mathcal{Q}_{u,k,m}$ and over all m such that $2^m \ge u^a$ we have

$$\begin{aligned} I &:= 2^{2k}(\#\mathcal{E}_k)\log(u) \sum_{m:2^m\ge u^a} u^\epsilon \max\big[2^{-m/6+\epsilon}\min(u^{11/12+a/12}, u^{5/6+a/4}), 2^{-m/2+\epsilon}\min(u^{13/12+a/12}, u^{1+a/4})\big] \\ &\lesssim 2^{2k}(\#\mathcal{E}_k)u^\epsilon \max\big[u^{-a/6}\min(u^{11/12+a/12}, u^{5/6+a/4}), u^{-a/2}\min(u^{13/12+a/12}, u^{1+a/4})\big], \end{aligned} \tag{3-21}$$

and using (3-6) when $N_{R,Q} \ge 2^{m\epsilon} \min(u^{1/12+a/12}, u^{a/4})$ and summing over all Q and over all m such that $2^m \ge u^a$ we have

$$II := 2^{2k}(\#\mathcal{E}_k)u^\epsilon \sum_{m:2^m\ge u^a} 2^{-m\epsilon} \max(u^{11/12-a/12}, u^{1-a/4}) \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)u^\epsilon \max(u^{11/12-a/12}, u^{1-a/4}). \tag{3-22}$$

Combining (3-18), (3-19) and (3-20), we thus have the estimate

$$\|G_{u,k}\|_2^2 \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)\big[u^a + u^\epsilon \max\big\{u^{-a/6}\min(u^{11/12+a/12}, u^{5/6+a/4}), u^{-a/2}\min(u^{13/12+a/12}, u^{1+a/4})\big\} + u^\epsilon \max(u^{11/12-a/12}, u^{1-a/4})\big].$$

Choose $a = \frac{11}{13}$ to obtain

$$\|G_{u,k}\|_2^2 \lesssim_\epsilon 2^{2k}(\#\mathcal{E}_k)u^{11/13+\epsilon}$$

for every $\epsilon > 0$, which is (3-2). □

Incomparable radii. We now want to estimate $\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle|$. Our estimate will be much better than in the comparable radii case. In view of (3-1), we will in fact prove the following.

Lemma 3.5. *Let $\epsilon > 0$. For the choice $N(u) = 100\epsilon^{-1}\log_2(2+u)$, we have*

$$\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle| \lesssim_\epsilon \sum_k 2^{2k}\#\mathcal{E}_k. \tag{3-23}$$

Fix u and k. Similar to the case of comparable radii, the first step is to cover $\mathcal{E}_k(u)$ by a collection $\mathcal{Q}_{u,k}$ of almost-disjoint cubes Q of side length 2^{k+5}. By the almost-disjointness of the cubes, it is enough to estimate $|\langle G_{u,k'}, G_{u,k}\rangle|$ when we restrict our points in $\mathcal{E}_k(u)$ and $\mathcal{E}_{k'}(u)$ to points in a fixed Q^* and get an estimate in terms of $\#(\mathcal{E}_k \cap Q^*)$, after which we may sum in $Q \in \mathcal{Q}_{u,k}$. So fix such a cube Q, and let $N_{R,Q,k}$ denote the cardinality of $(\mathcal{E}_k \cap Q^*)_R$ and for a fixed k', let $N_{R,Q,k'}$ denote the cardinality of $(\mathcal{E}_{k'} \cap Q^*)_R$. Similarly, let $N_{Y,Q,k}$ denote the cardinality of $(\mathcal{E}_k \cap Q^*)_Y$ and for a fixed k', let $N_{Y,Q,k'}$ denote the cardinality of $(\mathcal{E}_{k'} \cap Q^*)_Y$. Next, we prove a lemma that plays a role similar to Lemma 3.4 in the comparable radii case.

Lemma 3.6. *For each $Q \in \mathcal{Q}_{u,k}$, we have the estimates*

$$\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle| \lesssim R^2\#(\mathcal{E}_k \cap Q^*)u(N_{R,Q,k'})^{-1} \tag{3-24}$$

and

$$\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle| \lesssim N_{R,Q,k'}(\#(\mathcal{E}_k \cap Q^*))2^k(k\log(u))\max(u^{5/6}2^{5k/6}, u2^{k/2}). \tag{3-25}$$

Proof of Lemma 3.6. We will first prove (3-24), which will be a good estimate in the case that $N_{R,Q,k'}$ is large. For each $(Y, R) \in (\mathcal{E}_k(u) \cap Q^*)$ we need only consider $y \in (\mathcal{E}_{k'}(u) \cap Q^*)_Y$ lying in an annulus of width $2^{k'+5}$ built upon the sphere of radius R centered at Y in $\mathbb{R}^3$. Cover the intersection of this annulus with $(\mathcal{E}_{k'}(u) \cap Q^*)_Y$ by a collection $\mathcal{C}$ of $\lesssim R^2 2^{-2k'}$ 3-dimensional cubes C of side length $2^{k'+3}$ in $\mathbb{R}^3$ such that each $C \cap (\mathcal{E}_{k'}(u) \cap Q^*)_Y$ is nonempty. For each $C \in \mathcal{C}$, let $\widetilde{C}$ denote the 4-dimensional cube $\widetilde{C} = C \times [2^{k'} - 2^{k'+2}, 2^{k'} + 2^{k'+2}]$, and let $\widetilde{\mathbb{C}}$ denote the corresponding collection of cubes $\widetilde{C}$. Now note that $C \cap (\mathcal{E}_{k'}(u) \cap Q^*)_Y$ nonempty implies that $(\widetilde{C} \cap \mathcal{E}_{k'} \cap Q^*)_R = (\mathcal{E}_{k'} \cap Q^*)_R$, and also that $\#(\widetilde{C} \cap \mathcal{E}_{k'}) \lesssim u2^{k'}$, and hence by the product structure of $\widetilde{C} \cap \mathcal{E}_{k'} \cap Q^*$,

$$\#\big((\widetilde{C} \cap \mathcal{E}_{k'} \cap Q^*)_Y\big) \lesssim \#(\widetilde{C} \cap \mathcal{E}_{k'})\big(\#(\widetilde{C} \cap \mathcal{E}_{k'} \cap Q^*)_R\big)^{-1} \lesssim u2^{k'}(N_{R,Q,k'})^{-1}. \tag{3-26}$$

Next, note that for a fixed $Y \in (\mathcal{E}_k \cap Q^*)_Y$, a fixed $R \in (\mathcal{E}_k \cap Q^*)_R$, and a fixed $y \in (\mathcal{E}_{k'} \cap Q^*)_Y$, Lemma 3.2 gives rapid decay for $|\langle F_{Y,R}, F_{y,r}\rangle|$ as r moves away from two possible values of r', that is, when r moves far away from $r' = R - |Y - y|$ and $r' = |Y - y| - R$. For these values of r' we have $|\langle F_{Y,R}, F_{y,r'}\rangle| \lesssim 2^{k'}$. Using (3-26) and our bound on the size of the collection $\mathcal{C}$, we thus have

$$\begin{aligned}
&\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*}\ \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle| \\
&\quad\lesssim \sum_{(Y,R)\in\mathcal{E}_k\cap Q^*}\Bigl(\sum_{\widetilde{C}\in\widetilde{\mathbb{C}}}\Bigl(\sum_{(y,r)\in\mathcal{E}_{k'}\cap Q^*\cap\widetilde{C}} |\langle F_{Y,R}, F_{y,r}\rangle|\Bigr)\Bigr) \\
&\quad\lesssim \sum_{(Y,R)\in\mathcal{E}_k\cap Q^*}\Bigl(\sum_{\widetilde{C}\in\widetilde{\mathbb{C}}}\Bigl(\sum_{y\in(\mathcal{E}_{k'}\cap Q^*\cap\widetilde{C})_Y}\Bigl(\sum_{a\in\mathbb{Z},a\ge 0}\Bigl(\sum_{\substack{r\in(\mathcal{E}_{k'}\cap Q^*)_R\\ \max(|r'-r+|Y-y'||,|r'+r-|Y-y||)\approx 2^a}} 2^{-aN}2^{k'}\Bigr)\Bigr)\Bigr)\Bigr)\Bigr) \\
&\quad\lesssim R^2\#(\mathcal{E}_k\cap Q^*)(N_{R,Q,k'})^{-1}u,
\end{aligned}$$

which is (3-24).

Now we prove (3-25), which is the estimate that we will use in the case that $N_{R,Q,k'}$ is small. This estimate is similar to (3-5), and the proof is very similar with only minor modifications, but we give all the details anyways.

By incurring a factor of $N_{R,Q,k} \cdot N_{R,Q,k'}$, to estimate

$$\sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*}\ \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y,r}\rangle|,$$

it suffices to estimate for a fixed pair $r_1 \in (\mathcal{E}_k \cap Q^*)_R$ and $r_2 \in (\mathcal{E}_{k'} \cap Q^*)_R$

$$\sum_{(Y,r_1)\in\mathcal{E}_k\cap Q^*}\ \sum_{(y,r_2)\in\mathcal{E}_{k'}\cap Q^*} |\langle F_{Y,r_1}, F_{y,r_2}\rangle|.$$

Similar to the proof of (3-5), for $s \ge 0$, let $N'_{Y,Q,k} = 2^s \le N_{Y,Q,k}$ be a given dyadic number. Fix $t \le 2^{k+10}$, and define $K_{k,k'}(Q, s, t)$ to be the number of points $y \in (\mathcal{E}_k(u) \cap Q^*)_Y$ such that there are $\ge N'_{Y,Q,k} = 2^s$ many points $y' \in (\mathcal{E}_{k'} \cap Q^*)_Y$ such that y' lies in the annulus of inner radius t and thickness 3 centered at y. That is, define

$$K_{k,k'}(Q, s, t) := \#\bigl\{y \in (\mathcal{E}_k(u) \cap Q^*)_Y : \text{there exist at least } 2^s \text{ many points } y' \in (\mathcal{E}_{k'} \cap Q^*)_Y \text{ such that } \bigl||y' - y| - \bigl(t + \tfrac{3}{2}\bigr)\bigr| \le \tfrac{3}{2}\bigr\}.$$

Also define

$$K^*_{k,k'}(Q, s) := \max_{0\le t\le 2^{k+10}} K_{k,k'}(Q, s, t).$$

Note that the product structure of $\mathcal{E}$ implies that if both $\mathcal{E}_k \cap Q^*$ and $\mathcal{E}_{k'} \cap Q^*$ are nonempty, then their $\mathcal{Y}$-projections are equal, and so (3-8) implies the bound

$$K_{k,k'}(Q, s, t) \lesssim \max\bigl[\min(u2^k N_{Y,Q,k}^{5/3} 2^{-2s}, N_{Y,Q,k}), \min(u2^{k/2} N_{Y,Q,k} 2^{-s}, N_{Y,Q,k})\bigr]. \tag{3-27}$$

Using Lemma 3.2, we may bound

$$\begin{aligned}\sum_{\substack{(Y,R)\in(\mathcal{E}_k(u)\cap Q^*)\\(y,r)\in(\mathcal{E}_{k'}(u)\cap Q^*)}} |\langle F_{Y,R}, F_{y,r}\rangle| &\lesssim \sum_{\substack{r_1\in(\mathcal{E}_k(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_{k'}(u)\cap Q^*)_R}} \Bigg(\sum_{\substack{Y\in(\mathcal{E}_k(u)\cap Q^*)_Y\\ y\in(\mathcal{E}_{k'}\cap Q^*)_Y}} |\langle F_{Y,r_1}, F_{y,r_2}\rangle|\Bigg)\\
&\lesssim 2^k \sum_{\substack{r_1\in(\mathcal{E}_k(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_{k'}(u)\cap Q^*)_R}} \Bigg(\sum_{0\le a\le m+10}\Bigg(\sum_{Y\in(\mathcal{E}_k(u)\cap Q^*)_Y} \sum_{\substack{y\in(\mathcal{E}_{k'}\cap Q^*)_Y\\ \min_{\pm,\pm}(1+|r_1\pm r_2\pm|y-y'||)\approx 2^a}} 2^{-aN}\Bigg)\Bigg)\\
&\lesssim 2^k \sum_{\substack{r_1\in(\mathcal{E}_k(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_{k'}(u)\cap Q^*)_R}} \Bigg(\sum_{0\le a\le m+10} 2^{-aN}\Bigg(\sum_{s\ge 0:2^s\le 2N_{Y,Q,k}} K^*_{k,k'}(Q,s)2^s\Bigg)\Bigg)\\
&\lesssim 2^k N_{R,Q,k}\, N_{R,Q,k'} \sum_{s\ge 0:2^s\le 2N_{Y,Q,k}} K^*_{k,k'}(Q,s)2^s. \qquad (3\text{-}28)\end{aligned}$$

Applying (3-27), we have

$$\begin{aligned}&\sum_{\substack{(Y,R)\in(\mathcal{E}_k(u)\cap Q^*)\\(y,r)\in(\mathcal{E}_{k'}(u)\cap Q^*)}} |\langle F_{Y,R}, F_{y,r}\rangle|\\
&\lesssim N_{R,Q,k}\, N_{R,Q,k'}\, 2^k \sum_{s\ge 0:2^s\lesssim N_{Y,Q,k}} \max\big[\min(u2^kN^{5/3}_{Y,Q,k}2^{-s},\ N_{Y,Q,k}2^s), \min(u2^{k/2}N_{Y,Q,k}, N_{Y,Q,k}2^s)\big]. \qquad (3\text{-}29)\end{aligned}$$

Now, note that $u2^kN^{5/3}_{Y,Q}2^{-s} \ge N_{Y,Q}2^s$ if and only if $2^s \le u^{1/2}2^{k/2}N^{1/3}_{Y,Q}$. Also note that $u2^{k/2}N_{Y,Q,k} \ge N_{Y,Q,k}2^s$ if and only if $2^s \le u2^{k/2}$. Thus choosing the better estimate in the term $\min(u2^mN^{5/3}_{Y,Q}2^{-s}, N_{Y,Q}2^s)$ depending on s and the better estimate in the term $\min(u2^{k/2}N_{Y,Q,k}, N_{Y,Q,k}2^s)$ yields that the left-hand side of (3-29) is bounded by

$$N_{R,Q,k}\, N_{R,Q,k'}\, 2^k\, N_{Y,Q,k}\log(N_{Y,Q,k})\max(N^{1/3}_{Y,Q,k}u^{1/2}2^{k/2},\ u2^{k/2}). \qquad (3\text{-}30)$$

Using $N_{Y,Q,k} \lesssim u2^k$, (3-30) is bounded by

$$\begin{aligned}&N_{R,Q,k}\, N_{R,Q,k'}\, 2^k\, N_{Y,Q,k}(k\log(u))\max(u^{5/6}2^{5k/6},\ u2^{k/2})\\
&\qquad\lesssim N_{R,Q,k'}(\#(\mathcal{E}_k\cap Q^*))2^k(k\log(u))\max(u^{5/6}2^{5k/6},\ u2^{k/2}),\end{aligned}$$

which completes the proof of (3-25). □

Proof of Lemma 3.5. Fix $\epsilon > 0$, and set $N(u) = 100\epsilon^{-1}\log_2(2+u)$. We apply (3-24) when $N_{R,Q,k'} \ge 2^{k'\epsilon}$ and (3-25) when $N_{R,Q,k'} \le 2^{k'\epsilon}$, and then we sum over $N(u) < k' < k$ for k fixed to obtain

$$\begin{aligned}&\sum_{\substack{N(u)<k'<k\\ k \text{ fixed}}} \sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y',r'}\rangle|\\
&\qquad\lesssim_\epsilon R^2\#(\mathcal{E}_k\cap Q^*)\max(1, \log(u)u^{5/6}2^{-k/6+\epsilon}, \log(u)u2^{-k/2+\epsilon}). \qquad (3\text{-}31)\end{aligned}$$

Next we sum over $Q \in \mathcal{Q}_{u,k}$ and $k > N(u)$ to obtain

$$\sum_k \sum_{Q\in\mathcal{Q}_{u,k}} \sum_{\substack{N(u)<k'<k \\ k \text{ fixed}}} \sum_{(Y,R)\in\mathcal{E}_k(u)\cap Q^*} \sum_{(y,r)\in\mathcal{E}_{k'}(u)\cap Q^*} |\langle F_{Y,R}, F_{y',r'}\rangle| \lesssim_\epsilon \sum_k 2^{2k}\#\mathcal{E}_k. \tag{3-32}$$

We have thus shown that for the choice $N(u) = 100\epsilon^{-1}\log_2(2+u)$, we have

$$\sum_{k>k'>N(u)} |\langle G_{u,k'}, G_{u,k}\rangle| \lesssim_\epsilon \sum_k 2^{2k}\#\mathcal{E}_k. \qquad \square$$

Putting it together. Combining (3-1), (3-2) and (3-23), we have that for every $\epsilon > 0$,

$$\|G_u\|_2^2 = \Big\|\sum_k G_{u,k}\Big\|_2^2 \lesssim_\epsilon \log_2(2+u)\sum_k 2^{2k}(\#\mathcal{E}_k)u^{11/13+\epsilon}. \tag{3-33}$$

This completes the proof of Lemma 2.4 and hence the proof of Proposition 2.2. Thus we have finished the proof of Theorem 1.2. The rest of the paper will be devoted to the (more technical) proof of Theorem 1.4.

4. Preliminaries and reductions: part II

Similarly to Section 2, we will collect necessary preliminary results and reductions to prove Theorem 1.4. Much of the proof of Theorem 1.4 will be similar to the proof of Theorem 1.2, but there are nontrivial additional technical difficulties to the proof of Theorem 1.4 that will make the proof more involved. The main reason for this is the fact that Theorem 1.4 is a full L^p characterization rather than a restricted strong type (p, p) result, and therefore we cannot simply assume that our discrete sets $\mathcal{E}$ have product structure as we were able to do in the proof of Theorem 1.2. The obstacle in applying these techniques to the 3-dimensional case is in fact the case of "incomparable radii". While this case is very easy to deal with in dimensions $d \geq 4$, we currently do not know how to handle it in three dimensions without the product structure assumption we are allowed to make when proving restricted strong type inequalities.

Discretization and density decomposition of sets. Again, the first step will be to discretize our problem, and as before we will first need to introduce some notation. Let $\mathcal{Y}$ be a 1-separated set of points in $\mathbb{R}^4$ and let $\mathcal{R}$ be a 1-separated set of radii ≥ 1. Let $\mathcal{E} \subset \mathcal{Y}\times\mathcal{R}$ be a finite set, and let

$$u \in \mathcal{U} = \{2^\nu : \nu = 0, 1, 2, \dots\}$$

be a collection of dyadic indices. For each k, let $\mathfrak{B}_k$ denote the collection of all 5-dimensional balls of radius $\leq 2^k$. For a ball B, let $\operatorname{rad} B$ denote the radius of B. Following [Heo et al. 2011], define

$$\begin{aligned}
\mathcal{R}_k &:= \mathcal{R}\cap[2^k, 2^{k+1}),\\
\mathcal{E}_k &:= \mathcal{E}\cap(\mathcal{Y}\times\mathcal{R}_k),\\
\widehat{\mathcal{E}}_k(u) &:= \{(y,r)\in\mathcal{E}_k : \exists B\in\mathfrak{B}_k \text{ such that } \#(\mathcal{E}_k\cap B)\geq u \operatorname{rad} B\},\\
\mathcal{E}_k(u) &= \widehat{\mathcal{E}}_k(u)\setminus\bigcup_{\substack{u'\in\mathcal{U}\\ u'>u}}\widehat{\mathcal{E}}_k(u').
\end{aligned}$$

We will refer to u as the *density* of the set $\mathcal{E}_k(u)$. Note that we have the decomposition

$$\mathcal{E}_k = \bigcup_{u\in\mathcal{U}} \mathcal{E}_k(u).$$

Let σ_r denote the surface measure on rS^3, the 3-sphere centered at the origin of radius r. Now fix a smooth, radial function ψ_0 which is supported in the ball centered at the origin of radius $\frac{1}{10}$ such that $\hat{\psi}_0$ vanishes to order 40 at the origin. Let $\psi = \psi_0 * \psi_0$. For $y \in \mathcal{Y}$ and $r \in \mathcal{R}$, define

$$F_{y,r} = \sigma_r * \psi(\cdot - y).$$

For a given function $\gamma : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ and finite set $\mathcal{E} \subset \mathcal{Y} \times \mathcal{R}$, further define

$$G^{\gamma,\mathcal{E}}_{u,k} := \sum_{(y,r)\in\mathcal{E}_k(u)} \gamma(y,r) F_{y,r}, \quad G^{\gamma,\mathcal{E}}_{u} := \sum_{k\geq 0} G^{\gamma,\mathcal{E}}_{u,k}, \quad G^{\gamma,\mathcal{E}}_{k} := \sum_{u\in\mathcal{U}} G^{\gamma,\mathcal{E}}_{u,k}.$$

The discretized L^p inequality. We will prove the following proposition, which implies our main result for compactly supported multipliers.

Proposition 4.1. *Let $1 < p < \frac{36}{29}$. Let $\gamma : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ be a function that is a tensor product; i.e., $\gamma(y,r) = \gamma_1(y)\gamma_2(r)$. For each $j \in \mathbb{Z}$, define*

$$\mathcal{E}^{\gamma,j} := \{(y,r) \in \mathcal{Y} \times \mathcal{R} : 2^j \leq |\gamma(y,r)| < 2^{j+1}\},$$
$$\mathcal{E}^{\gamma,j}_k := \{(y,r) \in \mathcal{Y} \times \mathcal{R} : r \in \mathcal{R}_k,\ 2^j \leq |\gamma(y,r)| < 2^{j+1}\}.$$

Then

$$\Bigl\| \sum_{(y,r)\in\mathcal{E}^{\gamma,j}} \gamma(y,r) F_{y,r} \Bigr\|_p^p \lesssim_p 2^{jp} \sum_{l\geq j} 2^{(l-j)/5} \sum_k 2^{3k} \#\mathcal{E}^{\gamma,l}_k. \tag{4-1}$$

Using the dyadic interpolation lemma (Lemma 2.1), we obtain the following corollary.

Corollary 4.2. *Let $\gamma : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ be a function that is a tensor product; i.e., $\gamma(y,r) = \gamma_1(y)\gamma_2(r)$. Let $h : \mathbb{R}^5 \to \mathbb{C}$ be a function that is a tensor product; i.e., $h(y,r) = h_1(y)h_2(r)$. Then for $1 < p < \frac{36}{29}$, we have*

$$\Bigl\| \sum_{(y,r)\in\mathcal{Y}\times\mathcal{R}} \gamma(y,r) F_{y,r} \Bigr\|_p \lesssim_p \Bigl(\sum_{(y,r)\in\mathcal{Y}\times\mathcal{R}} |\gamma(y,r)|^p r^3 \Bigr)^{1/p}. \tag{4-2}$$

Also

$$\Bigl\| \int_{\mathbb{R}^3} \int_1^\infty h(y,r) F_{y,r}\, dr\, dy \Bigr\|_p \lesssim_p \Bigl(\int_{\mathbb{R}^3} \int_1^\infty |h(y,r)|^p r^3\, dr\, dy \Bigr)^{1/p}. \tag{4-3}$$

Proof that Proposition 4.1 implies Corollary 4.2. Apply Lemma 2.1 with $F_j = \sum_{(y,r)\in\mathcal{E}^{\gamma,j}} \gamma(y,r) F_{y,r}$, M^p the implied constant from (4-1), and

$$s_j = \sum_{l\geq j} 2^{(l-j)/5} \sum_k 2^{3k} \#\mathcal{E}^{\gamma,l}_k$$

to obtain (4-2).

Now we prove (4-3). Let $y = z + w$ for $z \in \mathbb{Z}^4$ and $w \in Q_0 := [0, 1)^4$ and $r = n + \tau$ for $n \in \mathbb{N}$ and $0 \le \tau < 1$. By Minkowski's inequality and (4-2),

$$\begin{aligned}\left\| \int_{\mathbb{R}^4} \int_1^\infty h(y, r) F_{y,r}\, dr\, dy \right\|_p &\lesssim_p \iint_{Q_0 \times [0,1)} \left\| \sum_{z \in \mathbb{Z}^4} \sum_{n=1}^\infty h_2(n+\tau) h_1(z+w) F_{z+w,n+\tau} \right\|_p dw\, d\tau \\ &\lesssim_p \iint_{Q_0 \times [0,1)} \left(\sum_{z \in \mathbb{Z}^4} \sum_{n=1}^\infty |h_2(n+\tau) h_1(z+w)|^p (n+\tau)^3 \right)^{1/p} dw\, d\tau \\ &\lesssim_p \left(\int_{\mathbb{R}^3} \int_1^\infty |h(y, r)|^p r^3\, dr\, dy \right)^{1/p},\end{aligned}$$

where in the last step we have used Hölder's inequality. □

Support-size estimates vs. L^2 inequalities. Fix a function $\gamma : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ that is a tensor product, i.e., $\gamma(y, r) = \gamma_1(y)\gamma_2(r)$, and fix $j \in \mathbb{Z}$. Let

$$\widetilde{\mathcal{E}}^{\gamma,j} := \{(y, r) \in \mathcal{Y} \times \mathcal{R} : 2^{j-5} \le |\gamma(y, r)| \le 2^{j+5}\},$$

and recall the density decomposition

$$\widetilde{\mathcal{E}}_k^{\gamma,j} = \bigcup_{u \in \mathcal{U}} \widetilde{\mathcal{E}}_k^{\gamma,j}(u)$$

defined previously. Define a function $\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}} : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ to be the restriction of the function $G_{u,k}^{\gamma,\widetilde{\mathcal{E}}^{\gamma,j}}$ to the set $\mathcal{E}_k^{\gamma,j}$, i.e.,

$$\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}(y, r) = \begin{cases} G_{u,k}^{\gamma,\widetilde{\mathcal{E}}^{\gamma,j}}(y, r) & \text{if } (y, r) \in \mathcal{E}_k^{\gamma,j}, \\ 0 & \text{if } (y, r) \notin \mathcal{E}_k^{\gamma,j}. \end{cases}$$

Similarly define

$$\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}} = \sum_{k \ge 0} \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}} \quad \text{and} \quad \widetilde{G}_k^{\gamma,\mathcal{E}^{\gamma,j}} = \sum_{u \in \mathcal{U}} \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}.$$

Note that $\widetilde{G}_k^{\gamma,\mathcal{E}^{\gamma,j}} = G_k^{\gamma,\mathcal{E}^{\gamma,j}}$, and $\sum_k G_k^{\gamma,\mathcal{E}^{\gamma,j}}$ appears on the left-hand side of the inequality in Proposition 4.1. Similarly to [Heo et al. 2011], we will show that the functions $\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{j,\gamma}}$ either have relatively small support size or satisfy relatively good L^2 bounds. As in the previous part of the paper, we begin with a support-size bound which follows immediately from the similar bound in [Heo et al. 2011] that improves as the density u increases.

Lemma C′. *For all $u \in \mathcal{U}$, the Lebesgue measure of the support of $\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}$ is $\lesssim u^{-1} 2^{3k} \#\big(\bigcup_{l:|l-j| \le 10} \mathcal{E}_k^{\gamma,l}\big)$.*

We will prove the following L^2 inequality, which in some sense an improved version of Lemma 3.6 from [Heo et al. 2011], although the hypotheses are different since it is crucial that we assume that the underlying set is of the form $\mathcal{E}^{\gamma,j}$, i.e., the $\approx 2^j$ level set of some function $\gamma(y, r) = \gamma_1(y)\gamma_2(r)$. This inequality improves as the density u decreases. In [Heo et al. 2011], the analogous L^2 inequality proved is

$$\|\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 \lesssim u^{2/(d-1)} \log(2+u) 2^{2j} \sum_k 2^{k(d-1)} \# \bigg(\bigcup_{l:|l-j| \le 10} \mathcal{E}_k^{\gamma,j} \bigg). \tag{4-4}$$

We use geometric methods to improve on (4-4) in four dimensions, and our argument will rely on Lemma A.1 proved later in the Appendix.

Lemma 4.3. *Let $\mathcal{E}^{\gamma,j}$, $\mathcal{E}_k^{\gamma,j}$, and $\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}$ be as above. Then for every $\epsilon > 0$,*

$$\|\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 \lesssim_\epsilon u^{11/18+\epsilon} 2^{2j} \sum_{l\ge j} 2^{(l-j)/10} \sum_k 2^{3k} \# \mathcal{E}_k^{\gamma,l}.$$

Combining Lemma C′ and Lemma 4.3, we obtain the following L^p bound.

Lemma 4.4. *For $p \le 2$, for every $\epsilon > 0$,*

$$\|\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}\|_p \lesssim_{\epsilon,p} u^{-(1/p-29/36-\epsilon)} 2^j \biggl(\sum_{l\ge j} 2^{(l-j)/10} \sum_k 2^{3k} \# \mathcal{E}_k^{\gamma,l}\biggr)^{1/p}.$$

Proof of Lemma 4.4 given Lemma C′ and Lemma 4.3. By Hölder's inequality,

$$\begin{aligned}\|\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}\|_p &\lesssim_p (\operatorname{meas}(\operatorname{supp}(\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}})))^{1/p-1/2} \|\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}}\|_2 \\ &\lesssim_{\epsilon,p} u^{29/36-1/p+\epsilon} 2^j \biggl(\sum_{l\ge j} 2^{(l-j)/10} \sum_k 2^{3k} \# \mathcal{E}_k^{\gamma,l}\biggr)^{1/p}.\end{aligned}$$ □

Summing over $u \in \mathcal{U}$, we obtain Proposition 4.1. Thus to prove Proposition 4.1 it suffices to prove Lemma 4.3. One may deduce Theorem 1.4 from Corollary 4.2 in the same way as one deduces Theorem 1.2 from Corollary 2.3.

5. Proof of the L^2 inequality: part II

We have shown in Section 4 that to prove our main result Theorem 1.4 it remains to prove Lemma 4.3, and the goal of this section is to prove Lemma 4.3. The intuition and reasoning behind our arguments will be loosely as follows. Unlike the previous case where we worked with characteristics functions, the level sets $\mathcal{E}^{\gamma,j}$ in Lemma 4.3 are no longer product sets in $\mathbb{R}^d \times \mathbb{R}$ since we no longer have the assumption that we are working with characteristic functions. However, they are still very well structured, since they are level sets of tensor products $g(y)h(r)$ of functions, where $y \in \mathbb{R}^d$ and $r \in \mathbb{R}$. The dyadic level sets $\mathcal{E}^{\gamma,j}$ may be written as a sum of product sets, and if there are not too many of them (e.g., logarithmic in the relevant parameters) then we may simply crudely sum over the total number of product sets and proceed with the same argument as in the characteristic function case. On the other hand, if there are a large number of such product sets, then this forces the underlying function to take on values much larger than 2^j, and we may then control sums over the sets $\mathcal{E}^{\gamma,j}$ by cardinalities of sets $\mathcal{E}_k^{\gamma,l}$ with $l \ge j$.

Another preliminary reduction. Recall that our goal is to estimate the L^2 norm of $\widetilde{G}_u^{\gamma,\mathcal{E}^{\gamma,j}} = \sum_{k\ge 0} \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}$. Let $N(u)$ be a sufficiently large number to be chosen later (it will be some harmless constant depending on u that is essentially $O(\log(2+u))$). We split the sum in k as $\sum_{k\le N(u)} \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}} + \sum_{k>N(u)} \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}$ and apply Cauchy–Schwarz to obtain

$$\Big\|\sum_k \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\Big\|_2^2 \lesssim N(u)\bigg(\sum_k \|\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 + \sum_{k>k'>N(u)} |\langle \widetilde{G}_{u,k'}^{\gamma,\mathcal{E}^{\gamma,j}}, \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\rangle|\bigg). \tag{5-1}$$

We may thus separately estimate $\sum_k \|\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2$ and

$$\sum_{k>k'>N(u)} |\langle \widetilde{G}_{u,k'}^{\gamma,\mathcal{E}^{\gamma,j}}, \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\rangle|,$$

which divides the proof of the L^2 estimate into two cases, the first being the case of "comparable radii" and the second being the case of "incomparable radii".

Comparable radii. We will first estimate $\sum_k \|\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2$. Our goal will be to prove the following lemma.

Lemma 5.1. *For every $\epsilon > 0$,*

$$\|\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 \lesssim_\epsilon u^{11/18+\epsilon} 2^{2j} \sum_{l\ge j} 2^{(l-j)/10} 2^{3k} (\#\mathcal{E}_k^{\gamma,l}). \tag{5-2}$$

Fix k and u. As in [Heo et al. 2011], we first observe that for $(y,r), (y',r') \in \widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap \mathcal{E}_k^{\gamma,j}$, we have $\langle F_{y,r}, F_{y',r'}\rangle = 0$ unless $|(y,r)-(y',r')| \le 2^{k+5}$. To estimate $\|\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2$ for a fixed k, we would thus like to bound

$$2^{2j} \sum_{\substack{(y,r),(y',r')\in\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap\mathcal{E}_k^{\gamma,j}\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|$$

for all $0 \le m \le k+4$.

Now fix $m \le k+4$. Let $\mathcal{Q}_{u,j,k,m}$ be a collection of almost disjoint cubes $Q \subset \mathbb{R}^5$ of side length 2^{m+5} such that

$$\widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap \mathcal{E}_k^{\gamma,j} \subset \bigcup_{Q\in\mathcal{Q}_{u,k,j,m}} Q$$

and so that every Q has nonempty intersection with $\widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap \mathcal{E}_k^{\gamma,j}$. Let Q^* denote the 2^5-dilate of Q and $\mathcal{Q}^*_{u,k,j,m}$ the corresponding collection of dilated cubes. Observe that

$$\begin{aligned}\|\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2 &\lesssim 2^{2j} \sum_{0\le m\le k+4} \bigg(\sum_{\substack{(y,r),(y',r')\in\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap\mathcal{E}_k^{\gamma,j}\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| + \sum_{(y,r)\in\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap\mathcal{E}_k^{\gamma,j}} \|F_{y,r}\|_2^2 \bigg) \\ &\lesssim 2^{2j} \sum_{0\le m\le k+4} \bigg(\sum_{Q\in\mathcal{Q}_{u,k,j,m}} \bigg(\sum_{\substack{(y,r),(y',r')\in(\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap\mathcal{E}_k^{\gamma,j}\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \bigg) + \sum_{(y,r)\in\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap\mathcal{E}_k^{\gamma,j}} \|F_{y,r}\|_2^2 \bigg).\end{aligned} \tag{5-3}$$

For each integer $b \in \mathbb{Z}$ define

$$\mathcal{E}_k^{\gamma,j,b} := \{(y,r) \in \mathcal{Y}\times\mathcal{R}_k : 2^{b-3} \le \gamma_1(y) \le 2^{b+3},\ 2^{j-b-3} \le \gamma_2(r) \le 2^{j-b+3}\}.$$

Note that

$$\mathcal{E}_k^{\gamma,j} \subset \bigcup_{b\in\mathbb{Z}} \mathcal{E}_k^{\gamma,j,b} \subset \widetilde{\mathcal{E}}_k^{\gamma,j}.$$

Note also that each set $\mathcal{E}_k^{\gamma,j,b}$ is a product, that is, a set of the form $Y \times R$, where $Y \subset \mathcal{Y}$ and $R \subset \mathcal{R}$. It follows that $\mathcal{E}_k^{\gamma,j,b} \cap Q$ is a product for any cube $Q \subset \mathbb{R}^{d+1}$.

We also define some parameters associated with a fixed $Q \in \mathcal{Q}_{u,k,j,m}$ and $b \in \mathbb{Z}$. Let $N_{R,Q,b}$ be the cardinality of the $\mathcal{R}$-projection of $\mathcal{E}_k^{\gamma,j,b} \cap Q^*$; i.e.,

$$N_{R,Q,b} := \#((\mathcal{E}_k^{\gamma,j,b} \cap Q^*)_R) = \#\{r : \exists (y,r) \in \mathcal{E}_k^{\gamma,j,b} \cap Q^*\}.$$

Similarly define

$$N_{Y,Q,b} := \#((\mathcal{E}_k^{\gamma,j,b} \cap Q^*)_Y) = \#\{y : \exists (y,r) \in \mathcal{E}_k^{\gamma,j,b} \cap Q^*\}.$$

We also note the following important observation, which we will use repeatedly. Using the definition of the sets $\widetilde{\mathcal{E}}_k^{\gamma,j}(u)$ and the fact that for each $b \in \mathbb{Z}$, the set $\mathcal{E}_k^{\gamma,j,b}$ has product structure, one may see that if $Q \in \mathcal{Q}_{u,k,j,m}$ is such that $\widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap \mathcal{E}_k^{\gamma,j,b} \cap Q^*$ is nonempty, then

$$|N_{Y,Q,b} \cdot N_{R,Q,b}| \lesssim \#(\mathcal{E}_k^{\gamma,j,b} \cap Q^*) \lesssim \#(\widetilde{\mathcal{E}}_k^{\gamma,j} \cap Q^*) \lesssim u2^m. \tag{5-4}$$

Now, we will organize our sets $\mathcal{E}_k^{\gamma,j,b}$ as follows. For a fixed m, given $Q \in \mathcal{Q}_{u,k,j,m}$, we would like to group together those $b \in \mathbb{Z}$ for which $\#(\mathcal{E}_k^{\gamma,j,b} \cap Q)$ has essentially equal cardinality and for which the ratio $N_{Y,Q,b}/N_{R,Q,b}$ is essentially equal. For each pair of integers $(c,d) \in \mathbb{Z}^2$, we define

$$\mathcal{B}_{Q,c,d} := \{b \in \mathbb{Z} : 2^{c-1} \le \#(\mathcal{E}_k^{\gamma,j,b} \cap Q^*) < 2^c,\ 2^{d-1} \le N_{Y,Q,b}/N_{R,Q,b} < 2^d\}.$$

Now with (5-3) in mind, we will prove the following lemma.

Lemma 5.2. *For each $Q \in \mathcal{Q}_{u,k,j,m}$ and each quadruple $(c,d,c',d') \in \mathbb{Z}^4$, we have the estimates*

$$\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim 2^{\max((c-d)/2,(c'-d')/2)}(\max(\#\mathcal{B}_{Q,c,d},\#\mathcal{B}_{Q,c',d'}))^2 \times 2^{\max(c,c')}2^{3(k-m/2)}(m\log(u))\max(u^{5/6}2^m, u2^{m/2}) \tag{5-5}$$

and

$$\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim 2^{3(k-m/2)}2^{\max(c,c')}(\max(\#\mathcal{B}_{Q,c,d},\#\mathcal{B}_{Q,c',d'}))^2 \times u2^m(2^{\max((c-d)/2,(c'-d')/2)})^{-1}. \tag{5-6}$$

Notice that (5-5) is the better estimate when $2^{\max((c-d)/2,(c'-d')/2)}$ is small and (5-6) is the better estimate when $2^{\max((c-d)/2,(c'-d')/2)}$ is large. We will use (5-5) when $2^{\max((c-d)/2,(c'-d')/2)} \le u^{1/12}$ and (5-6) when $2^{\max((c-d)/2,(c'-d')/2)} > u^{1/12}$. This yields the following corollary.

Corollary 5.3.
$$\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim_\epsilon I+II, \tag{5-7}$$

where

$$I := 2^{3k}2^{\max(c,c')}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))^2 u^\epsilon 2^{m\epsilon}\max(u^{11/12}2^{-m/2}, u^{13/12}2^{-m}), \tag{5-8}$$
$$II := 2^{3k}2^{\max(c,c')}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))^2 u^\epsilon 2^{m\epsilon} u^{11/12}2^{-m/2}. \tag{5-9}$$

Now note that if $(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'})) > 10000m\log(u)$, then for some l such that $l > j+\frac{1}{10}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))$, we have $\#(Q^*\cap\mathcal{E}_k^{\gamma,l})\ge 1$, because this implies there must be (y,r) such that $\gamma(y,r)\ge 2^l$, where $l > j+\frac{1}{10}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'})$. Since $2^{\max(c,c')}\lesssim\#(Q^*\cap\mathcal{E}_k^{\gamma,j})\lesssim u2^m\lesssim 2^{(l-j)/20}$, this implies

$$2^{\max(c,c')}(\max(\#\mathcal{B}_{Q,c,d}, \#\mathcal{B}_{Q,c',d'}))^2\lesssim 2^{(l-j)/10}\#(\mathcal{E}_k^{\gamma,l}\cap Q^*).$$

Thus Corollary 5.3 implies the following.

Corollary 5.4.
$$\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim I+II, \tag{5-10}$$

where

$$I := 2^{3k}\sum_{l\ge j}2^{(l-j)/10}\#(\mathcal{E}_k^{\gamma,l}\cap Q^*)u^\epsilon 2^{m\epsilon}\max(u^{11/12}2^{-m/2},\ u^{13/12}2^{-m}), \tag{5-11}$$
$$II : = 2^{3k}\sum_{l\ge j}2^{(l-j)/10}\#(\mathcal{E}_k^{\gamma,l}\cap Q^*)u^\epsilon 2^{m\epsilon}u^{11/12}2^{-m/2}. \tag{5-12}$$

By (5-4) there are $\lesssim m^4\log(u)^4$ quadruples (c,d,c',d') for which both $\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)$ and $\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)$ are nonempty, so Corollary 5.4 implies the following.

Corollary 5.5.
$$\sum_{\substack{(y,r),(y',r')\in\bigcup_{b\in\mathbb{Z}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \lesssim I+II, \tag{5-13}$$

where

$$I := 2^{3k}\sum_{l\ge j}2^{(l-j)/10}\#(\mathcal{E}_k^{\gamma,l}\cap Q^*)u^\epsilon 2^{m\epsilon}\max(u^{11/12}2^{-m/2},\ u^{13/12}2^{-m}) \tag{5-14}$$
$$II := 2^{3k}\sum_{l\ge j}2^{(l-j)/10}\#(\mathcal{E}_k^{\gamma,l}\cap Q^*)u^\epsilon 2^{m\epsilon}u^{11/12}2^{-m/2}. \tag{5-15}$$

Proof of Lemma 5.2. We will first prove (5-5). Fix $b\in\mathcal{B}_{Q,c,d}$ and $b'\in\mathcal{B}_{Q,c',d'}$. Set

$$N_{Y,Q,b,b'} = \max(N_{Y,Q,b}, N_{Y,Q,b'})\approx 2^{\max((c+d)/2,(c'+d')/2)}.$$

It suffices to prove

$$\sum_{\substack{(y,r)\in\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^* \\ (y',r')\in\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^* \\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r},\, F_{y',r'}\rangle| \lesssim N_{R,Q,b}N_{R,Q,b'}N_{Y,Q,b,b'}\times 2^{3(k-m/2)}(m\log(u))\max(u^{5/6}2^m,\; u2^{m/2}). \tag{5-16}$$

After incurring a factor of $N_{R,Q,b}N_{R,Q,b'}$, to estimate the left-hand side of (5-16) it suffices to estimate for a fixed pair r_1, r_2,

$$\sum_{\substack{(y,r_1)\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)^\times \\ (y',r_2)\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)^\times \\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r},\, F_{y',r'}\rangle|, \tag{5-17}$$

i.e., to restrict (y, r) and (y', r') to lie in fixed rows of the product-extensions of $(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)^\times$ and $(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)^\times$, respectively. (Our estimates will not depend on the particular choice of r_1 and r_2.)

Now, referring to the estimate in Lemma 3.2, we see that for a fixed y, r_1, r_2 we have that $|\langle F_{y,r_1}, F_{y',r_2}\rangle|$ decays rapidly as y' moves away from the set $\{y' : |y-y'| = |r_1-r_2| \text{ or } |y-y'| = r_1+r_2\}$, which is contained in a union of two annuli of thickness 2 and radii $|r_1-r_2|$ and r_1+r_2 centered at y.

Let $s\ge 0$, fix $t\le 2^{m+10}$, and define $K_k^{\gamma,j,b,b'}(Q, s, t)$ to be the number of points $y\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y$ such that there are $\ge 2^s$ many points $y'\in(\mathcal{E}_k^{\gamma,j,b'}\cap Q^*)_Y$ such that y' lies in the annulus of inner radius t and thickness 3 centered at y. That is, define

$$K_k^{\gamma,j,b,b'}(Q,s,t) := \#\big\{y\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y : \text{there exist at least } 2^s \text{ many points } y'\in(\mathcal{E}_k^{\gamma,j,b'}\cap Q^*)_Y \text{ such that } \big||y'-y|-\big(t+\tfrac{3}{2}\big)\big|\le\tfrac{3}{2}\big\}.$$

In view of the observation in the previous paragraph, for a given s sufficiently large but smaller than $u2^m$, say $s>m+100$, and a fixed number $t\le 2^{m+10}$, we would like to prove a bound on $K_k^{\gamma,j,b,b'}(Q, s, t)$. Our bound will depend on s and m but be independent of the choice of $t\le 2^{m+10}$. For this reason, we define the quantity

$$K_k^{\gamma,j,b,b',*}(Q,s) := \max_{0\le t\le 2^{m+10}} K_k^{\gamma,j,b,b'}(Q,s,t).$$

We will prove

$$K_k^{\gamma,j,b,b',*}(Q,s) \lesssim \max\big[u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-2s},\; u2^{m/2}N_{Y,Q,b,b'}2^{-s}\big], \quad s>m+100. \tag{5-18}$$

Combining this with the trivial bound $K_k^{\gamma,j,b,b',*}(Q,s)\lesssim N_{Y,Q,b,b'}$ yields

$$K_k^{\gamma,j,b,b',*}(Q,s) \lesssim \max\big[\min(u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-2s}, N_{Y,Q,b,b'}), \min(u2^{m/2}N_{Y,Q,b,b'}2^{-s}, N_{Y,Q,b,b'})\big], \quad s>m+100. \tag{5-19}$$

Note that (5-18) gives decay in the number of points $K_k^{\gamma,j,b,b',*}(Q,s)$ (i.e., $K_k^{\gamma,j,b,b'*}(Q,s)\ll N_{Y,Q,b,b'}$) if we have that both

(1) $N_{Y,Q,b,b'}^{5/3}2^{-2s}u2^{4m/3}\ll N_{Y,Q,b,b'}$, that is, if $2^s\gg N_{Y,Q,b,b'}^{1/3}u^{1/2}2^{2m/3}$, and

(2) $N_{Y,Q,b,b'}2^{-s}u2^{m/2}\ll N_{Y,Q,b,b'}$, that is, if $2^s\gg u2^{m/2}$.

Using Lemma 3.2, we may bound

$$
\begin{aligned}
&\sum_{\substack{(y,r)\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\
&\lesssim \sum_{\substack{r_1\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R}} \Bigg(\sum_{\substack{y,y'\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y\\ 2^m\le|(y,r_1)-(y',r_2)|\le 2^{m+1}}} |\langle F_{y,r_1}, F_{y',r_2}\rangle| \Bigg) \\
&\lesssim 2^{3(k-m/2)} \sum_{\substack{r_1\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R}} \Bigg(\sum_{0\le a\le m+10} \Bigg(\sum_{y\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y} \sum_{\substack{y'\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y\\ \min_{\pm,\pm}(1+|r_1\pm r_2\pm|y-y'||)\approx 2^a}} 2^{-aN} \Bigg)\Bigg) \\
&\lesssim 2^{3(k-m/2)} \sum_{\substack{r_1\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R\\ r_2\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R}} \Bigg(\sum_{0\le a\le m+10} 2^{-aN} \Bigg(\sum_{s\ge 0: 2^s\le 2N_{Y,Q,b}} K_k^{\gamma,j,b,b',*}(Q,s) 2^s \Bigg)\Bigg) \\
&\lesssim 2^{3(k-m/2)} N_{R,Q,b}\, N_{R,Q,b'} \sum_{s\ge 0:2^s\le 2N_{Y,Q,b}} K_k^{\gamma,j,b,b',*}(Q,s)2^s.
\end{aligned} \tag{5-20}
$$

Assuming (5-19) holds, we have

$$
\begin{aligned}
&\sum_{\substack{(y,r)\in(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\
&\lesssim N_{R,Q,b}N_{R,Q,b'}2^{3(k-m/2)} \sum_{\substack{s\ge 0\\ 2^s\lesssim N_{Y,Q,b,b'}}} \max\big[\min(u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-s}, N_{Y,Q,b,b'}2^s), \\
&\qquad\qquad\qquad\qquad\qquad\qquad \min(u2^{m/2}N_{Y,Q,b,b'}, N_{Y,Q,b,b'}2^s)\big] \\
&\lesssim N_{R,Q,b}N_{R,Q,b'}2^{3(k-m/2)} \Bigg[m2^m N_{Y,Q,b,b'} + \max\Bigg\{ \sum_{\substack{s\ge 0\\ 2^s\lesssim N_{Y,Q,b,b'}}} \min(u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-s}, N_{Y,Q,b,b'}2^s), \\
&\qquad\qquad\qquad\qquad \sum_{\substack{s\ge 0\\ 2^s\lesssim N_{Y,Q,b,b'}}} \min(u2^{m/2}N_{Y,Q,b,b'}, N_{Y,Q,b,b'}2^s) \Bigg\}\Bigg].
\end{aligned} \tag{5-21}
$$

Now, note that $u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-s} \ge N_{Y,Q,b,b'}2^s$ if and only if $2^s \le u^{1/2}2^{2m/3}N_{Y,Q,b,b'}^{1/3}$. Thus choosing the better estimate in the term $\min(u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-s}, N_{Y,Q,b,b'}2^s)$ depending on s yields that

$$
\sum_{s\ge 0:2^s\lesssim N_{Y,Q,b,b'}} \min(u2^{4m/3}N_{Y,Q,b,b'}^{5/3}2^{-s}, N_{Y,Q,b,b'}2^s) \lesssim u^{1/2}2^{2m/3}N_{Y,Q,b,b'}^{4/3}.
$$

Note that $u2^{m/2}N_{Y,Q,b,b'} \geq N_{Y,Q,b,b'}2^s$ if and only if $2^s \leq u2^{m/2}$. Thus choosing the better estimate in the term $\min(u2^{m/2}N_{Y,Q,b,b'},\, N_{Y,Q,b,b'}2^s)$ depending on s yields that

$$\sum_{s\geq 0: 2^s \lesssim N_{Y,Q,b,b'}} \min(u2^{m/2}N_{Y,Q,b,b'},\, N_{Y,Q,b,b'}2^s) \lesssim \log(N_{Y,Q,b,b'})N_{Y,Q,b,b'}\, u2^{m/2}.$$

Using that $N_{Y,Q,b,b'}^{1/3} \lesssim u^{1/3}2^{m/3}$, it follows that the left-hand side of (5-21) is bounded by

$$\begin{aligned} N_{R,Q,b}N_{R,Q,b'}2^{3(k-m/2)}N_{Y,Q,b,b'}\big[m2^m + \log(N_{Y,Q,b,b'})\max(N_{Y,Q,b,b'}^{1/3}u^{1/2}2^{2m/3},\ u2^{m/2})\big] \\ \lesssim N_{R,Q,b}N_{R,Q,b'}2^{3(k-m/2)}N_{Y,Q,b,b'}(m\log(u))\max(u^{5/6}2^m,\ u2^{m/2}) \\ \lesssim \max(N_{R,Q,b},\, N_{R,Q,b'})2^{\max(c,c')}2^{3(k-m/2)}(m\log(u))\max(u^{5/6}2^m,\ u2^{m/2}), \quad (5\text{-}22) \end{aligned}$$

which proves (5-5). This will be a good estimate when $\max(N_{R,Q,b},\, N_{R,Q,b'})$ is small.

Thus to prove (5-5) it remains to prove (5-18). We will in fact prove (5-18) with $K_k^{\gamma,j,b,b',*}(Q,s)$ replaced by $K_k^{\gamma,j,b,b'}(Q,s,t)$, uniformly in $t \leq 2^{m+10}$. Fix $t \leq 2^{m+10}$ and let $\alpha = \lceil \log_2(t) \rceil$ and cover $(\mathcal{E}_k^{\gamma,j,b} \cap \widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap Q^*)_Y$ by $\lesssim 2^{4(m-\alpha)}$ many 4-dimensional almost-disjoint balls of radius $2^{\alpha+5}$; denote this collection of balls as $\mathfrak{B} = \{B_i\}$. For each i, we define a collection of "special" points $A_{k,i}^{\gamma,j,b,b'}(Q,s,t)$ to be the set of all points

$$y \in (\mathcal{E}_k^{\gamma,j,b} \cap \widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap Q^*)_Y \cap B_i$$

such that there are $\geq 2^s$ many points $y' \in (\mathcal{E}_k^{\gamma,j,b'} \cap Q^*)_Y$ such that y' lies in the annulus of radius t and thickness 3 centered at y. That is, we define

$$\begin{aligned} A_{k,i}^{\gamma,j,b,b'}(Q,s,t) := \big\{ y \in (\mathcal{E}_k^{\gamma,j,b} \cap \widetilde{\mathcal{E}}_k^{\gamma,j}(u) \cap Q^*)_Y \cap B_i :\ & \text{there exist at least } 2^s \text{ many points} \\ & y' \in (\mathcal{E}_k^{\gamma,j,b'} \cap Q^*)_Y \text{ such that } \big| |y'-y| - \big(t+\tfrac{3}{2}\big) \big| \leq \tfrac{3}{2} \big\}. \end{aligned}$$

Let $K_{k,i}^{\gamma,j,b,b'}(Q,s,t)$ denote the cardinality of $A_{k,i}^{\gamma,j,b,b'}(Q,s,t)$. Now cover each B_i with $\lesssim 2^{4(\alpha-l)}$ many almost disjoint 4-dimensional balls $B_{i,\alpha}$ of radius 2^l for some $l \leq \alpha$. Each such ball contains at most $u2^l$ many points of $A_{k,i}^{\gamma,j,b,b'}(Q,s,t)$, so for a fixed i there must be at least $\gtrsim K_{k,i}^{\gamma,j,b,b'}(Q,s,t)(u2^l)^{-1}$ many balls $B_{i,\alpha}$ that contain at least one point in $A_{k,i}^{\gamma,j,b,b'}(Q,s,t)$. Thus there must be at least $\gtrsim K_{k,i}^{\gamma,j,b,b'}(Q,s,t)(u2^l)^{-1}$ many such points in $B_i \cap A_{k,i}^{\gamma,j,b,b'}(Q,s,t)$ spaced apart by $\gtrsim 2^l$; call this set $D_{k,i}^{\gamma,j,b,b'}(Q,s,t)$. But by Corollary A.2, which we prove later in the Appendix, the size of 3-fold intersections of 4-dimensional annuli of radius $t \approx 2^\alpha$ and thickness 3 spaced apart by $\gtrsim 2^l$ with centers lying in a ball of radius $2^{\alpha-5}$ is bounded above by $\lesssim 2^{3(\alpha-l)}2^\alpha$ provided that $l \geq \alpha/2 + 20$. This is true because for each point p in $D_{k,i}(Q,s,t)$, there will be at least $\gtrsim 2^s$ many points contained in the t-annulus centered at p that are contained in no more than two other different t annuli centered at different points p' and p'' in $D_{k,i}(Q,s,t)$, so the total number of points is $\gtrsim 2^s \times \mathrm{card}(D_{k,i}(Q,s,t))$.

It follows that if $l \geq \alpha/2 + 20$, then for each of these $\approx K_{k,i}^{\gamma,j,b,b'}(Q,s,t)(u2^l)^{-1}$ many points $p \in D_{k,i}^{\gamma,j,b,b'}(Q,s,t)$, there can be

$$\lesssim K_{k,i}^{\gamma,j,b,b'}(Q,s,t)^2(u2^l)^{-2}2^{3(\alpha-l)}2^\alpha$$

points lying inside the t-annulus centered at p that are simultaneously contained in at least two other different t-annuli centered at points in $D_{k,i}^{\gamma,j,b,b'}(Q,s,t)$. This implies that if $N_{Y,Q,b',i}$ denotes the cardinality of $(\mathcal{E}_k^{\gamma,j,b'}\cap Q^*)_Y\cap B_i^*$, where $B_i^*=10B_i$, then we have

$$N_{Y,Q,b',i}\gtrsim K_{k,i}^{\gamma,j,b,b'}(Q,s,t)(u2^l)^{-1}2^s, \tag{5-23}$$

which is essentially 2^s times the number of points in $D_{k,i}^{\gamma,j,b,b'}(Q,s,t)$, provided that 2^s is much bigger than the total number of points lying inside a t-annulus centered at p that are simultaneously contained in at least two other different t-annuli centered at points in $D_{k,i}^{\gamma,j,b,b'}(Q,s,t)$, i.e., provided that

$$K_{k,i}^{\gamma,j,b,b'}(Q,s,t)^2(u2^l)^{-2}2^{3(\alpha-l)}2^\alpha\ll 2^s \tag{5-24}$$

and

$$l\ge \alpha/2+20.$$

Solving for 2^l in (5-24) yields

$$2^l\gg K_{k,i}^{\gamma,j,b,b'}(Q,s,t)^{2/5}2^{4\alpha/5}u^{-2/5}2^{-s/5}. \tag{5-25}$$

Since $s\gg m$, we may choose a minimal l such that

$$2^l\gg \max\bigl[K_{k,i}^{\gamma,j,b,b'}(Q,s,t)^{2/5}2^{4\alpha/5}u^{-2/5}2^{-s/5},\ 2^{\alpha/2}\bigr]$$

for a sufficiently large implied constant. Substituting into (5-23) yields

$$K_{k,i}^{\gamma,j,b,b'}(Q,s,t)\lesssim \max\bigl[u2^{4m/3}N_{Y,Q,b',i}^{5/3}2^{-2s},\ u2^{m/2}N_{Y,Q,b',i}2^{-s}\bigr], \tag{5-26}$$

and summing over all i and using the almost-disjointness of the B_i^* gives

$$K_{k}^{\gamma,j,b,b'}(Q,s,t)\lesssim \max\bigl[u2^{4m/3}N_{Y,Q,b'}^{5/3}2^{-2s},\ u2^{m/2}N_{Y,Q,b'}2^{-s}\bigr]. \tag{5-27}$$

Taking the maximum over all $0\le t\le 2^{m+10}$ proves (5-18) and hence also (5-5).

It remains to prove (5-6), which we reproduce again below for convenience:

$$\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ (y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}}|\langle F_{y,r},F_{y',r'}\rangle| \lesssim 2^{3(k-m/2)}2^{\max(c,c')}(\max(\#\mathcal{B}_{Q,c,d},\#\mathcal{B}_{Q,c',d'}))^2u2^m(2^{\max((c-d)/2,(c'-d')/2)})^{-1}. \tag{5-28}$$

This will be a good estimate in the case that $2^{\max((c-d)/2,(c'-d')/2)}$ is large. Without loss of generality, assume that $c'-d'\ge c-d$. For a fixed $(y,r)\in Q^*$ and a fixed

$$y'\in(\mathcal{E}_k^{\gamma,j,b'}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y,$$

there are at most two values of r' away from which $\langle F_{y,r}, F_{y',r'}\rangle$ decays rapidly. Thus using Lemma 3.2 we may estimate

$$\sum_{\substack{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\(y',r')\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)\\2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle|$$
$$\lesssim \sum_{0\le a\le m+10}\Biggl(\sum_{(y,r)\in\bigcup_{b\in\mathcal{B}_{Q,c,d}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)}\Biggl(\sum_{y'\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y}$$
$$\Biggl(\sum_{\substack{r'\in\bigcup_{b\in\mathcal{B}_{Q,c',d'}}(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_R\\2^m\le|(y,r)-(y',r')|\le 2^{m+1}\\ \min_{\pm,\pm}(1+|r\pm r'\pm|y-y'||)\approx 2^a}} 2^{-Na}2^{3(k-m/2)}\Biggr)\Biggr)\Biggr)$$
$$\lesssim 2^{3(k-m/2)}(\max(\#\mathcal{B}_{Q,c,d},\#\mathcal{B}_{Q,c',d'}))^2\max_{b\in\mathcal{B}_{Q,c,d}}(\#(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*))$$
$$\times\max_{b'\in\mathcal{B}_{Q,c,d}}(\#(\mathcal{E}_k^{\gamma,j,b}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap Q^*)_Y)$$
$$\lesssim 2^{3(k-m/2)}(\max(\#\mathcal{B}_{Q,c,d},\#\mathcal{B}_{Q,c',d'}))^2 2^{\max(c,c')}u2^m(2^{(c'-d')/2})^{-1}, \tag{5-29}$$

and the proof of (5-6) is complete. □

We will now use Lemma 5.2 to prove Lemma 5.1.

Proof of Lemma 5.1. Fix an $a>0$ to be determined later. Similar to [Heo et al. 2011], we let $\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}=\sum_\mu \widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}$, where for each positive integer μ we set

$$I_{k,\mu}=[2^k+(\mu-1)u^a, 2^k+\mu u^a),$$
$$\mathcal{E}_{k,\mu}=\mathcal{Y}\times I_{k,\mu},$$
$$\widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}=\sum_{(y,r)\in\mathcal{E}_{k,\mu}\cap\mathcal{E}_k^{\gamma,j}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)}\gamma(y,r)F_{y,r}\quad\text{and}\quad \widetilde{G}_{u,k,\mu,r}^{\gamma,\mathcal{E}^{\gamma,j}}=\sum_{y:(y,r)\in\mathcal{E}_{k,\mu}\cap\mathcal{E}_k^{\gamma,j}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)}\gamma(y,r)F_{y,r}.$$

We have

$$\|\widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2\lesssim\Bigl\|\sum_\mu\widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\Bigr\|_2^2\lesssim\sum_\mu\|\widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2+\sum_{\mu'>\mu+10}|\langle\widetilde{G}_{u,k,\mu'}^{\gamma,\mathcal{E}^{\gamma,j}}, G_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\rangle|. \tag{5-30}$$

By Cauchy–Schwarz,

$$\|\widetilde{G}_{u,k,\mu}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2\lesssim u^a\sum_{r\in\mathcal{I}_{k,\mu}\cap\mathcal{R}}\|\widetilde{G}_{u,k,\mu,r}^{\gamma,\mathcal{E}^{\gamma,j}}\|_2^2.$$

Write

$$\widetilde{G}_{u,k,\mu,r}^{\gamma,\mathcal{E}^{\gamma,j}}=\Biggl(\sum_{y:(y,r)\in\mathcal{E}_{k,\mu}\cap\mathcal{E}_k^{\gamma,j}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)}\gamma(y,r)\psi_0(\cdot-y)\Biggr)*(\sigma_r*\psi_0).$$

By the Fourier decay of σ_r and the order of vanishing of ψ_0 at the origin, we have

$$\|\hat{\sigma}_r \hat{\psi}_0\|_\infty \lesssim r^{3/2}.$$

Since

$$\Big\| \sum_{y:(y,r)\in\mathcal{E}_{k,\mu}\cap\mathcal{E}_k^{\gamma,j}\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)} \gamma(y,r)\psi_0(\cdot - y)\Big\|_2^2 \lesssim 2^{2j}\,\#\{y\in\mathcal{Y} : (y,r)\in\mathcal{E}_{k,\mu}\cap\mathcal{E}_k^{\gamma,j}\cap\widetilde{\mathcal{E}}^{\gamma,j}(u)\},$$

we have

$$\sum_\mu \|\widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k,\mu}\|_2^2 \lesssim u^a \sum_\mu \sum_{r\in\mathcal{I}_{k,\mu}\cap\mathcal{R}} \|\widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k,\mu,r}\|_2^2 \lesssim 2^{2j}u^a 2^{3k}\,\#\mathcal{E}_k^{\gamma,j}. \tag{5-31}$$

By (5-30), it remains to estimate $\sum_{\mu'>\mu+10} |\langle \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k,\mu'}, \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k,\mu}\rangle|$.

Note that we have the bound

$$\begin{aligned}\sum_{\mu'>\mu+10} |\langle \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k,\mu'}, \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k,\mu}\rangle| &\lesssim 2^{2j} \sum_{\substack{m:2^m\ge u^a\\ (y,r),(y',r')\in\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap\mathcal{E}_k^{\gamma,j}\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \\ &\lesssim 2^{2j} \sum_{m:2^m\ge u^a} \Bigg(\sum_{Q\in\mathcal{Q}_{u,k,j,m}} \Bigg(\sum_{\substack{(y,r),(y',r')\in Q\cap\widetilde{\mathcal{E}}_k^{\gamma,j}(u)\cap\mathcal{E}_k^{\gamma,j}\\ 2^m\le|(y,r)-(y',r')|\le 2^{m+1}}} |\langle F_{y,r}, F_{y',r'}\rangle| \Bigg)\Bigg).\end{aligned} \tag{5-32}$$

To estimate the inner sum above, we will use Corollary 5.5. Summing over all $Q\in\mathcal{Q}_{u,k,j,m}$ and over all m such that $2^m\ge u^a$ we have

$$\sum_{\mu'>\mu+10} |\langle \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k,\mu'}, \widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k,\mu}\rangle| \lesssim_\epsilon 2^{2j}(I+II), \tag{5-33}$$

where

$$I := 2^{3k}\sum_{l\ge j} 2^{(l-j)/10}(\#\mathcal{E}_k^{\gamma,l})u^\epsilon \max(u^{11/12-a/2},\ u^{13/12-a}), \tag{5-34}$$

$$II := 2^{3k}\sum_{l\ge j} 2^{(l-j)/10}(\#\mathcal{E}_k^{\gamma,l})u^\epsilon u^{11/12-a/2}. \tag{5-35}$$

Combining (5-30), (5-31) and (5-33), we thus have the estimate

$$\|\widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\|_2^2 \lesssim_\epsilon 2^{2j}2^{3k}\sum_{l\ge j} 2^{(l-j)/10}(\#\mathcal{E}_k^{\gamma,l})(u^a + u^{11/12-a/2+\epsilon} + u^{13/12-a+\epsilon}).$$

Choose $a=\frac{11}{18}$ to obtain

$$\|\widetilde{G}^{\gamma,\mathcal{E}^{\gamma,j}}_{u,k}\|_2^2 \lesssim_\epsilon 2^{2j}2^{3k}\sum_{l\ge j} 2^{(l-j)/10}(\#\mathcal{E}_k^{\gamma,l})u^{11/18+\epsilon}$$

for every $\epsilon>0$, which is (5-2). □

Incomparable radii. We now want to estimate

$$\sum_{k>k'>N(u)} |\langle \widetilde{G}_{u,k'}^{\gamma,\mathcal{E}^{\gamma,j}}, \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}} \rangle|.$$

Our estimate will be much better than in the comparable radii case. In fact, since $d = 4$, we may simply use the estimate proved for incomparable radii in [Heo et al. 2011], which is more than sufficient for our purposes. We restate this estimate using our notation as follows.

Lemma 5.6. *Let $\epsilon > 0$. For the choice $N(u) = 100\epsilon^{-1}\log_2(2+u)$, we have*

$$\sum_{k>k'>N(u)} |\langle \widetilde{G}_{u,k'}^{\gamma,\mathcal{E}^{\gamma,j}}, G_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}} \rangle| \lesssim_\epsilon 2^{2j} \sum_k 2^{3k} \sum_{l:|l-j|\le 10} \#\mathcal{E}_k^{\gamma,l}. \tag{5-36}$$

For a proof of Lemma 5.6, see [Heo et al. 2011].

Putting it together. Combining (5-1), (5-2) and (5-36), we have that for every $\epsilon > 0$,

$$\Big\| \sum_k \widetilde{G}_{u,k}^{\gamma,\mathcal{E}^{\gamma,j}} \Big\|_2^2 \lesssim_\epsilon u^{11/18+\epsilon} \sum_k 2^{3k} 2^{2j} \sum_{l \ge j} 2^{(l-j)/10} (\#\mathcal{E}_k^{\gamma,l}). \tag{5-37}$$

This completes the proof of Lemma 4.3 and hence the proof of Proposition 4.1.

Appendix: A geometric lemma

In this section we prove the geometric lemma used in the previous section.

Lemma A.1. *Fix integers j, l with $l \le j$. Let $2^{j-1} \le t \le 2^{j+1}$. Then the size of the intersection of three annuli in $\mathbb{R}^3$ of thickness 4 and inner radius t such that the distance between the centers of any pair is at least 2^l and no greater than $2^j/10$ is $\lesssim 2^{3(j-l)}$, provided that $l \ge j/2 + 10$.*

We will use the following basic lemma which gives an estimate on the size of intersections of 2-dimensional annuli. This is an immediate corollary of Lemma 3.1 in [Wolff 1999].

Lemma D. *Let A_1 and A_2 be two annuli in $\mathbb{R}^2$ of thickness 1 built upon circles C_1 and C_2 of radius R, and let d denote the distance between the centers of C_1 and C_2. If $d \le R/5$, then $A_1 \cap A_2$ is contained in the 10-neighborhood of an arc of C_1 of length $\lesssim R/d$.*

Proof of Lemma A.1. Let A_1, A_2, A_3 denote the three annuli. Let $\ell_{1,2}$ denote the line through the centers of A_1 and A_2, and let $\ell_{1,3}$ denote the line through the centers of A_1 and A_3. Let P be any plane containing both $\ell_{1,2}$ and $\ell_{1,3}$. Then $A_1 \cap A_2$ is the 3-dimensional solid formed by rotating the intersection of the two (circular) annuli $A_1 \cap P$ and $A_2 \cap P$ about the line $\ell_{1,2}$. Similarly, $A_1 \cap A_3$ is the 3-dimensional solid formed by rotating the intersection of the two (circular) annuli $A_1 \cap P$ and $A_3 \cap P$ about the line $\ell_{1,3}$.

Now, by Lemma D, $A_1 \cap A_2 \cap P$ is contained in the 10-neighborhood of two arcs of length $\lesssim 2^{j-l}$ of the circle that $A_1 \cap P$ is built upon. Rotating $A_1 \cap A_2 \cap P$ about the line $\ell_{1,2}$ to get $A_1 \cap A_2$, this implies that $A_1 \cap A_2$ is the union of $\lesssim 2^{j-l}$ many 10-neighborhoods of circular annuli of radius $\lesssim 2^j$ lying in a plane normal to the line $\ell_{1,2}$. The same holds for $A_1 \cap A_3$ with $\ell_{1,2}$ replaced by $\ell_{1,3}$. Suppose first

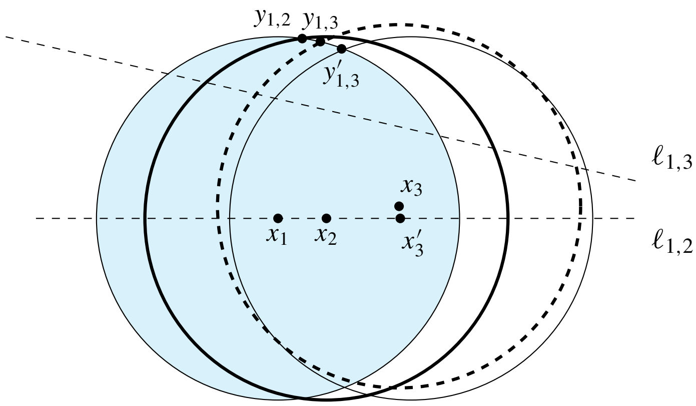

Figure 1. The circles C_1, C_2, C_3, and C'_3 in the plane P, from the proof of Lemma A.1. The shaded-in circle is C_1, the thick circle is C_2, the dashed circle is C_3, and the remaining circle is C'_3.

that the angle between $\ell_{1,2}$ and $\ell_{1,3}$ is $\geq 2^{l-j-3}$, in radians. Then $|A_1 \cap A_2 \cap A_3|$ is bounded by $\lesssim 2^{2(j-l)}$ times the largest possible size of the intersection of two 10-neighborhoods of circular annuli, where the first lies in a plane normal to $\ell_{1,2}$ and the second lies in a plane normal to $\ell_{1,3}$. One computes that the largest possible size of such an intersection is $\lesssim 2^{j-l}$.

It remains to consider the case when the angle between $\ell_{1,2}$ and $\ell_{1,3}$ is $< 2^{l-j-3}$, in radians. We now define the following coordinates associated to the lines $\ell_{1,2}$ and $\ell_{1,3}$. Let x_1, x_2, x_3 denote the centers of A_1, A_2, A_3 respectively. For $x \in \mathbb{R}^3$, we define the $\ell_{1,2}$-coordinate

$$(x)_{1,2} = \frac{\langle x - x_1, x_2 - x_1 \rangle}{|x_2 - x_1|}.$$

Similarly define the $\ell_{1,3}$-coordinate

$$(x)_{1,3} = \frac{\langle x - x_1, x_3 - x_1 \rangle}{|x_3 - x_1|}.$$

By interchanging the order of A_1, A_2, A_3, we may assume without loss of generality that $(x_3)_{1,2} \geq (x_2)_{1,2} = 1$. We will show that $l \geq j/2 + 10$ implies that $A_1 \cap A_2$ and $A_2 \cap A_3$ are actually disjoint. Observe that since the angle between $\ell_{1,2}$ and $\ell_{1,3}$ is $< 2^{l-j-3}$, we have that $(x_3 - x_2)_{1,2} \geq 2^{l-1}$. Now, let x'_3 be the closest point on the line $\ell_{1,2}$ whose distance from x_1 is the same as the distance from x_1 to x_3. Clearly, we also have $(x'_3 - x_2)_{1,2} \geq 2^{l-1}$. Let C_3 be the circle in P with center at x_3 and radius t and let C'_3 be the circle in P with center at x'_3 and radius t. Then if $y'_{1,3}$ denotes either of the two points in $C_1 \cap C'_3$ and $y_{1,2}$ either of the two points in $C_1 \cap C_2$, then $(x'_3 - x_2)_{1,2} \geq 2^{l-1}$ implies that $(y'_{1,3} - y_{1,2})_{1,2} \geq 2^{l-2}$. This is because with respect to the $\ell_{1,2}$-coordinate, $y'_{1,3}$ lies at the midpoint of x_1 and x_3 and $y_{1,2}$ lies at the midpoint of x_1 and x_2. Note that $C_1 \cap C_3$ is the rotation within P of $C_1 \cap C'_3$ by an angle of $< 2^{l-j-3}$, where the rotation is based at x_1. This implies that if $y_{1,3}$ is either of the two points in $C_1 \cap C_3$, then $|y'_{1,3} - y_{1,3}| \leq 2^{l-3}$. It follows that $(y_{1,3} - y_{1,2})_{1,2} \geq (y'_{1,3} - y_{1,2})_{1,2} - |y'_{1,3} - y_{1,3}| \geq 2^{l-2} - 2^{l-3} = 2^{l-3}$.

But by Lemma D, $A_1 \cap A_2$ is the rotation in $\mathbb{R}^3$ of a 10-neighborhood of an arc of C_1 of length $\lesssim 2^{j-l}$ that contains $y_{1,2}$ about $\ell_{1,2}$, and so $A_1 \cap A_2$ lives in the slab $\{z \in \mathbb{R}^3 : |(z - y_{1,2})_{1,2}| \le 2^{j-l+4}\}$. Similarly, $A_1 \cap A_3$ is the rotation in $\mathbb{R}^3$ of a 10-neighborhood of an arc of C_1 of length $\lesssim 2^{j-l}$ that contains $y_{1,3}$ about $\ell_{1,3}$, and so $A_1 \cap A_3$ lives in the half-infinite slab $\{z \in \mathbb{R}^3 : (z - y_{1,3})_{1,2} \ge -2^{j-l+4}\}$, and since $l \ge j/2+10$, we have $j-l+4 \le l-10$. Since $(y_{1,3} - y_{1,2})_{1,2} \ge 2^{l-3}$, it follows that $A_1 \cap A_2$ and $A_2 \cap A_3$ are disjoint. □

Corollary A.2. *Fix integers j, l with $l \le j$. Let $2^{j-1} \le t \le 2^{j+1}$. Then the size of the intersection of three annuli in $\mathbb{R}^4$ of thickness* 4 *and inner radius t such that the distance between the centers of any pair is at least 2^l and no greater than $2^j/10$ is $\lesssim 2^{3(j-l)} 2^j$, provided that $l \ge j/2 + 10$.*

Proof. Let P be a hyperplane in $\mathbb{R}^4$ containing the centers of the three annuli, and for each $t \in \mathbb{R}$, let P_t be the one-parameter family of hyperplanes with normals parallel to the normal to P. For each t, the intersection of each annulus with P_t is a 3-dimensional annulus of a fixed radius depending on t that is $\lesssim 2^j$ and a fixed width depending on t that is $\lesssim 1$, and with centers spaced apart by $\gtrsim 2^l$. By Lemma A.1, $A_1 \cap A_2 \cap A_3 \cap P_t$ has size $\lesssim 2^{3(j-l)}$. It follows that $A_1 \cap A_2 \cap A_3$ has size $\lesssim 2^{3(j-l)} 2^j$. □

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CONTINUUM LIMIT AND STOCHASTIC HOMOGENIZATION OF DISCRETE FERROMAGNETIC THIN FILMS

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We study the discrete-to-continuum limit of ferromagnetic spin systems when the lattice spacing tends to zero. We assume that the atoms are part of a (maybe) nonperiodic lattice close to a flat set in a lower-dimensional space, typically a plate in three dimensions. Scaling the particle positions by a small parameter $\varepsilon > 0$, we perform a Γ-convergence analysis of properly rescaled interfacial-type energies. We show that, up to subsequences, the energies converge to a surface integral defined on partitions of the flat space. In the second part of the paper we address the issue of stochastic homogenization in the case of random stationary lattices. A finer dependence of the homogenized energy on the average thickness of the random lattice is analyzed for an example of a magnetic thin system obtained by a random deposition mechanism.

1. Introduction

Polymeric magnets are known to be lighter and more flexible than conventional magnets. They can be easily manufactured to form thin films made of few layers and are currently considered one of the main building blocks of the future generations of electronic devices. Under external magnetic fields they form Weiss domains, whose wall energy is influenced by the thickness and the roughness of the film, which in turn depends on the physical and chemical properties of the specific material at use. A fairly large amount of experimental results reconstruct the relation between film thickness and interfacial domain wall energy for different ferromagnetic materials, see [Klein and Smith 1951], but no rigorous explanation has appeared so far in this direction. Among the reasons for such an unsatisfactory analysis, we single out

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one which has a geometric flavor: depositing magnetic particles on a substrate to obtain a thin film leads to disordered arrangements of particles and rough film surfaces, which makes it very difficult to formulate a suitable ansatz leading to the correct (and simpler) continuum model. In this paper we look at this problem from a different perspective: we single out a simple Ising-type model for a thin film obtained by random deposition of magnetic particles on a flat substrate, for which the geometric part of the problem is still nontrivial, and propose an ansatz-free variational analysis of such a film. Combining Γ-convergence and percolation theory, we finally obtain a rigorous explanation of the relation between film thickness and domain-wall energy in some asymptotic regimes.

A simple way to model thin ferromagnetic polymeric materials at the microscale first requires the definition of a polymeric matrix made of magnetic cells and then that of an interaction energy between those cells; see [Vollath 2013] for further details. The polymeric matrix of such a system can be seen as a random network whose nodes are the cross-linkers molecules of the three-dimensional polymeric magnet, which are supposed to entail the local magnetic properties of the system and to interact as magnetic elementary cells via a ferromagnetic Potts-type coupling. The system is supposed to be thin in the sense that the nodes of the matrix are within a small distance, of the order of the average distance between the nodes themselves, from a two-dimensional plane. In the presence of an external magnetic field or of proper boundary conditions, the ferromagnetic coupling induces the system to form mesoscopic Weiss domains, i.e., regions of constant magnetization.

In this paper we aim at upscaling the system described above from its microscopic description to a mesoscopic one in a variational setting. This consists in performing the limit of its energy as the average distance between the magnetic cells, which we denote by ε, goes to zero with respect to the macroscopic size of the system. Such a limit will have two main effects: it will allow us to describe the original discrete system as a continuum, while at the same time it will reduce its dimension from three to two (or more generally from d to k with $2 \leq k < d$).

The discrete-to-continuum analysis in this paper is also part of a general study of the effects of discreteness in lattice systems on their macroscopic description. It is directly related to a series of papers describing the overall behavior of spin energies [Caffarelli and de la Llave 2005; Alicandro et al. 2006; Braides and Piatnitski 2013; Braides and Cicalese 2017; Alicandro and Gelli 2016]. Moreover, discrete-to-continuum analyses for thin elastic objects in a deterministic setting have also been considered, e.g., in [Alicandro et al. 2008; Schmidt 2008; Lazzaroni et al. 2015], and the behavior of full-dimensional random lattices is dealt with in [Alicandro et al. 2011]; see also [Blanc et al. 2007]. For dimension-reduction problems for continuum elastic objects we also refer to [Le Dret and Raoult 1995; Braides et al. 2000], the latter introducing a dimensionally reduced localization argument similar to the one used in the present paper.

Using the same model as in [Alicandro et al. 2015] we describe the polymeric matrix as a random network whose nodes $\mathcal{L} \subset \mathbb{R}^d$ form a thin *admissible stochastic lattice*, meaning that the matrix is thin; i.e., there exist $k \in \mathbb{N}$ with $2 \leq k < d$ and $M > 0$ such that, identifying $\mathbb{R}^k$ with a linear subspace of $\mathbb{R}^d$,

$$\operatorname{dist}(x, \mathbb{R}^k) \leq M \qquad \text{for all } x \in \mathcal{L}$$

and it is admissible according to the following standard definition; see [Ruelle 1989] and also [Alicandro et al. 2011; Blanc et al. 2007] in the framework of rubber elasticity. We say that $\mathcal{L}$ is an admissible set of points if the following two requirements are satisfied:

(i) There exists $r > 0$ such that $|x - y| \geq r$ for all $x \neq y,\ x, y \in \mathcal{L}$,

(ii) There exists $R > 0$ such that $\operatorname{dist}(x, \mathcal{L}) \leq R$ for all $x \in \mathbb{R}^k$.

Within this definition we may include "slices" of periodic lattices [Alicandro et al. 2008], and also aperiodic geometries [Braides et al. 2012].

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable $\mathcal{L} : \Omega \to (\mathbb{R}^d)^{\mathbb{N}}$ is called an *admissible stochastic lattice* if $\mathcal{L}(\omega)$ is an admissible set of points uniformly with respect to $\omega \in \Omega$.

We assume that the magnetization takes only finitely many values; that is to say, we consider configurations $u : \varepsilon\mathcal{L} \to \mathcal{S}$ with a state-space $\mathcal{S} = \{s_1, \dots, s_q\}$ that we embed in the euclidean space $\mathbb{R}^q$. We have in mind the case of spin systems, where $u_i \in \{1, -1\}$. Note that even in that case it is sometimes necessary to use a larger set of parameters $\mathcal{S}$ if frustration forces the formation of texture; see [Braides and Cicalese 2017]. Note that if we have more than two parameters, we may have concentration phenomena of a third phase on the interfaces between two phases. A finer description of this phenomenon can be found in [Alicandro et al. 2012].

Associating a Voronoi tessellation $\mathcal{V}(\mathcal{L})$ to the lattice $\mathcal{L}$, one introduces the set of nearest neighbors $\mathcal{NN}(\mathcal{L})$ as the set of those pairs of points in $\mathcal{L}$ whose Voronoi cells share a $(d-1)$-dimensional edge. This allows us to distinguish between long-range and short-range interactions, introducing the ($\mathcal{L}$-dependent) interactions

$$f_\varepsilon(x, y, s_i, s_j) = \begin{cases} f^\varepsilon_{nn}(x, y, s_i, s_j) & \text{if } (x, y) \in \mathcal{NN}(\mathcal{L}), \\ f^\varepsilon_{lr}(x, y, s_i, s_j) & \text{otherwise,} \end{cases}$$

which we assume to be nonnegative and to satisfy the following coerciveness and growth assumptions.

Hypothesis 1. *There exist $c > 0$ and a decreasing function $J_{lr} : [0, +\infty) \to [0, +\infty)$ with*

$$\int_{\mathbb{R}^k} J_{lr}(|x|)|x| \, \mathrm{d}x = J < +\infty$$

such that, for all $\varepsilon > 0,\ x, y \in \mathbb{R}^d$ and $s_i, s_j \in \mathcal{S}$,

$$c|s_i - s_j| \leq f^\varepsilon_{nn}(x, y, s_i, s_j) \leq J_{lr}(|x - y|)|s_i - s_j|, \quad f^\varepsilon_{lr}(x, y, s_i, s_j) \leq J_{lr}(|x - y|)|s_i - s_j|.$$

We remark that the decay of J_{lr} is needed to control the effect of long-range interactions and we use the same bound for short-range interactions only to save notation.

We fix $D \subset \mathbb{R}^k$ and denote by $P_k : \mathbb{R}^d \to \mathbb{R}^k$ the projection onto $\mathbb{R}^k$. For a given configuration $u : \varepsilon\mathcal{L} \to \mathcal{S}$ we consider the energy per unit ($(k-1)$-dimensional) surface of D to have the ferromagnetic Potts form; see also [Alicandro et al. 2006; 2012; 2015; Braides and Cicalese 2017] given by

$$E_\varepsilon(u) = \sum_{\substack{x, y \in \mathcal{L} \\ \varepsilon x, \varepsilon y \in P_k^{-1} D}} \varepsilon^{k-1} f_\varepsilon(x, y, u(\varepsilon x), u(\varepsilon y)).$$

Since the sets $\varepsilon\mathcal{L}$ will eventually shrink to a k-dimensional set, we conveniently describe the system in terms of an *average spin order parameter* $Pu : \varepsilon P_k\mathcal{L} \to \mathrm{co}(\mathcal{S})$ defined on the k-dimensional set $\varepsilon P_k\mathcal{L}$ by

$$Pu(z) := \frac{1}{\#(P_k^{-1}(z) \cap \varepsilon\mathcal{L})} \sum_{\varepsilon x \in P_k^{-1}(z)\cap\varepsilon\mathcal{L}} u(\varepsilon x).$$

We then embed the energies E_ε in $L^1(D)$ by identifying Pu with a function piecewise constant on the cells of the Voronoi tessellation of $P_k\mathcal{L}$, define the convergence $u_\varepsilon \to u$ in D in the sense that the piecewise constant functions Pu_ε converge to u strongly in $L^1(D)$ and perform the Γ-convergence analysis with respect to this notion (see Section 2 for further details).

In Theorem 3.2 we prove a compactness and integral representation result for the Γ-limit E of E_ε, stating that, up to subsequences, this is finite only on $\mathrm{BV}(D, \mathcal{S})$, where it takes the integral form

$$E(u) = \int_{S_u} \phi^\omega(x, u^+, u^-, \nu_u)\, \mathrm{d}\mathcal{H}^{k-1}.$$

In this formula S_u is the jump set of u, the functions u^+ and u^- represent the traces on both sides of the jump set, $\nu_u \in S^{k-1}$ is the measure-theoretical normal to S_u and $\mathcal{H}^{k-1}$ is the $(k-1)$-dimensional Hausdorff measure. The function ϕ^ω is interpreted as the domain-wall interaction energy (per unit $(k-1)$-dimensional area) between Weiss domains.

The dependence of such an energy on the randomness of the lattice is studied in Section 5 in the context of stochastic homogenization, assuming the thin random lattice to be stationary (or ergodic) in the directions of the flat subspace to which it is close and the interaction coefficients to be invariant under translation in these directions. More precisely we assume that there exists a measure-preserving group action $(\tau_z)_{z\in\mathbb{Z}^k}$ on Ω such that, almost surely in Ω, we have $\mathcal{L}(\tau_z\omega) = \mathcal{L}(\omega) + z$ (if in addition $(\tau_z)_{z\in\mathbb{Z}^k}$ is ergodic, then also the lattice $\mathcal{L}$ is said to be ergodic) and the following structural assumption:

Hypothesis 2. *There exist functions* $f_{nn}, f_{lr} : \mathbb{R}^k \times \mathbb{R}^{2(d-k)} \times \mathcal{S}^2 \to [0, +\infty)$ *such that, setting* $\Delta_k(x, y) = (y_1 - x_1, \dots, y_k - x_k, x_{k+1}, y_{k+1}, \dots, x_d, y_d)$, *it holds that*

$$f_{nn}^\varepsilon(x, y, s_i, s_j) = f_{nn}(\Delta_k(x, y), s_i, s_j), \qquad f_{lr}^\varepsilon(x, y, s_i, s_j) = c_{lr}(\Delta_k(x, y), s_i, s_j).$$

In Theorem 5.8 we prove that under Hypotheses 1 and 2 and assuming the stationarity (or ergodicity) in the sense specified above, the Γ-limit of E_ε as $\varepsilon \to 0$ exists and is finite only on $\mathrm{BV}(D, \mathcal{S})$ where it takes the form

$$E_{\mathrm{hom}}^\omega(u) = \int_{S_u} \phi_{\mathrm{hom}}^\omega(u^+, u^-, \nu_u)\, \mathrm{d}\mathcal{H}^{k-1}.$$

The energy density is given by an asymptotic homogenization formula which is averaged in the probability space under ergodicity assumptions on $\mathcal{L}$, thus turning the stochastic domain wall energy into a deterministic one.

The result is proved by the abstract methods of Γ-convergence, first showing an abstract compactness result, and then giving an integral representation of the limit, as described in detail for deterministic bulk elastic thin films in [Braides et al. 2000]; for other applications of this method in a discrete-to-continuum setting see, e.g., [Alicandro and Cicalese 2004; Le Dret and Raoult 2017; Braides and Cicalese 2017]. The proof makes use of two main ingredients: the integral-representation theorem in [Bouchitté et al. 2002] and the subadditive ergodic theorem in [Akcoglu and Krengel 1981]. They are combined following a scheme introduced in [Alicandro et al. 2011] in the context of random discrete systems with limit energy on Sobolev spaces, see also [Dal Maso and Modica 1986], and recently extended to sets of finite perimeter in [Alicandro et al. 2015]. Section 6 is devoted to extending the result above to the case of a volume constraint on the phases.

An interesting issue in the theory of thin magnetic composite polymeric materials is the dependence of the domain wall energy on the random geometry of the polymer matrix. We devote the second part of the paper to this problem. We consider a specific model of a discrete system in which the state-space is $\mathcal{S} = \{\pm 1\}$ and the stochastic lattice is generated by the random deposition of magnetic particles on a two-dimensional flat substrate. For simplicity we limit ourselves to a simple deposition model with vertical order and suppose that the magnetic interactions have finite range. We are interested in the dependence of the domain wall energy on the average thickness of the thin film. Even though a complete picture would need a more extended treatment, thanks to percolation arguments we are able to attack the problem in the asymptotic cases when the thickness of the film is either small or very large.

More specifically, we model the substrate (where the particles are deposited) by taking a two-dimensional deterministic lattice, which we choose for simplicity as $\mathcal{L}^0 = \mathbb{Z}^2 \times \{0\}$. We then consider an independent random field $\{X_i^p\}_{i\in\mathbb{Z}^3}$, where the X_i^p are Bernoulli random variables with $\mathbb{P}(X_i^p = 1) = p \in (0, 1)$. For fixed $M \in \mathbb{N}$ we construct the random point set

$$\mathcal{L}_p^M(\omega) := \big\{(i_1, i_2, i_3) \in \mathbb{Z}^3 : 0 \le i_3 \le \textstyle\sum_{k=1}^M X^p_{(i_1,i_2,k)}(\omega)\big\},$$

which means that we successively deposit particles M times independently onto the flat lattice $\mathcal{L}^0$ and stack them over each other (the point set constructed is stationary with respect to translations in $\mathbb{Z}^2$ and ergodic). Moreover, given $u : \varepsilon\mathcal{L}_p^M(\omega) \to \{\pm 1\}$, we consider an energy of the form

$$E^p_{\varepsilon,M}(\omega)(u, A) = \sum_{\substack{x,y\in\mathcal{L}_p^M(\omega)\\ \varepsilon P_2(x),\varepsilon P_2(y)\in A}} \varepsilon\, c(x-y)|u(\varepsilon x) - u(\varepsilon y)|,$$

where the interaction constant $c : \mathbb{R}^3 \to [0, +\infty)$ has finite range, is bounded from above and is coercive on nearest neighbors, so that the Hypotheses 1 and 2 above are satisfied. As a result Theorem 5.8 guarantees the existence of a surface tension, say $\phi^p_{\text{hom}}(M; \nu)$, given by an asymptotic cell formula.

The main issue now is the dependence of $\phi^p_{\text{hom}}(M; \nu)$ on p and M.

A first result in this direction is proved in Proposition 7.3, where we show that, for every direction $\nu \in S^1$, the wall energy density is linear in the average thickness pM as $M \to +\infty$; that is,

$$\lim_{M\to+\infty} \frac{\phi^p_{\text{hom}}(M; \nu)}{pM} = \phi^1(\nu), \tag{1-1}$$

with $\phi^1(\nu)$ given in Lemma 7.2 being the wall energy per unit thickness of the deterministic problem obtained for $p = 1$.

A second and more delicate result is contained in Theorem 7.5 and concerns a percolation-type phenomenon which can be roughly stated as follows: when the deposition probability p is sufficiently low (below a certain critical percolation threshold) the domain wall energy is zero for M small enough. At this stage it is worth noticing that our energy accounts for the interactions between the deposited particles and the substrate. On one hand this assumption might be questionable from a physical point of view in the case one assumes to grow thin films on neutral media, thus expecting the properties of the film to be independent of the substrate. On the other hand removing such an interaction leads to a dilute model similar to the one considered in [Braides and Piatnitski 2012]. An adaption of this analysis would require a lot of additional work like the extension of fine percolation results to the (range-1)-dependent case, which goes far beyond the scope of the present paper (see also Remark 7.4). We prove the percolation result for nearest-neighbor positive interactions. Setting the interaction with the substrate to be $\eta > 0$, we can prove that if $p < 1 - p_{\mathrm{site}}$ (here p_{site} is the critical site percolation threshold in $\mathbb{Z}^2$), the limit energy $\phi_{\mathrm{hom}}^{p,\eta}(M;\nu)$ is bounded above (up to a constant) by η for M small enough. This result suggests the absence of a positive domain wall energy in the thin film on a neutral substrate ($\eta = 0$ case). In the limit as M diverges, (1-1) holds with $\phi_{\mathrm{hom}}^{p,\eta}(M;\nu)$, which is independent of η, thus showing that the contribution of the first layer does not affect the asymptotic average domain wall energy as expected. The proof of these results needs the extension to the dimension reduction framework of a result by Caffarelli and de la Llave [2005] about the existence of plane-like minimizers for discrete systems subject to periodic Ising-type interactions at the surface scaling. This is contained in Appendix A.

As a final remark, we mention that we prove all our results in the case when the flat object is at least two-dimensional. Most of the results can be extended to one-dimensional objects (with the proof being much simpler), except the ones contained in Section 6, which fail in dimension one as can be seen by simple examples and the percolation-type phenomenon in Section 7, as no percolation can occur in (essentially) one-dimensional lattices.

2. Modeling discrete disordered thin sets and spin systems

This section is devoted to the precise description of the model we are going to study. We start with the notation we are going to use in the sequel.

As we are concerned with dimension-reduction issues, there will be two geometric dimensions k and d with $2 \le k < d$. Given a measurable set $A \subset \mathbb{R}^k$, we denote by $|A|$ its k-dimensional Lebesgue measure, while more generally $\mathcal{H}^m(A)$ stands for the m-dimensional Hausdorff measure. We denote by $\mathbb{1}_A$ the *characteristic function* of A. Given $x \in \mathbb{R}^k$ and $r > 0$, we denote by $B_r(x)$ the open ball around x with radius r. By $|x|$ we denote the usual euclidean norm of x. Moreover, we set $\mathrm{d}_{\mathcal{H}}(A, B)$ to be the Hausdorff distance between the sets A and B and $\dim_{\mathcal{H}}(A)$ to be the Hausdorff dimension of A. If it is clear from the context we will use the same notation as above also in $\mathbb{R}^d$ (otherwise we will indicate the dimension by sub/superscript indices). Given an open set $D \subset \mathbb{R}^k$, we denote by $\mathcal{A}(D)$ the family of all bounded open subsets of D and by $\mathcal{A}^R(D)$ the family of those sets in $\mathcal{A}(D)$ with Lipschitz boundary. Given a unit vector $\nu \in S^{k-1}$, let $\nu = \nu_1, \dots, \nu_k$ be an orthonormal basis. We define the open cube in $\mathbb{R}^k$

$$Q_\nu = \left\{x \in \mathbb{R}^k : |\langle x, \nu_i\rangle| < \tfrac{1}{2} \text{ for all } i\right\},$$

and, for $x \in \mathbb{R}^k$, $\rho > 0$, we set $Q_\nu(x, \rho) := x + \rho\, Q_\nu$. We call $\nu \in S^{k-1}$ a rational direction if $\nu \in \mathbb{Q}^k$. We denote by $P_k : \mathbb{R}^d \to \mathbb{R}^k$ the projection onto $\mathbb{R}^k$.

For $q \in \mathbb{N}$ we let $\mathrm{BV}(D, \mathbb{R}^q)$ be the space of $\mathbb{R}^q$-valued *functions of bounded variation*, that is, those functions $u \in L^1(D, \mathbb{R}^q)$ such that their distributional derivative Du is a matrix-valued Radon measure. Given a set $\mathcal{S} \subset \mathbb{R}^q$, we denote by $\mathrm{BV}(A, \mathcal{S})$ the space of those functions $u \in \mathrm{BV}(A, \mathbb{R}^q)$ such that $u(x) \in \mathcal{S}$ almost everywhere. If $\mathcal{S}$ is a finite set, then the distributional derivative of u can be represented on any Borel set $B \subset D$ as $Du(B) = \int_{B \cap S_u} (u^+(x) - u^-(x)) \otimes \nu_u(x)\, \mathrm{d}\mathcal{H}^{k-1}(x)$ for a countably $\mathcal{H}^{k-1}$-rectifiable set S_u in D which coincides $\mathcal{H}^{k-1}$-almost everywhere with the complement in D of the Lebesgue points of u. Moreover $\nu_u(x)$ is a unit normal to S_u, defined for $\mathcal{H}^{k-1}$-almost every x and $u^+(x)$, $u^-(x)$ are the traces of u on both sides of S_u. Here the symbol $\otimes$ stands for the tensorial product of vectors; that is, for any $a, b \in \mathbb{R}^k$, we have $(a \otimes b)_{ij} := a_i b_j$. A measurable set B is said to have finite perimeter in D if its characteristic function belongs to $\mathrm{BV}(D)$. We refer the reader to [Ambrosio et al. 2000] for an introduction to functions of bounded variation. The letter C stands for a generic positive constant that may change every time it appears.

We want to describe (possibly nonperiodic) particle systems, where the particles themselves are located very close to a lower-dimensional linear subspace. To this end we make the following assumptions: Let $\mathcal{L} \subset \mathbb{R}^d$ be a countable set. We assume that there exists $M > 0$ such that, after identifying $\mathbb{R}^k \sim \mathbb{R}^k \times \{0\}^{d-k}$, we have

$$\mathrm{dist}(x, \mathbb{R}^k) \le M \quad \text{for all } x \in \mathcal{L}. \tag{2-2}$$

Moreover, adapting ideas from [Alicandro et al. 2011; 2015; Blanc et al. 2007] we assume that the point set is regular in the following sense:

Definition 2.1. A countable set $\mathcal{L} \subset \mathbb{R}^d$ is a *thin admissible lattice* if (2-2) holds and

(i) there exists $r > 0$ such that $|x - y| \ge r$ for all $x \ne y$, $x, y \in \mathcal{L}$,

(ii) there exists $R > 0$ such that $\mathrm{dist}(x, \mathcal{L}) \le R$ for all $x \in \mathbb{R}^k$.

We associate to such a lattice a truncated Voronoi tessellation $\mathcal{V}(\mathcal{L})$, where the corresponding d-dimensional cells $\mathcal{C} \in \mathcal{V}(\mathcal{L})$ are defined by

$$\mathcal{C}(x) := \big\{z \in \mathbb{R}^k \times [-2M, 2M]^{d-k} : |z - x| \le |z - x'| \text{ for all } x' \in \mathcal{L}\big\},$$

and we introduce the set of nearest neighbors accordingly by setting

$$\mathcal{NN}(\mathcal{L}) := \big\{(x, y) \in \mathcal{L}^2 : \dim_{\mathcal{H}}(\mathcal{C}(x) \cap \mathcal{C}(y)) = d - 1\big\}.$$

As usual in the passage from atomistic to continuum theories, we scale the point set $\mathcal{L}$ by a small parameter $\varepsilon > 0$. We assume that the magnetization of the particles takes values in a finite set $\mathcal{S} = \{s_1, \dots, s_q\} \subset \mathbb{R}^q$. Fix a k-dimensional reference set $D \in \mathcal{A}^R(\mathbb{R}^k)$. Given $A \in \mathcal{A}^R(D)$ and $u : \varepsilon\mathcal{L} \to \mathcal{S}$, we consider a localized (on A) pairwise interaction energy

$$E_\varepsilon(u, A) = \sum_{\substack{x, y \in \mathcal{L} \\ \varepsilon x, \varepsilon y \in P_k^{-1} A}} \varepsilon^{k-1} f_\varepsilon(x, y, u(\varepsilon x), u(\varepsilon y)),$$

where the ($\mathcal{L}$-dependent) interactions distinguish between long- and short-range interactions and are of the form

$$f_\varepsilon(x, y, s_i, s_j) = \begin{cases} f^\varepsilon_{nn}(x, y, s_i, s_j) & \text{if } (x, y) \in \mathcal{NN}(\mathcal{L}), \\ f^\varepsilon_{lr}(x, y, s_i, s_j) & \text{otherwise.} \end{cases}$$

For our analysis we make the following assumptions on the measurable functions $f^\varepsilon_{nn}, f^\varepsilon_{lr} : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}^2 \to [0, +\infty)$:

Hypothesis 1. There exist $c > 0$ and a decreasing function $J_{lr} : [0, +\infty) \to [0, +\infty)$ with

$$\int_{\mathbb{R}^k} J_{lr}(|x|)|x| \,\mathrm{d}x = J < +\infty$$

such that, for all $\varepsilon > 0$, $x, y \in \mathbb{R}^d$ and $s_i, s_j \in \mathcal{S}$,

$$c \le c^\varepsilon_{nn}(x, y) \le J_{lr}(|x - y|), \quad c^\varepsilon_{lr}(x, y) \le J_{lr}(|x - y|).$$

Since the sets $\varepsilon\mathcal{L}$ shrink to a k-dimensional set as ε vanishes, we want to define a convergence of discrete variables on shrinking domains.To that end, denoting by $\mathrm{co}(\mathcal{S})$ the convex hull of $\mathcal{S}$, we define the averaged and projected spin variable $Pu : \varepsilon P_k\mathcal{L} \to \mathrm{co}(\mathcal{S})$ via

$$Pu(\varepsilon z) := \frac{1}{\#(P_k^{-1}(z) \cap \mathcal{L})} \sum_{x \in P_k^{-1}(z) \cap \mathcal{L}} u(\varepsilon x). \tag{2-3}$$

The projected lattice $P_k\mathcal{L} \subset \mathbb{R}^k$ inherits property (ii) from Definition 2.1, but (i) might fail after projection. Nevertheless, due to (2-2) the projected lattice is still locally finite and the following uniform bound on the number of points holds true: there exists a constant $C = C_{\mathcal{L}} > 0$ such that, given a set $A \in \mathcal{A}(D)$ with $|\partial A| = 0$, we have

$$\varepsilon^k \#\{\varepsilon z \in \varepsilon P_k\mathcal{L} \cap A\} \le C|A| \tag{2-4}$$

for ε small enough. We now associate the corresponding k-dimensional Voronoi tessellation $\mathcal{V}(P_k\mathcal{L}) = \{\mathcal{C}_k(z)\}$ in $\mathbb{R}^k$ to the lattice $P_k\mathcal{L}$ and we identify Pu with a piecewise-constant function belonging to the class

$$\mathcal{PC}_\varepsilon(\mathcal{L}) := \{v : \mathbb{R}^k \to (\mathcal{S}) : v_{|\varepsilon\mathcal{C}_k(z)} \text{ is constant for all } z \in P_k\mathcal{L}\}$$

Note that we can embed $\mathcal{PC}_\varepsilon(\mathcal{L})$ in $L^1(D)$ since the intersection of two Voronoi cells always has zero k-dimensional Lebesgue measure.

For the sake of illustration, in Figure 1 we picture the construction in the simple case $d = 2$, $k = 1$ and $\mathcal{S} = \{\pm 1\}$. In the picture above, we draw a portion of the truncated Voronoi diagram of the lattice $\mathcal{L}$ represented by the dots, black for $u = -1$ and white for $u = +1$. At the bottom of the Voronoi diagram we include the projected points $P_1\mathcal{L}$ and the values of the variable $Pu \in [-1, 1]$ (range reflected by the gray scale in the figure). The dashed lines indicate the exceptional set of projection points where $|Pu| \neq 1$. The picture below represents the piecewise-constant function on the Voronoi intervals subordinated to $P_1\mathcal{L}$.

To deal with convergence of sequences $u_\varepsilon : \varepsilon\mathcal{L} \to \mathcal{S}$, we adopt the idea of [Braides et al. 2012]. We will see in Section 6 that this notion of convergence is indeed meaningful for variational problems in a random environment.

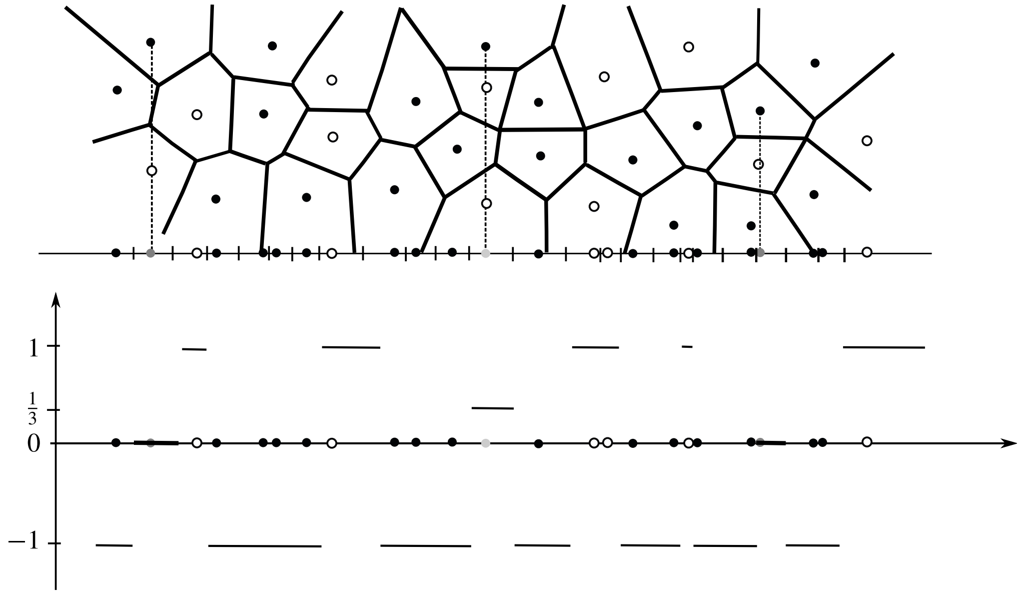

Figure 1. Construction of the piecewise-constant interpolation for $d = 2$, $k = 1$ and $\mathcal{S} = \{\pm 1\}$.

Definition 2.2. Let $A \in \mathcal{A}(D)$. We say that a sequence $u_\varepsilon : \varepsilon\mathcal{L} \to \mathcal{S}$ *converges* in A to $u : A \to \mathbb{R}^q$ if the piecewise-constant functions Pu_ε converge to u strongly in $L^1(A)$.

For our variational analysis we introduce the lower and upper Γ-limits $E', E'' : L^1(D, \mathbb{R}^q) \times \mathcal{A}^R(D) \to [0, +\infty]$ setting

$$\begin{aligned} E'(u, A) &:= \inf\Big\{\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, A) : u_\varepsilon \to u \text{ in } D\Big\}, \\ E''(u, A) &:= \inf\Big\{\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, A) : u_\varepsilon \to u \text{ in } D\Big\}. \end{aligned} \tag{2-5}$$

Remark 2.3. The functionals E', E'' are not Γ-lower/upper limits in the usual sense since they are not defined on the same space as E_ε. However, if we define the functionals $\widetilde{E}_\varepsilon : L^1(D, \mathbb{R}^q) \times \mathcal{A}^R(D) \to [0, +\infty]$ as

$$\widetilde{E}_\varepsilon(u, A) := \begin{cases} \inf_v E_\varepsilon(v, A) & \text{if } u = Pv \text{ for some } v : \varepsilon\mathcal{L} \to \mathcal{S}, \\ +\infty & \text{otherwise,} \end{cases}$$

then E', E'' agree with the Γ-lower/upper limit of $\widetilde{E}_\varepsilon$ in the strong $L^1(D)$-topology. Therefore we will refer to the equality of E' and E'' as Γ-convergence. Moreover, one can show that

$$\begin{aligned} E'(u, A) &= \inf\Big\{\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, A) : u_\varepsilon \to u \text{ in } A\Big\}, \\ E''(u, A) &= \inf\Big\{\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, A) : u_\varepsilon \to u \text{ in } A\Big\}. \end{aligned}$$

By the properties of Γ-convergence, this implies that both functionals $u \mapsto E'(u, A)$ and $u \mapsto E''(u, A)$ are $L^1(A)$-lower semicontinuous and hence local in the sense of Theorem 3.1(ii).

We now prove several properties of the convergence introduced in Definition 2.2. We start with an equicoercivity property.

Lemma 2.4. *Assume Hypothesis 1 holds. Let $A \in \mathcal{A}(D)$ and let $u_\varepsilon : \varepsilon\mathcal{L} \to \mathcal{S}$ be such that*

$$\sup_\varepsilon E_\varepsilon(u_\varepsilon, A) < +\infty.$$

Then, up to subsequences, the functions Pu_ε defined as in (2-3) *converge strongly in $L^1(A)$ to some $u \in \mathrm{BV}(A, \mathcal{S})$.*

Proof. Fix $A' \Subset A$ such that $A' \in \mathcal{A}^R(D)$. We start by estimating the measure of the set $\{Pu_\varepsilon \notin \mathcal{S}\} \cap A'$. Note that if $Pu_\varepsilon(\varepsilon z) \notin \mathcal{S}$ for some $z \in P_k\mathcal{L}$ such that $\varepsilon\, \mathcal{C}_k(z) \cap A' \neq \varnothing$, then there exist $x_1, x_2 \in P_k^{-1}(z) \cap \mathcal{L}$ such that $u_\varepsilon(\varepsilon x_1) \neq u_\varepsilon(\varepsilon x_2)$. As a preliminary step we show that we can find a path of nearest neighbors in $\mathcal{L}$ joining x_1 and x_2, that is, a finite collection of points $\{x^1, \dots, x^m\} \subset \mathcal{L}$ such that $x^1 = x_1$ and $x^m = x_2$ and $(x^i, x^{i+1}) \in \mathcal{NN}(\mathcal{L})$ for all $i = 1, \dots, m-1$. Moreover this path will be chosen such that it does not vary too much from the segment between x_1 and x_2. To this end, fix $0 < \delta \ll 1$ and consider the collection of segments

$$\mathcal{G}_\delta(x_1, x_2) = \{x + \lambda(x_2 - x_1) : x \in B_\delta(x_1),\ 0 \le \lambda \le 1\}. \tag{2-6}$$

We argue that there exists a segment $g^* = \{x^* + \lambda(x_2 - x_1) :\ 0 \le \lambda \le 1\} \subset \mathcal{G}_\delta$ satisfying the implication

$$g^* \cap \mathcal{C}(x) \cap \mathcal{C}(x') \neq \varnothing \quad \Longrightarrow \quad (x, x') \in \mathcal{NN}(\mathcal{L}). \tag{2-7}$$

Indeed, assume by contradiction that the implication is false for all $x^* \in B_\delta(x_1)$. Since the number of d-dimensional Voronoi cells $\mathcal{C}(x) \in \mathcal{V}(\mathcal{L})$ such that $\mathcal{C}(x) \cap \mathcal{G}_\delta \neq \varnothing$ is uniformly bounded, we can then find finitely many Voronoi facets of dimension less than $d-1$ whose projection onto the hyperplane containing x_1 and orthogonal to $x_2 - x_1$ covers a $(d-1)$-dimensional set. Since projections onto hyperplanes are Lipschitz continuous, we obtain a contradiction.

The path connecting x_1 and x_2 is then given by the set $G(x_1, x_2) := \{x \in \mathcal{L} : g^* \cap \mathcal{C}(x) \neq \varnothing\}$, provided that δ is small enough. Observe that there exist $x, y \in G(x_1, x_2)$ such that $(x, y) \in \mathcal{NN}(\mathcal{L})$ and $u_\varepsilon(\varepsilon x) \neq u_\varepsilon(\varepsilon y)$. From the coercivity assumption in Hypothesis 1, we thus deduce that each path contributes to the energy. Moreover, by (2-2) and the local construction of the paths, for any pair $(x, y) \in \mathcal{NN}(\mathcal{L})$ it holds that

$$\#\big\{z \in P_k\mathcal{L} : G(x_1, x_2) \cap \{x, y\} \neq \varnothing\big\} \le C.$$

From these two facts we infer that

$$\varepsilon^{k-1}\#\big\{\varepsilon z : \varepsilon\mathcal{C}_k(z) \cap A' \neq \varnothing,\ Pu_\varepsilon(\varepsilon z) \notin \mathcal{S}\big\} \le C E_\varepsilon(u_\varepsilon, A) \le C, \tag{2-8}$$

where we have used that $\varepsilon G(x_1, x_2) \subset (P_k^{-1}A) \cap \varepsilon\mathcal{L}$ for ε small enough. Since the measure of a Voronoi cell in $P_k\mathcal{L}$ can be bounded uniformly by a constant, by rescaling we deduce that

$$|\{Pu_\varepsilon \notin \mathcal{S}\} \cap A'| \le C\varepsilon. \tag{2-9}$$

We continue bounding the total variation $|DPu_\varepsilon|(A')$. Since Pu_ε is equibounded and piecewise constant, it is enough to provide a bound for $\mathcal{H}^{k-1}(S_{Pu_\varepsilon} \cap A')$. Note that the jump set S_{Pu_ε} is contained in the facets of the Voronoi cells of the lattice $\varepsilon P_k\mathcal{L}$. Since $\mathcal{L}$ is thin admissible in the sense of Definition 2.1 and property (ii) is preserved by projection, for each such facet F it holds that

$$\mathcal{H}^{k-1}(F) \leq C\varepsilon^{k-1}.$$

For ε small enough, we conclude that

$$\mathcal{H}^{k-1}(S_{Pu_\varepsilon} \cap A') \leq C\varepsilon^{k-1}\#\big\{(z,z') \in \mathcal{NN}(P_k\mathcal{L}) : Pu_\varepsilon(\varepsilon z) \neq Pu_\varepsilon(\varepsilon z'),\ \varepsilon z, \varepsilon z' \in A' + B_{R\varepsilon}(0)\big\}.$$

Given $\varepsilon z, \varepsilon z' \in A' + B_{R\varepsilon}(0)$ such that $(z,z') \in \mathcal{NN}(P_k\mathcal{L})$ and $Pu_\varepsilon(\varepsilon z) \neq Pu_\varepsilon(\varepsilon z')$, again we may find a path of nearest neighbors

$$G(z,z') = \{x^0 \in P_k^{-1}(z), x^1, \dots, x^m \in P_k^{-1}(z')\}$$

with $u_\varepsilon(\varepsilon x^0) \neq u_\varepsilon(\varepsilon x^m)$ and the paths are local in the sense that

$$\#\big\{(z,z') \in \mathcal{NN}(P_k\mathcal{L}) : G(z,z') \cap \{x,y\} \neq \varnothing\big\} \leq C$$

for all $(x,y) \in \mathcal{NN}(\mathcal{L})$. Reasoning as in the first part of the proof we find that

$$\varepsilon^{k-1}\#\big\{(z,z') \in \mathcal{NN}(P_k\mathcal{L}) : Pu_\varepsilon(\varepsilon z) \neq Pu_\varepsilon(\varepsilon z'),\ \varepsilon z, \varepsilon z' \in A' + B_{R\varepsilon}(0)\big\} \leq CE_\varepsilon(u_\varepsilon, A) \leq C.$$

By well-known compactness properties of BV-functions (see, for example, [Ambrosio et al. 2000, Corollary 3.49]) and (2-9), there exists a subsequence (not relabeled) such that $Pu_\varepsilon \to u$ in $L^1(A')$ for some $u \in BV(A', \mathcal{S})$. Since A' was arbitrary, the claim follows by a diagonal argument combined with equiboundedness, which rules out concentrations close to the boundary. □

We will also use the following auxiliary result about the convergence introduced in Definition 2.2.

Lemma 2.5. *Let $A \in \mathcal{A}(D)$ be such that $|\partial A| = 0$ and let $u_\varepsilon, v_\varepsilon : \varepsilon\mathcal{L} \to \mathcal{S}$ both converge in A to u in the sense of Definition 2.2 and assume both have equibounded energy on A. Then*

$$\lim_{\varepsilon\to 0} \sum_{\substack{\varepsilon x \in \varepsilon\mathcal{L} \\ \varepsilon P_k(x) \in A}} \varepsilon^k |u_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon x)| = 0.$$

Proof. Fix a set $A' \Subset A$ such that $A' \in \mathcal{A}^R(D)$. By (2-4) and equiboundedness of u_ε and v_ε it is enough to show that

$$\lim_{\varepsilon\to 0} \sum_{\substack{\varepsilon x \in \varepsilon\mathcal{L} \\ \varepsilon P_k(x) \in A'}} \varepsilon^k |u_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon x)| = 0.$$

Using the fact that $u_\varepsilon, v_\varepsilon$ both have finite energy in A, we can argue as in the derivation of (2-8) to show

$$\#\big\{\varepsilon x \in \varepsilon\mathcal{L} : \varepsilon P_k(x) \in A',\ Pu_\varepsilon(\varepsilon P_k(x)) \neq u_\varepsilon(\varepsilon x) \text{ or } Pv_\varepsilon(\varepsilon P_k(x)) \neq v_\varepsilon(\varepsilon x)\big\} \leq C\varepsilon^{1-k}.$$

Inserting this estimate and using that $\mathcal{L}$ satisfies (2-2) we obtain

$$\sum_{\substack{\varepsilon x \in \varepsilon\mathcal{L} \\ \varepsilon P_k(x) \in A'}} \varepsilon^k |u_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon x)| \leq C \sum_{\substack{\varepsilon z \in \varepsilon P_k\mathcal{L} \\ \varepsilon z \in A'}} \varepsilon^k |Pu_\varepsilon(\varepsilon z) - Pv_\varepsilon(\varepsilon z)| + C\varepsilon.$$

Thus it is enough to control the last sum. Since the Voronoi cells in the projected lattice may become degenerate, we can only use bounds on the number of cells. To this end fix $L > 1$ large enough such that, for all $z_L \in L\mathbb{Z}^k$, we have

$$1 \leq \#\big(\varepsilon P_k\mathcal{L} \cap (\varepsilon z_L + [0, L\varepsilon)^k)\big) \leq C. \tag{2-10}$$

Define $I_\varepsilon := \{z_L \in L\mathbb{Z}^k : (\varepsilon z_L + [0, L\varepsilon)^k) \cap A' \neq \varnothing\}$ and subdivide this set again as

$$\begin{aligned} I_\varepsilon^1 &:= \{z_L \in I_\varepsilon : Pu_\varepsilon \text{ is not constant on } \varepsilon z_L + [0, L\varepsilon)^k\},\\ I_\varepsilon^2 &:= \{z_L \in I_\varepsilon : Pv_\varepsilon \text{ is not constant on } \varepsilon z_L + [0, L\varepsilon)^k\},\\ I_\varepsilon^3 &:= I_\varepsilon \backslash (I_\varepsilon^1 \cup I_\varepsilon^2). \end{aligned}$$

Since every scaled k-dimensional Voronoi cell $\varepsilon\mathcal{C}_k(z)$ can only intersect finitely many cubic cells $\varepsilon z_L + [0, L\varepsilon)^k$ with a uniform bound on the cardinality, we can again use the energy bound in A and argue as for (2-8) to conclude

$$\#(I_\varepsilon^1 \cup I_\varepsilon^2) \leq C\varepsilon^{1-k}. \tag{2-11}$$

Combining (2-10) and (2-11) we infer from the definition of the set I_ε^3 that

$$\begin{aligned} \sum_{\substack{\varepsilon z \in \varepsilon P_k\mathcal{L}\\ \varepsilon z \in A'}} \varepsilon^k |Pu_\varepsilon(\varepsilon z) - Pv_\varepsilon(\varepsilon z)| &\leq C\varepsilon + \sum_{z_L \in I_\varepsilon^3} \sum_{\substack{\varepsilon z \in \varepsilon P_k\mathcal{L}\\ \varepsilon z \in \varepsilon z_L + [0, L\varepsilon)^k}} \varepsilon^k |Pu_\varepsilon(\varepsilon z) - Pv_\varepsilon(\varepsilon z)|\\ &\leq C\varepsilon + C\sum_{z_L \in I_\varepsilon^3} \int_{\varepsilon z_L + [0, L\varepsilon)^k} |Pu_\varepsilon(s) - Pv_\varepsilon(s)|\,\mathrm{d}s\\ &\leq C\varepsilon + C\|Pu_\varepsilon - Pv_\varepsilon\|_{L^1(A)}. \end{aligned}$$

This concludes the proof, since the last term tends to 0 by assumption. □

Following some ideas in [Alicandro et al. 2011] we introduce an auxiliary deterministic square lattice on which we will rewrite the energies E_ε. This lattice, shown in Figure 2, will turn out to be a convenient way to control the long-range interactions.

On setting $r' = r/\sqrt{d}$ it follows that $\#\{\mathcal{L} \cap \{\alpha + [0, r')^d\}\} \leq 1$ for all $\alpha \in r'\mathbb{Z}^d$. We now set

$$\begin{aligned} \mathcal{Z}_{r'}(\mathcal{L}) &:= \big\{\alpha \in r'\mathbb{Z}^d : \#\big(\mathcal{L} \cap \{\alpha + [0, r')^d\}\big) = 1\big\},\\ x_\alpha &:= \mathcal{L} \cap \{\alpha + [0, r')^d\}, \quad \alpha \in \mathcal{Z}_{r'}(\mathcal{L}), \end{aligned}$$

and, for $\xi \in r'\mathbb{Z}^d$, $U \subset \mathbb{R}^k$ and $\varepsilon > 0$,

$$R_\varepsilon^\xi(U) := \{\alpha : \alpha, \alpha + \xi \in \mathcal{Z}_{r'}(\mathcal{L}),\ \varepsilon x_\alpha, \varepsilon x_{\alpha+\xi} \in P_k^{-1}U\}.$$

Note that by (2-2), enlarging M if necessary, it is enough to consider

$$\xi \in r'\mathbb{Z}_M^d := r'\mathbb{Z}^d \cap (\mathbb{R}^k \times [-2M, 2M]^{d-k}).$$

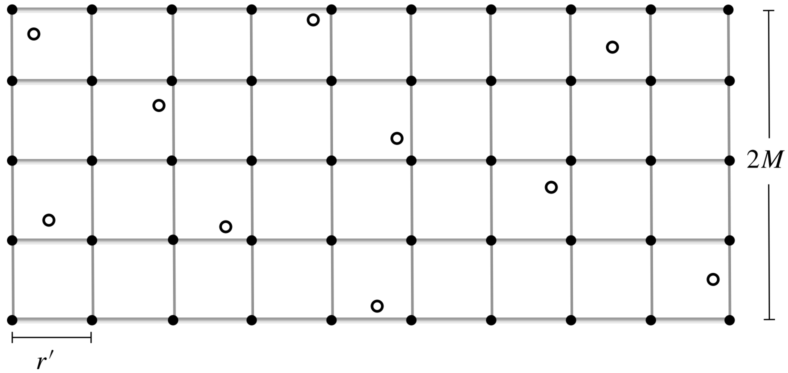

Figure 2. The particles in $\mathcal{L}$ (circles) and the auxiliary lattice $r'\mathbb{Z}^d$ (black dots).

We can then rewrite the localized energy as

$$E_\varepsilon(u, A) = \sum_{\xi\in r'\mathbb{Z}^d_M} \sum_{\alpha\in R^\xi_\varepsilon(A)} \varepsilon^{k-1} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, u(\varepsilon x_\alpha), u(\varepsilon x_{\alpha+\xi})).$$

Remark 2.6. Observe that we can write

$$\{\xi \in r'\mathbb{Z}^d_M\} = \bigcup_{\substack{z\in r'\mathbb{Z}^{d-k}\\ |z|_\infty\le 2M}} \{\xi = (\xi_k, z_1, \dots, z_{d-k}) : \xi_k \in r'\mathbb{Z}^k\}.$$

Hence the monotonicity assumption from Hypothesis 1 allows to transfer the decay of long-range interactions to the discrete environment as follows: given $\delta > 0$, there exists $L_\delta > 0$ such that

$$\sum_{\substack{\xi\in r'\mathbb{Z}^d_M\\ |\xi|>L_\delta}} J_{lr}(|\hat{\xi}|)|\xi| \le \delta, \tag{2-12}$$

where $\hat{\xi} \in \xi + [-r', r']^d$ is such that $|\hat{\xi}| = \operatorname{dist}\big([0, r')^d, [0, r')^d + \xi\big)$. This decay property along with Lemma 2.7 below will be crucial to control the long-range interactions. However note that L_δ in general depends on M.

The following lemma asserts that on convex domains we can essentially control the long-range interactions by considering only nearest neighbors.

Lemma 2.7. *Let $B \subset \mathcal{A}(\mathbb{R}^k)$ be convex and $B^\varepsilon = \{x \in \mathbb{R}^k : \operatorname{dist}(x, B) < 3(R + M)\varepsilon\}$. Then there exists a constant C depending only on r, R, M in Definition 2.1 such that for every $\xi \in r'\mathbb{Z}^d_M$ and every $u : \varepsilon\mathcal{L} \to \mathcal{S}$ it holds that*

$$\sum_{\alpha\in R^\xi_\varepsilon(B)} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, u(\varepsilon x_\alpha), u(\varepsilon x_{\alpha+\xi})) \le C J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y)\in\mathcal{NN}(\mathcal{L})\\ \varepsilon x,\varepsilon y\in P_k^{-1}B^\varepsilon}} f_\varepsilon(x, y, u(\varepsilon x), u(\varepsilon y)).$$

Proof. Let $\alpha \in R^\xi_\varepsilon(B)$. As in the proof of Lemma 2.4 we consider the collection of segments $\mathcal{G}_\delta(x_\alpha, x_{\alpha+\xi})$ defined as in (2-6). By the same argument, there exists a segment $g^* \subset \mathcal{G}_\delta(x_\alpha, x_{\alpha+\xi})$ satisfying (2-7).

Consider then the set $G(\alpha,\xi)=\{x\in\mathcal{L}:\ g^*\cap\mathcal{C}(x)\neq\varnothing\}$. By construction we can number $G(\alpha,\xi)=\{x_\alpha=x^0,\dots,x^N=x_{\alpha+\xi}\}$ such that $(x^i,x^{i+1})\in\mathcal{NN}(\mathcal{L})$. By the bounds of Hypothesis 1 it holds that

$$f_\varepsilon(x_\alpha,x_{\alpha+\xi},u(\varepsilon x_\alpha),u(\varepsilon x_{\alpha+\xi}))\leq J_{lr}(|\hat{\xi}|)|u(\varepsilon x_\alpha)-u(\varepsilon x_{\alpha+\xi})|\leq J_{lr}(|\hat{\xi}|)\sum_{\substack{(x,y)\in\mathcal{NN}(\mathcal{L})\\ x,y\in G(\alpha,\xi)}}|u(\varepsilon x)-u(\varepsilon y)|$$

$$\leq C\,J_{lr}(|\hat{\xi}|)\sum_{\substack{(x,y)\in\mathcal{NN}(\mathcal{L})\\ x,y\in(1/\varepsilon)P_k^{-1}B^\varepsilon\cap G(\alpha,\xi)}}f_\varepsilon(x,y,u(\varepsilon x),u(\varepsilon y)),\tag{2-13}$$

where we used that by convexity we have $G(\alpha,\xi)\subset\frac{1}{\varepsilon}P_k^{-1}B^\varepsilon$ provided δ is small enough. Now given $(x,y)\in\mathcal{NN}(\mathcal{L})\cap\frac{1}{\varepsilon}P_k^{-1}B^\varepsilon$ we set

$$T_\varepsilon^\xi(x,y):=\bigl\{\alpha\in R_\varepsilon^\xi(B):\{x,y\}\cap G(\alpha,\xi)\neq\varnothing\bigr\}.$$

Note that if $\alpha\in T_\varepsilon^\xi(x,y)$, then

$$x_\alpha\in\{z+t\xi:|z-x|\leq C,\ |t|\leq C\}$$

for some $C>0$, and hence $\#T_\varepsilon^\xi(x,y)\leq C|\xi|$ by Definition 2.1. The claim now follows by summing (2-13) over all $\alpha\in R_\varepsilon^\xi(B)$. □

3. Integral representation on the flat set

Our first aim is to characterize all possible variational limits of energies E_ε that satisfy Hypothesis 1. As for the case $k=d$ and $\mathcal{S}=\{\pm1\}$ treated in [Alicandro et al. 2015], the following version of Theorem 3 in [Bouchitté et al. 2002] will be the key ingredient:

Theorem 3.1. *Let $\mathcal{F}:\mathrm{BV}(D,\mathcal{S})\times\mathcal{A}(D)\to[0,+\infty)$ satisfy the following hypotheses*:

(i) *$\mathcal{F}(u,\cdot)$ is the restriction to $\mathcal{A}(D)$ of a Radon measure.*

(ii) *$\mathcal{F}(u,A)=\mathcal{F}(v,A)$ whenever $u=v$ a.e. on $A\in\mathcal{A}(D)$.*

(iii) *$\mathcal{F}(\cdot,A)$ is $L^1(D)$ lower semicontinuous for every $A\in\mathcal{A}(D)$.*

(iv) *There exists $c>0$ such that*

$$\frac{1}{c}\mathcal{H}^{k-1}(S_u\cap A)\leq\mathcal{F}(u,A)\leq c\,\mathcal{H}^{k-1}(S_u\cap A)$$

for every $(u,A)\in\mathrm{BV}(D,\mathcal{S})\times\mathcal{A}(D)$.

Then for every $u\in\mathrm{BV}(D,\mathcal{S})$ and $A\in\mathcal{A}(D)$,

$$\mathcal{F}(u,A)=\int_{S_u\cap A}g(x,u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{k-1},$$

with

$$g(x_0,s_i,s_j,\nu)=\limsup_{\rho\to0}\frac{m(u^{ij}_{x_0,\nu},Q_\nu(x_0,\rho))}{\rho^{k-1}},$$

where, for all $s_i, s_j \in \mathcal{S}$,

$$u^{ij}_{x_0,\nu} := \begin{cases} s_i & \text{if } \langle x - x_0, \nu\rangle \geq 0, \\ s_j & \text{otherwise,} \end{cases}$$

and for any $(v, A) \in \mathrm{BV}(D, \mathcal{S}) \times \mathcal{A}(D)$ *we set*

$$m(v, A) = \inf\bigl\{\mathcal{F}(u, A) : u \in \mathrm{BV}(A, \mathcal{S}),\ u = v \text{ in a neighborhood of } \partial A\bigr\}.$$

The following theorem is the main result of this section.

Theorem 3.2. *Let* $\mathcal{L}$ *be a thin admissible lattice and let* f^{ε}_{nn} *and* f^{ε}_{lr} *satisfy Hypothesis 1. For every sequence of* $\varepsilon \to 0^+$ *there exists a subsequence* ε_n *such that the functionals* E_{ε_n} Γ*-converge with respect to the convergence of Definition 2.2 with* $A = D$ *to a functional* $E : L^1(D, \mathbb{R}^q) \to [0, +\infty]$ *of the form*

$$E(u) = \begin{cases} \int_{S_u} \phi(x, u^+, u^-, \nu_u)\, \mathrm{d}\mathcal{H}^{k-1} & \text{if } u \in \mathrm{BV}(D, \mathcal{S}), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, a local version of the statement above holds: for all $u \in \mathrm{BV}(D, \mathcal{S})$ *and all* $A \in \mathcal{A}^R(D)$,

$$\Gamma\text{-}\lim_n E_{\varepsilon_n}(u, A) = \int_{S_u \cap A} \phi(x, u^+, u^-, \nu_u)\, \mathrm{d}\mathcal{H}^{k-1},$$

with respect to the same convergence as above.

Remark 3.3. If $k = 1$, then a similar result holds. In this case we obtain a limit energy finite for $u \in \mathrm{BV}(D, \mathcal{S})$ and of the form

$$E(u) = \sum_{x \in S_u} \phi(x, u^+, u^-).$$

The proof of Theorem 3.2 will be given later and it is based on Theorem 3.1. We now start proving several propositions that allow us to apply Theorem 3.1.

We start with the growth condition (iv) of Theorem 3.1. Using the lower semicontinuity of the perimeter of level sets in $\mathrm{BV}(D, \mathcal{S})$, one can use the same argument as for Lemma 2.4 to prove the following lower bound for $E'(u, A)$ defined in (2-5):

Proposition 3.4. *Assume that Hypothesis 1 holds. Then* $E'(u, A) < +\infty$ *only if* $u \in \mathrm{BV}(A, \mathcal{S})$ *and there exists a constant* $c > 0$ *independent of* A *such that*

$$\frac{1}{c}\mathcal{H}^{k-1}(S_u \cap A) \leq E'(u, A).$$

In the next step we provide a suitable upper bound for $E''(u, A)$ defined in (2-5).

Proposition 3.5. *Assume Hypothesis 1 holds. Then there exists a constant* $c > 0$ *such that, for all* $A \in \mathcal{A}^R(D)$ *and all* $u \in \mathrm{BV}(D, \mathcal{S})$,

$$E''(u, A) \leq c\, \mathcal{H}^{k-1}(S_u \cap A).$$

Proof. First, assume that u is a polyhedral function on $\mathbb{R}^k$, which means that all level sets have boundaries that coincide (up to $\mathcal{H}^{k-1}$-null sets) with a finite union of $(k-1)$-dimensional simplexes. We define a sequence $u_\varepsilon : \varepsilon\mathcal{L} \to \mathcal{S}$ by setting

$$u_\varepsilon(\varepsilon x) := u(\varepsilon P_k(x)).$$

Note that $u_\varepsilon \to u$ in the sense of Definition 2.2. Given $\delta > 0$, we choose $L_\delta > 0$ such that (2-12) holds. We further set $A^\delta = A + B_\delta(0)$. For $|\xi| \leq L_\delta$, we can argue as in the proof of Lemma 2.7 to show that, for ε small enough, it holds that

$$\sum_{\alpha\in R_\varepsilon^\xi(A)} \varepsilon^{k-1} f_\varepsilon\big(x_\alpha, x_{\alpha+\xi}, u_\varepsilon(\varepsilon x_\alpha), u_\varepsilon(\varepsilon x_{\alpha+\xi})\big) \leq C J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y)\in\mathcal{NN}(\mathcal{L})\\ \varepsilon x,\varepsilon y\in P_k^{-1}A^\delta}} \varepsilon^{k-1}|u_\varepsilon(\varepsilon x) - u_\varepsilon(\varepsilon y)|$$
$$\leq C J_{lr}(|\hat{\xi}|)|\xi|\mathcal{H}^{k-1}(S_u \cap A^\delta), \tag{3-14}$$

where the last estimate follows from the regularity of S_u. Next we consider the interactions where $|\xi| > L_\delta$. Let u be a polyhedral function; applying Lemma 2.7 we deduce for any $\varepsilon > 0$ the weaker bound

$$\sum_{\alpha\in R_\varepsilon^\xi(A)} \varepsilon^{k-1} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, u_\varepsilon(\varepsilon x_\alpha), u_\varepsilon(\varepsilon x_{\alpha+\xi})) \leq \sum_{\alpha\in R_\varepsilon^\xi(\mathbb{R}^k)} \varepsilon^{k-1} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, u_\varepsilon(\varepsilon x_\alpha), u_\varepsilon(\varepsilon x_{\alpha+\xi}))$$
$$\leq C J_{lr}(|\hat{\xi}|)|\xi|\mathcal{H}^{k-1}(S_u). \tag{3-15}$$

Combining (3-14),(3-15) and (2-12) and the integrability assumption from Hypothesis 1, we deduce that

$$E''(u, A) \leq \limsup_{\varepsilon} E_\varepsilon(u_\varepsilon, A) \leq C\mathcal{H}^{k-1}(S_u \cap A^\delta) + C\delta\mathcal{H}^{k-1}(S_u).$$

As $\delta > 0$ was arbitrary we obtain

$$E''(u, A) \leq C\mathcal{H}^{k-1}(S_u \cap \bar{A}). \tag{3-16}$$

Now we use locality and a density argument. Indeed, for every $u \in \mathrm{BV}(D, \mathcal{S})$ we can find a function $\tilde{u} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^k, \mathcal{S})$ such that $u = \tilde{u}$ on A and $\mathcal{H}^{k-1}(S_{\tilde{u}} \cap \partial A) = 0$; see Lemma 2.7 in [Braides et al. 2017]. From Remark 2.3 it follows that $E''(u, A) = E''(\tilde{u}, A)$. Then, by [Braides et al. 2017, Corollary 2.4] there exists a sequence $u_n \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^k, \mathcal{S})$ of polyhedral functions such that $u_n \to \tilde{u}$ in $L^1(D)$ and $\mathcal{H}^{k-1}(S_{u_n} \cap D) \to \mathcal{H}^{k-1}(S_{\tilde{u}} \cap D)$. By the $L^1(D)$-lower semicontinuity of $E''(\,\cdot\,, A)$ stated in Remark 2.3 and (3-16), we obtain

$$E''(u, A) \leq \liminf_n E''(u_n, A) \leq C \limsup_n \mathcal{H}^{k-1}(S_{u_n} \cap \bar{A}) \leq C\mathcal{H}^{k-1}(S_{\tilde{u}} \cap \bar{A}) = C\mathcal{H}^{k-1}(S_u \cap A),$$

where the last inequality is a consequence of the $L^1(D)$-lower semicontinuity of $u \mapsto \mathcal{H}^{k-1}(S_u \cap D\backslash\bar{A})$ for $u \in \mathrm{BV}(D, \mathcal{S})$. □

As is usual for applying integral-representation theorems, we next establish a weak subadditivity property of $A \mapsto E''(u, A)$.

Proposition 3.6. *Let f_{nn}^ε and f_{lr}^ε satisfy Hypothesis 1. Then, for every $A, B \in \mathcal{A}^R(D)$, every $A' \subset \mathcal{A}^R(D)$ such that $A' \Subset A$ and every $u \in \mathrm{BV}(D, \mathcal{S})$,*

$$E''(u, A' \cup B) \leq E''(u, A) + E''(u, B).$$

Proof. We may assume that $E''(u, A)$ and $E''(u, B)$ are both finite. Let $u_\varepsilon, v_\varepsilon : \varepsilon\mathcal{L} \to \mathcal{S}$ both converge to u in the sense of Definition 2.2 such that

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, A) = E''(u, A), \quad \limsup_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon, B) = E''(u, B). \tag{3-17}$$

Step 1: extensions to convex domains. Let Q_D be a cube containing $\bar{D}$. Since $D \in \mathcal{A}^R(D)$, we can extend u (without relabeling) to a function $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^k, \mathcal{S})$. We first show that we can modify u_ε and v_ε on $\varepsilon\mathcal{L}\backslash A$ and $\varepsilon\mathcal{L}\backslash B$ respectively, such that they converge to u on $L^1(Q_D)$ and such that they have equibounded energy on the larger set Q_D. We will show the argument for u_ε. Take another cube Q' such that $Q_D \Subset Q'$. Arguing as in the proof of Proposition 3.5, we find a sequence $\tilde{u}_\varepsilon : \varepsilon\mathcal{L} \to \mathcal{S}$ such that $\tilde{u}_\varepsilon \to u$ on Q' and $\limsup_{\varepsilon\to 0} E_\varepsilon(\tilde{u}_\varepsilon, Q') \le C\mathcal{H}^{k-1}(S_u \cap Q')$. We then set $\bar{u} \in \mathcal{PC}_\varepsilon(\mathcal{L})$ as

$$\bar{u}(\varepsilon x) = \mathbb{1}_A(P_k(\varepsilon x))u_\varepsilon(\varepsilon x) + (1 - \mathbb{1}_A(P_k(\varepsilon x)))\tilde{u}_\varepsilon(\varepsilon x).$$

Then $\bar{u}_\varepsilon \to u$ on Q_D and applying Lemma 2.7 combined with Hypothesis 1 and (2-2) yields

$$\begin{aligned} E_\varepsilon(\bar{u}_\varepsilon, Q_D) &\le C \sum_{\xi\in r'\mathbb{Z}^d_M} J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y)\in\mathcal{NN}(\mathcal{L})\\ \varepsilon x,\varepsilon y\in Q'}} \varepsilon^{k-1} f_\varepsilon(x, y, \bar{u}_\varepsilon(\varepsilon x), \bar{u}_\varepsilon(\varepsilon y)) \\ &\le C\Big(E_\varepsilon(u_\varepsilon, A) + E_\varepsilon(\tilde{u}_\varepsilon, Q'\backslash A) + \frac{1}{\varepsilon}|\partial A + B_{4R\varepsilon}(0)|\Big). \end{aligned}$$

The first and second terms remain bounded by construction, while the third term converges to a multiple of the Minkowski content of ∂A which agrees with $\mathcal{H}^{k-1}(\partial A)$ as $A \in \mathcal{A}^R(D)$.

Step 2: energy estimates. Again, given $\delta > 0$ we choose L_δ such that (2-12) holds. Fix $d' \le \frac{1}{2}\operatorname{dist}(A', \partial A)$ and let

$$N_\varepsilon := \left\lfloor \frac{d'}{\varepsilon(L_\delta + 2r)} \right\rfloor,$$

where $\lfloor\cdot\rfloor$ denotes the integer part. For $j \in \mathbb{N}$ we define

$$A_{\varepsilon,j} := \{x \in A : \operatorname{dist}(x, A') < j\varepsilon(L_\delta + 2r)\}.$$

We let $w^j_\varepsilon \in \mathcal{PC}_\varepsilon(\mathcal{L})$ be the interpolation defined by

$$w^j_\varepsilon(\varepsilon x) = \mathbb{1}_{A_{\varepsilon,j}}(P_k(\varepsilon x))u_\varepsilon(\varepsilon x) + (1 - \mathbb{1}_{A_{\varepsilon,j}}(P_k(\varepsilon x)))v_\varepsilon(\varepsilon x).$$

Note that for each fixed $j \in \mathbb{N}$, we have $w^j_\varepsilon \to u$ on D in the sense of Definition 2.2. We set

$$S^{\xi,\varepsilon}_j := \big\{x = y + tP_k(\xi') : y \in \partial A_{\varepsilon,j},\ |t| \le \varepsilon,\ \xi' \in \xi + [-r', r']^d\big\} \cap (A \cup B).$$

For $j \le N_\varepsilon$ we have

$$\begin{aligned} E_\varepsilon(w^j_\varepsilon, A'\cup B) &\le E_\varepsilon(u_\varepsilon, A_{\varepsilon,j}) + E_\varepsilon(v_\varepsilon, B\backslash A_{\varepsilon,j}) + \sum_{\xi\in r'\mathbb{Z}^d_M} \sum_{\alpha\in R^\xi_\varepsilon(S^{\xi,\varepsilon}_j)} \underbrace{\varepsilon^{k-1} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, w^j_\varepsilon(\varepsilon x_\alpha), w^j_\varepsilon(\varepsilon x_{\alpha+\xi}))}_{=:\rho^{\xi,\varepsilon}_j(\alpha)} \\ &\le E_\varepsilon(u_\varepsilon, A) + E_\varepsilon(v_\varepsilon, B) + \sum_{\xi\in r'\mathbb{Z}^d_M} \sum_{\alpha\in R^\xi_\varepsilon(S^{\xi,\varepsilon}_j)} \rho^{\xi,\varepsilon}_j(\alpha). \end{aligned} \tag{3-18}$$

We now distinguish between two types of interactions depending on L_δ. If $|\xi| > L_\delta$, we use Lemma 2.7. Since $A \cup B \Subset Q_D$, we deduce that

$$\sum_{|\xi|>L_\delta} \sum_{\alpha\in R^\xi_\varepsilon(S^{\xi,\varepsilon}_j)} \rho^{\xi,\varepsilon}_j(\alpha) \le C \sum_{|\xi|>L_\delta} J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y)\in\mathcal{NN}(\mathcal{L})\\ \varepsilon x,\varepsilon y\in P_k^{-1}Q_D}} \varepsilon^{k-1} f_\varepsilon(x, y, w^j_\varepsilon(\varepsilon x), w^j_\varepsilon(\varepsilon y)).$$

We have $P_k^{-1}Q_D \subset P_k^{-1}A_{\varepsilon,j} \cup P_k^{-1}(Q_D \backslash A_{\varepsilon,j})$. Nearest-neighbor interactions between those two sets are contained in $P_k^{-1}S_k^{\xi,\varepsilon}$ for some $\xi \in r'\mathbb{Z}^d_M$ with $|\xi| \le 4R$. Therefore, we can further estimate the last inequality via

$$\sum_{|\xi|>L_\delta} \sum_{\alpha\in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha) \le C\delta\bigg(E_\varepsilon(u_\varepsilon, A) + E_\varepsilon(v_\varepsilon, Q_D) + \sum_{|\xi|\le L_\delta} \sum_{\alpha\in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha)\bigg). \tag{3-19}$$

Now we treat the interactions when $|\xi| \le L_\delta$. Consider any points $\varepsilon x, \varepsilon y \in \varepsilon\mathcal{L}$. If $w_\varepsilon^j(\varepsilon x) \ne w_\varepsilon^j(\varepsilon y)$ then $\varepsilon x, \varepsilon y \in A_{\varepsilon,j}$, $\varepsilon x, \varepsilon y \notin A_{\varepsilon,j}$ or $\varepsilon x \in A_{\varepsilon,j}$ but $\varepsilon y \notin A_{\varepsilon,j}$ (the reverse case can be treated similarly). In the last case we have a contribution only if $u_\varepsilon(\varepsilon x) \ne v_\varepsilon(\varepsilon y)$. Then either $u_\varepsilon(\varepsilon y) = v_\varepsilon(\varepsilon y)$ or $f_\varepsilon(x, y, u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon y)) \le C|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)|$. Summarizing all cases we obtain the inequality

$$\rho_j^{\xi,\varepsilon}(\alpha) \le \varepsilon^{k-1} f_\varepsilon(x, y, u_\varepsilon(\varepsilon x), u_\varepsilon(\varepsilon y)) + \varepsilon^{k-1} f_\varepsilon(x, y, v_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon y)) + C\varepsilon^{k-1}|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)|.$$

By our construction we have $S_j^{\varepsilon,\xi} \subset (A_{\varepsilon,j+1} \backslash A_{\varepsilon,j-1}) =: S_j^\varepsilon$. We deduce that

$$\sum_{|\xi|\le L_\delta} \sum_{\alpha\in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha) \le E_\varepsilon(u_\varepsilon, S_j^\varepsilon) + E_\varepsilon(v_\varepsilon, S_j^\varepsilon) + C_\delta \sum_{\substack{y\in\mathcal{L} \\ \varepsilon P_k(y)\in S_j^\varepsilon}} \varepsilon^{k-1}|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)|,$$

where C_δ depends only on L_δ. Observe that by definition every point can only lie in at most two sets $S_{j_1}^\varepsilon, S_{j_2}^\varepsilon$. Thus averaging combined with (3-19), Step 1 and the last inequality yields

$$\begin{aligned} I_\varepsilon := \frac{1}{N_\varepsilon}\sum_{j=1}^{N_\varepsilon} \sum_{\xi\in r'\mathbb{Z}^d_M} \sum_{\alpha\in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha) &\le \frac{2}{N_\varepsilon}\sum_{j=1}^{N_\varepsilon} \sum_{|\xi|\le L_\delta} \sum_{\alpha\in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha) + C\delta \\ &\le \frac{4}{N_\varepsilon}\big(E_\varepsilon(u_\varepsilon, Q_D) + E_\varepsilon(v_\varepsilon, Q_D)\big) + C_\delta \sum_{\substack{y\in\mathcal{L} \\ \varepsilon P_k(y)\in D}} \varepsilon^d|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)| + C\delta \\ &\le \frac{C}{N_\varepsilon} + C_\delta \sum_{\substack{y\in\mathcal{L} \\ \varepsilon P_k(y)\in D}} \varepsilon^d|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)| + C\delta. \end{aligned}$$

Due to Step 1 we can apply Lemma 2.5 to deduce that $\limsup_{\varepsilon\to0} I_\varepsilon \le C\delta$. For every $\varepsilon > 0$, let $j_\varepsilon \in \{1, \dots, N_\varepsilon\}$ be such that

$$\sum_{\xi\in r'\mathbb{Z}^d_M} \sum_{\alpha\in R_\varepsilon^\xi(S_{j_\varepsilon}^{\xi,\varepsilon})} \rho_{j_\varepsilon}^{\xi,\varepsilon}(\alpha) \le I_\varepsilon \tag{3-20}$$

and set $w_\varepsilon := w_\varepsilon^{j_\varepsilon}$. Note that, as a convex combination, w_ε still converges to u on D. Hence, using (3-18) and (3-20), we conclude that

$$E''(u, A' \cup B) \le \limsup_{\varepsilon\to0} E_\varepsilon(w_\varepsilon, A' \cup B) \le E''(u, A) + E''(u, B) + C\,\delta.$$

The arbitrariness of δ proves the claim. □

Proof of Theorem 3.2. From Propositions 3.5 and 3.6 it follows by standard arguments that $E''(u,\cdot)$ is inner regular on $\mathcal{A}^R(D)$; see, for example, Proposition 11.6 in [Braides and Defranceschi 1998]. Therefore, given a sequence $\varepsilon_n \to 0^+$ we can use Remark 2.3 and the compactness property of Γ-convergence, see [Braides 2002, Section 1.8.2], to construct a subsequence ε_n (not relabeled) such that

$$\Gamma\text{-}\lim_n E_{\varepsilon_n}(u, A) =: \widetilde{E}(u, A)$$

exists for every $(u, A) \in L^1(D) \times \mathcal{A}^R(D)$. By Proposition 3.4 we know that $\widetilde{E}(u, A)$ is finite only if $u \in \mathrm{BV}(A, \mathcal{S})$. We extend $\widetilde{E}(u,\cdot)$ to $\mathcal{A}(D)$ setting

$$E(u, A) := \sup\{\widetilde{E}(u, A') : A' \Subset A,\ A' \in \mathcal{A}^R(D)\}.$$

To complete the proof, it is enough to show that E satisfies the assumptions of Theorem 3.1. Again by standard arguments $E(u,\cdot)$ fulfills the assumptions of the De Giorgi–Letta criterion [Braides 2002, Section 16] so that $E(u,\cdot)$ is the trace of a Borel measure. By Proposition 3.5, it is indeed a Radon measure. The locality property follows from Remark 2.3. By the properties of Γ-limits and again Remark 2.3 we know that $\widetilde{E}(\cdot, A)$ is $L^1(D)$-lower semicontinuous and so is $E(\cdot, A)$ as the supremum of lower semicontinuous functions. The growth conditions (iv) in Theorem 3.1 follow from Propositions 3.4 and 3.5, which still hold for E in place of $\widetilde{E}$. The local version of the theorem is a direct consequence of our construction. □

4. Convergence of boundary value problems

In this section we consider the convergence of minimum problems with Dirichlet-type boundary data. In order to model boundary conditions in our discrete setting we need to introduce a suitable notion of trace, taking into account possible long-range interactions; see also [Alicandro et al. 2015]. In what follows we will further assume a continuous spatial dependence of the integrand of the limit continuum energy. Without such a condition we can still obtain a weaker result stated in Lemma 4.3. On the other hand continuity assumptions are always fulfilled in the case of the homogenization problem that we are going to treat in Section 5.

Consider $A \in \mathcal{A}^R(D)$ and fix boundary data $u_0 \in \mathrm{BV}(\mathbb{R}^k_{\mathrm{loc}}, \mathcal{S})$. We assume the boundary data are well-prepared in the sense that, setting $u_{\varepsilon,0} \in \mathcal{PC}_\varepsilon(\mathcal{L})$ as $u_{\varepsilon,0}(\varepsilon x) = u_0(P_k(\varepsilon x))$, we have $u_{\varepsilon,0} \to u_0$ on D and

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_{\varepsilon,0}, B) \le C\mathcal{H}^{k-1}(S_{u_0} \cap \bar{B}), \quad \mathcal{H}^{k-1}(S_{u_0} \cap \partial A) = 0, \tag{4-21}$$

with C independent of $B \in \mathcal{A}^R(\mathbb{R}^k)$. Observe that as in the proof of Proposition 3.5 we may allow for any polyhedral function such that $\mathcal{H}^{k-1}(S_{u_0} \cap \partial A) = 0$, but more generally it suffices that all level sets are Lipschitz sets.

We define a *discrete trace constraint* as follows: let $l_\varepsilon > 0$ be such that

$$\lim_{\varepsilon\to 0} l_\varepsilon = +\infty, \quad \lim_{\varepsilon\to 0} l_\varepsilon \varepsilon = 0. \tag{4-22}$$

We set $\mathcal{PC}^{l_\varepsilon\varepsilon}_{\varepsilon,u_0}(\mathcal{L},A)$ as the space of those u that agree with u_0 at the discrete boundary of A, by setting

$$\mathcal{PC}^{l_\varepsilon\varepsilon}_{\varepsilon,u_0}(\mathcal{L},A):=\big\{u:\varepsilon\mathcal{L}\to\mathcal{S}:u(\varepsilon x)=u_0(P_k(\varepsilon x))\text{ if }\operatorname{dist}(P_k(\varepsilon x),\partial A)\le l_\varepsilon\varepsilon\big\}.$$

For $\varepsilon>0$ and $l_\varepsilon>0$ we consider the restricted functional $E^{l_\varepsilon}_{\varepsilon,u_0}(\,\cdot\,,A):\mathcal{PC}^{l_\varepsilon\varepsilon}_{\varepsilon,u_0}(\mathcal{L},A)\to[0,+\infty]$ defined as

$$E^{l_\varepsilon}_{\varepsilon,u_0}(u,A):=E_\varepsilon(u,A). \tag{4-23}$$

We need some further notation. Given $u\in\mathrm{BV}(D,\mathcal{S})$, we set $u_{A,0}:\mathbb{R}^k\to\mathcal{S}$ as

$$u_{A,0}(x):=\begin{cases}u(x) & \text{if } x\in A,\\ u_0(x) & \text{otherwise.}\end{cases}$$

Since A is regular we have $u_{A,0}\in\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^k,\mathcal{S})$. The following convergence result holds:

Theorem 4.1. *Let $\mathcal{L}$ be a thin admissible lattice and let f^ε_{nn} and f^ε_{lr} satisfy Hypothesis 1. For every sequence converging to* 0*, let ε_n and ϕ be as in Theorem 3.2. Assume that the limit integrand ϕ is continuous on $D\times\mathcal{S}^2\times S^{k-1}$. Then, for every set $A\in\mathcal{A}^R(D)$, $A\Subset D$, the functionals $E^{l_{\varepsilon_n}}_{\varepsilon_n,u_0}(\,\cdot\,,A)$ defined in* (4-23) *Γ-converge with respect to the convergence on A in Definition 2.2 to the functional*

$$E_{u_0}(\,\cdot\,,A):L^1(D,\mathbb{R}^q)\to[0,+\infty]$$

that is finite only for $u\in\mathrm{BV}(A,\mathcal{S})$, where it takes the form

$$E_{u_0}(u,A)=\int_{S_{u_{A,0}}\cap\bar{A}}\phi(x,u^+_{A,0},u^-_{A,0},\nu_{u_{A,0}})\,\mathrm{d}\mathcal{H}^{k-1}.$$

Proof. By Proposition 3.4 we know that the limit energy is finite only for $u\in\mathrm{BV}(A,\mathcal{S})$. To save notation, we replace the subsequence ε_n again by ε.

Lower bound: Without loss of generality let $u_\varepsilon\to u$ on A in the sense of Definition 2.2 be such that

$$\liminf_\varepsilon E^{l_\varepsilon}_{\varepsilon,u_0}(u_\varepsilon,A)\le C. \tag{4-24}$$

Passing to a subsequence, we may assume $u_\varepsilon\in\mathcal{PC}^{l_\varepsilon\varepsilon}_{\varepsilon,u_0}(\mathcal{L},A)$. We define a new sequence $v_\varepsilon:\varepsilon\mathcal{L}\to\mathcal{S}$ by

$$v_\varepsilon(\varepsilon x)=\mathbb{1}_A(P_k(\varepsilon x))u_\varepsilon(\varepsilon x)+(1-\mathbb{1}_A(P_k(\varepsilon x)))u_0(\varepsilon P_k(x)).$$

Note that by our assumptions on u_0 we have $v_\varepsilon\to u_{A,0}$ on D in the sense of Definition 2.2. Now fix $A_1\Subset A\Subset A_2$ such that $A_1,A_2\in\mathcal{A}^R(D)$. Setting

$$S^{\xi,\varepsilon}:=\{\alpha\in R^\xi_\varepsilon(A_2):\varepsilon x_\alpha\in P_k^{-1}A,\ \varepsilon x_{\alpha+\xi}\notin P_k^{-1}A\text{ or vice versa}\},$$

it holds that

$$E_\varepsilon(v_\varepsilon,A_2)\le E^{l_\varepsilon}_{\varepsilon,u_0}(u_\varepsilon,A)+E_\varepsilon(u_{\varepsilon,0},A_2\backslash\bar{A}_1)+\sum_{\xi\in r'\mathbb{Z}^d_M}\sum_{\alpha\in S^{\xi,\varepsilon}}\varepsilon^{k-1}f_\varepsilon(x_\alpha,x_{\alpha+\xi},v_\varepsilon(\varepsilon x_\alpha),v_\varepsilon(\varepsilon x_{\alpha+\xi})), \tag{4-25}$$

Given $\delta > 0$, let $L_\delta > 0$ be such that (2-12) holds. To bound the long-range interactions, we fix again a large cube Q_D containing $\overline{D}$. Then Lemma 2.7 and the coercivity assumption in Hypothesis 1 yield

$$\begin{aligned}\sum_{|\xi|>L_\delta}\sum_{\alpha\in S^{\xi,\varepsilon}} &\varepsilon^{k-1} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, v_\varepsilon(\varepsilon x_\alpha), v_\varepsilon(\varepsilon x_{\alpha+\xi}))\\ &\leq C\sum_{|\xi|>L_\delta} J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y)\in\mathcal{NN}(\mathcal{L})\\ \varepsilon x,\varepsilon y\in P_k^{-1}Q_D}} \varepsilon^{k-1} f_\varepsilon(x,y,v_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon y))\\ &\leq C\delta\Big(E_\varepsilon(u_\varepsilon, A) + E_\varepsilon(u_{\varepsilon,0}, Q_D) + \sum_{|\xi|\leq L_\delta}\sum_{\alpha\in S^{\xi,\varepsilon}} \varepsilon^{k-1} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, v_\varepsilon(\varepsilon x_\alpha), v_\varepsilon(\varepsilon x_{\alpha+\xi}))\Big). \quad (4\text{-}26)\end{aligned}$$

For interactions with $|\xi| \leq L_\delta$ and ε small enough, we have $S^{\xi,\varepsilon} \subset A_2\backslash\bar{A}_1$. Moreover, if $l_\varepsilon > L_\delta + 2r$, then by the boundary conditions on u_ε we get

$$\sum_{|\xi|\leq L_\delta}\sum_{\alpha\in S^{\xi,\varepsilon}} \varepsilon^{k-1} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, v_\varepsilon(\varepsilon x_\alpha), v_\varepsilon(\varepsilon x_{\alpha+\xi})) \leq E_\varepsilon(u_{\varepsilon,0}, A_2\backslash\bar{A}_1).$$

From the local version of Theorem 3.2, (4-21), (4-24), (4-25) and (4-26) we infer

$$E(u_{A,0}, A_2) \leq \liminf_{\varepsilon} E^{l_\varepsilon}_{\varepsilon,u_0}(u_\varepsilon, A) + C\,\delta(1+\mathcal{H}^{d-1}(S_{u_0}\cap\bar{Q}_D)) + C\mathcal{H}^{d-1}(S_{u_0}\cap\bar{A}_2\backslash A_1).$$

The lower bound follows by letting $A_2 \downarrow \bar{A}$ and $A_1 \uparrow A$ combined with (4-21) and the arbitrariness of δ.

Upper bound: We first provide a recovery sequence in the case when $u = u_0$ in a neighborhood of ∂A. Let $u_\varepsilon : \varepsilon\mathcal{L} \to \mathcal{S}$ converge to u on D in the sense of Definition 2.2 and be such that

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, A) = E(u, A). \quad (4\text{-}27)$$

Again, given $\delta > 0$ we let $L_\delta > 0$ be such that (2-12) holds. Now choose regular sets $A_1 \Subset A_2 \Subset A$ such that

$$u = u_0 \quad \text{on } A\backslash\bar{A}_1. \quad (4\text{-}28)$$

The remaining argument is similar to the proof of Proposition 3.6 and therefore we only sketch it. Fix $d' \leq \frac{1}{2}\operatorname{dist}(A_1, \partial A_2)$ and set $N_\varepsilon = \lfloor d'/(\varepsilon(L_\delta + 2r))\rfloor$. For $j \in \mathbb{N}$ we define the sets

$$A_{\varepsilon,j} := \{x \in A : \operatorname{dist}(x, A_1) < j\varepsilon(L_\delta + 2r)\}.$$

We further define $u^j_\varepsilon : \varepsilon\mathcal{L} \to \mathcal{S}$ setting

$$u^j_\varepsilon(\varepsilon x) = \begin{cases} u_0(\varepsilon x) & \text{if } P_k(\varepsilon x) \notin A_{\varepsilon,j},\\ u_\varepsilon(\varepsilon x) & \text{otherwise.}\end{cases}$$

It holds that

$$E_\varepsilon(u^j_\varepsilon, A) \leq E_\varepsilon(u_\varepsilon, A) + E_\varepsilon(u_{\varepsilon,0}, A\backslash\bar{A}_1) + \sum_{\xi\in r'\mathbb{Z}^d_M} \varepsilon^{k-1} \sum_{\alpha\in R^\xi_\varepsilon(S^{\xi,\varepsilon}_j)} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, u^j_\varepsilon(\varepsilon x_\alpha), u^j_\varepsilon(\varepsilon x_{\alpha+\xi})),$$

where the set $S^{\xi,\varepsilon}_j$ is defined as

$$S^{\xi,\varepsilon}_j := \big\{x = y + tP_k(\xi') : y \in \partial A_{\varepsilon,j},\ |t| \leq \varepsilon,\ \xi' \in \xi + [-r', r']^d\big\} \cap A.$$

As for (4-26), using (4-21) and (4-27) we can show that

$$\sum_{\xi\in r'\mathbb{Z}^d_M}\varepsilon^{k-1}\sum_{\alpha\in R^\xi_\varepsilon(S^{\xi,\varepsilon}_j)} f_\varepsilon(x_\alpha,x_{\alpha+\xi},u^j_\varepsilon(\varepsilon x_\alpha),u^j_\varepsilon(\varepsilon x_{\alpha+\xi}))$$
$$\le C\delta+C\sum_{|\xi|\le L_\delta}\sum_{\alpha\in^\xi_\varepsilon(S^{\xi,\varepsilon}_j)}\varepsilon^{k-1}f_\varepsilon(x_\alpha,x_{\alpha+\xi},u^j_\varepsilon(\varepsilon x_\alpha),u^j_\varepsilon(\varepsilon x_{\alpha+\xi})).$$

To estimate the interactions where $|\xi|\le L_\delta$, note that due to (4-28) we can use the averaging technique like in Step 2 of Proposition 3.6 to obtain $j_\varepsilon\in\{1,\dots,N_\varepsilon\}$ and the corresponding sequence $u^{j_\varepsilon}_\varepsilon$ satisfying the boundary conditions, at least for small ε because of (4-22), such that

$$\limsup_n E^{l_\varepsilon}_{\varepsilon,u_0}(u^{j_\varepsilon}_\varepsilon,A)\le E(u,A)+C\mathcal{H}^{k-1}(S_{u_0}\cap(\bar{A}\backslash A_1))+C\delta,$$

where we used (4-21). Moreover, due to the assumptions on u_0 and (4-28) we know that $u^{j_\varepsilon}_\varepsilon\to u$ on A. Letting first $\delta\to 0$ and then $A_1\uparrow A$ we finally get

$$\Gamma\text{-}\limsup_\varepsilon E^{l_\varepsilon}_{\varepsilon,u_0}(u,A)\le E(u,A)=E_{u_0}(u,A).$$

For a general function $u\in BV(A,\mathcal{S})$ we argue by approximation. To this end we take any $B\in\mathcal{A}^R(D)$ such that $A\Subset B$. By Lemma B.1 we obtain a sequence $u_n\in BV(D,\mathcal{S})$ such that $u_n=u_0$ in a neighborhood of ∂A and moreover $u_n\to u_{A,0}$ in $L^1(B)$ and $\mathcal{H}^{k-1}(S_{u_n}\cap B)\to\mathcal{H}^{k-1}(S_u\cap B)$. By $L^1(A)$-lower semicontinuity and the previous argument we obtain

$$\Gamma\text{-}\limsup_\varepsilon E^{l_\varepsilon}_{\varepsilon,u_0}(u,A)\le\liminf_n E(u_n,A)\le\liminf_n E(u_n,B)=E(u_{A,0},B).$$

In the last step we used the continuity assumption on the integrand and a Reshetnyak-type continuity result for functionals defined on partitions that is proven in [Ruf 2017]. Letting $B\downarrow\bar{A}$ we obtain the claim. □

Remark 4.2. (i) It is a direct consequence of our proof, that if we have only finite range of interactions, that is $f^\varepsilon_{lr}(x,y)=0$ for $|x-y|\ge L$, then it is enough to take $l_\varepsilon\ge L$.

(ii) By Remark 2.3 the above Theorem 4.1 implies the usual convergence of minimizers in the spirit of Γ-convergence.

Finally we prove an auxiliary result about convergence of boundary value problems that holds without any continuity assumptions. This result will be useful to treat homogenization problems as in Section 5. To this end we replace the discrete width l_ε by a macroscopic value η and then take first the limit when $\varepsilon\to 0$ and let $\eta\to 0$ in a second step. Given $\eta>0$ and $A\in\mathcal{A}^R(D)$, we set

$$\partial A_\eta=\{x\in A:\operatorname{dist}(x,\partial A)\le\eta\}.$$

We let u_0 be as before. Using a similar notation to that in Theorem 3.1 we define the quantities

$$m^\eta_\varepsilon(u_0,A)=\inf\{E_\varepsilon(v,A):v\in\mathcal{PC}^\eta_{\varepsilon,u_0}(\mathcal{L},A)\},$$
$$m(u_0,A)=\inf\{E(v,A):v=u_0\text{ in a neighborhood of }\partial A\},$$

where the limit functional E is given (up to subsequences) by Theorem 3.2. Note that the mapping $\eta\mapsto m^\eta_\varepsilon(u_0,A)$ is nondecreasing. Then we have the following weak version of Theorem 4.1.

Lemma 4.3. *Let ε_n and E be as in Theorem 3.2. Then it holds that*

$$\lim_{\eta\to 0}\liminf_n m^{\eta}_{\varepsilon_n}(u_0,A)=\lim_{\eta\to 0}\limsup_n m^{\eta}_{\varepsilon_n}(u_0,A)=m(u_0,A).$$

Proof. First note that by monotonicity the limits for $\eta\to 0$ are well-defined. Moreover, by the first assumption in (4-21) we have that $m^{\eta}_{\varepsilon}(u_0,A)$ is equibounded. Now for any $n\in\mathbb{N}$ let $u_n\in\mathcal{PC}^{\eta}_{\varepsilon_n,u_0}(\mathcal{L},A)$ be such that $m^{\eta}_{\varepsilon_n}(u_0,A)=E_{\varepsilon_n}(u_n,A)$. By Lemma 2.4 we know that, up to a subsequence (not relabeled), $u_n\to u$ on A and by the assumptions on u_0 it follows that $u=u_0$ on ∂A_η. Extending u we can assume that u is admissible in the infimum problem defining $m(u_0,A)$ and using Theorem 3.2 we obtain

$$m(u_0,A)\le E(u,A)\le\liminf_n E_{\varepsilon_n}(u_n,A)\le\liminf_n m^{\eta}_{\varepsilon_n}(u_0,A).$$

Since η is arbitrary, we conclude that $m(u_0,A)\le\lim_{\eta\to 0}\liminf_n m^{\eta}_{\varepsilon_n}(u_0,A)$.

In order to prove the remaining inequality, given $\gamma>0$ we let $u\in \mathrm{BV}(A,\mathcal{S})$ be such that $u=u_0$ in a neighborhood of ∂A and $E(u,A)\le m(u_0,A)+\gamma$. Now let $u_n:\varepsilon\mathcal{L}\to\mathcal{S}$ be a recovery sequence for u. Repeating the argument for the upper bound in Theorem 4.1, given $\delta>0$ we can modify u_n to a function $\bar{u}_n\in\mathcal{PC}^{\eta}_{\varepsilon_n,u_0}(\mathcal{L},A)$ for some $\eta=\eta(\delta)>0$ such that

$$\limsup_n E_{\varepsilon_n}(\bar{u}_n,A)\le E(u,A)+\delta.$$

By the choice of u we obtain

$$\lim_{\eta\to 0}\limsup_n m^{\eta}_{\varepsilon_n}(u_0,A)\le\limsup_n E_{\varepsilon_n}(\bar{u}_n,A)+\delta\le m(u_0,A)+\gamma+\delta.$$

The claim now follows letting first $\delta\to 0$ and then $\gamma\to 0$. □

5. Homogenization results for stationary lattices

We now replace the deterministic lattice $\mathcal{L}$ by a random point set. In what follows we introduce the probabilistic framework. To this end let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a complete σ-algebra $\mathcal{F}$.

Definition 5.1. We say that a family $(\tau_z)_{z\in\mathbb{Z}^k}$, $\tau_z:\Omega\to\Omega$, is an *additive group action* on Ω if

$$\tau_{z_1+z_2}=\tau_{z_2}\circ\tau_{z_1}\quad\text{for all } z_1,z_2\in\mathbb{Z}^k.$$

Such an additive group action is called *measure preserving* if

$$\mathbb{P}(\tau_z B)=\mathbb{P}(B)\quad\text{for all } B\in\mathcal{F},\ z\in\mathbb{Z}^k.$$

Moreover $(\tau_z)_{z\in\mathbb{Z}^k}$ is called *ergodic* if, in addition, for all $B\in\mathcal{F}$ we have the implication

$$(\tau_z(B)=B\ \text{ for all } z\in\mathbb{Z}^k)\quad\Longrightarrow\quad\mathbb{P}(B)\in\{0,1\}.$$

For general $m\in\mathbb{N}$ we denote by $[a,b):=\{x\in\mathbb{R}^m: a_i\le x_i<b_i\ \text{ for all } i\}$ the m-dimensional coordinate parallelepiped with opposite vertices a and b, and we set $\mathcal{I}_m=\{[a,b):a,b\in\mathbb{Z}^m,\ a\neq b\}$. Next, we introduce the notion of regular families and discrete subadditive stochastic processes:

Definition 5.2. Let $\{I_n\} \subset \mathcal{I}_m$ be a family of sets. Then $\{I_n\}$ is called *regular* if there exists another family $\{I'_n\} \subset \mathcal{I}_m$ and a constant $C > 0$ such that

(i) $I_n \subset I'_n$ for all n,

(ii) $I'_{n_1} \subset I'_{n_2}$ whenever $n_1 < n_2$,

(iii) $0 < \mathcal{H}^m(I'_n) \leq C\,\mathcal{H}^m(I_n)$ for all n.

Moreover, if $\{I'_n\}$ can be chosen such that $\mathbb{R}^m = \bigcup_n I'_n$, then we write $\lim_{n\to\infty} I_n = \mathbb{R}^m$.

Definition 5.3. A function $\mu : \mathcal{I}_m \to L^1(\Omega)$ is said to be a *discrete subadditive stochastic process* if the following properties hold $\mathbb{P}$-almost surely:

(i) For every $I \in \mathcal{I}_m$ and for every finite partition $(I_j)_{j\in J} \subset \mathcal{I}_m$ of I we have

$$\mu(I,\omega) \leq \sum_{j\in J} \mu(I_j,\omega).$$

(ii) $\inf\left\{(\mathcal{H}^m(I))^{-1} \int_\Omega \mu(I,w)\,\mathrm{d}\mathbb{P}(\omega) : I \in \mathcal{I}_m\right\} > -\infty$.

One of the key ingredients for our stochastic homogenization result will be the following pointwise ergodic theorem; see Theorem 2.7 in [Akcoglu and Krengel 1981].

Theorem 5.4. *Let $\mu : \mathcal{I}_m \to L^1(\Omega)$ be a discrete subadditive stochastic process and let I_n be a regular family in $\mathcal{I}_m$. If μ is stationary with respect to a measure-preserving group action $(\tau_z)_{z\in\mathbb{Z}^m}$, that is,*

$$\textit{for all } I \in \mathcal{I}_m,\ z \in \mathbb{Z}^m, \quad \mu(I+z,\omega) = \mu(I,\tau_z\omega) \textit{ almost surely},$$

then there exists $\mu^\infty : \Omega \to \mathbb{R}$ such that, for $\mathbb{P}$-almost every ω,

$$\lim_{n\to+\infty} \frac{\mu(I_n,\omega)}{\mathcal{H}^m(I_n)} = \mu^\infty(\omega).$$

The statement is written for a generic m since in this section we will use Theorem 5.4 for $m = k-1$, while in the next one we use it for $m = k$. We require some geometric and probabilistic properties of the random point set.

Definition 5.5. A random variable $\mathcal{L} : \Omega \to (\mathbb{R}^d)^{\mathbb{N}}$, $\omega \mapsto \mathcal{L}(\omega) = \{\mathcal{L}(\omega)_i\}_{i\in\mathbb{N}}$, is called a *stochastic lattice*. We say that $\mathcal{L}$ is a *thin admissible lattice* if $\mathcal{L}(\omega)$ is a thin admissible lattice in the sense of Definition 2.1 and the constants M, r, R can be chosen independent of ω $\mathbb{P}$-almost surely. The stochastic lattice $\mathcal{L}$ is said to be *stationary* if there exists a measure-preserving group action $(\tau_z)_{z\in\mathbb{Z}^k}$ on Ω such that, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\mathcal{L}(\tau_z\omega) = \mathcal{L}(\omega) + z.$$

If in addition $(\tau_z)_{z\in\mathbb{Z}^k}$ is ergodic, then $\mathcal{L}$ is called *ergodic*, too.

In order to prove a homogenization result we make the following *structural assumption*:

Hypothesis 2. There exist functions $f_{nn}, f_{lr} : \mathbb{R}^k \times \mathbb{R}^{2(d-k)} \to [0, +\infty)$ such that, setting $\Delta_k(x, y) = (y_1 - x_1, \dots, y_k - x_k, x_{k+1}, y_{k+1}, \dots, x_d, y_d)$, it holds that

$$f^{\varepsilon}_{nn}(x, y) = f_{nn}(\Delta_k(x, y)), \quad f^{\varepsilon}_{lr}(x, y) = f_{lr}(\Delta_k(x, y)).$$

Note that nearest-neighbor and long-range interaction coefficients are deterministic, but the set of nearest neighbors becomes now random. In the following we let $E_\varepsilon(\omega)$ be the discrete energy defined in the previous section, with the stochastic lattice $\mathcal{L}(\omega)$ in place of $\mathcal{L}$. As a general rule we will replace $\mathcal{L}$ by ω to indicate the dependence on the stochastic lattice $\mathcal{L}(\omega)$.

In view of Theorem 3.1 and Lemma 4.3 we can further characterize the Γ-limits of the family $E_\varepsilon(\omega)$ by investigating the quantities $m^{\eta}_{\varepsilon}(u_0, Q)$ for suitable oriented cubes and $u_0 = u^{ij}_{x,\nu}$. Due to the decay assumptions of Hypothesis 1 it will be enough to consider truncated interactions. To this end, for fixed $L \in \mathbb{N}$ we will replace the long-range coefficients by

$$f^{L}_{lr}(x, y) := f_{lr}(\Delta_k(x, y))\mathbb{1}_{|x-y|\le L}$$

and denote the corresponding energy by $E^L_\varepsilon(\omega)(u, A)$. By Remark 4.2 the Γ-limit of the truncated energies is characterized by the minimum problem defined below: for $s_i, s_j \in \mathcal{S}$, $\nu \in S^{k-1}$ and a cube $Q_\nu(x, \rho)$ we set

$$m^{\eta,L}_1(\omega)(u^{ij}_{x,\nu}, Q_\nu(x, \rho)) := \inf\bigl\{E^L_1(\omega)(u, Q_\nu(x, \rho)) : u \in \mathcal{PC}^{\eta}_{1,u^{ij}_{x,\nu}}(\omega, Q_\nu(x, \rho))\bigr\}. \tag{5-29}$$

The following technical auxiliary result will be used several times.

Lemma 5.6. *Let $Q = Q_\nu(z, \rho) \subset \mathbb{R}^k$ be a cube and let $\{Q_n = Q_\nu(z_n, \rho_n)\}_n$ be a finite family of disjoint cubes with the following properties*:

(i) $\min_n \rho_n \ge 4L$.

(ii) $z_n - z_1 \in \{\nu\}^\perp$.

(iii) $\operatorname{dist}(z_1, \{\nu\}^\perp + z) \le \frac{1}{4} \min_n \rho_n$.

(iv) $\bigcup_n Q_n \subset Q$.

(v) *Either* $\operatorname{dist}\bigl(\partial \bigcup_n Q_n, \partial Q\bigr) > \eta$ *or* $z_1 - z \in \{\nu\}^\perp$.

Then there exists $C = C_L > 0$ such that for all $\eta \ge L$,

$$m^{\eta,L}_1(\omega)(u^{ij}_{z,\nu}, Q) \le \sum_n m^{\eta,L}_1(\omega)(u^{ij}_{z_n,\nu}, Q_n) + C\mathcal{H}^{k-1}\Bigl(\Bigl(Q\setminus \bigcup_n \overline{Q}_n\Bigr) \cap (\{\nu\}^\perp + z)\Bigr) + C\sum_n \bigl(\mathcal{H}^{k-2}\bigl((\partial Q_n\setminus\partial Q) \cap (\{\nu\}^\perp + z_1)\bigr) + \mathcal{H}^{k-1}(\partial Q_n \cap S_\nu(z, z_1))\bigr),$$

where $S_\nu(z, z_1)$ is the infinite (possibly, flat) stripe enclosed by the two hyperplanes $\{\nu\}^\perp + z$ and $\{\nu\}^\perp + z_1$.

Proof. During this proof, given $y \in \mathbb{R}^k$, we denote by $P_{\nu,y}$ the projection onto the affine space $\{\nu\}^\perp + y$. For each n let u_n be a minimizer for the problem in (5-29) with $Q_\nu(x, \rho) = Q_n$. By assumptions (ii) and (v), the function $v : \mathcal{L}(\omega) \to \mathcal{S}$ defined as

$$v(x) = \begin{cases} u_n(x) & \text{if } P_k(x) \in \overline{Q}_n \text{ for some } n, \\ u^{ij}_{z,\nu}(P_k(x)) & \text{otherwise} \end{cases}$$

is well-defined and belongs to $\mathcal{PC}^{\eta}_{1,u^{ij}_{z,\nu}}(\omega, Q)$. For $x, y \in \mathcal{L}(\omega) \cap Q$ with $|x - y| \le L$, we say that

- (I) holds if $P_k(x) \in \overline{Q}_n$ and $P_k(y) \in \overline{Q}_m$ for $n \neq m$ or if $P_k(x), P_k(y) \in \partial Q_n$,
- (II) holds if $P_k(x) \in Q \setminus \bigcup_n \overline{Q}_n$ and $P_k(y) \in \overline{Q}_n$ for some n.

By (iv) and Hypothesis 1 we can estimate

$$m^{\eta,L}_1(\omega)(u^{ij}_{z,\nu}, Q) \le E^L_1(\omega)(v, Q) \le \sum_n m^{\eta,L}_1(\omega)(u^{ij}_{z_n,\nu}, Q_n) + E^L_1(\omega)\Big(v, Q \setminus \bigcup_n \overline{Q}_n\Big) + C \sum_{\substack{|x-y|\le L \\ \text{(I) or (II) hold}}} |v(x) - v(y)|. \tag{5-30}$$

We start with estimating the contribution of $x, y \in Q \setminus \bigcup_n \overline{Q}_n$. Suppose $v(x) \neq v(y)$. Then $P_k(x)$ and $P_k(y)$ lie on different sides of the hyperplane $\{\nu\}^{\perp} + z$. Then it holds true that $P_{\nu,z}(P_k(x)) \in Q \setminus \bigcup_n \overline{Q}_n$; otherwise assumptions (i) and (iii) would imply

$$L \ge |P_k(x) - P_k(y)| \ge |P_k(x) - P_{\nu,z}(P_k(x))| \ge \tfrac{1}{2}\rho_n - \tfrac{1}{4}\rho_n \ge 2L.$$

Thus $\operatorname{dist}\big(P_k(x), (Q \setminus \bigcup_n \overline{Q}_n) \cap (\{\nu\}^{\perp} + z)\big) \le L$ and, using the properties of Definition 2.1, it follows that

$$E^L_1(\omega)\Big(v, Q \setminus \bigcup_n \overline{Q}_n\Big) \le C\mathcal{H}^{k-1}\Big(\Big(Q \setminus \bigcup_n \overline{Q}_n\Big) \cap (\{\nu\}^{\perp} + z)\Big). \tag{5-31}$$

Next we have to control the interactions in Case (I). Given such x, y with $|x - y| \le L$, we know that by the definition of v, the boundary conditions on the smaller cubes and (ii) that $v(x) = u^{ij}_{z_1,\nu}(P_k(x))$ and $v(y) = u^{ij}_{z_1,\nu}(P_k(y))$, so that if they contribute to the energy we conclude from assumption (ii) that $P_k(x)$ and $P_k(y)$ must lie on different sides of the hyperplane $\{\nu\}^{\perp} + z_1$. We deduce that $|P_{\nu,z_1}(P_k(x)) - P_k(x)| \le L$. Since by (iv) the segment $[P_{\nu,z_1}(P_k(x)), P_{\nu,z_1}(P_k(y))]$ intersects the $(k-2)$-dimensional set $(\partial Q_n \setminus \partial Q) \cap (\{\nu\}^{\perp} + z_1)$, it follows that

$$\operatorname{dist}\big(P_k(x), (\partial Q_n \setminus \partial Q) \cap (\{\nu\}^{\perp} + z_1)\big) \le 2L.$$

Again, by Definition 2.1 and the above inequality we derive the estimate

$$\sum_{\substack{|x-y|\le L \\ \text{(I) holds}}} |v(x) - v(y)| \le C \sum_n \mathcal{H}^{k-2}\big((\partial Q_n \setminus \partial Q) \cap (\{\nu\}^{\perp} + z_1)\big). \tag{5-32}$$

It remains to estimate the contributions coming from Case (II). For such x, y with $|x - y| \le L$, due to the boundary conditions on the smaller cubes, a positive energy contribution implies $u^{ij}_{z,\nu}(P_k(x)) \neq u^{ij}_{z_1,\nu}(P_k(y))$. Thus the segment $[P_k(x), P_k(y)]$ intersects ∂Q_n in (at least) one point x_n and also $S_\nu(z, z_1)$ in (at least) one point x_S. Denote by $x_{n,S}$ the projection of x_S onto the facet of the cube Q_n containing x_n. Since this facet cannot be parallel to $\{\nu\}^{\perp}$ by (i) and (iii), it holds that $x_{n,S} \in \partial Q_n \cap S_\nu(z, z_1)$ and

$$|P_k(x) - x_{n,S}| \le |P_k(x) - x_S| + |x_S - x_{n,S}| \le L + |x_S - x_n| \le 2L,$$

which yields the estimate

$$\operatorname{dist}(P_k(x), \partial Q_n \cap S_\nu(z, z_1)) \leq 2L. \tag{5-33}$$

This set may not be $(k-1)$-dimensional in the second possibility of (v). In this case one can bound the interactions by the right-hand side of (5-31). Otherwise, using (5-33) we obtain the estimate

$$\sum_{\substack{|x-y|\leq L \\ \text{(II) holds}}} |v(x) - v(y)| \leq C \sum_n \mathcal{H}^{k-1}(\partial Q_n \cap S_\nu(z, z_1)). \tag{5-34}$$

In any case the claim now follows from (5-30), (5-31), (5-32) and (5-34). □

Remark 5.7. Lemma 5.6 still holds if we replace cubes by k-parallelepipeds of the type $I_\nu(z, \{\rho_m\}_m) = z + \{x \in \mathbb{R}^k : |\langle x, \nu_m\rangle| < \frac{1}{2}\rho_m\}$. Then the cubes Q_n are replaced by the collection $I_n = I_\nu(z_n, \{\rho_m^n\}_m)$ and in the assumptions (i) and (iii) we have to replace ρ_n by $\min_m \rho_m^n$.

The next theorem is the main result of this section.

Theorem 5.8. *Let $\mathcal{L}$ be a stationary, thin admissible stochastic lattice and let f_{nn} and f_{lr} satisfy Hypotheses 1 and 2. For $\mathbb{P}$-almost every ω and for all $s_i, s_j \in \mathcal{S}$ and $\nu \in S^{k-1}$ there exists*

$$\phi_{\text{hom}}(\omega; s_i, s_j, \nu) := \lim_{\eta\to 0} \limsup_{t\to+\infty} \frac{1}{t^{k-1}} \inf\bigl\{E_1(\omega)(u, Q_\nu(0, t)) : u \in \mathcal{PC}^{\eta t}_{1,u^{ij}_{0,\nu}}(\omega, Q_\nu(0, t))\bigr\}.$$

The functionals $E_\varepsilon(\omega)$ Γ-converge with respect to the convergence of Definition 2.2 to the functional $E_{\text{hom}}(\omega) : L^1(D, \mathbb{R}^q) \to [0, +\infty]$ defined by

$$E_{\text{hom}}(\omega)(u) = \begin{cases} \int_{S_u} \phi_{\text{hom}}(\omega; u^+, u^-, \nu_u)\, d\mathcal{H}^{k-1} & \text{if } u \in \mathrm{BV}(D, \mathcal{S}), \\ +\infty & \text{otherwise.} \end{cases}$$

If $\mathcal{L}$ is ergodic, then $\omega \mapsto \phi_{\text{hom}}(\omega, s_i, s_j, \nu)$ is almost-surely constant.

Proof. Fix any sequence $\varepsilon \to 0$. According to Theorem 3.2, for all $\omega \in \Omega$ such that $\mathcal{L}(\omega)$ is admissible, there exists a (ω-dependent) subsequence ε_n such that

$$\Gamma\text{-}\lim_n E_{\varepsilon_n}(\omega)(u, A) = \int_{S_u\cap A} \phi(\omega; x, u^+, u^-, \nu)\, d\mathcal{H}^{k-1}$$

for all $u \in \mathrm{BV}(D, \mathcal{S})$ and every $A \in \mathcal{A}^R(D)$. According to Theorem 3.1 and Lemma 4.3, for any $x \in D$, $s_i, s_j \in \mathcal{S}$ and $\nu \in S^{k-1}$ it holds that

$$\begin{aligned}\phi(\omega; x, s_i, s_j, \nu) &= \limsup_{\rho\to 0} \frac{1}{\rho^{k-1}} m(\omega)(u^{ij}_{x,\nu}, Q_\nu(x, \rho)) \\ &= \limsup_{\rho\to 0} \frac{1}{\rho^{k-1}} \lim_{\eta\to 0} \limsup_n m^\eta_{\varepsilon_n}(\omega)(u^{ij}_{x,\nu}, Q_\nu(x, \rho)).\end{aligned}$$

If we change the variables via $t_n = \varepsilon_n^{-1}$ and $v(x) = u(t_n^{-1}x)$, the above characterization reads as

$$\phi(\omega; x, s_i, s_j, \nu) = \limsup_{\rho\to 0} \lim_{\eta\to 0} \limsup_n \frac{1}{(\rho t_n)^{k-1}} m_1^{\eta t_n}(\omega)(u^{ij}_{t_n x,\nu}, t_n Q_\nu(x, \rho)).$$

Except for the claim on ergodicity, due to the Urysohn property of Γ-convergence (recall Remark 2.3) it is enough to show that for a set of full probability the limit in ρ can be neglected and the remaining limits do not depend on x or the subsequence t_n. We divide the proof into several steps.

Step 1: truncating the range of interactions. First we show that it is enough to consider the case of finite range interactions. We argue that it is enough to prove that there exists $\phi^L_{\mathrm{hom}}(\omega;\nu)$ and a set Ω_L of full probability such that for all $\omega\in\Omega_L$, $x\in D$, every cube $Q_\nu(x,\rho)$ and every sequence $t_n\to+\infty$ it holds that

$$\phi^L_{\mathrm{hom}}(\omega;s_i,s_j,\nu)=\lim_{\eta\to0}\limsup_n\frac{1}{(\rho t_n)^{k-1}}m_1^{\eta t_n,L}(\omega)(u^{ij}_{t_nx,\nu},t_nQ_\nu(x,\rho)),\tag{5-35}$$

where $m_1^{\eta t_n,L}(\omega)$ is defined in (5-29). Indeed, if (5-35) is proven, then for all $\omega\in\bigcap_L\Omega_L$ we find a configuration $v_n^L:\mathcal{L}(\omega)\to\mathcal{S}$ with the correct boundary conditions (extended to the whole space) that minimizes $E_1^L(\omega)(\,\cdot\,,t_nQ_\nu(x,\rho))$ in (5-29). Using Lemma 2.7 we obtain the estimate

$$\begin{aligned}0&\le\frac{m_1^{\eta t_n}(\omega)(u^{ij}_{t_nx,\nu},t_nQ_\nu(x,\rho))-m_1^{\eta t_n,L}(\omega)(u^{ij}_{t_nx,\nu},t_nQ_\nu(x,\rho))}{(\rho t_n)^{k-1}}\\&\le\frac{E_1(\omega)(v_n^L,t_nQ_\nu(x,\rho))-E_1^L(\omega)(v_n^L,t_nQ_\nu(x,\rho))}{(\rho t_n)^{k-1}}\\&\le\frac{C}{(\rho t_n)^{k-1}}\sum_{2|\xi|>L}J_{lr}(|\hat\xi|)|\xi|\sum_{\substack{(x,y)\in\mathcal{NN}(\omega)\\x,y\in(t_nQ_\nu(x,\rho))^{3(R+M)}}}f_{nn}(x,y,v_n^L(x),v_n^L(y)).\end{aligned}$$

The inner sum can be bounded by the energy plus interactions close to $\partial t_nQ_\nu(x,\rho)$. Due to the boundary conditions, these are of order $(\rho t_n)^{k-2}$. Using the trivial a priori bound $m_1^\eta(\omega)(u^{ij}_{t_nx,\nu},t_nQ_\nu(x,\rho))\le C(\rho t_n)^{k-1}$ we deduce that

$$0\le\frac{m_1^{\eta t_n}(\omega)(u^{ij}_{t_nx,\nu},t_nQ_\nu(x,\rho))-m_1^{\eta t_n,L}(\omega)(u^{ij}_{t_nx,\nu},t_nQ_\nu(x,\rho))}{(\rho t_n)^{k-1}}\le C\sum_{2|\xi|>L}J_{lr}(|\hat\xi|)|\xi|.$$

Due to the integrability assumption of Hypothesis 1, we infer that $\phi^L_{\mathrm{hom}}(\omega;s_i,s_j,\nu)$ is a Cauchy sequence with respect to L and moreover, in combination with (5-35), we deduce that

$$\lim_L\phi^L_{\mathrm{hom}}(\omega;s_i,s_j,\nu)=\lim_{\eta\to0}\limsup_n\frac{1}{(\rho t_n)^{k-1}}m_1^{\eta t_n}(\omega)(u^{ij}_{t_nx,\nu},t_nQ_\nu(x,\rho))$$

exists and is independent of x, ρ and the sequence t_n. Therefore it remains to show (5-35). For clarity of the argument we first consider an auxiliary problem where we replace the varying boundary width ηt_n by L. As an intermediate result we show that there exists

$$\phi^L_{ij}(\omega;\nu)=\lim_n\frac{1}{(\rho t_n)^{k-1}}m_1^{L,L}(\omega)(u^{ij}_{t_nx,\nu},t_nQ_\nu(x,\rho))\tag{5-36}$$

and this limit does not depend on x, ρ or the sequence t_n.

Step 2: existence of ϕ_{ij}^L for $x=0$ and rational directions. Fix $L\in\mathbb{N}$. We have to show that, for $\mathbb{P}$-almost every $\omega\in\Omega$ and every $s_i, s_j\in\mathcal{S}$ and $\nu\in S^{k-1}$, there exists the limit in (5-36). We start with the case $x=0$ and $\nu\in S^{k-1}\cap\mathbb{Q}^k$. For this choice we can use the subadditive ergodic theorem in $(k-1)$-dimensions.

Substep 2.1: defining a stochastic process. We need a few preliminaries: given $\nu\in S^{k-1}$ there exists an orthogonal matrix $A_\nu\in\mathbb{R}^{k\times k}$ such that $A_\nu e_k=\nu$, the mapping $\nu\mapsto A_\nu e_i$ is continuous on $S^{k-1}\backslash\{-e_k\}$ and if $\nu\in\mathbb{Q}^k$ then $A_\nu\in\mathbb{Q}^{k\times k}$ (it suffices to consider the orthogonal transformation that keeps the vector $\nu+e_k$ fixed and reverses the orthogonal complement). We now fix a rational direction $\nu\in S^{k-1}\cap\mathbb{Q}^k$. Then there exists an integer $N=N(\nu)>4L$ such that $NA_\nu(z,0)\in\mathbb{Z}^k$ for all $z\in\mathbb{Z}^{k-1}$. We now define a discrete stochastic process (see Definition 5.3). To $I=[a_1,b_1)\times\cdots\times[a_{k-1},b_{k-1})\in\mathcal{I}_{k-1}$ we associate the set $Q_I\subset\mathbb{R}^k$ defined by

$$Q_I:=NA_\nu\big(\operatorname{int} I\times\big(-\tfrac{1}{2}s_{\max},\tfrac{1}{2}s_{\max}\big)\big),$$

where $s_{\max}=\max_i|b_i-a_i|$ is the maximal side length. Then we define the process $\mu:\mathcal{I}_{k-1}\to L^1(\Omega)$ as

$$\mu(I,\omega):=\inf\big\{E_1^L(\omega)(v,Q_I):\, v\in\mathcal{PC}^L_{1,u^{ij}_{0,\nu}}(\omega,Q_I)\big\}+C_\mu\mathcal{H}^{k-2}(\partial I),\tag{5-37}$$

where C_μ is a constant to be chosen later. We first have to show that $\mu(I,\cdot)$ is an $L^1(\Omega)$-function. Testing the $\mathcal{L}(\omega)$-interpolation of $u_{0,\nu}$ as a candidate in the infimum problem, one can use the growth assumptions from Hypothesis 1 and Definition 2.1 to show that there exists a constant $C>0$ such that

$$\mu(I,\omega)\le CN^{k-1}\mathcal{H}^{k-1}(I)\tag{5-38}$$

for all $I\in\mathcal{I}_{k-1}$ and almost every $\omega\in\Omega$ so that $\mu(I,\cdot)$ is essentially bounded. $\mathcal{F}$-measurability can be proven similar to [Alicandro et al. 2015, Lemma A.2].

We continue with proving lower-dimensional stationarity of the process. Let $z\in\mathbb{Z}^{d-1}$. Note that $Q_{I-z}=Q_I-z_\nu^N$, where $z_\nu^N:=NA_\nu(z,0)\in\{\nu\}^\perp\cap\mathbb{Z}^k$. By the stationarity of $\mathcal{L}$ it holds that $v\in\mathcal{PC}^L_{1,u^{ij}_{0,\nu}}(\omega,Q_{I-z})$ if and only if $u(\cdot)=v(\cdot-z_\nu^N)\in\mathcal{PC}^L_{1,u^{ij}_{0,\nu}}(\tau_{z_\nu^N}\omega,Q_I)$. Moreover, by the definition of nearest neighbors, Hypothesis 2 and again the stationarity of $\mathcal{L}$, we obtain $E_1^L(\omega)(v,Q_{I-z})=E_1^L(\tau_{z_\nu^N}\omega)(u,Q_I)$. By the shift invariance of the Hausdorff measure we conclude that $\mu(I-z,\omega)=\mu(I,\tau_{z_\nu^N}\omega)$. Setting $\tilde{\tau}_z=\tau_{-z_\nu^N}$ we obtain a measure-preserving group action on $\mathbb{Z}^{k-1}$ such that $\mu(I,\tilde{\tau}_z\omega)=\mu(I+z)(\omega)$, which yields stationarity.

To show subadditivity, let $I\in\mathcal{I}_{k-1}$ and let $\{I_n\}_n\subset\mathcal{I}_{k-1}$ be a finite disjoint family such that $I=\bigcup_n I_n$. Note that Q_I and the family $\{Q_{I_n}\}_n$ fulfill the assumptions of Lemma 5.6 (in the sense of Remark 5.7). We conclude

$$m_1^{L,L}(\omega)(u^{ij}_{0,\nu},Q_I)\le\sum_n m_1^{L,L}(\omega)(u^{ij}_{0,\nu},Q_{I_n})+C\sum_n\mathcal{H}^{k-2}((\partial Q_{I_n}\backslash\partial Q_I)\cap\{\nu\}^\perp).$$

Applying the definition of $\mu(I,\omega)$ yields

$$\begin{aligned}\mu(I,\omega)&=m_1^{L,L}(\omega)(u^{ij}_{0,\nu},Q_I)+C_\mu\mathcal{H}^{k-2}(\partial Q_I\cap\{\nu\}^\perp)\\&\le\sum_n\mu(I_n,\omega)+(C-C_\mu)\sum_n\mathcal{H}^{k-2}((\partial Q_{I_n}\backslash Q_I)\cap\{\nu\}^\perp),\end{aligned}$$

which yields subadditivity if we choose $C_\mu > C$. Property (ii) in Definition 5.3 is trivial since $\mu(I,\omega)$ is always nonnegative. By Theorem 5.4 there exists $\phi^L_{ij}(\omega;\nu)$ such that almost surely, for rational directions $\nu\in S^{k-1}$, it holds that

$$\phi^L_{ij}(\omega;\nu)=\lim_{n\to+\infty}\frac{1}{(2Nn)^{k-1}}m_1^{L,L}(\omega)(u^{ij}_{0,\nu},Q_\nu(0,2Nn)),$$

where we used that the term $C_\mu\mathcal{H}^{k-2}(\partial I)$ is negligible for the limit.

Substep 2.2: from integer sequences to all sequences. Next we consider an arbitrary sequence $t_n\to+\infty$. From the previous step we know that

$$\phi^L_{ij}(\omega;\nu)=\lim_{n\to+\infty}\frac{1}{(2N\lfloor t_n\rfloor)^{k-1}}m_1^{L,L}(\omega)(u^{ij}_{0,\nu},Q_\nu(0,2N\lfloor t_n\rfloor))$$

exists almost surely. To shorten notation we set $\Lambda_n=2Nt_n$ and $\lambda_n=2N\lfloor t_n\rfloor$. For n large enough, we can apply Lemma 5.6 to the cube $Q_\nu(0,\Lambda_n)$ and singleton family $\{Q_\nu(0,\lambda_n)\}$ and obtain

$$\begin{aligned}&m_1^{L,L}(\omega)(u^{ij}_{0,\nu},Q_\nu(0,\Lambda_n))\\&\quad\le m_1^{L,L}(\omega)(u^{ij}_{0,\nu},Q_\nu(0,\lambda_n))+\mathcal{H}^{k-2}\big(\partial(Q_\nu(0,\lambda_n))\cap\{\nu\}^\perp\big)+C\mathcal{H}^{k-1}\big((Q_\nu(0,\Lambda_n)\setminus\overline{Q_\nu(0,\lambda_n)})\cap\{\nu\}^\perp\big)\\&\quad\le m_1^{L,L}(\omega)(u_{0,\nu},Q_\nu(0,\lambda_n))+C\Lambda_n^{k-2},\end{aligned}$$

which yields

$$\limsup_{j\to+\infty}\frac{1}{\Lambda_n^{k-1}}m_1^{L,L}(\omega)(u^{ij}_{0,\nu},Q_\nu(0,\Lambda_n))\le\phi^L_{ij}(\omega;\nu).\tag{5-39}$$

Similarly, one can prove that

$$\phi^L_{ij}(\omega;\nu)\le\liminf_{n\to+\infty}\frac{1}{\Lambda_n^{k-1}}m_1^{L,L}(\omega)(u^{ij}_{0,\nu},Q_\nu(0,\Lambda_n)).\tag{5-40}$$

Combining (5-39) and (5-40) yields almost surely the existence of the limit for arbitrary sequences.

Substep 2.3: shift invariance in the probability space. Up to neglecting a countable union of null sets, we may assume that the limit defining $\phi^L_{ij}(\omega;\nu)$ exists for all rational directions ν. We next prove that the function $\omega\mapsto\phi^L_{ij}(\omega;\nu)$ is invariant under the entire group action $\{\tau_z\}_{z\in\mathbb{Z}^k}$. This will be important to treat the ergodic case but also for the shift invariance in the physical space. Given $z\in\mathbb{Z}^k$ there exists $R=R(L,z)>0$ such that for all $t>0$

$$Q_\nu(0,t)\subset Q_\nu(-z,R+t),\quad 2L\le\operatorname{dist}\big(\partial Q_\nu(0,t),\partial Q_\nu(-z,R+t)\big).\tag{5-41}$$

Similar to the stationarity of the stochastic process we have

$$\begin{aligned}\phi^L_{ij}(\tau_z\omega;\nu)&\le\limsup_{t\to+\infty}\frac{1}{(R+t)^{k-1}}m_1^{L,L}(\omega)(u^{ij}_{-z,\nu},Q_\nu(-z,R+t))\\&=\limsup_{t\to+\infty}\frac{1}{t^{k-1}}m_1^{L,L}(\omega)(u^{ij}_{-z,\nu},Q_\nu(-z,R+t)).\end{aligned}$$

Due to (5-41) we can apply Lemma 5.6 to the cube $Q_\nu(-z, R+t)$ and the singleton family $\{Q_\nu(0,t)\}$ and deduce that there exists a constant $C = C(R, z)$ such that

$$m_1^{L,L}(\omega)(u^{ij}_{-z,\nu}, Q_\nu(-z, R+t)) \leq m_1^{L,L}(\omega)(u^{ij}_{0,\nu}, Q_\nu(0,t)) + Ct^{k-2}.$$

Hence we get $\phi^L_{ij}(\tau_z\omega;\nu) \leq \phi^L_{ij}(\omega;\nu)$. The other inequality can be proven similarly so that the limit indeed exists (which we implicitly assumed with our notation) and, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\phi^L_{ij}(\tau_z\omega;\nu) = \phi^L_{ij}(\omega;\nu). \tag{5-42}$$

Step 3: shift invariance in the physical space. In this step we prove the existence of the limit defining $\phi^L_{ij}(\omega;\nu)$ when we blow up a cube not centered in the origin. We further show that it agrees with the one already considered. We start with considering a cube $Q_\nu(x,\rho)$ with rational direction ν, $x \in \mathbb{Z}^k\backslash\{0\}$ and $\rho \in \mathbb{Q}$. Given $\varepsilon > 0$ and $N \in \mathbb{N}$ (not the same one of Step 2.1) we define the events

$$\mathcal{Q}_N := \Big\{\omega \in \Omega : \sup_{t \geq N/2} \big|(t\rho)^{1-k} m_1^{L,L}(\omega)(u^{ij}_{0,\nu}, Q_\nu(0, t\rho)) - \phi^L_{ij}(\omega;\nu)\big| \leq \varepsilon\Big\}.$$

By Step 2 we know that the function $\mathbb{1}_{\mathcal{Q}_N}$ converges almost surely to $\mathbb{1}_\Omega$ when $N \to +\infty$. Denote by $\mathcal{J}_x$ the σ-algebra of invariant sets for the measure-preserving map τ_x. Fatou's lemma for the conditional expectation yields

$$\mathbb{1}_\Omega = \mathbb{E}[\mathbb{1}_\Omega|\mathcal{J}_x] \leq \liminf_{N\to+\infty} \mathbb{E}[\mathbb{1}_{\mathcal{Q}_N}|\mathcal{J}_x]. \tag{5-43}$$

By (5-43), given $\delta > 0$, almost surely we find $N_0 = N_0(\omega,\delta)$ such that

$$1 \geq \mathbb{E}[\mathbb{1}_{\mathcal{Q}_{N_0}}|\mathcal{J}_x](\omega) \geq 1-\delta.$$

Now due to Birkhoff's ergodic theorem, almost surely, there exists $n_0 = n_0(\omega,\delta)$ such that, for any $n \geq \frac{1}{2}n_0$,

$$\left|\frac{1}{n}\sum_{l=1}^{n} \mathbb{1}_{\mathcal{Q}_{N_0}}(\tau_{lx}\omega) - \mathbb{E}[\mathbb{1}_{\mathcal{Q}_{N_0}}|\mathcal{J}_x](\omega)\right| \leq \delta.$$

Note that the set we exclude will be a countable union of null sets provided $\varepsilon \in \mathbb{Q}$.

For fixed $n \geq \max\{n_0, N_0\}$ we denote by R the maximal integer such that for all $l = n+1, \dots, n+R$ we have $\tau_{lx}(\omega) \notin \mathcal{Q}_{N_0}$. In order to bound R, let $\tilde{n}$ be the number of ones in the sequence $\{\mathbb{1}_{\mathcal{Q}_{N_0}}(\tau_{lx}(\omega))\}_{l=1}^n$. By the definition of R we have

$$\delta \geq \left|\frac{\tilde{n}}{n+R} - \mathbb{E}[1_{\mathcal{Q}_{N_0}}|\mathcal{J}_x](\omega)\right| = \left|1 - \mathbb{E}[1_{\mathcal{Q}_{N_0}}|\mathcal{J}_x](\omega) + \frac{\tilde{n}-n-R}{n+R}\right| \geq \frac{R+n-\tilde{n}}{n+R} - \delta.$$

Since $n - \tilde{n} \geq 0$ and without loss of generality $\delta \leq \frac{1}{4}$, this provides an upper bound by $R \leq 4n\delta$.

So for any $n \geq \max\{n_0, N_0\}$ and $\widetilde{R} = 6n\delta$, we find $l_n \in [n+1, n+\widetilde{R}]$ such that $\tau_{l_n x}(\omega) \in \mathcal{Q}_{N_0}$. Then by (5-42) and stationarity we have for all $t \geq \frac{1}{2}N_0$ that

$$\big|(t\rho)^{1-k} m_1^{L,L}(\omega)(u^{ij}_{-l_n x,\nu}, Q_\nu(-l_n x, t\rho)) - \phi^L_{ij}(\omega;\nu)\big| \leq \varepsilon. \tag{5-44}$$

Define $\beta_n = n + c_L\rho^{-1}|x|(l_n - n)$, where $c_L \in \mathbb{N}$ is chosen such that $Q_\nu(-nx, n\rho) \subset Q_\nu(-l_n x, \beta_n\rho)$ and $\operatorname{dist}\big(\partial Q_\nu(-nx, n\rho), \partial Q_\nu(-l_n x, \beta_n\rho)\big) > L$. Observe that such c_L exists as $l_n - n \geq 1$. Then each face of the cube $Q_\nu(-nx, n\rho)$ has at most distance $(\beta_n - n)\rho = c_L|x|(l_n - n)$ to the corresponding face in $Q_\nu(-l_n x, \beta_n\rho)$. Then, for n large enough, we can apply Lemma 5.6 to the cube $Q(-l_n x, \beta_n\rho)$ and the singleton family $\{Q_\nu(-nx, n\rho)\}$ to obtain

$$\begin{aligned}\frac{m_1^{L,L}(\omega)(u^{ij}_{-l_n x,\nu}, Q_\nu(-l_n x, \beta_n\rho))}{(\beta_n\rho)^{k-1}} &\leq \frac{m_1^{L,L}(\omega)(u^{ij}_{-nx,\nu}, Q_\nu(-nx, n\rho))}{(\beta_n\rho)^{k-1}} + C\widetilde{R}(\beta_n\rho)^{-1}\\ &\leq \frac{m_1^{L,L}(\omega)(u^{ij}_{-nx,\nu}, Q_\nu(-nx, n\rho))}{(n\rho)^{k-1}} + 6C\delta. \qquad (5\text{-}45)\end{aligned}$$

On the other hand we can define $\theta_n = n - c'_L\rho^{-1}|x|(l_n - n)$ for a suitable $c'_L \in \mathbb{N}$ and deduce from a similar reasoning that

$$\frac{m_1^{L,L}(\omega)(u^{ij}_{-nx,\nu}, Q_\nu(-nx, n\rho))}{(n\rho)^{k-1}} \leq \frac{m_1^{L,L}(\omega)(u^{ij}_{-l_n x,\nu}, Q_\nu(-l_n x, \theta_n\rho))}{(\theta_n\rho)^{k-1}} + 6C\delta. \qquad (5\text{-}46)$$

Now if δ is small enough (depending only on x, L and ρ) we have $\beta_n \geq \theta_n \geq \frac{1}{2}n \geq \frac{1}{2}N_0$. Combining (5-45),(5-46) and (5-44) we infer

$$\limsup_{n\to+\infty}\left|\frac{m_1^{L,L}(\omega)(u^{ij}_{-nx,\nu}, Q_\nu(-nx, n))}{n^{k-1}} - \phi^L_{ij}(\omega;\nu)\right| \leq 6C\delta + \varepsilon,$$

which yields the claim in (5-36) for $Q_\nu(x,\rho)$ with $x \in \mathbb{Z}^k$ and rational ν and ρ. The extension to arbitrary sequences $t_n \to +\infty$ (and thus to rational centers x) can be achieved again by Lemma 5.6, comparing first the minimal energy on the two cubes $Q_\nu(\lfloor t_n\rfloor x, \lfloor t_n\rfloor\rho)$ and $Q_\nu(\lfloor t_n\rfloor x, t_n\rho)$, similar to Substep 2.2, and then the energy on the latter cube with the one on $Q_\nu(t_n x, t_n\rho)$, as in Substep 2.3. Eventually the convergence of irrational ρ follows from the estimate

$$m_1^{L,L}(\omega)(u^{ij}_{t_n x,\nu}, Q_\nu(t_n x, t_n\rho)) \leq m_1^{L,L}(\omega)(u^{ij}_{t_n x,\nu}, Q_\nu(t_n x, t_n(\rho-\delta)) + Ct_n\delta(t_n\rho)^{k-2},$$

which is a consequence of Lemma 5.6 applied to the cube $Q_\nu(t_n x, t_n\rho)$ and $\{Q_\nu(t_n x, t_n(\rho-\delta))\}$, when one neglects lower-order terms. Choosing $0 < \delta_l \to 0$ such that $\rho - \delta_l \in \mathbb{Q}$ then yields

$$\limsup_n \frac{m_1^{L,L}(\omega)(u^{ij}_{t_n x,\nu}, Q_\nu(t_n x, t_n\rho))}{(t_n\rho)^{k-1}} \leq \phi^L_{ij}(\omega;\nu).$$

Using the same argument for the cube $Q_\nu(t_n x, t_n(\rho+\delta))$ and the family $\{Q_\nu(t_n x, t_n\rho)\}$, we find that the limit exists and agrees with $\phi^L_{ij}(\omega;\nu)$. Finally, for irrational centers we can again use a perturbation argument based on Lemma 5.6 as we did for proving (5-45) and (5-46). We omit the details.

Step 4: from rational to irrational directions. Now we extend the convergence from rational directions to all $\nu \in S^{k-1}$. As the argument is purely geometric similar to Lemma 5.6, we assume without loss of generality that $x = 0$. First note that the set of rational directions is dense in S^{k-1} (as the inverse of the stereographic projection maps rational points to rational directions). Given $\nu \in S^{k-1}$ and a sequence $t_n \to +\infty$ we define

$$\bar{\phi}^L_{ij}(\omega;\nu) = \limsup_{n\to+\infty} \frac{1}{t_n^{k-1}} m_1^{L,L}(\omega)(u^{ij}_{0,\nu}, Q_\nu(0,t_n)),$$
$$\underline{\phi}^L_{ij}(\omega;\nu) = \liminf_{n\to+\infty} \frac{1}{t_n^{k-1}} m_1^{L,L}(\omega)(u^{ij}_{0,\nu}, Q_\nu(0,t_n)).$$

Let $\nu \in S^{k-1}\backslash\mathbb{Q}^k$. By the construction of the matrix A_ν in Substep 2.1 we can assume that there exists a sequence of rational directions ν_l such that $A_{\nu_l} \to A_\nu$. Therefore, given $\delta > 0$ we find $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$ the following properties hold:

(i) $Q_\nu(0,(1-2\delta)) \Subset Q_{\nu_l}(0,1-\delta) \Subset Q_\nu(0,1)$.

(ii) $0 < \mathrm{d}_{\mathcal{H}}(\{\nu\}^\perp \cap B_2(0), \{\nu_l\}^\perp \cap B_2(0)) \leq \delta$.

For a fixed $l \geq l_0$ and $n \in \mathbb{N}$, we let $u_{n,l} : \mathcal{L}(\omega) \to \mathcal{S}$ be an admissible minimizer for the problem $m_1^{L,L}(\omega)(u^{ij}_{0,\nu_l}, Q_{\nu_l}(0,(1-\delta)t_n))$. We define a test function $v_n : \mathcal{L}(\omega) \to \mathcal{S}$ setting

$$v_n(x) := \begin{cases} u_{n,l}(x) & \text{if } x \in Q_{\nu_l}(0,(1-\delta)t_n), \\ u_{0,\nu}(x) & \text{otherwise.} \end{cases}$$

Note that if $P_k(x), P_k(y) \in Q_\nu(0,t_n)\backslash Q_{\nu_l}(0,(1-\delta)t_l)$ are such that $|x-y| \leq L$ and $v_n(x) \neq v_n(y)$, then by the choice of l_0 and (i), for l large enough we have

$$\mathrm{dist}\big(P_k(x), \big(Q_\nu(0,t_n)\backslash Q_\nu(0,(1-2\delta)t_n)\big) \cap \{\nu\}^\perp\big) \leq L. \tag{5-47}$$

If $P_k(x) \in Q_\nu(0,t_n)\backslash Q_{\nu_l}(0,(1-\delta)t_n)$ and $P_k(y) \in Q_{\nu_l}(0,(1-\delta)t_n)$ with $|x-y| \leq L$ and $v_n(x) \neq v_n(y)$, then, for l large enough one can show that by (ii) either $P_k(x)$ or $P_k(y)$ must lie in the cone

$$\mathcal{K}(\nu,\nu_l) = \{x \in \mathbb{R}^k : \langle x,\nu\rangle \cdot \langle x,\nu_l\rangle \leq 0\}.$$

As the segment $[P_k(x), P_k(y)]$ intersects $\partial Q_{\nu_l}(0,(1-\delta)t_n)$, we conclude that

$$\mathrm{dist}\big(P_k(x), (\mathcal{K}(\nu,\nu_l) + B_L(0)) \cap \partial Q_{\nu_l}(0,(1-\delta)t_n)\big) \leq L. \tag{5-48}$$

By (i) it holds that $v_n \in \mathcal{PC}^L_{1,u^{ij}_{0,\nu}}(\omega, Q_\nu(0,t_n))$ for n large enough. From (5-47), (5-48) and the choice of l_0 we deduce that for l large enough

$$m_1^{L,L}(\omega)(u^{ij}_{0,\nu}, Q_\nu(0,t_n) \leq m_1^{L,L}(\omega)\big(u^{ij}_{0,\nu_l}, Q_{\nu_n}(0,(1-\delta)t_n)\big) + C\delta t_n^{k-1}.$$

Dividing the last inequality by t_n^{k-1} and taking the lim sup as $n \to +\infty$, we deduce

$$\bar{\phi}^L_{ij}(\omega;\nu) \leq \phi^L_{ij}(\omega;\nu_l) + C\delta.$$

Letting first $l \to +\infty$ and then $\delta \to 0$ yields $\bar{\phi}^L_{ij}(\omega;\nu) \leq \liminf_l \phi^L_{ij}(\omega;\nu_l)$. By a similar argument we can also prove that $\limsup_l \phi^L_{ij}(\omega;\nu_l) \leq \underline{\phi}^L_{ij}(\omega;\nu)$. Hence, we get almost surely the existence of the limit in (5-36) for all directions ν and the limit does not depend on x, ρ or the sequence t_n.

Step 5: proof of (5-35). We claim that $\phi^L_{ij}(\omega;\nu)=\phi^L_{\rm hom}(\omega;s_i,s_j,\nu)$. By the preceding steps this concludes the proof. First observe that by monotonicity it is enough to show that $\phi^L_{\rm hom}(\omega;s_i,s_j,\nu)\le\phi^L_{ij}(\omega;\nu)$. Let $t_n\to+\infty$ and fix a cube $Q_\nu(x,\rho)$. By a trivial extension argument, for η small enough (depending on ρ) it holds that

$$m_1^{\eta t_n,L}(\omega)(u^{ij}_{t_nx,\nu},Q(t_nx,t_n\rho))\le m_1^{L,L}(\omega)(u^{ij}_{t_nx,\nu},Q(t_nx,t_n\rho-\eta t_n))+C\eta t_n^{k-1}.$$

Dividing by $(t_n\rho)^{k-1}$ and letting first $n\to+\infty$ and then $\eta\to0$ we obtain the claim.

When the group action is ergodic, the additional statement in Theorem 5.8 follows from (5-42) since in this case all the functions $\omega\mapsto\phi^L_{ij}(\omega;\nu)$ are constant and so is the pointwise limit when $L\to+\infty$. □

Remark 5.9. One can show that the surface tension can be obtained by one single limit procedure. Indeed, referring to (4-22) and repeating Steps 1 and 5 of the proof of Theorem 5.8, it follows that

$$\phi_{\rm hom}(\omega;s_i,s_j,\nu)=\lim_{t\to+\infty}\frac{1}{t^{k-1}}\inf\Bigl\{E_1(\omega)(u,Q_\nu(0,t)):u\in\mathcal{PC}^{l_{1/t}}_{1,u^{ij}_{0,\nu}}(\omega,Q_\nu(0,t))\Bigr\}.$$

6. Volume constraints in the stationary case

In this section we will discuss the variational limit of the energies $E_\varepsilon(\omega)$ when, for all $i=1,\dots,q$, we fix the number of lattice points where the configuration takes the value s_i. For general thin admissible lattices this energy might not converge without passing to a further subsequence, so we treat only the case of stationary lattices in the sense of Definition 5.5. In order to formulate the result, given $A\in\mathcal{A}^R(D)$ and a family $V_\varepsilon=\{V_{i,\varepsilon}\}_{i=1}^q\in\mathbb{N}^q$, we introduce the class

$$\mathcal{PC}^{V_\varepsilon}_\varepsilon(\omega):=\bigl\{u:\varepsilon\mathcal{L}(\omega)\to\mathcal{S}:\#\{\varepsilon x\in\varepsilon\mathcal{L}(\omega)\cap P_k^{-1}D:u(\varepsilon x)=s_i\}=V_{i,\varepsilon}\bigr\}.$$

Beside the natural compatibility condition $\sum_i V_{i,\varepsilon}=\#(\varepsilon\mathcal{L}(\omega)\cap P_k^{-1}D)$, we assume that for all $i=1,\dots,q$ there exists $V_i>0$ such that

$$\lim_{\varepsilon\to0}\frac{V_{i,\varepsilon}}{\#(\varepsilon\mathcal{L}\cap P_k^{-1}D)}=V_i.$$

Note that we exclude the case $V_i=0$ for some i. This case contains some nontrivial aspects which are related to the concept of (B)-convexity studied in [Ambrosio and Braides 1990b]. Such conditions are not necessarily satisfied by our discrete energies. Of course the extreme case $V_{i,\varepsilon}=0$ for all $\varepsilon>0$ can be treated by changing the set $\mathcal{S}$ and thus the whole model.

The following lemma describes how the volume constraint behaves for sequences with finite energy.

Lemma 6.1. *For $\mathbb{P}$-almost all $\omega\in\Omega$ the following statement holds true: for all $u\in\mathrm{BV}(D,\mathcal{S})$ such that there exists a sequence $u_\varepsilon:\varepsilon\mathcal{L}(\omega)\to\mathcal{S}$ with $u_\varepsilon\to u$ in the sense of Definition 2.2 and*

$$\sup_{\varepsilon>0}E_\varepsilon(\omega)(u_\varepsilon)\le C,\qquad\lim_{\varepsilon\to0}\frac{\#\{\varepsilon x\in\varepsilon\mathcal{L}(\omega)\cap P_k^{-1}D:u_\varepsilon(\varepsilon x)=s_i\}}{\#\{\varepsilon x\in\varepsilon\mathcal{L}(\omega)\cap P_k^{-1}D\}}=V_i',$$

we have

$$|\{u=s_i\}|=V_i'|D|.$$

Proof. Up to the transformation $T(s_i) = e_i$ we may assume that the vectors s_i form a basis. For $\omega \in \Omega$ we consider the sequence of nonnegative Borel measures $\gamma_\varepsilon(\omega)$ on D defined as

$$\gamma_\varepsilon(\omega) = \sum_{z \in P_k(\mathcal{L}(\omega)) \cap D/\varepsilon} \varepsilon^k \#(P_k^{-1}(z) \cap \mathcal{L}(\omega))\delta_{\varepsilon z}.$$

As $\gamma_\varepsilon(\omega)(D) \leq C|D|$, up to subsequences we know that $\gamma_\varepsilon(\omega) \overset{*}{\rightharpoonup} \gamma(\omega)$ in the sense of measures. We now identify the limit measure. To this end we define a discrete stochastic process $\gamma : \mathcal{I}_k \to L^1(\Omega)$ as

$$\gamma(I)(\omega) := \sum_{y \in P_k(\mathcal{L}(\omega)) \cap I} \#(P_k^{-1}(y) \cap \mathcal{L}(\omega)) = \#(x \in \mathcal{L}(\omega) : P_k(x) \in I). \tag{6-49}$$

It follows from (2-4) that $\gamma(I)$ is essentially bounded for every $I \in \mathcal{I}_k$. In addition it can be checked that $\gamma(I)$ is $\mathcal{F}$-measurable; thus we infer that $\gamma(I) \in L^\infty(\Omega)$. Upon redefining the group action as $\tilde{\tau}_z = \tau_{-z}$, the process γ is stationary and (sub)additive. By Theorem 5.4 there exists $\gamma_0(\omega)$ such that for almost every $\omega \in \Omega$ and all $I \in \mathcal{I}_k$ we have

$$\lim_{n \to +\infty} \frac{\gamma(nI)(\omega)}{n^k |I|} = \gamma_0(\omega).$$

It is straightforward to extend this result to all sequences $t_n \to +\infty$ and then to all cubes in $\mathbb{R}^k$ by a continuity argument. Now let $a, b \in \mathbb{R}^k$ and let $Q = [a, b)$. Then by definition

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon(\omega)(Q) = \lim_{\varepsilon \to 0} \sum_{z \in P_k(\mathcal{L}(\omega)) \cap (1/\varepsilon) Q} \varepsilon^k \#(P_k^{-1}(z) \cap \mathcal{L}(\omega)) = \gamma_0(\omega)|Q|. \tag{6-50}$$

Given any open set $A \in \mathcal{A}(D)$, for $\delta > 0$ we consider the following interior approximation:

$$A_{\text{int}}(\delta) = \bigcup_{z \in \delta\mathbb{Z}^k : z + [0,\delta)^k \subset A} z + [0, \delta)^k.$$

It can be checked by monotone convergence that $\lim_{\delta \to 0} |A(\delta)| = |A|$. By (6-50) and additivity we obtain

$$\liminf_{\varepsilon \to 0} \gamma_\varepsilon(\omega)(A) \geq \liminf_{\varepsilon \to 0} \gamma_\varepsilon(\omega)(A(\delta)) = \gamma_0(\omega)|A(\delta)|.$$

Letting $\delta \to 0$ we obtain $\liminf_\varepsilon \gamma_\varepsilon(\omega)(A) \geq \gamma_0(\omega)|A|$. By the portmanteau theorem we conclude that $\gamma(\omega)(B) = \gamma_0(\omega)|B|$ for all Borel sets $B \subset D$. In particular the whole sequence converges in the sense of measures. On the other hand, if $A \in \mathcal{A}(D)$ is such that $|\partial A| = 0$, then the outer approximation

$$A_{\text{out}}(\delta) = \bigcup_{z \in \delta\mathbb{Z}^k : z + [0,\delta)^k \cap A \neq \varnothing} z + [0, \delta)^k$$

also fulfills $\lim_{\delta \to 0} |A(\delta)| = |A|$; hence

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon(\omega)(A) = \gamma_0(\omega)|A| \tag{6-51}$$

for all $A \in \mathcal{A}(D)$ such that $|\partial A| = 0$. Given now $\delta > 0$, we take any polyhedral function $u_\delta \in \mathrm{BV}_{\text{loc}}(\mathbb{R}^k, \mathcal{S})$ such that $\|u - u_\delta\|_{L^1(D)} \leq \delta$. As u_δ is Borel-measurable, we have

$$\int_D P u_\varepsilon \, \mathrm{d}\gamma_\varepsilon(\omega) = \int_D (P u_\varepsilon - u_\delta) \, \mathrm{d}\gamma_\varepsilon(\omega) + \int_D u_\delta \, \mathrm{d}\gamma_\varepsilon(\omega).$$

Since u_δ is a polyhedral function, we can use (6-51) to obtain

$$\lim_{\varepsilon\to 0}\int_D u_\delta\,\mathrm{d}\gamma_\varepsilon(\omega)=\gamma_0(\omega)\int_D u_\delta\,\mathrm{d}x. \tag{6-52}$$

Concerning the first term, by (2-2) and the regularity of S_{u_δ} and ∂D we have

$$\left|\int_D (Pu_\varepsilon-u_\delta)\,\mathrm{d}\gamma_\varepsilon(\omega)\right|\le C\sum_{z\in P_k(\mathcal{L}(\omega))\cap D/\varepsilon}\varepsilon^k|Pu_\varepsilon(\varepsilon z)-u_\delta(\varepsilon z)|. \tag{6-53}$$

Now using the fact that u_ε has equibounded energy, one can reason as in the proof of Lemma 2.5 to show that

$$\limsup_{\varepsilon\to 0}\sum_{z\in P_k(\mathcal{L}(\omega))\cap D/\varepsilon}\varepsilon^k|Pu_\varepsilon(\varepsilon z)-u_\delta(\varepsilon z)|\le C\|u-u_\delta\|_{L^1(D)}\le C\delta.$$

Combining the above inequality with (6-52) and (6-53) we finally obtain by the arbitrariness of δ that

$$\lim_{\varepsilon\to 0}\int_D Pu_\varepsilon\,\mathrm{d}\gamma_\varepsilon(\omega)=\gamma_0(\omega)\int_D u\,\mathrm{d}x=\gamma_0(\omega)\sum_{i=1}^q s_i|\{u=s_i\}|$$

On the other hand, plugging in the definition and using again (6-51), it holds

$$\lim_{\varepsilon\to 0}\int_D Pu_\varepsilon\,\mathrm{d}\gamma_\varepsilon(\omega)=\lim_{\varepsilon\to 0}\sum_{i=1}^q s_i\#\big\{\varepsilon x\in\varepsilon\mathcal{L}(\omega)\cap D:\ u_\varepsilon(\varepsilon x)=s_i\big\}\varepsilon^k=\sum_{i=1}^q s_i V_i'|D|\gamma_0(\omega).$$

Since we assumed the s_i form a basis, we conclude the proof. □

In order to include the volume constraint in the functional, for almost every $\omega\in\Omega$ we introduce $E_\varepsilon^{V_\varepsilon}(\omega):\mathcal{PC}_\varepsilon(\omega)\to[0,+\infty]$ as

$$E_\varepsilon^{V_\varepsilon}(\omega)(u)=\begin{cases}E_\varepsilon(\omega)(u) & \text{if } u\in\mathcal{PC}_\varepsilon^{V_\varepsilon}(\omega),\\ +\infty & \text{otherwise.}\end{cases}$$

With the help of Lemma 6.1 we can now prove the following theorem.

Theorem 6.2. *Let $\mathcal{L}$ be a stationary stochastic lattice and let f_{nn} and f_{lr} satisfy Hypotheses 1 and 2. For $\mathbb{P}$-almost every ω the functionals $E_\varepsilon^{V_\varepsilon}(\omega)$ Γ-converge with respect to the convergence of Definition 2.2 to the functional $E_{\mathrm{hom}}^V(\omega):L^1(D,\mathbb{R}^q)\to[0,+\infty]$ defined by*

$$E_{\mathrm{hom}}^V(\omega)(u)=\begin{cases}\int_{S_u}\phi_{\mathrm{hom}}(\omega;u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{k-1} & \text{if } u\in\mathrm{BV}(D,\mathcal{S}) \text{ and } |\{u=s_i\}|=V_i|D| \text{ for all } i,\\ +\infty & \text{otherwise.}\end{cases}$$

Proof. The lower bound follows from Theorem 5.8 and Lemma 6.1. In order to prove the upper bound, for the moment assume that $u\in\mathrm{BV}(D,\mathcal{S})$ satisfies the volume constraint and that each level set $\{u=s_i\}$ contains an interior point. In particular, in each level set we find q disjoint open balls $B_\eta(x_i^l)\Subset\{u=s_i\}$ with $\eta\ll 1$. By Theorem 5.8 we can find a sequence $u_\varepsilon:\varepsilon\mathcal{L}(\omega)\to\mathcal{S}$ such that u_ε converges to u in the sense of Definition 2.2 and

$$\lim_{\varepsilon\to 0}E_\varepsilon(\omega)(u_\varepsilon)=E_{\mathrm{hom}}(\omega)(u). \tag{6-54}$$

Repeating the argument used for proving Proposition 3.6, one can show that without loss of generality we may assume that $u_\varepsilon(\varepsilon x) = s_i$ for all $\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap B_\eta(x_i^l)$ and that u_ε has equibounded energy on a large cube Q_D containing $\overline{D}$. For each i set $\widetilde{V}_{i,\varepsilon} = \#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap P_k^{-1}D : u_\varepsilon(\varepsilon x) = s_i\}$. Applying Lemma 6.1 we deduce that

$$\lim_{\varepsilon\to 0} \frac{\widetilde{V}_{i,\varepsilon} - V_{i,\varepsilon}}{\#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap P_k^{-1}D\}} = 0. \tag{6-55}$$

We now adjust the sequence u_ε so that it belongs to $\mathcal{PC}_\varepsilon^{V_\varepsilon}(\omega)$. This will be done locally on the balls $B_\eta(x_i^l)$. First we change the values on $B_\eta(x_1^1)$ and $B_\eta(x_2^1)$ so that the sequence satisfies the constraint for $i = 1$. In general, for $i < q$ we change the sequence on $B_\eta(x_i^i)$ and $B_\eta(x_{i+1}^i)$ so that it satisfies the constraints for all $j \le i$. At the end the constraint for $i = q$ follows by the compatibility assumption. Each modification will be such that L^1-convergence and convergence of the energies is conserved. We will provide the construction only for the first step. In what follows we consider the case $\widetilde{V}_{1,\varepsilon} > V_{1,\varepsilon}$. We set $h_\varepsilon = (\widetilde{V}_{1,\varepsilon} - V_{1,\varepsilon})^{1/k}$. Up to modifying u_ε on a set of lattice points contained in the complement of the union of the balls $B_\eta(x_i^l)$ and with diverging cardinality much less than ε^{1-k}, we may assume that $h_\varepsilon \to +\infty$. Note that such a modification still yields a recovery sequence.

Observe that (6-55) and the properties of a thin admissible lattice imply

$$\lim_{\varepsilon\to 0} h_\varepsilon \varepsilon = 0. \tag{6-56}$$

We already know from the proof of Lemma 6.1 that, almost surely, we can write

$$q^\omega(x_1^1, h_\varepsilon) := \#\big\{x \in \mathcal{L}(\omega) : P_k(x) \in Q_{e_1}(x_1^1, \gamma_0(\omega)^{-1}h_\varepsilon)\big\} = h_\varepsilon^k + h_\varepsilon^{k-1}\gamma_\varepsilon$$

for some sequence $\gamma_\varepsilon = \gamma_\varepsilon(\omega, x_1^1)$ such that $\lim_{\varepsilon\to 0}(\gamma_\varepsilon/h_\varepsilon) = 0$. In the following we assume that $\gamma_\varepsilon \le 0$, but with a similar argument we can also treat the case $\gamma_\varepsilon > 0$. As $\mathcal{L}(\omega)$ is thin admissible in the sense of Definition 2.1, one can show that for some appropriate $c = c(R) > 0$ it holds true that

$$\frac{1}{C} h_\varepsilon^{k-1} \le q^\omega(x_0, h_\varepsilon + n + c) - q^\omega(x_0, h_\varepsilon + n) \le C h_\varepsilon^{k-1}$$

for any $0 \le n \le h_\varepsilon$. In particular, there exist $n_\varepsilon = \mathcal{O}(\gamma_\varepsilon)$ and nonnegative equibounded c_ε such that

$$q^\omega(x_0, h_\varepsilon + n_\varepsilon) = h_\varepsilon^k + c_\varepsilon h_\varepsilon^{k-1}. \tag{6-57}$$

Now choose any set $G_\varepsilon \subset \mathbb{R}^d$ such that $P_k G_\varepsilon \subset B_\eta(x_2^1)$ and $\#(G_\varepsilon \cap \mathcal{L}(\omega)) = c_\varepsilon h_\varepsilon^{k-1}$. To reduce notation, set $Q_\varepsilon := Q_{e_1}(x_1^1, \gamma_0(\omega)^{-1}\varepsilon(h_\varepsilon + n_\varepsilon))$. We define

$$\bar{u}_\varepsilon(\varepsilon x) = \begin{cases} s_2 & \text{if } \varepsilon P_k(x) \in Q_\varepsilon, \\ s_1 & \text{if } \varepsilon x \in G_\varepsilon, \\ u_\varepsilon(\varepsilon x) & \text{otherwise.} \end{cases}$$

Note that by (6-56) we have $Q_\varepsilon \Subset B_\eta(x_1^1)$ for ε small enough and therefore

$$\#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap P_k^{-1}D : \bar{u}(\varepsilon x) = s_1\} = V_{1,\varepsilon}.$$

Again by (6-56) we still have $\bar{u}_\varepsilon \to u$ in the sense of Definition 2.2. From Hypothesis 1 we deduce

$$E_\varepsilon(\omega)(\bar{u}_\varepsilon) \le E_\varepsilon(\omega)(u_\varepsilon) + C \sum_{\xi \in r'\mathbb{Z}^d_M} J_{lr}(|\hat{\xi}|)\#(G_\varepsilon \cap \varepsilon\mathcal{L}(\omega))\varepsilon^{k-1} + \sum_{\xi \in r'\mathbb{Z}^d_M} \sum_{\substack{\alpha \in R^\xi_\varepsilon(D) \\ \varepsilon P_k([x_\alpha, x_{\alpha+\xi}]) \cap \partial Q_\varepsilon \neq \varnothing}} \varepsilon^{k-1} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, \bar{u}_\varepsilon(\varepsilon x_\alpha), \bar{u}_\varepsilon(\varepsilon x_{\alpha+\xi})).$$

It remains to bound the last term since the second one vanishes by (6-56) and the integrability of J_{lr}. We split the interactions according to (2-12). By Lemma 2.7 and Hypothesis 1, for ε small enough we have by construction

$$\sum_{|\xi| \le L_\delta} \sum_{\substack{\alpha \in R^\xi_\varepsilon(D) \\ \varepsilon P_k([x_\alpha, x_{\alpha+\xi}]) \cap \partial Q_\varepsilon \neq \varnothing}} \varepsilon^{k-1} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, \bar{u}_\varepsilon(\varepsilon x_\alpha), \bar{u}_\varepsilon(\varepsilon x_{\alpha+\xi}))$$
$$\le C \sum_{|\xi| \le L_\delta} J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ \varepsilon x, \varepsilon y \in B_\eta(x^1_1)}} \varepsilon^{k-1} f_\varepsilon(x, y, \bar{u}_\varepsilon(\varepsilon x), \bar{u}_\varepsilon(\varepsilon y)) \le C\mathcal{H}^{k-1}(\partial Q_\varepsilon) \le C(\varepsilon h_\varepsilon)^{k-1}, \quad \text{(6-58)}$$

so that the left-hand side vanishes when $\varepsilon \to 0$. To control the remaining interactions, recall that u_ε has finite energy on the larger cube Q_D. Hence Lemma 2.7 and Hypothesis 1 yield

$$\sum_{|\xi| > L_\delta} \sum_{\substack{\alpha \in R^\xi_\varepsilon(D) \\ \varepsilon P_k([x_\alpha, x_{\alpha+\xi}]) \cap \partial Q_\varepsilon \neq \varnothing}} \varepsilon^{k-1} f_\varepsilon(x_\alpha, x_{\alpha+\xi}, \bar{u}_\varepsilon(\varepsilon x_\alpha), \bar{u}_\varepsilon(\varepsilon x_{\alpha+\xi}))$$
$$\le C\delta \sum_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ \varepsilon x, \varepsilon y \in Q_D}} \varepsilon^{k-1} f_\varepsilon(x, y, \bar{u}_\varepsilon(\varepsilon x), \bar{u}_\varepsilon(\varepsilon y))$$
$$\le C\delta\big(E_\varepsilon(\omega)(u_\varepsilon, Q_D) + \mathcal{H}^{k-1}(\partial Q_\varepsilon) + \#(G_\varepsilon \cap \varepsilon\mathcal{L}(\omega))\varepsilon^{k-1}\big) \le C\delta.$$

As $\delta > 0$ was arbitrary, we infer from (6-54), (6-58) and (6-58) that

$$\limsup_{\varepsilon \to 0} E_\varepsilon(\omega)(\bar{u}_\varepsilon) = \limsup_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon) = E_{\mathrm{hom}}(\omega)(u).$$

The case when $V'_\varepsilon \le V_\varepsilon$ can be treated by an almost symmetric argument. Repeating this construction for the remaining phases as described at the beginning of this proof, we obtain

$$\Gamma\text{-}\limsup_{\varepsilon \to 0} E^{V_\varepsilon}_\varepsilon(\omega)(u) = E_{\mathrm{hom}}(\omega)(u).$$

Now for a general $u \in \mathrm{BV}(D, \mathcal{S})$ such that $|\{u = s_i\}| = V_i|D|$, the statement follows by density. This procedure is classical, see [Ambrosio and Braides 1990a], and therefore we omit the details. □

7. A model for random deposition

The general homogenization result proved in Section 5 describes only the qualitative phenomenon that interfaces may form on the flat subspace. In this final section we investigate the asymptotic behavior of the limit energy as a function of the average thickness. To simplify matters, we consider a three-dimensional to two-dimensional dimension-reduction problem in which magnetic particles are deposited with vertical order on a two-dimensional flat substrate and interact via finite-range ferromagnetic interactions of Ising-type, which means in particular that $\mathcal{S} = \{\pm 1\}$. We obtain information on the dependence of the limit energy on the average thickness when the latter is very small or very large.

In order to model the substrate where the particles are deposited, we take a two-dimensional deterministic lattice, which we choose for simplicity to be $\mathcal{L}^0 = \mathbb{Z}^2 \times \{0\}$. We then consider an independent random field $\{X_i^p\}_{i\in\mathbb{Z}^3}$, where the X_i^p are Bernoulli random variables with $\mathbb{P}(X_i^p = 1) = p \in (0, 1)$ and, for fixed $M \in \mathbb{N}$, we define the random point set

$$\mathcal{L}_p^M(\omega) := \bigl\{(i_1, i_2, i_3) \in \mathbb{Z}^3 :\ 0 \le i_3 \le \textstyle\sum_{k=1}^M X^p_{(i_1,i_2,k)}(\omega)\bigr\}, \tag{7-59}$$

which means that we successively deposit particles M times independently on the flat lattice $\mathcal{L}^0$ and stack them over each other (see Figure 3). Note that the point set constructed in (7-59) is stationary with respect to integer translations in $\mathbb{Z}^2$ and ergodic by the independence assumption. Given $u : \varepsilon\mathcal{L}_p^M(\omega) \to \{\pm 1\}$, we consider an energy of the form

$$E^p_{\varepsilon,M}(\omega)(u, A) = \sum_{\substack{x,y\in\mathcal{L}_p^M(\omega)\\ P_2(x),P_2(y)\in A/\varepsilon}} \varepsilon c(x-y)|u(\varepsilon x) - u(\varepsilon y)|, \tag{7-60}$$

where the interaction $c : \mathbb{R}^3 \to [0, +\infty)$ fulfills

(i) $c(z) \le C$ for all $z \in \mathbb{R}^3$,

(ii) $c(z) = 0$ if $|z| \ge L$,

(iii) $c(z) \ge c_0 > 0$ if $|z| = 1$.

Remark 7.1. The coefficients above satisfy Hypothesis 2, but in general are not coercive, as required in Hypothesis 1. However, the results obtained in the first part of this paper still hold true. This is due to the vertical order of the deposition model, which makes the proof of coercivity much simpler. However, note that, for instance, the constant in Lemma 2.7 now depends strongly on M.

Due to Remark 7.1 we can apply Theorem 5.8 and thus we know that there exists the effective (deterministic) surface tension

$$\phi^p_{\mathrm{hom}}(M; \nu) := \lim_{t\to+\infty} \frac{1}{t} \inf\bigl\{E^p_{1,M}(\omega)(v, Q_\nu(0,t)) : v(x) = u_{0,\nu}(P_2(x)) \text{ if } \operatorname{dist}(P_2(x), \partial Q_\nu(0,t)) \le 2L\bigr\},$$

where we used the alternative formula in Remark 5.9 and Remark 4.2. Note that, due to symmetry reasons, the surface tension does not depend on the traces; see also [Alicandro et al. 2015].

We are interested in the asymptotic behavior of $\phi^p_{\mathrm{hom}}(M; \nu)$ when $M \to +\infty$. First, we define some auxiliary quantities. Given $p \in (0, 1],\ 0 \le N < M$ and $u : \mathbb{Z}^3 \to \{\pm 1\}$, we set

$$E^p_{[N,M]}(\omega)(u, O) := \sum_{\substack{x,y\in\mathcal{L}_p^M(\omega)\\ x,y\in O\times[N,M]}} c(x-y)|u(x) - u(y)|$$

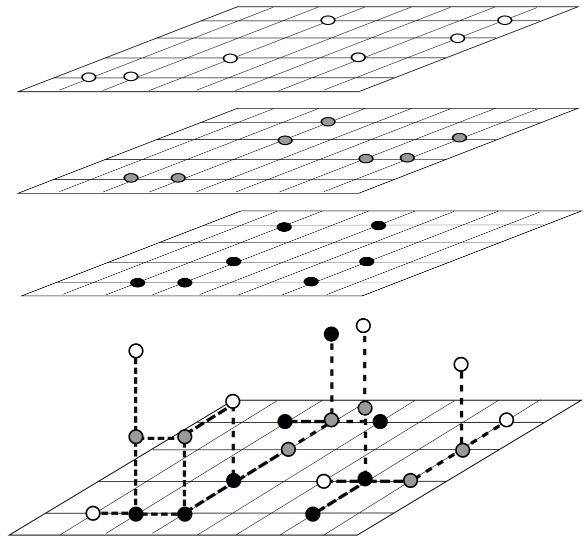

Figure 3. Three successive deposition steps (black, gray and white) in the construction of $\mathcal{L}_p^M(\omega)$. The dashed bonds connect nearest neighboring particles.

and omit the dependence on ω of $E^p_{[N,M]}$ when $p = 1$. In that case, given $\nu \in S^1$ we further introduce the corresponding surface tension

$$\phi^{1,M}(\nu) = \lim_{t\to+\infty} \frac{1}{t} \inf\bigl\{E^1_{[0,M]}(u, Q_\nu(0,t)) : v(x) = u_{0,\nu}(P_2(x)) \text{ if } \operatorname{dist}(P_2(x), \partial Q_\nu(0,t)) \le 2L\bigr\}.$$

Note that the existence of this limit follows by standard subadditivity arguments. The next lemma shows that the auxiliary surface tensions converge when $M \to +\infty$.

Lemma 7.2. *For any $\nu \in S^1$ there exists the limit*

$$\phi^1(\nu) := \lim_{M\to+\infty} \frac{1}{M}\phi^{1,M}(\nu).$$

Proof. We define a sequence $a_k = \phi^{1,k-1}(\nu)$. It is enough to show that a_k is superadditive. To reduce notation, similar to (5-29) we introduce

$$m_{[N,M]}(u_{0,\nu}, Q_\nu(x,\rho)) := \inf\bigl\{E^1_{[N,M]}(u, Q_\nu(x,\rho)) : u \in \mathcal{PC}^{2L}_{1,u_{0,\nu}}(Q_\nu(x,\rho))\bigr\}.$$

Note that by periodicity, $m_{[N,M]}(u_{0,\nu}, Q_\nu(x,\rho)) = m_{[N+k,M+k]}(u_{0,\nu}, Q_\nu(x,\rho))$ for every $k \in \mathbb{N}$. For fixed $t \gg 1$ one can take any admissible configuration for $m_{[0,M+M'-1]}(u_{0,\nu}, Q_\nu(0,t))$ and restrict it to the sets $Q_\nu(0,t) \times [0, M-1]$ and $Q_\nu(0,t) \times [M, M+M'-1]$ to obtain the inequality

$$\begin{aligned}\frac{1}{t} m_{[0,M+M'-1]}(u_{0,\nu}, Q_\nu(0,t)) &\ge \frac{1}{t} m_{[0,M-1]}(u_{0,\nu}, Q_\nu(0,t)) + \frac{1}{t} m_{[M,M+M'-1]}(u_{0,\nu}, Q_\nu(0,t)) \\ &= \frac{1}{t} m_{[0,M-1]}(u_{0,\nu}, Q_\nu(0,t)) + \frac{1}{t} m_{[0,M'-1]}(u_{0,\nu}, Q_\nu(0,t)),\end{aligned}$$

where we neglected the interactions between the two cubes and used periodicity in the last equality. Letting $t \to +\infty$, we obtain superadditivity of the sequence a_k. □

The next result shows the asymptotic behavior of the surface tension when the average number of layers pM diverges.

Proposition 7.3. *Let ϕ^1 be defined as in the previous lemma. For $\nu \in S^1$ it holds that*

$$\lim_{M\to+\infty} \frac{\phi^p_{\mathrm{hom}}(M;\nu)}{pM} = \phi^1(\nu).$$

Proof. Throughout this proof we assume without loss of generality that $L \in \mathbb{N}$ and we set $\mathbb{Z}^2_M = \mathbb{Z}^2 \times \{0, \dots, M\}$. Fix $\nu \in S^1$ (we will drop the dependence on ν for several quantities). We separately show two inequalities. For the moment we also fix M. Consider a sequence of minimizing configurations u_N such that $\lim_N (1/N) E^1_{[0,M]}(u_N, Q_\nu(0,N)) = \phi^{1,M}(\nu)$. As we show now, we can assume that u_N is a plane-like configuration, as given by Theorem A.3. Indeed, applying that theorem we find a plane-like ground state u_ν for the energy

$$E_M(u, Q_\nu(0,N)) := \sum_{\substack{x\in\mathbb{Z}^2_M\\ P_2(x)\in Q_\nu(0,N)}} \sum_{y\in\mathbb{Z}^2_M} c(x-y)|u(x)-u(y)|.$$

To reduce notation, we set

$$S_\nu(N,\lambda) = \{x \in \mathbb{R}^2 : x \in Q_\nu(0,N),\ \mathrm{dist}(x, \{\nu\}^\perp) \le 4(\lambda+L)\}$$

so that the energy of u_ν is concentrated on $S_\nu(N,\lambda)\times[0,M]$ with $\lambda \le CM$ (see Theorem A.3). For any $N \in \mathbb{N}$ we define two configurations $\bar u_N, \tilde u_N : \mathbb{Z}^2_M \to \{\pm 1\}$ via

$$\bar u_N(x) = \begin{cases} u_{0,\nu}(P_2(x)) & \text{if } \mathrm{dist}(P_2(x), \mathbb{R}^2\backslash Q_\nu(0,N)) \le 2L,\\ u_\nu(x) & \text{otherwise,}\end{cases}$$

$$\tilde u_N(x) = \begin{cases} u_\nu(x) & \text{if } \mathrm{dist}(P_2(x), \mathbb{R}^2\backslash (Q_\nu(0,N)) \le L,\\ u_N(x) & \text{otherwise.}\end{cases}$$

Then $\bar u_N$ is a plane-like configuration whose energy is again concentrated on $S_\nu(N,\lambda)\times[0,M]$. Using the boundary conditions and the finite-range assumptions one can prove that

$$\begin{aligned} E^1_{[0,M]}(u_N, Q_\nu(0,N)) &\le E^1_{[0,M]}(\bar u_N, Q_\nu(0,N)) \le E_M(u_\nu, Q_\nu(0,N)) + CM^2\\ &\le E_M(\tilde u_N, Q_\nu(0,N)) + CM^2 \le E^1_{[0,M]}(u_N, Q_\nu(0,N)) + 2CM^2.\end{aligned}$$

Dividing by N and letting $N \to +\infty$ we see that asymptotically we can replace u_N by the plane-like configuration $\bar u_N$. From now on we denote by $u_{N,M}$ a plane-like minimizer whose energy is concentrated on $S_\nu(N,\lambda)\times[0,M]$ with $\lambda \le CM$ and such that

$$\phi^{1,M}(\nu) = \lim_N \frac{1}{N} E^1_{[0,M]}(u_{N,M}, Q_\nu(0,N)).$$

We extend $u_{N,M}$ to $\mathbb{Z}^3$ setting $u_{N,M}(x) = u_{0,\nu}(P_2(x))$ for $x_3 \notin \{0,\dots,M\}$. For $\delta > 0$ small enough, we separate the contribution of the bottom and the first $M^p_\delta := \lceil (p+\delta)M\rceil$ random layers and estimate the remaining interactions. This leads to

$$\begin{aligned}\frac{1}{M}\phi^p_{\mathrm{hom}}(M;\nu) &\le \frac{1}{M}\liminf_{N\to+\infty}\frac{1}{N}\mathbb{E}\big[E^p_{1,M}(\omega)(u_{N,M^p_\delta},Q_\nu(0,N))\big]\\ &\le \frac{1}{M}\liminf_{N\to+\infty}\frac{1}{N}\mathbb{E}\big[E^1_{[0,M^p_\delta]}(u_{N,M^p_\delta},Q_\nu(0,N))\big]\\ &\qquad\qquad + \frac{C}{M}\limsup_{N\to+\infty}\frac{1}{N}\mathbb{E}\big[\#\big\{x\in\mathcal{L}^M_p(\omega): x\in S_\nu(N,\lambda)\times(M^p_\delta-L,M]\big\}\big]\\ &\le \frac{1}{M}\phi^{1,M^p_\delta}(\nu)+C\mathbb{E}\big[\#\big\{x\in\mathcal{L}^M_p(\omega): x\in\{(0,0)\}\times(M^p_\delta-L,M]\big\}\big]\\ &\le \frac{1}{M}\phi^{1,M^p_\delta}(\nu)+C\sum_{k=M^p_\delta-L}^{M}(k-M^p_\delta+L)\binom{M}{k}p^k(1-p)^{M-k},\end{aligned}$$

where in the last step we have used that the probability of having k points in $\{(0,0)\}\times(M^p_\delta-L,M]$ is the same as having $k+M^p_\delta-L$ successes out of M trials in a Bernoulli experiment. In order to bound the last sum, we use Hoeffding's inequality, which yields, for M large enough depending on L,δ,

$$\mathbb{P}\Big(\sum_{i=1}^M X^p_{(0,0,i)}\ge k+M^p_\delta-L\Big)\le\mathbb{P}\Big(\sum_{i=1}^M X^p_{(0,0,i)}\ge k+\Big(p+\frac{\delta}{2}\Big)M\Big)\le\exp\Big(-2M\Big(\frac{\delta}{2}+\frac{k}{M}\Big)^2\Big).$$

From this bound we infer the estimate

$$\sum_{k=M^p_\delta-L}^{M}(k-M^p_\delta+L)\binom{M}{k}p^k(1-p)^{M-k}\le\sum_{k=1}^M k\exp\big(-\tfrac{1}{2}M\delta^2\big)\exp(-2\delta k).$$

Since the right-hand side vanishes when $M\to+\infty$, by Lemma 7.2 we deduce $\limsup_M(1/M)\phi^p_{\mathrm{hom}}(M;\nu)\le(p+\delta)\phi^1(\nu)$. Since δ was arbitrary, the first inequality is proven.

It remains to show the reverse inequality. Given any admissible function $v_N:\mathcal{L}^M_p(\omega)\to\{\pm1\}$, we can neglect the interactions coming from $Q_\nu(0,N)\times[M^p_{-\delta}+1,M]$, which yields the estimate

$$E^p_{1,M}(\omega)(v_N,Q_\nu(0,N))\ge E^p_{[0,M^p_{-\delta}]}(\omega)(v_N,Q_\nu(0,N)).$$

Minimizing on both sides and dividing by N, we obtain in the limit that

$$\frac{1}{M}\phi^p_{\mathrm{hom}}(M;\nu)\ge\frac{1}{M}\phi^{p,M^p_{-\delta}}(\nu).\tag{7-61}$$

Now the idea is to estimate the error when we replace $\phi^{p,M^p_{-\delta}}(\nu)$ by $\phi^{1,M^p_{-\delta}}(\nu)$. Let u_N be a sequence of plane-like configurations as in the first part of the proof. We also consider an optimal sequence $u^{p,\delta}_N=u^{p,\delta}_N(\omega)$ such that

$$\phi^{p,M^p_{-\delta}}(\nu)=\lim_{N\to+\infty}\frac{1}{N}\mathbb{E}[E^p_{[0,M^p_{-\delta}]}(\omega)(u^{p,\delta}_N,Q_\nu(0,N))].$$

Since the deterministic surface tension dominates the random one, we have

$$\begin{aligned}0 &\le \phi^{1,M^p_{-\delta}}(\nu)-\phi^{p,M^p_{-\delta}}(\nu)=\lim_N \frac{1}{N}\mathbb{E}\big[E^1_{[0,M^p_{-\delta}]}(u_N,S_\nu(N,\lambda))-E^p_{[0,M^p_{-\delta}]}(\omega)(u_N^{p,\delta}(\omega),Q_\nu(0,N))\big]\\ &\le \limsup_N \frac{1}{N}\mathbb{E}\big[E^1_{[0,M^p_{-\delta}]}(u_N^{p,\delta},S_\nu(N,\lambda))-E^p_{[0,M^p_{-\delta}]}(\omega)(u_N^{p,\delta}(\omega),S_\nu(N,\lambda))\big]\\ &\le C\limsup_N \frac{1}{N}\mathbb{E}\big[\#\{x\in\big(S_\nu(\lambda,N)\times[1,M^p_{-\delta}]\big)\cap\mathbb{Z}^3 : x\notin\mathcal{L}^M_p(\omega)\}\big]\\ &\le CM\mathbb{E}\bigg[\max\bigg\{M^p_{-\delta}-\sum_{i=1}^M X^p_{(0,0,i)},0\bigg\}\bigg]\le CM\sum_{k=1}^{M^p_{-\delta}}k\,\mathbb{P}\bigg(M^p_{-\delta}-\sum_{i=1}^M X^p_{(0,0,i)}\ge k\bigg).\end{aligned}$$

Here we used that the number of missing interactions can be estimated by the number of missing lattice points since each point can only interact with finitely many others. Now we apply again Hoeffding's inequality, which yields

$$\mathbb{P}\bigg(M^p_{-\delta}-\sum_{i=1}^M X^p_{(0,0,i)}\ge k\bigg)\le \mathbb{P}\bigg(M\Big(p-\frac{\delta}{2}\Big)-k\ge\sum_{i=1}^M X^p_{(0,0,i)}\bigg)\le \exp\Big(-2M\Big(\frac{\delta}{2}+\frac{k}{M}\Big)^2\Big).$$

We conclude the bound

$$\sum_{k=1}^{M^p_{-\delta}}k\mathbb{P}\bigg(M^p_{-\delta}-\sum_{i=1}^M X^p_{(0,0,i)}\ge k\bigg)\le\sum_{k=1}^{M^p_{-\delta}}k\exp\big(-\tfrac{1}{2}M\delta^2\big)\exp(-2\delta k).$$

Again the right-hand side vanishes when $M\to+\infty$ and thus $\lim_M(1/M)|\phi^{1,M^p_{-\delta}}(\nu)-\phi^{p,M^p_{-\delta}}(\nu)|=0$, so that Lemma 7.2 and (7-61) imply the estimate

$$\liminf_{M\to+\infty}\frac{1}{M}\phi^p_{\mathrm{hom}}(M;\nu)\ge\lim_{M\to+\infty}\frac{1}{M}\phi^{1,M^p_{-\delta}}(\nu)=(p-\delta)\phi^1(\nu).$$

Again the desired estimate follows by the arbitrariness of $\delta>0$. □

Remark 7.4. If we had not included the initial layer $\mathcal{L}^0$, then Proposition 7.3 would still hold. However, then the surface tension may not be related to an appropriate Γ-limit since the compactness of sequences with bounded energy becomes a nontrivial issue. We refer to [Braides and Piatnitski 2012] for a possible approach to this problem in the case of nearest-neighbor interactions and bond-percolation models.

A percolation-type phenomenon. We close this final section with a result on the growth of the averaged surface tension when the number of layers increases. We let $\mathcal{L}^M_p(\omega)$ be defined as in (7-59) but restrict the analysis to nearest-neighbor interactions and make them nonperiodic in the sense that their magnitude is very small when one of the particles belongs to the initial layer $\mathcal{L}^0$. More precisely, given $0<\eta\ll 1$ we consider functions of the form

$$c_\eta(\Delta_2(x,y))=\begin{cases}0 & \text{if } |x-y|>1,\\ \eta & \text{if } |x-y|=1 \text{ and } x_3\cdot y_3=0,\\ c(x-y) & \text{otherwise,}\end{cases}$$

where Δ_2 is defined in Hypothesis 2 and $x \mapsto c(x)$ is strictly positive on the unit circle. Then the coefficients satisfy Hypothesis 2 and fulfill (a slightly weaker version of) Hypothesis 1. We define $E^{p,\eta}_{\varepsilon,M}$ as in (7-60) with c replaced by c_η. According to Theorem 5.8, again there exists the limit

$$\phi^{p,\eta}_{\mathrm{hom}}(M;\nu) := \lim_{t\to+\infty} \frac{1}{t} \inf\bigl\{E^{p,\eta}_{1,M}(\omega)(v, Q_\nu(0,t)) : v(x) = u_{0,\nu}(P_2(x)) \text{ if } \operatorname{dist}(P_2(x), \partial Q_\nu(0,t)) \le 2\bigr\}.$$

In contrast to Proposition 7.3, for this model we also consider the case of small M. We will show that if $p < 1 - p_{\mathrm{site}}$, where p_{site} is the critical site percolation probability on $\mathbb{Z}^2$, then it holds that

$$\phi^{p,\eta}_{\mathrm{hom}}(1;\nu) \le C_p\,\eta,$$

where C_p may blow up only for $p \to 1 - p_{\mathrm{site}}$. Note that we do not claim here that p_{site} is the optimal bound. We can actually improve the result in the sense that for all $M \in \mathbb{N}$ such that $(1-p)^M > p_{\mathrm{site}}$, we have

$$\phi^{p,\eta}_{\mathrm{hom}}(M;\nu) \le C_p\,\eta.$$

This shows that when the probability is very small but finite, the surface tension can be arbitrary small depending on the strength of the interaction in the substrate layer, on the other hand we will establish an analogue of Proposition 7.3 asserting that if the average number of layers increases further, even the normalized surface tension approaches a value independent of η. This result can be interpreted as the equivalent to the percolation phenomenon described in the introduction of the paper for the model without initial layer ($\eta = 0$). Before proving this result, we introduce the typical energy of one slice. Given $q \in (0,1]$ and $u : \mathbb{Z}^2 \to \{\pm 1\}$, we set

$$E^q_{sl}(\omega)(u, A) := \sum_{\substack{x,y \in \mathcal{L}^1_q(\omega)\setminus\mathcal{L}^0 \\ P_2(x), P_2(y) \in A}} c(x-y)|u(x) - u(y)|$$

and omit the dependence on ω if $q = 1$. We further introduce the corresponding surface tension

$$\phi^q_{sl}(\nu) = \lim_{t\to+\infty} \frac{1}{t} \inf\bigl\{E^q_{sl}(\omega)(u, Q_\nu(0,t)) : v(x) = u_{0,\nu}(x) \text{ if } \operatorname{dist}(x, \partial Q_\nu(0,t)) \le 2\bigr\}.$$

Note that the existence of this deterministic limit follows again from the subadditive ergodic theorem as in the proof of Theorem 5.8, since we used the coercivity only for passing from finite range to decaying interactions in Step 4. In general the random variables $\omega \mapsto E^q_{sl}(\omega)(u, A)$ are not defined on the same probability space but we will use them only for slices of the large set $\mathcal{L}^M_p(\omega)$.

Theorem 7.5. *Let $p \in (0,1)$ and $M \in \mathbb{N}$ be such that $(1-p)^M > p_{\mathrm{site}}$. There exists a constant $C_{p,M}$ locally bounded for $(1-p)^M \in (p_{\mathrm{site}}, 1)$ such that*

$$\phi^{p,\eta}_{\mathrm{hom}}(M;\nu) \le C_{p,M}\,\eta.$$

On the other hand, for any $p \in (0,1)$ it holds that

$$\lim_{M\to+\infty} \frac{1}{M} \phi^{p,\eta}_{\mathrm{hom}}(M;\nu) = 2p\bigl((c(e_1) + c(-e_1))|\nu_1| + (c(e_2) + c(-e_2))|\nu_2|\bigr).$$

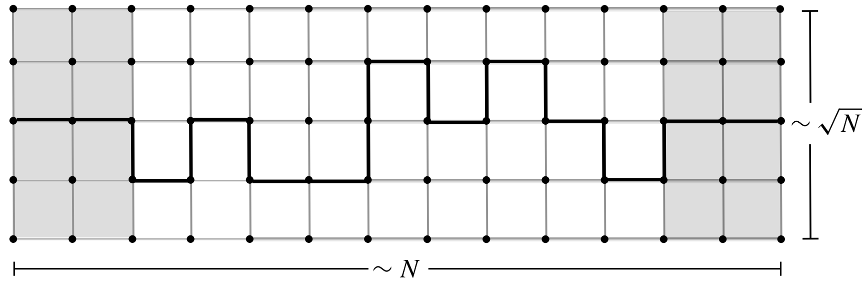

Figure 4. The different sets in the construction of u_N. R_N^+ and R_N^- correspond to the white regions above and below the bold line, respectively. In the light gray region, $u_N(x)$ agrees with $u_{0,e_2}(P_2(x))$.

Proof. In order to prove the first statement, we start with the case $\nu = e_2$ and use results from percolation theory which show that the contribution from the random layers is negligible: for $q := (1-p)^M > p_{\text{site}}$, we consider the so-called Bernoulli site percolation on $\mathbb{Z}^2$; that is, we assign independently a weight $X_i(\omega) \in \{\pm 1\}$ to all the vertices $i \in \mathbb{Z}^2$ such that $\mathbb{P}(X_i = 1) = q$. We say that $i_0, \dots, i_k$ is an occupied path if $|i_n - i_{n+1}| = 1$ and $X_{i_n}(\omega) = 1$ for all $n = 0, \dots, k$. Theorem 11.1 in [Kesten 1982] yields that there exist universal constants c_j, d_j such that

$$\begin{aligned}\mathbb{P}\big(&\text{at least } c_1(q-p_{\text{site}})^{d_1} n \text{ disjoint occupied paths from}\\ &\{0\}\times[0,n] \text{ to } \{m\}\times[0,n] \text{ and contained in } [0,m]\times[0,n] \text{ exist}\big) \geq 1 - c_2(m+1)\exp(-c_3(q-p_{\text{site}})^{d_2} n).\end{aligned}$$

Given $N \in \mathbb{N}$, we first combine this estimate with the Borel–Cantelli lemma and, using stationarity, we obtain that for almost every $\omega \in \Omega$ there exists $N_0 = N_0(\omega)$ such that for all $N \geq N_0$ we find at least $c_1(q - p_{\text{site}})^{d_1} 2\sqrt{N}$ disjoint occupied paths connecting the vertical boundary segments of the rectangle

$$R_N := \big[-\lfloor \tfrac{1}{2}N \rfloor + 2, \lfloor \tfrac{1}{2}N \rfloor - 2\big] \times \big[-\lceil \sqrt{N} \rceil, \lceil \sqrt{N} \rceil\big].$$

As the paths are disjoint and are contained in R_N, at least one of them uses at most $(2/c_1)(q - p_{\text{site}})^{-d_1} N$ vertices. Now we come back to the actual proof. By the definition of the random lattice in (7-59), using the above considerations in the layer $\mathbb{Z}^2 \times \{1\}$, for $N \geq N_0$ we can find a path connecting the vertical boundary segments of the rectangle $R_N \times \{1\}$, contained in $R_N \times \{1\}$, using at most $c_{p,M} N$ vertices with none of them belonging to $\mathcal{L}_p^M(\omega)$. This path separates $R_N \times \{1\}$ into two subregions $R_N^- \times \{1\}$ and $R_N^+ \times \{1\}$. As depicted in Figure 4, for $N \geq N_0$ we define a (random) configuration $u_N : \mathcal{L}_p^M(\omega) \to \{\pm 1\}$ as

$$u_N(x) = \begin{cases} u_{0,e_2}(P_2(x)) & \text{if } P_2(x) \notin R_N, \\ +1 & \text{if } P_2(x) \in R_N^+, \\ -1 & \text{otherwise.} \end{cases}$$

Up to possibly exchanging the roles of $R_N^{\pm}$, we can assume that $u_N \in \mathcal{PC}^2_{1,u_{0,e_2}}(\omega, Q_{e_2}(0, N))$. Hence by the definition of $\phi_{\text{hom}}^{p,\eta}(e_2)$ and the fact that u_N does not depend on the z-direction, it holds that

$$\begin{aligned}\phi_{\mathrm{hom}}^{p,\eta}(e_2) &\le \liminf_{N\to+\infty}\frac{1}{N}E_{1,M}^{p,\eta}(\omega)(u_N, Q_{e_2}(0,N))\\ &\le \limsup_{N\to+\infty}\frac{1}{N}\sum_{\substack{x,y\in Q_{e_2}(0,N)\cap\mathbb{Z}^2\\ |x-y|=1}}\eta|u_N(x)-u_N(y)|\\ &\qquad+\limsup_{N\to+\infty}\frac{1}{N}\sum_{k=1}^{M}\sum_{\substack{x,y\in\mathcal{L}_p^M(\omega)\\ x,y\in Q_{e_2}(0,N)\times\{k\}}} c(x-y)|u_N(x)-u_N(y)|.\end{aligned}\tag{7-62}$$

We now estimate each of the two terms on the right-hand side. Concerning the second one, we observe that if $x, y \in (Q_{e_2}(0, N) \times \{k\}) \cap \mathcal{L}_p^M(\omega)$ are such that $|x-y| = 1$ and $u_N(x) \neq u_N(y)$, then either $P_2(x), P_2(y) \in \pm\frac{1}{2}Ne_1 + \big([-4,4]\times[-2\sqrt{N}, 2\sqrt{N}]\big)$ or, without loss of generality, $P_2(x) \in R_N^-$ and $P_2(y) \in R_N^+$. In the second case, we note that either $(P_2(x), 1)$ or $(P_2(y), 1)$ has to be a vertex of the path constructed above; hence either $x \notin \mathcal{L}_p^M(\omega)$ or $y \notin \mathcal{L}_p^M(\omega)$. We then rule out the existence of such interactions and we may bound the second term via

$$\limsup_{N\to+\infty}\frac{1}{N}\sum_{k=1}^{M}\sum_{\substack{x,y\in\mathcal{L}_p^M(\omega)\\ x,y\in Q_{e_2}(0,N)\times\{k\}}} c(x-y)|u_N(x)-u_N(y)| \le \limsup_{N\to+\infty}\frac{CM}{\sqrt{N}} = 0. \tag{7-63}$$

Applying the same arguments for the first term, we may use the fact that the separating path uses at most $c_{p,M}N$ vertices and we deduce that

$$\limsup_{N\to+\infty}\frac{1}{N}\sum_{\substack{x,y\in Q_{e_2}(0,N)\cap\mathbb{Z}^2\\ |x-y|=1}}\eta|u_N(x)-u_N(y)| \le 4c_{p,M}\eta.$$

From this estimate, the first claim in the case $\nu = e_2$ follows by (7-62) and (7-63). The above argument can be adapted to the cases $\nu = -e_2$ and $\nu = \pm e_1$. By L^1-lower semicontinuity, the one-homogeneous extension of $\phi_{\mathrm{hom}}^{p,\eta}$ must be convex; see [Ambrosio and Braides 1990b]. For general $\nu \in S^1$ the claim then follows upon multiplying the constant by a factor $\sqrt{2}$.

In order to prove the second claim, we need to show two inequalities. Given a sequence of admissible configurations u_N such that

$$\lim_N \frac{1}{N}E_{sl}^1(u_N, Q_\nu(0,N)) = \phi_{\mathrm{sl}}^1(\nu),$$

we define an admissible configuration $\bar{u}_N : \mathcal{L}_p^M(\omega) \to \{\pm 1\}$ via

$$\bar{u}_N(x) = u_N(P_2(x)).$$

Arguing as in the proof of Proposition 7.3, we may assume that u_N is a plane-like configuration and its energy is concentrated in a stripe

$$S_\nu(N,\lambda) = \{x\in\mathbb{R}^2 : x\in Q_\nu(0,N),\ \mathrm{dist}(x,\{\nu\}^\perp) \le 4(\lambda+1)\},$$

where now λ is independent of N, M. By definition and the fact that $\bar{u}_N$ gives no interaction in the z-direction, we obtain that for any $\delta > 0$ small enough

$$\begin{aligned}
\frac{\phi^{p,\eta}_{\mathrm{hom}}(M;\nu)}{M} &\le \frac{1}{M}\liminf_{N\to+\infty}\frac{1}{N}\mathbb{E}[E^{p,\eta}_{1,M}(\omega)(\bar{u}_N, Q_\nu(0,N))]\\
&\le \left(\liminf_{N\to+\infty}\frac{1}{M}\sum_{k=1}^{M}\frac{1}{N}\mathbb{E}[E^{p_k}_{sl}(\omega)(u_N, Q_\nu(0,N))]\right)+\frac{C}{M}\limsup_{N\to+\infty}\frac{1}{N}\#\{z\in\mathbb{Z}^2\cap S_\nu(N,\lambda)\}\\
&\le \liminf_{N\to+\infty}\frac{1}{N}\left((p+\delta)E^1_{sl}(u_N, Q_\nu(0,N))+\frac{1}{M}\sum_{k>\lfloor(p+\delta)M\rfloor}^{M}\mathbb{E}[E^{p_k}_{sl}(\omega)(u_N, Q_\nu(0,N))]\right)+\frac{C\lambda}{M}\\
&= (p+\delta)\phi^1_{\mathrm{sl}}(\nu)+\sup_{k>\lfloor(p+\delta)M\rfloor}\liminf_{N\to+\infty}\frac{1}{N}\mathbb{E}[E^{p_k}_{\mathrm{sl}}(\omega)(u_N, Q_\nu(0,N))]+\frac{C\lambda}{M},
\end{aligned}$$

where $p_k = \sum_{l=k}^{M}\binom{M}{l}p^l(1-p)^{M-l}$ is the probability of having at least k successes out of M trials in a Bernoulli experiment. Note that here the new random variables are indeed defined on the same probability space and are coupled to the variables generating the stochastic lattice $\mathcal{L}^M_p(\omega)$. As λ is independent of M, the third term vanishes when $M\to+\infty$, so that we are left to show that also the second one converges to zero. In order to estimate the second term we use the fact that u_N is a plane-like configuration, so that

$$\frac{1}{N}\mathbb{E}[E^{p_k}_{\mathrm{sl}}(\omega)(u_N, Q_\nu(0,N))] = \frac{1}{N}\mathbb{E}[E^{p_k}_{\mathrm{sl}}(\omega)(u_N, S_\nu(N,\lambda))] \le p_k C\lambda.$$

For any $k > \lfloor(p+\delta)M\rfloor$, by the law of large numbers it holds that $p_k \to 0$ when $M\to+\infty$. Hence we deduce $\limsup_M (1/M)\phi^{p,\eta}_{\mathrm{hom}}(M;\nu) \le (p+\delta)\,\phi^1_{\mathrm{sl}}(\nu)$. As $\delta > 0$ was arbitrary, we finally obtain

$$\limsup_M \frac{1}{M}\phi^{p,\eta}_{\mathrm{hom}}(M;\nu) \le p\,\phi^1_{\mathrm{sl}}(\nu).$$

We next show the reverse inequality. Given any admissible function $\bar{u}_N : \mathcal{L}^M_p(\omega)\to\{\pm1\}$ we can neglect the interactions in the z-direction and the lowest layer $\mathcal{L}^0$ and obtain the estimate

$$E^{p,\eta}_{1,M}(\omega)(\bar{u}_N, Q_\nu(0,N)) \ge \sum_{k=1}^{M} E^{p_k}_{\mathrm{sl}}(\omega)(\bar{u}_N(\,\cdot\,,k), Q_\nu(0,N)) \ge \sum_{k=1}^{\lceil(p-\delta)M\rceil} E^{p_k}_{\mathrm{sl}}(\omega)(\bar{u}_N(\,\cdot\,,k), Q_\nu(0,N)).$$

Since $\bar{u}_N(\,\cdot\,,k)$ fulfills the correct boundary condition in every layer, we deduce that

$$\frac{1}{M}\phi^{p,\eta}_{\mathrm{hom}}(M;\nu) \ge (p-\delta)\inf_{k\le\lceil(p-\delta)M\rceil}\phi^{p_k}_{\mathrm{sl}}(\nu).$$

Again by the law of large numbers for an independent Bernoulli experiment it remains to show that the function $q\mapsto\phi^q_{sl}(\nu)$ is continuous in $q=1$, which means we can pass from a random to a deterministic lattice. This will be the last step.

In order to prove continuity let u_N be a plane-like sequence of configurations as in the first part of the proof and consider an optimal sequence $u^q_N(\omega)$ such that

$$\phi^q_{\mathrm{sl}}(\nu) = \lim_{N\to+\infty}\frac{1}{N}\mathbb{E}[E^q_{\mathrm{sl}}(\omega)(u^q_N(\omega), Q_\nu(0,N))].$$

Similar to the proof of Proposition 7.3 we obtain

$$\begin{aligned}0 \le \phi^1_{\mathrm{sl}}(\nu) - \phi^q_{\mathrm{sl}}(\nu) &= \lim_N \frac{1}{N}\mathbb{E}\big[E^1_{\mathrm{sl}}(u_N, S_\nu(\lambda, N)) - E^q_{\mathrm{sl}}(\omega)(u^q_N(\omega), Q_\nu(0, N))\big] \\ &\le \limsup_N \frac{1}{N}\mathbb{E}\big[E^1_{\mathrm{sl}}(u^q_N(\omega), S_\nu(\lambda, N)) - E^q_{sl}(\omega)(v^q_N(\omega), S_\nu(\lambda, N))\big] \\ &\le C \lim_N \frac{1}{N}\mathbb{E}\big[\#\{z \in (S_\nu(\lambda, N) \cap \mathbb{Z}^2) \times \{1\} : z \notin \mathcal{L}^1_q(\cdot)\}\big] = C(1-q)\lambda.\end{aligned}$$

The estimate above clearly implies convergence of the surface tensions when $q \to 1$, which shows that $\limsup_M (1/M)\phi^{p,\eta}_{\mathrm{hom}}(M; \nu) \ge p\,\phi^1_{\mathrm{sl}}(\nu)$.

It remains to identify $\phi^1_{\mathrm{sl}}(\nu)$. We just sketch the argument. Any admissible configuration asymptotically has an interface containing at least $|\nu_1|$ interactions along the two directions $\pm e_1$ and $|\nu_2|$ interactions along the directions $\pm e_2$. Since any pair of interacting points is counted twice with reversing direction and $|u(x) - u(y)| \in \{0, 2\}$, we find that $\phi^1_{\mathrm{sl}}(\nu) \ge 2(c(e_1) + c(-e_1))|\nu_1| + 2(c(e_2) + c(-e_2))|\nu_2|$. On the other hand, a suitable discretization of a plane attains this value; hence

$$\phi^1_{\mathrm{sl}}(\nu) = 2(c(e_1) + c(-e_1))|\nu_1| + 2(c(e_2) + c(-e_2))|\nu_2|. \qquad \square$$

Appendix A: Plane-like minimizers for one-periodic dimension-reduction problems

We prove that the results about plane-like minimizers for periodic interactions in [Caffarelli and de la Llave 2005] can be extended to dimension-reduction problems. We restrict the analysis to one-periodic interactions, which is the case when the coefficients depend only on the difference, as in Hypothesis 2. Moreover, we focus on the physical case of reducing from three dimensions to two dimensions. To fix notation, for any set $\Gamma \subset \mathbb{Z}^2$, we write $\Gamma_M = \Gamma \times (\mathbb{Z} \cap [0, M])$. In contrast to the main part of this paper, here we consider an interaction energy that takes into account also interactions outside the domain. To be more precise, given $u : \mathbb{Z}^2_M \to \{\pm 1\}$ we investigate finite-range energies of the form

$$E_M(u, \Gamma) = \sum_{x \in \Gamma_M} \sum_{y \in \mathbb{Z}^2_M} c(x - y)|u(x) - u(y)|,$$

where the coefficients fulfill the following assumptions:

(i) $0 \le c(z) \le C$ for all $z \in \mathbb{R}^3$ and $\min_i c(\pm e_i) \ge c_0 > 0$.

(ii) There exists $L > 0$ such that $c(z) = 0$ for all $|z| \ge L$.

Before stating and proving the main theorem we need some definitions.

Definition A.1. We say that $u : \mathbb{Z}^2_M \to \{\pm 1\}$ is a ground state for the energy E_M whenever $E_M(u, \Gamma) \le E_M(u, \Gamma)$ for all finite sets $\Gamma \subset \mathbb{Z}^2$ and all $v : \mathbb{Z}^2_M \to \{\pm 1\}$ such that $u = v$ on $\{z \in \mathbb{Z}^2_M : \exists z' \in (\mathbb{Z}^2 \backslash \Gamma)_M$ with $|z - z'| \le L\}$.

Remark A.2. When u and Γ are such that $E_M(u, \Gamma) \le E_M(v, \Gamma)$ for all v such that $u = v$ on $\{z \in \mathbb{Z}^2_M : \exists z' \in (\mathbb{Z}^2 \backslash \Gamma)_M$ with $|z - z'| \le L\}$, the same conclusion holds for every subset $\Gamma' \subset \Gamma$. Indeed, take any v

such that $u = v$ on $\{z \in \mathbb{Z}^2_M : \exists z' \in (\mathbb{Z}^2 \backslash \Gamma')_M \text{ with } |z - z'| \le L\}$. Then for any two points x, y with $x \in (\Gamma \backslash \Gamma')_M$ and $y \in \mathbb{Z}^2_M$ with $|x - y| \le L$, it holds that $u(x) = v(x)$ and $u(y) = v(y)$. Hence it follows that

$$E_M(u, \Gamma') - E_M(v, \Gamma') = E_M(u, \Gamma) - E_M(v, \Gamma) \le 0.$$

Using the same notation as for the stochastic group action, for $k \in \mathbb{Z}^2$ we denote by τ_k the shift operator acting on sets Γ and configurations $u : \mathbb{Z}^2_M \to \{\pm 1\}$ via

$$\tau_k \Gamma = \Gamma + k, \quad \tau_k u(x) = u(x - (k, 0)).$$

Then the following formula holds true:

$$E_M(\tau_k u, \tau_k \Gamma) = E_M(u, \Gamma). \tag{A-64}$$

The remaining part of this appendix will be devoted to the proof of the next theorem.

Theorem A.3. *There exists $\lambda > 0$ such that for all $\nu \in S^1$ there exists a ground state u_ν of E_M such that $u(x) \ne u(y)$ implies* $\operatorname{dist}(x, \{\nu\}^\perp) \le \lambda$. *Such a ground state is called plane-like. Moreover we can choose $\lambda \le CM$ for some constant C independent of ν, M.*

The proof of this theorem is very similar to [Caffarelli and de la Llave 2005; Cozzi et al. 2017]. We first construct a particular minimizer among periodic configurations that enjoys several geometric properties. To this end, we need further notation; see [Caffarelli and de la Llave 2005] for more details. Fix a rational direction $\nu \in S^1 \cap \mathbb{Q}^2$. We define the $\mathbb{Z}$-module $\mathbb{Z}_\nu = \{z \in \mathbb{Z}^2 : \langle z, \nu \rangle = 0\}$ and, given $m \in \mathbb{N}$, we let $\mathcal{F}_{m,\nu}$ be any fundamental domain of the quotient $\mathbb{Z}^2_{/m\mathbb{Z}_\nu}$; that is, for every $z \in \mathbb{Z}^2$ there exist unique $z_1 \in m\mathbb{Z}_\nu$ and $z_2 \in \mathcal{F}_{m,\nu}$ such that $z = z_1 + z_2$. Given real numbers θ and λ, with $\theta < \lambda$, we further introduce

$$\mathcal{F}^{\theta,\lambda}_{m,\nu} = \{z \in \mathcal{F}_{m,\nu} : \langle \nu, z \rangle \in [\theta, \lambda]\}.$$

Now we define an admissible class of periodic configurations: A function $u : \mathbb{Z}^2_M \to \{\pm 1\}$ is called (m, ν)-*periodic* if $u(x) = u(x + m(z, 0))$ for every $x \in \mathbb{Z}^2_M$ and every $z \in \mathbb{Z}_\nu$. We set

$$\mathcal{A}^{\theta,\lambda}_{m,\nu} = \big\{u : u \text{ is } (m, \nu)\text{-periodic}, \ u = +1 \text{ if } \langle P_2(z), \nu \rangle < \theta, \ u(z) = -1 \text{ if } \langle P_2(z), \nu \rangle > \lambda\big\}.$$

We start with a very elementary lemma that shows how for periodic functions any translation gives the same energy.

Lemma A.4. *Let u be (m, ν)-periodic and $k \in \mathbb{Z}^2$. Then it holds that*

$$E_M(\tau_k u, \mathcal{F}_{m,\nu}) = E_M(u, \mathcal{F}_{m,\nu}).$$

Proof. Given $x \in (\tau_{-k} \mathcal{F}_{m,\nu})_M$, we find $z_1(x) \in m\mathbb{Z}_\nu$ and $z_2(x) \in \mathcal{F}_{m,\nu}$ such that $P_2(x) = z_1(x) + z_2(x)$. By (m, ν)-periodicity, for any $y \in \mathbb{Z}^2_M$ it holds that

$$|u(x) - u(y)| = |u(x - (z_1(x), 0)) - u(y - (z_1(x), 0))|,$$
$$c(x - y) = c(x - (z_1(x), 0) - y + (z_1(x), 0)).$$

Now assume that there exist another $x' \in (\tau_{-k}\mathcal{F}_{m,\nu})_M\setminus\{x\}$ with $\langle x-x', e_3\rangle = 0$ and $z_2(x) = z_2(x')$. Then $\tau_k P_2(x) - \tau_k P_2(x') = z_1(x) - z_1(x') \in m\mathbb{Z}_\nu\setminus\{(0,0)\}$. As $\tau_k P_2(x), \tau_k P_2(x') \in \mathcal{F}_{m,\nu}$, this contradicts the fact that $\mathcal{F}_{m,\nu}$ is a fundamental domain. Using (A-64) we conclude by comparison that

$$E_M(\tau_k u, \mathcal{F}_{m,\nu}) = E_M(u, \tau_{-k}\mathcal{F}_{m,\nu}) \leq E_M(u, \mathcal{F}_{m,\nu}).$$

Applying the above inequality to τ_{-k} and $\tilde{u} := \tau_k u$, which is also (m, ν)-periodic, we obtain the claim. □

We define the class of minimizers for the energy $E_M(\,\cdot\,, \mathcal{F}_{m,\nu})$ on $\mathcal{A}^{\theta,\lambda}_{m,\nu}$ via

$$\mathcal{M}^{\theta,\lambda}_{m,\nu} = \big\{u \in \mathcal{A}^{\theta,\lambda}_{m,\nu} : E_M(u, \mathcal{F}_{m,\nu}) \leq E_M(v, \mathcal{F}_{m,\nu}) \text{ for all } v \in \mathcal{A}^{\theta,\lambda}_{m,\nu}\big\}.$$

As the set $\mathcal{A}^{\theta,\lambda}_{m,\nu}$ is finite, the class of minimizers is nonempty. Next we define the so-called infimal minimizer, which has several useful properties:

$$u^{\theta,\lambda}_{m,\nu} = \min\{u \in \mathcal{M}^{\theta,\lambda}_{m,\nu}\} \in \mathcal{A}^{\theta,\lambda}_{m,\nu}.$$

We next show that the infimal minimizer also belongs to the class of minimizers. This follows from the following elementary observation; see Lemma 2.1 and also Lemma 2.3 in [Cozzi et al. 2017].

Lemma A.5. *Given any $u : \mathbb{Z}^2_M \to \{\pm 1\}$ and $\Gamma \in \mathbb{Z}^2$ finite, it holds that*

$$E_M(\min\{u, v\}, \Gamma) + E_M(\max\{u, v\}, \Gamma) \leq E_M(u, \Gamma) + E_M(v, \Gamma).$$

Iterating the above lemma finitely many times we find that $u^{\theta,\lambda}_{m,\nu} \in \mathcal{M}^{\theta,\lambda}_{m,\nu}$.

We now turn to the first property of the infimal minimizer. This is the so-called absence of symmetry breaking, which says that the infimal minimizer does not depend on the length m of the period.

Lemma A.6. *For any $m \in \mathbb{N}$ it holds that $u^{\theta,\lambda}_{m,\nu} = u^{\theta,\lambda}_{1,\nu}$.*

Proof. We define an auxiliary configuration via $u = \min\{\tau_k u^{\theta,\lambda}_{m,\nu} : k \in \mathbb{Z}_\nu\}$. By elementary arguments it follows that $u \in \mathcal{A}^{\theta,\lambda}_{1,\nu}$, while Lemma A.4 implies that $\tau_k u^{\theta,\lambda}_{m,\nu} \in \mathcal{M}^{\theta,\lambda}_{m,\nu}$ and by iterating Lemma A.5 we obtain $u \in \mathcal{M}^{\theta,\lambda}_{m,\nu}$. Since $u \leq u^{\theta,\lambda}_{m,\nu}$, by the definition of infimal minimizer we obtain $u = u^{\theta,\lambda}_{m,\nu}$. Moreover, as u and $u^{\theta,\lambda}_{1,\nu}$ are both $(1, \nu)$-periodic it follows that

$$E_M(u, \mathcal{F}_{1,\nu}) = \frac{1}{m} E_M(u, \mathcal{F}_{m,\nu}) \leq \frac{1}{m} E_M(u^{\theta,\lambda}_{1,\nu}, \mathcal{F}_{m,\nu}) = E_M(u^{\theta,\lambda}_{1,\nu}, \mathcal{F}_{1,\nu}). \tag{A-65}$$

In particular we deduce that $u \in \mathcal{M}^{\theta,\lambda}_{1,\nu}$ and thus $u \geq u^{\theta,\lambda}_{1,\nu}$. On the other hand, (A-65) must be an equality, so that $u^{\theta,\lambda}_{1,\nu} \in \mathcal{M}^{\theta,\lambda}_{m,\nu}$ and therefore $u^{\theta,\lambda}_{1,\nu} \geq u$. This proves the claim. □

We next establish the so-called Birkhoff property of the infimal minimizer, which will be the main ingredient for the proof of Theorem A.3.

Lemma A.7. *Let $k \in \mathbb{Z}^2$. Then $\tau_k u^{\theta,\lambda}_{1,\nu} \leq u^{\theta,\lambda}_{1,\nu}$ if $\langle k, \nu\rangle \leq 0$ and $\tau_k u^{\theta,\lambda}_{1,\nu} \geq u^{\theta,\lambda}_{1,\nu}$ if $\langle k, \nu\rangle \geq 0$.*

Proof. We start with the case $\langle k, \nu\rangle \leq 0$ and define the two configurations $m = \min\{u^{\theta,\lambda}_{1,\nu}, \tau_k u^{\theta,\lambda}_{1,\nu}\}$ and $M = \max\{u^{\theta,\lambda}_{1,\nu}, \tau_k u^{\theta,\lambda}_{1,\nu}\}$. By elementary considerations one can prove $m \in \mathcal{A}^{\theta+\langle k,\nu\rangle,\lambda+\langle k,\nu\rangle}_{1,\nu}$ and $M \in \mathcal{A}^{\theta,\lambda}_{1,\nu}$. Using Lemma A.5 we obtain

$$E_M(m, \mathcal{F}_{1,\nu}) + E_M(u^{\theta,\lambda}_{1,\nu}, \mathcal{F}_{1,\nu}) \leq E_M(m, \mathcal{F}_{1,\nu}) + E_M(M, \mathcal{F}_{1,\nu}) \leq E_M(\tau_k u^{\theta,\lambda}_{1,\nu}, \mathcal{F}_{1,\nu}) + E_M(u^{\theta,\lambda}_{1,\nu}, \mathcal{F}_{1,\nu}),$$

which yields

$$E_M(m, \mathcal{F}_{1,\nu}) \le E_M(\tau_k u^{\theta,\lambda}_{1,\nu}, \mathcal{F}_{1,\nu}).$$

We claim that $\tau_k u^{\theta,\lambda}_{1,\nu} = u^{\theta+\langle k,\nu\rangle,\lambda+\langle k,\nu\rangle}_{1,\nu}$. Indeed, as $\tau_k u^{\theta,\lambda}_{1,\nu} \in \mathcal{A}^{\theta+\langle k,\nu\rangle,\lambda+\langle k,\nu\rangle}_{1,\nu}$, this configuration is admissible and minimality follows by Lemma A.4. Now assume it is not the infimal minimizer; then also $u^{\theta,\lambda}_{1,\nu}$ is not the infimal minimizer, as we could construct a smaller one by translation of the other infimal minimizer.

By definition of the infimal minimizer we infer that $m \ge \tau_k u^{\theta,\lambda}_{1,\nu}$, which proves the claim by definition of m. The case $\langle k, \nu\rangle \ge 0$ follows upon applying the translation τ_k to the inequality $\tau_{-k} u^{\theta,\lambda}_{1,\nu} \le u^{\theta,\lambda}_{1,\nu}$, which holds by the first part of the proof. □

In the next lemma we deduce a powerful property of configurations fulfilling the Birkhoff property.

Lemma A.8. *Let $u : \mathbb{Z}^2_M \to \{\pm 1\}$ satisfy the* Birkhoff property *with respect to $\nu \in S^1 \cap \mathbb{Q}^2$; that is, $\tau_k u \le u$ if $\langle k, \nu\rangle \le 0$, and $\tau_k u \ge u$ if $\langle k, \nu\rangle \ge 0$. Assume further that $u(x_0) = -1$ for some $x_0 \in \mathbb{Z}^2_M$. Then $u(x) = -1$ for all $x \in \mathbb{Z}^2_M$ such that $\langle x - x_0, e_3\rangle = 0$ and $\langle P_2(x - x_0), \nu\rangle \ge 0$.*

Proof. Every such x can be written as $x = x_0 - (k, 0)$ with $k \in \mathbb{Z}^2$ such that $\langle k, \nu\rangle \le 0$. Hence Lemma A.7 implies that $u(x) = \tau_k u(x_0) \le u(x_0) = -1$, so that $u(x) = -1$. □

We are now in a position to prove that the infimal minimizer becomes unconstrained when we take $\theta = 0$ and λ large enough. To reduce notation, from now on we set $u^\lambda_\nu := u^{0,\lambda}_{1,\nu}$.

Lemma A.9. *There exists $\lambda_0 > 0$ (depending on M in such a way that $\lambda_0 \le CM$) such that for all $\lambda \ge \lambda_0$ it holds that $u^\lambda_\nu(x) = -1$ for all $x \in \mathbb{Z}^2_M$ such that $\langle P_2(x), \nu\rangle \ge \lambda - \sqrt{2}$.*

Proof. By Lemma A.8 it is enough to show that for large enough λ, in every layer $\mathbb{Z}^2 \times \{l\}$ with $l \in \{0, \dots, M\}$ there exists some x_l such that $\langle P_2(x_l), \nu\rangle \le \lambda - \sqrt{2}$ and $u^\lambda_\nu(x_l) = -1$. We will show that this is always the case provided λ is large enough.

Assume that there exists a layer $\mathbb{Z}^2 \times \{l\}$ such that $u^\lambda_\nu(x) = 1$ for all $x \in \mathbb{Z}^2 \times \{l\}$ with $\langle P_2(x), \nu\rangle \le \lambda - \sqrt{2}$. We argue that in this case there must exists a second layer $\mathbb{Z}^2 \times \{l'\}$ and a point $x_{l'} \in \mathbb{Z}^2 \times \{l'\}$ with $\langle P_2(x_{l'}), \nu\rangle \le \sqrt{2}$ and $u^\lambda_\nu(x_{l'}) = -1$. Indeed, if this would be false, then the function $\tau_k u^\lambda_\nu$ with any $k \in \{0, \pm 1\}^2$ such that $\langle k, \nu\rangle < 0$ fulfills $\tau_k u^\lambda_\nu \in \mathcal{A}^{0,\lambda}_{1,\nu}$. By Lemma A.7 we further know that $\tau_k u^\lambda_\nu \le u^\lambda_\nu$. On the other hand, by Lemma A.4 we have $\tau_k u^\lambda_\nu \in \mathcal{M}^{0,\lambda}_{1,\nu}$; hence by the definition of infimal minimizer we obtain $\tau_k u^\lambda_\nu = u^\lambda_\nu$. This contradicts the boundary conditions by the choice of k. Now applying Lemma A.8 in the second layer $\mathbb{Z}^2 \times \{l'\}$, we obtain $u^\lambda_\nu(x) = -1$ for all $x \in \mathbb{Z}^2 \times \{l'\}$ such that $\langle P_2(x), \nu\rangle \ge \sqrt{2}$. As we will see now, for fixed M this will cost too much energy.

Without loss of generality we assume that $l > l'$, the other case can be treated almost the same way. For every $r \in \{1, \dots, M\}$ there exists $x \in \mathbb{Z}^2 \times \{r\}$ such that $u^\lambda_\nu(x_r) = -1$. Let x_r be one of such points that minimizes $\langle P_2(x), \nu\rangle$ among all such points. According to Lemma A.8 we obtain $u^\lambda_\nu(x) = -1$ for all $x \in \mathbb{Z}^2 \times \{r\}$ with $\langle P_2(x), \nu\rangle \ge \langle P_2(x_r), \nu\rangle =: p_r$. Note that

$$\left|\sum_{r=l'}^{l-1} (p_{r+1} - p_r)\right| \ge \lambda - 2\sqrt{2}. \tag{A-66}$$

On the other hand, just counting the interactions between neighboring layers, we obtain by the coercivity of the interactions and (A-66) that

$$E_M(u_\nu^\lambda, \mathcal{F}_{1,\nu}) \geq c\sum_{r=1}^{M}|p_r - p_{r-1}| \geq c(\lambda - 2\sqrt{2}).$$

Testing a discretized plane as a possible minimizer, by the finite-range assumption we know an a priori bound of the form $E_M(u_\nu^\lambda, \mathcal{F}_{1,\nu}) \leq CM$. Hence our assumption can only hold as long as $\lambda \leq CM$ for some constant C depending neither on ν nor on M and the claim follows upon setting $\lambda_0 = 2CM$. □

The next (and last) lemma bounds the oscillation of the jump set of the infimal minimizer $u_\nu^{\lambda_0}$.

Lemma A.10. *Let λ_0 be as in Lemma A.9. Then $u_\nu^{\lambda_0} \in \mathcal{M}_{m,\nu}^{-n,\lambda_0+n}$ for any $n, m \in \mathbb{N}$.*

Proof. We first claim that $u_\nu^{\lambda_0} = u_\nu^{\lambda_0+l}$ for any $l \in \mathbb{N}$. This will be done iteratively. First note that for any $\lambda \geq \lambda_0$ it holds that $u_\nu^\lambda \in \mathcal{A}_{1,\nu}^{0,\lambda+1}$ and by Lemma A.9 it also holds that $u_\nu^{\lambda+1} \in \mathcal{A}_{1,\nu}^{0,\lambda}$. Then

$$E_M(u_\nu^{\lambda+1}, \mathcal{F}_{1,\nu}) = E_M(u_\nu^\lambda, \mathcal{F}_{1,\nu})$$

and both are infimal minimizers. Hence they must agree. This proves the first claim.

Give an arbitrary configuration $v \in \mathcal{A}_{m,\nu}^{-n,\lambda_0+n}$, we choose a vector $k \in \mathbb{Z}^2$ such that $\langle k, \nu\rangle \geq n$ and $\langle k, \nu\rangle \in \mathbb{N}$. Then

$$\tau_k v \in \mathcal{A}_{m,\nu}^{-n+\langle k,\nu\rangle,\lambda_0+n+\langle k,\nu\rangle} \subset \mathcal{A}_{m,\nu}^{0,\lambda_0+n'}$$

with $n' \in \mathbb{N}$. Using the first claim and Lemmata A.4 and A.6 we obtain

$$E_M(u_\nu^{\lambda_0}, \mathcal{F}_{m,\nu}) \leq E_M(\tau_k v, \mathcal{F}_{m,\nu}) = E_M(v, \mathcal{F}_{m,\nu}).$$

As $u_\nu^{\lambda_0} \in \mathcal{A}_{m,\nu}^{-n,\lambda_0+n}$ we proved the claim. □

Proof of Theorem A.3. First assume that $\nu \in S^1 \cap \mathbb{Q}^2$. We show that $u_\nu^{\lambda_0}$ is a ground state. To this end let $\Gamma \subset \mathbb{Z}^2$ be finite and let $v : \mathbb{Z}_M^2 \to \{\pm 1\}$ be such that $v = u_\nu^{\lambda_0}$ on $\{z \in \mathbb{Z}_M^2 : \exists z' \in (\mathbb{Z}^2\backslash\Gamma)_M \text{ with } |z-z'| \leq L\}$. Then we find $m \in \mathbb{N}$ such that, for a suitable fundamental domain, $\Gamma \subset \mathcal{F}_{m,\nu}$. By Lemma A.10 we have $E_M(u_\nu^{\lambda_0}, \mathcal{F}_{m,\nu}) \leq E_M(v, \mathcal{F}_{m,\nu})$ and the claim then follows by Remark A.2.

For general directions $\nu \in S^1$ we argue by approximation. Take a sequence $\nu_j \to \nu$ of rational directions and consider the sequence $u_j := u_{\nu_j}^{\lambda_j}$, where λ_j is uniformly bounded in j. By Tychonoff's theorem we can assume that $u_j \to u$ for some $u : \mathbb{Z}_M^2 \to \{\pm 1\}$. It holds that u is a plane-like configuration. By the definition of the topology, given any finite set $\Gamma \subset \mathbb{Z}^2$ we can find an index j_0 such that $u_j(x) = u(x)$ for all $x \in \Gamma_M$ and all $j \geq j_0$. Since we assume a finite range of interaction, the previous convergence property implies that u is also a ground state. □

Appendix B: Density results for trace-constraints on partitions

In this second appendix we show the density result needed in the proof of Theorem 4.1.

Lemma B.1. *Let $A \Subset B$ both be bounded open sets with Lipschitz boundary. Given $v, w \in \mathrm{BV}(B, \mathcal{S})$ such that $\mathcal{H}^{k-1}(S_w \cap \partial A) = 0$, we set $u = \mathbb{1}_A v + (1 - \mathbb{1}_A)w$. Then there exists a sequence $A_n \Subset A$ of* *sets of finite perimeter such that* $u_n := \mathbb{1}_{A_n} v + (1 - \mathbb{1}_{A_n}) w$ *converges to* u *in* $L^1(B)$ *and additionally* $\mathcal{H}^{k-1}(S_{u_n} \cap B) \to \mathcal{H}^{k-1}(S_u \cap B)$.

Proof. We define the mapping $T : \mathcal{S} \to \mathbb{R}^q$ by $T(s_i) = e_i$. As a special case of Proposition 4.1 in [Schmidt 2015], applied to the bounded BV-function $\alpha := T(w) - T(v)$, for every $\varepsilon > 0$ we find an open set A_ε of finite perimeter such that $A_\varepsilon \Subset A$, $|A \backslash A_\varepsilon| \le \varepsilon$ and

$$\int_{\partial A_\varepsilon} |\alpha^+_{|\partial A_\varepsilon}| \, \mathrm{d}\mathcal{H}^{k-1} \le \int_{\partial A} |\alpha^+_{|\partial A}| \, \mathrm{d}\mathcal{H}^{k-1} + \varepsilon. \tag{B-67}$$

With the same arguments as in [Schmidt 2015], the sets A_ε can be constructed in such a way that for all $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$

$$\{x \in A : \mathrm{dist}(x, \partial A) > \delta\} \subset A_\varepsilon. \tag{B-68}$$

We show that the sets A_ε fulfill the required properties. As a first step we claim that $T(u_\varepsilon)$ converges strictly to $T(u)$. We have that $T(u_\varepsilon)$ converges to $T(u)$ in $L^1(B)$. By lower semicontinuity of the total variation it is enough to show that

$$\limsup_{\varepsilon \to 0} |DT(u_\varepsilon)|(B) \le |DT(u)|(B). \tag{B-69}$$

By definition we have $|DT(u_\varepsilon)|(B \backslash \bar{A}) = |DT(u)|(B \backslash \bar{A})$, so that we can reduce the analysis to $\bar{A}$. By Theorem 3.84 in [Ambrosio et al. 2000] it holds that

$$DT(u_\varepsilon) = DT(v) \llcorner A_\varepsilon^{(1)} + DT(w) \llcorner A_\varepsilon^{(0)} + (T(v)^+_{|\partial A_\varepsilon} - T(w)^-_{|\partial A_\varepsilon}) \otimes \nu \, \mathcal{H}^{k-1} \llcorner \partial A_\varepsilon,$$

where in general $A_\varepsilon^{(t)}$ is defined for $t \in [0, 1]$ via

$$A_\varepsilon^{(t)} = \left\{ x \in \mathbb{R}^k : \lim_{\rho \to 0} \frac{|A_\varepsilon \cap B_\rho(x)|}{|B_\rho(x)|} = t \right\}.$$

Since $A_\varepsilon \Subset A$ and A_ε is open we infer $A_\varepsilon^{(1)} \subset A$ and $A_\varepsilon^{(0)} \subset \mathbb{R}^k \backslash A_\varepsilon$, so that

$$\begin{aligned}
|DT(u_\varepsilon)|(\bar{A}) &\le |DT(v)|(A) + |DT(w)|(\bar{A} \backslash A_\varepsilon) + \int_{\partial A_\varepsilon} |T(v)^+_{|\partial A_\varepsilon} - T(w)^-_{|\partial A_\varepsilon}| \, \mathrm{d}\mathcal{H}^{k-1} \\
&\le |DT(v)|(A) + |DT(w)|(\bar{A} \backslash A_\varepsilon) + \int_{\partial A_\varepsilon} |T(w)^+_{|\partial A_\varepsilon} - T(w)^-_{|\partial A_\varepsilon}| \, \mathrm{d}\mathcal{H}^{k-1} \\
&\qquad + \int_{\partial A_\varepsilon} |T(v)^+_{|\partial A_\varepsilon} - T(w)^+_{|\partial A_\varepsilon}| \, \mathrm{d}\mathcal{H}^{k-1}.
\end{aligned}$$

By the assumption on w we have $|DT(w)|(\partial A) = 0$, so that by (B-68) the second and the third terms vanish when $\varepsilon \to 0$. For the fourth one we use (B-67) and infer

$$\begin{aligned}
\limsup_{\varepsilon \to 0} |DT(u_\varepsilon)|(\bar{A}) &\le |DT(v)|(A) + \int_{\partial A} |T(v)^+_{|\partial A} - T(w)^+_{|\partial A}| \, \mathrm{d}\mathcal{H}^{k-1} \\
&= |DT(v)|(A) + \int_{\partial A} |T(v)^+_{|\partial A} - T(w)^-_{|\partial A}| \, \mathrm{d}\mathcal{H}^{k-1} = |DT(u)|(\bar{A}),
\end{aligned}$$

where we used that inner and outer trace of $T(w)$ agree for $\mathcal{H}^{k-1}$-almost every $x \in \partial A$. By the structure of the set $T(\mathcal{S})$, strict convergence implies that

$$\mathcal{H}^{k-1}(S_{T(u_\varepsilon)} \cap B) = \frac{1}{\sqrt{2}}|DT(u_\varepsilon)| \to \frac{1}{\sqrt{2}}|DT(u)| = \mathcal{H}^{k-1}(S_{T(u)} \cap B).$$

As for every $u \in \mathrm{BV}(B, \mathcal{S})$ it holds that $\mathcal{H}^{k-1}(S_u \cap B) = \mathcal{H}^{k-1}(S_{T(u)} \cap B)$ and also L^1-convergence is conserved, we conclude the proof. □

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